Collective excitations of trapped binary mixtures of Bose-condensed gases

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(December 1, 2018)

Abstract

The linearised time dependent coupled Gross-Pitaevskii equations describing the long wavelength excitations of Bose-condensed binary mixtures are solved, in the bulk and in harmonic traps for the case where only the binary phase is present. In the former case we obtain two zero-sound branches. In the latter case the dispersion law also contains two branches whose dependence on the quantum numbers of the modes is the same as for a one-component condensate, but with different prefactors, depending on the ratios of the three s-wave scattering lengths of the two atomic species. In the general case where the binary phase in the trap coexists with one or both one-component phases, the mode spectrum depends on the geometry of the interphase boundaries due to the boundary conditions there. Measurements of the oscillation frequencies as in recent experiments with modulated traps would yield very detailed information on this system.
The recent dramatic advances in the ability to create Bose condensates have now reached the stage where Bose condensates of binary mixtures of two atomic species only distinguished in their angular momentum quantum numbers can be manufactured in a magnetic trap at effectively zero temperature [1]. The ground state of such binary Bose-condensed mixtures have recently been studied theoretically in the Thomas-Fermi approximation [2]. To learn more about their properties it seems interesting to study their long wavelength excitations. Recent experiments for one-component condensates have been reported in [3,4], where collective excitations have been excited by changes in the trap potential. Theoretical work on collective excitations, again for a single-component condensate, has been presented in [5], where the linearised Gross-Pitaevskii equation was solved numerically, and in [6], where an analytical solution was obtained in the long wavelength limit. Very good agreement between the theoretical predictions for the low lying mode frequencies [5,6] and the experimental results [3,4] was obtained. As shown by Stringari [6], in the Thomas Fermi approximation, the frequencies of the long wavelength excitations of a trapped Bose-Einstein condensate are independent of the microscopic properties of the condensate, in particular of the two-body scattering length, and only depend on the frequency $\omega_0$ of the single-particle oscillation in the trap.

In the present paper we shall consider the two coupled linearised Gross-Pitaevskii equations at zero temperature describing the excitations of the binary phase of a mixture of two Bose condensates. Considering the spatially homogeneous system first we obtain the Bogoliubov spectrum for this case with two branches of zero sound for $k \to 0$, where the sound velocities depend on the ratios of the three different scattering lengths in the problem. We also derive the coupled linearised hydrodynamic equations and their boundary conditions which govern the long wavelength excitations of Bose condensed binary mixtures in a trap coexisting with the one-component condensates. For the special case of a pure binary-phase condensate in a spherically symmetric trap we determine the mode-spectrum analytically in the long-wavelength and Thomas-Fermi approximations. We find that the dependence of the mode spectrum on the radial and angular momentum quantum numbers $n$ and $l$ is the
same as in the case of a single-component mixture,

$$\omega_{+, -} = \omega_{b+, -} \left(2n^2 + 2nl + 3n + l\right)^{1/2}, \tag{1}$$

but the prefactor $$\omega_{b+, -}^2$$ is no longer simply given by the trap frequency, $$\omega_0$$, but it has two solutions, corresponding to the two sound branches in the bulk, each of which depends on the two ratios of the two-particle scattering lengths.

The coupled Gross-Pitaevskii equations describing a binary mixture of Bose-Einstein condensates with equal particle masses are given by

$$i\hbar \dot{\psi}_1 = -\frac{\hbar^2}{2m} \nabla^2 \psi_1 + \psi_1 \left(V_1 |\psi_1|^2 + V_{12} |\psi_2|^2\right) + U_1(x) \psi_1, \tag{2a}$$

$$i\hbar \dot{\psi}_2 = -\frac{\hbar^2}{2m} \nabla^2 \psi_2 + \psi_2 \left(V_2 |\psi_2|^2 + V_{12} |\psi_1|^2\right) + U_2(x) \psi_2. \tag{2b}$$

We shall assume in this paper that the conditions

$$V_1 > 0, V_2 > 0, V_1 V_2 - V_{12}^2 > 0 \tag{3}$$

are satisfied. Here $$V_i = \frac{4\pi\hbar^2}{m}a_i$$, with $$i = 1, 2, 12$$, are the interaction parameters, $$a_i$$ being the s-wave scattering lengths. Let us first consider a spatially homogeneous system, i.e. the case of vanishing trap potentials $$U_1(x) = U_2(x) = 0$$, where the solutions of (2) of the form $$\psi_i \sim e^{-i\mu_i t/\hbar}$$ are given by

$$|\psi_1|^2 = |\psi_1^{(12)}|^2 = \frac{V_2 \mu_1 - V_{12} \mu_2}{V_1 V_2 - V_{12}^2}, \quad |\psi_2|^2 = |\psi_2^{(12)}|^2 = \frac{V_1 \mu_2 - V_{12} \mu_1}{V_1 V_2 - V_{12}^2}, \tag{4a}$$

$$|\psi_1|^2 = |\psi_1^{(1)}|^2 = \frac{\mu_1}{V_1}, \quad |\psi_2|^2 = |\psi_2^{(1)}|^2 = 0, \tag{4b}$$

$$|\psi_1|^2 = |\psi_1^{(2)}|^2 = 0, \quad |\psi_2|^2 = |\psi_2^{(2)}|^2 = \frac{\mu_2}{V_2}. \tag{4c}$$

The $$\mu_i$$, with $$i = 1, 2$$, are the chemical potentials of the two particle species to be determined by normalising $$|\psi_i|^2$$ to the particle numbers $$N_i$$. The stability regions of the three solutions have been discussed in [4]. In particular, the solution (2) for the binary phase is stable if its right-hand sides are both positive.

Putting now
\[ \psi_i = \sqrt{|\psi_i^{(12)}|^2 + \rho_i e^{-i(\mu_i t/\hbar - \varphi_i)}}, \quad (5) \]

and linearizing Eqs. (2) in \( \rho_i \) and \( \varphi_i \) we obtain for the excitations in the stability region of the binary condensate,

\[ \dot{\varphi}_1 = -\frac{1}{\hbar} (V_1 \rho_1 + V_{12} \rho_2) + \frac{\hbar}{4m |\psi_1^{(12)}|^2} \nabla^2 \rho_1, \quad (6a) \]
\[ \dot{\rho}_1 = -\frac{\hbar}{m} |\psi_1^{(12)}|^2 \nabla^2 \varphi_1, \quad (6b) \]

and similarly for \( \varphi_2, \rho_2 \) simply by interchanging indices. The dispersion law for solutions \( \varphi_i, \rho_i \) proportional to \( e^{i(\omega t - k \cdot x)} \) is easily obtained as \( \omega^2 = \lambda_{+,-}^2(k^2) k^2 \) with

\[ \lambda_{+,-}^2(k^2) = \frac{\hbar^2 k^2}{(2m)^2} + c_{+,-}^2, \quad (7a) \]
\[ c_{+,-}^2 = \frac{2\pi \hbar^2}{m^2} \left\{ a_1 |\psi_1^{(12)}|^2 + a_2 |\psi_2^{(12)}|^2 \pm \left[ (a_1 |\psi_1^{(12)}|^2 - a_2 |\psi_2^{(12)}|^2)^2 + 4a_{12}^2 |\psi_1^{(12)}\psi_2^{(12)}|^2 \right]\right\}^{1/2} \quad (7b) \]

For large \( k \) the two branches of the dispersion law approach that for the free particles like

\[ \omega = \frac{\hbar k^2}{2m} + \frac{mc_{+,-}^2}{\hbar} \quad (8) \]

The constant shift from the free-particle dispersion law would show up e.g. in light scattering as discussed for the case of a single component condensate in [7]. For \( k \to 0 \), on the other hand, we obtain two zero-sound branches of collective excitations with sound velocities \( c_{+,-} \) where the densities of the two components oscillate out of phase and in phase respectively.

Let us now turn to the case of a trapped binary condensate, \( U_1(x), U_2(x) \neq 0 \), which we consider in the Thomas-Fermi approximation. Then the time-dependent solutions (4) still apply with the replacements \( \mu_i \to \mu_i - U_i(x) \) (see [2]). Linearising Eqs. (2) with the ansatz (3) and making a long-wavelength approximation by keeping only the lowest order spatial derivatives we obtain in the place of (3)

\[ \dot{\varphi}_1 = -\frac{1}{\hbar} (V_1 \rho_1 + V_{12} \rho_2), \quad (9a) \]
\[ \dot{\rho}_1 = -\frac{\hbar}{m} \nabla \cdot |\psi_1^{(12)}(x)|^2 \nabla \varphi_1, \quad (9b) \]
and similarly for $\varphi_2, \rho_2$ by interchanging indices. Eliminating the phases $\varphi_i$ we obtain the two coupled equations for the long-wavelength density fluctuations,

$$\ddot{\rho}_1 = \frac{1}{m} \nabla \cdot |\psi_1^{(12)}(x)|^2 \nabla (V_1 \rho_1 + V_{12} \rho_2),$$  \(10a\)

$$\ddot{\rho}_2 = \frac{1}{m} \nabla \cdot |\psi_2^{(12)}(x)|^2 \nabla (V_2 \rho_2 + V_{12} \rho_1).$$  \(10b\)

In the general case, where the [12]-phase coexists with the one-component phases $k = [1], [2]$ Eqs. (10) have to be solved simultaneously with the equations

$$\ddot{\rho}_i = \frac{1}{m} \nabla \cdot |\psi_i^{(k)}(x)|^2 \nabla V_i \rho_i$$  \(11\)

and boundary conditions. These are dictated by number conservation and require the continuity of $\rho_i$ and the normal derivative of $\varphi_i$ across the interphase boundary between the phases [12] and $k$ on which $\psi_i^{(k)} \neq 0$. In general, the spectrum of collective excitations of the total system will therefore depend on the details of the geometry of the interphase boundaries, and solutions of the coupled equations can only be obtained numerically. Here we shall consider a special case where no interphase boundaries are present, for which an analytical solution can be given similar to the case of a one-component condensate considered in ref. [6]: Let us consider the case where $U_i = \frac{1}{2} m \Omega_i^2 r^2$ represent two concentric isotropic harmonic traps. It follows from the results in [2] that in the special case where $\mu_2/\mu_1 = (\Omega_2/\Omega_1)^2$ with

$$\max \left( \frac{V_2}{V_{12}}, \frac{V_{12}}{V_1} \right) > \frac{\Omega_2^2}{\Omega_1^2} > \min \left( \frac{V_2}{V_{12}}, \frac{V_{12}}{V_1} \right)$$  \(12a\)

no inter-phase boundaries occur in the trap and only the binary phase is present there. The condition for the chemical potentials requires that the particle numbers are chosen in the ratio $N_2/N_1 = (V_1 \Omega_2^2 - V_{12} \Omega_1^2)/(V_2 \Omega_2^2 - V_{12} \Omega_1^2)$, which can be satisfied mathematically if (12) holds, but can be satisfied physically only if the ratios of the three scattering lengths is known. Under these conditions we can solve Eqs. (10) by an ansatz $\rho_i = e^{i\omega t} \sum_{k=0}^{n} C_{ik} r^{i+k} Y_{lm}(\theta, \phi)$. The coefficients $C_{ik}, i = 1, 2, k = 0, 1, \cdots, n$, are determined by a recursion law. The condition $C_{i,n+1} = 0$ determines $\omega^2$ by Eq. (11) with
\[
\omega_{b,+,-}^2 = \frac{1}{2(a_1a_2 - a_{12}^2)} \left\{ \Omega_1^2a_2(a_1 - a_{12}) + \Omega_2^2a_1(a_2 - a_{12}) \right\} \\
\pm \left[ \left( \Omega_1^2a_2(a_1 + a_{12}) - \Omega_2^2a_1(a_2 + a_{12}) \right)^2 + 4a_2^2 \left( a_2\Omega_1^2 - a_{12}\Omega_2^2 \right) \left( a_1\Omega_2^2 - a_{12}\Omega_1^2 \right) \right]^{1/2} \right\}.
\]

For \(a_{12} = 0\) the dispersion law (13) reduces to Stringari’s result \(\omega_{b,+,-}^2 = \Omega_{1,2}^2\) for each particle species [6]. The result (13) is plotted in fig.1a (upper branch) and fig.1b (lower branch). As long as the inequalities (12) are satisfied the squared frequencies (13) are both positive. At the borders of this region in parameter space the frequency \(\omega_{b,-}\) vanishes and \(\omega_{b,+}\) approaches \(\Omega_1\) or \(\Omega_2\) depending on which of the two borders has been reached. This can be seen happening in figs.1a,b, where \(\omega_{b,+} = \Omega_2\) at the stability border defined by the vanishing of \(\omega_{b,-}\) for the choice \(\Omega_2^2 = 2\Omega_1^2\) made in these figures.

For axially symmetric trap potentials,

\[
U_i(x) = \frac{m}{2} \left[ \Omega_{\perp i}^2 + \left( \Omega_{\parallel i}^2 - \Omega_{\perp i}^2 \right) \cos^2 \theta \right] r^2,
\]

with \(\Omega_{\parallel 2}/\Omega_{\parallel 1} = \Omega_{\perp 2}/\Omega_{\perp 1}\) special solutions of Eqs. (10) can still be obtained, as in the single component case [6]. Solutions \(\propto r^l Y_{lm}(\theta, \phi)\) can be found for

\[
m = \pm l: \quad \omega_{+, -}^2 = l\omega_{\perp, -}^2,
\]

where

\[
\omega_{\perp, +,-}^2 = \omega_{b, +,-}|_{\Omega_2^2=\Omega_1^2};
\]

and

\[
m = \pm(l - 1): \quad \omega_{+, -}^2 = (l - 1)\omega_{\perp, +,-}^2 + \omega_{\parallel, +,-}^2,
\]

where

\[
\omega_{\parallel, +,-}^2 = \omega_{b, +,-}|_{\Omega_2^2=\Omega_1^2}.
\]

For the four coupled \(m = 0\) modes with \(n = 0, l = 2\) and \(n = 1, l = 0\) one obtains the four frequencies,
\[ \omega^2_{+,-1} = \omega^2_{\perp,-} \left( 2 + \frac{3}{2} \lambda^2_{+,-} + \frac{1}{2} \sqrt{9\lambda^4_{+,-} - 16\lambda^2_{+,-} + 16} \right), \]
\[ \omega^2_{+,-2} = \omega^2_{\perp,-} \left( 2 + \frac{3}{2} \lambda^2_{+,-} - \frac{1}{2} \sqrt{9\lambda^4_{+,-} - 16\lambda^2_{+,-} + 16} \right), \]

(19a)

with \( \lambda^2_{+,-} = \omega^2_{\parallel,+,-}/\omega^2_{\perp,+,-} \).

Generalisations of our results for mixtures with more than two components can be easily made. All that changes is the expressions for the prefactors \( \omega^2_b \) in the dispersion law (1). E.g. for condensates of ternary mixtures in isotropic concentric potentials \( U_i(\mathbf{x}) \) one would obtain three branches Eq. (1), whose squared frequencies \( \omega^2_{b,1,2,3} \) are the roots of a cubic equation and depend on the ratios of all six scattering lengths \( a_1, a_2, a_3, a_{12}, a_{13}, a_{23} \).

Mathematically, all explicit results (1), (15), (17), (19) are the same as the results for the one-component condensate, if there the oscillation frequency \( \omega^2_0 \) in the trap is replaced by the two frequencies (13). Physically, this difference is of interest, because these frequencies are determined dynamically by the coupling between the two components in the binary phase.

Measuring the frequency of the low lying excitations of the binary condensate phase in coexistence with one or both single-component phases with basically the same experimental set-up as employed in Refs. [5,6] seems very feasible. Because of the coupling of the coexisting phases by particle number conservation the mode spectrum is predicted to depend in general on the geometry of the interphase boundaries. To check our prediction for the simple mode spectrum in the pure binary phase the following requirements have to be met: The ratios of the scattering lengths must be known and inequality (12) must happen to be satisfied for the atomic species under consideration. Then in principle a pure binary phase condensate can be produced in concentric spherically or axially symmetric traps assuming that the ratios of the particle numbers can be chosen appropriately. To obtain concentric traps, their potentials must be made sufficiently stiff to suppress the offset of their centers due to gravity. Satisfying all these conditions one should be able to observe that the mode spectrum reduces to the simple form (1) with the scale factors (13).
ACKNOWLEDGEMENTS

We would like to thank Scott Parkins for assistance with the figures. This research was supported by the Marsden Fund of the Royal Society of New Zealand and the New Zealand Lottery Grants Board. One of us (R.G.) wishes to thank the Deutsche Forschungsgemeinschaft for financial support through the SFB 237 Unordnung und grosse Fluktuationen.
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FIGURES

FIG. 1. The frequency scale factors $\omega_{0+}^2/\Omega_1^2$ (figure 1.a) and $\omega_{0-}^2/\Omega_1^2$ (figure 1.b) as a function of the ratio of the scattering lengths $a_{12}/a_1$, for different values of $a_2/a_1$, for $\Omega_2^2/\Omega_1^2 = 2$. The chosen value of the ratio $\Omega_2^2/\Omega_1^2$ corresponds to the experiment [1].