Minimization Problems Based on a Parametric Family of Relative Entropies II: Reverse Projection

M. Ashok Kumar and Rajesh Sundaresan

Abstract

In part I of this two-part work, certain minimization problems based on a parametric family of relative entropies (denoted $I_\alpha$) were studied. Such minimizers were called forward $I_\alpha$-projections. Here, a complementary class of minimization problems leading to the so-called reverse $I_\alpha$-projections are studied. Reverse $I_\alpha$-projections, particularly on log-convex or power-law families, are of interest in robust estimation problems ($\alpha > 1$) and in constrained compression settings ($\alpha < 1$). Orthogonality of the power-law family with an associated linear family is first established and is then exploited to turn a reverse $I_\alpha$-projection into a forward $I_\alpha$-projection. The transformed problem is a simpler quasiconvex minimization subject to linear constraints.

Index Terms

Best approximant; exponential family; information geometry; Kullback-Leibler divergence; linear family; power-law family; projection; Pythagorean property; relative entropy; Rényi entropy; robust estimation; Tsallis entropy.

I. INTRODUCTION

This paper is a continuation of our study of minimization problems based on a parametric generalization of relative entropies, denoted $I_\alpha$. See [2] for the definition of $I_\alpha(P, Q)$, where $P$ and $Q$ are probability measures on an alphabet set $X$. We say “parametric generalization of relative entropy” because $\lim_{\alpha \to 1} I_\alpha(P, Q) = I(P\|Q)$, the usual relative entropy or Kullback-Leibler divergence. In part I [3], we showed how $I_\alpha$ arises and studied the problem of a forward $I_\alpha$-projection, namely

$$\min_{P \in \mathcal{E}} I_\alpha(P, R),$$

where $R$ is a fixed probability measure on $X$ and $\mathcal{E}$ is a convex set of probability measures on $X$. In this paper, we shall study reverse $I_\alpha$-projection, namely

$$\min_{P \in \mathcal{E}} I_\alpha(R, P).$$

The minimization now is with respect to the second argument of $I_\alpha$. Such problems arise in robust parameter estimation and constrained compression settings. The family $\mathcal{E}$ is usually a parametric family such as the exponential family, or its generalization, called the $\alpha$-power-law family.

We shall bring to light the geometric relation between the $\alpha$-power-law family and a linear family of probability measures. We shall turn the reverse $I_\alpha$-projection problem on an $\alpha$-power-law family into a forward $I_\alpha$-projection problem on a linear family. The latter turns out to be a minimization of a quasiconvex objective function subject to linear constraints.

The outline of the paper is as follows. In Section II we motivate reverse $I_\alpha$-projections for the cases $\alpha > 1$ and $\alpha < 1$. In Section III we define the required terminologies and highlight the contributions of the paper. In Section IV we study the existence of a reverse $I_\alpha$-projection on general log-convex sets. In Section V we provide simplified proofs of some essential results from II on the forward $I_\alpha$-projection. Our simplified proofs also serve the purpose of keeping this paper self-contained. In Section VI we explore the geometric relation between the $\alpha$-power-law and the linear families, and then exploit it to study reverse $I_\alpha$-projection on $\alpha$-power-law families. The paper ends with some concluding remarks in Section VII.

II. MOTIVATIONS

The purpose of this section is to motivate reverse $I_\alpha$-projections. The motivation for $\alpha > 1$ comes from robust statistics. The motivation for $\alpha < 1$ comes from information theory as well as from a strong similarity of the outcomes with the $\alpha = 1$ (relative entropy) setting.

M. Ashok Kumar was supported by a Council for Scientific and Industrial Research (CSIR) fellowship and by the Department of Science and Technology. A part of this material in this paper (Section V alone) is under consideration for presentation at the National Conference on Communications (NCC 2015), Mumbai, India, to be held during February 2015 [1].

M. Ashok Kumar and R. Sundaresan are with the ECE Department, Indian Institute of Science, Bangalore 560012, India.

Example linear families are (1) the set of probability measures $P$ on $X$ such that $\sum_x P(x) f(x) = 0$ for some $f : X \to \mathbb{R}$, and (2) finite intersections of such sets. If there is an additive structure on $X$, a concrete example is the set of all probability measures with a fixed mean.
A. Reverse $\mathcal{I}$-projection

Let $\mathbb{X}$ be a finite alphabet set and let $\mathcal{E} = \{P_\theta : \theta \in \Theta\}$ denote a family of probability measures on $\mathbb{X}$ indexed by the elements of the index set $\Theta \subseteq \mathbb{R}^k$ for some $k$. Let $x_1, x_2, \ldots, x_n$ be $n$ samples drawn independently and with replacement from $\mathbb{X}$ according to an unknown probability measure $P_\theta$ belonging to $\mathcal{E}$. The maximum likelihood estimate (MLE) of $\theta$, denoted $\hat{\theta}$, is the element of the index set $\Theta$ that maximizes the likelihood (if it exists), i.e.,

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \prod_{i=1}^n P_\theta(x_i).$$

(1)

Let $\hat{P}$ denote the empirical measure of the $n$ samples $x_1, \ldots, x_n$, i.e.,

$$\hat{P} := \frac{1}{n} \sum_{i=1}^n \delta_{x_i},$$

where $\delta_a$ denotes the Dirac mass at $a$. One may then write

$$\frac{\prod_{i=1}^n P_\theta(x_i)}{\prod_{i=1}^n P(x_i)} = \prod_{i=1}^n \frac{P_\theta(x_i)}{P(x_i)} = \prod_{x \in \mathbb{X}} \left( \frac{P_\theta(x)}{P(x)} \right)^{n \hat{P}(x)} = \exp\{-n \mathcal{I}(\hat{P}||P_\theta)\},$$

where

$$\mathcal{I}(P||Q) := \sum_{x \in \mathbb{X}} P(x) \log \frac{P(x)}{Q(x)}$$

is the relative entropy of $P$ with respect to $Q$. Hence the MLE is the minimizer (if it exists)

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \mathcal{I}(\hat{P}||P_\theta),$$

(2)

and the corresponding probability measure $P_\hat{\theta}$ is known as the reverse $\mathcal{I}$-projection of $\hat{P}$ on the family $\mathcal{E}$. Such reverse projections, particularly those related to robustifications of the MLE, are the subject matter of this paper.

Observe that the MLE depends on the samples only through their empirical measure. Let us write the MLE as a function of the empirical measure in a different way. Assume that the family $\mathcal{E}$ is sufficiently smooth in the parameter $\theta$ on account of which we can define the score function as $s(\cdot; \theta) := \nabla_\theta \log P_\theta(\cdot)$, the gradient of $\log P_\theta(\cdot)$ with respect to $\theta$. The first order optimality criterion applied to (1) after taking logarithms yields the so-called estimating equation for the MLE:

$$\frac{1}{n} \sum_{i=1}^n s(x_i; \theta) = 0;$$

the MLE $\hat{\theta}$ solves this equation. Write $E_P[\cdots]$ for expectation with respect to $P$. Noting that the score function satisfies

$$E_P[s(X; \theta)] = 0 \quad \forall P_\theta,$$

the estimating equation for the MLE can be rewritten as

$$\frac{1}{n} \sum_{i=1}^n s(x_i; \theta) = E_{P_\theta}[s(X; \theta)],$$

(3)

which is the same as

$$E_{\hat{P}}[s(X; \theta)] = E_{P_\theta}[s(X; \theta)].$$

(4)

If we write $T(\hat{P})$ for the $\theta$ that solves (4), we then have $\hat{\theta} = T(\hat{P})$. The estimator $T(\hat{P})$ is Fisher consistent\(^2\), a fact that can be easily checked using (4).

\(^2\)The usual convention is $p \log \frac{1}{p} = 0$ if $p = 0$ and $+\infty$ if $p > q = 0$.

\(^3\)An estimator that maps an empirical measure to an element in $\Theta$ is Fisher consistent if it is continuous and maps $P_\theta$ to the true parameter $\theta$. See \([3, \text{Sec. 5c.1}]\)
B. Reverse $\mathcal{I}_\alpha$-projection: $\alpha > 1$

Though the MLE is known to possess many good properties, asymptotic efficiency being an example, it is not appropriate when some of the data entries $(x_i)$ are contaminated by outliers. To achieve robustness, one may consider scaling the scores $s(x_i; \theta)$ in the left-hand side of (3) by weights $w(x_i; \theta)$ that weigh down outlying observations “relative to the model” (see for example Basu et al. [4]). This type of robustification, along with the requirement of Fisher consistency, is accomplished by the estimator that maps the empirical measure $\hat{P}$ to the $\theta$ that solves the equation

$$E_{\hat{P}}[w(X; \theta)s(X; \theta)] = E_{P_\theta}[w(X; \theta)s(X; \theta)].$$

(5)

Basu et al. [4] proposed the natural weighting $w(x; \theta) = P_\theta(x)^c$ where $c > 0$. As another robustification procedure, Basu et al. [4] proposed a weighting of the model by itself, motivated by the works of Field and Smith [5] and Windham [6], prior to solving the estimating equation. Their procedure is as follows. Given a measure $P$, their procedure is as follows. Given a measure $P$, the weighting parameters are $c$ and $\theta$, and $(P_\theta)^{(c, \theta)}$ is the probability measure proportional to $P_\theta^{c+1}$. The Basu et al. procedure is to find the $\theta$ that solves the equation

$$E_{(P_\theta)^{(c, \theta)}}[s(X; \theta)] = E_{(P_\theta)^{(c, \theta)}}[s(X; \theta)];$$

(6)

the $\hat{P}$ and $P_\theta$ of (4) are replaced by the model reweighted $(\hat{P})^{(c, \theta)}$ and $(P_\theta)^{(c, \theta)}$, respectively. It is clear that the corresponding estimator is Fisher consistent. Now (6) can be rewritten as

$$\frac{1}{n} \sum_{i=1}^{n} w(x_i; \theta)s(x_i; \theta) = \frac{\mathbb{E}_{P_\theta}[w(X; \theta)s(X; \theta)]}{\mathbb{E}_{P_\theta}[w(X; \theta)]},$$

which expands to

$$\sum_{i=1}^{n} P_\theta(x_i)^c s(x_i; \theta) = \sum_{x \in X} P_\theta(x)^{c+1} s(x; \theta) \frac{\sum_{i=1}^{n} P_\theta(x_i)^c}{\sum_{x \in X} P_\theta(x)^{c+1}}.$$

(7)

Jones et al. [7] compare the robustness properties of estimators arising from (5) and (7). According to Jones et al. [7] p. 866], the former is more efficient, but the latter has better robustness with respect to a mixture model of contamination with outliers. Equation (7) can be recognized as an estimating equation arising from the first order optimality criterion for the maximization

$$\max_{\theta \in \Theta} \left[ \frac{1}{c} \log \left( \frac{1}{n} \sum_{i=1}^{n} P_\theta(x_i)^c \right) - \frac{1}{1+c} \log \sum_{x \in X} P_\theta(x)^{1+c} \right].$$

(8)

We shall soon see why it ought to be a maximization. The objective function in (8) is called mean power likelihood [8]. The corresponding estimator is called the maximum mean power likelihood estimate (MMPE) by Eguchi and Kato [3], we shall denote it $\hat{\theta}_{c+1}$. (The appearance of 1 in the subscript $\hat{\theta}_{c+1}$ will soon become clear.) When $c = 0$, we see that $\hat{\theta}_1$ becomes the MLE $\hat{\theta}$. The parameter $c$ in (8) can thus be used to trade-off robustness for asymptotic efficiency as observed in [6], [7].

This procedure may be viewed as a generalization of the self-weighting procedure suggested by Windham [6] p. 604].

To see why the objective function in (8) is called mean power likelihood, verify that (8) is equivalent to

$$\frac{1}{n} \sum_{i=1}^{n} s_c(x_i; \theta) = 0$$

where

$$s_c(x; \theta) := P_\theta(x)^c \left[ s(x; \theta) - \frac{1}{1+c} \nabla_{\theta} \log \sum_{x \in X} P_\theta(x)^{c+1} \right].$$

The quantity $s_c(x_i; \theta)$ is a generalization of the power-weighted and centered score function. The centering ensures Fisher consistency. As $c \downarrow 0$, we have $s_c(x; \theta) \to s(x; \theta)$. 

\[\text{\textsuperscript{4}}\]
Let us now bring in the connection to a parametric family of relative entropies. Recall that \( \hat{P} \) is the empirical measure of the data. Maximizing the objective in (8) is the same as minimizing

\[
-\frac{c+1}{c} \log \left( \frac{1}{n} \sum_{i=1}^{n} P_{\theta}(x_{i})^{c} \right) + \frac{1}{c} \log \sum_{x \in \mathcal{X}} \hat{P}(x)^{c+1} + \log \sum_{x \in \mathcal{X}} P_{\theta}(x)^{c+1}
\]

\[
= -\frac{c+1}{c} \log \sum_{x \in \mathcal{X}} \hat{P}(x)^{c} + \frac{1}{c} \log \sum_{x \in \mathcal{X}} \hat{P}(x)^{c+1} + \log \sum_{x \in \mathcal{X}} P_{\theta}(x)^{c+1}
\]

\[
= : \mathcal{J}_{c+1}(\hat{P}, P_{\theta})
\]

over \( \theta \in \Theta \), where \( \mathcal{J}_{c+1} \) in (9) is a parametric extension of relative entropies already studied in our companion paper [2]. We thus have

\[
\hat{\theta}_{c+1} = \arg \min_{\theta \in \Theta} \mathcal{J}_{c+1}(\hat{P}, P_{\theta}),
\]

(10)

and the probability measure \( P_{\hat{\theta}_{c+1}} \) corresponding to the MMLE \( \hat{\theta}_{c+1} \) is called the reverse \( \mathcal{J}_{c+1} \)-projection of the empirical measure \( \hat{P} \) on the family \( \mathcal{E} \). It is known (see for example [2] Lemma 1-b)) that \( \lim_{c \downarrow 0} \mathcal{J}_{c+1}(P, Q) = \mathcal{J}(P \| Q) \), as it should be, for we already saw that \( c = 0 \) yields \( \hat{\theta}_{1} = \hat{\theta} \), the MLE, which is also the reverse \( \mathcal{J} \)-projection of the empirical measure \( \hat{P} \) on \( \mathcal{E} \). This operational continuity intuitively suggests that we must have minimization in (10) and maximization in (8).

Let us now use large sample asymptotics to justify the minimization in (10) (and maximization in (8)). Let \( \theta \) be, for we already saw that \( c = 0 \) yields \( \hat{\theta}_{1} = \hat{\theta} \), the MLE, which is also the reverse \( \mathcal{J} \)-projection of the empirical measure \( \hat{P} \) on \( \mathcal{E} \). This operational continuity intuitively suggests that we must have minimization in (10) and maximization in (8).

Some historical remarks are now called for. Basu et al. [4] studied a nonnormalized version of the estimating equation (7), namely (5) with equation (7) and identified a divergence associated with (7), see [7, Eq. (2.8)]. Fujisawa and Eguchi [9] found that

\[
\hat{\theta}_{c+1}(P, P_{\theta}) = \text{arg } \min_{\theta \in \Theta} \mathcal{J}_{c+1}(\hat{P}, P_{\theta}),
\]

where the last inequality follows from the fact that \( \mathcal{J}_{\alpha}(P_{\theta}^*, P_{\theta}^*) \geq 0 \) with equality if and only if \( \theta = \theta^* \) [2] Lem. 1-a]). From this, it is clear that one must minimize over \( \theta \in \Theta \) (and not maximize) in (10) in order to identify the true parameter \( \theta^* \).

Earlier Sundaresan [12] and [13] arrived at \( \mathcal{J}_{\alpha} \)-divergences in the context of redundancy in compression and guessing problems (for \( \alpha < 1 \). Let us now turn to this.

**C. Reverse \( \mathcal{J}_{\alpha} \)-projection: \( \alpha < 1 \)**

We now motivate reverse \( \mathcal{J}_{\alpha} \)-projection for \( \alpha < 1 \). Rényi entropies play a role similar to Shannon entropy when one wishes to minimize the normalized cumulant of compressed lengths as opposed to expected compressed lengths. More precisely, with \( \rho = \alpha^{-1} - 1 > 0 \), Campbell [14] showed that

\[
\min_{\theta \in \Theta} \frac{1}{n \rho} \log \mathbb{E}[\exp\{\rho L_{\alpha}(X^{n})\}] \rightarrow H_{\alpha}(\hat{P}) \quad (n \rightarrow \infty)
\]

for an iid source with marginal \( \hat{P} \). The minimization is taken over all length functions \( L_{n} \) that satisfy the Kraft inequality. \( \rho \) is the cumulant parameter. As \( \alpha \uparrow 1 \), we have \( \rho \downarrow 0 \), and it is well known that \( \lim_{n \uparrow 1} H_{\alpha}(\hat{P}) = H(\hat{P}) \), the Shannon entropy, so that Rényi entropy can be viewed as an operational generalization of Shannon entropy.

Suppose now that the compressor is forced to use for compression, not the true probability measure \( \hat{P} \), but a probability measure \( P_{\theta} \) from a family parameterized by \( \theta \in \Theta \). Let us denote, as before, \( \mathcal{E} = \{P_{\theta}: \theta \in \Theta\} \). As an example, \( \hat{P} \) may be a generic measure on \( \mathcal{X} = \{0, 1, \ldots, L\} \), but the compressor may wish to pick the best representation of \( \hat{P} \) among binomial distributions \( P_{\theta} \) having \( \theta \in (0, 1) \) as parameter [2]. If the compressor picks \( P_{\theta} \) instead of the true \( \hat{P} \), then the gap in the resulting normalized cumulant from the optimal value is \( \mathcal{J}_{\alpha}(\hat{P}, P_{\theta}) \) [13]. It follows that the best compressor from within \( \mathcal{E} \) has parameter

\[
\hat{\theta}_{n} = \text{arg } \min_{\theta \in \Theta} \mathcal{J}_{\alpha}(\hat{P}, P_{\theta})
\]

(11)

\[\text{The dependence of } \hat{P} \text{ on } n \text{ is understood and suppressed.}\]

\[\text{The outliers are generated using a mixture model.}\]

\[\text{More sophisticated examples are possible. Take } \mathcal{X} = \{0, 1\}^{2}, \hat{P} \text{ any fixed, stationary, and ergodic probability measure on } \mathcal{X}, \text{ and } \mathcal{E} \text{ the class of stationary Markov measures on } \mathcal{X} \text{ of fixed Markov order. Since this } \mathcal{X} \text{ is not finite, such examples are beyond the scope of this paper.}\]
and the probability measure \( P_\alpha \) is the reverse \( \mathcal{I}_\alpha \)-projection of \( \hat{P} \) on the family \( \mathbb{E} \). While (10) defines reverse \( \mathcal{I}_\alpha \)-projection for \( \alpha > 1 \), (11) defines such a projection for \( \alpha < 1 \). As one expects, \( \lim_{\alpha \uparrow 1} \mathcal{I}_\alpha(\hat{P}, P_\theta) = \mathcal{I}(\hat{P}||P_\theta) \), the penalty for mismatch in compression when expected lengths are considered, and one has the operational continuity that \( \mathcal{I}(\hat{P}||P_\theta) \) is the usual limiting penalty for mismatch as \( \alpha \uparrow 1 \).

\( \mathcal{I}_\alpha \) also arises as the gap from optimality due to mismatch in performance of guessing schemes (Arikan [15], Hanawal and Sundaresan [16], Sundaresan [13]) and more recently in the performance of coding for tasks (Bunte and Lapidoth [17]).

III. THE SETTING AND CONTRIBUTIONS

In this section, we formalize the notions projections and the families of interest. We then and highlight our contributions.

We begin by recalling the definition of \( \mathcal{I}_\alpha \) and its alternate expressions. We define

\[
\mathcal{I}_\alpha(P, Q) := \frac{\alpha}{1 - \alpha} \log \left[ \sum_x P(x)Q(x)^{\alpha - 1} \right] - \frac{1}{1 - \alpha} \log \sum_x P(x) + \log \sum_x Q(x)^\alpha
\]

(12)

and

\[
\mathcal{I}_\alpha(P, Q) := \frac{\alpha}{1 - \alpha} \log \left[ \sum_x \frac{P(x)}{\|P\|} \left( \frac{Q(x)}{\|Q\|} \right)^{\alpha - 1} \right],
\]

(13)

where

\[
\|Q\| := \left[ \sum_x Q(x)^\alpha \right]^{1/\alpha}.
\]

For convenience we suppress the dependence of \( \| \cdot \| \) on \( \alpha \); but this dependence should be borne in mind. Equation (12) is the same as (9) but with the parameter space extended to \( \alpha > 0, \alpha \neq 1 \). Equation (13) follows after regrouping of terms using the definition of \( \|P\| \) and \( \|Q\| \). For any \( \tau > 0 \), since \( Q/\|Q\| = \tau Q/\|\tau Q\| \), it follows that (13) can be extended to any pair of positive measures \( P \) and \( Q \) on \( \mathbb{X} \), and not just probability measures on \( \mathbb{X} \).

For each \( \alpha > 0, \alpha \neq 1 \), \( \mathcal{I}_\alpha(P, Q) \geq 0 \) with equality iff \( P = Q \).

Note that \( \mathcal{I}_\alpha(P, Q) = \infty \) if and only if either

- \( \alpha < 1 \) and \( P \) is not absolutely continuous with respect to \( Q \) (notation \( P \nless Q \)), or
- \( \alpha > 1 \) and \( P \) and \( Q \) are singular, i.e., the supports of \( P \) and \( Q \) are disjoint.

Let \( \mathcal{P}(\mathbb{X}) \) be the set of all probability measures on \( \mathbb{X} \). For a probability measure \( P \) on \( \mathbb{X} \), let \( \text{Supp}(P) = \{ x : P(x) > 0 \} \) denote the support of \( P \). For a set \( \mathbb{E} \) of probability measures, write \( \text{Supp}(\mathbb{E}) \) for the union of the supports of the members of \( \mathbb{E} \).

Let us now formally define what we mean by a reverse \( \mathcal{I}_\alpha \)-projection for \( \alpha > 0, \alpha \neq 1 \).

**Definition 1 (Reverse \( \mathcal{I}_\alpha \)-projection):** Let \( R \) be a probability measure on \( \mathbb{X} \). Let \( \mathbb{E} \) be a set of probability measures on \( \mathbb{X} \) such that \( \mathcal{I}_\alpha(R, P) < \infty \) for some \( P \in \mathbb{E} \). A probability measure \( Q \in \mathbb{E} \) satisfying

\[
\mathcal{I}_\alpha(R, Q) = \inf_{P \in \mathbb{E}} \mathcal{I}_\alpha(R, P) =: \mathcal{I}_\alpha(R, \mathbb{E})
\]

(14)

is called a reverse \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{E} \). If there is no such \( Q \in \mathbb{E} \), a probability measure \( Q \) in the closure of \( \mathbb{E} \) satisfying (14) is called a generalized reverse \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{E} \).

In a previous paper [2], we studied the forward \( \mathcal{I}_\alpha \)-projection of a probability measure \( R \) on a family. We reproduce [2, Defn. 6] here for they play a crucial role in this paper.

**Definition 2 (Forward \( \mathcal{I}_\alpha \)-projection):** Let \( R \) be a probability measure on \( \mathbb{X} \). Let \( \mathbb{E} \) be a set of probability measures on \( \mathbb{X} \) such that \( \mathcal{I}_\alpha(P, R) < \infty \) for some \( P \in \mathbb{E} \). A probability measure \( Q \in \mathbb{E} \) satisfying

\[
\mathcal{I}_\alpha(Q, R) = \inf_{P \in \mathbb{E}} \mathcal{I}_\alpha(P, R) =: \mathcal{I}_\alpha(\mathbb{E}, R)
\]

(15)

is called a forward \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{E} \).

In Definition 1 the minimization is with respect to the second argument, while in Definition 2 the minimization is with respect to the first argument. The focus in [2] was on forward projection on convex families and general alphabet spaces. We provided sufficient conditions for existence of the forward projection and argued that if the forward projection exists then it is unique. Convex families arise naturally from constraints placed by measurements of linear statistics. Examples of such families are linear families which we now define.

**Definition 3 (Linear family):** A linear family characterized by \( k \) functions \( f_i: \mathbb{X} \to \mathbb{R} \), \( 1 \leq i \leq k \), is the set of probability measures given by

\[
\mathbb{L} := \left\{ P \in \mathcal{P}(\mathbb{X}) : \sum_x P(x)f_i(x) = 0, \ i = 1, \ldots, k \right\}.
\]
Reverse $\mathcal{I}_\alpha$-projections, however, correspond to maximum likelihood or robust estimations, and are often on exponential families which we now define.

**Definition 4 (Exponential family):** An exponential family characterized by a probability measure $R$ and $k$ functions $f_i : \mathbb{X} \to \mathbb{R}$, $1 \leq i \leq k$, is the set of probability measures given by

$$\mathcal{M} := \{ P_\theta : \theta \in \Theta \subset \mathbb{R}^k \},$$

where

$$P_\theta(x)^{-1} := \frac{1}{Z(\theta)} \exp \left[ \log (R(x)^{-1}) + \sum_{i=1}^k \theta_i f_i(x) \right]$$

$$= \frac{Z(\theta) R(x)^{-1}}{\exp \left[ \sum_{i=1}^k \theta_i f_i(x) \right]} \quad \forall x \in \mathbb{X}$$

with $Z(\theta)$ being the normalization constant and $\Theta$ being the subset of $\mathbb{R}^k$ for which $P_\theta$ is a valid probability measure.

Examples of exponential families include
- Bernoulli distribution ($\mathbb{X} = \{0, 1\}$, $\Theta = (0, 1)$),
- Binomial distribution ($\mathbb{X} = \{0, 1, \ldots, L\}$, $\Theta = (0, 1)$),
- Poisson distribution ($\mathbb{X} = \{0, 1, \ldots\}$, $\Theta = (0, \infty)$), and
- Gaussian distribution ($\mathbb{X} = \mathbb{R}^d$, the parameter $\theta$ denotes the pair of mean and covariance).

The last two are given only as illustrative examples for they do not satisfy the finite $\mathbb{X}$ assumption of this paper. We will take up the study of reverse $\mathcal{I}_\alpha$-projection on the more general log-convex families which we now define.

**Definition 5 (Log-convex family):** A set $\mathcal{E}$ of probability measures on a finite alphabet set $\mathbb{X}$ is said to be log-convex if for any two probability measures $P$ and $Q$ in $\mathcal{E}$ that are not singular, and any $t \in [0, 1]$, the probability measure $P^t Q^{1-t}$ defined by

$$P^t Q^{1-t}(x) := \frac{P(x)^t Q(x)^{1-t}}{\sum_y P(y)^t Q(y)^{1-t}} \quad (17)$$

also belongs to $\mathcal{E}$.

Exponential families are log-convex, a fact that is easily checked.

We will also take up reverse projections on analogs of exponential families. To define these analogs, let us first define the generalized logarithm and the generalized exponential functions [18]. Let $\mathbb{R}_+ = \mathbb{R} \cup \{+\infty\}$ and let $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$.

**Definition 6:** For $\alpha > 0$, the $\alpha$-logarithm function, denoted $\ln_\alpha : \mathbb{R}_+ \to \mathbb{R}$, is defined to be

$$\ln_\alpha(u) := \begin{cases} \frac{u^{1-\alpha} - 1}{1-\alpha} & \alpha \neq 1 \\ \log(u) & \alpha = 1 \end{cases}$$

where the log function is the natural logarithm. Its functional inverse, the $\alpha$-exponential function, denoted $e_\alpha : \mathbb{R} \to \mathbb{R}_+$, is defined to be

$$e_\alpha(u) := \begin{cases} (\max\{1 + (1-\alpha)u, 0\})^{1/(1-\alpha)} & \alpha \neq 1 \\ \exp(u) & \alpha = 1. \end{cases}$$

It is easy to check that $e_\alpha(\ln_\alpha(u)) = u$ for $u > 0$ and that $\ln_\alpha(e_\alpha(u)) = u$ whenever $0 < e_\alpha(u) < \infty$.

The analogs of exponential families are the so-called $\alpha$-power-law families which we now define. (Compare Definitions 4 and 7.)

**Definition 7 (\(\alpha\)-power-law family):** Let $R$ be a probability measure such that if $\alpha > 1$ then $\text{Supp}(R) = \mathbb{X}$. An $\alpha$-power-law family characterized by the probability measure $R$ and $k$ functions $f_i : \mathbb{X} \to \mathbb{R}$, $1 \leq i \leq k$, is the set of probability measures given by

$$\mathcal{M}^{(\alpha)} := \{ P_\theta : \theta \in \Theta \subset \mathbb{R}^k \},$$

where

$$P_\theta(x)^{-1} := \frac{Z(\theta)}{\exp \left[ \ln_\alpha (R(x)^{-1}) + \sum_{i=1}^k \theta_i f_i(x) \right]} \quad \forall x \in \mathbb{X}, \quad (18)$$

*If $R(x)$ equals 0, then so does $P_\theta(x)$. \(\square\)
provided

\[ 1 + (1 - \alpha) \left[ \ln \alpha \left( R(x)^{-1} \right) + \sum_{i=1}^{k} \theta_i f_i(x) \right] > 0 \quad \forall x \in \mathbb{X}, \]

with \( Z(\theta) \) being the normalization constant and \( \Theta \) being the subset of \( \mathbb{R}^k \) for which \( P_\theta \) is a valid probability measure. Equivalently

\[ P_\theta(x)^{\alpha - 1} = Z(\theta)^{1 - \alpha} \left[ R(x)^{\alpha - 1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x) \right] > 0 \quad \forall x \in \mathbb{X}. \]  

(19)

When we wish to be explicit about the characterizing entities, we shall write \( \mathcal{M}(\alpha)(R, f_1, \ldots, f_k) \) for the family. In Appendix A we show that \( \mathcal{M}(\alpha) \) depends on \( R \) in only a weak manner. Any member \( P_{\theta^*} \in \mathcal{M}(\alpha) \) may equally well play the role of \( R \) and this merely corresponds to translation and scaling of the parameter space.

\( \mathcal{M}(\alpha) \) is not closed. Sometimes it will be required to consider its closure \( \text{cl}(\mathcal{M}(\alpha)) \).

One has the more general notion of \( \ln_\alpha \)-convex family as well (see van Erven and Harremoës [19]).

**Definition 8** (\( \ln_\alpha \)-convex family): A set \( \mathbb{E} \) of probability measures is said to be \( \ln_\alpha \)-convex if for any two probability measures \( P \) and \( Q \) in \( \mathbb{E} \) (that are not singular when \( \alpha \leq 1 \)), and any \( t \in [0, 1] \), the probability measure \( R \) defined by

\[ R^{-1} := Z e_\alpha \left( t \ln_\alpha(P^{-1}) + (1 - t) \ln_\alpha(Q^{-1}) \right) \]  

also belongs to \( \mathbb{E} \). The quantity \( Z \) is the normalization constant that makes \( R \) a probability measure.

Substitution of the definitions of \( e_\alpha \) and \( \ln_\alpha \) indicate that the probability measure \( R \) defined in (20) can be rewritten as

\[ Z^{-1} \left[ t P^{\alpha - 1} + (1 - t) Q^{\alpha - 1} \right] \overset{\triangle}{=} . \]

(21)

When \( \alpha = 1 \), \( \ln_\alpha \)-convexity is just log-convexity, thereby justifying that \( \ln_\alpha \)-convexity is an extension of log-convexity. Just as exponential families are log-convex, \( \alpha \)-power-law families are \( \ln_\alpha \)-convex, a fact that can be easily checked using (21).

While forward projections of interest are on convex families, reverse projections of interest, particularly those arising in estimation problems, are on log-convex, and by analogy, on \( \ln_\alpha \)-convex families. Log-convex or \( \ln_\alpha \)-convex families are not necessarily convex in the usual sense.

**Definition 8** is given only to complete the picture. We shall restrict attention in this paper to the \( \alpha \)-power-law family.

### A. closer look at our contributions.

For a given \( R \) and a given \( \mathbb{E} \) with some \( P \) such that \( \mathcal{J}_\alpha(R, P) < \infty \), we obviously have \( \mathcal{J}_\alpha(R, \mathbb{E}) < \infty \). If we consider a sequence \( (P_n) \subset \mathbb{E} \) such that \( \lim_{n \to \infty} \mathcal{J}_\alpha(R, P_n) = \mathcal{J}_\alpha(R, \mathbb{E}) \), by virtue of the continuity of \( \mathcal{J}_\alpha(R, \cdot) \) in the second argument (see [22] Rem. 5), all subsequential limits of \( (P_n) \) are generalized reverse \( \mathcal{J}_\alpha \)-projections. In this paper, we study example settings when the generalized reverse \( \mathcal{J}_\alpha \)-projection is unique, when it is not, and how one may characterize it, sometimes, as a forward \( \mathcal{J}_\alpha \)-projection. Specifically we do the following.

- **In Section IV** we study reverse \( \mathcal{J}_\alpha \)-projections on log-convex families. We show an example of nonuniqueness of generalized reverse \( \mathcal{J}_\alpha \)-projections on an exponential family when \( \alpha > 1 \). However uniqueness holds for \( \alpha < 1 \).

- **In Section V** our focus will be on the forward \( \mathcal{J}_\alpha \)-projection on certain convex families, in particular, linear families. We identify the form of the forward \( \mathcal{J}_\alpha \)-projection on a linear family \( \mathbb{L} \) and prove a necessary and sufficient condition for a \( Q \in \mathbb{L} \) to be the forward \( \mathcal{J}_\alpha \)-projection on \( \mathbb{L} \). We consider the cases \( \alpha > 1 \) and \( \alpha < 1 \) separately in two subsections. The proof for the \( \alpha < 1 \) case is similar to Csiszár and Shields’ proof for \( \alpha = 1 \) case [20]. For the proof of the \( \alpha > 1 \) case, we resort to the Lagrange multiplier technique. The structure of the forward \( \mathcal{J}_\alpha \)-projection naturally suggests a statistical model, namely the \( \alpha \)-power-law family \( \mathcal{M}(\alpha) \).

- **In Section VI** we study reverse \( \mathcal{J}_\alpha \)-projections on \( \mathcal{M}(\alpha) \), and show uniqueness of the generalized reverse projection for all \( \alpha > 0, \alpha \neq 1 \). To show this, we establish an orthogonality relationship between \( \mathcal{M}(\alpha) \) and an associated linear family. We then use this geometric property to turn a reverse \( \mathcal{J}_\alpha \)-projection on \( \mathcal{M}(\alpha) \) into a forward \( \mathcal{J}_\alpha \)-projection on the linear family. It will turn out that, sometimes, we may need to consider a larger family than just \( \text{cl}(\mathcal{M}(\alpha)) \).

\(^8\) A definition such as [18] is fraught with pesky issues of well-definedness. We have verified the equivalence of [19]. But a skeptical reader may simply take [19] as the starting point to define \( \mathcal{M}(\alpha) \). The definition in [18] is given only to highlight its similarity with Definition 4. Observe that, from [19], if \( \alpha < 1 \), \( R(x) = 0 \) implies \( P_{\theta}(x) = 0 \).

\(^9\) van Erven and Harremoës [19] gave a different name to what we call \( \ln_\alpha \)-convex family; they called this \( (\alpha - 1) \)-convex family. Our convention follows the notation for and parametrization of the generalized logarithm.
IV. REVERSE PROJECTION ONTO LOG-CONVEX SETS

We consider the cases $\alpha > 1$ and $\alpha < 1$ separately in the next two subsections. Before that, we present a lemma of some independent interest. This is an extension of a result for relative entropy ($\alpha = 1$); see Csiszár and Matúš [21, Eq. (3)], where (22) below is an equality.

**Lemma 9:** Let $P$ and $Q$ be probability measures on $\mathbb{X}$ that are mutually absolutely continuous. Let $R$ be any probability measure on $\mathbb{X}$ that is not singular with respect to $P$ or $Q$. Let $t \in [0, 1]$.

(a) If $\alpha < 1$, then
\[
 t \mathcal{J}_\alpha(R, P) + (1 - t) \mathcal{J}_\alpha(R, Q) \geq \mathcal{J}_\alpha(R, R^t Q^{1-t}) - \log \sum_x P'(x)^\alpha Q'(x)^{1-t},
\]
(22)
where $P'$ is the escort probability measure associated with $P$ given by
\[
P'(x) := \frac{P(x)^\alpha}{\sum_y P(y)^\alpha}
\]
and $Q'$ is the escort probability measure associated with $Q$.

(b) If $\alpha > 1$, the inequality in (22) is reversed.

**Proof:** Let us first observe that if $\alpha < 1$ and $R \not\ll R^t Q^{1-t}$, then, by the assumption that $P$ and $Q$ are mutually absolutely continuous, both sides of (22) are $+\infty$, and so (22) holds. We may thus assume that $R \ll R^t Q^{1-t}$ when $\alpha < 1$. Also, notice that the hypotheses imply that $R$ is not singular with respect to $P^t Q^{1-t}$. Hence, for both $\alpha < 1$ and $\alpha > 1$, we may take all the terms in (22) to be finite.

Let us write
\[
\sum_y P(y)^t Q(y)^{1-t} = \frac{P(x)^t (Q(x))^{1-t}}{\sum_y (P(y)^t (Q(y))^{1-t})}.
\]
Using this in (13) we get
\[
\mathcal{J}_\alpha(R, R^t Q^{1-t}) = \frac{\alpha}{1 - \alpha} \log \sum_x \frac{R(x)}{\|R\|} \left( \frac{P(x)}{\|P\|} \right)^t \left( \frac{Q(x)}{\|Q\|} \right)^{1-t} + \log \sum_x \left( \frac{P(x)}{\|P\|} \right)^{\alpha t} \left( \frac{Q(x)}{\|Q\|} \right)^{\alpha(1-t)}
\]
\[
\geq \frac{\alpha}{1 - \alpha} \log \sum_x \frac{R(x)}{\|R\|} \left( \frac{P(x)}{\|P\|} \right)^{\alpha - 1} \left( \frac{Q(x)}{\|Q\|} \right)^{\alpha - 1} \left( \log \sum_x \frac{R(x)}{\|R\|} \left( \frac{Q(x)}{\|Q\|} \right)^{\alpha - 1} \right)^{\alpha - 1} + \log \sum_x P'(x)^t Q'(x)^{1-t}
\]
\[
= t \mathcal{J}_\alpha(R, P) + (1 - t) \mathcal{J}_\alpha(R, Q) + \log \sum_x P'(x)^t Q'(x)^{1-t},
\]
for $\alpha < 1$, where the penultimate inequality follows by applying Hölder’s inequality to the inner-product within the first logarithm term, with exponents $1/t$ and $1/(1 - t)$. For $\alpha > 1$, the inequality is obviously reversed because the multiplication factor $\alpha/(1 - \alpha)$ is negative.

**A. Reverse $\mathcal{J}_\alpha$-projection for $\alpha > 1$**

Recall that the MMPLS on a log-convex family is the reverse $\mathcal{J}_\alpha$-projection of the empirical measure on the family for the case when $\alpha > 1$. For log-convex families, it is possible that multiple reverse $\mathcal{J}_\alpha$-projections may exist, and we provide an explicit example.

**Example 1:** Let $\mathbb{X} = \{0, 1, 2\}$, let $R$ be the uniform probability measure on $\mathbb{X}$, and let $E$ be the log-convex family of binomial distributions on $\mathbb{X}$ with parameter $\theta \in (0, 1)$. A member $P_\theta$ of the family is given by
\[
P_\theta(0) = (1 - \theta)^2, \quad P_\theta(1) = 2\theta(1 - \theta), \quad P_\theta(2) = \theta^2.
\]
such that, for every sequence $\theta$ of probability measures on $\mathcal{X}$ such that $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_\theta) < \infty$. Under these conditions, there exists a unique probability measure $\mathcal{Q}$ such that, for every sequence $(\mathcal{P}_n)$ in $\mathcal{E}$ satisfying $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n) \to \mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$, we have $\mathcal{P}_n \to \mathcal{Q}$ and $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{Q}) = \mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$.

**Proof:** The proof broadly follows the proof of Csiszár's [21, Th. 1].

Consider a sequence $(\mathcal{P}_n) \subset \mathcal{E}$ such that $\lim_n \mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n) = \mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$. Since $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$ is finite, we may assume without loss of generality that $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n)$ is finite for all $n$. Hence, for all $n$, $\mathcal{R}$ is not singular with respect to $\mathcal{P}_n$; indeed, $\mathcal{R} \ll \mathcal{P}_n$ and $\mathcal{P}_n \ll \mathcal{Q}$.

Eguchi and Kato [8] consider the problem of spontaneous clustering for a Gaussian mixture model with an unknown number of components, and put the possibility of multiple minima to good use. Very briefly, their procedure operates on the data as follows, and we refer the interested reader to [8] for further details. They first choose the parameter $\alpha$ using either the maximum range of the data or the Akaike information criterion. They then identify the resulting minima of $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n)$ for even $\alpha$, and put the possibility of multiple minima to good use. Very briefly, their procedure operates on the data as follows, and we refer the interested reader to [8] for further details. They first choose the parameter $\alpha$ using either the maximum range of the data or the Akaike information criterion. They then identify the resulting minima of $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n)$ for even $\alpha$, and put the possibility of multiple minima to good use.

![Figure 1](image)

**Fig. 1.** Multiple reverse $\mathcal{I}_\alpha$-projections are possible when $\alpha > 1$.

Figure 1 plots $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_\theta)$ as a function of $\theta$ for $\alpha = 2$ (plot on the left-hand side) and $\alpha = 4$ (plot on the right-hand side). Since $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_\theta)$ has mirror-symmetry around the point $\theta = 1/2$, a fact that can be easily checked, if there is a global minimum at $\theta^* \in (0, 1/2)$, then we have another global minimum at $1 - \theta^* \in (1/2, 1)$. This is the situation with the plot on the right-hand side.

Example 1 suggests a sequence $\mathcal{P}_n$ that converges to $\mathcal{P}_\infty$ in $\mathcal{E}$. Since $\mathcal{P}_n \ll \mathcal{Q}$ for all $n$, we have $\mathcal{P}_\infty \ll \mathcal{Q}$, and yet $\mathcal{P}_\infty = \mathcal{P}_{\infty^*}$ for even $\alpha$. All subsequential limits are of course generalization reverse $\mathcal{I}_\alpha$-projections.

**B. Reverse $\mathcal{I}_\alpha$-projection for $\alpha < 1$**

For $\alpha < 1$, the generalized reverse $\mathcal{I}_\alpha$-projection is unique, unlike the situation in the previous subsection.

**Theorem 10:** Let $\alpha < 1$. Let $\mathcal{E}$ be a log-convex set of mutually absolutely continuous probability measures on $\mathcal{X}$. Let $\mathcal{R}$ be a probability measure on $\mathcal{X}$ such that $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{E}) < \infty$. Under these conditions, there exists a unique probability measure $\mathcal{Q}$ such that, for every sequence $(\mathcal{P}_n)$ in $\mathcal{E}$ satisfying $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n) \to \mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$, we have $\mathcal{P}_n \to \mathcal{Q}$ and $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{Q}) = \mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$.

**Proof:** The proof broadly follows the proof of Csiszár’s [21, Th. 1].

Consider a sequence $(\mathcal{P}_n) \subset \mathcal{E}$ such that $\lim_n \mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n) = \mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$. Since $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{E})$ is finite, we may assume without loss of generality that $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_n)$ is finite for all $n$. Hence, for all $n$, $\mathcal{R}$ is not singular with respect to $\mathcal{P}_n$; indeed, $\mathcal{R} \ll \mathcal{P}_n$.

12The empirical measure $\mathcal{R}$ and the Gaussian $\mathcal{P}_\theta$ are singular. Following the formal definition in [2] Sec. II, strictly speaking, we have the relative $\alpha$-entropy $\mathcal{I}_\alpha(\mathcal{R}, \mathcal{P}_\theta) = \infty$. The expansion however does provide a valid expression for optimization although one cannot interpret it as the relative $\alpha$-entropy, and Eguchi and Kato [8] minimize the expression to get the MMPE.
for all \( n \). Apply Lemma \([2]\) with \( P = P_m, Q = P_n \), to get

\[
(t \mathcal{I}_\alpha(R, P_m) + (1 - t) \mathcal{I}_\alpha(R, P_n)) \geq \mathcal{I}_\alpha(R, \frac{P_m P_n^{1 - t}}{m, n}) - \log \sum_x P'_m(x)^t P'_n(x)^{1 - t} \]

\[
\geq \mathcal{I}_\alpha(R, E) - \log \sum_x P'_m(x)^t P'_n(x)^{1 - t},
\]

where last inequality follows from the hypothesis that \( \frac{P_m}{m, n} P_n^{1 - t} \in E \). Also observe that, by Hölder’s inequality,

\[
\sum_x P'_m(x)^t P'_n(x)^{1 - t} \leq \left( \sum_x P'_m(x)^t \right) \left( \sum_x P'_n(x)^{1 - t} \right) = 1.
\]

Let \( m, n \to \infty \) in (24) and use (25) to get

\[
\lim_{m, n \to \infty} \log \sum_x P'_m(x)^t P'_n(x)^{1 - t} = 0.
\]

Set \( t = 1/2 \) in this limit and undo the logarithm to get

\[
\lim_{m, n \to \infty} \sum_x \sqrt{P'_m(x) P'_n(x)} = 1
\]

so that

\[
\sum_x \left( \sqrt{P'_m(x)} - \sqrt{P'_n(x)} \right)^2 = 2 - 2 \sum_x \sqrt{P'_m(x) P'_n(x)} \to 0 \quad \text{as } m, n \to \infty.
\]

Thus \( (P'_n) \) is a Cauchy sequence. It must converge to some \( Q' \), an escort of some probability measure \( Q \). Given our finite alphabet assumption, we must then have \( P_n \to Q \).

If \( (Q_n) \subset E \) is another sequence such that \( \mathcal{I}_\alpha(R, Q_n) \to \mathcal{I}_\alpha(R, E) \), then since \( P_n \) and \( Q_n \) can be merged together, \( (Q_n) \) must also converge to the same \( Q \). The generalized reverse \( \mathcal{I}_\alpha \)-projection is therefore unique.

By continuity of \( \mathcal{I}_\alpha(R, \cdot) \), see [2] Rem. 5, we also have \( \mathcal{I}_\alpha(R, Q) = \mathcal{I}_\alpha(R, E) \).

The proof fails for \( \alpha > 1 \) because the inequality in (24) is in the opposite direction, and one cannot conclude that \( (P'_n) \) is a Cauchy sequence. Indeed, the previous subsection provides a counterexample for noncovergence and nonuniqueness of reverse \( \mathcal{I}_\alpha \)-projection on a log-convex family, when \( \alpha > 1 \).

V. FORWARD \( \mathcal{I}_\alpha \)-PROJECTION

In this section, we will recall some results on forward \( \mathcal{I}_\alpha \)-projection from [2] along with some refinements for our restricted finite alphabet setting. The proofs here use elementary tools and exploit the finite alphabet assumption. The results will then be used to turn a reverse \( \mathcal{I}_\alpha \)-projection on an \( \alpha \)-power-law family into a forward \( \mathcal{I}_\alpha \)-projection on a linear family.

A. \( \alpha < 1 \):

The result for \( \alpha < 1 \) is the following. It establishes the form of the forward \( \mathcal{I}_\alpha \)-projection on a linear family.

Theorem 11: Let \( \alpha < 1 \). Let \( \mathbb{L} \) be a linear family characterized by \( f_i, i = 1, \ldots, k \). Let \( R \) be a probability measure with full support. Then the following hold.

(a) \( R \) has a forward \( \mathcal{I}_\alpha \)-projection on \( \mathbb{L} \). Call it \( Q \).

(b) \( \text{Supp}(Q) = \text{Supp}(\mathbb{L}) \) and the Pythagorean equality holds (see Figure [2]):

\[
\mathcal{I}(P, R) = \mathcal{I}(P, Q) + \mathcal{I}(Q, R) \quad \forall P \in \mathbb{L}.
\]

(c) The forward \( \mathcal{I}_\alpha \)-projection \( Q \) satisfies

\[
Z^{\alpha - 1} Q(x)^{\alpha - 1} = R(x)^{\alpha - 1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i^* f_i(x) \quad \forall x \in \text{Supp}(\mathbb{L}),
\]

where \( \theta_1^*, \ldots, \theta_k^* \) are scalars and \( Z \) is the normalization constant that makes \( Q \) a probability measure.

(d) The forward \( \mathcal{I}_\alpha \)-projection is unique.

Proof: (a) The mapping \( P \mapsto \mathcal{I}(P, R) \) is continuous [2] Rem. 5] and \( \mathbb{L} \) is compact. Hence the forward \( \mathcal{I}_\alpha \)-projection exists.
(b) This was proved in greater generality for general measure spaces and for any convex set of probability measures, not just linear families. We reproduce the proof here to make the paper self-contained.

Let \( P \in \mathbb{L} \) and let \( P_t = (1 - t)Q + tP, \ 0 \leq t \leq 1 \). Since \( P_t \in \mathbb{L} \), the mean value theorem says that for each \( t \in (0,1) \), there exists \( \tilde{t} \in (0,t) \) such that

\[
0 \leq \frac{1}{t} \mathcal{J}_\alpha(P_t, R) - \mathcal{J}_\alpha(Q, R) = \frac{d}{ds} \mathcal{J}_\alpha(P_s, R)|_{s=\tilde{t}}.
\] **(28)**

The first inequality follows from the fact that \( Q \) is the projection. Using \((12)\), we see that

\[
\frac{d}{ds} \mathcal{J}_\alpha(P_s, R) = \frac{\alpha}{1 - \alpha} \left[ \sum_x (P(x) - Q(x))R(x)^{\alpha - 1} - \frac{\sum_x (P(x) - Q(x))P_s(x)^{\alpha - 1}}{\sum_x P_s(x)^\alpha} \right].
\] **(29)**

As \( t \downarrow 0 \), from \((29)\) and from the inequality in \((28)\), we get

\[
\frac{\sum_x (P(x) - Q(x))R(x)^{\alpha - 1}}{\sum_x Q(x)R(x)^{\alpha - 1}} \geq \frac{\sum_x (P(x) - Q(x))Q(x)^{\alpha - 1}}{\sum_x Q(x)^\alpha},
\] **(30)**

which, using \((12)\), can be seen to be equivalent to \((26)\), but with inequality \(\geq\) in place of equality.

Suppose \( Q(x) = 0 \) for an \( x \in \text{Supp}(P) \). Then \( \alpha < 1 \) implies that right-hand side of \((29)\) goes to \(-\infty\) as \( t \downarrow 0 \), which contradicts the nonnegativity requirement in \((23)\). Hence \( P \ll Q \). Since \( P \) was arbitrary, we have \( \text{Supp}(Q) = \text{Supp}(\mathbb{L}) \).

We will now establish equality in \((26)\). Once again, let \( P \in \mathbb{L} \). Since \( \text{Supp}(Q) = \text{Supp}(\mathbb{L}) \), we can find a new \( \tilde{t} < 0 \) such that \( P_t = (1 - t)Q + tP \) is a valid probability measure for all \( t \in (\tilde{t}, 0) \). Hence \( P_t \in \mathbb{L} \) for all \( t \in (\tilde{t}, 0) \). Since \( t < 0 \), we have

\[
0 \geq \frac{1}{t} [\mathcal{J}_\alpha(P_t, R) - \mathcal{J}_\alpha(Q, R)] \ \forall t \in (\tilde{t}, 0).
\]

An argument similar to the one that led to \((30)\) now proves that \((30)\) holds with \(\leq\) as well. This proves \((26)\) and completes the proof of (b).

(c) This follows the proof of Csiszár and Shields proof for the case \( \alpha = 1 \) \([20, \text{Th. 3.2}]\).

From \((16)\), it is clear that the probability measures \( P \in \mathbb{L} \), when considered as \( |\text{Supp}(\mathbb{L})|\)-dimensional vectors, belong to the orthogonal complement \( \mathcal{F}^\perp \) of the subspace \( \mathcal{F} \) of \( \mathbb{R}^{|	ext{Supp}(\mathbb{L})|} \) spanned by the vectors \( f_i(\cdot), i = 1, \ldots, k \), restricted to \( \text{Supp}(\mathbb{L}) \).

These \( P \in \mathbb{L} \) actually span \( \mathcal{F}^\perp \). (This follows from the fact that if a subspace of \( \mathbb{R}^{|	ext{Supp}(\mathbb{L})|} \) contains a vector all of whose components are strictly positive, here \( Q \), then it is spanned by the probability vectors of that space.) Using \((26)\), which is the same as \((30)\) with equality, we have

\[
\sum_x P(x) \left( \frac{R(x)^{\alpha - 1}}{\sum_a Q(a)R(a)^{\alpha - 1}} - \frac{Q(x)^{\alpha - 1}}{\sum_a Q(a)^\alpha} \right) = 0 \ \forall P \in \mathbb{L}.
\]

Consequently, the vector

\[
\frac{R(\cdot)^{\alpha - 1}}{\sum_a Q(a)R(a)^{\alpha - 1}} - \frac{Q(\cdot)^{\alpha - 1}}{\sum_a Q(a)^\alpha}
\]

belongs to \((\mathcal{F}^\perp)^\perp = \mathcal{F} \), that is,

\[
\frac{R(x)^{\alpha - 1}}{\sum_a Q(a)R(a)^{\alpha - 1}} - \frac{Q(x)^{\alpha - 1}}{\sum_a Q(a)^\alpha} = \sum_{i=1}^k \lambda_i f_i(x) \ \forall x \in \text{Supp}(\mathbb{L})
\]

for some scalars \( \lambda_i, i = 1, \ldots, k \). This verifies \((27)\) for obvious choices of \( Z \) and \( \theta_i^* \).
(d) Let \( Q_1 \) and \( Q_2 \) be two forward projections of \( R \) on \( \mathbb{L} \). Then \( \mathcal{I}_\alpha(Q_1, R) = \mathcal{I}_\alpha(Q_2, R) \). By (b), we have
\[
\mathcal{I}_\alpha(Q_2, R) = \mathcal{I}_\alpha(Q_2, Q_1) + \mathcal{I}_\alpha(Q_1, R).
\]
Canceling \( \mathcal{I}_\alpha(Q_2, R) \) and \( \mathcal{I}_\alpha(Q_1, R) \), we get \( \mathcal{I}_\alpha(Q_2, Q_1) = 0 \) which implies \( Q_1 = Q_2 \). \( \blacksquare \)

One can also state a converse.

**Theorem 12:** Let \( \alpha < 1 \). Let \( Q \in \mathbb{L} \) be a probability measure of the form \( (27) \). Then \( Q \) satisfies \( (26) \) and is the forward \( \mathcal{I}_\alpha \)-projection \( R \) on \( \mathbb{L} \).

**Proof:** If \( Q \in \mathbb{L} \) is of the stated form, then since every \( P \in \mathbb{L} \) satisfies
\[
\sum_{i=1}^{k} P(x)f_i(x) = \sum_{i=1}^{k} Q(x)f_i(x) = 0, \quad i = 1, \ldots, k,
\]
we have
\[
Z^{\alpha-1} \sum_x P(x)Q(x)^{\alpha-1} = \sum_x P(x)R(x)^{\alpha-1},
\]
and
\[
Z^{\alpha-1} \sum_x Q(x)^\alpha = \sum_x Q(x)R(x)^{\alpha-1}.
\]
Combining the above two equations to eliminate \( Z^{\alpha-1} \), we get
\[
\sum_x P(x)R(x)^{\alpha-1} = \sum_x \frac{Q(x)R(x)^{\alpha-1}}{\sum_x Q(x)^\alpha} \cdot \sum_x P(x)Q(x)^{\alpha-1},
\]
which, using \( (12) \), can be seen to be equivalent to \( (26) \). Thus, for any \( P \in \mathbb{L} \), we have
\[
\mathcal{I}_\alpha(P, R) = \mathcal{I}_\alpha(P, Q) + \mathcal{I}_\alpha(Q, R)
\]
\[
\geq \mathcal{I}_\alpha(Q, R),
\]
which implies that \( Q \) is the forward \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{L} \). \( \blacksquare \)

**B. \( \alpha > 1 \):**

We now establish the form of the forward \( \mathcal{I}_\alpha \)-projection on a linear family when \( \alpha > 1 \). The following result may be seen as a refinement of \( [2, \text{Th. 10}(a)] \).

**Theorem 13:** Let \( \alpha > 1 \). Let \( \mathbb{L} \) be a linear family characterized by \( f_i, i = 1, \ldots, k \). Let \( R \) be a probability measure with full support. Then the following hold.
(a) \( R \) has a forward \( \mathcal{I}_\alpha \)-projection on \( \mathbb{L} \). Call it \( Q \).
(b) The forward \( \mathcal{I}_\alpha \)-projection \( Q \) satisfies
\[
Z^{\alpha-1}Q(x)^{\alpha-1} = \left[ R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i^* f_i(x) \right]_+, \quad \forall x \in \mathbb{X},
\]
where \( \theta_1^*, \ldots, \theta_k^* \) are scalars, \( Z \) is the normalization constant that makes \( Q \) a probability measure, and \( \lfloor u \rfloor_+ = \max\{ u, 0 \} \).
(c) The Pythagorean inequality holds:
\[
\mathcal{I}_\alpha(P, R) \geq \mathcal{I}_\alpha(P, Q) + \mathcal{I}_\alpha(Q, R) \quad \forall P \in \mathbb{L}.
\]
(d) The forward \( \mathcal{I}_\alpha \)-projection is unique.
(e) If \( \text{Supp}(Q) = \text{Supp}(\mathbb{L}) \), then \( (32) \) holds with equality.

**Proof:** (a) The mapping \( P \mapsto \mathcal{I}_\alpha(P, R) \) is continuous \( [2, \text{Prop. 2}] \) and \( \mathbb{L} \) is compact. Hence the forward \( \mathcal{I}_\alpha \)-projection exists.
(b) The optimization problem for the forward $\mathcal{F}_\alpha$-projection is
\[
\min_P \mathcal{F}_\alpha(P, R)
\]
subject to
\[
\sum_x P(x)f_i(x) = 0, \quad i = 1, \ldots, k
\]
\[
\sum_x P(x) = 1
\]
\[
P(x) \geq 0 \quad \forall x \in \mathcal{X}.
\]  
(33)  
(34)  
(35)  
(36)

We will proceed in a sequence of steps.

(i) Observe that $\mathcal{F}_\alpha(\cdot, R)$, in addition to being continuous, is also continuously differentiable. Indeed, we have
\[
\frac{\partial}{\partial P(x)} \mathcal{F}_\alpha(P, R) = \frac{\alpha}{1 - \alpha} \left[ \frac{R(x)^{\alpha-1}}{\sum_a P(a)R(a)^{\alpha-1}} - \frac{P(x)^{\alpha-1}}{\sum_a P(a)^{\alpha}} \right].
\]  
(37)

Both denominators are bounded away from zero because for any $P \in \mathbb{L}$, we have $\max_x P(x) \geq 1/|\mathcal{X}|$, and therefore
\[
\sum_a P(a)R(a)^{\alpha-1} \geq \frac{1}{|\mathcal{X}|} \cdot \min_a R(a)^{\alpha-1} > 0,
\]
and
\[
\sum_a P(a)^{\alpha} \geq \frac{1}{|\mathcal{X}|^{\alpha}} > 0.
\]

Consequently, the partial derivative (37) exists everywhere on $\mathbb{R}^{|\mathcal{X}|}$, and is continuous because the terms involved are continuous. (The numerator of the second term in (37) is continuous because $\alpha > 1$).

(ii) Since the equality constraints in (34) and (35) arise from affine functions, and the inequality constraints in (36) arise from linear functions, we may apply [22, Prop. 3.3.7] to conclude that there exist Lagrange multipliers ($\lambda_i, i = 1, \ldots, k$, $\nu$, and $(\mu(x), x \in \mathcal{X})$ associated with the constraints (34), (35), and (36), respectively, that satisfy:
\[
\frac{\alpha}{1 - \alpha} \left[ \frac{Q(x)^{\alpha-1}}{\sum_a Q(a)^{\alpha}} - \frac{R(x)^{\alpha-1}}{\sum_a Q(a)R(a)^{\alpha-1}} \right] = \sum_{i=1}^k \lambda_i f_i(x) - \mu(x) + \nu \quad \forall x
\]
\[
\mu(x) \geq 0 \quad \forall x
\]
\[
\mu(x)Q(x) = 0 \quad \forall x.
\]  
(38)  
(39)  
(40)

In writing (38), we have substituted (37) for $\frac{\partial}{\partial P(x)} \mathcal{F}_\alpha(P, R)$.

(iii) Multiplying (38) by $Q(x)$, summing over all $x \in \mathcal{X}$, using $Q \in \mathbb{L}$, and using (40), we see that $\nu = 0$.

(iv) If $Q(x) > 0$, we must have $\mu(x) = 0$ from (40), and its substitution in (38) yields, for all such $x$,
\[
Q(x)^{\alpha-1} = \frac{R(x)^{\alpha-1}}{\sum_a Q(a)R(a)^{\alpha-1}} + \frac{1 - \alpha}{\alpha} \sum_{i=1}^k \lambda_i f_i(x).
\]  
(41)

If $Q(x) = 0$, (38) implies that
\[
\frac{R(x)^{\alpha-1}}{\sum_a Q(a)R(a)^{\alpha-1}} + \frac{1 - \alpha}{\alpha} \sum_{i=1}^k \lambda_i f_i(x) = \frac{(1 - \alpha)}{\alpha} \mu(x) \leq 0,
\]  
(42)

where the last inequality holds because of (39) and $\alpha > 1$. Therefore, (41) and (42) may be combined as
\[
Z^{\alpha-1}Q(x)^{\alpha-1} = \left[ R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^k \theta_i^* f_i(x) \right]_+ \quad \forall x \in \mathcal{X},
\]
where the choices of $Z$ and $\theta_i^*$ are obvious. This verifies (31) and completes the proof of (b).

(c) Using (31), for any $P \in \mathbb{L}$, we have
\[
\sum_x P(x)Q(x)^{\alpha-1} \geq Z^{1-\alpha} \sum_x P(x) \left[ R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^k \theta_i^* f_i(x) \right] = Z^{1-\alpha} \sum_x P(x)R(x)^{\alpha-1},
\]  
(43)
where the first inequality follows because the terms on the right-hand side corresponding to \( x \) with \( Q(x) = 0 \) are nonpositive, and the last equality follows because \( P \in \mathbb{L} \). Similarly,

\[
\sum_x Q(x)^\alpha = \sum_x Q(x)Q(x)^{\alpha - 1}
\]

\[
= Z^{1-\alpha} \sum_x Q(x) \left[ R(x)^{\alpha - 1} + (1 - \alpha) \sum_{i=1}^k \theta_i^* f_i(x) \right]
\]

\[
= Z^{1-\alpha} \sum_x Q(x)R(x)^{\alpha - 1},
\]

(44)

where the last equality follows because \( Q \in \mathbb{L} \). Combining (43) and (44) we get

\[
\sum_x P(x)R(x)^{\alpha - 1} \leq \frac{\sum_x Q(x)R(x)^{\alpha - 1}}{\sum_x Q(x)^\alpha} \cdot \sum_x P(x)Q(x)^{\alpha - 1},
\]

(45)

which, using (12), is equivalent to (32). This completes the proof of (c).

(d) Same as proof of Theorem 11 (d), with inequalities instead of equalities.

(e) If \( \text{Supp}(Q) \neq \text{Supp}(\mathbb{L}) \), then we have equality in (43), hence equality in (45), and hence equality in (32). This completes the proof of (e) and concludes the proof of the theorem.

As in the \( \alpha < 1 \) case, one has a converse.

**Theorem 14:** Let \( \alpha > 1 \). Let \( Q \in \mathbb{L} \) be a probability measure of the form (31). Then \( Q \) satisfies (32) for every \( P \in \mathbb{L} \), and \( Q \) is the forward \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{L} \).

**Proof:** The proof of Theorem 13 (c) shows that (32) holds. Now, for any \( P \in \mathbb{L} \), we have

\[
\mathcal{I}_\alpha(P, R) \geq \mathcal{I}_\alpha(P, Q) + \mathcal{I}_\alpha(Q, R)
\]

\[
\geq \mathcal{I}_\alpha(Q, R),
\]

which implies that \( Q \) is the forward \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{L} \).

When \( \alpha > 1 \), in general, \( \text{Supp}(Q) \neq \text{Supp}(\mathbb{L}) \) as shown by the following counterexample, and the Pythagorean inequality (32) may be strict.

**Example 2:** Let \( \alpha = 2 \). Let \( \mathbb{X} = \{1, 2, 3, 4\} \). Write \( P = (p_1, p_2, p_3, p_4) \) for a probability measure on \( \mathbb{X} \). Define the linear family \( \mathbb{L} \) to be

\[
\mathbb{L} = \{ P \in \mathcal{P}(\mathbb{X}) : 8p_1 + 4p_2 + 2p_3 + p_4 = 7 \}.
\]

Let \( R \) be the uniform probability measure on \( \mathbb{X} \). We claim that the forward \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{L} \) is \( Q = (3/4, 1/4, 0, 0) \). First, \( Q \in \mathbb{L} \) because \( 8q_1 + 4q_2 + 2q_3 + q_4 = 8 \times 3/4 + 4 \times 1/4 + 0 = 7 \). Second, \( Q \) is of the form (31). To see this, let us note that \( f_1(\cdot) = (1, -3, -5, -6) \). Take \( \theta^*_1 = -1/20 \) and \( Z = 2/5 \). Then

\[
\left[ R(\cdot)^{\alpha - 1} + (1 - \alpha)\theta^*_1 f_1(\cdot) \right]_+ = \left[ R(\cdot) - \theta^*_1 f_1(\cdot) \right]_+
\]

\[
= \left[ \left[ 1/4 + 1/20 \right]_+, \left[ 1/4 - 3/20 \right]_+, \left[ 1/4 - 5/20 \right]_+, \left[ 1/4 - 6/20 \right]_+ \right]
\]

\[
= (6/20, 3/20, 0, 0)
\]

\[
= Z \cdot Q(\cdot).
\]

That \( Q \) is the forward \( \mathcal{I}_\alpha \)-projection now follows from Theorem 14.

Clearly \( \text{Supp}(Q) \subsetneq \text{Supp}(\mathbb{L}) \). Also for \( P = (0.8227, 0.0625, 0.0536, 0.0612) \in \mathbb{L} \), numerical calculations yield a strict inequality in (32) since the left-hand side and the right-hand side of (32) evaluate to 1.0114 and 0.9871, respectively. See also [2] Rem. 13] where this counterexample showed that transitivity of projections does not hold for \( \alpha > 1 \). In both situations, the issue is that \( \text{Supp}(Q) \neq \text{Supp}(\mathbb{L}) \).

**VI. ORTHOGONALITY BETWEEN THE \( \alpha \)-POWER-LAW FAMILY AND THE LINEAR FAMILY**

The focus of this section is on the geometry of the \( \alpha \)-power-law family with respect to its associated linear family, and its exploitation. See Figure 3. We treat the cases \( \alpha < 1 \) and \( \alpha > 1 \) separately. Theorems 17 and 21 are the main contributions.
A. \( \alpha < 1 \):

This case is the simpler of the two. The core result of this section, one on which the main result Theorem 17 hinges, is the following that shows that the case \( \alpha < 1 \) is similar to \( \alpha = 1 \) [20, Th. 3.2].

**Theorem 15:** Let \( \alpha < 1 \). Let \( \mathbb{L} \) be a linear family characterized by \( f_i, i = 1, \ldots, k \), as in [16]. Let \( R \) be a probability measure with full support. Let \( M(\alpha) \) be the \( \alpha \)-power-law family, as in Definition 7 characterized by \( R \) and the same \( k \) functions \( f_i, i = 1, \ldots, k \). Let \( Q \) be the forward \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{L} \). Then the following hold.

(a) \( \mathbb{L} \cap \text{cl}(M(\alpha)) = \{Q\} \).
(b) For every \( P \in \mathbb{L} \), we have

\[
\mathcal{I}_\alpha(P, R) = \mathcal{I}_\alpha(P, Q) + \mathcal{I}_\alpha(Q, R). \tag{46}
\]

(c) If \( \text{Supp}(\mathbb{L}) = \mathbb{X} \), then \( \mathbb{L} \cap M(\alpha) = \{Q\} \).

**Proof:** Statement (b) is the same as Theorem 11(c). Let us observe from Theorem 11 that when \( \text{Supp}(\mathbb{L}) = \mathbb{X} \), the forward \( \mathcal{I}_\alpha \)-projection \( Q \) of \( R \) on \( \mathbb{L} \) satisfies

\[
Z^{\alpha-1}Q(x)^{\alpha-1} = R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i^* f_i(x) \quad \forall x \in \mathbb{X}
\]

for some scalars \( Z, \theta_1^*, \ldots, \theta_k^* \). Hence \( Q \in M(\alpha) \). Since \( Q \) is also in \( \mathbb{L} \), we have \( Q \in \mathbb{L} \cap M(\alpha) \).

Thus, in general, when \( \text{Supp}(\mathbb{L}) \) is not necessarily \( \mathbb{X} \), if we can show that (i) every member of \( \mathbb{L} \cap \text{cl}(M(\alpha)) \) satisfies (46), and (ii) \( \mathbb{L} \cap \text{cl}(M(\alpha)) \) is nonempty, then, since any member satisfying (46) is also forward \( \mathcal{I}_\alpha \)-projection and since the forward \( \mathcal{I}_\alpha \)-projection is unique, the theorem will be established. We now proceed to show (i) and (ii).

(i) Every \( \tilde{Q} \in \mathbb{L} \cap \text{cl}(M(\alpha)) \) satisfies (46).

Let \( (Q_n) \subset M(\alpha) \) be such that \( Q_n \to \tilde{Q} \). Then, for each \( n \), there exist \( \theta^{(n)} = \left( \theta_1^{(n)}, \ldots, \theta_k^{(n)} \right) \in \mathbb{R}^k \) and a constant \( Z_n \) such that

\[
Z_n^{\alpha-1}Q_n(x)^{\alpha-1} = R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i^{(n)} f_i(x) \quad \forall x \in \mathbb{X}. \tag{48}
\]

Since, for any \( P \in \mathbb{L} \), we have

\[
\sum_x P(x) f_i(x) = \sum_x \tilde{Q}(x) f_i(x) = 0, \quad i = 1, \ldots, k,
\]

by taking expectation with respect to \( P \) and \( \tilde{Q} \) on both sides of (48), we get

\[
Z_n^{\alpha-1} \sum_x P(x) Q_n(x)^{\alpha-1} = \sum_x P(x) R(x)^{\alpha-1}
\]

and

\[
Z_n^{\alpha-1} \sum_x \tilde{Q}(x) Q_n(x)^{\alpha-1} = \sum_x \tilde{Q}(x) R(x)^{\alpha-1},
\]

respectively. Using the above two equations to eliminate \( Z_n^{\alpha-1} \), we get

\[
\sum_x P(x) R(x)^{\alpha-1} = \frac{\sum_x \tilde{Q}(x) R(x)^{\alpha-1}}{\sum_x \tilde{Q}(x) Q_n(x)^{\alpha-1}} \sum_x P(x) Q_n(x)^{\alpha-1}.
\]
Letting $n \to \infty$, and then by using (12), we get (46) with $Q$ replaced by $\tilde{Q}$. This proves (i).

(ii) $\mathbb{L} \cap \text{cl}(\mathbb{M}^{(\alpha)})$ is nonempty.

Let
\[ \tau_i^{(n)} := \frac{1}{n} \sum_x R(x) f_i(x) \]
\[ \hat{f}_i(\cdot) := f_i(\cdot) - \tau_i^{(n)} R(\cdot)^{\alpha-1}, \quad i = 1, \ldots, k, \]
and define the sequence of linear families
\[ \mathbb{L}_n := \{ P \in \mathcal{P}(\mathbb{X}) : \sum_x P(x) \hat{f}_i(x) = 0, i = 1, \ldots, k \}. \]

The $\tau_i^{(n)}$'s are chosen so that $(1 - \frac{1}{n}) Q + \frac{1}{n} R \in \mathbb{L}_n$, and so $\text{Supp}(\mathbb{L}_n) = \mathbb{X}$. Let $Q_n$ be the forward $\mathcal{I}_\alpha$-projection of $R$ on $\mathbb{L}_n$. Then, by virtue of Theorem 11-(b), we have $\text{Supp}(Q_n) = \mathbb{X}$, and by virtue of Theorem 11-(c), we have
\[ Z_n^{\alpha-1} Q_n(x)^{\alpha-1} = R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^k \theta_i^{(n)} \hat{f}_i(x) \]
\[ = \left[ 1 - (1 - \alpha) \sum_{i=1}^k \theta_i^{(n)} \tau_i^{(n)} \right] R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^k \theta_i^{(n)} f_i(x) \quad \forall x \in \mathbb{X}. \quad (49) \]

Taking expectation with respect to $Q$ on both sides, and using $\sum_x Q(x) f_i(x) = 0$, $i = 1, \ldots, k$, we get
\[ Z_n^{\alpha-1} \sum_x Q(x) Q_n(x)^{\alpha-1} = \left[ 1 - (1 - \alpha) \sum_{i=1}^k \theta_i^{(n)} \tau_i^{(n)} \right] \sum_x Q(x) R(x)^{\alpha-1}. \]

As the summations on either side are finite and strictly positive for each $n$, the term within square brackets in the above equation is also strictly positive for each $n$. Rescaling (49) appropriately, we see that $Q_n \in \mathbb{M}^{(\alpha)}$. Note also that $\tau_i^{(n)} \to 0$ as $n \to \infty$ for $i = 1, \ldots, k$. Hence the limit of any convergent subsequence of $(Q_n)$ belongs to $\mathbb{L} \cap \text{cl}(\mathbb{M}^{(\alpha)})$. This verifies (ii) and concludes the proof of the theorem.

We now argue that the family $\text{cl}(\mathbb{M}^{(\alpha)})$ and $\mathbb{L}$ are “orthogonal” to each other, in a sense made precise in the statement of the next result.

**Corollary 16**: Under the hypotheses of Theorem 15 the following additional statements hold.

(a) For every $P \in \mathbb{L}$ and every $S \in \text{cl}(\mathbb{M}^{(\alpha)})$, we have
\[ \mathcal{I}_\alpha(P, S) = \mathcal{I}_\alpha(P, Q) + \mathcal{I}_\alpha(Q, S). \quad (50) \]

(b) For any $S \in \text{cl}(\mathbb{M}^{(\alpha)})$, the forward $\mathcal{I}_\alpha$-projection of $S$ on $\mathbb{L}$ is $Q$.

**Proof**: Since any member of $\mathbb{M}^{(\alpha)}$ can play the role of $R$, and since, by Theorem 15 $\mathbb{L} \cap \text{cl}(\mathbb{M}^{(\alpha)}) = \{ Q \}$, $Q$ is the forward $\mathcal{I}_\alpha$-projection of any member of $\mathbb{M}^{(\alpha)}$ on $\mathbb{L}$. Therefore (50) holds for every $P \in \mathbb{L}$ and every $S \in \mathbb{M}^{(\alpha)}$. Furthermore, (50) holds for the limit of any sequence of members of $\mathbb{M}^{(\alpha)}$, and hence (a) and (b) hold for members of $\text{cl}(\mathbb{M}^{(\alpha)}) \setminus \mathbb{M}^{(\alpha)}$ as well.

Let us now return to the compression problem discussed in Section II-C and show the connection between the reverse $\mathcal{I}_\alpha$-projection on an $\alpha$-power-law family and a forward $\mathcal{I}_\alpha$-projection on a linear family.

**Theorem 17**: Let $\alpha < 1$. Let $\hat{P}$ be a probability measure on $\mathbb{X}$. Let $\mathbb{M}^{(\alpha)}$ be characterized by the probability measure $R$ and the functions $f_i, i = 1, \ldots, k$. Let $\mathbb{L}$ be the associated linear family characterized by $f_i, i = 1, \ldots, k$, and assume that it is nonempty. Let $R$ have full support.

Define $\hat{\mathbb{L}}$ as
\[ \hat{\mathbb{L}} := \{ P \in \mathcal{P}(\mathbb{X}) : \sum_x P(x) \hat{f}_i(x) = 0 \}, \quad (51) \]
where
\[ \hat{f}_i(\cdot) = f_i(\cdot) - \tau_i^R R(\cdot)^{\alpha-1} \quad (52) \]
with
\[ \tau_i^R = \frac{\sum_x \hat{P}(x)f_i(x)}{\sum_x \hat{P}(x)R(x)^{\alpha-1}}, \quad i = 1, \ldots, k. \] (53)

Let \( Q \) be the forward \( \mathcal{J}_\alpha \)-projection of \( R \) on \( \hat{L} \).

(a) If \( \text{Supp}(Q) = \mathbb{X} \), then \( Q \) is the unique reverse \( \mathcal{J}_\alpha \)-projection of \( \hat{P} \) on \( \mathbb{M}^{(\alpha)} \).

(b) If \( \text{Supp}(Q) \neq \mathbb{X} \), then \( \hat{P} \) does not have reverse \( \mathcal{J}_\alpha \)-projection on \( \mathbb{M}^{(\alpha)} \). However, \( Q \) is the unique reverse \( \mathcal{J}_\alpha \)-projection of \( \hat{P} \) on \( \text{cl}(\mathbb{M}^{(\alpha)}) \).

\textbf{Proof:} \( \hat{L} \) is constructed so that \( \hat{P} \in \hat{L} \) (which is easy to check) and, further, \( \hat{L} \) is orthogonal to \( \mathbb{M}^{(\alpha)} \) in the sense of Corollary \[16\]. We now verify the latter statement. For concreteness, we will index the \( \alpha \)-power-law family by its characterizing entities. By Corollary \[16\], \( \hat{L} \) is orthogonal to \( \mathbb{M}^{(\alpha)}(R, \hat{f}_1, \ldots, \hat{f}_k) \). It therefore suffices to show that \( \mathbb{M}^{(\alpha)}(R, \hat{f}_1, \ldots, \hat{f}_k) = \mathbb{M}^{(\alpha)}(R, f_1, \ldots, f_k) \). Take any \( P_\theta \in \mathbb{M}^{(\alpha)}(R, f_1, \ldots, f_k) \). Then, for each \( x \in \mathbb{X} \), we have
\[
Z(\theta)^{\alpha-1}P_\theta(x)^{\alpha-1} = R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^k \theta_i f_i(x)
= (1 + (1 - \alpha)\theta_i R_i^R) R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^k \theta_i \hat{f}_i(x).
\]
Taking expectation with respect to \( \hat{P} \) on both sides, and using \( \sum_x \hat{P}(x)\hat{f}_i(x) = 0 \), \( i = 1, \ldots, k \), we get
\[
Z(\theta)^{\alpha-1} \sum_x \hat{P}(x)P_\theta(x)^{\alpha-1} = [1 + (1 - \alpha)\theta_i R_i^R] \cdot \sum_x \hat{P}(x)R(x)^{\alpha-1}.
\]
Since \( P_\theta \) and \( R \) have full support, it follows that \( [1 + (1 - \alpha)\theta_i R_i^R] > 0 \), and hence \( P_\theta \in \mathbb{M}^{(\alpha)}(R, \hat{f}_1, \ldots, \hat{f}_k) \). This shows \( \mathbb{M}^{(\alpha)}(R, f_1, \ldots, f_k) \subset \mathbb{M}^{(\alpha)}(R, \hat{f}_1, \ldots, \hat{f}_k) \). Similarly, using the assumption that \( L \) is nonempty, one can show that \( \mathbb{M}^{(\alpha)}(R, \hat{f}_1, \ldots, \hat{f}_k) \subset \mathbb{M}^{(\alpha)}(R, f_1, \ldots, f_k) \).

By Corollary \[16\] we have
\[ \mathcal{J}_\alpha(\hat{P}, S) = \mathcal{J}_\alpha(\hat{P}, Q) + \mathcal{J}_\alpha(Q, S) \quad \forall S \in \text{cl}(\mathbb{M}^{(\alpha)}). \] (54)

(a) If \( \text{Supp}(Q) = \mathbb{X} \), then \( Q \in \mathbb{M}^{(\alpha)} \), and from \[54\], the minimum of \( \mathcal{J}_\alpha(\hat{P}, S) \) over \( S \in \mathbb{M}^{(\alpha)} \) is attained at \( S = Q \). To prove the uniqueness, let \( P_{\theta^*} \in \mathbb{M}^{(\alpha)} \) also attain the minimum. Then, from \[54\], we have
\[ \mathcal{J}_\alpha(\hat{P}, P_{\theta^*}) = \mathcal{J}_\alpha(\hat{P}, Q) + \mathcal{J}_\alpha(Q, P_{\theta^*}). \] (55)

Since \( \mathcal{J}_\alpha(\hat{P}, P_{\theta^*}) = \mathcal{J}_\alpha(\hat{P}, Q) \), we have \( \mathcal{J}_\alpha(Q, P_{\theta^*}) = 0 \), and so \( P_{\theta^*} = Q \).

(b) Let \( \text{Supp}(Q) \neq \mathbb{X} \). Then \( Q \in \text{cl}(\mathbb{M}^{(\alpha)}) \setminus \mathbb{M}^{(\alpha)} \). Uniqueness on the closure follows just as in (a) immediately above. If \( \hat{P} \) has a reverse \( \mathcal{J}_\alpha \)-projection on \( \mathbb{M}^{(\alpha)} \), say \( P_{\theta^*} \), then by continuity of \( \mathcal{J}_\alpha(\hat{P}, \cdot) \) (\[2\] Rem. 5), we have \( \mathcal{J}_\alpha(\hat{P}, Q) = \mathcal{J}_\alpha(\hat{P}, P_{\theta^*}) \).

This contradicts the uniqueness.

\[ \blacksquare \]

\textbf{B.} \( \alpha > 1 \): Let us begin with a counterexample to Theorem \[15\] that illustrates that when \( \alpha > 1 \), \( \text{cl}(\mathbb{M}^{(\alpha)}) \) need not intersect the associated \( \mathbb{L} \).

\textbf{Example 3:} Let \( \alpha, X, L \), and \( R \) be as in Example \[2\] The associated \( \alpha \)-power-law family and its closure are
\[ \mathbb{M}^{(\alpha)} = \left\{ P_\theta : \theta \in (-1/24, 1/4) \right\}, \]
and
\[ \text{cl}(\mathbb{M}^{(\alpha)}) = \left\{ P_\theta : \theta \in [-1/24, 1/4] \right\}, \]
where
\[ P_\theta = \frac{1}{1 + 13\theta} \left( 1/4 - \theta, 1/4 + 3\theta, 1/4 + 5\theta, 1/4 + 6\theta \right). \]
We assert that no such \( P_\theta \), either of \( \mathbb{M}^{(\alpha)} \) or \( \text{cl}(\mathbb{M}^{(\alpha)}) \), is in \( L \). Furthermore, the forward \( \mathcal{J}_\alpha \)-projection of every member in \( \text{cl}(\mathbb{M}^{(\alpha)}) \) on \( L \) is \( Q = (3/4, 1/4, 0, 0) \) which, of course, is not in \( \text{cl}(\mathbb{M}^{(\alpha)}) \).

One must therefore extend \( \mathbb{M}^{(\alpha)} \) beyond its closure to identify the family that is orthogonal to \( L \) and intersects \( L \) at \( Q \). A natural guess, but one that does not work in general, is the following.
Definitions 7 and 18, respectively, characterized by inequality (56) holds.

Even though $\text{Supp}\{P_i\} \not\subseteq \text{Supp}(Q)$, we see that $Z(1/2) = 31/4$ and $P_{1/2} = (0, 7/31, 11/31, 13/31) \in M_+^{(\alpha)}$. Its forward $\mathcal{I}_\alpha$-projection on $\mathbb{L}$ is not $Q$ but is close to $Q_1 = (0.7755, 0.1735, 0.0510, 0) \in \mathbb{L}$. Indeed, numerical calculations show that $\mathcal{I}_\alpha(Q, P_{1/2}) = 4.2368 > \mathcal{I}_\alpha(Q_1, P_{1/2}) = 4.2223$.

The issue here is that $\text{Supp}(P_{1/2}) \not\subseteq \text{Supp}(Q)$.

Example 4: Consider the setting of Example 2 again. Setting $\theta_1 = 1/2$, we see that $Z(1/2) = 31/4$ and $P_{1/2} = (0, 7/31, 11/31, 13/31) \in M_+^{(\alpha)}$. Its forward $\mathcal{I}_\alpha$-projection on $\mathbb{L}$ is not $Q$ but is close to $Q_1 = (0.7755, 0.1735, 0.0510, 0) \in \mathbb{L}$. Indeed, numerical calculations show that $\mathcal{I}_\alpha(Q, P_{1/2}) = 4.2368 > \mathcal{I}_\alpha(Q_1, P_{1/2}) = 4.2223$.

The issue here is that $\text{Supp}(P_{1/2}) \not\subseteq \text{Supp}(Q)$.

Example 5: Consider the setting of Example 2 once again. Setting $\theta = -1/16$, we see that $Z(-1/16) = 3/8$ and $P_{-1/16} = (5/6, 1/6, 0, 0) \in M_+^{(\alpha)}$. Its forward $\mathcal{I}_\alpha$-projection on $\mathbb{L}$ is not $Q$, yet again, but near $Q_2 = (0.781, 0.171, 0.02, 0.028) \in \mathbb{L}$. Numerical calculations show that $\mathcal{I}_\alpha(Q, P_{-1/16}) = 0.0155 > \mathcal{I}_\alpha(Q_2, P_{-1/16}) = 0.0022$.

Even though $\text{Supp}(P_{-1/16}) \not\subseteq \text{Supp}(Q)$, this time, the issue is that the Pythagorean inequality is violated.

An appropriate extension of $M_+^{(\alpha)}$ turns out to be the following middle ground.

Definition 19: The family $\hat{M}_+^{(\alpha)}$ characterized by a probability measure $R$ and $k$ functions $f_i : X \to \mathbb{R}, i = 1, \ldots, k$, is defined as follows. Let $Q$ be the forward $\mathcal{I}_\alpha$-projection of $R$ on $\mathbb{L}$. We know that $Q = P_\theta \in \hat{M}_+^{(\alpha)}$ for some $\theta^*$. Define

$$\hat{M}_+^{(\alpha)} := \{ P_\theta \in M_+^{(\alpha)} \text{ satisfying } (a) \text{ and } (b) \text{ below} \},$$

where

(a) $\text{Supp}(P_{\theta^*}) \subseteq \text{Supp}(P_\theta)$;

(b) $\sum_{i=1}^k \theta_i f_i(x) \leq \sum_{i=1}^k \theta_i^* f_i(x) \ \forall x \notin \text{Supp}(P_\theta)$.

The following is the analog of the combined Theorem [15] and Corollary [16].

Theorem 20: Let $\alpha > 1$. Let $\mathbb{L}$ be a linear family characterized by $f_i, i = 1, \ldots, k$ as in [16]. Let $M_+^{(\alpha)}$ and $\hat{M}_+^{(\alpha)}$ be as in Definitions 7 and 18 respectively, characterized by $R$ and the $k$ functions $f_i, i = 1, \ldots, k$. Let $Q$ be the forward $\mathcal{I}_\alpha$-projection of $R$ on $\mathbb{L}$. Let $\hat{M}_+^{(\alpha)}$ be the extension of $M_+^{(\alpha)}$ as in Definition 19. We then have the following.

(a) $\mathbb{L} \cap \hat{M}_+^{(\alpha)} = \{ Q \}$ and

$$\mathcal{I}_\alpha(P, P_\theta) \geq \mathcal{I}_\alpha(P, Q) + \mathcal{I}_\alpha(Q, P_\theta) \quad (56)$$

for every $P \in \mathbb{L}$ and every $P_\theta \in \hat{M}_+^{(\alpha)}$.

(b) If $Q \in \text{cl}(\hat{M}_+^{(\alpha)})$, then $\mathbb{L} \cap \text{cl}(\hat{M}_+^{(\alpha)}) = \{ Q \}$ and (56) holds with equality for every $P \in \mathbb{L}$ and every $P_\theta \in \text{cl}(M_+^{(\alpha)})$.

(c) If $Q \in M_+^{(\alpha)}$, then $\mathbb{L} \cap M_+^{(\alpha)} = \{ Q \}$ and (56) holds with equality for every $P \in \mathbb{L}$ and every $P_\theta \in M_+^{(\alpha)}$.

Proof: (a) By virtue of Theorem [13](b), we have $Q \in \mathbb{L} \cap \hat{M}_+^{(\alpha)}$. Furthermore, by Theorem [14] any member of $\mathbb{L} \cap \hat{M}_+^{(\alpha)}$ is a forward $\mathcal{I}_\alpha$-projection of $R$ on $\mathbb{L}$. Since the forward projection is unique, $\mathbb{L} \cap \hat{M}_+^{(\alpha)}$ must be the singleton $\{ Q \}$.

Let $P_\theta \in \hat{M}_+^{(\alpha)}$. We claim that $P_\theta$ has $P_{\theta^*} = Q$ as its forward projection on $\mathbb{L}$. Assuming the claim, by Theorem [13](a), inequality (56) holds.
Let us now proceed to show the claim. By Theorem 13, it suffices to verify that \( P_{\theta^*} \) can be written as

\[
\hat{Z}(\hat{\theta})^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} = \left[ P_{\theta^*}(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \hat{\theta}_i f_i(x) \right]^+ \quad \forall x
\]  

(57)

for some \( \hat{Z}(\hat{\theta}) \) and \( \hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_k) \). To see this, by definition of \( P_{\theta^*} \), we have

\[
Z(\theta)^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} = \left[ R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x) \right]^+ \quad \forall x,
\]

(58)

and, by Theorem 13(b), we have

\[
Z(\theta^*)^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} = \left[ R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x) \right]^+ \quad \forall x.
\]

(59)

Let \( x \in \text{Supp}(P_{\theta^*}) \). By Definition 19(a), \( x \in \text{Supp}(P_{\theta^*}) \) as well. Hence, we can remove the \([\cdot]^+\) operation in (58) and (59) to get

\[
Z(\theta)^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} = R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x).
\]

\[
Z(\theta^*)^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} = R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x),
\]

Eliminating \( R(x)^{\alpha-1} \) from the preceding equations, we get

\[
Z(\theta^*)^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} = Z(\theta)^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} (\theta_i^* - \theta_i) f_i(x),
\]

equivalently,

\[
\left( \frac{Z(\theta^*)}{Z(\theta)} \right)^{\alpha-1} P_{\theta^*}(x)^{\alpha-1} = P_{\theta^*}(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \frac{(\theta_i^* - \theta_i)}{Z(\theta)^{\alpha-1}} f_i(x).
\]

(60)

This suggests that \( \hat{Z}(\hat{\theta}) = Z(\theta^*)/Z(\theta) \) and \( \hat{\theta}_i = (\theta_i^* - \theta_i)/Z(\theta)^{\alpha-1} \) soul work. Let us now verify that they do, that is, that (57) holds for all \( x \) with these choices of \( \hat{Z} \) and \( \hat{\theta} \).

The foregoing shows (57) holds for all \( x \in \text{Supp}(P_{\theta^*}) \). Next, let \( x \in \text{Supp}(P_{\theta^*}) \setminus \text{Supp}(P_{\theta^*}) \). The right-hand side of (60), upon substitution of (58) without the \([\cdot]^+\) operation, becomes

\[
\frac{R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x)}{Z(\theta)^{\alpha-1}}^+ = \frac{R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x)}{Z(\theta)^{\alpha-1}} \leq 0,
\]

as is required for \( x \notin \text{Supp}(P_{\theta^*}) \). Hence (57) holds for \( x \in \text{Supp}(P_{\theta^*}) \setminus \text{Supp}(P_{\theta^*}) \) as well, and therefore for all \( x \in \text{Supp}(P_{\theta^*}) \).

Finally, when \( x \notin \text{Supp}(P_{\theta^*}) \),

\[
R(x)^{\alpha-1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x) \leq 0.
\]

The right-hand side of (60) then satisfies

\[
(1 - \alpha) \sum_{i=1}^{k} \frac{(\theta_i^* - \theta_i)}{Z(\theta)^{\alpha-1}} f_i(x) \leq 0
\]

because of condition (b) in Definition 19 and \( \alpha > 1 \). This establishes that \( P_{\theta^*} \) is of the form (57), and is therefore the forward \( \mathcal{S}_\alpha \)-projection of \( P_{\theta^*} \) on \( \mathbb{R} \).

Proofs of (b) and (c) are the same as in \( \alpha < 1 \) case considered in Theorem 15.

Having established the orthogonality between a linear family and its associated \( \alpha \)-power-law family, let us now return to the problem of robust estimation discussed in section II-B. As in the case of \( \alpha < 1 \), we show a connection between the MMPLE on the extended \( \alpha \)-power-law family \( \mathcal{M}_{\alpha}^{(\alpha)} \), which is a reverse \( \mathcal{S}_\alpha \)-projection on \( \mathcal{M}_{\alpha}^{(\alpha)} \), and the forward \( \mathcal{S}_\alpha \)-projection on the related linear family.
Theorem 21: Let \( \alpha > 1 \). Let \( \hat{P} \) be a probability measure on \( X \). Let \( \mathbb{M}(\alpha) \) be characterized by the probability measure \( R \) and the functions \( f_i, i = 1, \ldots, k \). Let \( R \) have full support. Let \( \mathbb{L} \) be the associated linear family characterized by \( f_i, i = 1, \ldots, k \), and assume that it is nonempty. Define \( \mathbb{L} \) as in (51) using \( \hat{f}_i \) and \( \tau_i^R \) as defined in (52) and (53), respectively. Let \( Q \) be the forward \( \mathcal{I}_\alpha \)-projection of \( R \) on \( \mathbb{L} \). Then the following hold.

(a) If \( Q \in \mathbb{M}(\alpha) \), then \( Q \) is the unique reverse \( \mathcal{I}_\alpha \)-projection of \( \hat{P} \) on \( \mathbb{M}(\alpha) \).

(b) If \( Q \in \text{cl}(\mathbb{M}(\alpha)) \setminus \mathbb{M}(\alpha) \), then \( \hat{P} \) does not have a reverse \( \mathcal{I}_\alpha \)-projection on \( \mathbb{M}(\alpha) \). However, \( Q \) is the unique reverse \( \mathcal{I}_\alpha \)-projection of \( \hat{P} \) on \( \text{cl}(\mathbb{M}(\alpha)) \).

(c) If \( Q \notin \text{cl}(\mathbb{M}(\alpha)) \), then

(i) \( \hat{P} \) does not have a reverse \( \mathcal{I}_\alpha \)-projection on \( \mathbb{M}(\alpha) \).

(ii) \( \mathbb{M}(\alpha) \) can be extended to \( \mathbb{M}_+^\alpha (R, f_1, \ldots, f_k) \), and \( Q \) is the unique reverse \( \mathcal{I}_\alpha \)-projection of \( \hat{P} \) on \( \mathbb{M}_+^\alpha (R, f_1, \ldots, f_k) \).

Proof: Only (c)-(i) needs a proof. Proofs of all others follow the same arguments in the proof of Theorem 17, but now one uses Theorem 20 instead of Corollary 16.

Let us now prove (c)-(i) by contradiction. Suppose \( \hat{P} \) has a reverse \( \mathcal{I}_\alpha \)-projection on \( \mathbb{M}(\alpha) \). Call it \( P_{\theta^*} \). Since \( P_{\theta^*} \) has full support, there is a neighborhood \( N \) of \( \theta^* \) such that \( \theta \in N \) implies \( P_{\theta^*} \in \mathbb{M}(\alpha) \). The first order optimality condition applies, namely

\[
\frac{\partial}{\partial \theta_i} \mathcal{I}_\alpha(\hat{P}, P_{\theta^*}) \bigg|_{\theta = \theta^*} = 0, \quad i = 1, \ldots, k.
\]

We claim that this implies

\[
\sum_x P_{\theta^*}(x) \hat{f}_i(x) = 0, \quad i = 1, \ldots, k.
\]

(61)

But then \( P_{\theta^*} \in \mathbb{L} \) and so \( P_{\theta^*} = Q \), a contradiction to \( Q \notin \text{cl}(\mathbb{M}(\alpha)) \).

We now proceed to prove the claim (61). Observe that

\[
Z(\theta)^{\alpha-1} P_{\theta}(x)^{\alpha-1} = R(x)^{\alpha-1} + \sum_j \theta_j \hat{f}_j(x)
\]

and so

\[
Z(\theta)^{\alpha-1} \sum_x \hat{P}(x) P_{\theta}(x)^{\alpha-1} = \sum_x \hat{P}(x) R(x)^{\alpha-1} + \sum_j \theta_j \left( \sum_x \hat{P}(x) \hat{f}_j(x) \right)
\]

\[
= \sum_x \hat{P}(x) R(x)^{\alpha-1},
\]

where the last equality holds because \( \hat{P} \in \mathbb{L} \). Also,

\[
\sum_x P_{\theta}(x)^\alpha = \sum_x \left[ P_{\theta}(x)^{\alpha-1} \right]^{\frac{\alpha}{\alpha-1}}
\]

\[
= Z(\theta)^{-\alpha} \sum_x \left[ R(x)^{\alpha-1} + \sum_j \theta_j \hat{f}_j(x) \right]^{\frac{\alpha}{\alpha-1}}.
\]

Substituting (63) and (64) into (12) and taking the partial derivative, we get

\[
\frac{\partial}{\partial \theta_i} \mathcal{I}_\alpha(\hat{P}, P_{\theta^*}) = \frac{\alpha}{1-\alpha} \frac{\partial}{\partial \theta_i} \log Z(\theta)^{1-\alpha} + \frac{\partial}{\partial \theta_i} \log Z(\theta)^{-\alpha} + \frac{\partial}{\partial \theta_i} \log \sum_x \left[ R(x)^{\alpha-1} + \sum_j \theta_j \hat{f}_j(x) \right]^{\frac{\alpha}{\alpha-1}}
\]

\[
= \frac{\partial}{\partial \theta_i} \log \sum_x \left[ R(x)^{\alpha-1} + \sum_j \theta_j \hat{f}_j(x) \right]^{\frac{\alpha}{\alpha-1}}
\]

\[
= \frac{\alpha}{1-\alpha} \sum_x \left[ R(x)^{\alpha-1} + \sum_j \theta_j \hat{f}_j(x) \right]^{\frac{\alpha}{\alpha-1}} \hat{f}_i(x)
\]

\[
= \frac{1}{1-\alpha} Z(\theta) \sum_x P_{\theta}(x) \hat{f}_i(x),
\]

where \( [\ldots] \) stands for \( \sum_x \left[ R(x)^{\alpha-1} + \sum_j \theta_j \hat{f}_j(x) \right]^{\frac{\alpha}{\alpha-1}} \), and the last equality follows from (62). Thus,

\[
\frac{\partial}{\partial \theta_i} \mathcal{I}_\alpha(\hat{P}, P_{\theta^*}) \bigg|_{\theta = \theta^*} = 0 \implies \sum_x P_{\theta^*}(x) \hat{f}_i(x) = 0,
\]

thereby proving the claim.
We now provide some concluding remarks. Our focus has primarily been on the geometric relation between the $\alpha$-power-law and the linear families. This geometric relation enabled us to characterize the reverse $\mathcal{S}_\alpha$-projection on an $\alpha$-power-law family $\mathcal{M}^{(\alpha)} := \mathcal{M}^{(\alpha)}(R, f_1, \ldots, f_k)$ as a forward $\mathcal{S}_\alpha$-projection on a linear family. The procedure is as follows.

“Given the family $\mathcal{M}^{(\alpha)}$, sweep through a collection of linear families $\{\tilde{M}_i, 1 \leq i \leq k\}$ orthogonal to $\mathcal{M}^{(\alpha)}$ by varying $\tau_i$, $i = 1, \ldots, k$, and find the linear family $\tilde{L}$ that contains $\tilde{P}$. Then find the forward $\mathcal{S}_\alpha$-projection of $\tilde{P}$ on $\tilde{L}$. Call it $\tilde{Q}$. If $\tilde{Q} \in \mathcal{M}^{(\alpha)}$, then $\tilde{Q}$ is the reverse $\mathcal{S}_\alpha$-projection of $\tilde{P}$ on the $\mathcal{M}^{(\alpha)}$. If $\tilde{Q} \notin \text{cl}(\mathcal{M}^{(\alpha)})$, then $\tilde{P}$ does not have a reverse $\mathcal{S}_\alpha$-projection on $\mathcal{M}^{(\alpha)}$. When $\tilde{Q}$ intersects $\mathcal{M}^{(\alpha)}$, then $\tilde{Q}$ attains the minimum in the closure.”

The cases $\alpha < 1$ and $\alpha > 1$ have different characteristics. The $\alpha < 1$ case is similar to $\alpha = 1$ and one always has $\tilde{L} \cap \text{cl}(\mathcal{M}^{(\alpha)}) = \{Q\}$. On the other hand, when $\alpha > 1$, it is possible that $\tilde{L} \cap \text{cl}(\mathcal{M}^{(\alpha)}) = \emptyset$, and $\tilde{Q} \notin \text{cl}(\mathcal{M}^{(\alpha)})$. Then $\tilde{P}$ does not have a reverse $\mathcal{S}_\alpha$-projection on $\mathcal{M}^{(\alpha)}$. One then needs to extend $\mathcal{M}^{(\alpha)}$ to make it intersect $\tilde{L}$. We showed that the extension $\tilde{M}_+^{(\alpha)}$ is just right and satisfies $\tilde{L} \cap \tilde{M}_+^{(\alpha)} = \{Q\}$. However, $\tilde{Q}$, in the intersection $\tilde{L} \cap \tilde{M}_+^{(\alpha)}$, is no longer the reverse $\mathcal{S}_\alpha$-projection of $\tilde{P}$ on $\text{cl}(\mathcal{M}^{(\alpha)})$. It would be interesting to see if $\tilde{Q}$ can be used to simplify the computation of the true reverse $\mathcal{S}_\alpha$-projection of $\tilde{P}$ on $\text{cl}(\mathcal{M}^{(\alpha)})$.

Our characterization has algorithmic benefits since the forward $\mathcal{S}_\alpha$-projection is a minimization of a quasiconvex function subject to linear constraints. Standard techniques are available to solve such problems, for example, via a sequence of convex feasibility problems [23, Sec. 4.2.5], or via a sequence of simpler forward projections on single-constraint linear families [2 Th. 16, Rem. 13].

**APPENDIX A**

**WEAK DEPENDENCE OF THE $\alpha$-POWER-LAW FAMILY ON $R$**

The following result shows that the $\alpha$-power-law family depends on $R$ only in a weak manner, and that any member of $\mathcal{M}^{(\alpha)}$ could equally play the role of $R$. The same result is well-known for an exponential family.

**Proposition 22:** If $\alpha > 1$, let $R$ have full support. Consider the $\mathcal{M}^{(\alpha)}(R, f_1, \ldots, f_k)$ as in Definition 7. Fix $P_{0^*} \in \mathcal{M}^{(\alpha)}(R, f_1, \ldots, f_k)$. Then $\mathcal{M}^{(\alpha)}(P_{0^*}, f_1, \ldots, f_k) = \mathcal{M}^{(\alpha)}(R, f_1, \ldots, f_k)$.

**Proof:** Write $M^{(\alpha)}$ for $\mathcal{M}^{(\alpha)}(R, f_1, \ldots, f_k)$ and $\tilde{M}^{(\alpha)}$ for $\tilde{M}_+^{(\alpha)}(P_{0^*}, f_1, \ldots, f_k)$. We will check that an arbitrary element $P_0 \in \mathcal{M}^{(\alpha)}$ is an element of $\tilde{M}^{(\alpha)}$. This will establish $\mathcal{M}^{(\alpha)} \subseteq \tilde{M}^{(\alpha)}$. The converse holds by symmetry.

From the formula for $P_{0^*}$, observe that

\[
P_{0^*}(x)^{\alpha - 1} = Z(\theta^*)^{1 - \alpha} \left[ R(x)^{\alpha - 1} + (1 - \alpha) \sum_{i=1}^{k} \theta_i^* f_i(x) \right] \quad \forall x,
\]

and so

\[
R(x)^{\alpha - 1} = Z(\theta^*)^{\alpha - 1} P_{0^*}(x)^{\alpha - 1} \left[ (1 - \alpha) \sum_{i=1}^{k} \theta_i^* f_i(x) \right] \quad \forall x.
\]

Substitute this into the formula for $P_0$ in (19) to get

\[
P_0(x)^{\alpha - 1} = Z(\theta)^{1 - \alpha} \left[ Z(\theta^*)^{\alpha - 1} P_{0^*}(x)^{\alpha - 1} - (1 - \alpha) \sum_{i=1}^{k} \theta_i^* f_i(x) + (1 - \alpha) \sum_{i=1}^{k} \theta_i f_i(x) \right]
\]

\[
= \left( \frac{Z(\theta^*)}{Z(\theta)} \right)^{\alpha - 1} \left[ P_{0^*}(x)^{\alpha - 1} + (1 - \alpha) \sum_{i=1}^{k} \frac{\theta_i - \theta_i^*}{Z(\theta^*)^{\alpha - 1}} f_i(x) \right]
\]

\[
= \tilde{Z}(\xi)^{1 - \alpha} \left[ P_{0^*}(x)^{\alpha - 1} + (1 - \alpha) \sum_{i=1}^{k} \xi_i f_i(x) \right],
\]

where $\xi = (\theta - \theta^*)/Z(\theta^*)^{\alpha - 1}$, and $\tilde{Z}(\xi) = Z(\theta)/Z(\theta^*)$. Thus, $P_0 \in \tilde{M}^{(\alpha)}$.

Change of reference from $R$ to $P_{0^*}$ merely amounts to a translation and rescaling of the parameter space.
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