Least Squares Approximations of Measures via Geometric Condition Numbers*

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Abstract

For a probability measure $\mu$ on a real separable Hilbert space $H$, we are interested in “volume-based” approximations of the $d$-dimensional least squares error of $\mu$, i.e., least squares error with respect to a best fit $d$-dimensional affine subspace. Such approximations are given by averaging real-valued multivariate functions which are typically scalings of squared $(d+1)$-volumes of $(d+1)$-simplices in $H$. Specifically, we show that such averages are comparable to the square of the $d$-dimensional least squares error of $\mu$, where the comparison depends on a simple quantitative geometric property of $\mu$. This result is a higher dimensional generalization of the elementary fact that the double integral of the squared distance $s$ between points is proportional to the variance of $\mu$. We relate our work to two recent algorithms, one for clustering affine subspaces and the other for Monte-Carlo SVD based on volume sampling.

1 Introduction

Our setting includes a real separable Hilbert space $H$ (with dot product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$), a Borel probability measure $\mu$ on $H$ and a fixed intrinsic dimension $d \in \N$. We assume that the support of $\mu$ is bounded. Let $AG_d(H)$ denote the affine Grassmannian on $H$, that is, the set of all $d$-flats (i.e., $d$-dimensional affine subspaces) in $H$. The $d$-dimensional least squares (LS) error for $\mu$ is

$$e_2^2(\mu, d) = \inf_{L \in AG_d(H)} \int \text{dist}^2(x, L) \, d\mu(x),$$

where $\text{dist}(x, L)$ denotes the distance of $x \in H$ to $L$.

We form functions $c : H^{d+2} \to \R$, whose integrals approximate $e_2^2(\mu, d)$. Denoting an arbitrary element of $H^{d+2}$ by $X = (x_0, \ldots, x_{d+1})$ and viewing it as a $(d+1)$-simplex in $H$, we express the desired comparison as follows.

$$e_2^2(\mu, d) \approx \int_{H^{d+2}} c^2(X) \, d\mu^{d+2}(X)$$

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(i.e., the ratios of the LHS and RHS of (2) are bounded by constants, which may depend on \( \mu \)). Some of these functions are obtained by appropriate scaling of \((d+1)\)-volumes. We denote by \( M_{d+1}(X) \) the \((d+1)\)-volume of any of the parallelopipeds generated by the vertices of \( X \). We also denote the diameter of \( X \) by \( \text{diam}(X) \), i.e., the maximal edge length. An example of such a function \( c \) is obtained by scaling \( M_{d+1}(X) \) by a power of the diameter, i.e.

\[
c_{\text{vol}}(X) = \frac{M_{d+1}(X)}{\text{diam}^d(X)}.
\]

We refer to such functions as geometric condition numbers (GCNs), since they measure the geometric conditioning of the simplex \( X \) by a quantity that scales like the diameter of the simplex. The smaller they are the flatter, i.e., better-conditioned, the simplex is.

When \( d = 0 \), (2) reduces to an elementary though useful identity, which we exemplify for the GCN \( c_{\text{vol}} \). In this case, the best approximating 0-flat (i.e., best approximating point) is the mean, \( \int x \, d\mu \), and \( e_2^2(\mu, 0) \) is the variance of \( \mu \), that is,

\[
e_2^2(\mu, 0) = \int \left\| x - \int x \, d\mu(x) \right\|^2 \, d\mu(x) = \frac{1}{2} \int \left\| x_1 - x_2 \right\|^2 \, d\mu(x_1) \, d\mu(x_2).
\]

Moreover,

\[
c_{\text{vol}}(x_1, x_2) = \|x_1 - x_2\|
\]

and consequently,

\[
e_2^2(\mu, 0) = \frac{1}{2} \int c_{\text{vol}}^2(x_1, x_2) \, d\mu(x_1) \, d\mu(x_2).
\]

Since our GCNs (of \( d + 2 \) variables) are constant multiples of the pairwise distance when \( d = 0 \), this identity extends to all of them (with possibly a different multiplicative constant).

This paper generalizes (4) to higher dimensional approximations and obtains estimates like (2) for various GCNs. This generalization restricts the type of measure \( \mu \) by various conditions (depending on the GCN). Our weakest condition, which we refer to as \( d \)-separation tries to avoid the concentration of \( \mu \) around a subspace of dimension lower than \( d \) (see Section 5.1 for precise definition).

This investigation is partly motivated by the analysis of a recent spectral clustering method for data sampled from multiple subspaces [4, 5]. The goodness of clustering for this method depends on the averaged GCN within each cluster and the theory developed here interprets this dependence in terms of the \( d \)-dimensional LS errors within clusters. We also relate our study to some aspects of volume-based sampling for fast SVD [7, 8].

Many of our techniques are rooted in the theory of uniform rectifiability [6]. In particular, notions similar to the \( d \)-separation condition have appeared before for \( d \)-regular or upper \( d \)-regular measures (see Section 6 for their definitions) in [6] Lemma 5.8, [13] Lemma 2.3, [22] Lemma 8.2 and [15] Proposition 3.1. Moreover, differently scaled functions of \( d + 2 \) variables, referred to as discrete curvatures, were studied in [18, 13, 14, 15] for \( d \)-regular measures. For example, while \( M_{d+1}(X) \) is scaled by \( \text{diam}^d(X) \) to produce the geometric condition number \( c_{\text{vol}} \), it can be scaled differently to obtain the following discrete curvature:

\[
C_{\text{vol}}(X) = \frac{M_{d+1}(X)}{\text{diam}^{(d+1)^2}(X)}.
\]
It follows from [14, 15] that for \(d\)-regular measures the integral of \(C_{\text{vol}}^2\) is comparable to the Jones-type flatness, which adds up appropriately normalized \(d\)-dimensional LS errors of certain balls of different radii centered at different locations. Another type of scaling of \(M_{d+1}\) (or more precisely, an equivalent variant of it) appeared in [21] for exploring different geometric properties of the underlying measure.

### 1.1 Structure of This Paper and Additional Results

In Section 2 we introduce notational conventions. In Section 3 we verify the existence of a LS \(d\)-flat minimizing the error \(e_2(\mu, d)\) and construct it in terms of the singular value decomposition of a special operator, which we refer to as the data-to-features operator. In Section 4 we introduce \(d\)-dimensional GCNs of \(d+2\) variables, in addition to \(c_{\text{vol}}\). Section 5 controls \(e_2^2(\mu, d)\) from above by integrals of these GCNs, whereas Section 6 bounds \(e_2^2(\mu, d)\) from below by these integrals and thus concludes the desired comparisons. In Section 7, we form \(d\)-dimensional GCNs of both \(d+1\) and \(d\) variables, and we establish their comparisons. We also relate there our work to that of Deshpande et al. [7, 8]. Section 8 puts this work in a statistical context by relating our results to clustering affine subspaces as well as extending some of the previous comparisons with high probability to the corresponding empirical quantities estimated from i.i.d. samples from \(\mu\). We discuss further implications and possible extensions in Section 9.

### 2 Notational Conventions

#### 2.1 Comparisons

For real-valued functions \(f\) and \(g\), we let \(f \preceq g\) denote the existence of \(C > 0\) such that \(f \leq C \cdot g\). Similarly, \(f \approx g\) if \(f \preceq g\) and \(g \preceq f\). The constants may depend on some arguments of \(f\) and \(g\), which we indicate if they are unclear from the context.

#### 2.2 Simplices

Fixing \(n \in \mathbb{N}, n \geq 2\), we represent \(n\)-simplices in \(H\) by ordered \((n+1)\)-tuples of the product space, \(H^{n+1}\). We denote an element of \(H^{n+1}\) by \(X = (x_0, \ldots, x_n)\) and for \(0 \leq i \leq n\): \((X)_i = x_i\) denotes the projection of \(X\) onto its \(i^{\text{th}}\) \(H\)-valued coordinate (or vertex). For \(0 \leq i < j \leq n, y, z \in H\) and \(X \in H^{n+1}\) as above, we form the following elements:

\[
X(i) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),
\]

\[
X(y, i) = (x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n),
\]

The minimal edge length of \(X\) is denoted by \(\min(X)\). We define the following quantities of \(X\) with respect to its zeroth coordinate \(x_0\):

\[
\max_{x_0}(X) = \max_{1 \leq j \leq n} \|x_j - x_0\| \quad \text{and} \quad \min_{x_0}(X) = \min_{1 \leq j \leq n} \|x_j - x_0\|.
\]

For \(X\) such that \(\min(X) \neq 0\), let

\[
scale_{x_0}(X) = \frac{\min_{x_0}(X)}{\max_{x_0}(X)}.
\]
We say that a simplex \( X \) is well-scaled at \( x_0 \) (for \( \lambda > 0 \)) if \( \min(X) > 0 \) and \( \text{scale}_{x_0}(X) \geq \lambda \).

We let \( L[X] \) denote the affine subspace of \( H \) of minimal dimension containing the vertices of \( X \). We recall that for \( n \in \mathbb{N} \), \( M_n(X) \) is the \( n \)-volume of any of the paralleloptopes generated by the vertices of \( X \). We note that

\[
M_n(X) = \text{dist}(x_i, L[X(i)]) \cdot M_{n-1}(X(i)) \quad \text{for all} \ 0 \leq i \leq d + 1.
\]

### 3 Least Squares \( d \)-Flats and Their Construction

Formally, a LS \( d \)-flat for \( \mu \) is a \( d \)-flat \( L \in \text{AG}_d(H) \), for which the RHS of (1) obtains its minimal value. We show here that such \( d \)-flats exist, i.e., the function

\[
F(L) = \int \text{dist}^2(x, L) \, d\mu(x)
\]

obtains its minimum among all \( d \)-flats \( L \) in \( \text{AG}_d(H) \). Moreover we show how to construct a LS \( d \)-flat given the singular value decomposition (SVD) of the data-to-features operator described next.

#### 3.1 The Data-to-Features Operator

We define the center of mass of \( \mu \), \( x_{cm} \), by

\[
x_{cm} = \int x \, d\mu(x)
\]

and denote by \( L_2(\mu) \) the set of functions \( f : H \to \mathbb{R} \) such that \( \int |f(x)|^2 \, d\mu(x) < \infty \). The data-to-features operator \( A_{\mu} : H \to L_2(\mu) \) is

\[
(A_{\mu}y)(x) = \langle y, x - x_{cm} \rangle \quad \text{for all} \quad x, y \in H.
\]

We use the name “data-to-features” operator since if \( \mu \) is an atomic measure supported on \( N \) “data points” in \( H = \mathbb{R}^D \), then \( A_{\mu} \) is represented by an \( N \times D \) matrix whose rows are the data points, shifted by their center of mass. Therefore, in this case \( A_{\mu} \) maps data points in \( \mathbb{R}^D \) into \( N \)-dimensional feature vectors (containing coefficients according to the dictionary of shifted data points). We remark that the dependence of \( A_{\mu} \) on \( \mu \) is not only due to the use of \( x_{cm} \), but also because the range of \( A_{\mu} \) is in \( L_2(\mu) \).

Next, we specify a kernel associated with \( A_{\mu} \) and use it to conclude that \( A_{\mu} \) is Hilbert-Schmidt. Let us arbitrarily fix an orthonormal basis of \( H \), \( \{e_n\}_{n \in \mathbb{N}} \), and express \( A_{\mu} \) as follows:

\[
(A_{\mu}y)(x) = \sum_{n \in \mathbb{N}} \langle y, e_n \rangle \langle e_n, x - x_{cm} \rangle \quad \text{for all} \quad x, y \in H.
\]

We can thus view it as operator from \( \ell_2 \) (with the counting measure \( \mu_\sharp \)) to \( L_2(\mu) \) with the kernel \( k(x, n) = \langle e_n, x - x_{cm} \rangle \). We note that this kernel is in \( L_2(\mu_\sharp \times \mu) \), indeed, using the fact that the support of \( \mu \) is bounded we obtain that

\[
\int \sum_{n \in \mathbb{N}} |\langle e_n, x - x_{cm} \rangle|^2 \, d\mu(x) = \int \|x - x_{cm}\|^2 \, d\mu(x) < \infty.
\]
We thus conclude that $A_\mu$ is Hilbert-Schmidt and in particular compact (see e.g., [12, Section 4]). Since $A_\mu$ is compact, we can apply its SVD [23, Section 1.6.2]. We denote the singular values of $A_\mu$ repeated according to multiplicities by $\{\sigma_i\}_{i \in \mathbb{N}}$. Their corresponding right vectors are denoted by $\{v_i\}_{i \in \mathbb{N}}$. Equivalently, these are the orthonormal eigenvectors of $A_\mu^* A_\mu$ ($A_\mu^*$ is the adjoint of $A_\mu$) with eigenvalues $\{\sigma_i^2\}_{i \in \mathbb{N}}$. In Section 3.2 we apply the finiteness of $\sum_{i \in \mathbb{N}} \sigma_i^2$, which is equivalent to the Hilbert-Schmidt property of $A_\mu$.

### 3.2 Least Squares $d$-Flats by SVD of the Data-to-Features Operator

We use the SVD of $A_\mu$ to construct a LS $d$-flat and express its corresponding error as follows:

**Proposition 3.1.** A LS $d$-flat for $\mu$ exists and is obtained by

\[ x_{cm} + \text{Sp}\{v_1, \ldots, v_d\}, \]

where $v_1, \ldots, v_d$ are the top right vectors of the data-to-features operator $A_\mu$. It is unique if and only if $\sigma_d > \sigma_{d+1}$. Moreover,

\[ e_2(\mu, d) = \sqrt{\sum_{i>d} \sigma_i^2}. \]  

(16)

**Proof.** We express the function $F(L)$ of (11) in terms of a shift vector $c \in H$, a linear subspace $V \subseteq H$ and also in terms of the orthogonal projection of $H$ onto the orthogonal complement of $V$, which we denote by $P_V$. That is,

\[ F(L) \equiv F(c, V) = \int \text{dist}^2(x, c + V) \, d\mu(x) = \int \|P_V(x - c)\|^2 \, d\mu(x). \]  

(17)

We further note that

\[ F(c, V) = \int \|P_V(x - x_{cm})\|^2 \, d\mu(x) + \|P_V(c - x_{cm})\|^2 \, d\mu(x). \]  

(18)

We thus conclude that the vector $c = x_{cm}$ minimizes $F(c, V)$ independently of $V$ (more generally, the set of minimizers is $x_{cm} + V$).

We next note that

\[ \min_V \int \|P_V(x - x_{cm})\|^2 \, d\mu(x) = \int \|x - x_{cm}\|^2 \, d\mu(x) - \max_V \int \|P_V(x - x_{cm})\|^2 \, d\mu(x), \]  

(19)

where $P_V$ is the projection operator of $H$ onto $V$. Therefore, instead of minimizing $F(x_{cm}, V)$, we maximize the function

\[ G(V) = \int \|P_V(x - x_{cm})\|^2 \, d\mu(x) = \text{trace}(P_V A_\mu^* A_\mu P_V^*). \]  

(20)

The last equality in (20) is evident due to the following expression of the adjoint operator $A_\mu^*$: $L_2(\mu) \to H$:

\[ A_\mu^* f = \int (x - x_{cm}) f(x) \, d\mu(x) \text{ for all } f \in L_2(\mu). \]  

(21)
Indeed, if \( \{e_n\}_{n=1}^{\dim(V)} \) is an orthonormal basis of \( V \) and \( 1 \leq n \leq \dim(V) \), then
\[
\langle e_n, P_V A^* A \mu P_V^* e_n \rangle = \int \langle e_n, x - x_{cm} \rangle^2 d\mu(x).
\] (22)

Thus, summing both the LHS and RHS over \( n = 1, \ldots, \dim(V) \) we obtain the desired equality.

At last, we apply a theorem by Ky-Fan \[9\] (see also \[10, Theorem 3.5\]) to conclude that the maximum of \( G \) is attained at \( V := \text{Sp}\{v_1, \ldots, v_d\} \), where \( v_1, \ldots, v_d \) are the top eigenvectors of \( A^* A \mu \) and it is unique if and only if \( \sigma_d > \sigma_{d+1} \). That is, \( x_{cm} + \text{Sp}\{v_1, \ldots, v_d\} \) is a LS \( d \)-flat and unique whenever \( \sigma_d > \sigma_{d+1} \). Furthermore,
\[
e^2_2(\mu, d) = \min_{c, V} F(c, V) = \text{trace}(A^* A \mu) - \max_{V} \text{trace}(P_V A^* A \mu P_V^*) = \sum_{i>d} \sigma_i^2.
\]

4 Examples of Geometric Condition Numbers on \( H^{d+2} \)

In addition to the GCN \( c_{vol} \) defined in (3), we suggest four other GCNs of \( d + 2 \) variables. Two of these squared GCNs are also scaled versions of this volume. The first one has the form
\[
c_{vol,\mu}(X) = \frac{M_{d+1}(X)}{\text{diam}^2(\mu)}.
\] (23)

The second one uses the \( d \)-dimensional polar sine \[16\]. For \( 0 \leq i \leq d + 1 \), the polar sine of \( X = (x_0, \ldots, x_{d+1}) \) with respect to the coordinate \( x_i \) is
\[
p_d\sin_{x_i}(X) = \begin{cases} \frac{M_{d+1}(X)}{\prod_{0\leq j\leq d+1 \atop j \neq i} \|x_j - x_i\|}, & \text{if } \min(X) > 0; \\ 0, & \text{otherwise}. \end{cases}
\] (24)

The corresponding polar GCN has the form:
\[
c_{pol}(X) = \text{diam}(X) \sqrt{\frac{\sum_{i=0}^{d+1} p_d\sin_{x_i}^2(X)}{d+2}}.
\] (25)

Another GCN is obtained by the \( d \)-dimensional LS error of the empirical measure associated with \( X \) as follows:
\[
c_{dls}(X) = \min_{L \in A G_d(H)} \sqrt{\frac{\sum_{i=0}^{d+2} \text{dist}^2(x_i, L)}{d+2}}.
\] (26)

At last, we form the minimal height GCN:
\[
c_{ht}(X) = \min_{0 \leq i \leq d+1} \text{dist}(x_i, L[X(i)]).
\] (27)

We note that this GCN is practically comparable to an \( \ell_\infty \) version of the \( \ell_2 \) GCN, \( c_{dls} \). One can also form \( \ell_p \) versions of such GCNs for all \( 1 \leq p < \infty \), i.e., taking the \( p \)-th root of the average of \( p \)-th powers of the distances.
The five GCNs on $H^{d+2}$ of this paper satisfy a variety of pointwise comparisons. For example, via the product formula of [10], as well as [14] eqs. (16), (118) for arbitrary $X$ we have that

$$c_{\text{vol,}\mu}(X) \leq c_{\text{vol}}(X) \leq c_{\text{hit}}(X) \leq \frac{(d + 2)^{\frac{3}{2}}}{\sqrt{2}} \cdot c_{\text{dls}}(X).$$  \tag{28}

Furthermore, from the definitions above we also have that

$$c_{\text{vol,}\mu}(X) \leq c_{\text{vol}}(X) \leq c_{\text{pol}}(X).$$  \tag{29}

In order to control integrals of $c_{\text{pol}}$ by integrals of $c_{\text{dls}}$, we will use the following inequality of [14, Proposition 3.2]:

$$\text{diam}(X) \cdot \text{p$_{d\sin_0}$}(X) \leq \sqrt{2} \cdot (d + 1) \cdot (d + 2)^{\frac{3}{2}} \cdot \frac{1}{\text{scale}_{x_0}(X)} \cdot c_{\text{dls}}(X),$$  \tag{30}

where $\text{scale}_{x_0}(X)$ was defined in [9].

5 Upper Bounds on $e_2^2(\mu, d)$

5.1 On $d$-Separated Measures

The $d$-separated measures form the weakest class of probability measures for which we can bound $e_2^2(\mu, d)$ by integrals of squared GCNs. Let $\text{supp}(\mu)$ denote the support of $\mu$ and $\text{diam}(\mu)$ denote the diameter of this support. We say that a $d$-simplex $X = (x_0, \ldots, x_d) \in \text{supp}(\mu)^{d+1}$ is $d$-separated (for $\omega > 0$) if

$$\text{M}_d(X) \geq \omega \cdot \text{diam}(\mu)^d. \tag{31}$$

We say that the measure $\mu$ is $d$-separated (with positive constants $\omega$ and $\epsilon$) if there exist sets $V_i \subseteq \text{supp}(\mu)$, $0 \leq i \leq d$, that support $d$-separated $d$-simplices in the following way:

1. $\mu(V_i) \geq \epsilon$ for each $0 \leq i \leq d$. \hspace{1cm} (32)
2. $\prod_{i=0}^{d} V_i \subseteq \left\{ X \in \text{supp}(\mu)^{d+1} : \text{M}_d(X) \geq \omega \cdot \text{diam}(\mu)^d \right\}. \tag{33}$

The sets $V_i$ can be taken to be balls but this is not necessary and can be too restrictive.

We also say that $\mu$ is $d$-separated with respect to the center of mass of $\mu$, $x_{\text{cm}}$, or equivalently centrally $d$-separated, if there exist sets $V_i \subseteq \text{supp}(\mu)$, $1 \leq i \leq d$, satisfying (32) for $1 \leq i \leq d$ as well as the following modification of (33):

$$\prod_{i=1}^{d} V_i \times \{ x_{\text{cm}} \} \subseteq \left\{ X \in \text{supp}(\mu)^{d+1} : \text{M}_d(X) \geq \omega \cdot \text{diam}(\mu)^d \right\}. \tag{34}$$

The following lemma shows that $d$-separation is a very general quantitative property in terms of information about $e_2(\mu, d - 1)$. 

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Lemma 5.1.1. If $\mu$ is an arbitrary Borel probability measure on $H$, then the following statements are equivalent:

1. $\mu$ is $d$-separated.
2. There exists a $d$-simplex $X \in \text{supp}(\mu)^{d+1}$ such that $M_d(X) > 0$.
3. $e_2(\mu, d-1) > 0$.
4. $\mu$ is centrally $d$-separated.

Proof. The equivalence of the first two statements of the lemma immediately follows from the continuity of $M_d$ and the following elementary observation (where $B(x, r)$ is the closed ball centered at $x$ of radius $r$):

$$\text{supp}(\mu) = \{x \in H : \mu(B(x, r)) > 0 \text{ for all } r > 0\}.$$ 

To establish the equivalence of the second and third statements we first note that one direction is trivial. That is, $e_2(\mu, d-1) = 0$ implies that $M_d(X) = 0$ for all $X \in \text{supp}(\mu)^{d+1}$, since all vertices will be trapped in a $(d-1)$-dimensional minimizing space $L$.

The other direction, i.e., $e_2(\mu, d-1) > 0$ implies that $M_d(X) > 0$ for some $X \in \text{supp}(\mu)^{d+1}$, can be established by noting that for any affine space $L$ of dimension lesser than or equal to $d-1$ we have that

$$\int \text{dist}^2(x, L) \, d\mu(x) \geq e_2^2(\mu, d-1).$$

Using this observation we can construct $d$ points, $x_0, \ldots, x_{d-1} \in \text{supp}(\mu)$ such that for the $(d-1)$-simplex $X(d) = (x_0, \ldots, x_{d-1})$ we have that $M_{d-1}(X(d)) > 0$. Since

$$\int \text{dist}^2(x, L[X(d)]) \, d\mu(x) > 0,$$

we can select another point $x_d \in L[X(d)]^c \cap \text{supp}(\mu)$ (where $L[X(d)]^c$ is the complement of $L[X(d)]^c$) and taking $X = X(x_d, d) \in \text{supp}(\mu)^{d+1}$ we conclude that $M_d(X) > 0$.

The equivalence between the third and the fourth statements is proven in exactly the same way (recalling that $x_{cm}$ is contained in any LS $d$-flat).

5.2 The Main Theorem for Upper Bounds on $e_2^2(\mu, d)$

Since all GCNs with $d+2$ variables suggested here control $c_{\text{vol}, \mu}$ (see (28)), we only need to bound $e_2^2(\mu, d)$ by an integral of $c_{\text{vol}, \mu}$.

Theorem 5.1. If $\mu$ is $d$-separated for the positive constants $\omega$ and $\epsilon$, then

$$e_2^2(\mu, d) \leq \frac{1}{\omega^2 \cdot \epsilon} \int_{H^{d+2}} c_{\text{vol}, \mu}^2(X) \, d\mu^{d+2}(X).$$

(34)

Proof. We arbitrarily fix $\bar{X}(d+1) = (\bar{x}_0, \ldots, \bar{x}_d) \in \prod_{i=0}^d V_i$, where $\{V_i\}_{i=0}^d$ are the sets of (32) and (33) defining the $d$-separated measure $\mu$ (with constants $\omega$ and $\epsilon$). It follows from both (33) and (10) that for $\bar{X}(d+1)$ and $\bar{X}(y, d+1)$ as in (6) and (7),

$$c_{\text{vol}, \mu}^2(\bar{X}(y, d+1)) = \left(\frac{M_{d+1}(\bar{X}(y, d+1))}{\text{diam}^d(\mu)}\right)^2 \geq \omega^2 \cdot \text{dist}^2(y, L[\bar{X}(d+1)])$$

(35)
and consequently
\[ \frac{1}{\omega^2} \int_H c_{\text{vol}}^2(\tilde{X}(y, d + 1)) \, d\mu(y) \geq \int_H \text{dist}^2(y, L[\tilde{X}(d + 1)]) \, d\mu(y) \geq c_2^2(\mu, d). \]  
(36)

For \(0 < \rho < \infty\) let
\[ \mathcal{E}_\rho = \left\{ \tilde{X}(d + 1) \in \prod_{i=0}^d V_i : \int_H c_{\text{vol}}^2(\tilde{X}(y, d + 1)) \, d\mu(y) \leq \rho \int_H c_{\text{vol}}^2(X) \, d\mu^{d+2}(X) \right\}. \]  
(37)

By Chebyshev’s inequality we have
\[ \mu^{d+1}(\mathcal{E}_\rho) \geq \mu^{d+1}(\prod_{i=0}^d V_i) - \frac{1}{\rho}. \]

Then, taking \(\rho > \frac{1}{c_{\text{vol}}}\) forces \(\mathcal{E}_\rho \neq \emptyset\). Thus, restricting \(\tilde{X}(d + 1)\) to \(\mathcal{E}_\rho\) for \(\rho > \frac{1}{c_{\text{vol}}}\), and combining equations (36) and (37), we conclude that the inequality of Theorem 5.1 holds with the controlling constant \(\rho/\omega^2\). Since this holds for arbitrary such \(\rho\) we obtain the constant given in Theorem 5.1.

6 Lower Bounds for \(c_2^2(\mu, d)\)

We first verify a lower bound on \(c_2^2(\mu, d)\) by an integral of \(c_{\text{dls}}^2\). Since the GCNs \(c_{\text{vol}}, c_{\text{vol}}, c_{\text{ht}}\) are controlled by \(c_{\text{dls}}\) (see (29)), this bound also holds for all of these GCNs.

**Proposition 6.1.** If \(\mu\) is an arbitrary Borel probability measure on \(H\), then
\[ \int_{H^{d+2}} c_{\text{dls}}^2(X) \, d\mu^{d+2}(X) \leq c_2^2(\mu, d). \]  
(38)

**Proof.** For any fixed \(d\)-flat \(L \in \text{AG}_d(H)\), by the definition of the GCN \(c_{\text{dls}}(X)\) and a subsequent application of Fubini’s Theorem we obtain that
\[ \int_{H^{d+2}} c_{\text{dls}}^2(X) \, d\mu^{d+2}(X) \leq \frac{1}{d + 2} \sum_{i=0}^{d+1} \int_{H^{d+2}} \text{dist}^2((X)_i, L) \, d\mu^{d+2}(X) = \int_H \text{dist}^2(x, L) \, d\mu(x). \]

The proposition is concluded by taking the infimum over all \(L \in \text{AG}_d(H)\).

A lower bound on \(c_2^2(\mu, d)\) in terms of an integral of the GCN \(c_{\text{pol}}^2\) requires the following notions of regularity of \(\mu\). For \(\gamma > 0\), we say that \(\mu\) is \(\gamma\)-regular if there exists a \(C \geq 1\) such that
\[ \frac{t^\gamma}{C} \leq \mu(B(x, t)) \leq C \cdot t^\gamma \quad \text{for all } x \in \text{supp}(\mu), \quad 0 < t \leq \text{diam}(\mu). \]  
(39)

We say that \(\mu\) is \(\gamma\)-upper-regular if the upper bound of (39) holds. We call the minimal such constant \(C\) satisfying (39) (or its right hand side for upper-regular measures) the regularity constant of \(\mu\).

Using these notions we formulate the following lower bound on \(c_2(\mu, d)\) and verify it in the following section.
Theorem 6.1. If \( \mu \) satisfies either one of the following conditions: \( \mu \) is \( \gamma \)-upper-regular for \( \gamma > 2 \) or \( \mu \) is \( \gamma \)-regular for \( \gamma > 1 \) with \( d = 1 \), then

\[
\int_{H^{d+2}} c_{\text{pol}}^2(X) \, d\mu^{d+2}(X) \lesssim e^2_2(\mu, d),
\]

where the comparison only depends on \( d \), \( \text{diam}(\mu) \) and the regularity constant of \( \mu \).

6.1 Proof of Theorem 6.1

The proof of this proposition is technically detailed, however it is based on few elementary ideas. It starts by replacing the integral of \( c_{\text{pol}}^2(X) \) with the integral of \( \text{diam}^2(X) \cdot p_d \sin^2 x_0(X) \) by using a change of variables. Next, in view of (30), \( \text{diam}(X) \cdot p_d \sin x_0(X) \) is controlled by \( c_{\text{dls}}(X) \) for well-scaled simplices at \( x_0 \) (recalling that these are simplices for which the minimal edge length at \( x_0 \) is comparable to the maximal edge length at \( x_0 \); see Section 2). Therefore, by applying Proposition 6.1, the integral of \( \text{diam}^2(X) \cdot p_d \sin^2 x_0(X) \) over well-scaled simplices is controlled by \( e^2_2(\mu, d) \).

The proof thus only requires the control of the integral of \( \text{diam}^2(X) \cdot p_d \sin^2 x_0(X) \) on poorly scaled simplices in \( H^{d+2} \) (i.e., simplices which are not well-scaled). The idea follows the procedure of geometric multipoles \( \text{[14 Section 9]} \), which uses a multiscale decomposition of the integral and finds local control according to the goodness of approximation by \( d \)-flats at different scales and locations. While in \( \text{[14]} \) we sought local control in terms of multiscale best fit \( d \)-flats, in the current work we seek local control in terms of a global best fit \( d \)-flat.

6.1.1 Preliminary Notation and Conventions

For simplicity, we assume throughout the proof that \( \text{diam}(\mu) = 1 \) and thus suppress estimates depending on \( \text{diam}(\mu) \).

For \( X \in H^{d+2} \) with \( x_0 = (X)_0 \), we frequently refer to \( \text{min}(X) \), \( \text{min}_{x_0}(X) \), \( \text{max}_{x_0}(X) \) and \( \text{scale}_{x_0}(X) \) defined in Section 2. We decompose the set of simplices with non-zero edge lengths according to the following sets indexed by \( k, i \in \mathbb{N}_0 \):

\[
S_{i,k} = \left\{ X = (x_0, \ldots, x_{d+1}) \in H^{d+2} : \text{max}_{x_0}(X) \in (1/2^{i+1}, 1/2^i] \right. \\
\left. \quad \quad \quad \quad \text{and scale}_{x_0}(X) \in (1/2^{k+1}, 1/2^k] \right\}. \tag{41}
\]

We will mainly use their following subsets:

\[
S'_{i,k} = \left\{ X \in S_{i,k} : \text{min}_{x_0}(X) = \|x_1 - x_0\| \text{ and } \text{max}_{x_0}(X) = \|x_2 - x_0\| \right\}.
\]

For \( x_0 \in H \) and \( \ell \in \mathbb{N}_0 \) we denote the annulus centered at \( x_0 \) and of “radius” \( 1/2^\ell \) by

\[
A(x_0, \ell) = \{ x \in H : 1/2^{\ell+2} < \|x - x_0\| \leq 1/2^\ell \}, \tag{42}
\]

and we note that for all \( X \in S_{i,k} \) and fixed \( 1 \leq j \leq d + 1 \) we have that

\[
(X)_j \in A(x_0, \ell) \text{ for some } i \leq \ell \leq i + k \text{ depending on } X \text{ and } j. \tag{43}
\]
6.1.2 Case I: $\mu$ Upper-Regular for $\gamma > 2$

We decompose the integral of $c_{\text{pol}}$ by using the sets of (11) and applying symmetry properties of the polar sine:

$$\int_{H^{d+2}} c_{\text{pol}}^2(X) \, d\mu^{d+2}(X) = \int_{H^{d+2}} \diam^2(X) \, p_d \sin^2 x_0(X) \, d\mu^{d+2}(X) =$$

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \int_{S_{i,k}} \diam^2(X) \, p_d \sin^2 x_0(X) \, d\mu^{d+2}(X) =$$

$$d \cdot (d+1) \int_{\cup_{i=0}^{\infty} S'_{i,0}} \diam^2(X) \, p_d \sin^2 x_0(X) \, d\mu^{d+2}(X). \quad (44)$$

The elements of the last double sum of (44) that correspond to $k = 0$ can be controlled by combining (30) (where here scale$_{x_0}(X) \geq 1/2$) and Proposition 6.1, thus obtaining

$$\sum_{i=0}^{\infty} \int_{S'_{i,0}} \diam^2(X) \, p_d \sin^2 x_0(X) \, d\mu^{d+2}(X) = \int_{\cup_{i=0}^{\infty} S'_{i,0}} \diam^2(X) \, p_d \sin^2 x_0(X) \, d\mu^{d+2}(X) \leq c_2^2(\mu, d). \quad (45)$$

We now find sufficient bounds for the other terms in the last double sum (44) to obtain convergence in $i$ and $k$. Applying (30) to a fixed term on the last double sum of (44) we obtain the following bound for an arbitrary $d$-flat $L$:

$$\int_{S'_{i,k}} \diam^2(X) \cdot p_d \sin^2 x_0(X) \, d\mu^{d+2}(X) \leq$$

$$\sum_{j=0}^{d+1} \int_{S'_{i,k}} \frac{\dist^2(x_j, L)}{\scale^2_{x_0}(X)} \, d\mu^{d+2}(X) \leq 2^{2k} \cdot \sum_{j=0}^{d+1} \int_{S'_{i,k}} \dist^2(x_j, L) \, d\mu^{d+2}(X). \quad (46)$$

We claim that for all $0 \leq j \leq d+1$:

$$\inf_{L \in \AG_d(H)} \int_{S'_{i,k}} \dist^2(x_j, L) \, d\mu^{d+2}(X) \leq (1/2^{k+1})^\gamma (1/2^d)^{\gamma - d} \cdot c_2^2(\mu, d). \quad (47)$$

To see this it is sufficient to integrate with respect to $x_j$ last, and depending on the index $j$, to vary the order of integration of the other variables slightly. If $j > 0$ then we take the integration with respect to $x_0$ as the second to last integration. If $j > 1$, then we integrate with respect to $x_1$, then $x_0$ and then finally $x_j$. Following this procedure (47) clearly follows from the combination of (43) with the upper-regularity. Indeed, the factor $(1/2^{k+1})^\gamma$ arises from the integration over the coordinate $x_1$ if $j \neq 1$ and $x_0$ if $j = 1$, $(1/2^d)^{\gamma - d}$ from the rest of coordinates excluding $x_j$ and clearly $c_2^2(\mu, d)$ from the coordinate $x_j$.

Applying (47) to the RHS of (46), we obtain that

$$\int_{S'_{i,k}} \diam^2(X) \cdot p_d \sin^2 x_0(X) \, d\mu^{d+2}(X) \leq 1/2(\gamma - 2)^k \cdot 1/2^{\gamma(d+1)-d} \cdot c_2^2(\mu, d). \quad (48)$$
Finally, combining (44), (45) and (48) we conclude that

\[
\int_{H^{d+2}} \text{diam}^2(X) p_d \sin_{x_0}^2 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \int_{S_{i,k}} \text{diam}^2(X) p_d \sin_{x_0}^2 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left(1 + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2} \gamma^{-2} \cdot 1/2 \gamma^{(d+1)\cdot i} \right) \cdot e_2^2(\mu, d).
\]

Since the coefficient on the RHS above is clearly finite for \( \gamma > 2 \), the proposition is thus proved for the current case.

### 6.1.3 Case II: \( \mu \) is \( \gamma \)-Regular for \( \gamma > 1 \) and \( d = 1 \)

Since \( d = 1 \) we work with triangles \( X = (x_0, x_1, x_2) \in H^3 \) and we need to prove that

\[
\int_{H^3} \text{diam}^2(X) \sin_{x_0}^2 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \int_{S_{i,k}} \text{diam}^2(X) p_d \sin_{x_0}^2 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left(1 + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2} \gamma^{-2} \cdot 1/2 \gamma^{(d+1)\cdot i} \right) \cdot e_2^2(\mu, 1). \tag{49}
\]

The procedure here is similar to that of Section 6.1.2 however we must use an inequality for the sine function that holds with high probability for a \( \gamma \)-regular \( \mu \) with \( \gamma > 1 \). We clarify this as follows.

For fixed \( X = (x_0, x_1, x_2) \in H^3 \), \( X(u, 1) \) as in (7), and \( C \geq 1 \), let

\[
U(X, C) := \{ u \in H : |\sin_{x_0}(X)| \leq C \cdot |\sin_{x_0}(X(u, 1))| \}, \tag{50}
\]

and for \( \alpha > 0 \) let \( A_{\alpha}(X, C) \) denote the restriction of \( U(X, C) \) to an annulus:

\[
A_{\alpha}(X, C) := U(X, C) \cap B(x_0, \max_{x_0}(X)) \setminus B(x_0, \alpha \cdot \max_{x_0}(X)). \tag{51}
\]

The following lemma shows that the defining inequality of (50) occurs with high probability (we delay its proof to Section 6.1.4).

**Lemma 6.1.1.** If \( \mu \) is \( \gamma \)-regular for \( \gamma > 1 \) with regularity constant \( C_\mu \), and the constants \( C_0 \) and \( \alpha_0 \) are such that

\[
C_0 \geq \frac{1}{2} \cdot \left(4 \cdot 5^{\gamma/2} \cdot C_\mu^2\right)^{1/\gamma} \quad \text{and} \quad 0 < \alpha_0 \leq (4 \cdot C_\mu^2)^{-1/\gamma}, \tag{52}
\]

then the following inequality holds uniformly for all \( X \in \text{supp}(\mu)^3 \):

\[
\mu(A_{\alpha_0}(X, C_0)) \geq \frac{1}{2} \cdot \mu(B(x_0, \max_{x_0}(X))).
\]

For the rest of this section we use the optimal values of the constants \( C_0 \) and \( \alpha_0 \) in (52) (i.e., the lower bound for \( C_0 \) and upper bound for \( \alpha_0 \)). We decompose all triangles with non-zero edge lengths into the sets

\[
S_k = \{ X \in H^3 : \text{scale}_{x_0}(X) \in (\alpha_0^{k+1}, \alpha_0^k) \},
\]

for \( k \geq 1 \), and we denote

\[
S'_k = \{ X \in S_k : \max_{x_0}(X) = \|x_2 - x_0\| \} \subset S_k.
\]
By the symmetry of $|\sin_{x_0}(X)|$ with respect to $x_1$ and $x_2$, we note that

$$
\int_{H^3} \text{diam}^2(X) \cdot \sin^2_{x_0}(X) \, d\mu^3(X) = \sum_{k=0}^{\infty} \int_{S_k} \text{diam}^2(X) \cdot \sin^2_{x_0}(X) \, d\mu^3(X) \leq 2 \sum_{k=0}^{\infty} \int_{S'_k} \text{diam}^2(X) \cdot \sin^2_{x_0}(X) \, d\mu^3(X). \quad (53)
$$

We note that $X \in S'_0$ is well-scaled (for $\alpha_0$), and by combining (30) and Proposition 6.1 we see that

$$
\int_{S'_0} \text{diam}^2(X) \cdot \sin^2_{x_0}(X) \, d\mu^3(X) \lesssim c_2^2(\mu, 1).
$$

We now use Lemma 6.1.1 to control the individual terms for $k \geq 1$ on the RHS of (53). We arbitrarily fix $X \in S'_k$ and define the probability measure

$$
\tilde{\mu}_X := \frac{\mu_{|A_{\alpha_0}(X, C_0)}}{\mu(A_{\alpha_0}(X, C_0))}.
$$

We note that for any $y \in A_{\alpha_0}(X, C_0)$:

$$
\text{diam}^2(X) \cdot \sin^2_{x_0}(X) \leq C_0^2 \cdot \text{diam}^2(X(y, 1)) \cdot \sin^2_{x_0}(X(y, 1)),
$$

and consequently for $X(y, 1)$ as in (7),

$$
\text{diam}^2(X) \cdot \sin^2_{x_0}(X) \leq C_0^2 \int_{A_{\alpha_0}(X, C_0)} \text{diam}^2(X(y, 1)) \cdot \sin^2_{x_0}(X(y, 1)) \, d\tilde{\mu}_X(y). \quad (54)
$$

Since the triangle $X(y, 1)$ is well-scaled for each $y \in A_{\alpha_0}(X, C_0)$, we can apply the inequality of (30) to the integrand on the RHS of (54) to obtain

$$
\text{diam}^2(X) \cdot \sin^2_{x_0}(X) \lesssim \text{dist}^2(x_0, L) + \int_{A_{\alpha_0}(X, C_0)} \text{dist}^2(y, L) \, d\tilde{\mu}_X(y) + \text{dist}^2(x_2, L) \quad (55)
$$

for any $L \in \text{AG}_1(H)$, where the constant of the inequality is independent of $k$.

Fixing the line $L$, the middle term on the RHS of (55) trivially has the bound

$$
\int_{A_{\alpha_0}(X, C_0)} \text{dist}^2(y, L) \, d\tilde{\mu}_X(y) \leq \frac{1}{\mu(A_{\alpha_0}(X, C_0))} \int_{H} \text{dist}^2(x, L) \, d\mu(x). \quad (56)
$$

Thus, applying (55) to (53), and then (56) to (55), for an arbitrary line $L$ we have

$$
\int_{S'_k} \text{diam}^2(X) \cdot \sin^2_{x_0}(X) \, d\mu^3(X) \lesssim
\int_{S'_k} \text{dist}^2(x_0, L) \, d\mu^3(X) + \int_{H} \text{dist}^2(x, L) \, d\mu(x) \cdot \int_{S'_k} \frac{d\mu^3(X)}{\mu(A_{\alpha_0}(X, C_0))} + \int_{S'_k} \text{dist}^2(x_2, L) \, d\mu^3(X). \quad (57)
$$
We bound the terms of (57) separately. The first term satisfies
\[
\int_{S_k'} \frac{\text{dist}^2(x_0, L)}{\mu^3(A_{\alpha_0}(X, C_0))} \leq \int_{H^2} \text{dist}^2(x_0, L) \left( \int_{B(x_0, \alpha_0 \cdot \|x_2-x_0\|)} \frac{d\mu(x_1)}{\mu^2(A_{\alpha_0}(X, C_0))} \right) d\mu(x_2) d\mu(x_0) \lesssim \alpha_0^{k-\gamma} \cdot \int_H \text{dist}^2(x, L) d\mu(x). (58)
\]
A similar computation gives the same bound for the third term.

Then, by Lemma 6.1.1 and the regularity of \(\mu\) we have that \(\mu(A_{\alpha_0}(X, C_0)) \geq \max_{x_0}(X)^\gamma\), and thus the second term of (57) satisfies
\[
\int_H \text{dist}^2(x, L) d\mu(x) \cdot \int_{S_k'} \frac{\mu^3(X)}{\mu(A_{\alpha_0}(X, C_0))} \lesssim \int_H \text{dist}^2(x, L) d\mu(x) \cdot \left( \int_{S_k'} \frac{\mu^3(X)}{\|x_2-x_0\|^\gamma} \right) \lesssim \alpha_0^{k-\gamma} \cdot \int_H \text{dist}^2(x, L) d\mu(x). (59)
\]
Applying (58) and (59) to the terms of (57) we have the bound
\[
\int_{S_k'} \text{diam}^2(X) \cdot \sin^2_{x_0}(X) d\mu^3(X) \lesssim \alpha_0^{k-\gamma} \cdot \int_H \text{dist}^2(x, L) d\mu(x). (60)
\]
Taking an infimum over \(L \in \text{AG}_1(H)\) on the RHS of (60), and then summing this inequality over \(k \geq 1\) we see that the proposition holds.

6.1.4 **Proof of Lemma 6.1.1**

Equation (52) is a direct consequence of the following two equations:
\[
\mu(B(x_0, \max_{x_0}(X)) \setminus B(x_0, \alpha_0 \cdot \max_{x_0}(X))) \geq \frac{3}{4} \cdot \mu(B(x_0, \max_{x_0}(X))) \geq \frac{3}{4} \cdot \mu(B(x_0, \max_{x_0}(X))) \geq \frac{3}{4} \cdot \mu(B(x_0, \max_{x_0}(X))) (61)
\]
and
\[
\mu(U(X, C_0)) \geq \frac{3}{4} \cdot \mu(B(x_0, \max_{x_0}(X)). (62)
\]
The inequality of (61) follows from the \(\gamma\)-regularity of \(\mu\) and the constant \(\alpha_0\). Indeed,
\[
\mu(B(x_0, \max_{x_0}(X)) \setminus B(x_0, \alpha_0 \cdot \max_{x_0}(X))) \geq \mu(B(x_0, \max_{x_0}(X))) - C_\mu \cdot \alpha_0^\gamma \cdot \max_{x_0}(X)^\gamma \geq \frac{3}{4} \cdot \mu(B(x_0, \max_{x_0}(X)).
\]
We conclude by proving (62). We form the tube of radius \(\max_{x_0}(X)/C_0\) on the line \(L[X(1)],\)
\[
T_{\text{ube}}(L[X(1)], \max_{x_0}(X)/C_0) = \{ y : \text{dist}(y, L[X(1)]) \leq \max_{x_0}(X)/C_0 \},
\]
then
\[
\mu(U(X, C_0)) \geq \frac{3}{4} \cdot \mu(B(x_0, \max_{x_0}(X)).
\]
and note that
\[(T_{\text{ube}} (L[X(1)], \max_{x_0}(X)/C_0))^c \cap B(x_0, \max_{x_0}(X)) \subseteq U(X, C_0). \tag{63}\]
Indeed, since \(|\sin_{x_0}(X(u, 1))| \cdot \|u - x_0\| = \text{dist}(u, L[X(1)])\) we have the following lower bound for any \(u \in B(x_0, \max_{x_0}(X)):\)
\[|\sin_{x_0}(X(u, 1))| \cdot \max_{x_0}(X) \geq \text{dist}(u, L[X(1)]). \tag{64}\]
Applying \(64\) to \(u \in (T_{\text{ube}} (L[X(1)], \max_{x_0}(X)/C_0))^c \cap B(x_0, \max_{x_0}(X))\), we obtain that
\[C_0 \cdot |\sin_{x_0}(X(u, 1))| \geq 1 \geq |\sin_{x_0}(X)|,\]
i.e., \(u \in U(X, C_0)\) and \(63\) is concluded.

At last, we show that
\[\mu (T_{\text{ube}} (L[X(1)], \max_{x_0}(X)/C_0) \cap B(x_0, \max_{x_0}(X))) \leq \frac{1}{4} \cdot \mu (B(x_0, \max_{x_0}(X)) \tag{65}\]
and combining it with \(63\) we establish \(62\). We first note that the intersection of the tube \(T_{\text{ube}} (L[X(1)], \max_{x_0}(X)/C_0)\) with \(B(x_0, \max_{x_0}(X))\) can be covered by at most \(2 \cdot C_0\) balls of radius \(\sqrt{5} \cdot \max_{x_0}(X)/(2 \cdot C_0)\) and thus
\[\mu (T_{\text{ube}} (L[X(1)], \max_{x_0}(X)/C_0) \cap B(x_0, \max_{x_0}(X))) \leq C_0^2 \cdot 5^{1/2} \cdot (2 \cdot C_0)^{1-\gamma} \cdot \mu (B(x_0, \max_{x_0}(X)). \tag{66}\]
Equation \(65\) and consequently \(62\) follows by combining \(52\) with \(60\).

### 7 GCNs on \(H^{d+1}\) and \(H^d\) and their Corresponding Comparisons

#### 7.1 \(d\)-Dimensional GCNs of Only \(d + 1\) Variables

If one knows a point that lies on a LS \(d\)-flat, then any of the above GCNs can be reduced to a function of only \(d + 1\) variables by arbitrarily fixing one of the original variables at that point. We exemplify this idea with the center of mass, \(x_{\text{cm}}\), which lies on the LS \(d\)-flat (see Proposition 3.1) and later explain how to extend it to other points.

We consider the set of \((d + 1)\)-simplices with a fixed vertex at \(x_{\text{cm}}\), that is, we define the set \(H_{x_{\text{cm}}}^{d+1} = \{x_{\text{cm}}\} \times H_{x_{\text{cm}}}^{d+1}\), and we restrict our attention to the elements
\[X = (x_{\text{cm}}, x_1, \ldots, x_{d+1}) \in H_{x_{\text{cm}}}^{d+1}. \tag{67}\]
As such, we can replace any GCN, \(c : H^{d+2} \rightarrow \mathbb{R}\), by \(c(X) : H_{x_{\text{cm}}}^{d+1} \rightarrow \mathbb{R}\) and establish the relevant comparisons as follows, where \(\mu^{d+1}\) on \(H_{x_{\text{cm}}}^{d+1}\) is clearly on the set \(H^{d+1}\).

**Proposition 7.1.** If \(\mu\) is centrally \(d\)-separated (for \(\omega\) and \(\epsilon\)) then
\[e_2^2(\mu, d) \leq \frac{1}{\omega^2 \cdot \epsilon^d} \int_{H_{x_{\text{cm}}}^{d+1}} c_{\text{vol}, \mu}^2(X) \, d\mu^{d+1}(X). \tag{68}\]
If on the other hand \( \mu \) satisfies either one of the following conditions: \( \mu \) is \( \gamma \)-upper-regular for \( \gamma > 2 \) or \( \mu \) is \( \gamma \)-regular for \( \gamma > 1 \) with \( d = 1 \), then

\[
\int_{H^{d+1}_{x_{cm}}} c_{\text{pol}}^2(X) \, d\mu^{d+1}(X) \lesssim c_2^2(\mu, d),
\]

where the comparison only depends on \( d, \text{diam}(\mu) \) and the regularity constant of \( \mu \).

Moreover, if \( \mu \) is an arbitrary Borel probability measure on \( H \), then

\[
\int_{H^{d+1}_{x_{cm}}} c_{\text{dis}}^2(X) \, d\mu^{d+1}(X) \leq c_2^2(\mu, d).
\]

Proof. The proofs of (68) and (69) are identical to those of (34) and (40) respectively, while they also use the fact that \( x_{cm} \) lies in the LS \( d \)-flat.

In order to prove (70), we apply (28) and obtain the following for any fixed \( L \in \text{AG}_d(H) \):

\[
\int_{H^{d+1}_{x_{cm}}} c_{\text{dis}}^2(X) \, d\mu^{d+1}(X) \leq \frac{1}{d+2} \left( \sum_{i=1}^{d+1} \int_{H^{d+1}_{x_{cm}}} \text{dist}^2(x_i, L) \, d\mu^{d+1}(X) + \int_{H^{d+1}_{x_{cm}}} \text{dist}^2(x_{cm}, L) \, d\mu^{d+1}(X) \right).
\]

Since the function \( \text{dist}^2(\cdot, L) \) is convex for fixed \( L \), we apply Jensen’s Inequality to the last term on the RHS and then Fubini’s Theorem to all terms and thus conclude (70).

We note that when using a fixed point \( y \) on the LS \( d \)-flat instead of \( x_{cm} \), then few modifications are needed. First of all, the minimizations defining both \( e_2(\mu, d) \) and \( c_{\text{dis}} \) need to be restricted to subspaces in \( \text{AG}_d(H) \) containing the point \( y \). Also, \( d \)-separation needs to be defined w.r.t. \( y \) (instead of \( x_{cm} \)). When clustering \( d \)-dimensional linear subspaces, the LSCC algorithm (linear SCC) \[5, 4\] applies such a strategy with \( y = 0 \), which obviously lies on all linear subspaces.

7.2 A \( d \)-Dimensional GCN of Only \( d \) Variables

The work of Deshpande et al. \[7, 8\] suggests a GCN on \( H^d \), which we denote by \( c_{\text{Dsh}}^2 \). The idea is to look at the geometry of \( d \)-simplices having the center of mass, \( x_{cm} \), as a fixed vertex. As such, we make the definition \( H^d_{x_{cm}} = \{x_{cm}\} \times H^d \), and we consider \( d \)-simplices

\[
\tilde{X} = (x_{cm}, x_1, \ldots, x_d) \in H^d_{x_{cm}}.
\]

We note that \( \tilde{X} \in H^d_{x_{cm}} \) is simply the projection of some \( X \in H^{d+1}_{x_{cm}} \), i.e. \( \tilde{X} = X(d+1) \).

We define the square of \( c_{\text{Dsh}}^2(\tilde{X}) \) in the following way:

\[
c_{\text{Dsh}}^2(\tilde{X}) = \frac{M_d^2(\tilde{X}) \int_H \text{dist}^2(y, L[\tilde{X}]) \, d\mu(y)}{\int_{H^d_{x_{cm}}} M_d^2(\tilde{Y}) \, d\mu^d(\tilde{Y})},
\]

where \( \mu^d \) on \( H^d_{x_{cm}} \) is clearly taken on the set \( H^d \). For a fixed \( \tilde{X} \in H^d_{x_{cm}} \), the GCN \( c_{\text{Dsh}}^2(\tilde{X}) \) is simply the average squared volume of \((d+1)\)-simplices having \( \tilde{X} \) as a \( d \)-dimensional face, divided by the
average squared volume of \( d \)-simplices with the center of mass as a vertex. This follows directly from (10).

The comparison of \( e_2^2(\mu, d) \) and the integral of \( c_{\text{Dsh}}^2 \) is established as follows.

**Theorem 7.1.** If \( \mu \) is centrally \( d \)-separated with compact support, then

\[
e_2^2(\mu, d) \leq \frac{1}{\omega^2 \cdot e^d} \int_{H^d_{\text{xcm}}} c_{\text{Dsh}}^2(\tilde{X}) \, d\mu^d(\tilde{X}).
\]

If on the other hand \( \mu \) is an arbitrary Borel probability measure on \( H \), then

\[
\int_{H^d_{\text{xcm}}} c_{\text{Dsh}}^2(\tilde{X}) \, d\mu^d(\tilde{X}) \leq e_2^2(\mu, d).
\]

**Proof.** In order to simplify the argument, we introduce a related GCN on \((d + 1)\)-simplices with a vertex fixed at \( x_{\text{cm}} \). That is we look at \((d + 1)\)-simplices per (67), i.e., \( X \in H_{x_{\text{cm}}}^{d+1} \), and the GCN \( c_{\text{vol,Dsh}}^2(\tilde{X}) \), whose square is defined by

\[
c_{\text{vol,Dsh}}^2(\tilde{X}) = \frac{M_{d+1}^2(X)}{\int_{H_{x_{\text{cm}}}^{d+1}} M_d^2(\tilde{Y}) \, d\mu^d(\tilde{Y})}.
\]

Per (10) and Fubini’s Theorem we see that the corresponding integrals of the two squared GCNs, \( c_{\text{Dsh}}^2(\tilde{X}) \) and \( c_{\text{vol,Dsh}}^2(\tilde{X}) \), are equal, i.e.,

\[
\int_{H^d_{\text{xcm}}} c_{\text{Dsh}}^2(\tilde{X}) \, d\mu^d(\tilde{X}) = \int_{H_{x_{\text{cm}}}^{d+1}} c_{\text{vol,Dsh}}^2(\tilde{X}) \, d\mu^{d+1}(\tilde{X}).
\]

We will thus prove Theorem 7.1 with the simpler GCN \( c_{\text{vol,Dsh}}^2(\tilde{X}) \). Theorem 9.1 will immediately follow from our estimates below.

We first note that (71) follows from (68) and (74), as well as the following fact:

\[
c_{\text{vol}}(X) \leq c_{\text{vol,Dsh}}(X) \quad \forall X \in H_{x_{\text{cm}}}^{d+1}.
\]

In order to prove (72) we generalize [7, Lemma 3.1] to our continuous setting by proving the identity:

\[
\int_{H_{x_{\text{cm}}}^{d+1}} M_{d+1}^2(X) \, d\mu^{d+1}(X) = \sum_{1 \leq t_1 < ... < t_{d+1}} \sigma_{t_1}^2 \cdots \sigma_{t_{d+1}}^2,
\]

where \( \{\sigma_i\}_{i \in \mathbb{N}} \) are the singular values of the data-to-features operator \( A_\mu \). We obtain (75) by expanding the expression \( \det(I + \lambda A_\mu A_\mu^*) \) in \( \lambda \in \mathbb{R} \) in two different ways and equating the corresponding coefficients.

We first apply [10, Theorem IV.6.1] to obtain that

\[
\det(I + \lambda A_\mu A_\mu^*) = \prod_{j \in \mathbb{N}} (1 + \lambda \sigma_j^2) = 1 + \sum_{k \in \mathbb{N}} \sum_{j_1 < ... < j_k \in \mathbb{N}} \sigma_{j_1}^2 \cdots \sigma_{j_k}^2 \cdot \lambda^k.
\]
Next, in view of (21) we express the operator $A_\mu A_\mu^*: L^2(\mu) \to L^2(\mu)$ as follows:

$$\left( A_\mu A_\mu^* f \right)(y) = \int (x - x_{cm}, y - x_{cm}) f(x) \, d\mu(x), \text{ for all } f \in L^2(\mu) \text{ and } y \in H$$

(77)

so that it has the kernel

$$k(y, x) = (x - x_{cm}, y - x_{cm}).$$

(78)

By adapting the proof of [10, Theorem VI.1.1] to the operator $\lambda A_\mu A_\mu^*$ with the kernel $\lambda k(y, x)$ and the compactly supported measure $\mu$ we obtain that

$$\det(I + \lambda A_\mu A_\mu^*) = 1 + \sum_{m \in \mathbb{N}} \frac{\lambda^m}{m!} \int_{H^m} \det(\{k(x_i, x_j)\}_{i,j=1}^m) \, d\mu(x_1) \ldots \, d\mu(x_m) =$$

$$1 + \sum_{m \in \mathbb{N}} \lambda^m \int_{H_{x_{cm}}^m} M^2_m(Y) \, d\mu^m(Y),$$

(79)

where $H_{x_{cm}}^m = \{ x_{cm} \} \times H^m$. Equation (75) thus immediately follows from both (76) and (79).

We will also use the following immediate estimate:

$$\sum_{1 \leq t_1 < \ldots < t_{d+1}} \sigma^2_{t_1} \ldots \sigma^2_{t_{d+1}} \leq \sum_{1 \leq t_1 < \ldots < t_d} \sigma^2_{t_1} \ldots \sigma^2_{t_d} \sum_{j=d+1}^\infty \sigma^2_j.$$  

(80)

Now, combining (16), (75) and (80), we get that

$$\int_{H_{x_{cm}}^{d+1}} M^2_{d+1}(X) \, d\mu^{d+1}(X) \leq \int_{H_{x_{cm}}^d} M^2_d(\tilde{X}) \, d\mu^d(\tilde{X}) \cdot e_2^2(\mu, d),$$

that is,

$$\int_{H_{x_{cm}}^{d+1}} c_{vol,Dsh}^2(X) \, d\mu^{d+1}(X) \leq e_2^2(\mu, d)$$

and combining it with (74) we conclude (72) and thus Theorem 7.1.

8 Statistical Relevance of This Work

8.1 Application to the Problem of Clustering Subspaces

The identity of (14) is useful for clustering algorithms based on pairwise distances (see e.g., [3]). Similarly, the approximate identities of this paper are also useful for clustering algorithms based on higher-order correlations [11, 1, 20, 5, 4, 2]. The latter algorithms are designed to cluster intersecting subspaces or manifolds, where the former algorithms fail. For example, the Spectral Curvature Clustering (SCC) [5, 4] is an algorithm for clustering $d$-dimensional affine subspaces. It assigns to any $d + 2$ data points, $x_1, \ldots, x_{d+2}$, the affinity, $e^{-c_{pol}(x_1, \ldots, x_{d+2})/2\sigma_2}$, where $c_{pol}$ is the polar GCN and $\sigma$ is a positive tuning parameter that can be estimated from the data. It then organizes these affinities in a matrix whose spectral properties provide the clusters. We remark
that \( c_{\text{pol}} \) was referred to in [3, 4] as curvature, instead of GCN, and this resulted in the algorithm’s name SCC.

The results of the current paper have been used to justify the SCC algorithm [4]. More precisely, [4] assumed data sampled from a mixture of subspaces corrupted by sufficiently small noise and showed that the underlying subspaces could be recovered with sufficiently large probability and small error. This error was controlled by two terms: a sum of within-clusters errors scaled by \( \sigma^2 \) (where \( \sigma \) is the tuning parameter used to define the affinities) and between-clusters interaction.

The control of the first term (involving within-clusters errors) was established by some of the theory proved here. This theory is simpler and more general than the one referred to in [4, Section 2.3].

8.2 From Estimates in Expectation to Estimates in High Probability

We extend the comparisons of the two expected quantities (i.e, LS error, which is the expectation of \( \text{dist}^2(x, L) \), and the expectation of squared GCNs) to comparisons of their estimators obtained by i.i.d. samples from \( \mu \). That is, assume \( N \)-valued i.i.d. random variables drawn from \( \mu \), denoted by \( X_1, \ldots, X_N \). We can estimate the LS error and any of the integrals of squared GCNs (assume for simplicity \( c_{\text{dls}}^2 \)) as follows:

\[
e^2_2(X_1, \ldots, X_N; d) = \frac{1}{N} \min_{L \in \mathcal{A}_d(H)} \sum_{i=1}^{N} \text{dist}^2(X_i, L)
\]

and

\[
c_{\text{dls}}^2(X_1, \ldots, X_N; d) = \frac{1}{N^{d+2}} \sum_{X=(x_1, \ldots, x_{d+2})} c_{\text{dls}}^2(X),
\]

The following theorem shows that these two quantities are comparable to each other with high probability of sampling.

**Theorem 8.1.** If \( \mu \) is \( d \)-separated (for \( \omega \) and \( \epsilon \)), \( X_1, \ldots, X_N \) are \( N \)-valued i.i.d. random variables drawn from \( \mu \), then for any \( 0 < \delta < 1 \) and

\[
\kappa = \frac{\delta}{(d+2) \cdot \text{diam}(\mu)^2} \int_{H^{d+2}} c_{\text{dls}}^2(X) \, d\mu^{d+2}(X),
\]

the following estimate holds with probability \( 1 - 2 \cdot e^{-2N\kappa^2} \):

\[
c_{\text{dls}}^2(X_1, \ldots, X_N; d) \leq e^2_2(X_1, \ldots, X_N; d) \leq \frac{1 + \delta}{1 - \delta} \cdot \frac{1}{\omega^2 \cdot e^{d+1}} \cdot c_{\text{dls}}^2(X_1, \ldots, X_N; d).
\]

Moreover, for any \( 0 < \delta < \epsilon \) the following estimate holds with probability \( 1 - (d + 1) \cdot e^{-2N\delta^2} \):

\[
c_{\text{dls}}^2(X_1, \ldots, X_N; d) \leq e^2_2(X_1, \ldots, X_N; d) \leq \frac{1}{\omega^2 \cdot (\epsilon - \delta)^{d+1}} \cdot c_{\text{dls}}^2(X_1, \ldots, X_N; d).
\]

**Proof.** The LHS inequality of both (83) and (84) is proved identically to (38) and in fact is a deterministic inequality.
We first verify the RHS inequality of (83) by estimating with probability the integral quantities by their discrete counterparts (via concentration inequalities). In order to estimate the integral of \( c_{\text{dls}}^2(x_1, \ldots, x_N; d) \) we fix \( 1 \leq i \leq N \) and note that the number of additive terms in \( c_{\text{dls}}^2(x_1, \ldots, x_N; d) \) that contain \( x_i \) is \((d + 2) \cdot P(N - 1, d + 1)\), where \( P(N - 1, d + 1) \) denotes the permutations of \( d + 1 \) elements out of \( N - 1 \). Moreover, each of these terms is between 0 and \( \text{diam}(\mu)^2/N^{d+2} \). Consequently,

\[
\sup_{x_1, \ldots, x_N, \hat{x}_i} |c_{\text{dls}}^2(x_1, \ldots, x_i, \ldots, x_N; d) - c_{\text{dls}}^2(x_1, \ldots, \hat{x}_i, \ldots, x_N; d)| \leq (d + 2) \cdot \text{diam}(\mu)^2/N.
\] (85)

Applying McDiarmid’s inequality [19] with the underlying condition expressed in (85) we obtain that for any \( \beta > 0 \):

\[
\mu^N \left( \int_{H^{d+2}} c_{\text{dls}}^2(X) \text{d}\mu^{d+2}(X) - c_{\text{dls}}^2(x_1, \ldots, x_N; d) \geq \beta \right) \leq e^{-2N\beta^2/((d+2)^2\cdot\text{diam}(\mu)^4)}.
\] (86)

Setting

\[
\beta = \delta \int_{H^{d+2}} c_{\text{dls}}^2(X) \text{d}\mu^{d+2}(X),
\] (87)

we rewrite (86) as follows:

\[
\mu^N \left( c_{\text{dls}}^2(x_1, \ldots, x_N; d) \leq (1 - \delta) \int_{H^{d+2}} c_{\text{dls}}^2(X) \text{d}\mu^{d+2}(X) \right) \leq e^{-2N\beta^2/((d+2)^2\cdot\text{diam}(\mu)^4)}.
\] (88)

In order to estimate \( e_2(\mu, d) \) by \( e_2(x_1, \ldots, x_N; d) \) we note that

\[
e_2^2(x_1, \ldots, x_N; d) \leq \frac{1}{N} \sum_{i=1}^{N} \text{dist}^2(x_i, \hat{L}),
\] (89)

where \( \hat{L} \) is a fixed LS \( d \)-flat for \( \mu \). Applying Hoeffding’s inequality to the function on the RHS of (89), we obtain that

\[
\mu^N \left( \frac{1}{N} \sum_{i=1}^{N} \text{dist}^2(x_i, \hat{L}) - e_2^2(\mu, d) \geq \beta \right) \leq e^{-2N\beta^2/\text{diam}(\mu)^4}.
\] (90)

By further use of (88) and (87), we reduce (90) to the following probabilistic inequality:

\[
\mu^N \left( \frac{1}{N} \sum_{i=1}^{N} \text{dist}^2(x_i, \hat{L}) \geq (1 + \delta) \cdot e_2^2(\mu, d) \right) \leq e^{-2N\beta^2/\text{diam}(\mu)^4}.
\] (91)

The RHS inequality of (83) thus follows from the combination of (84), (88), (89) and (91).

Next, we prove the RHS inequality of (84) by showing that \( d \)-separation of \( \mu \) is maintained (for \( \omega \) and \( \epsilon' \), where \( \epsilon' < \epsilon \)) with overwhelming probability by i.i.d. random variables sampled from \( \mu \). We arbitrarily fix \( j = 1, \ldots, d + 1 \) and \( i = 1, \ldots, N \) and form the random variable \( I_{i,j} \) by the formula:

\[
I_{i,j}(x) = I_{x_i \in V_j}(x) \quad \text{for all} \ x \in H,
\]
where \( I \) is indicator function and \( \{ V_j \}_{j=1}^{d+1} \) are the sets used in defining the \( d \)-separation of \( \mu \). We note that
\[
\int \mathcal{I}_{i,j}(x) \, d\mu(x) = \mu(V_j) > \epsilon.
\]
Combining this observation with Hoeffding’s inequality we obtain that
\[
\mu^N \left( - \sum_{i=1}^{N} \mathcal{I}_{i,j} / N + \epsilon \geq \delta \right) \leq \mu^N \left( - \sum_{i=1}^{N} \mathcal{I}_{i,j} / N + \mu(V_j) \geq \delta \right) \leq e^{-2N\delta^2}.
\]
Consequently,
\[
\mu^N \left( \bigcap_{j=1}^{d+1} \left( \sum_{i=1}^{N} \mathcal{I}_{i,j} / N > \epsilon - \delta \right) \right) \geq 1 - (d + 1) \cdot e^{-2N\delta^2}.
\]
That is, with probability \( 1 - (d + 1) \cdot e^{-2N\delta^2} \) the empirical measure \( \mu_N(A) = \sum_{i=1}^{N} I_A(X_i) / N \) is \( d \)-separated for the parameters \( \omega \) and \( \epsilon - \delta \) and the same sets \( \{ U_j \}_{j=1}^{d+1}, \{ V_j \}_{j=1}^{d+1} \). For each such instance of \( d \)-separation of the empirical measure, we apply Theorem 5.1 to \( \mu_N \). That is, for a fixed sample \( X_1, \ldots, X_N \) whose empirical measure \( \mu_N \) is \( d \)-separated with these sets and constants \( \omega \) and \( \epsilon - \delta \), we have the following inequality which is simply Theorem 5.1 applied to \( \mu_N \):
\[
e_2^2(X_1, \ldots, X_N; d) \leq \frac{1}{\omega^2 \cdot (\epsilon - \delta)^{d+1} \cdot e_{dl}^2(X_1, \ldots, X_N; d)}.
\]
This inequality holds for all samples with probability \( 1 - (d + 1) \cdot e^{-2N\delta^2} \) and the RHS inequality of (84) is thus concluded.

9 Discussion

We presented examples of \( d \)-dimensional geometric condition numbers whose integrals are comparable to the \( d \)-dimensional least squares error for certain classes of measures. We related these results to the problem of clustering subspaces and to volume-based sampling for Monte-Carlo SVD. We discuss here further implications and open directions.

9.1 Comparisons of \( L_p \) Errors

For simplicity we only discussed LS errors, i.e., \( L_2 \) errors. Nevertheless, \( L_p \) errors for \( 1 \leq p < \infty \) can also be estimated using \( p \)-th powers of the GCNs.

9.2 Approximate Identities for Singular Values

Some of the approximate identities established in this paper can be translated to approximate identities involving singular values of certain operators. We exemplify this claim for the data-to-features operator as follows.
Theorem 9.1. If $\mu$ is centrally $d$-separated (for $\omega$ and $\epsilon$) with compact support and $\{\sigma_i\}_{i \in \mathbb{N}}$ are the singular values of the data-to-features operator, then

$$\omega^2 \cdot \epsilon^d \sum_{j=d+1}^{\infty} \sigma_j^2 \leq \sum_{1 \leq i_1 < \cdots < i_{d+1}} \sigma_{i_1}^2 \cdots \sigma_{i_{d+1}}^2 \leq \sum_{j=d+1}^{\infty} \sigma_j^2. \quad (92)$$

We note that the inequality on the RHS of (92) is trivial for any set of numbers $\{\sigma_i\}_{i \in \mathbb{N}}$. The LHS comparability is an immediate corollary of Theorem 7.1 (in view of (73)-(75)).

9.3 More Robust Notion of $d$-Separation

Our notion of $d$-separation is not sufficiently “robust to outliers” since it depends on $\text{diam}(\mu)$. Assume, e.g., a probability measure which is a mixture of one component supported in the unit ball and another component of an atomic measure supported on an arbitrarily far point with a sufficiently small weight. The diameter of this measure is mainly determined by the outlying atomic measure. However, for $X \in H^d_{xcm}$ and the GCN $c_{Dsh}(X)$ (or $X \in H^d_{xcm}$ and the GCN $c_{vol,Dsh}(X)$) we can weaken the effect of outliers by replacing the condition $M_d(X) \geq \omega \cdot \text{diam}(\mu)^d$ with

$$M_d^2(X) \geq \omega \int_{H^d_{xcm}} M_d^2(\tilde{Y}) \, d\mu^d(\tilde{Y}). \quad (93)$$

9.4 On $d$-Separation w.r.t. $(d+1)$-Simplices and Its Implications

A different notion of $d$-separation was previously used in the setting of $d$-regular measures on $H$ \cite{13,15}. It is based on $d$-separation of $(d+1)$-simplices (instead of $d$-simplices). We adapt this notion to the current setting and explain its relation with $d$-separation defined here, we also describe its implications.

We say that a $(d+1)$-simplex $X = (x_0, \ldots, x_{d+1}) \in \text{supp}(\mu)^{d+2}$ is $d$-separated (for $\omega$) if all of its faces are $d$-separated as $d$-simplices (for $\omega$). That is,

$$\min_{0 \leq i \leq d+1} M_d(X(i)) \geq \omega \cdot \text{diam}(\mu)^d. \quad (94)$$

We say that $\mu$ is $d$-separated w.r.t. $(d+1)$-simplices (with positive constants $\omega$, $\epsilon$ and $\tau$) if there exist sets $V_i \subseteq U_i \subseteq \text{supp}(\mu), 0 \leq i \leq d+1$, such that for each $0 \leq i \leq d+1$:

1. $\mu(V_i) \geq \epsilon$.
2. $\text{dist}_\mu(V_i, U_i^c) := \inf_{x \in V_i \cap \text{supp}(\mu)} \inf_{y \in U_i^c \cap \text{supp}(\mu)} \|x - y\| \geq \tau \cdot \text{diam}(\mu)$.
3. $\prod_{i=0}^{d+1} U_i \subseteq \{X \in \text{supp}(\mu)^{d+2} : \min_{0 \leq i \leq d+1} M_d(X(i)) \geq \omega \cdot \text{diam}(\mu)^d \}$.

In view of Lemma 5.1.1 and its proof $d$-separation is almost identical to $d$-separation w.r.t. $(d+1)$-simplices. The typical example of a $d$-separated measure which is not $d$-separated with respect to $(d+1)$-simplices is a measure supported on $d+1$ atoms with positive $d$-volume. One can add another part of the support lying on a $(d-1)$-flat containing $d$ of these atoms and provide this way additional examples.
Nevertheless, the extra care taken in defining \(d\)-separation w.r.t. \(d\)-simplices is necessary in formulating the following stronger version of Theorem 5.1, which restricts the integral of \(c_{\text{vol,} \mu}^2(X)\) to the following set of simplices with sufficiently large edge lengths (with respect to \(\tau\)):

\[
LE_\tau(\mu) = \left\{ X \in \text{supp}(\mu)^{d+2} : \min(X) \geq \tau \cdot \text{diam}(\mu) \right\}.
\]

**Theorem 9.2.** If \(\mu\) is \(d\)-separated (for \(\omega\), \(\epsilon\) and \(\tau\)) w.r.t. \((d+1)\)-simplices, then

\[
e_{2}^{2}(\mu, d) \leq \frac{4}{\omega^2 \cdot \epsilon^{d+1}} \left( 1 + 4 \cdot (d + 1)^2 + \frac{4 \cdot (d + 1)}{\omega^2 \cdot \epsilon} \right) \int_{LE_\tau(\mu)} c_{\text{vol,} \mu}^2(X) \, d\mu^{d+2}(X).
\]

The proof of this theorem follows the one of [15, Theorem 1.1]. This type of control was necessary in [13, 15] since singular curvature functions were used instead of GCNs and they had to be further integrated along various “scales” \(t\) w.r.t. the measure \(dt/t\). Clearly, it is not necessary in the current context.

### 9.5 Extension to Metric Spaces

It will be interesting to extend some of our results to metric spaces. In particular, by choosing appropriate metric GCNs one can obtain a corresponding notion of an approximate best-fit subspace. This task is considered in [17] for the purpose of clustering \(d\)-dimensional smooth structures in metric spaces.

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