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NONSINGULAR POISSON SUSPENSIONS

ALEXANDRE I. DANILENKO, ZEMER KOSLOFF, AND EMMANUEL ROY

Abstract. The classical Poisson functor associates to every infinite measure preserving dynamical system \((X, \mu, T)\) a probability preserving dynamical system \((X^*, \mu^*, T_*)\) called the Poisson suspension of \(T\). In this paper we generalize this construction: a subgroup \(\text{Aut}_2(X, \mu)\) of \(\mu\)-nonsingular transformations \(T\) of \(X\) is specified as the largest subgroup for which \(T_*\) is \(\mu^*\)-nonsingular. Topological structure of this subgroup is studied. We show that a generic element in \(\text{Aut}_2(X, \mu)\) is ergodic and of Krieger type III\(_1\). Let \(G\) be a locally compact Polish group and let \(A: G \to \text{Aut}_2(X, \mu)\) be a \(G\)-action. We investigate dynamical properties of the Poisson suspension \(A_*\) of \(A\) in terms of an affine representation of \(G\) associated naturally with \(A\). It is shown that \(G\) has property (T) if and only if each nonsingular Poisson \(G\)-action admits an absolutely continuous invariant probability. If \(G\) does not have property (T) then for each generating probability \(\kappa\) on \(G\) and \(t > 0\), a nonsingular Poisson \(G\)-action is constructed whose Furstenberg \(\kappa\)-entropy is \(t\).

1. Introduction

1.1. Poisson suspensions: measure preserving and nonsingular. In this paper we initiate a systematic study of nonsingular Poisson suspensions in the framework of ergodic theory. Poisson point processes have convenient mathematical properties and are often used as mathematical models for seemingly random phenomena. Say, in statistical physics they provide a model for ideal gas consisting of countably many randomly moving noninteracting points (particles) of a standard \(\sigma\)-finite infinite nonatomic measure space \((X, \mathcal{A}, \mu)\). The space of states \(X^*\) of the gas consists of countable subsets (configurations) of \(X\). It is endowed with the natural Borel \(\sigma\)-algebra \(\mathcal{A}^*\) and a probability measure \(\mu^*\) such that for each subset \(A \in \mathcal{A}\), the \(\mu^*\)-expected value of the number of particles in \(A\) has the Poisson distribution with parameter \(\mu(A)\). Given a transformation \(T: X \to X\), the Poisson suspension \(T_*: X^* \to X^*\) of \(T\) models the motion of the particles by \(T\). In other words, \(T_*\omega = \{Tx : x \in \omega\}\) for all \(\omega \in X^*\). If \(\mu^* \circ T_* \sim \mu^*\) then we call the system \((X^*, \mathcal{A}^*, \mu^*, T_*)\) the nonsingular Poisson suspension of \((X, \mathcal{A}, \mu, T)\).

In ergodic theory, only the measure preserving case \(\mu \circ T = \mu\) (and hence \(\mu^* \circ T_* = \mu^*\)) of Poisson suspensions has been considered so far. We refer the reader to the classical sources [11] studying the dynamical properties.

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of the ideal gas and to [28] providing a Poisson model for Bernoulli actions of locally compact groups\(^1\). Over the last 15 years we observe a boost of interest to Poisson suspensions in ergodic theory. The work [29] describes dynamical properties of \( T \) such as ergodicity, weak mixing, mixing, rigidity, \( K \)-property, etc. in terms of \( T \) and \( \mu \). Spectral properties, entropic properties, similarity and asymmetry, properties of joinings of the measure preserving Poisson suspensions are studied extensively in [29], [12, §8], [25], [15], [19], [24], etc. Summarizing this progress we can say that the Poisson functor \( T \to T^* \) from the category of infinite measure preserving actions to the probability preserving actions is similar (and of similar importance in ergodic theory) to the Gaussian functor from the category of orthogonal representations to the probability preserving actions.

Our global task is to find some extensions of the aforementioned results to the nonsingular Poisson suspensions as well as to investigate purely nonsingular properties of them such as dissipativeness, Krieger’s type, associated flow, Furstenberg entropy. We note that nonsingular Poisson suspensions are widely used in the representation theory to construct unitary representations of large groups such as diffeomorphism groups of non-compact manifolds and current groups (see the surveys [32], [33] and references therein and Chapter 10 of the book [27]). The fundamental (in fact, the only dynamical) property of the nonsingular Poisson suspensions of these group actions exploited there is the ergodicity, because it implies irreducibility of the associated unitary representation. In contrast to that, in this work we investigate dynamical properties of nonsingular Poisson suspensions for individual transformations or, more generally, locally compact group actions. For them, the aforementioned implication does not hold.

1.2. Main results. We now list the main results of this paper. The following fact can be deduced easily from Takahashi’s version of Kakutani dichotomy for Poisson point processes:

The set of \( \mu \)-nonsingular transformations \( T \) of \((X, A, \mu)\) such that \( T^* \) is \( \mu^* \)-nonsingular is exactly the group

\[
\text{Aut}_2(X, A, \mu) := \left\{ T : \mu \circ T^{-1} \sim \mu, \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1 \in L^2(\mu) \right\}.
\]

We note that \( \text{Aut}_2(X, A, \mu) \) contains (properly) a subgroup

\[
\text{Aut}_1(X, A, \mu) := \left\{ T : \mu \circ T^{-1} \sim \mu, \frac{d\mu \circ T^{-1}}{d\mu} - 1 \in L^1(\mu) \right\},
\]

\(^1\)The seemingly simpler Bernoulli model as the shiftwise \( G \)-action on the product space \( A^G \), for a probability space \( A \), drops out from the category of standard measure spaces if the group \( G \) is not countable.
which is the largest subgroup of nonsingular transformations for which the Poisson suspensions were defined in the literature ([32], [33], [27]) so far. Let \( \mathcal{U}_R(L^2(\mu)) \) denote the group of unitary operators in \( L^2(X, \mu) \) preserving the real functions and let \( \text{Aff}_R(L^2(\mu)) := L^2(\mathbb{R}, X, \mu) \rtimes \mathcal{U}_R(L^2(\mu)) \) denote the group of affine operators in \( L^2(X, \mu) \) preserving the real functions. We consider two natural representations of \( \text{Aut}_2(X, \mathcal{A}, \mu) \) in \( L^2(X, \mu) \): the unitary (well known) Koopman representation \( U : \text{Aut}_2(X, \mathcal{A}, \mu) \ni T \mapsto U_T \in \mathcal{U}_R(L^2(\mu)) \) and the affine representation \( A^{(2)} : \text{Aut}_2(X, \mathcal{A}, \mu) \ni T \mapsto A_T^{(2)} \in \text{Aff}_R(L^2(\mu)) \), given by the formulas

\[
(1.1) \quad U_T f := f \circ T^{-1} \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} \quad \text{and} \quad A_T^{(2)} f := U_T f + \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1. 
\]

It appears, surprisingly for us, that the nonsingular Poisson suspensions are related closely to geometrical (affine) properties of the underlying Hilbert space.

**Theorem A.** Let \( C_{-1} := \{ f \in L^2(\mu) : f \geq -1 \} \). Then \( \{ A \in \text{Aff}_R(L^2(\mu)) : AC_{-1} = C_{-1} \} = \left\{ \left( \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1, U_T \right) : T \in \text{Aut}_2(X, \mathcal{A}, \mu) \right\} . \)

A similar results holds also for \( \text{Aut}_1(X, \mathcal{A}, \mu) \) and the corresponding natural isometric and affine representations of this group in \( L^1(\mu) \).

Let \( W : \text{Aff}_R(L^2(\mu)) \ni A \mapsto W_A \in \mathcal{U}(F(L^2(\mu))) \) stand for the well known unitary representation of \( \text{Aff}_R(L^2(\mu)) \) by the Weyl operators \( W_A \) in the Fock space \( F(L^2(\mu)) \) constructed over \( L^2(\mu) \) [22]. The operators \( W_A \) are of fundamental importance in representation theory and quantum probability. They also appear naturally in description of the unitary Koopman operators associated with nonsingular Poisson suspensions.

**Theorem B.** Under the natural identification of \( L^2(\mu^*) \) with \( F(L^2(\mu)) \), for each transformation \( T \in \text{Aut}_2(X, \mathcal{A}, \mu) \), the Koopman operator \( U_T \), generated by \( T \), equals \( W_{A_T^{(2)}} \).

Theorem A enables us to define a Polish topology, denoted by \( d_2 \), on \( \text{Aut}_2(X, \mathcal{A}, \mu) \). In a similar way, utilizing the aforementioned analogue of Theorem A for \( \text{Aut}_1(X, \mathcal{A}, \mu) \) we also introduce a Polish topology \( d_1 \) on \( \text{Aut}_1(X, \mathcal{A}, \mu) \). We show that the homomorphism \( \chi : \text{Aut}_1(X, \mathcal{A}, \mu) \to \mathbb{R} \), defined in [27] by the formula \( \chi(T) := \int_X \left( \frac{d\mu \circ T^{-1}}{d\mu} - 1 \right) d\mu \), is \( d_1 \)-continuous. We then prove the following results.

**Theorem C.**

- The weak topology is strictly weaker than \( d_2 \) and \( d_2 \) is strictly weaker than \( d_1 \).
- \( \text{Aut}_p(X, \mu) \) endowed with \( d_p \) is a Polish group for \( p = 1, 2 \).
- The set \( \{ T_s : T \in \text{Aut}_2(X, \mathcal{A}, \mu) \} \) of nonsingular Poisson suspensions is a weakly closed subgroup of nonsingular transformations of \( (X^*, \mathcal{A}^*, \mu^*) \).

\(^{2}\)Neretin’s notation for \( \text{Aut}_1(X, \mathcal{A}, \mu) \) is \( \text{Gms}_{\text{sc}} \).
• $\text{Aut}_1(X, A, \mu)$ is isomorphic to a semidirect product $\text{Ker} \chi \times \mathbb{R}$ such that $\chi$ corresponds to the projection onto the second coordinate.

• Every conservative transformation in $\text{Aut}_1(X, A, \mu)$ belongs to $\text{Ker} \chi$.

The latter result (as well as the following theorem) shows that the group $\text{Ker} \chi$ is a more natural object than $\text{Aut}_1(X, A, \mu)$ from the ergodic theory point of view.

**Theorem D.**

• $\text{Aut}_2(X, A, \mu)$ endowed with $d_2$ has the Rokhlin property\(^3\). The subset of ergodic transformations of Krieger’s type III\(_1\) is a dense $G_\delta$ in $\text{Aut}_2(X, \mu)$.

• $\text{Ker} \chi$ endowed with $d_1$ has the Rokhlin property (Aut\(_1\)(X, A, µ) does not have it). The subset of ergodic transformations of Krieger’s type III\(_1\) is a dense $G_\delta$ in Ker \chi.

It is well known that the group of all $\mu$-nonsingular transformations of $(X, A, \mu)$ endowed with the weak topology also possesses similar properties (see [23], [9], [10], [16]). However the proof of Theorem D is more difficult for several reasons. The first is that $d_2$ and $d_1$ are stronger than the weak topology. The second is that we have no freedom to replace $\mu$ by an arbitrary equivalent measure any more. Indeed, if $\nu \sim \mu$ but $\sqrt{\frac{d\nu}{d\mu}} - 1 \notin L^2(\mu)$ then $\text{Aut}_2(X, A, \nu) \neq \text{Aut}_2(X, A, \mu)$.

In a subsequent work [14] we show that the subset of $T \in \text{Aut}_2(X, A, \mu)$ such that $T_\ast$ is ergodic and type III\(_1\) is a dense $G_\delta$. Combined with Theorem D this implies the existence of a type III\(_1\) ergodic transformation whose Poisson suspension is also ergodic and of type III\(_1\).

Let $G$ be a locally compact second countable group. The affine isometric representations of $G$ is a fundamental tool in geometric group theory connecting Kazhdan property (T), the Haagerup property, operator algebras, harmonic analysis, etc. (see [7], [17], [18], [5]). The Poisson $G$-actions deliver natural non-trivial examples of such representations. Indeed, if $T : G \ni g \mapsto T_g \in \text{Aut}_2(X, A, \mu)$ is a measurable $G$-action then the mapping $G \ni g \mapsto A^{(2)}_{T_g}$ (see (1.1)) is a continuous affine representation of $G$ in $L^2(\mu)$.

**Theorem E.** Let $T : G \ni g \mapsto T_g \in \text{Aut}_2(X, A, \mu)$ be a measurable $G$-action.

• The Poisson suspension $T_\ast := \{ (T_g)_\ast \}_{g \in G}$ of $T$ has an absolutely continuous invariant probability measure if and only if the $L^2(\mu)$-cocycle $c_T : G \ni g \mapsto c_T(g) := \sqrt{\frac{d\mu(T_g)^{-1}}{d\mu}} - 1$ is bounded.

• If $\int_G e^{-\frac{1}{2} \| c_T(g) \|_2^2} \, d\lambda(g) < \infty$ then $T_\ast$ is totally dissipative.

• $T_\ast$ is of zero type if and only if $\| c_T(g) \|_2 \to \infty$ as $g \to \infty$.

\(^3\)i.e. there is a dense conjugacy class in this group.
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The aforementioned zero type is a nonsingular analogue of the mixing in the probability preserving case (see [16]). We could not find a criterion for ergodicity of $T_*$ because it depends in a subtle way not only on the ergodic properties of $T$ but also on a “right” choice of a measure inside the equivalence class of $\mu$: we construct an example of an ergodic $T$ admitting a $\mu$-equivalent invariant probability such that $T_*$ is totally dissipative.

Let $\kappa$ be a probability on $G$. A nonsingular $G$-action $S = (S_g)_{g \in G}$ on a probability space $(Y, \mathcal{B}, \nu)$ is called $\kappa$-stationary if $\int_G \kappa(g) \nu \circ S_g \, d\kappa(g) = \nu$ (see a survey [20] for properties and applications of the stationary actions).

**Theorem F.** Let $T$ be as in the previous theorem. If $T_*$ is $\kappa$-stationary for a generating probability $\kappa$ on $G$ then $T_*$ preserves $\mu^*$ and $T$ preserves $\mu$.

There are several equivalent characterizations of property (T) for $G$ [6]. We provide one more in terms of the nonsingular Poisson suspensions.

**Theorem G.** $G$ has property (T) if and only if each nonsingular Poisson $G$-action $T_*$ admits an absolutely continuous invariant probability.

One more characterization was obtained in [8]: if a countable discrete group $G$ does not have property (T) then for each generating probability $\kappa$ on $G$, the Furstenberg $\kappa$-entropy $h_\kappa(\cdot)$ has no gap on the set of purely infinite ergodic nonsingular $G$-actions. This was refined in [13]: $h_\kappa(\cdot)$ takes all possible positive values on the subset of ergodic $G$-actions of type $II_1$.

We extend this result to arbitrary locally compact groups and partly refine it by considering only the nonsingular Poisson suspensions.

**Theorem H.** Let a locally compact $G$ do not have property (T) and let $\kappa$ be a probability on $G$. Then there is a nonsingular $G$-action $T$ on an infinite measure space $(X, \mu)$ such that the Poisson suspension $T_*$ of $T$ is $\mu^*$-nonsingular and $\{h_\kappa(T_*, \mu^*_t) : t \in (0, +\infty)\} = (0, +\infty)$, where $\mu_t = t\mu$.

During the course of work on this paper we learnt about [2], where it was constructed a functor from the affine $G$-representations to the nonsingular Gaussian systems. Certain nonsingular Gaussian counterparts of Theorems G and H are proved there. We believe that there should be some interplay between the theory of nonsingular Gaussian actions and the nonsingular Poisson suspensions.

### 1.3. Sections overview.

In Section 2 we introduce a model $(X^*, A^*, \mu^*, T_*)$ for the Poisson suspension of a dynamical system $(X, A, \mu, T)$, consider $L^2(\mu^*)$ as a Fock space over $L^2(\mu)$ and extend the exponential map $\mathcal{E} : L^2(\mu) \to L^2(\mu^*)$ to some non-square integrable functions. In Section 3 we extend and refine Takahashi’s theorem on equivalence and orthogonality of the Poisson suspensions of equivalent measures [31]. In Section 4 we introduce and study the topological groups $\text{Aut}_2(X, A, \mu)$, $\text{Aut}_1(X, A, \mu)$, the unitary and affine representations $U$ and $A^{(2)}$ of $\text{Aut}_2(X, A, \mu)$ and their analogues for $\text{Aut}_1(X, A, \mu)$, and prove Theorems A, B, C and related results. Section 5
is devoted to generic properties of \( \text{Aut}_2(X,\mathcal{A},\mu) \) and \( \text{Aut}_1(X,\mathcal{A},\mu) \). Theorem D is proved there. In Section 6, for arbitrary locally compact groups \( G \), we characterize some basic dynamical properties of the nonsingular Poisson suspensions \( T_s \) of \( G \)-actions \( T \) in terms of the underlying system \( (X,\mu,T) \). Theorem E is proved there. In Section 7 we consider stationary \( G \)-actions, prove Theorem F and compute the Furstenberg \( \kappa \)-entropy of \( T_s \) in terms of the underlying system \( (X,\mu,T) \). Section 8 is devoted to property (T). We prove Theorems G and H and related results there. The paper ends with Appendix which is devoted to infinitely divisible variables and stochastic integration. This material is used in the course of the proofs of Theorems 3.4 and 3.6.

2. Poisson suspensions

2.1. Space of point processes. Let \((X,\mathcal{A},\mu)\) be a \( \sigma \)-finite Lebesgue space with a non-atomic measure, that is \((X,\mathcal{A},\mu)\) is mod 0 isomorphic to the real line if \( \mu(X) = \infty \) or to a bounded closed interval if \( \mu(X) < \infty \), endowed with Lebesgue measure and Lebesgue measurable sets.

The space of point processes over \( X \) is defined as the set \( X^* \) of all measures \( \omega \) of the form \( \omega = \sum_{i \in I} \delta_{x_i} \) where \( I \) is at most countable. \( X^* \) is endowed with the smallest \( \sigma \)-algebra such that the \( \mathbb{Z}_+ \cup \{+\infty\} \)-valued maps \( N_A : \omega \mapsto \omega(A) \) are measurable, for all \( A \in \mathcal{A} \). We denote this \( \sigma \)-algebra on \( X^* \) by \( \mathcal{A}^* \).

2.2. Poisson measures. Let \( \mathcal{A}^\mu_f \) denote the collection of sets \( A \in \mathcal{A} \) of finite \( \mu \)-measure. There exists a unique probability measure \( \mu^* \) on \((X^*,\mathcal{A}^*)\) such that:

- For all \( k \geq 1 \) and pairwise disjoint sets \( A_1, \ldots, A_k \) in \( \mathcal{A}^\mu_f \), the random variables \( N_{A_i}, 1 \leq i \leq k \), are independent.
- For any \( A \in \mathcal{A}^\mu_f \), \( N_A \) is Poisson distributed with parameter \( \mu(A) \).

The probability space \((X^*,\mathcal{A}^*,\mu^*)\) is called the Poisson space over the base \((X,\mathcal{A},\mu)\). When completed with respect to \( \mu^* \), \((X^*,\mathcal{A}^*,\mu^*)\) is a Lebesgue space. The random measure \( A \mapsto N_A, A \in \mathcal{A} \) distributed as \( \mu^* \) is called a Poisson point process of intensity \( \mu \). In most cases this object is presented on \( \mathbb{R}^d \) (or on a subset of it) with Lebesgue measure as intensity and then called homogeneous Poisson point process. As Lebesgue spaces with a continuous measure are either isomorphic to \( \mathbb{R} \) or to a bounded closed interval with Lebesgue measure depending on whether the measure is finite or not, there is essentially no loss in generality in dealing with homogeneous Poisson point process. Observe the following three important features of a Poisson measure:

- \( \mu^* \) is supported on simple counting measures, that is:
  \[ \mu^*(\{\omega \in X^* : \forall x \in X, \omega(\{x\}) = 0 \text{ or } 1\}) = 1. \]
• The intensity of the random measure \( A \mapsto N_A \) is \( \mu \), that is \( \mathbb{E}_{\mu^*}[N_A] = \mu(A) \).

We shall also make use of the following important theorem:

**Theorem 2.1.** (Rényi) Let \( m \) be a probability measure on \((X^*, A^*)\) supported on simple counting measures. If for all \( A \in A_0^f \),

\[
m(\{\omega : N_A(\omega) = 0\}) = \mu^*(\{\omega : N_A(\omega) = 0\}),
\]

then \( m = \mu^* \).

2.3. **Poisson suspensions.** At the core of this paper is an easy yet fundamental observation: If \( \varphi \) is a measurable map between two Lebesgue spaces \((X, A, \mu)\) and \((Y, B, \nu)\) such that \( \mu \circ \varphi^{-1} = \nu \), then \( \varphi^* \) also acts measurably between \((X^*, A^*)\) and \((Y^*, B^*)\) by \( \varphi^* \omega = \omega \circ \varphi^{-1} \) and satisfies \( \mu^{*} \circ \varphi^{-1} = \nu^{*} \).

In particular, if \( T \) is an invertible transformation of \((X, A, \mu)\), we have the following picture:

\[
(X, A, \mu) \xrightarrow{T} (X, A, \mu \circ T^{-1})
\]

\[
(X^*, A^*, \mu^{*}) \xrightarrow{T} (X^*, A^*, \mu^{*} \circ T^{-1}).
\]

We will be interested in the situation where \( T \) is a non-singular automorphism, that is \( \mu \sim \mu \circ T^{-1} \). It is not always true that \( \mu^{*} \sim (\mu \circ T^{-1})^{*} \). We will recall necessary and sufficient conditions to get the equivalence of measures. When it is the case, the non-singular dynamical system \((X^*, A^*, \mu^{*}, T^*)\) will be called the Poisson suspension over \((X, A, \mu, T)\).

2.4. **Fock space structure of** \( L^2(\mu^{*}) \) **and coherent vectors.** We recall a very important structural feature of Poisson measures (see [3] or [27]): there is a canonical isometry between \( L^2(\mu^{*}) \) and the symmetric Fock space \( F(L^2(\mu)) \) over \( L^2(\mu) \). We recall that

\[
F(L^2(\mu)) := \bigoplus_{n=0}^{\infty} L^2(\mu)^{\odot n},
\]

where \( L^2(\mu)^{\odot 0} := \mathbb{C} \) and each factor \( L^2(\mu)^{\odot n} \) is equipped with the normalized scalar product \( n! \langle \cdot, \cdot \rangle_{L^2(\mu)^{\odot n}} \). The Hilbert space \( L^2(\mu)^{\odot n} \) considered as a subspace of \( L^2(\mu^{*}) \) is called the chaos of order \( n \). We now explain how to construct the canonical isometry. For that, we choose a distinguished family of vectors in \( F(L^2(\mu)) \), called the coherent vectors, defined, for \( f \in L^2(\mu) \), by

\[
\mathcal{E}(f) := \sum_{k=0}^{\infty} \frac{1}{n!} f^{\odot n} \in F(L^2(\mu)).
\]

They form a total family in \( F(L^2(\mu)) \) and satisfy the exponential relation

\[
\langle \mathcal{E}(f), \mathcal{E}(g) \rangle_{F(L^2(\mu))} = e^{\langle f, g \rangle_{L^2(\mu)}}.
\]
Denote by $\mathcal{B}_0(X)$ the subspace of $L^2(\mu)$-functions with finite $\mu$-measure support. Then the family $\{\mathcal{E}(f) : f \in \mathcal{B}_0(X)\}$ is also total in $F(L^2(\mu))$ (see e.g. [27], where it is shown that even a subspace of finitely valued functions from $\mathcal{B}_0(X)$ generates a total family in $F(L^2(\mu))$). On the other hand, for $f \in \mathcal{B}_0(X)$, define a bounded function $\exp(f)$ on $X^*$ by setting:

\[(2.2) \quad \exp(f)(\omega) = e^{-\int_X f d\mu} \prod_{\{x \in X : \omega(x) = 1\}} (1 + f(x)), \quad \omega \in X^*.
\]

In particular, for any set $A \in \mathcal{A}_f^\mu$,

\[(2.3) \quad \exp(-1_A) = e^{\mu(A)}1_{\{\omega : N_A(\omega) = 0\}}.
\]

A standard calculation shows that

\[(2.4) \quad \langle \exp(f), \exp(g) \rangle_{L^2(\mu^*)} = e^{\langle f, g \rangle_{L^2(\mu)}}.
\]

Due to the Rényi Theorem, the family $\{\exp(f) : f \in \mathcal{B}_0(X)\}$ is total in $L^2(\mu^*)$. Hence we deduce from (2.1) and (2.4) the map $\mathcal{E}(f) \mapsto \exp(f)$ extends to an isometry between $F(L^2(\mu))$ and $L^2(\mu^*)$. In the sequel, we will not distinguish between $\mathcal{E}(f)$ and $\exp(f)$. We will use the following properties of coherent vectors: $\mathcal{E}(f) = \mathcal{E}(g)$, $\mathcal{E}(f) \in L^1(\mu^*)$ and $\mathbb{E}_{\mu^*}[\mathcal{E}(f)] = 1$ for all $f \in L^2(\mu)$.

**2.5. Product formula and extended coherent vectors.** For every two functions $f$ and $g$ in $L^2(\mu)$, we define a function $f \cdot g$ by setting $f \cdot g := (1 + f)(1 + g) - 1$.

We now define a function space

\[\mathcal{L}(\mu) := \{\varphi : X \to \mathbb{R} : \exists f, g \in L^2(\mu), \varphi = f \cdot g\}.
\]

Clearly, $L^2(\mu) \subset \mathcal{L}(\mu)$. If $f, g \in \mathcal{B}_0(X)$, $f \cdot g$ is integrable and has finite measure support. Moreover, one can deduce from (2.2) the following product formula:

\[(2.5) \quad \mathcal{E}(f)(\omega) \mathcal{E}(g)(\omega) = e^{\int_X f g d\mu} e^{-\int_X f \cdot g d\mu} \prod_{\{x \in X : \omega(x) = 1\}} (f \cdot g(x) + 1).
\]

This formula enables us to extend the definition of coherent vectors to functions in $\mathcal{L}(\mu)$. Namely, we set

\[(2.6) \quad \mathcal{E}(f \cdot g) := e^{-\int_X f g d\mu} \mathcal{E}(f) \mathcal{E}(g)
\]

for all $f, g \in L^2(\mu)$. We have to verify that this formula is well defined. To that end, we first define an auxiliary map $\Psi : L^2(\mu) \times L^2(\mu) \to L^1(\mu^*)$ by setting

\[(2.7) \quad \Psi(f, g) := e^{-\int_X f g d\mu} \mathcal{E}(f) \mathcal{E}(g).
\]

Since the map $L^2(\mu) \ni f \mapsto \mathcal{E}(f) \in L^2(\mu^*)$ is continuous, it follows that $\Psi$ is continuous. Now we consider $f, g, f', g' \in L^2(\mu)$ such that $f \cdot g = f' \cdot g'$. Select an increasing sequence $A_1 \subset A_2 \subset \cdots$ of subsets of finite measure.
in $X$ such that $\bigcup_{n=1}^{\infty} A_n = X$. In we now set $f_n := f1_{A_n}$, $g_n := g1_{A_n}$, $f'_n := f'1_{A_n}$ and $g'_n := g'1_{A_n}$ then $f_n, g_n, f'_n, g'_n \in B_0(X)$ for each $n \in \mathbb{N}$ and $f_n \rightarrow f$, $g_n \rightarrow g$, $f'_n \rightarrow f'$, $g'_n \rightarrow g'$ in $L^2(\mu)$ as $n \rightarrow \infty$. Moreover, for all $n \in \mathbb{N}$,

$$f_n \cdot g_n = (f + g + fg)1_{A_n} = (f' + g' + f'g')1_{A_n} = f'_n \cdot g'_n.$$  

It now follows from (2.7) and (2.5) that $\Psi(f_n, g_n) = \Psi(f'_n, g'_n).$ Taking limits as $n \rightarrow \infty$ and using the continuity of $\Psi$, we obtain that

$$\Psi(f, g) = \Psi(f', g'),$$

as desired. Thus, utilizing (2.6), we can define $\mathcal{E}(\phi)$ for each $\phi \in \mathcal{L}(\mu)$. We call such $\mathcal{E}(\phi)$ the extended coherent vectors. If $\phi \in L^2(X)$ then $\phi = \phi \cdot 0$ and (2.6) implies that the extended coherent vector $\mathcal{E}(\phi)$ coincides with the “standard” coherent vector defined by $\phi$. We also note that $\mathbb{E}_{\mu^*}(\mathcal{E}(\varphi)) = 1$ for each $\varphi \in \mathcal{L}(\mu)$.

2.6. More properties of coherent vectors.

**Proposition 2.2.** An extended coherent vector $\mathcal{E}(\varphi), \varphi \in \mathcal{L}(\mu)$, is in $L^2(\mu^*)$ if and only if $\varphi \in L^2(\mu)$.

**Proof.** If $\varphi \in L^2(\mu)$ then the extended coherent vector $\mathcal{E}(\varphi)$ is the classical coherent vector and hence it belongs to $L^2(\mu^*)$.

Now we prove the converse. Let $f, g \in L^2(\mu)$ and $\mathcal{E}(f \cdot g) \in L^2(\mu^*)$. Denote by $\mathcal{S}$ the subspace of finitely valued functions from $L^2(\mu)$. Then for each $h \in \mathcal{S}$, the function $g \cdot h$ is in $L^2(\mu)$. We now have:

$$\mathbb{E}_{\mu^*}[\mathcal{E}(f \cdot g)\mathcal{E}(h)] = \mathbb{E}_{\mu^*}[e^{-\int_X f \cdot g \, d\mu} \mathcal{E}(f) \mathcal{E}(g) \mathcal{E}(h)]$$

$$= e^{-\int_X f \cdot g \, d\mu} \mathbb{E}_{\mu^*}[\mathcal{E}(f) \mathcal{E}(g) \mathcal{E}(h)]$$

$$= e^{-\int_X f \cdot g \, d\mu} \mathbb{E}_{\mu^*}[\mathcal{E}(f) \mathcal{E}(g \cdot h) \mathcal{E}(h)]$$

$$= e^{-\int_X f \cdot g \, d\mu} \mathbb{E}_{\mu^*}[\mathcal{E}(f \cdot g) \mathcal{E}(h) \mathcal{E}(h)] = e^{-\int_X f \cdot g \, d\mu} \mathbb{E}_{\mu^*}[\mathcal{E}(f \cdot g) \mathcal{E}(h)]$$

Hence a linear functional $L : \mathcal{S} \ni h \mapsto \int_X (f \cdot g) \cdot h \, d\mu \in \mathbb{C}$ is well defined. It is continuous at 0. Indeed, if a sequence $(h_n)_{n=1}^{\infty}$ with $h_n \in \mathcal{S}$, $n \in \mathbb{N}$, goes to 0 in $L^2(\mu)$ as $n \rightarrow \infty$ then $\mathcal{E}(h_n) \rightarrow \mathcal{E}(0) = 1$ as $n \rightarrow \infty$ in $L^2(\mu^*)$. Hence

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mu^*}[\mathcal{E}(f \cdot g)\mathcal{E}(h_n)] = \mathbb{E}_{\mu^*}[\mathcal{E}(f \cdot g)] = 1$$

as $n \rightarrow \infty$.

Therefore $e^{\int_X (f \cdot g) \cdot h_n \, d\mu} \rightarrow 1$, i.e. $\int_X (f \cdot g) \cdot h_n \, d\mu \rightarrow 0$ as $n \rightarrow \infty$. Since $L$ is linear, it follows that $L$ is continuous on the entire $\mathcal{S}$. Since $\mathcal{S}$ is dense in $L^2(\mu)$, we deduce that $L$ extends uniquely to a continuous linear functional on $L^2(\mu)$. In view of the Riesz representation theorem, we conclude that

$$f \cdot g \in L^2(\mu).$$

**Lemma 2.3.** Let a function $\varphi \in \mathcal{L}(\mu)$ take only real values. Then the function $\varphi := |1 + \varphi| - 1$ belong to $\mathcal{L}(\mu)$ and

$$|\mathcal{E}(\varphi)| = e^{-2\int_{x \in \mathbb{R} : \varphi(x)+1 < 0} \varphi(x) \, d\mu \mathcal{E}(\varphi)}.$$  

(2.8)
In particular, $\mathcal{E}(\varphi)$ is non-negative $\mu^\ast$-a.s. if and only if $\varphi \geq -1$ $\mu$-almost everywhere.

**Proof.** We consider separately three cases. Suppose first that $\varphi \in \mathcal{B}_0(X)$. Then $|1 + \varphi| - 1 \in \mathcal{B}_0(X)$ and, in view of (2.2),

$$
|\mathcal{E}(\varphi)(\omega)| = e^{-\int_X \varphi \, d\mu} \prod_{\{x \in X : \omega(\{x\}) = 1\}} |1 + \varphi(x)|
$$

$$
= e^{-\int_X \varphi \, d\mu} \prod_{\{x \in X : \omega(\{x\}) = 1\}} (1 + \tilde{\varphi}(x))
$$

(2.9)

$$
= e^{-\int_X (\varphi - \tilde{\varphi}) \, d\mu} e^{-\int_X \tilde{\varphi} \, d\mu} \prod_{\{x \in X : \omega(\{x\}) = 1\}} (1 + \tilde{\varphi}(x))
$$

$$
= e^{-\int_X (\varphi - \tilde{\varphi}) \, d\mu} \mathcal{E}(\tilde{\varphi})(\omega)
$$

$$
= e^{-2\int_{\{x \in X : \varphi(x) + 1 < 0\}} (\varphi(x) + 1) \, d\mu} \mathcal{E}(\tilde{\varphi})(\omega),
$$

as desired.

Suppose now that $\varphi \in L^2(\mu)$. We let $A_\varphi := \{x \in X : \varphi(x) < -1\}$. Then $\mu(A_\varphi) < \infty$. Select a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of functions $\varphi_n \in \mathcal{B}_0(X)$ such that

1. $\varphi_n \to \varphi$ in $L^2(\mu)$ as $n \to \infty$,
2. $A_{\varphi_n} = A_\varphi$ for each $n \in \mathbb{N}$ and
3. $\varphi_n(x) = \varphi(x)$ if $x \in A_\varphi$.

It is straightforward to verify that $|\tilde{\varphi}| \leq |\varphi|$ and $|\tilde{\varphi}_n - \tilde{\varphi}| \leq |\varphi_n - \varphi|$. This yields that $\tilde{\varphi} \in L^2(\mu)$ and $\tilde{\varphi}_n \to \tilde{\varphi}$ in $L^2(\mu)$ as $n \to \infty$. Therefore $\mathcal{E}(\tilde{\varphi}_n) \to \mathcal{E}(\tilde{\varphi})$ in $L^2(\mu^\ast)$ as $n \to \infty$. By the first case and the properties of $\varphi_n$,

$$
|\mathcal{E}(\varphi_n)| = e^{-2\int_{A_{\varphi_n}} (\varphi_n + 1) \, d\mu} \mathcal{E}(\tilde{\varphi}_n) = e^{-2\int_{A_\varphi} (\varphi + 1) \, d\mu} \mathcal{E}(\tilde{\varphi}_n).
$$

Passing to the limit as $n \to \infty$, we obtain (2.8), as desired. Before we proceed to the general case, we rewrite (2.8) in the following equivalent form:

(2.10)

$$
|\mathcal{E}(\varphi)| = e^{-\int_X (\varphi - \tilde{\varphi}) \, d\mu} \mathcal{E}(\tilde{\varphi}).
$$

Now, in the general case, let $\varphi = f \cdot g$ for arbitrary vectors $f, g \in L^2(\mu)$. A straightforward verification shows that $\tilde{\varphi} = \tilde{f} \cdot \tilde{g}$. We deduce from (2.6) and (2.10) that

$$
|\mathcal{E}(\varphi)| = e^{-\int_X fg \, d\mu} |\mathcal{E}(f)||\mathcal{E}(g)|
$$

$$
= e^{-\int_X fg \, d\mu} e^{\int_X (\tilde{f} - f) \, d\mu} \mathcal{E}(\tilde{f}) e^{\int_X (\tilde{g} - g) \, d\mu} \mathcal{E}(\tilde{g})
$$

$$
= e^{-\int_X (fg - \tilde{f} + f - \tilde{g} + g - \tilde{g}) \, d\mu} \mathcal{E}(\tilde{f} \cdot \tilde{g})
$$

$$
= e^{-\int_X (\varphi - \tilde{\varphi}) \, d\mu} \mathcal{E}(\tilde{\varphi}),
$$

and (2.8) follows.
To prove the second claim of the lemma, we assume first that $E(\varphi) \geq 0$ $\mu^*$-a.s. for some $\varphi \in L(\mu)$. Then

$$1 = \mathbb{E}[E(\varphi)] = \mathbb{E}[|E(\varphi)|] = \mathbb{E}\left[e^{-2 \int_{x : \varphi(x) + 1 < 0} (\varphi + 1) \, d\mu \, E(\varphi)}\right] = e^{-2 \int_{x : \varphi(x) + 1 < 0} (\varphi + 1) \, d\mu}.$$ 

Therefore $\int_{x : \varphi(x) + 1 < 0} (\varphi + 1) \, d\mu = 0$, which implies that $\varphi \geq -1$, $\mu$-almost everywhere. Conversely, if $\varphi \geq -1$ $\mu$-almost everywhere then $\int_{x : \varphi(x) + 1 < 0} (\varphi + 1) \, d\mu = 0$ and $\tilde{\varphi} = \varphi$. Therefore $|E(\varphi)| = E(\tilde{\varphi}) = E(\varphi). \quad \Box$

3. Absolute continuity and equivalence of Poisson measures

Let $(X, \mathcal{A}, \mu)$ be a non-atomic standard $\sigma$-finite measure space. We single out four important sets of measures:

- $M_{\mu,2}^+$ is the set of $\sigma$-finite measures $\nu$ on $(X, \mathcal{A})$ such that $\nu \ll \mu$ and $\sqrt{\frac{d\nu}{d\mu}} - 1 \in L^2(\mu)$,
- $M_{\mu,2}^{0,+} := \{\nu \in M_{\mu,2}^+ : \nu \sim \mu\}$,
- $M_{\mu,1}^+$ is the set of $\sigma$-finite measures $\nu$ on $(X, \mathcal{A})$ such that $\nu \ll \mu$ and $\frac{d\nu}{d\mu} - 1 \in L^1(\mu)$ and
- $M_{\mu,1}^{0,+} := \{\nu \in M_{\mu,1}^+ : \nu \sim \mu\}$.

Since $(\sqrt{x} - 1)^2 \leq |x - 1|$ for each $x > 0$, it follows that $M_{\mu,1}^+ \subset M_{\mu,2}^+$ and hence $M_{\mu,1}^{0,+} \subset M_{\mu,2}^{0,+}$.

Remark 3.1. (1) Let $\mu$ be a finite measure. Then $\nu \in M_{\mu,2}^+$ if and only if it is a finite measure and $\nu \ll \mu$. Hence $M_{\mu,1}^+ = M_{\mu,2}^+$ and $M_{\mu,1}^{0,+} = M_{\mu,2}^{0,+}$.

(2) If $\nu \in M_{\mu,2}^{0,+}$, then $M_{\nu,2}^{0,+} = M_{\mu,2}^{0,+}$. In a similar way, if $\nu \in M_{\mu,1}^{0,+}$, then $M_{\nu,1}^{0,+} = M_{\mu,1}^{0,+}$.

(3) If $\mu$ is infinite and $\nu \in M_{\mu,2}^+$ then $\mu\left(\left\{x \in X : \frac{d\nu}{d\mu}(x) = 0\right\}\right) < \infty$.

Lemma 3.2. Let $\nu \in M_{\mu,2}^+$. Then for any set $A \in \mathcal{A}$, we have that $\mu(A) < \infty$ if and only if $\nu(A) < \infty$, that is $A^\mu = A^\nu$.

Proof. We set $\phi := \frac{d\nu}{d\mu}$. If $\mu(A) < \infty$, then $\sqrt{\phi} - 1 \in L^2(\mu|A) \subset L^1(\mu|A)$. Since $L^1(\mu) \ni (\sqrt{\phi} - 1)^2 = (\phi - 1) - 2(\sqrt{\phi} - 1)$, we now obtain that $\phi - 1 \in L^1(\mu|A)$, which implies that $\nu(A) < \infty$.

Now if $\nu(A) < \infty$ then $\sqrt{\phi}$ is in $L^2(\mu|A)$. As $\sqrt{\phi} - 1 \in L^2(\mu|A)$, this implies that the constant function 1 is in $L^2(\mu|A)$ too. Hence $\mu(A) < \infty. \quad \Box$
The following theorem is the ground for the rest of the paper. The necessary and sufficient condition for the absolute continuity for Poisson measures was found by Takahashi in [31]. However he did not write the explicit formula for the Radon-Nikodym derivative as a coherent vector. We only prove this formula and show that it generalizes the formula obtained by Neretin in [27] for the smaller class of measures $\mathcal{M}_{\mu,1}^+.$

**Theorem 3.3.** Let $\nu$ be a $\sigma$-finite measure on $(X,A)$. Then $\nu^* \ll \mu^*$ if and only if $\nu \in \mathcal{M}_{\mu,2}^+$. If $\nu \in \mathcal{M}_{\mu,2}^+$ then $\frac{d\nu^*}{d\mu^*} = \mathcal{E}(\frac{d\nu}{d\mu} - 1)$. If $\nu \notin \mathcal{M}_{\mu,2}^+$, then $\nu^* \perp \mu^*$.

**Proof.** Assume $\nu \in \mathcal{M}_{\mu,2}^+$ and set $\phi := \frac{d\nu}{d\mu}$. We now observe that

$$\phi - 1 = (\sqrt{\phi} - 1) \ast (\sqrt{\phi} - 1)$$

with $\sqrt{\phi} - 1 \in L^2(\mu)$. Hence $\mathcal{E}(\phi - 1)$ is well defined. Take a subset $A \in \mathcal{A}^\mu_f$. Then $A \in \mathcal{A}_f^\mu$ by Lemma 3.2. Applying the product formula (2.6) three times, we obtain that

$$\mathcal{E}(-1_A)\mathcal{E}(\phi - 1) = \mathcal{E}(-1_A)e^{-\int_X (\sqrt{\phi} - 1)^2 d\mu} \mathcal{E}(\sqrt{\phi} - 1)\mathcal{E}(\sqrt{\phi} - 1)$$

$$= e^{-\int_X ((\sqrt{\phi} - 1)^2 + (\sqrt{\phi} - 1)1_A) d\mu} \mathcal{E}((-1_A) \ast (\sqrt{\phi} - 1))\mathcal{E}(\sqrt{\phi} - 1)$$

$$= e^{-\int_X (\phi - 1)1_A d\mu} \mathcal{E}(\sqrt{\phi} - 1)\mathcal{E}(1_A - (\phi - 1)).$$

Taking the mathematical expectation and using (2.3) twice, we obtain that

$$\mathbb{E}_{\mu^*}[1_{\omega \in X^*: N_A(\omega) = 0}] \mathcal{E}(\phi - 1) = e^{-\nu(A)} = \nu^*(\{\omega \in X^*: N_A(\omega) = 0\}).$$

Hence, by the Rényi’s theorem, $\nu^* \ll \mu^*$ and $\frac{d\nu^*}{d\mu^*} = \mathcal{E}(\frac{d\nu}{d\mu} - 1)$. The second claim of the theorem was proved in [31].

We note that Theorem 3.3 highlights the connection between extended coherent vectors and Radon-Nikodym derivatives of equivalent Poisson point process measures. The following result can be seen as an explicit description of $\frac{d\nu^*}{d\mu^*}$ for $\nu \in \mathcal{M}_{\mu,2}^{0,+}$ as a function from $X^*$ to $\mathbb{R}$.

**Theorem 3.4.** Let $\nu \in \mathcal{M}_{\mu,2}^{0,+}$ and set $\phi := \frac{d\nu}{d\mu}$. Then

1. We can represent $\log \frac{d\nu^*}{d\mu^*}$ as the following limit in probability:

$$\log \frac{d\nu^*}{d\mu^*}(\omega) = \lim_{\epsilon \to 0} \left( \int_{\{x \in X: |\log \phi(x)| > \epsilon\}} \log \phi \, d\omega - \int_{\{x \in X: |\log \phi(x)| > \epsilon\}} (\phi - 1) \, d\mu \right).$$

2. Moreover, $\log \frac{d\nu^*}{d\mu^*}$ is an infinitely divisible random variable whose Lévy measure is the image of $\mu$ by $\log \phi$, restricted to $\mathbb{R} \setminus \{0\}$. 
We have that
\[ \mathbb{E}_{\mu^*} \left[ \log \frac{d\nu^*}{d\mu^*} \right] = -\int_X (\phi - 1 - \log \phi) \, d\mu \in [-\infty, 0]. \]

It is finite if and only if
\[ \int \{ x \in X : |\log \phi(x)| > 1 \} |\log \phi| \, d\mu < \infty. \]

Proof. Given \( \epsilon > 0 \), we let \( X_\epsilon := \{ x \in X : |\log \phi(x)| > \epsilon \} \). As usual, \( X_\epsilon^c \) denotes the compliment to \( X_\epsilon \). We first prove three auxiliary claims.

Claim A: \( \mu(X_\epsilon) < \infty \) for each \( \epsilon > 0 \).
Indeed,
\[ \mu(X_\epsilon) = \mu(\{ x \in X : \phi(x) > e^\epsilon \} \cup \{ x \in X : \phi(x) < e^{-\epsilon} \}) \leq \mu(\{ x \in X : |\sqrt{\phi(x)} - 1| > \alpha \}) \leq \frac{1}{\alpha^2} \int_X (\sqrt{\phi} - 1)^2 \, d\mu < +\infty, \]
where \( \alpha = \min(e^\epsilon - 1, 1 - e^{-\epsilon}) \).

Claim B: \( (\log \phi)^2 \wedge 1 \leq \kappa (\sqrt{\phi} - 1)^2 \) for some constant \( \kappa > 0 \).
This claim follows from the standard inequality \( \log t \leq t - 1 \) for \( t > 0 \).

Claim C: \( \phi - 1 - \log \phi \cdot 1_{X_1^c} \in L^1(\mu) \).
To prove this claim we first write the function \( \phi - 1 - \log \phi \cdot 1_{X_1^c} \) as the following sum:
\[ (\sqrt{\phi} - 1)^2 + 2(\sqrt{\phi} - 1 - \log \sqrt{\phi})1_{X_1^c} + 2(\sqrt{\phi} - 1)1_{X_1}. \]

The first term in this sum is in \( L^1(\mu) \) because \( \sqrt{\phi} - 1 \in L^2(\mu) \). Since
\[ 0 \leq \sqrt{\phi} - 1 - \log \sqrt{\phi} \leq (\sqrt{\phi} - 1)^2, \]
it follows that the second term is in \( L^1(\mu) \) too. The Cauchy-Schwarz inequality and Claim A yield that the third term is also integrable. Claim C follows.

It follows from Claim B that the stochastic integral \( I_\mu(\log \phi) : X^* \to \mathbb{R} \) is well defined (see Appendix). Claim C implies that the real
\[ \beta := -\int_X (\phi - 1 - \log \phi \cdot 1_{X_1^c}) \, d\mu \]
is well defined. It follows that
\begin{equation}
I_\mu(\log \phi)(\omega) + \beta = \lim_{\epsilon \to 0} \left( \int_{X_\epsilon} \log \phi \, d\omega - \int_{X_\epsilon} (\phi - 1) \, d\mu \right),
\end{equation}
where the limit means the convergence in probability (see Appendix). Our purpose now is to identify the lefthand side of this formula as a Radon-Nikodym derivative. It is straightforward to verify that for each subset \( B \in \mathcal{A} \), we have that \( (\phi - 1)1_B = ((\sqrt{\phi} - 1)1_B) \cdot ((\sqrt{\phi} - 1)1_B) \). Since
\( \mathcal{E}((\sqrt{\phi} - 1)^{1}X_{\epsilon}) \rightarrow \mathcal{E}(\sqrt{\phi} - 1) \) in \( L^{2}(\mu^{*}) \), we can apply (2.6) to obtain that
\[
\mathcal{E}((\phi - 1)^{1}X_{\epsilon}) = e^{-\int X_{\epsilon}(\sqrt{\phi} - 1)^{2}d\mu} \mathcal{E}((\sqrt{\phi} - 1)^{2})^{2}
\]
\[
\rightarrow e^{-\int X(\sqrt{\phi} - 1)^{2}d\mu} \mathcal{E}(\sqrt{\phi} - 1)^{2}
\]
\[= \mathcal{E}(\phi - 1) \]
in \( L^{1}(\mu^{*}) \) as \( \epsilon \to 0 \). Since \((\sqrt{\phi} - 1)^{1}X_{\epsilon} \in \mathcal{B}_{0}(X)\), it follows from (2.2) that for a.e. \( \omega \in X^{*} \),
\[
\mathcal{E}((\phi - 1)^{1}X_{\epsilon}) = e^{-\int X_{\epsilon}(\sqrt{\phi} - 1)^{2}d\mu} \prod_{\{x \in X_{\epsilon}: \omega(x) = 1\}} \phi(x)
\]
\[= e^{\int X_{\epsilon} \log \phi d\omega - \int X_{\epsilon}(\phi - 1)d\mu} \]

From this and (3.4) we deduce that
\[
\lim_{n \to \infty} \left( \int X_{\epsilon} \log \phi d\omega - \int X_{\epsilon}(\phi - 1)d\mu \right) = \log \mathcal{E}(\phi - 1)
\]
where the limit is in \( \mu^{*} \)-probability. This formula, (3.3) and Theorem 3.3 yield that
\[
(3.5) \quad I_{\mu}(\log \phi) + \beta = \log \frac{d\nu^{*}}{d\mu^{*}}.
\]
Thus, (1) is proved. Moreover, \( I_{\mu}(f) \) is infinitely divisible (see Appendix) so (3.5) implies (2).

Since \( \phi - 1 - \log \phi \geq 0 \), the integral \( \int X(\phi - 1 - \log \phi)d\mu \) is always well defined. Combining this observation with Claim C, we obtain that
\[
(3.6) \quad \int X(\phi - 1 - \log \phi)d\mu < \infty \iff \int X_{1}|\log \phi|d\mu < \infty.
\]

By Proposition 9.2, the latter inequality is equivalent to the fact that \( I_{\mu}(\log \phi) \in L^{1}(\mu^{*}) \). The latter, in turn, is equivalent to \( \log \frac{d\nu^{*}}{d\mu^{*}} \in L^{1}(\mu^{*}) \) in view of (3.5).

Firstly we consider the case where \( \log \frac{d\nu^{*}}{d\mu^{*}} \in L^{1}(\mu^{*}) \). Then it follows from (3.5), Proposition 9.2 and the definition of \( \beta \) that
\[
\mathbb{E}_{\mu^{*}} \left[ \log \frac{d\nu^{*}}{d\mu^{*}} \right] = \mathbb{E}_{\mu^{*}}(I_{\mu}(\log \phi)) + \beta
\]
\[= \int X_{1} \log \phi d\mu + \beta
\]
\[= \int X(\log \phi - \phi - 1)d\mu
\]

Consider now the second case, where \( \int X_{1}|\log \phi|d\mu = +\infty \). Then from (3.6) we deduce that \( \int X(\phi - 1 - \log \phi)d\mu = +\infty \). On the other hand, by the Jensen inequality, \( \mathbb{E}_{\mu^{*}}[-\log \frac{d\nu^{*}}{d\mu^{*}}] \geq 0 \). Hence the fact \( \log \frac{d\nu^{*}}{d\mu^{*}} \notin L^{1}(\mu^{*}) \) implies \( \mathbb{E}_{\mu^{*}}[-\log \frac{d\nu^{*}}{d\mu^{*}}] = +\infty \). The proof of (3) is now complete. \( \square \)
Remark 3.5. With additional efforts, using the martingale convergence theorem, it is possible to prove the almost sure convergence in (3.2) instead of the convergence in probability.

The special case where \( \phi - 1 \in L^1(\mu) \) has been considered in [31]. In this case we have the following results.

**Theorem 3.6.** Let \( \nu \in \mathcal{M}_{\mu,1}^{0} \) and set \( \phi := \frac{d\nu}{d\mu} \). Then

1. \( \log \phi \in L^1(\omega) \) for \( \mu^* \)-almost every \( \omega \in X^* \) and

\[
\frac{d\nu^*}{d\mu^*}(\omega) = e^{-\int_X (\phi-1) \, d\mu} \prod_{\{x \in X : \omega(\{x\}) = 1\}} \phi(x),
\]

where the infinite product converges absolutely.

2. The integral \( \int_X \log \phi \, d\mu \) is well defined and takes values in the extended interval \( [\infty, \int_X (\phi-1) \, d\mu] \). Moreover,

\[
\mathbb{E}_{\mu^*}\left[ \log \frac{d\nu^*}{d\mu^*} \right] = -\int_X (\phi - 1) \, d\mu + \int_X \log \phi \, d\mu.
\]

**Proof.** In the course of proof we will use the notation \( X_\epsilon \) and \( X_1^\epsilon \) and refer to Claims B and C from the proof of Theorem 3.4.

If \( \nu \in \mathcal{M}_{\mu,1}^{0} \) then \( \phi - 1 \in L^1(\mu) \) and hence by Claim C, \( (\log \phi) 1_{X_1^\epsilon} \in L^1(\mu) \). By Claim B, \( |\log \phi|^2 1_{X_1^\epsilon} \in L^1(\mu) \), thus the stochastic integral \( I_\mu(|\log \phi|) \) is well defined as

\[
(3.7) \quad I_\mu(|\log \phi|) = \lim_{\epsilon \to 0} \left( \int_{X_\epsilon} |\log \phi| \, d\omega - \int_{X_\epsilon} |\log \phi| 1_{X_1^\epsilon} \, d\mu \right),
\]

where the limit is in probability. In particular, \( I_\mu(|\log \phi|) \) is finite \( \mu^* \)-almost surely. By the monotone convergence theorem and the integrability of \( |\log \phi| 1_{X_1^\epsilon} \),

\[
\lim_{\epsilon \to 0} \int_{X_\epsilon} |\log \phi| 1_{X_1^\epsilon} \, d\mu = \int_X |\log \phi| 1_{X_1^\epsilon} \, d\mu < \infty.
\]

It follows from this and (3.7) that there exists \( \lim_{\epsilon \to 0} \int_{X_\epsilon} |\log \phi| \, d\omega < \infty \) for a.e. \( \omega \in X^* \). By the monotone convergence theorem,

\[
\lim_{\epsilon \to 0} \int_{X_\epsilon} |\log \phi| \, d\omega = \int_X |\log \phi| \, d\omega.
\]

Thus, \( \log \phi \in L^1(\omega) \) for \( \mu^* \)-a.e. This fact combined with the integrability of \( \phi - 1 \) imply the almost everywhere convergence in (3.2). Passing to this limit, we obtain now that

\[
\log \frac{d\nu^*}{d\mu^*}(\omega) = \int_X \log \phi \, d\omega - \int_X (\phi - 1) \, d\mu
\]

for a.e. \( \omega \). This proves (1). The second claim follows from Theorem 3.4(3). \( \square \)
Remark 3.7. • Firstly, we note that the formula for the Radon-Nikodym derivative in Theorem 3.6 is well known and follows immediately from Theorem 3.4(1) and the fact that \( \phi - 1 \in L^1(\mu) \). However the absolute convergence of the infinite product (or the fact that \( \phi = L^1(\omega) \) for a.e. \( \omega \)) requires an additional reasoning.

• Secondly, it is worth mentioning that the combination of \( \nu \in M_{\mu,2}^{\circ,+} \) and \( \log \phi \in L^1(\mu) \) implies that \( \nu \in M_{\mu,1}^{\circ,+} \) and \( \frac{d\nu^*}{d\mu} \in L^1(\mu^*) \). Indeed, from \( \nu \in M_{\mu,2}^{\circ,+} \) we get that \( \phi - 1 - (\log \phi)1_X \in L^1(\mu) \) (see Claim C). On the other hand, the fact \( \log \phi \in L^1(\mu) \) implies that \( (\log \phi)1_X \in L^1(\mu) \). Therefore \( \phi - 1 \in L^1(\mu) \), i.e. \( \nu \in M_{\mu,1}^{\circ,+} \). The integrability of \( \log \frac{d\nu^*}{d\mu} \) follows now from Theorem 3.6(2).

4. Poisson suspensions of nonsingular transformations and related Koopman representations

4.1. The unitary Koopman representation of the group of nonsingular transformations. Let \((Y, B, \rho)\) be a \(\sigma\)-finite Lebesgue space. Denote by \(U(L^2(\rho))\) the group of unitary operators in \(L^2(\rho)\) and by \(U_2(L^2(\rho))\) the subgroup of unitaries that preserve invariant the \(\mathbb{R}\)-subspace \(L^2_{\mathbb{R}}(\rho)\) of real valued functions in \(L^2(\mu)\). Let \(\text{Aut}(Y, B, \rho)\) stand for the group of all nonsingular transformations of \((Y, B, \rho)\). For each \(S \in \text{Aut}(Y, B, \rho)\), we set \(S' := \frac{d\rho S^{-1}}{d\rho}\) and define a unitary operator \(U_S \in U_2(L^2(\rho))\) by setting \(U_S f := \sqrt{S'} f \circ S^{-1}\). Then the mapping \(U : \text{Aut}(Y, B, \rho) \ni S \mapsto U_S \in U_2(L^2(\rho))\)
is a unitary one-to-one representation of \(\text{Aut}(Y, B, \rho)\) in \(L^2(Y, \rho)\). It is called the unitary Koopman representation of \(\text{Aut}(Y, B, \rho)\).

Let \(C_0 := \{f \in L^2(\rho) : f \geq 0\}\). Then \(C_0\) is a closed cone in \(L^2(\rho)\). It is well known that

\[
\{V \in U(L^2(\rho)) : VC_0 = C_0\} = \{U_S : S \in \text{Aut}(Y, B, \rho)\}.
\]

Endow \(U_2(L^2(\rho))\) with the weak (equivalently, strong) operator topology. We recall that the weak topology on \(\text{Aut}(Y, B, \rho)\) is the weakest topology in which \(U\) is continuous. It follows from (4.1) that \(\text{Aut}(Y, B, \rho)\) furnished with the weak topology is a Polish group.

4.2. Nonsingular Poisson suspensions and related transformation groups. Let the measure space \((X, A, \mu)\) be as in the previous section and let \(T\) be a nonsingular (invertible) transformation of this space. Theorem 3.3 provides us with an “if and only if” criteria for when \(T_x\) is non-singular and Theorems 3.4 and Theorem 3.6 give an explicit pointwise description of the Radon-Nikodym derivative of \(T_x\) as follows.

**Corollary 4.1.** \(T_x\) is a nonsingular automorphism of \((X^*, A^*, \mu^*)\) if and only if \(\sqrt{T'} - 1 \in L^2(\mu)\). In this case \(T_x' = \mathcal{E}(T' - 1)\). Moreover,
We can represent \( \log(T_\epsilon)' \) as the following limit in probability:

\[
\log(T_\epsilon)'(\omega) = \lim_{\epsilon \to 0} \left( \int_{\{x \in X : |\log T'(x)| > \epsilon\}} \log T' \, d\omega - \int_{\{x \in X : |\log T'(x)| > \epsilon\}} (T' - 1) \, d\mu \right).
\]

(2) The function \( X^* \ni \omega \mapsto \log(T_\epsilon)'(\omega) \in \mathbb{R} \) is an infinitely divisible random variable whose Lévy measure is the (restriction to \( \mathbb{R} \setminus \{0\} \) of) the image of \( \mu \) by \( \log T' \).

(3) If \( T' - 1 \in L^1(\mu) \), then \( \log T' \in L^1(\omega) \) for \( \mu^* \)-almost every \( \omega \in X^* \) and

\[
(T_\epsilon)'(\omega) = e^{-\int_{X} (T' - 1) \, d\mu} \prod_{\{x \in X : \omega(x) = 1\}} T'(x),
\]

where the infinite product converges absolutely.

Our purpose in this paper is to study nonsingular Poisson suspensions. Therefore in view of Corollary 4.1 we introduce some special subgroups of nonsingular transformations that are related naturally to these suspensions.

**Definition 4.2.** We set

\[
\operatorname{Aut}_2(X, A, \mu) := \left\{ T \in \operatorname{Aut}(X, A, \mu), \sqrt{T'} - 1 \in L^2(\mu) \right\},
\]

\[
\operatorname{Aut}_1(X, A, \mu) := \left\{ T \in \operatorname{Aut}(X, A, \mu), T' - 1 \in L^1(\mu) \right\}
\]

and

\[
\operatorname{Aut}_\mathcal{P}(X^*, A^*, \mu^*) := \left\{ T_\epsilon \in \operatorname{Aut}(X^*, A^*, \mu^*), T \in \operatorname{Aut}_2(X, A, \mu) \right\}.
\]

Of course, \( \operatorname{Aut}_1(X, A, \mu) \subset \operatorname{Aut}_2(X, A, \mu) \). By Remark 3.1(2), the two objects are subgroups of \( \operatorname{Aut}(X, A, \mu) \). Since the map \( T \mapsto T_\epsilon \) is a homomorphism from \( \operatorname{Aut}_2(X, A, \mu) \) to \( \operatorname{Aut}(X^*, A^*, \mu^*) \), the set \( \operatorname{Aut}_\mathcal{P}(X^*, A^*, \mu^*) \) is a subgroup of \( \operatorname{Aut}(X^*, A^*, \mu^*) \).

**Definition 4.3.** The map \( \operatorname{Aut}_2(X, A, \mu) \ni T \mapsto T^* \in \operatorname{Aut}(X^*, A^*, \mu^*) \) will be called the Poisson homomorphism.

In the next three subsections we study \( \operatorname{Aut}_2(X, A, \mu) \), \( \operatorname{Aut}_1(X, A, \mu) \) and \( \operatorname{Aut}_\mathcal{P}(X^*, A^*, \mu^*) \) respectively in more detail.

**4.3. Polish group \( \operatorname{Aut}_2(X, A, \mu) \) and the associated affine Koopman representation.** We denote by \( \operatorname{Aff}_\mathbb{R}(L^2(\mu)) \) the subgroup of invertible affine operators in \( L^2(\mu) \) that preserve invariant the \( \mathbb{R} \)-subspace \( L^2_\mathbb{R}(\mu) \). Then

\[
\operatorname{Aff}_\mathbb{R}(L^2(\mu)) := L^2_\mathbb{R}(\mu) \times U_\mathbb{R}(L^2(\mu)).
\]

We recall that an operator \( \mathcal{A} = (f, V) \in \operatorname{Aff}_\mathbb{R}(L^2(\mu)) \) acts on \( L^2(\mu) \) by the formula \( \mathcal{A}h := f + Vh \). One can verify that the multiplication law in \( \operatorname{Aff}_\mathbb{R}(L^2(\mu)) \) is given by:

\[
(f, V)(f', V') := (f + Vf', VV').
\]

\( \operatorname{Aff}_\mathbb{R}(L^2(\mu)) \) is a Polish group when endowed with the product of the norm topology on \( L^2(\mu) \) and the weak operator topology on \( U_\mathbb{R}(L^2(\mu)) \). We now
let

\[ C_{-1} := \{ f \in L^2(\mu) : f \geq -1 \}. \]

Then \( C_{-1} \) is a closed semispace in \( L^2_{\mathbb{R}}(\mu) \). We now establish an “affine” analogue of (4.1).

**Theorem 4.4.**

\[ \{ A \in \text{Aff}_{\mathbb{R}}(L^2(\mu)) : AC_{-1} = C_{-1} \} = \{ (\sqrt{S^2} - 1, U_S) : S \in \text{Aut}_2(X, \mathcal{A}, \mu) \}. \]

**Proof.** If \( A := (\sqrt{S^2} - 1, U_S) \) for some transformation \( S \in \text{Aut}_2(Y, \mathcal{B}, \rho) \) then

\[ Ah = (h \circ S^{-1} + 1)\sqrt{S^2} - 1 \geq -1 \text{ for each } h \in C_{-1}. \]

Hence \( AC_{-1} \subseteq C_{-1} \). The same is true if we take \( S^{-1} \) in place of \( S \). Therefore we obtain that \( A^{-1}C_{-1} \subseteq C_{-1} \). Hence \( AC_{-1} = C_{-1} \), as desired.

Conversely, let \( A = (f, V) \in \text{Aff}_{\mathbb{R}}(L^2(\mu)) \) and \( AC_{-1} = C_{-1} \). The following properties are verified straightforwardly:

- \( C_0 + C_{-1} = C_{-1} \).
- If \( a + C_{-1} \subseteq C_{-1} \) for some \( a \in L^2(\mu) \) then \( a \in C_0 \).

Then \( A(C_0 + C_{-1}) = C_{-1} \). On the other hand,

\[ A(C_0 + C_{-1}) = AC_0 + AC_{-1} - A0 = VC_0 + C_{-1}. \]

Therefore \( VC_0 + C_{-1} = C_{-1} \). Hence \( VC_0 \subseteq C_0 \). Since \( A^{-1}C_{-1} = C_{-1} \), a similar reasoning yields that \( V^{-1}C_0 \subseteq C_0 \). Therefore \( VC_0 = C_0 \). In view of (4.1), there is \( S \in \text{Aut}(X, \mathcal{A}, \mu) \) such that \( V = U_S \). Hence

\[ VC_{-1} = \{ h \circ S\sqrt{S^2} + f | h \in C_{-1} \} = C_{-1}. \]

Let \( L(\mu) \) stand for the space of all measurable real valued functions on \( X \). Endowed with the natural order, \( L(\mu) \) is an ordered vector space. Considering the semispace \( C_{-1} \) as a subset of \( L(\mu) \), we deduce from (4.2) that

\[ L(\mu) \ni -1 = \inf C_{-1} = \inf VC_{-1} = -\sqrt{S^2} + f \in L(\mu). \]

Thus, \( \sqrt{S^2} - 1 = f \in L^2_{\mathbb{R}}(\mu) \). \( \square \)

We note that Theorem 4.4 provides an alternative characterization of \( \text{Aut}_2(X, \mathcal{A}, \mu) \). This characterization is not related straightforwardly to Poisson suspensions. Moreover, Theorem 4.4 determines a one-to-one representation of \( \text{Aut}_2(X, \mathcal{A}, \mu) \) in \( \text{Aff}_{\mathbb{R}}(L^2(\mu)) \).

**Definition 4.5.** We call the homomorphism

\[ A^{(2)} : \text{Aut}_2(X, \mathcal{A}, \mu) \ni S \mapsto A^{(2)}_S := (\sqrt{S^2} - 1, U_S) \in \text{Aff}_{\mathbb{R}}(L^2(\mu)) \]

the **affine Koopman representation** of \( \text{Aut}_2(X, \mathcal{A}, \mu) \). We call the weakest topology on \( \text{Aut}_2(X, \mathcal{A}, \mu) \) in which the affine Koopman representation is continuous the **d}_2\)-topology.

Thus, a sequence \( (T_n)_{n=1}^{\infty} \) of transformations \( T_n \in \text{Aut}_2(X, \mathcal{A}, \mu) \) converges in \( d}_2 \) to a transformation \( T \in \text{Aut}_2(X, \mathcal{A}, \mu) \) as \( n \to \infty \) if and only if \( T_n \to T \) weakly and \( \| \sqrt{T_n} - \sqrt{T} \|_2 \to 0 \) as \( n \to \infty \). It follows from Theorem 4.4 that the image of \( \text{Aut}_2(X, \mathcal{A}, \mu) \) under \( A^{(2)} \) is closed in
Aff($L^2(\mu)$). Hence Aut$_2(X, \mathcal{A}, \mu)$ endowed with the $d_2$-topology is a Polish group. We state the next proposition without proof. It follows easily from Remark 3.1(2).

**Proposition 4.6.** Let $\nu \in \mathcal{M}_{\mu,2}^\infty$. Then Aut$_2(X, \mathcal{A}, \mu) = \text{Aut}_2(X, \mathcal{A}, \mu)$ as topological groups furnished with the corresponding $d_2$-topologies.

### 4.4. Polish group Aut$_1(X, \mathcal{A}, \mu)$, the associated affine Koopman representation and the structure of semidirect product.

Let $U(L^1(\mu))$ stand for the group of isometries in $L^1(\mu)$ and let $U_{\mathbb{R}}(L^1(\mu))$ stand for the subgroup of isometries that preserve invariant the $\mathbb{R}$-subspace $L^1_{\mathbb{R}}(\mu)$ of real valued functions in $L^1(\mu)$. We denote by $\text{Aff}_{\mathbb{R}}(L^1(\mu)) := L^1_{\mathbb{R}}(\mu) \ltimes U_{\mathbb{R}}(L^1(\mu))$ the group of invertible affine operators in $L^1(\rho)$ that preserve invariant $L^1_{\mathbb{R}}(\mu)$. The multiplication law in $\text{Aff}_{\mathbb{R}}(L^1(\mu))$ is given by the same formula as the multiplication law in $\text{Aff}_{\mathbb{R}}(L^2(\mu))$. We also note that $\text{Aff}_{\mathbb{R}}(L^1(\mu))$ is a Polish group when endowed with the product of the norm topology on $L^1_{\mathbb{R}}(\mu)$ and the strong (not the weak!) operator topology on $U_{\mathbb{R}}(L^1(\mu))$. We now let

$$C_{-1}^{(1)} := \{f \in L^1(\mu) : f \geq -1\}.$$ Then $C_{-1}^{(1)}$ is a closed semispace in $L^1_{\mathbb{R}}(\mu)$. Given $S \in \text{Aut}_1(X, \mathcal{A}, \mu)$, we define an isometric invertible operator $U_S^{(1)}$ on $L^1(\mu)$ by setting $U_S^{(1)}f := f \circ S^{-1} \cdot S'$. Then we call the one-to-one homomorphism

$$U^{(1)} : \text{Aut}_1(X, \mathcal{A}, \mu) \ni S \mapsto U_S^{(1)} \in U_{\mathbb{R}}(L^1(\mu))$$

the isometric Koopman representation of Aut$_1(X, \mathcal{A}, \mu)$. The following theorem is an analogue of Theorem 4.4.

**Theorem 4.7.**

$$\{A \in \text{Aff}_{\mathbb{R}}(L^1(\mu)) : AC_{-1}^{(1)} = C_{-1}^{(1)}\} = \{(S' - 1, U_S^{(1)}) : S \in \text{Aut}_1(X, \mathcal{A}, \mu)\}.$$ We do not provide a proof of this theorem because it is very similar to the proof of Theorem 4.4.

**Definition 4.8.** We call the one-to-one homomorphism

$$A^{(1)} : \text{Aut}_1(X, \mathcal{A}, \mu) \ni S \mapsto A_S^{(1)} := (S' - 1, U_S^{(1)}) \in \text{Aff}_{\mathbb{R}}(L^1(\mu))$$

the affine Koopman representation of Aut$_1(X, \mathcal{A}, \mu)$. We call the weakest topology on Aut$_1(X, \mathcal{A}, \mu)$ in which the affine Koopman representation is continuous the $d_1$-topology.

Thus, a sequence $\{T_n\}_{n=1}^{\infty}$ of transformations $T_n \in \text{Aut}_1(X, \mathcal{A}, \mu)$ converges in $d_1$ to a transformation $T \in \text{Aut}_1(X, \mathcal{A}, \mu)$ as $n \to \infty$ if and only if $T_n \to T$ weakly and $\|T_n - T\|_1 \to 0$ as $n \to \infty$. It follows from Theorem 4.7 that the image of Aut$_1(X, \mathcal{A}, \mu)$ under $A^{(1)}$ is closed in $\text{Aff}_{\mathbb{R}}(L^1(\mu))$. Hence Aut$_1(X, \mathcal{A}, \mu)$ endowed with the $d_1$-topology is a Polish group. We state the next proposition without proof. It follows easily from Remark 3.1(2).
Proposition 4.9. Let \( \nu \in \mathcal{M}_{\mu,1}^{\circ,+} \). Then \( \text{Aut}_1(X,A,\nu) = \text{Aut}_1(X,A,\mu) \) as topological groups furnished with the corresponding \( d_1 \)-topologies.

The following important group homomorphism was introduced in [27]:

\[
\chi : \text{Aut}_1(X,A,\mu) \ni T \mapsto \chi(T) := \int_X (T' - 1) \, d\mu \in \mathbb{R}.
\]

Of course, \( \chi \) depends on \( \mu \). However, we now show that \( \chi \) does not depend on the choice of measure within the class \( \mathcal{M}_{\mu,1}^{\circ,+} \).

Proposition 4.10. Let \( \nu \in \mathcal{M}_{\mu,1}^{\circ,+} \). Then \( \chi(T) = \int_X (d\nu \circ T^{-1} - 1) \, d\nu \) for each \( T \in \text{Aut}_1(X,A,\mu) \).

Proof. By Remark 3.1, \( \mu \) and \( \nu \) share the same family of subsets of finite measure. Let \( \{A_n\}_{n \in \mathbb{N}} \) be an increasing sequence of sets of finite measure such that \( \bigcup_{n=1}^{\infty} A_n = X \). Then

\[
b := \int_X \left( \frac{d\nu}{d\mu} - 1 \right) d\mu = \lim_{n \to +\infty} \int_{A_n} \left( \frac{d\nu}{d\mu} - 1 \right) d\mu = \lim_{n \to +\infty} (\nu(A_n) - \mu(A_n)).
\]

In a similar way, \( b = \lim_{n \to +\infty} (\nu(T^{-1}A_n) - \mu(T^{-1}A_n)) \). Hence

\[
0 = \lim_{n \to +\infty} (\nu(A_n) - \mu(A_n) - \nu(T^{-1}A_n) + \mu(T^{-1}A_n))
\]

\[
= \lim_{n \to +\infty} (\mu(T^{-1}A_n) - \mu(A_n)) - \lim_{n \to +\infty} (\nu(T^{-1}A_n) - \nu(A_n))
\]

\[
= \chi(T) - \int_X \left( \frac{d\nu \circ T^{-1}}{d\nu} - 1 \right) d\nu.
\]

\( \square \)

Theorem 4.11. The homomorphism \( \chi \) is \( d_1 \)-continuous and the quotient group \( \text{Aut}_1(X,A,\mu)/\text{Ker} \chi \) is isomorphic to \( \mathbb{R} \). In other words, the following short sequence of Polish groups is exact

\[
\{1\} \to \text{Ker} \chi \to \text{Aut}_1(X,A,\mu) \to \mathbb{R} \to \{0\}
\]

Moreover, this sequence splits, i.e. there is a continuous one-to-one homomorphism \( \sigma : \mathbb{R} \to \text{Aut}_1(X,A,\mu) \) such that \( \chi \circ \sigma = id_{\mathbb{R}} \).

Proof. Of course, \( \chi \) is continuous. Next, there is no loss in generality if we take \( (X,A,\mu) = (\mathbb{R},\mathcal{B},m) \) where \( m \) is defined by \( \frac{dm}{dx} = 1_{\mathbb{R}^+} + 2 \times 1_{\mathbb{R}^+} \). For \( t \in \mathbb{R} \), denote by \( T_t : \mathbb{R} \to \mathbb{R} \) the translation by \( t \). Then it is easy to verify that \( T_t \in \text{Aut}_1(X,A,\mu) \) and \( \chi(T_{-t}) = t \). Of course, the homomorphism \( \sigma : \mathbb{R} \ni t \mapsto T_{-t} \in \text{Aut}_1(X,A,\mu) \) is continuous. \( \square \)

It follows from the second claim of the theorem that there is a topological isomorphism \( \theta : \text{Aut}_1(X,A,\mu) \to \text{Ker} \chi \times_{\sigma} \mathbb{R} \) such that the following diagram
commutes:
\[
\begin{array}{ccc}
\{1\} & \longrightarrow & \ker \chi \\
\downarrow \text{id} & & \downarrow \text{id} \\
\{1\} & \longrightarrow & \ker \chi \\
\end{array}
\]
\[
\begin{array}{ccc}
i & \longrightarrow & \operatorname{Aut}_1 (X, A, \mu) \\
\downarrow \theta & & \downarrow \theta \\
\chi & \longrightarrow & \mathbb{R} \\
\end{array}
\]
\[
\begin{array}{ccc}
i & \longrightarrow & \ker \chi \\
\downarrow \text{id} & & \downarrow \text{id} \\
\{1\} & \longrightarrow & \ker \chi \times_\sigma \mathbb{R} \\
\end{array}
\]

Thus, we have showed that \( \operatorname{Aut}_1 (X, A, \mu) \) has a natural structure of semidirect product of \( \ker \chi \) and \( \mathbb{R} \).

In order to state one more property of \( \chi \) we need to recall a definition of conservativeness.

**Definition 4.12.** A transformation \( T \in \operatorname{Aut} (X, A, \mu) \) is called **conservative** if for each subset \( A \in \mathcal{A} \) of positive measure, there is \( n > 0 \) such that \( \mu (T^{-1} A \cap A) > 0 \).

We recall that a nonsingular transformation \( T \) is conservative if and only if \( \sum_{k=0}^{\infty} U_T^{(1)} f = +\infty \) a.e. for each measurable function \( f > 0 \) [1].

**Proposition 4.13.** Let \( T \in \operatorname{Aut}_1 (X, A, \mu) \). If \( T \) is conservative then \( \chi (T) = 0 \).

**Proof.** Let \( \phi := T^r - 1 \). Then \( \phi \in L^1 (\mu) \) and \( \int_X \phi d\mu = \chi (T) \). Take \( g \in L^1 (\mu) \) such that \( g > 0 \) and \( \int_X g d\mu = 1 \). By the Hurewicz ratio ergodic theorem, there exists \( \psi \in L^1 (\mu) \) such that \( \psi \circ T = \psi \), \( \int_X \psi d\mu = \int_X \phi d\mu = \chi (T) \) and

\[
\frac{\sum_{k=0}^{n} U_T^{(1)} \phi}{\sum_{k=0}^{n} U_T^{(1)} g} \to \psi \quad \text{almost everywhere as} \quad n \to \infty.
\]

However \( \sum_{k=0}^{n} U_T^{(1)} \phi = (T^{n+1})^r - 1 \geq -1 \) while \( \sum_{k=0}^{n} U_T^{(1)} g \to +\infty \) as \( n \to \infty \). Consequently, \( \psi \geq 0 \) a.e. and hence \( \chi (T) \geq 0 \). However the transformation \( T^{-1} \) is also conservative and the above reasoning yields that \( \chi (T^{-1}) \geq 0 \). As \( \chi (T^{-1}) = -\chi (T) \), we obtain that \( \chi (T) = 0 \). \( \square \)

We conclude this subsection with a discussion about relationship among \( d_1 \), \( d_2 \) and the weak topology. Since \( \| \sqrt{T} - 1 \|_2^2 \leq \| T - 1 \|_1 \), it follows that \( d_1 \) is stronger than \( d_2 \).

**Proposition 4.14.**

- \( d_1 \) is strictly stronger than \( d_2 \) restricted to \( \operatorname{Aut}_1 (X, A, \mu) \) and \( d_2 \) is strictly stronger than the weak topology restricted to \( \operatorname{Aut}_2 (X, A, \mu) \).
- \( \operatorname{Aut}_1 (X, A, \mu) \) is dense and meager in \( \operatorname{Aut}_2 (X, A, \mu) \) endowed with \( d_2 \) and \( \operatorname{Aut}_2 (X, A, \mu) \) is dense and meager in \( \operatorname{Aut} (X, A, \mu) \) endowed with the weak topology.
- \( \chi \) is not \( d_2 \)-continuous. Hence \( \chi \) does not extend by continuity to \( \operatorname{Aut}_2 (X, A, \mu) \).
- If \( \mathcal{B}_1 \), \( \mathcal{B}_2 \) and \( \mathcal{B} \) stand for the Borel \( \sigma \)-algebras generated by \( \tau_1 \), \( \tau_2 \) and the weak topology respectively then \( \mathcal{B}_2 \upharpoonright \operatorname{Aut}_1 (X, A, \mu) = \mathcal{B}_1 \) and \( \mathcal{B} \upharpoonright \operatorname{Aut}_2 (X, A, \mu) = \mathcal{B}_2 \).
We do not provide a proof of this proposition because it will not be used below in the paper. We only note that it can be deduced from the general theorems of the descriptive topology combined with several facts that are proved in the next section: Propositions 5.2, 5.4, Theorem 5.8. The interested reader can also prove it independently of the next section by constructing appropriate concrete counterexamples.

4.5. **Unitary Koopman representations of** \( \text{Aut}_\mathcal{P}(X^*, A^*, \mu^*) \). Our objective in this subsection is to clarify relationship between the Koopman operators associated to \( T \) and \( T_* \) respectively.

Given an affine operator \( A = (f, V) \in \text{Aff}_\mathbb{R}(L^2(\mu)) \), we define an operator \( W_A \) in \( L^2(\mu^*) \) by setting

\[
(4.3) \quad W_A \mathcal{E}(h) := e^{-\frac{\|f\|_2^2}{2} - (f, Vh)_{L^2(\mu)}} \mathcal{E}(Ah)
\]

for all \( h \in L^2(\mu) \) and then extending \( W_A \) by linearity and continuity to the entire \( L^2(\mu^*) \). It is shown in [22, §2.2] that \( W_A \) is well defined and \( W_A \in \mathcal{U}(L^2(\mu^*)) \). It is called a Weyl operator. We observe that \( W_A \in \mathcal{U}_\mathbb{R}(L^2(\mu^*)) \) and \( W_A W_B = W_{AB} \) for all \( A, B \in \text{Aff}_\mathbb{R}(L^2(\mu)) \). It is possible to define \( W_A \) for arbitrary \( V \in \mathcal{U}(L^2(\mu)) \) and \( f \in L^2(\mu) \) by the same formula (4.3). Then \( W_A \in \mathcal{U}_\mathbb{R}(L^2(\mu^*)) \) if and only if \( A \in \text{Aff}_\mathbb{R}(L^2(\mu)) \). We leave the proof of this fact as an exercise for the reader. We also need one more auxiliary result that follows easily from [22, Theorem 2.1].

**Theorem.** The map \( W : \text{Aff}_\mathbb{R}(L^2(\mu)) \ni A \mapsto W_A \in \mathcal{U}_\mathbb{R}(L^2(\mu^*)) \) is a continuous one-to-one group homomorphism. Its image \( W := \{W_A : A \in \text{Aff}_\mathbb{R}(L^2(\mu))\} \) is a Polish subgroup in the induced topology.

We will call \( W \) the Weyl homomorphism. It follows from the above auxiliary theorem and [4, Proposition 1.2.1] that \( W \) is closed in \( \mathcal{U}_\mathbb{R}(L^2(\mu^*)) \) endowed with the weak operator topology.

It is well known that that if a transformation \( T \in \text{Aut}_2(X, \mathcal{A}, \mu) \) preserves \( \mu \) then the associated unitary Koopman operator \( U_T \) can be written as \( U_T = W_{0, U_T} = W_{A_T^{(2)}} \). We now extend this result to arbitrary (nonsingular) elements of \( \text{Aut}_2(X, \mathcal{A}, \mu) \).

**Theorem 4.15.** Let \( T \in \text{Aut}_2(X, \mathcal{A}, \mu) \). Under the natural identification of \( F(L^2(\mu)) \) with \( L^2(\mu^*) \) described in §2.4, we obtain that \( U_T = W_{A_T^{(2)}} \). In other words, the composition of the Poisson homomorphism of \( \text{Aut}_2(X, \mathcal{A}, \mu) \) with the unitary Koopman representation of \( \text{Aut}_\mathcal{P}(X^*, A^*, \mu^*) \) equals the composition of the affine Koopman representation of \( \text{Aut}_2(X, \mathcal{A}, \mu) \) with the Weyl homomorphism.

**Proof.** It is sufficient to verify that \( U_T \mathcal{E}(f) = W_{A_T^{(2)}} \mathcal{E}(f) \) for every simple (i.e. finite valued) function \( f \) from \( \mathcal{B}_0(X) \). We note that \( f \circ T^{-1} \) is also a
simple function from $B_0(X)$ and
\[\mathcal{E}(f \circ T^{-1})(\omega) = e^{-\int_X f \circ T^{-1} \, d\mu} \prod_{\{x \in X : \omega(x) = 1\}} (1 + f(T^{-1} x))\]
\[= e^{\int_X (f - f \circ T^{-1}) \, d\mu} \mathcal{E}(f)(T^{-1}_* \omega)\]
at a.e. $\omega \in X^*$. Using this and Corollary 4.1 we obtain that
\[U_T \mathcal{E}(f) = \sqrt{(T^*_f)^2 \mathcal{E}(f)} \circ T^{-1}_*\]
\[= \sqrt{\mathcal{E}(T^*_f - 1)} e^{\int_X (f \circ T^{-1} - f) \, d\mu} \mathcal{E}(f \circ T^{-1})\].
By the product formula (2.6) (see also (3.1)),
\[\mathcal{E}(T^*_f - 1) = e^{-\|\sqrt{T^*_f - 1}\|^2_2} \mathcal{E}((\sqrt{T^*_f - 1})^2)\]
Due to Lemma 2.3, $\mathcal{E}(\sqrt{T^*_f - 1}) \geq 0$ and hence
\[\sqrt{\mathcal{E}(T^*_f - 1)} = e^{-\frac{1}{2}\|\sqrt{T^*_f - 1}\|^2_2} \mathcal{E}((\sqrt{T^*_f - 1})^2)\]
By a straightforward computation, $(\sqrt{T^*_f - 1} \cdot (f \circ T^{-1}) = A^{(2)}_T f$. Hence, in view of (2.6),
\[\mathcal{E}(\sqrt{T^*_f - 1}) \mathcal{E}(f \circ T^{-1}) = e^{\int_X (\sqrt{T^*_f - 1}) f \circ T^{-1} \, d\mu} \mathcal{E}(A^{(2)}_T f)\]
Substituting first (4.5) and then (4.6) into (4.4), we obtain that
\[U_T \mathcal{E}(f) = e^{-\frac{1}{2}\|\sqrt{T^*_f - 1}\|^2_2 + \int_X (-f + U_T f) \, d\mu} \mathcal{E}(A^{(2)}_T f)\]
\[= e^{-\frac{1}{2}\|\sqrt{T^*_f - 1}\|^2_2 - (\sqrt{T^*_f - 1} U_T f)_{L^2(\mu)}} \mathcal{E}(A^{(2)}_T f)\]
\[= W_{A^{(2)}_T} \mathcal{E}(f)\]
Thus, the group $W$ contains the unitary Koopman operator generated by every transformation from $\text{Aut}_R(X^*, \mathcal{A}^*, \mu^*)$. We now show that it does not contain any Koopman operator generated by transformations from the set theoretical difference $\text{Aut}(X^*, \mathcal{A}^*, \mu^*) \setminus \text{Aut}_R(X^*, \mathcal{A}^*, \mu^*)$.

**Proposition 4.16.** $W \cap \{U_S : S \in \text{Aut}(X^*, \mathcal{A}^*, \mu^*)\} = \{W_{A^{(2)}_T} : T \in \text{Aut}_2(X, \mathcal{A}, \mu)\}$.

**Proof.** Suppose that for some operator $A \in \text{Aff}_R(L^2(\mu))$, the unitary $W_A$ is the Koopman operator generated by a nonsingular transformation of $(X^*, \mathcal{A}^*, \mu^*)$. Then, according to (4.1), $W_A$ preserves invariant the cone $L^2_+(\mu^*)$ of non-negative functions in $L^2(\mu^*)$. It follows from Lemma 2.3 that
\[L^2_+(\mu^*) \cap \{\mathcal{E}(h) : h \in L^2_+(X, \mu)\} = \{\mathcal{E}(h) : h \in C_{-1}\}\]
Hence $W_A(\{\mathcal{E}(h) : h \in C_{-1}\}) \subset L^2_+(\mu^*)$. In view of (4.3) and Lemma 2.3 this is equivalent to $AC_{-1} \subset C_{-1}$. Since $A^{-1}$ is also a Koopman operator, a similar reasoning yields that $A^{-1}C_{-1} \subset C_{-1}$. Hence $AC_{-1} = C_{-1}$. By Theorem 4.4, $A = A^{(2)}_T$ for some $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$. Thus, we showed that
W ∩ \{ U_S : S ∈ \text{Aut}(X^*, A^*, \mu^*) \} ⊂ \{ W_{\lambda^2} : T ∈ \text{Aut}_2(X, \mu) \}. The converse inclusion was established in Theorem 4.15. □

Since W is a closed subgroup of U(\lambda^2(\mu^*)), we obtain the following corollary from the above proposition.

**Corollary 4.17.** Aut_P(X^*, A^*, \mu^*) is weakly closed in Aut(X^*, A^*, \mu^*).

5. **Generic properties in \text{Aut}_2(X, \mu) and \text{Aut}_1(X, \mu)**

As the groups \text{Aut}_2(X, \mu) and \text{Aut}_1(X, \mu) are Polish, it is natural to ask: which dynamical properties (or, more rigorously, the subsets of elements possessing these properties) are generic in these groups in the Baire category sense? Recall that a set in a Polish space is generic if it contains a subset which is a dense G_δ in this space.

We first list the well known generic properties for Aut(X, \mu), the definitions of these (and other) properties will be given just below the Theorem.

**Theorem ([9], [10]).** The following subsets of nonsingular transformations:
- \text{Cons}(X, A, \mu) := \{ T ∈ \text{Aut}(X, A, \mu) : T \text{ is conservative} \},
- \text{Erg}(X, A, \mu) := \{ T ∈ \text{Aut}(X, A, \mu) : T \text{ is ergodic} \},
- \text{Erg}_{III_1}(X, A, \mu) := \{ T ∈ \text{Aut}(X, A, \mu) : T \text{ is of type III}_1 \}
are dense G_δ in Aut(X, A, \mu) endowed with the weak topology.

Recall that a transformation T ∈ Aut(X, A, \mu) is conservative if every wandering set W ∈ A for T, that is a set such that \{T^n W\}_{n \in \mathbb{Z}} are pairwise disjoint, is a null set. The transformation T is ergodic if every T-invariant subset is either \emptyset or X modulo null sets. The transformation T is aperiodic if there is a conull subset X′ ⊂ X such that for all x ∈ X′ and n > 0, T^n x ≠ x. There are several ways to define the type III_1 property, in this paper we will use the definition involving the Maharam extension. The Maharam extension of T ∈ Aut(X, A, \mu), is the transformation \tilde{T} on (X × \mathbb{R}, A⊗B_{\mathbb{R}}) defined by

\tilde{T}(x,y) := \left( T x, y - \log \frac{d\mu \circ T}{d\mu}(x) \right).

For every T ∈ Aut(X, A, \mu), its Maharam extension preserves the measure \tilde{\mu} defined by the formula \tilde{\mu}(A × I) := \mu(A)\int_I e^t dt for every A ∈ A and each interval I ⊂ \mathbb{R}. The transformation T is of type III_1 if its Maharam extension is ergodic.

The proof of genericity results usually consists of two steps: the first one is to show that the set under consideration is G_δ which may involve making use of ergodic theorems and countably many conditions defining the set. The second step is to prove that this set is dense. For the second step the following conjugacy lemma is often a key ingredient.

**Lemma.** For every aperiodic T ∈ Aut(X, A, \mu), the conjugacy class

\{ ST S^{-1} : S ∈ \text{Aut}(X, A, \mu) \}
of $S$ is weakly dense in $\text{Aut}(X,\mathcal{A},\mu)$ [9].

We also recall that a Polish group $G$ has the Rokhlin property if it has a dense conjugacy class. For instance, by the above lemma, $\text{Aut}(X,\mathcal{A},\mu)$ has the Rokhlin property.

We note that $\text{Aut}_1(X,\mathcal{A},\mu)$ appeared in [27] as a generalization of earlier work on representation theory of groups of diffeomorphisms on non-compact manifolds which are the identity outside a compact set (see [33], [21]). In this connection, it seems natural to introduce the following definition: a transformation $T \in \text{Aut}(X,\mathcal{A},\mu)$ is local if there is a set $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $Tx = x$ for all $x \notin A$. Denote by $\hat{\text{Aut}}_0(X,\mathcal{A},\mu)$ the set of all local transformations. Of course, $\hat{\text{Aut}}_0(X,\mathcal{A},\mu)$ is a group.

Since each transformation in $\text{Aut}_2(X,\mathcal{A},\mu)$ preserves the class of subsets of finite measure, $\hat{\text{Aut}}_0(X,\mathcal{A},\mu)$ is a normal subgroup of $\text{Aut}_1(X,\mathcal{A},\mu)$ and $\text{Aut}_2(X,\mathcal{A},\mu)$. (However it is not normal in $\text{Aut}(X,\mathcal{A},\mu)$.) The following consequence of Theorem 4.11 (specifically the topological isomorphism to a semidirect product) shows that $\hat{\text{Aut}}_0(X,\mathcal{A},\mu)$ is “too small” in $\text{Aut}_1(X,\mathcal{A},\mu)$.

**Corollary 5.1.** The group $\text{Aut}_1(X,\mathcal{A},\mu)$ does not have the Rokhlin property. The subgroup of local transformations, being a subset of $\ker \chi$, is $d_1$-nowhere dense in $\text{Aut}_1(X,\mathcal{A},\mu)$.

The situation with $\text{Aut}_2(X,\mathcal{A},\mu)$ is different. We preface the statement of the corresponding result with the following notation that will be of wide use in this section. Given $A, B \subset X$ which are both of finite measure, let $\tau_{A,B}$ denote a $\mu$-nonsingular bijection from $A$ to $B$ such that for all $x \in A$,

$$
\frac{d\mu \circ \tau_{A,B}}{d\mu}(x) = \frac{\mu(B)}{\mu(A)}.
$$

**Proposition 5.2.** $\hat{\text{Aut}}_0(X,\mathcal{A},\mu)$ is $d_2$-dense in $\text{Aut}_2(X,\mathcal{A},\mu)$.

**Proof.** Let $T \in \text{Aut}_2(X,\mathcal{A},\mu)$. There exists an increasing sequence $A_n \in \mathcal{A}$ of finite measure subsets satisfying $\bigcup_{n=1}^{\infty} A_n = X$ and

$$
\int_{X \setminus A_n} \left( 1 - \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} \right)^2 d\mu =: \epsilon_n \xrightarrow{n \to \infty} 0.
$$

Select subsets $B_n \subset X$ of finite measure such that $A_n \cup T A_n \subset B_n$ and

$$
\sqrt{\mu(B_n \setminus A_n)} + \sqrt{\mu(B_n \setminus T^{-1} A_n)} \geq n \left( \mu(A_n) + \mu(T^{-1} A_n) \right).
$$

We define $T_n \in \hat{\text{Aut}}_0(X,\mathcal{A},\mu)$ via

$$
T^{-1}_n x := \begin{cases} 
T^{-1} x, & x \in A_n \\
\tau_{B_n \setminus A_n, B_n \setminus T^{-1} A_n} x, & x \in B_n \setminus A_n \\
x, & x \notin B_n.
\end{cases}
$$
We will now show that $T_n$ converges in $d_2$ to $T$. Since $T_n^{-1}x = T^{-1}x$ for $x \in A_n$ we see that $T_n$ converges weakly to $T$ as $n \to \infty$. As

\begin{equation}
\frac{d\mu \circ T_n^{-1}}{d\mu}(x) := \begin{cases} \frac{d\mu \circ T^{-1}}{d\mu}(x), & x \in A_n \\ \frac{\mu(B_n \setminus T^{-1}A_n)}{\mu(B_n \setminus A_n)}, & x \in B_n \setminus A_n \\ 1, & x \notin B_n, \end{cases}
\end{equation}

we deduce that

\[
\left\| \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - \sqrt{\frac{d\mu \circ T_n^{-1}}{d\mu}} \right\|_{2} \leq \left( \left\| \left( \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1 \right) 1_{X \setminus A_n} \right\|_{2} + \left\| \left( \sqrt{\frac{d\mu \circ T_n^{-1}}{d\mu}} - 1 \right) 1_{X \setminus A_n} \right\|_{2} \right)^{1/2} 
\]

\[
\leq \sqrt{\epsilon_n} + \left( \int_{A_n} \left( \frac{\mu(B_n \setminus T^{-1}A_n)}{\mu(B_n \setminus A_n)} - 1 \right)^2 d\mu \right)^{1/2} 
\]

\[
= \sqrt{\epsilon_n} + \left( \frac{\mu(B_n \setminus A_n) - \mu(B_n \setminus T^{-1}A_n)}{\sqrt{\mu(B_n \setminus A_n)} + \sqrt{\mu(B_n \setminus T^{-1}A_n)}} \right)^2 
\]

\[
\leq \sqrt{\epsilon_n} + \frac{\mu(A_n) + \mu(T^{-1}A_n)}{\sqrt{\mu(B_n \setminus A_n)} + \sqrt{\mu(B_n \setminus T^{-1}A_n)}} 
\]

\[
\leq \sqrt{\epsilon_n} + \frac{1}{n}.
\]

It follows that $\lim_{n \to \infty} \left\| \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - \sqrt{\frac{d\mu \circ T_n^{-1}}{d\mu}} \right\|_{2} = 0$. Thus we have shown that $T_n$ converges to $T$ in $d_2$ as $n \to \infty$. \qed

For $A \in \mathcal{A}$, we can consider the group $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$ as the subset of $S \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that $S^{-1}A = A$ and $S|_{X \setminus A} = \text{id}|_{X \setminus A}$. We note that if $\mu(A) < \infty$ then the map $S \mapsto S|_A$ is a topological isomorphism of $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$ with the $d_2$-topology onto $\text{Aut}(A, \mathcal{A} \cap A, \mu|_A)$ with the weak topology.

**Corollary 5.3.**

1. The subset of periodic transformations from $\text{Aut}_0(X, \mathcal{A}, \mu)$ is $d_2$-dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$.
2. The subset of conservative transformations is a dense $G_\delta$-subset of $\text{Aut}_2(X, \mathcal{A}, \mu)$.

**Proof.** (1) By [23] and the metric isomorphism mentioned above, for every subset $A \in \mathcal{A}$ of finite measure, the subset of periodic transformations in $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$ is $d_2$-dense in $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$. This implies that the subset of periodic transformations is $d_2$-dense in $\text{Aut}_0(X, \mathcal{A}, \mu)$. The claim (1) follows from this and Proposition 5.2.

(2) Since $d_2$ is stronger than the weak topology and the subset of conservative transformations is a $G_\delta$ in $\text{Aut}(X, \mathcal{A}, \mu)$, it follows that the subset
of conservative transformations is a $G_\delta$ in $\text{Aut}_2(X, \mathcal{A}, \mu)$. As every periodic transformation is conservative, we deduce from (1) that subset of conservative transformations is $d_2$-dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$.

We say that a transformation $T \in \text{Aut}_0(X, \mathcal{A}, \mu)$ is **locally aperiodic** if there exists $A \in \mathcal{A}$ of positive finite measure such that $T \in \text{Aut}_2(A, A \cap A, \mu|_A)$ and $T|_A$ is aperiodic.

**Proposition 5.4.** Let $T \in \text{Aut}_0(X, \mathcal{A}, \mu)$ be locally aperiodic. Then the conjugacy class of $T$ is $d_2$-dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$. In particular, $\text{Aut}_2(X, \mathcal{A}, \mu)$ has the Rokhlin property.

**Proof.** In view of Proposition 5.2, it is enough to show that each local transformation is in the $d_2$-closure of the conjugacy class of $T$. Select a subset $A \in \mathcal{B}$ of positive finite measure such that $T|_A$ is aperiodic. It follows from [9, Theorem 2] that $\{S^{-1}TS : S \in \text{Aut}_2(A, A \cap A, \mu|_A)\}$ is $d_2$-dense in $\text{Aut}_2(A, A \cap A, \mu|_A)$. Since for every subset $B \subset X$ of positive finite measure, the map $\text{Aut}_2(A, A \cap A, \mu|_A) \ni T \mapsto (\tau_{B,A})^{-1}T \tau_{B,A} \in \text{Aut}_2(B, A \cap B, \mu|_B)$ is a topological group isomorphism and $\tau_{B,A} \in \text{Aut}_2(X, \mathcal{A}, \mu)$, every local transformation is in the $d_2$-closure of the conjugacy class of $T$. □

We now state one of the main result of this section.

**Theorem 5.5.** The subset

$$\text{Erg}_{III1}^2(X, \mathcal{A}, \mu) := \{ T \in \text{Aut}_2(X, \mathcal{A}, \mu) : T \text{ is ergodic of type III}_1 \}$$

is a dense $G_\delta$ in $\text{Aut}_2(X, \mathcal{A}, \mu)$.

The proof of Theorem 5.5 relies on the method of inducing which we now describe. Given a transformation $T \in \text{Aut}(X, \mathcal{A}, \mu)$, a subset $A \in \mathcal{A}$ is called $T$-sweeping out if $\mu(X \setminus \bigcup_{n=1}^{\infty} T^{-n}A) = 0$. If $T$ is ergodic then each subset $A \in \mathcal{A}$ of positive measure is $T$-sweeping out. Given a $T$-sweeping out subset $A$, we can define the *induced map* also known as the first return map, to $A$ as a nonsingular transformation $T_A$ of the space $(\mathcal{A}, A \cap A, \mu|_A)$ defined on a full measure subset of $A$ by

$$T_A x := T^{\varphi_A(x)} x,$$

where $\varphi_A(x) := \inf\{ n \in \mathbb{N} : T^n x \in A \}$ is the first return time function to $A$. We note $T_A$ is well defined because $A$ is $T$-sweeping out. The following facts will be used in the proof of Theorem 5.5:

- If $A$ is $T$-sweeping out then $T$ is ergodic if and only if $T_A$ is ergodic [1, Proposition 1.5.2].
- If $T$ is ergodic and $A$ is of positive measure then $T$ is of type III$_1$ if and only if $T_A$ is of type III$_1$.\(^4\)

\(^4\)This can be proved via inducing in the Maharam extension to the subset $A \times \mathbb{R}$ and noticing that $A$ is $T$-sweeping out if and only if $A \times \mathbb{R}$ is sweeping out for the Maharam extension of $T$. 


Proof of Theorem 5.5. Since Erg$^{III_1}(X, \mathcal{A}, \mu)$ is a $G_\delta$ in Aut$(X, \mathcal{A}, \mu)$ and $d_2$ is stronger than the weak topology, it follows that Erg$^{III_2}(X, \mathcal{A}, \mu)$ is a $G_\delta$ in Aut$_2(X, \mathcal{A}, \mu)$. It remains to show that Erg$^{III_1}(X, \mathcal{A}, \mu)$ is dense in Aut$_2(X, \mathcal{A}, \mu)$. For that, take a local transformation $T \in$ Aut$_2(X, \mathcal{A}, \mu)$. Our purpose is to find a sequence of transformations from Erg$^{III_1}(X, \mathcal{A}, \mu)$ that converges to $T$ in $d_2$ and apply Proposition 5.2. Let $A$ be a subset of finite measure such that $T|_{X \setminus A} = \text{id}_{X \setminus A}$. Take a sequence $\{B_n\}_{n=1}^\infty$ of subsets of finite measure and a sequence $\{P_n\}_{n=1}^\infty$ of countable partitions of $X \setminus A$ into subsets of finite measure such that the following are satisfied:

- $A \subset B_n \subset B_{n-1}$, $\mu(B_n \setminus A) > 0$ for each $n$ and $\lim_{n \to \infty} \mu(B_n) = \mu(A)$,
- $P_n = \{p_{l,j}^{(n)} : l \in \mathbb{N}, 1 \leq j \leq J_l^{(n)}\}$ for some $J_l^{(n)} > n$ for all $n, l > 0$,
- $\mu(p_{l,j}^{(n)}) = \mu(p_{l+1,j}^{(n)}) = \cdots = \mu(p_{l,j_l}^{(n)})$ for all $n, l > 0$,
- $B_n \setminus A = \bigsqcup_{l=1}^\infty p_{l,1}^{(n)}$,
- if $Q_n := \{\bigsqcup_{l=1}^{J_l^{(n)}} p_{l,j}^{(n)} : l \in \mathbb{N}\}$ then $Q_1 \subset Q_2 \subset \cdots$ and $Q_n \to A|_{X \setminus A}$ as $n \to \infty$.

Fix also a sequence $\{S_n\}_{n=1}^\infty$ of transformations $S_n \in$ Erg$^{III_1}(B_n, \mathcal{A} \cap B_n, \mu|B_n)$ that weakly converges to $T|_{A}$. By this we mean that $\|U_{S_n} f - U_f\|_2 \to 0$ as $n \to \infty$ for each $f \in L^2(A)$. We now can construct, for each $n > 0$, a nonsingular transformation $T_n$ of $X$ satisfying the following conditions:

- $T_n x = S_n x$ for all $x \in A$,
- $T_n p_{l,j}^{(n)} = \begin{cases} p_{l,j+1}^{(n)}, & \text{if } j \neq J_l^{(n)} \\ S_n p_{l,1}^{(n)}, & \text{if } j = J_l^{(n)} \end{cases}$
- $T'_n(x) = 1$ for each $x \notin \bigsqcup_{l=1}^\infty S_n p_{l,1}^{(n)}$.

It follows straightforwardly from the definition of $T_n$ that $B_n$ is $T_n$-sweeping out and $S_n$ is induced by $T_n$. Hence $T_n$ is ergodic of type III$_1$ for each $n \in \mathbb{N}$. Of course, $T_n \in$ Aut$_2(X, \mathcal{A}, \mu)$. We claim that $T_n \to T$ in $d_2$ as $n \to \infty$.

Take an atom $q$ of $Q_n$. Then $q = \bigsqcup_{l=1}^{J_l^{(n)}} p_{l,j}^{(n)}$ for some $l > 0$ and

$$\|U_T 1_q - U_{T_n} 1_q\|_2 = \left\| 1_{p_{l,1}^{(n)}} - U_{T_n} 1_{p_{l,j_l^{(n)}}} \right\|_2 \leq \|1_{p_{l,1}^{(n)}}\|_2 \leq \frac{2\|1_q\|_2}{\sqrt{J_l^{(n)}}}.$$  

Hence for each function $f \in L^2(X \setminus A)$ which is $Q_m$-measurable for some $m > 0$, we have that

$$\|U_T f - U_{T_n} f\|_2 \leq \frac{2\|f\|_2}{\sqrt{n}}$$

whenever $n \geq m$. Hence $U_{T_n} f \to U_T f$ as $n \to \infty$. Since $Q_n \to A|_{X \setminus A}$ as $n \to \infty$, it follows that $U_{T_n} f \to U_T f$ as $n \to \infty$ for each function $f \in L^2(X \setminus A)$. On the other hand, $U_{T_n} g = U_{S_n} g$ for each $g \in L^2(A)$ and $U_{S_n} g \to U_T g$ weakly as $n \to \infty$. We deduce that $T_n \to T$ weakly as $n \to \infty$.
Since $T'_n(x) \neq 1$ only if $x \in S_n A \cup \bigcup_{l=1}^{\infty} S_n p_{l,1}^{(n)} = S_n B_n = B_n$, we obtain that
\[
\|\sqrt{T^l} - \sqrt{T_n^l}\|_2 = \|((\sqrt{T^l} - \sqrt{T_n^l})1_{B_n})\|_2
\]
\[
= \|U_T 1_{B_n} - U_{T_n} 1_{B_n} + \sqrt{T_n^l}(1_{T_n} B_n - 1_{B_n})\|_2
\]
\[
\leq \|(U_T - U_{T_n}) 1_{B_n}\|_2 + \sqrt{\mu(B_n \triangle T_n^{-1} B_n)}
\]
and $\mu(B_n \triangle T_n^{-1} B_n) = \mu\left(\left(\bigcup_{l=1}^{\infty} p_{l,1}^{(n)}\right) \triangle \left(\bigcup_{l=1}^{\infty} p_{l,1}^{(n)}\right)\right) = 2\mu(B_n \setminus A) \to 0$ as $n \to \infty$. It follows that $\|\sqrt{T^l} - \sqrt{T_n^l}\|_2 \to 0$, as desired. \hfill $\Box$

**Remark 5.6.** Let $B$ be a subset of positive finite measure in the standard $\sigma$-finite non-atomic measure space $(X, \mathcal{A}, \mu)$. Arguing as in the proof of the above theorem, we see that given an ergodic nonsingular transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$, we can construct an ergodic transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that $S = T_B$. It is well known that given an ergodic nonsingular flow $W$ (i.e. an $\mathbb{R}$-action $(W(t))_{t \in \mathbb{R}}$), there is an ergodic nonsingular transformation whose associated flow is isomorphic to $W$. On the other hand, the associated flow of $T$ is isomorphic to the associated flow of each transformation induced by $T$. It follows from these facts that given an ergodic nonsingular flow $W$, there is $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that the associated flow of $T$ is $W$. In particular, for each $\lambda \in [0,1]$, the group $	ext{Aut}_2(X, \mathcal{A}, \mu)$ contains an ergodic transformation of Krieger’s type $\text{III}_\lambda$. For the definition of the associated flow, Krieger’s type and other concepts of orbit theory we refer to [16].

The following assertion is an analogue of Proposition 5.2 for $\text{Aut}_1(X, \mathcal{A}, \mu)$ furnished with $d_1$. It can not hold for the entire group $\text{Aut}_1(X, \mathcal{A}, \mu)$ because $\text{Aut}_0(X, \mathcal{A}, \mu)$ is a subgroup of the proper closed subgroup $\text{Ker} \chi$ of $\text{Aut}_1(X, \mathcal{A}, \mu)$. However, it holds for $\text{Ker} \chi$.

**Proposition 5.7.** $\text{Aut}_0(X, \mathcal{A}, \mu)$ is $d_1$-dense in $\text{Ker} \chi$.

*Proof.* Let $T \in \text{Ker} \chi$. Then there exists an increasing sequence $A_n \in \mathcal{A}$ of finite measure subsets satisfying $\bigcup_{n=1}^{\infty} A_n = X$,
\[
\int_{X \setminus A_n} \left| \frac{d\mu}{d\mu} \right| d\mu := \epsilon_n \xrightarrow{n \to \infty} 0.
\]
and
\[
|\mu(T^{-1} A_n) - \mu(A_n)| < \epsilon_n.
\]
Select subsets $B_n \subset X$ of finite measure such that $A_n \cup T^{-1} A_n \subset B_n$. As in the proof of Proposition 5.2, we define a transformation $T_n \in \text{Aut}_0(X, \mathcal{A}, \mu)$ by setting
\[
T_n^{-1} x := \begin{cases} T^{-1} x, & x \in A_n \\ \tau_{B_n \setminus A_n, B_n \setminus T^{-1} A_n} x, & x \in B_n \setminus A_n \\ x, & x \notin B_n. \end{cases}
\]
We will now show that \( T_n \) converges in \( d_1 \) to \( T \) as \( n \to \infty \). Since \( T_n^{-1}x = T^{-1}x \) for \( x \in \mathcal{A}_n \), it follows that \( T_n \) converges weakly to \( T \). Next, as in the proof of Proposition 5.2, we see that (5.1) holds and hence

\[
\left\| \frac{dm \circ T^{-1}}{d\mu} - \frac{d\mu \circ T_n^{-1}}{d\mu} \right\|_1 \leq \int_{X \setminus \mathcal{A}_n} \left| \frac{dm \circ T^{-1}}{d\mu} - 1 \right| d\mu \\
+ \int \left( \frac{\mu(B_n \setminus T^{-1}A_n)}{\mu(B_n \setminus A_n)} - 1 \right) d\mu \\
\leq \epsilon_n + \left| \mu(T^{-1}A_n) - \mu(A_n) \right| \leq 2\epsilon_n.
\]

\( \square \)

Proceeding from here in a similar way as we have done for \( \text{Aut}_2(X,\mathcal{A},\mu) \) but utilizing Proposition 5.7 instead of Proposition 5.2, we arrive at the following analogues of Corollary 5.3, Proposition 5.4 and Theorem 5.5 for \( \text{Ker} \chi \).

**Theorem 5.8.**

(1) The subset of periodic transformations from \( \text{Aut}_0(X,\mathcal{A},\mu) \) is a \( d_1 \)-dense subset of \( \text{Ker} \chi \).

(2) For each locally aperiodic transformation \( T \), the conjugacy class of \( T \) (in \( \text{Aut}_1(X,\mathcal{A},\mu) \)) is a \( d_1 \)-dense subset of \( \text{Ker} \chi \). In particular, \( \text{Ker} \chi \) has the Rokhlin property.

(3) The following three sets:

\[
\{ T \in \text{Aut}_1(X,\mathcal{A},\mu) : T \text{ is conservative} \}, \\
\{ T \in \text{Aut}_1(X,\mathcal{A},\mu) : T \text{ is ergodic} \} \text{ and} \\
\{ T \in \text{Aut}_1(X,\mathcal{A},\mu) : T \text{ is of type III}_1 \}
\]

are dense \( G_\beta \)-subsets of \( \text{Ker} \chi \) endowed with \( d_1 \).

### 6. Basic dynamical properties of Poisson suspensions for locally compact group actions

#### 6.1. Unitary and affine Koopman representations

Let \( G \) be a locally compact non-compact second countable group. A nonsingular \( G \)-action on a standard \( \sigma \)-finite measure space \((X,\mathcal{A},\mu)\) is a Borel map

\[
G \times X \ni (g,x) \mapsto T_gx \in X
\]

such that the mapping \( T_g : X \ni x \mapsto T_gx \in X \) is a \( \mu \)-nonsingular bijection of \( X \) for each \( g \in G \). Equivalently, a nonsingular \( G \)-action can be defined as a continuous group homomorphism \( T : G \ni g \mapsto T_g \in \text{Aut}(X,\mathcal{A},\mu) \), where the later group is furnished with the weak topology. The corresponding unitary Koopman representation \( G \ni g \mapsto U_{T_g} \in \mathcal{U}_\mathbb{R}(L^2(\mu)) \) of \( G \) is continuous in the weak (and strong) operator topology. Since the real Hilbert space \( L^2_{\mathbb{R}}(\mu) \) is invariant under the unitary Koopman representation, we can consider \( L^2_{\mathbb{R}}(\mu) \) as a \( G \)-module. Denote by \( Z^1(G,L^2_{\mathbb{R}}(\mu)) \) the vector space of all continuous
1-cocycles of $G$ in $L^2_{\mathbb{R}}(\mu)$, i.e. the mappings $c : G \ni g \mapsto c(g) \in L^2_{\mathbb{R}}(\mu)$ such that $c(gh) = c(g) + U_{T_g}c(h)$ for all $g, h \in G$. By $B^1(G, L^2_{\mathbb{R}}(\mu))$ we denote the subspace of 1-coboundaries, i.e. those 1-cocycles $c \in Z^1(G, L^2_{\mathbb{R}}(\mu))$ for which there is $f \in L^2_{\mathbb{R}}(\mu)$ such that $c(g) = U_{T_g}f - f$. We also recall that a 1-cocycle $c \in Z^1(G, L^2_{\mathbb{R}}(\mu))$ is called proper if $\|c(g)\|_2 \to \infty$ as $g \to \infty$.

Suppose now that we are given a nonsingular $G$-action $T$ such that $T_g \in \text{Aut}_2(X, A, \mu)$ for each $g \in G$. Since the $d_2$-topology is stronger than the weak topology, it follows that the restriction (to $\text{Aut}_2(X, A, \mu)$) of the Borel structure generated by the weak topology coincides with the Borel structure generated by $d_2$. Hence the map $T$ considered as a homomorphism from $G$ to $\text{Aut}_2(X, A, \mu)$ is $d_2$-Borel. Since each Borel homomorphism from a Polish group to another Polish group is continuous, we obtain that $T$ is continuous as a map from $G$ to $\text{Aut}_2(X, A, \mu)$ furnished with $d_2$. We recall that $\text{Aut}_2(X, A, \mu)$ embeds continuously into $\text{Aff}_R(L^2(\mu))$ via the affine Koopman representation $A^{(2)}$ (see Definition 4.5). Thus, we obtain the following proposition.

**Proposition 6.1.** If $T$ is a nonsingular $G$-action such that $T_g \in \text{Aut}_2(X, A, \mu)$ for each $g \in G$ then the 1-cocycle

$$c_T : G \ni g \mapsto c_T(g) := \sqrt{T'_g} - 1 \in L^2_{\mathbb{R}}(\mu)$$

of $G$ in $L^2_{\mathbb{R}}(\mu)$ is continuous.

Thus, $c_T \in Z^1(G, L^2_{\mathbb{R}}(\mu))$. It follows that under the condition of the above proposition, a (weakly) continuous affine representation $A_T : G \ni g \mapsto A_{T_g} \in \text{Aff}_R(L^2(\mu))$ of $G$ in $L^2(\mu)$ is well defined by the restriction of $A^{(2)}$ to $G$, i.e.

$$A_{T_g} h := U_{T_g}h + c_T(g).$$

**Definition 6.2.** We call $A_T$ the affine Koopman representation of $G$ generated by $T$.

In the next proposition we compare the property to have a fixed vector for the unitary and affine Koopman representations.

**Proposition 6.3.**

1. The unitary Koopman representation has a non-trivial fixed vector if and only if $T$ admits a non-trivial absolutely continuous invariant probability measure.

2. The affine Koopman representation has a fixed vector if and only if $T$ admits an absolutely continuous invariant measure belonging to $M^+_{\mu, 2}$.

**Proof.** (1) is standard. We leave its proof to the reader.

(2) If there is $h \in L^2(\mu)$ such that $A_g h = h$ for all $g \in G$ then

$$\sqrt{T'_g} (h \circ T^{-1}_g + 1) = h + 1.$$
We now define a measure $\nu$ which is absolutely continuous with respect to $\mu$ and such that $\frac{d\nu}{d\mu} := (h + 1)^2$. Then $\nu$ is invariant under $T$. Since $\sqrt{\frac{d\nu}{d\mu}} - 1 = |h + 1| - 1 \in L_2^+(\mu)$, it follows that $\nu \in M_{\mu, 2}^+$, as desired. The converse assertion is proved in a similar way by “reversing” the argument. □

6.2. Existence of an equivalent probability measure. We now examine when the Poisson suspension $T_* := \{(T_g)_*\}_{g \in G}$ of $T$ admits a $\mu^*$-absolutely continuous invariant probability measure.

**Proposition 6.4.** Let $T := \{T_g\}_{g \in G}$ be a nonsingular $G$-action such that $T_g \in \text{Aut}_2(X, A, \mu)$ for every $g \in G$. Then the following assertions are equivalent:

1. The 1-cocycle $c_T$ is bounded, i.e. there is $d > 0$ such that $\|c_T(g)\|_2 \leq d$ for each $g \in G$.
2. $c_T \in B^1(G, L_2^+(\mu))$.
3. There exists a $T$-invariant measure $\nu \in M_{\mu, 2}^+$ (hence the probability measure $\nu^*$ is $T_*$-invariant and $\nu^* \ll \mu^*$).
4. There exists a $T_*$-invariant probability measure $\rho \ll \mu^*$.

**Proof.** (1) $\iff$ (2) is classical, see [6, Proposition 2.2.9], for a proof.

(2) $\implies$ (3) because if $c_T(g) = U_{T_g}h - h$ for some $h \in L_2^+(\mu)$ and all $g \in G$ then $h$ is a fixed vector for the affine Koopman representation of $G$ generated by $T$. It remains to apply Proposition 6.3(2).

(3) $\implies$ (4) is obvious if we set $\rho := \nu^*$.

(4) $\implies$ (1) It follows from (4.5) that for each $g \in G$,

$$\sqrt{\mathcal{E}(T_g') - 1} = e^{-\frac{1}{2}\|c_T(g)\|^2_2} \mathcal{E}(\sqrt{\mathcal{T}_{g'} - 1}).$$

Integrating this equality and using Corollary 4.1 we obtain that

$$\langle U_{(T_g)_*}1_A, 1_B \rangle_{L^2(\mu^*)} = e^{-\frac{1}{2}\|c_T(g)\|^2_2}.$$

Therefore if $c_T$ were unbounded then there would exist a sequence $\{g_n\}_{n=1}^\infty$ of elements of $G$ such that $\langle U_{(T_g)_*}1_A, 1_B \rangle_{L^2(\mu^*)} \to 0$ as $n \to \infty$. Hence for all subsets $A, B$ of finite measure in $(X^*, \mu^*)$, we have that

$$\langle U_{(T_{g_n})_*}1_A, 1_B \rangle_{L^2(\mu^*)} \leq \langle U_{(T_g)_*}1_A, 1_B \rangle_{L^2(\mu^*)} \to 0 \quad \text{as } n \to \infty.$$

This implies, in turn, that $U_{(T_{g_n})_*}h \to 0$ for each $h \in L^2(X^*, \mu^*)$ in the weak topology. However, taking $h := \sqrt{\frac{d\rho}{d\mu^*}}$, we obtain that $U_{(T_{g_n})_*}h = h$, a contradiction. □

**Corollary 6.5.** Assume that $\mu$ is infinite and $T$ is ergodic. There exists a $T_*$-invariant probability measure $\rho \sim \mu^*$ if and only if there exists a $T$-invariant measure $\nu \in M_{\mu, 2}^+$. In this case $\rho = \nu^*$ and $T_*$ is ergodic (and weakly mixing).
Proof. The ergodicity of $T^*$ when $\nu^*$ is invariant follows from the lack of $T$-invariant set of non-zero and finite $\nu$-measure which is the classical ergodicity criteria in the measure preserving case. □

6.3. Conservativeness and zero type for Poisson $G$-actions. We first recall some standard definitions for locally compact group actions.

**Definition 6.6.** Let $T = \{T_g\}_{g \in G}$ be a nonsingular $G$-action on $(X, \mathcal{A}, \mu)$.

- $T$ is called **conservative** if for each subset $A$ of positive measure and a compact subgroup $K \subset G$, there is $g \in G \setminus K$ such that $\mu(A \cap T_gA) > 0$.
- $T$ is called **dissipative** if it is not conservative.
- $T$ is called **totally dissipative** if the restriction of $T$ to every $T$-invariant subset of $X$ is dissipative.
- $T$ is called **of zero type** if $\langle U(T_g)^* \rangle \to 0$ as $g \to \infty$ in the weak operator topology.

In case $G = \mathbb{Z}$, these definitions for conservativeness and dissipativeness are equivalent to those given above. We will need the following lemma from [2, Proposition A.34].

**Lemma 6.7.** Let $\lambda$ be a left Haar measure on $G$. If there is $s \in \mathbb{R}$ such that $\int_G (T_g)^s \, d\lambda(g) < \infty$ then $T$ is totally dissipative.

The next corollary follows from Lemma 6.7 and (6.1).

**Corollary 6.8.** If $T = \{T_g\}_{g \in G}$ is a nonsingular $G$-action such that $T_g \in \text{Aut}_2(X, \mu)$ for each $g \in G$ and

$$\int_G e^{-\frac{1}{2} \|c_T(g)\|_2^2} \, d\lambda(g) < \infty$$

then $T^*$ is totally dissipative.

Let $T_g \in \text{Aut}_2(X, \mu)$ for all $g \in G$ and let $A_T = \{A_{T_g}\}_{g \in G}$ stand for the affine Koopman representation of $G$ generated by $T$.

**Proposition 6.9.** $T^*$ is of zero type if and only if $c_T$ is proper.

**Proof.** We first note that $T^*$ is of zero type if and only if $\langle U(T_g) , 1, 1 \rangle \to 0$ as $g \to \infty$. This fact was shown (for arbitrary nonsingular $G$-actions) in the proof of Proposition 6.4. It remains to apply (6.1). □

A class of groups having property $(\text{BP}_0)$ was introduced in [18]. This class includes groups with property $(T)$, solvable groups (in particular, Abelian groups), connected Lie groups, linear algebraic groups over a local field of characteristic zero, etc. We need only the following fact: if $G$ has property $(\text{BP}_0)$ and $T$ is of zero type then $c_T$ is either proper or bounded [18]. This fact and Propositions 6.9 and 6.4 imply the next corollary.
Corollary 6.10. Let $G$ have property (BP). Let $T = \{T_g\}_{g \in G}$ be a non-singular $G$-action such that $T_g \in \text{Aut}_2(X, A, \mu)$ for every $g \in G$. Suppose that there is no any $T$-invariant measure in $M^+_{\mu, 2}$. If $T$ is of zero type then $T_*$ is of zero type.

7. Furstenberg entropy and Stationarity

7.1. Furstenberg entropy. Let $G$ be a locally compact group second countable group and let $\kappa$ be a generating probability measure on $G$ (i.e. the support of $\kappa$ generates a dense subgroup of $G$). The Furstenberg $\kappa$-entropy of a non-singular action $S = (S_g)_{g \in G}$ on a probability space $(Y, B, \nu)$ is the quantity
\[
h_\kappa(S, \nu) := -\int_G \left( \int_Y \log S_g' \, d\nu \right) d\kappa(g) \in [0, +\infty].
\]
We note that $h_\kappa(S, \nu) = 0$ if and only if $S$ preserves $\nu$. We will need the following corollary from Theorems 3.4 and 3.6.

Corollary 7.1. Let $T$ be in $\text{Aut}_2(X, A, \mu)$. We have that
\[
\mathbb{E}_\mu \left[ \log \frac{d\mu^* \circ T_*^{-1}}{d\mu^*} \right] = -\int_X (T' - 1 - \log T') \, d\mu,
\]

it is finite if and only if $\int_{\{x \in X : |\log T'(x)| > 1\}} |\log T'| d\mu < \infty$. In particular, if $T \in \text{Aut}_1(X, A, \mu)$, then $\int_X \log T' d\mu$ is well defined and takes its value in the extended interval $[-\infty, \chi(T)]$. We get in this case:
\[
\mathbb{E}_\mu \left[ \log (T') \right] = -\chi(T) + \int_X \log T' \, d\mu.
\]

Given $c > 0$, we denote by $\mu_c$ the $c$-scaling of $\mu$, i.e. $\mu_c(B) = c\mu(B)$ for each $B \in A$.

Corollary 7.2. Let $T$ be in $\text{Aut}_2(X, A, \mu)$. Then for each $c > 0$, the following formula holds:
\[
\mathbb{E}_{\mu_c} \left[ \log \frac{d\mu^* \circ T_*^{-1}}{d\mu^*} \right] = c \mathbb{E}_\mu \left[ \log \frac{d\mu^* \circ T_{c}^{-1}}{d\mu^*} \right].
\]

Proof. Since $\frac{d\mu_c \circ T_{c}^{-1}}{d\mu} = \frac{d\mu \circ T^{-1}}{d\mu}$, we deduce from Corollary 7.1 that
\[
\mathbb{E}_{\mu_c} \left[ \log \frac{d\mu^* \circ T_*^{-1}}{d\mu^*} \right] = -c \int_X \left( \frac{d\mu \circ T^{-1}}{d\mu} - 1 - \log \frac{d\mu \circ T^{-1}}{d\mu} \right) \, d\mu
= c \mathbb{E}_\mu \left[ \log \frac{d\mu^* \circ T_{c}^{-1}}{d\mu^*} \right].
\]

□

In view of the above results, we obtain the following.
**Proposition 7.3.** Let $T = (T_g)_{g \in G}$ be a non-singular $G$-action on $(X, \mathcal{A}, \mu)$ such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for each $g \in G$. Then:

$$h_\kappa(T_*, \mu^*) = - \int_{G} \left( \int_{X} \left( \log T'_g - T'_g + 1 \right) d\mu \right) d\kappa(g).$$

Moreover, $h_\kappa(T_*, \mu^*) = c h_\kappa(T_*, \mu^*)$, for any $c > 0$. If $T_g \in \text{Aut}_1(X, \mathcal{A}, \mu)$ for each $g \in G$ then

$$h_\kappa(T_*, \mu^*) = \int_{G} \left( \chi(T_g) - \int_{X} \log T'_g d\mu \right) d\kappa(g).$$

**Corollary 7.4.** Let $T = (T_g)_{g \in G}$ be a non-singular $G$-action on $(X, \mathcal{A}, \mu)$ such that $T_g \in \text{Aut}_1(X, \mathcal{A}, \mu)$ for each $g \in G$. If one of the following conditions is satisfied:

1. $T_g$ is conservative for each $g \in G$,
2. $\kappa$ is symmetric,

then we have

$$h_\kappa(T_*, \mu^*) = - \int_{G} \left( \int_{X} \log T'_g d\mu \right) d\kappa(g).$$

**Proof.** The first point follows from Proposition 4.13. For the second, take an increasing sequence of compact symmetric sets $\{A_n\}_{n \in \mathbb{N}}$ such that $G = \bigcup_{n \in \mathbb{N}} A_n$. Then

$$\int_{G} \left( \chi(T_g) - \int_{X} \log T'_g d\mu \right) d\kappa(g) = \lim_{n \to \infty} \int_{A_n} \left( \chi(T_g) - \int_{X} \log T'_g d\mu \right) d\kappa(g).$$

Since $G \ni g \mapsto \chi(T_g) \in \mathbb{R}$ is a continuous group homomorphism, it is integrable on $A_n$. Moreover, $\chi(T_g^{-1}) = -\chi(T_g)$ and hence, by the symmetry of $\kappa$ and $A_n$, we obtain that $\int_{A_n} \chi(T_g)d\kappa(g) = 0$. The result follows. \qed

**7.2. Stationarity for Poisson suspensions.** Let $\kappa$ be a generating probability measure on $G$. We recall that a non-singular action $(S_g)_{g \in G}$ on a probability space $(Y, \mathcal{B}, \nu)$ is called $\kappa$-stationary if $\int_{G} \nu \circ S_g d\kappa(g) = \nu$.

**Proposition 7.5.** Let $T = (T_g)_{g \in G}$ be a non-singular $G$-action on $(X, \mathcal{A}, \mu)$ such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for each $g \in G$. The Poisson suspension $T_* = ((T_g)_*)_{g \in G}$ of $T$ acting on the space $(X^*, \mathcal{A}^*, \mu^*)$ is stationary if and only if $T_*$ preserves $\mu^*$.

**Proof.** Assume that $T_*$ is $\kappa$-stationary. Then

$$\mu^* = \int_{G} \mu^* \circ (T_g)_* d\kappa(g) = \int_{G} (\mu \circ T_g)^* d\kappa(g).$$

Computing the intensity of the two sides of this equation we obtain that

$$\mu = \int_{G} \mu \circ T_g d\kappa(g).$$

(7.1)
Take a set $A \in \mathcal{A}$ with $0 < \mu(A) < +\infty$ and compute the void probabilities of $A$ using $\mu^*$ and $\int_G (\mu \circ T_g)^* d\kappa(g)$. We obtain that

$$e^{-\mu(A)} = \int_G e^{-\mu \circ T_g(A)} d\kappa(g).$$

From this and the Jensen inequality we deduce that

$$\mu(A) = -\log \int_G e^{-\mu \circ T_g(A)} d\kappa(g) \leq \int_G \mu \circ T_g(A) d\kappa(g).$$

In view of (7.1) we see that the equality case takes place in the Jensen inequality. This happens if and only if $\mu(A) = \mu(T_g A)$ for $\kappa$-a.e. $g \in G$. Thus $\mu = \mu \circ T_g$ for $\kappa$-a.e. $g \in G$. Since $\kappa$ is generating, $\mu$ is invariant under $T_g$ for $g$ belonging to a dense subgroup in $G$. Therefore $\mu = \mu \circ T_g$ for every $g \in G$. \hfill $\square$

8. Property (T), Poisson Suspensions and Furstenberg entropy

Let $G$ be a locally compact second countable group. The group has Kazhdan property (T) if every unitary representation of $G$ which has almost invariant vectors admits a non zero invariant vector. There are numerous equivalent characterizations of property (T), among them is that for every unitary representation of $G$ every cocycle is a coboundary [6]. Theorem 8.2 is a new characterization of property (T) in terms of nonsingular Poisson suspensions.

**Definition 8.1.** A nonsingular action $R = \{R_g\}_{g \in G}$ of $G$ is called Poisson if there is a $\sigma$-finite standard measure space $(X, \mathcal{A}, \mu)$ and a nonsingular action $T = \{T_g\}_{g \in G}$ on $X$ such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ and the Poisson suspension $T^*_g = \{(T_g)^*_g\}_{g \in G}$ of $T$ is isomorphic to $R$.

We establish the following characterization of property (T) in terms of nonsingular Poisson suspensions.

**Theorem 8.2.** $G$ has property (T) if and only if each nonsingular Poisson $G$-action admits an absolutely continuous invariant probability measure.

**Proof.** ($\Rightarrow$) Let $G$ has property (T) and let $R = \{R_g\}_{g \in G}$ be a nonsingular Poisson $G$-action. Then there is a $\sigma$-finite standard measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = \infty$ and a $G$-action $T: G \ni g \mapsto T_g \in \text{Aut}_2(X, \mu)$ such that the Poisson suspension $T^*_g = \{(T_g)^*_g\}_{g \in G}$ of $T$ is isomorphic to $R$. By the Delorme-Guichardet theorem (see [6]), $c_T$ is a coboundary. Hence there is $f \in L^2(X, \mu)$ such that

$$c_T(g) = f \circ T_g^{-1} \cdot \sqrt{T_g^* - f}.$$  \hfill (8.1)

We define a measure $\nu$ on $X$ by setting $\frac{d\nu}{d\mu} = (1 - f)^2$. Of course, $\nu(X) = \infty$ and $\nu \in \mathcal{M}_{\mu,2}^+$. The equation (8.1) yields that $\nu$ is invariant under $T_g$. $\nu \in \mathcal{M}_{\mu,2}^+$. Hence $\nu^*$ is a $T_g$-invariant probability absolutely continuous with respect to $\mu^*$.
Suppose now that $G$ does not have property (T). Then there is a weakly mixing measure preserving action $S = \{S_g\}_{g \in G}$ on a standard probability space $(X, \mathcal{B}, m)$ which is not strongly ergodic. The latter implies that there is a sequence $\{A_n\}_{n=1}^\infty$ of measurable subsets in $X$ such that $m(A_n) = \frac{1}{2}$ for every $n \in \mathbb{N}$ and for each compact subset $K \subset G$,

$$\lim_{n \to \infty} \sup_{g \in K} m(A_n \triangle S_g A_n) = 0.$$  

Fix an increasing sequence $\{K_n\}_{n=1}^\infty$ of symmetric compact subsets of $G$ with $1 \in K_1$ and $G = \bigcup_{n=1}^\infty K_n$. By passing to a subsequence in $\{A_n\}_{n=1}^\infty$, we can assume without loss of generality that for all $n \in \mathbb{N}$ and $g \in K_n$,

$$m(A_n \triangle S_g A_n) \leq 2^{-n}. \tag{8.2}$$

Let $(\Omega, \mathcal{C}, \mathbb{P}) := (X^\mathbb{N}, \mathcal{B}^\mathbb{N}, m^\mathbb{N})$. We will consider the diagonal (measure preserving) action $T: G \ni g \mapsto T_g := S_g \times S_g \times \cdots \in \text{Aut}(\Omega, \mathcal{C}, \mathbb{P})$ of $G$ on $\Omega$. Since $S$ is weakly mixing, so is $T$. For each $n \in \mathbb{N}$, we let

$$B_n := \left\{ x = (x_j)_{j=1}^\infty \in \Omega : \forall j \in \left[ \frac{n(n-1)}{2}, \frac{n(n+1)}{2} \right], x_j \in A_n \right\}.$$  

Of course, $B_n \in \mathcal{C}$ and $\mathbb{P}(B_n) = 2^{-n}$. Now we define a function $F: \Omega \to [1, +\infty)$ by setting

$$F(x) = \sqrt{1 + \sum_{n=1}^\infty 2^n 1_{B_n}(x)}.$$  

Since $\sum_{n=1}^\infty \mathbb{P}(B_n) < \infty$, it follows from the Borel-Cantelli Lemma that $F(x) < \infty$ at a.e. $x \in \Omega$. We note that $F \notin L^2(\mathbb{P})$.

**Claim.** We claim that $\|F^2 - F^2 \circ T_g\|_1 < \infty$ for each $g \in G$. To prove this inequality, we choose $N \in \mathbb{N}$ such that $g^{-1} \in K_n$ for all $n \geq N$. Then we have in view of (8.2):

$$\|1_{B_n} - 1_{B_n} \circ T_g\|_1 = \mathbb{P}(B_n \triangle T_g^{-1} B_n)$$

$$= \prod_{k = \frac{n(n-1)}{2}}^{n(n+1)-1} m(A_n \triangle S_g A_n) \leq 2^{-n^2} \tag{8.3}$$

for all $n \geq N$. Hence

$$\|F^2 - F^2 \circ T_g\|_1 \leq \sum_{n=1}^{\infty} 2^n \|1_{B_n} - 1_{B_n} \circ T_g\|_1$$

$$\leq \sum_{n=1}^{N-1} 2^n + \sum_{n=N}^{\infty} 2^n 2^{-n^2} < \infty,$$

as claimed.
Hence we define a new measure $\mu$ on $(\Omega, C)$ by setting: $\mu \sim P$ and $\int \frac{d\mu}{d\nu} := F^2$. Then $\mu$ is $\sigma$-finite and $\mu(X) = \infty$. For each $g \in G$,

$$\int_X \left| \frac{d\mu \circ T_g}{d\mu} - 1 \right| d\mu = \int_X \left| F^2 \circ T_g - F^2 \right| d\mathbb{P} < \infty \quad (8.4)$$

in view of Claim. Thus, $T_g \in \text{Aut}_1(\Omega, C, \mu)$. Hence the nonsingular Poisson $G$-action $T_\ast = \{(T_g)_g \}_g \in G$ is well defined on $(\Omega^*, C^*, \mu^*)$. If $T_\ast$ had an invariant $\mu^*$-absolutely continuous probability measure then by Proposition 6.4 there would exist an infinite $T$-invariant measure $\nu \in \mathcal{M}_{\mu^*}^+$, this contradicts to the fact that $T$ is ergodic and has an equivalent invariant probability measure.

$$\square$$

We are interested now in the computation of the Furstenberg entropy of Poisson $G$-actions. For that, we first prove the following proposition.

**Proposition 8.3.** Let $(\Omega, \mu, T)$ be as in the proof of Theorem 8.2. Then $\log T_g' \in L^1(\Omega, \mathbb{P})$ for each $g \in G$.

**Proof.** We first let $B_0 := \Omega$ and define a sequence $\{E_n\}_{n=0}^\infty$ of subsets of $\Omega$ by setting $E_n := B_n \setminus \left( \bigcup_{k=n+1}^\infty B_k \right)$. It follows from the definition of $F^2$ that

$$E_n = \{ x \in \Omega : 2^n < F(x)^2 \leq 2^{n+1} \}.$$ 

Hence $E_0, E_1, \ldots$ form a countable partition of $\Omega$. Fix $g \in G$ and note that

$$\int_{\Omega} \log \left| \frac{d\mu \circ T_g}{d\mu} \right| d\mu = \sum_{n=0}^\infty \int_{E_n} F^2 \log \left( \frac{F^2 \circ T_g}{F^2} \right) d\mathbb{P}.$$ 

We will show that the right hand side of this equality is finite. If for some $n \in \mathbb{N}$ and $x \in \Omega$, we have that $x \in D_{n,g} := E_n \cap T_g^{-1}(\bigcup_{k=n}^\infty E_k)$ then $F(T_gx)^2 > 2^n \geq F(x)^2 / 2$. Since $|\log y| \leq 2|y - 1|$ for each $y > 1 / 2$, we see that

$$\sum_{n=0}^\infty \int_{D_{n,g}} F^2 \log \left( \frac{F^2 \circ T_g}{F^2} \right) d\mathbb{P} \leq \sum_{n=0}^\infty 2 \int_{D_{n,g}} \left| F^2 \circ T_g - F^2 \right| d\mathbb{P}$$

$$\leq 2 \sum_{n=0}^\infty \int_{E_n} \left| F^2 \circ T_g - F^2 \right| d\mathbb{P}$$

$$= 2 \left\| F^2 - F^2 \circ T_g \right\|_1 < \infty$$

in view of (8.4). Now if $x \in E_n \setminus T_g^{-1}(\bigcup_{k=n}^\infty E_k)$ then

$$F^2(x) \log \left( \frac{F(T_gx)^2}{F(x)^2} \right) \leq 2^{n+1} \log \left( 2^{-(n+1)} \right).$$

As $E_n \subset B_n$ and $\bigcup_{k=n}^\infty E_k \supset B_n$, we obtain that $E_n \setminus T_g^{-1}(\bigcup_{k=n}^\infty E_k) \subset B_n \Delta T_g^{-1}B_n$. Let $N$ be the smallest natural number such that $g^{-1} \in K_n$ for
all $n \geq N$ then,
\[
\sum_{n=0}^{\infty} \int_{E_n \setminus D_{n,g}} F^2 \left| \log \left( \frac{F^2 \circ T_g^\ast}{F^2} \right) \right| \, dp \leq \sum_{n=0}^{\infty} (n+1)2^{n+1} \int_{B_n \Delta T_g^{-1}B_n} \, dp \leq \sum_{n=1}^{N-1} (n+1)2^{n+1} + \sum_{n=N}^{\infty} (n+1)2^{n+1}2^{-n^2},
\]
where the latter bound follows from (8.3). We conclude that
\[
\int_\Omega \left| \log \frac{d\mu \circ T_g}{d\mu} \right| \, d\mu < \infty,
\]
as desired.  

We can now prove the following result.

**Theorem 8.4.** Let $G$ do not have property (T) and let $\kappa$ be a probability measure on $G$. Then there is a nonsingular $G$-action $T$ on an infinite measure space $(\Omega,\mu)$ such that the Poisson suspension $T^\ast$ of $T$ is $\mu^\ast$-nonsingular and \{(h_\kappa(T^\ast,\mu^\ast) : t \in (0, +\infty)) \in (0, +\infty).

**Proof.** For each $N > 0$, we let
\[
C_N := \sum_{n=1}^{N-1} (n+3)2^{n+1} + \sum_{n=N}^{\infty} (n+2)2^{n+1}2^{-n^2}.
\]
Then we choose an increasing sequence $(K_n)_{n=1}^{\infty}$ of compacts in $G$ such that $\bigcup_{n=1}^{\infty} K_n = G$ and $\kappa(K_n^{-1}) > 1 - \frac{1}{2^{\gamma}C_n}$ for each $n$. By $K_n^{-1}$ we mean the subset $\{k^{-1} : k \in K_n\}$. Using this sequence, we construct the dynamical system $(\Omega,\mu,T)$ exactly as in the proof of Theorem 8.2. It follows from the proof of Proposition 8.3 that
\[
\sup_{g^{-1} \in K_N} \left| \int_\Omega \log \frac{d\mu \circ T_g}{d\mu} \, d\mu \right| < C_N
\]
for each $N > 0$. Since $T^\ast$ preserves a probability measure equivalent to $\mu$, it follows that $T^\ast$ is conservative for each $g \in G$. Therefore, by Proposition 7.3,
\[
h_\kappa(T^\ast,\mu^\ast) = \sum_{N=1}^{\infty} \int_{K_n^{-1} \setminus K_{n-1}^{-1}} \left( - \int_\Omega \log \frac{d\mu \circ T_g}{d\mu} \, d\mu \right) \, d\kappa(g)
\]
\[
\leq \sum_{N=1}^{\infty} C_N \kappa(K_n^{-1} \setminus K_{n-1}^{-1})
\]
\[
\leq \sum_{N=1}^{\infty} \frac{1}{2^{N}} = 1.
\]
By Corollary 7.4,
\[
\{h_\kappa(T^\ast,\mu^\ast) : t \in (0, +\infty)\} = \{th_\kappa(T^\ast,\mu^\ast) : 0 < t < +\infty\} = (0, +\infty).
\]
8.1. **A remark on ergodicity of $T_*$**. It is well known that if a transformation $T$ preserves $\mu$ then $T_*$ is ergodic if and only if there are no $T$-invariant sets of strictly positive finite measure. This is no longer true for general $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$. Moreover, using the techniques developed in this section, we show in the following example that even when $T$ is ergodic, $T_*$ can be non-ergodic.

**Proposition 8.5.** There exists an ergodic transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that $T_*$ is totally dissipative.

**Proof.** Fix a standard probability space $(X, \mathcal{B}, m)$ and select a mixing $m$-preserving transformation $S$ of $X$. Let $(\Omega, \mathcal{C}, \mathbb{P}) := (X^N, \mathcal{B}^N, m^N)$ and let $T := S \times S \times \cdots$. Then $T$ is an ergodic $\mathbb{P}$-preserving transformation of $\Omega$.

Suppose that we have a function $F : \Omega \rightarrow [0, +\infty)$ such that

- $F \notin L^2(\mathbb{P})$,
- $F - F \circ T^n \in L^2(\mathbb{P})$ for all $n \in \mathbb{Z}$ and
- $\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}\|F - F \circ T^n\|^2_2} < \infty$.

Then, as in the proof of Theorem 8.2, we define a new measure $\mu$ on $(\Omega, \mathcal{C})$ by setting $\mu \sim \mathbb{P}$ and $d\mathbb{P} := F^2$ to obtain that $T \in \text{Aut}_2(X^N, \mathcal{B}^N, \mu)$. Moreover,

$$\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}\|\sqrt{(T^n)^{-1}}\|^2_2} = \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}\|F - F \circ T^n\|^2_2} < \infty.$$ 

Hence $T_*$ is dissipative by Corollary 6.8 and the proposition is proved. Thus, it remains to build such an $F$. For that, we first apply the Rokhlin lemma to find a sequence $(A_n)_{n \in \mathbb{N}}$ of measurable subsets of $X$ such that $m(A_n) = \frac{1}{2}$ and

$$m(A_n \triangle S^j A_n) < 2^{-n} \text{ whenever } |j| \leq n.$$ 

Then we construct inductively an increasing sequence $(l_n)_{n=1}^\infty$ of positive integers. Let $l_1 := 1$. Suppose that we have already defined $l_n$ for some $n$. Since $S$ is mixing, there exists $l_{n+1} > l_n$ such that if $|j| > l_{n+1}$ then

$$(1)_{A_{l_n} \triangle S^j A_{l_n}} \geq 2 \left( m(A_{l_n}) - m(A_{l_n} \cap S^j A_{l_n}) \right) = \frac{1}{4}.$$

In this way we define $(l_n)_{n=1}^\infty$ entirely. For every $k > 1$, we now let

$$B_k := \left\{ w = (x_j)_{j=1}^\infty \in \Omega : x_j \in A_{l_{k-1}}, \text{ if } \sum_{i=1}^k l_i < j \leq \sum_{i=1}^{k+1} l_i \right\}.$$ 

Note that $\mathbb{P}(B_k) = m(A_{l_{k-1}}) l_{k+1} = 2^{-l_{k+1}}$. If $l_{k-1} > |j|$ then in view of (8.5),

$$(8.7) \quad \mathbb{P}(B_k \triangle T^j B_k) = m(A_{l_{k-1}} \triangle S^j A_{l_{k-1}}) l_{k+1} \leq 2^{-l_{k-1} - l_{k+1}}.$$

Since $\sum_{k=1}^\infty \mathbb{P}(B_k) < \infty$, it follows from the Borel-Cantelli lemma that the function $F := 1 + \sum_{k=1}^\infty 2^{l_{k+1}} l_{k+1} 1_{B_k}$ takes values in $[1, +\infty)$ almost everywhere. It is straightforward to verify that $\int_{\Omega} F \, d\mathbb{P} = +\infty$. Hence
$F \notin L^2(\mathbb{P})$. For each $n \in \mathbb{Z}$, we have that $F - F \circ T^n = \sum_{k=1}^{\infty} \Upsilon_{n,k}$, where 
$\Upsilon_{n,k} := 2l_{k+1}^2 T_{k+1} (1_{B_k} - 1_{B_k} \circ T^n)$. Since \{$\Upsilon_{n,k}\}_{k=1}^{\infty}$ is an independent sequence of centered (mean 0) random variables,

$$\|F - F \circ T^n\|_2^2 = \sum_{k=1}^{\infty} \|\Upsilon_{n,k}\|_2^2.$$ 

If $l_{k-1} > |n|$ then in view of (8.7),

$$\|\Upsilon_{n,k}\|_2^2 = 2^{2l_{k+1}} \mathbb{P}(B_k \triangle T^{-n} B_k) \leq 2^{2l_{k+1}} \ell_{k+1}^{-l_{k-1}l_{k+1}}.$$ 

As $l_k \geq 4$ for all $k > 3$ then $2^{2l_{k+1}} \ell_{k+1}^{-l_{k-1}l_{k+1}} = o(2^{-l_{k+1}})$. This implies that $\sum_{l_{k-1} > |n|} \|\Upsilon_{n,k}\|_2^2 < \infty$ and hence $F - F \circ T^n \in L^2(\mathbb{P})$.

On the other hand, if $l_k < |n| \leq l_{k+1}$ then

$$\|F - F \circ T^n\|_2^2 \geq \|\Upsilon_{n,k}\|_2^2 = 4^{l_{k+1}} \ell_{k+1}^2 \left\|1_{A_{l_{k-1}}} - 1_{A_{l_{k-1}}} \circ \mathcal{S}^n\right\|_{L^2(\mathbb{P})} \geq \ell_{k+1}^2$$

in view of (8.6). Hence

\[
\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2} \|F - F \circ T^n\|_2^2} \leq 3 + \sum_{k=1}^{\infty} \sum_{l_k < |n| \leq l_{k+1}} e^{-\frac{1}{2} \|F - F \circ T^n\|_2^2} \\
\leq 3 + \sum_{k=1}^{\infty} 2(l_{k+1} - l_k) e^{-l_{k+1}/2} < \infty,
\]

as desired. \hfill \Box

9. Appendix

9.1. Infinitely divisible random variables. We recall that a real valued random variable $X$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is infinitely divisible if the distribution $p$ of $X$ satisfies that, for each $k \geq 1$, there exists a probability distribution $p_k$ on $\mathbb{R}$ such that $p$ is the $k$-th convolution power of $p_k$. Equivalently, there exists $\delta, \kappa \geq 0$ and a $\sigma$-finite Borel measure $\sigma$ on $\mathbb{R}$ satisfying the following conditions: $\sigma(\{0\}) = 0$, $\int_{\mathbb{R}} x^2 \wedge 1 d\sigma(x) < \infty$ and such that

$$\mathbb{E}[e^{iaX}] = \exp \left(-\frac{\alpha^2 \kappa^2}{2} + ia\delta + \int_{\mathbb{R}} (e^{i\alpha x} - 1 - i\alpha x 1_{\{|y| \leq 1\}}(x)) d\sigma(x) \right)$$

The measure $\sigma$ is called the Lévy measure of $X$. Conversely, each $\sigma$-finite measure $\sigma$ such that $\sigma(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 \wedge 1 d\sigma < \infty(x)$ is the Lévy measure of some infinitely divisible random variable. For the proof of the following proposition we refer to [30, page 39, below formula (8.8)].
Proposition 9.1. Let $X$ be infinitely divisible. Then $X \in L^1(\mathbb{P})$ if and only if $\int_{\mathbb{R}} |x| 1_{\{y \in \mathbb{R}: |y| > 1\}}(x) \, d\sigma(x) < \infty$. In this case the characteristic function of $X$ can be written as

$$
\mathbb{E}[e^{iaX}] = \exp\left(-\frac{a^2 \kappa^2}{2} + ia\gamma + \int_{\mathbb{R}} (e^{iax} - 1 - iax) \, d\sigma(x)\right), \quad a \in \mathbb{R},
$$

and $\mathbb{E}[X] = \gamma$.

9.2. Stochastic integrals against a Poisson measure. Let $(X^*, \mathcal{A}^*, \mu^*)$ be the Poisson space over a base $(X, \mathcal{A}, \mu)$. If $f : X \to \mathbb{R}$ is a measurable function satisfying $\int_X f^2 \wedge 1 \, d\mu < \infty$ then a so-called stochastic integral $I_\mu(f) : X^* \to \mathbb{R}$ is well defined (see [26], up to a slight change) by the following formula

$$
I_\mu(f)(\omega) = \lim_{\epsilon \to 0} \left( \int_{\{x \in X : |f(x)| > \epsilon\}} f \, d\omega - \int_{\{x \in X : |f(x)| \leq \epsilon\}} f 1_{\{x \in X : |f(x)| \leq 1\}} \, d\mu \right),
$$

where the limit means the convergence in probability. It appears that the random variable $I_\mu(f)$ is infinitely divisible. The Lévy measure of $I_\mu(f)$ is $(\mu \circ f^{-1}) | \mathbb{R} \setminus \{0\}$. The characteristic function of $I_\mu(f)$ is

$$
(9.1) \quad \mathbb{E}_{\mu^*}[e^{iaI_\mu(f)}] = \exp\left(\int_X (e^{iaf} - 1 - iaf) 1_{\{x \in X : |f(x)| \leq 1\}} \, d\mu\right),
$$

for each $a \in \mathbb{R}$.

Proposition 9.2. Let a function $f : X \to \mathbb{R}$ satisfy $\int_X f^2 \wedge 1 \, d\mu < \infty$. Then $I_\mu(f) \in L^1(\mu^*)$ if and only if $f 1_{\{x \in X : |f(x)| > 1\}} \in L^1(\mu)$. In this case $\mathbb{E}_{\mu^*}(I_\mu(f)) = \int_{\{x \in X : |f(x)| > 1\}} f \, d\mu$.

Proof. The integrability criteria for $I_\mu(f)$ follows from Proposition 9.1 since $(\mu \circ f^{-1}) | \mathbb{R} \setminus \{0\}$ is the Lévy measure of $I_\mu(f)$. If the latter happens, we can rewrite (9.1) as

$$
\mathbb{E}_{\mu^*}[e^{iaI(f)}] = \exp\left(ia \int_{\{x \in X : |f(x)| > 1\}} f \, d\mu + \int_X (e^{iaf} - 1 - iaf) \, d\mu\right).
$$

By Proposition 9.1, $\mathbb{E}_{\mu^*}(I_\mu(f)) = \int_{\{x \in X : |f(x)| > 1\}} f \, d\mu$. \hfill \square

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