Quantum Groups and Cabibbo Mixing

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Treating the issue of hadron masses and mass relations by the use of quantum groups \( U_q(su_n) \) taken as hadron flavor symmetries suggests, at least in the case of baryons, a direct connection of the deformation parameter \( q \) with the Cabibbo angle. We discuss possible manifestations of the Cabibbo mixing implied by such connection, including unusual ones.

1 Introduction

The standard model of fundamental particles and forces incorporates the important concept of quark mixing \([1]\) usually described by the CKM matrix which in the Wolfenstein’s form looks as

\[
V_{\text{CKM}} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix} = \begin{pmatrix}
1 & \frac{1}{2}q^2 & \lambda \\
-\lambda & 1 & q \lambda^2 \\
-q \lambda^2 & -\lambda & 1
\end{pmatrix}.
\]

When keeping the first two quark families, only the parameter \( \lambda \approx 0.22 \) (the sine of Cabibbo angle \( \theta_C \)) persists. Cabibbo mixing \([1]\) is known to play basic role in describing weak decays of mesons and baryons by means of flavor changing (e.g., strangeness changing) quark currents. Our goal is to discuss some other, unusual implications of the Cabibbo mixing which are deduced on the base of adopting instead of the flavor groups \( SU(n) \) their quantum counterparts \( U_q(su(n)) \). Note that the idea of using \([2]\) in place of \( SU(n) \) the quantum groups (or \( q \)-deformed algebras) \( U_q(su_n) \) and their representations \([3]\) in order to treat hadronic flavor symmetries is natural, easily justifiable and very fruitful, leading to many interesting implications \([2, 4]\). The most important one is the possibility to link the \( q \)-parameter with the angle \( \theta_C \), and this enables to find novel connections for the concept of Cabibbo mixing. Say, the earliest modified (improved) version of the Gell-Mann - Okubo (GMO) mass relation for the \( SU(3) \) octet baryons \([5]\) derived in a particular version \([5]\) of ‘diquark-quark’ model (the LTK model), involves certain parameters characterizing diquark and separate quark. The alternative improved version of GMO for octet baryons is the extremely precise mass sum rule (MSR) obtained with \( U_q(su_n) \). It exploits fixed value of the \( q \)-parameter linked in a simple way to \( \theta_C \), and this fact allows to find direct connection, through the fixed \( q \), of the parameters of LTK diquark-quark model to the angle \( \theta_C \). Besides this, we discuss the connections of the \( \theta_C \) with such concepts as anyonic statistics parameter from the anyonic picture, as intercepts of the two- and three-pion (-kaon) correlation functions drawn in recent experiments on relativistic heavy ion collisions.

2 Baryon mass sum rules from \( q \)-deformation

2.1 Mass relations for octet and decuplet baryons

Consider first \( \frac{1}{2} \) \( q \)-baryons from \( SU(3) \) octet. Using \( U_q(su_n) \), \( n = 2, 3, 4 \), and \( U_q(u_{4,1}) \) (the latter playing the role of dynamical \( q \)-algebra), the \( q \)-deformed mass relation for octet baryons

\[
[2]M_N + \frac{[2]M_\Xi}{[2] - 1} = [3]M_\Lambda + \left( \frac{[2]^2}{[2] - 1} - [3] \right) M_\Sigma + \frac{A_q}{B_q} (M_\Xi + [2]M_N - [2]M_\Sigma - M_\Lambda) \quad (1)
\]
was obtained \([6]\). The pair of polynomials \(A_q, B_q\) (of the \(q\)-deuce \([2]\) \(\equiv [2]_q = q + q^{-1}\)), with non-overlapping sets of zeros, results from calculation in a particular dynamical representation (irrep) of \(U_q(u_{4,1})\); different irreps lead to differing pairs \(A_q, B_q\) in \([1]\). Each \(A_q\) possesses the mandatory factor \(([2]_q - 2)\), i.e. 'classical’ zero \(q = 1\), and nontrivial zeros. As particular cases, the \(q\)-analog \([1]\) yields the familiar \([7]\) Gell-Mann–Okubo (GMO) mass relation \(M_N + M_\Xi = \frac{3}{2} M_\Lambda + \frac{1}{2} M_\Sigma\) in the 'classical’ limit \(q = 1\), and two new mass sum rules of improved accuracy \([4, 8]\)

\[
M_N + \frac{3\sqrt{3}}{2} M_\Xi = \frac{2M_\Lambda}{\sqrt{3}} + \frac{2\sqrt{3}}{6} M_\Sigma \quad (0.22\%)
\]

\[
M_N + \frac{M_\Xi}{[2]_{irr} - 1} = \frac{M_\Lambda}{[2]_{irr} - 1} + M_\Sigma \quad (0.07\%)
\]

which correspond to \(A_q\) and \(\tilde{A}_q\) with respective zeros \(q_6 = e^{i\pi/6}\) and \(q_7 = e^{i\pi/7}\). Among the admissible irreps of \(U_q(u_{4,1})\) or its 'compact' counterpart \(U_q(u_5)\) there exist infinite series of irreps producing infinite set of MSRs numbered by integer \(m\) \((6 \le m < \infty)\), each given by \([1]\) with \(q_m\) put for \(q\), where \(q_m = e^{i\pi/m}\) guarantees vanishing of \(\frac{\Delta}{B_q}\). Each MSR in this set agrees better with data than the standard GMO one. Thus, a 'discrete choice' (instead of usual fitting) becomes possible; the \(q\)-polynomial \(A_q\) due to zero \(q_m\) serves as defining polynomial for the corresponding MSR. It is the value \(q_7 = e^{i\pi/7}\) and the MSR \([3]\) that provides the best choice.

For decuplet baryons \(\frac{3}{2}^+\), 1st order \(SU(3)\) breaking yields well-known \([7]\) equal spacing rule (ESR) for isolpets from \(10\)-plet. However, \(M_{\Sigma^*} - M_\Delta, M_\Xi - M_{\Sigma^*}\) and \(M_\Omega - M_{\Xi^*}\) significantly deviate from ESR: 152.6 \(\text{MeV} \leftrightarrow 148.8 \text{MeV} \leftrightarrow 139.0 \text{MeV}\). The other known relation \([9, 5]\)

\[
(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*})/2 = M_{\Sigma^*} - M_{\Sigma^*}
\]

(4)

accounts 1st, 2nd orders in \(SU(3)\)-breaking and holds only slightly better than the ESR.

On the contrary, the use of \(q\)-algebras \(U_q(su_n)\) gives nice improvement. Evaluation of decuplet masses with particular irreps of the dynamical \(U_q(u_{4,1})\) yields the \(q\)-deformed mass relation \([10]\)

\[
(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*})/(2 \cos \theta_{10}) = M_{\Sigma^*} - M_{\Sigma^*}, \quad q = \exp(i\theta_{10}).
\]

(5)

This mass relation is universal - it results from any admissible irrep (containing \(U_q(su_3)\)-decuplet embedded in \(2\theta\)-plet of \(U_q(su_4)\) ) of the dynamical \(U_q(u_{4,1})\). With empirical masses, eq. (5) holds remarkably for \(\theta_{10} \simeq \frac{\pi}{2}\). It is argued that \(\theta_{10} = \theta_C\), see \([4]\) and also next subsection.

2.2 Linking the \(q\)-parameter to the Cabibbo angle

We compare the generalization \([11]\) \(f_\pi^2 m_\pi^2 + 3 f_\eta^2 m_\eta^2 = 4 f_K^2 m_K^2\) of GMO-formula \(m_\pi^2 + 3m_\eta^2 = 4m_K^2\) for pseudoscalar (PS) mesons, with \(f_\pi^2 + 3 f_\eta^2 = 4 f_K^2\) imposed, and our \(q\)-deformed analog

\[
m_\pi^2 + [3]^{-1}(2 [2]_q - [3]_q)m_\eta^2 = 2 [2]_q(2 [2]_q - [3]_q)^{-1}m_K^2, \quad [3]_q = [2]_q^2 - 1,
\]

(6)

(found in \([2]\)) of PS-mesonic GMO, with duly fixed \(q = q_{PS}\) and physical \(\eta\)-meson put in place of \(\eta_8\) (that is, no need for explicit mixing). We find, using \(\xi_{\pi,K} \equiv (4 f_K^2 / f_\pi^2)^{-1}\), that

\[
f_K^2 / f_\pi^2 \leftrightarrow \frac{1}{2} [2] / ([2][2] - [3]), \quad [2]_\pm = 1 - \xi_{\pi,K} \pm (1 - \xi_{\pi,K})^2 + 1)^{1/2}.
\]

Since the ratio \(f_K / f_\pi\) is expressible through the Cabibbo angle (see e.g., \([12]\)), we infer: the deformation parameter \(q_{PS}\) is directly connected with the Cabibbo angle.

In another way of reasoning, we use the result of \([13]\) where the Lagrangian for quantum-group valued gauge field analog of the Weinberg - Salam (WS) model was constructed and the relation \(\tan \theta = h(q) \equiv (1 - q^2)/(1 + q^2)\) found. It provides proper mixing of the \(U(1)\)-component \(B_\mu\) and
the nonabelian component $A^3_{\mu}$. Physical photon $A_{\mu}$ and $Z$-boson of WS model appear through the Weinberg angle $\theta_W = \theta = \arctan h(q)$. At $\theta = 0$ the potentials $B_{\mu}$ and $A^3_{\mu}$ get unmixed, but the $q$-deformation with $\theta \neq 0$ provides mixing inherent in the WS model. So, the mixing of (electro)weak gauge fields is adequately modelled by the $q$-deformation. Due to the latter, the weak angle and the $q$-parameter are explicitly linked. The relation \[ \theta_W = 2(\theta_{12} + \theta_{23} + \theta_{13}), \] on the other hand, connects $\theta_W$ with the Cabibbo angle $\theta_{12} \equiv \theta_C$ (we neglect the 3rd family’s $\theta_{13}, \theta_{23}$). Thus, the apparently different mixing angles, in the (bosonic) interaction and in the fermionic (matter) sectors of the electroweak model, are related.

We conclude that the Cabibbo angle is linked with $q$-parameter of a quantum-group (or $q$-algebra) based symmetry structure applied in the fermion sector. The explicit connection is remarkably simple: $\theta_{10} = \theta_C$, $\theta_8 = 2\theta_C$. With $\theta_8 = \frac{2}{3}$ this suggests the exact value $\frac{2}{3}$ for $\theta_C$.

### 2.3 Nonpolynomiality in $SU(3)$-breaking and Michel’s statement

The universality of $q$-analog [15] concerns all admissible irreps of the ’compact’ dynamical $U_q(su_5)$, too. Say, calculation in the dynamical irrep $[4000]$ of $U_q(su_5)$ yields $M_{\Delta} = M_{10} + \beta$, $M_{\Sigma} = M_{10} + [2]3 + \alpha$, $M_{\Xi} = M_{10} + [3]\beta + [2]\alpha$, $M_{\Omega} = M_{10} + [4]\beta + [3]\alpha$, from which [15] stems. With hypercharge $Y$, all the decuplet masses $M_{D_i} \equiv M(Y(D_i))$ are comprised by single formula

$$M_{D_i} = M_{10} + \alpha[1 - Y(D_i)]_q + \beta[2 - Y(D_i)]_q \tag{7}$$

of explicit $Y$-dependence. The limit $q \to 1$ reduces it to $M_{D_i} = M_{10} + a Y(D_i)$ with $a = -\alpha - \beta$, $M_{10} = M_{10} + \alpha + 2\beta$, i.e., to linear dependence on hypercharge (or strangeness $S = Y - 1$).

Formula (7) involves highly nonlinear dependence of mass on hypercharge: for decuplet, $Y$ alone causes $SU(3)$-breaking. Since for any $N$ its $q$-number is $[N]_q = (q^N - q^{-N})/(q - q^{-1}) = q^{N-1} + q^{N-3} + \ldots + q^{-N+3} + q^{-N+1} (N$ terms), this shows exponential $Y$-dependence of masses. Such high nonlinearity makes (5) and (7) crucially different from the result (4) of traditional treatment accounting linear and quadratic effects in $Y$.

For octet baryon masses, nonpolynomiality in $SU(3)$-breaking effectively accounted [8] by the model is embodied in the expressions for isoset masses, with explicit dependence on $Y$ as well as isospin $I$, through $I(I + 1)$. Matrix elements contributing to octet baryon masses contain e.g., the terms $([Y/2]_q[Y/2+1]_q) - [I]_q[I+1]_q$ or $([Y/2-1]_q[Y/2-2]_q) - [I]_q[I+1]_q$, with multipliers depending on irrep labels $m_{15}, m_{55}$, that show explicit dependence on hypercharge and on the $q$-analog $[I]_q[I+1]_q$ of $SU(2)$ Casimir. The $q$-bracket $[n]_q$ means $[n]_q = \frac{\sinh(q)\sinh(n)}{\sinh(n)}$, $q = \exp(ih)$, so we see that octet baryon masses depend on hypercharge $Y$ and isospin $I$ (hence, on $SU(3)$-breaking effects) also in highly nonlinear - nonpolynomial - fashion.

We note finally that the conclusion made in the preceding subsection about the ability to find, due to the link $q \leftrightarrow \theta_C$, the exact value of Cabibbo angle is in accord with Michel’s statement [15] that only account of higher-order breaking effects enables gaining of such result.

### 3 Cabibbo angle and the anyonic statistics parameter

From $N$ sorts of lattice fermions $c_i(x)$, $c_i^+(x)$, $i = 1, ..., N$, with usual lattice anticommutation relations (ACRs), using the lattice angle functions [16] $\theta_\gamma(x, y)$ and $\theta_\delta(x, y)$ for two opposite ($\gamma$- and $\delta$-) types of cuts, the related ordering on the lattice ($x > y$ or $y > x$), and the two types of statistical operators $K_i(x)$ and $\tilde{K}_i(x)$,

$$K_j(x) = \exp(i\nu \sum_{y \neq x} \theta_\gamma(x, y)c_j^+(y)c_j(y)), \quad K_j(x) = \exp(i\nu \sum_{y \neq x} \theta_\delta(x, y)c_j^+(y)c_j(y)), \tag{8}$$

one defines [16] the anyonic oscillators $a_i(x) = K_i(x)c_i(x)$ and $a_i(x) = \tilde{K}_i(x)c_i(x)$ involving the anyonic statistics parameter $\nu$. From this definition and ACR’s for lattice fermions, the
relations of permutation for anyonic oscillators then follow. Some of them, e.g.

\[ a_i(x) a_i(y) + q^{-\text{sgn}(x-y)} a_i(y) a_i(x) = 0, \quad a_i(x) a_i^\dagger(y) + q^{\text{sgn}(x-y)} a_i^\dagger(y) a_i(x) = 0, \]

involve the deformation parameter \( q \) connected with the statistics parameter \( \nu \) of [8] as \( q = \exp(i \pi \nu) \). The generating elements \( A_{j,j+1}, A_{j+1,j} \) and \( H_j \) of the quantum algebra \( U_q(su_N) \) are realized bilinearly through anyonic oscillators \( a_i(x), a_i^\dagger(y) \) and shown to satisfy [16] the defining relations [3] of the quantum algebra \( U_q(su_N) \). Dual realization in terms of \( a_i(x), a_i^\dagger(y) \) is also valid. Then, in anyonic realization of \( U_q(su_N) \), both hadron mass operator \( M \) and basis vectors for hadronic irreps are explicitly constructed [17]. Say, for the irrep \( \{4000\} \) of 'dynamical' \( U_q(su_5) \), in accordance with the chain of \( q \)-algebras \( U_q(su_3) \subset U_q(su_4) \subset U_q(su_5) \) and respective chain of irreps \([30] \subset [300] \subset [4000]\), all basis state vectors \( |n_1 n_2 n_3 n_4\rangle \equiv a_{n_1}^\dagger(x_1) a_{n_2}^\dagger(x_2) a_{n_3}^\dagger(x_3) a_{n_4}^\dagger(x_4)|0\rangle \) of baryons \( \frac{2}{3}^+ \) are constructed by acting with lowering generators upon the highest weight vector. E.g., for isosuqartet baryon \( |\Delta^{++}\rangle \) we get \( |\Delta^{++}\rangle = [4]^{-1/2}(|5111\rangle + q^{-1}|1511\rangle + q^{-2}|1151\rangle + q^{-3}|1115\rangle) \), and similarly for \( |\Sigma^\star\rangle, |\Xi^\star\rangle, |\Omega^-\rangle \). The dual basis vectors \( |\Delta^{++}\rangle \) etc., are also given. Masses \( M_{D_1} \) of baryons \( D_1 \) in the dynamical \( U_q(su_5) \)-irrep \( \{4000\} \) are calculated with mass operator \( \hat{M} \) as \( M_{D_1} = [D_1]|M|D_1 \rangle \) to yield: \( M_\Delta = M_{10} + \beta, M_{\Sigma^\star} = M_{10} + [2]_q \alpha + [2]_q \beta, M_{\Xi^\star} = M_{10} + [2]_q [3]_q \alpha + [4]_q \beta \), and \( M_{1^\star} = M_{10} + [2]_q [3]_q \alpha + [4]_q \beta \), from which the relation [5] follows. This proves applicability [17] of quantum algebras and their irreps for deriving hadron mass relations by using their anyonic realization.

Since \( \theta_{10} = \theta_C \), we have the connection: Cabibbo angle \( \leftrightarrow \) anyonic statistics parameter \( \nu \).

## 4 Diquark-quark model parameters and the Cabibbo angle

The LTK diquark-quark model [5] uses, besides the \( SU(3) \) invariant masses \( m_t, m_s \) and \( m_q \) of the \( SU(3) \) diquark triplet, diquark sextet, and 3rd quark triplet (so the subscripts 't', 's', 'q'), also the mass parameters \( \delta_t, \delta_s \) and \( \delta_q \) measuring \( SU(3) \) violation in the respective multiplets.

The improved form of GMO obtained in the LTK model is

\[ \frac{3}{2} m_\Lambda + \frac{1}{2} m_\Sigma - m_N - m_\Xi = C_{\text{LTK}} \equiv \mu_s (3 \xi_{ts}^2 + 18 \xi_{ts} - 13) \]  

(9)

where \( \xi_{ts} = \frac{\delta_t - \delta_q}{\delta_s - \delta_q} \), \( \mu_s = V_{01}^2 \), \( \gamma_s = \frac{(\delta_s - \delta_q)}{\delta_s} \). The \( \mu_s \) must be positive being also involved [5] in the decuplet mass combination: \( 8 \mu_s = 2 m_\Xi - m_\Omega - m_\Sigma^\star > 0 \). As result, the LTK model gave the value \( \xi_{ts}^{\text{LTK}} = -3 \) which implied: \( \delta_t \) must be respectively greater (less) than \( \delta_q \) with the parameter \( \delta_s \) being less (greater) than \( \delta_q \). However, the value \( \xi_{ts}^{\text{LTK}} = -3 \) contradicts data as it renders the r.h.s. of [9] to be negative thus correcting the GMO in wrong direction.

On the other hand, our \( q \)-MR [3] for which \( q_t = \exp(i \hat{\phi}), \hat{\phi} = 2\theta_C \), can be rewritten as

\[ \frac{3}{2} m_\Lambda + \frac{1}{2} m_\Sigma - m_N - m_\Xi = C_{q_t} \equiv 2 - [2]_q [m_\Xi - m_\Lambda] - \frac{1}{2} (m_\Sigma - m_\Lambda). \]  

(10)

Comparison of the two improved versions [9], [10] leads to the relation [18]

\[ ([2]_q - 1)^{-1} = \frac{9 \xi_{ts}^2 - 6 \xi_{ts} + 5}{4(\xi_{ts}^2 - 6 \xi_{ts} + 5)}, \quad \bar{\xi}_{ts} = \frac{\delta_t - \delta_q}{\delta_s - \delta_q}, \]  

(11)

which connects the value \( q_t = \exp(i \hat{\phi}) \) of \( q \)-parameter in the \( q \)-GMO [11] with the ratio \( \xi_{ts} \) of the LTK model. Remark that the values of \( \delta_t, \delta_s \) and \( \delta_q \) in this relation are understood as the optimized ones reflecting, through the found connection, all-order account by [10] of \( SU(3) \) symmetry breaking in octet baryon masses - just this fact is denoted by tildas over \( \delta_t, \delta_s, \delta_q \).
5.1 Intercept \( \lambda \)

To obtain explicitly the intercept \( \lambda \) commutation relations. Deviation of seen to tend towards the Boltzmann one (reduced quantum statistical effects). For kaons, whose mass distribution \( \langle \lambda \rangle \) that for AC-type q-bosons, the \( \langle \lambda \rangle \)-deformed distribution function for the BM-type \([19]\) may be used to describe unusual behaviour of 2-particle correlations of identical pions or kaons produced in relativistic heavy ion collisions.

The model of ideal gas of q-bosons based on the algebra of q-deformed oscillators of Arik-Coon (AC) or Biedenharn-Macfarlane (BM) type \([19]\), may be used to describe unusual behaviour of 2-particle correlations of identical pions or kaons produced in relativistic heavy ion collisions.

The approach yields explicit expressions and clear predictions \([20]\) for the intercept \( \lambda \) (dependent on the temperature, particle mass, pair mean momentum, and the deformation parameter \( q \)).

Physical observables are evaluated as thermal averages \( \langle A \rangle = Sp(\rho(\beta))/Sp(\rho) \), \( \rho = e^{-\beta H} \), where the Hamiltonian is \( H = \sum \omega_i N_i \) and \( \beta = 1/T \). With \( b_i^\dagger b_i = [N_i q][2]q = 2 \cos \theta \), the q-deformed distribution function for the BM-type q-bosons results (see e.g. \([20]\)) as

\[
\langle b_i^\dagger b_i \rangle = (e^{\beta \omega_i} - 1)/(e^{2\beta \omega_i} - 2 \cos \theta e^{\beta \omega_i} + 1).
\]

At \( \theta = 0 \) (or \( q = 1 \)), it yields Bose-Einstein (BE) distribution, as \( q = 1 \) recovers usual bosonic commutation relations. Deviation of \( q \)-distribution \([14]\) from the quantum BE distribution is seen to tend towards the Boltzmann one (reduced quantum statistical effects). For kaons, whose mass \( m_K > 3m_\pi \), analogous curve gets closer (than pion’s one) to the BE distribution. Note that for AC-type q-bosons, the \( q \)-distribution looks more simple: \( \langle b_i^\dagger b_i \rangle = (e^{\beta \omega_i} - q)^{-1} \).

5 Cabibbo mixing in multiparticle correlations?

To obtain explicitly the intercept \( \lambda \) of two-particle correlations and Cabibbo angle

5.1 Intercept \( \lambda \) of two-pion (two-kaon) correlations and Cabibbo angle

To obtain explicitly the intercept \( \lambda \) of two-particle correlations one calculates the two-particle distribution \( \langle b_i^\dagger b_j^\dagger b_i b_j \rangle \) and normalizes it by \( \langle b_i b_j \rangle^2 \). The result, see \([20]\), for AC-type q-bosons reads

\[
\lambda = q - \frac{q(l - q^2)}{e^{\beta q} - q}, \quad -1 \leq q \leq 1, \quad \text{and for BM-type q-bosons,} \quad \mathcal{F}(\beta \omega) \equiv \cosh(\beta \omega), \quad \text{it is}
\]

\[
\lambda + 1 \equiv \langle b_i^\dagger b_j^\dagger b_i b_j \rangle / (\langle b_i b_j \rangle^2) = \frac{2 \cos \theta (\mathcal{F}(\beta \omega) - \cos \theta)^2}{(\mathcal{F}(\beta \omega) - 1)(\mathcal{F}(\beta \omega) - 2 \cos^2 \theta + 1)}.
\]
K over three intervals of transverse momenta temperature is give (see discussion in [22]) the values λ. Then, experimentally there should be a tendency of merging q. As suggested in [21, 22], correlations of pions and kaons are characterized by the same value of λ and θ that just the angle 2. 5. Three-particle correlations of pions (viewed as q-bosons) are fixed temperature, the intercept T = 180 MeV and fixed θ angle 2. Insisting on the asymptotical coincidence λasymp = q for the AC-type q-bosons (q real), and for the BM-type q-bosons (q is a phase factor) to the constant λBMasymp = 2 cos θ − 1, θ = −i ln q. (16)

As suggested in [21, 22], correlations of pions and kaons are characterized by the same value of q (a kind of universality). Then, experimentally there should be a tendency of merging λ(π) and λ(K) at large enough mean momenta: λasymp(π) = λasymp(K). Recent RHIC/STAR data give (see discussion in [22]) the values λ1(π), λ2(π) and λ3(π) for π−-intercept, as averaged over three intervals of transverse momenta Kt and also over rapidity y, −0.5 ≤ y ≤ 0.5.

In Fig.1, we show three curves for the intercept λ which correspond to fixing in [15] certain values of the deformation angle θ, along with the data λj(π), j = 1, 2, 3 (with error bars). The temperature is T = 180 MeV for all curves. We find remarkable agreement with data for the curve C (i.e., θ = 28.5°). Another interesting case is the curve B at θ = 28.5°, At T = 180 MeV, the curve B agrees, within error bars, with the points λ2(π) and λ3(π). However, slightly higher effective temperature T ≃ 198 MeV makes the curve marked by θ = 20°C respecting all the three error bars. Among various mixing angles known for hadrons, only the angle 20°C seems to be relevant to the issue of intercept λ(π). Hence, it is tempting to suggest that just the angle 20°C embodies the assumed universality, to be seen in 2-particle correlations:

λBMasymp(π)|θ=π/7 = λBMasymp(K)|θ=π/7 = 2 cos π/7 − 1 ≃ 0.80194. (17)

Insisting on the asymptotical coincidence λasymp(π) = λasymp(K), we predict for kaons: the intercept of 2-kaon correlations at any transverse momenta should obey: λ(K) ≤ 0.80194.

5.2 Three-particle correlations of pions (viewed as q-bosons) and θC

Now consider higher order (or multi-particle) correlations, first in the case of AC-type q-oscillators. It is not difficult to derive [23] the following 3-particle monomode correlation function

\[ \langle a_i^\dagger a_j^\dagger a_i a_j a_i \rangle = \frac{(1 + q)(1 + q + q^2)}{(e^{\eta_i} - q)(e^{\eta_i} - q^2)(e^{\eta_i} - q^3)} \] (18)

with ηi ≡ βωi. From this, and 1-particle distribution \( \langle a_i^\dagger a_i \rangle = \frac{1}{e^{\eta_i} - q^3} \), the intercept of 3-particle correlations for AC-type q-bosons in monomode case, dropping the subscript "i", reads:

\[ \lambda^{(3)}_{AC} + 1 = \frac{\langle a_i^\dagger a_i a_i^\dagger a_i a_i a_i^\dagger a_i a_i \rangle}{\langle a_i^\dagger a_i \rangle^3} = \frac{(1 + q)(1 + q + q^2)(e^{\eta_i} - q^2)}{(e^{\eta_i} - q)(e^{\eta_i} - q^2)(e^{\eta_i} - q^3)}, \quad -1 \leq q \leq 1. \] (19)
The $n$-particle generalization of (18) is obtained [23] in the form
\[
\langle (a_i^n)^n \rangle = \frac{[n]_q!}{\prod_{r=1}^n (e^m - q^r)} , \quad [m]_q \equiv \frac{1 - q^m}{1 - q} = 1 + q + q^2 + ... + q^{m-1}
\]
(20)

where $[n]_q! = [1]_q [2]_q ... [n]_q$. Moreover, this result admits direct two-parameter extension [23] to the model of $pq$-Bose gas, based on the use of so-called $pq$-oscillators, in the form
\[
\langle (A_i^n)^n \rangle = \frac{[n]_q!}{\prod_{r=1}^n (e^m - q^r)} , \quad [m]_{pq} \equiv \frac{q^m - p^m}{q - p} .
\]
(21)

With the use of one-particle distribution $\langle A_i^\dagger A_i^\dagger \rangle = \frac{e^w - 1}{(e^w - q)(e^w - p)}$ of $qp$-bosons (see also [24]) we obtain general formula [23] for the $qp$-deformed $n$-th order intercept $\lambda_{q,p}^{(n)}$:
\[
\lambda_{q,p}^{(n)} \equiv -1 + \frac{\langle A_i^n A_i^n \rangle}{\langle A_i^\dagger A_i^\dagger \rangle} = -1 + [n]_{pq}! \frac{(e^w - p)^n (e^w - q)^n}{(e^w - 1)^{n-1} \prod_{k=0}^{n-1} (e^w - q^{n-k}p^k)} .
\]
(22)

From this, the result [19] for AC-type $q$-bosons follows if $p = 1$. Similarly, putting $p = q^{-1}$ reduces to the important case of BM-type $q$-bosons which for the 3-particle intercept yields
\[
\lambda_{BM}^{(3)} \equiv -1 + \frac{\langle a_i^3 a_i^3 \rangle}{\langle a_i^\dagger a_i^\dagger \rangle^3} = -1 + [2]_q [3]_q \frac{(e^w - 2 \cos \theta e^w + 1)^2}{(e^w - 1)^2 (e^w - 2 \cos (3\theta) e^w + 1)} .
\]
(23)

(compare it with eq. [19]). Again, like the intercept $\lambda_{\text{asympt}}^{(2),BM}$ in [16], at large $w$ (large momenta or low $T$) we get $\lambda_{\text{asympt}}^{(3),BM}$ dependent solely on the deformation parameter $q = \exp(i\theta)$, i.e.
\[
\lambda_{\text{asympt}}^{(3),BM} = -1 + [2]_q [3]_q = -1 + 2 \cos (2 \cos \theta - 1)(2 \cos \theta + 1) .
\]
(24)

To confront the results [22], [24] with experimental data, consider the combination [25]
\[
r_0 = (\lambda^{(3)} - 3\lambda^{(2)})^{1/3} .
\]
(25)

Then, the $r_0$ that follows from [15] and [24] (or $r_0,\text{asympt}$ stemming from [16] and [24]) is at fixed $T$ a decreasing function of $\theta$ for $0 \leq \theta < \pi/3$. What about the Cabibbo angle? If we take the value $\theta = 2\theta_C$ of deformation angle as we did in the preceding subsection we get $r_0|_{\theta = 2\theta_C} \simeq 0.8955$. The universality conjecture dictates now that all possible values of the $r_0$ (composed of intercepts $\lambda^{(2)}$ and $\lambda^{(3)}$) for either pions or kaons should respect the value 0.8955. The presently available data [24] extracted in Pb-Pb and Au-Au collisions seems to be yet insufficient to prove or disprove this assertion.

6 Conclusion

The use of quantum groups (quantum algebras) in the context of baryon phenomenology implies important fact that the deformation parameter $q$ is linked very simply to the basic (fermion) Cabibbo mixing angle $\theta_C$. In turn, this leads to unexpected connections for $\theta_C$ and for the whole concept of Cabibbo mixing, e.g., with the parameters of diquark-quark model of baryons, with the statistics parameter of anyonic picture, with the experimentally testable intercept parameters of multi-pion (-kaon) correlation functions, etc.. Of course, it would be highly desirable to (re)obtain the considered connections of $\theta_C$ in a more strict way, and we hope this will be realized in a not very distant future.
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