Chapter 9.
Theory of Differential Inclusions and Its Application in Mechanics

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Abstract The following chapter deals with systems of differential equations with discontinuous right-hand sides. The key question is how to define the solutions of such systems. The most adequate approach is to treat discontinuous systems as systems with multivalued right-hand sides (differential inclusions). In this work three well-known definitions of solution of discontinuous system are considered. We will demonstrate the difference between these definitions and their application to different mechanical problems. Mathematical models of drilling systems with discontinuous friction torque characteristics are considered. Here, opposite to classical Coulomb symmetric friction law, the friction torque characteristic is asymmetrical. Problem of sudden load change is studied. Analytical methods of investigation of systems with such asymmetrical friction based on the use of Lyapunov functions are demonstrated. The Watt governor and Chua system are considered to show different aspects of computer modeling of discontinuous systems.

9.1 Introduction

Two hundred and thirty years ago, after numerous experiments, Coulomb has formulated a law of dry friction (Coulomb friction, see Fig. 9.1). Since then various problems stimulated the development of theory of mechanical systems with dry friction.

First of all, it is important to mention the well-known Penleve paradoxes [66], which provoked interesting discussions and showed contradiction of Coulomb’s law with Newton’s laws of classical mechanics. Nowadays, an independent research
branch named “tribology” has grown out of these classical problems. Many researchers contributed to this branch of science, among them there are such famous scientists as F.P. Bowden and D. Tabor [10], E. Rabinovich [74], P.J. Blau [8], K.C. Ludema [63], I.G. Goryacheva [25, 26], V.I. Kolesnikov [35].

From a mathematical point of view the problem of investigation of dynamics in models with dry friction is closely connected with the theory of differential inclusions and dynamical systems with discontinuous right-hand sides. Nowadays this theory is being actively developed and applied to investigation of different applications by such famous scientists as S.V. Emelyanov, A.S. Poznyak, V.I. Utkin and others (see e.g. [1, 6, 7, 12, 18, 19, 34, 67, 72, 73, 78, 78]).

The following work is motivated by the problem of investigation of a drilling system. This problem was studied by the research group from the Eindhoven University of Technology [13, 65]. In these papers, the interaction of the drill with the bedrock is described by symmetric discontinuous characteristics. In the paper [49] a more precise model of friction is considered for simplified mathematical model of drilling system actuated by induction motor. Here the following assumption is made: the moment of resistance force with asymmetric characteristics (see Fig. 9.2, \( M \) is assumed to be large enough) is used instead of classical Coulomb friction with symmetric discontinuous characteristics. Such an asymmetric characteristic has a “locking” property – it allows rotation of the drill in one direction only. The considered simplified model corresponds to an ordinary hand electric drill. In this case it is naturally to assume that the drilling takes place in one direction only.

The study of discontinuous systems with dry friction is a challenging task due to the need for a special theory for discontinuous systems to be developed. In particular, a proper definition of the solution on discontinuity surface is required. Now there are many definitions of solutions of discontinuous system, here three of them are considered following the works [22, 24, 31, 50]. Analytical investigation of stability of simplified drilling systems will be performed. Additional examples of numerical modeling theory of differential inclusions and its application in discontinuous
mechanical systems will be considered. It will be explained why it is necessary to use special methods of investigation for discontinuous systems.

9.2 Differential Equation with Discontinuous Right-Hand Sides and Differential Inclusions: Definitions of Solutions

The starting point of studies in theory of differential inclusions is usually connected with the works of French mathematician A. Marchaud and Polish mathematician S. K. Zaremba published in 1934-1936. They were studying equations of the form

\[ Dx \subset f(t, x), \]

where \( t \in \mathcal{D}_t \subset \mathbb{R} \), \( x \in \mathcal{D}_x \subset \mathbb{R}^n \) and \( f(t, x) \) is a multivalued vector function that maps each point \((t, x)\) of some region \( \mathcal{D} = \mathcal{D}_t \times \mathcal{D}_x \) to the set \( f(t, x) \) of points from \( \mathbb{R}^n \). For operator \( D \) the notions of contingent and paratingent were introduced by Marchaud and Zaremba respectively.

**Definition 9.1.** Contingent of vector function \( x(t) \) at the point \( t_0 \) is a set \( \text{Cont} x(t_0) \) of all limit points of sequences \( \frac{x(t_i) - x(t_0)}{t_i - t_0} \), \( t_i \rightarrow t_0 \), \( i = 1, 2, ... \)

**Definition 9.2.** Paratingent of vector function \( x(t) \) at the point \( t_0 \) is a set \( \text{Parat} x(t_0) \) of all limit points of sequences \( \frac{x(t_i) - x(t_j)}{t_i - t_j} \), \( t_i \rightarrow t_0 \), \( t_j \rightarrow t_0 \), \( i = 1, 2, ... \)

Wazhewski continued investigations of Marchaud and Zaremba and proved \([80]\) that if \( x(t) \) is a solution of differential inclusion \( (9.1) \) in the sense of Marchaud then vector function \( x(t) \) is absolutely continuous.
Definition 9.3. Let $I \subset \mathbb{R}_t \subset \mathbb{R}$ be an interval of time. Function $x(t) : I \rightarrow \mathbb{R}^n$ is absolutely continuous on $I$ if for every positive number $\varepsilon$ there is a positive number $\delta$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(t_{1k}, t_{2k})$ of $I$ with $t_{1k}, t_{2k} \in I$ satisfies
\[
\sum_k (t_{2k} - t_{1k}) < \delta
\]
then
\[
\sum_k ||x(t_{2k}) - x(t_{1k})|| < \varepsilon.
\]

Important property of absolutely continuous function $x(t)$ is that $x(t)$ has derivative $\dot{x}(t)$ almost everywhere on $I$ (see, e.g. [75]). This property played a key role in the development of theory of differential inclusions and equations with discontinuous right-hand side since it allowed to avoid artificial constructions in Definition 9.1 and 9.2 and to consider usual derivative almost everywhere.

In 1960 paper [20] was published by A. F. Filippov, where he considered solutions of differential equations with discontinuous right-hand side as absolutely continuous functions. Filippov approach is one of the most popular among other notions of solutions of systems with discontinuous right-hand sides. Following [20], consider a system
\[
\dot{x} = f(t, x), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \quad (9.2)
\]
where $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a piecewise continuous function such that measure of the set of discontinuity points is assumed to be zero.

Definition 9.4. Vector function $x(t)$, defined on an interval $(t_1, t_2)$, is called a solution of (9.2) if it is absolutely continuous and for almost all $t \in (t_1, t_2)$ vector $\dot{x}(t)$ is within minimal closed convex set, which contains all $f(t, x')$ when $x'$ is within almost all $\delta$-neighbourhood of the point $x(t)$ in $\mathbb{R}^n$ (for fixed $t$), i.e.
\[
\dot{x} \in \prod_{\delta > 0} \prod_{\mu = 0} \text{conv} f(t, U(x(t), \delta) - N). \quad (9.3)
\]

Consider the case when system (9.2) is autonomous and vector function $f(x)$ is discontinuous on some smooth surface $S$ in $\mathbb{R}^n$ and continuous in the neighbourhood of this surface. Let there exist limits $f_+(x)$ and $f_-(x)$ of vector function $f(x)$ when a point $x$ approaches $S$ from one or another side. Suppose that the vectors $f_+(x)$ and $f_-(x)$ are both pointing towards the discontinuity surface $S$. Then the so-called sliding mode appears. According to Definition 9.4, the vector field of sliding mode on the discontinuity surface can be defined as follows. The plane tangent to the surface $S$ at the point $x$ and the segment $l$, which connects the terminal points of vectors $f_+(x)$ and $f_-(x)$, are constructed. Then the vector with initial point at $x$ and terminal point at the point of intersection of the segment and tangent plane is constructed: $f_0 = f_0(x)$. According to Definition 9.4, vector $f_0(x)$ defines vector field at the point $x$. 
The obtained solution of (9.2) satisfies Definition 9.4 but nevertheless there are important applied problems for which Definition 9.4 is unsuitable. As an example of such problem we consider a problem of synthesis of controls $u_1$ and $u_2$, which are limited, $|u_1| \leq 1$, $|u_2| \leq 1$, and which transform optimally fast each point $(x_1(0), x_2(0))$ of the system

\[
\dot{x}_1 = x_2u_1, \quad \dot{x}_2 = u_2
\]  

(9.4)

to the origin of coordinates. It is well-known [9] that synthesis of such control is possible for the whole plane $(x_1, x_2)$. For example, for the first quadrant of the plane the optimal control is as follows

\[
\begin{align*}
  u_1 &= \begin{cases} 
    1, & x_1 < 0.5x_2^2, \\
    -1, & x_1 \geq 0.5x_2^2,
  \end{cases} \\
  u_2 &= \begin{cases} 
    -1, & x_1 \leq 0.5x_2^2, \\
    1, & x_1 > 0.5x_2^2.
  \end{cases}
\end{align*}
\]  

(9.5)

In particular, the trajectory $x_1 = 0.5x_2^2$ is optimal and for this trajectory system (9.4) takes the form $\dot{x}_1 = -x_2$, $\dot{x}_2 = -1$. Let us take the point $x = (x_1, x_2)$ on this trajectory and approach to this trajectory from the side $x_1 < 0.5x_2^2$. The limit value of the right-hand sides of system (9.4) is $f(x) = (x_2, -1)$. If we approach the trajectory from the side $x_1 > 0.5x_2^2$, then the limit is $f(x) = (-x_2, 1)$. Since $f_+(x) = -f_-(x)$, in this particular case the segment $l$ passes through the point $x$, i.e. $f_0(x) = 0$ and according to definition 9.4 the solution on sliding mode is equilibrium state. At the same time $(-x_2, -1)$ is a velocity vector on optimal trajectory. Thus, optimal trajectory is not a solution in the sense of Definition 9.4 by Filippov.

M.A. Aizerman and E.S. Pyatnitskiy [2] offered other definition of solution of equations with discontinuous right-hand sides which allows one to deal with usual derivative. We consider their approach in the particular case when $f(t, x)$ is discontinuous on the surface $\Sigma$. Consider a sequence of continuous vector functions $f_\varepsilon(t, x)$, which coincide with $f(t, x)$ outside of $\varepsilon$-neighbourhood of surface $\Sigma$, and tend to $f(t, x)$ for $\varepsilon \to 0$ at each point, which does not belong to $\Sigma$. Let $x_\varepsilon(t)$ be a solution of the system

\[
\dot{x} = f_\varepsilon(t, x).
\]  

(9.6)

Then the solution of system (9.2) in the sense of Aizerman and Pyatnitskiy is a limit of any uniformly converging subsequence of solutions $x_{\varepsilon_k}(t)$:

\[
x_{\varepsilon_k}(t) \Rightarrow x(t).
\]

In general, there may exist more than one such limit. Nevertheless this notion of solution, introduced in [2], does not always suitable for applications.

For example, consider a system

\[
\dot{x} = Ax + b\phi(\sigma), \quad \sigma = c^t x,
\]  

(9.7)
where $\phi(\sigma)$ is a dry friction characteristic, shown in Fig.9.1 or in Fig.9.3, i.e.

$$
\phi(\sigma) = \begin{cases} 
\text{sign } \sigma, & \sigma \neq 0, \\
[-1, 1], & \sigma = 0,
\end{cases} \quad \text{or} \quad
\phi(\sigma) = \begin{cases} 
\text{sign } \sigma, & \sigma \neq 0, \\
[-\alpha, \alpha], & \sigma = 0.
\end{cases}
$$

(9.8)

Since the definitions suggested by Filippov and by Aizerman and Pyatnitskiy deal only with those values of a nonlinearity for which $\sigma \neq 0$, the solutions of system (9.7) with dry friction characteristics, shown in Fig.9.1 and in Fig.9.3 coincide. This result does not match physics of this phenomena.

To take into account dynamics on the discontinuity surface, the most adequate approach is to consider system with discontinuous right-hand side (9.2) as system with multivalued right-hand side, called differential inclusion [21, 24]:

$$
\dot{x} \in f(t, x),
$$

(9.9)

where $t \in \mathcal{D}_t \subset \mathbb{R}$, $x \in \mathcal{D}_x \subset \mathbb{R}^n$ and $f(t, x)$ is a multivalued vector function that maps each point $(t, x)$ of some region $\mathcal{D} = \mathcal{D}_t \times \mathcal{D}_x$ to the set $f(t, x)$ of points from $\mathbb{R}^n$.

**Definition 9.5.** Vector function $x(t)$ is called a solution of differential inclusion (9.9), if it is absolutely continuous and for those $t$ for which derivative $\dot{x}(t)$ exists, the following inclusion holds:

$$
\dot{x}(t) \in f(t, x(t)).
$$

(9.10)

To build a substantive theory it is assumed that multivalued function $f(t, x)$ is semicontinuous. Filippov approach [22] requires additionally that $\forall (t, x) \in \mathcal{D}$ the set $f(t, x)$ is a minimal closed bounded set. This conditions coincide with Definition 9.4.

**Definition 9.6.** Function $f(t, x)$ is called *semitious* (upper semicontinuous, $\beta$-continuous) at the point $(t_0, x_0)$ if for any $\epsilon > 0$ there exists $\delta(\epsilon, t, x)$ such that
the set $f(t,x)$ is contained in the $\varepsilon$-neighbourhood of set $f(x_0,x_0)$, provided that the point $(t,x)$ belongs to $\delta$-neighbourhood of the point $(t_0,x_0)$.

As was shown above for some physical problems Filippov definition may give wrong results, thus a more general class of multivalued functions $f(t,x)$ was considered by A. Kh. Gelig, G. A. Leonov and V. A. Yakubovich [24] (Gelig-Leonov-Yakubovich approach): $\forall (t,x) \in \mathcal{D}$ the set $f(t,x)$ is a bounded, closed, and convex set.

The following local theorem on the existence of solutions of differential inclusion holds true [24].

**Theorem 9.1.** Suppose that multivalued function $f(t,x)$ is semicontinuous at every point $(t_1,x_1)$ of a region

$$\mathcal{D}_1 \subset \mathcal{D} : \quad |t_1 - t_0| \leq \alpha, \quad |x_1 - a| \leq \rho,$$

and set $f(t_1,x_1)$ is bounded, closed, and convex. In addition, suppose

$$\sup |y| = c \quad \text{for} \quad y \in f(t_1,x_1), \quad (t_1,x_1) \in \mathcal{D}_1.$$

Then for $|t - t_0| \leq \tau = \min(\alpha, \rho/c)$ there exist at least one solution $x(t)$ with initial condition $x(t_0) = a$, which satisfies (9.9) in the sense of Definition 9.5.

For differential inclusion (9.9) theorem on continuation of solution remaining in bounded region holds true. Also the theorem, which states that for every $\omega$-limiting point of trajectory $x(t)$ there exists at least one trajectory that entirely consists of $\omega$-limiting points and some other theorems of qualitative theory, are valid [22, 24, 77].

For generalization of classic results of stability theory on solutions of differential inclusion (9.9) in Gelig-Leonov-Yakubovich approach the existence of procedure of determination of discontinuous right-hand side according to a chosen solution (i.e. existence of extended nonlinearity, which allows one to replace differential inclusion with differential equation) was proved by B.M. Makarov specially for the monograph [24].

Let us demonstrate now the methods of theory of differential equations with discontinuous right-hand sides, described above, in concrete problems.

**9.3 Analytical methods of investigation of discontinuous systems:**

**an example of mathematical model of drilling system with “locking friction”**

Consider the simplified mathematical model of drilling system actuated by induction motor (here we follow the works [28, 49]). Assume that the drill is absolutely rigid
body stiffly connected to the rotor, which rotates by means of the magnetic field created by the stator of the induction motor. The value of interaction of the drill with the bedrock is defined as a value of resistance torque, which appears during the drilling process. Such a system experiences rapidly changing loads during the drilling, thus it is necessary to investigate the behaviour of induction motor during load jumps, i.e. when resistance torque acting on the drill suddenly changes.

The following problem of stability is urgent since decrease of drilling systems failures plays important role in the oil and gas industry \[76, 79\].

As the equations of electromechanical model of the drilling system we consider the equations of induction motor, proposed in \[36,52\], supplemented with the resistance torque \(M_f\) of drilling:

\[
\begin{align*}
L \frac{di_1}{dt} + Ri_1 &= SB \sin \theta \dot{\theta}, \\
L \frac{di_2}{dt} + Ri_2 &= SB \cos \theta \dot{\theta}, \\
I \ddot{\theta} &= -\beta SB(i_1 \sin \theta + i_2 \cos \theta) + M_f \left( \frac{R}{L} + \dot{\theta} \right).
\end{align*}
\]

\(\text{(9.11)}\)

Here \(\theta\) is a rotation angle of the drill about the magnetic field created by the stator, which rotates with a constant angular speed \(\frac{R}{L}\). \(i_1(t), i_2(t)\) are currents in rotor windings, \(R\) is resistance of windings, \(L\) is inductance of windings, \(B\) is the induction of magnetic field, \(S\) is an area of one wind, \(I\) is an inertia torque of drill, \(\beta\) is a proportionality factor, \(\omega = \dot{\theta} + \frac{R}{L}\) is an angular velocity of the drill rotation with respect to a fixed coordinate system. The resistance torque \(M_f\) is assumed to be of the Coulomb type \[24,66\]. Unlike the classic Coulomb friction law with symmetrical discontinuous characteristic the friction torque \(M_f\) has non-symmetrical discontinuous characteristics shown in Fig. 9.2.:

\[
M_f(\omega) = \begin{cases} 
-T_0 & \text{for } \omega > 0 \\
[-T_0, MT_0] & \text{for } \omega = 0 \\
MT_0 & \text{for } \omega < 0.
\end{cases}
\]

For \(T_0 \geq 0\) the number \(M > 0\) is assumed to be large enough. That means that the drilling process only takes place when \(\omega > 0\). Such characteristics does not allow for \(\omega\) to switch from positive to negative values during the transient process in real drilling systems. In this case the system only gets stuck for \(\omega = 0\) for a long enough period of time. These effects happen frequently during drilling operation and are studied by the analysis of system \(9.11\).

Performing the nonsingular change of variables
we reduce system (9.11) to the following one:

\[ \dot{s} \in ay + \tilde{M}_f(s), \]
\[ \dot{y} = -cy - s - xs, \]
\[ \dot{x} = -cx + ys, \]

where \( a = \frac{\beta (SB)^2}{IL} \), \( c = \frac{R}{L} \). Here variables \( x, y \) define electric values in rotor windings and the variable \( s \) defines the sliding of the rotor. \( \tilde{M}_f \) has the following form

\[ \tilde{M}_f(s) = \begin{cases} 
\gamma, & s < c; \\
[ -\gamma M, \gamma], & s = c; \\
-\gamma M, & s > c; 
\end{cases} \]

where \( \gamma = \frac{T_0}{T} \).

According to Makarov’s theorem for any solution of (9.12) in the sense of the Gelig-Leonov-Yakubovich approach there exists extended nonlinearity \( \tilde{M}_f_0 \) such that the following system is valid

\[ \dot{s}(t) = ay(t) + \tilde{M}_f_0(t), \]
\[ \dot{y}(t) = -cy(t) - s(t) - x(t)s(t), \]
\[ \dot{x}(t) = -cx(t) + ys(t), \]

for almost all \( t \).

Let us conduct local analysis of equilibrium states of system (9.13).

**Proposition 9.1.** For \( 0 \leq \gamma < \frac{a}{2} \) system (9.13) has a unique asymptotically stable equilibrium state.

Indeed, for \( \gamma = 0 \) system (9.13) has one asymptotically stable equilibrium state \( s = 0, y = 0, x = 0 \), which occurs when the rotation of drill with constant angular speed is congruent to the rotation speed of the magnetic field (idle speed operation).

For \( \gamma \in (0, \frac{a}{2}) \) system (9.13) has one equilibrium state

\[ s_0 = \frac{c(a - \sqrt{a^2 - 4\gamma^2})}{2\gamma}, \quad y_0 = -\frac{\gamma}{a}, \quad x_0 = -\frac{\gamma s_0}{ac}, \]
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where $s_0$ is the smallest root of the equation

$$\frac{acs}{c^2 + s^2} = \gamma.$$ 

In this case the direction of rotation of the drill and the magnetic field are the same, but the drill rotates with a lower angular speed $s_0 < c$.

Assume that there is a sudden change in load at the moment $t = \tau$ from value $\gamma_0$ to value $\gamma_1$, where $0 < \gamma_0 < \gamma_1$. This occurs at the moment when the drill comes in contact with the bedrock. For $\gamma = \gamma_0$ the system experiences a unique state of stable equilibrium

$$s_0 = \frac{c(a - \sqrt{a^2 - 4\gamma_0^2})}{2\gamma_0}, \quad y_0 = -\frac{\gamma_0}{a}, \quad x_0 = -\frac{\gamma_0s_0}{ac}.$$ 

It is essential that after the transient process the solution $s(t), x(t), y(t)$ of the system (9.12) with $\gamma = \gamma_1$ and the initial data $s(\tau) = \frac{c(a - \sqrt{a^2 - 4\gamma_0^2})}{2\gamma_0}, y(\tau) = -\frac{\gamma_0}{a}, x(\tau) = -\frac{\gamma_0s_0}{ac}$ tends to the equilibrium state

$$s_1 = \frac{c(a - \sqrt{a^2 - 4\gamma_1^2})}{2\gamma_1}, \quad y_1 = -\frac{\gamma_1}{a}, \quad x_1 = -\frac{\gamma_1s_1}{ac}$$

as $t \to +\infty$.

The following theorem holds.

**Theorem 9.2.** Let the following conditions be fulfilled

$$\gamma_0 < \frac{a}{2}, \quad (9.14)$$

$$\gamma_1 < \min \left\{ \frac{a}{2}, \ 2c^2 \right\}, \quad (9.15)$$

$$3(M^2 + 2M)\gamma_1^2 - 8c^2\gamma_1 + 3ac^2 \geq 0. \quad (9.16)$$

Then the solution of system (9.13) with $\gamma = \gamma_1$ and the initial data $s(\tau) = \frac{c(a - \sqrt{a^2 - 4\gamma_0^2})}{2\gamma_0}, y(\tau) = -\frac{\gamma_0}{a}, x(\tau) = -\frac{\gamma_0s_0}{ac}$ tends to an equilibrium state of this system as $t \to +\infty$.

Let us give the scheme of the proof of this theorem. We consider the region $\{s(t) < c\}$ of the phase space of system (9.13).

Performing change of variables
we reduce system (9.13) to the following system
\[
\dot{s}(t) = \eta(t), \\
\dot{\eta}(t) = -c\eta(t) + az(t)s(t) - \psi(s(t)), \quad \text{for almost all } t \\
\dot{z}(t) = -cz(t) - \frac{1}{a}s(t)\eta(t) - \frac{\gamma_1}{ac}\eta(t).
\] (9.17)

Here \(\psi(s) = -\frac{\gamma_1}{c}s^2 + as - c\gamma_1\).

Introduce a function
\[
V(s, \eta, z) = \frac{a^2}{2}z^2 + \frac{1}{2}\eta^2 + \int_{s_1}^{s} \psi(s')ds.
\]

For every solution of system (9.17) from region \(s(t) < c\) the following condition
\[
V(s(t), \eta(t), z(t)) = -a^2cz(t)^2 - \frac{\gamma_1}{c}\eta(t)z(t) - c\eta(t)^2 \leq 0 \quad \text{for almost all } t
\] (9.18)
is fulfilled.

Quadratic form in the right-hand side of system (9.18) definitely negative under condition (9.15).

We introduce a set
\[
\Omega = \left\{ V(s, \eta, z) \leq \int_{s_1}^{c} \psi(s')ds + \frac{(1+M)^2}{2}\gamma_1^2, s \in [s_2, c] \right\},
\]
where the point \(s_2 < c\) is such that
\[
\int_{s_2}^{c} \psi(s')ds + \frac{(1+M)^2}{2}\gamma_1^2 = 0.
\]
The set \(\Omega\) is limited and for \(s = c\) it becomes
\[
\frac{a^2}{2}z^2 + \frac{1}{2}\eta^2 \leq \frac{(1+M)^2}{2}\gamma_1^2.
\]

Returning to the initial coordinates \((x, y, s)\), we obtain
\[
(x + \frac{\gamma_1}{a})^2 + (y + \frac{\gamma_1}{a})^2 \leq \frac{(1+M)^2}{a^2}\gamma_1^2.
\]
Note that this circle is below the upper boundary $y = \frac{M}{a} \gamma_1$ of the sliding region
\[ \Delta = \left\{ s = c, -\frac{\gamma_1}{a} \leq y \leq \frac{M}{a} \gamma_1 \right\} \] of system (9.13).

In the sliding region $\Delta$ system (9.13) can be reduced to the system of ordinary
differential equations
\begin{align*}
\dot{y}(t) &= -cy(t) - c - cx(t), \\
\dot{x}(t) &= -cx(t) + cy(t),
\end{align*}
which is reduced by replacement of time $t = \frac{t_1}{c}$ to
\begin{align*}
\dot{y}(t) &= -y(t) - x(t) - 1, \\
\dot{x}(t) &= -x(t) + y(t).
\end{align*}

(9.19)

We introduce a function
\[ W(x,y) = (x + \frac{\gamma_1}{a})^2 + (y + \frac{\gamma_1}{a})^2. \]

Semicircles \( \left\{ W(x,y) = R^2, y \leq -\frac{\gamma_1}{a} \right\} \), where \( R \leq \frac{M + 1}{a} \gamma_1 \), are noncontact
for system (9.19). Indeed, for the solutions of the system (9.19) under condition
(9.15) the following relation
\[ \frac{1}{2} W(x(t),y(t)) = -y^2(t) - y(t) - \frac{\gamma_1}{a} + \frac{\gamma_1^2}{a^2} - (x(t) + \frac{\gamma_1}{a})^2 \\
= (\frac{2\gamma_1}{a} - 1)y(t) + \frac{\gamma_1}{a}(\frac{2\gamma_1}{a} - 1) - R^2 < 0 \]
is valid.

The solution, which falls into the sliding region, necessarily comes out through
the lower boundary $y = -\frac{\gamma_1}{a}$ into the region $s < c$ due to the fact that $\dot{s} < 0$ for
$s = c, y < -\frac{\gamma_1}{a}$. From condition (9.18) it follows that this solution proves to be
inside the region \( \left\{ V(s,\eta,z) \leq \int_{s_1}^c \psi(s) ds \right\} \). Then it does not fall further into the
sliding region, and tends to a unique equilibrium state $(s_1,y_1,x_1)$ of the system due
to the limitation of $\Omega$. It is obvious that the trajectories, which fall into $\Omega$, but not
existing in the sliding region, also tend to the equilibrium state.

This allows one to prove that the system is dichotomic if condition (9.15) is
fulfilled (for details about classical results of Lyapunov see [24]).

\[ ^1 \text{System is called dichotomic if every solution bounded for } t > 0 \text{ tends to stationary set for } t \to +\infty. \]
The set \( \Omega \) contains the point \( s = s_0, \eta = \gamma_1 - \gamma_0, z = \frac{\gamma_0 - \gamma_1}{ac} s_0 \) if

\[
\frac{(\gamma - \gamma_0)^2}{2c^2}s_0^2 + \frac{(\eta - \gamma_0)^2}{2} \leq \int_{s_0}^{c} \psi(s)ds + \frac{(1 + M)^2}{2} \eta^2.
\] (9.20)

For \( \gamma_0 < \gamma_1 \) and condition (9.16) we have

\[
\frac{(\gamma - \gamma_0)^2}{2} \leq \int_{0}^{c} \psi(s)ds + \frac{(1 + M)^2}{2} \eta^2.
\] (9.21)

Let us show that

\[
\frac{(\gamma - \gamma_0)^2}{2c^2}s_0^2 \leq \int_{s_0}^{0} \psi(s)ds.
\] (9.22)

Indeed, taking into account that \( \gamma_0 \leq \frac{a}{2c^2} \gamma_1 \), we obtain:

\[
\frac{\gamma_1}{3c} s_0^2 - \frac{a}{2} s_0 - \frac{(\gamma - \gamma_0)^2}{2c^2}s_0 + c\gamma_1 = \frac{1}{12c^2} \gamma_0^2 \left( c^2(a - \sqrt{a^2 - 4\gamma_0^2})^2 \gamma_1 - 3a^2 
\right.
\]

\[
\left. 2c^2 \gamma_0 + 3a^2 c \sqrt{a^2 - 4\gamma_0^2} \gamma_0 - 3(\gamma - \gamma_0)^2 (a - \sqrt{a^2 - 4\gamma_0^2}) \gamma_0 + 12c^2 \gamma_1 \gamma_0^2 \right) \geq
\]

\[
\frac{1}{12c^2} \gamma_0^2 \left( 2a^2 c^2 - 2a c^2 \sqrt{a^2 - 4\gamma_0^2} \gamma_0 + 3ac^2 \sqrt{a^2 - 4\gamma_0^2} - 3a^2 c^2 \gamma_0 + 3\sqrt{a^2 - 4\gamma_0^2} \gamma_0 \gamma_1^2 - 3a \gamma_1^2 \gamma_0 + 8c^2 \gamma_0^2 \gamma_1 \right) \geq 0.
\]

Hence, from inequalities (9.21) and (9.22) we obtain condition (9.20).

Thus solution \( s(t), \eta(t), z(t) \) with the initial data \( s(\tau) = s_0, \eta(\tau) = \gamma_1 - \gamma_0, z(\tau) = \frac{\gamma_0 - \gamma_1}{ac} s_0 \) tends to equilibrium state of the system.

Let \( M \) be a reasonably large number such that condition (9.16) of the theorem is fulfilled. In this case the following statement is valid.

**Corollary 9.1.** Let \( \gamma_0 = 0 \) and

\[
\gamma_1 < \min \left\{ \frac{a}{2}, \frac{2c^2}{a} \right\}.
\] (9.23)

Then the solution of system (9.12) with \( \gamma = \gamma_1 \) and the initial data \( s(\tau) = 0, y(\tau) = 0, x(\tau) = 0 \) tends to equilibrium state of this system as \( t \to +\infty \).

For the values \( \gamma_1 \in \left\{ \frac{2c^2}{a}, \frac{a}{2} \right\} \) (i.e., condition (9.23) is not fulfilled) the computer modeling of system (9.12) (region 2 in Fig. 9.4), which shows that the statement of consequence is retained, is carried out. Further we will discuss the aspects of modeling of systems with multivalued right-hand sides.

It can be checked that the extended nonlinearity can be written down in explicit form.
Fig. 9.4 Safe load region: 1 – due to the theorem, 2 – obtained by numerical modeling of the system

Corollary 9.2. For system (9.12) the extended nonlinearity is of the following form:

$$\tilde{M}_{f0} = \begin{cases} 
\gamma, & \text{if } s = c, y < -\frac{\gamma}{a} \text{ or } s < c; \\
-\gamma M, & \text{if } s = c, y > \frac{M \gamma}{a} \text{ or } s > c; \\
-ay, & \text{if } s = c, -\frac{\gamma}{a} \leq y \leq \frac{M \gamma}{a}.
\end{cases}$$

In the works [29, 30, 32, 33, 51, 55] more complex models of drilling systems were studied. Analytical investigation of such models is a challenging task, so it is necessary to use numerical methods. Let us further describe some aspects of numerical modeling in two other applied systems – Watt governor and Chua circuit.

9.4 Numerical Methods of Investigation of Discontinuous Systems

9.4.1 Difficulties of numerical modeling of discontinuous systems

Numerical modeling is one of the tools of investigation of differential equations with discontinuous equations with right-hand sides. Let us first show why it is important to use special methods developed for discontinuous systems. Consider the I.A. Vyshnegradsky problem. The following system of differential equations describes dynamics of Watt governor with dry friction

$$\begin{align*}
\dot{y}_1 &= -Ay_1 + y_2 - \text{sign}(y_1), \\
\dot{y}_2 &= -By_1 + y_3, \\
\dot{y}_3 &= -y_1.
\end{align*}$$

(9.24)

Let sign be understood here in ordinary sense:
Let us consider the values of parameters $A = 1.5$, $B = 1.1$ and conduct numerical modeling of trajectory of system (9.24) with initial data $y_1(0) = -0.5$, $y_2 = 1$, $y_3(0) = 1.2$, using standard Matlab build-in function ode45 for solving ordinary differential equations. As can be seen in Fig. 9.5 numerical modeling shows that there are oscillations in system (9.24).

This nonlinear system was studied by A.A. Andronov and A.G. Mayer [5]. In particular, they proved that sliding segment of this system is globally stable if the following inequalities

$$
A > 0, \quad B > 0, \quad AB > 1.
$$

are satisfied.

Thus, the result of modeling with standard build-in Matlab functions may lead to wrong results. Moreover, the notation (9.24) is wrong and right notation is as follows

$$
\dot{y}_1 \in -Ay_1 + y_2 - \text{Sign}(y_1), \\
\dot{y}_2 = -By_1 + y_3, \\
\dot{y}_3 = -y_1,
$$

and the model of dry friction is described in the following way

$$
\text{Sign}(y_1) = \begin{cases} 
1, & \text{if } y_1 > 0; \\
[-1, 1], & \text{if } y_1 = 0; \\
-1, & \text{if } y_1 < 0.
\end{cases}
$$
Let us conduct the numerical modeling using Filippov definition \cite{71}. The results of the modeling of system (9.26) correspond to theoretical results and are shown in Fig. 9.6.

\textbf{9.4.2 Numerical Modeling of Chua System}

We showed an example of numerical modeling of a discontinuous system based on Filippov approach. Let us compare this method with modeling based on the Aizerman-Pyatnitskiy approach.

Consider the following example of discontinuous system – modified Chua system with discontinues characteristic \cite{40,41,50}

\begin{align*}
\dot{x}_1 & = -\alpha(m_1 + 1)x_1 + \alpha x_2 - \alpha(m_0 - m_1)\text{Sign}(x_1), \\
\dot{x}_2 & = x_1 - x_2 + x_3, \\
\dot{x}_3 & = -\beta x_2 - \gamma x_3,
\end{align*}

(9.27)

where $\alpha$, $\beta$, $\gamma$, $m_0$, $m_1$ are parameters of the system.
For parameters $\alpha = 8.4562$, $\beta = 12.0732$, $\gamma = 0.0052$, $m_0 = -0.1768$, $m_1 = -1.1468$ system (9.27) has a so-called hidden attractor \cite{38, 43, 54, 57–59}.

So as to model system (9.27) with the help of both Filippov and Gelig-Leonov-Yakubovich definitions, the special event-driven numerical method, described in \cite{71}, was used. For modeling the system, using Aizerman-Pyatnitskiy approach, one needs to replace $\text{sign}(x_1)$ by $\text{sat}_\varepsilon(x_1) = \frac{1}{2}\left(\frac{x_1}{\varepsilon} + 1 - \left|\frac{x_1}{\varepsilon} - 1\right|\right)$, where $\varepsilon > 0$. Decrease of parameter $\varepsilon$ allows one to obtain Aizerman-Pyatnitskiy solution ($\text{sat}_\varepsilon(x_1) \Rightarrow \text{sign}(x_1)$ for $\varepsilon \neq 0$, see Fig. 9.7). In Fig 9.8 hidden attractor modeled using Filippov definition method is drawn in red colour and hidden attractor modeled using Aizerman and Pyatnitskiy definition method is drawn in green. As one can see, the more $\varepsilon$ is decreased, the more solutions (attractors) coincide with each other. This fact meets theorem proved in \cite{31, 50}.

**Conclusion**

We have discussed Filippov, Aizerman-Pyatnitskiy, and Gelig-Leonov-Yakubovich approaches to the study of differential equation with discontinuous right-hand sides and differential inclusions. While for a wide range of dynamical models with these three approaches give the same result (see, e.g. \cite{31, 50}), there are models, where the
difference between these definitions is essential. As examples, we have considered the Chua circuit with discontinuous characteristics, and mechanical systems with classical Coulomb symmetric friction law and the asymmetrical friction torque characteristic.

Acknowledgements This work was supported by the Russian Science Foundation (project 14-21-00041)

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