Onset of ferromagnetism for strongly correlated electrons in one-dimensional chains

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The existence of the Nagaoka ferromagnetism is examined in the context of the one-dimensional $U = \infty$ Hubbard model. We construct the exact quantum partition function to describe the physics of such a regime. Our calculation reveals that, while the ground-state in an open chain is always spin-degenerate, in a finite size closed chain with at least one vacancy, the ground-state is ferromagnetic whenever the number of electrons is less or equal to three. Our results shed more light on a very recent experimental verification of the Nagaoka ferromagnetism in a quantum dots set up.

Introduction.—The interplay between strong correlations and the onset of ferromagnetism in itinerant electronic systems has long intrigued the many-body physics community. In this respect, a rigorous result obtained from a single band Hubbard model (HM) in the early 1960s by Nagaoka [1] stands as one of the most prominent theoretical landmarks. In a nutshell, the Nagaoka theorem (NT) establishes that, for certain lattices, in the infinitely coupled regime, the presence of a single vacancy in the almost half-filled system yields a totally polarized ferromagnetic (FM) ground-state. Unfortunately, until very recently, the experimental verification of itinerant magnetism in such a regime seemed unattainable. Despite the great success of the quantum simulations of the HM in cold atoms [2], no observation of such a FM ground-state has been reported so far in those systems. Conversely, semiconductor based quantum dot arrays are systems that have also attracted a lot of interest as viable alternatives to realize experimentally the physics of the HM [3]. As a result, it was not surprising that the first experimental verification of the Nagaoka result was produced in a small scale quantum dot set up [4]. This new experiment, especially prepared for such a purpose, consists of a 4-site quantum dot plaquette filled with 3 electrons.

In such small quantum systems, it is perfectly possible to approach this problem by means of an exact diagonalization of the Hamiltonian for a fixed number of electrons. Indeed, by taking into account both distant-neighbor hopping and Coulomb couplings, another theoretical work has provided further evidence that the FM ground-state of 3 electrons in the 4-site plaquette is robust in the presence of long range Coulomb interactions [5]. In addition to this they also consider the case of 4 electrons in a 5-site ring, which no longer displays Nagaoka ferromagnetism. However, notwithstanding the simplicity of the HM, we are still not able to make general claims about what happens to the Nagaoka result if the Hubbard $U$ coupling is reduced from its infinite value, or if the number of vacancies is kept finite in the thermodynamic regime [6]. On top of that, it is well known that most of the conventional mean-field approximations and perturbation schemes are both ineffective to deal with such a strongly correlated regime. Particularly in one-dimension, the NT is not directly applicable, and for a long time it was believed that such a FM ground-state might even contradict the Lieb-Mattis theorem (LMT) [7]. However this is not the case. In view of the fact that the LMT only applies to open chains, the existence of itinerant ferromagnetism in closed chains was left entirely open until now.

In this letter, we present an alternative calculation for the exact quantum partition function of the $U = \infty$ HM in one-dimensional chains of finite size. Since our analysis already encompasses scenarios for different numbers of electrons and lattice sizes, we are able to make precise statements about the condition for the onset of itinerant ferromagnetism in closed chains. Our results are in full agreement with the experimental observations reported in [4]. Moreover we make new predictions which can also be tested experimentally.

Method.—We will consider the infinite coupling regime of the HM in one spatial dimension. To begin with, the HM describes, otherwise, free band electrons interacting via a on-site repulsive interaction of strength $U$ [8]. For a single conduction band, the corresponding HM Hamiltonian reads

$$H_{\text{HM}} = - \sum_{ij} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}. \quad (1)$$

where $c_{i\sigma}^\dagger$ ($c_{i\sigma}$) is the fermionic operator that creates (annihilates) an electron on the lattice site $i$ with spin projection $\sigma = \uparrow, \downarrow$ and the operator $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$ stands for the on-site spin-$\sigma$ electron number. The quantum dynamics of the Hamiltonian (1) preserves the numbers of spin-up and spin-down electrons separately since $[H, N_{\sigma}] = 0$, where $N_{\sigma} = \sum_i n_{i\sigma}$. This implies that both the total electron number $N = N_{\uparrow} + N_{\downarrow}$ and the total spin projection $J_z = \frac{1}{2} (N_{\uparrow} - N_{\downarrow})$ are conserved in the system. Henceforth, the eigenenergies of (1) can always be labelled by the quantum numbers of $N$ and $J_z$, or, equivalently, of $N_{\uparrow}$ and $N_{\downarrow}$. We restrict ourselves to the so called hole-doped scenario with $N \leq L$, $L$ being the total number of sites. This can be done without loss of
generality since, by performing an appropriate particle-hole transformation in (1), one can always recover the electron-doped energy solutions as well.

Now we turn to the large coupling limit of the HM. If the $U$ coupling becomes the dominant energy scale in the system, the doubly-occupied electron states immediately fall into disfavour. Indeed, in the $U = \infty$ limit, they are removed altogether from the set of available on-site states, and the hole-doped HM Hamiltonian (1) is then reduced to the projected hopping term

$$H = - \sum_{ij} t_{ij} X^a_{i0} X^a_{j0}. \quad (2)$$

Here $X^{ab} = |a\rangle \langle b|$ with $\{|a\rangle\} = \{|\downarrow\rangle, |\uparrow\rangle, |0\rangle\}$ are the standard on-site Hubbard operators [9]. From now on, for simplicity, the hopping amplitudes $t_{ij}$ are assumed to be non zero only for nearest neighbor sites $i$ and $j$. We work in the grand-canonical ensemble, and the quantum partition function associated with the Hamiltonian (2) is given by

$$Z = \text{tr} \ e^{-\beta(H - \mu \sum_i X^a_{i0})}. \quad (3)$$

Here $\beta$ is the inverse temperature, and the trace is taken over a complete set of $su(2|1)$ space states [10]. Notice that a chemical potential $\mu$ was introduced to keep track of the number of vacancies. Naturally, this number is always equal to the difference between the number of sites $L$ and the total number of electrons $N$.

The open chain.—In an open chain, the spectrum of the Hamiltonian (2) is completely degenerate with respect to the spin configurations, and the partition function can be evaluated with relative ease. The physical intuition behind this degeneracy is quite simple to understand. The existence of boundaries, in addition to the impossibility of exchanging their relative ordering, automatically prevents the projected electrons to access different spin configurations. For example, although the states $|\downarrow 00 \downarrow\rangle$ and $|\uparrow 00 \uparrow\rangle$, representing two electron states in four sites, belong to the same $N$ and $J$ subspaces, they are dynamically inaccessible to each other. Consequently, for each spin configuration, these projected electrons behave essentially as spinless fermions. Hence, the resultant partition function for the open chain is just

$$Z^o(\beta, z) = \prod_{p} \left( z + 2 e^{-\beta t_p} \right), \quad (4)$$

where $z = e^{\beta \mu}$ is the fugacity, $t_p = -2t \cos p$ is the electron dispersion and $\Omega$ is a set of momenta defined as

$$\Omega = \{p : p_0 = \frac{\pi n}{L}, n = 1, \ldots, L \}. \quad$$

It is worthwhile to notice that, when we turn off the hopping ($t = 0$), formula (4) reduces straightforwardly to $Z = (2 + z)^L$. This is a key property, since, if we further remove the chemical potential by taking $z = 1$, one is able to recover the appropriate number of degrees of freedom associated with the projected Hamiltonian (2).

From the partition function formula (4) one can have access, not only to the full spectra of the system, but also to some interesting finite temperature effects. For instance, if one computes the occupation number as a function of the chemical potential and temperature, the result is no longer the conventional Fermi-Dirac distribution. This simplifies that, despite its simplicity, the system never ceases to have a strongly correlated nature. Nevertheless, in the zero temperature regime ($\beta \to \infty$), $Z^o(\beta, z)$ has a quite simple asymptotic behavior and the expression

$$E^{GS} = -\lim_{\beta \to \infty} \left( \frac{\beta}{z} \ln Z^o(\beta, z) \right)_z$$

for the corresponding ground-state energy reduces to

$$E^{GS} = t - t \csc \frac{\pi}{L+1} \left( \frac{1}{2} \right) \sin \frac{\pi}{L+1} \left( \frac{2N+1}{2} \right). \quad (5)$$

Moreover, in the thermodynamic limit ($L \to \infty, N/L \to n_e$), the formula above assumes the form

$$\frac{E^{GS}}{L} = -\frac{2t}{\pi} \sin(\pi n_e), \quad n_e \leq 1, \quad (6)$$

which coincides with the exact result obtained earlier by Ogata and Shiba [11] making use of the Bethe ansatz.

The closed chain.—In a closed chain the situation is different. Without the boundaries to restrain the moving particles, they become free to jump around the loop and to permute cyclically their spin positions. Certainly, not all spin configurations are equivalent to each other and, in view of that, the spin degeneracy is partially lifted. While a fully polarized spin state, e.g., $|\uparrow\uparrow\downarrow\downarrow\rangle$, only has translational degrees of freedom, which can be related solely to the number of vacancy positions, the same does not generally hold to other spin states. One example of that is the state $|\uparrow\uparrow\downarrow\downarrow\downarrow\rangle$, which can also access dynamically all the other states which are cyclic permutations of these spins, e.g., $|\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\rangle$ and $|\downarrow\downarrow\downarrow\uparrow\downarrow\rangle$.

Thanks to this feature, the HM in a closed chain can be considered as an example of a quantum necklace. Indeed, the number of dynamically disconnected subspaces $D_s(N)$ in this model is equal to the number of distinct necklaces that can be made with the $N$ projected fermions of spin-$s$. Using the Burnside’s lemma [12], it turns out that

$$D_s(N) = \frac{1}{N} \sum_{d | N} \varphi \left( \frac{N}{d} \right) (2s + 1)^d, \quad (7)$$

where $\varphi(z)$ is the Euler’s totient function, which is defined as the number of positive integers between 1 and $x$ that are coprime to $x$. Here $d | N$ stands for a sum over the natural divisors of $N$. In particular, the first few
values of $D_s(N)$ for projected electrons are

$$
D_{1/2}(1) = 2, \quad D_{1/2}(2) = 3, \quad D_{1/2}(3) = 4,
D_{1/2}(4) = 6, \quad D_{1/2}(5) = 8, \quad D_{1/2}(6) = 14,
D_{1/2}(7) = 20, \quad D_{1/2}(8) = 36, \quad D_{1/2}(9) = 60.
$$

(8)

To find the corresponding spectra, now we just need to diagonalize separately each one of those distinct necklaces. The important point is that these necklaces can be further classified according to their irreducible cyclic symmetry of the spin configuration $C_d$, where $d$ is a natural divisor of $N$. For such a $C_d$ necklace, the projected electron momenta are quantized in the form

$$
p_n = \frac{\pi}{L}(2n + \eta) + \frac{2\pi}{NL}\nu,
$$

(9)

where $n = 0, \ldots, L - 1$, $\eta = (N + 1) \text{mod} \ 2$, and $\nu = 0, \frac{N}{d}, \ldots, (d - 1)\frac{N}{d}$. The formula (9) is essentially made of the contributions $\frac{2\pi}{L} n$, which results from translational invariance; $\frac{2\pi}{T}\eta$, which is directly associated with the fermionic total antisymmetry; and, finally, $\frac{2\pi}{NL}\nu$, which corresponds to the $C_d$ cyclic invariance of the spin configuration. The physical intuition behind this momentum shift is the relative movement of the “spin background” as the vacancies move along the sites. Making use of all these ingredients, we are in a position to write the partition function as

$$
Z^c(\beta, z) = \sum_{N=0}^{L} z^{L-N} Z^c_N,
$$

(10)

where $Z^c_N$ is the canonical partition function for the $N$ electrons,

$$
Z^c_N = \sum_{\nu=0}^{N-1} M^N_\nu \sum_{p_1 < p_2 < \cdots < p_N} e^{-\beta(t_{p_1} + t_{p_2} + \cdots + t_{p_N})}.
$$

Here $\Gamma(\nu)$ is the set defined as $\Gamma(\nu) = \{ p : p_n = \frac{\pi}{L}(2n + \eta) + \frac{2\pi}{T}\nu, n = 0, \ldots, L - 1 \}$ while the symbol $M^N_\nu$ denotes the degeneracy factor. The set $\{ M^N_\nu \}$ for each $N$ can be determined by identifying each one of these symbols with the number of distinct necklaces contributing to a particular solution $\nu$ [13]. For our present purposes, it suffices to extract from (9) the fact that $M^N_0 = D_s(N)$ and $M^N_{\nu} \leq D_s(N) - (2s + 1)$ for all $\nu > 0$. In particular, whenever $N$ is a prime number, all the inequalities are saturated and $Z^c_N$ can be explicitly written as

$$
Z^c_N = D_s(N) \sum_{\Gamma(0)} e^{-\beta(t_{p_1} + t_{p_2} + \cdots + t_{p_N})} + [D_s(N) - (2s + 1)] \sum_{\nu=1}^{N-1} \sum_{\Gamma(\nu)} e^{-\beta(t_{p_1} + t_{p_2} + \cdots + t_{p_N})}.
$$

(11)

Notice that, for $N = L$, as the projected electrons cannot move, the energy $E = t_{p_1} + \cdots + t_{p_N}$ in (11) invariably vanishes, and the canonical partition function formula produces the correct spin degeneracy: $Z^c_{N=L} = 2^L$.

Similarly to what happens with the open chain, $Z^c(\beta, z)$ also reduces to $Z = (2 + z)^L$ at $t = 0$. Additionally to that, it looses its spin dependence as the system approaches the thermodynamic limit. This takes place because, as $L \to \infty$, all the $\Gamma(\nu)$ sets become isomorphic to $S^L$. In fact, this is no surprise since the solutions for the open and the closed boundary conditions must approach each other in this limit, i.e.,

$$
\lim_{L \to \infty} \frac{Z^c(\beta, z)}{Z^o(\beta, z)} = 1.
$$

(13)

Nagaoka ferromagnetism in one dimensional chains.— Following Nagaoka [1], the ground-state is said to be FM only if it consists solely of the maximum total spin states, and there is no other lower total spin state which is degenerate to it. Thus, since in the $U = \infty$ limit, the spectra is completely degenerate with respect to spin in an open chain, its ground-state is always non-magnetic. Certainly, as discussed above, in real systems, the $t/U$ corrections will lift such a degeneracy. However, the absence of Nagaoka ferromagnetism, even in this finite coupling limit, is reassured by the LMT [7].

In contrast with that, for a closed chain, the itinerant ground-state degeneracy is equal to the number of disconnected subspaces $D_{1/2}(N)$. This, together with the fact that the maximum total spin states always contribute to the ground-state [14], implies that the ground-state is FM only if $D_{1/2}(N)$ is equal to the number of maximum total spin $J_{\text{max}} = \frac{1}{2} N$ states, i.e., if $D_{1/2}(N) = N + 1$. Therefore, from (8), we can conclude that the ground-state of the $U = \infty$ HM is FM for any finite closed chain as long as the number of electrons is no greater than three and there is at least one vacancy [15]. For instance, while a system with $N = 3$ electrons has only the four polarization states of total spin $\frac{3}{2}$ in the ground-state, for $N = 4$, the system has five polarization states of total spin-2 together with a single state of total spin-0 in the ground-state. Using the same reasoning, a closed chain of size $L = 5$ with two vacancies exhibits a FM ground-state, although the same chain has a non-magnetic ground-state.
The key point is that, differently from the standard NT condition which is characterized by the presence of a single vacancy, for closed one-dimensional chains, the existence or not of such a FM ground-state is directly determined by the total number of electrons in the system. The present result can be considered as a natural extension of the NT for one dimensional chains. In fact, these two concepts are complementary to each other and, for the $L = 4$ case, which can be interpreted either as a $2 \times 2$ array or a closed chain, they equally predict a FM ground-state if only a single vacancy is present in the system.

However, the feasibility to observe such phenomenon in closed chains will certainly depend on the precision of the instruments to measure the energy difference between the FM ground-state and the first unpolarized excited-state. In particular, for $N = 3$ electrons, the expected energy gap between the lowest $J = \frac{3}{2}$ and $J = \frac{1}{2}$ spin states is

$$E_{1/2} - E_{3/2} = 4t \sin^2 \left( \frac{\pi}{3L} \right) \left[ 1 + 2 \cos \left( \frac{2\pi}{L} \right) \right].$$

In Figure 1, we plot the values of this energy gap as a function of the chain size $L$, for $3 \leq L \leq 30$. As displayed there, the energy gap decreases as $L$ increases and eventually approaches zero in the $L \rightarrow \infty$ limit. This is a clear indication that there is no FM ground-state in the thermodynamic limit. Thus such a FM state is most likely to be detected in chains with a small number of sites. In particular, since the $L = 4$ case was already verified experimentally [4], the $L = 5$ case, which produces the maximal energy gap for $N = 3$ electrons can, in principle, be readily detected by similar experimental set ups.

**Robustness of the solution for finite values of $U$.** —We use a particle-hole transformation in (1) to write the ground-state energy density in the thermodynamic limit

$$\frac{E^{GS}}{L} = U(n_e - 1) + \frac{2t}{\pi} \sin(\pi n_e), \quad 1 \leq n_e \leq 2.$$  

In Figure 2, we plot $E^{GS}$ for all electronic densities by asserting a finite scale for $U$. At this point, one may wonder about the robustness of such a solution in a situation where $U$ is large but still finite. In this case, equation (6) for the ground-state energy is, in fact, the leading order correction of a series expansion in powers of $t/U$. However, even for $U \gg t$, such an approximation is only acceptable for electronic densities in which $|\frac{2t}{\pi} \sin(\pi n_e)|$ is much larger than any other energy contributions produced by the next order correction terms. To illustrate this point, we can consider a well-known scenario. At half-filling ($n_e = 1$), the kinetic energy contribution vanishes and, thanks to an emergent second order Heisenberg term, the system is naturally led to an antiferromagnetic ground-state.

**Connection to heavy-fermions.** —The same arguments are applicable to some heavy-fermion systems. This is suggested in view of the equivalence between the infinitely coupling regime of the Kondo lattice model (KLM) and the $U = \infty$ HM [16]. In the infinite Kondo coupling regime, the itinerant electrons become completely trapped into on-site singlet configurations with the localized spins in the lattice. Within this scenario, these composite singlet states can hop around the array of localized spins in the same way as the vacancies would do in a related electronic background. As a matter of fact, our conclusions can be easily adapted for those compounds and are in agreement with Sigrist et al. [17].

**Conclusion.** —We study the quantum dynamics of a projected hopping Hamiltonian in one-dimension and construct the corresponding exact partition function. Our approach not only enables us to extract a precise condition for the onset of itinerant ferromagnetism in closed chains. It also provides detailed information about the excited energy levels of those systems. Such exact
complementary results paves the way for a better understanding of quantum magnetism and strong electronic correlations in low dimensions.

This work was supported by the CAPES agency – Brazil – Finance Code 001. One of us (AF) also acknowledges the financial support from the Ministry of Education (MEC) and from the CNPq agency – Brazil.

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[13] See supplementary text for additional details in how to determine such degeneracy factors.
[14] Since the maximal $J^z = \pm sN$ states are certainly in the ground-state, the global spin $SU(2)$ symmetry tell us that, in fact, all the $2sN + 1$ maximum total spin states are in the ground-state.
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Supplementary material for: Onset of ferromagnetism for strongly correlated electrons in one-dimensional chains

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In this supplementary material, we show in more detail how to determine the degeneracy factors that appeared in our discussion of the closed chain. To clarify our reasoning, we also provide here a generalization of the condition for itinerant ferromagnetism and include some explicit examples at the end.

Necklaces and degeneracy factors for the closed chain.—Clearly, by taking into account the total number of spin states, the degeneracy factors $M_N^d$ must satisfy the condition $(2s+1)^N = \sum_{\nu=0}^{N-1} M_N^\nu$. However, the essential point to understand here is their relation with the number of necklaces that contribute for a particular solution $\nu$ in the momenta quantization. Following the general properties of the cyclic permutation group $C_N$, each one of the $D_s(N)$ necklaces formed with the $N$ projected particles can be further classified according to their irreducible cyclic symmetry $C_d$, where $d$ is a natural divisor of $N$. In general, as given in the main text, for a spin necklace of irreducible cyclic symmetry $C_d$, the momenta of the projected particles is quantized as

$$C_d: \quad p_n = \frac{\pi}{L} (2n + \eta) + \frac{2\pi}{NL} \nu,$$

where $n = 0, \ldots, L - 1$, $\eta = (N + 1) \mod 2$, and $\nu = 0, \frac{N}{d_1}, \ldots, (d - 1) \frac{N}{d}$. From it, we can see already that all the necklaces will contribute to the itinerant ground-state ($\nu = 0$), and, thus, its degeneracy factor, in this case, is equal to the total number of distinct necklaces, i.e., $M_0^N = D_s(N)$. The remaining $M_N^d$ factors can be determined in the same manner, and depend on the particular divisor structure of the cyclic group $C_N$. In fact, one can also extract from (S1) that, if $\nu$ and $\nu'$ share the same greatest common divisor with $N$, i.e., $\gcd(N, \nu) = \gcd(N, \nu')$, their degeneracy factors are equal $M_N^\nu = M_N^{\nu'}$ as well. At this point we highlight a case of particular simplicity. If $N$ is a prime number, the necklaces are only irreducible symmetric to $C_1$ or $C_N$, and, therefore, it turns out that $M_0^N = D_s(N) = \frac{1}{L} \left( (2s+1)^N - (N - 1)(2s + 1) \right)$, and $M_0^N = D_s(N) - (2s + 1)$.

On the condition for itinerant ferromagnetism of spin-$s$ projected fermions.—Due to the global spin $SU(2)$ symmetry, the energies corresponding to the same total spin multiplets are all degenerate. Moreover, since the $J^z = \pm s N$ states (which are $C_1$ irreducible necklaces) certainly are in the ground-state, in fact, there will be a total of $(2sN + 1)$ states of maximum total spin in the ground-state. Therefore, this ground-state can only be FM if there is no other total spin contribution there, i.e., if the condition $D_s(N) = 2s N + 1$ is satisfied. As in the main text, from this formula, follows that the ground-state of this system with at least one vacancy is FM whenever the number of electrons is no greater than three. Furthermore, it does not take much to see that, if the projected particle has a higher value for the spin, $s > \frac{1}{2}$, the same equation only provides physical solutions for $N = 1$. Some values of $D_s(N)$ for $s > \frac{1}{2}$ are given below:

$$D_1(1) = 3, \quad D_1(2) = 6, \quad D_1(3) = 11, \quad D_1(4) = 24;$$
$$D_{3/2}(1) = 4, \quad D_{3/2}(2) = 10, \quad D_{3/2}(3) = 24, \quad D_{3/2}(4) = 70;$$
$$D_2(1) = 5, \quad D_2(2) = 15, \quad D_2(3) = 45, \quad D_2(4) = 165. \quad (S2)$$

In particular, for two projected fermions of spin-1, the itinerant ground-state is $D_1(2) = 6$ degenerate, but only 5 of those states correspond to $J = 2$ total spin states.

![FIG. S1: The six distinct necklaces that can be formed with $N = 4$ projected electrons. The vacancies are omitted in here because, in spite of their movement, the projected hopping between nearest neighbors is unable to change the topology of the spin configuration. Each colour represents one of the two spin polarization states.](image-url)

To conclude these brief notes, let us solve explicitly some examples for spin-$\frac{1}{2}$ fermions. First, consider the scenario with just $N = 4$ electrons. In addition to $C_4$, there are also necklaces whose irreducible cyclic symmetry are $C_1$ or...
As one can identify in Figure S1, 2 of those necklaces are irreducible symmetric to \( C_1 \), and just 1 is irreducible symmetric to \( C_2 \). Thus, from the identities given above, \( M_1^3 = 6 \), \( M_1^4 = M_2^3 = 3 \) and \( M_2^5 = 4 \). If we now move to the case with \( N = 8 \) electrons, the relevant cyclic groups are \( C_8 \), \( C_4 \), \( C_2 \) and \( C_1 \). However, besides the number of electrons has changed, the number of necklaces irreducible symmetric related to a particular \( C_n \) follow the same pattern as before. Namely, there still are only 2 necklaces irreducible symmetric to \( C_1 \), 1 to \( C_2 \), and 3 to \( C_4 \). Therefore, \( M_8^3 = 36 \), \( M_8^4 = M_8^5 = M_8^6 = M_8^7 = 30 \), \( M_8^8 = 33 \) and \( M_8^9 = 34 \). Following this prescription [S1], we write down the explicit formulas of \( Z_N^C \) for the first few values of \( N \):

\[
Z_0^C = 1, \quad (S3a)
\]

\[
Z_1^C = 2 \sum_p \Gamma(0) e^{-\beta t_p}, \quad (S3b)
\]

\[
Z_2^C = 3 \sum_{p_1 < p_2} \Gamma(0) e^{-\beta (t_{p_1} + t_{p_2})} + \sum_{p_1 < p_2} \Gamma(1) e^{-\beta (t_{p_1} + t_{p_2})}, \quad (S3c)
\]

\[
Z_3^C = 4 \sum_{p_1 < p_2 < p_3} \Gamma(0) e^{-\beta (t_{p_1} + t_{p_2} + t_{p_3})} + 2 \sum_{p_1 < p_2 < p_3} \Gamma(1) e^{-\beta (t_{p_1} + t_{p_2} + t_{p_3})} + 2 \sum_{p_1 < p_2 < p_3} \Gamma(2) e^{-\beta (t_{p_1} + t_{p_2} + t_{p_3})}, \quad (S3d)
\]

\[
Z_4^C = 6 \sum_{p_1 < p_2 < p_3 < p_4} \Gamma(0) e^{-\beta (t_{p_1} + \cdots + t_{p_4})} + 3 \sum_{p_1 < p_2 < p_3 < p_4} \Gamma(1) e^{-\beta (t_{p_1} + \cdots + t_{p_4})} + 4 \sum_{p_1 < p_2 < p_3 < p_4} \Gamma(2) e^{-\beta (t_{p_1} + \cdots + t_{p_4})} \quad (S3e)
\]

\[
Z_5^C = 8 \sum_{p_1 < \cdots < p_5} \Gamma(0) e^{-\beta (t_{p_1} + \cdots + t_{p_5})} + 6 \sum_{p_1 < \cdots < p_5} \Gamma(1) e^{-\beta (t_{p_1} + \cdots + t_{p_5})} + 6 \sum_{p_1 < \cdots < p_5} \Gamma(2) e^{-\beta (t_{p_1} + \cdots + t_{p_5})} \quad (S3f)
\]

\[
Z_6^C = 14 \sum_{p_1 < \cdots < p_6} \Gamma(0) e^{-\beta (t_{p_1} + \cdots + t_{p_6})} + 9 \sum_{p_1 < \cdots < p_6} \Gamma(1) e^{-\beta (t_{p_1} + \cdots + t_{p_6})} + 11 \sum_{p_1 < \cdots < p_6} \Gamma(2) e^{-\beta (t_{p_1} + \cdots + t_{p_6})} \quad (S3g)
\]

\[
Z_7^C = 20 \sum_{p_1 < \cdots < p_7} \Gamma(0) e^{-\beta (t_{p_1} + \cdots + t_{p_7})} + 18 \sum_{p_1 < \cdots < p_7} \Gamma(1) e^{-\beta (t_{p_1} + \cdots + t_{p_7})} + 18 \sum_{p_1 < \cdots < p_7} \Gamma(2) e^{-\beta (t_{p_1} + \cdots + t_{p_7})} \quad (S3h)
\]

\[
+ 18 \sum_{p_1 < \cdots < p_7} \Gamma(3) e^{-\beta (t_{p_1} + \cdots + t_{p_7})},
\]

\[
+ 18 \sum_{p_1 < \cdots < p_7} \Gamma(4) e^{-\beta (t_{p_1} + \cdots + t_{p_7})},
\]

\[
+ 18 \sum_{p_1 < \cdots < p_7} \Gamma(5) e^{-\beta (t_{p_1} + \cdots + t_{p_7})},
\]

\[
+ 18 \sum_{p_1 < \cdots < p_7} \Gamma(6) e^{-\beta (t_{p_1} + \cdots + t_{p_7})},
\]
\[ Z_8^c = 36 \sum_{p_1 < \cdots < p_8} e^{-\beta(t_{p_1} + \cdots + t_{p_8})} + 30 \sum_{p_1 < \cdots < p_8} e^{-\beta(t_{p_1} + \cdots + t_{p_8})} \]
\[ + 30 \sum_{p_1 < \cdots < p_8} e^{-\beta(t_{p_1} + \cdots + t_{p_8})} = 0, \ldots, L - 1 \]  
\[ \text{(S3i)} \]

\[ Z_9^c = 60 \sum_{p_1 < \cdots < p_9} e^{-\beta(t_{p_1} + \cdots + t_{p_9})} + 56 \sum_{p_1 < \cdots < p_9} e^{-\beta(t_{p_1} + \cdots + t_{p_9})} \]
\[ + 58 \sum_{p_1 < \cdots < p_9} e^{-\beta(t_{p_1} + \cdots + t_{p_9})} = 0, \ldots, L - 1 \]  
\[ \text{(S3j)} \]

As specified in the main text, all the \( \Gamma \)-sets are defined as \( \Gamma(\nu) = \{ p : p_n = \pi L (2n + \eta) + 2\pi NL \nu, n = 0, \ldots, L - 1 \} \).

[S1] \( M_{d-1}^N = \sum_{d\|N} M_{d-1}^d \) is a helpful identity to determine some of the degeneracy factors.