SOME REMARKS ABOUT ANALYTIC FUNCTIONS DEFINED ON AN ANNULUS

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Abstract. Some notes and observations on analytic functions defined on an annulus

1. Introduction

In [BMM(PC)R2008], given \( f : \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \rightarrow \mathbb{C} \) analytic, \( f(0) = 0 \), we considered different ways of measuring \( f(r\mathbb{D}) \) for \( 0 < r < 1 \).

- Maximum modulus: \( \text{Rad} f(r\mathbb{D}) = \max_{|z| \leq r} |f(z)| \) (Schwarz)
- Diameter: \( \text{Diam} f(r\mathbb{D}) \) (Landau-Toeplitz)
- \( n \)-Diameter: \( n \)-Diam \( f(r\mathbb{D}) \)
- Capacity: \( \text{Cap} f(r\mathbb{D}) \)
- Area: \( \text{Area} f(r\mathbb{D}) \)
- What else? Perimeter, eigenvalues of the Laplacian, etc...

Note that if \( f'(0) \neq 0 \), then \( f \) is univalent on \( |z| < r_0 \) for some \( r_0 > 0 \).

Let \( M(f(r\mathbb{D})) \) be a measurement as above. Define:

\[
\phi_M(r) := \frac{M(f(r\mathbb{D}))}{M(r\mathbb{D})}.
\]

Theorem A ([BMM(PC)R2008]). Let \( M \) be Radius, \( n \)-Diam, or Capacity. Then \( \phi_M(r) \) is increasing and its log is a convex function of log \( r \). Actually, it is strictly increasing unless \( f \) is linear. In particular,

\[
M(f(r\mathbb{D})) \leq \phi_M(R)M(r\mathbb{D}) \quad 0 < r < R
\]

When \( M \) is Area (\( f \) not univalent) “logconvexity” might fail but strict monotonicity persists.

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These notes are inspired by the recent paper [IKO].
Given a Jordan curve $J$ let $G_1$ be its interior and $G_2$ be its exterior in $\mathbb{C} \cup \{\infty\}$, and assume that $0 \in G_1$. Compute the reduced modulus (defined below) $M_1$ of $J$ with respect to $0$ in $G_1$ and the reduced modulus $M_2$ of $J$ with respect to $\infty$ in $G_2$. Then $M_1 + M_2 \leq 0$ with equality if and only if $J$ is a circle of the form $\{|z| = r\}$. In fact, the sum $M_1 + M_2$ can be thought of as a measure of how far $J$ is from being a circle. Teichmüller’s famous Modulsatz says that if $-\delta \leq M_1 + M_2 \leq 0$, for $\delta$ sufficiently small, then the oscillation of $J$ is controlled in the sense that, for some finite constant $C$ (independent of $J$)

$$\sup_J |z| \leq 1 + C \sqrt{\delta \log \frac{1}{\delta}}.$$ 

See Chap. V.4 in [GM2005].

Even though a definition of reduced modulus can be given using modulus of path families, it turns out that

$$M_1 = \frac{1}{2\pi} \log |g_1'(0)|, \quad M_2 = \frac{1}{2\pi} \log \left| \left( \frac{1}{g_2} \right)'(0) \right|$$

where $g_1$ is a conformal map of $\mathbb{D}$ onto $G_1$ with $g_1(0) = 0$ and $g_2$ is a conformal map of $\mathbb{D}$ onto $G_2$ with $g_2(0) = \infty$. Note also that the conformal map $\psi := 1/g_2^{-1}$ sends $G_2$ to $\{|z| > 1\} \cup \{\infty\}$ and $\psi(\infty) = \infty$, so $M_2$ is related to the usual concept of logarithmic capacity of $J$:

$$M_2 = -\frac{1}{2\pi} \log \text{Cap}(J).$$

In fact, letting $1/J = \{z : 1/z \in J\}$, we find that

$$- (M_1 + M_2) = \frac{1}{2\pi} \log (\text{Cap}(J) \text{Cap}(1/J)) \geq 0.$$ 

Now consider a conformal map $f$ on $\mathbb{D}$ with $f(0) = 0$ and let $J(r) = f(r\partial\mathbb{D})$. Let $T(r) = -(M_1(r) + M_2(r))$, as above, measure how different $J(r)$ is from a circle centered at the origin. Note that $g_1(z) := f(zr)$ maps $\mathbb{D}$ conformally onto the interior of $J(r)$, so $M_1(r) = (1/(2\pi)) \log r |f'(0)|$. On the other hand, see for instance Example III.1.1 in [GM2005], $M_2(r) = -(1/(2\pi)) \log \text{Cap}(f(r\mathbb{D}))$. Thus we have

$$T(r) = \frac{1}{2\pi} \log \frac{\text{Cap} f(r\mathbb{D})}{|f'(0)| \text{Cap} r\mathbb{D}}.$$ 

By Theorem A, we now know that $T(r)$, as defined above, is increasing and convex, in fact strictly increasing unless $f$ is linear.
The proof of this Theorem A consisted in establishing the “increasing” and “log-convexity” part first and then deducing the “Moreover part” from the behavior of Area $f(r\partial D)$ and Pólya’s inequality relating area and capacity. More specifically, Pólya’s inequality gives the following relation:

$$\phi_{\text{Area}}(r) \leq \phi_{\text{Cap}}^2(r).$$

To show that $\phi_{\text{Cap}}$ is strictly increasing, by log convexity it’s enough to show that $\phi_{\text{Area}}(r)$ is strictly increasing at 0.

After [BMM(PC)R2008] appeared in print, C. Pommerenke pointed out that in [Pom1961] he had already shown the following result:

**Theorem B.** If $f$ is one-to-one and analytic in the annulus \{a < |z| < b\}, then $\log \text{Cap}\{f(r\partial D)\}$ when expressed as a function of $\log r$ is a convex function for $a < r < b$.

For the reader’s convenience we replicate Pommerenke’s proof here.

**Proof of Theorem B.** Fix $r$ as above and find $w_1, \ldots, w_n \in \partial D$ so that

$$n-\text{Diam } f(r\partial D) = \prod_{j<k} |f(w_j r) - f(w_k r)|^{2/(n(n-1))}.$$

Then consider the function

$$H(z) := \prod_{j<k} (f(w_j z) - f(w_k(z)))$$

which is analytic in the annulus \{a < |z| < b\}.

Fix $a < r_1 < r < r_2 < b$. Then by Hadamard’s three-circles theorem,

$$\log \max_{|z|=r} |H(z)| \leq \frac{\log(r_2/r)}{\log(r_2/r_1)} \log \max_{|z|=r_1} |H(z)| + \frac{\log(r_1/r)}{\log(r_2/r_1)} \log \max_{|z|=r_2} |H(z)|,$$

i.e.,

$$\log n-\text{Diam } f(r\partial D) \leq \frac{\log(r_2/r)}{\log(r_2/r_1)} \log n-\text{Diam } f(r_1\partial D) + \frac{\log(r_1/r)}{\log(r_2/r_1)} \log n-\text{Diam } f(r_2\partial D).$$

Now let $n$ tend to infinity. □

In this note we study the case of analytic functions defined on an annulus.

For $R > 1$, consider the family $S(R)$ of analytic and one-to-one functions $f$ that map the annulus $A(1, R) = \{1 < |z| < R\}$ onto a topological annulus $\mathcal{A}$ such that the bounded component of $\mathbb{C} \setminus \mathcal{A}$
coincides with the unit disk $\mathbb{D}$, and so that as $|z| \downarrow 1$ we have $|f(z)| \to 1$. By the Schwarz Reflection Principle, the conformal map $f$ extends analytically across $|z| = 1$, so the family $S(R)$ can be defined more compactly as the set of analytic and one-to-one maps on $A(1, R)$ such that $|f(z)| > 1$ for all $z \in A(1, R)$ and $|f(z)| = 1$ for $|z| = 1$.

We prove the following monotonicity result.

**Theorem 1.1.** Let $f \in S(R)$ and let $T(r)$ be defined as above for $1 \leq r < R$. Then, $T(r)$ is a convex function of $\log r$ and is strictly increasing, unless $f$ is the identity.

Also, in analogy with the local case, for a function $f \in S(R)$ we define the ratios:

$$
\psi_{n\text{-Diam}}(r) := \frac{n\text{-Diam}(f(A(1, r)) \cup \overline{\mathbb{D}})}{n\text{-Diam}(r\mathbb{D})} \quad \text{and} \quad \psi_{\text{Cap}}(r) := \frac{\text{Cap}(f(A(1, r)) \cup \overline{\mathbb{D}})}{\text{Cap}(r\mathbb{D})}.
$$

Then by Theorem B, both $\log \psi_{n\text{-Diam}}$ and $\log \psi_{\text{Cap}}$ are convex functions of $\log r$. It turns out that they are also strictly increasing unless $f$ is the identity.

**Theorem 1.2.** Let $f \in S(R)$ and let $\psi_{n\text{-Diam}}(r), \psi_{\text{Cap}}(r)$ be defined as above for $1 \leq r < R$. Then, $\log \psi_{\text{Cap}}$ is a convex function of $\log r$ and $\psi_{\text{Cap}}$ is strictly increasing, unless $f$ is the identity.

In particular,

$$
\text{Cap}(f(A(1, r)) \cup \overline{\mathbb{D}}) \leq r \frac{\text{Cap}(f(A(1, R)) \cup \overline{\mathbb{D}})}{R}.
$$

One line of proof is similar to the disk case in that looking at the area turns out to be the crucial ingredient. However, we show that Theorem 1.2 can also be deduced from Theorem 1.1.

2. **Proof of Theorem 1.1**

By Theorem B and (1.2) we see that $T(r)$ is a convex function of $\log r$, for $1 < r < R$. Therefore, since $T(r) \geq 0$ and $T(0) = 0$, we get that $T(r)$ is an increasing function of $r$.

Assume that $f$ is not the identity, then $f(|z| = r)$ is not a circle for $1 < r < R$. For if it were, then $f$ would be a conformal map between circular annuli and hence would be linear and hence the identity. Therefore, by Teichmüller’s Modulsatz, $T(r) > 0$ for $1 < r < R$. 
Suppose $T(r)$ fails to be strictly increasing. Then by monotonicity it would have to be constant on an interval $[s, t]$ with $1 < s < t < R$. By convexity, it would then have to be constant and equal to 0 on the interval $[1, t]$, but this would yield a contradiction. So Theorem 1.1 is proved.

**Remark 2.1.** Note that if $F(r) = G(\log r)$ for some convex function $G$ and $F'(1) \geq 0$, then $G'(0) \geq 0$ and by convexity $G'(t) \geq 0$ for all $t \geq 0$, i.e., $F'(r) \geq 0$ for all $r \geq 1$.

We now turn to Theorem 1.2. First we show how it can be deduced from Theorem 1.1.

### 3. Consequences of the serial rule

On one hand by (1.1) we have

$$\text{Cap} \left( f(\mathcal{A}(1, r) \cup \overline{D}) \right) = e^{-2\pi M_2(r)}.$$  

On the other hand, by the serial rule, see (V.4.1) of [GM2005],

$$M_1(r) \geq M_1(1) + \text{Mod}(f(\mathcal{A}(1, r))).$$

However, $M_1(1) = 0$ and by conformal invariance $\text{Mod}(f(\mathcal{A}(1, r))) = \frac{1}{2\pi} \log r$. So

$$\frac{1}{r} \leq e^{-2\pi M_1(r)}.$$

Putting this together, we get

$$\frac{\text{Cap} \left( f(\mathcal{A}(1, r) \cup \overline{D}) \right)}{r} \geq e^{-2\pi (M_1(r) + M_2(r))} = e^{2\pi T(r)}.$$

i.e.,

$$T(r) \leq \frac{1}{2\pi} \log(\psi_{\text{Cap}}(r)).$$

(3.1)

Now assume that $f$ is not linear. By Teichmüller's Modulsatz, $T(r) > 0$ for $1 < r < R$. So by (3.1), $\psi_{\text{Cap}}(r) > 1$ and by Theorem B, $\psi_{\text{Cap}}(r)$ is a convex function of $\log r$. Therefore, we can conclude as above that $\psi_{\text{Cap}}(r)$ is strictly increasing.

Teichmüller’s Modulsatz is based on the so-called Area-Theorem. Alternatively, Theorem 1.2 can be proved using “area” and Pólya’s inequality, in the spirit of [BMM(PCR)2008], as we will show next.
4. FROM AREA TO CAPACITY

Recall Pólya’s inequality:

$$\text{Area } E \leq \pi (\text{Cap } E)^2.$$ 

It implies that

$$\psi_{\text{Area}}(r) \leq \psi_{\text{Cap}}^2(r)$$

for all $1 \leq r < R$.

Lemma 5.1 below will establish that

$$\psi_{\text{Area}}(\rho) > 1$$

for $1 < \rho < R$, unless $f$ is linear.

Moreover $\psi_{\text{Cap}}(1) = 1$. So the derivative

$$\frac{d}{dr}|_{r=1} \log \psi_{\text{Cap}}(r) \geq 0.$$ 

Hence, by “convexity”, $\psi_{\text{Cap}}(r)$ is an increasing function of $r$.

In fact, suppose $\psi_{\text{Cap}}(r)$ fails to be strictly increasing. Then by monotonicity it would have to be constant on an interval $[s, t]$ with $1 < s < t < R$. By “convexity”, it would then have to be constant and equal to 1 on the interval $[1, t]$, but this would yield a contradiction in view of (4.1) and (4.2).

So Theorem 1.2 will be proved if we can establish (4.2).

5. AREA CONSIDERATIONS

Each map $f \in S(R)$ can be expanded in a Laurent series

$$f(z) = a_0 + \sum_{n \neq 0} a_n z^n.$$ 

The key now is to study the area function $h(\rho) := \text{Area } f(A(1, \rho))$. We use Green’s theorem to compute the area enclosed by the Jordan curve $\gamma_\rho(t) = f(\rho e^{it})$, $t \in [0, 2\pi]$. Thus

$$h(\rho) + \pi = \frac{i}{2} \int_{\gamma_\rho} \bar{w} dw = \frac{-i}{2} \int_0^{2\pi} \bar{f}(\rho e^{it}) f_\theta(\rho e^{it}) dt = \pi \sum_{n \neq 0} n |a_n|^2 \rho^{2n}.$$
In particular, when $\rho = 1$, $h(\rho) = 0$, so

\begin{equation}
\sum_{n \neq 0} n|a_n|^2 = 1
\end{equation}

The following lemma can be deduced from problem 83 in [PolS1972].

**Lemma 5.1.** For all $f \in S(R)$, except the identity, we have for $1 < \rho < R$,

$$\text{Area } f(A(1, \rho)) > \text{Area } A(1, \rho).$$

**Proof.** Let $1 < \rho < R$. Then, by (5.1),

$$h(\rho) = -\pi + \pi \sum_{n \neq 0} n|a_n|^2 \rho^{2n}$$

$$= \pi(\rho^2 - 1) + \pi \sum_{n \neq 0} n|a_n|^2 (\rho^{2n} - \rho^2)$$

$$= \text{Area } A(1, \rho) + \pi \rho^2 \sum_{n \neq 0} n|a_n|^2 (\rho^{2n-2} - 1)$$

But $n(\rho^{2n-2} - 1) \geq 0$ for all integers. \hfill \Box

This concludes the proof of Theorem 1.2.

6. **Principal frequency**

Another measure for $f(rD)$ is to consider:

$$M_0(f(rD)) := \frac{1}{\Lambda_1(f(rD))}.$$

Recall that given a bounded domain $\Omega \subset \mathbb{C}$,

$$\Lambda_1^2(\Omega) = \inf \frac{\int_{\Omega} |\nabla u|^2 dA}{\int_{\Omega} u^2 dA}$$

where the infimum ranges over all functions $u \in C^1(\overline{\Omega})$ vanishing on $\partial \Omega$ and is attained by a function $w \in C^2(\overline{\Omega})$ which is characterized as being the unique solution to

$$\Delta w + \Lambda_1^2 w = 0, w > 0 \text{ on } \Omega, w = 0 \text{ on } \partial \Omega.$$

It follows from [PolS1951] p. 98 (5.8.5) that

$$\phi_{M_0}(r) \left( = \frac{\Lambda_1(rD)}{\Lambda_1(f(rD))} \right) > |f'(0)|.$$

**Problem:** Show that $\phi_{M_0}(r)$ is strictly increasing when $f$ is not linear.
This problem turns out to have been solved already by work of Laugesen and Morpurgo [LM1998]. Although, I’m not sure if essentially different techniques are required in the case of the annulus.

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