HOMOTOPY INVARIANTS OF GAUSS WORDS

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Abstract. By defining combinatorial moves, we can define an equivalence relation on Gauss words called homotopy. In this paper we define a homotopy invariant of Gauss words. We use this to show that there exist Gauss words that are not homotopically equivalent to the empty Gauss word, disproving a conjecture by Turaev. In fact, we show that there are an infinite number of equivalence classes of Gauss words under homotopy.

1. Introduction

A Gauss word is a sequence of letters with the condition that any letter appearing in the sequence does so exactly twice. A Gauss word can be obtained from an oriented virtual knot diagram. Given a diagram, label the real crossings and arbitrarily pick a base point on the curve somewhere away from any of the real crossings. Starting from the base point, we follow the curve and read off the labels of the crossings as we pass through them. When we return to the base point we will have a sequence of letters in which each label of a real crossing appears exactly twice. Thus this sequence is a Gauss word.

If we give the real crossings different labels we will get a different Gauss word from the diagram. We wish to consider all such Gauss words equivalent, so we introduce the idea of an isomorphism Gauss words. Two Gauss words are isomorphic if there is a bijection between the sets of letters in the Gauss words, which transforms one Gauss word into the other. Diagrammatically, an isomorphism corresponds to relabelling the real crossings.

Depending on where we introduce the base point, we may get different Gauss words from a single diagram. To remove this dependence we introduce a combinatorial move on a Gauss word. The move allows us to remove the initial letter from the word and append it to the end of the word. We call this move a shift move. Diagrammatically, the shift move corresponds to the base point being moved along the curve through a single real crossing. We can then say that, modulo the shift move and isomorphism, the representation of an oriented virtual knot diagram by a Gauss word is unique.

We can define an equivalence relation on virtual knot diagrams by defining diagrammatic moves called generalized Reidemeister moves. Two diagrams are defined to be equivalent if there exists a finite sequence of such moves transforming one diagram into the other. Virtual knots are defined to be the equivalence classes of this relation.

By analogy, we can define combinatorial moves on Gauss words which correspond to the generalized Reidemeister moves. We can then use these moves, the shift move and isomorphism to define an equivalence relation on Gauss words which we call homotopy. We define the moves in such a way that if two virtual knot diagrams

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represent the same virtual knot then the Gauss words obtained from the diagrams must be equivalent under our combinatorial moves. This means that if we have two virtual knot diagrams for which the associated Gauss words are not equivalent, we can immediately say that the diagrams represent different virtual knots.

If we disallow the shift move, we can define another kind of homotopy of Gauss words which we call open homotopy. It is clear from the definition that if two Gauss words are open homotopic, they must also be homotopic. In this paper we will show that the opposite conclusion is not necessarily true. In other words, we will show that homotopy and open homotopy of Gauss words are different.

In [11], Turaev introduced the idea of nanowords which are defined as Gauss words with some associated data. By introducing moves on nanowords, different kinds of homotopy can be defined. From this viewpoint, homotopy of Gauss words is the simplest kind of nanoword homotopy. In fact, any Gauss word homotopy invariant is an invariant for any kind of homotopy of nanowords.

In [11], Turaev defined several invariants of nanowords. However, all these invariants are trivial in the case of Gauss words. This led Turaev to conjecture that open homotopy of Gauss words is trivial. That is, he conjectured that every Gauss word is open homotopically equivalent to the empty Gauss word.

In this paper we define a homotopy invariant of Gauss words called $z$ which takes values in an abelian group. This invariant was inspired by Henrich’s smoothing invariant for virtual knots defined in [4]. In fact, our invariant can be viewed as a version of Henrich’s invariant, weakened sufficiently to remain invariant under homotopy of Gauss words.

We give an example of a Gauss word for which $z$ takes a different value to that of the empty Gauss word. This shows that Turaev’s conjecture is false. We state this result as follows.

**Corollary 4.4.** There exist Gauss words that are not homotopically trivial.

Using the idea of a covering of a Gauss word, which was originally introduced by Turaev in [11], we define the height of a Gauss word. This is a homotopy invariant and we use it to prove the following proposition.

**Proposition 5.9.** There are an infinite number of homotopy classes of Gauss words.

We also give an invariant for open homotopy of Gauss words called $z_o$ which is defined in a similar way to $z$. The invariant $z_o$ itself can be viewed as an open homotopy invariant. However, we show that $z_o$ is a stronger invariant than $z$.

The rest of this paper is arranged as follows. In Section 2 we give a formal definition of Gauss words, homotopy and open homotopy. In Section 3 we describe Gauss phrases and recall the $S$ invariant which was defined in [2].

In Section 4 we give the definition of $z$ and prove its invariance under homotopy. We then use $z$ to give an example of a Gauss word that is not homotopically trivial. In Section 5 we recall the definition of the covering invariant from [11]. We use this invariant to show how we can construct infinite families of Gauss words which are mutually non-homotopic.

In Section 6 we describe the open homotopy invariant $z_o$. We use this invariant to show that open homotopy is different from homotopy.

In Section 7 we interpret the existence of Gauss words that are not homotopically trivial in terms of moves on virtual knot diagrams.

Having written this paper, we discovered a paper by Manturov [7] which studies objects called free knots. From Lemma 1 in Manturov’s paper and the discussion in Section 7 it is clear that an oriented free knot is equivalent to a homotopy class of Gauss words. Manturov shows the existence of non-trivial free knots which implies
the result we give in Corollary 4.4. Our Proposition 5.9 can be deduced from his results. In Manturov’s paper objects corresponding to open homotopy classes of Gauss words are not considered. However, it is clear that his invariant can be generalized to this case.

We also found a second paper on free knots written by Manturov [8]. In this paper he defines an invariant which is essentially the same as our $z$-invariant.

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2. Gauss words

An alphabet is a finite set and its elements are called letters. A word on an alphabet $A$ is a map $m$ from an ordered finite set $\{1, \ldots, n\}$ to $A$. Here $n$ is called the length of the word. It is a non-negative integer. We usually write a word as sequence of letters $m(1)m(2)\ldots m(n)$ from which the map can be deduced if needed. For example, $ABBB$ and $CABAACA$ are both words on the alphabet $\{A, B, C\}$. For any alphabet there is a unique empty word of length 0 which we write $\emptyset$.

A Gauss word on an alphabet $A$ is a word on $A$ such that every letter in $A$ appears in the word exactly twice. We define the rank of a Gauss word to be the size of $A$. This means that the rank of a Gauss word is always half its length.

For example, $ABAB$ and $ABBA$ are both Gauss words on the alphabet $\{A, B\}$. They both have length 4 and rank 2.

There is a unique trivial Gauss word $\emptyset$ on an empty alphabet which has length and rank 0.

By definition, the alphabet that a Gauss word is defined on is the set of letters appearing in the Gauss word. Therefore, we do not need to explicitly state the alphabet that the Gauss word is defined on.

Let $u$ be a Gauss word on $A$ and $v$ be a Gauss word on $B$. An isomorphism of Gauss words $u$ and $v$ is a bijection $f$ from $A$ to $B$ such that $f$ applied letterwise to $u$ gives $v$. If such an isomorphism exists we say that $u$ and $v$ are isomorphic.

We now define some combinatorial moves on Gauss words. If we have a Gauss word matching the pattern on the left of the move we may transform it to the pattern on the right or vice-versa. In each move $t$, $x$, $y$ and $z$ represent possibly empty, arbitrary sequences of letters and $A$, $B$ and $C$ represent individual letters. The moves are

\begin{align*}
\text{Shift:} & \quad AxAy \leftrightarrow xAyA, \\
H1: & \quad xAAy \leftrightarrow xy, \\
H2: & \quad xAByBAz \leftrightarrow xyz, \\
H3: & \quad xAByACzBCt \leftrightarrow xBAyCAzCBt
\end{align*}

and are collectively known as homotopy moves and were originally defined by Turaev in [11].

Two Gauss words are homotopic if there exists a finite sequence of isomorphisms and homotopy moves which transforms one into the other. This relation is an equivalence relation which we call homotopy. It divides the set of Gauss words into homotopy classes. We define the homotopy rank of a Gauss word $w$ to be the minimum rank of all the Gauss words that are homotopic to $w$. We say that a Gauss word is homotopically trivial if it is homotopic to the trivial Gauss word $\emptyset$. Such a Gauss word has homotopy rank 0.

If we disallow the shift move, we get a potentially different kind of homotopy which we call open homotopy. It is easy to see that if two Gauss words are open
homotopic, they must be homotopic. We will show later that the reverse is not necessarily true.

Homotopy of Gauss words is the simplest kind of homotopy of nanowords. Turaev defined nanowords in [11]. A nanoword is a Gauss word with a map, called a projection, from its alphabet to some set \( \alpha \). An isomorphism of nanowords is an isomorphism of Gauss words which preserves this projection. A particular homotopy of nanowords is determined by fixing \( \alpha \) and specifying some other information known collectively as homotopy data (see [11] for full details). Moves on nanowords are defined in the same way as moves on Gauss words. However, restrictions dependent on the projection, homotopy data and \( \alpha \) limit when the moves can be applied.

Homotopy on nanowords is defined analogously to homotopy of Gauss words. That is, two nanowords are homotopic if there exists a finite sequence of isomorphisms and homotopy moves transforming one nanoword into the other. In [11], Turaev defines homotopy of nanowords without allowing the shift move. In this paper we call this kind of homotopy open homotopy of nanowords.

In this general setting, homotopy of Gauss words is a homotopy of nanowords where the set \( \alpha \) is a single element.

In [11], Turaev derived some other moves from the homotopy moves \( H_1, H_2 \) and \( H_3 \) for nanowords. These hold for Gauss words and are

\[
\begin{align*}
H_{2a}: \quad xAByABz & \leftrightarrow xyz \\
H_{3a}: \quad xAByCAzBCt & \leftrightarrow xBAyACzCBt \\
H_{3b}: \quad xAByCAzCBt & \leftrightarrow xBAyACzBCt \\
H_{3c}: \quad xAByACzCBt & \leftrightarrow xBAyCAzCBt.
\end{align*}
\]

3. Gauss phrases

A phrase is a finite sequence of words \( w_1, \ldots, w_n \) on some alphabet. We call each word in the sequence a component of the phrase. If the concatenation \( w_1 \ldots w_n \) of all words in a phrase gives a Gauss word, we say that the phrase is a Gauss phrase. A Gauss phrase with only one component is necessarily a Gauss word.

In this paper we write Gauss phrases as a sequence of letters, using a \( | \) to separate components. So, for example, \( ABA|B \) is a Gauss phrase written in this way.

Let \( p \) and \( q \) be Gauss phrases with \( n \) components. We write \( p \) as \( u_1|\ldots|u_n \) and \( q \) as \( v_1|\ldots|v_n \). Then \( p \) and \( q \) are isomorphic if there exists a bijection \( f \) from the alphabet of \( p \) to the alphabet of \( q \) such that \( f \) applied letterwise to \( u_i \) gives \( v_i \) for all \( i \).

We define the homotopy moves \( H_1, H_2 \) and \( H_3 \) for Gauss phrases in the same way as we did for Gauss words. We modify the meaning of the letters \( t, x, y \) and \( z \) in these moves to allow for the inclusion of one or more occurrences of the component separator \( | \). Note that a move cannot be applied if the component separator \( | \) appears between the letters in the subwords \( AA, AB, BA, AB, AC \) and \( BC \) that are explicitly shown in the moves. For example, given the Gauss phrase \( AB|BAC|C \), we may apply the move \( H_2 \) to get the Gauss phrase \( \emptyset|C|C \), but we cannot apply the move \( H_1 \) to remove the letter \( C \).

We define a shift move for Gauss phrases which can be applied to a single component of the Gauss phrase. Suppose \( p \) is a Gauss phrase with \( i \)-th component of the form \( Ax \) for some letter \( A \) and some letter sequence \( x \). The shift move applied to the \( i \)-th component of \( p \) gives a Gauss phrase \( q \) where the \( i \)-th component has the form \(xA \) and every other component of \( q \) matches that of \( p \).

We say two Gauss phrases are homotopic if there is a finite sequence of isomorphisms, shift moves and the moves \( H_1, H_2 \) and \( H_3 \) which transform one Gauss
phrase into the other. This relation is an equivalence relation on Gauss phrases called homotopy.

None of the moves on Gauss phrases allows a component to be added or removed. Thus, the number of components of a Gauss phrase is a homotopy invariant. As Gauss words are one component Gauss phrases, we can see that homotopy of Gauss phrases is a generalization of the homotopy of Gauss words.

As we did for Gauss words, we can define open homotopy of Gauss phrases by disallowing the shift move. In fact we can define various kinds of homotopy on \( n \)-component Gauss phrases by only allowing shift moves on a subset of the components. A component for which the shift move is permitted is called closed and one for which the shift move is not permitted is called open. Thus under homotopy of Gauss phrases all components are closed and under open homotopy all components are open. In this paper we use the term mixed homotopy to mean the homotopy on 2-component Gauss phrases where the first component is closed and the second one is open.

By allowing permutations of components of a Gauss phrase we can define another kind of homotopy. In this paper we only consider this kind of homotopy when all the components are closed. We call this homotopy unordered homotopy.

We studied homotopy of Gauss phrases in [2]. In that paper we defined a homotopy invariant of Gauss phrases called the \( S \) invariant. We recall the definition here.

Let \( p \) be an \( n \)-component Gauss phrase. We write \( K_n \) for \((\mathbb{Z}/2\mathbb{Z})^n\). Given a vector \( \vec{v} \) in \( K_n \) we can define a map \( c_{\vec{v}} \) from \( K_n \) to itself as follows

\[
c_{\vec{v}}(\vec{x}) = \vec{v} - \vec{x} \mod 2.
\]

In [2] we showed that \( c_{\vec{v}} \) is either the identity map or an involution. This means that the orbits of \( K_n \) under \( c_{\vec{v}} \) all contain at most two elements. We define \( K(\vec{v}) \) to be the set of orbits of \( K_n \) under \( c_{\vec{v}} \).

For any subword \( u \) of a single component in \( p \) we define the linking vector of \( u \) to be a vector \( \vec{v} \) in \( K_n \). The \( i \)th element of \( \vec{v} \) is defined to be, modulo 2, the number of letters that appear once in \( u \) and for which the other occurrence appears in the \( i \)th component of \( p \).

Let \( w_k \) be the \( k \)th component of \( p \). As \( w_k \) can be considered a subword of the \( k \)th component, we define the linking vector of the \( k \)th component to be the linking vector of \( w_k \). We write this vector \( \vec{l}_k \).

For any letter \( A \) that appears twice in the same component of \( p \), that component must have the form \( xAyAz \) for some, possibly empty, arbitrary sequences of letters \( x, y \) and \( z \). We define the linking vector of \( A \) to be the linking vector of the subword \( y \).

Write \([0]\) for the orbit of \( \vec{0} \) in \( K(\vec{l}_k) \). For the \( k \)th component in \( p \) we define a subset \( O_k(p) \) of \( K(\vec{l}_k) - \{[0]\} \) as follows. Let \( A_k \) be the set of letters which appear twice in the \( k \)th component of \( p \). Then an orbit \( v \) in \( K(\vec{l}_k) - \{[0]\} \) is in \( O_k(p) \) if there are an odd number of letters in \( A_k \) for which the linking vector of the letter is in \( v \).

We define \( S_k(p) \) to be the pair \( (\vec{l}_k, O_k(p)) \). We then define \( S \) to be the \( n \)-tuple where the \( k \)th element of \( S \) is \( S_k(p) \). In [2] we showed that \( S \) is a homotopy invariant of \( p \).

We can represent \( S_k(p) \) as a matrix. To do this, we first define an order on \( K_n \) as follows. Let \( \vec{u} \) and \( \vec{v} \) be vectors in \( K_n \). Let \( j \) be the smallest integer for which the \( j \)th elements of \( \vec{u} \) and \( \vec{v} \) differ. Then \( \vec{u} \) is smaller than \( \vec{v} \) if the \( j \)th element of \( \vec{u} \) is 0.
We define the \textit{representative vector} of an orbit \( v \) in \( K(\mathbf{l}_k) \) to be the smallest vector in that orbit. Let \( R \) be the set of representative vectors of orbits which are in \( O_k(p) \). Let \( r \) be the number of vectors in \( R \).

We construct a matrix with \( n \) columns and \( r + 1 \) rows for \( S_k(p) \). The first row is the vector \( \mathbf{i}_k \). The remaining \( r \) rows are given by the elements of \( R \) written out in ascending order. In \cite{2} we observed that this construction is canonical in the following sense. Given \( n \)-component Gauss phrases \( p \) and \( q \), \( S_k(p) \) and \( S_k(q) \) are equivalent if and only if their matrix representations are equal. Thus we can write \( S \) as an \( n \)-tuple of matrices.

As preparation for Section 4 we consider what happens to the invariant \( S \) under unordered homotopy of 2-component Gauss phrases.

Let \( p \) be a 2-component Gauss phrase. Then \( p \) has the form \( w_1|w_2 \). Define \( q \) to be the 2-component Gauss phrase \( w_2|w_1 \). Under unordered homotopy, \( p \) and \( q \) are equivalent. We compare \( S(p) \) and \( S(q) \).

We can write \( S(p) \) as a pair of matrices \((M_1, M_2)\) where both \( M_1 \) and \( M_2 \) have 2 columns. Given a two column matrix \( M \) we define a new matrix \( T(M) \) by

\[
T(M) = M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The matrix \( T(M) \) is the same size as \( M \) but has its columns in the opposite order. Using this notation it is easy to check that we can write \( S(q) \) as \((T(M_2), T(M_1))\). We say that \( S(p) \) and \( S(q) \) are related by a \textit{transposition}. If we consider the invariant \( S \) modulo transposition, we get an unordered homotopy invariant of 2-component Gauss phrases.

\textit{Remark 3.1.} It is possible to define a similar equivalence relation on the \( S \) invariant to obtain an unordered homotopy invariant in the general \( n \)-component case. However, as we do not need to use such an invariant in this paper, we do not give a definition here.

In \cite{2} we also defined an open homotopy invariant of Gauss phrases similar to \( S \) which we called \( S_m \). We can construct a hybrid of the two invariants, called \( S_m \), which is a mixed homotopy invariant of 2-component Gauss phrases. Recall that under mixed homotopy the first component is closed and the second component is open. We now give a definition of \( S_m \).

Given a 2-component Gauss phrase \( p \), we define \( S_m \) to be the pair of pairs \((\mathbf{i}_1, O_1(p)), (\mathbf{i}_2, B_2(p))\) where \( \mathbf{i}_1 \), \( \mathbf{i}_2 \) and \( O_1(p) \) are defined as for \( S \). We define \( B_2(p) \) to be a subset of \( K_n - \{\emptyset\} \) as follows. Let \( A_2 \) be the set of letters which appear twice in the second component. Then a vector \( \bar{v} \) in \( K_n - \{\emptyset\} \) is in \( B_2(p) \) if there are an odd number of letters in \( A_2 \) for which the linking vector of the letter is \( \bar{v} \).

With reference to \cite{2} it is easy to check that \( S_m \) is a mixed homotopy invariant. We will use \( S_m \) in Section 4.

4. Homotopy invariant

Let \( \mathcal{GP}(2) \) be the set of equivalence classes of 2-component Gauss phrases under unordered homotopy. Let \( G \) be the free abelian group generated by \( \mathcal{GP}(2) \). We then define \( Z \) to be \( G/2G \). Let \( \theta \) be the natural homomorphism from \( G \) to \( Z \).

Let \( w \) be a Gauss word. For each letter \( A \) in \( w \) we can derive a 2-component Gauss phrase \( p(w, A) \) as follows. As \( A \) must appear twice in \( w \), \( w \) has the form \( xAyAz \) for some, possibly empty, sequences of letters \( x, y \) and \( z \). Then \( p(w, A) \) is the Gauss phrase \( y|xz \). We then define \( u(w, A) \) to be the unordered homotopy class
of \( p(w, A) \), an element in \( \mathcal{GP}(2) \). We define \( t(w) \) to be the element in \( \mathcal{GP}(2) \) given by the unordered homotopy class of \( \emptyset|w \).

We define a map \( g \) from the set of Gauss words to \( G \) as follows. For each Gauss word \( w \), \( g(w) \) is given by

\[
g(w) = \sum_{A \in w} (u(w, A) - t(w)).
\]

Then \( z(w) \), defined to be \( \theta(g(w)) \), gives a map \( z \) from the set of Gauss words to \( Z \).

We have the following theorem.

**Theorem 4.1.** The map \( z \) is a homotopy invariant of Gauss words.

**Proof.** We need to prove that if two Gauss words \( w_1 \) and \( w_2 \) are homotopic, \( z(w_1) \) is equal to \( z(w_2) \). In order to do this, it is sufficient to prove that \( z \) is invariant under isomorphism, the shift move and the moves \( H1, H2 \) and \( H3 \). We note that if \( w_1 \) and \( w_2 \) are homotopic, \( t(w_1) \) is equal to \( t(w_2) \) by definition.

Suppose \( w_1 \) and \( w_2 \) are isomorphic Gauss words. Then each letter \( A \) in \( w_1 \) is mapped to some letter \( A' \) in \( w_2 \) by some isomorphism \( f \). So \( p(w_2, A') \) is isomorphic to \( p(w_1, A) \) under the isomorphism \( f \) restricted to all the letters in \( w_1 \) except for \( A \). This means that \( u(w_2, A') \) is equal to \( u(w_1, A) \). As this is the case for every letter in \( w_1 \) we can conclude that \( g(w_1) \) is equal to \( g(w_2) \). In particular, this means that \( z(w_1) \) is equal to \( z(w_2) \) and so \( z \) is invariant under isomorphism.

Suppose \( w_1 \) and \( w_2 \) are related by a shift move. Then \( w_1 \) is of the form \( Av \) and \( w_2 \) is of the form \( vA \).

Let \( B \) be some other letter appearing in \( w_1 \). Then \( w_1 \) has the form \( AxByBzA \) and \( w_2 \) has the form \( xByBzA \). Now \( p(w_1, B) \) is \( y|Axz \) and \( p(w_2, B) \) is \( y|xzA \). Applying a shift move to the second component of \( p(w_1, B) \) we get \( p(w_2, B) \). This means \( u(w_1, A) \) is equal to \( u(w_2, B) \).

We now turn our attention to the letter \( A \). We can write \( w_1 \) in the form \( AxAy \) and \( w_2 \) in the form \( xAyA \). Then \( p(w_1, A) \) is \( x|y \) and \( p(w_2, A) \) is \( y|x \). As Gauss phrases, \( x|y \) and \( y|x \) are not necessarily equal. However, they are related by a permutation and so are equivalent under unordered homotopy. Thus \( u(w_1, A) \) is equal to \( u(w_2, A) \).

As \( u(w_1, X) \) equals \( u(w_2, X) \) for each letter \( X \) in \( w_1 \), we can conclude that \( g(w_1) \) is equal to \( g(w_2) \). Thus \( z \) is invariant under the shift move.

Suppose \( w_1 \) and \( w_2 \) are related by an \( H1 \) move. Then \( w_1 \) has the form \( xAAz \) and \( w_2 \) has the form \( xz \). Now for any letter \( B \) in \( w_1 \) other than \( A \), \( p(w_1, B) \) will contain the subword \( AA \) in one or other of its two components. Thus this subword can be removed from by \( p(w_1, B) \) an \( H1 \) move. The result is the Gauss phrase \( p(w_2, B) \) which implies that \( u(w_1, B) \) equals \( u(w_2, B) \). Therefore, if we subtract \( g(w_2) \) from \( g(w_1) \) we get \( u(w_1, A) - t(w_1) \). Now as \( p(w_1, A) \) is \( \emptyset|xz \) which is homotopic to \( \emptyset|w_1 \), \( u(w_1, A) \) is equal to \( t(w_1) \). Thus \( g(w_2) \) is equal to \( g(w_1) \) and so \( z \) is invariant under the \( H1 \) move.

Suppose \( w_1 \) and \( w_2 \) are related by an \( H2 \) move. Then \( w_1 \) has the form \( xAByBAz \) and \( w_2 \) has the form \( xB \). It is easy to see that for any letter \( C \) in \( w_1 \), other than \( A \) or \( B \), \( p(w_1, C) \) and \( p(w_2, C) \) will be related by an \( H2 \) move involving \( A \) and \( B \). Thus \( u(w_1, C) \) equals \( u(w_2, C) \) for all such letters \( C \). Now \( p(w_1, A) \) is \( ByB|xz \). By applying a shift move to the first component and then an \( H1 \) move, we can remove the letter \( B \) and get a Gauss phrase \( y|xz \). On the other hand, \( p(w_1, B) \) is \( y|xAAz \). Applying an \( H1 \) move to the second component, we get the Gauss phrase \( y|xz \). Thus \( p(w_1, A) \) is homotopic to \( p(w_1, B) \) and so \( u(w_1, A) \) equals \( u(w_1, B) \). We can conclude that if we subtract \( g(w_2) \) from \( g(w_1) \) we get \( 2u(w_1, A) - 2t(w_1) \). As \( 2u(w_1, A) - 2t(w_1) \) is in the kernel of \( \theta \), \( z(w_1) \) equals \( z(w_2) \) and \( z \) is invariant under the \( H2 \) move.
Finally, suppose $w_1$ and $w_2$ are related by an H3 move. Then $w_1$ has the form $tABxAyBCz$ and $w_2$ has the form $tBAxCyCBz$.

Let $D$ be some letter in $w_1$ other than $A$, $B$ or $C$. There are 10 possible cases depending on where the two occurrences of $D$ occur in relation to the subwords $AB$, $AC$ and $BC$. In Table 1 we show that in all 10 cases, $p(w_1, D)$ is homotopic to $p(w_2, D)$ by the homotopy move shown in the column furthest to the right. Thus in every case, $u(w_1, D)$ is equal to $u(w_2, D)$.

| Case | $w_1$ | $p(w_1, D)$ | $p(w_2, D)$ | Move |
|------|-------|-------------|-------------|------|
| 1    | $rsABDtACx|BCy|Dz$ | $srtABxAyBCz$ | $srtBAxCyCBz$ | H3 |
| 2    | $rsABDtACx|BCy|Dz$ | $sABt|rxACyBCz$ | $sBAtx|rxCAyCBz$ | H3c |
| 3    | $rsABtACxDy|BCz$ | $sABtACx|yBCz$ | $sBACx|ACyBCz$ | H3 |
| 4    | $rsABtACx|BCy|Dz$ | $sABtACx|BCy|rz$ | $sBACx|ACyBCz$ | H3c |
| 5    | $rABS|DtACx|BCy|Dz$ | $t|ABxAyBCz$ | $t|BAxCyCBz$ | H3c |
| 6    | $rABS|DtACx|BCy|Dz$ | $tACx|yABsCyBCz$ | $tCAXy|BAyBCz$ | H3 |
| 7    | $rABS|DtACx|BCy|Dz$ | $tACx|yABsCyBCz$ | $tCAXy|BAyBCz$ | H3 |
| 8    | $rABS|DtACx|BCy|Dz$ | $x|ABxAyBCz$ | $x|BAxCyCBz$ | H3 |
| 9    | $rABS|DtACx|BCy|Dz$ | $x|ABxAyBCz$ | $x|BAxCyCBz$ | H3 |
| 10   | $rABS|DtACx|BCy|Dz$ | $y|ABxAyBCz$ | $y|BAxCyCBz$ | H3 |

Table 1. Invariance under homotopy of a Gauss phrase associated with a letter uninvolved in an H3 move.

We now turn our attention to the letters $A$, $B$ and $C$.

In the case of $A$, $p(w_1, A)$ is $Bx|tCyBCz$ and $p(w_2, A)$ is $xC|tCyBCz$ which is isomorphic to $p(w_2, A)$. Thus $u(w_1, A)$ is equal to $u(w_2, A)$.

In the case of $B$, $p(w_1, B)$ is $xACy|tACz$. By applying an H2a move to remove the letters $A$ and $C$ we get the Gauss phrase $xy|tz$. On the other hand, $p(w_2, B)$ is $AxCAyC|tz$. We can apply a shift move to the first component followed by an H2a move to remove the letters $A$ and $C$. We again get the Gauss phrase $xy|tz$. Thus $p(w_1, B)$ and $p(w_2, B)$ are homotopic and so $u(w_1, B)$ is equal to $u(w_2, B)$.

In the case of $C$, $p(w_1, C)$ is $yB|tABxAz$ and $p(w_2, C)$ is $yA|tBAzBz$. Applying a shift move to the first component of $p(w_2, C)$ we get the Gauss phrase $yA|tBAzBz$ which is isomorphic to $p(w_1, C)$. Thus $u(w_1, C)$ is equal to $u(w_2, C)$.

So we have seen that $u(w_1, X)$ is equal to $u(w_2, X)$ for all letters $X$ in $w_1$. Therefore $g(w_1)$ is equal to $g(w_2)$ and we can conclude that $z$ is invariant under the H3 move.

We now calculate this invariant for two examples.

Example 4.2. Consider the trivial Gauss word $\emptyset$. As there are no letters in $\emptyset$, $g(\emptyset)$ is 0 and so $z(\emptyset)$ is 0.

Example 4.3. Let $w$ be the Gauss word $ABACDCEBED$. We calculate $z(w)$.

The Gauss phrase $p(w, A)$ is $B|CDCEBED$. By using a shift move on the second component and applying an H2a move, we can remove $C$ and $D$. After applying another shift move to the second component we can use an H1 move to remove $E$. Thus $p(w, A)$ is homotopic to $B|B$.

The Gauss phrase $p(w, B)$ is $ACDCE|AED$. Applying two shift moves to the first component gives $DCE|AC|AED$. We can then remove $A$ and $E$ by an H2 move and then, using an H1 move, remove $C$. This means that $p(w, B)$ is homotopic to $D|D$. 


The Gauss phrase \( p(w, C) \) is \( D|ABAEBED \). We can apply an H3 move to the letters \( A \), \( B \) and \( E \). The result is \( D|BAEAEBD \). The letters \( A \) and \( E \) can be removed by an H2a move and then the letter \( B \) can be removed by an H1 move. This shows that \( p(w, C) \) is homotopic to \( D|D \).

The Gauss phrase \( p(w, D) \) is \( CEBE|ABAC \).

The Gauss phrase \( p(w, E) \) is \( B|ABACDCD \). Applying a shift move to the second component gives \( B|BACDCDA \). We can then use an H2a move and an H1 move to remove the letters \( C \), \( D \) and \( A \). This shows that \( p(w, E) \) is homotopic to \( B|B \).

From these calculations we can see that \( p(w, A) \), \( p(w, B) \), \( p(w, C) \) and \( p(w, E) \) are all mutually homotopic. Thus we can write \( g(w) \) as \( 4u(w, A) + u(w, D) - 5t(w) \).

Now observe that \( \theta(4u(w, A)) \) is in the kernel of \( \theta \) and that \( \theta(-5t(w)) \) is equal to \( \theta(t(w)) \). So \( z(w) \) is equal to \( \theta(u(w, D)) + \theta(t(w)) \).

We calculate the \( S \) invariant for \( p(w, D) \). We find that it is the pair of matrices

\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]

The transposition of \( S(p(w, D)) \) gives the same pair of matrices. This is unsurprising because swapping the components in \( CEBE|ABAC \) gives \( ABAC|CEBE \) which, after applying three shift moves to the first component and one to the second, is isomorphic to \( CEBE|ABAC \).

Calculating \( S(\emptyset|w) \) and \( S(w|\emptyset) \), we find they are both given by

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

As the pairs of matrices in (4.1) and (4.2) are not equal, we can conclude that \( u(w, D) \) is not equal to \( t(w) \). Thus \( u(w, D) \) is not equal to \( 0 \). As we have already shown that \( z(\emptyset) \) is 0, we can conclude that \( w \) is not homotopically trivial.

From these examples we can make the following conclusion.

**Corollary 4.4.** There exist Gauss words that are not homotopically trivial.

If two nanowords are homotopic then their associated Gauss words must be homotopic. This means that the \( z \) invariant is an invariant of any homotopy of nanowords. As virtual knots can be represented as homotopy classes of a certain homotopy of nanowords [10], this shows that there exist non-trivial virtual knots for which non-triviality can be determined by just considering their associated Gauss word homotopy class.

![Figure 1. A non-trivial virtual knot](image-url)
a Gauss word from the diagram. We find it is the Gauss word considered in Example 4.3. From that example we know that the Gauss word is not homotopically trivial. Thus we can conclude that the virtual knot is non-trivial.

When we calculate the \( z \) invariant for a nanoword, we do so without reference to its associated projection. However, using information in the projection it should be possible to define stronger invariants based on \( z \). Indeed, as we mentioned in the introduction, Henrich’s smoothing invariant [4] is an example of such an invariant in the case of virtual knots.

5. Coverings

In Section 5 of [11] Turaev introduced an open homotopy invariant of nanowords called a covering. This invariant is a map from the set of open homotopy classes of nanowords to itself. In this section we give an alternative but equivalent definition of this invariant for Gauss words and show that it is a homotopy invariant. We will then use this invariant to construct infinite families of mutually non-homotopic Gauss words.

Given a Gauss word \( w \) and a letter \( A \) appearing in \( w \), \( w \) has the form \( xAyz \) for some, possibly empty, arbitrary sequences of letters \( x \), \( y \) and \( z \). We say that \( A \) has odd parity if \( y \) has an odd number of letters and \( A \) has even parity if \( y \) has an even number of letters. We define the covering of \( w \) to be a copy of \( w \) where all the odd parity letters have been removed. We denote the covering of \( w \) by \( e(w) \).

Example 5.1. Let \( w \) be the Gauss word \( ABCADBECED \). Then the letters \( B \) and \( E \) have odd parity and the other letters are even parity. So \( e(w) \) is the Gauss word \( ACADCD \).

Proposition 5.2. The homotopy class of the covering of a Gauss word \( w \) is a homotopy invariant of \( w \).

Proof. As Turaev has already proved this fact for open homotopy in Lemma 5.2.1 of [11], it is sufficient to prove invariance under the shift move.

Given a Gauss word \( u \) of the form \( AxAy \), the shift move transforms it to a Gauss word of the form \( xAyA \) which we label \( v \). Since the length of \( u \) is even, the lengths of \( x \) and \( y \) have the same parity. This means that the parity of \( A \) is the same in \( u \) and \( v \). For any letter other than \( A \), it is clear that the parity of the letter is the same in \( u \) and \( v \). Thus the parity of any letter in \( u \) is invariant under the shift move.

Suppose \( A \) has even parity. Then \( e(u) \) is given by \( Ax'Ay' \) for some words \( x' \) and \( y' \) derived from \( x \) and \( y \) by deleting odd parity letters. The covering of \( v \), \( e(v) \), is then \( x'Ay' A \). So \( e(v) \) can be obtained from \( e(u) \) by a shift move which means they are homotopic.

Suppose \( A \) has odd parity. Then \( e(u) \) is given by \( x'y' \) for some words \( x' \) and \( y' \) and \( e(v) \) is also given by \( x'y' \). Thus \( e(u) \) and \( e(v) \) are equal and therefore homotopic.

Remark 5.3. In fact, we can define this kind of covering for any homotopy of nanowords. This is because the parity of a letter in a Gauss word can be calculated without any reference to the projection of the nanoword. In this general setting we call this covering the even parity covering. We first gave a definition of the even parity covering of nanowords in [3].

We have the following lemma.
Lemma 5.4. Let \( w \) be a Gauss word. If \( w \) is not homotopic to \( e(w) \) then the homotopy rank of \( e(w) \) is strictly less than the homotopy rank of \( w \).

Proof. Let \( m \) be the homotopy rank of \( w \). Then we can find a Gauss word \( w' \) which is homotopic to \( w \) and has rank \( m \). Consider \( e(w') \). If the rank of \( e(w') \) is \( m \), it means that \( e(w') \) is the same as \( w' \). However, as \( e(w') \) is homotopic to \( e(w) \), this implies \( e(w) \) is homotopic to \( w \), contradicting the assumption of the lemma. Thus the rank of \( e(w') \) cannot be \( m \) and \( e(w') \) is different from \( w' \). As, by definition, we derive \( e(w') \) from \( w' \) by removing letters, we must conclude that the rank of \( e(w') \) is less than \( m \). Thus, the homotopy rank of \( e(w) \), which is less than or equal to the rank of \( e(w') \), is less than \( m \).

Remark 5.5. In fact, we can say that the rank of \( e(w') \) must be less than \( m - 1 \). For if the rank of \( e(w') \) was \( m - 1 \), it would mean that we obtained \( e(w') \) by removing a single letter from \( w' \). This in turn would imply that \( w' \) had a single odd parity letter. However, this would contradict Lemma 5.2 of [2], which states that any Gauss word has an even number of odd parity letters.

As the covering of a Gauss word is itself a Gauss word, we can repeatedly take coverings to form an infinite sequence of Gauss words. That is, given a Gauss word \( w \), define \( w_0 \) to be \( w \) and define \( w_i \) to be \( e(w_{i-1}) \) for all positive integers \( i \). As \( w \) has a finite number of letters, Lemma 5.4 shows that there must exist an \( n \) for which \( w_{n+1} \) is homotopic to \( w_n \). Let \( m \) be the smallest such \( n \). We define the height of \( w \), \( \text{height}(w) \), to be \( m \) and the base of \( w \), \( \text{base}(w) \) to be the homotopy class of \( w_m \). The height and base of \( w \) are homotopy invariants of \( w \).

In [1] we defined height and base invariants for virtual strings in the same way. We showed that the base invariants are non-trivial for virtual strings in that paper. However, we do not know whether the base invariant we have defined here is non-trivial for Gauss words. In other words, we have not yet found a Gauss word \( w \) for which we can prove \( \text{base}(w) \) is not homotopically trivial.

Given a Gauss word \( w \), we can define a new Gauss word \( v \) such that \( e(v) \) is \( w \). We start by taking a copy of \( w \). Then for each odd parity letter \( A \) in \( w \), we replace the first occurrence of \( A \) with \( XAX \) for some letter \( X \) not already appearing in \( w \). Note that this replacement changes the parity of \( A \) to make it even. The parity of any other letter in the word is unchanged because we replace a subword of length 1 with a subword of length 3. Note also that the introduced letter \( X \) has odd parity. After making the change for each odd parity letter in \( w \), we call the final Gauss word \( l(w) \).

By construction, all the letters in \( l(w) \) that were originally in \( w \) have even parity and all the letters that were introduced have odd parity. Thus, when we take the covering of \( l(w) \) we remove all the letters that we introduced and we are left with the letters in \( w \). Since we did not change the order of the letters, we can conclude that \( e(l(w)) \) is equal to \( w \).

Example 5.6. Let \( w \) be the Gauss word \( ABCADBECED \) from Example 5.1. There are only two odd parity letters in \( w \), \( B \) and \( E \). We replace the first occurrence of \( B \) with \( XBX \) and the first occurrence of \( E \) with \( YEY \). The result is the Gauss word \( AXBXCADBEYCED \) which we label \( l(w) \). The covering of \( l(w) \) is \( w \).

We remark that if a Gauss word \( w \) contains no odd parity letters, then \( l(w) \) is the same as \( w \). Even if \( w \) and \( l(w) \) are not equal as Gauss words, they may be homotopic. We provide an example to show this.

Example 5.7. Consider the Gauss word \( w \) given by \( ABAB \). Then \( l(w) \) is given by \( XAXYBYAB \). By move \( H_3c \) on \( X \), \( A \) and \( Y \), \( l(w) \) is homotopic to \( AXYXBAYB \). Applying a shift move we get \( XYXBAYBA \) which can be reduced to \( XYXY \) by
an H2a move involving A and B. This Gauss word is isomorphic to w. Thus \( l(w) \) and \( w \) are homotopic.

Given a Gauss word \( w \) we can make an infinite family of Gauss words \( w_i \) by repeated use of this construction. We define \( w_0 \) to be \( w \). Then inductively we define \( w_i \) to be \( l(w_{i-1}) \) for all positive integers \( i \).

**Lemma 5.8.** Let \( w \) be a Gauss word such that \( e(w) \) and \( w \) are not homotopic. Let \( w_i \) be the infinite family of Gauss words defined from \( w \) as above. Then the \( w_i \) are all mutually non-homotopic.

**Proof.** Suppose, for some \( i \), \( w_{i+1} \) is homotopic to \( w_i \). By construction, starting from \( w_{i+1} \) and taking the covering \( i + 1 \) times, we get \( w \). Similarly, starting from \( w_i \), taking the covering \( i + 1 \) times gets \( e(w) \). Since the covering of a Gauss word is a homotopy invariant, our supposition implies that \( w \) is homotopic to \( e(w) \). However this contradicts the assumption in the statement of the lemma. Therefore, for all \( i \), \( w_{i+1} \) is not homotopic to \( w_i \).

As \( e(w_{i+1}) \) is \( w_i \), this implies height(\( w_{i+1} \)) is equal to height(\( w_i \)) + 1 for all \( i \). It is now simple to prove that height(\( w_i \)) is height(\( w \)) + \( i \) by induction.

As each Gauss word \( w_i \) has a different height we can conclude that they are all mutually non-homotopic.

**Proposition 5.9.** There are an infinite number of homotopy classes of Gauss words.

**Proof.** By Lemma 5.8 we just need to give an example of a Gauss word which is not homotopic to its cover. Consider the Gauss word \( w \) given by \( ABACDCEBED \). Then \( e(w) \) is \( DD \) which is homotopic to the trivial Gauss word. On the other hand, in Example 4.3 we saw that \( w \) is not homotopic to the trivial Gauss word. Thus \( w \) and \( e(w) \) are not homotopic.

## 6. Open homotopy

Although the invariant \( z \) is an invariant for open homotopy of Gauss words, we can use a similar construction to make a stronger invariant for open homotopy. We call this invariant \( z_o \).

Let \( \mathcal{GPM}(2) \) be the set of equivalence classes of 2-component Gauss phrases under mixed homotopy. Recall that we defined this to be the homotopy where the first component is closed and the second is open. Let \( H \) be the free abelian group generated by \( \mathcal{GPM}(2) \). We then define \( Z_o \) to be \( H/2H \). Let \( \phi \) be the natural homomorphism from \( H \) to \( Z_o \).

For a Gauss word \( w \) and a letter \( A \) appearing in \( w \), we define \( u_m(w, A) \) to be the equivalence class in \( \mathcal{GPM}(2) \) which contains \( p(w, A) \). Here \( p(w, A) \) is the Gauss phrase defined in Section 4. We define \( t_m(w) \) to be the element in \( \mathcal{GPM}(2) \) which contains \( \emptyset \| w \).

For each Gauss word \( w \), we define \( h(w) \) by

\[
h(w) = \sum_{A \in w} (u_m(w, A) - t_m(w)).
\]

Then \( h \) is a map from the set of Gauss words to \( H \). We then define \( z_o(w) \) to be \( \phi(h(w)) \). Thus \( z_o \) is a map from the set of Gauss words to \( Z_o \).

We have the following theorem.

**Theorem 6.1.** The map \( z_o \) is an open homotopy invariant of Gauss words.

**Proof.** The fundamental difference between the definitions of \( z_o \) and \( z \) is the type of homotopy we use to determine equivalence of Gauss phrases. For \( z \) we consider
elements in $G\mathcal{P}(2)$ whereas for $z_o$ we consider elements in $G\mathcal{P}\mathcal{M}(2)$. Note that we can consider $G\mathcal{P}(2)$ to be $G\mathcal{P}\mathcal{M}(2)$ modulo permutation of the two components and allowing shift moves on the second component.

Looking at the proof of the invariance of $z$ (Theorem 4.1) we note the following two facts. Firstly, the permutation of components is only used for the proof of invariance under the shift move. Secondly, we only need to apply a shift move to the second component of a Gauss phrase in the proof of invariance under the shift move.

Thus, by changing the notation appropriately and omitting the section about invariance under the shift move, the proof of Theorem 4.1 becomes a proof of the invariance of $z_o$ under open homotopy. □

We now give an example of a Gauss phrase which is trivial under homotopy but non-trivial under open homotopy. This shows that homotopy and open homotopy of Gauss phrases are different.

**Example 6.2.** Let $w$ be the Gauss word $ABACDCBD$. By an H3c move applied to $A$, $B$ and $C$, $w$ is homotopic to $BACADBCD$. Applying a shift move we get $ACDC|AD$ which is homotopic to $ACAC$ by an H2a move. Applying another H2a move gives the empty Gauss word. Thus $ABACDCBD$ is trivial under homotopy.

We now calculate $z_o(w)$ to show that $w$ is not trivial under open homotopy. We start by calculating the Gauss phrases for each letter. We find that $p(w,A)$ is $B|CDCBD$, $p(w,B)$ is $ACDC|AD$, $p(w,C)$ is $D|D$ and $p(w,D)$ is $CB|ABAC$. Using an H2a move, we see that $p(w,C)$ is homotopic to $D|D$.

We then calculate $S_m$ for each of the four Gauss phrases. We find that

$$S_m(B|CDCBD) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_m(ACDC|AD) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$S_m(D|D) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$S_m(CB|ABAC) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore they are mutually distinct under mixed homotopy.

Noting that $4t_m(w)$ is in the kernel of $\phi$, we have $z_o(w) = \phi(B|CDCBD) + \phi(ACDC|AD) + \phi(D|D) + \phi(CB|ABAC) \neq 0$, where $(g)$ represents the mixed homotopy equivalence class of a Gauss phrase $g$. As $z_o(w)$ is not zero, we can conclude that $w$ is not trivial under open homotopy.

Note that we can define the height and base of Gauss words under open homotopy in the same way as we did for Gauss words under homotopy in Section 5. We write $\text{height}_o(w)$ for the height of $w$ and $\text{base}_o(w)$ for the base of $w$ under open homotopy. For a given Gauss word $w$, $\text{height}(w)$ and $\text{height}_o(w)$ are not necessarily the same. For example, if $w$ is the Gauss word $ABACDCBD$ in Example 6.2, $\text{height}(w)$ is 0 but $\text{height}_o(w)$ is 1. We do not know whether there exists a Gauss word $w$ for which $\text{base}(w)$ and $\text{base}_o(w)$ are different.

We end this section by remarking that there are an infinite number of open homotopy classes of Gauss words. This follows from Proposition 5.1 and the fact that if two Gauss words are not homotopic, they cannot be open homotopic.
7. Virtual knots

We have shown that homotopy of Gauss words is non-trivial. In this section we interpret this fact in terms of virtual knot diagrams. We start by briefly recalling some definitions.

A virtual knot diagram is an immersion of an oriented circle in a plane with a finite number of self-intersections. These self-intersections are limited to transverse double points. We call them crossings. There are three types of crossings which we draw differently in order to distinguish them. These crossing types are shown in Figure 2.

![Figure 2. The three types of crossing in a virtual knot diagram: real positive (left), real negative (middle) and virtual (right)](image)

Virtual knots can be defined as the equivalence classes of virtual knot diagram under a set of diagrammatic moves. These moves include the Reidemeister moves of classical knot theory and some other similar moves in which some crossings are virtual. Definitions of all these moves are given, for example, in [5]. In this paper we collectively call these moves the generalized Reidemeister moves.

We define two moves on a single real crossing in a virtual knot diagram. The first is called the crossing change. It allows us to replace a positive real crossing with a negative one or vice-versa. This move is shown in Figure 3.

![Figure 3. The crossing change](image)

In classical knot theory this move gives an unknotting operation. That is, any classical knot diagram can be reduced to a diagram with no crossings by a sequence of Reidemeister moves and crossing changes. On the other hand, this move is not an unknotting operation for virtual knots. This is because considering virtual knots modulo this move is equivalent to considering virtual strings and we know that non-trivial virtual strings exist (see for example [9]).

The second move we define is called a virtual switch. It is shown in Figure 4. Kauffman first defined this move in [5] and he used the name virtual switch for it in [6]. In [5] he showed that the involutary quandle, a virtual knot invariant, is invariant even under virtual switches. Since there exist virtual knots with different involutary quandles, we may conclude that the virtual switch is not an unknotting operation for virtual knots.

![Figure 4. The virtual switch](image)

Note that both the crossing change and the virtual switch do not change the Gauss word associated with the diagram. The generalized Reidemeister moves do
change the Gauss word associated with the diagram. However, the Gauss words before and after a generalized Reidemeister move are equivalent under the moves we gave in Section 2. We have seen that there exist Gauss words that are not homotopically trivial. Therefore, we can conclude that there exist virtual knots which cannot be unknotted even if we allow the use of both the crossing change and the virtual switch.

In fact, by considering the nanoword representation of virtual knots [10], it is easy to show that the set of homotopy classes of Gauss words is equivalent to the set of virtual knots modulo the crossing change and the virtual switch.

References

[1] A. Gibson, Coverings, composites and cables of virtual strings, preprint, Tokyo Institute of Technology, 2008, arXiv:math.GT/0808.0396.
[2] ———, Homotopy invariants of Gauss phrases, preprint, Tokyo Institute of Technology, 2008, arXiv:math.GT/0810.4389.
[3] ———, Enumerating virtual strings, Master’s thesis, Tokyo Institute of Technology, 2008.
[4] A. Henrich, A sequence of degree one Vassiliev invariants for virtual knots, preprint, 2008, arXiv:math.GT/0803.0754.
[5] L. H. Kauffman, Virtual knot theory, European J. Combin. 20 (1999), no. 7, 663–690.
[6] ———, Detecting virtual knots, Atti Sem. Mat. Fis. Univ. Modena 49 (2001), suppl. 241–282.
[7] V. O. Manturov, On free knots, arXiv:math.GT/0901.2214.
[8] ———, On free knots and links, arXiv:math.GT/0902.0127.
[9] V. Turaev, Virtual strings, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 7, 2455–2525.
[10] ———; Knots and words, Int. Math. Res. Not. (2006), Art. ID 84098, 23.
[11] ———; Topology of words, Proc. Lond. Math. Soc. (3) 95 (2007), no. 2, 360–412.

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