Spectral transitions for Aharonov-Bohm Laplacians on conical layers

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We consider the Laplace operator in a tubular neighbourhood of a conical surface of revolution, subject to an Aharonov-Bohm magnetic field supported on the axis of symmetry and Dirichlet boundary conditions on the boundary of the domain. We show that there exists a critical total magnetic flux depending on the aperture of the conical surface for which the system undergoes an abrupt spectral transition from infinitely many eigenvalues below the essential spectrum to an empty discrete spectrum. For the critical flux, we establish a Hardy-type inequality. In the regime with an infinite discrete spectrum, we obtain sharp spectral asymptotics with a refined estimate of the remainder and investigate the dependence of the eigenvalues on the aperture of the surface and the flux of the magnetic field.

Keywords: Schrödinger operator; quantum layers; existence of bound states; spectral asymptotics; conical geometries

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1. Introduction

1.1. Motivation and state of the art

Various physical properties of quantum systems can be explained through a careful spectral analysis of the underlying Hamiltonian. In this paper, we consider
the Hamiltonian of a quantum particle constrained to a tubular neighbourhood of a conical surface by hard-wall boundary conditions and subjected to an external Aharonov-Bohm magnetic field supported on the axis of symmetry. It turns out that the system exhibits a *spectral transition*: depending on the geometric aperture of the conical surface, there exists a critical total magnetic flux which suddenly switches from infinitely many bound states to an empty discrete spectrum.

The choice of such a system requires some comments. First, the existence of infinitely many bound states below the threshold of the essential spectrum is a common property shared by Laplacians on various conical structures. This was first found in [6, 13], revisited in [16], and further analysed in [10] for the Dirichlet Laplacian in the tubular neighbourhood of the conical surface. In agreement with these pioneering works, in this paper, we use the term *layer* to denote the tubular neighbourhood. Later, the same effect was observed for other realizations of Laplacians on conical structures [1, 3–5, 24, 27]. Second, the motivation for combining Dirichlet Laplacians on conical layers with magnetic fields has a clear physical importance in quantum mechanics [30]. Informally speaking, magnetic fields act as ‘repulsive’ interactions whereas the specific geometry of the layer acts as an ‘attractive’ interaction. Therefore, one expects that if a magnetic field is not too strong to change the essential spectrum but strong enough to compensate the binding effect of the geometry, the number of eigenvalues can become finite or the discrete spectrum can even fully disappear.

Our main goal is to demonstrate this effect for an idealized situation of an infinitely thin and long solenoid put along the axis of symmetry of the conical layer, which is conventionally realized by a singular Aharonov-Bohm-type magnetic potential. First of all, we prove that the essential spectrum is stable under the geometric and magnetic perturbations considered in this paper. As the main result, we establish the occurrence of an abrupt spectral transition regarding the existence and number of discrete eigenvalues. In the *sub-critical* regime, when the magnetic field is weak, we prove the existence of infinitely many bound states below the essential spectrum and obtain a precise accumulation rate of the eigenvalues with a refined estimate of the remainder. The method of this proof is inspired by [10], see also [24]. In the case of the critical magnetic flux, we obtain a global Hardy inequality which, in particular, implies that there are no bound states in the sup-critical regime.

A similar phenomenon is observed in [26] where it is shown that a sufficiently strong Aharonov-Bohm point interaction can remove finitely many bound states in the model of a quantum waveguide laterally coupled through a window [17, 32]. There are also many other models where a sort of competition between binding and repulsion caused by different mechanism occurs. For example, bending of a quantum waveguide acts as an attractive interaction [8, 12] whereas twisting of it acts as a repulsive interaction [14, 21]. Thus, bound states in such a waveguide exist only if the bending is in a certain sense stronger than twisting. It is also conjectured in [31, Sec. IX] (but not proven so far) that a similar effect can arise for atomic many-body Hamiltonians at specific critical values of the nucleus charge. Here, both binding and repulsive forces are played by Coulombic interactions.
1.2. Aharonov-Bohm magnetic Dirichlet Laplacian on a conical layer

Given an angle \( \theta \in (0, \pi/2) \), our configuration space is a \( \pi/2 \)-tubular neighbourhood of a conical surface of opening angle \( 2\theta \). Such a domain will be denoted here by \( \Lambda_\theta \) and called a conical layer. Because of the rotational symmetry, it is best described in cylindrical coordinates.

To this purpose, let \((x_1, x_2, x_3)\) be the Cartesian coordinates on the Euclidean space \( \mathbb{R}^3 \) and \( \mathbb{R}_+^2 \) be the positive half-plane \( (0, +\infty) \times \mathbb{R} \). We consider cylindrical coordinates \((r, z, \phi)\) \(\in \mathbb{R}_+^2 \times S^1\) defined \textit{via} the following standard relations

\[
\begin{align*}
    x_1 &= r \cos \phi, \\
    x_2 &= r \sin \phi, \\
    x_3 &= z.
\end{align*}
\]  

For further use, we also introduce the axis of symmetry \( S := \{(r, z, \phi) \in \mathbb{R}_+^2 \times S^1 : r = 0\} \). We abbreviate by \((e_r, e_\phi, e_z)\) the moving frame

\[
e_r := (\cos \phi, \sin \phi, 0), \quad e_\phi := (-\sin \phi, \cos \phi, 0), \quad e_z := (0, 0, 1),
\]

associated with the cylindrical coordinates \((r, z, \phi)\).

To introduce the conical layer \( \Lambda_\theta \) with half-opening angle \( \theta \in (0, \pi/2) \), we first define its meridian domain \( \Gamma_\theta \subset \mathbb{R}_+^2 \) (see figure 1) by

\[
\Gamma_\theta := \left\{(r, z) \in \mathbb{R}_+^2 : -\frac{\pi}{\sin \theta} < z, \max(0, z \tan \theta) < r < z \tan \theta + \frac{\pi}{\cos \theta}\right\}.
\]

Then the conical layer \( \Lambda_\theta \) associated with \( \Gamma_\theta \) is defined in cylindrical coordinates (1.1) by

\[
\Lambda_\theta := \Gamma_\theta \times S^1.
\]

The layer \( \Lambda_\theta \) can be seen as a sub-domain of \( \mathbb{R}^3 \) constructed \textit{via} rotation of the meridian domain \( \Gamma_\theta \) around the axis \( S \).

For later purposes, we split the boundary \( \partial \Gamma_\theta \) of \( \Gamma_\theta \) into two parts defined as

\[
\partial_0 \Gamma_\theta := \{(0, z) : -\pi < z \sin \theta < 0\}, \quad \partial_1 \Gamma_\theta := \partial \Gamma_\theta \setminus \partial_0 \Gamma_\theta.
\]

The distance between the two connected components of \( \partial_1 \Gamma_\theta \) is said to be the \textit{width} of the layer \( \Lambda_\theta \). We point out that the meridian domain is normalized so that the width of \( \Lambda_\theta \) equals \( \pi \) for any value of \( \theta \). This normalization simplifies notations significantly and it also preserves all possible spectral features without
loss of generality, because the problem with an arbitrary width is related to the present setting by a simple scaling.

In order to define the \textit{Aharonov-Bohm magnetic field (AB-field)} we are interested in, we introduce a real-valued function $\omega \in L^2(S^1)$ and the vector potential $A_\omega : \mathbb{R}_+^2 \times S^1 \rightarrow \mathbb{R}^3$ by

$$A_\omega(r, z, \phi) := \frac{\omega(\phi)}{r} e_\phi. \quad (1.4)$$

This vector potential is naturally associated with the singular AB-field

$$B_\omega = \nabla \times A_\omega = 2\pi \Phi_\omega \delta_S e_z, \quad (1.5)$$

where $\delta_S$ is the $\delta$-distribution supported on $S$ and $\Phi_\omega$ is the magnetic flux

$$\Phi_\omega := \frac{1}{2\pi} \int_0^{2\pi} \omega(\phi) \, d\phi.$$  

Note that to check identity (1.5) it suffices to compute $\nabla \times A_\omega$ in the distributional sense [25, Chap. 3].

We introduce the usual cylindrical $L^2$-spaces on $\mathbb{R}^3$ and on $\Lambda_\theta$

$$L^2_{\text{cyl}}(\mathbb{R}^3) := L^2(\mathbb{R}_+^2 \times S^1; r \, dr \, dz \, d\phi), \quad L^2_{\text{cyl}}(\Lambda_\theta) := L^2(\Gamma_\theta \times S^1; r \, dr \, dz \, d\phi).$$

For further use, we also introduce the cylindrical Sobolev space $H^1_{\text{cyl}}(\Lambda_\theta)$ defined as

$$H^1_{\text{cyl}}(\Lambda_\theta) := \left\{ u \in L^2_{\text{cyl}}(\Lambda_\theta) : \int_{\Lambda_\theta} \left( |\partial_r u|^2 + |\partial_z u|^2 + \frac{|\partial_\phi u|^2}{r^2} \right) r \, dr \, dz \, d\phi < +\infty \right\}.$$  

The space $H^1_{\text{cyl}}(\Lambda_\theta)$ is endowed with the norm $\| \cdot \|_{H^1_{\text{cyl}}(\Lambda_\theta)}$ defined, for all $u \in H^1_{\text{cyl}}(\Lambda_\theta)$, by

$$\|u\|^2_{H^1_{\text{cyl}}(\Lambda_\theta)} := \|u\|^2_{L^2_{\text{cyl}}(\Lambda_\theta)} + \int_{\Lambda_\theta} \left( |\partial_r u|^2 + |\partial_z u|^2 + \frac{|\partial_\phi u|^2}{r^2} \right) r \, dr \, dz \, d\phi.$$  

Now, we define the non-negative symmetric densely defined quadratic form on the Hilbert space $L^2_{\text{cyl}}(\Lambda_\theta)$ by

$$Q_{\omega, \theta, 0}[u] := \|(i\nabla - A_\omega)u\|^2_{L^2_{\text{cyl}}(\Lambda_\theta)}, \quad \text{dom } Q_{\omega, \theta, 0} := C^\infty_0(\Lambda_\theta). \quad (1.6)$$

The quadratic form $Q_{\omega, \theta, 0}$ is closable (see [19, theorem VI.1.27]) and in the sequel, it is convenient to have a special notation for the closure of $Q_{\omega, \theta, 0}$

$$Q_{\omega, \theta} := \overline{Q_{\omega, \theta, 0}}. \quad (1.7)$$

Now we are in a position to introduce the main object of this paper.

\textbf{Definition 1.1.} The self-adjoint operator $H_{\omega, \theta}$ in $L^2_{\text{cyl}}(\Lambda_\theta)$ associated with the form $Q_{\omega, \theta}$ via the first representation theorem [19, theorem VI.2.1] is regarded as the Aharonov-Bohm magnetic Dirichlet Laplacian on the conical layer $\Lambda_\theta$. 
The Hamiltonian $H_{\omega, \theta}$ can be seen as an idealization for a more physically realistic self-adjoint Hamiltonian $H_{\omega, \theta, W}$ associated with the closure of the quadratic form

$$u \in C_0^\infty (\mathbb{R}_+^2 \times S^1) \mapsto \|(i\nabla - A_\omega)u\|_{L_2^{\text{cyl}}(\mathbb{R}^3)}^2 + (Wu, u)_{L_2^{\text{cyl}}(\mathbb{R}^3)}$$

where the potential $W: \mathbb{R}_+^2 \times S^1 \to \mathbb{R}$ is a piecewise constant function given by

$$W(r, z, \phi) = \begin{cases} 0, & (r, z, \phi) \in \Lambda_\theta, \\ W_0, & (r, z, \phi) \not\in \Lambda_\theta. \end{cases}$$

The strong resolvent convergence of $H_{\omega, \theta, W}$ to $H_{\omega, \theta}$ in the limit $W_0 \to +\infty$ follows from the monotone convergence for quadratic forms [29, § VIII.7].

Without loss of generality, we reduce to the case of constant $\omega \in (0, 1/2]$. Indeed, performing an adequate gauge transform, one notices that for $\omega \in L^2(S^1)$ the operator $H_{\omega, \theta}$ is unitarily equivalent to $H_{\Phi_{\omega} + k, \theta}$ (for any $k \in \mathbb{Z}$) and for symmetry reason, we are left with the study for constant $\omega \in [0, 1/2]$. However, the case $\omega = 0$ reduces to the problem analysed in [10, 13, 16] so we exclude it from our considerations.

For $\omega \in (0, 1/2]$ the quadratic form $Q_{\omega, \theta}$ associated with $H_{\omega, \theta}$ simply reads

$$Q_{\omega, \theta}[u] = \int_{\Lambda_\theta} \left( |\partial_r u|^2 + |\partial_z u|^2 + \frac{|i\partial_\phi u - \omega u|^2}{r^2} \right) r dr dz d\phi.$$  

Decomposing along the usual Hilbert space basis $\{((2\pi)^{-1/2} e^{im\phi})_{m \in \mathbb{Z}} \}$ of $L^2(S^1)$ and according to the terminology of [28, § XIII.16], $H_{\omega, \theta}$ rewrites as

$$H_{\omega, \theta} \cong \bigoplus_{m \in \mathbb{Z}} F_{\omega, \theta}^{[m]}, \quad (1.8)$$

where the symbol $\cong$ stands for the unitary equivalence relation and, for all $m \in \mathbb{Z}$, the operators $F_{\omega, \theta}^{[m]}$ acting on $L^2(\Gamma_\theta; r dr dz)$ are the fibers of $H_{\omega, \theta}$. They are associated through the first representation theorem with the closed, densely defined, symmetric non-negative quadratic forms

$$f_{\omega, \theta}^{[m]}[u] := \int_{\Gamma_\theta} \left( |\partial_r u|^2 + |\partial_z u|^2 + \frac{(m - \omega)^2}{r^2} |u|^2 \right) r dr dz. \quad (1.9)$$

The form domains, being naturally deduced from the decomposition in fibers (see [2, § II.3.a]).

Finally, we introduce the unitary operator $U: L^2(\Gamma_\theta; r dr dz) \to L^2(\Gamma_\theta), Uu := \sqrt{r}u$. This unitary operator allows to transform the quadratic forms $f_{\omega, \theta}^{[m]}$ into other ones expressed in a flat metric. Indeed, the quadratic form $f_{\omega, \theta}^{[m]}$ is unitarily equivalent
We would like to emphasize that (1.11) does not hold for \( \omega = 0 \) but we excluded this case from our considerations.

It will be handy in what follows to drop the superscript [0] for \( m = 0 \) and to set
\[
F_{\omega,\theta} := f_{\omega,\theta}^0, \quad f_{\omega,\theta} := f_{\omega,\theta}^0, \quad q_{\omega,\theta} := q_{\omega,\theta}^0.
\]

### 1.3. Main results

We introduce a few notations before stating the main results of this paper. The set of positive integers is denoted by \( \mathbb{N} := \{1, 2, \ldots \} \) and the set of natural integers is denoted by \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Let \( T \) be a semi-bounded self-adjoint operator associated with the quadratic form \( t \). We denote by \( \sigma_{\text{ess}}(T) \) and \( \sigma_{\text{disc}}(T) \) the essential and the discrete spectrum of \( T \), respectively. By \( \sigma(T) \), we denote the spectrum of \( T \) (i.e. \( \sigma(T) = \sigma_{\text{ess}}(T) \cup \sigma_{\text{disc}}(T) \)).

Let \( t_1 \) and \( t_2 \) be two quadratic forms of domains \( \text{dom}(t_1) \) and \( \text{dom}(t_2) \), respectively. We say that we have the form ordering \( t_1 \prec t_2 \) if
\[
\text{dom}(t_2) \subset \text{dom}(t_1) \quad \text{and} \quad t_1[u] \leq t_2[u], \text{ for all } u \in \text{dom}(t_2).
\]

We set \( E_{\text{ess}}(T) := \inf \sigma_{\text{ess}}(T) \) and, for \( k \in \mathbb{N} \), \( E_k(T) \) denotes the \( k \)-th Rayleigh quotient of \( T \), defined as
\[
E_k(T) = \sup_{u_1, \ldots, u_{k-1} \in \text{dom } t} \inf_{u \in \text{span}(u_1, \ldots, u_{k-1}) \setminus \{0\}} \frac{t[u]}{\|u\|^2}.
\]

From the min-max principle (see e.g. [28, Chap. XIII]), we know that if \( E_k(T) \in (-\infty, E_{\text{ess}}(T)) \), the \( k \)-th Rayleigh quotient is a discrete eigenvalue of finite multiplicity. Especially, we have the following description of the discrete spectrum below \( E_{\text{ess}}(T) \)
\[
\sigma_{\text{disc}}(T) \cap (-\infty, E_{\text{ess}}(T)) = \{ E_k(T) : k \in \mathbb{N}, E_k(T) < E_{\text{ess}}(T) \}.
\]

Consequently, if \( E_k(T) \in \sigma_{\text{disc}}(T) \), it is the \( k \)-th eigenvalue with multiplicity taken into account. We define the counting function of \( T \) as
\[
\mathcal{N}_E(T) := \#(k \in \mathbb{N} : E_k(T) < E), \quad E \in E_{\text{ess}}(T).
\]

When working with the quadratic form \( t \), we use the notations \( \sigma_{\text{ess}}(t) \), \( \sigma_{\text{disc}}(t) \), \( \sigma(t) \), \( E_{\text{ess}}(t) \), \( E_k(t) \) and \( \mathcal{N}_E(t) \) instead.
Our first result gives a description of the essential spectrum of $H_{\omega,\theta}$.

**Theorem 1.2.** Let $\theta \in (0, \pi/2)$ and $\omega \in (0, 1/2]$. There holds,

$$\sigma_{\text{ess}}(H_{\omega,\theta}) = [1, +\infty).$$

The minimum at 1 of the essential spectrum is a consequence of the normalization of the width of $\Lambda_{\theta}$ to $\pi$. The method of the proof of theorem 1.2 relies on a construction of singular sequences as well as on form decomposition techniques. A similar approach is used e.g. in [6, 13, 16] for Dirichlet conical layers without magnetic fields and in [1] for Schrödinger operators with $\delta$-interactions supported on conical surfaces. In this paper, we simplify the argument by constructing singular sequences in the generalized sense [23] on the level of quadratic forms.

Now we state a proposition that gives a lower bound on the spectra of the fibers $F_{\omega,\theta}^m$ with $m \neq 0$.

**Proposition 1.3.** Let $\theta \in (0, \pi/2)$ and $\omega \in (0, 1/2]$. There holds

$$\inf \sigma(F_{\omega,\theta}^m) \geq 1, \quad \forall \ m \neq 0.$$  

Relying on this proposition and on theorem 1.2, we see that the investigation of the discrete spectrum of $H_{\omega,\theta}$ reduces to the axisymmetric fibre $F_{\omega,\theta}$ of decomposition (1.8). When there is no magnetic field ($\omega = 0$) this result can be found in [16, proposition 3.1]. An analogous statement holds also for $\delta$-interactions supported on conical surfaces [24, proposition 2.5].

Now, we formulate a result on the ordering between Rayleigh quotients.

**Proposition 1.4.** Let $0 < \theta_1 \leq \theta_2 < \pi/2$, $\omega_1 \in (0, 1/2]$, and $\omega_2 \in [\cos \theta_2 (\cos \theta_1)^{-1}\omega_1, 1/2]$. Then

$$E_k(F_{\omega_1,\theta_1}) \leq E_k(F_{\omega_2,\theta_2}).$$

holds for all $k \in \mathbb{N}$.

If the Rayleigh quotients in proposition 1.4 are indeed eigenvalues, we get immediately an ordering of the eigenvalues for different apertures $\theta$ and values of $\omega$. In particular, if $\omega_1 = \omega_2$, we obtain that the Rayleigh quotients are non-decreasing functions of the aperture $\theta$. The latter property is reminiscent of analogous results for broken waveguides [9, proposition 3.1] and for Dirichlet conical layers without magnetic fields [10, proposition 1.2]. A similar claim also holds for $\delta$-interactions supported on broken lines [15, proposition 5.12] and on conical surfaces [24, proposition 1.3]. The new aspect of proposition 1.4 is that we obtain a monotonicity result with respect to two parameters. Proposition 1.4 implies that the eigenvalues are non-decreasing if we weaken the magnetic field and compensate by making the aperture of the conical layer smaller and *vice versa*.

The next theorem is the first main result of this paper.
Theorem 1.5. Let $\theta \in (0, \pi/2)$ and $\omega \in [0, 1/2]$. The following statements hold.

(i) For $\cos \theta \leq 2\omega$, $\#\sigma_{\text{disc}}(F_{\omega, \theta}) = 0$.

(ii) For $\cos \theta > 2\omega$, $\#\sigma_{\text{disc}}(F_{\omega, \theta}) = \infty$ and

$$ N_{1-E}(F_{\omega, \theta}) = \frac{\sqrt{\cos^2 \theta - 4\omega^2}}{4\pi \sin \theta} |\ln E| + O(1), \quad E \to 0 +. $$

For a fixed $\theta \in (0, \pi/2)$, theorem 1.5 yields the existence of a critical flux

$$ \omega_{cr} = \omega_{cr}(\theta) := \frac{\cos \theta}{2} $$

(1.13)

at which the number of eigenvalues undergoes an abrupt transition from infinity to zero. This is, to our knowledge, the first example of a geometrically non-trivial model that exhibits such a behaviour. In comparison, in the special case $\omega = 0$, this phenomenon arises at $\theta = \pi/2$ which is geometrically simple because the domain $\Lambda_\pi/2$ can be seen in the Cartesian coordinates as the layer between two parallel planes at distance $\pi$.

The spectral asymptotics proven in theorem 1.5 (ii) is reminiscent of [10, theorem 1.4]. However, it can be seen that the magnetic field enters the coefficient in front of the main term. As a slight improvement upon [10, theorem 1.4], in theorem 1.5 we explicitly state that the remainder in this asymptotics is just $O(1)$.

The main new feature in theorem 1.5, compared with the previous publications on the subject is the absence of the discrete spectrum $F_{\omega, \theta}$ for strong magnetic fields stated in theorem 1.5 (i). This result is achieved by proving a Hardy-type inequality for the quadratic form $q_\theta := q_{\omega_{cr}, \theta}$. This inequality is the second main result of this paper. It is also of independent interest in view of potential applications in the context of the associated heat semigroup (cf. [7, 22]) and Hardy inequalities in conical domains (cf. [11]).

Theorem 1.6 (Hardy-type inequality). Let $\theta \in (0, \pi/2)$. There exists $c > 0$ such that

$$ q_\theta[u] - \|u\|^2_{L^2(\Gamma_\theta)} \geq c \int_{\Gamma_\theta} \frac{r \cos \theta - z \sin \theta}{1 + \frac{r^2}{\sin^2 \theta} \ln \left( \frac{r}{\cos \theta} \frac{r - z \sin \theta}{\cos \theta - z \sin \theta} \right)} |u|^2 \, drdz \quad (1.14) $$

holds for any $u \in C^\infty_0(\Gamma_\theta)$.

Finally, we point out that theorem 1.6 implies that for any $V \in C^\infty_0(\Lambda_\theta)$

$$ \#\sigma_{\text{disc}}(H_{\omega_{cr}, \theta} - \mu V) = 0 \quad (1.15) $$

holds for all sufficiently small $\mu > 0$. This observation can be extended to some potentials $V \in C^\infty_0(\overline{\Lambda_\theta})$, but we cannot derive (1.15) for any $V \in C^\infty_0(\overline{\Lambda_\theta})$ from theorem 1.6, because the weight on the right-hand side of (1.14) vanishes on the part of $\partial \Gamma_\theta$ satisfying $r = z \tan \theta$. It is an open question whether a global Hardy inequality with weight non-vanishing on the whole $\partial \Gamma_\theta$ can be proven.
1.4. Structure of the paper

In §2, we prove theorem 1.2 about the structure of the essential spectrum. In §3, we reduce the analysis of the discrete spectrum of $H_{\omega,\theta}$ to the discrete spectrum of its axisymmetric fibre, prove proposition 1.4 about inequalities between the Rayleigh quotients, and theorem 1.5 (ii) on infiniteness of the discrete spectrum and its spectral asymptotics. Theorem 1.5 (i) on the absence of discrete spectrum and theorem 1.6 on a Hardy-type inequality are proven in §4.

2. Essential spectrum

In this section, we prove theorem 1.2 on the structure of the essential spectrum of $H_{\omega,\theta}$. Observe that for any $m \neq 0$ the form ordering $f_{\omega,\theta} < f^{|m|}_{\omega,\theta}$ follows directly from (1.9). Hence, according to decomposition (1.8), to prove theorem 1.2 it suffices only to verify $\sigma_{\text{ess}}(f_{\omega,\theta}) = [1, +\infty)$ which is equivalent to checking that $\sigma_{\text{ess}}(q_{\omega,\theta}) = [1, +\infty)$.

To simplify the argument, we reformulate the problem in another set of coordinates performing the rotation

$$s = z \cos \theta + r \sin \theta, \quad t = -z \sin \theta + r \cos \theta,$$

that transforms the meridian domain $\Gamma_\theta$ into the half-strip with corner $\Omega_\theta$ (see figure 2) defined by

$$\Omega_\theta = \{(s, t) \in \mathbb{R} \times (0, \pi) : s > -t \cot \theta \}. \quad (2.2)$$

In the sequel of this subsection, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and the norm on $L^2(\Omega_\theta)$, respectively.

Rotation (2.1) naturally defines a unitary operator

$$U_\theta : L^2(\Omega_\theta) \to L^2(\Gamma_\theta), \quad (U_\theta u)(r, z) := u(z \cos \theta + r \sin \theta, -z \sin \theta + r \cos \theta), \quad (2.3)$$

and induces a new quadratic form

$$h_{\omega,\theta}[u] := q_{\omega,\theta}[U_\theta u] = \int_{\Omega_\theta} \left( \left| \partial_s u \right|^2 + \left| \partial_t u \right|^2 - \frac{\gamma |u|^2}{(s + t \cot \theta)^2} \right) dsdt,$$

$$\text{dom} \ h_{\omega,\theta} := H^1_0(\Omega_\theta), \quad (2.4a)$$

where $\gamma = \gamma(\omega, \theta) := \frac{1/4 - \omega^2}{\sin^2 \theta}. \quad (2.4b)$

Since the form $h_{\omega,\theta}$ is unitarily equivalent to $q_{\omega,\theta}$, proving theorem 1.2 is equivalent to showing that $\sigma_{\text{ess}}(h_{\omega,\theta}) = [1, +\infty)$. We split this verification into checking the two inclusions.

2.1. The inclusion $\sigma_{\text{ess}}(h_{\omega,\theta}) \supset [1, +\infty)$

We verify this inclusion by constructing singular sequences for $h_{\omega,\theta}$ in the generalized sense [23, App. A] for every point of the interval $[1, +\infty)$. Let us start by
fixing a function $\chi \in C^\infty_0(1,2)$ such that $\|\chi\|_{L^2(1,2)} = 1$. For all $p \in \mathbb{R}_+$, we define the functions $u_{n,p}: \Omega_\theta \to \mathbb{C}$, $n \in \mathbb{N}$, as

$$u_{n,p}(s,t) := \left( \frac{1}{\sqrt{n}} \chi\left( \frac{s}{n} \right) \exp(ip s) \right) \left( \sqrt{\frac{2}{\pi}} \sin(t) \right).$$ (2.5)

According to (1.11) it is not difficult to check that $u_{n,p} \in \text{dom} h_{\omega,\theta}$. It is also convenient to introduce the associated functions $v_{n,p}, w_{n,p}: \Omega_\theta \to \mathbb{C}$, $n \in \mathbb{N}$, as

$$v_{n,p}(s,t) := \left( \frac{1}{n^{3/2}} \chi'\left( \frac{s}{n} \right) \exp(ip s) \right) \left( \sqrt{\frac{2}{\pi}} \sin(t) \right),$$

$$w_{n,p}(s,t) := \left( \frac{1}{\sqrt{n}} \chi\left( \frac{s}{n} \right) \exp(ip s) \right) \left( \sqrt{\frac{2}{\pi}} \cos(t) \right).$$

First, we get

$$\|u_{n,p}\|^2 = \frac{2}{\pi} \int_0^\pi \int_{-n}^{2n} \frac{1}{n} \left| \chi\left( \frac{s}{n} \right) \right|^2 \sin^2(t)dsdt = 1.$$ (2.6)

$$\|v_{n,p}\|^2 = \frac{2}{\pi n^2} \int_0^\pi \int_{-n}^{2n} \frac{1}{n} \left| \chi'\left( \frac{s}{n} \right) \right|^2 \sin^2(t)dsdt = \frac{1}{n^2} \|\chi'\|^2_{L^2(1,2)} \to 0, \quad n \to \infty.$$ (2.7)

Further, we compute the partial derivatives $\partial_s u_{n,p}$ and $\partial_t u_{n,p}$

$$(\partial_s u_{n,p})(s,t) = ipu_{n,p}(s,t) + v_{n,p}(s,t), \quad (\partial_t u_{n,p})(s,t) = w_{n,p}(s,t),$$ (2.8)

and we define an auxiliary potential by

$$V_{\omega,\theta}(s,t) := \frac{\gamma(\omega,\theta)}{(s + t \cot \theta)^2}.$$ (2.9)
For any $\phi \in \mathfrak{h}_{\omega, \theta}$, we have
\[
I_{n, p}(\phi) := \mathfrak{h}_{\omega, \theta}[\phi, u_{n, p}] - (1 + p^2)\langle \phi, u_{n, p} \rangle
\]
\[
= \langle \nabla \phi, \nabla u_{n, p} \rangle - \langle V_{\omega, \theta} \phi, u_{n, p} \rangle - (1 + p^2)\langle \phi, u_{n, p} \rangle
\]
\[
= \left( \langle \nabla \phi, (i p u_{n, p}) \rangle \right) - (1 + p^2)\langle \phi, u_{n, p} \rangle
\]
\[
=: J_{n, p}(\phi)
\]
\[
+ \left( \langle \nabla \phi, (\nu_{n, p} \sqrt{\omega_{n, p}}) \rangle \right) - \langle V_{\omega, \theta} \phi, u_{n, p} \rangle.
\]
Integrating by parts and applying the Cauchy-Schwarz inequality, we obtain
\[
|J_{n, p}(\phi)| = \left| -\langle \phi, ip \delta_s u_{n, p} + \partial_t \nu_{n, p} \rangle - (1 + p^2)\langle \phi, u_{n, p} \rangle \right|
\]
\[
= \left| \langle \phi, p^2 u_{n, p} + u_{n, p} \rangle - (1 + p^2)\langle \phi, u_{n, p} \rangle - \langle \phi, ip \nu_{n, p} \rangle \right| = |\langle \phi, ip \nu_{n, p} \rangle|
\]
\[
\leq p\|\phi\|\|\nu_{n, p}\|.
\]
Applying the Cauchy-Schwarz inequality once again and using (2.6) and (2.8), we get
\[
|K_{n, p}(\phi)| \leq \|\phi\| \sup_{(s, t) \in (n, 2n) \times (0, \pi)} |V_{\omega, \theta}(s, t)| + \|\nabla \phi\|\|\nu_{n, p}\|
\]
\[
= \frac{\gamma}{n^2} \|\phi\| + \|\nabla \phi\|\|\nu_{n, p}\|.
\]
Let us define the norm $\| \cdot \|_{+1}$ as
\[
\|\phi\|_{+1} := \mathfrak{h}_{\omega, \theta}[\phi] + \|\phi\|^2, \quad \phi \in \text{dom} \mathfrak{h}_{\omega, \theta}.
\]
Clearly, $\|\phi\|_{+1} \geq \|\phi\|$ and, moreover, for sufficiently small $\varepsilon > 0$, it holds
\[
\omega(\varepsilon) := \sqrt{1/4 + (1 - \varepsilon)^{-1}(\omega^2 - 1/4)} \in (0, 1/2]
\]
and
\[
\|\phi\|^2_{+1} \geq \mathfrak{h}_{\omega, \theta}[\phi] = \varepsilon\|\nabla \phi\|^2 + (1 - \varepsilon)\mathfrak{h}_{\omega(\varepsilon), \theta}[\phi] \geq \varepsilon\|\nabla \phi\|^2,
\]
where we used $\mathfrak{h}_{\omega(\varepsilon), \theta}[\phi] \geq 0$ in the last step. Therefore, for any $\phi \in \text{dom} \mathfrak{h}_{\omega, \theta}$, $\phi \neq 0$, we have by (2.7)
\[
\frac{|I_{n, p}(\phi)|}{\|\phi\|_{+1}} \leq \frac{|J_{n, p}(\phi)|}{\|\phi\|_{+1}} + \frac{|K_{n, p}(\phi)|}{\|\phi\|_{+1}} \leq p\|\nu_{n, p}\| + \frac{\gamma}{n^2} + \varepsilon^{-1/2}\|\nu_{n, p}\| \to 0, \quad n \to \infty.
\]
(2.10)
Here, the upper bound on $((I_{n, p}(\phi))/\|\phi\|_{+1})$ is given by a vanishing sequence which is independent of $\phi$.

Since the supports of $u_{2k, p}$ and $u_{2k+1, p}$ with $k \neq 1$ are disjoint, the sequence $\{u_{2k, p}\}$ converges weakly to zero. Hence, (2.6) and (2.10) imply that $\{u_{2k, p}\}$ is a singular sequence in the generalized sense [23, App. A] for $\mathfrak{h}_{\omega, \theta}$ corresponding to the point $1 + p^2$. Therefore, by [23, theorem 5], $1 + p^2 \in \sigma_{\text{ess}}(\mathfrak{h}_{\omega, \theta})$ for all $p \in \mathbb{R}_+$ and it follows that $[1, +\infty) \subset \sigma_{\text{ess}}(\mathfrak{h}_{\omega, \theta})$. 
2.2. The inclusion $\sigma_{\text{ess}}(h_{\omega,\theta}) \subset [1, +\infty)$

We check this inclusion using the form decomposition method. For $n \in \mathbb{N}$, we define two subsets of $\Omega_{\theta}$

$$\Omega_+^n := \{(s, t) \in \Omega_{\theta} : s < n\}, \quad \Omega_-^n := \{(s, t) \in \Omega_{\theta} : s > n\},$$

as shown in figure 3. For the sake of simplicity, we do not indicate the dependence of $\Omega_+^n$ on $\theta$. We also introduce

$$U_n := \{(s, t) \in \Omega_{\theta} : s = n\}.$$  

For $u \in L^2(\Omega_{\theta})$, we set $u^\pm := u|_{\Omega_{\pm}^n}$. Further, we introduce the Sobolev-type spaces

$$H^{1}_{0,N}(\Omega_{\pm}^n) := \{u \in H^1(\Omega_{\pm}^n) : u|_{\partial \Omega_{\pm}^n \setminus U_n} = 0\}$$

and consider the following quadratic forms

$$h_{\omega,\theta,n}^\pm[u] := \int_{\Omega_{\pm}^n} (|\partial_s u^\pm|^2 + |\partial_t u^\pm|^2 - V_{\omega,\theta}|u^\pm|^2) \, ds\, dt,$$

$$\text{dom} \ h_{\omega,\theta,n}^\pm := H^{1}_{0,N}(\Omega_{\pm}^n),$$

where $V_{\omega,\theta}$ is as in (2.9).

One can verify that the form $h_{\omega,\theta,n}^\pm$ is closed, densely defined, symmetric and semibounded from below in $L^2(\Omega_{\pm}^n)$.

Due to the compact embedding of $H^{1}_{0,N}(\Omega_{\pm}^n)$ into $L^2(\Omega_{\pm}^n)$ the spectrum of $h_{\omega,\theta,n}^\pm$ is purely discrete. The spectrum of $h_{\omega,\theta,n}^\pm$ can be estimated from below as follows

$$\inf \sigma(h_{\omega,\theta,n}^-) \geq 1 - \sup_{(s,t) \in \Omega_{\pm}^n} V_{\omega,\theta}(s, t) = 1 - \frac{\gamma}{n^2}. \quad (2.14)$$

The discreteness of the spectrum for $h_{\omega,\theta,n}^\pm$ and the estimate (2.14) imply that

$$\inf \sigma_{\text{ess}}(h_{\omega,\theta,n}^\pm) = 1 - \frac{\gamma}{n^2}.$$  

Notice that the ordering $h_{\omega,\theta,n}^\pm \equiv h_{\omega,\theta,n}^- \prec h_{\omega,\theta}$ holds. Hence, by the min-max principle, we have

$$\inf \sigma_{\text{ess}}(h_{\omega,\theta}) \geq \inf \sigma_{\text{ess}}(h_{\omega,\theta,n}^\pm) \geq 1 - \frac{\gamma}{n^2},$$

and passing to the limit $n \to \infty$ we get $\inf \sigma_{\text{ess}}(h_{\omega,\theta}) \geq 1$. 

3. Discrete spectrum

The aim of this section is to discuss the properties of the discrete spectrum of \( H_{\omega, \theta} \), which has the physical meaning of quantum bound states. In § 3.1, we reduce the study of the discrete spectrum of \( H_{\omega, \theta} \) to its axisymmetric fibre \( F_{\omega, \theta} \) introduced in (1.12). Then, in § 3.2, we prove proposition 1.4 about the ordering of the Rayleigh quotients. Finally, in § 3.3, we are interested in the asymptotics of the counting function in the regime \( \omega \in (0, \omega_{cr}(\theta)) \) and we give a proof of theorem 1.5 (ii).

3.1. Reduction to the axisymmetric operator

The goal of this subsection is to prove proposition 1.3. In the proof, we use the strategy developed in [10, 16] for Dirichlet conical layers without magnetic fields.

Consider the quadratic forms in the flat metric \( q_{\omega, \theta}^{[m]} \) given in (1.10). For all \( m \neq 0 \) and \( \omega \in (0, 1/2] \), we have \((m - \omega)^2 \geq 1/4\). Consequently, for any \( u \in H^1_0(\Gamma_\theta) \), we get
\[
q_{\omega, \theta}^{[m]}[u] \geq \| \nabla u \|^2_{L^2(\Gamma_\theta)}.
\]
(3.1)

Any function \( u \in H^1_0(\Gamma_\theta) \) can be extended by zero to the strip
\[
\mathcal{S}_\theta := \left\{ (r, z) \in \mathbb{R}^2 : z \tan \theta < r < z \tan \theta + \frac{\pi}{\cos \theta} \right\},
\]
defining a function \( u_0 \in H^1_0(\mathcal{S}_\theta) \). Hence, inequality (3.1) can be re-written as
\[
q_{\omega, \theta}^{[m]}[u] \geq \| \nabla u_0 \|^2_{L^2(\mathcal{S}_\theta)}.
\]
The right-hand side of the last inequality is the quadratic form of the two-dimensional Dirichlet Laplacian in a strip of width \( \pi \). The spectrum of this operator is only essential and equals \([1, +\infty)\). Hence, by the min-max principle, we get
\[
q_{\omega, \theta}^{[m]}[u] \geq \| u_0 \|^2_{L^2(\mathcal{S}_\theta)} = \| u \|^2_{L^2(\Gamma_\theta)}.
\]
Finally, applying the min-max principle to the quadratic form \( q_{\omega, \theta}^{[m]} \), we obtain
\[
\inf \sigma(q_{\omega, \theta}^{[m]}) \geq 1.
\]
This achieves the proof of proposition 1.3.

3.2. Rayleigh quotients inequalities

The aim of this subsection is to prove proposition 1.4. This proof follows the same strategy as the proof of a related statement about broken waveguides developed in [9, § 3].

It will be more convenient to work with the quadratic form \( f_{\omega, \theta} \) in the non-flat metric. Let the domain \( \Omega_\theta \) be defined as in (2.2) through rotation (2.1). This rotation induces a unitary operator \( R_\theta : L^2(\Gamma_\theta; r dr dz) \to L^2(\Omega_\theta; (s \sin \theta + t \cos \theta) ds dt) \).
For \( u \in \text{dom} f_{\omega, \theta} \), we set \( \tilde{u}(s, t) = u(r, z) \) and obtain the identity \( f_{\omega, \theta}[u] = \tilde{f}_{\omega, \theta}[\tilde{u}] \) with the new quadratic form

\[
\tilde{f}_{\omega, \theta}[\tilde{u}] := \int_{\Omega_\theta} \left( |\partial_s \tilde{u}|^2 + |\partial_t \tilde{u}|^2 + \frac{\omega^2 |\tilde{u}|^2}{(s \sin \theta + t \cos \theta)^2} \right) (s \sin \theta + t \cos \theta) d \omega d \theta,
\]

\[
\text{dom} \tilde{f}_{\omega, \theta} := \mathcal{R}_\theta (\text{dom} f_{\omega, \theta}),
\]

which is unitarily equivalent to \( f_{\omega, \theta} \). Now, in order to get rid of the dependence on \( \theta \) of the integration domain \( \Omega_\theta \), we perform the change of variables \((s, t) \mapsto (\hat{s}, \hat{t}) = (s \tan \theta, t)\) that transforms the domain \( \Omega_\theta \) into \( \Omega := \Omega_{\pi/4} \). Setting \( \tilde{u}(\hat{s}, \hat{t}) = \tilde{u}(s, t) \) we get for the Rayleigh quotients

\[
\frac{f_{\omega, \theta}[u]}{\|u\|_{L^2(F_{\omega, \theta})}^2} = \frac{\int_{\Omega} (\tan^2 \theta |\partial_s \hat{u}|^2 + |\partial_t \hat{u}|^2 + \omega^2 \cos^{-2} \theta (\hat{s} + \hat{t})^{-2} |\hat{u}|^2) (\hat{s} + \hat{t}) \cos \theta \cot \theta d \hat{s} d \hat{t}}{\int_{\Omega} |\hat{u}|^2 (\hat{s} + \hat{t}) d \hat{s} d \hat{t}}
\]

\[
= \frac{\int_{\Omega} \tilde{f}_{\omega, \theta}[\tilde{u}]}{\int_{\Omega} |\tilde{u}|^2 (\hat{s} + \hat{t}) d \hat{s} d \hat{t}}.
\]

The domain of the quadratic form \( \tilde{f}_{\omega, \theta} \) does not depend on \( \theta \). However, we transferred the dependence on \( \theta \) into the expression of \( \tilde{f}_{\omega, \theta}[\tilde{u}] \). Now, let \( 0 < \theta_1 < \theta_2 < \pi/2 \), \( \omega_1 \in (0, 1/2) \) and \( \omega_2 \in \left[ \cos \theta_2 \left( \cos \theta_1 \right)^{-1} \omega_1, 1/2 \right] \). Then we get

\[
\tilde{f}_{\omega_2, \theta_2}[\tilde{u}] - \tilde{f}_{\omega_1, \theta_1}[\tilde{u}] = \int_{\Omega} \left[ (\tan^2 \theta_2 - \tan^2 \theta_1) |\partial_s \hat{u}|^2 + \frac{\omega_2^2}{\cos^2 \theta_2} - \frac{\omega_1^2}{\cos^2 \theta_1} \right] |\hat{u}|^2 (\hat{s} + \hat{t}) d \hat{s} d \hat{t}.
\]

Since the tangent is an increasing function, the first term on the right-hand side is non-negative. As \( \omega_2 \) is chosen, the second term is also non-negative. Therefore, for any \( k \in \mathbb{N} \), the min-max principle and (3.2) yield \( E_k(\tilde{f}_{\omega_1, \theta_1}) \leq E_k(\tilde{f}_{\omega_2, \theta_2}) \) which is equivalent to

\[
E_k(F_{\omega_1, \theta_1}) \leq E_k(F_{\omega_2, \theta_2}).
\]

This achieves the proof of proposition 1.4.

### 3.3. Asymptotics of the counting function

This subsection is devoted to the proof of theorem 1.5 (ii). All along this subsection, \( \theta \in (0, \pi/2) \) and \( \omega \in (0, \omega_{\text{cr}}(\theta)) \) with \( \omega_{\text{cr}}(\theta) = (1/2) \cos \theta \) as in (1.13). The proof follows the same steps as in [10, §3]. However, in the presence of a magnetic field the proof simplifies because instead of working with the form \( f_{\omega, \theta} \) introduced...
in (1.9), we can work with the unitarily equivalent quadratic form \( h_{\omega, \theta} \) defined in (2.4a). In particular, we avoid using IMS localization formula.

The main idea is to reduce the problem to the known spectral asymptotics of one-dimensional operators. To this aim, first, we recall the result of [20], later extended in [18]. Further, let \( \gamma > 0 \) be fixed. We are interested in the spectral properties of the self-adjoint operators acting on \( L^2(1, +\infty) \) associated with the closed, densely defined symmetric and semi-bounded quadratic form,

\[
q_N^D[f] := \int_1^\infty |f'(x)|^2 - \frac{\gamma |f(x)|^2}{x^2} \, dx, \quad \text{dom} q_N^D := H^1(1, +\infty),
\]

and with its restriction

\[
q_N^D[f] := q_N^N[f], \quad \text{dom} q_N^D := H^1_0(1, +\infty).
\]

It is well known that \( \sigma_{\text{ess}}(q_N^D) = \sigma_{\text{ess}}(q_N^N) = [0, +\infty) \) and it can be shown by a proper choice of test functions that \( \# \sigma_{\text{disc}}(q_N^D) = \# \sigma_{\text{disc}}(q_N^N) = \infty \) for all \( \gamma > 1/4 \).

**Theorem 3.1** ([20, theorem 1], [18, theorem 1]). As \( E \to 0^+ \) the counting functions of \( q_N^D \) and \( q_N^N \) with \( \gamma > 1/4 \) satisfy

\[
N_{-E}(q_N^D) = \frac{1}{2\pi} \sqrt{\sqrt{\gamma - \frac{1}{4}} |\ln E| + O(1)}, \quad N_{-E}(q_N^N) = \frac{1}{2\pi} \sqrt{\sqrt{\gamma - \frac{1}{4}} |\ln E| + O(1)}.
\]

In proposition 3.2, we establish a lower bound for \( N_{1-E}(h_{\omega, \theta}) \) while an upper bound is obtained in proposition 3.3. Together with theorem 3.1 these bounds yield theorem 1.5 (ii).

Let the sub-domains \( \Omega^\pm := \Omega_1^\pm \) (for \( n = 1 \)) of \( \Omega_\theta \) be as in (2.11) and the Sobolev-type spaces \( H^1_{0, \Omega}(\Omega^\pm) \) be as in (2.12). Let also the quadratic forms \( h_{\omega, \theta}^\pm := h_{\omega, \theta, 1}^\pm \) be as in (2.13). Define the restriction \( h_{\omega, \theta, D}^\pm \) of \( h_{\omega, \theta}^\pm \) by

\[
h_{\omega, \theta, D}^\pm [u] := h_{\omega, \theta}^\pm [u], \quad \text{dom} h_{\omega, \theta, D}^\pm := H^1_0(\Omega^-).
\]

To obtain a lower bound, we use a Dirichlet bracketing technique.

**Proposition 3.2.** Let \( \theta \in (0, \pi/2) \), \( \omega \in (0, \omega_{\text{cr}}(\theta)) \) be fixed and let \( \gamma = \gamma(\omega, \theta) \) be as in (2.4b). For any \( E > 0 \) set \( \hat{E} = (1 + \pi \cot \theta)^2 E \). Then the bound

\[
N_{-E}(q_N^D) \leq N_{1-E}(h_{\omega, \theta}),
\]

holds for all \( E > 0 \).
Proof. Any \( u \in H^1_0(\Omega^-) \) can be extended by zero in \( \Omega_\theta \), defining \( u_0 \in H^1_0(\Omega_\theta) \) such that \( \mathfrak{h}_{\omega,\theta,D}[u] = \mathfrak{h}_{\omega,\theta}[u_0] \). Then, the min-max principle yields

\[
\mathcal{N}_{1-\varepsilon}(\mathfrak{h}_{\omega,\theta,D}) \leq \mathcal{N}_{1-\varepsilon}(\mathfrak{h}_{\omega,\theta}). \tag{3.3}
\]

Now, we bound \( (s + t \cot \theta)^2 \) from above by \( (s + \pi \cot \theta)^2 \) and for any \( u \in H^1_0(\Omega^-) \), we get

\[
\mathfrak{h}_{\omega,\theta,D}[u] \leq \int_{\Omega^-} |\partial_s u|^2 + |\partial_t u|^2 - \frac{\gamma|u|^2}{(s + \pi \cot \theta)^2} \, dsdt. \tag{3.4}
\]

Further, we introduce the quadratic forms for one-dimensional operators

\[
\tilde{q}^D_Y[f] := \int_1^{+\infty} |f'(x)|^2 - \frac{\gamma|f(x)|^2}{(x + \pi \cot \theta)^2} \, dx, \quad \text{dom} \tilde{q}^D_Y := H^1_0(1, +\infty),
\]

\[
q^D_{(0,\pi)}[f] := \int_0^\pi |f'(x)|^2 \, dx, \quad \text{dom} q^D_{(0,\pi)} := H^1_0(0, \pi).
\]

The right-hand side of (3.4) can be represented as \( \tilde{q}^D_Y \otimes i_2 + i_1 \otimes q^D_{(0,\pi)} \) with respect to the tensor product decomposition \( L^2(\Omega^-) = L^2(1, +\infty) \otimes L^2(0, \pi) \) where \( i_1, i_2 \) are the quadratic forms of the identity operators on \( L^2(1, +\infty) \) and on \( L^2(0, \pi) \), respectively. The eigenvalues of \( q^D_{(0,\pi)} \) are given by \( \{k^2\}_{k \in \mathbb{N}} \) and hence

\[
\mathcal{N}_{1-\varepsilon}^{(3.4)}(\tilde{q}^D_Y) \leq \mathcal{N}_{1-\varepsilon}(\mathfrak{h}_{\omega,\theta,D}). \tag{3.5}
\]

Finally, we perform the change of variables \( y = (1 + \pi \cot \theta)^{-1}(x + \pi \cot \theta) \). For all functions \( f \in \text{dom} \tilde{q}^D_Y \), we denote \( g(y) = f(x) \). We get

\[
\frac{\tilde{q}^D_Y[f]}{\int_1^{+\infty} |f'(x)|^2 \, dx} = (1 + \pi \cot \theta)^{-2} \frac{q^D_{(0,\pi)}[g]}{\int_1^{+\infty} |g(y)|^2 \, dy}.
\]

Finally, using (3.3), (3.5) and the min-max principle, we get the desired bound on \( \mathcal{N}_{1-\varepsilon}(\mathfrak{h}_{\omega,\theta}) \). \( \square \)

To obtain an upper bound, we use a Neumann bracketing technique.

**Proposition 3.3.** Let \( \theta \in (0, \pi/2) \) and \( \omega \in (0, \omega_{cr}(\theta)) \) be fixed and let \( \gamma = \gamma(\omega, \theta) \) be as in (2.4b). Then there exists a constant \( C = C(\omega, \theta) > 0 \) such that

\[
\mathcal{N}_{1-\varepsilon}(\mathfrak{h}_{\omega,\theta}) \leq C + \mathcal{N}_{1-\varepsilon}(\mathfrak{h}_{\omega,\theta}^N)
\]

holds for all \( \varepsilon > 0 \).

To prove proposition 3.3, we will need the following two lemmas whose proofs are postponed until the end of the subsection.
Spectral transitions for Aharonov-Bohm Laplacians on conical layers

**Lemma 3.4.** Let $\theta \in (0, \pi/2)$ and $\omega \in (0, \omega_{cr}(\theta))$ be fixed. Then there exists a constant $C = C(\omega, \theta) > 0$ such that

$$N_{1 - E}(h^{+}_{\omega, \theta}) \leq C$$

holds for all $E > 0$.

**Lemma 3.5.** Let $\theta \in (0, \pi/2)$ and $\omega \in (0, \omega_{cr}(\theta))$ be fixed and let $\gamma = \gamma(\omega, \theta)$ be as in (2.4b). Then

$$N_{1 - E}(h^{-}_{\omega, \theta}) \leq N_{- E}(q^{N}_{\gamma})$$

holds for all $E > 0$.

**Proof of Proposition 3.3.** Note that we have the following form ordering

$$h^{+}_{\omega, \theta} \oplus h^{-}_{\omega, \theta} \prec h_{\omega, \theta}$$

and the min-max principle gives

$$N_{1 - E}(h_{\omega, \theta}) \leq N_{1 - E}(h^{+}_{\omega, \theta}) + N_{1 - E}(h^{-}_{\omega, \theta}). \tag{3.6}$$

The statement follows directly combining (3.6), lemmas 3.4 and 3.5. \qed

We conclude this part by the proofs of lemmas 3.4 and 3.5.

**Proof of Lemma 3.4.** Recall that the space $\mathcal{H}^{1}_{0, \mathcal{N}}(\Omega^{+})$ is compactly embedded into $L^{2}(\Omega^{+})$. Consequently, $\sigma(h^{+}_{\omega, \theta})$ is purely discrete and consists of a non-decreasing sequence of eigenvalues of finite multiplicity that goes to $+\infty$. In particular, there exists a constant $C = C(\omega, \theta) > 0$ such that

$$N_{1 - E}(h^{+}_{\omega, \theta}) \leq N_{1}(h^{+}_{\omega, \theta}) \leq C. \quad \Box$$

**Proof of Lemma 3.5.** In $\Omega^{-}$, we can bound $(s + t \cot \theta)^{2}$ from below by $s^{2}$. For any $u \in \text{dom} h^{-}_{\omega, \theta}$, we get

$$\int_{\Omega^{-}} |\partial_{s} u|^{2} + |\partial_{t} u|^{2} - \frac{\gamma |u|^{2}}{s^{2}} \, ds \, dt \leq h^{-}_{\omega, \theta}[u].$$

The left-hand side can be seen as the tensor product $q^{N}_{\gamma} \otimes i_{2} + i_{1} \otimes q^{D}_{(0, \pi)}$ with respect to the decomposition $L^{2}(\Omega^{-}) = L^{2}(1, +\infty) \otimes L^{2}(0, \pi)$ where the form $q^{D}_{(0, \pi)}$ is defined in the proof of proposition 3.2. Since the eigenvalues of $q^{D}_{(0, \pi)}$ are given by $\{k^{2}\}_{k \in \mathbb{N}}$, we deduce that

$$N_{1 - E}(h^{-}_{\omega, \theta}) \leq N_{- E}(q^{N}_{\gamma}). \quad \Box$$
Proof of Theorem 1.5 (ii). Combining propositions 3.2 and 3.3, for any $E > 0$ we get
\[
\mathcal{N}_{-(1 + \pi \cot \theta)^2 E}(q^D_D) \leq \mathcal{N}_{1-E}(h_{\omega,\theta}) \leq C + \mathcal{N}_{-E}(q^N_N). \tag{3.7}
\]
For the lower and upper bounds on $\mathcal{N}_{1-E}(h_{\omega,\theta})$ given in (3.7), theorem 3.1 implies that as $E \to 0+$ holds
\[
C + \mathcal{N}_{-E}(q^N_N) = \frac{1}{2\pi} \sqrt{\gamma - \frac{1}{4}} \ln E + O(1),
\]
\[
\mathcal{N}_{-(1 + \pi \cot \theta)^2 E}(q^D_D) = \frac{1}{2\pi} \sqrt{\gamma - \frac{1}{4}} |\ln((1 + \pi \cot \theta)^2 E)| + O(1)
\]
\[
= \frac{1}{2\pi} \sqrt{\gamma - \frac{1}{4}} |\ln E| + O(1).
\]
Hence, theorem 1.5 (ii) follows from the identity
\[
\sqrt{\gamma - \frac{1}{4}} = \frac{\sqrt{\cos^2 \theta - 4\omega^2}}{2 \sin \theta}.
\]

4. A Hardy-type inequality

The aim of this section is to prove theorem 1.6. Instead of working with the quadratic form $q_{\omega,\theta}$ which is used in the formulation of theorem 1.6 it is more convenient to work with $h_{\omega,\theta}$ defined in (2.4a). We go back to the form $q_{\omega,\theta}$ only at the end of this section. Recall that we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively, the inner product and the norm in $L^2(\Omega_\theta)$.

In this section, we are only interested in the critical case $\omega = \omega_{cr}(\theta) = (1/2) \cos \theta$ for which $\gamma(\omega_{cr}(\theta), \theta) = 1/4$ holds where $\gamma(\omega, \theta)$ is defined in (2.4b). To make the notations more handy, we define $h_{\theta} := h_{\omega_{cr},\theta}$. For further use, for any $(s, t) \in \Omega_\theta$, we introduce
\[
\rho := \rho(s, t) = s + t \cot \theta, \quad \rho_0 := \rho_0(t) = \frac{1}{2} t \cot \theta.
\]
With this notation the domain $\Omega_\theta$ can be represented as
\[
\Omega_\theta = \{(s, t) \in \mathbb{R} \times (0, \pi): s > -2\rho_0(t)\}
\]
and the quadratic form $h_{\theta}$ can be written as
\[
h_{\theta}[u] = \int_{\Omega_\theta} |\partial_s u|^2 + |\partial_t u|^2 - \frac{|u|^2}{4\rho^2} ds dt, \quad \text{dom } h_{\theta} = H^1_0(\Omega_\theta).
\]
The emptiness of the discrete spectrum stated in theorem 1.5 (i) is an immediate consequence of theorem 1.6 and of the min-max principle because for any $\omega \geq \omega_{cr}$ the form ordering $h_{\theta} < h_{\omega,\theta}$ holds. Another consequence of theorem 1.6 is the non-criticality of $H_{\omega,\theta}$ as stated in (1.15).

To prove theorem 1.6, we adapt the strategy developed in [7, § 3]. First, in § 4.1 we prove a local Hardy-type inequality for the quadratic form $h_{\theta}$ taking advantage
of the usual one-dimensional Hardy inequality. Second, in §4.2, we obtain a refined lower bound that allows us, in §4.3, to prove theorem 1.6.

4.1. A local Hardy inequality

Let us introduce the triangle $T_\theta$ (see figure 4), which is a sub-domain of $\Omega_\theta$ defined as

$$T_\theta := \{(s, t) \in \Omega_\theta : s < -\rho_0(t)/2\} = \{(s, t) \in \mathbb{R} \times (0, \pi) : -2\rho_0(t) < s < -\rho_0(t)/2\}.$$

We also need to define the auxiliary function

$$f(t) := \frac{\pi^2}{(\pi - t/4)^2} - 1. \quad (4.1)$$

Note that $f(t) \geq 0$ in $T_\theta$.

**Proposition 4.1.** For any $u \in C_0^\infty(\Omega_\theta)$ the inequality

$$\int_{\Omega_\theta} |\partial_t u|^2 ds dt - \|u\|^2 \geq \int_{T_\theta} f(t)|u|^2 ds dt,$$

holds with $f(\cdot)$ as in (4.1).

Before going through the proof of proposition 4.1, we notice that

$$\eta_\theta[u] - \|u\|^2 = \int_{\Omega_\theta} |\partial_t u|^2 ds dt - \|u\|^2 + \int_{t=0}^\pi \int_{s_- = -t \cot \theta}^{s_+} |\partial_s u|^2 ds dt - \frac{|u|^2}{4\rho^2} ds dt.$$

In fact, the last term on the right-hand side is positive. It can be seen by performing, in the $s$-integral, the change of variable $\sigma = \rho(s, t)$ for any fixed $t \in (0, \pi)$ and using the classical one-dimensional Hardy inequality (see e.g. [19, §VI.4., eq. (4.6)]). Together with proposition 4.1, it gives the following corollary.

**Corollary 4.2** (Local Hardy inequality). For any $u \in C_0^\infty(\Omega_\theta)$ the inequality

$$\eta_\theta[u] - \|u\|^2 \geq \int_{T_\theta} f(t)|u|^2 ds dt,$$

holds with $f(\cdot)$ as in (4.1).
Proof of Proposition 4.1. Let \( u \in C_0^\infty(\Omega_\theta) \). For fixed \( s \in (-\pi \cot \theta, 0) \) the function 

\[ (-s \tan \theta, \pi) \ni t \mapsto u(s, t) \]

satisfies Dirichlet boundary conditions at \( t = -s \tan \theta \) and \( t = \pi \). Let 

\[ \lambda_1(s) := \frac{\pi^2}{(\pi - |s| \tan \theta)^2} \]

be the first eigenvalue of the Dirichlet Laplacian on the interval \((-s \tan \theta, \pi)\). Hence, we get

\[
\int_{\Omega_\theta} |\partial_t u|^2 \mathrm{d}t \mathrm{d}s - \|u\|^2 \geq \int_{\Omega_\theta} (h(s) - 1)|u|^2 \mathrm{d}s \mathrm{d}t,
\]

with

\[
h(s) := \begin{cases} 
\lambda_1(s), & s \in (-\pi \cot \theta, 0), \\
1, & s \in [0, +\infty).
\end{cases}
\]

Particularly, we remark that for any \( s > -\pi \cot \theta \), we have \( h(s) - 1 \geq 0 \). It yields

\[
\int_{\Omega_\theta} |\partial_t u|^2 \mathrm{d}s \mathrm{d}t - \|u\|^2 \geq \int_{T_0} (h(s) - 1)|u|^2 \mathrm{d}s \mathrm{d}t.
\]

Finally, as \( h(\cdot) \) is non-increasing, we obtain

\[
\int_{\Omega_\theta} |\partial_t u|^2 \mathrm{d}s \mathrm{d}t - \|u\|^2 \geq \int_{T_0} (h(s) - 1)|u|^2 \mathrm{d}s \mathrm{d}t
\]

\[
= \int_{t=0}^{\pi} \int_{s=-2\rho_0}^{-\rho_0/2} (h(s) - 1)|u|^2 \mathrm{d}s \mathrm{d}t
\]

\[
\geq \int_{t=0}^{\pi} \int_{s=-2\rho_0}^{-\rho_0/2} (\lambda_1(-\rho_0/2) - 1)|u|^2 \mathrm{d}s \mathrm{d}t
\]

\[
= \int_{T_0} (\lambda_1(-\rho_0/2) - 1)|u|^2 \mathrm{d}s \mathrm{d}t = \int_{T_0} f(t)|u|^2 \mathrm{d}s \mathrm{d}t. \qed
\]

4.2. A refined lower-bound

In this subsection, we prove the following statement.

PROPOSITION 4.3. For any \( \varepsilon \in (0, \pi^{-3}) \)

\[
\int_{\Omega_\theta} |\partial_s u|^2 - \frac{1}{4\rho^2} |u|^2 \mathrm{d}s \mathrm{d}t \geq \frac{\varepsilon}{16} \int_{\Omega_\theta} \frac{t^3}{1 + \rho^2 \ln^2(\rho/\rho_0)} |u|^2 \mathrm{d}s \mathrm{d}t
\]

\[
- \varepsilon \int_{T_0} t^3 \left( \frac{4}{\rho^2_0} + \frac{1}{8} \right) |u|^2 \mathrm{d}s \mathrm{d}t
\]

holds for all \( u \in C_0^\infty(\Omega_\theta) \).

To prove proposition 4.3 we need the following lemma whose proof follows the same lines as the one of [7, lemma 3.1]. However, we provide it here for the sake of
Lemma 4.4. For any fixed $t \in (0, \pi)$ the inequality

$$\int_{s > -\rho_0(t)} |g'(s)|^2 \cdot \frac{1}{4 \rho^2} |g(s)|^2 \, ds \geq \frac{1}{4} \int_{s > -\rho_0(t)} \frac{|g(s)|^2}{\rho^2 \ln^2(\rho/\rho_0)} \, ds$$

holds for all $g \in H^1_0(-\rho_0(t), +\infty)$.

Proof. Let $t \in (0, \pi)$ and $g \in C^\infty_0(-\rho_0(t), +\infty)$ be fixed. We notice that for any $\alpha > 0$

$$\int_{s > -\rho_0} |(\rho^{-1/2} g)' - \frac{\alpha \rho^{-1/2} g'}{\rho \ln(\rho/\rho_0)}|^2 \, ds$$

$$= \int_{s > -\rho_0} |(\rho^{-1/2} g)'|^2 \, ds + \alpha^2 \int_{s > -\rho_0} \frac{|g|^2}{\rho^2 \ln^2(\rho/\rho_0)} \, ds - \alpha \int_{s > -\rho_0} \frac{|(\rho^{-1/2} g)|^2}{\ln(\rho/\rho_0)} \, ds. \quad (4.3)$$

For the first term on the right-hand side in (4.3), we get by (4.2) that

$$\int_{s > -\rho_0} |(\rho^{-1/2} g)'|^2 \, ds = \int_{s > -\rho_0} |g'|^2 - \frac{1}{4 \rho^2} |g|^2 \, ds. \quad (4.4)$$

Performing an integration by parts in the last term of the right-hand side in (4.3), we obtain

$$\int_{s > -\rho_0} \frac{|(\rho^{-1/2} g)|^2}{\ln(\rho/\rho_0)} \, ds = \int_{s > -\rho_0} \frac{|g|^2}{\rho^2 \ln^2(\rho/\rho_0)} \, ds. \quad (4.5)$$

Combining (4.3), (4.4), and (4.5), we get

$$\int_{s > -\rho_0} |g'|^2 - \frac{1}{4 \rho^2} |g|^2 \, ds \geq (\alpha - \alpha^2) \int_{s > -\rho_0} \frac{|g|^2}{\rho^2 \ln^2(\rho/\rho_0)} \, ds.$$

It remains to set $\alpha = 1/2$.

The extension of this result to $g \in H^1_0(-\rho_0(t), +\infty)$ relies on the density of $C^\infty_0(-\rho_0(t), +\infty)$ in $H^1_0(-\rho_0(t), +\infty)$ with respect to the $H^1$-norm and a standard continuity argument.

□
Applying lemma 4.4 and using (4.2), we get

\[ \int_{s > -\rho_0} \frac{|u|^2}{1 + \rho^2 \ln^2(\rho/\rho_0)} \, ds \leq 2 \int_{s > -\rho_0} \frac{|\xi u|^2}{\rho^2 \ln^2(\rho/\rho_0)} \, ds + 2 \int_{s > -\rho_0} |(1 - \xi)u|^2 \, ds, \]

where in both integrals we increased the integrands by making the denominators smaller. Note that for fixed \( a, b \in \mathbb{R} \), we get

\[ \int_{s > -\rho_0} \frac{|u|^2}{1 + \rho^2 \ln^2(\rho/\rho_0)} \, ds \leq 2 \int_{s > -\rho_0} \frac{|\xi u|^2}{\rho^2 \ln^2(\rho/\rho_0)} \, ds + 2 \int_{s > -\rho_0} |(1 - \xi)u|^2 \, ds, \]

Further, for any \( u \in C^\infty_0(\Omega_\theta) \) and fixed \( t \in (0, \pi) \) using \( (a + b)^2 \leq 2a^2 + 2b^2 \), \( a, b \in \mathbb{R} \), we get

\[ \int_{s = -\rho_0}^{\rho_0} \frac{|u|^2}{1 + \rho^2 \ln^2(\rho/\rho_0)} \, ds \leq 8 \int_{s = -\rho_0}^{\rho_0} \frac{|\xi u|^2}{4\rho^2} \, ds + \int_{s = -\rho_0}^{\rho_0} |(1 - \xi)u|^2 \, ds, \]

which is equivalent to

\[ \int_{s > -\rho_0} \left( \partial_s u \right)^2 \, ds \leq \frac{1}{16} \int_{s > -\rho_0} \frac{|u|^2}{1 + \rho^2 \ln^2(\rho/\rho_0)} \, ds \]

Now we have all the tools to prove proposition 4.3.

Proof of Proposition 4.3. First, we define the cut-off function \( \xi : \Omega_\theta \to \mathbb{R} \) by

\[ \xi(s, t) := \begin{cases} 0, & s \in (-2\rho_0(t), -\rho_0(t)), \\ 2\rho_0(t)^{-1}(s + \rho_0(t)), & s \in (-\rho_0(t), -\rho_0(t)/2), \\ 1, & s \in (-\rho_0(t)/2, +\infty). \end{cases} \]

The partial derivative of \( \xi \) with respect to the \( s \)-variable is given by

\[ (\partial_s \xi)(s, t) = \begin{cases} 2\rho_0(t)^{-1}, & s \in (-\rho_0(t), -\rho_0(t)/2), \\ 0, & s \in (-2\rho_0(t), -\rho_0(t)) \cup (-\rho_0(t)/2, +\infty), \end{cases} \]

where in both integrals we increased the integrands by making the denominators smaller. Note that for fixed \( t \in (0, \pi) \), we have \( s \mapsto \xi(s, t)u(s, t) \in H^1_0(-\rho_0(t), +\infty) \). Applying lemma 4.4 and using (4.2), we get

\[ \int_{s > -\rho_0} \frac{|u|^2}{1 + \rho^2 \ln^2(\rho/\rho_0)} \, ds \leq 8 \int_{s > -\rho_0} \frac{|\partial_s (\xi u)|^2}{4\rho^2} \, ds + 2 \int_{s = -\rho_0}^{\rho_0} |(1 - \xi)u|^2 \, ds, \]
Finally, we multiply each side by $\varepsilon t^3$ and integrate for $t \in (0, \pi)$

$$
\int_{\Omega_\varepsilon} \varepsilon t^3 \left( |\partial_s u|^2 - \frac{|u|^2}{4\rho^2} \right) ds dt \geq \frac{\varepsilon}{16} \int_{\Omega_\varepsilon} t^3 \frac{1}{1 + \rho^2 \ln^2(\rho/\rho_0)} |u|^2 ds dt \\
- \varepsilon \int_{\mathcal{T}_\varepsilon} t^3 \left( \frac{4}{\rho_0^2} + \frac{1}{8} \right) |u|^2 ds dt.
$$

Since for any $\varepsilon \in (0, \pi^{-3})$ holds $0 < \varepsilon t^3 < 1$, the inequality in proposition 4.3 follows. □

4.3. Proof of Theorem 1.6

By propositions 4.1 and 4.3, we have

$$
\mathcal{H}_\varepsilon[u] - \|u\|^2 = \int_{\Omega_\varepsilon} \left( |\partial_s u|^2 - \frac{|u|^2}{4\rho^2} \right) ds dt + \int_{\Omega_\varepsilon} |\partial_t u|^2 ds dt \\
\geq \frac{\varepsilon}{16} \int_{\Omega_\varepsilon} t^3 \frac{1}{1 + \rho^2 \ln^2(\rho/\rho_0)} |u|^2 ds dt (4.7)
$$

for all $u \in C_0^\infty(\Omega_\varepsilon)$. For the second term on the right-hand side of (4.7) to be positive it suffices to verify that for all $t \in (0, \pi)$

$$
\mathcal{H}_\varepsilon(t) := f(t) - \frac{16}{\cot^2 \theta} \varepsilon t - \frac{1}{8} \varepsilon t^3 \geq 0. (4.8)
$$

By definition, $f$ in (4.1) is a $C^\infty$-smooth bounded function on $(0, \pi)$ and for any $a \in (0, \pi)$ and all $t \in (a, \pi)$ we have $f(t) \geq f(a) > 0$. Moreover, $f(t) = (2\pi)^{-1} t + \mathcal{O}(t^2)$ when $t \to 0^+$. Consequently, we can find $\varepsilon_0 > 0$ small enough such that for all $\varepsilon \in (0, \varepsilon_0)$ inequality (4.8) holds. Going back to the form $q_\varepsilon[u]$, we get that there exists $c > 0$ such that for any $u \in C_0^\infty(\mathcal{R}_\varepsilon)$ holds

$$
q_\varepsilon[u] - \|u\|_{L^2(\mathcal{R}_\varepsilon)}^2 = \mathcal{H}_\varepsilon[U^{-1}_\varepsilon u] - \|U^{-1}_\varepsilon u\|^2 \\
\geq c \int_{\mathcal{R}_\varepsilon} \frac{t^3}{1 + \rho^2 \ln^2(\rho/\rho_0)} |(U^{-1}_\varepsilon u)(s, t)|^2 ds dt \\
- c \int_{\mathcal{R}_\varepsilon} \frac{r \cos \theta - z \sin \theta}{\sin^2 \theta} \ln^2 \left( \frac{r \cos \theta}{r \cos \theta - z \sin \theta} \right) |u|^2 dr dz,
$$

where we used the unitary transform $U_\varepsilon$ defined in (2.3). This finishes the proof of theorem 1.6.

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