Precept 2: Classification
COS 484

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2/7/2025
You train an n-gram model on some training corpus $D$ using counts

$$P(w_n | w_1, \ldots, w_{n-1}) = \frac{c(w_1, \ldots, w_n)}{\sum_{v \in V} c(w_1, \ldots, w_{n-1}, v)}.$$  

To prevent (possible) infinite perplexity on the test corpus $D_t$, you apply Laplace smoothing. Let $P(D)$, $P(D_t)$ be the unsmoothed probabilities and $P'(D)$, $P'(D_t)$ be the smoothed probabilities.
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$$P(w_n | w_1, \ldots, w_{n-1}) = \frac{c(w_1, \ldots, w_n)}{c(w_1, \ldots, w_{n-1})}.$$To prevent (possible) infinite perplexity on the test corpus $D_t$, you apply Laplace smoothing. Let $ppl(D), ppl(D_t)$ be perplexities of the unsmoothed model and $ppl'(D), ppl'(D_t)$ of the smoothed model. Is the following T, F or undetermined (depends on model, data, n, etc)?

1. $ppl'(D) \geq ppl(D)$

2. $ppl'(D_t) < ppl(D_t)$

3. $ppl(D) < ppl(D_t)$
Review Question

You train an n-gram model on some training corpus $D$ using counts

$$P(w_n \mid w_1, \ldots, w_{n-1}) = \frac{c(w_1, \ldots, w_n)}{c(w_1, \ldots, w_{n-1})}.$$ To prevent (possible) infinite perplexity on the test corpus $D_t$, you apply Laplace smoothing. Let $ppl(D)$, $ppl(D_t)$ be perplexities of the unsmoothed model and $ppl'(D)$, $ppl'(D_t)$ of the smoothed model. Is the following T, F or undetermined (depends on model, data, n, etc)?

1. $ppl'(D) \geq ppl(D)$
   - This is true! Remember that setting the probability using counts (above) is the MLE estimate, which means that $P(D)$ cannot increase under any other distribution for $P(w_n \mid w_1, \ldots, w_{n-1})$

2. $ppl'(D_t) < ppl(D_t)$

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1. $ppl'(D) \geq ppl(D)$ - T

2. $ppl'(D_t) < ppl(D_t)$
   
   This is undetermined! It’s not clear that the test corpus will have infinite perplexity. It is possible that the test corpus is very similar to the train corpus, and smoothing will cause its probability to drop.

3. $ppl(D) < ppl(D_t)$
You train an n-gram model on some training corpus $D$ using counts

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1. $ppl'(D) \geq ppl(D) - T$

2. $ppl'(D_t) < ppl(D_t) - U$

3. $ppl(D) < ppl(D_t)$

Undetermined. A test corpus consisting solely of high-frequency n-grams might have a higher probability
Given a document $d = w_1, \ldots, w_K$ and a set of classes $C = \{c_1, \ldots, c_m\}$, we want to find the class $c_i$ that maximizes $P(c | d)$. Two ways to do this:

**Naive Bayes** ← Covered in lecture in great detail!

Logistic Regression
Todays Topics

Given a document $d = w_1, \ldots, w_K$ and a set of classes $\mathcal{C} = \{c_1, \ldots, c_m\}$, we want to find the class $c_i$ that maximizes $P(c \mid d)$. Two ways to do this:

Naive Bayes

**Logistic Regression** ← Focus for today’s Precept
Logistic Regression: Intuition

Given a document $d = w_1, \ldots, w_K$ and a set of classes $\mathcal{C} = \{c_1, \ldots, c_m\}$, we want to find the class $c_i$ that maximizes $P(c \mid d)$
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Compared to NB, with LR we take a more direct approach: directly compute $P(c \mid d)$ given a set of features constructed from the input document $d$. 
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![Diagram](image-url)
Logistic Regression: Features

Given a document \( d = w_1, \ldots, w_K \) and a set of classes \( \mathcal{C} = \{c_1, \ldots, c_m\} \), we want to find the class \( c_i \) that maximizes \( P(c \mid d) \).

Compared to NB, with LR we take a more direct approach: directly compute \( P(c \mid d) \) given a set of features constructed from the input document \( d \).

This is the feature vector \( x \) for some input document \( d \).
Logistic Regression: Features

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Compared to NB, with LR we take a more direct approach: directly compute \( P(c \mid d) \) given a set of features constructed from the input document \( d \).

The features to use is a design decision. A natural default is to use a vector \( x \in \mathbb{R}^{|V|} \) where each dim is the counts of one word in the vocabulary. (BOW)
Given a document $d = w_1, \ldots, w_K$ and a set of classes $\mathcal{C} = \{c_1, \ldots, c_m\}$, we want to find the class $c_i$ that maximizes $P(c \mid d)$.

Now given some feature vector $x$ how do we turn this to a probability?
Logistic Regression: LR Model

Given a document \( d = w_1, \ldots, w_K \) and a set of classes \( \mathcal{C} = \{ c_1, \ldots, c_m \} \), we want to find the class \( c_i \) that maximizes \( P(c | d) \).

Now given some feature vector \( x \) how do we turn this to a probability?

1. Convert the features to a number. The higher the number, the more confident we are that the document belongs to a class. We call these numbers **logits**.

\[
d = w_1, \ldots, w_K
\]

\[
\mathcal{C} = \{ c_1, \ldots, c_m \}
\]

\[
P(c | d) = \frac{1}{1 + e^{-(w \cdot x + b)}}
\]
Logistic Regression: LR Model

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1. Convert the features to a number. The higher the number, the more confident we are that the document belongs to a class. We call these numbers **logits**.
2. Normalize the logits using sigmoid so we get a well-defined probability distribution.
   1. For more than 2 classes we use the softmax, which is the \( m > 2 \) generalization of sigmoid
Logistic Regression: How do we set $w, b$?

**Summary:** we want to estimate $P(c | d)$ using a model $\sigma(w \cdot x + b)$
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We use GD. But what should we make the loss?
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Let our train dataset be $\mathcal{D} = \{(d_1, c_1), \ldots, (d_n, c_n)\}$. Let’s find the probability of seeing these documents and labels. Assume that each datapoint is independent of the other.
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How to set $\theta = (w, b)$?
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How to set $\theta = (w, b)$? Use the MLE principle! Set $\theta$ such that $P(\mathcal{D})$ is maximized. This is analogous to setting the n-gram probabilities such that the probability of the train corpus is maximal.
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Since $\log$ is monotonic, this is equivalent to minimizing: $- \sum_i \log P(c_i | d_i) \leftarrow$ this is just CE loss!
Logistic Regression: How do we set \( w, b \) ?

**Summary:** we want to estimate \( P(c \mid d) \) using a model \( \sigma(w \cdot x + b) \)

Want to minimize: 
\[
- \sum \log P(c_i \mid d_i) \quad \leftarrow \text{this is just CE loss!}
\]

- Loss: 
\[
- \log \prod_{i=1}^{n} P(y_i \mid x_i) = - \sum_{i=1}^{n} \log P(y_i \mid x_i)
\]

\[
L_{CE} = - \sum_{i=1}^{n} [y_i \log \hat{y}_i + (1 - y_i) \log (1 - \hat{y}_i)]
\]
Logistic Regression: Gradient of Cross-entropy

Summary: in our binary logistic regression using a model \(\sigma(w \cdot x + b)\), our cross-entropy loss is

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L_{CE}(w, b) = -\frac{1}{n} \sum_{i=1}^{n} [y_i \log \hat{y}_i + (1 - y_i)\log(1 - \hat{y}_i)]
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How do we differentiate this with gradient descent?
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How do we differentiate this with gradient descent? We need to determine \( \frac{d\mathcal{L}}{dw}, \frac{d\mathcal{L}}{db} \)
Logistic Regression: Gradient of Cross-entropy

First we calculate gradient of $\mathcal{L}$ with respect to a specific $\hat{y}_i$ using the chain rule.
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$$

$$
= - \frac{1}{n} \left[ \frac{y_i}{\hat{y}_i} - \frac{1 - y_i}{1 - \hat{y}_i} \right]
$$

$$
\frac{d}{dx} \log(x) = \frac{1}{x}
$$
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$$\frac{d\hat{y}_i}{dz_i} = \sigma(z_i)(1 - \sigma(z_i)) = \hat{y}_i(1 - \hat{y}_i)$$
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$$= - \frac{1}{n} \left[ \frac{y_i}{\hat{y}_i} - \frac{1 - y_i}{1 - \hat{y}_i} \right]$$

Let $z_i = w \cdot x_i + b$, so $\hat{y}_i = \sigma(z_i)$. Differentiating with respect to $z_i$ gives

$$\frac{d\hat{y}_i}{dz_i} = \sigma(z_i)(1 - \sigma(z_i)) = \hat{y}_i(1 - \hat{y}_i)$$

All in all, the derivative with respect to $z_i$ is

$$\frac{d\mathcal{L}}{dz_i} = \frac{d\mathcal{L}}{d\hat{y}_i} \cdot \frac{d\hat{y}_i}{dz_i} = - \frac{1}{n} \left[ \frac{y_i}{\hat{y}_i} - \frac{1 - y_i}{1 - \hat{y}_i} \right] \hat{y}_i(1 - \hat{y}_i)$$
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Logistic Regression: Gradient of Cross-entropy

Now take derivative with respect to $w$ and $b$ for the final update equations.
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Now take derivative with respect to $\mathbf{w}$ and $b$ for the final update equations.

$$
\frac{dz_i}{d\mathbf{w}} = \frac{d}{d\mathbf{w}}(\mathbf{w} \cdot \mathbf{x}_i + b) = \mathbf{x}_i
$$
Logistic Regression: Gradient of Cross-entropy

Now take derivative with respect to \( w \) and \( b \) for the final update equations.

\[
\frac{dz_i}{dw} = \frac{d}{dw}(w \cdot x_i + b) = x_i
\]

\[
\frac{dz_i}{db} = \frac{d}{db}(w \cdot x_i + b) = 1
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$$

$$
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$$

Combining all together gives

$$
\frac{d\mathcal{L}_{CE}}{dw} = \frac{1}{n} \sum_{i=1}^{n} [\hat{y}_i - y_i]x_i
$$

$$
\frac{d\mathcal{L}_{CE}}{db} = \frac{1}{n} \sum_{i=1}^{n} [\hat{y}_i - y_i]
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Logistic Regression: Summary
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1. Given a document \(d = w_1, \ldots, w_K\) and a set of classes \(\mathcal{C} = \{c_1, \ldots, c_m\}\), we want to find the class \(c\) that maximizes \(P(c \mid d)\). Let’s say we estimating \(P(d \mid c)\) reliably is hard, we will need to estimate \(P(c \mid d)\) directly.
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2. Want to turn $d$ into a vector $x$ because then we can operate on it more conveniently.
   1. We can use a BOW, where each dim in $x \in \mathbb{R}^{|V|}$ is the # of times a word in $V$ appears
   2. We can also be creative and add additional features we think are important (e.g. # of emojis in text)
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4. Oh no! The linear combination might not be in \([0,1]\), so we normalize using sigmoid: \( \sigma(x) = (1 + e^{-x})^{-1} \)
   1. The probability for one class is \( \sigma(w \cdot x + b) \), so the other class must have prob \( 1 - \sigma(w \cdot x + b) \)
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5. Given our model, we can estimate the probability of a train set under the model $P(\mathcal{D})$
   1. We will set $w, b$ so that $P(\mathcal{D}) = \prod_i P(c_i \mid d_i)$ is maximal (MLE principle)
   2. For stability and convenience we can take the log to minimize $- \sum_i \log P(c_i \mid d_i)$ this is CE loss
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   2. We can also be creative and add additional features we think are important (e.g. # of emojis in text)

3. Somehow we need to turn \( x \) into a single number, because \( P(c \mid d) \) is a single number.
   1. Let’s be as lazy as possible and just take a linear combination of the features: \( w \cdot x + b \)

4. Oh no! The linear combination might not be in \([0,1]\), so we normalize using sigmoid: \( \sigma(x) = (1 + e^{-x})^{-1} \)
   1. The probability for one class is \( \sigma(w \cdot x + b) \), so the other class must have prob \( 1 - \sigma(w \cdot x + b) \)

5. Given our model, we can estimate the probability of a train set under the model \( P(\mathcal{D}) \)
   1. We will set \( w, b \) so that \( P(\mathcal{D}) = \Pi_i P(c_i \mid d_i) \) is maximal (MLE principle)
   2. For stability and convenience we can take the log to minimize \( -\sum_i \log P(c_i \mid d_i) \) this is CE loss

6. We can then use GD to minimize the CE loss! Since the function is convex, we will converge to the optimum.
Logistic Regression: what’s good and what’s not

- More freedom in designing features
- No strong independence assumptions like Naive Bayes

- Can even have the same feature twice! (why?)
- May not work well on small datasets (compared to Naive Bayes)
- Interpreting learned weights can be challenging
Multiclass Classification

- Supervised learning task (e.g. input-output pairs: \( \vec{x}, y \))
- Predict one of \( k \) categories (i.e. classes)
- Typically, \( y \in \{0, 1, 2, \ldots, k - 1\} \)
- Examples:
  - Blood typing: Medical information → \{A, B, AB, O\}
  - Digit recognition: image → \{0, 1, ..., 9\}
  - Object recognition: image → \{“golden retriever”, “laptop”, …\}
  - Weather prediction: weather metrics → \{“sunny”, “cloudy”, “rainy”, “snowy”\}

MNIST dataset
From Binary to Multiclass Classification

Extension of logistic regression to multiclass setting

Given $\vec{x} \in \mathbb{R}^d$, learn $k$ vectors $\theta_1, \theta_2, \ldots, \theta_k \in \mathbb{R}^d$:

$$\Pr[y = i \text{ on } \vec{x}] = \text{softmax}(\vec{z}) = \frac{\exp(\theta_i \cdot \vec{x})}{\sum_{j=1}^k \exp(\theta_j \cdot \vec{x})}, \text{ where } \vec{z} = \theta_i \cdot \vec{x}$$

Note about softmax function: $\sum_{i=1}^k \text{softmax}(\theta_i \cdot \vec{x}) = 1$

(e.g. sum of softmax probabilities for all $k$ classes is 1).

$\theta$: \texttt{\theta} in Latex

$\theta_1, \ldots, \theta_k$ are generalizations of $\vec{w}$ in logistic regression for binary classification.
Multinomial Logistic Regression

Logistic regression
Given $\vec{x} \in \mathbb{R}^d$ and $y \in \{1, -1\}$, learn $\vec{w} \in \mathbb{R}^d$.

$\Pr[y \text{ given } \vec{x}] = \sigma(z)$

$$= \frac{1}{1 + \exp(-y(\vec{w} \cdot \vec{x}))}$$

where $z = y(\vec{w} \cdot \vec{x})$.

$\sigma(z) = \frac{1}{1 + e^{-z}} \in [0, 1]$
Multinomial Logistic Regression

Logistic regression
Given $\mathbf{x} \in \mathbb{R}^d$ and $y \in \{1, -1\}$, learn $\mathbf{w} \in \mathbb{R}^d$.

$$
\Pr[y \text{ given } \mathbf{x}] = \sigma(z) = \frac{1}{1 + \exp(-y(\mathbf{w} \cdot \mathbf{x}))}
$$

where $z = y(\mathbf{w} \cdot \mathbf{x})$.

$$
\sigma(z) = \frac{1}{1 + e^{-z}} \in [0,1]
$$

Multinomial logistic regression
Given $\mathbf{x} \in \mathbb{R}^d$ and $y \in \{0, \ldots, k - 1\}$, learn $k$ vectors $\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k-1)} \in \mathbb{R}^d$.

$$
\Pr[y = i \text{ given } \mathbf{x}] = \text{softmax}(\tilde{z}) = \frac{\exp(\mathbf{w}^{(i)} \cdot \mathbf{x})}{\sum_{j=0}^{k-1} \exp(\mathbf{w}^{(j)} \cdot \mathbf{x})}
$$

where $\tilde{z} = \mathbf{w}^{(i)} \cdot \mathbf{x}$.

- $\text{softmax}(\tilde{z}) \in [0,1]$
- $\sum_{i=0}^{k-1} \text{softmax}(\theta_i \cdot \mathbf{x}) = 1$
- (In CN: $\mathbf{w}^{(i)} = \theta^i$, $\theta$: \text{\texttheta in Latex}$