Adaptive Bayesian multivariate density estimation with Dirichlet mixtures

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Abstract

We consider Bayesian multivariate density estimation using a Dirichlet mixture of normal kernel as the prior distribution. By representing a Dirichlet process as a stick-breaking process, we are able to extend convergence results beyond finitely supported mixtures priors to Dirichlet mixtures. Thus our results have new implications in the univariate situation as well. Assuming that the true density satisfies Hölder smoothness and exponential tail conditions, we show the rates of posterior convergence are minimax-optimal up to a logarithmic factor. This procedure is fully adaptive since the priors are constructed without using the knowledge of the smoothness level.

1 Introduction

Kernel methods for density estimation has been well studied in the past fifty years ([25]). In the nonparametric Bayesian literature, the study of asymptotic properties of posterior distributions received a lot of interest since the development of efficient Markov chain Monte Carlo (MCMC) methods ([19] and [22]). A general result on posterior consistency was established in [17] and [23] and then applied on the univariate Dirichlet mixture of normal prior. General posterior convergence rate theorems were obtained in [22] and [19]. Ghosal and van
der Vaart \cite{10} considered univariate Bayesian density estimation problem using Dirichlet mixture of normal kernel and studied the case when the true density is a location-scale mixture type while its standard deviation is bounded away from zero and infinity. Although the posterior rate is nearly the parametric rate $n^{-1/2}$, the assumption of “super smooth” true density with the bounded range of standard deviation is quite restrictive. Using a new general rate theorem, Ghosal and van der Vaart \cite{11} obtained posterior convergence rate of univariate Dirichlet mixture of normal kernel when the true density is only twice continuously differentiable. Though the number of mixture components increases, the minimax rate is still obtained. These results need a prior on the bandwidth parameter that scales appropriately with increasing sample size.

In recent studies, rate-adaptive estimators based on posterior distributions have been constructed to accommodate different levels of smoothness of the underlying true function of interest. Belitser and Ghosal \cite{1} considered the problem of estimating a signal with Gaussian white noise and showed that the posterior rate automatically adapts to the unknown smoothness condition if the “smoothness parameter” only takes values in a discrete set. Huang \cite{12} and Ghosal, Lember and van der Vaart \cite{9} showed that appropriate mixture of priors based on spline expansions or wavelets yield optimal posterior rates for a finite or countable range of smoothness parameters for density estimation and nonparametric regression problems. Alternatively, \cite{24} constructed a prior based on a randomly rescaled smooth Gaussian process, which automatically adapts for a continuous range of smoothness parameters. They treated the multidimensional case as well. A technical challenge in proving adaptation of the
posterior distribution is to find an approximation of the true function within the model, whose accuracy increases appropriately with increasing smoothness level of the true density. An interesting approximation idea proposed by [20] in the context of beta mixtures prior turns out to be very helpful for constructing required approximation and subsequent adaptive posterior distributions. A similar idea for normal mixtures was proposed by [13]. An analogous approximation in the multi-dimension situation was constructed recently in [3]. They used a special type of Gaussian process to construct an adaptive procedure. However, their constructions apply only to compactly supported densities. The issue of unboundedness of the support was resolved in [13] for univariate Gaussian mixtures by imposing appropriate tail conditions on the true density.

The adaptation results in [13] used a prior based on finite mixture of the normal kernel in a univariate setting. In practice, Dirichlet mixture priors are popularly used in the univariate density estimation problems ([5] and [14]), as well as in the multivariate situations ([16]). Posterior consistency results in terms of the $L_1$-distance were studied in [26] under a multivariate setting. An extension to multivariate mixed-scale density estimation was discussed in [2].

In this paper, we study the posterior convergence rates for Bayesian multivariate density estimation. We extend the approximation result in [13] to the multi-dimension setting assuming local $\beta$-Hölder smoothness and exponential tail conditions. Using the stick-breaking representation ([15]), we approximate a Dirichlet process by a finite sum of mixtures while the error is controlled within a pre-determined level, which helps us construct appropriate sieves for the problem. Similar technique has been used in [17] to prove posterior consis-
tency for conditional density estimation. We calculate the entropy and prior concentration rate around the true density. The posterior rate is shown to be $n^{-\beta/(2\beta+d)}(\log n)^\kappa$, where $\kappa$ is determined by the smoothness level, the dimension of the sample space and the tail behavior of the true density. The rate coincides with the minimax rate up to a logarithmic factor.

To the best of our knowledge, most frequentist approaches for adaptive estimation are focused on using wavelets under a regression model setting ([1] and [18]). The performance of adaptive multivariate kernel density estimation depends heavily on the choice of the bandwidth matrix and the smoothing kernel ([21]). Our model considers kernel based Bayesian adaptive estimation procedure that achieves optimal rates using product kernel.

The paper is organized as follows. In Section 2, some notations and assumptions on the true density are introduced. The main results on posterior convergence rates are presented in Section 3. Approximation results are given in Section 4. Section 5 gives the proof of the main rate theorem. A few auxiliary lemmas and their technical proofs are presented in the Appendix.

2 Notations and assumptions

2.1 Notations

Throughout the paper, we consider estimating a density $f$ on $\mathbb{R}^d$ based on $n$ independent and identically distributed (i.i.d) samples $X_1, \ldots, X_n$ taking values in $\mathbb{R}^d$. Let $X = (X_1, \ldots, X_d)$ stand for a generic observation from density $f$. We define marginal density functions of $f$ for $X_i$ as $f_i(x_i), i = 1$.
1, . . . , d. Let \( N = \{0, 1, 2, \ldots\} \) and let \( \Delta_k \) be a \( k \)-dimensional unit simplex. For \( k \in \mathbb{N}^d, \ x \in \mathbb{R}^d \), let \( k_i = k_1 + k_2 + \ldots + k_d \), \( k! = k_1! \cdots k_d! \) and \( x^k = x_1^{k_1} \cdots x_d^{k_d} \). Similarly, for a real-valued function \( f \) on \( \mathbb{R}^d \), let \( f(x)^k = f(x_1^{k_1} \cdots f(x_d^{k_d}) \).

We define partial order for \( j \) and \( k \) as \( j \geq k \) if \( j_i \geq k_i \) for \( i = 1, \ldots, d \). Let \( \|x\|_p = \left( \sum_{i=1}^d |x_i|^p \right)^{1/p} \) stand for the \( \ell_p \)-norm of a vector \( x \in \mathbb{R}^d \); \( 1 \leq p < \infty \) and \( \|x\|_\infty = \max_{1 \leq i \leq d} |x_i| \). Moreover, for \( p = 2 \), we simply write \( \|x\|_2 \) as \( \|x\| \).

For \( b > 0 \), let \( r_b \) stand for the largest integer strictly smaller than \( b \).

We use \( \sigma = (\sigma_1, \ldots, \sigma_d)' \in \mathbb{R}_+^d \) as the scale parameter and define a \( d \times d \) diagonal matrix \( \Sigma = \text{diag}(\sigma) \). Let \( \phi(x) = (2\pi)^{-1/2} \exp(-x^2/2) \) be the standard normal density and \( \phi_{\sigma}(x) = \sigma^{-1} \phi(x/\sigma) \). The corresponding multivariate normal density with independent components is denoted by \( \phi_{\sigma}(x) = \prod_{i=1}^d \phi_{\sigma_i}(x_i) \).

We use \( \lesssim \) for inequality up to a constant multiple, where the underlying constant of proportionality is universal or not important for our purposes. We define a linear operator \( K_{\sigma_i} \) as

\[
(K_{\sigma_i} f)(x) = \int_{-\infty}^{\infty} f(x_1, \ldots, x_i, x_i - y_i, x_{i+1}, \ldots, x_d) \phi_{\sigma_i}(y_i) dy_i. \quad (2.1)
\]

Then a composition operator is defined as \( K^{m_i}_{\sigma_i} = K_{\sigma_i} (K^{m_{i-1}}_{\sigma_i} f) \). Note that these convolution operators commute with each other. We extend this notation to the multivariate case as \( K^m_{\sigma} f = (K^{m_1}_{\sigma_1} \cdots K^{m_d}_{\sigma_d}) f \). For simplicity, we define \( K_{\sigma} = K^{(1, \ldots, 1)}_{\sigma} \).

We use \( D(\epsilon, T, d) \) to denote the packing number, which is defined as the maximum cardinality of an \( \epsilon \)-dispersed subset of \( T \) with respect to distance \( d \). Similarly, we write \( N(\epsilon, T, d) \) for the covering number, the minimal cardinality of an \( \epsilon \)-net for \( T \) in terms of the distance \( d \). We define \( \log_+(x) = \max(\log x, 0) \).
2.2 Assumptions on the true density

Let $f_0$ stand for the true density. We assume the following conditions on $f_0$.

- **(C1) Smoothness**: The function $\log f_0$ is assumed to be locally $\beta$-Hölder with derivatives $l_j(x) = \frac{\partial \log f(x)}{\partial x_1 \cdots \partial x_d}$. We assume the existence of a polynomial $L$ and a constant $\gamma > 0$, such that for $r = r_\beta$,

  $$|l_k(x) - l_k(y)| \leq r!L(x) \|x - y\|^{\beta - r}$$

  for all $k.$ and $x, y$ satisfying $\|x - y\| \leq \gamma$. Moreover, there exists a constant $\xi_0 > 0$ such that for all $j. \leq r$,

  $$\int f_0(x)|l_j(x)|^{(2\beta + \xi_0)/\beta} \, dx < \infty, \quad \int f_0(x)|L(x)|^{2+\xi_0/\beta} \, dx < \infty. \quad (2.2)$$

- **(C2) Marginal-joint relationship**: There exist a constant $C_0$ and density functions $g_1, \ldots, g_d$ such that $f_0(x_1, \ldots, x_d) \geq C_0 \prod_{i=1}^d g_i(x_i)$, such that

  $$\int f_0(x)(1/g(x))^\xi \max(1, \|x\|^2) \, dx < \infty$$

  for some $\xi > 0$, where $g(x) = \prod_{i=1}^d g_i(x_i)$.

- **(C3) Tail monotonicity**: On a region $D = [-a, b]^d$, where $a, b > 0$, we have that $\inf_{x \in D} g(x) = c_0 > 0$, $g_i$ is nondecreasing on $x_i < -a$ and nonincreasing on $x_i > b$ for $i = 1, \ldots, d$.

- **(C4) Tail decay**: The true density $f_0$ has exponential tails on $D^c$, i.e., there exist constants $C > 0$ and $\tau_1, \tau_2 > 0$, which only depend on $f_0$, such that

  $$f_0(x) \leq Ce^{-\tau_1 \|x\|^{\tau_2}}, \quad x \in D^c. \quad (2.3)$$
**Remark 1** Conditions (C2) and (C4) imply $\int f_0 \left( \log_+ \left( \frac{f_0}{g} \right) \right)^p < \infty$ for any $p > 0$. Conditions (C1), (C3) and (C4) imply $\int f_0 \left( \log_+ f_0 \right)^p < \infty$ for any $p > 0$.

A wide range of multivariate density functions satisfy Condition (C2), e.g., nonsingular multivariate normal distribution and their finite mixtures. To see this, consider $k$ multivariate normal densities $f_j$, $j = 1, \ldots, k$, with mean $0$ and covariance matrix $\Sigma_j$. For any convex combination of $f_j$’s $f^* = \sum_{j=1}^k \omega_j f_j$, there exists $\lambda > 0$ such that $f^*(x) \gtrsim \exp\{-\lambda \|x\|^2/2\}$. Define density $g^* = (\lambda/2\pi)^{d/2} \exp\{-\lambda \|x\|^2/2\}$, then $f^* \gtrsim g^*$. To see this, choose $\lambda$ to be the smallest eigenvalue of all $\Sigma_j^{-1}$s. Then for any $0 < \xi < 1$, $\int f^*(1/g^*)^\xi \max(1, \|x\|^2) < \infty$. Hence Condition (C2) holds for $f^*$.

Condition (C2) also holds for product type densities $f_0(x) = \prod_{j=1}^d f_j(x_j)$ with $g = f_0$, if $\int f_0^\xi \max(1, \|x\|^2) dx < \infty$ for some $0 < \xi < 1$.

**Remark 2** Condition (C2) is used to lower bound $K_\sigma f_0$ as in Lemma 2. Condition (C3) generalizes the monotone tail condition in [13] to the multivariate case.

### 3 Main results

We construct a prior for $f$ as follows:

- $p_{F,\sigma} = \int_{\mathbb{R}^d} \phi_\sigma(x - \mu) dF(\mu)$;

- $F$ follows a Dirichlet process $D_\alpha$ with base measure $\alpha$. Denote $\bar{\alpha} = \alpha/\alpha(\mathbb{R}^d)$. We assume that there exist constants $a_1, a_2 > 0$ such that $1 - \bar{\alpha}([-x, x]^d) \leq \exp\{-a_1 z^{a_2}\}$ for sufficiently large $x > 0$.  

7
\( \sigma_i \overset{iid}{\sim} G \) for \( i = 1, \ldots, d \), where \( G \) is a fixed probability distribution satisfying \( G(x) \preceq \exp\{-C_1x^{-a_3}\} \) as \( x \to 0 \) and \( 1 - G(x) \preceq x^{a_3} \) as \( x \to \infty \), where \( C_1 > 0 \) and \( a_3 \geq 1 \) are fixed constants. This condition allows a wide class of distributions, e.g., an inverse gamma distribution for \( \sigma_2^2 \) when \( a_3 = 2 \) or an inverse gamma distribution for \( \sigma \) when \( a_3 = 1 \).

We have the following result for posterior convergence rates:

**Theorem 1** Suppose that the true density \( f_0 \) satisfies Conditions (C1)–(C4). Then the posterior rate of convergence with respect to Hellinger or \( L_1 \)-distance is given by \( \epsilon_n = n^{-\beta/(d+2\beta)}(\log n)^t \), where \( t > \left( \frac{d}{\tau_2} + d + 1 \right) \frac{\beta}{2\beta + d} \).

The assumption on the base measure \( \bar{\alpha} \) is analogous to (11) of [13]. Our tail conditions on the prior of \( \sigma \) is weaker than the one in [11]. Both sets of conditions are needed to control the prior probability of the model.

For simplicity, we let \( \sigma_1 = \cdots = \sigma_d \) in the discussion. However, our results also hold for independently, not identically distributed \( \sigma_1, \ldots, \sigma_d \) as long as \( \max_i \sigma_i \leq C_2 \{\min_i \sigma_i\}^{C_3} \) for some constants \( C_2, C_3 > 0 \).

Our result also applies for finite-mixture priors. We consider the prior for \( f \) as follows:

- \( m(x; k, \mu, \omega, \sigma) = \sum_{j=1}^{k} \omega_j \phi_{\sigma}(x - \mu_j) \);

- There exists constants \( c_1 > c_2 > 0 \) and \( c_3 > 0 \) such that

\[
\exp\{-c_1 k (\log k)^{c_3}\} \preceq \Pi(k) \succeq \exp\{-c_2 k (\log k)^{c_3}\}.
\]
• Given \( k, \mu_1, \ldots, \mu_k \) are i.i.d realizations from a distribution, which satisfies \( \Pi(\mu \notin [-z, z]^d) < \exp\{-c_4 z^5\} \) for sufficiently large \( z > 0 \) and constants \( c_4, c_5 > 0 \).

• Given \( k \), the prior on weights \( \omega = (\omega_1, \ldots, \omega_k)' \) satisfies

\[
\Pi(||\omega - \omega_0||_1 \leq \epsilon) \gtrsim \exp\{-c_6 k (\log k)^{a_4} \log_+ \epsilon\}
\]

for any \( \omega_0 \in \Delta_k \) and constants \( a_4, c_6 > 0 \) and \( 0 < \epsilon < 1/k \).

• Bandwidth \( \sigma_1, \ldots, \sigma_d \) (i.i.d) follow inverse gamma distributions.

Then we have the following rate theorem, which is a generalization of Theorem 2 of [13].

**Theorem 2** Suppose that the true density \( f_0 \) satisfies Conditions (C1)–(C4). Then the posterior rate of convergence with respect to Hellinger or \( L_1 \)-distance is given by \( \epsilon_n = n^{-\beta/(d+2\beta)}(\log n)^t \), where \( t > \frac{\beta}{2\beta+d+\max\{c_3, 1+a_4, \frac{c_5}{\tau_2}\}} + \max\{0, (1 - c_3)/2\} \).

## 4  Approximation results

The following proposition helps prove the main theorem on posterior convergence rates. It is also of interest on its own as it bounds the Kullback-Leibler (KL) divergence between \( f_0 \) and its approximation. The proof is given in Appendix.
Proposition 1 Let $f_0$ be the true density satisfying Conditions (C1)–(C4). Then there exists a density $h_\beta$ such that for all sufficiently small $\sigma$,

\[
\int f_0(x) \log \frac{f_0(x)}{K_\sigma h_\beta(x)} dx = O(\sigma^{2\beta}), \quad (4.1)
\]

\[
\int f_0(x) \left( \log \frac{f_0(x)}{K_\sigma h_\beta(x)} \right)^2 dx = O(\sigma^{2\beta}). \quad (4.2)
\]

In order to prove approximation result, we use the expansion technique in [13] and its multivariate modification described by [3].

Let $r$ and $\beta$ be defined as in Condition (C1). For $k \in \mathbb{N}^d$, we define moments $m_k = \int y_1^{k_1} \cdots y_d^{k_d} \phi(y) dy$. Then we recursively define two collections of numbers $c_n$ and $d_n$ as follows:

For $n \in \mathbb{N}^d$, if $n = 1$, then $c_n = d_n = 0$. For $n \geq 2$, define

\[
c_n = \sum_{n=l+k, l \geq 1, k \geq 1} \frac{(-1)^{k+1}}{k!} m_k d_l, \quad d_n = \frac{m_n}{n!} + c_n. \quad (4.3)
\]

Since the Gaussian kernel is symmetric about 0, all odd moments are 0. Hence $c_n$ can be simplified as $c_n = -\sum_{n=l+2k, l \geq 1, k \geq 1} m_{2k} d_l / 2!$.

Define $f_\beta = f - \sum_{j=1}^r \sum_{k=j} d_k \sigma^k (D_k f)$, where $D_k = \frac{\partial^k}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}$. Lemma 3.4 in [3] shows that the supremum distance between $f_0$ and $f_\beta$ is $O(\sigma^\beta)$.

However, this type of construction does not guarantee that $f_\beta$ is a density function because it may take negative values. To overcome the problem, we define a truncated version of $f_\beta$ and then standardize it to obtain a density function:

\[
h_\beta^*(x) = f_\beta 1\{f_\beta > \frac{1}{2} f_0(x)\} + \frac{1}{2} f_0(x) 1\{f_\beta \leq \frac{1}{2} f_0(x)\}
\]

\[
h_\beta(x) = h_\beta^*(x) / \int h_\beta^*(u) du \quad (4.4)
\]
Remark 3 From (4.4), we get $h_\beta \lesssim h_\beta^* \lesssim f_\beta + f_0$. Using the same arguments in [13], we can show $f_\beta \lesssim f_0$. Then combining these two facts, we conclude that $h_\beta$ is upper bounded by a multiple of $f_0$.

Remark 4 From the definition, $f_\beta$ can be expressed as a linear combination of $K^j_\sigma f_0$'s:

$$f_\beta = C_\beta f_0 - \sum_{j \geq 0} c_j K^j_\sigma f_0,$$

where $C_\beta$ and $c_j$ are constants determined by $f_0$ and $\beta$. The coefficients $c_j$ satisfy $\sum_{j \geq 0} c_j = C_\beta - 1$. Hence $K_\sigma f_\beta$ is also a linear combination of $K^j_\sigma f_0$'s.

The approximation mixture in Proposition [1] can be discretized without changing the order of the approximation error. The following lemma is a multivariate generalization of Lemma 4 in [13]. This will be used to lower bound the prior probability on the KL-ball around $f_0$. Its proof is given in Appendix.

Lemma 1 Let $f_0$ be a density satisfying Conditions (C1)–(C4). Then there exists a finitely supported probability measure $F$ with at most $C_4\sigma^{-d}|\log \sigma|^{d/2}+d$ support points from the set $\{x : f_0(x) \geq c\sigma^{H_1+2\beta}\}$, where $C_4 > 0$ is a constant such that

$$\int f_0 \log \frac{f_0}{p_{F,\sigma}} = O(\sigma^{2\beta}), \quad \int f_0 (\log \frac{f_0}{p_{F,\sigma}})^2 = O(\sigma^{2\beta}).$$

(4.6)

5 Proof of Theorems

5.1 Some useful results

We first state a few results that are helpful for proving Theorem [1].
Since a Dirichlet process $F \sim D_\alpha$ can be represented by a Sethuraman’s stick-breaking process as $\sum_{i=1}^{\infty} V_i \delta_{\theta_i}$, where $V_i = \prod_{j=1}^{i-1} (1 - Y_j) Y_i$, $\theta_i \overset{iid}{\sim} G$, $Y_i \overset{iid}{\sim} \text{Be}(1, M)$, $i = 1, 2, \ldots$, $G$ is the cumulative distribution function of $\bar{\alpha}$ and $M = \alpha(\mathbb{R})$. We truncate the stick-breaking procedure after a certain level such that the error is within a predetermined level. Define the number of terms needed in the finite mixture as $N_\epsilon = \inf \{ m \geq 1 : \sum_{i=1}^{m} V_i > 1 - \epsilon \}$. Let $F_\epsilon = \sum_{i=1}^{N_\epsilon} V_i \delta_{\theta_i} + \bar{V}_\epsilon \delta_{\theta_0}$, where $\bar{V}_\epsilon = \prod_{i=1}^{N_\epsilon} (1 - Y_i)$ and $\theta_0 \sim G$ independently of everything else. By Lemma 3 of [15], it follows that

\begin{align*}
  d_{\text{TV}}(F, F_\epsilon) &\leq \epsilon, \quad (5.1) \\
  N_\epsilon - 2 &\sim \text{Poi}(M \log \epsilon), \quad (5.2)
\end{align*}

where $d_{\text{TV}}$ stands for the total variation distance. It is easy to see from (5.1) that $\|p_{F, \sigma} - p_{F_\epsilon, \sigma}\|_1 \leq \epsilon$.

The following lemma lower bounds $K_{\sigma f_0}$.

**Lemma 2** Assume $f_0$ satisfy Conditions (C2) and (C3). Then given $\sigma$ sufficiently small, $K_{\sigma f_0} \geq C_g f$ for some constant $C_g$ and density function $g$ defined in (C2).

We need the following inequalities to help lower bound the prior probability in the KL-ball around $f_0$.

**Lemma 3** Let $\mathbb{R}^d = \bigcup_{j=0}^{N} U_j$ be a partition of $\mathbb{R}^d$ and $F' = \sum_{j=1}^{N} p_j \delta_{z_j}$ be a probability measure with $z_j \in U_j$ and $\|z_j - z_k\|_1 > 2\epsilon$ for $j, k = 1, \ldots, N$, $j \neq k$ and $\epsilon > 0$. Define $V(z_j, \epsilon) = [z_{j,1} - \epsilon, z_{j,1} + \epsilon] \times \cdots \times [z_{j,d} - \epsilon, z_{j,d} + \epsilon]$
and $x_{(1)} = \min_{i=1}^d x_i$ for $x \in \mathbb{R}^d$. Then for any probability measure $F$ on $\mathbb{R}^d$, $\sigma, \sigma' \in \mathbb{R}^d_+$, we have that

$$\|p_{F, \sigma} - p_{F', \sigma'}\|_1 \lesssim \max_{i=1, \ldots, d} \frac{|\sigma_i - \sigma'_i|}{\sigma_i \land \sigma'_i} + \epsilon + \frac{\epsilon}{\sigma_{(1)} \land \sigma'_{(1)}}^d + \sum_{j=1}^N |F(V(z_j, \epsilon)) - p_j|, \quad (5.3)$$

and

$$\|p_{F, \sigma} - p_{F', \sigma'}\|_{\infty} \lesssim \frac{\epsilon}{\sigma_{(1)} \land \sigma'_{(1)}}^d + \frac{1}{\sigma_{(1)} \land \sigma'_{(1)}} \sum_{j=1}^N |F(V(z_j, \epsilon)) - p_j|. \quad (5.4)$$

The following discretization result gives multidimensional extensions of Lemmas 3.1 and 3.3 of [10]. Their proofs are given in Appendix.

**Lemma 4** (1) Let $0 \leq \epsilon \leq 1/2$ be given. Fix $\sigma_0, \sigma'_0 \in [\sigma_0, \bar{\sigma}_0]^d$ satisfying $|\sigma'_0 - \sigma_0| < \bar{\sigma}_0^d \epsilon$, then for any probability measure $F$ on a region $D' = [-a_1, a_1] \times \cdots \times [-a_d, a_d]$, where $\max_i a_i \leq L(\log_\epsilon)^\gamma_0$, $\gamma_0 \geq 1/2$ and $L > 0$ are constants, there exists a discrete probability measure $F'$ on $D$ with at most $N \leq \bar{\sigma}_0^{-2d}(\log_\epsilon)^{2\gamma_0 d}$ support points such that $\|p_{F, \sigma_0} - p_{F', \sigma'_0}\|_{\infty} \lesssim \epsilon \bar{\sigma}_0^{d}/\bar{\sigma}_0^{2d}$.

(2) If $\sigma \to 0$, then for any probability measure $F$ on $[-a_\epsilon, a_\epsilon]^d$ with $a_\epsilon = L(\log_\epsilon)^{\gamma_1}$, where $\gamma_1 \geq 0$, $0 \leq \epsilon \leq 1/2$ and $L > 0$ are constants, there exists a discrete probability measure $F'$ on $[-a_\epsilon, a_\epsilon]^d$ with at most $N \lesssim \sigma^{-d}(\log_\sigma)^{\gamma_1 d + d}$ support points such that

$$\|p_{F, \sigma} - p_{F', \sigma'}\|_{\infty} \lesssim \sigma^{-d} \epsilon, \quad (5.5)$$

and

$$\|p_{F, \sigma} - p_{F', \sigma'}\|_1 \lesssim \sigma^d (\sigma(\log_\epsilon)^{1/2} \vee (\log_\epsilon)^{\gamma_1})^d \epsilon. \quad (5.6)$$
5.2 Proof of Theorem 1 (Part I)

We apply Theorem 5 of [11] for $\tilde{\epsilon}_n = n^{-\beta/(2\beta+d)} (\log n)^{t_1}$, $\epsilon_n = n^{-\beta/(2\beta+d)} (\log n)^{t_2}$ for $t_2 > t_1$. We construct appropriate sieves $\mathcal{F}_{n,j}$ and verify the following three conditions:

\[ \sum_{j=0}^{\infty} \sqrt{N(\tilde{\epsilon}_n, \mathcal{F}_{n,j}, d)} \sqrt{\Pi_n(\mathcal{F}_{n,j})} e^{-n\epsilon_n^2} \to 0 \]  
\[ \Pi_n(\mathcal{K}(f_0, \tilde{\epsilon}_n)) \geq e^{-n\epsilon_n^2} \]  
\[ \Pi_n(\mathcal{F}^c_n) \leq e^{-4n\epsilon_n^2} \]

where $\mathcal{K}(f_0, \epsilon) = \{ f : \int f_0 (\log f_0) < \epsilon, \int f_0 (\log f_0)^2 < \epsilon \}$ is the KL ball around $f_0$ of size $\epsilon$. Choose $\sigma_{n,d} = \tilde{\epsilon}_n^{1/\beta}$. Define

\[ \sigma_n = n^{-A}, \quad \bar{\sigma}_n = \exp\{n\epsilon_n^2 (\log n)^{\delta} \}, \quad r_n = [n^{d/(2\beta+d)} (\log n)^{t_r}] + 1 \]

and $b_n > n^{d/a_2(d+2\beta)}$ for $A > 1, a_2, t_r, \delta > 0$. First we consider the collection of finite mixtures:

\[ \mathcal{F}^*_n = \left\{ \sum_{i=1}^{k} \omega_i \phi_{\sigma}(x - \mu_i) : k \leq r_n, \mu_i \in [-b_n, b_n]^d, \sigma \in S_n, j = 1, \ldots, k \right\} \]

as in [13], where $S_n = [\sigma_n, \bar{\sigma}_n]^d$.

Define the sieve

\[ \mathcal{F}_n = \{ p_{F,\sigma} : \text{there exists } p_{F',\sigma} \in \mathcal{F}^*_n \text{ such that } d_{TV}(F, F') \leq \epsilon_n \} \].

Notice that $\mathcal{F}^*_n \subset \mathcal{F}_n$.

We first verify equation (5.9). From the construction of priors of $\sigma_i$ as in Section 3,

\[ \Pi_n(\sigma_i \in (\sigma_n, \bar{\sigma}_n)^c) \lesssim \exp\{-C_1 n^{A_3} \} + \exp\{a_3 n\epsilon_n^2 (\log n)^{\delta} \} \]

\[ \lesssim \exp\{-C_6 n\epsilon_n^2 \} \]
for some constant $C_6 > 0$ when $n$ is sufficiently large.

Given the number of mixtures $N_{\tilde{\varepsilon}_n}$ fixed, from the assumption, we have

$$
\Pi_n(\mu \notin [-b_n, b_n]^d | N_{\tilde{\varepsilon}_n} = k) \leq k \Pi_n(\mu_1 \notin [-b_n, b_n]^d) \lesssim k e^{-a_1 b_n^{a_2}}. \quad (5.12)
$$

Therefore,

$$
\Pi_n(\mu \notin [-b_n, b_n]^d) = \sum_{k=1}^{\infty} \Pi(N_{\tilde{\varepsilon}_n} = k) \Pi_n(\mu \notin [-b_n, b_n]^d | N_{\tilde{\varepsilon}_n} = k) 
\lesssim E(N_{\tilde{\varepsilon}_n}) e^{-a_1 b_n^{a_2}} 
\lesssim e^{-a_1 b_n^{a_2}} \log n. \quad (5.13)
$$

Using (5.12) and tail estimates of Poisson distribution $P(X > r) \lesssim \exp\{-r \log r\}$ if $X \sim \text{Poi}(\lambda)$ and $r > \lambda e$, we have the following results for $X = N_{\tilde{\varepsilon}_n}$ and $r = r_n$

$$
\Pi_n(N_{\tilde{\varepsilon}_n} > r_n) \lesssim \exp\{-r_n \log r_n\} \lesssim \exp\{-n^{d/(d+2\beta)}(\log n)^{t_r+1}\}. \quad (5.14)
$$

All three bounds together give

$$
\Pi_n(F_n^c) \leq \Pi_n(F_n^{*c}) \leq \Pi_n(S_n^c) + \Pi_n(N_{\varepsilon} > r_n) + \Pi_n(\mu \notin [-b_n, b_n]^d) 
\lesssim \exp\{-C_7 n^{d/(d+2\beta)}(\log n)^{t_r+1}\} \quad (5.15)
$$

for some constant $C_7 > 0$, which decreases faster than $e^{-4n^{t_2}}$ if $t_r + 1 > 2t_1$.

### 5.3 Proof of Theorem 1 (Part II)

In order to verify (5.7), we split $(\sigma_n, \tilde{\sigma}_n)$ into $J_n + 1$ disjoint subsets

$$
(\sigma_n, \tilde{\sigma}_n) = \bigcup_{i=1}^{J_n} (\sigma_n(1 + \tilde{\varepsilon}_n)^{i-1}, \sigma_n(1 + \tilde{\varepsilon}_n)^i) \cup (\sigma_n(1 + \tilde{\varepsilon}_n)^{J_n}, \tilde{\sigma}_n) \quad (5.16)
$$
for $J_n = \lfloor (\log \bar{\sigma}_n / \underline{\sigma}_n) / \log (1 + \bar{\epsilon}_n) \rfloor$. Hence we obtain a partition of $S_n$ with $(J_n + 1)^d$ subsets. Denote $S_{n, j} = \bigotimes_{i=1}^d [\underline{\sigma}_n(1 + \bar{\epsilon}_n)^{j_i - 1}, \sigma_n(1 + \bar{\epsilon}_n)^{j_i} \lor \bar{\sigma}_n]$, where $j_i = 1, \ldots, J_n + 1$.

Then define
\[
F^*_{n,j} = \left\{ \sum_{i=1}^k \omega_i \phi_\sigma (x - \mu_i) : k \leq r_n, \omega_i > 0, \sum_{i=1}^k \omega_i = 1, \mu_i \in [-b_n, b_n]^d, \sigma \in S_{n,j} \right\},
\]
and
\[
F_{n,j} = \{ p_{F', \sigma} : \text{there exist } p_{F', \sigma} \in F^*_{n,j} \text{ such that } d_{TV}(F, F') \leq \epsilon_n \}.
\]

We can bound the prior probability on $F_{n,j}$ by
\[
\Pi_n(F_{n,j}) \leq \Pi_n(S_{n,j}) \lesssim (1 + \bar{\epsilon}_n)^{j-1} \sigma_n^d \bar{\epsilon}_n.
\]

In order to calculate the entropy, we further decompose $F^*_{n,j}$ into
\[
F^*_{n,j} = \bigcup_{k=1}^{r_n} F^*_{n,j,k} = \bigcup_{k=1}^{r_n} \left\{ \sum_{i=1}^k \omega_i \phi_\sigma (x - \mu_i) : \mu_j \in [-b_n, b_n]^d, \sigma_i \in S_{n,j} \right\}.
\]

Using the following general results on bracketing numbers taken from [10] and [13],
\[
D(\epsilon, \Delta_k, \| \cdot \|_1) \leq \left( \frac{5}{k} \right)^{k-1},
\]
\[
D(\epsilon, \bigotimes_{i=1}^k [b_i, d_i], \| \cdot \|_1) \leq k! \prod_{i=1}^k (d_i - b_i + 2\epsilon) / (2\epsilon)^k,
\]
we obtain the following estimates of packing numbers
\[
D(\bar{\epsilon}_n, \Delta_{r_n}, \| \cdot \|_1) \leq \left( \frac{5}{\bar{\epsilon}_n} \right)^{r_n-1},
\]
\[
D(\bar{\epsilon}_n, [-b_n, b_n]^{r_n d}, \| \cdot \|_1) \leq (r_n d)! \left( \frac{2\bar{\epsilon}_n}{\bar{\epsilon}_n} \right)^{-r_n d} (2b_n + 2\bar{\epsilon}_n)^{r_n d},
\]
\[
D(\bar{\epsilon}_n, S_{n,j}, \| \cdot \|_1) \leq d!(2\bar{\epsilon}_n)^{-d} \prod_{i=1}^d (\underline{\sigma}_n(1 + \bar{\epsilon}_n)^{j_i} - \sigma_n(1 + \bar{\epsilon}_n)^{j_i-1} + 2\bar{\epsilon}_n),
\]
\[
D(\bar{\epsilon}_n, F^*_{n,j,k}, \| \cdot \|_1) \leq D(\bar{\epsilon}_n, \Delta_k, \| \cdot \|_1) D(\bar{\epsilon}_n, [-b_n, b_n]^{r_n d}, \| \cdot \|_1) D(\bar{\epsilon}_n, S_{n,j}, \| \cdot \|_1).
\]
Combining (5.21), (5.22), (5.23), and using the relationship between covering and packing numbers, we have

\begin{align*}
N(3\tilde{e}_n, \mathcal{F}_{n,j}, \| \cdot \|_1) \\
\leq D(3\tilde{e}_n, \mathcal{F}_{n,j}, \| \cdot \|_1) \\
\leq D(\tilde{e}_n, \mathcal{F}^*_{n,j}, \| \cdot \|_1) \\
\leq r_n D(\epsilon_n, \mathcal{F}^*_{n,j,r_n}, \| \cdot \|_1) \\
\lesssim r_n (r_n d)!(\bar{\epsilon}_n)^{1-r_n-r_n d-d}b_n^{r_n d/2}c_8^{r_n} \prod_{i=1}^d \left( n^{-A_i} (1 + \tilde{e}_n)^{j_i} \tilde{e}_n + 2\tilde{e}_n \right) \\
\lesssim r_n (r_n d)!(\bar{\epsilon}_n)^{1-r_n-r_n d-d}b_n^{r_n d/2}c_8^{r_n} \left( n^{-A} (1 + \tilde{e}_n)^{j-1} + 2 \right)^d
\end{align*}

(5.25)

for some constant $C_8 > 0$. Combining (5.17) and (5.25) and applying Stirling’s formula on $(r_n d)!$, we find that $\sqrt{N(\epsilon_n, \mathcal{F}_{n,j}, d)\sqrt{\Pi_n(\mathcal{F}_{n,j})}}$ is bounded by a multiple of

\begin{align*}
n^{-Ad/2} (1 + \tilde{e}_n)^{j/2} \\
\times (r_n)^{r_n/2+3/4} (\bar{\epsilon}_n)^{(1+d)(1-r_n)/2}b_n^{r_n d/2}c_8^{r_n/2} \left( n^{-A} (1 + \tilde{e}_n)^{j-1} + 2 \right)^{d/2} \\
\lesssim n^{-Ad/2} (1 + \tilde{e}_n)^{j/2} (r_n)^{r_n/2+3/4} \\
\times (\epsilon_n)^{(1+d)(1-r_n)/2}b_n^{r_n d/2}c_8^{r_n/2} \left( n^{-A} (1 + \tilde{e}_n)^{j-1} \vee 2 \right)^{d/2} \\
\lesssim \exp \{ C_9 r_n (log n) \} (1 + \tilde{e}_n)^{j/2} \left( n^{-A} (1 + \tilde{e}_n)^{j-1} \vee 2 \right)^{d/2},
\end{align*}

(5.26)

for some constant $C_9 > 0$. Observe that $n^{-A}(1 + \tilde{e}_n)^{j-1} \leq 2$ implies $(1 +
\(\tilde{\epsilon}_n^{d/2} \lesssim n^{A/2}\). Therefore from equation (5.20), we have the following:

\[
\sum_{i=1}^{d} \sum_{j_i=1}^{J_n} \sqrt{N(\tilde{\epsilon}_n, \mathcal{F}_{n,j})} \sqrt{\Pi_n(\mathcal{F}_{n,j})}
\]

\[
\lesssim \sum_{i=1}^{d} \sum_{j_i=1}^{J_n} \exp\{C_{10} r_n(\log n)\} n^{A/2} 2^{d/2}
\]

\[
+ \sum_{i=1}^{d} \sum_{j_i=1}^{J_n} C_9 \exp\{C_{11} r_n(\log n)\} n^{-Ad/2}(1 + \tilde{\epsilon}_n)^{(1+d)/2}
\]

\[
\lesssim \exp\{C_{12} r_n(\log n)\} \left(n^{A/2} J_n^d + n^{-Ad/2}(1 + \tilde{\epsilon}_n)^{(1+d)dJ_n/2J_n^d}\right) \tag{5.27}
\]

for some new constants \(C_{10}, C_{11}, C_{12}\). Since \(J_n\) is defined that \(n - A(1 + \tilde{\epsilon}_n) J_n = \exp\{n\tilde{\epsilon}_n^2(\log n)\delta\}\), the r.h.s of (5.27) is bounded by a multiple of

\[
\exp\{C_{13} r_n(\log n) + n\tilde{\epsilon}_n^2(\log n)\delta(1 + d) d/2\}. \tag{5.28}
\]

In order to let (5.28) increase slower than \(\exp\{n\tilde{\epsilon}_n^2\}\), we need \(2t_2 > \max(t_r + 1, 2t_1 + \delta)\).

Finally, we verify (5.8) using similar arguments as in [11]. For sufficiently large \(b > 0\),

\[
\bigcap_{i=1}^{d} \bigcap_{j=1}^{N_n} \left\{ (F, \sigma) : \sum_{j=1}^{N_n} |F(U_j) - p_j| \leq \tilde{\epsilon}_n^b, F(U_j) \geq \tilde{\epsilon}_n^{2b}, |\sigma_i - \sigma_{n,i}| \leq \tilde{\epsilon}_n^b/\sigma_n, \sigma_n \right\}
\]

\[
\subset \left\{ (F, \sigma) : P_0 \left( \log \frac{p_0}{P_{F,\sigma}} \right)^k \lesssim \sigma_{(d)n}^{2\beta}, k = 1, 2 \right\},
\]

where \(N_n \lesssim \sigma^{-d} \log \sigma |d/\tau_2 + d|\) is obtained using Lemma 1. Applying Lemma 10 of [11] with \(N = N_n\) and \(\epsilon = \tilde{\epsilon}_n^b\), the prior probability is lower bounded by a multiple of

\[
\exp\{-C_{14} N_n \log - \tilde{\epsilon}_n\} \gtrsim \exp\{-C_{14} n^{d/(2\beta + d)} (\log n)^{d/\tau_2 + d + 1 - t_1 d/\beta}\}, \tag{5.29}
\]
which decreases more slowly than $e^{-n\tilde{c}^2}$ if $\frac{d}{\tau_2} + d + 1 - \frac{d}{\beta} < 2t_1$. Combining with $t_2 > t_1$, $t_k + 1 > 2t_1$ and $2t_2 > \max(t_k, 2t_1 + \delta)$, we obtain $t_2 > \frac{\delta}{2} + \left(\frac{d}{\tau_2} + d + 1\right)\frac{\beta}{2\beta + d}$, where $\delta$ is an arbitrary positive number and hence can be absorbed in the remaining terms. □

5.4 Proof of Theorem 2

The proof uses a multivariate modification to the proof in [13]. We consider $\tilde{c}_n = n^{-\beta/(d+2\beta)}(\log n)^{t_1}$, $\tilde{c}_n = n^{-\beta/(d+2\beta)}(\log n)^{t_2}$, $\sigma_n = \tilde{c}_n^{1/\beta}$ for $t_2 > t_1 > 0$. Also, the number of finite mixture terms is $k_n = O\left(n^{d/(d+2\beta)}(\log n)^{d/\tau + d - t_1 d/\beta}\right)$.

Then in order to satisfy (5.8), we have $t_1 (2 + d/\beta) > d/\tau_2 + d + \max\{a_4 + 1, c_3, c_5/\tau_2\}$. Conditions (5.7) and (5.9) together give $t_2 > \max\{t_1, t_1 - c_3/2 + 1/2\}$. Combining these two constraints, we obtain $t > \frac{\beta}{2\beta + d} \left(\frac{d}{\tau_2} + d + \max\{c_3, 1 + a_4, \frac{c_5}{\tau_2}\}\right) + \max\{0, (1 - c_3)/2\}$. □

6 Appendix

The following three lemmas are helpful in controlling the KL divergence between $f_0$ and $K_\sigma h_\beta$.

Lemma 5 Given $\beta > 0$, let $f_0$ satisfy Condition (C1). Then for all sufficiently small $\sigma$ and all $x$ contained in the set

$$A_\sigma = \{x \in \mathbb{R}^d : |l_j(x)| \leq B\sigma^{-j}|\log \sigma|^{-j/2}, j = 1, 2, \ldots, r, \ |

\left|L(x)\right| \leq B\sigma^{-\beta}|\log \sigma|^{-\beta/2d^{-\beta/2}}\right\},$$

we have

$$K_\sigma f_\beta(x) = f_0(x) (1 + O(\sigma^\beta)R(x)) + O(\sigma^H)(1 + R(x)),$$  \hspace{1cm} (6.1)
where \( R(x) = s_{r+1}|L(x)| + \sum_{j=1}^{r} s_j |l_j(x)|^{\beta/j} \), \( H \) is a positive number that can be chosen arbitrarily large, and \( s_{r+1} \) and \( s_j \) are nonnegative constants.

**Proof** We follow the approach in Appendix (C) of [13]. By Condition (C1),

\[
\log f_0(y) \leq \log f_0(x) + \sum_{j=1}^{r} \frac{l_j(x)}{j!}(y - x)^j + L(x)\|y - x\|^\beta. \tag{6.2}
\]

\[
\log f_0(y) \geq \log f_0(x) + \sum_{j=1}^{r} \frac{l_j(x)}{j!}(y - x)^j - L(x)\|y - x\|^\beta. \tag{6.3}
\]

for all \( x \) and \( y \) with \( \|y - x\| \leq \gamma \).

Define \( B_{f_0, r}(x, y) = \sum_{j=1}^{r} \frac{l_j(x)}{j!}(y - x)^j + L(x)\|y - x\|^\beta \). First we assume \( \beta \in (1, 2] \) and \( r = 1 \). We want to demonstrate below that

\[
K_{\sigma} f_0(x) \leq \{1 + O((|L(x)| + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \log f_0(x))^{\beta}) \} f_0(x)
\]

\[
+ (1 + |L(x)| + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \log f_0(x))O(\sigma^H). \tag{6.4}
\]

To prove (6.4), we define for any \( x \in \mathbb{R}^d \),

\[
D_x = \{ y : |y_i - x_i| \leq k^\prime \sigma |\log \sigma|^{1/2}, i = 1, \ldots, d \}, \tag{6.5}
\]

where \( k^\prime \) is a sufficiently large constant to be chosen below. Assume that \( k^\prime \sigma |\log \sigma|^{1/2} \leq \gamma \) for \( \sigma \) as in Condition (C1). Then (6.2) can be written as

\[
K_{\sigma} f_0(x) \leq f_0(x) \int_{D_x} e^{B_{f_0, r}(x, y)} \phi_\sigma(y - x)dy + f_0(y) \phi_\sigma(y - x)dy. \tag{6.6}
\]
Furthermore, if \( x \in A_\sigma \) and \( y \in D_x \), we consider the Taylor expansion of 
\( \exp\{B_{f_0,r}(x,y)\} \) to the \( r \)-th degree. Then for a sufficiently large \( M \),
\[
e^{B_{f_0,r}(x,y)} \leq \sum_{m=0}^{r} \frac{1}{m!} B_{f_0,r}^m(x,y) + M|B_{f_0,r}|^{r+1}(x,y)
\]
\[
\leq \sum_{m=0}^{r} \frac{1}{m!} \left( \sum_{j=1}^{r} \frac{l_j(x)}{j!}(y-x)^j + L(x)\|y-x\|^\beta \right)^m
\]
\[
+ M \left| \sum_{j=1}^{r} \frac{l_j(x)}{j!}(y-x)^j + L(x)\|y-x\|^\beta \right|^{r+1}.
\]  
(6.7)

Since \( r = 1 \), (6.7) turns into
\[
1 + \sum_{i=1}^{d} \frac{\partial \log f_0(x)}{\partial x_i} (y_i - x_i) + L(x)\|y-x\|^\beta 
+ M \left\{ \sum_{i=1}^{d} \frac{\partial \log f_0(x)}{\partial x_i} (y_i - x_i) + L(x)\|y-x\|^\beta \right\}^2.
\]  
(6.8)

When integrating over \( D_x \), the terms with a factor \( y_i - x_i \) disappear. So the first term on the r.h.s of (6.6) is bounded by
\[
f_0(x) \int_{D_x} \phi_\sigma(y-x) \left\{ 1 + M \sum_{i=1}^{d} \left| \frac{\partial \log f_0(x)}{\partial x_i} (y_i - x_i) \right|^{\beta} (k'B)^{2-\beta}
+ L(x)\|y-x\|^\beta + Mk'^\beta BL(x)\|y-x\|^\beta \right\} dy.
\]  
(6.9)

where the following two inequalities are used for \( x \in A_\sigma \) and \( y \in D_x \):
\[
\left| \frac{\partial \log f_0(x)}{\partial x_i} (y_i - x_i) \right| \leq B\sigma^{-1}\|\log \sigma\|^{-1/2}k'^\beta \|\log \sigma\|^{1/2} = k'B,
\]  
(6.10)

\[
|L(x)||y-x|^\beta \leq (dk'^2\|\log \sigma\|)^{\beta/2} B\sigma^{-\beta} |\log \sigma|^{-\beta/2} d^{-\beta/2}.
\]  
(6.11)

Now \( \int_{D_x} \phi_\sigma(y-x)|y_i - x_i|\alpha dy = O(\sigma^H) \) for any \( \alpha \geq 0, i = 1, \ldots, d \), when \( k' \) in the definition of \( D_x \) is sufficiently large. By choosing constants
\( k_1 = M(k'B)^{2-\beta} \) and \( k_2 = 1 + Mk^\beta B \), we obtain

\[
K_\sigma f_0(x) \leq f_0(x) \int_{R^d} \phi_\sigma(y - x) \left\{ 1 + k_1 \sum_{i=1}^{d} \left| \frac{\partial \log f_0}{\partial x_i} (y_i - x_i) \right|^\beta \right. \\
\left. + k_2 |L(x)||y - x|^\beta \right\} dy \\
+ (1 + \|f_0\|_\infty + k_1 \sum_{i=1}^{d} \left| \frac{\partial \log f_0}{\partial x_i} \right|^\beta + k_2 |L(x)|) O(\sigma^H),
\]

(6.12)

which gives (6.3) for \( \beta \in (1, 2] \). Using similar arguments on the other direction from (6.3), we obtain Lemma 1 for \( \beta \in (1, 2] \).

Now consider the case when \( r > 1, \beta \in (r, r+1] \). We have the following result by doing similar calculation as in (6.2), (6.3), (6.6), (6.7) and (6.9):

\[
K_\sigma f_0(x) = f_0 \left( 1 + \sum_{i=1}^{\lfloor r/2 \rfloor} m_i(x) \sigma^{2i} + R(x) O(\sigma^\beta) \right) + (1 + R(x)) O(\sigma^H). \quad (6.13)
\]

This follows by controlling the integral of terms containing a factor \( \prod_{i=1}^{d} (y_i - x_i)^{k_i} \) over \( D_x \). Since the normal kernel is symmetric over \( D_x \), we only need to consider the case when \( k_i \)'s are even numbers. When \( k_i > r \), there exists a \( k^* \in \mathbb{N}^d \) satisfying \( k^* = r \) and \( k_i^* \leq k_i \) for \( i = 1, \ldots, d \). Since one of the inequalities is strict, we can choose \( k_i^* \) such that \( k_i^* < k_i \) and then define \( Q(x, y, \beta) = \min_{k^*} |(y - x)^{k^*}(y_i - x_i)|^{k_i^*+\beta-r} \). The integral is bounded by a multiple of \( \sigma^\beta \) when \( k_i > r \) by taking a factor \( Q(x, y, \beta) \) out and bounding the remaining term by a certain power of \( |l_j|'s \) and \( |L| \), which are denoted by \( m_i(x) \) in (6.13). If \( k_i \leq r \), then they can be bounded by a multiple of \( \sigma^{2u} \), where \( u \leq \lfloor r/2 \rfloor \).
We can substitute $f_0$ in (6.13) by $f_\beta$ because $\int \phi_\sigma(x)\|x\|^k dx < \infty$ for all $k \in \mathbb{R}^d$, $k_i < \infty$, $i = 1, \ldots, d$, and the fact that $\int_{\|x\| > k'} \log^{1/2} \phi_\sigma(x) \|x\|^k dx = O(\sigma^H)$ for all $k$ and $H$ taking arbitrary large values, provided that $k'$ is sufficiently large. Hence the proof is complete. □

Lemma 6 Define $E_\sigma = \{x : g(x) \geq \sigma^{H_1}\}$. Assume that $f_0$ satisfies Conditions (C1)–(C4). Then for all sufficiently small $\sigma$, all $i \in \mathbb{N}^d$ and $\epsilon > 0$:

$$\int_{A_\sigma} K_\sigma^i f_0(x) dx = O(\sigma^{2\beta+\epsilon}), \quad \int_{E_\sigma} K_\sigma^i f_0(x) dx = O(\sigma^{2\beta+\epsilon}) \quad (6.14)$$

provided that $H_1$ is sufficiently large.

Proof Observe $i = \sum_{k=1}^j \hat{j}_k$, $\hat{j}_k \in \mathbb{N}^d$, where each component of $\hat{j}_k$ only takes two values 0 and 1. If some components of $\hat{j}_k$ are 0, then we can remove these 0s away and consider a corresponding convolution operator in a low-dimension case. Therefore it is good enough to prove (6.14) when $i_1 = \ldots = i_d = m$ for $m \in \mathbb{N}$. The proof for other cases can proceed in a similar way. In order to bound the first integral in (6.14), we consider sets $A_{\sigma, \delta} = \{x : |j_j(x)| \leq \delta B \sigma^{-j_j} \log \sigma^{-j_j/2}, j = 1, \ldots, r, |L(x)| \leq \delta B \sigma^{-|L(x)|} \log \sigma^{-|L(x)|} \sigma^{-2d^{-1/2}} \}$ indexed by $\delta \leq 1$. Using Markov’s inequality and Condition (C3),

$$\int_{A_{\sigma, \delta}} (K_\sigma^0 f_0)(x) dx \leq \mathbb{P}\{|L(x)| \geq (\delta B)^{2\beta+2\epsilon}/\beta \sigma^{-2\beta-2\epsilon} \log \sigma^{-(2\beta+2\epsilon)/2}\}
+ \sum_{j=1}^r \mathbb{P}\{|j_j(x)|^{(2\beta+2\epsilon)/j_j} \geq (\delta B)^{(2\beta+2\epsilon)/j_j} \sigma^{-(2\beta+2\epsilon)} \log \sigma^{-(\beta+\epsilon)}\}
= O(\sigma^{2\beta+\epsilon}), \quad (6.15)$$
provided that \( \sigma^{-\epsilon} | \log \sigma |^{-\beta - \epsilon} > 1 \) and \( \epsilon > 0 \), which is the case if \( \sigma \) is sufficiently small. This completes the proof for \( m = 0 \).

If \( m = 1 \), consider independent random vectors \( X \) and \( U \) with densities \( f_0 \) and standard normal \( \phi \) respectively. Then \( X + \Sigma U \) has density \( K_\sigma f_0 \). We want to prove \( X \in A_{\sigma, \delta} \) together with \( \|U\| \leq k' | \log \sigma |^{1/2} \) are in contradiction with \( X + \Sigma U \in A_{\sigma}^c \) when \( \delta \) is sufficiently small.

We observe that \( X + \Sigma U \in A_{\sigma, 1}^c \) implies

\[
|L(X + \Sigma U)| \geq B \sigma^{-\beta} | \log \sigma |^{-\beta/2} d^{-\beta/2} \quad \text{or} \quad |l_i(X + \Sigma U)| \leq B \sigma^{-i} | \log \sigma |^{-i/2}
\]

for some \( i \) satisfying \( i \leq r \).

From Condition (C1), if \( \delta \) is sufficiently small, then for all \( i = 1, \ldots, r \),

\[
|l_i(X + \Sigma U)| \leq | \sum_{j \geq i} \frac{l_j(X)}{(j - 1)!} (\Sigma U)^{j-1} | + \sum_{j=1}^r \sum_{j \geq i} \frac{j!}{(j - i)!} |L(X)||\Sigma U|^{j-i} \\
\leq B \sigma^{-i} | \log \sigma |^{-i/2}. \quad (6.16)
\]

Therefore it has to be a large value of \( |L(X + \Sigma U)| \) that forces \( X + \Sigma U \) to be in \( A_{\sigma}^c \). Hence it suffices to show \( |L(X)| \leq \delta B \sigma^{-\beta} | \log \sigma |^{-\beta/2} d^{-\beta/2} \) and \( \|U\| \leq k' | \log \sigma |^{1/2} \) are in contradiction with \( |L(X + \Sigma U)| \geq B \sigma^{-\beta} | \log \sigma |^{-\beta/2} d^{-\beta/2} \).

From Condition (C1), we assume \( L \) is a polynomial of degree \( q \) and has roots \( z_1, \ldots, z_q \). Let \( \eta = (\max_i |z_{1i}|, \ldots, \max_i |z_{id}|) \). If \( |X_i| \leq \eta_i + 1 \) for \( i = 1, \ldots, d \), then each component of \( \|X + \Sigma U\| \) is bounded by corresponding component of \( \eta + 2 \) when \( \sigma \) is sufficiently small. As a result, \( |L(X + \Sigma U)| \leq B \sigma^{-\beta} | \log \sigma |^{-\beta/2} d^{-\beta/2} \). Alternatively, if there exists a \( 1 \leq i^* \leq d \) such that
\[|X_i| > \eta_i + 1,\] then we consider the Taylor expansion of \(L(X + \Sigma U):\)

\[
|L(X + \Sigma U)| \leq |L(X)| + \left| \sum_{j=1}^{q} \frac{\sigma^j U^j L^{(j)}(X)}{j!} \right| + \frac{\sigma^q \|U\|^q}{q!} \left| L^{(q)}(\eta) - L^{(q)}(X) \right|
\]

\[
\leq \delta B (d\sigma^2 |\log \sigma|)^{-\beta/2} + \sum_{j=1}^{q} O(\sigma^{j-\beta} |\log \sigma|^{j-\beta/2}), \tag{6.17}
\]

which is less than \(B\sigma^{-\beta} |\log \sigma|^{-\beta/2} d^{-\beta/2}\) when \(\sigma < 1\) and \(\delta < 1\) are small enough.

Because \(P(\|U\| \geq k' |\log \sigma|^{1/2}) = O(\sigma^{2\beta+\epsilon})\) for \(\epsilon > 0\) if \(k'\) is sufficiently large, we have

\[
P(X + \Sigma U \in A^c_{\sigma})
\]

\[
\leq P(X + \Sigma U \in A^c_{\sigma}, \|U\| \leq k' |\log \sigma|^{1/2}) + P(\|U\| \geq k' |\log \sigma|^{1/2})
\]

\[
= P(X + \Sigma U \in A^c_{\sigma}, X \in A_{\sigma,\delta}, \|U\| \leq k' |\log \sigma|^{1/2}) + O(\sigma^{2\beta+\epsilon})
\]

\[+ P(X + \Sigma U \in A^c_{\sigma}, \|U\| \leq k' |\log \sigma|^{1/2}) \leq 0 + O(\sigma^{2\beta+\epsilon}) + P(X \in A^c_{\sigma,\delta})
\]

\[
\leq O(\sigma^{2\beta+\epsilon}). \tag{6.18}
\]

This completes the proof of first equation in (6.14) for \(m = 1\). For \(m > 1\), we can redefine the density of \(X\) as \(K_{\sigma}^{m-1} f_0\) and apply the same arguments above with a decreasing sequence of \(\delta\)'s.

Now we bound the second integral in (6.14). If \(m = 0\), using Condition (C2), we have

\[
\int_{E_{\sigma}} f_0(x)dx = \int_{E_{\sigma}} f_0(x) \frac{1}{(g(x))^{\xi}} (g(x))^{\xi} dx \lesssim \sigma^{\xi H_1} = O(\sigma^{2\beta+\epsilon}) \tag{6.19}
\]

when \(H_1 \geq (2\beta + \epsilon)/\xi\).
Consider \( m = 1 \), we define sets \( E_{\sigma,\delta} = \{ x : f_0(x) \geq \sigma^\delta H_1 \} \) indexed by \( \delta \leq 1 \), random vectors \( X \) having density \( f_0 \) and \( U \) following standard normal distribution. Observe \( X + \Sigma U \in E_{\sigma}^c \cap A_\sigma \) contradicts with \( X \in E_{\sigma,\delta} \cap A_\sigma \): on one hand, \( X + \Sigma U \in E_\sigma^c \) and \( X \in E_{\sigma,\delta} \) imply \(|l(X + \Sigma U) - l(X)| \geq (1 - \delta)H_1 \log \sigma\). On the other hand, \( X, X + \Sigma U \in A_\sigma \) implies that \(|l(X + \Sigma U) - l(X)| \leq B\sigma^{-d}|\log \sigma|^{-1/2} \sigma^d k' |\log \sigma|^{1/2} = O(1)\).

Similarly with the previous treatment, for a sufficiently large constant \( k' \) and \( H_1 \geq (4\beta + 2\epsilon)/\delta \), we have

\[
\int_{E_\sigma^c} K_\sigma f_0(x) dx \\
\leq \int_{A_\sigma} K_\sigma f_0(x) dx + \int_{E_\sigma^c \cap A_\sigma} K_\sigma f_0(x) dx \\
\leq O(\sigma^{2\beta + \epsilon}) + P(X + \Sigma U \in E_\sigma^c \cap A_\sigma) \\
\leq O(\sigma^{2\beta + \epsilon}) + P(X + \Sigma U \in E_\sigma^c \cap A_\sigma, X \in E_{\sigma,\delta} \cap A_\sigma, \|U\| \leq k' |\log \sigma|^{1/2}) \\
+ P(X + \Sigma U \in E_\sigma^c \cap A_\sigma, X \in E_{\sigma,\delta} \cap A_\sigma, \|U\| \leq k' |\log \sigma|^{1/2}) \\
\leq O(\sigma^{2\beta + \epsilon}) + P(X \in E_{\sigma,\delta}) \\
= O(\sigma^{2\beta + \epsilon}). \tag{6.20}
\]

This completes the proof for \( m = 1 \). The above procedure can be done repeatedly in the same way when \( H_1 \) is chosen sufficiently large for \( m > 1 \). Hence we obtain \ref{6.14}. \( \square \)

**Lemma 7** Assume that \( f_0 \) satisfies Conditions (C1)–(C4). If \( \beta > 2 \), \( x \in A_\sigma \cap E_\sigma \) and \( \sigma \) is sufficiently small, then

\[
K_\sigma h_\beta(x) = f_0(x)(1 + O(\sigma^\beta R(x))) + O(\sigma^H)(1 + R(x)), \tag{6.21}
\]

26
where \( R(x) \) is defined in Lemma 5.

**Proof** For \( x \in A_\sigma \cap E_\sigma \), apply similar arguments on \( K_{\sigma j} \) as in Lemma 5 to obtain

\[
K_{\sigma j} f(x) = (1 + O(\sigma^{2j}) R^{(2j)}(x)) f_0(x) + O(\sigma^H)(1 + R^{2j}(x)) \quad (6.22)
\]

Using Remark 4 for constants \( u_i \):

\[
f_\beta = C_\beta f_0 - \sum_{j \geq 0} c_j K_{\sigma j} f_0 \\
= C_\beta f_0 - \sum_{i=1}^{C_\beta - 1} \left( 1 + O(\sigma^{u_i}) R^{(u_i)}(x) \right) f_0(x) + O(\sigma^H)(1 + R^{u_i}(x)) \\
> f_0/2 \quad (6.23)
\]

when \( \sigma \) is chosen to be sufficiently small. Therefore \( A_\sigma \cap E_\sigma \subseteq J = \{ x : f_\beta(x) > \frac{1}{2} f_0(x) \} \). Now since \( x \in A_\sigma \cap E_\sigma \)

\[
K_\sigma h(x) = K_\sigma f_\beta(x) = f_0(x) \left( 1 + O(\sigma^\beta) R(x) \right) + O(\sigma^H)(1 + R(x))
\]

Therefore

\[
K_\sigma h_\beta(x) = K_\sigma h(x) / \left( 1 + O(\sigma^{2\beta}) \right) \\
= f_0(x) \left( 1 + O(\sigma^\beta) R(x) \right) + O(\sigma^H)(1 + R(x))
\]

where \( H \) can be chosen to be arbitrarily large. \( \square \)

**Remark 5** The density function \( h_\beta(x) \) is lower bounded by a multiple of \( g(x) \)
because
\[
\int h(x) \, dx = 1 + \int_{x^0} \left( \frac{1}{2} f_0 - f_\beta \right) \, dx \\
= 1 + \int_{x^0} \left( \frac{1}{2} f_0 - C_\beta f_0 + \sum_{j \geq 0} c_j K_\sigma f_0 \right) \, dx \\
= 1 + O(\sigma^{23}) \tag{6.24}
\]

Therefore \( K_\sigma h_\beta \) is also lower bounded by a multiple of \( g(x) \).

**Proof of Proposition** \[1\] Using inequality \( \log x \leq x - 1 \) for all \( x > 0 \), we have
\[
\int_S p \log \frac{p}{q} \leq \int_S \frac{p - q}{q} = \int_S \frac{(p - q)^2}{q} + \int_S (p - q). \tag{6.25}
\]
for any densities \( p \) and \( q \), and any set \( S \). We apply this result for \( p = f_0(x) \), \( q = K_\sigma h_\beta(x) \) and \( S = A_\sigma \cap E_\sigma \):
\[
\int f_0(x) \log \frac{f_0(x)}{K_\sigma h_\beta(x)} \, dx \leq \int_{A_\sigma \cup E_\sigma} f_0(x) \log \frac{f_0(x)}{K_\sigma h_\beta(x)} \, dx \\
+ \int_{A_\sigma \cup E_\sigma} (K_\sigma h_\beta(x) - f_0(x)) \, dx \\
+ \int_{A_\sigma \cap E_\sigma} \frac{(f_0(x) - K_\sigma h_\beta(x))^2}{K_\sigma h_\beta(x)} \, dx. \tag{6.26}
\]

Using Remark \[5\] \( K_\sigma h_\beta(x) \) is lower bounded by a multiple of \( g(x) \). Using Hölder’s inequality and Remark \[1\] the first term of (6.26) is bounded by
\[
\int_{A_\sigma \cup E_\sigma} f_0(x) \log \frac{f_0(x)}{g(x)} \, dx \\
\leq \left\{ \int f_0(x) \left( \log_+ \frac{f_0(x)}{g(x)} \right)^p \, dx \right\}^{1/p} \left\{ \int_{A_\sigma \cup E_\sigma} f_0(x) \right\}^{1/q} \\
\leq C_1 \left\{ \int f_0 \left( \log_+ f_0 \right)^p \right\}^{1/p} + C_2 \int f_0 \left( \frac{1}{g(x)} \right)^p \, dx \left\{ \int_{A_\sigma \cup E_\sigma} f_0(x) \right\}^{1/q} \\
\]
28
for constants $C_1, C_2 > 0$, $q = (2\beta + \epsilon)/2\beta$, $p = (2\beta + \epsilon)/\epsilon$ and $\xi$ as defined in Condition (C2). By choosing $H_1$ such that equation (6.14) in Lemma 3 holds for $i = 0$, the first integral of the r.h.s of (6.26) is $O(\sigma^{2\beta})$.

Since $h_{\beta}$ is a linear combination of $K_{\sigma}^i f_0$’s, so is $K_{\sigma} h_{\beta}(x)$. Therefore by another application of (6.14), we obtain the second integral is a finite sum of $O(\sigma^{2\beta+\epsilon})$, which is still $O(\sigma^{2\beta+\epsilon})$.

For the last integral of the r.h.s of (6.26), we apply Lemma 7. Observe that when $x \in A_{\sigma} \cap E_{\sigma}$, $K_{\sigma}(x)$ is bounded by a multiple of $f_0$ given $H \geq H_1$.

\[
\left( \int_{A_{\sigma} \cap E_{\sigma}} f_0(x) R^2(x)dx \right) O(\sigma^{2\beta}) + \left( \int_{A_{\sigma} \cap E_{\sigma}} (1 + R(x))^2/f(x)dx \right) O(\sigma^{2H}) + 2\left( \int_{A_{\sigma} \cap E_{\sigma}} R(x)(1 + R(x))dx \right) O(\sigma^{3+H})
\]

(6.27)

Condition (C1) implies $\int_{A_{\sigma} \cap E_{\sigma}} f_0(x) R^k(x)dx = O(1)$ for $k = 1, 2$. By choosing $H$ satisfying $H \geq H_1 + \beta$ and using $f_0(x) \geq \sigma H_1$ on $E_{\sigma}$, these three integrals in (6.27) are $O(\sigma^{2\beta})$, hence (4.1) follows.

The integral in (4.2) can be treated in a similar way:

\[
\int f_0(\log \frac{f_0}{K_{\sigma} h_{\beta}})^2 \leq \int_{A_{\sigma} \cup E_{\sigma}} f_0(x) \left( \log \frac{f_0(x)}{K_{\sigma} h_{\beta}(x)} \right)^2 dx + \int_{A_{\sigma} \cap E_{\sigma}} (f_0(x) - K_{\sigma} h_{\beta}(x))^2 \frac{1}{K_{\sigma} h_{\beta}(x)} dx + \int_{A_{\sigma} \cap E_{\sigma}} (f_0(x) - K_{\sigma} h_{\beta}(x))^3 \frac{1}{(K_{\sigma} h_{\beta}(x))^2} dx,
\]

(6.28)

where the first two terms on r.h.s are shown to be $O(\sigma^{2\beta})$, and the last integral
can be bounded by a multiple of
\[
\int_{A_\sigma \cap E_\sigma} f_0(x)R^3(x)O(\sigma^{3\beta}) \, dx + 3 \int_{A_\sigma \cap E_\sigma} R^3(x)O(\sigma^{2\beta+H}) \\
+ 3 \int_{A_\sigma \cap E_\sigma} R^3(x)O(\sigma^{\beta+2H}) / f_0(x) + \int_{A_\sigma \cap E_\sigma} R^3(x)O(\sigma^{3H}) / f_0^2(x) \, dx \\
= O(\sigma^{3\beta})
\] (6.29)
by choosing \( H \geq H_1 + \beta \). \( \square \)

**Proof of Lemma 1** Define set \( E'_\sigma = \{ x : h_\beta(x) \geq \sigma^{H_2} \} \) with \( H_2 \geq H_1 \) and \( \bar{h}_\beta(x) = h_\beta 1_{E'_\sigma}(x) / \int_{E'_\sigma} h_\beta(x) \, dx \). Remark 4 implies \( E'_\sigma \supset E_\sigma \). Using Lemma 6 and Remark 4 we have \( \int_{E'_\sigma} h_\beta(x) \, dx = 1 - \int_{E'_\sigma} h_\beta(x) \, dx = 1 + O(\sigma^{2\beta}) \).

\[
\int f_0 \log \frac{f_0}{p_{F,\sigma}} = \int f_0 \log \frac{f_0}{K_\sigma h_\beta} + \int_{E_\sigma} f_0 \left( \log \frac{K_\sigma h_\beta}{K_\sigma h_\beta} + \log \frac{K_\sigma \bar{h}_\beta}{p_{F,\sigma}} \right) \\
+ \int_{E'_\sigma} f_0 \log \frac{K_\sigma h_\beta}{p_{F,\sigma}} \quad (6.30)
\]

From Theorem 11 the first term is \( O(\sigma^{2\beta}) \). Now observe

\[
\frac{K_\sigma h_\beta}{K_\sigma h_\beta} = \frac{K_\sigma h_\beta(x)}{K_\sigma h_\beta 1_{E'_\sigma}(x)} (1 + O(\sigma^{2\beta})) \\
= (1 + O(\sigma^{2\beta})) \left( 1 + \int_{E'_\sigma} \phi_\sigma(x - y) h_\beta(y) \, dy \right) \\
\int_{E'_\sigma} \phi_\sigma(x - y) h_\beta(y) \, dy \quad (6.31)
\]

For \( x \in E_\sigma \) and \( y \in E'_\sigma \), because \( \int_{E'_\sigma} \phi_\sigma(x - y) h_\beta(y) \, dy \leq \sigma^{H_2} \leq \sigma^{H_2-H_1}g(x) \) and \( \int_{E'_\sigma} \phi_\sigma(x - y) h_\beta(y) \, dy \geq C_0g(x) \) for constant \( C_0 > 0 \), (6.31) is upper bounded by \( (1 + O(\sigma^{2\beta})) (1 + C_0^{-1} \sigma^{H_2-H_1}) = 1 + O(\sigma^{2\beta}) \) when we choose \( H_2 \geq H_1 + 2\beta \). On the other hand, (6.31) is lower bounded by \( 1 + O(\sigma^{2\beta}) \). Hence \( \frac{K_\sigma h_\beta}{K_\sigma h_\beta} = 1 + O(\sigma^{2\beta}) \) and therefore \( \int_{E_\sigma} f_0 \log \frac{K_\sigma h_\beta}{K_\sigma h_\beta} = O(\sigma^{2\beta}) \).
Now we bound \( \int_{E_\sigma} f_0 \log \frac{K_\sigma \bar{h}_\beta}{p_{F,\sigma}} \). Apply Lemma 4 for \( \epsilon = e^{-C_1|\log \sigma|} \) for some constant \( C_1 \) and \( \gamma_1 = 1/\tau_2 \), let \( p_{F,\sigma} \) be the finitely supported mixture approximating \( \bar{h}_\beta \) such that \( \|K_\sigma \bar{h}_\beta - p_{F,\sigma}\|_{\infty} \leq \sigma^{-d}e^{-C_1|\log \sigma|} \) and \( F \) has at most \( K\sigma^{-d}|\log \sigma|^{d/\tau_2+d} \) many support points, which are all contained in \( E'_\sigma \) because of Condition (C3). Notice that these support points are also contained in \( \{x : f_0(x) \geq c\sigma H_2\} \) for sufficiently small \( c > 0 \) by Remark 3. Applying

\[
\log \frac{p(x)}{q(x)} \leq \frac{|p(x) - q(x)|}{\min\{p(x), q(x)\}} \leq \frac{\|p - q\|_{\infty}}{(\min_y p(y) - \|p - q\|_{\infty})}
\]

(6.32)

if \( \min_y p(y) - \|p - q\|_{\infty} > 0 \) for \( p = K_\sigma \bar{h}_\beta \) and \( q = p_{F,\sigma} \), we have

\[
\int_{E_\sigma} f_0 \log \frac{K_\sigma \bar{h}_\beta}{p_{F,\sigma}} \leq \int_{E_\sigma} \frac{\|K_\sigma \bar{h}_\beta - p_{F,\sigma}\|_{\infty}}{\|K_\sigma \bar{h}_\beta - p_{F,\sigma}\|_{\infty}} \lesssim \sigma^{-H_2-d}e^{-C_1|\log \sigma|} \quad (6.33)
\]

When \( \sigma \) is small enough and \( C_1 \) is large enough, the above estimate is \( O(\sigma^{2\beta}) \).

Finally, we bound the last term in (6.30). Using Lemma 3 we can add a mixture component with mean 0 and weight \( \sigma^{2\beta} \) without influencing approximation results. Combine this result with the fact that \( K_\sigma h_\beta \) is upper bounded by a constant \( C_2 \), we have

\[
\int_{E'_\sigma} f_0(x) \log \frac{K_\sigma h_\beta(x)}{p_{F,\sigma}} dx \leq \sigma^{H_1\xi} \int_{E'_\sigma} f_0(x) \frac{1}{g^\xi(x)} \log \frac{C_2}{\sigma^{2\beta} \phi_\sigma(x)} dx
\]

\[
\lesssim \sigma^{H_1\xi} \int_{E'_\sigma} f_0(x) \frac{1}{g^\xi(x)} \|x\|^2 \sigma^{-2d} dx
\]

= \( O(\sigma^{2\beta}) \)

(6.34)

when \( H_1 \) is chosen to be large enough. Hence the proof of the first equation in (1.6) is complete. The proof for the second equation proceeds in the same way as in Appendix E of [13].
Proof of Lemma 2 Choose $\sigma_0 = 2a/\Phi^{-1}(5/6)$ such that $N(0, \sigma_0)$ gives probability $1/3$ to $(0, 2a)$. Let $\sigma = (\sigma_0, \ldots, \sigma_0)'$. Then if $x \in D$

$$K_{\sigma} f_0(x) \geq \int_D f_0(\theta) \phi_{\sigma}(x - \theta) d\theta \geq c_0 \int_D \phi_{\sigma}(x - \theta) d\theta = c_0 \prod_{i=1}^d \left[ \Phi \left( \frac{a - x_i}{\sigma_0} \right) + \Phi \left( \frac{x_i + a}{\sigma_0} \right) - 1 \right] \geq \frac{c_0}{3^d}. \quad (6.35)$$

If $x \notin D$, then at least one of $x_i$'s are not in $[-a, a]$. We only consider the case $x_1 > a$, $x_2 < -a$ and $x_3, \ldots, x_d \in [-a, a]$. The calculation can be done for other cases in a similar way.

$$K_{\sigma} f_0(x) \geq c \int_{-a}^{x_1} \int_{x_2}^a \cdots \int_{x_{d-1}}^{x_d} g_1(\theta_1) \phi_{\sigma}(x - \theta) d\theta_1 \cdots d\theta_{d-1} \int_{x_d}^{a} g_2(x_d) \phi_{\sigma}(x_2 - \theta_2) d\theta_2 \frac{c_0}{3^{d-2}} \geq \frac{c_0 c}{3^{d-2}} g_1(x_1) g_2(x_2) \left( \Phi \left( \frac{2a}{\sigma_0} \right) - \frac{1}{2} \right) \left( \frac{1}{2} - \Phi \left( \frac{-2a}{\sigma_0} \right) \right) \geq C_0 g_1(x_1) g_2(x_2) \geq C_1 g(x) \quad (6.36)$$

for some positive constants $C_0$ and $C_1$. □

Proof of Lemma 3 By an easy multidimensional extension of Lemma 5 in [11], we have

$$\|p_{F, \sigma} - p_{F', \sigma}\|_1 \lesssim \frac{\epsilon}{(\sigma_1 \wedge \sigma_1')^d} + \sum_{j=1}^N |F(V(z_j, \epsilon)) - p_j| \quad (6.37)$$
\[ \|p_{F,\sigma} - p_{F',\sigma}\|_{\infty} \lesssim \frac{\epsilon}{(\sigma(1) \land \sigma'(1))^{2d}} + \frac{1}{(\sigma(1) \land \sigma'(1))^{d}} \sum_{j=1}^{N} |F(V(z_j, \epsilon)) - p_j| \quad (6.38) \]

Similarly, by a multidimensional extension of Lemma 3 in [13]

\[ \|p_{F',\sigma} - p_{F',\sigma'}\|_{1} \leq \|\phi_\sigma - \phi_{\sigma'}\|_{1} \leq \sum_{i=1}^{d} \|\phi_{\sigma_i} - \phi_{\sigma'_i}\|_{1} \lesssim \max_{i=1,\ldots,d} |\sigma_i - \sigma'_i| \quad (6.39) \]

\[ \|p_{F',\sigma} - p_{F',\sigma'}\|_{\infty} \lesssim \left| \prod_{i=1}^{d} \frac{1}{\sigma_i} - \prod_{i=1}^{d} \frac{1}{\sigma'_i} \right| \leq \frac{1}{(\sigma(1) \land \sigma'(1))^{2d}} \prod_{i=1}^{d} |\sigma_i - \prod_{i=1}^{d} \sigma'_i| \quad (6.40) \]

Using triangle inequality on (6.37) and (6.39) gives (5.3). Similarly, combining (6.38) and (6.40) gives (5.4). □

**Proof of Lemma 4** The proof of part 1 proceeds in the similar way with Lemma 3.1 in [10]. Subscript 0 in \( \sigma_0 \) is used to denote that \( \sigma_0 \) and \( \sigma'_0 \) are fixed here. For simplicity, we drop them in the proof.

We first observe the following:

\[ \|\phi_\sigma - \phi_{\sigma'}\|_{\infty} \leq \frac{2}{(\sqrt{2\pi})^{d}} \sigma^{-2d} \left| \prod_{i=1}^{d} \sigma_i - \prod_{i=1}^{d} \sigma'_i \right| \leq \frac{\sigma^d}{\sigma^{2d}(\sqrt{2\pi})^d} \epsilon \quad (6.41) \]

So \( \|p_{F,\sigma} - p_{F',\sigma}\|_{\infty} \lesssim \frac{\sigma^d}{\sigma^{2d}(\sqrt{2\pi})^d} \epsilon \). Define \( M = \max(2a_1, \ldots, 2a_d, \sigma\sqrt{8\log_\epsilon}) \).

\[ \sup_{|x| \geq M} |p_{F,\sigma}(x) - p_{F',\sigma}(x)| \leq 2 \prod_{i=1}^{d} \phi_{\sigma_i}(M - a_i) \leq \frac{2\epsilon^d}{(\sqrt{2\pi}\sigma)^d} \lesssim \sigma^{-d} \epsilon \quad (6.42) \]

Applying Taylor’s expansion on \( \phi_{\sigma_i}(x) \) and using the fact \( k! \geq k^k e^{-k} \), we get

\[ \left| \phi_{\sigma}(x) - \sum_{j=0}^{k-1} \left( -\sum_{i=1}^{d} \frac{x^2}{2\sigma_i^2} \right)^j \left( \frac{1}{\sigma\sqrt{2\pi}} \right)^d \right| \leq \frac{1}{(\sigma\sqrt{2\pi})^d} \left( \frac{e^{1/2} - 1/2} {k^k} \|x\|_2 \right)^{2k} \quad (6.43) \]
Therefore, for any $F \in \mathfrak{M}(D)$,

$$\sup_{|x_i| \leq M} |p_{F,\sigma}(x) - p_{F',\sigma}(x)| \leq \sup_{|x_i| \leq M} \left| \int \sum_{j=0}^{k-1} \frac{(2\pi)^{-d/2}}{j!} \left[ \sum_{j=0}^{k-1} \frac{(-1)^j}{2\sigma^2} \right]^j d(F - F')(z) \right| + 2 \sup_{|x_i| \leq M} \left| \phi_\sigma(x - z) - \sum_{j=0}^{k-1} \frac{(-1)^j}{\sqrt{2\pi} \sigma_j} \sum_{j=0}^{k-1} \frac{(x_{w_j} - z_{w_j})^2}{2\sigma_{w_j}^2} \right| \right. \quad (6.44)$$

We want to choose $F'$ such that $\int z_1^{a_1} \cdots z_d^{a_d} dF = \int z_1^{a_1} \cdots z_d^{a_d} dF'$ for all $a_1, \ldots, a_d$ satisfying $1 \leq \sum_{i=1}^d a_i \leq 2k - 2$. Thus $F'$ can be chosen on $D$ with at most $N$ support points, where

$$N = \sum_{l=1}^{2k-2} \frac{(l + d - 1)!}{l!(d-1)!} + 1 \leq \sum_{l=1}^{2k-2} (l + 1)^{d-1} + 1 \leq \frac{(2k)^d}{d} \quad (6.45)$$

As a result, the first term in (6.44) is canceled out. Now we want to bound the second term. Observe that $|x_i - z_i| \leq M + a_i \leq \max(3L, \sqrt{18\sigma})(\log_+ \epsilon)^\gamma_0$. Denote $c = e^{1/2}2^{-1/2}\sigma^{-1}\max(3L, \sqrt{18\sigma})$, then we bound the second term in (6.44) by

$$2 \left( \frac{1}{\sqrt{2\pi}} \right)^d c^{2k} (\log_+ \epsilon)^{2\gamma k} = 2 \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^d \exp\{-k(\log k - 2\log(c(\log_+ \epsilon)^\gamma_0))\} \quad (6.46)$$

By choosing $k = \lceil (1 + c^2)(\log_+ \epsilon)^{2\gamma_0} \rceil + 1$, the above term is bounded by $\sigma^{-d}\epsilon$.

Without loss of generality, consider $W = \{ |x_1| > M, |x_i| \leq M, i = 2, \ldots, d \}$. Then

$$\sup_W |p_{F,\sigma}(x) - p_{F',\sigma}(x)| \leq 2\phi_\sigma(M - a_1) \sup_{|x_i| \leq M} \left| \int \phi_\sigma(x_i - z_i) d(F - F') \right| \leq \frac{2\epsilon}{\sqrt{2\pi} \sigma} \sigma^{-d+1}\epsilon \lesssim \sigma^{-d}\epsilon \quad (6.47)$$
Finally, combining results in (6.44), (6.45), (6.46) and (6.47) completes the proof.

Now we prove the second part in a similar way as Lemma 2 of [11]. We partition intervals $[-a_\epsilon, a_\epsilon]$ into $\lfloor 2a_\epsilon/\sigma \rfloor$ disjoint, consecutive subintervals of length $\sigma$ and an interval of length less than $\sigma$. Then $D$ is divided into $k = \lfloor 2a_\epsilon/\sigma \rfloor^d$ disjoint, consecutive and equally spaced regions $I_1, \ldots, I_k$ and some final pieces that has area less than $\sigma^d$. Applying the result from first part on $[0, 1]^d$, we have approximation result as $\|p_{F,\sigma} - p_{F',\sigma}\|_\infty \lesssim \sigma^{-d}\epsilon$ while the number of support points of $F'$ is bounded by a multiple of $\prod_i (a_\epsilon \sigma^{-1} \lor 1)(\log_\epsilon \sigma)^d \lesssim \sigma^{-d}(\log_\epsilon \sigma)^{\gamma_1 d + d}$. Applying a multivariate version of Lemma 3.2 of [10], we get

$$\|p_{F,\sigma} - p_{F',\sigma}\|_1 \lesssim \|p_{F,\sigma} - p_{F',\sigma}\|_\infty \max \left\{ \sigma \sqrt{\log_\epsilon \|p_{F,\sigma} - p_{F',\sigma}\|_\infty \sigma^d, a_\sigma, 1} \right\} \lesssim \sigma^d (\sigma(\log_\epsilon \sigma)^{1/2} \lor (\log_\epsilon \sigma)^{\gamma_1})^d \epsilon. \quad (6.48)$$

Hence the proof is complete. □

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