Unitarily inequivalent quantum cosmological bouncing models

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By quantising the background as well as the perturbations in a simple one fluid model, we show that there exists an ambiguity in the choice of relevant variables, potentially leading to incompatible observational physical predictions. In a classical or quantum inflationary background, the exact same canonical transformations lead to unique predictions, so the ambiguity we put forward demands a semiclassical background with a sufficiently strong departure from classical evolution. The latter condition happens to be satisfied in bouncing scenarios, which may thus be having predictability issues. Inflationary models could evade such a problem because of the monotonic behavior of their scale factor; they do, however, initiate from a singular state which bouncing scenarios aim at solving.

I. INTRODUCTION

Cosmological perturbations are usually studied on a classical or semiclassical background in the framework of inflation [1]. Most models of bouncing alternatives are either based on a classical background [2, 3] or it is assumed that the semiclassical approximation ensures similar behaviour for the perturbations. The purpose of this paper is to show that there might be some important caveat that should be taken into account as an unsolved ambiguity can emerge in a quantum bouncing scenario. It is worth mentioning that already in classical backgrounds, the notion of the initial vacuum state depends on the choice of perturbation variables for quantization as noted e.g. in [4]. Herein, we show that once the background is quantised, the physical ambiguity gets much stronger and concerns the dynamics of mode functions as well. A similar point was considered in Refs. [5, 6] for an inflationary background, leading to a vanishingly small effect.

To illustrate our point, we examine a simple model based on canonical quantization of general relativity (GR) in which the matter content is represented by a perfect fluid with constant equation of state $w\in[0,1]$. We first recall the classical model in its Hamiltonian formulation both for the background (singular) universe and the perturbations, before moving to a quantum approach aiming at resolving the classical singularity.

The paper is organized as follows. Sec. II first introduces the classical model consisting of general relativity sourced by a simple constant equation-of-state fluid, for which we define the Hamiltonian version of both the singular background and the divergent perturbations. We then move, in Sec. III, to the quantum model by assuming a very general quantization procedure allowing to account for self-adjointness issues on the half line; the perturbations are then treated in the usual (canonical) way. The ensuing quantum dynamics is examined in Sec. IV where the semiclassical approximation for the background is found to yield two different, and incompatible, equations of motion for the perturbation modes, leading to a potential ambiguity in the observational predictions. Our conclusions are followed by an appendix showing an explicit example of quantization based on coherent states with definite fiducial vectors.

II. CLASSICAL MODEL

In this section we provide the definition of the classical model together with two different but physically equivalent parametrizations. The physical phase space for the model is introduced together with the physical Hamiltonian that generates its dynamics with respect to an internal clock. The solution to the classical dynamics is briefly discussed.

A. Hamiltonian formalism

We assume the universe to be spatially compact, $\mathcal{M}\simeq\mathbb{R}\times\mathbb{T}^3$, with coordinate volume we note $V_0$ below. Its evolution is supposed to be driven by a perfect fluid that satisfies a barotropic equation of state $p=w\rho$, with $-\frac{1}{3} < w < 1$. The fully canonical formalism for the perturbed Friedmann universe that can be easily adapted to the present case can be found in [7]; we start from the Einstein-Hilbert-Schutz action [8, 9]

$$S_{\text{EHS}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} P(w,\phi),$$

where $P=w\rho$ is the pressure of the cosmic fluid while $\phi$ defines its flow. The action $S_{\text{EHS}}$ is first expanded to second order around the flat Friedmann universe. Next the

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Hamiltonian description is obtained in which the truly physical degrees of freedom are identified and the remaining ones removed.

Let us consider the usual Einstein-Hilbert action \( S_{\text{EH}} \) at zeroth order, omitting the integrated term

\[
\frac{1}{16\pi G_n} \int d^4x \sqrt{-g} R = - \frac{1}{2\kappa} \int d\tau N a^3 \int \sqrt{\gamma d^3x} \frac{\dot{a}^2}{\dot{a}^2 N^2},
\]

in which we used the background isotropic and homogeneous flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

\[
ds^2 = -N^2(\tau) d\tau^2 + a^2(\tau) \gamma_{ij} dx^i dx^j,
\]
a dot meaning a derivative with respect to the coordinate time \( \tau \), later to be identified with the fluid clock variable. Written as \( S_{\text{EH}} = \int L^{(0)}(a, \dot{a}) \text{d}t \), with Lagrangian \( L^{(0)} = 3\sqrt{\gamma} \dot{a} \gamma_{ij} \gamma^{ij}/(N\kappa) \), this yields the canonically conjugate momentum \( p_a = \partial L^{(0)}/\partial \dot{a} = 6\sqrt{\gamma} a \ddot{a}/(\kappa N) \), and the gravitational Hamiltonian at zeroth order \( H^{(0)}_a \) reads

\[
H^{(0)}_a = -\kappa N \frac{p_a}{12\sqrt{\gamma} a^2},
\]

which can also be expressed in terms of the canonical variables,

\[
q = \frac{4\sqrt{6}}{3(1-w)\sqrt{1+w}} a^{3/2(1-w)} \equiv \gamma a^{3/2(1-w)},
\]

thereby defining the constant \( \gamma \), and

\[
p = \frac{\sqrt{6(1+w)}}{2\kappa_0} a^{3/2(1+w)} H,
\]

where \( H = \dot{a}/(Na) \) is the Hubble rate and \( \kappa_0 = \kappa/V_0 \). The Hamiltonian \( H^{(0)}_a \) reads

\[
H^{(0)}_a = -\frac{2\kappa_0 N}{(1+w) a^{3w} w^2} = -2\kappa_0 \dot{p}_a^2,
\]

where in the last equality, we made the choice of the lapse \( N = (1+w)\dot{a}^{3w} \). It can be shown that for this particular choice of the lapse the matter Hamiltonian \( H^{(0)}_m \) obtained from the Schutz action \( S_g \) equals the cosmic fluid conjugate momentum (see, e.g. [10] for details). Therefore, the total Hamiltonian generates a uniform motion in the cosmic fluid variable. It is a standard procedure at this point to promote the cosmic fluid variable to the role of internal clock while removing it and its conjugate momentum from the phase space. The physical Hamiltonian that generates the dynamics of the background geometry with respect to the fluid variable is thus simply \( H^{(0)}_a \). However, we find it convenient to inverse the direction of time with respect to the fluid variable in order to have the positive physical Hamiltonian,

\[
H^{(0)} = -H^{(0)}_a = 2\kappa_0 \dot{p}_a^2.
\]

We shall denote the internal clock by \( \tau \) and assume it coincides with the FLRW time set in (3) [1]. It can be shown that the Hamiltonian \( H^{(0)} = (1+w)E_{\text{fl}}=1 \) equals \((1+w)\times\) the energy of the fluid contained in the universe when \( a = 1 \) (we choose a dimensionless scale factor, so that the canonical variable \( q \) is also dimensionless).

After identification of the truly physical degrees of freedom also at linear order, we write the full Hamiltonian \( H_{\text{full}} \) as

\[
H_{\text{full}} = H^{(0)} - \sum_k H^{(2)}_k,
\]

where the second-order Hamiltonian \( H^{(2)}_k \), depending only on the discrete (recall the Universe considered is compact) wavevector \( k \), reads

\[
H^{(2)}_k = \frac{1}{2} |\pi_{\phi,k} |^2 + \frac{1}{2} w(1+w)^2 \left( \frac{q}{\gamma} \right)^2 k^2 |\phi_k|^2,
\]

with \( \gamma \) defined in (5) above. The Fourier component \( \phi_k \) of the perturbation field is a combination of the fluid perturbation \( \delta \phi_k \) and the intrinsic curvature perturbation \( \delta \kappa_k \), namely [7]

\[
\phi_k = \frac{1}{\sqrt{2w(1+w)\kappa_0}} \frac{\sqrt{3} a^{-3w/2}}{4k^2} \delta \kappa_k + \sqrt{\frac{3}{w\kappa_0}} \frac{1}{2} \sqrt{\frac{a^{-3w/2}}{4k^2}} \delta \kappa_k.
\]

with \( k \equiv |k| \) the amplitude of the wavevector; note that since the FLRW background (3) is isotropic, it is expected, as usual, that the initial conditions, and therefore the solutions of the perturbation evolution equation should depend only on the amplitude \( k \) and not on its direction \( k/k \). Given our conventions, the physical dimensions are \( [\dot{\phi}_k] = \sqrt{M} L \) and \( [\pi_{\phi,k}] = \sqrt{M} \). The Poisson bracket reads \( \{\phi_{k_1}, \pi_{\phi,-k_2}\} = \delta_{k_1,k_2} \). The equation of motion expressed in the conformal time \( \eta \) defined below [see Eq. (18)], is found to read

\[
\phi'' + \left( \frac{q}{\gamma} \right)^2 \frac{4(3w-1)}{3(1-w)} w(1+w)^2 k^2 \phi_k = 0.
\]

It shows that for radiation, i.e. for \( w = 1/3 \), the dynamics of \( \phi_k \) becomes decoupled from the dynamical background.

There exists infinitely many parametrizations of the reduced phase space of perturbations and the initial parametrization \( (\phi, \pi_\phi) \) is just one example. We shall call it the fluid-parameterization, as the relevant time for that description is \( \tau \), which stems from the fluid. As another example, let us consider a pair of canonical fields

\footnote{The background fluid time \( \tau \) is actually a combination of the fluid variable and its momentum, \( (1+w)\tau = \phi|p_\phi|^{-1/w} \). For more details, see e.g. [7].}
(v, π_v) that is commonly used for solving the dynamics of scalar perturbations

\[ v_k = Z \phi_k, \quad \pi_{v,k} = Z^{-1} \pi_{\phi,k} + \frac{\dot{Z}}{Z} \phi_k, \]  

(13)

where

\[ Z(\tau) = \sqrt{1 + w \left( \frac{q}{\gamma} \right)^{3(\omega + 1)/(1 - \omega)}} \]  

(14)

and τ, as mentioned above, denotes the internal fluid time. Note that in the comoving gauge δΦ_k = 0, and

\[ H_k^{(2)} = \frac{1}{2} (1 + w) \left( \frac{q}{\gamma} \right)^{\frac{2(\omega + 1)}{3(\omega + 1)/(1 - \omega)}} \left\{ |\pi_{v,k}|^2 + \left[ w k^2 - \frac{8}{9 q^2} \left( \frac{q}{\gamma} \right)^{\frac{4(1 - 3 \omega)}{3(1 - \omega)}} \frac{(2 \kappa_0)^2 (1 - 3 w) p^2}{(1 + w)^2 (1 - w)^2} \right] |v_k|^2 \right\}, \]  

(17)

where we used the background equations of motion by assuming p → const.; we shall call the set of variables \((v_k, \pi_{v,k})\) the conformal parametrization, as it involves naturally the conformal time. It differs from the fluid parametrization (10) by the nontrivial coefficient standing in front of the entire expression as well as the frequency that now depends on both background variables, q and p.

The coefficient in front of the Hamiltonian (17) can be removed by switching to the internal conformal time η [11]

\[ d\eta = Z^2 d\tau = (1 + w) \left( \frac{q}{\gamma} \right)^{\frac{2(\omega + 1)}{3(\omega + 1)/(1 - \omega)}} d\tau, \]  

(18)

in terms of which the potential (16) takes the simpler and usual form \(\mathcal{V}_{cl} = Z''/Z\), where a prime means a derivative with respect to the conformal time η \((Z' = dZ/d\eta)\). The second-order Hamiltonian \(Z^{-2} H_k^{(2)}\) is then found to generate

\[ v_k'' + \left[ w k^2 - \frac{8}{9 q^2} \frac{(2 \kappa_0)^2 (1 - 3 w)}{(1 - w)^2} p^2 \right] v_k = 0, \]  

(19)

which can be written in the usual Mukhanov-Sasaki form

\[ v_k'' + \left[ w k^2 - \mathcal{V}_{cl}(\eta) \right] v_k'' = v_k'' + \left( w k^2 - \frac{z''}{z} \right) v_k = 0, \]  

(20)

thereby identifying the classical potential

\[ \mathcal{V}_{cl}(\eta) = \frac{8}{9 q^2} \frac{(2 \kappa_0)^2 (1 - 3 w)}{(1 - w)^2} p^2 = \frac{z''}{z}, \]  

(21)

where the last equality is obtained by applying the classical equations of motion Eq. (24) below and we have thus \(v_k = -\sqrt{\frac{3(1 + w)}{16 \pi \kappa_0}} a \Psi_k\), where \(\Psi_k = -a^2 \delta R_k/k^2\) is the comoving curvature. We easily obtain the second-order Hamilton \(H_k^{(2)}\) in terms of \((v, \pi_v)\), namely

\[ H_k^{(2)} = \frac{1}{2} Z^2 \left\{ |\pi_{v,k}|^2 + [w k^2 - \mathcal{V}_{cl}(\tau)] |v_k|^2 \right\}, \]  

(15)

with the potential \(\mathcal{V}_{cl}\) defined through

\[ \mathcal{V}_{cl} = \frac{1}{Z^4} \left[ \frac{\dot{Z}}{Z} - 2 \left( \frac{\dot{Z}}{Z} \right)^2 \right], \]  

(16)

which can be written explicitly in terms of the background canonical variables q and p as

used the generic function z, as there are in fact two different and equivalent choices that can be made, namely \(z_1 = q^{r_1}\) and \(z_2 = q^{r_2}\), with

\[ r_1 = \frac{3 w - 1}{3 (1 - w)}, \quad r_2 = \frac{2}{3 (1 - w)}. \]  

(22)

These two power laws stem from the fact that although what enters into (15) is \(Z''/Z\), with \(Z \sim z_1\), one can then just as well choose the second solution of \(z''/z = Z''/Z\), namely \(z_2 \sim Z \int d\eta/Z^2 = Z \int d\tau = Z \tau\) which, taking the background solution \(q \sim \tau\) [see Eq. (25) below] yields \(z_2 \sim Z q = z_1 q = q^{r_1+1}\), as indeed one has \(r_2 = r_1 + 1\).

The internal conformal time provides a convenient form of the equation of motion for perturbations. We shall, however, quantize the dynamics of both the background and the perturbations reduced with respect to a unique internal time, the internal fluid time. The term \(z''/z\) is usually referred to as the potential for the perturbations, as Eq. (20) is mathematically identical to a time-independent Schrödinger equation in such a potential [12]. The potential

\[ \mathcal{V}_{cl} = \frac{z''}{z} = 1 - 3 w \mathcal{H}^2, \]  

(23)

has the clear physical meaning of the conformal Hubble rate \(\mathcal{H}\) squared \((w < \frac{1}{3})\). Therefore, the conformal Hubble rate determines the coordinate scale at which the amplification of perturbations starts to take place.

### B. Background solution

The background Hamilton equations,

\[ \frac{dq}{d\tau} = 4 \kappa_0 p \quad \text{and} \quad \frac{dp}{d\tau} = 0, \]  

(24)
have the solution
\[
q(\tau) = \sqrt{8\kappa H(0)(\tau - \tau_0)} \quad \text{and} \quad p(\tau) = \sqrt{\frac{H(0)}{2\kappa_0}},
\]
(25)
where \(H(0)\) is the value of the zeroth-order Hamiltonian, a constant by virtue of its definition (8) and the equation of motion (24). The phase space trajectories that either terminate at or emerge from the singularity at time \(\tau_0\) are straight lines in phase space \(\{q,p\}\) with constant \(p\) [11], shown as straight lines in Fig. 1. Note that in order to assign the correct trajectory to the background universe, one needs to know the value of the energy of the fluid in the whole universe when \(a = 1\). This value can be determined only when one knows the size of the universe, size which can be fixed by demanding that the volume of the observable patch be a given ratio (less than unity) of the size of the full universe.

Anticipating the quantum solution (46), we write the classical solution as \(q = q_0\omega \tau\) (setting the singularity time to \(\tau_0 = 0\)), and therefore \(p = q_0\omega / (4\kappa_0)\). Eq. (18) with this solution permits to integrate explicitly for the conformal time \(\eta\), also assuming \(\eta \rightarrow 0\) for \(\tau \rightarrow 0\). One finds the classical conformal time to read
\[
\eta = \frac{1 + w}{2r_1 + 1} \left(\frac{q_0\omega}{\gamma}\right)^{2r_1} \tau^{2r_1 + 1},
\]
which is straightforwardly inverted to yield \(\tau(\eta)\), and finally
\[
q(\eta) = q_0\omega \left[\frac{2r_1 + 1}{1 + w} \left(\frac{q_0\omega}{\gamma}\right)^{-2r_1} \eta\right]^{1/(2r_1 + 1)} \propto \eta^{3(1-w)}/a,
\]
(27)
and the classical potential then reads
\[
V_{cl} = \frac{(q'^2)''}{q'^2} = \frac{(q'^2)''}{q'^2} \propto \frac{2(1 - 3w)}{(1 + 3w)^2\eta^2},
\]
(28)
as usual for a background dominated by a perfect fluid with constant equation of state.

C. Solution for perturbations

The two parametrizations described above, \((\phi, \pi_\phi)\) and \((v, \pi_v)\), are physically equivalent and therefore it is sufficient to consider just one of them, e.g., the conformal one, in order to determine the dynamics of perturbations. Using the definition (18) to derive the power-law behaviour of \(q(\eta)\) in (25), the potential \(z''/z\) in Eq. (20) is found to yield the specific form (28) (independently of the choice \(z = z_1\) or \(z = z_2\)), so that the classical evolution of perturbation modes is
\[
\frac{d^2v_k}{d\eta^2} + \left[w k^2 - \frac{2(1 - 3w)}{(1 + 3w)^2\eta^2}\right] v_k = 0.
\]
(29)
Clearly, the potential \(V_{cl} \propto \eta^{-2}\) is singular at this point too. The solution can be expressed in terms of Hankel functions, namely
\[
v_k(\eta) = \sqrt{\eta} \left[c_1(k)H^{(1)}_{\nu}(\sqrt{\kappa}k\eta) + c_2(k)H^{(2)}_{\nu}(\sqrt{\kappa}k\eta)\right],
\]
(30)
where \(\nu = \frac{3(1-w)}{2(3w+1)}\) and \(c_1(k), c_2(k)\) are constants depending on the comoving wavevector \(k\) through the initial conditions; for isotropic initial conditions as those used for quantum vacuum fluctuation, they can depend only on the amplitude \(k\) and not on the direction \(k/k\).

The solution is finite but discontinuous at \(\eta = 0\). Therefore, the comoving curvature \(\Psi_k \propto v_k/a\) in general blows up at \(\eta = 0\) where the scale factor reaches the singularity \(a \rightarrow 0\); see Ref. [13] for a full treatment of the relevant cases.

III. FULL QUANTUM MODEL

In the present section we quantize the Hamiltonian (9) in the two parametrizations we introduced above. Next we apply some approximations in order to integrate the dynamics. We find that the two classically equivalent parametrizations lead to two unitarily inequivalent quantum theories. This dependence on parametrization is a natural consequence of the nonlinearity of the theory of gravity. Recall that Dirac’s “Poisson bracket → commutator” quantization rule [14] works only for simplest observables. It is well-known that there exists no quantization of any given classical system that is an isomorphism between the Poisson and commutator algebras (there is actually one known exception that nevertheless is irrelevant in the present context, see [15] for an exhaustive review). As a result, a quantized observable is in general unitarily inequivalent if its quantization is made with a different choice of basic observables. Note that this “obstruction” is absent when quantum perturbations evolve linearly in classical backgrounds as in the latter case all the possible sets of basic variables are related by linear transformations that enjoy unique unitary representations consistent with Dirac’s rule.

The phase space for the cosmological background is the half-plane rather than the full plane and hence the usual canonical quantization rules seem to be inadequate. There exist many quantization methods (see, e.g. [16] for a comprehensive review) some of which one could find more suitable in the present context. In order to account for this issue, we introduce a family of quantum models, all of which in correspondence with the underlying classical model. They are given by a set of free parameters that can be computed, for instance, in the framework of the so-called affine quantization (see [17] for details) that has been proposed for a consistent quantum gravity programme [18, 19]; we briefly recap what is relevant for our purposes of this method in Appendix A. This approach enables us to free ourselves from a particular quantization
method and thus to emphasise the universal character of the quantization ambiguity that we study below.

We then introduce a semiclassical approximation to the quantum dynamics of the background geometry, which is a standard step in such models. It should be noted that any ambiguous effect such as the one we obtain here at a semiclassical level may only be enhanced if a fully quantum description of the background were to be used. We carefully construct the semiclassical description with the use of coherent states.

A. Background

Given the existing ambiguities due to factor ordering when going from classical to quantum, we propose the following set of operators to replace the Hamiltonian (8):

\[ H^{(0)} \rightarrow \hat{H}^{(0)} = 2\kappa_0 \left( \hat{P}^2 + \hbar^2 c_0 \hat{Q}^{-2} \right), \]  

where \( c_0 \geq 0 \) is a free parameter. The value \( c_0 = 0 \) corresponds to the “canonical quantization”, whereas the values \( c_0 > 0 \) can be justified in various ways, for instance by using the affine group as the symmetry of quantization \( [10, 20] \). In the latter case, the repulsive potential \( \alpha \hat{Q}^{-2} \), of quantum geometric origin, prevents the universe from reaching the singular point \( q = 0 \) by reversing its motion from contraction to expansion. If \( c_0 \geq \frac{3}{4} \), then \( \hat{H}^{(0)} \) is essentially self-adjoint and no boundary condition needs to be imposed at \( \hat{Q} = 0 \) to ensure a unique and unitary dynamics. The only way to determine the right value of the parameter \( c_0 \) is to compare the predictions of the model with the actual observations of the Universe.

We will need quantum operators to replace other zeroth-order quantities, for those appear in the Hamiltonians relevant for describing perturbations (10) and (17). We propose the following replacements

\[ q^\alpha \mapsto l(\alpha)\hat{Q}^\alpha, \]  

\[ q^\alpha p^\beta \mapsto a(\alpha)\hat{Q}^\alpha \hat{P}^2 + i\hbar b(\alpha)\hat{Q}^{\alpha-1}\hat{P} + \hbar^2 c(\alpha)\hat{Q}^{\alpha-2}, \]  

where \( \hat{Q} \) and \( \hat{P} \) are the ‘position’ and ‘momentum’ operators on the half-line, satisfying the usual commutation relation \( [\hat{Q}, \hat{P}] = i\hbar \), and therefore \( [\hat{Q}^\alpha, \hat{P}] = i\hbar \alpha \hat{Q}^{\alpha-1} \), so that \( b(\alpha) = -\alpha a(\alpha) \) in order to ensure that the second-line operator (32b) is symmetric, i.e. so that it reads

\[ q^\alpha p^\beta \mapsto a(\alpha)\hat{P}\hat{Q}^\alpha \hat{P} + \hbar^2 c(\alpha)\hat{Q}^{\alpha-2}; \]  

the power-depending numbers \( (\alpha), a(\alpha) \) and \( c(\alpha) \) are assumed positive and dimensionless.

B. Perturbations

The canonical perturbation variables of the fluid parametrization satisfy the reality condition \( \phi_k^\alpha = \phi_{-k} \) and \( \pi_{\phi, k} = \pi_{\phi, -k} \) and it is possible to promote their real and imaginary parts to canonical operators in \( L^2[\mathbb{R}^2, \frac{1}{2}\delta_{\phi_k}d\phi_k^\alpha] \) for each direction \( k \). It is, however, more convenient to work with the Fock representation,

\[ \phi_k \mapsto \hat{\phi}_k = \sqrt{\frac{\hbar}{2}} \left[ a_k \phi_k^\alpha(\tau) + a_{-k}^\dagger \phi_k^\alpha(\tau) \right], \]  

where the time-dependent mode functions \( \phi_k(\eta) \) are assumed to be isotropic and \( a_k \) and \( a_k^\dagger \) are fixed annihilation and creation operators that satisfy \( [a_k, a_k^\dagger] = \delta_{k_1, k_2} \) (recall the compactness of space implies discrete eigenvectors \( k \)). As shown later, it follows that the mode functions must satisfy a suitable normalization condition. Note that the whole evolution of the operators \( \hat{\phi}_k \) and \( \hat{\pi}_{\phi, k} \) in the Heisenberg picture is encoded into the mode functions.

Combining the background quantization with the quantization of perturbations, using the definition (22) of the classical power laws, yields the quantized Hamiltonian (10) in the fluid parametrization (henceforth dubbed \( \hat{F} \)-parametrization)

\[ \hat{H}_k^{(2)} = \frac{1}{2} |\hat{\pi}_{\phi, k}|^2 + \frac{\Sigma_q}{2} w(1 + w)^2 \left( \frac{\hat{Q}}{\gamma} \right)^{4r_1} k^2 |\hat{\phi}_k|^2, \]  

where \( \Sigma_q = 1(4r_1) \) is a free parameter of the quantization.

We repeat the same quantization for the conformal parametrization (\( c \)-parametrization in what follows),

\[ v_k \mapsto \hat{v}_k = \sqrt{\frac{\hbar}{2}} \left[ a_k v_k(\tau) + a_{-k}^\dagger v_k(\tau) \right], \]  

and obtain the quantum \( c \) Hamiltonian derived from (17) as

\[ \hat{H}_k^{(2)} = \frac{1}{2} (1 + w) \left( \frac{\hat{Q}}{\gamma} \right)^{2r_1} \mathcal{M}_q H_k^{(2),\text{eff}}, \]  

with

\[ \hat{H}_k^{(2),\text{eff}} = |\hat{\pi}_{\phi, k}|^2 + \left[ w k^2 - \frac{8\mathcal{M}_q^{-1}(2\kappa_0)^2(1 - 3w)}{9Q^2} (1 - w)^2(1 + w)^2 \left( \frac{\hat{Q}}{\gamma} \right)^{-4r_1} \mathcal{M}_q(\hat{P}^2 + i\hbar \mathcal{M}_q \hat{Q}^{\alpha-1}\hat{P} + \hbar^2 \Sigma_q \hat{Q}^{\alpha-2}) \right] |\hat{v}_k|^2, \]
where $M_q = l(2r_1)$, $M_q = a(-2r_2)$, $R_q = b(2r_2)$ and $S_q = c(-2r_2)$ are free parameters in the quantization map. Note that there are more free parameters and hence more quantization ambiguities in the $C$ parametrization.

IV. QUANTUM DYNAMICS

A general approach to solving the dynamics of quantum perturbations in quantum spacetime was recently given in [21]. In what follows, we assume the full state vector to be a product of background and perturbation states, i.e.,

$$|\psi(\tau)\rangle = |\psi_{b}(\tau)\rangle \cdot |\psi_{p}(\tau)\rangle. \tag{39}$$

The canonical formalism for cosmological perturbations has been developed under the assumption that the perturbations do not backreact on the background spacetime. Therefore, the dynamics of $|\psi_{b}(\tau)\rangle$ should be determined independently of the state $|\psi_{p}(\tau)\rangle$.

A. Background semiclassical solution

It is very useful to have at disposal background solutions $|\psi_{b}(\tau)\rangle$ corresponding to various energies and with various spreads in $\hat{Q}$ and $\hat{P}$. One can find a wide class of solutions by approximating the Hilbert space with a family of coherent states built from a single wavefunction, the so-called fiducial vector; this construction is presented in Appendix A. For the present purpose, suffice it to note that any fixed family of coherent states is given by state vectors $(q, p) \mapsto |q, p\rangle$ in one-to-one correspondence with the phase space. In practice, from a fiducial state $|\xi\rangle$, for which $\langle \xi | \hat{Q} | \xi\rangle = 1$ (recall $q$, and therefore $\hat{Q}$, is dimensionless) and $\langle \xi | \hat{P} | \xi\rangle = 0$, one builds the coherent state through [19]

$$|q(\tau), p(\tau)\rangle = e^{\eta(\tau) \hat{Q} / \hbar} e^{-i q(\tau) \hat{P} / \hbar} |\xi\rangle, \tag{40}$$

where $\hat{D} = \frac{1}{2}(\hat{Q} \hat{P} + \hat{P} \hat{Q})$ is the dilation operator. The expectation values of $\hat{Q}$ and $\hat{P}$ in $|q(\tau), p(\tau)\rangle$ are respectively $q(\tau)$ and $p(\tau)$.

The dynamics confined to the vectors $|q(\tau), p(\tau)\rangle$ can be deduced from the quantum action

$$S_{\text{q}} = \int \langle q(\tau), p(\tau)\rangle \left(i \hbar \frac{\partial}{\partial \tau} - \hat{H}(0)\right) |q(\tau), p(\tau)\rangle d\tau, \tag{41}$$

which, upon using the properties of the state (40), can be transformed into

$$S_{\text{q}} = \int \{\dot{q}(\tau) p(\tau) - H_{\text{sem}} [q(\tau), p(\tau)]\} d\tau, \tag{42}$$

with the semiclassical Hamiltonian given by

$$H_{\text{sem}} = (q, p) \hat{H}(0) |q, p\rangle \langle q, p| \tag{43}$$

from which one derives the ordinary Hamilton equations

$$\dot{q} = \frac{\partial H_{\text{sem}}}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H_{\text{sem}}}{\partial q}. \tag{44}$$

Given the quantum Hamiltonian (31), we find that the semiclassical background Hamiltonian reads, by virtue of our ordering choice (33) (with $\alpha = 0$)

$$H_{\text{sem}} = 2\kappa_0 \left(p^2 + \frac{\hbar^2 \hat{R}}{q^2}\right), \tag{45}$$

where the constant $\hat{R}$ is positive ($\hat{R} > 0$), irrespective of whether $c_0 = 0$ or $c_0 > 0$. The actual value of $\hat{R}$ depends on the choice of family of coherent states, as illustrated in Appendix A. We find the solution to (44) to read

$$q = q_b \sqrt{1 + (\omega \tau)^2}, \tag{46a}$$

$$p = \frac{q_b \omega^2}{4 \kappa_0} \frac{\tau}{\sqrt{1 + (\omega \tau)^2}}, \tag{46b}$$

where $q_b^2 = 2\kappa_0 \hbar^2 \hat{R} / H_{\text{sem}}$ and $\omega = 2H_{\text{sem}} / (\hbar \sqrt{\hat{R}})$. We display in Fig. 1 a few trajectories in the phase space illustrating these solutions.

With this semiclassical solution, one can also integrate (18) to yield the conformal time $\eta$, as a function of $\tau$

$$\eta = (1 + w) \tau \left(\frac{q_b}{\gamma}\right)^{2r_1} \mathcal{F} \left[\frac{1}{2}, -r_1; \frac{3}{2}; - (\omega \tau)^2\right], \tag{47}$$

where $\mathcal{F}(a, b; c; z)$ is the hypergeometric function (see Sec. 15 of Ref. [22]). As expected, one recovers the classical power law (20) in the large-time classical limit $\tau \gg \omega^{-1}$, up to a constant depending on the equation of state $w$ and vanishing for $w = \frac{1}{3}$. Fig. 2 shows the classical and quantum relationships $\eta(\tau)$. 

Figure 1. Background phase space evolutions: the straight lines represent Eqs. (25), either going to or emerging from a singularity ($q \rightarrow 0$), while the curves are the solutions (46) leading to the same asymptotic classical lines. The semiclassical solution are seen to consist of a bounce smoothly joining expanding ($p > 0$) and contracting ($p < 0$) classical universes.
Figure 2. Conformal time $\eta$ as a function of $\tau$, for the classical (26) and quantum (47) solutions for $w = 0$ (thin line), $w = 0.1$ (thick), $w = 0.2$ (dashed) and $w = 0.3$ (dotted). The quantum conformal time tends to the classical one up to a constant factor, which vanishes for $= \frac{1}{4}$.

B. Perturbation modes

Given that the dynamics of the background state is fixed by $|\psi_b\rangle \rightarrow |q(\tau), p(\tau)\rangle$, the dynamics of the perturbation state $|\psi_p(\tau)\rangle$ can be deduced from the quantum action at second order $S^{(0) + (2)} = S_b + S_p$

$$S^{(0) + (2)} = \int \langle \psi(\tau) \rangle \left(i\hbar \frac{\partial}{\partial \tau} - \hat{H}^{(0)} + \sum_k \hat{H}_k^{(2)} \right) |\psi(\tau)\rangle d\tau,$$

with the state vector given by (39). Extracting the zeroth order action $S_b$, one finds

$$S_b = \int \langle \psi_b \rangle \left(i\hbar \frac{\partial}{\partial \tau} + \sum_k \hat{H}_k \right) |\psi_b\rangle d\tau,$$

and setting $|\psi_b\rangle = \prod_k |\psi_k\rangle$ with $|\psi_k\rangle, |\psi_{k_1, k_2}\rangle = \delta_{k_1, k_2}$, one gets the associated Schrödinger equation for each Fourier mode $|\psi_k\rangle$ (up to an irrelevant phase factor), namely

$$i\hbar \frac{\partial}{\partial \tau} |\psi_k\rangle = \hat{H}_k |\psi_k\rangle,$$

where the operator $\hat{H}_k = -(q, p)\hat{H}_k^{(2)} |q, p\rangle$ is obtained from either (35) or (37) depending on the choice of parametrization. In the former case, the second-order Hamiltonian generating the dynamics of perturbations in the fluid parametrization reads

$$\langle q, p | \hat{H}_k^{(2)} |q, p\rangle = \frac{1}{2} |\bar{\pi}_{\phi, k}|^2 + \frac{\mathcal{L}}{2} w(1+w)^2 \left( \frac{q}{q} \right)^{4r_1} k^2 |\phi_k|^2,$$

where the value of $\mathcal{L}$ depends on the value of $\mathcal{L}_q$ and the family of coherent states used to approximate the background dynamics.

The Heisenberg equations of motion read

$$\frac{d}{d\tau} \tilde{\phi}_k = -\tilde{\pi}_{\phi, k},$$

$$\frac{d}{d\tau} \tilde{\pi}_{\phi, k} = \mathcal{L} w(1+w)^2 \left( \frac{q}{q} \right)^{4r_1} k^2 |\phi_k|^2,$$

and it follows from (52a) that

$$\tilde{\pi}_{\phi, k} = \sqrt{\hbar} \left[ a_k \phi_k^{*}(\tau) + a_k^{\dagger} \phi_k(\tau) \right],$$

and hence the canonical commutation rule, namely $[\hat{q}_{-k}, \hat{p}_{\phi, k}] = i\hbar$, implies the normalization condition on the mode functions $\phi_k \phi_k^{*} - \phi_k^{*} \phi_k = 2i$. By combining the above equations, we may obtain the second-order dynamical equation for $\phi_k$, which must also be obeyed by the mode function $\phi_k$. We switch to the internal conformal clock given by Eq. (18) and rescale the mode functions, $v_k = Z \phi_k$, where $v_k$ is the Mukhanov-Sasaki variable. The superscript “f” indicates that its dynamics is generated by the fluid Hamiltonian. More specifically, we find that the dynamics of $v_k$ generated by the Hamiltonian (51) reads

$$\frac{d^2 v_k}{d\eta^2} + \left[k^2 - V_{\nu}(\eta)\right] v_k = 0,$$

with the effective wave number $k_{\nu} = \sqrt{\sum w k}$, and the fluid potential given by

$$V_{\nu} = \frac{8}{9q^2 Z^4} \frac{(2\kappa a)^2 (1 - 3w)}{(1 - w)^2} \left[ p^2 - \frac{3(1-w)\Re}{2q^2} \right].$$

Note that for large $q$, i.e. away from the bounce, the semiclassical correction becomes negligible so that the semiclassical potential (53) approaches the classical one (17). Indeed, using $Z/Z = r_1 \dot{q}/q$ and $q' = \dot{q}/Z^2$, one finds

$$\frac{Z''}{Z} = \frac{r_1}{Z^4} \left[ \dot{q} - (1 + r_1) \left( \frac{\dot{q}}{q} \right)^2 \right],$$

and replacing the function $q(\tau)$ by the solution (46) for the background semiclassical trajectory, it is straightforward to check that, for all times, the potential $V_\nu$ can be given the familiar form $V_\nu = Z''/Z = (q^*)''/q^*$. Since the semiclassical trajectory (46) is asymptotic to the classical one (25) for $\omega \tau \to \infty$, i.e. for $\eta \to \infty$, the fluid potential satisfies

$$\lim_{\eta \to \infty} V_\nu(\eta) = V_c(\eta),$$

where $V_c$ is given by (28); it is illustrated in Fig. 3.

The same procedure applied to the conformal parametrization yields

$$\langle q, p | \hat{H}_k^{(2)} |q, p\rangle = \frac{1}{2} Z^2 \mathcal{M}_n \left[ |\bar{\pi}_{\phi, k}|^2 + \Omega^2 |\phi_k|^2 \right],$$
where they are well-approximated by their classical counterpart given by $V_{cl} = \frac{2(1-3w)}{(1+3w)^2\eta}$ (dotted line) [cf. Eq. (29)].
the quantization concerns both the linear perturbations and the background variables, the transformation of the perturbation variables (13) is nonlinear, contrary to the situation of Ref. [23], and therefore, it leads to unitarily inequivalent theories.

V. CONCLUSION

We have suggested a finite cosmological model in which quantum gravitational effects play a leading role, resolving the classically expected singularity to a bouncing scenario. Our model consists in adding to general relativity a perfect fluid with constant equation of state $w$. Classically, the FLRW solution initiates out of or contracts to a singularity at which the scale factor $a$ vanishes. The perturbations around such a background also tend to diverge at the singularity.

Upon quantizing the background, factor ordering ambiguities permit to add to the zeroth order Hamiltonian a repulsive potential term. Assuming a coherent state to describe the semiclassical evolution, one can then calculate a phase space trajectory which, thanks to the quantum effective potential, smoothly connects the contracting and expanding solutions, avoiding the singularity in the process.

Most models then would identify these bouncing trajectories as classical, and would then go on to quantize the perturbations on top. By doing so, one would then be allowed whatever canonical transformation on the perturbation variables, leading to classically undistinguishable theories. Here however, we take seriously the quantum nature of the background time development and show that the classically harmless canonical transformations become unitarily inequivalent theories with potentially different physical predictions.

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 Appendix A: Affine coherent states and affine quantization

In what follows we discuss the affine coherent states and their application to affine quantization and to semiclassical description of dynamics [10, 24–26].
Coherent states and quantization

The background phase space \((q,p)\) is the half-plane that is not invariant under the usual group of \(q-\) and \(p-\) translations. For this reason the application of "canonical quantization" based on the unitary and irreducible representation of the group of translations, the Weyl-Heisenberg group, is problematic. It is however possible to consider a more general quantization that is based on any minimal group of canonical transformations that enjoys a nontrivial unitary representation, the so-called covariant integral quantization. In the case of the half-plane the natural choice is the affine group of a real line, \((q,p) \in \mathbb{R}^+ \times \mathbb{R},\)

\[
(q',p') \circ (q,p) = \left( q' q, \frac{p}{q} + p' \right).
\]
(A1)

Its unitary, irreducible and square-integrable representation in the Hilbert space \(\mathcal{H} = L^2(\mathbb{R}^+, dx)\) reads

\[
\langle x|U(q,p)|\psi\rangle = \langle x|q,p\rangle = \sqrt{\frac{e^{ipx/\hbar}}{2\pi}} \left( \frac{x}{q} \right),
\]
(A2)

where \(\psi(x) = \langle x|\psi\rangle \in \mathcal{H}.

Let us consider a particular example of the covariant integral quantization that is based on coherent states. In the present case, they are the affine coherent states, \(\mathbb{R}^+ \times \mathbb{R} \ni (q,p) \mapsto |q,pangle := U(q,p)|\xi\rangle \in \mathcal{H},\)

where \(|\xi\rangle\) is the so-called fiducial vector, a fixed normalized vector in Hilbert space such that \(\mathcal{N} = \rho(0) < \infty,\)

\[
\rho(\alpha) = \int \frac{|\xi(x)|^2}{x^\alpha + 1} \, dx,
\]

and the operator \(U(q,p)\) is given by Eq. (A1). The resolution of unity is

\[
\int \frac{dq dp}{2\pi \hbar \mathcal{N}} |q,p\rangle \langle q,p| = 1,
\]
(A3)

as can be verified in a straightforward manner using Eq. (A2) and applying the above operator on two arbitrary states \(|\phi_1\rangle\) and \(|\phi_2\rangle:\)

\[
\int \frac{dq dp}{2\pi \hbar \mathcal{N}} \langle \phi_1|q,p\rangle \langle q,p|\phi_2\rangle = \int dx \phi_1^*(x) \phi_2(x) = \langle \phi_1|\phi_2\rangle,
\]
(A4)

using the usual closure relation

\[
\int dx |x\rangle \langle x| = \mathbb{1}
\]

and the property

\[
\delta(x-y) = \int \frac{dp}{2\pi \hbar} e^{ip(x-y)/\hbar}
\]

for the Dirac distribution.

The affine coherent state quantization is obtained by substituting functions of \(q\) and \(p\) are replaced by \(f(q,p) \mapsto A_f := \int_{\mathbb{R}^+ \times \mathbb{R}} \frac{dq dp}{2\pi \hbar \mathcal{N}} |q,p\rangle f(q,p) \langle q,p|,\)

(A5)

with \(\mathcal{N}\) the normalization constant. Let us also introduce

\[
\sigma(\alpha) = \int \frac{d\xi(x)}{\sqrt{\mathcal{N}}} \frac{d\xi(x)}{dx} \left| \frac{x}{q}\right|^{\alpha - 1},
\]
(A6)

which is the same as \(\rho\) with the function \(\xi(x)\) replaced by its derivative \(\xi'(x)\).

One may easily find the affine coherent state quantization (A5) of the following observables through (see, e.g., Appendices of [26] or [27] for explicit computations)

\[
A_1 = 1,
\]
(A7a)

\[
A_q^a = a(\alpha) \hat{Q}^a,
\]
(A7b)

\[
A_p = \hat{P},
\]
(A7c)

\[
A_{q^2, p^2} = a(\alpha) \hat{Q}^a \hat{P}^2 - i a h a(\alpha) \hat{Q}^a \hat{P}^2 + c(\alpha) h^2 \hat{Q}^{a^2}
\]
(A7d)

where \(\hat{Q}\) and \(\hat{P}\) are the ‘position’ and ‘momentum’ operators on the half-line. Eqs. (A7b) and (A7c) are to be understood as \(|x| A_{q,p} |\phi\rangle = a(\alpha) x^a \phi(x)\) and \(|x| A_{p} |\phi\rangle = -i h d\phi/dx,\) where \(\phi(x) := \langle x|\phi\rangle.\)

The parameters

\[
a(\alpha) = \frac{\rho(\alpha)}{\rho(0)}\]

and

\[
c(\alpha) = \frac{1}{2} \alpha (1 - \alpha) a(\alpha) + \frac{\sigma(\alpha - 2)}{\rho(0)}
\]

are calculable for any real fiducial vector \(\xi(x)\), which should be chosen such that \(a(1) = 1,\) i.e. \(\rho(1) = \rho(0)\) in order to ensure that \(A_q = \hat{Q}\) so that Eqs. (A7b) and (A7c) implement the required usual commutation relation \([A_q, A_p] = [\hat{Q}, \hat{P}] = i h.\)

From the above, it follows that the application of the affine quantization (A5) to the background Hamiltonian (8) yields

\[
H^{(0)} \mapsto \hat{H}^{(0)} = 2 \kappa \left( \hat{P}^2 + h^2 c_0 \hat{Q}^{-2} \right),
\]
(A8)

with \(c_0 = c(0) = \sigma(-2)/\rho(0).\)

Furthermore, using again (22), one may easily calculate the various constants appearing in the perturbation Hamiltonians, namely

\[
\varphi_q = \frac{\rho(4r_1)}{\rho(0)},
\]
(A9)

for (35), as well as

\[
\varphi_R = \frac{\rho(2r_1)}{\rho(0)},
\]
\[
\varphi_R = \frac{\rho(-2r_2)}{\rho(0)},
\]
(A10)

\[
\varphi_R = 2 \hbar r_2 \varphi_q,
\]
and
\[ \mathcal{F}_q = -r_2(1 + 2r_2)\mathcal{N}_q + \frac{\sigma (-2r_2 - 2)}{\rho(0)}, \] (A11)
which appear in (37). Obviously, these parameters are to a large extent free as the affine quantization depends on the fiducial vector \( |\xi\rangle \). One might think about the coherent state quantization based on the fiducial vector as a convenient method for parameterizing natural ordering ambiguities.

### Semiclassical approximation

The most important application of the affine coherent states in the present work is to derive a useful semiclassical description. As discussed around Eq. (40), one needs to ensure the so-called physical centering condition \( \langle \hat{Q} \rangle = 1 \), where the expectation value is taken in the fiducial state. This condition may not be satisfied by the state \( |\xi\rangle \), already normalised to enforce the canonical commutation relation, and so we introduce a new real fiducial vector \( |\tilde{\xi}\rangle \) and the associated moments \( \tilde{\rho}(\alpha) = \int_{\mathbb{R}^+} \frac{dx}{2\pi} |\tilde{\xi}|^2 \) and \( \tilde{\sigma}(\alpha) = \int_{\mathbb{R}^+} \frac{dx}{2\pi} |\tilde{\xi}|^2 \). We find
\[
\langle q, p | \hat{Q}^\alpha \hat{P}^\beta | q, p \rangle = \tilde{\rho}(\alpha - 1)q^\alpha p^\beta + i\alpha \tilde{\rho}(\alpha)q^{\alpha - 1} p
+ \left[ \tilde{\sigma}(\alpha - 1) + \frac{\alpha(1 - \alpha)}{2} \tilde{\rho}(\alpha + 1) \right] q^{\alpha - 2},
\] (A12a)
\[
\langle q, p | \hat{Q}^\alpha \hat{P} | q, p \rangle = \tilde{\rho}(\alpha - 1)q^\alpha p + i\frac{\alpha}{2} \tilde{\rho}(\alpha)q^{\alpha - 1},
\] (A12b)
\[
\langle q, p | \hat{Q}^\alpha | q, p \rangle = \tilde{\rho}(\alpha - 1)q^\alpha.
\] (A12c)

Note that the special case \( \alpha = 0 \) in (A12c) yields the normalisation \( \langle q, p | \hat{Q}^\alpha | q, p \rangle = \tilde{\rho}(\alpha - 1) = \langle \xi | \xi \rangle = 1 \).

For the quantum Hamiltonian (A8), we introduce the following semiclassical Hamiltonian
\[
H_{\text{sem}} := \langle q, p | \hat{H}_{(0)} | q, p \rangle = 2\kappa_0 \left( p^2 + \frac{\hbar^2 R}{q^2} \right),
\] (A13)
where the new constant \( \mathcal{R} \) is given by \( \mathcal{R} = c_0 \mathcal{R}(1) + \mathcal{R}(-2) \).

As for perturbations, it is straightforward to determine the constant in (51), namely
\[
\mathcal{F}_q = \mathcal{N}_q \tilde{\rho}(4r_1 - 1),
\] (A14)
whereas one gets
\[
\mathcal{M}_q = \mathcal{N}_q \tilde{\rho}(-2r_1 - 1),
\mathcal{N}_q = \mathcal{N}_q \tilde{\rho}(2r_2 - 1),
\] (A15)
and
\[
\mathcal{F}_q = \mathcal{N}_q \tilde{\sigma}(2r_2 - 1) + \mathcal{N}_q \tilde{\rho}(2r_2 + 1)
\] (A16)
for those appearing in (55).

---

![Figure 6. Fiducial functions \( \xi_{\nu}(10x) \) and \( \tilde{\xi}_{\nu}(x/10) \) (blue), for \( \nu, \mu = 1 \) (thin line), 2 (full), 3 (dashed) and 4 (dotted). For better readability of the figure, the functions have been shifted so that \( \xi_{\nu} \) appears centered around 0.1 and \( \tilde{\xi}_{\nu} \) around 10. As functions of \( x \), they should all be centered around \( x = 1 \).](image)

### Fiducial vectors

For the sake of concreteness in the present discussion let us consider some examples of fiducial vectors and the specific values of \( \tilde{\rho}(\alpha) \), \( \tilde{\sigma}(\alpha) \), \( \tilde{\alpha}(\alpha) \) and \( \tilde{\sigma}(\alpha) \) that they produce. We use two distinct families of fiducial vectors for quantization and for semiclassical approximation. This is due to the fact that they satisfy special and distinct conditions. Namely, the fiducial vectors for quantization are such as to preserve the canonical commutation rule (on the half-line), whereas the fiducial vectors for semiclassical approximations are such as to yield the expectation values for the momentum and position operators in any coherent state, aligned with the phase space point to which a given coherent state is assigned.

We consider the following family of fiducial vectors for quantization
\[
\xi_{\nu}(x) = \left( \frac{\nu}{\pi} \right)^{\frac{1}{4}} \frac{1}{\sqrt{x}} \exp \left[ -\frac{\nu}{2} \left( \ln x - \frac{3}{4\nu} \right)^2 \right],
\] (A17)
where \( \nu > 0 \) is assumed, and for which we obtain the corresponding coefficients
\[
\rho_{\nu}(\alpha) = \exp \left[ \frac{(\alpha - 2)(\alpha + 1)}{4\nu} \right],
\sigma_{\nu}(\alpha) = \left[ \frac{\nu}{2} + \left( \frac{\alpha + 2}{2} \right)^2 \right] \exp \left[ \frac{\alpha(\alpha + 3)}{4\nu} \right],
\] (A18)
which are positive definite. As expected, one verifies that \( \rho_{\nu}(1) = \rho_{\nu}(0) = e^{-1/(2\nu)} \), as needed to ensure the correct commutation relation between the position variable and its associated canonical momentum. We also note that \( \langle \xi | \hat{Q} | \xi \rangle = \rho_{\nu}(-2) = e^{3/(2\nu)} \neq 1 \), so the physical centering condition is not fulfilled by this fiducial state.

As for the semiclassical description, we consider the
where now $\mu > 0$ is assumed. In this case, we obtain

$$\tilde{\rho}_\mu(\alpha) = \exp\left[\frac{(\alpha + 1)(\alpha + 2)}{4\mu}\right],$$

$$\tilde{\sigma}_\mu(\alpha) = \left[\frac{\mu}{2} + \left(\frac{(\alpha + 2)}{2}\right)^2\right] \exp\left[\frac{(\alpha + 3)(\alpha + 4)}{4\mu}\right].$$

(A20)

These are also positive definite as expected. It is now clear that $\tilde{\rho}_\mu(-2) = 1$, as expected for the semiclassical description, but that now $\tilde{\rho}(1) = e^{\beta/(2\nu)} \neq e^{\gamma/(2\nu)} = \tilde{\rho}(0)$ so that these fiducial vectors cannot be used for quantization.

Some example functions $\xi_\mu$ and $\tilde{\xi}_\mu$ are displayed in Fig. 6.

The above relations permit to actually calculate the various coefficients appearing in the previous sections. First, one finds that $\xi_0 = \nu/2$, so that it suffices to demand $\nu \geq \frac{3}{2}$ to ensure self-adjointedness of the Hamiltonian (31). As for its semiclassical counterpart (45), one finds

$$\Re = \left(\frac{\nu}{2} + \frac{2\mu + 1}{4}\right) \exp\left(\frac{3}{2\mu}\right),$$

whose minimum value $\Re_{\text{min}}$ is reached for $\nu = 0$ and $\mu_{\text{min}} = (3 + \sqrt{21})/4 \approx 1.89$, at which point one has $\Re_{\text{min}} \approx 2.64$.

Moving to the quantum corrections to the evolution of perturbations, we find

$$\frac{\mathbf{T}_n}{\Re_n} = \left[\frac{1}{4} + \frac{\mu + \nu}{2}\right] \exp\left[\frac{17 - 9w}{6\mu(1 - w)}\right],$$

so that the conformal potential can be cast into the usual $z^n/z$ form if the equation

$$(\mu + \nu + \frac{1}{2}) \exp\left[\frac{17 - 9w}{6\mu(1 - w)}\right] = \frac{3(1 - w)}{1 - 3w} \left[\nu + \left(\mu + \frac{1}{2}\right) \exp\left(\frac{3}{2\mu}\right)\right],$$

has non trivial solutions for $\mu, \nu > 0$. This is solved for $\nu$ as a function of $\mu$ and $w$ through

$$\nu(w, \mu) = \frac{\exp\left(\frac{3}{2\mu}\right) - \frac{1 - 3w}{3(1 - w)} \exp\left[\frac{17 - 9w}{6\mu(1 - w)}\right]}{\frac{1 - 3w}{3(1 - w)} \exp\left[\frac{17 - 9w}{6\mu(1 - w)}\right] - 1} \left(\mu + \frac{1}{2}\right).$$

(A21)

Fig. 7 illustrates the behavior of (A21) for various values of $w$. For the conformal radiation case $w = \frac{1}{3}$, Eq. (A21) may only be satisfied for $\nu < 0$, in contradiction to the assumption. As expected from the form (58) of the potential $V_c$, the limit $w = \frac{1}{3}$ yields an identically vanishing potential, and (60) is undefined unless $\Re$ vanishes, which does not happen with the basis used.

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