Dimension Walks on Generalized Spaces

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This paper is dedicated to Daryl J. Daley and Robert Schaback. Such beautiful minds.

Abstract

Let $d, k$ be positive integers. We call generalized spaces the cartesian product of the $d$-dimensional sphere, $S^d$, with the $k$-dimensional Euclidean space, $\mathbb{R}^k$. We consider the class $P(S^d \times \mathbb{R}^k)$ of continuous functions $\varphi : [-1,1] \times [0,\infty) \to \mathbb{R}$ such that the mapping $C : (S^d \times \mathbb{R}^k)^2 \to \mathbb{R}$, defined as $C((x,y),(x',y')) = \varphi(\cos \theta(x,x'),\|y-y'\|)$, $(x,y), (x',y') \in S^d \times \mathbb{R}^k$, is positive definite. We propose linear operators that allow for walks through dimension within generalized spaces while preserving positive definiteness.

Keywords: positive definite functions; Montée operators; Descente operators; spheres; Euclidean spaces; generalized product.

1 Introduction

1.1 Context

The paper deals with the problem of linear projection operators that map the set of positive definite functions on a given space into the set of positive
definite functions on lower or higher dimensional space. Specifically, we con-
sider continuous functions that are positive definite over generalized spaces,
that we define here as the Cartesian product of the $k$-dimensional Euclidean
space with a $d$-dimensional unit sphere.

There are several motivations to consider this setting, and we sketch them
as follows:

a) There has been an increasing interest from several branches of statist-
ics, machine learning, and finance, for positive definite functions defined
over these product spaces, and the reader is referred to the recent review by
Porcu et al. (2020). The applications to real cases are ubiquitous, ranging
from climate and atmospheric sciences until deep learning on manifolds.

As far as finance is concerned, Gaussian processes play a central role in fi-
nancial modelling. The field of econometrics has devoted much effort to the
modelling of financial time series (Hamilton, 1994). The Brownian motion is
essential to the pricing of financial derivatives (Hull, 2003). The Ornstein-
Uhlenbeck process is often used to develop investment and trading strategies
(Lipton et al., 2020). Gaussian processes are one-dimensional applications
of the general concept of Gaussian random field (GRF). We motivate some
of the uses of GRFs in finance. A key feature of financial datasets is time
and spatial dependence. Coetaneous observations from variables in close
proximity tend to be more similar. For example, returns from U.S. stocks
last month are more similar to returns from U.S. stocks this month than
returns from U.S. stocks one year ago, or returns from Chinese stocks last
month. Willinger et al. (1999) noted that a Brownian motion with drift
does not replicate the time dependence observed in asset returns. Not sur-
prisingly, GRFs have attracted considerable interest among researchers in-
terested in modelling the joint time-space dynamics of financial processes.

To cite a few examples, Kennedy (1994) and Goldstein (2000) modelled the
term structure of interest rates as a two-dimensional random field. In their
models, time increments are independent, while the correlation structure
between bond yields of different maturities can be modelled with great flex-
ibility. Kimmel (2004) enhanced this approach by adding a state-dependent
volatility. Albeverio et al. (2004) introduced Lévy fields to the modelling
of yield curves. Özkan and Schmidt (2005) applied random fields to incor-
porate credit risk to the modelling of yield curves. As important as the
term structure of interest rates is, it is not the only financial application of
Gaussian fields. At least two further applications stand out: Option pric-
ing and actuarial modelling. For example, Hainaut et al. (2017) proposes
an alternative model for asset prices with sub-exponential, exponential and
hyper-exponential autocovariance structures. Hainaut sees price processes as conditional Gaussian fields indexed by the time. Under this framework, option prices can be computed using the technique of the change of numeraire. Biffis and Millossovich (2006) applied random fields to modelling the intensity of mortality, in an attempt to incorporate cross-generation effects. Biagini et al. (2017) built on that work to price and hedge life insurance liabilities.

b) Several branches of spatial statistics and computer sciences are interested in the simulation of random processes defined over generalized spaces, and we refer the reader to Emery (2008). It turns out that the use of these operators becomes crucial when associated with turning bands techniques (Matheron, 1973), which allow for simulation of a given random process from projections on lower dimensional spaces.

c) Projection operators for radial positive definite functions allowed to build positive definite functions that are compactly supported over balls embedded in $k$-dimensional Euclidean spaces. This inspired a fertile literature from spatial statistics with the goal of achieving accurate estimates while allowing for computational scalability. For instance, the tapering approach (Furrer et al., 2006) is substantially based on this idea.

d) There is a fertile literature from projection operators for symmetric (or radially symmetric) distributions, where radial symmetry is intended with respect to the composition of a given candidate function with the classical $\alpha$-norms (Cambanis et al., 1983).

1.2 Literature Review

Let $\mathbb{R}^k$ denote the $k$-dimensional Euclidean space, and let $\mathbb{S}^d$ be the $d$-dimensional unit sphere embedded in $\mathbb{R}^{d+1}$. Let $\| \cdot \|$ denote Euclidean distance and $\theta(x, y) := \arccos(\langle x, y \rangle)$ denote the geodesic distance in $\mathbb{S}^d$, with $\langle \cdot, \cdot \rangle$ denoting the dot product in $\mathbb{R}^{d+1}$. A continuous function $C : \mathbb{R}^k \to \mathbb{R}$ is called radially symmetric if there exists a continuous function $f : [0, \infty) \to \mathbb{R}$ such that $C(x) = f \circ \|x\|$, $x \in \mathbb{R}^k$, with $\circ$ denoting composition. The function $f$ is called the radial part of $C$. Radial symmetry is known as isotropy in spatial statistics (Daley and Porcu, 2014). A function $C : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ is called geodesically isotropic if $C(x, y) = g \circ \theta(x, y)$ for some continuous function $g : [0, \pi] \to \mathbb{R}$.

Positive definite functions that are radially symmetric over $k$-dimensional Euclidean spaces have a long history that can be traced back to Schoenberg (1938). Projections operators that map a positive definite radial mapping
from $\mathbb{R}^k$ into $\mathbb{R}^{k \pm h}$, for $h$ a positive integer, have been considered in the Matheron’s clavier spherique (Matheron, 1973, 1970). Matheron coined the terms *descente* and *montée* to define special operators that will be described throughout. The terms originate from an appealing physical interpretation in a mining context. These projections operators have then be investigated by Eaton et al. (1981), and subsequently by Wendland (1995), Schaback and Wu (1996), and by Gneiting (2002) in the context of positive definite radial functions that are additionally compactly supported on balls embedded in $\mathbb{R}^k$ with given radii. The work by Daley and Porcu (2014) provides a general perspective of such operators, in concert with some generalizations of the previously mentioned works. These linear operators have turned to be very useful to establish criteria of the Pólya type for radially symmetric positive definite functions (Gneiting, 2001), as well as in the definition of multiradial positive definite functions (Porcu et al., 2007). In probability theory, similar projection operators turned useful in the seminal paper by Cambanis et al. (1983) and in Fang (2018).

Positive definite functions that are geodesically isotropic on $d$-dimensional spheres have been characterized in Schoenberg (1942). Projection operators for this class of functions have been studied to a limited extent only, and we refer to the recent papers by Beatson et al. (2014) and more recently to the same authors (Beatson and zu Castell, 2016, 2017). Properties of these operators have then been inspected in Trübner and Ziegel (2017).

1.3 The Problem, and Our Contribution

A characterization of projection operators on product spaces of the type $\mathbb{S}^d \times \mathbb{R}^k$ has been elusive so far. The only exception being Bingham and Symons (2019), who consider the product space $\mathbb{S}^d \times \mathbb{R}$, and projections that are defined marginally for the sphere only.

Our paper contributes to the literature as follows. In Section 2 we provide the notations and basic literature. In Section 3 we define the Descente and Montée operators on the generalized space $\mathbb{S}^d \times \mathbb{R}^k$. The main results are statement in Section 4 and their proofs in Appendix A.

2 Notations and Background

Let $X, Y$ be nonempty sets. A function $C : (X \times Y)^2 \to \mathbb{R}$ is called positive definite if, for any finite system $\{a_k\}_{k=1}^N \subset \mathbb{R}$ and points $\{(x_k, y_k)\}_{k=1}^N \subset \mathbb{R}^2$, we have

\[
\sum_{i=1}^N \sum_{j=1}^N C((x_i, y_i), (x_j, y_j)) a_i a_j \geq 0.
\]
We deal with the case \( X = S^d \) and \( Y = \mathbb{R}^k \), for \( d \) and \( k \) being positive integers. Additionally, we suppose \( C \) to be continuous, and that there exists a continuous function \( \varphi : [-1, 1] \times [0, \infty) \to \mathbb{R} \) such that

\[
C((x, y), (x', y')) = \varphi\left( \cos \theta(x, x'), \|y - y'\| \right), \quad (x, y), (x', y') \in S^d \times \mathbb{R}^k.
\]

We call \( P(S^d \times \mathbb{R}^k) \) the class of such functions, \( \varphi \). Analogously, we call \( P(\mathbb{R}^k) \) the class of continuous functions \( \psi : [-1, 1] \to \mathbb{R} \) such that for the function \( C \) in (2.1) it is true that, for \( y = y' \), \( C((x, y), (x', y)) = \varphi(\cos \theta(x, x'), 0) = \psi(\cos \theta(x, x')) \). The class \( P(\mathbb{R}^k) \) is defined analogously.

The classes \( P(S^d \times \mathbb{R}^k) \) and \( P(S^d) \) have been characterized by Schoenberg (1938) and Schoenberg (1942), respectively. The class \( P(S^d \times \mathbb{R}^k) \) has been characterized by Berg and Porcu (2017) through a uniquely determined expansions of the type

\[
\varphi(x, t) = \sum_{n=0}^{\infty} f_n^d(t) C_n^{(d-1)/2}(x), \quad (x, t) \in [-1, 1] \times [0, \infty),
\]

where the functions \( f_n^d \) belong to \( P(\mathbb{R}^k) \), \( n \in \mathbb{Z}_+ \), and \( \sum_{n=0}^{\infty} f_n^d(0) C_n^{(d-1)/2}(1) < \infty. \)

The expansion above is uniformly convergent on \([-1, 1] \times [0, \infty)\). The coefficients functions \( f_n^d \) are called \( d \)-Schoenberg functions of \( \varphi \). The functions \( C_n^{(d-1)/2} \) are the Gegenbauer polynomials of degree \( n \) associated to the index \( (d - 1)/2 \) (Szegő, 1959).

Proposition 3.8 in Berg and Porcu (2017) shows that if \( \varphi \) belongs to the class \( P(S^d \times \mathbb{R}^k) \), then is continuously differentiable with respect to the first variable.

It is also important to note that a continuous function \( x \in [-1, 1] \mapsto \varphi(x, t) \) has an Abel-summable expansion for each \( t \in [0, \infty) \) in the form (see the proof of Theorem 3.3 in Berg and Porcu, 2017)

\[
\varphi(x, t) \sim \sum_{n=0}^{\infty} f_n^d(t) C_n^{(d-1)/2}(x),
\]
where
\[ f_d^n(t) = \varsigma_d \int_{-1}^{1} \varphi(x,t) C_n^{(d-1)/2}(x)(1-x^2)^{d/2-1} dx, \quad (2.4) \]
and \( \varsigma_d \) are positive constants.

### 2.1 Some Useful Facts

Arguments in Schoenberg (1938) prove that, for every \( n = 0, 1, \ldots \), each function \( f_d^n \in P(\mathbb{R}^k) \) in (2.2) admits a uniquely determined Riemann-Stieltjes integral representation of the form
\[
 f_d^n(t) = \int_0^{\infty} \Omega_k(tr)dF_n(r), \quad t \in [0, \infty), \quad (2.5)
\]
where \( F_n \) is a non negative bounded measure on \([0, \infty)\). The function \( \Omega_k : [0, \infty) \to \mathbb{R} \) is given by
\[
 \Omega_k(t) = \Gamma\left(\frac{k}{2}\right) \left(\frac{2}{t}\right)^{(k-2)/2} J_{(k-2)/2}(t), \quad (2.6)
\]
where \( J_\nu \) is the Bessel function of the first kind of order \( \nu \) given by
\[
 J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{t}{2}\right)^{2m}. \]

We follow Daley and Porcu (2014) and we call \( F_n \) the \( k \)-Schoenberg measure of \( f_d^n \). We also note that we are abusing of notation when writing \( F_n \) instead of \( F_d^n \). This last notation will not be used unless explicitly needed.

Some technicalities will be exposed here to allow for a neater exposition. The derivative function of the function \( \Omega_k \) is uniformly bounded, and it is given by (see Daley and Porcu, 2014; Gneiting, 2002; Porcu et al., 2007)
\[
 \frac{d\Omega_k}{dt}(t) = \Omega'_k(t) = -\frac{1}{k} t \Omega_{k+2}(t), \quad t \geq 0. \quad (2.7)
\]

Also,
\[
 |\Omega_k(t)| < 1 = \Omega_k(0), \quad t > 0. \quad (2.8)
\]
Since \( \lim_{t \to \infty} \Omega_k(t) = 0 \) for \( k > 0 \) (see Daley and Porcu (2014)), we have
\[
 \int_t^{\infty} u \Omega_k(u) du = (k - 2) \Omega_{k-2}(t), \quad t \geq 0. \quad (2.9)
\]
Some properties of Gegenbauer polynomials will turn to be useful throughout. For instance, we can invoke 4.7.14 in Szegő (1959) to infer that

$$\frac{dC_n^\lambda}{dx}(x) = (C_n^\lambda)'(x) = \delta_\lambda C_{n-1}^{\lambda+1}(x), \quad -1 \leq x \leq 1,$$  \hspace{1cm} (2.10)

and, as a consequence

$$\int_{-1}^{x} C_n^\lambda(x) dx = \frac{1}{\delta_\lambda}(C_{n+1}^{\lambda-1}(x) - C_{n+1}^{\lambda-1}(-1)), \hspace{1cm} (2.11)$$

where

$$\delta_\lambda = \begin{cases} 2\lambda, & \lambda > -1/2 \ (\lambda \neq 0), \\ 2, & \lambda = 0. \end{cases}$$  \hspace{1cm} (2.12)

Theorem 7.32.1 and Equation 4.7.3 in Szegő (1959) show that, for $\lambda > -1/2$,

$$|C_n^\lambda(x)| \leq C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)\Gamma(2\lambda)}, \quad x \in [-1, 1].$$  \hspace{1cm} (2.13)

Also, it is true that

$$\frac{C_{n+1}^{\lambda-k}(1)}{C_n^\lambda(1)} \leq \frac{\Gamma(2\lambda)}{\Gamma(2\lambda-2k)} := \varrho_{\lambda,k}, \quad \forall n \in \mathbb{Z}_+. \hspace{1cm} (2.14)$$

The following inequality (see Berg et al., 1984) will be repeatedly used in the manuscript:

$$|f(t)| \leq f(0), \quad t \in [0, \infty), \quad f \in \mathcal{P}({\mathbb{R}^k}).$$

We will also make use of the following fact: if $\varphi : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ has a derivative $\varphi_x$ with respect to the first variable for each $t \in [0, \infty)$ and if both functions have Gegenbauer expansions of the form

$$\varphi(x, t) \sim \sum_{n=0}^{\infty} f_n^\lambda(t)C_n^\lambda(x), \quad \varphi_x(x, t) \sim \sum_{n=0}^{\infty} \tilde{f}_{n+1}^\lambda(t)C_n^{\lambda+1}(x),$$  \hspace{1cm} (2.15)

$(x, t) \in [-1, 1] \times [0, \infty)$, then

$$\tilde{f}_{n-1}^{\lambda+1}(t) = \delta_\lambda f_n^\lambda(t), \quad n \in \mathbb{Z}_+, \quad \lambda > 0.$$  \hspace{1cm} (2.16)

The proof is very similar to the proof of Lemma 2.4 in Beatson and zu Castell (2016) and we omit it for the sake of brevity.
3 An Historical Account on Montée and Descente Operators

Beatson and zu Castell (2017) defined the Descente and Montée operators for the class $\mathcal{P}(S^d)$. Specifically, the Descente $D$ is defined as

$$(Df)(x) = \frac{d}{dx}f(x) = f'(x), \quad x \in [-1, 1],$$

provided such a derivative exists. The Montée $I$ is instead defined as

$$(If)(x) = \int_{-1}^{x} f(u)du, \quad x \in [-1, 1].$$

Beatson and zu Castell (2017) have shown that $f \in \mathcal{P}(S^{d+2})$ implies that there exists a constant, $\kappa$, such that $\kappa + If \in \mathcal{P}(S^d)$. Also, $f \in \mathcal{P}(S^d)$ implies $Df \in \mathcal{P}(S^{d+2})$. The implication in terms of differentiability at $x = 1$ are nicely summarized therein.

The tour de force by Beatson and zu Castell (2017) has then been generalized by Bingham and Symons (2019): let $d \in \mathbb{N}$ and $\varphi : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous functions. The Montée $I$ and Descente $D$ operators are defined respectively by

$$I(\varphi)(x, t) := \int_{-1}^{x} \varphi(u, t)du, \quad (x, t) \in [-1, 1] \times [0, \infty) \quad (3.1)$$

when $f$ is integrable with respect to the first variable, and

$$D(\varphi)(x, t) := \frac{\partial \varphi}{\partial x}(x, t), \quad (x, t) \in [-1, 1] \times [0, \infty). \quad (3.2)$$

They prove that if $\varphi \in \mathcal{P}(S^d \times \mathbb{R})$, then $D\varphi \in \mathcal{P}(S^{d+2} \times \mathbb{R})$ and in their correction of Theorem 2.1 they provided conditions under $\varphi \in \mathcal{P}(S^{d+2} \times \mathbb{R})$ such that $I\varphi \in \mathcal{P}(S^d \times \mathbb{R})$.

Montée and Descente operators with the class $\mathcal{P}(\mathbb{R}^k)$ have been defined much earlier, and we follow Gneiting (2002) to summarize them here. The Descente and Montée operators are respectively defined as

$$D\varphi(t) = \begin{cases} 
1, & t = 0 \\
\frac{\varphi'(t)}{t\varphi''(0)}, & t > 0,
\end{cases} \quad (3.3)$$
where $\varphi''(0)$ denotes the second derivative of $\varphi$ evaluated at $t = 0$, and

$$
\tilde{I}\varphi(t) = \int_t^\infty u\varphi(u)du \left( \int_0^\infty u\varphi(u)du \right)^{-1}.
$$

(3.4)

Gneiting (2002) proved that if $\varphi \in \mathcal{P}(\mathbb{R}^k)$, $k \geq 3$, and $u\varphi(u)$ is integrable over $[0, \infty)$, then $\tilde{I}\varphi \in \mathcal{P}(\mathbb{R}^{k-2})$. Invoking standard properties of Bessel functions in concert with direct inspection, Gneiting (2002) proved that, if $\varphi \in \mathcal{P}(\mathbb{R}^k)$ and $\varphi''(0)$ exists, then $D\varphi \in \mathcal{P}(\mathbb{R}^{k+2})$. Under mild regularity conditions, the operator $D$ and $\tilde{I}$ are inverse operators:

$$
\tilde{I}(D\varphi) = D(\tilde{I}\varphi) = \varphi.
$$

3.1 Descente and Montée Operators on Generalized Spaces

We start by defining the following Descente and Montée operators. The first is actually taken from Bingham and Symons (2019): we define the derivate operator $D_1$ by

$$
D_1\varphi(x,t) := \varphi_x(x,t) = \frac{\partial \varphi}{\partial x}(x,t), \quad (x,t) \in [-1, 1] \times [0, \infty).
$$

(3.5)

The integral operator $I_1$ is given by

$$
I_1\varphi(x,t) := \int_{-1}^{x} \varphi(u,t)du, \quad (x,t) \in [-1, 1] \times [0, \infty),
$$

(3.6)

when $\varphi(u,t)$ is integrable over $[-1, 1]$ for each $t \in [0, \infty)$.

We define

$$
D_2\varphi(x,t) := \begin{cases} 
1, & (x,t) = (1,0) \\
\frac{\varphi_t(x,t)}{t\varphi_{tt}(1,0)}, & (x,t) \in [-1, 1) \times (0, \infty) 
\end{cases}
$$

(3.7)

whenever $\varphi_{tt}(1,0) := \frac{\partial^2 \varphi}{\partial t^2}(1,0)$ exists, and

$$
I_2\varphi(x,t) := \frac{\int_{-1}^{\infty} v\varphi(x,v)dv}{\int_{0}^{\infty} v\varphi(1,v)dv}, \quad (x,t) \in [-1, 1] \times [0, \infty),
$$

(3.8)
when $v\varphi(1,v)$ is integrable over $[0,\infty)$ and provided the denominator is not identically equal to zero.

The composition between the operators defined in Bingham and Symons (2019) and Gneiting (2002) provides a new operator, that we define here as

$$I_3\varphi(x,t) := \int_t^\infty \int_{-1}^x v\varphi(u,v)\,dudv, \quad (x,t) \in [-1,1] \times [0,\infty), \quad (3.9)$$

when $v\varphi(u,v)$ is integrable over $[-1,1] \times [0,\infty)$.

Given $\kappa \in \mathbb{Z}_+$, we define the operator $I_j^\kappa$ by recurrence as:

$$I_j^0 \varphi := \varphi, \quad I_j^1 \varphi := I_j \varphi \quad \text{and} \quad I_j^\kappa \varphi := I_j(I_j^{\kappa-1} \varphi), \quad j = 1, 2, 3.$$

4 Dimension Walks within the Class $\mathcal{P}(\mathbb{S}^d \times \mathbb{R}^k)$

This section contains our original findings. Proofs are deferred to the Appendix.

4.1 Descente Operators

We start with a simple result that is an extension of Theorem 2.3 in Beatson and zu Castell (2016). In the Appendix 4.2 we provide a quick sketch of the main steps.

**Theorem 4.1.** If $\varphi : [-1,1] \times [0,\infty) \to \mathbb{R}$ belongs to $\mathcal{P}(\mathbb{S}^d \times \mathbb{R}^k)$, then $D_1\varphi$ belongs to $\mathcal{P}(\mathbb{S}^{d+2} \times \mathbb{R}^k)$.

Next result requires instead a lengthy proof and relates about the operator $D_2$.

**Theorem 4.2.** Let $d, k \in \mathbb{Z}_+, \varphi : [-1,1] \times [0,\infty) \to \mathbb{R}$ be a function in $\mathcal{P}(\mathbb{S}^d \times \mathbb{R}^k)$ and let $F_n$ be the $k$-Schoenberg measures associated with the $d$-Schoenberg functions of $\varphi$. If

(i) $\int_0^\infty r^2 dF_n(r) < \infty$, for all $n \in \mathbb{Z}_+$;

(ii) $0 < \frac{\partial^2 \varphi}{\partial t^2}(1,0) = \sum_{n=0}^\infty \int_0^\infty r^2 dF_n(r) < \infty$,

then $D_2\varphi$ belongs to $\mathcal{P}(\mathbb{S}^{d+2} \times \mathbb{R}^k)$. 

10
4.2 Montée Operators

In this section we consider functions \( \varphi : [-1, 1] \times [0, \infty) \to \mathbb{R} \) belonging to \( \mathcal{P}(S^d \times \mathbb{R}^k) \) as in (2.2) such that

\[
\int_0^\infty (1/r^{2\kappa}) dF_n(r) < \infty \quad (\kappa \in \mathbb{Z}_+^*). \tag{4.1}
\]

Thus, the functions defined by

\[
g_n^\kappa(t) := \int_0^\infty \Omega_{k-2\kappa}(tr) \frac{1}{r^{2\kappa}} dF_n(r), \quad t \in [-1, 1], \quad n, \kappa \in \mathbb{Z}_+, \tag{4.2}
\]

belong to the class \( \mathcal{P}(\mathbb{R}^{k-2\kappa}) \).

The first finding relates to the operator \( I_1 \). Again, the proof is deferred to the Appendix.

**Theorem 4.3.** Let \( k, \kappa \in \mathbb{Z}_+^* \) and \( d \) be an integer such that \( d > 2\kappa \). If \( \varphi : [-1, 1] \times [0, \infty) \to \mathbb{R} \) is a function in \( \mathcal{P}(S^d \times \mathbb{R}^k) \) such that \( u \mapsto I_1^\kappa \varphi(u, t) \) is integrable over \([-1, 1]\) for each \( t \in [0, \infty) \). Then, the function \( I_1^\kappa \varphi \) has a representation in a form of Gegenbauer series:

\[
I_1^\kappa(x, t) = \sum_{n=0}^{\infty} \tilde{f}_n^d(t) C_n^{(d-2\kappa-1)/2}(x), \quad (x, t) \in [-1, 1] \times [0, \infty), \tag{4.3}
\]

where

\[
\tilde{f}_n^d(t) := \left\{ \begin{array}{ll}
\tau_{d, \kappa} \sum_{i=0}^{\kappa-1} (-1)^i \chi_{n, i}^d f_i(t), & n = 0, 1, \ldots, \kappa - 1, \\
\tau_{d, \kappa} f_n^d(t), & n \geq \kappa.
\end{array} \right. \tag{4.4}
\]

The functions \( f_n^d \) are the \( d \)-Schoenberg functions of \( \varphi \) as in (2.2), the positive constant \( \tau_{d, \kappa} := (\prod_{j=1}^{\kappa} \delta_{(d-2j+1)/2})^{-1} \) and the coefficients

\[
\left\{ \begin{array}{l}
\chi_i^{0, d, \kappa} := \sum_{j=1}^{\kappa-1} (-1)^{j+1} \chi_i^{j-1, d, \kappa-1} C_j^{(d-2\kappa-1)/2}(1) - (-1)^{\kappa+1} C_i^{(d-2\kappa-1)/2}(1), \\
\chi_i^{n, d, \kappa} := \chi_i^{n-1, d, \kappa-1}, & n = 1, 2, \ldots, \kappa - 1,
\end{array} \right. \tag{4.5}
\]
satisfy
\[
|\chi_{i}^{n,d,\kappa}| \leq \gamma^{n,d,\kappa}C_{i}^{(d-1)/2}(1), \quad n = 0, 1, \ldots, \kappa - 1, \quad i \in \mathbb{Z}_{+}, \quad (4.6)
\]
where, for each \( n = 0, 1, \ldots, \kappa - 1 \) and \( \kappa \in \mathbb{Z}_{+}^{*} \), \( \gamma^{n,d,\kappa} \) is a positive constant that depends only on \( d \). Moreover, \( \sum_{n=0}^{\infty} \tilde{f}_{n,d,\kappa}(0)C_{n}^{(d-2\kappa-1)/2}(1) < \infty \).

**Corollary 4.4.** Under the conditions of Theorem 4.3, there exists a bounded function \( H^{\kappa} \) on \([-1, 1] \times [0, \infty)\) such that \( H^{\kappa} + I_{1}^{\kappa}\varphi \) belongs to \( \mathcal{P}(S^{d-2\kappa} \times \mathbb{R}^{k}) \).

**Remark 4.5.** Direct inspection shows that \( \chi_{i}^{0,d,\kappa} \geq 0 \), for \( \kappa = 1, 2 \). Therefore, \( \chi_{i}^{1,d,\kappa}, \chi_{i}^{2,d,\kappa}, \ldots, \chi_{i}^{\kappa-1,d,\kappa} \geq 0 \) for all \( \kappa \geq 2 \) and \( i \in \mathbb{Z}_{+} \).

**Remark 4.6.** By Remark 4.5, if \( f_{n}^{d+1} \equiv 0 \) for all \( n \), then \( I_{1}^{\kappa}\varphi \), for \( \kappa = 1, 2 \), belongs to the class \( \mathcal{P}(S^{d-2\kappa} \times \mathbb{R}^{k}) \). Therefore, our result generalizes the corrected version of Theorem 2.1 in Bingham and Symons (2019).

We can modify the functions \( \tilde{f}_{d,\kappa} \), \( n = 0, 1, \ldots, \kappa - 1 \), in (4.3) so that the new quasi Monté operator belongs to \( \mathcal{P}(S^{d-2\kappa} \times \mathbb{R}^{k}) \). Theorem 4.7 sheds some light in this direction.

**Theorem 4.7.** Let the functions \( \tilde{f}_{n,d,\kappa} \in \mathcal{P}(\mathbb{R}^{k}), n \geq \kappa \), and \( h_{1,n}^{\kappa}, h_{2,n}^{\kappa} \in \mathcal{P}(\mathbb{R}^{k}) \) be as respectively defined at (4.4) and (0.16).

Let \( k, \kappa \in \mathbb{Z}_{+}^{*} \) and let \( d \) be an integer such that \( d > 2\kappa \). Let \( \varphi : [-1, 1] \times [0, \infty) \to \mathbb{R} \) be a function in \( \mathcal{P}(S^{d} \times \mathbb{R}^{k}) \) such that \( u \mapsto I_{1}^{\kappa-1}\varphi(u, t) \) is integrable over \([-1, 1] \) for each \( t \in [0, \infty) \). If

\[
(i) \sum_{n=0}^{\infty} C_{n}^{(d-1)/2}(1) \int_{0}^{\infty} dF_{n}(r) < \infty;
\]

\[(ii) \text{there exists constant } K > 0 \text{ such that } \sum_{n=0}^{\infty} C_{n}^{(d-1)/2}(1)dF_{n}(r) \leq K, 0 \leq r < \infty, \]

then there exist \( 2\kappa \) constants \( A^{n} \) and \( B^{n}, n = 0, \ldots, \kappa - 1 \), such that

\[
I_{1}^{\kappa-1,0,\ldots,\kappa-1,B^{0,\ldots,\kappa-1}} \varphi(x, t) := \sum_{n=0}^{\kappa-1} (A^{n}h_{1,n}^{\kappa}(t) - B^{n}h_{2,n}^{\kappa}(t))C_{n}^{(d-2\kappa-1)/2}(x)
+
\sum_{n=\kappa}^{\infty} \tilde{f}_{n,d,\kappa}(t)C_{n}^{(d-2\kappa-1)/2}(x), \quad (4.7)
\]

belongs to \( \mathcal{P}(S^{d-2\kappa} \times \mathbb{R}^{k}) \).
Remark 4.8. For any $A^n \geq 0$, $n = 1, \ldots, \kappa - 1$ the function $I^\kappa,0A^1..A^{\kappa-1},0,0 \varphi$ belongs to $\mathcal{P}(S^{d-2\kappa} \times \mathbb{R}^k)$. This also can be seen as a generalization of the correction of Theorem 2.1 in Bingham and Symons (2019) (to appear).

Next results is related to the operator $I_2$.

Theorem 4.9. Let $d, \kappa \in \mathbb{Z}^*_+$ and $k$ be an integer such that $k > 2\kappa$. If $\varphi : [-1,1] \times [0,\infty) \rightarrow \mathbb{R}$ is a function in $\mathcal{P}(S^d \times \mathbb{R}^k)$ such that

(i) $g_\nu^n(0) = \int_0^\infty (1/r^{2\nu})dF_n(r) < \infty$, for all $n \in \mathbb{Z}_+$ and $\nu \in \{1,2,\ldots,\kappa\}$.

(ii) $0 \neq \sum_{n=0}^\infty g_\nu^n(0)C_n^{(d-1)/2}(1) < \infty$, for $\nu \in \{1,2,\ldots,\kappa\}$,

then the function $I^\kappa_2 \varphi$ has a representation in Gegenbauer series in the form

$$I^\kappa_2 \varphi(x,t) = \frac{1}{\sum_{n=0}^\infty g_\nu^n(0)C_n^{(d-1)/2}(1)} \sum_{n=0}^\infty g_\nu^n(t)C_n^{(d-1)/2}(x).$$ (4.8)

The functions $g_\nu^n$ are defined in (4.2) and $F_n$ are the $k$-Schoenberg measures of the $d$-Schoenberg functions of $\varphi$.

Moreover, $I^\kappa_2 \varphi$ belongs to $\mathcal{P}(S^d \times \mathbb{R}^{k-2\kappa})$.

We finish this part with the Montée operator $I_3$.

Theorem 4.10. Let $\kappa \in \mathbb{Z}^*_+$, $d$ and $k$ be integers such that $d,k > 2\kappa$. If $\varphi : [-1,1] \times [0,\infty) \rightarrow \mathbb{R}$ is a function in $\mathcal{P}(S^d \times \mathbb{R}^k)$ such that

(i) $g_\nu^n(0) = \int_0^\infty \frac{1}{r^{2\nu}}dF_n(r) < \infty$, for all $n \in \mathbb{Z}_+$ and $\nu \in \{1,2,\ldots,\kappa\}$;

(ii) $\sum_{n=0}^\infty g_\nu^n(0) = \sum_{n=0}^\infty \int_0^\infty \frac{1}{r^{2\nu}}dF_n(r) < \infty$, for and $\nu \in \{1,2,\ldots,\kappa\}$,

(iii) $\sum_{n=0}^\infty g_\nu^n(0)C_{n+\nu}^{(d-2\nu-1)/2}(1) < \infty$, for and $\nu \in \{1,2,\ldots,\kappa\}$,

then the function $I^\kappa_3 \varphi$ has a representation in Gegenbauer series in the form

$$I^\kappa_3 \varphi(x,t) = \sum_{n=0}^\infty h_\nu^n(t)C_n^{(d-2\kappa-1)/2}(x), \quad (x,t) \in [-1,1] \times [0,\infty),$$ (4.9)
where

\[ h_n^\kappa(t) := \begin{cases} \gamma_{d,k,\kappa} \sum_{i=0}^{\infty} (-1)^i \chi_i n^{d,\kappa} g_i^\kappa(t), & n = 0, 1, \ldots, \kappa - 1, \\ \gamma_{d,k,\kappa} g_{n-\kappa}^\kappa(t), & n \geq \kappa, \end{cases} \tag{4.10} \]

with \( \gamma_{d,k,\kappa} := \prod_{j=1}^{\kappa} \frac{k-2j}{\delta_{d-2j+1}} > 0 \) and \( \sum_{n=0}^{\infty} h_n^\kappa(0) C_n^{(d-2\kappa-1)/2}(1) < \infty. \) The functions \( g_n^\kappa \) are defined in (4.2) and belong to the class \( \mathcal{P}(\mathbb{R}^k) \) and \( \chi_i n^{d,\kappa} \) are given in (4.5).

**Remark 4.11.** By Remark 4.5, if \( g_{n+1}^\kappa \equiv 0 \) for all \( n \), then \( I_3^\kappa \varphi \), for \( \kappa = 1, 2 \), belongs to the class \( \mathcal{P}(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa}) \).

**Corollary 4.12.** Under the conditions of Theorem 4.10, there exists a bounded function \( H^\kappa \) on \([-1, 1] \times [0, \infty)\) such that \( H^\kappa + I_3^\kappa \varphi \) belongs to \( \mathcal{P}(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa}) \).

As previously mentioned, we can replace the functions \( h_n^\kappa \), \( n = 0, 1, \ldots, \kappa - 1 \), with others such that the new quasi Montée operator belongs to \( \mathcal{P}(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa}) \). Theorem 4.13 provides a construction in this sense.

**Theorem 4.13.** Let \( \kappa \in \mathbb{Z}_+^* \), \( d \) and \( k \) be integers such that \( d, k > 2\kappa \). Let \( \varphi : [-1, 1] \times [0, \infty) \rightarrow \mathbb{R} \) be a function belongs to the class \( \mathcal{P}(\mathbb{S}^d \times \mathbb{R}^k) \) satisfying the hypotheses of Theorem 4.10. If additionally, the \( k \)-Schoenberg measures \( F_n \) of the \( d \)-Schoenberg functions of \( \varphi \) satisfy

\[(i) \sum_{n=0}^{\infty} g_n^{\nu}(0) C_n^{(d-1)/2}(1) = \sum_{n=0}^{\infty} C_n^{(d-1)/2}(1) \int_0^{\infty} \frac{1}{r^{2\nu}} dF_n(r) < \infty, \text{ for and } \nu \in \{1, 2, \ldots, \kappa\}, \]

\[(ii) \text{ there exists constant } K > 0 \text{ such that } \sum_{n=0}^{\infty} C_n^{(d-1)/2}(1) dF_n(r) \leq K, 0 \leq r < \infty; \]

then there exist \( 2\kappa \) constants \( A^n \) and \( B^n \), \( n = 0, \ldots, \kappa - 1 \), such that

\[ I_3^\kappa A^n A^{\kappa-1} B^n B^{\kappa-1} \varphi(x, t) := \sum_{n=0}^{\kappa-1} \left( A^n \tilde{h}_{1,n}^\kappa(t) - B^n \tilde{h}_{2,n}^\kappa(t) \right) C_n^{(d-2\kappa-1)/2}(x) \]

\[ + \sum_{n=\kappa}^{\infty} \tilde{h}_n^\kappa(t) C_n^{(d-2\kappa-1)/2}(x). \tag{4.11} \]
The functions $h_n^\kappa \in \mathcal{P}(\mathbb{R}^k)$, $n \geq \kappa$ and $\tilde{h}_{1,n}^\kappa, \tilde{h}_{2,n}^\kappa \in \mathcal{P}(\mathbb{R}^k)$ are defined respectively in (4.10) and (0.21).

**Remark 4.14.** For any $A^n \geq 0$, $n = 1, \ldots, \kappa-1$ the function $I_3^{\kappa,0A^1..A^{\kappa-1},0,0} \varphi$ belongs to $\mathcal{P}(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa})$.

### Declarations

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### Appendix A

**Proof of Theorem 4.1.** Since $\varphi$ belongs to $\mathcal{P}(\mathbb{S}^d \times \mathbb{R}^k)$, then $\varphi$ is continuously differentiable with respect to the first variable (see (Berg and Porcu, 2017, Proposion 3.8)) and has a Gegenbauer expansion as (2.3). Also $\varphi_x$ has a Gegenbauer expansion in the form

$$\varphi_x(x,t) \sim \sum_{n=0}^{\infty} \tilde{f}^{d+1}_n(t) C_n^{(d+1)/2}(x).$$

Using (2.15)-(2.16), remainder of the proof follows as in (Beatson and zu Castell, 2016, Theorem 2.3). \qed

**Proof of Theorem 4.2.** Let $\varphi$ be a function as in (2.2). By (2.7),

$$\frac{df_n^d}{dt}(t) = \int_0^\infty -\frac{1}{k} tr^2 \Omega_{k+2}(tr) dF_n(r).$$

Deriving term by term, we obtain

$$\frac{\partial \varphi}{\partial t}(x,t) = \sum_{n=0}^{\infty} \frac{df_n^d}{dt}(t) C_n^{(d-1)/2}(x)$$

$$= -\frac{1}{k} t \sum_{n=0}^{\infty} \left( \int_0^\infty \Omega_{k+2}(tr) r^2 dF_n(r) \right) C_n^{(d-1)/2}(x). \tag{0.12}$$

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By Lemma 3 in Gneiting (1999), we have
\[
\frac{d^2 f_n^d}{dt^2}(0) = -\frac{1}{k} \int_0^\infty r^2 dF(r).
\]
Thus,
\[
\frac{\partial^2 \varphi}{\partial t^2}(x, 0) = -\frac{1}{k} \sum_{n=0}^\infty \left( \int_0^\infty r^2 dF_n(r) \right) C_n^{(d-1)/2}(x), \quad x \in [-1, 1]. \tag{0.13}
\]
In particular,
\[
\frac{\partial^2 \varphi}{\partial t^2}(1, 0) = -\frac{1}{k} \sum_{n=0}^\infty \left( \int_0^\infty r^2 dF_n(r) \right) C_n^{(d-1)/2}(1). \tag{0.14}
\]
Thus, by (0.13) and (0.14), for \( x \in [-1, 1] \) and \( t > 0 \) we have
\[
D_2^2 \varphi(x, t) = \frac{\varphi_t(x, t)}{t} \varphi_{tt}(1, 0) = \frac{1}{\sum_{n=0}^\infty \int_0^\infty r^2 dF_n(r)} \sum_{n=0}^\infty g_n^d(t) C_n^{(d-1)/2}(x), \tag{0.15}
\]
where the functions \( g_n^d(t) : [0, \infty) \to \mathbb{R} \) are defined by
\[
g_n^d(t) := \int_0^\infty \Omega_{k+2}(r) r^2 dF_n(r).
\]
We invoke Hypothesis (i) to imply that \( g_n^d \in \mathcal{P}^k(\mathbb{R}^{k+2}) \). Thus, the series in the Equation (0.15) converges absolutely and uniformly on \([-1, 1] \times [0, \infty)\). Hence, letting \( x = 1 \) and \( t = 0 \) in the expression of the series in (0.15) provides
\[
\sum_{n=0}^\infty \int_0^\infty r^2 dF_n(r) \sum_{n=0}^\infty g_n^d(0) C_n^{(d-1)/2}(1) = 1 = D_2^2 \varphi(1, 0).
\]
Therefore, \( D_2^2 \varphi \) is a continuous function on \([-1, 1] \times [0, \infty)\) having a representation series as in (2.2) with \( d \)-Schoenberg functions \( g_n^d \in \mathcal{P}(\mathbb{R}^{k+2}) \). Since, by (ii),
\[
\sum_{n=0}^\infty g_n^d(0) = \sum_{n=0}^\infty \int_0^\infty r^2 dF_n(r) < \infty,
\]
we can conclude that \( D_2^2 \varphi \) belongs to \( \mathcal{P}(S^d \times \mathbb{R}^{k+2}) \).
Proof of Theorem 4.3. We prove the statement by induction on $\kappa \in \mathbb{Z}_+^*$. 

**Step $\kappa = 1$:** We have

$$I_1^1 \varphi(x, t) = I_1 \varphi(x, t) = \int_{-1}^{x} \varphi(u, t) du.$$ 

By (2.2) and (2.11), integrating term by term, we obtain

$$I_1^1 \varphi(x, t) = \sum_{n=0}^{\infty} f_n^d(t) \frac{1}{\delta(d-1)/2} \left( C_{n+1}^{(d-3)/2}(x) - C_{n+1}^{(d-3)/2}(-1) \right).$$

Since $C_{n+1}^{(d-3)/2}(-1) = (-1)^{n+1} C_{n+1}^{(d-3)/2}(1)$, we have

$$I_1^1 \varphi(x, t) = \sum_{n=0}^{\infty} \tilde{f}_n^d(t) \chi_{n+1}(x),$$

where

$$\tilde{f}_n^d(t) := \begin{cases} \frac{1}{\delta(d-1)/2} \sum_{i=0}^{\infty} (-1)^i \chi_i^d f_i(t), & n = 0 \\ \frac{1}{\delta(d-1)/2} f_{n-1}^d(t), & n \geq 1 \end{cases}$$

where $\chi_i^0 = C_{i+1}^{(d-3)/2}(1)$ and, by (2.14),

$$0 \leq \chi_i^0 \leq C_{i+1}^{(d-1)/2}(1) \frac{C_i^{(d-3)/2}(1)}{C_{i}^{(d-1)/2}(1)} \leq (d-1)/2 \frac{C_i^{(d-1)/2}(1)}{C_{i}^{(d-1)/2}(1)} \leq 1.$$ 

which implies

$$\left| (-1)^i \chi_i^0 f_i(t) \right| \leq \gamma^{0,1} f_i(0) C_{i}^{(d-1)/2}(1)$$

and by (2.2), the series in the definition of $\tilde{f}_n^d$ is uniformly convergent on $[0, \infty)$. Again by (2.14), for $n \geq 1$,

$$\left| \tilde{f}_n^d(0) C_{n}^{(d-3)/2}(1) \right| \leq \frac{1}{\delta(d-1)/2} \gamma^{0,1} f_{n-1}(0) C_{n-1}^{(d-1)/2}(1).$$
Thus, \( \sum_{n=0}^{\infty} \tilde{f}_n^d(0) C_n^{(d-3)/2}(1) < \infty. \)

**Step \( \kappa = 2 \):** By algebraic manipulation we have

\[
I_1^2 \varphi(x, t) = \sum_{n=0}^{\infty} \tilde{f}_n^d(t) C_n^{(d-5)/2}(x)
\]

where

\[
\tilde{f}_n^d(t) := \begin{cases} 
\tau_d^2 \sum_{i=0}^{\infty} (-1)^i n_i d f_i(t), & n = 0, 1, \\
\tau_d^2 f_i(t), & n \geq 2,
\end{cases}
\]

with \( \tau_d^2 = (\delta(d-1)/2 \delta(d-3)/2)^{-1} \) and

\[
\chi_{i}^{0,2} := C_{i+1}^{(d-3)/2}(1) C_1^{(d-5)/2}(1) - C_{i+2}^{(d-5)/2}(1),
\]

\[
\chi_{i}^{1,2} := C_{i+1}^{(d-3)/2}(1) = \chi_{i}^{0,1}.
\]

It is clear that \( 0 \leq \chi_{i}^{1,2} \leq \Upsilon_{1,2} C_i^{(d-1)/2}(1) \), with \( \Upsilon_{1,2} = \Upsilon_{0,1} \). It is not difficult to see that

\[
\chi_{i}^{0,2} = \frac{(d-5) \Gamma(i + d - 1)(i + 1)}{\Gamma(d-3)(i + d - 3)\Gamma(i + 3)} \geq 0
\]

and, by (2.14),

\[
|\chi_{i}^{0,2}| \leq \left( \frac{C_{i+1}^{(d-3)/2}(1)}{C_i^{(d-1)/2}(1)} C_1^{(d-5)/2}(1) \right) C_i^{(d-1)/2}(1)
\]

\[
\leq \left( \theta_{(d-1)/2,1} \frac{\Gamma(1 + d - 5)}{\Gamma(2)\Gamma(d - 5)} + \theta_{(d-1)/2,2} \right) C_i^{(d-1)/2}(1)
\]

\[
= \left( \theta_{(d-1)/2,1} (d - 5) + \theta_{(d-1)/2,2} \right) C_i^{(d-1)/2}(1)
\]

By the same argument of the step \( \kappa = 1 \) we can conclude that the series in the definition of \( \tilde{f}_n^d \) for \( n = 0, 1 \) are uniformly convergent on \([0, \infty)\) and also \( \sum_{n=0}^{\infty} \tilde{f}_n^d(0) C_n^{(d-5)/2}(1) < \infty. \)

**Induction step:** let us assume that the expression in (4.3) of \( I_1^\kappa \varphi \) holds up to \( \kappa \), and let us prove it holds for \( I_1^{\kappa+1} \varphi \). We have

\[
I_1^{\kappa+1} \varphi(x, t) = I_1(I_1^\kappa \varphi)(x, t) = \int_{-1}^{x} I_1^\kappa \varphi(u, t) du.
\]
Using the induction hypothesis and integrating term by term, for \((x,t) \in [-1,1] \times [0,\infty)\), we obtain

\[
I_1^{\kappa+1} \varphi(x,t) = \sum_{n=0}^{\kappa-1} \tilde{f}^{d,\kappa}(t) \int_{-1}^{x} C_n^{(d-2\kappa-1)/2}(u)\,du + \sum_{n=\kappa}^{\infty} \tilde{f}_n^{d,\kappa}(t) \int_{-1}^{x} C_n^{(d-2\kappa-1)/2}(u)\,du
\]

\[
= \sum_{n=0}^{\kappa-1} \tau^{d,\kappa} \sum_{i=0}^{\infty} (-1)^i \chi^i_n d,\kappa f_i^d(t) \frac{1}{\delta_{(d-2\kappa-1)/2}} \left( C_{n+1}^{(d-2\kappa-3)/2}(x) - C_{n+1}^{(d-2\kappa-3)/2}(-1) \right)
\]

\[
+ \sum_{n=\kappa}^{\infty} \tau^{d,\kappa} f_n^{d-k}(t) \frac{1}{\delta_{(d-2\kappa-1)/2}} \left( C_{n+1}^{(d-2\kappa-3)/2}(x) - C_{n+1}^{(d-2\kappa-3)/2}(-1) \right).
\]

Thus,

\[
I_1^{\kappa+1} \varphi(x,t) = \tau^{d,\kappa+1} \sum_{n=0}^{\kappa-1} \sum_{i=0}^{\infty} (-1)^i \chi^i_n d,\kappa f_i^d(t) \left( C_{n+1}^{(d-2\kappa-3)/2}(x) - C_{n+1}^{(d-2\kappa-3)/2}(-1) \right)
\]

\[
+ \tau^{d,\kappa+1} \sum_{n=\kappa}^{\infty} f_n^{d-k}(t) \left( C_{n+1}^{(d-2\kappa-3)/2}(x) - C_{n+1}^{(d-2\kappa-3)/2}(-1) \right).
\]

After some algebraic manipulation,

\[
I_1^{\kappa+1} \varphi(x,t) = \sum_{n=0}^{\infty} \tilde{f}_n^{d,\kappa+1}(t) C_n^{(d-2(\kappa+1)-1)/2}(x),
\]

where

\[
\tilde{f}_n^{d,\kappa+1}(t) = \begin{cases} 
\tau^{d,\kappa+1} \sum_{i=0}^{\infty} (-1)^i \chi^i_n d,\kappa+1 f_i^d(t), & n = 0, 1, \ldots, \kappa, \\
\tau^{d,\kappa+1} f_n^{d-k}(t), & n \geq \kappa + 1,
\end{cases}
\]

with

\[
\chi^0_{i,d,\kappa+1} := \sum_{j=1}^{\kappa} (-1)^j+1 \chi_{i-1,d,\kappa} C_{j}^{(d-2(\kappa+1)-1)/2}(1) - (-1)^{\kappa+1} C_{i+\kappa+1}^{(d-2(\kappa+1)-1)/2}(1)
\]

and

\[
\chi^1_{i,d,\kappa+1} := \chi^0_{i-1,d,\kappa}, \quad n = 1, 2, \ldots, \kappa.
\]
It is clear that $|\chi_{i}^{n,d,\kappa+1}| \leq \Upsilon^{n,d,\kappa+1}/2(1)$, with $\Upsilon^{n,d,\kappa+1} = \Upsilon^{n-1,d,\kappa}$.

Now, by (2.14),

$$|\chi_{i}^{0,d,\kappa+1}| \leq \left(\sum_{j=1}^{\kappa} \frac{|\chi_{i}^{j-1,d,\kappa}|}{\Upsilon_{j}^{d-1}} \right) C_{j}^{(d-2\kappa-3)/2}(1) + \frac{C_{i+\kappa+1}^{(d-2\kappa-3)/2}(1)}{C_{i}^{(d-1)/2}(1)} C_{i}^{(d-1)/2}(1).$$

By induction hypothesis (4.6) and by (2.14) we obtain

$$|\chi_{i}^{0,d,\kappa+1}| \leq \left(\sum_{j=1}^{\kappa} \Upsilon^{j-1,d,\kappa}(d-2\kappa-3) + \theta_{d-1,2,2(\kappa+1)}\right) C_{i}^{(d-1)/2}(1).$$

The convergence of the series in the definition of $\tilde{f}_{d,\kappa+1}$ for $n = 0, 1, \ldots, \kappa$ and $\sum_{n=0}^{\infty} \tilde{f}_{d,\kappa+1}(0) C_{n}^{(d-2(\kappa+1)-1)/2}(1)$ follow as in the previous steps. 

Proof of Corollary 4.4. Note that, for $n = 0, 1, \ldots, \kappa - 1$, we can rewrite $\tilde{f}_{n}^{d,\kappa}$ as

$$\tilde{f}_{n}^{d,\kappa}(t) = h_{1,n}^{\kappa}(t) - h_{2,n}^{\kappa}(t),$$

where

$$h_{1,n}^{\kappa}(t) := \tau_{d,\kappa} \sum_{i=0}^{\infty} \chi_{2i}^{n,d,\kappa} f_{2i}^{d}(t), \quad \text{and} \quad (0.16)$$

$$h_{2,n}^{\kappa}(t) := \tau_{d,\kappa} \sum_{i=0}^{\infty} \chi_{2i+1}^{n,d,\kappa} f_{2i+1}^{d}(t) \quad (0.17)$$

Define the function $H^{\kappa}$ on $[-1, 1] \times [0, \infty)$ by

$$H^{\kappa}(x, t) := \sum_{n=0}^{\kappa-1} h_{2,n}^{\kappa}(t) C_{n}^{(d-2\kappa-1)/2}(x) - h_{1,0}^{\kappa}(t) C_{0}^{(d-2\kappa-1)/2}(x)$$

which is bounded on $[-1, 1] \times [0, \infty)$ because, by (2.13), (4.6) and (2.2),

$$|H^{\kappa}(x, t)| \leq \tau^{d,\kappa} \sum_{n=0}^{\kappa-1} \Upsilon^{n,d,\kappa} \left(\sum_{i=0}^{\infty} f_{2i+1}^{d}(0) C_{2i+1}^{(d-1)/2}(1)\right) C_{n}^{(d-1)/2}(1)$$

$$+ \tau^{d,\kappa} \Upsilon^{0,d,\kappa} \left(\sum_{i=0}^{\infty} f_{2i}^{d}(0) C_{2i}^{(d-1)/2}(1)\right) C_{0}^{(d-2\kappa-1)/2}(1) < \infty,$$
for all \((x, t) \in [-1, 1] \times [0, \infty)\).

By Remark 4.5, it is clear that \(h_{\kappa,n}^\ell \in \mathcal{P}(\mathbb{R}^k)\), \(n = 1, 2, \ldots, \kappa - 1\), and also \(\tilde{f}_{\kappa,n}^d \in \mathcal{P}(\mathbb{R}^k)\) for \(n \geq \kappa\). Therefore,

\[
H_\kappa(x, t) + I_1^\kappa \varphi(x, t) = \sum_{n=1}^{\kappa-1} h_{\kappa,n}^\ell(t)C_n^{(d-2\kappa-1)/2}(x) + \tau_d^{\kappa,n} \sum_{n=\kappa}^{\infty} \tilde{f}_{\kappa,n}^d(t)C_n^{(d-2\kappa-1)/2}(x)
\]

has an expansion uniformly convergent as (2.2) due to Theorem 4.3. By Theorem 3.3 of Berg and Porcu (2017) (see (2.2)), we can conclude that the function \(H_\kappa + I_1^\kappa \varphi\) belongs to the class \(\mathcal{P}(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^k)\).

Proof of Theorem 4.7. By (2.5), for any constants \(A\) and \(B\),

\[
Ah_{1,n}^\kappa(t) - Bh_{2,n}^\kappa(t) = \tau_d^{\kappa,n} \sum_{i=0}^{\infty} \int_0^\infty \Omega_k(tr) \left( A \chi_{2i}^{n,d,k} dF_{2i}(r) - B \chi_{2i+1}^{n,d,k} dF_{2i+1}(r) \right).
\]

(0.18)

Since, by (4.6),

\[
\int_0^\infty A|\chi_i^{n,d,k}|dF_i(r) \leq AY^{n,d,k} \Omega_i^{(d-1)/2}(1) \int_0^\infty dF_i(r),
\]

we have

\[
\int_0^\infty \left( A \chi_{2i}^{n,d,k} dF_{2i}(r) - B \chi_{2i+1}^{n,d,k} dF_{2i+1}(r) \right) < \infty.
\]

By (4.6) and (i) the series in (0.18) converges absolutely and uniformly on \([0, \infty)\).

Thus,

\[
Ah_{1,n}^\kappa(t) - Bh_{2,n}^\kappa(t) = \tau_d^{\kappa,n} \int_0^\infty \Omega_k(tr)d \left( \sum_{i=0}^{\infty} A \chi_{2i}^{n,d,k} F_{2i}(r) - B \chi_{2i+1}^{n,d,k} F_{2i+1}(r) \right).
\]

By (4.6) and (ii), the series \(\sum_{i=0}^{\infty} \chi_{2i}^{n,d,k} F_{2i}\) and \(\sum_{i=0}^{\infty} \chi_{2i+1}^{n,d,k} F_{2i+1}\) are uniformly bounded on \([0, \infty)\). Then we can choose \(A^n, B^n\) such that the series \(\sum_{i=0}^{\infty} A^n \chi_{2i}^{n,d,k} F_{2i} - B^n \chi_{2i+1}^{n,d,k} F_{2i+1}\) is non negative, which allows us conclude that \(A^n h_{1,n}^\kappa - B^n h_{2,n}^\kappa \in \mathcal{P}(\mathbb{R}^k)\). The convergence uniform of the series (4.7) follows by Theorem 4.3 and the result by Theorem 3.3 of Berg and Porcu (2017) (see (2.2)).
Proof of Theorem 4.9. We will prove (4.8) by mathematical induction on $\kappa$.

**Step $\kappa = 1$:** We have

$$I_2\varphi(x, t) = \frac{1}{\int_0^\infty v\varphi(1, v)dv} \int_t^\infty v\varphi(x, v)dv.$$

By (2.2), integrating term by term, we obtain

$$\int_t^\infty v\varphi(x, v)dv = \sum_{n=0}^\infty \left( \int_t^\infty v \int_0^\infty \Omega_k(vr)dF_n(r)dv \right) C_n^{(d-1)/2}(x).$$

Using Fubini Theorem, we have

$$\int_t^\infty v\varphi(x, v)dv = \sum_{n=0}^\infty \left( \int_0^\infty \left( \int_0^\infty \frac{w}{r^2} \Omega_k(w)dw \right) dF_n(r) \right) C_n^{(d-1)/2}(x).$$

By (2.9), for $(x, t) \in [-1, 1] \times [0, \infty)$,

$$\int_{tr}^\infty \frac{w}{r^2} \Omega_k(w)dw = \frac{(k-2)}{r^2} \Omega_{k-2}(tr).$$

Hence, for $(x, t) \in [-1, 1] \times [0, \infty)$,

$$\int_t^\infty v\varphi(x, v)dv = (k-2) \sum_{n=0}^\infty g_n^1(t) C_n^{(d-1)/2}(x), \quad (0.19)$$

where $g_n^1$ is defined in (4.2). In particular,

$$\int_0^\infty v\varphi(1, v)dv = (k-2) \sum_{n=0}^\infty g_n^1(0) C_n^{(d-1)/2}(1), \quad (0.20)$$

which is nonzero and finite. By (0.19) and (0.20), $I_2\varphi$ has the representation given in (4.8).

**Induction step:** let assume the expression in (4.8) of $I_2^\kappa \varphi$ holds up to $\kappa$, and let us prove it holds for $I_2^{\kappa+1} \varphi$.

We have

$$I_2^{\kappa+1} \varphi(x, t) = I_2(I_2^\kappa \varphi)(x, t) = \frac{1}{\int_0^\infty vI_2^\kappa \varphi(1, v)dv} \int_t^\infty vI_2^\kappa \varphi(x, v)dv.$$
Note that the Hypothesis (i) guarantees that $g_n^\kappa \in \mathcal{P}(\mathbb{R}^{k-2\kappa})$ and consequently the series in (4.8) converges absolutely and uniformly.

Using the induction hypothesis, integrating term by term, using Fubini theorem and (2.9), for $(x, t) \in [-1, 1] \times [0, \infty)$, we obtain:

\[
\int_t^\infty vI_2^\kappa \varphi(x, v) dv = \sum_{n=0}^{\infty} \left[ \int_t^\infty v \Omega_{k-2\kappa}(vr) dv \right] \frac{1}{r^{2\kappa}} dF_n(r) \right] C_n^{(d-1)/2}(x)
\]

\[
= (k - 2\kappa - 2) \sum_{n=0}^{\infty} \left[ \int_0^\infty \Omega_{k-2(n+1)}(tr) \frac{1}{r^{2(n+1)}} dF_n(r) \right] C_n^{(d-1)/2}(x)
\]

\[
= (k - 2\kappa - 2) \sum_{n=0}^{\infty} g_{n+1}^\kappa(t) C_n^{(d-1)/2}(x).
\]

In particular,

\[
\int_0^\infty vI_2^\kappa \varphi(1, v) dv = (k - 2\kappa - 2) \sum_{n=0}^{\infty} g_{n+1}^\kappa(0) C_n^{(d-1)/2}(1),
\]

which is nonzero and finite by (ii). Therefore,

\[
I_2^{\kappa+1} \varphi(x, t) = \frac{1}{\sum_{n=0}^{\infty} g_{n+1}^\kappa(0) C_n^{(d-1)/2}(1)} \sum_{n=0}^{\infty} g_{n+1}^\kappa(t) C_n^{(d-1)/2}(x)
\]

and (4.8) is proved.

Finally, given $\kappa \in \mathbb{Z}^*_+$, by (i) the $d$-Schoenberg functions $g_n^\kappa$ of $I_2^\kappa \varphi$ belong to the class $\mathcal{P}(\mathbb{R}^{k-2\kappa})$ and together with (ii) we can conclude $0 < \sum_{n=0}^{\infty} g_n^\kappa(0) C_n^{(d-1)/2}(1) < \infty$. Therefore, Theorem 3.3 of Berg and Porcu (2017) (see (2.2)) allows us to infer that $I_2^\kappa \varphi$ belongs to $\mathcal{P}(\mathbb{S}^d \times \mathbb{R}^{k-2\kappa})$.

**Proof of Theorem 4.10.** We will prove (4.9) by mathematical induction on $\kappa$.

**Step $\kappa = 1$:** For each $(x, t) \in [-1, 1] \times [0, \infty)$,

\[
I_2^1 \varphi(x, t) = \int_t^\infty \int_{-1}^x v \varphi(u, v) du dv.
\]
Using (2.2) and (2.5),

\[
\int_t^\infty \int_{-1}^x v \varphi(u, v) dvdu = \int_t^\infty \int_{-1}^x v \sum_{n=0}^{\infty} \left( \int_0^\infty \Omega_k(vr) dF_n(r) \right) C_n^{(d-1)/2}(u) dvdu.
\]

Integrating term by term and by Fubbini Theorem, we have

\[
\int_t^\infty \int_{-1}^x v \varphi(u, v) dvdu = \sum_{n=0}^{\infty} \left[ \int_0^\infty \left( \int_t^\infty v \Omega_k(vr) dv \right) dF_n(r) \right] \left[ \int_{-1}^x C_n^{(d-1)/2}(u) du \right].
\]

By (2.9) and (2.11), we obtain

\[
\int_t^\infty \int_{-1}^x v \varphi(u, v) dvdu = \sum_{n=0}^{\infty} \left[ \int_0^\infty \frac{(k-2)}{r^2} \Omega_{k-2}(tr) dF_n(r) \right] \times \frac{1}{\delta(d-1)/2} \left( C_{n+1}^{(d-3)/2}(x) - C_{n+1}^{(d-3)/2}(-1) \right)
\]

Since \( C_{n+1}^{(d-3)/2}(-1) = (-1)^{n+1} C_{n+1}^{(d-3)/2}(1) \),

\[
\int_t^\infty \int_{-1}^x v \varphi(u, v) dvdu = \frac{(k-2)}{\delta(d-1)/2} \left[ \left( \sum_{n=0}^{\infty} (-1)^n C_{n+1}^{(d-3)/2}(1) g_n^1(t) \right) C_0^{(d-3)/2}(x) \right. \\
\left. + \sum_{n=0}^{\infty} g_n^1(t) C_{n+1}^{(d-3)/2}(x) \right],
\]

where \( g_n^1 \) is given in (4.2).

Therefore,

\[
I_3^1 \varphi(x, t) = \sum_{n=0}^{\infty} h_n^1(t) C_n^{(d-3)/2}(x),
\]

where

\[
h_n^1(t) = \begin{cases} 
\gamma^{d,k,1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{0,d,1} g_i^1(t), & n = 0 \\
\gamma^{d,k,1} g_{n-1}^1(t), & n \geq 1,
\end{cases}
\]
where \( \gamma^{d,k,1} = \frac{(k-2)}{\delta(d-1)/2} > 0 \). Moreover \( \sum_{n=0}^{\infty} h_n^1(0)C_n^{(d-3)/2}(1) < \infty \) because, by (4.6) and (ii)-(iii), we have
\[
\sum_{n=0}^{\infty} |h_n^1(0)C_n^{(d-3)/2}(1)| \leq T_0^{0.1} C_0^{(d-3)/2}(1) \sum_{i=0}^{\infty} g_i^1(0) + \sum_{n=1}^{\infty} g_{n-1}^1(0)C_n^{(d-3)/2}(1) < \infty.
\]

**Induction step:** let assume the expression in (4.9) of \( I_3^\kappa \varphi \) holds up to \( \kappa \), and let us prove it holds for \( I_3^{\kappa+1} \varphi \).

We have
\[
I_3^{\kappa+1} \varphi(x,t) = I_3(I_3^\kappa \varphi)(x,t) = \int_t^x \int_{-1}^x v I_3^\kappa \varphi(u,v) du dv.
\]

Using the induction hypothesis, integrating term by term, using Fubini theorem, Equations (4.2), (2.9), (2.11), and making algebraic manipulations similar to the previous ones, for \((x,t) \in [-1,1] \times [0,\infty)\), we obtain
\[
\begin{align*}
\int_t^x \int_{-1}^x v I_3^\kappa \varphi(u,v) du dv &= \gamma^{d,k,\kappa} \sum_{n=0}^{\kappa-1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\kappa} \int_t^x g_i^\kappa(v) dv \int_t^x C_n^{(d-2\kappa-1)/2}(u) du \\
&\quad + \gamma^{d,k,\kappa} \sum_{n=\kappa}^{\infty} \int_t^x g_n^\kappa(v) dv \int_t^x C_n^{(d-2\kappa-1)/2}(u) du \\
&= \gamma^{d,k,\kappa} \frac{(k-2\kappa-2)}{\delta(d-2\kappa-1)/2} \times \\
\left[ \sum_{n=0}^{\kappa-1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\kappa} g_i^{\kappa+1}(t) \left( C_n^{(d-2\kappa-3)/2}(x) - C_n^{(d-2\kappa-3)/2}(-1) \right) \\
+ \sum_{n=\kappa}^{\infty} g_n^{\kappa+1}(t) \left( C_n^{(d-2\kappa-3)/2}(x) - C_n^{(d-2\kappa-3)/2}(-1) \right) \right]
\end{align*}
\]

Thus, as in the proof of Theorem 4.3,
\[
I_3^{\kappa+1} \varphi(x,t) = \gamma^{d,k,\kappa+1} \sum_{n=0}^{\infty} h_n^{\kappa+1}(t)C_n^{(d-2(\kappa+1)-1)/2}(x),
\]
where
\[
h_n^{\kappa+1}(t) := \begin{cases} \\
\gamma^{d,k,\kappa+1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\kappa+1} g_i^{\kappa+1}(t), & n = 0, 1, \ldots, \kappa \\
\gamma^{d,k,\kappa+1} g_n^{\kappa+1}(\kappa+1)(t), & n \geq \kappa + 1
\end{cases}
\]
with $\gamma_{d,k,\kappa+1} > 0$. By (4.6) and (ii)-(iii), $\sum_{n=0}^{\infty} h_n^{\kappa+1}(0)C_n^{(d-2(\kappa+1)-1)/2}(1) < \infty$

**Proof of Corollary 4.12.** We can proceed as in the proof of Corollary 4.4 and rewrite $h_n^{\kappa}$, $n = 0, 1, \ldots, \kappa - 1$, as

$$h_n^{\kappa}(t) = \tilde{h}_{1,n}^{\kappa}(t) - \tilde{h}_{2,n}^{\kappa}(t),$$

where

$$\tilde{h}_{1,n}^{\kappa}(t) := \gamma_{d,k,\kappa} \sum_{i=0}^{\infty} \chi_{2i}^{n,d,\kappa} g_{2i}^{\kappa}(t),$$

and

$$\tilde{h}_{2,n}^{\kappa}(t) := \gamma_{d,k,\kappa} \sum_{i=0}^{\infty} \chi_{2i+1}^{n,d,\kappa} g_{2i+1}^{\kappa}(t).$$

Define the bounded function $H^{\kappa}$ on $[-1, 1] \times [0, \infty)$ by

$$H^{\kappa}(x, t) := \sum_{n=0}^{\kappa-1} \tilde{h}_{2,n}^{\kappa}(t)C_n^{(d-2\kappa-1)/2}(x) - \tilde{h}_{1,0}^{\kappa}(t)C_0^{(d-2\kappa-1)/2}(x)$$

By Remark 4.5, it is clear that $\tilde{h}_{1,n}^{\kappa} \in \mathcal{P}(\mathbb{R}^k)$, $n = 1, 2, \ldots, \kappa - 1$, and also $h_n^{\kappa} \in \mathcal{P}(\mathbb{R}^k)$ for $n \geq \kappa$. Therefore,

$$H^{\kappa}(x, t) + I_3^{\kappa}\varphi(x, t) = \sum_{n=1}^{\kappa-1} \tilde{h}_{1,n}^{\kappa}(t)C_n^{(d-2\kappa-1)/2}(x) + \sum_{n=\kappa}^{\infty} h_n^{\kappa}(t)C_n^{(d-2\kappa-1)/2}(x)$$

has an expansion as (2.2) with the series uniformly convergent on $[-1, 1] \times [0, \infty)$ due to Theorem 4.10. By Theorem 3.3 of Berg and Porcu (2017) (see (2.2)), we can conclude that the function $H^{\kappa} + I_1^{\kappa}\varphi$ belongs to the class $\mathcal{P}(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^k)$.

**Proof of Theorem 4.13.** As in the proof of Theorem 4.7, for any constants $A$ and $B$, by (4.2),

$$A\tilde{h}_{1,n}^{\kappa}(t) - B\tilde{h}_{2,n}^{\kappa}(t) = \sum_{i=0}^{\infty} \int_{0}^{\infty} \Omega_{k-2\kappa}(tr) \left( A\chi_{2i}^{n,d,\kappa} \frac{1}{r^{2\kappa}} dF_{2i}(r) - B\chi_{2i+1}^{n,d,\kappa} \frac{1}{r^{2\kappa}} dF_{2i+1}(r) \right).$$

(0.22)
Since, by (4.6),

\[
\int_0^\infty A|\chi_i^{n,d,\kappa}| \frac{1}{r^{2\kappa}} dF_i(r) \leq A^\gamma_i^{n,d,\kappa} \gamma_i^{(d-1)/2}(1) \int_0^\infty \frac{1}{r^{2\kappa}} dF_i(r),
\]

we have

\[
\int_0^\infty \left( A\chi_{2i}^{n,d,\kappa} \frac{1}{r^{2\kappa}} dF_{2i}(r) - B\chi_{2i+1}^{n,d,\kappa} \frac{1}{r^{2\kappa}} dF_{2i+1}(r) \right) < \infty
\]

and by Hypothesis (i) the series in (0.22) converges absolutely and uniformly on \([0, \infty)\).

Thus,

\[
A\tilde{h}_{1,n}(t) - B\tilde{h}_{2,n}(t) = \int_0^\infty \Omega_k(tr) d \left( \sum_{i=0}^\infty A\chi_{2i}^{n,d,\kappa} \frac{1}{r^{2\kappa}} F_{2i}(r) - B\chi_{2i+1}^{n,d,\kappa} \frac{1}{r^{2\kappa}} F_{2i+1}(r) \right).
\]

By (ii), the series \(\sum_{i=0}^\infty \chi_{2i}^{n,d,\kappa} \frac{1}{r^{2\kappa}} F_{2i}\) and \(\sum_{i=0}^\infty \chi_{2i+1}^{n,d,\kappa} \frac{1}{r^{2\kappa}} F_{2i+1}\) are uniformly bounded on \([0, \infty)\). Then we can choose \(A^n, B^n\) such that \(\sum_{i=0}^\infty A^n\chi_{2i}^{n,d,\kappa} \frac{1}{r^{2\kappa}} F_{2i} - B^n\chi_{2i+1}^{n,d,\kappa} \frac{1}{r^{2\kappa}} F_{2i+1}\) is non negative which allows us conclude that \(A^n\tilde{h}_{1,n} - B^n\tilde{h}_{2,n} \in \mathcal{P}(\mathbb{R}^k)\). The uniform convergence of the series (4.11) follows by Theorem 4.10 and the result by Theorem 3.3 of Berg and Porcu (2017) (see (2.2)).

\[\square\]

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