Thermodynamic Product Formula for Hořava Lifshitz Black Hole

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Abstract

We examine the thermodynamic properties of inner and outer horizons in the background of Hořava Lifshitz black hole. We compute the horizon radii product, the surface area product, the entropy product, the surface temperature product, the Komar energy product and the specific heat product for both the horizons. We show that surface area product, entropy product and irreducible mass product are universal (mass-independent) quantities, whereas the surface temperature product, Komar energy product and specific heat product are not universal quantities because they all depend on mass parameter. We further study the stability of such black hole by computing the specific heat for both the horizons. It has been observed that under certain condition the black hole possesses second order phase transition.

Keywords:
Entropy product, Area product, Hořava Lifshitz Black Hole, Kehagias-Sfetsos Black Hole.

1. Introduction

There has been a revival of interest in the physics of thermodynamic product formulae in recent years due to the work of Ansorg et. al. [1, 2] (see also [3, 4, 5, 6]) from the gravity site and due to the work of Cvetič et. al. [7] (see also [8, 9, 10]) from the string theory site. The main interest therein is the area product formula and the entropy product formula of event horizon and Cauchy horizons. For example, let us consider a regular axisymmetric and stationary spacetime of Einstein-Maxwell gravity with surrounding matter then the area product formula of event horizon($H^+$) and Cauchy horizons($H^-$) is [2]

$$ \frac{\mathcal{A}_+ \mathcal{A}_-}{(8\pi)^2} = J^2 + \frac{Q^4}{4}. $$

and consequently the entropy product formula of $H^\pm$ are

$$ \frac{S_+ S_-}{(2\pi)^2} = J^2 + \frac{Q^4}{4}. $$

In the absence of Maxwell gravity, these product formulae reduce to the following form [5]:

$$ \frac{\mathcal{A}_+ \mathcal{A}_-}{(8\pi)^2} = J^2. $$
and
\[ \frac{S_+ S_-}{(2\pi)^2} = f^2. \] (4)

In the above formulae, the common point is that they all are independent of the mass, so-called the ADM (Arnowitt-Deser-Misner) mass of the background spacetime. Thus they all are universal quantities in this sense. Now if we incorporate the BPS states, the area product formula should be \[ \frac{A_+ A_-}{(8\pi\ell_{pl})^2} = \sqrt{N_1} \pm \sqrt{N_2} = N, \quad N \in \mathbb{N}, N_1 \in \mathbb{N}, N_2 \in \mathbb{N}. \] (5)

where \( \ell_{pl} \) is the Planck length.

It is a well known fact that certain BH possesses inner horizon or Cauchy horizon (CH) in addition to the outer horizon or event horizon. Thus there might be a relevance of inner horizon in BH thermodynamics to understanding the microscopic nature of inner BH entropy in comparison with the outer BH entropy. It is also true that CH is a blue-shift region whereas event horizon is a red-shift region by its own right. Furthermore the CH is highly unstable due to the exterior perturbation. Despite the above features, the CH horizon is playing an important role in BH thermodynamics.

Thus in this note we wish to study various thermodynamic products of Kehagias-Sfetsos (KS) BH in Hořava Lifshitz (HL) gravity. We have considered both the inner horizon and outer horizons to understanding the microscopic nature of BH entropy. We compute various thermodynamic products like horizon radii product, surface area product, BH entropy product, surface temperature product and Komar energy product. By computing the specific heat, we also analyze the stability of such BHs.

The plan of the paper is as follows. In Sec. 2, we shall describe the basic properties of the HL BH and shall compute various thermodynamic products. In this section there are two subsections. In first subsection, we compute irreducible mass product for inner horizon and outer horizons. In second subsection, we discuss the stability analysis for this BH by computing the specific heat and also derived the product of specific heat for both the horizons. Finally, we conclude our discussions in Sec. 3.

2. Hořava Lifshitz BH:

In 2009, Hořava [13, 14, 15] proposed a field theory model for a UV complete theory of gravity which is a non-relativistic renormalizable theory of gravity and reduces to Einstein’s general relativity at large scales for the dynamical coupling constant \( \lambda = 1 \). Introducing the ADM formalism where the metric can be written as
\[ ds^2 = -N^2 dt^2 + g_{ij} (dx^i - N_i dt) (dx^j - N_j dt). \] (6)

and for a spacelike hypersurface with a fixed time, its extrinsic curvature \( K_{ij} \) is given by
\[ K_{ij} = \frac{1}{2N} \left( g_{ij} - \nabla_i N_j - \nabla_j N_i \right). \] (7)
where a dot denotes a derivative with respect to \( t \) and covariant derivatives defined with respect to the spatial metric \( g_{ij} \). The generalized action for Hořava Lifshitz theory is given by

\[
S = \int dt d^3x \sqrt{\left| g \right|} \left[ \frac{\kappa^2 \mu^2 (\Lambda w - 3 \Lambda w^3)}{8(1 - 3 \lambda)} + \frac{\kappa^2 \mu^2 (1 - 4 \lambda)}{32(1 - 3 \lambda)} R^2 \right.
\]

\[
- \frac{\kappa^2}{2w^3} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right)(C^{ij} - \frac{\mu w^2}{2} R^{ij}) + \mu^4 R \] .
\] (8)

where \( \kappa, \lambda, \mu, w \) and \( \Lambda \) are constant parameters and the cotton tensor \( C_{ij} \) is defined by

\[
C_{ij} = \epsilon^{ikl} \nabla_k \left( R_{jl} - \frac{1}{4} \epsilon^{ikj} \partial_k R \right) .
\] (9)

Comparing the action to that of general relativity, one could find that the speed of light, Newton’s constant and the cosmological constant are

\[
c = \frac{\kappa^2 \mu^2}{4} \sqrt{\frac{\Lambda w}{1 - 3 \lambda}},
\]

\[
G = \frac{\kappa^2}{32 \pi c},
\]

\[
\Lambda = \frac{3}{2} \Lambda w .
\] (10)(11)(12)

respectively. It may be noted that when \( \lambda = 1 \), the first three terms in Eq. (8) reduces to the usual one of Einstein’s general relativity. It should also be noted that \( \lambda \) is a dynamic coupling constant and for \( \lambda > \frac{1}{2} \), the cosmological constant must be a negative one. However, it could be made a positive one if we give a following transformation like: \( \mu \rightarrow i \mu \) and \( w^2 \rightarrow -i w^2 \).

In this work, we emphasized the BH solution in the limit of \( \Lambda w \rightarrow 0 \). For this reason, we set \( N_i = 0 \) and therefore in order to get the spherically symmetric solution we have considered the metric ansatz\[16, 17, 12, 18, 19, 20\]:

\[
ds^2 = -N^2(r) dt^2 + \frac{dr^2}{g(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .
\] (13)

In order to find the solution, putting the metric ansatz (13) into the action and we have the reduced Lagrangian given by

\[
L = \frac{\kappa^2 \mu^2 N}{8(1 - 3 \lambda) \sqrt{g}} \left[ (2 \lambda - 1) \left( \frac{g - 1}{r^2} \right)^2 - 2 \lambda g - \frac{1}{2} \frac{g' g + g - 1}{g} g'^2 \right.
\]

\[
- 2 \omega (1 - g - r g') \] .
\] (14)

where \( \omega = \frac{8 \mu^2 (3 \lambda - 1)}{3 \kappa^2} \). Since we are interested in this work to investigate the case \( \lambda = 1 \) i.e. \( \omega = \frac{16 \mu^2}{3 \kappa^2} \). Thus we get the solution of the metric:

\[
N^2(r) = g = 1 - \sqrt{4M \omega r + \omega^2 r^4 + \omega^2} .
\] (15)
where $M$ is an integration constant related to the mass parameter. Thus the static, spherically symmetric solution in this case is given by

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

(16)

For $r \gg (\frac{M}{\omega})^{\frac{1}{3}}$, we obtain the usual behavior of a Schwarzschild BH.

The BH horizons correspond to $g(r) = 0$:

$$r_{\pm} = M \pm \sqrt{M^2 - \frac{1}{2\omega}}.$$  

(17)

As is $r_+$ corresponds to event horizon and $r_-$ corresponds to Cauchy horizon. Their product yields

$$r_+ r_- = \frac{1}{2\omega}$$

(18)

which is clearly independent of mass.

The area of this BH is given by

$$A_{\pm} = \int_0^{2\pi} \int_0^{\pi} \sqrt{g_{\theta \theta} g_{\phi \phi}} d\theta d\phi = 4\pi r_{\pm}^2$$

(19)

$$= 4\pi \left(2Mr_{\pm} - \frac{1}{2\omega}\right)$$

(20)

$$= 4\pi \left(2M^2 - \frac{1}{2\omega} \pm 2M \sqrt{M^2 - \frac{1}{2\omega}}\right).$$

(21)

and their product yields

$$A_+ A_- = \frac{4\pi^2}{\omega^2}.$$  

(22)

It indicates that the area product formula of KS BH is clearly independent of BH mass. Thus the area product formula in HL gravity is universal in nature.

It should be noted that using Eq. (21), the mass could be expressed as in terms of area of both the horizons:

$$M^2 = \frac{A_+}{16\pi} + \frac{\pi}{4\omega^2 A_-} + \frac{1}{4\omega}.$$  

(23)

The BH entropy[21] computed at $\mathcal{H}^+$ corresponds to

$$S_+ = \pi \left(2Mr_+ - \frac{1}{2\omega}\right).$$

(24)

Thus the entropy product formula is given by

$$S_+ S_- = \frac{\pi^2}{4\omega^2}.$$  

(25)

It is also independent of mass. Thus the entropy product formula is also universal in HL gravity.

For our record, we also compute other thermodynamic products.
The surface gravity computed at $\mathcal{H}^\pm$ is given by

\[ \kappa_{\pm} = \frac{\omega(r_\pm - M)}{1 + \omega r_\pm^2}. \]  

(26)

and their product is given by

\[ \kappa_+ \kappa_- = \frac{2\omega(1 - 2M^2\omega)}{(1 + 16M^2\omega)}. \]  

(27)

The Hawking temperature on $\mathcal{H}^\pm$ reads off

\[ T_{\pm} = \frac{\omega(r_\pm - M)}{2\pi(1 + \omega r_\pm^2)}. \]  

(28)

and their product yields

\[ T_+ T_- = \frac{\omega(1 - 2M^2\omega)}{2\pi^2(1 + 16M^2\omega)}. \]  

(29)

It may be noted that surface gravity product and surface temperature product both depends on mass thus they are not universal. Finally, the Komar energy computed at $\mathcal{H}^\pm$ is given by

\[ E_{\pm} = \frac{(r_\pm - M)(4M\omega r_\pm - 1)}{(4M\omega r_\pm + 1)}. \]  

(30)

and their product reads off

\[ E_+ E_- = \frac{\frac{1}{4\omega} - M^2}{1 + 16M^2\omega}. \]  

(31)

which indicates that the product does depend on the mass parameter and thus it is not an universal quantity.

2.1. Irreducible mass product for KS BH in HL gravity:

Firstly, Christodoulou had shown that the irreducible mass $M_{irr}$ of a non-spinning BH which is related to the surface area $A$ given by

\[ M_{irr, +}^2 = \frac{A_+}{16\pi} = \frac{S_+}{4\pi}. \]  

(32)

where ‘+’ sign indicates for $\mathcal{H}^+$ and it is now established that this relation is valid for CH also. That means

\[ M_{irr, -}^2 = \frac{A_-}{16\pi} = \frac{S_-}{4\pi}. \]  

(33)

Here, ‘−’ indicates for $\mathcal{H}^−$.

Equivalently, the expressions for area on $\mathcal{H}^\pm$, in terms of $M_{irr, \pm}$ are:

\[ A_{\pm} = \frac{16\pi(M_{irr, \pm})^2}{5}. \]  

(34)
Thus for KS BH, the product of the irreducible mass at the horizons $\mathcal{H}^\pm$ are:

$$M_{\text{irr,}+}M_{\text{irr,-}} = \frac{1}{8\omega}. \quad (35)$$

The above product is an universal quantity because it does not depend upon the mass parameter. It should be noted that Eq. (23) can be written as in terms of irreducible mass:

$$M = M_{\text{irr,}+} + \frac{1}{8\omega}M_{\text{irr,-}}. \quad (36)$$

2.2. Heat Capacity $C_\pm$ on $\mathcal{H}^\pm$:

One can define the specific heat on $\mathcal{H}^\pm$ is given by

$$C_\pm = \frac{\partial M}{\partial T_\pm}. \quad (37)$$

which is an important measure to study the thermodynamic properties of a BH. To determine it we first calculate the partial derivatives of mass $M$ and temperature $T_\pm$ with respect to $r_\pm$ are:

$$\frac{\partial M}{\partial r_\pm} = \frac{2\omega r_\pm^2 - 1}{4\omega r_\pm^2}. \quad (38)$$

and

$$\frac{\partial T_\pm}{\partial r_\pm} = \frac{(1 + 5\omega r_\pm^2 - 2\omega^2 r_\pm^4)}{8\pi r_\pm^2(1 + \omega r_\pm^2)^2}. \quad (39)$$

where the temperature of the BH could be found via the following relation:

$$T_\pm = \frac{1}{4\pi} \left. \frac{dg(r)}{dr} \right|_{r=r_\pm}. \quad (40)$$

which comes out

$$T_\pm = \frac{2\omega r_\pm^2 - 1}{8\pi r_\pm^2(1 + \omega r_\pm^2)}. \quad (41)$$

The temperature $T_\pm$ of the black hole is positive, zero or negative, depending on whether $2\omega r_\pm^2 - 1 > 0$, $2\omega r_\pm^2 - 1 = 0$ or $2\omega r_\pm^2 - 1 < 0$ respectively.

Therefore the final expression for heat capacity $C_\pm = \frac{\partial M}{\partial T_\pm}$ at the $\mathcal{H}^\pm$ becomes:

$$C_\pm = \frac{2\pi (2\omega r_\pm^2 - 1)(1 + \omega r_\pm^2)^2}{\omega 1 + 5\omega r_\pm^2 - 2\omega^2 r_\pm^4}. \quad (42)$$

Let us analyze the above expression of specific heat for a different regime in the parameter space.

Case I: The specific heat $C_\pm$ is positive when $\omega r_\pm^2 > \frac{1}{2}$, in this case the BH is thermodynamically stable.

Case II: The specific heat $C_\pm$ is negative when $\omega r_\pm^2 < \frac{1}{2}$, in this case the BH is thermodynamically unstable.
Case III: The specific heat $C_\pm$ blows up when $1 + 5 \omega r^2_\pm - 2 \omega^2 r^4_\pm = 0$ i.e. $r_\pm = \pm \sqrt{\frac{2 + \sqrt{5}}{4 \omega}}$, in this case the BH undergoes a second order phase transition.

Interestingly, the product of specific heat on $H_\pm$ becomes

$$C_+ C_- = \frac{\pi^2 (1 - 2M^2 \omega)(1 + 16M^2 \omega)}{2\omega^2 \left(2 + 13M^2 \omega - 16M^4 \omega^2\right)}.$$  \hspace{1cm} (43)

It indicates that the product depends on mass parameter and coupling constant $\omega$. Thus the product of specific heat of both the horizons are not an universal quantity.

It should be noted that in the extremal limit $M^2 = \frac{1}{2 \omega}$, the temperature, Komar energy and specific heat of the BH becomes zero.

3. Discussion:

In this short note, we have studied the thermodynamic features of KS BH in HL gravity. We computed various thermodynamic product formula for this BH. We observed that the surface area product, BH entropy product and irreducible mass product are universal quantities, whereas the surface temperature product, Komar energy product and specific heat product are not universal quantities because they all are depends on mass parameter.

Finally, we studied the stability of such black hole by computing the specific heat for both the horizons. It has been shown that under certain condition the black hole possesses second order phase transition. In summary, these product formulae may somehow help us a little bit to understanding the microscopic nature of BH entropy both interior and exterior which is the main aim in quantum gravity.

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