On convergence and stability of a numerical scheme of coupled nonlinear Schrödinger equations

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Abstract

We consider numerical solution of Coupled Nonlinear Schrödinger Equation. We prove stability and convergence in the $L_2$ space for an explicit scheme which estimations is used for implicit scheme and compare both method. As a test we compare numerical solutions of Manakov system with known analytical solitonic solutions and as example of general system - evolution of two impulses with different group velocity (model of pulses interaction in optic fibers). Last example, a rectangular pulse evolution, shows asymptotic behavior typical for Nonlinear Schrödinger Equation asymptotics with the same initial condition.

1 Introduction

A growing interest to encoded electromagnetic pulses propagation for long distances and processing is noted [1, 2]. A lot of realistic models are based on the Coupled Nonlinear Schrödinger Equations (CNLSE): they are developed and used in many nonlinear optics problems such as polarization modes interaction [2, 3, 4]. For such modes some numerical schemes and codes are elaborated beginning from a celebrated integrable Ablowitz-Ladik one [5, 6].

In papers [7, 8, 9, 10, 11, 12] authors investigate numerical methods for solving CNLS equation based on finite difference schemes. Such difference schemes are applied in [13] to model vector spatial soliton behavior in a nonlinear waveguide arrays.

There are numerical schemes which are unconditionally unstable, conditionally stable and unconditionally stable. Generally could be said that we have two useful schemes: conditionally stable (Euler type) schemes and unconditionally stable (Crank-Nicolson type) schemes; we focus on these types of schemes. Most important thing is to use conservation laws for CNLS equations while the scheme constructed. There are a lot of possibilities to define conservation laws [8] but we choose in this paper the standard one given by [10, 9].

Authors of [9, 11, 7] compare explicit method with implicit one underlying that both of the methods have good and bad sides and there is no an uniform point of view. Conditionally stable methods do not need matrix formulation, but they need smaller space and time steps to assure stability. Unconditionally stable method need to solve system of equations, but bigger time step could be used. We would like to compare numerical solutions for implicit and explicit method under the scope of stability proof for explicit method. In paper [9] authors prove convergence and stability for NLS equation but do not analyze CNLS equations and how stability and convergence depends on initial amplitude (energy) of impulses.

We consider a finite-difference scheme for the CNLSE which extends in a sense the results concerned linear Schrödinger equation [14], following the ideas successfully applied for the coupled Korteveg-de Vries (KdV) systems in [15, 16].
Most important aim which we would like to achieve is to estimate stability and convergence parameters as a function of initial amplitude of pulses (energy) for the CNLS equations on a base of the norm originated from $L_2$ space. Such result allows us to estimate time and space steps for the convergence regime inside which we could consider a solution as stable.

The system of the CNLSE is considered in the form:

$$i \partial_t U + i\sigma \partial_x U + k \partial_{xx} U + \left[a|U|^2 + b|V|^2\right] U = 0,$$  \hspace{1cm} (1a)

$$i \partial_t V - i\sigma \partial_x V + k \partial_{xx} V + \left[c|V|^2 + d|U|^2\right] V = 0,$$  \hspace{1cm} (1b)

where functions $U$ and $V$ are differentiable up to the second order and the following notations are used

$$U = U(x,t), \hspace{1cm} (2a)$$
$$V = V(x,t), \hspace{1cm} (2b)$$
$$\partial_t V \equiv U_t \equiv \frac{\partial U(x,t)}{\partial t}, \hspace{1cm} (2c)$$
$$\partial_{xx} V \equiv U_{xx} \equiv \frac{\partial^2 U(x,t)}{\partial x^2}, \hspace{1cm} (2d)$$

and (for the example of two modes excited in the waveguide [17, 18]) the parameter $\sigma$ is defined as

$$\sigma = \frac{1}{2} \left(k'_{01} - k'_{11}\right).$$  \hspace{1cm} (3)

This parameter describes the difference between group velocities of the modes. For simplicity of notation, but without losses of generality, we take $a, b, c, d, k, \sigma \geq 0$ (in other case we should put everywhere in estimation of norm absolute value of this coefficients which could make equations more illegible).

### 2 Conservation laws

Let us prove first that the solutions of the system (1) satisfy the following conservation laws

$$\int_{-\infty}^{\infty} |U|^2 d\tau = \text{const}, \hspace{1cm} (4a)$$
$$\int_{-\infty}^{\infty} |V|^2 d\tau = \text{const}. \hspace{1cm} (4b)$$

If one writes the conjugate to the first of equations (1) (equation for $U$ amplitude)

$$- i \partial_t \bar{U} + i\delta \partial_x \bar{U} + k \partial_{xx} \bar{U} + \left[a|U|^2 + b|V|^2\right] \bar{U} = 0,$$  \hspace{1cm} (5)

next multiply the result (5) by $-U$ and the equation (1) by $\bar{U}$, it yields

$$i \bar{U} \partial_t U - i\delta \bar{U} \partial_x U + k \bar{U} \partial_{xx} U + \left[a|U|^2 + b|V|^2\right] \bar{U} U = 0,$$  \hspace{1cm} (6a)

$$i U \partial_t \bar{U} - i\delta U \partial_x \bar{U} - k U \partial_{xx} \bar{U} - \left[a|U|^2 + b|V|^2\right] \bar{U} U = 0,$$  \hspace{1cm} (6b)
with obvious relations
\[ U U = U U = |U|^2. \] (7)

As a next step, adding the equation (6a) to (6b)
\[ i \partial_\xi U U - i \delta \partial_\tau U + k \partial_\tau [U \partial_\tau U - U \partial_\tau U] = 0, \] (8)

and integrating the result, one arrives at:
\[ i \partial_\xi \int_{-\infty}^{\infty} (U U) d\tau - i \delta \int_{-\infty}^{\infty} \partial_\tau (U U) d\tau + k \int_{-\infty}^{\infty} \partial_\tau [U \partial_\tau U - U \partial_\tau U] d\tau = 0, \] (9)
\[ i \partial_\xi \int_{-\infty}^{\infty} |U|^2 d\tau \bigg|_0^{+\infty} + k \int_{-\infty}^{\infty} \partial_\tau [U \partial_\tau U - U \partial_\tau U] \bigg|_0^{+\infty} = 0, \] (10)

Suppose the boundary conditions at both infinities
\[ \lim_{\tau \to \pm \infty} U = 0, \] (11a)
\[ \lim_{\tau \to \pm \infty} V = 0, \] (11b)
are imposed,
\[ i \partial_\xi \int_{-\infty}^{\infty} |U|^2 d\tau = 0, \] (12)
we obtain the first conservation law in the form
\[ \int_{-\infty}^{\infty} |U|^2 d\tau = \text{const.} \] (13)

For \( V \) we built the conservation law in the same way
\[ \int_{-\infty}^{\infty} |V|^2 d\tau = \text{const.} \] (14)

Such conservation laws give us a possibility to define the basic norm in \( L_2 \) space as
\[ ||W||^2 = \int_{-\infty}^{\infty} (|U|^2 + |V|^2) d\tau = \text{const.} \] (15)

We would denote other norms by the same notation if it will not lead to a confusion.

3 Discretization of the CNLSE system

3.1 Explicit scheme

Choose the explicit scheme in a simple form [19] with a second order discretization with respect to time and a third order one to space:
\[ \frac{i U_{i+1}^j - U_i^j}{\tau} + i \sigma \frac{U_{i+1}^j - U_{i-1}^j}{2h} + k \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2} + \left( a |U_i^j|^2 + b |V_i^j|^2 \right) U_i^j = 0. \] (16)
Let us derive conservation law for discrete CNLSE system (where $\overline{U}$ is complex conjugate of $U$). Multiplying equation (16) by $(U_{j+1}^{i+1} + U_j^i)$ and complex conjugating of this equation by $(U_{j+1}^i + U_j^{i+1})$.

If we apply zero boundary conditions, we have

$$\sum_{i=1}^{N} |U_{j+1}^{i+1}|^2 = \sum_{i=1}^{N} |U_j^i|^2,$$

(17a)

$$\sum_{i=1}^{N} |V_{j+1}^{i+1}|^2 = \sum_{i=1}^{N} |V_j^i|^2.$$

(17b)

### 3.2 Implicit scheme

This six point scheme [10]

$$\frac{U_{j+1}^{n+1} - U_j^n}{\tau} + i\sigma \frac{U_{j+1}^{n+1} + U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^n + U_{j-1}^n}{4h} + k \frac{U_{j+1}^{n+1} + U_{j-1}^{n+1} - 2U_j^{n+1} - 2U_{j-1}^n + U_{j+1}^n + U_{j-1}^n}{2h^2} + \left( a|U_j^{n+1/2}|^2 + b|V_j^{n+1/2}|^2 \right) \frac{U_j^{n+1} + U_j^n}{2} = 0,$$

(18)

is exactly Crank-Nicolson one [19]. This implicit scheme bases on the same finite difference as explicit scheme, we expect that implicit scheme converges to exactly solution quicker than explicit scheme.

Elements in a node 1/2 are calculated by iterations.

### 4 Stability

We can separate real and imaginary part of $U$ and $V$ by put

$$U = \xi + i\eta,$$

(20a)

$$V = \alpha + i\beta.$$

(20b)

Substituting this into (11) yields in four equation with real amplitudes

$$\xi_t + \sigma \xi_t + k\eta_{xx} + \left[ a(\xi^2 + \eta^2) + b(\alpha^2 + \beta^2) \right] \eta = 0,$$

(21a)

$$-\eta_t - \sigma \eta_t + k\xi_{xx} + \left[ a(\xi^2 + \eta^2) + b(\alpha^2 + \beta^2) \right] \xi = 0,$$

(21b)

$$\alpha_t - \sigma \alpha_t + k\beta_{xx} + \left[ c(\alpha^2 + \beta^2) + d(\xi^2 + \eta^2) \right] \beta = 0,$$

(21c)

$$-\beta_t + \sigma \beta_t + k\alpha_{xx} + \left[ c(\alpha^2 + \beta^2) + d(\xi^2 + \eta^2) \right] \alpha = 0.$$

(21d)

If we apply explicit scheme (16) to these equations, we can build time evolution matrix as

$$T^{j+1} = \begin{pmatrix}
T_{11}^{j+1} & T_{12}^{j+1} & 0 & 0 \\
T_{21}^{j+1} & T_{22}^{j+1} & 0 & 0 \\
0 & 0 & T_{33}^{j+1} & T_{34}^{j+1} \\
0 & 0 & T_{43}^{j+1} & T_{44}^{j+1}
\end{pmatrix}.$$

(22)
where
\[
T_{11} = T_{22} = \delta_{i,r} - \frac{\tau \sigma}{2h} (\delta_{i+1,r} - \delta_{i-1,r}),
\]
\[
T_{12} = -T_{21} = -\frac{\tau k}{h^2} (\delta_{i+1,r} - 2\delta_{i,r} - \delta_{i-1,r}) - \tau \delta_{i,r} \left\{ a[(\xi^i)^2 + (\eta^i)^2] + b[(\alpha^j)^2 + (\beta^j)^2] \right\},
\]
\[
T_{33} = T_{44} = \delta_{i,r} + \frac{\tau \sigma}{2h} (\delta_{i+1,r} - \delta_{i-1,r}),
\]
\[
T_{34} = -T_{43} = -\frac{\tau k}{h^2} (\delta_{i+1,r} - 2\delta_{i,r} - \delta_{i-1,r}) - \tau \delta_{i,r} \left\{ c[(\alpha^j)^2 + (\beta^j)^2] + d[(\xi^i)^2 + (\eta^i)^2] \right\}
\]
that acts in the vector space $R_{uw}$ of the columns:
\[
W^j = \begin{pmatrix} \xi^i \\ \eta^i \\ \alpha^j \\ \beta^j \end{pmatrix},
\]
\[
W^{j+1} = T^{j+1}W^j = T^{j+1}T^jW^{j-1} = \cdots = \prod_k T^kW^0.
\]

Now we will prove a stability with respect to small perturbations of initial conditions $dW$ [15, 16]
\[
dW^{j+1} = \prod_k T^k dW^0.
\]

For the stability conditions we require that the operator $\prod_k T^k$ must be bounded by a constant
in a sense of Frobenius matrix norm
\[
\left\| \prod_k T^k \right\| \leq C.
\]

For stability, sufficient condition could be write in form
\[
\left\| T^k \right\| < \exp(\rho \sigma),
\]
where $\rho$ is a constant, but case with $|\rho(\tau, h)| \leq \text{const} < \infty$
\[
\left\| T^k \right\| < \exp(\rho(\tau, h)),
\]
is also sufficient for stability under condition $\tau, h \to 0$ and some dependence of $\tau$ and $h$.

In matrix $T$ all elements are matrix with index of spatial grid. Now we could upper estimate $\rho$, first we upper estimate all of matrix elements using spectral norm of matrix $T$ [16]:
\[
\begin{align*}
||T_{11}|| &\leq 1 + \frac{\tau \sigma}{h}, \\
||T_{12}|| &\leq \frac{4\tau k}{h^2} + \tau a \max_i [(\xi^i)^2 + (\eta^i)^2] + \tau b \max_i [(\alpha^j)^2 + (\beta^j)^2], \\
||T_{21}|| &\leq \frac{4\tau k}{h^2} + \tau a \max_i [(\xi^i)^2 + (\eta^i)^2] + \tau b \max_i [(\alpha^j)^2 + (\beta^j)^2], \\
||T_{22}|| &\leq 1 + \frac{\tau \sigma}{h}, \\
||T_{33}|| &\leq 1 + \frac{\tau \sigma}{h}, \\
||T_{34}|| &\leq \frac{4\tau k}{h^2} + \tau c \max_i [(\alpha^j)^2 + (\beta^j)^2] + \tau d \max_i [(\xi^i)^2 + (\eta^i)^2], \\
||T_{43}|| &\leq \frac{4\tau k}{h^2} + \tau c \max_i [(\alpha^j)^2 + (\beta^j)^2] + \tau d \max_i [(\xi^i)^2 + (\eta^i)^2], \\
||T_{44}|| &\leq 1 + \frac{\tau \sigma}{h}.
\end{align*}
\]
Taking into account (27) and (30) we can write

\[ ||T^{j+1}_j|| \leq 4 + \frac{\tau \sigma}{h} + \frac{16 \tau k}{h^2} + 2 \tau \max_i (\xi^j_i)^2 + (\eta^j_i)^2 \]

\[ + 2 \tau \max_i (\alpha^j_i)^2 + (\beta^j_i)^2 \]

\[ + 2 \tau \max_i (\xi^j_i)^2 + (\eta^j_i)^2. \]

(31)

Taking into account the finite-difference conservation laws (17a) we write

\[ I_u = \sum_{i=1}^{N} |U^j_i|^2 = \sum_{i=1}^{N} |U^0_i|^2, \]

(32a)

\[ I_v = \sum_{i=1}^{N} |V^j_i|^2 = \sum_{i=1}^{N} |V^0_i|^2, \]

(32b)

now we can upper estimate norm (31) as

\[ ||T^{j+1}_j|| \leq 4 + \frac{\tau \sigma}{h} + \frac{16 \tau k}{h^2} + \tau 2(aI_u + bI_v + dI_u + cI_v)) \]

\[ \leq \exp (\rho \tau). \]

(33)

Finally we can write \( \rho \) as

\[ \rho = 4 \frac{\sigma}{h} + \frac{16k}{h^2} + 4 \max(a,d)I_u + 4 \max(b,c)I_v. \]

(34)

This means that the fraction \( \tau/h \) must tend to 0 (stability of this system of equation depends on initial energy quantity). For a better stability this condition requires that \( \tau \to 0 \) much faster than \( h \to 0 \).

5 Convergence

In this section we proof that the terms at l.h.s of equations in finite difference form (16) converge to the equation (11), which solutions are differentiable with respect to time and (twice) to \( x \). We put into equation (16) a solution in the form

\[ W^j_i = ES^j_i + V^j_i = \begin{pmatrix} e\xi^j_i + d\xi^j_i \\
\sigma \eta^j_i + d\eta^j_i \\
e\alpha^j_i + d\alpha^j_i \\
e\beta^j_i + d\beta^j_i \end{pmatrix}, \]

(35)

where \( ES \) is an exact solution and \( V \) is a difference between a numerical solution and the exact solution of CNLS equations.

Below we show only one of the matrix component

\[ \frac{d\xi^{j+1}_i - d\xi^j_i}{\tau} + \frac{d\xi^{j+1}_i - d\xi^j_i}{\sigma} + \frac{2h}{2h} \frac{dn^j_i + d\eta^j_i}{h^2} \]

\[ + \frac{e\xi^{j+1}_i - e\xi^j_i}{\tau} + \frac{e\xi^{j+1}_i - e\xi^j_i}{\sigma} + \frac{2h}{2h} \frac{en^j_i + e\eta^j_i}{h^2} \]

\[ + \left\{ a \left[ (e\xi^j_i + d\xi^j_i)^2 + (en^j_i + d\eta^j_i)^2 \right] + b \left[ (e\alpha^j_i + d\alpha^j_i)^2 + (e\beta^j_i + d\beta^j_i)^2 \right] \right\} (en^j_i + d\eta^j_i) = 0. \]

(36)
Let us use the conservation laws \(17a\) for the equation \(1\) and the finite difference equation \(16\), that allows us to write
\[
I_{ev} = \sum_{i=1}^{N} [(e_{\xi i}^2) + (e_{\eta i}^2)] = \sum_{i=1}^{N} [(e_{\xi i}^0)^2 + (e_{\eta i}^0)^2], \tag{37a}
\]
\[
I_{ev} = \sum_{i=1}^{N} [(e\alpha_i^2) + (e\beta_i^2)] = \sum_{i=1}^{N} [(e\alpha_i^0)^2 + (e\beta_i^0)^2], \tag{37b}
\]
or, in terms of the relation \(35\)
\[
d\xi_j^{i+1} = \frac{\tau (d\xi_j^i)}{2h} + \tau \left\{ a \left[ (e_{\xi j}^i)^2 + (e_{\eta j}^i)^2 \right] + b \left[ (e\alpha_j^i)^2 + (e\beta_j^i)^2 \right] \right\} d\eta_j^i
\]
\[
- \tau \left\{ a \left[ (e_{\xi j}^i + d\xi_j^i)^2 + (e_{\eta j}^i + d\eta_j^i)^2 \right] + b \left[ (e\alpha_j^i + d\alpha_j^i)^2 + (e\beta_j^i + d\beta_j^i)^2 \right] \right\} (e\eta_j^i + d\eta_j^i) =
\]
\[
- \tau \left\{ \frac{(e\xi_j^i - e\xi_j^{i+1})}{\tau} - \sigma \frac{e\xi_j^{i+1} - e\xi_j^i}{2h} - \frac{e\eta_j^i - e\eta_j^{i+1}}{2h} \right\} (e\eta_j^i + d\eta_j^i).
\] \(\tag{38}\)

The right side of the equation for the differentiable solutions is of the order \(\Theta(\tau + h + h^2)\) \([19]\), while to represent the left side we use the matrix \(T\). Hence one arrives at
\[
d\xi_j^{i+1} = T_{ij}^{i+1} V_j^i + \tau \left\{ a \left[ (e_{\xi j}^i)^2 + (e_{\eta j}^i)^2 \right] + b \left[ (e\alpha_j^i)^2 + (e\beta_j^i)^2 \right] \right\} (e\eta_j^i + d\eta_j^i)
\]
\[
- \tau \left\{ a \left[ (e_{\xi j}^i + d\xi_j^i)^2 + (e_{\eta j}^i + d\eta_j^i)^2 \right] + b \left[ (e\alpha_j^i + d\alpha_j^i)^2 + (e\beta_j^i + d\beta_j^i)^2 \right] \right\} (e\eta_j^i + d\eta_j^i)
\]
\[
+ \Theta(\tau + h + h^2). \tag{39}\)

Let us define the norm of the numerical solution as
\[
\|U^j\| = h \left( \sum_i |U_j^i|^2 \right)^{\frac{1}{2}}. \tag{40}\)

Now we upper estimate one element of the vector \(V^j\)
\[
\|d\xi^{j+1}\| \leq \|T_{ij}^{j+1}\| \|V^j\| + \tau (aI_{ue} + bI_{ve}) \|\eta^j\| - \tau (aI_{ue} + bI_{ve}) \|\eta^j\| + \Theta(\tau + h + h^2)
\]
\[
\leq \exp (\tau \rho) \|V^0\| + \tau (aI_{ue} + bI_{ve}) \|U^j\| - \tau (aI_{ue} + bI_{ve}) \|U^j\| + \Theta(\tau + h + h^2), \tag{41}\)

Next we write the converge condition for all \(V^{j+1}\) matrix components using the Frobenius norm up to the choice of the initial error \(\|V^0\| = 0\).
\[
\|V^{j+1}\| \leq Q + \Theta(\tau + h + h^2), \tag{43}\)

where \(i = 1, 2, 3, 4\) and we define
\[
Q = 4 |\max(a, b, c, d)(I_{ue} + I_{ve})^{3/2} - \max(a, b, c, d)(I_{ue} + I_{ve})^{3/2}|. \tag{44}\)
6 Numerical results

6.1 Nonlinear Schrödinger equation

For simple test we put \( c = d = 0 \) and \( k = 0.5 \) in this case we have simple nonlinear Schrödinger equation. First we test NLS equation with initial condition \( U(0, x) = \text{sech}(x) \) which pulse should not change shape during propagation (picture 1b).

As a full initial condition we set:

\[
\begin{align*}
U(0, x) &= A_1 \text{sech}(x), \\
V(0, x) &= 0,
\end{align*}
\]

this lead to solve NLSE problem in form

\[
iU_t + \frac{1}{2} U_{xx} + |U|^2 U = 0.
\]

When amplitude \( A_1 = 1 \) the exact solution of equation (46) is

\[
U(t, x) = A_1 \text{sech}(x) \exp \left( \frac{it}{2} \right)
\]

(a)

Figure 1: NLS equation with initial amplitude \( A_1 \) a) 1 and b) 2

Next we test NLS equation with amplitude \( A_1 = 2 \); the results are the same as in [2] (picture 1b). For this case the exact solution of the equation (46) takes more complicated form [2]

\[
U(t, x) = \frac{[\cosh(3x) + \exp(4it) \cosh(x)] \exp(it/2)}{\cosh(4x) + 4 \cosh(2x) + 3 \cos(4t)}
\]

(b)

The numerical solution for the second case deviates more than the first one because it has bigger energy per pulse and we use for both cases the same time and space steps (see equation 34).

6.2 Manakov solitons

Let us consider the soliton solutions of the Manakov system as examples to test the stability and the convergence of the explicit scheme. We start with the stability of the explicit scheme. At the picture
six cases of energy conservation behaviour are shown for the values of $\rho \tau$ which are bigger than 0.1: we observe very unstable results. When we decrease the time step, our solution is stabilized. It is very important to remark that $\rho$ depends on initial amplitude of pulses. We choose the initial conditions as

$$U(0, x) = A_1 \text{sech}(x),$$ (49a)
$$V(0, x) = A_2 \text{sech}(x).$$ (49b)

Table 1: Parameters for the numerical experiment "Manakov solitons"

| Parameter | Value |
|-----------|-------|
| $x$       | -50 ... 50 |
| time     | 15 |
| sigma    | 0 |
| $a$      | 1 |
| $b$      | 1 |
| $c$      | 1 |
| $d$      | 1 |
| $A_1$    | 1 |
| $A_2$    | 1 |
| space steps | 1000 |
| time steps  | 10000 - 1000000 |

6.3 Collision of two solitons

We make this experiment to compare the results with the "experiment 1" of the paper [10]. General properties of such collisions of two solitons are well known [20], hence we do not focus on details of this type of soliton interaction.

6.4 Different group velocities

The examples to be studied in this section are most important for us, because we interest in describing modes interaction in a nonlinear waveguide of different modes excited (different modes have different
Figure 3: Stability of explicit method for Manakov examples

Figure 4: Manakov case: $A_1 = A_2 = 1$, nonlinear coefficients $(a,b,c,d) = 1$ and $\sigma = 1$.

Figure 5: Inelastic collision of two solitons
group velocities) \cite{17,18}. This investigation could be useful not only for the situation described below, but for soliton trains interactions as well \cite{21}.

As initial conditions we took two sech-impulses and each equation have different nonlinear coefficients. This could appear in a waveguide when two modes are excited with different group velocity. We set initial conditions for this case as:

\begin{align}
U(0, x) &= A_1 \text{sech}(x + D_1) \exp(i v_1 x), \\
V(0, x) &= A_2 \text{sech}(x + D_2) \exp(i v_2 x).
\end{align} \tag{50a, 50b}

Consider two solitons one with zero velocity and second with velocity greater than zero, we show this situations on picture 6. This two cases of pulses interaction differ only by group velocity (parameters for this pulses are in the table 6.4).

This two impulses start from the same position (e.g. two modes excited in the waveguide). When velocity of second pulse is 0.7 (for given parameters) first impulses intercept part of energy thought nonlinear interactions and move with second pulse with average velocity for both pulses (picture 6a). When the velocity of second impulses is high enough we have two pulses which moves with different group velocity see picture 6b (nonlinear interaction between pulses happen only at the start of propagation).

\begin{table}[h]
\centering
\caption{Parameters for numerical experiment "Different group velocity"}
\begin{tabular}{|c|c|c|}
\hline
 & (a) & (b) \\
\hline
x & -30 ... 30 & 40 \\
\hline
time & 40 & 40 \\
\hline
\sigma & 0 & 0 \\
\hline
a & 1 & 1 \\
\hline
b & 1/3 & 1/3 \\
\hline
c & 1 & 1 \\
\hline
d & 1/3 & 1/3 \\
\hline
A_1 & 1.2 & 1.2 \\
\hline
A_2 & 1.4 & 1.4 \\
\hline
v_1 & 0.7 & 0.95 \\
\hline
v_2 & 0 & 0 \\
\hline
h & 0.2 & 0.2 \\
\hline
\tau & 0.02 & 0.02 \\
\hline
\end{tabular}
\end{table}

6.5 Difference between explicit and implicit scheme

If we would like to have more space details (smaller $h$), in explicit scheme we must take adequate smaller time step to assure stability of numerical scheme. This have very big influence on calculation time. In an implicit scheme we could use an iteration method \cite{19,7} and shorter time of calculation is achieved. On the picture 7 we show numerical results for both methods. Note, the time step for the explicit scheme is much larger than for the implicit scheme but each step in the implicit scheme needs 2-4 iterations.

6.6 Rectangular pulse decay

In this subsection we engaged in asymptotic solution of Nonlinear Schrödinger Equation. As initial condition we choose rectangular pulse as in \cite{22} (see picture 8b).
Figure 6: Two cases of impulses with different group velocity (for parameters see table 6.4)

Figure 7: Compare between explicit and implicit method.
Figure 8: Rectangular pulse evolution. On picture a) there is a 3D plot on picture b),c) and d) there are intersection of plot a) in different time.
Table 3: Parameters for numerical experiment for implicit and explicit method

| Parameter | explicit | implicit |
|-----------|----------|----------|
| space step| 300      | 300      |
| time step | 1000000  | 2000     |
| time      | 40       | 40       |
| x         | -30...30 | -30...30 |
| A_1       | 1.5      | 1.5      |
| A_2       | 1.5      | 1.5      |
| a         | 1        | 1        |
| b         | 0.2      | 0.2      |
| c         | 1        | 1        |
| d         | 1.6      | 1.6      |
| σ         | 0.3      | 0.3      |

We test our method for this kind of initial pulse. First we made two calculations with different time steps (for the first $\tau = 2e^{-4}$ and $\tau = 2e^{-5}$ for the second one) and compare errors between (picture 9b). The difference between calculations is of the order determined by $\tau \rho$. Next we check the energy conservation (see picture 9b).

![Figure 9: (a) Difference between two numerical solutions with different time steps ($\tau = 2e^{-4}$ and $\tau = 2e^{-5}$). (b) The energy deviation (to the initial state) for the time step $\tau = 2e^{-5}$.

For the presented time ($t = 4$) we expect that the transient stage from rectangular shape of the pulse to the asymptotic behaviour of solution is realized. On the picture 10 the evolution (maximum amplitude) of rectangular pulses with different width is presented. If the pulse have smaller width then the decay is faster.

7 Conclusion

We study a convergence and stability conditions which would be useful to estimate other numerical schemes for NLS as well. Additionally these conditions could be compared with ones for numerical methods for other nonlinear equations [16] and show a character of CNLS equations (from numerical side). Most important results which we show in this paper is how to estimate a time step for the
explicit method due to initial energy of pulses. The results could be adapted to other methods based on this basic scheme which we use [8]. There is a possibility to enhance this method to third or high order in time and this method could be used as a starting point (to calculate lacked points of grid).

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