Abstract—The performance of Douglas-Rachford splitting and the alternating direction method of multipliers (ADMM) is sensitive to the choice of algorithm parameters and on conditioning of the problem data. For a restricted class of problems that enjoy a linear rate of convergence, we show in this paper how to select the algorithm parameters to optimize a bound on the convergence rate. We show that our rate improves and/or generalizes previously known linear convergence rate bounds for these algorithms. For smooth and strongly convex finite dimensional problems, we show that the linear convergence rate bounds depend on the metric that is used in the algorithm. We show how to select a metric for which this bound is optimized. Since most real-world problems are not both smooth and strongly convex, we also propose heuristic metric and parameter selection methods to improve the performance of a much wider class of problem. These heuristic methods can be applied to problems arising, e.g., in compressed sensing, statistical estimation, model predictive control, and medical imaging. The efficiency of the proposed heuristics is confirmed in a numerical example on a model predictive control problem, where improvements of more than one order of magnitude are observed.

I. INTRODUCTION

Optimization problems of the form

\[
\begin{align*}
\text{minimize} & \quad f(x) + g(y) \\
\text{subject to} & \quad Ax = y
\end{align*}
\]

where \( x \in \mathcal{H} \) is the variable, \( f \) and \( g \) are convex, and \( A \) is a bounded linear operator, arise in numerous applications ranging from compressed sensing [8] and statistical estimation [23] to model predictive control [37] and medical imaging [30]. There exist a variety of algorithms for solving convex problems of the form (1), many of which are treated in the monograph [33]. The methods include primal and dual forward-backward splitting methods [10] and their accelerated variants [4], the Arrow-Hurwicz method [11], Douglas-Rachford splitting [15] and Peaceman-Rachford splitting [35], the alternating direction method of multipliers (ADMM) [22], [18], [7] (which is Douglas-Rachford splitting applied to the dual problem [17], [16]), and linearized ADMM [9].

In this paper, we focus on generalized Douglas-Rachford splitting, which includes Douglas-Rachford splitting and Peaceman-Rachford splitting when applied to the primal and under- and over-relaxed ADMM when applied to the dual. These methods have long been known to converge under very general assumptions, [18], [29], [16]. However, the rate of convergence in the general case has just recently been shown to be \( O(1/k) \), [24], [13], [11]. For a restricted class of problems Lions and Mercier showed in [29] that the Douglas-Rachford algorithm enjoys a linear convergence rate. To the author’s knowledge, this was the sole linear convergence rate results for a long period of time for these methods. Recently, however, many works have shown linear convergence rates for Douglas-Rachford splitting, Peaceman-Rachford splitting and ADMM in different settings [25], [36], [13], [12], [14], [19], [34], [26], [27], [6], [59]. The works in [25], [13], [6], [36] concern local linear convergence under different assumptions. The works in [26], [27], [59] consider distributed formulations, while the works in [12], [14], [19], [34], [29] consider global convergence. The works in [12], [14], [19], [34], [29] are the only ones that provide computable convergence rate factors that can be optimized by selecting the algorithm parameter. In this paper, we generalize the settings and/or improve on the convergence rate estimates compared to the works [12], [14], [19], [34], [29]. We highlight the improvements compared to [29] in several places in the manuscript, and in Section IV-A we discuss and compare the generalizations and convergence rate improvements compared to [12], [14], [19], [34], [29].

When solving problems of the form (1) in finite dimensional settings, we can choose a Hilbert space with inner product \( \langle \cdot, \cdot \rangle_M \) and induced norm on which to apply the generalized Douglas-Rachford algorithm. The algorithm behaves differently for different choices of \( M \) and an appropriate choice can significantly speed up the algorithm, both in theory and in practice. Another contribution of this paper is to show how to select a metric \( M \) to optimize the linear convergence rate factor for problems where \( f \) is smooth and strongly convex, \( g \) is any proper, closed, and convex function, and \( A \) is surjective, i.e., has full row rank. These results are applied to both the primal and dual problems, and therefore apply both to Douglas-Rachford splitting and ADMM (which is Douglas-Rachford splitting on the dual). This generalizes, in several directions, the work in [19] in which corresponding results for ADMM applied to solve quadratic programs with linear inequality constraints are provided.

Real-world problems rarely have the properties needed to ensure a linear convergence of the generalized Douglas-Rachford algorithm or ADMM. Therefore, we provide heuristic metric and parameter selection methods for cases when some of these assumptions are not met. The heuristics cover most optimization problems that have a quadratic part which is not necessarily strongly convex. Such problems arise, e.g., in model predictive control [37], statistical estimation [23] using, e.g., lasso [40], and compressed sensing [8] which is used, e.g., in medical imaging [30]. A numerical example on a model predictive control problem is provided that shows the efficiency of the proposed metric selection heuristic. For the considered problem, the execution time is decreased with about one order of magnitude compared to when applying the algorithm on the Euclidean space.

This paper is an extended version of [20]. It extends the results in [20] in that we provide convergence rate results for

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general Hilbert spaces, as opposed to Euclidean spaces in \[20\]. Also, we provide much more detailed analysis that sheds light on why our linear convergence rates are better than most other rates available in the literature.

A. Notation

We denote by \(\mathbb{R}\) the set of real numbers, \(\mathbb{R}^n\) the set of real column-vectors of length \(n\), and \(\mathbb{R}^{m \times n}\) the set of real matrices with \(m\) rows and \(n\) columns. Further \(\mathbb{R} := \mathbb{R} \cup \{\infty\}\) denotes the extended real line. We use notation \(S^n\) for symmetric \(n \times n\)-matrices and \(S^n_{++}, [S^n_+]\) for positive [semi] definite matrices. Throughout this paper \(\mathcal{H}\) denotes a real Hilbert space and its inner product is denoted by \(\langle \cdot, \cdot \rangle\), the induced norm denoted by \(\| \cdot \|\), and the identity operator is denoted by \(I_d\). We specifically consider finite-dimensional Hilbert-spaces \(\mathbb{H}\) with inner product \(\langle x, y \rangle = x^T H y\) and induced norm \(\| x \| = \sqrt{x^T H x}\). Sometimes the notation \(\langle \cdot, \cdot \rangle_H\) and \(\| \cdot \|_H\) is used, while sometimes we use the generic notation where space they refer to should be clear by context. We also sometimes denote the Euclidean inner-product by \(\langle \cdot, \cdot \rangle_2\) and the induced norm by \(\| \cdot \|_2\) for clarity. Finally, the class of closed, proper, and convex functions \(A : \mathcal{H} \to \mathbb{R}\) is denoted by \(\Gamma_0(\mathcal{H})\).

II. OPERATOR THEORY

In this section, we introduce some definitions and preliminary results in operator theory that will be used later to prove convergence rate results for the Douglas-Rachford algorithm. The definitions stated here are standard and can be found, e.g., in [3, 38]. We supplement many definitions with graphical representations with the intention to clarify concepts and provide intuition to the obtained results.

A. Operator definitions and properties

In this section, suppose that \(X\) and \(Y\) are two nonempty sets. Then \(2^Y\) is the power set of \(Y\), that is the set of all subsets of \(Y\).

**Definition 1 (Operators):** An operator (or mapping) \(A : X \to Y\) maps each point in \(X\) to a point in \(Y\).

We denote by \(A x\) and \(A(x)\) the point in \(Y\) that results from applying the operator \(A\) on \(x\).

**Definition 2 (Set-valued operators):** A set-valued operator \(A : X \to 2^Y\) maps each element in \(X\) to a set in \(Y\).

For set-valued operators, we denote by \(A(x)\) and \(Ax\) the set in \(X\) that results from applying the operator \(A\) on \(x\). This set might be the empty-set since \(\emptyset \in 2^Y\). If \(Ax\) is a singleton or the empty-set for all \(x \in X\), the operator \(A\) is at most single-valued. In such cases, by letting \(D \subseteq X\) be a subset of \(X\), the set-valued operator \(A\) can be associated with an operator \(B : D \to Y\) that satisfies \(Bx = Ax\) for all \(x \in D\) and where \(Ax = \emptyset\) for the remaining \(x \in X \setminus D\). With slight abuse of notation, we treat the at most single-valued operator \(A\) and its associated operator \(B\) to be the same. That is, we let \(Ax\) and \(A(x)\) denote the point in \(Y\) that results from applying \(A\) on \(x \in D\) as well as the singleton set in \(2^Y\) that contains the point \(Bx\).

**Definition 3 (Graph of an operator):** The graph of a set-valued operator \(A : X \to 2^Y\) is defined as

\[
\text{gph}(A) := \{(x, u) \in X \times Y \mid u \in A(x)\}.
\]

Any set-valued operator is (uniquely) characterized by its graph.

**Definition 4 (Inverse operator):** The inverse operator of \(A : X \to 2^Y\) is denoted by \(A^{-1} : Y \to 2^X\) and is described through its graph

\[
\text{gph}(A^{-1}) := \{(u, x) \in Y \times X \mid (x, u) \in \text{gph}(A)\}.
\]

This definition implies that for any pair of points \((x, u)\), we have that \(u \in A(x)\) is equivalent to \(x \in A^{-1}(u)\).

**Definition 5 (Fixed-points):** The set of fixed-points for a mapping \(A : X \to X\) is defined as

\[
\text{fix}A = \{x \in X \mid x = Ax\}.
\]

**Definition 6 (Strong convergence):** A sequence of points \(\{x_k\}_{k=0}^\infty\) converges strongly to a point \(x\) if \(\|x^k - x\| \to 0\) as \(k \to \infty\).

In this paper, the term convergence always refers to strong convergence.

Next, we state some properties for operators. Graphical representations of these properties are provided in Figure 1.

**Definition 7 (Monotonicity):** An operator \(A : \mathcal{H} \to 2^\mathcal{H}\) is monotone if

\[
\langle u - v, x - y \rangle \geq 0
\]

for all \((x, u) \in \text{gph}(A)\) and \((y, v) \in \text{gph}(A)\).

**Definition 8 (Strong monotonicity):** An operator \(A : \mathcal{H} \to 2^\mathcal{H}\) is \(\sigma\)-strongly monotone if

\[
\langle u - v, x - y \rangle \geq \sigma \|x - y\|^2
\]

for all \((x, u) \in \text{gph}(A)\) and \((y, v) \in \text{gph}(A)\).

Strong monotonicity is depicted in Figure 1(a). For the graphical representation, we assume that the operator \(A : \mathbb{R}^2 \to \mathbb{R}^2\) is at most single-valued and has a fixed-point. Any of these fixed-points is the mid-point in Figure 1(a) and is denoted by \(\text{fix}A\). The circle in Figure 1(a) is the unit circle, and the point \(x\) is the point on which the operator \(A\) operates. The gray area depicts the region within which the set \(A(x)\) is contained. The graphical representation of monotonicity is obtained by letting the vertical line defining the border of the gray region intersect with the point \(y\) (i.e., when \(\beta = 0\)).

**Definition 9 (Maximal monotonicity):** A monotone operator \(A : \mathcal{H} \to 2^\mathcal{H}\) is maximal monotone if \(\text{gph}(A)\) is not a proper subset of the graph of any other monotone operator \(B : \mathcal{H} \to 2^\mathcal{H}\).

A way to guarantee maximal monotonicity is to ensure that no pair of points can be added to \(\text{gph}(A)\) without violating the monotonicity definition in Definition 7.

In the following definitions, we suppose that \(D \subseteq \mathcal{H}\) is a nonempty subset of \(\mathcal{H}\).

**Definition 10 (Lipschitz continuity):** A mapping \(A : D \to \mathcal{H}\) is \(\beta\)-Lipschitz continuous if

\[
\|A(x) - A(y)\| \leq \beta \|x - y\|
\]
holds for all $x, y \in \mathcal{D}$. If $\beta = 1$ then $A$ is nonexpansive and if $\beta \in (0, 1)$ then $A$ is $\beta$-contractive.

Lipschitz continuity is depicted in Figure 1(b). From the figure, we see that for $\beta < 1$, i.e., for contractions, the distance to a fixed-point from $Ax$ is strictly smaller than the distance to the fixed-point from $x$. It can be shown that when iterating a contraction mapping, we get a linear convergence towards the fixed-point. For non-expansive mappings, i.e., with $\beta = 1$, the distance is non-increasing, but there are no decrease guarantees. Thus, when iterating a nonexpansive operator, we are not guaranteed to reach a fixed-point. To find a fixed-point of a nonexpansive operator, averaged operators can be used.

Definition 11 (Averaged mappings): A mapping $A : \mathcal{D} \rightarrow \mathcal{H}$ is $\alpha$-averaged if there exist a nonexpansive mapping $B : \mathcal{D} \rightarrow \mathcal{H}$ and $\alpha \in (0, 1)$ such that $A = (1 - \alpha)Id + \alpha B$.

An averaged mapping (or operator) is depicted in Figure 1(c) for different $\alpha$. In an averaged operator, the point $Ax$ end up somewhere on a straight line between the points $x$ and $Bx$ (which can be at the unit circle) where the fraction of the distance is decided by the scalar $\alpha$. Thus, the point $Ax = ((1 - \alpha)Id + \alpha B)x$ is strictly inside the unit circle, i.e., the distance to the fixed-point from $Ax$ is strictly smaller than the distance from $x$. It can be shown [10] that when iterating an averaged nonexpansive operator according to $x^{k+1} = A(x^k)$, then $\{Bx^k - x^k\}$ converges towards 0.

Definition 12 (Cocoercivity): A mapping $A : \mathcal{D} \rightarrow \mathcal{H}$ is $\beta$-cocoercive if

$$\langle A(x) - A(y), x - y \rangle \geq \beta \|A(x) - A(y)\|^2$$

holds for all $x, y \in \mathcal{D}$.

Cocoercivity is depicted in Figure 1(d). A 1-cocoercive mapping is equivalent to a $\frac{1}{2}$-averaged mapping. This can be seen in Figure 1 for the two-dimensional case. Mappings that are 1-cocoercive (or equivalently $\frac{1}{2}$-averaged) are called firmly nonexpansive.

### B. Function definitions and properties

In this section, we introduce some functions and list different function properties. We start with convexity.

**Definition 13 (Convexity):** A function $f : \mathcal{H} \rightarrow \mathbb{R}$ is convex if

$$f(x) \geq f(y) + \langle u, x - y \rangle$$

hold for all $x, y \in \mathcal{H}$ and all $u \in \partial f(y)$.

Convex functions that are also closed and proper (i.e., not $\infty$ everywhere) play an important role in optimization. One reason is that the subdifferential $\partial f$ of a proper, closed, and convex function $f : \mathcal{H} \rightarrow \mathbb{R}$ is a maximal monotone operator [3, Theorem 21.2]. In this paper, we denote the class of proper, closed, and convex functions $f : \mathcal{H} \rightarrow \mathbb{R}$ by $\Gamma_0(\mathcal{H})$.

**Definition 14 (Strong convexity):** A function $f \in \Gamma_0(\mathcal{H})$ is $\beta$-strongly convex if

$$f(x) \geq f(y) + \langle u, x - y \rangle + \frac{\beta}{2} \|x - y\|^2$$

holds for all $x, y \in \mathcal{H}$ and all $u \in \partial f(y)$.

A strongly convex function has a minimum curvature that is decided by $\beta$. Functions with a maximal curvature are called smooth. Next, we present smoothness definitions for general (non-convex) functions and for convex functions.

**Definition 15 (Smoothness for general functions):** A general (nonconvex), closed, function $f \in \Gamma_0(\mathcal{H})$ is $\beta$-smooth if it is differentiable and

$$|f(x) - f(y) - \langle \nabla f(y), x - y \rangle| \leq \frac{\beta}{2} \|x - y\|^2$$

holds for all $x, y \in \mathcal{H}$.

For convex functions, this definition can be stated as follows by noting that the expression inside the absolute value is always nonnegative.

**Definition 16 (Smoothness for convex functions):** A function $f \in \Gamma_0(\mathcal{H})$ is $\beta$-smooth if it is differentiable and

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\beta}{2} \|x - y\|^2$$

holds for all $x, y \in \mathcal{H}$.

We conclude this section by defining the conjugate function.

**Definition 17 (Conjugate functions):** The conjugate function to $f \in \Gamma_0(\mathcal{H})$ is defined as

$$f^*(y) \triangleq \sup_x \{ \langle y, x \rangle - f(x) \}.$$
This conjugate function will play an important role when analyzing properties of the proximal operator that will be introduced in Section III. Note also that the definition of the conjugate function is dependent on which space the function $f$ is defined since the conjugate depends on the inner product.

C. Duality results

In this section, we will state some duality results that are instrumental in proving the linear convergence rate results for Douglas-Rachford splitting. We start with some properties for the conjugate function that are proven in [3] Corollary 13.33, Corollary 16.24.

Proposition 1: Assume that $f \in \Gamma_0(\mathcal{H})$. Then the following holds:

(i) The conjugate function $f^* \in \Gamma_0(\mathcal{H})$.
(ii) The bi-conjugate $(f^*)^* = f$.
(iii) The subdifferential of the conjugate function satisfies $\partial f^* = (\partial f)^{-1}$.

The first property implies that $\partial f^*$ is a maximal monotone operator if $f \in \Gamma_0(\mathcal{H})$. The third property says that this maximal monotone operator is the inverse operator of $\partial f$.

Next, we state some duality results for $A$ and its inverse $A^{-1}$.

Proposition 2: Consider the following list of properties for $A: \mathcal{H} \to 2^\mathcal{H}$ and its inverse $A^{-1}: \mathcal{D} \to \mathcal{H}$, where $\mathcal{D} \subseteq \mathcal{H}$ is a subset of $\mathcal{H}$:

(i) $A$ is $\beta$-strongly monotone.
(ii) $A^{-1}$ is $\beta$-cocoercive.
(iii) $A^{-1}$ is $\frac{1}{\beta}$-Lipschitz continuous.

We have (i)$\iff$(ii) and (ii)$\implies$(iii).

Proof. The equivalence (i)$\iff$(ii) follows directly from the definitions of strong monotonicity (Definition 8) and cocoercivity (Definition 4) and the definition of the inverse operator (Definition 12). The implication (ii)$\implies$(iii) follows directly from the Cauchy-Schwarz inequality and the definitions of cocoercivity (Definition 12) and Lipschitz continuity (Definition 10).

Note that $A^{-1}$ is defined as an operator $A^{-1}: \mathcal{D} \to \mathcal{H}$ instead of $A^{-1}: \mathcal{H} \to 2^\mathcal{H}$. This is done for convenience since cocoercivity and Lipschitz continuity implies that the operator is at most single-valued. Also note that the implication (ii)$\implies$(iii) can be seen to hold in the two-dimensional Euclidean case in Figure 1(d) since the gray $\frac{1}{\beta}$-cocoercivity circle fits inside the gray $\beta$-Lipschitz continuity circle in Figure 1(b).

In the following proposition, we show an implication of Lipschitz continuity.

Proposition 3: Suppose that $A: \mathcal{D} \to \mathcal{H}$ is $\beta$-Lipschitz continuous. Then $A^{-1}: \mathcal{H} \to 2^\mathcal{H}$ satisfies

$$\|u - v\| \geq \frac{1}{\beta} \|x - y\|$$

for all $(x, u) \in \text{gph} A^{-1}$ and $(y, u) \in \text{gph} A^{-1}$.

Proof. This result follows directly by the definition of the inverse operator (Definition 4) and the definition of Lipschitz continuity (Definition 10). □

This property, that we call inverse Lipschitz continuity, can be graphically represented using the representation of Lipschitz continuity in Figure 1(b). In the figure, the set $A^{-1}(x)$ ends up outside the gray Lipschitz circle, where the gray Lipschitz circle has radius $\frac{1}{\beta}$.

The properties in Propositions 2 and 3 can be sharpened when $A$ is the subdifferential of a proper, closed, and convex function.

Proposition 4: Suppose that $f \in \Gamma_0(\mathcal{H})$. Then the following are equivalent:

(i) $f$ is $\beta$-strongly convex.
(ii) $\partial f$ is $\beta$-strongly monotone.
(iii) $\nabla f^*$ is $\beta$-cocoercive.
(iv) $\nabla f^*$ is $\frac{1}{\beta}$-Lipschitz continuous.
(v) $f^*$ is $\frac{1}{\beta}$-smooth.

A proof is provided in [3, Theorem 18.15].

Corollary 1: The converse statement (i.e., with $f$ and $f^*$ interchanged) also holds for $f \in \Gamma_0(\mathcal{H})$ since $f = (f^*)^*$, see Proposition 1.

This result shows that the subdifferential operator is special, since Lipschitz continuity implies cocoercivity, i.e. (iv)$\implies$(iii) in Proposition 4. This result, which is due to Baillon and Haddad in [2], is not true for general $A$, i.e. (iii)$\not\implies$(ii) in Proposition 2. The implication (iv)$\implies$(iii) in Proposition 4 will be the key when we in this paper improve the convergence rate estimates when minimizing the sum of two convex functions using Douglas-Rachford splitting, compared to when finding a zero of the sum of two general maximal monotone operators, the convergence rate of which was provided in [29].

The final result of this section is that the equivalence (iv)$\iff$(v) in Proposition 4 holds also for general nonconvex functions.

Proposition 5: Suppose that $f: \mathcal{H} \to \mathbb{R}$. Then the following are equivalent:

(i) $\nabla f$ is $\beta$-Lipschitz continuous.
(ii) $f$ is $\beta$-smooth.

A proof to this is provided in Appendix A.

III. IMPORTANT OPERATORS

In this section, we will introduce some operators that are used in the Douglas-Rachford algorithm. We will also state properties of these operators that will allow us to show linear convergence of the algorithm.

Definition 18 (Resolvents): Let $\mathcal{D}$ be a subset of $\mathcal{H}$. Then the resolvent $J_A: \mathcal{D} \to \mathcal{H}$ of a monotone operator $A: \mathcal{H} \to 2^\mathcal{H}$ is defined as

$$J_A := (I + A)^{-1}.$$
for the resolvent under strong monotonicity assumptions and Lipschitz continuity assumptions.

**Proposition 6:** Suppose that $A : \mathcal{H} \to 2^\mathcal{H}$ is a $\sigma$-strongly monotone and maximal operator. Then the resolvent $J_A : \mathcal{H} \to \mathcal{H}$ is $(1+\sigma)$-cocoercive, and $\frac{1}{1+\sigma}$-Lipschitz continuous.

**Proof.** This follows directly from Proposition [3] by noting that $I + A$ is $(1+\sigma)$-strongly monotone. □

**Proposition 7:** Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a $\beta$-Lipschitz continuous and maximal operator. Then the resolvent $J_A : \mathcal{H} \to \mathcal{H}$ satisfies

$$\|J_A(x) - J_A(y)\| \geq \frac{1}{1+\beta}\|x - y\|$$

for all $x, y \in \mathcal{H}$.

**Proof.** This follows directly from Proposition [4] and by noting that $A + I$ is $(1+\beta)$-Lipschitz continuous. □

The cocoercivity property is graphically represented in Figure 1(d) and the inverse Lipschitz property is graphically represented as the region outside the Lipschitz circle in Figure 1(b). If the maximal monotone operator $A$ is both $\sigma$-strongly monotone, and $\beta$-Lipschitz continuous, Figure 2 shows in which area (the gray area) the resolvent can end up. The figure is obtained by intersecting the $(1-\sigma)$-cocoercivity circle and the $\frac{1}{1+\sigma}$-inverse Lipschitz region.

In the special case where the operator is a positive scalar times the subdifferential of a proper, closed, and convex function, the resolvent is called the proximal operator.

**Definition 19 (Proximal operators):** The proximal operator of a function $f \in \Gamma_0(\mathcal{H})$ is given by

$$\text{prox}_{\gamma f}(y) := \arg\min_x \left\{ f(x) + \frac{1}{2\gamma} \|x - y\|^2 \right\}.$$ 

To see that the prox operator is indeed the resolvent of $\gamma \partial f$, we let $x^* = \text{prox}_{\gamma f}(y)$ and state the optimality conditions:

$$0 \in \partial f(x^*) + \gamma^{-1}(x^* - y)$$

$$\Leftrightarrow$$

$$y \in (I + \gamma \partial f)x^*$$

$$\Leftrightarrow$$

$$x^* \in (I + \gamma \partial f)^{-1}y.$$ 

Since the subdifferential is a maximal monotone operator, \cite[Theorem 21.1]{5} shows that the proximal operator has full domain, i.e., $\text{prox}_{\gamma f} : \mathcal{H} \to \mathcal{H}$.

We will also use another description of the proximal operator, namely that it is the gradient of the conjugate of the function $f_\gamma : \mathcal{H} \to \mathbb{R}$ defined as

$$f_\gamma := \gamma f + \frac{1}{2}\| \cdot \|^2 \tag{4}$$

where $\gamma > 0$. The function $f_\gamma$ is the scaled original function with quadratic regularization. That the prox operator is indeed the gradient of the conjugate of $f_\gamma$ is shown in the following proposition.

**Proposition 8:** Assume that $f \in \Gamma_0(\mathcal{H})$, then $\text{prox}_{\gamma f}(y) = \nabla f_\gamma(y)$, where $f_\gamma$ is defined in (4).

**Proof.** We have $\text{prox}_{\gamma f}(y) = (I + \gamma \partial f)^{-1}y = (\partial f_\gamma)^{-1}y = \nabla f_\gamma(y)$, where the last step follows from Proposition [1]. Since $f_\gamma$ is 1-strongly convex, Proposition [4] implies that $f_\gamma^*$ is smooth, hence differentiable. □

This relation between the proximal operator and the gradient of the conjugate of a strongly convex function can be used to derive properties of the proximal operator. This is done next.

**Proposition 9:** Assume that $f \in \Gamma_0(\mathcal{H})$ is $\sigma$-strongly convex. Then $\text{prox}_{\gamma f} : \mathcal{H} \to \mathcal{H}$ is $(1+\gamma\sigma)$-cocoercive and $\frac{1}{1+\sigma\gamma}$-contractive.

**Proof.** By noting that $\gamma \partial f$ is $\gamma\sigma$-strongly monotone, this follows directly from Proposition [8] and Proposition [4]. □

**Proposition 10:** Assume that $f \in \Gamma_0(\mathcal{H})$ is $\beta$-smooth. Then $\text{prox}_{\gamma f} : \mathcal{H} \to \mathcal{H}$ is $\frac{1}{1+\gamma\beta}$-strongly monotone.

**Proof.** Since $f$ is $\beta$-smooth, $f_\gamma$ is $(1 + \gamma\beta)$-smooth. Apply Propositions [8] and [4] to get the result. □

Next, we state a result that shows properties of the prox operator if $f$ is both strongly convex and smooth.

**Proposition 11:** Assume that $f \in \Gamma_0(\mathcal{H})$ is $\sigma$-strongly convex and $\beta$-smooth. Then $\text{prox}_{\gamma f} - \frac{1}{1+\gamma\beta}f$ is $\frac{1}{1+\gamma\sigma \beta}$-cocoercive.

**Proof.** Since $f$ is $\sigma$-strongly convex and $\beta$-smooth, Propositions [8, 9, 10] and [4] imply that $f_\gamma$ is $\frac{1}{1+\gamma\sigma \beta}$-smooth and...
The region of points for which the prox operator can be iterated is maximally monotone, then $R_{\gamma} A$ has full domain in that case. Therefore, the proximal operator is $1$-cocoercive, or equivalently $\frac{1}{2}$-averaged. By multiplying the corresponding circles by two (radially outward from the fixed-point) and shifting by $-I$ gives a gray circle that covers the unit circle. In the case of $A$ being Lipschitz continuous and strongly monotone, we get the following tighter result.

**Proposition 12:** Suppose that $A : \mathcal{H} \to \mathcal{H}$ is $\sigma$-strongly monotone and $\beta$-Lipschitz continuous. Then $R_{\gamma} A$ is $\sqrt{1 - \frac{4\gamma}{(1 + \gamma^2)}}$-contractive.

**Proof.** We have

$$\| R_{\gamma} A x - R_{\gamma} y \|^2 = \| 2 J_{A} x - 2 J_{A} y - (x - y) \|^2$$

$$= 4 \| J_{A} x - J_{A} y \|^2 - 4 \langle J_{A} x - J_{A} y, x - y \rangle + \| x - y \|^2$$

$$\leq 4 \| J_{A} x - J_{A} y \|^2 - 4 (1 + \sigma) \| J_{A} x - J_{A} y \|^2 + \| x - y \|^2$$

$$= -4 \sigma \| J_{A} x - J_{A} y \|^2 + \| x - y \|^2$$

$$\leq (1 - \frac{4 \gamma}{(1 + \gamma^2)}) \| x - y \|^2$$

where the first inequality comes from Proposition 5 and the second from Proposition 7.

This is essentially the result on which the linear convergence rate in [29] is based. The region within which the reflected resolvent can end up when $A$ is both strongly monotone and Lipschitz continuous is shown in dashed in Figure 2(a). This region is obtained by multiplying the gray resolvent region in the figure by two and shifting by $-I$. The contraction factor is given by the distance between the intersection of the two dashed circles and the fixed-point.

Finally, we introduce the reflected proximal operator as a special case of the reflected resolvent.

**Definition 21:** The reflected proximal operator to $f \in \Gamma_{0}(\mathcal{H})$ is defined as

$$R_{\gamma, f} := 2 \text{prox}_{\gamma f} - I.$$ 

With slight abuse of notation, we use $R_{\gamma, f} = R_{\gamma, \partial f}$, i.e., both refer to the reflected proximal operator. In the general case, the reflected proximal operator and the reflected resolvent have the same properties, i.e., they are nonexpansive. However, if $\partial f$ is both strongly monotone and Lipschitz continuous, the contraction factor of the reflected proximal operator is significantly smaller. This is shown in the following proposition which is proven in Appendix B.

**Proposition 13:** Suppose that $f$ is $\sigma$-strongly convex and $\beta$-smooth. Then $R_{\gamma, f}$ is $\max (\frac{2}{\gamma (\gamma + 1)}, \frac{1}{\gamma} \sqrt{\frac{\gamma + 1}{\gamma}})$-Lipschitz continuous.

The dashed region in Figure 2(b) shows where the reflected resolvent can end up is $f$ is strongly convex and Lipschitz continuous. The contraction factor is given by the distance between the part of the dashed circle that is farthest away from the fixed-point and the fixed-point itself. We see that for the same values of $\beta$ and $\sigma$, the contraction factor in the subdifferential case in Figure 2(b) is significantly smaller than in the general monotone operator case in Figure 2(a). This is the reason why we can significantly improve the convergence rate estimates compared to the results in [29].

### IV. GENERALIZED DOUGLAS-RACHFORD SPLITTING

The generalized Douglas-Rachford algorithm can be applied to solve inclusion problems of the form

$$0 \in A(x) + B(x)$$

where $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ are maximal monotone operators. The solution to (5) is characterized by the following optimality conditions, [3] Proposition 25.1

$$z = R_{\gamma A} R_{\gamma B} z,$$

$$x = J_{\gamma B} z$$

where $\gamma > 0$. In other words, the solution to (5) is found by applying the resolvent of $A$ on $z$, where $z$ is a fixed-point to $R_{\gamma A} R_{\gamma B}$. One approach to find a fixed-point to $R_{\gamma A} R_{\gamma B}$ is to iterate the composition:

$$z^{k+1} = R_{\gamma A} R_{\gamma B} z^{k}.$$ 

This algorithm is known as Peaceman-Rachford splitting. However, since $R_{\gamma A}$ and $R_{\gamma B}$ are nonexpansive, so is their composition, and convergence of this algorithm cannot be guaranteed in the general case. The generalized Douglas-Rachford splitting algorithm is obtained by introducing averaging to the nonexpansive Peaceman-Rachford operator $R_{\gamma A} R_{\gamma B}$. That is, it is given by the iteration

$$z^{k+1} = ((1 - \alpha) I + \alpha R_{\gamma A} R_{\gamma B}) z^{k}$$

(6)
where \( \alpha \in (0, 1) \), or more explicitly
\[
x^k = J_{\gamma A}(z^k) \quad (7)
\]
\[
y^k = J_{\gamma B}(2x^k - z^k) \quad (8)
\]
\[
z^{k+1} = z^k + 2\alpha(y^k - x^k) \quad (9)
\]

Since (6) is an averaged iteration of a nonexpansive mapping (see Figure 1(c) for a graphical representation of averaged operators), the sequence \( \{z^k - R_{\gamma A}R_{\gamma B}z^k\} \) converges to 0, if a fixed-point to \( R_{\gamma A}R_{\gamma B} \) exists, see [3] Theorem 5.14. The algorithm known as Douglas-Rachford splitting is obtained by letting \( \alpha = \frac{1}{2} \) in (6).

For some problem classes, the convergence towards the fixed-point is linear. This is shown in the following proposition, which is a slight improvement and generalization of the corresponding result in [29, Proposition 4].

**Proposition 14.** Suppose that \( A : \mathcal{H} \to \mathcal{H} \) and \( B : \mathcal{H} \to 2^{\mathcal{H}} \) are maximal monotone operators and that \( A \) is \( \sigma \)-strongly monotone and \( \beta \)-Lipschitz continuous. Then the generalized Douglas Rachford algorithm (6) converges linearly to a point \( \bar{z} \in \text{fix}(R_{\gamma A}R_{\gamma B}) \) with rate \( 1 - \alpha \left( 1 - \sqrt{1 - \frac{4\sigma}{(1+\gamma\beta)^2}} \right) \), i.e.
\[
\|z^{k+1} - \bar{z}\| \leq \left( 1 - \alpha \left( 1 - \sqrt{1 - \frac{4\sigma}{(1+\gamma\beta)^2}} \right) \right)^k \|z^0 - \bar{z}\|. \tag{10}
\]

For a proof, see Appendix C.

The provided bound depends explicitly on the parameters \( \gamma \) and \( \alpha \). Thus, the parameters \( \gamma \) and \( \alpha \) can be chosen to optimize the bound. This is done in the following proposition.

**Proposition 15.** Suppose that \( A : \mathcal{H} \to \mathcal{H} \) and \( B : \mathcal{H} \to 2^{\mathcal{H}} \) are maximal monotone operators and that \( A \) is \( \sigma \)-strongly monotone and \( \beta \)-Lipschitz continuous. Then the parameters that optimize the rate in (10) are given by \( \theta = 1 \) and \( \gamma = \frac{1}{\beta} \).

Further, the optimal rate becomes \( \sqrt{1 - \frac{\sigma}{\beta}} \).

**Proof.** The expression \( 1 - \alpha \left( 1 - \sqrt{1 - \frac{4\sigma}{(1+\gamma\beta)^2}} \right) \) is decreasing in \( \alpha \). Hence, since \( \alpha \in (0, 1] \), \( \alpha = 1 \) optimizes the rate. The optimal \( \gamma \) is obtained by minimizing \( \frac{4\sigma}{(1+\gamma\beta)^2} \) w.r.t. \( \gamma \). Differentiation gives
\[
\frac{4\sigma}{(1+\gamma\beta)^2} - \frac{8\sigma\gamma\beta}{(1+\gamma\beta)^3} = 0
\]
which implies \( \gamma = \pm 1/\beta \). Since \( \gamma > 0 \), we have \( \gamma = 1/\beta \). Inserting these into the rate factor \( 1 - \alpha \left( 1 - \sqrt{1 - \frac{4\sigma}{(1+\gamma\beta)^2}} \right) \) gives the optimal rate factor \( \sqrt{1 - \frac{\sigma}{\beta}} \).

When we restrict ourselves to solve inclusion problems of the form (5) for maximal monotone operators \( A = \partial f \) and \( B = \partial g \), where \( f \in \Gamma_0(\mathcal{H}) \) and \( g \in \Gamma_0(\mathcal{H}) \), these complexity bounds can be significantly improved. Finding a point \( x \) that solves the inclusion
\[
0 \in \partial f(x) + \partial g(x)
\]
is equivalent to solving the following composite convex optimization problem
\[
\text{minimize} \quad f(x) + g(x). \tag{11}
\]
This implies that generalized Douglas-Rachford splitting can be applied to solve composite optimization problems. In the following theorem, we present the improved linear convergence rate results for generalized Douglas-Rachford splitting in the context of composite convex optimization.

**Theorem 1:** Suppose that \( f, g \in \Gamma_0(\mathcal{H}) \) and that \( f \) is \( \sigma \)-strongly convex and \( \beta \)-smooth. Then the generalized Douglas Rachford algorithm (6) converges linearly towards a \( \bar{z} \in \text{fix}(R_{\gamma f}R_{\gamma g}) \) with rate \( 1 - \alpha \left( 1 - \max \left( \frac{2\beta-1}{\gamma\beta+1}, \frac{1-\gamma\sigma}{1+\gamma\beta} \right) \right) \), i.e.
\[
\|z^{k+1} - \bar{z}\| \leq \left( 1 - \alpha \left( 1 - \max \left( \frac{2\beta-1}{\gamma\beta+1}, \frac{1-\gamma\sigma}{1+\gamma\beta} \right) \right) \right)^k \|z^0 - \bar{z}\|.
\]

**Proof.** The proof is identical to the proof to Proposition 14 provided in Appendix C by letting \( \delta \) be the contraction factor of \( R_{\gamma f} \), which in Proposition 13 is shown to be \( \max \left( \frac{2\beta-1}{\gamma\beta+1}, \frac{1-\gamma\sigma}{1+\gamma\beta} \right) \).

Also in this case, the parameters \( \gamma \) and \( \alpha \) can be chosen to optimize the bound on the convergence rate. This is done in the following proposition.
Proposition 16: Suppose that $f, g \in \Gamma_0(H)$ and that $f$ is $\sigma$-strongly convex and $\beta$-smooth. Then the optimal parameters for the generalized Douglas-Rachford algorithm are given by $\alpha = 1$ and $\gamma = \frac{1}{\sqrt{\beta \sigma}}$ and the optimal rate is given by $\frac{\sqrt{\beta / \sigma - 1}}{\sqrt{\beta / \sigma + 1}}$.

Proof. The rate factor $1 - \alpha \left(1 - \max \left(\frac{\gamma \beta - 1}{\gamma \beta + 1}, \frac{1 - \gamma \sigma}{1 + \gamma \sigma}\right)\right)$ is a decreasing function of $\alpha$. Since $\alpha \in (0, 1]$, the rate factor is optimized by $\alpha = 1$. To optimize the rate factor, the $\gamma$ parameter should be chosen to minimize the max-expression $\max \left(\frac{\gamma \beta - 1}{\gamma \beta + 1}, \frac{1 - \gamma \sigma}{1 + \gamma \sigma}\right)$. This is done by letting the arguments equal, which gives $\gamma = 1/\sqrt{\beta \sigma}$. Inserting these values into the rate factor expression gives $\frac{\sqrt{\beta / \sigma - 1}}{\sqrt{\beta / \sigma + 1}}$. \qed

Note that in Propositions 15 and 16 $\alpha = 1$ is optimal. That is, the Peaceman-Rachford algorithm gives the fastest convergence rate under the strong monotonicity and Lipschitz continuity assumptions, even though the Peaceman-Rachford algorithm is not guaranteed to converge in the general case.

A. Comparison to other methods

In this section, we discuss in what ways our result in Proposition 16 generalizes and/or improves on previously known linear convergence rate results in [12], [14], [19], [34], [29]. Since Douglas-Rachford splitting and ADMM are equivalent in the case where $A = \text{Id}$ (that is, Douglas-Rachford is self-dual, i.e., it gives equivalent algorithms if applied to the primal and the dual when $A = \text{Id}$) we can compare DR convergence rate results with ADMM convergence rate results by letting $A = \text{Id}$. In this section, we suppose that one of the two the maximal monotone operators is $\sigma$-strongly monotone and $\beta$-Lipschitz continuous.

In [29] Proposition 4, Remark 10], the linear convergence rate for Douglas-Rachford splitting, i.e., [1] with $\alpha = \frac{1}{2}$, when solving the general inclusion problem (5) is shown to be $\sqrt{1 - \frac{2\sigma}{1 + \gamma \sigma}}$. This rate is optimized by $\gamma = 1/\beta$, which gives a rate of $\sqrt{1 - \frac{2\sigma}{\beta}}$. This was generalized, in the setting of the operators being subdifferentials to proper, closed, and convex functions, to any $\alpha$ in [12, Theorem 6]. The rate in [12, Theorem 6] is $\sqrt{1 - \frac{4\alpha \gamma}{(1 + \gamma \beta)^2}}$. The optimal parameters are $\alpha = 1$ and $\gamma = 1/\beta$, and the optimal rate becomes $\sqrt{1 - \frac{2\sigma}{\beta}}$. It can be shown that the result in Proposition 15 is a slight improvement (except for the case $\alpha = 1$ when the estimates coincide) to the result in [12, Theorem 6]. It is also a generalization since Proposition 15 holds in the general case of maximal monotone operators. The result in Proposition 16 is a clear improvement compared to [12, Theorem 6]. This can be seen in Figure 3 since the optimal rate in [12, Theorem 6] is given the contraction factor for the reflected resolvent in Figure 3(a), while the optimal rate in Proposition 16 is given by the contraction rate factor for reflected resolvent in Figure 3(b). Also, Figure 4 shows that Proposition 16 gives a better rate than [12, Theorem 6] for all ratios $\beta / \sigma$.

The optimal convergence rate in [14, Corollary 3.6] is given by $\sqrt{1/(1 + 1/\sqrt{\beta / \sigma})}$, and the optimal parameter coincides with our choice in Proposition 16. Figure 4 shows that the convergence rate in Proposition 16 is faster than the one provided in [14, Corollary 3.6] for all values of the ratio $\beta / \sigma$. Proposition 16 also generalizes [14, Corollary 3.6] since [14, Corollary 3.6] is stated in the Euclidean setting. For problems considered in [14], we show in Sections IV-B and IV-A how to improve the convergence estimates further by selecting an appropriate space on which to apply the algorithm.

In [34], a new interpretation to Douglas-Rachford splitting is presented. They show that if $f$ is small enough ($\gamma < 1/\beta$) and if $f$ is a quadratic, then the Douglas-Rachford splitting is equivalent to a gradient method applied to a function named the Douglas-Rachford envelope. The Lipschitz continuity and strong convexity parameters for the envelope function can be computed from the corresponding values of the original function $f$. Convergence rate estimates follow from the convergence results for the gradient method, that are well understood. Since the Douglas-Rachford envelope is smooth, they also propose an accelerated Douglas-Rachford algorithm (under the same assumptions, i.e., $f$ quadratic and $\gamma < 1/\beta$). The convergence rate estimates of this also follow from the convergence rate estimates of fast gradient methods. In Figure 3 we also plot the convergence rate estimates for the standard Douglas-Rachford algorithm [34, Theorem 4] and the fast Douglas-Rachford algorithm in [34, Theorem 6], both with parameter $\gamma = (\sqrt{2} - 1)/\beta$. We see that Proposition 16 gives better rates for all values of the ratio $\beta / \sigma$. Proposition 16 is also more general in applicability.

The results provided in [19] coincide with the results provided in Proposition 16. However, the generality of our analysis makes Proposition 16 applicable to a much wider class of problems than the finite-dimensional quadratic problems with linear inequality constraints considered in [19].

Besides generalizing and/or improving on existing results, the results in Proposition 16 can guide us in choosing a space on which to perform the Douglas-Rachford algorithm when solving finite-dimensional problems. By selecting the space appropriately, this can significantly improve the convergence properties of the algorithm, both in theory and in practice.
This is the topic of the following section.

B. Metric selection

In this section, we consider finite-dimensional composite convex optimization problems of the form (11), where \( f \) and \( g \) satisfy:

Assumption 1:

(i) The function \( f \in \Gamma_0(\mathbb{H}_M) \) is 1-strongly convex if defined on \( \mathbb{H}_M \) and 1-smooth if defined on \( \mathbb{H}_L \).

(ii) The function \( g \in \Gamma_0(\mathbb{H}_M) \).

Examples of functions \( f \) that satisfy Assumption 1(i) are piece-wise quadratic functions with Hessians \( Q_i \) that are differentiable on the boundary between the regions. The matrix \( \nabla^2 f \) varies with \( B \). Metric selection is the topic of the following section.

Assumption 1:

(i) The function \( f \) is a standard quadratic function with Hessian \( \nabla^2 f \) on which the ratio \( \frac{\|\nabla f - \nabla f(y)\|}{\|f - f(y)\|} \) is as small as possible. The squared norm and the function \( f \) are as similar as possible. \( f \) is separable down to the component. In such cases, choosing \( M \) is 1-strongly convex if defined on \( \mathbb{H}_M \), we get

\[
\langle \nabla f(x) - \nabla f(y), x - y \rangle_M = \langle \nabla f(x) - \nabla f(y), x - y \rangle_H \\
\geq \|x - y\|_H^2 \geq \lambda_{\text{min}}(DHD^T)\|x - y\|_M^2.
\]

That is, \( \sigma_M(f) = \lambda_{\text{min}}(DHD^T) \). Further, since \( f \) is 1-smooth if defined on \( \mathbb{H}_L \), we get

\[
\|\nabla f(x) - \nabla f(y)\|_M = \|\nabla f(x) - \nabla f(y)\|_L \\
\leq \|x - y\|_L \leq \lambda_{\text{max}}(DLD^T)\|x - y\|_M^2.
\]

Thus, \( \beta_M(f) = \lambda_{\text{max}}(DLD^T) \) and the proof is complete. \( \square \)

This result indicates that, to optimize the rate in Proposition 16 we should select a metric \( M = (D^TD)^{-1} \) that solves

\[
\minimize \frac{\beta_M(f)}{\sigma_M(f)} = \frac{\lambda_{\text{max}}(DLD^T)}{\lambda_{\text{min}}(DHD^T)} \tag{12}
\]

and the parameter \( \gamma = \frac{1}{\sqrt{\lambda_{\text{max}}(DLD^T)\lambda_{\text{min}}(DHD^T)}} \). If we minimize a problem with a quadratic function \( f \) with Hessian \( H \), and run the generalized Douglas-Rachford algorithm on the Euclidean space with \( M = D = I \), we get the rate

\[
\frac{\lambda_{\text{max}}(H)}{\lambda_{\text{min}}(H)+1} \tag{16}
\]

If we instead apply the generalized Douglas-Rachford algorithm on \( \mathbb{H}_M \) with \( M = H \) and \( D = H^{-1/2} \), we get rate

\[
\frac{\lambda_{\text{max}}(H^{-1/2}HH^{-1/2})}{\lambda_{\text{min}}(H^{-1/2}HH^{-1/2})+1} = 0, \text{ i.e., super-linear convergence.}
\]

However, often the functions \( f \) and/or \( g \) are separable down to the component. In such cases, choosing a non-diagonal \( M \) would significantly increase the computational complexity to evaluate the prox-operator. Thus, to get an efficient algorithm both in terms of convergence rate and in terms of complexity within each iteration, the metric matrix \( M = (D^TD)^{-1} \) should be chosen to minimize (12), subject to \( M \) being diagonal. In [21, Section 6] methods to minimize (12) exactly as well as computationally cheap methods to reduce the ratio in (12) are presented.

An interpretation of this can be seen by explicitly stating the prox operation on \( \mathbb{H}_M \):

\[
\text{prox}_\gamma f(y) = \arg\min_x \left\{ f(x) + \frac{1}{2\gamma^2}\|x - y\|_M^2 \right\}.
\]

To get an optimized convergence rate bound, the analysis tells us that \( M \) should be chosen such that the corresponding squared norm and the function \( f \) are as similar as possible. Another way to put it is that we should measure the function \( f \) in a metric with level sets that as good as possible match the levels sets of the function \( f \).

C. Preconditioning

To apply the Douglas-Rachford algorithm on the space \( \mathbb{H}_M \) is equivalent to apply Douglas-Rachford splitting on the Euclidean space \( \mathbb{R}^n \) to the preconditioned problem

\[
\minimize f_D(x) + g_D(x) \tag{13}
\]
where \( M = (D^T D)^{-1} \) and
\[
\begin{align*}
    f_D(x) &:= f(D^T x) \\
    g_D(x) &:= g(D^T x)
\end{align*}
\]
and \( f, f_D, g, \) and \( g_D \) are defined on the Euclidean space \( \mathbb{R}^n \). This can be seen as follows. Let \( \{x^k_M\}, \{y^k_M\}, \) and \( \{z^k_M\} \) be the Douglas-Rachford iterates (7)-(9) when solving the preconditioned problem (13) on \( \mathbb{R}^n \) and let \( \{x^k\}, \{y^k\}, \) and \( \{z^k\} \) be the Douglas-Rachford iterates (7)-(9) when solving the original problem on \( \mathbb{H}_M \). Further assume that \( z^0_{pc} = D^{-T} z^0_M \). Then, by using notation \( f_M \) and \( g_M \) when functions \( f \) and \( g \) are defined on \( \mathbb{H}_M \), we get
\[
\begin{align*}
x^0_{pc} &= \prox_{f_M}(z^0_{pc}) = \arg\min_x \left\{ f_M(x) + \frac{1}{2\kappa} \|x - z^0_{pc}\|^2 \right\} \\
&= D^{-T} \arg\min_v \left\{ f(v) + \frac{1}{2\kappa} \|D^{-T} v - z^0_{pc}\|^2 \right\} \\
&= D^{-T} \arg\min_v \left\{ f(v) + \frac{1}{2\kappa} \|v - D T z^0_{pc}\|^2 \right\} \\
&= D^{-T} \arg\min_v \left\{ f_M(v) + \frac{1}{2\kappa} \|v - z^0_{pc}\|^2 \right\} \\
&= D^{-T} \prox_{g_M}(z^0_{pc}) = D^{-T} z^0_M.
\end{align*}
\]
The same thing can be shown for the \( y \)-updates, i.e., that \( y^0_{pc} = D^{-T} y^0_M \). Finally,
\[
\begin{align*}
z^1_{pc} &= z^0_{pc} + 2\theta(y^0_{pc} - x^0_{pc}) \\
&= D^{-T}(z^0_M + 2\theta(y^0_M - x^0_M)) = D^{-T} z^1_M.
\end{align*}
\]
Recursive application of these statements shows the equivalence.

The difference between the Douglas-Rachford algorithm applied on \( \mathbb{H}_M \) to solve the original problem, and the Douglas-Rachford algorithm applied on \( \mathbb{R}^n \) to solve the preconditioned problem, is that in the former case, the metric in the algorithm is chosen to fit the problem data, while in the second case, the problem data is preconditioned to fit the fixed algorithm metric.

V. ADMM

In this section, besides \( \mathcal{H} \) being a real Hilbert space, also \( \mathcal{K} \) denotes a real Hilbert space. Here, we consider solving problems of the form
\[
\begin{align*}
\text{minimize} & \quad f(x) + g(A x) \\
\text{subject to} & \quad x \in \mathcal{K},
\end{align*}
\]
that satisfy the following assumptions:

**Assumption 2:**

(i) The function \( f \in \Gamma_0(\mathcal{H}) \) is \( \beta \)-smooth and \( \sigma \)-strongly convex.

(ii) The function \( g \in \Gamma_0(\mathcal{K}) \).

(iii) The bounded linear operator \( A : \mathcal{H} \to \mathcal{K} \) is surjective. The assumption of \( A \) being a surjective bounded linear operator reduces to \( \mathcal{A} \) being a real matrix with full row rank in the Euclidean case. Problems of the form (14) cannot be directly efficiently solved using generalized Douglas-Rachford splitting. Therefore, we instead solve the (negative) Fenchel dual problem, which is given by (see [3] Definition 15.19)
\[
\begin{align*}
\text{minimize} & \quad d(\mu) + g^*(\mu) \\
\text{subject to} & \quad A^* \mu = 0,
\end{align*}
\]
where \( g^* \in \Gamma_0(\mathcal{K}) \). Further, \( d \in \Gamma_0(\mathcal{K}) \) is given by
\[
d(\mu) := f^*(-A^* \mu)
\]
where \( A^* : \mathcal{K} \to \mathcal{H} \) is the adjoint operator of \( A \), defined as the unique operator that satisfies \( (Ax, \mu) = (x, A^* \mu) \) for all \( x \in \mathcal{H} \) and \( \mu \in \mathcal{K} \). Applying Douglas-Rachford splitting (i.e. generalized Douglas-Rachford splitting with \( \alpha = \frac{1}{2} \)) to the dual is well known to be equivalent to applying ADMM to the primal, see [17], [16]. To apply generalized Douglas-Rachford splitting to the dual for other choices of \( \alpha \) is known as ADMM with over-relaxation for \( \alpha \in (\frac{1}{2}, 1] \) and ADMM with under-relaxation for \( \alpha \in (0, \frac{1}{2}) \). Therefore, the results we obtain in this section applies to relaxed ADMM.

To optimize the bound on the convergence rate in Proposition (16) when applied to solve the dual problem (15), we need to quantify the strong convexity and smoothness parameters for \( d \). This is done in the following proposition.

**Proposition 18:** Suppose that Assumption 2 holds. Then \( d \in \Gamma_0(\mathcal{K}) \) is \( \frac{1}{\beta} \)-smooth and \( \frac{2}{\sigma} \)-strongly convex, where \( \beta > 0 \) always exists and satisfies \( \|A^* \mu\| \geq \omega \|\mu\| \) for all \( \mu \in \mathcal{K} \).

**Proof.** Since \( f \) is \( \sigma \)-strongly convex, Proposition 4 gives that \( f^* \) is \( \frac{1}{\beta} \)-smooth. Thus, \( d \) satisfies
\[
\|\nabla d(\mu) - \nabla d(\nu)\| = \|A \nabla f^*(-A^* \mu) - A \nabla f^*(-A^* \nu)\|
\]
\[
\leq \frac{\|A\|}{\beta} \|A^* \mu - \nu\| \\
\leq \frac{\|A^*\|^2}{\sigma} \|\mu - \nu\|
\]
since \( \|A\| = \|A^*\| \). This is equivalent to that \( d \) is \( \|A^*\|^2 \)-smooth, see Proposition 4.

Next, we show the strong convexity result. The property that \( f \) is \( \beta \)-smooth implies through Proposition 4 that \( f^* \) is \( \frac{1}{\beta} \)-strongly convex. This implies that \( d \) satisfies
\[
\langle \nabla d(\mu) - \nabla d(\nu), \mu - \nu \rangle
\]
\[
= \langle -A(\nabla f^*(-A^* \mu) - \nabla f^*(-A^* \nu)), \mu - \nu \rangle \\
= \langle \nabla f^*(-A^* \mu) - \nabla f^*(-A^* \nu), -A^* \mu + A^* \nu \rangle \\
\geq \frac{1}{\beta} \|A^* \mu - \nu\|^2 \geq \frac{\sigma^2}{\beta} \|\mu - \nu\|^2
\]

which by Proposition 4 is equivalent to \( d \) being \( \frac{\sigma^2}{\beta} \)-strongly convex. That \( \beta > 0 \) follows from [3] Fact 2.18 and Fact 2.19. Specifically, Fact 2.18 says that \( \text{ker} A^* = (\text{ran} A)^\perp = \emptyset \), since \( A \) is surjective. Since \( \text{ran} A = \mathcal{K} \) (again by surjectivity), it is closed. Then Fact 2.19 states that there exists \( \beta > 0 \) such that \( \|A^* \mu\| \geq \omega \|\mu\| \) for all \( \mu \in (\text{ker} A^*)^\perp = (0)^\perp = \mathcal{K} \). This concludes the proof.

This result gives us the following immediate corollary.

**Corollary 2:** Suppose that Assumption 2 holds and that generalized Douglas-Rachford is applied to solve the dual problem (15), or equivalently, ADMM is applied to solve the primal (14). Then the algorithm parameters \( \gamma \) and \( \alpha \) that optimize the bound on the convergence rate are \( \alpha = 1 \) and \( \gamma = \frac{\beta}{\sqrt{\|A^*\|^2 \beta}} \) and the linear convergence rate bound factor is \( \frac{1}{\kappa + 1} \), where \( \kappa = \frac{\|A^*\|^2 \beta}{\sigma^2} \).

**Proof.** This follows directly from Propositions 18 and 16.
We see that the convergence rate depends on both the conditioning of the function \( f \) and the conditioning of the linear operator \( A^* \). The better the conditioning, the faster the rate. However, some of the parameters might be hard to compute or estimate, especially \( \theta \). In the following section, we show how to compute this in finite-dimensional Hilbert spaces \( \mathbb{H}_K \). We also show how to select the space \( \mathbb{H}_K \) (i.e., select matrix \( K \)) to optimize the bound on the convergence rate.

### A. Metric selection

In this section, we still consider problems of the form \((14)\) and the following assumptions hold:

**Assumption 3:**

(i) The function \( f \in \Gamma_0(\mathbb{H}_M) \) is 1-strongly convex if defined on \( \mathbb{H}_H \) and 1-smooth if defined on \( \mathbb{H}_L \).

(ii) The function \( g \in \Gamma_0(\mathbb{H}_K) \).

(iii) The bounded linear operator \( A : \mathbb{H}_H \to \mathbb{H}_K \) is surjective.

Items (i) and (ii) are the same as in Assumption \([11\)] and the assumption on the bounded linear operator is added due to the more general problem formulation treated here.

Also here, we solve \((14)\) by applying Douglas-Rachford splitting on the dual problem \((15)\), or equivalently by applying ADMM directly on the primal \((14)\). In this case, we have the possibility to select the space \( \mathbb{H}_K \) on which to define the dual problem and apply the algorithm. To aid in the selection of such a space, we show in the following proposition how the strong convexity modulus and smoothness constant of \( d \in \Gamma_0(\mathbb{H}_K) \) depend on the space on which it is defined.

**Proposition 19:** Suppose that Assumption \([3\)] holds, that \( A \in \mathbb{R}^{m \times n} \) satisfies \( Ax = Ax \) for all \( x \), and that \( K = ET \). Then \( d \in \mathbb{H}_K \) is \( \| EAH^{-1}AT \| \)-smooth and \( \lambda_{\min}(EAL^{-1}AT^2) \) strongly convex, where \( \lambda_{\min}(EAL^{-1}AT^2) \) > 0.

**Proof.** First, we relate \( A^* : \mathbb{H}_K \to \mathbb{H}_M \) to \( A, \lambda \) (is defined on \( \mathbb{H}_M \)), and \( K \). We have:

\[
\langle Ax, \mu \rangle_K = \langle Ax, K \mu \rangle_2 = \langle x, AT K \mu \rangle_2 = \langle x, H^{-1}AT K \mu \rangle_H \]

Thus, \( A^* \mu = H^{-1}AT K \mu \) for all \( \mu \in \mathbb{H}_K \).

Next, we show that the space \( \mathbb{H}_M \) on which \( f \) (and \( f^* \)) is defined does not influence the shape of \( d \). Denote by \( d_H := f_H^* \circ -A_H^* \) where \( f_H \) is defined on \( \mathbb{H}_H \) and \( A_H^* : \mathbb{H}_K \to \mathbb{H}_H, \), by \( d_L := f_L^* \circ -A_L^* \) where \( f_L \) is defined on \( \mathbb{H}_L \) and \( A_L^* : \mathbb{H}_K \to \mathbb{H}_L \), and by \( d_e := f_e^* \circ -A \), where \( f_e^* \) and \( A \) are defined on the Euclidean space. By these definitions both \( d_L \) and \( d_H \) are defined on \( \mathbb{H}_K \), while \( d_e \) is defined on \( \mathbb{R}^m \).

Next we show that \( d_L \) and \( d_H \) are identical for any \( \mu \):

\[
d_H(\mu) = f_H^*(-A_H^* \mu) = \sup_x \{ -A_H^* \mu, x \} - f_H(x) \]

where \( A_H^* \mu = H^{-1}AT K \mu \) is used. This implies that we can show properties of \( d \in \mathbb{H}_K \) by defining \( f \) on any space \( \mathbb{H}_M \).

**Proposition 18** gives that 1-strong convexity of \( f \) when defined on \( \mathbb{H}_H \) implies \( \| A^* \|^2 \)-smoothness of \( d \), where

\[
\| A^* \| = \sup_{\mu} \{ \| A^* \mu \| : \| \mu \| \leq 1 \}
\]

Thus, Proposition \([18\)] gives that 1-strong convexity of \( f \) when defined on \( \mathbb{H}_H \) implies \( \| A^* \|^2 \)-smoothness of \( d \), where \( \theta > 0 \) satisfies \( \| A^* \mu \| \geq \theta \| \mu \| \) for all \( \mu \in \mathbb{H}_K \), see Proposition \([18\]) Such a \( \theta \) is given by

\[
\| A^* \| = \| L^{-1}AT K \mu \| \leq \| L^{-1/2}AT E^2 \| E \mu \|_2^2 = \| E \mu \|_2^2 \leq \| E \mu \|_2^2 \leq \lambda_{\min}(EAL^{-1}AT^2) \| E \mu \|_2^2
\]

The smallest eigenvalue \( \lambda_{\min}(EAL^{-1}AT^2) > 0 \) since \( A \) is surjective, i.e., has full row rank, and \( E \) and \( L \) are positive definite matrices. This concludes the proof.

This result shows how the Lipschitz constant and strong convexity modulus of \( d \in \Gamma_0(\mathbb{H}_K) \) changes with the space \( \mathbb{H}_K \) on which it is defined. Combining this with Proposition \([16\)] we get that the bound on the convergence rate for Douglas-Rachford splitting applied to the primal problem \((15)\), or equivalently ADMM applied to the primal \((14)\), is optimized by choosing \( K = ET \) where \( E \) solves

\[
\text{minimize } \lambda_{\max}(EAH^{-1}AT^2) \text{ subject to } \lambda_{\min}(EAL^{-1}AT^2) \leq \gamma
\]

and choosing \( \gamma = \frac{1}{\sqrt{\lambda_{\max}(EAH^{-1}AT^2)\lambda_{\min}(EAL^{-1}AT^2)}} \). In the quadratic case, where the Hessian is given by \( H \), the optimal \( \| AH^{-1}AT \|^{-1/2} \), which gives \( K = (AH^{-1}AT)^{-1} \). As for Douglas-Rachford splitting applied to the primal problem, using a non-diagonal \( K \) usually gives prohibitively expensive prox-evaluations. Therefore, we propose to select a diagonal \( K = ET \) that minimizes \((17)\). The reader is referred to \([21\) Section 6]) for different methods to achieve this exactly and approximately.

The procedure of selecting the metric for the dual Douglas-Rachford splitting is slightly different compared to the primal version. In the primal, we assume that we are given matrices \( H \) and \( L \) that define spaces \( \mathbb{H}_H \) and \( \mathbb{H}_L \), on which the function \( f \) is 1-strongly convex and 1-smooth respectively. To get optimal convergence behaviour, we suppose that these metrics are as tight as possible. Then a diagonal metric \( M \) (that defines \( \mathbb{H}_M \) on which the problem is defined) is chosen that optimizes the bound on the convergence rate. This metric affects only the algorithm, while the problem is the same for any choice of metric matrix \( M \).
In the dual formulation, we are not given matrices on which the dual is  \( \beta \)-strongly convex and \( \sigma \)-smooth respectively. Tight estimates of these parameters are instead computed and we show how these depend on the space \( \mathbb{H}_K \) on which the dual problem is defined. Since the dual problem actually changes with the space on which it is defined, the choice of \( \mathbb{H}_K \) affects both the shape of the dual problem and the metric used in the algorithm. Therefore, the interpretation made for the primal that the algorithmic metric is chosen to well estimate the level-sets of problem is not exactly true in this case. When we select a space \( \mathbb{H}_K \) for the dual, instead we simultaneously manipulate the level-sets of the dual problem and the metric in the algorithm such that the metric well approximates the manipulated level sets of the function.

B. Preconditioning

Similarly to in the primal Douglas-Rachford case, it is equivalent to apply Douglas-Rachford splitting on the dual problem on the space \( \mathbb{H}_K \) and to solve the following preconditioned dual problem defined on the Euclidean space:

\[
\text{minimize } d_E(\nu) + g^*_E(\nu)
\]

where \( K = E^T E \),

\[
d_E(\nu) := d(E^T \nu), \quad g^*_E(\nu) := g^*(E^T \nu),
\]

and \( d, d_E, g^* \), and \( g^*_E \) are defined on the Euclidean space \( \mathbb{R}^n \). To distinguish on which space \( f \) and \( g \) are defined, we denote the functions \( d \) and \( g^* \) by \( d_{\mathbb{H}_K} \) and \( g^*_{\mathbb{H}_K} \) respectively when defined on \( \mathbb{H}_K \). We will also need the equality \( d_{\mathbb{H}_K}(\mu) = d(K\mu) \) and \( g^*_{\mathbb{H}_K}(\mu) = d(K\mu) \) which holds since

\[
d_{\mathbb{H}_K}(\mu) = f^*_E(\mu) = \sup_{x} \{ -A^*\mu, x \} = \sup_{x} \{ -AH^{-1}A^*K\mu, x \} = \sup_{x} \{ -A^*K\mu, x \}
\]

where \( A^*\mu = H^{-1}A^*K\mu \) has been used.

To see the equivalence, we let \( \{x^0_{pc}\}, \{y^0_{pc}\}, \) and \( \{z^0_{pc}\} \) be the Douglas-Rachford iterates \((7), (9)\) when solving the preconditioned problem \((18)\) on \( \mathbb{R}^n \) and let \( \{x^0_K\}, \{y^0_K\}, \) and \( \{z^0_K\} \) be the Douglas-Rachford iterates \((7), (9)\) when solving the original dual problem \((15)\) on \( \mathbb{H}_K \). Further assume that \( E^0_{pc} = z^0_K \). Then

\[
x^0_{pc} = \text{prox}_{d_{\mathbb{H}_K}}(z^0_{pc}) = \text{argmin}_{\mu} \{ d_{\mathbb{H}_K}(\mu) + \frac{1}{2\gamma} \| x - z^0_{pc} \|_2^2 \}
\]

The same thing can be shown for the \( y \)-updates, i.e., that \( y^0_{pc} = E^{-1}y^0_M \). Finally,

\[
z^1_{pc} = z^0_{pc} + 2\theta(y^0_{pc} - z^0_{pc}) = E^{-1}(z^0_{K} + 2\theta(y^0_K - z^0_K)) = E^{-1}z^1_K.
\]

Recursive application of these statements shows the equivalence.

The primal problem that gives rise to the preconditioned dual problem \((18)\) is given by

\[
\text{minimize } f(x) + g(y) \\
\text{subject to } EAx = Ey.
\]

Since solving the problem \((18)\) using generalized Douglas-Rachford splitting is equivalent to solving the primal problem \((19)\) using ADMM, the result presented for dual Douglas-Rachford splitting holds when applying ADMM to the preconditioned primal, which is given by the iterations:

\[
y^{k+1} = \text{argmin}_{y} \{ g(y) + \frac{\rho}{2} \| Ey - EAx^k + \lambda z^k \|_2^2 \}
\]

\[
v^{k+1} = 2\alpha D y^{k+1} + (1 - 2\alpha) DA x^k
\]

\[
x^{k+1} = \text{argmin}_{x} \{ f(x) + \frac{\rho}{2} \| EAx - v^{k+1} - \gamma \lambda^k \|_2^2 \}
\]

\[
\lambda^{k+1} = \lambda^k + \gamma(v^{k+1} - EA x^{k+1})
\]

Thus, the optimal preconditioner \( E \) in ADMM under Assumption \((3)\) is computed by solving \((17)\), subject to \( E \) diagonal, and the optimal choice of \( \gamma \) is \( \gamma = \frac{1}{\| \lambda \|_\infty \max(1, \lambda) \min(\lambda, 1) \| E^{-1} (A^*, E) \|_\infty} \), and the optimal \( \alpha = 1 \). Note that the \( \gamma \)-parameter in ADMM is in the numerator, while the same \( \gamma \)-parameter in dual Douglas-Rachford splitting is in the denominator.

Remark 1: This preconditioning result is a generalization of the result in \((19)\) where the restricted case of \( f(x) = \frac{1}{2} x^T H x + h^T x \) and \( g(y) = I_{y \leq d}(y) \) is considered.

VI. HEURISTIC METRIC SELECTION

In this section, we discuss metric and parameter selection when some of the assumptions needed to have linear convergence are not met. We focus here on quadratic problems of the form

\[
\text{minimize } \frac{1}{2} x^T Q x + q^T x + \hat{f}(x) + g(Ax)
\]

where \( Q \in \mathbb{S}^n_+ \), \( q \in \mathbb{R}^n \), \( \hat{f} \in \Gamma_0(\mathbb{R}^n) \), \( g \in \Gamma_0(\mathbb{R}^n) \) and \( A \in \mathbb{R}^{m \times n} \). One set of assumptions that guarantee linear convergence for Douglas-Rachford splitting applied to the primal or the dual is that \( Q \) is positive definite, \( \hat{f} \equiv 0 \), and that \( A \) has full row rank. Here, we consider situations in which some of these assumptions are not met. Specifically, we consider situations where (some of) the following items violate the linear convergence assumptions:

(i) \( Q \in \mathbb{S}^n_+ \) is not positive definite, but positive semi-definite.

(ii) \( \hat{f} \not\equiv 0 \), but instead the indicator function of a convex constraint set (or some other non-smooth function without curvature).
(iii) \( A \in \mathbb{R}^{m \times n} \) does not have full row rank.

In the first case, we loose strong convexity in the primal formulation and smoothness in the dual formulation. In the second case, we lose smoothness in the primal formulation and strong convexity in the dual formulation. The third case is not applicable to the primal case (since \( A = f \) there), but in the dual formulation this results in loss of strong convexity.

We first discuss the primal formulation and assume that the assumptions that give linear convergence are violated using both (i) and (ii). Then, we have quadratic curvature only in the range of \( Q \). In the null-space of \( Q \), the function \( f \) is governed by the function \( \hat{f} \) (which is either 0 of \( \infty \) if it is the indicator function of a convex constraint set). Thus, we propose to select a diagonal metric \( M = (D^T D)^{-1} \) that optimizes the conditioning on the range of \( Q \), i.e., that solves

\[
\text{minimize } \frac{\lambda_{\max}(D^T Q D)}{\lambda_{\min > 0}(D^T Q D)}
\]

where \( \lambda_{\min > 0} \) denotes the smallest non-zero eigenvalue. Also, we propose to select the \( \gamma \)-parameter to reflect the curvature on the range on \( Q \), i.e.,

\[
\gamma = \frac{1}{\sqrt{\lambda_{\max}(D^T Q D) \lambda_{\min > 0}(D^T Q D)}}.
\]

For the dual case, i.e., the ADMM case, we propose to select the metric as if \( f = 0 \) (which is if the assumptions to get linear convergence are not violated by point (ii)). We define the quadratic part of \( f \) in (20) to be \( f_{\text{pc}}(x) := \frac{1}{2}Q^T Q x + q^T x \) and introduce the dual function \( d_{\text{pc}} = f_{\text{pc}}^* - A^T x \) which we propose to base the metric selection on. We have that

\[
f_{\text{pc}}^*(y) = \sup_x \{ \langle y, x \rangle - f_{\text{pc}}(x) \}
\]

\[
= \begin{cases} \frac{1}{2}(y-q)^T Q^T (y-q) & \text{if } (y-q) \in \mathcal{R}(Q) \\ \infty & \text{else} \end{cases}
\]

where \( Q^\dagger \) is the pseudo-inverse of \( Q \) and \( \mathcal{R} \) denotes the range. This gives

\[
d_{\text{pc}}(\mu) = \begin{cases} \frac{1}{2}(A^T \mu + q)^T Q^\dagger (A^T \mu + q) & \text{if } (A^T \mu + q) \in \mathcal{R}(Q) \\ \infty & \text{else} \end{cases}
\]

Thus, the quadratic part of the approximated dual \( d_{\text{pc}} \) is given by \( A Q^\dagger A^T \), and is defined on a sub-space only (if \( Q \) is not positive definite). As in the primal case, we propose to select a diagonal metric \( K = E^T E \) such that the quadratic part of the, in some cases approximate, dual function is well conditioned on its domain. That is, we propose to select a metric such that the pseudo condition number of \( A Q^\dagger A^T \) is minimized, i.e., a diagonal metric \( E \) that is computed by solving

\[
\text{minimize } \frac{\lambda_{\max}(E A Q^\dagger A^T E^T)}{\lambda_{\min > 0}(E A Q^\dagger A^T E^T)}
\]

This reduces to the optimal metric choice in the case where linear convergence is achieved, i.e., where none of items (i), (ii), or (iii) are met, and can be used as a heuristic when any of the points (i), (ii), and/or (iii) violates the assumptions to get linear convergence. The \( \gamma \)-parameter is also chosen to in accordance with the above reasoning as

\[
\gamma = \frac{1}{\sqrt{\lambda_{\max}(E A Q^\dagger A^T E^T) \lambda_{\min > 0}(E A Q^\dagger A^T E^T)}}.
\]

In the particular case where \( \hat{f} \) in (20) is the indicator function for an affine subspace, i.e., when \( \hat{f} = I_{Bz=b} \). Then \( d \) can be written as

\[
d(\mu) = \frac{1}{2}\mu^T A P_{11} A^T \mu + \xi^T \mu + \chi
\]

where \( \xi \in \mathbb{R}^n \), \( \chi \in \mathbb{R} \), and

\[
\begin{bmatrix} Q & B^T \\ B & 0 \end{bmatrix}^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.
\]

Then we can choose metric by minimizing the pseudo condition number of \( A P_{11} A^T \), which is the Hessian of \( d \), and select \( \gamma \) as

\[
\gamma = \frac{1}{\sqrt{\lambda_{\max}(E A P_{11} A^T E) \lambda_{\min > 0}(E A P_{11} A^T E)}}.
\]

Minimization of the pseudo condition number \( \lambda_{\max}/\lambda_{\min > 0} \) can be posed as a convex optimization problem and be solved exactly, see [21] Section 6] which also contains computationally cheap heuristics to select \( E \) that reduce the pseudo condition number.

VII. NUMERICAL EXAMPLE

In this section, we evaluate the metric and parameter selection method by applying ADMM to the (small-scale) aircraft control problem from [28], [5]. As in [5], the continuous time model from [23] is sampled using zero-order hold every 0.05 s. The system has four states \( x = (x_1, x_2, x_3, x_4) \), two outputs \( y = (y_1, y_2) \), two inputs \( u = (u_1, u_2) \), and obeys the following dynamics

\[
x^+ = \begin{bmatrix} 0.999 & -3.008 & -0.113 & -1.608 \\ -0.000 & 0.986 & 0.048 & 0.000 \\ 0.000 & 2.083 & 1.099 & -0.000 \\ -0.000 & 0.053 & 0.050 & 1.000 \end{bmatrix} x + \begin{bmatrix} -0.080 & -0.635 \\ -0.029 & -0.014 \\ -0.368 & -0.092 \\ -0.022 & -0.002 \end{bmatrix} u,
\]

\[
y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x
\]

where \( x^+ \) denotes the state in the next time step. The system is unstable, the magnitude of the largest eigenvalue of the dynamics matrix is 1.313. The outputs are the attack and pitch angles, while the inputs are the elevator and flapher angles. The inputs are physically constrained to satisfy \( |u_1| \leq 25^o \), \( |u_2| \leq 60^o \), \( i = 1, 2 \). The outputs are soft constrained to satisfy \( -s_1 - 0.5 \leq y_1 \leq 0.5 + s_2 \) and \( -s_3 - 100 \leq y_2 \leq 100 + s_4 \) respectively, where \( s = (s_1, s_2, s_3, s_4) \geq 0 \) are slack variables. The cost in each time step is

\[
\ell(x, u, s) = \frac{1}{2}((x - x^r)^T Q(x - x^r) + u^T R u + s^T S s)
\]

where \( x^r \) is a reference, \( Q = \text{diag}(10^{-4}, 10^{2}, 10^{-3}, 10^{2}) \), \( R = 10^{-2} I \), and \( S = 10^{6} I \). This gives a condition number of \( 10^{10} \) of the full cost matrix. Further, the terminal cost is \( Q \), and the control and prediction horizon is \( N = 10 \). The numerical data in Figure 5 is obtained by following a reference trajectory on the output. The objective is to change the pitch angle from \( 0^o \) to \( 10^o \) and then back to \( 0^o \) while the angle of attack satisfies the output constraints \( -0.5 \leq y_1 \leq 0.5 \). The constraints on the angle of attack limits the rate on how fast the pitch angle can be changed. The full optimization problem can be written on the form

\[
\text{minimize } \frac{1}{2} z^T H z + r^T z + I_{Bz=b_{x^r}} + I_{d_{\leq \leq d}}(z')
\]

subject to

\[
C z = z'
\]
where $x_t$ and $r_t$ may change from one sampling instant to the next.

This is the optimization problem formulation discussed in Section [VI] where item (ii) violates the assumptions that guarantee linear convergence. In Figure 5 the performance of the ADMM algorithm for different values of $\gamma$ and for different metrics is presented. Since the numerical example treated here is a model predictive control application, we can spend much computational effort offline to compute a metric that will be used in all samples in the online controller. We compute a metric $K = E^T E$ that minimizes the condition number of $E C H^{-1} C^T E^T$ (minimization of the pseudo condition number of $E C P_{\alpha} C^T E^T$ gives about the same performance and is therefore omitted) that defines the space $\mathbb{H}_K$. In Figure 5 the performance of ADMM when applied on $\mathbb{H}_K$ with relaxations $\alpha = \frac{1}{2}$ and $\alpha = 1$, and ADMM applied on $\mathbb{R}^n$ with $\alpha = \frac{1}{2}$ is shown. In this particular example, improvements of about one order of magnitude are achieved when applied on $\mathbb{H}_K$ compared to when applied on $\mathbb{R}^n$. Figure 5 also shows that ADMM with over-relaxation performs better than standard ADMM with no relaxation. The empirically best average iteration count for ADMM on $\mathbb{H}_K$ with $\alpha = 1$ is 15.9 iterations, for ADMM on $\mathbb{H}_K$ with $\alpha = \frac{1}{2}$ is 24.9 iterations, and for ADMM on $\mathbb{R}^n$ with $\alpha = \frac{1}{2}$, (which is essentially the algorithm proposed in [32]), is 446.1 iterations. The proposed $\gamma$-parameter selection is denoted by $\gamma^*$ in Figure 5 ($E$ or $C$ is scaled to get $\gamma^* = 1$ for all examples). Figure 5 shows that $\gamma^*$ does not coincide with the empirically found best $\gamma$, but still gives a reasonable choice of $\gamma$ in all cases.

VIII. CONCLUSIONS

We have presented methods to select metric and parameters for Douglas-Rachford splitting, Peaceman-Rachford splitting and ADMM. We have also provided a numerical example in which the proposed metric and parameter selection methods give rise to performance improvements in ADMM of about one order of magnitude, compared to when ADMM is applied on the Euclidean space.

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Thus
\[ \langle \nabla f(y) - \nabla f(x), x - y \rangle \]
\[ = -(\langle \nabla f(x) + \nabla f(y), y - x \rangle + 2\langle \nabla f(y), x - y \rangle) \]
\[ \leq 2f(x) - 2f(y) - (\nabla f(y) + \nabla f(x), x - y) \leq \beta \|x - y\|^2 \]
where the last inequality is due to (22). The situation with \((- , +)\) gives the same inequality by interchanging the place of \(x\) and \(y\). Thus, for any of the four scenarios, we have
\[ |\langle \nabla f(x) - \nabla f(y), x - y \rangle| \leq \beta \|x - y\|^2. \]
Now, apply Cauchy-Schwarz inequality to get
\[ \| \nabla f(x) - \nabla f(y) \| \leq \beta \|x - y\|. \]
This completes the proof. \(\Box\)

APPENDIX B

PROOF OF PROPOSITION 13

Proof. Define \( \hat{f} = 2f_1^* - \frac{1}{2} \| \cdot \|^2 \) (where \( f_1 \) is defined in [4]). Through Proposition 8 we get that \( \nabla \hat{f} = 2\nabla f_1^* - I = 2\text{prox}_{\hat{f}} - I = R_{\hat{f}}f \). We get
\[ \langle \nabla \hat{f}(y), x - y \rangle = \langle 2\nabla f_1^*(y) - y, x - y \rangle \]
\[ \leq 2f_1^*(x) - f_1^*(y) - \frac{1}{2\gamma^2(\sigma+1)} \|x - y\|^2 \]
\[ = f(x) - f(y) + \frac{2\beta - 1}{2\gamma^2(\sigma+1)} \|x - y\|^2 \]
where Proposition 10 and Proposition 4 are used in the inequality. We also have
\[ \langle \nabla \hat{f}(y), x - y \rangle = \langle 2\nabla f_1^*(y) - y, x - y \rangle \]
\[ \geq 2f_1^*(x) - f_1^*(y) - \frac{1}{2\gamma^2(\sigma+1)} \|x - y\|^2 \]
\[ = f(x) - f(y) + \frac{2\beta - 1}{2\gamma^2(\sigma+1)} \|x - y\|^2 \]
where Proposition 9 and Proposition 4 are used in the inequality. This implies that
\[ \frac{2\beta - 1}{2\gamma^2(\sigma+1)} \|x - y\|^2 \leq \langle \nabla \hat{f}(y), x - y \rangle + \hat{f}(y) - \hat{f}(x) \]
\[ \leq \frac{2\beta - 1}{2\gamma^2(\sigma+1)} \|x - y\|^2 \]
or equivalently (by negating the first inequality)
\[ |\langle \nabla \hat{f}(y), x - y \rangle + \hat{f}(y) - \hat{f}(x) | \]
\[ \leq \frac{1}{\gamma} \max\left( \frac{2\beta - 1}{2\gamma^2(\sigma+1)} \right) \|x - y\|^2. \] (23)
Since \( \nabla \hat{f} = R_{\hat{f}}f \), the result follows from Proposition 5. \(\Box\)

APPENDIX C

PROOF OF PROPOSITION 14

Proof. By \([3]\) Corollary 23.10 \( R_B \) is nonexpansive and by Proposition 12 \( R_A \) is \( \delta \)-(1 - \( \frac{4\beta}{\gamma(\sigma+1)} \))-contractive. Thus the composition \( R_AR_B \) is also \( \delta \)-contractive since
\[ \| R_AR_B z_1 - R_AR_B z_2 \| \leq \delta \| R_B z_1 - R_B z_2 \| \leq \delta \| z_1 - z_2 \|. \] (24)
Now, let $T = (1 - \alpha)I + \alpha R_A R_B$ be the generalized Douglas-Rachford operator in (6). Since $\bar{z}$ is a fixed-point of $R_A R_B$ it is also a fixed-point of $T$, i.e., $\bar{z} = T \bar{z}$. Thus

$$\|z^{k+1} - \bar{z}\| = \|T z^k - T \bar{z}\|^2$$

$$= \|(1 - \alpha)(z^k - \bar{z}) + \alpha(R_A R_B z^k - R_A R_B \bar{z})\|$$

$$\leq \alpha\|R_A R_B z^k - R_A R_B \bar{z}\| + (1 - \alpha)\|z^k - \bar{z}\|$$

$$\leq (\alpha \delta + (1 - \alpha))\|z^k - \bar{z}\|$$

$$= (1 - \alpha + \alpha \sqrt{1 - \frac{4\gamma \sigma}{(1 + \gamma \beta)^2}})\|z^k - \bar{z}\|$$

where (24) is used in the second inequality. This concludes the proof. \qed