Brownian motion of a charged test particle driven by vacuum fluctuations near a dielectric half-space

Hongwei Yu\textsuperscript{1,2}, Xiangyun Fu\textsuperscript{1,2} and Puxun Wu\textsuperscript{3}

\textsuperscript{1} Department of Physics and Institute of Physics, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
\textsuperscript{2} Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
\textsuperscript{3} Department of Physics and Tsinghua Center for Astrophysics, Tsinghua University, Beijing 100084, People’s Republic of China

Received 30 May 2008, in final form 24 June 2008
Published 17 July 2008
Online at stacks.iop.org/JPhysA/41/335402

Abstract
We study the Brownian motion of a charged test particle driven by quantum electromagnetic fluctuations in the vacuum region near a non-dispersive and non-absorbing dielectric half-space and calculate the mean squared fluctuations in the velocity of the test particle. Our results show that a nonzero susceptibility of the dielectrics has its imprints on the velocity dispersions of the test particles. The most noteworthy feature in sharp contrast to the case of an idealized perfectly conducting interface is that the velocity dispersions in the parallel directions are no longer negative and do not die off in time, suggesting that the potentially problematic negativeness of the dispersions in those directions in the case of perfect conductors is just a result of our idealization and does not occur for real material boundaries.

PACS numbers: 12.20.Ds, 03.70.+k, 05.40.Jc

1. Introduction

A fundamental feature to be expected of any quantized field is the quantum vacuum fluctuations. Although the effects of electromagnetic vacuum fluctuations upon a test particle may be unobservable in free space in quantum electrodynamics, it is well known that changes in the vacuum fluctuations induced by the presence of boundaries can produce novel observable effects. Typical examples are the Casimir effect [1], the Lamb shift and modified spontaneous emission rates and energy shifts for an atom near a reflecting surface [2–4].

Since there always exist quantum electromagnetic vacuum fluctuations, one would expect that test particles under the influence of these quantum field fluctuations will no longer move on the classical trajectories, but undergo random motion around a mean path. Recently, this kind of random (Brownian) motion, as opposed to that driven by classical or thermal
fluctuations [5], has been investigated for a charged test particle near a perfectly reflecting plane and between parallel plates [6–9], and the effects have been calculated of the modified electromagnetic vacuum fluctuations due to the presence of the boundary upon the motion of a charged test particle. In particular, it has been shown that the mean squared fluctuations in the normal velocity and position of the test particle can be associated with an effective temperature which is possibly experimentally accessible [6]. To date all such field-theoretical calculations have been done for boundaries that are idealized as perfect conductors. However, it is obvious that no real material can ever really be perfectly conducting. Therefore it remains interesting to see what happens when the perfect conductors are replaced by imperfect conductors with finite refractivity.

In the present paper, we will examine a situation where a single interface is located at $z = 0$, and the region to the left of this interface ($z < 0$) is filled with a non-dispersive and non-absorbing dielectric while the region to the right ($z > 0$) is vacuum. We will compute the effects of quantum electromagnetic vacuum fluctuations upon the motion of a charged test particle in the vacuum region. Since the boundary now is no longer a perfect conductor, one needs a quantization scheme for the electromagnetic fields which takes into account the evanescent waves in addition to the incident and reflected ones. Fortunately, such a quantization procedure has been given by Carniglia and Mandel [10].

2. The orthonormal eigenmodes and the two-point function

For simplicity, we model the imperfect conductor by a uniform nondispersive and nonabsorbing dielectric half-space in the region $z < 0$. Thus, the dielectric medium is characterized by a single parameter, its refractive index, which is real and frequency-independent. The refractive index can then be written as

$$n(r) = 1 + \theta(-z)(\epsilon - 1).$$  (1)

Here $\epsilon$ is the dielectric constant of the medium which can be expressed in terms of the dielectric susceptibility $\chi$, as $\epsilon = 1 + \chi$ and $\theta(z)$ is the step function.

Let us now consider the motion of a charged test particle subject to quantum electromagnetic vacuum fluctuations in the vacuum region ($z > 0$) with a dielectric half-space. In the limit of small velocities and assuming that the particle is initially at rest and has a charge-to-mass ratio of $e/m$, the mean squared speed in the $i$-direction can be written as (no sum on $i$)

$$\langle \Delta v_i^2 \rangle = \frac{e^2}{m^2} \int_0^t \int_0^t \langle E_i(x, t_1)E_i(x, t_2) \rangle_R \, dt_1 \, dt_2$$  (2)

where $\alpha$ is the fine-structure constant and $\langle E_i(x, t_1)E_i(x, t_2) \rangle_R$ is the renormalized electric field two-point function obtained by subtracting the boundary-independent Minkowski vacuum term. We have, for simplicity, assumed that the distance does not change significantly in a time $t$, so that it can be treated approximately as a constant. If there is a classical, nonfluctuating field in addition to the fluctuating quantum field, then equation (2) describes the velocity fluctuations around the mean trajectory caused by the classical field. Note that when the initial velocity does not vanish, one also has to consider the influence of fluctuating magnetic fields on the velocity dispersions of the test particles. However, it has been shown that this influence is, in general, of the higher order than that caused by fluctuating electric fields and is thus negligible [11].
In order to compute the dispersions of the random motion of the test particle, we need the renormalized electric field two-point function, \( \langle E_i(x, t_1) E_j(x, t_2) \rangle_R \). Fortunately, this function can be calculated using the orthonormal eigenmodes and the quantization procedure with the evanescent waves taken into account, as given by Camiglia and Mandel in [10]. To calculate the renormalized electric field two-point function, let \( x_T = (x, y) \) denote directions tangential to the interface and \( k_T \) be the Fourier transform variable associated with it. If the wavenumber in the \( z \)-direction is \( k \) in the vacuum and is \( k_D \) in the dielectric, then

\[
\begin{align*}
k_D^2 + k_T^2 &= \varepsilon \omega^2 & \text{for} \quad z < 0, \\
k^2 + k_T^2 &= \omega^2 & \text{for} \quad z > 0.
\end{align*}
\]

Since we are interested in the random motion of test particles in the vacuum region, we will only need the electric modes on the vacuum side of the interface.

Following [10, 12], one can show that

\[
\langle E_z(x, t'), E_z(x, t'') \rangle = \frac{1}{(2\pi)^3} \int_{k_D > 0} d^3k_D \frac{k_T^2}{2\pi} \left( \frac{2k_D}{k_D + k} \right) \left( \frac{2k_D}{k_D + k} \right)^* e^{i(k-k')z} e^{i\omega \Delta t}
\]

\[
+ \frac{1}{(2\pi)^3} \int_{k_D > 0} d^3k_D \frac{k_T^2}{2\pi} \left( e^{-ikz} + \frac{\varepsilon k - k_D}{\varepsilon k + k_D} e^{ikz} \right) \left( e^{-ikz} + \frac{\varepsilon k - k_D}{\varepsilon k + k_D} e^{ikz} \right)^* e^{i\omega \Delta t},
\]

and

\[
\langle E_x(x, t'), E_x(x, t'') \rangle = \langle E_y(x, t'), E_y(x, t'') \rangle
\]

\[
= \frac{1}{(2\pi)^3} \int_{k_D > 0} d^3k_D \frac{k^2}{4\pi} \left( \frac{2k_D}{k_D + k} \right) \left( \frac{2k_D}{k_D + k} \right)^* e^{i(k-k')z} e^{i\omega \Delta t}
\]

\[
+ \frac{1}{4} \left( \frac{2k_D}{k_D + k} \right)^3 e^{i(k-k')z} e^{i\omega \Delta t},
\]

where \( i \Delta t = t' - t'' \). Adopting the method of [12], the two-point functions can be rewritten, after performing the angle integration, as

\[
\langle E_z(x, t'), E_z(x, t'') \rangle = \frac{1}{(2\pi)^3} \int_0^\infty d\omega \int_0^1 d\xi \omega^3 (1 - \xi^2) \left( 1 + \frac{\varepsilon \xi - \sqrt{\varepsilon^2 + \xi^2}}{\varepsilon \xi + \sqrt{\varepsilon^2 + \xi^2}} \cos 2\omega \xi \right) e^{i\omega \Delta t}
\]

\[
+ \frac{1}{(2\pi)^3} \int_0^\infty d\omega \int_0^1 d\xi \omega^3 (1 + \xi^2) \frac{2\varepsilon \sqrt{\varepsilon^2 + (1 - \xi^2)} \sqrt{1 + (\varepsilon^2 - 1)\xi^2}}{1 + (\varepsilon^2 - 1)\xi^2} e^{-2\omega \sqrt{\varepsilon^2 + \xi^2}} e^{i\omega \Delta t},
\]

and

\[
\langle E_x(x, t'), E_x(x, t'') \rangle = \langle E_y(x, t'), E_y(x, t'') \rangle
\]

\[
= \frac{1}{8\pi^2} \int_0^\infty d\omega \int_0^1 d\xi \omega^3 \left[ \xi^2 \left( 1 - \frac{\varepsilon \xi - \sqrt{\varepsilon^2 + \xi^2}}{\varepsilon \xi + \sqrt{\varepsilon^2 + \xi^2}} \right) \right.
\]

\[
\left. + \left( \frac{\varepsilon \xi - \sqrt{\varepsilon^2 + \xi^2}}{\xi \xi + \sqrt{\varepsilon^2 + \xi^2}} \right) \cos 2\omega \xi e^{i\omega \Delta t} \right],
\]

where \( \varepsilon \) is the dielectric constant, \( \omega \) is the frequency, and \( \xi \) is the wavenumber in the \( x \)-direction.
\[ + \frac{1}{8\pi^2} \int_0^\infty d\omega \int_0^1 d\xi \omega^4 \left( \frac{2\xi^3 \epsilon \chi^{1/2} \sqrt{1 - \xi^2}}{1 + (\epsilon^2 - 1)\xi^2} \right) \\
+ 2\xi \chi \sqrt{1 - \xi^2} e^{-2\omega \sqrt{\xi} \xi} e^{i\omega\Delta t}. \] 

(7)

Here, the first integral represents the contribution of plane waves and the second that of evanescent waves. Integrating over \( \omega \) gives

\[ \langle E_z(x, t'), E_z(x, t'') \rangle = \frac{1}{(2\pi)^2} \int_0^1 d\xi \left[ \frac{6(1 - \xi^2)}{\Delta t^4} \right. \\
+ \frac{6(1 - \xi^2) \epsilon \xi - \sqrt{\chi + \xi^2}}{\epsilon \xi + \sqrt{\chi + \xi^2}} \\
+ \frac{12\epsilon \chi^{1/2} \sqrt{1 - \xi^2}}{[1 + (\epsilon^2 - 1)\xi^2](2\epsilon \sqrt{\chi} \xi - i\Delta t)^2} \left. \right] \frac{16\epsilon \chi^3 \xi^2 \Delta t^2 + \Delta t^4}{(4\epsilon^2 \xi^2 - \Delta t^2)^4} + \frac{12\epsilon \chi^{1/2} \sqrt{1 - \xi^2}}{[1 + (\epsilon^2 - 1)\xi^2](2\epsilon \sqrt{\chi} \xi - i\Delta t)^2} \right]. \]

(8)

The two-point functions can be expressed as the sum of Minkowski vacuum term and a correction term due to the dielectric half-space boundary. The first term, after being integrated over \( \xi \), is \( 1/(\pi^2 \Delta t^4) \). So, it is the Minkowski vacuum term and shall be dropped because it is not expected to produce any observable consequences. Then we obtain the renormalized two-point function

\[ \langle E_z(x, t'), E_z(x, t'') \rangle = \frac{1}{(2\pi)^2} \int_0^1 d\xi \left[ \frac{6(1 - \xi^2) \epsilon \xi - \sqrt{\chi + \xi^2}}{\epsilon \xi + \sqrt{\chi + \xi^2}} \\
+ \frac{12\epsilon \chi^{1/2} \sqrt{1 - \xi^2}}{[1 + (\epsilon^2 - 1)\xi^2](2\epsilon \sqrt{\chi} \xi - i\Delta t)^2} \right. \left. \right] \frac{16\epsilon \chi^3 \xi^2 \Delta t^2 + \Delta t^4}{(4\epsilon^2 \xi^2 - \Delta t^2)^4} + \frac{12\epsilon \chi^{1/2} \sqrt{1 - \xi^2}}{[1 + (\epsilon^2 - 1)\xi^2](2\epsilon \sqrt{\chi} \xi - i\Delta t)^2} \right]. \]

(10)
3. The Brownian motion of the test particle

With the renormalized two-point functions found, we now start to compute the velocity dispersions using equation (2). Although the integrations can be performed into closed form, the general results are very tedious and not very illuminating. We will not give them here, but, instead, we will analyze some special cases of interest. The first is what happens to the velocity dispersions at later times when the dielectric susceptibility $\chi$ is very large (the dielectric deviates significantly from a perfect conductor), i.e., we will analyze the behavior of both $\langle \Delta v_z^2 \rangle$ and $\langle \Delta v_y^2 \rangle$ in the limit of $\chi \to \infty$ and $t \gg z$. In this limit, we have

$$
\langle \Delta v_z^2 \rangle \approx \frac{e^2}{4\pi^2m^2} \int_0^t dt' \int_0^{t'} dt'' \int_0^z dz' \left[ 6(1 - \xi^2) \frac{\xi^2 - \chi + \xi^2}{\xi^2 + \chi + \xi^2} \right. \\
\left. \times \frac{16 \xi^4 + 24 \xi^2 \Delta t^2 + \Delta t^4}{(4 \xi^2 - \Delta t^2)^2} + \frac{12 \sqrt{\xi}(1 + \chi \xi^2)\sqrt{1 - \xi^2}}{[1 + (\xi^2 - 1)\xi^2][2\sqrt{\xi} - i\Delta t]^2} \right]
$$

$$
\approx \frac{e^2}{4\pi^2m^2} \frac{1}{z^2} + \frac{e^2}{3\pi^2m^2} \frac{1}{t^2} + \frac{e^2}{4\pi^2m^2} \frac{2 \ln(4\chi)}{t^2 \sqrt{\chi}} + \frac{1 + \ln(4\chi)}{2z^2 \sqrt{\chi}}
$$

$$
\approx \frac{e^2}{4\pi^2m^2} \frac{1}{z^2} \frac{1}{2\sqrt{\chi}} + \frac{e^2}{3\pi^2m^2} \frac{1}{t^2} \frac{1}{2\sqrt{\chi}}.
$$

(12)

In the above expression, one sees clearly that the first two terms are just the result for a perfectly conducting interface (refer to [6]), while the $\chi$-dependent terms are corrections induced by finite refractivity.

Similarly, one has, for the velocity dispersions in the directions parallel to the interface, in the limit $\chi \to \infty$ and $t \gg z$

$$
\langle \Delta v_y^2 \rangle \approx \frac{e^2}{4\pi^2m^2} \frac{1}{z^2} \left( \frac{\ln(4\chi)}{2\sqrt{\chi}} - 1 \right) - \frac{e^2}{3\pi^2m^2} \left( 1 - \frac{1}{\sqrt{\chi}} \right) \frac{1}{t^2}.
$$

(13)

Once again, we can see that when $\chi$ approaches infinity, we recover the result found in [6] for the vacuum half-space outside a perfectly conducting plane. This is what one would expect since perfect conductors are formally identified with dielectrics of infinite susceptibility. However, the approach to this limit in terms of the velocity dispersions of charged test particles under Brownian motion driven by vacuum fluctuations is slow. Take the dispersion in the normal direction for an example, we need $\chi \approx 2632$ for $\langle \Delta v_z^2 \rangle$ to be within 10% of its limiting value, and $\chi \approx 14288$ to be within 5%. Equations (12) and (13) show that the corrections induced by finite refractivity are positive and hence the velocity dispersions of the test particle outside real materials would be larger than those outside an idealized perfect conductor. The most noteworthy feature in sharp contrast to an idealized perfectly conducting interface [6] is that the velocity dispersions in the parallel directions, $\langle \Delta v_z^2 \rangle$ and $\langle \Delta v_y^2 \rangle$, are no longer negative and do not die off in time. Therefore, it is no longer a transient effect. Finally, let us note that the velocity dispersions all approach nonzero constant values at late times in contrast to Brownian motion due to thermal noise, where dissipation is needed for $\langle \Delta v_z^2 \rangle$ to be bounded at late times. The fact that they do not continue to grow in time can be understood as a consequence of energy conservation and they may be absorbed into a renormalization of the mass of the test particles [13].

For the sake of completeness, let us now also give the results for the velocity dispersions when the dielectric susceptibility $\chi$ is vanishingly small (corresponding to the vacuum).
In this case, we have
\[
\langle \Delta v^2_z \rangle \approx \frac{e^2 \chi}{4\pi^2 m^2 z^2} \left( \frac{t^2 + 12z^2}{12z^2} + \frac{(t^4 - 8z^4)}{16t^2 z^2} \ln \left( \frac{2z + t}{2z - t} \right) \right)
\]
\[
\approx \begin{cases} 
\frac{e^2}{4\pi^2 m^2 z^2} \left( \frac{7}{12z^2} + \frac{5}{3t^2} \right) \chi, & t \to \infty, \\
\frac{9e^2 t^2 \chi}{320m^2 \pi z^4}, & t \to 0,
\end{cases}
\]
and
\[
\langle \Delta v^2_x \rangle = \langle \Delta v^2_y \rangle \approx \frac{e^2}{4\pi^2 m^2 z^2} \left( \frac{t^4 - 4t^2 z^2 + 24z^4}{t^2 (t^4 - 4t^2 z^2)} + \frac{(t^4 - 8z^4)}{32t^3 z^2} \ln \left( \frac{2z + t}{2z - t} \right) \right)
\]
\[
\approx \begin{cases} 
\frac{e^2}{4\pi^2 m^2 z^2} \left( \frac{1}{6z^2} + \frac{1}{3t^2} \right) \chi, & t \to \infty, \\
\frac{7e^2 t^2 \chi}{320m^2 \pi z^4}, & t \to 0.
\end{cases}
\]

As expected, all the above values of the velocity dispersions tend to zero (linearly) as \( \chi \to 0 \). The same as the case when \( \chi \to \infty \); here the velocity dispersions in the parallel directions are also positive. This suggests that the negative values of velocity dispersions in the parallel directions obtained in [6] are a result of idealization of the boundary as perfectly conducting. In reality, a real conductor would not be well approximated by a perfect conductor within atomic distances and its plasma frequency. Therefore, the potentially problematic negativeness of the dispersions in those directions in the case of perfect conductors does not occur for real material boundaries. It should be pointed out that the above expressions are singular at \( t = 2z \). This corresponds to a time interval equal to the round-trip light travel time between the particle and the interface. Presumably, this might be a result of our assumption of a rigid plane boundary, and would thus be smeared out in a more realistic treatment, where fluctuations in the position of the interface are taken into account.

4. Summary

In summary, we have studied the Brownian motion of a charged test particle driven by quantum electromagnetic fluctuations in the vacuum region near a non-dispersive and non-absorbing dielectric half-space and calculated the mean squared fluctuations in the velocity of the test particle. Our results show that a nonzero susceptibility of the dielectrics has its imprints on the velocity dispersions of the test particles. When the susceptibility is small, the dependence of the velocity dispersions on it is linear. However, when the susceptibility is extremely large, the dependence is a rather complicated function involving logarithmic and square root. In comparison with the case where the interface is an idealized perfectly conducting boundary (infinite susceptibility), the dispersions of the test particle near a dielectric half-space are larger than near a perfect conducting boundary and they approach that of a perfect conducting boundary very slowly as susceptibility increases. The most noteworthy feature in sharp contrast to the case of an idealized perfectly conducting interface [6] is that the velocity dispersions in the parallel directions are no longer negative and do not die off in time. Therefore, the potentially problematic negativeness of the dispersions in those directions in the case of perfect conductors is just a result of our idealization and does not occur for real material boundaries.
Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under grants no. 10575035, 10775050, the SRFDP under grant no. 20070542002, and the Programme for the Key Discipline in Hunan Province.

References

[1] Casimir H B G 1948 Proc. K. Ned. Akad. Wet. 51 793
[2] Meschede D, Jhe W and Hinds E A 1990 Phys. Rev. A 41 1587
[3] Yu H and Lu S 2005 Phys. Rev. D 72 064022
   Yu H and Lu S 2006 Phys. Rev. D 73 109901 (erratum)
[4] Yu H and Zhu Z 2006 Phys. Rev. D 74 044032
[5] Einstein A 1905 Ann. Phys. 17 549
   Einstein A 1906 Ann. Phys. 19 371
[6] Yu H and Ford L H 2004 Phys. Rev. D 70 065009
[7] Yu H and Chen Jun 2004 Phys. Rev. D 70 125006
[8] Yu H, Chen J and Wu P 2006 J. High Energy Phys. JHEP02(2006)058
[9] Seriu M and Wu C-H 2008 Phys. Rev. A 77 022107
[10] Carniglia C K and Mandel L 1971 Phys. Rev. D 3 280
[11] Tan M and Yu H 2005 Chin. Phys. Lett. 22 2165
[12] Helfer A D and Lang A S I D 1999 J. Phys. A: Math. Gen. 32 1937
[13] Yu H 2007 Proc. 7th Asia-Pacific Int. Conf. on Gravitation and Astrophysics ed J M Nester, C-M Chen and J-P Hsu (Singapore: World Scientific) p 177