Islands and Icebergs may contribute nothing to the Page curve

Harvendra Singh

Theory Division, Saha Institute of Nuclear Physics (HBNI)
1/AF Bidhannagar, Kolkata 700064, India

Abstract

We study the entanglement entropy of a subsystem in contact with symmetrical bath where the complete system lives on the boundary of AdS3 spacetime. The system-A is taken to be in the middle of the bath system-B and the full system is taken to be some fixed localized region of the boundary 2-dimensional CFT. We generally assume that the d.o.f.s in the total system remain fixed when we vary the size of the bath which is to be guided by the conservation laws. It is found that the island and the subleading (icebergs) contributions are inseparable, and in totality they contribute nothing to the Page-curve of the bath. As such they contribute only to the unphysical branch of the entropy. The quantum entropy formula of the bath may simply be written as $\min\{S[A], S[A] + Const\}$, including for the black holes.
1 Introduction

The holographic principle in string theory [1] has revolutionised our understanding of strongly coupled quantum field theories. We shall mainly focus on the phenomenon of entanglement between two similar looking quantum systems having a common interfaces. The quantum information sharing is a real time phenomenon as the subsystem states always remain entangled. The quantum mechanical states evolve over the time. In quantum systems the information contained in a given state cannot be destroyed, cloned or even mutated. For example, in simple bi-partite systems the information can either be found in one part of the Hilbert space or in the compliment [2, 3]. Generally these exchanges or sharings of quantum information is guided by the unitarity and locality. Under such claims the understanding of the formation of gravitational black holes and subsequent evaporation processes (via Hawking radiation) remains a long puzzle. It is generally believed that the whole process should still be unitary and all information can be recovered after the black hole has fully evaporated. There is a proposal that the entanglement entropy curve for the radiation should bend after the half Page-time is crossed [6]. This certainly holds good when a pure state is divided into two smaller subsystems. But for mixed states, or finite temperature CFT duals of the AdS-black holes, it is not so straightforward to answer this question. However, an important progress has been made in some recent models by coupling holographic CFT to an external bath (or radiation) system, and also by involving nonperturbative techniques such as replica, wormholes and islands [4, 5]. Some answers to these difficult questions have been attempted.

Particularly the recent proposal for generalised entanglement entropy [4] involves an hypothesis of the island ($I$) contribution, including the gravitational contribution from the island boundary ($\partial I$), such that the complete quantum entropy of radiation bath ($B$) can be expressed as

$$S_{\text{Rad}}[B] = \min \left[ \text{ext} \left\{ \frac{\text{Area}(\partial I)}{4G_N} + S[B \cup I] \right\} \right]$$

Thus it is a ‘hybrid model’ as it involves contribution of gravitational ‘islands’ in dual JT gravity. There are both field theoretic and gravitational contributions in it. The formula however suggests that one needs to pick the lowest contribution out of a set of many such possible extremas, which inevitably includes island entropy contributions. The complicated looking formula (1) seemingly reproduces a Page-curve for the radiation entropy [4]. But it ignores contributions of the infinitely many such sub-leading terms which we shall pronounce here together as ‘icebergs’ terms. One of the important feature of above

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1Also see a review on information paradox along different paradigms in [7] and for a list of related references therein; see also [8-11].

2In some extensions of the hybrid models one might also include wormhole contributions, see [5].
proposal entails in the appearance of islands inside the dual (JT) gravity, usually outside of black hole horizon. Although the island itself does not arise by means of a dynamical principle. These are supposed to be associated with the presence of a bath system. The problem we highlight is that other similar subleading contributions of icebergs have been neglected. In this work we present an alternative view that there would exist island term but we must not ignore important icebergs contributions to the entropy. If these are ignored we may end up with an incorrect picture of the Page curve.

We are able to systematically show that the island and icebergs alongwith the leading entropy term altogether add up nicely as a series to give just a constant contribution to the entropy of the bath. These contributions are thus naturally inseparable from each other and also they compensate each other perfectly no matter what their individual contributions might be. We explicitly show this for the limiting case of a quantum-dot at the interface of symmetrical CFT (bath) system. Correspondingly there would be geometric contributions to the total entropy arising out of island and the icebergs, for a subsystem with large bath, such that total entropy of both systems is

\[ S_{total}[AUB] = S_{bath}^{(0)} + S_{island} + S_{icebergs} = \text{Fixed} \]

Furthermore, the quantum formula for the entropy of bath subsystem may simply be written as

\[ S_{quantum}[B] = \min \{ S[A], S[B] \} \]

The above expressions reproduce the Page curve for bath, including for the finite temperature case. The formula is valid for the static (equilibrium) situations. In explicit time dependent cases, like black hole evaporation, if at all there is change in the total system size (i.e. due to net loss or gain of d.o.fs) the picture would be similar at any given instant of time, so it may be applicable for slow processes only.

The paper is organized as follows. In section-2 we introduce the new icebergs contributions and define the generalized entropy formulation for pure AdS$_3$ case. On the boundary we take a finite subsystem in contact with a symmetrical bath. We then discuss a limiting case when subsystem becomes a dot like and in contact with finite size symmetrical CFT bath. The situation emerges like that of $n$ quantum dots and a bath. We extend our results for the black holes case in section-3. The last section-4 contains a brief summary.

## 2 The Islands and Icebergs?

Let us take a subsystem (A) in contact with a symmetrical bath system (B), both of finite size, living on the boundary of AdS$_3$ spacetime. It is important that both systems are
made of identical species, i.e. have identical field content, for simplicity. Consider pure 
\(AdS_3\) spacetime geometry

\[ds^2 = \frac{L^2}{z^2}(-dt^2 + dx^2 + dz^2)\]  

where \(L\) is large radius of curvature. The coordinate ranges are \(-\infty \leq (t, x) \leq \infty\) and 
\(0 \leq z \leq \infty\) represents the full holographic range.\(^3\)

The \(CFT_2\) lives on entire 2-dimensional noncompact \((t, x)\) flat boundary of \(AdS_3\). The 
\(CFT\) bath system lives on the coordinate patches \([- (b + a), -a][a, (b + a)]\) along spatial 
\(x\) direction, with subsystem-A sandwiched in between \([-a, a]\); see the sketches in figure \((\text{I})\). The entire system setup is taken in a particular symmetrical way for the convenience. 
The states of the system-A and the bath are necessarily entangled.

It is clear that the entanglement entropy of an extremal \(CFT_2\) system of net size 
\(l(= 2a + 2b)\) is given by

\[S_{total}[AUB] = \frac{L}{2G_3} \ln \frac{2(a + b)}{\epsilon}\]  

where \(\epsilon\) represents the UV cut-off scale. We shall keep \(l\) sufficiently large and fixed 
through out, and would like to only vary \(b\) between 0 and \(l/2\). We, of course, assume any 
local exchanges, that is gain (loss) of d.o.f. (information) in system-A is compensated by 
equal loss (gain) in the size of bath and vice versa. For dynamical cases there might be 
some definite rate of change, but \(\dot{a} = -2\dot{b}\) would be true due to local conservation laws 
also. The exact rate of loss or gain and the mechanism is not important and the actual nature of 
the physical process is not required here. All we are considering is that local 
conservation laws are at work, for the entire system \(l\). We are not considering explicit 
time dependent processes here. Obviously we are assuming that subsystem and bath are 
made of identical (CFT) fields content. The complementary system includes the patches 
\([b + a, \infty]\) and \([- (b + a), -\infty]\).

We will only study the static situations. Consider now two extreme type of cases 
below.

\(^3\)The Kaluza-Klein compactification on a circle \((x \simeq x + 2\pi R)\) produces a 
\(nearly or rather conformally\) \(AdS_2\) solution, also well known as Jackiw-Teitelboim background \([12, 13]\), which we note down here

\[ds_{JT}^2 = \frac{L^2}{z^2}(-dt^2 + dz^2)\]

\[e^{-2(\phi - \phi_0)} = \sqrt{g_{xx}} = \frac{L}{z}\]  

where \(\phi\) is the 2d dilaton field of JT theory, written in standard convention (effective string coupling 
vanishes near the boundary). The two Newton’s constants get related as \(\frac{2\pi R}{G_3} = \frac{1}{G_2}\), 
with \(G_2\) being dimensionless in 2-dim. The anti-de Sitter solution without a dilaton remains topological spacetime in 
2d with no propagating degrees of freedom.
Case-1: When \( b \ll a \), i.e. the bath size is very small, the HEE of the system-A is given by respective extremal RT surface \([14], [15]\):

\[
S[A] = \frac{L}{2G_3} \ln \left( \frac{l - 2b}{\epsilon} \right)
\]

while that of full bath system-B (on both sides) becomes

\[
S[B] = \frac{L}{G_3} \ln \frac{b}{\epsilon}
\]

This expression is the standard bath entropy and no other extremization is needed. The entropy of the bath will follow the equation \([6]\) as \( b \) grows large. However, at some point for large size bath a different extremum situation emerges, as the new extremal surfaces will appear.
Case-2: For \( b \gg a \) situation, one finds that the bath subsystem-B the entanglement entropy instead becomes
\[
S[B] = \frac{L}{2G_3} \ln \frac{l}{\epsilon} + \frac{L}{2G_3} \ln \frac{l - 2b}{\epsilon} \tag{7}
\]
as there is now an RT surface connecting the two farther ends of the bath system. The first term in (7) is in fact the entropy \( S[AUB] \) of the full system 'subsystem-A and the bath-B'. However \( S[AUB] \) is independent of the individual sizes (\( b \) or \( a \)), and it is a fixed quantity also for given \( l \). While the subsystem-A entropy is given by,
\[
S[A] = \frac{L}{2G_3} \ln \frac{l - 2b}{\epsilon} \tag{8}
\]
note we can write \( (a = \frac{l}{2} - b) \) whenever required. Now since \( l \) is fixed from beginning, both \( S[A] \) and \( S[B] \) above qualify as two independent extremas for bath subsystem-B entropy. Note it is because these entropies in (7) and (8) have same \( b \)-dependences. However it is clear that \( S[B] > S[A] \), therefore the HEE of the large bath should be given by the following quantum minimality principle
\[
S_{quantum}[B] = \min \{S[A], S[B]\} = S[A] \tag{9}
\]
So the Page curve for the bath subsystem-B follows from the principle that the minimum entropy, if there exist many possible extremas having identical local dependences, will only be accepted. This is the main conclusion of eq.(9).

This forms complete result for extremal AdS case (i.e. zero temperature CFT) of quantum dot system in contact with symmetrical bath.

**Finding Island and the Icebergs?**

Note that we have not encountered any island or icebergs so far. Where are these contributions hidden in the above analysis? We have got the entropy Page curve without even knowing these individual quantities. So let us discuss the total entropy of the 'subsystem and its bath' which we have denoted by \( S[AUB] \). Making an expansion of the r.h.s of (11), in small ratio \( (\frac{a}{b}) \), for \( a \ll b \), one can find that
\[
S[AUB] = \frac{L}{2G_3} \left( \ln \frac{2b}{\epsilon} + \frac{a}{b} - \frac{1}{2} \left( \frac{a}{b} \right)^2 + \cdots \right) \equiv S_{bath}^{(0)} + S_{Island} + S_{Icebergs} \tag{10}
\]
where expressions in last equality can be identified as
\[
S_{bath}^{(0)} = \frac{L}{2G_3} \ln \frac{2b}{\epsilon}, \quad S_{Island} = \frac{L}{2G_3} \frac{a}{b}, \quad S_{Icebergs} = -\frac{L}{4G_3} \left[ (\frac{a}{b})^2 - \frac{2}{3} (\frac{a}{b})^3 + \cdots \right]. \tag{11}
\]
Figure 2: Plots of $\log[l/\epsilon] + \log[(l - 2b)/\epsilon]$, $\log[(l - 2b)/\epsilon]$, and $2\log[b/\epsilon]$ for the values $l = 10$, $\epsilon = .01$, for 2-dim CFT system. The falling curve (lower yellow) is preferred for entanglement entropy of a larger bath ($b \gg 0$) under quantum entropy proposal. The sole rising curve (green) is good for entropy of small size bath system only. We see that for large size baths entropy falls towards the end. The topmost graph valid in large bath region is unphysical. Two falling curves differ by an overall constant only in large $b$ region. We set $\frac{L}{2b} = 1$.

Note the very first term on r.h.s. of eq.(10) represents purely the bath system entropy of a box size $2b$, while subleading term $S_{Island}$ represents the interaction between bath and the subsystem and it may be recognized as twice of ‘gravitational’ entropy due to an ‘Island’ boundary (that is located at $z = b$ inside the AdS bulk), while it is proportional to the segment size $a$ of subsystem-A. Whereas the Icebergs entropy includes contributions from rest of the subleading terms in small $\frac{a}{b}$ expansion. But we immediately realize that all these terms in (10) are actually inseparable from each other. That is to say they are all equally important because the total sum of them depends only on single variable $l$ and nothing else. The total entropy of ‘bath plus system’ thus has a constant value, with size $l$ being fixed. Had we tried to ignore subleading icebergs contributions in (10), due to their smallness, we would find that total entropy $S[AUB]$ starts depending on $a$ or $b$ in a strange way. This might lead to wrong conclusions regarding the Page curve for bath entropy! Hence we conclude that $S_{Island}$ and $S_{Icebergs}$ should not be treated separate from leading bath entropy in any situation. Furthermore, due to this the islands and icebergs will actually remain invisible, as these contribute only in eq.(7) which represents an unphysical (higher value) extremum of bath entropy as per quantum entropy proposal [9]. It will get clarified further as we proceed.
2.1 Generalization: Entropy of multiple quantum dots

We now discuss a limiting case of the entropy in eq.(7). Considering very small size limit $a \to R$, where $R$ is Kaluza-Klein scale and $R \gg \epsilon$. So we can express $a = 2\pi n R$, with $n$ being discrete. Note we are simply assuming that there exists an intermediate Kaluza-Klein scale $(2\pi R)$ at shorter distances. That is the system-A appears essentially point or dot like. (So that it becomes plausible to consider a dual JT gravity interpretation of dot-like system-A.) For brevity we shall only take small $n$ values, also $R$ is taken really small (If there is any difficulty we will simply take $n = 1$). The system-A now can be treated as a quantum-dot (point) sandwiched between finite CFT bath of size $b$ on both sides. We have the situation resembling as in figure (3).

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4The quantum entropy proposal might not work if the RT formula in eq.(7) does not correctly represent the entropy of large CFT bath system in a multi-system entanglement. I thank S. Theisen for fruitful discussion over this point.
The expansion on the r.h.s of (7), for small ratio \( \frac{2\pi nR}{b} \ll 1 \), gives us

\[
S_{\text{bath}}[B] = \frac{L}{2G_3} \left( \ln \frac{2b}{\epsilon} + \frac{2\pi nR}{b} - \frac{1}{2} \left( \frac{2\pi nR}{b} \right)^2 + \cdots \right) + \frac{L}{2G_3} \ln \frac{4\pi nR}{\epsilon}
\]

\[
\equiv \left( S_{\text{bath}}^{(0)} + S_{\text{Island}} + S_{\text{Icebergs}} \right) + \frac{L}{2G_3} \ln \frac{4\pi nR}{\epsilon} \tag{12}
\]

where the expressions are identified as

\[
S_{\text{Island}} = \frac{L}{2G_2} \frac{n}{b},
\]

\[
S_{\text{Icebergs}} = -\frac{L}{4G_2} \left( \frac{2\pi R n^2}{b^2} \right) + O(n^3) \tag{13}
\]

The \( G_2 \) is 2-dimensional Newton’s constant. The islandic contribution, especially for \( n = 1 \), is precisely the gravitational entropy of an island boundary (located at \( z = b \)), as first discussed by [4], while the icebergs entropy includes rest all subleading contributions and there are infinitely many such terms in the series. We note that the net contribution of all iceberg terms is however overall negative. In reality it will not be possible to separate them from first two leading terms in (12) at all! The icebergs become important because the total contribution of these terms (within the parenthesis) does add up to a constant, \( l \) dependent quantity (as full size \( l \) is fixed). It is quite clear from starting line of the perturbative expansion in (12). While the entropy of the q-dot in the center is

\[
S_{\text{dot}} = \frac{L}{2G_2} \frac{1}{2\pi R} \ln \frac{4\pi R n}{\epsilon} \tag{14}
\]

Thus in conclusion, the island or icebergs individual contribution is of no physical significance, as the Page-curve of bath CFT matter (of bath-B) will be fully determined by the entropy of the system-A (quantum KK dot) only. It may be simply stated as

\[
S_{\text{bath}}^{\text{quantum}} = \min \{ S_{\text{dot}}, S_{\text{bath}} \} = S_{\text{dot}} = \frac{L}{2G_2} \frac{1}{2\pi R} \ln \frac{4\pi nR}{\epsilon}
\]

\[
\propto \left( \frac{L}{\pi R} \ln 2n + A_0 \right) \tag{15}
\]

where \( A_0 = \frac{L}{\pi R} \ln \frac{2\pi R}{\epsilon} \) is a known area constant. It is the net bath entropy near the end of the Page curve and it entirely gets contribution from the smallest RT surface corresponding to the KK q-dot. Note the dot is not exactly a point instead it has size \( \sim O(2\pi R) \). Importantly the entropy comes as quantized \( \sim \ln 2n \), where \( n = 1, 2, 3, \ldots \). Obviously we could trust our results for small \( n \) only. For very large \( n \) it would be better to go for noncompact geometry. If this is true then the Page curve will necessarily show discrete jumps as and when \( n \) value jumps. This can be taken as an example of strongly
coupled system of 2n q-dots at the centre of a large CFT bath. These conclusions will not alter even if we take an infinite bath limit ($b \to \infty$, $l \to \infty$). We conclude that we should not see islands and icebergs individually, as their total contributions leads to a constant value only.

3 Entropy at finite temperature

We now study the case of entropy at finite temperature when boundary CFT is in a mixed state. Here the bulk AdS geometry has a black hole horizon

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2)$$ (16)

The function $f(z) = (1 - \frac{z^2}{z_0^2})$ with $z = z_0$ is location of black hole horizon. So there is a finite temperature in the boundary field theory. Now the system-A of size $2a$ is taken to be in thermal equilibrium with symmetrical bath, of total size $2b$, and system-A is located in the middle of bath system-B of size $2b$.

We only discuss the case when $a \ll b$. The entropy of the finite temperature CFT bath system is then given by

$$S[B] = \frac{L}{2 G_3} \ln \sinh \left( \frac{l}{z_0} \right) + \frac{L}{2 G_3} \ln \sinh \left( \frac{l - 2b}{z_0} \right) + UV \text{ terms}$$ (17)

The first term on the r.h.s. is just a system constant. While the entropy for the system-A will be

$$S[A] = \frac{L}{2 G_3} \ln \sinh \left( \frac{l - 2b}{z_0} \right) + S_{UV}.$$ (18)

Once again we can make out that the two expressions (17) and (18) above differ by an over all constant term that depends upon the total size ($l$). Actually both these entropies are two extremas. But the quantum entropy of the bath subsystem would only come from the minimum of the two values

$$S_{\text{quantum}}^{b \gg a}[B] = \min \{S[A], S[B]\} = \frac{L}{2 G_3} \ln \sinh \left( \frac{l - 2b}{z_0} \right) + S_{UV}.$$ (19)

An interesting case arises when $a \simeq 2\pi n R$, i.e. $b = \frac{l}{2} - 2\pi n R$, we might wish to expand the r.h.s. of eq.(17) as

$$S[B] = \frac{L}{2 G_3} \ln \sinh \left( \frac{l}{z_0} \right) + \frac{L}{2 G_3} \ln \sinh \left( \frac{4\pi n R}{z_0} \right) + UV \text{ terms.}$$ (20)

5One may also take compact coordinate $x = L\phi$, with range $0 \leq \phi \leq 2\pi$ for the BTZ black holes background.
Figure 4: Plots of \((2S_{UV} + \text{Log} \sinh(l/z_0) + \text{Log} \sinh(l - 2b)/z_0), (2S_{UV} + \text{Log} \sinh(2b/z_0) + \frac{2a}{z_0 \tanh(2b/z_0)}), (S_{UV} + \text{Log} \sinh(l - 2b)/z_0), \) and \((2S_{UV} + 2\text{Log} \sinh(b/z_0))\) for parametric choice \(l = 10, \ z_0 = 20, \) and \(S_{UV} \approx 7.6.\) Only the lowermost falling graph (in red) is good for entropy of large bath at finite temperature as per quantum entropy proposal. The upper falling curve (which includes all island and icebergs entropies) represents an unphysical extremum for bath entropy. The topmost curve (in green) which includes only islandic contribution is rather a crude approximation and untrustworthy. The sole rising graph (in blue) is good for entropy of small size \((b \sim 0)\) baths only. We have set \(\frac{L}{2G_3} = 1.\)

Note we may always express

\[
\frac{L}{2G_3} \ln \sinh \frac{l}{z_0} \equiv \frac{L}{2G_3} \ln \sinh \frac{2b}{z_0} + S_{island} + S_{icebergs} = \text{Fixed}
\]  

(21)

where gravitational contribution from island boundary (at \(z = b\)) is namely

\[
S_{island} = \frac{L}{2G_3} \frac{2a}{z_0 \tanh \frac{2a}{z_0}} = \frac{nL}{2G_2} \frac{1}{z_0 \tanh \frac{2a}{z_0}} = n \cdot S_{n=1}
\]  

(22)

which is like entropy of \(n\) independent islands. While other subleading (icebergs) terms in the series are important for generalized entropy, as altogether these add up to give just a constant entropy, \(\frac{L}{2G_3} \ln \sinh \frac{l}{z_0}.\) If we ignored any one of them under any assumption we cannot arrive at this conclusion.

For the central dot-like system-A (which may also be thought of as dual of JT gravity) the entanglement entropy is

\[
S_{dot} = \frac{1}{2G_2} \frac{L}{2\pi R} \ln \sinh \frac{4\pi nR}{z_0} + S_{UV}
\]  

(23)

which is the smallest entropy value and it is also quantized, by virtue of KK scale. However we should trust this result for small and discrete \(n\) values only.

Note that in infinite bath limit, the geometric entropy of the island boundary tends to becomes the horizon entropy (for \(b \to \infty\))

\[
S_{island} \to \frac{nL}{2G_2 z_0} \equiv 2n \cdot S_{BH}
\]  

(24)
But we should not isolate it from the entropy of the icebergs, because in totality the
island and icebergs entropies along with the leading ‘pure’ bath entropy term give overall
constant quantity, depending only on the size \( l \). Note length \( l \) also gives a measure of total
energy of the bath and the subsystem, so it has to be conserved unless there is a leakage
from the bath to the outside. We have not considered that hypothesis here or other time
evolving cases. It is possible that \( n \) can jump in a time dependent process. The discrete
nature of the bath entropy is a consequence of the Kaluza-Klein scale at small distances.
If there is no such compactification at short distances then the bath entropy will vanish
smoothly when \( a \to 0 \).

Thus from (19) for large CFT bath (in contact with quantum dot system) the quantum
entropy at finite temperature can simply be written as

\[
S_{bath} \equiv S_{dot} = \frac{1}{2G_2} \frac{L}{2\pi R} \ln \sinh \frac{4\pi n R}{z_0} + S_{UV}.
\]

(25)

It is smallest of the entropies and has no contribution from an island or the icebergs. It
also implies that the island and icebergs cannot be independently observed. They would
remain fictitious because they contribute only to an unphysical higher entropy branch
arising from eq. (20).

4 Summary

We have proposed that quantum entropy of entanglement of a large CFT bath system
follows the minimality principle that

\[
S_{bath} = \min \{ S_{dot}, S_{dot} + \text{Const.} \} = \frac{L}{4\pi G_2 R} \ln \sinh \frac{4\pi n R}{z_0} + S_{UV}.
\]

and thus it realizes the Page curve for the entropy of thermal CFT matter in the bath.
This conclusion is based upon the observation that q-dot and the large bath entropies
differ only by an overall constant. The constant part of entropy depends only on total
system size \((l)\), and that remains fixed following the conservation laws.

We have found that islands and the icebergs only contribute to the unphysical ex-
trema of higher entanglement entropy. The actual physical extremum with lower entropy
however never gets contribution from these fictitious elements. Therefore the entropy for-
\(mula\ such as (1) may not give complete picture of entanglement as it ignores subleading
contributions, such as the iceberg terms. In fact there is an infinite series of them. We
also find that our results are quite generic and may be extended to higher dimensional
systems also. These would be reported separately.
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