FINITE GROUP ACTIONS ON HIGGS BUNDLE MODULI SPACES AND TWISTED EQUIVARIANT STRUCTURES

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Abstract. We consider the moduli space $\mathcal{M}(G)$ of $G$-Higgs bundles over a compact Riemann surface $X$, where $G$ is a semisimple complex Lie group, and study the action of a finite group $\Gamma$ on $\mathcal{M}(G)$ induced by a holomorphic action of $\Gamma$ on $X$ and $G$, and a character of $\Gamma$. The fixed-point subvariety for this action is given by a union of moduli spaces of $G$-Higgs bundles equipped with a certain twisted $\Gamma$-equivariant structure involving a 2-co-cycle of $\Gamma$ with values in the centre of $G$. This union is parameterized by the non-abelian first cohomology set of $\Gamma$ in the adjoint group of $G$. We also describe the fixed points in the moduli space of representations of the fundamental group of $X$ in $G$, via a twisted equivariant version of the non-abelian Hodge correspondence, which involves the $\Gamma$-equivariant fundamental group of $X$.

1. Introduction

Let $G$ be a complex, connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $X$ be a smooth irreducible projective curve defined over $\mathbb{C}$ or equivalently a connected compact Riemann surface with canonical line bundle $K_X$. A $G$-Higgs bundle over $X$ is a pair $(E, \varphi)$ consisting of a principal $G$-bundle $E$ over $X$ and a section $\varphi$ of $E(\mathfrak{g}) \otimes K_X$, where $E(\mathfrak{g})$ is the bundle associated to $E$ via the adjoint representation of $G$ in $\mathfrak{g}$. Let $\mathcal{M}(G)$ be the moduli space of isomorphism classes of polystable $G$-Higgs bundles. It is known that the connected components of $\mathcal{M}(G)$ are parameterized by the fundamental group $\pi_1(G)$ (see [25, 9, 15]).

Let $\theta$ be a group automorphism of $G$. Let $E$ be a principal $G$-bundle. We denote by $\theta(E) := E \times_{\theta} G$ the principal $G$-bundle obtained by the extension of structure group by $\theta$. Recall that there is a biholomorphism $\tilde{\theta} : E \to \theta(E)$ satisfying, for $e \in E$ and $g \in G$, that

$$\tilde{\theta}(eg) = \tilde{\theta}(e)\theta(g).$$

Note that a Higgs field $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K_X)$ produces a Higgs field $\theta(\varphi) \in H^0(X, \theta(E)(\mathfrak{g}) \otimes K_X)$. One easily shows that if $(E, \varphi)$ is polystable then $(\theta(E), \theta(\varphi))$ is also polystable, and we thus have an action of the group $\text{Aut}(G)$ of holomorphic automorphisms of $G$ on $\mathcal{M}(G)$. Because of gauge equivalence, this action depends only on the element defined by $\theta \in \text{Aut}(G)$ in the group $\text{Out}(G)$ of outer automorphisms of $G$. Recall that $\text{Out}(G)$ is defined as the quotient of $\text{Aut}(G)$ by the normal subgroup $\text{Int}(G)$ of inner automorphisms.

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We thus have a short exact sequence
\[ 1 \longrightarrow \text{Int}(G) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1. \]
By a result of de Siebenthal [30] this sequence splits and hence there is a homomorphism
\[ (1.1) \quad \text{Out}(G) \rightarrow \text{Aut}(G) \]
which composed with the projection is the identity, defining an action of Out(\(G\)) on \(\mathcal{M}(G)\).

The group Aut(\(X\)) of holomorphic automorphisms of \(X\) also acts by pull-back on the moduli space \(\mathcal{M}(G)\). And the group \(\mathbb{C}^*\) acts on \(\mathcal{M}(G)\) by
\[ \lambda \cdot (E, \varphi) := (E, \lambda \varphi), \lambda \in \mathbb{C}^*, \quad (E, \varphi) \in \mathcal{M}(G). \]
So we can consider the combined action of the group Aut(\(X\)) \(\times\) Out(\(G\)) \(\times\) \(\mathbb{C}^*\) and define
\[ (\gamma, a, \lambda) \cdot (E, \varphi) := (\gamma^* \theta(E), \lambda \gamma^* \theta(\varphi)), \]
for \((\gamma, a, \lambda) \in \text{Aut}(X) \times \text{Out}(G) \times \mathbb{C}^*\), and \(\theta \in \text{Aut}(G)\) is the image of \(a\) given by \((1.1)\). We can easily check that the above association defines a (right) action of Aut(\(X\)) \(\times\) Out(\(G\)) \(\times\) \(\mathbb{C}^*\) on \(\mathcal{M}(G)\).

Let \(\Gamma\) be a finite group and \(F : \Gamma \rightarrow \text{Aut}(X) \times \text{Out}(G) \times \mathbb{C}^*\) be a homomorphism. Then \(\Gamma\) acts on the moduli space \(\mathcal{M}(G)\) via the homomorphism \(F\). Note that if we write
\[ F(\gamma) = (\eta_{\gamma}, a_{\gamma}, \chi_{\gamma}), \]
then the associations \(\gamma \mapsto \eta_{\gamma}, \gamma \mapsto a_{\gamma}, \gamma \mapsto \theta_{\gamma}\), where \(\theta_{\gamma} \in \text{Aut}(G)\) is the image of \(a_{\gamma}\) given by \((1.1)\), and \(\gamma \mapsto \chi_{\gamma}\), define homomorphisms \(\eta : \Gamma \rightarrow \text{Aut}(X), a : \Gamma \rightarrow \text{Out}(G), \theta : \Gamma \rightarrow \text{Out}(G)\), and \(\chi : \Gamma \rightarrow \mathbb{C}^*\). Now we can explicitly write the action of \(\Gamma\) on \(\mathcal{M}(G)\) as
\[ (1.2) \quad \gamma \cdot (E, \varphi) := (\eta^*_{\gamma} \theta_{\gamma}(E), \chi(\gamma) \eta^*_{\gamma} \theta_{\gamma}(\varphi)). \]

In this article we describe the fixed-point subvariety \(\mathcal{M}(G)^F\) under the action defined by \(F\) in terms of moduli spaces of \(G\)-Higgs bundles equipped with certain twisted equivariant structures. Let \(Z = Z(G)\) be the centre of \(G\), and \(c\) be 2-cocycle in \(Z^2_0(\Gamma, Z)\), where \(\Gamma\) acts on \(Z\) via \(\theta\). We define a notion of \((\Gamma, \eta, \theta, c, \chi)\)-equivariant structure on a \(G\)-Higgs bundle \((E, \varphi)\) (see Definitions 3.1 and 3.5). We also define appropriate notions of stability, semistability and polystability for a \(G\)-Higgs bundle equipped with such a twisted equivariant structure, and study the relation of these notions with the usual stability criteria for the underlying \(G\)-Higgs bundle (Proposition 4.1). In particular, we show that if a twisted equivariant \(G\)-Higgs bundle is polystable then the underlying \(G\)-Higgs bundle is polystable. Using this, we prove a Hitchin–Kobayashi correspondence for these objects (Theorem 4.2). This also defines a map from the moduli space of polystable \((\Gamma, \eta, \theta, c, \chi)\)-equivariant \(G\)-Higgs bundles to the moduli space \(\mathcal{M}(G)\) of polystable \(G\)-Higgs bundles. In Proposition 5.1 we show that the image of this map is in the fixed-point subvariety \(\mathcal{M}(G)^F\).

One of the main results of this paper is a converse to Proposition 5.1. In Theorem 5.7 we show that within the smooth locus of the moduli space \(\mathcal{M}(G)\) the fixed-point subvariety is given by the image of the moduli spaces of \((\Gamma, \eta, \theta', c, \chi)\)-equivariant \(G\)-Higgs bundles where \(c\) is any representative of an element running over the group \(H^2(\Gamma, Z)\) and \(\theta'\) is any representative of an element running over the set \(H^1_0(\Gamma, \text{Int}(G))\); here \(\theta\) is a fixed lift of \(a : \Gamma \rightarrow \text{Out}(G)\), and \(\Gamma\) acts on \(\text{Int}(G)\) via \(\theta\).
Building upon the non-abelian Hodge correspondence (which we recall in Section 2), identifying the moduli space $\mathcal{M}(G)$ with the moduli space $\mathcal{R}(G)$ of representations of the fundamental group of $X$ in $G$, we prove a twisted equivariant version of this correspondence (see Theorem 6.3). This identifies the moduli space of twisted equivariant $G$-Higgs bundles with the moduli space of twisted equivariant representations of the $\Gamma$-equivariant fundamental group of $X$ into a group whose underlying set is $G \times \Gamma$, but where the group operation is twisted by $\theta$ and the 2-cocycle $c$ (here we are taking the character $\chi: \Gamma \to \mathbb{C}^*$ to be trivial). After this, we reinterpret the action of $\Gamma$ on $\mathcal{M}(G)$ in terms of $\mathcal{R}(G)$ and identify the fixed points in terms of moduli spaces of twisted equivariant representations (see Theorem 6.5).

We finish the paper giving some examples for a few simple groups, just mentioning the main ingredients in those situations, so that one can readily apply our main theorems.

This paper generalizes some of the results in [17] where the authors consider $\Gamma$ to be a finite cyclic group, but the action of $\Gamma$ on the Riemann surface $X$ is trivial, as well as some of the results in [18], where $\Gamma$ is any finite group, but now the action corresponding to $a$ is trivial (this relates also to the work of Schaffhauser [31] in the case of vector bundles). Our twisted structures are also related to the pseudoreal structures on Higgs bundles [3, 6, 7, 8]. A motivation in some of these papers was to identify hyperkähler or Lagrangian subvarieties of $\mathcal{M}(G)$, the support of what is referred often as branes in the context of mirror symmetry and Langlands duality [21, 22]. Our results certainly produce new examples of such subvarieties — the detailed investigation of this will be carried out somewhere else.

We should mention that the twisted equivariant structures introduced in this paper appear naturally in the recent work on Higgs bundles and other types of Higgs pairs on $X$ with a non-connected structure group $G$ [14]. It turns out that these pairs can be described in terms of pairs for the connected component of $G$ on a Galois covering of $X$, equipped with twisted equivariant structures. Here $\Gamma$ is the Galois group of the covering. The twisted structures show up also in relation to a Prym–Narasimhan–Ramanan construction of fixed points in $\mathcal{M}(G)$ under the action of a subgroup of the group of $Z$-bundles on $X$, the analogue of tensoring vector bundles by torsion line bundles [27, 16, 4].

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## 2. G-Higgs bundles

Let $G$ be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $X$ be a smooth irreducible projective curve defined over $\mathbb{C}$ with canonical line bundle $K_X$.

### 2.1. Moduli space of G-Higgs bundles

A $G$-Higgs bundle over $X$ is a pair $(E, \varphi)$ where $E$ is a principal $G$-bundle $E$ and $\varphi$ is a section of $E(\mathfrak{g}) \otimes K_X$, where $E(\mathfrak{g}) := E \times_G \mathfrak{g}$ is the associated bundle corresponding to the adjoint action of $G$ on the Lie algebra $\mathfrak{g}$. We will denote sometimes the adjoint bundle $E(\mathfrak{g})$ by $\text{ad}(E)$.

Two $G$-Higgs bundles $(E, \varphi)$ and $(F, \psi)$ are isomorphic if there is an isomorphism $f: E \to F$ such that the induced isomorphism $\text{Ad}(f) \otimes \text{Id}_{K_X}: E(\mathfrak{g}) \otimes K_X \to F(\mathfrak{g}) \otimes K_X$ sends $\varphi$ to $\psi$. 
A $G$-Higgs bundle $(E, \varphi)$ is said to be simple if $\text{Aut}(E, \varphi) \cong Z$ where $\text{Aut}(E, \varphi)$ is the group of Higgs bundle automorphisms of $(E, \varphi)$ and $Z \subset G$ is the centre of $G$.

Let us recall the definitions of stability, semistability and polystability of $G$-Higgs bundles over $X$. Let $P \subset G$ be a parabolic subgroup of $G$ and let $\sigma \in \Gamma(X, E/P)$ be a reduction of structure group of $E$ to $P$. Denote by $E_\sigma \subset E$ the holomorphic $P$-bundle corresponding to $\sigma$. Let $\chi : P \to \mathbb{C}^*$ be an antidominant character of $P$, meaning that the $\mathbb{C}^*$-bundle $G \times_{G/P} \mathbb{C}^*$ is negative ample. The degree of $E$ with respect to $\sigma$ and $\chi$, denoted by $\text{deg}(E)(\sigma, \chi)$, is the degree of the line bundle obtained by extending the structure group of $E_\sigma$ through $\chi$. In other words,

$$\text{deg}(E)(\sigma, \chi) := \text{deg}(E_\sigma \times_{\chi} \mathbb{C}^*).$$

**Definition 2.1.** A $G$-Higgs bundle $(E, \varphi)$ over $X$ is:

- **semistable** if $\text{deg}(E)(\sigma, \chi) \geq 0$, for any parabolic subgroup $P$ of $G$, any non-trivial antidominant character $\chi$ of $P$, and any reduction of structure group $\sigma$ of $E$ to $P$ such that $\varphi \in H^0(X, E_\sigma(p) \otimes K_X)$, where $p$ is the Lie algebra of $P$;

- **stable** if $\text{deg}(E)(\sigma, \chi) > 0$, for any non-trivial parabolic subgroup $P$ of $G$, any non-trivial antidominant character $\chi$ of $P$, and any reduction of structure group $\sigma$ of $E$ to $P$ such that $\varphi \in H^0(X, E_\sigma(p) \otimes K_X)$;

- **polystable** if it is semistable and if $\text{deg}(E)(\sigma, \chi) = 0$, for some parabolic subgroup $P \subset G$, some non-trivial strictly antidominant character $\chi$ of $P$ and some reduction of structure group $\sigma$ of $E$ to $P$ such that $\varphi \in H^0(X, E_\sigma(p) \otimes K_X)$, then there is a further reduction of structure group $\sigma_L$ of $E_\sigma$ to the Levi subgroup $L$ of $P$ such that $\varphi \in H^0(X, E_{\sigma_L}(l) \otimes K_X)$, where $l$ is the Lie algebra of $L$.

**Remark 2.2.** If $(E, \varphi)$ is a semistable (resp. stable) $G$-Higgs bundle then for all parabolic subgroup $P \subset G$ and all reduction of structure group $\sigma$ of $E$ to $P$ we have $\text{deg} E_\sigma(p) \leq 0$ (resp. $< 0$)

Let $\mathcal{M}(G)$ be the moduli space of isomorphism classes of polystable $G$-Higgs bundles. The space $\mathcal{M}(G)$ has the structure of a quasi-projective variety, as obtained from the Schmitt’s general Geometric Invariant Theory construction (cf. [35]). For related constructions see [28, 32, 33]. If we fix the topological class $c$ of $E$ we can consider $\mathcal{M}_c(G) \subset \mathcal{M}(G)$, the moduli space of $G$-Higgs bundles with fixed topological class $c$. Since $G$ is connected the topological class is given by an element $c \in \pi_1(G)$. In this situation it is well-known ([25, 9]) that $\mathcal{M}_c(G)$ is non-empty and connected. A Morse-theoretic proof of this fact has been given more recently in [15], where the connectedness and non-emptyness of $\mathcal{M}_c(G)$ is also proved when $G$ is a non-connected complex reductive Lie group.

### 2.2. $G$-Higgs bundles and Hitchin’s equation

As above, let $G$ be a connected complex semisimple group and let $K \subset G$ be a maximal compact subgroup of $G$. Let $(E, \varphi)$ be a $G$-Higgs bundle on $X$. Let $h$ be a reduction of structure group of $E$ from $G$ to $K$, and $F_h$ be the curvature of the Chern connection — the unique connection on $E$ compatible with $h$ and the holomorphic structure of $E$ (see Section 6.3). Let $\tau_h : \Omega^{1,0}(X, E(\mathfrak{g})) \to \Omega^{0,1}(X, E(\mathfrak{g}))$ be $\mathbb{C}$-antilinear map defined by the reduction $h$ and
the conjugation between $(1,0)$- and $(0,1)$-forms on $X$. Consider the Hitchin equation

\[(2.1)\quad F_h - [\varphi, \tau_h(\varphi)] = 0.\]

One has the following (see [21, 31, 13]).

**Theorem 2.3.** Let $(E, \varphi)$ be a $G$-Higgs bundle on $X$. Then $(E, \varphi)$ is polystable if and only if the $G$-Higgs bundle $(E, \varphi)$ admits a metric $h$ satisfying (2.1).

2.3. Higgs bundles and representations of the fundamental group. By a representation we mean a homomorphism $\rho : \pi_1(X) \to G$. The set of all such homomorphisms, $\text{Hom}(\pi_1(X), G)$, is an analytic variety [19]. The group $G$ acts on $\text{Hom}(\pi_1(X), G)$ by conjugation

$$g \cdot \rho(\gamma) := g^{-1}\rho(\gamma)g.$$

for $g \in G$, $\rho \in \text{Hom}(\pi_1(X), G)$ and $\gamma \in \pi_1(X)$. If we restrict the action to the subspace $\text{Hom}^+(\pi_1(X), G)$ consisting of reductive representations, the orbit space is Hausdorff. A representation $\rho \in \text{Hom}(\pi_1(X), G)$ is called reductive if composed with the adjoint representation of $G$ in the Lie algebra $g$ decomposes as a direct sum of irreducible representations. The moduli space of reductive representations is defined to be the orbit space

$$\mathcal{R}(G) := \text{Hom}^+(\pi_1(X), G)/G.$$

This orbit space coincides with the GIT quotient

$$\mathcal{R}(G) := \text{Hom}(\pi_1(X), G) \sslash G.$$

Thus it has a structure of an algebraic variety (see [29]). Let $\rho : \pi_1(X) \to G$ be a representation. Let $Z_G(\rho)$ be the centralizer in $G$ of $\rho(\pi_1(X))$. We say that $\rho$ is irreducible if and only if it is reductive and $Z_G(\rho) = Z$ where $Z$ is the centre of $G$.

Let $(E, \varphi)$ be a $G$-Higgs bundle, and let $K \subset G$ be a maximal compact subgroup of $G$. Theorem 2.3 states that the polystability of $(E, \varphi)$ is equivalent to a reduction $h$ satisfying the Hitchin equation. A simple computation shows that if $\nabla_h$ is the Chern connection of $h$,

$$D = \nabla_h + \varphi - \tau_h(\varphi)$$

is a flat connection on the $G$-bundle $E$, whose holonomy defines a representation of $\pi_1(X)$ in $G$ which is reductive. By a theorem of Donaldson [10] and Corlette [11], it follows that all reductive representations $\rho : \pi_1(X) \to G$ arise in this way. More concretely one has the non-abelian Hodge correspondence given by the following.

**Theorem 2.4.** Let $G$ be a complex semisimple Lie group. The moduli space $\mathcal{M}(G)$ of polystable $G$-Higgs bundles and the moduli space $\mathcal{R}(G)$ of reductive representations of $\pi_1(X)$ in $G$ are homeomorphic. Under this homeomorphism, the irreducible representations in $\mathcal{R}(G)$ are in correspondence with the stable and simple $G$-Higgs bundles.

3. Twisted Equivariant Higgs bundles

Let $G$ be a complex connected semisimple Lie group and $Z \subset G$ be the centre of $G$. Let $X$ be a smooth, irreducible, projective curve defined over $\mathbb{C}$, and $\Gamma$ be a finite group.
3.1. Twisted $\Gamma$-equivariant structures on $G$-bundles. Let $\eta : \Gamma \to \text{Aut}(X)$ and $\theta : \Gamma \to \text{Aut}(G)$ be homomorphisms, and fix a a 2-cocycle $c \in Z^2_\theta(\Gamma, Z)$, where $\Gamma$ acts on $Z$ via the homomorphism $\theta$. In this section we introduce the notion of twisted $\Gamma$-equivariant Higgs bundle. This is a variant of the notion of twisted equivariant Higgs bundle introduced in [17, 18], and relates to the theory of pseudoreal Higgs bundles studied in [5, 6, 7, 8].

**Definition 3.1.** Let $E$ be a principal $G$-bundle on $X$. A $(\Gamma, \eta, \theta, c)$-equivariant structure on $E$ consists of a collection of biholomorphisms $\{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma}$ of the total space $E$ which satisfy

(i) $\tilde{\eta}_\gamma$ covers the automorphisms $\eta_\gamma : X \to X$ induced by $\eta$ i.e., the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\tilde{\eta}_\gamma} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{\eta_\gamma} & X
\end{array}
$$

is commutative;

(ii) $\tilde{\eta}_\gamma(eg) = \tilde{\eta}_\gamma(e)\theta_\gamma(g)$, $e \in E$, $g \in G$;

(iii) $\tilde{\eta}_{\gamma_1\gamma_2} = c(\gamma_1, \gamma_2)\tilde{\eta}_{\gamma_1}\tilde{\eta}_{\gamma_2}$, $\gamma_1, \gamma_2 \in \Gamma$.

Since the action of $G$ on $E$ is on the right, condition (iii) in Definition 3.1 should be understood, for $e \in E$, as

$$
\tilde{\eta}_{\gamma_1\gamma_2}(e) = \tilde{\eta}_{\gamma_1}(\tilde{\eta}_{\gamma_2}(e))c(\gamma_1, \gamma_2).
$$

We say that a principal $G$-bundle together with a $(\Gamma, \eta, \theta, c)$-equivariant structure is a $(\Gamma, \eta, \theta, c)$-equivariant principal $G$-bundle. We denote a $(\Gamma, \eta, \theta, c)$-equivariant principal $G$-bundle by the pair $(E, \{\tilde{\eta}_\gamma\})$.

**Remark 3.2.** Sometimes, depending on the context, we refer to a $(\Gamma, \eta, \theta, c)$-equivariant structure simply as a twisted $\Gamma$-equivariant structure.

Let $\text{Aut}(E)$ be the group of automorphisms of $E$ covering the identity of $X$, and let $\text{Aut}_{\Gamma, \eta, \theta}(E)$ be the group of biholomorphic maps $f : E \to E$, so that $f$ covers the automorphism $\eta_\gamma : X \to X$ for some $\gamma \in \Gamma$, and satisfies (i) and (ii) in Definition 3.1. There is an exact sequence

$$
1 \to \text{Aut}(E) \to \text{Aut}_{\Gamma, \eta, \theta}(E) \to \Gamma.
$$

A $(\Gamma, \eta, \theta, c)$-equivariant structure on $E$ is simply a twisted representation $\Gamma \to \text{Aut}_{\Gamma, \eta, \theta}(E)$ with cocycle $c \in Z^2_\theta(\Gamma, Z)$, that is a map $\sigma : \Gamma \to \text{Aut}_{\Gamma, \eta, \theta}(E)$ such that

$$
\sigma(\gamma \gamma') = c(\gamma, \gamma')\sigma(\gamma)\sigma(\gamma').
$$
3.2. Twisted $\Gamma$-equivariant structures on associated bundles. Let $(E, \{\tilde{\eta}_\gamma\})$ be a twisted $\Gamma$-equivariant principal $G$-bundle. Let $G$ act on a manifold $M$ (on the left), with an action denoted by $gm$ for $g \in G$ and $m \in M$, and let $\Gamma$ also act on $M$ in such a way that

\[(3.3)\quad \gamma \cdot (gm) = \theta_\gamma(g)(\gamma \cdot m) \quad \text{for} \quad \gamma \in \Gamma, g \in G \quad \text{and} \quad m \in M.\]

Consider the associated fibre bundle $E(M) := E \times_G M$, where the action of $G$ is given by

\[(3.4)\quad (e, m)g = (eg, g^{-1}m) \quad \text{for} \quad g \in G, e \in E \quad \text{and} \quad m \in M.\]

Define for every $\gamma \in \Gamma$ a map $\tilde{\eta}_\gamma^M : E \times_G M \to E \times_G M$ given by

\[(3.5)\quad \tilde{\eta}_\gamma^M(e, m) := (\tilde{\eta}_\gamma(e), \gamma \cdot m).\]

**Proposition 3.3.** Let $(E, \{\tilde{\eta}_\gamma\})$ be a twisted $\Gamma$-equivariant principal $G$-bundle and let $M$ be a $G$-space acted by $\Gamma$ so that \((3.3)\) is satisfied. Assume that the action of the centre of $G$ on $M$ is trivial. Then the collection of maps $\{\tilde{\eta}_\gamma^M\}_{\gamma \in \Gamma}$ defines a $\Gamma$-equivariant structure on $E(M)$. In particular, there is an action of $\Gamma$ on the space of sections of $E(M)$.

**Proof.** We first need to show that the map \((3.5)\) descends to a map $\tilde{\eta}_\gamma^M : E(M) \to E(M)$. To do this consider for $\gamma \in \Gamma, e \in E, g \in G$ and $m \in M$

\[
\tilde{\eta}_\gamma^M(eg, g^{-1}m) = (\tilde{\eta}_\gamma(eg), \gamma \cdot (g^{-1}m)) \\
= (\tilde{\eta}_\gamma(e)\theta_\gamma(g), \theta_\gamma(g^{-1})(\gamma \cdot m)) \\
= (\tilde{\eta}_\gamma(e), \gamma \cdot m)\theta_\gamma(g),
\]

where we have used that, by \((3.3)\), $\gamma \cdot (g^{-1}m) = \theta_\gamma(g^{-1})(\gamma \cdot m)$.

Now we need to check that $\tilde{\eta}_{\gamma_1\gamma_2}^M = \tilde{\eta}_{\gamma_1}^M \tilde{\eta}_{\gamma_2}^M$ for $\gamma_1, \gamma_2 \in \Gamma$.

\[
\tilde{\eta}_{\gamma_1\gamma_2}^M(e, m) = (\tilde{\eta}_{\gamma_1\gamma_2}(e), (\gamma_1\gamma_2) \cdot m) \\
= ((\tilde{\eta}_{\gamma_1}(\tilde{\eta}_{\gamma_2}(e)))c(\gamma_1, \gamma_2), (\gamma_1 \cdot (\gamma_2 \cdot m)) \\
= (\tilde{\eta}_{\gamma_1}(\tilde{\eta}_{\gamma_2}(e)), c(\gamma_1\gamma_2)^{-1}(\gamma_1 \cdot (\gamma_2 \cdot m)) \\
= (\tilde{\eta}_{\gamma_1}(\tilde{\eta}_{\gamma_2}(e)), \gamma_1 \cdot (\gamma_2 \cdot m)),
\]

where in the last equality we have used that the centre of $G$ acts trivially on $M$. From here we immediately obtain that $\tilde{\eta}_{\gamma_1\gamma_2}^M = \tilde{\eta}_{\gamma_1}^M \tilde{\eta}_{\gamma_2}^M$, and hence a $\Gamma$-equivariant structure on $E(M)$.

The $\Gamma$-equivariant structure on $E(M)$, defines an action of $\Gamma$ on the space of sections of $E(M)$. To see this explicitly, recall that the sections of $E(M)$ are in bijective correspondence with the $G$-antiequivariant maps $\psi : E \to M$, meaning $\psi(eg) = g^{-1}\psi(e)$ for all $e \in E$ and $g \in G$. Now for $\gamma \in \Gamma$ and $\psi$ as above, and $e \in E$, define the map $\gamma \cdot \psi : E \to M$ by

\[(\gamma \cdot \psi)(e) := \gamma^{-1} \cdot (\psi(\tilde{\eta}_\gamma(e))).\]
Again, using (3.3), one can show that \( \gamma \cdot \psi \) is \( G \)-anti-equivariant, and from the trivial action of the centre of \( G \) on \( M \), one shows that this defines an action of \( \Gamma \) on the space of sections of \( E(M) \).

**Remark 3.4.** Important examples that satisfy the conditions of Proposition 3.3 include the following:

1. \( M = g \) with the adjoint action of \( G \) on \( g \), and the action of \( \Gamma \) on \( g \) induced by the action of \( \Gamma \) on \( G \) given by \( \theta : G \to \text{Aut}(G) \), namely \( \gamma \cdot v = d\theta_\gamma(v) \) for \( \gamma \in \Gamma \) and \( v \in g \). These two actions satisfy (3.3) and, of course, the centre of \( G \) acts trivially on \( g \). We thus have a \( \Gamma \)-equivariant structure on \( \text{ad}(E) = E(g) \), given by a collection of maps \( \{\tilde{\eta}_\gamma^g\}_{\gamma \in \Gamma} \) defined by (3.5).

2. \( M = G \) with the action of \( G \) on \( g \) given by inner automorphisms, and \( \Gamma \) acting via \( \theta : \Gamma \to \text{Aut}(G) \). Again, these two actions satisfy (3.3), with the centre of \( G \) acting trivially. Hence we have a \( \Gamma \)-equivariant structure on \( \text{Ad}(E) = E(G) \), given by a collection of maps \( \{\tilde{\eta}_\gamma^G\}_{\gamma \in \Gamma} \) defined by (3.5).

### 3.3. Twisted \( \Gamma \)-equivariant structures on Higgs bundles

Let \((E, \varphi)\) be a \( G \)-Higgs bundle on \( X \) and let \( \chi : \Gamma \to \mathbb{C}^* \) be a group homomorphism. If \((E, \{\tilde{\eta}_\gamma\})\) is a twisted \( \Gamma \)-equivariant bundle, from Remark 3.4, we have a holomorphic automorphism \( \tilde{\eta}_\gamma^\varphi : E(g) \to E(g) \) defined by (3.5), for every \( \gamma \in \Gamma \) defining a \( \Gamma \)-equivariant structure on \( E(g) \).

**Definition 3.5.** A \((\Gamma, \eta, \theta, c, \chi)\)-equivariant structure on \((E, \varphi)\) is a \((\Gamma, \eta, \theta, c)\)-equivariant structure \( \{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma} \) on \( E \) such that the Higgs field \( \varphi \) makes the following diagram commutative:

\[
\begin{array}{ccc}
E(g) \otimes K_X & \xrightarrow{\tilde{\eta}_\gamma^\varphi \otimes \tilde{\eta}_\gamma^K_X} & E(g) \otimes K_X \\
\downarrow{\varphi} & & \downarrow{\chi(\gamma)\varphi} \\
X & \xrightarrow{\tilde{\eta}_\gamma} & X
\end{array}
\]

where \( \tilde{\eta}_\gamma^K_X : K_X \to K_X \) is induced by the action of \( \Gamma \) on \( X \) via the homomorphism \( \eta \).

**Remark 3.6.** In other words in Definition 3.5, the Higgs field \( \varphi \) is \( \Gamma \)-invariant up to the action of the character \( \chi(\gamma) \).

From [6, 8, 18], we know that in order to define stability for a twisted \( \Gamma \)-equivariant Higgs bundle \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) we have to consider reductions \( \sigma \) of \( E \) to a parabolic subgroup \( P \subset G \) such that if \( E_\sigma \) is the corresponding \( P \)-bundle and \( E_\sigma(p) \) is the adjoint bundle of \( E_\sigma \), we must have

\[
\text{deg}(\sigma) \geq 0,
\]

where \( \tilde{\eta}_\gamma^\varphi \) is given in Remark 3.4.

**Definition 3.7.** A \((\Gamma, \eta, \theta, c)\)-equivariant Higgs bundle \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) is:

- **semistable** if \( \text{deg}(E)(\sigma, \chi) \geq 0 \), for any parabolic subgroup \( P \) of \( G \), any non-trivial antidominant character \( \chi \) of \( P \), and any reduction of structure group \( \sigma \) of \( E \) to \( P \), satisfying (3.7) such that \( \varphi \in H^0(X, E_\sigma(p) \otimes K_X) \);
stable if \( \deg(E)(\sigma, \chi) > 0 \), for any non-trivial parabolic subgroup \( P \) of \( G \), any non-trivial antidominant character \( \chi \) of \( P \) and any reduction of structure group \( \sigma \) of \( E \) to \( P \) satisfying (3.6) such that \( \varphi \in H^0(X, E_\sigma(p) \otimes K_X) \);

polystable if it is stable or there exists a parabolic subgroup \( P \subset G \) and a reduction of structure group \( \sigma \) of \( E \) to \( E_\sigma \) satisfying (3.6), and a further reduction of structure group \( \sigma_L \) of \( E_\sigma \) to the Levi subgroup \( L \) of \( P \) with \( \tilde{\eta}^g_\gamma(E_{\sigma_L}(l)) \subset E_{\sigma_L}(l) \), and \( \varphi \in H^0(X, E_{\sigma_L}(l) \otimes K_X) \), such that \( (E_{\sigma_L}, \varphi) \) is stable.

**Remark 3.8.** If \( (E, \{\tilde{\eta}_\gamma\}, \varphi) \) is polystable as defined above then the Levi subalgebra Higgs bundle \( (E_{\sigma_L}(l), \text{ad}(\varphi)) \) is stable in the following sense: for all parabolic subalgebra bundles \( \mathcal{Q} \) of \( E_{\sigma_L}(l) \) with \( \text{ad}(\varphi)(\mathcal{Q}) \subset \mathcal{Q} \otimes K_X \) and \( \tilde{\eta}^g_\gamma(\mathcal{Q}) = \mathcal{Q} \) we have \( \deg(\mathcal{Q}) < 0 \).

The moduli space of isomorphism classes of polystable \( (\Gamma, \eta, \theta, c, \chi) \)-equivariant \( G \)-Higgs bundles will be denoted by \( \mathcal{M}(G, \Gamma, \eta, \theta, c, \chi) \).

### 3.4. Local structure at isotropy points and moduli spaces.

Let \( x \in X \), and

\[
\Gamma_x := \{ \gamma \in \Gamma \mid \eta_\gamma(x) = x \}
\]

be the corresponding isotropy subgroup. Let \( \mathcal{P} := \{ x \in X \mid \Gamma_x \neq \{1\} \} \).

It is well-known that, when \( G \) acts on \( X \) faithfully and properly discontinuously, \( \mathcal{P} \) consists of a finite number of points \( \{x_1, \cdots, x_r\} \) and for each \( x_i \in \mathcal{P} \) the isotropy subgroup \( \Gamma_{x_i} \subset G \) is cyclic (see [20] for example).

Let \( (E, \{\tilde{\eta}_\gamma\}, \varphi) \) be a twisted \( (\Gamma, \eta, \theta, c, \chi) \)-equivariant \( G \)-Higgs bundle. The underlying \( (\Gamma, \eta, \theta, c) \)-equivariant structure on \( E \) determines the following. For each \( x \in \mathcal{P} \) and \( e \in E \) such that \( \pi(e) = x \), there is a map

\[
\sigma_e : \Gamma_x \to G
\]

defined by

\[
\tilde{\eta}_\gamma(e) = e\sigma_e(\gamma)
\]

One has the following.

**Proposition 3.9.**

1. \( \sigma_e(\gamma_1\gamma_2) = c(\gamma_1, \gamma_2)\sigma_e(\gamma_1)\theta_{\gamma_1}(\sigma_e(\gamma_2)) \), \( \gamma_1, \gamma_2 \in \Gamma \).

2. Let \( e' \in \pi^{-1}(x) \), with \( e' = eg \) for \( g \in G \). Then \( \sigma_{e'}(\gamma) = g^{-1}\sigma_e(\gamma)\theta_\gamma(g) \) for all \( \gamma \in \Gamma \).

**Proof.** First we check (1). We have \( \tilde{\eta}_1\tilde{\eta}_2(e) = e\sigma_e(\gamma_1\gamma_2) \), and

\[
\tilde{\eta}_1\tilde{\eta}_2(e) = \tilde{\eta}_1(e\sigma_e(\gamma_2)) = \tilde{\eta}_1(e)\theta_{\gamma_1}(\sigma_e(\gamma_2)) = e\sigma_e(\gamma_1)\theta_{\gamma_1}(\sigma_e(\gamma_2)).
\]

Since \( \tilde{\eta}_1\tilde{\eta}_2 = c(\gamma_1, \gamma_2)\tilde{\eta}_1\tilde{\eta}_2 \), we have

\[
e\sigma_e(\gamma_1\gamma_2) = ec(\gamma_1, \gamma_2)\sigma_e(\gamma_1)\theta_{\gamma_1}(\sigma_e(\gamma_2)) \text{ for every } e \in E.
\]

Thus, \( \sigma_e(\gamma_1\gamma_2) = c(\gamma_1, \gamma_2)\sigma_e(\gamma_1)\theta_{\gamma_1}(\sigma_e(\gamma_2)) \).

Next we check (2). We have \( \tilde{\eta}_{\gamma}(e') = \tilde{\eta}_{\gamma}(eg) \). From this we see that

\[
e\sigma_{e'}(\gamma) = e\sigma_e(\gamma)\theta_\gamma(g).
\]

Thus \( \sigma_{e'}(\gamma) = g^{-1}\sigma_e(\gamma)\theta_\gamma(g) \). \( \square \)
For each \(\Gamma\)-fixed point \(x \in X\), let us denote by \(Z^1_{c_x}(\Gamma_x, G)\) the set of all \(\rho : \Gamma_x \to G\) satisfying \(\rho(\gamma_1 \gamma_2) = c_x(\gamma_1, \gamma_2) \rho(\gamma_1) \theta_{\gamma_1}(\rho(\gamma_2))\) where \(c_x\) is a 2-cocycle induced by the restriction of \(c\) to \(\Gamma_x\). We call such a \(\rho : \Gamma \to G\) a twisted 1-cocycle for the action of \(\Gamma\) on \(G\) given by \(\theta\). Two twisted 1-cocycles \(\rho_1\) and \(\rho_2\) are related if there exists a \(g \in G\) such that \(\rho_1 = g^{-1} \rho_2 \theta_{\gamma_1}(g)\). We denote the set of all twisted 1-cocycles modulo the above defined relation by \(H^1_{c_x}(\Gamma, G)\).

Let \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) be a \((\Gamma, \eta, \theta, c, \chi)\)-equivariant \(G\)-Higgs bundle. From Proposition 3.9 we have that, for each \(x \in \mathcal{P}\), there is a unique equivalence class \(\sigma_x\) of a twisted 1-cocycle, defined up to isomorphism of \((E, \{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma})\).

We fix a \(\sigma_{x_i} \in H^1_{c_{x_i}}(\Gamma_{x_i}, G)\) for each \(x_i \in \mathcal{P}\). We say that a \((\Gamma, \eta, \theta, c, \chi)\)-equivariant \(G\)-Higgs bundle \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) has local type \(\sigma_{x_i}\) at a fixed point \(x_i \in X\) if the twisted 1-cocycle induced by the \((\Gamma, \eta, \theta, c, \chi)\)-equivariant structure on \(E\) is \(\sigma_{x_i}\).

We define \(\mathcal{M}(G, \Gamma, \eta, \theta, c, \chi, \{\sigma_{x_i}\})\) as the subvariety of the moduli space \(\mathcal{M}(G, \Gamma, \eta, \theta, c, \chi)\) with fixed local types \(\sigma_{x_i}, i = 1, \ldots, r\).

4. Twisted stability and Hitchin–Kobayashi correspondence

Let \(G\) be a complex, connected, semisimple group and \(Z \subset G\) be the centre of \(G\). Let \(X\) be a smooth, irreducible, projective curve defined over \(\mathbb{C}\), and \(\Gamma\) be a finite group.

4.1. Twisted stability versus ordinary stability.

**Proposition 4.1.** Let \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) be a twisted \(\Gamma\)-equivariant \(G\)-Higgs bundle.

1. If \((E, \varphi)\) is semistable then \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) is semistable.
2. If \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) is semistable then \((E, \varphi)\) is semistable.
3. If \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) is stable then \((E, \varphi)\) is polystable.
4. If \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) is polystable then \((E, \varphi)\) is polystable.

**Proof.** The statement (1) is obvious. For the statement (2) suppose \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) is semistable but \((E, \varphi)\) is not semistable. Note that, in char 0, \((E, \varphi)\) is semistable if and only if \((\text{ad}(E), \text{ad}(\varphi))\) is semistable (see [2] Lemma 2.10). Thus \((\text{ad}(E), \text{ad}(\varphi))\) is not semistable. Then there is a unique filtration of \((\text{ad}(E), \text{ad}(\varphi))\) by \(\text{ad}(\varphi)\) invariant subbundles

\[
0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \text{ad}(E)
\]

such that each \((F_i/F_{i-1}, \text{ad}(\varphi)|_{F_i/F_{i-1}})\) is semistable and \(\mu(F_i/F_{i-1}) < \mu(F_{i-1}/F_{i-2})\), for all \(i = 1, 2, \ldots, n\). Then \(n\) is odd and \(F_{n+1/2}\) is a parabolic subalgebra bundle of \(\text{ad}(E)\) (see proof of [3] Lemma 2.5 and [2] Lemma 2.11). Moreover, as \((\text{ad}(E), \text{ad}(\varphi))\) is not semistable we have \(\deg(F_{n+1/2}) > 0\). Let \(\text{Ad}(E) := E \times^G G\) be the group scheme associated to \(E\) for the action of \(G\) on itself. By [1] Lemma 4 there exists a parabolic sub group scheme \(P \subset \text{Ad}(E)\) such that the associated Lie algebra bundle is \(F_{n+1/2}\). By uniqueness of Harder Narashimhan filtration we get \(\tilde{\eta}_\gamma^0(F_{n+1/2}) = F_{n+1/2}\) for all \(\gamma \in \Gamma\). Now we can show that there exists a parabolic subgroup \(P \subset G\) and a reduction of structure group \(E_0 \subset E\) to \(P\) such that \(\text{Ad}(E_0) = P\) (see the proof [2] Lemma 2.11). Therefore, \(F_{n+1/2} = E_0(p)\). But this would contradict our assumption that \((E, \{\tilde{\eta}_\gamma\}, \varphi)\) is semistable. Therefore, \((\text{ad}(E), \text{ad}(\varphi))\) is semistable.
From (2) it follows that if \((E, \varphi, \{\tilde{\eta}_n\})\) is stable then \((E, \varphi)\) is semistable. Thus \((\text{ad}(E), \text{ad}(\varphi))\) is semistable. A semistable Higgs vector bundle \((V, \tilde{\varphi})\) has a unique maximal polystable Higgs subbundle \((F, \tilde{\varphi}')\) with
\[
\deg(F)/\text{rank}(F) = \deg(V)/\text{rank}(V).
\]
This \((F, \tilde{\varphi}')\) is called the Socle subbundle of \((V, \tilde{\varphi})\). Assume that \((\text{ad}(E), \varphi)\) is not polystable. Then there is a unique filtration of \((\text{ad}(E), \text{ad}(\varphi))\) by \(\text{ad}(\varphi)\) invariant subbundles
\[
0 = F_0 \subset F_1 \subset \cdots F_{n-1} \subset F_n = \text{ad}(E).
\]
such that for each \(i \in [1, n]\), \((F_i/F_{i-1}, \text{ad}(\varphi)|_{F_i/F_{i-1}})\) is the socle of \((\text{ad}(E)/F_{i-1}, \text{ad}(\varphi))\). Then \(n\) is odd, and \(F_{n+1/2}\) is a parabolic subalgebra \(p\)-bundle of \(\text{ad}(E)\). By uniqueness of the Socle filtration we conclude that \(\eta_\gamma^{\text{ad}}(F_{n+1/2}) = F_{n+1/2}\). Now we have \(\deg(F_{n+1/2}) = 0\). The parabolic subalgebra bundle \(F_{n+1/2}\) corresponds to a reduction of structure group \(\sigma\) of \(E\) to a parabolic subgroup \(P \subset G\). Clearly, \(\varphi \in H^0(X, E_\sigma(p) \otimes K_X)\). Now as \(\deg(E_\sigma(p)) = 0\) by definition \((E, \varphi, \{\tilde{\eta}_n\})\) is not stable, which is a contradiction. The above proof is similar to the arguments given in [6, Lemma 2.6].

We now prove (4). We give similar arguments to those given in the proof of [6, Lemma 3.3]. If \((E, \{\tilde{\eta}_n\}, \varphi)\) is stable then the result follows from (3). So, suppose that \((E, \{\tilde{\eta}_n\}, \varphi)\) is polystable but not stable. By (1), \((\text{ad}(E), \text{ad}(\varphi))\) is semistable. Suppose it is not polystable. Let \((F_1, \text{ad}(\varphi)|_{F_1}) \subset (\text{ad}(E), \text{ad}(\varphi))\) be the socle subbundle. Since, \((E, \{\tilde{\eta}_n\}, \varphi)\) is polystable then there exists a Levi subgroup \(L \subset P\) of a parabolic subgroup \(P\) and a reduction \(\sigma_L\) of structure group \(E\) to \(L\) such that \(\hat{\eta}_\gamma^{\text{ad}}(E_{\sigma_L}(l)) = E_{\sigma_L}(l)\) for all \(\gamma \in \Gamma\) and \((E_{\sigma_L}, \varphi)\) is stable where \(\varphi \in H^0(X, E_{\sigma_L}(l) \otimes K_X)\). This would imply that \((E_{\sigma_L}(l), \text{ad}(\varphi))\) is a polystable Higgs vector bundle. Since \((F_1, \text{ad}(\varphi)|_{F_1})\) is the Socle of \((\text{ad}(E), \text{ad}(\varphi))\) we have \((E_{\sigma_L}(l), \text{ad}(\varphi)) \subset (F_1, \text{ad}(\varphi)|_{F_1})\) i.e., \(E_{\sigma_L}(l) \subset F_1\) and \(\text{ad}(\varphi)|_{F_1}(E_{\sigma_L}(l)) \subset E_{\sigma_L}(l) \otimes K_X\).

Note that the Socle filtration is left invariant by the Higgs field \(\text{ad}(\varphi)\). Now \(F_{n-1/2}\) is the nilpotent radical bundle of the parabolic subalgebra bundle \(F_{n+1/2} \subset \text{ad}(E)\). Thus, for each \(x \in X\), the fibre \((F_{n-1/2})_x\) consists of nilpotent elements and since \(F_1 \subset F_{n-1/2}\) the fibres of \(F_1\) consists of nilpotent elements. But \(E_L(l) \subset F_1\) is a bundle of reductive algebras therefore it can not entirely contain nilpotent elements. Thus \((\text{ad}(E), \text{ad}(\varphi))\) is polystable and therefore, \((E, \varphi)\) is polystable.

4.2. Hitchin–Kobayashi correspondence for twisted equivariant Higgs bundles.
Let \((E, \varphi)\) be a \(G\)-Higgs bundle on \(X\). Fix a compact subgroup \(K \subset G\) invariant under the action of the group \(\Gamma\). We can always choose such maximal compact subgroup, for example consider the semidirect product \(\hat{G} := G \rtimes_{\theta} \Gamma\) and \(\hat{K}\) be the maximal compact subgroup of \(\hat{G}\). Now choose \(K\) to be \(\hat{K} \cap G\). Let \(E_K \subset E\) be a \(C^\infty\)-reduction of structure group to the maximal compact subgroup. The Chern connection on \(E\) for \(E_K\) will be denoted by \(\nabla\), and the curvature of \(\nabla\) will be denoted by \(F_{\nabla}\). Let \(\varphi^*\) be the adjoint of \(\varphi\) with respect to the reduction \(E_K\). Since \(G\) is semisimple \(Z \subset K\) thus \(Z\) acts on \(G/K\) trivially and by Proposition [3.3] we have an action on the space of \(C^\infty\) sections \(\Omega^0(X, E/K)\) i.e., on the space of reductions of structure group of \(E\) to \(K\). With the above notations the main theorem of this section is the following.
Theorem 4.2. Let \((E, \{ \tilde{\eta}_\gamma \}, \varphi)\) be twisted \(\Gamma\)-equivariant \(G\)-Higgs bundle. Then \((E, \{ \tilde{\eta}_\gamma \}, \varphi)\) is polystable if and only if \(E\) admits a \(\Gamma\)-invariant reduction of structure group \(h\) to \(K\) satisfying (2.7).

Proof. First suppose that \((E, \{ \tilde{\eta}_\gamma \}, \varphi)\) is a polystable \(G\)-Higgs bundle. By (4) in Proposition 4.1, \((E, \varphi)\) is polystable. By Theorem 2.3 there exists a reduction of structure group \(h\) of \(E\) to \(K\) satisfying (2.7). Since \(G\) is a connected semisimple group this reduction is unique and thus it is invariant under the action of \(\Gamma\) on the space \(\Omega^0(X, E(G/K))\).

For the converse, we suitably modify the arguments given in the proof of [6, Proposition 3.10]. Suppose \((E, \tilde{\eta}_\gamma, \varphi)\) is a twisted \(\Gamma\)-equivariant \(G\)-Higgs bundle on \(X\) admitting a \(\Gamma\)-invariant reduction of structure group \(h\) of \(E\) to \(K\). Then \(h\) defines a Hermitian metric on \(\text{ad}(E)\) satisfying the Hitchin equation for the Higgs bundle \((\text{ad}(E), \text{ad}(\varphi))\). Thus \((\text{ad}(E), \text{ad}(\varphi))\) is polystable and, in particular, semistable. This implies the semistability of \((E, \varphi)\) and hence, by (1) in Proposition 4.1, the semistability of the twisted equivariant \(G\)-Higgs bundle \((E, \{ \tilde{\eta}_\gamma \}, \varphi)\). If it is stable then obviously it is polystable. So we assume that \((E, \{ \tilde{\eta}_\gamma \}, \varphi)\) is not stable. Then there is a parabolic subgroup \(P \subset G\) and a reduction \(E_\sigma \subset E\) such that \(\tilde{\eta}_\gamma^\varphi(E_\sigma(p)) = E_\sigma(p)\), for every \(\gamma \in \Gamma\) and \(\varphi \in H^0(X, E_\sigma(p) \otimes K)\). Also we have

\[
\deg(E_\sigma(p)) = 0.
\]

Let \(\mathcal{P} \subset \text{ad}(E)\) be the smallest (with respect to the inclusion of parabolic subalgebra bundle) parabolic subalgebra bundle over \(X\) such that \(\tilde{\eta}_\gamma^\varphi(\mathcal{P}) = \mathcal{P}\), \(\deg(\mathcal{P}) = 0\), and \(\text{ad}(\varphi)(\mathcal{P}) \subset \mathcal{P} \otimes K_X\).

We claim that the Chern connection \(\nabla^{\text{ad}}\) defined by \(h\) on \(\text{ad}(E)\) preserves the subbundle \(\mathcal{P}\). Since \(\deg(\mathcal{P}) = 0\) and \((\text{ad}(E), \text{ad}(\varphi))\) is polystable there is an \(\text{ad}(\varphi)\) invariant holomorphic subbundle \(W\) such that \(\mathcal{P} \oplus W \simeq \text{ad}(E)\). As \((\text{ad}(E), \text{ad}(\varphi))\) is a polystable Higgs vector bundle \((\mathcal{P}, \text{ad}(\varphi))\) and \((W, \text{ad}(\varphi))\) are polystable of degree 0. Let \(\nabla'\) and \(\nabla''\) be the connections on \((\mathcal{P}, \text{ad}(\varphi))\) and \((W, \text{ad}(\varphi))\), respectively, solving the Hitchin equations. Then \(\nabla' \oplus \nabla''\) is a connection on \(\text{ad}(E)\) solving the Hitchin equations. By uniqueness of this connection we have \(\nabla^{\text{ad}} = \nabla' \oplus \nabla''\). Thus \(\nabla^{\text{ad}}\) preserves \(\mathcal{P}\). Let \(E_K \subset E\) be the principal \(K\)-bundle defined by \(\Gamma\) invariant reduction of structure group \(h\). Then \(\tilde{\eta}_\gamma^K(E_K) = E_K\), where \(\tilde{\eta}_\gamma^K\) induced by \(\tilde{\eta}_\gamma\). This in turn implies \(\tilde{\eta}_\gamma^\varphi(\text{ad}(E_K)) = \text{ad}(E_K)\). The adjoint vector bundle \(\text{ad}(E_K)\) is a totally real sub bundle of \(\text{ad}(E)\) meaning \(\text{ad}(E_K) \cap \sqrt{-1} \text{ad}(E_K) = 0\). Since \(\mathcal{P}\) and \(\text{ad}(E_K)\) are preserved by \(\nabla^{\text{ad}}\) it follows that the totally real sub bundle \(\mathcal{P} \cap \text{ad}(E_K)\) of \(\text{ad}(E)\) is preserved by \(\nabla^{\text{ad}}\). Let \(\mathcal{E}\) be the complexification of \(\mathcal{P} \cap \text{ad}(E_K)\). Then we have \(\tilde{\eta}_\gamma^\varphi(\mathcal{E}) = \mathcal{E}\). Note that since \(\text{ad}(\varphi)(\mathcal{P}) \subset \mathcal{P} \otimes K_X\) we have \(\text{ad}(\varphi)(\mathcal{E}) \subset \mathcal{E} \otimes K_X\). This holomorphic bundle \(\mathcal{E}\) is a Levi subalgebra bundle of \(\mathcal{P}\). Thus to show that \((E, \{ \tilde{\eta}_\gamma \}, \varphi)\) is polystable by Remark 3.3 it is left to show that \((\mathcal{E}, \tilde{\eta}_\gamma^g, \text{ad}(\varphi))\) is stable. Let \(Q\) be the parabolic subalgebra bundle such that \(\text{ad}(\varphi)(Q) \subset Q \otimes K_X\) and \(\tilde{\eta}_\gamma^g(Q) = Q\), violating the stability of \((\mathcal{E}, \tilde{\eta}_\gamma^g, \text{ad}(\varphi))\). Let \(R_u(\mathcal{P})\) be the nilpotent sub algebra bundle of \(\mathcal{P}\) then \(\mathcal{E} \hookrightarrow \mathcal{P} \twoheadrightarrow \mathcal{P}/R_u(\mathcal{P})\) is an isomorphism. Since, \(\deg(Q) = 0\) as \(\mathcal{E}\) is semistable we have \(\deg(Q \oplus R_u(\mathcal{P})) = 0\). Also \(Q \oplus R_u(\mathcal{P})\) is proper parabolic subalgebra bundle of \(\text{ad}(E)\) contained in \(\mathcal{P}\) and \(\tilde{\eta}_\gamma^g(Q \oplus R_u(\mathcal{P})) = Q \oplus R_u(\mathcal{P})\). Thus it contradicts the minimality of \(\mathcal{P}\). Hence, \(\mathcal{E}\) is stable.

\[\square\]
By Theorem 4.2 we immediately have the following.

**Corollary 4.3.** If \((E, \tilde{\eta}_\gamma, \varphi)\) is a twisted \(\Gamma\)-equivariant \(G\)-Higgs bundle such that the underlying \(G\)-Higgs bundle \((E, \varphi)\) is polystable, then \((E, \tilde{\eta}_\gamma, \varphi)\) is polystable.

5. **Action of \(\Gamma\) on the moduli space of \(G\)-Higgs bundles and fixed points**

Let \(G\) be a connected complex semisimple Lie group with Lie algebra \(g\). Let \(X\) be a smooth irreducible projective curve defined over \(\mathbb{C}\) with canonical line bundle \(K_X\).

Let \(\Gamma\) be a finite group and \(F : \Gamma \to \text{Aut}(X) \times \text{Out}(G) \times \mathbb{C}^*\) be a homomorphism. Then \(\Gamma\) acts on the moduli space \(\mathcal{M}(G)\) via \(F\). In this section we will describe the fixed-point subvariety \(\mathcal{M}(G)^F\) under the action \(F\) in terms of the moduli spaces of twisted equivariant Higgs bundles.

5.1. **Action of \(\Gamma\) on \(\mathcal{M}(G)\).** First let us explicitly describe the action of \(\Gamma\) on the moduli space \(\mathcal{M}(G)\). Note that if we write

\[ F(\gamma) = (\eta_\gamma, a_\gamma, \chi_\gamma) \]

then the associations \(\gamma \mapsto \eta_\gamma\), \(\gamma \mapsto a_\gamma\) and \(\gamma \mapsto \chi_\gamma\) define homomorphisms \(\eta : \Gamma \to \text{Aut}(X)\), \(a : \Gamma \to \text{Out}(G)\) and \(\chi : \Gamma \to \mathbb{C}^*\). Recall from the discussion at the Introduction that the extension

\[ 1 \to \text{Int}(G) \to \text{Aut}(G) \to \text{Out}(G) \to 1 \]

splits (3.6) and hence there is a homomorphism

\[ l : \text{Out}(G) \to \text{Aut}(G) \]  

which composed with the projection to \(\text{Out}(G)\) is the identity. Consider the homomorphism \(\theta := l \circ a : \Gamma \to \text{Aut}(G)\), and denote \(\theta_\gamma = \theta(\gamma)\). We can explicitly write the action of \(\Gamma\) on \(\mathcal{M}(G)\) (see the Introduction for details) as

\[ (5.2) \quad \gamma \cdot (E, \varphi) := (\eta_\gamma^* \theta_\gamma(E), \chi(\gamma) \eta_\gamma^* \theta_\gamma(\varphi)). \]

Let us denote the fixed point subvariety of \(\mathcal{M}(G)\), under the above defined action of \(\Gamma\) by \(\mathcal{M}(G)^F\). By Proposition 4.1 we see that the forgetful map

\[
(E, \{\tilde{\eta}_i\}, \varphi) \mapsto (E, \varphi)
\]

defines a morphism \(\mathcal{M}(G, \Gamma, \eta, \theta, c, \chi, \{\sigma_i\}) \to \mathcal{M}(G)\). We denote the image of the forgetful map inside \(\mathcal{M}(G)\) by \(\mathcal{M}(G, \Gamma, \eta, \theta, c, \chi, \{\sigma_i\})\).

The image of the forgetful map consists of those isomorphism classes of polystable \(G\)-Higgs bundles which admit a \((\Gamma, \eta, \theta, c, \chi)\)-equivariant structure. Now if a \(G\)-Higgs bundle \((E, \varphi)\) admits a \((\Gamma, \eta, \theta, c, \chi)\)-equivariant structure then, by definition of twisted equivariant structures, we have

\[ (E, \varphi) \cong (\eta_\gamma^* \theta_\gamma(E), \chi(\gamma) \eta_\gamma^* \theta_\gamma(\varphi)), \]

where \(\cong\) denotes isomorphism of \(G\)-Higgs bundles. As closed points of \(\mathcal{M}(G)\) consist of isomorphism classes of polystable \(G\)-bundles we immediately have the following.

**Proposition 5.1.** \(\tilde{\mathcal{M}}(G, \Gamma, \eta, \theta, c, \chi, \{\sigma_i\}) \subset \mathcal{M}(G)^F\).
Let $c, c' \in Z^2_0(\Gamma, Z)$ be two cohomologous 2-cocycles, that is, there exists a map $f : \Gamma \to Z$ such that
\begin{equation}
(5.3) \quad c'(\gamma_1, \gamma_2) = c(\gamma_1, \gamma_2) f(\gamma_1) \theta_{\gamma_1}(f(\gamma_2)) f(\gamma_1 \gamma_2)^{-1}, \quad \gamma_1, \gamma_2 \in \Gamma.
\end{equation}

Let $\sigma_{x_i} \in H^1_{c_{x_i}}(\Gamma_{x_i}, G)$. If $c'$ is cohomologous to $c$ then we have 1-cocycles $\sigma'_{x_i} \in H^1_{c'_{x_i}}(\Gamma_{x_i}, G)$ induced by $\sigma_{x_i}$.

One has the following.

**Proposition 5.2.** If $c$ and $c'$ are cohomologous cocycles in $Z^2_0(\Gamma, Z)$
\[\widetilde{M}(G, \Gamma, \eta, \theta, c, \chi, \{\sigma_{x_i}\}) = \widetilde{M}(G, \Gamma, \eta, \theta, c', \chi, \{\sigma'_{x_i}\}).\]

**Proof.** Let $(E, \varphi) \in \widetilde{M}(G, \Gamma, \eta, \theta, c, \chi, \{\sigma_{x_i}\})$. If $c$ is cohomologous to $c'$ then we can show that $(E, \varphi)$ admits a $(\Gamma, \eta, \theta, \chi, c')$-equivariant structure with local types $\{\sigma'_{x_i}\}$. To do this, define
\[\tilde{\eta}'(e) := \tilde{\eta}_c(e) f(\gamma)^{-1}.\]

Since $f(\gamma) \in Z$, we have $\tilde{\eta}'(eg) = \tilde{\eta}'(e) \theta_{\gamma}(g)$, and using (5.3) we check that $\tilde{\eta}'_{(\gamma_1 \gamma_2)} = c'(\gamma_1, \gamma_2) \tilde{\eta}'_{\eta_{\gamma_1}} \tilde{\eta}'_{\eta_{\gamma_2}}$. Thus if $c$ and $c'$ are cohomologous and $(E, \varphi) \in \widetilde{M}(G, \Gamma, \eta, \theta, c, \chi, \{\sigma_{x_i}\})$, then by the above observation we get a $(\Gamma, \eta, \theta, c', \chi)$-equivariant structure on $(E, \varphi)$ with local types $\{\sigma'_{x_i}\}$. Therefore,
\[\widetilde{M}(G, \eta, \theta, \Gamma, c, \chi, \{\sigma_{x_i}\}) \subseteq \widetilde{M}(G, \eta, \Gamma, \theta, c', \chi, \{\sigma'_{x_i}\}).\]

Similarly, we can show the other inclusion. Hence the proposition follows. \qed

5.2. Lifts and non-abelian cohomology. We follow [17] to describe the different equivalence classes of lifts to $\text{Aut}(G)$ of $a : \Gamma \to \text{Out}(G)$, in terms of non-abelian cohomology.

Let $\Gamma$ be a group and $A$ another group acted on by $\Gamma$ via a homomorphism $\alpha : \Gamma \to \text{Aut}(A)$, that is, every $\gamma \in \Gamma$ defines an automorphism of $A$ that we denote by $\alpha_\gamma$. We define a 1-cocycle of $\Gamma$ in $A$ as a map $\gamma \mapsto a_\gamma$ of $\Gamma$ to $A$ such that
\begin{equation}
(5.4) \quad a_{\gamma \gamma'} = a_\gamma \alpha_\gamma(a_{\gamma'}). \quad \text{for} \quad \gamma, \gamma' \in \Gamma.
\end{equation}
The set of cocycles is denoted by $Z^1_\alpha(\Gamma, A)$. Two cocycles $a, a' \in Z^1_\alpha(\Gamma, A)$ are said to be cohomologous if there is $b \in A$ such that
\begin{equation}
(5.5) \quad a'_{\gamma} = b^{-1} a_\gamma \alpha_\gamma(b).
\end{equation}
This is an equivalence relation in $Z^1_\alpha(\Gamma, A)$ and the quotient is denoted by $H^1_\alpha(\Gamma, A)$. This is the first cohomology set of $\Gamma$ in $A$.

Coming back to our problem, let $S_a$ be the set of lifts $\theta : \Gamma \to \text{Aut}(G)$ of $a : \Gamma \to \text{Out}(G)$. We define an equivalence relation in $S_a$ as follows. Denote for an element $h \in G$, the inner automorphism $\text{Int}_h$ of $G$ defined by $\text{Int}_h(g) = hgh^{-1}$, for $g \in G$. Let $\theta, \theta' \in S_a$. We say that $\theta \sim \theta'$ if there is an element $h \in G$ so that
\begin{equation}
(5.6) \quad \theta'_\gamma = \text{Int}_h \theta_\gamma \text{Int}_h^{-1}.
\end{equation}
We observe from (5.6) that $\theta'_\gamma = \text{Int}_{s_\gamma} \theta_\gamma$ where $s_\gamma = h \theta_\gamma(h^{-1})$. The relevance of this equivalence relation is given by the following.
Proposition 5.3. Let $(E, \varphi)$ be a $G$-Higgs bundle with a $(\Gamma, \eta, \theta, c, \chi)$-equivariant structure, and let $\theta' \in S_a$ so that $\theta' \sim \theta$. Then $(E, \varphi)$ admits a $(\Gamma, \eta, \theta', c, \chi)$-equivariant structure.

Proof. Let $(E, \varphi)$ be a $G$-Higgs bundle with a $(\Gamma, \eta, \theta, c, \chi)$-equivariant structure $\{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma}$. Let $\theta' \in S_a$ with $\theta' \sim \theta$. Let $h \in G$ so $\theta' = \text{Int}_{h^{-1}} \theta \text{Int}_h$. Consider the $G$-bundle $\text{Int}_h(E)$. Recall that there is a biholomorphism $\text{Int}_{h^{-1}} : E \to \text{Int}_h(E)$ satisfying for $e \in E$ and $g \in G$

$$\text{Int}_h(eg) = \text{Int}_h(e) \text{Int}_h(g).$$

We define the biholomorphisms $\{\tilde{\eta}'_\gamma\}_{\gamma \in \Gamma}$ of $\text{Int}_h(E)$ by

$$\tilde{\eta}'_\gamma = \text{Int}_h \tilde{\eta}_\gamma \text{Int}_h^{-1}.$$ 

Since $E$ and $\text{Int}_h(E)$ are isomorphic $G$-bundles and $\text{Int}_h^{-1} = \text{Int}_{h^{-1}}$, we can easily check that $\{\tilde{\eta}'_\gamma\}_{\gamma \in \Gamma}$ satisfies the conditions given in Definition 3.5 for $\theta'$. One verifies immediately that $\varphi$ satisfies the condition in Definition 3.5 for $\{\tilde{\eta}'_\gamma\}_{\gamma \in \Gamma}$. \hfill $\square$

A consequence of this is the following.

Proposition 5.4. Let $\theta, \theta' \in S_a$ such that $\theta \sim \theta'$, i.e., $\theta'_\gamma = \text{Int}_{h^{-1}} \theta \text{Int}_h$, for $h \in G$. Then $h$ gives rise to a canonical isomorphism between $\mathcal{M}(G, \Gamma, \eta, \theta, c, \chi)$ and $\mathcal{M}(G, \Gamma, \eta, \theta', c, \chi)$. Since the action of $\text{Int}(h)$ on $\mathcal{M}(G)$ is trivial we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(G, \Gamma, \eta, \theta, c, \chi) & \to & \mathcal{M}(G) \\
\downarrow & & \uparrow \\
\mathcal{M}(G, \Gamma, \eta, \theta', c, \chi).
\end{array}$$

In view of Propositions 5.3 and 5.4 we need to characterize the set $S_a/\sim$. To do that, we notice that the action of $\Gamma$ on $G$ defined by $\theta : \Gamma \to \text{Aut}(G)$ induces an action of $\Gamma$ on $\text{Int}(G)$, since $\text{Int}(G)$ is isomorphic to the adjoint group $G/Z$, and $\theta$ preserves $Z$. This action is given by $\gamma \cdot (\text{Int}_g) := \text{Int}_{\theta'_\gamma g}$, for $\gamma \in \Gamma$ and $g \in G$. We have the following.

Proposition 5.5. Let $\theta : \Gamma \to \text{Aut}(G)$ be a lift of $\alpha : \Gamma \to \text{Out}(G)$. Then the set $S_a/\sim$ is in bijective correspondence with the cohomology set $\tilde{H}^1_\theta(\Gamma, \text{Int}(G))$, where $\Gamma$ acts on $\text{Int}(G)$ by $\gamma \cdot (\text{Int}_g) := \text{Int}_{\theta'_\gamma g}$, for $\gamma \in \Gamma$ and $g \in G$.

Proof. Let $\theta' \in S_a$. This means that there is a map $s' : \Gamma \to G$ so that $\theta'_\gamma = \text{Int}_{s'_\gamma} \theta_\gamma$, where $s'_\gamma := s'(\gamma)$. Define $\tilde{s}' : \Gamma \to \text{Int}(G)$ by $\tilde{s}'_\gamma := \tilde{s}'(\gamma) := \text{Int}_{s'_\gamma}$. A simple computation shows that according to [5.4], $\theta'_\gamma$ being an automorphism of $G$ is equivalent to $\tilde{s}'$ being an element in $\tilde{Z}^1(\Gamma, \text{Int}(G))$. We thus have that the assignment $\theta' \mapsto \tilde{s}'$ defines a bijection

$$S_a \to \tilde{Z}^1(\Gamma, \text{Int}(G)).$$

Let now $\theta'' \in S_a$ so that $\theta' \sim \theta''$. This means that there exists an element $h \in G$ so that $\theta''_\gamma = \text{Int}_h \theta'_\gamma \text{Int}_{h^{-1}}$. Then, if $\theta'_\gamma = \text{Int}_{s'_\gamma} \theta_\gamma$, and $\theta''_\gamma = \text{Int}_{s''_\gamma} \theta_\gamma$. From this we deduce that

$$(5.7) \quad \text{Int}_{s''_\gamma} = \text{Int}_h \text{Int}_{s'_\gamma} \text{Int}_{\theta_\gamma(h^{-1})}.$$
Now, if \( \tilde{s}'_\gamma = \text{Int}_{s'_\gamma} \) and \( \tilde{s}''_\gamma = \text{Int}_{s''_\gamma} \), then (5.7) shows that the cocycles \( \tilde{s}' \) and \( \tilde{s}'' \) are cohomologous, and hence define the same element \( H^1_0(\Gamma, \text{Int}(G)) \). The converse follows reversing the argument. \( \square \)

5.3. Fixed points and simplicity. Recall that a \( G \)-Higgs bundle \((E, \varphi)\) is said to be simple if \( \text{Aut}(E, \varphi) = \{1\} \). Let \((E, \varphi)\) be a \( G \)-Higgs bundle. Note that an isomorphism \( E \cong \eta^* \theta_\gamma(E) \), for \( \gamma \in \Gamma \), produces a Higgs field \( \eta^* \theta_\gamma(\varphi) \) of \( \eta^* \theta_\gamma(E) \). We say that \((E, \varphi) \cong (\eta^* \theta_\gamma(E), \chi(\gamma) \eta^* \theta_\gamma(\varphi)) \) if \( E \cong \eta^* \theta_\gamma(E) \) and the Higgs field \( \varphi \) makes the following diagram commutative:

\[
\begin{array}{ccc}
E(g) \otimes K_X & \cong & \eta^* \theta_\gamma(E)(g) \otimes K_X \\
\varphi & \uparrow & \chi \gamma \eta^* \theta_\gamma(\varphi) \\
X & = & X
\end{array}
\]

**Proposition 5.6.** Let \( \theta : \Gamma \to \text{Aut}(G) \) be a lift of \( a : \Gamma \to \text{Out}(G) \), and let \((E, \varphi)\) be a simple \( G \)-Higgs bundle over \( X \) such that \((E, \varphi) \cong (\eta^* \theta_\gamma(E), \chi(\gamma) \eta^* \theta_\gamma(\varphi)) \). Then \((E, \varphi)\) admits a \((\Gamma, \eta, \theta', c, \chi)\)-equivariant structure, where \( \theta' : \Gamma \to \text{Aut}(G) \) is also a lift of \( a \), and \( c \in Z^1_0(\Gamma, Z) \).

**Proof.** Let \((E, \varphi)\) be a \( G \)-Higgs bundle over \( X \) such that

\[
(E, \varphi) \cong (\eta^* \theta_\gamma(E), \chi(\gamma) \eta^* \theta_\gamma(\gamma))
\]

for all \( \gamma \in \Gamma \). Let \( \text{Aut}(E, \varphi) \) be the group of automorphisms covering the identity of \( X \), and \( \text{Aut}_{\Gamma, \eta, \theta, \chi}(E, \varphi) \) be the subgroup of the group \( \text{Aut}_{\Gamma, \eta, \theta}(E) \) defined in Section 3.1 consisting of elements for which \( \varphi \) satisfies the condition in Definition 3.5. Since the actions on \( \mathcal{M}(G) \) by different lifts \( \theta \in S_a \) coincide, (5.8) implies the existence of a \( \theta' \in S_a \) so that (3.2) for \( \theta' \) induces an exact sequence

\[
1 \to \text{Aut}(E, \varphi) \to \text{Aut}_{\Gamma, \eta, \theta', \chi}(E) \to \Gamma \to 1.
\]

The simplicity of \((E, \varphi)\) implies that \( \text{Aut}(E, \varphi) \cong \{1\} \), and hence (5.9) gives an extension

\[
1 \to Z \to \text{Aut}_{\Gamma, \eta, \theta', \chi}(E, \varphi) \to \Gamma \to 1.
\]

This extension defines a cocycle \( c \in Z^1_0(\Gamma, Z) \), and a twisted homomorphism

\[
\Gamma \to \text{Aut}_{\Gamma, \eta, \theta', \chi}(E, \varphi)
\]

with cocycle \( c \), that is, a \((\Gamma, \eta, \theta', \chi, c)\)-equivariant structure on \((E, \varphi)\). \( \square \)

We have the following.

**Theorem 5.7.** Let \( \mathcal{M}_{ss}(G) \subset \mathcal{M}(G) \) be the subvariety of \( \mathcal{M}(G) \) consisting of those \( G \)-Higgs bundles which are stable and simple. Then

\[\mathcal{M}_{ss}(G)^F \subset \bigcup_{c \in H^1_0(\Gamma, Z), \sigma \in H^1_0(\Gamma, \text{Aut}(G)), [\theta'] \in H^1_0(\Gamma, \text{Int}(G))} \mathcal{\tilde{M}}(G, \Gamma, \theta', c, \chi, \{\sigma_x\}).\]

Here \( \theta : \Gamma \to \text{Aut}(G) \) is any lift of \( a : \Gamma \to \text{Out}(G) \) and cohomology is with respect to the corresponding actions defined by \( \theta \).
Proof. Let \((E, \varphi) \in \mathcal{M}_{ss}(G)^F\). Then, by Proposition 5.6 \((E, \varphi)\) admits a \((\Gamma, \eta, \theta', c, \chi)\)-equivariant structure where \(\theta'\) is a lift of \(a\) and \(c \in Z^2_{\theta}(\Gamma, Z)\), the set of all 2-cocycles where \(\Gamma\) acts on \(Z\) via \(\theta'\) (notice that the action of \(\Gamma\) on \(Z\) only depends on \(a\)). Thus \((E, \varphi) \in \overline{\mathcal{M}}(G, \Gamma, \eta, \theta', c, \chi, \{\{s_{x_i}\}\})\). It follows from Propositions 5.2, 5.4 and 5.5 that the union should run over \([c] \in H^2_{\theta}(\Gamma, Z)\), \([\sigma] \in H^1_{c_{x_i}}(G, G)\), and \([\theta'] \in H^1_{\theta}(\Gamma, \text{Int}(G))\). \(\square\)

6. Non abelian Hodge correspondence and twisted equivariant structures

Let \(G\) be a complex, connected, semisimple group and \(Z \subset G\) be the centre of \(G\). Let \(X\) be a smooth, irreducible, projective curve defined over \(\mathbb{C}\), and \(\Gamma\) be a finite group. Let \(F : \Gamma \rightarrow \text{Aut}(X) \times \text{Out}(G)\) be a homomorphism defined by homomorphism \(\eta : \Gamma \rightarrow \text{Aut}(X)\) and \(a : \Gamma \rightarrow \text{Out}(G)\). Let \(\theta : \Gamma \rightarrow \text{Aut}(G)\) be a lift homomorphism of \(a\), and fix a 2-cochain \(c \in Z^2_{\theta}(\Gamma, Z)\), where \(\Gamma\) acts on \(Z\) via the homomorphism \(\theta\) (notice that this action only depends on \(a\)).

6.1. Equivariant fundamental group and twisted equivariant representations. An equivariant base point is a \(\Gamma\)-equivariant map \(x : \Gamma \rightarrow X\), where \(\Gamma\) acts on itself by multiplication, and on \(X\) via \(\eta\). Suppose that \((X, x)\) has a universal \(\Gamma\)-equivariant covering \((\hat{X}, \hat{x}) \rightarrow (X, x)\). By the universal property of such covering, the group of automorphisms of the equivariant covering \(\hat{X}\) over \(X\) is uniquely determined by \(X\), up to unique isomorphism. This group is called the equivariant fundamental group of \(X\) with respect to the action of \(\Gamma\) and is denoted by \(\pi_1(X, \Gamma, x)\) or simply \(\pi_1(X, \Gamma)\) (see [23, Definition 3.1]).

Let \(1 \in \Gamma\) be the identity and, denote \(x_1 := x(1)\), and let \(\pi_1(X, x_1)\) be the fundamental group of \(X\) with base point \(x_1\). By [23 Proposition 3.2], \(\pi_1(X, \Gamma, x)\) fits into an exact sequence

\[
(6.1) \quad 1 \rightarrow \pi_1(X, x_1) \rightarrow \pi_1(X, \Gamma, x) \rightarrow \Gamma \rightarrow 1.
\]

Note that if \(\Gamma\) acts trivially on \(X\) (i.e., \(\eta\) is trivial) then \(\pi_1(X, \Gamma, x) = \pi_1(X, x)\) and if \(\Gamma\) acts freely on \(X\) then \(\pi_1(X, \Gamma, x) = \pi_1(X/\Gamma, \hat{x})\) where \(\hat{x}\) is the composite map of \(x\) and the quotient \(X \rightarrow X/\Gamma\) (see [23 Proposition 3.4]).

Suppose that \(\eta\) is non trivial. Choose \(x \in X\) so that it is not fixed by any \(\gamma \in \Gamma\) with \(\gamma \neq 1\). By the description of the equivariant fundamental group in terms of equivariant loops (see [23 Section 6]) we can identify the set with the set of all homotopy classes of maps \(\sigma : [0, 1] \rightarrow X\) such that \(\sigma(0) = x\) and \(\sigma(1) \in \{\eta_\gamma(x) \mid \gamma \in \Gamma\}\). Under this identification the surjective map \(\beta : \pi_1(X, \Gamma, x) \rightarrow \Gamma\) can be identified with \(\beta(\sigma) = \gamma\) if \(\sigma(1) = \eta_\gamma(x)\).

Let \(\hat{G}_{\theta, c}\) be the group whose underlying set is \(G \times \Gamma\) and the group operation is given by

\[
(g_1, \gamma_1)(g_2, \gamma_2) = (c(\gamma_1, \gamma_2)g_1\theta_{\gamma_1}(g_2), \gamma_1\gamma_2).
\]

A representation \(\hat{\rho} : \pi_1(X, \Gamma, x) \rightarrow \hat{G}_{\theta, c}\) is called \((\Gamma, \eta, \theta, c)\)-equivariant if it is an extension of a representation \(\rho : \pi_1(X, x_1) \rightarrow G\) fitting in a commutative diagram of
homomorphisms

\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi_1(X, x_1) & \longrightarrow & \pi_1(X, \Gamma, x) & \longrightarrow & \Gamma & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow \hat{\rho} & & \downarrow \Pi & & 1 \\
0 & \longrightarrow & G & \longrightarrow & \hat{G}_{\theta,c} & \longrightarrow & \Gamma & \longrightarrow & 0.
\end{array}
\]

Denote by \(\text{Hom}_c(\pi_1(X, \Gamma, x), \hat{G}_{\theta,c})\) the set of \((\Gamma, \eta, \theta, c)\)-equivariant representations \(\hat{\rho} : \pi_1(X, \Gamma, x) \longrightarrow \hat{G}_{\theta,c}\). We want to consider the variety \(\mathcal{R}(G, \Gamma, \eta, \theta, c)\) consisting of \(G\)-conjugacy classes of elements of \(\text{Hom}_c(\pi_1(X, \Gamma, x), \hat{G}_{\theta,c})\) whose restriction to \(\pi_1(X, x_1)\) is reductive, that is, its conjugacy class is an element of the character variety \(\mathcal{R}(G)\).

6.2. Twisted equivariant representations and flat connections. Let \(\pi : E \rightarrow X\) be a \(C^\infty\) \(G\)-bundle. We can define on \(E\) a \((\Gamma, \eta, \theta, c)\)-equivariant structure, similarly to what is done in Definition 3.1 in the holomorphic category, as a collection of diffeomorphisms \(\{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma}\) of the total space \(E\) satisfying (i), (ii) and (iii) in Definition 3.1.

Let \(e \in E\) with \(\pi(e) = x\), and let us denote \(V_e := \ker(\pi_*)\), with \(\pi_* : T_e E \rightarrow T_x X\). Note that \(V_e\) can be identified with tangent space of the fibre \(E_x\) at \(e\). The association \(e \mapsto V_e\) defines a smooth distribution on \(E\). Recall that a connection \(\nabla\) on \(E\) is a smooth distribution \(\nabla(e) := H_e\) such that

(1) \(T_x E \cong V_e \oplus H_e\) for all \(e \in E\),

(ii) \(H_{eg} = R_g(H_e)\) for all \(g \in G\), where \(R_g\) is the right translation defined by \(g\).

Note that a \((\Gamma, \eta, \theta, c)\)-equivariant structure on \(E\) defines endomorphisms \(d\tilde{\eta}_\gamma : T_e E \rightarrow T_{\tilde{\eta}_\gamma(e)}E\) for all \(\gamma \in \Gamma\). We say a connection \(\nabla\) on \(E\) is compatible with the twisted \(\Gamma\)-equivariant structure if \(d\tilde{\eta}_\gamma(V_e) = V_{\tilde{\eta}_\gamma(e)}\) and \(d\tilde{\eta}_\gamma(H_e) = H_{\tilde{\eta}_\gamma(e)}\) for all \(e \in E\).

Proposition 6.1. There exists a natural bijection between \(\text{Hom}_c(\pi_1(X, \Gamma, x), \hat{G}_{\theta,c})\) modulo conjugation by \(G\) and the space of flat connections compatible with a \((\Gamma, \eta, \theta, c)\)-equivariant structure on a \(C^\infty\) principal \(G\)-bundle over \(X\) modulo \(G\)-gauge equivalence.

Proof. Let \(\hat{\rho} \in \text{Hom}_c(\pi_1(X, \Gamma, x), \hat{G}_{\theta,c})\), and \(\rho\) be the restriction \(\hat{\rho}|_{\pi_1(X, x)}\). Let \(E_G := E_\rho\) be the principal \(G\)-bundle associated to the representation \(\rho\) and \(\nabla\) be a flat connection corresponding to \(\rho\). We will show that \(E_G\) admits a \((\Gamma, \eta, \theta, c)\)-equivariant structure for which \(\nabla\) is compatible. Moreover, if \(\hat{\rho}\) and \(\hat{\rho}'\) are \(G\)-conjugate then the corresponding connections are \(G\)-gauge equivalent.

Let \(\gamma \in \Gamma\) be an element of \(\Gamma\) different to the identity. Let \(\alpha \in \pi_1(X, \Gamma, x)\) such that \(\beta(\alpha) = \gamma\) where \(\beta : \pi_1(X, \Gamma, x) \rightarrow \Gamma\) is the surjective map. Then \(\alpha(1) = \eta_\gamma(x)\), in other words \(\alpha\) is a path between \(x\) and \(\eta_\gamma(x)\). Also by commutativity of the diagram (6.2) we get \(\hat{\rho}(\gamma) = (g_\alpha, \gamma)\) for some \(g_\alpha \in G\). Fix \(e \in (E_G)_x\), and let \(e_1 \in (E_G)_{\eta_\gamma(x)}\) be the parallel transport of \(e\) along \(\alpha\). Since \(G\) acts on the fibre \((E_G)_x\) transitively, for any \(e \in (E_G)_x\) there exists a unique \(g \in G\) such that \(e_1 = eg\). We define \((\tilde{\eta}_\gamma)_x(eg) := e_1g^{-1}_\alpha\theta_\gamma(g)\). We can easily check that \((\tilde{\eta}_\gamma)_x\) is independent of \(\alpha\). However, \((\tilde{\eta}_\gamma)_x\) depends on the choice of \(e\), but we can show that \((\tilde{\eta}_\gamma)_x\) is well-defined up to the conjugacy class of \(\hat{\rho}\) by \(G\). To extend the map \((\tilde{\eta}_\gamma)_x\) to all the fibres satisfying the twisted equivariant property we do the following. Take any point \(x_1 \in X\) if \(x_1\) is fixed by all \(\gamma \in \Gamma\) then \(\pi_1(X, \Gamma, x)\) is isomorphic to the
semidirect product \(\pi_1(X, x_1) \ltimes \Gamma\). Suppose that \(x_1\) is not fixed by any \(\gamma \in \Gamma\). Let \(t\) be any path between \(x_1\) and \(x\). Then we have an isomorphism between \(\pi_1(X, x)\) and \(\pi_1(X, x_1)\) given by \(\alpha \mapsto t^{-1} \cdot \alpha \cdot t\), where \(\cdot\) denotes the composition of paths. We can extend this isomorphism to an isomorphism between the equivariant fundamental groups \(\pi_1(X, \Gamma, x)\) and \(\pi_1(X, \Gamma, x_1)\) by sending \(\alpha \in \beta^{-1}(\gamma)\) to \(\eta_\gamma(t^{-1}) \cdot \alpha \cdot t\). This induces an isomorphism

\[
k : \text{Hom}_c(\pi_1(X, \Gamma, x), \hat{G}_{\theta, c}) \cong \text{Hom}_c(\pi_1(X, \Gamma, x_1), \hat{G}_{\theta, c}).\]

Thus the flat \(G\)-bundle obtained from the representation \(k(\hat{\rho})\) is canonically isomorphic to \(E_\rho\). Hence, by the previous construction, we obtain a map \(\eta_\gamma(x_1) : (E_\rho)_{x_1} \to (E_\rho)_{x(x_1)}\).

We now check that the smooth maps \(\tilde{\eta}_\gamma : E_G \to E_G\) satisfy \(\tilde{\eta}_{\gamma_1 \gamma_2} = c(\gamma_1, \gamma_2) \tilde{\eta}_\gamma \tilde{\eta}_\gamma\) for all \(\gamma_1, \gamma_2 \in \Gamma\). For this let us fix a point \(e \in (E_G)_x\). Let \(\beta(\alpha_1) = \gamma_1\) and \(\beta(\alpha_2) = \gamma_2\). Then \(\alpha_1(1) = \eta_{\gamma_1}(x)\) and \(\alpha_2(1) = \eta_{\gamma_2}(x)\). Also \(\beta(\alpha_1 \cdot \alpha_2) = \gamma_1 \gamma_2\). By the commutativity of diagram (5.2) we get \(\rho(\alpha_1 \cdot \alpha_2) = (g_{\alpha_1 \cdot \alpha_2}, \gamma_1 \gamma_2)\). Let \(e_2\) be the parallel transport of \(e\) along \(\alpha_1 \cdot \alpha_2\). Then we have by definition of the composition \(\alpha_1 \cdot \alpha_2\) that there exists a path \(\alpha\) from \(\eta_{\gamma_1}(x)\) to \(\eta_{\gamma_2}(\eta_{\gamma_1}(x))\) such that \(\alpha_1 \cdot \alpha_2 = \alpha_1 \cdot \alpha\) where \(\cdot\) denotes the composition of paths. By the natural identification between \(\pi_1(X, \Gamma, x)\) and \(\pi_1(X, \Gamma, \eta_{\gamma_1}(x))\) we get that \(\hat{\rho}(\alpha_1) = k(\hat{\rho})(\alpha) = (g_{\alpha_1}(\gamma_1), \gamma_1\gamma_2)\). Let \(e_1\) be the image of \(e\) under the parallel translation, along \(\alpha_2\). Then as \(\alpha_1 \cdot \alpha_2 = \alpha_1 \cdot \alpha\) clearly parallel transport of \(e_1\) along \(\alpha\) is \(e_2\). Let \(e' \in (E_G)_x\). Then there is a unique \(g \in G\) such that \(e' = eg\). We have

\[
\tilde{\eta}_{\gamma_1}(\tilde{\eta}_{\gamma_2}(eg)) = e_2g_{\alpha_1}(g_{\alpha_2})\theta_{\gamma_1}(\theta_{\gamma_2}(g)).
\]

As \(\theta\) is a homomorphism, \(g_{\alpha_1 \cdot \alpha_2} = g_{\alpha_1}(g_{\alpha_2})\) and \(\rho(\alpha_1 \cdot \alpha_2) = c(\gamma_1 \cdot \gamma_2) \rho(\alpha_1) \rho(\alpha_2)\) we have that \(\tilde{\eta}_{\gamma_1 \gamma_2}(e') = c(\gamma_1 \gamma_2)(\tilde{\eta}_{\gamma_1 \gamma_2}(e') \rho(\alpha_1 \cdot \alpha_2))\). Similarly, we can check the above identity for any \(x' \in X\). Therefore, \(\tilde{\eta}_{\gamma_1 \gamma_2} = c(\gamma_1, \gamma_2) \tilde{\eta}_\gamma \tilde{\eta}_\gamma\) for all \(\gamma, \gamma_2 \in \Gamma\).

Conversely, suppose \(E_G\) is a \(C^\infty\) twisted \(\Gamma\)-equivariant principal \(G\)-bundle on \(X\) equipped with a flat connection \(\nabla\) compatible with the twisted structure. We will show that there exists a representation \(\hat{\rho} : \pi_1(X, \Gamma, x) \to \hat{G}_{\theta, c}\) such that the associated bundle is isomorphic to \(E_G\). On the normal subgroup \(\pi_1(X, x) \subset \pi_1(X, \Gamma, x)\) \(\hat{\rho}\) is defined to be the holonomy representation determined by \(\nabla\). Let \(\alpha \in \pi_1(X, \Gamma, x) \setminus \pi_1(X, x)\) such that \(\beta(\alpha) = \gamma\) and hence \(\alpha(1) = \eta_{\gamma}(x)\). Fix \(e \in (E)_x\), let \(e^\alpha \in (E)_{\eta_{\gamma}(x)}\) be the image of \(e\) under the parallel transport along \(\alpha\) defined by \(\nabla\). Since \(E_G\) has a \((\Gamma, \eta, \theta, c)\)-equivariant structure \(\{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma}\), there exists a unique \(g^\alpha \in G\) such that \(e^\alpha = \tilde{\eta}_\gamma(e) g^\alpha\). Let \(\beta' : \hat{G}_{\theta, c} \to \Gamma\) be the surjective homomorphism. Under the natural identification of \(\beta'^{-1}(\gamma)\) with \(G\) we get a unique element \(g^{\alpha'} \in \hat{G}_{\theta, c}\). We define

\[
\hat{\rho}(\alpha) = g^{\alpha}\).
\]

Then clearly \(\hat{\rho}\) restricted to the fundamental group is the holonomy representation of \(\nabla\). Using similar arguments to those given in the proof of [6, Proposition 4.5] we can check that \(\hat{\rho} \in \text{Hom}_c(\pi_1(X, \Gamma, x), \hat{G}_{\theta, c})\).

\[\square\]

6.3. Chern correspondence and twisted equivariant structures. Let \(\pi : E \to X\) be a \(C^\infty\) principal \(G\)-bundle. Let \(\lambda : E \times G \to E\) be the action map. An almost complex structure on \(E\) is an endomorphism \(J : TE \to TE\) such that \(J^2 = -I\) and satisfies
\( (i) \) \( \pi_s \circ J = J_X \circ \pi_s \), where \( J_X \) is the natural almost complex structure of \( X \) defined by the complex structure of \( X \);

\( (ii) \) \( J(\lambda_e(V, v)) = \lambda_e(J(V), J_G(v)) \), for \( e \in E, V \in T_eE \) and \( g \in G, v \in T_gG \), where \( J_G \) is the natural almost complex structures defined by the complex structure of \( G \).

Note that, since \( X \) has complex dimension one, \( J \) is integrable and hence defines a structure of a holomorphic \( G \)-bundle on \( E \). We then refer to \( J \) as a complex structure on \( E \).

We say that an (almost) complex structure \( \tilde{\eta} \) on \( \Gamma \) is \( \Gamma \)-equivariant if \( df \circ J = J \circ df \).

Let now \( \{ \tilde{\eta}_\gamma \}_{\gamma \in \Gamma} \) be a \( (\Gamma, \eta, \theta, c) \)-equivariant structure on \( E \), as defined in Section \ref{equivariant_structure}

We say that an (almost) complex structure \( J \) of \( E \) is compatible with the twisted \( \Gamma \)-equivariant structure if \( df \tilde{\eta}_\gamma \circ J = J \circ df \tilde{\eta}_\gamma \). The compatibility of \( J \) is then equivalent to the holomorphicity of the maps \( \{ \tilde{\eta}_\gamma \}_{\gamma \in \Gamma} \).

Let \( K \subset G \) be a maximal compact subgroup and fix a reduction \( E_K \subset E \) of structure group of \( E \) to \( K \). Recall that the Chern correspondence \cite{Chern, Simpson} establishes a bijection between the set of complex structures on \( E \) and the set of connections on \( E_K \). This is defined by assigning to a complex structure \( J \) on \( E \) a connection \( \nabla \) defined on \( E_K \) by the horizontal distribution for \( e \in E_K \)

\[
H_e := T_eE_K \cap J(T_eE_K) \subset T_eE_K.
\]

Now assume that \( K \subset G \) is chosen to be invariant under the action of \( \Gamma \), and that the reduction \( E_K \) of structure group of \( E \) to \( K \) is also \( \Gamma \)-invariant. This means, in particular, that \( \tilde{\eta}_\gamma(E_K) \subset E_K \). We then say (in a similar way to the case of a connection on \( E \), discussed in Section \ref{equivariant_connection}), that a connection on \( E_K \) is compatible with the twisted \( \Gamma \)-equivariant structure if \( df \tilde{\eta}_\gamma(V_e) = V_{\tilde{\eta}_\gamma(e)} \) and \( df \tilde{\eta}_\gamma(H_e) = H_{\tilde{\eta}_\gamma(e)} \) for all \( e \in E_K \), where \( H_e \) is the horizontal distribution defined by the connection and \( T_eE_K \cong V_e \oplus H_e \).

One can check that if \( E \) is endowed with a twisted \( \Gamma \)-equivariant structure, and \( J \) is a compatible complex structure on \( E \), then the connection on \( E_K \) given by the Chern correspondence is also compatible with the twisted \( \Gamma \)-equivariant structure. We thus have the following.

**Proposition 6.2.** The Chern correspondence restricts to give a bijection between the space of compatible (almost) complex structures on \( E \) and the space of compatible connections on \( E_K \).

**6.4. Equivariant non-abelian Hodge correspondence.** Let \( \mathcal{M}(G, \Gamma, \eta, \theta, c) \) be the moduli space of \( (\Gamma, \eta, \theta, c) \)-equivariant \( G \)-Higgs bundles considered in Section \ref{equivariant_higgs_bundles}. Here, we are taking the character \( \chi : G \rightarrow \mathbb{C}^* \) to be trivial. Let \( \mathcal{R}(G, \Gamma, \eta, \theta, c) \) be the moduli space consisting of \( G \)-conjugacy classes of elements of \( \text{Hom}_c(\pi_1(X, \Gamma, x), \hat{G}_{\theta, c}) \) whose restriction to \( \pi_1(X, x_1) \) is reductive (see Section \ref{reductive}). We then have the following.

**Theorem 6.3.** There is a homeomorphism between \( \mathcal{M}(G, \Gamma, \eta, \theta, c) \) and \( \mathcal{R}(G, \Gamma, \eta, \theta, c) \).

**Proof.** Let \( (E, \{ \tilde{\eta}_\gamma \}, \varphi) \) be a polystable twisted \( \Gamma \)-equivariant \( G \)-Higgs bundle. Then, by Theorem \ref{equivariant_reduction} \( E \) admits a \( \Gamma \)-invariant reduction of structure group \( h \) to \( K \) satisfying the Hitchin equation \ref{hitchin}, where \( K \) is \( \Gamma \)-invariant. Let \( \nabla_h \) be the Chern connection defined by the holomorphic structure of \( E \) and \( h \). By Proposition \ref{compatibility} \( \nabla_h \) is compatible with the
is a flat connection on $E$ compatible with the twisted $\Gamma$-equivariant structure. Since, by Proposition 4.1, $(E, \varphi)$ is polystable, the holonomy representation of $D$ is completely reducible. Thus by Proposition 6.1 there exists a $\hat{\rho} \in \text{Hom}_c(\pi_1(X, \Gamma), \hat{G}_{\theta,c})$.

For the reverse construction, let $\hat{\rho} \in \text{Hom}_c(\pi_1(X, \Gamma), \hat{G}_{\theta,c})$, so that the induced representation $\rho : \pi_1(X) \to G$ is reducible. By Proposition 6.1 there exists a $(\Gamma, \eta, \theta, c)$-equivariant structure $\{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma}$ on the $C^\infty$ $G$-bundle $E_\rho$ with flat connection $D$ defined by $\rho$, for which $D$ is compatible. Since $\rho$ is reductive and $G$ is semisimple, by a theorem of Corlette [11] Theorem 3.3] it follows that there exists a unique harmonic reduction $h \in \Omega^0(X, E_\rho(G/K))$ of the structure group to $K$. By uniqueness, $h$ is $\Gamma$-invariant, and using $h$ we can write $D = \nabla_h + \psi$, where $\nabla_h$ is a connection on the reduced $K$-bundle defined by $h$ and $\psi \in \Omega^1(X, E_\rho(it))$. Since $D$ is compatible with the twisted $\Gamma$-equivariant structure so are $\nabla_h$ and $\psi$. By Proposition 6.2 there is a complex structure $J$ on $E_\rho$ making the maps holomorphic $\{\tilde{\eta}_\gamma\}_{\gamma \in \Gamma}$. Let $E$ denote the holomorphic $G$-bundle defined by $J$ on $E_\rho$. One can extract the Higgs field $\varphi$ from the relation $\psi = \varphi - \tau_h(\varphi)$. The compatibility of $\psi$ with the twisted $\Gamma$-equivariant structure implies that $\varphi$ is also compatible and hence $(E, \{\tilde{\eta}_\gamma\}, \varphi)$ is a $(\Gamma, \eta, \theta, c)$-equivariant $G$-Higgs bundle. By the usual non-abelian Hodge correspondence we have that $(E, \varphi)$ is polystable and by Corollary 4.3 we conclude that $(E, \{\tilde{\eta}_\gamma\}, \varphi)$ is polystable.

6.5. Action of $\Gamma$ on the character variety and fixed points. We study now the action of $\Gamma$ on the character variety $\mathcal{R}(G)$ and describe the fixed points in terms of twisted equivariant representations. Recall that we are given homomorphisms $\eta : \Gamma \to \text{Aut}(X)$, $a : \Gamma \to \text{Out}(G)$ and $\theta : \Gamma \to \text{Aut}(G)$, where $\theta$ is a lift of $a$.

Fix a point $x \in X$. The automorphism $\eta_\gamma$ of $X$ produces a homomorphism $\eta_{\gamma_*} : \pi_1(X, x) \to \pi_1(X, \eta_\gamma(x))$.

This induces an automorphism of $\mathcal{R}(G)$ since the quotient $\text{Hom}(\pi_1(X, x), G)/G$ is independent of the base point of $X$. Now, given the automorphism $\theta_\gamma$ of $G$ and $\rho \in \text{Hom}(\pi_1(X, x), G)$, there is another representation of $\pi_1(X, x)$ in $G$ given by $\theta_\gamma \circ \rho$. This defines an action of $\Gamma$ on $\mathcal{R}(G)$ that clearly depends only on $a : \Gamma \to \text{Out}(G)$. So for every $\gamma \in \Gamma$ and $\rho \in \text{Hom}(\pi_1(X), G)$ we have $\gamma \cdot \rho \in \text{Hom}(\pi_1(X), G)$ given by $\gamma \cdot \rho = \theta_\gamma \circ \rho \circ \eta_{\gamma_*}$.

It is straightforward to show (see [8, 17, 18] for a similar computation) that the action of $\Gamma$ on $\mathcal{R}(G)$ given by this coincides with the action of $\Gamma$ on $\mathcal{M}(G)$ defined in Section 5.1 via the non-abelian Hodge correspondence (recall that here we are taking the character $\chi : \Gamma \to \mathbb{C}^*$ to be trivial). In other words we have the following.

**Proposition 6.4.** Denote the action of $\Gamma$ on $\mathcal{M}(G)$ and $\mathcal{R}(G)$ by $\gamma_*$, and consider the non-abelian Hodge correspondence $\mathcal{M}(G) \cong \mathcal{R}(G)$ given by Theorem 2.4. Then, for every
\(\gamma \in \Gamma\) the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{M}(G) & \xrightarrow{\cong} & \mathcal{R}(G) \\
\gamma \downarrow & & \gamma \downarrow \\
\mathcal{M}(G) & \xrightarrow{\cong} & \mathcal{R}(G).
\end{array}
\]

Combining Theorem 2.4 and Proposition 6.4 with Theorems 6.3 and 5.7 and Proposition 5.1 we have the following.

**Theorem 6.5.** Let \(\mathcal{R}_{\text{irr}}(G) \subset \mathcal{R}(G)\) be the subvariety of \(\mathcal{R}(G)\) consisting of irreducible representations, and let \(\bar{\mathcal{R}}(G, \Gamma, \eta, \theta, c)\) be the image of \(\mathcal{R}(G, \Gamma, \eta, \theta, c)\) in \(\mathcal{R}(G)\) under the natural map defined by diagram (6.2). Let \(\mathcal{R}(G)^F\) be the fixed-point subvariety for the action of \(\Gamma\) defined by the homomorphism \(F = (\eta, a) : \Gamma \to \text{Aut}(X) \times \text{Out}(G)\). Then

\[\bar{\mathcal{R}}(G, \Gamma, \eta, \theta, c) \subset \mathcal{R}(G)^F\]

for every homomorphism \(\theta : \Gamma \to \text{Aut}(G)\) lifting \(a\),

and

\[\mathcal{R}_{\text{irr}}(G)^F \subset \bigcup_{[c] \in H^2(\Gamma, Z), [\theta'] \in H^1(\Gamma, \text{Int}(G))} \bar{\mathcal{R}}(G, \Gamma, \eta, \theta', c) .\]

Here \(\theta : \Gamma \to \text{Aut}(G)\) is any lift of \(a : \Gamma \to \text{Out}(G)\) and cohomology is with respect to the corresponding actions defined by \(\theta\).

**Remark 6.6.** We could have considered also a non-trivial character \(\chi : \Gamma \to \mathbb{C}^\ast\) with image the subgroup of \(\mathbb{C}^\ast\) given by \(\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}\). With a minor modification in the definition of the group \(\hat{G}_\theta, c\) in Section 6.1, one would obtain similar results (see [8, 17, 18] for an analogous situation).

### 7. Examples

We give the main ingredients in some particular cases to which one can apply our main results.

#### 7.1. Example 1.

Let \(G = \text{SL}(2, \mathbb{C})\) and \((X, \sigma)\) be a hyperelliptic curve together with the hyperelliptic involution \(\sigma\). In this case let \(\Gamma = \mathbb{Z}/2\mathbb{Z}\) and consider the homomorphism \(\eta : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(X)\) defined by sending \(-1 \mapsto \sigma\). In this case \(\text{Out}(G) = 1\) and \(Z = \mathbb{Z}/2\mathbb{Z}\), hence \(\text{Aut}(G) = \text{Int}(G)\), and therefore \(\text{Aut}(G)\) acts trivially on the centre \(Z\). So, in this case, we have \(H^2(\mathbb{Z}/2\mathbb{Z}, Z) = \mathbb{Z}/2\mathbb{Z}\). Also, there are only two characters \(\chi^\pm\), defined by \(\chi^\pm(-1) = \pm 1\). We can then define actions on the moduli space of \(\text{SL}(2, \mathbb{C})\)-Higgs bundles defined by \(\eta\) and \(\chi^\pm\). The case in which \(\eta\) is the trivial homomorphism from \(\Gamma\) to \(\text{Aut}(X)\) and \(\chi = \chi^-\) is studied in [21, 12, 16, 17].
7.2. Example 2. Let $G = \text{SL}(n, \mathbb{C})$, with $n > 2$ and $X$ a hyperelliptic curve together with the hyperelliptic involution $\sigma$, as above. Let $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $\eta : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(X)$ be the homomorphism defined by sending $-1 \mapsto \sigma$. In this case $\text{Out}(G) \cong \mathbb{Z}/2\mathbb{Z}$ and $Z \cong \mathbb{Z}/n\mathbb{Z}$. Let us denote the trivial homomorphism from $\Gamma \to \text{Out}(G)$ by $a^+$ and the homomorphism which sends $-1$ to $b$, where $\text{Out}(G) = \langle b \rangle$, by $a^-$. In the first case we have the trivial action of $\Gamma$ on the centre $Z$ via $a^+$. To compute the second group cohomology of $Z/2\mathbb{Z}$ with coefficients in $\mathbb{Z}/n\mathbb{Z}$ we will use the following fact: Let $C$ be a cyclic group order $r$ generated by $t$ and $A$ be a finite abelian group with a $C$ action. Let $N = 1 + t + \cdots + t^{r-1} \in \mathbb{Z}[\Gamma]$ then obviously $Na, a \in A$, is fixed by all $\alpha \in C$. With these notations we have $H^n(C, A) = \frac{\Delta^r}{NA}$, $p = 2, 4, 6, \ldots$. Thus we have, in this case, $H^2(\Gamma, Z) = 0$ when $n$ is odd and $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})$. On the other hand the action of the generator of $\mathbb{Z}/2\mathbb{Z}$ on $Z$ induced by $a^-$ sends $x \in Z$ to $x^{-1}$. In this case we have $H^2_{a^-}(\Gamma, Z) = \mathbb{Z}^{\mathbb{Z}/2\mathbb{Z}}$. Thus the action is trivial when $n$ is odd and hence $H^2_{a^+}(\Gamma, Z) = 0$, and if $n$ is even then $H^2_{a^-}(\Gamma, Z)$ consists of all order 2 elements of $\mathbb{Z}/n\mathbb{Z}$. As in the previous example, we have $\chi^\pm$ as possible characters.

The cases in which $\eta$ is the trivial homomorphism from $\Gamma$ to $\text{Aut}(X)$ and $\chi = \chi^\pm$ is studied in [12, 17] (see also [20, 34] for related work).

7.3. Example 3. Let $G = \text{Spin}(8, \mathbb{C})$. Then $Z = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\text{Out}(G) \cong S_3$. In [17] the authors consider various actions of cyclic subgroups $\Gamma$ of $\text{Out}(G)$, with $\Gamma$ acting trivially on $X$, and identify the fixed-point subvarieties.

Now in our situation the following three cases are relevant.

Case (I): Let $X$ be a compact Riemann surface of genus $g > 2$ and $\Gamma := S_3$. Let $\eta : \Gamma \to \text{Aut}(X)$ be an injective homomorphism in other words the action of $\Gamma$ on $X$ is faithful.

Let $\sigma$ and $\tau$ generate the group $\text{Out}(\text{Spin}(8, \mathbb{C})) \cong S_3$. Let $a : \Gamma \to \text{Out}(\text{Spin}(8, \mathbb{C}))$ be the automorphism defined by sending order 2 generator to $\sigma$ and order 3 generator to $\tau$. Let $\chi : \Gamma \to \mathbb{C}^*$ be a character of $S_3$. We know that $S_3$ has three non equivalent conjugacy classes. Let $\chi_i, i = 1, 2, 3$, be the corresponding characters. We define homomorphisms $F_i = (\eta, a, \chi_i) : \Gamma \to \text{Aut}(X) \times \text{Out}(G) \times \mathbb{C}^*$, $i = 1, 2, 3$. Then each $F_i$ determines an action on the moduli space of $G$-Higgs bundles. Let $H = \langle \tau \rangle$ be the normal subgroup of $G$ generated by $\tau$. Then by [37, Lemma 2.2.4]) $H^2(\Gamma, Z) = H^2(\Gamma/H, Z^H)$. As $\tau$ is an element of order 3 either $Z^\tau = (e)$ or $Z^\tau = Z$. So, we have either $H^2(\Gamma, Z) = 0$ or $H^2(\Gamma, Z) = \mathbb{Z}/2\mathbb{Z}$.

Case (II): Let $X$ be a hyperelliptic curve and $\Gamma := S_3$. We define a homomorphism $\eta : S_3 \to \text{Aut}(X)$ by sending $\sigma$ to $\text{Id}$ and $\tau$ to a order 2 hyperelliptic involution. Let $b_i \in \text{Out}(G)$ be the class of an order two automorphism of $G$ and $\theta_i : \Gamma \to \text{Out}(G)$ be the homomorphism defined by $\tau \mapsto 1$ and $\sigma \mapsto b_i$. We define $F_i := (\eta, \theta_i, \chi_i) : \Gamma \to \text{Aut}(X) \times \text{Out}(G)$, $i = 1, 2, 3$. Then the action of $\Gamma$ on the moduli space of $G$-Higgs bundles are determined by $F_i$. In this case the subgroup $H$ acts on $Z$ trivially, therefore $Z^H = Z$, and hence $H^2(\Gamma, Z) = \mathbb{Z}/2\mathbb{Z}$.

Case (III): $X$ is a cyclic trigonal curve. In other words we assume that the subgroup $\langle \tau \rangle$ acts trivially on $X$ and $f$ is an order 3 automorphism such that $X/\langle f \rangle \cong \mathbb{P}^1$. This case is related to the work of Oxbury and Ramanan [18], and the Galois $\text{Spin}(8, \mathbb{C})$-bundles. We define a homomorphism $\eta : S_3 \to \text{Aut}(X)$ by sending $\sigma$ to
Id and \( \tau \) to the order 3 automorphism \( f \). Let \( b \) be the class of unique order 3 automorphism of \( X \) and \( \theta : S_3 \to \text{Out}(G) \) is defined by sending \( \sigma \) to \( I \) and \( \tau \) to \( b \). As in the previous case we define \( F_i := (\eta, \theta_i, \chi_i) : \Gamma \to \text{Aut}(X) \times \text{Out}(G) \), \( i = 1, 2, 3 \). Then the action of \( \Gamma \) on the moduli space of \( G \)-Higgs bundles is determined by \( F_i \). Since \( \mathbb{Z}/3\mathbb{Z} \) and \( \mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) have coprime order, by [18, Lemma 2.2.4] \( H^2(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) = 0 \).

7.4. Example 4. Let \( G \) be a group of type \( E_n \). In this case \( \text{Out}(G) = \mathbb{Z}/2\mathbb{Z} \). Let \( X \) be a hyperelliptic curve together with a hyperelliptic involution \( \sigma \) and \( \Gamma = \mathbb{Z}/2\mathbb{Z} \). We define a homomorphism \( \eta : \Gamma \to \text{Aut}(X) \) by sending \(-1 \mapsto \sigma \). As in the case of example 1 we have two homomorphisms \( a^\pm : \Gamma \to \text{Out}(G) \) and two characters \( \chi^\pm \).

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