The Skyrme theory has played a very important role in physics, in particular in nuclear physics [1-3]. A remarkable feature of Skyrme theory is its rich topological structure [4]. It has been discovered that the theory allows not only the original skyrmion and the baby skyrmion but also the Faddeev-Niemi knot whose topology is fixed by $\pi_3(S^2)$ [4,5]. Similar knots have appeared almost everywhere recently, in atomic physics in two-component Bose-Einstein condensates [6,7], in condensed matter physics in multi-gap superconductors [8,9], in plasma physics in coronal loops [10], even in high energy physics in Weinberg-Salam model [11]. But at the center of all these knots lies the Faddeev-Niemi knot of Skyrme theory [6,8,11]. So we need a better understanding of these knots.

The purpose of this Letter is to provide a new interpretation of topological objects in Skyrme theory which could allow us to construct the Faddeev-Niemi knot in laboratories, in particular in two-component superfluids and two-gap superconductors. We show that the Faddeev-Niemi knot is nothing but a vortex ring made of a helical baby skyrmion, a twisted chromomagnetic flux which is periodic in $z$-coordinate, with two periodic ends connected together. This allows us to interpret the knot as two quantized magnetic flux rings linked together, the first one winding the second $m$ times and the second one winding the first $n$ times, whose linking number $mn$ is fixed by the Chern-Simon index of the magnetic potential. This interpretation strongly suggests that the Skyrme theory could also describe a very interesting low energy physics in a completely different environment, which puts the theory in a totally new perspective.

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We identify the Faddeev-Niemi knot in Skyrme theory as a vortex ring made of a helical baby skyrmion (a twisted chromomagnetic vortex which is periodic in $z$-coordinate) with the periodic ends connected together. This allows us to interpret the knot as two quantized magnetic flux rings linked together, the first one winding the second $m$ times and the second one winding the first $n$ times, whose linking number $mn$ is fixed by the Chern-Simon index of the magnetic potential. This interpretation strongly suggests that the Skyrme theory could also describe a very interesting low energy physics in a completely different environment, which puts the theory in a totally new perspective.
the Lagrangian has a local $U(1)$ symmetry as well as a global $SU(2)$ symmetry. A remarkable point of the Lagrangian is that $\omega = \pi$, independent of $\hat{n}$, becomes a classical solution \([4]\). So restricting $\omega$ to $\pi$, one can reduce \([2]\) to the Skyrme-Faddeev Lagrangian

$$\mathcal{L} \rightarrow -\frac{\mu^2}{2} (\partial_\mu \hat{n})^2 - \frac{\alpha}{4} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2, \quad (3)$$

whose equation of motion is given by

$$\hat{n} \times \partial^2 \hat{n} + \frac{\alpha}{\mu^2} (\partial_\mu N_{\mu\nu}) \partial_\nu \hat{n} = 0,$$

$$N_{\mu\nu} = \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}). \quad (4)$$

It is this equation that allows not only the baby skyrmion and the Faddeev-Niemi knot but also the non-Abelian monopole.

In fact one can argue that the Lagrangian \([4]\) describes a theory of monopole \([4, 12]\). To see this notice that \([4]\) can be put into a very suggestive form,

$$\mathcal{L} = -\frac{\alpha}{4} \tilde{H}^2_{\mu\nu} - \frac{\mu^2}{2} \tilde{C}^2_{\mu},$$

$$\tilde{H}_{\mu\nu} = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu + g \tilde{C}_\mu \times \tilde{C}_\nu, \quad (5)$$

where $\tilde{C}_\mu$ is the “Cho connection” \([13, 14, 15, 16]\)

$$\tilde{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}. \quad (6)$$

Clearly this demonstrates that the Skyrme theory is deeply related to QCD. Just like the $SU(2)$ QCD the Lagrangian has the non-Abelian monopole solution \([4]\)

$$\hat{n} = \hat{r}, \quad (7)$$

where $\hat{r}$ is the unit radial vector. Notice that the potential $\tilde{C}_\mu$, with \([4]\), becomes nothing but the well-known Wu-Yang monopole potential \([12, 13]\). Moreover, the above solution becomes a solution even without the non-linear interaction (i.e., with $\alpha = 0$), which justifies the interpretation that the Skyrme theory is indeed a theory of monopole (interacting with the Skyrme field $\omega$). But one has to keep in mind that this monopole is not an electromagnetic monopole, but rather a non-Abelian chromomagnetic one. Notice that

$$\tilde{H}_{\mu\nu} = H_{\mu\nu} \hat{n} = -\frac{1}{g} \partial_\mu \hat{n} \times \partial_\nu \hat{n} = -\frac{1}{g} N_{\mu\nu} \hat{n}, \quad (8)$$

so that in this picture $N_{\mu\nu}$ becomes nothing but the Abelian chromomagnetic field of the $U(1)$ gauge symmetry in \([8]\).

Now, we argue that the Lagrangian \([4]\) can also be viewed to describe a $CP^1$ model which describes a two-component superfluid \([6, 8]\). To see this let $\xi$ be a $CP^1$ field which forms an $SU(2)$ doublet and consider the $CP^1$ Lagrangian

$$\mathcal{L} = -\frac{\mu^2}{2} \left( |\partial_\mu \xi|^2 - |\xi|^2 |\partial_\mu \xi|^2 \right) - \frac{\alpha}{4} (\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi)^2, \quad (\xi^\dagger \xi = 1). \quad (9)$$

But with the identification

$$\hat{n} = \xi^\dagger \partial_\mu \xi, \quad (10)$$

we have

$$(\partial_\mu \hat{n})^2 = 4 (|\partial_\mu \xi|^2 - |\xi|^2 |\partial_\mu \xi|^2),$$

$$N_{\mu\nu} = \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = 2i (\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi) = \partial_\mu C_\nu - \partial_\nu C_\mu, \quad (11)$$

where $C_\mu$ is the velocity potential of the doublet $\xi$ \([8]\)

$$C_\mu = 2i \xi^\dagger \partial_\mu \xi. \quad (12)$$

This tells that the three Lagrangians \([3, 4, 8]\), and \([4]\) are all identical to each other, which confirms that the Skyrme-Faddeev theory could also be regarded as a theory of two-component superfluid. In this view, however, $N_{\mu\nu}$ in \([4]\) acquires a new meaning. It now describes the vorticity field of the superfluid $\xi$. It is really remarkable that the theory allows such different interpretations.

To understand the physical meaning of the Faddeev-Niemi knot one has to understand the helical baby skyrmion first. To construct the desired helical vortex we let $(\varphi, \varphi, z)$ the cylindrical coordinates, and choose the ansatz

$$\xi = \left( \begin{array}{c} \cos \left( \frac{f(\varphi)}{2} \right) \exp\left( -imkz - in\varphi \right) \\ \sin \left( \frac{f(\varphi)}{2} \right) \end{array} \right),$$

$$\hat{n} = \xi^\dagger \partial_\mu \xi = \left( \begin{array}{c} \sin \left( \frac{f(\varphi)}{2} \right) \cos \left( mkz + n\varphi \right) \\ \sin \left( \frac{f(\varphi)}{2} \sin \left( mkz + n\varphi \right) \end{array} \right),$$

$$C_\mu = (\cos f(\varphi) + 1) (mk \partial_\varphi z + n \partial_\varphi \varphi). \quad (13)$$

With this the equation \([4]\) is reduced to

$$\left( 1 + \left( m^2 k^2 + \frac{n^2}{q^2} \sin^2 \frac{f}{g} \right) \hat{f} + \left( \frac{1}{q} + \frac{2\hat{\rho}}{\rho} \right) \right) \hat{f} + \left( \frac{1}{g^2} \sin^2 \frac{f}{g} \right) \hat{f} + \frac{1}{g^2} \left( m^2 k^2 - \frac{n^2}{q^2} \sin^2 \frac{f}{g} \right) \hat{f} + \left( m^2 k^2 + \frac{n^2}{g^2} \right) \sin \cos f = 0. \quad (14)$$

So with the boundary condition

$$f(0) = \pi, \quad f(\infty) = 0, \quad (15)$$

we obtain the non-Abelian vortex solutions shown in Fig.1. There are three points that have to be emphasized here. First, when $m = 0$, the solution describes
the well-known baby skyrmion $\frac{\pi}{2}$. But when $m$ is not zero, it describes a helical vortex which is periodic in $z$-coordinate. In this case, the vortex has a non-vanishing velocity potential (not only around the vortex but also) along the $z$-axis. Secondly, the superfluid velocity starts from the second component at the core, but the first component takes over completely at the infinity. This is due to the boundary condition $f(0) = \pi$ and $f(\infty) = 0$, which assures that our solution describes a genuine non-Abelian vortex. Thirdly, $C_\mu$ and $N_{\mu\nu}$, here can also be interpreted as the chromomagnetic potential and field, so that one can view the helical vortex a twisted magnetic vortex confined along the $z$-axis. This allows us to identify the baby skyrmion as the magnetic flux line which connects the monopole-antimonopole pair separated infinitely apart.

Remarkably the helical vortex has two distinct chromomagnetic fluxes. To see this notice first that it has a quantized magnetic flux along the $z$-axis,

$$\phi_z = \int H_{\varepsilon z} d\varphi d\varphi = -\frac{4\pi i}{g} \int (\partial_\varphi \xi^1 \partial_\varepsilon \xi - \partial_\varepsilon \xi^1 \partial_\varphi \xi) d\varphi = \frac{4\pi n}{g}. \quad (16)$$

But due to its helical structure it also has a quantized magnetic flux around the $z$-axis (in one period section from $0$ to $2\pi/k$ in $z$-coordinate) given by

$$\phi_\varphi = -\int H_{\varepsilon \varphi} dz d\varphi = \frac{4\pi i}{g} \int (\partial_\varepsilon \xi^1 \partial_\varphi \xi - \partial_\varphi \xi^1 \partial_\varepsilon \xi) \frac{d\varphi}{k} = \frac{4\pi m}{g}. \quad (17)$$

Obviously these quantized magnetic fluxes come from the quantized magnetic potential $C_\mu$ in (16), which in turn originates from the twisted topology of the helical vortex.

The helical vortex will become unstable unless the periodicity condition is enforced by hand. But for our purpose it plays a very important role, because it allows us to construct the Faddeev-Niemi knot. To understand this, notice that we can make it a vortex ring by smoothly connecting two periodic ends (or by twisting the monopole-antimonopole flux and putting the monopole and antimonopole together). Remarkably, this vortex ring naturally acquires the topology of a knot, and thus becomes a knot itself. This is because by construction this knot carries two magnetic fluxes, $m$ unit of flux passing through the knot disk and $n$ unit of flux passing along the knot. Moreover the two fluxes can be thought of two unit flux rings linked together winding each other $m$ and $n$ times, whose linking number becomes $mn$. This is a dynamical manifestation of the knot quantum number. Notice that the knot topology has always been described by the Hopf mapping $\pi_3(S^2)$. When $\pi_3(S^2)$ is non-trivial the preimages of any two points in $S^2$ forms two rings linked together, whose linking number is described by the Chern-Simon index of the potential $C_\mu$.

$$Q = \frac{1}{32\pi^2} \int \epsilon_{ijk} C_i N_{jk} d^3x = -\frac{1}{4\pi^2} \int \epsilon_{ijk} \xi^1 \partial_\xi \partial_\varepsilon \xi d^3x = mn. \quad (18)$$

This is the mathematical definition of the knot quantum number. But here we have shown that this knot quantum number is precisely the linking number of two magnetic flux rings, which have nothing to do with the preimages of the Hopf mapping. This tells that the knot structure is manifest even at the dynamical level. This point has not been well appreciated so far. Notice that, with the Hopf fibering of $S^3$ to $S^2 \times S^1$, the knot quantum number can also be viewed to represent the mapping $\pi_3(S^2)$ of $\tilde{n}$ or $\pi_3(S^3)$ of $\hat{\xi}$.

Clearly the knot has a topological stability, because two flux rings linked together cannot be disjointed by a smooth deformation of the field configuration. But the above analysis tells that the topological stability is now backed up by the dynamical stability. To see this, notice that the quantized chromomagnetic flux of the rings can be thought to come from the chromoelectric supercurrent

$$j_\mu = \frac{1}{4\pi} \partial_\nu H_{\mu\nu} = \frac{1}{4\pi g} (\partial^2 C_\mu - \partial_\mu \partial_\nu C_\nu), \quad (19)$$

which also has two components, the component moving along the knot, and the one moving around the knot tube. Now it must be clear that the supercurrent moving along the knot generates an angular momentum around the $z$-axis which provides the centrifugal force preventing the vortex ring to collapse. Put it differently, the supercurrent generates the $m$ unit of the magnetic flux trapped in the knot disk which can not be squeezed out. And this flux provides a stabilizing repulsive force which prevent the collapse of the knot. This is how the knot acquires the dynamical stability. It is this remarkable interplay between topology and dynamics which assures the existence of the stable knot in Skyrme theory. The nontrivial
topology of the magnetic flux rings which provides the topological stability now expresses itself in the form of the supercurrent and angular momentum which provides the dynamical stability of the knot.

The above analysis also makes it clear that alternatively the knot can also be viewed as a two quantized vorticity rings linked together in a two component superfluid, whose linking number gives the knot quantum number.

The energy of the Faddeev-Niemi knot is known to have the following bound \[ c_1 Q^{3/4} \leq E_Q \leq c_2 Q^{3/4}, \] (20)
which implies that the energy is proportional to \( Q^{3/4} \). Numerically this has been confirmed up to \( Q = 8 \) \[ 18 \]
\[ E_Q \simeq 252 Q^{3/4} \sqrt{\alpha \mu} \] (21)
This means that knot with large \( Q \) can not decay to the knots with smaller \( Q \).

We close with the following remarks.
1. In this paper we have clarified the physical meaning of topological objects in Skyrme theory. In particular, we have shown that the Faddeev-Niemi knot in Skyrme theory is nothing but the chromomagnetic vortex ring made of a monopole-antimonopole flux, twisted and connected together. This picture allows us to interpret the knot as two quantized flux rings linked together, whose knot quantum number is given by the linking number of the rings. This interpretation follows from the fact that the Lagrangian \( \mathcal{L} \) can be viewed as a massive Yang-Mills Lagrangian, which emphasizes the deep connection between the Skyrme theory and QCD \[ 12 \].

2. Our analysis tells that the Lagrangian \( \mathcal{L} \) could also be understood to describe a theory of two-component superfluid. This implies that it could play an important role in condensed matter physics, which puts the Skyrme theory in a totally new perspective. The Skyrme theory has always been associated to nuclear and/or high energy physics. But now it becomes clear that the theory could also describe interesting low energy physics in a completely different environment, in two-component condensed matters \[ 6,8 \]. This is really remarkable.

3. In our analysis we have outlined how one can actually construct the Faddeev-Niemi knot (or a similar one) in laboratories. So the challenge now is to verify the existence of the topological knot experimentally. Constructing the knot may be a tricky task at present moment, but might have already been done in two-component Bose-Einstein condensates \[ 19,21 \]. We predict that similar knots could be constructed in laboratories in the near future.

A detailed discussion on the subject will be published elsewhere \[ 21 \].

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