Monotonicity of the Lebesgue constant for equally spaced knots

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January 14, 2013

Abstract

Let \( t_i = \frac{i}{n} \) for \( i = 0, \ldots, n \) be equally spaced knots in the unit interval \([0, 1]\). Let \( S_n \) be the space of piecewise linear continuous functions on \([0, 1]\) with knots \( \pi_n = \{ t_i : 0 \leq i \leq n \} \). Then we have the orthogonal projection \( P_n \) of \( L^2([0, 1]) \) onto \( S_n \). In Section 1 we collect a few preliminary facts about the solutions of the recurrence \( f_{k-1} - 4f_k + f_{k+1} = 0 \) that we need in Section 2 to show that the sequence \( a_n = \| P_n \|_1 \) of \( L^1 \)-norms of these projection operators is strictly increasing.

1 Solutions of \( f_{k-1} - 4f_k + f_{k+1} = 0 \) and their Properties

This section is to define and examine a few properties of the solutions of the recurrence \( f_{k-1} - 4f_k + f_{k+1} = 0 \), which we will use extensively in the sequel. For an arbitrary real number \( x \), let \( A_x := \cosh(\alpha x) \) and \( B_x := \sqrt{3} \sinh(\alpha x) \) with \( \alpha > 0 \) defined by \( \cosh \alpha = 2 \). For \( k \in \mathbb{N}_0 \), \( A_k \) and \( B_k \) can also be defined by the recurrence relations

\[
A_{k+1} = 2A_k + 3B_k \quad \text{with} \quad A_0 = 1, \quad (1.1)
\]
\[
B_{k+1} = A_k + 2B_k \quad \text{with} \quad B_0 = 0. \quad (1.2)
\]

This follows from the basic identities

\[
cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y, \quad (1.3)
\]
\[
\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y. \quad (1.4)
\]

The crucial fact about \( A_k \) and \( B_k \) is that they are independent solutions of the linear recursion \( f_{k-1} - 4f_k + f_{k+1} = 0 \) and this recursion in turn takes into account the special form of the Gram matrix for equally spaced knots (see (2.1)). We note that it is easy to see that the inequalities

\[
A_{k+1} \leq 4A_k \quad \text{for} \quad k \in \mathbb{N}_0, \quad (1.5)
\]
\[
B_{k+1} \leq 4B_k \quad \text{for} \quad k \in \mathbb{N} \quad (1.6)
\]

hold. Observe also that

\[
A_k = 2A_{k+1} - 3B_{k+1} \quad (1.7)
\]
\[
B_k = 2B_{k+1} - A_{k+1} \quad (1.8)
\]
for $k \in \mathbb{N}_0$. We also have the formulae

$$ A_x = \frac{1}{2}(\lambda^x + \lambda^{-x}), \quad B_x = \frac{1}{2\sqrt{3}}(\lambda^x - \lambda^{-x}), \quad x \in \mathbb{R} \quad (1.9) $$

with

$$ \lambda = 2 + \sqrt{3}, \quad \lambda^{-1} = 2 - \sqrt{3}. $$

We remark that $\alpha = \log \lambda$. For reference, we list the first few values of both $A_n$ and $B_n$:

$$(A_0, \ldots, A_4) = (1, 2, 7, 26, 97), \quad (B_0, \ldots, B_4) = (0, 1, 4, 15, 56)$$

**Lemma 1.1.** For $K \in \mathbb{N}_0$ we have the following formulae

$$ \sum_{k=0}^{K} B_k + B_{k+1} = A_{K+1} - 1, \quad 2 \sum_{k=0}^{K} A_k = 3B_{K+1} - A_{K+1} + 1, $$

$$ \sum_{k=0}^{K} A_k + A_{k+1} = 3B_{K+1}, \quad 2 \sum_{k=0}^{K} B_k = A_{K+1} - B_{K+1} - 1. $$

**Proof.** The proof uses induction and the recurrences (1.1), (1.2), (1.7) and (1.8) for $A_n$ and $B_n$. \qed

**Lemma 1.2.** For all $n \in \mathbb{N}$ and $0 \leq k \leq n$ the following equalities hold

$$ B_k A_{n-k} + A_k B_{n-k} = B_n, \quad B_n A_{n-k} - B_{n-k} A_n = B_k, $$

$$ A_k A_{n-k} + 3B_{n-k} B_k = A_n, \quad A_n A_{n-k} - 3B_n B_{n-k} = A_k. $$

**Proof.** This is only a different formulation of (1.3) and (1.4). \qed

### 2 Equally spaced knots on $[0, 1]$]

We let $t_i = i/n$ for $0 \leq i \leq n$ and view the partition of points $\pi_n = \{t_i : i \in \{0, \ldots, n\}\}$. Additionally we set $t_{-1} = 0$ and $t_{n+1} = 1$. If we define $\delta_i := t_i - t_{i-1}$ for $0 \leq i \leq n+1$, we get that $\delta_i = 1/n$ for $1 \leq i \leq n$ this case of equally spaced knots. Furthermore, we have for the entries $(a_{i,k})$ of the inverse of the Gram matrix $(b_{i,k}) = \langle N_i, N_k \rangle$ consisting of the pairwise scalar products of the piecewise linear, continuous B-spline functions corresponding to the partition $\pi_n$ (see [1] or [2])

$$ a_{i,k} = \frac{2n}{B_n}(-1)^{i+k}A_{i\land k}A_{n-i\land n-k} \quad \text{with } 0 \leq i, k \leq n \quad (2.1) $$

Observe that from formula (2.1) it follows that

$$ \frac{|a_{i,k}|}{|a_{i-1,k}|} = \begin{cases} \frac{A_i}{A_{i-1}}, & \text{for } 1 \leq i \leq k, \\ \frac{A_{i-1}}{A_n}, & \text{for } k < i \leq n. \end{cases} \quad (2.2) $$
Let \( S_n \) be the space of piecewise linear continuous functions with knots \( \pi_n \). For the \( L^1 \)-norm (or the \( L^\infty \)-norm) of the projection operator \( P_n : L^2([0,1]) \to S_n \) we have the formula (see for instance \cite{2})

\[
\|P_n\|_1 = \max_{0 \leq k \leq n} \sum_{i=1}^{n} p_{i,k} \varphi \left( \frac{|a_{i,k}|}{|a_{i-1,k}|} \right) =: \max_{0 \leq k \leq n} g_k(n),
\]

where \( \varphi(t) = \frac{t^2}{(1+t)^2} \) and \( p_{i,k} = \frac{1}{2}(|a_{i,k}| + |a_{i-1,k}|) \). We observe that for \( t > 1 \), \( \varphi \) is strictly increasing and \( \varphi(t) = \varphi(1/t) \) for \( t > 0 \). Furthermore, \( \varphi(2 + \sqrt{3}) = 2 \).

Using the definition of \( g_k(n) \), (2.2) and the properties of \( \varphi \), we obtain for \( 0 \leq k \leq n \)

\[
g_k(n) = \frac{1}{B_n} \left( A_{n-k} \sum_{j=0}^{k-1} (A_{j+1} + A_j) \varphi(A_{j+1}/A_j) + A_k \sum_{j=0}^{n-k-1} (A_{j+1} + A_j) \varphi(A_{j+1}/A_j) \right). \tag{2.3}
\]

**Theorem 2.1.** For all \( n \in \mathbb{N} \), we have that \( g_0(n+1) > g_0(n) \).

**Proof.** The expression \( Dg_0(n) := g_0(n+1) - g_0(n) \) equals

\[
\sum_{j=0}^{n} \frac{1}{B_{n+1}} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right) - \sum_{j=0}^{n-1} \frac{1}{B_{n}} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right) \tag{2.4}
\]

Thus, we get further

\[
Dg_0(n) = \frac{A_n + A_{n+1}}{B_{n+1}} \varphi \left( \frac{A_{n+1}}{A_n} \right) + \sum_{j=0}^{n-1} \frac{1}{B_{n+1}} - \frac{1}{B_{n}} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right)
\]

\[
= \frac{A_n + A_{n+1}}{B_{n+1}} \varphi \left( \frac{A_{n+1}}{A_n} \right) - \frac{B_{n+1} - B_{n}}{B_{n+1}B_{n}} \sum_{j=0}^{n-1} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right)
\]

\[
> \varphi \left( \frac{A_{n+1}}{A_n} \right) \left( \frac{A_n + A_{n+1}}{B_{n+1}} - 3 \frac{B_{n+1} - B_{n}}{B_{n+1}} \right)
\]

where for the inequality, we have used Lemma 1.1 and the fact that for \( j \leq n - 1 \) we have \( \varphi \left( \frac{A_{j+1}}{A_j} \right) < \varphi \left( \frac{A_{n+1}}{A_n} \right) \), since \( A_{j+1}/A_j < A_{n+1}/A_n \) for \( j \leq n - 1 \). Now we use the recurrences \( 1.1 \) and \( 1.2 \) to obtain that last term in the big bracket in the previous display equals zero and thus the theorem is proved. \qed

Theorem 2.2 will then show that it suffices to have Theorem 2.1 to conclude the monotonicity of the sequence of norms of the projection operators.

**Theorem 2.2.** For all \( n \in \mathbb{N} \) and all \( k \in \mathbb{N} \) with \( 1 \leq k \leq \lfloor n/2 \rfloor \), we have

\[
g_0(n) \geq g_k(n).
\]

**Remark.** Due to symmetry, we get this inequality for all \( 1 \leq k \leq n - 1 \), and in fact the equality \( g_0(n) = g_n(n) \).
Proof of Theorem 2.2. We first observe that for general \( k \) we have to consider \( h_k(n) := B_n(g_0(n) - g_k(n)) \) and show that this is greater or equal zero. Now by definition

\[
h_k(n) = \sum_{j=0}^{n-1} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right) - A_{n-k} \sum_{j=0}^{k-1} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right)
\]

Rearranging terms, this yields

\[
h_k(n) = -(A_k - 1) \sum_{j=0}^{n-k-1} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right) + \sum_{j=n-k}^{n-1} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right)
\]

Employing again the inequality \( \varphi(A_{j+1}/A_j) < \varphi(A_{k+1}/A_k) \) for \( j < k \), we get

\[
h_k(n) \geq \varphi \left( \frac{A_{n-k+1}}{A_{n-k}} \right) \left( -(A_k - 1) \sum_{j=0}^{n-k-1} (A_j + A_{j+1}) + \sum_{j=n-k}^{n-1} (A_j + A_{j+1}) \right)
\]

since \( k \leq \lfloor n/2 \rfloor \). Finally a calculation using Lemmas 1.1 and 1.2 shows that the big bracket equals zero and thus the theorem is proved. \( \square \)

Remark. We observe that we can use this theorem to conclude with Theorem 2.1 that \( \|P_n\|_1 < \|P_{n+1}\|_1 \). We also remark that we get a simple proof that \( \|P_n\|_1 < 2 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \|P_n\|_1 = 2 \), since with formula (2.3) and the last theorem we get

\[
\|P_n\|_1 = g_0(n) = \frac{1}{B_n} \sum_{j=0}^{n-1} (A_j + A_{j+1}) \varphi \left( \frac{A_{j+1}}{A_j} \right) < 2,
\]

since \( \varphi \left( \frac{A_{j+1}}{A_j} \right) < 2/3 \) for all \( j \in \mathbb{N}_0 \) and \( \sum_{j=0}^{n-1} (A_j + A_{j+1}) = 3B_n \) by Lemma 1.1. Additionally, for \( \varepsilon > 0 \) we choose \( m \) such that \( \varphi \left( \frac{A_{m+1}}{A_m} \right) > (2 - \varepsilon)/3 \), which is possible, since \( \lim_{m \to \infty} \varphi \left( \frac{A_{m+1}}{A_m} \right) = \varphi(2 + \sqrt{3}) = 2/3 \). For \( n > m \) we thus have

\[
\|P_n\|_1 = g_0(n) \geq \frac{1}{B_n} \sum_{j=m}^{n-1} (A_j + A_{j+1}) \varphi \left( \frac{A_{m+1}}{A_m} \right) > (2 - \varepsilon) \frac{B_n - B_m}{B_n} \to 2 - \varepsilon \quad \text{for} \ n \to \infty.
\]
References

[1] Z. Ciesielski. Properties of the orthonormal Franklin system. II. *Studia Math.*, 27:289–323, 1966.

[2] Z. Ciesielski and A. Kamont. The Lebesgue constants for the Franklin orthogonal system. *Studia Math.*, 164(1):55–73, 2004.