On Generalized Uncertainty Principle

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Abstract

We study generalized uncertainty principle through the basic concepts of limit and Fourier transformation to analyze the quantum theory of gravity or string theory from the perspective of a complex function. Motivated from the noncommutative nature of string theory, we have proposed a UV/IR mixing dependent function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$. We arrived at the string uncertainty principle from the analyticity condition of a newly introduced complex function which depends upon the UV cut-off. This non trivially modifies the quantum measurements, black hole physics and short distance geometries. Present analysis is based on the postulate that the Planck scale is the minimal length scale in nature and is in good agreement with the existence of maximum length scale in the nature. Both of these rely only on the analysis of the complex function and do not directly make use of any theory or the specific structure of the Hamiltonian. The Regge behaviour of the string spectrum with the quantization of area is also a natural consequence of our new complex function which may contain all the corrections operating in nature and reveal important clues to find the origins of the M-theory.

Keywords: generalized uncertainty principle; string theory; quantum gravity

PACS: 04.60.-m Quantum gravity; 04.60.Nc Lattice and discrete methods; 02.30.-f Function theory, analysis.

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1 Introduction

The analysis of the uncertainty principle in string theory have led to many new insights into the classical and quantum aspects of the general relativity, black hole physics and string theory. The question of the ultimate foundations and the ultimate reality of physics remain open. It is not known in what direction there will be it’s final solution or even whether a final objective answer can at all be expected.

We know that the short distance physics is not well understood. So in order to describe the small scale structure of spacetime adequately, we need to modify the usual classical continuum geometry for example, the Connes NCG [1]. In other words, an extension of quantum mechanics might be required in order to accommodate the gravity. Moreover, the existence of duality symmetries in non-perturbative string theory indicates that strings do not distinguish small spacetime scales from the large ones which requires a modification of the usual Heisenbergs uncertainty principle where beyond Planck scale energies the size of the string grows with momenta instead of the falling off. An introduction on this T-dual description of spacetime as a result of string theory is given by Witten [2],[3] where below the Planck length the very concept of spacetime changes it’s meaning and the Heisenberg uncertainty principle needs to be modified. The thrust of the present work is to explain the completely generalized uncertainty relations from the viewpoint of a complex function whose first order faithful interpolation in terms of the fundamental string length \( l_s \) is the following formula:

\[
\Delta x^\mu \geq \frac{\hbar}{\Delta p^\mu} + \frac{l_s^2}{8\pi} \Delta p^\mu,
\]

where henceforth the speed of light is taken to be \( c = 1 \).

On the same lines, Carlos Castro has conjectured [4] that the special theory of scale relativity recently proposed by Nottale [5] must play a fundamental role in string theory, specially it demonstrates that there is a universal, absolute and impassable scale in nature, which is invariant under dilatations and it’s lower limit is the Planck scale. The fundamental scales of nature are determined by constraints which are set at both small and large scales in perfect agreement with the string duality principles. Applying the scale relativity principle to the universe, one arrives at the proposition that there must exist an absolute, impassable, upper scale in nature which is invariant under dilatations (particularly the expansion of the universe) that holds all the properties of the infinity. This upper scale \( L \) defines the radius of the universe and when it is seen at its own resolution, it becomes invariant under dilations.

In recent years, the measurements in the quantum gravity are governed by generalized uncertainty principle. Evidences from string theory, quantum geometry and black hole physics suggest that the usual Heisenburg uncertainty principle needs certain modification(s). These evidences have an origin on the quantum fluctuations of the background metric. The generalized uncertainty principle provides the existence of a minimal length scale to the nature which is of the order of the Planck length. Adler et. al. [6],[7] considers the issue of black hole re-
manants in the framework of generalized uncertainty principle and shows that the generalized uncertainty principle may prevent small black hole’s total evaporation whereas in the Bekenstein Hawking approach, the total evaporation of a micro black hole is possible. The generalized uncertainty principle indicates the quantum gravitational corrections to the black hole thermodynamics.

The coordinates and the corresponding momenta cannot be simultaneously specified after the quantization due to the uncertainty relation. Moreover, phase space as the total physical states space must be modified for nonzero Planck constant. It is the Hilbert space of wave functions in quantum mechanics that is the space of states characterized by the principle of superposition of quantum mechanical states. Similarly, in view of the validity of the spacetime uncertainty relation, we may expect certain modifications of the notion of the spacetime. Thus, string theory may be taken as some sort of ‘quantum geometry’ along this line of thought but at present it is difficult to formulate this kind of ideas concretely. As before string theory, there have been several attempts to generalize local field theories based on similar ideas. For example, let the spacetime coordinates be operators instead of ordinary numbers, then the coordinates and momenta can be treated as operators in the quantum mechanics acting on certain Hilbert space. This idea in a certain limit has been seen recently in the context of string theory with some assumptions on certain background fields as a ‘noncommutative field theory’[8],[9],[10]. However, these limits neglect the crucial extendedness of the strings along the longitudinal directions so we do not have to date notable new insight on the spacetime uncertainties characterized by the string length parameter $l_s$.

As a matter of fact, due to the noncommutative nature of spacetime at Plank scale, the usual Heisenberg uncertainty principle should be reformulated. So as a consequence, there exists a minimal observable distance of the order of the Plank length where all the measurements in the limit of extreme quantum gravity are governed. In the context of string theory, this observable minimal distance is known as generalized uncertainty principle: $\Delta x \geq \frac{\hbar}{\Delta p} + \text{constant} \cdot G \Delta p$, which on using the minimal nature of $l_p$ can be written as [11],[12],[13],[14]: $\Delta x \geq \frac{\hbar}{\Delta p} + \frac{\alpha^2 \hbar^2}{\Delta p} \Delta p$. This is because the minimal length in the nature exists so there are possibilities to correct the usual Heisenberg uncertainty principle as generalized uncertainty principle. Though in any general theory, the higher order contributions in the above are non zero but the minimal length in the theory is just governed by the parameter $\alpha$, and the generalized uncertainty principle shows that there is a minimum dispersion $\Delta x$ for any value of $\Delta p$ at least as long as the first two terms on the right hand side are non zero. The generalized uncertainty principle or a minimum length is also motivated by the study of short distance behavior of strings theory [15],[16],[17],[18],[19],[20], black hole physics [21] and de Sitter spaces [22].

From the perspective of D-brane physics, the spacetime uncertainty principle
gives a simple qualitative characterization of nonlocal and/or noncommutative nature of short distance spacetime structure in string theory. For example, Tamiaki Yoneya [23] considers spacetime uncertainty and approaches to D-brane field theory where the recent approaches toward field theories for D-branes are briefly outlined and some key ideas lying in the background are put on the emphasis [24] and references therein. Further motivation comes from the quantum propagator of a bosonic p-brane obtained in the quenched minisuperspace approximation which suggests a possibility of novel and unified description of p-branes with different dimensionality [25]. In this case, the background metric has emerged as a quadratic form on a Clifford manifold where the substitution of Lorenzian metric with the Clifford line element changes the very structure of the spacetime fabric as the new metric is built out of a minimum length below which it is impossible to resolve the distance between two points. Furthermore the introduction of the Clifford line element extends the usual relativity of the motion to the case of relative dimensionalities of all p-branes which makes up the spacetime manifold near the Plank scale.

The stringy corrections to the original Heisenberg’s uncertainty principle also follow directly from the quantum mechanical wave equations on noncommutative Clifford manifolds where all dimensions and signatures of spacetime are on the same footing[26]. Castro has considered the new relativity principle to find a fully covariant formulation of the p-brane quantum mechanical loop wave equations where the string uncertainty relations arrises naturally. Infact, there is one to one correspondence between the nested hierarchy of p-loop histories encoded in terms of hypermatrices and wave equations written in terms of Clifford algebra valued multivector quantities which allows to write the quantum mechanical wave equations associated with the hierarchy of nested p-loop histories being embedded in a D dimensional target spacetime with a single quantum mechanical functional wave equation whose lines live in a noncommutative Clifford manifold of $2^D$ dimensions having $p = 0, 1, 2, 3 \ldots D - 1$ where $D - 1$ is the maximum value of $p$ that saturates the dimension of embedding spacetime [27]. In this C-spaces the $x, p$ must not be interpreted as ordinary vectors of spacetime but as one of the many components of the Clifford algebra valued multivectors that “coordinatize” the noncommutative Clifford manifold. The noncommutativity is encoded in the effective Plancks constant $\hbar_{eff}$ which modifies the Heisenberg Weyl $\hat{x}, \hat{p}$ commutation algebra and consequently on keeping the first two terms in the expansion of $\hbar_{eff}(k)$ generates the ordinary string uncertainty relation[27]:

$$\Delta x \geq \frac{\hbar}{2p} + \frac{\beta_1 l_4}{4n} (\Delta p).$$

This is an inherent noncommutative nature of the Clifford manifold which reshuffles a loop history into a membrane history, a membrane history into a p-brane history or more generally it can transform a p-brane history into suitable combinations of the other p-brane histories as building blocks. This bootstrap idea is taken from the point particle case to the p-branes case with each brane made out of all the other p-branes where the Lorentz transformations
in C-spaces involve hypermatrix changes of coordinates in the p-brane quantum mechanics [26].

The present paper is an effort to bridge up both string theory and quantum mechanics within the framework of Heisenberg's uncertainty principle and give a more general expression from the complex analysis than the one presented in past years [9], [11], [12], [13], [14], [16], [17], [19], [20], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33]. We shall notice, why the upper scale of nature must appear in the fundamental equation? The importance of noncontinuous maps in string theory has been discussed by Borde and Lizzi [34]. The space of string configurations in string theory required both continuous and noncontinuous square integrable maps in order to reproduce the results from the dual models. The size and shape of strings in their ground state in the lightcone gauge has been investigated long time ago by Susskind et. al. [35] where it is found that in two dimensions the extrinsic curvature is divergent. A regularization scheme is needed where the string is kept continuous. As the dimensionality of spacetime increases the string become smoother and have divergent average size. This is unphysical since their size cannot exceed the size of the Universe. It is because of this reason that the upper scale of nature must also appear in the fundamental equation. The four dimensional average curvature diverges due to kinks and cusps on a string and so it is important to further study and analyze the properties of correcting functions in the uncertainty principle. Having presented the reasons why uncertainty principle is an important relevant issue, we shall explain the importance of the stringy uncertainty principle and generalize it from the perspective of the complex function theory.

In the present article, we study the effects of higher derivative terms on the uncertainty principle from the analyticity condition of a complex function and give a simple explanation of the string uncertainty principle from the analysis of a holomorphic or anti-holomorphic function. Present work has been organized in several sections. The first section introduces the problem and it’s motivation. In section 2, we have reviewed usual uncertainty principle in finite dimensional quantum mechanics. On taking the account of shape and size we have illustrated the well known Heisenburg uncertainty principle for any arbitrary $L^2$ function. In section 3, motivated from the string theory with the emphasis on the concept of limit and Fourier transform of any complex function, we have proposed a function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ and given a resolution resolution criteria for the UV/IR mixings. Furthermore, we have perturbatively proved our proposition and outline a generalization on an arbitrary manifold. In section 4, we have explained that our completely generalized uncertainty principle renders at the string uncertainty principle with all order perturbative corrections, a resolution criteria for the UV/IR mixings, physics of quantum gravity, black hole physics, existance of minimal and maximal length scales in nature, short distance geometry versus string theory, Fourier transformation versus distribution and discretization of the spacetime. Moreover, our completely generalized uncertainty principle reveles all
these known physical and mathematical concepts nicely from a newly proposed single function \( \tilde{\delta}(\Delta x, \Delta k, \epsilon) \) that may reveal to find the geometric origin of the fundamental M-theory. Finally, the section 5 contains some concluding issues and remarks for the future.

2 Measurements and Quantum Mechanics.

We begin by considering some needful basic features of quantum mechanics and measurements of certain physical observables [36] needed for further developments in the later section. On the basis of dual nature of matter, it is well known that macroscopically it is possible to measure exactly the position of a moving particle at any instant and momentum of the particle at that position, but microscopically it is not possible to measure exactly (or with certainty) the position of a particle and its momentum simultaneously. According to the usual quantum mechanics, the behaviour of a moving particle can be defined by a wave packet moving with a group velocity \( v_g = \frac{d\omega}{dk} \) which in accordance with the Max Born, the particle can be found anywhere within the wave packet. In other words, the position of the particle is uncertain within the limit of wave packet. Furthermore as the wave packet advances with the group velocity \( v_g \), there exists uncertainty in the velocity or the momentum of the particle.

It is well known that the square of the amplitude of wave packet at a point represents the probability of finding the particle at that point. For example, periodic wave function with constant wave length has no uncertainty in the momentum but the amplitude is same everywhere, so probability is same for the particle to be found anywhere in the wave packet, i.e. uncertainty in the position is infinity. Moreover, the uncertainty principle for Fourier transform pairs follows immediately from the scaling property, which may be physically stated as follows: “Time Duration \( \times \) Frequency Bandwidth \( \geq C \)” where \( C \) is some constant determined by the precise definitions of “duration” in the time domain and “bandwidth” in the frequency domain. But from the definition of the Fourier transform and its inverse, we know that if duration and bandwidth are defined as the nonzero interval, then we obtain \( C = \infty \), which is not very useful in the physical situations.

To illustrate better the above defined idea, consider a wave packet formed by superposition of two progressive waves of infinitesimal different angular frequencies \( \omega, \omega + d\omega \) and propagation constants \( k, k + dk \) along with \( k = \frac{2\pi}{\lambda} \), where \( \lambda \) is wave length. Let the equations of the waves be \( \varphi_1 = A\sin(\omega t - kx), \varphi_2 = A\sin((\omega + \Delta\omega)t - (k + \Delta k)x) \). Then the wave packet obtained by their superposition is: \( \varphi = \varphi_1 + \varphi_2 = 2A\sin((\omega + \frac{\Delta\omega}{2})t - (k + \frac{\Delta k}{2})x) \cdot \cos((\frac{\Delta\omega}{2})t - (\frac{\Delta k}{2})x) \). Or \( \varphi = 2A\cos((\frac{\Delta\omega}{2})t - (\frac{\Delta k}{2})x) \cdot \sin(\omega t - kx) \). So the amplitude of the resultant wave is \( A_{res} = 2A\cos((\frac{\Delta\omega}{2})t - (\frac{\Delta k}{2})x) \) and the spread of each wave packet is equal to half of the wave length \( \lambda_m \) of the resultant wave. In other words, uncertainty
in the position of particle is $\Delta x = \frac{\lambda}{2\kappa}$. As propagation constant of the wave packet is $k_m = \Delta k$ and so with $k_m = \frac{2\pi}{\lambda}$ we have $\Delta x \Delta k = 2\pi$. But $k = \frac{2\pi}{\lambda}$ and $\lambda = \frac{\hbar}{p}$. Hence $k = \frac{2\pi}{\hbar}$ or $\Delta k = \frac{2\pi\Delta p}{\hbar}$ $\Rightarrow$ $\Delta x \Delta p = \hbar$. This is well known Heisenburg’s uncertainty principle in the quantum mechanics. Now we explain from the statistical deviation methods that the size and shape of the wave packets are important and they replace the above equality into the inequality, ie $\Delta x \Delta p \geq \frac{\hbar}{2}$.

In particular we consider the Heisenburg’s uncertainty principle in the regorous setting for a particle moving in $R^1$.

The usual uncertainty principle for the one dimensional quantum mechanical particle can be seen by the calculation of the uncertainties in the position and in the momentum from the standard statistical deviation method as follows: Let $\psi(x)$ be a normalized wave function of a particle in $R^1$ then probability of finding the particle in between the position $x$ and $x + dx$ is defined by $\psi^*(x)\psi(x)dx$. The expectation value of $x$ for normalized $\psi(x)$ is defined by $<x> := \int \psi^*(x)x\psi(x)dx$.

Then uncertainty in position of a particle in the $x$ direction is given by $\Delta x := \begin{array}{l} \{ x - < x > \}^2 \end{array}^{1/2}$. Similarly the uncertainty in momentum is given by $\Delta p := \begin{array}{l} \{ p - < p > \}^2 \end{array}^{1/2}$ where $<p> := \int \psi^*(x)(-i\hbar \frac{\partial}{\partial x})\psi(x)dx$. Without loss of generality, we can choose our coordinate system such that $<x> = 0$ and the average momentum is zero, i.e. we can assume that the normalized wave packet is initially centered at $<x> = 0$ and $<p> = 0$ so $(\Delta x)^2 = <x^2>$ and $(\Delta p)^2 = <p^2>$.

Let us now turn to the analysis of the following integral: $i\hbar \int_{-\infty}^{+\infty} \psi^*(x) \frac{d\psi}{dx} (x\psi(x))$. Integrating by parts from $-\infty$ to $+\infty$ with boundry condition $\psi\psi|_{x=\pm\infty} = 0$, we have, $i\hbar \int \frac{d\psi}{dx} x\psi dx = -i\hbar \int \psi^* \frac{d\psi}{dx} (x\psi(x))dx = -i\hbar [\int \psi^* x \frac{d\psi}{dx} dx + \int \psi^* \psi.1 dx]$. Since $\int \psi^* \psi dx = 1$ for normalized $\psi(x)$. Hence $i\hbar \int \frac{d\psi}{dx} x\psi dx + i\hbar \int \psi^* \frac{d\psi}{dx} x dx = -i\hbar$. Since the quantity on the right hand side is purely imaginary, so it must be equal to the imaginary quantity of the left hand side. I.e. Imaginary part of l.h.s. $(2i \int i\hbar \frac{d\psi}{dx} x\psi dx) = -i\hbar$. Now taking modulus of both side and squaring yields, $4|Im(\int i\hbar \frac{d\psi}{dx} x\psi dx)|^2 = \hbar^2$. As $|z| \geq |y|, \forall$ complex number $z = x + iy$. So $|\int i\hbar \frac{d\psi}{dx} x\psi dx| \geq |Im(\int i\hbar \frac{d\psi}{dx} x\psi dx)|$. In otherwords, $4|\int i\hbar \frac{d\psi}{dx} x\psi dx|^2 \geq \hbar^2$. But for any two complex functions $f$ and $g$ we know from the Schwarz inequality that $|\int f^*gdx|^2 \leq (\int f^*fdx)(\int g^*gdx)$, which gives particularly in our case, $|\int i\hbar \frac{d\psi}{dx} x\psi dx|^2 \leq (\int i\hbar \psi^*\psi dx)(\int x^2 \psi^*\psi dx)$. Hence above equation reads, $\int i\hbar \frac{d\psi}{dx} x\psi dx \leq \frac{\hbar^2}{4}$. As by the definition, $<x^2> = \int \psi^* x^2 \psi dx$ and $<p^2> = \int | -i\hbar \frac{d\psi}{dx}|^2 dx$. One then obtains, $(\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4}$. Or $(\Delta x) \geq \frac{\hbar}{2(\Delta p)}$ which is usual one dimensional Heisenburg’s uncertainty principle in ordinary quantum mechanics.

On other hands, in short distance regime, the notion of continum spacetime is drastically modified by the string theory which may finally unify the general relativity with quantum theory. In this article, we discuss the possible significance of short distance aspect of string theory from a perspective focusing on the uncertainty principles and complex analysis. One of the distinguishing feature of
any quantum theory compared to classical physics is that there exists non-zero quantum fluctuations. In classical physics, in principle a physical state can be exactly determined with sufficient knowledge of the state at a given time and one can exactly predict the precise values of various physical quantities at any other time just by solving the equations of motion, whereas in quantum theory, one can predict only the probabilities of possible values of physical quantities, even though one know the state at a given time as precisely as possible in the theory, which is well known Heisenberg uncertainty principle. Precisely, we can never make both uncertainties either $\Delta x$ and $\Delta p$ or $\Delta t$ and $\Delta E$ small beyond the following restriction for these errors: $\Delta x \Delta p \geq h$ or $\Delta t \Delta E \geq h$ which simply follow from the Heisenberg Weyl algebra: $\Delta x \Delta p \geq | \left[ \hat{x}, \hat{p} \right] |$ with $\left[ \hat{x}, \hat{p} \right] := i\hbar$. On other hand, in the general case of any $L^2$ function, we can arrive at the Heisenburg’s uncertainty principle of quantum mechanics by using the standard deviation of $|f|^2$ as a measure $\Delta(f)$ of the spread of $f$ and the same measure $\Delta(\hat{f})$ for $\hat{f}$ defined by $\langle f, g \rangle = \frac{1}{\sqrt{2\pi}} \int_R \overline{f}(x) g(x) dx$ with $\|f\|^2 = \langle f, f \rangle$. The Fourier transform is defined by $\hat{f}(k) = (\mathcal{F} f)(k) = \langle e^{ikx}, f(s) \rangle$ and so taking the centroid of $|f|^2$ and $|\hat{f}|^2$ to be zero, we have $\Delta(f) = \|f(x)\|^2_{\|f(x)\|^2_2}$ then the Heisenburg’s uncertainty principle is just $\Delta(f)\Delta(\hat{f}) \geq \frac{\hbar}{4\pi}$.

These quantum uncertainties are one of the most essential features of any quantum theory and resolve the apparent conflict between a particle and wave which unifies them as mutually complementary aspects of the physical degrees of freedom in the microscopic domain of nature. Furthermore, in any physical state of an arbitrary physical processes, the uncertainty principle says that there always exists quantum fluctuations on which we can never have complete control by any means and remains true even in the absolute vacuum at absolute zero temperature. For example, the fluctuations of energy and momentum can become arbitrarily large as we go to shorter and shorter spacetime intervals but of course our experimental apparatus always have limitations in their precision of measurements and hence cannot probe spacetime structure at arbitrarily short intervals. Because of the relationship among the results of various measurements, we can ignore the various quantum fluctuations associated with arbitrarily short intervals exceeding the order of the precision of our measuring apparatus. When we take into account the existence of gravity quantum mechanically the situation is completely ruined because gravity directly couples to the energy momentum tensor which increase without limit at arbitrarily short distance scales and directly affects the strength of gravitational interaction. Due to this reason when we apply the renormalization method to general relativity, we are forced to introduce an infinite number of undetermined constants, which must be fitted to the experimental data. So the renormalization method loses its power for the gravity, i.e. quantum gravity requires an infinite number of undetermined constants in order to make predictions itself.

On other hands, to determine physical quantities like $S$ matrix describing on-
mass-shell scattering amplitudes, one need to develop a theory whose ingredients can be deduced from certain quantum field theory. For example unitarity and maximal analyticity of the S matrix which basically encode the requirements of causality and non-negative probabilities. Actually, there are two possible questions: (i) Is it only the failure of the renormalization methods based on perturbation theory and not the failure of the general relativity theory itself? (ii) Should general relativity be modified in the short distance regime irrespective of the validity of perturbation theory such that the quantum fluctuations of energy and momentum tensor becomes large? The significance of the above conflict between general relativity and quantum theory is so profound that we can not prevent ourselves from these difficulties for concentrating towards its resolution. The resolution as by now understood is the string theory which can be regarded as a sort of the final outcome of many essential ideas springing from various attempts towards the fundamental theory of all the interactions. Although the final answer to the above fundamental question has not yet been obtained, it is at least true by exploring string theory that we are uncovering a multitude of facets of the theory which is useful for strengthening out our understanding of gauge theory and general relativity in quite unexpected way apart from the general understanding of string theories themselves. The basic reason to make it possible is that the both of gauge field theory and general relativity are inextricably intertwined in the same framework of string theory.

3 Stringy Uncertainty Principle.

What follows in this sections, we consider the stringy $\alpha'$ corrections from the perspective of complex analysis and show that there exists corrections from the holomorphic and anti-holomorphic sectors. It is well known fact that all string theories automatically contain gravity which has already provided a remarkable arena where various physical ideas and mathematical structures that were regarded as being entirely unrelated, are unified. So it seems that there is a great possibility to achieve the ultimate unified final theory of the nature. The string theory is astonishingly rich and has many features which are desirable in an ultimate unified theory. This is because after the establishment of the existence of gravity in any string theory, Scherk and Schwarz suggested that string theory should be regarded as a fundamental theory. The idea of string theory as the fundamental theory was taken more seriously after the failure of various attempts towards building a consistent theory of quantum gravity within the framework of the ordinary local field theories. Further the extreme self consistency of the string theory is not a defect but it is interpreted as the most important signature for the ultimate unification.

As string theory includes gravity, gauge like forces etc which in the low energy limit, when the length of the string can be ignored, are approximately
described by appropriate gauge theories of ordinary type like Maxwells electromagnetism. The gravitational interaction contained in string theory is described in the low energy limit by the usual supergravities which are actually constructed in the attempts towards a generalization of the general relativity by extending symmetries. The mathematical structure of the theory shows that all the parameters of the theory, apart from the fundamental string length, including the space time geometry itself can in principle be determined by the dynamics of the theory itself. The appearance of the critical spacetime dimensions can be regarded as a special case of this general feature of the theory. Unfortunately, one doesn’t actually think that the meaning and the content of string theory are fully grasped at the present stage of the developments. The string theory resolved the problem of the divergences associated with the earlier attempts at quantizing gravity but it is needed to understand non perturbatively along with the limitations of perturbative quantum gravity. This shows how deep string theory could be in general and how difficult it is to find the really appropriate mathematical language to formulate the principles behind the string theory. In my opinion, we need probably a new mathematical framework in order to satisfactorily express the whole content of the string theory and the principles behind it without using perturbation theory.

It is known, the fundamental difficulty of the divergences of quantum gravity are related to the quantum uncertainties which are resolved in string theory. In order to see the basic nature of the string dynamics, we need to understand a string which is simply a one dimensional extended object where energy density along a string in the fundamental string theory is assumed to be a universal constant given as $\frac{1}{2\pi \alpha'}$, where the new fundamental constant $\alpha'$ characterizes the underlying string theory \[38\]. So the energy per unit length remains constant as the string stretches or shrinks. In other words, the total mass of a string can be determined by its total length which means that the length of the string in the lowest energy states is at least classically zero and so these states are massless. But when a string is treated quantum mechanically, the quantum fluctuations have to be taken into account so one can not say in the classical sense that the length of the lowest energy states of the string is strictly zero. In string theory, the massless spin-2 close string state behaving as graviton is responsible for the universal gravitational force which in the low energy limit exactly coincides with the graviton that one expects from the quantization of general relativity and the spin-1 open string state coincides with the gauge particles like photon in quantum electrodynamics or gluons in quantum chromodynamics interactions \[38\], \[39\]. That’s why string theory in general contains gravity and/or gauge forces. In particular, even if we start only with open strings, the consistency of the theory requires that closed strings must always coexist with open strings and interact with open and closed strings in a manner which is dictated, at least in the low energy limit such as supergravity. The string interactions are such that divergences in any string diagrams are absent in the peculiar ways in contrast to
the ordinary Feynmann diagrams of the gauge theories [38], [39]. These string interactions are splitting or joining at the end points of open strings and the rejoining of two closed strings at arbitrary points along with both open and closed strings which amounts to the statement that sufficiently small portion of each worldsheet at an arbitrary point on it is always dynamically equivalent to the segment of a one sheeted plane. The uniformity of the worldsheet in this sense is mathematically formulated by a characteristic conformal invariance which is intimately connected to the universal nature of the energy density of the string.

One of the important proposal due to Chew and Frautschi is maximal analyticity in angular momentum [40], [41]. According to this proposal one can uniquely extend the partial wave amplitudes \( a_l(s) \) to an analytic function \( a(l, s) \) of \( l \) with isolated poles, the so called Regge poles. The Mandelstam invariant \( s \) is the square of the invariant energy of the scattering reaction. The position of a Regge pole is given by a Regge trajectory \( l = \alpha(s) \). The physical hadron states are determined by the values of \( s \) for which \( l \) takes a physical values. The necessity of branch points in the \( l \) plane associated with Regge cuts has been established by Mandelstam [42]. Phenomenologically there are many new hadrons discovered in experiments for which mass squared versus angular momentum plot with fixed values of other quantum numbers shows the Regge trajectories that are approximately linear with a common slope \( \alpha(s) = \alpha(0) + \alpha s, \alpha \sim 1.0\,(GeV)^2 \). One argue on the basis of crossing symmetry properties of analytically continued scattering amplitudes that the exchange of Regge poles in the t-channel, controlled at high-energy with fixed momentum transfer is given by the following asymptotic behavior of physical amplitudes:

\[
A(s,t) \sim \beta(t)(\frac{s}{s_0})^{\alpha(t)} \text{ with } s \to \infty, t < 0.
\]

An exact analytic formula exhibiting duality with linear Regge trajectories designed to give a good phenomenological description of the reaction \( \pi^+ \pi^- \to \pi^+ \pi^- \) or the decay \( \omega \to \pi^+ \pi^- + \pi^0 + \pi \) was obtained long time ago by Veneziano whose structure is just the sum of three Euler beta functions: \( T = A(s,t) + A(t,u) + A(u,s) \) with \( A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \) [43]. In string theory, there are several remarkable discoveries of an N-particle generalization of the Veneziano formula or Virasoro formula having a consistent factorization on a spectrum of single particle states which is described by an infinite number of harmonic oscillators \( \{a_m\}, \mu := 1, 2, \ldots, d-1; m = 1, 2, \ldots \) with one set of such oscillators in the Veneziano case and two sets in the Virasoro case [44], [45], [46], [47]. These results may be interpreted as describing the scattering modes of a relativistic string: open strings in the first case and closed strings in the second case. Furthermore, the branch points become poles for \( \alpha(0) = 1 \) and \( d = 26 \) [48]. These poles are interpreted as closed-string modes in a one-loop open-string amplitude which is referred to as open string- closed string duality.

On other hands, the theory of renormalization group is based on the fact that it is possible to organize physical phenomena according to the energy (or distance) scale, i.e. the short distance physics is not directly affected by the
qualitative features of the long distance physics and vice versa. This sort of separation of ultraviolet versus infrared physics holds good in usual quantum field theories. But there exists interrelations between UV and IR physics for the generalizations such as noncommutative field theory and quantum gravity particularly the string theory where one can explicitly demonstrate the UV/IR mixings\cite{10}, \cite{49}, \cite{50}, \cite{51}, \cite{52}. From the viewpoint of probing the short distance spacetime structure, the most decisive directions of distances or momentum in the string dynamics are along the strings themselves where the physical pictures are united in the properties of the string worldsheet which can be analyzed just by using complex analysis. Since the simplest model does not contain gravity explicitly, so the generalized uncertainty principle arises as a consequence of the discretization of space which may or may not be a property of the full quantum gravity. But such explanation of the generalized uncertainty principle in a simple models may be useful in understanding how the generalized uncertainty principle arises in more realistic physical situations. In particular, it is the complex analysis from which we have shown that our theorem is a resolution criteria of the UV/IR mixing problem with the existance of certain functions in our following proposition which contains all the effects of the quantum gravity at the all scales of nature.

Let us now turn to the analysis of the generalized uncertainty relations as-sociated with any quantum mechanical physical system. Recall the concept of $\epsilon, \delta$ limit and usual Fourier transformation for any complex function $f(x) \; \text{53}$. For given any $\epsilon > 0, \exists \delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Consider a fourier pair $(f, \hat{f})$ for a fourier conjugates $(x,k)$ with $f(x) = \frac{1}{2\pi} \int dk \hat{f}(k) e^{ikx}$ in the sense of $L^2$ convergence. This implies that $\Delta x = \frac{a}{\Delta k}$ where $a$ is some constant depending on the size and shape of the wave packet. Let us now consider various step sizes to be $\{\epsilon_i\}_{i=1}^N$ for some given $\epsilon$ as a certain sequence. Consider real lattice with variable step sizes $\{\epsilon_i - \epsilon_j\}$. Let $\epsilon = \max_{i,j\in\Lambda} |\epsilon_i - \epsilon_j|$ be the maximum step size and the $\Lambda = \{1,2,\ldots,N\}$ be some index set. That is the equation $\Delta x = \frac{a}{\Delta k}$ holds if $\epsilon \to 0$. We claim that there exist certain functions containing the minimum and maximum length scales in the perspective of quantum relativity which modifies the uncertainty principle of the usual quantum mechanics. Physically we explain that these functions are very important from the perspective of the string theory \cite{38}, \cite{39}.

**Proposition (Existance):** There exist a function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ such that $\tilde{\delta}(\Delta x, \Delta k, \epsilon) \to 0$ whenever $\epsilon \to 0$, then following equation holds:

$$\Delta x = \frac{a}{\Delta k} + \tilde{\delta}(\Delta x, \Delta k, \epsilon). \quad (1)$$

**Lemma:** For any function $f(x,y) \forall x, y \in \mathbb{R}$ a complex valued function
$F(z) := u(x,y) + iv(x,y); z = x + iy \in C$ where $u,v$ are two real functions. Or $\exists$ an isomorphism $f(x,y) \longrightarrow F(z)$.

**Proof:** Let $(x,y) \in R^2$ be an ordered pair of $x,y \in R$ then we can consider them as a cartesian representation and $\forall(x,y), (x,y)$ corresponds to some $(x,y) \rightarrow f(x,y)$ which can be represented as a vector $\vec{r} = xi + yj$. Now consider in argand plane, i.e. $z = x + iy; x = Re(z), y = Im(z)$. I.e. $z = Re(z) + iIm(z) = (Re(z), Im(z))$ corresponds to some uniquely determined $F(z)$, as desired. Hence $\forall f(x,y) \exists F(z)$ such that (i) $(0,0) \rightarrow 0 + i0$. (ii) $\forall z = (x + iy)$ we have $(x,y) \in R^2$ (iii) $\forall (x,y) \in R^2$, we have a $z = (x + iy) \in C$, uniquely. As $F(z)$ is a complex function, so it can trivially be expressed as $F(z) := u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are real functions. Hence $f(x,y) \longrightarrow F(z)$ is an isomorphism.

**Theorem:** $\exists \alpha \tau$ corrections iff the $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ factorizes into a analytic or anti-analytic function either with respect to $(\Delta k, \epsilon)$ or $(\Delta x, \epsilon)$.

**Proof:** Assume that $\exists \tilde{\delta}(\Delta x, \Delta k, \epsilon)$ such that

$$\lim_{\epsilon \rightarrow 0} \tilde{\delta}(\Delta x, \Delta k, \epsilon) \longrightarrow 0. \quad (2)$$

As in equation (2), $\Delta x = \frac{\partial}{\partial x} + \tilde{\delta}(\Delta x, \Delta k, \epsilon)$. That is to say that $\Delta x, \Delta k$ are dependent, so $\Delta x$ can be expressed in terms of $\Delta k$ and $\epsilon$. Hence it is enough to consider a real reduced function $\tilde{\delta}(\Delta k, \epsilon) = C\tilde{\delta}(\Delta x, \Delta k, \epsilon)$, where $C$ is some constant. Without ambiguity from above equation (3), we have $\tilde{\delta}(\Delta k, \epsilon) \longrightarrow 0$ as $\epsilon \rightarrow 0$.

From the complex function theory we know, for any complex function $\tilde{\delta}$ on an open set $U$ with $\tilde{\delta}(\Delta k + i\epsilon) = u(\Delta k, \epsilon) + iv((\Delta k, \epsilon))$, where the real functions $u(\Delta k, \epsilon)$ and $v(\Delta k, \epsilon)$ are real and imaginary parts of the $\tilde{\delta}$ that the differentiability condition of $\tilde{\delta}$ are seen in terms of the $u(\Delta k, \epsilon)$ and $v(\Delta k, \epsilon)$: Let us concentrate on the holomorphic sector and derive the equivalent conditions for the differentiability of $\tilde{\delta}$ on the $u(\Delta k, \epsilon)$ and $v(\Delta k, \epsilon)$. At a fix z, let $\tilde{\delta}(z) = a_1 + ia_2$ and $w = h_1 + ih_2; h_1, h_2 \in R$. Suppose $\tilde{\delta}(z + w) - \tilde{\delta}(z) = \tilde{\delta}(z)w + \sigma(w)w$ with $\lim_{w \rightarrow 0} \sigma(w) = 0$. Then $\tilde{\delta}(z)w = (a_1 + ia_2)(h_1 + ih_2) = a_1h_1 - a_2h_2 + i(a_2h_1 + a_1h_2)$. On other hand, let $\tilde{\delta} : U \rightarrow R^2$ be a vector field such that $\tilde{\delta}(x,y) = (u(\Delta k, \epsilon), v(\Delta k, \epsilon))$. We may call such $\tilde{\delta}$ real vector field associated with the $\tilde{\delta}$. Now we see, $\tilde{\delta}(\Delta k + h_1, \epsilon + h_2) - \tilde{\delta}(\Delta k, \epsilon) = a_1h_1 - a_2h_2 + i(a_2h_1 + a_1h_2) + \sigma_1(h_1, h_2)h_1 + \sigma_2(h_1, h_2)h_2$ with $\sigma_1(h_1, h_2)(h_1, h_2) \longrightarrow 0$; where $j = 1, 2$.

Assuming that $\tilde{\delta}$ is holomorphic then one can conclude that $\tilde{\delta}$ is differentiable in the sense of the real variables. In particular the derivatives are represented
by the Jacobian matrix
\[ J_\delta(\Delta k, \epsilon) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} u_{\Delta k} - u_\epsilon \\ v_{\Delta k} - v_\epsilon \end{pmatrix} \Rightarrow \tilde{\delta}'(z) = \frac{\partial u}{\partial \Delta k} - i\frac{\partial u}{\partial \epsilon} \text{ and } u_{\Delta k} = v_\epsilon = a; u_\epsilon = -v_{\Delta k} = -b, \text{ which are the calibrated Cauchy Riemann equations. Conversely, let } u(\Delta k, \epsilon) \text{ and } v(\Delta k, \epsilon) \text{ be two continuously differentiable real functions satisfying Cauchy Riemann equations. Define a complex function } \tilde{\delta}(z) = \tilde{\delta}(\Delta k + i\epsilon) = u(\Delta k, \epsilon) + iv(\Delta k, \epsilon) \text{ then reversing the above steps, immediately yields that } \tilde{\delta} \text{ is complex differentiable, i.e. } \tilde{\delta} \text{ is holomorphic. Further notic that the Jacobian determinant } \Delta_\tilde{\delta} = a^2 + b^2 = u_{\Delta k}^2 + v_{\Delta k}^2 = u_{\Delta k}^2 + u_\epsilon^2. \\
So } \Delta_\tilde{\delta} \geq 0 \text{ and is non zero iff } \tilde{\delta}'(z) \neq 0. \text{ We see simply that the relation between } \tilde{\delta}'(z) \text{ and Jacobian determinant of } \tilde{\delta} \text{ is: } \Delta_\tilde{\delta}(\Delta k, \epsilon) = |\tilde{\delta}'(z)|^2. \text{ So let } \\
\tilde{\delta}(\Delta k, \epsilon) = u(\Delta k, \epsilon) + iv(\Delta k, \epsilon) \text{ then from above equation } u(\Delta k, \epsilon) \to 0 \text{ and } v(\Delta k, \epsilon) \to 0 \text{ as } \epsilon \to 0. \text{ Conversely, if Cauchy- Riemann conditions are satisfied then } \tilde{\delta}(\Delta k, \epsilon) \text{ is analytic. Whence } u_{\Delta k} = v_\epsilon \text{ and } u_\epsilon = -v_{\Delta k}. \text{ Since } u(\Delta k, \epsilon), v(\Delta k, \epsilon) \to 0 \text{ as } \epsilon \to 0 \text{ so } v_\epsilon \text{ and } u_\epsilon \text{ exits and so the } u_{\Delta k} \text{ and } v_{\Delta k} \text{ exits too from the above Cauchy- Riemann equations. Hence } \exists \text{ an analytic function } \tilde{\delta}(\Delta k, \epsilon) \text{ and hence the } \alpha \text{ corrections. Further in the case of anti-analytic function, we have anti-Cauchy Riemann conditions along with the similar steps.}

**Proof of the Proposition:**

Let } \tilde{\delta}(\Delta k, \epsilon) \text{ be the analytic function, then we have } \tilde{\delta}(\Delta k, \epsilon) = \tilde{\delta}(0, 0) + \Delta k \frac{\partial \tilde{\delta}}{\partial \Delta k}|_0 + \Delta \epsilon \frac{\partial \tilde{\delta}}{\partial \Delta \epsilon}|_0 + \ldots \text{ which is usual Taylor series in the sense of a real valued function. For instance in the case of single variable it is just as usual: } f(\xi) = f(\xi_0) + (\xi - \xi_0)\frac{df(\xi)}{d\xi}|_{\xi = \xi_0} + \ldots. \text{ So from equation (2) we have, } \Delta x = \frac{a}{\Delta k} + \frac{\tilde{\delta}(0,0)}{C} + \frac{\alpha}{C} \Delta k + \frac{\gamma}{C} \Delta \epsilon + \ldots, \text{ where } \alpha := \frac{\partial \tilde{\delta}}{\partial \Delta k}|_0 \text{ and } \gamma := \frac{\partial \tilde{\delta}}{\partial \Delta \epsilon}|_0. \text{ Or } \Delta x = \frac{a}{\Delta k} + \frac{\tilde{\delta}(0,0)}{C} + \frac{\alpha}{C} \Delta k + \frac{\gamma}{C} \Delta \epsilon + \tilde{\delta}(0) + \ldots \text{ As } \tilde{\delta}(0) \text{ is arbitrary small constant so if it may be choosen to be zero, i.e. } \tilde{\delta}(0) = 0. \text{ And similarly for given } \{\epsilon_i\} \text{ if we can chose a sequence } \epsilon \text{ such that } \Delta \epsilon = 0 \text{ which is physically equispaced real lattice. Then we have, } \Delta x = \frac{a}{\Delta k} + \frac{\alpha}{C} \Delta k. \text{ By the dilation } k \to ak \text{ we can define a new } x \text{ in the Fourier transform such that } f(x) \text{ remains same after normalization, which is just the scaling, i.e., } \Delta x = \frac{a}{\Delta k} + \alpha \Delta k. \text{ Define a new variable } p = h \epsilon \text{ then } \Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha}{h} \Delta p; \text{ where } \alpha := \frac{\partial \tilde{\delta}}{\partial \epsilon}|_0. \text{ In other words, we have at leading order(s), } \tilde{\delta}(\Delta x, \Delta k; \epsilon) = C \alpha \Delta k. \text{ Further our analysis can be easily generalized to the case of non zero } \tilde{\delta}(0), \text{ i.e. } \tilde{\delta}(0) \neq 0 \text{ then } \Delta x \geq \frac{\hbar}{\Delta p} + \frac{\alpha}{h} \Delta p. \text{ It is obvious even if the } \Delta \epsilon \neq 0 \text{ yet this generalized stringy uncertainty principle holds good. This completes the perturbative proof of the proposition.}

**Remark cum Corollary:**
We notice for arbitrary quantum physical situation that the non perturbative spacetime corrections are contained in the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ whereas the perturbative corrections comes whenever there exist a factorization of the type $\tilde{\delta}(\Delta x, \Delta k, \epsilon) = C_1(\Delta x)\tilde{\delta}(\Delta k, \epsilon)$ or $\tilde{\delta}(\Delta x, \Delta k, \epsilon) = C_2(\Delta k)\tilde{\delta}_1(\Delta x, \epsilon)$ which is a resolution criteria for the UV/IR mixing problem contains in the functions $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ operating at all the scales. Moreover, we can observed that the first Taylor coefficient of $\tilde{\delta}(\Delta k, \epsilon)$ determines the physical quantities like Planck length scales or string length scales: $\tilde{\alpha} = \alpha h_p^2 = l_s^2$.

**Generalization:**

The generalized uncertainty principle in more realistic physical situations arises as follows: Consider a general $d$ dimensional manifold $(M, g)$ and a sequence $\{\varepsilon_i | i \in \Lambda\}$ with $\varepsilon_i := (\varepsilon_i^1, \varepsilon_i^2, \ldots, \varepsilon_i^d) \in M$. Define the norm by $\|\varepsilon_i - \varepsilon_j\| := \sum_{\alpha, \beta = 1}^{d} g_{\alpha \beta}(\varepsilon_i^\alpha - \varepsilon_j^\alpha)(\varepsilon_i^\beta - \varepsilon_j^\beta)$, where $d = \text{dim}(M)$. Then the various step sizes are $\{\varepsilon_i := \|\varepsilon_i - \varepsilon_j\|\}$ for some fixed $j \in \Lambda$. Let $\epsilon := \max_{i, j \in \Lambda}\{\|\varepsilon_i - \varepsilon_j\|\}$ be the maximum step size with given any index set $\Lambda$. Then $\forall i, j \in \Lambda$ we see that $\epsilon \geq \|\varepsilon_i - \varepsilon_j\|$. This local definition of $\epsilon$ is in perfect agreement with usual $d$-dimensional $R^d$ for $g_{\alpha \beta} = \delta_{\alpha \beta}$ with $\|\varepsilon_i - \varepsilon_j\|_2 = \sqrt{(\varepsilon_i^1 - \varepsilon_j^1)^2 + (\varepsilon_i^2 - \varepsilon_j^2)^2 + \ldots + (\varepsilon_i^d - \varepsilon_j^d)^2}$ which in the case of usual one dimensional situations reduces to $\epsilon := \max_{i, j \in \Lambda}\{|\varepsilon_i - \varepsilon_j|\}$. Notice that the usual metric is 2-norm given by $d\varepsilon^2 := \sum_{\alpha, \beta = 1}^{d} g_{\alpha \beta}d\varepsilon^\alpha d\varepsilon^\beta \leq \epsilon^2$ and $\epsilon = 0$ iff $\varepsilon_i = \varepsilon_j$. In particular, for any general $d$-dimensional lattice $Z^d$, one just need to restrict the $\varepsilon_i$ to $Z^d$. Furthermore, we can have two cases for the $\Lambda$: (i) when $\Lambda$ is a finite set then the underlying manifold $M$ can be finitely covered by suitable open sets $U_i$ and so is compact with $M = \bigcup_{i \in \Lambda} U_i$, (ii) when $\Lambda$ is infinite set then the $M$ can not be finitely covered and so is non compact. In either cases, one may conclude that there exists a function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ such that the known generalized uncertainty principles given before holds as the special cases which has been proved in the simple case of $d = 1$. It is the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ that contains in general all the stringy at corrections, UV/IR mixings, and the minimum and maximum length scales of nature from the perspective of a quantum spacetime or short distance geometries.

### 4 The function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$.

In this section, we have analyzed and compared our results and properties of the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ with the other known physical and mathematical very recent results. We have focused mainly the attention on the consequences coming from the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ to certain recent developements such as: quantum gravity effects, black holes and existance of minimal length scale in nature, string theory and short distance geometries, fourier transformation versus distributions and discretization of the spacetime. It has been shown that our results are in
perfect accordance with these developments and are the appropriate indications of the properties required for any physical theory at the short distance geometries. The physical and mathematical consequences of our completely generalized uncertainty principle arising from the properties of the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ are in order:

(i) **Quantum gravity effects:** The quantum mechanical uncertainty relations with the explanations of above sorts are useful for understanding some crucial qualitative properties of short distance divergences associated with the quantum gravity where string theory unifies gravity with other gauge forces and resolves the difficulty of divergence by exhibiting a promising new structures which could not be envisaged if one remains within the framework of the ordinary local field theories. The function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ contains the most general uncertainty relation(s) for example the time energy uncertainty relation formulated by reinterpreting the spacetime distance scales resolves the divergence problems and leads to the proposal of a new uncertainty relation in spacetime. In fact the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ reflects the existence of a minimum length in nature which is consistent with the discretization of spacetime at the Planck scale [54] and the replacement of ordinary Lorentzian group symmetries by quantum group symmetries as the spacetime at small scales which is not necessarily governed by the familiar Poincaré symmetries.

The term $\Delta x^\mu \geq \Delta p^\mu$ is due to the classical gravity effects emerging from nonpertubative quantum string dynamics which indicates long distance effects [16], [19], [20], [28], [29], [30]. The high energy particle scattering is dominated by graviton exchange [55] and with energies higher than $m_p c^2$ black hole is produced by the coherent emission of real gravitons. On other hand, the regime where $\Delta x^\mu \geq \frac{1}{\Delta p^\mu}$ is characterized by ordinary gauge interactions based on standard quantum mechanics. So there appears to be two physical regimes which are manifestations of the same quantum string dynamics: one where energies are much larger and other where the energies are much smaller than the Planck energy. Further within the context of scale relativity Carlos Castro has proposed that the $\Delta x^\mu \geq \Delta p^\mu$ behaviour originates from the existence of the upper scale in nature which is also a long distance effect pertaining to classical gravity interactions at cosmological scales; whereas the $\Delta x^\mu \geq \frac{1}{\Delta p^\mu}$ terms are the standard quantum mechanical results originating from the fractal structure of spacetime at microphysical scales: [4], [5], [56]. All these physical situations are described by the single function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$. In this fashion the dilation of the $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ merge with string theory where the upper length scale $L$, can have a connection to Planck scale physics along with the basis of Nottales proposal to resolve the cosmological constant problem.

It is this function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ that contains UV/IR mixing which is resolved if our remark of the factorization holds. In this case our theorem implies the existence of perturbative stringy $\alpha'$ corrections. Moreover, the relationship to string
theory is realized by the T-duality symmetry: $R \leftrightarrow \alpha' R$; where string theory does not distinguish large radius spacetime from the other small radius spacetime, see [57] for a review of non-perturbative string theory. The fractal structure of spacetime is a crucial question in any theory of quantum gravity. Actually, the idea of scaling property can be generalized for certain non-abelian discrete groups $\Gamma$ in a geometric model by considering a set of equivalence classes of unitary irreducible representations $\hat{\Gamma}$ of a finitely generated discrete group $\Gamma$. The structure of this dual object $\hat{\Gamma}$ can be analyzed from an algebraic and measure theoretic point of view by choosing the dimension $d$ of the manifold and the number of generators $r$ of $\Gamma$ independently from each other, in contrast to the usual Schrödinger operator case. Furthermore, there exists generating spectral gaps in an abstract geometric situation. In particular, let $\mathcal{N}$ be a non-compact Riemannian covering manifold with a discrete isometric group $\Gamma$ with the Laplacian $\Delta_\mathcal{N}$ such that the quotient $\mathcal{N}/\Gamma$ is a compact manifold. Then minimum/maximum principle implies that $\exists$ a finite number of spectral gaps for the operator $\Delta_\mathcal{N}$ associated with a suitable class of manifolds $\mathcal{N}$ with non-abelian covering transformation groups $\Gamma$ [58], [59]. Quantum mechanics requires a functional average over all possible equivalence classes of metrics and it is in this way how the effective dimension of spacetime is measured which has been shown by Nottale that quantum mechanics arises from the fractal nature of particle trajectories [5], [56].

(ii) **Black holes and existence of minimal length scale in nature:** On the same line, why the minimum scale of nature must appear in the fundamental equation? It was proposed by Susskind [60] that the size of the string increases instead of decreasing at energies higher than the Planck energy so the energy imparted to the string is used to break the strings into pieces. This statement is consistent with the Bekenstein Hawking bound [61], [62], [63] where the black hole entropy in terms of the horizons area is $S = \frac{A}{4G}$. This is to say that one cannot have more than one bit of information per unit area in Planck units. We are sure that the uncertainty principle will play an important role in the underlying fundamental principles to formulate string theory where the minimum length in nature are the arguments of nonpertubative quantum string dynamics pertaining to the nonexistence of black holes of sizes $2M$ smaller than $l_p$ and the subsequent fact that Hawking radiation must stop at some point when the Hawking temperature reaches the Hagedorn temperature: $T = \frac{1}{8\pi M} = T_H$. It was suggested that at this point conformal invariance breaks down and no consistent string propagation is possible [33]. We can see that the relation of $\tilde{\alpha}$ to the minimum length parameter of nature $l_p$ is given by $\tilde{\alpha} = \alpha l_p^2$. Actually the usual Lorentz transformations do not apply in the world of Planck scale physics as only at large scales the Riemannian continuum is recaptured. The minimum scale in string theory is implemented by the more fundamental Finsler Geometries which is Riemannian geometry without the restriction that the line element be quadratic [64]. The function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ not only reproduces the ordinary string
uncertainty relations but yields all the perturbative as well as non perturbative corrections operating in nature thereof in one single stroke, as we have shown. This is a positive sign that the function $\tilde{\delta}$ is on the right track to reveal the geometrical foundations of M-theory. Further the $\tilde{\alpha}$ shows a link between the Regge trajectories behaviour of the string spectrum and the quantization of black holes horizon area which is giving a signal of the microscopic counting of the microstates and forming a black hole.

The black hole thermodynamics with the generalized uncertainty principle has been considered by Nozari et. al [11]. In this case the generalized uncertainty principle modifies the usual black hole thermodynamics as follows. The small black holes emit black body radiation at the Hawking temperature, $T = \frac{M_c^2}{4\pi} \left[ 1 \pm \sqrt{1 - \frac{M_p^2}{M^2}} \right]$ with modified Benkenstein Hawking entropy $S = 2\pi \left[ \frac{M_p^2}{M^2} \left( 1 - \frac{M_p^2}{M^2} + \sqrt{1 - \frac{M_p^2}{M^2}} \right) \right]$, where in above expression of temperature $T$ the minus sign indicate the large mass limit. Further in their framework of generalized uncertainty principle, a black hole can evaporate until it reaches the Plank mass. So in this view black hole remnants are stable objects. Further application of generalized uncertainty principle to black hole thermodynamics strongly suggests the possible existence of black hole remnant. These black hole remnants have remnant entropy which at least can be related to background spacetime metric fluctuations. Further the above authors have shown: If we consider hydrogen atom and black hole in the generalized uncertainty principle, then hydrogen atom is unstable and collapse totally but a black hole evaporate until it reaches the Plank mass. The issue of stability of remnants can be considered in the framework of symmetry principle, in particular supergravity gives very good framework of providing such black hole remnants. According to the standard notion, the black hole evaporation would terminate when it reduces to a remnant as the graviton spectrum which should have a cutoff at the Plank mass that would have by now redshifted to $\sim 10^{14}$. As a consequence of generalized uncertainty principle, we can only measure the position of any object only up to the Plank length $l_p$. So one can not setup a measurement to find more accurate particle position than the Planck length itself which means that the notion of locality breaks down at the $l_p$ which is in complete accordance with the properties of the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$. In certain dynamical sense this equation is more compatible with physical grounds than the more generalization on the right hand side of the above generalized uncertainty equation. For example, consider $\Delta x = \Delta x = \frac{k}{2\pi}$ and $\Delta p = \hbar \Delta k$ with $\Delta x = \frac{k}{2\pi} + \tilde{\alpha} \Delta k$ and $\omega = \frac{\omega}{k} \Delta k$ then the dispersion relation in this situation is given by $\omega = \omega(k) = \frac{k_c}{(1+\tilde{\alpha}k^2)}$.

(iii) **String theory and short distance geometries:** From a perspective of quantum strings, the string dynamics obey the usual rules of quantum mechanics, so it is natural to seek a simple universal characterization of the stringy properties with respect to the spacetime distance scales just by using the uncertainty
relations with taking into account of the special characteristics of string theory. Although the coordinate of the center of mass and the corresponding momentum of the string satisfy the usual coordinate momentum uncertainty relation but the crucial role of the extendedness of the string can be seen properly by the energy time uncertainty relation which is valid for arbitrary dynamical processes. So let the uncertainty with respect to time is $\Delta t$ then the energy of strings is just given by the length of a string measured along the string. Now due to the universal nature of the constant energy density, it is natural to identify the precision scale $\Delta E$ of energy to the extension $\Delta x$ of the string along its longitudinal direction as $\Delta E \sim \frac{\Delta x}{\tilde{\alpha}}$ [38]. Therefore the time energy uncertainty relation can be reinterpreted as the uncertainty relation of the spacetime in the form $\Delta t \Delta x \geq \tilde{\alpha}$, where the new parameter $\tilde{\alpha}$ is defined in terms of the string length parameter $l_s$ by $l_s^2 = \tilde{\alpha}$. In this sense precisely how the function $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ determines all the stringy corrections contained in our completely generalized uncertainty principle for all the perturbative as well as nonperturbative stringy corrections. This spacetime uncertainty relation for non zero $\tilde{\delta}(\Delta x, \Delta k, \epsilon)$ indicates that the strings cannot probe short distance scales to arbitrary precision simultaneously with respect to both time like and space like distances. As string theory is self-contained so the spacetime structure should be determined self consistently by using the dynamics of strings themselves. So the above relation characterizes qualitatively the spacetime itself if we believe string theory as the fundamental unified theory of nature. We may express this situation by claiming that spacetime is ‘quantized’. In addition to this, the proportionality between energy and the longitudinal length indicates that the large quantum fluctuations of energies associated with short time measurement are actually reinterpreted as the fluctuations of long strings and hence turn into a long distance phenomenon. In other words, the structure of quantum fluctuations of string theory is drastically different from that of the ordinary quantum field theory of point particles. This is precisely the physical mechanism which is hidden behind the proof that the string perturbation theory has no divergences associated with the short distance spacetime structure [38],[39].

(iv) Fourier transformation and distributions: Now we focus on the relation between Fourier transform and uncertainty principle from a general perspective: Classically Fourier transformation in geometry is a duality map between functions on a vector space $V = \mathbb{R}^d$ and on its dual vector space $V^*$ which is given by: $f(\vec{x}) \rightarrow \hat{f}(\vec{k}) = \frac{1}{(2\pi)^d} \int_V f(\vec{x}) e^{i \vec{k} \cdot \vec{x}} d\vec{x}$. From the point of view of Fourier transform, the uncertainty principle asserts that a function $f$ and its Fourier transform $\hat{f}$ cannot both be sharply localized. For instance, let $E, \Sigma \subset \mathbb{R}^d$ be two measurable sets then we define: (i) $(E, \Sigma)$ to be a ‘weakly annihilating pair’ if the function $f \in L^2(\mathbb{R}^d)$ vanishes as soon as $f$ is supported in $E$ and $\hat{f}$ is supported in $\Sigma$. (ii) $(E, \Sigma)$ to be a ‘strongly annihilating pair’ if $\exists \mathcal{C}$ such
that $\forall f \in L^2(R^d)$ with spectrum in $\Sigma : \|f\|_2^2 \leq C \int_{E^c} |f|^2 dx$ holds. There are certain methods from complex analysis to obtain sharp results on $E$ in terms of the density for it's complement which determines a precise estimate of the constant $C$. See for example, Nazarov when $E$ and $\Sigma$ are finite measure in one dimension [65] and Kovrijkine when $\Sigma$ is a finite union of parallelopipeds [66]. To consider annihilating pairs, let us recall usual definition of $\varepsilon$- thin set in $R^d$. Let $0 < \varepsilon < 1$ and $B(x)$ is the ball centered at $x$ of radius $\text{min}(1, \frac{1}{|x|})$ then a set $E$ in $R^d$ is called $\varepsilon$- thin if $\forall x \in R^d : \frac{|E \cap B(x)|}{|B(x)|} < \varepsilon$. The strongly annihilating pairs can be expressed in terms of decomposition of the spacetime related to the level set of functions $\{|x|^{a_i}\}_{i=1}^d$ which is linked with Heisenburg type inequalities and shows the existence of spectral gaps for some operators. Furthermore, Hardy type uncertainty principle asserts that for any function $f$ satisfying following two inequalities: (i) $|\hat{f}(x)| \leq C(1 + |x|)^N e^{-\pi|x|^2}$ and (ii) $|\hat{f}(k)| \leq C(1 + |k|)^N e^{-\pi|k|^2}$. Then $\exists$ a polynomial $P(x)$ of degree atmost $N$ such that $f(x) = P(x)e^{-\pi|x|^2}$.

Let us consider a general problem of the uncertainty principle related with quadratic forms: Let $q, q' \in R^d$ be two non degenerate quadratic forms, then the distributions $f \in S'(R^d)$ with $e^{\pi q} f \in S'(R^d)$ and $e^{\pm \pi q'} f \in S'(R^d)$ describes the space of such distributions whenever it is possible and so find the conditions on $q, q'$ such that this space is reduced either to 0 or to singular distributions? In the case when one of the quadratic form is positive, or when one of them has signature $(d-1,1)$ the uncertainty principle is answered by following Lemma: Consider a function $f$ on $R^2$ satisfying: (i) $|f(x,y)| \leq Ce^{2\pi a |x|y}$ and (ii) $|\hat{f}(x,y)| \leq Ce^{2\pi b |k_1k_2|}$, with $ab > 1$, then $f = 0$ [67]. As it is not known apriori whether the two functions are in $L^1$ or $L^2$, so we need to take the Fourier transform in the sense of a distribution instead of functions. The classical complex analysis is the main tool appearing naturally once we transform the conditions by an integral transform, for example Bargmann’s representation theory. So we see that for some pairs $(q, q')$, the two sets $\{x \in R^d; |q(x)| < A\}$ and $\{k \in R^d; |q'(k)| < A\}$ form a weakly annihilating pair in the sense of distribution. The radar ambiguity functions generalizes all the results and defines the strongly or weakly annihilating sets which are given as follows: Let $u, v$ be two functions of $L^2(R^d)$ then the radar ambiguity function associated to $u$ and $v$ is defined by $A(u,v)(x,y) = \int_{R^d} u(k + \frac{x}{2})v(k - \frac{y}{2})e^{2\pi i<k,y>} dk, \forall x, y \in R^d$. Furthermore the radar ambiguity function can be generalized by elementary changes to Wigner transform, windowed Fourier transforms and the phase retrieval problems which all may be shown to be related to the function $\delta$ from the point of view of a distribution that would probably require the knowledge of full non perturbative behavior of quantum string or full quantum gravity itself. We leave these issues to the future.

The spectral properties of Laplacians on manifolds in comparison with the periodic Schrödinger operators can be investigated by defining a periodic manifold $\mathcal{N}$ or Riemannian covering space of $\mathcal{M}$ with covering transformation group $\Gamma$ as follows [68]: Let $\mathcal{N}$ be a non compact Riemannian manifold of dimension
$d \geq 2$ then (i) $\mathcal{N}$ is called periodic manifold if the action on $\mathcal{N}$ of $\Gamma$ of the isometries of $\mathcal{N}$ are such that the quotient $\mathcal{M} := \mathcal{N}/\Gamma$ is a $d$-dimensional compact Riemannian manifold. (ii) A fundamental domain $\mathcal{D}_{\gamma}$ is fixed by an open set $\mathcal{D} \subset \mathcal{N}$ such that $\gamma \mathcal{D} \cap \gamma' \mathcal{D} = \phi$ for all $\gamma \neq \gamma'$ with $\bigcup_{\gamma \in \Gamma} \gamma \mathcal{D} = \mathcal{N}$. What follows, we analyze geometrically a periodic operator as follows. Let $(g_{ij})$ be the metric tensor with the inverse metric tensor $(g^{ij})$ then the pointwise norm of the 1-form $d\xi$ in coordinate representation is given as $|d\xi|^2 = \sum_{i,j} g^{ij} \partial_i \xi \partial_j \xi$.

Consider an elliptic operator with Laplacian $\Delta_{\mathcal{N}}$ on $\mathcal{N}$ acting on a dense subspace of the Hilbert space $L^2(\mathcal{N})$ with norm $\|\cdot\|_{\mathcal{N}}$. Then a positive self-adjoint operator $\Delta_{\mathcal{N}}$ can be defined in terms of a suitable quadratic form $q_{\mathcal{N}}$ as follows [69]: $q_{\mathcal{N}}(\xi) := \|d\xi\|^2_{\mathcal{N}} = \int_{\mathcal{N}} |d\xi|^2 d\mathcal{N}$, where $\xi \in C_c^\infty$. Now because of the $\Gamma$-invariance of the metric on $\mathcal{N}$, the Laplacian $\Delta_{\mathcal{N}}$ commutes with the translations on $\mathcal{N}$, i.e., $(T_\gamma \xi)(x) := \xi(\gamma^{-1} x), \forall \xi \in L^2(\mathcal{N}), \forall \gamma \in \Gamma$. So the usual the operator $\Delta_{\mathcal{N}}$ is related with the quadratic form $q_{\mathcal{N}}$ by the formula $\langle \Delta_{\mathcal{N}} \xi, \xi \rangle = q_{\mathcal{N}}(\xi), \xi \in C_c(\mathcal{N})$. Hence the closure of $q_{\mathcal{N}}$ can be extended onto the Sobolev space $H^1(\mathcal{N}) := \{ \xi \in L^2(\mathcal{N}) | q_{\mathcal{N}}(\xi) < \infty \}$. Then one can find an order relation for the eigen values for the either Dirichlet Laplacian $\Delta^D_{\mathcal{N}}$ or Neumann Laplacian $\Delta^N_{\mathcal{N}}$ by the maximum minimum principle: $\lambda_k^+ = \inf_{\{\mathcal{L}_k\}} \sup_{\|\xi\|^2_{H^1(\mathcal{N})}} \frac{q_{\mathcal{N}}(\xi)}{\|\xi\|^2}$, where the infimum is taken over all $k$-dimensional subspaces $L_k$ of the corresponding quadratic form domain $\text{dom}(q^D_{\mathcal{N}})$ with $\text{dom}(q^D_{\mathcal{N}}) \subset \text{dom}(q^N_{\mathcal{N}})$. So in general for purely discrete spectrum leveled by $k \in \mathbb{N}$, the Dirichlet eigen values $\lambda_k^+$ are greater than the corresponding Neumann eigen values $\lambda_k^-$. On other hand, Mustard has developed a new family of measures that are invariant under the group of fractional Fourier transform obeying certain uncertainty principle [70]. In this case, the Heisenburg’s uncertainty principle asserts an inequality for some measure of joint uncertainty associated with a function and it’s Fourier transform. Mathematically, an uncertainty principle asserts a reciprocity relation between the spread of a function $f$ and the spread of it’s Fourier transform $\hat{f}$. Unless both $f$ and $\hat{f}$ decreases to zero at infinity faster than $|x|^2$, either $\Delta(f)$ or $\Delta(\hat{f})$ is infinite and then the Heisenburg’s uncertainty principle is uninformative about the minimum of the other. To get rid of the limitations of the Heisenburg’s uncertainty principle we have various generalizations to other physical situations then the usual quantum systems and other different measures for the spread of $\Delta(f)$ and $\Delta(\hat{f})$. One of the such nice situation can be illustrated in the signal analysis by considering the asymptotic dimension of a class of functions which are almost band limited and duration limited for which $\|f - \chi_T f\|^2 < \epsilon$ and $\|\hat{f} - \chi_\Omega \hat{f}\|^2 < \epsilon$ as $\Omega T \to \infty$. Then we have a fractional Fourier transform $\mathcal{F}_\theta$ invariant definition of the class of functions which are considerably interesting and appropriate. For example, in the case of $R^2$ Mustard conjectured [70] that $S(A, \epsilon) = \{ f | \min(E) \|W_f - \chi_{E(A)} W_f\|^2_{R^2} < \epsilon \}$, where $W_f$ is a Wigner distribution of $f$ and $\chi_{E(A)}$ is the characteristic function of $A$ on $R^2$. It is the measure theory of $f$ and $\hat{f}$ given respectively as $\Delta(f)$ and $\Delta(\hat{f})$ with our
completely generalized uncertainty principle: \( \Delta(\hat{f}) = a \Delta(f) + \tilde{\delta} \) that is interestingly important from the perspective of obtaining the relations between \( \tilde{\delta} \) with \( \mathcal{W}_f \) and the physical perspective of the existence of the minimal and maximal length scales in the nature, black hole physics and short distance geometries like noncommutative geometry, noncommutative Clifford manifolds, and discretization of the spacetime. For more detailed discussions like variants of Heisenberg’s inequality, local uncertainty inequalities, logarithmic uncertainty inequalities, results relating to Wigner distributions, qualitative uncertainty principles, theorems on approximate concentration, and decompositions of phase space, see a survey article [71].

(v) **Discretization of the spacetime:** The complete generalization of the uncertainty principle involves higher integer expectation values like \( <x^n>^m \) and so the other existing generalized uncertainty principles are only an approximation to our complete expression with the function \( \tilde{\delta}(\Delta x, \Delta k, \epsilon) \) containing all these corrections \( \forall n, m \in \mathbb{Z} \). For example when the underlying space is compactified on a circle of radius \( r \), then Taylor or Laurent expansion should be valid when \( r \) is large compared to the dispersion in the state \( \Delta x^2 \). By interchanging the roles of configuration space and momentum space one obtains a discrete spectrum for the position space. So let \( r_p \) be the compactification radius of momentum space, then by interchanging the roles of position and momentum, one obtains a discrete spectrum for position operator with consecutive eigenvalues separated by \( \frac{1}{r_p} \) such that position space is discretized. Further if we want to discretize configuration space on a certain scale, say \( l_P \), then \( r_p \) is determined in terms of the \( l_P \). The parameter \( \tilde{\alpha}_P \) characterizes the inequivalent discretizations of the position operator and the separation of it’s eigenvalues in terms of the compactification radius which is given by \( \frac{1}{r_p} (\hbar = 1) \). Moreover, \( \forall d \geq 2 \), the \( R^d \) Schrödinger operator \( H := -\Delta + V \) with a suitable periodic potential \( V \) has gaps in its spectrum which is ensured by following two facts: (a) The \( V \) is periodic \( \Rightarrow \exists \) a basis \( \{ \eta_i \}_{i=1}^d \) of \( R^d \) such that \( V(x + \eta_i) = V(x), \forall i = 1, \ldots, d \). (b) Let \( D \subset N \) be a fundamental domain for example usual parallelepiped with \( D = (0,1)\eta_1 + \ldots + (0,1)\eta_d \) then the potential \( V \) has a high barrier near the boundary of \( D \), i.e. \( V \) decouples the fundamental domain \( D \) from the neighbouring domains \( \{ \eta_i + D \}_{i=1}^d \). Furthermore, since the potential \( V \) is periodic so \( \exists \) an action of \( \mathbb{Z}^d \) on \( R^d \) which is completely specified on a fundamental domain \( D \subset N \).

The examples of states which do not have any analog with respect to the usual Heisenberg uncertainty principle are those states which are localized at discrete eigenvalues of the position operator. Our generalized uncertainty principle can describe the relationship between the dispersion in \( x \) and \( p \) in the limit where the discrete eigenvalues of the position operator are very finely spaced. The generalized uncertainty principle from a toy model of discretized space with quantum mechanical considerations, where the compactification involves the momentum
dependence which may be a useful models in exploring the ultraviolet limit, is
approached in more realistic models of discrete spacetime or models of quan-
tum gravity with a fundamental or minimum length. A generalized uncertainty
principle contains not only an infinite series of higher powers of the momentum
dispersion, but also typically involves contributions from higher order quantities
such as $< p^n >^m \forall n, m \in \mathbb{Z}$. Further the simple discretization of space implied
by a compactified momentum is enough to obtain the leading order terms in the
generalized uncertainty principle in a natural way and the detailed incorporation
of gravity does not seem to be necessary to obtain the generalized uncertainty
principle. In this way, our result is an improved understanding of the origin of
the generalized uncertainty principle in the theories of quantum gravity. So the
string theory exists since it is mathematically consistent, and it’s devil exists since
it’s non perturbative consistency can not be proved. However, we believe that
the investigations of the models of this sort may be useful for seeking the right
direction towards our final goal to develop appropriate mathematical framework
and to construct really the basis of the ultimate theory of everything.

5 Conclusion.

In this paper we have analyzed the complete extension and modification of the
Heisenberg uncertainty principle within the framework of the theory of complex
functions. In particular with the help of complex analysis, we have shown that
our theorem is a resolution criteria for the UV/IR mixing problem. This theorem
is obtained by proposing the existance of certain functions $\tilde{\delta}$ which includes the
effects of quantum gravity at all scales. In our analysis of stringy uncertainty prin-
ciple the size of the strings are bounded by the Planck scale and the size of the uni-
verse. It is the complex function $\tilde{\delta}$ that explains string uncertainty priciple with
all higher order as well as nonperturbative stringy correcctions, physics of quan-
tum gravity, black hole physics, existance of minimal and maximal length scales
in nature, short distance geometry versus string theory and Fourier transforma-
tion versus distributions etc. Our completely generalized uncertainty principle
from the discretization of the spacetime is in good agreement with short distance
geometries like noncommutative geometry, noncommutative Clifford manifolds,
noncommutative nature of spacetime at the Plank scale where all such known
physical concepts may be explained nicely just from a single complex function $\tilde{\delta}$
which may also reveals the geometric origin of the fundamental M-theory. Based
on the inherent lattice structures, we may say that the underlying fundamental
principle behind string theory should be based on an extension of the scaling of
the complex function $\delta$ which is to be incorporated on the same footing as that
of the dynamics.

It is shown that the generalized uncertainty principle based on the $\tilde{\delta}$ have
found strong supports from string theory, noncommutative geometry and loop
quantum gravity. Moreover, there are many implications originated from our generalized uncertainty principle for the rest of the physics. For example, generalized uncertainty principle changes, from the view point of the statistical mechanics, the volume of the fundamental cell of the phase space in a momentum dependent manner by the $\tilde{\delta}$ or the resolution of the UV/IR mixing problem of quantum field theories, etc are in good accordance with our results. These quantum gravity features have novel implications for statistical properties of a thermodynamical systems. Generalized uncertainty principle is a common feature of all promising candidates of quantum gravity. For example string theory, loop quantum gravity and noncommutative geometry that gives a deeper insight to the nature of spacetime at Planck scale, all indicate the modification of standard Heisenberg uncertainty principle. It has also been indicated that within quantum gravity scenarios, a modification of dispersion relation is unavoidable and there is a positive sign for the existence of black hole remanants because of the quantum gravitational induced corrections to black hole entropy and temperature.

In any theory of quantum gravity, it is expected on general grounds that our generalized uncertainty principle may apply when the momenta are of the order of Planck scale and gravitational effects become important. In the context of the quantum mechanics, the generalized uncertainty principle has been described as a general consequence of incorporating a minimal length by the first Taylor coefficient of the $\tilde{\delta}$ from a theory of quantum gravity. We can realize physically that if enough mass energy is confined to a small region of space then a black hole must form so there exists a minimal length that can be measured. For example one expects that the short distance effects to be hidden behind an event horizon if one increases the energy of colliding particles beyond the Planck energy. In fact, the size of this event horizon increases and an effective minimal length might arise through the discretization of spacetime as the energy is increased. So one may ask the following question for more proper resolution of the minimum length problem: Does the generic appearance of a minimal length in a low energy effective quantum gravity survives in the full quantum theory when the ultraviolet sector is completed?

On other hand, most of the modifications to the uncertainty relations are motivated by either what is a general property of any theory of quantum gravity or analyze the phenomenological implications of the generalized uncertainty principle. But in this paper we have explained, how the generalized uncertainty principle can be well understood just from a holomorphic or anti-holomorphic function and how is it completed when all the relevant terms are included form the Taylor or Laurant series expansion of $\tilde{\delta}$ that contain all the perturbative and non-perturbative information of quantum gravity. It is like a model of quantum theory with a position operator having discrete eigenvalues which can be considered as a theory with a minimum length associated with the difference between eigenvalues. So our generalized uncertainty principle obtained from the complex analysis the of function $\tilde{\delta}$ fits into a complete theory of discretized spacetime,
with the existence of minimum and maximal length scales in nature, quantum geometries and black hole physics. Furthermore we are interested in general: how to quantize a general classical systems with some phase space particularly like geometric quantizations which we leave for the future investigations.

6 Acknowledgements

The author would like to thank Bikash Bhattacharjya, Vinod Chandra, Ashok Garai, Pratyooosh Kumar, Ravindra Kumar, Pankaj Kumar Mishra, J. Prakash, V. Ravishankar, Ashoke Sen, Gautam Sengupta, Ravinder Singh, K. P. Yogendra for discussion and comments; and Mohd. Ashraf Bhat, Ashish Kumar Mishra, Nirbhay Kumar Mishra for reading the manuscript and making several helpful suggestions. This work was supported by CSIR, India, under the research grant: CSIR-SRF-9/92(343)/2004-EMR-I.

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