Evasion and prediction —
— the Specker phenomenon and Gross spaces

Jörg Brendle*

Department of Mathematics, Bar–Ilan University, 52900 Ramat–Gan, Israel

and

Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

Abstract

We study the set–theoretic combinatorics underlying the following two algebraic phenomena.

(1) A subgroup $G \leq \mathbb{Z}^\omega$ exhibits the Specker phenomenon iff every homomorphism $G \to \mathbb{Z}$ maps almost all unit vectors to 0. Let $se$ be the size of the smallest $G \leq \mathbb{Z}^\omega$ exhibiting the Specker phenomenon.

(2) Given an uncountably dimensional vector space $E$ equipped with a symmetric bilinear form $\Phi$ over an at most countable field $\mathbb{K}$, $(E, \Phi)$ is strongly Gross iff for all countably–dimensional $U \leq E$, we have $\dim(U^\perp) \leq \omega$.

Blass showed that the Specker phenomenon is closely related to a combinatorial phenomenon he called evading and predicting. We prove several additional results (both theorems of $ZFC$ and independence proofs) about evading and predicting as well as $se$, and relate a Luzin–style property associated with evading to the existence of strong Gross spaces.

* The author would like to thank the MINERVA-foundation for supporting him
Introduction

History and motivation. The goal of this work is the investigation of the set-theoretic combinatorics underlying two algebraic phenomena which do not seem related at first glance.

One of them, coming from abelian group theory, has been studied in recent work of Blass [Bl 2]. We say that $G \leq (\mathbb{Z}^\omega, +)$ exhibits the Specker phenomenon iff, given a group homomorphism $h : G \to \mathbb{Z}$, for all but finitely many unit vectors $e_n$ (i.e. $e_n \in \mathbb{Z}^\omega$ is defined by $e_n(n) = 1$ and $e_n(i) = 0$ for $i \neq n$), one has $h(e_n) = 0$. We let $\text{se} := \min\{|G|; \oplus_n \langle e_n \rangle \leq G \leq \mathbb{Z}^\omega \land \forall \text{homomorphisms } h : G \to \mathbb{Z}, h(e_n) = 0 \text{ for almost all } n\}$, the Specker–Eda number. Specker [S, Satz III] proved that $\mathbb{Z}^\omega$ exhibits the Specker phenomenon, and Eda [E] showed that $\text{se} = 2^\omega$ under Martin’s Axiom $\text{MA}$ as well as the consistency of $\text{se} < 2^\omega$. Thus the question — investigated in [Bl 2] — comes up whether any of the combinatorial cardinal invariants of the continuum equals $\text{se}$.

The other phenomenon comes from quadratic form theory, and has been intensively discussed in recent work of Spinas, Baumgartner, and Shelah (see [BSp], [Sp 1], [ShSp] and the survey [Sp 2]). Given a finite or countable field $\mathbb{K}$, an uncountably–dimensional $\mathbb{K}$–vector space $E$ equipped with a symmetric bilinear form $\Phi : E^2 \to \mathbb{K}$ is called (strongly) Gross iff for all countably–dimensional subspaces $U \leq E$, one has $\dim U^\perp < \dim E$ ($\dim U^\perp \leq \omega$) — here, $U^\perp := \{e \in E; \forall u \in U \ (\Phi(e, u) = 0)\}$ denotes the orthogonal complement of $U$ in $E$. Many of the results in the work mentioned above say roughly that, for certain cardinal invariants $\text{ci}$ of the continuum, if $\text{ci} = \omega_1$, then there is a strong Gross space over $\mathbb{K}$, and that for certain (different!) cardinals $\text{ci}'$, if $\text{ci}' > \omega_1$, then there are no strong Gross spaces over $\mathbb{K}$. Thus the question naturally arises whether there is a cardinal (defined in terms of set–theoretic combinatorics) such that its being $\omega_1$ is equivalent to the existence of strong Gross spaces.

Both phenomena turn out to be related to the following combinatorial concept which has been introduced as well in Blass’ work [Bl 2, section 4]. Given a finite or countable set $S$, an $S$–valued predictor (or simply: $S$–predictor) is a pair $\pi = (D_\pi, (\pi_n; \ n \in D_\pi))$ where $D_\pi \subseteq \omega$ is infinite and for each $n \in D_\pi$, $\pi_n$ is a function from $S^n$ to $S$. We say $\pi$ predicts $f \in S^\omega$ iff for all but finitely many $n \in D_\pi$, we have $f(n) = \pi_n(f|n)$; otherwise $f$ evade
We set
\[ e := \min\{|F|; F \subseteq \omega^\omega \land \forall \pi \exists f \in F \ (f \text{ evades } \pi)\}, \]
the evasion number. A \( Z \)-predictor is called linear iff for all \( n \in D_\pi \), the function \( \pi_n : \mathbb{Z}^n \to \mathbb{Q} \) is linear with rational coefficients. We let
\[ e_\ell := \min\{|F|; F \subseteq \mathbb{Z}^\omega \land \forall \text{ linear } \mathbb{Z} \text{-predictors } \pi \exists f \in F \ (f \text{ evades } \pi)\}, \]
the linear evasion number. Clearly \( e_\ell \leq e \).

Results. To be able to explain our main results, we shall need the definition of some of the classical cardinals associated with the continuum — for more on such cardinals as well as the forcing notions used in our proofs we refer the reader to section 1. Given a \( \sigma \)-ideal \( \mathcal{I} \) on the real line, the additivity of \( \mathcal{I} \), \( \text{add}(\mathcal{I}) \), is the smallest size of a family of members of \( \mathcal{I} \) the union of which is not in \( \mathcal{I} \). \( \mathcal{L} \) and \( \mathcal{M} \) denote the ideals of Lebesgue measure zero and meager sets, respectively. For \( A, B \subseteq \omega \), \( A \subseteq^* B \) (\( A \) is almost contained in \( B \)) means that \( A \setminus B \) is finite; and for \( f, g \in \omega^\omega \), we say \( f \leq^* g \) (\( g \) eventually dominates \( f \)) iff \( \forall \infty n \ (f(n) \leq g(n)) \) (where \( \forall \infty n \) stands for for all but finitely many \( n \); similarly \( \exists \infty n \) denotes there are infinitely many \( n \)). Using this notation, the pseudointersection number \( p \) is the smallest cardinality of a family \( F \) of subsets of \( \omega \) with the strong finite intersection property (i.e. given finitely many \( A_i \in F \), \( i < n \), we have \( |\bigcap_{i<n} A_i| = \omega \) so that \( \neg\exists B \in [\omega]^\omega \forall A \in F \ (B \subseteq^* A) \) (we say: \( F \) does not have a pseudointersection). Next, the unbounding number \( b \) is the smallest size of a family \( F \subseteq \omega^\omega \) so that \( \forall g \in \omega^\omega \exists f \in F \ \exists \infty n \ (g(n) \leq f(n)) \). Finally, the splitting number \( s \) is the smallest cardinality of a family \( F \) of subsets of \( \omega \) so that \( \forall B \in [\omega]^\omega \exists A \in F \ (|A \cap B| = |B \setminus A| = \omega) \) (we say: \( A \) splits \( B \)). It is well-known ([Fr], [vD, section 3]) that \( \omega_1 \leq \text{add}(\mathcal{L}) \), \( p \leq \text{add}(\mathcal{M}) \leq b \leq 2^\omega \) and \( p \leq s \leq 2^\omega \).

Blass [Bl 2, Theorem 12, Corollaries 11 and 14] proved that \( \text{add}(\mathcal{L}) \leq e_\ell \leq \text{add}(\mathcal{M}) \), as well as \( p \leq e_\ell \). It is well-known that using standard techniques (see 3.1. for details), the consistency of \( p, \text{add}(\mathcal{L}) < e_\ell \) with \( ZFC \) can be shown. We complete this cycle of independence results (in 2.1. and 2.2.) by proving the consistency of \( e_\ell < \text{add}(\mathcal{M}) \); more explicitly:

**Theorem A.** It is consistent that \( e = \omega_1 < b = \text{add}(\mathcal{M}) = 2^\omega = \kappa \) for any regular uncountable \( \kappa \).

Another result of Blass’ concerns the relationship between the Specker–Eda number and
the concept of evasion, namely $e_\ell \leq s_\epsilon$ [Bl 2, Corollary 8 and Theorem 10]. We shall see (in 2.4.) that an upper bound to $s_\epsilon$ can be given in terms of evasion as well (the cardinal $e'$, introduced in 2.3.), and derive from this:

**Theorem B.** $s_\epsilon \leq \text{unif}(\mathcal{L})$, the size of the smallest non-measurable set of reals; in particular $s_\epsilon < b$ is consistent.

The interest in the latter consistency stems from Blass’ $s_\epsilon \leq b$ [Bl 2, Theorem 2].

In the third section we look at the phenomenon of evading and predicting in general and in particular at the relation between various forms of evasion numbers (and some other cardinals as well). Namely, we consider the spaces $n^\omega$ ($n \geq 2$), $n$–valued predictors, and the corresponding evasion numbers $e_n$ — or, more generally, compact spaces of the form $\prod_{n \in \omega} f(n) = \{g \in \omega^\omega; \forall n (g(n) < f(n))\}$ for $f \in \omega^\omega$, and the corresponding predictors and evasion numbers $e_f$ (see 3.1. for exact definitions). We let $e_{\text{ubd}} := \min\{e_f; f \in \omega^\omega\}$ and $e_{\text{fin}} := \min\{e_n; n \in \omega\}$. We shall show in 3.2. and 3.3.:

**Theorem C.** (a) $e \geq \min\{b, e_{\text{ubd}}\}$ and $s \leq e_{\text{fin}} = e_n$ for all $n$.  
(b) Both $e < e_{\text{ubd}}$ and $e_{\text{ubd}} < e_{\text{fin}}$ are consistent.

The forth section deals with Luzin–style properties related to evading: given an arbitrary finite or countable field $\mathbb{K}$, an uncountably–dimensional subspace $(G, +) \leq \mathbb{K}^\omega$ is called a Luzin group iff for all linear $\mathbb{K}$–predictors $\pi$ all but countably many elements of $G$ evade $\pi$; $G$ is generalized Luzin of size $\kappa$ iff any linear $\mathbb{K}$–predictor predicts less than $\kappa$ many elements of $G$. We shall prove (in 4.3.–4.6.):

**Theorem D.** (dichotomy theorem) (a) It is consistent that there are no generalized Luzin groups $G \leq \mathbb{K}^\omega$, where $\mathbb{K}$ is any finite field.  
(b) For any countable field $\mathbb{K}$, there is a generalized Luzin group $G \leq \mathbb{K}^\omega$ of size $b$.

**Theorem E.** (equivalence theorem) For any finite or countable field $\mathbb{K}$, the following are equivalent: 
(a) there exists a strong Gross space $(E, \Phi)$ over $\mathbb{K}$;  
(b) there is a Luzin group $G \leq \mathbb{K}^\omega$.

Using these results one gets alternative proofs of the theorems of Baumgartner, Shelah and Spinas ([BSp], [ShSp]) as well as one new result: $\text{add}(\mathcal{L}) > \omega_1$ implies the non–existence of strong Gross spaces (4.7.).

We close our considerations with a list of questions in section 5.
Notation. We use standard set-theoretic notation and refer the reader to [Ku], [Je 1], [Je 2] and [Bau] for set theory in general and forcing in particular.

Given a finite sequence $s$ (i.e. $s \in \omega^\omega$), we let $lh(s) := \text{dom}(s)$ denote the length of $s$; for $\ell \in lh(s)$, $s|\ell$ is the restriction of $s$ to $\ell$. $\hat{}$ is used for concatenation of sequences; and $\langle \rangle$ is the empty sequence. Given a finite set $A \subseteq \kappa$ and $i < |A|$, $A(i)$ denotes the $i$–th element of $A$ under the inherited ordering. Given a p.o. $P$, we shall denote $P$-names by symbols like $\hat{f}$, $\hat{\pi}$, $\hat{D}$, ...

Acknowledgment. I would like to thank Myriam for drawing the diagram on the computer.

§ 1. Cardinals and forcing notions

1.1. Cardinals. In addition to the cardinals we have seen already, we define, for a given $\sigma$–ideal $\mathcal{I}$ on the reals,

\begin{align*}
\text{cov}(\mathcal{I}) & := \min\{|F|; F \subseteq \mathcal{I} \land \bigcup F = \omega^\omega\}, \text{ the covering number of } \mathcal{I}, \\
\text{unif}(\mathcal{I}) & := \min\{|F|; F \in P(\omega^\omega) \setminus \mathcal{I}\}, \text{ the uniformity of } \mathcal{I}, \text{ and} \\
\text{cof}(\mathcal{I}) & := \min\{|F|; F \subseteq \mathcal{I} \land \forall A \in \mathcal{I} \exists B \in F (A \subseteq B)\}, \text{ the cofinality of } \mathcal{I}.
\end{align*}

Furthermore we set

\begin{align*}
\text{d} & := \min\{|F|; F \subseteq \omega^\omega \land \forall g \in \omega^\omega \exists f \in F (g \leq^* f)\}, \text{ the dominating number, and} \\
\text{r} & := \min\{|F|; F \subseteq [\omega]^\omega \land \forall A \in [\omega]^\omega \exists B \in F (|A \cap B| < \omega \text{ or } |B \setminus A| < \omega)\}, \text{ the reaping number.}
\end{align*}

Most of these invariants come in pairs, i.e. one of them can be defined from the other essentially by taking negation and modifying the range of quantifiers. Compare, e.g., $\text{cov}(\mathcal{I}) = \min\{|F|; F \subseteq \mathcal{I} \land \forall x \in \omega^\omega \exists A \in F (x \in A)\}$ with $\text{unif}(\mathcal{I}) = \min\{|F|; F \subseteq \omega^\omega \land \forall A \in \mathcal{I} \exists x \in F (x \notin A)\}$, or $\text{b} = \min\{|F|; F \subseteq \omega^\omega \land \forall f \in \omega^\omega \exists f \in F (g \leq^* f)\}$ with $\text{d}$. Other pairs are $(\text{add}(\mathcal{I}), \text{cof}(\mathcal{I}))$ and $(\text{s}, \text{r})$. One effect of this duality is that $\text{ZFC}$–proofs of inequalities between cardinals dualize (e.g., $\text{b} \geq \text{add}(\mathcal{M})$ is proved the same way as $\text{d} \leq \text{cof}(\mathcal{M})$); another, that consistency proofs involving finite support iterations
dualize as well (see [Bl 1, in particular section 5], [BaJS, section 1] or [Br] for duality). The inequalities between these cardinals (as well as some others which are crucial for our investigations) which are provable in $\text{ZFC}$ are displayed in the diagram in subsection 3.5.

1.2. Forcing notions. Hechler forcing. The Hechler p.o. $\mathbb{D}$ is defined as follows:

$$ (s, f) \in \mathbb{D} \iff s \in \omega^{<\omega} \land f \in \omega^\omega \land s \subseteq f \land f \text{ strictly increasing} $$

$$ (s, f) \leq (t, g) \iff s \supseteq t \land \forall n \in \omega \ (f(n) \geq g(n)) $$

Following Baumgartner and Dordal ([BD, § 2]; see also [BrJS, § 1]), given $t \in \omega^{<\omega}$ strictly increasing and $A \subseteq \omega^{<\omega}$, we define by induction when the rank $rk(t, A)$ is $\alpha$.

(a) $rk(t, A) = 0$ iff $t \in A$.

(b) $rk(t, A) = \alpha$ iff for no $\beta < \alpha$ we have $rk(t, A) = \beta$, but there are $m \in \omega$ and $(t_k; k \in \omega)$ such that $\forall k \in \omega: t \subseteq t_k, t_k \in \omega^m, t_k(lh(t)) \geq k$, and $rk(t_k, A) < \alpha$.

Clearly, the rank is either $< \omega_1$ or undefined (in which case we say $rk = \infty$). The following result is the main tool in the proof of Theorem A (see 2.1.).

**Lemma** (Baumgartner–Dordal [BD, § 2]; see also [BrJS, 1.2.]) Let $I \subseteq \mathbb{D}$ be dense. Set $A := \{ t; \exists f \in \omega^\omega \text{ such that } (t, f) \in I \}$. Then $rk(t^*, A) < \omega_1$ for any $t^* \in \omega^{<\omega}$. □

Mathias forcing. The Mathias p.o. $\mathbb{M}$ is defined as follows [Je 2, part one, section 3]:

$$ (s, S) \in \mathbb{M} \iff s \in \omega^{<\omega} \land S \in [\omega]^\omega \land s \text{ strictly increasing} \land \max(ran(s)) < \min(S) $$

$$ (s, S) \leq (t, T) \iff s \supseteq t \land S \subseteq T \land \forall i \in (lh(s) \setminus lh(t)) \ (s(i) \in T) $$

Laver forcing. The Laver p.o. $\mathbb{L}$ is defined as follows [Je 2, part one, section 3]:

$$ T \in \mathbb{L} \iff T \subseteq \omega^{<\omega} \text{ is a tree} \land \exists \rho \in T \forall \sigma \in T \ (\sigma \subseteq \rho \lor [\rho \subseteq \sigma \land \exists \infty n \ (\sigma^*(n) \in T)]) $$

$$ T \leq S \iff T \subseteq S $$

The $\rho$ required to exist in the above definition is usually called the stem of $T$, $\text{stem}(T)$. Furthermore, for $\rho \in T$ we let $\text{succ}_T(\rho) := \{ n \in \omega; \rho^*(n) \in T \}$, the set of successors of $\rho$ in $T$.

Laver property. Both Laver and Mathias forcing as well as their countable support iterations have the following property of p.o.’s $\mathbb{P}$, sometimes referred to as Laver property. Given $p \in \mathbb{P}$, a function $f \in \omega^\omega$ and a $\mathbb{P}$-name $\dot{g}$ for an element of $\prod_n f(n)$, there is $\phi \in \prod_n [f(n)]^n$ and $q \leq p$ so that

$$ q \Vdash_{\mathbb{P}} \forall n \ (\dot{g}(n) \in \phi(n)). $$
See, e.g., [Bau, section 9] for details. One consequence of this is that $\mathbb{P}$ adds neither random nor Cohen reals.

§ 2. Evasion and the Specker phenomenon

2.1. To prove the consistency of $e < \text{add}(\mathcal{M})$ we shall iterate Hechler forcing $\kappa$ times with finite support over a model $V$ satisfying $CH$. To make the argument that $e$ is still $\omega_1$ at the end go through smoothly, we shall consider the following property of p.o.'s $P$:

$\ast\ast$ given $F \subseteq \omega^\omega \cap V$, $F \in V$, a family of functions below the identity (i.e. $\forall n \forall f \in F (f(n) \leq n)$), such that for any countable family of predictors $\Pi$ there is $f \in F$ evading all $\pi \in \Pi$, and $\langle \tilde{\pi}_n; n \in \omega \rangle$ a sequence of $\mathbb{P}$–names for predictors, we can find a sequence $\langle \pi_n; n \in \omega \rangle \in V$ of predictors such that whenever $f \in F$ evades all $\pi_n$, then

$$\vdash \mathbb{P}^\omega f \text{ evades all } \tilde{\pi}_n^\omega.$$ 

**Theorem.** $\mathbb{D}$ satisfies $\ast\ast$.

**Proof.** We shall use the notion of rank for $\mathbb{D}$ as explained in 1.2.

Let $F$ be a family satisfying the requirements in the definition of $\ast\ast$, and let $\langle \tilde{\pi}_n; n \in \omega \rangle$ be a sequence of $\mathbb{D}$–names for predictors. Associated with the name $\tilde{\pi}_n$ we have the name $\tilde{D}_n$ and the sequence of names $\langle \tilde{\pi}_m^n; m \in \omega \rangle$ such that

$$\vdash \mathbb{D}^\omega \tilde{\pi}_n \text{ predicts on the set } \tilde{D}_n; \tilde{\pi}_m^n \text{ is the predicting function on the } m\text{–th element of } \tilde{D}_n.$$ 

Fix $n, m$. Let $I_n^m := \{(t, f); (t, f) \text{ decides the } m\text{–th element of } \tilde{D}_n \text{ (and all preceding ones)}, say: } (t, f) \vdash_{\mathbb{D}} k \text{ is the } m\text{–th element of } \tilde{D}_n^\omega \text{; and } (t, f) \text{ decides } \tilde{\pi}_n^m \text{ (and all } \tilde{\pi}_i^n, i < m \text{) on all sequences of length } k \text{ below the identity } \}$. All $I_n^m$ are dense and $I_n^{m+1} \subseteq I_n^m$. Thus, if $A_n^m := \{t; \exists f \in \omega^\omega ((t, f) \in I_n^m)\}$, we can apply Lemma 1.2. to $A_n^m$, i.e. $rk(t, A_n^m) < \infty$ for all $t \in \omega^\omega$ strictly increasing.

Now we define by recursion on rank:

- when $t$ is $(n, m)$–happy, when it is $(n, m)$–sad, and when it is minimal $(n, m)$–sad;
- $k(t, n, m) \in \omega$ and $\pi(t, n, m)$, where $\text{dom}(\pi(t, n, m)) = \{\sigma \in \omega^\omega; \text{ lh}(\sigma) = k(t, n, m) \}$ and $\forall i \in k(t, n, m) (\sigma(i) \leq i) \}$ and $\text{ran}(\pi(t, n, m)) \subseteq k(t, n, m) + 1$, for $t$'s which are
Now suppose that $\forall i \in k(t, n, m) (\sigma(i) \leq i)$,
\[
\pi(t, n, m)(\sigma) = j \iff \begin{cases} 
(t, f) \vdash k(t, n, m) \text{ is the } m \text{-th element of } \tilde{D}_n^m. 
\end{cases}
\]
and $j \leq k(t, n, m)$ or
\[
\begin{cases}
(t, f) \vdash \tilde{\pi}_m^m(\sigma) > k(t, n, m) \text{ and } j = 0
\end{cases}
\]
This makes sense by definition of the set $I_n^m$.
\[
\Rightarrow rk(t, A_n^m) = \alpha.
\]
Then we have $(t_i; i \in \omega)$ and $\ell \in \omega$ such that for all $i$: $t \subseteq t_i$, $t_i \in \omega^\ell$, $t_i(\ell(t)) \geq i$ and $rk(t_i, A_n^m) < \alpha$. If $\exists i$ such that $t_i$ is $(n, m)$–sad, then $t$ is $(n, m)$–sad, but not minimal.
Now suppose that $\forall i \in \omega$, $t_i$ is $(n, m)$–happy.
If $\{k(t_i, n, m); i \text{ such that } t_i \text{ is } (n, m) \text{–happy} \}$ is infinite, $t$ is minimal $(n, m)$–sad. In this case we can without loss assume that $i < j$ implies $k(t_i, n, m) < k(t_j, n, m)$. Let
\[
\tilde{D}(t, n, m) = \{k(t_i, n, m); i \in \omega\}
\]
and define a predictor $\tilde{\pi}(t, n, m)$ as follows:
\begin{itemize}
  \item $D_{\tilde{\pi}(t, n, m)} = \tilde{D}(t, n, m);
  \item for $k \in \tilde{D}(t, n, m)$, let $i \in \omega$ be such that $k = k(t_i, n, m)$ and set $\tilde{\pi}_k(t, n, m)(\sigma) := \pi(t_i, n, m)(\sigma)$ for $\sigma$ of length $k$ below the identity; for other $\sigma$, $\tilde{\pi}_k(t, n, m)(\sigma)$ can be defined arbitrarily.
\end{itemize}
If $\{k(t_i, n, m); i \text{ such that } t_i \text{ is } (n, m) \text{–happy} \}$ is finite, $t$ is still $(n, m)$–happy. In this case we can without loss assume that the latter set contains just one element, $k(t, n, m)$; and that $\forall i \in \omega$, $\pi(t_i, n, m)$ is the same function which we call $\pi(t, n, m)$. — This concludes the definition of happiness and sadness.

Next, for each $n \in \omega$ and each $t$ such that $\forall^\infty m t$ is $(n, m)$–happy, we define a predictor $\hat{\pi}(t, n)$ as follows:
\begin{itemize}
  \item $D_{\hat{\pi}(t, n)} = \{k(t, n, m); m \in \omega \land t \text{ is } (n, m) \text{–happy} \}$ (note that this set must be infinite as $k(t, n, m) \geq m$);
  \item for $k \in D_{\hat{\pi}(t, n)}$, let $m \in \omega$ be minimal such that $k = k(t, n, m)$ and set $\hat{\pi}_k(t, n)(\sigma) := \pi(t, n, m)(\sigma)$ for $\sigma$ of length $k$ below the identity (for other $\sigma$, $\hat{\pi}_k(t, n)(\sigma)$ can be defined arbitrarily).
\end{itemize}

\[
(n, m)\text{–happy};
\]
— sets $\tilde{D}(t, n, m)$ and predictors $\tilde{\pi}(t, n, m)$ for $t$ which are minimal $(n, m)$–sad.

$\Rightarrow k(t, A_n^m) = 0.$

Then we say $t$ is $(n, m)$–happy. We choose $f$ such that $(t, f) \in I_n^m$. Let $k(t, n, m)$ be such that
\[
(t, f) \vdash k(t, n, m) \text{ is the } m \text{-th element of } \tilde{D}_n^m.
\]
Let $\pi(t, n, m)$ be such that, for $\forall i \in k(t, n, m) (\sigma(i) \leq i)$,
\[
\pi(t, n, m)(\sigma) = j \iff \begin{cases} 
(t, f) \vdash \tilde{\pi}_m^m(\sigma) = j
\end{cases}
\]
and $j \leq k(t, n, m)$ or
\[
\begin{cases}
(t, f) \vdash \tilde{\pi}_m^m(\sigma) > k(t, n, m)
\end{cases}
\]
This makes sense by definition of the set $I_n^m$.
\[
\Rightarrow rk(t, A_n^m) = \alpha.
\]
Let $\Pi := \{ \tilde{\pi}(t, n, m); t \text{ minimal } (n, m)\text{–sad} \} \cup \{ \hat{\pi}(t, n); \forall m, t \text{ is } (n, m)\text{–happy} \}$. This is a countable set of predictors. Choose $f \in F$ evading all $\pi \in \Pi$.

**Claim.** $\forces_{\bar{D}}'' f$ evades all $\tilde{\pi}_n, n \in \omega$.

**Proof of Claim.** By contradiction. Suppose there are $(t, g) \in \bar{D}, n \in \omega$ and $m_0 \in \omega$ such that

\[ (t, g) \forces_{\bar{D}}'' \forall m \geq m_0 \ (\tilde{\pi}_n^m(f|\check{k}_m) = f(\check{k}_m))'', \]

where $\forces_{\bar{D}}'' \check{k}_m$ is the $m$-th element of $\check{D}_n$.

We consider two cases.

1. $\forall^\infty m \ (t \text{ is } (n, m)\text{–happy}).$ Then we look at the predictor $\hat{\pi}(t, n)$. Let $m_1 \geq m_0$ be such that $\forall m \geq m_1, t \text{ is } (n, m)\text{–happy}$. As $f$ evades $\hat{\pi}(t, n)$, there is $m_2 \geq m_1$ and $k' \in D_{\hat{\pi}(t, n)}$ such that $m_2$ is minimal with $k' = k(t, n, m_2)$ and $\hat{\pi}_{k'}(t, n)(f|k') \neq f(k')$. By construction, $\hat{\pi}_{k'}(t, n)(f|k') = \pi(t, n, m_2)(f|k')$.

   **Subclaim 1.** There is $(t', g') \leq (t, g)$ such that

   \[ (t', g') \forces_{\bar{D}}'' k' = \check{k}_{m_2} \text{ and } \tilde{\pi}_n^{m_2}(f|k') = \pi(t, n, m_2)(f|k')'', \]

   contradicting $(+)$. 

   **Proof of Subclaim 1.** This is an easy induction on rank. If $rk(t, A_n^{m_2}) = 0$, let $t' = t$ and $g' \geq g$ such that $(t', g') \in I_n^{m_2}$. If $rk(t, A_n^{m_2}) > 0$, find $s$ such that $rk(s, A_n^{m_2}) < rk(t, A_n^{m_2})$, $s \supseteq t$, $\forall i \in dom(s) \ (s(i) \geq g(i))$, $s$ is $(n, m_2)$-happy, $k(s, n, m_2) = k(t, n, m_2) = k'$, and $\pi(s, n, m_2) = \pi(t, n, m_2)$. □

2. $\exists^\infty m \ (t \text{ is } (n, m)\text{–sad}).$ Choose $m_1 \geq m_0$ such that $t$ is $(n, m_1)$-sad. Next choose $(t', g') \leq (t, g)$ such that $t'$ is minimal $(n, m_1)$-sad (this is possible by construction). This time we look at the predictor $\tilde{\pi}(t', n, m_1)$. Choose $i_0$ such that $\forall i \geq i_0 \ \forall j \in dom(t_i) \ (t_i(j) \geq g'(j))$, where the sequence $(t_i; \ i \in \omega)$ is chosen for $t$ as in the definition of minimal $(n, m_1)$-sadness. As $f$ evades $\tilde{\pi}(t', n, m_1)$, there is $k' \in \check{D}(t', n, m_1)$, $k' \geq k(t_{i_0}, n, m_1)$, such that $\pi_{k'}(t', n, m_1)(f|k') \neq f(k')$. By construction, $\pi_{k'}(t', n, m_1)(f|k') = \pi(t, n, m_1)(f|k')$, where $i \geq i_0$ is such that $k' = k(t_i, n, m_1)$.

   **Subclaim 2.** There is $(t'', g'') \leq (t', g')$ such that
\[(t'', g'') \models \exists k' = \bar{k}_{m_1} \land \bar{\pi}^{m_1}_n(f|k') = \pi(t_i, n, m_1)(f|k''),\]

contradicting (+).

Proof of Subclaim 2. Again an easy induction on rank. If \(rk(t_i, A^{m_1}_n) = 0\), let \(t'' = t_i\) and \(g'' \geq g'\) such that \((t_i, g'') \in I^{m_1}_n\). Then clearly \((t'', g'') \leq (t', g')\). If \(rk(t_i, A^{m_1}_n) > 0\), we proceed as in the proof of subclaim 1. □

This concludes the proof of the claim and finishes the proof of the Theorem as well.

□ □

2.2. Now let \(D_\alpha\) denote the iteration of Hechler forcing of length \(\alpha\). We claim that \(D_\alpha\) still has property (**). By 2.1. this is a consequence of the following preservation result:

**Lemma.** Assume \(\langle P_\beta, \bar{Q}_\beta; \beta < \alpha \rangle\) is an \(\alpha\)–stage finite support iteration of ccc partial orders such that

\[\forall \beta < \alpha \quad \models \bar{P}_{\beta} \text{ satisfies (**)}\]

Then \(P_\alpha\) satisfies (**).

**Proof.** By induction on \(\alpha\). The successor step as well as the case \(cf(\alpha) > \omega\) are trivial. So assume \(cf(\alpha) = \omega\); without loss \(\alpha = \omega\).

Let \(\langle \bar{\pi}_n; n \in \omega \rangle\) be a sequence of \(P_\omega\)–names for predictors; for \(n \in \omega\), \(\bar{\pi}_n = (\hat{D}_n; (\bar{\pi}^k_n; k \in \omega))\). For each \(m \in \omega\) let \(\langle \bar{\pi}_{n,m}; n \in \omega \rangle = \langle (\hat{D}_{n,m}; (\bar{\pi}^k_{n,m}; k \in \omega)); n \in \omega \rangle\) be a sequence of \(P_m\)–names for predictors and \(\langle \bar{p}^k_m; k \in \omega \rangle\) a sequence of \(P_m\)–names for elements of \(\bar{P}_m\) such that

\[\models \bar{P}_m \bar{p}^{k+1}_m \leq \bar{p}^k_m \text{ and } \models \bar{P}_m \bar{p}^k_m \text{ the } k\text{-th elements of } \hat{D}_n \text{ and } \hat{D}_{n,m} \text{ are equal, say } \ell, \]

and \(\bar{\pi}^k_n\) equals \(\bar{\pi}^k_{n,m}\) on all sequences of length \(\ell\) below the identity”.

By induction hypothesis find \(\langle \pi_{n,m}; n, m \in \omega \rangle \in V\) a sequence of predictors such that whenever \(f \in F\) evades all \(\pi_{n,m}\), then for all \(m:\)

\[\models \bar{P}_m \text{ ”f evades all } \pi_{n,m}, \text{ where } n \in \omega”\]

We claim that \(\models \bar{P}_\omega \text{ ”f evades all } \bar{\pi}_n”\).

For suppose there are a condition \(p \in P_\omega, n \in \omega \) and \(k_0 \in \omega\) such that

\[ (+) \quad p \models \bar{P}_\omega \forall k \geq k_0 \text{ ( } \bar{\pi}^k_n(f|\bar{k}) = f(\bar{k}))”\]

where \(\models \bar{P}_\omega \bar{k}\) is the \(k\)-th element of \(\hat{D}_n”\). Let \(m = \text{ supp}(p)\). By induction hypothesis we know that
$\vdash_{\mathcal{P}_m} f$ evades $\tilde{\pi}_{n,m}$.

Hence we can find $q \leq p$, $q \in \mathcal{P}_m$, and $k \geq k_0$ such that

$$q \vdash_{\mathcal{P}_m} f(\tilde{\ell}_m^k) \neq \tilde{\pi}_{n,m}^k(f|\tilde{\ell}_m^k),$$

where $\vdash_{\mathcal{P}_m} \tilde{\ell}_m^k$ is the $k$-th element of $\tilde{D}_{n,m}$. Thus, by definition of the name $\tilde{p}_m^k$,

$$q \vdash_{\mathcal{P}_m} \tilde{p}_m^k \vdash_{\tilde{\mathcal{P}}_{m,\omega}} f(\ell_m^k) \neq \tilde{\pi}_{n,m}^k(f|\ell_m^k) = \tilde{\pi}_n^k(f|\ell_m),$$

contradicting $(+)$.

**Applying $(**)$ to $D_\kappa$ we get that $F = (\omega^\omega)^V$ is a family of functions of size $\omega_1$ such that for every predictor $\pi \in V^{D_\kappa}$, there is $f \in F$ evading $\pi$. Thus $V^{D_\kappa} \models e = \omega_1$. It is well–known that $V^{D_\kappa} \models add(M) = \kappa$. This ends the proof of Theorem A. Using the methods of [Br] one can in fact show the consistency of $e = \kappa$ and $b = add(M) = \lambda$ for any regular uncountable $\kappa < \lambda$.

2.3. We consider the following more general notion of predicting: we are given two sets $D_\pi = \{k_n; n \in \omega\} \subseteq \omega$ and $E_\pi = \{\ell_n; n \in \omega\} \subseteq \omega$ such that $k_n \leq \ell_n < k_{n+1}$ for all $n \in \omega$; we also have for each $n$ a function $\pi_n : \omega^{\ell_n \setminus \{k_n\}} \to \omega$; we say the predictor $\pi = (D_\pi, E_\pi, (\pi_n; n \in \omega))$ predicts $f \in \omega^\omega$ iff $\forall \infty n (\pi_n(f|\ell_n \setminus \{k_n\})) = f(k_n))$. Let $e'$ be the smallest size of a set of functions $F$ from $\omega$ to $\omega$ such that given a countable set of such predictors $\Pi$, there is $f \in F$ evading all $\pi \in \Pi$. Clearly, $e' \geq e$. Also, a set predicted by countably many predictors is necessarily contained in the union of countably many closed measure zero sets. Thus $e' \leq unif(M), unif(L)$.

2.4. We shall give an upper bound to $se$ in terms of evading by showing:

**Theorem.** $se \leq e'$.

Note that, by the remarks in 2.3., this finishes the proof of Theorem B: to get the consistency of $se < b$ simply add $\omega_1$ random reals over a model for MA; then $se \leq e' \leq unif(L) = \omega_1$, whereas $b = 2^\omega$.

**Proof of Theorem.** Let $\{p_n; n \in \omega\}$ be an enumeration of all primes. Let $\mathcal{F} = \{f_\alpha; \alpha < e'\} \subseteq \omega^\omega$ be a family of functions evading all families of countably many predictors (in the sense of 2.3., of course). For $\alpha < e'$, we define $x_\alpha \in \omega^\omega$ as follows:

$$x_\alpha(0) = 1$$

$$x_\alpha(1) = p_0^{f_\alpha(0)}$$

11
\[ x_\alpha(2) = p_0^{f_\alpha(0)+1} \cdot p_1^{f_\alpha(1)} \]
\[ \ldots \]
\[ x_\alpha(n) = \prod_{i<n} p_i^{f_\alpha(i)+n-i-1} \]

We let \( G \leq \mathbb{Z}^\omega \) be the pure subgroup of \( \mathbb{Z}^\omega \) generated by the \( x_\alpha \) and the unit vectors \( e_n \). Clearly \( |G| = \mathbf{e}' \). We claim that \( G \) exhibits the Specker phenomenon.

For suppose not, and assume \( h : G \to \mathbb{Z} \) is a homomorphism such that there are infinitely many \( n \) with \( h(e_n) \neq 0 \). Fix \( z \in \mathbb{Z} \). We shall introduce a predictor \( \pi = \pi_z \) of the required sort. \( D_\pi = \{ k_n; \ n \in \omega \} \) and \( E_\pi = \{ \ell_n; \ n \in \omega \} \) are such that

- \( h(e_{k_n}) \neq 0, |h(e_{k_n})| > 2 \cdot \sum_{i<k_n} |h(e_i)|, \) and \( \forall \alpha < \mathbf{e}' \forall \infty n \ (|h(e_{k_n})| > f_\alpha(k_n), x_\alpha(k_n)) \)

[Note that we can indeed choose the \( k_n \) in such a way by the argument of the proof of [Bl 2, Theorem 2] (going over to \( G' \geq G \) still of size \( \mathbf{e}' \), if necessary).]

- \( k_0 \) is such that \( |h(e_{k_0})| > |z| \) and \( k_n < \ell_n < k_{n+1} \) are such that \( \ell_n - k_n > 2 \cdot |h(e_{k_n})|^2 \).

To motivate the predictor, assume there are \( f_\alpha, f_\beta \in \mathcal{F} \) such that \( h(x_\alpha) = h(x_\beta) = z, \)
\( f_\alpha \ell_n \setminus \{k_n\} = f_\beta \ell_n \setminus \{k_n\}, \) and \( f_\alpha(k_n) \neq f_\beta(k_n) \), where \( n \) is large enough (so that \( |h(e_{k_n})| > f_\gamma(k_n), x_\gamma(k_n) \), where \( \gamma = \alpha \) or \( \beta \)). We put

\[ a := h(x_\alpha|_{(e_{k_n+1}, \infty)}) = z - \sum_{i<k_n} h(e_i) \cdot x_\alpha(i) - h(e_{k_n}) \cdot x_\alpha(k_n) \neq 0, \]

because the absolute value of the last term is larger than the absolute values of the others together. Also note that \( a = h(x_\beta|_{(e_{k_n+1}, \infty)}) \). For \( j \) with \( k_n + 1 \leq j \leq \ell_n \), we let
\( b_j = \prod_{i<k_n \vee k_n<i<j} p_i^{f_\alpha(i)+j-i-1}; \) and, letting \( p := p_{k_n} \), we define by recursion on such \( j \) the numbers \( \hat{x}^\gamma_j \) where \( \gamma = \alpha \) or \( \beta \):

\[ \hat{x}^\gamma_{k_n+1} \cdot b_{k_n+1} \cdot p^{f_\gamma(k_n)} = a = h(e_{k_n+1}) \cdot b_{k_n+1} \cdot p^{f_\gamma(k_n)} + \hat{x}^\gamma_{k_n+2} \cdot b_{k_n+2} \cdot p^{f_\gamma(k_n)+1} \]

\[ \hat{x}^\gamma_j \cdot b_j \cdot p^{f_\gamma(k_n)+j-k_n-1} = h(e_j) \cdot b_j \cdot p^{f_\gamma(k_n)+j-k_n-1} + \hat{x}^\gamma_{j+1} \cdot b_{j+1} \cdot p^{f_\gamma(k_n)+j-k_n-1} \]

for \( k_n + 1 < j < \ell_n \). Note that, by purity, all the \( \hat{x}^\gamma_j \) must be integers. Furthermore \( \hat{x}^\gamma_{k_n+1} \neq 0 \) because \( a \neq 0 \). Suppose without loss that \( f_\alpha(k_n) < f_\beta(k_n) \). Let \( m \) be maximal so that \( \hat{x}^\alpha_{k_n+1} \) is divisible by \( p^m \); then \( \hat{x}^\beta_{k_n+1} \) is divisible only by \( p^m' \) where \( m' = m - f_\beta(k_n) + f_\alpha(k_n) \).

In particular \( \hat{x}^\alpha_{k_n+1} - \hat{x}^\beta_{k_n+1} \neq 0 \). By assumption on \( n \), we must have \( m < \ell_n - k_n \) (in fact, \( |a| < \ell_n - k_n \)). We can now divide the above equations for \( \alpha \) and \( \beta \), respectively, by the appropriate power of \( p \) and products of other primes, subtract recursively one from the other and get
\[ \hat{x}^\alpha_{k_n+1} - \hat{x}^\beta_{k_n+1} = (\hat{x}^\alpha_{k_n+2} - \hat{x}^\beta_{k_n+2}) \cdot \prod_{i<k_n} p_i \cdot p_{n+1}^\alpha \cdot p \]
\[ \hat{x}^\alpha_j - \hat{x}^\beta_j = (\hat{x}^\alpha_{j+1} - \hat{x}^\beta_{j+1}) \cdot \prod_{i<k_n \land j<i} p_i \cdot p_j^\alpha \cdot p \]

Then \( \hat{x}^\alpha_{k_n+1} - \hat{x}^\beta_{k_n+1} \) is at most divisible by \( p^m \), hence \( \hat{x}^\alpha_{k_n+2} - \hat{x}^\beta_{k_n+2} \) is at most divisible by \( p^m-1 \) etc. Thus there must be a \( j \leq \ell_n \) so that \( \hat{x}^\alpha_j - \hat{x}^\beta_j \) is not an integer anymore, a contradiction.

This shows that given \( f_\alpha, f_\beta \in F \) with \( h(x_\alpha) = h(x_\beta) = z \) and \( n \) large enough, if \( f_\alpha| (\ell_n \setminus \{k_n\}) = f_\beta| (\ell_n \setminus \{k_n\}) \), then we must have \( f_\alpha(k_n) = f_\beta(k_n) \); so we simply let the predictor \( \pi_z \) predict this uniquely defined value. In the end we get a countable family \( \{\pi_z; z \in \mathbb{Z}\} \) of predictors so that each \( f_\alpha \) is predicted by one \( \pi_z \), a contradiction to the choice of the family \( F \). \( \square \)

§ 3. Towards a general theory of evasion and prediction

3.1. The proofs of the preceding section suggest that we generalize the notions of evading and predicting defined in the Introduction, and look at the corresponding cardinals.

One way of doing this goes as follows. Fix \( f \in (\omega + 1 \setminus 2)^\omega \), and let \( X := \prod_n f(n) \); i.e. \( X \) consists of the functions from the Baire space which are below \( f \) everywhere. An \( X \)–predictor (or: \( f \)–predictor) is a pair \( \pi = (D_\pi, (\pi_n; n \in D_\pi)) \) such that for every \( n \in D_\pi, \pi_n : \prod_{k<n} f(k) \rightarrow f(n) \); \( \pi \) predicts \( g \in X \) iff \( \forall \alpha g \in D_\pi \) (\( \pi_n (g|n) = g(n) \)); otherwise \( g \) evades \( \pi \). Let \( e_X \) (or: \( e_f \)) be the corresponding evasion number; i.e. the smallest size of a set of functions \( F \subseteq X \) such that every \( X \)–predictor is evaded by some \( g \in F \). In case \( f \) eventually equals \( \omega \), we get \( e_f = e \); in case \( X = n^\omega \) \( (n \geq 2) \), we talk about \( n \)–predictors (instead of \( n^\omega \)–predictors) and set \( e_n := e_X \). Finally we let \( e_{ubd} := \min \{e_f; f \in \omega^\omega\} \), the unbounded evasion number, and \( e_{bdd} = \min \{e_f; f \in \omega^\omega \text{ is bounded} \} = \min \{e_n; n \in \omega \} =: e_{fin} \), the bounded or finite evasion number. We trivially have \( e \leq e_{ubd} \leq e_{fin} \), and we shall see in the next two subsections that nothing else can be proved in \( ZFC \) about the relationship between these three cardinals. Furthermore, a set predicted by any predictor is easily seen to be the union of countably many closed measure
zero sets; thus \( e_{\text{fin}} \leq \text{unif}(\mathcal{E}) \leq \text{unif}(\mathcal{M}), \text{unif}(\mathcal{L}) \), where \( \mathcal{E} \) is the \( \sigma \)-ideal generated by the latter (this has been studied recently in [BS]).

The notion of \textit{linear predicting} can be generalized as well. Let \( K \) be a finite or countable field. A \( K \)-predictor \( \pi = (D_\pi, (\pi_n; \ n \in D_\pi)) \) is called \textit{linear} iff for every \( n \in D_\pi \), \( \pi_n : K^n \to K \) is a linear function. We let \( e_K \) be the smallest size of a set of functions \( F \subseteq K^\omega \) such that every linear \( K \)-predictor is evaded by some \( f \in F \). As it is easier to evade just linear predictors, we have \( e_K \leq e_{|K|} \), in particular \( e_K \leq e \) for countable fields. If \( \mathbb{Q} \) denotes the field of the rationals, \( e_\mathbb{Q} = e_\ell \), the cardinal studied by Blass [Bl2, section 4]. His results are easily seen to carry over to the other \( e_K \)'s, and we have, e.g., \( e_K \geq \text{add}(\mathcal{L}), p \) for any finite or countable field \( K \) and \( e_K \leq \text{add}(\mathcal{M}) \) for any countable field \( K \) (cf Introduction).

Turning to consistency results, the following are known. Iterating a Borel \( \sigma \)-centered forcing notion adding a generic linear predictor (see 3.3. for similar forcing notions) with finite support, we easily get \( \text{CON}(e, e_K > \text{add}(\mathcal{L}), s, p) \) \( [e_K > \text{add}(\mathcal{L}) \) holds because iterations of \( \sigma \)-centered forcing notions do not add random reals (as far as I know this is due to Miller and implicit in [Mi 1, section 5]); \( e_K > s, p \) holds by easy definability of the forcing notions (see [JS 1])]. We finally note that in the proof of Theorem A (2.1. and 2.2.) we proved in fact that \( e_{id} = \omega_1 \) in the Hechler real model; however, the choice of the identity function was arbitrary, and it is easily read off from the proof that all evasion numbers defined above equal \( \omega_1 \) in the latter model.

3.2. \textit{Some ZFC-results.} We start the proof of Theorem C with a series of Lemmata.

**Lemma 1.** \( e \geq \min\{b, e_{\text{ubd}}\} \), and thus \( \min\{b, e\} = \min\{b, e_{\text{ubd}}\} \).

**Proof.** Let \( F \subseteq \omega^\omega \) be of size \( \min\{b, e_{\text{ubd}}\} \). We have to find a predictor predicting every function in \( F \).

By \( |F| < b \) find \( x \in \omega^\omega \) such that \( \forall f \in F \ \forall^\infty n \ (f(n) < x(n)) \). Given \( f \in F \), we let for \( n \in \omega \)

\[
 f'(n) := \begin{cases} 
 f(n) & \text{if } f(n) < x(n) \\
 0 & \text{otherwise}, 
\end{cases}
\]

and let \( F' := \{f'; \ f \in F\} \). By \( |F'| < e_\mathbb{X} \) (where \( X = \prod_n x(n) \)) find an \( X \)-predictor \( \pi = (D_\pi, (\pi_k; \ k \in D_\pi)) \) predicting every function of \( F' \). We extend \( \pi \) to an \( \omega \)-predictor...
\[\pi^* \text{ as follows: given } k \in D_\pi \text{ and } \sigma \in \omega^k \setminus \prod_{n<k} x(n), \text{ we define for } n < k \]

\[\sigma'(n) := \begin{cases} 
\sigma(n) & \text{if } \sigma(n) < x(n) \\
0 & \text{otherwise.} 
\end{cases} \]

Next we let \(\pi^*_k(\sigma) := \pi_k(\sigma')\). We claim that \(\pi^* = (D_\pi, (\pi^*_k; k \in D_\pi))\) predicts every function of \(F\).

For, given \(f \in F\), there is \(n_0\) such that \(\forall n \geq n_0 \ (f'(n) = f(n))\); next there is \(n_1 \geq n_0\) such that for all \(k \geq n_1\), if \(k \in D_\pi\), then \(\pi_k(f'|k) = f'(k)\). Thus for all \(k \geq n_1\), if \(k \in D_\pi\), then \(f(k) = f'(k) = \pi_k(f'|k) = \pi^*_k(f|k)\).

**Lemma 2.** \(e_{fin} = e_n\) for all \(n \geq 2\).

**Proof.** It suffices to show by induction on \(n\) that \(e_n \geq e_2\). To this end assume \(e_n < e_2\) (\(n\) minimal), and let \(F = \{f_\alpha; \alpha < e_n\} \subseteq n^\omega\) be a family of functions evading every \(n\)–predictor.

We define for \(\alpha < e_n\)

\[g_\alpha(m) := \begin{cases} 
0 & \text{if } f_\alpha(m) \neq 1 \\
1 & \text{if } f_\alpha(m) = 1. 
\end{cases} \]

By assumption, there is a 2–predictor \(\pi = (D_\pi, (\pi_k; m \in \omega))\) (where \(D_\pi = \{k_m; m \in \omega\}\) is the increasing enumeration) predicting all \(g_\alpha\). Next define for \(\alpha < e_n\)

\[h_\alpha(m) := \begin{cases} 
0 & \text{if } f_\alpha(k_m) = 0 \text{ or } 1 \\
1 & \text{if } f_\alpha(k_m) - 1 = f_\alpha(k_m) \geq 2. 
\end{cases} \]

By assumption again, there is an \((n - 1)\)–predictor \(\pi' = (D_\pi', (\pi'_m; m \in D_\pi'))\) predicting all \(h_\alpha\). We can define an \(n\)–predictor \(\tilde{\pi}\) as follows: \(D_{\tilde{\pi}} = \{k_m; m \in D_{\pi'}\}\); given \(\sigma \in n^{k_m}\) we let

\[\sigma'(i) := \begin{cases} 
0 & \text{if } \sigma(k_i) = 0 \text{ or } 1 \\
\sigma(k_i) - 1 & \text{if } \sigma(k_i) \geq 2; 
\end{cases} \]

next, given \(m \in D_{\pi'}\), we define

\[\tilde{\pi}_{k_m}(\sigma) := \begin{cases} 
1 & \text{if } \pi_{k_m}(\sigma) = 1 \land \pi'_m(\sigma') = 0 \\
\ell & \text{if } \pi_{k_m}(\sigma) = 0 \land \pi'_m(\sigma') = \ell - 1 \text{ for } \ell \geq 2 \\
0 & \text{otherwise} 
\end{cases} \]

It is easy to check that \(\tilde{\pi}\) predicts all \(f_\alpha\), thus giving a contradiction. \(\Box\)
LEMMA 3.  \(e_{fin} \geq s\); also \(e_{\mathbb{K}} \geq s\) for all finite fields \(\mathbb{K}\).

Proof.  As \(e_{fin} \geq e_{\mathbb{K}}\), it suffices to show the second inequality. To this end, let \(\kappa < s\) and let \(\{f_\alpha; \alpha < \kappa\}\) be a family of functions from \(\omega\) to \(\mathbb{K}\). For each \(\alpha \in \mathbb{K}\) let \(A_{\alpha,\alpha} \coloneqq f_\alpha^{-1}(\{a\})\). As \(\{A_{\alpha,\alpha}; \alpha \in \mathbb{K} \land \alpha < \kappa\}\) is not a splitting family, there is an infinite \(B \subseteq \omega\) which is not split, i.e. \(\forall \alpha < \kappa \exists a_\alpha \in \mathbb{K}\) such that \(B \subseteq^* A_{\alpha,\alpha}\). Assume \(B = \{b_n; n \in \omega\}\) is the increasing enumeration of \(B\). Let \(D_\pi := \{b_{2n+1}; n \in \omega\}\) and define \(\pi_n : \mathbb{K}^{b_{2n+1}} \rightarrow \mathbb{K}\) by \(\pi_n(\sigma) = \sigma(b_{2n})\). \(\pi_n\) is trivially linear, and \(\pi = (D_\pi, (\pi_n; n \in \omega))\) is easily seen to predict all the \(f_\alpha\). \(\Box\)

Let us note that these results together with Blass’ \(\min\{e, b\} \leq add(M)\) [Bl 2, Theorem 13] already yield one consistency result concerning evasion numbers, namely \(CON(e_{fin} > e_{ubd})\). To see that this holds in the Mathias real model, note that the latter satisfies \(s = \omega_2\) (this is straightforward from the combinatorial properties of the Mathias generic real) and hence \(e_{fin} = \omega_2\), whereas \(\min\{e_{ubd}, b\} = \min\{e, b\} \leq add(M) \leq cov(M) = \omega_1\) by the fact that iterated Mathias forcing doesn’t add Cohen reals (see 1.2.). Thus \(b = \omega_2\) in the Mathias real model gives \(e_{ubd} = \omega_1\). In fact, a canonical application of the Laver property gives \(e_x = \omega_1\) for any \(x \in \omega^\omega\) converging to infinity in this model.

3.3. The goal of this subsection is to show:

**Theorem.** For any regular uncountable \(\kappa\), it is consistent that \(e_{ubd} = 2^\omega = \kappa\) and \(e = b = \omega_1\).

This will complete the proof of Theorem C.

Given \(x \in \omega^\omega\), the p.o. \(P_x\) for adding a predictor for the space \(X = \prod_n x(n)\) is defined as follows.

\[
(d, (\pi_k; k \in d), F) \in P_x \leftrightarrow \begin{cases} 
  d \subseteq \omega \text{ finite } \land \\
  \forall k \in d (\text{dom}(\pi_k) = \prod_{n < k} x(n) \land \text{ran}(\pi_k) \subseteq x(k)) \land \\
  F \text{ finite, } \forall f \in F (\text{dom}(f) \leq \omega \land f \in \prod_{n < \text{dom}(f)} x(n))
\end{cases}
\]

We order \(P_x\) by setting \((e, (\pi_k; k \in e), G) \leq (d, (\pi_k; k \in d), F)\) iff

1) \(e \supseteq d\) and \(\max(d) < \min(e \setminus d)\);

2) \(\pi_k = \pi_k\) for \(k \in d\);

3) \(\forall f \in F \exists g \in G (f \subseteq g)\) and \(\forall k \in (e \setminus d) \forall f \in F (k \in \text{dom}(f) \rightarrow \pi_k(f|k) = f(k))\).

Clearly \(P_x\) is a \(\sigma\)-centered p.o. (more explicitly, \((d, (\pi_k; k \in d), F)\) and \((d, (\pi_k; k \in d), G)\) are compatible with common extension \((d, (\pi_k; k \in d), F \cup G)\) for any choice of \(F\) and \(G\)).
Obviously, we can make \( e_{\text{ubd}} = \kappa \) by starting with a model for \( CH \), iterating p.o.’s of the form \( P_x \), where \( x \) is a real in some intermediate stage of the extension, \( \kappa \) many times with finite support (this is a standard enumeration argument as in the classical consistency proof of \( MA \)).

So we will be done if we can show that in the final model, the ground model reals are still unbounded, and there is no predictor predicting all ground model functions (in \( \omega^\omega \)); i.e. \( b = e = \omega_1 \) (by \( CH \)).

To this end we use a modification of a notion and some techniques of [BrJ, § 1]. Given a p.o. \( P \), a function \( h : P \rightarrow \omega \) is a height function iff \( p \leq q \) implies \( h(p) \geq h(q) \) for \( p, q \in P \).

A pair \((P, h)\) is soft iff \( P \) is a p.o., \( h \) is a height function on \( P \), and the following three conditions are met:

1. if \( \{p_n; \ n \in \omega\} \) is decreasing and \( \exists m \in \omega \ \forall n \in \omega \ (h(p_n) \leq m) \), then \( \exists p \in P \ \forall n \in \omega \ (p \leq p_n) \);
2. given \( m \in \omega \) and \( p, q \in P \) there is \( \{q_i; \ i \in \ell \} \subseteq P \) so that
   - (i) \( \forall i \in \ell \ (q_i \leq q \land q_i \perp p) \);
   - (ii) whenever \( q' \leq q \) is incompatible with \( p \) and \( h(q') \leq m \) then there exists \( i \in \ell \) so that \( q' \leq q_i \);
3. if \( p, q \in P \) are compatible, there is \( r \leq p, q \) so that \( h(r) \leq h(p) + h(q) \).

(II) given \( m \in \omega \) and \( p, q \in P \) there is \( \{q_i; \ i \in \ell \} \subseteq P \) so that
   - (i) \( \forall i \in \ell \ (q_i \leq q \land q_i \perp p) \);
   - (ii) whenever \( q' \leq q \) is incompatible with \( p \) and \( h(q') \leq m \) then there exists \( i \in \ell \) so that \( q' \leq q_i \);

(III) if \( p, q \in P \) are compatible, there is \( r \leq p, q \) so that \( h(r) \leq h(p) + h(q) \).

(Note that (by [BrJ, 1.7]) this notion is a strengthening of the one given in [BrJ, 1.1].) We say a p.o. \( P \) is soft iff there is \( P' \subseteq r.o.(P) \) dense and \( h : P' \rightarrow \omega \) so that \( (P', h) \) is soft.

Furthermore, a pair \((P, h)\) satisfies property (*) iff \( P \) is a p.o., and \( h \) is a height function on \( P \) satisfying (III) above and:

(*) given \( p \in P, a \) maximal antichain \( \{p_n; \ n \in \omega\} \subseteq P \) of conditions below \( p \) and \( m \in \omega \), there exists \( n \in \omega \) such that: whenever \( q \leq p \) is incompatible with \( \{p_j; \ j \in n\} \) then \( h(q) > m \).

A compactness argument shows:

**Lemma 1.** (cf [BrJ, 1.2]) If \((P, h)\) is soft, then \((P, h)\) has property (*).

**Proof.** Put together the arguments of 1.7 and 1.2 in [BrJ]. (This uses only (I) and (II) of softness.) \( \square \)

Our strategy to finish the proof of the Theorem is as follows: show that \( P_x \) is soft (Lemma 4) — and prove that iterating p.o.’s with property (*) doesn’t increase \( b \) and \( e \) (Lemmata 2 and 3). For the latter we need the following notion.
A pair \( \pi = ((A_k; k \in \omega), (\pi_k; k \in \omega)) \) is called a generalized predictor iff for all \( k \in \omega \), \( A_k \subseteq [k; \omega) \) is finite, and \( \pi_k \) is a function with \( \text{dom}(\pi_k) = \{ \sigma \in \omega^{<\omega}; \text{lh}(\sigma) \in A_k \} \) and \( \text{ran}(\pi_k) \subseteq [\omega]^{<\omega} \). \( \pi \) predicts \( f \in \omega^\omega \) iff \( \forall \ell \in A_k \ (f(\ell) \in \pi_k(f|\ell)) \); otherwise \( f \) evades \( \pi \). — The original definition of predicting is a special instance of this notion (in case \( A_k = \{ \ell_k \}, \ell_k < \ell_{k+1} \) and \( |\pi_k(\sigma)| = 1 \) for \( \sigma \in \omega^\ell_k \)).

Next let us consider the following property of p.o.'s \( \mathbb{P} \):

(++) given \( F \subseteq \omega^\omega \cap V \), \( F \in V \), such that for any countable family of generalized predictors \( \Pi \) there is \( f \in F \) evading all \( \pi \in \Pi \), and \( \langle \bar{\pi}_n; n \in \omega \rangle \) a sequence of \( \mathbb{P} \)-names for generalized predictors, we can find a sequence \( \langle \pi_n; n \in \omega \rangle \in V \) of generalized predictors such that whenever \( f \in F \) evades all \( \bar{\pi}_n \), then

\[ \| \neg \pi \| f \text{ evades all } \bar{\pi}_n. \]

(Note that this is almost the same as (**) in 2.1.) This more general version of predicting as well as (++) are needed for the following preservation results:

**Lemma 2.** Suppose \( \mathbb{P} \) is a ccc p.o., \( h \) is a height function on \( \mathbb{P} \), and \( (\mathbb{P}, h) \) satisfies property (*). Then:

(a) any unbounded family of functions in \( \omega^\omega \cap V \) is still unbounded in \( V[G] \), where \( G \) is \( \mathbb{P} \)-generic over \( V \);

(b) any family of functions in \( \omega^\omega \cap V \) which is not predicted by a single (countable family of) generalized predictor(s) still has this property in \( V[G] \) — and, in fact, \( \mathbb{P} \) satisfies (++)

**Lemma 3.** Assume \( (\mathbb{P}_\beta, \hat{\mathbb{Q}}_\beta; \beta < \alpha) \) is an \( \alpha \)-stage finite support iteration of ccc p.o.'s such that \( \| \neg \mathbb{P}_\beta \| h_\beta \text{ is a height function on } \hat{\mathbb{Q}}_\beta \). Then:

(a) if \( \forall \beta < \alpha \ \| \neg \mathbb{P}_\beta \| \hat{\mathbb{Q}}_\beta \text{ has property (*)} \), then \( \mathbb{P}_\alpha \text{ does not add dominating reals.} \)

(b) if \( \forall \beta < \alpha \ \| \neg \mathbb{P}_\beta \| \hat{\mathbb{Q}}_\beta \text{ has property (*)} \) or just \( \| \neg \mathbb{P}_\beta \| \hat{\mathbb{Q}}_\beta \text{ satisfies (++)} \), then \( \mathbb{P}_\alpha \text{ satisfies (++)} \) (and thus no \( \omega \)-predictor predicts all old reals in the Baire space).

Proof of Lemma 2. (a) [BrJ, 1.3].

(b) Let \( F \) be such a family in \( \omega^\omega \cap V \). Suppose \( \| \neg \mathbb{P} \| \bar{\pi} = ((\bar{A}_k; k \in \omega), (\bar{\pi}_k; k \in \omega)) \) is a generalized predictor. For \( k \in \omega \) let \( \{ p^k_n; n \in \omega \} \) be a maximal antichain deciding the set \( \bar{A}_k \). Choose \( n_k \) according to (*) so that: whenever \( p \) is incompatible with \( \{ p^j_k; j \in n_k \} \), then \( h(p) > k \). For \( j \in n_k \) let \( A^j_k \) be such that \( p^j_k \| \neg \mathbb{P} \| \bar{A}_k = A^j_k \) and let \( A_k := \bigcup_{j \in n_k} A^j_k \).

18
Fix $k \in \omega$ and $j \in n_k$. Let $A^j_k = \{ \ell^j_0, \ldots, \ell^j_{a(j,k)−1} \}$ be the increasing enumeration of $A^j_k$. By recursion on $m < a(j,k)$ we define conditions $p^{\hat{i}, \tau}_{k,\sigma}$ for $\sigma \in \omega^\ell_m$ and $\tau \in \omega^{m+1}$ as follows: in case $m = 0$, $\sigma \in \omega^\ell_0$, let $\{ p^{\hat{i},(n)}_{k,\sigma}; n \in \omega \}$ be a maximal antichain of conditions below $p^j_k$ deciding the set $\mathfrak{p}_k(\sigma)$; in case $m \geq 0$, $\sigma \in \omega^\ell_{m+1}$, $\tau \in \omega^{m+1}$, let $\{ p^{\hat{i},\tau}_{k,\sigma}; n \in \omega \}$ be a maximal antichain of conditions below $p^{\hat{i},\tau}_{k,\sigma|\ell_m}$ deciding the set $\mathfrak{p}_k(\sigma)$. Next, for $\sigma \in \omega^\ell_{m+1}$, $\tau \in \omega^m$, $m < a(j,k)$, choose $n^{\hat{i},\tau}_{k,\sigma} \in \omega$ according to (*) so that: whenever $p < p^{\hat{i},\tau}_{k,\sigma|\ell_m}$ (or $p < p^j_k$ in case $m = 0$) is incompatible with $\{ p^{\hat{i},\tau(i)}_{k,\sigma}; i \in n^{\hat{i},\tau}_{k,\sigma} \}$, then $h(p) \geq k + h(p^j_k) + \ldots + h(p^{\hat{i},\tau}_{k,\sigma|\ell_m})$. For $i \in n^{\hat{i},\tau}_{k,\sigma}$ let $\mathfrak{p}^{\hat{i},\tau(i)}_{k,\sigma}$ be such that $p^{\hat{i},\tau(i)}_{k,\sigma} \parallel p^n \mathfrak{p}(\hat{i}) = \mathfrak{p}^{\hat{i},\tau(i)}_{k,\sigma}$”. Unfixing $j$, we define, for $\sigma \in \omega^\omega$ with $\text{dom}(\sigma) \in A_k$, $\mathfrak{p}^{\hat{i},\tau} = \bigcup \{ \mathfrak{p}^{\hat{i},\tau(i)}_{k,\sigma} \}$ for $j \in n_k$ is such that $\text{dom}(\sigma) \in A^j_k$, e.g., $\text{dom}(\sigma) = \ell^j_m$ where $m < a(j,k)$, $\tau \in \omega^{m+1}$, and for all $i \leq m$, $\tau(i) \in n^{\hat{i},\tau}_{k,\sigma(\ell^j_m)}$. Then $\mathfrak{p} = (A_k; k \in \omega), (\mathfrak{p}_k; k \in \omega)$ is a generalized predictor. Choose $f \in F$ evading $\mathfrak{p}$.

Claim. $\parallel p^n f$ evades $\mathfrak{p}^n$.

Proof of Claim. Suppose there is a $p \in \mathbb{P}$ and a $k_0 \in \omega$ so that

$$p \parallel \mathbb{P}^n \forall k \geq k_0 \exists \ell \in A_k (f(\ell) \in \mathfrak{p}_k(f|\ell))$$.

Choose $k \geq k_0$ so that $h(p) \leq k$ and $\forall \ell \in A_k (f(\ell) \notin \mathfrak{p}_k(f|\ell))$. Then $p$ is compatible with $p^j_k$ for some $j \leq n_k$; let $q^j_k$ be a common extension such that $h(q^j_k) \leq h(p) + h(p^j_k) \leq k + h(p^j_k)$. Next construct recursively $\tau_m \in \omega^{m+1}$ and $q^{\hat{i},\tau_m}_{k,f|\ell^j_0}$ for $m < a(j,k)$ as follows: in case $m = 0$, find $i < n^{\hat{i},\tau}_{k,f|\ell^j_0}$ so that $q^j_k$ and $p^{\hat{i},(i)}_{k,f|\ell^j_0}$ are compatible, let $q^{\hat{i},(i)}_{k,f|\ell^j_0}$ be a common extension of height $\leq k + h(p^j_k) + h(p^{\hat{i},(i)}_{k,f|\ell^j_0})$ and let $\tau_0 = \langle i \rangle$; in case $m \geq 0$, find $i < n^{\hat{i},\tau}_{k,f|\ell^j_{m+1}}$
so that \(q^j,\tau_m\mid f\ell_m\) and \(p^j,\tau_m\mid f\ell_{m+1}\) are compatible, let \(q^j,\tau_m\mid f\ell_m\) be a common extension of height \(\leq k + \ldots + h(p^j,\tau_m\mid f\ell_{m+1})\) and let \(\tau_{m+1} = \tau_m\mid f\ell_m\). Note that for \(m = a(j, k) - 1\) we have
\[
q^j,\tau_m\mid f\ell_m \models \forall \ell \in \check{A}_k \left( f(\ell) \not\in \check{\pi}_k(f \mid \ell) \right),
\]
a contradiction. \(\Box\)

Note that a trivial modification of this argument proves the stronger version of Lemma 2 (b) as well. Notice also that we used property (III) of the definition of softness in the proof of the claim. \(\Box\)

**Proof of Lemma 3.** (a) [JS 2, 2.2]; see also [BrJ, 1.8].
(b) Simply rewrite the proof of Lemma 2.2 in the present context. \(\Box\)

Using these two preservation results as well as Lemma 1, we can finish the proof of the Theorem by showing:

**Lemma 4.** For any \(x \in \omega^\omega\), \(\mathbb{P}_x\) is soft.

**Proof.** Let \(h : \mathbb{P}_x \to \omega\) be defined by \(h(d, \langle \pi_k; k \in d \rangle, F) := \max\{\max(d), |F|\}\). \(h\) is trivially a height function; furthermore any decreasing sequence of \(\mathbb{P}_x\) which becomes eventually constant in height becomes eventually constant in the first two coordinates, and in the third coordinate, the size of the set of functions considered is eventually constant — so (I) of the definition of softness is obvious. (III) is easy, and we are left with (II).

We can assume \(p = (d, \langle \pi_k; k \in d \rangle, F)\) and \(q = (e, \langle \pi_k; k \in e \rangle, G)\) are compatible and \(q \not\subseteq p\). We now describe which conditions of height \(\leq m\) we put into our finite set.

(i) Assume \(d \subseteq e\). Then we must have: \(\exists f \in F \forall g \in G \ (f \not\subseteq g)\). We take all conditions of the form \((e', \langle \pi'_k; k \in e' \rangle, G)\) extending \(q\) such that \(\max(e') \leq m\) and for some \(f \in F\) (with \(\forall g \in G \ (f \not\subseteq g)\)) and some \(k \in e' \setminus e\) \((\pi'_k(f \mid k) \neq f(k))\).

(ii) Assume \(e \subset d\). We take all conditions of the form \((e', \langle \pi'_k; k \in e' \rangle, G')\) extending \(q\) such that \(\max(e') \leq m\), \(|G'| \leq m\) and either
(a) \(G = G'\) and \([d \cap \max(e') \not\subseteq e' \text{ or } e' \cap \max(d) \not\subseteq d]\) or
(b) \(G = G'\) and for some \(k \in (d \cap e') \setminus e\) \((\pi'_k \neq \pi_k)\) or
(c) \(G' = G \cup \tilde{G}\) where \(\forall g \in \tilde{G} \ (\text{dom}(g) \leq \max(d) + 1 \ \wedge \ g \in \prod_{n<\text{dom}(g)} x(n))\) and \(\exists g \in \tilde{G} \ \exists k \in d \ (\pi_k(g \mid k) \neq g(k))\) or
(d) \(G = G'\) and \(d \subset e'\) and for some \(f \in F\) (with \(\forall g \in G \ (f \not\subseteq g)\)) and some \(k \in e' \setminus d\) \((\pi'_k(f \mid k) \neq f(k))\) [this is like (i)].
In either case we have described a finite set of conditions and leave it to the reader to check that each condition of height \( \leq m \) below \( q \) incompatible with \( p \) is indeed below one of the conditions exhibited. This finishes the proof of the Lemma.

As in 2.2. we notice that we can in fact prove the consistency of \( e_{ubd} = \lambda \) and \( b = e = \kappa \) for arbitrary uncountable regular \( \kappa < \lambda \).

3.4. There is an even stronger result:

**THEOREM 1.** It is consistent that \( e_{ubd} = 2^\omega = \omega_2 \) and \( d = \omega_1 \).

To appreciate this recall that by [Bl 2, Theorem 13], we have \( e \leq d \). Our reason for nevertheless keeping the result in 3.3. is that it allows us to choose the size of \( e, b, \) and \( e_{ubd} \) arbitrarily. The proof of Theorem 1 goes as follows. Let

\[
\lambda^* := \text{min}\{|X|; \exists f \in (\omega \setminus 2)^\omega \ (X \subseteq \prod_n f(n) \land \forall \phi \in \prod_n [f(n)]^n \exists x \in X \exists^\infty n (x(n) \not\in \phi(n)))\}.
\]

Pawlikowski [Pa, Lemma 2.2.] proved that \( \lambda^* \) equals the transitive additivity of the ideal \( \mathcal{L} \). Rewriting Blass’ proof of \( \text{add}(\mathcal{L}) \leq e \) [Bl 2, Theorem 12] in our context, one easily sees \( \lambda^* \leq e_{ubd} \). Hence Theorem 1 follows from Shelah’s

**THEOREM 2.** [Sh 326, section 2] \( \text{CON}(\lambda^* = 2^\omega = \omega_2 \land d = \omega_1) \).

(Note that this uses only one of the forcing notions of Shelah’s result, namely the one from [Sh 326, Proposition 2.9]. Also notice that Pawlikowski’s p.o. [Pa, Theorem 2.4] is soft and thus his model for \( \lambda^* > b \) is an alternative to ours for showing Theorem 3.3. — this wouldn’t shorten the argument, however, for we still would have to prove that iterating soft p.o.’s doesn’t increase \( e \).)

3.5. *Evasion ideals and duality.* With the concepts of evading and predicting we can associate \( \sigma \)-ideals on the reals as follows. Fix a space \( X = \prod_n f(n) \), where \( f \in (\omega + 1 \setminus 2)^\omega \). Let \( \mathcal{I}_X = \{A \subseteq X; \text{there is a countable set of } X \text{-predictors } \Pi \text{ such that for all } g \in A \text{ there is } \pi \in \Pi \text{ predicting } g\} \). Note that the ideals \( \mathcal{I}_X \) are subideals of the ideal \( \mathcal{E} \) (see 3.1.). We shall study the cardinal coefficients related to these ideals (see § 1).

**PROPOSITION.** \( \text{add}(\mathcal{I}_X) = \omega_1, \text{cof}(\mathcal{I}_X) = 2^\omega \) and \( \text{unif}(\mathcal{I}_X) \geq e_X \).

**Proof.** Let \( \{D_\alpha; \alpha < 2^\omega\} \) be an a.d. family of subsets of \( \omega \). Let \( \pi_\alpha \) be an arbitrary \( X \)-predictor with \( D_{\pi_\alpha} = D_\alpha \). Let \( A_\alpha = \{g \in X; \pi_\alpha \text{ predicts } g\} \), and note that whenever \( \Gamma \subseteq 2^\omega \) is uncountable then \( \bigcup_{\alpha \in \Gamma} A_\alpha \not\in \mathcal{I}_X \). This shows the two equalities; the inequality is trivial. \( \square \)
It is unclear whether $\text{unif}(\mathcal{I}_X) = e_X$ (cf section 5, question (4)); we note, however, that all known ZFC-results as well as all known consistency results about $e_X$ carry over to $\text{unif}(\mathcal{I}_X)$. Also, the cardinal $e'$ defined in 2.3. can be viewed as the uniformity of a (slightly larger) ideal.

Similarly, dualizing these results (cf 1.1.), we get corresponding results about $\text{cov}(\mathcal{I}_X)$, the smallest size of a set of predictors needed to predict all reals from $X$. The only problem occurs when dualizing the consistency results gotten from countable support iterations. Concerning this we note that the Mathias real model satisfies $b = \omega_2$ and $\text{cov}(\mathcal{I}_X) = \omega_1$ for any space of the form $X = \prod_n f(n)$, where $f \in (\omega \setminus 2)^\omega$ (this forms part of the proof of Theorem D, see 4.3.), and thus is dual to the one of 3.4., Theorem 1. On the other hand, to show the consistency of $\text{cov}(\mathcal{I}_{n\omega}) = \omega_1$ and $\text{cov}(\mathcal{I}_X) = \text{unif}(\mathcal{M}) = \omega_2$ for $X = \prod_n f(n)$, $f \in (\omega \setminus 2)^\omega$ converging (fast enough) to infinity (dual to the Mathias real model, see the remark at the end of subsection 3.2.), use the model gotten by iterating with countable support the forcing of [BaJS, section 3] (we leave the details of this to the reader; just note that the generic real is not predicted by $X$-predictors from the ground model and thus $\text{cov}(\mathcal{I}_X) = \omega_2$ after the iteration, and that $\text{cov}(\mathcal{I}_{n\omega}) = \omega_1$ can be shown by an argument similar to the one that the iteration doesn’t add random reals [BaJS, 3.6, 3.8 and 3.15]).

The relationship between the cardinals associated with evading and predicting as well as many other cardinals can be displayed in the following diagram (where the invariants grow larger when moving up along the lines).

put the diagram here

Here $\mathcal{I} = \mathcal{I}_{\omega\omega}$, and $\mathcal{I}_\ell$ is the ideal associated with linear predicting as defined by Blass; both $\mathcal{I}_\ell$ and $e_\ell$ could be replaced by $\mathcal{I}_K$ and $e_K$, respectively, where $K$ is a countable field. To ease the reading we did not include several inequalities not related to evading and predicting; these are $\text{add}(\mathcal{M}) \leq \text{unif}(\mathcal{E})$ and $\text{cov}(\mathcal{E}) \leq \text{cof}(\mathcal{M})$, $b \leq \text{unif}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq d$, $\text{cov}(\mathcal{L}) \leq \text{unif}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq \text{unif}(\mathcal{L})$, as well as $b \leq r$ and $s \leq d$. All inequalities are proved in [vD, section 3], [Fr], [BS], [Bl 2] or our work (see also [Va] for other references). Almost all inequalities are consistently strict (see [vD, section 5], [BaJS] or our work); this is unclear only for $e_\ell \leq e$ and dually $\text{cov}(\mathcal{I}) \leq \text{cov}(\mathcal{I}_\ell)$; see question (2) in section 5.

22
§ 4. Luzin sets of evading functions, Luzin groups, and Gross spaces

4.1. Recall (cf, e.g., [Mi 2, p. 206]) that given a $\sigma$–ideal $I$ on the real line, an $I$–Luzin set is defined to be an uncountable subset of the reals with at most countable intersection with every member of $I$. One of the goals of this section is to study this notion in case of the ideals introduced in 3.5.

Let $X = \prod_n f(n)$, where $f \in (\omega + 1 \setminus 2)^\omega$, be one of the spaces studied in the latter subsection. We say an uncountable $F \subseteq X$ is a Luzin set of evading functions iff $F$ is $I_X$–Luzin iff for all $X$–predictors $\pi$ at most countably many $f \in F$ are predicted by $\pi$. More generally, given $\kappa$ with $\text{cf}(\kappa) > \omega$, $F \subseteq X$ is a generalized Luzin set of evading functions of size $\kappa$ iff $F$ is generalized $I_X$–Luzin iff for all $X$–predictors $\pi$ less that $\kappa$ many $f \in F$ are predicted by $\pi$.

It turns out, however, that in case of linear prediction the following notion is more useful. Let $K$ be a finite or countable field. An additive group $G = (G, +) \leq (K^\omega, +)$ which is closed under multiplication with elements from $K$ (i.e., it is a subspace of the vector space $K^\omega$) is called a Luzin group iff for all $K$–linear predictors $\pi$ at most countably many $g \in G$ are predicted by $\pi$. Similarly, for $\kappa$ with $\text{cf}(\kappa) > \omega$, we say $G \leq K^\omega$ is a generalized Luzin group of size $\kappa$ iff for all $K$–linear predictors $\pi$ less than $\kappa$ many $g \in G$ are predicted by $\pi$.

4.2. Obviously, the existence of a Luzin set of evading functions for a space $X$ implies that the corresponding evasion number equals $\omega_1$. Using ideas from [JS 3], we proceed to show that the converse need not hold.

**Theorem.** It is consistent that $e_{\text{fin}} = \omega_1$, but there are no Luzin sets of evading functions for any space $n^\omega$ (where $n \in \omega$) and no Luzin groups $G \leq K^\omega$, where $K$ is the two–element field. Actually, the conclusion holds in Laver’s model for the Borel conjecture.

**Proof.** $e_{\text{fin}} = \omega_1$ follows from $e_{\text{fin}} \leq \text{unif}(\mathcal{L})$ and the fact that $\text{unif}(\mathcal{L}) = \omega_1$ holds in Laver’s model [JS 2, section 1].

Using a similar argument as in 3.2. (Lemma 2) it is easy to see that the existence of a Luzin set of evading functions for $n^\omega$ ($n \geq 2$) is equivalent to the existence of a Luzin set of evading functions for the Cantor space $2^\omega$. Furthermore, as any such Luzin set of size
would lie in an intermediate extension of Laver’s model, it suffices to show that adding one Laver real destroys the ground model’s Luzin sets.

So assume $F \in V$ is a Luzin set of evading functions (for $2^\omega$). We introduce an $\mathbb{L}$–name $\check{\pi}$ for a predictor as follows: if $\check{\ell}$ is the $\mathbb{L}$–name for the generic real, let $\check{D}_{\pi} := \{\check{\ell}(i); i \in \omega\}$ be the name for the set of numbers on which we predict, and let $\check{\pi}_k(\sigma) = 0$ for all $\sigma \in 2^k$ and $k \in \check{D}_{\pi}$.

**Claim.** $\not\models_{\mathbb{L}} \check{\pi}$ predicts uncountably many elements from $F$.

**Proof of Claim.** Let $T \in \mathbb{L}, N \prec \langle H(\kappa), ... \rangle$ countable with $T \in N$. Let $f \in F$ be such that $f$ evades all predictors from $N$ (note that all but countably many elements of $F$ have this property). We shall construct $S \leq T, S \in \mathbb{L}$ such that

$$S \models_{\mathbb{L}} \check{\pi} \text{ predicts } f.$$

Clearly this is enough (it shows that no $T \in \mathbb{L}$ forces that $\check{\pi}$ predicts only countably many elements).

For $\rho \in T$ such that $\text{succ}_T(\rho)$ is infinite, let

$$A_\rho := \{n \in \text{succ}_T(\rho); f(n) = 0\}.$$ 

Note that $A_\rho$ must be infinite (otherwise, define a predictor $\pi$ by: $D_\pi := \text{succ}_T(\rho)$ and $\pi_k(\sigma) = 1$ for $\sigma \in 2^k$ ($k \in D_\pi$) iff $\sigma(\max(D_\pi \cap k)) = 1$; then $\pi \in N$ and $\pi$ predicts $f$, a contradiction). Now define $S$ by recursion on its levels:

(i) $\text{stem}(S) = \text{stem}(T)$;

(ii) assume $\rho \in S$, then: $\rho \langle n \rangle \in S \iff n \in A_\rho$.

Note that $S \models_{\mathbb{L}} \check{\pi}$ predicts $f$. This concludes the proof of the claim. $\square$

To see that there are no Luzin groups over $\mathbb{K}$, note that both the predictor defined from the Laver real in the extension and the predictor defined in the proof of the claim are in fact linear. $\square$

4.3. We start the proof of the dichotomy theorem (Theorem D) and recall (an extended version of) the statement of its first part.

**Theorem.** It is consistent that $2^\omega = \omega_2$ and there are no generalized Luzin sets of evading functions for any space $n^{\omega}$ (where $2 \leq n \in \omega$). Furthermore there are no generalized Luzin groups $G \leq \mathbb{K}^\omega$, where $\mathbb{K}$ is a finite field.

**Proof.** This is true in the Mathias model. That there are no Luzin sets (Luzin groups) of size $\omega_1$ follows from $e_{fin} \geq e_\mathbb{K} \geq s$ (3.2., Lemma 3) and $s = \omega_2$. 

24
To see that there are no generalized Luzin sets (Luzin groups) of size $\omega_2$ we apply the Laver property (see 1.2.). In fact, we show something slightly more general: the ground model predictors predict all new reals of the space $X = \prod_n f(n)$, where $f \in \omega^\omega$ is arbitrary (i.e. $\text{cov}(I_X) = \omega_1$ in the language of 3.5.). Thus, any set of size $\omega_2$ contains a subset of same size predicted by one predictor.

Let $P$ be a proper forcing notion satisfying the Laver property. Let $\dot{g}$ be a $P$–name for an element of $X$. Partition $\omega$ into disjoint intervals $\{I_n; n \in \omega\}$ such that $|I_n| = n^2$, and $\max(I_n) + 1 = \min(I_{n+1})$. Using the Laver property, we can find for given $p \in P$ a $q \leq p$ and a sequence $\{\sigma_n^k; n \in \omega \land k \in n\}$ such that

$$\forall n \in \omega \forall k \in n \left( \sigma_n^k \in \prod_{m \in I_n} f(m) \right)$$

and

$$q \forces \forall n \exists k \in n \left( \dot{g}\restriction I_n = \sigma_n^k \right).$$

Fix $n$. As $|I_n| = n^2$ there must be $m_n \in I_n$ such that for $k_1, k_2 \in n$: whenever $\sigma_n^{k_1} \rest m_n = \sigma_n^{k_2} \rest m_n$, then $\sigma_n^{k_1}(m_n) = \sigma_n^{k_2}(m_n)$. Let $D_\pi = \{m_n; n \in \omega\}$ and define the predictor $\pi$ by: for $n \in \omega$ and $\sigma \in \prod_{i \in m_n} f(i)$,

$$(*) \quad \pi_{m_n}(\sigma) = \begin{cases} \sigma_n^k(m_n) & \text{if } \sigma \rest (I_n \cap m_n) = \sigma_n^k \rest m_n \\ 0 & \text{otherwise.} \end{cases}$$

By the choice of $m_n$, $\pi_n$ is well–defined. It follows from the construction that

$$q \forces \pi \text{ predicts } \dot{g}.$$ 

Finally, in case of generalized Luzin groups, we have to define a linear predictor in the ground model. To do this, we proceed as above, and take for fixed $n$ a maximal linearly independent subset of the $\sigma_n^k$, $k \in n$, and use this in the definition corresponding to $(*)$. \[\square\]

4.4. **End of proof of dichotomy theorem.** We got the idea to prove part (b) of Theorem D from [ShSp, section 3].

Let $\{\phi_n; n \in \omega\}$ enumerate all linear functions from some $\mathbb{K}^m$ to $\mathbb{K}$ ($m \in \omega$). Let $\mathbb{K} = \{a_n; n \in \omega\}$. Furthermore choose an unbounded, well–ordered (under $\leq^*$) family $\{f_\alpha; \alpha < \mathbf{b}\}$ of strictly increasing functions from $\omega$ to $\omega$.

For $\sigma \in \omega^{<\omega}$ we define $\sigma^* \in \omega^{<\omega}$ by recursion on $lh(\sigma)$ and $\sigma(lh(\sigma) - 1)$ as follows. Fix $n, m \in \omega$ and $\sigma \in \omega^n$ such that $\sigma(n - 1) = m$. For $i < n - 1$ we let
$$\sigma^*(i) := (\sigma^i(i+1))^*(i).$$

We let $\sigma^*(n-1)$ be the minimal $\ell$ such that

$$(+) \ a_\ell \not\in \{ \phi_k(a_\ell, \ldots) \mid k < m \land \ell \in \{ \text{ran}(\tau^i) \mid \tau \in m^{\leq n} \} \} \quad (\text{where } j < \dim(\text{dom}(\phi_k)))\}.$$

This concludes the definition of $\sigma^*$.

Next, given $\alpha < b$, we define $g_\alpha \in \mathbb{K}^\omega$ by:

$$(++) \ g_\alpha(n) := a(f_\alpha \mid (n+1))^*(n).$$

We claim that $G := \langle g_\alpha; \ \alpha < b \rangle$ is a Luzin group.

For suppose not. Then (without loss) there are $n \in \omega$, $A_\alpha \subseteq b$ ($\alpha < b$) a $\Delta$–system of sets of size $n$, $b_\ell \in \mathbb{K}$ ($\ell < n$) and a linear predictor $(D_\pi, (\pi_m; \ m \in D_\pi))$ such that

$$\forall m \in D_\pi \forall \alpha < b \ (\pi_m(\sum_{\ell < n} b_\ell g_{A_\alpha}(\ell) \mid m) = \sum_{\ell < n} b_\ell g_{A_\alpha}(\ell)(m),$$

\[(*) \quad \text{i.e.,} \quad \frac{1}{b_{n-1}} \cdot \pi_m(\sum_{\ell < n} b_\ell g_{A_\alpha}(\ell)(0), \ldots, \sum_{\ell < n} b_\ell g_{A_\alpha}(\ell)(m-1)) - \frac{1}{b_{n-1}} \sum_{\ell < n-1} b_\ell g_{A_\alpha}(\ell)(m) = g_{A_\alpha(n-1)}(m).\]

Without loss we may as well assume that for some $m_0 \in \omega$ for all $\alpha < b$, for all $m \geq m_0$ and for all $\ell < n - 1$, $f_{A_\alpha(n-1)}(m) > f_{A_\alpha(\ell)}(m)$. Now choose $m \geq m_0$, $m \in D_\pi$ such that

$$\{ f_{A_\alpha(n-1)}(m); \ \alpha < b \} \text{ is unbounded in } \omega \quad (\text{this is possible since } \{ f_{A_\alpha(n-1)}; \ \alpha < b \} \text{ is unbounded in } \omega^\omega, \text{ because } \{ f_\alpha; \ \alpha < b \} \text{ is unbounded in } \omega^\omega \text{ and well–ordered), and think of the expression } (*) \text{ as a linear function from } \mathbb{K}^{m-n+n-1} \text{ to } \mathbb{K}. \text{ Next choose } k \in \omega \text{ such that } \phi_k \text{ is the left–hand side of } (*) \text{ and } \alpha < b \text{ such that } k < f_{A_\alpha(n-1)}(m). \text{ Then, for } \ell < n - 1,$$

$$f_{A_\alpha(\ell)} \mid (m+1) \subseteq (f_{A_\alpha(n-1)}(m))^{m+1} \text{ and } f_{A_\alpha(n-1)} \mid m \in (f_{A_\alpha(n-1)}(m))^m,$$

and it is immediate from $(+)$ and $(++)$ that $g_{A_\alpha(n-1)}(m) \not\in \phi_k(g_{A_\alpha(0)}(0), \ldots, g_{A_\alpha(n-2)}(m), g_{A_\alpha(n-1)}(0), \ldots, g_{A_\alpha(n-1)}(m-1))$, contradicting equation $(*)$. \[\square\]

Note that the statement of the Theorem is a strong way of saying $e_\mathbb{K} \leq b$. For $\mathbb{K} = \mathbb{Q}$ this was proved (rather indirectly using various intermediate group–theoretical notions) by Blass (see Introduction, cf also 3.1.).

Granted the inequality $e_\mathbb{K} \leq b$, there is, in a sense, nothing peculiar about this result. It merely reflects the fact that there is (in ZFC) a set with Luzin–style properties associated with $b$ (this is the unbounded and well–ordered family $\{ f_\alpha; \ \alpha < b \}$ we started with) and that it is a (seemingly) general state of affairs that given such a Luzin–style set for one cardinal (in our case, $b$) we can construct a similar set (of same size) for a smaller cardinal.
(in our case, \( e_{K} \)) (see [Ci, Theorem 3.1.] for related results). The main difficulty then is to get a Luzin group and not just a Luzin set of evading functions for the family of linear \( K \)-valued predictors (cf § 5, question (5)).

4.5. We prove a more general version of the implication \((a) \implies (b)\) in the equivalence theorem (Theorem E).

**Lemma.** (A) If there is a strong Gross space \((E, \Phi)\) of dimension \( \kappa \) over the field \( K \), then there is a Luzin group \( G \leq K^{\omega} \) of size \( \kappa \).

(B) Assume \( cf(\kappa) > \omega \). If there is a Gross space \((E, \Phi)\) of dimension \( \kappa \) over the field \( K \), then there is a generalized Luzin group \( G \leq K^{\omega} \) of size \( \kappa \).

**Proof.** Both are similar. So we shall only prove (A) and leave (B) to the reader.

Let \((E, \Phi)\) be strongly Gross of dimension \( \kappa \) over \( K \). Assume \( \{e_{\alpha}; \alpha < \kappa\} \) is a basis of \( K \). For \( \alpha \geq \omega \) we define \( f_{\alpha}: \omega \rightarrow K \) by

\[
f_{\alpha}(n) := \Phi(e_{n}, e_{\alpha}).
\]

We claim that the subspace \( G \leq K^{\omega} \) generated by the \( f_{\alpha} \) (\( \omega \leq \alpha < \kappa \)) is a Luzin group.

For suppose not. Then there is a linear \( K \)-valued predictor \( \pi \) predicting \( \omega_{1} \) many \( g_{\alpha} \in G \) (\( \alpha < \omega_{1} \)). Let \( \pi = (D_{\pi}, (\pi_{m}; m \in D_{\pi})) \); without loss, for all \( m \in D_{\pi} \) for all \( \alpha < \omega_{1} \), we have \( \pi_{m}(g_{\alpha}|m) = g_{\alpha}(m) \). We can assume that there are \( n \in \omega, A_{\alpha} \subseteq \kappa \setminus \omega \) (\( \alpha < \omega_{1} \)) forming a \( \Delta \)-system of sets of size \( n \), and \( b_{\ell} \in K \) (\( \ell < n \)) such that

\[
g_{\alpha} = \sum_{\ell<n} b_{\ell} f_{A_{\alpha}(\ell)} \text{ for } \alpha < \omega_{1}.
\]

For \( m \in D_{\pi} \) and \( i \in m \), we let \( c_{im} := \pi_{m}(\sigma_{im}) \in K \) where \( \sigma_{im} \in K^{m} \) is such that

\[
\sigma_{im}(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}
\]

By linearity of \( \pi_{m} \), we have for any \( \sigma \in K^{m+1} \) satisfying \( \pi_{m}(\sigma|m) = \sigma(m) \):

\[
\sigma(m) = \sum_{j<m} c_{jm} \sigma(j),
\]

i.e.

\[
\sum_{j<m} c_{jm} \sigma(j) - \sigma(m) = 0.
\]

Thus we have for all \( m \in D_{\pi} \) and all \( \alpha < \omega_{1} \)

\[
0 = \sum_{j<m} c_{jm} \cdot g_{\alpha}(j) - g_{\alpha}(m) = \sum_{\ell<n} b_{\ell} \left( \sum_{j<m} c_{jm} \cdot f_{A_{\alpha}(\ell)}(j) - f_{A_{\alpha}(\ell)}(m) \right) = \]

27
Hence, if \( U \) is the subspace of \( E \) spanned by the vectors \( \sum_{j<m} c_{jm} \cdot e_j - e_m, \sum_{\ell<n} b_{\ell} \cdot e_{A_{\alpha}(\ell)} \), then \( U^\perp \geq (\sum_{\ell<n} b_{\ell} \cdot e_{A_{\alpha}(\ell)}; \alpha < \omega_1) \). Therefore \( (E, \Phi) \) cannot be strongly Gross, a contradiction. \( \square \)

4.6. The proof of \( (b) \implies (a) \) in the equivalence theorem (Theorem E). The following argument was heavily influenced by [ShSp, section 4, Theorem 4].

Let \( G \leq \mathbb{K}^\omega \) be a Luzin group of size \( \omega_1 \). Let \( \{ g_\alpha; \alpha < \omega_1 \} \) be a set of generators of \( G \) as a \( \mathbb{K} \)-vector space.

Choose recursively injective functions \( h_\alpha : \alpha \rightarrow \omega \) such that \( \omega \setminus \text{ran}(h_\alpha) \) is infinite and \( \forall \beta < \alpha \) the set \( \{ \gamma < \beta; h_\beta(\gamma) \neq h_\alpha(\gamma) \} \) is finite (this is one of the standard constructions of an Aronszajn tree, due to Todorcevic (cf [To, (2.2.)]).

Let \( E \) be a vector space of dimension \( \omega_1 \) over \( \mathbb{K} \); assume that \( E = \langle e_\alpha; \alpha < \omega_1 \rangle \). We define a symmetric bilinear form \( \Phi \) on \( E \) as follows:

\[
\Phi(e_\alpha, e_\beta) := g_\beta(h_\beta(\alpha)) \quad \text{for} \quad \alpha < \beta < \omega_1.
\]

We claim that \( (E, \Phi) \) is a Gross space. For if this were not the case, we could find (using standard thinning–out arguments) vectors \( y_k \) \( (k \in \omega) \) and \( z_\gamma \) \( (\gamma \in \omega_1) \), and \( \alpha^* \in \omega_1, n \in \omega, m_k \) \( (k \in \omega) \), \( B_\gamma \) \( (\gamma \in \omega_1) \), \( A_k \) \( (k \in \omega) \), \( b_i \) \( (i \in n) \) and \( a_{jk} \) \( (j \in m_k \) and \( k \in \omega) \) such that

(i) for all \( k \in \omega \) and \( \gamma \in \omega_1 \) we have \( \Phi(y_k, z_\gamma) = 0 \);

(ii) \( A_k \subseteq \alpha^*, |A_k| = m_k \), and \( a_{jk} \in \mathbb{K} \) such that \( y_k = \sum_{j \in m_k} a_{jk} e_{A_k(j)} \); furthermore \( k_1 < k_2 \) implies \( \text{max} h_{\alpha^*} (A_{k_1}) < \text{min} h_{\alpha^*} (A_{k_2}) \);

(iii) \( B_\gamma \subseteq \omega_1, |B_\gamma| = n \), and \( b_i \in \mathbb{K} \) such that \( z_\gamma = \sum_{i \in n} b_i e_{B_\gamma(i)} \); furthermore \( \gamma_1 < \gamma_2 \) implies \( \alpha^* < \text{min}(B_{\gamma_1}) < \text{max}(B_{\gamma_1}) < \text{min}(B_{\gamma_2}) \).

Next let us introduce a linear \( \mathbb{K} \)-valued predictor \( \pi \) as follows. Fix \( k \in \omega \). Let \( d_k := \text{max} h_{\alpha^*}(A_k) \); set \( D_\pi := \{ d_k; k \in \omega \} \); and let \( j_k \) be such that \( d_k = h_{\alpha^*}(A_k(j_k)) \). We set

\[
\pi_k(\sigma_{ik}) := \begin{cases} 
-\frac{a_{jk}}{a_{jk}} & \text{if } i = h_{\alpha^*}(A_k(j)) \\
0 & \text{if } i \notin h_{\alpha^*}(A_k),
\end{cases}
\]

where \( \sigma_{ik} \in \mathbb{K}^{d_k} \) is such that

\[
\sigma_{ik}(j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}.
\end{cases}
\]

28
We extend \( \pi_k \) linearly to \( \mathbb{K}^{d_k} \), and thus define a linear \( \mathbb{K} \)-valued predictor. By Luzinity of \( G \), there is a \( \gamma \in \omega_1 \) such that \( g := \sum_{i \in n} b_i g_{B}(i) \) evades \( \pi \). Choose \( k \in \omega \) such that

\begin{enumerate}
    \item \( h_{\alpha}(i)(A_k) = h_\alpha^*(A_k) \) for all \( i \in n \);
    \item \( \pi_k(g|d_k) \neq g(d_k) \).
\end{enumerate}

On the other hand, we have

\[ 0 = \Phi(y_k, z_\gamma) = \sum_{j \in m_k} \sum_{i \in n} b_i \cdot a_{jk} \cdot g_{B}(i)(h_{\alpha}(A_k)(j)) = \sum_{j \in m_k} \sum_{i \in n} b_i \cdot a_{jk} \cdot g_{B}(i)(h_\alpha^*(A_k)(j)). \]

I.e.

\[ \sum_{j \in m_k \setminus \{j_k\}} a_{jk} \cdot g(h_\alpha^*(A_k)(j)) = -a_{jk} \cdot g(d_k). \]

Thus

\[ \pi_k(g|d_k) = - \sum_{j \in m_k \setminus \{j_k\}} \frac{a_{jk}}{a_{jk}} \cdot g(h_\alpha^*(A_k)(j)) = g(d_k), \]

a contradiction. \( \square \)

4.7. Corollary. (Baumgartner, Shelah, Spinas; [BSp], [ShSp], see also [Sp 2]) Let \( \mathbb{K} \) be an arbitrary finite or countable field.

(a) Assume any of the following:

\begin{itemize}
    \item there is a Luzin group \( G \leq \mathbb{K}^\omega \)
    \item \( \text{cof}(\mathcal{M}) = \omega_1 \)
    \item \( b = \omega_1 \) (in case \( |\mathbb{K}| = \omega \))
\end{itemize}

Then there is a strong Gross space over \( \mathbb{K} \).

(b) Assume any of the following:

\begin{itemize}
    \item there is no Luzin group \( G \leq \mathbb{K}^\omega \)
    \item \( e_{\mathbb{K}} > \omega_1 \)
    \item \( \text{add}(\mathcal{L}) > \omega_1 \)
    \item \( p > \omega_1 \)
    \item \( s > \omega_1 \) (in case \( |\mathbb{K}| < \omega \))
\end{itemize}

Then there is no strong Gross space over \( \mathbb{K} \).

Proof. (a) 4.4. and 4.6.

We leave the construction of a Luzin group from \( \text{cof}(\mathcal{M}) = \omega_1 \) to the reader.
(b) 4.5. and [Bl, section 4] — see also § 3 and in particular 3.2. □

In a sense our results say that there is no cardinal invariant such that its being \( \omega_1 \) is equivalent to the existence of strong Gross spaces over \( K \) (cf the question in the Introduction). The natural candidate for such a cardinal would be \( e_K \), but, by 4.2., \( e_K \) may be \( \omega_1 \) and there may be no Luzin subgroup of \( K^{\omega} \) and hence no strong Gross space over \( K \).

We note in closing that we think of the dichotomy theorem as the basic result underlying the fact that it is much more difficult to get rid of Gross spaces over countable fields than over finite fields (in fact, the consistency of the non–existence of Gross spaces over countable fields is an open problem [Sp 2]). It follows from 4.5. and 4.3. that there are no Gross spaces over finite fields in Laver’s or Mathias’ models (this was known to be true previously [ShSp, section 4, Theorem 2] in a model constructed by Shelah [BSh] in which there are both \( P_{\omega_1} \)– and \( P_{\omega_2} \)–points — in this peculiar situation it is indeed not difficult to see that there are no such spaces; however we think that Laver’s or Mathias’ models are much easier to grasp combinatorially).

§ 5. Questions

We have introduced a multitude of cardinals and in spite of our ZFC– and consistency results many questions concerning the relationship between them remain open. We mention but a few.

The results of section 2 suggest a positive answer to the following.

(1) Is \( se \leq \text{add}(M) \) (or even \( \min\{e', b\} \leq \text{add}(M) \)) in ZFC? Is \( se < \text{add}(M) \) (or even \( e' < \text{add}(M) \)) consistent?

Recall that Blass proved \( \min\{e, b\} \leq \text{add}(M) \) [Bl 2, Theorem 13]. To calculate the value of \( e' \) in the Hechler real model may shed some light on the situation.

The most important problem is perhaps:

(2) (Blass [Bl 2, section 5, question (2)]) Clarify the relationship between \( e \) and \( b \) (and between \( e \) and \( e_\ell \)!) Or: is there a generalized Luzin set of evading functions for \( \omega^\omega \)
(cf 4.3. and 4.4.)?

Related is

(3) Clarify the relationship between the different $e_K$’s (and between $e_K$ and $e_{fin}$ for finite $K$)! Or: does the existence of a strong Gross space over some finite field imply the existence of a strong Gross space over every finite or countable field?

Concerning sections 3 and 4, the following additional questions may be of some interest:

(4) Is $e_X = \text{unif}(I_X)$ (cf 3.5.)?

(5) Does the existence of a Luzin or Sierpiński set imply the existence of a Luzin group (cf 4.4.)?

REFERENCES

[BaJS] T. Bartoszyński, H. Judah and S. Shelah, The Cichoń diagram, submitted to Journal of Symbolic Logic.

[BS] T. Bartoszyński and S. Shelah, Closed measure zero sets, Annals of Pure and Applied Logic, vol. 58 (1992), pp. 93-110.

[Bau] J. Baumgartner, Iterated forcing, Surveys in set theory (edited by A.R.D. Mathias), Cambridge University Press, Cambridge, 1983, pp. 1-59.

[BD] J. Baumgartner and P. Dordal, Adjoining dominating functions, Journal of Symbolic Logic, vol. 50 (1985), pp. 94-101.

[BSp] J. Baumgartner and O. Spinas, Independence and consistency proofs in quadratic form theory, Journal of Symbolic Logic, vol. 56 (1991), pp. 1195-1211.

[Bl 1] A. Blass, Simple cardinal characteristics of the continuum, in: Set theory of the reals, Proceedings of the Bar–Ilan conference in honour of Abraham Fraenkel.

[Bl 2] A. Blass, Cardinal characteristics and the product of countably many infinite cyclic groups, preprint.
A. Blass and S. Shelah, *There may be simple $P_{\aleph_1}$– and $P_{\aleph_2}$–points and the Rudin–Keisler ordering may be downward directed*, Annals of Pure and Applied Logic, vol. 33 (1987), pp. 213-243.

J. Brendle, *Larger cardinals in Cichoń’s diagram*, Journal of Symbolic Logic, vol. 56 (1991), pp. 795-810.

J. Brendle and H. Judah, *Perfect sets of random reals*, to appear in Israel Journal of Mathematics.

J. Brendle, H. Judah and S. Shelah, *Combinatorial properties of Hechler forcing*, Annals of Pure and Applied Logic, vol. 58 (1992), pp. 185-199.

J. Cichoń, *On two–cardinal properties of ideals*, Transactions of the American Mathematical Society, vol. 314 (1989), pp. 693-708.

K. Eda, *A note on subgroups of $Z^N$,* Abelian Group Theory, Proceedings, Honolulu 1982/83 (R. Göbel, L. Lady and A. Mader, eds.), Lecture Notes in Mathematics 1006, Springer–Verlag, 1983, pp. 371-374.

D. Fremlin, *Cichoń’s diagram*, Séminaire Initiation à l’Analyse (G. Choquet, M. Rogalski, J. Saint Raymond), Publications Mathématiques de l’Université Pierre et Marie Curie, Paris, 1984, pp. 5-01 - 5-13.

T. Jech, *Set theory*, Academic Press, San Diego, 1978.

T. Jech, *Multiple forcing*, Cambridge University Press, Cambridge, 1986.

H. Judah and S. Shelah, *Souslin forcing*, Journal of Symbolic Logic, vol. 53 (1988), pp. 1188-1207.

H. Judah and S. Shelah, *The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing)*, Journal of Symbolic Logic, vol. 55 (1990), pp. 909-927.

H. Judah and S. Shelah, *Killing Luzin and Sierpiński sets*, to appear.

K. Kunen, *Set theory*, North-Holland, Amsterdam, 1980.

A. Miller, *Some properties of measure and category*, Transactions of the American Mathematical Society, vol. 266 (1981), pp. 93-114.

A. Miller, *Special subsets of the real line*, Handbook of set–theoretic topology, K. Kunen and J. E. Vaughan (editors), North–Holland, Amsterdam, 1984, pp. 201-233.
[Pa] J. Pawlikowski, *Powers of transitive bases of measure and category*, Proceedings of the American Mathematical Society, vol. 93 (1985), pp. 719-729.

[Sh 326] S. Shelah, *Vive la différence I: nonisomorphism of ultrapowers of countable models*, Set Theory of the Continuum (H. Judah, W. Just and H. Woodin, eds.), Mathematical Sciences Research Institute Publications, Springer–Verlag, 1992, pp. 357-405.

[ShSp] S. Shelah and O. Spinas, *How large orthogonal complements are there in a quadratic space?* preprint.

[S] E. Specker, *Additive Gruppen von Folgen ganzer Zahlen*, Portugaliae Mathematica, vol. 9 (1950), pp. 131-140.

[Sp 1] O. Spinas, *Iterated forcing in quadratic form theory*, Israel Journal of Mathematics, vol. 79 (1992), pp. 297-315.

[Sp 2] O. Spinas, *Cardinal invariants and quadratic forms*, in: Set theory of the reals, Proceedings of the Bar–Ilan conference in honour of Abraham Fraenkel.

[To] S. Todorcevic, *Partitioning pairs of countable ordinals*, Acta mathematica, vol. 159 (1987), pp. 261-294.

[vD] E. K. van Douwen, *The integers and topology*, Handbook of set–theoretic topology, K. Kunen and J. E. Vaughan (editors), North–Holland, Amsterdam, 1984, pp. 111-167.

[Va] J. Vaughan, *Small uncountable cardinals and topology*, Open problems in topology (J. van Mill and G. Reed, eds.), North–Holland, 1990, pp. 195-218.
