Spectra of Tridiagonal Matrices over a Field

R. S. Costas-Santos*

Dpto. de Física y Matemáticas, Facultad de Ciencias, Universidad de Alcalá, 28871
Alcalá de Henares, Spain

C. R. Johnson

Department of Mathematics, College of William and Mary, Williamsburg, VA 23187

Abstract

We consider spectra of \( n \)-by-\( n \) irreducible tridiagonal matrices over a field and of their \( n - 1 \)-by-\( n - 1 \) trailing principal submatrices. The real symmetric and complex Hermitian cases have been fully understood: it is necessary and sufficient that the necessarily real eigenvalues are distinct and those of the principal submatrix strictly interlace. So this case is very restrictive.

By contrast, for a general field, the requirements on the two spectra are much less restrictive. In particular, in the real or complex case, the \( n \)-by-\( n \) characteristic polynomial is arbitrary (so that the algebraic multiplicities may be anything in place of all 1’s in the classical cases) and that of the principal submatrix is the complement of a lower dimensional algebraic set (and so relatively free). Explicit conditions are given.

Keywords: Eigenvalues, Irreducible, Orthogonal polynomials, Characteristic polynomial, Recurrence relation, Tridiagonal matrix.

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1. Introduction

An \( n \)-by-\( n \) matrix \( A = (a_{ij}) \) is called tridiagonal if \(|i - j| > 1\) implies \(a_{ij} = 0\). Such a matrix may have nonzero entries only on the sub-, super-,
and main diagonals.

\[
A = \begin{bmatrix}
        & & & & 0 \\
        & & & \downarrow & \\
        & & \downarrow & & \\
        & \downarrow & & & \\
        0 & & & & \\
    \end{bmatrix}
\]

We are interested in the eigenvalues of such a matrix and of its trailing \((n-1)\)-by-\((n-1)\) principal submatrix: \(A(1) = A[\{2, 3, \ldots, n\}]\), which we view in terms of their characteristic polynomials, over a general field \(F\). If \(\{\lambda_1, \ldots, \lambda_n\}\) occur as the eigenvalues of \(A\) and \(\{\mu_1, \ldots, \mu_{n-1}\}\) as the eigenvalues of \(A(1)\), they will also occur for a tridiagonal matrix with all super-diagonal entries nonzero. When the super-diagonal entries are all nonzero, they may be normalized to be all be 1’s, via diagonal similarity.

So, wlog, we consider normalized tridiagonal matrices. Our \(A\) looks like

\[
A = \begin{bmatrix}
    a_1 & 1 & 0 & \cdots & 0 \\
    b_1 & a_2 & 1 & \ddots & \vdots \\
    0 & b_2 & \ddots & \ddots & 0 \\
    \vdots & \ddots & \ddots & \ddots & 1 \\
    0 & \cdots & 0 & b_{n-1} & a_n
\end{bmatrix}
\]

In this case, some sub-diagonal entries may be 0, in which case \(A\) is reducible, or all may be nonzero, in which case \(A\) is irreducible. In the reducible case, \(A\) and \(A(1)\) must have eigenvalues in common.

The number of common eigenvalues is \(k\) (counting multiplicities) if and only if \(b_{n-k} = 0\) and no prior \(b_i's\) are 0. We focus upon the irreducible case, from which the reducible case may be deduced, in which case \(A\) and \(A(1)\) have no common eigenvalues.

Let \(p_n(t) = \det(tI - A)\) and \(p_{n-1}(t) = \det(tI - A(1))\), the characteristic polynomial of \(A\) and \(A(1)\), respectively. Generally let \(p_k(t)\) be the characteristic polynomial of the trailing \(k\)-by-\(k\) principal submatrix.

Via determinantal expansion, we have the following known relationships:

\[
p_n(t) = (t - a_1)p_{n-1}(t) - b_1p_{n-2}(t),
\]
and generally
\[ p_{k+1}(t) = (t - a_{n-k})p_k(t) - b_{n-k}p_{k-1}(t), \quad k = 0, \ldots, n-1, \] (3)
in which \( p_{-1}(t) = 0 \) and \( p_0(t) = 1. \)

From these, it is clear, in the irreducible case, that

i) \( p_{n-2}, p_{n-3}, \ldots, p_1 \) are uniquely determined by \( p_n \) and \( p_{n-1} \), and thus
the eigenvalues of all the trailing principal submatrices are determined by those of the first two;
ii) \( p_n \) and \( p_{n-1} \) are relatively prime, so that \( A \) and \( A(1) \) have no eigenvalues in common. The same is true for \( p_{k+1} \) and \( p_k \). However, \( p_{k+1} \) and \( p_{k-r} \) could have common roots for \( r \geq 1. \)

**Example 1.** The matrix
\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]
has spectra \( \sigma(A) = \{-\sqrt{2}, 0, \sqrt{2}\} \) and \( \sigma(A(1,2)) = \{0\}. \)

iii) Unlike the Hermitian case, \( p_n \) may have multiple roots but, like the Hermitian case, each of them has geometric multiplicity 1 as an eigenvalue of \( A \) (as \( \text{rank}(tI - A) \geq n - 1 \) for each \( t \)).
So the eigenvalues of \( A \) may be algebraically multiple, but not geometrically so.

Suppose now that \( p \) and \( q \) are given monic polynomials, i.e., the leading coefficient is 1, over \( \mathbb{F} \), of degree \( n \) and \( n-1 \), respectively. If there is an \( n \)-by-\( n \) tridiagonal matrix over \( \mathbb{F} \), such that \( p = p_n \) and \( q = p_{n-1} \), we call \( p \) and \( q \) a **tridiagonal pair** (TrP), and if the tridiagonal matrix may be taken to be irreducible, we call \( p \) and \( q \) an **irreducible tridiagonal pair** (ITrP).

We seek to understand which pairs are ITrP over \( \mathbb{F} \) and which polynomials \( p \) occur in an ITrP. Not all pairs are ITrP, but every monic \( p \) of degree \( n \) does occur as \( p_n \) when \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \) (not in general). So, over \( \mathbb{R} \) or \( \mathbb{C} \) an irreducible tridiagonal matrix may have any characteristic polynomial, and thus, any algebraic multiplicities for its eigenvalues, in stark contrast to the real symmetric or Hermitian cases.
2. Theorems and Examples

Given monic polynomial \( p \) and \( q \) over \( \mathbb{F} \) of degrees \( n \) and \( n - 1 \), when we apply the division algorithm to them, one of two things may happen: either a) the degree of the remainder drops by exactly 1 each time, so that the algorithm consumes \( n - 1 \) steps, or b) at some stage of the division algorithm, there is a drop in degree by more than 1.

In case a), which is generic over \( \mathbb{R} \) or \( \mathbb{C} \), we call \( p \) and \( q \) a \textbf{proper pair}. If \( p \) is given, we say that \( q \) is proper with \( p \), and we may refer to the set of such \( q \) as the \textbf{proper set for} \( p \).

If \( p \) and \( q \) form an ITrP, the proof that \( p \) and \( q \) is a proper pair is similar. We note that not every relatively prime pair \( p \) and \( q \) is an ITrP, even over \( \mathbb{R} \).

\begin{proof}
Suppose that \( p \) and \( q \) form a proper pair. Then, upon division of \( p \) by \( q \), according to (2), we may conclude what \( a_1, b_1 \) and \( p_{n-2} \) would have to be in order to have \( p_n = p \) and \( p_{n-1} = q \).

Since \( p, q \) is proper, \( b_1 \) is nonzero and well-defined, and \( \deg p_{n-2} = n - 2 \). However, also since \( p, q \) is proper, we may continue by applying (3) to \( p_{n-1} = q \) and \( p_{n-2} \) to get \( a_2, b_2 \) and \( p_{n-3} \) in the same way; \( b_2 \neq 0 \) and \( \deg p_{n-3} = n - 3 \). Again, as \( p, q \) is proper, we may continue to get \( a_3, b_3 \) and \( p_{n-4} \), and so on.

This allows us to construct the unique (normalized) irreducible tridiagonal matrix \( A \) for which \( p_n = p \) and \( p_{n-1} = q \), which shows that \( p, q \) is an ITrP.

If \( p \) and \( q \) form an ITrP, the proof that \( p \) and \( q \) is a proper pair is similar. We have \( p_n = p \) and \( p_{n-1} = q \), so that (2) and (3) imply that \( p, q \) is a proper pair, as \( b_1, b_2, \ldots, b_{n-1} \neq 0 \).

We note that when \( p, q \) is a proper pair (and thus an ITrP), the irreducible (normalized) tridiagonal matrix that realizes them is uniquely determined. So \( p_{n-2}, \ldots, p_1 \) (and their roots, the eigenvalues of the trailing principal submatrices) are fully determined. It is an interesting question how these roots are a function of the roots of \( p \) and \( q \).

We note that not every relatively prime pair \( p \) and \( q \) is an ITrP, even over \( \mathbb{R} \).
Example 2. Let \( n = 3 \) and \( 2, -3, -5 \) be the roots of monic \( p_3 \) and \( 1, -1 \) the roots of the monic \( p_2 \). Then \( p_2 \) and \( p_3 \) are relatively prime, but there is no tridiagonal matrix with eigenvalues \( 2, -3, -5 \) and with \( 1, -1 \) as the eigenvalues of the upper \( 2 \times 2 \) principal submatrix. We have

\[
p_3(t) = t^3 + 6t^2 - t - 30, \quad \text{and} \quad p_2(t) = t^2 - 1.
\]

Suppose there is a tridiagonal matrix \( A = (a_{ij}) \). Then

\[
p_3(t) = (t - a_1)p_2(t) - b_1p_1(t).
\]

This implies that \( a_1 = -6 \) and that \( p_1(t) = 24/b_1 \), which is a polynomial of degree 0. Therefore \( p_3, p_2 \) is not a proper pair and not an ITrP.

Example 3. Let \( p_4 \) have roots \( -1, -2, 3, 4 \) and \( p_3 \) have roots \( -3, 1, 2 \). Then \( p_3 \) and \( p_4 \) are relatively prime. Let us assume that there is a \( 4 \times 4 \) tridiagonal matrix \( A \) with \( p_A(t) = p_4(t) \) and \( p_A(1)(t) = p_3(t) \). If we apply the division algorithm to \( p_3 \) and \( p_4 \), we get

\[
p_4(t) = (t - 4)p_3(t) - 12t + 48 = (t - 4)p_3(t) - 12p_2(t).
\]

Therefore the degree of \( p_2 \) drops by 2; hence such a \( 4 \times 4 \) tridiagonal matrix does not exist.

However, because the proper set of a monic polynomial over \( \mathbb{R} \), or \( \mathbb{C} \) is the complement of a sufficiently low dimensional algebraic set, the proper set is necessarily nonempty.

Theorem 2. Suppose that \( p \) is a monic degree \( n \) polynomial over \( \mathbb{R} \) or \( \mathbb{C} \). Then, there is a monic polynomial \( q \) over the same field as \( p \) such that \( p \) and \( q \) form an ITrP.

Proof. For a given \( p \) the existence of such \( q \) is straightforward since the proper set of \( p \) is the complement an algebraic set which is strictly contained in \( \mathbb{R}^{n-1} \) or \( \mathbb{C}^{n-1} \).

Nevertheless, it may happen for other fields that the result for theorem 2 is not true.

Example 4. Over the field \( \mathbb{GF}_2 \), not every monic polynomial is attained as the characteristic polynomial of an irreducible \( 3 \times 3 \) tridiagonal matrix.
Let \( p(t) = t^3 + 1 \). If \( A \in M_3(GF_2) \) is irreducible and tridiagonal, then \( A \) is of the form

\[
A = \begin{bmatrix}
d_1 & 1 & 0 \\
1 & d_2 & 1 \\
0 & 1 & d_3
\end{bmatrix},
\]

with each \( d_i = 0 \) or \( 1 \), \( i = 1, 2, 3 \). Then

\[
p_A(t) = t^3 - (d_1 + d_2 + d_3)t^2 + (d_1d_2 + d_1d_3 + d_2d_3)t - (d_1d_2d_3 - d_1 - d_3).
\]

For \( d_1 + d_2 + d_3 = 0 \), either 0 or 2 of the \( d_i \)’s is 1. Then for \( d_1d_2 + d_1d_3 + d_2d_3 = 0 \), it must be that all \( d_i = 0 \) (if two are equal to 1, this expression is 1). But if all are 0, then \( d_1d_2d_3 - d_1 - d_3 = 0 \), not 1.

By a simple counting argument, over any finite field \( \mathbb{F} \) some polynomials do not occur as the characteristic polynomial of a normalized irreducible tridiagonal matrix. If \( \mathbb{F} \) has \( k \) elements, then there are \( (k - 1)^{n-1} \) such matrices, but \( k^n \) distinct monic polynomials. It is an interesting question which polynomials are realized.

**Corollary 3.** Over \( \mathbb{R} \) or \( \mathbb{C} \), an irreducible tridiagonal matrix may have any characteristic polynomial (and thus, any eigenvalues, counting multiplicities).

For real symmetric and complex Hermitian irreducible tridiagonal matrices, it is known [6] that the only multiplicity list that occurs for the eigenvalues is all 1’s. And, in general, the maximum geometric multiplicity is 1. However, for algebraic multiplicity, the situation is quite different.

For a further reading about the Inverse eigenvalue problems for band matrices see e.g. [1, 2, 4].

**Corollary 4.** Any partition of \( n \) may be the list of algebraic multiplicities of an irreducible tridiagonal matrix over \( \mathbb{R} \) or \( \mathbb{C} \).

The (undirected) graph of an irreducible tridiagonal matrix is simply a path. We conjecture that the same is true for other trees, i.e. any algebraic multiplicities may occur, and this is true for the star on \( n \) vertices [5, 8].

Though over \( \mathbb{R} \) or \( \mathbb{C} \) any polynomial occurs as the characteristic polynomial of an irreducible tridiagonal matrix, it is not easy to explicitly give a tridiagonal matrix realization. In the next section, we show how a realization may be given, using some ideas form orthogonal polynomials.
3. Tridiagonal Matrices and Orthogonal Polynomials

The theory of linear functionals is a natural tool to understand tridiagonal realizability. We first give some basic facts we need.

Given a linear functional \( L : \mathbb{F}[t] \to \mathbb{F} \) we denote by \( m_k = L(t^k) \), for all \( k = 0, 1, \ldots \), the moments of \( L \), and by \( H_k \) the \((k+1)\times(k+1)\) Hankel matrix

\[
H_k = \begin{bmatrix}
m_0 & m_1 & m_2 & \cdots & m_k \\
m_1 & m_2 & m_3 & \cdots & m_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_k & m_{k+1} & m_{k+2} & \cdots & m_{2k}
\end{bmatrix}.
\]

(4)

The linear functional \( L \) is said to be quasi-definite if \( \det(H_k) \neq 0 \) for all \( k = 0, 1, \ldots \).

Remark 5. In this work we fix an integer, \( n > 0 \), and, since we are interested in \( n \times n \) matrices, it is enough to suppose that \( \det(H_k) \neq 0 \) for \( k = 0, 1, \ldots, n-1 \). Therefore it is not an issue if there exists some \( N > n \) such that \( H_N \) is singular. So, we will say that \( L \) is quasi-definite if the matrices \( H_0, H_1, \ldots, H_{n-1} \) are all invertible.

The following result is well-known for orthogonal polynomial sequences:

**Proposition 6.** [3, p. 17] For any quasi-definite linear functional \( L \), there exists a polynomial sequence \( \{p_k\} \), unique up to a multiplicative constant, defined by

\[
p_k(t) = \begin{bmatrix}
m_0 & m_1 & m_2 & \cdots & m_k \\
m_1 & m_2 & m_3 & \cdots & m_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{k-1} & m_k & m_{k+1} & \cdots & m_{2k-1} \\
1 & t & t^2 & \cdots & t^k
\end{bmatrix}, \quad k = 0, 1, \ldots, \quad (5)
\]

that fulfills the property of orthogonality

\[
L(p_\ell p_k) = 0, \quad n \neq m, \quad \ell, k = 0, 1, \ldots,
\]

\[
L(p_k^2) \neq 0, \quad k = 0, 1, \ldots
\]
Note that, due to the normalization taken for our tridiagonal matrices, we need to consider the following normalization for the polynomials:

\[ P_0(t) = 1, \quad P_k(t) = \frac{1}{\det(H_{k-1})} p_k(t), \quad k = 1, 2, 3, \ldots \]

and \( P_k(0) = \det(\tilde{H}_{k-1})/\det(H_{k-1}) \), in which

\[
\tilde{H}_{k-1} = \begin{bmatrix}
m_1 & m_2 & \cdots & m_k \\
m_2 & m_3 & \cdots & m_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
m_k & m_{k+1} & \cdots & m_{2k-1}
\end{bmatrix}.
\]

**Remark 7.** Since we are considering tridiagonal matrices normalized so that the superdiagonal is all 1’s, it is more convenient to use monic polynomials. Observe that if \( (p_k) \) satisfies the recurrence relation (3) then \( (P_k) \) satisfies the following recurrence relation for \( k = 1, \ldots, n-1 \):

\[
tP_k(t) = \frac{\det(H_k)}{\det(H_{k-1})} P_{k+1}(t) + a_{n-k} P_k(t) + b_{n-k} \frac{\det(H_{k-2})}{\det(H_{k-1})} P_{k-1}(t), \quad (6)
\]

and since \( \mathcal{L} \) is quasi-definite, \( \det(H_k) \neq 0 \) for all \( k = 0, 1, \ldots, n-1 \).

Observe that \( p_k(x) \) has degree \( k \) if and only if \( H_k \) is regular for \( k = 0, 1, \ldots, n-1 \). Hence, the fact that \( \mathcal{L} \) is quasi-definite means that \( p_n \) and \( p_{n-1} \) form a proper pair.

For further reading on the existence of orthogonal polynomial sequences and this matrix representation we suggest [3, Chapter 3].

Taking all this into account now we can state an explicit result about when \( p \) and \( q \) form an ITrP.

**Theorem 8.** For any polynomials \( p \) and \( q \) of degree \( n \) and \( n-1 \), respectively, with coefficients over a field \( \mathbb{F} \), let us denote \( p \) by \( p_n \) and \( q \) by \( p_{n-1} \), and let us consider the following two linear functionals:

- If all the roots of \( p \) are different, then \( \mathcal{L}_1: \mathbb{F}[t] \to \mathbb{F} \)

\[
\mathcal{L}_1(f(t)) = \sum_{k=1}^{n} \frac{f(\lambda_k)}{p_n'(\lambda_k)p_{n-1}(\lambda_k)},
\]

where

\[
p_n(t) = \prod_{k=1}^{n} (t - \lambda_k), \quad \text{and} \quad p_{n-1}(t) = \prod_{k=1}^{n-1} (t - \mu_k).
\]
• If all the roots of \( p \) are the same, namely \( a \) with multiplicity \( n \), then

\[
\mathcal{L}_2 : \mathbb{F}[t] \to \mathbb{F}
\]

\[
\mathcal{L}_2(f(t)) = C \frac{d^{n-1}}{z^{n-1}} \left( \frac{f(z)}{p_{n-1}(z)} \right)(a),
\]

where \( C \) is a constant such that \( \mathcal{L}(1) = m_0 \).

Then, the following statements are equivalent:

1. \( p \) and \( q \) form a proper pair.
2. All the Hankel matrices \( H_k \) associated with the linear functional \( \mathcal{L} \) are invertible for \( k = 0, 1, \ldots, n-1 \).

Proof. WLOG we need to prove this result in the following two situations: i) when all the zeros of \( p \) are different, and ii) when \( p \) has one zero with multiplicity \( n \).

Let \( \lambda_1, \ldots, \lambda_n \) be the zeros of \( p \), all of them different, and \( \mu_1, \ldots, \mu_{n-1} \) be the zeros of \( q \) over the field \( \mathbb{F} \), such that \( \{\lambda_1, \ldots, \lambda_n\} \cap \{\mu_1, \ldots, \mu_{n-1}\} = \emptyset \).

By definition, the functional \( \mathcal{L}_1 \) is linear. Moreover, if it is quasi-definite then the Hankel matrices associated to it are invertible, i.e. \( \det(H_k) \neq 0 \) for \( k = 0, 1, \ldots, n-1 \).

Let us define the \( n \)-by-\( n \) tridiagonal matrix \( A \) defined in (1) where

\[
b_i = \frac{\mathcal{L}_1(p^2_{n-i})}{\mathcal{L}_1(p^2_{n-i-1})}, \quad i = 1, 2, \ldots, n-1,
\]

and

\[
a_i = -\frac{p_{n-i+1}(0) - b_ip_{n-i-1}(0)}{p_{n-i}(0)}, \quad i = 1, 2, \ldots, n,
\]

being \( p_{-1}(t) = 0 \) and \( p_0(t) = 1 \).

Remark 9. Note that we can assume \( p_k(0) \neq 0 \) for all \( k \), because if not we apply a linear change of variables \( y(x) = x + b, \ b \neq 0 \), so that the recurrence relation coefficients \( b_i \)'s remain the same and, since \( p_k(b) \neq 0 \) for all \( k = 1, \ldots, n-1 \), then

\[
a_i = -\frac{p_{n-i+1}(b) - b_ip_{n-i-1}(b)}{p_{n-i}(b)}
\]

are finite.
If we prove that \( p_A(t) = p_n(t) \) and \( p_A(1)(t) = p_{n-1}(t) \) and the matrix is irreducible, i.e., \( b_i \neq 0 \) for all \( i = 1, \ldots, n-1 \), then the necessary condition holds and, therefore, \( p_n(t), p_{n-1}(t) \) form a ITrP.

By construction we know there exists \( (q_k(t))_{k=0}^n \) a sequence of monic polynomials orthogonal with respect to \( \mathcal{L}_1 \), i.e., they fulfill the following property of orthogonality:

\[
\mathcal{L}_1(t^\ell q_k(t)) = 0, \quad \ell = 0, 1, \ldots, k-1, \quad k = 1, 2, \ldots, n,
\]
as well as the three-term recurrence relation

\[
tq_k(t) = q_{k+1}(t) + \alpha_{n-k}q_k(t) + \beta_{n-k}q_{k-1}(t), \quad k = 1, 2, \ldots, n-1. \tag{8}
\]

By using the previous recurrence relation and the orthogonality conditions for \( q_k \), it is straightforward to prove \( q_n(t) = p_n(t) \), as well as \( \beta_1 = b_1 \). Moreover, if we set \( t = 0 \) and \( k \mapsto n-1 \) in (8) we obtain that \( \alpha_1 = a_1 \). In order to prove that \( q_{n-1}(t) = p_{n-1}(t) \) we need the followig result.

**Lemma 10.** For any polynomial \( p(t) \) of degree \( m > 1 \), with different zeros \( x_1, x_2, \ldots, x_m \), the following identity holds true:

\[
\sum_{j=1}^m \frac{1}{p'(x_j)} = 0.
\]

So

\[
\mathcal{L}_1(p_{n-1}(t)) = \sum_{j=1}^n \frac{1}{p'_n(\lambda_j)} = 0,
\]

and if we consider, for \( \ell = 1, \ldots, n-2 \), the polynomials \( \pi_\ell(t) = (t-\lambda_1) \cdots (t-\lambda_\ell) \), then

\[
\mathcal{L}_1(p_{n-1}(t)\pi_\ell(t)) = \sum_{j=\ell+1}^n \frac{\pi_\ell(\lambda_j)}{p'_n(\lambda_j)} = \sum_{j=\ell+1}^n \frac{1}{(p_n/\pi_\ell)'(\lambda_j)} = 0,
\]

therefore, by unicity, \( q_{n-1}(t) = p_{n-1}(t) \).

In fact, since

\[
\mathcal{L}_1(q_k^2(t)) = \mathcal{L}_1(t^k q_k(t)) = \beta_k\mathcal{L}_1(t^{k-1} q_{k-1}(t)) = \mathcal{L}_1(1)\beta_1\beta_2 \cdots \beta_k \neq 0,
\]

we get that, by construction, for \( k = 1, 2, \ldots, n \), \( \beta_{n-k} = b_{n-k} \) and by the orthogonality conditions \( \alpha_{n-k} = a_{n-k} \). Therefore \( p_A(t) = p(t) \) and \( p_{A(1)}(t) = q(t) \).
Remember that, by construction, we have

\[ 0 \neq (\det(H_{k-1}))^2 \mathcal{L}_1(p_k^2) = \mathcal{L}_1(p_k^2) = b_k b_{k-1} \cdots b_1 \mathcal{L}_1(1). \]

And it is sufficient to have a tridiagonal pair, because in such a case there exists a matrix $A$ so that $p_A(t) = p_n(t)$ and $p_A(n)(t) = p_{n-1}(t)$. So we consider the same inner product and, by construction, the polynomial $p_k(t)$ has degree $k$ for $k = 0, 1, \ldots, n$, and they are monic.

Then if we establish the orthogonality conditions again, we check in a straightforward way that the leading coefficient of the matrix expression (5) is indeed $\deg(H_k)$ that must be non-zero so, the linear functional is quasi-definite and that completes the proof for this case.

If $p$ has one zero, namely $a$, with multiplicity $n$, then we consider the linear functional $L_2$.

Since the key to the proof is not about the expression for $L_2$ but about the fact that the operator is linear we leave this part of the proof to the reader. \hfill \Box

**Example 5.** If $p(x) = x(x-1)^3(x+2)(x-5)$, we need to consider the linear functional that is a linear combination of the ones presented in theorem 8, i.e.

\[ \mathcal{L}(f) = \frac{f(0)}{10} - \frac{f(-2)}{378} + \frac{f(5)}{80640} + \frac{1}{\pi i} \int_{|z-1|=1} \frac{f(z)}{(z-1)^3(z+1)^2} \, dz, \]

where we have considered for the construction of the coefficients of the first part the polynomial $(x+1)^2$, but any polynomial of degree 2 or greater, proper with $x(x+2)(x-5)$, can be chosen.
With this construction we get the following sequence of polynomials:

\[
\begin{align*}
p_0(x) & = 1 \\
p_1(x) & = x + \frac{811}{2193} \\
p_2(x) & = x^2 + \frac{4310x}{2199} + \frac{7097}{6597} \\
p_3(x) & = x^3 - \frac{121347x^2}{39845} + \frac{26393x}{7969} - \frac{37047}{39845} \\
p_4(x) & = x^4 - \frac{73660x^3}{33301} + \frac{11933x^2}{33301} + \frac{33874x}{33301} - \frac{40680}{33301} \\
p_5(x) & = x^5 - \frac{50023x^4}{8243} + \frac{14679x^3}{8243} + \frac{175435x^2}{8243} - \frac{236462x}{8243} + \frac{93312}{8243} \\
p_6(x) & = x(x-1)^3(x+2)(x-5) = p(x).
\end{align*}
\]

Observe that, by construction, \(q(x) = p_5(x)\) is proper with \(p(x)\). Moreover, we obtain the 6-by-6 tridiagonal matrix (1) where

\[
\begin{align*}
\vec{a} & = \left(-\frac{565}{8243}, \frac{1058636543}{274500143}, -\frac{1105993747}{1326878345}, \frac{438574003}{87619155}, -\frac{852049}{535823}, \frac{811}{2193}\right) \\
\vec{b} & = \left(-\frac{43158096}{67947049}, \frac{7882616040}{1108956601}, \frac{659060091}{1587624025}, -\frac{58253390}{4835601}, \frac{2345600}{4809249}\right).
\end{align*}
\]

**Remark 11.** Note that in the proper case there is an iterative algorithm to construct the realizing tridiagonal matrix computationally.

**Example 6.** Here, we want to give an example in which \(p\) has degree 3 and multiple roots, and \(q\) is of degree 2. We obtain conditions for them to be proper pair can.

Consider the polynomials \(p(x) = (x+1)(x-1)^2\), and \(q(x) = (x-a)(x-b)\), \(a, b \neq \pm 1\). With these polynomials we define the linear functional

\[
\mathcal{L}_{a,b}(f) = \frac{f(-1)}{4(1+a)(1+b)} + \frac{f'(1)}{1-b} + \frac{f(1)}{(1-b)^2}.
\]

After a straightforward calculation we get the determinant of Hankel matrices
for this linear functional

\[
\det(H_0) = -a + b^2 - 2b + 2 \quad \frac{1}{(a+1)(b-1)^2};
\]

\[
\det(H_1) = -a + 4b - 1 \quad \frac{1}{(a+1)(b-1)^2};
\]

\[
\det(H_2) = 16 \quad \frac{1}{(a+1)(b-1)^2} \neq 0.
\]

Therefore if \(a + b^2 - 2b + 2 \neq 0\), and \(-a + 4b - 1 \neq 0\), we obtain the polynomials

\[
p_1(x) = x - \frac{ab - b^2 + 3b - 1}{a + b^2 - 2b + 2},
\]

\[
p_2(x) = x^2 - \frac{2(a - 2b + 3)}{a - 4b + 1} x + \frac{a + 8b - 3}{a - 4b + 1},
\]

where \(p_2(x)\) is proper with \(p(x)\). Moreover, observe that when \(sa = 3\), \(b = 0\) we have \(p_2(x) = q(x)\).

Theorem 8 has some nice consequences, for example, by construction, as we pointed out in example 5, the polynomial \(p_{n-1}(x)\) in such construction is proper with the given \(p(x)\); moreover the following result connects our problem to the Gaussian quadrature formulae.

**Remark 12.** Observe that, in \(\mathbb{C}\), if \(x_1 = x_2 = \cdots = x_n = a\) then

\[
\int_\Gamma \frac{f(z)}{p_n(z)p_{n-1}(z)} \, dz = \frac{2\pi i}{(n-1)!} d^{n-1} \left( \frac{f(z)}{p_{n-1}(z)} \right) (a),
\]

where \(\Gamma\) is a Jordan curve such that \(a\) lies inside \(\Gamma\), and the roots of \(p_{n-1}\) lie outside of \(\Gamma\).

**Remark 13.** Taking into account theorem 8 and the Remark 12 if we have a field \(\mathbb{F}\) in which the derivative may not make sense, for example if \(n = 2\), then (7) becomes

\[
\mathcal{L}(f(t)) = C \left( \frac{f'(a)}{p_1(a)} - \frac{f(a)}{p_1(a)} \right),
\]

understanding that \(f(a)\) (resp. \(f'(a)\)) represents the coefficient of \((z-a)^0 \equiv 1\) (resp. \((z-a)\)) in the expansion of \(f(z)\) in terms of \(\{(z-a)^k\}_{k=0}^\infty\) in \(\mathbb{F}\).

We can proceed in an analogous way for the the \(n = 3\) case. In such a case we have

\[
\mathcal{L}(f(t)) = C \left( \frac{f''(a)}{p_2(a)} - 2 \frac{f'(a)p_2'(a)}{p_2^2(a)} + 2 \frac{f(a)(p_2^2)'(a)}{p_2^3(a)} + 2 \frac{f(a)(p_2^2)^2(a)}{p_2^3(a)} \right).
\]
4. Further observations

In this section we present some other results that are connected with the results presented previously.

**Theorem 14.** Let \( p \) and \( q \) be monic polynomials, of degree \( n \) and \( n - 1 \) respectively, over a field \( \mathbb{F} \).

If there exists an irreducible tridiagonal matrix \( A \) such that \( p_A(t) = p \) and \( p_{A(1)}(t) = q \), i.e. \( p \) and \( q \) are ITrP, then

a) \( S_2(A) - S_2(A(1)) - a_1S_1(A(1)) \neq 0. \)

b) For \( k = 2, 3, \ldots, n - 2, \)

\[
S_2(A(1, \ldots, k - 1)) - S_2(A(1, \ldots, k)) - a_kS_1(A(1, \ldots, k - 1)) \neq 0,
\]

where

\[
\det(A - \lambda I) = \sum_{k=0}^{n} (-1)^k S_{n-k}(A)\lambda^k.
\]

Conversely, if conditions a) and b) holds, then \( p_A(t) \) and \( p_{A(1)}(t) \) is a proper pair. Note that \( S_0(A) = 1 \), \( S_1(A) = \text{Tr}(A) \), \( S_n(A) = \det(A) \), \( a_1 = S_1(A) - S_1(A(1)) \), and for \( k = 2, \ldots, n - 2, \)

\[
a_k = S_1(A(1, \ldots, k - 1)) - S_1(A(1, \ldots, k)).
\]

In fact, the given conditions in theorem 14 b) can be expressed in terms of the coefficients of the characteristic polynomial of \( A \) and \( A(1) \). For example, if \( n = 4 \) such condition for \( k = 2 \) can be written as follows:

\[
(S_3(A) - S_3(A(1)) - a_1S_2(A(1)))^2 + b_1(S_3(A) - S_3(A(1)) - a_1S_2(A(1))) + b_1S_2(A(1)) - b_1(a_1S_3(A(1)) - S_4(A)) \neq 0,
\]

where \( b_1 = S_2(A(1)) - S_2(A) + a_1S_1(A(1)). \)

**Proof.** This result follows straightforwardly by using the fact that \( b_k \)'s in the matrix \( A \) can be computed as the coefficient of \( x^{n-k-1} \) in the polynomial \( \det(A(1, 2, \ldots, k - 1) - tI) - (t - a_k) \det(A(1, 2, \ldots, k - 1, k) - tI) \), and that they need to be nonzero. \( \square \)
Note that if
\[ P_{A(1,2,\ldots,k-1)}(t) = t^{n-k+1} - c_{k1}t^{n-k} + c_{k2}t^{n-k-1} + \cdots, \]
and
\[ P_{A(1,2,\ldots,k)}(t) = t^{n-k} - d_{k1}t^{n-k-1} + d_{k2}t^{n-k-2} + \cdots, \]
then the previous result can be written as follows.

**Theorem 15.** Let \( p_n \) and \( p_{n-1} \) be relatively prime monic polynomials over a field \( \mathbb{F} \) of degree \( n \) and \( n-1 \). Then \( p_n, p_{n-1} \) is a proper pair if and only if
\[(d_{k2} - c_{k2}) + (c_{k1} - d_{k1})d_{k1} \neq 0, \quad k = 1, 2, \ldots, n - 1.\]
In this event, \( p_n, p_{n-1} \) is an ITrP and the realizing normalized tridiagonal matrix is unique.

**Remark 16.** Note that in the relatively prime, proper case there is an iterative algorithm to construct the realizing tridiagonal matrix computationally.

The proof of theorem 15 follows from the fact that
\[ b_k = (d_{k2} - c_{k2}) + (c_{k1} - d_{k1})d_{k1} \quad \text{for} \quad k = 1, 2, \ldots n - 1. \]

Another interesting fact related with our problem is the following. We can find the values of the Hankel determinants for linear functionals in terms of the roots of \( p \). Here we present the \( n = 4 \) case:

**Lemma 17.** Let \( a, b, c \) and \( d \) be four different numbers, and let \( \omega_a, \omega_b, \omega_c, \omega_d \) be another four nonzero numbers. Then the Hankel determinants associated with the linear functional
\[ \mathcal{L}(f) = \omega_a f(a) + \omega_b f(b) + \omega_c f(c) + \omega_d f(d), \]
are
\[ \det(H_0) = \omega_a + \omega_b + \omega_c + \omega_d, \]
\[ \det(H_1) = \omega_a\omega_b(b - a)^2 + \omega_a\omega_c(c - a)^2 + \omega_a\omega_d(d - a)^2 + \omega_b\omega_c(c - b)^2 + \omega_b\omega_d(d - b)^2, \]
\[ \det(H_2) = \omega_a\omega_b\omega_c(c - a)^2(c - b)^2 + \omega_a\omega_b\omega_d(b - a)^2(d - a)^2(d - b)^2 + \omega_a\omega_c\omega_d(c - b)^2(d - b)^2(d - c)^2, \]
\[ \det(H_3) = \omega_a\omega_b\omega_c\omega_d(b - a)^2(c - a)^2(d - a)^2(c - b)^2(d - b)^2(d - c)^2, \]
\[ \det(H_k) = 0, \quad k = 4, 5, \ldots \]
In fact, we consider the following conjecture for such values for the determinant of the Hankel matrices.

**Conjecture:** For any different numbers \(x_1, x_2, \ldots, x_n\) and for any \(\omega_1, \ldots, \omega_n\), all different from zero, let us consider the linear functional

\[
\mathcal{L}(f) = \sum_{i=1}^{n} \omega_i f(x_i).
\]

Then, the determinant of the Hankel matrices associated with this linear functional can be computed explicitly as

\[
\det(H_k) = \sum_{\Omega \subseteq \{1,2,\ldots,n\}} S_{k+1}(\omega_{\Omega}) V^2(x_{\Omega}),
\]

where \(S_k\) is the \(k\)-th elementary symmetric function, \(\lambda_{\Omega}\) represents the set \(\{\lambda_j : j \in \Omega\}\), and \(V(\lambda_{\Omega})\) represents the Vandermonde determinant associated with the numbers of the set \(\lambda_{\Omega}\).

**Remark 18.** Note that if \(k + 1 > n\) then \(\det(H_k) = 0\), and \(V(\{x\}) = 1\).

We also have considered some cases in which \(p\) has a multiple zero.

**Lemma 19.** For any given number \(a\) and any nonzero value \(\omega\), let us define the linear functional (see Remark 13)

\[
\mathcal{L}(f) = \omega f'(a) - \omega^2 f(a).
\]

Then the first moments are \(m_0 = -\omega^2\), \(m_1 = \omega - \omega^2 a\), \(m_2 = 2\omega a - \omega^2 a^2\), and

\[
\det(H_0) = \det(H_1) = -\omega^2, \quad \det(H_k) = 0, \quad k = 2,3,\ldots.
\]

**Lemma 20.** For any given number \(a\) and any two nonzero values \(\omega_1, \omega_2\), let us define the linear form (see Remark 13)

\[
\mathcal{L}(f) = \omega_1 f''(a) - 2\omega_2 \omega_1^2 f'(a) - 2\omega_1^3 f(a) + 2\omega_2^2 \omega_1^3 f(a).
\]

Then the first moments are \(m_0 = -2\omega_1^3 + 2\omega_1^3 \omega_2^3\), \(m_1 = 2a\omega_2^2 \omega_1^3 - 2\omega_1^2 - 2\omega_2 \omega_1^3\), \(m_2 = 2a^2 \omega_2^2 \omega_1^3 - 2a^2 \omega_1^3 - 4a\omega_2 \omega_1^3 + 2\omega_1\), and

\[
\det(H_0) = 2\omega_1^2 (\omega_1 \omega_2 - 1),
\]
\[
\det(H_1) = -4\omega_1^2,
\]
\[
\det(H_2) = -8\omega_1^3,
\]
\[
\det(H_k) = 0, \quad k = 3,4,\ldots
\]
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