Hertz-Moriya-Millis theory with dynamical critical exponent $z = 2$ has been proposed to describe antiferromagnetic quantum criticality in itinerant electron systems. In this study we show that the dynamical critical exponent changes from $z = 2$ to $z = 3$ at low temperatures, which results from effective long-range interactions between spin fluctuations, generated by Fermi-surface fluctuations beyond the Eliashberg framework. We claim that the underlying physics for the $z = 3$ antiferromagnetic quantum criticality is the emergence of fermionized skyrmion excitations at low energies, which form a Fermi surface, referred as a skyrmion liquid, where the interplay between itinerant electrons and skyrmions is argued to allow fermionized skyrmions. We construct a dual field theory in terms of skyrmion excitations, and obtain the $z = 3$ antiferromagnetic quantum criticality. This demonstration suggests the expected but nontrivial consistency between weak-coupling and strong-coupling theories.

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I. INTRODUCTION

At present, it is almost impossible to understand non-perturbative physics based on the weak coupling approach, where physically relevant quantum corrections should be taken into account up to an infinite order. Actually, we do not know how to describe various strong coupling phenomena such as the Kondo effect, Mott transition, Anderson localization, and etc., starting from the Fermi liquid theory and performing the diagrammatic analysis. Dualities between weak-coupling and strong-coupling theories have been playing key roles in figuring out non-perturbative phenomena, where topologically nontrivial excitations instead of original variables are introduced to describe strong coupling physics. Domain wall excitations appear to describe quantum criticality in the dual field theory of Ising model, identified with fermions, where the transformation from the Ising variable to the fermion field is highly nonlocal and nonlinear [1]. Unfortunately, duality transformations are not well defined in most cases. It is also almost impossible to derive the corresponding dual theory. Even if a dual field theory can be constructed, it is hard to prove the equivalence of physics at quantum criticality between the original and dual field theories [2].

However, the situation becomes better in one dimension [3]. Starting from an interacting one-dimensional model, one can perform the weak coupling analysis to find self-consistent equations for self-energy corrections of electrons, renormalized interactions, and vertex corrections. Resorting to the Ward identity, one obtains asymptotically exact effective interactions, which allow us to find the electron Green’s function. It turns out not to have poles but to show two branch cuts, implying the spin-charge separation and identified with Luttinger liquid. This diagrammatic analysis can be reformulated in terms of topologically nontrivial excitations called spinons and holons, which carry only spin and charge quantum numbers, respectively. This duality transformation is called bosonization, where duality equations are highly nonlinear and nonlocal as the case of Ising model. Resorting to such dual variables, one can construct a dual field theory in terms of topological excitations, which results in asymptotically the same Green’s function as that of the diagrammatic approach.

In this study we address essentially the same issue on non-perturbative dynamics in the Fermi-surface problem. Before 2009, it has been believed that the $1/N_\sigma$ expansion, where $N_\sigma$ denotes the spin degeneracy to extend from $\uparrow, \downarrow$ to $1, 2, ..., N_\sigma$, is a controllable method in understanding quantum criticality of itinerant electron systems, sometimes referred as the Eliashberg framework or a self-consistent RPA (random phase approximation) theory [4]. This belief in suspect has been broken in 2009, where all planar diagrams turn out to be in the “same” order even for the $1/N_\sigma$ expansion [5]. This conclusion means that vertex corrections associated with planar diagrams should be introduced up to an infinite order in order to obtain reliable critical exponents near quantum criticality [6].

We investigate two-dimensional antiferromagnetic quantum criticality in itinerant electron systems. Hertz-Moriya-Millis theory with dynamical critical exponent $z = 2$ has been proposed to describe the antiferromagnetic quantum criticality. However, we show that the dynamical critical exponent changes from $z = 2$ to $z = 3$ at low temperatures, which results from effective long-range interactions between spin fluctuations, generated by Fermi-surface fluctuations beyond the Eliashberg framework. We argue that the underlying physics for the $z = 3$ antiferromagnetic quantum criticality is the emergence of fermionized skyrmion excitations at low energies, which form a Fermi surface, referred as a skyrmion liquid, where the interplay between itinerant electrons and skyrmions is argued to allow fermionized...
We construct a dual field theory in terms of skyrmion excitations, and obtain the \( z = 3 \) antiferromagnetic quantum criticality. This demonstration suggests the expected but nontrivial consistency between weak-coupling and strong-coupling theories.

II. HERTZ-MORIYA-MILLIS-CHUBUKOV THEORY REVISITED

A. Spin-fermion model

A “standard” model for the spin-density-wave quantum criticality is the spin-fermion model, given by

\[
Z = \int Dc_{n\sigma} D\phi e^{-\int_{\mathbb{R}^3} dt \int d^d r \mathcal{L}},
\]

\[
\mathcal{L} = c_{n\sigma}^+(\partial_\tau - \mu_c) c_{n\sigma} + \frac{1}{2m_e} |(\nabla - i\mathbf{A}) c_{n\sigma}|^2 + \lambda_\phi e^{i\mathbf{Q} \cdot \mathbf{r}} \phi \cdot c_{n\sigma} \sigma_{\alpha\beta} c_{n\beta} + \phi (\nabla - v_\phi^2 \nabla^2 + \xi^2) \phi,
\]

where \( c_{n\sigma} \) is an electron field with spin \( \sigma = \uparrow, \downarrow \) and \( n = 1, ..., N \), and \( \phi \) is a spin-density field with three components. \( \mu_c \) is an electron chemical potential, and \( m_e \) is an electron band mass. \( \mathbf{A} \) is an electromagnetic vector potential field, externally applied and not considered in this study. \( v_\phi \) is a spin-wave velocity, and \( \xi \) is a spin-spin correlation length. The antiferromagnetic quantum critical point is defined as the renormalized correlation length diverges, i.e., \( \xi \rightarrow \infty \). \( \lambda_\phi \) is a coupling constant between spin fluctuations and itinerant electrons, and \( \mathbf{Q} \) is a nesting wave vector. Although we focus on two-dimensional quantum criticality, we leave the dimension to be represented by \( d \).

B. Eliashberg theory

The spin-fermion model has been solved in the Eliashberg framework, where only self-energy corrections of electrons and spin fluctuations are taken into account self-consistently while vertex corrections are neglected \[7\]. Self-consistent equations are given by

\[
\Sigma(\mathbf{k}, i\omega) = 3\lambda_{\phi}^2 \frac{1}{\beta} \sum_{q} G(\mathbf{k} + \mathbf{q}, i\omega + i\Omega) \chi(\mathbf{q}, i\Omega),
\]

\[
\Pi(\mathbf{q}, i\Omega) = -N \frac{3\lambda_{\phi}^2}{2} \sum_{i\omega} \sum_{\mathbf{k}} G(\mathbf{k} + \mathbf{q}, i\omega + i\Omega) G(\mathbf{k}, i\omega),
\]

where \( \Sigma(\mathbf{k}, i\omega) \) and \( \Pi(\mathbf{q}, i\Omega) \) are self-energy corrections of electrons and spin fluctuations, respectively, and \( G(\mathbf{k}, i\omega) \) and \( \chi(\mathbf{q}, i\Omega) \) are renormalized electron Green’s function and spin susceptibility, given by Dyson equations

\[
G(\mathbf{k}, i\omega) = \frac{1}{i\omega + \mu_c - \epsilon_k - \Sigma(\mathbf{k}, i\omega)},
\]

\[
\chi(\mathbf{q}, i\Omega) = \frac{1}{\Omega^2 + v_\phi^2 q^2 + \xi^2 + \Pi(\mathbf{q}, i\Omega)},
\]

respectively. These equations can be solved easily at the antiferromagnetic quantum critical point, solutions of which are \( \Pi(\mathbf{q}, i\Omega) \propto N\lambda_{\phi}^2 v_F^{-1} |\Omega| \) for the spin-fluctuation self-energy and \( \Sigma(i\omega) \propto \lambda_{\phi}^2 N_F \sqrt{|\omega|} \) for the electron self-energy in two dimensions, respectively, where \( N_F \) is the density of states at the Fermi energy. We note that the scaling relation between momentum and frequency in the spin-fluctuation propagator changes from \( \Omega^2 \sim v_\phi^2 |\mathbf{q}|^2 \) to \( |\Omega| \sim v_\phi^2 v_F / (N\lambda_{\phi}^3) |\mathbf{q}|^2 \) at low frequencies due to the spin-fluctuation self-energy. As a result, we obtain the dynamical critical exponent of \( z = 2 \) for this antiferromagnetic quantum critical point.

C. Hertz-Moriya-Millis-Chubukov theory

The \( z = 2 \) quantum critical liquid state, described by the Eliashberg theory for spin fluctuations, has been proposed to be the zeroth-order mother state at the quantum critical point as the Fermi liquid state away from quantum critical points \[4\]. In order to go beyond the Eliashberg framework, we integrate over electronic degrees of freedom, their
dynamics of which is given by the Eliashberg theory, and expand the resulting logarithmic term up to the fourth order in spin fluctuations. As a result, we obtain an effective field theory for spin fluctuations,

\[
S_{\text{eff}} = \frac{1}{\beta} \sum_{i\Omega} \int \frac{d^dq}{(2\pi)^d} \phi_r(i\Omega, q) \left( \Omega^2 + v_0^2 q^2 + \gamma_\phi |\Omega| + \xi^{-2} \right) \phi_r(-i\Omega, -q)
+ \frac{1}{\beta} \sum_{i\Omega} \int \frac{d^dq}{(2\pi)^d} \frac{1}{\beta} \sum_{i\nu} \int \frac{d^dp}{(2\pi)^d} \frac{1}{\beta} \sum_{i\nu'} \int \frac{d^dp'}{(2\pi)^d} \lambda_4(i\Omega, q; i\nu, p; i\nu', p')
\[
\left\{ \left[ \phi_r(i\nu + i\Omega/2, p + q/2) \cdot \phi_r(-i\nu + i\Omega/2, -p + q/2) \right] \left[ \phi_r(i\nu' - i\Omega/2, p' - q/2) \cdot \phi_r(-i\nu' - i\Omega/2, -p' - q/2) \right] \right\}
+ \left\{ \phi_r(i\nu + i\Omega/2, p + q/2) \cdot \phi_r(-i\nu + i\Omega/2, -p + q/2) \right\} \phi_r(i\nu' - i\Omega/2, p' - q/2) \cdot \phi_r(-i\nu' - i\Omega/2, -p' - q/2),
\]
\]

(4)
sometimes referred as the Hertz-Moriya-Millis-Chubukov theory. \(\gamma_\psi = N\lambda_0^2 v_F^{-1}\) is the Landau damping coefficient, given by Eq. (2) with Eq. (3), and \(\xi\) is a renormalized correlation length. \(\lambda_4(i\Omega, q; i\nu, p; i\nu', p')\) is a renormalized interaction vertex with four legs, given by [2]

\[
\lambda_4(i\Omega, q; i\nu, p; i\nu', p') = \frac{1}{4!} \left\{ \frac{1}{\beta} \sum_{i\omega} \int \frac{d^dk}{(2\pi)^d} G(k, i\omega) G(k + p - q/2, i\omega + i\nu - i\Omega/2) G(k - q, i\omega - i\Omega) G(k - p' - q/2, i\omega - i\nu' - i\Omega/2) \right\},
\]

(5)
where \(G(k, i\omega)\) is the Eliashberg electron Green’s function, given by Eq. (3).

An essential aspect is that the renormalized vertex of \(\lambda_4(i\Omega, q; i\nu, p; i\nu', p')\) depends on the momentum of \(q\) in a singular way, giving rise to long range interactions effectively in real space. More precisely, such long range interactions occur when the frequency of \(\Omega\) becomes finite. If one considers the limit of \(\Omega \to 0\), this renormalized interaction recovers a constant interaction-vertex, referred as the Hertz-Moriya-Millis theory.

We investigate the role of such nonlocal correlations in non-perturbative dynamics of spin fluctuations, which originate from Fermi-surface fluctuations beyond the Eliashberg approximation. Actually, their effects have been examined perturbatively, where vertex corrections associated with the nonlocal term are introduced into the Eliashberg spin susceptibility [3]. The dynamical critical exponent turns out to get a correction term, resulting in \(z \neq 2\). On the other hand, we perform the RPA analysis beyond the previous study, where the renormalization for the \(\lambda_4(i\Omega, q; i\nu, p; i\nu', p')\) vertex is taken into account. This non-perturbative analysis gives rise to \(z = 3\) antiferromagnetic quantum criticality.

### D. Molecular approximation

In order to renormalize the interaction vertex of \(\lambda_4(i\Omega, q; i\nu, p; i\nu', p')\), it is necessary to simplify the nonlocal interaction term, but keeping an essential feature. An idea is to introduce “molecular” field variables, which consist of composites of original spin variables. Although this procedure can be regarded to be quite conventional, our main approximation is to neglect their internal structures and to keep their center-of-mass dynamics only.

Performing the Fourier transformation for Eq. (4), we start from the Hertz-Moriya-Millis-Chubukov theory in real space

\[
S_{\text{eff}} = \int_0^\beta d\tau \int d^d r \phi_r(r, \tau) \left( -\partial_r^2 - v_0^2 \nabla_r^2 + \gamma_\phi \sqrt{-\partial_r^2 + \xi^{-2}} \right) \phi_r(r, \tau)
+ \int_0^\beta d\tau_1 \int d^d r_1 \int_0^\beta d\tau_2 \int d^d r_2 \int_0^\beta d\tau_3 \int d^d r_3 \int_0^\beta d\tau_4 \int d^d r_4 \lambda_4([r_2 - r_1, \tau_2 - \tau_1, r_4 - r_3, \tau_4 - \tau_3, (r_4 + r_3)/2 - (r_2 + r_1)/2, (\tau_4 + \tau_3)/2 - (\tau_2 + \tau_1)/2]
\[
\left\{ \left[ \phi_r(r_1, \tau_1) \cdot \phi_r(r_2, \tau_2) \right] \left[ \phi_r(r_3, \tau_3) \cdot \phi_r(r_4, \tau_4) \right] \right\} \int \{ \phi_r(r_1, \tau_1) \cdot \phi_r(r_2, \tau_2) \} \cdot \{ \phi_r(r_3, \tau_3) \cdot \phi_r(r_4, \tau_4) \}
\]
\]

(6)
It is straightforward to rewrite this field theory in terms of center-of-mass and relative coordinates, given by

\[
S_{\text{eff}} = \int_0^\beta d\tau \int d^d r \phi_r(r, \tau) \left(-\partial_\tau^2 - v_\phi^2 \nabla^2 + \gamma_\psi \sqrt{-\partial_\tau^2 + \xi^2}\right) \phi_r(r, \tau) \\
+ \int_0^\beta d\tau_1 \int d^d r_1 \int_0^\beta d\tau_2 \int d^d r_2 \int_0^\beta d\tau_1 \int d^d R_1 \int_0^\beta d\tau_2 \int d^d R_2 \lambda, \phi_r(R_1 + r_1/2, \tau_1 + \tau_1/2) \cdot \phi_r(R_1 - r_1/2, \tau_1 - \tau_1/2)|[\phi_r(R_2 + r_2/2, \tau_2 + \tau_2/2) \cdot \phi_r(R_2 - r_2/2, \tau_2 - \tau_2/2)]
\]

where \( R_1 = \frac{r_1 + r_2}{2}, \tau_1 = \frac{\tau_1 + \tau_2}{2} \) and \( R_2 = \frac{r_1 + r_2}{2}, \tau_2 = \frac{\tau_1 + \tau_2}{2} \) are center-of-mass coordinates and \( r_2 - r_1 \rightarrow r_1, \tau_2 - \tau_1 \rightarrow \tau_1 \) and \( r_4 - r_3 \rightarrow r_2, \tau_4 - \tau_3 \rightarrow \tau_2 \) are relative coordinates.

We introduce molecular field variables,

\[
\left\langle \phi_r(R_1 + r_1/2, \tau_1 + \tau_1/2) \cdot \phi_r(R_1 - r_1/2, \tau_1 - \tau_1/2) \right\rangle \rightarrow V(r_1, \tau_1; R_1, \tau_1), \\
\left\langle \phi_r(R_1 + r_1/2, \tau_1 + \tau_1/2) \times \phi_r(R_1 - r_1/2, \tau_1 - \tau_1/2) \right\rangle \rightarrow X(r_1, \tau_1; R_1, \tau_1),
\]

where \( V(r_1, \tau_1; R_1, \tau_1) \) represents valence bond fluctuations, and \( X(r_1, \tau_1; R_1, \tau_1) \) expresses vector spin-chirality fluctuations.

Performing the Hubbard-Stratonovich transformation and doing the Fourier transformation, we obtain

\[
S_{\text{eff}} = \sum_{\alpha \Omega} \sum_q \phi_r(q, \alpha \Omega) \left(\Omega^2 + v_\phi^2 q^2 + \gamma_\psi |\Omega| + \xi^{-2}\right) \phi_r(-q, -\alpha \Omega) \\
+ \sum_{\alpha \Omega} \sum_q \frac{1}{\beta} \sum_{\alpha' \Omega'} \phi_r(q, \alpha' \Omega') \phi_r(q', \alpha' \Omega') V((q + q')/2, (\alpha + \alpha')/2; q_1, \alpha \Omega) + \lambda_q \left[-q_1; -q, -\alpha \Omega\right] \\
+ \frac{1}{4} X(q_1, \alpha \Omega; q_1, \alpha \Omega) \rightarrow V(q - q', \alpha \Omega - \alpha' \Omega) \rightarrow X(q - q', \alpha \Omega - \alpha' \Omega) + \frac{1}{4} X(q_1, \alpha \Omega; q_1, \alpha \Omega) \rightarrow X(q - q', \alpha \Omega - \alpha' \Omega)
\]

Up to now, all procedures from Eq. (4) to Eq. (9) are exact.

An essential approximation is to neglect internal dynamics of molecular field variables and to keep their center-of-mass dynamics. Actually, this approximation has been rather conventionally utilized because internal dynamics can be regarded as fast degrees of freedom. However, we confess that the validity of this approximation is not proven, particulary, at the quantum critical point. If sizes of molecules become enhanced and the probability for overlapping between them gets larger at the quantum critical point, our approximation to view such molecules as point particles will not be valid any more.

Performing the point-particle approximation for such composite field variables, we obtain

\[
S_{\text{eff}} = \sum_{\alpha \Omega} \sum_q \phi_r(q, \alpha \Omega) \left(\Omega^2 + v_\phi^2 q^2 + \gamma_\psi |\Omega| + \xi^{-2}\right) \phi_r(-q, -\alpha \Omega) \\
+ \sum_{\alpha \Omega} \sum_q \lambda_4(q, \alpha \Omega) \left[V(q, \alpha \Omega) V(-q, -\alpha \Omega) + \lambda_4(q, \alpha \Omega) \right] X(q, \alpha \Omega) \cdot X(-q, -\alpha \Omega) \\
- \frac{1}{\beta} \sum_{\alpha' \Omega'} \lambda_4(q, \alpha' \Omega') \int d^d \text{q} \left[ \phi_r(q, \alpha \Omega) \phi_r(-q, -\alpha \Omega) \right]
\]

Rescaling \( V(q - q', \alpha \Omega - \alpha' \Omega) \rightarrow V(q - q', \alpha \Omega - \alpha' \Omega) + \frac{1}{\beta} \sum_{\alpha' \Omega'} \sum_q \lambda_4(q, \alpha' \Omega') \cdot X(q - q', \alpha \Omega - \alpha' \Omega) \rightarrow X(q - q', \alpha \Omega - \alpha' \Omega) \), and \( \lambda_4(q, \alpha \Omega) \rightarrow \left(\frac{1}{\beta} \sum_{\alpha' \Omega'} \sum_q \lambda_4(q, \alpha' \Omega') \right)^2 \lambda_4(q, \alpha \Omega) \), we reach the following
expression

\[ Z_{\text{eff}} = \int D\phi_r(q, i\Omega) D\nu(q, i\Omega) D\mathcal{X}(q, i\Omega) \exp \left\{ -\beta S_{\text{eff}}[\phi_r(q, i\Omega), \nu(q, i\Omega), \mathcal{X}(q, i\Omega)] \right\}, \]

\[ S_{\text{eff}} = \sum_{q} \sum_{i\Omega} \phi_r(q, i\Omega) \left( \Omega^2 + v_0^2 q^2 + \gamma \nu q^2 + \xi^{-2} \right) \phi_r(-q, -i\Omega) \]

\[ + \sum_{q} \sum_{i\Omega} \left( \frac{\lambda_4(q, i\Omega)}{4} \nu(q, i\Omega) \nu(-q, -i\Omega) \right) \mathcal{X}(q, i\Omega) \cdot \mathcal{X}(-q, -i\Omega) \]

\[ - i \frac{1}{\beta} \sum_{q} \sum_{i\Omega} \nu(q - q', i\Omega - i\Omega') \left[ \phi_r(q', i\Omega') \cdot \phi_r(-q, -i\Omega) \right] - i \frac{1}{\beta} \sum_{q} \sum_{i\Omega} \mathcal{X}(q - q', i\Omega - i\Omega') \cdot \left[ \phi_r(q', i\Omega') \times \phi_r(-q, -i\Omega) \right] \right\}. \]

(11)

One can evaluate the four-point interaction vertex, given by

\[ \lambda_4(q, i\Omega) \approx \lambda_4 \left| \frac{\Omega}{q^2} \right|^2 \]  

 asymptotically, which implies that this interaction is marginal in two dimensions.

Integrating over molecular fields and performing the Fourier transformation in Eq. (11), it is not difficult to understand the reason why we are saying that only the center-of-mass dynamics is considered,

\[ S_{\text{eff}} = \int_{0}^{\beta} d\tau \int d^3r \phi_r(r, \tau) \left( -\partial_\tau^2 - \frac{\nu_0}{2} \mathbf{v}_r^2 + \gamma \mathbf{\nu} \sqrt{-\partial_\tau^2 + \xi^{-2}} \right) \phi_r(r, \tau) \]

\[ + \int_{0}^{\beta} d\tau_1 \int d^3r_1 \int_{0}^{\beta} d\tau_2 \int d^3r_2 \int_{0}^{\beta} d\tau_3 \int d^3r_3 \left\{ \frac{1}{\lambda_4(r_1 - r_2, \tau_1 - \tau_2)} \right\} \]

\[ \sum_{q} \sum_{i\Omega} \left[ \phi_r(r_1, \tau_1) \times \phi_r(r_3, \tau_3) \right] \cdot \left[ \phi_r(r_2, \tau_2) \times \phi_r(r_3, \tau_3) \right] \right\}. \]

(13)

An interesting point is that the interaction vertex appears in the denominator instead of the numerator. Of course, this is consistent with its dimension, considering how it rescales. This relation reminds us of the weak-coupling and strong-coupling duality.

### III. BEYOND THE ELIASHBERG FRAMEWORK

#### A. Renormalization for the interaction vertex in the RPA level

An important point beyond all previous studies is to introduce renormalization of the interaction vertex. We perform the renormalization within the RPA analysis. It is straightforward to see that no renormalization occurs from vector spin-chirality fluctuations, given by

\[ S'_{\text{int}} \approx \frac{1}{2} \sum_{\Omega} \sum_{q'} \frac{1}{\beta} \sum_{q''} \frac{1}{\beta} \sum_{q'''} \frac{1}{\beta} \sum_{q''''} \left\{ \left[ \phi_r(q', i\Omega') \times \phi_r(-q, -i\Omega) \right] \left[ \phi_r(q'', i\Omega'') \times \phi_r(-q'', -i\Omega'') \right] \right\}_c \]

\[ \mathcal{X}(q - q', i\Omega - i\Omega') \mathcal{X}(q'' - q'', i\Omega'' - i\Omega'') = 0, \]  

(14)

where the subscript “c” denotes “connected” for the diagrammatic expansion. On the other hand, valence bond fluctuations give rise to

\[ S'_{\text{int}} \approx \frac{1}{2} \sum_{\Omega} \sum_{q'} \frac{1}{\beta} \sum_{q''} \frac{1}{\beta} \sum_{q'''} \frac{1}{\beta} \sum_{q''''} \left\{ \left[ \phi_r(q', i\Omega') \cdot \phi_r(-q, -i\Omega) \right] \left[ \phi_r(q'', i\Omega'') \cdot \phi_r(-q'', -i\Omega'') \right] \right\}_c \]

\[ \mathcal{V}(q - q', i\Omega - i\Omega') \mathcal{V}(q'' - q'', i\Omega'' - i\Omega'') \]

\[ \approx \sum_{\Omega} \sum_{q'} \sum_{q''} \sum_{q'''} \left[ \phi_r(q, i\Omega) \cdot \phi_r(-q, -i\Omega) \right] \left[ \phi_r(q', i\Omega') \cdot \phi_r(-q', -i\Omega') \right] \]

\[ \mathcal{V}(q - q', i\Omega - i\Omega') \mathcal{V}(q + q', -i\Omega + i\Omega') \]

\[ \approx \frac{1}{4} \sum_{\Omega} \sum_{q'} \sum_{q''} \sum_{q'''} \left[ \phi_r(q + q', i\Omega + i\Omega') \rangle \langle \phi_r(q', i\Omega') \mathcal{V}(q, i\Omega) \mathcal{V}(q, -i\Omega), \right. \]

where \( g_\psi(q, i\Omega) = \left( \Omega^2 + v_0^2 q^2 + \gamma \nu q^2 + \xi^{-2} \right)^{-1} \) is the spin-fluctuation propagator.
B. Self-energy corrections to antiferromagnetic fluctuations

The renormalized interaction vertex causes the self-energy correction for spin fluctuations, given by

\[ S_{\text{int}}^{\phi} \approx 2 \sum_{i\Omega} \sum_q \frac{1}{\beta} \sum_{i\Omega'} \sum_{q'} \left\langle \nabla(q - q', i\Omega - i\Omega') \nabla(-q + q', -i\Omega + i\Omega') \right\rangle_c \left\langle \phi_r(q', i\Omega') \cdot \phi_r(-q', -i\Omega') \right\rangle_c \]

\[ \phi_r(q, i\Omega) \cdot \phi_r(-q, -i\Omega) = \sum_{i\Omega} \sum_q \left( \frac{1}{2} \beta \sum_{i\Omega'} \sum_{q'} G_V(q - q', i\Omega - i\Omega') g_\phi(q', i\Omega') \right) \phi_r(q, i\Omega) \cdot \phi_r(-q, -i\Omega), \quad (16) \]

where the RPA propagator for valence bond fluctuations is

\[ G_V(q, i\Omega) = \frac{1}{\lambda(q, i\Omega) + i \frac{1}{\beta} \sum_{i\Omega'} \sum_q g_\phi(q + q', i\Omega + i\Omega') g_\phi(q', i\Omega')} \]

(17)

As a result, the effective action for spin fluctuations reads

\[ S_{\text{eff}}^{\phi} = \sum_{i\Omega} \sum_q \phi_r(q, i\Omega) \left[ \Omega^2 + v_\phi^2 q^2 + \gamma_\phi |\Omega| \right] + \frac{1}{2} \sum_{i\Omega'} \sum_q g_\phi(q', i\Omega') \left( \lambda(q - q', i\Omega - i\Omega') + \frac{1}{2} \beta \sum_{i\Omega''} \sum_q g_\phi(q - q', i\Omega - i\Omega' + i\Omega'') g_\phi(q', i\Omega'') \right) \phi_r(-q, -i\Omega). \quad (18) \]

This expression shows the main point of our study. If the RPA renormalization is not introduced, the self-energy correction for spin fluctuations is reduced to that of the previous study [6]. Indeed, one can find logarithmic singularities from the “bare” interaction. However, we find that the self-energy correction in the RPA valence-bond fluctuation propagator overcomes the “bare” part. In other words, we obtain

\[ G_V(q, i\Omega) \approx \left\{ \frac{1}{\lambda(q, i\Omega) + i \frac{1}{\beta} \sum_{i\Omega'} \sum_q g_\phi(q + q', i\Omega + i\Omega')} \right\}^{-1}. \]

Inserting this expression into Eq. (18), we find

\[ S_{\text{eff}}^{\phi} \approx \sum_{i\Omega} \sum_q \phi_r(q, i\Omega) \left[ \gamma_\phi |\Omega| + v_\phi^2 q^2 - \frac{2 \lambda^2 \gamma_\phi^2 \Omega^2}{\beta} \ln(-i\Omega) \right] \phi_r(-q, -i\Omega). \quad (19) \]

Detailed calculations are shown in appendix. As shown in this expression, \( z = 3 \) antiferromagnetic quantum criticality emerges in \( q < q_c \), where \( q_c \) is a characteristic momentum given by the comparison between the first term and others. This leads us to conclude that the \( z = 3 \) Hertz-Moriya-Millis-Chubukov theory describes the antiferromagnetic quantum critical point at low temperatures.

IV. A STRONG-COUPLING APPROACH

A. An effective field theory

It is not easy to understand the underlying physics for the emergence of the \( z = 3 \) antiferromagnetic quantum criticality from the \( z = 2 \) Hertz-Moriya-Millis-Chubukov theory. Here, “to understand the underlying physics” means to identify elementary excitations, resulting in \( z = 3 \). As described before, the renormalization of the interaction vertex from valence bond fluctuations is the source of the \( z = 3 \) quantum criticality. In this respect our question is as follows. Can we find the dual field variable of valence bond fluctuations? We claim that fermionized skyrmion excitations play the role of valence bond fluctuations. We show that fermionized skyrmion excitations with their Fermi surface result in \( z = 3 \) quantum criticality.

We start from the following spin-fermion model without the kinetic energy term of spin fluctuations,

\[ Z = \int Dc_{\alpha} D\phi e^{-\int_0^t d\tau f d^d r L}, \]

\[ L = c_{\alpha}^\dagger (\partial_\tau - \mu_c) c_{\alpha} + \frac{1}{2m_c} (\mathbf{\nabla} - i A)^2 c_{\alpha} c_{\alpha} + \lambda_\phi e^{iQ \cdot r} \phi \cdot c_{\alpha}^\dagger \sigma_{\alpha \beta} c_{\beta}, \quad (20) \]

regarded as the strong coupling limit of Eq. (1). We introduce the CP1 representation for the angular field variable, where the amplitude field is assumed to be frozen in the low energy limit, given by

\[ \lambda_\phi e^{iQ \cdot r} \phi = \lambda_\phi e^{iQ \cdot r} \phi U_{\alpha \gamma} \sigma_{\gamma \delta} U_{\delta \beta}^\dagger. \quad (21) \]
$U = \begin{pmatrix} z \uparrow & z \downarrow^* \\ z \downarrow^* & -z \uparrow \end{pmatrix}$ is an SU(2) matrix field to represent the angular fluctuation, where $z \uparrow = e^{-i\frac{\phi}{2}} \cos \frac{\theta}{2}$ and $z \downarrow = e^{i\frac{\phi}{2}} \sin \frac{\theta}{2}$ denote complex boson fields, identified with spinons.

An essential ansatz in the strong coupling limit of $\lambda_\phi$ is that the spin of an electron field is screened from the angular fluctuation of the spin-fluctuation field, represented as follows

$$\psi_{na} = U_{a\beta}^\dagger c_{n\beta}. \quad (22)$$

$\psi_{na}$ is a holon field which carries the same charge quantum number as an electron while it does not carry the spin quantum number. Based on this ansatz, we rewrite the spin-fermion model in the strong coupling limit as follows

$$Z = \int D\psi_{na} D\psi_{n\alpha} D\psi_{n\sigma} \delta(U_{a\alpha} U_{\gamma\beta} - \delta_{a\beta}) e^{-\int_0^\beta f_0^\lambda d\tau \int f^2 d^2 L},$$

$$\mathcal{L} = \psi_{na}^\dagger \left[ (\partial_\tau - \mu_e - m_e^{-1}) \delta_{a\alpha} + U_{\alpha\gamma} \partial_\tau U_{\gamma\beta} \right] \psi_{n\beta} + \frac{1}{2m_e^2} \left( \partial_\tau \psi_{n\beta}^\dagger - \psi_{n\alpha}^\dagger U_{\alpha\gamma} \partial_\tau U_{\gamma\beta} \right) \left( \partial_\tau \psi_{n\beta} - (\partial_\tau U^\dagger)_{\gamma\beta} U_{\gamma\alpha} \psi_{n\alpha} \right)$$

$$+ \lambda_\phi e^{iQ \cdot \mathbf{r}} |\phi| \sigma_{\alpha\sigma} \psi_{n\sigma} + \frac{1}{2m_e} \psi_{n\alpha}^\dagger \left( \partial_\tau U^\dagger + [U^\dagger \partial_\tau U] U^\dagger \right) \alpha_{\alpha\beta} \left( \partial_\tau U + U [(\partial_\tau U^\dagger) U] \right)_{\gamma\beta} \psi_{n\beta} - \frac{1}{2m_e} \partial_\tau (\psi_{n}^\dagger \psi_{n}), \quad (23)$$

where holons and spinons appear to realize the spin-charge separation. Here, we omit the electromagnetic vector potential for the time being. We emphasize that this expression is just a reformulation of Eq. (20), regarded as the change of variables.

Introducing the nonabelian gauge field of $A_{a\beta}^\nu = -i [(\partial_\mu U^\dagger U)_{a\beta}]$, one can rewrite the above as follows

$$Z = \int D\psi_{na} D\psi_{n\alpha} D\psi_{n\sigma} \delta(U_{a\alpha} U_{\gamma\beta} - \delta_{a\beta}) \delta(A_{a\beta}^\nu + i [(\partial_\mu U^\dagger U)_{a\beta}]) \left( \partial_\tau A_{a\beta}^\nu \right) e^{-\int_0^\beta f_0^\lambda d\tau \int f^2 d^2 L},$$

$$\mathcal{L} = \psi_{na}^\dagger \left[ (\partial_\tau - \mu_e - m_e^{-1}) \delta_{a\alpha} + \lambda_\phi e^{iQ \cdot \mathbf{r}} |\phi| \sigma_{\alpha\beta} \right] \psi_{n\beta} + \frac{1}{2m_e^2} \left( \partial_\tau \psi_{n\beta}^\dagger - \psi_{n\alpha}^\dagger A_{a\beta}^\nu \right) \left( \partial_\tau \psi_{n\beta} - iA_{a\beta}^\nu \psi_{n\gamma} \right)$$

$$+ \frac{1}{2m_e} \psi_{n\alpha}^\dagger \left( \partial_\tau U^\dagger + iA_{a\alpha}^\nu U^\dagger \right) \alpha_{\alpha\beta} \left( \partial_\tau U + iU \zeta A_{a\beta}^\nu \right) \psi_{n\beta}, \quad (24)$$

where SU(2) gauge symmetry becomes shown explicitly.

Although exactness is still guaranteed in Eq. (24), it is rather complicated to handle this effective field theory. In this respect, we perform the U(1) projection, considering the subset of the SU(2) gauge theory, given by

$$Z = \int D\psi_{n\sigma} Dz_{\alpha} D\alpha_{\mu} \delta(|z_{\sigma}|^2 - 1) \delta(\partial_\tau z_{\sigma}) e^{-\int_0^\beta f_0^\lambda d\tau \int f^2 d^2 L}, \quad \mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_z,$$

$$\mathcal{L}_\psi = \psi_{n\alpha}^\dagger (\partial_\tau - \mu_e + \lambda_\phi e^{iQ \cdot \mathbf{r}} |\phi| \sigma_{\alpha\beta}) \psi_{n\sigma} + \frac{1}{2m_e^2} \left( \partial_\tau - i\sigma_\alpha \right) \left( \partial_\tau - i\sigma_\alpha \right) \psi_{n\sigma}^2,$$

$$\mathcal{L}_z = \frac{1}{2m_e} \left( \partial_\tau - i\alpha_\tau \right) |z_{\sigma}|^2 + \frac{\rho_s}{2m_e} \left( \partial_\tau - i\alpha_\tau \right) |z_{\sigma}|^2 \quad (25)$$

with $\mu_e = \mu_e + m_e^{-1}$. One may understand this procedure as the staggered-flux ansatz in the SU(2) slave-boson theory [3], where the SU(2) gauge symmetry is reduced to the U(1) symmetry. It is interesting to see that the internal gauge field $a_e$ couples to the spin current of the renormalized electron field while the electromagnetic field $A_\tau$ does to the charge current. The spinon part is reduced to the CP1 representation of the O(3) nonlinear $\sigma$ model [8] with a stiffness parameter of $\rho_s = \langle (\sigma_{n\alpha}^\dagger \psi_{n\sigma})^2 \rangle$, where the time derivative term is added explicitly. This time derivative term is expected to appear from quantum corrections, i.e., the self-energy correction to the spinon dynamics.

### B. Duality transformation

To pull out dynamics of skyrmions, we perform the duality transformation. Unfortunately, the duality transformation for the SU(2) case is not known. We take the easy plane approximation

$$z_{\sigma} = \frac{1}{\sqrt{2}} e^{i\phi_\sigma}. \quad (26)$$

In this representation the topological excitation corresponding to a vortex is a meron instead of a skyrmion [10]. Since a meron solution can be regarded as a half skyrmion, we will introduce a meron pair as a skyrmion [11].
Resorting to the easy plane approximation, we obtain
\[ \mathcal{L} = \psi^\dagger_{ns}(\partial_\tau - \mu_r + \lambda_\phi e^{iQ_r \cdot \sigma} \phi | \sigma - i\sigma \alpha_r \psi_{ns} + \frac{1}{2m_c} |(\delta_\tau - i\alpha_r - iA_r)\psi_{ns}|^2 + \frac{1}{2u_c}(\partial_\tau \phi_\sigma - a_r)^2 + \frac{\rho_s}{2m_c}(\partial_\tau \phi_\sigma - a_r)^2. \]  
(27)

Performing the duality transformation, we obtain the dual Lagrangian
\[ \mathcal{L}_D = \psi^\dagger_{ns}(\partial_\tau - \mu_r + \lambda_\phi e^{iQ_r \cdot \sigma} \phi | \sigma - i\sigma \alpha_r \psi_{ns} + \frac{1}{2m_c} |(\delta_\tau - i\alpha_r - iA_r)\psi_{ns}|^2 + \frac{u_c}{2}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 - \frac{m_c}{2\rho_s}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 - \frac{m_c}{2\rho_s}J^\dagger_D \mu c^\mu_\nu, \]  
(28)
where \( J^\dagger_D \mu \) is a meron current and \( c^\sigma_\mu \) is the dual gauge field representing spin wave excitations.

Introducing the second quantization for meron dynamics \([10]\), we obtain
\[ \mathcal{L}_D = \psi^\dagger_{ns}(\partial_\tau - \mu_r + \lambda_\phi e^{iQ_r \cdot \sigma} \phi | \sigma - i\sigma \alpha_r \psi_{ns} + \frac{1}{2m_c} |(\delta_\tau - i\alpha_r - iA_r)\psi_{ns}|^2 + |(\partial_\mu - i\phi \sigma)|^2 + m^2_D |\Phi^\dagger_D|^2 + \frac{u_c}{2}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 + \frac{m_c}{2\rho_s}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2, \]  
(29)
where \( \Phi^\dagger_D \) is a meron field and \( V_\text{an}[\Phi^\dagger_D] \) is an anisotropy potential for merons.

To find skyrmion dynamics, we rewrite the above meron Lagrangian in terms of meron pairs. This derivation is far from rigorous, but sometimes utilized in order to make physical excitations \([11]\). Considering \( \Phi^\dagger_D \rightarrow e^{i\theta_\tau} \), the meron kinetic energy term is
\[ (\partial_\mu \theta_\sigma - c^\mu_\sigma)^2 = (\partial_\mu \theta_\tau - c^\mu_\tau)^2 + (\partial_\mu \theta_\sigma - c^\mu_\sigma)^2 = \frac{1}{2} \left( [\partial_\mu \theta_\tau - \partial_\mu \theta_\sigma] - [c^\mu_\tau - c^\mu_\sigma] \right)^2. \]  
(30)
The spin wave term is rewritten as follows
\[ \frac{u_c}{2}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 + \frac{m_c}{2\rho_s}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 = \frac{u_c}{4}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 + \frac{m_c}{4\rho_s}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 \]  
(31)
As a result, we reach
\[ \mathcal{L}_D = \psi^\dagger_{ns}(\partial_\tau - \mu_r + \lambda_\phi e^{iQ_r \cdot \sigma} \phi | \sigma - i\sigma \alpha_r \psi_{ns} + \frac{1}{2m_c} |(\delta_\tau - i\alpha_r - iA_r)\psi_{ns}|^2 - \frac{m_c}{2\rho_s}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 \]  
\[ + |(\partial_\mu - i\phi \sigma)|^2 + m^2_D |\Phi^\dagger_D|^2 + \frac{u_c}{2}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2 + \frac{m_c}{2\rho_s}(\epsilon_{\tau\mu\nu}\partial_\mu c^\sigma_\nu)^2, \]  
(32)
where
\[ \theta_\pm = \theta_\tau \pm \theta_\sigma \rightarrow \Phi^\dagger_D, \]  
(33)
It is interesting to see that the \( \Phi^\dagger_D \) field does not couple to dynamics of renormalized electrons \( \psi_{ns} \) directly. Only the \( \Phi^\dagger_D \) field couples to fermions via the mutual Chern-Simons term. Integrating over \( \Phi^\dagger_D, c^\sigma_\mu \), and \( \alpha_r \) fields, we will obtain an effective field theory for renormalized electrons interacting with skyrmions. It is natural to expect the emergence of nonlocal interactions between spin currents of renormalized electrons (holons) and skyrmion currents. Integrating over holon excitations, we reach a dual filed theory for skyrmion dynamics. Unfortunately, this procedure cannot be performed rigorously. We propose the following skyrmion field theory
\[ S_{sk} = \int_0^{\beta_r} d\tau \int d^2r \left\{ \mu_s \Phi^\dagger_s(\partial_\tau - i\tau)\Phi_s + \frac{1}{2} \left( \Phi_s \right)^2, \right. \]  
(34)
where \( \Phi_D \) is replaced with \( \Phi_s \) and the \( - \) superscript in \( c_{\mu} \) is omitted.

It is our proposal to introduce the linear-time derivative term into Eq. (34). Then, the linear-time derivative term becomes more relevant than the second-time derivative term at low energies. Unfortunately, we cannot prove the emergence of the Galilean invariance (the linear-time derivative) from the relativistic invariance (the second-time derivative) at present. In this respect one can call the emergence of the Galilean invariance in the presence of itinerant electrons as our speculation.

There must be an underlying physical mechanism for this linear-time derivative term, which breaks the particle-hole symmetry for skyrmion excitations. It has been demonstrated that the density of skyrmion excitations is finite at an antiferromagnetic quantum critical point without itinerant electrons, i.e., in an insulating antiferromagnet \[12\]. Of course, only the second-time derivative term is allowed in this case, i.e., particle-hole symmetric, which means that an equal number of skyrmion and anti-skyrmion excitations exists at this quantum critical point. Our problem is what happens on the particle-hole symmetry in skyrmion excitations at the quantum critical point where itinerant charge carriers are introduced. This is a long-standing problem, where non-perturbative effects from interactions between itinerant electrons and “many” topologically nontrivial excitations should be taken into account on equal footing. Frankly speaking, we do not have any reliable mathematical tools for the description of such interactions.

Our speculation is that the presence of itinerant electrons will induce the particle-hole symmetry breaking in the skyrmion sector because itinerant electrons favor skyrmion excitations (or anti-skyrmions, i.e., one of the two). The physical mechanism is as follows. When itinerant electrons move in the background of skyrmions, they feel an effective magnetic flux, which quenches the kinetic energy of electrons. Our expectation is that the gain in the electron kinetic energy contribution can overcome the energy cost due to the particle-hole symmetry breaking in the skyrmion sector. As a result, the particle-hole symmetry breaking is favorable in the respect of the total energy. Actually, this mechanism has been realized in the system of frustrated magnets, where the presence of itinerant electrons leads the co-planar ordering in the triangular antiferromagnet to be ordered into an out-of-plane way, which corresponds to a spin chiral order \[13\].

Possibility of statistical transmutation for vortices has been discussed in the vortex liquid phase \[14\] and at quantum criticality in geometrically frustrated spin systems \[13\]. Fermionization of skyrmion excitations can be performed in the same way as that of vortices. The key point is that the Chern-Simons term becomes irrelevant at quantum criticality \[14\ \[17\], implying that the quantum statistics for skyrmions or vortices may not be well defined at quantum criticality. Actually, such excitations are strongly interacting at criticality, where we are not allowed to pin down elementary excitations clearly. We reach an effective field theory of fermionic skyrmions,

\[
\mathcal{S}_{\text{eff}} = \int_0^{\beta} d\tau \int d^2 r \left\{ \psi_s^\dagger (\partial_\tau - \mu_\psi - ic_\tau) \psi_s + \frac{1}{2m_{sk}} |(\nabla - ic)\psi_s|^2 + \frac{ue}{2}(\epsilon_{\tau\mu\nu} \partial_\mu \epsilon_{\nu\sigma})^2 + \frac{m_c}{2\rho_s}(\epsilon_{\tau\mu\nu} \partial_\mu \epsilon_{\nu\sigma})^2 \right\},
\]

where \( \psi_s \) represents the fermionic skyrmion field with the skyrmion chemical potential \( \mu_\psi \) and the band mass \( m_{sk} \propto \mu_{sk} \). We would like to emphasize that Eq. (35) can be derived from Eq. (34) at least formally via the Chern-Simons transformation \[14\ \[13\].

Another important point with the particle-hole symmetry breaking term is that the skyrmion chemical potential is assumed to be finite. As discussed before, the skyrmion density has been shown to be finite in the O(3) nonlinear \( \sigma \) model without itinerant electrons \[12\]. It is natural to expect that this will hold even in the presence of itinerant electrons. Then, Landau damping for gauge fluctuations emerges from particle-hole excitations of fermionic skyrmions near their Fermi surface, resulting in the \( z = 3 \) dynamics for spin fluctuations.

Although the \( z = 3 \) antiferromagnetic quantum criticality appears in the fermionized skyrmion ansatz with the skyrmion Fermi surface, it is still not clear how this \( z = 3 \) quantum criticality can be connected with that of the previous diagrammatic approach. It has been shown that the skyrmion field has the same symmetry as the valence bond field in an insulating antiferromagnet \[10\ \[16\ \[17\]. Resorting to the fermion representation for the spin operator, one can construct an effective Dirac theory with an enhanced emergent symmetry, compared with the Heisenberg model with the O(3) spin symmetry \[16\ \[18\]. Based on the Dirac theory, one can construct the valence bond operator and skyrmion operator in terms of the Dirac spinor. Since the symmetry property is well defined for Dirac fermions, it is straightforward to investigate symmetries of both valence bond and skyrmion operators. They turn out to be the same as each other. One can perform the same work in the boson representation for spin, where the role of the spin Berry phase is crucial to assign the valence bond quantum number to a skyrmion \[10\]. On the other hand, it is not verified yet whether skyrmion excitations can be identified with valence bond fluctuations in the presence of itinerant electrons. In particular, the role of Berry phase is not clarified yet. However, it is expected that valence bond fluctuations will be deeply related with skyrmion fluctuations. Actually, the skyrmion operator has been constructed in terms of fermion bilinear operators, identified with the valence bond operator in itinerant antiferromagnets \[19\].
V. DISCUSSION AND SUMMARY

In summary, we showed that the dynamical critical exponent can change from $z = 2$ to $z = 3$ at low temperatures in antiferromagnetic quantum criticality, where nonlocal interactions between spin fluctuations, which result from Fermi surface fluctuations, are responsible for the change of the dynamics of spin-fluctuations. In particular, we claimed that renormalization in the nonlocal interaction vertex should be introduced because the renormalized part turns out to be larger than the bare (unrenormalized) interaction. We performed the renormalization in the RPA level, where valence bond fluctuations are introduced explicitly and the nonlocal interaction vertex is identified with a bare propagator of valence bond fluctuations. As a result, the RPA renormalization in valence bond fluctuations gives rise to the $z = 3$ antiferromagnetic quantum criticality.

We tried to understand the underlying physics for the emergence of $z = 3$ antiferromagnetic quantum criticality. Hints from the fact that valence bond fluctuations can be identified with skyrmion excitations, we constructed a dual field theory in terms of fermionized skyrmion excitations interacting with spin-wave fluctuations. If the density of skyrmions is finite where the Chern-Simons term turns out to be irrelevant at the quantum critical point, we found a dual theory in terms of fermionized skyrmion excitations interacting with spin-wave fluctuations. Allowing the statistical transmutation for skyrmion fluctuations, where the Chern-Simons term turns out to be irrelevant at the quantum critical point, we showed that the dynamical critical exponent can change from $z = 2$ to $z = 3$ at low temperatures in antiferromagnetic quantum criticality.

An important missing point in our study is to neglect higher-order terms beyond the fourth order expansion. Such higher-order terms have been argued to be important because the two-dimensional spin-fluctuation theory turns out to allow infinite number of marginal interactions, which originate from singular momentum and frequency dependencies for interaction vertices. Based on this observation, it was claimed that an anomalous dimension for spin fluctuations appears beyond the Eliashberg framework [22]. We do not exclude this possibility. Actually, the spin susceptibility can be 

\[ \chi_{\gamma}(q, i\Omega) = \left( \gamma_{\psi}|\Omega| + v_{\psi}^2 q^2 - \frac{2v_{\psi}^2 \gamma_{\psi}^2 \Omega^2}{v_{\psi}^2 q^2 + \ln(-i\Omega)} \right)^{-1} \]

instead of \( \chi_{\psi}(q, i\Omega) = \left( \gamma_{\psi}|\Omega| + v_{\psi}^2 q^2 - \frac{2v_{\psi}^2 \gamma_{\psi}^2 \Omega^2}{v_{\psi}^2 q^2 + \ln(-i\Omega)} \right)^{-1} \) in Eq. (19), where \( \gamma \) is the anomalous dimension of spin fluctuations. We leave the study beyond the fourth order as a future work.

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Appendix: Self-energy corrections to antiferromagnetic fluctuations

The RPA self-energy correction for valence bond fluctuations is given by

\[
\frac{1}{\beta} \sum_{i\Omega''} g_{\phi}(q - q' + q'') \frac{1}{i\Omega'' + i\Omega'} g_{\phi}(q'', i\Omega'') \\
= \frac{1}{\beta} \sum_{i\Omega''} g_{\phi}(q - q' + q'') \frac{1}{i\Omega'' + i\Omega'} \frac{1}{\Omega'' + \Omega' + \Omega''} \\
\approx \frac{1}{(2\pi)^2} \sum_{i\Omega''} \int_0^\infty dq'' d\theta q'' \frac{1}{v_{\phi}^2 q'' + 2v_{\phi}^2 q'' + \gamma_{\phi} |\Omega''|} \\
\approx \frac{C}{4\pi v_{\phi}^2 \ln \frac{\Omega'''}{\Omega'''}} ,
\]

where we resorted to \( q \approx q' \) in the last line. \( C \) is a positive numerical constant. This renormalized contribution turns out to be much larger than the bare vertex of \( \lambda_{\phi}(q, i\Omega) \) because \( \frac{\Omega_{\phi}}{q_{\phi}} \ll 1 \) should be satisfied in the expansion. As a
result, the self-energy correction in spin fluctuations is given by

$$\frac{1}{\beta} \sum_{i\nu'} \sum_{q'} \lambda_i \lambda_{q',i\nu'} \frac{g_\phi(q',i\Omega')}{4} + \frac{1}{\beta} \sum_{i\nu'} \sum_{q'} g_\phi(q - q' + q'', i\Omega - i\Omega' + i\Omega'') g_\phi(q'', i\Omega'')$$

$$\approx \frac{1}{\beta} \sum_{i\nu'} \sum_{q'} \frac{1}{\beta} \sum_{i\nu''} g_\phi(q'' - q' + q''', i\Omega - i\Omega' + i\Omega'') g_\phi(q''', i\Omega''')$$

$$\approx \frac{1}{\beta} \sum_{i\nu'} \sum_{q'} \frac{1}{\beta} \sum_{i\nu'} g_\phi(q'' - q' + q''', i\Omega - i\Omega' + i\Omega'') g_\phi(q''', i\Omega''') \approx \frac{4\pi \gamma_\psi^2}{\beta} \sum_{i\nu'} \ln\left(\frac{1}{4\pi^2} \frac{1}{v_\phi^2 q^2 + \gamma_\psi |\Omega'|} \right)$$

$$= \frac{4\pi \nu^2}{\pi C} \int_0^\infty d\nu \ln(\nu - i\Omega) \frac{\gamma_\psi \nu}{\gamma_\psi \nu^2 + v_\phi^2 q^4}$$

$$= \frac{2\nu^2}{\pi C} \left\{ \ln(-i\Omega) \ln\left(\frac{-\gamma_\psi^2 q^2 + v_\phi^4 q^4}{v_\phi^4 q^4}\right) - \text{PolyLog}\left(2, \frac{\gamma_\psi \Omega}{\gamma_\psi \Omega + v_\phi^2 q^2}\right) - \text{PolyLog}\left(2, \frac{\gamma_\psi \Omega}{\gamma_\psi \Omega + v_\phi^2 q^2}\right) \right\}. \quad (A.2)$$

Considering that the $\text{PolyLog}$ function is not singular at low momenta and frequencies, we expand the first $\log$ term in $|\Omega|/q^2 \ll 1$ and find the expression of Eq. (19).

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