Two-loop gluon diagrams from string theory

Lorenzo Magnea* and Rodolfo Russo†

NORDITA*
Blegdamsvej 17, DK–2100 Copenhagen Ø, Denmark
Dipartimento di Fisica, Politecnico di Torino †
Corso Duca degli Abruzzi 24, I–10129 Torino, Italy

Abstract. We briefly review the string technology needed to calculate Yang-Mills amplitudes at two loops, and we apply it to the evaluation of two-loop vacuum diagrams.

INTRODUCTION

It is well known [1] that string theory is a powerful calculational tool for the evaluation of tree and one-loop amplitudes in Yang-Mills theory and gravity. During the past two years considerable progress has been made towards the extension of string-inspired techniques to more than one loop [2]. The calculation of the two-loop Yang-Mills vacuum diagrams, presented here, is the first simple application of this formalism to gauge theories beyond one loop.

The general features of string-inspired techniques remain unchanged to all orders in perturbation theory: the field theory limit is obtained by taking the string tension $T = 1/(2\pi\alpha')$ to infinity, decoupling all massive string modes; the corners of string moduli space contributing to the field theory result are those where the integrand of the string amplitude exhibits a singular behavior, and in these regions string moduli are naturally related to Schwinger proper times in field theory; finally, string-derived amplitudes are written in a form which is strongly reminiscent of the world-line formalism in field theory [3].

A special feature of calculations done using the bosonic string is the existence of contact terms, corresponding to four-point vertices in field theory, that arise as finite remainders of tachyon exchange. These contributions are present to all orders, however at one loop they can be eliminated by performing a partial integration at the string level. This simplifies calculations, but it obscures the connection

*) On leave from Università di Torino, Italy.
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between string-derived rules and field theory. At two loops, we find that the same prescription that was used at one loop to handle tachyon exchange (a ζ-function regularization) is sufficient to obtain the correct result.

It should be noted that the string derivation is not tied to any special choice of regularization scheme for ultraviolet and infrared divergences. While dimensional regularization is naturally implemented and useful for practical calculations, any regularization scheme that can be applied at the level of Schwinger parameter integrals is just as natural. Thus it is meaningful to calculate vacuum diagrams, although they vanish in dimensional regularization.

MULTILOOP GLUON AMPLITUDES

Let us begin by recalling the general expression for the color-ordered $h$-loop $M$-gluon planar amplitude in the open bosonic string [2],

\[
A_P^{(h)} = C_h N_0^M \int [dm]_h \prod_{i=1}^{M} \frac{dz_i}{dV_{abc}} \prod_{i<j} \left[ \exp \left( \frac{G^{(h)}(z_i, z_j)}{\sqrt{V_i'(0) V_j'(0)}} \right) \right]^{2\alpha' p_i \cdot p_j} \times \exp \left[ \sum_{i \neq j} \left( \sqrt{2\alpha' p_j \cdot \varepsilon_i} \partial_{z_i} G^{(h)}(z_i, z_j) + \frac{1}{2} \varepsilon_i \cdot \varepsilon_j \partial_{z_i} \partial_{z_j} G^{(h)}(z_i, z_j) \right) \right],
\]

where only terms linear in each polarization should be kept, and we omitted the color factor $N^h \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_M})$. The dimensionless string coupling constant $g_s$ is related to the $d$-dimensional gauge coupling $g_d = g g_{\mu}(4 - d)/2$ by

\[
g_s = \frac{g_d}{2} (2\alpha')^{1-d/4}.
\]

The fundamental ingredients of Eq. (1) are the bosonic Green function $G^{(h)}(z_i, z_j)$ (the correlator of two scalar fields on the $h$-loop string world sheet), and the measure of integration on moduli space $[dm]_h$. Both these quantities depend only on the genus $h$ of the surface and thus represent the building blocks for the calculation of all diagrams at $h$ loops with an arbitrary number of external states. In particular, the string Green function acts as a generator of the specific world-line Green functions found for particle diagrams of different topology, to which it reduces in the appropriate corners of moduli space. Explicit expressions for the Green function, for the projective transformations $V_i(z)$, which define local coordinate systems around the punctures, and for the normalization constants $C_h$ and $N_0$ can be found in Ref. [2]; here we want to focus on the measure of integration, since it encodes all the information needed for the derivation of the vacuum diagrams. It is given by

\[
[dm]_h = \prod_{\mu=1}^{h} \left[ \frac{dk_\mu d\xi_\mu d\eta_\mu}{k_\mu^2 (\xi_\mu - \eta_\mu)^2 (1 - k_\mu)^2} \left[ \det (-i\tau_{\mu\nu}) \right]^{-d/2} \right] \prod_{n=1}^{\infty} (1 - k_n^a)^{-d} \prod_{n=2}^{\infty} (1 - k_n^a)^2.
\]
The various ingredients of this formula have a geometric interpretation on a Riemann surface of genus \( h \). In particular, \( \tau_{\mu\nu} \) is the period matrix, while \( k_\mu, \xi_\mu \) and \( \eta_\mu \) are the moduli of the surface in the Schottky parametrization \[4\]; the primed product over \( \alpha \) denotes a product over conjugacy classes of elements of the Schottky group, where only elements that cannot be written as powers of other elements must be included. Three of \( 2h + M \) parameters \( \xi_\mu, \eta_\mu \) and \( z_i \) can be fixed using an overall projective invariance of the amplitude. The fixing of this invariance introduces the projective invariant volume element \( dV_{abc} \) in Eq. (1). Including the “multipliers” \( k_\mu \), one is left with \( 3h - 3 + M \) variables, the correct number of independent moduli for a Riemann surface of genus \( h \) with \( M \) punctures. In the field theory limit, only the region in moduli space in which the multipliers \( k_\mu \) are small gives finite contributions.

**YANG-MILLS VACUUM DIAGRAMS**

Now we turn to the study of vacuum diagrams at two loops, specializing the formulas of the previous section to the case \( M = 0, h = 2 \). First, we use projective invariance to fix \( \xi_1, \xi_2 \) and \( \eta_1 \) to \( \infty, 1 \) and 0 respectively. Then we evaluate Eq. (3) in the field theory limit. To this end, we expand it in power of \( k_\mu \), and we ignore all terms that are quadratic in one multiplier, since they correspond to the undesired exchange of a massive spin 2 state. In this approximation, Eq. (1) becomes

\[
A_2^0 = N^3 C_2 (2\pi)^d \int \frac{dk_1 dk_2 d\eta_2}{k_1^2 k_2^2 (1 - \eta_2)^2} \times \left[ 1 + (d - 2)(k_1 + k_2) + \left( (d - 2)^2 + d (1 - \eta_2)^2 \frac{1 + \eta_2^2}{\eta_2^2} \right) k_1 k_2 \right] \\
\times \left[ \ln k_1 \ln k_2 - \ln^2 \eta_2 + \frac{2(1 - \eta_2)^2}{\eta_2} (k_1 \ln k_1 + k_2 \ln k_2) + \frac{4 (1 - \eta_2)^4}{\eta_2^2} \left( 1 + \frac{1 + \eta_2}{1 - \eta_2} \ln \eta_2 \right) k_1 k_2 \right]^{-d/2}.
\]

The region of integration can be deduced by studying the Schottky representation of the two–annulus \[5\]. In the small \( k \) limit, it is given by \( 0 \leq \sqrt{k_2} \leq \sqrt{k_1} \leq \eta_2 \leq 1 \). The fixed point \( \eta_2 \) can be interpreted as the distance between the two loops, so that we can tentatively identify the region \( \eta_2 \to 1 \) as related to reducible diagrams, and the region \( \eta_2 \to 0 \) as related to irreducible ones.

In the region \( \eta_2 \to 1 \), in order to isolate the contribution of massless states, we must extract from the integrand of Eq. (4) the term proportional to \( k_1^{-1}k_2^{-1}(1 - \eta_2)^{-1} \). Then the field theory results are recovered if one introduces the Schwinger proper times \( t_i = \alpha' \ln k_i \), \( t_3 = \alpha' \ln(1 - \eta_2) \), which must remain finite as \( \alpha' \to 0 \). It can be checked that no terms survive in the limit \( \alpha' \to 0 \), which is the stringy way to say that the reducible diagram we are considering is zero. There is however
a contact interaction leftover from tachyon exchange in the limit \( \eta_2 \to 1 \). This is obtained by isolating the term in Eq. (4) that is proportional to \( k_1^{-1}k_2^{-1}(1 - \eta_2)^{-2} \), and requiring that the integrand be independent of \( \eta_2 \) except for the tachyon double pole. The double pole is then regularized using a \( \zeta \)-function regularization, as
\[
\int_0^\infty \frac{dx}{x^2} \sim \int_0^\infty dx \sum_{n=1}^\infty n e^{-nx} \sim \sum_{n=1}^\infty 1 \sim \zeta(0) = -\frac{1}{2} . \tag{5}
\]

The field theory contribution to the remaining integral over the proper times \( t_1 \) and \( t_2 \) is given by the term that does not depend on \( \alpha' \),
\[
A_{t_2 \to 1}^0 = -\frac{g_4^2}{(4\pi)^d} N^3 (d - 2)^2 \int_0^\infty dt_1 dt_2 (t_1 t_2)^{-d/2} . \tag{6}
\]

Let us now turn to the region \( \eta_2 \to 0 \). Here it is convenient to introduce the variables \( q_1 = k_2/\eta_2 \), \( q_2 = k_1/\eta_2 \), \( q_3 = \eta_2 \), which are directly related to the field theory proper times by \( t_i = \alpha' \ln q_i \). The region of integration now takes the form \( 0 \leq q_1 \leq q_2 \leq q_3 \leq 1 \), and the term that survives in the limit \( \alpha' \to 0 \) is given by
\[
A_{t_3 \to 0}^0 = \frac{g_4^2}{(4\pi)^d} N^3 (d - 2) \int_0^\infty dt_2 \int_0^{t_2} t_1 dt_1 \int_0^{t_1} \frac{t_1 + t_2 + 2t_3}{(t_1 t_2 + t_1 t_3 + t_2 t_3)^{1+d/2}} . \tag{7}
\]

The integrand of the last equation is not symmetric in the proper times \( t_i \). This prevents us from rewriting the integrations of Eq. (7) as independent, thus identifying this contribution with that of the irreducible field theory diagram with three propagators. However we expect that also as \( \eta_2 \to 0 \) there may be contributions with only two propagators, which might, and do, symmetrize Eq. (7). In this case the relevant contribution does not involve the regularization of a tachyon pole, but it simply comes from an integration region in which two moduli are kept very close to each other. In our case the only such region can be parametrized as \( q_3 = y q_2 \), where we do not associate any proper time to the variable \( y \), which is kept finite. This region contributes
\[
A_{t_3 \to q_2}^0 = -\frac{g_4^2}{(4\pi)^d} N^3 2(d - 2) \int_0^\infty dt_2 \int_0^{t_2} dt_1 (2t_1 t_2 + t_2^2)^{-d/2} , \tag{8}
\]
where the integrand can be rewritten as
\[
(2t_1 t_2 + t_2^2)^{-d/2} = (t_1 t_2)^{-d/2} + \int_0^{t_1} dt_3 \frac{t_1 + t_2}{(t_1 t_2 + t_1 t_3 + t_2 t_3)^{1+d/2}} . \tag{9}
\]

The \((-1)\) factor in Eq. (8) comes from the integration over \( y \), where only the contribution from the lower limit of integration is relevant in the field theory limit. Eq. (8) thus symmetrizes Eq. (7), as well as contributing to the contact term. It is easy to check that the sum of Eqs. (6), (7) and (8) correctly reproduces the sum of the corresponding Feynman diagrams in Yang-Mills theory. In particular, it is amusing to notice that, after doing the \( t_3 \) integral, the sum of Eq. (7) and Eq. (8) vanishes, so that the entire result for the two-loops vacuum diagrams actually comes from the contact term arising from tachyon exchange, given by Eq. (6).
CONCLUSIONS

We have reviewed some of the results of string perturbation theory that lead to an efficient organization of multiloop Yang-Mills amplitudes, and we have described the simplest application of the method at two loops. The next natural step is the evaluation of the two-loop two-point function, which is expected to contain the two-loop Yang-Mills $\beta$ function, since string amplitudes lead to background field gauges. Work in this direction is in progress.

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