About one method of parallelization of calculations during the reconstruction of a tomographic image

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Abstract

A new method for solving systems of linear algebraic equations of a special type arising in solving problems of image reconstruction has been proposed. This method, due to a certain symmetry of the matrix and the choice of the voxel numeration method for two-dimensional problems, allows us to divide the initial system of algebraic equations into two independent systems, which enables us to carry out the calculation in parallel. The dimension of the matrices of the resulting systems is 4 times less than the dimension of the original matrix, and these matrices are less dispersed.

Keywords: system matrix, algorithmic computational speedup, cluster computing, tomosynthesis, image reconstruction.

1. Introduction

An important method of non-invasive imaging widely used in medical diagnostics is X-ray computed tomography. Image reconstruction is performed both by direct and iterative methods. Iterative methods are preferred when using a limited set of projections, for example, in digital tomosynthesis. Digital tomosynthesis is a new method of three-dimensional reconstruction of an object using projections at limited viewing angles. Currently, the method

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of simultaneous algebraic reconstruction (SART) is a generally accepted iterative reconstruction method [1]. The disadvantage of SART is its high computational complexity and cost. In the works [2, 3, 4], the possibility of improving the SART image reconstruction speed was investigated. The use of parallel methods to speed up the SART reconstruction was proposed in [5, 6, 7, 8, 9, 10, 11].

In [12] the method of partitioning the reconstruction region into an uneven voxel grid was considered. This allows one to split the original three-dimensional problem into a chain of two-dimensional independent tasks.

In this paper, we propose a method that, by virtue of a certain symmetry of the matrix and the choice of the numbering method for voxels for two-dimensional problems, allows us to split the initial system of algebraic equations into two independent systems, which makes it possible to perform a parallel calculation. The dimension of the matrices of the resulting systems is 4 times less than the dimension of the original matrix, and these matrices are less dispersed.

2. Methods

Let \( f = (f_1, \ldots, f_N) \in R^N \) be a discrete representation of an unknown image (\( N \) is the total number of image voxels) to be reconstructed, \( p = (p_1, \ldots, p_M) \in R^M \) be the measured projection data (\( M \) is the total number of rays), and \( A = (w_{i,j}) \) be a known system matrix, whose element \( w_{i,j} \) is the weighting factor that represents the contribution of the \( j \)th voxel of the \( i \)th ray integral. Hereinafter we shall assume that the numbers \( N \) and \( M \) are even (as a rule, it takes place in practice). The image reconstruction problem can be formulated as a system of linear equations

\[
Af = p
\]  

(1)

Suppose that during the motion of the source, the irradiation is performed in positions that are pairwise symmetrical about the axis OZ. The reconstruction region is divided into layer-by-layer voxels, and each of the \( j \) voxels is characterized by the parameter \( f_j \). However, voxels are numbered in a special order (according to the vertical snake rule) as shown in Fig. 1 for the case \( N = 32 \).

In this case, the matrix \( A \) is symmetric in the sense

\[
w_{i,j} = w_{M-i+1,N-j+1}.
\]

(2)
Consider two systems of linear algebraic equations that are independent of each other

\[ A^{(1)} f^{(1)} = p^{(1)}, \quad (3) \]

\[ A^{(2)} f^{(2)} = p^{(2)}, \quad (4) \]

where the elements of matrices \( A^{(1)} \) and \( A^{(2)} \) depend on the elements of the matrix \( A \), and are found by the formulas

\[ w_{i,j}^{(1)} = w_{i,j} - w_{M-i+1,j}, \quad w_{i,j}^{(2)} = w_{i,j} + w_{M-i+1,j}, \quad i = 1, \ldots, \frac{M}{2}, \quad j = 1, \ldots, \frac{N}{2}, \]

and the elements of vectors \( p^{(1)} \) and \( p^{(2)} \) depend on the elements of vector \( p \)

\[ p_{i}^{(1)} = p_{i} - p_{M-i+1}, \quad p_{i}^{(2)} = p_{i} + p_{M-i+1}, \quad i = 1, \ldots, \frac{M}{2}. \]

**Theorem 1.** Let the matrix of system (1) satisfies the symmetry conditions (2), whereas \( f^{(1)} \) and \( f^{(2)} \) are the solutions of systems (3) and (4) respectively, then \( f = (f_1, \ldots, f_N) \), where

\[ f_j = \frac{f_j^{(2)} + f_j^{(1)}}{2}, \quad f_{N-j+1} = \frac{f_j^{(2)} - f_j^{(1)}}{2}, \quad j = 1, \ldots, \frac{N}{2}, \quad (5) \]

is a solution to equation (1).
**Proof.** We introduce the notation

\[ f_j^{(1)} = f_j - f_{N-j+1}, \quad f_j^{(2)} = f_j + f_{N-j+1}, \quad j = 1, \ldots, N. \]

It’s obvious that

\[ f_j^{(1)} = -f_{N-j+1}, \quad f_j^{(2)} = f_{N-j+1}, \quad j = 1, \ldots, N. \]  

(6)

From equation (1), by virtue of the equalities \( f = \frac{f^{(1)} + f^{(2)}}{2} \) and \( p = \frac{p^{(1)} + p^{(2)}}{2} \), we get

\[ A \left( \frac{f^{(1)} + f^{(2)}}{2} \right) = \frac{1}{2} A f^{(1)} + \frac{1}{2} A f^{(2)} = \frac{1}{2} p^{(1)} + \frac{1}{2} p^{(2)}, \]

(7)

where \( p^{(1)} \) and \( p^{(2)} \) are extended for all \( i = 1, \ldots, N \) so that the symmetry conditions are satisfied (6).

By virtue of the symmetry condition (2) for matrix \( A \), vector \( Af^{(1)} \) satisfies the first symmetry condition from (6), whereas vector \( Af^{(2)} \) satisfies the second symmetry condition from (6). Since the vector representation as the sum of two vectors satisfying the symmetry conditions (6) is unique, then (7) splits into two systems

\[ Af^{(1)} = p^{(1)}, \]

(8)

\[ Af^{(2)} = p^{(2)}. \]

(9)

System (8) can be represented as

\[
\begin{align*}
A^{(1)} f^{(1)} &= p^{(1)}, \\
-A^{(1)} f^{(1)} &= -p^{(1)},
\end{align*}
\[
\Leftrightarrow \quad A^{(1)} f^{(1)} = p^{(1)}. \]  

(10)

And system (9) can be represented as

\[
\begin{align*}
A^{(2)} f^{(2)} &= p^{(2)}, \\
A^{(2)} f^{(2)} &= p^{(2)},
\end{align*}
\[
\Leftrightarrow \quad A^{(2)} f^{(2)} = p^{(2)}. \]  

(11)

Theorem 1 is proved.

The dimension of the matrices \( A^{(1)} \) and \( A^{(2)} \) is 4 times less than the dimension of the original matrix \( A \). In addition, the matrices \( A^{(1)} \) and \( A^{(2)} \) are less dispersed. The independence of these matrices makes it possible to find solutions of systems (3) and (4) by the method of parallel calculations. Obviously, if the number of arithmetic operations required for solving system
is proportional to \( N^\alpha \cdot M^\beta \) \((\alpha > 0, \beta > 0)\), then the time spent on solving system (1) using the same method with the help of (4) will be \( 2^{\alpha+\beta} \) times less.

**Theorem 2.** Let \( \bar{f}^{(1)} = (\bar{f}_1^{(1)}, \ldots, \bar{f}_{N/2}^{(1)}) \) be the normal pseudo-solution of system (3), and \( \bar{f}^{(2)} = (\bar{f}_1^{(2)}, \ldots, \bar{f}_{N/2}^{(2)}) \) be the normal pseudo-solution of system (4), then

\[
\bar{f}_j = \frac{\bar{f}_j^{(2)} + \bar{f}_j^{(1)}}{2}, \quad \bar{f}_{N-j+1} = \frac{\bar{f}_j^{(2)} - \bar{f}_j^{(1)}}{2}, \quad j = 1, \ldots, \frac{N}{2},
\]

is a normal pseudo-solution of equation (1).

The proof of Theorem 2 follows from the following obvious equalities

\[
\|f\|_2^2 = \sum_{j=1}^{N} f_j^2 = \sum_{j=1}^{N/2} \left( \left( \frac{f_j^{(2)} + f_j^{(1)}}{2} \right)^2 + \left( \frac{f_j^{(2)} - f_j^{(1)}}{2} \right)^2 \right) =
\]

\[
\frac{1}{2} \sum_{j=1}^{N/2} (f_j^{(1)})^2 + \frac{1}{2} \sum_{j=1}^{N/2} (f_j^{(2)})^2 = \frac{1}{2} \|f^{(1)}\|_2^2 + \frac{1}{2} \|f^{(2)}\|_2^2.
\]

**Example.** Consider the case \( N = 6, M = 4 \)

\[
\begin{pmatrix}
1 & 3 & 5 & 7 & 9 & 1 \\
2 & 4 & 6 & 8 & 3 & 7 \\
7 & 3 & 8 & 6 & 4 & 2 \\
1 & 9 & 7 & 5 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{pmatrix} =
\begin{pmatrix}
5 \\
6 \\
8 \\
7
\end{pmatrix}.
\]

In order to solve system (13) consider the following two systems

\[
\begin{pmatrix}
0 & -6 & -2 \\
-5 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
f_1^{(1)} \\
f_2^{(1)}
\end{pmatrix} = \begin{pmatrix}
-2 \\
-2
\end{pmatrix},
\]

\[
\begin{pmatrix}
2 & 12 & 12 \\
9 & 7 & 14
\end{pmatrix}
\begin{pmatrix}
f_1^{(2)} \\
f_2^{(2)} \\
f_3^{(2)}
\end{pmatrix} = \begin{pmatrix}
12 \\
14
\end{pmatrix}.
\]
The general solution of systems (14) and (15) respectively has the form
\[ f_1^{(1)} = -\frac{7}{15}x + \frac{7}{15}, \quad f_2^{(1)} = -\frac{1}{3}x + \frac{1}{3}, \quad f_3^{(1)} = x, \quad \text{where } x \in \mathbb{R}, \quad (16) \]
\[ f_1^{(2)} = -\frac{42}{47}y + \frac{42}{47}, \quad f_2^{(2)} = -\frac{40}{47}y + \frac{40}{47}, \quad f_3^{(2)} = y, \quad \text{where } y \in \mathbb{R}. \quad (17) \]
From (16) and (17) based on Theorem 1, we obtain the general solution of system (13)
\[ f = \frac{1}{1410} \begin{pmatrix} -329x - 630y + 959 \\ -235x - 600y + 835 \\ 705x + 705y \\ -705x + 705y \\ 235x - 600y + 365 \\ 329x - 630y + 301 \end{pmatrix}. \]
Normal pseudo-solutions, up to four decimal places, of systems (14) and (15) respectively, have the form
\[ f^{(1)} = (0.3512, 0.2508, 0.2475)^T, \quad \|f^{(1)}\|_2 = 0.4975, \]
\[ f^{(2)} = (0.3542, 0.3373, 0.6036)^T, \quad \|f^{(2)}\|_2 = 0.7769. \]
Normal pseudo solutions, up to four decimal places, for the initial system (13) have the form
\[ f = (0.3527, 0.2941, 0.4256, 0.1781, 0.0433, 0.0015)^T, \quad \|f\|_2 = 0.6523. \]
For this example, the validity of Theorem 2 is confirmed.
Numerous calculations for square matrices show the validity of the following equality
\[ \det(A) = \det(A^{(1)}) \cdot \det(A^{(2)}). \quad (18) \]
If the matrix $A$ is not square and satisfies symmetry conditions in the sense of (2), then the matrix $B = A^T \cdot A$ will also be symmetric in the sense (2). In addition, the following equalities hold true:
\[ B^{(1)} = (A^{(1)})^T \cdot A^{(1)}, \quad B^{(2)} = (A^{(2)})^T \cdot A^{(2)}. \]
The proposed method is applicable for an arbitrary symmetric motion geometry of the source and detector, which ensures the symmetry of the reconstruction matrix. For example, in paper [13] it was considered the
problem of recovering images of nuclear waste in a cylindrical container using tomographic Gamma Scanning. Each reconstructed layer was divided into a 72 voxel using the polar coordinate system, and the voxel numbering was chosen in the order shown in Fig. 2a. If we redesign the voxel numbering by our proposed method, as shown in Fig. 2b, then the system of linear algebraic equations from [13] can be divided into two independent systems with 36 unknowns and the number of equations two times smaller than the original one.

![Figure 2: a) voxel numbering proposed in [13], b) voxel numbering according to our proposed method.](image)

3. Conclusion

A new method for solving systems of linear algebraic equations of a special type arising in solving problems of image reconstruction has been proposed. This method, due to a certain symmetry of the matrix and the choice of voxel numbering method for two-dimensional problems, allows us to divide the initial system of algebraic equations into two independent systems, which makes it possible to carry out the calculation in parallel. The dimension of the matrices of the resulting systems is 4 times less than the dimension of the original matrix, and these matrices are less dispersed.

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reconstruction matrix.

The combined use of this method and efficient algorithms, such as those proposed in [6, 11, 12, 13], for reconstructing a tomographic image will increase the speed of these algorithms at least 4 times, while maintaining the accuracy of the reconstruction.

This parallelization method can also be applied when solving multidimensional problems of mathematical physics considered in symmetric domains, the approximation of which leads to matrices symmetric in the sense of [2].

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