Point sets that minimize \((\leq k)\)-edges, 3-decomposable drawings, and the rectilinear crossing number of \(K_{30}\)

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Abstract

There are two properties shared by all known crossing-minimizing geometric drawings of \(K_n\), for \(n\) a multiple of 3. First, the underlying \(n\)-point set of these drawings minimizes the number of \((\leq k)\)-edges, that means, has exactly \(3\left(\frac{k+2}{2}\right)\leq k\)-edges, for all \(0 \leq k < n/3\). Second, all such drawings have the \(n\) points divided into three groups of equal size; this last property is captured under the concept of 3-decomposability. In this paper we show that these properties are tightly related: every \(n\)-point set with exactly \(3\left(\frac{k+2}{2}\right)\leq k\)-edges for all \(0 \leq k < n/3\), is 3-decomposable. The converse, however, is easy to see that it is false. As an application, we prove that the rectilinear crossing number of \(K_{30}\) is 9726.

Keywords: \(k\)-edges, 3-decomposability, rectilinear crossing number

1. Introduction

The rectilinear crossing number \(\text{cr}(G)\) of a graph \(G\), is the minimum number of edge crossings in a geometric drawing of \(G\) in the plane, that is, a drawing of \(G\) in the plane where the vertices are points in general position and the edges are straight segments. Determining \(\text{cr}(K_n)\), where \(K_n\) is the complete graph with \(n\) vertices, is a well-know open problem in combinatorial geometry initiated by Guy \([11]\).

The rectilinear crossing number problem is related with the concept of \(k\)-edges. A \(k\)-edge of an \(n\)-point set \(P\), with \(0 \leq k \leq n/2 - 1\), is a line through two points of \(P\) leaving exactly \(k\) points on one side. A \((\leq k)\)-edge is an \(i\)-edge with \(0 \leq i \leq k\). Let \(E_k(P)\) denote the number of \(k\)-edges of \(P\) and \(E_{\leq k}(P)\) denotes the number of \((\leq k)\)-edges, that is, \(E_{\leq k}(P) = \sum_{j=0}^{k} E_j(P)\). Finally, \(E_{\leq k}(n)\) denotes the minimum of \(E_{\leq k}(P)\) taken over all \(n\)-point sets \(P\).

The exact determination of \(E_{\leq k}(n)\) is another notable open problem in combinatorial geometry. In 2005 \([6]\), Aichholzer et al. gave the following lower bound for \(E_{\leq k}(n)\):

\[
E_{\leq k}(n) \geq 3\left(\frac{k+2}{2}\right) + 3\left(\frac{k+2}{2} - \frac{n/3}{2}\right) - \max\{0, (k+1 - \lfloor n/3\rfloor)(n-3\lfloor n/3\rfloor)\},
\]

later, in 2007 \([7]\), Aichholzer et al. proved that this lower bound is tight for \(k \leq \lfloor 5n/12\rfloor - 1\).

The number of crossings in a geometric drawing of \(K_n\) and the number of \(k\)- and \((\leq k)\)-edges in the underlying \(n\)-point set \(P\) are closely related by the following equality, independently proved by Lóvász et al. \([12]\) and Abrego and Fernández-Merchant \([8]\). For any set \(P\) of \(n\) points
Another concept that plays a central role in this paper is the $3$–decomposability, which is a property shared by all known crossing-minimizing geometric drawings of $K_n$, for $n$ a multiple of $3$. Formally, we say that a finite point set $P$ is $3$–decomposable if it can be partitioned into three equal sized sets $A$, $B$ and $C$ such that there exists a triangle $T$ enclosing the point set $P$ and the orthogonal projection of $P$ onto the three sides of $T$ show $A$ between $B$ and $C$ on one side, $B$ between $C$ and $A$ on the second side, and $C$ between $A$ and $B$ on the third side. We say that a geometric drawing of $K_n$ is $3$-decomposable if its underlying point set is $3$-decomposable.

In the following result we establish the relationship between $3$–decomposability and the number of $(\leq k)$-edges

**Theorem 1 (Main Theorem).** Let $P$ be an $n$–point set, for $n$ a multiple of $3$, with exactly $3\binom{n+2}{2}$ $(\leq k)$-edges for all $0 \leq k < n/3$, then $P$ is $3$-decomposable.

In fact, in [2] Ábrego et al. conjectured that for each positive integer $n$ multiple of $3$, all crossing–minimal geometric drawings of $K_n$ are $3$–decomposable.

As an application of the Main Theorem we prove that a $30$-point set that minimize the crossing number is $3$-decomposable. Aichholzer established $9726$ as the upper bound for $\overline{\text{cr}}(K_{30})$ [5], moreover we have the following theorem

**Theorem 2 (The Rectilinear Crossing Number of $K_{30}$).** $\overline{\text{cr}}(K_{30})$ is $9726$.

All the results of this paper are proved in the more general context of generalized configuration of points. In this scope we define by analogy the pseudolinear crossing number $\overline{\text{cr}}(K_n)$.

Our main tool are the allowable sequences which will be formally define in Section [2] and we mention some preliminary results due to Lóvasz et al. in [12]. In Section [3] we prove the Main Theorem. In Section [4] we use the Main Theorem to establish that a configuration with $30$ points that minimize the crossing number is $3$–decomposable and we give some implications of the $3$–decomposability. Finally, in Section [5] is the formal proof of Theorem 2.

2. Allowable Sequences

An allowable sequence $\Pi$ is a doubly infinite sequence . . . , $\pi_{-1}$, $\pi_0$, $\pi_1$, . . . of permutations of $n$ elements, where consecutive permutations differ by a transposition of neighboring elements, and $\pi_1$ is the reverse permutation of $\pi_{i+\binom{n}{2}}$. Thus $\Pi$ has period $2\binom{n}{2}$, and the hole information of $\Pi$ is contained in any of its $n$–half-periods, which we call $n$–half-periods. We usually denote by $P$ an $n$-half-period of $\Pi$.

It is known that if $P$ is a set of $n$ points in the plane in general position, then all the combinatorial information of $P$ can be encoded by an allowable sequence $\Pi_P$ on the set $P$, called circular sequence associated to $P$ [10]. It is important to note that most allowable sequence are not circular sequences, however there is a one-to-one correspondence between allowable sequences and generalized configurations of points [10].

We have the following definitions and notations for allowable sequences. A transposition that occurs between elements in sites $i$ and $i+1$ is an $i$–transposition, and we say that moves through the $i$th–gate. In this new setting an $i$–transposition, or $(n-i)$–transposition corresponds to an $(i-1)$–edge. For $i \leq n/2$, an $i$–critical transposition is either an $i$–transposition or an $(n-i)$–transposition, and a $(\leq k)$–critical transposition is a transposition that is $i$–critical for some $1 \leq i \leq k$. If $\Pi$ is an $n$–half-period, then $N_k(\Pi)$ and $N_{\leq k}(\Pi)$ denote the number of $k$–critical transpositions and $(\leq k)$–critical transpositions in $\Pi$, respectively. Therefore $N_k(\Pi) = E_{k-1}(\Pi)$, $N_{\leq k}(\Pi) = E_{\leq k-1}(\Pi)$. When $n$ is even an $n/2$–transposition is also called halving and $h(\Pi)$ denotes the number of halvings, and thus $h(\Pi) = E_{n/2-1}(\Pi)$.
The identity (2) relating $k$-edges to crossing number was originally proved for allowable sequences. All these definitions and functions coincide with their original counterparts for $P$ when $\Pi$ is the circular sequence of $P$. However, when $\mathcal{T}(n)$, and $E_{\leq k}(n)$ are minimized over all allowable sequences on $n$ points rather than over all sets of $n$ points, the corresponding quantities may change so we define the notation $\tilde{c}(n)$ and $\tilde{E}_{\leq k}(n)$. But it is clear that $\tilde{c}(n) \leq \mathcal{T}(n)$ and $\tilde{E}_{\leq k}(n) \leq E_{\leq k}(n)$. Ábrego et al. [11] proved that the lower bound (1) on $E_{\leq k}(n)$ is also a lower bound on $\tilde{E}_{\leq k}(n)$ and use it to extend the lower bound on $\mathcal{T}(n)$ to $\tilde{c}(n)$.

Let $\Pi = (\pi_0, \pi_1, \ldots, \pi_{n\choose 2})$ be an $n$–half-period. For each $k < n/2$, define $m = m(k, n) := n - 2k$. In order to keep track of $(\leq k)$–critical transpositions in $\Pi$, it is convenient to label the points so that the starting permutation is

$$\pi_0 = (a_k, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_k).$$

Sometimes it will be necessary to say who an element is moving, so we will say that an element $x$ exits (respectively, enters) through the $i$th $A$–gate if it moves from the position $k - i + 1$ to the position $k - i + 1$ (respectively, from the position $k - i + 2$ to the position $k - i + 1$) during a transposition with another element. Similarly, $x$ exits (respectively, enters) through the $i$th $C$–gate if it moves from the position $m + k + i$ to the position $m + k + i - 1$ (respectively, from $m + k + i - 1$ to $m + k + i$) during a transposition.

An $a \in \{a_1, \ldots, a_k\}$ (respectively, $c \in \{c_1, \ldots, c_k\}$) is confined until the first time it exits through the first $A$–gate (respectively, $C$–gate); then it becomes free. A transposition is confined if both elements involved are confined.

The following results, from Proposition 3 to Proposition 7, are due to Lovász et al. in the paper [12]:

**Proposition 3.** Let $\Pi_0$ be an $n$–half-period, and let $k < n/2$. Then there is an $n$–half-period $\Pi$, with the same number of $(\leq k)$–critical transpositions as $\Pi_0$, and with no confined transpositions.

In view of this statement, for the rest of the section we assume that the $n$–half-period $\Pi$ under consideration has no confined transpositions.

The liberation sequence $\sigma(\Pi)$ (or simply $\sigma$ if no confusion arises) of $\Pi$ contains all the $a$’s and all the $c$’s, in the order in which they become free in $\Pi$. Since $\Pi$ has no confined transpositions, the $a$’s appear in increasing order, as do the $c$’s. We let $T(a_i)$ (respectively $T(c_i)$) denote the set of all those $c$’s (respectively $a$’s) that appear after $a_i$ (respectively $c_i$) in $\sigma$.

A transposition that swaps elements in the positions $i$ and $i + 1$ occurs in the $A$–Zone (respectively, $C$–Zone) if $i \leq k$ (respectively, $i \geq k + m$). Such transpositions are of obvious relevance: a transposition is $(\leq k)$–critical if and only if it occurs either in the $A$–Zone or in the $C$–Zone.

For $1 \leq i \leq j \leq k$, the $i$th $A$–gate is a compulsory exit–gate for $a_j$, and the $i$th $C$–gate is a compulsory entry–gate for $a_i$: that is, $a_j$ has to exit through the $i$th $A$–gate at least once, and to enter the $i$th $C$–gate at least once. Analogous definitions and observations hold for $c_j$: the $i$th $A$–gate is a compulsory entry–gate for $c_j$, and the $i$th $C$–gate is a compulsory exit–gate for $c_j$. A transposition in which an element enters (respectively, exits) one of its compulsory entry (respectively, exit)–gates for the first time is a discovery transposition for the element. A transposition is a discovery transposition if it is a discovery transposition for at least one of the elements involved. If it is a discovery transposition for both elements, then it is a double–discovery transposition (for the reader familiar with [12], what we call double–discovery transpositions are the transpositions represented by a directed edge in the savings digraph of [12]).

Discovery and double–discovery transpositions play a central role in [12]. The key results are the following, which hold for any $n$–half-period with no confined transpositions (the first statement is a straightforward counting, whereas the second does definitely require a proof).

**Observation 4.** There are (exactly) $2^{k+1\choose 2}$ transpositions that are discovery transpositions for some $a$, and (exactly) $2^{k+1\choose 2}$ transpositions that are discovery transpositions for some $c$.

**Proposition 5.** There are at most $k+1\choose 2$ double–discovery transpositions.

Since each discovery transposition is $(\leq k)$–critical, these statements immediately imply the following.

**Proposition 6.** There are at least $3k+1\choose 2$ $(\leq k)$–critical transpositions.
An \( n \)-half-period \( II \) with no confined transpositions is perfect if the following hold:

(a) Each transposition in \( II \) that occurs in the \( A \)-Zone or in the \( C \)-Zone is a discovery transposition.

(b) \( a_r \) is involved in (exactly) \( \min \{i, |T(a_r)|\} \) double–discovery transpositions in the \( C \)-Zone.

(c) Each \( c_i \) is involved in (exactly) \( \min \{i, |T(c_i)|\} \) double–discovery transpositions in the \( A \)-Zone.

The following result is implicit in the proof of Theorem 10 in [12].

**Proposition 7.** If \( II \) is perfect, then it has exactly \( 3\binom{k+1}{2} \leq k \)–critical transpositions for all \( k \leq m \). Conversely, if \( II \) has no confined transpositions, and has exactly \( 3\binom{k+1}{2} \leq k \)–critical transpositions for all \( k \leq m \), then it is perfect.

3. Proof of Main Theorem

The concept of 3-decomposability for \( n \)-point sets is also generalized in the setting of allowable sequences. An \( n \)-half-period \( II \) of an allowable sequence \( II \) is \( 3 \)-decomposable if the elements in \( II \) can be labeled \( A = \{a_{n/3}, a_{n/3-1}, \ldots, a_1\}, B = \{b_1, b_2, \ldots, b_{n/3}\}, C = \{c_1, c_2, \ldots, c_{n/3}\} \) and if \( \tau_0 = (a_{n/3}, a_{n/3-1}, \ldots, a_1, b_1, b_2, \ldots, b_{n/3}, c_1, c_2, \ldots, c_{n/3}) \) is the first permutation of \( II \), thus, all transpositions between an element of \( A \) and an element of \( B \) occur before that the transpositions between \( C \) and \( A \cup B \), after occur all transposition between \( A \) and \( C \) prior that the transposition between \( B \) and \( C \) and later occur all transposition between \( C \) and \( B \). In particular, there are some indices \( 0 < s < t < \binom{n}{2} \), such that \( \tau_{s+1} \) shows all the \( a \)-elements followed by all the \( c \)-elements followed by all \( b \)-elements, and \( \tau_{t+1} \) shows all the \( b \)-elements followed by all the \( a \)-elements followed by all the \( c \)-elements. An allowable sequence is \( 3 \)-decomposable if contains an \( n \)-half-period \( 3 \)-decomposable.

Before proving the Main Theorem, we must first state two propositions:

**Proposition 8.** Suppose that \( II \) is perfect. Then, in the liberation sequence \( \sigma \) of \( II \), either all the \( a \)’s occur consecutively or all the \( c \)’s occur consecutively.

**Proof.** The last entry in \( \sigma \) is either \( a_k \) or \( c_k \), and by symmetry we may assume without any loss of generality that it is \( a_k \). Our strategy is to suppose that \( a_{\ell-1} \ldots a_{\ell} \ldots a_k \) is a suffix of \( \sigma \), where \( \ell > 1 \) and \( 2 \leq t \leq k \), and derive a contradiction.

We claim that \( a_{\ell-1} \) swaps with \( c_k \) in the \( C \)-Zone. We start by noting that since \( II \) is perfect, and \( |T(a_{\ell-1})| = k - \ell + 1 \geq 1 \), it follows that \( a_{\ell-1} \) is involved in a double–discovery transposition in the \( C \)-Zone with at least one \( c \). If this transposition involves \( (a_{\ell-1} \text{ and } c_k) \), then our claim obviously holds. Thus suppose that it involves \( (a_{\ell-1} \text{ and } c_i) \) for some \( i < k \). Then, right after \( a_{\ell-1} \) and \( c_i \) swap, \( c_k \) is to the right of \( a_{\ell-1} \), since no confined transpositions occur in \( II \). Note that all transpositions that swap \( a_{\ell-1} \) to the left involve an \( a_j \) with \( j > t - 1 \). On the other hand, since \( a_j \) (moreover, every \( a_j \) with \( j \geq t \)) gets freed after \( c_k \), it follows that before any transposition can move \( a_{\ell-1} \) left, \( c_k \) must be freed (and before that it must transpose with \( a_{\ell-1} \)). This shows that the transposition \( \mu \) that swaps \( a_{\ell-1} \) with \( c_k \) occurs in the \( C \)-Zone.

Thus, right after \( \mu \) occurs, \( a_{\ell-1} \) is at position \( r \), where \( r \geq k + m + 1 \). We claim that \( \max\{r, k + m + t - 1\} < 2k + m \). Since \( t - 1 \geq k \), then \( k + m + t - 1 < 2k + m \), and so it suffices to show that if \( r > k + m + t - 1 \), then \( r < 2k + m \). So suppose that \( r > k + m + t - 1 \). Note that the final position in \( II \) (that is, the position in \( \tau_{t+1} \)) of \( a_{\ell-1} \) is \( k + m + t - 1 \), and so by the time \( \mu \) occurs there has been a transposition \( \tau \) that moves \( a_{\ell-1} \) to the right of its final position (we remark that possibly \( \tau = \mu \)). Since \( \tau \) occurs in the \( C \)-Zone and clearly is not a discovery step for \( a_{\ell-1} \), and \( II \) is perfect, it follows that \( \tau \) is a discovery step for a \( c_i \). Moreover, \( |T(a_{\ell-1})| = k - \ell + 1 \) is greater than \( t - 1 \), as otherwise (by the perfectness of \( II \)) the transposition between \( a_{\ell-1} \) and \( c_i \) would have to be a double–discovery step. Thus \( |T(a_{\ell-1})| > t - 1 \), and again invoking the perfectness of \( II \) we get that \( a_{\ell-1} \) is involved with (exactly) \( t - 1 \) double–discovery steps in the \( C \)-Zone, each with an element in \( \{c_1, \ldots, c_k\} \). Therefore the number of possible transpositions that move \( a_{\ell-1} \) to the right of its final position \( k + m + t - 1 \) is at most \( k - \ell + 1 - (t - 1) \). Thus the rightmost position of \( a_{\ell-1} \) throughout \( II \) (and consequently \( \tau \)) is at most \( k + m + t - 1 + k + m + 1 - \ell - 1 = 2k + m + 1 - \ell < 2k + m \).

Let \( R \) be the set of the points that occupy the positions \( r + 1, r + 2, \ldots, 2k + m \) immediately after \( \mu \) occurs. Since at this time every \( a_j \) with \( j > t - 1 \) is confined, it follows that each point in \( R \) is either a \( b \), a free \( c \) (this follows
Proof. In particular, at this point it follows that every transposition in the \( L \)–Zone, contradicting (precisely) our assumption that \( a_{t-1} \) did not move back to the \( B \)–Zone.

Thus, after \( \mu \) occurs, \( a_{t-1} \) eventually re-enters the \( B \)–Zone, and since its final position is \( k + m + t - 1 \), afterwards it has to re-enter the \( C \)–Zone via a transposition \( \lambda \) that moves \( a_{t-1} \) to the right and an element \( x \in R \) to the left. Since \( \lambda \) occurs in the \( C \)–Zone, and \( \Pi \) is perfect, then \( \lambda \) must be a discovery transposition. We complete the proof by arriving to a contradiction: \( \lambda \) cannot be a discovery transposition. Indeed, \( \lambda \) cannot be discovery for \( a_{t-1} \) (since it had already been in the \( C \)–Zone), so it must be a discovery step for \( x \). On the other hand, since each \( x \in R \) is either a \( b \), a free \( c \), or an \( a_i \) with \( j < t - 1 \), \( \lambda \) it follows that \( \lambda \) cannot be a discovery transposition for \( x \) either.

Our next statement shows that we can actually go a bit further: there is a perfect \( n \)–half-period \( \Pi' \) whose liberation sequence has all \( a \)'s followed by all \( c \)'s or vice versa.

**Proposition 9.** Suppose that \( \Pi \) is a perfect \( n \)–half-period of an allowable sequence \( \Pi \). Then \( \Pi \) contains a perfect \( n \)–half-period \( \Pi' \), with initial permutation \( a'_0a'_{k-1} . . . a'_1b'_1 . . . b'_m c'_1 . . . c'_k \), and whose liberation sequence is either \( a'_0a'_2 . . . a'_k c'_1 . . . c'_k \) or \( c'_0c'_2 . . . c'_k a'_1 . . . a'_k \).

**Proof.** Let \( \Pi = (\pi_0, \pi_1, \ldots, \pi_{n/2}) \) be any perfect \( n \)–half-period, with initial permutation \( \pi_0 = (a_ka_{k-1} . . . a_1 b_1 . . . b_mC_2 . . . c_k) \), and let \( \sigma \) be the liberation sequence associated to \( \Pi \). Thus the last entry of \( \sigma \) is either \( a_k \) or \( c_k \), and a straightforward symmetry argument shows that we may assume without loss of generality that last entry in \( \sigma \) is \( a_k \). If \( \sigma = c_1c_2 . . . c_k a_1a_2 . . . a_k \), then we are done. Thus we may assume that there is a \( t, 2 \leq t \leq k \), such that \( a_{t-1}, c_1, c_2, . . . , c_{t-1}, a_t, a_{t+1}, . . . , a_k \) is a suffix of \( \sigma \).

In order to define the \( n \)–half-period \( \Pi' \) claimed by the proposition, we establish some facts regarding \( \Pi \).

\[ \text{(A) Let } \pi_{i+1} \text{ be the permutation where } c_i \text{ becomes free. Then } \pi_i \text{ is of the form } (a_k, a_{k-1}, . . . , a_t, d_1, d_2, . . . , d_p c_1, c_2, . . . , c_k) \text{ where } p = t - 1 + m \text{ and each } d_j \text{ is either a } b \text{ or a free } a. \]

**Proof of (A).** The perfectness of \( \Pi \) readily implies that every transposition in the \( A \)–Zone that involves an element in \( L := \{a_t, a_{t+1}, . . . , a_k\} \) is a double–discovery transposition. In particular, the first element that moves an element in \( L \) must involve a \( \sigma \). Therefore, as long as no \( c \) becomes free, all the elements in \( L \) must stay in their original position. Finally, we observe that when \( c_1 \) becomes free, \( a_1, a_2, . . . , a_{t-1} \) are already free, so each \( d_j \) is either a \( b \) or a free \( a \), as claimed.

\[ \text{(B) No element in } \{a_ka_{k-1} . . . a_t d_1, . . . , d_{t-1}\} \text{ (these are the elements that are in the \( A \)–Zone, in the given order, in } \pi_i \text{) leaves the } A \text{–Zone before } c_k \text{ becomes free.} \]

**Proof of (B).** Seeking a contradiction, let \( e \) be the first element in \( \{a_ka_{k-1} . . . a_t d_1, . . . , d_{t-1}\} \) that moves out of the \( A \)–Zone before \( c_k \) becomes free. The perfectness of \( \Pi \) readily implies that the element that takes \( e \) out of the \( A \)–Zone is some \( c_j \) (where by assumption \( j \neq k \)). Now right after \( c_j \) swaps with \( e, c_j \) and \( c_k \) are in the \( A \)– and \( C \)–Zones, respectively. In particular, at this point \( c_j \) and \( c_k \) have not swapped. Now as we observed above, every transposition in the \( A \)–Zone involving an element in \( L \) is double–discovery, and so it follows that \( c_j \) never gets beyond (to the left of) the position \( k - j + 1 \). No matter where the \((c_j, c_k) \Rightarrow (c_k, c_j)\) transposition occurs, this implies that \( c_j \) must
at some point be in a position \( r \), with \( k - j + 1 \leq r \leq k \), and then move (right) to position \( r + 1 \). Now in order to reach its final position, \( c_j \) must eventually move back to the position \( r \), via some transposition \( \varepsilon = (x, c_j) \mapsto (c_j, x) \). Since \( \Pi \) is perfect, and \( \varepsilon \) occurs in the \( \mathcal{A} \)-Zone, \( \varepsilon \) is a discovery transposition. But it clearly cannot be discovery for \( c_j \), since \( c_j \) is re-visiting the position \( r \). Now \( x \in \{a_k, a_{k-1}, \ldots, a_t, d_1, \ldots, d_{t-1}\} \), since these were the elements to the left of \( c_j \) when it first entered the \( \mathcal{A} \)-Zone. Clearly \( x \) cannot be a \( d \), since each \( d \) is either an \( a \) or a free \( a \), and \( \varepsilon \) must be discovery for \( x \). Thus \( x \) must be in \( L = \{a_k, a_{k-1}, \ldots, a_t\} \). But this is also impossible, since (see Proof of (A)) every transposition that involves an element in \( L \) must be a double–discovery transposition.

(C) Suppose that two elements that are in the \( \mathcal{A} \)-Zone (respectively, \( \mathcal{C} \)-Zone) in \( \pi_r \) transpose with each other in the \( \mathcal{A} \)-Zone (respectively, \( \mathcal{C} \)-Zone) after \( \pi_{r+1} \). Then at least one of these elements leaves the \( \mathcal{A} \)-Zone (respectively, \( \mathcal{C} \)-Zone) after \( \pi_r \) and before this transposition occurs.

Proof of (C). First we note that the elements that are in the \( \mathcal{A} \)-Zone in \( \pi_r \) are \( a_1, a_2, \ldots, a_k \), in this order, and that if two of them transpose before at least one of them leaves the \( \mathcal{C} \)-Zone, this transposition would be confined, contradicting the assumption that \( \Pi \) is perfect. That takes care of the \( \mathcal{C} \)-Zone part of (C).

Now we recall that the elements that are in the \( \mathcal{A} \)-Zone in \( \pi_r \) are \( a_1, a_2, \ldots, a_t, d_1, d_2, \ldots, d_{t-1} \), in this order. Suppose that two such elements transpose in the \( \mathcal{A} \)-Zone after \( \pi_{r+1} \), and that between \( \pi_r \) and this transposition (call it \( \lambda \)) none of them leaves the \( \mathcal{A} \)-Zone. It follows from the perfectness of \( \Pi \) that, for each \( a_j \), every move of \( a_j \) until it leaves the \( \mathcal{A} \)-Zone must involve some \( c_j \). Thus none of the elements involved in \( \lambda \) can be an \( a_j \), that is, both must be \( d_j \)'s. But such a transposition would clearly not be discovery (recall that each \( d \) is a free \( a \) or a \( b \)), contradicting the perfectness of \( \Pi \). This completes the proof of (C).

(D) After \( \pi_{r+1} \), the elements in the \( \mathcal{A} \)-Zone leave it in the order \( d_{t-1}, d_{t-2}, \ldots, d_1, a_t, \ldots, a_{k-1}, a_k \), and the elements in the \( \mathcal{C} \)-Zone leave it in the order \( c_1, c_2, \ldots, c_k \).

Proof of (D). This is an immediate corollary of (C).

Now define \( \Pi' := (\pi_1, \pi_{r+1}, \ldots, \pi_{t-1}) = (\pi_0^{-1}, \pi_1^{-1}, \ldots, \pi_{t-1}^{-1}, \pi_{t}^{-1}). \) It is straightforward to check that \( \Pi' \) is an \( n \)-half-period. Define the relabeling \( a_i \mapsto a_i' \) for \( i = t, t+1, \ldots, k; d_s \mapsto d_s' \) for \( s = 1, \ldots, t-1; d_s \mapsto b_s' \) for \( s = t, t+1, \ldots, p; \) and \( c_i \mapsto c_i' \) for \( i = 1, \ldots, k, \) so that the initial permutation of \( \Pi' \) (namely \( \pi = (a_1a_{k-1} \ldots a_t d_1 d_2 \ldots, d_p c_1 c_2 \ldots c_k) \) is \( (a_1' a_{k-1}' \ldots a_t' b_1' b_2' \ldots b_p' c_1' c_2' \ldots c_k') \).

To complete the proof, we check that (i) the liberation sequence of \( \Pi' \) is \( c_1' c_2' \ldots c_p' a_1' a_2' \ldots a_t' \), and that (ii) \( \Pi' \) is perfect. We note that (i) follows immediately from (B) and (D). Now in view of Proposition[7] in order to prove that \( \Pi' \) is perfect it suffices to show that it has no confined transpositions, and that it has exactly \( 3^{(k+1)/2} \) \( \leq k \)–critical transpositions. From (C) it follows that \( \Pi' \) has no confined transpositions. On the other hand, an application of Proposition[7] to \( \Pi \) (which is perfect) yields that \( \Pi \) has \( 3^{(k+1)/2} \) \( \leq k \)–critical transpositions. The construction of \( \Pi' \) clearly reveals that \( \Pi \) and \( \Pi' \) have the same number of \( \leq k \)–critical transpositions, and so \( \Pi' \) has \( 3^{(k+1)/2} \) \( \leq k \)–critical transpositions, as required.

Proof of Theorem[7]

Let \( \Pi \) be an \( n \)-half-period of \( \Pi_P \), for \( n \) a multiple of 3. By the hypothesis of the Main Theorem and the fact \( E_{\leq k-1}(P) = N_{\leq k}(\Pi) \), we have \( N_{\leq k}(\Pi) = 3^{(k+1)/2} \) for each \( 1 \leq k \leq n/3 \). This equality and Proposition[8] guarantee that \( \Pi_P \) contains an \( n \)-half-period, say \( \Pi_P \), that satisfies the hypothesis of Proposition[7]. Thus \( \Pi_P \) is perfect, and using Proposition[9] we get an \( n \)-half-period which behaves as we need for \( \Pi_P \) to be 3-decomposable.

4. On Allowable Sequences That Minimize The Crossing-Number of \( K_{30} \)

This section is devoted to study of allowable sequences which come from configurations of 30 points that minimize the crossing-number. In particular, each result presented in this section is focused on establish features of such sequences. Later, in Section[5] each of these properties will be used in the proof of Theorem[2].

We begin by proving, with the help of Theorem[1] that all optimal sequence of \( K_{30} \) are 3-decomposable.
We have the following bounds given by Ábrego et al. [4] for any \( n \)-half-period \( \Pi \) of an allowable sequence.

\[
N_{(n/2)}(\Pi) \leq \begin{cases} \left\lfloor \frac{7}{15} n(n+30) - 3 \right\rfloor , & \text{if } n \text{ is even,} \\ \left\lfloor \frac{7}{15} n(n+30) - 2 \right\rfloor + \frac{3}{5} , & \text{if } n \text{ is odd.} \end{cases}
\]

(3)

and

\[
N_{\leq (n/2) - 1}(\Pi) \geq \begin{cases} \left\lfloor \frac{7}{15} n(n+30) - 3 \right\rfloor , & \text{if } n \text{ is even,} \\ \left\lfloor \frac{7}{15} (n-3)(n+45) + \frac{3}{5} \right\rfloor , & \text{if } n \text{ is odd.} \end{cases}
\]

(4)

Now, if \( \Pi \) is a 30–half-period associated to a generalized configuration \( P \) of 30 points, then from (3) we know that \( N_{15}(\Pi) \leq 72 \) and if we combine (1) and (4) we get that \( N_{14}(\Pi) \geq 72 \). With this bounds in (2) we have 9723 as a lower bound for \( \widetilde{c}(K_{30}) \). Moreover, if for some \( k = 0, \ldots, 12 \), (11) is not tight, then a simple calculation in (3) shows that \( \widetilde{c}(P) \geq 9727 \) and therefore \( P \) will be worse than the best known configuration given implicitly by Aichholzer and Krasser in [8], which establishes 9726 as an upper bound. Besides \( 72 \leq N_{14}(\Pi) \leq 75 \) or \( \widetilde{c}(P) \geq 9727 \). So, in an optimal configuration with 30 points, (11) must be tight for each \( k = 0, \ldots, 12 \) and so, by the Main theorem, \( P \) is 3–decomposable.

For the remainder of this subsection, let us assume that \( \Pi = (\pi_0, \pi_1, \ldots, \pi_{(n/2)}) \) is a 3–decomposable 30–half-period, with initial permutation \( \pi_0 = (a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1) \) and \( A = \{a_{10}, a_9, \ldots, a_1\} \), \( B = \{b_1, b_2, \ldots, b_{10}\} \) and \( C = \{c_1, c_2, \ldots, c_{10}\} \).

In order to count the number of \( (\leq k) \)-critical transpositions in \( \Pi \), we define two types of transpositions. A transposition is monochromatic if it occurs between two elements of the same set \( A, B \) or \( C \), otherwise is called bichromatic. We denote the number of monochromatic (respectively, bichromatic) \((\leq k)\)-critical transpositions in \( \Pi \) by \( N_{\leq k}(\Pi) \) (respectively, \( N_{\leq k}^{\text{mono}}(\Pi) \)). Note that \( N_{\leq k}(\Pi) = N_{\leq k}^{\text{mono}}(\Pi) + N_{\leq k}^{\text{bi}}(\Pi) \).

From (2) we get the next account for bichromatic transpositions on a 3–decomposable \( n \)-half-period \( \Pi' \):

\[
N_{\leq k}^{\text{bi}}(\Pi') = \left\lfloor \frac{3(k+1)}{2} \right\rfloor + (k - n/3)n \quad \text{if } k \leq n/3, \quad \text{if } n/3 < k < n/2.
\]

(5)

As a consequence of (5) we have the next two Corollaries:

**Corollary 10.** \( N_k^{\text{bi}}(\Pi) = 3k \) for \( k = 1, 2, \ldots, 10 \).

**Corollary 11.** \( N_k^{\text{bi}}(\Pi) = 30 \) for \( k = 11, 12, 13, 14 \).

**Lemma 12.** \( N_{15}(\Pi) = 15 \).

**Proof.** The number of bichromatic transpositions between \( A \) and \( B \) is 100 because there is, exactly, one bichromatic transposition for each element of \( A \times B \). For the same reason there are 100 bichromatic transpositions between \( A \) and \( C \) and 100 between \( B \) and \( C \). So \( N_{\leq 15}(\Pi) = 300 \). The desired result it follows from Corollaries 10 and 11 and the fact that \( N_{15}(\Pi) = 300 - \sum_{k=1}^{14} N_k^{\text{bi}}(\Pi) \). \( \square \)

From the above discussion, Corollary 11 and Theorem 1 it follows that all monochromatic transpositions occur in the middle third. Where the middle third is the space from the 11th–position to 20th–position.

**4.1. Digraphs**

Let \( \Pi \) be a 3–decomposable \( n \)-half-period of an allowable sequence \( \Pi \). A transposition between elements in the positions \( i \) and \( i + 1 \) with \( k < i < n - k \) is called a \((\geq k)\)-transposition. All these transpositions are said to occur in the \( k \)-center. Let us denote the number of monochromatic transpositions that occur in the \( k \)-center and are of the kinds \( aa, bb \), and \( cc \) by \( N_{a}^{aa}(\Pi), N_{b}^{bb}(\Pi), \) and \( N_{c}^{cc}(\Pi) \), respectively. Since each monochromatic transposition is an \( aa \)-or \( bb \)-or \( cc \)-transposition, then \( N_{a}^{aa}(\Pi) + N_{b}^{bb}(\Pi) + N_{c}^{cc}(\Pi) \) is the total number of monochromatic transpositions that occur in the \( k \)-center.

Let \( D_k \) be the digraph with vertex set \( \{i/3, i/3 - 1, \ldots, 1\} \), and such that there is a directed edge from \( i \) to \( j \) if and only if \( i > j \) and the transposition \( a_ia_j \) occurs in the \( k \)-center. Note that the number of edges of \( D_k \) is exactly \( N_{a}^{aa}(\Pi) \).
In order to count the edges in \( D_k \), let \( D_{v,m} \) be the class of all digraphs on \( v \) vertices, say \( v, v-1, \ldots, 1 \), satisfying that \([i]^+ \leq m + [i]^−\) for all \( v \geq i \geq 1 \), where \([i]^+\) and \([i]^−\) denote the outdegree and the indegree of the vertex \( i \), respectively, and if we have an edge from \( i \) to \( j \), \( i \to j \), then \( i > j \). Let \( D_0(v, m) \) be the graph in \( D_{v,m} \) with vertices \( v, v-1, \ldots, 1 \) recursively defined by

- \([v]^− = 0\),
- \([i]^+ = \min\{[i]^− + m, i − 1\}\) for each \( v \geq i \geq 1 \), and
- for all \( v \geq i > j \geq 1 \), \( i \to j \) if and only if \( i − 1 > j \geq i − 1 − [i]^−\).

Balogh and Salazar prove in [9] that the maximum number of edges of a digraph in \( D_{v,m} \) is attained by \( D_0(v, m) \). We note that \( D_k \) is in \( D_{n/3, n−2k−1} \), and hence the number of edges in \( D_k \) is bounded above by the number of edges in \( D_0(n/3, n−2k−1) \).

\[
\begin{array}{ccccccccccc}
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array}
\]

Figure 1: Digraphs \( D_0(10, 1) \)

From the preceding information, we can deduce that the number of edges in \( D_{14} \) is at most 20 (Figure 1). This means that \( N^{aa}(\Pi) \leq 20 \), \( N^{bb}(\Pi) \leq 20 \), and \( N^{cc}(\Pi) \leq 20 \). Similarly, the number of edges in \( D_{13} \) is at most 33 and we know that \( \binom{30}{2} - N^{\leq 13}(\Pi) = 144 \) because all the bounds for \((\leq k)\)-sets, for \( k = 1, \ldots, 13 \), are tight. Thus \( N_{14}(\Pi) + h(\Pi) = 144 \), besides from Corollary[11] and Lemma[12] we get that \( N^{bb}_{13}(\Pi) + N^{hh}_{12}(\Pi) = 45 \). This implies that \( N^{\text{mono}}_{13}(\Pi) = 99 \) and therefore there are exactly 33–monochromatic transpositions in the 13–center per each set \( A, B \) and \( C \).

**Lemma 13.** If \( D \) is a digraph in \( D_{10,3} \) with 33 edges, then for \( i, j = 10, 9, 8, 7 \) and \( i > j \) there is an edge from \( i \) to \( j \).

**Proof.** Clearly, the number of edges with tail in \( \{10, 9, 8, 7\} \) and head in \( \{6, 5, \ldots, 1\} \) is at most 12 and the number of edges in the vertex set \( \{6, 5, \ldots, 1\} \) is at most 15 (this is attained by \( D_0(6, 3) \)). Then we need the 6 edges between the elements in \( \{10, 9, 8, 7\} \) in order to get the 33 edges in \( D \).

4.2. Restrictions in the monochromatic transpositions

From now on, we shall use \( \Pi = (\pi_0, \pi_1, \ldots, \pi_{30}) \) to denote a 3–decomposable 30–half-period of an optimal configuration for \( K_30 \) and \( \pi_0 = (a_{10}, a_9, a_8, a_7, b_{10}, b_9, b_8, c_1, c_2, \ldots, c_{10}) \) to denote its first permutation. Also we assume that \( A := \{a_1, a_2, \ldots, a_{10}\}, B := \{b_1, b_2, \ldots, b_{10}\} \) and \( C := \{c_1, c_2, \ldots, c_{10}\} \).

As \( \Pi \) is 3–decomposable and all monochromatic transpositions occur in the middle third, it follows that there is a unique element of \( B \) that reaches the position 1 (or 30). We shall denote by \( b_{10} \) such element of \( B \). For the same reasons, for \( i = 2, 3, \ldots, 10 \), there is a unique element of \( B \), which we denote by \( b_{10−i+1} \), that reaches the position \( i \) (or 30 − \( i + 1 \)) but not the position \( i − 1 \) (or 30 − \( i + 2 \)). Clearly, \( B = \{b_1, b_2, \ldots, b_{10}\} \).

In this subsection we use that in \( \Pi \) the lower bound given in (11) is tight for \( k = 0, \ldots, 12 \) in order to deduce some restrictions about the monochromatic transpositions.

**Remark 14.** Because \( \Pi \) is 3–decomposable (\( A \) can interchange the role with \( B \) or \( C \)), everything that we say for \( A \) is also valid for \( B \) or \( C \).

**Lemma 15.** Each transposition of \( \Pi \) that contributes to \( N^{\text{mono}}_{11}(\Pi) + N^{\text{mono}}_{12}(\Pi) + N^{\text{mono}}_{13}(\Pi) \) involves to some of \( a_{10}, a_9, a_8, b_{10}, b_9, b_8, c_1, c_2, c_3 \) or \( c_8 \)

**Proof.** Since we have exactly 33 monochromatic transpositions in the 13–center, then, by Lemma[13] mandatory the transpositions between elements of \( \{a_{10}, a_9, a_8, a_7\} \) occur in the 13–center.

From the equation (5) and the fact that (11) is tight for \( k = 0, \ldots, 12 \), we get that \( N^{\text{mono}}_{11}(\Pi) = 6, N^{\text{mono}}_{12}(\Pi) = 12 \) and \( N^{\text{mono}}_{13}(\Pi) = 18 \). Because no other \( a \) is behind \( a_{10} \), it is not possible to have more than one monochromatic
transpositions associated with a $N_{a}$ with a $c$.

Thus, the 2 · 3 monochromatic transpositions due to $a_{10}$, $b_{10}$ and $c_{10}$ are all the monochromatic transpositions associated with $N_{10}^{mono}(\Pi)$.

For the preceding, every monochromatic transposition involving $a_{9}$ occurs in 11–center. Since the swap between $a_{10}$ and $a_{9}$ occurs in the 13–center, thus $a_{9}$ contributes 2 to $N_{12}^{mono}(\Pi)$. So we have 4 different monochromatic transpositions due to $a_{10}$ and $a_{9}$. By Remark 14 we get 2 · 3 + 2 · 3 monochromatic transpositions due to $a_{10}, a_{9}, b_{10}, b_{9}$ and $c_{10}, c_{9}$ and they are all the monochromatic transpositions associated with $N_{12}^{mono}(\Pi)$.

So each monochromatic transposition involving $a_{8}$ occurs in the 12–center. Thus $a_{8}$ contributes 2 to $N_{13}^{mono}(\Pi)$. $a_{10}$ and $a_{9}$ also have others 2 monochromatic transpositions there, and all the transpositions are different because $a_{10}, a_{9}$ and $a_{8}$ change in the 13–center. Hence the 2 · 3 + 2 · 3 monochromatic transposition associated with $N_{13}^{mono}(\Pi)$ are generated by $a_{10}, a_{9}, b_{10}, b_{9}, b_{8}, c_{10}, c_{9}$ and $c_{8}$.

Let $k \in \{10, 11, \ldots, 14\}$. Note that every element in a 3–decomposable 30–half-period $\Pi'$ occupies each position of the 10–center at least once. From now on, if $\tau$ is the first (respectively, last) transposition in which $x \in A \cup C$ enters (respectively, leaves) the $k$–center, then we say that $\tau$ is the swap in which $x$ enters (respectively, leaves) the $k$–center of $\Pi'$.

**Lemma 16.** For $x \in \{a, c\}$, the elements $x_{1}, x_{2}, \ldots, x_{10}$ enter (respectively, leave) the 13–center of $\Pi$ in ascending (respectively, descending) order. Moreover, for $i = 1, 2, \ldots, 7$:

1. The swap between $a_{i}$ and $b_{7-i+1}$ occurs in the 13th–gate and it is precisely the swap in which $a_{i}$ enters (and $b_{7-i+1}$ leaves) the 13–center of $\Pi$,

2. The swap between $a_{7-i+1}$ and $c_{i}$ occurs in the 17th–gate and it is precisely the swap in which $a_{7-i+1}$ leaves (and $c_{i}$ enters) the 13–center of $\Pi$ and,

3. The swap between $b_{i}$ and $c_{7-i+1}$ occurs in the 13th–gate and it is precisely the swap in which $c_{7-i+1}$ leaves (and $b_{i}$ enters) the 13–center of $\Pi$.

It follows from (3) (respectively, (1)) that $b_{1}, b_{2}, \ldots, b_{7}$ also enter (respectively, leave) the 13–center of $\Pi$ in ascending (respectively, descending) order.

**Proof.** By Lemma 13 and the fact that there are exactly 33 monochromatic transpositions in the 13–center of $\Pi$, each transposition between elements of $\{x_{10}, x_{9}, x_{8}, x_{7}\}$ occurs in the 13–center. Also, by Lemma 15 each transposition between elements of $\{x_{7}, x_{6}, \ldots, x_{1}\}$ occurs in the 13–center. Together, these two conclusions, imply that the elements of $\{x_{10}, x_{9}, x_{1}, x_{2}\}$ enter (respectively, leave) the 13–center of $\Pi$ in ascending (respectively, descending) order.

We only show (1). The parts (2) and (3) are analogous.

Let $w \in \{a, b\}$. Because all monochromatic transpositions of $\Pi$ occur in the 10–center, then the elements of $\{w_{10}, w_{9}, \ldots, w_{1}\}$ enter (respectively, leave) the 10–center of $\Pi$ in ascending (respectively, descending) order.

For $t = 1, 2, 3$ we know (Lemma 15) that every monochromatic transposition involving $b_{10-t+1}$ occurs in the $(10 + t - 1)$–center. This and the fact that the $b$’s leave the 10–center in descending order imply that the swap between $a_{1}$ and $b_{10-t+1}$ occurs in the $(10 + t - 1)$th–gate.

Since (Lemma 15) each transposition between elements of $\{b_{7}, b_{6}, \ldots, b_{1}\}$ occurs in the 13–center and they leave the 10–center in descending order, then the swap where $a_{j}$ enters in the 13–center must be with $b_{7-j+1}$, where $j = 1, 2, \ldots, 7$.

**Lemma 17.** Let $\pi_{a_{10}}$ be the permutation of $\Pi$ where $a_{10}$ enters in the 13–center. Then $\pi_{a_{10}}$ looks like

$$(B, a_{\leq 4}, a_{\leq 5}, a_{\leq 6}, a_{10}, a_{1}, a_{k}, a_{k}, a_{\leq 5}, a_{\leq 5}, a_{\leq 4}, C)$$

where $a_{\leq p}$ is an $a_{u}$ with $1 \leq u \leq p$, further $\{i, j, k\} = \{7, 8, 9\}$.
Proof. For \( j = 7, 6, \ldots, 1 \) let \( \tau_j \) be the transposition in which \( a_j \) enters the 13–center. So, when \( \tau_5 \) occurs there is at least one \( r \in \{1, 2, 3, 4\} \) such that \( a_r \) is to the right hand side of the 13–center (without loss of generality, we assume that \( a_r \) is the rightmost \( a \) element). By Lemma 13 all the monochromatic transpositions between elements of \( \{a_7, a_6, \ldots, a_1\} \) or between elements of \( \{a_10, a_9, a_8, a_7\} \) occur in the 13–center. Thus \( a_r \) does not move to the left until after \( a_{10} \) exits of the 13–center. On the other hand, since all monochromatic transpositions occur in the middle third, thus, when \( a_{10} \) enters in the 13–center \( a_r \) must be at position 20. Using similar arguments with \( \tau_6 \) and \( \tau_7 \) we get the restriction on the right hand side.

Let \( a_l \) be the \( a \) that swaps with \( a_{10} \) in the \((14-j)\)th–gate (where \( j = 3, 2, 1 \)). Since each \( aa \) transposition that contributes to \( N_{11}(\Pi) + N_{12}(\Pi) + N_{13}(\Pi) \) involves to \( a_{10}, a_9 \) or \( a_8 \) and the transpositions between elements of \( \{a_{10}, a_9, a_8, a_7\} \) occur in the 13–center, then \( l \leq 6 \). Thus \( a_l \) needs \( j \) transpositions of kind \( a_l, c \) in order to move to 13–center. Hence \( a_l \) will remain to the left hand side of the 13–center until after \( c_j \) enters in the 13–center. But, by Lemma 13 when \( c_j \) enters in the 13–center all \( a_n \)’s with \( n \geq 8 - j \) have left from there. Hence \( l \leq 7 - j \).

Let \( \text{hal}(a_j) \) denote the number of \( a_i \) elements, \( i < j \), such that \( a_j \) changes with \( a_i \) in the 15th–gate. This means, the outdegree of the vertex \( a_j \) in the digraph \( D_{14} \) associated to \( N_{14}^{aa}(\Pi) \).

Some facts are easier to see in \( \Pi^* \), the reverse half-period of \( \Pi \). We define the reverse half-period of \( \Pi \) as \( \Pi^* = (\pi^*_0, \pi^*_1, \ldots, \pi^*_4) := (\pi^{-1}_0, \pi^{-1}_1, \ldots, \pi^{-1}_5, \pi^{-1}_6, \ldots, \pi^{-1}_9) \). It is clear that \( \Pi \) and \( \Pi^* \) have the same combinatorial properties.

Lemma 18. Let \( \pi_{a_{10}} \) be the permutation of \( \Pi \) where \( a_{10} \) enters in the 13–center. If \( a_i, 1 \leq i \leq 5 \), is at position \( 10 + l \) or at position \( 20 - l + 1 \), \( 1 \leq l \leq 3 \), then \( \text{hal}(a_i) \leq l \).

Proof. We just prove the case when \( a_i \) is at position \( 10 + l \), otherwise we look at \( \Pi^* \). Let \( B(a_i) \) be the set of \( l - 1 \) \( a \)’s that are behind of \( a_i \) in \( \pi_{a_{10}} \). Let \( j \) be the number of element in \( B(a_i) \) with index smaller than \( i \). This means that in \( \pi_{a_{10}} \), \( a_i \) has already changed with each element of \( B(a_i) \) with index smaller than \( i \). Note that these transpositions contribute at most \( j \) to \( \text{hal}(a_i) \). On the other hand, each element of \( B(a_i) \) with index greater than \( i \) moves \( a_i \) to the left one time, then \( a_i \) could make at most \((l - 1) + 1\) transpositions in the 15th–gate which involve an \( a \) with index smaller than \( i \). Thus \( \text{hal}(a_i) \leq j + ((l - 1) - j + 1) = l \).

Corollary 19. \( N_{15}^{bb}(\Pi) \leq 19, N_{15}^{bh}(\Pi) \leq 19 \) and \( N_{15}^{cb}(\Pi) \leq 19 \).

Proof. What we say for \( A \) also apply for \( B \) and \( C \). By Lemmas 17 and 18 \( \text{hal}(a_4) + \text{hal}(a_5) \leq 5 \) and hence the digraph \( D_{14} \) associated to \( N_{14}^{aa}(\Pi) \) has at most 19 edges: at most 5 edges with tail in \( \{a_4, a_5, a_3, a_2, a_1\} \), at most 6 edges between the elements of \( \{a_{10}, a_9, a_8, a_7, a_6\} \) and head in \( \{a_5, a_4, a_3, a_2, a_1\} \), at most 6 edges between the elements of \( \{a_{10}, a_9, a_8, a_7, a_6\} \), at most 5 edges with tail in \( \{a_5, a_4\} \), and at most 3 edges between the elements of \( \{a_3, a_2, a_1\} \).

Remark 20. In fact, if we want to have 19 halvings, thus \( \text{hal}(a_{10}) + \text{hal}(a_9) + \cdots + \text{hal}(a_6) \) must be 11, \( \text{hal}(a_5) + \text{hal}(a_4) \) must be 5 and \( \text{hal}(a_3) + \text{hal}(a_2) + \text{hal}(a_1) \) must be 3. The later means that \( a_3, a_2, a_1 \) have to change in the 15th–gate.

Corollary 21. If \( N_{15}^{bb}(\Pi) = 19 \), then in the permutation \( \pi_{a_{10}} \) of \( \Pi \) in which \( a_{10} \) enters in the 13–center, \( a_1 \) and \( a_2 \) are at positions 11 and 20, respectively, or vice versa.

Proof. From Lemma 13 and Remark 20 it follows that \( a_4 \) is at position 11 or 20 in \( \pi_{a_{10}} \). On the other hand, by Lemma 17 we know that \( a_6 \) is at position 13 (position 18), then \( a_8, a_5 \) occupy the positions 18 and 19 (positions 12 and 13) or they occupy the positions 12 and 18 (positions 13 and 19), not necessarily in that order. Because hal\((a_3)\) must be 2, then, by the Lemma 18 and with the prior discussion, \( a_3 \) must be at position 12 or 19. So we get that \( a_1, a_2 \) are at positions 11 and 20, not necessarily in that order.

Before proceeding with the proof of Theorem 2 we need to establish two more lemmas.

Lemma 22. Let \( \pi_{a_{10}}, \pi_{c_{10}} \) and \( \pi_{b_{10}} \) be the permutations of \( \Pi \) where \( a_{10}, c_{10} \) and \( b_{10} \) enter in the 13–center, respectively. If \( a_3 \) is at position 12 or 19 in \( \pi_{a_{10}} \), then \( N_{15}^{aa}(\Pi) < 19, N_{15}^{bb}(\Pi) < 19 \) and \( N_{15}^{cb}(\Pi) < 19 \).
Proof. Suppose that \( a_5 \) is at position 12 in \( \pi_{a_{10}} \) (the case when \( a_5 \) is at position 19 is the same if we see \( \Pi^* \)). So \( \pi_{a_{10}} \) looks like

\[
\pi_{a_{10}} = (B, a_{11}, a_{12}, a_{13}|a_{10} = - - |a_{12}, a_{14}, a_{15}, C).
\] (6)

Since there are no \( aa\)–transpositions after of \( \pi_{a_{10}} \) on the left hand side of the 13–center, \( a_5 \) moves to the 13–center by means of two \( ac\)–transpositions. By Lemma 19 the swap between \( a_5 \) and \( c_3 \) occurs in the 17th–gate, and hence, \( a_5 \) is moved from the position 12 to 13–center by \( c_3 \) and \( c_1 \). On the other hand, because all the transpositions between elements of \( \{c_1, c_2, \ldots, c_7\} \) or between elements of \( \{c_7, c_8, c_9, c_{10}\} \) occur in the 13–center, then when \( c_{10} \) enters in the 13–center, \( c_1 \) and \( c_2 \) are at positions 11 and 12, not necessarily in that order. So \( \pi_{c_{10}} \) looks like

\[
\pi_{c_{10}} = (B, c_1 \ u_2, c_2 \ u_1, c_1 | - - - |c_{10}|c_{12}, c_{14}, c_{13}, A),
\] (7)

and by Lemma 17 \( j_4 \in \{3, 4\} \).

Now we deduce some restrictions on \( \pi_{b_{10}} \). As before, since there are no \( cc\)–transpositions after of \( \pi_{c_{10}} \) on the right hand side of the 13–center, \( c_{j4} \) moves to the 13–center by means of three \( bc\)–transpositions. By Lemma 16 the swap between \( c_{j4} \) and \( b_{7-j+1} \) occurs in the 13th–gate, and hence, \( c_{j4} \) is moved from the position 20 to 13–center by three \( b\)’s, say \( b_{k1}, b_{k2}, \) and \( b_{k3}, \) such that \( k_1, k_2, k_3 < 7 - j_4 + 1 \leq 5 \). Thus, when \( \pi_{b_{10}} \) occurs, \( b_{k1}, b_{k2}, \) and \( b_{k3} \) are at positions 18, 19 and 20. So \( \pi_{b_{10}} \) looks like \( (C, b_{k6}, b_{k5}, b_{k4}|b_{10} = - - - |b_{k3}, b_{k2}, b_{k1}, A) \). Thus, by Lemma 17 \( k_4 = 6 \) and \( k_5 = 5 \) and \( \pi_{b_{10}} \) looks like

\[
\pi_{b_{10}} = (C, b_{k4}, b_{k5}, b_{k6}|b_{10} = - - - |b_{k3}, b_{k2}, b_{k1}, A).
\] (8)

In a similar way that (7) was obtained from (6), it is possible to obtain (9) (respectively, (11)) from (8) (respectively, (10)). (10) can be obtained from (9) like (8) was obtained from (7).

\[
\pi_{a_{10}}(\pi^{(2)}_{a_{10}}) = (C, a_1 \ u_2, a_2 \ u_1, a_{13}| - - - |a_{12}, a_5, a_{14}, B).
\] (9)

\[
\pi_{c_{10}}(\pi^{(2)}_{c_{10}}) = (A, c_{j4}, c_5, c_6|c_{10} = - - - |c_{j1}, c_2 \ u_1, c_{12}, B).
\] (10)

\[
\pi_{b_{10}}(\pi^{(2)}_{b_{10}}) = (A, b_1 \ u_2, b_2 \ u_1, b_9| - - - |b_{10}|b_{k6}, b_{k5}, C).
\] (11)

The desired result is immediate from (9), (10), (11) and Corollary 21.

Lemma 23. Let \( \pi_{a_{10}}, \pi_{c_{10}} \) and \( \pi_{b_{10}} \) as in Lemma 22 If \( N^{\alpha}_{15}(\Pi) = 19 \) and for \( x = a, b, c; x_j \) occupies the 11th– or 20th–position in \( \pi_{x_{10}} \), then \( j \in \{1, 2\} \).

Proof. We only prove the case \( x = c \) (the cases \( x = a \) and \( x = b \) are analogous). Suppose that \( c_j \) occupies the 11th– or 20th–position in \( \pi_{c_{10}} \).

CASE 1. \( c_j \) occupies the 11th–position in \( \pi_{c_{10}} \). Suppose that \( a_t \) occupies the 13th–position in \( \pi_{a_{10}} \), By Lemma 22 we know that \( t \in \{5, 6\} \).

By Lemma 16 the swap between \( a_t \) and \( c_{7-t+1} \) occurs in the 17th–gate, and hence, \( a_t \) is moved from the position 13 to 13–center by a \( c_r \) such that \( r \leq 7 - t \leq 2 \). On the other hand, by Lemma 15 we know that \( c_r \) does not have monochromatic transpositions on the left hand side of the 13–center until after \( \pi_{c_{10}} \) occurs. Thus \( c_r = c_j \).

CASE 2. \( c_j \) occupies the 20th–position in \( \pi_{c_{10}} \). Seeking a contradiction, suppose that \( j \notin \{1, 2\} \). So by Lemma 17 \( j \in \{3, 4\} \). Again, by Lemma 16 the swap between \( c_j \) and \( b_{7-j+1} \) occurs in the 13th–gate, and hence, \( c_j \) is moved from the position 20 to 13–center by three \( b \)’s, say \( b_{j1}, b_{j2}, \) and \( b_{j3}, \) such that \( j_1, j_2, j_3 < 7 - j + 1 \leq 5 \). It follows from Lemma 15 that none of \( b_{j1}, b_{j2}, \) and \( b_{j3} \) moves until after \( \pi_{b_{10}} \) occurs. This implies that \( b_{j1}, b_{j2}, \) and \( b_{j3} \) occupy the positions 18, 19 and 20 in \( \pi_{b_{10}} \). By Lemma 17 \( b_5 \) is in the 12th–position and by Remark 14 and Lemma 23 \( N^{\alpha}_{15}(\Pi) < 19 \).
5. The Rectilinear Crossing Number of $K_{30}$: Proof of Theorem 2

Let $\pi_{a_{10}}, \pi_{b_{10}}$ and $\pi_{c_{10}}$ as in Lemma 22. By Lemmas 22 and 23 if $N_{15}^{ao}(\Pi) = 19$ then, without loss of generality, $\pi_{a_{10}}$ looks like

$$\pi_{a_{10}} = (B, a_{i1}, a_{i2}, a_{i0}|a_{10} = - - |a_{5}, a_{i3}, a_{i4}, C),$$

with $\{i_1, i_2\} = \{1, 2\}$, otherwise we look $\Pi^*$, besides in the 13–center are $a_9, a_8, a_7$ in some order.

By Lemma 16 $a_6$ leaves the 13–center with $c_2$, so $a_6$ re-enters in the 13–center with the transposition with $c_1$. Thus $c_1$ occupies the 11th–position of $\pi_{c_{10}}$. So by Lemma 23 $\pi_{c_{10}}$ looks like

$$\pi_{c_{10}} = (B, c_{i1}, c_{i2}, c_{i0}| - - |c_{10}|c_{j1}, c_{j2}, c_{j4}, c_{j3}, c_{j2}, A).$$

Again, since $b_9$ enters in the 13–center with the swap with $c_2$, $\pi_{b_{10}}$ looks like $(C, b_{k1}, b_{k2}, b_{k3}|b_{10} = - - |b_{k4}, b_{k5}, b_{k6}, A)$ with $k_1, k_5, k_6 \leq 5$. Thus, by Lemmas 17 and 22 $\pi_{b_{10}}$ looks like

$$\pi_{b_{10}} = (C, b_{k1}, b_{k2}, b_{k3}|b_{10} = - - |b_{5}, b_{k5}, b_{k6}, A).$$

In a similar way that (13) was obtained from (12), it is possible to obtain (15) (respectively, (17)) from (14) (respectively, (16)). $\pi_{a_{10}}$ can be obtained from (15) like (14) was obtained from (13).

$$\pi_{a_{10}} + (\omega) = (C, a_{i1}, a_{i2}, a_{i0}| - - |a_{6}, a_{i2}, a_{2}, B).$$

$$\pi_{c_{10}} + (\omega) = (A, c_{j1}, c_{j2}, c_{j4}|c_{10} = - - |c_{5}, c_{j1}, c_{j2}, c_{1}, B).$$

$$\pi_{b_{10}} + (\omega) = (A, b_{1}, b_{k1}, b_{k5}| - - |b_{10}, b_{b}, b_{k2}, b_{2}, C).$$

So we have only two cases, when $i_2$ equals to 3 or 4.

CASE $i_2 = 4$. The permutation $\pi_{a_{10}}$ is $(B, a_{2}, a_{4}, a_{6}|a_{10} = - - |a_{5}, a_{3}, a_{1}, C)$. By Lemma 16 $a_4$ leaves the 13–center with $c_4$, then $a_4$ must re-enters to the 13–center with $c_4$ and therefore $\pi_{c_{10}}$ is $(B, c_{1}, c_{3}, c_{5}| - - |c_{10}|c_{6}, c_{4}, c_{2}, A)$, and for similar reasons, the permutation $\pi_{b_{10}}$ is $(C, b_{2}, b_{4}, b_{k}|b_{10} = - - |b_{5}, b_{3}, b_{1}, A)$.

Claim 24. If $\text{hal}(a_3) + \text{hal}(a_2) + \text{hal}(a_1) = 3$, then $\text{hal}(c_5) \leq 2$. Hence $N_{15}^{ac}(\Pi) \leq 18$

Proof of Claim 24. Since $N_{15}^{ac}(\Pi) = 19$, by Remark 20 $\text{hal}(a_3) + \text{hal}(a_2) + \text{hal}(a_1) = 3$. By Lemma 16 $a_3$ leaves the 13–center swapping with $c_5$, and the permutation is

$$(B, c_{1}, \{c_3, a_2\}|\{c_2, c_4, a_1\}|c_5|a_{3}, ...),$$

where the notation $\{\}$ means that $c_2, c_4, a_1$ occupy those positions, but not necessarily in that order, similarly for $a_2$ and $c_3$. Because $a_2$ must to change with $a_1$ in the 15th–gate, this is only possible if $c_5$ changes with $a_1$ in the 15th–gate, but then $c_5$ does not change with neither $c_2$ or $c_4$ in the 15th–gate, and therefore $\text{hal}(c_5) \leq 2$. $N_{15}^{ac}(\Pi) \leq 18$ is a consequence of the Remark 20. This completes the proof of Claim 24.

If $N_{15}^{ac}(\Pi) = 18$ and with the fact that $\text{hal}(c_5) \leq 2$, by Remark 20 we conclude that $\text{hal}(c_3) + \text{hal}(c_2) + \text{hal}(c_1) = 3$. Since $\pi_{a_{10}}$ has the same configuration as $\pi_{a_{10}}$, named $(B, c_{1}, c_{3}, c_{5}| - - |c_{10}|c_{6}, c_{4}, c_{2}, A)$ and also satisfies the hypotheses of Claim 24, we conclude that $N_{15}^{ac}(\Pi) \leq 18$. Now if $N_{15}^{bb}(\Pi) = 18$, $B$ satisfies the Claim 24 too and implies that $N_{15}^{ao}(\Pi) \leq 18$, which is a contradiction. Then $N_{15}^{ao}(\Pi) = 19, N_{15}^{ac}(\Pi) = 18$ and $N_{15}^{bb}(\Pi) \leq 17$.

So we suppose that $N_{15}^{ac}(\Pi) \leq 17$. The only case we have to worry about is when $N_{15}^{bb}(\Pi) = 19$, but recall that when $b_{10}$ enters in the 13–center, the permutation $\pi_{b_{10}}$ is

$$\pi_{b_{10}} = (C, b_{2}, b_{4}, b_{k}|b_{10} = - - |b_{5}, b_{3}, b_{2}, A)$$

and $B$ holds the hypotheses of Claim 24 which implies that $N_{15}^{ao}(\Pi) \leq 18$, and this is a contradiction. Thus $N_{15}^{ao}(\Pi) = 19, N_{15}^{ac}(\Pi) \leq 17$ and $N_{15}^{bb}(\Pi) \leq 18$.

CASE $i_2 = 3$. So, $\pi_{a_{10}} = (B, a_{2}, a_{3}, a_{6}|a_{10} = - - |a_{5}, a_{4}, a_{1}, C)$. By Lemma 16 $a_3$ leaves the 13–center with $c_5$, then $a_3$ re-enters to 13–center with $c_3$ or $c_4$.
Suppose that \( a_3 \) re-enters with \( c_4 \), then \( \pi_{c_{10}} \) looks like

\[
\pi_{c_{10}} = (B, c_1, c_3, c_5 | \ldots - c_{10} | c_6, c_4, c_2, A),
\]

but \( c_4 \) leaves the 13–center with \( b_4 \), then \( c_4 \) must re-enter with \( b_4 \), so we have

\[
\pi_{b_{10}} = (C, b_2, b_4, b_6 | b_{10} | \ldots - b_5, b_3, b_1, A),
\]

but again, \( b_4 \) leaves the 13–center with \( a_4 \), so \( b_4 \) re-enters with \( a_3 \), and then we get

\[
\pi_{a_{10}} + \{30_1\} = (C, a_1, a_3, a_5 | \ldots - a_{10} | a_6, a_4, a_2, B),
\]

which is a contradiction. Thus \( a_3 \) re-enters to the 13–center with \( c_4 \).

Here, just by convenience we work in \( \Pi^* \). Let \( \pi^*_{a_{10}} \) be the permutation of \( \Pi^* \) where \( a_{10} \) enters in the 13–center. So,

\[
\pi^*_{a_{10}} = (C, a_1, a_4, a_5 | a_{10} | \ldots - a_6, a_3, a_2, B).
\]

**Claim 25.** \( b_2 \) does not change with \( b_1 \) or, if \( \text{hal}(a_5) = 3 \) then \( b_3 \) does not change with \( b_1 \) in the 15th–gate. Moreover, in both cases \( N^a_{15}(\Pi) \leq 18 \).

**Proof of Claim 25** If \( b_2 \) does not change with \( b_1 \) in the 15th–gate, by Remark 20 \( N^a_{15}(\Pi) \leq 18 \).

So we assume that \( b_2 \) changes with \( b_1 \) in the 15th–gate. Like \( N^a_{15}(\Pi) = 19 \), by Remark 20 and Lemma 18 \( \text{hal}(a_5) \) is 3. When \( a_5 \) leaves the 13–center, this swap is with \( b_2 \), so in that moment we have the following situation

\[
(... | a_2, a_3, b_1) b_2 | a_6, ...),
\]

When \( b_2 \) changes with \( b_1 \) in the 15th–gate, we have the following

\[
(... | a_2, a_3, b_2, b_1, a_3, a_2 | ...),
\]

\( a_5 \) re-enters in the 13–center with \( b_3 \) and must to change with either \( a_2 \) or \( a_3 \) in the 15th–gate to complete 3 halvings because at most \( a_3 \) has changed in the 15th–gate with \( a_1 \) and \( a_4 \), this implies that that must be an \( a \) in the 16th–position and that is only possible if \( b_1 \) swaps with the leftmost \( a \) of the 13–center, and so when \( a_5 \) leaves the 13–center and \( b_3 \) enters in it, the permutation is

\[
(... | b_1, a_2, a_3, a_2, b_3 | a_5, ...),
\]

but \( a_4 \) re-enters in the 13–center with \( b_3 \), and there are no more \( b \)'s in the 13–center until after \( a_4 \) leaves it, thus no one moves \( b_1 \) from the 13th–position and therefore \( b_4 \) does not change with \( b_1 \) in the 15th–gate. This and Remark 20 imply \( N^a_{15}(\Pi) \leq 18 \). This completes the proof of Claim 25.

If \( N^a_{15}(\Pi) = 18 \) and knowing that \( \text{hal}(b_3) + \text{hal}(b_2) + \text{hal}(b_1) \leq 2 \), by Remark 20 we get that \( \text{hal}(b_3) \) is 3 and also we have the same configuration \( (C, b_2, b_3, b_6 | \ldots - b_{10} | b_5, b_3, b_1, A) \). Then the hypotheses of the Claim 25 are satisfied and consequently \( N^a_{15}(\Pi) \leq 18 \).

But again, if \( N^a_{15}(\Pi) = 18 \) and \( \text{hal}(c_3) + \text{hal}(c_2) + \text{hal}(c_1) \leq 2 \) then \( \text{hal}(c_5) \) is equal to 3 and, by Claim 25 \( N^{a+c}_{15}(\Pi) \leq 18 \), and this is a contradiction. So \( N^{a+c}_{15}(\Pi) = 19, N^{b+}\Pi \leq 18 \) and \( N^{a+c}_{15}(\Pi) \leq 17 \).

Now we suppose that \( N^a_{15}(\Pi) \leq 17 \). The only case we concern about is when \( N^a_{15}(\Pi) = 19 \). Since \( C \) satisfies the Claim 25 then in the moment that \( C \) changes with \( A \) we will get \( N^{a+c}_{15}(\Pi) \leq 18 \), which is a contradiction. Thus \( N^{a+c}_{15}(\Pi) = 19, N^{b+}\Pi \leq 17 \) and \( N^{a+c}_{15}(\Pi) \leq 18 \).

So, \( N_{15}(\Pi) = N^{a+c}_{15}(\Pi) + N_{15}(\Pi) = 69 \). This implies that \( N_{14}(\Pi) = 75 \), and by (3) we are done.

6. Concluding Remarks

In this paper we have presented a result that relate the number of \( (\leq k) \)-edges with 3-decomposability. That is, every set of points in the plane which has a certain number of \( (\leq k) \)-edges, can be grouped into three independent equal sized sets. Theorem 1 goes a step forward to the understanding of the structure of sets minimizing the number
of \((\leq k)\)-edges. Aichholzer et al. [6] established that such sets always have a triangular convex hull. Here we show that these sets also are 3-decomposable.

As an application of Theorem 1, we give a free computer-assisted proof that the rectilinear crossing number of \(K_{30}\) is 9726. This closes the gap between 9723 and 9726, the best lower and upper bounds previously known.

In view of Theorem 1 we now give a more precise version of the Conjecture 1 in [2]:

**Conjecture 26.** For each positive integer \(n\) multiple of 3, all crossing-minimal geometric drawings of \(K_n\) have exactly \(3\left\lfloor \frac{k+2}{2} \right\rfloor (\leq k)\)-edges for all \(0 \leq k \leq n/3\).

We believe that Conjecture 26 is one of the main problems to solve in order to understand the basic structure of the crossing-minimal geometric drawings of \(K_n\).

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