A simple counting argument of the irreducible representations of SU(N) on mixed product spaces

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Abstract: That the number of irreducible representations of SU(N) on $V^\otimes k$ (which is also the number of Young tableaux with $k$ boxes) is given by the number of involutions in $S_k$ is a well known result (see, e.g., [1] and other standard textbooks). In this paper, we present an alternative proof for this fact using a basis of projection and transition operators [2, 3] of the algebra of invariants of SU(N) on $V^\otimes k$. This proof easily generalizes to the irreducible representations of SU(N) on mixed product spaces $V^\otimes m \otimes (V^*)^\otimes n$, implying that the number of irreducible representation of SU(N) on a product space $V^\otimes m \otimes (V^*)^\otimes n$ remains unchanged if one exchanges factors $V$ for $V^*$ and vice versa, as long as the total number of factors remains unchanged.

1 Introduction

The aim of this paper is to provide an alternative, compact proof of Theorem 1, which states that the number of irreducible representations of the special unitary group SU(N) on a product space $V^\otimes k$ (for some positive integer $k$) is the same as the number of self-inverse permutations (involutions) in the symmetric group $S_k$; our proof is presented in section 2. We then go beyond Theorem 1 and give a counting argument for the irreducible representations of SU(N) on mixed product spaces $V^\otimes m \otimes (V^*)^\otimes n$, c.f. Corollary 1 in section 3.

Before presenting our proof of Theorem 1, we outline, without detail, the steps involved in the standard proof of this theorem (see, e.g., [1] and other standard textbooks). This standard proof critically involves Young tableaux [4], which are combinatorial objects that classify the irreducible representations of SU(N) on $V^\otimes k$. A Young tableau of size $k$ is an arrangement of $k$ boxes that are left justified and top justified, and each box is filled with a unique integer in $\{1, 2, \ldots, k\}$ such that the numbers increase from left to right and from top to bottom [5, 6]; for example

\[
\begin{array}{cccc}
1 & 3 & 4 & 9 \\
2 & 7 \\
5 & 8 \\
6 \\
\end{array}
\]

is a Young tableau. We denote the set of all Young tableaux consisting of $k$ boxes by $\mathcal{Y}_k$. As already stated, the Young tableaux in $\mathcal{Y}_k$ classify the irreducible representations of SU(N) on $V^\otimes k$ [4, 5, 7–9].

The Robinson-Schensted (RS) prescription [10, 11] defines a bijection between the symmetric group $S_k$ and the set $\mathcal{Y}_k^{PQ}$ of all pairs of Young tableaux of the same shape

\[
S_k \overset{\text{RS}}{=} \mathcal{Y}_k^{PQ} := \{(P, Q) \in \mathcal{Y}_k \times \mathcal{Y}_k | P \text{ and } Q \text{ have the same shape}\}
\]

(for the definition of the Robinson-Schensted bijection refer to, e.g., [6, 10, 11]). Ref. [10] refers to the unique pair of tableaux corresponding to a particular permutation $\rho \in S_k$ as the $P$-symbol and the $Q$-symbol of
Figure 1: This table shows the elements in $\mathcal{Y}^{PQ}_3$, the set of all ordered pairs $(P_\rho, Q_\rho)$ of tableaux of the same shape, where $P_\rho, Q_\rho \in \mathcal{Y}_3$. Each of the pairs $(P_\rho, Q_\rho)$ corresponds to a unique permutation $\rho \in S_3$ via the Robinson-Schensted correspondence; this permutation is captured together with the pair of tableaux here. The diagonal (shaded) elements are those that obey $P_\rho = Q_\rho$, which are clearly in 1-to-1 correspondence with the elements of $\mathcal{Y}_3$ (c.f. eq. (3)). Notice that all permutations corresponding to pairs $(P_\rho, P_\rho)$ on the diagonal are involutions, and none of the permutations on the off-diagonal are involutions, as is claimed in Theorem 1. Clearly, the block structure one obtains from arranging the pairs $(P_\rho, Q_\rho)$ in a table is the same as that obtained from arranging Hermitian projection and unitary transition operators in a table (c.f. [2, 3]).

The proof of Theorem 1 shown below clarifies that the here exemplified correspondence between the pairs of tableaux in $\mathcal{Y}^{PQ}_k$ and the Hermitian projection and unitary transition operators in $S_k$ is true in general.

$$\rho; \text{ we will denote the unique ordered pair of tableaux corresponding to } \rho \text{ by } (P_\rho, Q_\rho)\text{ in order to make the permutation } \rho \text{ explicit. Notice that the subset of } \mathcal{Y}^{PQ}_k \text{ in which } P_\rho = Q_\rho \text{ is isomorphic to } \mathcal{Y}^{PP}_k,$$

$$\mathcal{Y}_k \cong \{(P, P)|P \in \mathcal{Y}_k\} \subset \mathcal{Y}^{PQ}_k \quad (3)$$

(see Figure 1 for an example illustrating the set $\mathcal{Y}^{PQ}_k$ and eq. (3)). It remains to show that the permutations in $S_k$ that map to the set $\mathcal{Y}^{PP}_k$ are exactly those that are their own inverse.

It is shown in [12] that, under the RS prescription, the $P$-symbol of the inverse permutation $\rho^{-1}$ is $Q_\rho$, and the $Q$-symbol of $\rho^{-1}$ is $P_\rho$, that is

$$P_{\rho^{-1}} = Q_\rho \quad \text{and} \quad Q_{\rho^{-1}} = P_\rho,$$

such that

$$\rho \leftrightarrow_{RS} (P_\rho, Q_\rho) \quad \text{and} \quad \rho^{-1} \leftrightarrow_{RS} (P_{\rho^{-1}}, Q_{\rho^{-1}}) = (Q_\rho, P_\rho).$$

$$\rho = \text{id}$$

$$\rho = (23)$$

$$\rho = (132)$$

$$\rho = (123)$$

$$\rho = (12)$$

$$\rho = (13)$$
If $\rho$ is in involution, that is $\rho = \rho^{-1}$, then it’s $P$-symbol and $Q$-symbol must be equal. Since the Robinson-Schensted prescription mapping every permutation $\rho \in S_k$ to a unique pair of tableaux $(P_\rho, Q_\rho)$ (where $P_\rho$ and $Q_\rho$ have the same shape) is a bijective map to begin with, we have now obtained a bijection between the involutions of $S_k$ and the Young tableaux in $Y_k$. With all these prerequisites fully established, one obtains the following Theorem:

**Theorem 1 (Involutions and irreducible representations of $\text{SU}(N)$)**  
The number of irreducible representations of $\text{SU}(N)$ on a product space $V^\otimes k$ is given by the number of involutions in the symmetric group $S_k$.

The full details of the argument underlying Theorem 1, including proofs, is available in Donald Knuth’s book “The Art of Computer Programming, Volume 3” [1, section 5.1.4]. Despite its elegance, the proof is rather lengthy: it takes up approximately 10 pages. Knuth himself comments: “This connection between involutions and tableaux is by no means obvious, and there is probably no very simple way to prove it”. The present paper demonstrates that Knuth might have been more optimistic: We present a proof (leading to a more general statement, Corollary 1) that is less than one page long. Our argument exploits concepts of representation theory, in particular the theory of invariants.

### 2 Invariant theory of $\text{SU}(N)$ and the projector basis

Consider a product representation of $\text{SU}(N)$ constructed from its fundamental representation on a given vector space $V$ with $\dim(V) = N$, whose action will simply be denoted by $v \mapsto Uv$ for all $U \in \text{SU}(N)$ and $v \in V$. Choosing a basis $\{e_{(i)}| i = 1, \ldots, \dim(V)\}$ such that $v = \sum_{i} v^{(i)} e_{(i)}$, this becomes $v \mapsto U^{v} v$. This immediately induces a product representation of $\text{SU}(N)$ on $V^\otimes k$ if one uses this basis of $V$ to induce a basis on $V^\otimes k$ so that a general element $v \in V^\otimes k$ takes the form $v = \sum_{i} v^{(i)} e_{(i)} \otimes \cdots \otimes e_{(i_k)}$.

$$U \circ v = U \circ v^{\rho_1 \cdots \rho_k} e_{(i_1)} \otimes \cdots \otimes e_{(i_k)} := U^{\rho_1}_{i_1} \cdots U^{\rho_k}_{i_k} v^{\rho_1 \cdots \rho_k} e_{(i_1)} \otimes \cdots \otimes e_{(i_k)}. \tag{6}$$

The theory of invariants [13] exploits a set of functions (known as invariants) that leave the action of the group invariant. That is, $\sigma$ is said to be an invariant of $\text{SU}(N)$ if, for every $U \in \text{SU}(N)$,

$$\sigma \circ U = U \circ \sigma, \tag{7}$$

where $\circ$ denotes the composition of linear maps.

Since all the factors in $V^\otimes k$ are identical, the notion of permuting factors is a natural one that leads to a map from the permutation group $S_k$ to the space of invertible linear maps on $V^\otimes k$, $\text{GL}(V^\otimes k)$. In particular, we define the action of a permutation $\rho \in S_k$ on $V^\otimes k$ by

$$\rho \circ v = \rho \circ v^{\rho_1 \cdots \rho_k} e_{(i_1)} \otimes \cdots \otimes e_{(i_k)} := v^{\rho_1 \cdots \rho_k} e_{(\rho_{(i_1)})} \otimes \cdots \otimes e_{(\rho_{(i_k)})}. \tag{8}$$

If $\dim(V) = N \geq k$, the map $S_k \rightarrow \text{GL}(V^\otimes k)$ defined through (8) is bijective and we will simply refer to the the image of $\rho \in S_k$ by $\rho$. In the following, we therefore think of $S_k$ as a subgroup of $\text{GL}(V^\otimes k)$. Notice that the requirement $N \geq k$ is necessary to ensure that none of the linear maps in $S_k$ become linearly dependent.

From the definitions (6) and (8) one immediately infers that the product representation commutes with all permutations on any $v \in V^\otimes k$, $U \circ \rho \circ v = \rho \circ U \circ v$, implying that any such permutation $\rho \in S_k$ is an

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1. The theory of invariants is also described in many modern textbooks such as [7–9, 14], just to name a few.
2. Permuting the basis vectors instead involves $\rho^{-1}$: $v^{\rho_1 \cdots \rho_k} e_{(i_1)} \otimes \cdots \otimes e_{(i_k)} = v^{\rho_1 \cdots \rho_k} e_{(\rho_{(i_1)})} \otimes \cdots \otimes e_{(\rho_{(i_k)})}$. 

3
invariant of \(SU(N)\) in accordance with eq. (7). It can further be shown that these permutations in fact span the space of all linear invariants of \(SU(N)\) on \(V^\otimes k\) \[8\]. The permutations are thus referred to as the primitive invariants of \(SU(N)\) on \(V^\otimes k\). The full space of linear invariants (viewed as a subspace of the space of linear maps on \(V^\otimes k\), \(\text{Lin}(V^\otimes k)\)) is then given by

\[
\text{API}(SU(N),V^\otimes k) := \left\{ \sum_{\rho \in S_k} \alpha_\rho \sigma \mid \alpha_\rho \in \mathbb{C}, \rho \in S_k \right\} \subset \text{Lin}(V^\otimes k) .
\] (9)

In [3], we explained that there exists a basis of the algebra of invariants \(\text{API}(SU(N),V^\otimes k)\) in terms of Hermitian Young projection operators onto the irreducible representations of \(SU(N)\) [2] and the unitary transition operators between equivalent irreducible representations (transition operators are essentially intertwining operators acting on the whole space [3], c.f. eqns. (12)). For the sake of completeness, we provide a short argument here:

Let \(N \geq k\). Consider a set of Hermitian projection operators onto the irreducible representations of \(SU(N)\) on \(V^\otimes k\), and denote this set by \(\mathfrak{P}_k\). Suppose that the operators in \(\mathfrak{P}_k\) are mutually transversal (following the nomenclature of [15]), that is to say that the images of any two Hermitian projectors \(P_i, P_j \in \mathfrak{P}_k\) only intersect at 0,

\[
P_i P_j = \delta_{ij} .
\] (10)

Then, clearly, \(P_i\) and \(P_j\) are orthogonal under the scalar product

\[
\langle A | B \rangle := \text{tr}(A^\dagger B) .
\] (11)

Using Schur’s Lemma [16], we may complete the set \(\mathfrak{P}_k\) to a basis of \(\text{API}(SU(N),V^\otimes k)\) (we cite a combination of the versions of Schur’s Lemma given in [6, 16, 17]):

**Lemma 1 (Schur’s Lemma)** Let \(V_i\) and \(V_j\) be two irreducible \(G\)-modules of a group \(G\). Let \(T_{ij} : V_j \rightarrow V_i\) be a \(G\)-homomorphism. Then

1. \(T_{ij}\) is a \(G\)-isomorphism if and only if \(V_i\) and \(V_j\) correspond to equivalent representations of \(G\), or
2. \(T_{ij}\) is the zero map.

Since each projection operator \(P_i : V_i \rightarrow V_i\) defines an irreducible \(SU(N)\)-module \(V_i\) on \(V^\otimes k\), Lemma 1 ensures us that for each pair of Hermitian projection operators \(P_i : V_i \rightarrow V_i\) and \(P_j : V_j \rightarrow V_j\) corresponding to equivalent representations there exists a pair of transition operators \(T_{ij} : V_j \rightarrow V_i\) and \(T_{ji}\) satisfying

\[
T_{ij} = T_{ji}^\dagger
\]
\[
P_i T_{ij} = T_{ij} P_j
\]
\[
T_{ij} T_{ji}^\dagger = P_i
\] (eqns. (12) describe what it means for \(T_{ij}\) to be an \(SU(N)\)-isomorphism). We will denote the set of transition operators by \(\mathfrak{T}_k\). Thus, the set of projection operators \(\mathfrak{P}_k\) can be augmented by \(\mathfrak{T}_k\) to form a set of linearly independent operators \(\mathfrak{P}_k \cup \mathfrak{T}_k\), which are mutually orthogonal under the scalar product (11). From part 2 of Schur’s Lemma, we know that there are exist no further operators that are linearly independent from the
operators in $\Psi_k \cup \mathfrak{T}_k$. Thus, $\Psi_k \cup \mathfrak{T}_k$ constitutes a basis of $\text{API} (\text{SU}(N), V \otimes^k)$; we refer to this basis as the projector basis,\(^4\) and denote it by $\mathfrak{S}_k$,

$$\mathfrak{S}_k = \Psi_k \cup \mathfrak{T}_k .$$

(13)

Note that the requirement $N \geq k$ ensures that none of the operators in $\mathfrak{S}_k$ vanish.\(^5\)

For $N \geq k$, the symmetric group $S_k$ and the set of all Hermitian projection and unitary transition operators of $\text{SU}(N)$ on $V \otimes^k$, $\mathfrak{S}_k$, both constitute a basis of $\text{API} (\text{SU}(N), V \otimes^k)$, implying that these two sets must have the same size,

$$|S_k| = |\mathfrak{S}_k| .$$

(14)

We are now in a position to present a compact proof of Theorem 1:

**Proof of Theorem 1:** Consider the elements of $S_k$ to be linear maps on $V \otimes^k$ as given in (8). These linear maps are unitary with respect to the scalar product (11), that is $\rho^{-1} = \rho^\dagger$ for all $\rho \in S_k$.

Let $n_T (S_k)$ denote the number of non-Hermitian elements in $S_k$, and let $n_P (S_k)$ denote the number of Hermitian elements in $S_k$. Since $S_k$ is a group and all its elements are unitary, the elements of $S_k$ either are involutions $\rho^1 = \rho^{-1} = \rho$, or occur in Hermitian conjugate pairs $(\rho, \rho^1)$, forcing $n_T (S_k)$ to be an even number. For each pair $(\rho, \rho^1)$ (where $\rho$ is not an involution) we can construct a Hermitian element $h_\rho$ and an anti-Hermitian element $a_\rho$ as

$$h_\rho := \rho + \rho^\dagger \quad \text{and} \quad a_\rho := \rho - \rho^\dagger .$$

(15)

Clearly, the set of all Hermitian elements of $S_k$ together with the set of all $h_\rho$ and $a_\rho$ for all non-Hermitian $\rho \in S_k$ constitutes a basis of $\text{API} (\text{SU}(N), V \otimes^k)$, which will be denoted by $\tilde{S}_k$. In particular, $\tilde{S}_k$ splits into two disjoint subsets:

1. the set $H_k \subset \tilde{S}_k$ consisting of all Hermitian elements in $\tilde{S}_k$ with size

$$|H_k| = n_P (S_k) + \frac{n_T (S_k)}{2} ,$$

(16a)

2. the set $A_k \subset \tilde{S}_k$ containing all anti-Hermitian elements in $\tilde{S}_k$ with size

$$|A_k| = \frac{n_T (S_k)}{2} .$$

(16b)

Similarly, the projector basis of $\text{API} (\text{SU}(N), V \otimes^k)$, $\mathfrak{S}_k$, consists of Hermitian elements (the projection operators) and elements that occur in Hermitian conjugate pairs (the transition operators). Thus, we may again form a set $\tilde{\mathfrak{S}}_k$ from $\mathfrak{S}_k$, such that $\tilde{\mathfrak{S}}_k$ is the union of two disjoint sets $\tilde{\mathfrak{H}}_k$ and $\tilde{\mathfrak{A}}_k$ consisting of Hermitian and anti-Hermitian elements, respectively, analogous to eqns. (15) and (16). Since $H_k$ and $A_k$, respectively, $\tilde{\mathfrak{H}}_k$ and $\tilde{\mathfrak{A}}_k$ are disjoint, it follows that

$$|H_k| = |\tilde{\mathfrak{H}}_k| \quad \text{and} \quad |A_k| = |\tilde{\mathfrak{A}}_k|$$

(17)

for both sets $\tilde{\mathfrak{S}}_k = H_k \cup A_k$ and $\tilde{\mathfrak{S}}_k = \tilde{\mathfrak{H}}_k \cup \tilde{\mathfrak{A}}_k$ to constitute a basis of the algebra of invariants $\text{API} (\text{SU}(N), V \otimes^k)$.

\(^4\)In [15] such a basis is also called a multiplet basis.

\(^5\)For $N < k$, it is possible that elements of $\mathfrak{S}_k$ become zero. Such operators are referred to as dimensionally null operators and are discussed in more detail in [2, 3, 18].
Clearly, a linear combination of Hermitian objects will be Hermitian. Thus, only the elements of $A_k$ and $A_k$ can be used to construct the non-Hermitian elements of $\text{API} (\text{SU}(N), V^\otimes k)$. From eq. (17) it then follows that

$$n_T (\mathcal{G}_k) = |\mathcal{T}_k| = 2|A_k| = 2|A_k| = n_T (S_k) \ .$$

(18)

Knowing that the sets $\mathcal{G}_k$ and $S_k$ have the same size (c.f. eq. (14)) as do their subsets containing only the non-Hermitian elements (eq. (18)), it follows that

$$|\Psi_k| = n_P (\mathcal{G}_k) = |\mathcal{G}_k| - n_T (\mathcal{G}_k) \stackrel{\text{eq. (18)}}{=} |S_k| - n_S (S_k) = n_P (S_k) \ ,$$

(19)

thus concluding the proof of the theorem.

\[\square\]

3 The irreducible representations of $\text{SU}(N)$ on mixed product spaces

As explained in the previous section, the permutations in $S_k$ span the algebra of invariants $\text{API} (\text{SU}(N), V^\otimes k)$. These permutations may be depicted graphically as birdtracks [8, 19]: Consider a particular permutation $\rho \in S_k$. To obtain the birdtrack of $\rho$, we write two columns $(1, 2, 3, \ldots, k)$ next to each other, and then connect the entry $i$ of the right column to the value of $\rho(i)$ in the left column, marking each line with an arrow from right to left. We then delete the numbers from the diagram, retaining only the lines. For example,

the last image is the birdtrack of $\rho$. Birdtracks ideally lend themselves to be interpreted as linear maps on $V^\otimes k$ according to eq. (8), for example the equation

$$(123) \circ v_1 \otimes v_2 \otimes v_3 = v_3 \otimes v_1 \otimes v_2$$

(21)

is written in the birdtrack formalism as

where each term in the product $v_1 \otimes v_2 \otimes v_3$ (written as a tower $v_1 \otimes v_2 \otimes v_3$) can be thought of as being moved along the lines of $\ldots$. Furthermore, the birdtrack formalism allows for an efficient way to talk about antifundamental representations of $\text{SU}(N)$ on the mixed space $V^\otimes m \otimes (V^*)^\otimes n$ and the associated algebra of invariants:

As one considers the fundamental representation of $\text{SU}(N)$ on a vector space $V$, one may also consider the \textit{anti-fundamental} representation of $\text{SU}(N)$ on the dual space $V^*$. Again, the irreducible representations
of SU(N) on a mixed product space $V^\otimes m \otimes (V^*)^\otimes n$ can be classified through the invariants living in the algebra \([8, 9]\)

$$\text{API} \left( \text{SU}(N), V^\otimes m \otimes (V^*)^\otimes n \right) := \left\{ \sum_{\rho \in S_{m,n}} \alpha_\rho \rho \mid \alpha_\rho \in \mathbb{C} \right\} \subset \text{Lin} \left( V^\otimes m \otimes (V^*)^\otimes n \right), \quad (23)$$

where $S_{m,n}$ denotes the set of primitive invariants of SU(N) on $V^\otimes m \otimes (V^*)^\otimes n$ as described below (c.f. eqns. (24)). The elements of $S_{m,n}$ are in a 1-to-1 correspondence with the primitive invariants in $S_{m+n}$ as we can construct them graphically from the elements of $S_{m+n}$ by swapping the left and right endpoints on the specific $V$ in $V^\otimes (m+n)$ to be converted into is dual vector space $V^*$. An example will give clarity: The primitive invariants $S_3$ map onto those in $S_{2,1}$ as

$$S_3 : \quad \begin{array}{cccccc}\end{array} \quad S_{2,1} : \quad \begin{array}{cccccc}\end{array}\quad (24a)$$

Since the primitive invariants of SU(N) on $V^\otimes m \otimes (V^*)^\otimes n$ are Hermitian with respect to the scalar product (11) if and only if their birdtrack expressions are symmetric under a reflection about the vertical axis (c.f. \([2, 8]\))\footnote{This statement is only true for the \textit{primitive} invariants: The birdtrack formalism allows for a compact description of operators in the algebra of invariants, which, on the flip-side symmetric under a reflection about their vertical axis. However, once these operators are resolved into their constituent primitive invariants, their Hermiticity properties are once again determined through a reflection about the vertical axis \([2, 8, 18]\).}, the graphical procedure of transforming some of its fundamental legs into antifundamental legs does not affect the Hermiticity of the birdtrack. Thus, the subset of Hermitian elements in $S_{m,n}$ and $S_{m+n}$ have the same size.

It should be noted that $S_{m,n}$, unlike $S_{m+n}$, is not a group; for example, the last four elements of $S_{2,1}$ in (24b) do not have an inverse.

The algebra of invariants $\text{API} \left( \text{SU}(N), V^\otimes m \otimes (V^*)^\otimes n \right)$ again allows for an orthogonal basis in terms of Hermitian projection and unitary transition operators, as the argument based on Schur’s Lemma (Lemma 1) given in section 2 can immediately be adapted to this situation: Lemma 1 tells us that the set of projection and transition operators constitute the largest possible set of linearly independent operators. (Alternatively, an argument in terms of Clebsch-Gordan operators is given in \([20]\)). The projector basis of $\text{API} \left( \text{SU}(N), V^\otimes m \otimes (V^*)^\otimes n \right)$ will be denoted by $\mathcal{S}_{m,n}$.

Therefore, since $S_{m,n}$ and $S_{m+n}$ contain the same number of Hermitian elements, and since $\mathcal{S}_{m,n}$ and $S_{m,n}$ each constitute a basis of the algebra of invariants $\text{API} \left( \text{SU}(N), V^\otimes m \otimes (V^*)^\otimes n \right)$, the proof of Theorem 1 immediately translates to the irreducible representations of SU(N) on the mixed space $V^\otimes m \otimes (V^*)^\otimes n$. This gives rise to the following Corollary:

\textbf{Corollary 1 (Irreducible representations of SU(N) on $V^\otimes m \otimes (V^*)^\otimes n$)} The number of irreducible representations of SU(N) over a product space $V^\otimes m \otimes (V^*)^\otimes n$ does not depend on $m$ and $n$ individually, but only on the sum $(m + n)$. In particular, the number of irreducible representations of SU(N) on any space $V^\otimes m \otimes (V^*)^\otimes n$ satisfying $m + n = k$ is the same as the number of irreducible representations of SU(N) on $V^\otimes k$.
4 Conclusion

In the present paper, we provided an alternative, simple proof of Theorem 1, which states that the number of irreducible representations of $SU(N)$ on $V^\otimes k$ is the same as the number of involutions in the symmetric group $S_k$. The proof given in section 2 heavily utilizes the theory of invariants. In particular, we rely on the fact that the set of Hermitian projection operators on the irreducible representations of $SU(N)$ on $V^\otimes k$, and the unitary transition operators between equivalent representations, span the algebra of invariants $\text{API}(SU(N), V^\otimes k)$; this fact was previously proven in [3].

In section 3, we used the birdtrack formalism to show that the number of Hermitian primitive invariants of $SU(N)$ on $V^\otimes k$ is the same as the number of Hermitian primitive invariants on $V^\otimes m \otimes (V^*)^\otimes n$ for $m + n = k$ (we understand Hermiticity with respect to the scalar product (11)). This, together with the fact that the set of Hermitian projection and unitary transition operators $S_{m,n}$ also constitutes a basis for $\text{API}(SU(N), V^\otimes m \otimes (V^*)^\otimes n)$, allowed us to formulate Corollary 1, which states that the number of irreducible representations of $SU(N)$ on any product space consisting of factors $V$ and $V^*$ only depends on the total number of factors, but not on which of them are $V$ and which are its dual $V^*$.

Corollary 1 has interesting consequences: As was explained in section 1, the irreducible representations of $SU(N)$ on $V^\otimes k$ are classified by Young tableaux. Similarly, one may construct tableaux corresponding to the irreducible representations of $SU(N)$ on $V^\otimes m \otimes (V^*)^\otimes n$ using an algorithm that goes back to Littlewood and Richardson [21]. Corollary 1 states that the Littlewood-Richardson construction must yield the same number of tableaux on $V^\otimes m \otimes (V^*)^\otimes n$ and $V^\otimes m' \otimes (V^*)^\otimes n'$ if $m + n = m' + n'$. In particular, the number of Littlewood-Richardson tableaux of $SU(N)$ on $V^\otimes m \otimes (V^*)^\otimes n$ must be the same number as $|Y_{m+n}|$, the number of Young tableaux consisting of $m + n$ boxes.

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