GENERALIZATIONS OF THE DIRECT SUMMAND THEOREM OVER UFD-S FOR SOME BIGENERATED EXTENSIONS AND AN ASYMPTOTIC VERSION OF KOH’S CONJECTURE

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Abstract. This article deals with two different problems in commutative algebra. In the first part we give a proof of generalized forms of the Direct Summand Theorem (DST (or DCS)) for module-finite extension rings of mixed characteristic $R \subset S$ satisfying the following hypotheses: The base ring $R$ is a Unique Factorization Domain of mixed characteristic zero. We assume that $S$ is generated by two elements which satisfy, either radical quadratic equations, or general quadratic equations under certain arithmetical restrictions.

In the second part of this article, we discuss an asymptotic version of Koh’s Conjecture. We give a model theoretical proof using “non-standard methods”.

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1 Introduction

The Homological Conjectures have been a focus of research activity since Jean Pierre Serre introduced the theory of multiplicities in the early 1960s (22), and since the introduction of characteristic prime methods in commutative algebra by Peskine, Szpiro, and M. Hochster, in the mid 1970s [9], [18]. These conjectures relate the homological properties of a commutative rings to certain invariants of the ring structure, as, for instance, its Krull dimension and its depth. They have been settled for equicharacteristic rings (i.e., rings for which the characteristic of the ring coincides with that of its residue field) but many remain open in mixed characteristic [11]. Their validity in mixed

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characteristic, nonetheless, is known for rings of Krull dimension less than four.

Among these conjectures the **Direct Summand Conjecture** occupies a central place, implying, or actually being equivalent, to many of the other conjectures [10], [17], [19], [16].

**Direct Summand Conjecture (DSC):** Let \( R \subset S \) be a module-finite extension of noetherian rings (i.e., \( S \), regarded as an \( R \)-module is finitely generated) where \( R \) is assumed to be regular. Then, the inclusion map \( R \subset S \) splits as a map of \( R \)-modules; or equivalently, there is a retraction \( \rho : S \to R \) from \( S \) into \( R \). By a retraction we mean an \( R \)-linear homomorphism satisfying \( \rho(1) = 1 \). The central role of the DSC in Commutative Algebra as well as its relation to the other homological conjectures is comprehensively explained in [9].

Now, the Direct Summand Conjecture (or D. S. Theorem) was proved in the general setting by Yves Andrè, essentially by proving the (remaining) case of unramified complete regular local rings with the framework of perfectoids developed by Scholze [1], [2].

The problem of showing the existence of a retraction \( \rho : S \to R \) may be reduce to the case where \( R \) and \( S \) are complete local domains [9]. Therefore, one may assume that \( (R, \mathfrak{m}) \) is in particular a unique factorization domain (UFD) [4], page 483. If the Krull dimension of \( S \) is \( d \) one can always choose a system of parameters for \( S \) contained in \( \mathfrak{m} \). By a **system of parameters** in an arbitrary local ring \( (S, \mathfrak{n}) \) of Krull dimension \( d \) we mean a sequence of elements \( \{x_1, ..., x_d\} \) such that the radical of the ideal they generate in \( S \) is precisely \( \mathfrak{n} \), the unique maximal ideal of \( S \) ([4], page 222). It can be proved then that the DSC holds if and only if for any system of parameters in \( S \), and any natural number \( t > 0 \), the **socle element** \( (x_1 \cdots x_d)^t \) is not contained in the ideal in \( S \) generated by the \( t + 1 \) powers of the parameters. That is, if \( (x_1 \cdots x_d)^t \not\in (x_1^{t+1}, ..., x_d^{t+1})S \). This last statement is known as the **Monomial Conjecture**. More precisely:

**Monomial Conjecture (MC):** Let \( (S, \mathfrak{n}) \) be any local noetherian ring of dimension \( d \) and let \( \{x_1, ..., x_d\} \) be a system of parameters for \( S \). Then, for any positive integer \( t \), \( (x_1 \cdots x_d)^t \not\in (x_1^{t+1}, ..., x_d^{t+1})S \). This conjecture is equivalent to the (DSC) [8]. In low dimension, that is, for rings of Krull dimension \( \leq 2 \), the DSC, and consequently the MC, follows as a consequence of the existence of a *normalization* for \( S \). By mapping \( S \) into its normalization one may assume that \( S \) is a *normal* domain, and, for dimensional reasons, a Cohen-Macaulay ring ([4], pages 118, 420). Then the Auslander-Buchsbaum formula ([4], page 469) implies that \( S \) must have projective dimension zero. That is, \( S \) must be an \( R \)-free module, and consequently the inclusion map automatically splits.
On the other hand, the general equicharacteristic case, i.e., the case if which $R$ contains a field of zero characteristic, is handled by elementary methods: The trace map form the fraction field of $S$ to the fraction field of $R$ provides a natural retraction. In fact, the inclusion map splits as a map of $R$-modules under the much weaker hypothesis of $R$ being a normal domain [8, Lemma 2].

If $R$ is equicharacteristic, but contains a field of characteristic $p > 0$, a classical argument given by Hochster (actually, a precursor of his Tight Closure Theory) shows the validity of the MC by a method in which the properties of the iterated powers of the Frobenius map are exploited in a clever way.

It should be remarked that the existence of Big Cohen Macaulay Modules and Big Cohen Macaulay Algebras for equicharacteristic rings immediately implies the validity of the (MC) and the (DSC) [11], [12].

In the mixed characteristic case it is known that the DSC holds for regular rings $R$ of Krull dimension $\leq 3$. The dimension three case was proved quite recently by R. Heitmann, by means of a rather involved combinatorial argument [17], [19]. This is, undoubtedly, one of the most significant advances in commutative algebra of the last decades. As mentioned before, the DSC would follow from the existence of Big Cohen Macaulay Modules or Big Cohen Macaulay Algebras in mixed characteristic, an open problem in dimensions greater than three [11].

A natural generalization of the (DSC) was proposed by J. Koh in his doctoral dissertation [13]. Koh’s question replaces the condition of $R$ being regular for the weaker condition of $S$ having finite projective dimension as a module over $R$.

**Koh’s Conjecture:** Let $R$ be a noetherian ring, and let $R \subset S$ be a module-finite extension such that $S$, regarded as an $R$-module, has finite projective dimension. Then there is a retraction from $S$ into $R$.

It is known that many theorems fail when one weakens the hypothesis of $R$ being regular and replaces it by the condition that $R$ is just noetherian, even a Cohen Macaulay or a Gorenstein complete local domain (see Definition 4.6). Then one imposes the condition that the corresponding $R$-modules have finite projective dimension (if $R$ is regular this hypothesis is satisfied automatically, due to Serre’s Theorem, [4], Chapter 19). For instance, the Rigidity of Tor is no longer true in this context [6]. In a similar manner, the positivity of the intersection multiplicity $\chi_R(M, N)$ for modules $M, N$ of finite projective dimension over $R$, when $R$ is not regular, is no longer valid [3]. Notwithstanding, Koh’s conjecture is true for rings of equal characteristic zero. Unfortunately, it turned out to be false for equicharacteristic rings of prime characteristic as well as in the case of mixed characteristic [23].

The fact that this conjecture is true in characteristic zero suggests that it may be true “asymptotically”. By an asymptotic version we mean the following: Given any bound $b > 0$ for the “complexity” of the extension, a notion we
will define in a precise manner in (4.1), the set $S_b$ of prime numbers for which there are counterexamples whose characteristic lie in $S_b$ must be finite (Theorem 4.14). We will prove this asymptotic form for rings that are localization at prime ideals of affine $k$-algebras, where $k$ is an algebraically closed field. We achieve this by first formulating Koh’s Conjecture as a first order sentence in the language of rings and algebraically closed fields. Then we give a proof via Lefschetz’s Principle. A main reference for the model theoretical methods involved is [21]. Also [20] may be consulted for a more succinct account of “nonstandard methods in commutative algebra.

In the first part of this article we provide a proof of the DSC for module-finite extensions of rings $R \subset S$ satisfying certain conditions. We will assume $R$ to be a UFD., where the most interesting case will be when $R$ is a ring of mixed characteristic zero. On the other hand, $S$ will be a module-finite extension generated as a $R$-algebra by two elements satisfying, either radical quadratic equations (Theorem 1.4), or satisfying general quadratic equations, under certain arithmetical restrictions (Theorem 2.1).

In the second part of this article we discuss an asymptotic version of Koh’s Conjecture. We will develop a Model Theoretical approach using “nonstandard methods” similar to those developed by H. Schoutens [20]. The main result of this section is Theorem 4.14.

All rings will be commutative, with identity element 1, and all modules will be assumed to be unitary.

1. The DSC for some radical quadratic extensions

1.1. Some reductions. Let $R$ be a UFD, and let us denote by $L$ its fraction field. Let $S$ be a module-finite extension of $R$ such that $S$ is generated as an $R$-algebra by two elements $s_1$ and $s_2$ that satisfy monic polynomials $f_1(x)$ and $f_2(x)$ in $R[x]$, respectively.

Let us first see that, without loss of generality, we may assume that $f_1(x)$ and $f_2(x)$ have degree greater than one. For if one of them, for instance $f_1(x)$, had degree one then the element $s_1$ would be already in $R$. In this case $S = R[s_2]$. By mapping $C = R[x]/(f_2(x))$ onto $S$ we can represent $S$ as a quotient of the form $C/J$, where $J$ is an ideal of $C$ of height zero. This is because the Krull dimension of $S$ and $T$ is the same and equal to the Krull dimension of $R$, since both rings are module-finite extensions of $R$ ([14], Corollary 2.13, page 47).

Thus, $J$ would be contained in some minimal prime $P$ of $C$. Since $R[x]$ is also a UFD, $P$ is generated by a monic prime factor $p(x)$ of $f_2(x)$, hence $J \subset (p(x))$. But in order to find a $R$-retraction from $S$ into $R$ it suffices to find any retraction “further above”, $\rho: C/P \to R$. This is because the composition of the canonical map $S = C/J \to C/P$ with $\rho$ then provides a
retraction from $S$ into $R$. But the map $R \to C/P \simeq R[x]/(p(x))$ splits, since $R[x]/(p(x))$ is free as an $R$-module and consequently $\rho$ can be taken as the projection onto $R$.

Let us then assume that the degrees of $f_1$ and $f_2$ are greater that one and henceforth $S$ is minimally generated as an $R$-algebra by $s_1, s_2 \in S$. Set $T = R[x_1, x_2]/I$, with $I = (f_1(x_1), f_2(x_2))$, where $f_1(x_1)$ and $f_2(x_2)$ are monic polynomials for $s_1$ and $s_2$, respectively. It is easy to see that $T$ is a free $R$-module, because $T \cong R[x_1]/(f_1) \otimes_R R[x_2]/(f_2)$. In fact, an $R$-basis for $T$ consists of monomials of the form

$$B = \{t_1^{d_1} t_2^{d_2}, \text{ with } 0 \leq d_i < \deg f_i\}.$$  

Let $\varphi : T \to S$ be the surjective $R$-homomorphism defined by ending $x_i$ into $s_i$. Let $J$ denote its kernel, so that $S \cong T/J$, where the height of $J$ must be zero, and therefore $J$ must be contained in some minimal prime of $T$.

The next lemma analyzes the case when $T$ turns out to be a domain, i.e., when $J = (0)$. In this case the existence of a retraction $\rho : S \to R$ follows automatically, since $S = T$ is free as an $R$-module.

In what follows $i$ and $j$ denote a pair of indices $1 \leq i, j \leq 2$, with $i \neq j$.

**Lemma 1.1.** Let $R, T, f_1, f_2$ be as above. Let $E_i = L[x_i]/(f_i)$, and $F_j = E_i[x_j]/(f_j)$. Then $T$ is a domain if and only if both $E_i$ and $F_j$ are fields. That is, if and only if $f_i$ is irreducible in $L[x_i]$ and $f_j$ is irreducible in $E_i[x_j]$.

**Proof.** First, we observe that $L \otimes_R T \cong L[x_1, x_2]/I \cong E_i[x_j]/(f_j) = F_j$ and that the natural homomorphism $\mu : T \to L \otimes_R T$ is an injection. This is because $T$ is a torsion free $R$-module, since $R$ is a domain and $T$ is a free module. Therefore, $T$ is a subring of $F_j$, and if $F_j$ is a field, $T$ must be a domain. This gives the “only if” part of the lemma.

Conversely, let us assume that $T$ is a domain. Arguing by contradiction let us suppose that either $E_i$ or $F_j$ is not a field. In the first case there are monic polynomials of positive degree, $g_1$ and $g_2$ in $L[T_i]$, such that $f_i = g_1 g_2$ with $\deg g_1 < \deg f_i$. Now, let $\alpha \in R \setminus \{0\}$ be a common denominator for the coefficients of $g_1$ and $g_2$. The equality $\alpha^2 f_i = (\alpha g_1)(\alpha g_2)$ in $R[x_i]$ implies that $\alpha g_i$ are zerodivisors in $T$. Besides, $\alpha g_i = \alpha x_i^{\deg g_i} + \cdots$ cannot be zero because $g_i$, written in the $R$-basis of $T$ has at least one coefficient different from zero. Therefore, $T$ would not be a domain, a contradiction. Then we may assume that $E_i$ is a field.

On the other hand, if $f_j$ were reducible over $E_i[x_j]$ we could write $f_j = h_1 h_2$, where $h_1, h_2 \in E_i[x_j]$ are monic polynomials of degree less than $\deg f_j$.

Let us choose $h_1, h_2$ in $L[x_1, x_2]$ such that $\psi(h_s) = h_s$, $s = 1, 2$, where $\psi : L[x_1, x_2] \to E_i[x_j]$ is the natural homomorphism induced by the projection map $L[x_i] \to E_i$. In fact, we can choose each $h_s$, considered as a polynomial in $(L[x_i])[x_j]$, such that each of its coefficients in $L[x_i]$ is a polynomial in $x_i$ with degree less than $\deg f_i$. Hence, there exists $h_3$ in $L[x_1, x_2]$ such

\[ B = \{t_1^{d_1} t_2^{d_2}, \text{ with } 0 \leq d_i < \deg f_i\}. \]
that \( f_j - \tilde{h}_1 \tilde{h}_2 = \tilde{h}_3 f_i \). Choose any nonzero element \( c \in R \) such that \( \tilde{h}_r \in R[x_1, x_2] \), for \( r = 1, 2, 3 \). Then we have that \( \tilde{c}h_1 \tilde{h}_2 = c^2 f_j - c(\tilde{c}h_3) f_i \in I \) and consequently the classes of \( \tilde{c}h_1 \) and \( \tilde{c}h_2 \) in \( T \) must be different from zero. Thus, \( T \) would not be a domain, which is a contradiction. This proves \( f_j \) must be irreducible over \( E_i[x_j] \).

\[ \square \]

**Corollary 1.2.** Let \( R \) be a UFD where its field of fractions \( L \) has characteristic different from two. Assume that \( f_i = x_i^2 - a_i \) are irreducible polynomials in \( L[x_i] \), \( i = 1, 2 \). If \( T = R[x_1, x_2]/(f_1, f_2) \) is not a domain then there exist nonzero elements \( c, d, u \) in \( R \) such that \( a_1 = d^2 u, a_2 = c^2 u \), where \( c, d \) are relatively prime.

**Proof.** Since \( T \) is not a domain, by Lemma 1.1 we may assume without loss of generality that one of the polynomials \( f_i \), for instance \( f_2(x_2) \), is reducible in \( E_1[x_2] \). But this is equivalent to saying that \( f_2(x_2) \) has a root \( e \in E_1 \), that we may write as \( e = e_1 + e_2 x_1 \), where \( e_1, e_2 \in L \), and \( e_1^2 - e_1 = a_1 \). Hence

\[
a_2 = e^2 = (e_1^2 + e_2 x_1) + 2 e_1 e_2 x_1.
\]

Then \( a_2 = e_1^2 + e_2 x_1 + 2 e_1 e_2 \) and \( 2 e_1 e_2 = 0 \). But \( \text{char}(L) \neq 2 \) implies \( e_1 e_2 = 0 \). If \( e_2 = 0 \) then \( a_2 = e_1^2 \) and therefore \( f_2 = (x_2 + e_1)(x_2 - e_1) \), which is a contradiction. Thus, \( e_1 = 0 \) and \( a_2 = e_2 x_1 \).

Now write \( e_2 = e/d \), where \( c, d \neq 0 \) are relatively prime elements in \( R \). So \( d^2 a_2 = e^2 a_1 \); but \( d^2 \) does not divide \( e^2 \) and consequently \( d^2 \) divides \( a_1 \). Hence, there is \( u \in R \) such that \( a_1 = d^2 u \). Replacing \( a_1 \) in \( d^2 a_2 = e^2 a_1 \) gives the equation \( d^2 a_2 = c^2 d^2 u \). After dividing by \( d^2 \neq 0 \) we obtain \( a_2 = c^2 u \), which proves the corollary.

\[ \square \]

**Lemma 1.3.** Let \( R \) be a UFD, let \( B = R[x, y] \), and let \( u, c, d \) be elements in \( R \) different from zero. Define \( f_1 = x^2 - d^2 u, f_2 = y^2 - c^2 u \), and set \( I = (f_1, f_2) \), the ideal in \( B \) generated by \( f_1 \) and \( f_2 \). Assume that \( \{c, d\} \) is a regular sequence in \( R \) (page 173). Then the minimal prime ideals of \( I \) are \( P_r = (f_1, f_2, f_3, r, f_4, r) \), where \( f_3, r = dy + (-1)^r cx, f_4, r = xy + (-1)^r cdu, r = 0, 1 \).

**Proof.** Let us introduce a new variable \( z \), and let \( g(z) = z^2 - u \) in \( R[z] \). Clearly \( g \) has no roots in \( R \), since \( f_1 \) is irreducible. Therefore, \( g \) is also irreducible and consequently the ideal \( (g) \) is prime in \( R[z] \); this is because \( R \) is a UFD. Define \( \psi_r : B \to R[z]/(g) \) as the unique \( R \)-homomorphism that sends \( z \) into \( d\xi \) and \( y \) into \( (-1)^{r+1} c\xi \), where \( \xi \) denotes the class of \( z \) in \( R[z]/(g) \). We prove that \( \ker(\psi_r) = P_r \).

First, we observe that \( P_r \subset \ker(\psi_r) \), since:

\[
\psi_r(f_1) = d^2\xi^2 - d^2 u = d^2 u - d^2 u = 0,
\]

\[
\psi_r(f_2) = c^2\xi^2 - c^2 u = 0,
\]

\[
\psi_r(f_3, r) = \psi_r(dy + (-1)^r cx) = d(-1)^{r+1} c\xi + (-1)^r cdu = 0,
\]

\[
\psi_r(f_4, r) = \psi_r(xy + (-1)^r cdu) = d(-1)^{r+1} c\xi + (-1)^r cdu = 0.
\]
Let \( h(x, y) \) be a polynomial in \( \ker(\psi_r) \). If we regard \( h(x, y) \) as a polynomial in the variable \( y \) with coefficients in \( R[x] \) then, dividing by \( f_1 = x^2 - d^2u \) we can write \( h(x, y) \) as

\begin{equation}
(1.2) \quad h(x, y) = f_1 q(x, y) + q_1(y)x + q_0(y),
\end{equation}

where \( q(x, y) \in B \) and \( q_0(y), q_1(y) \in R[y] \).

Similarly, after dividing \( q_0(y) \) by \( f_2 = y^2 - c^2u \) we may find \( q_2(y) \in R[y] \), and \( a_1, a_2 \) elements in \( R \) such that

\begin{equation}
(1.3) \quad q_0(y) = f_2 q_2(y) + (a_1 y + a_2).
\end{equation}

Similarly, there exists polynomials \( q_3(y) \), \( q_4(y) \) in \( R[y] \), and \( b_1, b_2, \in R \) such that

\begin{equation}
(1.4) \quad q_1(y) x = f_4 r q_3(y) + q_4(y) + b_1 x + b_2.
\end{equation}

Dividing \( q_4(y) \) by \( f_2 \) we obtain:

\begin{equation}
(1.5) \quad q_4(y) = q_5(y) f_2 + e_1 y + e_2,
\end{equation}

for certain polynomial \( q_5(y) \), and certain elements \( e_1, e_2 \) in \( R \).

Replacing equations (1.3), (1.4) and (1.5) in (1.2) we can write \( h(x, y) \) as

\begin{equation}
(1.6) \quad h(x, y) = f_1 q(x, y) + f_4 r q_3(y) + (q_5(y) + q_2(y)) f_2 + l(x, y),
\end{equation}

where \( l(x, y) \) is the linear polynomial

\begin{equation}
(1.7) \quad l(x, y) = v_1 y + b_1 x + v_2,
\end{equation}

where \( v_1 = e_1 + a_1 \) and \( v_2 = e_2 + a_2 + b_2 \). From the condition \( \psi_r(h) = 0 \) we get:

\begin{equation}
\psi_r(h) = \psi_r(l) = v_1 (\psi_r)^{r+1} c + b_1 d \psi + v_2 = 0.
\end{equation}

Since \( R[z]/(g) \) is a \( R \)-free module with basis \( \{ 1, \psi \} \) we then must have:

\begin{equation}
(1.8) \quad (\psi_r)^{r+1} v_1 c + b_1 d = 0 \text{ and } v_2 = 0.
\end{equation}

But \( \{ c, d \} \) is a regular sequence in \( R \), hence there must be some \( u \in R \) such that \( b_1 = uc \). Since \( R \) is a domain we obtain from (1.8) that \( (\psi_r)^{r} d = v_1 \), and consequently

\begin{equation}
(1.9) \quad l(x, y) = v_1 y + b_1 x = u((\psi_r)^{r} dy + cx) = u(\psi_r)^{r} f_3.r.
\end{equation}

This shows \( h \in P_r \).

Clearly \( P_0 \) and \( P_1 \) are prime ideal of \( B \) because \( B/P_r \) is an integral domain isomorphic to \( R[z]/(g) \).
Finally, let us show that $P_0$ and $P_1$ are the the only minimal primes of $I$. Let $Q$ be any other minimal prime over $I$. Then

$$c^2 f_1 - d^2 f_2 = -(dy - cx)(cx + dy) \in Q,$$

so $f_{3,1} = dy - cx \in Q$, or $f_{3,0} = cx + dy \in Q$. □

**Proof.** Let us suppose $f_{3,1} \in Q$. Hence

$$-x(cx - dy) = dxy - cx^2 = dxy - cd^2 u = d(xy - cdu) \in Q.$$

Similarly,

$$y(cx - dy) = cxy - dy^2 = cxy - dc^2 u = c(xy - cdu) \in Q.$$

We claim that $f_{4,1} = xy - cdu$ is an element of $Q$. If we suppose otherwise, the elements $c, d$ would be contained in $Q$. Then we would have that $x^2 = (f_1 + d^2 u)$ and $y^2 = f_2 + c^2 u$ would also be contained in $Q$. Consequently, $x, y$ would also be elements of $Q$. Henceforth, the ideal $(c, d, x, y)B$ would be contained in $Q$. But each generator of $P_1$ is in $(c, d, x, y)B$ and consequently $P_1 \subset Q$. But this must be a proper inclusion, since $x \notin P_1$ (this is because every monomial containing $x$ in each one of the generator of $P_1$ is either quadratic, $x^2$ or $xy$, or linear of the form $cx$, but $x \notin (x^2, xy, cx)B$, since $1 \notin (x, y, c)B$). This contradicts the minimality of $Q$, and proves the claim.

Thus $Q$ must contain $f_1, f_2, f_{3,1}, f_{4,1}$ and consequently $P_1 = Q$.

The second case, when $f_{3,0} = cx + dy \in Q$, can be treated in a similar fashion. When this occurs we obtain that $P_0 = Q$. □

**Theorem 1.4.** Let $R \subset S$ be a module-finite extension of noetherian rings such that $R$ is a UFD and such that the characteristic of the fraction field $L$ of $R$ is different from two. Suppose $S$ is generated as an $R$-algebra by two elements $s_1, s_2 \in S$ satisfying monic radical quadratic polynomials $f_1 = x_1^2 - a_1$ and $f_2 = x_2^2 - a_2$, respectively. Then $R \subset S$ splits.

**Proof.** By the Going Up ([4], page 129) there exists a prime ideal $Q \subset S$ that contracts to zero in $R$. Therefore, we may replace $S$ by $S/Q$ without altering the relevant hypothesis. As observed in Section 1.1 it suffices to find a retraction from $S/Q$ into $R$. Hence, we may reduce to the case where $S$ is a domain.

As observed in that same section we can write $S$ as a quotient $T/J$, where $T = R[x_1, x_2]/I$, $I = (f_1, f_2)$, and $J \subset T$ is an ideal of height zero. Moreover, we can assume each $f_i$ to be irreducible in $\in R[x_i]$, otherwise, as noticed in that section, the splitting of $R \subset S$ would immediately follow from the fact that $S$ would be a free $R$-module.

On the other hand, if $T$ is a domain then $J = 0$, and so $S = T$ would be a free $R$-free (of rank four) and $R \subset S$ would also split. If on the contrary, $T$ is not a domain, then by Corollary 1.2 there exist $c, d, u$ nonzero elements of $R$ such that $a_1 = d^2 u$, $a_2 = c^2 u$, where $c, d$ are relatively prime. This
Lemma 2.2. Let $\mathcal{R}$ be a UFD such that the characteristic of the fraction field $L$ of $\mathcal{R}$ is different from two. Let us assume that the element 2 is a unit in $\mathcal{R}$, and that $S$ is generated as an $\mathcal{R}$-algebra by two elements $s_1, s_2 \in S$ satisfying monic quadratic polynomials $f_1 = x^2 - ax + b$ and $f_2 = y^2 - cy + d$, respectively. Then $\mathcal{R} \subset S$ splits.

Proof. We can reduce to the radical quadratic case, as in Theorem 1.3 by “completing the squares”. That is, define rings

$$T' = \mathcal{R}[u, v]/(u^2 + b - a^2/4, v^2 + c - d^2/4)$$

and

$$T = \mathcal{R}[x, y]/(x^2 - ax + b, y^2 - cy + d).$$

Let $\psi : T' \to T$ be the linear isomorphism that sends the variable $u$ to $x - a/2$ and the variable $v$ to $y - c/2$.

We already know that $S$ is isomorphic to a quotient of $T$ hence it is also isomorphic to a quotient of $T'$. Then, we may choose new generators of $S$ as $\mathcal{R}$-algebra satisfying monic radical quadratic equations: Namely, the classes of $\overline{\pi}$ and $\overline{r}$. Thus, by Theorem 1.3 the extension $\mathcal{R} \subset S$ must splits. □

2. The DSC for some nonradical quadratic extensions

Our next goal is to prove the following result.

Theorem 2.1. Let $\mathcal{R}$ be a UFD such that such that the characteristic of the fraction field $L$ of $\mathcal{R}$ is different from two. Let $\mathcal{R} \subset S$ be a module-finite extension such that $S$ is minimally generated as an $\mathcal{R}$-algebra by elements $s_1, s_2 \in S$. Let us assume $f(s_1) = g(s_2) = 0$, where $f(x) = x^2 - ax + b$ and $g(y) = y^2 - cy + d$, for some $a, b, c, d \in \mathcal{R}$. If $\gcd(2, c) = 1$ and $a^2 - 4b$ is square free, then $\mathcal{R} \subset S$ splits.

In order to prove this Theorem we need the following lemma:

Lemma 2.2. Let $\mathcal{R}$ be a UFD such that such that the characteristic of the fraction field $L$ of $\mathcal{R}$ is different from two. Let $T = \mathcal{R}[x, y]/(f(x), g(y))$, where $f(x) = x^2 - ax + b$ and $g(y) = y^2 - cy + d$, for some $a, b, c, d \in \mathcal{R}$. Suppose that $\gcd(2, c) = 1$, that the discriminant of $f(x)$, $a^2 - 4b \neq 0$, is square free in $\mathcal{R}$ and that $f(x)$ is irreducible. If $T$ is not a domain, then there exists $e \in \mathcal{R}$
such that \((c \pm ae)/2 \in R\). In this case the minimal primes of \(T\) are \(P_1 = (2)\) and \(P_2 = (h_2)\), where \(h_1\) and \(h_2\) are the classes in \(T\) of the polynomials \(h_1(x, y) = y - ex - (c - ae)/2\) and \(h_2(x, y) = y - ex - (c + ae)/2\).

**Proof.** If we assume that \(T\) is not a domain, then, by Lemma 1.1, \(g\) must be reducible in \(E[y]\), where \(E\) denotes the field \(L[x]/(f(x))\). It is clear that \(E\), as a field, is isomorphic to the extension field \(L(y^{1/2})\), for \(u = a^2 - 4b\). Therefore, \(g(y)\) has a root \(\gamma = \alpha + \beta u^{1/2}\) in \(E\), since \(T \cong E[y]/(g(x))\) is not a domain. But one can verify directly that the conjugate \(\bar{\gamma} = \alpha - \beta u^{1/2}\) is also a root of \(g\). Thus, \(g = (y - \gamma)(y - \bar{\gamma})\). By comparing coefficients we get \(a^2 - \beta^2 u = d\) and \(c = 2a\). Hence, \(4d = c^2 - 4\beta^2 u\).

Let us write \(\beta = q/r\), for \(q, r \in R\) such that \(\gcd(q, r) = 1\). From \(4d = c^2 - 4\beta^2 u\) we obtain \(4r^2 d = r^2 c^2 - 4q^2 u\). Then, \(4(r^2 d + q^2 u) = r^2 c^2\). This implies that \(4 \mid r^2 c^2\); but \(\gcd(2, c) = 1\), therefore \(4 \mid r^2\), and so \(2 \mid r\). Write \(r = 2t\), for some \(t \in R\setminus\{0\}\). Thus,

\[
(2.1) \quad 4(r^2 d + q^2 u) = 4t^2 c^2
\]

After dividing by 4 equation 2.1 we obtain: \(4t^2 d + q^2 u = t^2 c^2\). Or, equivalently, \(t^2(c^2 - 4d) = q^2 u\). From this, it follows that \(t^2 \mid q^2 u\), which in turn implies \(t^2 \mid u\), because \(\gcd(t, q) = 1\). But \(u\) is square free, therefore \(t\) must be a unit. Then we may assume that \(q/t \in R\). If we let \(e = q/t\), then the equation \(t^2(c^2 - 4d) = q^2 u\) can be rewritten as:

\[
(2.2) \quad c^2 - 4d = e^2(a^2 - 4b).
\]

We will now prove that \(2 \mid (c \pm ae)\). In fact, suppose that \(2 = \prod p_i^{k_i}\), and that \(c + ae = \prod p_i^{m_i}\) and \(c - ae = \prod p_i^{k_i}\) are factorizations into powers of prime elements. By allowing some exponents to be zero we may assume each product involves the same primes. We shall see that \(n_i \leq \min(m_i, k_i)\), for all \(i\). From 2.2 it follows that

\[
((c - ea)/2)((c + ea)/2) = d - e^2 b \in R.
\]

This implies that \(2n_i \leq m_i + k_i\), since \((c-ea)(c+ea)/2^2\) belongs to \(R\). Arguing by contradiction, suppose there is \(j\) such that \(n_j > \min(m_j, k_j)\). Without loss of generality, we may assume \(m_j = \min(m_j, k_j)\). Hence, \(n_j \leq k_j\), otherwise \(2n_j > m_j + k_j\), which is a contradiction. Therefore, \(p_j^{n_j} \mid (c - ae)\), and consequently

\[
p_j^{n_j} \mid (c - ae) + 2ae = c + ae
\]

which means that \(n_j \leq m_j\). Thus, \(n_j \leq \min(m_j, k_j)\). Summarizing, \(n_i \leq \min(m_i, k_i)\) for all \(i\). Hence, \(2 \mid (c \pm ae)\).

Now, let us see that \(P_1 = (h_1)\) and \(P_2 = (h_2)\) are the minimal primes of \(T\). Using the fact that \(((c - ea)/2)((c + ea)/2) = d - e^2 b\) we see by a direct computation that

\[
(2.3) \quad h_1 h_2 = e^2 f + g.
\]
Hence, any minimal prime in $T = R[x, y]/(f, g)$ must contain either $\overline{h}_1$ or $\overline{h}_2$. On the other hand, $R[x, y]/(f, g, h_1) \cong R[x]/(f)$, since we can eliminate the variable $y$ using $h_1(x, y) = y - ex - (c - ae)/2$, by sending $y$ into $ex + (c - ae)/2$. In a similar fashion we see that $R[x, y]/(f, g, h_2) \cong R[x]/(f)$. Since $R[x]/(f(x))$ is a domain, $P_1 = (\overline{h}_1)$ and $P_2 = (\overline{h}_2)$ must be prime ideals of $T$ and since each minimal prime of $T$ must contain either $\overline{h}_1$ or $\overline{h}_2$, then these must be the only minimal primes. □

Now, we are ready to prove Theorem 2.1.

**Proof.** As we observed in Section 1.1 we may assume $S$ is minimally generated as an $R$-algebra by certain elements $s_1$ and $s_2$ whose corresponding monic polynomials $f(x)$ and $g(y)$ over $R$ have degree greater than one. As we observed in that same section, this implies that $f(x)$ is irreducible.

We know that $S$ can be represented as a quotient of the form $T/J$, where $T = R[x, y]/(f(x), g(y))$ and $J \subset T$ is an ideal of $T$ of height zero. By Lemma 2.2, $J \subset P_1 = (\overline{h}_1)$ or $J \subset P_2 = (\overline{h}_2)$, the minimal prime ideals of $T$ defined in Lemma 2.2, with $h_1(x, y) = y - ex - (c - ae)/2$ and $h_2(x, y) = y - ex - (c + ae)/2$. We can obtain the desired retraction $\rho : S \to R$ as the composition of the following natural chain of $R$-homomorphisms:

$$S = T/J \to T/P_j \xrightarrow{\varphi} R[x]/(f(x)) \to R \oplus Rx \xrightarrow{\pi_1} R,$$

where $\varphi$ is the $R$-homomorphism defined by defining $x$ into $x$ and $y$ into $h_j - y$, and $\pi_1$ is canonical projection on $R$. □

3. An asymptotic form of Koh’s conjecture

3.1. Lefschetz’s Principle. In this second part of this article we give an asymptotic formulation of Koh’s Conjecture as well as its proof. Let us recall that this conjecture states that if $R$ is a Noetherian ring and that $R \subset S$ a module-finite extension of rings such that the projective dimension of $S$ as an $R$-module is finite, then there exists a retraction $\rho : S \to R$.

The asymptotic form of Koh’s conjecture is the following: given any bound $b > 0$ for the “complexity” of the extension (see Definition 4.1) the set $S_b$ of prime numbers for which there are counterexamples whose characteristic lie in $S_b$ must be finite. We will prove this asymptotic form for rings that are localization at prime ideals of affine $k$-algebras, where $k$ is an algebraically closed field. We refer to such rings by the shorter name of local $k$-algebras.

Let us begin by recalling Lefschetz’s principle:

**Theorem 3.1** (Lefschetz’s Principle). Let $\phi$ be a sentence in the language of rings. The following statements are equivalent.

1. The sentence $\phi$ is true in an algebraic closed field of characteristic zero.
(2) There exists a natural number \( m \) such that for every \( p > m \), \( \phi \) is true for every algebraically closed field of characteristic \( p \).

**Proof.** See [15], Corollary 2.2.10, page 42. \( \square \)

We state without proof one of the cornerstones of Model Theory, the Compactness Theorem. This theorem guarantees the existence of a model for a \( \mathcal{L} \)-theory \( T \) (we say in this case that \( T \) is satisfiable) if and only if there exists a model for each finite subset of \( T \).

**Theorem 3.2** (Compactness Theorem). Suppose \( T \) is a \( \mathcal{L} \)-theory. Then, \( T \) is satisfiable if and only if every finite subset of \( T \) is satisfiable.

As a consequence of the Compactness Theorem one can readily deduce the following proposition ([15], page 42).

**Proposition 3.3.** Let \( \phi \) be a first order sentence which is true in every field \( k \) of characteristic zero. Then, there exists a prime number \( p_0 \) such that \( \phi \) is true in each field \( F \) of characteristic \( q \), for \( q > p_0 \).

4. **Codes for polynomial rings and modules**

Throughout this discussion we will fix a field \( k \) and a monomial order in the polynomial ring \( A = k[x_1, \ldots, x_n] \).

**Definition 4.1.** Let \( R \) be a finitely generated \( k \)-algebra, and let \( I \) be an ideal of \( k[x_1, \ldots, x_n] \). We will say that:

1. The ideal \( I \) has **complexity at most** \( d \), if \( n \leq d \) and it is possible to choose generators for \( I, f_1, \ldots, f_s \), with \( \deg f_i \leq d \), for \( i = 1, \ldots, s \).
2. We say \( R \) has **complexity at most** \( d \) if there is a presentation of \( R \) as \( k[x_1, \ldots, x_n]/I \), with \( I \) of complexity at most \( d \).
3. If \( J \subset R \) is an ideal, we will say that \( J \) has **complexity at most** \( d \), if \( R \) has complexity less than or equal to \( d \), and there exists a lifting of \( J \) in \( k[x_1, \ldots, x_n] \), let us say \( J' \), with complexity at most \( d \).
4. If \( R \) is a local \( k \)-algebra, we say it has **complexity at most** \( d \) if \( R \) can be written as \( R = (k[x_1, \ldots, x_n]/I)/p \), for some prime ideal \( p \subset k[x_1, \ldots, x_n]/I \) such that the complexity of \( R \) and \( p \) is at most \( d \).
5. If \( M \) is any finitely generated \( R \)-module, we will say that \( M \) has **complexity at most** \( d \) if \( R \) is a \( k \)-algebra of complexity at most \( d \), and there exists an exact sequence \( R^t \xrightarrow{\Gamma} R^s \to M \to 0 \), with \( s, t \leq d \), where all the entries of the matrix \( \Gamma \) are polynomials (or quotients of polynomials, in the local case) with degree at most \( d \).
6. Let \( M \subset R^d \) be \( R \)-submodule. We will say that \( M \) has **degree type at most** \( d \) (written as \( gt(M) \leq d \)) if the complexity of \( R \) is at most \( d \), and \( M \) is generated by \( d \)-tuples with all its entries of
degree at most $d$. If $M$ is a finitely generated $R$-module, we will say that $M$ has complexity degree at most $d$ if there exist submodules $N_2 \subset N_1 \subset R^d$, both of degree type at most $d$, such that $M \cong N_1/N_2$.

Now, for any polynomial $f \in A$ we will denote by $a_f$ the tuple of all the coefficients of $f$ listed according to the fixed order. When the complexity of an ideal $f$ is at most $d$, and $I = (f_1, \ldots, f_n)$, then $I$ can be encoded by the tuple $a_f$ that consists of all the coefficients of the polynomials $f_i$. It is not difficult to see that the length of this tuple only depends on $d$ [20]. On the other hand, given one of those tuples $a$ we can always reconstruct the ideal where it comes from, an ideal we shall denote by $I(a)$. Similarly, if $R$ is a $k$-algebra with complexity at most $d$ then $R$ can be written as $k[x_1, \ldots, x_n]/I(a)$. We will write this fact as $R = R(a)$.

Let $M$ be an $R$-module. If the complexity degree of $M$ is at most $d$ then the minimal number of generator for $M$ is bounded in function of $d$. Hence $M$ can be encoded by a tuple $v = (n_1, n_2)$, where $n_1$ is a code for $N_1$ and $n_2$ is a code for $N_2$. We will write this as $M \cong M(v)$ (see [20]).

Finally, if $\phi(\xi)$ is a formula with free variable $\xi$ and parameters from a ring $R$, then by $a \in |\phi|_R$ we will mean $R \models \phi(a)$ ([15], Definition 1.1.6.)

The proof of the following theorem may be found in [20], Remark 2.3, and [21], Theorem 4.4.1, page 59.

**Theorem 4.2.** Given $d > 0$, there exists a formula $\text{IdMem}_d$ (Ideal Membership) such that for any field $k$, any ideal $I \subset k[x_1, \ldots, x_n]$, and any $k$-algebra $R$, both of complexity at most $d$ over $k$, it holds that $f \in IR$ if and only if $k \models \text{IdMem}_d(a_f, a_I)$. Here $a_f$ and $a_I$ denote codes for $f$ and $I$, respectively.

**Remark 4.3.** (1) Using Corollary 4.2 it is easy to get for each $d$ formulas $\text{Inc}_d$ (Inclusion) and $\text{Equal}_d$ (Equality) such that if $R$ is a finitely generated $k$-algebra with complexity at most $d$, and if $J$ and $I$ are ideals of $R$ with complexity less than $d$, then $(a_I, a_J) \in |\text{Inc}_d|_K$ (resp. $(a_I, a_J) \in |\text{Equal}_d|_K$) if and only if $I$ is included in $J$, $I \subset J$, (resp. $I = J$.)

(2) Given $d, n > 0$ there exists a formula $\text{MaxIdeal}_{d,n}$ such that for any algebraic closed field $k$ and any ideal $m \subset k[x_1, \ldots, x_n]$ of complexity at most $d$ we have: $m$ is a maximal ideal if and only if $k \models \text{MaxIdeal}_{d,n}(a_m)$, where $a_m$ is a code for $m$. In fact, by the Nullstellensatz $m$ is maximal if and only if there exist $b_1, \ldots, b_n \in k$ such that $m = (x_1 - b_1, \ldots, x_n - b_n)$. Let us call $J = (x_1 - b_1, \ldots, x_n - b_n)$.

Then, the required formula is:

$$\text{MaxIdeal}(\xi) = (\exists b_1, \ldots, b_n)(\text{Equal}_d(\xi, a_J)),$$

where $\xi$ and $a_J$ must be replaced by the codes $a_m$ of $m$, and $a_J$ of $J$, respectively.

We will also need the following lemma:


Lemma 4.4. (20, Lemma 3.2) For each \( d > 0 \) there is a bound \( D = D(d) \) with the following property: Let \( T \) be a local \( k \)-algebra of complexity at most \( d \). Let \( M \) and \( M' \) be submodules of \( T^d \) of degree type at most \( d \). Then the degree type of \((M :_T M') = \{ t \in T : tM' \subset M \}\) is bounded by \( D \).

In particular, if \( T \) is a local \( k \)-algebra of complexity at most \( d \) and \( J \subset T \) is an ideal of complexity at most \( d \) then \( \text{Ann}_T J \) must have complexity at most \( D = D(d) \), a bound that only depends on \( d \).

4.1. **Proof of the asymptotic form of Koh’s conjecture.** In this section we give an non standard proof of the asymptotic version of Koh’s conjecture. We start by defining the complexity of a ring extension (\( d \) will denote a positive integer).

**Definition 4.5.** Let \( R \subset S \) be a module-finite extension of local \( k \)-algebras. We say this extension has complexity \( \leq d \) if:

(1) The complexity over \( k \) of the local \( k \)-algebras \( R \) and \( S \) is at most \( d \).
(2) The minimal number of generators of \( S \) as an \( R \)-module is at most \( d \).
(3) The projective dimension of \( S \) as an \( R \)-module is less than or equal to \( d \).

We intend to prove the following: Given \( d > 0 \), there exists a prime \( p_d \) such that for any algebraically closed field \( k \) of characteristic \( p > p_d \), and any modulo-finite extension \( R \subset S \) of local \( k \)-algebras of complexity less than \( d \), there is a retraction \( \rho : S \to R \) and consequently \( R \subset S \) splits.

We start by recalling the following definition.

**Definition 4.6.** A local ring \((R, m)\) is called Gorenstein if \( R \) is Cohen Macaulay (CM), and if for any system of parameters \( \{x_1, \ldots, x_d\} \) in \( R \) the socle of \( \overline{R} = R/(x_1, \ldots, x_d) \), defined as \( \text{Ann}_{\overline{R}}(m) \), is a 1-dimensional \( R/m \)-vector space (4, page 526).

The following result will be of fundamental importance [24].

**Proposition 4.7.** Let \((R, m) \subset (T, n)\) be a module-finite extension of local rings with \( T \) a free \( R \)-module. If \( T/mT \) is Gorenstein, and if \( J \subset T \) is any ideal, then there exists a retraction \( \rho : T/J \to R \) if and only if \( \text{Ann}_T(J) \notin mT \).

We also need the following [23].

**Theorem 4.8** (Koh in characteristic zero). Let \( R \) be a ring containing a field of characteristic zero, and let \( R \subset S \) be a module-finite extension of rings such that the projective dimension of \( S \) as an \( R \)-module is finite. Then, there exists a retraction \( \rho : S \to R \).

**Remark 4.9.** Let \((R, m)\) be a local ring and let \( R \subset S \) be a module-finite extension. Let us take \( s_1, \ldots, s_n \in S \) generators of \( S \) as an \( R \)-algebra. For each \( s_i \in S \) choose an arbitrary monic polynomial with coefficients in \( R \),
\( f_i(t) = x_i^d_i + \sum_{r_i} x_i^{d_i-1} + \cdots + r_i \), that each \( s_i \) satisfies. Let \( T \) denote the quotient ring \( R[x_1, \ldots, x_n]/(f_1(x_1), \ldots, f_n(x_n)) \). As in Section 1.1 we may represent \( S \) as a quotient of \( T \) by defining a surjective \( R \)-homomorphism

\( \phi : T \to S \), sending the class of \( x_i \) into \( s_i \). If \( J \) denotes its kernel then \( \text{ht}(J) = 0 \), as showed in that same section.

The representation of \( S \) as the quotient \( T/J \) makes it possible to give a very useful criterion for the existence of a retraction. Let \( (R, \mathfrak{m}) \) be a local ring and let \( R \subset S \) be a module-finite extension. Then, the inclusion map \( R \subset T/J \) splits if and only if \( \text{Ann}_T(J) \) is not contained in \( \mathfrak{m}T \). This follows immediately from Proposition 4.7.

The following theorem states that given \( i \geq 0 \) and \( d > 0 \) there exists a formula \((\text{Tor}_i)_d\) such that for any \( k \)-algebra \( R \) and \( R \)-modules \( M, N, V \), all of complexity at most \( d \), then \( \text{Tor}_d^R(M, N) \cong V \) if and only if \((\text{Tor}_i)_d\) evaluated in codes of \( R, M, N \) and \( V \) is true over \( k \) (analogously for \( \text{Ext} \)).

**Theorem 4.10.** Given \( i \geq 0 \), \( d > 0 \), there exist formulas \((\text{Tor}_i)_d\) and \((\text{Ext}_i)_d\) with the following properties: Let \( k \) be any field; then, if a tuple \((a, m, n, v)\) is in \((\text{Tor}_i)_d)k\) (respectively, in \((\text{Ext}_i)_d)k\), then \( M(v) \) is isomorphic to

\[ \text{Tor}_d^A(k)(M(m), M(n)) \]

(respectively to \( \text{Ext}_d^A(k)(M(m), M(n)) \)). Moreover, for each tuple \((a, m, n)\) we can find at least one \( v \) such that \((a, m, n, v)\) belongs to \((\text{Tor}_i)_d)k\) (respectively, to \((\text{Ext}_i)_d)k\).

**Proof.** See [20], Corollary 4.4, page 150. \( \square \)

We recall the following standard result ([4], page 167, Theorem 6.8).

**Theorem 4.11.** Let \( (R, \mathfrak{m}) \) be a local ring, and denote by \( k \) the residue field \( R/\mathfrak{m} \). If \( M \) is a finitely generated \( R \)-module, then \( \text{pd}_R(M) \leq n \) if and only if \( \text{Tor}_{n+1}^R(M, k) = 0 \).

**Remark 4.12.** It is clear from the previous theorems that there exists a formula \((\text{pd}_n)_d\) such that, if \( M = M(v) \) is an \( R = R(a) \)-module with complexity less than \( d \), where \((R, \mathfrak{m})\) is a local \( k \)-algebra with complexity less than \( d \), then \( k \models (\text{pd}_n)_d(a, v) \), if and only if, \( \text{pd}_R(M) \leq n \).

From these preliminaries we obtain the following main result:

**Theorem 4.13.** For each \( d > 0 \) there exists a first order formula \( \text{Koh}_d \) such that if \( R \subset S \) is a module-finite extension of local \( k \)-algebras such that the complexity of this extension is at most \( d \), then there exists a retraction \( \rho : S \to R \) if and only if \( k \models \text{Koh}_d(a, b) \), where \( R \cong R(a) \) and \( S \cong S(b) \).

**Proof.** As shown in Remark 4.9 we can represent \( S \) as \( S \cong T/J \), where the complexity of \( J \) and \( T \) is less than \( d \). We know there is a retraction \( \rho : S \to R \).
if and only if $\text{Ann}_T(J) \not\subseteq mT$.

As proved in Remark 4.12, there exists a formula $(\text{pd} < n)_d$ such that if $R = \mathcal{R}(a)$ and $S = \mathcal{R}(b)$ then $k \models (\text{pd} < n)(a, b)$ if and only if $\text{pd}_R(S) < n$.

Let $\text{Koh}_d$ be the formula which establishes the following: if $\text{pd}_R(S) < d$, then $\text{Ann}_T(J) \not\subseteq mT$. Explicitly:

$$\text{Koh}_d(\xi, \xi', \nu, \nu') : \bigvee_{i=0}^{d-1} Pd_R(\xi, \nu) = i \Rightarrow \neg \text{Inc}_d(\nu', \xi').$$

Here $\nu'$ and $\xi'$ are reserved for a code of $\text{Ann}_T(J)$ and $mT$, respectively. Then, it is clear that $k \models \text{Koh}_d(a, a', b, b')$ if and only if there exist a retraction $\rho : S \to R$. □

**Theorem 4.14.** Let $R \subset S$ be module-finite extensions of local $k$-algebras. Fix $d > 0$, an arbitrary positive integer. The set of prime numbers $p$ for which there are counterexamples to Koh’s Conjecture of complexity less than $d$ is finite.

**Proof.** From Theorem 4.8, we see that $K \models \text{Koh}_d(a, b)$ for any field $K$ of characteristic zero. Then, by Proposition 3.3, we deduce that $k \models \text{Koh}_d(a, b)$, for every field $k$ of prime characteristic $p$ sufficiently large. More precisely: Given $d > 0$, there exists a prime number $p_d$ such that for any field $k$ of characteristic $p > p_d$, and any modulo-finite extension $R \subset S$ of local $k$-algebras with complexity at most $d$ there exists a retraction $\rho : S \to R$. □

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