RATIONAL POLYHEDRA AND PROJECTIVE
LATTICE-ORDERED ABELIAN GROUPS WITH ORDER UNIT

LEONARDO CABRER ‡ AND DANIELE MUNDICI†

Abstract. An \( \ell \)-group \( G \) is an abelian group equipped with a translation invariant lattice order. Baker and Beynon proved that \( G \) is finitely generated projective iff it is finitely presented. A \textit{unital} \( \ell \)-group is an \( \ell \)-group \( G \) with a distinguished \textit{order unit}, i.e., an element \( 0 \leq u \in G \) whose positive integer multiples eventually dominate every element of \( G \). While every finitely generated projective unital \( \ell \)-group is finitely presented, the converse does not hold in general. Classical algebraic topology (à la Whitehead) will be combined in this paper with the Wlodarczyk-Morelli solution of the weak Oda conjecture for toric varieties, to describe finitely generated projective unital \( \ell \)-groups.

1. Introduction: Unital \( \ell \)-groups and rational polyhedra

A main reason of interest in unital \( \ell \)-groups is that Elliott classification yields a one-one correspondence \( \kappa \) between isomorphism classes of unital AF \( C^* \)-algebras whose Murray-von Neumann order of projections is a lattice, and isomorphism classes of countable unital \( \ell \)-groups: \( \kappa \) is an order-theoretic enrichment of Grothendieck \( K_0 \) functor, [14, 3.9, 3.12].

A unital \( \ell \)-group \((G,u)\) is \textit{projective} if whenever \( \psi : (A,a) \to (B,b) \) is a surjective unital \( \ell \)-homomorphism and \( \phi : (G,u) \to (B,b) \) is a unital \( \ell \)-homomorphism, there is a unital \( \ell \)-homomorphism \( \theta : (G,u) \to (A,a) \) such that \( \phi = \psi \circ \theta \). As usual, \textit{unital \( \ell \)-homomorphisms} between unital \( \ell \)-groups are group homomorphisms that also preserve the order unit and the lattice structure, [5].

Baker [2] and Beynon [4, Theorem 3.1] (also see [10, Corollary 5.2.2]) gave the following characterization: An \( \ell \)-group \( G \) is finitely generated projective iff it is finitely presented. While the \( \Rightarrow \)-direction still holds for every unital \( \ell \)-group \((G,u)\) (see, e.g., [16, Proposition 5]), in this paper we will show that various arithmetical, geometrical and topological conditions must be imposed to ensure that a finitely presented \((G,u)\) is projective.

For \( n = 1, 2, \ldots \) we let \( M_n \) denote the unital \( \ell \)-group of all continuous functions \( f : [0,1]^n \to \mathbb{R} \) having the following property: there are linear polynomials \( p_1, \ldots, p_m \) with integer coefficients, such that for all \( x \in [0,1]^n \) there is \( i \in \{1, \ldots, m\} \) with \( f(x) = p_i(x) \). \( M_n \) is equipped with the pointwise operations

\begin{itemize}
    \item \textit{Date:} July 17, 2009.
    \item 2000 Mathematics Subject Classification. Primary: 06F20, 52B20. Secondary: 08B30, 14M25, 20F60, 52B11, 54C15, 54C55, 54D05, 55U10, 57Q05, 57Q10.
    \item Key words and phrases. Lattice-ordered abelian group, order unit, projective, rational polyhedron, regular fan, desingularization, blow-up, weak Oda conjecture, retract, contractibility, collapsibility, Whitehead theorem.
\end{itemize}
Proposition 1.1. ([14] 4.16]) \( M_n \) is generated by the coordinate maps \( \xi_i : [0, 1]^n \to \mathbb{R} \) and the order unit 1. For every unital \( \ell \)-group \((G, u)\) and \( 0 \leq g_1, \ldots, g_n \leq u \), if the set \( \{g_1, \ldots, g_n, u\} \) generates \( G \) there is a unique unital \( \ell \)-homomorphism \( \psi \) of \( M_n \) onto \( G \) such that \( \psi(\xi_i) = g_i \) for each \( i = 1, \ldots, n \).

An ideal \( i \) of a unital \( \ell \)-group \((G, u)\) is the kernel of a unital \( \ell \)-homomorphism of \( G \), ([11] p.8 and 1.14]). \( i \) is principal if it is singly (finitely) generated.

A unital \( \ell \)-group \((G, u)\) is finitely presented if for some \( n = 1, 2, \ldots \), \((G, u)\) is unitally \( \ell \)-isomorphic to the quotient of \( M_n \), by some principal ideal \( j \), in symbols, \( (G, u) \cong M_n / j \).

For every nonempty closed set \( X \subseteq [0, 1]^n \) we introduce the notation

\[
M_n \upharpoonright X = \{ f \upharpoonright X \mid f \in M_n \}
\]

for the unital \( \ell \)-group of restrictions to \( X \) of the functions in \( M_n \).

Following [15] 1.1, by a polyhedron \( P \) in \([0, 1]^n\) we mean a finite union of (always closed) simplexes \( P = S_1 \cup \cdots \cup S_t \) in \([0, 1]^n\). If the coordinates of the vertices of every simplex \( S_i \) are rational numbers, \( P \) is said to be rational.

For any rational point \( v \in \mathbb{R}^n \), the least common denominator of the coordinates of \( v \) is called the denominator of \( v \) and is denoted \( \text{den}(v) \).

The relationship between rational polyhedra and finitely presented unital \( \ell \)-groups is given by the following result:

Proposition 1.2. ([10] Propositions 4 and 5]) Let \((G, u)\) be a unital \( \ell \)-group.

(a) \((G, u)\) is finitely presented iff there is \( n = 1, 2, \ldots \) and a rational polyhedron \( P \subseteq [0, 1]^n \) such that \((G, u)\) is unitally \( \ell \)-isomorphic to \( M_n \upharpoonright P \).

(b) If \((G, u)\) is finitely generated projective then \((G, u)\) is finitely presented.

One may now naturally ask for which rational polyhedron \( P \subseteq [0, 1]^n \) the unital \( \ell \)-group \( M_n \upharpoonright P \) is projective. In Theorem 5.2 and Corollary 5.1 it is shown that \( P \) satisfies the following necessary conditions: (i) \( P \) is contractible, (ii) \( P \) contains a vertex of the \( n \)-cube \([0, 1]^n\) and (iii) \( P \) has a regular triangulation \( \Delta \) (as defined in Section 2 following 20) such that for every maximal simplex \( T \in \Delta \), the denominators of the vertices of \( T \) are coprime. As proved in Corollaries 4.4 and 5.1, these three conditions are sufficient for \( M_n \upharpoonright P \) to be projective in case \( P \) is one-dimensional. Further, if \( P \) is an \( n \)-dimensional rational polyhedron satisfying (ii) and (iii), then \( M_n \upharpoonright P \) is a projective unital \( \ell \)-group, provided Condition (i) is strengthened to the collapsibility ([19] 8 [18]) of at least one triangulation of \( P \).

We refer to [5] [9] [10] [11] for \( \ell \)-groups, to [12] for algebraic topology, to [13] for polyhedral topology, and to [8] for regular fans—the homogeneous correspondents of rational polyhedra. Their desingularization procedures yield a key tool for our results.

2. Regular triangulations, Farey mediants and blow-ups

For every (always finite) simplicial complex \( K \) the point-set union of the simplexes of \( K \) is denoted \( |K| \); \( K \) is said to be a triangulation of \( |K| \). A simplicial
complex is said to be a \textit{rational} if the vertices of all its simplexes are rational. Given simplicial complexes $K$ and $H$ with $|K| = |H|$ we say that $H$ is a \textit{subdivision} of $K$ if every simplex of $H$ is a union of simplexes of $K$. For any rational point $v \in \mathbb{R}^n$ the integer vector

$$\bar{v} = \text{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$$

is called the \textit{homogeneous correspondent} of $v$. An $m$-simplex $\text{conv}(w_0, \ldots, w_m) \subseteq [0,1]^n$ is said to be \textit{regular} if its vertices are rational and the set of integer vectors $\{\tilde{w}_0, \ldots, \tilde{w}_m\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$. Following \cite{20}, a simplicial complex $K$ is said to be a \textit{regular triangulation (of $|K|$)} if all its simplexes are regular. Regular triangulations are called “unimodular” in \cite{10}. Given a regular triangulation $\Delta$ with $|\Delta| \subseteq [0,1]^n$, the homogeneous correspondents of its vertices are the generating vectors of a complex of cones in $\mathbb{R}^{n+1}$, which is a regular (also known as “nonsingular”) fan \cite{8}.

\textbf{Lemma 2.1.} (\cite{10} Proposition 1) For every rational polyhedron $P$ there is a regular triangulation $\Delta$ such that $P = |\Delta|$.

\textbf{Lemma 2.2.} Let $S = \text{conv}(v_1, \ldots, v_k) \subseteq [0,1]^n$ be a regular $(k-1)$-simplex and \{w_1, \ldots, w_k\} a set of rational points in $[0,1]^m$. Then the following conditions are equivalent:

(i) $\text{den}(w_i)$ is a divisor of $\text{den}(v_i)$, for each $i = 1, \ldots, k$.

(ii) For some integer matrix $M \in \mathbb{Z}^{m \times n}$ and vector $b \in \mathbb{Z}^m$ we have $Mv_i + b = w_i$ for each $i = 1, \ldots, k$.

\textit{Proof.} For the nontrivial direction, suppose $\text{den}(w_i)$ is a divisor of $\text{den}(v_i)$, for each $i = 1, \ldots, k$. With reference to \cite{2}, let $\{\tilde{v}_1, \ldots, \tilde{v}_k, b_{k+1}, \ldots, b_{n+1}\}$ be a basis of the free abelian group $\mathbb{Z}^{n+1}$, for suitable vectors $b_{k+1}, \ldots, b_{n+1} \in \mathbb{Z}^{n+1}$. Let $D$ be the $(n+1) \times (n+1)$ integer matrix whose columns are the vectors $\tilde{v}_1, \ldots, \tilde{v}_k, b_{k+1}, \ldots, b_{n+1}$. Then $D^{-1} \in \mathbb{Z}^{(n+1) \times (n+1)}$. By hypothesis, for each $i = 1, \ldots, k$, the vector $c_i = \text{den}(v_i)(w_i, 1)$ belongs to $\mathbb{Z}^{m+1}$. Let $d_{k+1}, \ldots, d_{n+1}$ be vectors in $\mathbb{Z}^{m+1}$ such that for each $j = k+1, \ldots, n+1$ the $(m+1)$th coordinate of $d_j$ coincides with the $(n+1)$th coordinate of $b_j$. Let $C \in \mathbb{Z}^{(n+1) \times (n+1)}$ be the matrix whose columns are given by the vectors $c_1, c_2, d_{k+1}, \ldots, d_{n+1}$. Since the $(n+1)$th row of $D$ equals the $(m+1)$th row of $C$, we have

$$CD^{-1} = \left( \begin{array}{cc} M & b \\ 0, & 0 \\ 1 \end{array} \right)$$

for some $m \times n$ integer matrix $M$ and integer vector $b \in \mathbb{Z}^m$. For each $i = 1, \ldots, k$ we then have $(CD^{-1})\tilde{v}_i = (CD^{-1})\text{den}(v_i)(v_i, 1) = \text{den}(v_i)(Mv_i + b, 1)$. By definition, $(CD^{-1})\tilde{v}_i = c_i = \text{den}(v_i)(w_i, 1)$, whence $Mv_i + b = w_i$ as desired. \qed

\textbf{Blow-up and Farey mediant.} Let $\Delta$ be a simplicial complex and $p \in |\Delta| \subseteq \mathbb{R}^n$. Then the (Alexander, \cite{11}) \textit{blow-up} $\Delta(p)$ of $\Delta$ at $p$ is the subdivision of $\Delta$ which is obtained by replacing every simplex $T \in \Delta$ that contains $p$ by the set of all simplexes of the form $\text{conv}(F \cup \{p\})$, where $F$ is any face of $T$ that does not contain $p$. (We are using the terminology of \cite{20, p.376}. Synonyms of “blow-up” are “stellar subdivision” and “elementary subdivision”, \cite{8} III, 2.1.)
The notation $\Delta_{(w_1, \ldots, w_m)}$ stands for the final outcome of a sequence of blow-ups of $\Delta$ at points $w_1, \ldots, w_m$, i.e.,

$$\Delta_{(w_1, \ldots, w_{m+1})} = \Delta_{(w_1, \ldots, w_{l+1})}.$$

(3)

For any regular $m$-simplex $E = \text{conv}(v_0, \ldots, v_m) \subseteq \mathbb{R}^n$, the Farey mediant of (the vertices of) $E$ is the rational point $v$ of $E$ whose homogeneous correspondent $\tilde{v}$ equals $v_0 + \cdots + \tilde{v}_m$. This is in agreement with the classical terminology in case $E = [0, 1]$. If $E$ belongs to a regular triangulation $\Delta$ and $v$ is the Farey mediant of $E$ then the blow-up $\Delta(v)$ is a regular triangulation. Also the converse is true, (a proof can be obtained from [8, V, 6.2]). $\Delta(v)$ will be called the Farey blow-up of $\Delta$ at $v$. By a (Farey) blow-down we understand the inverse operation of a (Farey) blow-up.

The proof of the “weak Oda conjecture” by Wlodarczyk [20] and Morelli [13] immediately yields:

**Lemma 2.3.** Let $P$ be a rational polyhedron. Then any two regular triangulations of $P$ are connected by a finite path of Farey blow-ups and Farey blow-downs.

**Definition 2.4.** A triangulation $\Delta$ of a rational polyhedron $P \subseteq [0, 1]^n$ is said to be strongly regular if it is regular and the greatest common divisor of the denominators of the vertices of each maximal simplex of $\Delta$ is equal to 1.

**Lemma 2.5.** Let $\Delta$ and $\nabla$ be regular triangulations of a rational polyhedron $P \subseteq [0, 1]^n$. Then $\Delta$ is strongly regular iff $\nabla$ is.

**Proof.** In view of Lemma 2.3 it is enough to argue in case $\Delta$ is the blow-up at the Farey mediant $v$ of an $m$-simplex $S = \text{conv}(v_0, \ldots, v_m) \in \nabla$. Let $M \in \nabla$ be a maximal $(m+k)$-simplex such that $S \subseteq M$. There are $w_1, \ldots, w_k \in M$ such that $M = \text{conv}(v_0, \ldots, v_m, w_1, \ldots, w_k)$. Since $\text{den}(v)$ is equal to $\sum_{j=0}^m \text{den}(v_j)$, the greatest common divisor of the integers $\text{den}(v_0), \ldots, \text{den}(v_m), \text{den}(w_1), \ldots, \text{den}(w_k)$ coincides with the greatest common divisor of

$$\text{den}(v_0), \ldots, \text{den}(v_{i-1}), \text{den}(v), \text{den}(v_{i+1}), \ldots, \text{den}(v_m), \text{den}(w_1), \ldots, \text{den}(w_k).$$

\[ Q.E.D. \]

**Lemma 2.6.** If $T = \text{conv}(v_1, \ldots, v_l) \subseteq [0, 1]^n$ is a regular $(t-1)$-simplex and the denominators of its vertices are coprime, then for all large integers $l$ there is a rational point $v \in T$ such that $\text{den}(v)$ is a divisor of $l$.

**Proof.** Let $\tilde{v}_1, \ldots, \tilde{v}_t, w_{i+1}, \ldots, w_{n+1}$ be a basis $B$ of the free abelian group $\mathbb{Z}^{n+1}$. For each $i = 1, \ldots, t$, let $d_i = \text{den}(v_i)$. Since $\gcd(d_1, \ldots, d_t) = 1$, without loss of generality the $(n+1)$th coordinate of each vector $w_j$ can be assumed to be 0. Let further $C = \mathbb{R}_{\geq 0} \tilde{v}_1 + \cdots + \mathbb{R}_{\geq 0} \tilde{v}_t + \mathbb{R}_{\geq 0} w_{i+1} + \cdots + \mathbb{R}_{\geq 0} w_{n+1}$ denote the cone positively spanned by $B$ in the vector space $\mathbb{R}^{n+1}$. Let the vector $s = (s_1, \ldots, s_{n+1}) \in \mathbb{Z}^{n+1}$ be defined by $s = \tilde{v}_1 + \cdots + \tilde{v}_t + w_{i+1} + \cdots + w_{n+1}$. Let $\mathbb{R}_{\geq 0} s$ denote the ray of $s$, i.e., the positive real span of the vector $s$ in $\mathbb{R}^{n+1}$. For every integer $l = 1, 2, \ldots$, let the hyperplane $H_l$ be defined by

$$H_l = \{ (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} \mid y_{n+1} = l \}.$$
The vanishing of the last coordinate of each $w_j$ is to the effect that $s_{n+1} = d_1 + \cdots + d_k > 0$, whence the set $H_1 \cap \mathbb{R}_{\geq 0} s$ contains a single point, denoted $h_1$. This is a rational point lying in the interior of $C$. In particular, for some $0 < \epsilon \in \mathbb{R}$ the point $h_1$ lies in a closed $n$-cube of side length $\epsilon$ contained in $C \cap H_1$. Consequently, for all large integers $l$, the rational point $h_l$ lies in some closed unit $n$-cube $D_l$ contained in the convex set $H_l \cap C$. Necessarily $D_l$ contains an integer point $p = (p_1, \ldots, p_n, l)$.

To conclude the proof, as noted in [S V, 1.11], there are integers $m_1, \ldots, m_{n+1} \geq 0$ such that $p = m_1 \cdot \cdot \cdot + m_t \cdot \cdot \cdot + m_{n+1} w_{n+1} + \cdots + m_{n+1} w_{n+1}$. Let the vector $q \in \mathbb{Z}^{n+1}$ be defined by $q = m_1 \cdot \cdot \cdot + m_t \cdot \cdot \cdot$. Since the $(n+1)$th coordinate of $p$ and of $q$ are equal, $m_1 d_1 + \cdots + m_t d_t = l$. Let $v$ be the only rational point of $[0,1]^n$ whose homogeneous correspondent $\tilde{v}$ lies on the ray $\mathbb{R}_{\geq 0} q$ of $q$. Then $v$ belongs to $T$, and $r$ is a positive integer multiple of $\tilde{v}$. Thus the $(n+1)$th coordinate $\text{den}(v)$ of $\tilde{v}$ is a divisor of the $(n+1)$th coordinate $l$ of $q$. □

Our next result essentially follows from Cauchy’s 1816 analysis of the Farey sequence, (Oeuvres, II Série, Tome VI, 1887, pp.146–148, or Tome II, 1958, pp.207–209), and is also a consequence of the De Concini-Procesi theorem on elimination of points of indeterminacy, [S] p.252. For the sake of completeness we give the elementary proof here:

**Proposition 2.7.** If $T \subseteq [0,1]^n$ is a regular simplex then for every rational point $v \in T$ there is a sequence of regular complexes $\Delta_0 = \{ T \text{ and its faces} \}, \Delta_1, \ldots, \Delta_u$ such that $\Delta_{i+1}$ is a Farey blow-up of $\Delta_i$, and $v$ is a vertex of (some simplex of) $\Delta_u$.

**Proof.** Let $\omega$ be a fixed but otherwise arbitrary well-ordering of the set of all pairs of distinct rational points (=edges) in $[0,1]^n$. We now inductively define the regular triangulation $\Delta_{i+1}$ of $T$ by

\[ \Delta_{i+1} = \text{the blow-up of } \Delta_i \text{ at the Farey mediant of the } \omega \text{-first edge} \]

\[ \text{conv}(w_1, w_2) \text{ of } \Delta_i \text{ such that } \text{den}(w_1) + \text{den}(w_2) \leq \text{den}(v). \]

This sequence must terminate after a finite number $u$ of steps, just because there are only finitely many rational points $w$ in $[0,1]^n$ satisfying $\text{den}(w) \leq \text{den}(v)$. Let $F$ be the smallest simplex of $\Delta_u$ containing $v$. In other words, $F$ is the intersection of all simplexes of $\Delta_u$ containing $v$. It follows that $v$ belongs to the relative interior of $F$. By way of contradiction, suppose $v$ is not a vertex of $F$. Then, for some $w_1, \ldots, w_r \in [0,1]^n$ with $r \geq 2$, we have $F = \text{conv}(w_1, \ldots, w_r)$ and $\text{den}(v) \geq \text{den}(w_1) + \cdots + \text{den}(w_r)$. The inequality is strict, unless $v$ is the Farey mediant of $F$. (See e.g., [S V, 1.11].) A fortiori, $\text{den}(v) \geq \text{den}(w_1) + \text{den}(w_2)$, whence the Farey blow-up $\Delta_{u+1}$ of $\Delta_u$ exists, against our assumption about $u$. □

3. Z-retracts and Projective Unital $\ell$-groups

Given rational polyhedra $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ together with a map $\eta: P \rightarrow Q$, we say that $\eta$ is a Z-map if there is a triangulation $K$ of $P$ such that over every simplex $T$ of $K$, $\eta$ coincides with a linear map $\eta_T$ with integer coefficients.

Since the intersection of any two simplexes of $K$ is again a (possibly empty) simplex of $K$, the continuity of $\eta$ follows automatically. The assumed properties
of the finite set of maps \( \{ \eta_T \mid T \in \mathcal{K} \} \), jointly with the rationality of \( P \), are to the effect that \( \mathcal{K} \) can be assumed rational, without loss of generality. It follows that \( \eta(P) \) is a rational polyhedron in \([0,1]^m\).

A \( \mathbb{Z} \)-map \( \theta: P \to Q \) is said to be a \( \mathbb{Z} \)-homeomorphism (of \( P \) onto \( Q \)) if it is one-one onto \( Q \) and the inverse \( \theta^{-1} \) is a \( \mathbb{Z} \)-map.

A \( \mathbb{Z} \)-map \( \sigma: P \to P \) is a \( \mathbb{Z} \)-retraction of \( P \) if it is idempotent, \( \sigma \circ \sigma = \sigma \). The rational polyhedron \( R = \sigma(P) \subseteq [0,1]^m \) is said to be a \( \mathbb{Z} \)-retract of \( P \).

If \( U, V, W \) are rational polyhedra in \([0,1]^n\), \( \mu \) is a \( \mathbb{Z} \)-retraction of \( U \) onto \( V \), and \( \nu \) is a \( \mathbb{Z} \)-retraction of \( V \) onto \( W \), then the composite map \( \nu \circ \mu \) is a \( \mathbb{Z} \)-retraction of \( U \) onto \( W \).

The relationship between \( \mathbb{Z} \)-retracts of cubes and finitely generated projective unital \( \ell \)-groups is given by the following

**Theorem 3.1.** A unital \( \ell \)-group \((G,u)\) is finitely generated projective iff it is unitally \( \ell \)-isomorphic to \( \mathcal{M}_n \upharpoonright P \) for some \( n = 1,2,\ldots \) and some \( \mathbb{Z} \)-retract \( P \) of \([0,1]^n\).

**Proof.** In [14, 3.9] a categorical equivalence \( \Gamma \) is established between unital \( \ell \)-groups and \( MV \)-algebras—those algebras satisfying the same \((\oplus,\neg)\)-equations as the unit interval \([0,1]\) equipped with truncated addition \( x \oplus y = \min(x+y,1) \) and involution \( \neg x = 1 - x \). By definition, \( \Gamma(G,u) = \{ g \in G \mid 0 \leq g \leq u \} \). Further, for every unital \( \ell \)-homomorphism \( \theta: (G,u) \to (G',u') \), \( \Gamma(\theta) \) is the restriction of \( \theta \) to \( \Gamma(G,u) \). The preservation properties of \( \Gamma \) are to the effect that \((G,u)\) is finitely generated projective iff so is \( \Gamma(G,u) \), (see [14, 3.4, 3.5]). Now apply [6, Theorem 1.2]. \( \square \)

Let, as above, \( P \subseteq [0,1]^n \) and \( Q \subseteq [0,1]^m \) be rational polyhedra, together with a \( \mathbb{Z} \)-map \( \eta: P \to Q \). Then for every rational point \( v \in P \),

\[
\text{den}(\eta(v)) \quad \text{is a divisor of} \quad \text{den}(v).
\]  
(4)

Conversely, we have

**Lemma 3.2.** Let \( P \subseteq [0,1]^n \) be a rational polyhedron, \( \Delta \) a regular triangulation of \( P \), and \( \mathcal{V} \) the set of vertices of \( \Delta \). Let the map \( f: \mathcal{V} \to [0,1]^m \) be such that \( \text{den}(f(v)) \) is a divisor of \( \text{den}(v) \) for every \( v \in \mathcal{V} \). Then \( f \) can be uniquely extended to a \( \mathbb{Z} \)-map \( \eta: P \to [0,1]^m \) which is linear on each simplex of \( \Delta \).

**Proof.** By Lemma 2.2, for each \( S \in \Delta \) there is a linear map with integer coefficients \( \eta_S: S \to [0,1]^m \) such that \( \eta_S(v) = f(v) \). The uniqueness of each \( \eta_S \) ensures \( \eta = \bigcup \{ \eta_S \mid S \in \Delta \} \) is well defined. Since \( \eta \) coincides with \( \eta_S \) over every simplex \( S \in T \), it is the desired \( \mathbb{Z} \)-map. \( \square \)

The following result states that the property of being a \( \mathbb{Z} \)-retract of some cube is invariant under \( \mathbb{Z} \)-homeomorphisms:

**Lemma 3.3.** Let \( \eta: [0,1]^n \to P \) be a \( \mathbb{Z} \)-retraction onto \( P \), and \( \theta: P \to Q \subseteq [0,1]^m \) a \( \mathbb{Z} \)-homeomorphism of \( P \) onto \( Q \). Then \( Q \) is a \( \mathbb{Z} \)-retract of \([0,1]^m\).
Lemma 3.4. Let $b$ and $\eta$.

Claim. There is a regular triangulation $\Delta$ of $[0,1]^m$ such that the set $\Delta_Q = \{ T \in \Delta \mid T \subseteq Q \}$ is a triangulation of $Q$, and $\theta^{-1}$ is linear over each simplex of $\Delta_Q$.

As a matter of fact, let $K$ be a rational triangulation of $Q$ such that the $\mathbb{Z}$-map $\theta^{-1}$ is linear over each simplex of $K$. Then the affine counterpart of [8 III, 2.8] provides a rational triangulation $\nabla$ of $[0,1]^m$ such that $K \subseteq \nabla$. The desingularization procedure of [15, Theorem 1.2] now yields a regular subdivision $\Delta$ of $\nabla$ having the desired properties to settle our claim.

Let $o$ denote the origin of $\mathbb{R}^n$. By Lemma 3.2 we have a uniquely determined $\mathbb{Z}$-map $\mu: [0,1]^m \to [0,1]^n$ satisfying

$$\mu(v) = \begin{cases} \theta^{-1}(v) & \text{if } v \in Q \\ o & \text{if } v \notin Q \end{cases}$$

for each vertex $v$ of $\Delta$. By definition, $\mu|Q = \theta^{-1}$, whence $\mu(Q) = P$. From $\eta|P = I_{d_P}$ it follows that $\theta \circ \eta \circ \mu|Q = I_{d_Q}$ and $\theta \circ \eta \circ \mu([0,1]^m) = \theta(P) = Q$. In conclusion, the map $\theta \circ \eta \circ \mu: [0,1]^m \to Q$ is a $\mathbb{Z}$-retraction onto $Q$. \hfill $\square$

Lemma 3.4. Let $\text{conv}(v,w) \subseteq [0,1]^n$ be a regular 1-simplex such that $a = \text{den}(v)$ and $b = \text{den}(w)$ are coprime. Then for each integer $m > 0$ there is a rational point $z \in \text{conv}(v,w)$ such that $m$ is a divisor of $\text{den}(z)$.

Proof. By hypothesis, there exist integers $p, q$ satisfying

(a) $qa - pb = 1$,
(b) $0 \leq p < a$ and $0 < q \leq b$.

By (a), the two vectors $(p, a)$ and $(q, b)$ form a basis of $\mathbb{Z}^2$. Stated otherwise, $[p/a, q/b]$ is a regular 1-simplex. By (b), $[p/a, q/b] \subseteq [0,1]$. Again by (a), $p$ and $a$ are coprime, whence $\text{den}(p/a) = a = \text{den}(v)$. Similarly, $\text{den}(q/b) = \text{den}(w)$. Lemma 3.2 now yields a $\mathbb{Z}$-homeomorphism $\eta$ of $[p/a, q/b]$ onto $\text{conv}(v,w)$. Let $s \in [p/a, q/b]$ be a rational point such that $m$ is a divisor of $\text{den}(s)$. A trivial density argument shows the existence of $s$. By Lemma 3, $m$ is a divisor of $\text{den}(\eta(s)) = \text{den}(s)$ and $\eta(s)$ is the desired rational point of $\text{conv}(v,w)$.

\hfill $\square$

Theorem 3.5. If the polyhedron $P$ is a $\mathbb{Z}$-retract of $[0,1]^n$ then

(i) $P$ is contractible,
(ii) $P$ contains a vertex of $[0,1]^n$, and
(iii) $P$ has a strongly regular triangulation (Definition 2.4).

Proof. The proof of (i) is a routine exercise in algebraic topology, showing that $[0,1]^n$ is contractible, and a retract of a contractible space is contractible.

Concerning (ii), let $\eta: [0,1]^n \to P$ be a $\mathbb{Z}$-retraction onto $P$. By (i), $\eta$ must send every vertex of $[0,1]^n$ into some vertex of $[0,1]^n$.

To prove (iii), let $\Delta$ be a regular triangulation of $P$, as given by Lemma 2.1. Let the $r$-simplex $T = \text{conv}(v_0, \ldots, v_r)$ be maximal in $\Delta$. Let us write $d = \text{gcd}(\text{den}(v_0), \ldots, \text{den}(v_r))$, with the intent of proving $d = 1$. Let $z$ be a rational point lying in the relative interior of $T$, say for definiteness $z = \text{the Farey mediant}$ of $T$. Since $T$ is maximal there is an open set $U \subseteq [0,1]^n$ such that $z \in U$ and
Let \( U \cap P \subseteq T \). Let \( \eta : [0, 1]^n \to P \) be a \( \mathbb{Z} \)-retraction onto \( P \). Then \( \eta^{-1}(U) \) is an open set. Let \( w \) be a rational point in \( \eta^{-1}(U) \) whose denominator is a prime \( p > d \).

Since \( \eta(w) \) lies in the regular simplex \( T \), by Proposition 2.7 \( \eta(w) \) can be obtained via a finite sequence of Farey blow-ups starting from the regular complex given by \( T \) and its faces. One immediately verifies that \( d \) is a common divisor of the denominators of all Farey mediants thus obtained. In particular, \( d \) is a divisor of \( \text{den}(\eta(w)) \). Since \( p \) is prime, from (4) it follows that \( \text{den}(\eta(w)) \in \{1, p\} \). Since \( p > d \), \( d = 1 \). We have proved that \( \Delta \) is strongly regular. \( \Box \)

**Remark 3.6.** By Lemmas 2.1 and 2.5, Condition (iii) above is equivalent to

(iii') Every regular triangulation of \( P \) is strongly regular.

Condition (i) has the following equivalent reformulations (for definitions see the references given in the proof):

**Proposition 3.7.** For \( P \subseteq [0, 1]^n \) a rational polyhedron, the following conditions are equivalent:

- \( (\alpha) \) \( P \) is contractible.
- \( (\beta) \) \( P \) is \( n \)-connected, i.e., the homotopy group \( \pi_i(P) \) is trivial for each \( i = 0, \ldots, n \).
- \( (\gamma) \) \( P \) is a deformation retract of \([0, 1]^n\).
- \( (\delta) \) \( P \) is a retract of \([0, 1]^n\).
- \( (\epsilon) \) \( P \) is an absolute retract for the class of metric spaces.

**Proof.** \( (\alpha) \Rightarrow (\beta) \), [14] p.405.

\( (\beta) \Rightarrow (\alpha) \), [12] p.359.

\( (\alpha) \Rightarrow (\gamma) \) is a consequence of Whitehead theorem, [12] 346.

\( (\gamma) \Rightarrow (\delta) \), trivial.

\( (\delta) \Rightarrow (\alpha) \), because a retract of a contractible space (like \([0, 1]^n\)) is contractible.

\( (\alpha) \Leftrightarrow (\epsilon) \), [7] 15.2 together with [12] p.522. \( \Box \)

4. A converse of Theorem 3.5

In Theorem 4.3 below we will prove that Conditions (ii) and (iii) of Theorem 3.5, together with a stronger form of Condition (i), known as collapsibility [8, p.97], [18, 6.6], are also sufficient for a polyhedron \( P \subseteq [0, 1]^n \) to be a \( \mathbb{Z} \)-retract of \([0, 1]^n\). The necessary notation and terminology are as follows:

An \( m \)-simplex \( T \) of a simplicial complex \( \nabla \) in \([0, 1]^n\) is said to have a free face \( F \) if \( F \) is a facet (= maximal proper face) of \( T \), but is a face of no other \( m \)-simplex of \( \nabla \). It follows that \( T \) is a maximal simplex of \( \nabla \), and the removal from \( \nabla \) of both \( T \) and \( F \) results in the subcomplex \( \nabla' = \nabla \setminus \{T, F\} \) of \( \nabla \). The transition from \( \nabla \) to \( \nabla' \) is called an elementary collapse in [8 III.7.2] ("elementary contraction" in [19] p.247). If a simplicial complex \( \Delta \) can be obtained from \( \nabla \) by a sequence of elementary collapses we say that \( \nabla \) collapses to \( \Delta \). We say that \( \nabla \) is collapsible if it collapses to (the simplicial complex consisting of) one of its vertices.
Given a rational polyhedron \( P \subseteq [0,1]^n \) and a point \( a \in [0,1]^n \), following tradition we let
\[
aP = \bigcup \{ \text{conv}(a,x) : x \in P \},
\]
and we say that \( aP \) is the join of \( a \) and \( P \). If \( P = \emptyset \) we let \( aP = a \). Further, for any simplex \( S \) we use the notation
\[
\hat{S} = \bigcup \{ F \subseteq S : F \text{ is a facet of } S \} \quad \text{ and } \quad a\hat{S} = \bigcup \{ aT : T \text{ a facet of } S \}.
\]
Finally, we denote by \( o \) the origin, and by
\[
e_1, \ldots, e_n
\]
the standard basis vectors in \( \mathbb{R}^n \).

**Lemma 4.1.** Let \( m_1, \ldots, m_n \) be coprime integers \( \geq 1 \) and \( s \in \{ 2, \ldots, n \} \). Let
\[
M = \text{conv}(e_1/m_1, \ldots, e_n/m_n), \quad F = \text{conv}(e_1/m_1, \ldots, e_{s-1}/m_{s-1}), \quad p = e_s/m_s.
\]
Then there is a \( \mathbb{Z} \)-retraction of \( M \cup o(PF) \) onto \( M \cup o(pF) \).

**Proof.** First of all, both simplexes \( M \) and \( o(PF) \) are regular. In the light of Lemma 2.6 let us fix an integer \( k \geq 1 \) such that for every integer \( l \geq k \) there exists a point in \( M \) whose denominator is a divisor of \( l \).

Let the regular triangulation \( \Phi \) of \( oP \) consist of \( oF \) together with its faces. Let \( t_1 \) be the Farey mediant of \( oF \). Proceeding inductively, for each \( i = 1, \ldots, k \) let \( t_{i+1} \) be the Farey mediant of \( t_i F \). Let \( \Psi = \Phi(t_1, \ldots, t_{k+1}) \). By construction,
\[
den(t_{k+1}) = 1 + (k + 1) \sum_{i=1}^{s-1} m_i > den(t_k) = 1 + k \sum_{i=1}^{s-1} m_i > k.
\]
Since the 1-simplex \( \text{conv}(t_k, t_{k+1}) \) satisfies the hypotheses of Lemma 5.1, there is a rational point \( p^* \in \text{conv}(t_k, t_{k+1}) \) such that \( m_s \) is divisor of \( \text{den}(p^*) \), in symbols,
\[
m_s = \text{den}(p) \big| \text{den}(p^*).
\]

Let \( \Lambda \) be the regular triangulation consisting of the 1-simplex \( \text{conv}(t_k, t_{k+1}) \) together with its faces. By Proposition 2.7 there is a finite sequence of Farey blow-ups \( \Lambda(\omega_1, \omega_2), \ldots, \Lambda(\omega_{w_1}, \ldots, \omega_{w_n}) \) such that \( p^* \) is a vertex of some simplex of \( \Lambda(\omega_1, \ldots, \omega_{w_n}) \). The sequence of consecutive Farey blow-ups of \( \Psi \) at \( w_1, \ldots, w_n \) yields the regular triangulation \( \Delta = \Psi(\omega_1, \ldots, \omega_{w_n}) \) of the polyhedron \( oF \). Let \( w \) be an arbitrary point in the set \( V = \{ w_1, \ldots, w_n, t_{k+1} \} \). By (8) we can write \( \text{den}(w) > \text{den}(t_{k+1}) > k \), whence our initial stipulation about the integer \( k \) yields a rational point \( x_w \in M \) such that \( \text{den}(x_w) \) is a divisor of \( \text{den}(w) \), in symbols,
\[
\forall w \in V \exists x_w \in M \text{ such that } \text{den}(x_w) \big| \text{den}(w).
\]

To conclude the proof, let the regular triangulation \( \nabla \) of \( M \cup o(PF) \) consist of all faces of \( M \) together with the set of simplexes \( \{ pS : S \in \Delta \} \cup \Delta \). The vertices of \( \nabla \) are \( e_1/m_1, \ldots, e_{s-1}/m_{s-1}, e_s/m_s = p, e_{s+1}/m_{s+1}, \ldots, e_n/m_n \in M \), together with \( o, t_1, \ldots, t_{k+1} \in oF \) and \( w_1, \ldots, w_n \in \text{conv}(t_k, t_{k+1}) \). Let \( \xi \) be the unique continuous map of \( M \cup o(PF) \) into \([0,1]^n\) which is linear over each simplex of \( \nabla \), and for each vertex \( v \) of \( \nabla \) satisfies
Next we define \( \xi \).

By Lemma 3.2, recalling (9) and (10), \( \xi \) is a \( \mathbb{Z} \)-map. For some (uniquely determined) permutation \( \beta \) of \( \{1, \ldots, u\} \) let the list \( t_k, w_{\beta(1)}, \ldots, w_{\beta(u)}, t_{k+1} \) display the vertices of \( \nabla \) lying on \( \text{conv}(t_k, t_{k+1}) \) in the order of increasing distance from \( t_k \).

The maximal simplexes of \( \nabla \) are \( M, p(t_{k+1}F) \) and \( p(\text{conv}(a, b, S)) \), where \( S \) ranges over the set \( F \) of facets of \( F \), and \( a, b \in oF \) are consecutive vertices in the list

\[ o, t_1, t_2, \ldots, t_k, w_{\beta(1)}, \ldots, w_{\beta(u)}, t_{k+1}. \]

For any such maximal simplex \( T \in \nabla \), a tedious but straightforward perusal of (11) shows that \( \xi(T) \) is contained in the rational polyhedron \( M \cup o(p\tilde{F}) \), whence

\[ \xi(M \cup o(pF)) \subseteq M \cup o(p\tilde{F}). \]

Further, for every vertex \( v \in \{o, e_1/m_1, \ldots, e_n/m_n\} \) the last line of (11) shows that \( \xi(v) = v \). It follows that \( \xi(w) = w \) for each \( w \in M \cup o(p\tilde{F}) \).

In conclusion, \( \xi \) is a \( \mathbb{Z} \)-retraction of \( M \cup o(pF) \) onto \( M \cup o(p\tilde{F}) \).

For the proof of Theorem 4.3 below, given any regular complex \( \Lambda \) in \([0, 1]^n\) with vertex set \( V = \{v_1, \ldots, v_u\} \), we will construct a \( \mathbb{Z} \)-homeomorphic copy \( \Lambda^- \) of \( \Lambda \) in \([0, 1]^u\) in two steps as follows:

For any \((k-1)\)-simplex \( T = \text{conv}(v_{i(1)}, \ldots, v_{i(k)}) \) in \( \Lambda \), recalling the notation (7) we first set

\[ T^\perp = \text{conv}(\frac{e_{i(1)}}{\text{den}(v_{i(1)})}, \ldots, \frac{e_{i(k)}}{\text{den}(v_{i(k)})}) \subseteq [0, 1]^u. \]

Next we define

\[ \Lambda^- = \{T^\perp \mid T \in \Lambda\}. \]

It follows that \( \Lambda^- \) is a regular complex, whose symbiotic relation with \( \Lambda \) is given by the following

**Lemma 4.2.** For every rational polyhedron \( P \subseteq [0, 1]^n \) and regular triangulation \( \Lambda \) of \( P \) there is a \( \mathbb{Z} \)-homeomorphism \( \eta \) of \( P \) onto \( |\Lambda^-| \) which is linear on each simplex of \( \Lambda \).

**Proof.** Letting, as above, \( \{v_1, \ldots, v_u\} \) be the vertices of \( \Lambda \), the proof immediately follows from (12)-(13), upon noting that the map

\[ f : v_j \mapsto \frac{e_j}{\text{den}(v_j)}, \quad (j = 1, \ldots, u) \]

as well as its inverse \( f^{-1} \), satisfy the hypotheses of Lemma 3.2 \( \square \)

**Theorem 4.3.** Let \( P \subseteq [0, 1]^n \) be a polyhedron. Suppose

(i') \( P \) has a collapsible triangulation \( \nabla \);

(ii) \( P \) contains a vertex \( v \) of \([0, 1]^n\);

(iii) \( P \) has a strongly regular triangulation \( \Delta \).
Then $P$ is a $\mathbb{Z}$-retract of $[0,1]^n$.

Proof. By (iii), $P$ is a rational polyhedron. By [3], it is no loss of generality to assume that $\nabla$ is rational. The desingularization process of [15, 1.2], yields a regular subdivision $\tilde{\nabla}$ of $\nabla$ via finitely many blow-ups. By (i') and [19, Theorem 4], $\tilde{\nabla}$ is collapsible. By Lemma 2.8, $\tilde{\nabla}$ is strongly regular, since so is $\Delta$ by (iii).

Let $v_1, \ldots, v_n$ be the vertices of $\tilde{\nabla}$. Defining the regular complex $\tilde{\nabla}^\perp$ as in [13], we have a rational polyhedron $Q = [\tilde{\nabla}^\perp] \subseteq [0,1]^n$ which, by Lemma 4.2, is a $\mathbb{Z}$-homeomorphic copy of $P$. Thus in view of Lemma 3.3 it is sufficient to prove that

$$Q \text{ is a } \mathbb{Z}\text{-retract of } [0,1]^n. \quad (14)$$

To this purpose, let us first note that $\nabla^\perp$ inherits from $\tilde{\nabla}$ both properties of strong regularity and collapsibility. The vertices of $\nabla^\perp$ are $v_1^\perp, \ldots, v_n^\perp$, where $v_i^\perp = \frac{v_i}{\text{den}(v_i)}$, as given by (12). By (ii), it is no loss of generality to assume $v = v_1$, whence $\text{den}(v_1) = 1$ and $v_1^\perp = e_1$.

Following Whitehead [19, p.248], let

$$\nabla^\perp = \Delta_0, \Delta_1, \ldots, \Delta_m \quad (15)$$

be a sequence of regular triangulations such that for each $i = 1, \ldots, m$, $\Delta_i$ is obtained from $\Delta_{i-1}$ via an elementary collapse, and $\Delta_m$ only consists of the 0-simplex $\{e_1\}$. For each $i = 1, \ldots, m$, we then have uniquely determined simplexes $p_i, F_i, E_i \in \Delta_{i-1}$ such that

(a) $p_i$ is a vertex of $E_i$ not in $F_i$, and $E_i = p_iF_i$;

(b) $F_i$ is a proper face of no other simplex of $\Delta_{i-1}$ but $E_i$;

(c) $\Delta_i = \Delta_{i-1} \setminus \{E_i, F_i\}$.

Letting $o$ denote the origin in $\mathbb{R}^n$, the join $oQ$ is star-shaped at $o$, [12, p.38], in the sense that for every $z \in oQ$ the set $\text{conv}(o,z)$ is contained in $oQ$. The proof of [6, Theorem 1.4] shows that $oQ$ is a $\mathbb{Z}$-retract of $[0,1]^n$. We will construct a sequence

$$oQ = |\Lambda_0| \xrightarrow{n_1} |\Lambda_1| \xrightarrow{n_2} \cdots \xrightarrow{n_m} |\Lambda_m| = Q \cup \text{conv}(o,e_1) \quad (16)$$

of $\mathbb{Z}$-retractions of rational polyhedra in $[0,1]^n$, together with a $\mathbb{Z}$-retraction $\phi$ of $Q \cup \text{conv}(o,e_1)$ onto $Q$. To this purpose, for each $j = 0, \ldots, m$ we set

$$\Lambda_j = \{o\} \cup \Delta_0 \cup \{oT \mid T \in \Delta_j\}. \quad (17)$$

Every $\Lambda_j$ is a regular complex, and

$$|\Lambda_j| = |\Delta_0| \cup \{oT \mid T \in \Delta_j\} = Q \cup o\{T \mid T \in \Delta_j\} = Q \cup o|\Delta_j|. \quad (18)$$

Recalling (c) we immediately have

$$\Lambda_i = \Lambda_{i-1} \setminus \{oE_i, oF_i\} \quad \text{and} \quad |\Lambda_{i-1}| = oE_i \cup |\Lambda_i|, \quad (19)$$

for each $i = 1, \ldots, m$. From (a) we obtain

$$oE_i \cap |\Lambda_i| = \bigcup \{oF \mid F \text{ is a facet of } E_i \text{ and } F \neq F_i\} = o(p_iF_i), \quad (20)$$

for all $i = 1, \ldots, m$. As the reader will recall from (6), $F_i$ denotes the pointset union of facets of $F_i$. By (a), $p_iF_i$ is the pointset union of the facets of $E_i$ different from $F_i$. We now choose a maximal simplex $M_i$ of $\Delta_0 = \nabla^\perp$ such that $E_i \subseteq M_i$. Since $\Delta_0$ is strongly regular, the denominators of the vertices of $M_i$ are coprime.
An application of Lemma 4.1 yields a \( \mathbb{Z} \)-retraction \( \xi_i \) of \( M_i \cup oE_i = M_i \cup o(p_iF_i) \) onto \( M_i \cup o(p_iF_i) \). By (19) and (20), the map \( \eta_i : |A_{i-1}| \to |A_i| \) defined by

\[
\eta_i(w) = \begin{cases} 
  w & \text{if } w \in |A_i| \\
  \xi_i(w) & \text{if } w \in oE_i 
\end{cases}
\]

is a \( \mathbb{Z} \)-retraction of \( |A_{i-1}| \) onto \( |A_i| \), as promised in (16). The composite map \( \eta = \eta_m \circ \ldots \circ \eta_1 : oQ \to |A_m| \) is a \( \mathbb{Z} \)-retraction of \( oQ \) onto \( Q \cup \text{conv}(o,e_1) \).

Next let the map \( \phi : |A_m| \to Q \) be defined by

\[
\phi(w) = \begin{cases} 
  w & \text{if } w \in Q \\
  e_1 & \text{if } w \in \text{conv}(o,e_1) 
\end{cases}
\]

Then \( \phi \) is the promised \( \mathbb{Z} \)-retraction of \( |A_m| \) onto \( Q \), thus showing that \( Q \) is a \( \mathbb{Z} \)-retract of \( oQ \).

As already observed, we have a \( \mathbb{Z} \)-retraction \( \mu \) of \( [0,1]^n \) onto the star-shaped polyhedron \( oQ \).

Summing up, the composite map \( \phi \circ \eta \circ \mu \) is a \( \mathbb{Z} \)-retraction of \( [0,1]^n \) onto \( Q \), as required to settle (14). The proof is complete. \( \square \)

**Corollary 4.4.** Let \( P \subseteq [0,1]^n \) be a one-dimensional polyhedron. Then \( P \) is a \( \mathbb{Z} \)-retract of \( [0,1]^n \) iff \( P \) satisfies Conditions (i)-(iii) of Theorem 3.5. In the present case, Condition (i) is equivalent to saying that \( P \) is connected and simply connected, i.e., \( P \) is a tree.

**Proof.** \( P \) necessarily is a rational polyhedron. Let \( \nabla \) be a regular triangulation of \( P \), as given by Lemma 2.1. In the light of Theorems 3.5 and 4.3 with Proposition 3.7 we have only to check that \( P \) is connected and simply connected iff \( \nabla \) is collapsible. \((\Rightarrow)\) : Then \( P \) contains no simple closed curve, whence \( P \) is a tree. All triangulations of \( P \) are collapsible. \((\Leftarrow)\) : Suppose \( \nabla \) collapses to its vertex \( v \). Then \( v \) is a deformation retract of \( P \), and \( P \) is contractible, i.e., \( P \) is connected and simply connected. \( \square \)

## 5. Finitely generated projective unital \( \ell \)-groups

We now apply the results of the previous sections to finitely generated projective unital \( \ell \)-groups. In contrast to what Baker and Beynon proved for \( \ell \)-groups in [2] and [4], finitely generated projective unital \( \ell \)-groups form a very special subclass of finitely presented unital \( \ell \)-groups:

**Corollary 5.1.** Suppose \((G,u)\) is a finitely presented unital \( \ell \)-group, and write \((G,u) \cong M_n \upharpoonright P\) for some \( n = 1, 2, \ldots \) and some rational polyhedron \( P \subseteq [0,1]^n \) as given by Proposition 1.2.

(I) If \((G,u)\) is projective then \( P \) satisfies Conditions (i)-(iii) of Theorem 3.5.

(II) If \( P \) is one-dimensional and satisfies Conditions (i)-(iii) then \((G,u)\) is projective.

(III) More generally, if \( P \) satisfies Conditions (ii)-(iii) and has a collapsible triangulation then \((G,u)\) is projective.
Proof. (I) In the light of Theorems 3.5 and 3.1 we must only prove that $P$ is a $\mathbb{Z}$-retract of $[0, 1]^n$. By Lemma 3.3 it suffices to settle the following

Claim: If a rational polyhedron $Q \subseteq [0, 1]^m$ satisfies $\mathcal{M}_n \upharpoonright P \cong \mathcal{M}_m \upharpoonright Q$ then $P$ and $Q$ are $\mathbb{Z}$-homeomorphic.

To this purpose, let $\iota: \mathcal{M}_n \upharpoonright P \cong \mathcal{M}_m \upharpoonright Q$ be a unital $\ell$-isomorphism. Let $\xi_1, \ldots, \xi_n: [0, 1]^n \to [0, 1]$ be the coordinate maps. Each element $\xi_i \uparrow P \in \mathcal{M}_n \upharpoonright P$ is sent by $\iota$ to some element $h_i \uparrow Q$ of $\mathcal{M}_m \upharpoonright Q$. Since each $\xi_i$ belongs to the unit interval of $\mathcal{M}_n \upharpoonright P$, then $h_i$ belongs to the unit interval of $\mathcal{M}_m \upharpoonright Q$, i.e., the range of $h_i$ is contained in the unit interval $[0, 1]$. Then the map $\eta: [0, 1]^m \to [0, 1]^n$ defined by

$$\eta(x_1, \ldots, x_m) = (h_1(x_1, \ldots, x_m), \ldots, h_n(x_1, \ldots, x_m)),$$

is a $\mathbb{Z}$-map. Let $f$ be an arbitrary function in $\mathcal{M}_n$. Arguing by induction on the number of operations in $f$ in the light of Proposition 1.1 we get

$$\iota(f \upharpoonright P) = (f \circ \eta) \upharpoonright Q.$$  \hspace{1cm} (21)

Since $\iota$ is a unital $\ell$-isomorphism,

$$f \upharpoonright P = 0 \iff \iota(f \upharpoonright P) = 0 \iff f \circ \eta \upharpoonright Q = 0 \iff f \upharpoonright Q(0) = 0.$$  \hspace{1cm} 

By [14] Proposition 4.17, $P = \eta(Q)$. Interchanging the roles of $\iota$ and $\iota^{-1}$ we obtain a $\mathbb{Z}$-map $\mu: [0, 1]^n \to [0, 1]^m$ such that $\iota^{-1}(g \upharpoonright Q) = (g \circ \mu) \upharpoonright P$ and $\mu(P) = Q$. By (21), $f \upharpoonright P = f \circ (\eta \circ \mu) \upharpoonright P$ and $g \upharpoonright Q = g \circ (\mu \circ \eta) \upharpoonright \mathcal{M}_m$. Again by [14] Proposition 4.17, the composition $\eta \circ \mu$ is the identity map on $P$, and $\mu \circ \eta$ is the identity map on $Q$. Thus $P$ and $Q$ are $\mathbb{Z}$-homeomorphic, as required to settle our claim and also to complete the proof of (I).

(II) From Corollary 4.4 and Theorem 3.1

(III) From Theorems 4.8 and 3.1 \hfill \Box

Our final result in this paper, Theorem 5.3 below, will give (intrinsic) necessary and sufficient conditions for $(G, u)$ to be finitely generated projective, in terms of the spectral properties of $G$. To this purpose, we denote by

$$\text{MaxSpec}(G)$$

the set of maximal ideals of $G$, equipped with the spectral topology. [5, §10], [10] 5.7: a basis of closed sets for MaxSpec$(G)$ is given by all sets of the form

$$\{p \in \text{MaxSpec}(G) \mid a \in p\},$$

where $a$ ranges over all elements of $G$.

A maximal ideal $m$ is discrete if the ordering of the totally ordered quotient $(G, u)/m$ is discrete (non-dense). In this case, by the Hion–Hölder theorem [9] p.45-47, [9] 2.6], $(G, u)$ is unitaly $\ell$-isomorphic to $(\mathbb{Z}_n, 1)$ for a unique integer $n \geq 1$, called the rank of $m$ and denoted $\rho(m)$. In case $m$ is not discrete we set

$$\rho(m) = +\infty \quad \text{and} \quad \gcd(n, +\infty) = +\infty, \forall n = 1, 2, \ldots.$$  \hspace{1cm} (22)

Lemma 5.2. For every $n = 1, 2, \ldots$ and nonempty closed set $X \subseteq [0, 1]^n$ we have:

(a) The map $\alpha: x \in X \mapsto m_x = \{f \in \mathcal{M}_n \upharpoonright X \mid f(x) = 0\}$ is a homeomorphism of $X$ onto MaxSpec$(\mathcal{M}_n \upharpoonright X)$. The inverse map sends every $m \in \text{MaxSpec}(\mathcal{M}_n \upharpoonright X)$ to the only member $x_m$ of the set $\{g^{-1}(0) \mid g \in m\}$. 

(b) For every \( m \in \text{MaxSpec}(M_n | X) \) there is a unique pair \((\iota_m, R_m)\), where \( R_m \) is a unital \( \ell \)-subgroup of \((\mathbb{R}, 1)\), and \( \iota_m \) is a unital \( \ell \)-isomorphism of the quotient \( \text{MaxSpec}(M | X)/m \) onto \( R_m \). For every \( x \in X \) and \( f \in M_n | X \), \( f(x) = \iota_m(f/m_x) \).

(c) Suppose \( x \in X \) and \( m = \alpha(x) \). Then \( x \) is rational iff \( m \) is discrete. If this is the case, \( \rho(m) = \text{den}(x) \).

Proof. The proof of (a) follows from a classical result due to Yosida [21], because the functions in \( M_n | X \) separate points, [14, 4.17]. (See [14, 8.1] for further details). (b) is a reformulation of the time-honored Hion-Hölder theorem [9, p.45-47], [5, 2.6]. Finally, (c) follows from (a) and (b).

Theorem 5.3. Let \((G, u)\) be a unital \( \ell \)-group.

(1) The following conditions are necessary for \((G, u)\) to be finitely generated projective:

(A) \((G, u)\) is finitely presented.

(B) For every discrete maximal ideal \( m \) of \( G \), and open neighborhood \( N \) of \( m \) in \( \text{MaxSpec}(G) \), there is \( n \in N \) such that \( \gcd(\rho(n), \rho(m)) = 1 \).

(C) \( G \) has a maximal ideal of rank 1.

(D) The topological space \( \text{MaxSpec}(G) \) is compact Hausdorff, metrizable, finite-dimensional and contractible.

(2) If \( \text{MaxSpec}(G) \) is one-dimensional, the four conditions (A)-(D) are also sufficient for \((G, u)\) to be finitely generated projective. Actually, Condition (D) can be replaced by the requirement that \( \text{MaxSpec}(G) \) is connected and simply connected.

(3) More generally, if \((G, u)\) satisfies Conditions (B)-(C), and \((G, u) \cong M_n | P \) for some rational polyhedron \( P \) having a collapsible triangulation, then \((G, u)\) is finitely generated projective.

Proof. (1) Condition (A) holds by Proposition 1.2.

To prove Condition (B), by (A) jointly with Corollary 5.11(I) we can write \((G, u) = M_n | P \) for some rational polyhedron \( P \subseteq [0, 1]^n \) satisfying Conditions (i)-(iii) of Theorem 3.5. Using the homeomorphism \( \alpha \) of Lemma 5.2, ranks of discrete maximal ideals of \( G \) coincide with denominators of their corresponding rational points in \( P \). If \( P \) is a singleton, then by Condition (ii) it coincides with some vertex of \([0, 1]^n\) and we have nothing to prove. Otherwise, let \( \Delta \) be a strongly regular triangulation of \( P \) as given by Condition (iii). Let \( x \) be a rational point of \( P \). The proof of Theorem 3.5 shows that every open neighborhood of \( x \) contains rational points \( q \) of arbitrarily large prime denominator, whence \( \gcd(\text{den}(x), \text{den}(q)) = 1 \), and (B) is proved.

Using \( \alpha \) we see that Condition (C) holds, because \( P \) satisfies Condition (ii) of Theorem 3.5.

To conclude the proof of (1), we must show that \( \text{MaxSpec}(G) \) has all the properties listed in Condition (D). In the light of Lemma 5.2, this is equivalent to checking that \( P \) has all these properties. (The invariance of contractibility under homeomorphisms follows, e.g., from Proposition 3.7.) The first three properties
are trivially verified. \( P \) is contractible because it satisfies Condition (i) of Theorem 3.5.

(2) Since \((G, u)\) satisfies Condition (A), it can be identified with \(M_n \upharpoonright P\), for some \( n = 1, 2, \ldots \) and rational polyhedron \( P \subseteq [0, 1]^n \). The homeomorphism \( \alpha \) of Lemma 5.2 again ensures that ranks of discrete maximal ideals of \( G \) coincide with denominators of their corresponding rational points of \( P \). Thus Condition (C) is to the effect that \( P \) must contain some vertex of \([0, 1]^n\), whence \( P \) satisfies Condition (ii) of Theorem 3.5.

We next prove that every regular triangulation \( \Delta \) of \( P \) is strongly regular. By Remark 3.6 this is an equivalent reformulation of Condition (iii). Suppose \( \Delta \) is a counter-example, and let \( T \) be a maximal simplex of \( \Delta \) such that the gcd of the denominators of the vertices of \( T \) is \( d > 1 \). Pick a rational point \( q \) in the relative interior of \( P \) and observe that, by Proposition 2.7, \( d \) is a divisor of \( \text{den}(q) \). By the assumed maximality of \( T \), for every rational point \( q' \) in a suitably small open neighborhood of \( q \), \( d \) is a divisor of \( \text{den}(q') \). The maximal ideal \( \alpha(q) \) of \( G \) falsifies the assumed Condition (B). We have shown that \( P \) satisfies Condition (iii) of Theorem 3.5.

Further, \( P \) satisfies Condition (i) because its homeomorphic copy \( \text{MaxSpec}(G) \) is contractible, by Condition (D).

Having thus shown that \( P \) satisfies Conditions (i)-(iii) of Theorem 3.5 an application of Corollary 5.1(II) proves the first statement in (2).

For the second statement, since \( G \) has an order unit, \( \text{MaxSpec}(G) \) is a nonempty compact Hausdorff space, (for a proof see [5, 10.2.2], where the order unit is called “unité forte”). The homeomorphism \( \alpha \) of Lemma 5.2 ensures that there is no ambiguity in defining the dimension of the compact Hausdorff metrizable space \( P \) and of its homeomorphic copy \( \text{MaxSpec}(G) \). Condition (A) ensures that \( \text{MaxSpec}(G) \) is finite-dimensional and metrizable. Thus Condition (D) equivalently states that \( \text{MaxSpec}(G) \) is contractible. By Proposition 3.7, this is in turn equivalent to stating that the one-dimensional space \( \text{MaxSpec}(G) \) is connected and simply connected. This completes the proof of (2).

(3) By Proposition 1.2, \((G, u)\) is finitely presented. Given the map \( \alpha \) of Lemma 5.2 Condition (C) is equivalent to stating that \( P \) satisfies Condition (ii) of Theorem 3.5. Condition (iii) now follows from Condition (B) arguing as in (2) above. An application of Corollary 5.1(III) concludes the proof.

Acknowledgment. The authors are grateful to Professor Marco Grandis and to Dr. Bruno Benedetti for introducing them to the main results of algebraic topology used in this paper. We also thank Professor Vincenzo Marra for drawing our attention to reference 3.

References

[1] Alexander, J. W.: The combinatorial theory of complexes. Ann. of Math. 31, 292–320 (1930)
[2] Baker, K. A.: Free vector lattices. Canad. J. Math. 20, 58–66 (1968)
[3] Beynon, W. M.: On rational subdivisions of polyhedra with rational vertices. Canad. J. Math. 29, 238–242 (1977)
[4] Beynon, W. M.: Applications of duality in the theory of finitely generated lattice-ordered abelian groups. Canad. J. Math. 29, 243–254 (1977)
[5] Bigard, A., Keimel, K., Wolfenstein, S.: Groupes et Anneaux Réticulés. Springer Lecture Notes in Mathematics 608, Springer-Verlag, Berlin and New York (1977).
[6] Cabrer, L. M., Mundici, D.: Projective MV-algebras and rational polyhedra. Algebra Universalis, Special issue in memoriam P. Conrad, (J. Martinez, Ed.) (2009)
[7] Dugundji, J.: Absolute neighborhood retracts and local connectedness in arbitrary metric spaces. Compos. Math. 13, 229–246 (1956-1958)
[8] Ewald, G.: Combinatorial Convexity and Algebraic Geometry. Springer-Verlag, New York (1996)
[9] Fuchs, L.: Partially ordered algebraic systems. Pergamon Press, Oxford, (1963)
[10] Glass, A. M. W.: Partially ordered groups. World Scientific, Singapore (1999)
[11] Goodearl, K. R.: Partially Ordered Abelian Groups with Interpolation. Math. Surveys and Monogr. 20, Amer. Math. Soc. (1986).
[12] Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
[13] Morelli, R.: The birational geometry of toric varieties. J. Algebraic Geom. 5, 751–782 (1996)
[14] Mundici, D.: Interpretation of AF $C^*$-algebras in Lukasiewicz sentential calculus. J. Funct. Anal. 65, 15–63 (1986)
[15] Mundici, D.: Farey stellar subdivisions, ultrasimplicial groups, and $K_0$ of AF $C^*$-algebras. Adv. Math. 68, 23–39 (1988)
[16] Mundici, D.: The Haar theorem for lattice-ordered abelian groups with order unit. Discrete Contin. Dyn. Syst. 21, 537–549 (2008)
[17] Spanier, E. H.: Algebraic Topology. McGraw-Hill, (1966). Corrected reprint, Springer-Verlag, New York-Berlin (1981/1986).
[18] Stallings, J. R.: Lectures on Polyhedral Topology. Tata Institute of Fundamental Research, Bombay, (1967)
[19] Whitehead, J. H. C.: Simplicial spaces, nuclei and m-groups. Proc. Lond. Math. Soc. 45, 243–327 (1939)
[20] Włodarczyk, J.: Decompositions of birational toric maps in blow-ups and blow-downs. Trans. Amer. Math. Soc. 349, 373–411 (1997)
[21] Yosida, K.: On the representation of the vector lattice. Proc. Imp. Acad. Tokyo 18, 339–343 (1942)

(L.Cabrer) CONICET, Dep. de Matemáticas – Facultad de Ciencias Exactas, Universidad Nacional del Centro de la Provincia de Buenos Aires, Punto 399 – Tandil (7000), Argentina
E-mail address: lcabrer@exa.unicen.edu.ar

(D.Mundici) Dipartimento di Matematica “Ulisse Dini”, Università degli Studi di Firenze, viale Morgagni 67/A, I-50134 Firenze, Italy
E-mail address: mundici@math.unifi.it