On a finite trigonometric sum related to Dedekind sum

Mouloud Goubi

Abstract. Finite trigonometric sums appear in various branches of Physics, Mathematics and their applications. For \( p, q \) to coprime positive integers and \( r \in \mathbb{Z} \) we consider the finite trigonometric sums involving the product of three trigonometric functions

\[
S(p, q, r) = \sum_{k=1}^{p-1} \cos \left( \frac{\pi k r}{p} \right) \csc \left( \frac{\pi k q}{p} \right) \csc \left( \frac{\pi k}{p} \right).
\]

In this work we establish a reciprocity law satisfied by \( S(p, q, r) \) in some particular cases. And we compute explicitly the value of the sum

\[
S(p, 1, 0) = \sum_{k=1}^{p-1} \csc^2 \left( \frac{\pi k}{p} \right).
\]

1. Introduction and main results

Let us remember the cosecant function \( \csc \) defined by

\[
\csc \theta = \frac{1}{\sin \theta}.
\]

From which the cosecant numbers are extracted, as the rational coefficients of it’s power series expansion, a generalization of these numbers is given in the paper [6] of V. Kowalenko.

For \( p, q \) two coprime positive integers and \( r \in \mathbb{Z} \) we consider the following finite trigonometric sum.

\[
S(p, q, r) = \sum_{k=1}^{p-1} \cos \left( \frac{\pi k r}{p} \right) \csc \left( \frac{\pi k q}{p} \right) \csc \left( \frac{\pi k}{p} \right)
\]

Let \( 0 \leq r \leq 2p - 1 \) the represent of the class of \( r \) in \( \mathbb{Z}/2p\mathbb{Z} \), then

\[
S(p, q, r) = S(p, q, \overline{r})
\]

2010 Mathematics Subject Classification. Primary 33B10, 11L03.

Key words and phrases. Dedekind sum, finite trigonometric sums, cosecant function.
For example if $r \equiv 0 \pmod{2p}$;

$$S(p, q, r) = \sum_{k=1}^{p-1} \csc \left( \frac{\pi kq}{p} \right) \csc \left( \frac{\pi k}{p} \right)$$

The periodicity of cosine stays that there are exactly $2p$ finite trigonometric sums of this kind for a fixed positive integers $p, q$.

If $q = \overline{q} \pmod{2p}$ then

$$S(p, q, r) = S(p, \overline{q}, r)$$

and for a fixed $p$ and $q, r$ are arbitrary integers such that $q$ is not multiple of $p$ their exist exactly $2p$ finite trigonometric sums $S(p, q, r)$.

Now if $r \equiv 0 \pmod{p}$ and $q \equiv 1 \pmod{2p}$, we get

$$S(p, q, r) = \sum_{k=1}^{p-1} \csc^2 \left( \frac{\pi k}{p} \right),$$

and if $q = 1$ and $r \equiv 1 \pmod{p}$ then

$$S(p, q, 0) = \sum_{k=1}^{p-1} (-1)^k \csc^2 \left( \frac{\pi k}{p} \right).$$

Finally for $r = q$ the considered trigonometric sum will be written in the following form

$$\sum_{k=1}^{p-1} \cot \left( \frac{\pi kq}{p} \right) \csc \left( \frac{\pi k}{p} \right).$$

The problem of evaluation of the sum

$$\sum_{k=1}^{p-1} \csc^{2m} \left( \frac{\pi k}{p} \right)$$

steal open. A partial answer is given in the work of the physician N. Gauthier and Paul S. Bruckman [5] at order $Q$, where $Q = \frac{p-1}{2}$ if $p$ odd and $Q = \frac{p-2}{2}$ otherwise.

$$\sum_{k=1}^{Q} \csc^{2m} \left( \frac{\pi k}{p} \right) = \sum_{k=1}^{m} (2k-1)! \varphi_{k-1, m-1} J_{2k} (p), \ m \geq 1$$

where

$$\varphi_{r, m} = \frac{s_{r, m}}{(2m+1)!}; \ 0 \leq r \leq m; m \geq 1,$$
s_{r,m} is the sum of all the possible distinct products of the following numbers \(4.1^2, 4.2^2, \ldots, 4.2^m\) and

\[ J_{2k} (p) = \left( \frac{p}{\pi} \right)^{2k} \sum_{r=1}^{Q} \sum_{n} (r - np)^{-2k}. \]

But the value of the sum of maximal order of even integral powers of the secant

\[ \sum_{k=1}^{p-1} \sec^{2m} \left( \frac{\pi k}{p} \right) \] is expressed by [3, p. 1]

\[ \sum_{k=1}^{p-1} \sec^{2m} \left( \frac{\pi k}{p} \right) = p \sum_{k=1}^{2m-1} (-1)^{m+k} \left( \frac{m - 1 + kp}{2m - 1} \right) \sum_{j=k}^{2m-1} \left( \frac{2m}{j+1} \right). \]

The problem of finite sums with negative powers of cosecant or secant is completely resolved in [4] and [3, Theorem 2.1 p. 4], we copy here the result for cosecant

\[ \sum_{k=0}^{p-1} \csc^{-2m} \left( \frac{\pi k}{p} \right) = \begin{cases} 2^{1-2m} \left( \frac{2m-1}{m-1} \right) + \sum_{n=1}^{[m/p]} \left( -1 \right)^{p-n} \left( \frac{2m}{m-pn} \right), & \text{if } m \geq p, \\ 2^{1-2m} \left( \frac{2m-1}{m-1} \right), & \text{if } m < p. \end{cases} \]

Let \( m, n, l \in \mathbb{Z} \), the last finite trigonometric sums can be extended to the general case

\[ S_{n,m,l}(p, q, r) = \sum_{k=1}^{p-1} \cos^n \left( \frac{\pi kr}{p} \right) \csc^m \left( \frac{\pi kq}{p} \right) \csc^l \left( \frac{\pi k}{p} \right), \]

and then

\[ S(p, q, r) = S_{1,1,1}(p, q, r). \]

Since \( p - k \) runs all the numbers from 1 to \( p - 1 \) for \( k \in \{1, 2, 3, \ldots, p - 1\} \) then the sum \( S_{n,m,l}(p, q, r) \) can be written in the following form

\[ S_{n,m,l}(p, q, r) = \sum_{k=1}^{p-1} \cos^n \left( \frac{\pi r - \pi kr}{p} \right) \csc^m \left( \frac{\pi q - \pi kq}{p} \right) \csc^l \left( \frac{\pi - \pi k}{p} \right). \]

but \( \cos (\pi k - \theta) = (-1)^k \cos \theta \) and \( \sin (\pi k - \theta) = (-1)^{k+1} \sin \theta \) then

\[ S_{n,m,l}(p, q, r) = \sum_{k=1}^{p-1} (-1)^{rn+(q+1)m} \cos^n \left( \frac{\pi kr}{p} \right) \csc^m \left( \frac{\pi kq}{p} \right) \csc^l \left( \frac{\pi k}{p} \right), \]

which means that

\[ \left( 1 - (-1)^{rn+(q+1)m} \right) S_{n,m,l}(p, q, r) = 0 \]

and for \( rn + (q+1)m \) odd number we get

\[ S_{n,m,l}(p, q, r) = 0. \]

On general, evaluating the reciprocity law of \( S_{n,m,l}(p, q, r) \) steal an open problem, for which we need more tools to resolve.
For more background about finite trigonometric sums in literature we refer to [1], [2] and [8]. The methods of computation based only on the residue theorem from complex analysis. The well-known classical Dedekind sum is

\[ s(q, p) = \sum_{k=1}^{p-1} \left( \left( \frac{kq}{p} \right) \left( \frac{k}{p} \right) \right) \]

where \( q, p \) are coprime positive integers and

\[ ((x)) := \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{otherwise.} \end{cases} \]

In terms of fractional part function \( \{x\} \) (i.e. \( \{x\} = x - [x] \)) we can write

\[ s(q, p) = \frac{1}{4} - \frac{p}{2} + \sum_{k=1}^{p-1} \left\{ \left( \frac{kq}{p} \right) \left( \frac{k}{p} \right) \right\} \]

From the property that \( \{x\} + \{-x\} = 1 \), If \( 0 < q < p \), we get

\[ s(p - q, p) = \frac{p - 1}{2} - s(q, p). \]

\( s(q, p) \) satisfies the following reciprocity law [7, p.4]

\[ s(q, p) + s(p, q) = -\frac{1}{4} + \frac{1}{12} \left( \frac{q}{p} + \frac{p}{q} + \frac{1}{pq} \right) \]

The reciprocity law inducts an expression for \( s(p - q, p) \) on function of \( s(q, p) \).

\[ s(p, p - q) = s(q, p) + \frac{1}{4} + \frac{1}{12p(p - q)} (-6p^3 + 6p^2q + 2p^2q + q^2 + 2pq + 1) \]

This sum is interesting, since it can only be traduced to a finite trigonometric sum of product of cotangent functions [7, p.18]

\[ s(q, p) = \frac{1}{4p} \sum_{k=1}^{p-1} \cot \left( \frac{\pi kq}{p} \right) \cot \left( \frac{\pi k}{p} \right) \]

The objective of this paper is to establish a reciprocity law for the trigonometric sums \( S(p, q, q \pm 1) \) and \( S(p, p - q, p - q \pm 1) \), and get explicit evaluation of the sum

\[ S(p, 1, 0) = \sum_{k=1}^{p-1} \csc^2 \left( \frac{\pi k}{p} \right). \]

**Theorem 1.1.** For \( p, q \) two coprime positive integers the following statement is true

\[ q[S(p, q, q + 1) + S(p, q, q - 1)] + p[S(q, p, p + 1) + S(q, p, p - 1)] = -2pq + \frac{2}{3} (p^2 + q^2 + 1) \]
And if \( q < p \) we get

\[
p[S(q,p,p+1) + S(q,p,p-1)] - q[S(p,p-q,p-q+1) + S(p,p-q,p-q-1)] = 2pq - 4p^2q + \frac{2}{3}(p^2 + q^2 + 1)
\]

**Corollary 1.1.** Let \((p,q) = 1\) if \( q < p \) then

\[
S(p,q,q+1) + S(p,q,q-1) + S(p,p-q,p-q+1) + S(p,p-q,p-q-1) = 4pq(p-1)
\]

The following theorem gives another reciprocity law and compute the sum of the maximal order of the square of cosecant.

**Theorem 1.2.** Let \( q \equiv 1 \pmod{p} \) then

\[
S(q,p,p+1) + S(q,p,p-1) = \frac{2}{3} \left( q + \frac{p^2}{q} + \frac{1}{q} - p^2 - 2 \right)
\]

and

\[
\sum_{k=1}^{p-1} \csc^2 \left( \frac{\pi k}{p} \right) = \frac{p^2 - 1}{3}
\]

### 2. Proof of main results

**Lemma 2.1.**

\[
\cot q\beta \cot \beta = \frac{\cos (q+1)\beta + \cos (q-1)\beta}{2 \sin q\beta \sin \beta}
\]

**Proof.** From the well-known trigonometric formula

\[
\cos \alpha \cos \beta = \frac{1}{2} (\cos (\alpha + \beta) + \cos (\alpha - \beta))
\]

we get

\[
\cot \alpha \cot \beta = \frac{\cos \alpha \cos \beta}{\sin \alpha \sin \beta} = \frac{\cos (\alpha + \beta) + \cos (\alpha - \beta)}{2 \sin \alpha \sin \beta}
\]

and the result is deduced by taking \( \alpha = q\beta \).

**Lemma 2.2.**

\[
\sum_{k=1}^{p-1} \cot \left( \frac{\pi k q}{p} \right) \cot \left( \frac{\pi k}{p} \right) = \frac{1}{2} \{ S(p,q,q+1) + S(p,q,q-1) \}
\]

**Proof.**

\[
\sum_{k=1}^{p-1} \cot \left( \frac{\pi k q}{p} \right) \cot \left( \frac{\pi k}{p} \right) = \frac{1}{2} \sum_{k=1}^{p-1} \left( \cos \left( \frac{\pi k (q+1)}{p} \right) + \cos \left( \frac{\pi k (q-1)}{p} \right) \right) \csc \left( \frac{\pi k q}{p} \right) \csc \left( \frac{\pi k}{p} \right)
\]

The decomposition of the right hand of the equality on two sums gives the result.
Using the relation (2.2) we deduce that

\[(2.3) \quad s(q, p) = \frac{1}{8p} \{S(p, q, q + 1) + S(p, q, q - 1)\}\]

**Lemma 2.3.** Let \( q \equiv 1 \pmod{p} \) then

\[(2.4) \quad s(q, p) = \frac{(p - 1)(p - 2)}{12p}\]

and

\[(2.5) \quad s(p, q) = \frac{1}{12} \left(\frac{q - 2}{p} + \frac{p}{q} + \frac{1}{pq} - p\right)\]

**Proof.** For \( q \equiv 1 \pmod{p} \), \( s(q, p) = s(1, p) \) is explicitly evaluated and

\[s(q, p) = \sum_{k=1}^{p-1} \left(\frac{k}{p} - \frac{1}{2}\right)^2\]

which becomes

\[s(q, p) = \frac{p - 1}{4} - \frac{1}{p} \sum_{k=1}^{p-1} k + \frac{1}{p^2} \sum_{k=1}^{p-1} k^2\]

It’s well-known that

\[\sum_{k=1}^{p-1} k = \frac{(p - 1)p}{2}\]

and

\[\sum_{k=1}^{p-1} k^2 = \frac{p(p - 1)(2p - 1)}{6}\]

and the result (2.4) follows.

The second result (2.5) is the consequence of the relation (2.4) and the reciprocity law (1.6). \(\Box\)

2.0.1. **Proof of Theorem 1.1.** From the relation (2.2) Lemma 2.2 and the expression (1.8) we get

\[s(q, p) = \frac{1}{8p} \{S(p, q, q + 1) + S(p, q, q - 1)\}\]

By symmetry we get the similar expression

\[s(p, q) = \frac{1}{8q} \{S(q, p, p + 1) + S(q, p, p - 1)\}\]

The reciprocity formula (1.6) inducts

\[\frac{1}{8p} \{S(p, q, q + 1) + S(p, q, q - 1)\} + \frac{1}{8q} \{S(q, p, p + 1) + S(q, p, p - 1)\} = -\frac{1}{4} + \frac{1}{12} \left(\frac{q}{p} + \frac{p}{q} + \frac{1}{pq}\right)\].
Multiplying this equality by $8pq$ we get the result (1.9) Theorem1.1.

Only we have

$$s(p - q, p) = \frac{1}{8p} \{S(p, p - q, p - q + 1) + S(p, p - q, p - q - 1)\}$$

and from the relation (1.5) we get

$$S(q, p) = \frac{p - 1}{2} - \frac{1}{8p} \{S(p, p - q, p - q + 1) + S(p, p - q, p - q - 1)\},$$

and then

$$\frac{1}{8p} \{S(p, q, q + 1) + S(p, q, q - 1)\} = \frac{p - 1}{2} - \frac{1}{8p} \{S(p, p - q, p - q + 1) + S(p, p - q, p - q - 1)\},$$

thus

$$q \{S(p, q, q + 1) + S(p, q, q - 1)\} = 4pq (p - 1) - q \{S(p, p - q, p - q + 1) + S(p, p - q, p - q - 1)\},$$

and then

$$p \{S(q, p, p + 1) + S(q, p, p - 1)\} - q \{S(p, p - q, p - q + 1) + S(p, p - q, p - q - 1)\}$$

$$= 2pq - 4p^2q + \frac{2}{3} \left(p^2 + q^2 + 1\right).$$

2.0.2. Proof of Corollary1.1. The reciprocity theorem (1.11) of Corollary1.1 is result of the relation (1.9) minus the relation (1.10)

2.0.3. Proof of Theorem1.2. From the reciprocity theorem

$$s(p, q) = -\frac{1}{4} + \frac{1}{12} \left(\frac{q}{p} + \frac{p}{q} + \frac{1}{pq}\right) - s(q, p)$$

using the relation (2.4) we deduce that

$$s(p, q) = -\frac{1}{4} + \frac{1}{12} \left(\frac{q}{p} + \frac{p}{q} + \frac{1}{pq}\right) - \frac{(p - 1)(p - 2)}{12p}$$

and

$$s(p, q) = -\frac{1}{4} + \frac{1}{12} \left(\frac{q}{p} + \frac{p}{q} + \frac{1}{pq} - p + 3 - \frac{2}{p}\right).$$

Thus

$$s(p, q) = \frac{1}{12} \left(\frac{q - 2}{p} + \frac{p}{q} + \frac{1}{pq} - p\right).$$

From the relation (2.3) we get

$$\frac{1}{8p} \{S(q, p, p + 1) + S(q, p, p - 1)\} = \frac{1}{12} \left(\frac{q - 2}{p} + \frac{p}{q} + \frac{1}{pq} - p\right)$$

and the result (1.12) follows.

Taking $q = 1$, and combining the relation (2.3) and equality (2.4) Lemma2.3 we get

$$S(p, 1, 2) + S(p, 1, 0) = \frac{2}{3} (p - 1)(p - 2),$$
then
\[ \sum_{k=1}^{p-1} \frac{\cos \left( \frac{2\pi k}{p} \right)}{\sin^2 \left( \frac{2k}{p} \right)} + \frac{1}{3} = 2p^2 - 6p + 4. \]

But
\[ \cos \left( \frac{2\pi k}{p} \right) = 1 - 2\sin^2 \left( \frac{\pi k}{p} \right) \]
then
\[ \sum_{k=1}^{p-1} 2 - 2\sin^2 \left( \frac{2k}{p} \right) = 2p^2 - 6p + 4. \]
and
\[ \sum_{k=1}^{p-1} \frac{2}{\sin^2 \left( \frac{2k}{p} \right)} = \frac{2p^2 - 6p + 4}{3} + 2(p - 1). \]
Thus
\[ \sum_{k=1}^{p-1} \frac{2}{\sin^2 \left( \frac{2k}{p} \right)} = \frac{p^2 - 3p + 2}{3} + p - 1 \]
Finally
\[ \sum_{k=1}^{p-1} \frac{1}{\sin^2 \left( \frac{\pi k}{p} \right)} = \frac{p^2 - 1}{3}. \]

3. Additional result

They are several different sums extracted from the classical Dedekind sum which satisfy a reciprocity law for a particular case. An example of a generalized cosecant sum (1.2) is given in the following proposition

Proposition 3.1.

(3.1) \[ 2S_{3,1,1} (p, 2, 1) - S_{1,1,1} (p, 2, 1) = \frac{1}{6} p^2 - p + \frac{5}{2} \]

Proof. It’s trivial that \( s(p, 2) = 0 \) and from the reciprocity law (1.6) we get

\[ s(2, p) = -\frac{1}{4} + \frac{1}{12} \left( \frac{p}{2} + \frac{5}{2p} \right). \]

Form the relation (2.3) we get

\[ s(2, p) = \frac{1}{8p} (S (p, 2, 3) + S (p, 2, 1)) \]
and
\[ S (p, 2, 3) + S (p, 2, 1) = \sum_{k=1}^{p-1} \left( \cos \left( \frac{3\pi k}{p} \right) + \cos \left( \frac{\pi k}{p} \right) \right) \csc \left( \frac{2\pi k}{p} \right) \csc \left( \frac{\pi k}{p} \right). \]
Using the trigonometric relation
\[ \cos 3\theta = 4\cos^3\theta - 3\cos\theta \]

\[
S(p, 2, 3) + S(p, 2, 1) = 4 \sum_{k=1}^{p-1} \cos^3 \left( \frac{\pi k}{p} \right) \csc \left( \frac{2\pi k}{p} \right) \csc \left( \frac{\pi k}{p} \right) - 2 \sum_{k=1}^{p-1} \cos \left( \frac{\pi k}{p} \right) \csc \left( \frac{2\pi k}{p} \right) \csc \left( \frac{\pi k}{p} \right).
\]

We deduce that
\[ 4S_{3,1,1}(p, 2, 1) - 2S_{1,1,1}(p, 1, 1) = 8s(p, 2, p). \]

Replacing \( s(2, p) \) by its value we get the result (3.1) Proposition3.1. □

Exploitation of \( S(2, p) = 0 \) conduct to proof of the following trigonometric formula
\[ \cos \left( \frac{\pi (p + 1)}{2} \right) + \cos \left( \frac{\pi (p - 1)}{2} \right) = 0 \]

The argument is that
\[
\frac{1}{16} \{ S(2, p, p + 1) + S(2, p, p - 1) \} = s(2, p) = 0
\]

and then
\[
\left\{ \cos \left( \frac{\pi (p + 1)}{2} \right) + \cos \left( \frac{\pi (p - 1)}{2} \right) \right\} \csc \left( \frac{\pi}{2} \right) = 0
\]

References

[1] B. C. Berndt, B. P. Yeap and B. Pin Explicit evaluations and reciprocity theorems for finite trigonometric sums, Adv. Appl. Math. 29 (2002) no.3 358–385
[2] C. Datta and P. Agrawal A Few Finite Trigonometric Sums, Mathematics 5 (2017) no.1, Article ID 13,11 p.
[3] da Fonseca, Carlos M.,M. Lawrance, V. Kowalenko, Basic trigonometric power sums with applications Ramanujan J.42 (2017) no.2, 401–428
[4] C. M. da Fonseca and V. Kowalenko, On a finite sum with power of cosines, Appl. Anal. Discrete Math., 7,(2013),354–377
[5] N. Gauthier and P. S. Bruckman, Sums of the even integral powers of the cosecant and secant,Fibonacci Q. 44 (2006) no.3 264–273
[6] V. Kowalenko, Generalized cosecant numbers and the Hurwitz zeta function arXiv: 1702.04000v1 [Math, NT] 14 Feb, 2017
[7] H. Rademacher and E. Grosswald, Dedekind Sums. Carus Mathematical Monograph, 16, Mathematical Association of America, Washington D.C. 1972.
[8] K. S. Williams and N. Y. Zhang Evaluation of two trigonometric sums Mathematica Slovaca, 44, (1994) no.5, 575–583

Mouloud Goubi, Department of Mathematics, University of UMMTO RP, 15000, Tizi-ouzou, Algeria, Laboratoire d’Algèbre et Théorie des Nombres, USTHB Alger E-mail address: mouloud.goubi@ummto.dz