Period Estimates for Autonomous Evolution Equations with Lipschitz Nonlinearities

Aleksander Ćwiszewski, Władysław Klinikowski,
Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, Poland
e-mail: aleks@mat.umk.pl, wklin@mat.umk.pl

October 13, 2020

Abstract

We derive an estimate for the minimal period of autonomous strongly damped hyperbolic problems. Our result corresponds to the works by Yorke [12], Busenberg et al. [2] for ordinary differential equations as well as Robinson and Vidal-Lopez [9] and [10] for parabolic problems. A general approach is developed for treating both hyperbolic and parabolic problems. An example of application to a class of beam equations is provided.

1 Introduction

The known estimates for ordinary differential equations comes from Yorke who proved in [12] that if $T > 0$ is the minimal period of a solution to an ordinary differential equation

$$\ddot{u}(t) = f(u(t)),$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz with constant $L > 0$, then $T \geq 2\pi/L$. The same estimate was showed in the infinite dimensional phase space by Busenberg et al. in [2], where (1) is considered in a Hilbert space. For (1) in a general Banach space, the estimate $T \geq 6/L$ was proved therein. A natural question arises if there is any period estimate in case of partial differential equations and systems. In case of the abstract parabolic problems of the form

$$\ddot{u}(t) + Au(t) = f(u(t)), t > 0,$$

where $A$ is a self-adjoint positive operator in a separable Hilbert space $X$ and a Lipschitz function $f : X^\beta \to X$ is defined on the fractional space $X^\beta$ with $\beta \in [0, 1)$ (associated to the operator $A$—see e.g. [7]), Robinson and Vidal-Lopez in [10] (see also [9]) obtained the following lower bound

$$T \geq (2^{1-\beta} + (\beta/e)^\beta/(1 - e^{-1/2})(1 - \beta))^{-1/(1-\beta)} \cdot L^{-1/(1-\beta)}.$$  

In this paper we shall revisit the parabolic problem, slightly improving the above estimate (3) and deal with the minimal period for hyperbolic problems of the form

$$\ddot{u}(t) + \alpha \dddot{u}(t) + Au(t) = f(u(t), \dot{u}(t)), t > 0,$$
where $A$ is as above and $\alpha > 0$ is a damping coefficient. In the hyperbolic case we assume, additionally, that $A$ has compact resolvent, which means that the spectrum $\sigma(A)$ consists of positive $\lambda_n, n \in \mathbb{N}$, such that $\lambda_n \to +\infty$ as $n \to +\infty$. The nonlinear term $f : X^{1/2} \times X \to X$ ($X^{1/2}$ is the fractional space determined by $A$) is Lipschitz, i.e. there exists $L > 0$ such that, for any $u_1, u_2 \in X^{1/2}$ and $v_1, v_2 \in X$,
\[\|f(u_1, v_1) - f(u_2, v_2)\| \leq L \left(\|u_1 - u_2\|^{1/2}_{1/2} + \|v_1 - v_2\|^{1/2}_{1/2}\right)^{1/2}\]  
(5)

where $\| \cdot \|_{1/2}$ stands for the norm in $X^{1/2}$. It is an abstract model for many physical (systems of) equations, including systems with the so-called strongly damped beam that is fixed at both ends (see Section 6).

Let us make general comments on the hyperbolic case. When $\alpha = 0$ there is no lower bound for periods of periodic solutions of (4). Indeed, if $e \in X \setminus \{0\}$ is the eigenvector for $A$, corresponding to an eigenvalue $\lambda > 0$ and $f(u,v) := L \cdot u$, with $L < \lambda$, then $u(t) := \sin(t\sqrt{\lambda - L}) \cdot e$ is a well-defined periodic solutions of (4) with the minimal period $2\pi/\sqrt{\lambda - L}$. Therefore, if the eigenvalues of the operator $A$ make an unbounded set, then we have a periodic solution of (4) of arbitrarily small period. Another case where one can not expect any minimal period estimate is when (4) is gradient-like, that is there exists a Lyapunov functional, i.e. a functional that decreases/increases along nontrivial trajectories (see e.g. [6]). Then obviously (4) has no nonstationary periodic solutions. However, like in the parabolic case, there are classes of nonlinearities $f$ for which hyperbolic partial differential equations with strong damping are not gradient-like (see Section 6) and when periodic solutions occur. Note that also systems of hyperbolic equations are not gradient-like if only the linear perturbation field $f$ does not come from a gradient field.

In order to study (4) we rewrite the problem as a system
\[
\begin{aligned}
\dot{u} &= v \\
\dot{v} &= -A(\alpha \cdot v + u) + f(u,v),
\end{aligned}
\]
which in turn can be represented as the first order equation
\[\dot{z} = Az + F(z), \quad t > 0,\]  
(6)

where $z = (u,v)$ and the operator $A : D(A) \to X$ in $X := X^{1/2} \times X$ is given by
\[A(u,v) := (v, -A(\alpha \cdot v + u)), \quad (u,v) \in D(A),\]  
(7)

with $D(A) := \{(u,v) \in X \mid \alpha \cdot v + u \in D(A), v \in X^{1/2}\}$. The nonlinear term $F : X \to X$ is given by $F(u,v) = (0, f(u,v))$ and is also Lipschitz with the constant $L$ if $X$ is equipped with the norm given by the scalar product $\langle (u_1, v_1), (u_2, v_2) \rangle = \langle u_1, u_2 \rangle^{1/2}_{1/2} + \langle v_1, v_2 \rangle$. It is well-known that $-A$ is sectorial (see e.g. [4], [5] or [6]). One cannot apply the result by Robinson and Vidal-Lopez from [10] due to the fact that $A$ is not symmetric (as it is assumed therein), neither a direct application of the idea sketched in remarks after the proof of Theorem 3.2 in [10] will provide satisfactory estimates. In this paper we develop an approach that can be effectively applied to the hyperbolic case.

We come up with a unified approach allowing to consider both parabolic and hyperbolic partial differential systems/equations. We first consider a general operator $A : D(A) \subset X \to X$
being an infinitesimal generator of a $C_0$ semigroup of bounded linear operators $e^{t\mathcal{A}} : \mathbb{X} \to \mathbb{X}$, $t \geq 0$, on a Banach space $(\mathbb{X}, \| \cdot \|_\mathbb{X})$. Assume that $(\mathcal{V}, \| \cdot \|_\mathcal{V})$ is a Banach space such that 

(V1) $D(\mathcal{A}) \subset \mathcal{V} \subset \mathbb{X}$;

(V2) $\mathcal{V}$ is embedded continuously into $\mathbb{X}$;

(V3) $e^{t\mathcal{A}}(\mathbb{X}) \subset \mathcal{V}$, for all $t > 0$, and the family $\{e^{t\mathcal{A}}|_\mathcal{V} : \mathcal{V} \to \mathcal{V}\}_{t \geq 0}$, of operators restricted to $\mathcal{V}$, is a $C_0$ semigroup of bounded linear operators on $\mathcal{V}$.

We say that $\mathcal{A}$ has the uniformly half-bounded decomposition property, in short property (UHBD), whenever there exists $\mu_0 \geq 0$, $M > 0$ and a continuous $m : (0, +\infty) \to (0, +\infty)$ such that, for any $\mu > \mu_0$, there exists a decomposition $\mathbb{X} = \mathbb{X}_\mu^- \oplus \mathbb{X}_\mu^+$ into closed subspaces such that 

(D$_{\mu,+}$) $\mathbb{X}_\mu^+ \subset D(\mathcal{A})$, $\mathcal{A}\mathbb{X}_\mu^+ \subset \mathbb{X}_\mu^+$ and $\|\mathcal{A}w\|_{\mathcal{V}} \leq \mu M\|w\|_{\mathcal{V}}$, for all $w \in \mathbb{X}_\mu^+$;

(D$_{\mu,-}$) $\mathcal{A}(\mathbb{X}_\mu^- \cap D(\mathcal{A})) \subset \mathbb{X}_\mu^-$, $\|e^{t\mathcal{A}}w\|_{\mathbb{X}} \leq e^{-\mu t}\|w\|_{\mathbb{X}}$, $\|e^{t\mathcal{A}}w\|_{\mathcal{V}} \leq m(t)e^{-\mu t}\|w\|_{\mathbb{X}}$, for all $w \in \mathbb{X}_\mu^-$ and $t > 0$, and $\|e^{t\mathcal{A}}w\|_{\mathcal{V}} \leq e^{-\mu t}\|w\|_{\mathcal{V}}$, for all $w \in \mathbb{X}_\mu^- \cap \mathcal{V}$ and $t > 0$;

(D$_{\mu,0}$) $\int_0^t m(s)e^{-\mu s}\, ds < +\infty$ for all $t > 0$.

A continuous function $z : (T_1, T_2) \to \mathcal{V}$, $T_1 < T_2$, is said to be a mild solution of 

\[ \dot{z} = \mathcal{A}z + \mathcal{F}(z) \]  

(8)

if and only if, for any $t, t_0 \in (T_1, T_2)$ with $t_0 < t$,

\[ z(t) = e^{(t-t_0)\mathcal{A}}z(t_0) + \int_{t_0}^t e^{(t-s)\mathcal{A}}\mathcal{F}(z(s))\, ds. \]  

(9)

In this abstract setting we get the following result.

**Theorem 1.** Suppose that the generator $\mathcal{A}$ of a $C_0$-semigroup of bounded linear operators in a Banach space $\mathbb{X}$ satisfies (V1)–(V3) and has property (UHBD) with some Banach space $\mathcal{V} \subset \mathbb{X}$ and $\mathcal{F} : \mathcal{V} \to \mathbb{X}$ is Lipschitz with constant $L > 0$. If there exists a nonstationary $T$-periodic mild solution $z : \mathbb{R} \to \mathcal{V}$ of (8), then either $T \geq 1/\mu_0 M$ or, for all $\mu \in (\mu_0, 1/MT)$,

\[ 1 \leq TL \left[ \frac{K^+_\mu}{1 - \mu MT} + \frac{K^-_\mu}{\mu T} \left( e^{-\mu T} m(T) + \int_0^{\mu T} m(s/\mu) \cdot e^{-s} \, ds \right) \right]. \]  

(10)

with $K^+_\mu := \|\mathbb{P}^+_\mu\|_{\mathcal{L}(\mathbb{X}, \mathcal{V})}$ and $K^-_\mu := \|\mathbb{P}^-_\mu\|_{\mathcal{L}(\mathbb{X}, \mathbb{X})}$, where $\mathbb{P}^+_\mu : \mathbb{X} \to \mathbb{X}_\mu^+$ and $\mathbb{P}^-_\mu : \mathbb{X} \to \mathbb{X}_\mu^-$ are the projections.

The above estimate does not provide an explicit estimate for $T$, however in a special case we get the following result, which applies in the case of damped hyperbolic problems (\ref{ injections}) (see Theorem \ref{ corollary 2.1} below).

**Corollary 2.** If additionally to the assumptions of Theorem \ref{ theorem 1} we assume that $\mu_0 > 0$, $m \equiv 1$ and

\[ K^-_\mu \leq (1 - \mu_0/\mu)^{-1}, \quad K^+_\mu \leq (1 - \mu_0/\mu)^{-1}, \quad \text{for all } \mu > \mu_0, \]  

(11)

then the period of any nontrivial periodic solution of (8) satisfies the inequality

\[ T \geq 1/L \left( 1 + \sqrt{M(1 + \mu_0/L)} \right)^2. \]
The proof of Theorem 1 and Corollary 2 is presented in Section 2.

In Section 3 we shall apply the above setting to the parabolic evolution problem (2) and provide an estimate for the minimal period of parabolic systems.

**Theorem 3.** If \( u : [0, T] \to X^{\beta}, \beta \in [0, 1) \) is a nonstationary periodic solution of (2) with the period \( T > 0 \), then

\[
T \geq \frac{1}{L K_\beta} \left( 1^{(1-\beta)} \right)
\]

where \( K_\beta := \min_{\eta \in (0,1)} H(\eta) \) and \( H : (0,1) \to (0, +\infty) \) is given by

\[
H(\eta) := \frac{\eta^\beta}{1 - \eta} + \frac{M_\beta}{\eta} \left( e^{-\eta} + \eta^\beta \int_0^\eta s^{-\beta} e^{-s} \, ds \right)
\]

with \( M_\beta = (\beta/e)^\beta \) if \( \beta \in (0, 1) \) and \( M_\beta = 1 \) if \( \beta = 0 \).

Consequently, one has

\[
T \geq \frac{1}{L} \left( 1^{(1-\beta)} \right) \cdot \left( 2^{1-\beta} + 2M_\beta/e^{1/2} + M_\beta/(1 - \beta) \right)^{1/(1-\beta)}
\]

for \( \beta \in (0, 1) \),

and,

\[
T \geq \frac{1}{4L} \quad \text{for} \quad \beta = 0.
\]

The above result improves (3) obtained by Robinson and Vidal-Lopez in [10] (see Remark 12).

In Section 4 we study the spectral properties of the hyperbolic operator \( A \) given by (7) and in Section 5 we prove that \( A \) has the property \( (UHBBD) \) with constants \( \mu_0 = 2/\alpha \) and \( M = 1 + \sqrt{2} \) (see Corollary 23) as well as the inequalities (11) hold (see Corollary 22). Therefore, in view of Corollary 2, we get the following minimal period estimate for damped hyperbolic equations, which is the main result of the paper.

**Theorem 4.** If \( u : [0, T] \to X \) is a nonstationary periodic solution of (1) with the period \( T > 0 \), then

\[
T \geq \frac{1}{L} \left( 1 + \sqrt{(1 + \frac{1}{\sqrt{2}})}(1 + 2/(\alpha L)) \right)^{2}.
\]

We shall illustrate the result in case of a damped beam equation in Section 6.

## 2 General period estimate – proof of Theorem 1

Assume that the operator \( A \) and \( F : V \to X \) is as in Theorem 1. Suppose that \( z : \mathbb{R} \to \mathbb{V} \) is a mild solution of

\[
\dot{z} = Az + F(z) \quad \text{on} \quad \mathbb{R}
\]

with minimal period \( T > 0 \). In particular, \( z(t) = z(t + T) \), for all \( t \in \mathbb{R} \), and one has the Duhamel formula

\[
z(t) = e^{(t-t_0)A}z(t_0) + \int_{t_0}^t e^{(t-s)A}F(z(s)) \, ds
\]

for any \( t_0 \in \mathbb{R} \) and all \( t > t_0 \). This yields, for any \( t \in \mathbb{R} \),

\[
z(t) = z(t + T) = e^{T A}z(t) + \int_t^{t+T} e^{(t+s-T)A}F(z(s)) \, ds,
\]
which, after change of variables in the integral, yields

\[(I - e^{Tk})z(t) = \int_0^T e^{(T-s)\lambda}F(z(t+s))\,ds.\]  

(17)

Let us take \(\mu > \mu_0\) such that

\[\mu MT < 1\]  

(18)

and use property \((UHBD)\) to obtain the decomposition

\[\mathcal{X} = \mathcal{X}_\mu^- + \mathcal{X}_\mu^+.

**Remark 5.** By \((D_{\mu,+})\) and \((D_{\mu,-})\), one has \(\mathbb{A}\mathcal{P}_\mu^+ w = \mathcal{P}_\mu^+ aw\) for all \(w \in D(A)\) and \(\mathbb{A}\mathcal{P}_\mu^- w = \mathcal{P}_\mu^- aw\) for all \(w \in D(A) \cap \mathcal{X}_\mu^-\).

We shall also need the following estimates.

**Lemma 6.** Under the above assumptions, for any \(\mu > \mu_0\) and \(T > 0\), define \(R_{\mu,T} : \mathcal{X}_\mu^- \cap \mathcal{V} \to \mathcal{X}_\mu^- \cap \mathcal{V}\) by \(R_{\mu,T} u := u - e^{Tk}u, u \in \mathcal{X}_\mu^- \cap \mathcal{V}\). Then

(i) \[\|R_{\mu,T}^{-1} w\|_\mathcal{V} \leq (1 - e^{-\mu T})^{-1}\|w\|_\mathcal{V} \text{ for all } w \in \mathcal{X}_\mu^- \cap \mathcal{V},\]

(ii) \[\|R_{\mu,T}^{-1} w - w\|_\mathcal{V} \leq e^{-\mu T}(1 - e^{-\mu T})^{-1}m(T)\|w\|_\mathcal{X} \text{ for all } w \in \mathcal{X}_\mu^- \cap \mathcal{V}.

**Proof.** (i) By \((D_{\mu,-})\) we have \(e^{Tk}\mathcal{X}_\mu^- \subset \mathcal{X}_\mu^-\) and

\[\|e^{Tk}\|_{\mathcal{L}(\mathcal{X}_\mu^- \cap \mathcal{V}, \mathcal{X}_\mu^- \cap \mathcal{V})} \leq e^{-\mu T} < 1,\]  

(19)

where in the closed subspace \(\mathcal{X}_\mu^- \cap \mathcal{V}\) of \(\mathcal{V}\) we consider the norm from \(\mathcal{V}\). Hence, we can infer that the operator \(R_{\mu,T}\) is invertible and

\[\|R_{\mu,T}^{-1}\|_{\mathcal{L}(\mathcal{X}_\mu^- \cap \mathcal{V}, \mathcal{X}_\mu^- \cap \mathcal{V})} \leq (1 - \|e^{Tk}\|_{\mathcal{L}(\mathcal{X}_\mu^- \cap \mathcal{V}, \mathcal{X}_\mu^- \cap \mathcal{V})}^{-1} \leq (1 - e^{-\mu T})^{-1}.\]  

(20)

(ii) According to (i) and \((D_{\mu,-})\), for any \(w \in \mathcal{X}_\mu^- \cap \mathcal{V},\) we get

\[\|R_{\mu,T}^{-1} w - w\|_\mathcal{V} = \left| \sum_{k=1}^{\infty} e^{kT}w \right|_\mathcal{V} = \left| \sum_{k=0}^{\infty} e^{kT}e^{Tk}w \right|_\mathcal{V} \]

\[\leq \left| R_{\mu,T}e^{Tk}w \right|_\mathcal{V} \leq (1 - e^{-\mu T})^{-1}\|e^{Tk}w\|_\mathcal{V} \leq (1 - e^{-\mu T})^{-1}e^{-\mu T}m(T)\|w\|_\mathcal{X},\]

which completes the proof.

Now choose \(\tau \in (0,T)\) and consider \(D : [0, +\infty) \to \mathcal{X}\) given by

\[D(t) := z(t + \tau) - z(t), \ t \geq 0.\]

Clearly, \(D\) is a nonzero function as \(T\) is the minimal period of \(z\) and we have the following estimates.
Lemma 7. Under the above assumptions if \( \mu_0 < 1/MT \), then, for any \( \mu \in (\mu_0, 1/MT) \),

(i) \[ \| \mathbb{P}_\mu^+ D \|_{L^\infty(0,T;\mathbb{V})} \leq TLK_\mu(1 - \mu MT)^{-1}\| D \|_{L^\infty(0,T;\mathbb{V})}. \]

(ii) \[ \| \mathbb{P}_\mu^- D \|_{L^\infty(0,T;\mathbb{V})} \leq K_\mu L \left( \mu^{-1} e^{-\mu T} m(T) + \int_0^T m(s) \cdot e^{-\mu s} ds \right) \| D \|_{L^\infty(0,T;\mathbb{V})}. \]

Proof. Observe that, in view of (16),

\[ D(t) = z(t + \tau) - z(t) = e^{tA} D(0) + \int_0^t e^{(t-s)A}[\mathbb{F}(z(s + \tau)) - \mathbb{F}(z(s))] ds \quad \text{for all } t \geq 0. \]

Acting with \( \mathbb{P}_\mu^+ \) on both sides, one gets

\[ \mathbb{P}_\mu^+ D(t) = e^{tA} \mathbb{P}_\mu^+ D(0) + \int_0^t e^{(t-s)A} \mathbb{P}_\mu^+ w(s) ds \]

with \( w(s) := \mathbb{F}(z(s + \tau)) - \mathbb{F}(z(s)) \). Since \( A \) is bounded on \( X_\mu^+ \) we get

\[ (\mathbb{P}_\mu^+ D)'(t) = A \mathbb{P}_\mu^+ D(t) + \mathbb{P}_\mu^+ w(t), \quad \text{for any } t \geq 0. \] (21)

On the other hand

\[ \int_0^T \mathbb{P}_\mu^+ D(s) ds = \mathbb{P}_\mu^+ \left( \int_0^T z(s + \tau) ds - \int_0^T z(s) ds \right) = 0, \]

which implies

\[ \mathbb{P}_\mu^+ D(t) = \frac{1}{T} \int_0^T \mathbb{P}_\mu^+ D(t) dr = \frac{1}{T} \int_0^T \left( \mathbb{P}_\mu^+ D(r) + \int_r^t (\mathbb{P}_\mu^+ D)'(s) ds \right) dr \]

\[ = \frac{1}{T} \int_0^T \left( \int_r^t (\mathbb{P}_\mu^+ D)'(s) ds \right) dr. \]

Observe that \( X_\mu^+ \subset D(A) \subset \mathbb{V} \), therefore

\[ \| \mathbb{P}_\mu^+ D(t) \|_\mathbb{V} \leq \int_0^t \| (\mathbb{P}_\mu^+ D)'(s) \|_\mathbb{V} ds. \] (22)

By use of (21), (22) and \((D_{\mu,+})\), we obtain, for all \( t \in [0, T] \),

\[ \| \mathbb{P}_\mu^+ D(t) \|_\mathbb{V} \leq \int_0^t \| A \mathbb{P}_\mu^+ D(s) \|_\mathbb{V} ds + \int_0^t \| \mathbb{P}_\mu^+ w(s) \|_\mathbb{V} ds \]

\[ \leq \mu M \int_0^T \| \mathbb{P}_\mu^+ D(s) \|_\mathbb{V} ds + K_\mu L \int_0^T \| D(s) \|_\mathbb{V} ds \]

\[ \leq \mu MT \| \mathbb{P}_\mu^+ D \|_{L^\infty(0,T;\mathbb{V})} + TLK_\mu \| D \|_{L^\infty(0,T;\mathbb{V})} \]

and, in consequence,

\[ \| \mathbb{P}_\mu^+ D \|_{L^\infty(0,T;\mathbb{V})} \leq TLK_\mu(1 - \mu MT)^{-1}\| D \|_{L^\infty(0,T;\mathbb{V})}, \]
which proves (i).

In order to show (ii), we use (I) and the invariance of $X_\mu^-$ with respect to $e^{T\mu}$ to get

$$\mathbb{P}_\mu^- D(t) = (I - e^{T\mu})^{-1} \int_0^T e^{(T-s)\mu} \mathbb{P}_\mu^- w(t+s) \, ds.$$  

In view of $(D_{\mu,-})$ and $(D_{\mu,0})$, the integral

$$\int_0^T e^{(T-s)\mu} \mathbb{P}_\mu^- w(t+s) \, ds$$  

which, by definition, is an element of the space $X_\mu^-$, is also convergent in the space $V$, since, by use of $(D_{\mu,-})$ and the Lipschitz property of $F$, we get

$$\int_0^T \|e^{(T-s)\mu} \mathbb{P}_\mu^- w(t+s)\|_V \, ds \leq \int_0^T \int_0^T m(T-s) e^{-\mu(T-s)} \|\mathbb{P}_\mu^- [F(z(t+s+\tau)) - F(z(t+s))]\|_X \, ds \, ds$$

$$\leq K^-_\mu L \int_0^T m(T-s) e^{-\mu(T-s)} \|D(t+s)\|_V \, ds$$

$$\leq K^-_\mu L \left( \int_0^T m(s) e^{-\mu s} \, ds \right) \|D\|_{L^\infty(0,T;V)} < \infty.$$  

Hence, by Lemma 6(ii) and $(D_{\mu,-})$,

$$\|\mathbb{P}_\mu^- D(t)\|_V \leq \left\| (R_{\mu,T}^{-1} - I) \int_0^T e^{(T-s)\mu} \mathbb{P}_\mu^- w(t+s) \, ds \right\|_V + \left\| \int_0^T e^{(T-s)\mu} \mathbb{P}_\mu^- w(t+s) \, ds \right\|_V$$

$$\leq e^{-\mu T} (1 - e^{-\mu T})^{-1} m(T) \left\| \int_0^T e^{(T-s)\mu} \mathbb{P}_\mu^- w(t+s) \, ds \right\|_V$$

$$+ K^-_\mu L \left( \int_0^T m(s) e^{-\mu s} \, ds \right) \|D\|_{L^\infty(0,T;V)}$$

$$\leq e^{-\mu T} (1 - e^{-\mu T})^{-1} m(T) K^-_\mu L \left( \int_0^T e^{-\mu(T-s)} \, ds \right) \|D\|_{L^\infty(0,T;V)} +$$

$$+ K^-_\mu L \left( \int_0^T m(s) e^{-\mu s} \, ds \right) \|D\|_{L^\infty(0,T;V)}$$

$$\leq K^-_\mu L \left( \mu^{-1} e^{-\mu T} m(T) + \int_0^T m(s) \cdot e^{-\mu s} \, ds \right) \|D\|_{L^\infty(0,T;V)},$$

which ends the proof of (ii).  

\textbf{Proof of Theorem 1.} Since

$$\|D\|_{L^\infty(0,T;V)} \leq \|\mathbb{P}_\mu^+ D\|_{L^\infty(0,T;V)} + \|\mathbb{P}_\mu^- D\|_{L^\infty(0,T;V)}$$

we see that, by use of Lemma 7,

$$\|D\|_{L^\infty(0,T;V)} \leq TL \cdot \left[ \frac{K^+_\mu}{1 - \mu MT} + \frac{K^-_\mu}{\mu T} \left( e^{-\mu T} m(T) + \int_0^T m(s/\mu) \cdot e^{-s} \, ds \right) \right] \|D\|_{L^\infty(0,T;V)}$$

Since $\|D\|_{L^\infty(0,T;X)} \neq 0$, we obtain the assertion (**) .

\hfill \Box
Remark 8. In the proof of Lemma 7, another possibility is to use (i) to get
\[
\| \mathbb{P}_\mu D(t) \|_V \leq (1 - e^{-\mu T})^{-1} \left( \int_0^T e^{(T-s)h} \mathbb{P}_\mu w(t + s) \, ds \right) \leq (1 - e^{-\mu T})^{-1} K_\mu^- L \left( \int_0^T m(s) \cdot e^{-\mu s} \, ds \right) \| D \|_{L^\infty(0,T;V)},
\]
which is an alternative for the inequality (ii) in Lemma 7. Following the proof of Theorem 1 we get the following version of its assertion: if \( z : \mathbb{R} \to V \) is a nonstationary \( T \)-periodic mild solution of (8), then either \( T \geq 1/\mu_0 M \) or, for all \( \mu \in (\mu_0, 1/MT) \),
\[
1 \leq TL \cdot \left( \frac{K_\mu^+}{1 - \mu MT} + \frac{K_\mu^-}{\mu T (1 - e^{-\mu T})} \right) \int_0^T m(s/\mu) \cdot e^{-s} ds,
\]
This means that, if only we knew that
\[
\frac{1}{(1 - e^{-\mu T})} \int_0^T m(s/\mu) \cdot e^{-s} ds \geq e^{-\mu T} m(T) + \int_0^T m(s/\mu) \cdot e^{-s} ds,
\]
i.e. equivalently
\[
\int_0^T m(\tau) \cdot e^{-\mu \tau} d\tau \geq m(T) \cdot (1 - e^{-\mu T})/\mu.
\]
then the estimate (10) implies (23). Observe that the (24) holds if \( m \) is decreasing, e.g. in the parabolic case when \( m(\eta) = M_0 \eta^{-\beta} \) with \( \beta \in (0, 1) \). This will imply that our estimates provide stronger results for parabolic problems than those obtained in [10] – see Remark 12. In another interesting case, with \( m \) being a constant function, one has an equality between both sides of (24) and it is clear that (10) and (23) provide the same results.

Proof of Corollary 2. We assume that the assumption (11) holds. If \( T \) is the period of a non-trivial \( T \)-periodic solution of (8), then either \( T \geq 1/\mu_0 M \) or the inequality, coming from (10), holds
\[
1 \leq TL(1 - \mu_0 / \mu)^{-1} \cdot [(1 - \mu MT)^{-1} + 1/\mu T],
\]
which, after setting \( \eta = \mu MT \), yields
\[
1 - \mu_0 MT/\eta \leq TL \left( (1 - \eta)^{-1} + M/\eta \right).
\]
Consequently
\[
T \geq 1/G(\eta)
\]
where \( G : (0, 1) \to (0, +\infty) \) is given by
\[
G(\eta) := L/(1 - \eta) + LM/\eta + \mu_0 M/\eta = L/(1 - \eta) + C/\eta
\]
with \( C := M(L + \mu_0) \). A direct computation shows that \( G \) attains the minimal value
\[
G_0 = (\sqrt{L} + \sqrt{C})^2
\]
at the point \( \eta_0 = \sqrt{C}/(\sqrt{L} + \sqrt{C}) \). Now observe that either \( \eta_0/MT \leq \mu_0 \), which is equivalent to the inequality

\[
T \geq \eta_0/\mu_0 M = \sqrt{C}/\mu_0 M(\sqrt{L} + \sqrt{C}) = C/\mu_0 M(\sqrt{LC} + C) = (1 + L/\mu_0)/(\sqrt{LC} + C)
\]

or for \( \mu = \eta_0/MT \) we get, by use of (25),

\[
T \geq 1/G(\eta_0) = 1/G_0 = 1/(\sqrt{L} + \sqrt{C})^2 = 1/(L + C + 2\sqrt{LC}).
\]

Hence taking into consideration that

\[
(1 + L/\mu_0)/(\sqrt{LC} + C) \geq 1/(L + C + 2\sqrt{LC})
\]

we see that

\[
T \geq 1/(L + C + 2\sqrt{LC}) = 1/L \left( 1 + \sqrt{M(1 + \mu_0/L)} \right)^2.
\]

\[\square\]

**Remark 9.** If instead of the global Lipschitzianity of \( F \) in Corollary 2 (comp. Theorem 1) we assume that for any \( R > 0 \) there exists \( L_R > 0 \) such that

\[
\|F(z_1) - F(z_2)\|_X \leq L_R \|z_1 - z_2\|_V \quad \text{for all} \quad z_1, z_2 \in \{ z \in X \mid \|z\|_V \leq R \},
\]

then we can slightly refine the assertion. Namely, if \( z : \mathbb{R} \to V \) is a \( T \)-periodic solution and \( R := \max\{\|z(t)\| \mid t \in \mathbb{R}\} \), then, due to Corollary 2 one has

\[
T \geq 1/L_R \left( 1 + \sqrt{M(1 + \mu_0/L_R)} \right)^2.
\]

### 3 Parabolic equations – proof of Theorem 3

Assume that \( A : D(A) \to X \) is as in Section 1, i.e. \( A \) is a positive self-adjoint operator on a separable Hilbert space \( X \) with the norm \( \| \cdot \| \). Let \( X^\beta \) with the fractional norm given by

\[
\|u\|_\beta = \|A^\beta u\|, \quad u \in X^\beta,
\]

where \( A^\beta \) is the fractional power of the operator \( A \) (see e.g. [7]). It is well-known that \( -A \) generates an analytic \( C_0 \)-semigroup \( e^{-tA} \) such that \( e^{-tA}(X) \subset D(A) \) and

\[
\|e^{-tA}\|_{L(X,X^{\beta})} \leq M_\beta/t^\beta \quad \text{for all} \quad u \in X^\beta
\]

with \( M_\beta = (\beta/e)^\beta \) if \( \beta \in (0,1) \) and \( M_\beta = 1 \) if \( \beta = 0 \). Let us collect below facts concerning spectral properties of such operators that can be obtained by use of spectral theory for self-adjoint operators (see Lemma 3.1 in [10]).

**Proposition 10.** Under the above assumptions, for any \( \beta \in [0,1) \) and \( \mu > 0 \), there exists a decomposition \( X = X^+_\mu \oplus X^-_\mu \) with \( X^+_\mu \subset D(A) \) and such that:

(i) \( \|Au\| \leq \mu \|u\| \), for all \( u \in X^+_\mu \), and \( \|P^+_\mu u\|_\beta \leq \mu^\beta \|u\|_\beta \), for all \( u \in X \);

(ii) \( \|e^{-tA}u\|_\beta \leq e^{-\mu t}\|u\|_\beta \), for all \( u \in X^-_\mu \), and \( \|e^{-tA}u\|_\beta \leq e^{-\mu t}\|u\|_\beta \) for all \( u \in X^\beta \cap X^-_\mu \);

(iii) \( \|e^{-tA}u\|_\beta \leq M_\beta t^{-\beta} e^{-\mu t}\|u\|_\beta \) for all \( t > 0 \) and \( u \in X^-_\mu \).

As an immediate consequence we get the following conclusion.

**Proposition 11.** If \( X := X \), \( V := X^\beta \) with the fractional norm \( \| \cdot \|_\beta \) and \( A := -A \), then \( A \) satisfies the assumptions \((V_1) - (V_3)\) and has the \((UHBD)\) property with \( \mu_0 = 0 \), \( M = 1 \), \( K^+_\mu = \mu^\beta \), \( K^-_\mu = 1 \), \( m(t) = M_\beta/t^\beta \), \( t > 0 \).
The proof involves spectral calculus for positive self-adjoint operators in Hilbert spaces.

**Proof of Theorem 3.** Assume that \( u \in C(\mathbb{R}, X^\beta) \) is a nontrivial \( T \)-periodic solution of (2) with \( T > 0 \) (it is clear that \( u \) is also a mild solution). By Proposition 11, we may apply Theorem 1 and obtain for all \( \mu \in (0, 1/T) \)

\[
1 \leq TL \cdot \left[ \frac{\mu^\beta}{1 - \mu T} + \frac{M_\beta}{\mu T} \left( e^{-\mu T/\beta} + \int_0^{\mu T} (s/\mu)^{-\beta} \cdot e^{-s} \, ds \right) \right] = T^{1-\beta} L \cdot \left[ \frac{(\mu T)^\beta}{1 - \mu T} + \frac{M_\beta}{\mu T} \left( e^{-\mu T} + (\mu T)^\beta \int_0^{\mu T} s^{-\beta} \cdot e^{-s} \, ds \right) \right],
\]

i.e.

\[
1 \leq T^{1-\beta} L \cdot H(\mu T)
\]

where \( H : (0, 1) \to (0, +\infty) \) is given by (13). It is clear that \( 1 \leq T^{1-\beta} \cdot L \cdot K_\beta \).

Now in order to prove (14) taking \( \mu > 0 \) such that \( \mu T = 1/2 \) we get

\[
1 \leq T^{1-\beta} L \cdot H(1/2) = 2^{1-\beta} + 2M_\beta \left( e^{-1/2} + (1/2)^\beta \int_0^{1/2} s^{-\beta} \, ds \right)
\]

Finally note that, for \( \beta \in (0, 1) \),

\[
H(1/2) \leq 2^{1-\beta} + 2M_\beta \left( e^{-1/2} + (1/2)^\beta \int_0^{1/2} s^{-\beta} \, ds \right) = 2^{1-\beta} + 2M_\beta/2^{1/2} + M_\beta/(1 - \beta),
\]

which implies (14). If \( \beta = 0 \), then \( H \) attains minimum at \( \eta = 1/2 \) and

\[
1 \leq TL \cdot H(1/2) = T \cdot 4L,
\]

which ends the proof.

**Remark 12.** (i) If we apply in the above proof the inequality (23) instead of (10), then we get

\[
1 \leq TL \cdot \left( \frac{\mu^\beta}{1 - \mu T} + \frac{M_\beta}{\mu T} \int_0^{\mu T} (s/\mu)^{-\beta} \cdot e^{-s} \, ds \right) = T^{1-\beta} L \cdot \left( \frac{(\mu T)^\beta}{1 - \mu T} + \frac{M_\beta}{\mu T} \int_0^{\mu T} s^{-\beta} \cdot e^{-s} \, ds \right) = T^{1-\beta} L \cdot \tilde{H}(\mu T),
\]

where \( \tilde{H} : (0, 1) \to (0, +\infty) \) is given by

\[
\tilde{H}(\eta) := \frac{\eta^\beta}{1 - \eta} + \frac{M_\beta}{\eta^{1-\beta}(1 - \eta)} \int_0^\eta s^{-\beta} \cdot e^{-s} \, ds.
\]

Observe that

\[
\tilde{H}(1/2) = 2^{1-\beta} + \frac{M_\beta}{(1/2)^{1-\beta}(1 - e^{-1/2})} \int_0^{1/2} s^{-\beta} \cdot e^{-s} \, ds < 2^{1-\beta} + \frac{M_\beta}{(1/2)^{1-\beta}(1 - e^{-1/2})} \int_0^{1/2} s^{-\beta} \, ds = 2^{1-\beta} + M_\beta/(1 - e^{-1/2})(1 - \beta),
\]

which completes the proof.
which allows us to deduce the estimate (3) that was provided originally in [10].

(ii) The estimate (12) is stronger than (3). Indeed, reasoning as in Remark 8, we see that
\[ \tilde{H}(\eta) > H(\eta) \]
for all \( \eta \in (0, 1) \). Therefore, in view of (28),
\[
2^{1-\beta} + M_\beta/(1 - e^{-1/2})(1 - \beta) > \tilde{H}(1/2) \geq \min_{\eta \in (0,1)} H(\eta) > K_\beta,
\]
which shows the relation between the estimates.

(iii) To see that also (14) is stronger than (3) one can directly verify the inequality
\[
2^{1-\beta} + M_\beta/(1 - e^{-1/2})(1 - \beta) > 2^{1-\beta} + 2M_\beta/e^{1/2} + M_\beta/(1 - \beta)
\]
that is equivalent to \( 4 > e \).

4 Spectrum of hyperbolic operator

Let us assume that \( A : D(A) \to X \) is a sectorial operator in a Banach space \( X \) and that the so-called hyperbolic operator \( A = A_\alpha : D(A_\alpha) \to X \) in \( X := X^{1/2} \times X \), where \( X^{1/2} \) is the fractional space related to \( A \), is defined by
\[
A(u, v) := (v, -A(\alpha \cdot v + u)), \ (u, v) \in D(A), \tag{29}
\]
with
\[
D(A) := \{(u, v) \in X \mid \alpha \cdot v + u \in D(A), \ v \in X^{1/2}\}.
\]
Without loss of generality we may assume that \( X \) is a complex space, so is \( X \). Let us start with the following observation.

Lemma 13. Suppose that \( (u, v), (g, h) \in X \) and \( \xi \in \mathbb{C} \). If \( \xi \neq -1/\alpha \) then
\[
(u, v) \in D(A) \quad \text{and} \quad (\xi I - A)(u, v) = (g, h) \tag{30}
\]
if and only if \( w := u + \alpha \cdot v \in D(A) \),
\[
u = \frac{\xi}{1 + \alpha \xi} \cdot w - \frac{1}{1 + \alpha \xi} \cdot g, \quad v = \frac{\xi}{1 + \alpha \xi} \cdot w - \frac{1}{1 + \alpha \xi} \cdot g \tag{31}
\]
and
\[
(A - s(\xi)I)w = \frac{\xi}{1 + \alpha \xi} \cdot g + h \tag{32}
\]
where the mapping \( s : \mathbb{C} \setminus \{-1/\alpha\} \to \mathbb{C} \) is given by
\[
s(\xi) := -\frac{\xi^2}{1 + \alpha \xi}.
\]
If \( \xi = -1/\alpha \) then (30) is equivalent to the following condition
\[
g \in D(A), \quad u = -\alpha \cdot (-\alpha^2 \cdot Ag + g - \alpha \cdot h) \quad \text{and} \quad v = -\alpha \cdot (\alpha \cdot Ag + h). \tag{33}
\]
Proof. Suppose that (30) holds. Then, by the definition of $A$, $w = u + \alpha \cdot v \in D(A)$ and
\[ \xi \cdot u - v = g \quad \text{and} \quad \xi \cdot v + Aw = h. \] (34)
Substituting $u = w - \alpha \cdot v$ in the first equality of (34) one gets
\[ \xi \cdot w - (1 + \alpha \xi) \cdot v = g \] (35)
and $v = (1 + \alpha \xi)^{-1} \cdot w - (1 + \alpha \xi)^{-1} \cdot g$, i.e. the second part of (31). Applying it in $u = w - \alpha \cdot v$, one has $u = (1 + \alpha \xi)^{-1} \cdot w + \alpha(1 + \alpha \xi)^{-1} \cdot g$, i.e. the first part of (31), and applying second part of (31) in the second equality of (31) we arrive at (32).

Now suppose that $w = u + \alpha \cdot v \in D(A)$ and both (31) and (32) hold. Then, obviously $v = \alpha^{-1} \cdot (w - u) \in X^{1/2}$ and a direct calculation shows that
\[ (\xi I - A)(u, v) = (\xi \cdot u - v, \xi \cdot v + Aw) = (g, -s(\xi) \cdot w - \xi(1 + \alpha \xi)^{-1} \cdot g + Aw) = (g, h), \]
which ends the proof of the first equivalence.

The proof of the second one is similar and therefore omitted. \(\square\)

Remark 14. Suppose $B : D(B) \to X$ is an arbitrary closed operator in a Banach space $X$. Recall that $0 \in \rho(B)$ if and only if for any $g \in X$ there exists a unique $u \in D(B)$ such that $Bu = g$.

The next result provides the characterization of the spectrum of $A$.

Proposition 15. The operator $A$ defined by (29) has the following properties
(i) $A$ is closed.
(ii) $\rho(A) \setminus \{-1/\alpha\} = s^{-1}(\rho(A))$.
(iii) $\sigma(A) \setminus \{-1/\alpha\} = s^{-1}(\sigma(A))$.
(iv) If $\xi \neq -1/\alpha$, then $\text{Ker}(\xi I - A) = \{(u, \xi \cdot u) \mid u \in \text{Ker}(s(\xi) I - A)\}$.

Proof. (i) Suppose $(u_n, v_n) \in D(A)$, $n \geq 1$, and $(u_n, v_n) \to (u, v)$ and $A(u_n, v_n) \to (g, h)$ in $X$. It follows directly that $u_n \to u$ in $X^{1/2}$, $v_n \to v$ in $X$ and that $v_n \to g$ in $X^{1/2}$ and $A(u_n + \alpha \cdot v_n) \to h$, which implies $v = g \in X^{1/2}$ and that $u_n + \alpha \cdot v_n \to u + \alpha \cdot v$. By the closedness of $A$, $u + \alpha \cdot v \in D(A)$ and $A(u + \alpha \cdot v) = -h$. It means that $(u, v) \in D(A)$ and $A(u, v) = (v, -A(u + \alpha \cdot v)) = (g, h)$, which shows (i).

(ii) If $\xi \in \rho(A) \setminus \{-1/\alpha\}$, then, in particular, for $g = 0$ and any $h \in X$, there is a unique $(u, v) \in D(A)$ such that $(\xi I - A)(u, v) = (0, h)$, which due to Lemma 13 means that $w := u + \alpha \cdot v \in D(A)$ is the only solution of $(A - s(\xi)I)w = h$, which, in view of Remark 14 means that $s(\xi) \in \rho(A)$. On the other hand, if $s(\xi) \in \rho(A)$ and we take any $(g, h) \in X$, then we get a unique $w \in D(A)$ satisfying (32). Then $(u, v)$ given by (31) solves $(\xi I - A)(u, v) = (g, h)$. The uniqueness of solutions comes immediately from the uniqueness of $w$. Hence, again by Remark 14 we infer that $\xi \in \rho(A)$.

(iii) Observe that, by use of (ii),
\[ \sigma(A) \setminus \{-1/\alpha\} = (C \setminus \rho(A)) \setminus \{-1/\alpha\} = \\
= (C \setminus \{-1/\alpha\}) \setminus (\rho(A) \setminus \{-1/\alpha\}) = (C \setminus \{-1/\alpha\}) \setminus s^{-1}(\rho(A)) \\
= s^{-1}(C \setminus \rho(A)) = s^{-1}(\sigma(A)). \]

(iv) follows directly from Lemma 13. \(\square\)
Corollary 16. If $A$ is an unbounded operator with $\sigma(A) \subset (0, +\infty)$, then
\[ \sigma(A) \setminus \{-1/\alpha\} = \xi_-(\sigma(A)) \cup \xi_+(\sigma(A)) \]
and
\[ \sigma_p(A) = \xi_-(\sigma_p(A)) \cup \xi_+(\sigma_p(A)) \]
where if $\lambda > (2/\alpha)^2$
\[ \xi_-(\lambda) = \frac{-\alpha\lambda - \sqrt{\Delta(\lambda)}}{2} = -\frac{2}{\alpha} \cdot \frac{1}{1 - \sqrt{1 - (2/\alpha)^2/\lambda}} \]
\[ \xi_+(\lambda) = \frac{-\alpha\lambda + \sqrt{\Delta(\lambda)}}{2} = -\frac{2}{\alpha} \cdot \frac{1}{1 + \sqrt{1 - (2/\alpha)^2/\lambda}} \]
if $0 < \lambda < (2/\alpha)^2$
\[ \xi_-(\lambda) = \frac{-\alpha\lambda - i\sqrt{\Delta(\lambda)}}{2} = -\frac{2}{\alpha} \cdot \frac{1}{1 - i\sqrt{(2/\alpha)^2/\lambda} - 1} \]
\[ \xi_+(\lambda) = \frac{-\alpha\lambda + i\sqrt{\Delta(\lambda)}}{2} = -\frac{2}{\alpha} \cdot \frac{1}{1 + i\sqrt{(2/\alpha)^2/\lambda} - 1} \]
and if $\lambda = (2/\alpha)^2$
\[ \xi_-(\lambda) = \xi_+(\lambda) = \frac{-\alpha\lambda}{2} \]
with $\Delta(\lambda) = (\alpha\lambda)^2 - 4\lambda$. Moreover
(i) For any $\lambda > 0$, $\xi_-(\lambda)$ and $\xi_+(\lambda)$ are the roots of
\[ \xi^2 + \alpha\lambda \xi + \lambda = 0, \quad (36) \]
in particular
\[ \xi_-(\lambda) + \xi_+(\lambda) = -\alpha\lambda \quad \text{and} \quad \xi_-(\lambda) \cdot \xi_+(\lambda) = \lambda. \]
(ii) $\sigma(A) \subset \{z \in \mathbb{C} \mid \text{Re } z < 0\}$.
(iii) If $\lambda > (2/\alpha)^2$ then
\[ \xi_-(\lambda) < -2/\alpha < \xi_+(\lambda) < -1/\alpha. \]
(iv) If $0 < \lambda < (2/\alpha)^2$ then
\[ -2/\alpha < \text{Re } \xi_+(\lambda) < 0. \]
(v) $\xi_-(\lambda) \to -\infty$ as $\lambda \to \infty$ and $\xi_-$ is decreasing on $((2/\alpha)^2, +\infty)$.
(vi) $\xi_+(\lambda) \to -1/\alpha$ as $\lambda \to \infty$ and $\xi_+$ is increasing on $((2/\alpha)^2, +\infty)$.

Proof. Observe that $s^{-1}(\{\lambda\})$ consists of $\xi \in \mathbb{C}$ solving (36), i.e. $s^{-1}(\{\lambda\}) = \{\xi_-(\lambda), \xi_+(\lambda)\}$.

Assertions (i)-(vi) are immediate. \qed
In order to estimate the norms of projections and $A$ on eigenspaces we shall need the following elementary fact.

**Lemma 17.** For any $z \in \mathbb{C}$ one has the following equalities

(i) \[ \max\{|z \cdot z_1 + z_2| \mid z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\} = \sqrt{1 + |z|^2}; \]

(ii) \[ \max\{|z_1|^2 + |2 \cdot z \cdot z_1 + z_2|^2 \mid z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\} = (|z| + \sqrt{1 + |z|^2})^2. \]

The following proposition will be crucial when computing the norms of projection onto spectral decomposition components.

**Proposition 18.** Suppose that $\lambda \in \sigma_p(A) \setminus \{(2/\alpha)^2\}$ and $e_0$ be element of $\text{Ker}(A - \lambda I)$ with $\|e_0\| = 1$. Then, for any $u, v \in \mathbb{C}$,

\[ u \cdot (e_0, 0) + v \cdot (0, e_0) = p_\lambda^-(u, v) \cdot e_\lambda^- + p_\lambda^+(u, v) \cdot e_\lambda^+ \tag{37} \]

with $e_\lambda^- := (e_0, \xi_-(\lambda) \cdot e_0)$, $e_\lambda^+ := (e_0, \xi_+(\lambda) \cdot e_0)$, being the eigenvalues of $A$ corresponding to $\xi_+(\lambda)$ and $\xi_-(\lambda)$, respectively, and

\[ p_\lambda^-(u, v) := \frac{\xi_+(\lambda)u - v}{\xi_+(\lambda) - \xi_-(\lambda)}, \quad p_\lambda^+(u, v) := \frac{\xi_-(\lambda)u - v}{\xi_-(\lambda) - \xi_+(\lambda)}. \]

Let $X_{e_0} := \text{span}\{(e_0, 0), (0, e_0)\}$, $P_\lambda^- : X_{e_0} \to \mathbb{C} \cdot e_\lambda^-$ and $P_\lambda^+ : X_{e_0} \to \mathbb{C} \cdot e_\lambda^+$ be the projections, i.e.

\[ P_\lambda^-(u \cdot e_0, v \cdot e_0) = p_\lambda^-(u, v) \cdot e_\lambda^-, \quad P_\lambda^+(u \cdot e_0, v \cdot e_0) = p_\lambda^+(u, v) \cdot e_\lambda^+ \]

If $\lambda > (2/\alpha)^2$, then

\[ \|P_\lambda^-\| = \|P_\lambda^+\| = 1/\sqrt{1 - (2/\alpha)^2/\lambda} \tag{38} \]

and, if $0 < \lambda < (2/\alpha)^2$, then

\[ \|P_\lambda^-\| = \|P_\lambda^+\| = 1/\sqrt{1 - \lambda/(2/\alpha)^2}. \tag{39} \]

**Proof.** The equality \[ (37) \] can be verified by a direct algebraic computation.

By use of Lemma 17 (i) one has

\[ \|P_\lambda^-\| = \max\{\|P_\lambda^-(u \cdot e_0, v \cdot e_0)\|_{X} \mid \|(u \cdot e_0, v \cdot e_0)\|_{X} = 1\} = \max\{|p_\lambda^-(u, v)| \cdot \|(e_0, \xi_-(\lambda) \cdot e_0)\|_{X} \mid \lambda|u|^2 + |v|^2 = 1\} = \max\{\|(\xi_+/(\sqrt{\lambda}))z_1 - z_2\| \mid |z_1|^2 + |z_2|^2 = 1\} \cdot \sqrt{\lambda + |\xi_-(\lambda)|^2} \cdot |\xi_+(\lambda) - \xi_-(\lambda)|^{-1} = \sqrt{1 + |\xi_+(\lambda)|^2/\lambda} \cdot \sqrt{\lambda + |\xi_-(\lambda)|^2} \cdot |\xi_+(\lambda) - \xi_-(\lambda)|^{-1}. \]

Hence, if $\lambda > (2/\alpha)^2$ then $\xi_-(\lambda), \xi_+(\lambda) \in \mathbb{R}$ and, in view of (i) in Corollary 16 one gets

\[ \|P_\lambda^-\|^2 = (1 + \xi_+(\lambda)^2/\lambda) \cdot (\lambda + \xi_-(\lambda)^2)/(\xi_+(\lambda) - \xi_-(\lambda))^2 \]

\[ = \lambda^{-1} \cdot (\xi_+(\lambda)\xi_-(\lambda) + \xi_+(\lambda)^2) \cdot (\xi_+(\lambda)\xi_-(\lambda) + \xi_-(\lambda)^2)/\Delta(\lambda) \]

\[ = (\xi_-(\lambda) + \xi_+(\lambda))^2/\Delta(\lambda) = (\alpha\lambda)^2/((\alpha\lambda)^2 - 4\lambda) = \frac{1}{1 - (2/\alpha)^2/\lambda}. \]
In the same way one computes the norm of \( P_+^k \). When \( 0 < \lambda < (2/\alpha)^2 \), then \( \xi_+^k(\lambda) = \xi_-^k(\lambda) \) and
\[
\|P_+^k\| = (1 + |\lambda|^k(\lambda)/\lambda) \cdot (\lambda + |\lambda|^k(\lambda)/\lambda) / |\lambda|^k(\lambda) - |\lambda|^k(\lambda) = 4\lambda/(4\lambda - (\alpha\lambda)^2) = 1 - 1/2(2/\alpha)^2.
\]
As before, the computation for \( P_-^k \) is similar. \( \square \)

**Proposition 19.** Suppose that \( \lambda \in \sigma_p(A) \), \( e_0 \in \text{Ker}(A - \lambda I) \) with \( \|e_0\| = 1 \) and \( X_{e_0} := \text{span}\{(e_0, 0), (0, e_0)\} \) and let \( A_{e_0} : X_{e_0} \to X_{e_0} \) be the restriction of \( A \) to \( X_{e_0} \). Then
\[
\|A_{e_0}\| = (2/\alpha) \cdot g ((2/\alpha)^2/\lambda)
\]
where \( g : (0, +\infty) \to (0, +\infty) \) is given by
\[
g(r) := 1 + \sqrt{1 + r}, r > 0.
\]

**Proof.** Observe that, for any \( u, v \in \mathbb{C} \),
\[
A(u \cdot e_0, v \cdot e_0) = (v \cdot e_0, -A(u \cdot e_0 + \alpha v \cdot e_0)) = (v \cdot e_0, -\lambda(u + \alpha v) \cdot e_0).
\]

Hence, by use of Lemma \ref{lem:lemma}(ii),
\[
\|A_{e_0}\|^2 = \|A(u \cdot e_0, v \cdot e_0)\|_X^2 = \max\{\|A(u \cdot e_0, v \cdot e_0)\|_X \mid \|u \cdot e_0, v \cdot e_0\|_X = 1\}
= \lambda \cdot \max\{|v|^2 + \lambda|u + \alpha v|^2 \mid \lambda|u|^2 + |v|^2 = 1\}
= \lambda \cdot \max\{|z_1|^2 + |\lambda^{1/2}\alpha z_1 + z_2|^2 \mid |z_1|^2 + |z_2|^2 = 1\}
= \lambda(\lambda^{1/2}\alpha/2 + \sqrt{1 + \lambda(\alpha/2)^2})^2
= (2/\alpha)^2 \left(\lambda(\alpha/2)^2 + \sqrt{\lambda(\alpha/2)^2 + \lambda^2(\alpha/2)^4}\right)^2
= (2/\alpha)^2 \cdot \left(r^{-1} + \sqrt{r^{-1} + r^{-2}}\right)^2
\]
where \( r = (2/\alpha)^2/\lambda \). \( \square \)

5 Spectral decomposition and property \((UHB\&D)\) for hyperbolic operator

Assume that \( A : D(A) \subset X \to X \) is a self-adjoint operator in a separable Hilbert space \( X \) (endowed with the scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \|\cdot\| \)) with the spectrum \( \sigma(A) \) consisting of positive eigenvalues \( \lambda_k, k \geq 1 \), of finite multiplicities such that \( \lambda_k \to +\infty \) as \( k \to +\infty \). The corresponding eigenvectors we denote by \( e_k, k \geq 1 \). Then it is clear (see Theorem 13.36 in \cite{11}) that \( D(A) \neq X \) and therefore, in view of Lemma \ref{lem:lemma} \(-1/\alpha \in \sigma(A)\).

In view of Corollary \ref{cor:corollary} we infer that
\[
\sigma(A) = \{-1/\alpha\} \cup \bigcup_{k=1}^{\infty} \{\xi_k^-, \xi_k^+\}
\]
where \( \xi_k^- := \xi_-(\lambda_k), \xi_k^+ := \xi_+(\lambda_k) \). It is also clear that if \( \xi_k^-, \xi_k^+ \in \mathbb{R} \) then
\[
\xi_k^- < -2/\alpha < \xi_k^+ < -1/\alpha,
\]
\( \xi_k^- \to -\infty \) and \( \xi_k^+ \to -1/\alpha \). Moreover, the set \( \sigma(A) \setminus \mathbb{R} \) is finite and
\[
\sigma(A) \setminus \mathbb{R} \subset \{ z \in \mathbb{C} \mid -2/\alpha < \text{Re} \, z < 0 \}.
\]

Using the notation from Proposition 18 we define
\[
e_k^- := e_{\lambda_k}^-(e_k, \xi_k^- \cdot e_k), \quad e_k^+ := e_{\lambda_k}^+(e_k, \xi_k^+ \cdot e_k), \quad k \in \mathbb{N} \setminus N^0,
\]
where \( N^0 := \{ k \in \mathbb{N} \mid \lambda_k = (2/\alpha)^2 \} \) (it is clearly a finite set) and
\[
e_k^- := (e_k, 0), \quad e_k^+ := (0, e_k), \quad k \in N^0.
\]

Observe that, in view of (37),
\[
X = \bigoplus_{k \in \mathbb{N}} \text{span} \{ (e_k, 0), (0, e_k) \} = \bigoplus_{k \in \mathbb{N} \setminus N^0} \text{span} \{ e_k^-, e_k^+ \} \oplus X^0
\]
and
\[
X^0 := \bigoplus_{k \in N^0} \text{span} \{ e_k^-, e_k^+ \}.
\]

Obviously, for any \( k, l \in \mathbb{N} \) such that \( k \neq l \),
\[
\text{span} \{ e_k^-, e_k^+ \} \perp \text{span} \{ e_l^-, e_l^+ \}.
\]

**Proposition 20.** For any \((u, v) \in X\)
\[
(u, v) = \sum_{k=1}^{\infty} p_k^- (u_k, v_k) \cdot e_k^- + \sum_{k=1}^{\infty} p_k^+ (u_k, v_k) \cdot e_k^+ \text{ in } X
\]
where \( p_k^- := p_{\lambda_k}^-, p_k^+ := p_{\lambda_k}^+ \) if \( k \in \mathbb{N} \setminus N^0 \) and \( p_k^+(z_1, z_2) := z_1, p_k^+(z_1, z_2) := z_2 \) if \( k \in N^0 \) and for any \( k \in \mathbb{N} \) \( u_k := \langle u, e_k \rangle \), \( v_k := \langle v, e_k \rangle \). In consequence
\[
X = \bigoplus_{k \in \mathbb{N} \setminus N^0} \mathbb{C} \cdot e_k^- \oplus \bigoplus_{k \in N^0} \mathbb{C} \cdot e_k^+ \oplus X^0.
\]

**Proof.** Applying Proposition 18 for large \( k \), we obtain
\[
\|p_k^- (u_k, v_k) \cdot e_k^- \|_X^2 \leq 1/(1 - (2/\alpha)^2/\lambda_k) \cdot \|(u_k \cdot e_k, v_k \cdot e_k)\|_X^2 = 1/(1 - (2/\alpha)^2/\lambda_k) (\lambda_k |u_k|^2 + |v_k|^2).
\]
The same estimate we get for \( \|p_k^+ (u_k, v_k) \cdot e_k^+ \|_X^2 \). Since, \( 1/(1 - (2/\alpha)^2/\lambda_k) \to 1 \) as \( k \to \infty \) and the series with terms \( \lambda_k |u_k|^2 + |v_k|^2 \) is convergent, we see that the series with the terms \( \|p_k^- (u_k, v_k) \cdot e_k^- \|_X^2 \) and \( \|p_k^+ (u_k, v_k) \cdot e_k^+ \|_X^2 \) are convergent as well. \( \square \)
Now take any $\mu > 2/\alpha$ and consider the spectral decomposition into

$$X = X^- \oplus X^+$$

corresponding to the spectral sets (see [4]) $\sigma^-_\mu$ and $\sigma^+_\mu$ given by

$$\sigma^-_\mu := \{ z \in \sigma(A) \mid \text{Re} z \leq -\mu \}, \quad \sigma^+_\mu := \sigma(A) \setminus \sigma^-_\mu.$$

The following proposition, being a straightforward consequence of Lemma 13, Corollary 16 and Proposition 20, provides explicitly the components of the above decomposition.

**Proposition 21.** If $\mu > 2/\alpha$ then

$$X^- = \bigoplus_{k \in N^-_\mu} \mathbb{C} \cdot e^-_k,$$

$$X^+ = \bigoplus_{k \in N^\mu \setminus N^0} \mathbb{C} \cdot e^-_k \oplus \bigoplus_{k \in N^\mu \setminus N^0} \mathbb{C} \cdot e^+_k \oplus X^0,$$

where $N^-_\mu := \{ k \in \mathbb{N} \mid \lambda_k > (2/\alpha)^2, \xi_k \leq -\mu \}.$

Next we shall estimate the norms of the projections $P^-_\mu$ and $P^+_\mu$ in $X$ onto $X^-_\mu$ and $X^+_\mu$, respectively.

**Proposition 22.** If $\mu > 2/\alpha$ then

$$\| P^-_\mu \|_{\mathcal{L}(X,X)} \leq (1 - (2/\alpha)/\mu)^{-1} \quad \text{and} \quad \| P^+_\mu \|_{\mathcal{L}(X,X)} \leq (1 - (2/\alpha)/\mu)^{-1}.$$  

**Proof.** Let $P^-_k := P^-_{\lambda_k} \circ P_k$ where $P^-_{\lambda_k} : \text{span}\{(e_k,0),(0,e_k)\} \to \mathbb{C} \cdot e^-_k$ as in Proposition 18 and $P_k : X \to \text{span}\{(e_k,0),(0,e_k)\}$ is the projection defined as follows: for any $(u,v) \in X$$P_k(u,v) := (u_k \cdot e_k, v_k \cdot e_k)$. Since all of the one dimensional components of $X^-_\mu$ are orthogonal with respect to each other and

$$1/\sqrt{1 - (2/\alpha)^2/\lambda_k} \leq 1/(1 - (2/\alpha)/\mu)$$

for $k \in N^-_\mu$, we infer that, for any $(u,v) \in X$,

$$\| P^-_\mu(u,v) \|^2 = \sum_{k \in N^-_\mu} \| P^-_k(u,v) \|^2 \leq \sum_{k \in N^-_\mu} 1/(1 - (2/\alpha)^2/\lambda_k) \cdot \| (u_k \cdot e_k, v_k \cdot e_k) \|^2_X$$

$$\leq 1/(1 - (2/\alpha)/\mu)^2 \sum_{k \in N^-_\mu} \| (u_k \cdot e_k, v_k \cdot e_k) \|^2_X \leq 1/(1 - (2/\alpha)/\mu)^2 \cdot \| (u,v) \|^2_X.$$  

In order to estimate the norm of $P^+_\mu$ observe that $X^+_\mu$ can be split into four orthogonal parts

$$X^+_\mu = X^+_{\mu,R} \oplus X^+_{\mu,C} \oplus X^+_{\mu,-} \oplus X^0$$ (40)

where

$$X^+_{\mu,R} := \bigoplus_{k \in N^+_{\mu,R}} \text{span}\{e^-_k, e^+_k\}, \quad X^+_{\mu,C} := \bigoplus_{k \in N^+_{\mu,C}} \text{span}\{e^-_k, e^+_k\}, \quad X^+_{\mu,-} := \bigoplus_{k \in N^-_\mu} \mathbb{C} \cdot e^+_k$$
Clearly, in view of the orthogonality of these components, for any \((u, v) \in X\),

\[
\|P^+_{\mu}(u, v)\|_X^2 = \|P^+_{\mu,R}(u, v)\|_X^2 + \|P^+_{\mu,C}(u, v)\|_X^2 + \|P^+_{\mu,-}(u, v)\|_X^2 + \|P^0(u, v)\|_X^2
\]

where \(P^+_{\mu,R}, P^+_{\mu,C}, P^+_{\mu,-}\) and \(P^0\) are projections onto the proper components. Therefore

\[
\|P^+_{\mu,R}(u, v)\|_X^2 = \sum_{k \in N^+_{\mu,R}} \|(u_k \cdot e_k, v_k \cdot e_k)\|_X^2, \quad \|P^+_{\mu,C}(u, v)\|_X^2 = \sum_{k \in N^+_{\mu,C}} \|(u_k \cdot e_k, v_k \cdot e_k)\|_X^2
\]

and

\[
\|P^0(u, v)\|_X^2 = \sum_{k \in N^0} \|(u_k \cdot e_k, v_k \cdot e_k)\|_X^2.
\]

Reasoning as in the case of \(P^-\) and using Proposition 18, we get

\[
\|P^+_{\mu,-}(u, v)\|_X \leq 1/(1 - (2/\alpha)/\mu)^2 \sum_{k \in N^+_{\mu}} \|(u_k \cdot e_k, v_k \cdot e_k)\|_X^2.
\]

Now applying all the estimates for the components of \((\text{II})\) we get

\[
\|P^+_{\mu}(u, v)\|_X^2 \leq 1/(1 - (2/\alpha)/\mu)^2 \sum_{k \in N^+} \|(u_k \cdot e_k, v_k \cdot e_k)\|_X^2 = 1/(1 - (2/\alpha)/\mu)^2 \|(u, v)\|_X^2,
\]

which ends the proof.

**Proposition 23.** For any \(\mu > 2/\alpha\) the following properties hold

1. \(X^+_{\mu} \subset D(A), \ A(X^+_{\mu}) \subset X^+_{\mu}\);
2. \(A(X^+_{\mu} \cap D(A)) \subset X^+_{\mu}\);
3. \(\|A(u, v)\|_X \leq \mu \cdot (1 + \sqrt{2})\|(u, v)\|_X\) for any \((u, v) \in X^+_{\mu}\);
4. \(\langle A(u, v), (u, v)\rangle_X \leq -\mu \|(u, v)\|_X^2\) for all \((u, v) \in X^-_{\mu} \cap D(A)\) and, in consequence,
   \[
   \|e^{tA}(u, v)\|_X \leq e^{-\mu t}\|(u, v)\|_X \quad \text{for any } (u, v) \in X^-_{\mu}.
   \]

**Proof.** (i) Note that the space \(X^+_{\mu}\) is the closure of a sum of invariant spaces on which the operator \(A\) is bounded with the norms estimated by the same constant. Then the completeness of \(X\) and the closedness of \(A\) show that \(X^+_{\mu} \subset D(A)\). The invariance is immediate.

(ii) Take any \((u, v) \in X^-_{\mu} \cap D(A)\). Then

\[
(u, v) = \sum_{k \in N^-_{\mu}} 2^k \cdot e^{-k}
\]

(42)
where \( \alpha_k \in \mathbb{C}, k \in N_\mu^- \). In particular

\[
\alpha = \sum_{k \in N_\mu^-} \alpha_k \cdot e_k \quad \text{and} \quad \nu = \sum_{k \in N_\mu^-} \alpha_k \xi_k^- \cdot e_k.
\]

Since \( v \in X^{1/2} \) and \( u + \alpha \cdot v \in D(A) \) we have, in view of Corollary [10] (i),

\[
\sum_{k \in N_\mu^-} \lambda_k |\alpha_k \xi_k^-|^2 < +\infty \quad \text{and} \quad \sum_{k \in N_\mu^-} \lambda_k^2 |1 + \alpha \xi_k^-|^2 |\alpha_k|^2 = \sum_{k \in N_\mu^-} |\xi_k^-|^4 |\alpha_k|^2 < +\infty. \tag{43}
\]

Observe also that

\[
\|A(\alpha_k \cdot e_k^-)\|_X^2 = |\alpha_k \xi_k^-|^2 \|(e_k, \xi_k^- \cdot e_k)\|_X^2 = (\lambda_k + |\xi_k^-|^2) |\xi_k^-|^2 |\alpha_k|^2,
\]

which, in view of [13], implies the convergence of the series

\[
\sum_{k \in N_\mu^-} A(\alpha_k \cdot e_k^-) \quad \text{in} \quad X.
\]

Hence, by use of the closedness of \( A \), we get

\[
A(u, v) = \sum_{k \in N_\mu^-} A(\alpha_k \cdot e_k^-) \in X_\mu^- \tag{44}
\]

This completes the proof of the invariance.

(iii) Take any \((u, v) \in X_\mu^+\). Using [40], we have

\[
\begin{align*}
P_{\mu,\mathbb{R}}^+(u, v) &= \sum_{k \in N_{\mu,\mathbb{R}}^+} (u_k \cdot e_k, v_k \cdot e_k), \\
P_{\mu,\mathbb{C}}^+(u, v) &= \sum_{k \in N_{\mu,\mathbb{C}}^+} (u_k \cdot e_k, v_k \cdot e_k), \\
P_{\mu,-}^+(u, v) &= \sum_{k \in N_{\mu}^-} p_k^+(u_k, v_k) \cdot e_k^+, \\
P^0(u, v) &= \sum_{k \in N^0} (u_k \cdot e_k, v_k \cdot e_k).
\end{align*}
\]

By the invariance and orthogonality of the components

\[
\|AP_{\mu,\mathbb{R}}^+(u, v)\|_X^2 = \|AP_{\mu,\mathbb{R}}^+(u, v)\|_X^2 + \|AP_{\mu,\mathbb{C}}^+(u, v)\|_X^2 + \|AP_{\mu,-}^+(u, v)\|_X^2 + \|AP^0(u, v)\|_X^2. \tag{45}
\]

Clearly, by use of Proposition [19] and the fact that \( g \) is decreasing and

\[
(2/\alpha)^2/\lambda_k > 1 - (1 - (2/\alpha)/\mu)^2 \quad \text{for all} \quad k \in N_{\mu,\mathbb{R}}^+,
\]

we obtain

\[
\|AP_{\mu,\mathbb{R}}^+(u, v)\|_X^2 \leq \sum_{k \in N_{\mu,\mathbb{R}}^+} \|A(u_k \cdot e_k, v_k \cdot e_k)\|_X^2 \leq \mu^2 \cdot \|P_{\mu,\mathbb{R}}^+(u, v)\|_X^2 \leq \mu^2 \cdot (1 + \sqrt{2})^2 \cdot \|P_{\mu,\mathbb{R}}^+(u, v)\|_X^2
\]

19
where $(0, 1) \ni \varrho := (2/\alpha)/\mu$. Next, again using Proposition 19 and the inequality
\[
(2/\alpha)^2/\lambda_k > 1 \quad \text{for all} \quad k \in N_{\mu,\mathcal{C}}^+
\]
we have
\[
\|\mathcal{P}_{\mu,\mathcal{C}}(u, v)\|_X^2 = \sum_{k \in N_{\mu,\mathcal{C}}^+} \|\mathcal{A}(u_k \cdot e_k, v_k \cdot e_k)\|_X^2 \leq [(2/\alpha) \cdot g(1)]^2 \|\mathcal{P}_{\mu,\mathcal{C}}(u, v)\|_X^2.
\]
Furthermore, taking into considerations the inequality
\[
0 \geq \xi_k^+ \geq -\frac{2}{\alpha} \cdot \frac{1}{2 - (2/\alpha)/\mu} \quad \text{for all} \quad k \in N_{\mu}^-,
\]
we get
\[
\|\mathcal{P}_{\mu,-}(u, v)\|_X^2 = \sum_{k \in N_{\mu}^-} \|\mathcal{A}(p_k^+(u_k, v_k) \cdot e_k^+)\|_X^2 = \sum_{k \in N_{\mu}^-} |\xi_k^+|^2 \|p_k^+(u_k, v_k) \cdot e_k^+\|^2_X
\]
\[
\leq [(2/\alpha)/(2 - (2/\alpha)/\mu)]^2 \cdot \|\mathcal{P}_{\mu,-}(u, v)\|_X^2 \leq (2/\alpha)^2 \|\mathcal{P}_{\mu,-}(u, v)\|_X^2.
\]
Observe also that, by use of Proposition 19, one has
\[
\|\mathcal{P}_0(u, v)\|_X^2 = \sum_{k \in N_0} \|\mathcal{A}(u_k \cdot e_k, v_k \cdot e_k)\|_X^2 \leq [(2/\alpha) \cdot g(1)]^2 \cdot \|\mathcal{P}_0(u, v)\|_X^2.
\]
Combining the above inequalities together with (15) we arrive at the desired assertion.

(iv) Take any $(u, v) \in X_{\mu}^- \cap D(A)$. Clearly, there are $\alpha_k \in \mathbb{C}$, $k \in N_{\mu}^-$, such that (12) holds. In view of (14) and the orthogonality of the components, we have
\[
\langle \mathcal{A}(u, v), (u, v) \rangle_X = \sum_{k \in N_{\mu}^-} \langle \mathcal{A}(\alpha_k \cdot e_k), \alpha_k \cdot e_k \rangle_X = \sum_{k \in N_{\mu}^-} \xi_k \alpha_k \cdot e_k \|\alpha_k \cdot e_k\|_X^2 \leq -\mu \|(u, v)\|_X^2,
\]
which ends the proof. \qed

**Corollary 24.** The operator $\mathcal{A}$ has property (UHBD).

Thus, we can apply directly Corollary 2 with $\mu_0 := 2/\alpha$ and $M := 1 + \sqrt{2}$ to derive the estimate in Theorem 4.

**Remark 25.** We can refine our result by use of Remark 9. Instead of the global Lipschitzianity of $f$, assume that, for any $R, R' > 0$, there exists $L = L_{R,R'} > 0$ such that (5) holds whenever $\|u_1\|_{1/2}, \|u_2\|_{1/2} \leq R$ and $\|u_1\|, \|v_2\| \leq R'$. Then if $u : \mathbb{R} \to X^{1/2}$ is a $T$-periodic solution of (4), then $T \geq 1/L_{R,R'} \left(1 + \sqrt{(1 + 1/\sqrt{2})(1 + 2/\alpha L_{R,R'})}\right)^2$. 

20
6 Application to strongly damped beam equation

As an illustration, let us consider the following damped beam equation

$$\begin{cases}
    u_t + \alpha u_{xxxx} + \beta u_t + u_{xxxx} = h(x,u,u_t,u_x,u_{xx}), & x \in (0,l), \ t \geq 0, \\
    u(0,t) = u(l,t) = 0, & t > 0 \\
    u_{xx}(0,t) = u_{xx}(l,t) = 0, & t > 0
\end{cases} \tag{46}$$

where $l > 0$, $\alpha, \beta > 0$ and $h : [0,l] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ is a continuous function such that

$$|h(x,z_1) - h(x,z_2)| \leq L|z_1 - z_2|$$

for some fixed $L > 0$ and all $z_1, z_2 \in \mathbb{R}^4$ and $x \in [0,l]$. Here $| \cdot |$ stands for either the Euclidean norm (in $\mathbb{R}^4$) or the absolute value (in $\mathbb{R}$). It is the so-called beam equation with strong damping coming from Ball’s model for extensible beam coming from [1]. The boundary conditions above can be replaced by $u(0,t) = u(l,t) = 0$ and $u_x(0,t) = u_x(l,t) = 0$, which corresponds to the way the beam’s ends are fixed.

If we define $A : D(A) \rightarrow L^2(0,l)$ with

$$D(A) := \{ u \in L^2(0,l) \mid u \in W^{4,2}(0,l), u(0) = u(l) = 0, u''(0) = u''(l) = 0\}$$

or $D(A) := \{ u \in L^2(0,l) \mid u \in W^{4,2}(0,l), u(0) = u(l) = 0, u'(0) = u'(l) = 0\}$

(depending on the boundary conditions) and

$$Au = u''', \ u \in D(A).$$

The space $X := L^2(0,l)$ is equipped with the standard scalar product $\langle u,v \rangle_{L^2} := \int_0^l u(s)v(s) \, ds$ and the norm $\|u\|_{L^2} := \sqrt{\langle u,u \rangle_{L^2}}$, $u,v \in X$. The operator $A$ (in both versions) has compact resolvent and using its spectral representation one may show that $A^{1/2}u = -u''$, for $u \in X^{1/2} = D(A^{1/2})$ and $\|A^{1/4}u\|_{L^2} = \|u''\|_{L^2}$ for each $u \in X^{1/4}$. The mapping $f : X^{1/2} \times X \rightarrow X$ we define by

$$\langle f(u,v)\rangle(x) := h(x,u(x),v(x),u_x(x),u_{xx}(x)) - \beta v(x), \text{ for a. e. } x \in [0,l].$$

We denote eigenvalues of $A$ by $\lambda_k$, $k \in \mathbb{N}$. Observe that for all $k \in \mathbb{N}$ $\lambda_{k+1} \geq \lambda_k > 0$ and $\lambda_k \rightarrow \infty$. One can directly verify that for any $u_1, u_2 \in X^{1/2}$ and $v_1, v_2 \in X$

$$\|f(u_1,v_1) - f(u_2,v_2)\|_X \leq \tilde{L} \cdot \|(u_1,v_1) - (u_2,v_2)\|_{X^{1/2} \times X}.$$
[2] Busenberg S. N., Fisher D. C., Martelli M.: *Better bounds for periodic solutions of differential equations in Banach spaces*, Proc. Amer. Math. Soc. 98 (1986), 376–378.

[3] Ćwiszewski A., Rybakowski K. P.: *Singular dynamics of strongly damped beam equation*, J. Differential Equations 247 (2009), 3202–3233.

[4] Dunford N., Schwartz J. T.: *Linear Operators*, Parts I and II, Wiley-Interscience, New York 1966.

[5] Fitzgibbon W. E.: *Strongly damped quasilinear evolution equations*, J. Math. Anal. Appl. 79 (1981), 536-550.

[6] Hale J.: *Asymptotic behavior of dissipative system*, Mathematical Surveys and Monographs 25, American Mathematical Society 2007.

[7] Henry D.: *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin 1981.

[8] Massatt P.: *Limiting behavior for strongly damped nonlinear wave equations*, J. Differential Equations 48 (1983), 334–349.

[9] Robinson J. C., Vidal-Lopez A.: *Minimal periods of semilinear evolution equations with Lipschitz nonlinearity*, J. Differential Equations 220 (2006), 396-406.

[10] Robinson J. C., Vidal-Lopez A.: *Minimal periods of semilinear evolution equations with Lipschitz nonlinearity revisited*, J. Differential Equations 254 (2013), 4279–4289.

[11] Rudin W.: *Functional Analysis*, McGraw-Hill, 1991.

[12] Yorke J. A.: *Periods of periodic solutions and the Lipschitz constant*, Proc. Amer. Math. Soc. 22 (1969), 509–512.