RATE OF CONVERGENCE OF EULER APPROXIMATIONS OF
SOLUTION TO MIXED STOCHASTIC DIFFERENTIAL
EQUATION INVOLVING BROWNIAN MOTION AND
FRACTIONAL BROWNIAN MOTION

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Abstract. We consider a mixed stochastic differential equation involving
both standard Brownian motion and fractional Brownian motion with Hurst
parameter $H > 1/2$. The mean-square rate of convergence of Euler approxi-
mations of solution to this equation is obtained.

Introduction

The main object of this paper is the following mixed stochastic differential equa-
tion involving independent Wiener process $B$ and fractional Brownian motion $B^H$
with Hurst index $H \in (1/2, 1)$:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t c(X_s)dB^H_s, \quad t \in [0, T],$$

where the integral w.r.t. Wiener process is the standard Itô integral, and the inte-
gral w.r.t. fBm is the forward stochastic integral. The questions of existence and
uniqueness of solution for equations of such type were considered in [7, 9, 5, 11].

Such mixed equations arise in different applied areas. In financial mathematics,
for example, it is often natural to assume that the underlying random noise consists
of two parts: a “fundamental” part, describing the economical background for a
stock price, and a “trading” part, coming from the randomness inherent for the
stock market. In this case the fundamental part of the noise should have a long
memory, while the second part is likely to be a white noise.

Due to a wide area of applications of equation (1), it is important to consider
certain numerical methods to solve it. We use here the most popular and probably
the simplest method of Euler approximations: one takes a uniform partition of
the interval, where the equation is being solved, and replaces differentials by a
correspondent finite differences. There is a vast literature dedicated to numerical
methods for stochastic differential equations driven by the Wiener process, we refer
to classical monographs [8] and [9] for an overview of the subject. There are also
several papers dealing with discrete time approximations for stochastic differential
equations with fractional Brownian motion, for example, [10] [12] [3].

The main difficulty when considering equation (1) lies in the fact that the ma-
chinery behind the two stochastic integrals is very different. The Itô integral is
treated usually in a mean square sense, while the integral with respect to fractional

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Brownian motion is understood and controlled in a pathwise sense. The mixture of two integrals makes things a lot harder, forcing us to consider very smooth coefficients and to make delicate estimates.

The paper is organized as follows. In Section 1, we give basic facts about forward and Skorokhod integration with respect to fractional Brownian motion and formulate main hypotheses. In Section 2, we define Euler approximations of (1) and establish some uniform integrability results for them. Section 3 contains the main result about rate of convergence of Euler approximations for equation (1). Unsurprisingly, the rate of convergence appears to be equal to the worst of the rates for corresponding “pure” equations, i.e. the mean-square distance between true and approximate solutions is of order $\delta^{1/2} \lor \delta^{2H-1}$, where $\delta$ is the mesh of the partition.

1. Preliminaries

1.1. Fractional Brownian motion and stochastic integration. In this section we give basic facts about the stochastic calculus for fractional Brownian motion. A more extensive exposition can be found e.g. in [4, 1].

Fractional Brownian motion (fBm) $B^H$ is by definition a centered Gaussian process with the covariance

$$\mathbb{E} [B^H_t B^H_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right), \ t, s \geq 0.$$ 

It has a version with almost surely $\kappa$-Hölder continuous paths for any $\kappa < H$. For $H \in (1/2, 1)$ (the case we consider here) it exhibits a property of long-range dependence.

Let $L^2_H[0, T]$ be the completion of the space of continuous functions with respect to the scalar product

$$\langle f, g \rangle_H = \int_{[0,T]^2} f(t) g(s) \psi(t, s) ds \, dt,$$

where $\psi(t, s) = H(2H - 1) |t - s|^{2H-2}$. Denote also by $\|f\|_H = \sqrt{\langle f, f \rangle_H}$ the corresponding norm.

Now we recall the notion of stochastic derivative. Let infinitely differentiable function $F: \mathbb{R}^n \to \mathbb{R}$ be bounded along with derivatives. For a smooth functional $G = F(B^H_{t_1}, \ldots, B^H_{t_n})$, where $t_1, \ldots, t_n \in [0, T]$ the stochastic derivative is defined as

$$D_x G = \sum_{k=1}^n F'_x(B^H_{t_1}, \ldots, B^H_{t_n}) \mathbb{1}_{[0,t_k]}(s).$$

The Sobolev space $D^{1,2}$ is the closure of the space of smooth functionals with respect to the norm

$$\|G\|^2_{1,2} = \mathbb{E} [G^2] + \mathbb{E} \left[ \|DG\|^2_H \right].$$

The Skorokhod, or divergence, stochastic integral is the adjoint to the stochastic derivative in the following sense. Let the domain $\text{dom} \delta$ of the divergence integral be the space of random processes $u \in L^2(\Omega, L^2_H[0, T])$ such that

$$\mathbb{E} [\langle DG, u \rangle_H] \leq C_u \|G\|_{L^2(\Omega)}.$$
for all $G \in \mathbb{D}^{1,2}$. Then the divergence integral
\[
\delta(u) = \int_0^T u_t \delta B^H_t
\]
is defined as the unique element of $L_2(\Omega)$ such that
\[
E[\langle DG, u \rangle_H] = E[G \delta(u)]
\]
for all $G \in \mathbb{D}^{1,2}$. It is worth to remark that $\text{dom} \delta$ contains the space $\mathbb{D}^{1,2}(L^2[0,T])$ of processes such that
\[
\|u\|_{H;1,2}^2 = E[\|u\|_H^2] + \iint_{[0,T]^2} E[D_s u_t D_s u_z \psi(t, s) \psi(v, z) ds dt dz dv]
\]
is finite. Moreover, for such processes
\[
E[\delta(u)^2] \leq \|u\|_{H;1,2}^2.
\]
The forward integral with respect to fBm is defined as the uniform limit in probability
\[
\int_0^t u_s dB^H_s = \lim_{\varepsilon \to 0} \int_0^t u_s \frac{B^H_{t+s} - B^H_s}{\varepsilon} ds,
\]
provided this limit exists. It is well-known (see e.g. [1]) that if $u \in \mathbb{D}^{1,2}(L^2_H [0,T])$
\[
\iint_{[0,T]^2} |D_s u_t| \psi(t, s) ds dt < \infty,
\]
then the forward integral exists and is equal to
\[
\int_0^T u_t dB^H_t = \int_0^T u_t \delta B^H_t + \iint_{[0,T]^2} D_s u_t \psi(t, s) ds dt.
\]

1.2. Assumptions. The following hypotheses on the ingredients of equation (1) will be assumed throughout the paper.

(A) The functions $a$ and $b$ are bounded together with their derivatives $a'_x$, $b'_x$:
\[
|a(t, x)| + |b(t, x)| + |a'_x(t, x)| + |b'_x(t, x)| \leq K;
\]
(B) the functions $a$ and $b$ are uniformly $(2H - 1)$-Hölder continuous in time:
\[
|a(t, x) - a(s, x)| + |b(t, x) - b(s, x)| \leq K |t - s|^{2H - 1};
\]
(C) the coefficient $c$ is bounded together with its first and second derivatives and uniformly positive:
\[
0 \leq c(x) + c(x)^{-1} + |c'(x)| + |c''(x)| \leq K.
\]

Here $K$ is a constant independent of $x$, $t$ and $s$;

(D) the Wiener process $W$ and the fractional Brownian motion $B^H$ are independent.

In what follows $C$ will denote a generic constant, whose value might change from line to line. To emphasize dependence on some variables, we will put them into subscript. For a random process $X$ we denote its increments by $X_{t,s} = X_t - X_s$. 
2. Euler approximations and auxiliary results

For \( N \geq 1 \) consider the following partition of the fixed interval \( [0, T] : \{0 = \nu_0 < \nu_1 < \cdots < \nu_N = T, \ \delta = T/N, \ \nu_k = k\delta \). The Euler approximation for equation (1) is defined recursively as

\[
X^\delta_{\nu_k+1} = X^\delta_{\nu_k} + a(\nu_k, X^\delta_{\nu_k}) \delta + b(\nu_k, X^\delta_{\nu_k}) \Delta W_k + c(X^\delta_{\nu_k}) \Delta B^H_k,
\]
where \( \Delta W_k = W_{\nu_{k+1}, \nu_k}, \ \Delta B^H_k = B^H_{\nu_{k+1}, \nu_k} \). The initial value of approximations is \( X^\delta_{\nu_0} = X_0 \).

Set \( n^\delta_n = \max\{n : \nu_n \leq u\}, \ t^\delta_u = \nu_{n^\delta_n} \), and define continuous interpolation by

\[
X^\delta_u = X^\delta_{t^\delta_u} + \int_{t^\delta_u}^u a(t^\delta, X^\delta_{t^\delta}) (u - t^\delta) \, dt + \int_{t^\delta_u}^u b(t^\delta, X^\delta_{t^\delta}) \, dW_t + \int_{t^\delta_u}^u c(t^\delta, X^\delta_{t^\delta}) \, dB^H_t,
\]
or, in the integral form,

\[
(5) \quad X^\delta_u = X_0 + \int_0^u a(t^\delta, X^\delta_t) \, dt + \int_0^u b(t^\delta, X^\delta_t) \, dW_t + \int_0^u c(t^\delta, X^\delta_t) \, dB^H_t.
\]

The following lemma is a discrete analogue of the Gronwall inequality.

**Lemma 2.1.** If a non-negative sequence \( \{x_n, n \geq 1\} \) satisfies

\[
x_{n+1} \leq x_n (1 + K\delta) + K\delta.
\]

Then

\[
x_n \leq (x_0 + 1)e^{Kn\delta}.
\]

The following two lemmas are technical.

**Lemma 2.2.** For \( s < \nu_n, \ n \geq 1, \) one has

\[
(6) \quad D_s X^\delta_{\nu_n} = c(X^\delta_{t^\delta_s}) \prod_{k=\nu^\delta_n}^{n-1} (1 + a'_x(\nu_k, X^\delta_{\nu_k}) \delta + b'_x(\nu_k, X^\delta_{\nu_k}) \Delta W_k + c'(X^\delta_{\nu_k}) \Delta B^H_k).
\]

(The product is set to 1 when the upper limit is smaller than the lower).

**Proof.** Clearly, \( D_s X^\delta_{\nu_n} = 0 \) if \( \nu_n < s \). Now observe that \( D_s \Delta B^H_{\nu_n} = \mathbb{I}_{n=n^\delta_n} \). Hence, for \( n = n^\delta_n \), we have

\[
D_s X^\delta_{\nu_n} = D_s \left( X^\delta_{\nu_{n-1}} + a(\nu_{n-1}, X^\delta_{\nu_{n-1}}) \delta + b(\nu_{n-1}, X^\delta_{\nu_{n-1}}) \Delta W_{n-1} + c(X^\delta_{\nu_{n-1}}) \Delta B^H_{n-1} \right) = c(X^\delta_{\nu_{n-1}}) D_s \Delta B^H_{n-1} = c(X^\delta_{\nu_{n-1}}).
\]

Further, for \( n > n^\delta_n \) we can write

\[
D_s X^\delta_{\nu_n} = D_s \left( X^\delta_{\nu_{n-1}} + a(\nu_{n-1}, X^\delta_{\nu_{n-1}}) \delta + b(\nu_{n-1}, X^\delta_{\nu_{n-1}}) \Delta W_{n-1} + c(X^\delta_{\nu_{n-1}}) \Delta B^H_{n-1} \right) = (1 + a'_x(\nu_{n-1}, X^\delta_{\nu_{n-1}}) \delta + b'_x(\nu_{n-1}, X^\delta_{\nu_{n-1}}) \Delta W_{n-1} + c'(X^\delta_{\nu_{n-1}}) \Delta B^H_{n-1}) D_s X^\delta_{\nu_{n-1}},
\]
and deduce (6) by induction. \( \square \)

**Lemma 2.3.** For any \( M > 0 \) it holds

\[
\mathbb{E} \left[ \exp \left\{ M \sum_{k=0}^{N-1} ((\Delta W_k)^2 + (\Delta B^H_k)^2) \right\} \right] < C_M
\]
for all \( N \) large enough with \( C_M \) independent of \( N \).
Proof. Using independence of $W$ and $B^H$, we then can write

$$
E \left[ \exp \left\{ M \sum_{k=0}^{N-1} ((\Delta W_k)^2 + (\Delta B^H_k)^2) \right\} \right]
$$

$$
= E \left[ \exp \left\{ M \sum_{k=0}^{N-1} (\Delta W_k)^2 \right\} \right] E \left[ M \exp \left\{ \sum_{k=0}^{N-1} (\Delta B^H_k)^2 \right\} \right]
$$

$$
\leq \prod_{k=0}^{N-1} E \left[ \exp \left\{ M (\Delta W_k)^2 \right\} \right] \left( \prod_{k=0}^{N-1} E \left[ \exp \left\{ MN (\Delta B^H_k)^2 \right\} \right] \right)^{1/N}
$$

$$
= C(1 - 2M\delta)^{-N/2} (1 - 2M \cdot 2\delta) - 1/2
$$

$$
= C(1 - 2MT/N)^{-N/2} (1 - 2MT^{2H}N^{1-2H})^{-1/2},
$$

where the last equalities hold provided $2MT/N < 1$ and $2MT^{2H}N^{1-2H} < 1$, which is true for all $N$ large enough. Observing that

$$
C(1 - 2MT/N)^{-N/2} (1 - 2MT^{2H}N^{1-2H})^{-1/2} \rightarrow Ce^{MT}, \quad N \rightarrow \infty,
$$

we get the desired boundedness. \hfill \square

Now we are ready to prove that the moments of Euler approximations as well as of their stochastic derivatives are uniformly bounded.

**Lemma 2.4.** For any $p > 0$ one has

$$
(7) \quad E \left[ \left| D_s X^{\delta}_{\nu_k} \right|^p \right] < C_p
$$

for all $s \in [0, T]$, $n \leq N$, with $C_p$ independent of $\delta$.

**Proof.** It is easy to see from (6) that the left-hand side of (7) is finite. Therefore, it suffices to establish boundedness only for $N$ large enough.

Introduce the following notation:

$$
a_k = a'_\nu (\nu_k, X_{\nu_k}^{\delta}), \quad b_k = b'_\nu (\nu_k, X_{\nu_k}^{\delta}), \quad c_k = c'(X_{\nu_k}^{\delta}),
$$

$$
\Theta_k = \left| 1 + a_k \Delta + b_k \Delta W_k + c_k \Delta B^H_k \right|, \quad \gamma_k = |\Delta W_k| + |\Delta B^H_k|,
$$

$$
d_k = a(\nu_k, X_{\nu_k}^{\delta}) \delta + b(\nu_k, X_{\nu_k}^{\delta}) \Delta W_k, \quad \Delta_k = d_k + c(X_{\nu_k}^{\delta}) \Delta B^H_k = X_{\nu_k+1, \nu_k}^{\delta}.
$$

Fix a small positive constant $\gamma$ (its value will be specified later to satisfy our needs). Put $A = \{ \forall k \gamma_k \leq \gamma \}$.

$$
E \left[ \left| D_s X^{\delta}_{\nu_k} \right|^p \right] = E \left[ \left| D_s X^{\delta}_{\nu_k} \right|^p \mathbb{I}_A \right] + E \left[ \left| D_s X^{\delta}_{\nu_k} \right|^p \mathbb{I}_{\Omega\setminus A} \right].
$$

**Step 1.** First we estimate $E \left[ \left| D_s X^{\delta}_{\nu_k} \right|^p \mathbb{I}_{\Omega\setminus A} \right]$. Write

$$
E \left[ \left| D_s X^{\delta}_{\nu_k} \right|^p \mathbb{I}_{\Omega\setminus A} \right] = \sum_B E \left[ c(X_{\delta}^{\delta})^p \left( \prod_{k \notin B} \Theta_k \mathbb{I}_{\gamma_k \leq \gamma} \prod_{k \in B} \Theta_k \mathbb{I}_{\gamma_k > \gamma} \right) \right],
$$

where the outer sum is taken over all non-empty $B \subset \{ \nu_k, \nu_k + 1, \ldots, n - 1 \}$.

Observe that

$$
|a_k \delta + b_k \Delta W_k + c_k \Delta B^H_k| \leq K(\delta + \gamma_k),
$$

so this expression does not exceed 1 whenever $\delta < 1/(2K)$ and $\gamma_k \leq \gamma < 1/(2K)$, and we can write

$$
\Theta_k \mathbb{I}_{\gamma_k \leq \gamma} \leq \exp \{ a_k \delta + b_k \Delta W_k + c_k \Delta B^H_k \}.
$$
For $\gamma_k > \gamma$ we estimate simply $\Theta_k < \exp \{ K(\delta + \gamma_k) \}$, therefore,

\[
E \left[ \left| D_s X_n \right| \mathbb{I}_{\Omega \setminus A} \right] \\
\leq C_p \sum_{B} E \left[ \left( \exp \left\{ \sum_{k \in B} (a_k \delta + b_k \Delta W_k + c_k \Delta B^H_k) \right\} \prod_{k \in B} \Theta_k \mathbb{I}_{\gamma_k > \gamma} \right)^p \right] \\
\leq C_p E \left[ \exp \left\{ p \sum_{k \notin B} (K \delta + b_k \Delta W_k + K | \Delta B^H_k |) \right\} \prod_{k \in B} e^{pK(\delta + \gamma_k)} \mathbb{I}_{\gamma_k > \gamma} \right] \\
\leq C_p \left( E \left[ \exp \left\{ 3p \sum_{k \notin B} b_k \Delta W_k \right\} \right] \right) \left( E \left[ \exp \left\{ 3pK \sum_{k=0}^{N-1} | \Delta B^H_k | \right\} \right] \right) \\
\quad \times \left( E \left[ \prod_{k \in B} e^{3pK\gamma_k} \mathbb{I}_{\gamma_k > \gamma} \right] \right)^{1/3}.
\]

By the standard properties of the stochastic integral with respect to $W$,

\[
E \left[ \exp \left\{ 3p \sum_{k \notin B} b_k \Delta W_k \right\} \right] = E \left[ \exp \left\{ \sum_{k \notin B} 9p^2 b_k^2 \delta / 2 \right\} \right] \\
\leq \exp \left\{ 5p^2 K^2 N \delta \right\} = \exp \left\{ 5p^2 K^2 T \right\}.
\]

Now estimate, using the Hölder inequality,

\[
E \left[ \exp \left\{ 3p \sum_{k \notin B} b_k \Delta W_k \right\} \right] \leq E \left[ \exp \left\{ 3pK \sum_{k=0}^{N-1} | \Delta B^H_k | \right\} \right] \\
\leq \left( \prod_{k=0}^{N-1} E \left[ e^{3pK N | \Delta B^H_k |} \right] \right)^{1/N} \\
\leq \left( \prod_{k=0}^{N-1} E \left[ e^{3pK N 2^H} \right] \right)^{1/N} \\
\leq C e^{3p^2 K^2 N^2 \delta 2^H} = C e^{3p^2 K^2 T 2^H N^2 2^H}.
\]

Further,

\[
E \left[ \prod_{k \in B} e^{3pK\gamma_k} \mathbb{I}_{\gamma_k > \gamma} \right] \leq E \left[ \prod_{k \in B} e^{3pK\gamma_k/\gamma} \mathbb{I}_{\gamma_k > \gamma} \right] \\
\leq \left( E \left[ \exp \left\{ \frac{6pK}{\gamma} \sum_{k=0}^{N-1} \gamma_k^2 \right\} \right] E \left[ \prod_{j \in B} \mathbb{I}_{\gamma_j > \gamma} \right] \right)^{1/2} \leq C_{p, \gamma} \left( E \left[ \prod_{j \in B} \mathbb{I}_{\gamma_j > \gamma} \right] \right)^{1/2},
\]

where the last inequality hold for all $N$ large enough thanks to Lemma 2.3. To estimate the last expectation, recall that $W$ and $B^H$ are independent and take first
the expectation with respect to $W$:

$$
\mathbb{E} \left[ \prod_{j \in B} \mathbb{1}_{\gamma_j > \gamma} \right] \leq \mathbb{E} \left[ \prod_{j \in B} 2\Phi \left( (|\Delta B_j^H| - \gamma)\delta^{-1/2} \right) \right] \\
\leq 2^{n(B)} \left( \prod_{j \in B} \mathbb{E} \left[ \Phi \left( (|\Delta B_j^H| - \gamma)\delta^{-1/2} \right)^{n(B)} \right] \right)^{1/n(B)} \\
\leq 2^{n(B)} \mathbb{E} \left[ \Phi \left( (|\Delta B_0^H| - \gamma)\delta^{-1/2} \right)^{n(B)} \right],
$$

(10)

where $n(B)$ is the number of elements of $B$. We split the inner expectation into parts where $|\Delta B_0^H| \leq \gamma/2$ and $|\Delta B_0^H| > \gamma/2$. For $|\Delta B_0^H| \leq \gamma/2$ it holds $\Phi \left( (|\Delta B_0^H| - \gamma)\delta^{-1/2} \right) \leq \Phi(-\gamma\delta^{-1/2}/2)^{n(B)}$, also we have $P(|\Delta B_0^H| > \gamma/2) \leq 2\Phi(-\gamma\delta^{-1/2}/2)$, hence

$$
\mathbb{E} \left[ \Phi \left( (|\Delta B_0^H| - \gamma)\delta^{-1/2} \right)^{n(B)} \right] \leq \Phi(-\gamma\delta^{-1/2}/2)^{n(B)} + 2\Phi(-\gamma\delta^{-1/2}/2) \\
\leq e^{-\gamma\delta^{-1}n(B)/8} + e^{-\gamma\delta^{-2}H/2} \leq e^{-C_\gamma n(B)} + e^{-C_\gamma N^{2H}}.
$$

Plugging this into (10), we get

$$
\mathbb{E} \left[ \prod_{j \in B} \mathbb{1}_{\gamma_j > \gamma} \right] \leq \mathbb{E} \left[ \prod_{j \in B} 2\Phi \left( (|\Delta B_j^H| - \gamma)\delta^{-1/2} \right) \right] \\
\leq e^{C(1-C_\gamma N)n(B)} + e^{Cn(B)-C_\gamma N^{2H}}
$$

and combining this with (8) and (9), we arrive to

$$
\mathbb{E} \left[ |D_x X_{\nu_1}^A|^p \mathbb{1}_{\Omega \setminus \Lambda} \right] \\
\leq C_{p,\gamma} e^{C_p N^{2-2H}} \left( \sum_B e^{C(1-C_\gamma N)n(B)} + e^{-C_\gamma N^{2H}} \sum_B e^{Cn(B)} \right).
$$

For $N$ large enough it holds $C(C_\gamma N - 1) \geq C_\gamma N$ (naturally, with different constants $C_\gamma$ in the left-hand and in the right-hand sides), so the first sum is bounded from above by

$$
\sum_B e^{-C_\gamma Nn(B)} = (1 + e^{-C_\gamma N})^{n-n^0} - 1 \\
\leq (1 + e^{-C_\gamma N})^N - 1 = \exp \left\{ N \log (1 + e^{-C_\gamma N}) \right\} - 1 \\
\leq \exp \left\{ CN e^{-C_\gamma N} \right\} - 1 \leq \exp \left\{ C_\gamma e^{-C_\gamma N} \right\} - 1 \leq C_\gamma e^{-C_\gamma N}.
$$

Similarly, the second sum is bounded by

$$
e^{-C_\gamma N^{2H}} \sum_B e^{Cn(B)} = e^{-C_\gamma N^{2H}} (1 + e^C)^{n-n^0} \leq C_\gamma e^{CN-C_\gamma N^{2H}}.
$$

Since $H \in (1/2, 1)$, this implies

$$
\mathbb{E} \left[ |D_x X_{\nu_1}^A|^p \mathbb{1}_{\Omega \setminus \Lambda} \right] \leq C_{p,\gamma} e^{C_p N^{2-2H}} \left( e^{-C_\gamma N} + e^{CN-C_\gamma N^{2H}} \right) \\
\leq C_{p,\gamma} \left( e^{-C_\gamma N} + e^{-C_\gamma N^{2-4H}} \right)
$$
for all $N$ large enough. This expression vanishes as $N \to \infty$, hence we get the boundedness of $E \left[ \left| D_s X^\delta_{v_n} \right|^p I_{[A]} \right]$.

Step 2. Now we turn to $E \left[ \left| D_s X^\delta_{v_n} \right|^p I_{A} \right]$. If we take $\gamma < K^{-3}/3$ and $\delta < K^{-3}/3$, then $|\Delta_k| < 2K^{-2}/3$ on $A$ and $\left| c(X^\delta_{v_{k+1}}) - c(X^\delta_{v_k}) \right| < 2K^{-1}/3$. But $c(X^\delta_{v_k}) > K^{-1}$, so $c(X^\delta_{v_{k+1}})/c(X^\delta_{v_k}) \in (1/3, 5/3)$, which allows us to write by the Taylor formula

$$
\log \frac{c(X^\delta_{v_{k+1}})}{c(X^\delta_{v_k})} = c'(X^\delta_{v_k}) \Delta_k + R'_k = c_k \Delta B^H_k + \frac{c_k}{c(X^\delta_{v_k})} d_k + R'_k,
$$

where $|R'_k| \leq C \Delta_k^2$. Similarly, on $A$

$$
\log \Theta_k = \log \left( 1 + a_k \delta + b_k \Delta W_k + c_k \Delta B^H_k \right) = a_k \delta + b_k \Delta W_k + c_k \Delta B^H_k + R''_k
$$

with $|R''_k| \leq C \Delta_k^2$. Plugging into this formula the expression for $c_k \Delta B^H_k$ from (11), we get

$$
\Theta_k = \frac{c(X^\delta_{v_{k+1}})}{c(X^\delta_{v_k})} \exp \{ a_k \delta + b_k \Delta W_k + R_k \},
$$

where $R_k = R''_k - R'_k$, $a_k = a - c_k a(X^\delta_{v_k})/c(X^\delta_{v_k})$, $b_k = b_k - c_k b(X^\delta_{v_k})/c(X^\delta_{v_k})$.

Now we can estimate

$$
E \left[ \left| D_s X^\delta_{v_n} \right|^p I_{A} \right] = E \left[ c(X^\delta_{v_n})^p \prod_{k=n_k}^{n-1} \Theta_k^p I_{A} \right]
= E \left[ c(X^\delta_{v_{n-1}})^p \exp \left\{ p \sum_{k=n_k}^{n-1} \left( a_k \delta + b_k \Delta W_k + R_k \right) \right\} \right]
\leq C_p \left( E \left[ \exp \left\{ 2p \sum_{k=n_k}^{n-1} \beta_k \Delta W_k \right\} \right] E \left[ \exp \left\{ 2p \sum_{k=0}^{N-1} |R_k| \right\} \right] \right)^{1/2}
\leq C_p \left( E \left[ \exp \left\{ 2p^2 \sum_{k=n_k}^{n-1} \beta_k^2 \delta \right\} \right] E \left[ \exp \left\{ \sum_{k=0}^{N-1} \Delta_k^2 \right\} \right] \right)^{1/2}
\leq C_p E \left[ \exp \left\{ \sum_{k=0}^{N-1} \beta_k^2 \right\} \right] \leq C_p,
$$

where the last holds for all $N$ large enough due to Lemma 2.3. This completes the proof. \hfill \Box

**Lemma 2.5.** For any $p > 0$ one has

$$
E \left[ \left| X^\delta_t \right|^p \right] \leq C_p
$$

for all $t \in [0, T]$.

Moreover,

$$
E \left[ \left| X^\delta_t - X^\delta_{t^k} \right|^p \right] \leq C_p \delta^{p/2}
$$

for any $t \in [0, T]$. 

Proof. It is enough to prove this for \( p = 2m, m \in \mathbb{N} \). We first prove \((12)\) for \( t = \nu_n \), using an induction by \( m \).

Start with \( m = 1 \).

Denote \( a_n = a(\nu_n, X_{\nu_n}^\delta), b_n = b(\nu_n, X_{\nu_n}^\delta), c_n = c(X_{\nu_n}^\delta) \) and write for \( \delta \in (0, 1/2) \) by Jensen’s inequality
\[
\mathbb{E} \left[ (X_{\nu_{n+1}}^\delta)^2 \right] \leq \mathbb{E} \left[ (X_{\nu_n}^\delta + b_n \Delta W_n + c_n \Delta B_n^H)^2 \right] (1 - \delta)^{-1} + 2\mathbb{E} \left[ (a_n \delta)^2 \right] \delta^{-1}
\]
\[
\leq \left( \mathbb{E} \left[ (X_{\nu_n}^\delta)^2 \right] + \mathbb{E} \left[ (b_n \Delta W_n)^2 \right] + \mathbb{E} \left[ (c_n \Delta B_n^H)^2 \right] \right)
+ 2\mathbb{E} \left[ X_{\nu_n}^\delta b_n \Delta W_n \right] + 2\mathbb{E} \left[ b_n c_n \Delta W_n \Delta B_n^H \right] + 2\mathbb{E} \left[ X_{\nu_n}^\delta \Delta B_n^H \right] \epsilon^{2\delta} + C\delta
\]
\[
\leq \left( \mathbb{E} \left[ (X_{\nu_n}^\delta)^2 \right] + C\delta + C\delta^{2H} + 2\mathbb{E} \left[ X_{\nu_n}^\delta \Delta B_n^H \right] \right) \epsilon^{2\delta} + C\delta
\]
\[
\leq \mathbb{E} \left[ (X_{\nu_n}^\delta)^2 \right] \epsilon^{2\delta} + \mathbb{E} \left[ X_{\nu_n}^\delta \Delta B_n^H \right] \epsilon^{2\delta} + C\delta.
\]

By \((2)\) and \((7)\), we can write
\[
\mathbb{E} \left[ X_{\nu_n}^\delta \Delta B_n^H \right] = \alpha_H \int_{\nu_n}^{\nu_{n+1}} \int_{\nu_n}^{\nu_{n+1}} \mathbb{E} \left[ D_s X_{\nu_n}^\delta \right] \delta^{2H-2} dt \, ds
\]
\[
\leq C \int_{\nu_n}^{\nu_{n+1}} (t - \nu_n)^{2H-1} dt \leq C\delta.
\]

Then by Lemma \(2.1\)
\[
\mathbb{E} \left[ (X_{\nu_n}^\delta)^2 \right] \leq X_0^2 e^{C\delta n} \leq C e^{C\delta N} \leq C,
\]
as required.

Now let \( m \geq 2 \) and for \( l \leq m \)
\[
\mathbb{E} \left[ (X_{\nu_n}^\delta)^{2l} \right] \leq C_{2l}.
\]

In the further estimates constants may depend on \( m \), but not on \( n \).

Observe that by the Jensen inequality for \( \delta < 1 \)
\[
(a + b)^{2m} \leq (1 - \delta)^{1-2m} a^{2m} + \delta^{1-2m} b^{2m},
\]
whence
\[
(a + b)^{2m} \leq a^{2m} (1 + C_m \delta) + C_m b^{2m} \delta^{1-2m},
\]
therefore
\[
\mathbb{E} \left[ (X_{\nu_{n+1}}^\delta)^{2m} \right] \leq \mathbb{E} \left[ (X_{\nu_n}^\delta + b_n \Delta W_n + c_n \Delta B_n^H)^{2m} \right] (1 + C\delta)
\]
\[
+ \mathbb{E} \left[ (a_n \delta)^{2m} \right] \delta^{1-2m} \leq \mathbb{E} \left[ (X_{\nu_n}^\delta + b_n \Delta W_n + c_n \Delta B_n^H)^{2m} \right] (1 + C\delta) + C\delta.
\]
Expand the power in the first term and consider a generic term of this expansion (without a coefficient):

\[
E \left[ (X_{\nu_n}^\delta)^{2m-i-k} (b_n \Delta W_n)^i (c_n \Delta B_n^{H_i})^k \right]
\]

\[
= E \left[ (X_{\nu_n}^\delta)^{2m-i-k} b_n^i c_n^k E \left[ (\Delta W_n)^i (\Delta B_n^{H_i})^k \mid \mathcal{F}_{\nu_n} \right] \right]
\]

\[
= E \left[ (X_{\nu_n}^\delta)^{2m-i-k} b_n^i c_n^k \right] E \left[ (\Delta W_n)^i \mathcal{F}_{\nu_n} \right]
\]

\[
= E \left[ (X_{\nu_n}^\delta)^{2m-i-k} b_n^i c_n^k (\Delta B_n^{H_i})^k \right] \frac{i!}{(i/2)!} \delta^{i/2} \mathbb{I}_{i \text{ even}}.
\]

Thus, we can write

\[
E \left[ (X_{\nu_n}^\delta + b_n \Delta W_n + c_n \Delta B_n^{H_i})^{2m} \right]
\]

\[
= \sum_{k=0}^{2m} \sum_{j=0}^{m} \binom{2m}{k, j, 2m-k-2j} E \left[ (X_{\nu_n}^\delta)^{2m-2j-k} b_n^j c_n^k (\Delta B_n^{H_i})^k \right] \frac{(2j)!}{2^j j!} \delta^j,
\]

where

\[
\left( a + b + c \right) \left( a, b, c \right) = \frac{(a + b + c)!}{a! b! c!}
\]

is a trinomial coefficient.

For \( k = 0, j \geq 1 \), the terms of this sum are bounded by \( C \delta \) by the induction hypothesis and boundedness of \( b_n, c_n \).

Further, for \( k \geq 2 \)

\[
E \left[ (X_{\nu_n}^\delta)^{2m-2j-k} b_n^j c_n^k (\Delta B_n^{H_i})^k \right] \leq C E \left[ \left| X_{\nu_n}^\delta \right|^{2m-2j-k} (\Delta B_n^{H_i})^k \right]
\]

\[
\leq C \left( 1 + E \left[ (X_{\nu_n}^\delta)^{2m} \right] \right) \delta^{2H} \leq C \left( 1 + E \left[ (X_{\nu_n}^\delta)^{2m} \right] \right) \delta,
\]

where \( \lambda = 2m/(2m-2j-k) \), \( \eta = \lambda/(\lambda-1) \); here we have used an obvious estimate

\[
(15) \quad \left( E \left[ (X_{\nu_n}^\delta)^{2m} \right] \right)^{1/\lambda} \leq 1 + E \left[ (X_{\nu_n}^\delta)^{2m} \right].
\]

Now estimate the term with \( k = 1, j = 0 \), using formula (2):

\[
E \left[ (X_{\nu_n}^\delta)^{2m-1} c_n \Delta B_n^{H_i} \right] = \int_{t_{\nu_n}}^{t_{\nu_n}+1} \int_{s_{\nu_n}}^{s_{\nu_n}+1} \mathbb{E} \left[ D_s \left( (X_{\nu_n}^\delta)^{2m-1} c(X_{\nu_n}^\delta) \right) \psi(t, s) \right] ds \left( t \right) dt ds
\]

\[
= \int_{t_{\nu_n}}^{t_{\nu_n}+1} \mathbb{E} \left[ \left( (2m-1)(X_{\nu_n}^\delta)^{2m-2} c(X_{\nu_n}^\delta) + (X_{\nu_n}^\delta)^{2m-1} c'(X_{\nu_n}^\delta) \right) D_s X_{\nu_n}^\delta \right]
\]

\[
\times H(2H-1) (t-s)^{2H-2} dt ds
\]

\[
\leq C \left( 1 + E \left[ (X_{\nu_n}^\delta)^{2m} \right] \right) \int_{t_{\nu_n}}^{t_{\nu_n}+1} (t-s)^{2H-1} dt \leq C \left( 1 + E \left[ (X_{\nu_n}^\delta)^{2m} \right] \right) \delta.
\]

Here, as above we have used the Hölder inequality, inequality (15) and boundedness of moments of the stochastic derivative. The terms with \( k = 1, j \geq 1 \) are estimated similarly.

Collecting the estimates, we get

\[
E \left[ (X_{\nu_n+1}^\delta)^{2m} \right] \leq E \left[ (X_{\nu_n}^\delta)^{2m} \right] (1 + C \delta) + C \delta,
\]
so by Lemma 2.1 we get the desired boundedness.

Now write for $s \in [\nu_n, \nu_{n+1})$

$$E\left[|X_s^\delta - X_{\nu_n}^\delta|^p\right] \leq C_p \left(E[|u_n(s - \nu_n)|^p] + E[|b_n W_{s, \nu_n}|^p] + E\left[|c_n B_{s, \nu_n}^H|^p\right]\right)$$

$$\leq C_p \left((s - \nu_n)^p + (s - \nu_n)^{p/2} + (s - \nu_n)^{pH}\right) \leq C_p(s - \nu_n)^{p/2},$$

which gives (13) and together with (12) for $t = \nu_n$ implies (12) for all $t \in [0, T]$.

3. Rate of convergence

Now we are ready to prove the main result about the mean-square rate of convergence of Euler approximations.

**Theorem 3.1.** Euler approximations [5] for the solution of equation (1) satisfy

$$E\left[(X_t - X_0)^2\right] \leq C(\delta + \delta^{4H-2}).$$

**Proof.** Define $\psi(x) = \int_0^x c(z)^{-1} dz$. It is clear that

$$K^{-1} |x - y| \leq |\psi(x) - \psi(y)| \leq K |x - y|.$$

Write by the Itô formula

$$\psi(X_t) = \psi(X_0) + \int_0^t \left(\alpha(s, X_s) ds + \beta(s, X_s) dW_s\right) + B_t^H,$$

where

$$\alpha(s, x) = \frac{a(s, x)}{c(x)} - \frac{b(s, x)^2 c'(x)}{2c(x)^2}, \quad \beta(s, x) = \frac{b(s, x)}{c(x)}.$$

Similarly,

$$\psi(X_t^\delta) = \psi(X_0^\delta) + \int_0^t \left(\alpha(s, X_s^\delta) ds + \beta(s, X_s^\delta) dW_s\right) + B_t^H - G_t^\delta,$$

where

$$G_t^\delta = \int_0^t \left[c^{-1}(X_s^\delta)\left((a(s, X_s^\delta) - a(t_s^\delta, X_{t_s^\delta}^\delta)) ds + (b(s, X_s^\delta) - b(t_s^\delta, X_{t_s^\delta}^\delta)) dW_s\right)\right.$$

$$\left.+c^{-1}(X_s^\delta)(c(X_s^\delta) - c(X_s^\delta)) dB_s^H + \frac{c'(X_s^\delta)}{2c^2(X_s^\delta)}(b(s, X_s^\delta) - b(t_s^\delta, X_{t_s^\delta}^\delta))^2 ds\right]$$

$$= \int_0^t \left(a_s^\delta ds + b_s^\delta dW_s + c_s^\delta dB_s^H + d_s^\delta ds\right).$$

Estimate

$$E\left[\left(\int_0^t a_s^\delta ds\right)^2\right] \leq C \int_0^t E\left[(a_s^\delta)^2\right] ds$$

$$\leq C \int_0^t E\left[c(X_s^\delta)^2(a(s, X_s^\delta) - a(t_s^\delta, X_{t_s^\delta}^\delta))^2 + (a(s, X_s^\delta) - a(s, X_{t_s^\delta}^\delta))^2\right] ds$$

$$\leq C \left(\delta^{2H-1} + \int_0^t E\left[(X_s^\delta)^2\right] ds\right) \leq C \delta^{2H-1}.$$
and using the Itô isometry,
\[ E \left[ \left( \int_0^t b^\delta_s \, dW_s \right)^2 \right] = \int_0^t E \left[ (b^\delta_s)^2 \right] \, ds \leq C \delta^{2H-1}. \]

Further, by the chain rule for the stochastic derivative,
\[ u \left( \frac{\partial}{\partial x} \right) X_{s,t}^\delta = \frac{\partial}{\partial x} X_{s,t}^\delta + \frac{\partial}{\partial y} X_{s,t}^\delta \frac{\partial}{\partial y} X_{s,t}^\delta + \frac{\partial}{\partial z} X_{s,t}^\delta \frac{\partial}{\partial z} X_{s,t}^\delta \]
\[ + \cdots + \frac{\partial}{\partial y} X_{s,t}^\delta \frac{\partial}{\partial y} X_{s,t}^\delta + \frac{\partial}{\partial z} X_{s,t}^\delta \frac{\partial}{\partial z} X_{s,t}^\delta \]
\[ + \cdots + \frac{\partial}{\partial x} X_{s,t}^\delta \frac{\partial}{\partial x} X_{s,t}^\delta + \frac{\partial}{\partial y} X_{s,t}^\delta \frac{\partial}{\partial y} X_{s,t}^\delta + \frac{\partial}{\partial z} X_{s,t}^\delta \frac{\partial}{\partial z} X_{s,t}^\delta \]
\[ = C \delta. \]

Similarly,
\[ E \left[ D_3(u,s)^2 \right] \leq C \delta. \]

Further, for $u \leq t^\delta_s$,
\[ X_{s,t^\delta_s}^\delta = D_3(u,s) \left( \frac{\partial}{\partial x} X_{s,t^\delta_s}^\delta \left( s - t^\delta_s \right) + \frac{\partial}{\partial y} X_{s,t^\delta_s}^\delta \left( s - t^\delta_s \right) + \frac{\partial}{\partial z} X_{s,t^\delta_s}^\delta \left( s - t^\delta_s \right) \right) \]
and
\[ E \left[ D_3(u,s)^2 \right] \leq E \left[ (D_3(u,s))^4 \right]^{1/2} \]
\[ \times E \left[ \left( \frac{\partial}{\partial x} X_{s,t^\delta_s}^\delta \left( s - t^\delta_s \right) + \frac{\partial}{\partial y} X_{s,t^\delta_s}^\delta \left( s - t^\delta_s \right) + \frac{\partial}{\partial z} X_{s,t^\delta_s}^\delta \left( s - t^\delta_s \right) \right)^4 \right]^{1/2} \leq C \delta; \]
for $u \in [t^\delta_s, s)$
\[ D_3(u,s) = c(X_{t^\delta_s}^\delta) \]
and $D_3(u,s) = 0$ for $u > s$. Thus
\[ E \left[ I''(\delta,t)^2 \right] \leq C \int_{[0,t]}^t E \left[ D_1(u,s)^2 + D_2(u,s)^2 \right] \]
\[ + D_3(u,s)^2 \mathbb{1}_{[t^\delta_s]}(u) \psi(s,u) \, du \, ds + CE \left[ \left( \int_0^t \int_{t^\delta_s}^s \left| c(X_{t^\delta_s}^\delta) \psi(s,u) \right| \, du \, ds \right)^2 \right] \]
\[ \leq C \left( \delta + \left( \int_0^t \left| s - t^\delta_s \right|^{2H-1} \, ds \right)^2 \right) \leq C(\delta + \delta^{4H-2}). \]
By (33) \[ E \left[ (\delta, t)^2 \right] \leq \int_{[0, t]^2} E \left[ (\delta_u)^2 \right] \psi(s, u) ds du + \int_{[0, t]^4} E \left[ D_u \delta_u D_v \delta_v \right] \psi(s, u) dv ds dz. \]

The first term is estimated by \( C\delta \) using that
\[ E \left[ (\delta_u)^2 \right] \leq E \left[ (\delta_u)^2 \right] + CE \left[ (X_{s, t}^\delta)^2 \right] + E \left[ (X_{u, t}^\delta)^2 \right] \leq C\delta. \]

In the second, we write \( D_u \delta_u = D_1(u, v) + D_2(u, v) + D_3(u, v) \) and similarly for \( D_v \delta_v \). For \( D_1, D_2 \) we use the Cauchy inequality and the above estimates to get a bound of \( C\delta \). This also works for \( D_3 \) when \( u \notin [t^\delta_v, v) \) and \( s \notin [t^\delta_s, z) \). Two remaining terms for \( D_3 \) are similar, take e.g.
\[ \int_0^t \int_0^t \int_0^t u \left[ D_u \delta_u D_v \delta_v \right] |s - u|^{2H-2} |z - v|^{2H-2} du dv ds dz \leq C\delta^{2H-1} \int_0^t \int_0^t \int_0^t u \left[ D_v \delta_v \right] |z - v|^{2H-2} dv ds dz. \]

Again, if \( s \notin [t^\delta_s, z) \), the integral can be estimated by \( C\delta^{1/2} \); for \( s \in [t^\delta_s, z) \) we get \( \delta^{2H-1} \). Ultimately,
\[ E \left[ (\delta, t)^2 \right] \leq C(\delta + \delta^{4H-2}) \]
and adding this to the previous estimates, we get
\[ E \left[ (G^\delta_t)^2 \right] \leq C(\delta + \delta^{4H-2}). \]

So we can write
\[ E \left[ (\psi(X_t) - \psi(X_{t}^\delta))^2 \right] \leq C \left( \int_0^t E \left[ (\alpha(s, X_s) - \alpha(s, X_{s}^\delta))^2 + (\beta(s, X_s) - \beta(s, X_{s}^\delta))^2 \right] ds + E \left[ (G^\delta_t)^2 \right] \right) \]
\[ \leq C \int_0^t E \left[ (X_s - X_{s}^\delta)^2 \right] ds + C(\delta + \delta^{4H-2}) \]
\[ \leq C \int_0^t E \left[ (\psi(X_s) - \psi(X_{s}^\delta))^2 \right] ds + C(\delta + \delta^{4H-2}). \]

By the Gronwall lemma,
\[ E \left[ (\psi(X_t) - \psi(X_{t}^\delta))^2 \right] \leq C(\delta + \delta^{4H-2}), \]

hence
\[ E \left[ (X_t - X_{t}^\delta)^2 \right] \leq C(\delta + \delta^{4H-2}), \]
as required.

**Remark 3.1.** The obtained estimate can also be written as \( E \left[ (X_t - X_{t}^\delta)^2 \right] \) \( \leq C(\delta^{1/2} + \delta^{2H-1}) \), so the mean-square rate of convergence for the mixed equation is the worst of the two rates for “pure” stochastic differential equation with Brownian motion, \( C\delta^{1/2} \), and with fractional Brownian motion, \( C\delta^{2H-1} \). As long as these estimates for pure equations are sharp (see [6, 12]), we get that in our case the estimate is sharp as well.
An interesting observation is that the value of the Hurst index where the rate of convergence changes is $H = 3/4$. From [2] it is known that the measure induced by the mixture of Brownian motion and independent fractional Brownian motion is equivalent to the Wiener measure iff $H > 3/4$. So in this case it is perhaps natural to expect that the rate of convergence of Euler approximations is the same as for Brownian motion, and this is exactly what we see here.

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