Anomalous Hall Effect in Weyl Metals

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We present a theory of the anomalous Hall effect (AHE) in a doped Weyl semimetal, or Weyl metal, including both intrinsic and extrinsic (impurity scattering) contributions. We demonstrate that a Weyl metal is distinguished from an ordinary ferromagnetic metal by the absence of the extrinsic and the Fermi surface part of the intrinsic contributions to the AHE, as long as the Fermi energy is sufficiently close to the Weyl nodes. The AHE in a Weyl metal is thus shown to be a purely intrinsic, universal property, fully determined by the location of the Weyl nodes in the first Brillouin zone.

An exciting recent development in condensed matter physics is the emerging extension of the concepts of nontrivial electronic structure topology, which have long been confined exclusively to insulators, to gapless metallic states. These ideas, pioneered some time ago by Volovik [1], have recently been brought to the forefront of condensed matter research, with specific solid-state realizations of the first topologically-nontrivial metallic state, a Weyl semimetal, proposed [2–5]. The recent observation of the closely related Dirac semimetals [6–11] paves the way for the realization of Weyl semimetals in the near future.

The electronic structure of a Weyl semimetal contains points in momentum space, at which two nondegenerate bands touch at the Fermi energy. Such points, called Weyl nodes, can occur generically (but not necessarily at the Fermi energy) in three-dimensional (3D) band structures, as long as either time-reversal (TR) or inversion (I) symmetries are violated, which is needed to create non-degenerate bands, otherwise prohibited by the Kramers theorem. These points are topologically-nontrivial objects, characterized by an integer topological charge, and are monopole sources of the Berry curvature, momentum-space dual of the magnetic field in real space.

Apart from the appearance of the Weyl nodes themselves, which is generically possible in 3D, Weyl semimetal requires the Fermi energy to be aligned with the nodes. This situation is not generic, but is a special case of a Weyl metal: a metal is which the Fermi surface is broken up into disjoint pieces, each surrounding, in the simplest case, a single Weyl node. Such individual sheets of the Fermi surface may be characterized by a Chern number, which is equal to the topological charge, enclosed by the Fermi surface sheet. One may then ask the following question: what are, if any, observable consequences of such a topologically-nontrivial character of the Fermi surface of a Weyl metal?

The purpose of this paper is to describe one such phenomenon, which is characteristic of a specific subclass of Weyl metals, namely the ferromagnetic (FM) Weyl metals, in which the nodes owe their existence to broken TR. Any FM metal exhibits anomalous Hall effect (AHE), i.e. an antisymmetric contribution to the off-diagonal resistivity, “proportional” to the magnetization rather than to the applied magnetic field. As has been clearly demonstrated in recent work [12, 13], geometrical properties of the electronic structure play an important role in this effect.

Perhaps the most controversial part of the AHE story, that has emerged in recent years, has been the relative role played by the intrinsic properties of the electronic structure of the material (intrinsic AHE) and impurity scattering (extrinsic AHE). In general, both are present and are of the same order of magnitude, making it very difficult to disentangle intrinsic and extrinsic contributions experimentally. In this paper we show that, in contrast to a generic FM metal, in a FM Weyl metal the extrinsic contribution is essentially absent and the AHE is of purely intrinsic origin. Moreover, the intrinsic part of the AHE is fully determined only by the relative location and topological charge of the Weyl nodes, and is (almost) independent of the properties of the Fermi surface, as long as individual Fermi surface sheets have nonzero Chern numbers. This property becomes increasingly more precise as the Fermi energy approaches the Weyl nodes. As we show below, this is closely related to the topology of the Weyl nodes.

We start from a simple model of a ferromagnetic Weyl metal, motivated by the topological insulator multilayer model, introduced by us before [4]. The model has the advantage of being general enough to capture all the essential features of the electronic structure of a generic metallic FM, yet simple enough to be amenable to analytic calculations. The momentum-space Hamiltonian we start from is given by

\[ H_t(k) = v_F (\hat{z} \times \vec{\tau}) \cdot \vec{k} + m_t(k_z)\tau^z. \]  

Here the Pauli matrices, \( t = \pm \), \( m_{\pm} = b \pm \Delta(k_z) \), \( b \) is the mean-field spin splitting and \( \Delta(k_z) \) is the band dispersion along the magnetization direction in the first Brillouin zone (BZ), \( -\pi/d \leq k_z < \pi/d \), and we will use \( \hbar = c = 1 \) units throughout. The specific form of \( \Delta(k_z) \) is unimportant, but we will assume that it is a nonnegative function with a single minimum and a single maximum in the first BZ at \( k_z = 0 \) and \( k_z = \pi/d \). This guarantees the simplest situation with a single pair of Weyl nodes.
nally, this may be thought of as a Hamiltonian of a pair of 2D Dirac fermions, with the masses \( m_{\pm}(k_z) \), which depend on parameter \( k_z \). In the paramagnetic state, when \( b = 0 \), Eq. (1) describes two pairs of Kramers-degenerate bands with dispersions \( \pm \sqrt{\varepsilon_{-1}^2(k_x^2 + k_y^2) + \Delta^2(k_z^2)} \). When \( b \) becomes sufficiently large (taking \( b > 0 \) for concreteness), the mass \( m_{-}(k_z) \) may change sign at a minimum of two points in the BZ, given by the solutions of the equation \( \Delta(k_z) = b \). These points are the Weyl nodes. Eq. (1) may be regarded as a minimal model of the electronic structure of a 3D metallic FM.

We would like to evaluate the anomalous Hall conductivity of this model FM, in the presence of impurity potential \( V(\mathbf{r}) = V_0 \sum \delta(\mathbf{r} - \mathbf{r}_a) \), which we will assume for simplicity to be gaussian, with only second order correlators present: \( \langle V(\mathbf{r})V(\mathbf{r}') \rangle = \gamma^2 \delta(\mathbf{r} - \mathbf{r}') \), where \( \gamma^2 = n_i V_0^2 \) and \( n_i \) is the impurity density. We will, however, later comment on the influence of higher-order correlators, which are known to be important for AHE (skew-scattering) in principle, but do not in fact affect our results. We will also assume that the impurity potential is diagonal in both the pseudospin \( \sigma \) and the \( t = \mp \) index. Again, this assumption is used only for computational simplicity and does not affect the essence of our results. To find the anomalous Hall conductivity, we will use a somewhat nonstandard method, which we find to be the most convenient one for our purposes, as it allows us to do this calculation separately for each pair of bands, labeled by \( t \), and then simply sum the individual contributions. We will thus omit the \( t \) index in what follows, until we come to the final results. The retarded impurity averaged one-particle Green’s functions have the following general form

\[
G^R_{\sigma_1\sigma_2}(\mathbf{k},\epsilon) = \frac{\hat{s}_\sigma A_{\sigma_1} A_{\sigma_2}}{\epsilon - \xi^s_k + i\pi \rho_s/2},
\]

Here \( s = \pm \) labels the two bands, obtained by diagonalizing Eq. (1) for a specific \( t \) (the sum over \( s \) is made implicit above), \( \xi^s_k = s \epsilon_k - \epsilon_F = s \sqrt{\varepsilon_{-1}^2(k_x^2 + k_y^2) + m^2(k_z) - \epsilon_F} \) are the band energies, counted from the Fermi energy \( \epsilon_F \), and \( |\xi^s_k| = \frac{1}{\sqrt{2}} \left( \sqrt{1 + \frac{m(k_z)}{\epsilon_F}}, -i se^{i\phi} \sqrt{1 - \frac{m(k_z)}{\epsilon_F}} \right) \) is the corresponding eigenvector with \( e^{i\phi} = \frac{k_x + ik_y}{\sqrt{k_x^2 + k_y^2}} \).

In what follows we will assume, for concreteness, that \( \epsilon_F > 0 \), i.e. the Weyl metal is electron-doped. The impurity scattering rates \( 1/\tau_{\pm} \) are given, in the Born approximation, by

\[
\frac{1}{\tau_s(k_z)} = \frac{1}{\tau} \left[ 1 + s \frac{m(k_z)}{\epsilon_F} \right],
\]

where \( 1/\tau = \pi \gamma^2 g(\epsilon_F) \) and \( g(\epsilon_F) = \int \frac{d^3k}{(2\pi)^3} \delta(\epsilon_k - \epsilon_F) = \frac{\epsilon_F}{4\pi^2} \int_{-\pi/4}^{\pi/4} dk_z \Theta(\epsilon_F - |m(k_z)|) \) is the density of states at Fermi energy. We have also defined the average of \( m(k_z) \) over the Fermi surface as

\[
\langle m \rangle = \frac{1}{g(\epsilon_F)} \int \frac{d^3k}{(2\pi)^3} m(k_z) \delta(\epsilon_k - \epsilon_F).
\]

The impurity averaged response function, analytically continued to real frequency as \( \Pi(\mathbf{q},\Omega) = \Pi(\mathbf{q},i\Omega \to \Omega + i\eta) \), is given, in the self-consistent non-crossing approximation, by the sum of ladder diagrams, which gives

\[
\Pi(\mathbf{q},\Omega) = \Pi^{I}(\mathbf{q},\Omega) + \Pi^{II}(\mathbf{q},\Omega),
\]

where

\[
\Pi^{I}(\mathbf{q},\Omega) = 2e^2 v_F \int_{-\infty}^{\infty} \frac{de}{2\pi i} \frac{dn_F(e)}{de} P_{0x}(\mathbf{q},\epsilon - i\eta,\epsilon + \Omega + i\eta),
\]

and

\[
\Pi^{II}(\mathbf{q},\Omega) = 4ie^2 v_F \int_{-\infty}^{\infty} \frac{de}{2\pi i} n_F(e) \text{Im} P_{0x}(\mathbf{q},\epsilon + i\eta,\epsilon + \Omega + i\eta).
\]
The 4×4 matrix $P$, whose 0x component we are interested in, is given by $P(q, -i\eta, \Omega + i\eta) = \gamma^{-2} I_{\alpha A}(q, \Omega) D(q, \Omega)$, $P(q, \epsilon + i\eta, \epsilon + \Omega + i\eta) = I_{\alpha A}(q, \Omega)$, where $D = (1 - I_{\alpha A})^{-1}$ is the diffusion propagator and $I_{\alpha A}(q, \Omega) = \frac{i^2}{\pi} \int_{q_0}^{q_{\text{max}}} G_{\sigma_1, \sigma_2}^{RA}(k + q, \Omega) G_{\sigma_1, \sigma_2}^{A}(k, 0)$, $I_{\alpha A}(q, \Omega) = \frac{1}{2} \int_{q_0}^{q_{\text{max}}} \int_{q_0}^{q_{\text{max}}} G_{\sigma_1, \sigma_2}^{A}(k + q, \Omega)$, the physical meaning of the two distinct contributions to the response function $\Pi^{\text{II}}(q, \Omega)$ is clear from Eq. $[9]$. $\Pi^{\text{II}}(q, \Omega)$ describes the non-equilibrium part of the response that happens at the Fermi surface. This response is diffusive when $\Omega \tau \ll 1$ and ballistic in the opposite limit. We will discuss this in more detail below. In contrast, $\Pi^{\text{II}}(q, \Omega)$ is an equilibrium, non-dissipative contribution to the overall response, to which all states below the Fermi energy contribute [14]. These response functions give rise to two physically distinct contributions to the anomalous Hall conductivity, which behave very differently as a function of the Fermi energy, as we will show below.

We will start by evaluating the nonequilibrium part of the response function, $\Pi^{\text{II}}(q, \Omega)$. Computing the matrix elements $I_{\alpha A}(q, \Omega)$ is easily done in the standard way, assuming $\epsilon_F \tau \gg 1$. One obtains

$$\Pi^{\text{II}}(q, \Omega) = i e^2 v_F \Omega \tau g(\epsilon_F) [I_{00}^{RA} D_{0x} + I_{0x}^{RA} D_{xx} + I_{0z}^{RA} D_{zz}].$$

where we have taken into account that $D_{yz} = 0$ by symmetry. The relevant matrix elements of the diffusion propagator can be found analytically to first order in $q$. One obtains

$$\Pi^{\text{II}}(q, \Omega) = i e^2 v_F \Omega \tau g(\epsilon_F) \frac{I_{0z}^{RA} (1 - I_{zz}^{RA}) + I_{0z}^{RA} I_{zz}^{RA}}{\Gamma(1 - I_{zz}^{RA})},$$

where $\Gamma(q, \Omega) = (1 - I_{00}^{RA}) (1 - I_{zz}^{RA}) - I_{00}^{RA} I_{zz}^{RA}$ is the determinant of the 0z block of the diffusion propagator, which corresponds to the diffusion of the charge density, a conserved quantity (this block decouples from the rest of the diffusion when $q \to 0$). This means, in particular, that $\Gamma$ must satisfy an exact Ward identity $\Gamma(0, 0) = 0$.

Explicitly, the relevant matrix elements of $I_{\alpha A}(q, \Omega)$ to first order in $q$ are given by

$$I_{00}^{RA} = \left\langle \frac{\tau_+}{1 - i\Omega \tau_+} \right\rangle, \quad I_{0x}^{RA} = \frac{i v_F q}{2 \epsilon_F} \left\langle \frac{\tau_+}{1 - i\Omega \tau_+} \right\rangle,$$

$$I_{0z}^{RA} = \frac{m^2}{\epsilon_F} \left\langle \frac{\tau_+}{1 - i\Omega \tau_+} \right\rangle,$$

$$I_{xx}^{RA} = \frac{m^2}{\epsilon_F} \left\langle \frac{\tau_+}{1 - i\Omega \tau_+} \right\rangle,$$

$$I_{xx}^{RA} = \frac{m^2}{\epsilon_F} \left( 1 - \frac{m^2}{\epsilon_F} \right) \left\langle \frac{\tau_+}{1 - i\Omega \tau_+} \right\rangle,$$

and

$$\Pi^{\text{II}, \text{diff}}(q, 0) = -\frac{i q e^2 v_F^2 g(\epsilon_F)}{2 \epsilon_F} \langle m \tau_+ \rangle F[m].$$

FIG. 1. (Color online). (a) Plot of the band edges along the z-direction in momentum space for the two bands that touch at the Weyl nodes, using specific expression for $\Delta(k_z)$ from the multilayer model of Ref. [4]. (b) Field lines of the Berry curvature in the $k_y = 0$ plane, for the same band structure as in (a). Corresponding Fermi surface section is shown by the two contours, enclosing the Weyl nodes.

where the average over the Fermi surface is defined in the same way as in Eq. $[6]$. The charge conservation Ward identity then takes the following explicit form

$$\Gamma(0, 0) = \left( 1 - \frac{\tau_+}{\tau} \right) \left( 1 - \frac{m^2 \tau_+}{\epsilon_F^2 \tau} \right) - \frac{m \tau_+}{\epsilon_F \tau} = 0.$$

The correctness of Eq. $[13]$ may be easily checked using Eq. $[5]$ and expanding in Taylor series in $m/\epsilon_F$. Expanding $\Gamma(0, \Omega)$ to first order in $\Omega$ and taking the limit $\Omega \to 0$ at fixed $\tau$, which corresponds to the diffusive limit, we finally obtain

$$\Pi^{\text{II}, \text{diff}}(q, 0) = -\frac{i q e^2 v_F^2 g(\epsilon_F)}{2 \epsilon_F} \langle m \tau_+ \rangle F[m].$$
The plateau-like feature in $\sigma_{xy}$ correlates with the range of the Fermi energies, for which the Fermi surface consists of two separate sheets, each enclosing a single Weyl node.

where

$$\Pi[m] = \frac{1 + \frac{1}{2} \left( \frac{1 - \frac{m^2}{\epsilon_F^2}}{1 - \frac{m^2}{\epsilon_F^2}} \right) \tau^z}{1 - \frac{m^2}{\epsilon_F^2}} \left[ \frac{\partial \Gamma(0, \Omega)}{\partial (\Omega \tau)} \right]_{\Omega=0}^{-1}. \quad (15)$$

The explicit form of the functional $F[m]$ is in fact not that important for our purposes, except for the evenness property, easily seen from Eq. (15). $F[m] = F[-m]$. As a consequence, $\Pi_{diff}$ is an odd functional of $m$, which will play an important role below. It is important to note that the charge conservtion, whose mathematical consequence is the presence of the diffusion pole in $\Pi^f(q, \Omega)$, is crucial in obtaining a nonzero result in the diffusive limit in Eq. (14). The analogous quantity in the calculation of the spin Hall conductivity, for example, would vanish in the diffusive limit [17].

It is also of interest to examine the ballistic limit of $\Pi^f$, which corresponds to the case of a clean Weyl metal. In this case we send both $\Omega$ and $1/\tau$ to zero in such a way that $\Omega \tau \to \infty$. In this case we obtain

$$\Pi_{bal}(q, 0) = -\frac{i q}{2 \epsilon_F} \frac{\epsilon_F^2}{\epsilon_F^2} \left\langle \frac{m}{\epsilon_F} \right\rangle, \quad (16)$$

which agrees with the clean Weyl metal result, obtained by us before [15] [16]. As seen from Eqs. (14) and (16), the difference between $\Pi_{diff}$ and $\Pi_{bal}$ is only quantitative. In the extensive AHE literature, this difference is said to arise from the so-called side-jump processes [18] [22].

The final step of the calculation is to evaluate the equilibrium part of the response function, $\Pi^{II}(q, \Omega)$. In the limit $\epsilon_F \tau \gg 1$ one finds that this part of the response function is unaffected by the impurity scattering and is given by

$$\Pi^{II}(q, \Omega) = \frac{e^2}{2 \epsilon_F} \frac{q |e_v^2|}{2 \epsilon_F} \left\langle \frac{m}{\epsilon_F} \right\rangle, \quad (17)$$

where summation over the band indices $s, s'$ is again implicit. Eq. (17) contains both interband terms, with $s \neq s'$, and an intraband term, with $s = s' = +$. Evaluating them explicitly one finds that the contribution of the intraband term is exactly cancelled by part of the inter band contribution [15]. The remaining interband contribution gives

$$\Pi^{II}(q, 0) = \frac{-i q e^2}{8 \pi^2} \int_{-\pi/d}^{\pi/d} dk_z \text{sign}[m(k_z)] \times \{1 - \Theta(\epsilon_F - |m(k_z)|)\}. \quad (18)$$

The first term in Eq. (18) arises from the completely filled bands, while the second is the contribution of the incompletely filled bands.

We can now finally evaluate the anomalous Hall conductivity. We will focus on the diffusive limit results, as ballistic limit is qualitatively similar. At this point we will also explicitly include the contribution of both $t = \pm$ pairs of bands, which simply amounts to restoring the index $t$ in $m_t$, and summing over $t$. Using Eq. (3) and remembering that $A_G \to iA_0$ upon Wick rotation to the real time, we obtain

$$\sigma_{xy} = \frac{e^2}{2 \epsilon_F} \sum_t q_t(\epsilon_F) \left\langle \frac{m_t \tau_{tt}}{\epsilon_F \tau_t} \right\rangle F[m_t], \quad (19)$$

and

$$\sigma_{xy}^{II} = \frac{e^2}{8 \pi^2} \int_{-\pi/d}^{\pi/d} dk_z \text{sign}[m_t(k_z)] \times \{1 - \Theta(\epsilon_F - |m_t(k_z)|)\}. \quad (20)$$

Since $m_+(k_z)$ is positive throughout the first BZ, while $m_-(k_z)$ changes sign at the Weyl nodes, the first term in Eq. (20), which comes from completely filled bands, gives a universal (almost) quantized contribution

$$\sigma_{xy}^{\text{quant}} = \frac{e^2 \mathcal{K}}{4 \pi^2}, \quad (21)$$

where $\mathcal{K}$ is the distance between the Weyl nodes. This equation also describes the cases when the Weyl nodes are absent, in which case $\sigma_{xy}^{\text{quant}}$ is truly quantized since $\mathcal{K} = 0$, $G$, where $G = 2\pi/d$ is a reciprocal lattice vector.

We are now ready to state our main result. This comes from examining the remaining, non-quantized parts of $\sigma_{xy}$. Suppose we have a situation when the Weyl nodes are present and $\epsilon_F$, while not zero, is not too far from it (what this means precisely will become clear below), as shown in Fig. 1. Recall that at the location of the Weyl node $m_-(k_z) = b - \Delta(k_z)$ changes sign. This implies that, as long as $\mathcal{K} \frac{d\epsilon_F}{dk_z} |_{k_z = k_0} \ll \epsilon_F$, where $k_0$ is the location of a given Weyl node, the average of any odd function of $m_-(k_z)$ over the Fermi surface will vanish. This means that in such a situation, which we call...
Weyl metal, all contributions to the anomalous Hall conductivity, associated with incompletely filled bands, will vanish and $\sigma_{xy}$ attains a universal value, characteristic of Weyl semimetal $\sigma_{xy} = \sigma_{xy}^{\text{quant}}$, where $\sigma_{xy}^{\text{quant}}$ is given by Eq. (21). Note that the linear dispersion sufficiently close to Weyl nodes \(^{23}\) is a topological property, in the sense that it follows directly and exclusively from the existence of a nonzero topological charge by the so-called Atiyah-Bott-Shapiro construction \(^{24}\).

To understand this result physically, it is helpful to recall that the Weyl nodes are monopole sources of the Berry curvature $\Omega_k$. In a clean metal, the anomalous Hall conductivity $\sigma_{xy}$ is given by the integral of the $z$-component of the Berry curvature over all occupied states $\sigma_{xy} = e^2 \int d^2k n_F(\epsilon_k) \Omega_{kz}$. However, as clear from Fig. 1 when the Fermi surface breaks up into disconnected sheets, enclosing individual nodes, the contribution of the states, enclosed by the Fermi surface, to this integral will always be very small, vanishing exactly in the limit when the band dispersion away from the nodes may be taken to be exactly linear. An obvious analogy here is with the electric field of a dipole. A pair of Weyl nodes is like a dipole of two topological charges. Its field has a well-defined and nonzero on average $z$-component at large distances from the dipole. At short distances, however, the field is that of individual charges, which winds around the location of each charge and thus any particular component of it averages to zero.

We have so far confined ourselves to the model of Gaussian-distributed disorder, which only contains second-order correlators of the impurity potential. This model misses a sometimes significant contribution to the AHE, known as skew-scattering \(^{12}\). This arises from third-order scattering processes, which give a contribution to $\sigma_{xy}^{I}$ of the following form \(^{20}\)

$$
\sigma_{xy}^{I,\text{skew}} = \frac{e^2 v_F^2}{2\epsilon_F} \sum_i g_i(\epsilon_F) \left( \frac{m_{t+1}}{n_i V_0 \tau_i} \right) \tilde{F}[m_i],
$$

where the functional $\tilde{F}[m]$ is again even under the change of sign of $m$, which follows simply from symmetry considerations. It is then clear that $\sigma_{xy}^{I,\text{skew}}$ vanishes in the Weyl metal by exactly the same arguments as above.

In conclusion, we have demonstrated that the AHE in Weyl metals has a purely intrinsic origin and can be associated entirely with the Weyl nodes, just as in the case of a Weyl semimetal, when the Fermi energy coincides with the nodes and the Fermi surface is absent. This is in contrast to an ordinary FM metal, in which the anomalous Hall conductivity always has both a significant Fermi surface contribution and an extrinsic contribution.

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