In this paper we study the fractional Laplacian \((-\Delta)^{\sigma/2}\) on the \(n\)-dimensional torus \(\mathbb{T}^n\), \(n \geq 1\). Our approach is based on the semigroup language.

We obtain a pointwise integro-differential formula for \((-\Delta)^{\sigma/2}f(x)\), \(0 < \sigma < 2\), \(x \in \mathbb{T}^n\), that is derived via the heat kernel on \(\mathbb{T}^n\). The limits as \(\sigma \to 0^+\) and \(\sigma \to 2\) are computed. Regularity estimates on Hölder, Lipschitz and Zygmund spaces are deduced.

We also present a general extension problem to characterize any fractional power of an operator \(L^\gamma\), \(\gamma > 0\). Here \(L\) is a general nonnegative selfadjoint operator defined in an \(L^2\)-space. In particular, \(L\) can be taken to be the Laplace–Beltrami operator on a Riemannian manifold or a divergence form elliptic operator with measurable coefficients on a bounded domain. This generalizes to all \(\gamma > 0\) and a large class of operators \(L\) previous known results by Caffarelli–Silvestre. This extension result is of independent interest. The extension problem and the theory of degenerate elliptic equations is applied to prove interior and boundary Harnack’s inequalities for \((-\Delta)^{\sigma/2}\).

1. Introduction

Very recently, there has been an increasing interest in the study of nonlinear partial differential equations involving fractional operators. Such problems arise naturally in applications, like Fluid Dynamics [5, 10], Strange Kinetics and Anomalous Transport [18], Financial Mathematics [3, 19], among many others. Fractional operators are very well-known from the Functional Analysis point of view, see the classical book by K. Yosida [26]. Nevertheless, there are some issues in these nonlinear nonlocal fractional problems, not covered by the general theory, in which tools like pointwise formulas, Hölder estimates and Harnack’s inequalities are needed [2, 3, 5, 10, 19, 22].

We study properties and regularity estimates of fractional powers of the Laplacian \(-\Delta = -\Delta_{\mathbb{T}^n}\) on the \(n\)-dimensional torus \(\mathbb{T}^n\), \(n \geq 1\). Applying an approach based on the heat semigroup, we obtain, in Theorem 3.4 below, a pointwise integro-differential formula for \((-\Delta)^{\sigma/2}f(x)\), \(0 < \sigma < 2\), of the form

\[
(-\Delta)^{\sigma/2}f(x) = \text{P. V.} \int_{\mathbb{T}^n} (f(x) - f(y)) K^{\sigma/2}(x-y) \, dy,
\]

where \(K^{\sigma/2}(x)\) is a suitable singular kernel. By using this nonlocal formula, we show in Propositions 3.5 and 3.6 that for all \(x \in \mathbb{T}^n\),

\[
\lim_{\sigma \to 0^+} (-\Delta)^{\sigma/2}f(x) = f(x) - \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(y) \, dy
\]

and

\[
\lim_{\sigma \to 2} (-\Delta)^{\sigma/2}f(x) = -\Delta f(x),
\]

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respectively. Observe the contrast of the identity (1.2) above with the case of the Laplacian $-\Delta_{\mathbb{R}^n}$ on $\mathbb{R}^n$, where $\lim_{\sigma\to 0^+} (-\Delta_{\mathbb{R}^n})^{\sigma/2} f(x) = f(x)$, see [22, Proposition 2.5]. Taking into account the multiple Fourier series definition of the fractional Laplacian on $\mathbb{T}^n$,
\begin{equation}
(-\Delta)^{\sigma/2} f(x) = \sum_{\nu \in \mathbb{Z}^n} |\nu|^\sigma c_\nu (f) e^{i\nu \cdot x}, \quad x \in \mathbb{T}^n,
\end{equation}
see Section 2, the identities in (1.2) and (1.3) are obvious, but as limits in $L^2(\mathbb{T}^n)$. Here we prove that the limits actually hold in the pointwise sense for a large class of functions. A crucial step for it is to compute all the constants appearing in the kernel $K^{\sigma/2}(x)$ exactly. We manage to do it avoiding the computation of inverse multiple Fourier series.

Regularity properties of the fractional Laplacian on Hölder, Lipschitz and Zygmund spaces are also analyzed, see Theorem 4.4 below. Our idea here is, first, to characterize all these spaces of smooth functions with the heat semigroup as done in Proposition 4.4, and, secondly, to express the fractional Laplacian on $\mathbb{T}^n$ again with the semigroup:
\begin{equation}
(-\Delta)^{\sigma/2} f(x) = \frac{1}{\Gamma(-\sigma/2)} \int_0^\infty \left( e^{t\Delta} f(x) - f(x) \right) \frac{dt}{t^{1+\sigma/2}}, \quad 0 < \sigma < 2.
\end{equation}

In this way we avoid the rather cumbersome computations with the pointwise formula (1.1). Moreover, these ideas can be applied in other contexts as well. We mention that Schauder and regularity estimates for the fractional Laplacian on $\mathbb{R}^n$ had been previously derived by L. Silvestre in [19] with the aid of the pointwise formula. Hölder estimates for the fractional harmonic oscillator $(-\Delta + |x|^2)^{\sigma/2}$ can be found in [22, 24].

On the other hand, by using an extension problem due to Stinga and Torrea [23, Theorem 1.1] we obtain, in Theorems 7.1 and 7.3, interior and boundary Harnack’s inequalities for $(-\Delta)^{\sigma/2}$ on $\mathbb{T}^n$. Recall that in [4] L. Caffarelli and L. Silvestre showed that any fractional power of the Laplacian on $\mathbb{R}^n$ can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. Namely, let $u$ be the solution to the boundary value problem
\begin{equation}
\begin{cases}
\Delta_{\mathbb{R}^n} u + \frac{1-\sigma}{y} u_y + u_{yy} = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x,0) = f(x), & \text{on } \mathbb{R}^n;
\end{cases}
\end{equation}
then there exists a constant $c_\sigma > 0$ such that
\begin{equation}
- \lim_{y \to 0^+} u_y(x,y) = c_\sigma (-\Delta_{\mathbb{R}^n})^{\sigma/2} f(x), \quad x \in \mathbb{R}^n.
\end{equation}

This was used, together with local PDE techniques, to give a novel proof of Harnack’s inequalities for $(-\Delta_{\mathbb{R}^n})^{\sigma/2}$. Since the extension equation is a degenerate elliptic equation in $\mathbb{R}^{n+1}_+$, the theory by E. B. Fabes, D. Jerison, C. Kenig and R. Serapioni [11, 12] was applied. We point out that interior Harnack’s inequality for the fractional Laplacian on $\mathbb{R}^n$ is classical [15]. Boundary Harnack’s estimates were first proved by K. Bogdan in [11] by using probabilistic techniques. In [23] the Caffarelli–Silvestre extension problem was generalized to apply to general fractional operators $L^\gamma$, $0 < \gamma < 1$, where $L$ is a nonnegative self-adjoint linear operator on a Hilbert space. The main idea in [23] was to introduce the semigroup language in order to study fine properties of fractional operators. The extension problem technique was used to prove interior Harnack’s inequality for the fractional harmonic oscillator (see [23, Theorem 1.2]) and more general fractional divergence form elliptic operators $(-\div a(x)V + V)^{\sigma/2}$ on domains of $\mathbb{R}^n$ (see [25, Theorem A]). In such cases the extension equation becomes a degenerate elliptic Schrödinger equation, so Harnack’s inequality proved by C. E. Gutiérrez in [14] is needed. The advantage is that the extension equation is a local PDE for which well-known techniques from the Calculus of Variations can be applied, see [4, 23, 25].

In the present paper the extension result of [23] is extended to fractional operators $L^\gamma$, where $\gamma$ is any noninteger positive number. The result is contained in Theorem 6.1 below and it is of independent interest. In this way we answer a question raised by Ricardo G. Durán about how to characterize higher-order fractional Laplacians via an extension problem. More comments in this direction can be found in the remarks of Section 6.
We would like to stress that the semigroup language we adopt here is the most adequate for our purposes. In particular, it allows us to compute all the constants exactly, to study regularity properties in a simple and general way and to have an explicit solution for the extension problem in terms of the underlying semigroup.

We would also like to point out the following. At first glance, it seems that one of the main difficulties to overcome when using \( (1.5) \) to treat the fractional Laplacian is that the heat kernel \( W_t(x) \) on \( \mathbb{T}^n \) is given as a series \( (2.3) \). There is not, up to our knowledge, a closed expression for \( W_t(x) \). The lack of it (as it happens, for instance, in a general Riemannian manifold) may well block the way to develop the analysis. It is clear that the singular kernel \( K^{\sigma/2}(x) \) must be expressed as a series too, see \( (3.3) \) below. It came as a surprise to us that, thanks to the Poisson summation formula and the general semigroup language, we were still able to develop the analysis and manage the multiple Fourier series. With this extra work, the results that one would expect with a closed formula follow here too.

If we would like to make the full analysis with a closed expression for a semigroup kernel on the torus, we could use the Poisson semigroup in \( \mathbb{T} \). Indeed, the Poisson kernel on \( \mathbb{T} \) has the tractable closed formula \( (5.2) \). But now the problem is that \( (1.5) \) is not the suitable definition for \( (-\Delta)^{\sigma/2} \) since it involves the heat kernel. We need to use a different equivalent definition to make appear the Poisson semigroup, see \( (5.3) \) below. In this way more close formulas for \( K^{\sigma/2}(x) \) can be obtained and the results follow as well, see Section \( 5 \).

We close this circle of ideas by pointing out that any of the two approaches described above (heat semigroup –with kernel given as a series– or Poisson semigroup –with kernel given by a closed formula, though only in one dimension–) are equally effective to establish the Hölder regularity theory, see Section \( 4 \) and Subsection \( 5.4 \).

The paper is divided into two parts. In the first one we focus on the definition of the fractional Laplacian on the \( n \)-torus, the pointwise formula \( (1.1) \), the limits \( (1.2) \) and \( (1.3) \) and the regularity estimates in Hölder, Lipschitz and Zygmund spaces for \( (-\Delta)^{\sigma/2} \) in \( \mathbb{T}^n \). Also, the one dimensional case is treated from the point of view of the Poisson semigroup. Part 2 contains the generalization of the extension problem to any positive power \( L^\gamma \), \( \gamma > 0 \) and the interior and boundary Harnack’s inequalities for \( (-\Delta)^{\sigma/2} \).

Throughout this paper the letters \( c \) and \( C \) denote positive constants that may change at each occurrence. For two positive quantities \( A \) and \( B \) we write \( A \sim B \) to denote that there exist constants \( 0 < c \leq C \) independent of \( \sigma \) such that \( c \leq A/B \leq C \).

**Part 1. Pointwise formulas and Hölder spaces**

2. Definition of the fractional Laplacian on \( \mathbb{T}^n \)

Denote by \( \mathbb{S} \) the unit circle in the complex plane \( C \). Let \( \mathbb{T}^n = \mathbb{S} \times \cdots \times \mathbb{S} \) (\( n \)-times), \( n \geq 1 \), be the \( n \)-dimensional torus. Consider the lattice \( \mathbb{Z}^n \) of points in \( \mathbb{R}^n \) having integer coordinates. There is a natural identification \( \mathbb{R}^n / (2\pi \mathbb{Z}^n) \equiv \mathbb{T}^n \) given by the mapping \( (x_1, \ldots, x_n) \mapsto (e^{ix_1}, \ldots, e^{ix_n}) \). From here we can identify functions defined on \( \mathbb{T}^n \) with \( 2\pi \mathbb{Z}^n \)-periodic functions \( f \) on \( \mathbb{R}^n \), that is, functions on \( \mathbb{R}^n \) for which \( f(x + 2\pi \nu) = f(x) \), for all \( x \in \mathbb{R}^n \) and \( \nu \in \mathbb{Z}^n \). This type of periodic functions on \( \mathbb{R}^n \) are completely determined once their values at the so-called fundamental cube

\[
Q_n := (-\pi, \pi]^n
\]

are fixed. Hence we will sometimes identify the \( n \)-torus and functions on \( \mathbb{T}^n \) with the fundamental cube \( Q_n \) and \( 2\pi \mathbb{Z}^n \)-periodic functions on \( \mathbb{R}^n \), respectively. Measurability of functions defined on \( \mathbb{T}^n \) is given through this identification. In the same way, \( f \in C^\infty(\mathbb{T}^n) \) if and only if \( f \) has a \( 2\pi \mathbb{Z}^n \)-periodic extension to \( \mathbb{R}^n \) which belongs to \( C^\infty(\mathbb{R}^n) \). Also, integration over \( \mathbb{T}^n \) can be described in terms of Lebesgue integration over \( Q_n \):

\[
\int_{\mathbb{T}^n} f(x) \, dx = \int_{Q_n} f(x) \, dx,
\]
where in the right hand side the integrand is the restriction to $Q_n$ of the periodic extension of $f$. In this way we have a natural way to define $L^p$ spaces on $\mathbb{T}^n$, $1 \leq p \leq \infty$.

Let $\Delta$ be the Laplace–Beltrami operator on the $n$-torus. It is well-known that $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, where $x_j$ is the variable describing each circle $S_j = \{ e^{i x_j} : -\pi < x_j \leq \pi \}$. Given an integrable function $f$ on the torus $\mathbb{T}^n$, we write its multiple Fourier series expansion as

$$f(x) = \sum_{\nu \in \mathbb{Z}^n} c_\nu(f) e^{i \nu \cdot x}, \quad c_\nu(f) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(x) e^{-i \nu \cdot x} \, dx,$$

where $x \cdot \nu = x_1 \nu_1 + \cdots + x_n \nu_n$, $x \in \mathbb{T}^n$. The heat semigroup generated by $\Delta$ is defined by

$$e^{t\Delta} f(x) = T_t f(x) := \sum_{\nu \in \mathbb{Z}^n} e^{-t|\nu|^2} c_\nu(f) e^{i \nu \cdot x}, \quad f \in L^2(\mathbb{T}^n), \quad t \geq 0.$$

For $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$, set $|\nu| = (\nu_1^2 + \cdots + \nu_n^2)^{1/2}$. Then $T_t f(x)$ is the solution of the heat equation $\partial_t v = \Delta v$ in $\mathbb{T}^n \times (0, \infty)$, with initial condition $v(x,0) = f(x)$ on $\mathbb{T}^n$. We have the convolution formula

$$T_t f(x) = \int_{\mathbb{T}^n} W_t(x-y) f(y) \, dy, \quad x \in \mathbb{T}^n,$$

where, for $x \in \mathbb{T}^n$ and $t > 0$,

$$W_t(x) = \frac{1}{(2\pi)^n} \sum_{\nu \in \mathbb{Z}^n} e^{-t|\nu|^2} e^{i \nu \cdot x} = \frac{1}{(4\pi t)^{n/2}} \sum_{\nu \in \mathbb{Z}^n} e^{-\frac{|x-\nu|^2}{4t}},$$

is the heat kernel on $\mathbb{T}^n$. This follows from Poisson summation formula, see [21, Chapter VII, Corollary 2.6]. Observe that $\frac{1}{(4\pi t)^{n/2}}$ is in fact the periodization of the classical heat kernel on $\mathbb{R}^n$.

Let $f \in C^\infty(\mathbb{T}^n)$. For $0 < \sigma < 2$, we define the fractional powers of the Laplacian on $\mathbb{T}^n$ as in [1.4]. If in the definition [1.4] we put $\sigma = 2$, then we recover the Laplacian $-\Delta f$, for $f \in C^\infty(\mathbb{R}^n)$. Note that, since $f$ is smooth, for any $N \in \mathbb{N}$ there exists a constant $C_{N,f}$ such that $|c_\nu(f)| \leq C_{N,f} |\nu|^{-N}$, for all $\nu \in \mathbb{Z}^n$, $\nu \neq 0$. Therefore, the series that defines $(-\Delta)^{\sigma/2} f$ is absolutely convergent and it is a $C^\infty(\mathbb{T}^n)$-function. We also have the symmetry property $((-\Delta)^{\sigma/2} f, g)_{L^2(\mathbb{T}^n)} = (f, (-\Delta)^{\sigma/2} g)_{L^2(\mathbb{T}^n)}$, $g \in C^\infty(\mathbb{T}^n)$. In fact, the series in [1.4] converges in $L^2(\mathbb{T}^n)$ whenever $f$ has the property that $\sum_{\nu \in \mathbb{Z}^n} |\nu|^{2\sigma} |c_\nu(f)|^2 < \infty$, that is, when $f$ is in the Sobolev space $H^\sigma = \text{Dom}((-\Delta)^{\sigma/2})$. This allows us to extend the definition of $(-\Delta)^{\sigma/2}$ to this class. The sum in [1.4] is over $\mathbb{Z}^n \times \{(0, \ldots, 0)\}$ and it is clear that the limits in [1.2] and [1.3] hold in $L^2(\mathbb{T}^n)$.

It is easy to check that for any $\lambda > 0$ and $0 < \sigma < 2$ we have the integral identity

$$\lambda^{\sigma/2} = \frac{1}{\Gamma(-\sigma/2)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+\sigma/2}}.$$

Plugging this into [1.4] with $\lambda = |\nu|^2$ and interchanging the summation with the integration, we get

$$(-\Delta)^{\sigma/2} f \equiv \sum_{\nu \in \mathbb{Z}^n} |\nu|^{2\sigma} c_\nu(f)\delta_\nu = f$$

for $f \in C^\infty(\mathbb{T}^n)$. It can be easily checked that the constant appearing in [1.5] satisfies

$$\frac{(\sigma - 2)^{-1}}{\Gamma(-\sigma/2)} \rightarrow \frac{1}{2} \quad \text{as} \quad \sigma \rightarrow 2^-, \quad \frac{-2/\sigma}{\Gamma(-\sigma/2)} \rightarrow 1, \quad \text{as} \quad \sigma \rightarrow 0^+.$$

### 3. Pointwise formula for the fractional Laplacian on $\mathbb{T}^n$

In this section we obtain the pointwise formula for $(-\Delta)^{\sigma/2} f(x)$ when $f$ belongs to the Hölder spaces. We also prove the pointwise limits [1.2] and [1.3].

**Definition 3.1** (Hölder spaces on $\mathbb{T}^n$). Let $0 < \alpha \leq 1$ and $k \in \mathbb{N}_0$. A continuous real function $f$ defined on $\mathbb{T}^n$ belongs to the Hölder space $C^{k,\alpha}(\mathbb{T}^n)$, if $f \in C^k(\mathbb{T}^n)$ and

$$[f^{(k)}]_{C^{k,\alpha}(\mathbb{T}^n)} := \sup_{x,y \in \mathbb{T}^n \atop x \neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} < \infty.$$

We define the norm in the spaces $C^{k,\alpha}(\mathbb{T}^n)$ to be $\|f\|_{C^{k,\alpha}(\mathbb{T}^n)} := \sum_{0 \leq \ell \leq k} \|f^{(\ell)}\|_{L^\infty(\mathbb{T}^n)} + [f^{(k)}]_{C^{k,\alpha}(\mathbb{T}^n)}$. 

Remark 3.2. The condition \( f \in C^{k,\alpha}(\mathbb{T}^n) \) is equivalent to ask for \( f \) to have a \( 2\pi \mathbb{Z}^n \)-periodic extension in \( C^k(\mathbb{R}^n) \) that satisfies
\[
|f^{(k)}(x) - f^{(k)}(y)| \leq C_{k,\alpha} \left| \left( \frac{x_1 - y_1}{2}, \ldots, \frac{x_n - y_n}{2} \right) \right|^{\alpha},
\]
for all \( x, y \in \mathbb{R}^n, x \neq y \). The least constant \( C_{k,\alpha} \) for which the inequality above holds is equivalent to the value of \( [f^{(k)}]_{C^\alpha} \) in Definition 3.1.

Consider the test space \( C^\infty(\mathbb{T}^n) \) endowed with the family of norms
\[
\|\phi\|_k := \|(I-\Delta)^k \phi\|_{L^2(\mathbb{T}^n)} = \sum_{\nu \in \mathbb{Z}^n} (1 + |\nu|^2)^k |\nu(\phi)|^2, \quad k \geq 1,
\]
A real linear functional \( S \) on \( C^\infty(\mathbb{T}^n) \) is a periodic distribution if it satisfies the following continuity property: if \( \phi_j \in C^\infty(\mathbb{T}^n) \), \( \|\phi_j\|_k \to 0 \) as \( j \to \infty \) for every \( k \in \mathbb{N} \), then \( S(\phi_j) \to 0 \). Note that if \( f \in L^1(\mathbb{T}^n) \) then \( f \) defines a periodic distribution by \( f(\phi) = \int_{\mathbb{T}^n} f \phi \). See Schwartz [17, Chapter VII]. The fractional Laplacian on the torus is a continuous linear operator on \( C^\infty(\mathbb{T}^n) \). We remark that this is a difference with respect to the fractional Laplacian on \( \mathbb{R}^n \), which is not continuous on the natural test space for the Fourier transform, namely, the Schwartz class \( S(\mathbb{R}^n) \), see [19]. The following result will allow us to extend the definition of the fractional Laplacian from \( C^\infty(\mathbb{T}^n) \) to the Hölder classes.

Lemma 3.3. Suppose that \( S \) is a continuous linear operator on \( C^\infty(\mathbb{T}^n) \), such that \( \langle S\phi, \psi \rangle_{L^2(\mathbb{T}^n)} = \langle \phi, S\psi \rangle_{L^2(\mathbb{T}^n)} \), for all \( \phi, \psi \in C^\infty(\mathbb{T}^n) \), and
\[
S\phi(x) = \int_{\mathbb{T}^n} (\phi(x) - \phi(y)) K(x - y) \, dy, \quad \phi \in C^\infty(\mathbb{T}^n), \quad x \in \mathbb{T}^n.
\]
Assume that the kernel \( K \) above extends to a \( 2\pi \mathbb{Z}^n \)-periodic function on \( \mathbb{R}^n \) with
\[
|K(x)| \leq \frac{C_{n,\gamma}}{|x|^{n+\gamma}}, \quad x \in Q_n,
\]
for some \( 0 \leq \gamma < 1 \). Let \( f \in C^{0,\gamma+\varepsilon}(\mathbb{T}^n) \), with \( 0 < \gamma + \varepsilon \leq 1, \varepsilon > 0 \). Then \( Sf \) is well defined as a periodic distribution and it coincides with the continuous function
\[
Sf(x) = \int_{\mathbb{T}^n} (f(x) - f(y)) K(x - y) \, dy, \quad x \in \mathbb{T}^n.
\]
Proof. By (3.1) and the assumption on \( f \), the integral in (3.2) is absolutely convergent. Indeed, for each \( x \in \mathbb{T}^n \),
\[
\int_{\mathbb{T}^n} |f(x) - f(y)| |K(x - y)| \, dy \leq C \int_{Q_n} |x - y|^{-n} \, dy < \infty.
\]
As \( f \in L^1(\mathbb{T}^n) \), we can define \( Sf \) as a periodic distribution by using the symmetry of \( S \), that is, \( (Sf)(\phi) := \int_{\mathbb{T}^n} fS\phi \, d\phi \in C^\infty(\mathbb{T}^n) \). Let \( f_j(x) = T_{1/j} f(x), j \in \mathbb{N}, x \in \mathbb{T}^n \), where \( T_t \) is the heat semigroup [2,2]. It is well known that \( f_j \in C^\infty(\mathbb{T}^n) \) and that \( f_j \to f, j \to \infty \), in \( L^p(\mathbb{T}^n), 1 \leq p \leq \infty \) (the latter is a consequence of [21, Chapter VII, Theorem 2.11]). It is easy to check that \( [f_j]_{C^{\gamma+\varepsilon}(\mathbb{T}^n)} \leq [f]_{C^{\gamma+\varepsilon}(\mathbb{T}^n)} \), for all \( j \). Now, from the \( L^p(\mathbb{T}^n) \)-convergence of \( f_j \) to \( f \), we can see that \( Sf_j \to Sf \) as periodic distributions, which is to say \( \lim_{j \to \infty} Sf_j(\phi) = \lim_{j \to \infty} \int_{\mathbb{T}^n} f_j S\phi = \int_{\mathbb{T}^n} f(S\phi) = (Sf)(\phi) \), for each \( \phi \in C^\infty(\mathbb{T}^n) \). Let \( \eta > 0 \) be arbitrary. There exists \( \delta > 0 \) such that
\[
C_{n,\gamma} [f]_{C^{\gamma+\varepsilon}(\mathbb{T}^n)} \int_{|x-y| < \delta, y \in Q_n} |x - y|^{-n} \, dy < \frac{\eta}{3}.
\]
Then, for all \( j \),
\[
\left| \int_{|x-y| < \delta, y \in \mathbb{T}^n} (f_j(x) - f_j(y)) K(x - y) \, dy \right| + \left| \int_{|x-y| < \delta, y \in \mathbb{T}^n} (f(x) - f(y)) K(x - y) \, dy \right| < \frac{2}{3} \eta.
\]
On the other hand, for all sufficiently large \( j \), uniformly in \( x \) in a compact subset of \( \mathbb{T}^n \). Therefore, the right hand side of (3.2) with \( f_j \) converges uniformly on compact subsets of \( \mathbb{T}^n \) to the right hand side of (3.2) with \( f \), and the limit is a continuous function. By uniqueness of the limits, (3.2) holds.

Next we derive the pointwise formula (1.1). The kernel \( K^{\sigma/2}(x) \) is given in (3.3) below. Formally, from (1.5), one can see that this is the correct kernel. The point is that (3.4) and (3.5) are valid for a large class of continuous functions—not just \( C^\infty(\mathbb{T}^n) \)—and in the pointwise sense—not just in \( L^2(\mathbb{T}^n) \). As a consequence of this, the pointwise semigroup formula (1.5) will also be true for Hölder continuous functions, see Remark 3.8 below.

**Theorem 3.4** (Pointwise formula). For \( 0 < \sigma < 2 \) we define the following positive kernel on \( \mathbb{T}^n \):

\[
K^{\sigma/2}(x) := \frac{1}{\Gamma(-\sigma/2)} \int_0^\infty W_t(x) \frac{dt}{t^{1+\sigma/2}} = \frac{2^\sigma \Gamma(n+\sigma/2)}{\Gamma(-\sigma/2) \pi^{n/2}} \sum_{\nu \in \mathbb{Z}^n} \frac{1}{|x-2\pi\nu|^n + \sigma}, \quad x \in \mathbb{T}^n, \ x \neq 0.
\]

(3.3)

1) Suppose that \( 0 < \sigma < 1 \). If \( f \in C^0,\sigma+\varepsilon(\mathbb{T}^n) \), for some \( \varepsilon > 0 \) such that \( 0 < \sigma + \varepsilon \leq 1 \), then \( (-\Delta)^{\sigma/2} f \) is a continuous function and, for all \( x \in \mathbb{T}^n \),

\[
(-\Delta)^{\sigma/2} f(x) = \int_{\mathbb{T}^n} (f(x) - f(y)) K^{\sigma/2}(x-y) \ dy.
\]

The integral above is absolutely convergent.

2) Suppose that \( 1 \leq \sigma < 2 \). If \( f \in C^{1,\sigma+\varepsilon}(\mathbb{T}^n) \), for some \( \varepsilon > 0 \) such that \( 0 < \sigma + \varepsilon - 1 \leq 1 \), then

\[
(-\Delta)^{\sigma/2} f(x) = \int_{\mathbb{T}^n} (f(x) - f(y) - \nabla f(x) \cdot (x-y)) K^{\sigma/2}(x-y) \ dy
\]

(3.4)

\[
= P.V. \int_{\mathbb{T}^n} (f(x) - f(y)) K^{\sigma/2}(x-y) \ dy
\]

(3.5)

\[
= \lim_{\delta \to 0^+} \int_{|x-y| \geq \delta, y \in \mathbb{T}^n} (f(x) - f(y)) K^{\sigma/2}(x-y) \ dy.
\]

**Proof.** The second identity in (3.3) follows from (2.3), Tonelli’s theorem and the change of variables \( |x-2\pi\nu|^2/(4t) = s \). Indeed,

\[
\int_0^\infty W_t(x) \frac{dt}{t^{1+\sigma/2}} = \sum_{\nu \in \mathbb{Z}^n} \int_0^\infty e^{-|x-2\pi\nu|^2/(4t)} \frac{dt}{t^{1+\sigma/2}}
\]

\[
= \frac{2^\sigma}{\pi^{n/2}} \left( \int_0^\infty e^{-s} \frac{ds}{s} \right) \sum_{\nu \in \mathbb{Z}^n} \frac{1}{|x-2\pi\nu|^n + \sigma}.
\]

To prove (3.4), suppose for the moment that \( f \in C^\infty(\mathbb{T}^n) \). By Tonelli’s theorem,

\[
T_{11}(x) = \int_{\mathbb{T}^n} W_t(x,y) \ dy = \sum_{\nu \in \mathbb{Z}^n} \int_{Q_{\nu}} e^{-|x-y|^2/(4t)} \ dy
\]

(3.6)

\[
= \sum_{\nu \in \mathbb{Z}^n} \int_{Q_{\nu}} e^{-|x-y|^2/(4t)} \ dy = \int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)} \ dy = 1 \equiv 1,
\]

\[
\int_{|x-y| \geq \delta, y \in \mathbb{T}^n} e^{-|x-y|^2/(4t)} \ dy = \int_{|x-y| \geq \delta, y \in \mathbb{T}^n} e^{-|x-y|^2/(4t)} \ dy.
\]
for all $x \in \mathbb{T}^n$, $t > 0$. Then, by the formula with the heat semigroup in (1.5),

$$(-\Delta)^{\sigma/2} f(x) = \frac{1}{\Gamma(\sigma/2)} \int_0^\infty \int_{\mathbb{T}^n} W_t(x-y)(f(x) - f(y)) \, dy \, \frac{dt}{t^{1+\sigma/2}}. \tag{3.7}$$

Since $f \in C^\infty(\mathbb{T}^n)$, by Tonelli’s theorem and (3.3),

$$\int_0^\infty \int_{\mathbb{T}^n} |W_t(x-y)(f(x) - f(y))| \, dy \, \frac{dt}{t^{1+\sigma/2}} = C \sum_{\nu \in \mathbb{Z}^n} \int_{Q_n} \frac{|f(x) - f(y + 2\pi \nu)|}{|x-y - 2\pi \nu|^{n+\sigma}} \, dy, \tag{3.8}$$

In the identities above we are identifying $\mathbb{T}^n$ with $Q_n$ and $f$ with its periodic extension. The last integral in (3.8) is absolutely convergent because the periodic extension of $f$ is bounded (which gives integrability at infinity) and Hölder continuous (which gives integrability at $x \sim y$). Hence we can apply Fubini’s theorem in (3.7) to obtain (3.4) for $f \in C^\infty(\mathbb{T}^n)$. Observe that for $\nu \neq 0$ and $x \in Q_n$, $|x-2\pi \nu| \geq 2\pi |\nu|/2$, so

$$0 \leq K^{\sigma/2}(x) \leq 2^\sigma \Gamma(\frac{n+\sigma}{2}) \frac{1}{\Gamma(-\sigma/2)|\pi|^{n/2}} \frac{1}{|\nu|^{n+\sigma}} + \sum_{\nu \neq 0} \frac{1}{|\pi \nu|^{n+\sigma}} \leq \frac{2^\sigma \Gamma(\frac{n+\sigma}{2})}{\sigma \Gamma(-\sigma/2)|\pi|^{n/2}} C_n, \quad x \in Q_n, \tag{3.9}$$

Above we applied the asymptotic behavior to the Gamma function to see that

$$\sum_{\nu \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|\nu|^{n+\sigma}} \leq C_n \sum_{k=1}^{\infty} \frac{1}{k^{1+\sigma}} \frac{\Gamma(k+n)}{\Gamma(k) k^n} \leq C_n \sum_{k=1}^{\infty} \frac{1}{k^{1+\sigma}} \leq C_n \sigma^{-1}.$$  

Therefore we can apply Lemma 3.3 to get (3.4) and the continuity of $(-\Delta)^{\sigma/2} f$ for $f \in C^{0,\sigma+\varepsilon}(\mathbb{T}^n)$.

Now we derive (3.5). Suppose again that $f \in C^\infty(\mathbb{T}^n)$. Using (1.6) and (3.6),

$$(-\Delta)^{\sigma/2} f(x) = \frac{1}{\Gamma(-\sigma/2)} \int_0^\infty \int_{\mathbb{T}^n} (f(x) - f(y)) W_t(x-y) \, dy \, \frac{dt}{t^{1+\sigma/2}}$$

$$= \frac{1}{\Gamma(-\sigma/2)} \int_0^\infty \int_{Q_n} (f(x) - f(x-z)) W_t(z) \, dz \, \frac{dt}{t^{1+\sigma/2}},$$

where in the last identity we applied that $\int_{Q_n} z_i W_t(z) \, dz = 0$, for all $i = 1, \ldots, n$. Since $f \in C^{1,\sigma+\varepsilon-1}(\mathbb{T}^n)$, we have that $|f(x) - f(x-z) - \nabla f(x) \cdot z| \leq C \|f\|_{C^{1,\sigma+\varepsilon-1}(\mathbb{T}^n)} |z|^{\sigma+\varepsilon}$. This and a computation parallel to (3.8) allow us to see that the double integral above is absolutely convergent. Therefore, for smooth functions $f$,

$$(-\Delta)^{\sigma/2} f(x) = \int_{\mathbb{T}^n} (f(x) - f(y) - \nabla f(x) \cdot (x-y)) K^{\sigma/2}(x-y) \, dy.$$

with $K^{\sigma/2}(x)$ as in (3.3). Noticing that the approximation argument in the proof of Lemma 3.3 can be applied also here –one just has to carry on the gradient in the computations–, we get the identity above for any $f \in C^{1,\sigma+\varepsilon-1}(\mathbb{T}^n)$, and the integral is absolutely convergent. For the principal value, note that $\int_{Q_n} z_i K^{\sigma/2}(z) \, dz = 0$.

We now consider the pointwise limit in (1.2).
Proposition 3.5. Let \( f \in C^{0,\alpha}(\mathbb{T}^n) \), for some \( 0 < \alpha \leq 1 \). Then, for each \( x \in \mathbb{T}^n \),
\[
\lim_{\sigma \to 0^+} (-\Delta)^{\sigma/2} f(x) = f(x) - \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(y) \, dy.
\]

Proof. We must check that
\[
(-\Delta)^{\sigma/2} f(x) - f(x) + \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(y) \, dy = \int_{\mathbb{T}^n} (f(x) - f(y)) \left[ K^{\sigma/2}(x-y) - \frac{1}{(2\pi)^n} \right] dy \to 0,
\]
as \( \sigma \to 0^+ \). Take any \( 0 < \sigma < \alpha \). Let us call \( d_\sigma := -1/\Gamma(-\sigma/2) > 0 \). By (3.3),
\[
K^{\sigma/2}(x) - \frac{1}{(2\pi)^n} = d_\sigma \int_0^1 W_t(x) \, \frac{dt}{t^{1+\sigma/2}} + d_\sigma \int_1^\infty W_t(x) \, \frac{dt}{t^{1+\sigma/2}} - \frac{1}{(2\pi)^n}
\]
(3.10)\[
= d_\sigma \int_0^1 W_t(x) \, \frac{dt}{t^{1+\sigma/2}} + d_\sigma \int_1^\infty \left( W_t(x) - \frac{1}{(2\pi)^n} \right) \, \frac{dt}{t^{1+\sigma/2}} + \frac{1}{(2\pi)^n} \left( 2d_\sigma - 1 \right)
\]
=: \( I_\sigma + II_\sigma + III_\sigma \).

As at the beginning of the proof of Theorem 3.4,
\[
0 \leq I_\sigma = d_\sigma \frac{2^\sigma}{\pi^{n/2}} \sum_{\nu \in \mathbb{Z}^n} \left( \int_0^\infty e^{-s} s^{n+\sigma} ds \right) \int_0^\infty e^{s|t-2\pi\nu|^2/4} \frac{ds}{s} \left[ 1 - \frac{1}{2\pi\nu} \right]^{n+\sigma}.
\]
\[
\leq d_\sigma \frac{2^\sigma}{\pi^{n/2}} \left( \int_0^\infty e^{-s/2} s^{n+\sigma} ds \right) \sum_{\nu \in \mathbb{Z}^n} e^{-c|t-2\pi\nu|^2} \left[ 1 - \frac{1}{2\pi\nu} \right]^{n+\sigma}.
\]

The constant in front of the sum above behaves like \( \sigma/2 \) as \( \sigma \to 0^+ \), see (2.4). For \( II_\sigma(x) \) note that, from (2.3), the Fourier coefficient of \( W_t(x) \) corresponding to the zero eigenvalue \( \nu = (0,\ldots,0) \) is exactly \( (1/(2\pi)^n) \). Then, using that for nonzero \( \nu \) we have \( |\nu| \geq 1 \),
\[
\left| W_t(x) - \frac{1}{(2\pi)^n} \right| \leq C \sum_{\nu \in \mathbb{Z}^n \setminus \{0\}} e^{-c|\nu|^2} \leq Ce^{-1/2} \sum_{\nu \in \mathbb{Z}^n \setminus \{0\}} e^{-\nu^2/2} \leq C e^{-t/2}, \quad t \geq 1.
\]

Hence, for a constant \( C \) independent of \( \sigma \),
\[
\left| II_\sigma \right| \leq c_\sigma C \int_1^\infty e^{-t/2} \, dt = d_\sigma C.
\]

This estimate and (2.4) give that \( II_\sigma \to 0 \) as \( \sigma \to 0^+ \). Also, \( III_\sigma \to 0 \) as \( \sigma \to 0^+ \) because of (2.4).

Collecting terms in (3.10),
\[
\int_{\mathbb{T}^n} \left| f(x) - f(y) \right| K^{\sigma/2}(x-y) - \frac{1}{(2\pi)^n} \, dy \leq d_\sigma \frac{2^{n+\sigma}}{\pi^{n/2}} \Gamma\left( \frac{n+\sigma}{2} \right) \int_{\mathbb{T}^n} \left| f(x) - f(y) \right| \left[ \sum_{\nu \in \mathbb{Z}^n} \frac{e^{-c|t-2\pi\nu|^2}}{|x - y - 2\pi\nu|^{n+\sigma}} \right] \, dy + \| f \|_{L^\infty(\mathbb{T}^n)} F(\sigma),
\]

where \( F(\sigma) \) is a function of \( \sigma \) (containing the bounds for the \( II_\sigma \) and \( III_\sigma \) that tends to 0 as \( \sigma \to 0^+ \)).

Also, the first term above goes to 0 as \( \sigma \to 0^+ \). Indeed, by the smoothness of \( f \) and (2.4),
\[
d_\sigma \int_{\mathbb{T}^n} \left| f(x) - f(y) \right| \left[ \sum_{\nu \in \mathbb{Z}^n} \frac{e^{-c|t-2\pi\nu|^2}}{|x - y - 2\pi\nu|^{n+\sigma}} \right] \, dy \leq d_\sigma C \sum_{\nu \in \mathbb{Z}^n} \int_{Q_\alpha} \frac{|x - y|^{\alpha}}{|x - y - 2\pi\nu|^{n+\sigma}} e^{-c|t-2\pi\nu|^2} \, dy
\]
\[
= d_\sigma C \int_{\mathbb{R}^n} \frac{e^{-c|x|^2}}{|x|^{n+\sigma-\alpha}} \, dx = d_\sigma C \frac{\Gamma(\frac{n+\sigma}{2})}{\Gamma(\frac{n+\alpha}{2})} \to 0, \quad \text{as} \ \sigma \to 0^+.
\]
Let us compute the pointwise limit in (1.3).

**Proposition 3.6.** Let $f \in C^2(\mathbb{T}^n)$. Then, for each $x \in \mathbb{T}^n$,

$$\lim_{\sigma \to 2^-} (-\Delta)^{\sigma/2} f(x) = -\Delta f(x).$$

**Proof.** Recall the pointwise formula (3.5) in Theorem 3.4. To shorten the notation, we let $Rf(x, z) := f(x + z) - f(x) - \nabla f(x) \cdot z$, and $\sin (\frac{z}{2}) := (\sin \frac{x_1}{2}, \ldots, \sin \frac{x_n}{2})$. Then, for any $1 \leq \sigma < 2$,

$$(-\Delta)^{\sigma/2} f(x) = -\int_{\mathbb{T}^n} \left( Rf(x, z) - \frac{1}{2} z^T D^2 f(x) z \right) K^{\sigma/2}(z) \, dz$$

$$- \int_{\mathbb{T}^n} \left( \frac{1}{2} z^T D^2 f(x) z - 2 \sin \left( \frac{z}{2} \right) z^T D^2 f(x) \sin \left( \frac{z}{2} \right) \right) K^{\sigma/2}(z) \, dz$$

$$- 2 \int_{\mathbb{T}^n} \sin \left( \frac{z}{2} \right) z^T D^2 f(x) \sin \left( \frac{z}{2} \right) K^{\sigma/2}(z) \, dz$$

$$=: J_{1,\sigma} + J_{2,\sigma} + J_{3,\sigma}.$$

Let us show first that $J_{1,\sigma}$ and $J_{2,\sigma}$ tend to 0 as $\sigma \to 2^-$. Let $\varepsilon$ be any positive number. Since $f \in C^2(\mathbb{T}^n)$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|D^2 f(x) - D^2 f(y)| < \varepsilon$ for all $y \in \mathbb{T}^n$ with $|x - y| < \delta$. Hence $|Rf(x, z) - \frac{1}{2} z^T D^2 f(x) z| \leq C_{n,f} |z|^2 \varepsilon$, if $|z| < \delta$. Then, by (3.9) and (2.4),

$$|J_{1,\sigma}| \leq \frac{C_{n,f} 2^{\sigma} \Gamma \left( \frac{n+\sigma}{2} \right)}{\Gamma \left( -\sigma/2 \right) \pi^{n/2} \sigma} \left[ \varepsilon \int_{|z|<\delta} |z|^{2-n-\sigma} \, dz + \int_{|z|>\delta} |z|^{-n-\sigma} \, dz \right]$$

$$\leq \frac{C_{n,f} 2^{\sigma} \Gamma \left( \frac{n+\sigma}{2} \right)}{\Gamma \left( -\sigma/2 \right) \pi^{n/2} \sigma} \left( \frac{\delta^{2-\sigma} \varepsilon}{2-\sigma} + \frac{1}{\sigma \delta^\sigma} \right) \rightarrow C \varepsilon, \quad \text{as} \quad \sigma \to 2^-.$$

Since $\varepsilon$ was arbitrary, $J_{1,\sigma} \to 0$ as $\sigma \to 2^-$. Let us continue with $J_{2,\sigma}$. We have

$$J_{2,\sigma} = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} (x) \int_{\mathbb{T}^n} \left( \frac{1}{2} z_i^2 - 2 \sin^2 \left( \frac{z_i}{2} \right) \right) K^{\sigma/2}(z) \, dz.$$

By using the Maclaurin series of $\cos z_i$ and (3.9),

$$|J_{2,\sigma}| \leq C_{n,f} \int_{\mathbb{T}^n} \left| \frac{1}{2} z_i^2 - 1 + \cos z_i \right| K^{\sigma/2}(z) \, dz$$

$$\leq \frac{C_{n,f} 2^{\sigma} \Gamma \left( \frac{n+\sigma}{2} \right)}{\Gamma \left( -\sigma/2 \right) \pi^{n/2} \sigma} \int_{Q_\delta} |z|^{1-n-\sigma} \, dz \leq \frac{C_{n,f} 2^{\sigma} \Gamma \left( \frac{n+\sigma}{2} \right)}{\Gamma \left( -\sigma/2 \right) \pi^{n/2} \sigma (4-\sigma)}$$

and, due to (2.4), the last expression tends to 0 as $\sigma \to 2^-$. We finally prove that $J_{3,\sigma} = -\Delta f(x)$. By taking into account (3.3), (2.3), Tonelli’s theorem and the orthogonality of the trigonometric system.
on the torus,
\[ J_{3,\sigma} = -2 \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x) \int_{\mathbb{T}^n} \sin^2 \left( \frac{z_i}{2} \right) K^{\sigma/2}(z) \, dz \]
\[ = 2 \Delta f(x) \int_{\mathbb{T}^n} \sin^2 \left( \frac{z_i}{2} \right) W_t(z) \, dz \frac{dt}{t^{1+\sigma/2}} \]
\[ = 2 \Delta f(x) \int_{\mathbb{T}^n} \sin^2 \left( \frac{z_i}{2} \right) \left[ \sum_{\nu \in \mathbb{Z}^n} e^{-t|\nu|^2} e^{i\nu \cdot z} \right] \, dz \frac{dt}{t^{1+\sigma/2}} \]
\[ = \frac{2^{n+1} \Delta f(x)}{\Gamma(-\sigma/2)(2\pi)^n} \int_{\mathbb{T}^n} \left( 1 - \cos z_1 \right) \prod_{k=1}^{n} \left( \frac{1}{2} + \sum_{\nu \in \mathbb{Z}^n} e^{-t\nu^2} \cos \nu_k z_k \right) \, dz \frac{dt}{t^{1+\sigma/2}} \]
\[ = \frac{\Delta f(x)}{\Gamma(-\sigma/2)} \int_{0}^{\infty} (1 - e^{-t}) \frac{dt}{t^{1+\sigma/2}} = -\Delta f(x). \]

Remark 3.7. When \( f \in C^{2,\alpha}(\mathbb{T}^n) \), \( 0 < \alpha \leq 1 \), we also have \( \lim_{\sigma \to 2^+} (-\Delta)^{\sigma/2} f(x) = -\Delta f(x). \) Indeed, for \( 2 < \alpha < 3 \) we can write \( \sigma = 2 + \epsilon \) for some \( \epsilon > 0 \). Then, by Proposition 3.5,
\[ \lim_{\sigma \to 2^+} (-\Delta)^{\sigma/2} f(x) = \lim_{\epsilon \to 0^+} (-\Delta)^{(2+\epsilon)/2} f(x) = \lim_{\epsilon \to 0^+} (-\Delta)(f(x) - \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} (-\Delta) f(y) \, dy) = -\Delta f(x). \]

This and Proposition 3.6 yield (1.3).

Remark 3.8. Under the hypothesis of Theorem 3.4, formula (1.5) holds. Indeed, just write down the kernel in (3.4) and (3.5) in terms of the heat kernel and apply Fubini’s theorem, by taking into account that \( J_{Q_n}(x-y)W_t(x-y) \, dy = 0 \) in the second case.

4. Regularity estimates in Hölder spaces

The following theorem is analogous to the results in [19] for \( (-\Delta_{\mathbb{R}^n})^{\sigma/2} \), Theorem A in [24] for \( (-\Delta_{\mathbb{R}^n} + |x|^2)^{\sigma/2} \), and Theorem 1.2 in [16] for general fractional Schrödinger operators \( (-\Delta + V)^{\sigma/2} \) on \( \mathbb{R}^n \), with \( V \geq 0 \) in the reverse Hölder class. It explains how the operators \( (-\Delta)^{\sigma/2} \) interact with the Hölder spaces \( C^{k,\alpha}(\mathbb{T}^n) \). Our proof relies on semigroup ideas and is much simpler and general than the proofs in [19] [24] that are based on the manipulation of the pointwise formulas. Note also that in [16] the Poisson semigroup is used instead of the heat semigroup. We will explain this approach in Section 5 in the one dimensional case.

Theorem 4.1 (Interaction with Hölder spaces). Let \( \alpha \in (0,1] \) and \( 0 < \sigma < 2 \).

1. Let \( f \in C^{0,\alpha}(\mathbb{T}^n) \) and \( \alpha < \sigma \). Then \( (-\Delta)^{\sigma/2} f \in C^{0,\alpha-\sigma}(\mathbb{T}^n) \) and
\[ \|(-\Delta)^{\sigma/2} f\|_{C^{0,\alpha-\sigma}(\mathbb{T}^n)} \leq C \|f\|_{C^{0,\alpha}(\mathbb{T}^n)}. \]

2. Let \( f \in C^{1,\alpha}(\mathbb{T}^n) \) and \( \alpha < \sigma \). Then \( (-\Delta)^{\sigma/2} f \in C^{1,\alpha-\sigma}(\mathbb{T}^n) \) and
\[ \|(-\Delta)^{\sigma/2} f\|_{C^{1,\alpha-\sigma}(\mathbb{T}^n)} \leq C \|f\|_{C^{1,\alpha}(\mathbb{T}^n)}. \]

3. Let \( f \in C^{1,\alpha}(\mathbb{T}^n) \) and \( \alpha \geq \sigma \), with \( \alpha - \sigma + 1 \neq 0 \). Then \( (-\Delta)^{\sigma/2} f \in C^{0,\alpha-\sigma+1}(\mathbb{T}^n) \) and
\[ \|(-\Delta)^{\sigma/2} f\|_{C^{0,\alpha-\sigma+1}(\mathbb{T}^n)} \leq C \|f\|_{C^{1,\alpha}(\mathbb{T}^n)}. \]

4. Let \( f \in C^{k,\alpha}(\mathbb{T}^n) \) and assume that \( k + \alpha - \sigma \) is not an integer. Then \( (-\Delta)^{\sigma/2} f \in C^{0,\beta}(\mathbb{T}^n) \), where \( \beta = k + \alpha - \sigma - l \).
In Theorem 4.1 (4) is a consequence of (1)–(3) by iteration. As we mentioned, Theorem 4.1 could be proved by following the ideas of [19] or [21] and taking into account that the Riesz transforms are bounded on $C^{0,\alpha}(\mathbb{T}^n)$, $0 < \alpha < 1$ (see Zygmund [27], Chapter III, (13.29)) for the one dimensional case, and Calderón–Zygmund [6, Theorem 11] for the multidimensional case. This method essentially uses the pointwise formulas (3.4) and (3.5).

We present a unified and more general way to prove Theorem 4.1. It stands on the semigroup characterization of $C^{k,\alpha}(\mathbb{T}^n)$ given in Proposition 4.4 below. The method has several advantages. First, we only need to use the semigroup formula for the fractional powers (1.5), so the idea could be extended to other fractional operators. Secondly, Zygmund’s classes can be also considered. The Zygmund space $\Lambda_1(\mathbb{T}^n)$ contains the Lipschitz space $C^{0,1}(\mathbb{T}^n)$, see [27], Chapter II and [20, p. 148]. Taking into account Proposition 4.4 below, we readily see that Theorem 4.1 is a direct corollary of Theorem 4.5.

**Definition 4.2.** Let $\beta > 0$ and $k = [\beta/2] + 1$. We define

\[
\Lambda_\beta(\mathbb{T}^n) := \left\{ f \in C(\mathbb{T}^n) : \| t^k \partial_t^k T_t f(x) \|_{L^\infty(\mathbb{T}^n)} \leq A t^{\beta/2}, \ t > 0 \right\}.
\]

If $A_k$ is the least constant appearing above, then the norm in $\Lambda_\beta(\mathbb{T}^n)$ is $\| f \|_{\Lambda_\beta(\mathbb{T}^n)} = \| f \|_{L^\infty(\mathbb{T}^n)} + A_k$.  

**Remark 4.3.** Observe that the estimate $\| t^k \partial_t^k T_t f(x) \|_{L^\infty(\mathbb{T}^n)} \leq A t^{\beta/2}$ is relevant only for $t$ near zero. This is so because for $t$ large we have a stronger inequality that follows from $f$ being bounded and (4.2) below. Thus, if $\beta < \beta'$ then we have the inclusion $\Lambda_{\beta'}(\mathbb{T}^n) \subset \Lambda_{\beta}(\mathbb{T}^n)$.

Next we relate the spaces $\Lambda_\beta(\mathbb{T}^n)$ with the Hölder spaces $C^{k,\alpha}(\mathbb{T}^n)$ and the Zygmund class $\Lambda_\ast$.

**Proposition 4.4.** Let $\beta > 0$.

(i) Let $f \in C(\mathbb{T}^n)$ and $k, \ell > \beta/2$ be two integers. Then the two conditions

\[
\| t^k \partial_t^k T_t f \|_{L^\infty(\mathbb{T}^n)} \leq A_k t^{\beta/2}, \quad \| t^\ell \partial_t^\ell T_t f \|_{L^\infty(\mathbb{T}^n)} \leq A_\ell t^{\beta/2}, \quad \text{for } t > 0,
\]

are equivalent. The least constants $A_k$ and $A_\ell$ that satisfy the inequalities above are comparable.

(ii) If $0 < \beta < 1$ then $\Lambda_\beta(\mathbb{T}^n) = C^{0,\beta}(\mathbb{T}^n)$, with equivalent norms.

(iii) We have $\Lambda_1(\mathbb{T}^n) = \Lambda_\ast(\mathbb{T}^n)$, the Zygmund class defined as the set of continuous functions $f$ on $\mathbb{T}^n$ such that $|f(x+h) + f(x-h) - 2f(x)| \leq C|h|$, for all $x \in \mathbb{T}^n$ and $h \in \mathbb{R}^n$. The quantity

\[
\| f \|_{\Lambda_\ast(\mathbb{T}^n)} = \| f \|_{L^\infty(\mathbb{T}^n)} + \sup_{|h| > 0} \frac{\| f(x+h) + f(x-h) - 2f(x) \|_{L^\infty(\mathbb{T}^n)}}{|h|},
\]

is equivalent to $\| f \|_{\Lambda_1(\mathbb{T}^n)}$. Consequently, $C^{0,1}(\mathbb{T}^n) \subset \Lambda_1(\mathbb{T}^n)$ and $\| f \|_{\Lambda_1(\mathbb{T}^n)} \leq C\| f \|_{C^{0,1}(\mathbb{T}^n)}$.

(iv) If $1 < \beta < 2$ then $f \in \Lambda_\beta(\mathbb{T}^n)$ if and only if $f$ is differentiable and $\nabla f \in \Lambda_{\beta-1}(\mathbb{T}^n)$. Moreover, $\| f \|_{\Lambda_{\beta}(\mathbb{T}^n)}$ is equivalent to $\| f \|_{L^\infty(\mathbb{T}^n)} + \| \nabla f \|_{\Lambda_{\beta-1}(\mathbb{T}^n)}$. Similarly, $\Lambda_2(\mathbb{T}^n) = \{ f : \nabla f \in \Lambda_\ast(\mathbb{T}^n) \}$.

(v) If $\beta$ is not an integer, then $\Lambda_\beta(\mathbb{T}^n) = C^{[\beta],\beta-[\beta]}(\mathbb{T}^n)$ with equivalent norms. Similarly, for $\beta = j \in \mathbb{N}$, we have

\[
\Lambda_j(\mathbb{T}^n) = \{ f : D^k f \in \Lambda_\ast, \text{ for all } k = (k_1, \ldots, k_n) \in \mathbb{N}^n \text{ such that } k_1 + \cdots + k_n = j \}.
\]

**Proof.** Note that (v) follows from (ii)–(iv) by iteration. Item (ii) of this proposition in the case when $\mathbb{T}^n$ and $T_t$ are replaced by $\mathbb{R}^n$ and the heat semigroup on $\mathbb{R}^n$ is already known. Though we think (ii) belongs to the folklore, we provide a proof here for completeness. In fact we can follow the ideas of the parallel description of the Hölder spaces on $\mathbb{R}^n$ in terms of the Poisson semigroup given in [20].

(i) This is consequence of the semigroup property of $T_t$ and the following simple estimate

\[
|\partial_t^k W_t(x)| \leq C_{n,k} \sum_{\nu \in \mathbb{Z}^n} e^{-c_k|x-2\nu t|^2/t} t^{n/2+k}.
\]
Indeed, assume first that $k > \ell$. Then, with a computation as in (3.6),

$$|t^k \partial_t^k T_t f(x)| = |t^k \partial_t^{k-\ell} T_{t/2} (\partial_t^\ell T_{t/2} f)(x)| = t^k \int_{\mathbb{R}^n} \frac{\partial_t^{k-\ell} W_{t/2} (x-y) \partial_t^\ell T_{t/2} f(y) \, dy}{t^{n/2+k-\ell}} \leq C t^{k+\beta/2-\ell} \sum_{\nu \in \mathbb{Z}^n} e^{-c|x-y-2\nu|^2/t} \, dy = C t^{\beta/2}.$$ 

Suppose now that $k < \ell$. Let $m$ be the integer for which $k < \ell = k + m$. Then,

$$|t^k \partial_t^k T_t f(x)| \leq t^k \int_0^\infty \cdots \int_0^\infty |\partial_t^m T_{m} f(x)| \, ds_m \cdots ds_2 \, ds_1 \leq C t^k \int_0^\infty \cdots \int_0^\infty s^{-\beta/2-(m+k)} ds_m \cdots ds_2 \, ds_1 = C t^{\beta/2}.$$ 

For (ii), suppose that $f \in \Lambda\beta(T^0)$. We can write

$$|f(x) - f(x+h)| \leq |f(x) - T_{|h|^2} f(x)| + |T_{|h|^2} f(x) - T_{|h|^2} f(x+h)| + |T_{|h|^2} f(x+h) - f(x+h)|.$$ 

Then, since $T_0 f(x) = f(x)$, the first term above is bounded by

$$\int_0^{|h|^2} |\partial_s T_s f(x)| \, ds \leq \|f\|_{\Lambda\beta(T^0)} \int_0^{|h|^2} s^{-1+\beta/2} \, ds = C \|f\|_{\Lambda\beta(T^0)} |h|^{\beta/2}.$$ 

The third term is estimated analogously. For the second term, we need to show that

$$\|\nabla T_t f(\cdot)\|_{L^\infty(T^0)} \leq C \|f\|_{\Lambda\beta(T^0)} t^{\beta/2-1/2}.$$ 

Because with (4.3) we would obtain that the second term is bounded by

$$C \sup_{\xi} |\nabla T_{|h|^2} f(\xi)| |h| \leq C \|f\|_{\Lambda\beta(T^0)} (|h|^2)^{\beta/2-1/2} |h| = C \|f\|_{\Lambda\beta(T^0)} |h|^{\beta}.$$ 

In order to prove (4.3), observe first that the simple estimate

$$|\nabla W_t(x)| \leq C \sum_{\nu \in \mathbb{Z}^n} e^{-c|x-2\nu|^2/t} \frac{1}{t^{n/2+1/2}};$$

implies

$$\|\nabla W_t\|_{L^1(T^0)} \leq C t^{-1/2}.$$ 

Since $W_t = W_{t_1} * W_{t_2}$, $t = t_1 + t_2$, $t_j > 0$, we get $T_t f(x) = W_{t_1} * T_{t_2} f(x)$. Taking $t_1 = t_2 = t/2$, we have $\partial_t \nabla T_{t} f = \nabla W_{t/2} * (\partial_s T_{s} f)_{s=t/2}$. In this way, (4.4) and the assumption $\|\partial_t T_t f\|_{L^\infty(T^0)} \leq \|f\|_{\Lambda\beta(T^0)} t^{\beta/2-1}$ give

$$\|\partial_t \nabla T_{t} f\|_{L^\infty(T^0)} \leq C \|f\|_{\Lambda\beta(T^0)} t^{\beta/2-3/2}.$$ 

Nevertheless, $\|\nabla T_{t} f\|_{L^\infty(T^0)} = \|\nabla W_t * f\|_{L^\infty(T^0)} \leq \|\nabla W_t\|_{L^1(T^0)} \|f\|_{L^\infty(T^0)} \leq C t^{-1/2} \|f\|_{L^\infty(T^0)}$. Therefore $\nabla T_{t} f \to 0$ as $t \to \infty$, thus we can write $\nabla T_{t} f = - \int_0^t \partial_t \nabla T_{s} f \, ds$. From here, and in view of (4.5), we obtain (4.3).
Next let us assume that \( f \in C^{0,\beta}(T^n) \). Clearly, from (5.6), we have \( \int_{T^n} \partial_t W_t(x) \, dx = 0 \). Thus, using (4.2),
\[
\| \partial_t T_t f(x) \|_{L^\infty(T^n)} \leq C \int_{T^n} |\partial_t W_t(h)||f(x + h) - f(x)| \, dh
\]
\[
\leq C \int_{Q_n} \sum_{\nu \in \mathbb{Z}^n} \left| e^{-\epsilon(h - 2\pi \nu)^2/t} \right| |f(x + h - 2\pi \nu) - f(x)| \, dh
\]
\[
\leq C \| f \|_{C^{0,\beta}(T^n)} \int_{Q_n} \sum_{\nu \in \mathbb{Z}^n} \left| h - 2\pi \nu \right|^\beta \, dh
\]
\[
\leq C \| f \|_{C^{0,\beta}(T^n)} \int_{Q_n} \sum_{\nu \in \mathbb{Z}^n} \left| h - 2\pi \nu \right|^\beta \frac{t^\beta/2}{t^{n/2+1}} \, dh = C \| f \|_{C^{0,\beta}(T^n)} t^{\beta/2-1}.
\]

For the proof of (iii) we need the trivial facts that \( \int_{T^n} \partial_t W_t(x) \, dx = 0 \) and \( \partial_t W_t(x) = \partial_t W_t(-x) \). With these, if \( f \in \Lambda_t(T^n) \), we see that
\[
\partial_t T_t f(x) = \frac{1}{2} \int_{T^n} \partial_t W_t(h)(f(x + h) + f(x - h) - 2f(x)) \, dh,
\]
and so, by (4.2),
\[
\| \partial_t T_t f(x) \|_{L^\infty(T^n)} \leq C \int_{Q_n} \sum_{\nu \in \mathbb{Z}^n} \left| e^{-\epsilon(h - 2\pi \nu)^2/t} \right| |f(x + h - 2\pi \nu) + f(x - h + 2\pi \nu) - f(x)| \, dh
\]
\[
\leq C \| f \|_{\Lambda_t(T^n)} \int_{Q_n} \sum_{\nu \in \mathbb{Z}^n} \left| h - 2\pi \nu \right|^\beta \, dh
\]
\[
\leq C \| f \|_{\Lambda_t(T^n)} \int_{Q_n} \sum_{\nu \in \mathbb{Z}^n} \left| h - 2\pi \nu \right|^\beta \frac{t^{1/2}}{t^{n/2+2}} \, dh = C \| f \|_{\Lambda_t(T^n)} t^{1/2-2}.\]

In order to prove that \( \Lambda_t(T^n) \subset \Lambda_s(T^n) \) in (iii), one can follow the ideas in [20, Chapter V, Section 4.3, Proposition 8], by taking the heat semigroup in \( T^n \) instead of the Poisson in \( \mathbb{R}^n \). Let us sketch here the main steps. First we observe that, for a function \( F \) with two continuous derivatives,\n\[
(4.6) \quad \| F(x + h) + F(x - h) - 2F(x) \|_{L^\infty(T^n)} \leq C \| h \|^2 \| D^2 F \|_{L^\infty(T^n)}.
\]
By the inclusion \( \Lambda_t(T^n) \subset \Lambda_s(T^n) \), for \( \alpha < 1 \), see Remark 4.3, we have \( \| \partial_t T_t f(x) \|_{L^\infty(T^n)} \leq C t^{\alpha/2-1} \), so, in particular, \( t \| \partial_t T_t f(x) \|_{L^\infty(T^n)} \to 0 \) as \( t \to 0 \). Hence, we can write\n\[
(4.7) \quad f(x) = T_0 f(x) = \int_0^t \partial_t T_t f(x) \, ds - t \partial_t T_t f(x) + T_t f(x).
\]
However, by following an argument similar to the one in the proof of (ii), we can prove that the inequality \( \| \partial_t T_t f(x) \|_{L^\infty(T^n)} \leq C \| f \|_{\Lambda_t(T^n)} t^{1/2-2} \) implies the estimates \( \| D^2 T_t f \|_{L^\infty(T^n)} \leq C \| f \|_{\Lambda_t(T^n)} t^{-1/2} \), and \( \| \partial_t D^2 T_t f \|_{L^\infty(T^n)} \leq C \| f \|_{\Lambda_t(T^n)} t^{-3/2} \). Therefore, by plugging (4.7) into (4.6),
\[
\| f(x + h) + f(x - h) - 2f(x) \|_{L^\infty(T^n)} \leq C \| f \|_{\Lambda_t(T^n)} \left[ \int_0^t ss^{1/2-2} \, ds + (t \cdot t^{-3/2} + t^{-1/2}) \left| h \right|^2 \right].
\]
Take \( t = \left| h \right|^2 \) and the result follows.

Finally, item (iv) follows analogous ideas from [20, Chapter V, Section 4.3, Proposition 9]. Indeed, take \( f \in \Lambda_b(T^n) \). By using the same technique as in items (ii) and (iii) we have that \( \| \partial_{ttt} \nabla T_t f \|_{L^\infty(T^n)} \leq C t^{\beta/2-3} \) implies the estimate \( \| \partial_{tt} \nabla T_t f \|_{L^\infty(T^n)} \leq C t^{\beta/2-5/2} \). With this, we can prove that \( \nabla f \in L^\infty(T^n) \) and \( f \in \Lambda_{10^{-1}}(T^n) \) with the equivalence of the norms, just following the same steps as in [20, Page 148]. The proof of the converse implication works in the same way. We omit further details.
Theorem 4.5. Let $\beta > 0$ and $0 < \sigma < 2$ with $\sigma < \beta$. If $f \in \Lambda_\beta(T^n)$ then $(-\Delta)^{\sigma/2} f \in \Lambda_{\beta-\sigma}(T^n)$, and
\[
\|(-\Delta)^{\sigma/2} f\|_{\Lambda_{\beta-\sigma}(T^n)} \leq C \|f\|_{\Lambda_\beta(T^n)}.
\]

Proof. Suppose first that $0 < \sigma < 1$. We have to prove that, for $f \in \Lambda_\beta(T^n)$, $(-\Delta)^{\sigma/2} f$ is a continuous function on $T^n$ (which is true by Theorem 5.4) and
\[
\|t^{k} \partial_t^k T_t (-\Delta)^{\sigma/2} f(x)\|_{L^\infty(T^n)} \leq C \|f\|_{\Lambda_\beta(T^n)} t^{\frac{\sigma-\sigma'}{2}}, \quad \text{for } k = \left[\frac{\beta-\sigma}{2}\right] + 1.
\]
By Remark 4.8
\[
(-\Delta)^{\sigma/2} f(x) = \frac{1}{\Gamma(-\sigma/2)} \int_0^\infty (T_s f(x) - f(x)) \frac{ds}{s^{1+\sigma/2}}
\]
\[
= \frac{1}{\Gamma(-\sigma/2)} (J_1(x,t) + J_2(x,t)),
\]
where $J_1(x,t)$ denotes the part of the integral running from 0 to $t$. By using the semigroup property, the hypothesis and Proposition 4.4 (i),
\[
|t^k \partial_t^k T_t J_1(x,t)| = \left| t^k \partial_t^k T_t \int_0^t \partial_t^r T_r f(x) \frac{dr ds}{s^{1+\sigma/2}} \right|
\]
\[
\leq t^k \int_0^t \left| \partial_t^{k+1} T_w f(x) \right|_{w=t+r} \frac{ds}{s^{1+\sigma/2}}
\]
\[
\leq t^k \|f\|_{\Lambda_\beta(T^n)} \int_0^t \left| (t+r)^{\beta/2-k-1} \right| \frac{ds}{s^{1+\sigma/2}}
\]
\[
= t^{\beta/2} \|f\|_{\Lambda_\beta(T^n)} \int_0^t \left| (1+u)^{\beta/2-k-1} \right| du \frac{ds}{s^{1+\sigma/2}}
\]
\[
\leq C t^{\beta/2} \|f\|_{\Lambda_\beta(T^n)} \int_0^t \frac{ds}{s^{1+\sigma/2}} = C \|f\|_{\Lambda_\beta(T^n)} t^{\frac{\beta-\sigma}{2}}.
\]

On the other hand, by the semigroup property and Proposition 4.4 (i),
\[
|t^k \partial_t^k T_t J_2(x,t)| \leq t^k \int_0^\infty \left| \partial_t^k T_s f(x) \right|_{w=t+s} \frac{ds}{s^{1+\sigma/2}} + \int_0^t \left| t^k \partial_t^k T_t f(x) \right| \frac{ds}{s^{1+\sigma/2}}
\]
\[
\leq C \|f\|_{\Lambda_\beta(T^n)} \left( t^k \int_0^\infty \left| (t+s)^{\beta/2-k} \right| \frac{ds}{s^{1+\sigma/2}} + t^{\frac{\sigma-\sigma'}{2}} \right)
\]
\[
= C \|f\|_{\Lambda_\beta(T^n)} t^{\frac{\beta-\sigma}{2}} \left( \int_1^\infty (1+u)^{\beta/2-k} \frac{du}{u^{1+\sigma/2}} + 1 \right) = C \|f\|_{\Lambda_\beta(T^n)} t^{\frac{\beta-\sigma}{2}}.
\]

Consider now the situation $1 \leq \sigma < 2$. We can write $(-\Delta)^{\sigma/2} f = (-\Delta)^{\sigma/2-1/2} (-\Delta)^{1/2} f = (-\Delta)^{\sigma/2-1/2} R \nabla f$, where $R = \nabla (-\Delta)^{-1/2}$ are the Riesz transforms on $T^n$. Observe that, by Proposition 4.4 (iv), if $f \in \Lambda_\beta(T^n)$, $\beta > \sigma \geq 1$, then $\nabla f \in \Lambda_{\beta-1}(T^n)$. Therefore, the result follows from the boundedness of the Riesz transforms on the spaces $\Lambda_{\gamma}(T^n)$, $\gamma > 0$, (see Zygmund [27], Chapter III, (13.29)) for the one dimensional case, and Calderón–Zygmund [6, Theorem 11] for the multidimensional case), and also from the case just proved above ($0 < \sigma/2 - 1/2 < 1$).

5. THE ONE DIMENSIONAL TORUS $T$ AND THE EXPONENTIAL POISSON KERNEL

All the previous analysis could be carried out with the Poisson semigroup on $T^n$ in place of the heat semigroup. The particular feature we would have again is that the Poisson kernel is given as a series. Nevertheless, in the case of the one dimensional torus $T$, the Poisson kernel is given by a closed expression (see [5,2] below), so more explicit formulas for the fractional Laplacian can be obtained. In particular, we avoid the use of an infinite series to represent its kernel $K^{\sigma/2}(x)$. Let us explain how to do this and which are the differences with the general $n$-dimensional case.
5.1. The fractional Laplacian on $\mathbb{T}$ via the Poisson semigroup. The Poisson semigroup generated by $\Delta$ on $\mathbb{T}$ is defined by

$$e^{-t\sqrt{-\Delta}} f(x) \equiv P_t f(x) := \sum_{k \in \mathbb{Z}} e^{-t|k|} c_k f(x) e^{ikx}, \quad f \in L^2(\mathbb{T}), \ t \geq 0.$$ 

Then $P_t f(x)$ is the solution of the harmonic extension $(\partial_t + \Delta)v = 0$ in $\mathbb{T} \times (0, \infty)$, with boundary condition $v(x,0) = f(x)$ on $\mathbb{T}$. The convolution formula

$$P_t f(x) = \int_{\mathbb{T}} P_t(x-y) f(y) \, dy,$$

holds, where for $x \in \mathbb{T}$ and $t > 0,$

$$P_t(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-t|k|} e^{ikx} = \frac{1}{\pi} \left( \frac{1}{2} + \sum_{k=1}^{\infty} e^{-tk} \cos kx \right),$$

is the Poisson kernel on $\mathbb{T}$. See Zygmund [27, Chapter III, (6.2)]. To get a formula for the fractional Laplacian with this Poisson semigroup, we note that for any $\lambda > 0$ and $0 < \sigma < 2$ we have the identity

$$\lambda^{\sigma/2} = \frac{1}{c_\sigma} \int_0^\infty (e^{-t\lambda^{1/\sigma}} - 1)^{\sigma+1} \frac{dt}{t^{1+\sigma}},$$

where

$$c_\sigma = \int_0^\infty (e^{-s} - 1)^{\sigma+1} \frac{ds}{s^{1+\sigma}}.$$ 

Plugging this into the definition of the fractional Laplacian given by (1.4) (remember that the sum runs over $\mathbb{Z}$ instead of $\mathbb{Z}^n$), and interchanging the summation with the integration, we obtain

$$(-\Delta)^{\sigma/2} f(x) = \frac{1}{c_\sigma} \int_0^\infty (e^{-t\sqrt{-\Delta}} - 1)^{\sigma+1} f(x) \frac{dt}{t^{1+\sigma}}, \quad f \in C^\infty(\mathbb{T}), \ x \in \mathbb{T},$$

where $I$ is the identity operator. It is not difficult to prove that the constant $c_\sigma$ in (5.3) satisfies the following asymptotic estimates:

$$\frac{1}{c_\sigma} \sim (2-\sigma), \quad \text{as } \sigma \to 2^-,$n

$$\frac{1}{c_\sigma} \sim -\sigma, \quad \text{as } \sigma \to 0^+.$$ 

These two relations, together with (5.6) and (5.9) below, will be useful to see (1.2) and (1.3).

5.2. The pointwise formulas with the Poisson kernel. Observe that to get a pointwise formula for the fractional Laplacian in $\mathbb{T}$ from (5.4) we need to take into account the value of $[\sigma]$, the integer part of $\sigma \in (0,2)$. 

Theorem 5.1 (Pointwise formula for $0 < \sigma < 1$). Let $0 < \sigma < 1$ and $f \in C^{0,\sigma+\varepsilon}(\mathbb{T})$, for some $\varepsilon > 0$ such that $0 < \sigma + \varepsilon \leq 1$. Then $(-\Delta)^{\sigma/2} f$ is a continuous function and

$$(-\Delta)^{\sigma/2} f(x) = \int_{\mathbb{T}} (f(x) - f(y)) K^{\sigma/2}(x-y) \, dy,$$

where

$$K^{\sigma/2}(x) = \frac{1}{c_\sigma} \int_0^\infty P_t(x) \frac{dt}{t^{1+\sigma}} \geq 0,$$

with $c_\sigma$ as in (5.3). Moreover, there exist universal constants $C,c > 0$ such that

$$-c_\sigma (1-\sigma)|\sin(x/2)|^{1+\sigma} \leq K^{\sigma/2}(x) \leq \frac{C}{c_\sigma \sigma(1-\sigma)|\sin(x/2)|^{1+\sigma}}, \quad x \in \mathbb{T}.$$ 

Besides, the integral in (5.6) is absolutely convergent.
Proof. It is clear from (5.4) that (5.6) is true with the kernel given by (5.7). Indeed, for smooth $f$, just write down (5.1) into (5.4), use the fact that the integral of the kernel (5.2) is 1, and apply Fubini’s theorem. Then invoke Lemma 3.3 to arrive to (5.6) for $f \in C^{0,\sigma + \varepsilon}({\mathbb{T}})$. Recall that the same path was taken in the proof of (5.4) in Theorem 3.4. We only need to prove (5.8). Recall (5.2). Let us split the integral in (5.7) into two parts

$$J := \int_0^\infty \frac{1 - e^{-2t}}{2\pi(1 - e^{-t})^2 + 4e^{-t}\sin^2(x/2)} \frac{dt}{t^{1+\sigma}} = \int_0^1 + \int_1^\infty =: J_1 + J_2.$$  

For $J_1$, 

$$J_1 \sim \int_0^1 \frac{t}{t^2 + C\sin^2(x/2)} \frac{dt}{t^{1+\sigma}} = \frac{1}{\sin^2(x/2)} \int_0^{1/|\sin(x/2)|} s^{-\sigma} s^2 + C \, ds.$$ 

There exist constants $c, C > 0$, independent of $\sigma$, such that

$$\int_0^{1/|\sin(x/2)|} s^{-\sigma} s^2 + C \, ds \leq \frac{1}{C} \int_0^1 s^{-\sigma} \, ds + \int_1^\infty \frac{ds}{s^2 + C} \leq \frac{C}{1 - \sigma},$$

and

$$\int_0^{1/|\sin(x/2)|} s^{-\sigma} s^2 + C \, ds \geq \frac{1}{1 + C} \int_0^1 s^{-\sigma} \, ds = \frac{c}{1 - \sigma}.$$ 

Thus, $J_1 \sim \frac{1}{(1 - \sigma)|\sin(x/2)|^{1+\sigma}}$. We also have $J_2 \leq C \int_1^\infty \frac{dt}{t^{1+\sigma}} = \frac{C}{\sigma}$. Therefore, (5.8) follows. □

Next we take $1 \leq \sigma < 2$ in (5.4).

Theorem 5.2 (Pointwise formula for $1 \leq \sigma < 2$). Let $1 \leq \sigma < 2$ and $f \in C^{1,\sigma + \varepsilon}({\mathbb{T}})$, for some $\varepsilon > 0$ such that $0 < \sigma + \varepsilon - 1 \leq 1$. Then

$$(-\Delta)^{\sigma/2} f(x) = \int_{\mathbb{T}} (f(x) - f(y) - f'(x)(x-y)) K^{\sigma/2}(x-y) \, dy$$

(5.9)

where

$$K^{\sigma/2}(x) = \frac{1}{c_\sigma} \int_0^\infty (2P_2(t) - P_2(t)) \frac{dt}{t^{1+\sigma}} \geq 0,$$

and $c_\sigma$ is as in (5.3). Moreover, there exist universal constants $C, c > 0$ such that

$$\frac{c}{c_\sigma|\sin(x/2)|^{1+\sigma}} \leq K^{\sigma/2}(x) \leq \frac{C}{c_\sigma|\sin(x/2)|^{1+\sigma}}.$$

Proof. Let us begin with the proof of (5.11). Let $D_t(x) := 2P_t(x) - P_2(x)$. Then, by (5.2),

$$D_t(x) = \frac{(1 - e^{-2t})(1 - e^{-t})^2 \frac{2(1 + e^{-t})^2 - (1 + e^{-2t}) - 4e^{-t}|\sin(x/2)|^2}{2\pi((1 - e^{-t})^2 + 4e^{-t}|\sin(x/2)|^2 ((1 - e^{-2t}) + 4e^{-2t}|\sin(x/2)|^2)} \geq 0.$$

We split the integral in (5.10) as $\int_0^1 + \int_1^\infty$. On one hand,

$$\int_0^1 D_t(x) \frac{dt}{t^{1+\sigma}} \sim \int_0^1 \frac{t^3}{t^{1+\sigma}} \frac{dt}{t^{1+\sigma}} \sim \frac{1}{\sin^2(x/2)} \int_0^{1/|\sin(x/2)|} s^{-\sigma} s^2 + 1 \, ds \sim \frac{1}{|\sin(x/2)|^{1+\sigma}}.$$

The last equivalence is true because there exist constants $C, c > 0$, independent of $\sigma$, such that

$$\int_0^{1/|\sin(x/2)|} s^{-\sigma} s^2 + 1 \, ds \leq \int_0^1 s^{-\sigma} \, ds + \int_1^\infty s^{-\sigma} \, ds \leq C,$$
and
\[ \int_0^{1/|\sin(x/2)|} s^{2-\sigma} ds \geq \frac{1}{2} \int_0^1 s^{2-\sigma} ds \geq c, \]
for all \( 1 \leq \sigma < 2 \). On the other hand, \( \int_1^\infty D_t(x) \frac{dt}{t^{1+\sigma}} \leq C \int_1^\infty \frac{dt}{t^{1+\sigma}} \leq C \), for all \( 1 \leq \sigma < 2 \). Hence, \( 5.11 \) holds.

Now we derive the pointwise formula for \((-\Delta)^{\sigma/2} f(x)\). As in the proof of Theorem 3.4 we only need to show (5.9) for smooth \( f \) and then invoke (a slight modification of) Lemma 3.3. We have \((P_t - 1)^2 f(x) = (P_t - 1)(P_t f(x) - f(x)) = P_t f(x) - 2P_t f(x) + f(x)\). Using (5.4) and the fact that the Poisson kernel has integral 1,
\[ (-\Delta)^{\sigma/2} f(x) = \frac{1}{c_\sigma} \int_0^\infty \int_{-\pi}^{\pi} (f(x) - f(y)) D_t(x - y) dx \frac{dt}{t^{1+\sigma}}, \]
where in the last identity we used that \( zD_t(z) \) is an odd function. Fubini’s theorem yields the conclusion. \( \Box \)

5.3. The pointwise limits with the Poisson kernel.

**Proposition 5.3.** Let \( f \in C^{0,\alpha} (\mathbb{T}) \), for some \( 0 < \alpha \leq 1 \). Then, for each \( x \in \mathbb{T} \),
\[ \lim_{\sigma \to 0^+} (-\Delta)^{\sigma/2} f(x) = f(x) - \frac{1}{2\pi} \int_{\mathbb{T}} f(y) dy, \quad x \in \mathbb{T}. \]

**Proof.** The proof is parallel to the proof of Proposition 3.5. We need to replace (3.10) by
\[ (5.12) \quad K^{\sigma/2}(x) - \frac{1}{2\pi} = \frac{1}{-c_\sigma} \int_0^\infty \int_{-\pi}^{\pi} P_t(x) \frac{dt}{t^{1+\sigma}} \cdot \frac{1}{-c_\sigma} \int_1^\infty \left( P_t(x) - \frac{1}{2\pi} \right) \frac{dt}{t^{1+\sigma}} + \frac{1}{2\pi} \left( \frac{1}{-c_\sigma} - 1 \right). \]
As we did in the proof of Theorem 3.1, we can see that the first integral in (5.12) is bounded from above by \( C \left( -c_\sigma (1 - \sigma) |\sin(x/2)|^{1+\sigma} \right) \). The second summand of (5.12) tends to zero as \( \sigma \to 0^+ \) because \( -c_\sigma = -\sigma \Gamma(-\sigma) = \Gamma(1 - \sigma) \). As for the second summand in (5.12), note that
\[ \left| P_t(x) - \frac{1}{2\pi} \right| \leq C e^{-t} \left( 1 - e^{-t} \right)^2 + 4 e^{-t} \sin^2(x/2). \]
Hence,
\[ \frac{1}{-c_\sigma} \int_1^\infty \left| P_t(x) - \frac{1}{2\pi} \right| \frac{dt}{t^{1+\sigma}} \leq \frac{C}{-c_\sigma} \int_1^\infty e^{-t} dt = \frac{C}{-c_\sigma}. \]
Collecting terms in (5.12),
\[ \left| K^{\sigma/2}(x) - \frac{1}{2\pi} \right| \leq \frac{C}{-c_\sigma (1 - \sigma) |\sin(x/2)|^{1+\sigma} + F(\sigma), \]
where \( F(\sigma) \to 0 \), as \( \sigma \to 0^+ \). Therefore,
\[ \int_\mathbb{T} \left| f(x) - f(y) \right| K^{\sigma/2}(x - y) - \frac{1}{2\pi} dy \leq \frac{C}{-c_\sigma (1 - \sigma)} \int_\mathbb{T} \left| f(x) - f(y) \right| \sin \frac{x-y}{2}^{\alpha-1-\sigma} dy + \| f \|_{L^\infty(\mathbb{T})} F(\sigma). \]
For the first term above, by applying (5.5),
\[ \frac{C}{-c_\sigma (1 - \sigma)} \int_\mathbb{T} \left| f(x) - f(y) \right| \sin \frac{x-y}{2}^{\alpha-1-\sigma} dy \leq \frac{C}{-c_\sigma (1 - \sigma)} \int_\mathbb{T} \left| \sin \frac{x-y}{2} \right|^{\alpha-1-\sigma} dy \leq \frac{C}{-c_\sigma (1 - \sigma)} \to 0, \quad \text{as } \sigma \to 0^+. \]
\( \Box \)
Proposition 5.4. Let \( f \in C^2(\mathbb{T}) \). Then
\[
\lim_{\sigma \to -2} \left( -\Delta \right)^{\sigma/2} f(x) = -f''(x), \quad x \in \mathbb{T}.
\]

Proof. The computation here is analogous to the proof of Proposition 3.6. The difference is in the integral \( J_{3,\sigma} \). Indeed, by using the orthogonality of the trigonometric system on the torus,
\[
\int_{-\pi}^{\pi} \left| \sin \frac{z}{2} \right|^2 P_t(z) \, dz = \int_{-\pi}^{\pi} \frac{1 - \cos z}{2} P_t(z) \, dz = \frac{1 - e^{-t}}{2}.
\]
With this, (5.10) and Tonelli’s theorem we compute the new \( J_{3,\sigma} \):
\[
J_{3,\sigma} = -2\frac{f''(x)}{c_\sigma} \int_{-\pi}^{\pi} \left| \sin \frac{z}{2} \right|^2 \int_{0}^{\infty} \left( 2P_t(z) - P_{2t}(z) \right) \frac{dt}{t^{1+\sigma}} \, dz
\]
\[
= -2\frac{f''(x)}{c_\sigma} \int_{0}^{\infty} \left( 2 \int_{-\pi}^{\pi} \left| \sin \frac{z}{2} \right|^2 P_t(z) \, dz - \int_{-\pi}^{\pi} \left| \sin \frac{z}{2} \right|^2 P_{2t}(z) \, dz \right) \frac{dt}{t^{1+\sigma}}
\]
\[
= -\frac{f''(x)}{c_\sigma} \int_{0}^{\infty} \left( 1 - e^{-t} \right)^2 \frac{dt}{t^{1+\sigma}} = -f''(x).
\]

5.4. The interaction with Hölder and Zygmund spaces via the Poisson semigroup. We begin by noticing that, for \( f \) as in the hypothesis of Theorems 5.1 and 5.2, formula (5.4) holds. Indeed, just write down the kernels in (5.6) and (5.9) and apply Fubini’s theorem.

Certainly, Theorem 4.1 continues to hold on \( \mathbb{T} \). But, if we want to prove it by using (5.4), we need to characterize the Hölder and Zygmund spaces in \( \mathbb{T} \) by using the Poisson semigroup. In this case, Definition 4.2 must be replaced by the following

Definition 5.5. Let \( \beta > 0 \) and \( k = [\beta] + 1 \). We define \( \tilde{\Lambda}_{\beta}(\mathbb{T}) := \{ f \in C(\mathbb{T}) : \| k^k \partial^k_x P_x f \|_{L^1(\mathbb{T})} \leq Bt^\beta \} \).

With this new definition, Proposition 4.4 holds with \( \tilde{\Lambda}_{\beta}(\mathbb{T}) \) and \( P_t \) in place of \( \Lambda_{\beta}(\mathbb{T}^n) \) and \( T_t \), respectively. The structure of the proof is the same. Items (i)–(iv) of that proposition, in the case when \( \mathbb{T} \) and \( P_t \) are replaced by \( \mathbb{R}^n \) and the Poisson semigroup on \( \mathbb{R}^n \), respectively, is contained in Stein [20 Chapter V, Section 4].

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The proof there relies on appropriate estimates and cancelations of the Poisson kernel on \( \mathbb{R}^n \), the semigroup property of \( e^{-t\sqrt{-\Delta}} \) and the equation \( (\partial_{tt} + \Delta_{\mathbb{R}^n})(e^{-t\sqrt{-\Delta}}f)(x) = 0 \). The same ingredients are present in the case of the one-dimensional torus and the arguments from Stein follow line by line here. We just mention the estimates for the Poisson kernel on \( \mathbb{T} \):
\[
|\partial^k_x P_t(x)| \leq C_k \left| \sin \frac{x}{2} \right|^{-(k+1)}, \quad |\partial^k_x P_t(x)| \leq C_k e^{-t} (1 - e^{-t})^{-(k+1)},
\]
for any \( x \in \mathbb{T}, t > 0 \), and \( k \geq 1 \), see [9 Lemma 3]. The derivatives with respect to \( x \) satisfy the same bounds. With this new space \( \tilde{\Lambda}_{\beta}(\mathbb{T}) \) and the description of the fractional Laplacian as in (5.4), the parallel of Theorem 4.3 can be proved in a similar way, thus yielding the interaction of \( (-\Delta)^{\sigma/2} \) with the Hölder spaces \( C^{k,\alpha}(\mathbb{T}) \).

Remark 5.6. The ideas of the Poisson semigroup approach explained in this subsection were also used in [10], to study the interaction of fractional Schrödinger operators of the form \( L^{\sigma/2} = (-\Delta + V)^{\sigma/2} \) with adapted Hölder spaces \( C^{0,\alpha}_{L^{\sigma/2}}(\mathbb{R}^n) \).

Part 2. Extension problem and Harnack’s inequalities

6. The general extension problem

Let \( L \) be a nonnegative, densely defined and self-adjoint operator on some space \( L^2(\Omega, d\eta) = L^2(\Omega) \). To fix ideas, we take \( \Omega \) to be an open subset of, say, \( \mathbb{R}^n \), \( n \geq 1 \), and \( d\eta \) is a positive measure on \( \Omega \), see Remark 6.3. There is a unique resolution of the identity \( E \), supported on the spectrum of \( L \), such that
\[
\langle Lf, g \rangle = \int_{\Omega} \lambda dE_{f,g}(\lambda), \quad f \in \text{Dom}(L), g \in L^2(\Omega).
\]
Here $dE_{f,g}(\lambda)$ is a regular Borel complex measure of bounded variation. Throughout this section we use the notation $\langle f, g \rangle = \int_{\Omega} f(x)g(x) \, d\eta(x)$. The heat-diffusion semigroup generated by $L$ is given by

$$\langle e^{-tL}f, g \rangle = \int_0^\infty e^{-\lambda t} \, dE_{f,g}(\lambda), \quad f, g \in L^2(\Omega), \quad t \geq 0.$$  

Fix any $\gamma > 0$. The fractional operators $L^\gamma$ are defined by

$$\langle L^\gamma f, g \rangle = \int_0^\infty \lambda^\gamma \, dE_{f,g}(\lambda), \quad f \in \text{Dom}(L^\gamma), \quad g \in L^2(\Omega),$$

with domain

$$\text{Dom}(L^\gamma) = \left\{ f \in L^2(\Omega) : \int_0^\infty \lambda^{2\gamma} \, dE_{f,g}(\lambda) < \infty \right\} \supset \text{Dom}(L^\gamma).$$

**Theorem 6.1 (Extension problem).** Let $\gamma \in (0, \infty) \setminus \mathbb{N}$, and $f$ in the domain of $L^\gamma$. A solution $u \in C^\infty((0, \infty) \setminus \mathbb{N}; \text{Dom}(L)) \cap C([0, \infty); L^2(\Omega))$ of the extension problem

$$
\begin{cases}
-L_x u + \frac{1-2\gamma}{y} u_y + u_{yy} = 0, & \text{in } \Omega \times (0, \infty), \\
u(x,0) = f(x), & \text{on } \Omega,
\end{cases}
$$

is given by

$$u(x,y) = \frac{y^{2\gamma}}{4^\gamma \Gamma(\gamma)} \int_0^\infty e^{-\lambda t} f(x) e^{-\frac{\lambda}{4^\gamma}} \frac{dt}{t^{1+\gamma}},$$

and

$$\lim_{y \to 0^+} y^{1-2(\gamma-[\gamma])} \partial_y \left( (y^{-1} \partial_y)^{[\gamma]} u(x,y) \right) = \mu_\gamma L^\gamma f(x),$$

where $[\gamma]$ is the integer part of $\gamma$ and

$$\mu_\gamma = \frac{4^{\gamma-[\gamma]} \Gamma(\gamma-[\gamma])}{2(\gamma-[\gamma])! \Gamma(-\gamma-[\gamma])} \cdot \frac{1}{2^{\gamma}(\gamma-[\gamma])(\gamma-[\gamma]+1) \cdots (\gamma-1)}.$$  

**Remark 6.2.** Theorem [6.1] for $0 < \gamma < 1$ is already stated in [23]. Notice that there the result is written for second order differential operators, but the proof relies only on the spectral theorem. Therefore, [23, Theorem 1.1] is as general as the one presented here. When $L = -\Delta_\mathbb{R}^n$ and $0 < \gamma < 1$ we recover in [6.1] and [6.3] the Caffarelli–Silvestre extension problem (1.6)–(1.7). Also in the Laplacian case, but for general noninteger $\gamma > 0$, condition (6.3) was obtained by Chang and González in [8]. They derive the result by relating the fractional Laplacian ($-\Delta_{\mathbb{R}^n}$)$^\gamma$ with a class of conformally covariant operators defined at the boundary of the hyperbolic space. In turn, our approach here is more general and it is based on the explicit formula for $u$ of (6.2), already found in [23] when $0 < \gamma < 1$, that involves the semigroup $e^{-tL}$.

**Remark 6.3.** Observe that in Theorem [6.1] we are considering $L$ as an operator defined on $L^2(\Omega)$. Nevertheless, we can replace this assumption by a more general one, that is, we can take $L$ to be a normal operator acting on an abstract Hilbert space. Indeed, the main analytic tool used in the proof is the spectral theorem. Hence, important examples like weighted Laplace–Beltrami operators on weighted Riemannian manifolds or Lie groups, divergence form elliptic operators on domains of $\mathbb{R}^n$, pseudo-differential operators of even order, among others, are covered by Theorem [6.1]. See also [23, 25].

**Remark 6.4.** We can push further the class of operators $L$ for which Theorem [6.1] is valid. The most general extension result we know holds for generators of *integrated semigroups* and can be found in [13]. In particular, Theorem 1.1 of [13] applies for generators of semigroups in Banach spaces like $L^p$, or operators with complex spectrum. Here examples include fractional powers of $i\Delta$ or $\partial_x^3$. Moreover, in [13, Appendix], the case of complex powers $L^\gamma$, Re $\gamma > 0$, is considered.
Proof of Theorem 6.1. As in [23], the equality (6.2) means that $u(\cdot, y) \in \text{Dom}(L)$ for any $y > 0$ and
\[
\langle u(\cdot, y), g(\cdot) \rangle = \frac{y^{2\gamma}}{4\Gamma(\gamma)} \int_0^\infty \langle e^{-tL} f, g \rangle e^{-\frac{y^2}{4t}} \left( \frac{2}{t^{2+\gamma}} - \frac{y^2}{2t^{3+\gamma}} \right) \, dt,
\]
for all $g \in L^2(\Omega)$. It is proved in [23] Theorem 1.1] that, when $0 < \gamma < 1$, the function $u$ given in (6.2) and interpreted as above, is well defined and satisfies (6.1) and (6.3). If we consider $\gamma > 1$, then, by following exactly the same arguments as in [23], it is easy to see that $u$ as in (6.2) is well defined and verifies (6.1). It remains to prove (6.3). We proceed by induction on $\gamma$. As we have just said, (6.3) is valid for $[\gamma] = 0$. Assume (6.3) for $j < \gamma < j + 1$, $j \in \mathbb{N}$. Let us check (6.3) for $j + 1 < \gamma + 1 < j + 2$. Take $f \in \text{Dom}(L^{\gamma+1})$ and $g \in L^2(\Omega)$.

\[
\langle (y^{-1} \partial_y) u(\cdot, y), g(\cdot) \rangle = \frac{1}{4^{(\gamma+1)} \Gamma(\gamma + 1)} \int_0^\infty \langle e^{-tL} f, g(y^{-1} \partial_y) \left( \frac{y^{2(\gamma+1)}}{4t} \right) \rangle \left( \frac{2}{t^{2+\gamma}} - \frac{y^2}{2t^{3+\gamma}} \right) \, dt
\]

\[
= \frac{-yg^{2\gamma}}{4^{(\gamma+1)} \Gamma(\gamma + 1)} \int_0^\infty \langle e^{-tL} f, g \partial_y (e^{-\frac{y^2}{4t}} t^{-(1+\gamma)}) \rangle \, dt
\]

\[
= \frac{2}{4\gamma} \cdot \frac{y^{2\gamma}}{4\Gamma(\gamma)} \int_0^\infty \langle e^{-tL} f, g \partial_y (e^{-\frac{y^2}{4t}}) \rangle \left( \frac{2}{t^{2+\gamma}} - \frac{y^2}{2t^{3+\gamma}} \right) \, dt = \frac{1}{2\gamma} \langle w(\cdot, y), g(\cdot) \rangle.
\]

Observe that $v$ is a solution to (6.1) with initial data $v(x, 0) = Lf(x)$. By the induction hypothesis,
\[
\lim_{y \to 0^+} \langle (y^{-1} \partial_y) u(\cdot, y), g(\cdot) \rangle = \frac{1}{2\gamma} \lim_{y \to 0^+} \langle (y^{-1} \partial_y) v(\cdot, y), g(\cdot) \rangle
\]

\[
= \frac{-yg^{2\gamma}}{4^{(\gamma+1)} \Gamma(\gamma + 1)} \cdot \frac{1}{2^j(\gamma - j)(\gamma - j + 1) \cdots (\gamma - 1)} \langle L^j(Lf), g \rangle
\]

\[
= \frac{2^{(\gamma+1)-(j+1)}}{(\gamma + 1) - (j + 1) \Gamma(\gamma + 1) - (j + 1)} \langle L^j+1 f, g \rangle,
\]

which is (6.3). The theorem is proved. \hfill \Box

Remark 6.5. As done in [23], more properties of the general extension problem could be established, like Poisson formulas, fundamental solutions, Cauchy–Riemann equations, conjugate Poisson kernels and $L^p$ estimates.

Remark 6.6. If we assume that $f \in \text{Dom}(L^{[\gamma]+1})$, $\gamma > 0$, then
\[
L^\gamma f = \frac{1}{d_\gamma} \int_0^\infty (e^{-tL} - 1)^{[\gamma]+1} f \, dt \left( \frac{2}{t^{2+\gamma}} - \frac{y^2}{2t^{3+\gamma}} \right), \text{ in } L^2(\Omega),
\]

with $d_\gamma = \int_0^\infty (e^{-t} - 1)^{[\gamma]+1} \, dt \left( \frac{2}{t^{2+\gamma}} - \frac{y^2}{2t^{3+\gamma}} \right)$. Therefore, in the case that $L$ has an explicit heat kernel, one could obtain parallel pointwise formulas for $L^\gamma f(x)$ like those of Theorem 3.4 (just follow a similar argument as the one presented in the proof of Theorem 5.2).

7. Interior and boundary Harnack’s inequalities for $(-\Delta)^{\sigma/2}$ on $\mathbb{T}^n$

In this section we apply the extension problem to prove interior and boundary Harnack’s inequalities for $(-\Delta)^{\sigma/2}$, $0 < \sigma < 2$. We follow the ideas contained in [1] Section 5.

Let us see first that the extension problem for the fractional Laplacian on the torus admits a classical solution. We show this by using the classical Fourier method.
Take $f \in \text{Dom}(-\Delta)$. We first claim that a solution $u : \mathbb{T}^n \times [0, \infty) \to \mathbb{R}$ to the extension problem \[6.1\] for $f$ can be written as

$$
\begin{align*}
    u(x, y) &= \frac{y^\sigma}{4\pi^2/\Gamma(\sigma/2)} \int_0^\infty e^{t\Delta} f(x)e^{-t^2/2} \frac{dt}{t^{1+\sigma/2}} \\
    &= \frac{y^\sigma}{4\pi^2/\Gamma(\sigma/2)} \sum_{\nu \in \mathbb{Z}^n} c_\nu(f)e^{i\nu\cdot x} \int_0^\infty e^{-t|\nu|^2} e^{-t^2/2} \frac{dt}{t^{1+\sigma/2}}.
\end{align*}
\tag{7.1}
$$

Indeed, to see \eqref{7.1}, observe that the series in \eqref{2.1} converges uniformly in \eqref{6.1} for \eqref{7.1} by dominated convergence.

Secondly, let us also note that, if $\mu$ is the constant in Theorem 6.1, we first claim that a solution \[2.1\] in the constant in Theorem 6.1, equal to $\frac{y^\sigma}{4\pi^2/\Gamma(\sigma/2)} \sum_{\nu \in \mathbb{Z}^n} e^{-t|\nu|^2} |c_\nu(f)| \leq \|f\|_{L^2(\mathbb{T}^n)} \left( \sum_{\nu \in \mathbb{Z}^n} e^{-2t|\nu|^2} \right)^{1/2} \leq C\|f\|_{L^2(\mathbb{T}^n)} \left( \sum_{k \geq 0} k^ne^{-2tk^2} \right)^{1/2} \leq C\|f\|_{L^2(\mathbb{T}^n)} t^{-n/2} \left( \sum_{k \geq 0} e^{-2tk^2} \right)^{1/2} \leq C\|f\|_{L^2(\mathbb{T}^n)} t^{-n/2-1/4}.

Since

$$
\begin{align*}
    \int_0^\infty \sum_{\nu \in \mathbb{Z}^n} e^{-t|\nu|^2} c_\nu(f)e^{i\nu\cdot x} \frac{dt}{t^{1+\sigma/2}} \leq C_f \int_0^\infty e^{-t/2} t^{-n/2-1/4} \frac{dt}{t^{1+\sigma/2}} < \infty,
\end{align*}
$$

Fubini’s theorem can be applied to obtain the second equality of \eqref{7.1}.

Secondly, $u(\cdot, y) \in C^2(\mathbb{T}^n)$, for every $y > 0$. Indeed, for $h > 0$ and $e_j$ the $j$-th coordinate unit vector in $\mathbb{Z}^n$, $j = 1, \ldots, n$,

$$
\begin{align*}
    \frac{u(x + he_j, y) - u(x, y)}{h} &= \frac{y^\sigma}{4\pi^2/\Gamma(\sigma/2)} \sum_{\nu \in \mathbb{Z}^n} c_\nu(f)e^{i\nu\cdot (x+he_j) - i\nu\cdot x} \int_0^\infty e^{-t|\nu|^2} e^{-t^2/2} \frac{dt}{t^{1+\sigma/2}}.
\end{align*}
$$

As

$$
\sum_{\nu \in \mathbb{Z}^n} |c_\nu(f)| \int_0^\infty |\nu|e^{-t|\nu|^2} e^{-t^2/2} \frac{dt}{t^{1+\sigma/2}} \leq \sum_{\nu \in \mathbb{Z}^n} |c_\nu(f)| \int_0^\infty e^{-t|\nu|^2} e^{-t^2/2} \frac{dt}{t^{1+\sigma/2}} < \infty,
$$

by dominated convergence, $u$ is differentiable with respect to $x$ and the derivative can be taken inside the series in \eqref{7.1}. A similar argument for $\nabla_y u$ shows that $u(\cdot, y) \in C^2(\mathbb{T}^n)$.

Finally, let us see that, for $\mu_\sigma$ the constant in Theorem 6.1, \[7.2\] \[\|y^{1-\sigma} u_{y}(x, y)\|_{L^2(\mathbb{T}^n)} \to \mu_{\sigma/2}\|(-\Delta)^{\sigma/2} f\|_{L^2(\mathbb{T}^n)}, \quad \text{as } y \to 0^+.

To prove \eqref{7.2} we use \eqref{7.1}, the cancelation

$$
\int_0^\infty e^{-t^2/2} \left( \sigma - \frac{y^2}{2t} \right) \frac{dt}{t^{1+\sigma/2}} = 0, \quad y > 0,
$$

and dominated convergence, as follows:

$$
\begin{align*}
    \|y^{1-\sigma} u_{y}(x, y)\|_{L^2(\mathbb{T}^n)} &= \sum_{\nu \in \mathbb{Z}^n} |c_\nu(f)|^2 \left( \frac{1}{4\pi^2/\Gamma(\sigma/2)} \int_0^\infty e^{-t|\nu|^2} e^{-t^2/2} \left[ \sigma - \frac{y^2}{2t} \right] \frac{dt}{t^{1+\sigma/2}} \right)^2 \\
    &= \sum_{\nu \in \mathbb{Z}^n} |c_\nu(f)|^2 \left( \frac{\sigma}{4\pi^2/\Gamma(\sigma/2)} \int_0^\infty (e^{-t|\nu|^2} - 1)e^{-t^2/2} \left[ \sigma - \frac{y^2}{2t} \right] \frac{dt}{t^{1+\sigma/2}} \right)^2 \\
    &= C_{\sigma/2} \sum_{\nu \in \mathbb{Z}^n} |\nu|^{2\sigma}|c_\nu(f)|^2 = C_{\sigma/2}\|(-\Delta)^{\sigma/2} f\|_{L^2(\mathbb{T}^n)}^2.
\end{align*}
$$

Let us also note that, if $f \geq 0$, then $u \geq 0$. 
Theorem 7.1 (Interior Harnack’s inequality). Let $0 < \sigma < 2$ and let $O \subseteq \mathbb{R}^n$ be an open set. Fix a compact subset $K \subset O$. There exists a positive constant $C$ depending only on $n$, $\sigma$ and $K$ such that

$$\sup_K f \leq C \inf_K f,$$

for all solutions to

$$\begin{cases}
\Delta u = 0, & \text{in } L^2(O), \\
f \geq 0, & \text{on } \mathbb{T}^n, \\
f \in \text{Dom}(\Delta).
\end{cases}$$

As a consequence, any solution $f$ of the problem above is a Hölder continuous function in $K$.

Proof. Set $\tilde{u}(x, y) = u(x, |y|)$, $x \in \mathbb{T}^n$, $y \in \mathbb{R}$, where $u$ is as in (7.1). Let us verify that $\tilde{u}$ is a nonnegative weak solution of

$$\text{div}(|y|^{1-\sigma} \nabla \tilde{u}) = 0, \quad \text{in } \mathcal{C} := O \times (-R, R), \quad R > 0. \quad (7.3)$$

Indeed, for any $\varphi \in C_c^\infty(\mathcal{O} \times (-R, R))$ and $\delta > 0$, by applying the divergence theorem,

$$\int_{O \times (-\delta, \delta)} |y|^{1-\sigma} \varphi \nabla \tilde{u} \cdot \nabla \varphi dx dy = \int_{O \times \{y \geq \delta\}} \text{div}(|y|^{1-\sigma} \varphi \nabla \tilde{u}) dx dy + \int_{O \times \{y < \delta\}} |y|^{1-\sigma} \nabla \tilde{u} \cdot \nabla \varphi dx dy$$

$$= \int_{\mathcal{O} \times (-\delta, \delta)} \varphi(x, \delta) \delta^{1-\sigma} \tilde{u}_y(x, \delta) dx dy + \int_{O \times (-\delta, \delta)} |y|^{1-\sigma} \nabla \tilde{u} \cdot \nabla \varphi dx dy.$$

The first term above is bounded by $\|\varphi\|_{L^\infty((-R, R), L^2(\mathcal{O}))} \|\delta^{1-\sigma} \tilde{u}_y(x, \delta)\|_{L^2(\mathcal{O})}$, which tends to $0$ as $\delta \to 0^+$ because of (7.2). As for the second term, we write $\nabla \tilde{u} \cdot \nabla \varphi = \sum_{k=1}^n \partial_{x_k} \tilde{u} \partial_{x_k} \varphi + \partial_y \tilde{u} \partial_y \varphi$, so the integral splits into $\sum_{k=1}^n J_k + J$. To deal with $J_k$, we see that, as $f \in \text{Dom}(\Delta)$, the degeneracy weight in the extension equation $\partial_x \tilde{u} \in L^2(\mathbb{T}^n)$. Next we check that $\|\partial_x \tilde{u}(x, y)\|_{L^2(\mathbb{T}^n)} \to \|\partial_x f\|_{L^2(\mathbb{T}^n)}$, as $y \to 0^+$. This is proved by using (7.1), a change of variables and dominated convergence:

$$\|\partial_x \tilde{u}(x, y)\|_{L^2(\mathbb{T}^n)}^2 = \sum_{\nu \in \mathbb{Z}^n} \nu^2_k |c_{\nu}(f)|^2 \left( \frac{1}{\Gamma(\sigma/2)} \int_0^\infty e^{-t}|\nu|^2 e^{-\frac{t^2}{s^2 - \sigma^2/4}} \frac{dt}{t^{1+\sigma/2}} \right)^2$$

$$= \sum_{\nu \in \mathbb{Z}^n} \nu^2_k |c_{\nu}(f)|^2 \left( \frac{\Gamma(\sigma/2)}{\Gamma(\sigma/2)} \int_0^\infty e^{-\frac{t^2}{s^2 - \sigma^2/4}} \frac{ds}{s^{1+\sigma/2}} \right)^2$$

$$\to \sum_{\nu \in \mathbb{Z}^n} \nu^2_k |c_{\nu}(f)|^2 = \|\partial_x f\|_{L^2(\mathbb{T}^n)}^2, \quad \text{as } y \to 0^+.$$

Thus there exists a constant $C(f)$ such that $\|\partial_x \tilde{u}(x, y)\|_{L^2(\mathbb{T}^n)} < C(f)$ for all sufficiently small $y$. Hence, $|J_k| \leq C(f, \varphi) \delta^{2-\sigma} \to 0$, as $\delta \to 0$. In order to estimate $J$, by using (7.2), there exists $C$ such that $\|y|^{1-\sigma} \tilde{u}_y(x, y)\|_{L^2(\mathcal{O})} \leq C$ for all sufficiently small $y$. Therefore,

$$|J| \leq \int_{-\delta}^{\delta} \|y|^{1-\sigma} \tilde{u}_y(x, y)\|_{L^2(\mathcal{O})} \|\partial_y \varphi\|_{L^2(\mathcal{O})} dy \leq C \varphi \delta \to 0, \quad \delta \to 0.$$

Hence, $\tilde{u}$ is a nonnegative weak solution to (7.3) in $\mathcal{C} = O \times (-R, R)$. The equation in (7.3) is a degenerate elliptic equation with $A_2$ weight $|y|^{1-\sigma}$. By applying Harnack’s inequality of [12], Theorems 2.3.8 and 2.3.12 to $\tilde{u}$, we get the conclusions for $f$. \qed

Remark 7.2. In view of Theorems 6.1 and 7.1, a natural question that arises is how to apply the extension problem to get interior Harnack’s inequality for $(-\Delta)^{\sigma/2}$ with $\sigma > 2$. First, we must note that some extra hypothesis on $f$ should be added. Indeed, Harnack’s inequality for the biharmonic operator $(-\Delta_{R^2})^2$ holds if we also know that $(-\Delta_{R^2})^2 f \geq 0$, the counterexample being $f(x) = x_1^2$ in $B_2(0)$, see [7]. Secondly, if $\sigma > 2$, the degeneracy weight in the extension equation $\text{div}(y^{1-\sigma} \nabla u) = 0$ does not belong to any $A_p$ class and, up to our knowledge, Harnack’s inequality in this case is not known.
**Theorem 7.3** (Boundary Harnack’s inequality). Let $0 < \sigma < 2$ and $f_1, f_2 \in \text{Dom}(-\Delta)$ be two nonnegative functions on $\mathbb{T}^n$. Suppose that $(-\Delta)^{\sigma/2} f_j = 0$ in $L^2(\mathcal{O})$, for some open set $\mathcal{O} \subseteq \mathbb{T}^n$. Let $x_0 \in \partial \mathcal{O}$ and assume that $f_j = 0$ for all $x \in B_r(x_0) \cap \mathcal{O}$. Assume also that $\partial \mathcal{O} \cap B_r(x_0)$ is a Lipschitz graph in the direction of $x_1$. Then, there is a constant $C$ depending on $\mathcal{O}$, $x_0$, $r$, $n$ and $\sigma$, but not on $f_1$ or $f_2$, such that

$$\sup_{x \in \partial \mathcal{O} \cap B_{r/2}(x_0)} f_1(x) - f_2(x) \leq C \inf_{x \in \partial \mathcal{O} \cap B_{r/2}(x_0)} f_1(x).$$

Moreover, $f_j / f_2$ is $\alpha$-Hölder continuous in $\overline{\mathcal{O} \cap B_{r/2}(x_0)}$, for some universal $0 < \alpha < 1$.

**Proof.** We take $r = 1/2$, the proof for a general $r > 0$ is the same. Let $\tilde{u}_j(x, y) = u_j(x, |y|)$, where $u_j$ is the extension of $f_j$ as in Theorem 6.1. As in the proof of Theorem 7.1, $\tilde{u}_j$ satisfies the degenerate elliptic equation $\text{div}(|y|^{1-\sigma} B \nabla \tilde{u}_j) = 0$ in the weak sense in $\mathcal{O} \times \mathbb{R}$. Moreover, $\tilde{u}_j$ verifies the equation in the weak sense in $(\mathbb{T}^n \times \mathbb{R}) \setminus \{(x_0) : x \in \mathcal{O}\}$ and $\tilde{u}_j(x_0) = f_j(x)$ for all $x \in \mathcal{O}$. Let us take a bilipschitz map $\Psi : \mathbb{T}^n \rightarrow \mathbb{T}^n$ that flattens $\partial \mathcal{O} \cap B_{1/2}(x_0)$, that is, such that $\Psi(x_0) = 0$ and $\Psi(\mathcal{O}) \cap B_{1/2}(0) = \{x_1 > 0\} \cap B_{1/2}(0)$. We can extend this map to $\mathcal{O} \times \mathbb{R}$ as a constant in the variable $y$. Then, the functions $v_j = \tilde{u}_j \circ \Psi^{-1}$ are also solutions of an equation in the same class, namely, $\text{div}(|y|^{1-\sigma} B \nabla v_j) = 0$ in $(\mathbb{T}^n \times \mathbb{R}) \setminus \{(x, 0) : x \in \Psi(\mathcal{O})\}$. Indeed, for any test function $\varphi \in C_c^{\infty}(\mathcal{O})$,

$$\int (\nabla \varphi)^T \nabla v_j \, dx = \int (\nabla \psi)^T (D\Psi)^T (D\Psi) \nabla v_j \, \frac{dz}{|\det D\Psi|},$$

where $\psi(z) = \varphi(\Psi^{-1}(z))$ and $D\Psi$ denotes the Jacobian matrix of the transformation. Then matrix $B$ is $(D\Psi)^T (D\Psi)$, which is uniformly elliptic because $\Psi$ is a bilipschitz transformation. See Figure 1.

![Figure 1](image-url)

*Figure 1. The equation for $v_j$*

For $(x, y) = (x_1, \ldots, x_n, y) \in \mathbb{T}^n \times \mathbb{R}$ we can write, by using polar coordinates, $(x_1, x_2, \ldots, x_n, y) = (\rho \cos \theta, x_2, \ldots, x_n, \rho \sin \theta), \rho > 0, \theta \in (-\pi, \pi)$. Consider now the map

$$\Phi : (\mathbb{T}^n \times \mathbb{R}) \setminus \{(x, 0) : x_1 \leq 0\} \rightarrow (\mathbb{T}^n \times \mathbb{R}) \cap \{(x, y) : x_1 > 0\},$$

defined to be constant in the variables $x_2, \ldots, x_n$ and such that

$$(\rho \cos \theta, x_2, \ldots, x_n, \rho \sin \theta) \mapsto (\rho \cos \frac{\theta}{2}, x_2, \ldots, x_n, \rho \sin \frac{\theta}{2}) := (X_1, x_2, \ldots, x_n, Y) := (X, Y).$$
We see that
\[
D \Phi = \begin{pmatrix}
\frac{\partial}{\partial x_1} X_1 & \frac{\partial}{\partial x_2} X_1 & \cdots & \frac{\partial}{\partial x_n} X_1 & \frac{\partial}{\partial y} X_1 \\
\frac{\partial}{\partial x_1} X_2 & \frac{\partial}{\partial x_2} X_2 & & \cdots & \frac{\partial}{\partial x_n} X_2 & \frac{\partial}{\partial y} X_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{\partial}{\partial x_1} X_n & \frac{\partial}{\partial x_2} X_n & \cdots & \frac{\partial}{\partial x_n} X_n & \frac{\partial}{\partial y} X_n & \\
\frac{\partial}{\partial y} Y & \frac{\partial}{\partial x_2} Y & \cdots & \frac{\partial}{\partial x_n} Y & \frac{\partial}{\partial y} Y & \\
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\theta}{2} & 0 & \cdots & 0 & -\rho / 2 \sin \frac{\theta}{2} \\
0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\sin \frac{\theta}{2} & 0 & \cdots & 0 & \rho / 2 \cos \frac{\theta}{2} \\
\end{pmatrix}.
\]

Therefore, if we denote by $I_n$ the identity matrix of size $n \times n$, $(D \Phi)^T (D \Phi) = \begin{pmatrix} I_n & 0 \\ 0 & \rho^2 / 4 \end{pmatrix}$.

Then, the singular values of $D \Phi$ are equal to one, except for the one in the direction of $\frac{\partial}{\partial \theta}$, that is $\rho / 2$. Also, $\det D \Phi = \rho / 2 = \sqrt{X_1^2 + Y^2} / 2$. Define $w_j = v_j \circ \Phi^{-1}$, in $T^1 \times R \setminus \{(x, y) : x_1 \leq 0\}$. Then $w_j$ is a nonnegative weak solution of $\text{div}(C \nabla w_j) = 0$, in $B_{1/2}(0) \cap \{(x, y) : x_1 > 0\}$. Here $C = \frac{(D \Phi)^T B(D \Phi)}{\det D \Phi} m(X, Y)$, and $m(X, Y) = \frac{2X_1 Y}{\sqrt{X_1^2 + Y^2}}^{1-\sigma}$. See Figure 2.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The equation for $w_j$}
\end{figure}

The equation for $w_j$ above is a degenerate elliptic equation with $A_2$ weight. Therefore, we can apply the theory in [11, Section 2] to get boundary Harnack’s inequality
\[
\sup_{B_{1/4}(0) \setminus \{(x, y) : x_1 > 0\}} \frac{w_1}{w_2} \leq C \inf_{B_{1/4}(0) \setminus \{(x, y) : x_1 > 0\}} \frac{w_1}{w_2},
\]
and the Hölder continuity of $w_1 / w_2$. Go back to $\tilde{u}_1$ and $\tilde{u}_2$ and restrict them to $y = 0$ for the conclusion. \hfill \Box

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