HOLOMORPHIC SPHERES AND FOUR-DIMENSIONAL SYMPLECTIC PAIRS

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Abstract. We classify four-dimensional manifolds endowed with symplectic pairs admitting embedded symplectic spheres with non-negative self-intersection, following the strategy of McDuff’s classification of rational and ruled symplectic four manifolds.

1. Introduction

A symplectic pair on a smooth manifold [2, 3, 11] is a pair of non-trivial closed two-forms \((\omega, \eta)\), of constant and complementary ranks, for which \(\omega\) restricts to a symplectic form on the leaves of the kernel foliation of \(\eta\), and vice versa. On a four-manifold \(M\), a symplectic pair \((\omega, \eta)\) can be equivalently defined as a pair of symplectic forms \((\Omega_+, \Omega_-)\) satisfying

\[
\Omega_+ \wedge \Omega_- = 0.
\]

In particular \(\Omega_+\) and \(\Omega_-\) induce opposite orientations.

Symplectic pairs appear naturally in the study of Riemannian metrics for which the product of harmonic forms is still harmonic [10] and in the investigation of the group cohomology of symplectomorphism groups [11]. In [3] several interesting examples and constructions are given, especially on closed four-manifolds. Among them we have manifold carrying Thurston geometries, flat symplectic bundles and Gompf’s sum for symplectic pairs. Moreover it is proven that every \(T^2\)-bundles over \(T^2\) carry a symplectic pair. Also in [11], flat symplectic bundles are used to prove the existence of symplectic pairs on some closed four-manifold with non-vanishing signature.

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At present the only known obstructions to the existence of a symplectic pair on closed manifolds are the obvious ones due to the cohomology classes determined by the symplectic pair, the existence of two transverse and complementary foliations and the fact that those manifolds are symplectic for both orientations.

This paper aims at being a first step toward the search of more refined obstructions to the existence of a symplectic pair and the classification of manifolds carrying such a structure. This is achieved by using the theory of $J$-holomorphic curves. To do that, we have to adapt the theory to our setting, meaning that we must consider almost complex structures for which the foliations have pseudoholomorphic leaves.

Note that a symplectic pair provides no canonical way to choose an orientation over the other. For this reason, every theorem we will state for $\Omega^+$ will also hold for $\Omega^-$.

Making use of some Bott-Baum formulas proved by Muñoz and Presas [16], we first prove the following:

**Theorem 1.1.** Let $M$ be a closed 4-manifold admitting a symplectic pair $(\omega, \eta)$. Then $(M, \Omega^+_\omega, \eta^+_\omega)$ is minimal.

By applying the strategy of McDuff [12], we can classify four-dimensional manifolds carrying symplectic pairs which admit embedded symplectic spheres with nonnegative self-intersection:

**Theorem 1.2.** Let $M$ be a closed four-manifold admitting a symplectic pair $(\omega, \eta)$ and $S \hookrightarrow (M, \Omega^+_\omega, \eta^+_\omega)$ a symplectically embedded sphere.

1. If $S \cdot S = 0$, then $M$ is the total space of a flat symplectic sphere bundle over a surface $\Sigma$, the fibres of $M \to \Sigma$ are the leaves of one of the the foliations and $S$ is isotopic to a fibre.
2. If $S \cdot S > 0$, then $M = S^2 \times S^2$ and $(\omega, \eta)$ is the symplectic pair induced by the product.

Finally, we prove a converse to Theorem 1.2 by determining which sphere bundles over a surface carry a symplectic pair.

**Theorem 1.3.** Let $M$ be the total space of a sphere bundle over a surface $\Sigma$. If the bundle is trivial or $\Sigma$ has positive genus, then $M$ carries a symplectic pair such that the fibres are leaves of one of the characteristic foliations.

This article is organised as follows. In Sections 2 and 3 we will recall some basic facts on symplectic pairs and discuss the version of the
Bott-Baum formulas we will use. In Section 4, we will develop the technical details useful to adapt the theory of $J$-holomorphic curves to symplectic pairs. Section 5 contains the proofs of the main theorems. In Section 6 we study four-manifolds admitting a pair of symplectic forms inducing opposite orientations using Seiberg-Witten theory rather than $J$-holomorphic curves. In the Appendix we slightly depart from the main theme of this article and prove that two orthogonal foliations with minimal leaves induce a symplectic pair.

2. Preliminaries on symplectic pairs

In this section we recall the main objects studied in this article and some basic results needed in the next sections.

**Definition 2.1** ([2] [3] [11]). Let $M$ be a $2n$-dimensional manifold. A pair of closed 2-forms $(\omega, \eta)$ is called a symplectic pair of type $(k, n-k)$ if they have constant ranks $2k$ and $2(n-k)$ respectively, and moreover $\omega^{2k} \wedge \eta^{2(n-k)}$ is a volume form.

A symplectic pair gives rise to two symplectic forms

$$\Omega_+ = \omega + \eta, \quad \Omega_- = \omega - \eta$$

on $M$ and on $(-1)^{n-p}M$ respectively, where $-M$ denotes the oriented manifold obtained by reversing the orientation of $M$. To make the definition interesting, we will assume that $k > 0$ and $n > k$. Then, when $M$ has dimension four — the case of interest in the present article —, a symplectic pair on $M$ can only be of type $(1, 1)$ and, in particular, $M$ is symplectic for both orientations.

Moreover, in dimension four, a symplectic pair $(\omega, \eta)$ can be equivalently defined by a pair of symplectic forms $(\Omega_+, \Omega_-)$ satisfying

$$\Omega_+^2 = -\Omega_-^2, \quad \Omega_+ \wedge \Omega_- = 0.$$

The symplectic pair is then given by

$$\left(\frac{\Omega_+ + \Omega_-}{2}, \frac{\Omega_+ - \Omega_-}{2}\right).$$

**Remark 2.2.** If $M$ is a closed four-manifold which is symplectic for both orientations, then $b_+ (M) > 0$ by Equation [11]. In particular, $\mathbb{C}P^2$ does not admit a symplectic pair.
The kernels of $\omega$ and $\eta$ are integrable complementary distributions and therefore integrate to a pair of transverse foliations $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$, called characteristic foliations, such that

$$T\mathcal{F}_\omega = \ker \omega \quad \text{and} \quad T\mathcal{F}_\eta = \ker \eta.$$  

See [3] for example. Each form is symplectic on the leaves of the foliation induced by the other form and moreover $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$ are symplectically orthogonal with respect to the symplectic form $\Omega_\pm$.

Since this article deals with symplectic spheres in $(M, \Omega_\pm)$, the Reeb stability theorem will play a crucial role. We recall its statement.

**Theorem 2.3** (Reeb stability theorem – see [6, Theorem 2.4.3]). If $L$ is a compact leaf of a foliated manifold $(M, \mathcal{F})$ and if $L$ is diffeomorphic to a sphere $S^k$, $k \geq 2$, then there is a foliated neighbourhood $V \subset M$ containing $L$ such that $V \cong S^k \times D^{n-k}$ and $\mathcal{F}|_V$ is the foliation by spheres induced by the product.

### 3. Adjunction Formulas

In this section we collect some results about $J$-holomorphic curves in four-manifolds admitting $J$-holomorphic foliations. By a $J$-holomorphic curve we will mean a close, connected and embedded surface $S \subset M$ such that $J(TS) = TS$. Let $(M, \mathcal{F})$ be a four-manifold endowed with a codimension two foliation. Throughout this section we will assume that $J$ is an almost complex structure on $M$ which preserves the tangent distribution $T\mathcal{F}$.

Now we study how $J$-holomorphic curves which are not leaves of $\mathcal{F}$ intersect $\mathcal{F}$. The starting point is the following lemma.

**Lemma 3.1.** Let $S \subset M$ be a $J$-holomorphic curve which is not a leaf of $\mathcal{F}$. Then the set $T_S \subset S$ defined as

$$T_S = \{x \in S : T_x S \subset T_x \mathcal{F}\}$$

is finite.

**Proof.** We follow the proof of [15, Lemma 2.4.3] very closely to show that every point $x \in S$ has a neighbourhood which either is contained in a leaf, or intersects $T_S$ in a finite set. Since $S$ is compact and connected and $T_S$ is closed, this will prove the lemma.

Given $x \in S \setminus \text{int}(T_S)$, after choosing coordinates $z = s + it$ in $S$ and $(w_1, w_2)$ in $M$ around $x$, we can assume that:
(i) $S$ around $x$ is parametrised by a smooth map $u: \Omega \to \mathbb{C}^2$ for
\[ \Omega \subset \mathbb{C} \text{ an open neighbourhood of } 0 \text{ and } u(0) = 0, \]
(ii) the leaves of $\mathcal{F}$ are the planes with constant coordinate $w_2$,
(iii) $u = (u_1, u_2)$ with $u_2$ not identically zero, and
(iv) $J(w_1, 0) = J_0$ for any $w_1 \in \mathbb{C}$, where $J_0$ is the canonical almost
complex structure on $\mathbb{C}^2$.

Note that we consider here $\mathbb{C}^2$ as a real manifold with an almost complex
structure $J: \mathbb{C}^2 \to GL(4, \mathbb{R})$.

By condition (iv) we can write
\[ J(w_1, w_2) = J_0 + I(w_1, w_2) w_2 \]
for a smooth map $I: \mathbb{C}^2 \to \text{Hom}_\mathbb{R}(\mathbb{C}, \text{End}_\mathbb{R}(\mathbb{C}^2))$. Using this decomposition,
we can show that $u_i, i = 1, 2$, satisfies the equation
\[ \frac{\partial u_i}{\partial s} + J_0 \frac{\partial u_i}{\partial t} + C_i u_2 \]
for a smooth function $C_i: \Omega \to \text{End}_\mathbb{R}(\mathbb{C})$. If we apply $\partial_s - J_0 \partial_t$ to the
above equation for $i = 2$ and estimate the terms of order zero and one,
we obtain the inequality
\[ |\Delta u_2| \leq c(|u_2| + |\nabla u_2|), \]
which holds in a possibly smaller open neighbourhood of of 0 in $\Omega$. Then Aronszajn’s theorem [15, Theorem 2.3.4] implies that the Taylor
expansion of $u_2$ at 0 is non trivial, and thus there exist real polynomials
$(p_1, p_2)$ of degree $l > 0$ such that $u_i(s, t) = p_i(s, t) + o(|z|^l)$ and $p_2 \neq 0$
is homogeneous. Since $C_i u_2 = o(|z|^l)$, Equation (2) implies that $p_1$ and
$p_2$ are complex polynomials; in particular $p_2(z) = az^l$ for $a \in \mathbb{C} \setminus \{0\}$.

Tangencies between $S$ and $\mathcal{F}$ are the same as critical points of $u_2$,
so it will be enough to show that, in some neighbourhood of 0, the
function $u_2$ has no critical point other than (possibly) 0 itself. In fact, $u_2'(z) = laz^{l-1} + o(|z|^{l-1})$ and therefore, for $|z| < \varepsilon$ sufficiently small,
$|u_2'(z) - laz^{l-1}| < la|z|^{l-1}$. This implies that $u_2'(z) \neq 0$ if $0 < |z| < \varepsilon$,
and therefore proves the lemma.

The following formulas, proved by Muñoz and Presas [16], generalise
on the one hand the intersection theory for $J$-holomorphic curves origi-
nally due to McDuff (see [15, Appendix E] for a comprehensive treat-
ment), and on the other hand some results of Brunella [5] in the context
of holomorphic foliations on compact complex surfaces. Brunella, as
well as Muñoz and Presas, consider also singular foliations. We will
not need that level of generality.

Given an embedded $J$-holomorphic curve $S$ which is not contained in
a leaf of $\mathcal{F}$, one can define an integer number $\sigma(x, S, \mathcal{F})$ for each point
$x \in S$ which verifies the following properties:
(1) \( \sigma(x, S, F) \geq 0 \), and
(2) \( \sigma(x, S, F) = 0 \) if and only if \( S \) and \( F \) are transverse at \( x \).

Note that \( \sigma(x, S, F) > 0 \) only at finitely many points by Lemma \( \ref{3.1} \).

The number \( \sigma(x, S, F) \) is defined as
\[
\sigma(x, S, F) = I(x, S, F_x) - 1,
\]
where \( I(x, S, F_x) \) is the local intersection number at \( x \) between \( S \) and the leaf of \( F \) passing through \( x \) defined by McDuff; see [15, Appendix E.2].

The tangency number of \( S \) with respect to \( F \) is defined as
\[
\Sigma_F(S) = \sum_{x \in S} \sigma(x, S, F).
\]

Denoting by \( N_F \) the normal bundle of \( F \), the value of the first Chern class of \( N_F \) on \( S \) can be computed as follows.

**Proposition 3.2** ([16, Lemma 4.2]). For a compact embedded \( J \)-holomorphic curve \( S \) in \( M \) which is not a leaf of \( F \) we have:
\[
< c_1(N_F), S > = \chi(S) + \Sigma_F(S).
\]

Now we come back to the case of symplectic pairs on four-manifolds, where we have a pair of transverse foliations \( F_\eta \) and \( F_\omega \).

**Proposition 3.3.** Let \( M \) be a four-manifold equipped with a pair of transverse codimension two foliations \( F_\eta \) and \( F_\omega \). If \( J \) is an almost complex structure on \( M \) preserving the distributions \( TF_\eta \) and \( TF_\omega \) and \( S \) is a \( J \)-holomorphic curve which is not a leaf of either \( F_\omega \) or \( F_\eta \), then
\[
S \cdot S \geq \chi(S).
\]

**Proof.** The tangent bundle of \( M \) splits as \( TM = TF_\omega \oplus TF_\eta \). Then for the normal bundles \( N_{F_\omega} \) and \( N_{F_\eta} \) we have
\[
N_{F_\omega} \cong TF_\eta, \quad N_{F_\eta} \cong TF_\omega.
\]

Applying Proposition 3.2 we get:
\[
< c_1(TM), S > = < c_1(TF_\omega), S > + < c_1(TF_\eta), S >
= < c_1(N_{F_\eta}), S > + < c_1(N_{F_\omega}), S >
= 2\chi(S) + \Sigma_{F_\eta}(S) + \Sigma_{F_\omega}(S).
\]

On the other hand, by the adjontion formula we have:
\[
< c_1(TM), S > = \chi(S) + S \cdot S.
\]

\(^1\)It seems to us that the argument given in [16], based on the local intersection of \( S \) with the leaves of \( F \), is not sufficient.
Comparing the two formulas we obtain:
\[ S \cdot S = \chi(S) + \Sigma G(S) + \Sigma G(S). \]
Since the indices of tangency are nonnegative, we obtain
\[ S \cdot S \geq \chi(S), \]
as desired. \qed

4. Moduli spaces of $J$-holomorphic spheres

In this section we prove a generic transversality result for $J$-holomorphic curves, when $J$ is compatible with a symplectic pair. We denote by $J(\omega, \eta)$ the set of $\Omega$-compatible almost complex structures on $M$ which make the leaves of $F_\omega$ and $F_\eta$ $J$-holomorphic.

Lemma 4.1. $J(\omega, \eta)$ is nonempty and contractible.

Proof. Let $TF_\omega$ and $TF_\eta$ be the tangent distributions to $F_\omega$ and $F_\eta$ respectively. They are symplectic sub-bundles of $TM$, and we denote by $J(F_\omega)$ and $J(F_\eta)$ the sets of compatible almost complex structures on them, which are nonempty and contractible by [14, Proposition 2.63]. Since $TF_\omega$ and $TF_\eta$ are symplectic orthogonal, there is a symplectic bundle isomorphism $TM \cong TF_\omega \oplus TF_\eta$ which induces a bijection between $J(\omega, \eta)$ and $J(F_\omega) \times J(F_\eta)$. This proves the lemma. \qed

Given $J \in J(\omega, \eta)$, we say that a smooth map $u: S^2 \to M$ is $J$-holomorphic if
\[ du \circ i = J \circ du, \]
where $i$ is the standard complex multiplication on $TS^2$ coming from the identification $S^2 \cong \mathbb{C}P^1$. For any $A \in H_2(M; \mathbb{Z})$ we denote by $\tilde{\mathcal{M}}(A, J)$ the space of $J$-holomorphic maps $u: S^2 \to M$ such that $u_*[S^2] = A$, and define the moduli space $\mathcal{M}(A, J)$ as the quotient of $\tilde{\mathcal{M}}(A, J)$ by the group $PSL(2, \mathbb{C})$ of holomorphic reparametrisations of $S^2$. The topology on $\tilde{\mathcal{M}}(A, J)$ is the $C^\infty$-topology and the topology on $\mathcal{M}(A, J)$ is the quotient topology.

A $J$-holomorphic map $u: S^2 \to M$ is multiply covered if there is a $J$-holomorphic map $\tilde{v}: S^2 \to M$ and a nontrivial holomorphic branched covering $\varphi: S^2 \to S^2$ such that $u = \tilde{v} \circ \varphi$. If $u$ is not multiply covered we say that it is simple. We denote by $\tilde{\mathcal{M}}^s(A, J)$ the subset of $\tilde{\mathcal{M}}(A, J)$ consisting of simple maps, and by $\mathcal{M}^s(A, J)$ its quotient by holomorphic reparametrisations of $S^2$. 
For every map $u \in W^{1,p}(S^2, M)$ with $p > 2$ there is a linearised Cauchy-Riemann operator

$$D_u : W^{1,p}(u^*TM) \to L^p(\text{Hom}_C(TS^2, u^*TM)),$$

where $\text{Hom}_C(TS^2, u^*TM)$ denotes the bundle of anti-$C$-linear homomorphisms from $TS^2$ to $u^*TM$; see [15, Proposition 3.1.1]. By [15, Theorem C.1.10] $D_u$ is a Fredholm operator of index $2\langle c_1(M), A \rangle + 4$.

**Definition 4.2.** An almost complex structure $J \in J(\omega, \eta)$ is regular for $A \in H^2(A; \mathbb{Z})$ if $D_u$ is surjective for every $u \in \mathcal{M}^s(A, J)$. It is regular if it is regular for all $A \in H^2(M; \mathbb{Z})$.

The following is a standard result in the theory of $J$-holomorphic maps: see [15, Theorem 3.1.5(i)].

**Theorem 4.3.** If $J$ is regular for $A$, then

- $\mathcal{M}^s(A, J)$ is a smooth manifold of dimension $2\langle c_1(M), A \rangle - 2$ if $\langle c_1(M), A \rangle \geq 1$, or
- $\mathcal{M}^s(A, J) = \emptyset$ if $\langle c_1(M), A \rangle \leq 0$.

Regular almost complex structures are generic in the set of compatible almost complex structures by [15, Theorem 3.1.5(ii)]. However we will need to work with almost complex structures of a more restricted type, and therefore we will need to make some minor changes to the statement and the proof of the basic transversality result. In order to simplify the statement we introduce the following terminology: we say that a property holds for a generic $J \in J(\omega, \eta)$ if it holds for every $J$ in a countable intersection of open dense subsets.

Given $J \in J(\omega, \eta)$, let $\text{Sym}_J(TF_*)$, for $* \in \{\omega, \eta\}$, be the space of smooth sections $Y_*$ of $\text{End}(TF_*)$ such that

$$Y_* J_* + J_* Y_* = 0 \quad \text{and} \quad Y_*^* = Y_*,$$

where $J_*$ denotes the restriction of $J$ to $F_*$ and $Y_*^*$ is the adjoint of $Y_*$ with respect to the metric on $TF_*$ induced by $J_*$ and $\Omega_+$. It is clear that, if $Y = (Y_\omega, Y_\eta) \in \text{Sym}_J(TF_\omega) \oplus \text{Sym}_J(TF_\eta)$, then $e^Y J e^{-Y} \in J(\omega, \eta)$.

Given a sequence $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ with $\varepsilon_n \to 0$, we define Floer’s $C_\varepsilon$-space $\text{Sym}_{J,\varepsilon}(TF_*)$ as the set of all $Y_* \in \text{Sym}_J(TF_*)$ such that

$$\sum_{n=0}^{\infty} \varepsilon_n \|Y_*\|_{C^n} < +\infty.$$

We refer to [1, Section 6.3.2] and [19, Section 4.4.1] for more details.
Floer’s $C_\varepsilon$-space is a separable Banach space and, for suitable $\varepsilon$, it contains bump sections with arbitrarily small support and arbitrary values at any point of $M$: see [19, Lemma 4.52]. We fix an arbitrary almost complex structure $J_0 \in J(\omega, \eta)$ and define

$$J_\varepsilon = \{ e^Y J_0 e^{-Y} : Y \in \text{Sym}_{J_0, \varepsilon}(T\mathcal{F}_\omega) \oplus \text{Sym}_{J_0, \varepsilon}(T\mathcal{F}_\eta) \}.$$  

Then $J_\varepsilon$ is a separable Banach manifold, admits a continuous inclusion into $J(\omega, \eta)$, and its tangent space at $J \in J_\varepsilon$ is

$$T_J J_\varepsilon = \text{Sym}_{J_\varepsilon}(T\mathcal{F}_\omega) \oplus \text{Sym}_{J_\varepsilon}(T\mathcal{F}_\eta).$$

We denote by $\tilde{\mathcal{M}}(A, J)$ the subspace of simple maps in $\tilde{\mathcal{M}}(A, J)$ which are not contained in a leaf of $\mathcal{F}_\omega$ or $\mathcal{F}_\eta$.

**Proposition 4.4.** For a generic $J \in J_\varepsilon$, the linearised operator $D_u$ is surjective for every $J$-holomorphic map $u \in \tilde{\mathcal{M}}(A, J)$.

**Proof.** The proof will follow [15, Section 3.2] very closely, except that we will use Floer’s $C_\varepsilon$-spaces instead of spaces of $l$ times differentiable almost complex structures.

We denote by $\mathcal{B}$ the set of maps $u \in W^{1,p}(S^2, M)$, with $p > 2$, such that $u_*[S^2] = A$. This is a Banach manifold whose tangent space at $u$ is $W^{1,p}(u^*TM)$. We define the Banach bundle $\mathcal{E} \to \mathcal{B} \times J_\varepsilon$ whose fibre at $(u, J)$ is

$$\mathcal{E}_{u, J} = L^p(\text{Hom}_J(TS^2, u^*TM)),$$

where the complex multiplication on $u^*TM$ is induced by $J$.

The map $G(u, J) = du + J(du \circ i)$ defines a smooth section of $\mathcal{E} \to \mathcal{B} \times J_\varepsilon$. We consider the universal moduli space

$$\tilde{\mathcal{M}}(A) = \{ (u, J) \in \mathcal{B} \times J_\varepsilon : G(u, J) = 0 \}$$

and its subset $\tilde{\mathcal{M}}(A)$ consisting of pairs $(u, J)$ such that $u$ is simple and is not contained in a leaf of $\mathcal{F}_\omega$ or $\mathcal{F}_\eta$. We will prove that $\tilde{\mathcal{M}}(A)$ is a smooth Banach submanifold of $\mathcal{B} \times J_\varepsilon$.

Let $D_{u, J}G : T_u \mathcal{B} \times T_J \mathcal{J}_\varepsilon \to \mathcal{E}_{(u, J)}$ be the vertical differential of $G$, i.e. the composition of $d_{(u, J)}G$ with the projection to the tangent space to the fibres. Then we need to show that $D_{(u, J)}G$ is surjective if $(u, J) \in \tilde{\mathcal{M}}(A)$.

Given $(\xi, Y) \in T_u \mathcal{B} \times T_J \mathcal{J}_\varepsilon$, we have

$$D_{(u, J)}G(\xi, Y) = D_u \xi + Y(u) \circ du \circ i.$$
It is enough to prove that, for every \((u, J) \in \tilde{\mathcal{M}}(A)\), the image of \(D_{(u,J)}\mathcal{G}\) has codimension zero.

Assume, to the contrary, that the image of \(D_{(u,J)}\mathcal{G}\) has positive codimension. Since \(D_u\) is a Fredholm operator, and therefore its image is closed and has finite codimension, the operator \(D_{(u,J)}\mathcal{G}\) has closed image. Then there exists a nontrivial \(\zeta \in L^q(\text{Hom}_J(TS^2, u^*TM))\) such that

\[
\int_{S^2} \langle D_u \xi, \zeta \rangle = 0
\]

for all \(\xi \in T_u\mathcal{B} = W^{1,p}(u^*TM)\) and

\[
\int_{S^2} \langle Y(u) \circ du \circ i, \zeta \rangle = 0
\]

for every \(Y \in T_J\mathcal{J}_\varepsilon\). In both equations \(\langle \cdot, \cdot \rangle\) denotes a pointwise scalar product on \(T^0_1S^2 \otimes \mathbb{C} u^*TM\) and the integral is computed with respect to some volume form on \(S^2\).

Elliptic regularity and Equation (4) imply that \(\zeta\) is smooth. Therefore, if \((u, J) \in \tilde{\mathcal{M}}(A)\) and \(\zeta \neq 0\), then by Lemma 3.1 and [15, Proposition 2.5.1] the set \(\mathcal{U}\) of the points \(z \in S^2\) such that

(i) \(\zeta(z) \neq 0\),
(ii) \(u^{-1}(u(z)) = \{z\}\), and
(iii) \(du(z) \notin T_{u(z)}\mathcal{F}_\omega \cup T_{u(z)}\mathcal{F}_\eta\) for every \(v \in T_zS^2 \setminus \{0\}\)

is open and nonempty. The idea of the proof is that we compensate the smaller set of almost complex structures with a stronger somewhere injectivity property.

We write \(du \circ i = \phi_\omega + \phi_\eta\), where \(\phi_* \in \text{Hom}_J(TS^2, T\mathcal{F}_*)\) for \(* \in \{\omega, \eta\}\). By [15] Lemma 3.2.2 the maps

\[
\text{Sym}_J(T_{u(z)}\mathcal{F}_*) \to \text{Hom}_J(T_zS^2, T_{u(z)}\mathcal{F}_*), \quad a_* \mapsto a_* \circ \phi_*(z),
\]

for \(* \in \{\omega, \eta\}\), are surjective when \(\phi_*(z) \neq 0\).

Take \(z_0 \in \mathcal{U}\). Since \(\zeta(z_0) \neq 0\), by the surjectivity of the maps (6) there is

\[
a = (a_\omega, a_\eta) \in \text{Sym}_J(T_{u(z_0)}\mathcal{F}_\omega) \oplus \text{Sym}_J(T_{u(z_0)}\mathcal{F}_\eta)
\]

such that

\[
\langle a \circ du(z_0) \circ i, \zeta(z_0) \rangle > 0.
\]

By (ii) there exist a neighbourhood \(\mathcal{U}_0\) of \(z_0\) in \(\mathcal{U}\) and a neighbourhood \(\mathcal{V}\) of \(u(z_0)\) in \(M\) such that \(u^{-1}(\mathcal{V}) \subset \mathcal{U}_0\), and \(Y \in \text{Sym}_J(\mathcal{F}_\omega) \oplus \text{Sym}_J(\mathcal{F}_\eta)\).
$\text{Sym}_{J, \varepsilon}(TF_0^\perp)$, with support in $V$ and $Y(u(z_0)) = a$, such that $\langle Y(u(z)) \circ d_z u \circ i, \zeta(z) \rangle > 0$ for all $z \in U_0$. Then
\[
\int_{S^2} \langle Y(u) \circ du \circ i, \zeta \rangle > 0,
\]
contradicting Equation (5). This proves that $\zeta(z_0) = 0$, contradicting $z_0 \in U$, and thus a section $\zeta \in L^2(\text{Hom}_{J}(TS^2, u^*T M))$ satisfying Equations (4) and (5) vanishes everywhere. This proves that $D_{(u, J)} G$ is surjective whenever $(u, J) \in \tilde{M}^\varepsilon(A)$, and therefore the universal moduli space $\tilde{M}^\varepsilon(A)$ is a Banach manifold.

From now on, the proof proceed as in the proof of [15, Theorem 3.1.5(ii)]: if $J$ is a regular value of the projection $\pi: \tilde{M}^\varepsilon(A) \to \mathcal{J}^l$, for every $u \in \tilde{M}^\varepsilon(A, J)$ the linearised Cauchy-Riemann operator $D_u$ is surjective. By Sard-Smale theorem, the regular values of $\pi$ are generic in $\mathcal{J}^\varepsilon$. □

**Remark 4.5.** The conclusion of Proposition 4.4 also holds for $J$-holomorphic maps from higher genus Riemann surfaces. The proof needs only the standard modifications to take into account the variations of the complex structure at the source.

Now we consider $J$-holomorphic maps whose image is everywhere tangent to a leaf.

**Lemma 4.6.** Let $u: S^2 \to M$ be a simple $J$-holomorphic map. If the image of $u$ is contained in a leaf of $\mathcal{F}_*$, $* \in \{\eta, \omega\}$, then $u$ parametrises a leaf of $\mathcal{F}_*$.

**Proof.** Suppose then that the image of $u$ is contained in a leaf $F$. If $F$ is a noncompact leaf, then $u_*[S^2] = 0$ in $H_2(M)$ because $H_2(F) = 0$, and therefore $u$ is constant. Then $F$ is compact and therefore $u: S^2 \to F$ is a holomorphic branched covering. Since $u$ is simple, the covering has degree one, and therefore $u$ is an embedding. □

We will use the automatic transversality result of Hofer, Lizan and Sikorav to deal with $J$-holomorphic maps $u: S^2 \to M$ whose image in contained in a leaf of either $\mathcal{F}_\omega$ or $\mathcal{F}_\eta$. The following theorem is a reformulation of [9, Theorem 1].

**Theorem 4.7.** Let $u: S^2 \to M$ be a $J$-holomorphic map for some almost complex structure $J$. If $u$ is an embedding and $u_*[S^2] = A$ with $A \cdot A \geq -1$, then $D_u$ is surjective.
Corollary 4.8. A generic $J \in \mathcal{J}(\omega, \eta)$ is regular.

Proof. First we observe that a simple $J$-holomorphic maps $u : S^2 \to M$ with values in a leaf of either $\mathcal{F}_\eta$ or $\mathcal{F}_\omega$ satisfies automatic transversality. In fact, by Lemma 4.6 $u$ parametrises a leaf $F$ and by Reeb’s stability theorem $2.3 \ F \cdot F = 0$. Then Theorem 4.7 implies that $D_u$ is surjective. This together with Proposition 4.2 implies that a generic almost complex structure in $\mathcal{J}_\varepsilon$ is regular for $A$, for any $A \in H_2(M; \mathbb{Z})$.

Since genericity is preserved by countable intersections, it follows that a generic almost complex structure in $\mathcal{J}_\varepsilon$ is regular. In particular this implies that the almost complex structure $J_0$ used to define $\mathcal{J}_\varepsilon$ can be approximated by regular almost complex structures in the $C^\infty$ topology. Since $J_0$ was chosen arbitrarily, this proves that regular almost complex structures are dense in $\mathcal{J}(\omega, \eta)$. Finally, using an argument due to Taubes (and explained in detail in [19, Section 4.4.2]) we conclude that a generic almost complex structure in $\mathcal{J}(\omega, \eta)$ is generic. □

5. Proof of the main theorems

In this section we prove the main theorems of the article.

Proof of Theorem 1.1. Let $S \hookrightarrow (M, \Omega_+)$ be an embedded symplectic sphere with $S \cdot S = -1$. By [19, Theorem 5.1] and Corollary 4.8 there exists a generic $J \in \mathcal{J}(\omega, \eta)$ for which $S$ is isotopic to the image of a $J$-holomorphic embedding $u : S^2 \to M$. From now on we denote by $S$ the image of $u$.

By the Reeb stability Theorem 2.3, if $S$ were a leaf of either $\mathcal{F}_\omega$ of $\mathcal{F}_\eta$, then it would satisfy $S \cdot S = 0$. Since $S$ is not a leaf, we are in position to apply Proposition 3.2 which gives

$$S \cdot S \geq \chi(S) = 2.$$

Then $S$ cannot have self-intersection $-1$ and therefore $M$ is minimal. □

Proof of Theorem 1.2(i). By Theorem 1.1 $(M, \Omega_+)$ is minimal. If $S$ is an embedded symplectic sphere in $(M, \Omega_+)$ with $S \cdot S = 0$, then by [12] (see also [19] for a more modern treatment) if $J$ is an $\Omega_+$-compatible regular almost complex structure $J$, then there is a fibration

$$\pi : M \to \Sigma$$

over a surface $\Sigma$ whose fibres are $J$-holomorphic spheres which are isotopic to $S$. By Corollary 4.8 we can assume that $J \in \mathcal{J}(\omega, \eta)$. 

\
Suppose now that $S$ is a fibre of $\pi$. Since it violates the inequality (3), it must be a leaf of either $\mathcal{F}_\omega$ or $\mathcal{F}_\eta$. We will assume without loss of generality that it is a leaf of $\mathcal{F}_\omega$. Let us consider the following subset of $\Sigma$:

$$X = \{ x \in \Sigma \mid \pi^{-1}(x) \text{ is a leaf of } \mathcal{F}_\omega \}.$$ 

Since $S$ is both a leaf and a fibre, $X$ is non empty. We shall prove that $X$ is both open and closed.

To prove that it is closed, let us consider a sequence $\{x_n\}$ in $X$ converging to $x \in \Sigma$, and $F_n = \pi^{-1}(x_n)$. Since $F_n$ is a leaf for all $n$, we have that $\omega|_{TF_n} = 0$. By a limiting argument we obtain $\omega|_{TF} = 0$, which means that the fibre over $x$ is a leaf of $\mathcal{F}_\omega$.

Now we prove that $X$ is open. For $x \in X$, the preimage $F = \pi^{-1}(x)$ is both a fibre of $\pi$ and a leaf of $\mathcal{F}_\omega$. Since $F$ is a sphere, by Reeb's Stability Theorem [2,3] there it is an open foliated neighborhood $U$ of $F$ where every leaf is a sphere. Let $V$ be an open neighborhood of $x$ in $\Sigma$ such that $\pi^{-1}(V) \subset U$.

Consider $y \in V$ and $z \in \pi^{-1}(y) = S_y$. Let $F_z$ be the leaf of $\mathcal{F}$ passing through $z$. We know that $F_z$ is a $J$-holomorphic sphere. By McDuff’s positivity of intersection [15, Theorem E.1.5], we have either $F_z = S_y$ or $S_y \cdot F_z > 0$ because $F_z$ and $S_y$ intersect in $z$.

But both $S_y$ and $F_z$ have the same homology class of $F$ and then we have:

$$F_z \cdot S_y = F \cdot F = 0.$$ 

We conclude from this that $F_z = S_y$, and therefore $y \in X$ for all $y \in V$. Then $X$ is open, so we have $X = \Sigma$. □

Proof of Theorem 1.2(ii). The symplectic manifold $(M, \Omega_\pm)$ is minimal by Theorem 1.1. If $S \cdot S > 0$, then [12, Corollary 1.6] implies that $(M, \Omega_\pm)$ is symplectomorphic to either $\mathbb{CP}^2$ or $S^2 \times S^2$ with a product symplectic form. Since $\mathbb{CP}^2$ carries no symplectic pair because $b_-(\mathbb{CP}^2) = 0$, only the latter possibility remains. The same arguments of the proof of (i) applied to the spheres $S^2 \times \{\ast\}$ and $\{\ast\} \times S^2$ show that the foliations $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$ are given by the product structure. □

Before proving Theorem 1.3 we recall two well known results.

**Lemma 5.1.** For every closed, oriented surface $\Sigma$ there are exactly two oriented $S^2$-bundles over $\Sigma$ up to isomorphism, and they have non diffeomorphic total spaces.
The symplectic pairs in Theorem 1.3 will be constructed via flat $SO(3)$-bundles. Isomorphism classes of flat $SO(3)$-bundles over a surface $\Sigma$ are in bijection with the conjugacy classes of representations $\rho : \pi_1(\Sigma) \to SO(3)$.

**Lemma 5.2.** The flat bundle with holonomy representation $\rho: \pi_1(\Sigma) \to SO(3)$ is trivial if and only if $\rho$ can be lifted to a representation $\tilde{\rho}: \pi_1(\Sigma) \to SU(2)$.

Finally, we can prove Theorem 1.3.

**Proof of Theorem 1.3.** A trivial $S^2$-bundle over a surface always admits the product symplectic pair, so we only need to consider nontrivial bundles over positive genus surfaces.

Let $\Sigma$ be a closed surface of positive genus. Since the standard area form of $S^2$ is $SO(3)$-invariant, if $\rho: \pi_1(\Sigma) \to SO(3)$ is a representation, then the total space $M$ of the flat $S^2$-bundle over $\Sigma$ with holonomy $\rho$ carries a symplectic pair by the construction given in Section 3 of [3].

It remains to find a representation inducing the nontrivial $S^2$-bundles over $\Sigma$. First, we choose two $\tilde{A}, \tilde{B} \in SU(2)$ such that $\tilde{A}\tilde{B} = -\tilde{B}\tilde{A}$.

To find such elements, we can identify $SU(2)$ with the quaternions of norm one and take $\tilde{A} = i$ and $\tilde{B} = j$. We denote by $A$ and $B$ the images of $\tilde{A}$ and $\tilde{B}$ in $SO(3)$. Then $AB = BA$.

We choose generators $a_1, b_1, \ldots, a_g, b_g$ of $\pi_1(\Sigma)$ satisfying the relation

$$[a_1, b_1] \cdots [a_g, b_g] = 1$$

and define $\rho: \pi_1(\Sigma) \to SO(3)$ such that $\rho(a_1) = A$, $\rho(b_1) = B$, and $\rho(a_i) = \rho(b_i) = I$ for $i = 2, \ldots, g$. By construction $\rho$ does not lift to a representation in $SU(2)$, and therefore the associated flat $S^2$-bundle is nontrivial by Lemma 5.2.

6. **Amphisymplectic manifolds**

Some of the previous results can be generalized to the case where the manifold admits symplectic forms with opposite orientations.

**Definition 6.1.** We say that a manifold $M$ is amphisymplectic if it admits symplectic forms $\Omega_+$ and $\Omega_-$ inducing opposite orientations.
We will fix the convention that an amphisymplectic manifold is oriented by $\Omega_+$. Since $\Omega_+$ and $\Omega_-$ can be exchanged, on the one hand this convention is arbitrary, but on the other hand it loses no generality.

Clearly a 4-manifold endowed with a symplectic pair is amphisymplectic. Another source of amphisymplectic manifolds comes from symplectic manifolds admitting an orientation-reversing self-diffeomorphism. In particular, symplectic surface bundles over surfaces are amphisymplectic because the base of bundle admits an orientation-reversing self-diffeomorphism which is covered by a bundle map which inverts the orientation of the total space. Thus, from Theorem 1.1 we obtain the following corollary.

**Corollary 6.2.** The non-trivial sphere bundle over the sphere obtained by blowing up $\mathbb{CP}^2$ at one point is amphisymplectic but admits no symplectic pair.

**Remark 6.3.** Observe that $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ fulfills the cohomological properties required for the existence of a symplectic pair, has a splitting of the tangent bundle into rank-two subbundles and is symplectic for both orientations.

Now we analyse some topological properties of amphisymplectic manifolds. The following proposition states a particular case of the adjunction inequality, which is proved using the Seiberg-Witten equations and the nonvanishing of the Seiberg-Witten invariants of symplectic manifolds.

**Proposition 6.4.** Let $(M, \Omega)$ be a symplectic four-manifold with $b_+(M) > 1$. If $S \subset M$ is a smoothly embedded sphere representing an essential element in $H_2(M; \mathbb{Z})$, then $S \cdot S < 0$.

Observing that the intersection form of $M$ changes sign if the orientation of $M$ is changed, we immediately obtain the following corollary.

**Corollary 6.5.** Let $(M, \Omega_+, \Omega_-)$ be an amphisymplectic four-manifold. If $S \subset M$ is a smoothly embedded sphere with $E \cdot E \leq -1$, then $b_-(M) = 1$.

As the example of $\mathbb{CP}^2$ blown up once shows, this is the closest we can have to Theorem 1.1 in the amphisymplectic world.

**Question 6.6.** If $(M, \Omega_+, \Omega_-)$ is amphisymplectic and $E$ as in the above corollary, is $M$ an $S^2$-bundle over $S^2$?
The following corollary is the amphisymplectic correspondent of Theorem 1.2.

**Corollary 6.7.** Let \((M, \Omega^+, \Omega^-)\) be an amphisymplectic manifold. If there exists an embedded symplectic sphere \(S \subset (M, \Omega^+\)) with non-negative self-intersection, then \(M\) is a \(\mathbb{C}P^1\)-bundle over a surface \(\Sigma\). Moreover, if \(S \cdot S > 0\), then \(\Sigma = S^2\).

**Proof.** By McDuff [12, Theorem 1.4] we have two possibilities: either \(M = \mathbb{C}P^2 \# k \mathbb{C}P^2\) or \(M = N \# k \mathbb{C}P^2\), where \(N\) is an \(S^2\)-bundle over a surface \(\Sigma\).

In the first case we exclude \(k = 0\) because \(\mathbb{C}P^2\) is not amphisymplectic, as \(b_-(\mathbb{C}P^2) = 0\), and we exclude \(k > 1\) by Corollary 6.5. Thus it remains \(\mathbb{C}P^2 \# \mathbb{C}P^2\), which is an \(S^2\)-bundle over \(S^2\), and therefore is amphisymplectic.

In the second case, a simple homological computation yields

\[
b_+(N) = b_+(N \# k \mathbb{C}P^2) = 1, \quad b_-(N \# k \mathbb{C}P^2) = 1 + k.
\]

By Corollary 6.5 we conclude that \(k = 0\). Finally, an \(S^2\)-bundle over a surface \(\Sigma\) contains a spherical homology class with positive self-intersection if and only if \(\Sigma = S^2\), so we have proved the corollary. \(\square\)

**APPENDIX A. TAUT FOLIATIONS AND SYMPLECTIC PAIRS**

By Rummmler and Sullivan’s criterion (see [8]), the characteristic foliations of a symplectic pair are taut, which means that there is a Riemannian metric for which the leaves are minimal. Moreover, it is possible to construct a Riemannian metric making the foliations orthogonal and both with minimal leaves [3]. It is shown in [17] that “two taut make one symplectic”. In this section we prove that “two taut make a symplectic pair”. Then Theorem 1.1 can be rephrased in term of taut foliations.

**Proposition A.1.** Let \(M\) be an orientable four-dimensional manifold endowed with two transverse and complementary orientable foliations \(\mathcal{F}\) and \(\mathcal{G}\) of dimension 2. If \(\mathcal{F}\) and \(\mathcal{G}\) are orthogonal and have minimal leaves for some Riemannian metric on \(M\), then they are the characteristic foliations of a symplectic pair.
Proof. Let \( g \) be a metric for which \( \mathcal{F} \) and \( \mathcal{G} \) are orthogonal and have minimal leaves. Consider \( g \)-orthogonal almost complex structures \( J_1, J_2 \) respectively on \( T\mathcal{F} \) and \( T\mathcal{G} \). Then we have two almost complex structures on \( TM \) given by

\[
J_\pm = J_1 \oplus (\pm J_2).
\]

Let \( \Omega_\pm(X, Y) = g(X, J_\pm Y) \). For the Levi-Civita connection \( \nabla \) of \( g \) and \( X, Z \) vector fields on \( M \), we have (see [17, Appendix] for a proof):

\[
d\Omega_\pm(X, J_\pm X, Z) = g([X, J_\pm X], J_\pm Z) - g(\nabla_X X + \nabla_{J_\pm X} J_\pm X, Z).
\]

To prove that \( d\Omega_\pm = 0 \) it is enough to prove that \( d\Omega_\pm \) vanishes when calculated on 3 linearly independent vector fields. We can choose a local basis such that \( X, J_\pm X \) are tangent to \( \mathcal{F} \) and \( Z, J_\pm Z \) are tangent to \( \mathcal{G} \). Any triple of vectors of the local basis is of the form \( (U, J_\pm U, V) \).

By the minimality of the leaves we have \( g(\nabla_X X + \nabla_{J_\pm X} J_\pm X, Z) = 0 \) (see [4] for example). Frobenius theorem and the orthogonality of \( \mathcal{F} \) and \( \mathcal{G} \) implies \( g([X, J_\pm X], Z) = 0 \). This implies that the 2-forms \( \Omega_\pm \) are symplectic.

Since \( \Omega_\pm \) are symplectic, there is a unique isomorphism \( A \) of the tangent bundle of \( M \), called recursion operator, such that

\[
\Omega_+(X, Y) = \Omega_-(AX, Y).
\]

In fact \( A \) is the composition of the usual musical isomorphisms.

In our case the recursion operator \( A \) is the identity on one foliation and minus the identity on the other one. In particular \( A \) is not the identity itself, but its square is the identity. Thus, by [4] the pair \( (\Omega_+, \Omega_-) \) arises from a symplectic pair. \( \square \)

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