Fractional elliptic systems with critical nonlinearities

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Abstract

This paper deals with existence, uniqueness and multiplicity of positive solutions to the following nonlocal system of equations:

\[
\begin{align*}
\left( -\Delta \right)^s u & = \frac{\alpha}{2s} |u|^{\alpha-2}u|v|^{\beta} + f(x) \quad \text{in} \ \mathbb{R}^N, \\
\left( -\Delta \right)^s v & = \frac{\beta}{2s} |v|^{\beta-2}v|u|^{\alpha} + g(x) \quad \text{in} \ \mathbb{R}^N,
\end{align*}
\]

\((S)\)

where \(0 < s < 1, N > 2s, \alpha, \beta > 1, \alpha + \beta = 2N/(N - 2s)\), and \(f, g\) are nonnegative functionals in the dual space of \(H^s(\mathbb{R}^N)\), i.e., \(\langle f, u \rangle_{H^s} \geq 0\), whenever \(u\) is a nonnegative function in \(H^s(\mathbb{R}^N)\). When \(f = 0 = g\), we show that the ground state solution of \((S)\) is unique. On the other hand, when \(f\) and \(g\) are nontrivial nonnegative functionals with \(\ker(f) = \ker(g)\), then we establish the existence of at least two different positive solutions of \((S)\) provided that \(\|f\|_{H^s}^s\) and \(\|g\|_{H^s}^s\) are small enough. Moreover, we also provide a global compactness result, which gives a complete description of the Palais–Smale sequences of the above system.

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1. Introduction

In this article we study existence, uniqueness and multiplicity of positive solutions to the following fractional nonhomogeneous elliptic system in \( \mathbb{R}^N \)

\[
\begin{align*}
(-\Delta)^s u &= \frac{\alpha}{2^*_s} |u|^{\alpha-2}u|v|^\beta + f(x) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s v &= \frac{\beta}{2^*_s} |v|^{\beta-2}v|u|^\alpha + g(x) \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

(S)

where \( 0 < s < 1, N > 2s, \alpha, \beta > 1, \alpha + \beta = 2^*_s := 2N/(N-2s), \) and \( f, g \) are nonnegative functionals in the dual space of \( \dot{H}^s(\mathbb{R}^N) \). Here \( (-\Delta)^s \) denotes the fractional Laplace operator which can be defined for the Schwartz class functions \( S(\mathbb{R}^N) \) as follows

\[
(-\Delta)^s u(x) := c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy,
\]

which can be defined for the Schwartz class functions \( S(\mathbb{R}^N) \) as follows

\[
\hat{H}^s(\mathbb{R}^N) := \left\{ u \in L^{2^*_s}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy < \infty \right\},
\]

be the homogeneous fractional Sobolev space, endowed with the inner product \( \langle \cdot, \cdot \rangle_{\hat{H}^s} \) and corresponding Gagliardo norm

\[
\|u\|_{\hat{H}^s} := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2}.
\]

It is well-known that \( u \in \hat{H}^s(\mathbb{R}^N) \) implies \( u \in L^p_{\text{loc}}(\mathbb{R}^N) \) for any \( p \in [2, 2^*_s] \).

In the vectorial case, as described in [2], the natural solution space for (S) is the Hilbert space \( \hat{H}^s(\mathbb{R}^N) \times \hat{H}^s(\mathbb{R}^N) \), equipped with the inner product

\[
\langle (u, v), (\phi, \psi) \rangle_{\hat{H}^s,\hat{H}^s} := \langle u, \phi \rangle_{\hat{H}^s} + \langle v, \psi \rangle_{\hat{H}^s},
\]

and the norm

\[
\|(u, v)\|_{\hat{H}^s,\hat{H}^s} := \left( \|u\|^2_{\hat{H}^s} + \|v\|^2_{\hat{H}^s} \right)^{1/2}.
\]

In general, given any two Banach spaces \( X \) and \( Y \), the product space \( X \times Y \) is endowed with the following product norm

\[
\|(x, y)\|_{X \times Y} := \left( \|x\|^2_X + \|y\|^2_Y \right)^{1/2}.
\]

For instance, \( L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \) \( (p > 1) \) is equipped with the product norm

\[
\|(u, v)\|_{L^p \times L^p} := \left( \|u\|^2_{L^p} + \|v\|^2_{L^p} \right)^{1/2}.
\]
Definition 1.1. A pair \((u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)\) is said to be a positive (weak) solution of (S) if \(u > 0\) and \(v > 0\) in \(\mathbb{R}^N\) and for every \((\phi, \psi) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)\) it holds
\[
\langle (u, v), (\phi, \psi) \rangle_{H^s \times H^s} = \frac{\alpha}{2} \int_{\mathbb{R}^N} |v|^{\alpha-2} u |\phi|^2 \, dx + \frac{\beta}{2} \int_{\mathbb{R}^N} |v|^{\beta-2} u |\psi|^2 \, dx
\]
\[
+ \langle (v, 0), (\phi, \psi) \rangle_{H^s} + \langle (w, 0), (\phi, \psi) \rangle_{H^s},
\]
where \(\langle \cdot, \cdot \rangle_{H^s}\) denotes the duality bracket between the dual space \(H^s(\mathbb{R}^N)\)' of \(H^s(\mathbb{R}^N)\) and \(H^s(\mathbb{R}^N)\) itself.

Define
\[
S = S_{\alpha+\beta} := \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^2_{H^s} + \|v\|^2_{H^s}}{\left( \int_{\mathbb{R}^N} |u|^\alpha \, dx \right)^{2/\alpha}}
\]
and
\[
S_{\alpha, \beta} := \inf_{(u,v) \in H^s(\mathbb{R}^N) \setminus \{(0,0)\}} \frac{\|u\|^2_{H^s} + \|v\|^2_{H^s}}{\left( \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx \right)^{2/\alpha}}.
\]
(1.2)

In the celebrated paper [13], Chen, Li and Ou prove that the best Sobolev constant \(S_{\alpha+\beta} = S\) is achieved by \(u\), where \(u\) is the unique positive solution (up to translations and dilations) of
\[
(-\Delta)^{s} u = u^{2^*_s - 1} \quad \text{in} \ \mathbb{R}^N, \quad u \in H^s(\mathbb{R}^N).
\]
(1.3)

Next, we recall a result from [24] ([1] in the local case) which states the relation between \(S_{\alpha, \beta}\) and \(S_{\alpha+\beta}\).

Lemma 1.1. [24, lemma 5.1] In all cases \(\alpha > 1, \beta > 1, \text{ with } \alpha + \beta \leq 2^*_s\), it results
\[
S_{\alpha, \beta} = \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\alpha+\beta}} + \left( \frac{\beta}{\alpha} \right)^{\frac{\beta}{\alpha+\beta}} \right] S_{\alpha+\beta}.
\]

Moreover, if \(w\) achieves \(S_{\alpha+\beta}\) then \((Bw, Cw)\) achieves \(S_{\alpha, \beta}\) for all positive constants \(B\) and \(C\) such that \(B/C = \sqrt{\alpha/\beta}\).

The scalar version of (S) has been considered by Bhakta and Pucci in [4], where they prove existence of at least two positive solutions. This class of problems in the scalar and local cases, involving Sobolev critical exponents was treated in the pioneering paper [9]. Then existence was extended in [39] to multiplicity results. These kind of problems were studied in several directions. Let us mention [11, 12, 23, 37, 41] for more general perturbations and [17] for existence of sign changing solutions. Versions for systems were extended, for instance, in [7, 29, 30, 40] and in the references therein.

Elliptic systems arise in biological applications (e.g. population dynamics) or physical applications (e.g. models of a nuclear reactor) and have been drawn a lot of attention (see [1, 18, 32, 36] and references therein). For systems in bounded domains with nonhomogeneous terms we refer to [6]. Problems involving the fractional Laplace operator appear in several areas such as phase transitions, flames propagation, chemical reaction in liquids, population dynamics, finance, etc, see for e.g. [10, 21].

In the nonlocal case, there are not so many papers, in which weakly coupled systems of equations have been studied. We refer to [3, 14, 19, 24, 28, 30], where Dirichlet systems of equations in bounded domains have been treated. For the nonlocal systems of equations in the entire space \(\mathbb{R}^N\), we cite [2, 25, 26] and the references therein. In the very recent work [2], the
first, second and fourth authors of this current paper have proved existence of one solution to \((S)\) when \(f\) and \(g\) are nontrivial but \(\|f\|_{H^s} \) and \(\|g\|_{H^s} \) are small enough. To the best of our knowledge, so far there have been no papers in the literature, where uniqueness/multiplicity of positive solutions have been established for \((S)\), with the fractional Laplacian and the critical exponents in \(\mathbb{R}^N\). The main results in the paper are new even in the local case \(s = 1\).

First of all, we say that a pair \((u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)\) is a ground state solution or least energy solution for \((S)\), with \(f = 0 = g\), if \((u, v)\) is a minimizer of \(S_{a, b}\).

Lemma 1.1 poses a natural question: are all the ground state solutions of \((S)\), with \(f = 0 = g\), of the form \((Bw, Cw)\), where \(w\) is the unique positive solution of \((1.3)\)?

We answer this question affirmatively in our first main theorem which is here below stated.

**Theorem 1.1 (Uniqueness of ground state for homogeneous system).** Let \((u_0, v_0)\) be a minimizer of \(S_{a, b}\). Then there exist \(\tau, B > 0\) such that

\[
(u_0, v_0) = (Bw, Cw), \quad \text{with } C = B\tau, \quad \tau = \sqrt{\frac{\beta}{\alpha}},
\]

where \(w\) is the unique positive solution of \((1.3)\).

The above result partially extends the uniqueness theorem due to Chen et al [13] from the scalar case \((1.3)\) to the system \((S)\) with \(f = 0 = g\). Theorem 1.1 proves the uniqueness of ground state solution of the system \((S)\) when \(f = 0 = g\) and also generalizes [24, lemma 5.1], whereas in [13] uniqueness has been established among all positive solutions of \((1.3)\).

Our next main result is the multiplicity of solutions for the nonhomogeneous system \((S)\).

**Theorem 1.2 (Multiplicity for nonhomogeneous system).** Assume that \(f, g\) are nontrivial nonnegative functions in the dual space of \(H^s(\mathbb{R}^N)\) with \(\ker(f) = \ker(g)\) and

\[
\max\{\|f\|_{H^s}, \|g\|_{H^s}\} < C_0 S_{\alpha, \beta}^{\frac{4s}{N + 2s} - 1 - \frac{N - 2s}{2s}}, \quad \text{where } C_0 := \left(\frac{4s}{N + 2s}\right) (2^*_c - 1) \frac{N - 2s}{2s},
\]

then \((S)\) admits at least two positive solutions.

Furthermore, if \(f \equiv g\), then the solution \((u, v)\) of \((S)\) has the property that \(uw\), whenever \(\alpha \neq \beta\). Finally, if \(\alpha = \beta\) but \(f \neq g\), then \(u \neq v\).

Theorem 1.2 complements the mentioned work [2] on \((S)\).

The proof of the uniqueness theorem 1.1 is inspired by some arguments made in [15, 35] (also see [16]). The main difference is that in our case the nontrivial solution \((u, v)\) has both components nontrivial, that is \(u \neq 0\) and \(v \neq 0\), and in the proof it was necessary to deal with a non symmetric system.

To prove the multiplicity theorem 1.2, the main difficulty is the lack of compactness of the Sobolev space \(H^s(\mathbb{R}^N)\) into the Lebesgue space \(L^2(\mathbb{R}^N)\). For this reason the functional associated to system \((S)\) may fail to satisfy the Palais–Smale condition at some critical levels. To overcome this, it is necessary to look for a nice energy range where the (PS) condition holds in order to use variational arguments. Classification of (PS) sequences associated with a scalar equation (local/nonlocal) has been done in many papers, to quote a few, we cite [4, 20, 31, 33, 34, 38]. To the best of our knowledge, the (PS) decomposition associated to systems of equations has not been studied much. We quote the recent work [35], where in the local case the (PS) decomposition was done for systems of equations in bounded domains.

Again to the best of our knowledge, in both the local and nonlocal cases, proposition 3.1 (see section 3) is the first result where the (PS) decomposition has been established for systems of equations in the whole space \(\mathbb{R}^N\). Next, to prove multiplicity of solutions, we decompose
the space $\dot{H}^s(\mathbb{R}^N)$ into three disjoint components. The first solution is constructed using a minimization argument in one of the components. Another solution is obtained by combining the Ekeland variational principle with a careful analysis of the critical levels by using the homogeneous unique solution with some estimates in a slightly larger Morrey space.

The paper has been organized as follows. In section 2, we prove the uniqueness for the ground state solution of the homogeneous system, namely theorem 1.1. Section 3 deals with the Palais–Smale decomposition associated with the functional of $(\mathcal{S})$. In section 4, we prove theorem 1.2. In the appendix A we discuss a few elementary embeddings of the product of Morrey spaces.

**Notation:** $u_+ := \max\{u, 0\}, u_- := -\min\{u, 0\}$. By $\text{ker}(f)$ we denote the kernel of $f$.

**Remark 1.1.** Adapting the arguments in the proof of theorems 1.1 and 1.2, the results of uniqueness and multiplicity can be obtained for the following system of equations:

(a)  
\[
\begin{aligned}
(-\Delta)^s u + u &= \frac{\alpha}{\alpha + \beta} |u|^{\alpha-2} u |v|^\beta + f(x) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s v + v &= \frac{\beta}{\alpha + \beta} |v|^{\beta-2} v |u|^\alpha + g(x) \quad \text{in } \mathbb{R}^N, \\
u, v &> 0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]  
where $N > 2s$, $\alpha, \beta > 1$ and $\alpha + \beta < 2^*_s$, and $f, g$ are nonnegative functionals in the dual space of $H^s(\mathbb{R}^N)$ (see [2] for existence of solutions). It is known that the scalar equation
\[
(-\Delta)^s u + u = |u|^{\alpha+\beta-2} u \quad \text{in } \mathbb{R}^N
\]  
has a unique ground state solution (see [27]). If $\omega$ denotes the unique ground state solution of (1.5), then it can be shown that $(r\omega, t\omega)$ is a ground state solution of (1.4) when $f = 0 = g$ and $r/t = \sqrt{\alpha/\beta}$. Next, following an argument similar to theorem 1.1, with obvious modifications, it can be shown that any ground state solution of (1.4) with $f = 0 = g$ is of the form $(r\omega, t\omega)$ where $r/t = \sqrt{\alpha/\beta}$.

(b)  
\[
\begin{aligned}
(-\Delta)^s u &= \frac{\alpha}{2^*_s} a(x) |u|^{\alpha-2} u |v|^\beta + f(x) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s v &= \frac{\beta}{2^*_s} b(x) |v|^{\beta-2} v |u|^\alpha + g(x) \quad \text{in } \mathbb{R}^N, \\
u, v &> 0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]  
where $\alpha, \beta, f, g$ are as in $(\mathcal{S})$ and the potentials $a, b$ are continuous functions in $\mathbb{R}^N$ with $a, b \geq 1$ and $a(x), b(x) \to 1$ as $|x| \to \infty$. See for instance [4] in the scalar case.

(c) One can also try to adopt the methodology of this paper in order to study the system of equations involving the Hardy operator i.e., if $(-\Delta)^s$ is replaced by the Hardy operator $(-\Delta)^s - \frac{\lambda}{|x|^2}$, where $\gamma \in (0, \gamma_{N,s})$ and $\gamma_{N,s}$ is the best Hardy constant in the fractional Hardy inequality. The multiplicity question in the scalar case was already solved for this problem in the recent paper [5].

**Remark 1.2.** Theorem 1.1 proves uniqueness of ground state solutions of $(\mathcal{S})$ with $f = 0 = g$. Therefore, it is interesting to ask if any positive solution of $(\mathcal{S})$ with $f = 0 = g$ is of the form $(r\omega, t\omega)$, where $r/t = \sqrt{\alpha/\beta}$ and $\omega$ is the unique positive solution of (1.3).
2. Uniqueness for the homogeneous system

First we need an auxiliary lemma which will be used to prove theorem 1.1. Consider the following system with a parameter $\mu > 0$

\[
\begin{aligned}
(-\Delta)^s u &= \mu |u|^{\alpha_2-2}u + \frac{\alpha}{2s} |u|^{\alpha-2}u|v|^\beta \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s v &= \frac{\beta}{2s} |v|^{\beta-2}v|u|^{\alpha} \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

(2.1)

Associated to (2.1), we define

\[
S_{\mu,\alpha,\beta} := \inf_{(u,v) \in H^s(\mathbb{R}^N) \times \{(0,0)\}} \frac{\|(u,v)\|_{H^s(\mathbb{R}^N)}^2}{\mu \int_{\mathbb{R}^N} |u|^{2s} \, dx + \int_{\mathbb{R}^N} |u|^\alpha |v|^{\beta} \, dx}^{2/2s}.
\]  

(2.2)

**Lemma 2.1.**

(a) Let $h(\tau) := \frac{1+\tau^2}{(\mu+\tau^2)^{2s}}$, $\tau > 0$. Then there exists $\mu_0 > 0$ such that for $\mu \in (0, \mu_0)$,

\[S_{\mu,\alpha,\beta} = h(\tau_0), \quad \text{where } h(\tau_0) = \min_{\tau > 0} h(\tau).\]

Furthermore, $\tau_0 = \tau_0(\mu, \alpha, \beta, N, s) > 0$.

(b) For any $r > 0$, $(ru_0, rv_0)$ achieves $S_{\mu,\alpha,\beta}$, where $w$ is the unique positive solution of (1.3).

**Proof.** Let $\{u_n, v_n\}$ be a minimizing sequence for $S_{\mu,\alpha,\beta}$. Choose $\tau_n > 0$ such that $\|v_n\|_{L^{2s}(\mathbb{R}^N)} = \tau_n\|u_n\|_{L^{2s}(\mathbb{R}^N)}$. Now set, $z_n = \frac{u_n}{v_n}$. Therefore, $\|u_n\|_{L^{2s}(\mathbb{R}^N)} = \|z_n\|_{L^{2s}(\mathbb{R}^N)}$ and applying Young’s inequality,

\[
\int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta \, dx \leq \frac{\alpha}{2s} \int_{\mathbb{R}^N} |u_n|^{2s} \, dx + \frac{\beta}{2s} \int_{\mathbb{R}^N} |z_n|^{2s} \, dx = \int_{\mathbb{R}^N} |u_n|^{2s} \, dx = \int_{\mathbb{R}^N} |z_n|^{2s} \, dx.
\]

Hence,

\[
S_{\mu,\alpha,\beta} + o(1) = \frac{\|u_n\|_{H^s}^2 + \|v_n\|_{H^s}^2}{(\mu \int_{\mathbb{R}^N} |u_n|^{2s} \, dx + \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^{\beta} \, dx)^{2/2s}}
\]

\[
= \frac{\|u_n\|_{H^s}^2}{\left(\mu \int_{\mathbb{R}^N} |u_n|^{2s} \, dx + \tau_n^\beta \int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^{\beta} \, dx\right)^{2/2s}}
\]

\[
+ \frac{\tau_n^2 \|z_n\|_{H^s}^2}{\left(\mu \int_{\mathbb{R}^N} |z_n|^{2s} \, dx + \tau_n^\beta \int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^{\beta} \, dx\right)^{2/2s}}
\]

\[
\geq \frac{1}{(\mu + \tau_n^2)^{2/2s}} \left(\frac{\|u_n\|_{H^s}^2}{(\mu + \tau_n^2)^{2/2s}} + \frac{\|z_n\|_{H^s}^2}{(\mu + \tau_n^2)^{2/2s}}\right) + \frac{\tau_n^2}{(\mu + \tau_n^2)^{2/2s}}
\]

\[
\geq 1 + \frac{\tau_n^2}{(\mu + \tau_n^2)^{2/2s}} S \geq \min_{\tau > 0} h(\tau) S.
\]
Note that $h$ is a $C^1$ function with $h(\tau) > 0$ for all $\tau \geq 0$, $h(\tau) \to \infty$ as $\tau \to \infty$ and $h(\tau) \to \mu \frac{1}{\tau^2}$ as $\tau \to 0$. Therefore, there exists $\tau_0 \geq 0$ such that $\min_{\tau \geq 0} h(\tau) = h(\tau_0)$. Next, we claim that $\tau_0 > 0$, if we choose $\mu > 0$ small enough. To prove the claim, first we note that $h(0) = \mu \frac{1}{\tau^2}$ and $h(1) = 2(1 + \mu)^{-2/2}$. Therefore, we can choose $\mu_0 > 0$ small enough such that for $\mu \in (0, \mu_0)$, $h(0) > h(1)$. Thus, $h$ cannot attain global minimum at 0, if $\mu \in (0, \mu_0)$. Hence $\tau_0 > 0$.

Consequently, $S_{0,0,0} + o(1) \geq h(\tau_0)S$, and as $o(1) \to 0$ as $n \to \infty$, we get $S_{0,0,0} \geq h(\tau_0)S$. On the other hand, choosing $(u, v) = (\tau_0 u, \tau_0 v)$, we easily see that $S_{0,0,0} \leq h(\tau_0)S$. Hence $S_{0,0,0} = h(\tau_0)S$.

Since $\tau_0$ is the minimum point for $h$, clearly $h(\tau_0) = 0$. Thus $\tau_0$ satisfies

$$\tau \left( \mu 2^* + \alpha \tau^\beta - \beta \tau^{\beta-2} \right) = 0.$$ 

But $\tau_0 > 0$, and so $\tau_0$ satisfies $\mu 2^* + \alpha \tau^\beta - \beta \tau^{\beta-2} = 0$. This proves (a).

(b) Note that for $(u, v) = (r w, r \tau_0 w)$, an easy computation yields

$$\frac{(\mu \|u\|_{H^s})^2}{\mu \|w\|_{H^s}} + \frac{\|v\|_{H^s}}{\|w\|_{H^s}} \frac{\|u\|_{H^s}}{\|w\|_{H^s}} = h(\tau_0)S.$$ 

Hence using (a), we conclude that $S_{0,0,0}$ is achieved by $(r w, r \tau_0 w)$.

\[\square\]

**Proof of theorem 1.1.**

**Proof.** Suppose that $(u_0, v_0)$ and $w$ achieves $S_{\alpha, \beta}$ and $S$ respectively. We are going to prove that there are $r, t > 0$ such that

$$(u_0, v_0) = (r w, t w).$$

Claim.

(a) 

$$\int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \, dx = r^\alpha t^\beta \int_{\mathbb{R}^N} w^{2s} \, dx,$$

whenever $\frac{r}{t} = \sqrt{\frac{\alpha}{\beta}}$.

(b) There exists $r > 0$ such that

$$\int_{\mathbb{R}^N} |u_0|^{2s} \, dx = r^{2s} \int_{\mathbb{R}^N} w^{2s} \, dx.$$ 

Assuming the claim for a while, first we complete the proof.

Indeed, fix $r$ as found in claim (b) and set $t = r \sqrt{\beta/\alpha}$. Therefore, by lemma 1.1, $(r w, t w)$ achieves $S_{\alpha, \beta}$. Consequently, $(r w, t w)$ solves $(S)$ with $f = 0 = g$ and

$$\frac{\alpha}{2s} r^{\alpha-2} \beta \rho^\beta = 1 = \frac{\beta}{2s} r^{\beta-2}. \quad (2.3)$$

Now define $(u_1, v_1) = (\frac{\|u_0\|_{H^s}}{r^\alpha t^\beta}, \frac{\|v_0\|_{H^s}}{r^\alpha t^\beta})$. Then, by claim (a) we have

$$\|u_1\|_{H^s}^2 = \frac{1}{r^2} \|u_0\|_{H^s}^2 = \frac{\alpha}{2r^2} \int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \, dx = \frac{\alpha r^\alpha t^\beta}{2s r^2} \int_{\mathbb{R}^N} w^{2s} \, dx = \|w\|_{H^s}^2.$$
where for the last equality we have used (2.3). Similarly, it follows that
\[ \|v_1\|_{H^s}^2 = \|w\|_{H^s}^2. \]
Therefore
\[ \|u_1\|_{H^s}^2 = \|w\|_{H^s}^2 = \|v_1\|_{H^s}^2. \]
(2.4)
Further, using claim (b) in the definition of \(u_1\) yields
\[ \int_{\mathbb{R}^N} |u_1|^2 \, dx = \int_{\mathbb{R}^N} |w|^2 \, dx. \]
(2.5)
Combining (2.4) and (2.5), by the uniqueness result in the scalar case, see [16], we conclude that
\[ u_1 = w, \quad \text{that is} \quad u_0 = rw. \]
Now we prove that \(v_1 = w\). Indeed, by claim (a)
\[ \int_{\mathbb{R}^N} |w|^2 \, dx = \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx \leq \left( \int_{\mathbb{R}^N} |u_1|^2 \, dx \right)^{\frac{\alpha}{2}} \left( \int_{\mathbb{R}^N} |v_1|^2 \, dx \right)^{\frac{\beta}{2}} = \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{\alpha}{2}} \left( \int_{\mathbb{R}^N} |v|^2 \, dx \right)^{\frac{\beta}{2}}. \]
Consequently, \( \|w\|_{L^2}^2 \leq \|v_1\|_{L^2}^2 \). Combining this with (2.4) and the fact that \(w\) achieves \(S\), we obtain
\[ S^{-1/2} \|v_1\|_{H^s} = S^{-1/2} \|u\|_{H^s} = \|w\|_{L^2} \leq \|v_1\|_{L^2} \leq S^{-1/2} \|v_1\|_{H^s}. \]
Hence the inequality becomes equality in the above expression, i.e., \(v_1\) achieves \(S\). Again by the uniqueness result in the scalar case, we conclude that
\[ v_1 = w, \quad \text{that is} \quad v_0 = tw. \]
This proves theorem 1.1. Now we are going to prove the claim. First, we prove claim (a).
Consider the following problem with a parameter \(\mu > 0\)
\[ \begin{cases} (-\Delta)^su = \frac{\mu \alpha}{2^s} |u|^\alpha - 2 |v|^\beta \quad \text{in } \mathbb{R}^N, \\ (-\Delta)^sv = \frac{\mu \beta}{2^s} |v|^\beta - 2 |u|^\alpha \quad \text{in } \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N. \end{cases} \]
\((S_\mu)\)
Associated to \((S_\mu)\), define the following min-max problem
\[ B(\mu) := \inf_{(u,v) \in H^s \times H^s \setminus \{(0,0)\}} \max_{t>0} E_{\mu}(tu, tv), \]
where
\[ E_{\mu}(u, v) := \frac{1}{2} \|(u, v)\|^2_{H^s \times H^s} - \frac{\mu}{2^s} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx. \]
Note that there exists \( t_\mu > 0 \) such that
\[
\max_{t \leq 0} E_{t_\mu}(t u_0, t v_0) = E_{t_\mu}(t_\mu u_0, t_\mu v_0),
\]
where \( t_\mu \) satisfies
\[
H(\mu, t_\mu) = 0 \quad \text{and} \quad H(\mu, t) := C - \mu D_\mu^2 - 2,
\]
with
\[
C = \|(u_0, v_0)\|_{H^{1, r}}^2 \quad \text{and} \quad D = \int_{\mathbb{R}^N} |u_0|^{p} |v_0|^q \, dx.
\]
Since \((u_0, v_0)\) is a least energy solution of \((S)\) with \( f = 0 = g \),
\[
H(1, 1) = 0, \quad \frac{\partial}{\partial t} H(1, 1) < 0 \quad \text{and} \quad H(\mu, t_\mu) = 0.
\]
By the implicit function theorem \( t_\mu \) is a \( C^1 \) function near of \( \mu = 1 \), and
\[
t_\mu(\mu) = 1 + t'_\mu(1)(\mu - 1) + O(|\mu - 1|^2)
\]
Consequently,
\[
t_\mu^2(\mu) = 1 + 2t'_\mu(1)(\mu - 1) + O(|\mu - 1|^2).
\]
Further, as \( H(\mu, t_\mu) = C - \mu D_\mu^2 - 2 = 0 \) and \( C = D \), we have \( t_\mu^2 - 2 = -1 \). Therefore,
\[
B(\mu) \leq E_{t_\mu}(t_\mu u_0, t_\mu v_0) = \frac{1}{2} \left( \frac{1 - \mu t_\mu^2 - 2}{2^*} \right) \|(u_0, v_0)\|_{H^{1, r}}^2
\]
\[
= \frac{t_\mu^2}{N} (\|(u_0, v_0)\|_{H^{1, r}}^2) = \frac{t_\mu^2}{N} B(1)
\]
\[
= B(1) - \frac{2}{2^* - 2} B(1)(\mu - 1) + O(|\mu - 1|^2). \tag{2.6}
\]
From the definition of \( B(1) \), a direct computation yields
\[
B(1) = \inf_{(u, v) \in H^{1, r}(\mathbb{R}^N) \setminus \{(0, 0)\}} E_1(u_0, v_0), \quad \text{where} \quad i = \left( \frac{\|(u, v)\|_{H^{1, r}}^2}{\int_{\mathbb{R}^N}|u|^p |v|^q \, dx} \right)^{1/(2^* - 2)}
\]
\[
= \inf_{(u, v) \in H^{1, r}(\mathbb{R}^N) \setminus \{(0, 0)\}} \frac{s}{N} \left( \frac{\|(u_0, v_0)\|_{H^{1, r}}^2}{\int_{\mathbb{R}^N}|u_0|^p |v_0|^q \, dx} \right)^{2^*/(2^* - 2)}
\]
\[
= \frac{s}{N} \left( \frac{\|(u_0, v_0)\|_{H^{1, r}}^2}{\int_{\mathbb{R}^N}|u_0|^p |v_0|^q \, dx} \right)^{2^*/(2^* - 2)} = s \frac{D}{N}. \tag{2.7}
\]
Substituting the above value of $B(1)$ in (2.6) yields

$$B(\mu) \leq B(1) - \frac{D}{2s}(\mu - 1) + O(|\mu - 1|^2).$$

Thus

$$B(\mu) - B(1) = -\frac{D}{2s}(\mu - 1) + O(|\mu - 1|) \text{ if } \mu > 1 \quad \text{(2.8)}$$

The first inequality in (2.8) implies $B'(1) \leq -\frac{D}{2s}$ and the second inequality in (2.8) implies $B'(1) \geq -\frac{D}{2s}$. Hence,

$$B'(1) = -\frac{D}{2s} = -\frac{1}{2s} \int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \, dx. \quad (2.9)$$

On the other hand, proceeding as in (2.7), we derive that

$$B(\mu) = \frac{s}{N} \frac{1}{\mu^s - t} \inf_{(u,v) \in H^\alpha \times H^\beta \setminus \{(0,0)\}} \left( \frac{\| (u,v) \|_{H^\alpha \times H^\beta}^2}{\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx} \right)^{\frac{2}{s}}$$

$$= \frac{s}{N} \frac{1}{\mu^s - t} S_{r,t}^{* \mu, \beta}.$$

Since, $(rw, tw)$ (for any $r, t > 0$ with $r/t = \sqrt{\alpha/\beta}$) is also a ground state solution of $(S)$ with $f = 0 = g$, from the above expression of $B(\mu)$, we obtain

$$B(\mu) = \frac{s}{N} \frac{1}{\mu^s - t} r^\alpha t^\beta \int_{\mathbb{R}^N} |w|^{2s} \, dx \quad \Rightarrow \quad B'(1) = -\frac{r^\alpha t^\beta}{2s} \int_{\mathbb{R}^N} |w|^{2s} \, dx.$$

Comparing this with (2.9), we conclude that

$$\int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \, dx = r^\alpha t^\beta \int_{\mathbb{R}^N} |w|^{2s} \, dx,$$

where $r, t > 0$ are arbitrary with $r/t = \sqrt{\alpha/\beta}$. This proves claim (a).

Let us turn to the proof of claim (b). Let $\mu_0$ be as in lemma 2.1. Consider the system (2.1) with $\mu \in (0, \mu_0)$ and define the following min-max problem

$$\bar{B}(\mu) := \inf_{(u,v) \in H^\alpha \times H^\beta \setminus \{(0,0)\}} \max_{t > 0} E_\mu(tu, tv),$$

where

$$E_\mu(u, v) := \frac{1}{2} \| (u, v) \|^2_{H^\alpha \times H^\beta (\mathbb{R}^N)} - \frac{\mu}{2s} \int_{\mathbb{R}^N} |u|^{2s} \, dx - \frac{1}{2s} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx.$$

Note that there exists $t_\mu > 0$ such that

$$\max_{t > 0} E_\mu(tu_0, tv_0) = E_\mu(t_\mu u_0, t_\mu v_0).$$
where \( t_\mu \) satisfies
\[
\tilde{H}(\mu, t_\mu) = 0 \quad \text{and} \quad \tilde{H}(\mu, t) = C - (\mu G + D)t^{2^*_s - 2}
\]
with
\[
C = \| (u_0, v_0) \|_{H^s_x L^p}^2 = \int_{\mathbb{R}^N} |u_0|^{2^*_s} \, dx \quad \text{and} \quad D = \int_{\mathbb{R}^N} |u_0|^{\alpha} |v_0|^{\beta} \, dx.
\]
Since \((u_0, v_0)\) is a ground state solution of \((S)\) with \( f = g = 0 \),
\[
\tilde{H}(0, 1) = C - D = 0, \quad \frac{\partial}{\partial t} \tilde{H}(0, 1) = -(2^{s^*_s} - 2)D \quad \text{and} \quad \frac{\partial}{\partial \mu} \tilde{H}(0, 1) = -G,
\]
evaluated at \( t = 1 \) and \( \mu = 0 \).

By the implicit function theorem \( t_\mu \) is a \( C^1 \) function near of \( \mu = 0 \), and
\[
t'_\mu |_{\mu=0} = - \frac{\partial^2 \tilde{H}}{\partial \mu \partial t} |_{\mu=0, t=1} = - \frac{\partial \tilde{H}}{\partial t} |_{\mu=0, t=1} = \frac{G}{(2^{s^*_s} - 2)D}.
\]
The Taylor formula around \( \mu = 0 \) and \( t_\mu = 1 \) yields
\[
t_\mu(\mu) = 1 + \mu t'_\mu(0) + O(|\mu|^2),
\]
consequently,
\[
t^2_\mu(\mu) = 1 + 2 \mu t'_\mu(0) + O(|\mu|^2).
\]
Now \( \tilde{B}(0) = B(1) \), where \( B(.) \) is as defined in the proof of claim (a). Therefore, \( \tilde{B}(0) = \frac{\partial \tilde{B}}{\partial \mu} |_{\mu=0} \).

Since \( \tilde{H}(\mu, t_\mu) = C - (\mu G + D)t^{2^*_s - 2} = 0 \), and \( C = D \) using an argument as before, it follows that
\[
\tilde{B}(\mu) \leq \tilde{E}_\mu(t_\mu u_0, t_\mu v_0) = \frac{t^{2^*_s}_\mu}{2} C - \frac{t^{2^*_s}_\mu}{2} (\mu G + D) = \frac{t^{2^*_s}_\mu}{2} \frac{3D}{N} = \frac{t^{2^*_s}_\mu \tilde{B}(0)}{}
\]
\[
= \tilde{B}(0) - \frac{2G}{(2^{s^*_s} - 2)D} \mu \tilde{B}(0) + O(|\mu|^2)
\]
\[
= \tilde{B}(0) - \frac{1}{2^{s^*_s}} G \mu + O(|\mu|^2).
\]
Then
\[
\tilde{B}'(0) = \lim_{\mu \to 0} \frac{\tilde{B}(\mu) - \tilde{B}(0)}{\mu} = - \frac{G}{2^{s^*_s}} = - \frac{1}{2^{s^*_s}} \int_{\mathbb{R}^N} |u_0|^{2^*_s} \, dx.
\]
On the other hand, from the definition of \( \tilde{B}(\mu) \), a straightforward computation yields
\[
\tilde{B}'(0) = \lim_{\mu \to 0} \frac{\tilde{B}(\mu) - \tilde{B}(0)}{\mu} = - \frac{G}{2^{s^*_s}} = - \frac{1}{2^{s^*_s}} \int_{\mathbb{R}^N} |u_0|^{2^*_s} \, dx.
\]
\[ \tilde{B}(\mu) = \inf_{(u, v) \in H^s \times H^s \setminus \{(0, 0)\}} \tilde{E}_\mu(\tilde{u}, \tilde{v}) \quad \text{where} \quad \tilde{r} = \left( \frac{\|(u, v)\|_{H^s}^2}{\mu \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N} |u| |v|^2 \, dx} \right)^{1/(2s^2 - 2)}. \]

\[ = \frac{s}{N} \inf_{(u, v) \in H^s \times H^s \setminus \{(0, 0)\}} \left[ \frac{\|(u, v)\|^2}{\left(\mu \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N} |u| |v|^2 \, dx\right)^{2/(2s^2)}} \right]^{2/(2s^2 - 2)}. \]

Since by lemma 2.1, \( S_{\mu, \alpha, \beta} \) is achieved by \((rw, \tau_0 r \alpha)\), an easy computation yields

\[ \tilde{B}(\mu) = \frac{s}{N} \left( \frac{1 + \tau_0^2}{\left(\mu + \tau_0^2\right)^{2s^2}} \right) \int_{\mathbb{R}^N} |u|^2 \, dx. \]

As a consequence,

\[ \tilde{B}'(0) = -\frac{1}{2s^2} \left( \frac{1 + \tau_0^2}{\tau_0^2} \right)^{2/(2s^2 - 2)} \int_{\mathbb{R}^N} |u|^2 \, dx. \]

Now set

\[ \tilde{r} = \left( \frac{1 + \tau_0^2}{\tau_0^2} \right)^{1/(2s^2 - 2)}. \]

Therefore, \( \tilde{B}'(0) = -\frac{2s^2}{2s^2} \int_{\mathbb{R}^N} |u|^2 \, dx. \) Comparing this with (2.10) yields

\[ \int_{\mathbb{R}^N} |u_0|^2 \, dx = \tilde{r}^{2s^2} \int_{\mathbb{R}^N} |u|^2 \, dx. \]

This proves claim (b). Thus, we conclude the proof of theorem 1.1. \( \square \)

3. The Palais–Smale decomposition

In this section we study the Palais–Smale sequences (in short, (PS) sequences) of the functional associated to \((S)\), namely,

\[ I_{f,g}(u, v) := \frac{1}{2} \|(u, v)\|_{H^s}^2 - \frac{1}{2s^2} \int_{\mathbb{R}^N} |u| |v|^2 \, dx - \langle \langle f, u \rangle \rangle_{H^s} - \langle \langle g, v \rangle \rangle_{H^s}. \]

We say that the sequence \( \{(u_n, v_n)\} \subset \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \) is a (PS) sequence for \( I_{f,g} \) at level \( \beta \) if \( I_{f,g}(u_n, v_n) \to \beta \) and \( I_{f,g}'(u_n, v_n) \to 0 \) in \((H^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N))'\). It is easy to see that the weak limit of a (PS) sequence of \( I_{f,g} \) solves \((S)\) except the positivity.

However the main difficulty is that the (PS) sequence may not converge strongly and hence the weak limit can be zero even if \( \beta > 0 \). The main purpose of this section is to classify (PS) sequences for the functional \( I_{f,g} \).

**Proposition 3.1.** Let \( \{(u_n, v_n)\} \subset \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \) be a (PS) sequence for \( I_{f,g} \) at a level \( \gamma \). Then there exists a subsequence (still denoted by \( \{(u_n, v_n)\} \)) for which the following
hold: there exist an integer $k \geq 0$, sequences $\{x_n^i\}_n \subset \mathbb{R}^N$, $r_n^i > 0$ for $1 \leq i \leq k$, pair of functions $(u, v), (\tilde{u}, \tilde{v}) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ for $1 \leq i \leq k$ such that $(u, v)$ satisfies $(S)$ without the sign restrictions and

\[
\begin{cases}
(-\Delta)^s \tilde{u}_i = \frac{\alpha}{\Sigma} |\tilde{u}_i|^{\alpha-2} \tilde{u}_i |\tilde{v}_i|^{\beta} \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s \tilde{v}_i = \frac{\beta}{2^{*}} |\tilde{v}_i|^{\beta-2} \tilde{v}_i |\tilde{u}_i|^{\alpha} \quad \text{in } \mathbb{R}^N,
\end{cases}
\]

(3.2)

\[(u_n, v_n) = (u, v) + \sum_{i=1}^{k} (\tilde{u}_i, \tilde{v}_i)^\gamma x_n^i + o(1),\]

where $(\tilde{u}_i, \tilde{v}_i)^\gamma := r^{-\frac{\alpha \beta}{2^{*}}} \left( \tilde{u}_i \left( \frac{x-y}{r} \right), \tilde{v}_i \left( \frac{x-y}{r} \right) \right)$

and $o(1) \to 0$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$,

\[
\gamma = I_{f,\gamma}(u, v) + \sum_{i=1}^{k} I_{0,0}(\tilde{u}_i, \tilde{v}_i) + o(1),
\]

\[r_n^i \to 0 \quad \text{and either } x_n^i \to x^i \in \mathbb{R}^N \quad \text{or } |x_n^i| \to \infty, \quad 1 \leq i \leq k,
\]

\[
\left| \log \left( \frac{r_n^i}{r_n^j} \right) \right| + \frac{x_n^i - x_n^j}{r_n^i} \to \infty \quad \text{for } i \neq j, \quad 1 \leq i, j \leq k,
\]

(3.3)

where in the case $k = 0$ the above expressions hold without $(\tilde{u}_i, \tilde{v}_i), x_n^i$ and $r_n^i$.

**Remark 3.1.** In the case (PS) condition holds for $I_{f,\gamma}$ for all levels $\gamma$ which cannot be decomposed as $\gamma = I_{f,\gamma}(u, v) + \sum_{i=1}^{k} I_{0,0}(\tilde{u}_i, \tilde{v}_i)$ where $k \geq 1$ and $(\tilde{u}_i, \tilde{v}_i)$ is a solution of (3.2). This can be seen, for instance with $k = 0$, proceeding as in the proof of step 3 of the proposition 3.1.

Before starting the proof of this proposition, we prove some lemmas which will be used in proving proposition 3.1.

**Lemma 3.1.** [8, theorem 2] Let $j : \mathbb{C} \to \mathbb{C}$ be a continuous function with $j(0) = 0$ and satisfy the following hypothesis that for every $\varepsilon > 0$, there exists two continuous functions $\varphi_\varepsilon$ and $\psi_\varepsilon$ such that

\[
|j(a + b) - j(a)| \leq \varepsilon \varphi_\varepsilon(a) + \psi_\varepsilon(b) \quad \forall a, b \in \mathbb{C}.
\]

Further, let $f_n = f + g_n$ be a sequence of measurable functions from $\mathbb{R}^N$ to $\mathbb{C}$ such that

(a) $g_n \to 0$ a.e.
(b) $j(f) \in L^1(\mathbb{R}^N)$.
(c) $\int_{\mathbb{R}^N} \varphi_\varepsilon(g_n(x)) \, dx \leq C < \infty$, for some constant $C$, independent of $\varepsilon$ and $n$.
(d) $\int_{\mathbb{R}^N} \psi_\varepsilon(f(x)) \, dx < \infty$ for all $\varepsilon > 0$.

Then

\[
\int_{\mathbb{R}^N} |j(f + g_n) - j(f) - j(g_n)| \, dx \to 0, \quad \text{as } n \to \infty.
\]
Lemma 3.2. Let $\alpha, \beta > 1$. Then for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\[
\|x + a|\alpha|y + b|\beta - |x|\alpha|y|\beta\| \leq \varepsilon (|x|^{\alpha+\beta} + |y|^{\alpha+\beta}) + C_\varepsilon (|a|^{\alpha+\beta} + |b|^{\alpha+\beta})
\]
holds for all $x, y, a, b \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$ be arbitrary. Then there exists $C_\varepsilon > 0$ such that
\[
|x + a|\alpha|y + b|\beta - |x|\alpha|y|\beta = (|y + b|\beta(|x + a|\alpha - |x|\alpha) + |x|\alpha(|y + b|\beta - |y|\beta)
\]
\[
\leq 2^{\beta-1}(|y|\beta + |b|\beta) \left( \frac{\varepsilon}{2^{\beta-1}} - \frac{1}{2} |x|\alpha + C_\varepsilon |a|\alpha \right)
\]
\[
+ |x|\alpha \left( \frac{\varepsilon}{2} |x|\beta + C_\varepsilon |b|\beta \right),
\]
\[
\leq \varepsilon \left( \frac{|x|\alpha|y|\beta + 1}{2} |b|\beta |x|\alpha \right) + C_\varepsilon \left( |x|\alpha|b|\beta
\]
\[
+ |y|\beta |a|\alpha + |a|\alpha |b|\beta \right)
\]
\[
\leq \varepsilon (|x|^{\alpha+\beta} + |y|^{\alpha+\beta}) + C_\varepsilon (|a|^{\alpha+\beta} + |b|^{\alpha+\beta}),
\]
where in the last inequality we have used Young’s inequality with different $\varepsilon$. This completes the proof. \qed

Lemma 3.3. If $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $\dot{H}^s(\mathbb{R}^N)$. Then
\[
\int_{\mathbb{R}^N} \left( |u_n|^\alpha |v_n|^{\beta} - |u|\alpha |v|^{\beta} - |u_n - u|\alpha |v_n - v|^{\beta} \right) \, dx = o(1).
\]

Proof. Define $j : \mathbb{R}^2 \to \mathbb{R}$ given by $j(x, y) = |x|\alpha |y|\beta$. Then $j$ satisfies the hypothesis of lemma 3.1. Next considering
\[
f_n := (u_n, v_n), \quad f = (u, v), \quad g_n = (u_n - u, v_n - v),
\]
we see that all the hypothesis of lemma 3.1 are satisfied. Hence the lemma follows from lemma 3.1. \qed

Lemma 3.4. Let \{$(u_n, v_n)$\} weakly converge to $(u, v)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ and pointwise a.e. in $\mathbb{R}^N \times \mathbb{R}^N$, then
\[
\int_{\mathbb{R}^N} |u_n|^{\alpha-2} u_n |v_n|^{\beta} \phi \, dx \longrightarrow \int_{\mathbb{R}^N} |u|^{\alpha-2} u |v|^{\beta} \phi \, dx \quad {\text{as}} \; n \to \infty, \quad (3.5)
\]
and
\[
\int_{\mathbb{R}^N} |u_n|^{\alpha} |v_n|^{\beta-2} v_n \psi \, dx \longrightarrow \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta-2} v \psi \, dx \quad {\text{as}} \; n \to \infty, \quad (3.6)
\]
for all $(\phi, \psi) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$.

Proof. Set
\[
M := \max \left\{ \|u_n\|_{L^{2^*_s}(\mathbb{R}^N)}^{\alpha-1}, \|v_n\|_{L^{2^*_s}(\mathbb{R}^N)}^{\beta}, \|u\|_{L^{2^*_s}(\mathbb{R}^N)}^{\alpha-1}, \|v\|_{L^{2^*_s}(\mathbb{R}^N)}^{\beta} \right\}.
\]

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Proof. We divide the proof into few steps. Proof of proposition

Using the Sobolev inequality we see that $M$ is well-defined. Let $\phi \in \dot{H}^s(\mathbb{R}^N)$ and $\varepsilon > 0$ be arbitrary. Then, there exists $R = R(\varepsilon) > 0$ such that \((\int_{B(0,R)} |\phi|^2 \, dx)^{1/2} < \frac{\varepsilon}{2M}\). Note that,

\[
\int_{\mathbb{R}^N} \left( |u_n|^{\alpha-2} u_n |v_n|^3 - |u|^{\alpha-2} |v|^3 \right) \phi \, dx
\]

\[
= \left( \int_{B(0,R)} + \int_{B(0,R)^c} \right) \left( |u_n|^{\alpha-2} u_n |v_n|^3 - |u|^{\alpha-2} |v|^3 \right) \phi \, dx
\]

and using the Hölder inequality

\[
\int_{B(0,R)^c} \left( |u_n|^{\alpha-2} u_n |v_n|^3 - |u|^{\alpha-2} |v|^3 \right) \phi \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^N} |u_n|^{2s} \, dx \right) \left( \int_{\mathbb{R}^N} |v_n|^{2s} \, dx \right)^{(\alpha-1)/2s} \left( \int_{B(0,R)} |\phi|^2 \, dx \right)^{\alpha/2s} \left( \int_{\mathbb{R}^N} |\phi|^2 \, dx \right)^{1/2s}
\]

\[
+ \left( \int_{\mathbb{R}^N} |u|^{2s} \, dx \right) \left( \int_{\mathbb{R}^N} |v|^{2s} \, dx \right)^{(\alpha-1)/2s} \left( \int_{B(0,R)^c} |\phi|^2 \, dx \right)^{\alpha/2s} \left( \int_{\mathbb{R}^N} |\phi|^2 \, dx \right)^{1/2s}
\]

\[
< \varepsilon.
\]

On the other hand, using the Hölder inequality as above, it is also easily checked that \((|u_n|^{\alpha-2} u_n |v_n|^3 - |u|^{\alpha-2} |v|^3) \phi\) is equi-integrable in $B(0, R)$. Therefore, applying the Vitali convergence theorem it follows that

\[
\lim_{n \to \infty} \int_{B(0,R)} \left( |u_n|^{\alpha-2} u_n |v_n|^3 - |u|^{\alpha-2} |v|^3 \right) \phi \, dx = 0.
\]

Hence the lemma follows. \(\square\)

Proof of proposition 3.1.

Proof. We divide the proof into few steps.

Step 1 Using standard arguments it follows that (PS) sequences for $I_{f,g}$ are bounded in $H^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$. More precisely, as $n \to \infty$

\[
\gamma + o(1) + o(1)(\|(u_n, v_n)\|_{H^s \times \dot{H}^s})
\]

\[
\geq I_{f,g}(u_n, v_n) - \frac{1}{2s} \|f\|_{\dot{H}^s \times \dot{H}^s} \|u_n\|_{H^s} \|v_n\|_{\dot{H}^s}
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{2s} \right) \|u_n\|_{H^s}^2 + \left( \frac{1}{2} - \frac{1}{2s} \right) \|g\|_{\dot{H}^s} \|v_n\|_{\dot{H}^s}
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{2s} \right) \|u_n\|_{H^s}^2 + \left( \frac{1}{2} - \frac{1}{2s} \right) \|g\|_{\dot{H}^s} \|v_n\|_{\dot{H}^s}
\]

\[
- \left( \frac{1}{2} - \frac{1}{2s} \right) \|f\|_{\dot{H}^s} \|u_n\|_{H^s} \|v_n\|_{\dot{H}^s}
\]

\[
- \left( \frac{1}{2} - \frac{1}{2s} \right) \|f\|_{\dot{H}^s} \|v_n\|_{\dot{H}^s} \|u_n\|_{H^s}
\]

\[
- \left( \frac{1}{2} - \frac{1}{2s} \right) \|g\|_{\dot{H}^s} \|u_n\|_{H^s} \|v_n\|_{\dot{H}^s}.
\]
This immediately implies that \( \{(u_n, v_n)\} \) is bounded in \( \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \). Consequently, up to a subsequence, \((u_n, v_n) \to (u, v)\) in \( \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \). Further, 
\[
(\dot{u}, \dot{v})_{\dot{H}^s \times \dot{H}^s} = 0
\]
implies
\[
\langle (u_n, v_n), (\phi, \psi) \rangle_{\dot{H}^s \times \dot{H}^s} - \frac{\alpha}{2s} \int_{\mathbb{R}^N} |u_n|^{\alpha-2} u_n |v_n|^2 \phi \, dx - \frac{\beta}{2s} \int_{\mathbb{R}^N} |v_n|^{\beta-2} v_n |u_n|^2 \psi \, dx
\]
\[
- \langle \dot{u}, \dot{\phi} \rangle_{\dot{H}^s} - \langle \dot{v}, \dot{\psi} \rangle_{\dot{H}^s} = o(1).
\]

Passing to the limit using lemma 3.4, we see that \((u, v)\) satisfies (S) without signed restrictions.

**Step 1** We show that \( \{(u_n - u, v_n - v)\} \) is a (PS) sequence for \( I_{0,0} \) at the level \( \gamma = I_{f,g}(u, v) \).

To see this, first we observe that as \( n \to \infty \)
\[
\|u_n - u\|_{\dot{H}^s}^2 = \|u_n\|_{\dot{H}^s}^2 - \|u\|_{\dot{H}^s}^2 + o(1), \quad \|v_n - v\|_{\dot{H}^s}^2 = \|v_n\|_{\dot{H}^s}^2 - \|v\|_{\dot{H}^s}^2 + o(1).
\]

Using this along with the fact that \((u_n, v_n) \to (u, v), f, g \in (\dot{H}^s(\mathbb{R}^N))'\) and lemma 3.3 yields
\[
I_{0,0}(u_n - u, v_n - v)
\]
\[
= \frac{1}{2} \|u_n - u, v_n - v\|_{\dot{H}^s \times \dot{H}^s}^2 - \frac{1}{2} \alpha \int_{\mathbb{R}^N} |u_n|^{\alpha} |v_n|^{\beta} \, dx + \frac{1}{2} \beta \int_{\mathbb{R}^N} |v_n|^{\beta} |u_n|^{\alpha} \, dx - \frac{1}{2} \alpha \int_{\mathbb{R}^N} |u_n|^{\alpha} |v_n|^{\beta} \, dv
\]
\[
+ \frac{1}{2} \beta \int_{\mathbb{R}^N} |v_n|^{\beta} |u_n|^{\alpha} \, dw + \frac{1}{2} \alpha \int_{\mathbb{R}^N} |u_n|^{\alpha} |v_n|^{\beta} \, dx - \|u_n - u\|_{\dot{H}^s}^2 - \|v_n - v\|_{\dot{H}^s}^2 + o(1)
\]
\[
= I_{f,g}(u_n, v_n) - I_{f,g}(u, v) + o(1).
\]

Next, as \((u_n - u, v_n - v) \to (0, 0)\) in \( \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \), applying lemma 3.4, we obtain
\[
(\dot{u} \times \dot{v})_{\dot{H}^s \times \dot{H}^s} = \langle (u_n - u, v_n - v) , (\phi, \psi) \rangle_{\dot{H}^s \times \dot{H}^s}
\]
\[
= \frac{\alpha}{2s} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^2 |u|^{\alpha-2} (u_n - u) v_n^\beta \phi \, dx
\]
\[
- \frac{\beta}{2s} \int_{\mathbb{R}^N} |v_n|^\beta |u_n|^2 |u_n - u|^{\alpha-2} v_n^\alpha \psi \, dx
\]
\[
= o(1).
\]

This completes step 2.

**Step 3** Rescaling of \( \{(u_n, v_n)\} \) in the nontrivial case.
If \((u_n, v_n) \to (u, v)\) in \(H'(\mathbb{R}^N) \times H'(\mathbb{R}^N)\), then the theorem is proved with \(k = 0\). Therefore, we assume \((u_n, v_n) \not\to (u, v)\) in \(H'(\mathbb{R}^N) \times H'(\mathbb{R}^N)\). Set,
\[
\tilde{u}_n := u_n - u, \quad \tilde{v}_n := v_n - v.
\]
Therefore, we are in the case where \((\tilde{u}_n, \tilde{v}_n) \not\to (0, 0)\) in \(H'(\mathbb{R}^N) \times H'(\mathbb{R}^N)\). Since, by step 2, \(
\{(\tilde{u}_n, \tilde{v}_n)\}\) is a bounded \((PS)\) sequence for \(I_{0,0}\), we have \(I'_{0,0}(\tilde{u}_n, \tilde{v}_n)\tilde{u}_n, \tilde{v}_n) = o(1)\). Therefore, up to a subsequence
\[
0 < \| (\tilde{u}_n, \tilde{v}_n) \|^2_{H'^2(\mathbb{R}^N) \times H'^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\tilde{u}_n|^2 \, dx \leq \int_{\mathbb{R}^N} |\tilde{u}_n|^2 \, dx + \int_{\mathbb{R}^N} |\tilde{v}_n|^2 \, dx \leq \| (\tilde{u}_n, \tilde{v}_n) \|^2_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)}.
\]
Thus \((\tilde{u}_n, \tilde{v}_n) \not\to (0, 0)\) in \(L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\). Consequently,
\[
\inf_n \| (\tilde{u}_n, \tilde{v}_n) \|_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)} \geq \delta \quad \text{for some } \delta > 0.
\]
Hence, applying lemma A.1,
\[
\delta \leq C \| (\tilde{u}_n, \tilde{v}_n) \|^\theta_{H'^2(\mathbb{R}^N) \times H'^2(\mathbb{R}^N)} \| (\tilde{u}_n, \tilde{v}_n) \|^{\frac{4-\theta}{2(N-2)}}_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)} \leq C \| (\tilde{u}_n, \tilde{v}_n) \|^\theta_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)},
\]
that is,
\[
\| (\tilde{u}_n, \tilde{v}_n) \|^\theta_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)} \geq C_1 \quad \text{for some } C_1 > 0.
\]
Comparing the above inequality with (A.3) yields existence of some \(C > 0\) such that
\[
\tilde{C} \leq \| (\tilde{u}_n, \tilde{v}_n) \|^\theta_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)} \leq \tilde{C}^{-1},
\]
that is
\[
\tilde{C} \leq \sup_{x \in \mathbb{R}^N, R > 0} R^{N-2s} \int_{B(x,R)} (|\tilde{u}_n|^2 + |\tilde{v}_n|^2) \, dy \leq \tilde{C}^{-1}.
\]
As a result, for every \(n \in \mathbb{N}\), there exists \(x_n \in \mathbb{R}^N\) and \(r_n > 0\) such that
\[
r_n^{N-2s} \int_{B(x_n, r_n)} (|\tilde{u}_n|^2 + |\tilde{v}_n|^2) \, dy \geq \| (\tilde{u}_n, \tilde{v}_n) \|^\theta_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)} = \frac{C}{2n} \geq \tilde{C} \geq 0.
\]
Now define
\[
\tilde{u}_n := \frac{N-2s}{r_n} \tilde{u}_n(r_n x + x_n), \quad \tilde{v}_n := \frac{N-2s}{r_n} \tilde{v}_n(r_n x + x_n).
\]
In view of the scaling invariance of the \(H'(\mathbb{R}^N)\) norm and \(L^2(\mathbb{R}^N)\) norm, \((\tilde{u}_n, \tilde{v}_n)\) is a bounded sequence in \(H'(\mathbb{R}^N) \times H'(\mathbb{R}^N)\) and up to a subsequence
\[
(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v}) \quad \text{in } H'(\mathbb{R}^N) \times H'(\mathbb{R}^N) \quad \text{and}
\]
\((\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v})\) in \(L^2_{0, \text{loc}}(\mathbb{R}^N) \times L^2_{0, \text{loc}}(\mathbb{R}^N)\).

Therefore, using change of variable, we observe from (3.8) that

\[
0 < r_n^{-2s} \int_{B(0, r_n)} (|\tilde{u}_n|^2 + |\tilde{v}_n|^2) \, dy = \int_{\mathbb{R}^1} (|\tilde{u}_n(z)|^2 + |\tilde{v}_n(z)|^2) \, dz \\
\to \int_{\mathbb{R}^1} (|\tilde{u}|^2 + |\tilde{v}|^2) \, dx.
\]

Hence \((\tilde{u}, \tilde{v}) \neq (0, 0)\). Clearly, up to a subsequence, either \(x_n \to x_0 \in \mathbb{R}^N\) or \(|x_n| \to \infty\). Further, as \((\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v}) \neq (0, 0)\) in \(\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)\) and \((\tilde{u}_n, \tilde{v}_n) \to (0, 0)\) in \(H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)\), we infer that \(r_n \to 0\).

Step 4 In this step we prove that \((\tilde{u}, \tilde{v})\) solves

\[
\left\{ \begin{array}{l}
(-\Delta)^s \tilde{u} = \frac{\alpha}{2s} |\tilde{u}|^{\alpha-2} \tilde{u} \tilde{v}^\beta \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s \tilde{v} = \frac{\beta}{2s} |\tilde{v}|^{\beta-2} \tilde{v} \tilde{u} \quad \text{in } \mathbb{R}^N.
\end{array} \right.
\]

(3.9)

To this aim, it is enough to show that for arbitrary \((\varphi, \psi) \in C_0^{\infty}(\mathbb{R}^N) \times C_0^{\infty}(\mathbb{R}^N)\) it holds

\[
(\tilde{u}, \varphi)_{\dot{H}^s} + (\tilde{v}, \psi)_{\dot{H}^s} = \frac{\alpha}{2s} \int_{\mathbb{R}^N} |\tilde{u}|^{\alpha-2} \tilde{u} \varphi \, dx + \frac{\beta}{2s} \int_{\mathbb{R}^N} |\tilde{v}|^{\beta-2} \tilde{v} \psi \, dx.
\]

Let \(\varphi, \psi \in C_0^{\infty}(\mathbb{R}^N)\) be arbitrary. As \((\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v})\) in \(\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)\), using change of variables and step 2, that is \((\tilde{u}_n, \tilde{v}_n)\) is a (PS) sequence for \(I_{0,0}\), we deduce

\[
\begin{align*}
(\tilde{u}, \varphi)_{\dot{H}^s} + (\tilde{v}, \psi)_{\dot{H}^s} &= \lim_{n \to \infty} \left( (\tilde{u}_n, \varphi)_{\dot{H}^s} + (\tilde{v}_n, \psi)_{\dot{H}^s} \right) \\
&= \lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{r_n^{2s}}{r_n^{2s}} \left( \tilde{u}_n(r_n x + x_n) - \tilde{u}_n(y) \right) \left( \varphi \left( \frac{x-y}{r_n} \right) - \varphi(y) \right) \, dx \\
&+ \lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{r_n^{2s}}{r_n^{2s}} \left( \tilde{v}_n(r_n x + x_n) - \tilde{v}_n(y) \right) \left( \psi \left( \frac{x-y}{r_n} \right) - \psi(y) \right) \, dx \\
&= \lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{r_n^{2s}}{r_n^{2s}} \left( \tilde{u}_n(x) - \tilde{u}_n(y) \right) \left( \varphi \left( \frac{x-y}{r_n} \right) - \varphi(y) \right) \, dx \\
&+ \lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{r_n^{2s}}{r_n^{2s}} \left( \tilde{v}_n(x) - \tilde{v}_n(y) \right) \left( \psi \left( \frac{x-y}{r_n} \right) - \psi(y) \right) \, dx \\
&= \lim_{n \to \infty} \left[ \frac{\alpha}{2s} \int_{\mathbb{R}^N} |\tilde{u}_n|^{\alpha-2} |\tilde{u}_n|^{\beta} \varphi_n \, dx + \frac{\beta}{2s} \int_{\mathbb{R}^N} |\tilde{v}_n|^{\beta-2} |\tilde{v}_n|^{\beta} \psi_n \, dx \right],
\end{align*}
\]

(3.10)
where 
\[ \tilde{\varphi}_n(x) := r_n \tilde{\varphi} \left( \frac{x - x_n}{r_n} \right) \quad \text{and} \quad \tilde{\psi}_n(x) := r_n \tilde{\psi} \left( \frac{x - x_n}{r_n} \right). \]

Again applying a change of variables to (3.10) yields us
\[
\text{RHS of (3.10)} = \lim_{n \to \infty} \left[ \frac{\alpha}{2^s} \int_{\mathbb{R}^N} |\tilde{u}_n|^{\alpha - 2} |\tilde{u}_n| \varphi \, dx + \beta \int_{\mathbb{R}^N} |\tilde{\varphi}_n|^{\beta - 2} |\tilde{\varphi}_n| \psi \, dx \right]
= \frac{\alpha}{2^s} \int_{\mathbb{R}^N} |\tilde{u}|^{\alpha - 2} |\tilde{u}| \varphi \, dx + \beta \int_{\mathbb{R}^N} |\tilde{\varphi}|^{\beta - 2} \varphi \, dx,
\]
where the last equality is obtained by lemma 3.4. This completes step 4.

Now define,
\[
\begin{align*}
 w_n(x) &:= \tilde{u}_n(x) - r_n \tilde{\varphi} \left( \frac{x - x_n}{r_n} \right) \quad \text{and} \\
 z_n(x) &:= \tilde{v}_n(x) - r_n \tilde{\psi} \left( \frac{x - x_n}{r_n} \right).
\end{align*}
\tag{3.11}
\]

Step 5 In this step we show that \( \{ (w_n, z_n) \} \) is a (PS) sequence for \( I_{0,0} \) at the level \( \gamma - I_{f}(u, v) - I_{0,0}(\tilde{u}, \tilde{v}). \)

To prove that, first we set
\[
\tilde{w}_n := r_n \tilde{w}(r_n x + x_n), \quad \text{and} \quad \tilde{z}_n := r_n \tilde{z}(r_n x + x_n).
\tag{3.12}
\]

Combining (3.11) and (3.12) yields
\[
\tilde{w}_n = \tilde{u}_n - \tilde{u}, \quad \tilde{z}_n = \tilde{v}_n - \tilde{v},
\]
and from the scaling invariance in the norm of \( \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \) gives
\[
\| (w_n, z_n) \|_{\dot{H}^s \times \dot{H}^s} = \| (\tilde{w}_n, \tilde{z}_n) \|_{\dot{H}^s \times \dot{H}^s} = \| (\tilde{u}_n - \tilde{u}, \tilde{v}_n - \tilde{v}) \|_{\dot{H}^s \times \dot{H}^s}.
\]

A straightforward computation using the above equality, change of variables and lemma 3.3 yields
\[
\begin{align*}
I_{0,0}(w_n, z_n) &= \frac{1}{2} \| \tilde{u}_n - \tilde{u} \|_{\dot{H}^s}^2 + \frac{1}{2} \| \tilde{v}_n - \tilde{v} \|_{\dot{H}^s}^2 - \frac{1}{2} \int_{\mathbb{R}^N} |w_n|^{\alpha} |z_n|^{\beta} \, dx \\
&= \frac{1}{2} \left( \| \tilde{u}_n \|_{\dot{H}^s}^2 - \| \tilde{u} \|_{\dot{H}^s}^2 + \| \tilde{v}_n \|_{\dot{H}^s}^2 - \| \tilde{v} \|_{\dot{H}^s}^2 + o(1) \right) \\
&\quad - \frac{1}{2^s} \int_{\mathbb{R}^N} |\tilde{u}_n - \tilde{u}|^{\alpha} |\tilde{v}_n - \tilde{v}|^{\beta} \, dx \\
&= \frac{1}{2} \| (\tilde{u}_n, \tilde{v}_n) \|_{\dot{H}^s \times \dot{H}^s}^2 - \frac{1}{2} \| (\tilde{u}, \tilde{v}) \|_{\dot{H}^s \times \dot{H}^s}^2 + o(1)
\end{align*}
\]
where in the last equality we have used step 2. Now, to complete the proof of step 5, it remains to show that \( \langle I_{0,0}(w_n, z_n)(\varphi, \psi) \rangle = 0 \) for all \((\varphi, \psi) \in C_{c}^{\infty}(\mathbb{R}^N) \times C_{c}^{\infty}(\mathbb{R}^N)\). Let \((\varphi, \psi) \in C_{c}^{\infty}(\mathbb{R}^N) \times C_{c}^{\infty}(\mathbb{R}^N)\) be arbitrary and set
\[
\varphi_n := r_n^{-\frac{\alpha}{2}} \varphi(r_n x + x_n), \quad \psi_n := r_n^{-\frac{\beta}{2}} \psi(r_n x + x_n).
\]
Thus \( \varphi_n \rightarrow 0 \) and \( \psi_n \rightarrow 0 \) in \( H'(\mathbb{R}^N) \) as \( r_n \rightarrow 0 \). Observe that applying change of variables,
\[
\langle (w_n, z_n), (\varphi, \psi) \rangle_{H^s \times H^s} = \langle w_n, \varphi \rangle_{H^s} + \langle z_n, \psi \rangle_{H^s},
\]
\[
= \langle \tilde{u}_n, \varphi_n \rangle_{H^s} + \langle \tilde{z}_n, \psi_n \rangle_{H^s},
\]
\[
= \langle \tilde{u}_n - \tilde{u}, \varphi_n \rangle_{H^s} + \langle \tilde{v}_n - \tilde{v}, \psi_n \rangle_{H^s}.
\]
Therefore,
\[
\langle I_{0,0}'(w_n, z_n)(\varphi, \psi) \rangle = \langle \tilde{u}_n - \tilde{u}, \varphi_n \rangle_{H^s} + \langle \tilde{v}_n - \tilde{v}, \psi_n \rangle_{H^s},
\]
\[
- \frac{\alpha}{2} \int_{\mathbb{R}^N} |\tilde{u}_n - \tilde{u}|^{\alpha-2} (\tilde{u}_n - \tilde{u}) |\tilde{v}_n - \tilde{v}|^{\beta} \varphi_n \, dx,
\]
\[
- \frac{\beta}{2} \int_{\mathbb{R}^N} |\tilde{v}_n - \tilde{v}|^{\beta-2} (\tilde{v}_n - \tilde{v}) |\tilde{v}_n - \tilde{v}|^{\beta} \varphi_n \, dx,
\]
\[
= o(1),
\]
where the last equality follows by change of variables and an argument similar to step 2. This concludes step 5.

Now, starting from a (PS) sequence \( \{ (\tilde{u}_n, \tilde{v}_n) \} \) for \( I_{0,0} \) we have extracted another (PS) sequence \( \{ (w_n, z_n) \} \) at a level which is strictly lower than the previous one, with a fixed minimum amount of decrease (as it is easy to check that \( I_{0,0}(\tilde{u}, \tilde{v}) \geq \frac{N}{2s} \| u \|_{H^s}^2 \)). On the other hand, as \( \sup_p \| (\tilde{u}_n, \tilde{v}_n) \|_{H^s \times H^s} \leq C \) (finite), this process terminates after a finite number of steps and the last (PS) sequence strongly converges to 0. Further, \( \log \left( \frac{1}{r_n} \right) \left| 1 + \frac{\alpha r_n^s}{||} \right| \rightarrow \infty \) for \( i \neq j, \ 1 \leq i, j \leq m \) (see [34, theorem 1.2]). This completes the proof. \( \square \)

### 4. Multiplicity in the nonhomogeneous case

In this section we aim to prove theorem 1.2. For that first we would like to establish existence of two positive critical points of the functional
\[
J_{f,g}(u,v) = \frac{1}{2} \| (u, v) \|_{H^s \times H^s}^2 - \frac{1}{2} \int_{\mathbb{R}^N} u^3 v^4 \, dx - \langle \mathcal{H}_{f,g}(u), u \rangle_{H^s} - \langle \mathcal{H}_{f,g}(v), v \rangle_{H^s}
\]
(4.1)
where \( f, g \) are nontrivial nonnegative functionals on \( (H^s(\mathbb{R}^N))' \) with \( \ker(f) = \ker(g) \).

**Remark 4.1.** If \((u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)\) is a nontrivial critical point of \( J_{f,g} \) then \((u, v)\) solves

\[
\begin{aligned}
(\Delta)^{\gamma} u &= \frac{\alpha}{2\gamma} u^{\alpha-1} v^\beta + f(x) \quad \text{in } \mathbb{R}^N, \\
(\Delta)^{\gamma} v &= \frac{\beta}{2\gamma} u^\alpha v^{\beta-1} + g(x) \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]  

(4.2)

Note that taking \((\phi, \psi) = (u_-, v_-)\) as a test function in (4.2), we obtain

\[
-\|\langle u_-, v_-\rangle\|^2_{H^s \times H^s} = \int_{\mathbb{R}^N} \frac{[u_+(y)u_-(x) + u_+(x)u_-(y)]}{|x-y|^{N+2\gamma}} \, dx \, dy - \int_{\mathbb{R}^N} \frac{[v_+(y)v_-(x) + v_+(x)v_-(y)]}{|x-y|^{N+2\gamma}} \, dx \, dy
\]

which in turn implies \( u_- = 0 \) and \( v_- = 0 \). Therefore \( u, v \geq 0 \) and \((u, v)\) is a solution of (S) without strict positivity condition.

Next, we assert that \((u, v) \neq (0, 0)\) implies \( u \neq 0 \) and \( v \neq 0 \). Suppose not, that is assume for instance that \( u \neq 0 \) but \( v = 0 \). Then taking \((\phi, \psi) = (u, 0)\) as test function we get \( \|u\|^2_{H^s(\mathbb{R}^N)} = \langle i_{H^s}(f, u)_{H^s}, u \rangle_{H^s} \). Further choosing \((\phi, \psi) = (0, u)\) as test function, we have \( \langle i_{H^s}(g, u)_{H^s}, u \rangle_{H^s} = 0 \). These together with the hypothesis that \( \ker(f) = \ker(g) \) implies \( \|u\|_{H^s} = 0 \). This contradicts \((u, v) \neq (0, 0)\). Similarly we can show that if \( u = 0 \) then \( v = 0 \) too. Hence our assertion follows. Next, we claim that \( u > 0 \), and \( v > 0 \) in \( \mathbb{R}^N \). Taking \((\phi, 0)\) as test function where \( \phi \geq 0 \) in \( H^s(\mathbb{R}^N) \) we get,

\[
\langle u, \phi \rangle_{H^s} = \frac{\alpha}{2\gamma} \int_{\mathbb{R}^N} u^{\alpha-1} v^\beta \phi \, dx + \langle i_{H^s}(f, \phi)_{H^s}, \phi \rangle \geq 0.
\]

This implies \( 0 \leq u \in H^s(\mathbb{R}^N) \) is a weak supersolution to \((-\Delta)^{\gamma} u = 0 \). Therefore applying maximum principle [22, theorem 1.2(2)], with \( \epsilon = 0 \) and \( p = 2 \) there, we obtain \( u > 0 \) in \( \mathbb{R}^N \). Similarly we can show that \( v > 0 \) in \( \mathbb{R}^N \). Hence, if \((u, v)\) is a critical point of \( J_{f,g} \) then \((u, v)\) is a solution of (S).

To prove, existence of two critical points for \( J_{f,g} \), first we decompose the space \( H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \) into three disjoint sets via the function \( \Psi : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R} \) defined by

\[
\Psi(u, v) := \|(u, v)\|^2_{H^s \times H^s} - (2^{*}_s - 1) \int_{\mathbb{R}^N} |u|^{2^*_s} |v|^{2^*_s} \, dx.
\]

Set

\[
\begin{align*}
\Omega_1 &:= \{(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) : (u, v) = (0, 0) \text{ or } \Psi(u, v) > 0\}, \\
\Omega_2 &:= \{(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) : \Psi(u, v) < 0\}, \\
\Omega &:= \{(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \setminus \{(0, 0)\} : \Psi(u, v) = 0\}.
\end{align*}
\]
Put
\[ c_0 := \inf_{\Omega} J_{f,g}(u, v), \quad c_1 := \inf_{\Omega} J_{f,g}(u, v). \quad (4.3) \]

**Remark 4.2.** Note that for all \( \lambda > 0 \) and \((u, v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^p(\mathbb{R}^N)\)
\[ \Psi(\lambda u, \lambda v) = \lambda^2 \| (u, v) \|_{\dot{H}^s, \dot{H}^p}^2 - \lambda^2 (2^*_s - 1) \int_{\mathbb{R}^N} |u|^a |v|^b \, dx. \]
Moreover, \( \Psi(0, 0) = 0 \) and \( \lambda \mapsto \Psi(\lambda u, \lambda v) \) is a strictly concave function in \( \mathbb{R}^+ \). Thus for any \((u, v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^p(\mathbb{R}^N) \) with \( \| (u, v) \|_{\dot{H}^s, \dot{H}^p} = 1 \), there exists a unique \( \lambda \) (\( \lambda \) depends on \( (u, v) \)) such that \((\lambda u, \lambda v) \in \Omega \). Moreover as
\[ \Psi(\lambda u, \lambda v) = (\lambda^2 - \lambda^2 s^*_s) \| (u, v) \|_{\dot{H}^s, \dot{H}^p}^2 \quad \text{for all} \quad (u, v) \in \Omega, \]
\( (\lambda u, \lambda v) \in \Omega_1 \) for all \( \lambda \in (0, 1) \) and \((\lambda u, \lambda v) \in \Omega_2 \) for all \( \lambda > 1 \).

**Lemma 4.1.** Assume \( C_0 \) is defined as in theorem 1.2 and \( c_0 \) and \( c_1 \) are defined as in (4.3).
Further, if
\[ \inf_{(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^p(\mathbb{R}^N) \quad f_{g}|a|v|^b \quad dx=1} \left\{ C_0 \| (u, v) \|_{\dot{H}^s, \dot{H}^p}^{n+2s} - d \inf \left( \int_{\mathbb{R}^N} |u|^a |v|^b \, dx \right)^{\frac{n+2s}{n}} - d \right. \]
\[ \left. - \frac{\| (u, v) \|_{\dot{H}^s, \dot{H}^p}^{n+2s}}{\| (u, v) \|_{\dot{H}^s, \dot{H}^p}^{n+2s}} \right) > 0. \quad (4.4) \]
then \( c_0 < c_1 \).

**Proof.**

Step 1 First we assert that there exists \( \delta > 0 \) such that
\[ \left. \frac{d}{dr} J_{f,g}(tu, tv) \right|_{r=1} \geq \delta \quad \forall (u, v) \in \Omega. \]

Doing a straightforward computation, it is easy to see that for any \((u, v) \in \Omega\)
\[ \left. \frac{d}{dr} J_{f,g}(tu, tv) \right|_{r=1} = \frac{4s}{N+2s} \| (u, v) \|_{\dot{H}^s, \dot{H}^p}^2 \bigg( \int_{\mathbb{R}^N} |u|^a |v|^b \, dx \bigg)^{\frac{n+2s}{n+2b}} \]
\[ = C_0 \| (u, v) \|_{\dot{H}^s, \dot{H}^p}^{n+2s} \bigg( \int_{\mathbb{R}^N} |u|^a |v|^b \, dx \bigg)^{\frac{n+2s}{n}} \left( \int_{\mathbb{R}^N} f_{g}|a|v|^b \, dx \right)^{\frac{n+2s}{n}} \]
\[ - \| (u, v) \|_{\dot{H}^s, \dot{H}^p}^2 \bigg( \int_{\mathbb{R}^N} f_{g}|a|v|^b \, dx \bigg)^{\frac{n+2s}{n}} - \left( \int_{\mathbb{R}^N} f_{g}|a|v|^b \, dx \right)^{\frac{n+2s}{n}}. \quad (4.5) \]

Further, (4.4) implies there exists \( d > 0 \) such that
\[ \inf_{(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^p(\mathbb{R}^N) \quad f_{g}|a|v|^b \quad dx=1} \left\{ C_0 \| (u, v) \|_{\dot{H}^s, \dot{H}^p}^{n+2s} - \left( \int_{\mathbb{R}^N} f_{g}|a|v|^b \, dx \right)^{\frac{n+2s}{n}} - \right. \]
\[ \left. - \left( \int_{\mathbb{R}^N} f_{g}|a|v|^b \, dx \right)^{\frac{n+2s}{n}} \right) \geq d. \quad (4.6) \]
(4.6) \iff C_0 \frac{\| (u, v) \|^N_{H^s} \| v \|^2}{\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx} \geq d, \quad \text{with } \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx = 1.

\iff C_0 \frac{\| (u, v) \|^N_{H^s} \| v \|^2}{\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx} \geq d \left( \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx \right)^{1/2},

\forall (u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \setminus \{ (0, 0) \}.

Observe that \( \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx \) is bounded away from 0 for all \( (u, v) \in \Omega \). Therefore, plugging back the above estimate into (4.5) proves step 1.

Step 2 Let \( \{ (u_n, v_n) \} \) be a minimizing sequence for \( J_{f,g} \) on \( \Omega \), i.e., \( J_{f,g}(u_n, v_n) \to c_1 \) and \( \| (u_n, v_n) \|^2_{H^s} = (2^*_s - 1) \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \, dx \). Therefore, for large \( n \)

\[ c_1 + o(1) \geq J_{f,g}(u_n, v_n) \geq I_{f,g}(u_n, v_n) \geq \frac{1}{2} \left( 1 - \frac{1}{2^*_s(2^*_s - 1)} \right) \| (u_n, v_n) \|^2_{H^s} - \| f \|_{H^s} + \| g \|_{H^s} \| (u_n, v_n) \|_{H^s}.

This implies that \( \{ I_{f,g}(u_n, v_n) \} \) is a bounded sequence and \( \{ \| (u_n, v_n) \|_{H^s} \} \) and \( \{ \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \, dx \} \) are bounded.

Claim: \( c_0 < 0 \).

Observe that to prove the claim, it is sufficient to show that there exists \( (u, v) \in \Omega_1 \) such that \( J_{f,g}(u, v) < 0 \). Using remark 4.2, we can choose \( (u, v) \in \Omega \) such that \( \| f \|_{H^s} + \| g \|_{H^s} > 0 \). Therefore,

\[ J_{f,g}(u, v) = t^2 \left[ 2^*_s - 1 - \frac{2^*_s - 2}{2^*_s} \right] \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx - t \| f \|_{H^s} - t \| g \|_{H^s} < 0 \]

for \( t \ll 1 \). Moreover, \( (u, v) \in \Omega_1 \) by remark 4.2. Hence the claim follows.

Due to the above claim, \( J_{f,g}(u_n, v_n) \to 0 \) for large \( n \). Consequently, for large \( n \)

\[ 0 > J_{f,g}(u_n, v_n) \geq \frac{1}{2} \left( 1 - \frac{1}{2^*_s(2^*_s - 1)} \right) \| (u_n, v_n) \|^2_{H^s} - \| f \|_{H^s} + \| g \|_{H^s} \| (u_n, v_n) \|_{H^s}.

This in turn implies \( \| f \|_{H^s} + \| g \|_{H^s} > 0 \) for all large \( n \). Consequently, \( \frac{d}{dt} I_{f,g}(u_n, v_n) < 0 \) for \( t > 0 \) small enough. Thus, by step 1, there exists \( t_n \in (0, 1) \) such that
We decompose the proof into few steps.

Step 1 In this step we show that
\[
\lim_{n \to \infty} \{ J_{f,g}(u_n, v_n) - J_{f,g}(\bar{u}, \bar{v}) \} > 0.
\] (4.7)

Observe that if \( J_{f,g}(u_n, v_n) - J_{f,g}(\bar{u}, \bar{v}) \geq \frac{1}{2^{s^*}(2^* - 1)} \int_{\mathbb{R}^N} |u_n|^2 |v_n|^2 \ dx \)
and that for all \( n \in \mathbb{N} \)
there is \( \xi_n > 0 \) such that \( t_n \in (0, 1 - 2\xi_n) \) and \( \frac{d}{dt} J_{f,g}(t_n) \geq \delta/2 \) for \( t \in [1 - \xi_n, 1] \).

To establish (4.7), it is enough to show that \( \xi_n > 0 \) can be chosen independent of \( n \in \mathbb{N} \). This is possible as \( \frac{d}{dt} J_{f,g}(t_n, (1 - t_n)\bar{v}) \geq \frac{\delta}{2} \) for \( t \in [1 - \xi_n, 1] \) and \( \{(u_n, v_n)\} \) is bounded in \( H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \), so that for all \( n \in \mathbb{N} \) and \( t \in [0, 1] \)
\[
\frac{d^2}{dt^2} J_{f,g}(t_n, (1 - t_n)\bar{v}) \geq \left| \left| (u_n, v_n) \right|_{H^s \times H^s}^2 - \left( 2^* - 1 \right) t_n^{2s-2} \int_{\mathbb{R}^N} |u_n|^2 |v_n|^2 \ dx \right|
= \left| 1 - t_n^{2s-2} \right| \left| (u_n, v_n) \right|_{H^s \times H^s}^2 \leq C,
\]
for all \( n \geq 1 \) and \( t \in [0, 1] \).

Step 4 From the definition of \( J_{f,g} \) and \( I_{f,g} \), it immediately follows that \( \frac{d}{dt} J_{f,g}(t_n, (1 - t_n)\bar{v}) \geq \frac{d}{dt} I_{f,g}(t_n, (1 - t_n)\bar{v}) \) for all \( (u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \) and for all \( t > 0 \). Hence,
\[
J_{f,g}(u_n, v_n) - J_{f,g}(\bar{u}, \bar{v}) = \int_{\mathbb{R}^N} \left( \frac{d}{dt} J_{f,g}(t_n, (1 - t_n)\bar{v}) - \frac{d}{dt} I_{f,g}(t_n, (1 - t_n)\bar{v}) \right) dt
\geq \int_{\mathbb{R}^N} \frac{d}{dt} I_{f,g}(t_n, (1 - t_n)\bar{v}) dt
= I_{f,g}(u_n, v_n) - I_{f,g}(\bar{u}, \bar{v}).
\]
Since \( \{(u_n, v_n)\} \subset \Omega \) is a minimizing sequence for \( J_{f,g} \) on \( \Omega \), and \( (t_n u_n, t_n v_n) \in \Omega_1 \), we conclude using (4.7) that
\[
c_0 = \inf_{(u, v) \in \Omega_1} J_{f,g}(u, v) < \inf_{(u, v) \in \Omega} J_{f,g}(u, v) = c_1.
\]

\[\square\]

**Proposition 4.1.** Assume that (4.4) holds. Then \( J_{f,g} \) has a critical point \((u_0, v_0) \in \Omega_1\), with
\[J_{f,g}(u_0, v_0) = c_0.\] In particular, \((u_0, v_0)\) is a positive weak solution to (S).

**Proof.** We decompose the proof into few steps.

Step 1 \( c_0 > -\infty \).

Since \( J_{f,g}(u) > J_{f,g}(v) \), it is enough to show that \( I_{f,g} \) is bounded from below. From the definition of \( \Omega_1 \), it immediately follows that for all \( (u, v) \in \Omega_1 \),
\[
I_{f,g}(u, v) \geq \left[ \frac{1}{2} - \frac{1}{2^s(2^* - 1)} \right] \left\| (u, v) \right\|_{H^s \times H^s}^2 - \left\| f \right\|_{H^s'} \left\| (u, v) \right\|_{H^s \times H^s},
\]
\[\text{for all } u, v, f \in \mathbb{R}^N. \]

\[\boxed{\text{(4.8)}}\]
Step 2 In this step we show that there exists a bounded nonnegative (PS) sequence 
\( \{(u_n, v_n)\} \subset \Omega_1 \) for \( J_{f,g} \) at level \( c_0 \).

Let \( \{(u_0, v_0)\} \subset \Omega_1 \) such that \( J_{f,g}(u_0, v_0) \to c_0 \). Since lemma 4.1 implies that \( c_0 < c_1 \), without restriction we can assume \( \{(u_n, v_n)\} \subset \Omega_1 \). Further, using Ekeland’s variational principle from \( \{(u_0, v_0)\} \), we can extract a (PS) sequence in \( \Omega_1 \) for \( J_{f,g} \) at level \( c_0 \). We again call it by \( \{(u_n, v_n)\} \). Moreover, as \( J_{f,g}(u,v) \geq J_{f,g}(u,v) \), from (4.8) it follows that \( \{(u_n, v_n)\} \) is a bounded sequence. Therefore, up to a subsequence \((u_n, v_n) \to (u_0, v_0)\) in \( H'(\mathbb{R}^N) \times \dot{H}'(\mathbb{R}^N) \) and \((u_n, v_n) \to (u_0, v_0)\) a.e. in \( \mathbb{R}^N \). In particular, \((u_n)_+ \to (u_0)_+\), \((v_n)_+ \to (v_0)_+\) and \((u_n)_- \to (u_0)_-\), \((v_n)_- \to (v_0)_-\) a.e. in \( \mathbb{R}^N \). Moreover, as \( f, g \) are nonnegative functionals, a straightforward computation yields

\[
\begin{align*}
\sigma(1) &= \|J'_{f,g}(u_0, v_0) - J'_{f,g}(u_{n}, v_{n})\|_{H'\times\dot{H}'} \\
&= \frac{1}{2} \|\{(u_{n}, v_{n}) - (u_{0}, v_{0})\}\|^2_{H'\times\dot{H}'} \\
&
\leq -\frac{1}{2} \|\{(u_{n}, v_{n}) - (u_{0}, v_{0})\}\|^2_{H'\times\dot{H}'}.
\end{align*}
\]

Therefore, \((u_{n}, v_{n}) \to (0, 0)\) in \( H'(\mathbb{R}^N) \times \dot{H}'(\mathbb{R}^N) \), which in turn implies up to a subsequence \((u_{n})_+ \to 0\) and \((v_{n})_+ \to 0\) a.e. in \( \mathbb{R}^N \). Thus \((u_0)_+ = 0\) and \((v_0)_+ = 0\) a.e. in \( \mathbb{R}^N \). Consequently, without loss of generality, we can assume that \( \{(u_n, v_n)\} \) is a nonnegative sequence. This completes the proof of step 2.

Step 3 In this step we show that \( (u_n, v_n) \to (u_0, v_0) \) in \( H'(\mathbb{R}^N) \times \dot{H}'(\mathbb{R}^N) \) and \( (u_0, v_0) \in \Omega_1 \).

Applying proposition 3.1, we get

\[
(u_n, v_n) - \left( (u_0, v_0) + \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j) \right) \to (0, 0) \quad \text{in} \quad H'(\mathbb{R}^N) \times \dot{H}'(\mathbb{R}^N). \tag{4.9}
\]

with \( J'_{f,g}(u_0, v_0) = 0 \), \((\tilde{u}_j, \tilde{v}_j)\) is a nonnegative solution of (3.2) \((u_n, v_n)\) is (PS) sequence for \( J_{f,g} \) implies \((\tilde{u}_j, \tilde{v}_j)\) a solution of (4.2) with \( f = 0 = g \) and therefore by remark 4.1, \((\tilde{u}_j, \tilde{v}_j)\) is a nonnegative solution of (3.2), and \( \{x_k^i\} \subset \mathbb{R}^N \), \( \{x_k^j\} \subset \mathbb{R}^+ \) are some appropriate sequences such that \( x_k^i \xrightarrow[n \to \infty]{} x^i \) or \( |x_k^i| \xrightarrow[n \to \infty]{} \infty \). To prove step 3, we need to show that \( m = 0 \). Arguing by contradiction, suppose that \( j \neq 0 \) in (4.9). Therefore,

\[
\begin{align*}
\Psi \left((\tilde{u}_j, \tilde{v}_j)\right)^{\dot{u}_j, \dot{v}_j} &= \|((\tilde{u}_j, \tilde{v}_j))^2_{H' \times \dot{H}'} - (2^* - 1) \int_{\mathbb{R}^N} |\tilde{u}_j|^{2^*} |\tilde{v}_j|^2 \, dx \\
&= (2 - 2^*) \|((\tilde{u}_j, \tilde{v}_j))^2_{H' \times \dot{H}'} \, < \, 0. \tag{4.10}
\end{align*}
\]

From proposition 3.1, we also have

\[
c_0 = \lim_{n \to \infty} J_{f,g}(u_n) = J_{f,g}(u_0, v_0) + \sum_{j=1}^{m} J_{0,0}(\tilde{u}_j, \tilde{v}_j).
\]

Since \((\tilde{u}_j, \tilde{v}_j)\) is a solution to (3.2), it is easy to see that \( J_{0,0}(\tilde{u}_j, \tilde{v}_j) = \frac{\mu}{2} \|((\tilde{u}_j, \tilde{v}_j))^2_{H' \times \dot{H}'} \) and \( S_{(u, v)} \leq \|((u, v))^2_{H' \times \dot{H}'} \), which in turn implies \( J_{0,0}(\tilde{u}_j, \tilde{v}_j) = \frac{\mu}{2} S_{(\tilde{u}_j, \tilde{v}_j)} \). Consequently,
\[ J_{f,a}(u_0, v_0) < c_0. \] Therefore, \((u_0, v_0) \notin \Omega_1 \) and

\[ \Psi(u_0, v_0) \leq 0. \] \hspace{1cm} (4.11)

Next, we evaluate \( \Psi \left( (u_0, v_0) + \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right) \). We observe that \((u_n, v_n) \in \Omega_1 \) implies \( \Psi(u_n, v_n) \geq 0 \). Combining this with the uniform continuity of \( \Psi \) and (4.9) yields

\[ 0 \leq \liminf_{n \to \infty} \Psi(u_n, v_n) = \liminf_{n \to \infty} \Psi \left( (u_0, v_0) + \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right). \] \hspace{1cm} (4.12)

Note that from step 2, we already have \( u_0, v_0 \geq 0 \) and \((\tilde{u}_j, \tilde{v}_j)\) is nonnegative for all \( j \) (see the paragraph after (4.9)). Therefore, as \( \alpha, \beta > 1 \)

\[
\Psi \left( (u_0, v_0) + \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right) \\
= \left\| (u_0, v_0) \right\|_{H^{r_{j}^{\alpha}, s_{j}^{\alpha}}}^2 + \left\| \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right\|_{H^{r_{j}^{\alpha}, s_{j}^{\alpha}}}^2 \\
+ 2 \left( (u_0, v_0), \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right)_{H^{r_{j}^{\alpha}, s_{j}^{\alpha}}} \\
- (2r^+ - 1) \int_{\mathbb{R}^N} \left[ u_0 + \sum_{j=1}^{m} \tilde{u}_j^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right]^\alpha \left[ v_0 + \sum_{j=1}^{m} \tilde{v}_j^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right]^\beta \, dx \\
\leq \left\| (u_0, v_0) \right\|_{H^{r_{j}^{\alpha}, s_{j}^{\alpha}}}^2 + \sum_{j=1}^{m} \left\| (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right\|_{H^{r_{j}^{\alpha}, s_{j}^{\alpha}}}^2 \\
+ 2 \sum_{j=1}^{m} \sum_{j=1}^{m} \left( (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}}, (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right)_{H^{r_{j}^{\alpha}, s_{j}^{\alpha}}} \\
+ 2 \left( (u_0, v_0), \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right)_{H^{r_{j}^{\alpha}, s_{j}^{\alpha}}} \\
- (2r^+ - 1) \left( \int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \, dx + \sum_{j=1}^{m} \int_{\mathbb{R}^N} |\tilde{u}_j^{r_{j}^{\alpha}, s_{j}^{\alpha}}|^\alpha |\tilde{v}_j^{r_{j}^{\alpha}, s_{j}^{\alpha}}|^\beta \, dx \right) \\
= \Psi(u_0, v_0) + \sum_{j=1}^{m} \Psi \left( (\tilde{u}_j, \tilde{v}_j)^{r_{j}^{\alpha}, s_{j}^{\alpha}} \right) + \text{the above inner products.} \hspace{1cm} (4.13)

We now prove that all the inner products in the rhs of (4.13) approaches 0 as \( n \to \infty \). As \( r_{n}^+ \xrightarrow{n \to \infty} 0 \), it follows that \( u_{n}^{r_{j}^{\alpha}, s_{j}^{\alpha}} \to 0 \) and \( v_{n}^{r_{j}^{\alpha}, s_{j}^{\alpha}} \to 0 \) in \( H^r(\mathbb{R}^N) \) as \( n \to \infty \) (see [33, lemma 3]).
Therefore, \((u_0, u_j^{l, n_i})_{H'} \xrightarrow{n \to \infty} 0\) and \((v_0, v_j^{l, n_i})_{H'} \xrightarrow{n \to \infty} 0\) for all \(j = 1, \ldots, m\). Hence,

\[
2 \left\langle u_0, v_0, \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{l, n_i} \right\rangle_{H' \times H'} = o(1) \quad \text{as } n \to \infty.
\]

Next,

\[
\left\langle (\tilde{u}_j, \tilde{v}_j)^{l, n_i}, (\tilde{u}_j, \tilde{v}_j)^{l, n_i} \right\rangle_{H' \times H'} = \left( r_j^l \right)^{N-2} \left( r_j^l \right)^{N-2} \times \int_{\mathbb{R}^{2N}} \left( \tilde{u}_j \left( \frac{\tilde{x}^{\prime} - \tilde{x} \prime}{r^l_{n_i}} \right) - \tilde{u}_j \left( \frac{x^{\prime} - x \prime}{r^l_{n_i}} \right) \right) \left( \tilde{v}_j \left( \frac{\tilde{x}^{\prime} - \tilde{x} \prime}{r^l_{n_i}} \right) - \tilde{v}_j \left( \frac{x^{\prime} - x \prime}{r^l_{n_i}} \right) \right) \, dx \, dy
\]

where \(\tilde{u}_j^{l}(x) := \left( \frac{\tilde{u}_j}{r^l_{n_i}} \right) \tilde{u}_j \left( \frac{\tilde{x}^{\prime} - \tilde{x} \prime}{r^l_{n_i}} \right)\), and \(\tilde{v}_j^{l}(x) := \left( \frac{\tilde{v}_j}{r^l_{n_i}} \right) \tilde{v}_j \left( \frac{\tilde{x}^{\prime} - \tilde{x} \prime}{r^l_{n_i}} \right)\). Further, we observe that using the following

\[
\left| \log \left( \frac{r^l_{n_i}}{r^l_{n_i}} \right) \right| + \left| \frac{x^{\prime} - x \prime}{r^l_{n_i}} \right| \to \infty
\]

from proposition 3.1, it is easy to see that \(\tilde{u}_j^{l} \to 0\) and \(\tilde{v}_j^{l} \to 0\) in \(\dot{H}^l(\mathbb{R}^N) \times \dot{H}^l(\mathbb{R}^N)\) as \(n \to \infty\) for each fixed \(i\) and \(j\) (see [33, lemma 3]). Hence

\[
\left\langle (\tilde{u}_j, \tilde{v}_j)^{l, n_i}, (\tilde{u}_j, \tilde{v}_j)^{l, n_i} \right\rangle_{H' \times H'} = o(1).
\]

Substituting this back into (4.13) and using (4.10) and (4.11) gives a contradiction to (4.12). Therefore, \(m = 0\) in (4.9). Hence, \((u_0, v_0) \to (u_0, v_0)\) in \(\dot{H}^l(\mathbb{R}^N) \times \dot{H}^l(\mathbb{R}^N)\). Consequently, \(\Psi(u_0, v_0) \to \Psi(u_0, v_0)\), which in turn implies \((u_0, v_0) \in \Omega_1\). But, since \(c_0 < c_1\), we conclude \((u_0, v_0) \in \Omega_1\). Thus step 3 follows.

Step 4 From the previous steps we see that \(J_{f,q}(u_0, v_0) = c_0\) and \(J_{f,q}'(u_0, v_0) = 0\). Therefore, \((u_0, v_0)\) is a weak solution to (4.2). Combining this with remark 4.1, we end the proof of the proposition.

\[\Box\]
Proposition 4.2. Assume that (4.4) holds. Then, \( J_{f,\beta} \) has a second critical point \((u_0, v_0)\), where \((u_0, v_0)\) is the positive solution to \((S)\) obtained in proposition 4.1. In particular, \((u_1, v_1)\) is a second positive solution to \((S)\).

Proof. Let \((u_0, v_0)\) be the critical point obtained in proposition 4.1 and \((Bw, Cw)\) (with \(C = B \sqrt{\frac{2}{N}}\)) be a positive ground state solution of (3.2) described as in theorem 1.1. A standard computation yields that \(I_{0,0}(Bw, Cw) = \frac{1}{N} \Sigma_{\alpha, \beta}^{\infty} \). For \( t > 0 \), define

\[
    u_{t}(x) := u \left( x \right), \quad \tilde{u}_{t}(x) := Bw_{t}(x), \quad \tilde{v}_{t}(x) := Cw_{t}(x).
\]

Claim 1. \((u_0 + \tilde{u}_t, v_0 + \tilde{v}_t) \in \Omega_2 \) for \( t > 0 \) large enough.

Indeed, as \((u_0, v_0)\) and \((\tilde{u}, \tilde{v})\) are positive and \( \alpha, \beta > 1 \), using Young’s inequality with \( \varepsilon > 0 \), we have

\[
    \begin{align*}
        \Psi(u_0 + \tilde{u}_t, v_0 + \tilde{v}_t) \\
        &= \| (u_0 + \tilde{u}_t) \|_{L^p}^2 + \| (v_0 + \tilde{v}_t) \|_{L^p}^2 - (2^*_p - 1) \int_{\mathbb{R}^N} \| u_0 + \tilde{u}_t \|^{\alpha} |v_0 + \tilde{v}_t|^2 \ dx \\
        &
    \end{align*}
\]

Hence the claim follows.

Claim 2. \( J_{f,\beta} \left( (u_0 + \tilde{u}_t, v_0 + \tilde{v}_t) \right) < J_{f,\beta}(u_0, v_0) + J_{0,0}(\tilde{u}, \tilde{v}) \) \( \forall t > 0 \).

Indeed, since \( u_0, v_0, u_1, B > 0 \), taking \((\tilde{u}, \tilde{v})\) as the test function for (4.2) yields

\[
    \langle f(u_0, v_0), (\tilde{u}, \tilde{v}) \rangle_{L^p}^2 \geq \frac{\alpha}{2^*} \int_{\mathbb{R}^N} u_0^{\alpha - 2} \tilde{u}_t \ dx + \frac{\beta}{2^*} \int_{\mathbb{R}^N} v_0^{\beta - 2} \tilde{v}_t \ dx \geq \frac{\alpha}{2^*} \int_{\mathbb{R}^N} u_0^{\alpha - 2} \tilde{u}_t \ dx + \frac{\beta}{2^*} \int_{\mathbb{R}^N} v_0^{\beta - 2} \tilde{v}_t \ dx
\]

Consequently, using the above expression, we obtain

\[
    \begin{align*}
        J_{f,\beta} \left( (u_0 + \tilde{u}_t, v_0 + \tilde{v}_t) \right) \\
        &= \frac{1}{2} \| (u_0, v_0) \|_{L^p}^2 + \frac{1}{2} \| (\tilde{u}, \tilde{v}) \|_{L^p}^2 + \langle f(u_0, v_0), (\tilde{u}, \tilde{v}) \rangle_{L^p}^2 \\
        &\geq \frac{1}{2} \int_{\mathbb{R}^N} (u_0 + \tilde{u}_t)^2 + \frac{1}{2} \int_{\mathbb{R}^N} (v_0 + \tilde{v}_t)^2 \ dx - \langle f(u_0 + \tilde{u}_t, v_0 + \tilde{v}_t) \rangle_{L^p}^2
\end{align*}
\]

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where

\[ \lim_{t \to \infty} J_{f,t}(\bar{u}, \bar{v}) = -\infty, \]  

and

\[ \sup_{t > 0} J_{0,0}(\bar{u}, \bar{v}) = J_{0,0}(\bar{u}_\rho, \bar{v}_\rho), \quad \text{where } t' = \left( \frac{B^2 + C^2}{B^0 C^3} \right)^{\frac{1}{2}}. \]

Therefore, doing a straightforward computation and using lemma 1.1, we get that

\[ \sup_{t > 0} J_{0,0}(\bar{u}, \bar{v}) = \frac{s}{N} \left( \frac{B^2 + C^2}{B^0 C^3} \right)^{\frac{d}{2N}} \leq \frac{s}{N} \frac{\tilde{N}}{\alpha_{\lambda}}, \]

Combining this with claim 2 and (4.14) yields

\[ J_{f,t}(u_0 + \bar{u}, v_0 + \bar{v}) < J_{f,t}(u_0, v_0) + \frac{s}{N} \frac{s}{\alpha_{\lambda}} \quad \forall t > 0 \]

and

\[ J_{f,t}(u_0 + \bar{u}, v_0 + \bar{v}) < J_{f,t}(u_0, v_0) \quad \text{for } t \text{ large enough.} \]

Fix \( t_0 > 0 \) large enough such that (4.15) and claim 1 are satisfied.

Next, we set

\[ \eta := \inf_{\gamma \in [0,1]} \max_{\gamma \in [0,1]} J_{f,\gamma}(\gamma(r)), \]

where

\[ \Gamma := \left\{ \gamma \in C([0,1], H^0(\mathbb{R}^N) \times H^0(\mathbb{R}^N)) : \gamma(0) = (u_0, v_0), \right. \]

\[ \left. \gamma(1) = (u_0 + \bar{u}_0, v_0 + \bar{v}_0) \right\}. \]

As \((u_0, v_0) \in \Omega_1\) and \((u_0 + \bar{u}_0, v_0 + \bar{v}_0) \in \Omega_2\), for every \( \gamma \in \Gamma \), there exists \( r_\gamma \in (0,1) \) such that \( \gamma(r_\gamma) \in \Omega \). Therefore,

\[ \max_{r \in [0,1]} J_{f,\gamma}(\gamma(r)) \geq J_{f,\gamma}(\gamma(r_\gamma)) \geq \inf_{\Omega} J_{f,\gamma}(u, v) = c_1. \]
Thus, \( \eta \geq c_1 > c_0 = J_{f,g}(u_0, v_0) \). Here in the last inequality we have used lemma 4.1.

**Claim 3.** \( J_{f,g}(u_0, v_0) < \eta < J_{f,g}(u_0, v_0) + \frac{S_{\alpha,\beta}^N}{N} \).

Since \( \lim_{t \to 0} \|u(t)\|_{H^1(\mathbb{R}^N)} = 0 \), we also have \( \lim_{t \to 0} \|v(t)\|_{H^1(\mathbb{R}^N)} = 0 \). Thus, if we define \( \tilde{\gamma}(r) := (u_0, v_0) + (\tilde{u}_{m}, \tilde{u}_{m}), \) then \( \lim_{r \to 0} \|\tilde{\gamma}(r) - (u_0, v_0)\|_{H^1(\mathbb{R}^N)} = 0 \). Consequently, \( \tilde{\gamma} \in \Gamma \).

Therefore, using (4.15), we obtain

\[
\eta \leq \max_{r \in [0,1]} J_{f,g}(\tilde{\gamma}(r)) = \max_{r \in [0,1]} J_{f,g}(u_0 + \tilde{u}_{m}, v_0 + \tilde{v}_{m}) < J_{a,f}(u_0, v_0) + \frac{S_{\alpha,\beta}^N}{N}.
\]

Hence claim 3 follows.

Using Ekeland’s variational principle, there exists a (PS) sequence \( \{u_n, v_n\} \) for \( J_{f,g} \) at level \( \eta \). Arguing as before we see that \( \{u_n, v_n\} \) is a bounded sequence. Further, since claim 3 holds, from proposition 3.1 we conclude that \( (u_n, v_n) \to (u_1, v_1) \), for some \( (u_1, v_1) \in H^1(\mathbb{R}^N) \) such that \( J_{f,g}(u_1, v_1) = 0 \) and \( J_{f,g}(u_1, v_1) = \eta \). On the other hand, as \( J_{f,g}(u_0, v_0) < \eta \), we conclude \( (u_0, v_0) \neq (u_1, v_1) \).

\( J_{f,g}(u_1, v_1) = 0 \Rightarrow (u_1, v_1) \) is a weak solution to (4.2). Combining this with remark 4.1, we complete the proof of the proposition. \( \square \)

**Lemma 4.2.** Let \( C_0 \) be as defined in theorem 1.2. If \( \max \{\|f\|_{H^s},\|g\|_{H^s}\} < C_0 S_{\alpha,\beta}^N \) then (4.4) holds.

**Proof.**

**Assertion 1:**

\[
\frac{4s}{N + 2s} \|u, v\|_{H^s} \geq C_0 S_{\alpha,\beta}^N \quad \forall (u, v) \in \Omega.
\]

To see this, we fix \( (u, v) \in \Omega \). Therefore, using the definition of \( S_{\alpha,\beta} \) we have

\[
\|u, v\|_{H^s} \geq \sqrt[2s]{\int_{\mathbb{R}^N} |u|^2 |v|^2 \, dx} \geq \frac{1}{C_0 S_{\alpha,\beta}^N} \|u, v\|_{H^s}^{2s/(2s - 1)}.
\]

From here, using the definition of \( C_0 \), the assertion follows.

Note that by the given hypothesis, there exists \( \varepsilon > 0 \) such that

\[
\|f\|_{H^s} + \|g\|_{H^s} < C_0 S_{\alpha,\beta}^N - \varepsilon.
\]

Combining this with the above assertion 1, for all \( (u, v) \in \Omega \), it holds

\[
\|f\|_{H^s} + \|g\|_{H^s} < C_0 S_{\alpha,\beta}^N - \varepsilon.
\]

Consequently,

\[
\liminf_{(u, v) \to \Omega} \left[ \frac{4s}{N + 2s} \|u, v\|_{H^s}^2 - \|f\|_{H^s} - \|g\|_{H^s} \right] < C_0 S_{\alpha,\beta}^N - \varepsilon.
\]

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Since \( \|(u, v)\|_{H^s \times H^s} \) is bounded away from 0 on \( \Omega \), the above expression implies that

\[
\inf_{(u, v) \in \Omega} \left[ \frac{4s}{N + 2s} \|(u, v)\|_{H^s \times H^s}^2 - \langle H^s \langle f, u \rangle_{H^s} - \langle H^s \langle g, v \rangle_{H^s} \right] > 0.
\]

(4.16)

On the other hand,

\[
(4.4) \ Leftrightarrow C_0 \frac{\|(u, v)\|_{H^s \times H^s}^{N + 2s}}{\int_{\mathbb{R}^N} |u| |v|^{\frac{2}{s}} \, dx} \frac{1}{4s} - \langle H^s \langle f, u \rangle_{H^s} - \langle H^s \langle g, v \rangle_{H^s} > 0
\]

for \( \int_{\mathbb{R}^N} |u| |v|^{\frac{2}{s}} \, dx = 1 \)

\[
\Leftrightarrow C_0 \frac{\|(u, v)\|_{H^s \times H^s}^{N + 2s}}{\int_{\mathbb{R}^N} |u| |v|^{\frac{2}{s}} \, dx} \frac{1}{4s} - \langle H^s \langle f, u \rangle_{H^s} - \langle H^s \langle g, v \rangle_{H^s} > 0 \quad \text{for} \,(u, v) \in \Omega
\]

\[
\Leftrightarrow \frac{4s}{N + 2s} \|(u, v)\|_{H^s \times H^s}^2 - \langle H^s \langle f, u \rangle_{H^s} - \langle H^s \langle g, v \rangle_{H^s} > 0 \quad \text{for} \,(u, v) \in \Omega.
\]

(4.17)

Clearly, (4.16) ensures that the rhs of (4.17) holds. The lemma now follows. \( \square \)

End of proof of theorem 1.2 combining propositions 4.1 and 4.2 with lemmas 4.2 and 4.1, we obtain two positive solutions of theorem 1.2. The last assertion of the theorem follows as proved in [2, theorem 1.1].

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Appendix A. Product of Morrey spaces

First we recall the definition of the homogeneous Morrey spaces \( L^{r, \gamma}(\mathbb{R}^N) \), introduced by Morrey as a refinement of the usual Lebesgue spaces. A measurable function \( u : \mathbb{R}^N \rightarrow \mathbb{R} \) belongs to the Morrey space \( L^{r, \gamma}(\mathbb{R}^N) \), with \( r \in [1, \infty) \) and \( \gamma \in [0, N] \) if and only if

\[
\|u\|_{L^{r, \gamma}(\mathbb{R}^N)} := \sup_{R > 0, x \in \mathbb{R}^N} R^{-\gamma} \int_{B(x, R)} |u|^r \, dy < \infty.
\]

(A.1)
Note that if $\gamma = N$ then $L^{r,N}(\mathbb{R}^N)$ coincides with the usual Lebesgue space $L^r(\mathbb{R}^N)$ for any $r \geq 1$ and similarly $L^{r,N}(\mathbb{R}^N)$ coincides with $L^{\infty}(\mathbb{R}^N)$. Also we observe that $L^{r,N}(\mathbb{R}^N)$ experiences same translation and dilation invariance as in $L^{2,N}(\mathbb{R}^N)$ and therefore of $H^r(\mathbb{R}^N)$ if $\Theta = \frac{N}{2,N-2}$. Let $(u)^{\Theta}_{2,N}$ be the function defined by (3.3). By change of variable formula, one can see that the following equality holds

$$
\|(u)^{\Theta}_{2,N}\|_{L^{\frac{N}{2,N}}^r} = \|u\|_{L^{\frac{N}{2,N}}^r},
$$

for any $r \in [1,2]$. We recall that there exists a constant $C = C(N,s)$ such that

$$
\|u\|_{L^{r,N-2s}} \leq C\|u\|_{L^{2,N}}^{\frac{N}{2,N}} \quad \text{for all } u \in L^{2,N}(\mathbb{R}^N),
$$

(A.2) see [33, theorem 1] (also see [4, (A.2)]). For further discussion on Morrey spaces, we refer the reader to [33]. Next we define the product space $L^{2,N-2\theta}(\mathbb{R}^N) \times L^{2,N-2\theta}(\mathbb{R}^N)$ in the standard way with

$$
\|(u,v)\|_{L^{2,N-2\theta}} = \left(\|u\|_{L^{2,N-2\theta}}^2 + \|v\|_{L^{2,N-2\theta}}^2\right)^\frac{1}{2}.
$$

Therefore, using the Sobolev inequality and (A.2), it follows that

$$
\hat{H}^r \times \hat{H}^r \hookrightarrow L^{2,N} \times L^{2,N} \hookrightarrow L^{2,N-2\theta} \times L^{2,N-2\theta},
$$

(A.3)

where the embeddings are continuous.

**Lemma A.1.** For any $0 < s < N/2$ there exists a constant $C = C(N,s)$ such that, for any $2/2_s \leq \theta < 1$ and for any $1 \leq r < 2_s$

$$
\|(u,v)\|_{L^{2,N}}^{\frac{N}{2,N}} \leq C\|(u,v)\|_{H_s \times H_s}^{\theta} \|v\|_{L^{2,N-2s}}^{1-\theta}
$$

for all $(u,v) \in H^r(\mathbb{R}^N) \times H^r(\mathbb{R}^N)$.

**Proof.** Using [33, theorem 1],

$$
\|(u,v)\|_{L^{2,N}}^{\frac{N}{2,N}} = \left(\|u\|_{L^{2,N}}^2 + \|v\|_{L^{2,N}}^2\right)^\frac{1}{2}
\leq \|u\|_{L^{2,N}} + \|v\|_{L^{2,N}}
\leq C \left(\|(u,v)\|_{H_s \times H_s}^\theta \|u\|_{L^{2,N-2s}}^{1-\theta} + \|v\|_{L^{2,N-2s}}^{1-\theta}\right)
\leq C \left(\|(u,v)\|_{H_s \times H_s}^\theta \|u\|_{L^{2,N-2s}}^{1-\theta} + \|(u,v)\|_{H_s \times H_s}^{1-\theta}\right)
\leq C \left(\|(u,v)\|_{H_s \times H_s}^\theta \|u\|_{L^{2,N-2s}}^{1-\theta} + \|v\|_{L^{2,N-2s}}^{1-\theta}\right)
\leq C \left(\|(u,v)\|_{H_s \times H_s}^\theta \|u\|_{L^{2,N-2s}}^{1-\theta} + \|(u,v)\|_{L^{2,N-2s}}^{1-\theta}\right)
\leq 2C \|(u,v)\|_{H_s \times H_s}^{1-\theta} \leq 2C \|(u,v)\|_{H_s \times H_s}^{1-\theta} \|u\|_{L^{2,N-2s}}^{1-\theta}.
$$

□

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