JACOB’S LADDERS AND THE MULTIPLICATIVE ASYMPTOTIC FORMULA FOR SHORT AND MICROSCOPIC PARTS OF THE HARDY-LITTLEWOOD INTEGRAL

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Abstract. The elementary geometric properties of Jacob’s ladders lead to a class of new asymptotic formulae for short and microscopic parts of the Hardy-Littlewood integral. This class of asymptotic formulae cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic.

Dedicated to 110 anniversary of E.C. Titchmarsh.

1. FORMULATION OF THE THEOREM

In the papers [4] and 5 I obtained the following additive formula

\begin{equation}
\int_T^{T+U} Z^2(t) dt = U \ln \left( \frac{\varphi(T)}{2} e^{-a} \right) \tan [\alpha(T, U)] + O \left( \frac{1}{T^{1/3 - 4\epsilon}} \right),
\end{equation}

where

\begin{equation}
U \in (0, U_0], \quad U_0 = T^{1/3 + 2\epsilon}, \quad a = \ln 2\pi - 1 - c,
\end{equation}

that holds true for short parts of the Hardy-Littlewood integral. In the present work I prove a multiplicative formula, which is asymptotic at \( T \to \infty \) also if \( U \to 0 \).

Namely, the following theorem takes place

**Theorem.**

\begin{equation}
\int_T^{T+U} Z^2(t) dt = U \ln T \tan [\alpha(T, U)] \begin{cases} 1 + O \left( \frac{\ln \ln T}{\ln T} \right) \end{cases}, \quad U \in \left( 0, \frac{T}{\ln T} \right],
\end{equation}

for \( \mu[\varphi] = 7\varphi \ln \varphi \).

The main idea of the proof of the Theorem is to use the formula

\begin{equation}
Z^2(t) = \Phi'[\varphi(T)] \frac{d\varphi(T)}{dT}, \quad \Phi' = \Phi'_{\varphi},
\end{equation}

where

\begin{equation}
\Phi'[\varphi] = \frac{2}{\varphi^2} \int_0^{\mu[\varphi]} te^{-\frac{2}{\varphi^2} t} Z^2(t) dt + Z^2\{\mu[\varphi]\} e^{-\frac{2}{\varphi^2} \mu[\varphi]} \frac{d\mu[\varphi]}{d\varphi},
\end{equation}

(see [4], (3.5), (3.9)).

**Remark 1.** In the proof of the formula (1.2) we shall get also the multiplicative formula

\begin{equation}
Z^2(T) = \frac{1}{2} \ln T \frac{d\varphi(T)}{dT} \begin{cases} 1 + O \left( \frac{\ln \ln T}{\ln T} \right) \end{cases}, \quad T \to \infty.
\end{equation}

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Remark 2. Our new method leads to asymptotic formulae (see, e.g. (2.3), (2.4)) for short and microscopic parts of the Hardy-Littlewood integral. These asymptotic formulae cannot be derived within complicated methods of Balasubramanian, Heath-Brown and Ivic (see [1] and estimates in [3], pp. 178 and 191).

2. Consequences of the Theorem

2.1. First of all, we will show a canonical equivalence that follows from (1.2). Let us remind that we call the chord binding the points

\[ T, \frac{1}{2}\varphi(T), \quad T + U_0, \frac{1}{2}\varphi(T + U_0) \], \[ \tan[\alpha(T, U_0)] = 1 + O(1) \ln T \]

of the Jacob’s ladder \( y = \frac{1}{2}\varphi(T) \) the fundamental chord (see [4]).

Definition. The chord binding the points

\[ N, \frac{1}{2}\varphi(N), \quad M, \frac{1}{2}\varphi(M), \quad [N, M] \subset [T, T + U_0], \]

for which the property

\[ \tan[\alpha(N, M - N)] = 1 + o(1), \quad T \to \infty \]

is fulfilled, is called the almost parallel chord to the fundamental chord. This property we will denote by the symbol /\.

Corollary 1. Let \([N, M] \subset [T, T + U_0] \]. Then

\[ \frac{1}{M - N} \int_N^M Z^2(t)dt \sim \ln T \leftrightarrow /\].

Remark 3. We see that the analytic property

\[ \frac{1}{M - N} \int_N^M Z^2(t)dt \sim \ln T \]

is equivalent to the geometric property /\ of Jacob’s ladder \( y = \frac{1}{2}\varphi(T) \).

2.2. Next, for example, similarly to the case of the paper [5], Cor. 1, we obtain from our Theorem

Corollary 2. There is continuum of intervals \([N, M] \subset [T, T + U_0] \) for which the following asymptotic formula

\[ \int_N^M Z^2(t)dt = (M - N) \ln T \left(1 + O\left(\frac{\ln \ln T}{\ln T}\right)\right) \]

holds true.

Remark 4. Especially, there is continuum of intervals \([N, M] : 0 < M - N < 1 \) for which the asymptotic formula \( (2.3) \) holds true (this follows from the elementary mean value theorem of differentiation).

And similarly to [5], Cor. 3, part A, we obtain from our Theorem

Corollary 3. For every sufficiently big zero \( T = \gamma \) of the function \( \zeta(\frac{1}{2} + iT) \) there is continuum of intervals \([\gamma, U(\gamma, \alpha)] \) such that the following is true

\[ \int_\gamma^{\gamma + U(\gamma, \alpha)} Z^2(t)dt = U \ln \gamma \tan \alpha \left(1 + O\left(\frac{\ln \ln \gamma}{\ln \gamma}\right)\right) \]

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where \( \tan \alpha \in [\eta, 1-\eta] \), and \( \alpha \) is the angle of the rotating chord binding the points
\[
\left[ \gamma, \frac{1}{2} \varphi(\gamma) \right], \left[ \gamma+U, \frac{1}{2} \varphi(\gamma+U) \right],
\]
and \( 0 < \eta \) is an arbitrarily small number.

**Remark 5.** For example, in the case \( \alpha = \pi/6 \) we have, as a special case of eq. (2.4),
\[
\int_{\gamma}^{\gamma+U(\gamma,\pi/6)} Z^2(t) dt \sim \frac{1}{\sqrt{3}} U \ln \gamma.
\]

**Remark 6.** It is obvious that
\[
U(\gamma, \alpha) < T^{1/3+2\epsilon}.
\]
Moreover, the following is also true
\[
U(\gamma, \alpha) < T^\omega, \omega \in \left[ \frac{1}{4} + \epsilon, \frac{1}{3} + 2\epsilon \right),
\]
(compare the Good’s \( \Omega \)-Theorem [2]), where \( \omega \) can attain every value for which the formula
\[
\int_{0}^{T} Z^2(t) dt = T \ln T + (2c - 1 - \ln 2\pi) T + O(T^\omega)
\]
will be proved.

### 3. An estimate for \( \Phi''_{y^2}[\varphi] \)

The following lemma is true

**Lemma 1.** If \( \mu[\varphi] = 7 \varphi \ln \varphi \) then

(3.1) \[
\Phi''_{y^2}[\varphi] = O \left( \frac{1}{\varphi} \ln \varphi \ln \ln \varphi \right).
\]

**Proof.** Since \( \mu(y) = 7y \ln y \) we have
\[
\mu(y) \to y = \varphi(\mu) = \varphi(T) = \varphi,
\]
see [1] and (1.4),

(3.2) \[
\Phi''_{y^2}[\varphi] = 4 \varphi^3 \int_{0}^{\mu[\varphi]} t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi^2}} Z^2(t) dt + Q[\varphi],
\]

(3.3) \[
Q[\varphi] = e^{-\frac{2}{\varphi^2}} \mu[\varphi] \left\{ \frac{2 \varphi^2 Z^2 \{\mu[\varphi]\} \mu[\varphi] \frac{d\mu[\varphi]}{d\varphi}}{\varphi^2} + \frac{2}{\varphi^2} Z^2 \{\mu[\varphi]\} \frac{d\mu[\varphi]}{d\varphi} \right\} + Z^2 \{\mu[\varphi]\} \left( \frac{d\mu[\varphi]}{d\varphi} \right)^2 + Z^2 \{\mu[\varphi]\} \frac{d^2\mu[\varphi]}{d\varphi^2} \right\}.
\]

Let
\[
g(t) = t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi^2}}, \quad t \in [0, \mu[\varphi]].
\]
We apply the following elementary facts
\[ g(0) = g(\varphi) = 0, \quad g' \left[ \left( 1 - \frac{1}{\sqrt{2}} \right) \varphi \right] = g' \left[ \left( 1 + \frac{1}{\sqrt{2}} \right) \varphi \right] = 0, \]
and the Hardy-Littlewood formula (1918)
\[ \int_T^{T+U} Z^2(t)dt \sim T \ln T, \quad T \to \infty. \]
First of all we have
\[ \frac{4}{\varphi} \int_0^{\varphi \ln \ln \varphi} = \mathcal{O} \left( \frac{1}{\varphi^2} \int_0^{\varphi \ln \ln \varphi} Z^2(t)dt \right) = \mathcal{O} \left( \frac{1}{\varphi} \ln \varphi \ln \ln \varphi \right), \]
\[ \frac{4}{\varphi^3} \int_0^{\varphi \ln \ln \varphi} = \mathcal{O} \left( \frac{1}{\varphi^3} \varphi \left( \frac{\ln \varphi}{\ln \varphi} \right)^2 \ln^2 \varphi \right) = \mathcal{O} \left( \frac{1}{\varphi} (\ln \varphi)^2 \right) \]
by (3.4), (3.5). Next we have (see (3.3))
\[ Q[\varphi] = \mathcal{O}(\varphi^{-13}) \to 0, \quad T \to \infty. \]
Finally, we obtain (3.1) from (3.2) by (3.6) and (3.7).

**Remark 7.** It is quite evident that our Lemma (i.e. also our Theorem) is true for continual class of functions
\[ \mu[\varphi] = 7\varphi^{\omega_1} \ln^{\omega_2} \varphi, \quad \omega_1, \omega_2 \geq 1. \]

4. **Proof of the Theorem**

By (1.3) we have
\[ \int_T^{T+U} Z^2(t)dt = \Phi'_y[\varphi(t_1)] \int_T^{T+U} d\varphi = \Phi'_y[\varphi(t_1)] \{ \varphi(T+U) - \varphi(T) \}, \]
i.e.
\[ \int_T^{T+U} Z^2(t)dt = 2U \Phi'_y[\varphi(t_1)] \tan[\alpha(T, U)], \quad t_1 = t_1(U) \in (T, T+U), \]
\[ \tan[\alpha(T, U)] = \frac{1}{2} \frac{\varphi(T+U) - \varphi(T)}{U}. \]
Next we have
\[ \int_T^{T+U_0} Z^2(t)dt = 2U_0 \Phi'_y[\varphi(t_2)] \left\{ 1 + \mathcal{O} \left( \frac{1}{\ln T} \right) \right\}, \quad t_2 = t_2(U_0) \in (T, T+U_0), \]
by \((2.1), (4.1)\). Let us remind that \(\varphi(T)/2 \sim T\) by the formula
\[
(4.3) \quad \pi(T) \sim \frac{1}{1-c} \left\{ T - \frac{\varphi(T)}{2} \right\}, \quad T \to \infty,
\]
(see \[4], (6.2)). Hence by comparison of the formulae \((1.1)\) \(U = U_0\) (see \((2.1)\)) and \((4.2)\) we obtain
\[
(4.4) \quad \Phi_y'[\varphi(t_2)] = \frac{1}{2} \ln T + O(1).
\]
We have by \((4.3)\)
\[
\varphi(t_1) - \varphi(t_2) = 2(t_1 - t_2) + O\left( \frac{T}{\ln T} \right) = O\left( \frac{T}{\ln T} \right), \quad U \in \left( 0, \frac{T}{\ln T} \right),
\]
and subsequently
\[
(4.5) \quad \Phi_y'[\varphi(t_1)] - \Phi_y'[\varphi(t_2)] = O \left\{ |\Phi_y''(T)| \cdot |\varphi(t_1) - \varphi(t_2)| \right\} = O(\ln \ln T),
\]
and therefore
\[
(4.6) \quad \Phi_y'[\varphi(t_1)] = \frac{1}{2} \ln T + O(\ln \ln T),
\]
by \((4.3), (4.5)\). Finally, \((1.2)\) follows \((1.1), (4.6)\).
Similarly to \((4.5)\) we have
\[
(4.7) \quad \Phi_y'[\varphi(t_1)] - \Phi_y'[\varphi(T)] = O(\ln \ln T).
\]
Then we obtain \((1.5)\) by \((4.6), (4.7)\).

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