INTERMITTENT SYMMETRY BREAKING AND STABILITY OF THE SHARP AGMON–HÖRMANDER ESTIMATE ON THE SPHERE

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Abstract. We compute the optimal constant and characterise the maximisers at all spatial scales for the Agmon–Hörmander $L^2$-Fourier adjoint restriction estimate on the sphere. The maximisers switch back and forth from being constants to being non-symmetric at the zeros of two Bessel functions. We also study the stability of this estimate and establish a sharpened version in the spirit of Bianchi–Egnell. The corresponding stability constant and maximisers again exhibit a curious intermittent behaviour.

1. Introduction

In [1, Theorem 2.1], Agmon and Hörmander established an estimate for compact manifolds, which reads as follows when applied to the unit sphere $S^{d-1} \subset \mathbb{R}^d$, for $d \geq 2$:

$$\frac{1}{\rho} \int_{B_\rho} |\hat{\sigma}(x)|^2 \frac{dx}{(2\pi)^d} \leq C_d(\rho) \int_{S^{d-1}} |f(\omega)|^2 \ d\sigma(\omega).$$

Here, $\sigma$ denotes the standard surface measure on $S^{d-1}$ and $B_\rho \subset \mathbb{R}^d$ denotes the ball of radius $\rho > 0$ centred at the origin. Also, $\hat{\sigma}$ denotes the Fourier transform; see the forthcoming (2.2).

In the aforementioned paper, it is shown that the quantity $C_d(\rho)$ is uniformly bounded in $\rho$, but no explicit value is given for it. The first purpose of this note is to provide a proof of (1.1) that yields the optimal value $C_d(\rho)$, as well as the functions that attain it. We will see that, as $\rho$ increases, these maximising functions change intermittently at the zeros of the Bessel functions $J_{\nu}(\rho)$ and $J_{\nu+1}(\rho)$, where $\nu = \frac{d}{2} - 1$; recall (e.g. from Appendix A) that these functions do not have common positive zeros. For $k \in \mathbb{N}_{\geq 0}$, we introduce the function

$$\Lambda_{k,d}(\rho) := \frac{\rho}{2} J_{\nu+k}(\rho) - \frac{\rho}{2} (J_{\nu+k-1} J_{\nu+k+1})(\rho).$$

Finally, we let $\mathcal{H}_k$ denote the vector subspace of $L^2(S^{d-1})$ consisting of the spherical harmonics of degree $k \in \mathbb{N}_{\geq 0}$; in particular, $\mathcal{H}_0$ consists of the constants. See §2 for details.

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Theorem 1. For each $\rho > 0$, the optimal constant $C_d(\rho)$ equals
\[
\max_{0 \neq f \in L^2(S^{d-1})} \frac{1}{\rho} \int_{B_\rho} |\hat{\sigma}(x)|^2 \frac{dx}{(2\pi)^d} = \frac{\|f\|^2_{L^2(S^{d-1})}}{\Lambda_{0,d}(\rho)} = \begin{cases} 
\Lambda_{0,d}(\rho), & (J_\nu J_{\nu+1})(\rho) > 0, \\
\Lambda_{1,d}(\rho), & (J_\nu J_{\nu+1})(\rho) \leq 0.
\end{cases}
\]

The maximum is attained if and only if $f \in \mathcal{M}_d(\rho) \setminus \{0\}$, where $\mathcal{M}_d(\rho)$ equals

(i) $\mathcal{H}_0$, $\quad (J_\nu J_{\nu+1})(\rho) > 0$;
(ii) $\mathcal{H}_1$, $\quad (J_\nu J_{\nu+1})(\rho) < 0$;
(iii) $\mathcal{H}_0 \oplus \mathcal{H}_1$, $\quad J_\nu(\rho) = 0$;
(iv) $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, $\quad J_{\nu+1}(\rho) = 0$.

In case (ii), the constants $\mathcal{H}_0$ are not maximisers; this is remarkable, since both sides of (1.1) are rotationally invariant, hence one might expect the maximisers to be invariant functions (the constants being the only such functions on the sphere). In the literature, this phenomenon is sometimes called symmetry breaking (see, e.g. [9, p. 25]), and it is especially studied in the context of inequalities related to elliptic PDE. We find it striking that in our context, involving oscillatory operators, symmetry breaking occurs even at the relatively elementary level of (1.1), which is an $L^2$-estimate and as such can be studied via orthogonality methods.

Another noteworthy feature of Theorem 1 is that the space of maximisers $\mathcal{M}_d(\rho)$ does not vary continuously with $\rho$, but instead is locally constant, except for a sequence of jumps at the zeroes of $J_\nu$ and $J_{\nu+1}$. This lies at the root of another interesting fact, which follows from the next result. Letting $\delta(f, \mathcal{M}_d(\rho)) = \inf\{\|f - m\|_{L^2(S^{d-1})} : m \in \mathcal{M}_d(\rho)\}$,
Theorem 2. For each $f$, there is equality in the left-hand inequality if and only if
\[ \delta_d(f; \rho) \geq 0, \]
with $\delta_d(f; \rho) = 0$ if and only if $f \in \mathcal{M}_d(\rho)$. Letting $\delta_d(f; \rho) \leq 0$ in the right-hand inequality only in the trivial case $\rho = 0$. Note that $\delta_d(f; \rho) \geq 0$, with $\delta_d(f; \rho) = 0$ if and only if $f \in \mathcal{M}_d(\rho)$.

**Corollary 3.** Let $d \geq 2$ and $\rho \in (0, \infty)$. Then:

- The function $\rho \mapsto C_d(\rho)$ is not differentiable at each positive zero of $J_\nu J_{\nu+1}$. It defines a Lipschitz function on $(0, \infty)$ which is real-analytic between any two consecutive zeroes of $J_\nu J_{\nu+1}$.

1The symbol $\parallel$ is to be replaced by either of the values above or the below, which coincide in each case.
Figure 2. The sharpened Agmon–Hörmander estimate on $S^1 \subset \mathbb{R}^2$: plot of the stability constant $S_2(\rho)$, for $0 < \rho \leq 15$.

- The function $\rho \mapsto S_2(\rho)$ has a jump discontinuity at each positive zero of $J_\nu J_{\nu+1}$. It defines a piecewise real-analytic function between any two consecutive zeroes of $J_\nu J_{\nu+1}$, which fails to be differentiable at each positive zero of $J_{\nu+2}$.

Some of the loss-of-regularity phenomena exhibited by $C_d(\rho)$ and $S_d(\rho)$ are depicted in Figures 1 and 2 respectively. The behaviour of optimal constants has been studied in the context of the non-oscillatory Brascamp–Lieb inequalities, see [3, 4], where it found numerous applications, in particular to Fourier restriction theory.

In the $L^2$-setting of the present paper, estimate (1.1) has been generalised [2], and sharp and sharpened inequalities in the context of smoothing and trace estimates have been extensively investigated; see [5, 6], and the references therein. Within this general framework, the main new contributions of Theorems 1 and 2 lie in the complete solutions to both the sharp and the sharpened problems, in terms of the explicit values for the optimal constants $C_d(\rho), S_d(\rho)$ and the full characterisation of the spaces of maximisers $\mathcal{M}_d, \mathcal{E}_d$.

We finish the Introduction with some brief remarks on sharp Fourier restriction theory. Inequality (1.1) can be regarded as the most basic example of an adjoint Fourier restriction estimate on the sphere [19, §5]. The corresponding optimal constant and maximisers are known only for a few such estimates; most notably, Foschi [10] proved that constant functions maximise the endpoint Stein–Tomas estimate on $S^2$. It turns out that constant functions maximise the $L^2(S^{d-1}) - L^{2n}(\mathbb{R}^d)$ adjoint restriction estimate for every $d \in \{3, 4, 5, 6, 7\}$ and integer $n \geq 2$; see [8, 17]. Moreover, sharpened Fourier restriction inequalities have been recently established in [12, 13, 14, 15, 16]. We refer the interested reader to the survey [11] for a more extended discussion and further references.
Structure of the paper. We prove Theorem 1, Theorem 2, and Corollary 3 in §2, §3, and §4, respectively. For every \( f \in L^2(S^{d-1}) \), our methods also yield the limit
\[
\lim_{\rho \to \infty} \frac{1}{\rho} \int_{B_\rho} |\hat{f}\sigma(x)|^2 \frac{dx}{(2\pi)^d} = \frac{1}{\pi} \int_{S^{d-1}} |f(\omega)|^2 \, d\sigma(\omega),
\]
in accordance with Agmon–Hörmander [1, Theorem 3.1]. We discuss this limit in §5. An immediate corollary is that \( C_d(\rho) \to \frac{1}{\pi} \) and \( S_d(\rho) \to 0 \), as \( \rho \to \infty \); see Figures 1 and 2. Finally, we collect all the relevant facts concerning Bessel functions in Appendix A.

2. THE OPTIMAL CONSTANT \( C_d(\rho) \) AND ITS MAXIMISERS: PROOF OF THEOREM 1

Throughout the paper, set \( \nu = d/2 - 1 \). We will always use the notation \( Y_k(\omega) \) to denote a spherical harmonic of degree \( k \in \mathbb{N}_{>0} \), which by definition is a complex-valued homogeneous harmonic polynomial in \( \omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d \), of degree \( k \), considered as a function on \( S^{d-1} \). In particular, \( Y_k \) is never the zero function. As stated in the Introduction, we denote
\[
\mathcal{H}_k := \{ Y_k : Y_k \text{ is a spherical harmonic of degree } k \} \cup \{0\},
\]
which is a finite-dimensional vector subspace of \( L^2(S^{d-1}) \). Spherical harmonics of different degrees are mutually orthogonal and form a complete system, see [18, Chapter IV, Corollary 2.3], meaning that to each nonzero \( f \in L^2(S^{d-1}) \) there uniquely correspond \( F[f] \subseteq \mathbb{N}_{>0} \) and \( \{ Y_k[f] \in \mathcal{H}_k \setminus \{0\} : k \in F[f] \} \) such that
\[
f = \sum_{k \in F[f]} Y_k[f], \quad \text{thus } \|f\|_{L^2(S^{d-1})}^2 = \sum_{k \in F[f]} \|Y_k[f]\|_{L^2(S^{d-1})}^2.
\]
For notational convenience, we will leave out the dependence on \( f \), writing \( F \) and \( Y_k \) in place of \( F[f] \) and \( Y_k[f] \), respectively.

For a single spherical harmonic, we have the Fourier transform formula
\[
\hat{Y}_k(\xi) = \int_{S^{d-1}} Y_k(\omega) e^{-i\omega \cdot \xi} \, d\sigma(\omega) = \frac{(2\pi)^{\frac{d}{2}}}{i^k |\xi|^\nu} J_{\nu+k}(|\xi|) Y_k \left( \frac{\xi}{|\xi|} \right), \quad \xi \in \mathbb{R}^d;
\]
see, for example, [18, Chapter IV, Theorem 3.10]. Using the decomposition (2.1) and integrating in polar coordinates, we obtain the diagonal form of the left-hand side in (1.1):
\[
\frac{1}{\rho} \int_{B_\rho} |\hat{f}\sigma(x)|^2 \frac{dx}{(2\pi)^d} = \sum_{k \in F} \frac{1}{\rho} \int_{B_\rho} \int_{S^{d-1}} |\hat{Y}_k(\omega)|^2 r^{d-1} \frac{dr \, d\sigma(\omega)}{(2\pi)^d}
\]
\[
= \sum_{k \in F} \frac{1}{\rho} \int_0^\rho J_{\nu+k}^2(r^2 \rho^2) \frac{dr}{2\rho} \|Y_k\|_{L^2(S^{d-1})}^2.
\]
As we will see in Lemma 5 below, the latter integral can be evaluated explicitly, yielding
\[
\frac{1}{\rho} \int_{B_\rho} |\hat{f}\sigma(x)|^2 \frac{dx}{(2\pi)^d} = \sum_{k \in F} \Lambda_{k,\nu}(\rho) \|Y_k\|_{L^2(S^{d-1})}^2,
\]
where the coefficients \( \Lambda_{k,\nu} \) have been introduced in (1.2). In turn, the coefficients \( \Lambda_{k,\nu} \) are related to the optimal constant \( C_d(\rho) \) via the following simple observation.
Lemma 4. For each \( \rho > 0 \),
\[
(2.4) \quad C_d(\rho) := \sup_{0 \neq f \in L^2(S^{d-1})} \frac{\frac{1}{\rho} \int_{B_\rho} |\widehat{f}(x)|^2 \frac{dx}{(2\pi)^d}}{\|f\|_{L^2(S^{d-1})}^2} = \sup \{ \Lambda_{k,d}(\rho) : k \in \mathbb{N}_{\geq 0} \}.
\]

Letting \( K := \{ k \in \mathbb{N}_{\geq 0} : \Lambda_{k,d}(\rho) = \sup_{h \in \mathbb{N}_{\geq 0}} \Lambda_{h,d}(\rho) \} \), we have that \( f \in L^2(S^{d-1}) \setminus \{0\} \) attains the supremum in (2.4) if and only if
\[
(2.5) \quad f(\omega) = Y_{k_1}(\omega) + Y_{k_2}(\omega) + \ldots + Y_{k_n}(\omega), \quad k_j \in K.
\]

We remark that, a priori, \( K \) could be empty, or it could be infinite; we will prove that this is never the case.

Proof of Lemma 4. From (2.1) and (2.3) it follows that
\[
\frac{1}{\rho} \int_{B_\rho} |\widehat{f}(x)|^2 \frac{dx}{(2\pi)^d} \frac{\|f\|_{L^2(S^{d-1})}^2}{\sum_{k \in F} \Lambda_{k,d}(\rho) \|Y_k\|_{L^2(S^{d-1})}^2} = \frac{\sum_{k \in F} \|Y_k\|_{L^2(S^{d-1})}^2}{\sum_{k \in F} \|Y_k\|_{L^2(S^{d-1})}^2}.
\]

It is clear that the supremum of this quotient is \( \sup_{k \geq 0} \Lambda_{k,d}(\rho) \), with equality if and only if \( F \subseteq K \), verifying (2.5) and concluding the proof of the lemma.

Lemma 5. For each \( \rho > 0 \) and \( k \in \mathbb{N}_{\geq 0} \),
\[
(2.6) \quad \frac{1}{\rho} \int_0^\rho J_{\nu+k}^2(r) dr = \Lambda_{k,d}(\rho) = \frac{\rho}{2} J_{\nu+k}^2(\rho) - \frac{\rho}{2} J_{\nu+k-1}(\rho) J_{\nu+k+1}(\rho).
\]

Moreover,
\[
(2.7) \quad \Lambda_{k,d}(\rho) - \Lambda_{k+1,d}(\rho) = J_{\nu+k}(\rho) J_{\nu+k+1}(\rho),
\]
\[
(2.8) \quad \Lambda_{k,d}(\rho) - \Lambda_{k+2,d}(\rho) = \frac{2(\nu + k + 1)}{\rho} J_{\nu+k+1}^2(\rho).
\]

In particular, \( \Lambda_{k,d}(\rho) \geq \Lambda_{k+2,d}(\rho) \), with equality if and only if \( J_{\nu+k+1}(\rho) = 0 \).

Proof. Identity (2.6) is due to Lommel; see [20] §5.11 (11). We apply this identity, together with the Bessel recursion (A.3) in Appendix A, to obtain
\[
\Lambda_{k,d}(\rho) - \Lambda_{k+1,d}(\rho) = \frac{\rho}{2} \left[ J_{\nu+k}(\rho) (J_{\nu+k}(\rho) + J_{\nu+k+2}(\rho)) - J_{\nu+k+1}(\rho) (J_{\nu+k-1}(\rho) + J_{\nu+k+1}(\rho)) \right]
\]
\[
= (\nu + k + 1)(J_{\nu+k} J_{\nu+k+1}(\rho)) - (\nu + k)(J_{\nu+k} J_{\nu+k+1}(\rho))
\]
\[
= (J_{\nu+k} J_{\nu+k+1}(\rho)).
\]
respectively, and it is attained by linear combinations of spherical harmonics of degree 0 or 1 in Cases (i) and (ii), indeed applying the Bessel recursions \((A.3)\)–\((A.4)\), we have that

\[
\Lambda_{k,d}(\rho) - \Lambda_{k+2,d}(\rho) = \frac{1}{\rho} \int_0^\rho (J_{\nu+k}^2(r) - J_{\nu+k+2}^2(r)) \, dr
\]

\[
= \frac{4(\nu + k + 1)}{\rho} \int_0^\rho J_{\nu+k+1}(r)J'_{\nu+k+1}(r) \, dr
\]

\[
= \frac{2(\nu + k + 1)}{\rho} J_{\nu+k+1}^2(\rho),
\]

where in the last computation we used the fact that \(J_{\nu+k+1}(0) = 0\). \(\square\)

Having settled these classical preliminaries, we now start with the actual proof of Theorem 1. By Lemma 5, we obtain the following chains of inequalities.

(i) Case \((J_\nu J_{\nu+1})(\rho) > 0\). We have

\[
\begin{align*}
\Lambda_{0,d}(\rho) &> \Lambda_{1,d}(\rho) \geq \Lambda_{3,d}(\rho) \geq \Lambda_{5,d}(\rho) \geq \ldots \\
\Lambda_{0,d}(\rho) &> \Lambda_{2,d}(\rho) \geq \Lambda_{4,d}(\rho) \geq \Lambda_{6,d}(\rho) \geq \ldots
\end{align*}
\]

The strict inequalities follow from \((2.7)\) and \((2.8)\), respectively.

(ii) Case \((J_\nu J_{\nu+1})(\rho) < 0\). We have

\[
\begin{align*}
\Lambda_{1,d}(\rho) &> \Lambda_{0,d}(\rho) \geq \Lambda_{2,d}(\rho) \geq \Lambda_{4,d}(\rho) \geq \ldots \\
\Lambda_{1,d}(\rho) &> \Lambda_{3,d}(\rho) \geq \Lambda_{5,d}(\rho) \geq \Lambda_{7,d}(\rho) \geq \ldots
\end{align*}
\]

The strict inequalities are obtained as before, noting that \(J_{\nu+2}(\rho) \neq 0\); see Lemma 7.

(iii) Case \(J_\nu(\rho) = 0\). Note that \(J_{\nu+1}(\rho) \neq 0\) and \(J_{\nu+2}(\rho) \neq 0\), by Lemma 8. Reasoning as in the previous steps, we obtain

\[
\begin{align*}
\Lambda_{0,d}(\rho) &> \Lambda_{2,d}(\rho) \geq \Lambda_{4,d}(\rho) \geq \Lambda_{6,d}(\rho) \geq \ldots \\
\Lambda_{1,d}(\rho) &> \Lambda_{3,d}(\rho) \geq \Lambda_{5,d}(\rho) \geq \Lambda_{7,d}(\rho) \geq \ldots
\end{align*}
\]

(iv) Case \(J_{\nu+1}(\rho) = 0\). Since \(J_{\nu+2}(\rho) \neq 0 \neq J_{\nu+3}(\rho)\), we have

\[
\begin{align*}
\Lambda_{0,d}(\rho) = \Lambda_{2,d}(\rho) > \Lambda_{4,d}(\rho) \geq \Lambda_{6,d}(\rho) \geq \ldots \\
\Lambda_{1,d}(\rho) = \Lambda_{3,d}(\rho) \geq \Lambda_{5,d}(\rho) \geq \Lambda_{7,d}(\rho) \geq \ldots
\end{align*}
\]

We conclude that \(C_d(\rho) = \Lambda_{0,d}(\rho)\) in Case (i), and \(C_d(\rho) = \Lambda_{1,d}(\rho)\) in Case (ii), while \(C_d(\rho) = \Lambda_{k,d}(\rho)\) for \(k \in \{0, 1\}\) in Case (iii), and for \(k \in \{0, 1, 2\}\) in Case (iv). The optimal constant is attained by single spherical harmonics of degree 0 or 1 in Cases (i) and (ii), respectively, and it is attained by linear combinations of spherical harmonics of degrees \(\{0, 1\}\) or \(\{0, 1, 2\}\) in Cases (iii) and (iv), respectively. The proof of Theorem 1 is complete.
3. Sharp stability: Proof of Theorem 2

Here we prove Theorem 2. Reasoning as in the previous section, we write the deficit functional as follows:

\[
\delta_d(f; \rho) = C_d(\rho) \|f\|_{L^2(S^d-1)}^2 - \frac{1}{\rho} \int_{B_\rho} |\hat{f}(x)|^2 \frac{dx}{(2\pi)^d}
\]

\[
= \sum_{k: \Lambda_k,d(\rho) < C_d(\rho)} (C_d(\rho) - \Lambda_k,d(\rho)) \|Y_k\|_{L^2(S^d-1)}^2,
\]

and we notice that

\[
\sum_{k: \Lambda_k,d(\rho) < C_d(\rho)} \|Y_k\|_{L^2(S^d-1)}^2 = d(f, M_d(\rho))^2.
\]

We immediately infer the inequalities

\[
S_d(\rho) d(f, M_d(\rho))^2 \leq \delta_d(f; \rho) \leq \tilde{C}_d(\rho) d(f, M_d(\rho))^2,
\]

where, letting \(A := \{C_d(\rho) - \Lambda_k,d(\rho) : k \in \mathbb{N}_{\geq 0} \text{ such that } \Lambda_k,d(\rho) < C_d(\rho)\}\),

\[
S_d(\rho) := \inf A, \quad \tilde{C}_d(\rho) := \sup A.
\]

By the Bessel limit (A.2), we see that \(\Lambda_k,d(\rho) \to 0\) as \(k \to \infty\), for all \(\rho > 0\). In particular, we infer that \(\tilde{C}_d(\rho) = C_d(\rho)\), and that the right-hand inequality in (3.1) is always strict, except in the trivial case when \(d(f, M_d(\rho)) = 0\), i.e. \(f \in M_d(\rho)\). This completes the proof of the right-hand inequality in Theorem 2.

To complete the proof of the left-hand inequality, we need to compute \(S_d(\rho)\) and the set of indices \(k \in \mathbb{N}_{\geq 0}\) that attain the infimum in (3.1); the space \(E_d(\rho)\) will then coincide with the direct sum of the corresponding \(H_k\). In the following, we will repeatedly make use of Lemma 5 and the index \(h\) will always range over \(\mathbb{N}_{\geq 0}\).

Case (1): \((J_{\nu} J_{\nu+1})(\rho) > 0\). We have \(C_d(\rho) = \Lambda_{0,d}(\rho)\). By the inequalities (2.9),

\[
S_d(\rho) = \Lambda_{0,d}(\rho) - \max_{k \in \{1, 2\}} \Lambda_k,d(\rho).
\]

Now we recall that

\[
\Lambda_{1,d}(\rho) - \Lambda_{2,d}(\rho) = (J_{\nu+1} J_{\nu+2})(\rho).
\]

So, if \((J_{\nu+1} J_{\nu+2})(\rho) > 0\), then \(S_d(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{1,d}(\rho)\). Moreover, by (2.8) we see that \(\Lambda_{1,d}(\rho) > \Lambda_{3,d}(\rho) \geq \Lambda_{3+2h,d}(\rho)\), thus \(E_d(\rho) = H_1\).

On the other hand, if \((J_{\nu+1} J_{\nu+2})(\rho) < 0\), then \(S_d(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{2,d}(\rho)\), and \(J_{\nu+3}(\rho) \neq 0\) by Lemma 7, so \(\Lambda_{2,d}(\rho) > \Lambda_{4,d}(\rho) \geq \Lambda_{4+2h,d}(\rho)\), therefore \(E_d(\rho) = H_2\).

The only remaining alternative is that \(J_{\nu+2}(\rho) = 0\); but then, both \(J_{\nu+3}(\rho) \neq 0\) and \(J_{\nu+4}(\rho) \neq 0\) by Lemma 8.

So

\[
\Lambda_{0,d}(\rho) > \Lambda_{2,d}(\rho) > \Lambda_{4,d}(\rho) \geq \Lambda_{4+2h,d}(\rho)
\]

or

\[
\Lambda_{1,d}(\rho) = \Lambda_{3,d}(\rho) > \Lambda_{5,d}(\rho) \geq \Lambda_{5+2h,d}(\rho)
\]
and we conclude that
\[ S_d(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{1,d}(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{2,d}(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{3,d}(\rho), \]
\[ \mathcal{E}_d(\rho) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \]

Case (ii): \((J_\nu J_{\nu+1})(\rho) < 0\). We have \( C_d(\rho) = \Lambda_{1,d}(\rho) \). By the inequalities \(2.10\),
\[ S_d(\rho) = \Lambda_{1,d}(\rho) - \max_{k \in \{0,3\}} \Lambda_{k,d}(\rho), \]
so we are led to define \( \mathcal{J}_\nu(\rho) \) as follows, where we also use the Bessel recursion \( \Lambda.3 \):
\[ \Lambda_{0,d}(\rho) - \Lambda_{3,d}(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{1,d}(\rho) + \Lambda_{1,d}(\rho) - \Lambda_{3,d}(\rho) \]
\[ = (J_\nu J_{\nu+1})(\rho) + \frac{2}{\rho^2}(\nu + 2)J_{\nu+2}^2(\rho) \]
\[ = (J_\nu J_{\nu+1})(\rho) + (J_{\nu+1}J_{\nu+2})(\rho) + (J_{\nu+2}J_{\nu+3})(\rho) = : \mathcal{J}_\nu(\rho). \]
If \( \mathcal{J}_\nu(\rho) > 0 \) then \( S_d(\rho) = \Lambda_{1,d}(\rho) - \Lambda_{0,d}(\rho) \) and so, since \( J_{\nu+1}(\rho) \neq 0 \), it follows that \( \Lambda_{0,d}(\rho) > \Lambda_{2,d}(\rho) \geq \Lambda_{2+2\nu,d}(\rho) \), thus \( \mathcal{E}_d(\rho) = \mathcal{H}_0 \).

On the other hand, if \( \mathcal{J}_\nu(\rho) < 0 \) then \( S_d(\rho) = \Lambda_{1,d}(\rho) - \Lambda_{3,d}(\rho) \). In this case, we must have \( \Lambda_{3,d}(\rho) > \Lambda_{5,d}(\rho) \geq \Lambda_{5+2\nu,d}(\rho) \). Indeed, assuming towards a contradiction that \( \Lambda_{3,d}(\rho) = \Lambda_{5,d}(\rho) \), we would have \( J_{\nu+4}(\rho) = 0 \) and so \( (J_{\nu+2}J_{\nu+3})(\rho) > 0 \). But this contradicts \( \mathcal{J}_\nu(\rho) < 0 \); indeed, using the Bessel recursion as before, we see that
\[ \mathcal{J}_\nu(\rho) = \frac{2}{\rho^2}(\nu + 1)J_{\nu+1}^2(\rho) + (J_{\nu+2}J_{\nu+3})(\rho), \]
and the right-hand side is positive. We conclude that \( \mathcal{E}_d(\rho) = \mathcal{H}_3 \).

Finally, if \( \mathcal{J}_\nu(\rho) = 0 \), then
\[ S_d(\rho) = \Lambda_{1,d}(\rho) - \Lambda_{0,d}(\rho) = \Lambda_{1,d}(\rho) - \Lambda_{3,d}(\rho). \]
Now, \( \Lambda_{0,d}(\rho) > \Lambda_{2,d}(\rho) \geq \Lambda_{2+2\nu,d}(\rho) \), because \( J_{\nu+1}(\rho) \neq 0 \). On the other hand, we have \( \Lambda_{3,d}(\rho) > \Lambda_{5,d}(\rho) \geq \Lambda_{5+2\nu,d}(\rho) \) because \( J_{\nu+4}(\rho) \neq 0 \), which is proved by contradiction as we did in the previous paragraph. We conclude that \( \mathcal{E}_d(\rho) = \mathcal{H}_0 \oplus \mathcal{H}_3 \).

Case (iii): \( J_\nu(\rho) = 0 \). By the inequalities \(2.11\),
\[ S_d(\rho) = \Lambda_{0,d}(\rho) - \max_{k \in \{2,3\}} \Lambda_{k,d}(\rho). \]
We have that \( \Lambda_{2,d}(\rho) - \Lambda_{3,d}(\rho) = (J_{\nu+2}J_{\nu+3})(\rho) \); if the latter is strictly positive, then \( S_d(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{2,d}(\rho) \) and \( \Lambda_{2,d}(\rho) > \Lambda_{4,d}(\rho) \geq \Lambda_{4+2\nu,d}(\rho) \) because \( J_{\nu+3}(\rho) \neq 0 \), so \( \mathcal{E}_d(\rho) = \mathcal{H}_2 \). If, on the other hand, \( (J_{\nu+2}J_{\nu+3})(\rho) < 0 \), which implies \( J_{\nu+4}(\rho) \neq 0 \), then \( S_d(\rho) = \Lambda_{0,d}(\rho) - \Lambda_{3,d}(\rho) \) and \( \Lambda_{3,d}(\rho) > \Lambda_{5,d}(\rho) \geq \Lambda_{5+2\nu,d}(\rho) \), and we conclude that \( \mathcal{E}_d(\rho) = \mathcal{H}_3 \). The case \( (J_{\nu+2}J_{\nu+3})(\rho) = 0 \) does not occur, as it would contradict Lemma 8.

Case (iv): \( J_{\nu+1}(\rho) = 0 \). By the inequalities \(2.12\),
\[ S_d(\rho) = \Lambda_{0,d}(\rho) - \max_{k \in \{3,4\}} \Lambda_{k,d}(\rho). \]
The proof follows the exact same steps of the previous case upon replacing \( \nu \) by \( \nu + 1 \).

The proof of Theorem 2 is complete.
4. Regularity of $\mathbf{C}_d(\rho)$ and $\mathbf{S}_d(\rho)$: Proof of Corollary 3

Recall that $\{j_{\nu,k}\}_{k \geq 1}$ denotes the sequence of positive zeroes of $J_{\nu}$; see Appendix A. To prove Corollary 3 we start by showing that $\mathbf{C}_d(\rho)$ is not differentiable at $j_{\nu,k}$, where $k \geq 1$ is arbitrary. The argument for $j_{\nu+1,k}$ is entirely analogous. From the proof of Theorem and identity (2.7), respectively, we have that

$$\mathbf{C}_d(\rho) = \max\{\Lambda_{0,d}(\rho), \Lambda_{1,d}(\rho)\} \quad \text{and} \quad \Lambda_{0,d}(\rho) - \Lambda_{1,d}(\rho) = (J_{\nu}J_{\nu+1})(\rho).$$

If $\mathbf{C}_d(\rho)$ were differentiable at $\rho = j_{\nu,k}$, then necessarily $\Lambda'_{0,d}(j_{\nu,k}) = \Lambda'_{1,d}(j_{\nu,k})$. Instead,

$$\Lambda'_{0,d}(j_{\nu,k}) - \Lambda'_{1,d}(j_{\nu,k}) = J'_{\nu}(j_{\nu,k})J_{\nu+1}(j_{\nu,k}) + J_{\nu}(j_{\nu,k})J''_{\nu+1}(j_{\nu,k}) = J'_{\nu}(j_{\nu,k})J_{\nu+1}(j_{\nu,k}) \neq 0.$$  

Indeed, $J'_{\nu}(j_{\nu,k}) \neq 0$ since all the zeroes of $J_{\nu}$ are simple, and $J_{\nu+1}(j_{\nu,k}) \neq 0$ in light of Lemma 8.

We now show that $\mathbf{C}_d(\rho) = \max\{\Lambda_{0,d}(\rho), \Lambda_{1,d}(\rho)\}$ defines a Lipschitz function on the positive half-line $(0, \infty)$. Since the maximum of two Lipschitz functions is Lipschitz, it will suffice to show that $\Lambda_{0,d}(\rho)$ is Lipschitz; the proof for $\Lambda_{1,d}(\rho)$ is entirely analogous. In fact, the derivative of

$$\Lambda_{0,d}(\rho) = \frac{\rho}{2} J''_{\nu}(\rho) - \frac{\rho}{2} (J_{\nu-1}J_{\nu+1})(\rho)$$

is uniformly bounded on $(0, \infty)$. To prove this, we invoke the Bessel recursion (A.4) to obtain,

$$\Lambda_{0,d}(\rho) = \frac{1}{2} (J_{\nu}^2 - J_{\nu-1}J_{\nu+1})(\rho) + \frac{\rho}{4} (J_{\nu-1}J_{\nu} - J_{\nu-2}J_{\nu+1} - J_{\nu}J_{\nu+1} + J_{\nu-1}J_{\nu+2})(\rho).$$

In particular, a direct computation shows that the right derivative at 0 satisfies $\Lambda'_{0,2}(0^+) = \frac{1}{2}$, and $\Lambda'_{0,2}(0^+) = 0$, for $d \in \{3, 4, 5\}$. For $d \geq 6$, the fact that $\Lambda'_{0,d}(0^+) = 0$ follows at once from the behaviour of the Bessel functions at the origin, which in turn can be read off from (A.1). On the other hand, the asymptotic (A.5) of the Bessel functions at infinity guarantees that (4.1) remains bounded, as $\rho \to \infty$.

Finally, recall that the zeroes of $J_{\nu}$ and $J_{\nu+1}$ interlace. On each interval $(j_{\nu,k}, j_{\nu,k+1})$, resp. $(j_{\nu+1,k}, j_{\nu+1,k+1})$, we have that $\mathbf{C}_d(\rho)$ equals $\Lambda_{0,d}(\rho)$, resp. $\Lambda_{1,d}(\rho)$, and so the claimed real-analyticity follows from identity (2.6).

Now we turn to the analysis of $\mathbf{S}_d(\rho)$. Start by noting that, from the proof of Theorem 2, it follows that $\mathbf{S}_d(\rho) > 0$, for every $\rho > 0$. In order to show that $\mathbf{S}_d(\rho)$ has a jump discontinuity at each positive zero of $J_{\nu}J_{\nu+1}$, it suffices to check that

$$\lim_{\rho \to j_{\nu,k}} \mathbf{S}_d(\rho) = 0 = \lim_{\rho \to j_{\nu+1,k}} \mathbf{S}_d(\rho),$$

whenever $k \geq 1$. We verify the first identity in (4.2), since the second one can be dealt with in an analogous way. Choose $k \geq 1$, set $\rho^* := j_{\nu,k}$, and note that from Theorem 1 it follows that $\mathcal{M}_d(\rho) \subseteq \mathcal{M}_d(\rho^*)$, whenever $\rho \neq \rho^*$ is sufficiently close to $\rho^*$. Let $f_\ast \in \mathcal{M}_d(\rho^*) \backslash \mathcal{M}_d(\rho)$, for every such $\rho \neq \rho^*$. By Theorem 2

$$0 \leq \mathbf{S}_d(\rho) d^2(f_\ast, \mathcal{M}_d(\rho)) \leq \delta_d(f_\ast; \rho) = \mathbf{C}_d(\rho) ||f_\ast||^2_{L^2(\mathbb{R}^d)} \frac{1}{\rho} \int_{B_{\rho^*}} |\tilde{f_\ast}(x)|^2 \frac{dx}{(2\pi)^d}.$$
In the last equation we recalled the definition of $\delta_d(f_\ast; \rho)$ to facilitate the proof of its continuity in the variable $\rho > 0$; indeed, we already proved that $C_d(\rho)$ defines a continuous function of $\rho$, and the other (integral) term is also seen to be continuous in $\rho$ by dominated convergence. So $\delta_d(f_\ast; \rho) \to 0$, as $\rho \to \rho^*$, and since $d^2(f_\ast, M_d(\rho)) \neq 0$ is independent of $\rho$ for all $\rho \neq \rho^*$ under consideration, identity (4.2) follows. This establishes the first claim about $S_d(\rho)$. The second claim follows similarly to the corresponding claim about $C_d(\rho)$, with an entirely analogous proof, which is therefore omitted.

We finish by showing that $S_d(\rho)$ is not differentiable at $j_{\nu+2,k}$, for arbitrary $k \geq 1$. If $\rho$ is sufficiently close to $j_{\nu+2,k}$, then necessarily $(J_{\nu+1}(\rho)) > 0$. This follows by continuity from $(J_{\nu+1}(j_{\nu+2,k})) > 0$, which in turn is a consequence of Lemma 7. By Case (i) of Theorem 2 and identity (2.7) respectively, we then have that

$$S_d(\rho) = \Lambda_{0,d}(\rho) - \max\{\Lambda_{1,d}(\rho), \Lambda_{2,d}(\rho)\} \quad \text{and} \quad \Lambda_{1,d}(\rho) - \Lambda_{2,d}(\rho) = (J_{\nu+1}J_{\nu+2})(\rho).$$

The rest of the argument is similar to what we did before: If $S_d(\rho)$ were differentiable at $j_{\nu+2,k}$, then necessarily $\Lambda_{1,d}(j_{\nu+2,k}) = \Lambda_{2,d}(j_{\nu+2,k})$, but instead we have

$$\Lambda_{1,d}(j_{\nu+2,k}) - \Lambda_{2,d}(j_{\nu+2,k}) = J_{\nu+1}(j_{\nu+2,k})J'_{\nu+2}(j_{\nu+2,k}) \neq 0,$$

in light of Lemma 8 and the simplicity of the zeroes of $J_{\nu+2}$.

This concludes the proof of Corollary 3.

5. The $\rho \to \infty$ Limit of the Agmon–Hörmander Estimate

The following result is the case of [11, Theorem 3.1] which is relevant to the present paper. To facilitate the reading, we provide here an adaptation of the original proof. We will then conclude by recovering the same result via the method of the previous sections.

**Theorem 6.** For each $f \in L^2(S^{d-1})$,

$$\lim_{\rho \to \infty} \frac{1}{\rho} \int_{B_\rho} |\widehat{f}\sigma(x)|^2 \frac{dx}{(2\pi)^d} = \frac{1}{\pi} \|f\|_{L^2(S^{d-1})}^2.$$

**Proof.** Let $1_B$ denote the indicator function of $B := B_1$, the unit ball in $\mathbb{R}^d$ centred at the origin. By Plancherel’s Theorem,

$$\frac{1}{\rho} \int_{B_\rho} |\widehat{f}\sigma(x)|^2 \frac{dx}{(2\pi)^d} = \rho^{d-1} \int_{S^{d-1}} f(x) \overline{\int_{B(\rho(\xi - \eta))} \frac{d\sigma(\xi) d\sigma(\eta)}{(2\pi)^d}}.$$

We will use the formula $d\sigma(\xi) = 2 \delta(1 - |\xi|^2) \, d\xi$, where $\delta(\cdot)$ denotes the one-dimensional Dirac distribution; see, for example, the appendix to [11] for more details on this and other formulae of this kind. Before letting $\rho \to \infty$, we apply the change of variables

$$\begin{cases} y = \rho(\xi - \eta), \\ z = \eta, \end{cases} \quad \text{so} \quad d\sigma(\xi) d\sigma(\eta) = \frac{2}{\rho^d} \delta\left(1 - \left|z - \frac{y}{\rho}\right|^2\right) dy d\sigma(z).$$

We also observe that, for $|z| = 1$,

$$\delta\left(1 - \left|z - \frac{y}{\rho}\right|^2\right) = \frac{\rho}{2} \delta\left(z \cdot y - \frac{|y|^2}{2\rho}\right).$$
Thus we see that \(5.1\) equals
\[
2 \rho \int_{\mathbb{R}^d \times S^{d-1}} f \left( z + \frac{y}{\rho} \right) \hat{1}_B(y) \delta \left( 1 - \left| \frac{z - y}{\rho} \right|^2 \right) \frac{dy \, d\sigma(z)}{(2\pi)^d} \\
\rightarrow \int_{S^{d-1}} |f(z)|^2 \left( \int_{\mathbb{R}^d} \hat{1}_B(y) \delta(z \cdot y) \frac{dy}{(2\pi)^d} \right) \, d\sigma(z), \text{ as } \rho \to \infty.
\]

We conclude by evaluating the latter inner integral. Since \(1_B\) is radially symmetric, that integral is independent on \(z \in S^{d-1}\), so we assume that \(z = (0, \ldots, 0, 1)\) and obtain
\[
\int_{\mathbb{R}^d} \hat{1}_B(y) \delta(z \cdot y) \frac{dy}{(2\pi)^d} = \int_{S^{d-1}} \hat{1}_B(y_1, \ldots, y_{d-1}, 0) \frac{dy_1 \ldots dy_{d-1}}{(2\pi)^d} \\
= \int_{-\infty}^{\infty} 1_B(0, \ldots, 0, \xi_d) \frac{d\xi_d}{2\pi} = \frac{1}{\pi}.
\]

The proof is complete. \(\square\)

As we saw in §2, Theorem 6 is equivalent to the statement that
\[
\lim_{\rho \to \infty} \Lambda_{k,d}(\rho) = \frac{1}{\pi}, \text{ for every } k \in \mathbb{N}_{\geq 0}.
\]

We give a direct proof of this statement. Applying the Bessel recursion (A.3), we obtain the following alternative expression for \(\Lambda_{k,d}(\rho)\):
\[
\Lambda_{k,d}(\rho) = \frac{\rho}{2} \left( J_{\nu+k}^2(\rho) - \frac{2(\nu + k)}{\rho} (J_{\nu+k} J_{\nu+k+1})(\rho) + J_{\nu+k+1}^2(\rho) \right).
\]

By the Bessel asymptotic (A.5), we see that
\[
\Lambda_{k,d}(\rho) = \frac{1}{\pi} \left( \cos^2 \left( \rho - \frac{2\nu + 2k + 1}{4} \pi \right) + \cos^2 \left( \rho - \frac{2\nu + 2k + 3}{4} \pi \right) \right) + O \left( \rho^{-1} \right) \\
= \frac{1}{\pi} \left( \cos^2 \left( \rho - \frac{2\nu + 2k + 1}{4} \pi \right) + \sin^2 \left( \rho - \frac{2\nu + 2k + 1}{4} \pi \right) \right) + O \left( \rho^{-1} \right) \\
= \frac{1}{\pi} + O \left( \rho^{-1} \right),
\]

which proves (5.2).

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APPENDIX A. BESSEL FUNCTIONS

Bessel functions can be defined in a number of ways. We follow the classical treatise [20] and define, for \( \alpha > -1 \) and \( \Re(z) > 0 \),

\[
J_\alpha(z) = \left( \frac{z}{2} \right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{z}{2} \right)^{2n}}{n! \Gamma(\alpha + n + 1)}.
\]

When \( \alpha \geq 0 \), since \( \Gamma(\alpha + n + 1) \geq \Gamma(\alpha + 1) \) for \( n \in \mathbb{N}_{>0} \), we have the crude bound

\[
|J_\alpha(z)| \leq \left( \frac{|z|}{2} \right)^\alpha e^{\frac{|z|^2}{4}}, \quad \text{so in particular } \lim_{\alpha \to \infty} J_\alpha(z) = 0.
\]

The Bessel function \( J_\alpha \) satisfies the following recursion relations:

\[
J_{\alpha-1}(z) + J_{\alpha+1}(z) = \frac{2\alpha}{z} J_\alpha(z),
\]

\[
J_{\alpha-1}(z) - J_{\alpha+1}(z) = 2J'_\alpha(z).
\]

As a consequence of (A.3), we have the following simple fact, which is used several times throughout the text. The proof is immediate.

**Lemma 7.** If \( \alpha, z > 0 \) and \( J_{\alpha+1}(z) = 0 \), then \( (J_{\alpha-1} J_\alpha)(z) > 0 \).

For any fixed \( \alpha \geq 0 \), and \( r > 0 \), one has the following asymptotic at infinity:

\[
J_\alpha(r) = \left( \frac{\pi r}{2} \right)^{-\frac{1}{2}} \cos \left( r - \frac{2\alpha + 1}{4} \pi \right) + O \left( r^{-\frac{3}{2}} \right), \quad \text{as } r \to \infty.
\]

The Bessel function \( J_\alpha \) is entire if \( \alpha \) is an integer, otherwise it is a multivalued function with a singularity at the origin. However, when \( \alpha \) is a half-integer, then \( J_\alpha \) is an elementary function, and one easily checks that it is real-analytic on the positive half-line \((0, \infty)\).

Finally we discuss zeroes of Bessel functions. The function \( J_\alpha \) has infinitely many positive zeros, all of which are simple, isolated, and denoted by

\[ 0 < j_{\alpha,1} < j_{\alpha,2} < j_{\alpha,3} < \ldots \]

The zeroes \( \{j_{\alpha,k}\}_{k \geq 1} \) and \( \{j_{\alpha+1,k}\}_{k \geq 1} \) are well-known to interlace. The following result is known as Bourget hypothesis; see [20 §15.28].

**Lemma 8 (Bourget hypothesis).** Let \( \alpha \geq 0 \) be rational and \( m \geq 1 \) be an integer. Then the functions \( J_\alpha(r) \) and \( J_{\alpha+m}(r) \) have no common zeros other than the one at \( r = 0 \).

The reader will have noticed that, for the purposes of the present paper, only the considerably easier cases \( m \in \{1, 2, 3, 4\} \) of Lemma 8 were needed.

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