Proof of W.M. Schmidt’s conjecture  
concerning successive minima of a lattice

Moshchevitin N.G. 1

Abstract

We prove W.M. Schmidt’s conjecture about a one-parameter family of lattices related to simultaneous Diophantine approximations.

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1 Introduction

Consider real numbers \( \xi_j \in [0, 1) \), \( 1 \leq j \leq n \). For a real \( x \) we denote by \( |x| \) the absolute value of \( x \). We know since Dirichlet that for any real \( N > 1 \) the inequalities

\[
|x| \leq N, \quad \max_{1 \leq j \leq n} |x \xi_j - y_j| \leq N^{-1/n}
\]

have a solution in integers \( x \neq 0, y_1, \ldots, y_n \). Similarly for any real \( N > 1 \) the inequalities

\[
|x| \leq N, \quad \left( \sum_{j=1}^{n} |x \xi_j - y_j|^2 \right)^{1/2} \leq 2w_n^{-1/n}N^{-1/n}
\]

have a solution in integers \( x \neq 0, y_1, \ldots, y_n \) (here \( w_n \) stands for the volume of the unit ball in the \( n \)-dimensional Euclidean space).

In this paper we work with Euclidean space \( \mathbb{R}^{n+1} \) with coordinates \((x, y_1, \ldots, y_n)\) and with Euclidean space \( \mathbb{R}^n \) with coordinates \((y_1, \ldots, y_n)\).

Consider an \((n + 1)\)-dimensional vector \( \xi = (1, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1} \).

For a vector \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) we define \( |y| \) to be the Euclidean norm of \( y \). So, \( |y| = \sqrt{y_1^2 + \cdots + y_n^2} \). We also use the notation \( |y|_s = \max_{1 \leq j \leq n} |y_j| \) for the sup-norm of a vector \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \).

For a real \( N \geq 1 \) and a vector \( \xi \) we define a matrix

\[
A(\xi, N) = \begin{pmatrix}
N^{-1} & 0 & 0 & \cdots & 0 \\
N^{\frac{1}{n}}\xi_1 & -N^{\frac{1}{n}} & 0 & \cdots & 0 \\
N^{\frac{1}{n}}\xi_2 & 0 & -N^{\frac{1}{n}} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
N^{\frac{1}{n}}\xi_n & 0 & 0 & \cdots & -N^{\frac{1}{n}}
\end{pmatrix}
\]

and a lattice

\[
\Lambda(\xi, N) = A(\xi, N)\mathbb{Z}^{n+1}.
\]

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Consider the \((n+1)\)-dimensional unit cube
\[
\mathcal{U} = \{ z = (x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : \max(|x|, |y|) \leq 1 \}
\]
and a convex 0-symmetric body
\[
\mathcal{W} = \{ z = (x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : \max(|x|, |y|) \leq 1 \}
\]

For a natural \(l, 1 \leq l \leq n + 1\) let \(\lambda_l(\xi, N)\) be the \(l\)-th successive minimum of \(\mathcal{U}\) with respect to \(\Lambda(\xi, N)\) and let \(\mu_l(\xi, N)\) be the \(l\)-th successive minimum of \(\mathcal{W}\) with respect to \(\Lambda(\xi, N)\). By the second Minkowski theorem for convex body (see [1], Ch. VIII or [2], Ch.IV) we have
\[
\frac{1}{(n+1)!} \leq \prod_{l=1}^{n+1} \lambda_l(\xi, N) \leq 1, \quad \frac{2^n}{w_n(n+1)!} \leq \prod_{l=1}^{n+1} \mu_l(\xi, N) \leq \frac{2^n}{w_n}.
\]

In the case \(n = 1\) we have \(\mu_l(\xi, N) = \lambda_l(\xi, N), l = 1, 2\). Suppose that \(\xi_1/\xi_2 \notin \mathbb{Q}\). Since there are arbitrary large values of \(N\) with \(\mu_l(\xi, N) = \mu_2(\xi, N)\), it may never happen that \(\mu_1(\xi, N) \to 0, N \to +\infty\).

Consider the general case. Suppose that numbers \(\xi_1, ..., \xi_n\) are linearly independent over \(\mathbb{Z}\) together with 1. Then for \(1 \leq k \leq n\) there exist arbitrary large values of \(N\) such that \(\mu_k(\xi, N) = \mu_{k+1}(\xi, N)\). But in the case \(n > 1\) it may happen that \(\mu_1(\xi, N) \to 0, N \to +\infty\). Moreover from A.Khintchine’s result [3] (see also [4]) it follows that it may happen that \(\mu_{n-1}(\xi, N) \to 0, N \to +\infty\).

In this paper we prove the following theorem.

**Theorem 1.** Let \(1 \leq k \leq n - 1\). Then there exist real numbers \(\xi_j \in [0, 1), 1 \leq j \leq n\), such that
- 1, \(\xi_1, ..., \xi_n\) are linearly independent over \(\mathbb{Z}\);
- \(\mu_k(\xi, N) \to 0\) as \(N \to \infty\);
- \(\mu_{k+2}(\xi, N) \to \infty\) as \(N \to \infty\).

We make two remarks.

**Remark 1.** The analogous result for \(\lambda_l(\xi, N)\) was conjectured by W.M. Schmidt in [5]. In this paper we consider the Euclidean norm only for simplicity reasons. We must note that the main result is valid not only for the Euclidean norm but also for the sup-norm \(|\cdot|_s\) (as it was conjectured in W.M. Schmidt’s paper [5]).

**Remark 2.** It is shown in Section 3 that Theorem 1 becomes trivial without the condition on 1, \(\xi_1, ..., \xi_n\) to be linearly independent over \(\mathbb{Z}\).

The construction in the proof of Theorem 1 in the case \(k = 1\) is very simple. It is close to the construction from [6], where the author gives a counterexample to J. Lagarias’ conjecture concerning the behavior of consecutive best simultaneous Diophantine approximations (see [7]). We give a complete proof of Theorem 1 in the case \(k = 1\) in Section 2.

The case \(k > 1\) the construction in the proof of Theorem 1 is a little bit more difficult. It is close to procedures from [8], [9] (See the author’s review [10] for related topics). We give a complete proof of Theorem 1 in the case \(k > 1\) in Sections 3-5.

In the proof we shall need the following notation.

By \(\mu_l(C; L)\) we denote the \(l\)-th successive minimum of a convex 0-symmetric set \(C\) with respect to a lattice \(L\).

Let \(w_l\) denote the volume of the unit ball in the \(l\)-dimensional Euclidean space.

For a set \(\mathcal{M} \subset \mathbb{R}^{n+1}\) we denote by \(\overline{\mathcal{M}}\) the smallest closed set containing \(\mathcal{M}\). We also denote the smallest linear and affine subspaces of \(\mathbb{R}^{n+1}\) containing \(\mathcal{M}\), by \(\text{span}\mathcal{M}\) and \(\text{aff}\mathcal{M}\), respectively.

Consider a sublattice \(\Lambda \subset \mathbb{Z}^{n+1}\). By \(\text{dim} \Lambda\) we denote the dimension of the linear subspace \(\text{span} \Lambda\). A sublattice \(\Lambda \subset \mathbb{Z}^{n+1}\) is defined to be complete if
\[
\Lambda = (\text{span} \Lambda) \cap \mathbb{Z}^{n+1},
\]
that is in the linear subspace span $\Lambda$ there is no integer points different from points of $\Lambda$.

For every positive $Q$ and $\sigma$ we define a cylinder $\mathcal{C}(Q, \sigma) \subset \mathbb{R}^{n+1}$ as follows:

$$
\mathcal{C}(Q, \sigma) = \left\{ z = (x, y) \in \mathbb{R}, \; y = (y_1, \ldots, y_n) \in \mathbb{R}^n : \; |x| < Q, \; \left( \sum_{j=1}^{n} |y_j - \xi_j x| \right)^{1/2} < \sigma \right\}.
$$

The quantities $\mu_l(\xi, N)$ coincide with the successive minima of $\mathbb{Z}^{n+1}$ with respect to $\mathcal{C}(N, N^{-1/n})$, that is

$$
\mu_l(\xi, N) = \mu_l(\mathcal{C}(N, N^{-1/n}); \mathbb{Z}^{n+1}).
$$

## 2 Proof of Theorem 1: case $k = 1$

We need two auxiliary results - Lemmas A and B.

**Lemma A.** Let $\xi = \left(1, \frac{a_1}{q}, \ldots, \frac{a_n}{q}\right) \in [0, 1]^{n+1}$ be a rational vector. Suppose that for integers $q, a_1, \ldots, a_n \in \mathbb{Z}$ we have

$$
q \geq 1, (q, a_1, \ldots, a_n) = 1.
$$

Then for any positive $U > 0$ and any natural $i$ there exists a positive real number

$$
\eta = \eta(\xi, i, U) > 0
$$

such that for every real vector $\xi' = (1, \xi'_1, \ldots, \xi'_n)$ under condition $|\xi' - \xi| < \eta$ the inequalities

$$
\mu_1(\xi', N) \leq i^{-1}, \; \mu_2(\xi', N) \geq i
$$

are valid for all $N$ in the interval

$$
(2qi\sqrt{n+1})^n \leq N \leq U.
$$

Proof. First of all we note that for $N \geq q$ we obviously have

$$
\mu_1(\xi, N) \leq qN^{-1}.
$$

Besides that, the Euclidean distance between the one-dimensional subspace span $\xi$ and the set $\mathbb{Z}^{n+1} \setminus \text{span}\, \xi$ is not less than $(q\sqrt{n+1})^{-1}$. Thus, in order to catch an integer point, independent with $\xi$, in the cylinder $t\mathcal{C}(N, N^{-1/n})$, we should take $t$ to be not less than $N^{1/n}(q\sqrt{n+1})^{-1}$. Hence

$$
\mu_2(\xi, N) \geq N^{\frac{1}{n}} q^{-1}.
$$

From the hypothesis of Lemma A we deduce that the inequalities

$$
\mu_1(\xi, N) \leq (2i)^{-1}, \; \mu_2(\xi, N) \geq 2i
$$

hold for all $N \geq (2qi\sqrt{n+1})^n$. Now Lemma A follows from the observation that for any $l$ the function $\mu_l(\xi, N)$ is a continuous function in $\xi$ and $N$. Lemma A is proved.

**Lemma B.** Let $\Gamma$ be a two-dimensional complete sublattice of $\mathbb{Z}^{n+1}$. Let $R$ be the two-dimensional fundamental volume of $\Gamma$ and let $\rho = \rho(\Gamma) > 0$ be the Euclidean distance between span $\Gamma$ and $\mathbb{Z}^{n+1} \setminus \Gamma$. Suppose that $\xi = (1, \xi_1, \ldots, \xi_n) \in \text{span}\, \Gamma$. Then for any positive $N$ we have the following estimates:

$$
\mu_1(\xi, N) \leq N^{\frac{1-n}{2n}} R^{\frac{1}{2}}, \; \mu_3(\xi, N) \geq N^{\frac{1}{n}} \rho.
$$
Proof. First of all we prove the upper bound for $\mu_1(\xi, N)$. The intersection of the cylinder $C_\xi(N, N^{-1/n})$ with $\text{span} \Gamma$ is an $0$-symmetric parallelogram, whose two-dimensional volume greater than or equal to $4N^{\frac{n-1}{n}}$. Suppose that $4t^2N^{\frac{n-1}{n}} > 4R$ for some $t > 0$. Then, by the Minkowski convex body theorem there is a nonzero point of $\Gamma$ inside the parallelogram $tC_\xi(N, N^{-1/n}) \cap \text{span} \Gamma$. So, for any $t > N^{\frac{n-1}{n}}$ the cylinder $tC_\xi(N, N^{-1/n})$ contains a nonzero integer point and the upper bound for $\mu_1(\xi, N)$ is proved.

To prove the lower bound for $\mu_3(\xi, N)$ we need to take into account that if the cylinder $tC_\xi(N, N^{-1/n})$ contains more than two linearly independent integer points, then one of these points does not belong to $\Gamma$ and $tN^{-1/n} \geq \rho$. Lemma B is proved.

Now we describe the inductive procedure which gives the proof of Theorem 1 in the case $k = 1$.

The set of all $n$-dimensional sublattices of $\mathbb{Z}^{n+1}$ is countable. We fix an enumeration of this set and let

$L_1, L_2, ..., L_i, ...$

be the sequence of all $n$-dimensional complete sublattices of $\mathbb{Z}^{n+1}$. Set $\pi_i = \text{span} L_i$. Suppose that

$\pi_1 = \{z = (x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : x = 0\}$.

We construct a sequence of rational vectors

$\xi_i = \left(1, \frac{a_{1,i}}{q_i}, ..., \frac{a_{n,i}}{q_i}\right), \ q_i, a_{1,i}, ..., a_{n,i} \in \mathbb{Z}, \ q_i \geq 1, (q_i, a_{1,i}, ..., a_{n,i}) = 1, \ i = 1, 2, 3, ...$

with $q_i \to \infty$ as $i \to \infty$, a sequence of two-dimensional complete sublattices $\Gamma_{i+1} \subset \mathbb{Z}^{n+1}, \ i = 1, 2, 3, ..., $ and a sequence of positive real numbers

$\eta_1, \eta_2, \eta_i, ...$

satisfying the following conditions (i) - (iv).

(i) For every $i \geq 1$ we have

$\xi_i, \xi_{i+1} \in \text{span} \Gamma_{i+1}$.

(ii) The closed ball $\overline{B}_i$ of radius $\eta_i$ centered at $\xi_i$ and has no common points with the subspace $\pi_i$:

$\overline{B}_i \cap \pi_i = \emptyset$.

(iii) The balls $\overline{B}_i$ form a nested sequence:

$\overline{B}_1 \supset \overline{B}_2 \supset ... \supset \overline{B}_i$.

(iv) Let $H_0 = 1$ and

$H_i = (4q_i(i+1)\sqrt{n+1})^n, \ i = 1, 2, 3, ...$

Then for any $i \geq 1$, for any $\xi \in \overline{B}_i$ and for any $N$, such that

$H_{i-1} \leq N < H_i,$

the following inequalities holds:

$\mu_1(\xi, N) \leq i^{-1}, \mu_3(\xi, N) \geq i$. 
Suppose all the objects are already constructed. Then from (ii), (iii) one can easily see that 
 \[ \lim_{i \to \infty} \eta_i = 0. \]
 Then the intersection \( \bigcap_{i \in \mathbb{N}} B_i \) consists of the only one point. Note that for every \( i \) the center \( \xi_i \) of the ball \( B_i \) has its first coordinate equal to one. So the unique point from the intersection \( \bigcap_{i \in \mathbb{N}} B_i \) is of the form 
 \[ \xi = (1, \xi_1, \ldots, \xi_n) \]
 (as \( \xi_i \to \xi, i \to \infty \)). Then

\[ \mu_1(\xi, N) \leq i^{-1}, \quad \mu_3(\xi, N) \geq i, \quad H_{i-1} \leq N < H_i. \]

Hence
\[ \mu_1(\xi, N) \to 0, \quad \mu_3(\xi, N) \to \infty, \quad N \to \infty, \]

and it follows from the conditions (ii) and (iii) that \( 1, \xi_1, \ldots, \xi_n \) are linearly independent over \( \mathbb{Z} \). This proves Theorem 1 in the case \( k = 1 \).

We start our inductive procedure with the vector 
\[ \xi_1 = (1, 0, \ldots, 0), \]

Then \( H_1 = (8\sqrt{n + 1})^n \). The sublattice \( \Gamma_1 \) is not defined yet and we do not care about the condition (i) at this stage. The condition (ii) is obviously satisfied for any choice of \( \eta_1 < 1 \). The conditions (iii) is empty. Recall that \( H_0 = 1 \). To satisfy the condition (iv) we take \( \eta_1 \) to be small enough.

Now we pass to the inductive step. Suppose that all the objects \( \xi_i, \Gamma_i, \eta_i, i \leq t \) satisfying conditions (i) – (iv) are already constructed. We describe how to construct the \((t+1)\)-th set of objects.

First of all we can take a two-dimensional complete sublattice \( \Gamma_{t+1} \) satisfying the conditions

\[ q_t \xi_t = (q_t, a_{1,t}, \ldots, a_{n,t}) \in \Gamma_{t+1}, \quad \Gamma_{t+1} \not\subset \pi_{t+1}. \]

Let \( R_t \) be the two-dimensional fundamental volume of \( \Gamma_{t+1} \) and let \( \rho_t \) be the Euclidean distance between span \( \Gamma_{t+1} \) and \( \mathbb{Z}^{n+1} \setminus \Gamma_{t+1} \).

Set
\[ U_t = \max \left( (2(t+1)\rho_t^{-1})^n, (2(t+1)R_t^{1/2}) \frac{n}{n-1} \right). \]

Now we apply Lemma A with \( \xi = \xi_t, i = 2(t+1) \) and \( U = U_t \). We get a positive \( \eta_t' \), such that for every \( \xi' \) under the condition \( |\xi' - \xi_t| < \eta_t' \) one has
\[ \mu_1(\xi', N) \leq (2t + 2)^{-1}, \quad \mu_2(\xi', N) \geq 2t + 2 \]
for every \( N \) in the interval 
\[ H_t = (4q_t(t+1)\sqrt{n + 1})^n \leq N \leq U_t. \]

Obviously, there is an integer point
\[ (q_{t+1}, a_{1,t+1}, \ldots, a_{n,t+1}) \in \Gamma_{t+1} \setminus \pi_{t+1}, \quad q_{t+1} \geq q_t, \quad (q_{t+1}, a_{1,t+1}, \ldots, a_{n,t+1}) = 1, \]

such that for
\[ \xi_{t+1} = \left( 1, \frac{a_{1,t+1}}{q_{t+1}}, \ldots, \frac{a_{n,t+1}}{q_{t+1}} \right) \]
we have
\[ |\xi_{t+1} - \xi_t| < \frac{\min(\eta_t, \eta_t')}{2}. \]
Since $\xi_{t+1} \in \Gamma_{t+1}$, we can apply Lemma B with $\xi = \xi_{t+1}, \Gamma = \Gamma_{t+1}$. This gives that for any $N$ under the condition
\[ N \geq U_t \] (1)
one has
\[ \mu_1(\xi_{t+1}, N) \leq (2(t+1))^{-1}, \quad \mu_3(\xi_{t+1}, N) \geq 2(t+1). \] (2)
But $|\xi_{t+1} - \xi_t| < \eta'_t$ and $\mu_3(\xi_{t+1}, N) \geq \mu_2(\xi_{t+1}, N)$. So, the inequalities (2) are valid not only for $N$ in the interval (1) but also for $N$ in the interval $N \geq H_t$. Having constructed $\xi_{t+1}$, we define $H_{t+1}$ from the condition (iv) of the $(t+1)$-th step of the inductive process. Now we take into account that for any $l$ the function $\mu_1(\xi, N)$ is a continuous function in $\xi$ and $N$. This means that we can find a number $\eta_{t+1} < \min(\eta_t, \eta'_t)/2$, such that
\[ \mu_1(\xi, N) \leq (t+1)^{-1}, \quad \mu_2(\xi, N) \geq t + 1 \]
for all $\xi$ under the condition
\[ |\xi - \xi_{t+1}| \leq \eta_{t+1} \]
and all $N$ in the interval
\[ H_t \leq N < H_{t+1}. \]
Moreover, since $\xi_{t+1} \not\in \pi_{t+1}$, we can take $\eta_{t+1}$ to be small enough, so that the ball $\overline{B}_{t+1}$ of radius $\eta_{t+1}$ centered at $\xi_{t+1}$ and has no common points with $\pi_{t+1}$. The $(t + 1)$-th step of the inductive procedure is described completely and hence Theorem 1 in the case $k = 1$ is proved.

3 Lemmas concerning successive minima and badly approximable numbers

In this section we start the proof of Theorem 1 in the general case.

Let $l$ be an integer and $1 \leq l \leq n$. First of all we should say that everywhere in the sequel we consider $l$-dimensional lattices $L$ such that for some real vector $\xi = (1, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ one has
\[ \xi \in \text{span } L. \]
This condition leads to the following corollary concerning linear subspace span $L$. Consider affine subspace
\[ \mathcal{P} = \{ z = (x, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} : x = 1 \} \subset \mathbb{R}^{n+1}. \]
Then the intersection
\[ \text{span } L \cap \mathcal{P} \]
is an affine subspace of dimension $l - 1$.

We prove some auxiliary results on successive minima and badly approximable numbers.

For positive integer $l$ and positive $\sigma \in (0, 1)$ define
\[ Q_1 = Q_1(\sigma, l, s) = 2^{l-1}w_{l-1}^{-1}\sigma^{1-l}s, \quad Q_2 = Q_2(\sigma, l, s) = 2^{l-1}Q_1 = 2^{2(l-1)}w_{l-1}^{-1}\sigma^{1-l}s. \] (3)

**Lemma 1.** Consider an integer $l$, such that $2 \leq l \leq n + 1$. Consider a complete sublattice $L \subseteq \mathbb{Z}^{n+1}$ and suppose that $\dim L = l$ and $\xi \in \text{span } L$. Suppose that the fundamental $l$-dimensional volume of the lattice $L$ is equal to $s$. Consider a cylinder $\mathcal{C} = \mathcal{C}_\xi(Q, \sigma)$. Suppose also that for some positive $Q$ and $\sigma$ we have $\mathcal{C} \cap L = \{0\}$. Then the following upper bounds are valid:
\[ \mu_1(\mathcal{C}) \leq Q_1Q^{-1}, \quad \mu_m(\mathcal{C}) \leq Q_2Q^{-1}, \quad 2 \leq m \leq l. \]
Proof. Consider the cylinder
\[ C^{(1)} = C_{\xi}(Q_1, \sigma) \cap \text{span} \ L. \]
As \( \xi \in \text{span} \ L \) we see that each section
\[ C^{(1)} \cap \{ z = (x, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} : x = x_0 \}, \ |x_0| \leq Q_1 \]
is a \((l-1)\)-dimensional ball of the volume \( w_{l-1} \sigma^{l-1} \). Let \( H_1 \) be the distance between \((l-1)\)-dimensional facets of the cylinder \( C^{(1)} \). Then the \( l \)-volume of \( C^{(1)} \) is equal to
\[ w_{l-1} H_1 \sigma^{l-1} . \]
But \( H_1 \geq 2Q_1 \). So the \( l \)-volume of \( C^{(1)} \) is
\[ \geq 2w_{l-1}Q_1 \sigma^{l-1} = 2^l s. \]
(in the equality here we take into account the definition of \( Q_1 \) from (30)). By the Minkowski’s theorem for convex body, there is a nonzero integer points \( \zeta^{(1)} \in C^{(1)} \cap L \). So, \( \mu_1(C) \leq Q_1Q^{-1} \) and the bound for the first successive minimum is proved.

Here we should note that as \( \sigma < 1 \) the first coordinate of \( \zeta^{(1)} \) is not equal to zero.

Now we describe an inductive process of constructing linearly independent integer point \( \zeta^{(1)}, \ldots, \zeta^{(l)} \) with non-zero first coordinates which ensure the upper bound for the successive minima under consideration.

Suppose that linearly independent integer points \( \zeta^{(1)}, \ldots, \zeta^{(\nu)} \in \text{span} \ L \) with \( 1 \leq \nu \leq l - 1 \) are already constructed. Set \( \pi = \text{span}(\zeta^{(1)}, \ldots, \zeta^{(\nu)}) \subset \text{span} \ L \). As all the points \( \zeta^{(1)}, \ldots, \zeta^{(\nu)} \) are linearly independent we see that \( \dim \pi = \nu < l \). Note that the dimension of the affine subspace \( \pi' = \pi \cap \{ z = (x, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} : x = Q \} \) is equal to \( \dim \pi' - 1 < l - 1 \). Consider the facet \( B = \overline{\pi} \cap \{ x = Q \} \). This facet is an \( n \)-dimensional ball of radius \( \sigma \) centered at \( Q\xi \).

Note that as \( \xi \in \text{span} \ L \) we see that the intersection \( \text{span} \ L \cap \{ x = Q \} \) is a \((l-1)\)-dimensional affine subspace. Consider the intersection \( B \cap \text{span} \ L \). As \( Q\xi \in \text{span} \ L \) we see that this intersection is a \((l-1)\)-dimensional ball centered at \( Q\xi \).

We have the following situation. In the \((l-1)\)-dimensional affine subspace \( \text{span} \ L \cap \{ x = Q \} \) there are the ball \( B \cap \text{span} \ L \) of dimension \( l - 1 \) and the affine subspace \( \pi' \) of dimension \( \dim \pi' < l - 1 \). So there exists a \( n \)-dimensional ball \( B' \subset B \subset \{ x = Q \} \) of radius \( \sigma/2 \) centered at a certain point \( \Xi \in \text{span} \ L \cap \{ x = Q \} \) and such its \((l-1)\)-dimensional section \( B' \cap \text{span} \ L \) does not intersect with \( \pi' \):
\[ B' \cap \text{span} \ L \cap \pi = \emptyset. \]

In fact as \( \pi' \subset \text{span} \ L \cap \{ x = Q \} \), it means that
\[ B' \cap \pi = \emptyset. \]

Put \( \zeta^{(\nu+1)} = \Xi \) and consider the cylinder
\[ C^{(\nu+1)} = C_{\xi^{(\nu+1)}}(Q_2, \sigma/2) \cap \text{span} \ L. \]
As \( \xi^{(\nu+1)} \in \text{span} \ L \) from (30) we see that the \( l \)-volume of \( C^{(\nu+1)} \) is equal to
\[ w_{l-1} H_2 \left( \frac{\sigma}{2} \right)^{l-1}, \]
where $H_2 \geq 2Q_2$ is the distance between $(l - 1)$-dimensional facets of the cylinder $C^{(ν+1)}$. So the $l$-volume of $C^{(ν+1)}$ is

$$\geq 2w_{l-1}Q_2 \left(\frac{σ}{2}\right)^{l-1} = 2^l s.$$  

Applying again the Minkowski theorem for convex body we get a nonzero integer point $ζ^{(ν+1)} \in C^{(ν+1)} \cap L$.

As $σ < 1$ we see that the first coordinate of $ζ^{(ν+1)}$ is not equal to zero. Moreover the first coordinate of the point $ζ^{(ν+1)}$ is greater than $Q$ as there is no nonzero integer points in $C \cap \text{span}(L)$ and

$$C^{(ν+1)} \cap \{z = (x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : |x| \leq Q\} \subset C = C_ξ(Q, σ).$$

So

$$ζ^{(ν+1)} \notin C.$$

But

$$π \cap C^{(ν+1)} \subset C.$$

So $ζ^{(ν+1)} \notin π$. It means that $ζ^{(ν+1)}$ is independent of $ζ^{(1)}, ..., ζ^{(ν)}$.

To conclude the proof we make two following observations:

1. each point $z = (x, y_1, ..., y_n) \in C^{(ν+1)}$ satisfies $|x| \leq Q_2$;
2. for the section $C^{(ν+1)} \cap \{x = Q\}$ one has

$$C^{(ν+1)} \cap \{x = Q\} \subset C \cap \{x = Q\},$$

and the section $C \cap \{x = Q\}$ is a ball of radius $σ$ centered at $Qξ$.

So

$$C^{(ν+1)} \subset C_ξ(Q, σ).$$

Now $μ_m(C) \leq Q_2Q^{-1}$ for $2 \leq m \leq l$. Lemma 1 is proved.

**Lemma 2.** Let $2 \leq l \leq n + 1$. Consider a complete sublattice $L \subseteq \mathbb{Z}^{n+1}$ and suppose that $\text{dim}(L) = l$ and $ξ \in \text{span}(L)$. Suppose that the fundamental $l$-dimensional volume of $L$ is equal to $s$. Suppose also that for some $Q, σ > 0$ we have

$$C_ξ(Q, σ) \cap L = \{0\}.$$

Then for any $M, δ > 0$ the following upper bound is valid:

$$μ_l(C_ξ(M, δ)) \leq Q_2Q^{-1} \max(QM^{-1}, σδ^{-1}).$$

**Corollary.** Suppose that the conditions of Lemma 2 are satisfied. Then for the cylinder $C_ξ(N, N^{-1/n})$ we have

$$μ_l(ξ, N) \leq Q_2Q^{-1} \max(QN^{-1}, σN^{1/n}).$$

Proof of Lemma 2. Put $t = \max(QM^{-1}, σδ^{-1})$. Then

$$C_ξ(Q, σ) \subset tC_ξ(M, δ),$$

and applying Lemma 1 we see that

$$μ_l(C_ξ(M, δ)) = tμ_l(tC_ξ(M, δ)) \leq tμ_l(C_ξ(Q, σ)) \leq Q_2Q^{-1}t.$$
Lemma 2 is proved.

Put \( \xi_0 = 1 \). For a real vector \( \xi = (\xi_0, \xi_1, ..., \xi_n) = (1, \xi_1, ..., \xi_n) \in \mathbb{R}^{n+1} \) we define \( \dim_\mathbb{Q} \xi \) to be the maximal integer \( t \), such that the components \( \xi_j, ..., \xi_n, 0 \leq j_1, ..., j_t \leq n + 1 \) are linearly independent over \( \mathbb{Q} \). For example, the equality \( \dim_\mathbb{Q} \xi = 1 \) occurs only if \( \xi \in \mathbb{Q}^{n+1} \setminus \{0\} \) and the equality \( \dim_\mathbb{Q} \xi = n + 1 \) occurs only if all the components \( 1, \xi_1, ..., \xi_n \) are linearly independent over \( \mathbb{Q} \). Obviously, if \( \dim_\mathbb{Q} \xi = l \), \( 1 \leq l \leq n + 1 \), then there is a complete sublattice \( L \subseteq \mathbb{Z}^{n+1} \), such that \( \dim L = l \) and \( \xi \in \text{span} \ L \). Moreover,

\[
\dim_\mathbb{Q} \xi = \min\{l \in \mathbb{N} : \text{there exists a sublattice } L \subseteq \mathbb{Z}^{n+1}, \text{ such that } \dim L = l \text{ and } \xi \in \text{span} \ L\}.
\]

Let us now we consider a complete sublattice \( L \subseteq \mathbb{Z}^{n+1} \), such that \( \dim L = l \geq 2 \) and let us consider a vector \( \xi = (1, \xi_1, ..., \xi_n) \in \text{span} \ L \) (then \( \dim_\mathbb{Q} \xi \leq l \)). We shall say that \( \xi \) is \( \gamma \)-badly approximable with respect to \( L \) (briefly \( (L, \gamma) \)-BAD) if for any nonzero integer point \( \zeta = (q, a) = (q, a_1, ..., a_n) \in L \) with \( q \neq 0 \) one has

\[
|q\xi - \zeta| \geq \gamma |q|^{-1/(l-1)}.
\]

(4)

We should note that for any \( (L, \gamma) \)-BAD vector \( \xi \) and any \( Q \geq 1 \) the cylinder

\[
C_\xi(Q, \sigma_Q) \cap \text{span} \ L, \quad \sigma_Q = \gamma Q^{-1/(l-1)}
\]

contains no nonzero integer points inside. A vector \( \xi \in \text{span} \ L \) is defined to be badly approximable with respect to \( L \) (briefly \( L \)-BAD) if (4) holds with some positive \( \gamma \). It is easy to see that if vector \( \xi \) is badly approximable with respect to \( L \) and \( \dim L = l \) then \( \xi \in \text{span} \ L \) and \( \dim_\mathbb{Q} \xi = l \).

Let \( W \geq 1 \). It is necessary for us to consider vectors \( \xi \in \text{span} \ L \), such that the cylinder (5) contains no nonzero integer point inside only for \( Q \geq W \). We define such vectors to be \( (\gamma, W) \)-badly approximable with respect to \( L \) (briefly \( (L, \gamma, W) \)-BAD). A vector \( \xi \in \text{span} L \) is \( (\gamma, W) \)-badly approximable with respect to \( L \) iff (4) holds for all \( \zeta \) with \( |q| \geq W \) and for all \( q \) under the condition \( 1 \leq |q| \leq W \) the following inequality holds instead of (4):

\[
|q\xi - \zeta| \geq \gamma W^{-1/(l-1)}.
\]

It is obvious that a vector \( \xi \in \text{span} \ L \) is \( (\Lambda, \gamma) \)-BAD iff it is \( (\Lambda, \gamma, 1) \)-BAD.

Example 1. Consider the space \( \mathbb{R}^{n+1} \) related to coordinates \( x, y_1, ..., y_n \). Consider the case when real algebraic integers \( 1, \alpha_1, ..., \alpha_{l-1} \) form a basis of a real algebraic field \( \mathcal{K} \) of degree \( l \geq 2 \). Then there exists a constant \( \gamma = \gamma(\mathcal{K}) \), such that for all natural \( q \) we have

\[
\left( \sum_{j=1}^{l-1} \|q\alpha_j\|^2 \right)^{1/2} \geq \gamma q^{-1/(l-1)}
\]

(see [11], Chapter V, §3) and hence the \((n + 1)\)-dimensional vector

\[
(1, \alpha_1, ..., \alpha_{l-1}, 0, ..., 0)
\]

is \( (L, \gamma(\mathcal{K})) \)-BAD where \( L = \mathbb{Z}^{n+1} \cap \{y_1 = ... = y_n = 0\} \).

Define

\[
G_1 = G_1(l, s, \gamma) = 2^{2(l-1)} w_{l-1}^{-1} s^{-1} \gamma^{-l}, \quad G_2 = G_2(l, s, \gamma) = 2^{2(l-1)} w_{l-1}^{-1} s^{-2} \gamma^{-2(l-1)}.
\]
Lemma 3. Let \( L \subseteq \mathbb{Z}^{n+1} \) be a complete sublattice, such that \( \dim L = l \geq 2 \) and let \( \xi = (1, \xi_1, \ldots, \xi_n) \in \text{span} L \) be an \((L, \gamma, W)\)-BAD vector. Let \( s \) be the \( l \)-dimensional fundamental volume of \( L \). Consider positive \( M, \delta \) and the cylinder \( C = C_\xi(M, \delta) \). Then the following statements hold.

1) If
\[
(M\gamma\delta^{-1})^{\frac{l-1}{n}} \leq W
\] (6)
then
\[
\mu_l(C) \leq G_1 WM^{-1}.
\] (7)

2) If
\[
(M\gamma\delta^{-1})^{\frac{l-1}{n}} \geq W
\] (8)
then
\[
\mu_l(C) \leq G_2 M^{-\frac{n}{n-1}}\delta^{\frac{n}{n-1}}.
\] (9)

Remark. We actually construct in the proof \( l \) nonzero linearly independent integer points \( \zeta_j \in L \) lying in the cylinder \( \mu_l(C) \cdot \overline{C} \). It is seen from the construction that in the case 2) of Lemma 3 each ray \([0, \zeta_j), 1 \leq j \leq l\) intersects the facet \( \{x = M\} \) of the cylinder \( C = C_\xi(M, \delta) \).

Proof of Lemma 3. For \( Q \geq W \) the cylinder \( \overline{C} \) has no nonzero integer points. By Lemma 2 for any \( Q \geq W \) we have
\[
\mu_l(C) \leq G_1 \max(QM^{-1}, \frac{\gamma Q^{-1/2}}{l-1} \delta^{-1}).
\]
Consider
\[
m(M, \delta, W) = \min_{Q \geq W} \max(QM^{-1}, \frac{\gamma Q^{-1/2}}{l-1} \delta^{-1}).
\]
If (6) holds we have
\[
m(M, \delta, W) = WM^{-1}.
\]
If (8) holds we see that
\[
m(M, \delta, W) = \gamma^{\frac{l-1}{n}} \delta^{\frac{n}{n-1}} M^{-\frac{n}{n-1}}.
\]
Lemma 3 follows.

Lemma 3 applied to the cylinder \( C_\xi(N, N^{-1/n}) \) gives the following

Corollary 1. Let \( \xi = (1, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1} \). Let \( L \subseteq \mathbb{Z}^{n+1} \) be a complete sublattice such that \( \dim L = l \geq 2 \) and let \( \xi \in \text{span} L \) be a \((L, \gamma, W)\)-BAD vector. Let \( s \) be the \( l \)-dimensional fundamental volume of \( L \). Then

1) for any positive \( N \) under the condition
\[
N \leq \gamma^{-\frac{n}{n+1}} W^{\frac{ln}{(n+1)(l-1)}}
\] (10)
one has
\[
\mu_l(\xi, N) \leq G_1 WN^{-1}.
\] (11)

2) for any \( N \) under the condition
\[
N \geq \gamma^{-\frac{n}{n+1}} W^{\frac{ln}{(n+1)(l-1)}}
\] (12)
one has
\[
\mu_l(\xi, N) \leq G_2 N^{\frac{l-n-1}{n}}.
\] (13)
Corollary 2. Let $2 \leq l \leq n$. Let $\xi \in \mathbb{R}^{n+1}$. Let $L \subseteq \mathbb{Z}^{n+1}$ be a complete sublattice such that $\dim L = l$ and let $\xi \in \text{span} L$ be a $L$-BAD vector. Then

$$
\mu_l(\xi, N) \to 0, \ \mu_{l+1}(\xi, N) \to +\infty, \ N \to \infty.
$$

Remark 1. Obviously, for $l = 1$ in the case $\dim \omega \xi = 1$ we have

$$
\mu_1(\xi, N) \to 0, \ \mu_2(\xi, N) \to +\infty, \ N \to \infty.
$$

Remark 2. Of course, the assertion of Corollary 2 enforces the components $1, \xi_1, \ldots, \xi_n$ to be linearly dependent over $\mathbb{Q}$.

Proof of Corollary 2. The statement about $\mu_l(\xi, N)$ follows immediately from Corollary 1 of Lemma 3 as the exponent in the right hand side of (13) is negative. We prove the statement about $\mu_{l+1}(\xi, N)$.

Suppose that $f_1, \ldots, f_l \in L$ form a basis of $L$. Then it can be completed to a basis $f_1, \ldots, f_l, g_{l+1}, \ldots, g_{n+1}$ of the entire integer lattice $\mathbb{Z}^{n+1}$. Let $L'$ be the sublattice generated by $g_{l+1}, \ldots, g_{n+1}$. Then $\mathbb{Z}^{n+1} = L \oplus L'$, $\dim L' = n + 1 - l$ and span $L \cap \text{span} L' = \{0\}$. Consider the $n + 1 - l$ dimensional linear subspace $\pi \subseteq \mathbb{R}^{n+1}$, orthogonal to span $L$. Then $\pi \oplus \text{span} L = \mathbb{R}^{n+1}$ and any two vectors $u \in \pi, v \in \text{span} L$ are orthogonal. Hence the orthogonal projection of $L'$ onto $\pi$ is a lattice $L''$, such that span $L'' = \pi$. Let $\omega = \omega(L) > 0$ be the length of the shortest nonzero vector in $L''$. Then for any integer point $\xi \in \mathbb{Z}^{n+1} \setminus \pi$ the Euclidean distance from $\xi$ to span $L$ is not less than $\omega$. Suppose that $\xi \in \text{span} L$ and that the cylinder $C_\xi(t N, t N^{-1/n})$ contains $l + 1$ linearly independent integer points. Then at least one of these points belongs to $\mathbb{Z}^{n+1} \setminus L$. Hence $t N^{-1/n} > \omega(L)$ and $\mu_{l+1}(\xi, N) \geq \omega(L) N^{1/n} \to +\infty, \ N \to \infty$. The Corollary is proved.

Lemma 4. Let $2 \leq l \leq n + 1$. Let $s$ be the $l$-dimensional fundamental volume of a lattice $L$, $\dim L = l$. Suppose that a vector $\xi = (1, \xi_1, \ldots, \xi_n) \in \text{span} L$ and positive numbers $\gamma$ and $T \geq 1$ satisfy the equality

$$
C_\xi(T, \gamma T^{-1/(l-1)}) \cap L = \{0\}.
$$

Let

$$
\gamma^* = \gamma^*(\gamma, L) = \min \left(3^{-2} \gamma, 3^{l-2} (2w_{l-1} l!)^{-1/(l-1)} s^{-1/(l-1)} \right).
$$

Then there exists an ($L, \gamma^*, T$)-BAD vector $\xi^*(1, \xi_1^*, \ldots, \xi_n^*) \in \text{span} L$, such that

$$
|\xi^* - \xi| < \gamma T^{-1/(l-1)}.
$$

Proof. Put $T_\nu = 3^{(l-1)\nu} T$, $\nu = 0, 1, 2, \ldots$. To prove Lemma 4 it suffices to construct a sequence of cylinders

$$
C^{(\nu)} = C_{\xi^{(\nu)}}(T_\nu, 9\gamma^* T_\nu^{-1/(l-1)}), \ \xi^{(\nu)} \in \text{span} L
$$

such that

(i) for every $\nu$ we have $C^{(\nu)} \cap L = \{0\}$;

(ii) the section $\{x = T_\nu\} \cap C^{(\nu+1)}$ of the cylinder $C^{(\nu+1)}$ lies inside the facet $\{x = T_\nu\} \cap \overline{C^{(\nu)}}$ of the preceding cylinder $C^{(\nu)}$; moreover, the distance between the centers of $B$ and $B'$ does not exceed $3\gamma^* T_\nu^{-1/(l-1)}$.

If such cylinders $C^{(\nu)}$ are constructed and $C^{(0)} = C_\xi(T, \gamma T^{-1/(l-1)})$ then the vector $\xi^* = \lim_{\nu \to +\infty} \xi^{(\nu)}$ satisfies (16). Moreover, it follows from (ii) that $\xi^*$ is an ($L, \gamma^*, T$)-BAD vector. Indeed, for an integer point $\zeta = (q, a_1, \ldots, a_n)$ with $T_\nu \leq q \leq T_{\nu+1} = 3^{l-1} T_\nu$ we have

$$
|q \xi^* - \zeta| \geq 3\gamma^* T_{\nu+1}^{-1/(l-1)} \geq \gamma^* q^{-1/(l-1)}.
$$
We now describe the inductive process, which constructs the sequence of cylinders $C^{(\nu)}$. Suppose that $C^{(\nu)}$ is already constructed. Consider the cylinder

$$C' = C_{\zeta^{(\nu)}}(T_{\nu+1}, 3^{l+1}\gamma^*T_{\nu}^{-(l-1)})$$

We prove that there exists a linear subspace $L \subset \text{span } L$ of dimension $\dim L = l - 1$ containing all the integer points $\zeta \in L \cap C'$. Suppose that there are $l$ linearly independent integer points

$$\zeta^{(1)}, \ldots, \zeta^{(l)} \in \Gamma \cap C'.$$

Then the $l$-dimensional volume $V$ of the convex hull $\text{conv } (0, \zeta^{(1)}, \ldots, \zeta^{(l)})$ is bounded from below by the fundamental volume of $L$:

$$V \geq s(l)!^{-1}.$$  \hfill (17)

On another hand, the volume of $\text{conv } (0, \zeta^{(1)}, \ldots, \zeta^{(l)})$ admits an upper bound based on the relation $\text{conv } (0, \zeta^{(1)}, \ldots, \zeta^{(l)}) \subset C'$. Taking into account (15) we see that

$$V \leq T_{\nu+1} \left(\frac{3^{l+1}\gamma^*T_{\nu}^{-(l-1)}}{l-1}\right)^{l-1} \leq s(2l)!^{-1}$$  \hfill (18)

Relations (17,18) contradict each other, which means that all the integer points from the cylinder under consideration lie in a subspace $L$.

Let $B$ be the $(l-1)$-dimensional facet $\{x = T_\nu\}$ of $C^{(\nu)} \cap \text{span } L$. In fact $B$ is an $(l-1)$-dimensional open ball of radius $3\gamma^*T_{\nu}^{-(l-1)}$ centered at $T_\nu \zeta^{(\nu)} \in \text{span } L$. There is an $(l-1)$-dimensional open ball $B' \subset B$ of radius $\gamma^*T_{\nu+1}^{-(l-1)} = 3\gamma^*T_{\nu}^{-(l-1)}$ and centered at a certain point of the affine subspace $\{x = T_\nu\} \cap \text{span } L$, such that $B' \cap L = \emptyset$ and the point $T_\nu \zeta^{(\nu)}$ lies on the boundary of $B'$. Let $(T_\nu, \Xi_1, \ldots, \Xi_n)$ be the center of $B'$. Put

$$\zeta^{(\nu+1)} = \left(1, \frac{\Xi_1}{T_\nu}, \ldots, \frac{\Xi_n}{T_\nu}\right) \in \text{span } L.$$

As $C^{(\nu+1)} \subset C'$, we see that there are no nonzero points of $L$ in $C^{(\nu+1)}$ and (i) is valid with $\nu$ replaced by $\nu + 1$. From the construction we see that (ii) is also valid for $\nu + 1$. Lemma 4 is proved.

Remark 1. Lemma 4 is obtained by well-known arguments (see [2], Chapter 3, §2). The constant $3^{l-2}(2w_{l-1}l!)^{-1} s^{l-1}$ in (15) may be slightly improved but this is of no importance for the proof of our main result.

4 Lemmas concerning two sublattices

Lemma 5. Let $\Gamma \subset \mathbb{Z}^{n+1}$ be a complete lattice, such that $\dim \Gamma = k + 1 \geq 3$. Let $R$ be the $(k+1)$-dimensional fundamental volume of $\Gamma$. Let vector $\xi = (1, \xi_1, \ldots, \xi_n) \in \text{span } \Gamma$, $\xi_j \in (0, 1)$, be $(\Gamma, \gamma, W)$-BAD. Consider a positive number $\kappa$, such that

$$\kappa \leq \gamma W^{-\frac{k+1}{k}}.$$ \hfill (19)

Then there is a sublattice $\Lambda \subset \Gamma$, $\dim \Lambda = k$ satisfying the following two conditions:

1) the Euclidean distance from $\xi \in \mathbb{R}^{n+1}$ to $\text{span } \Lambda \cap \{x = 1\}$ does not exceed $\kappa$;

2) the $k$-dimensional fundamental volume $r$ of $\Lambda$ admits the following upper bound:

$$r \leq G(\gamma, \Gamma)\kappa^{-\frac{1}{k+1}},$$
where
\[ G(\gamma, \Gamma) = 2^{2k^2}w_k^{-k}w_{k-1}k!\gamma^{-\frac{k}{k+1}}R^k. \]

Proof. We take
\[ M = (\gamma_{k-1})^{\frac{k}{k+1}}, \quad \delta = M\kappa. \]

By (19) we have
\[ (M\gamma\delta^{-1})^{\frac{k}{k+1}} = (\gamma_{k-1})^{\frac{k}{k+1}} \geq W. \]

We now can apply the statement 2) of Lemma 3 for \( l = k + 1, L = \Gamma \). Then (9) gives the inequality
\[ \mu = \mu_k(C_\varepsilon(M, \delta)) \leq \mu_{k+1}(C_\varepsilon(M, \delta)) \leq 2^{2k}w_k^{-1}R\gamma^{-k}. \]

We see now that the cylinder \( \mu C_\varepsilon(M, \delta) \) has \( k \) linearly independent integer points \( \zeta_1, \ldots, \zeta_k \). Define \( \Lambda = \text{span}(\zeta_1, \ldots, \zeta_k) \cap \mathbb{Z}^{n+1} \). The remark after Lemma 3 shows that the condition 1) is satisfied. Let us obtain the needed upper bound for the fundamental volume \( r \) of \( \Lambda \). As
\[ \text{conv}(0, \zeta_1, \ldots, \zeta_k) \subset \text{span}\Lambda \bigcap \mu C_\varepsilon(M, \delta) \]
(here \( \text{conv}(0, \zeta_1, \ldots, \zeta_k) \) stands for the convex hull of the points \( 0, \zeta_1, \ldots, \zeta_k \in \mathbb{R}^{n+1} \)) we see that
\[ r \leq k!\mu^k w_{k-1} \delta^{k-1} M, \]
and the required upper bound follows from (21). Lemma is proved.

Let \( \Gamma \) be a sublattice as in Lemma 5. For each lattice \( \Gamma \) and for every \( \gamma \) small enough there exist \((\Gamma, \gamma, W)\)-BAD vectors (see Example 1). Consider a \((\Gamma, \gamma, W)\)-BAD vector \( \xi = (1, \zeta_1, \ldots, \zeta_n) \in \text{span} \Gamma \). Then for any \( T \geq W \) the cylinder \( C_\xi(T, \gamma T^{-1/k}) \) contains no nonzero points of \( \Gamma \). Let \( \mathcal{B} \) be the facet \( \{x = T\} \) of the cylinder \( C_\xi(T, \gamma T^{-1/k}) \) \( \cap \text{span} \Gamma \). This facet is a \( k \)-dimensional ball of radius \( \gamma T^{-1/k} \) centered at \( T\xi \). Lemma 5 with
\[ \kappa = \frac{\gamma}{4n} T^{-\frac{k+1}{k}} \]
implies that there is a \( k \)-dimensional complete sublattice \( \Lambda \subset \Gamma \) with \( k \)-dimensional fundamental volume \( r \) satisfying the condition
\[ r \leq G(\gamma, \Gamma)(4n\gamma^{-1})^{\frac{1}{k+1}} T_{\frac{1}{k}}, \]
and such that the intersection of \( \text{span} \Lambda \) with \( \mathcal{B} \) is a \((k - 1)\)-dimensional ball \( \mathcal{B}' \subset \mathcal{B} \) with the center \( \Xi' \) and radius \( \geq \gamma T^{-1/k} / 2 \). Take another \( k - 1 \)-dimensional ball \( \mathcal{B}'' \subset \mathcal{B}' \) of radius \( 2^{-3}\gamma T^{-1/k} \) centered at \( \Xi' \). Then the distance from \( \mathcal{B}'' \) to the boundary of \( \mathcal{B} \) is greater than \( 2^{-3}\gamma T^{-1/k} \). Put \( \xi' = \Xi' / T \). Then the cylinder
\[ C_{\xi'}(T, 2^{-3}\gamma T^{-1/k}) \cap \text{span} \Lambda = C_{\xi'}(T, \gamma' T^{-1/(k-1)}) \cap \text{span} \Lambda, \quad \gamma' = 2^{-3}\gamma T^{-\frac{1}{k(k-1)}} \]
contains no nonzero points of \( \Lambda \) and we can apply Lemma 2 with \( l = k \) and \( L = \Lambda \). Thus we obtain a \((\Lambda, \dot{\gamma}, T)\)-BAD vector \( \xi' \) with
\[ \dot{\gamma} = \dot{\gamma}(\gamma, T, \Lambda) = \gamma^* (2^{-3}\gamma T^{-\frac{1}{k(k-1)}}), \Lambda). \]

As the cylinder (23) has no nonzero points of \( \Lambda \) we see from the Minkowski theorem on convex bodies that
\[ 2T \cdot w_{k-1} \left( 2^{-3}\gamma T^{-\frac{1}{k}} \right)^{k-1} < 2^k r. \]
\[ r > 2^{-2(k-1)}T^\frac{1}{k}w_{k-1}^k. \]  

(25)

Note that from (24,15,25) we see that

\[ \hat{\gamma} \geq C(k)\gamma T^{\frac{1}{k(k-1)}}, \]

where

\[ C(k) = \min \left( 3^{-5}, 2^{-1-\frac{1}{k-1}}3^{-k-1}(k!)^{-\frac{1}{k-1}} \right). \]  

(27)

Now we put

\[ Z_1(\gamma, k) = (C(k)\gamma)^{\frac{1}{n+1-k}}, \]

(28)

\[ Z_2(i, \gamma, \Gamma) = \left( i \cdot 2^{2(k-1)}w_{k-1}^{-1}G(\gamma, \Gamma)(4n\gamma^{-1})^{\frac{1}{n+1}}(C(k)\gamma)^{-\frac{(k-1)^2}{n+1-k}} \right)^{\frac{nk}{n+1-k}}. \]  

(29)

Lemma 6. For the vector \( \hat{\xi} \) defined above and for \( N \) under the condition

\[ N \geq Z(i, \gamma, \Gamma, T) = \max(Z_1(\gamma, k)T^{\frac{n(k+1)}{n+1-k}}, Z_2(i, \gamma, \Gamma)T^{\frac{n}{k(n+1-k)}}) \]  

(30)

we have

\[ \mu_k(\hat{\xi}, N) \leq i^{-1}. \]

Proof. As \( N \geq Z_1(\gamma, k)T^{\frac{n(k+1)}{n+1-k}} \), we see from (28,27,26) that the condition (12) of the case 2) of Corollary 1 to Lemma 3 is satisfied. Then due to (13) we have

\[ \mu_k(\hat{\xi}, N) \leq 2^{2(k-1)}w_{k-1}^{-1}r\hat{\gamma}^{-\frac{(k-1)^2}{k}}N^{k-n-1nk}. \]

It remains to make use of the inequality \( N \geq Z_2(i, \gamma, \Gamma)T^{\frac{n}{k(n+1-k)}} \) and of the formulas (29,25,26,27). Lemma 6 follows.

We shall need the following notation related to a pair of sublattices.

Let \( \Lambda \subset \Gamma \in \mathbb{Z}^{n+1} \) be complete sublattices such that

\[ \dim \Gamma \geq \dim \Lambda. \]

Then \( \Gamma \) can be partitioned into classes \( (\mod \Lambda) \):

\[ \Gamma = \bigcup_{\alpha \in \mathbb{Z}^v} \Gamma_\alpha, \quad \Gamma_0 = \Lambda, \quad v = \dim (\text{span} \Gamma) - \dim (\text{span} \Lambda), \]

so that the affine subspaces \( \text{aff} \Gamma_\alpha \) are parallel \( \text{span} \Lambda = \text{aff} \Gamma_0 \). Here by \( \text{aff} \Omega \) we mean the smallest affine subspace of \( \mathbb{R}^{n+1} \) containing \( \Omega \).

Denote by \( R = R(\Lambda, \Gamma) > 0 \) the minimal distance between points \( z^{(1)}, z^{(2)} \), where \( z^{(1)} \in \Gamma \setminus \Lambda \) and \( z^{(2)} \in \text{span} \Lambda \). For our purpose we need not the \( R(\Lambda, \Gamma) \) itself but a little bit different distance \( \rho = \rho(\Lambda, \Gamma) \) which we define now. Recall that as it was pointed out in the very beginning of Section 3 all \( k \)-dimensional lattices \( \Lambda \) under the consideration admit the property

\[ \dim (\Lambda \cap \mathcal{P}) = k - 1 \]

(here \( \mathcal{P} = \{ z = (x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : x = 1 \} \)). For such a lattice \( \Lambda \) and for a lattice \( \Gamma \supset \Lambda \) of dimension \( \dim \Gamma \geq \dim \Lambda \) we consider the following objects. Put \( \mathcal{L} = \text{span} \Lambda \cap \mathcal{P} \). Let \( \mathcal{G} \) be the parallel
projection of $\Gamma \setminus \Lambda$ along $\text{span} \Lambda$ onto $\mathcal{P}$. Then we define $\rho = \rho(\Lambda, \Gamma)$ to be the minimal distance between points $z^{(1)}, z^{(2)}$, where $z^{(1)} \in \mathcal{G}$ and $z^{(2)} \in \mathcal{L} = \text{span} \Lambda \cap \mathcal{P}$. We see that $\rho = \rho(\Lambda, \Gamma) > 0$. In fact, $\rho(\Lambda, \Gamma) \geq \rho(\Lambda, \mathbb{Z}^{n+1}) > 0$.

Now we give two more lemmas.

**Lemma 7.** Let $\Lambda \subset \Gamma \subset \mathbb{Z}^{n+1}$ be complete sublattices, such that

$$\dim \Gamma = k + 1 = \dim \Lambda + 1,$$

and $\rho = \rho(\Lambda, \Gamma)$. Let $\xi = (1, \xi_1, ..., \xi_n) \in \text{span} \Lambda$ be a $(\Lambda, \gamma, W)$-BAD vector with some positive $\gamma$ and $W \geq 1$. Put

$$\gamma' = \gamma'(\gamma, \Lambda, \Gamma) = \gamma^{\frac{k+1}{k-1}} \frac{2}{\gamma} \rho^{\frac{1}{k-1}}$$

and

$$A_1 = A_1(\gamma, \Lambda, \Gamma) = \max \left( (\rho(2\gamma')^{-1} \frac{k}{k-1}, (2\gamma' \rho^{-1})^k \right).$$

Suppose that

$$T \geq A_1 W^{\frac{1}{k-1}} \geq \max \left( (\rho(2\gamma')^{-1} W)^{\frac{1}{k-1}}, (2\gamma' \rho^{-1})^k \right).$$

Let $\xi' = (1, \xi_1', ..., \xi_n') \in \text{span} \Gamma$ satisfy the following two conditions:

1) the orthogonal projection of vector $\xi'$ on the subspace $\text{span} \Lambda$ is of the form $\lambda \xi$ with some positive $\lambda$;

2) for the Euclidean norm we have $|\xi' - \xi| = (2T)^{-1} \rho$.

Then

$$C_{\xi'}(T, T^{1/k}) \cap \Gamma = \{0\}.$$

Proof. The $(k + 1)$-dimensional linear subspace $\text{span} \Gamma$ contains parallel $k$-dimensional affine subspaces $\text{aff} \Gamma_i, i \in \mathbb{Z}$. Each such subspace $\text{aff} \Gamma_i$ divides the subspace $\text{span} \Gamma$ into two “half-subspaces” with the common boundary $\text{aff} \Gamma_i$. The situation with the $k$-dimensional affine subspace $\text{span} \Gamma \cap \{x = T\}$ and $(k - 1)$-dimensional affine subspaces $\text{aff} \Gamma_i \cap \{x = T\}, i \in \mathbb{Z}$ is quite similar. We should note that the Euclidean distance between neighboring subspaces $\text{aff} \Gamma_i \cap \{x = T\}$ and $\text{aff} \Gamma_i \cap \{x = T\}$ is exactly $\rho$.

Without loss of generality we may suppose that the point $T \xi' \in \text{span} \Gamma \cap \{x = T\}$ lies in the same “half-subspace” (with respect to $\text{aff} \Gamma_0 \cap \{x = T\} = \text{span} \Lambda \cap \{x = T\}$) as the set $\text{span} \Gamma_1 \cap \{x = T\}$. From the conditions 1), 2) we see that the distance from the point $T \xi'$ to the subspace $\text{aff} \Gamma_0 \cap \{x = T\}$ is equal to the distance from $T \xi'$ to the subspace $\text{aff} \Gamma_1 \cap \{x = T\}$ and is equal to $\rho/2$.

Define $H = 2\gamma' \rho^{-1} T^{\frac{k+1}{k}}$. Then by definition of $H$ and (33), we have $H \leq T$. Note that the distance from each point of the form $t \xi, H \leq t \leq T$ to the corresponding affine subspace $\text{aff} \Gamma_1 \cap \{x = t\}$ is greater than $\gamma' T^{-1/k}$. So

$$C_{\xi'}(T, T^{1/k}) \cap \{z = (x, y_1, ..., y_n) : |x| \geq H\} \cap \text{span} \Lambda = \emptyset.$$

It means that the cylinder $C_{\xi'}(T, T^{1/k})$ intersected with the domain $\{H \leq x \leq T\}$ has no points of the lattice $\Lambda$. But the distance from each point of the form $t \xi, H \leq t \leq T$ to the corresponding affine subspace $\text{aff} \Gamma_1 \cap \{x = t\}$ is greater than the distance from $t \xi$ to $\text{span} \Lambda \cap \{x = t\}$. So the distance from each point of the form $t \xi, H \leq t \leq T$ to the any affine subspace $\text{aff} \Gamma_i \cap \{x = t\}, i \neq 0$ is greater than $\gamma' T^{-1/k}$ also. So

$$C_{\xi'}(T, T^{1/k}) \cap \{z = (x, y_1, ..., y_n) : |x| \geq H\} \cap \Gamma = \emptyset.$$
From the other hand if $0 \leq t \leq H$ then the distance from $t\xi$ to any aff $\Gamma_i \cap \{x = t\}, i \neq 0$ is again greater than $\gamma'T^{-1/k}$. Hence
\[ C_{\xi'}(T, \gamma'T^{-1/k}) \bigcap \Gamma = C_{\xi'}(H, \gamma'T^{-1/k}) \bigcap \Lambda. \]
But (31) implies that $\gamma'T^{-1/k} = \gamma H^{-(k-1)}$. As $\xi$ is a $(\Lambda, \gamma, W)$-BAD vector we see that
\[ C_{\xi'}(H, \gamma'T^{-1/k}) \bigcap \Lambda \subseteq C_{\xi'}(H, \gamma H^{-(k-1)}) \bigcap \Lambda = \{0\}. \]
(Note that from (33) it follows that $H \geq W$.) Lemma 7 is proved.

**Lemma 8.** In the notation of Lemma 5, let $r$ be the $k$-dimensional fundamental volume of the lattice $\Lambda$, vector $\xi'$ be defined in Lemma 7 and let $\xi'' = (1, \xi''_1, ..., \xi''_n) \in \text{span} \Gamma$ be a vector satisfying
\[ |\xi'' - \xi'| < \rho(4T)^{-1}. \] (34)

Set
\[ A_2 = A_2(\gamma, \Lambda, \Gamma) = 3\rho\gamma^{-1/4}, \] (35)
\[ B_1 = B_1(\gamma) = (\sqrt{2\gamma})^{-\frac{k}{k-1}}, \quad B_2 = B_2(\Lambda, \Gamma) = \left(\frac{2\sqrt{2}}{3\rho}\right)^{\frac{n}{n+1}}, \] (36)
\[ C_1 = C_1(\gamma, \Lambda) = 2^{2k-2}w_{k-1}^{-1}r\gamma^{1-k}, \quad C_2 = C_2(\gamma, \Lambda) = 2^{\frac{k^2-k}{2k} - \frac{k}{k-1}}w_{k-1}^{-1}r\gamma^{-\frac{(k-1)^2}{k}}, \] (37)
\[ C_3 = C_3(\gamma, \Lambda, \Gamma) = 2^{\frac{k^2-k}{2k} - \frac{k}{k-1}}3^{\frac{3k}{2}}w_{k-1}^{-1}r\gamma^{-\frac{(k-1)^2}{k}}\rho^{\frac{k}{4}}. \]
Suppose that
\[ T \geq A_2W^{\frac{k}{k-1}}. \] (38)
Then the following statements are valid:
1) for $N$ in the interval
\[ N \leq B_1W^{\frac{k}{(k-1)(n+1)}} \] (39)
we have
\[ \mu_k(N, \xi'') \leq C_1WN^{-1}; \] (40)
2) for $N$ in the interval
\[ B_1W^{\frac{k}{(k-1)(n+1)}} \leq N \leq B_2T^{\frac{n}{n+1}} \] (41)
we have
\[ \mu_k(N, \xi'') \leq C_2N^{\frac{k-n-1}{nk}}; \] (42)
3) for $N$ in the interval
\[ N \geq B_2T^{\frac{n}{n+1}} \] (43)
we have
\[ \mu_k(N, \xi'') \leq C_3T^{-\frac{n}{k}}N^{\frac{1}{2}}. \] (44)

**Corollary.** Under the conditions of Lemma 8, for $N$ in the interval
\[ H(i, \gamma, \Lambda, W) = \max \left( (C_1(\gamma, \Lambda)iW), (C_2(\gamma, \Lambda)i)\frac{n}{n+1} \right) \leq N \leq (iC_3(\gamma, \Lambda, \Gamma))^{-n}T^\frac{n}{n+1} \] (45)
we have the following inequality:
\[ \mu_k(N, \xi'') \leq i^{-1}. \] (46)
Proof of Lemma 8. First of all, let us consider the case 3).

Set

\[ M = B_2^{n+1} T N^{-\frac{1}{2}} \leq N, \ \delta = N^{-\frac{1}{2}}/\sqrt{2}. \]

It follows from the definition of \( \xi' \) and (34) that

\[ C_{\xi''}(N, N^{-1/n}) \supset C_{\xi}(M, \delta) \cap \text{span} \Lambda. \] (47)

Hence

\[ \mu_k(N, \xi'') \leq \mu_k(C_{\xi}(M, \delta)), \]

so it is suffices to obtain the corresponding upper bound for the latter successive minimum. We observe that by (38) we have

\[ (M\delta^{-1}\gamma)^\frac{k+1}{k} = \left( \frac{4}{3} T \rho^{-1}\gamma \right)^\frac{k+1}{k} \geq W. \]

Applying the statement 2) of Lemma 3 we obtain (9) with \( l = k, s = r \). Now (44) follows from (9).

Consider the case 2). Since \( M \geq N \), the relation (47) may be false, so we have

\[ C_{\xi''}(N, N^{-1/n}) \supset C_{\xi}(N, \delta) \cap \text{span} \Lambda. \] (48)

Hence

\[ \mu_k(N, \xi'') \leq \mu_k(C_{\xi}(N, \delta)), \]

It follows from (41) that

\[ (N\delta^{-1}\gamma)^\frac{k+1}{k} \geq W. \]

Let us apply the statement 2) of Lemma 3 for the cylinder \( C_{\xi}(N, \delta) \) from (48). Then the conclusion (9) of Lemma 3 with our parameters leads to (42).

Finally, we consider the case 1). Again, we have \( M \geq N \). So we must use the relation (48). But (39) implies that

\[ (N\delta^{-1}\gamma)^\frac{k+1}{k} \leq W. \]

Applying the statement 1) of Lemma 3 we get from (7) the desired inequality (40).

Lemma 8 is proved.

5 Proof of Theorem 1: general case

Now we are able to give a proof of Theorem 1 in the case \( k \geq 2 \). We begin with the same consideration of the countable set of all the \( n \)-dimensional complete sublattices of the integer lattice \( \mathbb{Z}^{n+1} \). We fix an enumeration of this set and let

\[ L_1, L_2, ..., L_i, ... \]

be all these lattices. Set \( \pi_i = \text{span} L_i \). Suppose that

\[ \pi_1 = \{ z = (x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : x = 0 \}. \]

Let \( 2 \leq k \leq n - 1 \). We construct a sequence of real numbers

\[ \eta_1 > \eta_2 > ... > \eta_i > ... \]
decreasing to zero, a sequence of positive real numbers

\[ \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots, \]

two sequences of real numbers

\[ W_1, W_2, \ldots, W_i, \ldots, \]
\[ H_1, H_2, \ldots, H_i, \ldots, \]

\[ W_i \geq 1, \quad W_i, H_i \to +\infty, \quad i \to +\infty, \]
two sequences of complete sublattices

\[ \Lambda_1, \Lambda_2, \ldots, \Lambda_{i-1}, \Lambda_i, \ldots, \]
\[ \Gamma_2, \Gamma_3, \ldots, \Gamma_i, \Gamma_{i+1}, \ldots, \]

and a sequence of vectors

\[ \xi_i = (1, \xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{R}^{n+1} \]
satisfying the following conditions (i) – (vii). Further, suppose \( r_i \) be the \( k \)-dimensional fundamental volume of \( \Lambda_i \) and let \( R_i \) be the \( (k + 1) \)-dimensional fundamental volume of \( \Gamma_i \).

(i) For every \( i \in \mathbb{N} \) we have

\[ \Lambda_i \subset \mathbb{Z}^{n+1}, \quad \dim \Lambda_i = k; \]
\[ \Gamma_{i+1} \subset \mathbb{Z}^{n+1}, \quad \dim \Gamma_{i+1} = k + 1; \]
\[ \Lambda_i, \Lambda_{i+1} \subset \Gamma_{i+1}. \]

(ii) For every \( i \in \mathbb{N} \) the vector \( \xi_i \) is \((\Lambda_i, \gamma_i, W_i)\)-BAD.

(iii) The \( n \)-dimensional closed ball \( \overline{B}_i \subset \{ z = (x, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} : \ x = 1 \} \) of radius \( \eta_i \) is centered at \( \xi_i \) and has no common points with \( \pi_i \).

(iv) The balls defined in (iii) form a nested sequence

\[ \overline{B}_1 \supset \overline{B}_2 \supset \ldots \supset \overline{B}_i. \]

(v) For every \( i \geq 2 \) the following inequality holds:

\[ H_i \geq \max \left( H(2(i + 1), \gamma_i, \Lambda_i, W_i), \frac{4(i + 1)}{\rho(\Lambda_i, \mathbb{Z}^{n+1})} \right) \quad (49) \]

(here the value of \( \rho(\cdot, \cdot) \) for two lattices is defined in Section 4 before Lemma 7 and \( H(\cdot, \cdot, \cdot, \cdot) \) is defined in [15]).

(vi) For every \( i \geq 2 \), every \( \xi \in \overline{B}_i \) and for every real \( N \) in the interval \( H_{i-1} \leq N < H_i \) one has

\[ \mu_k(\xi, N) \leq i^{-1}. \]

(vii) For every \( i \geq 2 \), every \( \xi \in \overline{B}_i \) and every real \( N \) in the interval \( H_{i-1}^n \leq N < H_i^n \) one has

\[ \mu_{k+2}(\xi, N) \geq i. \]

Suppose that all these objects are already constructed. Then we have Theorem 1 proved in the case \( k \geq 2 \). Indeed, if we consider the unique vector \( \xi = (1, \xi_1, \ldots, \xi_n) \) from the intersection \( \cap_{i \in \mathbb{N}} \overline{B}_i \), then the components \( 1, \xi_1, \ldots, \xi_n \) are linearly independent over \( \mathbb{Z} \) due to (iii), and

\[ \lim_{N \to +\infty} \mu_k(\xi, N) = 0, \quad \lim_{N \to +\infty} \mu_{k+2}(\xi, N) = +\infty \]
We also need one more quantity $E$. Take $\xi_1$). We do not define $\Gamma_C$ where due to (vi) and (vii).

First of all, put $W_1 = H_1 = 1$,

$$\Lambda_1 = \mathbb{Z}^{n+1} \cap \{z = (x, y_1, ..., y_n) \in \mathbb{R}^{n+1}: y_k = ... = y_n = 0\},$$

Take $\xi$ to be a $(\Lambda, \gamma_1, 1)$-BAD vector with some positive $\gamma_1$ (we can take such a vector from Example 1). We do not define $\Gamma_1$.

Obviously, $\rho(\Lambda_1, \mathbb{Z}^{n+1}) = 1$. The conditions (i) – (vii) for $i = 1$ are satisfied (note that the conditions (v) – (vii) are empty).

Assume that all the objects $\eta_i, \gamma_i, W_i, H_i, \Lambda_i, \Gamma_i, \xi_i$ for every natural $i$ up to $t$ are constructed to satisfy the conditions (i) – (vii). Let us describe the construction for $i = t + 1$.

Consider the integer vector $q \in \mathbb{Z}^n$ orthogonal to the subspace $\pi_{t+1}$. For any small positive $\varepsilon$ there exists an integer vector $q'$ such that the angle between $q$ and $q'$ is less than $\varepsilon$ and $q' \notin \Lambda_t$. Put

$$\Gamma_{t+1} = \mathbb{Z}^{n+1} \cap \text{span} (\Lambda_t \cup q').$$

Then $\Gamma_{t+1}$ is a complete sublattice of dimension $\dim \Gamma_{t+1} = k + 1$, $\Gamma_{t+1} \supset \Lambda_t$, and \[ \text{span} \Gamma_{t+1} \not\subset \pi_{t+1}. \]

Moreover for the vector $q' \in \Gamma_{t+1}$ and any nonzero vector $p \in \pi_{t+1}$ the angle between $p$ and $q'$ is greater than $\frac{\pi}{2} - \varepsilon$.

Let $R_{t+1}$ be the $(k + 1)$-dimensional fundamental volume of $\Gamma_{t+1}$. Set

$$\rho_t^{(1)} = \rho(\Lambda_t, \mathbb{Z}^{n+1}), \quad \rho_t^{(2)} = \rho(\Lambda_t, \Gamma_{t+1}), \quad \rho_t^{(3)} = \rho(\Gamma_{t+1}, \mathbb{Z}^{n+1}).$$

Set

$$E_j(t) = A_j(\gamma_t, \Lambda_t, \Gamma_{t+1})W_t^{k+1}, \quad j = 1, 2,$$

where the right hand sides are defined by (32,35), and set

$$E_3(t) = \frac{3\rho_t^{(2)}}{4\eta_k}, \quad E_4(t) = \frac{2^{n+3}(t + 1)^{n+1}\rho_t^{(2)}}{\rho_t^{(1)}(\rho_t^{(3)})^n}.$$  

We also need one more quantity $E_5(t)$ defined as follows. First, we put

$$Z_1(t) = Z_1(\gamma^*(\gamma_t^{k-1}2^{-\frac{k-1}{k}}(\rho_t^{(2)})^\frac{1}{k}, \Gamma_{t+1}), k),$$

$$Z_2(t) = Z_2(2(t + 1), \gamma^*(\gamma_t^{k-1}2^{-\frac{k-1}{k}}(\rho_t^{(2)})^\frac{1}{k}, \Gamma_{t+1}), \Gamma_{t+1}),$$

where $Z_1(\cdot, \cdot)$, $Z_2(\cdot, \cdot, \cdot)$ are defined by (28,29) and $\gamma^*(\cdot, \cdot)$ is defined by (15). Then we put

$$E_5(t) = \max \left( (Z_1(t))^{\frac{(n+1)k}{n(n-k)}}(2(t + 1)C_3(\gamma_t, \Lambda_t, \Gamma_{t+1}))^{\frac{(n+1)k}{n(n-k)}}, (Z_2(t))^{\frac{(n+1)k}{n(n-k)}}(2(t + 1)C_3(\gamma_t, \Lambda_t, \Gamma_{t+1}))^{\frac{(n+1)k}{n(n-k)}} \right).$$

where $C_3(\cdot, \cdot, \cdot)$ is defined by (37). Note that as $k \leq n - 1$ all the exponents are positive (particulary, $n - k \geq 1$ and all the denominators in the exponents are nonzero).

Put

$$T_t = \max_{1 \leq j \leq 5} E_j(t).$$

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Since \( T_t \geq E_1(t), E_2(t) \), we can apply Lemmas 7,8 to the lattices \( \Lambda = \Lambda_t, \Gamma = \Gamma_{t+1} \). Denote by \( \xi_t' \) the real \((n+1)\)-dimensional vector satisfying the conditions 1), 2) of Lemma 7. Consider the ball 
\[
\mathcal{B}_t' = \{ \xi = (1, \xi_1, ..., \xi_n) : |\xi - \xi_t'| < \rho_t^{(2)}(4T_t)^{-1} \}.
\]
Note that for \( \xi \in \mathcal{B}_t' \) we have \( |\xi - \xi_t| < \frac{\rho_t^{(2)}(2T_t)}{4} \). Since \( T_t \geq E_3(t) \), we have
\[
\mathcal{B}_t' \subset \mathcal{B}_t.
\]
Note that
\[
H_t \geq H(2(t+1), \gamma_t, \Lambda_t, W_t)
\]
by the inductive conjecture (v). So by Corollary of Lemma 8 we see that for every \( N \) in the interval \( H_t \leq N \leq H_t' \), where
\[
H_t' = (2(t+1)C_3(\gamma_t, \Lambda_t, \Gamma_{t+1}))^{-n} T_t^{\frac{n}{2}},
\]
and every \( \xi \in \mathcal{B}_t' \cap \text{span} \Gamma_{t+1} \) we have
\[
\mu_k(\xi, N) \leq 2(t+1)^{-1}.
\]
Let us prove that for any \( N \geq H_t^n \) and for any \( \xi \in \mathcal{B}_t' \cap \text{span} \Gamma_{t+1} \) we have
\[
\mu_{k+2}(\xi, N) \geq 2(t+1).
\]
To do this let us put
\[
U = T_t \cdot \frac{\rho_t^{(1)}}{3(t+1)\rho_t^{(2)}}.
\]
Recall that
\[
|\xi_t - \xi_t'| = \frac{\rho_t^{(2)}}{2T_t}
\]
and \( \xi_t \in \text{span} \Lambda_t \). So for every \( N \) in the interval \( H_t^n \leq N \leq U \) we have
\[
|N\xi - N\xi_t| \leq U \cdot \frac{3\rho_t^{(2)}}{4T_t} \leq \frac{\rho_t^{(1)}}{4(t+1)},
\]
and so the distance between \( N\xi \) and \( \text{span} \Lambda \) does not exceed \( \frac{\rho_t^{(1)}}{4(t+1)} \). But it follows from the condition (v) of the \( t \)-th step that for the considered values of \( N \) we have
\[
N^{-\frac{1}{n}} \leq H_t^{-1} \leq \frac{\rho_t^{(1)}}{4(t+1)}.
\]
We see that the maximal distance between a point of the cylinder \( C_\xi(N, N^{-1/n}) \) and the subspace \( \text{span} \Lambda_t \) is \( \leq \frac{\rho_t^{(1)}}{2(t+1)} \). Recall that \( \dim \Lambda_t = k \). Hence the cylinder \( 2(t+1) \cdot C_\xi(N, N^{-1/n}) \) cannot contain \( k+1 \) linearly independent integer points inside for \( H_t^n \leq N \leq U \), so in this case we have the inequality
\[
\mu_{k+1}(\xi, N) \geq 2(t+1)
\]
(and thus, the inequality (51)).
Suppose that \( N \geq U \). Then we deduce from the inequality \( T_t \geq E_4(t) \) that

\[
N^{-\frac{1}{n}} \leq U^{-\frac{1}{n}} \leq \frac{\rho_t^{(3)}}{2(t+1)}.
\]

So the distance between a point of \( C_{\xi}(N, N^{-1/n}) \) and the linear subspace \( \text{span} \Gamma_{t+1} \) is \( \leq \frac{\rho_t^{(3)}}{2(t+1)} \). Hence the cylinder \( 2(t+1) \cdot C_{\xi}(N, N^{-1/n}) \) cannot contain \( k+2 \) linearly independent integer points inside. This implies \( \gamma^{(3)} \) in the case \( N \geq U \).

We have proved the following statement: for any \( \xi \in B_t' \cap \text{span} \Gamma_{t+1} \) we have

\[
\mu_{k+2}(\xi, N) \geq 2(t+1), \quad N \geq H_t^n,
\]

\[
\mu_k(\xi, N) \leq (2(t+1))^{-1}, \quad H_t \leq N \leq H_t',
\]

where \( H_t' \) is defined by \( \gamma^{(3)} \). Moreover, if

\[
\gamma_t' = \gamma'(\gamma_t, \Lambda_t, \Gamma_{t+1}),
\]

where \( \gamma'(\cdot, \cdot, \cdot) \) is defined by \( \gamma^{(3)} \) then it follows from Lemma 7 that the cylinder

\[
C_{\xi_{t}'}(T_t, \gamma_t'T_t^{1/k})
\]

contains no nonzero points of \( \Gamma_{t+1} \). Recall that we constructed \( \Gamma_{t+1} \) to satisfy the condition \( \text{span} \Gamma_{t+1} \not\subset \pi_{t+1} \). Moreover for the vector \( q' \in \Gamma_{t+1} \) and any nonzero vector \( p \in \pi_{t+1} \) the angle between \( p \) and \( q' \) is greater than \( \frac{\pi}{2} - \varepsilon \). We can find a \( k \)-dimensional ball \( B'' \) of radius \( \gamma_t'T_t^{-1/k}/2 \) inside the facet \( \{x = T_t\} \cap \text{span} \Gamma_{t+1} \) of the cylinder \( C_{\xi_{t}'}(T_t, \gamma_t'T_t^{-1/k}) \cap \text{span} \Gamma_{t+1} \) (in fact, this facet is a \( k \)-dimensional ball of radius \( \gamma_t'T_t^{-1/k} \)) such that \( B'' \cap (\text{span} \Gamma_{t+1} + \pi_{t+1} + \{x = T_t\}) = \emptyset \). Let \( \Xi'' \) be the center of \( B'' \). Put \( \xi''_t = \Xi''/T_t \). Then \( \xi''_t \) is inside \( \text{span} \Gamma_{t+1} \). From the construction we see that \( n \)-dimensional ball with the center at point \( \Xi'' \) and radius \( \gamma_t'T_t^{-1/k}/4 \) has no common points with the subspace \( \pi_{t+1} \).

We get a cylinder

\[
C_{\xi''_{t}'}(T_t, \gamma''_tT_t^{-1/k}/4)
\]

with no nonzero points of \( \Gamma_{t+1} \) inside it and with \( \xi''_{t} \) inside \( \text{span} \Gamma_{t+1} \). By Lemma 4 we construct a \( (\Gamma_{t+1}, \gamma_t', T_t) \)-BAD vector \( \xi_{t}'' \in \text{span} \Gamma_{t+1} \) with

\[
\gamma_{t}'' = \gamma''(\gamma_t'/4, \Gamma_{t+1})
\]

(\( \gamma''(\cdot, \cdot, \cdot) \) defined by \( \gamma^{(3)} \)), such that the facet \( \{x = T_t\} \) of

\[
C_{\xi''_{t}'}(T_t, \gamma_{t}''T_t^{-1/k})
\]

lies inside the facet \( \{x = T_t\} \) of \( C_{\xi_{t}'}(T_t, \gamma_t'T_t^{1/k}) \) and does not intersect \( \pi_{t+1} \). Hence the ball

\[
B_t^* = \{\xi = (1, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1} : |\xi - \xi_{t}''| \leq \gamma_{t}''T_t^{-(k+1)/k}\}
\]

enjoys the following properties:

\[
B_t^* \subset B_t', \quad B_t^* \cap \pi_{t+1} = \emptyset.
\]

Recall that \( \xi_{t}^* \) is a \( (\Gamma_{t+1}, \gamma_{t}^*, T_t) \)-BAD vector. Applying Lemma 5 to the lattice \( \Gamma = \Gamma_{t+1}, \Gamma_{t+1}, \gamma_{t}^*, T_t \)-BAD vector \( \xi_{t}^* \) and \( \kappa = \gamma_{t}^*T_t^{-(k+1)/k}/(4n) \) we get a complete \( k \)-dimensional lattice \( \Lambda_{t+1} \) with fundamental volume

\[
r_{t+1} \leq G(\gamma_{t}^*, \Gamma_{t+1})(\gamma_{t}^*/4n)^{-\frac{1}{k-1}}(T_t)^{\frac{1}{4}}
\]

(55)
(here $G(\cdot, \cdot)$ is defined by (20)), such that $\Lambda_{t+1} \subset \Gamma_{t+1}$, and the Euclidean distance between $\xi_t^*$ and span $\Lambda_{t+1} \cap \{x = 1\}$ does not exceed $\gamma_t^*(T_t)^{-(k+1)/4n}$.

Next, we apply the construction described in Section 4 after Lemma 5 and obtain a $(\Lambda_{t+1}, \gamma_{t+1}, T_{t+1})$-BAD vector

$$\xi_{t+1} \in \text{span} \Lambda_{t+1}.$$ 

We set

$$W_{t+1} = T_t.$$ 

In the notation of Section 4 we have

$$\xi_{t+1} = \hat{\xi}_t^*,$$

$$\gamma_{t+1} = \hat{\gamma}(\gamma_t^*, T_t, \Lambda_{t+1}) = \gamma^*(2^{-3} \gamma_t^* T_t^{1/4n}, \Lambda_{t+1})$$

(here $\hat{\gamma}(\cdot, \cdot, \cdot)$ is defined by (24) and $\gamma^*(\cdot, \cdot)$ is defined by (15)).

Now we set

$$H_{t+1} = \max \left( Z(2(t+2), \gamma_t^*, \Gamma_{t+1}, T_t), H(2(t+1), \gamma_{t+1}, \Lambda_{t+1}, T_{t+1}), \frac{4(t+2)}{\rho(\Lambda_{t+1}, \mathcal{Z}^{n+1})} \right),$$

(56)

where $Z(\cdot, \cdot, \cdot, \cdot)$ is defined in (30) and $H(\cdot, \cdot, \cdot, \cdot)$ is defined in (45).

Note that due to (53) we have

$$\mu_k(\xi_{t+1}, N) \leq (2(t+1))^{-1}, \quad H_t \leq N \leq H_t'$$

$H_{t+1}$ may be greater than $H_t'$. But it follows from the inequality $T_t \geq E_5(t)$ and the definition (50) of $H_t'$ that

$$Z(2(t+1), \gamma_t^*, \Gamma_{t+1}, T_t) \leq H_t'.$$

Lemma 6 implies that for

$$N \geq Z(2(t+1), \gamma_t^*, \Gamma_{t+1}, T_t)$$

we have

$$\mu_k(\xi_{t+1}, N) \leq (2(t+1))^{-1}.$$ 

Hence

$$\mu_k(\xi_{t+1}, N) \leq (2(t+1))^{-1}, \quad H_t \leq N \leq H_{t+1}.$$

On the other hand, it follows from (52) that

$$\mu_{k+2}(\xi_{t+1}, N) \geq 2(t+1), \quad N \geq H_t^n.$$ 

For every $l$ the function $\mu_l(\xi, N)$ is a continuous function in $\xi$ and $N$. So, there exists $\eta_{t+1} > 0$, such that

$$\mu_k(\xi, N) \leq (t+1)^{-1}, \quad \forall \xi: |\xi - \xi_{t+1}| \leq \eta_{t+1}, \quad \forall N: H_t \leq N \leq H_{t+1},$$

(57)

$$\mu_{k+2}(\xi, N) \geq t+1, \quad \forall \xi: |\xi - \xi_{t+1}| \leq \eta_{t+1}, \quad \forall N: H_t^n \leq N \leq H_{t+1}^n,$$

(58)

and

$$\mathcal{B}_{t+1} = \{ \xi = (1, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1} : |\xi - \xi_{t+1}| \leq \eta_{t+1} \} \subset \mathcal{B}_t^* \subset \mathcal{B}_t$$

(59)

Now for the objects $\eta_i, \gamma_i, W_i, H_i, \Lambda_i, \Gamma_i, \xi_i$ with $i = t+1$ we have the following statements.

The condition (i) is satisfied by the construction.

The condition (ii) is satisfied since $\xi_{t+1}$ is a $(\Lambda_{t+1}, \gamma_{t+1}, W_{t+1})$-BAD vector.

The condition (iii) follows from (54).

The condition (iv) follows from (59).
The condition (v) follows from the definition $H_{t+1}$ of $H_{t+1}$.
The condition (vi) follows from (53).
The condition (vii) follows from (58).
The inductive procedure is described completely and Theorem 1 for $k \geq 2$ is proved.

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References

[1] J.W.S. Cassels, An introduction to the geometry of numbers. Springer-Verlag., 1959.

[2] W.M. Schmidt, Diophantine approximations., Lecture Notes in Mathematics., 785, Springer-Verlag., 1980.

[3] A.Ya. Khinchin, Uber eine klasse linear Diophantinen Approximationen. // Rendiconti Circ. Math. Palermo, 1926, 50, p.170 - 195.

[4] H.Davenport, W.M.Schmidt, A theorem on linear forms // Acta Arithmetica, 1968, 14, p. 209 - 223.

[5] W. M. Schmidt, Open problems in Diophantine approximations. // ”Approximations Diophantiennes et nombres transcendants” Luminy, 1982, Progress in Mathematics, Birkhäuser (1983), p.271 - 289.

[6] N.G. Moshchevitin, On best simultaneous approximations. // Russian Mathematical Surveys, 1996, V. 51, No.6, P. 213 - 214.

[7] J.S. Lagarias, Best simultaneous Diophantine approximation II. // Pac. J. Math., 1982, V. 102, No. 1, p. 61 -88.

[8] R.K. Akhunzhanov, N.G. Moshchevitin, Vectors of given Diophantine type. // Mathematical Notes, 2006, V. 80, No.3, p. 318 - 328.

[9] N.G. Moshchevitin, On simultaneous diophantine approximations. Vectors of given Diophantine type. // Mathematical Notes, 1997, V. 61, No. 5, p. 590 - 599.

[10] N.G. Moshchevitin, Best Diophantine approximations: the phenomenon of degenerate dimension. // LMS Lecture Notes Series. V. 338, 2007, p. 162 - 186.

[11] J.W.S. Cassels, An introduction to Diophantine approximations., Cambridge Univ. Press., 1957.

author: Nikolay Moshchevitin

e-mail: moshchevitin@mech.math.msu.su, moshchevitin@rambler.ru