No New Quantum Thermal Effect of Dirac Particles
in a Charged Vaidya - de Sitter Black Hole

S. Q. Wu* and X. Cai†

Institute of Particle Physics, Hua-Zhong Normal University, Wuhan 430079, China
(November 10, 2018)

ABSTRACT

It is shown that Hawking radiation of Dirac particles does not exist for $P_1, Q_2$ components but for $P_2, Q_1$ components in a charged Vaidya - de Sitter black hole. Both the location and the temperature of the event horizon change with time. The thermal radiation spectrum of Dirac particles is the same as that of Klein-Gordon particles. Our result demonstrates that there is no new quantum effect in the thermal radiation of Dirac particles in any spherically symmetry black holes.

Key words: Hawking effect, Dirac equation, evaporating black hole
PACS numbers: 97.60.Lf, 04.70.Dy

*E-mail: sqwu@iopp.ccnu.edu.cn
†E-mail: xcai@ccnu.edu.cn
I. INTRODUCTION

In the last decades, the Hawking radiation [1] of Dirac particles had been largely investigated in some spherically symmetric and non-static black holes [2]. However, most of these studies concentrated on the spin state \( p = 1/2 \) of the four-component Dirac spinors. Recently, the Hawking radiation of Dirac particles of spin state \( p = -1/2 \) attracted a little more attention [3]. In a series of papers, Li et al. [3,4] claimed that they had discovered a kind of new quantum effect for the Vaidya-Bonner-de Sitter black hole. On the base of the generalized Teukolsky-type master equation [5] for fields of spin (\( s = 0, 1/2, 1 \) and 2 for the scalar, Dirac, electromagnetic and gravitational field, respectively) in the Vaidya-type space-times [3,4], they showed that this effect is depicted by

\[
\omega_0 = \frac{2(1 - s - p)\mu_0 \lambda r_H}{\lambda^2 + (2\mu_0 r_H)^2}, \quad (p = \pm s)
\]

\[
= \begin{cases} 
-\frac{(\ell+1/2)\mu_0 \omega_H^2}{(\ell+1/2)^2 + (\mu_0 r_H)^2}, & (\mu_0 \neq 0, s = -p = 1/2), \\
0, & (\text{otherwise}) \).
\end{cases}
\]

(1)

where \( \mu_0 \) is the mass of fields with spin-s.

Among their master equations, as the mass of Klein-Gordon particle and that of Dirac particle are nonzero, so their argument sounds only for the massive spin-1/2 particles. They found that the massive Dirac field of spin state \( p = 1/2 \) differs greatly from that of spin state \( p = -1/2 \) in radiative mechanism, and suggested that it originate from the variance of Dirac vacuum near the event horizon in the non-static space-times caused by spin state. Further, they conjectured that this effect originates from the quantum ergosphere [6], that is, the quantum ergosphere can influence the radiative mechanism of a black hole.

In this paper, we re-investigate the Hawking effect of Dirac particles in the Vaidya-type black hole by means of the generalized tortoise transformation (GTCT) method. We consider simultaneously the limiting forms of the first order form and the second order form of Dirac equation near the event horizon because the Dirac spinors should satisfy both of them. From the former, we can obtain the event horizon equation, while from the latter, we can derive the Hawking temperature and the thermal radiation spectrum of electrons. Our results are in accord with others. With our new method, we can prove rigorously that the Hawking radiation takes place only for \( P_2, Q_1 \) but not for \( P_1, Q_2 \) components of Dirac spinors. The origin of this asymmetry of the Hawking radiation of different spinorial component maybe stem from the asymmetry of space-time in the advanced Eddington-Finkelstein coordinate.
system. As a by-product, we show that there could not have new quantum effect in the Hawking radiation of Dirac particles in any spherically symmetric black hole whether it is static or non-static. Our conclusion is contrary completely to that of Li’s [3,4] who argued that the radiative mechanism of massive spin fields depends on the spin state.

The paper is organized as follows: In section 2, we write out the spinor form of Dirac equation in the Vaidya-type black hole, then, we obtain the event horizon equation in Sec. 3. The Hawking temperature and the thermal radiation spectrum are derived in Sec. 4 and 5, respectively. Finally we present some discussions.

II. DIRAC EQUATION

The metric of a charged Vaidya - de Sitter black hole with the cosmological constant $\Lambda$ is given in the advanced Eddington-Finkelstein coordinate system by

$$ds^2 = 2dv(Gdv - dr) - r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$  \hfill (2)

and the electro-magnetic one-form is

$$A = \frac{Q}{r}dv$$  \hfill (3)

where $2G = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^4$, in which both mass $M(v)$ and electric charge $Q(v)$ of the hole are functions of the advanced time $v$.

We choose such a complex null-tetrad $\{l, n, m, \overline{m}\}$ that satisfies the orthogonal conditions $l \cdot n = -m \cdot \overline{m} = 1$. Thus the covariant one-forms can be written as

$$l = dv, \quad m = \frac{r}{\sqrt{2}} (d\theta + i \sin \theta d\varphi),$$

$$n = Gdv - dr, \quad \overline{m} = \frac{r}{\sqrt{2}} (d\theta - i \sin \theta d\varphi).$$  \hfill (4)

and their corresponding directional derivatives are

$$D = -\frac{\partial}{\partial r}, \quad \delta = \frac{1}{\sqrt{2r}} \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right),$$

$$\Delta = \frac{\partial}{\partial \varphi} + G \frac{\partial}{\partial r}, \quad \overline{\delta} = \frac{1}{\sqrt{2r}} \left( \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right).$$  \hfill (5)

The non-vanishing Newman-Penrose complex coefficients [7] in the above null-tetrad are easily obtained as follows

$$\mu = \frac{G}{r}, \quad \gamma = -\frac{G_r}{2} = -\frac{dG}{2dr}, \quad \beta = -\alpha = \frac{\cot \theta}{2\sqrt{2r}}.$$  \hfill (6)
Inserting for the following relations among the Newman-Penrose spin-coefficients

\[
\begin{align*}
\epsilon - \rho &= -\frac{1}{r}, \\
\tilde{\pi} - \alpha &= \cot \frac{\theta}{2\sqrt{2r}}, \\
\mu - \gamma &= \frac{G}{r} + \frac{G_r}{2}, \\
\beta - \tau &= \cot \frac{\theta}{2\sqrt{2r}},
\end{align*}
\]

and the electro-magnetic potential

\[
A \cdot \ell = 0, \quad A \cdot n = Q/r, \quad A \cdot m = -A \cdot m = 0,
\]

into the spinor form of the coupled Chandrasekhar-Dirac equation [8], which describes the dynamic behavior of spin-1/2 particles, namely

\[
\begin{align*}
(D + \epsilon - \rho + iqA \cdot \ell)F_1 + (\delta + \tilde{\pi} - \alpha + iqA \cdot m)F_2 &= \frac{i\mu_0}{\sqrt{2}} G_1, \\
(\Delta + \mu - \gamma + iqA \cdot n)F_2 + (\delta + \beta - \tau + iqA \cdot m)F_1 &= \frac{i\mu_0}{\sqrt{2}} G_2, \\
(D + \epsilon^* - \rho^* + iqA \cdot \ell)G_1 - (\delta + \tilde{\pi}^* - \alpha^* + iqA \cdot m)G_1 &= \frac{i\mu_0}{\sqrt{2}} F_1, \\
(\Delta + \mu^* - \gamma^* + iqA \cdot n)G_1 - (\delta + \beta^* - \tau^* + iqA \cdot m)G_2 &= \frac{i\mu_0}{\sqrt{2}} F_2,
\end{align*}
\]

where \(\mu_0\) and \(q\) is the mass and charge of Dirac particles, one obtains

\[
\begin{align*}
-\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) F_1 + \frac{1}{\sqrt{2r}} L_{1/2} F_2 &= \frac{i\mu_0}{\sqrt{2}} G_1, \\
\frac{1}{2r^2} \mathcal{D} F_2 + \frac{1}{\sqrt{2r}} L_{1/2}^\dagger F_1 &= \frac{i\mu_0}{\sqrt{2}} G_2, \\
-\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) G_2 - \frac{1}{\sqrt{2r}} L_{1/2}^\dagger G_1 &= \frac{i\mu_0}{\sqrt{2}} F_2, \\
\frac{1}{2r^2} \mathcal{D} G_1 - \frac{1}{\sqrt{2r}} L_{1/2} G_2 &= \frac{i\mu_0}{\sqrt{2}} F_1,
\end{align*}
\]

in which we have defined operators

\[
\mathcal{D} = 2r^2 \left(\frac{\partial}{\partial v} + G \frac{\partial}{\partial r}\right) + (r^2 G)_r + 2iqQr, \\
L_{1/2} = \frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta - \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi}, \\
L_{1/2}^\dagger = \frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi}.
\]

By substituting

\[
F_1 = \frac{1}{\sqrt{2r}} P_1, \quad F_2 = P_2, \quad G_1 = Q_1, \quad G_2 = \frac{1}{\sqrt{2r}} Q_2,
\]

into Eq. (11), they have the form

\[
\begin{align*}
-\frac{\partial}{\partial r} P_1 + L_{1/2} P_2 &= i\mu_0 r Q_1, \\
\mathcal{D} P_2 + L_{1/2}^\dagger P_1 &= i\mu_0 r Q_2, \\
-\frac{\partial}{\partial r} Q_2 - L_{1/2}^\dagger Q_1 &= i\mu_0 r P_2, \\
\mathcal{D} Q_1 - L_{1/2} Q_2 &= i\mu_0 r P_1.
\end{align*}
\]
III. EVENT HORIZON

An apparent fact is that the Chandrasekhar-Dirac equation (11) could be satisfied by identifying $Q_1, Q_2, qQ$ with $P_2^*,-P_1^*, -qQ$, respectively. So one may deal with a pair of components $P_1, P_2$ only. As to the thermal radiation, one may concern about the behavior of Eq. (11) near the horizon. Though Eq. (11) can be decoupled in a spherically symmetric space-time such as Vaidya black hole, we do not separate it in advance into a radial part and an angular one. As the Vaidya-type space-time is spherically symmetric, let’s introduce a generalized tortoise coordinate transformation [9] as

$$r_\ast = r + \frac{1}{2\kappa} \ln[r - r_H(v)], \quad v_\ast = v - v_0,$$

where $r_H = r_H(v)$ is the location of the event horizon, $\kappa$ is an adjustable parameter and is unchanged under tortoise transformation. The parameter $v_0$ is an arbitrary constant. From formula (12), we can deduce some useful relations for the derivatives as follows:

$$\frac{\partial}{\partial r} = \left[1 + \frac{1}{2\kappa(r - r_H)} \right] \frac{\partial}{\partial r_\ast}, \quad \frac{\partial}{\partial v} = \frac{\partial}{\partial v_\ast} - \frac{r_{H,v}}{2\kappa(r - r_H)} \frac{\partial}{\partial r_\ast}.$$

Under the transformation (12), Eq. (11) with respect to a pair of components $(P_1, P_2)$ can be reduced to the following limiting form near the event horizon

$$\frac{\partial}{\partial r_\ast} P_1 = 0, \quad \frac{\partial}{\partial r_\ast} P_2 = 0,$$

after being taken the $r \rightarrow r_H(v_0)$ and $v \rightarrow v_0$ limits. A similar form holds for $Q_1, Q_2$.

From Eq. (13), we know that $P_1$ is independent of $r_\ast$ and regular on the event horizon. If the derivative $\frac{\partial}{\partial r_\ast} P_2$ in Eq. (13) does not be equal to zero, the existence condition of a non-trial solution for $P_2$ is then (as for $r_H \neq 0$)

$$2G(r_H) - 2r_{H,v} = 0,$$

which determines the location of horizon. The event horizon equation (14) can be inferred from the null hypersurface condition, $g^{ij}\partial_i F \partial_j F = 0$, and $F(v, r) = 0$, namely $r = r(v)$. Because $r_H$ depends on time $v$, so the location of the event horizon and the shape of the black hole change with time.
IV. HAWKING TEMPERATURE

To investigate the Hawking radiation of $P_1, P_2$ components of spin-1/2 particles, one need consider the behavior of their second order form of Dirac equation near the event horizon. A direct calculation gives the second order equation for $(P_1, P_2)$

$$
2r^2 \frac{\partial^2}{\partial v \partial r} P_1 + 2r^2 G \frac{\partial^2}{\partial r^2} P_1 + \mathcal{L}_{1/2}^\dagger P_1 - \mu_0^2 r^2 P_1 \\
+ \left(2iqQr + r^2 G_r + 2rG\right) \frac{\partial}{\partial r} P_1 + 2ir^2 G\mu_0 Q_1 = 0 , \quad (15)
$$

where

$$
\mathcal{L}_{1/2}^\dagger = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \theta^2} + \frac{i \cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \phi} - \frac{1}{4 \sin^2 \theta} - \frac{1}{4} ,
$$

$$
\mathcal{L}_{1/2} = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \theta^2} - \frac{i \cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \phi} - \frac{1}{4 \sin^2 \theta} - \frac{1}{4} .
$$

The angular parts of $P_1, P_2$ are spinorial spherical harmonics $\pm \frac{1}{2} Y_{\ell m}(\theta, \phi)$ [10]. One easily show that

$$
\mathcal{L}_{1/2}^\dagger P_1 = - (\ell + 1/2)^2 P_1 , \quad \mathcal{L}_{1/2} P_2 = - (\ell + 1/2)^2 P_2 .
$$

Given the GTCT in Eq. (12) and after some calculations, the limiting form of Eqs. (15,16), when $r$ approaches $r_H(v_0, \theta_0)$ and $v$ goes to $v_0$, reads

$$
\left\{ \begin{array}{l}
\frac{A}{2\kappa} + r_H^2 [4G(r_H) - 2r_{H,v}] \frac{\partial^2}{\partial r_*^2} P_1 + 2r_H^2 \frac{\partial^2}{\partial r_* \partial v_*} P_1 = 0 , \\
\end{array} \right.
$$

and

$$
\left\{ \begin{array}{l}
\frac{A}{2\kappa} + r_H^2 [4G(r_H) - 2r_{H,v}] \frac{\partial^2}{\partial r_*^2} P_2 + 2r_H^2 \frac{\partial^2}{\partial r_* \partial v_*} P_2 \\
+ \left\{ -A + 3r_H^2 G_r(r_H) + 2iqQr_H + r_H[6G(r_H) - 4r_{H,v}] \right\} \frac{\partial}{\partial r_*} P_2 = 0 . \\
\end{array} \right.
$$

In deriving Eq. (17) we have used the relation $\frac{\partial}{\partial r_*} P_1 = 0$.

With the aid of the event horizon equation (14), namely

$$
2G(r_H) = 2r_{H,v} ,
$$
we know that the coefficient $A$ is an infinite limit of $\frac{0}{0}$ type. By use of the L’Hôpital rule, we get the following result

$$A = \lim_{r \to r_H(v_0)} \frac{2r^2(G - r_{H,v})}{r - r_H} = 2r_H^2G,_{r}(r_H). \quad (19)$$

Now let us select the adjustable parameter $\kappa$ in Eqs. (17, 18) such that

$$r_H^2 \equiv \frac{A}{2\kappa} + 2r_H^2[2G(r_H) - r_{H,v}] = \frac{r_H^2G,_{r}(r_H)}{\kappa} + 2r_H^2G(r_H), \quad (20)$$

which gives the temperature of the horizon

$$\kappa = \frac{G,_{r}(r_H)}{1 - 2G(r_H)} = \frac{G,_{r}(r_H)}{1 - 2r_{H,v}}. \quad (21)$$

With such a parameter adjustment and using the event horizon equation (14), we can reduce Eqs. (17, 18) to

$$\frac{\partial^2}{\partial r^2} P_1 + 2 \frac{\partial^2}{\partial r \partial v} P_1 = 0, \quad (22)$$

and

$$\frac{\partial^2}{\partial r^2} P_2 + 2 \frac{\partial^2}{\partial r \partial v} P_2 + \left[ G,_{r}(r_H) + 2i\omega_0 \frac{G(r_H)}{r_H} \right] \frac{\partial}{\partial r} P_2 = 0. \quad (23)$$

Eqs. (22, 23) are standard wave equations near the horizon. As the angular parts of $P_1, P_2$ have no relation to Hawking radiation, so we can omit them in the following section. The radial parts $R_1, R_2$ satisfy the same equations as that of $P_1, P_2$

$$\frac{\partial^2}{\partial r^2} R_1 + 2 \frac{\partial^2}{\partial r \partial v} R_1 = 0, \quad \frac{\partial}{\partial r} R_1 = 0, \quad (24)$$

$$\frac{\partial^2}{\partial r^2} R_2 + 2 \frac{\partial^2}{\partial r \partial v} R_2 + 2 (C + i\omega_0) \frac{\partial}{\partial r} R_2 = 0, \quad (25)$$

where $\omega_0, C$ will be regarded as finite real constants,

$$\omega_0 = \frac{qQ}{r_H}, \quad C = \frac{1}{2} G,_{r}(r_H) + \frac{r_{H,v}}{r_H}.$$

V. THERMAL RADIATION SPECTRUM

From Eqs. (24), we know that $R_1$ is a constant on the event horizon. The solution $R_1 = R_{10}e^{-i\omega_0 v_\ast}$ means that Hawking radiation does not exist for $P_1, Q_2$. 

Now separating variables to Eq. (23) as follows

\[ R_2 = R_2(r_*) e^{-i\omega v_*} \]

one gets

\[ R''_2 = 2|i(\omega - \omega_0) - C|R'_2, \quad (26) \]

The solution is

\[ R_2 \sim e^{2|\omega(\omega - \omega_0) - C|r_*}, \quad (27) \]

The ingoing wave and the outgoing wave to Eq. (25) are

\[ R_2^\text{in} = e^{-i\omega v_*}, \quad (28) \]

\[ R_2^\text{out} = e^{-i\omega v_*} e^{2|\omega(\omega - \omega_0) - C|r_*}, \quad (r > r_H). \]

Near the event horizon, we have

\[ r_* \sim \frac{1}{2\kappa} \ln(r - r_H). \]

Clearly, the outgoing wave \( R_2^\text{out}(r > r_H) \) is not analytic at the event horizon \( r = r_H \), but can be analytically extended from the outside of the hole into the inside of the hole through the lower complex \( r \)-plane

\[ (r - r_H) \rightarrow (r_H - r)e^{-i\pi} \]

to

\[ R_2^\text{out} = e^{-i\omega v_*} e^{2|\omega(\omega - \omega_0) - C|r_*} e^{i\pi C/\kappa} e^{\pi(\omega - \omega_0)/\kappa}, \quad (r < r_H). \quad (29) \]

So the relative scattering probability of the outgoing wave at the horizon is easily obtained

\[ \left| \frac{R_2^\text{out}}{R_2^\text{out}} \right|^2 = e^{-2\pi(\omega - \omega_0)/\kappa}. \quad (30) \]

According to the method of Damour-Ruffini-Sannan’s [11], the thermal radiation Fermionic spectrum of Dirac particles from the event horizon of the hole is given by

\[ \langle N_\omega \rangle = \frac{1}{e^{(\omega - \omega_0)/T_H} + 1}, \quad (31) \]

with the Hawking temperature being

\[ T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_H} \cdot \frac{Mr_H - Q^2 - \Lambda r_H^4/3}{Mr_H - Q^2/2 - \Lambda r_H^4/6}. \quad (32) \]

It follows that the temperature depends on the time, because it is determined by the surface gravity \( \kappa \), a function of \( v \). The temperature coincides with that derived from the investigating of the thermal radiation of Klein-Gordon particles [3].
VI. CONCLUSIONS

In this paper, we have studied the Hawking radiation of Dirac particles in a black hole whose mass and electric charge change with time. We have dealt with the asymptotic behavior of Dirac equation near the event horizon, not only its first order form but also its second order form. We find that not all component of Dirac spinors but for $P_2, Q_1$ displays the property of thermal radiation. The asymmetry of Hawking radiation with respect to the four-component Dirac spinors maybe originate from the asymmetry of space-times in the advanced Eddington-Finkelstein coordinate.

Equations (14) and (21) give the location and the temperature of event horizon, which depend on the advanced time $v$. They are just the same as that obtained in the discussing on thermal radiation of Klein-Gordon particles in the same space-time. Eq. (31) shows the thermal radiation spectrum of Dirac particles in a charged Vaidya black hole with a cosmological constant $\Lambda$. These results are consistent with others.

From the thermal spectrum (31) of Dirac particles, we know that there is no other interaction energy except the Coulomb energy $\omega_0$ in a Vaidya-type space-time. This inferres that there is no new quantum effect called by Li [3,4] in any spherically symmetric black hole whether it is static or non-static.

The discussion in this paper is easily extended to the case of the Hawking radiation of photon in the Vaidya-Bonner-de Sitter spacetime. Taking into account of the restrict of the asymptotic behavior of the first order coupled Maxwell equations near the event horizon (namely, $\frac{\partial}{\partial r^*}\phi_1 = \frac{\partial}{\partial r^*}\phi_2 = 0$), one can show that only the complex scalar $\phi_0$ takes part in the Hawking radiation and Li’s conclusion does not hold in the case of particles with spin-1. The results derived from the Hawking effect of photon are consistent with the present paper. Details will be published elsewhere.

Acknowledgment

S.Q. Wu is indebted to Dr. Jeff Zhao at Motomola Company for his longterm helps. This work is supported in part by the NSFC in China.

[1] S. W. Hawking, Nature, 248 (1974) 30; Commun. Math. Phys. 43 (1975) 199.
[2] Z. Zhao, C. Q. Yang and Q. A. Ren, Gen. Rel. Grav. 26 (1994) 1055; Z. H. Li and Z. Zhao, Chin. Phys. Lett. 10 (1993) 126; Y. Ma and S. Z. Yang, Int. J. Theor. Phys. 32 (1993) 1237; J. Y. Zhu, J. H. Zhang and Z. Zhao, Int. J. Theor. Phys. 33 (1994) 2137; L. C. Zhang, Y. Q. Wu and R. Zhao, Int. J. Theor. Phys. 38 (1999) 665.

[3] Z. H. Li, Chin. Phys. Lett. 15 (1998) 553; Z. H. Li and Z. Zhao, Journal of Beijing Normal University (Natural Science), 34: 3 (1998) 345.

[4] Z. H. Li, Y. Liang and L. Q. Mi, Int. J. Theor. Phys. 38 (1999) 925; IL Nuovo Cimento 114B (1999) 555.

[5] S. A. Teukolsky, Astrophys. J. 185 (1973) 635.

[6] J. M. Jr. York, Phys. Rev. D28 (1983) 292.

[7] E. Newman and R. Penrose, J. Math. Phys. 3 (1962) 566.

[8] S. Chandrasekhar, The Mathematical Theory of Black Holes, (New York: Oxford University Press, 1983); D. Page, Phys. Rev. D14 (1976) 1509.

[9] Z. Zhao and X. X. Dai, Mod. Phys. Lett. A7 (1992) 1771.

[10] J. N. Goldberg, A. J. Macfarane, E. T. Newman, F. Rohrlich and E. C. G. Sudarshan, J. Math. Phys. 8 (1968) 2155.

[11] T. Damour and R. Ruffini, Phys. Rev. D14 (1976) 332; S. Sannan, Gen. Rel. Grav. 20 (1988) 239.