Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials

Dae San Kim¹ and Taekyun Kim²*

¹Correspondence: tkim@kw.ac.kr
²Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract
In this paper, we consider higher-order Frobenius-Euler polynomials, associated with poly-Bernoulli polynomials, which are derived from polylogarithmic function. These polynomials are called higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials. The purpose of this paper is to give various identities of those polynomials arising from umbral calculus.

1 Introduction
For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order $\alpha$ ($\alpha \in \mathbb{R}$) are defined by the generating function to be

$$\left( \frac{1 - \lambda}{e^t - \lambda} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda) \frac{t^n}{n!} \quad \text{(see [1–5]).} \tag{1.1}$$

When $x = 0$, $H_n^{(\alpha)}(\lambda) = H_n^{(\alpha)}(0|\lambda)$ are called the Frobenius-Euler numbers of order $\alpha$. As is well known, the Bernoulli polynomials of order $\alpha$ are defined by the generating function to be

$$\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} \quad \text{(see [6–8]).} \tag{1.2}$$

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ is called the $n$th Bernoulli number of order $\alpha$. In the special case, $\alpha = 1$, $B_n^{(1)}(x) = B_n(x)$ is called the $n$th Bernoulli polynomial. When $x = 0$, $B_n = B_n(0)$ is called the $n$th ordinary Bernoulli number. Finally, we recall that the Euler polynomials of order $\alpha$ are given by

$$\left( \frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!} \quad \text{(see [9–13]).} \tag{1.3}$$

When $x = 0$, $E_n^{(\alpha)} = E_n^{(\alpha)}(0)$ is called the $n$th Euler number of order $\alpha$. In the special case, $\alpha = 1$, $E_n^{(1)}(x) = E_n(x)$ is called the $n$th ordinary Euler polynomial. The classical polylogarithmic function $L_k(x)$ is defined by

$$L_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (k \in \mathbb{Z}) \quad \text{(see [7]).} \tag{1.4}$$
As is known, poly-Bernoulli polynomials are defined by the generating function to be

\[
\frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \quad (\text{cf. [7]}).
\]

(1.5)

Let \( \mathbb{C} \) be the complex number field, and let \( \mathcal{F} \) be the set of all formal power series in the variable \( t \) over \( \mathbb{C} \) with

\[
\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbb{C} \right\}.
\]

(1.6)

Now, we use the notation \( \mathbb{P} = \mathbb{C}[x] \). In this paper, \( \mathbb{P}^* \) will be denoted by the vector space of all linear functionals on \( \mathbb{P} \). Let us assume that \( (L|p(x)) \) be the action of the linear functional \( L \) on the polynomial \( p(x) \), and we remind that the vector space operations on \( \mathbb{P}^* \) are defined by \( (L + M|p(x)) = (L|p(x)) + (M|p(x)) \), \( (cL|p(x)) = c(L|p(x)) \), where \( c \) is a complex constant. The formal power series

\[
f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}
\]

defines a linear functional on \( \mathbb{P} \) by setting

\[
(f(t)|x^n) = a_n, \quad \text{for all } n \geq 0 \text{ (see [14, 15])}.
\]

(1.8)

From (1.7) and (1.8), we note that

\[
(t^k|x^n) = n! \delta_{n,k} \quad \text{(see [14, 15])},
\]

(1.9)

where \( \delta_{n,k} \) is the Kronecker symbol.

Let us consider \( f_i(t) = \sum_{k=0}^{\infty} \frac{(x|t)}{k!} t^k \). Then we see that \( (f_i(t)|x^n) = (L|x^n) \), and so \( L = f_i(t) \) as linear functionals. The map \( L \mapsto f_i(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) will denote both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional (see [14]). We shall call \( \mathcal{F} \) the umbral algebra. The umbral calculus is the study of umbral algebra. The order \( o(f(t)) \) of a nonzero power series \( f(t) \) is the smallest integer \( k \), for which the coefficient of \( t^k \) does not vanish.

A series \( f(t) \) is called a delta series if \( o(f(t)) = 1 \), and an invertible series if \( o(f(t)) = 0 \). Let \( f(t), g(t) \in \mathcal{F} \). Then we have

\[
(f(t)g(t)|p(x)) = (f(t)|g(t)p(x)) = (g(t)|f(t)p(x)) \quad \text{(see [14]).}
\]

(1.10)

For \( f(t), g(t) \in \mathcal{F} \) with \( o(f(t)) = 1 \), \( o(g(t)) = 0 \), there exists a unique sequence \( S_n(x) \) (deg \( S_n(x) = n \)) such that \( \langle g(t)f(t)^k|S_n(x) \rangle = n! \delta_{n,k} \) for \( n, k \geq 0 \). The sequence \( S_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \), which is denoted by \( S_n(x) \sim (g(t), f(t)) \) (see [14, 15]). Let \( f(t) \in \mathcal{F} \) and \( p(t) \in \mathbb{P} \). Then we have

\[
f(t) = \sum_{k=0}^{\infty} \frac{[f(t)|x^k]}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{(t^k|x^k)}{k!}.
\]

(1.11)
From (1.11), we note that
\[ p^{(k)}(0) = \langle t^k | p(x) \rangle = [1 | p^{(k)}(x) \rangle]. \quad (1.12) \]

By (1.12), we get
\[ t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad (\text{see [14, 15]}). \quad (1.13) \]

From (1.13), we easily derive the following equation
\[ e^{yt} p(x) = p(x + y), \quad \langle e^{yt} | p(x) \rangle = p(y). \quad (1.14) \]

For \( p(x) \in \mathbb{P}, f(t) \in \mathcal{F} \), it is known that
\[ \langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle = [f'(t) | p(x) \rangle \quad (\text{see [14]}). \quad (1.15) \]

Let \( S_n(x) \sim (g(t), f(t)) \). Then we have
\[ \frac{1}{g(\bar{f}(t))} e^{\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(y) \frac{t^n}{n!} \quad \text{for all } y \in \mathbb{C}, \quad (1.16) \]

where \( \bar{f}(t) \) is the compositional inverse of \( f(t) \) with \( \bar{f}(f(t)) = t \), and
\[ f(t) S_n(x) = n S_{n-1}(x) \quad (\text{see [14, 15]}). \quad (1.17) \]

The Stirling number of the second kind is defined by the generating function to be
\[ (e^t - 1)^m = m! \sum_{l=0}^{m} S_2(l, m) \frac{t^n}{n!} \quad (m \in \mathbb{Z}_{\geq 0}). \quad (1.18) \]

For \( S_n(x) \sim (g(t), t) \), it is well known that
\[ S_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) S_n(x) \quad (n \geq 0) \quad (\text{see [14, 15]}). \quad (1.19) \]

Let \( S_n(x) \sim (g(t), f(t)), r_n(x) \sim (h(t), l(t)) \). Then we have
\[ S_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \quad (1.20) \]

where
\[ C_{n,m} = \frac{1}{m!} \left\{ \frac{h'(\bar{f}(t))}{g'(\bar{f}(t))} l(\bar{f}(t))^m \right\} x^n \quad (\text{see [14, 15]}). \quad (1.21) \]

In this paper, we study higher-order Frobenius-Euler polynomials associated with poly-Bernoulli polynomials, which are called higher-order Frobenius-Euler and poly-Beroulli mixed-type polynomials. The purpose of this paper is to give various identities of those polynomials arising from umbral calculus.
2 Higher-order Frobenius-Euler polynomials, associated poly-Bernoulli polynomials

Let us consider the polynomials $T_{n}^{(r,k)}(x|\lambda)$, called higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials, as follows:

\[
\left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} T_{n}^{(r,k)}(x|\lambda) \frac{x^n}{n!}, \tag{2.1}
\]

where $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, $r, k \in \mathbb{Z}$.

When $x = 0$, $T_{n}^{(r,k)}(\lambda) = T_{n}^{(r,k)}(0|\lambda)$ is called the $n$th higher-order Frobenius-Euler and poly-Bernoulli mixed type number.

From (1.16) and (2.1), we note that

\[
T_{n}^{(r,k)}(x|\lambda) \sim \left(\frac{e^t-\lambda}{1-\lambda}\right)^r \frac{1-e^{-t}}{Li_k(1-e^{-t})} t. \tag{2.2}
\]

By (1.17) and (2.2), we get

\[
t T_{n}^{(r,k)}(x|\lambda) = n T_{n-1}^{(r,k)}(x|\lambda). \tag{2.3}
\]

From (2.1), we can easily derive the following equation

\[
T_{n}^{(r,k)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda) B_{l}^{(k)}(x)
= \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(x|\lambda) B_{l}^{(k)}. \tag{2.4}
\]

By (1.16) and (2.2), we get

\[
T_{n}^{(r,k)}(x|\lambda) = \frac{1}{g_{r,k}(t)} x^n = \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^n. \tag{2.5}
\]

In [7], it is known that

\[
\frac{Li_k(1-e^{-t})}{1-e^{-t}} x^n = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x-j)^n. \tag{2.6}
\]

Thus, by (2.5) and (2.6), we get

\[
T_{n}^{(r,k)}(x|\lambda) = \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^n
= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(\frac{1-\lambda}{e^t-\lambda}\right)^r (x-j)^n
= \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} H_{n}^{(r)}(x-j|\lambda). \tag{2.7}
\]
By (1.1), we easily see that

\[ H^{(r)}_n(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} H^{(r)}_{n-l}(\lambda)x^l. \]  

(2.8)

Therefore, by (2.7) and (2.8), we obtain the following theorem.

**Theorem 2.1** For \( r, k \in \mathbb{Z}, n \geq 0 \), we have

\[
T^{(r,k)}_n(x|\lambda) = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{l=0}^{n} \binom{n}{l} H^{(r)}_{n-l}(\lambda)(x-j)^l.
\]

\[
= \sum_{j=0}^{n} \left\{ \binom{n}{l} H^{(r)}_{n-l}(\lambda) \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \right\}(x-j)^l.
\]

In [7], it is known that

\[
\frac{L \lambda^{(1-e^t)}}{1-e^{-t}} x^n = \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m!S_2(n-j,m) \right\} x^j.
\]  

(2.9)

By (2.5) and (2.9), we get

\[
T^{(r,k)}_n(x|\lambda) = \left( \frac{1-\lambda}{e^{-t}-\lambda} \right)^r \frac{L \lambda^{(1-e^t)}}{1-e^{-t}} x^n
\]

\[
= \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m!S_2(n-j,m) \right\} \left( \frac{1-\lambda}{e^{-t}-\lambda} \right)^r x^j
\]

\[
= \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m!S_2(n-j,m) \right\} H^{(r)}_j(x|\lambda).
\]  

(2.10)

Therefore, by (2.8) and (2.10), we obtain the following theorem.

**Theorem 2.2** For \( r, k \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0} \), we have

\[
T^{(r,k)}_n(x|\lambda) = \sum_{l=0}^{n} \left\{ \sum_{j=0}^{n-l} \binom{n}{j} \left( \frac{1}{(m+1)^k} \binom{m}{l} H^{(r)}_{m-j}(\lambda)S_2(n-j,m) \right) \right\} x^l.
\]

From (1.19) and (2.2), we have

\[
T^{(r,k)}_{n+1}(x|\lambda) = (x - \frac{\partial}{\partial t}^{r,k}(t)) T^{(r,k)}_n(x|\lambda).
\]  

(2.11)
Thus, by (2.12), we get

\[
\frac{g'_{r,k}(t)}{g_{r,k}(t)} = \left( r \log(e^r - \lambda) - r \log(1 - \lambda) + \log(1 - e^{-t}) - \log L_i_k(1 - e^{-t}) \right)'
\]

\[
= r + \frac{r \lambda}{e^r - \lambda} + \left( \frac{t}{e^r - 1} \right) \frac{L_i_k(1 - e^{-t}) - L_i_{k-1}(1 - e^{-t})}{tL_i_k(1 - e^{-t})}.
\]

(2.12)

By (2.11) and (2.12), we get

\[
T_{n+1}^{(r,k)}(x|\lambda) = xT_n^{(r,k)}(x|\lambda) - rT_n^{(r,k)}(x|\lambda) - \frac{r \lambda}{1 - \lambda} \left( \frac{1 - \lambda}{e^r - \lambda} \right)^{r+1} \frac{L_i_k(1 - e^{-t})}{1 - e^{-t}} x^n
\]

\[
\quad \cdot \left[ \frac{L_i_k(1 - e^{-t}) - L_i_{k-1}(1 - e^{-t})}{t(1 - e^{-t})} x^n \right] + \left( x - r \right) T_n^{(r,k)}(x|\lambda) - \frac{r \lambda}{1 - \lambda} T_{n}^{(r+1,k)}(x|\lambda)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \left( \frac{1 - \lambda}{e^r - \lambda} \right)^l \frac{L_i_k(1 - e^{-t}) - L_i_{k-1}(1 - e^{-t})}{t(1 - e^{-t})} x^n.
\]

(2.13)

It is easy to show that

\[
\frac{L_i_k(1 - e^{-t}) - L_i_{k-1}(1 - e^{-t})}{1 - e^{-t}} = \frac{1}{1 - e^{-t}} \sum_{n=1}^{\infty} \left\{ \frac{(1 - e^{-t})^n}{\mu^k} - \frac{(1 - e^{-t})^n}{\mu^{k+1}} \right\}
\]

\[
= \left( \frac{1 - e^{-t}}{2^k} - \frac{1 - e^{-t}}{2^{k+1}} \right) + \ldots
\]

\[
= \left( \frac{1}{2^k} - \frac{1}{2^{k+1}} \right) t + \ldots.
\]

(2.14)

For any delta series \( f(t) \), we have

\[
\frac{f(t)}{t} x^n = f(t) \frac{1}{n + 1} x^{n+1}.
\]

(2.15)

Thus, by (2.13), (2.14) and (2.15), we get

\[
T_{n+1}^{(r,k)}(x|\lambda) = (x - r) T_n^{(r,k)}(x|\lambda) - \frac{r \lambda}{1 - \lambda} T_{n}^{(r+1,k)}(x|\lambda)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \left( \frac{1 - \lambda}{e^r - \lambda} \right)^l \frac{L_i_k(1 - e^{-t}) - L_i_{k-1}(1 - e^{-t})}{1 - e^{-t}} x^n
\]

\[
\quad \cdot \left[ \frac{L_i_k(1 - e^{-t}) - L_i_{k-1}(1 - e^{-t})}{t(1 - e^{-t})} x^n \right] + \left( x - r \right) T_n^{(r,k)}(x|\lambda) - \frac{r \lambda}{1 - \lambda} T_{n}^{(r+1,k)}(x|\lambda)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \left[ \eta_{i+1}^{(r,k)}(x|\lambda) - T_{i+1}^{(r,k-1)}(x|\lambda) \right]
\]

\[
= (x - r) T_n^{(r,k)}(x|\lambda) - \frac{r \lambda}{1 - \lambda} T_{n}^{(r+1,k)}(x|\lambda).
\]
Thus, by (2.19), we get

\[- \frac{1}{n+1} \sum_{l=0}^{n+1} \left( \frac{n+1}{l} \right) B_{n+1-l} \left\{ T_l^{(r,k)}(x|\lambda) - T_l^{(r-1,k)}(x|\lambda) \right\} \]

\[= (x - r) T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1 - \lambda} T_n^{(r+1,k)}(x|\lambda) \]

\[- \frac{1}{n+1} \sum_{l=0}^{n+1} \left( \frac{n+1}{l} \right) B_{n+1-l} \left\{ T_l^{(r,k)}(x|\lambda) - T_l^{(r-1,k)}(x|\lambda) \right\} \]

\[= (x - r) T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1 - \lambda} T_n^{(r+1,k)}(x|\lambda) \]

\[- \frac{1}{n+1} \sum_{l=0}^{n+1} \left( \frac{n+1}{l} \right) B_{l} \left\{ T_{n+1-l}^{(r,k)}(x|\lambda) - T_{n+1-l}^{(r-1,k)}(x|\lambda) \right\}. \tag{2.16} \]

Therefore, by (2.16), we obtain the following theorem.

**Theorem 2.3** For \( r, k \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}, \) we have

\[ T_{n+1}^{(r,k)}(x|\lambda) = (x - r) T_{n}^{(r,k)}(x|\lambda) - \frac{r\lambda}{1 - \lambda} T_{n}^{(r+1,k)}(x|\lambda) \]

\[- \frac{1}{n+1} \sum_{l=0}^{n+1} \left( \frac{n+1}{l} \right) B_{l} \left\{ T_{n+1-l}^{(r,k)}(x|\lambda) - T_{n+1-l}^{(r-1,k)}(x|\lambda) \right\}. \]

**Remark 1** If \( r = 0, \) then we have

\[ \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \frac{L_i(1 - e^{-t})}{(1 - e^{-t})} = \sum_{n=0}^{\infty} T_n^{(0,k)}(x|\lambda) \frac{t^n}{n!}. \tag{2.17} \]

Thus, by (2.17), we get \( B_n^{(k)}(x) = T_n^{(0,k)}(x|\lambda). \)

From (2.4), we have

\[ t x T_n^{(r,k)}(x|\lambda) = t \left( x \sum_{l=0}^{\infty} \left( \frac{n+1}{l} \right) H_{l-1}^{(r)}(\lambda) B_l^{(k)}(x) \right) \]

\[ = \sum_{l=0}^{n} \left( \frac{n}{l} \right) H_{n-l}^{(r)}(\lambda) \left\{ l \cdot B_l^{(k)}(x) + B_l^{(k)}(x) \right\} \]

\[ = n x \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) H_{n-l-1}^{(r)}(\lambda) B_l^{(k)}(x) + \sum_{l=0}^{n} \left( \frac{n}{l} \right) H_{n-l}^{(r)}(\lambda) B_l^{(k)}(x) \]

\[ = n x T_{n-1}^{(r,k)}(x|\lambda) + T_n^{(r,k)}(x|\lambda). \tag{2.18} \]

Applying \( t \) on both sides of Theorem 2.3, we get

\[ (n+1) T_n^{(r,k)}(x|\lambda) \]

\[ = n x T_{n-1}^{(r,k)}(x|\lambda) + T_n^{(r,k)}(x|\lambda) - r n T_{n}^{(r,k)}(x|\lambda) - \frac{r\lambda}{1 - \lambda} T_{n+1}^{(r+1,k)}(x|\lambda) \]

\[- \frac{1}{n+1} \sum_{l=0}^{n+1} \left( \frac{n+1}{l} \right) B_{l} \left\{ (n+1 - l) T_{n+1-l}^{(r,k)}(x|\lambda) - (n+1 - l) T_{n+1-l}^{(r-1,k)}(x|\lambda) \right\}. \tag{2.19} \]
Thus, by (2.19), we have

\[(n + 1)T^{(r,k)}_n(x|\lambda) + n\left(r - \frac{1}{2} - x\right)T^{(r,k)}_{n-1}(x|\lambda) + \sum_{l=0}^{n-2} \binom{n}{l} B_{n-l} T^{(r,k)}_l(x|\lambda)\]

\[= - \frac{r\lambda n}{1 - \lambda} T^{(r+1,k)}_{n-1}(x|\lambda) + \sum_{l=0}^{n} \binom{n}{l} B_{n-l} T^{(r,k-1)}_l(x|\lambda). \quad (2.20)\]

Therefore, by (2.20), we obtain the following theorem.

\section*{Theorem 2.4}

For \(r, k \in \mathbb{Z}, n \in \mathbb{Z}\) with \(n \geq 2\), we have

\[(n + 1)T^{(r,k)}_n(x|\lambda) + n\left(r - \frac{1}{2} - x\right)T^{(r,k)}_{n-1}(x|\lambda) + \sum_{l=0}^{n-2} \binom{n}{l} B_{n-l} T^{(r,k)}_l(x|\lambda)\]

\[= - \frac{r\lambda n}{1 - \lambda} T^{(r+1,k)}_{n-1}(x|\lambda) + \sum_{l=0}^{n} \binom{n}{l} B_{n-l} T^{(r,k-1)}_l(x|\lambda).\]

From (1.14) and (2.5), we note that

\[T^{(r,k)}_n(y|\lambda) = \left(\frac{1 - \lambda}{e^\lambda - \lambda}\right)^r \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} |x^n - 1|\]

\[= \left(\frac{1 - \lambda}{e^\lambda - \lambda}\right)^r \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} |x^{n-1}|. \quad (2.21)\]

By (1.15) and (2.21), we get

\[T^{(r,k)}_n(y|\lambda) = \left\{ \begin{array}{lr}
\partial_t \left(\left(\frac{1 - \lambda}{e^\lambda - \lambda}\right)^r \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} |x^n - 1|\right) \\
\left(\left(\frac{1 - \lambda}{e^\lambda - \lambda}\right)^r \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} |x^{n-1}|\right) \\
\left(\left(\frac{1 - \lambda}{e^\lambda - \lambda}\right)^r \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \partial_t e^{\lambda t} |x^{n-1}|\right) \\
\left(\left(\frac{1 - \lambda}{e^\lambda - \lambda}\right)^r \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \partial_t e^{\lambda t} |x^{n-1}|\right).
\end{array} \right\} \quad (2.22)\]

Therefore, by (2.22), we obtain the following theorem.

\section*{Theorem 2.5}

For \(r, k \in \mathbb{Z}, n \geq 1\), we have

\[T^{(r,k)}_n(x|\lambda) = (x - r)T^{(r,k)}_{n-1}(x|\lambda) - \frac{r\lambda n}{1 - \lambda} T^{(r+1,k)}_{n-1}(x|\lambda)\]

\[+ \sum_{l=0}^{n-1} \left\{ (-1)^{n-l} \binom{n-1}{l} \sum_{m=0}^{n-1} (-1)^m \frac{(m + 1)!}{(m + 2)^l} S_2(n - 1 - l, m) \right\} H^{(r)}_l(x - 1|\lambda).\]

Now, we compute \(\left(\frac{1 - \lambda}{e^\lambda - \lambda}\right)^r Li_k(1 - e^{-t}) |x^{r+1}|\) in two different ways.
On the one hand,

\[
\left| \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \text{Li}_k(1 - e^{-t}) \right| x^{n+1}
\]

\[
= \left| \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} \right| (1 - e^{-t}) x^{n+1}
\]

\[
= \left| \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} \right| x^{n+1} - (x - 1)^{n+1}
\]

\[
= \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} x^m
\]

\[
= \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} \left( 1 \right) H_{n+1}^{(r)}(x|\lambda)
\]

\[
= \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} T_m^{(r,k)}(\lambda).
\] (2.23)

On the other hand, we get

\[
\left| \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \text{Li}_k(1 - e^{-t}) \right| x^{n+1}
\]

\[
= \left| \text{Li}_k(1 - e^{-t}) \right| \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r x^{n+1}
\]

\[
= \int_0^t \left( \text{Li}_k(1 - e^{-s}) \right)' ds \left| H_{n+1}^{(r)}(x|\lambda) \right|
\]

\[
= \int_0^t e^{-s} \text{Li}_k(1 - e^{-s}) ds \left| H_{n+1}^{(r)}(x|\lambda) \right|
\]

\[
= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} B_m^{(k-1)} \frac{1}{l!} \int_0^t s^l ds \left| H_{n+1}^{(r)}(x|\lambda) \right|
\]

\[
= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} \frac{B_m^{(k-1)}}{(l+1)!} \left| t^{l+1} H_{n+1}^{(r)}(x|\lambda) \right|
\]

\[
= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} \binom{n+1}{l+1} (-1)^{l-m} B_m^{(k-1)} H_{n-l}^{(r)}(\lambda).
\] (2.24)

Therefore, by (2.23) and (2.24), we obtain the following theorem.

**Theorem 2.6** For \( r, k \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0} \), we have

\[
\sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} T_m^{(r,k)}(\lambda)
\]

\[
= \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B_m^{(k-1)} H_{n-l}^{(r)}(\lambda).
\]
Now, we consider the following two Sheffer sequences:

\[ T^{(r,k)}_n(x|\lambda) \sim \left( \left( \frac{e^r - \lambda}{1 - \lambda} \right)^x \frac{1 - e^{-t}}{L ik(1 - e^{-t})} \right)^n t^m, \]

\[ E^{(s)}_n \sim \left( \left( \frac{e^s - 1}{t} \right)^x, t \right), \]

where \( s \in \mathbb{Z}_{\geq 0}, r, k \in \mathbb{Z} \) and \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \). Let us assume that

\[ T^{(r,k)}_n(x|\lambda) = \sum_{m=0}^{n} C_{n,m} B^{(s)}_m(x). \]  

By (1.21) and (2.26), we get

\[
C_{n,m} = \frac{1}{m!} \left( \left( \frac{e^r - 1}{t} \right)^x \left( \frac{1 - \lambda}{e^s - \lambda} \right)^x \frac{1}{L ik(1 - e^{-t})} \right)^n \left( \frac{t^m}{\lambda^m} \right) \]

\[
= \frac{1}{m!} \left( \left( \frac{e^r - 1}{t} \right)^x \left( \frac{1 - \lambda}{e^s - \lambda} \right)^x \frac{1}{L ik(1 - e^{-t})} \right)^n \left( \frac{t^m}{\lambda^m} \right)
\]

\[
= \left( \frac{n}{m} \right) \sum_{l=0}^{n-m} \frac{s!}{(l+s)!} S_2(l+s,s) \left( \left( \frac{1 - \lambda}{e^s - \lambda} \right)^x \frac{1}{L ik(1 - e^{-t})} \right) \left( \frac{t^m}{\lambda^m} \right)
\]

\[
= \left( \frac{n}{m} \right) \sum_{l=0}^{n-m} \frac{s(l+n-m)}{(l+s)!} S_2(l+s,s) \left( \frac{1 - \lambda}{e^s - \lambda} \right)^x \frac{1}{L ik(1 - e^{-t})} \left( \frac{t^m}{\lambda^m} \right)
\]

\[
= \left( \frac{n}{m} \right) \sum_{l=0}^{n-m} \frac{(n-m)}{(l+s)!} \left( \frac{s}{l} \right) S_2(l+s,s) T^{(r,k)}_{n-m-\lambda}(\lambda).
\]

Therefore, by (2.26) and (2.27), we obtain the following theorem.

**Theorem 2.7** For \( r, k \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0} \), we have

\[ T^{(r,k)}_n(x|\lambda) = \sum_{m=0}^{n} \left( \left( \frac{n}{m} \right) \sum_{l=0}^{n-m} \left( \frac{s}{l} \right) S_2(l+s,s) T^{(r,k)}_{n-m-\lambda}(\lambda) \right) B^{(s)}_m(x). \]

From (1.3) and (2.1), we note that

\[ T^{(r,k)}_n(x|\lambda) \sim \left( \left( \frac{e^r - \lambda}{1 - \lambda} \right)^x \frac{1 - e^{-t}}{L ik(1 - e^{-t})} \right)^n t^m, \]

\[ E^{(r,k)}_n(x) \sim \left( \left( \frac{e^s + 1}{2} \right)^x, \right), \]

where \( r, k \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0} \).

By the same method, we get

\[ T^{(r,k)}_n(x|\lambda) = \frac{1}{2^{\lambda}} \sum_{m=0}^{n} \left( \left( \frac{n}{m} \right) \sum_{j=0}^{\frac{s}{2}} \left( \frac{s}{2} \right) T^{(\lambda)}_{n-m-\lambda}(\lambda) \right) E^{(s)}_m(x). \]
From (1.1) and (2.1), we note that

\begin{align}
T_n^{(r,k)}(x|\lambda) & \sim \left(\frac{e^t - \lambda}{1 - \lambda}\right)^r \frac{1 - e^{-t}}{L_i(k(1 - e^{-t}))} t^r, \\
H_n^{(s)}(x|\mu) & \sim \left(\frac{e^t - \mu}{1 - \mu}\right)^s t^s,
\end{align}

(2.30)

where \( r, k \in \mathbb{Z} \) and \( \lambda, \mu \in \mathbb{C} \) with \( \lambda \neq 1, \mu \neq 1, s \in \mathbb{Z}_{\geq 0} \).

Let us assume that

\[ T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\mu). \]  

(2.31)

By (1.21) and (2.31), we get

\[ C_{n,m} = \frac{1}{m!} \left( \frac{e^t - \mu}{1 - \mu} \right)^s \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \frac{1 - e^{-t}}{1 - e^{-t}} \left| x^n \right|^{e^t - 1} \]

(2.32)

Therefore, by (2.31) and (2.32), we obtain the following theorem.

**Theorem 2.8** For \( r, k \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0} \), we have

\[ T_n^{(r,k)}(x|\lambda) = \frac{1}{(1 - \mu)^s} \sum_{m=0}^{n} \binom{n}{m} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} T_{n-m}^{(r,k)}(x|\lambda) \]  

(2.33)

\[ H_m^{(s)}(x|\mu). \]

It is known that

\[ T_n^{(r,k)}(x|\lambda) \sim \left(\frac{e^t - \lambda}{1 - \lambda}\right)^r \frac{1 - e^{-t}}{L_i(k(1 - e^{-t}))} t^r, \]

(2.33)

\[ (x)_n \sim (1, e^t - 1). \]

Let

\[ T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} C_{n,m}(x)_m. \]

(2.34)

Then, by (1.21) and (2.34), we get

\[ C_{n,m} = \frac{1}{m!} \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \frac{1 - e^{-t}}{1 - e^{-t}} \left| x^m \right|^{e^t - 1} \]

(2.33)

where \( r, k \in \mathbb{Z}, \lambda, \mu \in \mathbb{C} \) with \( \lambda \neq 1, \mu \neq 1, s \in \mathbb{Z}_{\geq 0} \).
\[
= \sum_{l=0}^{n-m} S_2(l + m, m) \left( \frac{n}{l + m} \right) \frac{(n - m)!}{(l + m)!} \left( 1 - \lambda \right)^r L_i(1 - e^{-t}) x^{n-m-l}
\]
\[
= \sum_{l=0}^{n-m} \left( \begin{array}{c} n \\l + m \end{array} \right) S_2(l + m, m) T_{n-m-l}(\lambda).
\]

(2.35)

Therefore, by (2.34) and (2.35), we obtain the following theorem.

**Theorem 2.9** For \( r, k \in \mathbb{Z} \), we have

\[
T^{(r,k)}_n(x \mid \lambda) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \left( \begin{array}{c} n \\l + m \end{array} \right) S_2(l + m, m) T_{n-m-l}(\lambda) \right\} (x)_m.
\]

Finally, we consider the following two Sheffer sequences:

\[
T^{(r,k)}_n(x \mid \lambda) \sim \left( \left( \frac{e^t - \lambda}{1 - \lambda} \right)^r \frac{1 - e^{-t}}{L_i(1 - e^{-t})}, t \right),
\]
\[
x^{[n]} \sim (1, 1 - e^{-t}),
\]

where \( x^{[n]} = x(x + 1) \cdots (x + n - 1) \).

Let us assume that

\[
T^{(r,k)}_n(x \mid \lambda) = \sum_{m=0}^{n} C_{n,m} x^{[m]}.
\]

(2.37)

Then, by (1.21) and (2.37), we get

\[
C_{n,m} = \frac{1}{m!} \left( \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r \frac{L_i(1 - e^{-t})}{1 - e^{-t}} \right)^m x^{[m]}.
\]

\[
= \sum_{l=0}^{\infty} (-1)^l S_2(l + m, m) \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r L_i(1 - e^{-t}) x^{n-m-l}
\]
\[
= \sum_{l=0}^{n-m} \left( \begin{array}{c} n \\l + m \end{array} \right) S_2(l + m, m) T_{n-m-l}(\lambda).
\]

(2.38)

Therefore, by (2.37) and (2.38), we obtain the following theorem.

**Theorem 2.10** For \( r, k \in \mathbb{Z} \), \( n \geq 0 \), we have

\[
T^{(r,k)}_n(x \mid \lambda) = \sum_{m=0}^{n} \left( \sum_{l=0}^{n-m} (-1)^l \left( \begin{array}{c} n \\l + m \end{array} \right) S_2(l + m, m) T_{n-m-l}(\lambda) \right) x^{[m]}.
\]

**Competing interests**

The authors declare that they have no competing interests.
Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details
1 Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea. 2 Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.

Acknowledgements
This work was supported by the National Research Foundation of Korea (NRF) grant, funded by the Korea government (MOE) (No. 2012R1A1A2003786).

Received: 11 July 2013 Accepted: 6 August 2013 Published: 20 August 2013

References
1. Araci, S, Acikgoz, M: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math. 22(3), 399-406 (2012)
2. Kim, T: An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant \( q \)-integrals on \( \mathbb{Z}_p \). Rocky Mt. J. Math. 41(1), 239-247 (2011)
3. Kim, T: Identities involving Frobenius-Euler polynomials arising from non-linear differential equations. J. Number Theory 132(1), 2854-2865 (2012)
4. Ryoo, C: A note on the Frobenius-Euler polynomials. Proc. Jangjeon Math. Soc. 14(4), 495-501 (2011)
5. Ryoo, CS, Agarwal, RP: Exploring the multiple Changhee \( q \)-Bernoulli polynomials. Int. J. Comput. Math. 82(4), 483-493 (2005)
6. Kim, DS, Kim, T, Kim, YH, Lee, SH: Some arithmetic properties of Bernoulli and Euler numbers. Adv. Stud. Contemp. Math. 22(4), 467-480 (2012)
7. Kim, DS, Kim, T: Poly-Bernoulli polynomials arising from umbral calculus (communicated)
8. Kim, T: Power series and asymptotic series associated with the \( q \)-analogue of the two-variable \( p \)-adic L-function. Russ. J. Math. Phys. 12(2), 186-196 (2005)
9. Can, M, Cenkci, M, Kurt, V, Simsek, Y: Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler L-functions. Adv. Stud. Contemp. Math. 18(2), 135-160 (2009)
10. Ding, D, Yang, J: Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials. Adv. Stud. Contemp. Math. 20(1), 7-21 (2010)
11. Kim, T, Choi, J: A note on the product of Frobenius-Euler polynomials arising from the \( p \)-adic integral on \( \mathbb{Z}_p \). Adv. Stud. Contemp. Math. 22(2), 215-223 (2012)
12. Kurt, B, Simsek, Y: On the generalized Apostol-type Frobenius-Euler polynomials. Adv. Differ. Equ. 2013, 1 (2013)
13. Simsek, Y, Yardimci, D, Kurt, V: On interpolation functions of the twisted generalized Frobenius-Euler numbers. Adv. Stud. Contemp. Math. 15(2), 187-194 (2007)
14. Roman, S: The Umbral Calculus. Pure and Applied Mathematics, vol. 111. Academic Press, New York (1984)
15. Roman, S, Rota, G-C: The umbral calculus. Adv. Math. 27(2), 95-188 (1978)

doi:10.1186/1687-1847-2013-251
Cite this article as: Kim and Kim: Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials. Advances in Difference Equations 2013 2013:251.