Poisson–Lie T–plurality as canonical transformation

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Abstract

We generalize the prescription realizing classical Poisson–Lie T–duality as canonical transformation to Poisson–Lie T–plurality. The key ingredient is the transformation of left–invariant fields under Poisson–Lie T–plurality. Explicit formulae realizing canonical transformation are presented and the preservation of canonical Poisson brackets and Hamiltonian density is shown.

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1 Introduction

Poisson–Lie T–duality and T–plurality is already quite an old subject. It was introduced in 1995 when C. Klimčík and P. Ševera in [1, 2, 3] proposed Poisson–Lie T–duality as an approach solving certain problems in T–duality with respect to non–Abelian groups of isometries (especially that the original T–duality worked only in one direction). Already in [1, 3] they considered the possibility of what is now called Poisson–Lie T–plurality. This is related to the fact that the Lie algebra of Drinfel’d double may be decomposable into more than one pair of subalgebras whose transposition corresponds to duality. On the other hand, further development (like the explicit formulation of canonical transformation in [4, 5]) focused only on Poisson–Lie T–duality and almost no explicit formulae and no examples of genuine Poisson–Lie T–plurality were known until 2002 when R. von Unge considered T–plurality of conformal quantum sigma models (on one–loop level) in [6] and coined
the current phrase “Poisson–Lie T–plurality”. By that time classifications of Drinfeld’s doubles in low dimension, e.g. [7], became available, facilitating construction of more examples and study of their properties (see [8] and references therein).

Gradually, the need arose for generalization of formulae previously derived in the Poisson–Lie T–duality context to the general plurality case. One of these is the formulation of Poisson–Lie T–plurality as canonical transformation (derived for the duality case by K. Sfetsos in [4, 5]).

In this paper we shall derive the explicit canonical formulation of Poisson–Lie T–plurality. As we shall show, the key point is the transformation of the extremal left and right–invariant fields, which can be derived in a direct way, and which will enable us to find the transformation of the canonical variables of the dualizable \( \sigma \)–models and prove that they really constitute a canonical transformation.

One of possible applications of our results is in the study of the worldsheet boundary conditions. Recently, Poisson–Lie T–duality transformation of worldsheet boundary conditions of the dualizable \( \sigma \)–models was derived in [9]. The key formulae there were the transformations of left–invariant fields by the Poisson–Lie T–duality obtained from canonical formulation of T–dual \( \sigma \)–models [4, 5]. Using the formulae derived in this paper one can easily generalize the results of [9] to the T–plurality case. Detailed discussion of them shall be the subject of future work.

A note concerning the conventions: we are using in the current paper the conventions introduced in [4, 5] in order to be able to compare with results therein. Unfortunately, this notation is not the same as the one used in [1, 2] and all our previous papers. The two notations are equivalent upon substitutions \( g, l, \ldots \leftrightarrow g^{-1}, l^{-1}, \ldots \) accompanied by the worldsheet parity transformation \( x_+ \leftrightarrow x_- \).

\section{Elements of Poisson–Lie T–plurality}

For simplicity we shall consider the \( \sigma \)–models without spectator fields, i.e. with target manifold isomorphic to the group of generalized isometries. The inclusion of spectators is straightforward, see [4, 6]. The classical action of \( \sigma \)–model without spectators reads

\[ S_{\mathcal{E}}[\phi] = \frac{1}{2} \int d^2 x \, \partial_+ \phi^\mu \mathcal{E}_{\mu\nu}(\phi) \partial_- \phi^\nu \]  

where \( \mathcal{E} \) is a tensor on a Lie group \( G \) and the functions \( \phi^\mu : V \subset \mathbb{R}^2 \to \mathbb{R}, \mu = 1, 2, \ldots, \dim G \) are obtained by the composition \( \phi^\mu = y^\mu \circ g \) of a map \( g : V \subset \mathbb{R}^2 \to G \) and a coordinate map \( y \) of a neighborhood \( U_g \) of an element \( g(x_+, x_-) \in G \). Further on we shall use formulation of the \( \sigma \)–models in terms of left–invariant fields \( g^{-1} \partial_\pm g \). The tensor \( \mathcal{E} \) can be written as

\[ \mathcal{E}_{\mu\nu} = \epsilon_{\mu}^{\nu a}(g) E_{ab}(g) e_{\nu}^{L b}(g) \]  

where

- \( \epsilon_{\mu}^{L a} \) are components of left–invariant forms (vielbeins) \( g^{-1} dg = dy^\mu \epsilon_{\mu}^{L a}(g) T_a \),
- \( T_a \) are basis elements of \( g \), i.e. Lie algebra of \( G \),
\[ E_{ab}(g) \text{ are matrix elements of a } G \text{-dependent bilinear nondegenerate form on } g \text{ in the basis } \{ T_a \}. \]

The action of the \( \sigma \)-model then reads\(^{1}\)

\[ S[g] = \frac{1}{2} \int d^2x L_+(g) \cdot E(g) \cdot L_-(g) \quad (3) \]

where

\[ L_\pm(g)^a = (g^{-1}\partial_\pm g)^a = \partial_\pm \phi^0 e^\alpha_L(g), \quad g^{-1}\partial_\pm g = L_\pm(g) \cdot T. \quad (4) \]

The \( \sigma \)-models that can be transformed by the Poisson–Lie T–plurality are formulated (see \([1,2]\)) on a Drinfel’d double \( D \equiv (G|\tilde G) \) – a Lie group whose Lie algebra \( \mathfrak{d} \) admits a decomposition \( \mathfrak{d} = \mathfrak{g} + \tilde{\mathfrak{g}} \) into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form \( \langle \cdot, \cdot \rangle \). The matrices \( E(g) \) for such \( \sigma \)-models are of the form

\[ E(g) = (E_0^{-1} + \Pi(g))^{-1}, \quad \Pi(g) = b^t(g) \cdot a(g) = -\Pi(g)^t, \quad (5) \]

where \( E_0 \) is a constant matrix and \( a(g), b(g) \) are submatrices of the adjoint representation of the subgroup \( G \) on the Lie algebra \( \mathfrak{d} \) satisfying

\[ gTg^{-1} = Ad(g) \triangleright T = a^{-1}(g) \cdot T, \quad g\tilde{T}g^{-1} = Ad(g) \triangleright \tilde{T} = b^t(g) \cdot T + a^t(g) \cdot \tilde{T}, \quad (6) \]

and \( \tilde{T}^a \) are elements of dual basis in the dual algebra \( \tilde{\mathfrak{g}} \), i.e. \( \langle T_a, \tilde{T}^b \rangle = \delta^b_a \). The matrix \( a(g) \) also relates the left– and right–invariant fields on \( G \)

\[ \partial_\pm gg^{-1} = R_\pm(g) \cdot T, \quad L_\pm(g) = R_\pm(g) \cdot a(g). \quad (7) \]

The equations of motion of the dualizable \( \sigma \)-models can be written as Bianchi identities for the left–invariant fields \( \tilde{L}_\pm(h) \) on the dual algebra \( \tilde{\mathfrak{g}} \)

\[ \tilde{L}_+(h) \cdot \tilde{T} \equiv \tilde{h}^{-1}\partial_+ \tilde{h} = L_+(g) \cdot E(g) \cdot a^t(g) \cdot \tilde{T}, \]

\[ \tilde{L}_-(h) \cdot \tilde{T} \equiv \tilde{h}^{-1}\partial_- \tilde{h} = -L_-(g) \cdot E^t(g) \cdot a^t(g) \cdot \tilde{T}. \quad (8) \]

This is a consequence of the fact that the equations of motion of the dualizable \( \sigma \)-model can be formulated as the equations on the Drinfel’d double \( \mathfrak{d} \)

\[ \langle l^{-1}\partial_\pm l, \mathcal{E}^\mp \rangle = 0, \quad (9) \]

where \( l = \tilde{h}g \in D, \quad \tilde{h} \in \tilde{G}, \quad g \in G \) and

\[ \mathcal{E}^+ = \text{span} \left( T + E_0 \cdot \tilde{T} \right), \quad \mathcal{E}^- = \text{span} \left( T - E_0^t \cdot \tilde{T} \right) \]

are two orthogonal subspaces in \( \mathfrak{d} \). (The unique decomposition \( l = \tilde{h}g \) on \( D \) exists for a general Drinfel’d double only in the vicinity of the group unit. For the so–called perfect Drinfel’d doubles it is defined globally and we shall consider only these. Otherwise all the constructions considered would hold only locally.)

\(^{1}\)Central dot means matrix multiplication and we consider \( L_\pm \) as a row vector whereas \( T \) is a column vector with components \( T_a \). \( L^t \) denotes transposition. Later on we shall also use the notation \( M^{-1} = (M^{-1})^t \) for matrices.
In general there are several decompositions (Manin triples) of a Drinfel’d double. Let \( \hat{\mathfrak{g}} + \bar{\mathfrak{g}} \) be another decomposition of the Lie algebra \( \mathfrak{d} \) into maximal isotropic subalgebras. Then another \( \sigma \)–model can be defined. The dual bases of \( \mathfrak{g}, \bar{\mathfrak{g}} \) and \( \hat{\mathfrak{g}}, \bar{\mathfrak{g}} \) are related by the linear transformation
\[
\begin{pmatrix}
T \\
\bar{T}
\end{pmatrix} =
\begin{pmatrix}
K & Q \\
R & S
\end{pmatrix}
\begin{pmatrix}
\hat{T} \\
\bar{T}
\end{pmatrix},
\]
where the matrices \( K, Q, R, S \) are chosen in such a way that the structure of the Lie algebra \( \mathfrak{d} \)
\[
[T_a, T_b] = f_{abc} T_c,
\]
\[
[\hat{T}^a, \hat{T}^b] = \hat{f}^{ab} c \hat{T}^c,
\]
\[
[\bar{T}^a, T_b] = f_{bc}^a \hat{T}^c - \hat{f}^{ac} b \hat{T}^c
\]
transforms to the similar one where \( T \rightarrow \hat{T}, \bar{T} \rightarrow \bar{T} \) and the structure constants \( f, \hat{f} \) of \( \mathfrak{g} \) and \( \hat{\mathfrak{g}} \) are replaced by the structure constants \( \bar{f}, \bar{f} \) of \( \mathfrak{g} \) and \( \bar{\mathfrak{g}} \). The duality of both bases requires
\[
(K Q R S)^{-1} = (S^t Q^t R^t K^t).
\]
The other \( \sigma \)–model is defined analogously to (3-5) where
\[
\hat{E}(\hat{g}) = (\hat{E}_0^{-1} + \hat{\Pi}(\hat{g}))^{-1}, \quad \hat{\Pi}(\hat{g}) = \hat{b}(\hat{g}) \cdot \hat{a}(\hat{g}) = -\hat{\Pi}(\hat{g})^t,
\]
\[
\hat{E}_0 = (K + E_0 \cdot R)^{-1} \cdot (Q + E_0 \cdot S) = (S^t \cdot E_0 - Q^t) \cdot (K^t - R^t \cdot E_0)^{-1},
\]
and classical solutions of the two \( \sigma \)–models are related by two possible decompositions of \( l \in D \),
\[
l = \hat{h} g = \bar{h} \bar{g}.
\]
The explicit examples of solutions of the \( \sigma \)–models related by the Poisson–Lie T–plurality are given in [10].

3 Poisson–Lie transformation of extremal left–invariant fields

As mentioned in the Introduction, the formulae for transformation of left–invariant fields evaluated on solutions of equations of motion (hence extremal) by the Poisson–Lie T–duality were found in [9]. We are going to derive the extension of these formulae in an alternative way.

Let us write the left–invariant field \( l^{-1} \partial_+ l \) on the Drinfel’d double in terms of \( L_+(g) \) and \( \bar{L}_+ (\bar{h}) \)
\[
l^{-1} \partial_+ l = (\hat{h} g)^{-1} (\partial_+ (\hat{h} g)) = L_+(g) \cdot T + \bar{L}_+ (\bar{h}) \cdot g^{-1} \bar{T} g
\]
\[
= L_+(g) \cdot T + \bar{L}_+ (\bar{h}) \cdot \left[ b(g) \cdot T + a^{-1}(g) \cdot \bar{T} \right]
\]
where \( a(g) \) and \( b(g) \) are the matrices introduced in [6].
Using the equations of motion (8) and the expression (5) for $E(g)$ we get

$$\ell^{-1} \partial_\tau \ell = L_+(g) \cdot T + L_+(g) \cdot E(g) \cdot \left[a^t(g) \cdot b(g) \cdot T + \hat{T}\right]$$

$$= L_+(g) \cdot E(g) \cdot \left[E_0^{-1} \cdot T + \hat{T}\right].$$

(17)

On the other hand, from the decomposition $\ell = \tilde{h} \hat{g}$ we find by a similar procedure

$$\ell^{-1} \partial_\tau \ell = \hat{L}_+(\hat{g}) \cdot \hat{E}(\hat{g}) \cdot \left[\hat{E}_0^{-1} \cdot \hat{T} \right].$$

(18)

Inserting (10) and (14) into (17) and comparing coefficients of $\hat{T}$ and $\tilde{T}$ with those in (18) we obtain the formula for transformation of the left–invariant fields under the Poisson–Lie T–plurality

$$\hat{L}_+(\hat{g}) = L_+(g) \cdot E(g) \cdot \left[S + E_0^{-1} \cdot Q\right] \cdot \hat{E}^{-1}(\hat{g}).$$

(19)

In the same way we can derive

$$\hat{L}_-(\hat{g}) = L_-(g) \cdot E^t(g) \cdot \left[S - E_0^{-t} \cdot Q\right] \cdot \hat{E}^{-t}(\hat{g}).$$

(20)

This agrees with the formulae obtained in [9] for Poisson–Lie T–duality, i.e. for $Q = R = 1$, $K = S = 0$, $L_{\pm}(\tilde{g}) = \tilde{L}_{\pm}(\hat{g})$, which in our notation (i.e. $L_{\pm}$ rows) read

$$\hat{L}^t_+(\hat{g}) = \hat{E}^{-t}(\hat{g}) \cdot E_0^{-t} \cdot E^t(g) \cdot L^t_+(g).$$

(21)

The transformations of right–invariant extremal fields can be easily obtained from the relation (7).

4 Transformation of canonical variables

In the present section we are going to generalize the formulae for canonical transformation obtained in [4, 5] for the Poisson–Lie T–duality to the general T–plurality case.

Recall that the time and space coordinates on the worldsheet are $\tau = x_+ + x_-$, $\sigma = x_+ - x_-$, i.e. $\partial_\tau = \frac{1}{2}(\partial_+ + \partial_-)$, $\partial_\sigma = \frac{1}{2}(\partial_+ - \partial_-)$. The canonical momentum is defined by

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\sigma \phi^\mu)} = \frac{1}{2} (\mathcal{E}_{\mu \nu} \partial_- \phi^\nu + \mathcal{E}_{\nu \mu} \partial_+ \phi^\nu).$$

(22)

It turns out, similarly as above, to be advantageous to use a momentum in local frame, defined as

$$\mathcal{P}_a = v_a^{L\mu}(g) \mathcal{P}_\mu$$

(23)

where $v^L = (e^L)^{-1}$. We shall denote by $\mathcal{P}$ the column vector with the components $\mathcal{P}_a$ so that

$$\mathcal{P} = \frac{1}{2} \left(E(g) \cdot L^t_-(g) + E^t(g) \cdot L^t_+(g)\right).$$

(24)
We also define

\[ L_\sigma = \frac{1}{2} (L_+(g) - L_-(g)). \]  

(25)

For the future reference let us quote the inverse relations

\[
\begin{align*}
L_+(g) &= 2 \left( \mathcal{P}^t + L_\sigma \cdot E^t(g) \right) \cdot \left( E(g) + E^t(g) \right)^{-1}, \\
L_-(g) &= 2 \left( \mathcal{P}^t - L_\sigma \cdot E(g) \right) \cdot \left( E(g) + E^t(g) \right)^{-1}.
\end{align*}
\]

Defining the similar quantities \( \hat{\mathcal{P}}, \hat{L}_\sigma \) for the model after the Poisson–Lie T–plurality transformation and using \([19,20]\) we find

\[
\begin{align*}
\hat{\mathcal{P}} &= \frac{1}{2} \left( (Q^t \cdot E_0^{-t} + S^t) \cdot E^t(g) \cdot L_+(g) - (Q^t \cdot E_0^{-1} - S^t) \cdot E(g) \cdot L_+(g) \right), \\
\hat{L}_\sigma &= \frac{1}{2} \left( L_+(g) \cdot E(g) \cdot (E_0^{-1} \cdot Q + S) \cdot \hat{E}(\hat{g})^{-1} + L_-(g) \cdot E^t(g) \cdot (E_0^{-t} \cdot Q - S) \cdot \hat{E}(\hat{g})^{-1} \right),
\end{align*}
\]

which, as we shall show, becomes the transformation of the canonical variables\(^2\)

\[
\begin{align*}
\hat{\mathcal{P}} &= \left( Q^t \cdot \Pi(g) + S^t \right) \cdot \mathcal{P} + Q^t \cdot L_\sigma, \\
\hat{L}_\sigma &= \mathcal{P}^t \cdot \left[ (S - \Pi(g) \cdot Q) \cdot \hat{\Pi}(\hat{g}) + R - \Pi(g) \cdot K \right] + L_\sigma \cdot \left( Q \cdot \hat{\Pi}(\hat{g}) + K \right).
\end{align*}
\]

(29)

(30)

This agrees with the formulae obtained in [4] for Poisson–Lie T–duality\(^3\), i.e. for \(Q = R = 1, K = S = 0\), but generalizes the results from [4,5] to any T–plurality transformation.

In order to deduce \([29,30]\) we shall first list a few useful formulae. Because of their complexity we shall suppress the \(g, \hat{g}\)–dependence in the proof (i.e. till the end of this section) and also the explicit dot · for matrix multiplication. This doesn’t lead to any difficulty because the derivation of \([29,30]\) from \([27,28]\) is purely algebraic.

We have matrix identities valid for any matrix \(A\) (whenever the expressions make sense)

\[
\begin{align*}
A^{-1}(A^{-1} + A^{-t})^{-1} &= (A + A^t)^{-1} A^t, \\
A^{-t}(A^{-1} + A^{-t})^{-1} &= (A + A^t)^{-1} A, \\
A^{-1}(A^{-1} + A^{-t})^{-1} A^{-t} &= (A + A^t)^{-1} = A^{-t} (A^{-1} + A^{-t})^{-1} A^{-1}.
\end{align*}
\]

(31)

Directly from the definition \([32]\) of \(E\) we have

\[ E^{-1} + E^{-t} = E_0^{-1} + E_0^{-t} \]

(32)

and its consequences due to \([31]\)

\[
\begin{align*}
E(E + E^t)E^t &= (E^{-1} + E^{-t})^{-1} = (E_0^{-1} + E_0^{-t})^{-1}, \\
E^t(E + E^t)E &= (E^{-1} + E^{-t})^{-1} = (E_0^{-1} + E_0^{-t})^{-1}.
\end{align*}
\]

(33)

\(^2\)We slightly abuse the terminology here: strictly speaking the canonical variables are \(P_\mu, \phi^\mu\) and \(\hat{\mathcal{P}}_\mu, \hat{\phi}^\mu\), respectively. Because the plurality transformation of \(\phi^\mu\) defined via \([15]\) (where \(h, \hat{h}\) are constructed via \([5]\)) is nonlocal, we write instead the transformation of its space derivative \(\partial_\sigma \phi^\mu\) and also we use for convenience the local frame versions instead of coordinate versions of these. Nevertheless, as we show later on, this doesn’t lead to any nonlocalities in the Hamiltonian or the Poisson brackets.

\(^3\)and, as another consistency check, reduces to identity transformation when \(K = S = 1, R = Q = 0\).
Finally, combining (33) and (31) (using first $A = E$ and then $A = E_0^{-1}$) together with (30) (in the second equality) we get
\[
(E + E')^{-1}(EE_0^{-1} - E'EE_0^{-1}) = E^{-1}(E_0^{-1} + E_0^{-1})^{-1}E_0^{-1}E_0^{-1} + E^{-1}(E_0^{-1} + E_0^{-1})^{-1}E_0^{-1} = \\
E_0^{-1}(E_0^{-1} + E_0^{-1})^{-1}E_0^{-1} - E_0^{-1}(E_0^{-1} + E_0^{-1})^{-1}E_0^{-1}E_0^{-1} = \\
-\Pi(E_0^{-1} + E_0^{-1})^{-1}(E_0^{-1} + E_0^{-1}) = \\
-\Pi
\] (34)
and similarly its transpose
\[
(E_0^{-1}E' - E_0^{-1}E) (E + E')^{-1} = \Pi.	ag{35}
\]

Now, when we substitute the relations (26) into the formula (27), we get
\[
\hat{P} = \left[ (Q' E_0^{-t} + S') E(E + E')^{-1} - (Q' E_0^{-1} - S') E(E + E')^{-1} \right] P + \\
+ Q' \left[ E_0^{-t} E(E + E')^{-1} E + E_0^{-1} E(E + E')^{-1} E' \right] L_0^{t}
\]
(the terms involving $S'(\ldots)L_0^{t}$ cancel each other). Using (33) we simplify the coefficient of $L_0^{t}$, getting the desired $Q'L_0^{t}$ term in (24). The terms of the form $S'(\ldots)P$ give $S'P$. The remaining $Q'(\ldots)P$ terms are simplified using (35)
\[
Q' \left( E_0^{-t} E' - E_0^{-1} E \right) (E + E')^{-1} P = Q' \Pi P.
\]

Therefore, the formula (24) is proven.

Similarly, we substitute the relations (26) together with the definition of $\hat{E}$
\[
\hat{E} = \left( (Q + E_0 S)^{-1}(K + E_0 R) + \hat{\Pi} \right)^{-1} = \left( (E_0 S - Q)^{-1}(K - E_0 R) - \hat{\Pi} \right)^{-t}
\]
into the formula (28). We get
\[
\hat{L}_0 = \left[ P'(E + E')^{-1} \left( E(E_0^{-1} Q + S) \right) \left( (Q + E_0 S)^{-1}(K + E_0 R) + \hat{\Pi} \right) + \\
+ E'(E_0^{-1} Q - S) \left( (E_0 S - Q)^{-1}(K - E_0 R) - \hat{\Pi} \right) \right] + \\
+ L_0 \left[ E'(E + E')^{-1} E(E_0^{-1} Q + S) \left( (Q + E_0 S)^{-1}(K + E_0 R) + \hat{\Pi} \right) - \\
- E(E + E')^{-1} E'(E_0^{-1} Q - S) \left( (E_0 S - Q)^{-1}(K - E_0 R) - \hat{\Pi} \right) \right].
\]

We note that
\[
(E_0^{-1} Q + S)(Q + E_0 S)^{-1} = E_0^{-1}, \quad (E_0^{-1} Q - S)(E_0 S - Q)^{-1} = -E_0^{-1} \]
and using relations (31) (34) we simplify the expression for $\hat{L}_0$ to the desired form (30) which finishes the proof of the formulae (24 30).
5 Poisson–Lie T–plurality as canonical transformation

In order to show that (29,30) is really a canonical transformation we shall use the expressions for Poisson brackets of $P_a$ and $J_a = L^a_\sigma + \Pi(g)_{ab}P_b$, i.e. $J = L^b_\sigma + \Pi(g) \cdot P$ (36) introduced in [5], namely

\[\{J^a, J^b\} = \tilde{f}^{ab}_c J^c \delta(\sigma - \sigma'), \]
\[\{P_a, P_b\} = f^{ab}_c P_c \delta(\sigma - \sigma'), \]
\[\{J^a, P_b\} = (f^{ac}_b J^c - \tilde{f}^{ac}_b P_c) \delta(\sigma - \sigma') + \delta^a_b \delta'(\sigma - \sigma'). \]

These Poisson brackets are equivalent to the canonical ones

\[\{P_\mu, P_\nu\} = \{\partial_\sigma \phi^\mu, \partial_\sigma \phi^\nu\} = 0, \]
\[\{\partial_\sigma \phi^\mu, P_\nu\} = \delta^\mu_\nu \delta'(\sigma - \sigma'). \]

(37) (38)

Further, using the definition (36) we note that the transformation of the canonical momentum (29) can be written as

\[\hat{P} = S^t \cdot P + Q^t \cdot J \]

which reminds of the inverse of the transformation (10) of the basis elements of the Drinfel’d double. From this one can conjecture that

\[\hat{J} = R^t \cdot P + K^t \cdot J. \]

(39) (40)

and a simple calculation using the definition (36) of $J$ proves that (40) is indeed equivalent to (30).

To prove the invariance of the Poisson brackets (37) (and thus of (38)) under the Poisson–Lie T–plurality transformations (29,30) or (39,40) it is useful to note that their structure strongly reminds of the Lie structure (11) of the Drinfel’d double, namely that the Poisson brackets (37) can be written in the compact form

\[\{Y_\alpha, Y_\beta\} = \mathcal{F}_{\alpha\beta\gamma} Y_\gamma \delta(\sigma - \sigma') + B_{\alpha\beta} \delta'(\sigma - \sigma') \]

where $\alpha, \beta, \gamma = 1, \ldots, \dim \mathfrak{d}$,

\[Y = \begin{pmatrix} P \\ J \end{pmatrix}, \]

(41) (42)

$\mathcal{F}_{\alpha\beta\gamma}$ are structure coefficients of the Drinfel’d double and $B_{\alpha\beta}$ are matrix elements of the bilinear form $\langle \ldots \rangle$ in the basis $T_{\alpha}, \tilde{T}^\alpha$ of $\mathfrak{d}$. From this compact form it is clear that the Poisson brackets (41) are form–invariant under the transformation (39,40) that is an analog of the transformation (10) of bases of the Drinfel’d double which transforms $f, \tilde{f}$ to $\hat{f}, \bar{f}$ and preserves the duality of bases, i.e. $B_{\alpha\beta}$. Consequently,
the canonical Poisson brackets are invariant, i.e. (38) is transformed by Poisson–Lie T–plurality to

\[ \{ \hat{P}_{\mu}, \hat{P}_{\nu} \} = \{ \partial_{\sigma} \hat{\phi}^{\mu}, \partial_{\sigma} \hat{\phi}^{\nu} \} = 0, \]
\[ \{ \partial_{\sigma} \hat{\phi}^{\mu}, \hat{P}_{\nu} \} = \delta_{\nu}^{\sigma} (\sigma - \sigma'). \] (43)

Finally, we compute the Hamiltonian density

\[ \mathcal{H} = \partial_{\tau} \phi^{\mu} \mathcal{P}_{\mu} - \mathcal{L} \] (44)

where the Lagrangian density is deduced from the action (39)

\[ \mathcal{L} = \frac{1}{2} L_{+}(g) \cdot E(g) \cdot L_{+}^{\dagger}(g) = \frac{1}{4} (L_{+}(g) \cdot E(g) \cdot L_{-}^{\dagger}(g) + L_{-}(g) \cdot E^{\dagger}(g) \cdot L_{+}^{\dagger}(g)) \]

and we have used an obvious identity valid for any column vector \( x \) and matrix \( A \)

\[ x^{t} A x = x^{t} A^{t} x = \frac{1}{2} x^{t} (A + A^{t}) x. \] (45)

We recall that due to the definition (22,24) of the canonical momentum we have

\[ \partial_{\tau} \phi^{\mu} \mathcal{P}_{\mu} = \frac{1}{2} (\partial_{+} \phi^{\mu} + \partial_{-} \phi^{\mu}) \mathcal{P}_{\mu} = \frac{1}{4} (L_{+}(g) + L_{-}(g)) \cdot (E(g) \cdot L_{-}^{\dagger}(g) + E^{\dagger}(g) \cdot L_{+}^{\dagger}(g)) \]

Substituting into the definition of Hamiltonian density (44) we find

\[ \mathcal{H} = \frac{1}{4} (L_{-}(g) \cdot E(g) \cdot L_{-}^{\dagger}(g) + L_{+}(g) \cdot E(g) \cdot L_{+}^{\dagger}(g)) \] (46)

where the substitution for the left–invariant fields \( L_{-}(g), L_{+}(g) \) in terms of the canonical variables (26) is understood. Performing explicitly the substitution (26) we get for the Hamiltonian density the formula

\[ \mathcal{H} = \frac{1}{2} (P_{\tau} - L_{\sigma} \cdot B) \cdot G^{-1} \cdot (P + B \cdot L_{\sigma}^{\dagger}) + \frac{1}{2} L_{\sigma} \cdot G \cdot L_{\sigma}^{\dagger} \] (47)

used in (41,5) where \( G, B \) are symmetric and antisymmetric part of \( E(g) \), respectively, i.e.

\[ G = \frac{1}{2} (E(g) + E^{\dagger}(g)), \quad B = \frac{1}{2} (E(g) - E^{\dagger}(g)). \]

The Hamiltonian density of the \( \sigma \)–model obtained by T–plurality transformation can be written analogously as

\[ \hat{\mathcal{H}} = \frac{1}{4} \left( \hat{L}_{-}(\hat{g}) \cdot \hat{E}(\hat{g}) \cdot \hat{L}_{-}^{\dagger}(\hat{g}) + \hat{L}_{+}(\hat{g}) \cdot \hat{E}(\hat{g}) \cdot \hat{L}_{+}^{\dagger}(\hat{g}) \right) \]

where we assume, as above, that the left–invariant fields \( \hat{L}_{-}(\hat{g}), \hat{L}_{+}(\hat{g}) \) are expressed in terms of the new canonical variables. Using the transformation of the left–invariant fields (19,26) we find

\[ \hat{\mathcal{H}} = \frac{1}{4} \left( L_{-}(g) \cdot E^{\dagger}(g) \cdot (S - E_{0}^{-1} Q) \cdot \hat{E}(\hat{g})^{-\dagger} \cdot (S^{t} - Q^{t} E_{0}^{-1} \cdot E(g) \cdot L_{-}^{\dagger}(g) + L_{+}(g) \cdot E(g) \cdot (S + E_{0}^{-1} Q) \cdot \hat{E}(\hat{g})^{-\dagger} \cdot (S^{t} + Q^{t} E_{0}^{-1} \cdot E^{\dagger}(g) \cdot L_{+}^{\dagger}(g)) \right). \]
Due to the identity (45) we can replace \( \tilde{E}^{-t}(\hat{g}) \) by 
\[
\tilde{E}^{-t}(\hat{g}) + \tilde{E}^{-1}(\hat{g}) = \tilde{E}^{-t} + \tilde{E}^{-1}.
\]
From the definition of \( \tilde{E}_0 \) [14] and the duality of bases [12], i.e.
\[
QR^t = 1 - KS^t, \quad RS^t = -SR^t, \quad KQ^t = -QK^t, \quad RQ^t = 1 - SK^t
\]
we get
\[
\hat{\mathcal{H}} = \frac{1}{8} \left( L_-(g) \cdot E^t(g) \cdot (E_0^{-1} + E_0^{-t}) \cdot E(g) \cdot L'_-(g) 
+ L_+(g) \cdot E(g) \cdot (E_0^{-1} + E_0^{-t}) \cdot E^t(g) \cdot L'_+(g) \right).
\]
Using the relation (32) we replace \( E_0^{-1} + E_0^{-t} \) by \( E^{-1}(g) + E^{-t}(g) \) and employ once again the identity (45), getting the final result
\[
\hat{\mathcal{H}} = \frac{1}{4} \left( L_-(g) \cdot E(g) \cdot L'_-(g) + L_+(g) \cdot E(g) \cdot L'_+(g) \right).
\]
Consequently, we find that the Hamiltonian density is preserved under Poisson–Lie T–plurality transformation,
\[
\hat{\mathcal{H}} = \mathcal{H}.
\]
We could have equivalently used the form of the Hamiltonian density (47) together with the transformation of canonical variables (29,30). In the approach we used the computation of \( \hat{\mathcal{H}} \) in terms of original canonical variables \( P, L_\sigma \), or equivalently the left–invariant fields \( L_-(g), L_+(g) \), is significantly algebraically simpler.

6 Conclusions

We have derived a transformation of the canonical structure of dualizable \( \sigma \)–models, more precisely their (pseudo)canonical variables, Poisson brackets and Hamiltonian densities under the Poisson–Lie T–plurality. It turned out that by a suitable choice of the variables the Poisson brackets acquire a rather symmetric form that can be turned into the compact form (41). This expression is explicitly form–invariant with respect to the choice of basis in the Drinfel’d double on which the \( \sigma \)–models are defined. This proves the invariance of the canonical structure under the Poisson–Lie T–plurality because its transformations follow from various decompositions of the Drinfel’d double, i.e. special transformations of its bases that turn one decomposition into another.

The explicit formulae for transformations of extremal left and right–invariant fields (19,20) and canonical variables (29,30,39,40) can be used for further investigation of particular properties of \( \sigma \)–models related by the Poisson–Lie T–plurality transformations, for example their boundary conditions.

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