Quark and Gluon Propagators from Meson Data

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Abstract

We report robust calculations of various low energy QCD hadronic properties. We use a multi-rank separable expansion for the gluon propagator which greatly facilitates the numerical computations within the Global Colour Model for QCD. The parameters for the propagators are determined by fitting experimental values for $f_\pi$ and the $\pi$ and $a_1$ meson masses.

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The computation of the low energy properties of QCD is a difficult non-perturbative problem in quantum field theory. Here we report new results using the Global Colour Model (GCM) approximation to QCD. These robust computations are made feasible by the use of multi-rank separable expansions of the gluon propagator. While good progress has been made in computing the gluon propagator from first principles in QCD, here we adopt the procedure of determining an effective gluon propagator by using a convenient separable form, and selecting the parameters therein by fitting $f_\pi$.

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and the $\pi$ and $a_1$ meson masses to experimental values. The validity of both the GCM and the effective separable-form gluon propagator are demonstrated by then computing numerous other low energy hadronic properties. A non-separable translation-invariant form for the effective gluon propagator, appropriate to low energy hadronic states, is then reconstructed from the separable form.

An overview and an insight into the nature of the non-perturbative low energy hadronic regime of QCD is provided by the functional integral hadronization of QCD [3, 4]. This amounts to a dynamically determined change of functional integration variables, from quarks and gluons, to bare hadrons

$$
\int \mathcal{D}\pi \mathcal{D}N \cdots \exp(-S_{\text{had}}[\pi, \cdots, N, \cdots] + J_{\pi}[\eta, \eta] + J_\pi \cdots) \approx \int \mathcal{D}\pi \mathcal{D}N \cdots \exp(-S_{\text{qcd}}[A, \pi, q] + \pi q + \eta) + \eta)$$

(1)

The final functional integration over the hadrons gives the hadronic observables, and amounts to dressing each hadron by, mainly, lighter mesons. This functional integral transformation cannot yet be done exactly. The basic insight is that the quark-gluon dynamics, on the LHS of (1), is fluctuation dominated, whereas the RHS is not, and for example the meson dressing of bare hadrons is known to be almost perturbative. In performing the change of variables essentially normal mode techniques are used [3]. In practice this requires detailed numerical computation of the gluon propagator, quark propagators, and meson and baryon propagators. The mass-shell states of the latter are determined by covariant Bethe-Salpeter and Faddeev equations. The Faddeev computations are made feasible by using the diquark correlation propagators, which must also be determined.

The first and easiest formal transformation results from doing the gluon integrations, leaving an action for quarks of the form

$$
S[\bar{q}, q] = \int \bar{q}(x)(-\gamma \cdot \partial + \mathcal{M})q(x) + \frac{1}{2} \int j^a_{\mu}(x)j^a_{\nu}(y)D_{\mu\nu}(x-y) + \frac{1}{3!} \int j^a_{\mu}j^b_{\nu}j^c_{\rho}D^{abc}_{\mu\nu\rho} + \cdots
$$

(2)

where $j^a_{\mu}(x) = \bar{q}(x)\gamma_{\mu}q(x)$. The GCM is a model field theory for QCD based on a truncation of $S[\bar{q}, q]$ in which the higher order n-point ($n \geq 3$) functions are neglected, and only the gluon 2-point function $D_{\mu\nu}(x-y)$ is retained.

The GCM is thus a quantum field theory that can also be considered to be defined
by the action

\[ S_{gcm}[\bar{q}, q, A^{\mu}_{\alpha}] = \int \left( \overline{\psi}(x)(-\gamma_{\cdot}\partial + \mathcal{M} + iA^{\mu}_{\mu} \frac{\Lambda^{\alpha}}{2} \gamma_{\mu})\delta(x - y)q(y) + \right. \]
\[ \left. + \frac{1}{2} A^{\mu}_{\mu}(x)D^{-1}_{\mu\nu}(i\partial)\delta(x - y)A^{\nu}_{\nu}(y) \right) \]

where the matrix \( D^{-1}_{\mu\nu}(p) \) is the inverse of \( D_{\mu\nu}(p) \), which in turn is the Fourier transform of \( D_{\mu\nu}(x) \). This action has a global colour symmetry. The GCM is thus analogous to QED except for colour currents and the non-quadratic phenomenological form for \( D^{-1}_{\mu\nu}(p) \) in the pure gluon sector. The main purpose of this work is to report a robust and effective procedure for determining \( D_{\mu\nu}(p) \) and to demonstrate its general validity for a range of hadronic phenomena. This procedure is to use the separable expansion technique which proved very effective in the 1960’s in studying non-relativistic few particle systems.

Having made the GCM truncation in (2) it is possible to proceed further and to transform the quark functional integrations into the hadronic functional integrations, as in (1). If the additional approximation \( D_{\mu\nu}(x - y) \to g\delta_{\mu\nu}\delta(x - y) \) is made in (2), i.e. a contact coupling of the quark currents, then the NJL type models are obtained. If in (1) a derivative expansion of the complete non-local hadronic effective action is performed, then the Chiral Perturbation Theory (CPT) phenomenology is obtained. However in the GCM, with appropriate \( D_{\mu\nu}(x) \), all computations are finite and no cutoffs or renormalisation procedures are used. As well, using a mean field approximation, the soliton phenomenology for the baryons may be derived, and has been studied in [5].

As shown here and elsewhere this approximation is surprisingly effective in describing the low energy QCD determined hadronic properties, and suggests that some particular mechanism is responsible for its success. There are indications that the neglected terms in (2) may only be significant in those bound states which are not colour singlets. In colour singlet states some colour neutrality may render the higher order terms ineffective. In the GCM the remaining gluon 2-point function is now regarded as an effective 2-point function: \( D_{\mu\nu}(x - y) \to D_{\mu\nu}(x - y)_{eff} \). There are then two key steps in the GCM: (1) the determination of this effective 2-point function, and the demonstration that the same function does well in determining a variety of hadronic observables, so that in some sense it is universal, and (2) the comparison of \( D_{eff} \) with
the true one obtained from QCD. This would establish to what extent the effects of the higher order terms have been accounted for by $D_{\text{eff}}$. From now on we shall call this $D$.

Having specified $D(p)$ the usual procedure would be to first determine the (constituent) quark propagators, in which $m$ is the quark current mass,

$$G(q) = (iA(q)q.\gamma + B(q) + m)^{-1} = -iq.\gamma\sigma_v(q) + \sigma_s(q)$$

using the non-linear Dyson-Schwinger equations (DSE), in Euclidean metric,

$$B(p) = \frac{16}{3} \int \frac{d^4q}{(2\pi)^4} D(p - q). \frac{B(q) + m}{q^2A(q)^2 + (B(q) + m)^2},$$

where (5), using the convolution theorem for Fourier transforms, has a particularly simple form in coordinate space

$$B(x) = \frac{16}{3} D(x)\sigma_s(x).$$

Eqn.(5) is the more sensitive one of (5) and (6): the meson form factors are closely related to the extent of $B(p)$, whereas $A(p)$ is changing more slowly. For simplicity we have used a Feynman-like gauge, and the perturbative quark-gluon vertex function. One can choose to use the GCM in other gauges. If one chooses a Landau gauge form

$$D_{\mu\nu}(p) = (\delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2})\Delta(p^2)$$

then from the $O(4)$ invariance of $B(p)$, we obtain that $D(p^2) \equiv \frac{3}{4}\Delta(p^2)$ in (5). A similar effect occurs in the BSE because for the low mass states, with the confining quark propagator, the form factors are almost $O(4)$ invariant wrt the relative momentum dependence $|s|$.

The DSE only have the forms in (5) and (6) for a translation invariant action, as in (2) or (3). To solve these equations for various $D(p)$ and to analytically continue $A(s)$ and $B(s)$ into the complex $s$-plane, where $s = q^2$, when solving for meson, diquark and baryon propagators is particularly difficult. Hence we have studied the well known separable expansion technique. This greatly facilitates the solutions of the SDE, the BSE and the Faddeev equations.
We first imagine expanding $D(p - q)$ into $O(4)$ hyperspherical harmonics

$$D(p - q) = D_0(p^2, q^2) + q.p D_1(p^2, q^2) + ...$$  \hspace{1cm} (8)

where, for example,

$$D_0(p^2, q^2) = \frac{2}{\pi} \int_0^\pi d\beta \sin^2 \beta D(p^2 + q^2 - 2pq \cos \beta).$$  \hspace{1cm} (9)

However note that putting $q = 0$ in (9) gives $D(p) = D_0(p^2, 0)$. Thus the full $D(p)$ may be easily reconstructed from the first term in the RHS of (8). Note also that only the first two terms in (8) are needed in (5) and (6); higher order terms do not contribute as they are orthogonal to the measures therein.

To facilitate the numerical computations we then introduce multi-rank separable expansions for each term

$$D_0(p^2, q^2) = \sum_{i=1,n} \Gamma_i(p^2)\Gamma_i(q^2).$$  \hspace{1cm} (10)

Introduction of the separable expansion clearly breaks translational invariance and must be regarded purely as a numerical procedure, much like a lattice breaks translation invariance. Translation invariance is restored as the rank of the separability is increased. The infrared hadronic region appears to be well described by a rank $n = 2$ form for $D_0$.

The DSE then have solutions of the form

$$B(s) = \sum b_i \Gamma_i(s), ...$$  \hspace{1cm} (11)

Equations for the $b_i, ..$ are easily determined by substituting these forms back into (5) and (6), giving coupled transcendental non-linear equations easily solved by iteration. Then $\sigma_s$ and $\sigma_v$ are seen to have the form of sums

$$\sigma_s(s) = \sum_{i=1,n} \sigma_s(s)_i, \quad \sigma_v(s) = \sum_{i=1,k} \sigma_v(s)_i.$$  \hspace{1cm} (12)

However this, in principle, procedure suffers from the defect that since $D$ is unknown then some ad hoc choice of the parametrised $\Gamma_i$ must be made. The resulting $\sigma_s(s)_i$ and $\sigma_v(s)_i$ develop spurious singularities which impede the use of the quark propagator in meson and baryon computations.

A much more robust and physically sensible procedure is to specify a parametrised form for the $\sigma_s(s)_i$ and $\sigma_v(s)_i$ of the chiral-limit quark propagator with known analyticity properties. We then use the DSE in an inverse manner to compute the $\Gamma_i(s)$. 

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By using entire functions the much speculated but as yet unproven quark confinement property can be ensured. Suitable forms are based on simple model solutions of the DSE. These forms are

\[
\sigma_s(s) = c_i \exp(-d_i s), \quad \sigma_v(s) = \frac{2s - \beta^2(1 - \exp(-2s/\beta^2))}{2s^2},
\]

where \(\sigma_v(s)\) has only a rank \(k = 1\) expansion here.

From these parametrised forms we can easily determine the various parameters \(\{b_i,\ldots\}\), which will depend on the basic, \(m = 0\), chiral-limit parameter set \(P_0 = \{c_1, c_2, d_1, d_2, \beta\}\). For example from (5) we obtain

\[
b^2_i = \frac{16}{3} \pi^2 \int_0^\infty sdsB(s)_i \frac{B(s)}{sA(s)^2 + B(s)^2} \]

in which \(B(s) = B(s)_1 + B(s)_2\), and

\[
B(s)_i = \sigma_s(s)_i/(s\sigma_v(s)^2 + \sigma_s(s)^2).
\]

The basic procedure is to find those values of these parameters \(P_0\) which allow the fitting of some small set of hadronic observables to the experimental values. We have chosen \(f_\pi\) (see [8] for the expression), which probes exclusively the space-like region of the quark propagators, and the \(a_1\) meson mass which extends the probe into the time-like region. The \(a_1\) meson is chosen because all two-quark states considered here, except for the \(1^-\) diquark correlation, have lower mass. Thus the region in the complex \(s\)-plane where the quark propagator is needed has been probed by our fitting procedure. A space-like only fitting procedure would require unreliable and untested extrapolations into the time-like region when computing other hadronic observables. The \(\pi\) mass is needed to mainly determine the averaged \(u\) and \(d\) current masses.

The \(a_1\) and \(\pi\) masses are computed using the Bethe-Salpeter equations (BSE), keeping only the dominant Lorentz amplitude [9]. The BSE are solved with the loop integration in the Euclidean metric, while the meson momentum, in the rest frame \(P = (0, iM)\), is in the time-like region of the Minkowski metric. This mixed metric is typical of hadron calculations in the GCM, and ensures that the quark propagators stay as close as possible to the real positive \(s\)-axis, which is the region most studied for the quark and gluon propagators. As is well known a separable expansion for the kernel
of an integral equation, here the gluon propagator in the BSE, reduces that equation to algebraic form. The solution of the BSE is thus rendered almost trivial.

We now consider the inclusion of quark current masses in the computations. For any set of $P_0$ values, which implicitly parametrise the gluon propagator, we can compute the non-chiral quark propagator by returning to solve (5) and (6) with $m \neq 0$. The resulting $\sigma_s$ and $\sigma_v$ now depend on $m$. However the approximate DSE is only robust in the space-like region. Hence we have fitted, in this region, $\sigma_s(s; m)$ and $\sigma_v(s; m)$ to the forms in (13). This ensures that the confinement ansatz continues to hold. It is this non-chiral quark propagator which is needed in the $\pi$ and $a_1$ mass computations. The value of the average u/d quark current mass $m$ is varied, along with the $P_0$ values, during the fitting procedure. Once the $P_0$ values were known we were able to use the DSE for the strange quark. Various values of the strange quark current mass were used until the K-meson BSE produced the experimental K meson mass.

The determined parameter values $P_0$ are shown in Table 1, and the values of the fitted observables are shown in Table 2. These three observables are sufficient to give a robust determination of the parameter set and the u/d quark mass. Table 2 also shows some of the predicted observables which test the universality of the quark and gluon propagators. Figs. 1 and 2 show the form of the chiral-limit $\sigma_s$ and $\sigma_v$.

Now we consider the task of reconstructing the translation invariant form for the effective gluon propagator. We can easily do this using $D(p) = D_0(p^2, 0)$ or (7) for $D(x)$. Hence if the translation invariant constituent quark propagator is known, and the separability technique does not compromise that property, then the full $D(p)$ or its Fourier transform $D(x)$, determined finally by the values of the parameter set $P_0$, may be constructed. It is this $D(p)$, but with the GCM analysis done in the appropriate gauge, which should be compared with future direct computations of the gluon propagator. Using the separable technique an explicit expression for $D(p)$ may be obtained. We have

$$D(p) = D_0(p^2, 0) = \sum_i \Gamma_i(p^2) \Gamma_i(0) = \sum_i \frac{1}{b_i^2} B(p^2)_i B(0)_i$$

$$= \sum_i \frac{1}{b_i^2} \frac{\sigma_s(0)_i}{\sigma_s(p^2)_i} \frac{\sigma_s(p^2)_i}{\sigma_s(p^2)_i} + \frac{\sigma_s(p^2)_i}{\sigma_s(p^2)_i},$$

an expression which becomes increasingly more accurate as the rank of the separable
expansion is increased. With the parameter set in Table 1, (14) gives $b_1 = 0.02672\, GeV^2$ and $b_2 = 0.02395\, GeV^2$, and the resulting $D(p)$ is shown in Fig.3. Note that the meson data mandates that the gluon propagator has a strong IR component (corresponding to a large distance confining effect) and a longer range component (corresponding to a medium distance effect), as also seen in the QCD studies in [2].

The predicted hadronic observables in Table 2 are mainly self-evident. The details of the GCM computational techniques have all been discussed in the literature. This set of hadronic predictions is obtained without cutoffs or renormalisation procedures. The meson and diquark masses are from BSE computations, while the nucleon-core mass (equivalent to the quenched approximation in lattice QCD) is from a covariant Faddeev computation [3, 10] keeping only the $0^+$ diquark correlation. The separable expansion for the gluon propagator leads to a separable form for the diquark correlation propagator. The original nucleon-core Faddeev computations [11] assumed such a separable form, see [3] for details.

Later computations of the nucleon core [11, 12, 13, 14] usually had free parameters that are adjusted to give the experimental nucleon mass. A full computation of the nucleon mass in the GCM requires the inclusion of the $1^+$ diquark correlation, and the dressing of this nucleon core by mesons (equivalent to including quark loops in lattice models).

The constituent quark masses arise from the most probable value of the quark running masses $(B(s, m) + m)/A(s)$ in the BSE kernel integrations, which is at $s \approx 0.3 GeV^2$. Expressions for the $\pi - \pi$ scattering lengths and the pion charge radius $r_\pi$ (without re-scattering corrections) are given in [5]. In [5] the MIT bag phenomenology was derived from the GCM using a mean field soliton approximation. This gave an expression for the MIT bag constant in terms of $B(s)$ and $A(s)$.

The good values of the predicted observables suggest that the GCM is capable of accurately describing low energy QCD, and that the gluon propagator multi-rank separable expansion is a particularly useful and robust computational tool. By including more experimental data in the fitted observables larger rank expansions could be used. This would result in a more accurate mapping out of the gluon and constituent quark propagators in the momentum range appropriate to low energy hadronic physics.
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Addendum: Frank and Roberts [19] have reported model gluon propagator results also from a fit to meson data using the separable expansion technique. Burden et al. [20] have also used the separable expansion technique for the gluon propagator, but in particular they have studied the meson BSE equation beyond the dominant Lorentz amplitude approximation. The results of both works should be compared with the results reported here.
Figure 1: Plot of extracted $\sigma_s(p) \ (GeV^{-1})$ plotted against $p^2 \ (GeV^2)$.
Figure 2: Plot of extracted $\sigma_v(p)$ plotted against $p^2 \ (GeV^2)$.
Figure 3: Plot of extracted $D(p) \ (GeV^{-2})$ plotted against $p^2 \ (GeV^2)$.

| Table 1: Quark Propagator Parameters: $\mathcal{P}_0$ |
|------------------------------------------------------|
| $c_1$ 0.5200GeV$^{-1}$                              | $c_2$ 1.1794GeV$^{-1}$ |
| $d_1$ 2.0737GeV$^{-2}$                              | $d_2$ 4.7214GeV$^{-2}$ |
| $\beta$ 0.5082GeV                                   |
| Observable                                              | Theory  | Expt./Theory[Ref] |
|----------------------------------------------------------|---------|-------------------|
| **Fitted observables**                                   |         |                   |
| $f_\pi$                                                   | 93.00MeV| 93.00MeV          |
| $a_1$ meson mass                                         | 1230MeV | 1230MeV           |
| $\pi$ meson mass                                         | 138.5MeV| 138.5MeV          |
| $K$ meson mass (for $m_s$ only)                          | 496MeV  | 496MeV            |
| **Predicted observables**                                |         |                   |
| $(m_u + m_d)/2$                                           | 6.5MeV  | 6.0               |
| $m_s$                                                    | 135MeV  | 130MeV            |
| $\omega$ meson mass                                      | 804MeV  | 782MeV            |
| $a_0^0\pi - \pi$ scattering length                      | 0.1634  | 0.21 ± 0.01       |
| $a_0^0\pi - \pi$ scattering length                      | -0.0466 | -0.040 ± 0.003    |
| $a_1^0\pi - \pi$ scattering length                      | 0.0358  | 0.038 ± 0.003     |
| $a_2^0\pi - \pi$ scattering length                      | 0.0017  | 0.0017 ± 0.003    |
| $a_2^2\pi - \pi$ scattering length                      | -0.0005 | not measured      |
| $r_\pi$ pion charge radius                               | 0.55fm  | 0.66fm            |
| nucleon-core mass                                        | 1390MeV | ~1300MeV          |
| constituent quark rms size                               | 0.59fm  | -                 |
| chiral quark constituent mass                            | 270MeV  | -                 |
| u/d quark constituent mass                                | 300MeV  | ~340MeV           |
| s quark constituent mass                                  | 525MeV  | ~510MeV           |
| $0^+$ diquark rms size                                   | 0.78fm  | -                 |
| $0^+$ diquark constituent mass                           | 692MeV  | >400MeV           |
| $1^+$ diquark constituent mass                           | 1022MeV | -                 |
| $0^-$ diquark constituent mass                           | 1079MeV | -                 |
| $1^-$ diquark constituent mass                           | 1369MeV | -                 |
| MIT bag constant                                         | (154MeV)$^4$ | (146 MeV)$^4$ |
| MIT nucleon-core mass (no cm corr.)                      | 1500MeV | ~1300MeV          |
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Fig. 2
Fig. 3