Classification of finite dimensional representations of the q-deformed Heisenberg algebra $H_q(3)$ is made by the help of Clifford algebra of polynomials and generalized Grassmann algebra. Special attention is paid when $q$ is a primitive $n$ root of unity. As a further application we obtain finite dimensional representations of $sl(2)_q$ using its embedding into $H_q(3)$. 
Symmetries have always played a central role in physics, however under special assumptions it is interesting to make some deformations of the basic relations of the group of symmetry. These deformations should be understood as perturbations of the symmetry. There are two ways of making such deformations: on the group of symmetry, either on its associated universal enveloping algebra [1]; or directly on the algebra itself [2] (see also [3]). The first type of deformations, also called quantum groups, has a Hopf algebra structure [1], whereas for the second type (quantum algebras) it is not necessarily the case. The consideration of such algebras is motivated, for example, by the resolution of the Yang-Baxter equation, which appears in particle or statistical physics [4] amongst others. The quantum algebras, deformed by an $n$ primitive root of unity $\omega$ should have a $\mathbb{Z}_n$ graded structure [2]. It is known that superalgebras [5] ($\mathbb{Z}_2$-graded algebras) are subtended by Clifford algebras or Grassmann algebras or, in other words, by quadratic forms. Similarly the $\omega$-deformed algebras may be understood within the framework of Clifford algebras of polynomials [6] $C_f$, or $n$-exterior algebra [7] $\Lambda_n$. These algebras were introduced to allow a linearization of the $n$-degree polynomial $f$ for the former, and to take the $n$-root of the null polynomial for the latter. But, in contrast to the usual Clifford or Grassmann cases, these algebras are not finite [6,7,8]. However, it has been shown that a finite (but not faithful) representation can be obtained for cubic polynomials [9] or even general ones [10]. The main property of the $C_f$ algebra is that it can be connected naturally to the $n$-exterior algebra [11]. This algebra has, as a finite (not faithful) representation, the generalized Grassmann algebra [10,12], which is the fundamental tool to obtain finite dimensional representations of $H_\omega(3)$, the $\omega$-deformed Heisenberg algebra [13]. Let us mention that an $n$-dimensional representation has already been obtained in [14]. The content of this paper is organised as follows. In sect.1 we recall the basic and useful properties of Clifford algebras of polynomials. Construction of finite dimensional representations of $H_q(3)$ is carried out in sect.2, and that of $sl(2)_q$ in sect.3.

1 – Clifford algebras of polynomials

Let $f$ be a homogeneous polynomial of degree $n$ with $p$ variables. Its associated $C_f$ is generated by the $p$ elements $g_i$ ($i = 1 \cdots p$) allowing the linearization of $f$ [6]

$$f(x) = \sum_{\{i\}=1}^{p} x_{i_1} \cdots x_{i_n} g_{i_1 \cdots i_n}, \tag{1.1.a}$$

$$= (x_1 g_1 + \cdots + x_p g_p)^n$$

where $g_{i_1 \cdots i_n}$ is the symmetric tensor associated with $f$. By developing the $n$-th power, the result is that the generators $g_i$ of $C_f$ are submitted to the constraints

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} g_{i_{\sigma(1)}} \cdots g_{i_{\sigma(n)}} = g_{i_1 \cdots i_n} \tag{1.1.b}$$

where $\Sigma_n$ is the group of permutation of $n$ elements. In spite of the fact that all the $C_f$ algebras are not equivalent, we can naturally associate the $n$-exterior algebra
\( \Lambda_n^{(p)} \) to \( C_f \) [11]. It is defined by taking \( f(x) = 0 \) in (1.1.a), and hence by the following constraints

\[
\sum_{\sigma \in \Sigma_n} g_{i_{\sigma(1)}} \cdots g_{i_{\sigma(n)}} = 0 \tag{1.2}
\]

Like \( C_f \), \( \Lambda_n^{(p)} \) is of infinite dimension, but a finite dimensional representation can be obtained through the homomorphism of \( \Lambda_n^{(p)} \) on \( G_n^{(p)} \) which maps \( g_i \) in \( \theta_i \) in such a way that \( 0 \neq g_i g_j - \omega g_i g_j (i < j) \) is sent to zero [10,12]. The \( n^p \)-dimensional algebra so obtained is generated by the \( p \) canonical generators fulfilling

\[
\theta_i \theta_j = \omega \theta_j \theta_i \quad (i < j)
\]

\[
\theta_i^n = 0 \tag{1.3}
\]

It can be proved that the \( \theta_i \)'s satisfy (1.2) as it should for a representation of \( \Lambda_n^{(p)} \), but (1.2) does not lead to (1.3). It has also been observed that the minimal faithful representation of \( G_n^{(p)} \) is obtained by \( n^p \times n^p \)-matrices [10,12]. This algebra, also called paragrassmann* appears quite naturally in the frame of parastatistics [15], parasupersymmetric quantum mechanics, parasuperalgebras [16] and even in 2D conformal field theory [17]. The correspondence between \( C_f \) and \( \Lambda \) can also be obtained by an explicit calculation as we will see. The fundamental property which leads to an explicit matrix representation of \( C_f \) is based on the following property: to linearize any polynomial, only two basic polynomials have to be considered - the sum \( S(x) = x_1^n + \cdots + x_p^n \) and the product \( P(x) = x_1 \cdots x_n \) - which are linearized by matrices that turn out to be a representation of the generalized Clifford algebra [10] \( C_n^{(p)} \). This \( n^p \)-dimensional algebra is defined by \( p \) canonical generators \( \gamma_i \) satisfying

\[
\gamma_i \gamma_j = \omega \gamma_j \gamma_i \quad (i < j)
\]

\[
\gamma_i^n = 1 \tag{1.4}
\]

and has a large range of applications in physics (see [10,12] for references), and leads to a finite dimensional quantum mechanics [19] (see also refs.[8,9] of [12]). This last property has to be compared with the next sect. where we study the \( \omega \)-deformed Heisenberg algebra. The minimal faithful representation of the \( \gamma_i \) is obtained with \( n^k \times n^k \) matrices, \( k = E(p/2) \)

\[
\gamma_{2i-1} = \sigma_3^{\otimes(i-1)} \otimes \sigma_1 \otimes 1^{\otimes(k-i)}
\]

\[
\gamma_{2i} = \sigma_3^{\otimes(i-1)} \otimes \sigma_2 \otimes 1^{\otimes(k-i)} \tag{1.5}
\]

\[
\gamma_{2p+1} = \sigma_3^{\otimes(k)}
\]

where \( i = 1 \cdots p, 1 \) is the \( n \times n \) identity matrix and

* Strictly speaking, for a paragrassmann algebra, there are cubic constraints instead of (1.3) [15]
\[
\sigma_1 = \begin{pmatrix} 0 & 1 & & & \\
 & & & & \\
 & & & & \\
 & & & & \\
1 & & & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \omega & & & 0 \\
 & \omega^2 & & \\
 & & \ddots & \\
0 & & & 1 \end{pmatrix}
\]

\[
\sigma_2 = \begin{cases} \frac{\sigma_3 \sigma_1}{\sqrt{\omega \sigma_3 \sigma_1}} & (n \text{ odd}) \\
(\text{e even}) & \text{with } \omega = \exp \left( \frac{2i\pi}{n} \right). \end{cases}
\]

Similar formulas can be found in [10,12,18]. From the generalized Clifford algebra \(C_n^{(2p)}\), one can build \(n\) explicit \(G_{n,l}^{(p)}\) generalized Grassmann algebras \((l = 0 \ldots n-1)\) \([10,12]\)

\[
\theta_i^{(l)} = \gamma_{2i-1} + \omega^{l+\frac{1}{2}} \gamma_{2i}
\]

and the reciprocal formulas are

\[
\begin{align*}
\gamma_{2i-1} &= \sum_{l=0}^{n-1} \theta_i^{(l)} \\
\gamma_{2i} &= \sum_{l=0}^{n-1} \omega^{-l-\frac{1}{2}} \theta_i^{(l)}
\end{align*}
\]

Due to the property stating that finite dimensional representations of \(C_f\) are sub-
tended by \(C_n^{(p)}\), and due to (1.7), we see that any polynomial can be naturally
associated to \(G_n^{(p)}\) by the following sequence \(f \rightarrow C_f \rightarrow \Lambda_n \rightarrow G_n\). The purpose of
the next sect. is to connect \(G_n\) with the \(\omega\)-deformed Heisenberg algebra \(H_\omega(3)\).

It is worth noting that generalized Clifford or Grassmann algebras constitute a
representation of the quantum hyperplane \(\mathbb{R}_q^p\), with an appropriate \(R\) matrix \([20]\]

\[
R_{i_1i_2j_1j_2} = \delta_{i_2j_1} \delta_{i_1j_2} (1 + (\omega - 1)\delta_{i_1i_2}) + (\omega - \omega^{-1})\delta_{i_1j_1}\delta_{i_2j_2} \Theta(i_2 - i_1)
\]

where \(\Theta(i_2 - i_1)\) is equal to 1 if \(i_2 > i_1\), otherwise 0.

In term of the \(R\) matrix, the equations (1.3.a) and (1.4.a) become

\[
(\omega^{-1}R_{12} - 1_{12})x_1x_2 = 0
\]

where \(x\) belongs to \(\mathbb{R}_q^p\). \(R\) is understood as a matrix which represents an application
of \(\mathbb{R}_q^p \otimes \mathbb{R}_q^p \otimes \mathbb{R}_q^p\) in itself, and \(1_{12}\) is the identity matrix. A direct calculation shows that
\(R\) is a solution of the Yang-Baxter equation

\[
R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}
\]

which indicates how applications of \(\mathbb{R}_q^p \otimes \mathbb{R}_q^p \otimes \mathbb{R}_q^p\) can be treated.
It is possible to construct an analysis compatible with the $Z_n$ graded structure of $G_n$, the results obtained in [14] for $p=1$ should be recalled briefly. The $Z_n$ graded Leibnitz rule

$$\partial(ab) = \partial(a)b + \omega^{gr(a)}a\partial(b)$$

(1.12)

where $gr(\theta^p) = p, gr(\partial^p) = -p$, leads to

$$\partial(\theta^p) = \{p\}\theta^{p-1}$$

(1.13)

with $\{p\} = \frac{(1-\omega^p)}{(1-\omega)}$. Using (1.5) and (1.7), introducing $\partial = F\theta F^{-1}$ (finite Fourier transformation) and utilizing (1.8), we get a matrix representation of $H_\omega(3)$

$$\partial \theta - \omega \theta \partial = 1$$

(1.14)

2 - $\alpha$- deformed Heisenberg algebras

The $\alpha$-deformed Heisenberg algebra is generated by two generators $P$ and $Q$ satisfying [13]

$$PQ - qQP = 1$$

(2.1)

For $q = 1$, with $Q = x$ and $P = \partial_x$, the usual number and derivative, the Heisenberg algebra is obtained, which has only infinite dimensional representations in accordance with Bose-Einstein statistics. For $q = -1$ with $Q = \theta$ and $P = \partial_\theta$, two Grassmann variables, we obtain two dimensional representation in accordance with the Pauli exclusion principle. It is known however that we can construct statistics which describe neither fermions nor bosons but parafermions or parabosons [15]; among those two statistics only the former has finite dimensional representations. This feature has been exploited to build a parasupersymmetric extension of quantum mechanics [16]. So it seems natural to study finite dimensional representations of $H_\omega(3)$ induced by $G_n^{(1)}$; an idea which is stressed by the results of H. Weyl [19] connected to eqs.(1.4-7). Our starting point is then the resolution of

$$Q^n = 0$$

$$P^n = 0$$

$$PQ - \omega QP = 1$$

(2.2)

where we want $Q = \theta$ to belong to $G_n^{(1)}$ and $P = \partial$, the canonical variable associated with $\theta$. $n$ is the minimum power of $\theta$ and $\partial$ which is equal to zero. It is obvious that the minimal size of $\theta$ is a $n \times n$ matrix (see [10,12] for more details).

a – $n$-dimensional representations.

We are looking for $n \times n$ matrices satisfying (2.2). If $\theta^n = 0$, it is obvious that all its eigenvalues are zero. Using the Jordan decomposition of matrices with degenerated eigenvalues, the only non-zero matrix elements of $\theta$ are those which are under the
principal diagonal. Arguing that \( \theta^p \neq 0 (p < n) \), all those elements are one. Thus \( \theta \) is always equivalent to

\[
\theta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}
\]  

(2.3)

Having defined \( \theta \), we are looking for a derivative satisfying (2.1) and which is conjugated to \( \theta : \partial = F\theta F^{-1} \) where \( F \) is the finite Fourier transformation which must be specified. To obtain \( F \), we proceed in two steps; the first one allows to go from \( \theta \) to \( \theta^+ \), or in other words to \( \sigma_1^+ \) to \( \sigma_3 \), the second one allows a reproduction of the derivative properties (1.13). The first step is nothing more than the product of two Sylvester matrices which transforms \( \sigma_1^+ \) to \( \sigma_3 \) [21] and then \( \sigma_3 \) to \( \sigma_1 \). Finally we obtain

\[
F_{ij} = \frac{1}{{(i-1)!}} \delta_{i+j-n-1,0}
\]

(2.4)

where \( \{a\}! = \{a\}\{a-1\}\cdots\{1\} \) and

\[
\partial_{ij} = \{i\} \delta_{i+1j}
\]

(2.5)

It is now easy to verify that \( \theta \) and \( \partial \) generate \( \mathcal{H}_\omega(3) \) as it should be. At this stage, we have recaptured the matrix representation obtained in [14], but we know now that it is the only \( n \)-dimensional representation. All the other representations \( \theta', \partial' \), are related to \( \theta \) and \( \partial \) by an invertible transformation \( P: \theta' = P\theta P^{-1}, \partial' = P\partial P^{-1} \) and \( F' = PF \).

b– \( k > n \)-dimensional representations.

Now we look for \( k \times k \) matrices which satisfy (2.2). Using the results obtained in the previous sub-section as well as the fact that the minimal representation of \( \theta \) is \( n \)-dimensional, we see that we have solutions of (2.2) iff \( k \) is a multiple of \( n \) (\( k = ln \))

\[
\theta = 1 \otimes \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}
\]

(2.6.a)

\[
\partial = 1 \otimes \begin{pmatrix} 0 & \{1\} & \cdots & 0 \\ 0 & 0 & \{2\} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \{n-1\} \\ 0 & 0 & \cdots & 0 \end{pmatrix}
\]

(2.6.b)

1 is the unit \( l \times l \) matrix. Thus any higher dimensional representations are not irreducible and consist of \( l \) copies of the smallest representation.

Until now we have considered the \( \omega \)-deformed Heisenberg algebra when \( \omega \) is an \( n \)-th primitive root of unity. If it is assumed that \( n = n_1n_2 \), using the results
obtained previously, the n-dimensional representation for $H_{\omega_{n_1}}(3)$ can be built. All the results can be used, but now $\partial$ is a matrix constructed with $\{p\} = \frac{(1-\omega^{n_1})}{(1-\omega)}$ so

$$\partial = \begin{pmatrix} 0 & \{1\} & \cdots & 0 \\ 0 & 0 & \{2\} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \{n_2-1\} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \otimes 1 \quad (2.7)$$

where 1 is the $n_1 \times n_1$ unit matrix. Thus we have

$$\partial^{n_2} = 0$$
$$\theta^{n_1,n_2} = 0$$
$$\partial \theta - \omega^{n_1} \theta \partial = 0 \quad (2.8)$$

instead of (2.2). In addition, $\partial$ and $\theta$ are not conjugated in this case because $F$ is singular (see (2.4)). We see that some kind of singular representations are spanned by non-prime numbers. Some differences between prime and non-prime numbers have been noticed in [14].

**c- Other representations of $H_q(3)^*$

Until now we have simply focused on representations of the $q$-deformed Heisenberg algebra induced by generalized Grassmann algebra. Now let us consider other representations: it is easy to check that (2.1) leads to

$$PQ^a = \omega^a Q^a P + \{a\} Q^{a-1}$$
$$P^a Q = \omega^a P^a Q + \{a\} P^{a-1} \quad ,$$

so when $a = n$ we have

$$PQ^n = Q^n P$$
$$P^n Q = Q P^n \quad , (2.10)$$

and considering Shur’s lemma, $P^n, Q^n$, both are proportionnal to the identity matrix for irreducible representation. One can then find a basis where $P$ is diagonal (Cayley-Hamilton theorem)

$$P = \begin{pmatrix} p_1. I_{r_1} \\ p_2. I_{r_2} \\ \vdots \\ p_k. I_{r_k} \end{pmatrix} \quad , (2.11)$$

where $p_1, \cdots, p_k$, are the eigenvalues of $P$, which are degenerated $r_1, \cdots, r_k$ times. We have assumed that the dimension of the representation is $r_1 + \cdots + r_k$. All the

---

* This sub-section was accomplished with the help of M. Rosso
\( p_i^n \) are equal, and in the special case of \( p_i = 0 \), we can recapture the representation of the previous sub-section. Inserting this \( P \) into (2.1) resulting in \( Q \) (\( p_i \neq 0 \))

\[
Q_{ii} = \frac{1}{(1 - \omega)P_{ii}}.
\]

\[
0 = Q_{ij}(P_{ii} - \omega P_{jj}).
\] (2.12)

We see that the diagonal elements of \( Q \) are related to those of \( P \) and \( Q_{ij} \) is arbitrary if \( P_{ii} = \omega P_{jj} \), otherwise zero. Finally we can find as many inequivalent representations of the dimension as we want for \( H_\omega(3) \), and in general \( P \) and \( Q \) are not conjugated one from the other.

Of course, all that has been done here is equally valid for an arbitrary \( q \), but because, in this instance, \( Q \) or \( P \) are not necessarily proportional to the identity, other representations for \( H_q(3) \) than those obtained in eqs. (2.11) and (2.12) can be found. For \( q = 1 \), however it is known that \( P \) or \( Q \) can have a diagonal form, thus from (2.12) we see that finite dimensional representations cannot be obtained.

Finally, it should mentioned that any invertible matrix \( P \) leads to an appropriate \( Q = 1/(1 - q)P^{-1} \), but such a representation is not irreducible (see (2.10)).

d- The case of more than two generators

Returning to \( H_\omega \), it can be observed that along the same lines, representations of the Heisenberg algebra generated by \( p \theta \) and \( \partial \), \( H_\omega(2p + 1) \), induced by generalized Grassmann algebra can now be obtained. But the interesting feature already obtained in [14] is the possibility to get a matrix representation of the quantum hyperplan, the natural structure which emerges by covariance principles [20]. The price to be paid is however to use the singular representation with \( \omega ^2 \) when \( n \) is even (\( \partial \) is expressed with \( \omega ^2 \) c.f. eq.(2.5)).

\[
\theta_i = 1 \otimes (i-1) \otimes \theta \otimes \sigma_3^{\otimes (p-i)}
\]

\[
\partial_i = 1 \otimes (i-1) \otimes \partial \otimes \sigma_3^{\otimes (p-i)}
\] (2.13.a)

\( i = 1 \cdots p, \)

generate the algebra

\[
\theta_i \theta_j = \omega \theta_j \theta_i \ \ i < j
\]

\[
\partial_i \partial_j = \omega^{-1} \partial_j \partial_i \ \ i < j
\]

\[
\partial_i \theta_j = \omega \theta_j \partial_i \ \ i \neq j
\]

\[
\partial_i \theta_i = 1 + \omega^2 \theta_i \partial_i + (-1 + \omega^2) \sum_{k > i} \theta_k \partial_k
\] (2.13.b)

We see that the asymmetry between the \( \theta_i \)'s comes from (2.13.a) and from \( \partial \theta - \theta \partial = \sigma_3^2 \).

Let us make some final remarks on this section. From (1.7), with \( l = 0 \), we can build the \( \theta_i \) of (2.13.a), but, in contrast to the usual Clifford case, we cannot obtain the \( \partial_i \) from the \( \theta_i^{(l)} \) (\( l \neq 0 \)) because of (2.4) and (1.12). So, whereas for the case
$n = 2$, we can build from $\theta_i = \theta_i^{(0)}$ and $\partial_i = \theta_i^{(1)}$ the spinorial representation of $O(2p)$, which is nothing more than $H_{-1}(2p+1)$, when $n > 2$ the quantum hyperplan cannot be obtained from (1.7).

3 – Finite dimensional representations of $sl(2)_q$

In this section, we want to construct some finite dimensional representations of $sl(2)_q$ from $H_q(3)$. It should be observed that in [22], a classification of the finite dimensional representations of quantum groups has been addressed when $q$ is not a primitive $n-$root of unity. Deformations of $sl(2)$ have been classified in [2], and among all those deformations, we consider that which is introduced in [23], corresponding to the second Witten’s quantum deformation [24]. This type of algebra is generated by $J^+, J^-, J^0$ which fulfil the relations

\begin{align*}
J^- J^0 - q J^0 J^- &= J^-
J^0 J^+ - q J^+ J^0 &= J^+
J^- J^+ - q^2 J^+ J^- &= (q + 1) J^0
\end{align*}

(3.1)

It is known that $sl(2)_q$ is embedded into the $q-$deformed Heisenberg algebra $H_q(3)$ through the following identification [25]

\begin{align*}
J^+ &= (1 - 2\alpha(1 - q))^{-1/2}(QQP - 2\alpha Q)
J^0 &= (1 - 2\alpha(1 - q))^{-1} \left[ \left( 1 + \frac{2\alpha(q^2 - q)}{1 + q} \right) QP - \frac{2\alpha}{1 + q} \right]
J^- &= (1 - 2\alpha(1 - q))^{-1/2} P
\end{align*}

(3.2)

where $\alpha$ is an arbitrary number and $P, Q$ are the generators of the $q-$deformed Heisenberg algebra. So, particularly when $P, Q$ are the $d-$dimensional generators constructed in the previous part, a $d$-dimensional representation for $sl(2)_q$ is obtained: any finite dimensional representations of $H_q(3)$ induce a finite dimensional representation for $sl(2)_q$. Of course, the converse is not true, and infinite dimensional representation of $H_q(3)$ may lead to a finite one for $sl(2)_q$. That property is already true for the usual $sl(2)$ ($q = 1$), where $d+1$ dimensional representation, corresponding to polynomials of degree $d$, could be obtained from $H(3)$ through (3.2), in spite of the fact that the latter algebra does not admit finite dimensional representations [25]. This means that taking infinite dimensional matrix representations for $H(3)$, using (3.2) we arrive at infinite matrices for the generators of $sl(2)$, into a block diagonal form, where blocks describe a finite dimensional representation for $sl(2)$.

Returning to the construction of representations for $sl(2)_q$ induced by $H_q(3)$, we see that we get two kinds of representations:

(a) those coming from (2.11) and (2.12) for any $q$;
(b) those built from generalized Grassmann algebra when $q = \omega$. 


The first one is peculiar, as, using (2.11), we see that $J^-$ cannot be understood as a usual annihilation operator because it is diagonal; thus it is not nilpotent and we do not have a highest weight state annihilated by $J^-$. Of course, such a property is also valid for $J^+$. In fact the states which describe the representation, due to the special form of $J^-$, which is diagonal, are eigenvectors of that annihilation operator: thus in using its action on any vector states we never span the whole representation. This never occurs in the second type of representation, because the generators of generalized Grassmann algebras are nilpotent and so $J^+, J^-$. They are thus interpreted as usual creation and annihilation operators.

So using (3.2), we get various inequivalent representations of $sl(2)_q$ of the same dimension, specified by the representation we choose for $H_q(3)$ and by $\alpha$. Notice that when $2\alpha = \{d\}$ one has

$$(J^+)^{d+1} = (1 - \{d\}(1 - q))^{-(d+1)/2}q^{d(d+1)/2}P^{d+1}Q^{2d+2}$$

This property has already been exploited in ref.[25] to build $d+1$ dimensional representation of $sl(2)_q$ from an infinite dimensional representation of $H_q(3)$, in a similar way as when one constructs finite dimensional representation for $sl(2)$. This representation is nothing but the polynomial of degree $d$, where $Q = x$, the usual number, and $P = D$, the Jackson symbol for the derivative

$$Df(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

It can be easily shown that $D$ fulfils the $q-$ graded Leibnitz rule.

Finally, it should be stressed again that unlike the usual Lie algebras, the situation is quite different for the $q-$ deformed algebras, because in the usual Lie algebras, representations of the type (a) does not exist. This property is essentially due to the $q-$deformed commutation relations, which allow 1-dimensional representations which are not the trivial ones.

4 – Conclusion

We have studied all the finite dimensional representations of the $q-$deformed Heisenberg algebras, with special attention to $q$ as an $n$-root of unity. The most interesting representations are basically related to $n-$degree polynomials, similar in the way that fermionic oscillators are related to Grassmann variables. This type of study should be useful, as $q-$deformed Heisenberg algebras provide an appropriate tool for the construction of quantum algebras. Furthermore, we can easily give an oscillator-type interpretation for the $q-$ deformed Heisenberg algebras, and it can be seen that $q-$deformations ($q-$oscillators) can be used in the framework of quantum algebras in a way analogous to the classical harmonic oscillator. This feature has already been exploited in several papers, with various extensions of the classical Heisenberg algebra [26] (this algebra, after some transformations on the generators, is related to the one we have studied); with twisted second quantization for boson [20] (which is equivalent to the quantum hyperplan) or fermions [27]; by considering the $q-$analogues of the Clifford and Weyl algebras [28] (an extension which cannot
be connected to generalised Clifford algebras or the quantum hyperplan) and with superoscillators [29].

Acknowledgment.

I would like to acknowledge N. Fleury for helpful discussions, M. Rosso for useful remarks and advice especially in sect.2, and A. Turbiner who motivated this work and was a great help for the work in sect.3.

REFERENCES

[1] V. G. Drinfel’d Proc. Int. Cong. of Mathematicians (MSRI, Berkley 1986) 798.
[2] C.Zachos Paradigms of Quantum Algebras, Proceedings of the Conference on Deformation Theory of Algebras and Quantization with Applications to Physics, Contemporary Mathematics, AMS 1991, J. Stasheff and M. Gerstenhaber (ed)
[3] S. Majid Int. J. Mod. Phys. A5 (1990) 1.
[4] C. N. Yang Phys. Rev. Lett 19 (1967) 1312; R. J. Baxter Exactly Solved Models in Statistical Mechanics. Acad. Press, London, 1982; M. Jimbo Int. J. Mod Phys. A4 (1989) 3759.
[5] L. Corwin, Y. Ne’eman and S. Sternberg Rev. Mod. Phys. 47 (1975) 573.
[6] N. Roby C.R. Acad. Sc. Paris 268(1969) 484
[7] N. Roby Bull. Sc. Math. 94(1970) 49
[8] L. N. Childs Lin. and Mult. Alg. 5 (1978) 267
[9] M. van den Bergh J. Alg. 109 (1987) 172
[10] N. Fleury and M. Rausch de Traubenberg J.Math. Phys. 33 (1992) 3356; Finite dimensional representations of Clifford Algebras of Polynomials preprint CRN 94-03
[11] Ph. Revoy J. Alg. 46 (1977) 268
[12] A. K. Kwasnievski J. Math.Phys. 26 (1985) 2284
[13] D. B. Fairlie and C. K. Zachos Phys. Lett. B256 (1991) 43
[14] A.T. Filipov, A. P. Isaev and A. B. Kurdikov Mod. Phys. Lett. A7 (1992) 2129
[15] Y. Ohnuki and S. Kamefuchi, Quantum Field Theory and Parastatistics, Univ. of Tokyo press, 1982
[16] V. Rubakov and V. P. Spiridonov Mod. Phys. Lett. A3 (1988) 1337; S. Durand, R. Floreanini and L. Vinet Phys. Lett. 233B (1989) 523; S. Durand and L. Vinet J. Phys. A23 3661 (1990); N. Fleury, M. Rausch de Traubenberg and R. M. Yamaleev Matricial Representation of Rational Power of Operators and Paragrassmann Extension of Quantum Mechanics, to appear in Int. J. Mod. Phys. A.
[17] A. B. Zamoloddchikov and V. I. Fateev Sov. Phys. JETP 62 (1985) 215; V. I. Pasquier and H. Saleur Nucl. Phys. B330 (1990) 523.
[18] K. Morinaga and T. Nono J. of Sc. Hiroshima Univ. Ser. A16 (1952) 13; A. O. Morris Quart. J. Math. Oxford 18 (1967) 7; 19 (1968) 289.
[19] H. Weyl The theory of Groups and Quantum Mechanics, E. P. Dutton pp. 272-
280, 1932 (Reprinted, Dover, New York, 1950)

[20] W. Pusz and S. L. Woronowicz Rep. Math. Phys 27 (1989) 231; J. Wess and B. Zumino Nucl. Phys. (proc. Suppl.) 18B(1990) 302

[21] R. Balain and C. Itzykson C. R. Acad. Sc. Paris 303(1986) 773

[22] M. Rosso Com. Math. Phys. 117 (1988) 581

[23] O. Ogievetsky and A. Turbiner sl(2, R)_q and Quasi Exact-Solvable Problems, Preprint CERN-TH 6212/91 ,1991 (unpublished)

[24] E. Witten Nuc. Phys. B330 (1990) 285.

[25] A. Turbiner Lie Algebras and Linear Operators with Invariant Subspace, To appear in Lie Algebras Cohomologies and New Findings in Quantum Mechanics Contemporary Mathematics, AMS,1993, N. Kamran and P. Olver (ed); A.Turbiner and G. Post J. Phys. A27 (1994) L9; N. Fleury and A. Turbiner About the problem of Normal Ordering preprint CRN 94-08

[26] A. J. Macfarlane J. Phys. A22 (1989) 4581; L. C. Biedenharn J. Phys. A22 (1989) L873;P. P. Kulish and E. V. Damaskinsky J. Phys. A23 (1990) L415

[27] W. Pusz Rep. Math. Phys 27 (1989) 349

[28] T. Hayashi Com. Math. Phys. 127 (1990) 129

[29] M. Chaichain, P. Kulish and J. Lukierski Phys. Lett B262 (1991) 43