THE NEXT-TO-LADDER APPROXIMATION FOR LINEAR DYSON-SCHWINGER EQUATIONS

ISABELLA BIERENBAUM\textsuperscript{1}, DIRK KREIMER\textsuperscript{2} AND STEFAN WEINZIERL

1. Introduction

Ladder approximations have been one of the most basic attempts to simplify and truncate Dyson–Schwinger equations in field theory in a still meaningful way \cite{Smit}. From a mathematical viewpoint they simplify the combinatorics of the forest formula considerably, and are solvable by a scaling Ansatz for sufficiently simple kinematics.

Here, we discuss such a scenario, but iterate one- and two-loop skeletons jointly, combining some analytic progress with a thorough discussion of the underlying algebraic properties.

1.1. Purpose of this paper. The main purpose is to sum an infinite series of graphs based on the iteration of two underlying skeleton graphs. We progress in a manner such that our methods can be generalized to any countable number of skeletons. We restrict to linear Dyson Schwinger equations, a case relevant for theories at a fixpoint of the renormalization group. We proceed using one-dimensional Mellin transforms, a privilege of linearity of which we make full use. See \cite{Bierenbaum:2006wa, Bierenbaum:2006ye, Bierenbaum:2006ft, Bierenbaum:2006ru} for the general approach.

2. The Dyson–Schwinger Equation

2.1. The integral equation. The equation which we consider is in massless Yukawa theory in four-dimensional Minkowski space $\mathbb{M}$, for pedagogical purposes. We define a renormalized Green function describing the coupling of a scalar particle to a fermion line by

$$
G_R(a, \ln(-q^2/\mu^2)) = 1 - a \int_M \frac{d^4k}{i\pi^2} \left\{ \frac{1}{k} G_R(a, \ln(-k^2/\mu^2)) \frac{1}{q^2} \right\}_- \\
+ a^2 \int_M \frac{d^4k}{i\pi^2} \int_M \frac{d^4l}{i\pi^2} \left\{ \frac{f(l+k')G_R(a, \ln(-(k+l)^2/\mu^2))(l+k')(k+q')}{[(k+l)^2]^{1/2}(k+q)^2k^2(l-q)^2} \right\}_- ,
$$

where $\{\}_-$ indicates subtraction at $\mu^2 = -q^2$, so that $G_R(a,0) = 1$:

$$
\{G_R(a, \ln(-k^2/\mu^2))\}_- = G_R(a, \ln(-k^2/\mu^2)) - G_R(a, \ln(-k^2/\mu^2)) - G_R(a, \ln(-k^2/\mu^2)).
$$

The kinematics are such that the fermion has momentum $q$ and the external scalar particle carries zero momentum. The equation can graphically be represented as
where the blob represents the unknown Green function \( G_R(a, \ln(-q^2/\mu^2)) \). This linear Dyson–Schwinger equation can be solved by a scaling solution, \( L = \ln(-q^2/\mu^2) \),

\[
G_R(a, L) = \exp\{-\gamma_G(a)L\}. 
\]

Indeed, this satisfies the desired normalization and leads to the equation

\[
\exp\{-\gamma_G(a)L\} = 1 + (\exp\{-\gamma_G(a)L\} - 1) \left[ aF_1(\gamma_G) + a^2F_2(\gamma_G) \right],
\]

where the two Mellin transforms are the functions determined by

\[
F_1 : \rho \to - \int \frac{d^4k}{i\pi^2} \frac{1}{k^2(k - q)^2} \left( \frac{k^2}{q^2} \right)^{-\rho},
\]

and similarly

\[
F_2 : \rho \to \int_M \frac{d^4k}{i\pi^2} \int_M \frac{d^4l}{i\pi^2} \frac{\delta(k' + q)}{(k + l)^2l^2(k + q)^2(l - q)^2} \left[ \frac{(l + k)^2}{q^2} \right]^{-\rho}.
\]

Clearing the factor \( \exp\{-\gamma_G(a)L\} - 1 \) in this equation gives

\[
1 = aF_1(\gamma_G) + a^2F_2(\gamma_G).
\]

It remains to determine \( F_1, F_2 \) explicitly and solve this implicit equation for \( \gamma_G \) in terms of \( a \). We do so in the next sections but first discuss the perturbative structure behind this solution.

### 2.2. The algebraic structure

We can identify any graph in this resummation with a word in two letters \( u, v \) say, for example:

\[
\text{graph}
\]

We have renormalized Feynman rules \( \phi_R \) such that

\[
\phi_R(u)(L) = -L \lim_{\rho \to 0} \rho F_1(\rho),
\]

and

\[
\phi_R(v)(L) = -L \lim_{\rho \to 0} \rho F_2(\rho).
\]

The Green function \( G_R(a, L) \) is obtained as the evaluation by \( \phi_R \) of the fixpoint of the combinatorial Dyson Schwinger equation

\[
X(a) = 1 + aB_+^u(X(a)) + a^2B_+^v(X(a)).
\]

We have

\[
X(a) = 1 + au + a^2(2u + v) + \ldots = \exp\gamma [au + a^2v],
\]

where \( \gamma \) is the shuffle

\[
H_{\text{lin}} \times H_{\text{lin}} \to H_{\text{lin}},
\]

\[
B_+^i(w_1) \gamma B_+^j(w_2) = B_+^i(w_1 \gamma B_+^j(w_2)) + B_+^j(w_2 \gamma B_+^i(w_1)),
\]

\( \forall i, j \in \{u, v\} \). Note that for example \( u \gamma u = 2uu \).
The two maps $B^i_\pm$ are Hochschild one-cocycles, and $X(a)$ is group-like:

$$\Delta X(a) = X(a) \otimes X(a).$$

Correspondingly, decomposing $X(a) = 1 + \sum_{k \geq 1} a^k c_k$, we have

$$\Delta c_k = \sum_{j=0}^k c_j \otimes c_{k-j},$$

with $c_0 = 1$. This is a decorated version of the Hopf algebra of undecorated ladder trees $t_n$ with coproduct $\Delta t_n = \sum_{j=0}^n t_j \otimes t_{n-j}$. Feynman rules become iterated integrals as

$$\phi_R(B^i_\pm(w))(L) = \int \{\phi(w) (\ln k^2/\mu^2) d\mu_i(k)\}_-, $$

where $d\mu_i$ is the obvious integral kernel for $i \in u,v$, cf. \[1\]. Apart from the shuffle product, we have disjoint union as a product which makes the Feynman rules into a character

$$\phi(w_1 \cdot w_2) = \phi(w_1)\phi(w_2).$$

These two commutative products $\gamma, \cdot$ allow to express the primitive elements associated with shuffles of letters $u,v$, see for example \[7\]:

**Theorem 1.** The primitive elements are given by polarization of the primitive elements $p_n$ of the undecorated ladder trees $t_n$. These are given by $p_n = \frac{1}{|S*Y|}(t_n)$.

Here, $Y$ denotes the grading operator, defined by $Y(t_k) = k t_k$ and the star product is defined as usual by $O_1 * O_2 = \cdot \circ (O_1 \otimes O_2) \circ \Delta$. Polarization of the undecorated primitive elements $p_n$ means that we decorate each vertex of $p_n$ with $u+v$.

The set $\mathcal{P}(u,v)$ of primitive elements is hence spanned by elements $p_{i_u+i_v}$, where the integers $i_u, i_v$ count the number of letters $u$ and $v$ in the polarization of $t_{i_u+i_v}$. For example the primitive element corresponding to the undecorated ladder tree $t_2$ is $p_2 = t_2 - \frac{1}{2} t_1 t_1$. Polarization yields

$$p_{2,0} = \frac{1}{2} (u \gamma u - u \cdot u) = uu - \frac{1}{2} u \cdot u, \quad p_{0,2} = \frac{1}{2} (v \gamma v - v \cdot v) = vv - \frac{1}{2} v \cdot v,$$

$$p_{1,1} = u \gamma v - u \cdot v = uv + vu - u \cdot v.$$

3. The Mellin transforms

The general structure of the Mellin transform can be obtained from quite general considerations. The crucial input comes from powercounting and conformal symmetry.

**Theorem 2.** The Mellin transforms above are invariant under the transformation $\rho \to 1 - \rho$.

Proof: Explicit computation. We give it here for $F_1$. We assume $\Re \rho > 0$ so that $F_i$ is well defined as a function. Then, the conformal inversion $k_\mu \to k'_{\mu} = k_\mu/k^2$ gives explicitly

$$-\int \frac{d^4k'}{i\pi^2} \frac{1}{k'^2(k' - q)^2} \left(\frac{k^2}{q^2}\right)^{-1+\rho}$$

for $F_1$. $F_2$ can be treated similarly by conformal inversion in both Minkowski spaces. \[\square\]

3.1. The Mellin transform of the one-loop kernel. This Mellin transform is readily integrated to deliver

$$F_1(\rho) = -\frac{1}{\rho(1-\rho)},$$

exhibiting the expected conformal symmetry.
3.2. The Mellin transform of the two-loop kernel. Determining this Mellin transform is the core part of this paper. We proceed by making use of the advantage that we remain in four dimensions, and use results of \cite{6}. We are interested in the integral
\begin{equation}
F_2(\rho) = (-q^2)^\rho \int_M \frac{d^4 k}{i \pi^2} \int_M \frac{d^4 l}{i \pi^2} \frac{\Gamma(\nu + q)[-\nu - (k - l)]^\rho}{(k + l)^2 \Gamma(k + q)^2 (l - q)^2}.
\end{equation}
Integration is over the eight dimensional cartesian product of two Minkowski spaces furnished with a quadratic form
\begin{equation}
a^2 = a_0^2 - a_1^2 - a_2^2 - a_3^2.
\end{equation}
A simple tensor calculus delivers
\begin{equation}
F_2(\rho) = \frac{1}{2} \{-2G_4(1, 1 + \rho)G_4(1, 1 + \rho) + I_6(1, 1, \rho, 1, 1, 2 - \rho) + I_6(1, 1, 1 + \rho, 1, 1, 1 - \rho)\},
\end{equation}
where
\begin{equation}
G_D(a, b) = \frac{\Gamma(a + b - D/2)\Gamma(D/2 - a)\Gamma(D/2 - b)}{\Gamma(a)\Gamma(b)\Gamma(D - a - b)},
\end{equation}
so that
\begin{equation}
G_4(1, 1 + \rho) = \frac{1}{\rho (1 - \rho)}.\end{equation}

We use the notation of \cite{6} for $I_6$. In this notation, we have $I_6 = T_6$. Setting $u \to 0$ and $v = -\rho$ or $v = 1 - \rho$ we can determine the two $I_6$ integrals as a limit $u \to 0$ in Eq.\,(19)(op.cit.) as
\begin{equation}
I_6(1, 1, 1 - v, 1, 1, 1 + v) = 8 \sum_{n=1}^{\infty} n \zeta_{2n+1}(1 - 2^{-2n})v^{2n - 2},
\end{equation}
and similarly for $v = 1 - \rho$. We hence find the above DSE in the form
\begin{equation}
1 = -a\frac{1}{\gamma_G(1 - \gamma_G)} - a^2 \left\{\frac{1}{\gamma_G^2(1 - \gamma_G)^2} - 4 \sum_{n=1}^{\infty} n \zeta_{2n+1}(1 - 2^{-2n}) \left[\gamma_G^{2n-2} + (1 - \gamma_G)^{2n-2}\right]\right\}.
\end{equation}

4. The solution

We can solve for $\gamma_G$ in the above in two different ways, expressing the solution as an infinite product or via the logarithmic derivative of the $\Gamma$ function.

4.1. Solution as an infinite product. We have:
\begin{equation}
G_R(a, L) = \exp \left\{ \sum_{p \in \mathcal{P}(u, v)} a^{[p]} \phi_R(p)(L) \right\}.
\end{equation}
Here, the sum is over all primitives $p \in \mathcal{P}(u, v)$, where $\mathcal{P}$ is the set of primitives assigned to any tree $t_n$ decorated arbitrarily by letters in the alphabet $u, v$, as described above. The proof is an elementary exercise in the Taylor expansion of the two Mellin transforms. Note that $\phi_R(p)(L)$ is linear in $L$ for primitive $p$,
\begin{equation}
\frac{\partial^2}{L^2} \phi_R(L) = 0.\end{equation}

We hence find for $\gamma_G(a)$
\begin{equation}
\gamma_G(a) = \left. \frac{\partial \ln G}{\partial L} \right|_{L=0} = -\sum_p a^{[p]} \phi_R(p)/L.
\end{equation}
Convergence of the sum is covered by the implicit function theorem, which provides for $\gamma_G$ through the two Mellin transforms in the DSE. We hence proceed to the second way of expressing the solution.
4.2. Solution via the $\psi$-function. We can express the DSE using the logarithmic derivative of the $\Gamma$ function and we obtain

\begin{equation}
1 = -a \frac{1}{\gamma G (1-\gamma G)} - a^2 \left\{ \frac{1}{\gamma G^2 (1-\gamma G)^2} + \frac{1}{\gamma G} \left[ \psi' \left( 1 + \frac{\gamma G}{2} \right) - \psi' \left( 1 - \frac{\gamma G}{2} \right) \right] - \frac{1}{2 \gamma G} \left[ \psi' \left( 2 - \frac{\gamma G}{2} \right) - \psi' \left( 2 + \frac{\gamma G}{2} \right) \right] \right\}.
\end{equation}

Here

\begin{equation}
\psi(x) = \frac{d^2}{dx^2} \ln \Gamma(x).
\end{equation}

Again, the two-loop solution shows explicitly the conformal symmetry $\gamma G \to 1 - \gamma G$. Note that the apparent second order poles at $\gamma G = 0$ and $\gamma G = 1$ on the rhs are only first order poles upon using standard properties of the logarithmic derivative of the $\Gamma$ function, as it has to be. This provides an implicit equation for $\gamma G$, which can be solved numerically.

5. Conclusions

We determined the Mellin transform of the two-loop massless vertex in Yukawa theory. We used it to resum a linear Dyson–Schwinger equation. Following [2, 3, 4], more complete applications to non-linear Dyson–Schwinger equations will be provided elsewhere. Techniques to deal with non-linearity have indeed been developed recently [2, 3, 4], and involve transcendental functions even upon resummation of terms from the first Mellin transform [2]. In the non-linear case one gets indeed results very different from scaling, as has been demonstrated early on in field theory [8]. Finally, we note that the same two-loop Mellin transform also appears in setting up the full DSE in other renormalizable theories [5].

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Isabella.Bierenbaum@desy.de, DESY, Zeuthen; kreimer@ihes.fr, IHES (http://www.ihes.fr) and Boston U. (http://math.bu.edu); stefanw@thep.physik.uni-mainz.de, ThEP, Institut für Physik, Universität Mainz