Likelihood Inference for Possibly Non-Stationary Processes via Adaptive Overdifferencing

Maryclare Griffin\textsuperscript{1}, Gennady Samorodnitsky\textsuperscript{2}, David S. Matteson\textsuperscript{3}

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Abstract

We make a simple observation that facilitates valid likelihood-based inference for the parameters of the popular ARFIMA or FARIMA model without requiring stationarity by allowing the upper bound $\bar{d}$ for the memory parameter $d$ to exceed 0.5. We observe that estimating the parameters of a single non-stationary ARFIMA model is equivalent to estimating the parameters of a sequence of stationary ARFIMA models. This enables improved inference because many standard methods perform poorly when estimates are close to the boundary of the parameter space. It also allows us to leverage the wealth of likelihood approximations that have been introduced for estimating the parameters of a stationary process. We explore how estimation of the memory parameter $d$ depends on the upper bound $\bar{d}$ and introduce adaptive procedures for choosing $\bar{d}$. Via simulations, we examine the performance of our adaptive procedures for estimating the memory parameter when the true value is as large as 2.5. Our adaptive procedures estimate the memory parameter well, can be used to obtain confidence intervals for the memory parameter that achieve nominal coverage rates, and perform favorably relative to existing alternatives.

Keywords: long memory; ARFIMA; FARIMA

\textsuperscript{1}Department of Mathematics and Statistics, University of Massachusetts Amherst, Amherst, MA 01003 (maryclaregri@umass.edu).
\textsuperscript{2}School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14853 (gs18@cornell.edu).
\textsuperscript{3}Department of Statistics and Data Science, Cornell University, Ithaca, NY 14853 (matteson@cornell.edu).
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1 Introduction

Consider time series data, a sequence of \( n \) random variables \( y = (y_1, \ldots, y_n) \). Such data is prevalent and many methods for analyzing it have been developed. Stationary autoregressive moving average (ARMA) models and their non-stationary generalizations predominate. A stationary ARMA\((p,q)\) model assumes that the deviation of each observation \( y_t \) from its mean \( \mu_t \) is a linear function of past deviations and stochastic errors,

\[
\phi(B) (y_t - \mu_t) = \theta(B) z_t, \quad z_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2),
\]

where \( \phi(B) = 1 - \sum_{\ell=1}^{p} \phi_{\ell} B^\ell, \theta(B) = 1 + \sum_{\ell=1}^{q} \theta_{\ell} B^\ell, \) and \( B \) is the shift \( B^\ell y_t = y_{t-\ell} \). The mean \( \mu_t \) is a specified function of a small number of parameters, and may assumed to be constant, a low degree polynomial function in time \( t \), or a linear function of a small number of predictors. Equation (1) has a stationary, causal, and invertible solution when all of the roots of the autoregressive polynomial \( \phi(z) \) and all of the roots of the moving average polynomial \( \theta(z) \) lie outside of the unit circle. Stationarity ensures that the mean and variance of the deviations from the mean are constant over time and that correlations between observations depend only on how far apart they are in time. Further, the autocorrelation function uniquely determines the ARMA\((p,q)\) parameters. An autoregressive moving average (ARIMA) model generalizes the ARMA model to allow for certain types of non-stationarity. In particular, an ARIMA model allows for the presence of certain deterministic time trends without needing to estimate them [Box and Jenkins 1970]. An ARIMA\((p,d,q)\) model assumes that there is a nonnegative integer \( d \) such that the \( d \)-th differences \( (1 - B)^d (y_t - \mu_t) \) satisfy the ARMA\((p,q)\) equation. Thus, the ARIMA\((p,d,q)\) model is equivalent to assuming an ARMA\((p,q)\) model for deviations of a simple function of observations from their means \( (1 - B)^d \mu_t \), which can allow for the presence of certain deterministic trends without estimating them because \( (1 - B)^d \mu_t = 0 \) when \( \mu_t = \sum_{j=0}^{d-1} t^j \lambda_j \).

However, stationary ARMA\((p,q)\) models are not well suited to data with correlations that decay very slowly over time because slowly decaying correlations require ARMA\((p,q)\) models
with an increasing number of parameters as the length of the time series grows (Granger, 1980). For this purpose, long memory or autoregressive fractionally differenced moving average (ARFIMA or FARIMA) models have been developed (Hosking, 1981; Granger, 1980). An ARFIMA model assumes that a fractional difference of the deviations \((1 - B)^d(y_t - \mu_t)\) is distributed according to a stationary ARMA\((p,q)\) model where

\[
(1 - B)^d = \sum_{\ell=0}^{\infty} \binom{d}{\ell} (-1)^{\ell-1} B^\ell.
\]

(2)

A process for which (2) is well defined and which solves the ARMA\((p,q)\) equation (1) is said to follow the ARFIMA\((p,d,q)\) model. When \(-0.5 < d < 0.5\), a stationary and invertible ARFIMA\((p,d,q)\) process exists. This ARFIMA\((p,d,q)\) process has slowly decaying correlations, due to the fact that each deviation \(y_t - \mu_t\) depends on infinitely many past deviations, and the corresponding weights decay slowly. This is achieved with a single parameter \(d\). The ARIMA\((p,d,q)\) model is a special case, when \(d\) is an integer. Such models are fit to the data by first differencing the data an appropriate number of times, and fitting a stationary model to the differenced deviations.

Accordingly, one common approach for maximum likelihood estimation of possibly non-stationary ARFIMA\((p,d,q)\) models is to decide how much to difference the data first by finding the smallest integer \(k\) for which \((1 - B)^k(y_t - \mu_t)\) appears to be stationary, and assuming a stationary ARFIMA\((p,d,q)\) model for the differenced deviations \((1 - B)^k(y_t - \mu_t)\) for \(-0.5 < d < 0.5\). This approach is described in Hualde and Robinson (2011) and intuitively called the “difference-and-add-back” approach by Johansen and Nielsen (2016); it has been recommended by Box-Steffensmeier and Smith (1998) and used in practice (Byers et al., 1997, 2000; Dolado et al., 2003). Although useful, this procedure can lead to practical challenges in the presence of nearly non-stationary differenced ARFIMA\((p,d,q)\) processes, which are processes that are well represented by values of \(d\) close to the boundary of stationarity of the differenced processes, e.g. \(d \approx 0.5\), \(d \approx 1.5\), or \(d \approx 2.5\). Furthermore, it can be difficult to decide whether or not a stationary model is reasonable for an observed time
Figure 1: Simulated length $n = 500$ time series and their sample autocorrelation functions (ACFs). Both time series satisfy the ARFIMA$(0,d,0)$ model with $\mu = 0$ and the same stochastic errors with memory parameter $d = 0.45$ or $d = 0.55$. For reference, approximate 95% intervals for sample autocorrelations of a white noise process are provided with ACFs.

Figure 1 illustrates this with two time series of length $n = 500$. Both time series are simulated according to an ARFIMA$(0,d,0)$ model $(1 - B)^d y_t = z_t$ using the same stochastic errors $z_t$ simulated from a standard normal distribution. The first is simulated according to a stationary process with $d = 0.45$ and the second is stimulated according to a non-stationary process with $d = 0.55$. The latter time series is simulated by taking cumulative sums of time series simulated according to a stationary process. Although the first time series is simulated from a stationary model and the second is not, both observed time series and their corresponding sample autocorrelation functions look similar.

Despite the fact that the stationary and non-stationary time series shown in Figure 1 look similar, the likelihoods of the two time series under an ARIMA$(0,d-k,0)$ model for the deviations $y_t - \mu$ obtained using the “difference-and-add-back” approach of fitting a stationary ARIMA$(0,d-k,0)$ model to the $k$-th differenced data $(1 - B)^k (y_t - \mu)$ for $-0.5 < d - k < 0.5$ are not continuous at $d = k + 0.5$; see Figure 2. This is because the data changes from the $n$ observed time series values to the $n-1$ observed differences when we evaluate the log-likelihood for $d > 0.5$. For this reason, log-likelihood values obtained in this way are of limited utility. They are only comparable across subsets of $d$ values that correspond to
the same amount of differencing. Furthermore, the presence of boundaries and discontinuities can produce misleading standard errors and confidence intervals for estimates of the memory parameter $d$, whether they are based on asymptotic or bootstrap methods. A possible solution is suggested by recognizing that $(1 - B)^d (y_t - \mu) = (1 - B)^d - k (1 - B)^k (y_t - \mu)$. Thus, if $y_t - \mu$ is an ARFIMA$(0, d, 0)$ process, then the $k$-th difference $(1 - B)^k (y_t - \mu)$ is an ARFIMA$(0, d - k, 0)$ process. However, an ARFIMA$(0, d - k, 0)$ process is only stationary and invertible for values of $d$ in the narrow range of $[k - 1, k + 0.5]$ (Odaki, 1993).

Several existing papers have noted this problem and offered possible alternative solutions. The conditional sum-of-squares (CSS) approximate likelihood, as described in Beran (1995), Hualde and Robinson (2011), and Hualde and Nielsen (2020), uses the truncated fractional difference $(1 - B)^d_+ (y_t - \mu) = \sum_{t=0}^{t-1} (\ell) (-1)^{t-1} B^\ell (y_t - \mu)$ to approximate $(1 - B)^d (y_t - \mu)$ and obtain an approximate likelihood that is continuous in $d$. Hualde and Robinson (2011) and Hualde and Nielsen (2020) proved that the CSS approximate likelihood provides consistent, asymptotically normal parameter estimates under a slightly modified model where the truncated fractional difference of the data $(1 - B)^d_+ (y_t - \mu_t) = \sum_{t=0}^{t-1} (\ell) (-1)^{t-1} B^\ell (y_t - \mu_t)$ is distributed according to a stationary ARMA$(p, q)$ model with $\mu_t = \mu_0 \left(t \mathbb{1}_{\{t \geq 1\}} \right)^{\gamma_0}$ for known and unknown $\mu_0$ and $\gamma_0$, respectively. However, it is known that these CSS approximate likelihood based estimates can be more biased than exact likelihood based estimates in finite
samples if the data is generated according to an ARFIMA$(p, d, q)$ process (Johansen and Nielsen, 2016). Many other leading alternatives involve the specification of additional tuning parameters.

Velasco and Robinson (2000) introduced spectral methods for $-0.5 < d$ and Hurvich and Chen (2000) introduced spectral methods that have the added benefit of being invariant to the presence of linear trends for $-0.5 < d < 1.5$. However the estimators introduced in Velasco and Robinson (2000) depend on the tapering applied to the sample periodogram and both the estimators introduced in both Velasco and Robinson (2000) and Hurvich and Chen (2000) require specification of the number of periodogram ordinates used for estimation of the ARFIMA$(p, d, q)$ parameters. More recently, Mayoral (2007) introduced a moment-based method for $d > -0.75$ based on the first $k$ sample autocorrelations, however it also requires the specification of the number of sample autocorrelations to be considered.

In this paper, we show that given an upper bound $\bar{d}$ for the memory parameter $d$, it is possible to define an exact likelihood for differenced data that is continuous for all $d < \bar{d}$. This makes standard errors and confidence intervals straightforward to obtain. Our approach is motivated by the earlier observation that the $k$-th difference of an ARFIMA$(0, d, 0)$ process is a ARFIMA$(0, d - k, 0)$ process, and makes use of alternative representations of non-invertible ARFIMA$(0, d - k, 0)$ models. The key idea is that given an upper bound $\bar{d}$, we can difference the data before estimation and reduce the problem of estimating the parameters of a single non-stationary ARFIMA$(p, d, q)$ model to the problem of estimating the parameters of a sequence of stationary ARFIMA$(p, d, q)$ models with constrained moving average parameters. Having shown that it is possible to define an exact likelihood that is continuous for all $d < \bar{d}$, we also provide adaptive procedures for selecting the upper bound $\bar{d}$. Approximate inference can be performed by substituting any likelihood approximation in place of the stationary ARFIMA$(p, d, q)$ likelihoods throughout.

This paper proceeds as follows. First, we explain how the problem of estimating the parameters of a possibly non-stationary ARFIMA$(p, d, q)$ model with memory parameter $d$
bounded above by a fixed value $\bar{d}$ can be transformed to a simpler problem of estimating the parameters of a sequence of stationary ARFIMA($p, d, q$) models with constrained moving average parameters that correspond to a non-invertible moving average process. We then introduce adaptive procedures for choosing $\bar{d}$. We demonstrate the need for and performance of the adaptive procedures based on the exact likelihood and the approximate Whittle likelihood in simulations (Whittle, 1953). We show that the adaptive procedures can produce estimates of $d$ with low bias and, when based on the exact likelihood, confidence intervals with nominal coverage. We also compare the bias of adaptive exact likelihood estimators to the two competitors described in Beran (1995) and introduced in Mayoral (2007). We observe comparable performance to competitors when $n$ is relatively small and better performance than the competitors as $n$ increases when long memory is present. We conclude by applying the methods to several real datasets.

2 Methodology

2.1 Relating Non-stationary to Stationary Problems Given $\bar{d}$

Let $y_t - \mu_t$ be an possibly non-stationary ARFIMA($p, d, q$) process with autoregressive parameters $\phi = (\phi_1, \ldots, \phi_p)$ and moving average parameters $\theta = (\theta_1, \ldots, \theta_q)$ satisfying

$$\phi(B)(1 - B)^d(y_t - \mu_t) = \theta(B)z_t, \quad z_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2),$$

(3)

where all roots of $\phi(z)$ and $\theta(z)$ lie outside of the unit circle and $d < \bar{d}$ for some $\bar{d} \geq 0.5$. Deviations of the differenced process $x_t^{(m)} = (1 - B)^m y_t$ from their means $\mu_t^{(m)} = (1 - B)^m \mu_t$ are stationary for any $m$ satisfying $\bar{d} - m < 0.5$ and can be computed exactly for $t > m$ if $m$ is an integer. Let $m_{\bar{d}}$ be the smallest integer that satisfies $\bar{d} - m_{\bar{d}} < 0.5$.

It is well known that, having found an integer $m_{\bar{d}}$ which satisfies $-0.5 \leq \bar{d} - m_{\bar{d}} \leq 0.5$, if the true parameter $d$ satisfies $-0.5 \leq d - m_{\bar{d}} < 0.5$, then we can find the best model in the range $d \in [\bar{d} - 1, \bar{d}]$ by evaluating the likelihood of the differenced data $l_{\bar{d}}(x^{(m_{\bar{d}})} | d, \mu_t, \sigma, \theta, \phi)$.
for \( d \in [\bar{d} - 1, \bar{d}] \). What is less often noticed is that we can also easily evaluate the likelihood of the differenced deviations for \( d \) such that \( d - m_\bar{d} < -0.5 \). Given a positive integer \( j \) such that \(-0.5 \leq d - m_\bar{d} + j < 0.5 \) we can rewrite the model as

\[
\phi(B)(1 - B)^{d - m_\bar{d} + j}\left(x_t^{(m_\bar{d})} - \mu_t^{(m_\bar{d})}\right) = (1 - B)^j \theta(B) z_t, \quad z_t \sim i.i.d. \mathcal{N}(0, \sigma^2).
\] (4)

This is a stationary ARFIMA\((p, d - m_\bar{d} + j, j + q)\) model for the deviations with \( j + q \) constrained moving average parameters \( \tilde{\theta}_1^{(j)}, \ldots, \tilde{\theta}_{j+q}^{(j)} \) obtained by expanding out \((1 - B)^j \theta(B)\).

Then given a choice of upper bound \( \bar{d} \) and a choice of lower bound \(-0.5 + m_\bar{d} - k \) for some \( k \geq 0 \), a maximum likelihood based procedure for estimating the parameters of an ARFIMA\((p, d, q)\) model for \(-0.5 + m_\bar{d} - k \leq d < \bar{d} \) can be obtained by:

1. Finding the integer \( m_\bar{d} \) which satisfies \(-0.5 < m_\bar{d} - \bar{d} \leq 0.5\);
2. Computing the differenced data \( x_t^{(m_\bar{d})} = (1 - B)^{m_\bar{d}} y_t \) and mean \( \mu_t^{(m_\bar{d})} = (1 - B)^{m_\bar{d}} \mu_t \);
3. For \( j = 0, \ldots, k \):
   
   (a) Maximizing the log-likelihood of the differenced time series data \( x_t^{(m_\bar{d})} \) under the stationary constrained ARFIMA\((p, d - m_\bar{d} + j, j + q)\) model given by \(4\), constraining \( d \) to the range \([-0.5 + m_\bar{d} - j, 0.5 + m_\bar{d} - j]\) using the methods described in Sowell (1992), Doornik and Ooms (2003), and Durham et al. (2019).

   Record the maximum value of the log-likelihood \( l_d^{*(j)} \) and the parameters at the maximum \( \mu_{m_d}^{*(j)}, \sigma_d^{*(j)}, d_d^{*(j)}; \phi_d^{*(j)}, \theta_d^{*(j)} \).

4. Finding \( j^* = \arg\max_j l_d^{*(j)} \), and setting maximum likelihood estimates of the parameters to \( \bar{\mu}_{m_d} = \mu_{m_d}^{*(j^*)}, \bar{\sigma}_d = \sigma_d^{*(j^*)}, \bar{d}_d = d_d^{*(j^*)}, \bar{\phi}_d = \phi_d^{*(j^*)}, \) and \( \bar{\theta}_d = \theta_d^{*(j^*)} \).

In this paper, we set \( k \) to satisfy \(-0.5 + m_\bar{d} - k = -2.5 \). We find that this ensures that upper and lower 95\% quantiles of \( \hat{d}_d \) are contained in the interval \((-0.5 + m_\bar{d} - k, \bar{d})\) in the simulation settings and applications we consider. We note that this procedure has the potential benefit of being able to produce estimates of the differencing parameter over the
interval \((-0.5 + m_d - k, \bar{d})\) that are invariant to polynomial trends of degree \(m\) or lower, depending on the assumed mean \(\mu_t\). For instance, the estimator obtained by setting \(\bar{d} = 1.5\) and \(\mu_t = \lambda_0 + \lambda_1 t\) would be invariant to linear trends, and the estimator obtained by setting \(\bar{d} = 2.5\) and \(\mu_t = \sum_{j=0}^{2} t^j \lambda_j\) would be invariant to quadratic trends.

This approach maximizes a log-likelihood \(l_{\bar{d}}(x^{(m_d)}|d, \mu_t, \sigma, \theta, \phi)\) that is continuous for \(-0.5 + m_d - k \leq d < \bar{d}\). A proof of this claim is provided in Section A of the Appendix. We can obtain approximate standard errors and confidence intervals by numerically differentiating the exact log-likelihood at the maximizing parameter values. We can also maximize an easier- and faster-to-compute approximate likelihood by substituting an arbitrary likelihood approximation for the exact likelihood in 3. (a), although continuity of the resulting approximate likelihood for \(-0.5 + m_d - k \leq d < \bar{d}\) may not be preserved. Pipiras and Taqqu (2017) provides a review of the exact likelihood, Whittle likelihood, and other likelihood approximations for ARFIMA\((p, d, q)\) processes. In this paper, we consider two approximations, which we refer to as Whittle and stationary conditional sum of squares (SCSS) likelihood. The former is obtained by substituting the Whittle log-likelihood, as described in Beran (1995), for the exact log-likelihood of an ARFIMA\((p, d - m_d + j, j + q)\) process. The latter is obtained by assuming that the finite differences \((1 - B)^{d - m_d + j} (x_t^{(m_d)} - \mu_t^{(m_d)})\) are distributed according to an ARFIMA\((p, 0, j + q)\) process. The SCSS likelihood is not generally equivalent to the CSS likelihood, which assumes that the finite differences \((1 - B)^{d} (y_t - \mu_t)\) are distributed according to an ARFIMA\((p, 0, q)\) process.

### 2.2 Data-Adaptive Choice of Upper Bound \(\bar{d}\)

It is desirable to set \(\bar{d}\) as small as possible because additional differencing has a price. Taking the \(m_d\)-th difference \(x_t^{(m_d)}\) necessarily reduces the amount of available data from \(n\) observations to \(n - m_d\) observations. As \(\bar{d}\) and, accordingly, \(m_d\) increase, standard errors may increase and confidence intervals may widen. For instance, if we choose the value \(\bar{d} = 2.5\), then it is necessary to use \(m_d = 2\). This effectively yields \(n - 2\) observations. At the same
time, it is desirable to set $d$ large enough that the estimator $\hat{d}$ is not too close to the boundary $\bar{d}$, because many likelihood- and bootstrap-based methods for estimating standard errors for $\hat{d}$, estimating confidence intervals for $\hat{d}$, and performing tests using $\hat{d}$ are likely to perform poorly when the estimator $\hat{d}$ is close to the boundary $\bar{d}$.

We suggest the following adaptive procedure for selecting $\bar{d}$. We adaptively find the smallest $\bar{d}$ such that both the maximizing value $\hat{d}$ and the approximate $(1 - \epsilon) \times 100\%$ percentile of the distribution of $\hat{d}$ is less than $\bar{d}$ for some small $\epsilon > 0$. We achieve the former by checking whether the profile likelihood of the differenced data $l_d(x^{(m\bar{d})}|d)$ is decreasing as $d$ approaches the upper boundary $\bar{d}$. Crucially, this procedure avoids maximization of the log-likelihood for values of $\bar{d}$ that have a local maximum at the boundary, and is a stepwise procedure that stops once a suitable value $\bar{d}$ is reached.

Starting with $\bar{d} = 0.5$ and given $\delta > 0$ and $\epsilon > 0$, our procedure proceeds as follows:

1. Approximate the derivative of the profile log-likelihood $l_d(x^{(m\bar{d})}|d)$ at $d = \bar{d} - 2\delta$.

$$l'_d(x^{(m\bar{d})}|d - 2\delta) \approx \frac{l_d(x^{(m\bar{d})}|d - \delta) - l_d(x^{(m\bar{d})}|d - 2\delta)}{\delta}.$$

- If $l'_d(x^{(m\bar{d})}|d - 2\delta) > 0$, set $\bar{d} = \bar{d} + 1$ and return to 1. Otherwise, proceed to 2.

2. Compute $\hat{d}$, the value of the memory parameter $d$ that maximizes the log-likelihood $l_d(x^{(m\bar{d})}|\bar{d})$, and compute an approximate $(1 - \epsilon) \times 100\%$ percentile according to $\hat{d} + z_{1-\epsilon}/\sqrt{l''_d(x^{(m\bar{d})}|\hat{d}) (n - m_{\bar{d}} - p - q)}$, where $z_{1-\epsilon}$ is the $(1 - \epsilon) \times 100\%$ percentile of a standard normal distribution.

- If the $(1 - \epsilon) \times 100\%$ percentile exceeds $\bar{d}$, set $\bar{d} = \bar{d} + 1$ and return to 1.

- If the $(1 - \epsilon) \times 100\%$ percentile is less than $\bar{d}$, stop.

When it is not feasible to approximate the sampling distribution of $\hat{d}$ well, a simpler procedure which just verifies that $\hat{d}$ is achieved on the interior of the parameter space, i.e. $\hat{d} < \bar{d}$ can be obtained by setting $\epsilon = 0.5$. This procedure requires choices of $\delta$ and $\epsilon$. We find that $\delta = 0.01$ performs well in practice; it is large enough to avoid numerical instability.
near the boundary and small enough to give us a meaningful quantification of the behavior of the log-likelihood at the boundary. We explore the choice of $\epsilon$ in simulations.

3 Simulations

First, we use simulations to explore the performance of exact likelihood, Whittle, and SCSS likelihood estimators for fixed upper bounds $\bar{d} \in \{0.5, 1.5, 2.5, 3.5\}$. For simplicity, we fix $\mu_t = \mu$. We focus on ARFIMA$(0, d, 0)$ models in simulations because computation for fitting ARFIMA$(p, d, q)$ models with $p > 0$ or $q > 0$ is much slower than computation for fitting ARFIMA$(0, d, 0)$ models. ARFIMA$(p, d, q)$ models are fit in two applications in Section 4. Also for simplicity, we consider interval estimation and sampling distribution based adaptive estimators for exact likelihood based estimators only.

We consider the same set of simulations performed in Beran (1995) and Mayoral (2007). We examine the performance of alternative estimators of $d$ for true memory parameter values $d \in \{-0.7, -0.3, -0.2, 0.0, 0.2, 0.4, 0.7, 0.8, 1.0, 1.2, 1.4, 2.0, 2.2\}$ and sample sizes $n \in \{100, 200, 400, 500\}$. For each set of $d$ and $n$ values, we simulate 10,000 ARFIMA$(0, d, 0)$ time series, fixing $\mu = 0$ and $\sigma^2 = 1$. For $d \geq 0.5$, simulated time series are obtained by taking cumulative sums of simulated stationary ARFIMA time series.

3.1 Point Estimation of the Memory Parameter for Fixed $\bar{d}$

The first row of Figure 3 shows the average estimates of $d$ obtained by maximizing the exact likelihood with respect to $d$, $\mu$, and $\sigma^2$. Note that we consider upper bounds $\bar{d}$ that exceed the true data generating value of the memory parameter $d$. We estimate $d$ well at all sample sizes as long as $\bar{d}$ exceeds the true value of $d$. Surprisingly, we estimate $d$ well even when the true value of $d$ is much smaller than $\bar{d}$, e.g., when the true value of $d$ is negative and $\bar{d} > 0.5$. This suggests that we do not pay a high price for overdifferencing the data when we are interested in obtaining a point estimate of the memory parameter $d$ by maximizing
Figure 3: Average estimates of $d$ across 10,000 ARFIMA($0,d,0$) time series with $\mu = 0$ and $\sigma^2 = 1$ from maximizing exact, Whittle, or SCSS likelihoods with respect to $d$, $\mu$, and $\sigma^2$.

The second row of Figure 3 shows the average estimates of $d$ obtained by maximizing the Whittle likelihood with respect to $d$, $\mu$, and $\sigma^2$. Whittle estimates of $d$ are much more sensitive to the choice of $\bar{d}$; they are biased if $\bar{d}$ is too large or too small. Not only do they underestimate the memory parameter $d$ when the true value of the memory parameter exceeds $\bar{d}$, they also systematically overestimate $d$ when $\bar{d}$ exceeds the true value of $d$, especially when $\bar{d}$ is much larger than the true value of $d$. Systematic overestimation of $d$ when the true value $d \leq -0.5$ is consistent with the literature. Hurvich and Ray (1995) demonstrated that the log periodogram is biased when the differencing parameter $d \leq -0.5$ in such a way that leads to overestimation of the differencing parameter, and Hurvich and Chen (2000) observed this phenomenon in simulations.

The last row of Figure 3 shows the average estimates of $d$ obtained by maximizing the SCSS likelihood with respect to $d$, $\mu$, and $\sigma^2$. Like the estimates obtained by maximizing the exact and Whittle likelihoods, they underestimate the memory parameter when the true
value of the memory parameter exceeds \( \bar{d} \). They also tend to overestimate \( d \) when \( \bar{d} \) is large, i.e. we pay a high price for overdifferencing when using the Whittle or SCSS likelihood. Both approximate likelihood estimators of \( d \) perform relatively well only when the true value \( d \) satisfies \( \bar{d} - 1.5 \leq d \leq \bar{d} \).

### 3.2 Interval Estimation of the Memory Parameter for Fixed \( \bar{d} \)

In practice, we may also be interested in assessing the uncertainty of our estimate of the memory parameter \( d \). Accordingly, for exact likelihood based estimators of the long memory parameter, Figure 4 shows the coverage of 95% confidence intervals for the memory parameter \( d \), average standard errors of \( \hat{d} \), and approximate standard deviations of \( \hat{d} \). Unsurprisingly, coverage is poor when the true value of the memory parameter \( d \) exceeds the maximum value \( \bar{d} \). Also unsurprisingly, standard errors are larger and 95% confidence intervals are wider for larger values of \( \bar{d} \), especially when \( n \) is small. Thus, we pay a small price for overdifferencing when we are interested in assessing the uncertainty of our estimate of the memory parameter \( d \) when \( \bar{d} \), as long as \( \bar{d} \) is not too large.

However, we pay a high price for overdifferencing if \( \bar{d} \) is especially large relative to the true value of the long memory parameter, specifically if the difference between \( \bar{d} \) exceeds the
true value of the long memory parameter by three or more, even when \( n \) is large. Specifically, we obtain 95\% intervals with near zero coverage and standard errors that vastly underestimate the variability of \( \hat{d} \). This is caused by numerical instability of the likelihood when \( d \in (-0.5 + m_{\bar{d}} - j, 0.5 + m_{\bar{d}} - j) \) and \( j \geq 4 \). We evaluate the likelihood in this range by computing the likelihood of a stationary ARFIMA\((p, d - m_{\bar{d}} + j, j + q)\) model with \( j + q \) constrained moving average parameters \( \tilde{\theta}_1^{(j)}, \ldots, \tilde{\theta}_{j+q}^{(j)} \) obtained by expanding out \((1 - B)^j \theta (B)\).

Although this corresponds to the likelihood of a stationary process and thus corresponds to a positive definite variance-covariance matrix of the differenced data, we find that the variance-covariance matrix of the differenced data can become numerically positive semidefinite in practice, with eigenvalues that are numerically indistinguishable from zero. This leads to failure of the adjusted version of Durbin’s algorithm used to evaluate the likelihood, especially when \( n \) is large. A more detailed exploration of this phenomenon is provided in Section E of the Appendix.

### 3.3 Adaptive Point Estimation of the Memory Parameter

The results of the previous section, in particular the second two rows of Figures 3 and 4, motivate the need for the adaptive methods introduced in Section 2.2. We abbreviate the procedure obtained by setting \( \epsilon = 0.5 \) as BND because it checks that the log-likelihood is decreasing in \( d \) at the upper boundary \( \bar{d} \). We abbreviate procedures obtained by setting \( \epsilon < 0.5 \) as BFR(\( \epsilon \)) because they ensure that a buffer between \( \hat{d} \) and the upper boundary \( \bar{d} \) by requiring that the approximate \((1 - \epsilon) \times 100\%\) percentile of \( d \) be less than the upper boundary \( \bar{d} \). We consider three different values of \( \epsilon = 5 \times 10^{-s} \) corresponding to \( s \in \{2, 4, 16\} \). We note that \( s = 16 \) is the largest integer that returned a finite \((1 - 5 \times 10^{-s}) \times 100\%\) standard normal percentile, given the version of R we were using.

Figure 5 shows the average estimates of \( d \) obtained using the BND and BFR(\( \epsilon \)) procedures. The BND procedure and all three BFR(\( \epsilon \)) procedures estimate \( d \) well, regardless of the value of \( \epsilon \) chosen or true value of the memory parameter. Figure 5 also shows the average
estimates of $d$ obtained using the BND Whittle and SCSS procedures, and indicates that both BND procedures produces excellent estimates of $d$ regardless of the true value of the memory parameter, especially when $n$ is large.

### 3.4 Adaptive Interval Estimation of the Memory Parameter

In our final simulation study, we examine the coverage of BND and BFR($\epsilon$) 95% confidence intervals for $d$ obtained from the exact likelihood maximizing estimates. Figure 6 shows that the best coverage rates were obtained by setting $\epsilon = 5 \times 10^{-16}$. This suggests the simple strategy of choosing the smallest possible $\epsilon$. The BND and BFR($\epsilon$) 95% confidence intervals generally behave as we would hope; they produce interval estimates whose coverage converges to 0.95 as $n$ increases. Additionally, the performance of BFR($\epsilon$) 95% confidence intervals does not seem very sensitive to the choice of $\epsilon$, especially when $n$ is large.

The greatest gains of BFR($\epsilon$) over BND confidence intervals are obtained when the true memory parameter $d$ is near boundary values 0.5, 1.5, 2.5 and $n$ is small. The BND procedure can yield estimates $\hat{d}_d$ that are very close to the boundary $\bar{d}$, where the asymptotic normal
Figure 6: Coverage of 95% confidence intervals for $d$ across 10,000 ARFIMA(0, $d$, 0) time series with $\mu = 0$ and $\sigma^2 = 1$ for different values of $\epsilon$, which determines the percentile used to choose $\bar{d}$. When estimating $d$, $\mu$ and $\sigma^2$ are treated as unknown.

approximations to the sampling distribution of $\hat{d}_{\bar{d}}$ used to construct approximate confidence intervals are known to be poor. In contrast, the BFR($\epsilon$) procedure is more likely to yield estimates $\hat{d}_{\bar{d}}$ that are far from the boundary $\bar{d}$. The same patterns are observed when comparing BFR($\epsilon$) intervals computed using smaller versus larger values of $\epsilon$.

### 3.5 Comparison with Competitors

We compare the absolute bias of the adaptive exact, SCSS, and Whittle likelihood estimators of $d$ to the absolute bias of two competitors, one which treats the mean $\mu$ and variance $\sigma^2$ as unknown and one which treats the mean $\mu$ as known and equal to its true value and treats $\sigma^2$ as unknown. Specifically, we compare to a CSS approximate likelihood estimator as described in Beran (1995), Hualde and Robinson (2011), and Hualde and Nielsen (2020) which treats the mean $\mu$ and variance $\sigma^2$ as unknown, and we compare to the most favorable oracle implementation of the generalized minimum distance (GMD) estimator introduced by Mayoral (2007), which treats the mean $\mu$ as known and equal to its true value and treats $\sigma^2$ as unknown. The GMD estimator requires specification of the number of sample autocorrelations to include in the objective function, denoted by $k$. We consider the most favorable oracle implementation of the GMD estimator which we call “Best GMD” by using the the number of sample autocorrelations $k$ that yields the least biased GMD estimator for each value of the true memory parameter $d$ and sample size $n$. We do not consider the estimator introduced by Velasco and Robinson (2000) because simulation results from
Mean Absolute Bias $|\hat{d} - d|$ of CSS and Best GMD Estimators vs. BND Exact, BND SCSS, BND Whittle, and BFR Exact Estimators

Figure 7: Average absolute bias of CSS, BND, and BFR($5 \times 10^{-16}$) estimators and the Best GMD estimator of $d$ across 10,000 and 5,000 ARFIMA$(0, d, 0)$ time series with mean $\mu = 0$ and variance $\sigma^2 = 1$, respectively, reprinted from Mayoral (2007) for Best GMD.

Mayoral (2007) show that it is more biased and more variable than the GMD estimator for these simulation settings.

Figure 7 shows the average absolute bias of the BND exact estimator, the BFR($5 \times 10^{-16}$) exact estimator, the BND SCSS estimator, and the BND Whittle estimator compared to the average absolute bias of the CSS estimator and the Best GMD estimator. Average absolute bias for the Best GMD estimator is reprinted from Mayoral (2007). As in this paper, Mayoral (2007) simulates ARFIMA($p, d, q$) time series with mean 0 and error variance $\sigma^2 = 1$, but differs in the number of simulations performed for each pair of memory parameter $d$ and sample size $n$ values. Mayoral (2007) performs 5,000.

Figure 7 shows that our adaptive exact and SCSS estimators, especially the BFR($5 \times 10^{-16}$) estimator, are less biased in many settings, whereas our adaptive Whittle likelihood estimator is the most biased estimator when $d \geq 0.5$ and performs similarly to our adaptive SCSS estimator otherwise. In comparison to the Best GMD estimator, the BFR($5 \times 10^{-16}$) performs better when $d \geq 0$ regardless of the sample size. As the sample size increases, the BND exact estimator outperforms the Best GMD estimator for all $d \geq 0$ and the BND SCSS estimator outperforms the Best GMD estimator for most $d \geq 0$. This is especially
noteworthy given that the BFR($\epsilon$) estimator summarized in Figure 7 treats the overall mean $\mu$ and variance $\sigma^2$ as unknown, whereas the Best GMD estimator treats the overall mean $\mu$ as constant and known to be equal to 0 and the variance $\sigma^2$ as unknown.

Comparison to the CSS estimator suggests systematically poorer performance of the CSS estimator versus most adaptive estimators when the true value of $d$ is close to the boundaries of stationarity, 0.5, and 1.5, even as the sample size increases. When the sample size is smaller, the BFR($5 \times 10^{-16}$) estimator performs better than the CSS estimator when $-0.3 \leq d \leq 1.4$, the BND exact estimator performs better than the CSS estimator when $d \in \{-0.2, 0, 1.4\}$, and the BND SCSS estimator performs better than the CSS estimator when $0 \leq d \leq 1$ and $d = 1.4$. As the sample size increases, the BFR($5 \times 10^{-16}$) estimator outperforms the CSS estimator when $d \in \{-0.7, 0, 0.2, 0.4, 1.4\}$, the BND exact estimator outperforms the CSS estimator when $d \in \{-0.7, 0, 1.4\}$, and the BND SCSS estimator outperforms the CSS estimator when $-0.7 \leq d \leq -0.2$, $0.2 \leq d \leq 1$, and $d = 1.4$.

We also find that the variability of our adaptive estimators is comparable to the variability of the CSS estimator and the Best GMD estimator for most sample sizes and true values of $d$. Additionally, we find that our adaptive estimators of the noise variance $\sigma^2$, especially the BND exact and BFR($5 \times 10^{-16}$) exact estimators of $\sigma^2$, tend to be less biased than the CSS estimator of $\sigma^2$ in the presence of long memory, even as the sample size increases. More details regarding variability of estimators and estimation of the noise variance are provided in Sections B and C of the Appendix. We did not explore estimation of the overall mean $\mu$ because, like other estimators of based on integer differenced data, the BND and BFR($\epsilon$) estimators of $\mu$ are not unique after differencing.
4 Applications

4.1 Chemical Process Concentration and Temperature

Because Beran (1995) considered the same problem as this paper, we begin by applying our methods to fit ARFIMA\((0,d,0)\) models with \(\mu_t = \mu\) to the examples discussed in Beran (1995): the chemical process concentration readings (Series A) and the chemical process temperature readings (Series C). Descriptive plots of Series A and Series C are provided in Section F of the Appendix. For both, we consider exact likelihood estimation of the memory parameter because we are interested in obtaining point and interval estimates. Based on the results of the simulation study which suggest that choosing the smallest possible \(\epsilon\) yields the best performance, we compute \(\text{BFR}(\epsilon = 5 \times 10^{-16})\) exact likelihood estimates. Analogous results for the Whittle and SCSS approximate likelihoods and corresponding BND estimates are provided in Section F of the Appendix.

Table 1, which shows the exact estimates of \(d\) for \(\tilde{d} \in \{0.5, 1.5, 2.5\}\), with the BFR(\(\epsilon\)) exact likelihood estimates highlighted in gray. Exact profile log-likelihood curves for \(\tilde{d} \in \{0.5, 1.5, 2.5\}\), with \(\mu\) and \(\sigma^2\) profiled out are provided in Section F of the Appendix. We observe that the exact likelihood curves retain the same shape as \(\tilde{d}\) increases; both the maximizing value of \(d\) and the curvature about the maximizing value change little as \(\tilde{d}\) increases. When computing estimates using the BFR(\(\epsilon\)) procedure, we set \(\epsilon = 5 \times 10^{-16}\) based on the simulation results discussed in Section 3.4.

| Series | \(n\) | \(\tilde{d}\) | \(\hat{d}_{\tilde{d}}\) | 95\% Interval for \(d\) | | Series | \(\tilde{d}\) | \(\hat{d}_{\tilde{d}}\) | 95\% Interval for \(d\) |
|--------|------|-------------|-----------------|-----------------| |        |------|-------------|-----------------|-----------------| |
| A      | 197  | 0.5         | 0.400           | (0.304, 0.496)  | C      | 226 | 1.5  | 1.500           | –               | |
|        |      | 1.5         | 0.427           | (0.319, 0.534)  |        |    |      | 2.5             | 1.788           | (1.659, 1.918) |
|        |      | 2.5         | 0.436           | (0.326, 0.545)  |        |    |      |                 |                 |                 |

Table 1: Estimates and intervals for \(d\) for Series A and C, treating \(\mu\) and \(\sigma^2\) are treated as unknown. BFR(\(\epsilon = 5 \times 10^{-16}\)) estimates given in gray. Intervals provided when the log-likelihood is decreasing at \(\tilde{d}\).

From Table 1 we again see that the choice of \(\tilde{d}\) affects the exact likelihood estimates of \(d\) minimally and the 95\% intervals for the exact likelihood estimates of \(d\) moderately but in
Table 2: Estimates and intervals for ARFIMA(0, \(d, 1\)) and ARFIMA(1, \(d, 0\)) models with \(\mu_t = \mu\) for Series A and C, respectively, treating \(\mu\) and \(\sigma^2\) as unknown. BFR(\(\epsilon = 5 \times 10^{-16}\)) estimates given in gray. Intervals provided when the log-likelihood is decreasing at \(\tilde{d}\).

| Series | \(\tilde{d}\) | Estimate | 95% Interval |
|--------|--------------|----------|--------------|
|        |              | \(d\)    | \(\theta_1\) | \(\phi_1\) |
| A      | 0.5          | 0.419    | -0.037 -     | -            |
|        |              | (0.286, 0.553) \(\tilde{d} = 1\).5, the BFR(\(\epsilon = 5 \times 10^{-16}\)) exact likelihood estimate of \(d\), and \(\tilde{d} = 2.5\), for the Series A data contain 0.5, which suggests that the Series A data may not be stationary. This is consistent with the findings of [Beran (1995)](https://www.jstor.org/stable/2335087), which used another approximate likelihood method to obtain an estimate \(\hat{d} = 0.41\) and a 95% confidence interval of (0.301, 0.519).

To further explore the utility of the methods we have introduced in this paper, and to provide a complete comparison to [Beran (1995)](https://www.jstor.org/stable/2335087), we also fit an ARFIMA(0, \(d, 1\)) model with \(\mu_t = \mu\) to the Series A data and an ARFIMA(1, \(d, 0\)) with \(\mu_t = \mu\) to the Series C data for different values of \(\tilde{d}\). ARFIMA(\(p, d, q\)) models are more computationally challenging to fit than ARFIMA(0, \(d, 0\)) models because the log-likelihood surface can have many modes and evaluating the log-likelihood is much more computationally intensive.

Table 2 shows the corresponding parameter estimates we obtain for both data sets for \(\tilde{d} \in \{0.5, 1.5, 2.5, 3.5\}\), with the BFR(\(\epsilon = 5 \times 10^{-16}\)) exact likelihood estimates highlighted in gray. Examining the estimates of an ARFIMA(0, \(d, 1\)) model for Series A, we observe striking changes in the exact likelihood estimates of \(d\) and \(\theta_1\) as \(\tilde{d}\) changes, and substantial differences between several of the exact and approximate maximum likelihood estimates. Furthermore, the 95% intervals corresponding to the BFR(\(\epsilon = 5 \times 10^{-16}\)) exact estimates for \(d\) and \(\theta_1\) shown in Table 2 do not contain the exact likelihood estimates for \(\tilde{d} > 1.5\). At the same time, the BFR(\(\epsilon = 5 \times 10^{-16}\)) exact estimates for \(d\) and \(\theta_1\) align well with the estimates

\[
A \quad 0.5 \quad 0.419 \quad -0.037 \quad - \\
1.5 \quad 0.502 \quad -0.117 \quad - \\
2.5 \quad 1.314 \quad -0.923 \quad - \\
3.5 \quad 1.310 \quad -0.911 \quad - \\
C \quad 0.5 \quad 0.500 \quad - \quad 1.000 \quad - \\
1.5 \quad 0.950 \quad - \quad 0.850 \quad - \\
2.5 \quad 0.972 \quad - \quad 0.842 \quad - \\
3.5 \quad 0.971 \quad - \quad 0.852 \quad - 
\]
Figure 8: Exact profile log-likelihoods for ARFIMA(0,1,1) and ARFIMA(1,0,0) models with \( \mu_t = \mu \) for Series A and Series C data, respectively, with \( \mu \) and \( \sigma^2 \) profiled out.

\( \hat{d} = 0.445, \hat{\theta}_1 = -0.056 \) and corresponding 95% intervals \((0.261, 0.629)\) and \((-0.290, 0.179)\) found in Beran (1995). This warrants more careful investigation.

The first row of Figure 8 shows the exact joint log-likelihoods for the Series A data as a function of the moving average parameter \( \theta_1 \) and \( d \) for \( \bar{d} \in \{0.5, 1.5, 2.5, 3.5\} \). The exact log-likelihood is multimodal when \( \bar{d} > 0.5 \), one with \( d < 1 \) and another with \( d > 1 \). Which mode maximizes the likelihood depends on the choice of \( \bar{d} \); the maximum likelihood estimates for \( \bar{d} \leq 1.5 \) corresponds to the mode with \( d < 1 \), whereas the maximum likelihood estimate for \( \bar{d} > 1.5 \) corresponds to the mode with \( d > 1 \). This suggests possibly poor identifiability of the parameters of the ARIMA\((p,d,q)\) model even in simple cases with \( p = 0 \) and \( q = 1 \), which should be considered whenever an ARIMA\((p,d,q)\) is applied.

We revisit Table 2 to examine estimates of the parameters of an ARFIMA(1,0,0) model for Series C and see more stable estimates across values of \( \bar{d} \). The BFR(\( \epsilon = 5 \times 10^{-16} \)) exact likelihood estimates of \( \hat{d}_d = 0.972 \) and \( \hat{\phi}_{d,1} = 0.842 \), with corresponding 95% intervals of \((0.684, 1.261)\) and \((0.654, 1.031)\) are consistent with the approximate likelihood estimates \( \hat{d} = 0.905 \) and \( \hat{\phi}_1 = 0.864 \) and 95% confidence intervals \((0.662, 1.148)\) and \((0.708, 1.00)\) provided in Beran (1995). We do not observe evidence of multimodality of the exact log-likelihoods in Table 2, however the joint log-likelihoods in the second row of Figure 8 are banana shaped near the maximum. Again, this reflects somewhat poor identifiability of \( d \) and \( \phi_1 \) and further emphasizes the need for care when applying ARIMA\((p,d,q)\) models.
4.2 CO₂ Emissions

Barassi et al. (2018) uses long memory models to assess whether relative per capita CO₂ emissions of 28 OECD countries are consistent with mean reverting long memory processes that may converge to a linear trend over enough time. Letting \( y_{tc} \) be the relative per capita CO₂ emissions of country \( c \) at time \( t \) as defined in Barassi et al. (2018), they assume that deviations of the relative per capita CO₂ emissions of each country \( y_{tc} \) from an overall mean and country-specific linear trend \( \mu_{tc} = \mu_c + \beta_c t \) are distributed according to an ARFIMA\((0,d_c,0)\) process, with \( (1 - B)^{d_c} (y_{tc} - \mu_c - \beta_c t) = z_{tc}, \ z_{tc} \overset{i.i.d.}{\sim} N(0,\sigma_c^2) \), and error variances \( \sigma_c^2 \). Under this model, country \( c \)'s relative per capita CO₂ emissions are mean reverting if \( d_c < 1 \). Accordingly, they assess whether each country’s relative per capita CO₂ emissions are mean reverting by testing the null hypothesis \( d_c \geq 1 \) for each country.

Treating the country-specific means \( \mu_c \), slopes \( \beta_c \), and variances \( \sigma_c^2 \) as unknown, we perform level \( \alpha = 0.05 \) tests of the null hypotheses that \( d_c \geq 1 \) versus the alternative that \( d_c < 1 \) for each country by comparing the upper bound of the 90% confidence interval for the BFR(\( \epsilon = 5 \times 10^{-16} \)) exact likelihood estimate \( \hat{d}_c \) to 1 and rejecting if it fails to exceed 1. Figure 9 shows the 90% confidence intervals for the BFR(\( \epsilon = 5 \times 10^{-16} \)) exact likelihood estimates for all 28 OECD countries. We reject the null hypothesis that \( d_c \geq 1 \) for Iceland, Austria, Denmark, Ireland, Switzerland, Norway, and the Netherlands. This is largely consistent with Barassi et al. (2018), which reports strong evidence of mean reversion for Austria, Denmark, Finland, Iceland, Ireland, Israel, Norway and Switzerland based on a battery of alternative methods, many of which depend on the choice of several tuning parameters. One apparent disagreement between our findings and those of Barassi et al. (2018) is that Barassi et al. (2018) conclude that there is strong evidence that Israel’s per capita CO₂ emissions are mean reverting, whereas we are not able to estimate the memory parameter for Israel with enough precision to reject the null hypothesis that \( d_c \geq 1 \).
4.3 ECIS Measurements

Last, we compute BFR($\epsilon = 5 \times 10^{-16}$) exact likelihood estimates of the memory parameter for several observed time series of electric cell-substrate impedance sensing (ECIS) measurements, which is a relatively new tool for monitoring the growth and behavior of cells in culture (Giaever and Keese, 1991). Previous research has suggested that ARFIMA($0, d, 0$) models may be appropriate for ECIS measurements, and it has been hypothesized that the memory parameter $d$ may vary by cell type and contamination status (Giaever and Keese, 1991; Tarantola et al., 2010; Gelsinger et al., 2020; Zhang et al., 2020). We consider ECIS measurements from eight experiments, four conducted using Madin-Darby canine kidney (MDCK) cells and African green monkey kidney epithelial (BSC-1) cells. From each experiment, we examine 40 individual time series of about $n \approx 170$ measurements, which correspond to measurements from 40 cell filled wells on a single tray collected between 40 and 72 hours after the wells were filled and the experiment began. Of the 40 wells examined in each experiment, half were prepared using one medium referred to here as BSA and half were prepared using another medium referred to as gel. Of the 20 wells in each experiment prepared with the same medium, 12 contain cells contaminated by mycoplasma and 8 contain uncontaminated cells. Letting $y_{temwf}$ refer to ECIS measurements for well $w$ prepared with medium $m$ in experiment $e$ measured at frequency $f$, we assume that the
Figure 10: BFR(\(\epsilon = 5 \times 10^{-16}\)) exact likelihood estimates and 95% intervals for memory parameters \(d^{(0)}_{emf}\) (blue) and \(d^{(1)}_{emf}\) (red). When estimating \(d^{(0)}_{emf}\) and \(d^{(1)}_{emf}\), \(\mu_{emwf}\) and \(\sigma^2_{emwf}\) are treated as unknown.

The mean \(\mu_{temwf} = \mu_{emwf}\) is constant over time and

\[
(1 - B)^{d^{(1)}_{emf}} (y_{temwf} - \mu_{emwf}) = z_{temwf} \quad \text{for wells } w \text{ with contaminated cells and}
\]

\[
(1 - B)^{d^{(0)}_{emf}} (y_{temwf} - \mu_{emwf}) = z_{temwf} \quad \text{otherwise},
\]

where \(z_{temwf} \sim \mathcal{N}(0, \sigma^2_{emwf})\). Pooling over ECIS measurements with the same cells, contamination status, and preparation in the same experiment, we obtain pairs of estimates of the memory parameters \(d^{(0)}_{emf}\) and \(d^{(1)}_{emf}\) for each experiment, medium, and frequency.

Figure 10 shows some evidence that ECIS measurements may have long memory behavior that depends on contamination status. We see longer memory in measurements of contaminated MDCK cells prepared using BSA at 32,000 Hz, shorter memory in measurements of contaminated MDCK cells prepared using gel at 4,000 and 8,000 Hz, shorter memory in measurements of contaminated BSC-1 cells prepared using BSA at 32,000 Hz, and shorter
memory in measurements of contaminated BSC-1 cells prepared using gel at all frequencies, especially 32,000 Hz. This preliminary analysis suggests that long memory behavior of ECIS measurements may help distinguish contaminated from uncontaminated cells, although the sample sizes are small and the scope is limited.

5 Conclusion

In this paper, we make a simple but powerful observation that allows us to estimate the parameters of ARFIMA($p,d,q$) models without restricting the memory parameter $d$ to the range of values that correspond to a stationary ARFIMA($p,d,q$) process. We explain how this observation facilitates exact or approximate likelihood estimation, and introduce adaptive procedures that allow for fast and parsimonious inference. We demonstrate the utility of this method via simulation studies and applications, both to canonical datasets that have been explored in the ARFIMA($p,d,q$) literature and to more modern datasets.

There are several natural future directions to pursue. First, alternative likelihood approximations may provide better estimates of the memory parameter. In particular, we observe that Whittle estimates tend to be biased, which is a well known limitation of Whittle likelihood estimates (Contreras-Cristán et al., 2006). Using one of the novel approximations introduced in Jesus and Chandler (2017), Sykulski et al. (2019), or Das and Yang (2020) in place of the Whittle likelihood may yield less biased approximate estimators. Second, incorporating approximate sampling distributions for CSS and Whittle estimators, e.g. those described in Velasco and Robinson (2000) and Hualde and Nielsen (2020), may yield BFR($\epsilon$) SCSS and Whittle estimators that outperform the BND SCSS or Whittle estimators considered here. Third, alternative approaches to approximating standard errors and confidence intervals may provide better standard error and interval estimates. We have used numerical differentiation throughout this paper to approximate standard errors and confidence intervals. As shown in Section D of the Appendix, confidence intervals corresponding to BND and
BFR(\(\epsilon\)) exact likelihood estimators perform well on average across the simulations considered in this paper, however they can be unstable in individual simulations. Fortunately, the methods we introduce in this paper ease the use of modern techniques, e.g. the parametric bootstrap, for approximating the sampling distribution of the memory parameter \(d\) by ensuring that the estimate \(d\) is far from the upper boundary \(\bar{d}\). Fourth, replacing the adjusted version of Durbin’s algorithm used to evaluate the likelihood with an alternative algorithm that does not require that the variance-covariance matrix of the differenced data be positive definite may improve performance of the exact likelihood estimator for large maximum values \(\bar{d}\) when the true value of the differencing parameter \(d\) is much smaller than \(\bar{d}\), thus reducing the need for an adaptive approach altogether. Last, this approach may provide a useful framework for obtaining exact likelihood estimation for multivariate fractional differencing models, which may provide performance advantages in comparison to existing approximate methods such as the CSS approximate likelihood estimator introduced in Nielsen (2015).

6 Supplementary Materials

A stand-alone package for implementing the methods described in this paper can be downloaded from https://github.com/maryclare/ns1m.

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A Continuity of $l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | d, \mu_t, \sigma, \theta, \phi \right)$

Theorem A.1 The log-likelihood $l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | d, \mu_t, \sigma, \theta, \phi \right)$ is continuous for $-0.5 + m_{\tilde{d}} - k \leq d < \tilde{d}$.

Proof Continuity of $l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | d, \mu_t, \sigma, \theta, \phi \right)$ on the intervals $[-0.5 + m_{\tilde{d}} - j, 0.5 + m_{\tilde{d}} - j)$ for $j = 0, \ldots, k$ follows from continuity of the log-likelihood of a stationary ARFIMA$(p, d, q)$ process [Dahlhaus, 1989]. Note that Dahlhaus (1989) requires stationarity of the ARFIMA$(p, d, q)$, but does not require that roots of the moving average polynomial $\theta(x)$ lie strictly outside the unit circle.

Continuity of $l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | d, \mu_t, \sigma, \theta, \phi \right)$ at $d = -0.5 + m_{\tilde{d}} - j$ for $j = 0, \ldots, k - 1$ requires

$$\lim_{\varepsilon \to 0^+} l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | 0.5 + m_{\tilde{d}} - (j + 1) - \varepsilon, \mu_t, \sigma, \theta, \phi \right) = l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | -0.5 + m_{\tilde{d}} - j, \mu_t, \sigma, \theta, \phi \right).$$

Let $\Omega(d, \theta, \phi, \sigma)$ refer to the $(n - m_{\tilde{d}}) \times (n - m_{\tilde{d}})$ covariance matrix of the stationary differenced data $x^{(m_{\tilde{d}})}$ with elements $\omega(d, \theta, \phi, \sigma)_{ii}$, given by the autocovariance function $\omega(|i - i'|; d, \theta, \phi, \sigma)$. The log-likelihood $l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | d, \mu_t, \sigma, \theta, \phi \right)$ is

$$l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | d, \mu, \sigma, \theta, \phi \right) = -n \log (2\pi) / 2 - \log (|\Omega(d, \theta, \phi, \sigma)|) / 2 - (x^{(m_{\tilde{d}})} - \mu^{(m_{\tilde{d}})})' \Omega(d, \theta, \phi, \sigma)^{-1} (x^{(m_{\tilde{d}})} - \mu^{(m_{\tilde{d}})}) / 2.$$  

Because the log-likelihood depends on $d$ only through the autocovariance function and is a continuous function of the autocovariance function, the log-likelihood $l_{\tilde{d}} \left( x^{(m_{\tilde{d}})} | d, \mu_t, \sigma, \theta, \phi \right)$ is continuous at $d = -0.5 + m_{\tilde{d}} - j$ if the autocovariance function is continuous at $d = -0.5 + m_{\tilde{d}} - j$,

$$\lim_{\varepsilon \to 0^+} \omega(h; 0.5 + m_{\tilde{d}} - (j + 1) - \varepsilon, \theta, \phi, \sigma) = \omega(h; -0.5 + m_{\tilde{d}} - j, \theta, \phi, \sigma).$$

Letting $\gamma(h; d, \theta, \phi, \sigma)$ refer to the autocovariance function of a mean-zero stationary ARFIMA$(p, d, q)$ process with parameters $d, \theta, \phi$, and $\sigma^2$, and letting $\tilde{\theta}^{(j)}$ refer to the coefficients of the moving
average polynomial \((1 - B)^j \theta (B)\), we have

\[
\lim_{\varepsilon \to 0^+} \omega (h; 0.5 + m_\theta - (j + 1) - \varepsilon, \theta, \phi, \sigma) = \lim_{\varepsilon \to 0^+} \gamma (h; 0.5 - \varepsilon, \hat{\theta}^{(j+1)}, \phi, \sigma) = \\
\lim_{\varepsilon \to 0^+} \int_{-\pi}^{\pi} \exp \{ih\nu\} \left( \frac{\sigma^2 |\hat{\theta}^{(j+1)}(\exp \{-i\nu\})|^2}{2\pi |\phi(\exp \{-i\nu\})|^2} \right) |1 - \exp \{-i\nu\}|^{-2(0.5 - \varepsilon)} d\nu = \\
\lim_{\varepsilon \to 0^+} \int_{-\pi}^{\pi} \exp \{ih\nu\} \left( \frac{\sigma^2 (1 - \exp \{-i\nu\})^{j+1} \theta (\exp \{-i\nu\})^2}{2\pi |\phi(\exp \{-i\nu\})|^2} \right) |1 - \exp \{-i\nu\}|^{-2(0.5 - \varepsilon - 1)} d\nu = \\
\lim_{\varepsilon \to 0^+} \int_{-\pi}^{\pi} \exp \{ih\nu\} \left( \frac{\sigma^2 |\hat{\theta}^{(j)}(\exp \{-i\nu\})|^2}{2\pi |\phi(\exp \{-i\nu\})|^2} \right) |1 - \exp \{-i\nu\}|^{-2(-0.5 - \varepsilon)} d\nu = \\
\lim_{\varepsilon \to 0^+} \gamma (h; -0.5 - \varepsilon, \hat{\theta}^{(j)}, \phi, \sigma) = \gamma (h; -0.5, \hat{\theta}^{(j)}, \phi, \sigma) = \omega (h; -0.5 + m_\theta - j, \theta, \phi, \sigma).
\]

Throughout, we make use of derivations of the spectral density and autocovariance function of an ARFIMA\((p, d, q)\) process in Sowell (1992). Although Sowell (1992) focuses on stationary ARFIMA\((p, d, q)\) process with roots of the autoregressive and moving average polynomials \(\phi (x)\) and \(\theta (x)\) outside the unit circle and \(-0.5 < d < 0.5\), the derivations themselves do not require that the roots of the moving average polynomial \(\theta (x)\) lie strictly outside the unit circle and allow for \(-1 < d \leq -0.5\).

### B Variability Comparison with Competitors

Figure 11 shows that the variability of our adaptive estimators is comparable to the variability of the CSS estimator and the “Best GMD” estimator for most sample sizes and true values of \(d\). Again, this is especially noteworthy given that the adaptive BND CSS, BND exact, and BFR(\(\varepsilon\)) exact estimators summarized in Figure 11 treat the overall mean \(\mu\) and variance \(\sigma^2\) as unknown, whereas the version of Mayoral’s estimator summarized in Figure 11 treats the overall mean \(\mu\) as constant and known to be equal to 0 and the variance \(\sigma^2\) as unknown. There are some exceptions when the sample size is smaller; the adaptive BND CSS estimator is especially variable when \(d \geq 2\) and the adaptive BND exact estimator is especially variable when \(0.5 < d < 1\).
Figure 11: For CSS, BND, and BFR(\(\epsilon = 5 \times 10^{-16}\)), root mean squared error (RMSE) of estimates of the memory parameter \(\hat{d}\) is approximated from 10,000 simulated ARFIMA\((0, d, 0)\) time series with mean \(\mu = 0\) and variance \(\sigma^2 = 1\) for each sample size \(n\) and true value of the differencing parameter \(d\). Root mean squared error (RMSE) for the best GMD estimator is reprinted from Mayoral (2007) by choosing the smallest average RMSE across GMD estimators that use a different number of autocorrelations at each value of the true memory parameter \(d\) and sample size \(n\).
C  Estimation of $\sigma^2$ Comparison with Competitors

Absolute Bias $|\hat{\sigma}^2 - 1|$ of CSS Estimator vs.
BND Exact, BND SCSS, BND Whittle, and BFR Exact Estimators

Figure 12: Average absolute bias of estimates of the noise variance $\sigma^2$ across 10,000 simulated ARFIMA(0, $d$, 0) time series with mean $\mu = 0$ and variance $\sigma^2 = 1$ for each sample size $n$ and true value of the differencing parameter $d$. 
### D Estimation of Confidence Intervals

| Quantile       | BND          | BFR($\epsilon=5\times10^{-2}$) | BFR($\epsilon=5\times10^{-4}$) | BFR($\epsilon=5\times10^{-16}$) |
|----------------|--------------|---------------------------------|---------------------------------|---------------------------------|
| -0.5           | -0.5         | -0.5                            | -0.5                            | -0.5                            |
| 0.5            | 0.5          | 0.5                             | 0.5                             | 0.5                             |
| 1.5            | 1.5          | 1.5                             | 1.5                             | 1.5                             |

Figure 13: Comparison of lower 2.5% and upper 97.5% quantiles of BND and BFR($\epsilon$) exact likelihood estimators obtained by simulating 10,000 simulated ARFIMA(0,$d$,0) time series with mean $\mu = 0$ and variance $\sigma^2 = 1$ versus average lower 2.5% and average upper 97.5% quantiles corresponding to BND and BFR($\epsilon$) exact likelihood estimators obtained using numerical differentiation across 10,000 simulated ARFIMA(0,$d$,0) time series with mean $\mu = 0$ and variance $\sigma^2 = 1$ for each sample size $n$ and true value of the differencing parameter $d$. 

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Figure 14: Variance of upper 97.5% quantiles corresponding to BND and BFR(ε) exact likelihood estimators obtained using numerical differentiation across 10,000 simulated ARFIMA(0, d, 0) time series with mean \( \mu = 0 \) and variance \( \sigma^2 = 1 \) for each sample size \( n \) and true value of the differencing parameter \( d \).
E Likelihood Instability

Figure 15: The first row shows histograms of exact maximum likelihood estimates $\hat{d}$ of the differencing parameter $d$ obtained by setting $\bar{d} = 3.5$ across 10,000 simulated ARFIMA(0, $-0.7$, 0) time series with mean $\mu = 0$ and variance $\sigma^2 = 1$ for each sample size $n$. The second row shows selected profile log-likelihood curves obtained by setting $\bar{d} = 3.5$ for four out of 10,000 simulated ARFIMA(0, $-0.7$, 0) time series with mean $\mu = 0$ and variance $\sigma^2 = 1$ for each sample size $n$. 
Figure 16: Observed Series A and Series C time series and corresponding exact, Whittle, and SCSS profile log-likelihood curves for $\bar{d} \in \{0.5, 1.5, 2.5\}$, with the mean $\mu$ and variance $\sigma^2$ profiled out.
Table 3: Estimates and corresponding 95% confidence intervals for $d$ for the chemical process concentration readings (Series A) and chemical process temperature readings (Series C) for different values of $\bar{d}$. The BFR($\epsilon = 5 \times 10^{-16}$) exact likelihood, BND Whittle, and BND CSSS estimates are highlighted in gray. 95% intervals for exact likelihood estimates $\hat{d}$ are provided for values of $\bar{d}$ that correspond to log-likelihoods that are decreasing at $\hat{d}$. The mean $\mu$ and variance $\sigma^2$ are treated as unknown when estimating the differencing parameter $d$. 

| Data | n  | $\bar{d}$ | Exact $\hat{d}$ | $\bar{d}$ | 95% Interval for $\hat{d}$ | Whittle $\hat{d}$ | SCSS $\hat{d}$ |
|------|----|-----------|----------------|-----------|-------------------------|------------------|---------------|
|      |    |           | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ |
| Series A | 197 | 0.5 | 0.419 | -0.037 | - | 0.449 | -0.043 | - | 0.480 | -0.093 | - |
|         |    | 1.5 | 0.502 | -0.117 | - | 0.470 | -0.077 | - | 0.500 | -0.115 | - |
|         |    | 2.5 | 1.314 | -0.923 | - | 2.047 | -1.000 | - | 0.990 | -0.685 | - |
|         |    | 3.5 | 1.310 | -0.911 | - | 2.292 | -0.665 | - | 0.994 | -0.721 | - |
| Series C | 226 | 0.5 | 0.500 | - | 1.000 | 0.215 | - | 0.500 | - | - |
|         |    | 1.5 | 0.950 | - | 0.850 | 0.908 | - | 0.894 | - | - |
|         |    | 2.5 | 0.972 | - | 0.842 | 0.892 | - | 1.014 | - | - |
|         |    | 3.5 | 0.971 | - | 0.852 | 1.109 | - | 0.996 | - | - |

Table 4: Estimates of the parameters of ARFIMA(0, $d$, 1) and ARFIMA(1, $d$, 0) models both with $\mu_t = \mu$ for the chemical process concentration readings (Series A) and chemical process temperature readings (Series C) for different values of $d$. The BFR($\epsilon = 5 \times 10^{-16}$) exact likelihood estimates, BND Whittle estimates, and BND CSSS estimates are highlighted in gray. The mean $\mu$ and variance $\sigma^2$ are treated as unknown when estimating the differencing parameter $d$. 

| Data | n  | $\bar{d}$ | Exact $\hat{d}$ | $\bar{d}$ | 95% Interval for $\hat{d}$ | Whittle $\hat{d}$ | CSSS $\hat{d}$ |
|------|----|-----------|----------------|-----------|-------------------------|------------------|---------------|
|      |    |           | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ |
|      |    |           | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ |
|      |    |           | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ |
|      |    |           | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ | $d$ | $\theta_1$ | $\phi_1$ |

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Figure 17: Exact, Whittle, and SCSS joint profile log-likelihoods for Series A as a function of the moving average parameter $\theta_1$ and $d$ for $\bar{d} \in \{0.5, 1.5, 2.5, 3.5\}$, with the mean $\mu$ and variance $\sigma^2$ profiled out.

Figure 18: Exact, Whittle, and SCSS joint profile log-likelihoods for Series C as a function of the autoregressive parameter $\phi_1$ and $d$ for $\bar{d} \in \{0.5, 1.5, 2.5, 3.5\}$, with the mean $\mu$ and variance $\sigma^2$ profiled out.
Figure 19: CSS joint profile log-likelihood for Series A as a function of the moving average parameter $\theta_1$ and $d$, with the mean $\mu$ and variance $\sigma^2$ profiled out.

Figure 20: CSS joint profile log-likelihood for Series C as a function of the autoregressive parameter $\phi_1$ and $d$, with the mean $\mu$ and variance $\sigma^2$ profiled out.