A Study of Holographic Dark Energy Models in Chern-Simon Modified Gravity

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Abstract This paper is devoted to study some holographic dark energy models in the context of Chern-Simon modified gravity by considering FRW universe. We analyze the equation of state parameter using Granda and Oliveros infrared cut-off proposal which describes the accelerated expansion of the universe under the restrictions on the parameter $\alpha$. It is shown that for the accelerated expansion phase $-1 < \omega_{\Lambda} < -\frac{1}{3}$, the parameter $\alpha$ varies according as $1 < \alpha < \frac{3}{2}$. Furthermore, for $0 < \alpha < 1$, the holographic energy and pressure density illustrates phantom-like theory of the evolution when $\omega_{\Lambda} < -1$. Also, we discuss the correspondence between the quintessence, K-essence, tachyon and dilaton field models and holographic dark energy models on similar fashion. To discuss the accelerated expansion of the universe, we explore the potential and the dynamics of quintessence, K-essence, tachyon and dilaton field models.

Keywords CS modified gravity · Dark energy · Holographic dark energy models

1 Introduction

The physicists and cosmologists are facing two fundamental curious problems, the “dark energy (DE)” and “dark matter (DM)”. Since last decade, the astronomical observational data collected from large scale structures, type Ia Supernovae and the cosmic microwave background anisotropy supported that our universe is in accelerated expansion [1–6]. Type
Ia supernovae observational data provided the evidences that our universe is under accelerated expansion due to an exotic energy which has negative pressure and it is so-called DE. According to the astrophysical observations [7–9], more than 95 percent of the contents of our universe are consist of DM and DE while only about 4 percent is byronic matter with negligible amount of radiation. It is more interesting that about 70 percent of the energy density is DE which is responsible of accelerated expansion of the universe. Although, a huge number of efforts have been made to resolve these issues but there is no satisfactory answer obtained till now.

A number of DE models have been discussed on the holographic principle available in literature [10–17]. Gao et al. [18] purposed some cosmological constraints on the holographic Ricci dark energy models. Adabi et al. [19] discussed the correspondence between the ghost dark energy model and Chaplygin scalar field in the framework of general relativity (GR). They investigated FRW universe containing DE and DM. K. Karami and Fehri [20] found the evolution equation as well as equation of state (EoS) parameters using holographic dark energy (HDE) model with Granda and Oliveros cut-off. Jamil et al. [21] studied the HDE problem with a varying gravitational constant taking into account flat and non-flat universe. Along-with Setare [22], he discussed the HDE issues with a varying gravitational constant, in Hörava-Lifshitz gravity. With his collaborators [23], they investigated the model of interacting DE and derive its EoS and found the correspondence between the K-essence, tachyon and dilaton scalar fields with the interacting entropy corrected new agegraphic DE model in the non-flat case of FRW universe. Jamil et al. [24] also, using Granda-Oliveros cut-off, studied the holographic dark energy model in the framework of Brans-Dicke gravity theory. Many other DE models have been investigated in different theories, for example, DE modal with quintessence [25, 26], quintom field [27–29], K-essence field [30, 31], tachyon field [32–37], dilaton field [38–40], phantom field [41–43].

The cosmic baryon asymmetry is longstanding problem of cosmology which suggests a modification in the theory of GR by introducing Chern-Simons (CS) term in inflationary process [44]. The CS modified gravity is an extension to GR introduced by Jackiw and Pi [45]. In this theory, the gravitational field is coupled with a scalar field using a parity-violating CS term. Pasqua et al. [46] investigated the HDE model using Granda-Oliveros cut-off, modified holographic Ricci dark energy model as well as they investigated a model containing higher derivatives of the Hubble parameter in the context of CS modified gravity. Jamil and Sarfraz [47] found the Ricci dark energy of Amended FRW universe in the frame work of CS modified gravity.

We organize this paper in following order. The brief review of CS modified gravity is presented in Section 2. In Section 3, we investigate the HDE model and explore the EoS parameter in the framework of CS modified gravity. The Correspondence between holographic and scalar field models is studied in Section 4. The results are summarized in the last section.

## 2 Brief Review of CS Modified Gravity

The Einstein-Hilbert action for CS modified gravity theory is given by [45]

$$S = \int d^4x \sqrt{-g} [\kappa R + \frac{\zeta}{4} \Theta * RR - \frac{\eta}{2} (g^{\mu \nu} \nabla_\mu \Theta \nabla_\nu \Theta + 2V(\Theta))] + S_{mat},$$  \hspace{1cm} (1)

where $\kappa = \frac{1}{16 \pi G}$, $\nabla_\mu$ is the covariant derivative, $R$ is the Ricci scalar, $*RR$ is called Pontryagin term defined as $*RR = *R^a_{\ b} \ e^d R^b_{\ acd}$, is topological invariant. The $R^b_{\ a cd}$ is the
Reimann tensor and $^*R_{abcd}$ is the dual Reimann tensor defined as $^*R_{abcd} = \frac{1}{2} \epsilon^{cdef} R_{befc}$. The terms $\zeta$ and $\eta$ are defined as coupling constants and the function $\Theta$ is called CS coupling field, a function of spacetime using as a deformation function. If function $\Theta$ is taken to be a constant, CS modified theory reduces to GR identically.

Now, the variation of the action with respect to metric tensor $g_{\mu\nu}$ and scalar field $\Theta$ yields two field equations of CS modified gravity \[ G_{\mu\nu} + lC_{\mu\nu} = \kappa T_{\mu\nu}, \] \[ g^{\mu\nu} \nabla_\mu \nabla_\nu \Theta = -\frac{\zeta}{4} ^*RR, \]

where $G_{\mu\nu}$ is the Einstein tensor, the term $l$ is 4D coupling constant, $C_{\mu\nu}$ is the C-tensor defined as

\[ C_{\mu\nu} = \frac{1}{2\sqrt{-g}} \left[ \nu_\sigma \epsilon^{\sigma\mu\xi\eta} \nabla_\xi R^{\nu}_{\eta} + \frac{1}{2} \nu_{\sigma\tau} \epsilon^{\sigma\nu\xi\eta} R^{\tau}_{\xi\eta} \right] + (\mu \leftrightarrow \nu). \]

Here, $\nu_\sigma \equiv \nabla_\sigma \Theta$ and $\nu_{\sigma\tau} \equiv \nabla_\sigma \nabla_\tau \Theta$. The energy-momentum tensor $T_{\mu\nu}$ consists of the matter part $T_{\mu\nu}^m$ and the external field part $T_{\mu\nu}^\Theta$, defined respectively as

\[ T_{\mu\nu}^m = (\rho + p) U_\mu U_\nu - pg_{\mu\nu}, \]
\[ T_{\mu\nu}^\Theta = \eta(\partial_\mu \Theta)(\partial_\nu \Theta) - \frac{\eta}{2} g_{\mu\nu} (\partial^2 \Theta)(\partial_\lambda \Theta), \]

where $\rho$ is energy density, $p$ is pressure and $U$ is the four-vector velocity in co-moving coordinates of the spacetime.

3 HDE Model in CS Modified Gravity

Granda and Oliveros [55] proposed an infrared cut-off for the HDE which is the sum of the square of the Hubble scale parameter and its time derivative given by

\[ \rho_\Lambda = 3M_p^2 (\alpha H^2 + \beta \dot{H}), \]

where $\alpha$ and $\beta$ are constants which satisfy the restrictions of observational data and $H = \frac{\dot{a}}{a}$ is Hubble parameter. Now, we discuss the FRW universe defined by line element

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

where $\kappa$ is the curvature of the space. Here $\kappa = -1, 0, 1$ denotes open, flat and closed universe respectively. The 00-component of the field (2) of FRW metric and using (8), turns out to be

\[ H^2 + \frac{\kappa}{a^2} = (\alpha H^2 + \beta \dot{H}) + \frac{1}{6} \dot{\Theta}^2. \]

Now we calculate the value of $\Theta$ by using the (3). As for FRW universe the Pontryagin term $^*RR = ^*R^{a}{}_{b}{}^{cd} R^{b}{}_{a}{}^{cd}$ vanishes, so (3) takes the form

\[ g^{\mu\nu} \nabla_\mu \nabla_\nu \Theta = g^{\mu\nu} [\partial_\mu \partial_\nu \Theta - \Gamma^\tau_{\mu\nu} \partial_\tau \Theta] = 0. \]
The solution of this equation can be found, in terms of $\dot{\Theta}$, as
\[ \dot{\Theta} = C a^{-3}. \] (11)

Substituting the value of $\dot{\Theta}$ in (9) along with assumption $x = \ln a$, we arrive at
\[ \frac{dH^2}{dx} + \frac{2(\alpha - 1)}{\beta} H^2 + \frac{C^2}{3\beta} e^{-6x} - \frac{2\kappa}{\beta} e^{-2x} = 0. \] (12)

The solution of this differential equation is obtained using direct integration technique, given as
\[ H^2(x) = C_1 e^{-\frac{2(\alpha-1)x}{\beta}} - \frac{C^2}{6(\alpha - 3\beta - 1)} e^{-6x} + \frac{\kappa}{\alpha - \beta - 1} e^{-2x}. \] (13)

The conservation equation is given by [55]
\[ \dot{\rho}_\Lambda + 3H(\rho_\Lambda + p_\Lambda) = 0. \] (14)

The holographic energy and pressure densities are related by the barotropic equation of state (EoS) defined as $p_\Lambda = \omega_\Lambda \rho_\Lambda$, where $\omega_\Lambda$ is the EoS parameter. Then the last equation takes the form
\[ \omega_\Lambda = -1 - \frac{2\alpha \dot{H} + \beta \ddot{H}}{3(\alpha H^2 + \beta \dot{H})}. \] (15)

Using (13) in (15), we have
\[ \omega_\Lambda = -1 - \frac{1}{3} \left( \frac{\beta(3\beta - 2\alpha + 2)}{C_1 e^{-\frac{2(\alpha-1)x}{\beta}}} \right) \left[ C_1 e^{-\frac{2(\alpha-1)x}{\beta}} + \frac{\kappa(\alpha - \beta)}{(\alpha - \beta - 1)} e^{-2x} - \frac{C^2(\alpha - 3\beta)}{6(\alpha - 3\beta - 1)} e^{-6x} \right]. \] (16)

It is mentioned here that the EoS parameter is time dependent that can be transit from $\omega_\Lambda > -1$ to $\omega_\Lambda < -1$ [52]. Although, the recent studies [53, 54] of DE properties are mildly support the models with $\omega_\Lambda$ crossing $-1$. For the flat case, when $\kappa = 0$, by using the assumption $\alpha = 3\beta$, the last equation turns out to be
\[ \omega_\Lambda = \frac{\alpha - 2}{\alpha}, \] (17)
which describes the EoS parameter in term of constant $\alpha$ only. The accelerated expansion of the universe can be obtained with restrictions on $\alpha$ such that $1 < \alpha < \frac{3}{2}$, if the phase $-1 < \omega_\Lambda < -\frac{1}{3}$ is under consideration. If we consider $0 < \alpha < 1$ then the holographic energy and pressure density illustrates phantom-like theory of the evolution alongwith $\omega_\Lambda < -1$.

## 4 Correspondence Between Holographic and Scalar Field Models

Here we establish a correspondence between infrared cut-off proposed by Granda and Oliveros [55] for the holographic dark energy density and some of famous scalar field models, like quintessence model, tachyon model, K-essence model and dilaton model. We compare the holographic density defined by Granda and Oliveros with the density of corresponding scalar field model in the context of CS modified gravity. Further, we equate the barotropic EoS parameter, given in (17), with the EoS parameter of the corresponding scalar field models to find the scalar field and the potential energy.
4.1 Quintessence Model in CS Modified Gravity

Quintessence model can be described as canonical scalar field. This model was purposed to investigate the late-time cosmic acceleration. The pressure density and energy density of quintessence scalar field are defined as

\[ p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi), \]

\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \]

where the dot denotes derivative with respect to time. The dark energy EoS parameter for the quintessence scalar field is

\[ \omega_\phi = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}. \]

Now, we compare the new HDE modal \( \omega_\Lambda \), given in (16), with that of quintessence DE modal \( \omega_\phi \), given in (20), and obtain

\[ \frac{1}{3} \left( \frac{(3\beta-2\alpha+2)}{\beta} C_1 e^{\frac{-2(\alpha-1)x}{\beta}} + \kappa(\alpha - \beta) e^{-2x} + \frac{C^2(\alpha-3\beta)}{2(\alpha-3\beta-1)} e^{-6x} \right) = \frac{1}{2} \dot{\phi}^2 - V(\phi) \]

On comparing (7) and (19), it comes out that

\[ \frac{1}{2} \dot{\phi}^2 + V(\phi) = 3M_P^2 (\alpha H^2 + \beta \dot{H}). \]

Making use of (22) in (21) yields the explicit expression for \( \dot{\phi} \) and potential \( V(\phi) \)

\[ \dot{\phi}^2 = 2M_P^2 \left[ \frac{(\alpha - 1)}{\beta} C_1 e^{\frac{-2(\alpha-1)x}{\beta}} + \frac{\kappa(\alpha - \beta)}{(\alpha - \beta - 1)} e^{-2x} + \frac{C^2(\alpha - 3\beta)}{2(\alpha - 3\beta - 1)} e^{-6x} \right] \]

and

\[ V(\phi) = M_P^2 \left[ \frac{C_1(3\beta - \alpha + 1)}{\beta} e^{\frac{-2(\alpha-1)x}{\beta}} + \frac{\kappa(\alpha - \beta)}{\alpha - \beta - 1} e^{-2x} \right] \]

respectively. As we assumed \( x = \ln a \), it follows that \( \dot{\phi} = \phi \dot{H} \), where prime denotes the derivative with respect to \( x \). On substituting this value, (23) turns out to be

\[ \phi' = \sqrt{2} M_P \left[ \frac{(\alpha - 1)}{\beta} C_1 e^{\frac{-2(\alpha-1)x}{\beta}} + \frac{\kappa(\alpha - \beta)}{(\alpha - \beta - 1)} e^{-2x} + \frac{C^2(\alpha - 3\beta)}{2(\alpha - 3\beta - 1)} e^{-6x} \right]^{\frac{1}{2}}. \]

For the flat case, i.e., \( \kappa = 0 \), using the assumption \( \alpha = 3\beta \) and taking \( \phi(t_0) = 0 \) at initial time \( t_0 = 0 \), the (25) and (24) become

\[ \phi(t) = \sqrt{\frac{6(\alpha - 1)}{\alpha}} M_P \ln t \]

and

\[ V(\phi) = \frac{3}{\alpha} C_1 M_P^2 e^{-\sqrt{\frac{6(\alpha - 1)}{\alpha} \frac{\phi}{M_P}}}. \]

The potential \( V(\phi) \) becomes a source of accelerated expansion of the universe if \( \alpha < \frac{3}{2} \).

If we discuss the phase-space analysis, the potential \( V(\phi) \) corresponding to scalar field \( \phi \),
behaves like an attractor solution which is indication of accelerated expansion for $\alpha < \frac{3}{2}$, same conditions are followed in power law accelerated expansion. The results obtained for potential in the context of general relativity in [55, 56] in flat Friedmann background and in non flat scenario [20] and our findings in the framework of CS modified gravity are similar in exponential form.

4.2 New Holographic Tachyon Model in CS Modified Gravity

The tachyon modal is considered as a good candidate for dark energy. The idea of tachyon is 40 years old and attained much attention again after the research papers by Sen [48–50]. The tachyon scalar field $\phi$ is studied with Born-Infeld Lagrangian $V(\phi)\sqrt{1 - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}$ which have minimal coupling with gravity. In the tachyon model the energy and pressure densities are given by

$$\rho_T = \frac{V(\phi)}{\sqrt{1 - \dot{\phi}^2}}, \quad (28)$$

$$P_T = V(\phi)\sqrt{1 - \dot{\phi}^2}. \quad (29)$$

The EoS parameter for tachyon scalar field is

$$\omega_T = \frac{\dot{\phi}^2 - 1}{\rho_T}. \quad (30)$$

The comparison between barotropic EoS given in (16) and (30), yields

$$1 - \dot{\phi}^2 = \frac{1}{3}\left(\frac{3\beta - 2\alpha + 2}{\beta}C_1e^{-\frac{2(\alpha-1)x}{\beta}} + \frac{\kappa(\alpha-\beta)}{(\alpha-\beta-1)}e^{-2x} - \frac{C_2^2(3-3\beta)}{2(3-3\beta-1)}e^{-6x}\right), \quad (31)$$

which implies that

$$\dot{\phi}^2 = \frac{2}{3}\left[\frac{C_1e^{-\frac{2(\alpha-1)x}{\beta}}}{\beta} + \frac{\kappa(\alpha-\beta)}{(\alpha-\beta-1)}e^{-2x} - \frac{C_2^2(3-3\beta)}{2(3-3\beta-1)}e^{-6x}\right]. \quad (32)$$

Since $\dot{\phi} = \dot{\phi}'H$ and using corresponding values of $\phi$ and $H$, the evolutionary form of tachyon scalar field yields as

$$\phi(a) - \phi(0) = \int_0^{\ln a} \frac{1}{H} \sqrt{\frac{2(C_1e^{-\frac{2(\alpha-1)x}{\beta}} + \frac{\kappa(\alpha-\beta)}{(\alpha-\beta-1)}e^{-2x} - \frac{C_2^2(3-3\beta)}{2(3-3\beta-1)}e^{-6x})}{3(C_1e^{-\frac{2(\alpha-1)x}{\beta}} + \frac{\kappa(\alpha-\beta)}{(\alpha-\beta-1)}e^{-2x} - \frac{C_2^2(3-3\beta)}{6(3-3\beta-1)}e^{-6x})}} dx. \quad (33)$$

The analytic solution of this integral cannot be found explicitly. For approximate solution, assume that $\alpha = 3\beta$ and $\phi(0) = 0$, i.e., initial time and furthermore, we consider the flat universe, i.e., $\kappa = 0$, the last equation becomes

$$\phi(a) = \sqrt{2(1 - \frac{1}{\alpha})} \int_0^{\ln a} \frac{dx}{\sqrt{C_1e^{-\frac{6(\alpha-1)x}{\alpha}} + \frac{C_2^2}{\alpha}e^{-6x}}}. \quad (34)$$
The integral on R.H.S can be solved in term of hypergeometric function as

$$\phi(a) = 2\sqrt{\frac{\alpha - 1}{3\alpha}} e^{3x} \frac{\rho}{C} 2F_1 \left[ \frac{1}{2}, \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2}, -\frac{6e^{6x}C_1}{C^2} \right].$$  \hspace{1cm} (35)

Finally, by substituting $x = \ln a$, we obtain

$$\phi(a) = 2\sqrt{\frac{\alpha - 1}{3\alpha}} a^3 \frac{\rho}{C} 2F_1 \left[ \frac{1}{2}, \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2}, -\frac{6a^6C_1}{C^2} \right].$$  \hspace{1cm} (36)

Clearly, for $\alpha = 1$, it yields

$$\phi(a) = 0.$$  \hspace{1cm} (37)

Now, the comparison between Granda and Oliveros cut-off, given in (7), and holographic tachyon model density, given in (28), yields

$$\rho_{\Lambda} = 3M_P^2 (\alpha H^2 + \beta \dot{H}) = \frac{V(\phi)}{\sqrt{1 - \dot{\phi}^2}}.$$  \hspace{1cm} (38)

Substituting the values of $\phi$ and $H$, the tachyon potential energy turns to be

$$V(\phi) = \sqrt{3}M_P^2 \left[ C_1 e^{-\frac{2(\alpha - 1)x}{\beta}} + \frac{\kappa(\alpha - \beta)}{(\alpha - \beta - 1)} e^{-2x} - \frac{C^2(\alpha - 3\beta)}{6(\alpha - 3\beta - 1)} e^{-6x} \right]$$

$$\times \left[ C_1 \frac{3\beta - 2\alpha + 1}{\beta} e^{-\frac{2(\alpha - 1)x}{\beta}} + \frac{\kappa(\alpha - \beta)}{(\alpha - \beta - 1)} e^{-2x} - \frac{C^2(\alpha - 3\beta)}{2(\alpha - 3\beta - 1)} e^{-6x} \right].$$  \hspace{1cm} (39)

Again, for $\kappa = 0$ and $\alpha = 3\beta$, we have

$$V(\phi) = 3M_P^2 C_1 \sqrt{\frac{2 - \alpha}{\alpha}} e^{\frac{1 - \alpha}{\alpha} x}.$$  \hspace{1cm} (40)

For particular case $\alpha = 1$, the potential $V(\phi)$ is constant which corresponds to ghost condensate scenario discussed in [57].

### 4.3 New Holographic K-essence Modal in CS Modified Gravity

The concept of k-essence scalar field model was introduced by Armendariz and Mukhanov [51] to explain the accelerated expansion of the universe. The theory of k-essence deals with dynamical attractor solutions which acts as a cosmological constant. The scalar field action for K-essence modal is defined as

$$S = \int d^4x \sqrt{-g} p(\phi, X),$$  \hspace{1cm} (41)

where $p(\phi, X)$ denotes pressure density and most of time it corresponds to Lagrangian density defined as $p(\phi, X) = f(\phi)\psi(X)$. In string theory, the Lagrangian density is transformed into

$$p(\phi, X) = f(\phi)(-X + X^2).$$  \hspace{1cm} (42)
The energy density of the field $\phi$ corresponding to the Lagrangian density expression is given by

$$\rho(\phi, X) = f(\phi)(-X + 3X^2). \quad (43)$$

Using (42) and (43), one can easily obtain EoS parameter, given as

$$\omega_K = \frac{X - 1}{3X - 1}. \quad (44)$$

In particular $X < \frac{3}{2}$, the EoS $\omega_{\phi} < -\frac{1}{3}$ indicate the accelerated expansion. The comparison between (16) and new EoS parameter (44) yields

$$X = \frac{1}{3} \left[ \frac{3\beta - \alpha + 1}{\beta} C_1 e^{-\frac{2(a-1)x}{\rho}} + \frac{2\kappa(a-\beta)}{(\alpha-\beta-1)} e^{2x} + C^2(a-3\beta) e^{-6x} \right]. \quad (45)$$

The term $\dot{\phi}^2 = 2X$ defined in [30, 31] and $\dot{\phi} = \phi H$, using these expressions, the evolutionary form of K-essence scalar field takes the form

$$\phi(a) = \frac{\sqrt{2}}{\sqrt{3}} \times \int_0^{\ln a} \frac{1}{H} \left[ \frac{3\beta - \alpha + 1}{\beta} C_1 e^{-\frac{2(a-1)x}{\rho}} + \frac{2\kappa(a-\beta)}{(\alpha-\beta-1)} e^{2x} \right. \left. + \frac{\kappa(a-\beta)}{(\alpha-\beta-1)} e^{2x} + \frac{C^2(a-3\beta)}{6(a-3\beta-1)} e^{-6x} \right] dx. \quad (46)$$

The compassion of (16) and (44), alongwith values of $H$, yields the expression for $f(\phi)$ as

$$f(\phi) = \frac{3M_p^2 (1 - 3\omega) - 2(1 - \omega)}{2(1 - \omega)} \times \left[ C_1 e^{-\frac{2(a-1)x}{\rho}} + \frac{\kappa(a-\beta)}{(\alpha-\beta-1)} e^{2x} - \frac{C^2(a-3\beta)}{6(a-3\beta-1)} e^{-6x} \right]. \quad (47)$$

which can be further written as

$$f(\phi) = \frac{M_p^2}{2} \left[ \frac{2(2\beta - \alpha + 1)}{\beta} C_1 e^{-\frac{2(a-1)x}{\rho}} + \frac{2\kappa(a-\beta)}{(\alpha-\beta-1)} e^{2x} + \frac{C^2(a-3\beta)}{6(a-3\beta-1)} e^{-6x} \right]^2. \quad (48)$$

Solving the (45), (46) and (48) analytically, we consider the flat case, i.e., $\kappa = 0$, use the assumption $\alpha = 3\beta$ and $\phi(0) = 0$ for the initial time $t_0 = 0$, it turns out to be

$$X = \frac{1}{3 - \alpha}, \quad (49)$$

$$\phi(a) = \sqrt{\frac{2}{3 - \alpha}} \frac{e^{3x}}{C} 2F_1 \left[ \frac{1}{2}, \frac{\alpha}{2}, 1 + \frac{\alpha}{2}, -6e^{\frac{6x}{C^2}} C_1 \right], \quad (50)$$

$$f(\phi) = \frac{M_p^2}{2} \frac{(3 - \alpha)^2}{\alpha(\alpha - 1)} C_1 e^{-\frac{6(a-1)}{\alpha} x}. \quad (51)$$
For $\alpha = \frac{3}{2}$ the above equations turned into

$$X = \frac{2}{3},$$  \hspace{1cm} (52)$$

$$\phi(a) = \sqrt{\frac{4}{3} a^3} F_1 \left[ \frac{1}{2}, 3, 7, \frac{-6a^6}{C^2} C_1 \right],$$  \hspace{1cm} (53)$$

$$f(\phi) = M_p^2 C_1 a^{-2}.$$  \hspace{1cm} (54)$$

The potential $f(\phi)$ obeys the power law expansion analysed in [56]. Our result for the potential in the context of CS modified gravity is very similar to [56] which is in the framework of GR.

4.4 New Holographic Dilaton Field in CS Modified Gravity

The dilaton model of DE is described by a 4-dimensional effective low-energy limit action in string theory. It includes the higher order kinetic energy term which may be negative in the framework of Einstein relativity. It indicates that the dilaton model works like a phantom-type scalar field. The dilaton scalar field model is defined by the pressure density

$$P_d = -X + c_1 e^{\lambda \phi} X^2,$$  \hspace{1cm} (55)$$

where $c_1$ and $\lambda$ are positive constants. The corresponding dilaton energy density is given by

$$\rho_d = -X + 3c_1 e^{\lambda \phi} X^2,$$  \hspace{1cm} (56)$$

where $2X = \dot{\phi}^2$. The EoS parameter $\omega_d = \frac{P_d}{\rho_d}$ can be obtained from (55) and (56).

$$\omega_d = -1 + c_1 e^{\lambda \phi} \frac{X}{1 + 3c_1 e^{\lambda \phi} X}.$$  \hspace{1cm} (57)$$

Now, we compare (57) with new holographic EoS parameter, given in (16), i.e., $\omega_d = \omega_\Lambda$ to obtain

$$C' e^{\frac{\lambda \phi}{2}} X = \frac{1}{3} \left[ \frac{2a-2\beta}{\beta} C_1 e^{-\frac{2(a-1)x}{\beta}} + \frac{2x(\alpha-\beta)}{(\alpha-\beta-1)} e^{-2x} \right].$$  \hspace{1cm} (58)$$

Making use of $X = \frac{\dot{\phi}^2}{2}$ and $\dot{\phi} = \phi' H$ in the last equation and then integrating with respect to $x$, we get

$$e^{\frac{2\phi(a)}{\beta}} - e^{\frac{2\phi(0)}{\beta}} = \frac{\lambda}{\sqrt{6C}^\gamma}$$

$$\times \int_0^{\ln a} \frac{1}{H} \left[ \frac{(2-2a)\beta C_1 e^{-\frac{2(a-1)x}{\beta}} + \frac{2x(\alpha-\beta)}{(\alpha-\beta-1)} e^{-2x}}{2(2\beta-\alpha+1) \beta C_1 e^{-\frac{2(a-1)x}{\beta}} + \frac{2x(\alpha-\beta)}{(\alpha-\beta-1)} e^{-2x} + \frac{C_2 (\alpha-3\beta)}{5(\alpha-3\beta-1)} e^{-6x}} \right] dx$$  \hspace{1cm} (59)$$
To obtain the evolutionary form of dilaton field, we consider the flat universe, i.e., $\kappa = 0$, using assumption $\alpha = 3\beta$ and at initial time $t_0 = 0$, the $\phi(0) = 0$ and have

$$\phi(a) = \frac{\lambda}{6C'} \sqrt{\frac{1-\alpha}{3-\alpha}} e^{\frac{6\alpha}{C'}a} _2F_1\left[\frac{1}{2}, \frac{\alpha}{2}, 1 + \frac{\alpha}{2}, \frac{-6e^{\frac{6\alpha}{C'}a}}{C'^2-C_1}\right].$$  \hfill (60)

Re-substituting $x = \ln a$ in the last equation, we arrived at

$$\phi(a) = \frac{\lambda}{\sqrt{6C'}} \sqrt{\frac{1-\alpha}{3-\alpha}} a^{\frac{6\alpha}{C'^2-C_1}} _2F_1\left[\frac{1}{2}, \frac{\alpha}{2}, 1 + \frac{\alpha}{2}, \frac{-6a^{\frac{6\alpha}{C'}}}{C'^2-C_1}\right].$$ \hfill (61)

For the particular value of $\alpha = 3$ the dilaton $\phi \rightarrow \infty$ is effectively decoupled from gravity and it is so-called the runaway dilaton, also discussed in same style in the context of GR in [58]

5 Conclusion

The accelerated expansion of the universe is a most discussed issue in the recent past. In this paper, we found the EoS parameter $\omega/\Lambda_1$ which describe the accelerated expansion of universe under certain restrictions on the parameter $\alpha$. It is shown that for the accelerated expansion phase $-1 < \omega_\Lambda < -\frac{1}{3}$, the parameter $\alpha$ varies according as $1 < \alpha < \frac{3}{2}$. Furthermore, for $0 < \alpha < 1$, the holographic energy and pressure density illustrates phantom-like theory of the evolution when $\omega_\Lambda < -1$.

We explored the scalar field $\phi$ and potential $V(\phi)$ of different holographic dark energy models such that quintessence, techyon, K-essence and dilaton in the framework of CS modified gravity. The potential $V(\phi)$ in case of quintessence becomes a source of accelerated expansion of the universe if $\alpha < \frac{3}{2}$. When we discuss the phase-space analysis, the potential $V(\phi)$ corresponding to scalar field $\phi$ behaves like an attractor solution which is indication of accelerated expansion for $\alpha < \frac{3}{2}$, same conditions are followed in power law accelerated expansion. The results obtained for potential in the context of general relativity using flat Friedmann background in [55, 56] and in non flat scenario [20] and our findings in the framework of CS modified gravity are similar in exponential form. The solutions of techyon and K-essence models in the context of CS modified gravity are similar to those of GR results. The dilaton model for the particular choice of parameter $\alpha = 3$ is effectively decoupled from gravity and called runaway dilaton.

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References

1. Riess, A.G. et al.: Astron. J 116, 1009 (1998)
2. Perlmutter, S. et al.: Astrophys. J 517, 565 (1999)
3. Tegmark, M. et al.: Phys. Rev. D 69, 103501 (2004)
4. Abazajian, K. et al.: Astron. J 128, 502 (2004)
5. Spergel, D.N. et al.: Astrophys. J. Suppl. 148, 175 (2003)
6. Bennett, C.L. et al.: Astrophys. J. Suppl. 148, 1 (2003)
7. Tonry, J.L., et al.: Astrophys. J 1, 594 (2003)  
8. de Bernardis, P. et al.: Nature 404, 955 (2000)  
9. Hanany, S. et al.: Astrophys. J 545, 5 (2000)  
10. Li, M.: Phys. Lett. B 603, 1 (2004)  
11. Horvat, R.: Phys. Rev. D 70, 087301 (2004)  
12. Huang, Q.G., Li, M.: JCAP 0408, 013 (2004)  
13. Bouso, R.: Rev. Mod. Phys. 74, 825 (2002)  
14. Hsu, S.D.H.: Phys. Lett. B 603, 1 (2004)  
15. Kiran, M., Reddy, D.R.K., Rao, V.U.M.: Astrophy. and Space Sci. 356, 407 (2015)  
16. Zhang, J.F., Cui, J.L., Zhang, X.: Eur. Phys. J. C 74, 3100 (2014)  
17. Samanta, G.C.: Int. J. Theor. Phys. 55, 5095–5105  
18. Nojiri, S., Odintsov, S.D.: Phys. Lett. B 562, 147 (2003)  
19. Nojiri, S., Odintsov, S.D.: Phys. Lett. B 565, 1 (2003)  
20. Alexander, S.H.S., Peskin, M.E., Sheik-Jabbari, M.M.: Phys. Rev. Lett. 96, 081301 (2006)  
21. Jackiw, R., Pi, S.Y.: Phys. Rev. D 68, 104012 (2003)  
22. Pasqua, A., da Rocha, R., Chattopadhyay, S.: Eur. Phys. J. C 75, 44 (2015)  
23. Amir, M.J., Ali, S.: Int. J. Theor. Phys. 54, 1362 (2015)  
24. Sen, A.: J. High Energy Phys. 04, 048 (2002)  
25. Sen, A.: J. High Energy Phys. 07, 065 (2002)  
26. Wang, B., Gong, Y., Abdalla, E.: Phys. Rev. D 74, 083520 (2006)  
27. Wang, Y., Tegmark, M.: Phys. Rev. D 71, 103513 (2005)  
28. Alam, U., Sahni, V., Starobinsky, A.A.: J. Cosmol. Astropart. Phys. 06, 008 (2004)  
29. Granda, L.N., Oliveros, A.: Phys. Lett. B 671, 199 (2009)  
30. Copeland, E.J., Sami, M., Tsujikawa, S.: Int. J. Mod. Phys. D 15, 1751 (2006)  
31. Hamed, N.A., Creminelli, P., Mukohyama, S., Zaldarriaga, M.: J. Cosmol. Astropart. Phys. 04, 001 (2004)  
32. Piazza, F., Tsujikawa, S.: J. Cosmol. Astropart. Phys. 07, 004 (2004)  
33. Caldwell, R.R.: Phys. Lett. B 545, 23 (2002)  
34. Nojiri, S., Odintsov, S.D.: Phys. Rev. D 68, 104012 (2003)  
35. Alexander, S.H.S., Peskin, M.E., Sheik-Jabbari, M.M.: Phys. Rev. Lett. 96, 081301 (2006)  
36. Jackiw, R., Pi, S.Y.: Phys. Rev. D 68, 104012 (2003)  
37. Pasqua, A., da Rocha, R., Chattopadhyay, S.: Eur. Phys. J. C 75, 44 (2015)  
38. Amir, M.J., Ali, S.: Int. J. Theor. Phys. 54, 1362 (2015)  
39. Sen, A.: J. High Energy Phys. 04, 048 (2002)  
40. Sen, A.: J. High Energy Phys. 07, 065 (2002)  
41. Caldwell, R.R.: Phys. Lett. B 545, 23 (2002)  
42. Nojiri, S., Odintsov, S.D.: Phys. Rev. D 68, 104012 (2003)  
43. Alexander, S.H.S., Peskin, M.E., Sheik-Jabbari, M.M.: Phys. Rev. Lett. 96, 081301 (2006)  
44. Jackiw, R., Pi, S.Y.: Phys. Rev. D 68, 104012 (2003)  
45. Pasqua, A., da Rocha, R., Chattopadhyay, S.: Eur. Phys. J. C 75, 44 (2015)  
46. Amir, M.J., Ali, S.: Int. J. Theor. Phys. 54, 1362 (2015)  
47. Sen, A.: J. High Energy Phys. 04, 048 (2002)  
48. Sen, A.: J. High Energy Phys. 07, 065 (2002)  
49. Sen, A.: Mod. Phys. Lett. A 17, 1799 (2002)  
50. Armendariz-Picon, C., Mukhanov, V.: Phys. Rev. D 63, 103510 (2001)  
51. Wang, B., Gong, Y., Abdalla, E.: Phys. Rev. D 74, 083520 (2006)  
52. Wang, Y., Tegmark, M.: Phys. Rev. D 71, 103513 (2005)  
53. Alam, U., Sahni, V., Starobinsky, A.A.: J. Cosmol. Astropart. Phys. 06, 008 (2004)  
54. Granda, L.N., Oliveros, A.: Phys. Lett. B 671, 199 (2009)  
55. Copeland, E.J., Sami, M., Tsujikawa, S.: Int. J. Mod. Phys. D 15, 1751 (2006)  
56. Hamed, N.A., Creminelli, P., Mukohyama, S., Zaldarriaga, M.: JCAP 0404, 001 (2004)  
57. Gasperini, M., Piazza, F., Veneziano, G.: Phys. Rev. D 65, 023508 (2002)