Random Matrix Theory of Transition Strengths and Universal Magnetoconductance in the Strongly Localized Regime

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Random matrix theory of the transition strengths is applied to transport in the strongly localized regime. The crossover distribution function between the different ensembles is derived and used to predict quantitatively the universal magnetoconductance curves in the absence and in the presence of spin-orbit scattering. These predictions are confirmed numerically.

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Random matrix theory (RMT) has been used extensively and successfully in the field of nuclear physics,\textsuperscript{1,2} where the predictions of the theory concerning the distribution of excitation energies and the distribution of transition strengths agree well with experimental data. The results of RMT concerning the \textit{excitation-energy distribution} have been successfully applied also in condensed matter physics.\textsuperscript{3–6} In this work we demonstrate how RMT of \textit{transition strengths} can be applied in condensed matter theory, in particular in the study of transport in the variable-range-hopping regime. We obtain \textit{universal} behavior of the magnetoconductance (MC) in the deeply localized regime, which is determined by the crossover function between the orthogonal to the unitary ensemble, in the absence of spin-orbit scattering, and between the symplectic to the unitary ensemble in the presence of strong spin-orbit scattering. We derive this crossover function, and predict quantitatively the universal MC curves, which are verified numerically (Fig. 2).

The idea underlying the application of RMT rests on the assumption that the statistical behavior of a complicated system is determined by its symmetries. Accordingly, the Hamiltonian describing the system can be classified into one of three universality classes, characterized by a parameter $\beta$ which counts the number of degrees of freedom associated with each matrix element: The Gaussian orthogonal ensemble ($\beta = 1$), the Gaussian unitary ensemble ($\beta = 2$), where time-reversal symmetry is broken, e.g. by a magnetic field, and the Gaussian symplectic ensemble ($\beta = 4$), where rotational symmetry is broken, e.g. by spin-orbit interactions. For these different ensembles one can derive the distributions of level spacings and of overlap probabilities, which in the case of nuclear transitions are related to the distributions of excitation energies and of transition strengths, respectively.

The use of RMT in condensed matter problems has been confined so far to predictions concerning the level distribution. It was used for studying electronic properties of small metallic particles,\textsuperscript{3} and it has been also invoked\textsuperscript{4,5} to explain the theoretically predicted\textsuperscript{7} universal conductance fluctuations in the weakly localized
regime in terms of the rigidity of the level spectrum of the transfer matrix\(^4\) or the Hamiltonian.\(^5\) Altshuler and Shklovskii\(^5\) showed that when the Zeeman splitting is neglected, the amplitude of the conductance fluctuations is determined by a parameter \(1/\chi\), defined by \(\chi = 4\beta/s^2\), where \(s\) is the degeneracy of each level. Thus \(\chi = (a)\ 1,\ (b)\ 2,\ (c)\ 4,\ (d)\ 8\) for the cases (a) both time-reversal and rotational symmetries are conserved, (b) only time-reversal symmetry is broken, (c) only rotational symmetry is broken and (d) both symmetries are broken (\(\beta = 2\)). In particular, a magnetic field which breaks time-reversal symmetry decreases the magnitude of the fluctuations, independently of the amount of spin-orbit scattering present. This result was confirmed by experiments.\(^8\)

More recently Pichard et al.\(^9\) claimed that similar arguments can also be applied to the strongly localized regime. Here the conductance is determined by an equivalent resistor network\(^10\) in which each two impurities are connected by a conductance \(g_0 J(H) \exp \left[-r_{ij}/\xi(H) - \Delta\epsilon/kT\right]\), where \(g_0\) has units of conductance, \(r_{ij}\) is the distance between the impurities, \(\Delta\epsilon = (|\epsilon_i| + |\epsilon_j| + |\epsilon_i - \epsilon_j|)/2\), where \(\epsilon_i\) are the energies of the impurity states, and \(\xi\) is the localization length. Both \(\xi\) and the amplitude \(J\) depend upon the magnetic field. Using RMT predictions for the eigenvalue statistics, Pichard et al. concluded that \(\xi\) is doubled as time-reversal symmetry is broken and increases by a factor of four as rotational symmetry is broken, in agreement with earlier exact results\(^11\) in quasi-1\(d\) samples.\(^12,13\) However, a relatively large magnetic field (\(\equiv H_\xi\)) of a unit quantum flux through an area \(\xi^2\) is necessary to induce a change in the localization length. (In such magnetic fields other effects, such as the shrinking of the wave functions may be significant, especially in doped semiconductors, where \(\xi\) is on the order of the Bohr radius of the impurity state.) On the other hand the amplitude \(J(H)\) of the relevant hops is determined by the interference of all paths within a cigar shaped area of length \(R\) and width \(\sqrt{R\xi}\).\(^14,15\) Accordingly, the relevant magnetic field scale for a change in the amplitude, \(H_R\), is a unit flux through the much larger cigar shaped area. (Experimental values for the
ratio between these two areas, $(R/\xi)^{3/2}$, range from 5 to 100 and more, depending on temperature$^{16}$. Consequently, the MC is determined by the amplitude, for a wide range of magnetic fields. Moreover, in the presence of strong spin-orbit scattering the localization length is unaffected by a magnetic field,$^{11,13}$ and the MC is dominated by the magnetic field dependence of the amplitude. This amplitude is determined by the overlap between the impurity wavefunctions. The overlap distribution has been calculated using various numerical and analytic approximations.$^{14,15,17,18}$

In this work we use the fact that the overlap probability between two wavefunctions can be written in a form analogous to the transition strength in nuclear physics, to demonstrate how the overlap distribution can be obtained similarly to the calculation of the transition-strength distribution, using RMT. Thus one expects this distribution to be of a universal nature, leading to universal predictions for the MC in the variable-range-hopping regime.

In order to derive the crossover distribution we study a general interpolating $N \times N$ quaternionic matrix $M$,

$$
M = \sqrt{1 - \frac{\gamma^2}{2}} \left[ \sqrt{1 - \frac{3\delta^2}{4}} S_0 \otimes I + i \frac{\delta}{2} \sum_{i=1}^{3} A_i \otimes \sigma_i \right] \\
+ i \frac{\gamma}{\sqrt{2}} \left[ \sqrt{1 - \frac{3\delta^2}{4}} A_0 \otimes I + i \frac{\delta}{2} \sum_{i=1}^{3} S_i \otimes \sigma_i \right],
$$

(1)

where $S_i(A_i)$ are symmetric (antisymmetric) $N \times N$ random matrices, whose elements are normally distributed with a zero mean and unit variance, and $\sigma_i$ are the Pauli matrices. For $\gamma = \delta = 0$ the matrix $M$ belongs to the Gaussian orthogonal ensemble, for $\gamma = 1, \delta = 0$ it belongs to the Gaussian unitary ensemble, and for $\gamma = 0, \delta = 1$ it belongs to the Gaussian symplectic ensemble. Thus turning on the parameter $\gamma$ from zero to 1 interpolates between the Gaussian orthogonal and the Gaussian unitary ensembles when $\delta = 0$ and between the Gaussian symplectic and the Gaussian unitary ensembles when $\delta = 1$. Similarly changing $\delta$ from zero to 1, with $\gamma = 0$, interpolates between the Gaussian orthogonal and the Gaussian
symplectic ensembles. $M$ is normalized such that the average overlap is independent of the ensemble (i.e. of $\gamma$ and $\delta$). The overlap probability between sites $i$ and $j$, summed over all final spin states and averaged over all initial spin states is given by
\begin{equation}
y = \frac{1}{2} tr M_{ij}^\dagger M_{ij} = \left(1 - \frac{\gamma^2}{2}\right)\left(1 - \frac{3\delta^2}{4}\right)s^2_0 + \\
\left(1 - \frac{\gamma^2}{2}\right)\delta^2 \sum_{k=1}^{3} a_k^2 + \left(1 - \frac{3\delta^2}{4}\right)\gamma^2 a_0^2 + \frac{\delta^2 \gamma^2}{4} \sum_{k=1}^{3} s_k^2, \tag{2}
\end{equation}
where $M_{ij}$ is the $2 \times 2$ block of the matrix $M$, $s_k = (S_k)_{ij}$ and $a_k = (A_i)_{ij}$. The distribution of the overlap probability, $P(y)$, is readily expressed in terms of its Laplace transform $F(s)$,
\begin{equation}
F(s) = \int_0^\infty e^{-sy} P(y) \, dy \tag{3}
= 1/ \left\{ \left[1 + 2\left(1 - \frac{\gamma^2}{2}\right)\left(1 - \frac{3\delta^2}{4}\right)s\right]^{\frac{\chi}{2}} \left[1 + \gamma^2\left(1 - \frac{3\delta^2}{4}\right)s\right]^{\frac{\gamma}{2}} \left[1 + \delta^2\left(1 - \frac{\gamma^2}{2}\right)s\right]^{\frac{\delta}{2}} \left[1 + \frac{\delta^2 \gamma^2}{4} s\right]^{\frac{\gamma}{2}} \right\}.
\end{equation}
For the four pure symmetry cases discussed above, which correspond to (a) $\gamma = \delta = 0$, (b) $\gamma = 1$, $\delta = 0$, (c) $\gamma = 0$, $\delta = 1$, and (d) $\gamma = \delta = 1$, the overlap distribution function is given by
\begin{equation}
P(y) = \frac{(\chi/2)^{\chi/2}}{\Gamma(\chi/2)} y^{\chi/2 - 1} e^{-\chi y/2}. \tag{4}
\end{equation}
Eq. (4) is the exact result for the transition-strength distribution for large matrices belonging to the Gaussian orthogonal ($\chi = \beta = 1$), the Gaussian unitary ($\chi = \beta = 2$) and the Gaussian symplectic ($\chi = \beta = 4$) ensembles. However, when both time-reversal and rotational symmetries are broken, $P(y)$ is given by Eq. (4), with $\chi = 8$, even though the $2N \times 2N$ matrix $M$ [Eq. (1)] belongs to the Gaussian unitary ensemble. The reason, as in the weakly localized regime, is that even when time-reversal symmetry is broken, the broken rotational symmetry couples the two spin directions, so the overlap amplitude involves a sum over the two spins, leading to eight free variables [see Eq. (3)].

The distribution function can be expressed analytically for the crossover from the Gaussian orthogonal ensemble to the Gaussian unitary ensemble ($\delta = 0$, finite
\( P(y) = e^{-y/Z} I_0(\sqrt{1-Zy/Z}/\sqrt{Z}) \), with \( Z = 2\gamma^2(1-\gamma^2/2) \), which is the exact result for the transition-strength crossover function for this case, in the limit of large matrices.\(^2,\)\(^15\) This function has been verified numerically for a kicked rotor.\(^19\) Similarly, the crossover function from the Gaussian orthogonal ensemble to the Gaussian symplectic ensemble is given by

\[
P(y) = \left(1 - \frac{3\delta^2}{4}\right)^{-1/2} y \exp \left[-\frac{y(1-\delta^2)}{\delta^2(1-\frac{3\delta^2}{4})}\right] \left\{ I_0 \left[ \frac{y(1-\delta^2)}{\delta^2(1-\frac{3\delta^2}{4})} \right] - I_1 \left[ \frac{y(1-\delta^2)}{\delta^2(1-\frac{3\delta^2}{4})} \right] \right\},
\]

while the breaking of time-reversal symmetry in the Gaussian symplectic ensemble (\( \delta = 1 \)) gives rise to the crossover function

\[
P(y) = 4 \left[ ye^{-2y/(1-\frac{\gamma^2}{2})} + e^{-4y/\gamma^2} \right] \frac{\gamma^2}{2} \left(1 - \frac{\gamma^2}{2}\right) e^{-2y/(1-\frac{\gamma^2}{2})} - e^{-4y/\gamma^2} \right] \frac{(1-\gamma^2)^3}{(1-\gamma^2)^2}.
\]

For general \( \gamma \) and/or \( \delta \), \( P(y) \) can be written as a convolution of two functions. The small-\( y \) behavior, however, can be directly deduced from the Laplace transform \( F(s) \) and will be described by the power law corresponding to the lower symmetry ensemble, crossing over to the higher symmetry ensemble(s) behavior(s) at values of \( y \) which depend upon the values of \( \gamma \) and \( \delta \).

The most important consequence of Eq. \( (4) \) is that the power law that describes the small-\( y \) behavior, the region that contributes the most in the strongly localized regime, increases with a magnetic field (i.e. with \( \gamma \)), independently of the amount of spin-orbit scattering (i.e. of \( \delta \)). This leads to a positive MC in the presence and in the absence of spin-orbit scattering.\(^{13-15,17,18}\) We checked this prediction by calculating the distribution of the spin-averaged transmission probability \( T \), at energy \( E/V = 0.1 \), through a 5\( \times \)5 diamond, described by the Anderson Hamiltonian of band width 4\( V \), with on-site uniform disorder of width \( W \). We include an orbital magnetic field, characterized by the overall flux through the diamond \( \phi \), in units of \( \phi_0 \), the quantum flux, and spin-orbit scattering, characterized by the typical angle
of rotation per hop in spin-space, $\lambda$.\(^{20}\) In Fig. 1 we plot the resulting small-$T$ distributions for four cases (a) $\phi = 0$, $\lambda = 0$, (b) $\phi = 5.5$, $\lambda = 0$, (c) $\phi = 0$, $\lambda = 2\pi$ and (d) $\phi = 5.5$, $\lambda = 2\pi$, corresponding to the four cases discussed above. The two panels in Fig. 1 correspond to the choices (a) $W/V = 2$ and (b) $W/V = 4$. We also plot in each panel the four slopes resulting from Eq. (4). There is clearly a satisfactory agreement between the predictions and the numerical results. In particular, there is a clear increase in the slope upon application of a magnetic field. As the localization length only determines the average transmission probability, the small-$y$ behavior is insensitive to the value of disorder.

In order to calculate the conductance one has to solve the percolation problem of the random resistor network,$^{21,15}$ with resistors distributed according to $P(y)$. This procedure leads to positive or negative MC for a system close to the metal-insulator transition, depending on the system parameters.$^{15}$ However, away from the metal-insulator transition, the conductance of the system is given$^{14,18,22}$ by $G = G_0 e^{<\ln(y)>}$, except for an exponentially small region of small magnetic fields. $G_0$ denotes a typical conductance of a single hop, while $< ... >$ denotes an average over the distribution $P(y)$. Thus the relative MC, $\delta G = [G(H) - G(H = 0)] / G(H = 0)$, will be universal — depending only on the change in $P(y)$ as time-reversal symmetry is broken, namely as $\gamma$ increases from zero. In particular, this crossover function is given for the case of no spin-orbit scattering ($\delta = 0$) by $\delta G = \gamma^2 / (\sqrt{2} - \gamma^2)$. Since we expect the time-reversal breaking parameter $\gamma$ to be proportional to magnetic field, the MC is linear at small fields,$^{14,17,18}$ saturating at a value of 1 for a strong enough magnetic field.$^{15}$ For strong spin-orbit scattering ($\delta = 1$) the relative MC is given by

$$\delta G = (1 - \frac{\gamma^2}{2}) \left( 2 - \frac{\gamma^2}{\gamma^2} \right) \frac{3^4 \gamma^4 - 2 \gamma^6}{4^{(\gamma^2 - 1)^4}} \exp \left[ \frac{\gamma^2 - \gamma^4/2}{(\gamma^2 - 1)^2} \right] - 1. \quad (7)$$

At low fields the MC is now quadratic, saturating at a value of $\exp(5/6)/2 - 1 \simeq 0.15$ at high fields.

We checked numerically these universal behaviors by calculating the MC through
the same $5 \times 5$ diamond giving rise to Fig. 1. The conductance for a single realization was calculated from the transmission probability, using Landauer formula, and was logarithmically averaged over 100000 or 200000 realizations for $W/V = 25$, without spin-orbit scattering (Fig. 2a) and with strong spin-orbit scattering (Fig. 2b). The high disorder was chosen so that the localization length will be smaller than the size of the sample, giving rise to the separation of the magnetic field scale at which the localization length changes ($H_\xi$) from the magnetic field scale at which we expect our predictions to hold ($H_R$). As is clearly seen in Fig. 2, there is a range of magnetic fields, corresponding approximately to one unit flux through the sample, where the MC, as expected, saturates. At higher fields, the MC increases further as the localization length starts to increase significantly. This effect is less important in the presence of spin-orbit scattering because here, for large enough samples, the localization length remains unchanged by the magnetic field. We also plot in the figure the prediction of our theory, in excellent agreement with the numerical data. The only fitting parameter in both panels was the ratio between the time-reversal symmetry breaking parameter, $\gamma$, and the magnetic flux, $\phi$, which was chosen as $\gamma = 1.5\phi$. The shape of the curves is very similar to what one expects in experiments: the MC at small fields should follow our universal predictions, while at higher fields other effects, such as the change in the localization length and the shrinking of the wavefunction start to play a significant role, and one expects deviations from universality. The deeper in the localized regime, the more separated the magnetic scales $H_R$ and $H_\xi$ become and the more universal the MC should be.

To conclude we have derived the crossover distribution for the overlap probabilities (the transition strengths) between the different ensembles. This distribution reduces to all the exact random-matrix-theory results in the appropriate limits. This function has been used to predict the universal magnetoconductance curves deep in the localized regime, where the localization length is much smaller than the hopping length, and the percolation criterion for the conductance is equivalent to the log av-
eraging procedure. In that case there is a wide range of magnetic fields, where our universal predictions hold. In this regime we expect all the relative magnetoconductance curves, e.g. for different temperatures, to collapse onto a single curve, when expressed in terms of the scaled time-reversal breaking parameter, presumably the magnetic flux through the cigar shaped hopping area. So far experiments have been confined to the vicinity of the metal-insulator transition, since in this regime the conductivity is more easily measurable. While one has to use more sensitive tools to probe the deep localized regime, we hope that our work will stimulate further experimental effort in this direction.

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Figure Captions:

1. The distribution of the transmission probability through a $5 \times 5$ diamond, described by an on-site disordered Anderson Hamiltonian, in the presence of magnetic flux and spin-orbit scattering for two values of disorder. The four curves correspond to (a) no magnetic field and no spin-orbit scattering, (b) finite magnetic field (c) finite spin-orbit scattering and (d) finite magnetic field and finite spin-orbit scattering. Also depicted are the slopes (with arbitrary offsets) expected from Eq. (3), (a) $-1/2$, (b) 0, (c) 1 and (d) 3.

2. The magnetoconductance through a $5 \times 5$ diamond for disorder $W/V = 25$. The solid lines are the predictions of the theory, where the ratio $\gamma/\phi = 1.5$, between the time-reversal symmetry breaking parameter $\gamma$ and the magnetic flux $\phi$, is the only fitting parameter.
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