Bergman kernel and period map for curves

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Abstract
As for any symmetric space the tangent space to Siegel upper-half space is endowed with an operation coming from the Lie bracket on the Lie algebra. We consider the pull-back of this operation to the moduli space of curves via the Torelli map. We characterize it in terms of the geometry of the curve, using the Bergman kernel form associated to the curve. It is known that the second fundamental form of the Torelli map outside the hyperelliptic locus can be seen as the multiplication by a certain meromorphic form. Our second result says that the Bergman kernel form is the harmonic representative—in a suitable sense—of this meromorphic form.

Keywords  Period matrices · Variation of Hodge structures · Bergman kernel

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1 Introduction

1.1. Let X be a Riemannian symmetric space. For a fixed point $x \in X$ we have $X = G/K$, where $G$ is a Lie group (independent of $x$) and $K = G_x$ is the stabilizer of $x$. Moreover there is a Cartan decomposition $g = \mathfrak{t} \oplus \mathfrak{p}$ such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}$ and $[\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}$. Since $\mathfrak{p} \cong T_x X$, the Lie bracket on $\mathfrak{g}$ gives rise to an operation $B_\mathfrak{s}: T_x X \times T_x X \cong \mathfrak{p} \times \mathfrak{p} \to \mathfrak{t} = \mathfrak{g}_s$. Let $S = G \times_K \mathfrak{t}$ be the homogeneous bundle over $X$ corresponding to the adjoint representation of $K$. Then $S_x = \mathfrak{g}_s$ for any $x$, so $B$ is a section of $\Lambda^2 T^* X \otimes S$. Since the differential geometry of $X$ can be studied by means of Lie theory, the tensor $B$, which reflects Lie bracket, is of central importance.

1.2. Since the tensor $B$ is invariant by the action of $G$ it makes sense also on any locally symmetric space $X$. In this paper we consider the case where $X$ is $\mathfrak{A}_g$, the moduli space of principally polarized abelian varieties of dimension $g$ over $\mathbb{C}$, which is a locally symmetric space obtained as a quotient of the Siegel upper half-space $\mathfrak{S}_g$. Denote by $M_g$ the moduli space of curves of genus $g$. We are interested in the Torelli map $j: M_g \to \mathfrak{A}_g$, which associates to $[C] \in M_g$ its Jacobian variety $[JC] \in \mathfrak{A}_g$. Our motivation comes from the study of totally geodesic subvarieties of $\mathfrak{A}_g$ that are generically contained in $j(M_g)$ [2,5,7]. This is also connected to the Coleman–Oort conjecture [9]. The tensor $B$ controls the local geometry of $\mathfrak{A}_g$ and its pull-back $B = j^* B$ to $M_g$ should give important information on the extrinsic geometry of the inclusion $j(M_g) \subset \mathfrak{A}_g$. For example we expect that the study of $B$ will give constraints on the existence of Lie triples tangent to $M_g$.

1.3. The first step in this direction is the computation of $B$ at a moduli point $[C] \in M_g$ in terms of the geometry of the curve $C$. This is the first main result of this note. Let $\bar{C}$ denote the conjugate curve, i.e., with the opposite complex structure. We first show that the dual map of $B$ can be seen as map

$$B^*: H^0(K_C) \otimes H^0(K_{\bar{C}}) \to H^0(2K_C) \otimes H^0(2K_{\bar{C}}).$$

Secondly, we consider the algebraic surface $Z = C \times \bar{C}$. By Künneth formula $H^0(Z, K_Z) \cong H^0(K_C) \otimes H^0(K_{\bar{C}})$ and $H^0(2K_Z) \cong H^0(2K_C) \otimes H^0(2K_{\bar{C}})$. With these identifications we prove the following.

Theorem A The map

$$B^*: H^0(Z, K_Z) \to H^0(Z, 2K_Z)$$

coincides with the multiplication by $-i \mathbf{K}$, where $\mathbf{K} \in H^0(Z, K_Z)$ is the Bergman kernel of the curve $C$.

(See Theorem 3.7. See 2 for the definition of Bergman kernel in the sense we need.) In other words the Bergman kernel $\mathbf{K}$ governs the restriction of the Lie bracket to $d j(TM_g)$.

1.4. Another approach to the extrinsic geometry of inclusion $j(M_g) \subset \mathfrak{A}_g$ outside the hyperelliptic locus uses the second fundamental form. If $C$ is non-hyperelliptic, the second fundamental form at $[C]$ has been interpreted as the multiplication by a holomorphic section $\hat{\eta}$ of the line bundle $K_S(2\Delta)$, where $S = C \times \bar{C}$ and $\Delta \subset S$ is the diagonal (see [2,3]). This leads back the study of the behavior of the second fundamental form to the study of the $2$–form $\hat{\eta} \in H^0(S, K_S(2\Delta))$.

1.5. The form $\hat{\eta}$ has been further studied in [1] in relation with projective structures on compact Riemann surfaces. Section 5 of [1] is dedicated to the study of the cohomology class
of the form \( \hat{\eta} \) and contains a characterization of \( \hat{\eta} \) as the unique element (up to multiples) of \( H^0(S, K_S(2\Delta)) \) with cohomology class in \( H^2(S - \Delta) \) of pure type \((1, 1)\).

1.6. In our second result we give an explicit description of the harmonic representative of the cohomology class of \( \hat{\eta} \):

**Theorem B** The Bergman kernel is the \((1, 1)\)-harmonic representative of the cohomology class of \( \hat{\eta} \in H^0(S, K_S(2\Delta)) \) in \( H^2(S - \Delta) \). More precisely, there exists \( \alpha \in H^0(S, A^{1,0}(\Delta)) \) such that

\[
\hat{\eta} - 2\pi K = d\alpha,
\]

that is \( \partial\alpha = \hat{\eta} \) and \( \bar{\partial}\alpha = -2\pi K \).

(See Theorem 4.3.) It is quite hard to control the behaviour of \( \hat{\eta} \) outside of the diagonal. Only along \( \Delta \) its behaviour admits an algebraic description, via the second Gaussian map \( \mu_2 \), see [3]. We expect the above result to allow some better understanding of \( \hat{\eta} \) and the second fundamental form.

## 2 Bergman kernel

2.1. Let \( C \) be a smooth complex projective curve of genus \( g \geq 1 \). Set \( S := C \times C \) and let \( p, q : S \to C \) be the projections \( p(x, y) = x, q(x, y) = y \).

Let \( \bar{C} \) denote the conjugate variety and set \( Z := C \times \bar{C} \). \( Z \) coincides with \( S \) as a real manifold, but has a different complex structure. The projections \( p : Z \to C, q : Z \to \bar{C} \) are holomorphic.

Denote by \( h \) the Hodge Hermitian product on \( H^0(C, K_C) \), defined by

\[
h(\alpha, \beta) := i \int \alpha \wedge \bar{\beta}.
\]

**Definition 2.1** Let \( \omega_1, \ldots, \omega_g \) be a unitary basis for \( H^0(C, K_C) \). Then

\[
K := \sum_{j=1}^{g} p^*\omega_j \wedge q^*\bar{\omega}_j
\]

is a well-defined \((1, 1)\)-form on \( S \) independent of the choice of the unitary basis. It is called the **Bergman kernel form** of the algebraic curve \( C \).

This is the definition of the Bergman kernel form on an arbitrary complex manifold due to Kobayashi [8]. It generalizes the classical Bergman kernel on open domains in \( \mathbb{C}^n \). \( K \) can also be seen as a holomorphic 2-form on \( Z \). In particular it is a harmonic form with respect to any Kähler metric on \( Z \). If we consider it as a \((1,1)\)-form on \( S \), it is harmonic for any Kähler metric on \( S \) which is Kähler also on \( Z \). In particular it is harmonic for any product metric.

If \( x, y \in C \), \( T_{(x, y)}S = T_x C \oplus T_y C \). Thus elements of \( T_{(x, y)}S \) are pairs \((u, v)\) with \( u \in T_x C \) and \( v \in T_y C \). Since \( K \) is a \((1,1)\)-form, its behaviour is controlled by the values \( K((u, 0), (0, \bar{v})) \) for \( u \in T_x^{1,0} C \), \( v \in T_y^{1,0} C \).

2.2. Although not needed in the following, it is interesting to point out the following relation between the Bergman kernel and the period matrix associated to the algebraic curve \( C \) (cf. [11, eq. (2.4)]). Let \( Q \) denote the intersection form on \( H_1(C, \mathbb{Z}) \) and let \( \{a_i, b_j\} \) be a symplectic basis for \((H_1(C, \mathbb{Z}), Q)\). Consider a basis \( \omega_1, \ldots, \omega_g \) of \( H^0(C, K_C) \) normalized with respect
to \( \{a_i, b_i\} \) and the period matrix \( \mathcal{X} = (z_{ij}) \) with \( z_{ij} = \int_{b_j}^{a_i} \omega_i \). Then, with respect to this basis, the Bergman kernel has the form

\[
K = \frac{1}{2} \sum_{i,j} (\text{Im } \mathcal{X})^{ij} p^* \omega_j \wedge q^* \bar{\omega}_j
\]

(2.1)

where \((\text{Im } \mathcal{X})^{ij}\) denote the coefficients of \((\text{Im } \mathcal{X})^{-1}\).

To check this observe first that \( h(\omega_i, \omega_j) = 2 \text{ Im } z_{ij} \). Indeed let \( \mathcal{B} = \{a_i^*, b_j^*\} \) be the dual basis. If \( D : H^1(C) \to H_1(C) \) is Poincaré duality, then \( Da_i^* = b_j \) and \( Db_j^* = -a_i \), so \( \mathcal{B} \) is symplectic for \( Q^*(\alpha, \beta) = \int_C \alpha \cup \beta \). Since \( \omega_i = a_i^* + \sum_{k=1}^n z_{ik} b_k^* \), the result follows.

Now (2.1) is a consequence of the following general fact: if \( \alpha_1, \ldots, \alpha_8 \) is a basis of \( H^0(C, K_C) \) and \( A \) is the matrix with entries \( a_{ij} := h(\alpha_i, \alpha_j) \), then \( K = \sum_{i,j} a_{ij} p^* \alpha_i \wedge q^* \bar{\alpha}_j \).

2.3. Next we show how to recover the Bergman kernel using the so-called elementary potentials. Let \( (U, z) \) be a chart centered at \( x \in C \) and set \( u = \frac{\partial}{\partial z}(x) \). Classical results ensure the existence of a harmonic function \( f_u \in C^\infty(C - \{x\}) \) such that \( f_u = -\frac{1}{z} + g(z) \) on \( U - \{x\} \) for some \( g \in C^\infty(U) \). The function \( f_u \) is unique up to an additive constant and it is called elementary potential (see [2, §3]). We recall some of its properties that will be relevant in our analysis. It follows from the definition that \( \bar{\partial} f_u \) is smooth on \( C \) and that

\[
\int_C \omega \wedge (\bar{\partial} f_u) = 2\pi i \omega(u), \quad \text{for } \omega \in H^0(C, K_C).
\]

(2.2)

(See [2, Section 3] for more details.) This shows that elementary potentials are related to evaluation and the canonical map. Therefore they are clearly related to Bergman kernel as we show now.

2.4. For \( x \in C \) and \( u \in T_x C \), let \( \text{ev}_u : H^0(C, K_C) \to \mathbb{C} \) be the evaluation map and let \( k_u \in H^0(K_C) \) be such that

\[
\text{ev}_u = h(\cdot, k_u).
\]

Lemma 2.2 For \( u \in T_x^{1,0} C, v \in T_y^{1,0} C \), with \( x, y \in C \), we have

\[
K((u, 0), (0, \bar{v})) = h(k_v, k_u) = k_v(u).
\]

Proof Let \( \omega_1, \ldots, \omega_8 \) be a unitary basis for \( H^0(K_C) \). Then \( k_v = \sum_j \lambda_j \omega_j \), with \( \lambda_j = h(k_v, \omega_j) = \overline{\omega_j(v)} \). Thus \( h(k_v, k_u) = k_v(u) = \sum_j \overline{\omega_j(v)} \omega_j(u) = K((u, 0), (0, \bar{v})) \).

Lemma 2.3 Let \( x, y \in C \), and \( u \in T_x^{1,0} C, v \in T_y^{1,0} C \). If \( f_u \) is an elementary potential, then \( \bar{\partial} f_u = 2\pi \overline{k_u} \). In particular

\[
K((u, 0), (0, \bar{v})) = \frac{1}{2\pi} \bar{\partial} f_u(\bar{v}).
\]

Proof From (2.2) and (2.3) it follows that \( \int_C \omega \wedge (\bar{\partial} f_u) = -2\pi i \omega(u) = -2\pi i h(\omega, k_u) = 2\pi \int_C \omega \wedge \overline{k_u} \). So \( \bar{\partial} f_u \) and \( 2\pi \overline{k_u} \) have the same cohomology class. Since both are harmonic they coincide. Next by the previous Lemma \( K((u, 0), (0, \bar{v})) = h(k_v, k_u) = h(k_u, k_v) = k_u(v) = k_v(\bar{v}) = \frac{1}{2\pi} \bar{\partial} f_u(\bar{v}) \).

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3 Lie bracket

In this section we study the Lie bracket i.e. the tensor $B$ on Siegel space, introduced in 1, and prove Theorem A. We start by giving a suitable description of the Siegel upper half-space. Next we go through several identifications of the tangent space to $G$, and write down $B$ in terms of them (Proposition 3.4). Given a curve, we apply this to its Jacobian, i.e. we consider $B$ as in 1 (Proposition 3.6). We recall further identifications using the conjugate curve and the surface $Z = C \times \bar{C}$. This allows to understand the dual map $B^*$ as a map $H^0(Z, K_Z) \rightarrow H^0(Z, 2K_Z)$. Finally using the elementary potentials we prove our main result Theorem 3.7.

Let $(V, Q)$ be a real symplectic vector space. If $J \in \text{End } V$ satisfies $J^2 = -I_V$ and $J^* Q = Q$, the bilinear form $g_J(v, v') := Q(Jv, Jv')$ is symmetric. Siegel upper half-space is defined as

\[ \mathfrak{G}(V, Q) := \{ J \in \text{End } V : J^2 = -I_V, J^* Q = Q, g_J \gg 0 \}. \]

For every $J$ we denote $V_{-1,0}(J)$ and $V_{0,-1}(J)$ the $\pm i$-eigenspaces of $J$ on $V_C$. We also set

\[ H^{1,0}_J := \text{Ann } V_{0,-1}, \quad H^{0,1}_J := \text{Ann } V_{-1,0}(J). \]

We usually drop $J$ in the notation. When $V = \mathbb{R}^{2g}$ and $Q$ is the standard form, we write $\mathfrak{G}_g$. The symplectic group $\text{Sp} := \text{Sp}(V, Q)$ acts on $\mathfrak{G}$ by conjugation. This action is transitive and $\mathfrak{G}$ is a Hermitian symmetric space. For $X \in \text{End } V$ set $Q_X := Q(\cdot, X \cdot)$. Then $\mathfrak{sp} = \mathfrak{sp}(V, Q) = \{ X \in \text{End } V : Q_X \text{ is symmetric} \}$. If $\mathfrak{sp} = \mathfrak{sp}_J \oplus \mathfrak{p}$ is the Cartan decomposition at $J \in \mathfrak{G}$, then

\[ \mathfrak{p} = \{ X \in \mathfrak{sp} : XJ + JX = 0 \}, \quad \mathfrak{sp}_J = \{ X \in \mathfrak{sp} : [J, X] = 0 \}. \]

We endow $\mathfrak{p} \cong T_J \mathfrak{G}$ with the complex structure $i := (1/2) \text{ad } J$. Then

\[ \mathfrak{p}_C = \{ X \in \mathfrak{sp}_C : X(V_{-1,0}) \subset V_{0,-1} \text{ and } X(V_{0,-1}) \subset V_{-1,0} \}, \]

\[ \mathfrak{p}^{1,0} = \{ X \in \text{Hom}(V_{0,-1}, V_{-1,0}) : Q_X \text{ is symmetric} \}. \]

3.1. We have an isomorphism

\[ \varphi_Q : V_C \longrightarrow V_C^*, \quad \varphi_Q(v) := Q(\cdot, v). \]

Its inverse is denoted by $\psi_Q := \varphi_Q^{-1}$. For any Lagrangian subspace $L \subset V_C$ the isomorphism $\varphi_Q$ maps $L$ onto $\text{Ann}(L)$. Therefore $\varphi_Q$ gives an isomorphism $V_{0,-1} \cong H^{1,0}$.

As mentioned in 1 we are interested in the Lie bracket which can be seen as a section of a bundle over the symmetric space $\mathfrak{G}$. We wish to compute $B_J \in \Lambda^2 T^*_J \mathfrak{G} \otimes \mathfrak{sp}_J$. As usual it is useful to look at $B$ through its complexification

\[ B_J : (T_J \mathfrak{G})_C \times (T_J \mathfrak{G})_C \longrightarrow (\mathfrak{sp}_J)_C. \]

Recall that $(T_J \mathfrak{G})_C = \mathfrak{p}_C = \mathfrak{p}^{1,0} \oplus \mathfrak{p}^{0,1}$, where $\mathfrak{p}^{1,0}$ is given by (3.1) and

\[ (\mathfrak{sp}_J)_C = \{ X \in \text{End } V_C : X(V_{-1,0}) \subset V_{0,-1}, \quad X(V_{0,-1}) \subset V_{-1,0}, \quad Q_X \text{ is symmetric} \}. \]

Lemma 3.1 The map $B_J$ is of type $(1, 1)$, i.e. it vanishes on vectors of the same type.

**Proof** Since $B_J$ is real, it is enough to show that it vanishes on pairs of $(1, 0)$-vectors. If $X, Y \in \mathfrak{p}^{1,0}$, then $X(V_C) \subset V_{-1,0}$ and $Y|_{V_{-1,0}} = 0$, thus $XY = 0$. For the same reason also $XY = 0$. Thus $B_J(X, Y) = XY - YX = 0$. \qed
If $X \in \text{End} V_C$, let $X^* \in \text{End} V_C^*$ denote the transpose. The transposition map $X \mapsto X^*$ is a canonical isomorphism $\text{End} V_C \cong \text{End} V_C^*$. It is useful to reinterpret everything in terms of $\text{End} V_C^*$ rather than $\text{End} V_C$. Set

$$Q^* := \psi_Q^* Q.$$  

(Notation as in 3.1.) Then $Q^*$ is a symplectic form on $V_C^*$.

**Lemma 3.2** If $X \in \text{End} V_C$, then $Q_X$ is symmetric iff $Q_{X^*}$ is symmetric.

**Proof** Define $\tilde{X} \in \text{End} V_C$ by $Q(Xa, b) = Q(a, \tilde{X}b)$. Then

$$X^*\varphi_Q a = X^*Q(\cdot, a) = Q(X\cdot, a) = Q(\cdot, \tilde{X}a) = \varphi_Q \tilde{X}a.$$  

Given $\alpha, \beta \in V_C^*$ let $\varphi = \psi_Q a, b = \varphi_Q \beta$. Then

$$-Q_{X^*}(\alpha, \beta) = Q^*(X^*\alpha, \beta) = Q(\psi_X^*\varphi_Q a, \psi_Q^*\psi_Q b) = Q(\tilde{X}a, b) = Q(a, Xb) = Q_X(a, b).$$

So $-Q_{X^*}(\alpha, \beta) = Q_X(a, b)$. The statement follows. \hfill \Box

**Lemma 3.3** If $X \in \text{End} V_C$, then

$$X|_{V_{-1,0}} = 0 \implies \text{Im } X^* \subset H^{0,1} \text{ and } X \subset V_{-1,0} \implies X^*|_{H^{0,1}} = 0.$$  

**Proof** Recall that for any linear map $L : E \rightarrow F$ of vector spaces $\text{Ann } \text{Im } L = \ker L^*$. Now $X|_{V_{-1,0}} = 0 \implies V_{-1,0} \subset \ker X \implies H^{1,0} = \text{Ann } V_{-1,0} \supset \text{Ann } \ker X = \text{Im } X^*$. And $\text{Im } X \subset V_{-1,0} \implies \ker X^* = \text{Ann } \text{Im } X \supset \text{Ann } V_{-1,0} = H^{0,1}$. \hfill \Box

**Proposition 3.4** There are canonical isomorphisms

$$\mathfrak{sp}_1^0 \cong \{ t \in \text{Hom}(H^{1,0}, H^{0,1}) : Q_t^* \text{ is symmetric}, \} \quad (\mathfrak{sp}_J)_C \cong \text{End } H^{1,0}.$$  

Using these isomorphisms $B_J$ gets identified with the map

$$B_J : \mathfrak{p}^1 \times \overline{\mathfrak{p}}^1 \rightarrow \text{End } H^{1,0} \quad (s, \bar{t}) \mapsto \bar{t}s.$$  

**Proof** The first isomorphism is simply the restriction to $\mathfrak{p}^{1,0}$ of the map $X \mapsto X^*$. Lembmas 3.2 and 3.3 show that indeed the image of $\mathfrak{p}^{1,0}$ is the set of $X^* \in \text{End} V_C$ that vanish on $H^{0,1}$, have image in $H^{1,0}$ and such that $Q_{X^*}$ is symmetric. To describe the second isomorphism start from (3.2). Again the Lemmas show that the map $X \mapsto X^*$ sends $(\mathfrak{sp}_J)_C$ to the set of $X^* \in \text{End} V_C^*$ that preserve each $H^{p,q}$ and such that $Q_{X^*}$ is symmetric. The latter means that $Q^*(u, X^*v) = Q^*(v, X^*u)$. This identity is trivial if $u$ and $v$ have the same type, since in that case both term vanish. Hence $X^*|_{H^{1,0}}$ is an arbitrary endomorphism of $H^{1,0}$. On the contrary the identity shows that for any $v \in H^{0,1}$ the value $X^*v$ is determined by $X^*|_{H^{1,0}}$. Hence

$$(\mathfrak{sp}_J)_C \cong \text{End } H^{1,0}, \quad X \mapsto X^*|_{H^{1,0}}.$$  

is the desired isomorphism. Now let $X, Y \in \mathfrak{p}^{1,0}$ and set $s := X^*, t := Y^*$. Then $B(X, Y)^* = \bar{t}s - s\bar{t}$. Since $\bar{t}|_{H^{1,0}} = 0$, in the isomorphism (3.4) $B(X, Y) \in (\mathfrak{sp}_J)_C$ corresponds to $B(X, Y)^*|_{H^{1,0}} = \bar{t}s$. \hfill \Box

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Since $H^{1,0} = \text{Ann} \ V_{0,-1}$, we have $\text{Ann} \ H^{1,0} = V_{0,-1}$. So there is a canonical isomorphism $(H^{1,0})^* \cong V_C / \text{Ann} \ H^{1,0} = V_C / V_{0,-1} \cong V_{-1,0}$. We treat this isomorphism as an identity.

By 3 $\varphi_Q$ maps $V_{-1,0}$ isomorphically onto $H^{0,1}$. Thus $\psi_Q = \varphi_Q^{-1}$ restricts to an isomorphism

$$\psi_Q : H^{0,1} \xrightarrow{\cong} (H^{1,0})^*.$$  \hfill (3.5)

**Lemma 3.5** For $\tilde{\omega} \in H^{0,1}$, we have $\psi_Q(\tilde{\omega}) = Q^* (\tilde{\omega}, \cdot)$.

**Proof** First we claim that $\varphi_Q \ast \varphi_Q = - i d_{V_C}$ i.e. $\psi_Q = - \varphi_Q \ast$. Indeed fix $v \in V_C$. That $\varphi_Q \ast \varphi_Q (v) = -v$ means that $Q^* (\cdot, \varphi_Q (v)) = -v$, i.e. that $Q^* (\lambda, \varphi_Q (v)) = -\lambda (v)$ for any $\lambda \in V_C^\vee$. Assume $\lambda = \varphi_Q (w)$ for $w \in V_C$. Then $\lambda (v) = Q (v, w)$ and $Q^* (\lambda, \varphi_Q (v)) = Q^* (\varphi_Q (w), \varphi_Q (v)) = Q(w, v)$. \hfill $\square$

Now consider the period map $j : M_g \rightarrow A_g$. Let $x \in M_g$ be the moduli point of a curve $C : x = [C]$. If we fix a symplectic basis of $H_1 (C, \mathbb{Z})$ we get a symplectic isomorphism of $H_1 (C, \mathbb{R})$ with the intersection form onto $(\mathbb{R}^{2g}, Q)$. Thus the Hodge decomposition $H^1 (C, \mathbb{C}) = H^{1,0} (C) \oplus H^{0,1} (C)$ gives a complex structure on $H^1 (C, \mathbb{C})$, hence a point $J \in \mathbb{S}_g$. In the following we use $T_x M_g$ to denote the real tangent space i.e. the tangent space of $M_g$ as a differentiable manifold (and similarly for $A_g$). Thus $(T_x M_g)_{\mathbb{C}} = T_x^{1,0} M_g \oplus T_x^{0,1} M_g$ and $T_x^{0,1} M_g = H^1 (C, T_C)$, while $(T_x A_g)_{\mathbb{C}} = (T_J \mathbb{S}_g)_{\mathbb{C}} = \mathbb{P}^1.0 \oplus \mathbb{P}^{0,1}$. By a theorem of Griffiths the map $d j_x : H^1 (C, T_C) \rightarrow \mathbb{P}^{1,0}$ using the interpretation (3.3) is given by

$$d j_x (\xi) = \xi \cup : H^{1,0} \rightarrow H^{0,1}, \quad d j_x (\xi) (\omega) = \xi \cup \omega.$$  

(See e.g. [10, pp. 234ff].) Now $T_x^{0,1} M_g = H^1 (C, T_C)$ is the conjugate vector space, i.e. it has the same underlying real vector space as $H^1 (C, T_C)$ but multiplication by $i$ is replaced with multiplication by $-i$. Since $j$ is holomorphic, its differential is a direct sum of the maps $d j_x : H^1 (C, T_C) \rightarrow \mathbb{P}^{1,0}$ and its conjugate. Hence for $\tilde{\eta} \in \overline{H^1 (C, T_C)}$ and $\omega \in H^{1,0} (C)$ we have

$$d j_x (\tilde{\eta}) = \tilde{\eta} \cup : H^{1,0} \rightarrow H^{0,1}, \quad d j_x (\tilde{\eta}) (\omega) = \tilde{\eta} \cup \omega.$$  

As mentioned in the Introduction our goal is to study the map

$$B_{\xi} := d j^*_{\xi} B_{\tilde{\eta}} : H^1 (C, T_C) \times \overline{H^1 (C, T_C)} \rightarrow \text{End} H^{1,0} (C).$$

The following is a consequence of Proposition 3.4.

**Proposition 3.6** For $\xi, \eta \in H^1 (C, T_C)$ and $\omega \in H^{1,0} (C)$, we have

$$B (\xi, \tilde{\eta}) (\omega) = \tilde{\eta} \cup (\xi \cup \omega).$$

Once again it is useful to dualize. This time we dualize the map $B$ itself. Using (3.5) we can describe the domain of $B^*$ as follows:

$$(\text{End} \ H^{1,0})^* = (H^{1,0})^* \otimes H^{1,0})^* = H^{1,0} \otimes H^{1,0} \cong H^{1,0} \otimes H^{0,1}.$$  

More explicitly, let $\omega, \omega' \in H^{1,0}$ and $t \in \text{End} \ H^{1,0}$. Then $\omega \otimes \omega' \in H^{1,0} \otimes H^{0,1}$. Recalling Lemma 3.5 one easily verifies that the corresponding element of $(\text{End} \ H^{1,0})^*$ is the linear functional mapping $t$ to $Q^* (\omega', t \omega)$. The dual of $H^1 (C, T_C)$ is $H^0 (C, 2 K_C)$. Thus the dual of $B$ is defined on $H^0 (C, 2 K_C) \otimes H^0 (C, 2 K_C)$ and maps to $H^{1,0} (C) \otimes H^{1,0} (C)$. 

$\square$
Denoting by \( \tilde{C} \) the conjugate variety we have
\[
H^0(C, 2K_C) = H^0(\tilde{C}, 2K_{\tilde{C}}), \quad H^{1,0}(\tilde{C}) = H^{1,0}(C).
\]
Thus \( B^* \) is a map from \( H^{1,0}(C) \otimes H^{1,0}(\tilde{C}) \) to \( H^0(C, 2K_C) \otimes H^0(\tilde{C}, 2K_{\tilde{C}}) \). We further reinterpret domain and target of \( B^* \) as spaces of sections of appropriate bundles on \( Z = C \times \tilde{C} \). Denoting by \( p : Z \to C \) and \( q : Z \to \tilde{C} \) the projections and given bundles \( L \to C \) and \( M \to \tilde{C} \), set \( L \boxtimes M := p^*L \otimes q^*M \). The map
\[
H^0(C, L) \otimes H^0(\tilde{C}, M) \to H^0(Z, L \boxtimes M), \quad s \otimes t \mapsto p^*s \otimes q^*t,
\]
is an isomorphism. For any positive integer \( n \) there is a canonical isomorphism
\[
K^n_Z \cong K^n_C \boxtimes K^n_{\tilde{C}}, \tag{3.6}
\]
obtained as follows: if \( \alpha \in K_{C,x} \) and \( \beta \in K_{\tilde{C},y} \), denote by \( \alpha^n \in (K_{C,x})^\otimes n \) and \( \beta^n \in (K_{\tilde{C},y})^\otimes n \) the tensor powers. Then \( \alpha^n \otimes \beta^n \in (K^n_C \boxtimes K^n_{\tilde{C}})_{(x,y)} \), while \( (p^*\alpha \wedge q^*\beta)^n \in K_Z(x,y) \). The isomorphism (3.6) maps \( \alpha^n \otimes \beta^n \) to \( (p^*\alpha \wedge q^*\beta)^n \).

We now prove Theorem A.

**Theorem 3.7** The map
\[
B^* : H^0(Z, K_Z) \to H^0(Z, 2K_Z)
\]
coinsides with the multiplication by \(-iK\).

**Proof** Fix a point \( x \in C \) and a chart \((U, z)\) centered in \( x \). Set \( u = \frac{\partial}{\partial z}(x) \) and consider the Schiffer variation \( \xi_u \) at \( x \in C \). We recall that
\[
\xi_u \cup = -2\pi \text{ ev}_u \otimes \overline{k}_u. \tag{3.7}
\]
Indeed fix a Dolbeault representative \( \varphi = \tilde{\varphi} b \) \( \frac{\partial}{\partial z} \) where \( b \in C^\infty \) is a bump function which is equal to 1 in a neighbourhood of \( x \). For \( \omega \in H^0(K_C) \), with local expression \( \omega = h(z)dz \) on \( U \), it holds that
\[
dj_x(\xi_u)(\omega) = \xi_u \cup \omega = [\varphi \cdot \omega] = \left[ \tilde{\partial}(bh) \right] \frac{\partial z}{z}.
\]
If \( f_u \) is an elementary potential, the functions
\[
\frac{b}{z} + f_u \quad \text{and} \quad b \cdot \frac{h - h(0)}{z}
\]
are smooth on \( C \). Hence
\[
\tilde{\partial} \left( \frac{bh}{z} \right) = \tilde{\partial} \left( b \cdot \frac{h - h(0)}{z} \right) + h(0) \cdot \tilde{\partial} \left( \frac{b}{z} + f_u \right) - h(0) \partial f_u.
\]
Thus \( dj_x(\xi_u)(\omega) = h(0) \cdot [\tilde{\partial} f_u] \). As usual we identify \( H^{0,1}(C) \) with the space of antiholomorphic forms. Thus since \( h(0) = \omega(u) \), using Lemma 2.3 and the fact that \( \overline{k}_u \) is antiholomorphic, we get (3.7).

We also recall (see [2, Lemma 2.3]) that for \( \beta \in H^0(C, 2K_C) = H^1(C, T_C)^* \), we have \( \beta(\xi_u) = 2\pi i \beta(u) \). It follows that for \( \Phi \in H^0(Z, 2K_Z) \)
\[
\Phi((u, 0), (0, \overline{v})) = -\frac{1}{4\pi^2} \Phi(\xi_u \otimes \xi_{\overline{v}}).
\]
Now we can prove the statement. Without loss of generality we can assume that $\Omega = p^*\omega \wedge q^*\overline{\omega}$, with $\omega, \omega' \in H^0(K_C)$. Then

$$(B^*\Omega)_{(x, y)}((u, 0), (0, \overline{v})) = -\frac{1}{4\pi^2} B^*(p^*\omega \wedge q^*\overline{\omega})(\xi_u \otimes \xi_{\overline{v}})$$

$$= -\frac{1}{4\pi^2} (p^*\omega \wedge q^*\overline{\omega})(B(\xi_u \otimes \xi_{\overline{v}})).$$

It follows from (3.7) that

$$\xi_{\overline{v}} \cup (\xi_u \cup \omega) = -2\pi \omega(u) \cdot \xi_{\overline{v}} \cup \kappa_u = -2\pi \omega(u) \cdot \xi_{\overline{v}} \cup \kappa_u$$

$$= 4\pi^2 \omega(u) \cdot \kappa_u(v) \cdot k_v.$$

Using this and Lemma 3.5 we get

$$(B^*\Omega)_{(x, y)}((u, 0), (0, \overline{v})) = -\frac{1}{4\pi^2} Q^*(\overline{\omega'}, B(\xi_u \otimes \xi_{\overline{v}})\omega)$$

$$= -\frac{1}{4\pi^2} Q^*(\overline{\omega'}, \xi_{\overline{v}} \cup (\xi_u \cup \omega)) = -\omega(u) \cdot \kappa_u(v) \cdot Q^*(\overline{\omega'}, k_v).$$

Since $Q^*(\overline{\omega'}, k_v) = i \cdot \overline{\omega'(v)}$ and using Lemma 2.2 we finally get

$$(B^*\Omega)_{(x, y)}((u, 0), (0, \overline{v})) = -i \omega(u)\overline{\omega'(v)} \cdot \kappa_u(v)$$

$$= -i \cdot (\Omega \cdot K)_{(x, y)}((u, 0), (0, \overline{v})).$$

\[\square\]

4 The form $\hat{\eta}$

Fix a smooth complex projective curve $C$ of genus $g > 0$ and let $\Delta \subset S = C \times C$ be the diagonal. In this section we recall the definition of the meromorphic form $\hat{\eta} \in H^0(C, K_S(2\Delta))$ constructed in [2, 3], which governs the second fundamental form of the Torelli map with respect to the Siegel metric. Next we recall from [1] the analysis of its cohomology class. Finally we prove our second main result, i.e. Theorem B.

4.1. The construction of the form $\hat{\eta}$ goes as follows. For $x \in C$, let

$$j_x : H^0(C, K_C(2x)) \hookrightarrow H^1(C - \{x\}, \mathbb{C}) = H^1(C, \mathbb{C})$$

be the map that associates to $\omega \in H^0(C, K_C(2x))$ its de Rham cohomology class. This map is an injection since $C \neq \mathbb{P}^1$. As $H^{1,0}(C) \subset j_x(H^0(C, K_C(2x)))$ and $h^0(C, K_C(2x)) = g + 1$, the preimage $j_x^{-1}(H^{0,1}(C))$ is a line. Thus, fixed a chart $(U, z)$ centered at $x \in C$, there exists a unique element $\varphi$ in this line such that on $U - \{x\}$

$$\varphi = \left(\frac{1}{z^2} + h(z)\right) dz$$

with $h \in O_C(U)$. Set $u = \frac{\partial}{\partial z}(x)$, and define the map

$$\eta_x : T^1_{x, 0}C \rightarrow H^0(C, K_C(2x)), \quad \lambda u \mapsto \eta_x(\lambda u) := \lambda \varphi.$$ 

It is easy to see that $\eta_x$ does not depend on the choice of the local coordinate. In the following we will also use the fact that if $f_u$ is an elementary potential, then $\partial f_u = \eta_u$, see [2, Lemma 3.1].
Next consider the line bundle $L := K_S(2\Delta)$ on $S$ and set

$$V := p_*(q^*K_C(2\Delta)), \quad E := p_*L.$$  

By the projection formula $E = K_C \otimes V$. Also, since $q^*K_C(2\Delta)|_{\{x\} \times C} = q^*K_C(2x)$, we have that $H^0(p^{-1}(x), q^*K_C(2\Delta)) \simeq H^0(C, K_C(2x))$ and the fiber of the holomorphic vector bundle $V \to C$ on $x \in C$ is isomorphic to $H^0(C, K_C(2x))$. Thus $\eta_x \in E_x$. More precisely, the map $x \mapsto \eta_x$ is a holomorphic section of $E [2, Proposition 3.4].$

Finally, since $E = p_*L$, there is an isomorphism between $H^0(C, E)$ and $H^0(S, L)$ that associates to $\alpha \in H^0(C, E)$ the section $\hat{\alpha}$ of $L$ such that $\alpha_x = \hat{\alpha}|_{\{x\} \times C} \in E_x$. The form $\hat{\eta} \in H^0(S, K_S(2\Delta))$ is defined as the holomorphic section of $L$ corresponding to $\eta \in H^0(C, E)$. Note that, in particular, for $u \in T^1_1 C$ and $v \in T^1_1 C$ with $x \neq y$, it holds

$$\eta_x(u)(v) = \hat{\eta}(u, v). \quad (4.1)$$

4.2. The form $\hat{\eta}$ also appears in an unpublished book of Gunning [6] under the name of intrinsic double differential of the second kind.

4.3. The importance of $\hat{\eta}$ comes from the fact that the second fundamental form of the Torelli map outside the hyperelliptic locus coincides with the multiplication by $\hat{\eta} [2,3]$. The form $\hat{\eta}$ has been further studied in [1] in relation with projective structures on compact Riemann surfaces. Moreover Section 5 in [1] contains an analysis of the cohomology class of the form $\hat{\eta}$. Denoting $j : S - \Delta \hookrightarrow S$ the inclusion map, it follows from the exact sequence of homology groups for the pair $(S, \Delta)$ and Poincaré and Lefschetz dualities, that the homomorphism $j^* : H^2(S, \mathbb{Z}) \to H^2(S - \Delta, \mathbb{Z})$ is surjective and its kernel is generated by the (pure) class of the diagonal. Consequently, the Hodge decomposition of $H^2(S)$ induces a decomposition of $H^2(S - \Delta)$. In particular, for any $\zeta \in H^0(S, K_S(2\Delta))$ there is $[\gamma] \in H^2(S)$ such that $[\zeta] = j^* [\gamma] \in H^2(S - \Delta)$. Moreover the $(0, 2)$ part of $[\gamma]$ vanishes. So

$$[\gamma] = \gamma^{2,0} + \gamma^{1,1}$$

where $\gamma^{2,0}$ is holomorphic and $\gamma^{1,1}$ is harmonic of type $(1, 1)$. (Harmonicity is for any product metric.) In this context, the fundamental result on the cohomology of $\hat{\eta}$ is that

$$[\hat{\eta}] = j^*[\gamma^{1,1}] \in H^2(S - \Delta),$$

where $\gamma^{1,1}$ is a harmonic $(1, 1)$--form on $S$. That is, $\hat{\eta}$ has cohomology class in $H^2(S - \Delta)$ of pure type $(1, 1)$. In fact, this implies the characterization of $\hat{\eta}$ as the unique element (up to multiples) of $H^0(S, K_S(2\Delta))$ with cohomology class in $H^2(S - \Delta)$ of pure type $(1, 1)$. In the following we give an explicit description of the harmonic representative $\gamma^{1,1}$ of $\hat{\eta}$.

4.4. Denote by $\mathcal{A}^{p,q}$ the sheaf of smooth differential forms of type $(p, q)$ on $S$. Denote by $\mathcal{A}^{p,q}(n\Delta)$ the sheaf of $(p, q)$--forms having a pole of order at most $n$ on $\Delta$, i.e. those forms $\omega$ such that $x^n \omega$ is smooth of type $(p, q)$, where $x = 0$ is a local equation of $\Delta$.

4.5. The elementary potentials are defined up to an additive constant. We can normalize them as follows: if $f_u = -1/z + g(z)$, we impose that $g(0) = 0$. In the following we fix this normalization. For $u \in (T_x C)_{\mathbb{C}}$, denote by $u^{1,0}$ the $(1, 0)$--component of $u$ and set $f_u := f_u^{1,0}$. For $u \in (T_x C)_{\mathbb{C}}$ and $v \in (T_y C)_{\mathbb{C}}$, set

$$\alpha(u, v) := 2f_v(p) + f_u(q).$$

Lemma 4.1 The form $\alpha$ is a section of $\mathcal{A}^{1,0}(\Delta)$. 

\[ Springer\]
The Bergman kernel is the \( f_{\bar{z}_2}(z_1) = -\frac{1}{z_2 - z_1} + \varphi(z_1, z_2) \).

By the normalization we have \( \varphi(z_1, z_1) = 0 \). We show that \( \partial \varphi / \partial z_2 \) and \( \partial \varphi / \partial \bar{z}_2 \) are smooth functions on \( U \times U \). This will prove that \( \varphi \in C^\infty(U \times U) \) and hence the lemma. Indeed

\[
\frac{\partial \varphi}{\partial z_2} = \frac{\partial}{\partial z_2}\left(f_{\bar{z}_2}(z_1) + \frac{1}{z_2 - z_1}\right) = 2\pi K\left((\frac{\partial}{\partial z_1}, 0), (0, \frac{\partial}{\partial \bar{z}_2})\right),
\]

\[
\frac{\partial \varphi}{\partial \bar{z}_2} = \frac{\partial}{\partial \bar{z}_2}\left(f_{\bar{z}_2}(z_1) + \frac{1}{z_2 - z_1}\right) = \eta\left(\frac{\partial}{\partial z_2}\right) - \frac{1}{(z_2 - z_1)^2}.
\]

The second term is smooth since \( \hat{\eta} \in H^0(S, K_S(2\Delta)) \).

**Remark 4.2** The form \( \hat{\eta} \) is holomorphic on \( S \) and is symmetric in the sense that \( \hat{\eta}(u_0, (0, \nu)) = \hat{\eta}(v, 0, (0, u)) \), whereas \( K \) is of type \((1, 1)\) and \( \overline{K(u_0, (0, \nu))} = K((v, 0), (0, \overline{\nu})) \). As it will be clear in the proof below, the asymmetry in the definition of \( \alpha(u, v) \) is needed to accord these differences.

We can now prove Theorem B.

**Theorem 4.3** The Bergman kernel is the \((1, 1)\)-harmonic representative of the cohomology class of \( \hat{\eta} \in H^0(S, K_S(2\Delta)) \) in \( H^2(S - \Delta) \). More precisely,

\[
\hat{\eta} - 2\pi K = d\alpha,
\]

that is \( \partial \alpha = \hat{\eta} \) and \( \overline{\partial} \alpha = -2\pi K \).

**Proof** We first observe that for \( u \in (T_x C)_C \) and \( v \in (T_y C)_C \), denoted by \( U, V \) two vector fields on \( C \) such that \( U_x = u \) and \( V_y = v \), since \([U, 0], (0, V)\] = 0, we have that

\[
d\alpha((u, 0), (0, v)) = (U, 0)(\alpha(0, v)) - (0, V)(\alpha(u, 0)).
\]

Now assume \( u \in T_x^{1,0} C \) and \( v \in T_y^{1,0} C \) and that \( U \) and \( V \) are \((1, 0)\). For \( x \neq y \), \( \partial f_u = \eta_x(u) \) and we get

\[
\partial\alpha((u, 0), (0, v)) = 2(U, 0)f_v - (0, V)f_u = 2\partial f_v(u) - \partial f_u(v) = 2\eta_y(v)(u) - \eta_x(u)(v) = \hat{\eta}(u, 0), (0, v),
\]

where for the last equality we used (4.1) and the symmetry of \( \hat{\eta} \) (see [2, Lemma 3.5]).

Similarly using Lemma 2.3

\[
\overline{\partial}\alpha((u, 0), (0, \overline{v})) = -(0, \overline{V})f_u = -\overline{\partial} f_u(\overline{v}) = -2\pi K((u, 0), (0, \overline{v})).
\]

**Remark 4.4** A very intriguing question was asked by an anonymous referee: is there any connection between \( \hat{\eta} \) and the prime form? The latter, sometimes named after Schottky and Klein, is a classical object living on \( C \times C \), see [4, p. 16ff]. The easy answer is that this is impossible, because \( \hat{\eta} \) depends just on the Riemann surface, while the prime form depends also on the choice of a symplectic basis of \( H_1(C, \mathbb{Z}) \), see [4, (iv) p. 17]. Yet at a deeper level remains the problem of analyzing the relation between \( \hat{\eta} \) and the various bidifferentials and projective structures known in the classical literature, that usually depend on extra data, like theta characteristics.
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