RANDOM NETWORKS WITH HETEROGENEOUS RECIPROCITY

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Users of social networks display diversified behavior and online habits. For instance, a user’s tendency to reply to a post can depend on the user and the person posting. For convenience, we group users into aggregated behavioral patterns, focusing here on the tendency to reply to or reciprocate messages. The reciprocity feature in social networks reflects the information exchange among users. We study the properties of a preferential attachment model with heterogeneous reciprocity levels, give the growth rate of model edge counts, and prove convergence of empirical degree frequencies to a limiting distribution. This limiting distribution is not only multivariate regularly varying, but also has the property of hidden regular variation.

1. Introduction Social networks have grown rapidly and users are exhibiting different interaction and behavioral patterns on platforms like Facebook and Twitter. Reciprocity is one such pattern and helps characterize information exchange between social network users [12, 18]. For instance, consider the network of Facebook wall posts: a directed edge from user A to user B is formed when user A leaves a message on the Facebook wall of user B, and a reciprocal edge from user B to user A is created if and when user B replies to the message. Depending on both users’ behavioral features and networking habits (e.g. the closeness of friendship, whether user A is broadcasting, the obsessiveness of user B’s replying habits), the probability of generating a reciprocal edge may vary across different pairs of users, and network modeling should include such heterogeneity. A study in [11] shows that online social networks tend to have a high proportion of reciprocal edges, compared to other types of networks such as biological networks, communication networks, software call graphs and peer-to-peer networks. Due to the large and diversified user groups on social networks, it is essential to incorporate heterogeneous reciprocity levels into the modeling.

For the modeling of dynamic networks, the preferential attachment (PA) model [5, 14] is an appealing starting point. This model captures the scale-free property of complex networks, where both in- and out-degree distributions have Pareto-like tails [21, 22, 23, 24]. Recently, [25] show that for a wide and realistic range of model parameters, the standard PA model in [5] generates networks with very small proportion of reciprocal edges, deviating from the empirical observations in [11]. This led to two consecutive studies [6, 26] on the theoretical properties and the estimation of a PA model with homogeneous reciprocity, where reciprocal edges are generated by simply flipping a two-sided coin. To capture the heterogeneous reciprocal patterns, we extend here the model in [26] by dividing users into $K$ different behavioral groups, and the probability of generating a reciprocal edge from a user in group $m$ to a user in group $r$ is $\rho_{m,r} \in (0, 1)$, $m, r \in \{1, \ldots, K\}$.

We study three properties of the proposed PA model with heterogeneous reciprocity. Under modest assumptions, we first analyze the growth of the edge counts by identifying the almost sure limit for the scaled number of edges emanating from and pointing to nodes of...
group type $m$. Secondly, by embedding in- and out-degree sequences into a family of multitype branching processes whose particles are given group labels, we prove that the empirical frequencies of nodes with in-degree $i$ and out-degree $j$ converge to a limiting distribution $p_{i,j}$. Third, we show that the asymptotic limiting distribution, $p_{i,j}$, is multivariate regularly varying with limit measure concentrating on a ray and after removing large in- and out-degree pairs close to the concentrating ray, we also detect hidden regular variation [16].

The rest of the paper is organized as follows. We start with a detailed description of the PA model with heterogeneous reciprocity levels in Section 1.1. Section 2 studies the growth of edge counts, and in order to ensure the convergence of scaled edge counts, sufficient assumptions are imposed. Also, we extend the embedding technique in [26] to a family of multitype branching processes with different group labels, and derive the limit of empirical degree frequencies in Section 3. We then characterize the asymptotic dependence structure of large in- and out-degrees in Section 4, and give concluding remarks in Section 5. Technical proofs of results in Section 2 are collected in Section 6.

1.1. **PA Model with Heterogeneous Reciprocity**  

The proposed model extends the directed preferential attachment (PA) model studied in [5, 14, 23, 24] by amending a mechanism that generates a reciprocal edge with probabilities depending on characteristics of the node pair being connected by the edge. Because these characteristics can be specific to the node pair, the model incorporates heterogeneous responding patterns of users in social networks.

We now specify a growing sequence of graphs. Let $G(n)$ be the graph after $n$ steps with $V(n)$ being the set of nodes and $E(n)$ being the set of edges in $G(n)$. Attach to each node $v$ a communication type $W_v$, where $\{W_v, v \geq 1\}$ are iid random variables, independent from the growth mechanism of the graph, with

$$P(W_v = r) = \pi_r, \quad \text{for} \quad \sum_{r=1}^{K} \pi_r = 1.$$ 

We imagine that when a node is born, it flips a multi-sided coin to determine its communication type. Think of $\pi_r$ as the percent of the users in a social network that have communication habits labeled as type $r$. Let $W(n) := \{W_v : v \in V(n)\}$ denote the set of group types for all nodes in $G(n)$. Throughout we assume that the communication group of node $v$ is always observed upon its creation, and remains unchanged afterwards. In the rest of this paper, we assume $G(n) = (V(n), E(n), W(n))$, for $n \geq 0$.

Denote the cardinality of a discrete set $S$ by $|S|$ and initialize the model with graph $G(0)$, which consists of one node (labeled as node 1) and a self-loop, with $V(0) = \{1\}$, $|V(0)| = 1$, $W(0) = \{W_1\}$, and $E(0) = \{(1,1)\}$. For each new edge $(u,v)$ with $W_u = r, W_v = m$, the reciprocity mechanism adds its reciprocal counterpart $(v,u)$ instantaneously with probability $\rho_{m,r} \in [0,1]$, for $m, r \in \{1, 2, \ldots, K\}$. Here $\rho_{m,r}$ measures the probability of adding a reciprocal edge from a node in group $m$ to a node in group $r$. Note that the matrix $\rho := (\rho_{m,r})_{m,r}$ is not necessarily a stochastic matrix, but can be an arbitrary matrix in $M_{K \times K}([0,1])$, the set of all $K \times K$ matrices with entries belonging to $[0,1]$. Later in Section 2, we will give particular regularity conditions on $\rho$ to facilitate theoretical analysis.

Let $(D_v^{\text{in}}(n), D_v^{\text{out}}(n))$ be the in- and out-degrees of node $v \in V(n)$ in $G(n)$, and we use the convention that $D_v^{\text{in}}(n) = D_v^{\text{out}}(n) = 0$ if $v \notin V(n)$. Let $\delta > 0$ be an offset parameter. The evolution of the network $G(n+1)$ from $G(n)$ is described as follows.

1. With probability $\alpha \in (0,1)$, add a new node $|V(n)| + 1$ with a directed edge $(|V(n)| + 1, v)$, where $v \in V(n)$ is chosen with probability

$$\frac{D_v^{\text{in}}(n) + \delta}{\sum_{v \in V(n)} (D_v^{\text{in}}(n) + \delta)} = \frac{D_v^{\text{in}}(n) + \delta}{|E(n)| + \delta |V(n)|}, \quad (1)$$
and update the node set $V(n + 1) = V(n) \cup \{ |V(n)| + 1 \}$ and $W(n + 1) = W(n) \cup \{ W_{|V(n)| + 1} \}$. The new node $|V(n)| + 1$ belongs to group $r$ with probability $\pi_r$. If node $v$ belongs to group $m$, then a reciprocal edge $(v, |V(n)| + 1)$ is added with probability $\rho_{m,r}$. Upon reciprocation, update the edge set as $E(n + 1) = E(n) \cup \{ (|V(n)| + 1, v), (v, |V(n)| + 1) \}$. If the reciprocal edge is not created, set $E(n + 1) = E(n) \cup \{ (|V(n)| + 1, v) \}$.

2. With probability $\gamma \equiv 1 - \alpha \in (0, 1)$, add a new node $|V(n)| + 1$ with a directed edge $(v, |V(n)| + 1)$, where $v \in V(n)$ is chosen with probability

$$P \left( \frac{D^\text{out}(n) + \delta}{\sum_{v \in V(n)} (D^\text{out}(n) + \delta)} \right) = \frac{D^\text{out}(n) + \delta}{|E(n)| + \delta} |V(n)|,$$

and update the node set $V(n + 1) = V(n) \cup \{ |V(n)| + 1 \}$, $W(n + 1) = W(n) \cup \{ W_{|V(n)| + 1} \}$. The new node $|V(n)| + 1$ belongs to group $r$ with probability $\pi_r$. If node $v$ belongs to group $m$, then a reciprocal edge $(|V(n)| + 1, v)$ is added with probability $\rho_{r,m}$. Upon reciprocation, update the edge set as $E(n + 1) = E(n) \cup \{ (v, |V(n)| + 1), (|V(n)| + 1, v) \}$. If the reciprocal edge is not created, set $E(n + 1) = E(n) \cup \{ (v, |V(n)| + 1) \}$.

Note that $|V(n)| = n + 1$, $n \geq 0$, since a new node is added at each step, and the offset parameter $\delta$ is assumed to be the same for both in- and out-degrees.

2. Growth of Edge Counts Heterogeneous reciprocity levels make analysis of the convergence of $|E(n)|/n$ more complicated compared to the homogeneous case. In Theorem 2.2, we show the concentration of $|E(n)|$ around $\mathbb{E}[|E(n)|]$, and then prove the convergence of $\mathbb{E}[|E(n)|]/n$, thereby showing linear growth of the number of edges. We now prepare for this result.

Write

$$\rho_m := \sum_{r=1}^{K} \rho_{m,r} \pi_r = \mathbb{P}[\text{Type } m \text{ node sends reciprocal edge}],$$

$$\rho_m \ast := \sum_{r=1}^{K} \rho_{r,m} \pi_r = \mathbb{P}[\text{Type } m \text{ node receives reciprocal edge}].$$

Let $\mathcal{G}_n$ be denote the $\sigma$-field generated by observing the network evolution up to $n$ steps, i.e.

$$\mathcal{G}_n = \sigma \{ (V(k), E(k), W(k)) : k = 0, \ldots, n \}.$$

For $v \in V(n)$, we have

$$\mathbb{E}^{\mathcal{G}_n} \left( D^\text{in}(n + 1) - D^\text{in}(n) \right) = \alpha \frac{D^\text{in}(n) + \delta}{|E(n)| + \delta |V(n)|} + \gamma \sum_{m=1}^{K} \rho_m \ast \frac{D^\text{out}(n) + \delta}{|E(n)| + \delta |V(n)|} 1_{\{ W_v = m \}},$$

and

$$\mathbb{E}^{\mathcal{G}_n} \left( D^\text{out}(n + 1) - D^\text{out}(n) \right) = \gamma \frac{D^\text{out}(n) + \delta}{|E(n)| + \delta |V(n)|} + \alpha \sum_{m=1}^{K} \rho_m \ast \frac{D^\text{in}(n) + \delta}{|E(n)| + \delta |V(n)|} 1_{\{ W_v = m \}}.$$

Let $V_m(n) := \{ v \in V(n) : W_v = m \}$, and note that

$$\sum_{v \in V(n)} \mathbb{E}^{\mathcal{G}_n} \left( D^\text{in}(n + 1) - D^\text{in}(n) \right) 1_{\{ W_v = m \}}$$

$$= \mathbb{P}^{\mathcal{G}_n} \left( E(n + 1) = E(n) \cup \{ (|V(n)| + 1, v), v \in V_m(n) \} \right)$$
\[ + \mathbb{P}^{G_n}(E(n+1) = E(n) \cup \{(v, |V(n)| + 1, (|V(n)| + 1, v)\},
\]
\[ v \in V_m(n), (|V(n)| + 1, v) \text{ is due to reciprocation.} \]

\[ (5) \quad = \mathbb{P}^{G_n}(\{(|V(n)| + 1, v) \in E(n+1) \setminus E(n), v \in V_m(n)\}). \]

Write \[ E^{in}_m(n) := \{(u, v) \in E(n) : W_u = m\}, \]
and \[ E^{out}_m(n) := \{(v, u) \in E(n) : W_v = m\}. \]
We see that

\[
\mathbb{E}^{G_n}(|E^{in}_m(n+1)| - |E^{in}_m(n)|)
\]
\[ = \mathbb{P}^{G_n}(\|(|V(n)| + 1, v) \in E(n+1) \setminus E(n), v \in V_m(n)\))
\]
\[ + \mathbb{P}^{G_n}(\{(v, |V(n)| + 1) \in E(n+1) \setminus E(n), W_{|V(n)|+1} = m, v \in V(n)\) .
\]

and applying (3) and (5) gives

\[ = \sum_{v \in V(n)} \mathbb{E}^{G_n}(D^{in}_v(n+1) - D^{in}_v(n)) \mathbf{1}_{\{W_v = m\}}
\]
\[ + \mathbb{P}^{G_n}(\{(v, |V(n)| + 1) \in E(n+1) \setminus E(n), W_{|V(n)|+1} = m, v \in V(n)\) .
\]

\[
= \alpha \frac{|E^{in}_m(n)| + |\delta V_m(n)|}{|E(n)| + |\delta V(n)|} + \gamma \rho_m \frac{|E^{out}_m(n)| + |\delta V_m(n)|}{|E(n)| + |\delta V(n)|}
\]
\[ + \mathbb{P}^{G_n}(\{(v, |V(n)| + 1) \in E(n+1) \setminus E(n), W_{|V(n)|+1} = m, v \in V(n)\) .
\]

(6)

For the third term in (6), we have that

\[ \mathbb{P}^{G_n}(\{(v, |V(n)| + 1) \in E(n+1) \setminus E(n), W_{|V(n)|+1} = m, v \in V(n)\) .
\]
\[ = \mathbb{P}^{G_n}(\{(v, |V(n)| + 1) \in E(n+1) \setminus E(n), W_{|V(n)|+1} = m, v \in V(n)\),
\]
\[ (v, |V(n)| + 1) \text{ is not due to reciprocation}
\]
\[ + \mathbb{P}^{G_n}(\{(v, |V(n)| + 1) \in E(n+1) \setminus E(n), W_{|V(n)|+1} = m, v \in V(n)\),
\]
\[ (v, |V(n)| + 1) \text{ is due to reciprocation, } W_v = r, \text{ for some } r
\]

(7)

\[ = \gamma \pi_m + \alpha \pi_m \sum_{r=1}^{K} \rho_{r,m} \frac{|E^{in}_m(n)| + |\delta V_r(n)|}{|E(n)| + |\delta V(n)|} .
\]

Therefore, combining (6) and (7) shows that

\[ \mathbb{E}^{G_n}(|E^{in}_m(n+1)| - |E^{in}_m(n)|)
\]
\[ = \alpha \frac{|E^{in}_m(n)| + |\delta V_m(n)|}{|E(n)| + |\delta V(n)|} + \gamma \rho_m \frac{|E^{out}_m(n)| + |\delta V_m(n)|}{|E(n)| + |\delta V(n)|}
\]
\[ + \gamma \pi_m + \alpha \pi_m \sum_{r=1}^{K} \rho_{r,m} \frac{|E^{in}_m(n)| + |\delta V_r(n)|}{|E(n)| + |\delta V(n)|} .
\]

(8)

Following a similar argument, we use the relationship in (4) to obtain that

\[ \mathbb{E}^{G_n}(|E^{out}_m(n+1)| - |E^{out}_m(n)|)
\]
\[ = \gamma \frac{|E^{out}_m(n)| + |\delta V_m(n)|}{|E(n)| + |\delta V(n)|} + \alpha \rho_m \frac{|E^{out}_m(n)| + |\delta V_m(n)|}{|E(n)| + |\delta V(n)|} .
\]
\[ + \alpha \pi_m + \gamma \pi_m \sum_{r=1}^{K} \rho_{m,r} \frac{|E_r^{\text{out}}(n)| + \delta |V_r(n)|}{|E(n)| + \delta |V(n)|} , \]

In addition, summing over \( m \) in (8) gives
\[ \mathbb{E}_m^G \left( |E(n + 1)| - |E(n)| \right) = 1 + \alpha \sum_{m=1}^{K} \rho_{m,m} \frac{E_{m}(n)}{|E(n)| + \delta |V(n)|} + \gamma \sum_{m=1}^{K} \rho_{m,m} \frac{E_{m}^{\text{out}}(n)}{|E(n)| + \delta |V(n)|} . \] (9)

As remarked at the beginning of Section 2, the proof of Theorem 2.2 requires showing the convergence of \( \mathbb{E}[|E(n)|]/n \) and for this it suffices to check \( \mathbb{E}[|E_m(n)|]/n \rightarrow x_m \) and \( \mathbb{E}[|E_m^{\text{out}}(n)|]/n \rightarrow y_m \), for some \( x, y \in [0, 1] \), which is implied by the approximation \( \mathbb{E}_m^G \left( |E_m(n + 1)| - |E_m(n)| \right) \approx x_m \) and \( \mathbb{E}_m^G \left( |E_m^{\text{out}}(n + 1)| - |E_m^{\text{out}}(n)| \right) \approx y_m \). Therefore, we expect \( x_m, y_m, m = 1, \ldots, K \), to be solutions to the following system of equations:
\[ x_m = \alpha \frac{x_m + \delta \pi_m}{\sum_r x_r + \delta} + \gamma \rho_{m,m} \frac{y_m + \delta \pi_m}{\sum_r y_r + \delta} + \gamma \pi_m + \alpha \pi_m \sum_r \frac{x_r + \delta \pi_r}{\sum_r x_r + \delta} \] (10)
\[ y_m = \gamma \frac{y_m + \delta \pi_m}{\sum_r y_r + \delta} + \alpha \rho_{m,m} \frac{x_m + \delta \pi_m}{\sum_r x_r + \delta} + \alpha \pi_m \sum_r \frac{y_r + \delta \pi_r}{\sum_r y_r + \delta} \] (11)

for \( m = 1, \ldots, K \). Summing over \( m \) in (10) and (11) shows \( \sum_{m=1}^{K} x_m = \sum_{m=1}^{K} y_m \) and from (10) we get
\[ \sum_{m=1}^{K} x_m \geq \sum_{m=1}^{K} \left( \alpha \frac{x_m + \delta \pi_m}{\sum_r x_r + \delta} + \gamma \pi_m \right) = 1 , \]
and therefore \( \sum_{m=1}^{K} y_m \geq 1 \). Also, since
\[ \frac{x_m + \delta \pi_m}{\sum_r x_r + \delta} \leq 1 \quad \text{and} \quad \frac{y_m + \delta \pi_m}{\sum_r y_r + \delta} \leq 1 , \]
the right hand side of (10) is upper bounded by
\[ \alpha + \gamma \rho_{m,m} + \gamma \pi_m + \alpha \pi_m \max_r \rho_{r,m} \leq 2 . \]

Similarly, the right hand side of (11) is bounded by 2.

Lemma 2.1 shows that under some regularity conditions, the system of equations (10) and (11) has a unique solution, \((x_1, \ldots, x_K, y_1, \ldots, y_K)\) in
\[ \mathcal{Z} := \{ z \in [0, 2]^{2K} : \sum_{i=1}^{K} z_i \geq 1 , \sum_{i=K+1}^{2K} z_i \geq 1 \} . \]

The proof relies on the contraction mapping theorem [13, Theorem 1.2.2]. We proceed by defining a function \( f : \mathcal{Z} \rightarrow \mathcal{Z} \), using the right-hand-side expressions in (10) and (11). Applying the mean value theorem to \( f \), we deduce a sufficient condition for \( f \) to be a contraction by deriving an upper bound for the 1-norm of the Jacobian matrix. Due to the linked definition of \((x_m)\) and \((y_m)\) in (10) and (11), we write the Jacobian matrix as a block matrix, and find the upper bounds block by block.

**Lemma 2.1.** Let \( \otimes \) and \( \otimes \) denote the Hadamard and Kronecker products of matrices, respectively. Denote the identity matrix in \( \mathbb{R}^K \) as \( I_K \), and use \( 1_K \) to denote the vector in \( \mathbb{R}^K \).
with all entries being 1. Set π := (π1, . . . , πK), and the reciprocity matrix, ρ, whose (i, j)-th entry is ρi,j. Consider the four matrices:

\[ J^*(1, 1) = α(I_K + (π_T^1) ⊗ ρ^T), \quad J^*(1, 2) = γ(ρ^T π) ⊗ 1_K^T, \]
\[ J^*(2, 1) = α(ρπ) ⊗ 1^T_K, \quad J^*(2, 2) = γ(I_K + (π^T_1) ⊗ ρ), \]

and the block matrix

\[ J^* := \begin{bmatrix} J^*(1, 1) & J^*(1, 2) \\ J^*(2, 1) & J^*(2, 2) \end{bmatrix}. \]

Let \( ||J^*||_1 \) denote the 1-norm of the matrix \( J^* \). Then as long as

\[ \delta > \max\{||J^*||_1 - 1, 0\}, \]

the system of equations in (10) and (11) has a unique solution in \( Z \).

The proof of Lemma 2.1 is deferred to Section 6.1. Continuing to assume (12) holds, the next Theorem 2.2 gives the a.s. convergence of \( |E(n)|/n \), \( |E^{in}(n)|/n \) and \( |E^{out}(n)|/n \).

Define a constant

\[ C_δ = α \sum_{m=1}^{K} ρ_m ∀ x_m + δπ_m + γ \sum_{m=1}^{K} ρ_m ∀ y_m + δπ_m, \]

and a matrix

\[ H := \begin{bmatrix} \frac{α}{1+δ} (1 + υ_m ρ_m) & \frac{γ}{1+δ} V_m ρ_m & \frac{1}{1+δ} (α + C_δ) \\ \frac{α}{1+δ} V_m ρ_m & \frac{γ}{1+δ} (1 + υ_m ρ_m) & \frac{1}{1+δ} (γ + C_δ) \\ \frac{α}{1+δ} υ_m ρ_m & \frac{γ}{1+δ} V_m ρ_m & \frac{1}{1+δ} (C_δ) \end{bmatrix}. \]

The proof of Theorem 2.2 requires supposing λH, the largest eigenvalue of \( H \), satisfies \( λ_H < 1 \) to ensure the convergence of \( \mathbb{E}[|E(n)|]/n \), \( \mathbb{E}[|E^{in}(n)|]/n \) and \( \mathbb{E}[|E^{out}(n)|]/n \).

Define \( Δ_m^{in}(n) := |E^{in}_m(n)| + δ|V_m(n)| - n(x_m + δπ_m), Δ_m^{out}(n) := |E^{out}_m(n)| + δ|V_m(n)| - n(y_m + δπ_m) \), and \( Δ(n) := |E(n)| - n \sum_m x_m \). We will prove Theorem 2.2 by deriving iterative upper bounds such that for the matrix \( H \) given in (14),

\[ \begin{bmatrix} \sum_m |E(Δ_m^{in}(n + 1))| \\ \sum_m |E(Δ_m^{out}(n + 1))| \end{bmatrix} \leq \begin{bmatrix} I + \frac{1}{n} H \end{bmatrix} \begin{bmatrix} \sum_m |E(Δ_m^{in}(n))| \\ \sum_m |E(Δ_m^{out}(n))| \end{bmatrix}. \]

Hence, provided that all elements in \( H \) are strictly positive, applying the Perron-Frobenius theorem suggests restricting the largest eigenvalue of \( H \) will be sufficient to control the growth of the expected edge counts. In addition, the assumption of positive elements in \( H \) leads to the requirement of \( α, γ > 0 \), \( υ_m ρ_m > 0 \), and \( V_m ρ_m > 0 \).

Recall we may interpret \( ρ_m \) and \( υ_m \) as the likelihood of sending and attracting reciprocal edges for a user in group \( m \), respectively. The assumption \( υ_m ρ_m > 0 \) and \( V_m ρ_m > 0 \) requires that there exists at least one group whose probability of generating or attracting reciprocal edges is strictly positive.

**THEOREM 2.2.** Assume \( α, γ > 0 \), \( υ_m ρ_m > 0 \), \( V_m ρ_m > 0 \), \( δ > ||J^*||_F - 1 \) and \( λ_H < 1 \). Suppose that \((x_1, . . . , x_K; y_1, . . . , y_K)\) is the solution to the system of equations given in (10) and (11). We have:

(i) For \( m = 1, . . . , K \), \( |E^{in}_m(n)| \overset{a.s.}{\to} x_m \) and \( |E^{out}_m(n)| \overset{a.s.}{\to} y_m \).

(ii) In addition, \( \frac{|E(n)|}{n} \overset{a.s.}{\to} \sum_{m=1}^{K} x_m = \sum_{m=1}^{K} y_m \).
Note that since \((x_m, y_m)\) satisfies (10) and (11), the a.s. limit of \(|E(n)|/n\) depends on the value of \(\delta\) as well. This is different from the model with a homogeneous reciprocity level, where only the reciprocity parameter determines the limit of \(|E(n)|/n\).

The proof of Theorem 2.2 is in Section 6.2. Throughout the rest of the paper, assume the following regularity conditions:

\[
\alpha, \gamma > 0, \quad \delta > \|J^*\|_1 - 1, \quad \bigvee_m \rho_m > 0, \quad \bigvee_m \rho_m > 0, \quad \lambda_H < 1.
\]

### 3. Growth of Degree Counts

We next focus on the asymptotic behavior of the joint in- and out-degree counts:

\[
N_{k,l}(n) := \sum_{v \in V(n)} 1_{\{ (D^0_v(n), D^\omega_v(n)) = (k, l) \}}.
\]

We extend the homogeneous reciprocity techniques in [6, 26] to obtain the convergence of \(N_{k,l}(n)/n\) under heterogeneous reciprocity. The way forward is via embedding the in- and out-degree sequences in a family of multi-type Markov branching processes with immigration (MBI processes).

#### 3.1. Markov Branching with Immigration

To pursue count asymptotics using embedding, we need a linked family of two-type Markov branching processes ((3)) where each process is a Markov Branching with Immigration (MBI) process. See Section 2.1.1 of [26]. For each \(m \in \{1, \ldots, K\}\),

\[
\{ \xi_{\delta}(t, m) = (\xi_{\delta}^{(1)}(t, m), \xi_{\delta}^{(2)}(t, m)) : t \geq 0 \},
\]

is an MBI process whose branching structure depends on \(m\); this will be specified more precisely in (16) and (17). In the rest of the paper, we refer to \(\xi_{\delta}(\cdot, m)\) as a MBI process with group label \(m\). The process \(\xi_{\delta}(\cdot, m)\) is designed to mimic evolution of in- and out-degrees of a fixed node with communication type \(m\). The general setup of continuous-time multitype branching processes without immigration is reviewed in [2, Chapter V] and discussions on the MBI process are included, for instance, in [19, 26].

We assume life time parameters of \(\xi_{\delta}^{(1)}(\cdot, m)\) and \(\xi_{\delta}^{(2)}(\cdot, m)\) to be \(\alpha\) and \(\gamma \equiv 1 - \alpha\), respectively. For \(s = (s_1, s_2) \in [0, 1]^2\), the branching structure of \(\xi_{\delta}(\cdot, m)\) is specified through offspring generating functions:

\[
\begin{align*}
\xi_{\delta}^{(1)}(s, m) &= (1 - \rho_m) s_1^2 + \rho_m s_2^2 s_2, \\
\xi_{\delta}^{(2)}(s, m) &= (1 - \rho_m) s_2^2 + \rho_m s_1 s_2^2.
\end{align*}
\]

(16) According to (16), when a group-\(m\) type I particle’s lifetime ends, with probability \(1 - \rho_m\), it splits into two group-\(m\) type I particles, increasing the total number of group-\(m\) type I particles by 1, and with probability \(\rho_m\), it splits into two group-\(m\) type I particles and one group-\(m\) type II particle. This last eventuality increases the total numbers of group-\(m\) type I and II particles both by 1. Immigration events arrive following a homogeneous Poisson process with rate \(\delta > 0\). When an immigration event happens, \(\xi_{\delta}(\cdot, m)\) is incremented by (1, 0), (0, 1), or (1, 1) following the distribution

\[
p_0(x, m) = (\alpha(1 - \rho_m)) 1_{\{x = (1, 0)\}} \left(1_{\{x = (0, 1)\}} (\alpha \rho_m + \gamma \rho_m) \right)^{1_{\{x = (1, 1)\}}}.
\]

All immigrants are of group \(m\), and have the same branching structure as given in (16) and (17). Following the discussion in Chapter V.7.2 of [2], we use (16)–(17) to obtain a matrix that helps specify the branching structure of \(\xi_{\delta}(\cdot, m)\):

\[
A_m = \begin{bmatrix} \alpha & \alpha \rho_m \\ \gamma \rho_m & \gamma \end{bmatrix}, \quad m \in \{1, \ldots, K\}.
\]
Applying the Perron-Frobenius theorem, $A_m$ has a largest positive eigenvalue with multiplicity 1, i.e.

$$
\lambda_m = \frac{1}{2} \left( 1 + \sqrt{(\alpha - \gamma)^2 + 4\alpha\gamma\rho_{m\cdot}\rho_{m\cdot}} \right) =: \frac{1}{2} \left( 1 + \sqrt{D_0(m)} \right).
$$

Also, the smaller eigenvalue of $A_m$ is

$$
\lambda'_m = \frac{1}{2} \left( 1 - \sqrt{D_0(m)} \right),
$$

which plays an important role in the discussion of hidden regular variation in Theorem 4.2.

Let $v(m) \equiv (v^{(1)}(m), v^{(2)}(m))$, $u(m) \equiv (u^{(1)}(m), u^{(2)}(m))$ be the left and right eigenvectors associated with $\lambda_m$ respectively, with all coordinates strictly positive, and $u(m)^T 1 = 1$, $u(m)^T v(m) = 1$. Applying Theorem 1 in [26] gives that there exists some finite positive random variable $Z(m)$ such that

$$
e^{-\lambda_m t} \xi_\delta(t, m) \overset{a.s.}{\to} Z(m) v(m), \quad m \in \{1, \ldots, K\}.
$$

The PA model with heterogeneous reciprocity assumes that the communication group of a fixed node is determined at the node’s creation by flipping a $K$-sided coin and a corresponding feature must also be present in the embedding process. Let $L$ be a random variable with

$$
\mathbb{P}(L = m) = \pi_m, \quad m = 1, \ldots, K,
$$

and for $t \geq 0$, the generating function of $\xi_\delta(t, L)$ is

$$
\mathbb{E} \left( s_1^{(1)}(t, L) s_2^{(2)}(t, L) \right) = \sum_{m=1}^{K} \pi_m \mathbb{E} \left( s_1^{(1)}(t, m) s_2^{(2)}(t, m) \right), \quad s_1, s_2 \in [0, 1].
$$

Imagine $L$ assigns a label to a MBI which remains unchanged throughout the evolution of $\xi_\delta(\cdot, \cdot)$. The initial value, $\xi_\delta(0, L)$, is a random vector on $\{(0, 1), (1, 0), (1, 1)\}$, whose distribution depends on the label $L$ and will be specified during the embedding process. Conditioning on $(\xi_\delta(0, L), L)$, $\xi_\delta(\cdot, L)$ behaves like a MBI process with fixed group label $L$. In the sequel, we refer to $(\xi_\delta(t, L) : t \geq 0)$ as a two-type MBI process with random group label.

3.2. Embedding  Let $\{L_v : v \geq 1\}$ be random variables with common pmf’s as in (22). We now explain how to embed the degree sequence $\{D^\text{in}(n), D^\text{out}(n) : v \in V(n)\}_{n \geq 1}$ into a family of linked two-type MBI processes with random group labels, $\{\xi_{v, \delta}(t, L_v) : t \geq 0\}_{v \geq 1}$. We assume $\{\xi_{v, \delta}(\cdot, \cdot) : v \geq 1\}$ have the same parameters but possibly different initializations.

At $T_0 = 0$, flip a $K$-sided coin with outcome $L_1$. If $L_1 = m_1$, which occurs with probability $\pi_{m_1}$, initiate the MBI process $\xi_{1, \delta}(\cdot, m_1)$ with $\xi_{1, \delta}(0, m_1) = (1, 1)$. Let $T_1$ be the first jump time of $\xi_{1, \delta}(\cdot, m_1)$. Then for $t \geq 0$,

$$
\mathbb{P}(T_1 > t, L_1 = m_1) = \pi_{m_1} \exp \left\{ -t \left( \alpha \left( \xi_{1, \delta}(0, m_1) + \delta \right) + \gamma \left( \xi_{1, \delta}(0, m_1) + \delta \right) \right) \right\} = \pi_{m_1} e^{-t(1+\delta)},
$$

and $\mathbb{P}(T_1 > t) = e^{-t(\delta+1)}$.

At $T_1$, flip another $K$-sided coin with outcome $L_1$ and start the process $\{\xi_{2, \delta}(t - T_1, L_2) : t \geq T_1\}$. The initial value of $\xi_{2, \delta}(0, L_2)$ is set depending on which of the following four cases happens:

(i) If $L_1 = m_1$ and at time $T_1$ the process $\xi_{1, \delta}(\cdot, m_1)$ increases by $(1, 0)$, then we have $L_2 = m_2$ with (conditional) probability $\pi_{m_2}(1 - \rho_{m_1, m_2})/(1 - \rho_{m_1, \cdot})$, and set $\xi_{2, \delta}(0, m_2) = (0, 1).$
(ii) If the $\xi_{1,\delta}(:,L_1)$ with $L_1 = m_1$ is increased by $(0,1)$, then we have $L_2 = m_2$ with (conditional) probability $\pi_{m_2}(1-\rho_{m_2,m_1})/(1-\rho_{m_1})$, and set $\xi_{2,\delta}(0,m_2) = (1,0)$.

(iii) If one type I particle in $\xi_{1,\delta}(:,L_1)$ with $L_1 = m_1$ splits into 2 type I and 1 type II particles at $T_1$, then we have $L_2 = m_2$ with (conditional) probability $\pi_{m_2}\rho_{m_1,m_2}/\rho_{m_1}$, and set $\xi_{2,\delta}(0,m_2) = (1,1)$.

(iv) If one type II particle in $\xi_{1,\delta}(:,L_1)$ with $L_1 = m_1$ splits into 1 type I and 2 type II particles at $T_1$, then we have $L_2 = m_2$ with (conditional) probability $\pi_{m_2}\rho_{m_1,m_2}/\rho_{m_1}$, and set $\xi_{2,\delta}(0,m_2) = (1,1)$.

Therefore, we see that

$$\mathbb{P}(L_1 = m_1, L_2 = m_2, T_2 - T_1 > t)$$

$$= \pi_{m_1}e^{-t(1+\delta)} \left( \alpha(1-\rho_{m_1}) \frac{\pi_{m_2}(1-\rho_{m_1,m_2})}{1-\rho_{m_1}} + \alpha\rho_{m_1} \frac{\pi_{m_2}\rho_{m_1,m_2}}{\rho_{m_1}} + \gamma(1-\rho_{m_1}) \frac{\pi_{m_2}(1-\rho_{m_2,m_1})}{1-\rho_{m_1}} + \gamma\rho_{m_1} \frac{\pi_{m_2}\rho_{m_2,m_1}}{\rho_{m_1}} \right)$$

$$= \pi_{m_1} \pi_{m_2} e^{-t(1+\delta)},$$

which shows that $(L_1, L_2)$ are independent. Also, $\xi_{2,\delta}(0,L_2)$ is a 2-dimensional random vector whose generating function satisfies

$$\mathbb{E} \left( \frac{\xi_{2,\delta}^{(0,L_2)}}{s_1} \frac{\xi_{2,\delta}^{(0,L_2)}}{s_2} \bigg| L_1 \right) = \alpha(1-\rho_{L_1})s_2 + \gamma(1-\rho_{L_1})s_1 + (\alpha\rho_{L_1} + \gamma\rho_{L_1})s_1s_2,$$

so that for $\rho_0 := \sum_m \pi_{m}\rho_{m} = \sum_m \pi_{m}\rho_{m}$,

$$\mathbb{E} \left( \frac{\xi_{2,\delta}^{(0,L_2)}}{s_1} \frac{\xi_{2,\delta}^{(0,L_2)}}{s_2} \right) = \sum_m \pi_{m} \left( \alpha(1-\rho_{m})s_2 + \gamma(1-\rho_{m})s_1 + (\alpha\rho_{m} + \gamma\rho_{m})s_1s_2 \right)$$

$$= \alpha(1-\rho_0)s_2 + \gamma(1-\rho_0)s_1 + \rho_0s_1s_2,$$

where $s_1, s_2 \in [0,1]$.

Define $R_1 := 1_{\{\xi_{1,\delta}(T_1,L_1) = \xi_{1,\delta}(0,L_1) + (1,1)\}}$, so that

$$\mathbb{P}(R_1 = 1 | L_1 = m) = \alpha\rho_{m} + \gamma\rho_{m} = 1 - \mathbb{P}(R_1 = 0 | L_1 = m),$$

and $\mathbb{P}(R_1 = 1) = \rho_0 = 1 - \mathbb{P}(R_1 = 0)$. Set

$$\mathcal{F}_{T_1} := \sigma \left( \{L_k\}_{k=1,2} ; \{\xi_{k,\delta}(t-T_{k-1}, L_k) ; t \in [T_{k-1}, T_k] \}_{k=1,2} \right)$$

$$= \sigma \left( \{L_k\}_{k=1,2} ; \xi_{k,\delta}(t, L_1) , t \in [0, T_1] ; \xi_{2,\delta}(0,L_2) \right),$$

we see that $R_1$ is $\mathcal{F}_{T_1}$-measurable.

In general, for $n \geq 1$, suppose that we have initiated $n + 1$ MBI processes with group labels $\{L_k : 1 \leq k \leq n + 1\}$ at time $T_n$,

$$\{\xi_{k,\delta}(t-T_{k-1}, L_k) ; t \geq T_{k-1}\}_{1 \leq k \leq n+1}.$$

Define $T_{n+1}$ as the first time when one of the processes in (24) jumps, and let $J_{n+1}$ be the index of the process that jumps at $T_{n+1}$. Define the $\sigma$-field

$$\mathcal{F}_{T_n} := \sigma \left( \{L_k\}_{k=1}^{n+1} ; \{\xi_{k,\delta}(t-T_{k-1}, L_k) ; t \in [T_{k-1}, T_k] \}_{1 \leq k \leq n+1} \right),$$

and

$$R_{n+1} := 1_{\{\xi_{n+1,\delta}(T_{n+1} - T_{n+1} - 1 ; L_{n+1}) = \xi_{n+1,\delta}(0,T_{n+1} - T_{n+1} - 1 ; L_{n+1}) + (1,1)\}}.$$
(i) If the $\xi_{J_{n+1},\delta}(\cdot , L_{J_{n+1}})$ with $L_{J_{n+1}} = m_{n+1}$ is increased by $(1, 0)$, then we have $L_{n+2} = m_{n+2}$ with (conditional) probability $\pi_{m_{n+2}}(1 - \rho_{m_{n+1},m_{n+2}})/(1 - \rho_{m_{n+1},\bullet})$, and set $\xi_{n+2,\delta}(0, m_{n+2}) = (0, 1)$.

(ii) If the $\xi_{J_{n+1},\delta}(\cdot , L_{J_{n+1}})$ with $L_{J_{n+1}} = m_{n+1}$ is increased by $(1, 0)$, then we have $L_{n+2} = m_{n+2}$ with (conditional) probability $\pi_{m_{n+2}}(1 - \rho_{m_{n+2},m_{n+1}})/(1 - \rho_{m_{n+2},\bullet})$, and set $\xi_{n+2,\delta}(0, m_{n+2}) = (1, 0)$.

(iii) If one type I particle in $\xi_{J_{n+1},\delta}(\cdot , L_{J_{n+1}})$ with $L_{J_{n+1}} = m_{n+1}$ splits into 2 type I and 1 type II particles at $T_{n+1}$, then we have $L_{n+2} = m_{n+2}$ with (conditional) probability $\pi_{m_{n+2}}\rho_{m_{n+1},m_{n+2}}/\rho_{m_{n+1},\bullet}$, and set $\xi_{n+2,\delta}(0, m_{n+2}) = (1, 1)$.

(iv) If one type II particle in $\xi_{J_{n+1},\delta}(\cdot , L_{J_{n+1}})$ with $L_{J_{n+1}} = m_{n+1}$ splits into 1 type I and 2 type II particles at $T_{n+1}$, then we have $L_{n+2} = m_{n+2}$ with (conditional) probability $\pi_{m_{n+2}}\rho_{m_{n+1},m_{n+2}}/\rho_{m_{n+1},\bullet}$, and set $\xi_{n+2,\delta}(0, m_{n+2}) = (1, 1)$.

Since $\sum_{k=1}^{n+1} s_{k,\delta}(T_n - T_{k-1}, L_k) = \sum_{k=1}^{n+1} s_{k,\delta}(T_n - T_{k-1}) = n + 1 + \sum_{k=1}^{n} R_k$, we then have

$$P_{\mathcal{F}_{T_n}}(L_{n+2} = m, T_{n+1} - T_n > t)$$

$$= \sum_{r=1}^{K} P_{\mathcal{F}_{T_n}}(L_{n+2} = m, L_{J_{n+1}} = r, T_{n+1} - T_n > t)$$

$$= \exp \left\{-t \left(1 + \delta\right)(n + 1) + \sum_{k=1}^{n} R_k \right\}$$

$$\times \sum_{r=1}^{K} \frac{\sum_{k=1}^{n+1} s_{k,\delta}(T_n - T_{k-1}, L_k) + \delta 1_{\{L_k = r\}}}{\left(1 + \delta\right)(n + 1) + \sum_{k=1}^{n} R_k} \left(\alpha(1 - \rho_{\bullet,m})\pi_m(1 - \rho_{r,m}) + \alpha\rho_{\bullet} \pi_m\rho_{r,m} \overline{\rho_{\bullet,m}}\right)$$

$$+ \sum_{k=1}^{n+1} s_{k,\delta}(T_n - T_{k-1}, L_k) + \delta 1_{\{L_k = r\}} \left(\gamma(1 - \rho_{\bullet,r})\pi_m(1 - \rho_{m,r}) + \gamma\rho_{\bullet} \pi_m\rho_{m,r} \overline{\rho_{\bullet,r}}\right)$$

$$= \pi_m \exp \left\{-t \left(1 + \delta\right)(n + 1) + \sum_{k=1}^{n} R_k \right\}$$

$$= P_{\mathcal{F}_{T_n}}(L_{n+2} = m) P_{\mathcal{F}_{T_n}}(T_{n+1} - T_n > t).$$

Hence, $L_{n+2}$ and $T_{n+1} - T_n$ are independent under $P_{\mathcal{F}_{T_n}}$, and since

$$P_{\mathcal{F}_{T_n}}(L_{n+2} = m) = \pi_m = P(L_{n+2} = m),$$

$L_{n+2}$ is independent from $\{ L_k : 1 \leq k \leq n + 1 \}$.

In addition, from the four cases listed above, we also see that

$$P_{\mathcal{F}_{T_n}}(\xi_{n+2,\delta}(0, L_{n+2}) = (0, 1), L_{J_{n+1}} = m, L_{n+2} = r)$$

$$= \frac{\alpha\pi_r(1 - \rho_{m,r})\sum_{k=1}^{n+1} s_{k,\delta}(T_n - T_{k-1}, L_k) + \delta 1_{\{L_k = m\}}}{\alpha \sum_{k=1}^{n+1} s_{k,\delta}(T_n - T_{k-1}, L_k) + \gamma \sum_{k=1}^{n-1} s_{k,\delta}(T_n - T_{k-1}, L_k) + (n + 1)\delta}$$

$$= \alpha\pi_r(1 - \rho_{m,r})\frac{\sum_{k=1}^{n+1} s_{k,\delta}(T_n - T_{k-1}, L_k) + \delta 1_{\{L_k = m\}}}{\left(1 + \delta\right)(n + 1) + \sum_{k=1}^{n} R_k}.$$
Similarly, we have
\[
\mathbb{P}^{F_{\mathcal{T}}} (\xi_{n+2,\delta}(0, L_{n+2}) = (1, 0), L_{J_{n+1}} = m, L_{n+2} = r) = \gamma \pi_r(1 - \rho_{r,m}) \sum_{k=1}^{n+1} \left( \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \delta \right) 1 \{L_k = m\} \frac{(1 + \delta)}{(1 + \delta)(n + 1) + \sum_{k=1}^{n} R_k},
\]
and
\[
\mathbb{P}^{F_{\mathcal{T}}} (\xi_{n+2,\delta}(0, L_{n+2}) = (1, 1), L_{J_{n+1}} = m, L_{n+2} = r) = \alpha \pi_r \rho_{m,r} \sum_{k=1}^{n+1} \left( \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \delta \right) 1 \{L_k = m\} \frac{(1 + \delta)}{(1 + \delta)(n + 1) + \sum_{k=1}^{n} R_k} + \gamma \pi_r \rho_{r,m} \sum_{k=1}^{n+1} \left( \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \delta \right) 1 \{L_k = m\} \frac{(1 + \delta)}{(1 + \delta)(n + 1) + \sum_{k=1}^{n} R_k}.
\]
Therefore, under \( \mathbb{P}^{F_{\mathcal{T}}} \), \( \xi_{n+2,\delta}(0, L_{n+2}) \) is a random vector following the distribution
\[
p_{n+2,\delta}(x) = \left( \sum_{m=1}^{K} \frac{\alpha(1 - \rho_{m,m}) \sum_{k=1}^{n+1} \left( \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \delta \right) 1 \{L_k = m\} (1 + \delta)}{(1 + \delta)(n + 1) + \sum_{k=1}^{n} R_k} \right)^{1 \{x = (0, 1)\}} \times \left( \sum_{m=1}^{K} \frac{\gamma(1 - \rho_{m,m}) \sum_{k=1}^{n+1} \left( \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \delta \right) 1 \{L_k = m\} (1 + \delta)}{(1 + \delta)(n + 1) + \sum_{k=1}^{n} R_k} \right)^{1 \{x = (1, 0)\}} \times \left( \sum_{m=1}^{K} \frac{\alpha \rho_{m,m} \sum_{k=1}^{n+1} \left( \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \delta \right) 1 \{L_k = m\} (1 + \delta)}{(1 + \delta)(n + 1) + \sum_{k=1}^{n} R_k} + \frac{\gamma \rho_{m,m} \sum_{k=1}^{n+1} \left( \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \delta \right) 1 \{L_k = m\} (1 + \delta)}{(1 + \delta)(n + 1) + \sum_{k=1}^{n} R_k} \right) \right)^{1 \{x = (1, 1)\}}.
\]
Meanwhile, since \( \{L_k : 1 \leq k \leq n+1\} \) are \( F_{\mathcal{T}} \)-measurable, we then have
\[
\mathbb{P}^{F_{\mathcal{T}}} (R_{n+1} = 1, J_{n+1} = w, T_{n+1} - T_n > t) = \alpha \rho_{L,w} \left( \xi_{w,\delta}(T_n - T_{w-1}, L_w) + \delta \right) + \gamma \rho_{L,w} \left( \xi_{w,\delta}(T_n - T_{w-1}, L_w) + \delta \right) = \frac{\alpha}{n+1} \sum_{k=1}^{n+1} \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \gamma \rho_{L,w} \left( \xi_{w,\delta}(T_n - T_{w-1}, L_w) + \delta \right) \left( \alpha \sum_{k=1}^{n+1} \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \gamma \sum_{k=1}^{n+1} \xi_{k,\delta}(T_n - T_{k-1}, L_k) + (n + 1) \delta \right) \times \exp \left\{ -t \left( \alpha \sum_{k=1}^{n+1} \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \gamma \sum_{k=1}^{n+1} \xi_{k,\delta}(T_n - T_{k-1}, L_k) + (n + 1) \delta \right) \right\}
\]
\[
= \alpha \rho_{L,w} \left( \xi_{w,\delta}(T_n - T_{w-1}, L_w) + \delta \right) + \gamma \rho_{L,w} \left( \xi_{w,\delta}(T_n - T_{w-1}, L_w) + \delta \right) \frac{e^{-t(1 + \delta, L_k)} \exp \left\{ -t \left( \alpha \sum_{k=1}^{n+1} \xi_{k,\delta}(T_n - T_{k-1}, L_k) + \gamma \sum_{k=1}^{n+1} \xi_{k,\delta}(T_n - T_{k-1}, L_k) + (n + 1) \delta \right) \right\}}{(n + 1)(1 + \delta) + \sum_{k=1}^{n} R_k}
\]
(26)
\[
= \mathbb{P}^{F_{\mathcal{T}}} (R_{n+1} = 1, J_{n+1} = w) \mathbb{P}^{F_{\mathcal{T}}} (T_{n+1} - T_n > t).
\]
Therefore, \( (R_{n+1}, J_{n+1}) \) is independent from \( T_{n+1} - T_n \) under \( \mathbb{P}^{F_{\mathcal{T}}} \).
Define for $n \geq 0$,
\[
\xi_n^\delta(T_n, L_{\lfloor n + 1 \rfloor}) := (\xi_1, \delta(T_n, 1), \xi_2, \delta(T_n - T_1, L_2), \ldots, \xi_{n+1, \delta}(0, L_{n+1}), (0, 0), \ldots),
\]
then the embedding framework just described shows that $\{\xi_n^\delta(T_n, L_{\lfloor n + 1 \rfloor}) : n \geq 0\}$ is Markov with state space $(\mathbb{N}^2)$. The next theorem gives the embedding of in- and out-degree sequences in the PA model with heterogeneous reciprocity into a linked system of delayed MBI processes.

**Theorem 3.1.** In $(\mathbb{N}^2)\infty$, define the in- and out-degree sequences as
\[
D(n) := ((D_{\text{in}}(n), D_{\text{out}}(n)), \ldots, (D_{\text{in}}(n+1, n), D_{\text{out}}(n+1, n)), (0, 0), \ldots)
\]
Then for $\{T_k : k \geq 0\}$ and $\xi_{k, \delta}(t - T_k-1, L_k) : t \geq T_k-1$ $k \geq 1$ constructed above, we have that in $(\mathbb{N}^2)\infty$,
\[
\{D(n) : n \geq 0\} \overset{d}{=} \{\xi_n^\delta(T_n, L_{\lfloor n + 1 \rfloor}) : n \geq 0\}.
\]

**Proof.** By the model description in Section 1.1, $\{D(n) : n \geq 0\}$ is Markovian on $(\mathbb{N}^2)\infty$, so it suffices to check the transition probability from $D(n)$ to $D(n+1)$ agrees with that from $\xi_n^\delta(T_n, L_{\lfloor n + 1 \rfloor})$ to $\xi_{n+1}^\delta(T_{n+1}, L_{\lfloor n + 2 \rfloor})$. Write
\[
e^{(1)}_v := \left( (0, 0), \ldots, (0, 0), (1, 0), (0, 0), \ldots \right),
\]
\[
e^{(2)}_v := \left( (0, 0), \ldots, (0, 0), (0, 1), (0, 0), \ldots \right),
\]
\[
e^{(3)}_v := \left( (0, 0), \ldots, (0, 0), (1, 1), (0, 0), \ldots \right),
\]
and we have
\[
\mathbb{P}^{G_n} \left( D(n+1) = D(n) + e^{(1)}_v + e^{(2)}_{|V(n)|+1} \right) = \frac{\alpha(D_{\text{in}}(n) + \delta)}{|E(n)| + \delta|V(n)|} \sum_{m=1}^K (1 - \rho_{W_{m,m}})\\(1 - \rho_{W_{v,v}});\\(27)
\]
similarly,
\[
\mathbb{P}^{G_n} \left( D(n+1) = D(n) + e^{(2)}_v + e^{(1)}_{|V(n)|+1} \right) = \frac{\gamma(D_{\text{out}}(n) + \delta)}{|E(n)| + \delta|V(n)|} (1 - \rho_{W_{v,v}}),\\(28)
\]
\[
\mathbb{P}^{G_n} \left( D(n+1) = D(n) + e^{(3)}_v + e^{(3)}_{|V(n)|+1} \right) = \frac{\alpha \rho_{W_{v,v}}(D_{\text{in}}(n) + \delta) + \gamma \rho_{W_{v,v}}(D_{\text{out}}(n) + \delta)}{|E(n)| + \delta|V(n)|},\\(29)
Note that $|V(n)| = n + 1$ for all $n \geq 0$, and from (26), we have
\[
\mathbb{P}_{F_{n}}(\xi_{\delta}^{*}(T_{n+1}, L_{[n+2]}) = \xi_{\delta}^{*}(T_{n}, L_{[n+1]}) + e_{v}^{(3)} + e_{[V(n)]+1}^{(3)})
= \mathbb{P}_{F_{n}}(R_{n+1} = 1, J_{n+1} = v)
= \alpha \rho_{L_{v}} \cdot \left( \xi_{v, \delta}^{(1)}(T_{n+1} - T_{v-1}, L_{v}) + \delta \right) + \gamma \rho_{L_{v}} \cdot \left( \xi_{v, \delta}^{(2)}(T_{n} - T_{v-1}, L_{v}) + \delta \right)
(30)
\]

So it remains to check whether $\sum_{k=1}^{n} R_{k}$ has the same distribution as $|E(n)| - (n + 1)$. Again, applying (26) gives that
\[
\mathbb{P}_{F_{n}}(R_{n+1} = 1)
= \sum_{v=1}^{n} \alpha \rho_{L_{v}} \cdot \left( \xi_{v, \delta}^{(1)}(T_{n} - T_{v-1}, L_{v}) + \delta \right) + \gamma \rho_{L_{v}} \cdot \left( \xi_{v, \delta}^{(2)}(T_{n} - T_{v-1}, L_{v}) + \delta \right)
(31)
\]

By (25), we see that $\{L_{v}: v \geq 1\}$ are iid random variables with $\mathbb{P}(L_{v} = m) = \pi_{m}$, agreeing with the distributional property of $\{W_{v}: v \geq 1\}$. Therefore, we conclude from (31) that $\sum_{k=1}^{n} R_{k}$ has the same distribution as $|E(n)| - (n + 1)$, which implies the agreement between the transition probabilities in (29) and (30).

3.3. Asymptotic Growth of Empirical Degree Frequencies We now use the embedding results in Theorem 3.1 to prove the convergence of $N_{k,l}(n)/n$.

**Theorem 3.2.** Let $\{\tilde{\xi}_{\delta}(t, L^{*}): t \geq 0\}$ be a MBI process with random group label $L^{*}$, where $L^{*}$ satisfies $\mathbb{P}(L^{*} = m) = \pi_{m}$, $m = 1, \ldots, K$. Suppose that the regularity condition (15) holds, and the initialization, $\tilde{\xi}_{\delta}(0, L^{*})$, satisfies that for $s_{i} \in [0, 1]$, $i = 1, 2$,
\[
\mathbb{E}\left( s_{1}^{(1)}(0, L^{*}) s_{2}^{(2)}(0, L^{*}) \right)
= \sum_{r=1}^{K} \pi_{r} \left[ \alpha \left( 1 - \sum_{m=1}^{K} \rho_{m,r} \frac{x_{m} + \delta \pi_{m}}{s_{1}} \right) s_{2} + \gamma \left( 1 - \sum_{m=1}^{K} \rho_{m,r} \frac{y_{m} + \delta \pi_{m}}{s_{2}} \right) s_{1} \right. \right.
\]
\[
+ \left. \left( \alpha \sum_{m=1}^{K} \rho_{m,r} \frac{x_{m} + \delta \pi_{m}}{s_{1}} + \gamma \sum_{m=1}^{K} \rho_{m,r} \frac{y_{m} + \delta \pi_{m}}{s_{2}} \right) s_{1}s_{2} \right] .
(32)
\]
For $L^{*} = m$, the branching structure of $\{\tilde{\xi}_{\delta}(t, m): t \geq 0\}$ is given by $A_{m}$ (cf. (19)). Write
\[
\rho^{*} := \sum_{m=1}^{K} x_{m} - 1 = \sum_{m=1}^{K} y_{m} - 1 > 0 \text{ and } c^{*} := 1 + \rho^{*} + \delta.
(33)
\]
then as $n \to \infty$, we have for $k, l \geq 0$,

$$
(34) \quad \frac{N_{k,l}(n)}{n} \rightarrow \mathbb{P} \left( \xi_\delta(t, m) = (k, l) \right) dt.
$$

Let $T^*$ be an exponential random variable with rate $c^*$, independent from $\{\xi_\delta(\cdot, m) : m = 1, \ldots, K\}$, then the integral on the right hand side of (34) is

$$
P \left( \xi_\delta(T^*, m) = (k, l) \right), \quad m = 1, \ldots, K,
$$

representing the limiting in- and out-degree frequencies for nodes of communication group $m$. In the sequel, set $(I_m, O_m) = \xi_\delta(T^*, m)$ to denote the limit random variables so that (34) becomes

$$
(35) \quad \frac{N_{k,l}(n)}{n} \rightarrow \frac{p}{m} \sum_{m=1}^{K} \pi_m P \left[ (I_m, O_m) = (k, l) \right].
$$

**Proof.** By the embedding results in Theorem 3.1, we have

$$
(36) \quad \frac{N_{k,l}(n)}{n} = \frac{1}{n} \sum_{w=2}^{n+1} 1_{\{\xi_{w,\delta}(T_{n-T_w-1},L_w)=(k,l)\}} + \frac{1}{n} 1_{\{\xi_{1,\delta}(T_n,L_1)=(k,l)\}},
$$

where the second term goes to 0 a.s. as $n \to \infty$. Then we only need to consider the limit of the first term in (36).

Let $\{\xi_{w,\delta}(t, L_w) : t \geq 0\}_{w=1}^\infty$ be a sequence of iid MBI processes with random labels, which have the same distributional properties as $\xi_\delta(\cdot, L^*)$ and satisfy the initialization condition in (32). Then we divide the first term in (36) into different parts:

$$
\frac{1}{n} \sum_{w=2}^{n+1} 1_{\{\xi_{w,\delta}(T_{n-T_w-1},L_w)=(k,l)\}}
$$

$$
= \frac{1}{n} \sum_{w=2}^{n+1} \left( 1_{\{\xi_{w,\delta}(T_{n-T_w-1},L_w)=(k,l)\}} - 1_{\{\xi_{w,\delta}(\log(n/w)/c^*,L_w)=(k,l)\}} \right)
$$

$$
+ \frac{1}{n} \sum_{w=2}^{n+1} \left( 1_{\{\xi_{w,\delta}(\log(n/w)/c^*)=(k,l)\}} - \mathbb{P}_{F^{T_{w-1}}} \left[ \xi_{w,\delta} \left( \frac{1}{c^*} \log(n/w), L_w \right) = (k, l) \right] \right)
$$

$$
+ \frac{1}{n} \sum_{w=2}^{n+1} \left( \mathbb{P}_{F^{T_{w-1}}} \left[ \xi_{w,\delta} \left( \frac{1}{c^*} \log(n/w), L_w \right) = (k, l) \right] \right.
$$

$$
- \mathbb{P} \left[ \xi_{w,\delta} \left( \frac{1}{c^*} \log(n/w), \bar{L}_w \right) = (k, l) \right| \xi_{w,\delta}(0, \bar{L}_w), \bar{L}_w \right]
$$

$$
+ \left( \frac{1}{n} \sum_{w=2}^{n+1} \mathbb{P} \left[ \xi_{w,\delta} \left( \frac{1}{c^*} \log(n/w), \bar{L}_w \right) = (k, l) \right| \xi_{w,\delta}(0, \bar{L}_w), \bar{L}_w \right]
$$

$$
- \int_0^1 \mathbb{P} \left[ \xi_\delta \left( -\frac{1}{c^*} \log t, L^* \right) = (k, l) \right] dt
$$

$$
+ \int_0^1 \mathbb{P} \left[ \xi_\delta \left( -\frac{1}{c^*} \log t, L^* \right) = (k, l) \right] dt
$$

$$
=: A_1(n) + A_2(n) + A_3(n) + A_4(n) + A_5.
$$
Note that by a change of variable argument, \( A_5 \) is identical to the right hand side of (34), and we now show that \( A_j(n) \xrightarrow{p} 0 \), for \( j = 1, 2, 3, 4 \).

For \( A_1(n) \), we have

\[
\mathbb{E}|A_1(n)| \leq \frac{1}{n} \sum_{w=2}^{n+1} \mathbb{E} \left| 1 \{\xi^{(i)}_{w,d}(T_n-T_{w-1},L_w)=k\} - 1 \{\xi^{(i)}_{w,d}(\log(n/w)/c^*,L_w)=k\} \right|
\]

(37)

\[
+ \frac{1}{n} \sum_{w=2}^{n+1} \mathbb{E} \left| 1 \{\xi^{(i)}_{w,d}(T_n-T_{w-1},L_w)=l\} - 1 \{\xi^{(i)}_{w,d}(\log(n/w)/c^*,L_w)=l\} \right|
\]

Since both \( \xi^{(1)}_{w,d}(\cdot, L_w) \) and \( \xi^{(2)}_{w,d}(\cdot, L_w) \) have finite number of jumps in any finite time interval \([0, K] \) a.s., then applying Lemma 3.1 in [1] gives that for all \( K > 0 \) and \( w \geq 2 \),

\[
\lim_{\epsilon \downarrow 0} \sup_{t \in [0, K]} \mathbb{P} \left( \xi^{(i)}_{w,d}(t + \epsilon, L_w) - \xi^{(i)}_{w,d}(t - \epsilon, L_w) \right) = 0, \quad i = 1, 2.
\]

Also, we see from [1, Corollary 2.1(iii)] that for \( \eta > 0 \),

\[
\sup_{\eta \leq w \leq n} \left| T_n - T_{w-1} - \frac{1}{c^*} \log(n/w) \right| \xrightarrow{a.s.} 0.
\]

Then using techniques from [1, Theorem 1.2, pp 489–490] further gives

\[
\left| 1 \{\xi^{(i)}_{w,d}(T_n-T_{w-1},L_w)=k\} - 1 \{\xi^{(i)}_{w,d}(\log(n/w)/c^*,m)=k\} \right|
\]

\[
\leq \sup_{w \geq 1} \sup_{1 \leq m \leq K} \sup_{t \in [0, -\log \eta/c^*]} \mathbb{P} \left( \xi^{(1)}_{w,d}(t + \epsilon, m) - \xi^{(1)}_{w,d}(t - \epsilon, m) \right) \geq 1
\]

\[
+ \mathbb{P} \left( \sup_{\eta \leq w \leq n} \left| T_n - T_{w-1} - \frac{1}{c^*} \log(n/w) \right| \geq \epsilon \right) =: p_1(\epsilon, \eta).
\]

Similarly,

\[
\left| 1 \{\xi^{(i)}_{w,d}(T_n-T_{w-1},L_w)=l\} - 1 \{\xi^{(i)}_{w,d}(\log(n/w)/c^*,L_w)=l\} \right|
\]

\[
\leq \sup_{w \geq 1} \sup_{1 \leq m \leq K} \sup_{t \in [0, -\log \eta/c^*]} \mathbb{P} \left( \xi^{(2)}_{w,d}(t + \epsilon, m) - \xi^{(2)}_{w,d}(t - \epsilon, m) \right) \geq 1
\]

\[
+ \mathbb{P} \left( \sup_{\eta \leq w \leq n} \left| T_n - T_{w-1} - \frac{1}{c^*} \log(n/w) \right| \geq \epsilon \right) =: p_2(\epsilon, \eta).
\]

Then by (37), we see that

\[
\mathbb{E}|A_1(n)| \leq 2 \cdot \frac{1}{n} \cdot \eta \cdot n + \frac{1}{n} (1 - \eta) n (p_1(\epsilon, \eta) + p_2(\epsilon, \eta)),
\]

which implies \( \lim_{n \to \infty} \mathbb{E}|A_1(n)| = 0 \). Therefore, \( A_1(n) \xrightarrow{p} 0 \).

For \( A_2(n) \), define

\[
X_w := \mathbb{1} \{\xi_{w,d}(\frac{1}{c^*} \log(n/w), L_w) = (k, l)\} - \mathbb{P}F_{w-1} \left( \xi_{w,d} \left( \frac{1}{c^*} \log(n/w), L_w \right) = (k, l) \right),
\]

then we see that \( \mathbb{E}(X_w) = \mathbb{E} \left( \mathbb{E}F_{w-1}X_w \right) = 0 \). Also, for \( u < w \), since

\[
\mathbb{E}(X_w X_u) = \mathbb{E} \left( \mathbb{E}F_{w-1}X_w \mathbb{E}F_{w-1}X_u \right) = \mathbb{E} \left( \mathbb{E}F_{w-1}X_w \mathbb{E}F_{w-1} \right) = 0,
\]

\[
\mathbb{E}(X_w X_u) = \mathbb{E} \left( \mathbb{E}F_{w-1}(X_w X_u) \right) = \mathbb{E} \left( \mathbb{E}F_{w-1}(X_u) \right) = 0,
\]
then by the weak law of large numbers, we have $A_2(n) \overset{p}{\to} 0$.

For $A_3(n)$, we first note that for $w \geq 2$ and $t \geq 0$,

$$
\mathbb{P}^{S_{tw-1}}(\xi_{w,\delta}(t, L_w) = (k, l)) = \mathbb{P}(\xi_{w,\delta}(t, L_w) = (k, l) | \xi_{w,\delta}(0, L_w), L_w).
$$

Then we have

$$
|A_3(n)| \leq \frac{1}{n} \sum_{w=2}^{n+1} \mathbb{1}\{ (\xi_{w,\delta}(0, L_w), L_w) \neq (\tilde{\xi}_{w,\delta}(0, L_w), L_w) \}.
$$

Hence, to prove $A_3(n) \overset{p}{\to} 0$, it suffices to show

$$
(38) \quad \frac{1}{n} \sum_{w=2}^{n+1} \mathbb{P}\left[ (\xi_{w,\delta}(0, L_w), L_w) \neq (\tilde{\xi}_{w,\delta}(0, L_w), L_w) \right] \to 0.
$$

Let $\mathcal{X} := \{(0, 1), (1, 0), (1, 1)\} \times \{1, \ldots, K\}$, and we note that

$$
\mathbb{P}\left[ (\xi_{w,\delta}(0, L_w), L_w) \neq (\tilde{\xi}_{w,\delta}(0, L_w), L_w) \right]
\leq \sum_{x \in \mathcal{X}} \mathbb{P}[\{ (\xi_{w,\delta}(0, L_w), L_w) = x \} - \mathbb{P}[\{ (\xi_{w,\delta}(0, L_w), L_w) = x \}]
$$

$$
\leq 2\alpha \sum_{m=1}^{K} \rho_m \cdot \mathbb{E}\left( \frac{|E_{in}(w-1) + \delta |V_m(w-1)|}{|E(w-1)| + \delta |V(w-1)|} \right) - \frac{x_m + \delta \pi_m}{\sum_s x_s + \delta}
$$

$$
+ 2\gamma \sum_{m=1}^{K} \rho_m \cdot \mathbb{E}\left( \frac{|E_{out}(w-1) + \delta |V_m(w-1)|}{|E(w-1)| + \delta |V(w-1)|} \right) - \frac{y_m + \delta \pi_m}{\sum_s y_s + \delta}.
$$

From the proof of Theorem 2.2, we see that there exist constants $C_1(\delta), C_2(\delta) > 0$ such that

$$
\mathbb{E}\left( \frac{|E_{in}(w-1) + \delta |V_m(w-1)|}{|E(w-1)| + \delta |V(w-1)|} \right) - \frac{x_m + \delta \pi_m}{\sum_s x_s + \delta} \leq C_1(\delta) w^{\lambda_H - 1},
$$

and

$$
\mathbb{E}\left( \frac{|E_{out}(w-1) + \delta |V_m(w-1)|}{|E(w-1)| + \delta |V(w-1)|} \right) - \frac{y_m + \delta \pi_m}{\sum_s y_s + \delta} \leq C_2(\delta) w^{\lambda_H - 1}.
$$

Then the left hand side of (38) is bounded by

$$
2 \sum_{w=2}^{n+1} \sum_{m=1}^{K} (\rho_m \cdot C_1(\delta) + \rho_m \cdot C_2(\delta)) w^{\lambda_H - 1}, \quad \lambda_H < 1,
$$

which goes to 0 as $n \to \infty$, thus proving the claim in (38).

For $A_4(n)$, since $\{(\xi_{w,\delta}(0, L_w), L_w) : w \geq 1\}$ are iid random vectors in $\mathcal{X}$, then

$$
\frac{1}{n} \sum_{w=2}^{n+1} \mathbb{P}\left[ \tilde{\xi}_{w,\delta}\left( \frac{1}{c^*} \log(n/w), \tilde{L}_w \right) = (k, l) | \tilde{\xi}_{w,\delta}(0, \tilde{L}_w), \tilde{L}_w \right]
$$

$$
- \mathbb{P}\left[ \tilde{\xi}_{w,\delta}\left( \frac{1}{c^*} \log(n/w), \tilde{L}_w \right) = (k, l) \right] \overset{p}{\to} 0.
$$

Also, by the definition of $\tilde{\xi}_{w,\delta}$, $w \geq 1$, we see that

$$
\frac{1}{n} \sum_{w=2}^{n+1} \mathbb{P}\left[ \tilde{\xi}_{w,\delta}\left( \frac{1}{c^*} \log(n/w), \tilde{L}_w \right) = (k, l) \right] = \frac{1}{n} \sum_{w=2}^{n+1} \mathbb{P}\left[ \tilde{\xi}_{\delta}\left( \frac{1}{c^*} \log(n/w), L^x \right) = (k, l) \right].
$$
Since the function \( \mathbb{P}[\xi_\delta(t, L^*) = (k, l)] \) is bounded and continuous in \( t \), then we conclude that \( A_4(n) \xrightarrow{p} 0 \) by applying the Riemann integrability of \( \mathbb{P} \left[ \xi_\delta(- \log t/c^*, L^*) = (k, l) \right] \), which completes the proof of (34).

\[ \Box \]

4. Power Laws and Asymptotic Dependence of Degree Frequencies In this section, we study the dependence between large in- and out-degrees by examining the asymptotic behavior of the distribution \( \mathbb{P}[(T_m, O_m) \in \cdot] \), for each \( m = 1, \ldots, K \).

4.1. Multivariate and Hidden Regular Variation To formalize our analysis, we provide some useful definitions related to \textit{multivariate regular variation} (MRV) and \textit{hidden regular variation} (HRV) of distributions.

Suppose that \( C_0 \subset C \subset \mathbb{R}^2_+ \) are two closed cones, and we provide the definition of \( \mathbb{M} \)-convergence in Definition 4.1 (cf. [4, 7, 10, 15, 16]) on \( C \setminus C_0 \), which lays the theoretical foundation of regularly varying measures (cf. Definition 4.2).

**Definition 4.1.** Let \( \mathbb{M}(C \setminus C_0) \) be the set of Borel measures on \( C \setminus C_0 \) which are finite on sets bounded away from \( C_0 \), and \( C(C \setminus C_0) \) be the set of continuous, bounded, non-negative functions on \( C \setminus C_0 \) whose supports are bounded away from \( C_0 \). Then for \( \mu_n, \mu \in \mathbb{M}(C \setminus C_0) \), we say \( \mu_n \rightarrow \mu \) in \( \mathbb{M}(C \setminus C_0) \), if \( \int f \, d\mu_n \rightarrow \int f \, d\mu \) for all \( f \in C(C \setminus C_0) \).

Without loss of generality [16], we can and do take functions in \( C(C \setminus C_0) \) to be uniformly continuous as well. Denote the modulus of continuity of a uniformly continuous function \( f : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) by

\[
\Delta_f(\delta) = \sup\{|f(x) - f(y)| : d(x, y) < \delta\}
\]

where \( d(\cdot, \cdot) \) is an appropriate metric on the domain of \( f \). Uniform continuity means \( \lim_{\delta \to 0} \Delta_f(\delta) = 0 \). We now present the definition of multivariate regular variation with \( C = \mathbb{R}^2_+ \) and \( C_0 = \{0\} \).

Following Definition 2.1 in [20], we denote a regularly varying function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with index \( a \in \mathbb{R} \), as \( f \in RV_a \). Definition 4.2 gives the formal description of the MRV of distributions.

**Definition 4.2.** The distribution \( \mathbb{P}(Z \in \cdot) \) of a random vector \( Z \) on \( \mathbb{R}^2_+ \), is (standard) regularly varying on \( \mathbb{R}^2_+ \setminus \{\mathbf{0}\} \) with index \( c > 0 \) if there exists some regularly varying scaling function \( b(t) \in RV_{1/c} \) and a limit measure \( \nu(\cdot) \in \mathbb{M}(\mathbb{R}^2_+ \setminus \{\mathbf{0}\}) \) such that as \( t \to \infty \),

\[
t \mathbb{P}(Z/b(t) \in \cdot) \rightarrow \nu(\cdot), \quad \text{in } \mathbb{M}(\mathbb{R}^2_+ \setminus \{\mathbf{0}\}).
\]

\( It \text{ is convenient to write } \mathbb{P}(Z \in \cdot) \in MRV(c, b(t), \nu, \mathbb{R}^2_+ \setminus \{\mathbf{0}\}). \)

When analyzing the asymptotic dependence between components of a bivariate random vector \( Z \) satisfying (40), it is often informative to make a polar coordinate transform and consider the transformed points located on the \( L_1 \) unit sphere

\[
(x, y) \mapsto \left( \frac{x}{x+y}, \frac{y}{x+y} \right),
\]

after thresholding the data according to the \( L_1 \) norm.

When a limit measure concentrates on a subcone of the full state space, to improve estimates of probabilities in the complement of the subcone, we can seek a second hidden regular variation regime after removing the subcone.
**Definition 4.3.** The vector \( Z \) is regularly varying on \( \mathbb{R}_+^2 \setminus \{0\} \) and has hidden regular variation on \( \mathbb{R}_+^2 \setminus C_0 \) if there exist \( 0 < c \leq c_0 \), scaling functions \( b(t) \in RV_{1/c} \) and \( b_0(t) \in RV_{1/c_0} \) with \( b(t)/b_0(t) \to \infty \) and limit measures \( \nu, \nu_0 \), such that

\[
P(Z \in \cdot) \in MRV(\nu, b(t), \nu_0, \mathbb{R}_+^2 \setminus \{C_0\}).
\]

A convenient way to characterize HRV is through the generalized polar coordinate transformation for \( \mathbb{R}_+^2 \setminus C_0 \) and an associated metric \( d(\cdot, \cdot) \) satisfying \( d(cx, cy) = cd(x, y) \) for scalars \( c > 0 \). The metric \( d(\cdot, \cdot) \) that we use in practice is the \( L_1 \)-metric. When using generalized polar coordinates with respect to the forbidden zone \( C_0 \), we define \( \mathcal{N}_{C_0} := \{ x \in \mathbb{C} \setminus C_0 : d(x, C_0) = 1 \} \), the locus of points at distance 1 from \( C_0 \). Then the generalized polar coordinates are specified through the transformation, \( \text{GPOLAR} : \mathbb{R}_+^2 \setminus C_0 \leftrightarrow (0, \infty) \times \mathcal{N}_{C_0} \) with

\[
\text{GPOLAR}(x) = \left( \frac{x}{d(x, C_0)}, \frac{d(x, C_0)}{d(x, C_0)} \right).
\]

Let \( \nu_c(\cdot) \) be a measure in \( \mathcal{M}(\mathbb{R}_+ \setminus \{0\}) \) satisfying \( \nu_c(x, \infty) = x^{-c} \), \( c > 0 \), and \( S_0(\cdot) \) be a probability measure on \( \mathcal{N}_{C_0} \). Then generalized polar coordinates allow re-writing (42) as

\[
t \mathbb{P} \left[ \left( d(Z, C_0), \frac{Z}{b_0(t)} \right) \in \cdot \right] \to (\nu_c \times S_0)(\cdot)
\]

in \( \mathcal{M}((\mathbb{R}_+ \setminus \{0\}) \times \mathcal{N}_{C_0}) \). See [7] and [16] for details.

**4.2. Degree Frequencies and HRV** Let \( L^* \) be a random variable with pmf \( \mathbb{P}(L^* = m) = \pi_m, \ m = 1, \ldots, K \), independent from \( T^* \) and \( \{\xi_\delta(\cdot, m) : m = 1, \ldots, K\} \). Using (34), the limit in (35) becomes

\[
\sum_{m=1}^K \pi_m \mathbb{P}\left( I_m, O_m = (k, l) \right) = \mathbb{P}\left( \xi_\delta(T^*, L^*) = (k, l) \right) =: \mathbb{P}\left( (I, O) = (k, l) \right),
\]

the limiting empirical proportion of nodes with in-degree \( k \) and out-degree \( l \).

We now discuss the regular variation properties of this limit distribution. The branching structure of \( \xi_\delta(\cdot, m) \) is specified through the matrix \( A_m \) given in (19), and the largest eigenvalue of \( A_m \) is \( \lambda_m \) given in (20). We assume parameters \( \alpha, \gamma \) and the matrix \( \rho \), are chosen such that \( \lambda_m, m = 1, \ldots, K \), are all distinct values and without loss of generality, assume the behavioral group labels are chosen so that

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_K.
\]

Theorem 4.1 gives the multivariate regular variation of \( \mathbb{P}(I, O) \in \cdot \) on \( \mathbb{R}_+^2 \setminus \{0\} \). The proof is based on an extended Breiman’s theorem [26, Theorem 3], reviewed in Theorem A.1. Theorem 4.1 requires an additional assumption that \( \lambda_1 \geq \log 2 \) to guarantee that the moment condition in (69) is satisfied.

**Theorem 4.1.** Recall the definition of \( \rho^*, c^* \) in (33), suppose \( \lambda_1 \geq \log 2 \) and that the regularity conditions (15) hold. Then, as \( t \to \infty \),

\[
t \mathbb{P}\left[ \left( I, O \right) \in \cdot \right] \to \mu_1, \quad \text{in} \quad \mathcal{M}(\mathbb{R}_+^2 \setminus \{0\}),
\]

where the limiting measure \( \mu_1 \in \mathcal{M}(\mathbb{R}_+^2 \setminus \{0\}) \) satisfies that for \( f \in C(\mathbb{R}_+^2 \setminus \{0\}) \),

\[
\mu_1(f) = \mathbb{P}(L^* = 1) \int_0^\infty \mathbb{E} \left( f(y \tilde{Z}(1) \nu(1)) \right) \nu_{c^*/\lambda_1}(dy).
\]

\[
= \mathbb{P}(L^* = 1) \mathbb{E} \left( \tilde{Z}(1)^{c^*/\lambda_1} \right) \int_0^\infty f(s \nu(1)) \nu_{c^*/\lambda_1}(ds).
\]
Here \( \tilde{Z}(1) \) satisfies \( e^{-\lambda t} \tilde{\xi}_\delta(t, 1) \overset{a.s.}{\rightarrow} \tilde{Z}(1) \nu(1) \), and

\[
(44) \quad a(1) := \frac{v(2)}{v(1)}(1) = \frac{\gamma - \alpha + \sqrt{D_0(1)}}{2\gamma \rho_{11}}.
\]

Write

\[
(45) \quad a(m) := \frac{v(2)(m)}{v(1)(m)}, \quad m = 1, \ldots, K.
\]

Theorem 4.1 shows that the limiting measure \( \mu_1 \) concentrates Pareto mass on the ray \( \mathcal{L}(1) := \{(x, y) \in \mathbb{R}^2_+ : y = a(1)x\} \) and concentration on \( \mathcal{L}(1) \) suggests that at scale \( b(t) = t^{\lambda_1/c^*} \), large in- and out-degree pairs in the PA model with heterogeneous reciprocity levels satisfy \( \mathcal{O} \approx a(1) \mathcal{I} \).

**Proof.** Similar to the proof of Theorem 6 in [26], we first see that \( \mathbb{P}(\tilde{Z}_1 > 0) = 1 \) by the property of MBI processes. Also, since \( \lambda_1 > \lambda_r \) for \( r = 2, \ldots, K \),

\[
e^{-\lambda_1 t} \tilde{\xi}_\delta(t, 1) \overset{a.s.}{\rightarrow} 0.
\]

Therefore,

\[
e^{-\lambda_1 t} \tilde{\xi}_\delta(t, L^*) \overset{a.s.}{\rightarrow} \tilde{Z}(1) \nu(1) \mathbf{1}_{\{L^* = 1\}}
\]

Then the proof of Theorem 4.1 is an application of Theorem A.1 after making the identifications

\[
\xi(t) = t^{-1} \tilde{\xi}_\delta \left( \frac{1}{\lambda_1} \log t, L^* \right), \quad \xi_\infty = \tilde{Z}(1) \nu(1) \mathbf{1}_{\{L^* = 1\}}, \quad X = e^{\lambda_1 T^*},
\]

\[
b(t) = t^{\lambda_1/c^*}, \quad c = c^*/\lambda_1.
\]

The remaining piece is to show the moment condition (69) in this context and we will show any \( \delta \geq 0 \) and any \( q = 1, 2, \ldots, \) there exists some constant \( C(\delta, q) > 0 \) such that

\[
(46) \quad \sup_{t \geq 0} e^{-\lambda_1 q t} \mathbb{E} \left[ \left( \tilde{\xi}^{(1)}(t, 1) \right)^q \right] \leq C(\delta, q),
\]

which is true by Proposition 2 in [26]. \( \square \)

Next, Theorem 4.2 gives a second hidden regular variation (HRV) regime after removing \( \mathcal{L}(1) \) [7, 8, 9, 16, 20]. The existence of HRV has been detected empirically in network data [9], and here we theoretically prove HRV present in the PA model with heterogeneous reciprocity.

The limit measure given in Theorem 4.1 concentrates on \( \mathcal{L}(1) \). Thus, we may seek a regular variation property on \( \mathbb{R}^2_+ \setminus \mathcal{L}(1) \) using a weaker scaling function \( b_0(t) \). A convenient way to seek the hidden regular variation is by using generalized polar coordinates which in this case amount to the transformation

\[
x \rightarrow \left( d_1(x, \mathcal{L}(1)), \frac{x}{d_1(x, \mathcal{L}(1))} \right),
\]

where \( d_1(x, y) \) is a metric on \( \mathbb{R}^2_+ \setminus \{\mathbf{0}\} \) chosen for convenience to be the \( L_1 \)-metric. The \( L_1 \)-distance of a point \( (x, y) \) to the line \( \mathcal{L}(1) \) is readily computed to be

\[
d_1((x, y), \mathcal{L}(1)) = |y - a(1)x|/\max\{1, a(1)\},
\]

and we use a scaled version

\[
d'_1((x, y), \mathcal{L}(1)) = |y - a(1)x|.
\]
Define $\mathcal{N}_{L(1)} := \{x \in \mathbb{R}_+^2 \setminus \mathcal{L}(1) : d'_1(x, \mathcal{L}(1)) = 1\}$, which are 2 line segments in $\mathbb{R}_+^2$ parallel to $\mathcal{L}(1)$. Hidden regular variation will be present for $(\mathcal{I}, \mathcal{O}) \overset{d}{\sim} \tilde{\xi}_\delta(T^*, L^*)$ if

$$
 t^P \left[ \frac{d'_1(\tilde{\xi}_\delta(T^*, L^*), \mathcal{L}(1))}{b_0(t)}, \frac{\tilde{\xi}_\delta(T^*, L^*)}{d'_1(\tilde{\xi}_\delta(T^*, L^*), \mathcal{L}(1))} \right] \in \cdot
$$

converges to a limit measure in $\mathcal{M}(\mathbb{R}_+^2 \setminus \{0\} \times \mathcal{N}_{L(1)})$. The next theorem explains the convergence on $\mathcal{M}(\mathbb{R}_+^2 \setminus \mathcal{L}(1))$.

**Theorem 4.2.** Assume that $\lambda_2 > \lambda_1/2$, and $\lambda_2 \geq \log 2$. Let $\tilde{Z}(2)$ be the limiting random variable satisfying $e^{-t\lambda_2} \tilde{\xi}_\delta(t, 2) \overset{a.s.}{\rightarrow} \tilde{Z}(2)\nu(2)$ as $t \rightarrow \infty$. Then we have in $\mathcal{M}(\mathbb{R}_+^2 \setminus \mathcal{L}(1))$ that

$$
 t^P \left[ (\mathcal{I}, \mathcal{O}) \overset{t\lambda_2/c^*}{\sim} \cdot \right] \rightarrow \mu_2,
$$

where the limit measure $\mu_2 \in \mathcal{M}(\mathbb{R}_+^2 \setminus \mathcal{L}(1))$ concentrates on the ray $y = a(2)x$, $x > 0$ (recall (45)) in the first quadrant and satisfies for $g \in C(\mathbb{R}_+^2 \setminus \mathcal{L}(1))$,

$$
\mu_2(g) = \mathbb{P}(L^* = 2) \int_0^\infty \mathbb{E} \left( g(y\tilde{Z}(2)\nu(2)) \right) \nu_{c^*/\lambda_2}(dy)
$$

and

$$
\mathbb{P}(L^* = 2) \mathbb{E} \left( \tilde{Z}(2)\nu(2)/\lambda_2 \right) \int_0^\infty g(y\nu(2)) \nu_{c^*/\lambda_2}(dy).
$$

Theorem 4.2 suggests that after deleting large in- and out-degree pairs close to $\mathcal{L}(1)$, at the scale $t\lambda_2/c^*$ the remaining large observations of in- and out-degrees tend to concentrate around another line

$$
\mathcal{L}(2) := \{(x, y) \in (0, \infty)^2 : y = a(2)x\}.
$$

Define also that $\mathcal{L}(0) := \{0\}$, and combining Theorems 4.1 and 4.2 gives

$$
\mathbb{P}( (\mathcal{I}, \mathcal{O}) \in \cdot ) \in \bigcap_{m=1}^2 \text{MRV} \left( c^*/\lambda_m, t\lambda_m/c^*, \mu_m, \mathbb{R}_+^2 \setminus \left( \bigcup_{i=0}^{m-1} \mathcal{L}(i) \right) \right).
$$

**Remark 4.1.** Results in Theorem 4.2 can be extended as follows. Let $\lambda_{m_0}, m_0 \geq 2$, denote the $m_0$-th largest eigenvalue such that for all $m \in \{2, \ldots, m_0\}$,

$$
\lambda_m > \lambda_{m-1}/2, \quad \text{and} \quad \lambda_m \geq \log 2.
$$

For $m = 1, \ldots, K$, write $\mathcal{L}(m) := \{(x, y) \in (0, \infty)^2 : y = a(m)x\}$, and set $\tilde{Z}(m)$ to be the limiting random variable satisfying $e^{-t\lambda_m} \tilde{\xi}_\delta(t, m) \overset{a.s.}{\rightarrow} \tilde{Z}(m)\nu(m)$ as $t \rightarrow \infty$. Define also the measure $\mu_m \in \mathcal{M}(\mathbb{R}_+^2 \setminus (\bigcup_{i=1}^m \mathcal{L}(i)))$ such that for $g \in C(\mathbb{R}_+^2 \setminus (\bigcup_{i=1}^m \mathcal{L}(i)))$,

$$
\mu_m(g) = \mathbb{P}(L = m) \int_0^\infty \mathbb{E} \left( g(y\tilde{Z}(m)\nu(m)) \right) \nu_{c^*/\lambda_m}(dy).
$$

Then applying the proof technique of Theorem 4.2 for $m_0$ times gives

$$
\mathbb{P}( (\mathcal{I}, \mathcal{O}) \in \cdot ) \in \bigcap_{m=1}^{m_0} \text{MRV} \left( c^*/\lambda_m, t\lambda_m/c^*, \mu_m, \mathbb{R}_+^2 \setminus \left( \bigcup_{i=0}^{m-1} \mathcal{L}(i) \right) \right).
$$
PROOF. Define \( \theta_{(2)} := (1, a(2))/|a(1) - a(2)| \in \mathbb{R}_+ \). Applying the generalized polar transformation shows that verifying (48) is equivalent to justifying

\[
\mathbb{P} \left[ \left( \frac{d'(\xi_\delta(T^*, L^*), L_1)}{t^{\lambda_2/c}}, \quad \xi_\delta(T^*, L^*) \right) \in \cdot \right] = t \mathbb{P} \left[ \left( \frac{a(1)\xi_\delta^{(1)}(T^*, L^*) - \xi_\delta^{(2)}(T^*, L^*)}{t^{\lambda_2/c}}, \quad \xi_\delta(T^*, L^*) \right) \in \cdot \right]
\]

(49)

\[ \rightarrow C_2\nu_{c^*/\lambda_2} (\cdot) \times \epsilon_{\theta_{(2)}} (\cdot), \]

in \( \mathbb{M} \left((0, \infty) \times \mathbb{R}_{[L_1]} \right) \), where

\[ C_2 = \mathbb{P}(L^* = 2) \frac{c^*}{\lambda_2} \times \int_0^\infty z^{-1-c^*/\lambda_2} \mathbb{P} \left( Z(2) > \frac{1/z}{v(1)(2)|a(1) - a(2)|} \right) \, dz. \]

To prove (49), we first claim that for \( \lambda_2 > \lambda_1/2 \), as \( t \to \infty \),

\[ e^{-t\lambda_2} d'_1(\xi_\delta(T^*, L^*), L_1) = e^{-t\lambda_2} \left| a(1)\xi_\delta^{(1)}(t, L^*) - \xi_\delta^{(2)}(t, L^*) \right| \]

(50)

\[ \xrightarrow{a.s.} |a(1) - a(2)| v(1)(2) Z(2) 1_{\{L^* = 2\}}. \]

In addition, since

\[ \frac{\xi_\delta(t, m)}{a(1)\xi_\delta^{(1)}(t, m) - \xi_\delta^{(2)}(t, m)} \xrightarrow{a.s.} \frac{(1, a(m))}{|a(1) - a(m)|}, \]

we have

\[
\left( \frac{a(1)\xi_\delta^{(1)}(t, L^*) - \xi_\delta^{(2)}(t, L^*)}{e^{t\lambda_2}}, \quad \frac{\xi_\delta(t, L^*)}{a(1)\xi_\delta^{(1)}(t, L^*) - \xi_\delta^{(2)}(t, L^*)} \right) = \sum_{m=1}^K \left( \frac{a(1)\xi_\delta^{(1)}(t, m) - \xi_\delta^{(2)}(t, m)}{e^{t\lambda_2}}, \quad \frac{\xi_\delta(t, m)}{a(1)\xi_\delta^{(1)}(t, m) - \xi_\delta^{(2)}(t, m)} \right) 1_{\{L^* = m\}}
\]

(51)

\[ \xrightarrow{a.s.} \left| a(1) - a(2) \right| v(1)(2) Z(2) 1_{\{L^* = 2\}} \cdot \sum_{m=1}^K \frac{(1, a(m))}{|a(1) - a(m)|} 1_{\{L^* = m\}}. \]

Equation (51) further gives that in \( \mathbb{M} \left((0, \infty) \times (0, \infty) \times \mathbb{R}_{[L_1]} \right) \),

\[
\mathbb{P} \left[ \left( \frac{e^{T^*}}{t^{1/c}}, \quad \sum_{m=1}^K \frac{a(1)\xi_\delta^{(1)}(T^*, m) - \xi_\delta^{(2)}(T^*, m)}{e^{\lambda_2 T^*}}, \quad \frac{\xi_\delta(T^*, m)}{a(1)\xi_\delta^{(1)}(T^*, m) - \xi_\delta^{(2)}(T^*, m)} \right) 1_{\{L^* = m\}} \right] \in \cdot
\]

\[ \rightarrow \mathbb{P}(L^* = 2) \nu_{c^*/\lambda_2} (\cdot) \times \mathbb{P} \left( \left| a(1) - a(2) \right| v(1)(2) Z(2) \in \cdot \right) \times \epsilon_{\theta_{(2)}} (\cdot). \]

Hence, as long as we check the moment condition that for \( q = 1, 2, \ldots \) and \( \delta > 0 \),

\[ \sup_{t \geq 0} e^{-q t^{\lambda_2}} \mathbb{E} \left[ \left| a(1)\xi_\delta^{(1)}(t, 2) - \xi_\delta^{(2)}(t, 2) \right|^q \right] < \infty, \]

(52)
then applying the generalized Breiman’s theorem (cf. Theorem A.1) completes the proof of (49). To verify (52), we notice that since
\[
\left| a(1)\tilde{\xi}_\delta^{(1)}(t, 2) - \tilde{\xi}_\delta^{(2)}(t, 2) \right| \leq a(1)\tilde{\xi}_\delta^{(1)}(t, 2) + \tilde{\xi}_\delta^{(2)}(t, 2),
\]
it suffices to show
\[
\sup_{t \geq 0} e^{-qt\lambda_2} E \left[ \left( a(1)\tilde{\xi}_\delta^{(1)}(t, 2) + \tilde{\xi}_\delta^{(2)}(t, 2) \right)^q \right] < \infty, \quad \lambda_2 \geq \log 2,
\]
which is true by Proposition 2 in [26].

It remains to prove the claim in (50). Since
\[
|a(m)\tilde{\xi}_\delta^{(1)}(t, m) - \tilde{\xi}_\delta^{(2)}(t, m)| \leq a(m)\tilde{\xi}_\delta^{(1)}(t, m) + \tilde{\xi}_\delta^{(2)}(t, m),
\]
then for \( m \notin \{1, 2\} \), we have
\[
e^{-t\lambda_2} |a(m)\tilde{\xi}_\delta^{(1)}(t, m) - \tilde{\xi}_\delta^{(2)}(t, m)| \overset{a.s.}{\to} 0, \quad as \; t \to \infty.
\]
Hence, we only need to consider \( m = 1, 2 \). Consider the case when \( m = 1 \), and define \( \{\psi_i(t, 1) : t \geq 0\}_{i \geq 1} \) as a sequence of iid two-type branching processes (without immigration) with group label 1, whose branching structure is specified through \( A_1 \). Also, assume that \( \psi_i(0, 1) \) is a 2-dimensional random vector with distribution \( p_0(\mathbf{r}, 1) \) (cf. (18)). Let \( 0 < \tau_1 < \tau_2 < \ldots \) be arrival times of points in a homogeneous Poisson process with rate \( \delta > 0 \), which is independent from \( \{\psi_i(t, 1) : t \geq 0\}_{i \geq 1} \). Then by the distributional construction of the MBI process in [19], we write
\[
(53) \quad \tilde{\xi}_\delta(t, 1) = \sum_{i=1}^{\infty} \psi_i(t - \tau_i, 1) \mathbf{1}_{\{t \geq \tau_i\}}.
\]

Next, we use a Borel Cantelli argument to show
\[
(54) \quad e^{-t\lambda_2} |a(1)\tilde{\xi}_\delta^{(1)}(t, 1) - \tilde{\xi}_\delta^{(2)}(t, 1)| \overset{a.s.}{\to} 0,
\]
as \( t \to \infty \), i.e. we will show that
\[
(55) \quad \sum_{n=1}^{\infty} P \left( e^{-n\lambda_2} |a(1)\tilde{\xi}_\delta^{(1)}(n, 1) - \tilde{\xi}_\delta^{(2)}(n, 1)| > \epsilon \right) < \infty.
\]

By Markov’s inequality, we have for \( n \geq 1 \),
\[
P \left( e^{-n\lambda_2} |a(1)\tilde{\xi}_\delta^{(1)}(n, 1) - \tilde{\xi}_\delta^{(2)}(n, 1)| > \epsilon \right) \\
\leq e^{-2n\lambda_2} E \left[ e^{-2n\lambda_2} |a(1)\tilde{\xi}_\delta^{(1)}(n, 1) - \tilde{\xi}_\delta^{(2)}(n, 1)|^2 \right] \\
\leq e^{-2n(2\lambda_2 - \lambda_1)} \sup_{t \geq 0} E \left[ \left( e^{-t\lambda_1/2} \left( a(1)\tilde{\xi}_\delta^{(1)}(t, 1) - \tilde{\xi}_\delta^{(2)}(t, 1) \right) \right)^2 \right].
\]

Note that the vector \([a(1), -1]^T\) is the right eigenvector associated with the smaller eigenvalue of \( A_1 \). Recall the discussion in Section 3.1 that \( \lambda'_1 \) is the smaller eigenvalue of \( A_1 \), then the condition \( \lambda_1 \geq \log 2 \) guarantees that \( \lambda_1 > 2\lambda'_1 \). Hence, we apply Theorem V.7.1(i) in [2] to conclude that for \( i \geq 1 \),
\[
\sup_{t \geq 0} E \left[ \left( e^{-t\lambda_i/2} \left( a(1)\psi_i^{(1)}(t, 1) - \psi_i^{(2)}(t, 1) \right) \right)^2 \right] < \infty.
\]
We then conclude from (53) that
\[
\left( e^{-t\lambda_1/2} \left( a(1)\xi_\delta^{(1)}(t, 1) - \xi_\delta^{(2)}(t, 1) \right) \right)^2
\]
\[
= \left( e^{-t\lambda_1/2} \sum_{i=1}^\infty \left( a(1)\psi_i^{(1)}(t - \tau_i, 1) - \psi_i^{(2)}(t - \tau_i, 1) \right) \mathbf{1}_{\{t \geq \tau_i\}} \right)^2
\]
\[
\leq \left( \sum_{i=1}^\infty e^{-\tau_i\lambda_1/2} \right)
\times \left( \sum_{i=1}^\infty e^{-\tau_i\lambda_1/2} \left( a(1)\psi_i^{(1)}(t - \tau_i, 1) - \psi_i^{(2)}(t - \tau_i, 1) \right) \right)^2.
\]

Therefore,
\[
\mathbb{E} \left( e^{-t\lambda_1/2} \left( a(1)\xi_\delta^{(1)}(t, 1) - \xi_\delta^{(2)}(t, 1) \right) \right)^2
\leq \sup_{t \geq 0} \mathbb{E} \left[ \left( e^{-t\lambda_1/2} \left( a(1)\psi_i^{(1)}(t, 1) - \psi_i^{(2)}(t, 1) \right) \right)^2 \right]
\times \mathbb{E} \left[ \sum_{i=1}^\infty e^{-\tau_i\lambda_1/2} \left( \sum_{i=1}^\infty e^{-\tau_i\lambda_1/2} \right) \right] < \infty.
\]

Also, since we assume \( \lambda_2 > \lambda_1/2 \), then (55) follows from (56), leading to (54). Then the claim in (50) follows by realizing that
\[
e^{-t\lambda_2} \left( a(1)\xi_\delta^{(1)}(t, 2) - \xi_\delta^{(2)}(t, 2) \right) \xrightarrow{a.s.} \left( a(1)v^{(1)}(2) - v^{(2)}(2) \right) \tilde{Z}(2)
= (a(1) - a(2)) v^{(1)}(2) \tilde{Z}(2).
\]

\[\square\]

5. Concluding Remarks

In this paper, we propose a preferential attachment model with heterogeneous reciprocity levels and study its theoretical properties. Using the MBI embedding technique, we find that the distribution of large in- and out-degrees is jointly regularly varying, concentrating along a specific ray. Additionally, after deleting large in- and out-degree pairs close to the ray, we further have hidden regular variation with limit measure concentrating around another ray.

We now outline some open problems related to this model, which will be left as future research.

Estimation. The fitting of the proposed model remains open at this point. For a given dataset, we need to first decide how many communication groups (\( K \)) should be assumed. One may consult classical clustering methods such as \( k \)-means and \( k \)-nearest neighbors to determine a proper \( K \) beforehand, but their applications to the network framework needs rigorous justification. Once \( K \) is chosen, we can also derive proper tools (similar to those in [9]) to detect the existence of hidden regular variation under the network setup.

Varying group labels. So far we have assumed that the communication group of a user (node) is determined upon its creation and remains unchanged afterwards. For real-world applications, however, this may be a naive assumption, since users’ interaction patterns can change over time. A possible extension is to assume that the group label for each node follows another Markov chain with finite state space. One may consider applying variational Bayesian inference methods (cf. [17]) to these extended models.
Covariate-dependent reciprocity probabilities. The model studied in this paper assumes a deterministic matrix, \( \rho \), to characterize the reciprocation between different communication groups. If more node-specific information (covariates) is available, e.g., various demographic, socio-economic, and behavioral factors, then we can generalize the definition of the matrix \( \rho \) to be covariate-dependent. This may possibly lead to heterogeneous extremal behaviors.

6. Proofs of Results in Section 2

6.1. Proof of Lemma 2.1 Define a function \( f : \mathcal{Z} \mapsto \mathcal{Z} \) such that for \( i = 1, \ldots, K \),

\[
f_i(z) = \alpha \frac{z_i + \delta \pi_i}{\sum_{j=1}^{K} z_j + \delta} + \gamma \rho_{i,j} \frac{z_{K+i} + \delta \pi_i}{\sum_{j=1}^{K} z_{K+j} + \delta} + \gamma \pi_i + \alpha \pi_i \sum_{r=1}^{K} \sum_{j=1}^{K} \rho_{r,j} \frac{z_i + \delta \pi_i}{\sum_{j=1}^{K} z_j + \delta},
\]
and for \( i = K + 1, \ldots, 2K \),

\[
f_i(z) = \gamma \frac{z_i + \delta \pi_{i-K}}{\sum_{j=K+1}^{2K} z_j + \delta} + \alpha \rho_{(i-K),j} \frac{z_{i-K} + \delta \pi_{i-K}}{\sum_{j=1}^{K} z_j + \delta} + \alpha \pi_i - K \sum_{r=1}^{K} \sum_{j=K+1}^{2K} \rho_{r,j} \frac{z_i + \delta \pi_{i-K}}{\sum_{j=K+1}^{2K} z_j + \delta}.
\]

Let \( J(z') \) be the Jacobian matrix evaluated at some \( z' \) between \( z_1 \) and \( z_2 \). Use \( \| z \|_1 \) to denote the \( L_1 \)-norm for some \( z \in \mathcal{Z} \), and set \( \| \cdot \|_{(1, \infty)} \) to be the \( L_{1, \infty} \) norm of a matrix. Then by the mean value theorem, we see that for \( z_1, z_2 \in \mathcal{Z} \),

\[
\| f(z_1) - f(z_2) \|_1 \leq \sup_{z' \in \mathcal{Z}} \| J(z') \|_{(1, \infty)} \| z_1 - z_2 \|_\infty 
\]

(57)

If we can show \( \sup_{z' \in \mathcal{Z}} \| J(z') \|_{(1, \infty)} < 1 \), then by the contraction mapping theorem (cf. Theorem 1.2.2 in [13]) we are able to show the existence of a unique solution \( f(z) = z \).

To find an upper bound for \( \sup_{z \in \mathcal{Z}} \| J(z) \|_{(1, \infty)} \), we now give upper bounds for the absolute value of each entry in \( J(z) \). Let \( J_{i,j}(z) \) be the \( (i, j) \)-th entry in \( J(z) \), and we have for \( 1 \leq i, j \leq K \),

\[
|J_{i,j}(z)| = \left| \frac{\partial f_i}{\partial z_j}(z) \right| \leq \alpha (\pi_i \rho_{j,i} + 1_{\{i=j\}}) \frac{1}{\sum_{m=1}^{K} z_m + \delta} \leq \frac{\alpha (\pi_i \rho_{j,i} + 1_{\{i=j\}})}{1 + \delta}.
\]

(58)

For \( 1 \leq i \leq K \) and \( K + 1 \leq j \leq 2K \),

\[
|J_{i,j}(z)| = \left| \frac{\partial f_i}{\partial z_j}(z) \right| = \gamma \rho_{i,j} \left| \frac{z_{K+i} + \delta \pi_i}{\left( \sum_{m=K+1}^{2K} z_m + \delta \right)^2} + \frac{1_{\{i=K+i\}}}{\sum_{m=K+1}^{2K} z_m + \delta} \right| \leq \frac{\gamma \rho_{i,j}}{1 + \delta}.
\]

(59)

For \( K + 1 \leq i \leq 2K \) and \( 1 \leq j \leq K \), we see that

\[
|J_{i,j}(z)| = \left| \frac{\partial f_i}{\partial z_j}(z) \right| = \alpha \rho_{i,j} \left| \frac{z_{i-K} + \delta \pi_{i-K}}{\left( \sum_{m=1}^{K} z_m + \delta \right)^2} + \frac{1_{\{j=i-K\}}}{\sum_{m=1}^{K} z_m + \delta} \right| \leq \frac{\alpha \rho_{i,j}}{1 + \delta}.
\]

(60)

Also, for \( K + 1 \leq i, j \leq 2K \), we have

\[
|J_{i,j}(z)| = \left| \frac{\partial f_i}{\partial z_j}(z) \right| \leq \gamma (\pi_{i-K} \rho_{i,j} + 1_{\{i=j\}}) \frac{1}{\sum_{m=K+1}^{2K} z_m + \delta} \leq \frac{\gamma (\pi_{i-K} \rho_{i,j} + 1_{\{i=j\}})}{1 + \delta}.
\]

(61)
Let $J_{i,j}^*$ be the $(i,j)$-th entry of the $J^*$ matrix. Equations (58)–(61) imply that

$$\sup_{z' \in \mathcal{Z}} \|J(z')\|_{(1, \infty)} \leq \frac{1}{1 + \delta} \sqrt{\sum_{j=1}^{K} \frac{|J_{i,j}^*|}{\sum_{i=1}^{K} |J_{i,j}^*|}} = \frac{1}{1 + \delta} \|J^*\|_1.$$ 

Therefore, as long as

$$\delta > \|J^*\|_1 - 1,$$

Equation (57) gives

$$\|f(z_1) - f(z_2)\|_1 \leq \frac{1}{1 + \delta} \|J^*\|_1 \|z_1 - z_2\|_1 < \|z_1 - z_2\|_1,$$

indicating that $f(z) = z$ has a unique solution in $\mathcal{Z}$.

6.2. Proof of Theorem 2.2 We first show the concentration of $|E(n)|/n$ around its expectation, $\mathbb{E}[|E(n)|]/n$. Suppose we have a graph $G(n) = (V(n), E(n))$, constructed from the PA model with heterogeneous reciprocity levels. We claim that for $C_0 > 2$,

$$\mathbb{P} \left( |E(n)| - \mathbb{E}[|E(n)|] \geq C_0 \sqrt{n \log n} \right) \leq 2n^{-2}. \quad (62)$$

We prove (62) using the Azuma-Hoeffding inequality. For $k \leq n$, define $M_k := \mathbb{E}(|E(n)| | G(k))$, then $\mathbb{E}(M_{k+1} | G(k)) = M_k$. So we need to consider the martingale difference:

$$M_{k+1} - M_k = \mathbb{E}(|E(n)| | G(k+1)) - \mathbb{E}(|E(n)| | G(k)).$$

Define another graph $G'(n) \equiv (V'(n), E'(n))$ such that $G'(s) = G(s)$ for $s \leq k$, while $G'(s)$ evolves independently of $\{G(s)\}_{s \geq k+1}$ for $s \geq k+1$, according to the same evolution rules as in Section 1.1. Then we have

$$M_{k+1} - M_k = \mathbb{E}(|E(n)| | G(k+1)) - \mathbb{E}(|E'(n)| | G(k+1))$$

$$= \mathbb{E} \left[ \mathbb{E}(|E(n)| - |E'(n)| | G(k+1), G'(k+1)) | G(k+1) \right]$$

$$= \mathbb{E} \left[ |E(k+1)| - |E'(k+1)| | G(k+1) \right],$$

which gives

$$|M_{k+1} - M_k| \leq 1.$$ 

Since $|E(n)| - \mathbb{E}[|E(n)|] = \sum_{k=0}^{n-1} (M_{k+1} - M_k)$, then applying the Azuma-Hoeffding inequality gives that for $\varepsilon > 0$,

$$\mathbb{P} \left( |E(n)| - \mathbb{E}[|E(n)|] \geq \varepsilon \right) \leq 2e^{-\frac{\varepsilon^2}{2\mathbb{E}[|E(n)|]}},$$

and (62) follows by setting $\varepsilon = C_0 \sqrt{n \log n}$ for $C_0 > 2$.

Next, we study the convergence of $\mathbb{E}[|E(n)|]/n$. Since $x_m$ satisfies (10), then we see from (8) that

$$\mathbb{E}^{\mathcal{G}_n}(\Delta_m^{\text{in}}(n + 1)) = \Delta_m^{\text{in}}(n) + \left( \mathbb{E}^{\mathcal{G}_n}(|E_m^{\text{in}}(n + 1)| - |E_m^{\text{in}}(n)| - x_m \right)$$

$$= \Delta_m^{\text{in}}(n) + \alpha \left( \frac{|E_m^{\text{in}}(n)| + \delta |V_m(n)|}{|E(n)| + \delta |V(n)|} - \frac{x_m + \delta \pi_m}{\sum_r x_r + \delta} \right)$$

$$+ \gamma \rho_m \left( \frac{|E_m^{\text{out}}(n)| + \delta |V_m(n)|}{|E(n)| + \delta |V(n)|} - \frac{y_m + \delta \pi_m}{\sum_r y_r + \delta} \right)$$

$$+ \alpha \pi_m \sum_r \rho_{r,m} \left( \frac{|E_m^{\text{in}}(n)| + \delta |V_r(n)|}{|E(n)| + \delta |V(n)|} - \frac{x_r + \delta \pi_r}{\sum_r x_r + \delta} \right),$$
which implies

\[ |\mathbb{E}(\Delta_{m}^{in}(n + 1))| \leq |\mathbb{E}(\Delta_{m}^{in}(n))| + \alpha \left| \mathbb{E} \left( \frac{|E_{m}^{in}(n)| + \delta|V_{m}(n)|}{|E(n)| + \delta|V(n)|} - \frac{x_{m} + \delta \pi_{m}}{\sum_{r} x_{r} + \delta} \right) \right| + \gamma \rho_{m} \left| \mathbb{E} \left( \frac{|E_{m}^{out}(n)| + \delta|V_{m}(n)|}{|E(n)| + \delta|V(n)|} - \frac{y_{m} + \delta \pi_{m}}{\sum_{r} y_{r} + \delta} \right) \right| + \alpha \pi_{m} \sum_{r} \rho_{r,m} \left| \mathbb{E} \left( \frac{|E_{r}^{in}(n)| + \delta|V_{r}(n)|}{|E(n)| + \delta|V(n)|} - \frac{x_{r} + \delta \pi_{r}}{\sum_{r} x_{r} + \delta} \right) \right| \]

(63)

Also, we have that for \( m = 1, \ldots, K \),

\[ \left| \mathbb{E} \left( \frac{|E_{m}^{in}(n)| + \delta|V_{m}(n)|}{|E(n)| + \delta|V(n)|} - \frac{x_{m} + \delta \pi_{m}}{\sum_{r} x_{r} + \delta} \right) \right| \leq \frac{|\mathbb{E}(\Delta_{m}^{out}(n))|}{(1 + \delta)n} \right| + \frac{y_{m} + \delta \pi_{m}}{\sum_{r} y_{r} + \delta} \right| \]

Therefore, it follows from (63) that

\[ |\mathbb{E}(\Delta_{m}^{in}(n + 1))| \leq |\mathbb{E}(\Delta_{m}^{in}(n))| \left( 1 + \frac{\alpha}{(1 + \delta)n} \right) + |\mathbb{E}(\Delta_{m}^{out}(n))| \left( \frac{\gamma \rho_{m}}{(1 + \delta)n} + \frac{\alpha \pi_{m}}{(1 + \delta)n} \right) \sum_{r} \rho_{r,m} \left| \mathbb{E}(\Delta_{m}^{in}(n)) \right| + \left( \frac{\alpha \pi_{m}}{(1 + \delta)n} \right) \sum_{r} \rho_{r,m} \left| \mathbb{E}(\Delta_{m}^{in}(n)) \right| \frac{y_{m} + \delta \pi_{m}}{\sum_{r} y_{r} + \delta} \right| + \alpha \pi_{m} \sum_{r} \rho_{r,m} \frac{x_{r} + \delta \pi_{r}}{\sum_{r} x_{r} + \delta} \right) \]

Summing over \( m \) gives

\[ \sum_{m} |\mathbb{E}(\Delta_{m}^{in}(n + 1))| \leq \sum_{m} |\mathbb{E}(\Delta_{m}^{in}(n))| \left( 1 + \frac{\alpha}{(1 + \delta)n} \right) + \sum_{m} |\mathbb{E}(\Delta_{m}^{out}(n))| \frac{\gamma \sqrt{m} \rho_{m}}{(1 + \delta)n} + \frac{\alpha \sqrt{m} \rho_{m}}{(1 + \delta)n} \sum_{m} |\mathbb{E}(\Delta_{m}^{in}(n))| + \frac{|\mathbb{E}(\Delta(n))|}{(1 + \delta)n} (\alpha + C_{\delta}) \]

(64)

Following a similar reasoning, we have

\[ \sum_{m} |\mathbb{E}(\Delta_{m}^{out}(n + 1))| \leq \sum_{m} |\mathbb{E}(\Delta_{m}^{out}(n))| \left( 1 + \frac{\gamma}{(1 + \delta)n} \right) \left( 1 + \sqrt{\frac{\rho_{m}}{(1 + \delta)n}} \right) \]
Then by the Perron-Frobenius theorem, we see that

\[ \frac{\alpha V_m \rho_m}{(1 + \delta)n} + \frac{|E(\Delta(n))|}{(1 + \delta)n} (\gamma + C_\delta), \]

and

\[ \sum_m |E(\Delta(n + 1))| \]

\[ \leq |E(\Delta(n))| \left( 1 + \frac{\alpha}{(1 + \delta)n} \sum_m \rho_m \frac{x_m + \delta \pi_m}{x_r + \delta} + \frac{\gamma}{(1 + \delta)n} \sum_m \rho_m \frac{y_m + \delta \pi_m}{y_r + \delta} \right) \]

Combining Equations (64), (65) and (67) gives:

\[ \left[ \sum_m |E(\Delta_m(n + 1))| \right] \leq \left[ \sum_n H \left( \sum_m |E(\Delta_m(n))| \right) \right]. \]

Then by the Perron-Frobenius theorem, we see that

\[ \frac{1}{n} \sum_m |E(\Delta_m(n))| \rightarrow 0, \quad \frac{1}{n} \sum_m |E(\Delta_m(n))| \rightarrow 0, \quad \text{and} \quad \frac{1}{n} |E(\Delta(n))| \rightarrow 0, \]

if the largest eigenvalue of \( H, \lambda_H \), is less than 1, thus completing the proof of the theorem.

**APPENDIX A: GENERALIZED BREIMAN’S THEOREM**

**THEOREM A.1.** Suppose \( \{\xi(t) : t \geq 0\} \) is an \( \mathbb{R}_+^p \)-valued stochastic process for some \( p \geq 1 \). Let \( X \) be a positive random variable with regularly varying distribution satisfying for some scaling function \( b(t) \),

\[ \lim_{t \to \infty} t P(X/b(t) > x) = x^{-c} =: \nu_c((x, \infty)), \quad x > 0, c > 0. \]

Further suppose

1. For some finite and positive random vector \( \xi_\infty \),

\[ \lim_{t \to \infty} \xi(t) = \xi_\infty \quad (\text{almost surely}); \]

2. The random variable \( X \) and the process \( \xi(\cdot) \) are independent.

Then:

(i) In \( M(\mathbb{R}_+^p \times (\mathbb{R}_+ \setminus \{0\})) \),

\[ t P \left[ \frac{\xi(X)}{b(t)} \in \cdot \right] \to P(\xi_\infty \in \cdot) \times \nu_c(\cdot) =: \eta(\cdot). \]

If \( \xi_\infty \) is of the form \( \xi_\infty =: Lv \) where \( L > 0 \) almost surely and \( v \in (0, \infty)^p \), then \( \eta(\cdot) \) concentrates on the subcone \( \mathcal{L} \times (\mathbb{R}_+ \setminus \{0\}) \) where \( \mathcal{L} = \{\theta v : \theta > 0\} \).

(ii) If additionally, for some \( c' > c \) we have the condition

\[ k := \sup_{t \geq 0} E \left[ \left( \|\xi(t)\| \right)^{c'} \right] < \infty, \]

for some \( L_p \) norm \( \| \cdot \| \), then the product of components in (68), \( \xi(X)X \), has a regularly varying distribution with scaling function \( b(t) \) and in \( M(\mathbb{R}_+^p \setminus \{0\}) \),

\[ t P \left[ \frac{X\xi(X)}{b(t)} \in \cdot \right] \to (P(\xi_\infty \in \cdot) \times \nu_c) \circ h^{-1}, \]

where \( h(y, x) = xy \).
For the classical Breiman Theorem where \( p = 1 \) and \( \xi(t) \equiv \xi_{\infty} \), (69) is the expected moment condition.

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