Revisiting the spin-half bosons with mass dimension three-half

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This work is dedicated to deeply understanding a quite recent theoretical discovery about new particles that reside in the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation space, namely, spin-half bosons, introduced in [1]. We delve into the first principles of the quantum field theory, following Weinberg’s approach to causal fields, to bring out the main characteristics carried by all spin-half particles, seeking to connect the Weinberg formalism with a (supposed) evasion of spin-statistics theorem [2]. As we see, the local quantum fields associated with these spin-half bosons cannot be used to represent elementary particles with spin $1/2$ in a quantum field theory that satisfies Lorentz symmetry.

I. INTRODUCTION

Wigner’s classic works established the very concept of particle in physics [3, 4]. It has been demonstrated, in a well-posed mathematical formulation, that a particle is nothing more than an irreducible representation of the Poincaré group. First, in Ref. [3], Wigner developed all discussions within the orthochronous Lorentz subgroup itself, without taking into account the reflections carried out by discrete symmetries that lead to the complete Lorentz group. Later, in a more complete article [4], a general treatment taking into account the states of a particle including reflections was developed.

As important results coming from the Wigner’s works, we evince four different cases, namely Wigner classes. That is, one case standing for a spin one-half one-particle state with well defined transformation rules under action of parity, time reversal, charge conjugation operations, that is, the Dirac spinors, and also three quite non-usual cases with respect to their behaviour under action of the aforementioned transformations, encoding, thus, certain doubling states, i.e. degeneracy, under reflections. The results obtained by Wigner were re-examined in [5], bringing, thus, a concise interpretation of the results.

The foundations of Quantum Field Theory dictate that quantum fields are the result of engaging well defined one-particle states with well defined transformation rules under action of parity, time reversal, charge conjugation operations, that is, the Dirac’s quantum field from first principles of relativity and quantum mechanics is accomplished without invoking the Dirac equation and the work results in a no-go theorem: on the impossibility of constructing another spin one-half quantum field without violating Lorentz symmetries, and, consequently, locality. However, quite recently, new possibilities concerning spin one-half fermionic field have been investigated in [9], bypassing the rigidity of the Weinberg theorem. Such formalization may lead to a local and Lorentz invariant field endowed with a mass dimension one, thus evading the Weinberg’s no-go theorem for a freedom in the spinorial adjoint structure.

Recently, a series of results, such as the theoretical discovery of new spin one-half fermions endowed with a mass dimension one [11–13], widely explored in cosmology, mathematical physics and phenomenology framework [14]-[41], suggests that the fundamental structure of Quantum Field Theory (QFT) is certainly being improved and augmented. As far as we know, such mass dimension one fermions belongs to what is known as the Beyond the Standard Model (BSM) of particle physics. Certainly, understanding the fundamental

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aspects of the aforementioned particles will help us to understand what the content of the BSM particles could be.

In the second decade of the 2000s, a new theoretical discovery encompassing some new spin-half particles caught attention for their rather peculiar features [1]. It is claimed that such spinors belong to an entirely new class of spin-half particles that reside in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space, endowed with mass dimension three-half which evades the spin-statistics theorem \([2, 42, 43]\). In other words, the creator and annihilator operators of the associated local field satisfy commutation relations instead. Therefore, such a reported unusual features make it necessary to revisit the pillars of QFT and also the spin-statistics theorem \([8]\) in search of a possible freedom in the Weinberg’s formalism.

This article is organized as follows: in section II, we follow S. Weinberg in order to find the statistics that spin one-half fields, in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space must satisfy. We consider only general principles, such as Lorentz invariance of the S-matrix and the cluster decomposition. We generalize the Weinberg’s result to fields with expansion coefficients that are not eigenstates of any discrete symmetry. In section III, we consider in detail the quantum field proposed in Ref. [1] in order to deeply understand why this field apparently evades the spin-statistics theorem of the quantum field theory. Finally, in the last section, we present our conclusions.

II. SPIN ONE-HALF RELATIVISTIC QUANTUM FIELDS

In this section, we will consider relativistic spin one-half fields in order to show that this type of field satisfies the Fermi-Dirac statistic regardless of whether they are chosen as parity eigenstates, charge conjugation, or time inversion operators or not. To do this, we take into account two general principles: the Lorentz invariance of the S-matrix and the cluster decomposition as in Ref. [8]. The first one requires that the interaction term is the spacetime integral of a Hamiltonian scalar density, \(\mathcal{H}(x)\), that satisfies:

\[
U(\Lambda, a)\mathcal{H}(x)U(\Lambda, a)^{-1} = \mathcal{H}(\Lambda x + a), \text{ i.e. } \mathcal{H}(x) \text{ is a Lorentz scalar,}
\]

and

\[
[\mathcal{H}(x), \mathcal{H}(x')] = 0, \text{ for } (x - x')^2 \geq 0,
\]

where \(\Lambda\) is a Lorentz transformation, \(a\) is a general space-time translation and, \(U\) is the corresponding operator representing the Lorentz transformation in the Hilbert space. Also, note that throughout this work we use the metric signature \((-+, ++, +)\). Roughly speaking, the cluster decomposition principle imposes that \(\mathcal{H}(x)\) is constructed out the creation and annihilation operators. Because of the transformation of these operators under Lorentz transformations, \(\mathcal{H}(x)\) must be built out the fields with the following form:

\[
\psi^\ell_+(x) = \sum_{s, n} \int d^3 p \, u_\ell(x, p, s, n) a(p, s, n),
\]

\[
\chi^\ell_-(x) = \sum_{s, n} \int d^3 p \, v_\ell(x, p, s, n) b^\dagger(p, s, n),
\]

where \(n\) denotes internal quantum numbers and \(s\) runs over the spin \(z\)-components. From here on, we will omit the \(n\) index because these are not essential for our purposes.

Also, the coefficients \(u_\ell(x, p, s)\) and \(v_\ell(x, p, s)\) must be chosen such that the fields \(\psi^\ell_+(x)\) and \(\chi^\ell_-(x)\) transform under Lorentz transformations as

\[
U(\Lambda, a)\psi^\ell_+(x)U(\Lambda, a)^{-1} = \sum_\ell D_{\ell\ell}(\Lambda^{-1})\psi^\ell_+(\Lambda x + a),
\]

\[
U(\Lambda, a)\chi^\ell_-(x)U(\Lambda, a)^{-1} = \sum_\ell D_{\ell\ell}(\Lambda^{-1})\chi^\ell_-(\Lambda x + a),
\]

where the \(D\)-matrix furnishes a representation of the homogeneous Lorentz group.

More specific properties of the coefficients \(u_\ell(x, p, s)\) and \(v_\ell(x, p, s)\) can be deduced applying Eqs. (5) and (6) for pure translations, boosts and rotations. First, applying pure translations it is found (Ref. [8]):

\[
u_\ell(x, p, s) = (2\pi)^{-3/2} e^{ipx} u_\ell(p, s),
\]

\[
v_\ell(x, p, s) = (2\pi)^{-3/2} e^{-ipx} v_\ell(p, s),
\]
where the factor \((2\pi)^{-3/2}\) is conventional. In other words, because of the translational invariance, the dependence on \(x\) of \(u(x, p, s)\) and \(v(x, p, s)\) is only an exponential factor. We also have that due to the invariance under a general boost, \(L(p)\), these coefficients must satisfy

\[
u_x(p, s) = \sqrt{\frac{m}{p^0}} \sum_{\ell} D_{\ell\ell}(L(p)) \nu_{\ell}(0, s), \quad \nu_{\ell}(p, s) = \sqrt{\frac{m}{p^0}} \sum_{\ell} D_{\ell\ell}(L(p)) \nu_{\ell}(0, s),
\]

where \(D(L(p))\) is the matrix representation of a general Lorentz boost and, \(u(0, s)\) and \(v(0, s)\) are the corresponding zero momentum coefficients. Finally, applying Eqs. (5) and (6) with \(\Lambda = R\) (where \(R\) is a general rotation) to the coefficients \(u(0, s)\) and \(v(0, s)\) it is straightforward see that these coefficients have to satisfy the following fundamental conditions

\[
\sum_{\tilde{s}} u_{\ell}(0, \tilde{s}) J^{(j)}_{\ell\tilde{s}} = \sum_{\ell} J_{\ell\ell} u_{\ell}(0, s),
\]

\[
- \sum_{\tilde{s}} v_{\ell}(0, \tilde{s}) J^{(j)}_{\ell\tilde{s}} = \sum_{\ell} J_{\ell\ell} v_{\ell}(0, s),
\]

where \(J^{(j)}\) and \(J\) are the angular momentum matrices in the representations \(D^{(j)}(R)\) and \(D(R)\). The conditions in Eqs. (10) and (11) mean that if \(\psi^+(x)\) and \(\chi^-(x)\) are supposed to describe particles with a spin \(j\) then the representation \(D(R)\) must contain among its irreducible components the spin-\(j\) representation \(D^{(j)}(R)\).

In this point, it is important to remark that the conditions on the \(\psi^+(x)\) and \(\chi^-(x)\) fields given in Eqs. (5)-(6) and their respective consequences in Eqs. (7)-(11) are general in the sense that they do not depend on neither a particular Lagrangian (equations of motion) nor the using of a discrete symmetry such a parity, time reversal or charge conjugation.

Now, it is time to use the general framework above discussed to our specific case. To do that, we will use the Weyl basis to write \(J^{(j)}\) and \(J\) matrices in Eqs. (10)-(11) in the specific representations \(j = 1/2\) and \((3/2, 0) \oplus (0, 3/2)\), respectively. In this basis we have

\[
J^{(1/2)} = \frac{1}{2} \sigma, \quad -J^{(1/2)^*} = \frac{1}{2} \sigma_2 \sigma_2,
\]

and

\[
J_{\ell 0} = -\frac{i}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \quad J_{ij} = \frac{i}{2} \epsilon_{ijk} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix},
\]

where \(\sigma_k, k = 1, 2, 3\) are the Pauli matrices. Substituting the above matrix representations in conditions (10) and (11), we obtain

\[
\sum_{\tilde{s}} (u_{\pm}(0, \tilde{s}))_i J^{(1/2)}_{s \tilde{s}} = \sum_{j} \frac{1}{2} \sigma_{ij} (u_{\pm}(0, s))_j,
\]

\[
- \sum_{\tilde{s}} (v_{\pm}(0, \tilde{s}))_i J^{(1/2)^*}_{s \tilde{s}} = \sum_{j} \frac{1}{2} \sigma_{ij} (v_{\pm}(0, s))_j.
\]

where we have defined \(u(0, s) \equiv (u_+(0, s), u_-(0, s))^T\) and \(v(0, s) \equiv (v_+(0, s), v_-(0, s))^T\).

By considering \((u_{\pm}(0, s))_i\) and \((v_{\pm}(0, s))_i\) as the \((i, s)\) elements of corresponding matrices \(U_{\pm}\) and \(V_{\pm}\), we can rewrite conditions (10) and (11) in matrix form

\[
U_{\pm} J^{(1/2)} = \frac{1}{2} \sigma U_{\pm},
\]

\[
-V_{\pm} J^{(1/2)^*} = \frac{1}{2} \sigma V_{\pm}.
\]

Then, the Schur’s Lemma can be applied to straightforwardly show that there exist only two possible solutions for the \(U_{\pm}\) and \(V_{\pm}\) matrices. The first one is a trivial solution with vanishing matrices, and the second one
tell us that $U_\pm$ and $V_\pm \sigma_2$ are proportional to the identity matrix. Thus, we find that the most general zero-momentum $u_\ell(0, s)$ and $v_\ell(0, s)$ spinors can take only the following forms

\[
u \left( 0, \frac{1}{2} \right) = \begin{bmatrix} c_+ & 0 \\ 0 & c_- \end{bmatrix}, \quad \nu \left( 0, -\frac{1}{2} \right) = \begin{bmatrix} 0 & c_- \\ c_+ & 0 \end{bmatrix}, \quad \nu \left( 0, \frac{1}{2} \right) = \begin{bmatrix} 0 \\ 0 \\ d_+ \end{bmatrix}, \quad \nu \left( 0, -\frac{1}{2} \right) = -\begin{bmatrix} 0 \\ d_- \\ 0 \end{bmatrix}, \tag{18}\]

where $c_+$ and $d_\pm$ are arbitrary constants, which in general can be complex numbers or even zero.

Now, we turn our attention on the condition in Eq. (2) which comes from the Lorentz invariance of the S-matrix. As shown in [8] this condition requires that $\mathcal{H}(x)$ is a function of the $\Psi(x)$ (and its Hermitian conjugate) which is written as

\[
\Psi(x) = \kappa \psi^+(x) + \lambda \chi^-(x), \tag{20}\]

where $\kappa$ and $\lambda$ are constant which are chosen such that, for space-like intervals, the fields satisfy the following conditions

\[
\left[ \Psi_\ell(x), \Psi_\ell(y) \right] = 0, \quad \left[ \Psi_\ell(x), \psi^\dagger_\ell(y) \right] = 0, \tag{21}\]

where $(-, +)$ signs means commutator and anti-commutator, respectively.

The first condition in Eq. (21) is satisfied for any $\kappa$ and $\lambda$ constants because

\[
[a(p, s), a(p', s')]_\mp = [b(p, s), b(p', s')]_\mp = [a(p, s), b(p', s')]_\mp = 0, \tag{22}\]

and the corresponding relations for the Hermitian conjugate operators. However, the second condition in Eq. (21) is not satisfied in general, in this way imposing some constraints over the type of statistics that the field satisfies and, eventually, over the $\kappa$ and $\lambda$ constants. Using the canonical relations

\[
[a(p, s), a^\dagger(p', s')]_\mp = \delta^{(3)}(p - p')\delta_{ss'}, \quad [b(p, s), b^\dagger(p', s')]_\mp = \delta^{(3)}(p - p')\delta_{ss'}, \tag{23}\]

we obtain

\[
\left[ \Psi_\ell(x), \psi^\dagger_\ell(y) \right] = (2\pi)^{-3} \int d^3p \left[ |\kappa|^2 N_\ell(p) e^{ip(x-y)} \mp |\lambda|^2 M_\ell(p) e^{-ip(x-y)} \right], \tag{24}\]

where

\[
N_\ell(p) = \sum_s u_\ell(p, s) u_\ell^\dagger(p, s), \tag{25}\]

\[
M_\ell(p) = \sum_s v_\ell(p, s) v_\ell^\dagger(p, s). \tag{26}\]

Applying Eq. (9), we can write the above quantities as

\[
N_\ell(p) = \frac{m}{p^0} D(L(p)) N_\ell(0) D^\dagger(L(p)), \tag{27}\]

\[
M_\ell(p) = \frac{m}{p^0} D(L(p)) M_\ell(0) D^\dagger(L(p)), \tag{28}\]

where

\[
N_\ell(0) = \begin{bmatrix} |c_+|^2 & 0 & c_+ c_+^* & 0 \\ 0 & |c_+|^2 & 0 & c_+ c_-^* \\ c_- c_+^* & 0 & |c_-|^2 & 0 \\ 0 & c_- c_+^* & 0 & |c_-|^2 \end{bmatrix}, \quad M_\ell(0) = \begin{bmatrix} |d_+|^2 & 0 & d_+ d_+^* & 0 \\ 0 & |d_+|^2 & 0 & d_+ d_-^* \\ d_- d_+^* & 0 & |d_-|^2 & 0 \\ 0 & d_- d_+^* & 0 & |d_-|^2 \end{bmatrix}. \tag{29}\]
From Eq. (29), we can see that \( N_{\ell\ell}(0) \) and \( M_{\ell\ell}(0) \) have a similar form and, for that reason, we will explicitly consider the calculations only for \( N_{\ell\ell}(0) \), which can be spanned in terms of the gamma matrices as follows

\[
N_{\ell\ell}(0) = \frac{1}{2} (|c_+|^2 + |c_-|^2) \mathbb{1} + \frac{1}{2} (c_+ c_-^* + c_- c_+^*) i\gamma^0 + \frac{1}{2} (|c_+|^2 - |c_-|^2) \gamma^5 + \frac{1}{2} (c_+ c_-^* - c_- c_+^*) i\gamma^0 \gamma^5. \tag{30}
\]

It is worth mentioning that Eq. (30) is a general result, where no assumptions on parity symmetry or any other discrete symmetries have been used. Now, substituting Eq. (30) in Eq. (27), and using the following relations

\[
D(L(p))\mathbb{1} D^\dagger(L(p)) = \frac{1}{m} p_\mu \gamma^\mu \gamma^0, \tag{31}
\]

\[
D(L(p))i\gamma^0 D^\dagger(L(p)) = i\gamma^0, \tag{32}
\]

\[
D(L(p))\gamma_5 D^\dagger(L(p)) = \frac{1}{m} p_\mu \gamma^\mu \gamma_5 \gamma^0, \tag{33}
\]

\[
D(L(p))i\gamma^0 \gamma_5 D^\dagger(L(p)) = i\gamma_5 \gamma^0, \tag{34}
\]

we obtain

\[
N_{\ell\ell}(p) = \frac{1}{2p^0} \left[ -i (|c_+|^2 + |c_-|^2) p_\mu \gamma^\mu + m (c_+ c_-^* + c_- c_+^*) \mathbb{1} + m (c_+ c_-^* - c_- c_+^*) \gamma_5 - i (|c_+|^2 - |c_-|^2) p_\mu \gamma^\mu \gamma_5 \right] i\gamma^0. \tag{35}
\]

By replacing \( c_\pm \rightarrow d_\pm \) in Eq. (35) we get the expression for \( M_{\ell\ell}(p) \), namely

\[
M_{\ell\ell}(p) = \frac{1}{2p^0} \left[ -i (|d_+|^2 + |d_-|^2) p_\mu \gamma^\mu + m (d_+ d_-^* + d_- d_+^*) \mathbb{1} + m (d_+ d_-^* - d_- d_+^*) \gamma_5 - i (|d_+|^2 - |d_-|^2) p_\mu \gamma^\mu \gamma_5 \right] i\gamma^0. \tag{36}
\]

With these results we can finally write Eq. (24) as

\[
\left[ \Psi_\ell(x), \Psi_\ell^\dagger(y) \right] = \left( |\kappa|^2 \left[ -(|c_+|^2 + |c_-|^2) \gamma^\mu \partial_\mu + m (c_+ c_-^* + c_- c_+^*) \mathbb{1} + m (c_+ c_-^* - c_- c_+^*) \gamma_5 - (|c_+|^2 - |c_-|^2) \gamma^\mu \gamma_5 \partial_\mu \right] i\gamma^0 \Delta(x-y) + \mathbb{1} \left[ |\lambda|^2 \left[ -(|d_+|^2 + |d_-|^2) \gamma^\mu \partial_\mu + m (d_+ d_-^* + d_- d_+^*) \mathbb{1} + m (d_+ d_-^* - d_- d_+^*) \gamma_5 - (|d_+|^2 - |d_-|^2) \gamma^\mu \gamma_5 \partial_\mu \right] i\gamma^0 \Delta(y-x) \right] \right), \tag{37}
\]

where

\[
\Delta(x) \equiv \int \frac{d^4p}{2p^0(2\pi)^3} e^{ipx}. \tag{38}
\]

In order to satisfy the second condition in Eq. (21), it is necessary and sufficient that

\[
|\kappa|^2 |c_+|^2 = \mp |\lambda|^2 |d_+|^2, \quad |\kappa|^2 |c_-|^2 = \mp |\lambda|^2 |d_-|^2, \tag{39}
\]

and

\[
|\kappa|^2 c_+ c_-^* = \pm |\lambda|^2 d_+ d_-^*, \tag{40}
\]

where we have used that for \((x - y)\) space-like \(\Delta(x-y)\) and its first derivative are even and odd functions of \((x - y)\), respectively. We notice that the constraints in Eq. (39) have a non-trivial solution provided we choose the bottom sign, meaning that the spin-1/2 field \(\Psi(x)\) in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation must satisfy anti-commutation relations, i.e. the spin-1/2 field \(\Psi(x)\) is a fermion, in complete agreement with the spin-statistics theorem. This general result does not depend on the use of any discrete symmetry such as parity.
Once the statistics of the $\Psi(x)$ field is settled, Eqs. (39) and (40) simplify a little more and we are taken to the following useful relation

$$\frac{c_+}{c_-} = \frac{d_+}{d_-}. \tag{41}$$

Note that $c_\pm$ and $d_\pm$ are not completely determined from the condition in Eq. (41). Now, we can write $\kappa = |\kappa|e^{i\theta_x}$, $\lambda = |\lambda|e^{i\theta_x}$ and, redefining the creation and annihilation operators in Eqs. (3) and (4) as $a(p, s) \rightarrow e^{i\theta_x}a(p, s)$ and $b(p, s) \rightarrow e^{-i\theta_x}b(p, s)$ (note that these redefinitions do not modify the canonical anti-commutation relations of $a$ and $b$ operators in Eq. (23)), we obtain

$$\Psi(x) = |\kappa| \left[ \psi^+(x) + \frac{|\lambda|}{|\kappa|} \chi^-(x) \right]. \tag{42}$$

Absorbing the overall factor $|\kappa|$ in the normalization of the field $\Psi(x)$ and using the constraints in Eqs. (39) and (40), we get

$$\Psi(x) = \psi^+(x) + \frac{c_+}{|d_\pm|} \chi^-(x). \tag{43}$$

Needless to say, we are assuming that $|d_+| \neq 0$ or $|d_-| \neq 0$ in Eq. (43).

To determine completely $c_\pm$ and $d_\pm$ additional physical or mathematical conditions must be imposed. For example, a default choice is to impose parity symmetry on the fields in Eqs. (3) and (4) which lead us to

$$\left( \frac{c_+}{c_-} \right)^2 = \left( \frac{d_+}{d_-} \right)^2 = 1. \tag{44}$$

Using this relation and the freedom of the overall factor, the form of the $\Psi(x)$ field is the standard Dirac field, i.e. $\Psi(x) = \psi^+(x) + \chi^-(x)$. It is also possible consider the charge-conjugation and time reversal properties of the field in Eq. (43) to obtain, for instance, the Majorana fields c.f. [8].

### III. SPIN-HALF BOSONS WITH MASS DIMENSION THREE-HALF

We now turn our attention to the class of quantum fields presented in [1]. Roughly speaking, in this reference, it is claimed that this spin one-half field belongs to the $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$ representation, however it satisfies the Bose-Einstein statistics which is in clear contradiction with the spin-statistics theorem. Therefore, in this section we will briefly review the main ideas behind this field and show why it does not satisfy the mentioned theorem.

The key idea behind the proposed field in Ref. [1] is to expand the coefficients of the quantum field as a combination of the eigenvectors of one the sixteen different roots of the $1_{4\times 4}$ matrix. Note that $1_{4\times 4}$, $i\gamma_1$, $i\gamma_2$, $i\gamma_3$, $\gamma_0$, $i\gamma_2\gamma_3$, $i\gamma_1\gamma_2$, $\gamma_0\gamma_3$, $\gamma_0\gamma_2\gamma_3$, $i\gamma_0\gamma_1\gamma_2\gamma_3$, $i\gamma_0\gamma_1\gamma_2$, $\gamma_1\gamma_2\gamma_3$ and $i\gamma_0\gamma_1\gamma_2\gamma_3$ are all roots of the $1_{4\times 4}$ matrix. From here on these roots are denoted as $\Omega_i$ with $i = 1, \ldots, 16$. Then, the author chooses the following vectors

$$\lambda_1(0) = \sqrt{\frac{m}{2}} (\mu_1 + i\mu_2), \tag{45}$$
$$\lambda_2(0) = \sqrt{\frac{m}{2}} (\mu_1 - i\mu_2), \tag{46}$$

and

$$\lambda_3(0) = \sqrt{\frac{m}{2}} (\mu_3 + i\mu_4), \tag{47}$$
$$\lambda_4(0) = \sqrt{\frac{m}{2}} (\mu_3 - i\mu_4), \tag{48}$$

Where $m$ is the mass of the field. This choice of basis makes the field satisfy the Bose-Einstein statistics.
as the expansion coefficients of the quantum field at momentum \( \mathbf{p} = 0 \). Here \( \mu_1, \mu_2, \mu_3 \) and \( \mu_4 \) are the eigenvectors of the \( \Omega_4 = i\gamma_3 \) that are written as

\[
\mu_1 = \left[ 0, 0, 1 \right]^T, \quad \mu_2 = \left[ i, 0, 1 \right]^T, \quad \mu_3 = \left[ 0, -i, 1 \right]^T, \quad \mu_4 = -\left[ -i, 0, 1 \right]^T, \quad (49)
\]

Applying the Lorentz boost, \( D(L(\mathbf{p})) \), on the \( \lambda_i(0) \) it is straightforward to obtain the \( \lambda_i(\mathbf{p}) \) for any \( \mathbf{p} \neq 0 \). Once this is done, the quantum field is defined as

\[
\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \left[ \sum_{i=1,2} \lambda_i(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} a_i(\mathbf{p}) + \sum_{i=3,4} \lambda_i(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} b_i(\mathbf{p}) \right], \quad (51)
\]

and its adjoint field

\[
\overline{\Phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \left[ \sum_{i=1,2} \overline{\lambda}_i(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} a_i^\dagger(\mathbf{p}) + \sum_{i=3,4} \overline{\lambda}_i(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} b_i^\dagger(\mathbf{p}) \right]. \quad (52)
\]

The dual in Eq. (52) is not the standard one but \( \overline{\lambda}_i(\mathbf{p}) = [\mathcal{P}\lambda_i(\mathbf{p})]^\dagger \mathcal{P} \gamma_0 \) where \( \mathcal{P} = m^{-1}p^\mu \gamma_\mu \) is the parity operator in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation. This redefinition of the dual is necessary in this construction because of the invariant orthogonality relations. It is also important remark that in the momentum space the expansion coefficients of \( \Phi(x), \lambda_i(\mathbf{p}) \), satisfy Dirac-like equations, i.e.

\[
(a_\mu p^\mu - m \mathbb{1}) \lambda_{1,2}(\mathbf{p}) = 0, \quad (53)
\]

\[
(a_\mu p^\mu + m \mathbb{1}) \lambda_{3,4}(\mathbf{p}) = 0. \quad (54)
\]

in which \( a_\mu = i\gamma_\mu \) with \( \gamma = \frac{1}{4!} e^{\mu\nu\lambda\sigma} \gamma_\nu \gamma_\lambda \gamma_\sigma \). That means that the \( \lambda_i(\mathbf{p}) \) are eigenvectors of the \( a_\mu p^\mu \) operator however they are not eigenstate of the parity operator \( \mathcal{P} \).

In order to set the statistics that \( \Phi(x) \) field satisfies, the author applies the condition in Eq. (21) finding that the \( \Phi(x) \) is a bosonic field. Then, the author claims that \( \Phi(x) \) is a new of type bosonic field endowed with spin-half, that evades the spin-statistics theorem of the quantum field theory. However, the field in Eq. (51), or its adjoint in Eq. (52), does not create or annihilate particles with spin 1/2. There are at least two ways to see that. The first one is to note that the expansion coefficients, \( \lambda_i(0) \), do not satisfy any of the fundamental conditions in Eqs. (14) and (15) as they should be if the \( \Phi(x) \) was for describing particles with spin 1/2 as it was explained in Sec. II. The second way is to directly apply the spin operator \( \mathcal{J} \) in the specific representation \((\frac{3}{2}, 0) \oplus (0, \frac{1}{2})\), Eq. (13), to (an)a (anti-)particle state created by the \( \Phi(x) \) field. For instance, if the \( \Phi(x) \) field in Eq. (51) is contracted in the vacuum state, \( |0\rangle \), it creates an anti-particle (or a type-\(b\) particle), which we denoted as \( |1_\lambda\rangle \), with two possible spin projections given by \( \lambda_3(\mathbf{p}) \) and \( \lambda_4(\mathbf{p}) \). Then, if the spin-z operator, i.e., \( \mathcal{J}_{12} \), is applied to this one particle state \( |1_\lambda\rangle \) it is not obtained \( \pm \frac{\gamma}{2} |1_\lambda\rangle \) as it should be if the \( \Phi(x) \) field created particles/anti-particles with spin 1/2. Therefore, it is now absolutely clear that the field \( \Phi(x) \) in Eq. (51) is neither appropriated to describe particles with spin 1/2 in the representation \((\frac{3}{2}, 0) \oplus (0, \frac{1}{2})\) nor there is an exception to the fundamental spin-statistics theorem.

It is our understanding that the main cause of the field proposed in reference [1] fails to describe particles with spin 1/2 is that although this construction mimics in some points Dirac’s historical construction, it uses as expansion coefficients of the field a linear combination of the eigenstates of the \( \Omega_4 \) operator, one of the sixteen roots of the \( 4 \times 4 \) identity matrix. However, these are not eigenstates of the spin operator in the \((\frac{3}{2}, 0) \oplus (0, \frac{1}{2})\) representation.

\section*{IV. CONCLUSIONS}

In this paper, we first show that any spinorial field in the representation \((\frac{3}{2}, 0) \oplus (0, \frac{1}{2})\) of the Lorentz group satisfies the Fermi-Dirac statistics, that is, it is a fermionic field, regardless of whether its expansion coefficients are eigenstates of the parity operator or any other discrete symmetry operator. Next, we show that, despite the fact that the quantum field introduced in Ref. [1] satisfies some of the conditions arising
from the Lorentz invariance of the $S$-matrix, such as translation, Eqs. (7) and (8), and boost invariance, Eq. (9), it fails to satisfy the invariance under rotations, Eqs. (10) and (11), which causes that this proposed field does not create particles with spin $1/2$. Therefore, this field can not be used to represent elementary particles with spin $1/2$ in a quantum field theory that satisfies Lorentz symmetry.

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