Generalized Nambu system on $S^3$ and spinors

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Abstract: It is shown that the manifold $S^3$ can be equipped with a natural Nambu structure arising out of a cross product on the tangent space. Further, the group action of $SU(2)$ on $S^3$ is shown to be Nambu action. Moreover, we compare the action of $SU(2)$ on spinors with that of a Nambu system.

1. Introduction

There exist certain dynamical systems which resist the usual canonical description\cite{1,2}. Nambu dynamics and its various generalizations\cite{3,4,5,6,7,8,9} offer an appropriate possibility for the description of a certain class of non-canonical dynamical systems.

In\cite{10} we proposed a geometric formulation of generalized Nambu systems. The motivation behind that work was to provide a framework suitable from the point of view of dynamical systems. The framework proposed in\cite{10} involves a $3n$ dimensional manifold $M^{3n}$, together with a closed, strictly non-degenerate 3-form $\omega^{(3)}$. The time evolution is governed by a pair of Nambu functions $H_1, H_2$. It was seen that the formulation involving 2-forms provides a natural approach to a Nambu system. The connection between symmetries of Nambu systems and conserved 2-forms is established in\cite{11} by introducing the notion of a “Nambu momentum map”. A generalization of the symplectic Noether theorem for Nambu systems is carried out in\cite{11}.

In case of Hamiltonian systems the manifold $\mathbb{R} \times T^1$ plays a special role. We expect that the manifold $S^3$ should play similar role for Nambu systems. The purpose of the present paper is two fold. Firstly, we explicitly construct a Nambu system on a non-trivial (non vector space) manifold $S^3$ using the cross product. Secondly, we study the Nambu action of $SU(2)$ on $S^3$ and identify that the action of $SU(2)$ on the normalized spinor is equivalent to the action of natural Nambu vector field on $S^3$. The relevance of this observation is obvious.
In section 2 we list the main features of Nambu systems developed in [10, 11]. In section 3 we give natural Nambu structure on $S^3$ and show the equivalence between the action of Nambu vector field on $S^3$ and action of $SU(2)$ on normalized spinors.

2. Nambu Systems

In this section we briefly review the relevant definitions and results regarding Nambu systems [10,11].

**Definition 1. (Nambu Manifold)**: Let $M^{3n}$ be a $3n$-dimensional $C^\infty$ manifold and let $\omega^{(3)}$ be a 3-form field on $M^{3n}$ such that $\omega^{(3)}$ is completely anti-symmetric, closed and strictly non-degenerate (See [10]) at every point of $M^{3n}$ then the pair $(M^{3n}, \omega^{(3)})$ is called a Nambu manifold.

The Nambu structure preserving transformation (Nambu canonical transformation) is defined in [10,11]. Using this, a generalization of Darboux theorem is proved, which gives a Nambu-Darboux coordinates on Nambu manifold [10,11]. We define the analogs of raising and lowering operations. The map $\flat : \mathcal{X}(M^{3n}) \to \Omega^2_0(M^{3n})$ is defined by $X \mapsto X^\flat = i_X \omega^{(3)}$, where $\mathcal{X}(M^{3n})$ is space of vector fields on $M^{3n}$ and $\Omega^2_0(M^{3n})$ is space of 2-forms on $M^{3n}$. The map $\sharp : \Omega^2_0(M^{3n}) \to \mathcal{X}(M^{3n})$, is defined by the following prescription. Let $\alpha$ be a 2-form and $\alpha_{ij}$ be its components in Nambu-Darboux coordinates, then the components of $\alpha^\sharp$ are given by [10,11]

$$\alpha^\sharp_{3i+p} = \frac{1}{2} \sum_{l,m=1}^{3} \varepsilon_{plm} \alpha_{3i+l}^{3j+m}$$

These maps provide a bracket structure of 2-forms on Nambu manifold [10,11] as, let $\alpha, \beta \in \Omega^2_0(M^{3n})$, then the bracket is $\{\alpha, \beta\} = [\alpha^\sharp, \beta^\sharp]$, where $[,]$ is Lie bracket of vector fields [10,11].

The Nambu action and the Nambu momentum maps are defined in [11] as

**Definition 2. (Nambu action)**: Let $G$ be a Lie group and let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold. Let $\Phi_G$ be the action of $G$ on $M^{3n}$. $\Phi_G$ is called Nambu action if

$$\Phi_G^* \omega^{(3)} = \omega^{(3)}$$

i.e., $\Phi_G$ induces a Nambu canonical transformation.

A quantity similar to symplectic momentum map is defined in [11] as

**Definition 3. (Nambu momentum maps)**: Let $G$ be a Lie group and let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold, let $\Phi_G$ be a Nambu action on $M^{3n}$. Then the mapping $J \equiv (J_1, J_2) : M^{3n} \to g^* \times g^*$ is called Nambu momentum maps provided, for every $\xi \in g$

$$(d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi))^2 = \xi_{M^{3n}}$$

where $\hat{J}_1(\xi) : M^{3n} \to \mathbb{R}$, $\hat{J}_2(\xi) : M^{3n} \to \mathbb{R}$, defined by $\hat{J}_1(\xi)(x) = J_1(x)\xi$ and $J_2(\xi)(x) = J_2(x)\xi \ \forall x \in M^{3n}$ and $\xi_{M^{3n}}$ is infinitesimal generator of the action $\Phi_G$. 
**Definition 4.** *(Nambu G-space)*: The five tuple $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ is called Nambu G-space.

The consistency between the bracket of 2-forms and the Nambu momentum maps follows from the proposition established in [11] viz

**Proposition 1.** Let $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ be a Nambu G-space and $\xi, \eta \in g$ then
\[ (d\hat{J}_1([\xi, \eta]) \wedge d\hat{J}_2([\xi, \eta]))^\# = \{d\hat{J}_1(\eta) \wedge d\hat{J}_2(\eta), d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi)\} \]
i.e., The following diagram commutes

\[
\begin{array}{ccc}
\Omega^0_2(M^{3n}) & \xrightarrow{\#} & \mathcal{X}(M^{3n}) \\
\hat{J}_1, \hat{J}_2 & & \\
g & \xi \mapsto \xi_{M^{3n}}
\end{array}
\]

In analogy with the notion of symplectic symmetry for the Hamiltonian system, the idea of Nambu Lie symmetry is introduced in [11].

**Definition 5.** *(Nambu Lie symmetry)*: Consider a Nambu system $(M^{3n}, \omega^{(3)}, H_1, H_2)$.

Let $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ be a Nambu G-space. We call $\Phi_G$ a Nambu Lie symmetry of the Nambu system if
\[ \Phi_G^* (dH_1 \wedge dH_2) = (dH_1 \wedge dH_2) \]

The Noether theorem in this framework now reads as [11]

**Theorem 1.** Consider a Nambu system $(M^{3n}, \omega^{(3)}, H_1, H_2)$ where $H_1, H_2$ are so chosen that $dH_1 \wedge dH_2 = (dH_1 \wedge dH_2)^\#$. Let this system be a Nambu G-space $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ where $J_1, J_2$ are so chosen that $d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi) = (d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi))^\# \forall \xi \in g$. If $\Phi_G$ is Nambu Lie symmetry of this system then
\[ L_{(dH_1 \wedge dH_2)}^\# \left( d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi) \right) = 0 \]
i.e., $d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi)$ is conserved by the Nambu flow.

3. **Spinors and the Nambu system on $S^3$**

The manifold $\mathbb{R} \times T^1$ plays a special role in Hamiltonian systems. We expect that $S^3$ should play a similar role in case of Nambu systems. Here, we show that $S^3$ can be equipped with a Nambu structure, thus provide a non-trivial (other than $\mathbb{R}^{3n}$) example of Nambu manifold. We also demonstrate that the action of $SU(2)$ on $S^3$ is a Nambu action and notice the similarity with evolution of Pauli spinors.
3.1. SU(2) Spinors. The group SU(2) is a group of complex unimodular unitary $2 \times 2$ matrices. A standard parameterization of an element of SU(2) is

$$U_n(\theta) = \exp\left(-i\frac{\theta}{2} n \cdot \sigma\right) = 1 \cos\left(\frac{\theta}{2}\right) + i \cdot n \cdot \sigma \sin\left(\frac{\theta}{2}\right)$$

where $n$ is a unit vector and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. The elements of $\mathbb{C}^2$ are the SU(2) spinors.

For various reasons and especially in view of its relevance in quantum mechanics the action of SU(2) on the normalized spinors is of great interest (Here the normalization is carried out with the hermitian scalar product). We denote the space of normalized spinors by $SP$.

Remarks:
The space of normalized spinors is $S^3$.

Let $\xi \in SP$. So without loss of generality we write

$$\xi = \begin{pmatrix} x + iy \\ z + iw \end{pmatrix}$$

where $x, y, z, w \in \mathbb{R}$ and the condition $\xi^\dagger \xi = 1$ implies that $x^2 + y^2 + z^2 + w^2 = 1$, which is $S^3$.

3.2. Nambu system on $S^3$.

**Proposition 2.** The manifold $S^3$ is equipped with a natural Nambu structure.

**Proof:** Since $S^3$ is the group manifold of the Lie group SU(2) we define the 3-form at the identity and the by left action on can define it everywhere.

The Lie algebra of SU(2) is $su(2) = \mathbb{R}^3$. Every three dimensional space has vector cross product $[12]$. So we define 3-form in $T_eS^3$ as

$$\epsilon(a, b, c) = \omega^{(3)}(e)(a, b, c) = (a, b \times c) \text{ where } a, b, c \in T_eS^3$$

The 3-form $\epsilon$ is non-degenerate and hence strictly non-degenerate since the space is three dimensional $[10]$.

Now we define $\omega^{(3)}(g) = L_{g^{-1}}^*\epsilon$, where $L_g$ is the left action corresponding to $g \in SU(2)$, as the 3-form every where. Since $d\omega = 0$, the form $\omega^{(3)}$ is also closed. So we have the Nambu structure $\omega^{(3)}$ on $S^3$.

Since the 3-form $\omega^{(3)}$ is defined by the left action it is natural to expect that the action is a Nambu action.

**Proposition 3.** The left action of SU(2) on $S^3$ is Nambu action.
\[ \mathcal{L}_h \omega^{(3)}(g) = \mathcal{L}_h \mathcal{L}_{g^{-1}} \epsilon \]
\[ = \mathcal{L}_h \mathcal{L}_{g^{-1}} \epsilon \]
\[ = \omega^{(3)}(h^{-1} g) \]

By proposition 2, the left action is a Nambu action.

We construct Momentum maps corresponding to this action. Let \( f \) be a dual basis of \( g^* \) corresponding to \( e_1, e_2, e_3 \). We choose a Nambu-Darboux coordinate system on \( S^3 \). Let \( p \in S^3 \) be represented by \((x, y, z)\) in this coordinate system. Let \( r = xf_1 + yf_2 + zf_3, \rho = (y^2 + z^2)f_1 + (x^2 + z^2)f_2 + (y^2 + x^2)f_3 \in g^* \). Then the momentum maps are \( J_1(x, y, z) = r \) and \( J_2(x, y, z) = \rho \). The 2-forms obtained from these are

\[ d\hat{J}_1(e_1) \wedge d\hat{J}_2(e_1) = S_1 = 2ydx \wedge dy + 2zdx \wedge dz \]
\[ d\hat{J}_1(e_2) \wedge d\hat{J}_2(e_2) = S_2 = -2xdx \wedge dy + 2zdy \wedge dz \]
\[ d\hat{J}_1(e_3) \wedge d\hat{J}_2(e_3) = S_3 = -2xdx \wedge dz - 2ydy \wedge dz \]

Corresponding to these 2-forms we write vector fields. By the definition of momentum maps we write \( S_1 = i_{\hat{s}_1} \omega^{(3)} = i_{\hat{s}_1}(dx \wedge dy \wedge dz) \Rightarrow S_1^\rho = (0, -2z, 2y) \)

Similarly

\[ S_2^\rho = (2z, 0, -2x) \]
\[ S_3^\rho = (-2y, 2x, 0) \]

Using the definition of the Nambu bracket, clearly, \( \{S_1, S_2\} = i_{[S_1, S_2]} \omega^{(3)} = L_{S_1} i_{S_2} \omega^{(3)} - i_{S_1} L_{S_2} \omega^{(3)} \) Since \( \omega^{(3)}, S_1, S_2, S_3 \) are closed forms, \( \{S_1, S_2\} = d(i_{S_1} S_2) = -2S_3 \). Similarly

\[ \{S_2, S_3\} = -2S_1, \{S_3, S_1\} = -2S_2. \tag{1} \]

This establishes the consistency condition stated in Proposition 3.

As shown above the left action of \( SU(2) \) is Nambu action. Which implies that there are momentum maps which are generators of such an action. From equation (1) it is clear that the algebra of the generators of Nambu action is isomorphic to the algebra of Pauli spin matrices.

3.3. Spin systems and Nambu systems. In section 3.2, we have shown that the action of \( SU(2) \) on \( S^3 \) is Nambu action. In this section, through Proposition 4, we show that such a Nambu action is equivalent to the action of \( SU(2) \) on normalized spinors.
Proposition 4. Let \( h : S^3 \to SP \) defined by

\[
\begin{pmatrix}
 x + iy - z + iw \\
 z + iw \\
x - iy
\end{pmatrix} \mapsto
\begin{pmatrix}
 x + iy \\
 z + iw
\end{pmatrix}
\]

where \( x, y, z, w \in \mathbb{R}^4 \) satisfying condition \( x^2 + y^2 + z^2 + w^2 = 1 \). Let \( \Phi_U \) be the induced action of \( SU(2) \) on \( SP \). Then the Nambu action of \( SU(2) \) on \( S^3 \) is equivalent to the action \( \Phi_U \) of \( SU(2) \) on \( SP \). i.e. The following diagram commutes.

\[
\begin{array}{ccc}
S^3 & \xrightarrow{h} & SP \\
\downarrow{\mathcal{L}_y} & & \downarrow{\Phi_U} \\
S^3 & \xrightarrow{h} & SP
\end{array}
\]

**Proof:** Let \( \xi_M = \begin{pmatrix} x + iy - z + iw \\ z + iw \\ x - iy \end{pmatrix} \in S^3 \). The left action of \( SU(2) \) with the above parameterization takes this point to

\[
\begin{pmatrix}
x' \\
y' \\
z' \\
w'
\end{pmatrix} = \begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \cos(\theta) - \begin{pmatrix}
y \\
z \\
w
\end{pmatrix} \sin(\theta)
\]

Now

\[
\Phi_U : h(\xi_M) = U \left( \begin{pmatrix} x + iy \\
z + iw \end{pmatrix} \right) = \begin{pmatrix} x' + iy' \\
z' + iw' \end{pmatrix} = \xi'
\]

which is

\[
\xi' = h(\xi_M)
\]

\( \square \)

**Remarks:**

1. Consider quantum mechanical evolution of a charged spin \( \frac{1}{2} \) particle in the presence of a constant uniform external magnetic field. The time evolution of normalized Pauli spinor can be considered as action of \( SU(2) \) on the spinor. Hence, in view of Proposition 4, the time evolution can be considered as Nambu evolution.

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\( ^1 \) Since \( S^3 \) is group manifold of group \( SU(2) \) we are not distinguishing between a point in \( SU(2) \) and a point of \( S^3 \).
2. Classical pure spin systems are considered in Hamiltonian framework \[13\]. Proposition\[8\] suggests that Nambu framework with generalized Nambu-Poisson bracket is a suitable framework for classical pure spin systems.

4. Conclusions

In this paper we have constructed a Nambu system on $S^3$. Further the action of $SU(2)$ on such a system is identified as Nambu action. In fact, this observation has an obvious relevance for the time evolution of Pauli spinor in quantum mechanics. We hope that eventually such construction will find their use in construction of spin manifold \[14,15\].

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