Measures of shape for the generalized beta exponential distribution

V G Polosin
Medical Cybernetics and Computer Science Department, Penza State University, 40, Krasnay Str., Penza, 440026, Russia

E-mail: polosin-vitalij@yandex.ru

Abstract. This paper contains parametric and informational measures of shape for various families of the generalized beta exponential distribution since it is important to determination of the distribution shape for analysing an experimental data set. A logistic parameter is used to select independent types of beta exponential distributions, that it allows to combine the distributions of different subfamilies. In this paper the use of parametric shape measures to pre-define distribution shape is discusses. In particular, the initial and standard central moments for the main types of generalized beta exponential distribution are given. In the paper it is proposes to use the entropy coefficient of unshifted distribution as an independent information measure of the shape of unshifted generalized beta exponential distributions. In order to increase the reliability of the preliminary determination of the shape of the model, expressions for the entropy coefficient of shifted families both the generalized beta exponential distributions of the first and second types, and the generalized gamma exponential distribution were obtained. For practical applied the entropy coefficients of unshifted distributions for various subfamilies of generalized beta exponential distributions can be useful.

1. Introduction
Exponential models are extensively applied in experimental research in physics, economics, biology and other areas. Different families of distributions exponentials are commonly used for data analysis when observing complex systems. Probabilistic modeling plays a critical role in modern image processing problems [1]. The task of obtaining the result of a measurement experiment of real estimates of physical quantities is also often based on the choice of the distribution density [2]. Exponential distribution is also used in medicine for modeling response probability and dosage in quantum reaction bioanalysis [3]. Many distributions have been considered as descriptive models for the distributions of income [4]. Certain cumulative exponential distribution functions were obtained in the nineteenth century by B. Gompers [5] and P. F. Verhulst [6] when solving the equations of population growth and used to compare the known tables of changes in mortality. These models remain relevant to modern research.

Various distribution models provide the possibility of constructing models with sequential and parallel inclusion of elements. Generalized exponential distributions include such families as gamma, Weibull, lognormal distribution [7]. The versatile class of exponential families of distributions includes many well-known distributions such as Gaussian, Poisson, Dirichlet, polynomial, Gamma, Beta and many others distributions. The most general is the family of generalized beta exponential distributions, since it includes many known models as special or limiting cases.
With a wide variety of families of distributions, it becomes relevant to choose the shape of the model for the array of experimental data. In modern studies, the preliminary choice of the form of an alternative model is based on the use of probabilistic measures obtained on the basis of statistical moments of distributions. It is of interest to use information measures to increase the recognizability of models.

2. Density function of generalized beta exponential distribution

The generalized beta exponential distribution is pioneered by J. B. McDonald [8], where the c logistic parameter was used to combine and to select the independent distributions types. The generalized beta exponential distribution is built on the basis of the transformation of beta distributions of the first and second types. The essence of the transformation can be formulated as follows. A random variable \( Z \) corresponds to a generalized beta exponential distribution if a random variable \( X \) corresponds to a beta distribution of the first or second type and if the random variables \( X \) and \( Z \) are related by the relation

\[
\ln (Z) = -s - d
\]

where the letters of \( \delta \) and \( \sigma \) are location and scale parameters.

For the random variable \( Z \) that is defined by relation (1), the density function of the generalized beta exponential distribution is given as

\[
f_{\text{GBe}}(z) = \frac{1}{B(u,v)} \frac{1}{\sigma} \exp\left(u \frac{\delta - z}{\sigma}\right) \left(1 - (1 - c)\exp\left(-\frac{d - z}{\sigma}\right)\right)^{-1} \left(1 + c\exp\left(-\frac{d - z}{\sigma}\right)\right)^{-u-v}. \]

(2)

Where \( B(u,v) \) is the beta function, the letters of \( u \) and \( v \) are first and second shape parameters, respectively. This is under condition (1), a random variable \( Z \) given on the interval

\[
\ln (1-c)^{-1} - \sigma - \delta < z < \infty.
\]

(3)

The random variable \( Z \) is set in the interval \([(-\sigma-\delta), +\infty]\), if the value of the parameter \( c \) is equal to zero, and in the interval \([-\infty, +\infty]\) if the value of the parameter \( c \) is equal to one. The logistic function of \((\ln(1-c) - 1)\) allowed us to combine the two types of beta distribution. This makes also possible to include other types of distributions in the family of the generalized beta exponential distribution (2).

3. Parametric shape measures for the generalized beta exponential distribution

In the modern literature, probabilistic models are used to construct experimental regressions. The result of modeling the experimental data strongly depends on the flexibility of alternative models and the possibility of pre-selecting its shape. For this reason, the generalized beta exponential distribution is of interest, as this allows us to consider the continuous range of values of possible shape of the including exponential families of distributions as special cases. Usually, for the preliminary selection of the probabilistic model of the subfamily, various methods of moments are used, which are well were considered in the works [9, 10, 11, 12].

The theoretical moments of the probability distribution provide useful information about the flexibility of alternative models. Due to the fact that formula (2) unites families using a logical function, this contains properties of the generalized beta exponential for individual most complete subfamilies of distributions that we used often to analyze. For the subfamily of generalized beta exponential distribution, the expressions of the initial and standardized moments in [13, 14] is given. Formulas of the initial moments of the most significant subfamilies of distributions in table 1 are given. For these random variables, interval is given by inequality (3). In table 1, the initial and standardized moments are defined through polygamas functions of first, second, third and four order, that are denotes as \( \psi(\xi) \), \( \psi'(\xi) \), \( \psi''(\xi) \) and \( \psi'''(\xi) \), respectively.
Table 1. Properties of the generalized exponential beta distributions.

| Distributions subfamilies | Initial moments of distributions | Third and fourth standardized moments of distributions |
|---------------------------|----------------------------------|---------------------------------------------------|
| Generalized beta exponential of the first kind | $m_1 = \delta + \sigma (\psi(u+v) - \psi(u))$ | $Sk_{EGB}^I = \frac{\text{sign}(\sigma)(\psi'''(u+v) - \psi'''(u))}{(\psi'(u) - \psi'(u+v))^3/2}$ |
| | $m_2 = \sigma^2 \left( \psi'(u) - \psi'(u+v) + (\psi(u))^2 + \right.$ | $Ex_{EGB}^I = \frac{\psi'''(u) - \psi'''(u+v)}{(\psi'(u) - \psi'(u+v))^2} + 3$ |
| | $\left. + (\psi(u+v))^2 - 2\psi(u)\psi(u+v) + \right.$ | |
| | $\left. + \delta^2 + 2\delta\sigma(\psi(u+v) - \psi(u)) \right)$ | |
| Generalized beta exponential of the second kind | $m_1 = \delta + \sigma (\psi(v) - \psi(u))$ | $Sk_{EGB}^I = \frac{\text{sign}(\sigma)(\psi'''(v) - \psi'''(u))}{(\psi'(u) + \psi'(v))^3/2}$ |
| | $m_2 = \sigma^2 \left( \psi'(v) + \psi'(u) + (\psi(v))^2 - \right.$ | $Ex_{EGB}^I = \frac{\psi'''(u) + \psi'''(v)}{(\psi'(u) + \psi'(v))^2} + 3$ |
| | $\left. - 2\psi(u)\psi(v) + (\psi(u))^2 + \right.$ | |
| | $\left. + 2\delta\sigma(\psi(v) - \psi(u)) + \delta^2 \right)$ | |
| Generalized gamma exponential | $m_1 = \delta + \sigma \psi(u)$ | $Sk_{EGB}^I = (\psi'(u))^{3/2} \text{sign}(\sigma)\psi''(u)$ |
| | $m_2 = \sigma^2 \left( \psi'(u) + (\psi(u))^2 \right)$ | $Ex_{EGB}^I = (\psi'(u))^2 \psi'''(u) + 3$ |
| | $+ 2\psi(u) + \delta^2$ | |

As you can see at table 1, the standardized moments only depend on the first and second shape parameters. This allows us to use these moments as for a preliminary estimate of the shape of the generalized beta exponential distribution, and as well as for analyze the proximity of the distributions of different subfamilies.

An effective tool for analyzing distributions is the use of moment various diagrams, that were first proposed in the work [15]. Typically, moment ratio diagrams are used to preselect the distribution type. [15, 16, 17]. Standard moment diagrams allow you to investigate the closeness of shapes between different univariate distributions and create a short list of potential probabilistic models from a dataset [18]. The standardized moment of degree $k$ is the ratio of the $k$-th moment about the mean to the $k$-th power of the standard deviation. The third and fourth standardized moment are measures of skewness and kurtosis. For this reason, the coefficients of skewness and kurtosis for main subfamilies of beta distributions in table 1 is given.

As shown in [13], in the space of standard moments, there is an overlap of the distributions for the families included in the generalized beta exponential distribution. For this reason, these curves are only used to define the behavior of the models. Moreover, the location of the sampled moments in the moment ratio diagram can be used to pre-select alternative models.

4. Entropy coefficients for families of generalized beta exponential distribution

It was shown in [19, 20], the ratio of information and probability intervals allows one to obtain the entropy coefficients for shifted and non-shifted distributions, that are depend only on the shape parameters. New shape measures can be used for preliminary analysis of distributions shape together with known probabilistic shape measures such as asymmetry and kurtosis.

It was proposed in [21] to use the ratio of the $\Delta H(Y)$ entropy uncertainty interval to the square root of the $m_2(Y)$ second initial moment of the same random value as an informational characteristic of the shape for non-shifted distributions. For the entropy coefficients of the non-shifted generalized beta exponential distributions, the system of expressions is given as
At system of (4) the first and second expressions correspond to the entropy coefficients of non-shifted distributions of generalized beta exponential distributions of the first and second types, respectively, the third expression of system is the entropy coefficients of non-shifted distributions of the generalized exponential gamma distribution.

Table 2 and table 3 contains expressions for the entropy coefficients of non-shifted distributions, that were obtained by reducing the expressions of system (4). Table 2 is expressions for the subfamilies of the generalized beta exponential distribution of the first type. Table 3 is expressions for the generalized beta exponential distribution of the second type and for the generalized exponential gamma distribution. It is convenient to use simplified expressions for the entropy coefficients when you carrying out research work. As you can see, a number of special cases do not contain beta functions.

Table 2. Entropy coefficients of non-shifted distributions for the generalized beta exponential distribution of the first type.

| Distributions | Entropy coefficients of non-shifted distributions |
|---------------|--------------------------------------------------|
| Generalized beta exponential of the first kind | $K_{Ho:EB1}(u,v) = B(u,v) \cdot \frac{\exp\left(\left(\frac{u + v - 1}{\psi(v + u) - (v - 1)\psi(v) - u\psi(u)\right)}{\left(0\psi(u) - \psi'(u + v) + \left(\psi(u)\right)^2 - 2\psi(u)\psi(v) + \left(\psi(u)\right)^2\right)^{1/2}}\right.}$ |
| Generalized Gompertz-Verhulst | $K_{Ho:G-V}(v) = \frac{1}{v} \cdot \frac{\exp\left(\frac{v(\psi(v + 1) - (v - 1)\psi(v) - \psi(1))}{6^{-1}\pi^2 - \psi'(v + 1) + \left(\gamma + \psi(v + 1)\right)^2}\right)}{\left(6^{-1}\pi^2 - \psi'(v + 1) + \left(\gamma + \psi(v + 1)\right)^2\right)^{1/2}}$ |
| Generalized exponential distribution of the first kind | $K_{Ho:GE}(v) = \frac{1}{v} \cdot \frac{\exp\left(\frac{v(\psi(v + 1) - (v - 1)\psi(v) - \psi(1))}{6^{-1}\pi^2 - \psi'(v + 1) + \left(\gamma + \psi(v + 1)\right)^2}\right)}{\left(6^{-1}\pi^2 - \psi'(v + 1) + \left(\gamma + \psi(v + 1)\right)^2\right)^{1/2}}$ |
| Exponential | $K_{Ho:E}(u) = 1.9221$ |
| Beta exponential | $K_{Ho:BE}(u) = \frac{\exp(\psi(u + 1) - \psi(1) + 1)}{\sqrt{2}\left(3u^2 + 3u + 1\right)^{1/2}}$ |
| Standard Beta exponential of the first kind | $K_{Ho:SB1}(u,v) = B(u,v) \cdot \frac{\exp\left(\left(\frac{u + v - 1}{\psi(v + u) - (v - 1)\psi(v) - u\psi(u)}\right)}{\left(0\psi(u) - \psi'(u + v) + \left(\psi(u)\right)^2 - 2\psi(u)\psi(v) + \left(\psi(u)\right)^2\right)^{1/2}}\right.}$ |
| Nadarajah-Kotz distributions | $K_{Ho:N-K} = 1.376$ |
Table 3. Entropy coefficients of non-shifted distributions for the generalized beta exponential distribution of the second type and for the generalized exponential gamma distribution.

| Distributions                                             | Entropy coefficients of non-shifted distributions                                                                 |
|-----------------------------------------------------------|---------------------------------------------------------------------------------------------------------------|
| Generalized beta exponential of the second kind            | $K_{H_{EBE}_{II}}(u,v) = B(u,v) \exp\left( (u + v)\psi(v + u) - v\psi(v) - u\psi(u) \right) \left( \psi'(v) + \psi'(u) + (\psi(v) - \psi(u))^2 \right)^{1/2}$ |
| Generalized logistic distribution type IV                  | $K_{H_{Logit}_{IV}}(u,v) = B(u,v) \exp\left( (u + v)\psi(v + u) - v\psi(v) - u\psi(u) \right) \left( \psi'(v) + \psi'(u) + (\psi(v) - \psi(u))^2 \right)^{1/2}$ |
| Generalized logistic distribution type III                 | $K_{H_{Logit}_{III}}(u) = B(u,u) \left( 2\psi'(u) \right)^{-1/2} \exp\left( 2u(\psi(2u) - \psi(u)) \right)$                                           |
| Generalized logistic distribution type II                  | $K_{H_{Logit}_{II}}(u) = \frac{1}{u} \exp\left( \psi(1 + u) - \psi(1) + 1 \right) \left( \psi'(1) + \psi'(u) + (\psi(1) - \psi(u))^2 \right)^{1/2}$                     |
| Generalized logistic distribution type I                   | $K_{H_{Logit}_{I}}(v) = \frac{1}{v} \exp\left( \psi(v + 1) - \psi(1) + 1 \right) \left( \psi'(v) + \psi'(1) + (\psi(v) - \psi(1))^2 \right)^{1/2}$                       |
| Logistic distributions hyperbolic tangents                | $K_{H_{Logit}} = 4.074$                                                                                       |
| Hyperbolic secant                                         | $K_{H_{sesh}} = 4$                                                                                                |
| Gamma exponential                                         | $K_{H_{GE}}(u) = \Gamma(u) \left( \psi'(u) + (\psi(u))^2 \right)^{1/2} \exp(-u\psi(u) + u)$                          |
| Standard Gamma exponential                                 | $K_{H_{SGE}}(u) = \Gamma(u) \left( \psi'(u) + (\psi(u))^2 \right)^{1/2} \exp(-u\psi(u) + u)$                          |
| Chi square exponential                                     | $K_{H_{ChSqE}} = 2.631$                                                                                      |
| Generalized Gumbel                                         | $K_{H_{GG}}(n) = \Gamma(n) \left( \psi'(n) + (\psi(n))^2 \right)^{1/2} \exp(n(1 - \psi(n)))$                         |
| Minimum and maximum extreme value distributions            | $K_{H_{GE}} = 3.442$                                                                                           |

5. Conclusions

It follows from table 2 and table 3 that the entropy coefficient of non-shifted distributions for generalized beta exponential distributions is completely specified by two shape parameters. Wherein for most of the subfamilies, the entropy coefficients are determined by only one of the shape parameters or by a specific
numerical value. Therefore, it is possible to use the entropy coefficients of non-shifted distributions as an information measure of the shapes for non-shifted families of a generalized beta exponential distribution. It should be noted that for the entropy coefficients of non-shifted distributions equations system of (4) is subject to the condition according to which the position parameter is zero.

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