Analytically expanded and integrated results for massive fermion production in two-photon collisions and a high precision $\alpha_s$ determination

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The cross section for massive fermion production in two-photon collisions was examined at next-to-leading order in QCD/QED for general photon helicity. The delta function (virtual+soft) part of the differential cross section was analytically integrated over the final state phase space. Series expansions for the complete differential and total cross sections were given up to tenth order in the parameter $\beta$. These were shown to be of practical use and revealed much structure. Accurate parametrizations of the total cross sections were given, valid up to higher energies. The above results were applied to top quark production in the region not too far above threshold. The cross section was shown to be quite sensitive to $\alpha_s$ in the appropriate energy region.

I. INTRODUCTION

High energy photons may be produced by backscattering laser light off high energy $e^-$ or $e^+$ beams. In addition, high degrees of polarization are possible and the photons may carry a large fraction of the electron energy. Photon-photon collisions also arise naturally as a background in $e^+e^-$ collisions. One major motivation for constructing a $\gamma\gamma$ interaction region at a high energy next linear collider (NLC) is to produce Higgs bosons on resonance via $\gamma\gamma$ fusion, which also allows direct determination of the $H\gamma\gamma$ coupling, which is sensitive to possible non Standard Model charged particles of large mass that may enter in the triangle loop. Using polarized photons allows one to control the backgrounds arising from $\gamma\gamma \to b\bar{b}$, for an intermediate mass Higgs [1]. This background has now been studied including QCD [2–5] and electroweak [6] corrections.

In this paper we will consider in some detail the process $\gamma\gamma \to f\bar{f} + X$ in the region not too far above threshold, making use of the complete analytical results presented in [2], which include photon polarization. We will demonstrate the usefulness of $\gamma\gamma \to t\bar{t} + X$ in determining $\alpha_s$ precisely. We extend the analytical results presented in [2] by integrating and obtaining analytical results for the single integral (virtual+soft) part and by series expanding the entire differential and integrated cross section to order $\beta^{10}$, where $\beta$ is the massive fermion velocity in the soft radiation limit. Such an expansion is shown to be of practical use, not too far above threshold, and it demonstrates many interesting features of the corrected cross sections. We have also provided parametrizations of the total integrated cross sections valid up to higher energies.

As the diagrams involve only QED-like vertices, the process under consideration is quite fundamental in nature. The fact that complete analytical results have been absent, until recently, reflects the lack of experimental feasibility of directly colliding photons of high energy, although, as mentioned earlier, such collisions naturally arise as a background in $e^+e^-$ collisions. It also reflects the difficulty in obtaining and presenting in a compact form complete analytical results, including bremsstrahlung, when massive fermions are present. The task would be even more formidable for reactions such as $gg \to Q\bar{Q} + X$ [8]. Our hope is that the approach of using high order series expansions to simplify, and clarify, such results will become more widespread. As well, our analytical integration of the single integral part is an essential part of a complete analytical integration, which is likely to be performed sometime in the near future. In the meantime, our parametrizations provide sufficient accuracy to be useful, as do our series expansions closer to threshold.

The possibility of directly producing fermion pairs in $\gamma\gamma$ collisions has recently been realized at SLAC [9], where a high energy ($\sim 29.2$ GeV) photon beam was collided on a low energy (2.35 eV) laser beam. Since the center of mass energy was insufficient to produce a pair ($e^+e^-$, in this case), multiple photon fusion was required; a different mecha-
anism than that being considered here. The high energy beam was produced via backscattering (of the same 2.35 eV beam) of a 46.6 GeV electron beam and represents a first step towards the construction of a higher energy $\gamma\gamma$ collider with both beams produced via backscattering. Of course, many technical difficulties arise in such a machine and these have been investigated (see [10]).

II. GENERAL FORM AND DECOMPOSITION OF THE DIFFERENTIAL CROSS SECTION

The process under consideration is

$$\gamma(p_1, \lambda_1) + \gamma(p_2, \lambda_2) \rightarrow f(p_3) + \bar{f}(p_4) + [V(k)],$$

(1)

where $\lambda_1, \lambda_2$ denote helicities and the $p_i, k$ denote momenta. $f (= q, l)$ represents a fermion with mass $m$ and $V = g, \gamma$. The square brackets represent the fact that there may or may not be a gluon/photron in the final state. We have the following invariants,

$$s \equiv (p_1 + p_2)^2, \quad t \equiv T - m^2 \equiv (p_1 - p_3)^2 - m^2,$$

$$u \equiv U - m^2 \equiv (p_2 - p_3)^2 - m^2 \quad (2)$$

and

$$s_2 \equiv S_2 - m^2 \equiv (p_1 + p_2 - p_3)^2 - m^2 = s + t + u. \quad (3)$$

Defining

$$v \equiv 1 + \frac{t}{s}, \quad w \equiv \frac{-u}{s + t},$$

$$\beta \equiv \sqrt{1 - 4m^2/s}, \quad x \equiv \frac{1 - \beta}{1 + \beta}, \quad (4)$$

we may express

$$t = -s(1 - v), \quad u = -svw,$$

$$s_2 = sv(1 - w), \quad m^2 = \frac{s}{4}(1 - \beta^2). \quad (5)$$

Now introduce

$$\kappa(s) \equiv 2\pi \alpha^2 e^2_f [N_c], \quad C_1 \equiv [C_F] \frac{\alpha_V}{2\pi} \quad (6)$$

where the $N_c, C_F$ factors are present only for $f = quark$ and $V = gluon$, respectively and $e_f$ is the fermion's fractional charge. Here

$$\alpha_V = \begin{cases} \alpha_s, & V = g \\ \alpha, & V = \gamma \end{cases}. \quad (7)$$

Then

$$\frac{d\sigma}{dv dw} = \frac{d\sigma^{(0)}}{dv dw} + \frac{d\sigma^{(1)}}{dv dw}$$

$$= \kappa(s) \left[ \frac{1}{2\pi} \frac{df^{(0)}}{dv dw} + \frac{C_1}{\pi} \frac{df^{(1)}}{dv dw} \right]. \quad (8)$$

The $f$ functions are dimensionless functions of $v$ and $w$, which allow us to parametrize our cross sections in an exact fashion, without dependence on $\alpha_V$. We use the normalization convention of [11]. Since, in that normalization, the $f^{(i)}$ contain an overall factor of $\pi$, we consistently present analytical results for $f^{(i)}/\pi$ in order to cancel it. The unpolarized and polarized $f^{(i)}$ are given by

$$f^{(i)}_{\text{unp}} = \frac{1}{2} [f^{(i)}(+, +) + f^{(i)}(+, -)],$$

$$f^{(i)}_{\text{pol}} = \frac{1}{2} [f^{(i)}(+, +) - f^{(i)}(+, -)], \quad (10)$$

in the notation $f^{(i)}(\lambda_1, \lambda_2)$. Define

$$j = \begin{cases} 0 & \rightarrow f^{(i)}(j) = f^{(i)}(+, +) = f^{(i)}(+, -) \\ 2 & \rightarrow f^{(i)}(j) = f^{(i)}(+, -) = f^{(i)}(-, +) \end{cases}. \quad (14)$$

The LO term is given by

$$\frac{1}{2\pi} \frac{df^{(0)}(j)}{dv dw} = \delta(1 - w) \left\{ \frac{2m^2/s}{v^2(1 - v)^2} (1 - 2m^2/s) \right\}$$

$$+ j \left( \frac{1}{v(1 - v) - 2} \right) \left[ 1 - \frac{2m^2}{sv(1 - v)} \right] \right\}$$

$$= \delta(1 - w) \left\{ \frac{1 - \beta^2}{4\beta^2(1 - v)^2} \right\}$$

$$+ j \left( \frac{1}{v(1 - v) - 2} \right) \left[ 1 - \frac{1 - \beta^2}{2v(1 - v)} \right]. \quad (16)$$

The last form shows explicitly the polynomial structure of the leading order differential cross section in terms of $\beta$. This is somewhat misleading, however, as we shall see in the next section, since the phase space in $v$ itself depends on $\beta$.

From [4] we see that $df^{(1)}/dv dw$ has the form

$$\frac{1}{\pi} \frac{df^{(1)}}{dv dw} = F_h(v, w) + F_q(v, w) + F_g(v) \delta(1 - w) \quad (17)$$

$$= F_h(v, w) + \frac{F_q(v, w) - F_q(v, 1)}{1 - w}.$$
where

\[ w_1 = \frac{1 - \beta^2}{4v(1 - v)} \]

and

\[
\frac{1}{\pi} \frac{d\tilde{f}^{(1)}}{dvdw} = F_s(v, w) + \frac{1}{\pi} \frac{df^{(0)}}{dvdw} \]

When integrating from \( w_1 \) to 1, \( \frac{d\tilde{f}^{(1)}}{dvdw} \) makes no contribution, otherwise it contributes. The function \( F_\delta(\delta(1 - w)) \) has the form

\[
F_\delta(v)\delta(1 - w) = F_s(v, 1) \ln \left( \frac{sv}{m^2} \right) \delta(1 - w)
\]

and \( F_s(v, 1) \) has the form

\[
F_s(v, 1) = \frac{1}{2\pi} F_S^B(\beta) \frac{df^{(0)}}{dv} .
\]

Putting all these together yields

\[
\frac{1}{\pi} \frac{d\tilde{f}^{(1)}}{dvdw} = F_h(v, w) + \frac{1}{\pi} \frac{df^{(0)}}{dvdw} + \frac{F_\delta^B(\beta)}{2\pi} \frac{df^{(0)}}{dvdw} \ln \left( \frac{sv}{m^2} \right) + \frac{F_\delta^S(v)}{2\pi} \frac{df^{(0)}}{dvdw} + F_\delta^{NB}(v)\delta(1 - w) .
\]

Writing

\[
\frac{d\tilde{f}^{(1)}}{dvdw} = \left( \frac{df^{(1)}}{dvdw} \right)_\delta + \left( \frac{df^{(1)}}{dvdw} \right)_{N\delta} ,
\]

where the subscript \( \delta \) denotes the part proportional to \( \delta(1 - w) \), we have

\[
\frac{1}{\pi} \left( \frac{df^{(1)}}{dvdw} \right)_\delta = F_\delta^S(\beta) \frac{df^{(0)}}{dvdw} \ln \left( \frac{sv}{m^2} \right) + \frac{F_\delta^S(\beta)}{2\pi} \frac{df^{(0)}}{dvdw} + F_\delta^S(v)\delta(1 - w) + \frac{F_\delta^S(v)}{2\pi} \frac{df^{(0)}}{dvdw} + F_\delta^{NB}(v)\delta(1 - w) .
\]

The two simple \( F \)'s are given by

\[
F_S^B(\beta) = -4 \left[ \frac{1 + \beta^2}{2\beta} \ln x + 1 \right] ,
\]

\[
F_s^B(\beta) = -4 \left[ \frac{1 + \beta^2}{2\beta} \ln x + 1 \right] \left[ 2 \ln x \right] + 2 \left[ -2 \ln x \ln \left( 1 + 2\beta \right) - 2\beta \right] .
\]

The other three \( F \)'s are somewhat lengthy and will not be presented here as they can be directly inferred from the expressions given in [3]. \( F_{S+N} \) is the contribution from the virtual diagrams which is not proportional to the Born term. \( F_{h(1)} \) is proportional to the first bracketed term in Eq. (20) of [3] and \( F_{S+1}(v, w) \) is proportional to the second bracketed term of that same equation. Both arise from gluon/photon bremsstrahlung.

It is standard [3] to divide the cross section (i.e. \( f^{(1)} \)) into two parts. Firstly, there is the virtual plus soft part,

\[
\frac{df^{(1)}}{dvdw} = \frac{df^{(1)}}{dvdw} + \frac{df^{(1)}}{dvdw} ,
\]

where \( \frac{df^{(1)}}{dvdw} \) denotes the virtual contribution and \( \frac{df^{(1)}}{dvdw} \) is obtained by integrating the bremsstrahlung contribution to \( \frac{df^{(1)}}{dvdw} \) over the region

\[
1 \geq w \geq w_{1,soft} \equiv 1 - \frac{m^2}{sv} ,
\]

then multiplying by \( \delta(1 - w) \). We follow the definition of the soft parameter, \( \delta \), given in [3] such that the gluon/photon radiated becomes arbitrarily soft by making \( \delta \) arbitrarily small. Since this takes into account all virtual corrections and soft radiation, the hard radiation may be taken into account by integrating \( df^{(1)}/dvdw \) in the region \( w_1 \leq w \leq w_{1,soft} \). Since we never reach \( w = 1 \), \( F_{s(w, w)}(1 - w) \) in the hard radiation integration. We thus define \( df^{(1)}_H/dvdw \) as being \( df^{(1)}_H/\pi dvdw \) in the region \( w_1 \leq w \leq w_{1,soft} \), where the \( \delta(1 - w) \) terms do not contribute:

\[
\frac{df^{(1)}_H}{dvdw}(w \leq w_{1,soft}) .
\]

It is not necessary to define \( df^{(1)}_H/dvdw \) outside that region since it is never evaluated there.

The sum of the (integrated over some region) hard and soft contributions so defined is independent of \( \delta \) in the limit \( \delta \rightarrow 0 \) and this method of separation is referred to as the phase space slicing method. As one might expect, there is a close relation between \( df^{(1)}_{H+S}/dvdw \) and \( df^{(1)}/dvdw \) as well as between \( df^{(1)}_H/dvdw \) and \( df^{(1)}/dvdw \) as well. We now give explicitly the necessary conversion terms.
It is straightforward to show that \( d\sigma_S/dvdw \) is obtained from the term proportional to \( d\sigma_{LO}/dvdw \) in Eq. (30) of [3] by making the substitution

\[
\ln \left( \frac{\sqrt{s}u}{m_t^2} \right) \to \ln \delta.
\]

(32)

From this, we infer that the conversion term necessary to transform our result into \( f_{N+S}^{(1)} \) is

\[
\frac{1}{\pi} \frac{df_{S,conv}^{(1)}}{dvdw} = \frac{1}{2\pi} \frac{df_{0}^{(0)}}{dvdw} F_S^B(\beta) \left[ \ln \delta - \ln \left( \frac{4v}{1 - \beta^2} \right) - \ln(1 - w) \right].
\]

(33)

The transformation is simply

\[
\frac{1}{\pi} \frac{df_{H+\delta}^{(1)}}{dvdw} = \frac{1}{\pi} \left( \frac{df_{H}^{(1)}}{dvdw} \right)_{\delta} + \frac{1}{\pi} \frac{df_{H,conv}^{(1)}}{dvdw}
\]

(34)

To convert from \( df_{H}^{(1)}/dvdw \) to \( df_{H}^{(1)}/dvdw \) we must add the following conversion term,

\[
\frac{1}{\pi} \frac{df_{H}^{(1)}}{dvdw} = \frac{1}{\pi} \left( \frac{df_{H}^{(1)}}{dvdw} \right)_{N+\delta} + \frac{1}{\pi} \frac{df_{H,conv}^{(1)}}{dvdw}
\]

(35)

\[
= \frac{1}{\pi} \frac{df_{H}^{(1)}}{dvdw} \left( \frac{F_S^B(\beta)df_{0}^{(0)}/d\tau}{2\pi} \right) - 1 - w.
\]

Note that both soft and hard conversion terms are correctly determined by taking \( df^{(1)}/dvdw \) into account. Using (32), we reproduced the result of [3] for \( df_{S,\text{LO}}^{(1)}/dvdw \) and using (34), we reproduced the result of [3] for \( df_{S,\text{LO}}^{(1)}/dvdw \). Further checks on \( df^{(1)}/dvdw \) will be discussed in the next section.

The variables \( v, w \) are suitable for performing analytical integration of the cross section (at least for the single integral part). They are not, on the other hand, suitable for performing series expansions of the integrated cross section about \( \beta \approx 0 \). The reason is that, in these variables, the integration limits depend on \( \beta \) so that the series expansion of the integrated cross section does not follow straightforwardly from the series expansion of the differential cross section, and we only have complete analytical results for the differential cross sections. Otherwise, we could just expand the final integrated result. The above will become clear in the following sections. This approach also allows for cross checking; when one first expands the differential cross section and then integrates, the result should coincide with that obtained by directly expanding the analytically integrated cross section. We will check this requirement for the single integral part, for which we do have analytical results.

At this point, we introduce a new set of variables, \( \tau \) and \( \omega \), suitable for performing series expansions in \( \beta \). They are defined through

\[
v \equiv \frac{1}{2} (1 + \tau \beta), \quad w \equiv 1 - c_{\beta}(\tau)\beta^2(1 - \omega),
\]

(36)

where

\[
c_{\beta}(\tau) \equiv \frac{1 - \tau^2}{1 - \tau^2 \beta^2}.
\]

(37)

We note

\[
\frac{df^{(i)}}{d\tau d\omega} = \frac{\beta^3 c_{\beta}(\tau) df^{(i)}}{2}.
\]

(38)

Because of the factor \( c_{\beta}(\tau) \), \( df^{(i)}/d\tau d\omega \) will never get a part proportional to \( \delta(1 \pm \tau) \). Defining

\[
c_u \equiv c_{\beta}(\tau)(1 - \omega)(1 + \tau \beta),
\]

(39)

the invariants are given by

\[
t = -\frac{s}{2} (1 - \tau \beta), \quad u = -\frac{s}{2} (1 + \tau \beta - \beta^2 c_u),
\]

\[
s_2 = \frac{s}{2} c_u \beta^2, \quad T = -\frac{s}{4} (1 - 2 \tau \beta + \beta^2),
\]

\[
U = -\frac{s}{4} [1 + 2 \tau \beta + \beta^2 (1 - 2 c_u)],
\]

\[
S_2 = \frac{s}{4} [1 + \beta^2 (2 c_u - 1)].
\]

(40)

In terms of \( \tau \) and \( \omega \), the LO term has the form

\[
\frac{1}{2\pi} \frac{df^{(0)}(j)}{d\tau d\omega} = \frac{\delta(1 - \omega)}{1 - \tau^2 \beta^2} \left[ 2(1 - \beta^4) - j(1 + \tau^2 \beta^2) \times (1 + \tau^2 \beta^2 - 2 \beta^2) \right] 
\]

\[
= \delta(1 - \omega)[(2 - j) \beta + (2j + 4 \tau^2 - 4j \tau^2) \beta^3 + O(\beta^5)].
\]

(41)

We see explicitly that the \( j = 0, 1 \) differential cross sections, in terms of these variables, vanish by order \( \beta \) in the limit \( \beta \to 0 \), while the \( j = 2 \) differential cross section is order \( \beta^3 \).

The conversion term (33) becomes

\[
\frac{1}{\pi} \frac{df_{S,conv}^{(1)}}{d\tau d\omega} = \frac{1}{2\pi} \frac{df_{0}^{(0)}}{d\tau d\omega} F_S^B(\beta) \left[ \ln \delta - \ln \left( \frac{2(1 + \tau \beta)}{1 - \beta^2} \right) \right]
\]

\[
- \ln[\beta^2 c_{\beta}(\tau)]
\]

(43)

and the conversion term (33) becomes

\[
\frac{1}{\pi} \frac{df_{H,conv}^{(1)}}{d\tau d\omega} = \frac{1}{2\pi} \frac{F_S^B(\beta)df^{(0)}/d\tau}{1 - \omega}.
\]

(44)

III. ANALYTIC INTEGRATION OF THE DELTA FUNCTION PART

The only complete analytical results for the differential cross sections were presented in [3]. Analytical results for the virtual+soft part were presented in [3] for the unpolarized case and in [3] for the polarized case (where
the virtual and soft parts are given separately, in terms of various functions). Still, not even the virtual+soft part has previously been integrated (over fermion angle) analytically. In this section, we present such an analytical integration. We were not able to integrate the non delta function (or hard) part analytically, in a straightforward fashion, and reserve that for future work.

The integrated cross section (or \( f^{(1)} \)) is obtained via

\[
f^{(i)} = \int_{v_1}^{v_2} dv \int_{w_1}^{w_2} dw \frac{df^{(i)}}{dvdw} = \int_{-1}^{1} d\tau \int_{0}^{1} d\omega \frac{df^{(i)}}{d\tau d\omega},
\]

where

\[
v_1 = \frac{1}{2} (1 - \beta), \quad v_2 = \frac{1}{2} (1 + \beta).
\]

Let \( \theta_3 \) be the angle between \( p_3 \) and \( p_1 \) in the \( \gamma\gamma \) c.m. Then \( \theta_3 \) is given by

\[
\cos \theta_3 = -\frac{1 - v - vw}{\sqrt{(1 - v + vw)^2 + \beta^2 - 1}} = -\frac{2\tau - \beta c_\omega}{\sqrt{4 - c_\omega (4 - \beta^2 c_\omega)}}.
\]

Thus,

\[
\cos \theta_3 = -\frac{1 - 2\tau}{\beta} = \tau, \quad \text{for } w = \omega = 1.
\]

We see that for \( \beta \to 0 \), \( \cos \theta_3 \) varies rapidly with \( v \), while it is simply equal to \( \tau \). This is why the phase space in \( v \) becomes vanishingly small by order \( \beta \). Similarly, from (43) and (44), or (42), we see that the \( w \) phase space is order \( \beta^2 \). Thus, the double integration over \( v \) and \( w \) is order \( \beta^3 \), in accord with (48).

The integration of (49) or (41) is rather straightforward, yielding the LO term,

\[
\frac{f^{(0)}(j)}{2\pi} = 2\beta (1 + \beta^2) - 6\beta j - (1 - \beta^4 + 2j) \ln x.
\]

Since

\[
\ln x = -2 \sum_{k=0}^{\infty} \frac{\beta^{2k+1}}{2k+1} = -2\beta - 2\beta^3 - \cdots,
\]

we have

\[
\frac{f^{(0)}(j)}{2\pi} = 2(2 - j)\beta + \frac{4(2 + j)}{3} \beta^3 + 2 \sum_{k=2}^{\infty} \left( \frac{-1}{2k-3} + \frac{1 + 2j}{2k+1} \right) \beta^{2k+1},
\]

so that \( f^{(0)}(0, 1) \) are order \( \beta \) and \( f^{(0)}(2) \) is order \( \beta^3 \). Also, we see that only \( f^{(0)}(0) \) is finite in the limit \( \beta \to 1 \) and it approaches

\[
f^{(0)}(0) \to 8\pi, \quad \text{for } \beta \to 1.
\]

This is because the \( 1 - \beta^4 \) term in (41) keeps the \( j = 0 \) channel finite. For \( j = 2 \), the cross section vanishes for exactly \( \tau = \pm 1 \), as required by angular momentum conservation along the \( \gamma\gamma \) axis, but for \( \beta \to 1 \) the part proportional to \( j \) goes like \( (1 + \tau^2)/(1 - \tau^2) \) as soon as we move away from exactly \( \tau = \pm 1 \) and is hence not integrably finite at \( \beta = 1 \). In order that the \( j = 0 \) cross section be nonvanishing for \( \tau = \pm 1 \), where its maximum lies, the \( f \) and \( \tilde{f} \) must have opposite spins by angular momentum conservation, leading to \( m^2/s \sim 1 - \beta^2 \) suppression in the numerator. The fact that the LO \( j = 0 \) cross section continues to be \( 1 - \beta^2 \) suppressed for \( \tau \neq \pm 1 \) follows from symmetry arguments. This exactly compensates the \( t \)-channel singularity in the propagator, leading to a finite \( f^{(0)}(0) \) for \( \beta \to 1 \). Of course, for \( \tau \neq \pm 1 \), the \( j = 0 \) differential cross section will vanish like \( (1 - \beta^2)/(1 - \tau^2) \), making it unobservable in LO, for \( \beta \to 1 \). So, had we taken the limit \( \beta \to 1 \) from the beginning, the \( j = 0 \) cross section would have vanished identically. Hence the nonzero \( f^{(0)}(0) \) in the \( \beta \to 1 \) limit is a remnant of using the fermion mass as a “regulator”.

Near threshold, the \( 1 - \beta^2 \) suppression of the \( j = 0 \) channel will not be significant, hence the major constraint will come from angular momentum conservation in the forward and backward directions which will lead to suppression of the \( j = 2 \) cross section there. The \( j = 0 \) cross section reaches its maximum in those configurations, however. Thus, we can clearly understand the feature of the numerical results for top quark production in [2] which show that imposing angular cuts in the direction of the beam pipe has a greater effect on the \( j = 0 \) channel than on the \( j = 2 \) channel.

We denote the single and double integral contributions to \( f^{(1)} \) by

\[
f^{(1)} \equiv \int_{v_1}^{v_2} dv \int_{w_1}^{w_2} dw \frac{df^{(1)}}{dvdw} \delta/Nd, \quad f^{(1)} \equiv \int_{-1}^{1} d\tau \int_{0}^{1} d\omega \frac{df^{(1)}}{d\tau d\omega},
\]

We performed the single integration using (54), as opposed to (55). It turned out to be quite lengthy and involved. We did not check to see whether using (55) simplifies the calculation. This question is probably more relevant to the double integration, however. Our final (simplified) result is

\[
\frac{f^{(1)}}{\pi} = a_1 x^2/6 + a_2 Li_2(x) + a_3 Li_2(-x) + \ln(x)\beta [a_4 Li_2(x) + a_5 Li_2(-x)] + a_6 \ln[(1 + \beta)/2] [\pi^2/6 + 2 Li_2(x)] + a_7 \ln(x) [\pi^2/6 + a_8 \ln^2(3 + \beta^2)/4] \ln(x) + 5 \ln[(1 - \beta^2)/4] \ln[3 + \beta^2/4] \ln(x) + 2 Li_3[-2x/(1 + \beta)] - 2 Li_3[-2x/(1 - \beta)] + 2 Li_3[(1 - \beta^2)/(3 + \beta^2)] \ln[2/(x(1 - \beta))] - 2 Li_3[(1 + \beta^2)/(3 + \beta^2)] \ln[2x/(1 + \beta)] + a_9 Li_3[(1 + \beta)/2] - Li_3[(1 - \beta)/2].
\]
Hence we must perform the above mentioned numerical calculations. There have been no other analytical results presented quite in a form suitable for direct analytical comparison. The latter are not obtained in exactly, and similar expressions for the polarized case (also with a gluon mass as infrared regulator). The latter are not obtained in the polarized or unpolarized cases.

Two independent determinations of (56) were made using Mathematica [3] and REDUCE [3]. That software could not evaluate certain integrals which can be found in [1]. It was verified that the analytically integrated result agreed numerically with the numerically integrated result. In the next section, we will show how one can use the series expansion as a very solid check as well.

Perhaps the most convincing check of (56) and the analytical result for $df^{(1)}/dv_{dw}$ (or $dv_{NLQ}/dv_{dw}$) obtained in [3] is the excellent numerical agreement with tabulated results for $f^{(1)}$ existing in the literature. The only existing analytical results, aside from those in [2], are the expressions for $df^{(1)}_{V+S}/dv_{dw}$ (i.e. $(df^{(1)}/dv_{dw})_{\beta}$), $df^{(1)}_{S}/dv_{dw}$ given in [3] for the unpolarized case (using dimensional regularization), with which we agree exactly, and similar expressions for the polarized case in [3] (obtained using a gluon energy cut and a small gluon mass as infrared regulator). The latter are not quite in a form suitable for direct analytical comparison. There have been no other analytical results presented for $(df^{(1)}/dv_{dw})_{NS}$ in the polarized or unpolarized cases. Hence we must perform the above mentioned numerical checks.

Define

$$z \equiv \sqrt{s} = \frac{1}{\sqrt{1-\beta^2}} \leftarrow \beta = \sqrt{1-1/z^2}.$$  (58)

In Table III we give numerically computed values for $f^{(1)}_{unp}$, $f^{(1)}_{pol}$, $f^{(1)}_{unp}(+,+)$, $f^{(1)}_{unp}(+-)$ as well as the specific contributions from all the $f^{(1)}_{si}$ and $f^{(1)}_{di}$ to the corresponding $f^{(1)}$, for various values of $1.2 \leq z \leq 20$. The result at $z = 1$ is given exactly by the series expansions presented in the next section. We also indicate the number of significant figures, n.s., following the decimal point, in $f^{(1)}_{di}$ (and $f^{(1)}$).

We find it useful to describe how the values in Table III were obtained in order that one may see clearly which numbers have been rounded and how. The values for $f^{(1)}_{si,unp}$, $f^{(1)}_{si,pol}$ were obtained using [6] which permits arbitrary precision, using a package like Mathematica. The values of $f^{(1)}_{si}(+,+)$ and $f^{(1)}_{si}(+-)$ were obtained adding/subtracting the values of $f^{(1)}_{si,unp}$, $f^{(1)}_{si,pol}$ so obtained. The $f^{(1)}_{di,unp}$, $f^{(1)}_{di,pol}$ were obtained by numerical integration using [6], for the $N\delta$ part. From these, $f^{(1)}_{di}(+,+)$ and $f^{(1)}_{di}(+-)$ were obtained by adding/subtracting. Finally, $f^{(1)}(++,+)$, $f^{(1)}(+-,-)$ were obtained by adding the corresponding $f^{(1)}_{si}$, $f^{(1)}_{di}$ (rather than adding/subtracting $f^{(1)}_{unp}$, $f^{(1)}_{pol}$) similarly for $f^{(1)}_{pol}$. We did not check to see if [6] leads to any reduction in computational time, for the precision obtained. The general trend is that one needs more integration points as one goes to higher $z$.

The next issue is, of course, how well these values compare with other tabulated values for $f^{(1)}$. Two other such tables exist at present. The original one of [11] gave $f^{(1)}_{unp}$ for $z = 2, 3, 4, 5, 10$; the value at $z = 1$ being numerically equal to the known threshold result, as given in the next section. Their numerical values were obtained using the $f^{(1)}_{unp}$ given in [6] added numerically to $f^{(1)}_{H,unp}$ determined there using the same methodology as [6], which is equivalent to our method. We find numerical agreement with [11] to within the precision of those values, which is roughly at the order of one part in 10,000 or better. This can only be achieved with correct analytical results. Our calculation of $f^{(1)}_{pol}$ is identical in method (same integrals and structure) to that of $f^{(1)}_{unp}$ (at the differential and integrated level), the only difference arising from different traces due to the contraction with a polarized photonic tensor rather than an unpolarized one. As two independent determinations of these traces were performed, there is little room for any error in $f^{(1)}_{pol}$. Fortunately, we may directly check this assertion since the values of $f^{(1)}(+,+)$ and $f^{(1)}(+,-)$ for $z = 2, 3, 4, 5, 10, 20, 50$ were tabulated in [3]. There, Monte Carlo methods were used, leading to accuracy at the level of better than 1% in regions where the $f^{(1)}$ are sizable, but apparently not better than ±0.2 or so in absolute error. This absolute error is noticeable only for $f^{(1)}(+,+)$. Only for $z = 2, 3$, where $f^{(1)}(+,+) = 2.3, 4, 5, 10, 20, 50$ were tabulated in [3]. We may convert from (56) to $f^{(1)}_{V+S}$ by adding the following conversion term,

$$f^{(1)}_{V+S} = \frac{f^{(1)}_{si}}{\pi} + \frac{f^{(1)}_{S,conv}}{\pi}.$$  (59)
\[ f^{(1)}_S = \frac{f^{(1)}_s}{\pi} + F_S^B(\beta) \left\{ \ln \left( \frac{1 - \beta^2}{4} \right) + \ln \delta \right\} \frac{f^{(0)}}{2\pi} \]
\[ + \beta (1 + \beta^2 - 4j) + 2\beta (1 + \beta^2 - 3j) \ln \left( \frac{1 + \beta^2}{2\beta^2} \right) \]
\[ + \left[ \frac{3}{2} + \beta - 2\beta^2 + \beta^3 - \frac{\beta^4}{2} + j(4 - 3\beta + \beta^2) \right] \ln x \]
\[ + \left[ \frac{1}{4} \ln^2 x + 2 \ln x \ln \left( \frac{1 + \beta^2}{2} \right) - 2\text{Li}_2(x) - \text{Li}_2(-x) \right] \]
\[ + \frac{\pi^2}{4} (1 - \beta^4 + 2j) \],

where \( f^{(1)}_{S,\text{conv}} \) follows from integrating \( df^{(1)}_{S,\text{conv}}/dv \), given in (33), or from integrating \( df^{(1)}_{S,\text{conv}}/d\tau \), given in (13).

**IV. SERIES EXPANSION OF THE DELTA FUNCTION PART**

Besides providing a useful check of the analytical integration of the previous section, there are many reasons why it is useful and instructive to series expand the differential and integrated cross sections about \( \beta = 0 \). In the absence of complete analytically integrated results, only a series expansion about \( \beta = 0 \) can be used to make (very) high precision predictions in the \( \beta \ll 0 \) region. One also sees the structure of the cross section in a way that cannot be inferred from the non-expanded analytical results, which are somewhat complicated. From a practical viewpoint, having “simple” series expansions for the differential cross sections allows one to do complete numerical studies in the region not too far above threshold rather easily. This is because the resulting expansions only involve simple polynomials and simple logarithms. We will address the issue of the region of validity of the expansions as well.

The other issue is that of resummation. There are large correction terms at threshold which can be resummed. Having a series expansion of high enough order to be of practical use allows one to explicitly perform resummations up to some order in \( \beta \) while leaving the higher order terms the same. The net result would be an equally simple series, improved via resummation so as to allow one to go closer to threshold. This is beyond the scope of this paper as are other very near threshold effects. Suffice it to say that having the threshold series expansion will facilitate these studies for those interested.

Throughout, we will expand up to order \( \beta^{10} \) (including \( \beta^{11} \ln \beta \) terms). The expansion which exists in the literature (see [11]) is only for \( f^{(1)}_{\text{uni}} \) and only goes to order \( \beta \). Going to order \( \beta^{10} \) may seem excessive at first, but we found it to be a good stopping point for several reasons. Considerable structure arises beyond order \( \beta \) which allows us to see the general, all-orders in \( \beta \), structure of the various series. Also, one gains little in terms of precision by going to even higher orders in \( \beta \), without including several more terms. Then, the series would start to become lengthy and cumbersome, reducing the advantage over the analytical result in terms of ease of use. For certain series, going much beyond \( \beta^{10} \) would take a very large amount of computer memory and runtime, not justifying the extra effort, as going to order \( \beta^{10} \) was a considerable task in itself. Finally, by going to such a high order, we may stringently check the analytically integrated single integral result of the previous section as will be described below.

We find that \( df^{(1)}/d\tau d\omega \) may be expanded in the general form

\[ df^{(1)}/d\tau d\omega = \sum_{i=0}^{\infty} \sum_{j=0}^{1} c_{ij}(\tau, \omega) \beta^i \ln^j \beta. \]  

Therefore \( f^{(1)} \) may be expanded as

\[ f^{(1)} = \tilde{f}^{(1)} = \sum_{i=0}^{\infty} \sum_{j=0}^{1} d_{ij} \beta^i \ln^j \beta, \]  

where the \( d_{ij} \) are given by

\[ d_{ij} = \int_{-1}^{1} d\tau \int_{0}^{1} d\omega c_{ij}(\tau, \omega). \]  

With the variables \( v \) and \( w \), the integration limits depend on \( \beta \), hence the above arguments do not hold. So, one sees clearly the necessity of the change of variables.

We convert from \( (df^{(1)}/dvdw)_\delta \) to \( (df^{(1)}/d\tau d\omega)_\delta \) using (38), which modifies the overall factor via \( \delta(1 - w) \rightarrow \beta^2 c_{ij}(\tau)(1 - w)/2 = \beta \delta(1 - \omega)/2 \). Then, the results for the series expansions of \( (df^{(1)}/d\tau d\omega)_\delta \) are, for \( j = 0, \)

\[ \frac{1}{\pi} \left( \frac{df^{(1)}(+,+)}{d\tau d\omega} \right)_{\delta} = \]
\[ \delta(1-\omega) \left\{ 2\pi^2 + \beta(-20 + \pi^2) + 2\beta^2\pi^2(1 + 2\tau^2) + \beta^3 \left( -9\pi^2(2 - \tau^2) - 4(22 + 45\tau^2) \right) \right. \]
\[ + 96\{(1 - 9\tau^2) \ln(2) + \ln[4\beta^2(1 - \tau^2)]\}/9 + 2\beta^4[-3\pi^2 + 16\tau + 3\pi^2\tau^2(2 + 3\tau^2)]/3 \]
\[ + \beta^5 \left( 225\pi^2(1 - 14\tau^2 + 17\tau^4) + 4[13944 - 25(464\tau^2 - 381\tau^4)] + 480\{(34 + 205\tau^2 - 245\tau^4) \ln(2) \right. \]
\[ + 2(1 + 5\tau^2) \ln[4\beta^2(1 - \tau^2)]\}/225 + 2\beta^6[16\tau(6 + 35\tau^2)/45 - \pi^2(1 + \tau^2)(1 + \tau^2 - 4\tau^4)] \]
\[ + \beta^7 \left[ \pi^2\tau^2[7 - 34\tau^2 + 31\tau^4] \right. \]
\[ + 4\{-72244 + 49\tau^2(35068 - 5\tau^2(18680 - 12373\tau^2))\}/11025 \]
\[ + 32\left\{ (227 + 7\tau^2[-473 + \tau^2(1315 - 973\tau^2)]) \ln(2) + [-26 + 7\tau^2(4 + 15\tau^2)] \right. \]
\[ \times \ln[4\beta^2(1 - \tau^2)]\}/105 \right. \]
\[ + 2\beta^8\{\pi^2\tau^2(-1 - \tau^2)(2 + \tau^2 - 5\tau^4) \]
\[ + 32\tau[-39 + 7\tau^2(7 + 29\tau^2)]/315 \left. \right) + \beta^9 \left( 315\pi^2\tau^4[17 + \tau^2(-62 + 49\tau^2)] \right. \]
\[ + 4\{830486 + 9\tau^2\{2777328 + 7\tau^2[2146844 + 65\tau^2(-55762 + 29125\tau^2)]\}\}/315 \]
\[ + 32\left\{ (376 + 3\pi^2\{4899 + \tau^2[-26579 + \tau^2(46207 - 25243\tau^2)]\}) \ln(2) + 64\{-11 + 3\tau^2[-26 \right. \]
\[ + 7\tau^2(3 + 10\tau^2)] \ln[4\beta^2(1 - \tau^2)]\}/315 \left. \right) + 2\beta^{10} \left( \pi^2\tau^4(-1 - \tau^2)(3 + \tau^2 - 6\tau^4) \right. \]
\[ + 32\tau[-55 + \tau^2[-455 + \tau^2(406 + 1455\tau^2)]\}/1575 \right. \]
\[ + 64\beta^{11} \ln(2\beta) \left( -122 + 11\tau^2[-44 + \tau^2[-78 + 7\tau^2(8 + 25\tau^2)]\} \right. \}
\[ \left. /3465 \right) \] (63)

and, for \( j = 2 \),
\[ \frac{1}{\pi} \left( \frac{df^{(1)}(+,-)}{d\tau d\omega} \right)_{\delta} = \]
\[ 2(1 - \tau^2)\delta(1 - \omega) \left\{ 2\beta^2\pi^2 - 16\beta^3 + \beta^4\pi^2(1 + 5\tau^2) \right. \]
\[ + 4\beta^2\{14 - 159\tau^2 - 6(7 - 19\tau^2) \ln(2) \right. \]
\[ + 24\ln[4\beta^2(1 - \tau^2)]\}/9 - \beta^6\{32\tau/3 + \pi^2[1 \right. \]
\[ - \tau^2(3 + 8\tau^2)] \left. \right) - 2\beta^7 \left( 1102 - 21355\tau^2 + 37995\tau^4 \right. \]
\[ + 120\{(19 + 273\tau^2 - 382\tau^4) \ln(2) \right. \]
\[ + (1 - 25\tau^2) \ln[4\beta^2(1 - \tau^2)]\}/225 \right. \]
\[ - \beta^8\tau[16(3 - 85\tau^2)/45 + \pi^2\tau(2 - 5\tau^2 - 11\tau^4)] \]
\[ + 2\beta^9\{129611 - 805\tau^2(3721 - 2\tau^2(6781 - 6092\tau^2)] \right. \]
\[ + 840[-219 + \tau^2(5172 - 18803\tau^2 + 16134\tau^4)] \ln(2) \]
\[ + 3360(1 + 70\tau^4) \ln[4\beta^2(1 - \tau^2)]\}/11025 \right. \]
\[ + \beta^{10} \tau[\pi^2\tau^3(-3 + 7\tau^2 + 14\tau^4) \right. \]
\[ + 16(12 - 7\tau^2 + 105\tau^4)/315 \right. \]
\[ + 32\beta^{11} \ln(2\beta) \left( 13 + 51\tau^2 + 21\tau^4(1 + 55\tau^2)] \right. \}
\[ /315 \right) \] (64)

The remaining terms are of order \( \beta^{11} \). Here, and throughout, we group terms proportional to \( \ln(2\beta) \) rather than \( \ln \beta \) as in (53), (54) for purposes of compactness. Of course, the choice is quite arbitrary.

We notice that the cross section is isotropic up to order \( \beta \); the angular (\( \tau \)) dependence enters only at order \( \beta^2 \). The LO term, on the other hand, was isotropic up to order \( \beta^2 \). We also see that the step function threshold behaviour
arises entirely from the \( j = 0 \) channel, at the level of the differential cross section, since the \( j = 2 \) channel starts at order \( \beta^2 \). From the \( 1−\tau^2 \) overall factor, and using (43), we see that the delta function contribution to the \( j = 2 \) cross section vanishes at \( \cos \theta_3 = \pm 1 \) as did the LO cross section (11). This vanishing is not obvious from the exact analytical expressions, but simply reflects angular momentum conservation along the \( \gamma \gamma \) axis when \( \omega = 1 \) (2 \to 2 kinematics). The \( j = 0 \) channel, on the other hand, becomes infinite (but integrably finite) for \( \cos \theta_3 = \pm 1 \) due to the \( \ln(1−\tau^2) \) terms.

The expansions (63), (64) have rather simple structure in that, aside from the \( \ln(1−\tau^2) \) terms, the \( c_{ij}(\tau) \) are simply polynomial in \( \tau \). This amounts to considerable simplification and reduction in computational time relative to the exact expressions, especially after the non delta function part is added, where the simplification is even greater as we shall see in the next section. Adding the conversion term (44) gives \( \frac{1}{\pi} f_{N+\delta}^{(1)} \).

Two independent calculations of (63), (64) were performed using Mathematica and REDUCE. The expansions were also checked numerically by subtracting them from the exact expressions. The difference was checked to be of order \( \beta^{11} \). This is most straightforwardly done by taking rather small \( \beta \).

Assuming we are working at \( \beta \) where the series are sufficiently accurate, one could easily analytically integrate (63), (64) over a region of \( \tau \) (\( \cos \theta_3 \)) relevant to some experiment, if desired, and implement angular cuts analytically. After a suitable change of variables, the same could be done for the hard radiation part, either analytically or numerically. Cuts on additional observables may be made by subtracting off the unwanted configurations using the squared amplitudes given in (2) and Monte Carlo integration, for instance. Here, we simply present the total integrated results.

For the \( j = 0 \) channel, we find

\[
\frac{1}{\pi} f_{N}^{(1)}(+,+) = 2 \left\{ 2\pi^2 - (20 - \pi^2)\beta + 10/3\pi^2\beta^2 + \beta^3/3 [-340/3 + \pi^2 + 64\ln(2\beta)] + 8/15\pi^2\beta^4 + 4/15\beta^5 [-6343/45 - \pi^2 - 32\ln(2) + 256/3\ln(2\beta)] - 104/105\pi^2\beta^6 + 4/105\beta^7 [-39163/315 - \pi^2 + 208/3\ln(2\beta)] - 88/315\pi^2\beta^8 + 4/9\beta^9 \left[ 17[(-\pi^2/5 - 64/3\ln(2)) + 1/25[128\ln(2\beta) - 43903/315]] - 488/3465\pi^2\beta^{10} + 103232/51975\beta^{11}\ln(2\beta) \right] \right\}
\]

and for \( j = 2 \),

\[
\frac{1}{\pi} f_{N}^{(1)}(+,-) = 16/3 \left\{ \pi^2\beta^2 - 8\beta^3 + \pi^2\beta^4 + 32\beta^5 [-289/720 + \ln(2)/5 + \ln(2\beta)/3] + \pi^2/7\beta^6 + 6/5\beta^7 [-3947/945 + 16/7\ln(2) + 32/9\ln(2\beta)] + 29/105\pi^2\beta^6 + 4/15\beta^9 [-823/45 + 8\ln(2) + 16\ln(2\beta)] + 289/1155\pi^2\beta^{10} + 256/63\beta^{11}\ln(2\beta) \right\}.
\]

The results are indeed quite simple. We may obtain \( \frac{1}{\pi} f_{N+\delta}^{(1)} \) by adding the conversion term (59).

The strongest check comes from the fact that the expansions (65), (66) which come from integrating (63), (64) agree exactly with the expression obtained by expanding the analytically integrated result (64) directly. In this way, we simultaneously check all the above mentioned expressions, including our analytical integration (64). The expansions (65), (66) were also checked numerically by subtracting them from the exact expression (64) and verifying that the difference was order \( \beta^{11} \), as was done for (63), (64).

V. SERIES EXPANSION OF THE NON DELTA FUNCTION PART

Perhaps the most remarkable result of the series expansion is the simplification of the non delta function part, whose original form is the most lengthy part of the exact result, involving complicated logarithms, etc... Although the intermediate expressions were very lengthy and considerable computational time was required, a large degree of cancellation resulted in the following simple series. We convert \( (df^{(1)}/d\tau d\omega)_{N\delta} \) to \( (df^{(1)}/d\tau d\omega)_{N\delta} \) by multiplying by \( \beta^3 c_\beta(\tau)/2 \). Then, for the \( j = 0 \) channel we find

\[
\frac{1}{\pi} \left( \frac{df^{(1)}(+,+)}{d\tau d\omega} \right)_{N\delta} =
-8/3(1−\tau^2) \left\{ 4\beta^3 + 4\tau\beta^4 - 1/5\beta^5 [27 - 43\omega - \tau^2 (63 - 43\omega)] - 1/35\beta^7 [2(-87 - 108\omega + 179\omega^2) - \tau^2 (1133 - 2353\omega + 716\omega^2) + \tau^4 (1499 - 2137\omega + 358\omega^3)] + 8/35\beta^8 [70 - 259\omega + 199\omega^2 - 2\tau^2 (133 - 416\omega + 199\omega^2)] + \tau^4 (339 - 573\omega + 199\omega^2) \right\}
+2/105\beta^9 [338 - 1363\omega + 2222\omega^2 - 1189\omega^3 + \tau^2 (1339 - 11315\omega - 13399\omega^2 + 3567\omega^3)] + \tau^4 (104266 - 27244\omega - 20132\omega^2 + 3567\omega^3) + 2/105\beta^{10} [-462 + 4317\omega - 9396\omega^2 + 5221\omega^3] + 3\tau^2 (2529 - 12399\omega + 15371\omega^2 - 5221\omega^3) + 3\tau^4 (-6022 + 23659\omega - 21346\omega^2 + 5221\omega^3) + \tau^4 (13897 - 38097\omega + 27321\omega^2 - 5221\omega^3) \right\}
\]

and, for \( j = 2 \),


\[
\frac{1}{\pi} \left( \frac{df^{(1)}(+,-)}{d\tau d\omega} \right)_{\delta}^N = \\
-8/3(1-\tau^2) \left\{ 4\beta^5[1+\omega+\tau^2(1-\omega)] - 8\beta^6(1-\omega) \times (1-\tau^2) + 2/5\beta^7[3(1-6\omega+7\omega^2) - 2\tau^2(14-31\omega+21\omega^2) + \tau^4(73-44\omega+21\omega^2)] - 2/5\tau(1-\tau^2)\beta^8 \times [-53+30\omega-35\omega^2+\tau^2(169-154\omega+35\omega^2)] \\
-1/35\beta^9[152-226\omega+385\omega^2-415\omega^3] \\
+\tau^2(-2688+389\omega-345\omega^2+1245\omega^3) \\
-\tau^4(-7192+8782\omega-5747\omega^2+1245\omega^3) \\
+\tau^6(-5792+5118\omega-2681\omega^2+415\omega^3) \\
+2/35(1-\tau^2)\tau\beta^{10} [-420+1035\omega-1475\omega^2+904\omega^3] \\
+2\tau(2091-2569\omega+2215\omega^2-904\omega^3) \\
+\tau^4(-6390+6971\omega-2955\omega^2+904\omega^3) \right\}. \quad (68)
\]

We notice the absence of any logarithms, including powers of \(\ln \beta\). The structure is fairly predictable as well. We see that the series begin at order \(\beta^3\) and \(\beta^5\) respectively, so that their effect will be negligible very near to threshold. On the other hand, the large coefficients imply that they soon become noticeable for small \(\beta\). We may obtain \(\frac{d f^{(1)}_H}{d\tau d\omega}\) by adding to (67), (68) the conversion term (64).

Two independent determinations of (67), (68) were performed using Mathematica and REDUCE. These expressions were also checked numerically analogously to the delta function part of the differential cross section.

The integration of (67), (68) over \(\tau, \omega\) is straightforward and we obtain

\[
\frac{1}{\pi} f^{(1)}_{d_1}(+,-) = -\frac{128}{9} \beta^3 - \frac{448}{225} \beta^5 + \frac{34624}{2205} \beta^7 + \frac{42368}{3675} \beta^9, \quad (69)
\]

\[
\frac{1}{\pi} f^{(1)}_{d_1}(+,-) = -\frac{1024}{45} \beta^5 - \frac{2816}{525} \beta^7 - \frac{134656}{19845} \beta^9. \quad (70)
\]

These are remarkably simple results, which suggest that the exact integrated result for \(f^{(1)}_{d_1}\) is not too complicated. We notice the vanishing of the coefficients of the even powers of \(\beta\). This follows from the antisymmetry in \(\tau\) of the corresponding terms in the differential cross section.

To convert to \(f^{(1)}_H\), we must integrate the conversion term (42) (or (43)), over \(v\) and between \(w_1 \leq w \leq w_{1,\text{soft}}\). This yields

\[
\frac{f^{(1)}_H}{\pi} = \frac{f^{(1)}_{d_1}}{\pi} + \frac{f^{(1)}_{H,\text{conv}}}{\pi} = \frac{f^{(1)}_{d_1}}{\pi} - \frac{f^{(1)}_{S,\text{conv}}}{\pi}, \quad (71)
\]

where \(f^{(1)}_{S,\text{conv}}\) is given in (69). This verifies the cancellation of the \(\delta\) dependence of \(f^{(1)}_H + f^{(1)}_S\) in the limit \(\delta \to 0\). Implicitly we were working in this limit since we integrated the non delta function part over all \(\tau, \omega\).

We checked (63), (67) against the numerically integrated result and again found the difference was order \(\beta^{11}\). In the next section we will tabulate the numerical errors on the series expansions for \(f^{(1)}\) (total) for various values of \(\beta\), relative to the numerical result obtained from the exact expressions.

VI. TOTAL SERIES RESULTS AND NUMERICAL PARAMETRIZATIONS

We are now in a position to study the total cross section, by combining the results of the previous sections. Adding (63) and (69) gives the series for the \(j = 0\) total cross section

\[
\frac{1}{\pi} f^{(1)}(+,+) = \\
2 \left\{ 2\pi^2 - (20 - \pi^2)\beta + 10/3\pi^2\beta^2 + \beta^3/3[-404/3 + \pi^2] \\
+64 \ln(2\beta) + 8/15\pi^2\beta^4 + 4/15\beta^5[-6511/45 - \pi^2] \\
-32 \ln(2) + 256/3 \ln(2\beta)] - 104/105\pi^2\beta^6 \\
+4/105\beta^7[25757/315 - \pi^2 + 208/3 \ln(2\beta)] - 88/315\pi^2\beta^8 \\
+4/9\beta^9[1/7[-\pi^2/5 - 64/3 \ln(2)] + 1/25(128 \ln(2\beta) \\
+407639/2205)] - 488/3465\pi^2\beta^{10} \\
+103232/51975\beta^{11}\ln(2\beta) \right\}. \quad (72)
\]

and adding (66), (70) gives the series for the \(j = 2\) total cross section

\[
\frac{1}{\pi} f^{(1)}(+,-) = \\
16/3 \left\{ \pi^2\beta^2 - 8\beta^3 + \pi^2\beta^4 + 32\beta^5[-77/144 + \ln(2)/5] \\
+\ln(2\beta)/3] + \pi^2/7\beta^6 + 6/5\beta^7[-677/135 + 16/7 \ln(2) \\
+32/9 \ln(2\beta)] + 29/105\pi^2\beta^8 + 4/15\beta^9[-16949/735 \\
+8\ln(2) + 16 \ln(2\beta)] + 289/1155\pi^2\beta^{10} \\
+256/63\beta^{11}\ln(2\beta) \right\}. \quad (73)
\]

Such simple expressions indeed make numerical studies not too far above threshold rather straightforward. We can get an idea of how well these series work for typical \(\beta\) by comparing with numerically calculated values of \(f^{(1)}\).

In Table I we present the fractional error on the series for \(f^{(1)}(+,+), f^{(1)}(+,-), f^{(1)}_{\text{unp}}\) relative to the result obtained using numerical integration, for various values of \(z\) in the region 1.05 \(\leq z \leq 1.4\). For \(z \leq 1.05\), the series expansions are more accurate than the numerical results. At \(z = 1.05\), the errors are at the \(10^{-7} - 10^{-6}\) level. For \(z = 1.2\) they are at the \(10^{-4} - 10^{-3}\) level and for \(z = 1.4\) they are at the \(10^{-3} - 10^{-2}\) level. The errors
on $f^{(+,+)}$ are at the lower end, while the errors on $f^{(+,-)}$ are at the higher end and those for $f^{(1)}_{\text{up}}$ lie in between. This is good because, as we shall see in the next section, in determining $\alpha_s$ via top quark production at a $\gamma\gamma$ collider, it is the $j = 0$ and unpolarized channels which are of interest, the $j = 0$ channel being the most interesting one. With precision of better than one percent for $z \leq 1.4$, we have sufficient accuracy to use the series expansions (differential in particular) to perform easy numerical studies relevant to top quark production at a $\gamma\gamma$ collider of $\sqrt{s} \lesssim 500$ GeV. As we shall see, for the $\alpha_s$ determination, going to much higher energies is not useful since the determination is best done near $z = 1.2 (\sqrt{s} \approx 420$ GeV).

It is also useful to be able to parametrize $f^{(1)}$ to good accuracy for larger $\beta$, relevant for bottom and charm quark production at intermediate energies or top quark production at very high energies. This was done by fitting numerically computed values of $f^{(1)}$. We divide the parametrizations into 3 regions: a low energy region ($1 \leq z \leq 1.5$ or $0 \leq \beta \leq 0.7454$), an intermediate energy region ($1.5 < z \leq 5$) and a high energy region ($5 < z \leq 20$). We will denote the corresponding $f^{(1)}$ as $f^{(1),ie}$, $f^{(1),he}$ and $f^{(1),he}$, respectively.

The various forms for the parametrizations are

$$f^{(1),ie}(+,+) = 2\pi \left[ 2\pi^2 - (20 - \pi^2)\beta + \frac{10\pi^2}{3}\beta^2 \right] + \frac{64}{3}\beta^3 \ln \beta + \sum_{i=3}^{7} c_i \beta^i,$$

$$f^{(1),ie}(+,+) = \sum_{i=0}^{6} c_i (z - 1.5)^i,$$

$$f^{(1),he}(+,+) = \sum_{i=0}^{4} c_i (z - 5)^i \quad (74)$$

and

$$f^{(1),he}(+,+) = \frac{16\pi}{3} \left[ \frac{\pi^2}{3} \beta^2 - 8\beta^3 + \pi^2 \beta^4 \right] + \sum_{i=5}^{10} c_i \beta^i,$$

$$f^{(1),he}(+,+) = \sum_{i=0}^{4} c_i (z - 1.5)^i,$$

$$f^{(1),he}(+,+) = \sum_{i=0}^{3} c_i (z - 5)^i \quad (75)$$

The $c_i$ are given in Appendix B. In the low energy region, where high accuracy is required, the parametrizations are accurate to $\lesssim 0.01\%$, with the errors being the largest near the higher end of the region. The leading terms, given analytically, guarantee the correct threshold behaviour as they are just those in the exact series expansion. As mentioned earlier in connection with the series expansions, one can explicitly perform resummation on those terms. Thus one could modify the above parametrizations to include resummation effects without changing the higher order coefficients. Here, we simply present the one-loop corrections.

In the intermediate energy region, $f^{(1),ie}(+,+) = \sum 0.1\%$, $f^{(1),ie}(+,+) = \sum 0.1\%$, except very near $f^{(1),ie}(+,+) = 0$, which occurs for $z \approx 2.15, 3.15$. There, the absolute errors remain small, but of course the fractional error is larger. In the high energy region, $f^{(1),he}(+,+) = \sum 0.05\%$, while $f^{(1),he}(+,+) = \sum 0.5\%$. The above errors are rather conservative and one can not distinguish the parametrizations from the exact results for practical purposes.

The (exact) plots of $f^{(1)}$, $f^{(0)}$ in the three energy ranges are given in Figures 1–3. In Fig. 1 we plot $f^{(1)}$, $f^{(0)}$ in the low energy region versus $\beta$. $\beta$ is more suitable than $z$ in this region since the threshold region becomes compressed and $f^{(1)}$ varies quite rapidly with $z$ right at threshold. Fig. 1(a) highlights the fact that $f^{(1)}$ is most naturally decomposed into $f^{(1)}(+,+)$ and $f^{(1)}(+,0)$ since $f^{(1)}(+,+)$ is monotonically decreasing in the threshold region while $f^{(1)}(+,0)$ is monotonically increasing. $f^{(1)}$, on the other hand exhibits a rather sudden dip and peak which seems unnatural, until broken down into $(+,-)$ and $(+,+)$ components. From Fig. 1(b) we see that $f^{(0)}$ is monotonically increasing in this region for both helicity states.

In Fig. 2 we plot $f^{(1)}$, $f^{(0)}$ versus $z$ in the intermediate energy region. We notice that the $f^{(1)}$ cross just before $z = 1.5$ and the $f^{(0)}$ cross just before $z = 2$. $f^{(1)}(+,+)$ becomes negative and reaches a minimum near $z = 2.6$ and then monotonically increases, as does $f^{(1)}(+,0)$ throughout. $f^{(0)}(+,0)$ continues to grow while $f^{(0)}(+,+)$ levels off in accord with (73). In Fig. 3 we plot $f^{(1)}$, $f^{(0)}$ in the high energy region. The behaviour remains unchanged.

VII. PRECISION $\alpha_s$ DETERMINATION FROM TOP-QUARK PRODUCTION

A high energy $\gamma\gamma$ collider can be used as a “factory” for many interesting particles: Higgs bosons, $W^\pm$ bosons, top quarks etc... The beam polarization we be useful in producing Higgs bosons and reducing $QQ$ backgrounds. More specifically, the $j = 0$ channel will be of interest. This channel also turns out to be the channel of interest when trying to determine $\alpha_s$ via top quark production, making it complementary to the Higgs studies. The reason is that the cross section, and QCD corrections, are enhanced in this channel, thereby improving the statistics and the determination of $\alpha_s$, to which the cross section will be quite sensitive. The process $\gamma\gamma \rightarrow t\bar{t}X$ is more powerful than $e^+e^- \rightarrow Q\bar{Q}X$ in determining $\alpha_s$ because the QCD corrections are quite small in the latter, thus requiring an unreasonably large number of events for high precision; the corrections are suppressed
by \( \alpha_s / \pi \simeq 4\% \), relative to the Born term. In \( \gamma \gamma \to t \bar{t} + X \), we can “pick” our QCD correction by choosing the appropriate beam energy. Of course, as one gets too close to threshold, the perturbation series cannot be trusted, for reasons we will discuss below. Hence there are limitations.

To best illustrate the above idea, in Fig. 4 we have plotted the \( \gamma \gamma \to t \bar{t} + X \) cross section at LO and NLO, in the region \( 1 \leq z \leq 1.4 \), for the various helicity states. We took \( N_f = 5 \), \( m_t = 174 \text{ GeV} \) and used \( \Lambda = 230 \text{ MeV} \) in the two-loop expression for \( \alpha_s \), evaluated at \( \mu^2 = s \). One could also use \( N_f = 6 \), but since we are not far above threshold it is simpler to use \( N_f = 5 \) for evolution from \( \mu^2 = M_Z^2 \) to \( \mu^2 = s \). We notice that the \( j = 0 \) cross section is the largest, as are its QCD corrections, in this region. The region \( z \simeq 1.2 \) is nice in that the \( j = 0 \) cross section is near its maximum and the QCD corrections are sizable (\( \simeq 20\% \) of the total cross section), yet not so large that the perturbative expansion is unreliable. As one gets closer to threshold, other higher order effects, nonperturbative effects and top width effects may also become important. For these, and other reasons to be considered below, we will suggest \( z = 1.2 \) as being the optimal region for extracting \( \alpha_s \), and we will give a rough estimate of how precisely \( \alpha_s \) may be determined there. As well, we suggest the \( j = 0 \) channel as being the most powerful.

Firstly, we note that \( z = 1.2 \) corresponds to \( \sqrt{s_{\gamma \gamma}} \simeq 420 \text{ GeV} \), for top quark production. This energy should be accessible at a \( \sqrt{s_{\gamma \gamma}} \gtrsim 500 \text{ GeV} \) NLC. A typical \( \gamma \gamma \) luminosity assumed is \( 20 \text{ fb}^{-1} \). Since \( \sigma \simeq 1.4 \text{ pb} \), this corresponds to roughly 28,000 \( t \bar{t} \) events. Since the QCD correction is \( \simeq 20\% \) of the total cross section, this translates to \( \Delta \alpha_s / \alpha_s \simeq 3\% \), statistically. With a luminosity increase and, possibly, extended running, one could envision going to the percent level or better.

The above analysis was purely based on statistics and one-loop QCD corrections. Therefore, we will briefly discuss various theoretical systematic uncertainties. Clearly, one needs a two-loop analysis when dealing with one-loop corrections of order \( 20\% \), in order to determine \( \alpha_s \) at the level of a few percent. Threshold resummation can also be performed. One should also take into account the one-loop electroweak corrections. The QED ones are identical in form to the QCD ones, with the appropriate change in normalization, given by \( \alpha \). There will be a minor dependence on \( m_t \), which will be lessened with future Fermilab runs. The uncertainty on \( m_t \) translates to an uncertainty on \( z \). Since the \( j = 0 \) cross section is near its peak for \( z \simeq 1.2 \), minor variations in \( z \) will not appreciably affect the results.

Of some concern are resolved photon contributions, where a gluon or quark within the photon can participate directly in the interaction. Suppression of these contributions is a major reason for working close to threshold. Since the parton distributions within the photon drop steeply with increasing momentum fraction, \( x \), and since \( x \) must be large near threshold, such contributions are quite suppressed. Confirmation of this assertion may be inferred from the resolved contributions to \( b \) quark production near threshold presented in [13] from which we conclude that only very poor knowledge (if any) of the photon structure will be required, as such contributions will be a fraction of a percent of the cross section. One can further reduce those contributions by identifying outgoing jets collinear with one of the photon beams, which are a signature of resolved photon events. One can also require that the energy deposited in the detectors be equal to the total beam energy in order to account for missed jets of the type mentioned above.

From the experimental side, we are assuming only that \( t \bar{t} \) events can be clearly identified. With experience gained from Fermilab, this seems reasonable, especially considering the cleaner initial and final states in the \( \gamma \gamma \) case. Another experimental issue is that of normalization. In order to avoid normalization uncertainties, arising from luminosity uncertainties, we suggest the measurement of a ratio of cross sections denoted

\[
R_{Q}\frac{\gamma\gamma}{P} \equiv \frac{\sigma(\gamma\gamma \to Q\bar{Q} + X)}{\sigma(\gamma\gamma \to P\bar{P} + X)}, \quad P = W, l. \tag{76}
\]

The ratio of \( t \bar{t} \) to \( W^+W^- \) events is statistically quite powerful as over one million \( W^+W^- \) events are expected at such a “W factory”. This highlights the complementary nature of top quark and \( W^\pm \) production at a \( \gamma \gamma \) collider. As well, electroweak corrections to \( W^+W^- \) production have been studied. For the same reasons as for \( t \bar{t} \) production, the resolved photon contributions will be suppressed. If a \( b \bar{b} \) pair is produced in conjunction with the \( W^+W^- \), this will constitute a background to \( t \bar{t} \) production.

It is worth discussing the many advantages of determining \( \alpha_s \) via \( \gamma \gamma \to t \bar{t} + X \) relative to some of the options currently being used. The calculation is perturbative and avoids nonperturbative contributions arising in \( \alpha_s \) determinations from mass splittings and tau decays. Other determinations, based on evolution of hadronic structure functions, rely on the parton model and assumed knowledge of hadronic structure. No such assumptions are made here. Unlike the 3- to 2-jet ratio from \( e^+e^- \) annihilation, we avoid having to define the jet isolation criteria by measuring the total \( t \bar{t} \) cross section. Since we are at a large energy scale, not only does perturbation theory work well, but we automatically determine \( \alpha_s \) at (or above) the \( t \bar{t} \) threshold, without having to perform evolution or cross flavor thresholds. From a theoretical viewpoint, the most comparably clean determination comes from the ratio of hadrons to lepton pairs produced in \( e^+e^- \) annihilation at the \( Z \) pole. As mentioned earlier, the small QCD correction proves an insurmountable limiting factor in that case.

At this stage, our enthusiasm is dampened somewhat however by the need for a two-loop calculation. This need is highlighted by the fact that there is an arbitrariness in the choice of renormalization scale, \( \mu \), which can
only be compensated by the inclusion of two-loop corrections. The variation of \( \alpha_s \) with \( \ln \mu \) is order \( \alpha_s^2 \) though, so for a reasonable choice of \( \mu \) (i.e. \( \sqrt{s} \), \( m_t \), . . .) the two-loop scale dependent contribution should not be too large and should not change the value of \( \alpha_s \) radically. Nonetheless, as pointed out earlier, a two-loop calculation will eventually be required. In light of that fact, we see the importance of having simple analytical results for the one-loop corrections as they will be incorporated in the two-loop result. Also, we do not suggest that one consider this determination of \( \alpha_s \) in isolation. Rather, it should be combined with all other precision determinations, including the low energy ones, in order to minimize the error and provide an excellent test of QCD at the same time.

VIII. CONCLUSIONS

The analytical results for the one-loop QCD/QED corrections to massive fermion production presented in [3] were extended by analytically integrating the single integral (virtual+soft) part. The differential and integrated cross sections were series expanded to order \( \beta_{10} \) (including \( \beta_{11} \ln \beta \) terms) and were shown to be of practical use as well as being informative. Accurate parametrisations of the total cross section, valid up to \( \sqrt{s}/2m = 20 \), were presented. As an application, we showed how top quark production at a \( \gamma\gamma \) collider capable of reaching \( \sqrt{s} \approx 420 \) GeV could be used to precisely determine \( \alpha_s \), statistics permitting. Theoretical uncertainties were briefly discussed as were advantages over other \( \alpha_s \) determinations. Those advantages make this method of determination quite appealing.

The importance of performing the two-loop corrections was emphasized. As well, the best value of \( \alpha_s \) will still come from combining all determinations, including the low energy ones. The major strengths of this determination are the largeness of the QCD corrections and the potential to reduce the theoretical systematic errors via inclusion of higher order contributions, calculable using well-established perturbation theory. The major weakness being the tedious nature of the required two-loop calculation.

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APPENDIX A:

Here we present the coefficients \( a_i(j) \) appearing in the expression (10) for the analytically integrated single integral part, \( f_{y_1}^{(1)} \). They are

\[
\begin{align*}
a_1 &= (4 + 14 \beta - 6 \beta^2)(1 + \beta) + j(7 - 9 \beta - 19 \beta^2 + 3 \beta^3), \quad a_2 = 8[2 \beta^2(1 + \beta^2) - j(5 + 3 \beta^2)], \\
a_3 &= -16 + 8 \beta^4 + j(6 - 14 \beta^2), \quad a_4 = -8(1 - \beta^4 + 2j)(1 + \beta^2), \\
a_5 &= -\{(4(1 + \beta)^2 - \beta(1 - \beta^2))(1 + \beta) + [8 + \beta/2 + 14 \beta^2 - 7 \beta^3 - 2 \beta^4 + 5/2 \beta^5]\}, \\
a_6 &= (1 - \beta^2)^2 - j/2(1 - 14 \beta^2 + 5 \beta^4), \quad a_7 = 6(1 - \beta^2)^2 + j(25 - 3 \beta + 10 \beta^2 + \beta^3 - 9 \beta^4), \\
a_8 &= [(1 - \beta^2)^2 + j(15/2 + 3 \beta^2 - 5/2 \beta^4)]/2, \quad a_9 = -(1 - \beta^2)^2 + j/2(1 - \beta^2)(17 - 5 \beta^2), \\
a_{10} &= 4[2 - \beta + 2 \beta^2](1 + \beta) + j(3 - 10 \beta - 13 \beta^2 - 2 \beta^3 - 6 \beta^4)/(3 + \beta^2), \\
a_{11} &= [(1 + \beta^2)^2 - \beta(1 - \beta^2)](1 - \beta^2) + j/6(12 + 29 \beta + 39 \beta^2 - 10 \beta^3 - 9 \beta^4 - \beta^5), \\
a_{12} &= \{(1 + \beta^2)(9 + 9 \beta + 9 \beta^2 + 2 \beta^4) - 3(1 - \beta^2) - 2 \beta^3](1 - \beta) \\
&\quad + j/2(180 - 219 \beta + 165 \beta^2 - 219 \beta^3 + 95 \beta^4 - 73 \beta^5 + 35 \beta^6 - \beta^7 + 5 \beta^8)/(3 + \beta^2), \\
a_{13} &= -\{7(1 - \beta^2)^2 + j/2(73 + 58 \beta^2 - 35 \beta^4)]/2, \quad a_{14} = 8(1 + \beta^2 - 7/2j), \\
a_{15} &= -16[1 + \beta^2 - 3j], \quad a_{16} = -4(1 + \beta^2)(1 - \beta^4 + 2j), \\
a_{17} &= 8[1 + 2 \beta^2 - j(19 + 7 \beta^2)/(3 + \beta^2)], \\
a_{18} &= 4[9 + 2 \beta^2)(1 - \beta^2^2 - j/2(33 - 27 \beta^2 - 29 \beta^4 - 9 \beta^6)/(3 + \beta^2)]/(3 + \beta^2), \\
a_{19} &= \{8(1 + \beta^2)^2 - 3 \beta(1 - \beta^2)(1 - \beta^2) + j(16 - 37/2 \beta + 28 \beta^2 -11 \beta^3 - 4 \beta^4 + 15/2 \beta^5).\}
\end{align*}
\]

APPENDIX B:

Here we present the \( c_i \) entering in the parametrizations for \( f_{(1)}^{(1)}(+, +), f_{(1)}^{(1)}(+, -) \) in the various energy regions whose form is given in Equations (7), (7), respectively. For \( f_{(1)}^{(1), \text{ie}}(+, +) \), the coefficients are

\[
c_3 = -155, \quad c_4 = -125.68, \quad c_5 = 119.09, \quad c_6 = -540.34, \quad c_7 = 364.26.
\]

For \( f_{(1)}^{(1), \text{ie}}(+, +) \),

\[
c_0 = 53.502, \quad c_1 = -137.56, \quad c_2 = 102.57, \quad c_3 = -29.359, \\
c_4 = 3.3413, \quad c_5 = .21711, \quad c_6 = -.061446.
\]

For \( f_{(1)}^{(1), \text{ie}}(+, +) \),

\[
c_0 = 71.912, \quad c_1 = 47.622, \quad c_2 = -.67576, \\
c_3 = -2.1675 \times 10^{-2}, \quad c_4 = 1.1221 \times 10^{-3}.
\]

For \( f_{(1)}^{(1), \text{ie}}(+, +) \),

\[
c_5 = -667.218, \quad c_6 = 2252, \quad c_7 = -5395.85, \\
c_8 = 8137.55, \quad c_9 = -6654.48, \quad c_{10} = 2304.95.
\]

For \( f_{(1)}^{(1), \text{ie}}(+, +) \),

\[
c_0 = 55.267, \quad c_1 = 57.115, \quad c_2 = -7.3405, \\
c_3 = 1.3777, \quad c_4 = -.11197.
\]

For \( f_{(1)}^{(1), \text{ie}}(+, +) \),

\[
c_0 = 207.66, \quad c_1 = 37.016, \quad c_2 = -1.5617, \\
c_3 = -1.2899 \times 10^{-3}.
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FIG. 1. The functions (a) $f^{(1)}$; (b) $f^{(0)}$, versus $\beta$, in the low energy region, for the various helicity states.
FIG. 2. The functions (a) $f^{(1)}$; (b) $f^{(0)}$, versus $z$, in the intermediate energy region, for the various helicity states.

FIG. 3. The functions (a) $f^{(1)}$; (b) $f^{(0)}$, versus $z$, in the high energy region, for the various helicity states.

FIG. 4. The $\gamma\gamma \rightarrow t\bar{t} + X$ cross section at LO and NLO, versus $z$, for the various helicity states.
TABLE I. The various $f^{(1)}$ for values of $1.2 \leq z \leq 20$, and the corresponding single and double integral contributions. Here n.s. is the number of significant figures after the decimal point in $f^{(1)}$ and $f^{(1)}_{di}$.

| $z$ | n.s. | $f^{(1)}_{i, unp}$ | $f^{(1)}_{i, unp}$ (+,+) | $f^{(1)}_{u}$ | $f^{(1)}_{u}$ (+,+) | $f^{(1)}_{i, pol}$ | $f^{(1)}_{i, pol}$ (+,+) | $f^{(1)}_{d, pol}$ | $f^{(1)}_{d, pol}$ (+,+) | $f^{(1)}$ (+,+) |
|-----|------|-------------------|-------------------------|-----------|-----------------|----------------|----------------------|----------------|----------------------|----------------|
| 1.2 | 4    | 70.578894        | -5.47998017             | 65.0989   | 33.4162848      | -1.3766182     | 32.0397              | 4              | 103.9951788          | 33.4162848     |
| 1.2 | 4    | 103.9951788      | -6.85659845             | 97.1386   | 37.1626092      | -4.10336189    | 33.4162848          | 4              | 103.9951788          | 33.4162848     |
| 2   | 3    | 68.5516          | -24.064                 | 44.488    | -76.4447        | 38.792         | -37.653              | 8              | 32.0397              | 33.4162848     |
| 2   | 3    | -7.8931          | 14.728                  | 6.835     | 144.9963        | -62.856        | 82.140               | 3              | 32.0397              | 33.4162848     |
| 3   | 3    | 92.2075          | -29.594                 | 62.614    | -191.7554       | 125.852        | -65.903              | 3              | 32.0397              | 33.4162848     |
| 3   | 3    | -99.5479         | 96.259                  | -3.289    | 283.9629        | -155.447       | 128.516              | 3              | 32.0397              | 33.4162848     |
| 4   | 3    | 132.0495         | -33.0381                | 99.011    | -285.7603       | 215.4483       | -70.312              | 3              | 32.0397              | 33.4162848     |
| 4   | 3    | -153.7108        | 182.4102                | 28.699    | 417.8098        | -248.4864      | 169.323              | 3              | 32.0397              | 33.4162848     |
| 5   | 3    | 176.7014         | -36.802                 | 139.899   | -367.5540       | 299.902        | -67.652              | 3              | 32.0397              | 33.4162848     |
| 5   | 3    | -190.8526        | 263.100                 | 72.247    | 544.2554        | -336.704       | 207.551              | 3              | 32.0397              | 33.4162848     |
| 10  | 3    | 395.3262         | -55.3625                | 339.964   | -688.1880       | 639.5445       | -48.6435             | 3              | 32.0397              | 33.4162848     |
| 10  | 3    | -292.8618        | 584.182                 | 291.320   | 1083.5142       | -694.907       | 388.607              | 3              | 32.0397              | 33.4162848     |
| 20  | 1    | 749.8886         | -79.437                 | 670.45    | -1140.3966      | 1086.967       | -53.416              | 3              | 32.0397              | 33.4162848     |
| 20  | 1    | -390.5080        | 1007.530                | 617.02    | 1890.2852       | -1166.404      | 723.88               | 3              | 32.0397              | 33.4162848     |

TABLE II. The fractional errors on the various $f^{(1)}$ computed using the series expansions up to order $\beta^{10}$, for values of $1.05 \leq z \leq 1.4$.

| $z$ | $\beta$ | f. err (+,+) | f. err (+,−) | f. err unp | $\beta^{11}$ |
|-----|---------|-------------|-------------|-----------|-------------|
| 1.05| .3049   | $2.1 \times 10^{-7}$ | $2.5 \times 10^{-6}$ | $4.3 \times 10^{-7}$ | $2.1 \times 10^{-6}$ |
| 1.1 | .4166   | $-6.8 \times 10^{-6}$ | $2.3 \times 10^{-4}$ | $3.1 \times 10^{-5}$ | $6.6 \times 10^{-5}$ |
| 1.2 | .5528   | $-2.0 \times 10^{-4}$ | $3.3 \times 10^{-3}$ | $6.9 \times 10^{-4}$ | $1.5 \times 10^{-3}$ |
| 1.3 | .6390   | $-1.4 \times 10^{-3}$ | $1.3 \times 10^{-2}$ | $3.4 \times 10^{-3}$ | $7.3 \times 10^{-3}$ |
| 1.4 | .6999   | $-5.6 \times 10^{-3}$ | $2.9 \times 10^{-2}$ | $8.9 \times 10^{-3}$ | $2.0 \times 10^{-2}$ |