THE CLOSURE OF TWO-SIDED MULTIPLICATIONS ON C*-ALGEBRAS AND PHANTOM LINE BUNDLES

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Abstract. For a C*-algebra $A$ we consider the problem of when the set $TM_0(A)$ of all two-sided multiplications $x \mapsto axb$ ($a, b \in A$) on $A$ is norm closed, as a subset of $B(A)$. We first show that $TM_0(A)$ is norm closed for all prime C*-algebras $A$. On the other hand, if $A \cong \Gamma_0(E)$ is an $n$-homogeneous C*-algebra, where $E$ is the canonical $M_n$-bundle over the primitive spectrum $X$ of $A$, we show that $TM_0(A)$ fails to be norm closed if and only if there exists a $\sigma$-compact open subset $U$ of $X$ and a phantom complex line subbundle $L$ of $E$ over $U$ (i.e. $L$ is not globally trivial, but is trivial on all compact subsets of $U$). This phenomenon occurs whenever $n \geq 2$ and $X$ is a CW-complex (or a topological manifold) of dimension $3 \leq d < \infty$.

1. Introduction

Let $A$ be a C*-algebra and let $IB(A)$ (resp. $ICB(A)$) denote the set of all bounded (resp. completely bounded) maps $\phi : A \to A$ that preserve (closed two-sided) ideals of $A$ (i.e. $\phi(I) \subseteq I$ for all ideals $I$ of $A$). The most prominent class of maps $\phi \in ICB(A) \subseteq IB(A)$ are elementary operators, i.e. those that can be expressed as finite sums of two-sided multiplications $M_{a,b} : x \mapsto axb$, where $a$ and $b$ are elements of the multiplier algebra $M(A)$.

Elementary operators play an important role in modern quantum information and quantum computation theory. In particular, maps $\phi : M_n \to M_n$ ($M_n$ are $n \times n$ matrices over $\mathbb{C}$) of the form $\phi = \sum_{i=1}^{\ell} M_{a_i^*,a_i}$ ($a_i \in M_n$ such that $\sum_{i=1}^{\ell} a_i^*a_i = 1$) represent the (trace-duals of) quantum channels, which are mathematical models of the evolution of an ‘open’ quantum system (see e.g. [21]). Elementary operators also provide ways to study the structure of C*-algebras (see [2]).

Let $E(A)$, $TM(A)$ and $TM_0(A)$ denote, respectively, the sets of all elementary operators on $A$, two-sided multiplications on $A$ and two-sided multiplications on $A$ with coefficients in $A$ (i.e. $TM_0(A) = \{M_{a,b} : a, b \in A\}$).

The elementary operators are always dense in $IB(A)$ in the topology of pointwise convergence (by [23, Corollary 2.3]). However, more subtle considerations enter in when one asks if $\phi \in ICB(A)$ can be approximated pointwise by elementary operators of cb-norm at most $\|\phi\|_{cb}$ ([24] shows that nuclearity of $A$ suffices; see also [26]).
It is an interesting problem to describe those operators \( \phi \in \text{IB}(A) \) (resp. \( \phi \in \text{ICB}(A) \)) that can be approximated in operator norm (resp. cb-norm) by elementary operators. Earlier works, which we cite below, revealed that this is an intricate question in general, and can involve many and varied properties of \( A \) and \( \phi \). In this paper, we show that the apparently much simpler problems of describing the norm closures of \( \text{TM}(A) \) and \( \text{TM}_0(A) \) can have complicated answers even for rather well-behaved \( C^* \)-algebras.

In some cases \( \mathcal{E}\ell(A) = \text{IB}(A) \) (which implies \( \mathcal{E}\ell(A) = \text{ICB}(A) \)); or \( \mathcal{E}\ell(A) \) is norm dense in \( \text{IB}(A) \); or \( \mathcal{E}\ell(A) \subset \text{ICB}(A) \) is dense in cb-norm. The conditions just mentioned are in fact all equivalent for separable \( C^* \)-algebras \( A \). More precisely, Magajna [25] shows that for separable \( C^* \)-algebras \( A \), the property that \( \mathcal{E}\ell(A) \) is norm (resp. cb-norm) dense in \( \text{IB}(A) \) (resp. \( \text{ICB}(A) \)) characterizes finite direct sums of homogeneous \( C^* \)-algebras with the finite type property. Moreover, in this situation we already have the equality \( \text{ICB}(A) = \text{IB}(A) = \mathcal{E}\ell(A) \).

It can happen that \( \mathcal{E}\ell(A) \) is already norm closed (or cb-norm closed). In [13, 14], the first author showed that for a unital separable \( C^* \)-algebra \( A \), if \( \mathcal{E}\ell(A) \) is norm (or cb-norm) closed then \( A \) is necessarily subhomogeneous, the homogeneous subquotients of \( A \) must have the finite type property and established further necessary conditions on \( A \). In [14, 15] he gave some partial converse results.

There is a considerable literature on derivations and inner derivations of \( C^* \)-algebras. Inner derivations \( d_u \) on a \( C^* \)-algebra \( A \), (i.e. those of the form \( d_u(x) = ax - xa \) with \( a \in M(A) \)) are important examples of elementary operators. In [35] Corollary 4.6] Somerset shows that if \( A \) is unital, \( \{d_u : a \in A\} \) is norm closed if and only if \( \text{Orc}(A) < \infty \), where \( \text{Orc}(A) \) is a constant defined in terms of a certain graph structure on \( \text{Prim}(A) \) (the primitive spectrum of \( A \)). If \( \text{Orc}(A) = \infty \), the structure of outer derivations that are norm limits of inner derivations remains undescribed. In addition, if \( A \) is unital and separable, then by [19] Theorem 5.3] and [35] Corollary 4.6] \( \text{Orc}(A) < \infty \) if and only if the set \( \{M_{u,u^*} : u \in A, u \text{ unitary}\} \) of inner automorphisms is norm closed.

In [12, 15] the first author considered the problem of which derivations on unital \( C^* \)-algebras \( A \) can be cb-norm approximated by elementary operators. By [15] Theorem 1.5] every such a derivation is necessarily inner in a case when every Glimm ideal of \( A \) is prime. When this fails, it is possible to produce examples which have outer derivations that are simultaneously elementary operators ([12 Example 6.1]).

While considering derivations \( d \) that are elementary operators and/or norm limits of inner derivations, we realized that they are sometimes expressible in the form \( d = M_{a,b} - M_{b,a} \) even though they are not inner. We have not been able to decide when all such \( d \) are of this form, but this led us to the seemingly simpler question of considering the closures of \( \text{TM}(A) \) and \( \text{TM}_0(A) \). In this paper, we see that nontrivial considerations enter into these questions about two-sided multiplications. Of course the left multiplications \( \{M_{a,1} : a \in M(A)\} \) are already norm closed, as are the right multiplications. So \( \text{TM}(A) \) is a small subclass of \( \mathcal{E}\ell(A) \), and seems to be the basic case to study.

This paper is organized as follows. We begin in [22] with some generalities and an explanation that the set of elementary operators of length at most \( \ell \) has the same completion in the operator and cb-norms (for each \( \ell \geq 1 \)).
In [3] we show that for a prime $C^*$-algebra $A$, we always have $TM(A)$ and $TM_0(A)$ both norm closed.

In [4] we recall the description of $[n]$-homogeneous $C^*$-algebras $A$ as sections $\Gamma_0(\mathcal{E})$ of $M_n$-bundles $\mathcal{E}$ over $X = \text{Prim}(A)$ and some general results about $\text{IB}(A)$, $\text{ICB}(A)$ and $\mathcal{E}_\ell(A)$ for such $A$.

In [5], for homogeneous $C^*$-algebras $A = \Gamma_0(\mathcal{E})$, we consider subclasses $\text{IB}_1(A)$ and $\text{IB}_{0,1}(A)$ of $\text{IB}(A)$ that seem (respectively) to be the most obvious choices for the norm closure of $TM(A)$ and of $TM_0(A)$, extrapolating from fibrewise restrictions on $\phi \in TM(A)$. For each $\phi \in \text{IB}_1(A)$ we associate a complex line subbundle $L_\phi$ of the restriction $\mathcal{E}|_U$ to an open subset $U \subseteq X = \text{Prim}(A)$, where $U$ is determined by $\phi$ as the cozero set of $\phi$ ($U$ identifies the fibres of $\mathcal{E}$ on which $\phi$ acts by a nonzero operator). For separable $A$, the main result of this section is Theorem 5.15 where we characterize the condition $\phi \in TM(A)$ in terms of triviality of the bundle $L_\phi$.

We close [5] with Remark 5.24 comparing our bundle considerations to slightly similar results in the literature for innerness of $C(X)$-linear automorphisms when $X$ is compact, or for some more general unital $A$.

Our final [6] is the main section of this paper. For homogeneous $C^*$-algebras $A = \Gamma_0(\mathcal{E})$, we characterize operators $\phi$ in the norm closure of $TM_0(A)$ as those operators in $\text{IB}_{0,1}(A)$ for which the associated complex line bundle $L_\phi$ is trivial on each compact subset of $U$, where $U$ is as above (Theorem 6.14). As a consequence, we obtain that $TM_0(A)$ fails to be norm closed if and only if there exists a $\sigma$-compact open subset $U$ of $X$ and a phantom complex line subbundle $L$ of $\mathcal{E}|_U$ (i.e. $L$ is not globally trivial, but is trivial on each compact subset of $U$). Using this and some algebraic topological ideas, we show that $TM(A)$ and $TM_0(A)$ both fail to be norm closed whenever $A$ is $n$-homogeneous with $n \geq 2$ and $X$ contains an open subset homeomorphic to $\mathbb{R}^d$ for some $d \geq 3$ (Theorem 6.18).

2. Preliminaries

Throughout this paper $A$ will denote a $C^*$-algebra. By an ideal $A$ we always mean a closed two-sided ideal. As usual, by $Z(A)$ we denote the centre of $A$, by $M(A)$ the multiplier algebra of $A$, and by $\text{Prim}(A)$ the primitive spectrum of $A$ (i.e. the set of kernels of all irreducible representations of $A$ equipped with the Jacobson topology).

Every $\phi \in \text{IB}(A)$ is linear over $Z(M(A))$ and, for any ideal $I$ of $A$, $\phi$ induces a map

$$\phi_I : A/I \to A/I,$$

which sends $a + I$ to $\phi(a) + I$.

It is easy to see that the norm (resp. cb-norm) of an operator $\phi \in \text{IB}(A)$ (resp. $\phi \in \text{ICB}(A)$) can be computed via the formulae

$$\|\phi\| = \sup \{\|\phi_P\| : P \in \text{Prim}(A)\} \quad \text{resp.}$$

$$\|\phi\|_{cb} = \sup \{\|\phi_P\|_{cb} : P \in \text{Prim}(A)\}.$$ (2.2)

The length of a non-zero elementary operator $\phi \in \mathcal{E}_\ell(A)$ is the smallest positive integer $\ell = \ell(\phi)$ such that $\phi = \sum_{i=1}^{\ell} a_i b_i$ for some $a_i, b_i \in M(A)$. We also define $\ell(0) = 0$. We write $\mathcal{E}_{\ell}(A)$ for the elementary operators of length at most $\ell$. Thus $\mathcal{E}_{\ell_1}(A) = TM(A)$. 

We will also consider the following subsets of \( \text{TM}(A) \):

\[
\text{TM}_{\text{cp}}(A) = \{ M_{a,a^*} : a \in M(A) \}, \quad \text{InnAut}_{\text{alg}}(A) = \{ M_{a,a^{-1}} : a \in M(A), \ a \text{ invertible} \}, \quad \text{InnAut}(A) = \{ M_{u,u^*} : u \in M(A), \ u \text{ unitary} \}
\]

(2.3)

where \( \text{cp} \) and \( \text{alg} \) signify, respectively, completely positive and algebraic). Note that \( \text{InnAut}(A) = \text{TM}_{\text{cp}}(A) \cap \text{InnAut}_{\text{alg}}(A) \).

It is well known that elementary operators are completely bounded with the following estimate for their cb-norm:

\[
\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h,
\]

where \( \| \cdot \|_h \) is the Haagerup tensor norm on the algebraic tensor product \( M(A) \otimes M(A) \), i.e.

\[
\| u \|_h = \inf \left\{ \left( \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} : u = \sum_i a_i \otimes b_i \right) : u = \sum_i a_i \otimes b_i \right\}.
\]

By inequality (2.4) the mapping

\[
(M(A) \otimes M(A), \| \cdot \|_h) \to (\mathcal{E}\ell(A), \| \cdot \|_{cb}) \quad \text{given by} \quad \sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i},
\]

defines a well-defined contraction. Its continuous extension to the completed Haagerup tensor product \( M(A) \otimes_h M(A) \) is known as a canonical contraction from \( M(A) \otimes_h M(A) \) to \( \text{ICB}(A) \) and is denoted by \( \Theta_A \).

We have the following result (see [2 Proposition 5.4.11]):

**Theorem 2.1 (Mathieu).** \( \Theta_A \) is isometric if and only if \( A \) is a prime \( C^* \)-algebra.

The next result is a combination of [36 Corollary 3.8], [22] and the facts that for \( \phi = \sum_{i=1}^\ell M_{a_i,b_i} \), we have \( \| \phi \| = \| \phi_{\ker \pi} \| \) and \( \| \phi \|_{cb} = \| \phi_{\ker \pi} \|_{cb} \) where for irreducible representation \( \pi : A \to B(H_\pi) \), \( \pi = \sum_{i=1}^\ell M_{a_i(a_i),b_i(b_i)} = \sum_{i=1}^\ell \pi(a_i(a_i),\pi(b_i)) \in \mathcal{E}\ell(B(H_\pi)) \) (as in [36 §4]).

**Theorem 2.2 (Timoney).** For a \( C^* \)-algebra and arbitrary \( \phi \in \mathcal{E}\ell(A) \) of length \( \ell \) we have

\[
\| \phi \|_{cb} = \| \phi^{(\ell)} \| \leq \sqrt{\ell} \| \phi \|,
\]

where \( \phi^{(\ell)} \) denotes the \( \ell \)-th amplification of \( \phi \) on \( M_\ell(A) \), \( \phi^{(\ell)} : [x_{i,j}] \mapsto [\phi(x_{i,j})] \).

In particular, on each \( \mathcal{E}\ell_\ell(A) \) the metric induced by the cb-norm is equivalent to the metric induced by the operator norm.

3. Two-sided multiplications on prime \( C^* \)-algebras

If \( A \) is a prime \( C^* \)-algebra, we prove here (Theorem 5.3) that \( \text{TM}(A) \) and \( \text{TM}_0(A) \) must be closed in \( \mathcal{B}(A) \).

The crucial step is the following lemma.

**Lemma 3.1.** Let \( a, b, c \) and \( d \) be norm-one elements of an operator space \( V \). If

\[
\| a \otimes b - c \otimes d \|_h < \varepsilon \leq 1/3,
\]

then...
then there exists a complex number $\mu$ of modulus one such that

$$\max\{\|a - \mu c\|, \|b - \bar{\mu} d\|\} < 6\varepsilon.$$  

Proof. First we dispose of the simpler cases where $a$ and $c$ are linearly dependent or where $b$ and $d$ are linearly dependent. If $a$ and $c$ are dependent then $a = \mu c$ with $|\mu| = 1$. So $a \otimes b - c \otimes d = c \otimes (\mu b - d)$ and $\|a \otimes b - c \otimes d\|_h = \|\mu b - d\| < \varepsilon < 6\varepsilon$.

Similarly if $b$ and $d$ are dependent, $b = \bar{\mu} d$ with $|\mu| = 1$ and $\|\bar{\mu} a - c\| < \varepsilon$.

Leaving aside these cases, $a \otimes b - c \otimes d$ is a tensor of rank 2. By [5, Lemma 2.3] there is an invertible matrix $S \in \mathbb{M}_2$ such that

$$\| [a - c] S \| < \varepsilon \quad \text{and} \quad \| S^{-1} b \| < \varepsilon.$$  

Write $\alpha_{i,j}$ for the $i, j$ entry of $S$ and $\beta_{i,j}$ for the $i, j$ entry of $S^{-1}$.

Since $\alpha_{1,1}\beta_{1,1} + \alpha_{1,2}\beta_{2,1} = 1$, at least one of the absolute values $|\alpha_{1,1}|$, $|\beta_{1,1}|$, $|\alpha_{1,2}|$ or $|\beta_{2,1}|$ must be at least $1/\sqrt{2}$. We treat the four cases separately, by very similar arguments.

$|\alpha_{1,1}| \geq 1/\sqrt{2}$: Now from

$$[a - c] S = \begin{bmatrix} \alpha_{1,1}a - \alpha_{2,1}c & \alpha_{1,2}a - \alpha_{2,2}c \end{bmatrix}$$

we have $\|\alpha_{1,1}a - \alpha_{2,1}c\| < \varepsilon$, so

$$\left\| a - \frac{\alpha_{2,1}}{\alpha_{1,1}} c \right\| < \frac{\varepsilon}{|\alpha_{1,1}|} \leq \sqrt{2} \varepsilon.$$  

Let $\lambda = \alpha_{2,1}/\alpha_{1,1}$. Then

$$\frac{1 - |\lambda|}{|\lambda|} = \frac{|\lambda| (1 - |\lambda|)}{|\lambda|} \leq \frac{|a - \lambda c|}{\|a - \lambda c\|} \leq \sqrt{2} \varepsilon,$$

and so $|\lambda| \in \left[1 - \sqrt{2} \varepsilon, 1 + \sqrt{2} \varepsilon\right]$. Also

$$\left| \frac{\lambda}{|\lambda|} - 1 \right| = \frac{|\lambda| (1 - |\lambda|)}{|\lambda|} \leq \frac{\sqrt{2} \varepsilon}{1 - \sqrt{2} \varepsilon} < \frac{\sqrt{2} + 2}{2} \varepsilon$$  

(since $\varepsilon \leq 1/3$). So for $\mu = \frac{\lambda}{|\lambda|}$, we have $|\mu| = 1$ and

$$\|a - \mu c\| \leq \|a - \lambda c\| + |\lambda - \mu| \|c\| < \sqrt{2} \varepsilon + (\sqrt{2} + 2) \varepsilon < 5 \varepsilon.$$  

Then

$$a \otimes b - c \otimes d = a \otimes b - (\mu c \otimes \bar{\mu} d)$$

$$= (a \otimes b) - (a \otimes \bar{\mu} d) + (a \otimes \bar{\mu} d) - (\mu c \otimes \bar{\mu} d)$$

$$= (a \otimes (b - \bar{\mu} d)) + ((a - \mu c) \otimes \bar{\mu} d)$$

and thus

$$\|b - \bar{\mu} d\| = \|a \otimes (b - \bar{\mu} d)\|_h$$

$$\leq \|a \otimes b - c \otimes d\|_h + \|(a - \mu c) \otimes \bar{\mu} d\|_h$$

$$\leq \varepsilon + 5 \varepsilon = 6 \varepsilon.$$  

$|\alpha_{1,2}| \geq 1/\sqrt{2}$: We start now with $\|\alpha_{1,2}a - \alpha_{2,2}c\| < \varepsilon$ and proceed in the same way (with $\lambda = \alpha_{2,2}/\alpha_{1,2}$).
\[ |\beta_{1,1}| \geq 1/\sqrt{2}: \text{ In this case we use} \]
\[
S^{-1} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \beta_{1,1}b + \beta_{1,2}d \\ \beta_{2,1}b + \beta_{2,2}d \end{bmatrix}
\]
\[\text{and } \|\beta_{1,1}b + \beta_{1,2}d\| < \varepsilon, \text{ leading to a similar argument (with } \lambda = -\beta_{1,2}/\beta_{1,1} \text{ and } b \text{ taking the role of } a). \]
\[|\beta_{2,1}| \geq 1/\sqrt{2}: \text{ In this case use } \|\beta_{2,1}b + \beta_{2,2}d\| < \varepsilon. \]

**Corollary 3.2.** If \( V \) is an operator space, the set \( S_1 = \{a \otimes b : a, b \in V\} \) of all elementary tensors forms a closed subset of \( V \otimes_h V \).

**Proof.** Suppose \( a_n \otimes b_n \rightarrow u \in V \otimes_h V \) (for \( a_n, b_n \in V \)). If \( u = 0 \), certainly \( u \in S_1 \) and otherwise we may assume \( \|u\|_{h} = 1 \) and also that
\[
\|a_n \otimes b_n\|_{h} = 1 = \|a_n\| = \|b_n\| \ (n \geq 1).
\]
Passing to a subsequence, we may suppose
\[
\|a_n \otimes b_n - a_{n+1} \otimes b_{n+1}\|_{h} \leq \frac{1}{6 \cdot 2^n} \ (n \geq 1).
\]
By Lemma 3.1 we may multiply \( a_n \) and \( b_n \) by complex conjugate modulus one scalars chosen inductively to get \( a'_n \) and \( b'_n \) such that
\[
a_n \otimes b_n = a'_n \otimes b'_n, \quad \|a'_n - a'_{n+1}\| \leq 1/2^n \quad \text{and} \quad \|b'_n - b'_{n+1}\| \leq 1/2^n \ (n \geq 1).
\]
In this way we find \( a = \lim_{n \to \infty} a'_n \) and \( b = \lim_{n \to \infty} b'_n \) in \( V \) with \( u = a \otimes b \in S_1 \). \( \square \)

**Question 3.3.** If \( V \) is an operator space and \( \ell > 1 \), is the set
\[
S_{\ell} = \left\{ \sum_{i=1}^{\ell} a_i \otimes b_i : a_i, b_i \in V \right\}
\]
of all tensors of rank at most \( \ell \) closed in \( V \otimes_h V \)? In particular, can we extend Lemma 3.1 as follows.

Let \( V \) be an operator space and let \( a \otimes b \) and \( c \otimes d \) be two norm-one tensors of the same (finite) rank \( \ell \) in \( V \otimes_h V \), where \( a, c \) and \( b, d \) are, respectively, \( 1 \times \ell \) and \( \ell \times 1 \) matrices with entries in \( V \). Suppose that \( \|a \otimes b - c \otimes d\|_{h} < \varepsilon \) for some \( \varepsilon > 0 \). Can we find absolute constants \( C \) and \( \delta \) (which depend only on \( \ell \) and \( \varepsilon \)) so that \( \delta \to 0 \) as \( \varepsilon \to 0 \) with the following property:

There exists an invertible matrix \( S \in M_{\ell} \) such that
\[
\|S\|, \|S^{-1}\| \leq C, \quad \|aS^{-1} - c\| < \delta \quad \text{and} \quad \|bS - d\| < \delta.
\]

**Theorem 3.4.** If \( A \) is a prime \( C^* \)-algebra, then both \( TM(A) \) and \( TM_0(A) \) are norm closed.

**Proof.** By Theorem 2.2 we may work with the cb-norm instead of the (operator) norm. Since \( A \) is prime, by Mathieu’s theorem (Theorem 2.3) the canonical map \( \Theta : M(A) \otimes_h M(A) \to ICB(A), \Theta : a \otimes b \mapsto M_{a,b} \), is isometric. By Corollary 3.2 the set \( S_1 \) of all elementary tensors in \( M(A) \otimes_h M(A) \) is closed in the Haagerup norm. Therefore, \( TM(A) = \Theta(S) \) is closed in the cb-norm.

For the case of \( TM_0(A) \), we use the same argument but work with the restriction of \( \Theta \) to \( A \otimes_h A \). \( \square \)

**Corollary 3.5.** If \( A \) is a prime \( C^* \)-algebra, then the sets \( TM_{cp}(A), \text{InnAut}_{alg}(A) \), and \( \text{InnAut}(A) \) (see (2.3)) are all norm closed.
Proof. Suppose that an operator $\phi$ in the norm closure of any of these sets. Then, by Theorem 3.4 there are $b,c \in M(A)$ such that $\phi = M_{b,c}$. Let $\varepsilon > 0$.

Suppose that $\phi$ is in the closure of $\text{TM}_{cp}(A)$. By Theorem 3.2, we may work with the cb-norm instead of the (operator) norm. We may also assume that $||\phi||_{cb} = 1 = ||b|| = ||c||$. Then there is $a \in M(A)$ such that

$$||M_{b,c} - M_{a,a^*}||_{cb} = ||b \otimes c - a \otimes a^*||_{cb} < \varepsilon$$

(Theorem 2.1). If $\varepsilon \leq 1/3$, by Lemma 3.1 we can find a complex number $\mu$ of modulus one such that $||b - \mu a|| < 6\varepsilon$ and $||c - \overline{\mu} a^*|| < 6\varepsilon$. Then $||b - c^*|| \leq 12\varepsilon$.

Hence $c = b^*$, so $\phi = M_{b,c} \in \text{TM}_{cp}(A)$.

Suppose that $\phi$ is in the closure of $\text{InnAut}_{alg}(A)$. Then there is an invertible element $a \in M(A)$ such that $||M_{b,c} - M_{a,a^{-1}}|| < \varepsilon$. Since $A$ is an essential ideal in $M(A)$, this implies $||bxc - axa^{-1}|| < \varepsilon$ for all $x \in M(A)$, $||x|| \leq 1$. Letting $x = 1$ we obtain $||bc - 1|| < \varepsilon$. Hence $c = b^{-1}$, so $\phi = M_{b,c} \in \text{InnAut}_{alg}(A)$.

$\text{InnAut}(A)$ is norm closed as an intersection $\text{TM}_{cp}(A) \cap \text{InnAut}_{alg}(A)$ of two closed sets.

4. ON HOMOGENEOUS $C^*$-ALGEBRAS

We recall that a $C^*$-algebra $A$ is called $n$-homogeneous (where $n$ is finite) if every irreducible representation of $A$ acts on an $n$-dimensional Hilbert space. We say that $A$ is homogeneous if it is $n$-homogeneous for some $n$. We will use the following definitions and facts about homogeneous $C^*$-algebras:

Remark 4.1. Let $A$ be an $n$-homogeneous $C^*$-algebra. By [20] Theorem 4.2 $\text{Prim}(A)$ is a (locally compact) Hausdorff space. If there is no danger of confusion, we simply write $X$ for $\text{Prim}(A)$.

(a) A well-known theorem of Fell [10, Theorem 3.2], and Tomiyama-Takesaki [37, Theorem 5] asserts that for any $n$-homogeneous $C^*$-algebra, $A$, there is a locally trivial bundle $\mathcal{E}$ over $X$ with fibre $M_{n}$ and structure group $PU(n) = \text{Aut}(M_{n})$ such that $A$ is isomorphic to the $C^*$-algebra $\Gamma_0(\mathcal{E})$ of continuous sections of $\mathcal{E}$ which vanish at infinity.

Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with primitive spectra $X_i$ ($i = 1, 2$) are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ (the pullback bundle) as bundles over $X_1$ (see [37, Theorem 6]). Thus, we may identify $A$ with $\Gamma_0(\mathcal{E})$.

(b) For $a \in A$ and $t \in X$ we define $\pi_t(a) = a(t)$. Then, after identifying the fibre $\mathcal{E}_t$ with $M_{n}$, $\pi_t : a \mapsto \pi_t(a)$ (for $t \in X$) gives all irreducible representations of $A$ (up to the equivalence).

For a closed subset $S \subseteq X$ we define

$$I_S = \bigcap_{t \in S} \ker \pi_t = \{a \in A : a(t) = 0 \text{ for all } t \in S\}.$$  

By [11, VII 8.7.] any closed two-sided ideal of $A$ is of the form $I_S$ for some closed subset $S \subseteq X$. Further, by the the generalized Tietze Extension Theorem we may identify $A_S = A/I_S$ with $\Gamma_0(\mathcal{E}|_S)$ (see [11, II, 14.8. and VII 8.6]). If $S = \{t\}$ we just write $A_t$.

(c) If $\phi \in \text{IB}(A)$ and $S \subseteq X$ closed, we write $\phi_S$ for the operator $\phi_{|_S}$ on $A_S$ (see (2.1)). If $S = \{t\}$ we just write $\phi_t$. If $A$ is trivial (i.e. $A = C_0(X, M_{n})$),
Thus, we have

\[ \text{Proposition 4.2.} \]

Let \( A \in \text{IB}(A) \) be an operator \( : M_n \to M_n \) (after identifying \( A \) with \( M_n \) in the obvious way).

If \( U \subset X \) is open, we can regard \( B = \Gamma_0(\mathcal{E}_U) \) as the ideal \( I_{X\setminus U} \) of \( A \) (by extending sections to be zero outside \( U \)) and for \( \phi \in \text{IB}(A) \), we then have a restriction \( \phi_{|U} \in \text{IB}(B) \) of \( \phi \) to this ideal (with \( (\phi_{|U})_t = \phi_t \) for \( t \in U \)).

(d) \( \text{IB}(A) = \text{ICB}(A) \). Indeed, for \( \phi \in \text{IB}(A) \) and \( t \in X \) we have \( \|\phi_t\|_{cb} \leq n\|\phi_t\| \)

(\[29\] p. 114), so by \( (2.2) \) we have \( \|\phi\|_{cb} \leq n\|\phi\| \). Hence \( \phi \in \text{ICB}(A) \).

(e) Since each \( a \in Z(A) \) has \( a(t) \) a multiple of the identity in the fibre \( E_t \) for each \( t \in X \), we can identify \( Z(A) \) with \( C_0(X) \). Observe that \( A \) is quasi-central (i.e. no primitive ideal of \( A \) contains \( Z(A) \)).

(f) By \[25\] Lemma 3.2, we can identify \( M(A) \) with \( \Gamma_0(\mathcal{E}) \) (the \( C^* \)-algebra of bounded continuous sections of \( \mathcal{E} \)). As usual, we will identify \( Z(M(A)) \) with \( C_b(X) \) (using the Dauns-Hofmann theorem \[33\] Theorem A.34).

If \( A = C_0(X, M_n) \), it is well known that \( M(A) = C_b(X, M_n) = C(\beta X, M_n) \)

[11] Corollary 3.4, where \( \beta X \) denotes the Stone-Cech compactification.

(g) On each fibre \( E_t \), we can introduce an inner product \( \langle \cdot, \cdot \rangle_2 \) as follows.

Choose an open covering \( \{U_n\} \) of \( X \) such that each \( E_{|U_n} \) is isomorphic to \( U_n \times M_n \) (as an \( M_n \)-bundle), say via isomorphism \( \Phi_n \). Let

\[ \langle \xi, \eta \rangle_2 = \text{tr}(\Phi_n(\xi)\Phi_n(\eta)^*) \quad (\xi, \eta \in \mathcal{E}_t), \]

where \( \alpha \) is chosen so that \( t \in U_n \) and \( \text{tr}(\cdot) \) is the standard trace on \( M_n \).

This is independent of the choice of \( \alpha \) since all automorphisms of \( M_n \) are inner and \( \text{tr}(\cdot) \) is invariant under conjugation by unitaries. If \( a, b \in M(A) = \Gamma_b(\mathcal{E}) \) then \( t \mapsto \langle a(t), b(t)\rangle_2 \) is in \( C_b(X) \).

The norm \( \|\cdot\|_2 \) on \( \mathcal{E}_t \) associated with \( \langle \cdot, \cdot \rangle_2 \) satisfies

\[ \|\xi\| \leq \|\xi\|_2 \leq \sqrt{n}\|\xi\| \quad (\xi \in \mathcal{E}_t). \]

In the terminology of \[3\], \( (\mathcal{E}, \langle \cdot, \cdot \rangle_2) \) is a (complex continuous) Hilbert bundle of rank \( n^2 \) with fibre norms equivalent to the original \( C^* \)-norms (by \( (4.2) \)).

(h) \( A \) is said to have the \textbf{finite type property} if \( \mathcal{E} \) can be trivialized over some finite open cover of \( X \). By \[29\] Remark 3.3, \( M(A) \) is homogeneous if and only if \( A \) has the finite type property. When this fails, it is possible to have \( \text{Prim}(M(A)) \) non-Hausdorff \[11\] Theorem 2.1. On the other hand, \( M(A) \) is always quasi-standard (see \[3\] Corollary 4.10).

For completeness we include a proof of the following.

**Proposition 4.2.** Let \( X \) be a locally compact Hausdorff space and \( A = C_0(X, M_n) \).

(a) \( \text{IB}(A) \) can be identified with \( C_b(X, \mathcal{B}(M_n)) \) by a mapping which sends an operator \( \phi \in \text{IB}(A) \) to the function \( (t \mapsto \phi_t) \).

(b) Any \( \phi \in \text{IB}(A) \) can be written in the form

\[ \phi = \sum_{k,i=1}^{n} M_{e_{k,i}, a_{k,i}}, \]

where \( (e_{k,i})_{k,i=1}^{n} \) are standard matrix units of \( M_n \) (considered as constant functions in \( C_b(X, M_n) = M(A) \)) and \( a_{k,i} \in M(A) \) depend on \( \phi \).

Thus, we have

\[ \text{IB}(A) = \text{ICB}(A) = C_b(X, \mathcal{B}(M_n)) = \mathcal{E}(A) = \mathcal{E}e_n(A). \]
Proof. Let $φ ∈ \text{IB}(A)$.

(a) Suppose that the function $t ↦ φ_t : X → \mathcal{B}(\mathbb{M}_n)$ is discontinuous at some point $t_0 ∈ X$. Then there is a net $(t_α)$ in $X$ converging to $t_0$ such that $∥φ_{t_α} − φ_{t_0}∥ ≥ δ > 0$ for all $α$. So there is $u_α ∈ \mathbb{M}_n$ of norm at most 1 with $∥φ_{t_α}(u_α) − φ_{t_0}(u_0)∥ ≥ δ$. Passing to a subnet we may suppose $u_α → u$ and then (since $∥φ_{t_0}∥ ≤ ∥φ∥$ and $∥φ_{t_0}∥ ≤ ∥φ||$ we must have

$$∥φ_{t_α}(u) − φ_{t_0}(u)|| ≥ δ/2$$

for $α$ large enough.

Now choose $f ∈ C_0(X)$ equal to 1 on a neighbourhood of $t_0$ and put $a(t) = f(t)u$. We then have $a ∈ A$ and

$$π_{t_α}(φ(a)) = f(t_α)φ_{t_α}(u) = φ_{t_α}(u)$$

for large $α$ and this contradicts continuity of $φ(a)$ at $t_0$.

So $t ↦ φ_t$ must be continuous (and also bounded by $∥φ||$).

Conversely, assume that the function $t ↦ φ_t$ is continuous and uniformly bounded by some $M > 0$. Then for $a ∈ A$, $t ↦ φ_t(π_t(a))$ is continuous, bounded and vanishes at infinity, hence in $A$. So there is an associated mapping $φ : A → A$ which is easily seen to be bounded and linear. Moreover $φ ∈ \text{IB}(A)$ since all ideals of $A$ are of the form $I_S$ for some closed $S ⊂ X$.

(b) First assume that $A$ is unital, so that $X$ is compact. Then each $x ∈ A$ is a linear combination over $C(X) = Z(A)$ of the $e_{i,j}$ and since $φ$ is $C(X)$-linear, we have

$$x = \sum_{i,j=1}^n x_{i,j}e_{i,j} \Rightarrow φ(x) = \sum_{i,j=1}^n x_{i,j}φ(e_{i,j}).$$

We may write

$$φ(e_{i,j}) = \sum_{k,l=1}^n φ_{i,j,k,l}e_{k,l} = \sum_{k=1}^n e_{k,i}φ_{i,j,k} \left( \sum_{t=1}^n φ_{i,j,k,t}e_{j,t} \right)$$

where $φ_{i,j,k,l} ∈ \mathcal{C}(X)$. It follows that

$$φ(x) = \sum_{k,i=1}^n e_{k,i}x \left( \sum_{j,l=1}^n φ_{i,j,k,l}e_{j,l} \right) \Rightarrow φ = \sum_{k,i=1}^n M_{e_{k,i},a_{k,i}},$$

where $a_{k,i} = \sum_{j,l=1}^n φ_{i,j,k,l}e_{j,l} ∈ M(A)$.

Now suppose that $A$ is non-unital (so that $X$ is non-compact). By (a) we can identify $φ$ with the function $t ↦ φ_t : X → \mathcal{B}(\mathbb{M}_n)$, which can be then uniquely extended to a continuous function $βX → \mathcal{B}(\mathbb{M}_n)$. This extension defines an operator in $\text{IB}(\mathcal{C}(βX, \mathbb{M}_n)) = \text{IB}(M(A))$, which we also denote by $φ$. By the first part of the proof, $φ$ can be represented as (4.3).

Remark 4.3. In fact, in the case of general separable $C^*$-algebras $A$, Magajna [25] establishes the equivalence of the following properties:

(a) $\text{IB}(A) = \mathcal{Eℓ}(A)$.
(b) $\mathcal{Eℓ}(A)$ is norm dense in $\text{IB}(A)$.
(c) $A$ is a finite direct sums of homogeneous $C^*$-algebras with the finite type property.
Proof. Let \( \mathcal{I} \) be the maximal ideal of \( N \). Moreover, we can choose \( \mathcal{I} \) such that \( A \) has a nonempty open subset homeomorphic to (an open set in) \( \mathbb{R}^d \) with \( d \geq 3 \) (Corollary 5.16).

**Corollary 4.4.** If \( A \) is a homogeneous \( C^* \)-algebra, then for any \( \phi \in \mathcal{I} \) the function \( t \mapsto \| \phi_t \| \) is continuous on \( X \). Hence the cozero set \( \text{coz}(\phi) = \{ t \in X : \phi_t \neq 0 \} \) is \( \sigma \)-compact and open in \( X \).

5. Fibrewise length restrictions

Here we consider a homogeneous \( C^* \)-algebra \( A = \Gamma_0(\mathcal{E}) \) and operators \( \phi \in \mathcal{I} \) such that \( \phi_t \) is a two-sided multiplication on each fibre \( A_t \) (with \( t \in X \), and \( X = \text{Prim}(A) \) as usual). We will write \( \phi \in \mathcal{I} \) for this hypothesis. For separable \( A \), the main result in this section (Theorem 5.15) characterizes when all such operators \( \phi \) are two-sided multiplications, in terms of triviality of complex line subbundles of \( \mathcal{E}_U \) for \( U \subset X \) open.

In addition to \( \mathcal{I} \), we introduce various subsets \( \mathcal{I} \) which are designed to facilitate the description of \( TM(A) \), \( TM_0(A) \) and both of their norm closures in terms of complex line bundles. The sufficient condition that ensures \( \mathcal{I} \subset TM(A) \) is that \( X \) is paracompact with vanishing second integral Čech cohomology group \( \check{H}^2(X; \mathbb{Z}) \) (Corollary 5.11). For \( X \) compact of finite covering dimension \( d \) and \( A = C(X, M_n) \) we show that \( TM(A) \subset \mathcal{I} \) provided \( \check{H}^2(X; \mathbb{Z}) \neq 0 \) and \( n^2 \geq (d + 1)/2 \) (Proposition 5.12). We get the same conclusion \( TM(A) \subset \mathcal{I} \) for \( \sigma \)-unital \( n \)-homogeneous \( C^* \)-algebras \( A = \Gamma_0(\mathcal{E}) \) with \( n \geq 2 \) provided \( X \) has a nonempty open subset homeomorphic to (an open set in) \( \mathbb{R}^d \) with \( d \geq 3 \) (Corollary 5.16).

**Notation 5.1.** Let \( A \) be an \( n \)-homogeneous \( C^* \)-algebra. For \( \ell \geq 1 \) we write

\[
\mathcal{I}_\ell(A) = \{ \phi \in \mathcal{I} : \phi_t \in \mathcal{E}_\ell(A_t) \text{ for all } t \in X \}.
\]

**Lemma 5.2.** Let \( A = \Gamma_0(\mathcal{E}) \) be a homogeneous \( C^* \)-algebra and \( \phi \in \mathcal{I}_\ell(A) \).

If \( t_0 \in X \) is such that \( \phi_{t_0} \in \mathcal{E}_\ell(A_{t_0}) \setminus \mathcal{E}_\ell-1(A_{t_0}) \) (that is, such that \( \phi_{t_0} \) has length exactly the maximal \( \ell \)), then there are \( a_1, \ldots, a_\ell, b_1, \ldots, b_\ell \in A \) and a compact neighbourhood \( N \) of \( t_0 \) such that \( \phi \) agrees with the elementary operator \( \sum_{i=1}^\ell M_{a_i,b_i} \) modulo the ideal \( I_N \), that is

\[
\phi(x) - \sum_{i=1}^\ell a_ixb_i \in I_N \quad \text{for all } x \in A.
\]

Moreover, we can choose \( N \) so that \( \phi_t \in \mathcal{E}_\ell(A_t) \setminus \mathcal{E}_{\ell-1}(A_t) \) for all \( t \in N \) (that is, \( \phi_t \) is of the maximal length \( \ell \) for \( t \) in a neighbourhood of \( t_0 \)).

**Proof.** Choose a compact neighbourhood \( K \) of \( t_0 \) such that \( A_K \cong C(K, M_n) \) and let \( \phi_K \) be the induced operator (Remark 4.1 (b), (c)).

Then, for \( x \in A_K \) we have \( \phi_K(x) = \sum_{i=1}^n c_i x d_i \) for some \( c_i, d_i \in A_K \) (by Proposition 4.2 (b)). Moreover we can assume that \( \{ c_1, \ldots, c_n \} \) are linearly independent for each \( t \in K \), and even independent of \( t \). Since \( (\phi_K)^{t_0} = \phi_{t_0} \) has
length \( \ell \), we must be able to write (in \( \mathcal{M}_n \otimes \mathcal{M}_n \))

\[
\sum_{i=1}^{n^2} c_i(t_0) \otimes d_i(t_0) = \sum_{j=1}^{\ell} c_j' \otimes d_j'.
\]

We can choose \( d_1', \ldots, d_\ell' \) to be a maximal linearly independent subsequence of \( d_1(t_0), \ldots, d_{n^2}(t_0) \). Then, via elementary linear algebra, there is a matrix \( \alpha \) of size \( n^2 \times \ell \) and another matrix \( \beta \) of size \( \ell \times n^2 \) so that

\[
\begin{bmatrix}
  d_1(t_0) \\
  \vdots \\
  d_{n^2}(t_0)
\end{bmatrix} = \alpha
\begin{bmatrix}
  d_1' \\
  \vdots \\
  d_\ell'
\end{bmatrix},
\]

\[
\begin{bmatrix}
  d_1(t_0) \\
  \vdots \\
  d_{n^2}(t_0)
\end{bmatrix} = \beta
\begin{bmatrix}
  d_1' \\
  \vdots \\
  d_\ell'
\end{bmatrix}
\]

and \( \beta \alpha = \text{the identity.} \)

We have

\[
[c_1' \cdots c_{\ell}'] = [c_1(t_0) \cdots c_{n^2}(t_0)] \alpha.
\]

If we define

\[
\begin{bmatrix}
  d_1'(t) \\
  \vdots \\
  d_\ell'(t)
\end{bmatrix} = \beta
\begin{bmatrix}
  d_1(t) \\
  \vdots \\
  d_{n^2}(t)
\end{bmatrix}
\]

then \( d_1'(t), \ldots, d_\ell'(t) \) must be linearly independent for all \( t \) in some compact neighbourhood \( N \) of \( t_0 \). Thus for \( t \in N \) we have (in \( \mathcal{M}_n \otimes \mathcal{M}_n \))

\[
\sum_{i=1}^{n^2} c_i(t) \otimes d_i(t) = \sum_{i=1}^{n^2} c_i(t_0) \otimes d_i(t) = \sum_{j=1}^{\ell} c_j' \otimes d_j'(t).
\]

By Remark 4.1(b) we can find elements \( a_j, b_j \in A \) (\( 1 \leq j \leq \ell \)) such that \( a_j(t) = c_j' \) and \( b_j(t) = d_j'(t) \) for all \( t \in N \).

Since for each \( t \in N \) both of the sets \( \{a_1(t), \ldots, a_\ell(t)\} \) and \( \{b_1(t), \ldots, b_\ell(t)\} \) are linearly independent, we get that \( \phi_t = \sum_{j=1}^{\ell} M_{a_j(t), b_j(t)} \) has length exactly \( \ell \) for all \( t \in N \) as required.

\[\square\]

**Corollary 5.3.** Let \( A \) be a homogeneous \( C^* \)-algebra and \( \phi \in \text{IB}_1(A) \). If \( t_0 \in X \) is such that \( \phi_{t_0} \neq 0 \) then there is a compact neighbourhood \( N \) of \( t_0 \) and \( a, b \in A \) such that \( a(t) \neq 0 \) and \( b(t) \neq 0 \) for all \( t \in N \) and \( \phi \) agrees with \( M_{a,b} \) modulo the ideal \( I_N \).

**Remark 5.4.** Let \( A = \Gamma_0(\mathcal{E}) \) be a homogeneous \( C^* \)-algebra, \( a, b \in M(A) = \Gamma_b(\mathcal{E}) \) and \( \phi = M_{a,b} \).

We may replace \( a \) and \( b \) by

\[ t \mapsto \sqrt{\frac{\|b(t)\|}{\|a(t)\|}}a(t) \quad \text{and} \quad t \mapsto \sqrt{\frac{\|a(t)\|}{\|b(t)\|}}b(t) \]

without changing \( \phi \) so as to ensure that \( \|a(t)\| = \|b(t)\| \) for each \( t \in X \) and that \( \|\phi_t\| = \|a(t)\|^2 = \|b(t)\|^2 \) for \( t \in X \).

**Notation 5.5.** Let \( A \) be a homogeneous \( C^* \)-algebra. We write

\[ \text{IB}^n(A) = \{ \phi \in \text{IB}(A) : 0 \neq \phi_t \in TM(A_t) \text{ for all } t \in X \} \]

(where \( n \) signifies nowhere-vanishing).
We also use
\[
\begin{align*}
\text{IB}_0(A) &= \{ \phi \in \text{IB}(A) : (t \mapsto ||\phi_t||) \in C_0(X) \}, \\
\text{IB}_{0,1}(A) &= \text{IB}_0(A) \cap \text{IB}_1(A), \\
\text{IB}^\text{nv}_{0,1}(A) &= \text{IB}^\text{nv}_0(A) \cap \text{IB}_1(A), \\
\text{TM}^\text{nv}(A) &= \text{TM}(A) \cap \text{IB}^\text{nv}_1(A).
\end{align*}
\]

By Remark 5.4, TM_0(A) = TM(A) \cap \text{IB}_0(A).

**Proposition 5.6.** Let \(A = \Gamma_0(\mathcal{E})\) be a homogeneous \(C^*\)-algebra and suppose \(\phi \in \text{IB}^\text{nv}_{0,1}(A)\). Then there is a canonically associated complex line subbundle \(\mathcal{L}_\phi\) of \(\mathcal{E}\) with the property that
\[
\phi \in \text{TM}(A) \iff \mathcal{L}_\phi \text{ is a trivial bundle.}
\]

**Proof.** By Corollary 5.3 locally \(\phi\) is a two-sided multiplication. That is, given \(t_0 \in X\) there is a compact neighbourhood \(N_t = \{ x \in X : t_0 \cdot x < 2 \}\) of \(a\) for all \(t \in N\).

The definition we gave of \(\mathcal{L}_\phi \cap (\mathcal{E}|_N)\) shows that \(\mathcal{L}_\phi\) is a locally trivial complex line subbundle of \(\mathcal{E}\). The map
\[
(t, \lambda) \mapsto \mathcal{L}_\phi \cap (\mathcal{E}|_N)
\]
provides a local trivialization.

If \(\phi \in \text{TM}(A)\), then clearly \(\mathcal{L}_\phi\) is a trivial bundle. Conversely, if \(\mathcal{L}_\phi\) is a trivial bundle, choose a continuous nowhere vanishing section \(s : X \to \mathcal{L}_\phi\). Then for any neighbourhood \(N\) as above there is a continuous map \(\zeta : N \to C \setminus \{0\}\) such that \(a(t) = \zeta(t) s(t)\).

**Notation 5.7.** If \(A = \Gamma_0(\mathcal{E})\) is homogeneous and \(\phi \in \text{IB}_1(A)\), we consider the cozero set \(U = \text{coz}(\phi)\) (open by Corollary 4.4) and then, for \(B = \Gamma_0(\mathcal{E}|_U)\), \(\phi|_U \in \text{IB}^\text{nv}_{0,1}(B)\) (see Remark 4.4 (c)). We occasionally use \(\mathcal{L}_\phi\) for the subbundle \(\mathcal{L}_\phi|_U\) of \(\mathcal{E}|_U\).

**Proposition 5.8.** Let \(A = \Gamma_0(\mathcal{E})\) be an \(n\)-homogeneous \(C^*\)-algebra such that \(X\) is \(\sigma\)-compact. If \(\mathcal{L}\) is a complex line subbundle of \(\mathcal{E}\), then there is \(\phi \in \text{IB}^\text{nv}_{0,1}(A)\) with \(\mathcal{L}_\phi = \mathcal{L}\).

**Proof.** Let \(\langle \cdot, \cdot \rangle_2\) be as in Remark 4.4 (g). With respect to this inner product we have a complementary subbundle \(\mathcal{L}^\perp\) of \(\mathcal{E}\) such that \(\mathcal{L} \oplus \mathcal{L}^\perp = \mathcal{E}\).

Note that, by local compactness, \(X\) has a base consisting of \(\sigma\)-compact open sets. (If \(t_0 \in U \subseteq X\) with \(U\) open, choose a compact neighborhood \(N\) of \(t_0\) contained in \(U\) and a function \(f \in C_0(X)\) supported in \(N\) with \(f(t_0) = 1\). Take \(V = \{ t \in X : |f(t)| > 0 \}\).)
Since $X$ is $\sigma$-compact (and since every $\sigma$-compact space is Lindelöf), we can find a countable open cover $\{U_i\}_{i=1}^{\infty}$ of $X$ such that each restriction $\mathcal{E}|_{U_i}$ is trivial and each $U_i$ is $\sigma$-compact. Then we can find $n^2$ norm-one sections $(e_j^i)_{j=1}^{n^2}$ of $\Gamma_0(\mathcal{E}|_{U_i}) \cong C_0(U_i, \mathbb{M}_n)$ such that

$$\text{span}\{e_1^i(t), \ldots, e_{n^2}(t)\} = \mathcal{E}_i \cong \mathbb{M}_n \quad \text{for all } t \in U_i.$$  

By extending outside $U_i$ with $0$ we may assume that $e_j^i$ are globally defined, so that $e_j^i \in A$. Define $f_j^i(t)$ as the orthogonal projection of $e_j^i$ into the fibre $\mathcal{L}_t$, so that $f_j^i \in A$. We define

$$\phi : A \to A \quad \text{by} \quad \phi = \sum_{i=1}^{\infty} \frac{1}{2^i} \left( \sum_{j=1}^{n^2} M_{f_j^i(f_j^i)^*} \right).$$

Note that $\phi \in \text{IB}_0(A)$ as a sum of an absolutely convergent series of operators in $\text{IB}_0(A)$ (and $\text{IB}_0(A)$ is norm closed). We claim that $\phi \in \text{IB}_0^\text{nv}(A)$ and $\mathcal{L}_0 = \mathcal{L}$. Indeed, for an arbitrary point $t \in X$ choose a norm-one (in $C^*$-norm) vector $s \in \mathcal{L}_t$. Then there are scalars $\lambda_j^i$ with $f_j^i(t) = \lambda_j^i \cdot s$ and $|\lambda_j^i| = \|f_j^i(t)\| \leq \sqrt{n}\|e_j^i(t)\| = \sqrt{n}$ (by (1.2)). Then

$$\phi_t = \left( \sum_{i=1}^{\infty} \frac{1}{2^i} \left( \sum_{j=1}^{n^2} |\lambda_j^i|^2 \right) \right) \cdot M_{s,s^*}.$$  

This shows that $\phi \in \text{IB}_0^\text{nv}(A)$ and that for all $t \in X$ we have $\phi_t = M_{a(t),a^*(t)}$ for some $a(t) \in \mathcal{L}_t$. By the proof of Proposition 5.6 we conclude $\mathcal{L}_0 = \mathcal{L}$. \hfill $\square$

**Remark 5.9.** Let $\mathcal{L}$ be a complex line bundle over a locally compact Hausdorff space $X$.

(a) $\mathcal{L}$ is isomorphic to a subbundle of some $\mathbb{M}_2$-bundle $\mathcal{E}$. Indeed, let $\mathcal{F} = \mathcal{L} \oplus (X \times \mathbb{C})$. Then $\mathcal{E} = \text{Hom}(\mathcal{F}, \mathcal{F}) = \mathcal{F} \otimes \mathcal{F}^*$ is an $\mathbb{M}_2$-bundle with the desired property (see [22, Example 3.5]).

Further, if $X$ is $\sigma$-compact, then $A = \Gamma_0(\mathcal{E})$ (with $\mathcal{E}$ as above) is an example of a 2-homogeneous $C^*$-algebra with $\text{Prim}(A) = X$ that allows an operator $\phi \in \text{IB}_0^\text{nv}(A)$ such that $\mathcal{L}_0 \cong \mathcal{L}$ (by Proposition 5.6).

(b) Suppose that $\mathcal{L}$ is a subbundle of a trivial bundle $X \times \mathbb{C}^m$. If $p = \lfloor \sqrt{m} \rfloor$, then for each $n \geq p$ we can regard $\mathcal{L}$ as a subbundle of a trivial matrix bundle $X \times \mathbb{M}_n$, using some linear embedding $\mathbb{C}^m \hookrightarrow \mathbb{M}_n$.

**Remark 5.10.** If the space $X$ is paracompact, it is well-known that locally trivial complex line bundles over $X$ are classified by the homotopy classes of maps from $X$ to $\mathbb{C}P^\infty$ and/or by the elements of the second integral Čech cohomology $\check{H}^2(X; \mathbb{Z})$ (see e.g. [15, Corollary 3.5.6 and Theorem 3.4.7] and [33, Proposition 4.53 and Theorem 4.42].) By [18], we know that complex line bundles over $X$ are pullbacks of the canonical bundle over $\mathbb{C}P^\infty$ (via a map from $X$ to $\mathbb{C}P^\infty$).

In light of Proposition 5.6 and Remark 5.10, for a given a homogeneous $C^*$-algebra $A = \Gamma_0(\mathcal{E})$ we define a map

$$\theta : \text{IB}_1^\text{nv}(A) \to \check{H}^2(X; \mathbb{Z})$$

which sends an operator $\phi \in \text{IB}_1^\text{nv}(A)$ to the corresponding class of $\mathcal{L}_0$ in $\check{H}^2(X; \mathbb{Z})$. By Proposition 5.6 we have $\theta^{-1}(0) = \text{TM}^\text{nv}(A)$. As a direct consequence of this observation we have:
Corollary 5.11. Let $A$ be a homogeneous $C^*$-algebra such that $X$ is paracompact. If $H^2(X;\mathbb{Z}) = 0$ then $\text{IB}^{nv}_1(A) = \text{TM}^{nv}(A)$.

We will now give some sufficient conditions on a trivial homogeneous $C^*$-algebra $A$ that will ensure the surjectivity of the map $\theta$. To do this, first recall that a topological space $X$ is said to have the Lebesgue covering dimension $d < \infty$ if $d$ is the smallest non-negative integer with the property that each finite open cover of $X$ has a refinement in which no point of $X$ is included in more than $d + 1$ elements (see e.g. [9]). In this case we write $d = \dim X$.

Proposition 5.12. Let $X$ be a compact Hausdorff space with $\dim X \leq d < \infty$. For $n \geq 1$ let $A_n = C(X, M_n)$. If $p = \left\lceil \frac{\sqrt{(d+1)/2}}{n} \right\rceil$ then for any $n \geq p$ the mapping $\theta$ from (5.7) is surjective. In particular, if $H^2(X;\mathbb{Z}) \neq 0$, then $\text{TM}^{nv}(A_n) \subseteq \text{IB}^{nv}_1(A_n)$ for all $n \geq p$.

To prove this will use the following fact (which may be known) :

Lemma 5.13. Let $X$ be a CW-complex with $\dim X = d$. Then each complex line bundle $\mathcal{L}$ over $X$ is isomorphic to a line subbundle of $X \times \mathbb{C}^m$ with $m = \left\lceil (d+1)/2 \right\rceil$.

Proof. We consider $\mathbb{C}P^\infty$ as a CW-complex in the usual way (see [10] Example 0.6). Let $\Psi : X \to \mathbb{C}P^\infty$ be the classifying map of the bundle $\mathcal{L}$ (Remark 5.10). Using the cellular approximation theorem [10] Theorem 4.8 and Remark 5.10 we may assume that the map $\Psi$ is cellular, so that $\Psi$ takes the $k$-skeleton of $X$ to the $k$-skeleton of $\mathbb{C}P^\infty$ for all $k$. Since $\mathbb{C}P^\infty$ has one cell in each even dimension, $\Psi(X)$ is contained in the $d$-skeleton of $\mathbb{C}P^\infty$, which is the $(d-1)$-skeleton if $d$ is odd, and is $\mathbb{C}P^{m-1}$.

Hence $\mathcal{L}$ is isomorphic to the pullback $\Psi^*(\gamma)$ of the canonical line bundle $\gamma$ on $\mathbb{C}P^{m-1}$ (Remark 5.10), a subbundle of the trivial bundle $\mathbb{C}P^{m-1} \times \mathbb{C}^m$. \qed

Proof of Proposition 5.11. Let $\mathcal{L}$ be any complex line bundle over $X$. By the proof of [32] Lemma 2.3 there exists a finite complex $Y$ with $\dim Y \leq d$, a continuous function $f : X \to Y$, and a line bundle $\mathcal{L}'$ over $Y$ such that $\mathcal{L} \cong f^*(\mathcal{L}')$. By Lemma 5.13 we conclude that $\mathcal{L}'$ is isomorphic to a line subbundle of $Y \times \mathbb{C}^m$, with $m = \left\lceil (d+1)/2 \right\rceil$. Hence, $\mathcal{L}$ is isomorphic to a line subbundle of $X \times \mathbb{C}^m$. By Remark 5.9 (b) if $n \geq p = \left\lceil \sqrt{n/m} \right\rceil = \left\lceil \sqrt{(d+1)/2} \right\rceil$ ($\left\lceil \sqrt{n/m} \right\rceil = \left\lceil \sqrt{n} \right\rceil$ for all $n \geq 0$), we can assume that $\mathcal{L}$ is already a subbundle of $X \times M_n$. By the proof of Proposition 5.8 we can find an operator $\phi \in \text{IB}^{nv}_1(A_n)$ such that $\mathcal{L}_\phi = \mathcal{L}$. By Remark 5.10 we conclude that the map $\theta$ is surjective. That $\text{TM}^{nv}(A) \subseteq \text{IB}^{nv}_1(A_n)$ ($n \geq p$) when $H^2(X;\mathbb{Z}) \neq 0$ follows directly from previous observations and Proposition 5.9. \qed

Example 5.14. Note that if $X$ is either the 2-sphere, the 2-torus or the Klein bottle, then it is well-known that $H^2(X;\mathbb{Z}) \neq 0$. In particular, if $A = C(X, M_n)$ ($n \geq 2$) then Proposition 5.12 shows that $\text{TM}^{nv}(A) \subseteq \text{IB}^{nv}_1(A)$.

Theorem 5.15. Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous $C^*$-algebra. Consider the following conditions:

(a) For every open subset $U \subset X$, each complex line subbundle of $\mathcal{E}|_U$ is trivial.
(b) $\text{IB}_1(A) = \text{TM}(A)$.
(c) $\text{IB}_{0,1}(A) = \text{TM}_0(A)$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). If $A$ is separable, conditions (a), (b) and (c) are equivalent.
Remark 5.17. By a manifold of dimension $d$.

Proof. (a) $\Rightarrow$ (b): Assume (a) holds and $\phi \in \text{IB}_1(A)$. Let $U = \text{coz}(\phi)$ (open by Corollary 5.14). By Proposition 5.10 we may assume that $U \neq X$. Let $B = \Gamma_0(\mathcal{E}|_U)$ and let $\phi|_U$ be the restriction of $\phi$ to $B$. Then $\phi|_U \in \text{IB}_{0,1}^*(B)$. By (a), $\mathcal{L}_a$ is trivial (on $U$) and by Proposition 5.18 we have $\phi|_U \in \text{TM}(B)$, that is $\phi|_U = M_{c,d}$ for some $c,d \in M(B) = \Gamma_0(\mathcal{E}|_U)$. By Remark 5.16 we can suppose that $\|c(t)\|^2 = \|d(t)\|^2 = \|\phi_t\|$ for $t \in U$, so that $c,d \in B$. We can then define $a,b \in A$ by $a(t) = b(t) = 0$ for $t \in X \setminus U$ and, for $t \in U$, $a(t) = c(t), b(t) = d(t)$. Then we have $\phi = \phi_{a,b} \in \text{TM}_0(A) \subseteq \text{TM}(A)$.

(b) $\Rightarrow$ (c): Take intersections with $\text{IB}_0(A)$.

Now assume that $A$ is separable, so that $X$ is second-countable.

(c) $\Rightarrow$ (b): If $\phi \in \text{IB}_1(A)$, take a strictly positive function $f \in C_0(X)$ and define $\psi = f(t)^2 \phi_t$. By (c) and Remark 5.16 we have $\psi = M_{c,d}$ for $c,d \in A$ with $\|c(t)\|^2 = \|d(t)\|^2 = \|\psi_t\|$. We can define $a,b \in M(A) = \Gamma_b(\mathcal{E})$ by $a(t) = c(t)/f(t)$ and $b(t) = d(t)/f(t)$ to get $\phi = \phi_{a,b} \in \text{TM}(A)$.

(b) $\Rightarrow$ (a): Assume (b) holds. Let $U$ be an open subset of $X$ and $\mathcal{L}$ a complex line subbundle of $\mathcal{E}|_U$. By Proposition 5.8 applied to $B = \Gamma_0(\mathcal{E}|_U)$ ($U$ is $\sigma$-compact since $X$ is second-countable), there is $\psi \in \text{IB}_0(B)$ with $\mathcal{L}_a = \mathcal{L}$. Since $(t \mapsto \|\psi_t\|) \in C_0(U)$, we can define $\phi \in \text{IB}_0(A)$ by $\phi_t = \psi_t$ for $t \in U$ and $\phi_t = 0$ for $t \in X \setminus U$. By (b), $\phi = \phi_{a,b}$ for $a,b \in M(A) = \Gamma_b(\mathcal{E})$ and then $\phi|_U$ defines a nowhere vanishing section of $\mathcal{L}$. $\square$

Corollary 5.16. Let $A = \Gamma_0(\mathcal{E})$ be an $n$-homogeneous $C^*$-algebra with $n \geq 2$.

(a) If $X$ is second-countable with $\dim X < 2$, or if $X$ is (homeomorphic to) a subset of a non-compact connected 2-manifold, then

$$\text{IB}_{0,1}(A) = \text{TM}_0(A) \quad \text{and} \quad \text{IB}_1(A) = \text{TM}(A).$$

(b) If $X$ is $\sigma$-compact and contains a nonempty open subset homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then

$$\text{IB}_{0,1}(A) \setminus \text{TM}_0(A) \neq \emptyset \quad \text{and} \quad \text{IB}_1(A) \setminus \text{TM}(A) \neq \emptyset.$$

Remark 5.17. By a manifold of dimension $d$ we always mean a second-countable topological manifold of dimension $d$.

To prove this we will use the following facts (which are well-known to topologists).

Remark 5.18. If $X$ is a metrizable space with $\dim X = d < \infty$, then any locally trivial fibre bundle over $X$ can be trivialized over some open cover of $X$ consisting of at most $d + 1$ elements. This follows from Dowker’s and Ostrand’s theorems [9, Theorems 3.2.1 and 3.2.4].

Lemma 5.19. Let $Y$ be a metrizable space with $\dim Y = d < \infty$ and let $X$ be a closed subset of $Y$. Then any map $f : X \to \mathbb{C}P^\infty$ can be, up to homotopy, continuously extended to some open neighbourhood of $X$ in $Y$.

Proof. Let $\mathcal{L}$ be a complex line bundle over $X$ defined by $f$ (Remark 5.10). By [9, Theorem 3.1.4], we have $\dim X \leq \dim Y = d$. By Remark 5.15, $\mathcal{L}$ can be trivialized over some open cover of $X$ consisting of (at most) $d + 1$ elements. In particular, $\mathcal{L}$ is determined by some map $g : X \to \mathbb{C}P^d$ (see e.g. [18, § 3.5]) and by Remark 5.10 $g$ is homotopic to $f$. By [17, Theorem V.7.1], finite dimensional manifolds (in particular $\mathbb{C}P^d$) are ANR spaces and so by [17, Theorem III.3.2], $g$ extends (continuously) to some open neighbourhood of $X$ in $Y$. $\square$
Proposition 5.20. Suppose that $X$ is a locally compact subset of a non-compact connected 2-manifold $M$. Then $\hat{H}^2(X;\mathbb{Z}) = 0$.

Proof. First assume that $X = M$. Then by [28, Theorem 2.2], since every 2-manifold admits a smooth structure (a classical result for which we have failed to find a complete modern reference), $X$ is homotopy equivalent to a CW-complex of dimension $d < 2$. Using Lemma 5.13 (and Remark 5.10) we conclude that $\hat{H}^2(X;\mathbb{Z}) = 0$.

Now let $X$ be an open subset of $M$. Since the previous argument applies to each connected component of $X$, we again have $\hat{H}^2(X;\mathbb{Z}) = 0$.

If $X$ is a locally compact subset of $M$, then $X$ is open in its closure $\overline{X}$. Let $Y = \overline{X} \setminus X$. Then $N = M \setminus Y$ is open in $M$ and $X$ is closed in $N$. Suppose that $\hat{H}^2(X;\mathbb{Z}) \neq 0$ and let $f : X \to CP^{\infty}$ be any non-null-homotopic map (Remark 5.10). By Lemma 5.19 $f$ extends, up to homotopy, to a map defined on some open neighbourhood $U$ of $X$ in $M$. In particular, $\hat{H}^2(U;\mathbb{Z}) \neq 0$ which contradicts the second part of the proof. □

Proof of Corollary 5.16. For (a) it suffices to show that $\hat{H}^2(U;\mathbb{Z}) = 0$ for all open subsets $U$ of $X$ (by Theorem 5.15 and Remark 5.10).

By Proposition 5.20 this is true if $X$ is a subset of a non-compact connected 2-manifold. Suppose that $X$ is second-countable with dim $X < 2$. Then for each open subset $U$ of $X$ we have dim $U \leq$ dim $X$ (by the 'subset theorem' [9, Theorem 3.1.19]), so $\hat{H}^2(U;\mathbb{Z}) = 0$ (see e.g. [31, p. 94–95]).

For (b) we first choose an open subset $U \subset X$ for which $E|_{U} \cong U \times M_n$ and such that $U$ can be considered as an open set in $\mathbb{R}^d$ ($d \geq 3$). We use the simple fact that $U$ contains an open subset that has the homotopy type of the 2-sphere $S^2$. So, replacing $U$ by such a subset, we can find a non-trivial line subbundle $L$ of $U \times \mathbb{C}^2$. By Remark 5.9 (b) we may assume that $L$ is a subbundle of $U \times M_n \cong E|_{U}$. The assertion now follows from the proof of Theorem 5.15. □

Remark 5.21. In the literature there are somewhat similar phenomena that arise for unital $C^*$-algebras $A$ of sections of a $C^*$-bundle over a (second-countable) compact Hausdorff space $X$. The question was to describe when the set $\text{Aut}_{C(X)}(A)$ of all $C(X)$-linear automorphisms of such $A$ coincides with the inner automorphisms of $A$ (see e.g. [22, 34, 30, 31]). For example, if $A$ is any separable unital continuous trace $C^*$-algebra with (primitive) spectrum $X$, there always exists an exact sequence

$$0 \longrightarrow \text{InnAut}(A) \longrightarrow \text{Aut}_{C(X)}(A) \longrightarrow \hat{H}^2(X;\mathbb{Z})$$

of abelian groups. In general, $\eta$ does not need to be surjective unless $A$ is stable [30, Theorem 2.1]. If $A$ is $n$-homogeneous then the image of $\eta$ is contained in the torsion subgroup of $\hat{H}^2(X;\mathbb{Z})$ [20, 2.19]. In particular, $A = C(S^2, M_2)$ shows that it can happen that $TM^\text{inv}(A) \subseteq IB^\text{inv}_1(A)$ even though $\text{Aut}_{C(S^2)}(A) = \text{InnAut}(A)$ (since $\hat{H}^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$ is torsion free). Our Proposition 5.12 shows that the map $\theta$ from [5.11] is surjective in this case. In contrast to $\eta$, there is no obvious group structure on the domain $IB^\text{inv}_1(A)$ of $\theta$.

6. Closure of $TM(A)$ on homogeneous $C^*$-algebras

Here we continue to work with $n$-homogeneous algebras $A = \Gamma_0(E)$. The class $IB_1(A)$ considered in [44] is rather obviously designed to capture a restriction on the
closure of $\text{TM}(A)$ (and similarly $\text{IB}_{0,1}(A)$ should relate to the closure of $\text{TM}_0(A)$). We verify right away (Proposition 6.1) that $\text{IB}_1(A)$ and $\text{IB}_{0,1}(A)$ are indeed closed. However, further restrictions on the operators $\phi$ in the closure of $\text{TM}(A)$ arise because triviality of the line bundles $L_\phi$ associated with $\psi \in \text{TM}(A)$ is still present for the line bundle $L_\phi$ provided $U = \text{coz}(\phi)$ is compact (see Corollary 6.5). If $U$ is not compact, this triviality is evident on compact subsets of $U$ (see Theorem 6.9 where we characterize the closure of $\text{TM}_0(A)$). However, $L_\phi$ need not be trivial globally on $U$ (so that $\phi \not\in \text{TM}(A)$ is possible) and this led us to define the concept of a phantom bundle (Definition 6.11). The terminology is by analogy with the well known concept of a phantom map (see [27]). Thus, in Corollary 6.12 we see that finding $\phi$ in the norm closure of $\text{TM}_0(A)$ with $\phi \not\in \text{TM}_0(A)$ is directly related to finding suitable phantom complex line bundles.

For these to exist, we need $U$ to have a rather complicated algebraic topological structure, and we find examples with $\pi_1(U) \cong \mathbb{Q}$ (Proposition 6.17). In fact, we can also find such examples when $X$ contains a copy of an open subset of $\mathbb{R}^3$ with $d \geq 3$ and $n \geq 2$ (Theorem 6.18).

Proposition 6.1. Let $A$ be a homogeneous $C^*$-algebra. Then $\text{IB}_1(A)$ and $\text{IB}_{0,1}(A)$ are norm closed subsets of $\mathcal{B}(A)$.

Proof. If $(\phi_k)_{k=1}^\infty$ is a sequence in $\text{IB}_1(A)$ that converges in operator norm to $\phi \in \mathcal{B}(A)$, then it is clear that $\phi(I) \subset I$ for each ideal $I$ of $A$. Thus $\phi \in \mathcal{B}(A)$.

By Corollary 6.2 we have $|\phi - \phi_k| = \sup_{t \in X} ||\phi_t - (\phi_k)_t||$ and so $\lim_{k \to \infty} (\phi_k)_t = \phi_t \in \mathcal{B}(A_t)$ (for $t \in X$). Since $A_t \cong M_{n^2}$, invoking Theorem 3.4 we have $\phi_t \in \text{TM}(A_t)$ (for $t \in X$) and hence $\phi \in \text{IB}_1(A)$.

If $\phi_k \in \text{IB}_{0,1}(A)$ for each $k$, then $|(\phi_k)_t| \to ||\phi_t||$ uniformly for $t \in X$. As $(t \mapsto ||(\phi_k)_t||) \in \mathcal{C}_0(X)$, it follows that $(t \mapsto ||\phi_t||) \in \mathcal{C}_0(X)$ and so $\phi \in \text{IB}_0(A)$.

Lemma 6.2. Let $A$ be a homogeneous $C^*$-algebra, and let $\phi \in \text{IB}_1^{nv}(A)$. Then there is $\psi \in \text{IB}_1^{nv}(A)$ with $\psi_t = \phi_t / ||\phi_t||$ for each $t \in X$.

Moreover $\phi \in \text{TM}(A) \iff \psi \in \text{TM}(A)$.

Proof. Since $t \mapsto ||\phi_t||$ is continuous by Corollary 6.2 we can define $\psi_t = \phi_t / ||\phi_t||$ and get $\psi \in \text{IB}(A)$ via local applications of Proposition 6.1. Clearly $\psi \in \text{IB}_1^{nv}(A)$.

If $\phi = M_{a,b} \in \text{TM}(A)$ for $a, b \in A(A = \Gamma_0(\mathcal{E})$, then we can normalize $a$ and $b$ as in Remark 6.3 and then take $c, d \in A$ with $c(t) = a(t) / \sqrt{||\phi_t||}$, $d(t) = b(t) / \sqrt{||\phi_t||}$ to get $\psi = M_{c,d}$. So $\psi \in \text{TM}(A)$. We can reverse this argument.

Remark 6.3. Let $\text{TM}(A)$ denote the operator norm closure of $\text{TM}(A)$, and similarly for $\text{TM}_0(A)$. If $A$ is homogeneous, then Proposition 6.11 gives $\text{TM}(A) \subset \text{IB}_1(A)$ and $\text{TM}_0(A) \subset \text{IB}_{0,1}(A)$.

Proposition 6.4. Let $A = \Gamma_0(\mathcal{E})$ be an $n$-homogeneous $C^*$-algebra. Suppose that $\phi \in \text{TM}(A)$ such that $\inf_{t \in X} ||\phi_t|| = \delta > 0$. Then $\phi \in \text{TM}(A)$.

Proof. Let $(\phi_k)_{k=1}^\infty$ be a sequence in $\text{TM}(A)$ with $\lim_{k \to \infty} \phi_k = \phi \in \mathcal{B}(A)$. For $k$ large enough that $||\phi_k - \phi|| < \delta / 2$ we must have $||((\phi_k)_t) - (\phi_t)|| > \delta / 2$ for each $t \in X$ (and hence $\phi_k \in \text{IB}_1^{nv}(A)$). With no loss of generality we may assume that this holds for all $k \geq 1$.

Since $\sup_{t \in X} ||(\phi_k)_t - (\phi_t)|| \leq \sup_{t \in X} ||(\phi_k)_t - (\phi_t)|| = ||\phi_k - \phi||$, we conclude $\phi \in \text{TM}(A)$.
we may use Lemma 5.2 to normalise each $\phi_k$ and $\phi$ and assume that

$$1 = \|\phi\| = \|\phi_k\| = \|(\phi_k)_t\| = \|\phi_t\|$$

holds for all $k \geq 1$ and $t \in X$ (and still $\lim_{k \to \infty} \phi_k = \phi$).

We now write $\phi_k = M_{a_k,b_k}$ for $a_k, b_k \in M(A) = \Gamma_b(E)$ such that $\|a_k(t)\| = \|b_k(t)\| = 1$ (for all $t \in X$ and all $k$). We consider the line bundle $L_\phi$ associated with $\phi$ according to Proposition 5.6 which is locally expressible as $\{(t, \rho(t))\}$, where $\phi_t = M_{a(t), b(t)}$ locally. We assume, as we can, that $\|a(t)\| = \|b(t)\| = 1$ (locally).

Let $0 < \varepsilon < (18n)^{-1/2}$.

By Remark 4.1(d), for $k$ suitably large (but fixed) and $t \in X$ arbitrary, we have $\|(\phi(t)) - \phi_t\|_{cb} < \varepsilon$. Since, by Mathieu's theorem (Theorem 2.1), we locally have

$$\|(\phi(t)) - \phi_t\|_{cb} = \|M_{a(t), b(t)} - M_{a(t), b(t)}\|_{cb} = \|a(t) \otimes b(t) - a(t) \otimes b(t)\|_{cb},$$

by Lemma 3.3 we can locally find a scalar $\mu_k(t)$ of modulus 1 such that

$$\|a(t) - \mu_k(t)a(t)\| < 6\varepsilon$$

(note that $(18n)^{-1/2} < 1/3$ for all $n \geq 1$).

Consider the inner product $\langle \cdot, \cdot \rangle_2$ defined in Remark 4.1(g). We claim that locally $\langle a(t), a(t) \rangle_2 \neq 0$. Indeed, first note that (locally)

$$|\langle a(t), a(t) \rangle_2| = |\langle a(t), \mu_k(t)a(t) \rangle_2|$$

and by (4.2)

$$\|a(t)\|_2 \geq 1, \quad \|\mu_k(t)a(t)\|_2 \geq 1, \quad \|a(t) - \mu_k(t)a(t)\|_2 < 6\sqrt{n}\varepsilon.$$  

Since any two vectors $v$ and $w$ of norm at least 1 in a Hilbert space satisfy

$$\|v - w\|_2^2 \geq \|v\|_2^2 + \|w\|_2^2 - 2|\langle v, w \rangle_2| \geq 2(1 - \|v, w\|_2^2),$$

letting $v = a(t)$ and $w = \mu_k(t)a(t)$, we have (locally)

$$|\langle a(t), \mu_k(t)a(t) \rangle_2| \geq 1 - \frac{1}{2}\|a(t) - \mu_k(t)a(t)\|_2^2 > 1 - 18n\varepsilon^2 > 0.$$  

We can therefore define $a'(t)$ locally as the normalised (in operator norm) orthogonal projection

$$a'(t) = \frac{\langle a(t), a(t) \rangle_2}{|\langle a(t), a(t) \rangle_2|} \cdot a(t).$$

Then $t \mapsto a'(t)$ is locally well-defined and continuous (by Remark 4.1(g)). As $a'(t)$ is independent of multiplying $a(t)$ by unit scalars, it defines a nowhere vanishing global section of $L_\phi$. By Proposition 5.6 we must have $\phi \in TM(A)$, as required.

**Corollary 6.5.** Let $A$ be a unital homogeneous $C^*$-algebra. Then

$$TM(A) \cap \overline{IB}^\text{rev}(A) \subset TM(A).$$

**Proof.** Let $\phi \in TM(A) \cap \overline{IB}^\text{rev}(A)$. Since $t \mapsto \|\phi_t\|$ is continuous (Corollary 4.4) and never vanishing on $X$ (which is compact, as $A$ is unital), it has a minimum value $\delta > 0$. By Proposition 6.4 $\phi \in TM(A)$. \qed
Example 6.6. Let $A = C(X, M_n)$ ($n \geq 2$), where $X$ is any compact Hausdorff space with $\dim X \leq 7$ and $\check{H}^2(X; \mathbb{Z}) \neq 0$. Then $\text{TM}(A) \subseteq \text{IB}_1(A)$. Indeed, by Proposition 5.12 there exists $\phi \in \text{IB}_1^n(A) \setminus \text{TM}(A)$. By Corollary 5.2, $\phi \notin \text{TM}(A)$. (Since $A$ is unital, $\text{TM}_0(A) = \text{TM}(A)$ and $\text{IB}_{0,1}(A) = \text{IB}_1(A)$.)

Corollary 6.7. If $A = \Gamma_0(\mathcal{E})$ is a homogeneous $C^*$-algebra, then both $\text{InnAut}_{\text{alg}}(A)$ and $\text{InnAut}(A)$ (see (2.3)) are norm closed.

Proof. If $M_{a,n^{-1}} \in \text{InnAut}_{\text{alg}}(A)$, then for all $t \in X$ we have $\| (M_{a,n^{-1}}) \| = \| a(t) \| a(t)^{-1} \| \geq 1$. Hence if $\phi$ is in the norm closure of $\text{InnAut}_{\text{alg}}(A)$, we have $\| \phi \| \geq 1$ for each $t \in X$. By Proposition 5.1 $\phi = M_{b,c}$ for some $b, c \in M(A)$. Since $\phi(t) = 1$, $c(t) = d(t)^{-1}$ for each $t$ and so $c = b^{-1} \in M(A) = \Gamma_b(\mathcal{E})$.

The proof for the $\text{InnAut}(A)$ is similar. □

Remark 6.8. The results that $\text{InnAut}(A)$ is norm closed if the $C^*$-algebra $A$ is prime or homogeneous (in Corollaries 5.2 and 6.7) can also be deduced from [19] 3. To explain the deductions, we first identify $\text{InnAut}(A)$ with $\text{InnAut}(M(A))$.

If $A$ is prime, then $M(A)$ is also prime (by [2] Lemma 1.1.7). In particular, $\text{Orc}(M(A)) = 1$ (in the sense of [25] [22]), so by [25] Corollary 4.1 inner derivations of $M(A)$ are norm closed. Then [19] Theorem 5.3 implies that $\text{InnAut}(M(A))$ is also norm closed.

If $A$ is homogeneous (or more generally quasi-central and quasi-standard in the sense of [3]), then $M(A)$ is quasi-standard [3] Corollary 4.10. Thus we have $\text{Orc}(M(A)) = 1$, and we may conclude as in the prime case.

Theorem 6.9. Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous $C^*$-algebra. For an operator $\phi \in \mathcal{B}(A)$, the following two conditions are equivalent:

(a) $\phi \in \text{TM}_0(A)$.
(b) $\phi \in \text{IB}_{0,1}(A)$ and for $U = \text{coz}(\phi)$ (open by Corollary 4.4) $L_{\phi}$ is trivial on each compact subset of $U$.

Proof. (a) $\Rightarrow$ (b): Let $\phi \in \text{TM}_0(A)$, so that $\phi \in \text{IB}_{0,1}(A)$ (Remark 6.3). For each compact subset $K \subseteq U$, we have $\phi_K \in \text{TM}(A_K)$ (recall that $A_K = \Gamma(\mathcal{E}|_K)$ by Remark 4.1 (b)). By Corollary 6.5, we have $\phi_K \in \text{TM}(A_K)$, so that $L_{\phi}$ must be trivial on $K$ (by Proposition 5.5).

(b) $\Rightarrow$ (a): Let $\phi \in \text{IB}_{0,1}(A)$, so that $t \mapsto \| \phi_t \|$ is in $C_0(X)$.

For any sequence $\delta_n > 0$ decreasing strictly to 0 (for instance $\delta_n = 1/n$) let

$$K_n = \{ t \in X : \| \phi_t \| \geq \delta_n \}.$$  

Then each $K_n$ is compact, $K_n \subseteq K^n_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = U$. By Proposition 5.6, $\psi_{K_n} \in \text{TM}(A_{K_n}) \subseteq \text{TM}(\Gamma(\mathcal{E}|_{K_n}))$ and so there are $a_n, b_n \in A_{K_n}$ with $\psi_{K_n} = M_{a_n,b_n}$. Using Remark 5.4 we may assume $\| a_n(t) \| = \| b_n(t) \| = \sqrt{\| \phi_t \|}$ for $t \in K_n$.

By Remark 4.1 (b) we may extend $a_n$ to $c_n \in A$ with $c_n(t) = 0$ for $t \in X \setminus K^n_{n+1}$ and $\| c_n(t) \|^2 \leq \delta_n$ for all $t \in X \setminus K_n$. Similarly we extend $b_n$ to $d_n \in A$ supported in $K^n_{n+1}$ with $\| d_n(t) \|^2 \leq \delta_n$ for $t \in X \setminus K_n$. Then $(M_{c_n,d_n} - \phi)_{t}$ has norm at most $2\delta_n$ for all $t \in X$ and hence $\lim_{n \to \infty} M_{c_n,d_n} = \phi$. Thus $\phi \in \text{TM}_0(A)$.

Corollary 6.10. For a homogeneous $C^*$-algebra $A = \Gamma_0(\mathcal{E})$ the following conditions are equivalent:

(a) $\text{TM}_0(A) = \text{IB}_{0,1}(A)$. 
(b) $\text{InnAut}_{\text{alg}}(A)$ and $\text{InnAut}(A)$ (see (2.3)) are norm closed.
(b) For each σ-compact open subset $U$ of $X$, every complex line subbundle of $E|_U$ is trivial on all compact subsets of $U$.

Proof. (a) $\Rightarrow$ (b): Let $U$ be a σ-compact open subset of $X$, $B = \Gamma_0(E|_U)$ and $L$ a complex line subbundle of $E|_U$. By Proposition 5.8 we can find an operator $\phi \in IB_{0,1}^\mathrm{inv}(B)$ such that $L_\phi = L$. By extending $\phi$ to be zero outside $U$, we may assume that $\phi \in IB_{0,1}(A)$, so that $U = \mathrm{coz}(\phi)$. By assumption, $\phi \in TM_0(A)$, so by Theorem 6.9 $L$ is trivial on all compact subsets of $U$.

(b) $\Rightarrow$ (a): If $\phi \in IB_{0,1}(A)$ then $U = \mathrm{coz}(\phi)$ is an open, necessarily σ-compact subset of $X$ (since $t \mapsto \|\phi_t\|$ is in $C_0(X)$). By assumption, $L_\phi$ is trivial on every compact subset of $U$. Hence, $\phi \in TM_0(A)$ by Theorem 6.9

Definition 6.11. A locally trivial fibre bundle $F$ over a locally compact Hausdorff space $X$ is said to be a phantom bundle if $F$ is not globally trivial, but is trivial on each compact subset of $X$.

Corollary 6.12. Let $A = \Gamma_0(E)$ be a homogeneous $C^*$-algebra. Then $TM_0(A)$ fails to be norm closed in $B(A)$ if and only if there exists a σ-compact open subset $U$ of $X$ and a phantom complex line subbundle of $E|_U$.

If these equivalent conditions hold, then $TM(A)$ fails to be norm closed.

Proof. If $TM_0(A)$ fails to be norm closed, there is $\phi \in TM_0(A) \setminus TM_0(A)$. Note that $\phi \in IB_{0,1}(A)$ by Proposition 6.1. By Theorem 6.9 for $U = \mathrm{coz}(\phi)$ (open and σ-compact), $L_\phi$ is trivial on each compact subset of $U$. Moreover $\phi|_{U} \in IB_{0,1}^{\mathrm{inv}}(B)$ for $B = \Gamma_0(E|_U)$. By Proposition 6.6 if $L_\phi$ is globally trivial, then $\phi|_{U} \in TM(B) \cap IB_0(B) = TM_0(B)$. So $\phi|_{U} = M_{a,b}$ for $a,b \in B$. Since $B$ can be considered as an ideal of $A$ (Remark 4.1 (c)), we treat $a,b \in A$. Hence $\phi = M_{a,b} \in TM_0(A)$, a contradiction. Thus $L_\phi$ is a phantom bundle.

Conversely, suppose that $U \subset X$ is open and σ-compact and that $L$ is a phantom complex line subbundle of $E|_U$. Then, taking $B = \Gamma_0(E|_U)$, Proposition 5.8 provides $\psi \in IB_{0,1}^{\mathrm{inv}}(B)$ with $L!\psi = L$. As $L$ is a phantom bundle, by Proposition 5.6 $\psi \notin TM(B)$. We may define $\phi \in IB_{0,1}(A)$ by $\phi_t = \psi_t$ for $t \in U$ and $\phi_t = 0$ for $t \in X \setminus U$. From $\psi = \phi|_{U} \notin TM(B)$, we have $\phi \notin TM(A)$ but $\phi \in TM_0(A)$ by Theorem 6.9

We now describe below a class of homogeneous $C^*$-algebras $A$ for which $TM_0(A)$ and $TM(A)$ both fail to be norm closed. We first explain some preliminaries.

Remark 6.13. Let $G$ be a group and $n$ a positive integer. Recall that a space $X$ is called an Eilenberg-MacLane space of type $K(G,n)$, if its $n$-th homotopy group $\pi_n(X)$ is isomorphic to $G$ and all other homotopy groups trivial. If $n > 1$ then $G$ must be abelian (since for all $n > 1$, the homotopy groups $\pi_n(X)$ are abelian). We state some basic facts and examples about Eilenberg-MacLane spaces:

(a) There exists a CW-complex $K(G,n)$ for any group $G$ at $n = 1$, and abelian group $G$ at $n > 1$. Moreover such a CW-complex is unique up to homotopy type. Hence, by abuse of notation, it is common to denote any such space by $K(G,n)$ [10] p. 365-366.

(b) Given a CW-complex $X$, there is a bijection between its cohomology group $H^n(X;G)$ and the homotopy classes $[X,K(G,n)]$ of maps from $X$ to $K(G,n)$ [10] Theorem 4.57.
(c) $K(\mathbb{Z}, 2) \cong \mathbb{C}P^\infty$ [16 Example 4.50]. In particular, by (b) and Remark 5.10 for each CW-complex $X$ there is a bijection between $[X, K(\mathbb{Z}, 2)]$ and isomorphism classes of complex line bundles over $X$.

**Proposition 6.14.** If $X$ is a locally compact CW-complex of type $K(\mathbb{Q}, 1)$, then every non-trivial complex line bundle over $X$ is a phantom bundle. Moreover there are uncountably many non-isomorphic such bundles.

**Proof.** The standard model of $K(\mathbb{Q}, 1)$ is the mapping telescope $\Delta$ of the sequence

$$S^1 \xrightarrow{f_1} S^1 \xrightarrow{f_2} S^1 \xrightarrow{f_3} \cdots,$$

where $f_n : S^1 \to S^1$ is given by $z \mapsto z^{n+1}$ (see e.g. [7] Example 1.9 and [16] Section 3.F).

We first consider the case when $X = \Delta$. Applying $H_1(\cdot; \mathbb{Z})$ to the levels of the mapping telescope (6.1) gives the system

$$\mathbb{Z} \xrightarrow{(f_1)_*} \mathbb{Z} \xrightarrow{(f_2)_*} \mathbb{Z} \xrightarrow{(f_3)_*} \cdots,$$

where $(f_n)_* : \mathbb{Z} \to \mathbb{Z}$ is given by $k \mapsto (n+1)k$ (see [16] Section 3.F)). The colimit of this system is (by [16] Proposition 3.33) $H_1(\Delta; \mathbb{Z}) = \mathbb{Q}$ and all other integral homology groups are trivial. By the universal coefficient theorem for cohomology [16] Theorem 3.2] (see also [16] §3.F]) each integral cohomology group of $\Delta$ is trivial, except for $H^2(\Delta; \mathbb{Z})$ which is isomorphic to $\text{Ext}(\mathbb{Q}; \mathbb{Z})$. By [35] $\text{Ext}(\mathbb{Q}; \mathbb{Z})$ is isomorphic to the additive group of real numbers. Hence, by Remark 5.10 there exists uncountably many non-isomorphic complex line bundles over $\Delta$. We claim that each non-trivial such bundle $\mathcal{L}$ is a phantom bundle. Indeed, for $n \geq 1$ let $\Delta_n$ denote the $n$-th level of the mapping telescope (6.1). If $K$ is an arbitrary compact subset of $\Delta$ then $K$ is contained in some $\Delta_n$. Since all $\Delta_n$’s are homotopy equivalent to $S^1$, and since $\tilde{H}^2(S^1; \mathbb{Z}) = 0$, we conclude that $\mathcal{L}|_{\Delta_n}$ is trivial. Then $\mathcal{L}|_{K}$ is also trivial, since $K \subset \Delta_n$.

If $X$ is another locally compact CW-complex of of type $K(\mathbb{Q}, 1)$, then by Remark 6.13 (a), there are maps $f : \Delta \to X$ and $g : X \to \Delta$ such that $g \circ f$ and $f \circ g$ are homotopic (respectively) to the identity maps (on $\Delta$ and $X$, respectively). If $\mathcal{L}$ is a non-trivial complex line bundle over $\Delta$, then $g^*(\mathcal{L})$ is non-trivial over $X$ (Remark 5.10). Moreover $g^*(\mathcal{L})$ is a phantom bundle because $K \subset X$ compact implies $g(K) \subset \Delta$ compact and $g^*(\mathcal{L})|_K$ is a restriction of $g^*(\mathcal{L})|_{g^{-1}(g(K))} = g^*(\mathcal{L}|_{g(K)})$, which is a trivial bundle. Since $g$ is a homotopy equivalence, every non-trivial complex line bundle over $X$ must be isomorphic to $g^*(\mathcal{L})$ for some $\mathcal{L}$. \qed

**Remark 6.15.** With the same notation as in the proof of Proposition 6.14, one can show that for each compact subset $K$ of $\Delta$ we have $\tilde{H}^2(K; \mathbb{Z}) = 0$. To sketch the proof, choose an arbitrary complex line bundle $\mathcal{L}$ over $K$. Then using Lemma 5.10 (and Remark 5.10) $\mathcal{L}$ can be extended to an open neighbourhood $U$ of $K$. The assertion can now be established via an argument with triangulations of $\Delta$. There is a triangulation of $\Delta$ where $\Delta_1$ has 3 triangles and each $\Delta_{n+1}$ has $n + 3$ more triangles than $\Delta_n$. We may subdivide the triangles that touch $K$ to get finitely many that cover $K$ and are all contained in $U$. Now consider the union $T$ of the triangles that touch $K$. It is enough to show $\mathcal{L}|_T$ is trivial. We can deformation retract $T$ to a union of 1-simplices. To do so, work on one triangle (2-cell) at a time, starting with any 2-cell in $\Delta_1$ with a ‘free’ edge not in the boundary of $\Delta_1$ relative to $\Delta_2$ (where ‘free’ means the edge does not bound a second 2-cell). After
each step, consider the remaining 2-cells, edges and vertices. Move on to \( \Delta_2 \) once all 2-cells in \( \Delta_1 \) are exhausted, etc, so as to arrive at a 1-simplex after finitely many steps. As all complex line bundles over 1-simplices are trivial, we have that \( L|_T \) is trivial.

In private correspondence, Mladen Bestvina informed us that we can find phantom bundles even over some open subset of \( \mathbb{R}^3 \), and referred us to [6]. We outline the construction of such a subset.

**Proposition 6.16.** There exists an open subset \( \Omega \) of \( \mathbb{R}^3 \) of type \( K(\mathbb{Q}, 1) \).

**Proof.** In [6], a construction is given of dense open sets \( U \) in the 3-sphere \( S^3 \) with fundamental groups \( \pi_1(U) \) that are large subgroups of \( \mathbb{Q} \). Given a sequence \( n_i \) of natural numbers \( n_i > 1 \), \( \pi_1(U) \) can be \( \{p/q : p \in \mathbb{Z}, q = \prod_{i=1}^{k} n_i \text{ for some } k \} \). In particular we will take \( n_i = i + 1 \) and then \( \pi_1(U) = \mathbb{Q} \).

The construction defines \( U \) as a union of closed solid tori \( U = \bigcup_{i=1}^{\infty} S_i \). For each \( i \), both \( S_i \) and the complement of its interior \( T_i = S^3 \setminus S_i^o \) are solid tori with intersection \( S_i \cap T_i \) a (2-dimensional) torus. At each step, \( T_{i+1} \) is constructed inside \( T_i \) as an unknotted solid torus of smaller cross-sectional area that winds \( n_i \) times around the meridian circle of \( T_i \). Since \( T_{i+1} \) can be unfolded to a standard embedding of a torus via an ambient isotopy of \( S^3 \), \( S_1 \) must be a solid torus.

Let \( f : S^n \to U \) be an arbitrary map. Then \( f \) maps \( S^n \) into one of the solid tori \( S_i \) and these are homotopic to their meridian circle. In particular \( \pi_n(U) = 0 \) for all \( n > 1 \). By Remark 6.13 \( U \) has the type \( K(\mathbb{Q}, 1) \).

Choose any point \( t \in S^3 \setminus U \). Since \( S^3 \setminus \{t\} \) is homeomorphic to \( \mathbb{R}^3 \), say via the homeomorphism \( F \), then \( \Omega = F(U) \) is an open subset of \( \mathbb{R}^3 \) of the type \( K(\mathbb{Q}, 1) \). \( \square \)

**Proposition 6.17.** Let \( X \) be any locally compact \( \sigma \)-compact CW-complex of type \( K(\mathbb{Q}, 1) \) (e.g. \( X = \Delta \)). Then the \( C^* \)-algebra \( A = C_0(X, M_n) \) \((n \geq 2)\) has the following property:

There exists an operator \( \phi \in IB_{0,1}^w(A) \setminus TM(A) \) such that \( \phi \) is in the norm closure of \( TM_{cp}(A) \cap TM_0(A) = \{ M_{a,a^*} : a \in A \} \).

In particular, \( TM_0(A) \), \( TM(A) \) and \( TM_{cp}(A) \) all fail to be norm closed.

Further, if \( X = \Delta \), we have \( TM_0(A) = IB_{0,1}(A) \).

**Proof.** Choose any phantom complex line bundle \( L \) over \( X \) (Proposition 6.14). Since, by Remark 6.13 \((a)\), \( X \) has the same homotopy type as the space \( \Delta \) of Proposition 6.14 (which is a 2-dimensional complex), using Remark 5.10 and Lemma 5.13 we may assume that \( L \) is a subbundle of the trivial bundle \( X \times \mathbb{C}^2 \). We also realise \( \mathbb{C}^2 \) as a subset of \( M_n \) as \( \{ z_1 e_{1,1} + z_2 e_{1,2} : z_1, z_2 \in \mathbb{C} \} \), and in this way consider \( L \) a subbundle of \( X \times M_n \). By the proof of Proposition 5.8 we can find two sections \( a, b \) of \( L \) vanishing at infinity (so that \( a, b \in A \)) such that \( \text{span}\{a(t), b(t)\} = L_t \) for each \( t \in X \).

We define a map

\[ \phi : A \to A \text{ by } \phi = M_{a,a^*} + M_{b,b^*}. \]

Then \( \phi \) defines a completely positive elementary operator on \( A \) of length at most 2. Clearly, \( \phi_t \neq 0 \) for all \( t \in X, \text{ so } \phi \in IB_{0,1}^w(A) \). Also, \( L_\phi = L \). Since the bundle \( L \) is non-trivial, by Proposition 5.6 we have \( \phi \notin TM(A) \). On the other hand, since \( L \) is a phantom bundle, Theorem 6.2 implies \( \phi \in TM_0(A) \). Thus \( \phi \in TM_0(A) \setminus TM(A) \) and consequently \( \phi \) has length 2.
We have \( \phi_K \) completely positive on \( A_K = \Gamma(\mathcal{E}|_K) \) for each compact \( K \subset X \). Since \( \mathcal{E}|_K \) is a trivial bundle, \( \phi_K = M_{a,b} \) for some \( a,b \in A_K \) and we may suppose \( \|a(t)\| = \|b(t)\| \) holds for all \( t \in K \). It follows from positivity of \( \phi_t \) that \( b(t) = a(t)^* \) (for \( t \in K \)). By the proof of Theorem 6.9, \( \phi \) is in the norm closure of \( \TM_{cp}(A) \cap \TM_0(A) \).

Now suppose that \( X = \Delta \). Then by Remark 6.13 \( \hat{H}^2(K; \mathbb{Z}) = 0 \) for all compact subsets \( K \) of \( \Delta \). By Corollary 6.10 (and Remark 5.10) we conclude that \( \TM_0(A) = \IB_{0,1}(A) \).

**Theorem 6.18.** Let \( A = \Gamma_0(\mathcal{E}) \) be an \( n \)-homogeneous \( C^* \)-algebra with \( n \geq 2 \).

(a) If \( X \) is second-countable with \( \dim X < 2 \) or if \( X \) is (homeomorphic to) a subset of a non-compact connected 2-manifold, then both \( \TM_0(A) \) and \( \TM(A) \) are norm closed.

(b) If there is a nonempty open subset of \( X \) homeomorphic to (an open subset of) \( \mathbb{R}^d \) for some \( d \geq 3 \), then \( \TM_0(A) \) and \( \TM(A) \) both fail to be norm closed.

**Proof.** (a) follows from Corollary 6.10 (a) and Proposition 6.1.

For (b) we first choose an open subset \( U \subset X \) for which \( \mathcal{E}|_U \) is trivial and such that \( U \) can be considered as an open set in \( \mathbb{R}^d \) (\( d \geq 3 \)). Choose any open subset \( V \) of \( U \) that has the homotopy type of the set \( \Omega \) of Proposition 6.13. In particular, \( V \) is of type \( K(\mathbb{Q}, 1) \), so it allows a phantom complex line bundle (Proposition 6.14). Now apply Corollary 6.12.

**Remark 6.19.** Suppose that \( A = \Gamma_0(\mathcal{E}) \) is a separable \( n \)-homogeneous \( C^* \)-algebra with \( n \geq 2 \) such that \( \dim X = d < \infty \). By Remark 6.13 (applied to an \( \mathcal{M}_n \)-bundle \( \mathcal{E} \)) \( A \) has the finite type property. Hence, by [25, Theorem 1.1], we have \( \IB(A) = \mathcal{E}\ell(A) \). If \( X \) is either a CW-complex or a subset of a \( d \)-manifold, the following relations between \( \TM_0(A) \), \( \TM(A) \) and \( \IB_{0,1}(A) \) occur:

(a) If \( d < 2 \) we always have \( \TM_0(A) = \TM(A) = \IB_{0,1}(A) \) (Corollary 6.10 (a)).

(b) If \( d \geq 2 \) we have four possibilities:

(i) \( \TM_0(A) = \TM(A) = \IB_{0,1}(A) \). This happens e.g. whenever \( X \) is a subset of a non-compact connected 2-manifold (Corollary 6.10 (a)).

(ii) \( \TM_0(A) = \TM(A) \not\subseteq \IB_{0,1}(A) \). This happens e.g. for \( A = C(X, \mathcal{M}_n) \), where \( X = S^2 \) (by Example 5.14) and since any proper open subset \( U \) of \( S^2 \) is homeomorphic to an open subset of \( \mathbb{R}^2 \), so \( H^2(U; \mathbb{Z}) = 0 \) by Proposition 5.20.

(iii) \( \TM_0(A) \not\subseteq \TM(A) = \IB_{0,1}(A) \). This happens e.g. for \( A = C_0(X, \mathcal{M}_n) \), where \( X = \Delta \) is the standard model of \( K(\mathbb{Q}, 1) \) (Proposition 6.17).

(iv) \( \TM_0(A) \not\subseteq \TM(A) \not\subseteq \IB_{0,1}(A) \). This happens e.g. for \( A = C_0(X, \mathcal{M}_n) \), where \( X \) is the topological disjoint union \( S^2 \sqcup \Delta \) (by Proposition 6.17 Corollary 6.10) and Example 5.14).

(c) If \( d > 2 \) we always have \( \TM_0(A) \not\subseteq \TM(A) \not\subseteq \IB_{0,1}(A) \) (by Theorem 6.18 (b) and the fact that \( X \) must contain an open subset homeomorphic to \( \mathbb{R}^d \) — if \( X \) is a subset of a \( d \)-manifold, this follows from [9, Theorems 1.7.7, 1.8.9 and 4.1.9]).
Similar relations occur between $\text{TM}(A)$, $\text{TM}(A)$ and $\text{IB}_1(A)$ in parts (a) and (c) of the above cases.

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