Global Existence and Nonexistence of Solutions to a Cross Diffusion System with Nonlocal Boundary Conditions

Z. R. Rakhmonov¹, A. Khaydarov¹, J. E. Urunbaev²

¹Department of Applied Mathematics and Computer Analysis, Faculty of Mathematics, National University of Uzbekistan, Tashkent, 100174, Uzbekistan
²Department of Mathematical Modeling and Complex Programming, Faculty of Applied Mathematics and Computer Science, Samarkand State University, Samarkand, 140104, Uzbekistan

Received April 20, 2020; Revised May 21, 2020; Accepted June 23, 2020

Abstract Mathematical models of nonlinear cross diffusion are described by a system of nonlinear partial parabolic equations associated with nonlinear boundary conditions. Explicit analytical solutions of such nonlinearly coupled systems of partial differential equations are rarely existed and thus, several numerical methods have been applied to obtain approximate solutions. In this paper, based on a self-similar analysis and the method of standard equations, the qualitative properties of a nonlinear cross-diffusion system with nonlocal boundary conditions are studied. We are constructed various self-similar solutions to the cross diffusion problem for the case of slow diffusion. It is proved that for certain values of the numerical parameters of the nonlinear cross-diffusion system of parabolic equations coupled via nonlinear boundary conditions, they may not have global solutions in time. Based on a self-similar analysis and the comparison principle, the critical exponent of the Fujita type and the critical exponent of global solvability are established. Using the comparison theorem, upper bounds for global solutions and lower bounds for blow-up solutions are obtained.

Keywords Cross-diffusion, Critical Exponents, Global Solvability, Blow-up, Self-similar Analysis

1. Introduction

In this paper, we studied the qualitative properties of solutions of a nonlinear cross-diffusion system associated with nonlocal boundary conditions

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left( u^{m_i-1} \frac{\partial u^{p_i-2}}{\partial x} \right), \\
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial x} \left( v^{m_j-1} \frac{\partial v^{p_j-2}}{\partial x} \right), \quad x \in R_+, t > 0, \\
-u^{m_i-1} \frac{\partial u}{\partial x}(0,t) &= u^0 \left(0,t\right), \\
-u^{m_j-1} \frac{\partial v}{\partial x}(0,t) &= v^0 \left(0,t\right), t > 0, \\
u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in R_+, (3)
\end{align*}
\]

where \( p > \max\{m_i, m_j\} + 1 \), \( m_i > 1 \), \( m_j > 1 \), \( q > 0 \)(i = 1, 2), \( u_0 \) and \( v_0(x) \) – are non-negative continuous functions with compact support in \( R_+ \).

Recently, there has been a surge in the analysis and simulation of mathematical models of the reaction-diffusion type in the presence of the so-called cross-diffusion. Cross-diffusion is a process in which a concentration or density gradient of one chemical or biological type induces a flow (linear or non-linear) of another type. The concept of cross-diffusion also includes well-known cases of modeling chemo- and hypo taxis. The application of reaction-cross-diffusion of a system is easily found in literature and includes the pattern formation development in biology [16], electrochemistry, cancer motility [5, 8, 11, 29] and biofilms [10]. By introducing cross-diffusion into standard reaction-diffusion models, it has shown that there is no cross-diffusion when preventing explosion phenomena associated with such systems [4]. Explicit analytical
solutions to these complex and often nonlinearly coupled systems of partial differential equations rarely exist, and thus several numerical methods have been applied to obtain approximate solutions.

Cross-diffusion models are found in various fields of natural science. For example, in physical systems (plasma physics) [1, 12, 23], in chemical systems (dynamics of electrolytic solutions), in biological systems (cross-diffusion transport, dynamics of population systems), in ecology (dynamics of forest age structure), in seismology - Burridge-Knopoff model describing the tectonic plates interaction [13-15, 21, 29]. In recent years, in the study of biological population and the tectonic plates movement, mathematical models with cross-diffusion have been widely used [14, 15, 29-34].

The local existence (in time) of solutions to systems of parabolic equations was established by authors [5, 6, 24-27] in a series of important papers. For existence of solution, not much is known, however. Under the different condition on the cross-diffusion flow, or the initial values $u_0$ and $v_0$, obtained the global existence [25-27]. In recent years, by the many authors the condition for the global existence of solutions and the condition for the occurrence of a blow-up regime for various boundary value problems have been intensively studied (see [5-26]).

A general treatment of global blow-up, as well as lower bounds for regional and single point blow-up for arbitrary nonlinear equations and systems with the parabolic monotonicity property (including Neumann boundary conditions), was performed by the method of stationary states in [15] and [28], where some estimates of $L^r$ norms of the components and their supports are obtained.

It is known that systems of degenerate equations may not have a classical solution in the region, where $u, v = 0$. For system (1)-(3), the local existence and the comparison principle of weak solutions are defined in the usual integral way (see [18, 28]). In this paper, we assume that $u, v \neq 0$

In [16], the conditions of global solvability and insolubility in time of a solution were studied, and the solution was estimated near the explosion time of a nonlocal diffusion problem

$$ u_t = u_{xx}, \quad v_t = v_{xx}, \quad x > 0, \quad 0 < T < 0, \quad (4) $$

$$ -u_t \left( 0, t \right) = u^\alpha v^\beta, \quad -v_t \left( 0, t \right) = u^\alpha v^\beta, \quad 0 < t < T, \quad (5) $$

$$ u \left( x, 0 \right) = u_0 \left( x \right), \quad v \left( x, 0 \right) = v_0 \left( x \right), \quad x > 0. \quad (6) $$

It is proved that if $pq \leq (1-\alpha)\left( 1-\beta \right)$, then every solution to the problem (4)-(6) is global.

In [11], the systems of cross-diffusion equations on a stationary surface of the following form are studied

$$ \frac{\partial u_m}{\partial t} - \sum_{k=1}^{r} d_{mk} \Delta_t u_k = f_m(u_1, \ldots, u_r), \quad m \in \Gamma \times (0, T), $$

$$ u_m(x, 0) = u_{0,m}(x), \quad \forall x \in \Gamma, \quad m = 1, \ldots, r $$

where $r \geq 1$. They provide a fully-discrete scheme by applying the Implicit-Explicit Euler method. In addition, they provide sufficient conditions for the existence of polypotent invariant regions for the numerical solution after spatial and full discretization. Furthermore, they prove optimal error bounds for the semi- and fully-discrete methods, that is, the convergence rates are quadratic in the mesh size and linear in the time step.

In [24] the following problem is investigated

$$ u_t = \left( u^\alpha \right)_x, \quad v_t = \left( v^\beta \right)_x, \quad x \in \mathbb{R}, \quad t > 0, \quad (7) $$

$$ -\left( u^\alpha \right)_x \left( 0, t \right) = u^\alpha v^\beta \left( 0, t \right), $$

$$ -\left( v^\beta \right)_x \left( 0, t \right) = u^\alpha v^\beta \left( 0, t \right), \quad t > 0, \quad (8) $$

$$ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}. \quad (9) $$

It is shown that the solution to problem (7)-(8) is global if $pq \leq (n+1)(k+1)/4$. Conditions were obtained on the numerical parameters of systems (7) - (9) under which the solution to the problem explodes in a finite time.

The studies given in [10] should also be noted; there, system (7) was studied with the following boundary conditions

$$ -\left( u^\alpha \right)_x \left( 0, t \right) = u^\alpha v^\beta \left( 0, t \right), $$

$$ -\left( v^\beta \right)_x \left( 0, t \right) = u^\alpha v^\beta \left( 0, t \right), \quad t > 0. $$

It is shown that $\min \{ y_1 - r_1, y_2 - r_2 \} \geq 0$, where

$$ r_1 = \frac{2p + k + 1 - 2\beta}{4pq - (k + 1 - 2\alpha)(n + 1 - 2\beta)}, $$

$$ r_2 = \frac{2p + n + 1 - 2\beta}{4pq - (k + 1 - 2\alpha)(n + 1 - 2\beta)}, $$

$$ y_1 = \frac{1 - r_1 \left( n - 1 \right)}{2}, \quad y_2 = \frac{1 - r_2 \left( k - 1 \right)}{2}, $$

are the critical Fujita exponents.

In [23], Yongsheng Mi, Chunlai Mu, and Botao Chen considered the following problem

$$ \left\{ \begin{array}{l} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial u^{\alpha - 1}}{\partial x} \right), \quad x > 0, \quad 0 < T < 0, \\
\frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial v^{\beta - 1}}{\partial x} \right), \quad x > 0, \quad 0 < T < 0, \end{array} \right. $$

where $r \geq 1$. They provide a fully-discrete scheme by applying the Implicit-Explicit Euler method. In addition, they provide sufficient conditions for the existence of polypotent invariant regions for the numerical solution after spatial and full discretization. Furthermore, they prove optimal error bounds for the semi- and fully-discrete methods, that is, the convergence rates are quadratic in the mesh size and linear in the time step.
Global Existence and Nonexistence of Solutions to a Cross Diffusion System with Nonlocal Boundary Conditions

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial u^m}{\partial x} &= \nu^p (0, t), \quad 0 < T < 0, \\
\frac{\partial u}{\partial t} - \frac{\partial u^m}{\partial x} &= \nu^p (0, t), \quad 0 < T < 0, \\
u(x, 0) = u_0(x), \quad \nu(x, 0) = v_0(x), \quad x > 0.
\end{align*}
\]

They showed that the critical global existence exponent and critical Fujita exponent are

\[ q_1 q_2 = \frac{(2 p_1 - 1 + m_1)(2 p_2 - 1 + m_2)}{p_1 p_2} \]

and

\[ \min \{ l_1 - k_1, l_2 - k_2 \} = 0, \]

where

\[ k_1 = \frac{(p_1 - 1)p_1q_1 + (p_1 - 1)(2q_1 + m_1 + 1)}{q_1 q_2 p_1 p_2 - (2q_1 + m_1 + 1)(2q_2 + m_2 + 1)}, \]

\[ k_2 = \frac{(p_2 - 1)p_2q_2 + (p_2 - 1)(2q_2 + m_2 + 1)}{q_1 q_2 p_1 p_2 - (2q_1 + m_1 + 1)(2q_2 + m_2 + 1)}, \]

\[ l_1 = \frac{k_1 q_1 - k_1 (p_1 + m_1 - 2)}{p_1 - 1}, \]

\[ l_2 = \frac{k_2 q_2 - k_2 (p_2 + m_2 - 2)}{p_2 - 1}. \]

In [33] considered cross-diffusion induced pattern formation in a prey–predator model with Rosenzweig–MacArthur type reaction kinetics in a one-dimensional spatial domain. Authors this work investigated the bifurcation of travelling wave solution into Turing patterns and transition of one pattern into another in the presence of cross-diffusion.

Esther S. Daus, Ansgar Jüngel and Bao Quoc Tang [34] studied the large-time asymptotics of weak solutions to Maxwell–Stefan diffusion systems for chemically reacting fluids with different molar masses and reversible reactions. They obtained the exponential decay to the unique equilibrium with a rate that is constructive up to a finite-dimensional inequality.

The purpose of this study is to find the conditions of existence and nonexistence of solutions to problem (1)-(3) over time based on self-similar analysis. Various self-similar solutions to the problem (1)-(3) are constructed, estimates of the solutions are obtained, the critical Fujita exponents and critical exponents for the global existence of the solution are established.

Introduce the notation

\[ \beta = \frac{(q_1 - 1)(q_2 - 1) - (p - 2)(q_1 - 1) - (m_1 - 1)(q_1 - 1)}{p(q_1 - 1)(q_2 - 1) - (p - 2)(q_1 - 1) - (m_1 - 1)(q_1 - 1)}, \]

\[ \alpha_i = \frac{(p - 1)(q_1 - 1)}{l_i}, \quad \alpha_2 = \frac{(p - 1)(q_1 - 1)}{l_2}, \]

\[ l_1 = p(q_1 - 1)(q_2 - 1) - (p - 2)(q_1 - 1) - (m_1 - 1)(q_2 - 1), \]

\[ l_2 = p(q_1 - 1)(q_2 - 1) - (p - 2)(q_2 - 1) - (m_1 - 1)(q_1 - 1). \]

2. Main Results

**Theorem 2.1.** Let \( \min \{l_1, l_2\} > 0 \), then, any solution to problem (1)-(3) is unbounded for sufficiently large initial data.

**Proof.** Introduce new functions \( \tilde{u}(x, t) \) and \( \varphi(x, t) \) of the types:

\[ \tilde{u}(x, t) = (T - t)^{-\alpha} \phi(\xi), \]

\[ \varphi(x, t) = (T - t)^{-\beta} \xi, \quad x > 0, \]

which are self-similar ones. Theorem 1 can be proved at \( T > 0 \). Substituting these functions in (1)-(3), we obtain a self-similar problem consisting of the following systems of equations relative to \( \varphi(\xi) \) and \( \xi \):

\[ \begin{align*}
\left\{ \begin{aligned}
\frac{d}{d\xi} \left( \frac{d\varphi}{d\xi} \right)^{p-2} &\frac{d\varphi}{d\xi} - \beta \frac{d\varphi}{d\xi} - \alpha \varphi = 0, \\
\frac{d}{d\xi} \left( \frac{d\xi}{d\xi} \right)^{p-2} &\frac{d\xi}{d\xi} - \beta \frac{d\xi}{d\xi} - \alpha \xi = 0,
\end{aligned} \right.
\end{align*} \]

obtained by substituting (10) into (1)-(3) and some simplifications. Define the conditions under which (10) is an unbounded lower solution to problem (1)-(3). As the compared functions, we choose the following ones:

\[ \begin{align*}
\tilde{\phi}(\xi) &= A_i (a - \xi)^k, \\
\tilde{\xi}(\xi) &= A_i (a - \xi)^l,
\end{align*} \]

where \( A_i > 0 \) (\( i = 1, 2 \),

\[ k = \frac{(p - 1)(p - m_1 - 1)}{(p - 2)^2 - (m_1 - 1)(m_1 - 1)}, \quad \xi = \frac{(p - 1)(p - m_1 - 1)}{(p - 2)^2 - (m_1 - 1)(m_1 - 1)}. \]
To apply the comparison theorem, the following inequalities are required:
\[
\begin{align*}
\frac{d}{d\xi} \phi_m^{-1} + \frac{d\phi_m^{-1}}{d\xi} + \frac{d\phi}{d\xi} - \beta \phi_m^{-1} & \geq 0, \\
\frac{d}{d\xi} \phi_m^{-1} + \frac{d\phi_m^{-1}}{d\xi} + \frac{d\phi}{d\xi} - \beta \phi_m^{-1} & \geq 0,
\end{align*}
\]
where \(\beta=\alpha, \lambda, \gamma\).

Two systems of inequalities with respect to \(A_1\) and \(A_2\) are obtained
\[
\begin{align*}
A_1^{p-2}A_m^{-1} - \alpha_1 a (1 + k \xi) & \geq 0, \\
A_1^{p-2}A_m^{-1} - \alpha_2 a (1 + z \xi) & \geq 0, \\
A_2^{p-1}A_m^{-1}k^{p-1}a^k & \leq A_1^{q_1}a^{q_1}, \\
A_2^{p-1}A_m^{-1}z^{p-1}a^z & \leq A_2^{q_2}a^{q_2}. \\
\end{align*}
\]

We can find the condition for systems (II); the upper bound for parameter \(a\) is:
\[
a \leq \min \left\{ \frac{A_1^{p-2}A_m^{-1}k^{p-1}a^k}{\alpha_1}, \frac{A_1^{p-2}A_m^{-1}z^{p-1}a^z}{\alpha_2} \right\},
\]
and the initial data are sufficiently small, then any solution to problem (1) - (3) is global.

**Theorem 2.2.** Let \(\max \{\alpha_1 - \beta, \alpha_2 - \beta\} < 0\) and the initial data are sufficiently small, then any solution to problem (1)-(3) is global.

**Proof.** Constructing bounded upper solutions, we can determine the conditions of solvability in time in the following way:
\[
\begin{align*}
\frac{d}{d\xi} \phi_m^{-1} + \frac{d\phi_m^{-1}}{d\xi} + \frac{d\phi}{d\xi} - \beta \phi_m^{-1} & \geq 0, \\
\frac{d}{d\xi} \phi_m^{-1} + \frac{d\phi_m^{-1}}{d\xi} + \frac{d\phi}{d\xi} - \beta \phi_m^{-1} & \geq 0,
\end{align*}
\]
where \(T > 0\), \(f(\xi)\) and \(g(\xi)\) are the sought for functions, which, by the solution comparison theorem, must satisfy the system of inequalities:
\[
\begin{align*}
\frac{d}{d\xi} g^{-1} \frac{d\phi}{d\xi} + \frac{d\phi}{d\xi} - \beta \phi_m^{-1} & \geq f(\xi), \\
\frac{d}{d\xi} g^{-1} \frac{d\phi}{d\xi} + \frac{d\phi}{d\xi} - \beta \phi_m^{-1} & \geq g(\xi),
\end{align*}
\]
Thus, choosing the parameters \(A_1, A_2\), we can obtain the system of inequalities (I, II) under condition \(\min \{l_1, l_2\} > 0\). By the principle of comparing solutions, we have estimates for the initial data, with respect to the lower self-similar solutions (10), (13):
Theorem 1 shows that the critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux. J Math Anal Appl, 2004, 298: 308-324.

Note 1. Theorem 3 proves the method described in [27].

Note 2. Theorem 1 shows that the critical Fujita exponents are \( \min \{\alpha_1 - \beta, \alpha_2 - \beta\} = 0 \).

Acknowledgements

We are very grateful to experts for their appropriate and constructive suggestions to improve this template.

REFERENCES

[1] Aripov M.M. Methods of reference equations for solving nonlinear boundary value problems, Fan Publ., Tashkent, 1988, 137

[2] Amann H. Dynamic theory of quasilinear parabolic systems, III Global existence, Math Z., 1989, 202(2). 219-250. https://doi.org/10.1007/BF01215256

[3] Amann H. Dynamic theory of quasilinear parabolic equations-II. Reaction-diffusion systems, Differential Integral Equations, 1990, 3(1) 13-75. Zbl0729.35062

[4] Amann H. Non-homogeneous linear and quasilinear elliptic and parabolic boundary value problems, in H. Schmeisier and H. Triebel (eds) Function Spaces, Differential Operator and Nonlinear Analysis, 1993, 9-126.

[5] Aripov M.M., Matyakubov A.S. Self-similar solutions of a cross-diffusion parabolic system with variable density: explicit estimates and asymptotic behavior, Nanosystems: Physics, Chemistry, Mathematics, 2017, 8(1), 5-12. https://doi.org/10.17586/2220-8054-2017-8-1-5-12

[6] Aripov M.M., Matyakubov A.S. To the qualitative properties of solution of system equations not in divergence form of polytropic filtration in variable density Nanosystems: Physics, Chemistry, Mathematics, 2017, 8(3), 317-322. doi: https://doi.org/10.17586/2220-8054-2017-8-3-317-322

[7] Deuring P. An initial-boundary value problem for a certain density-dependent diffusion system Math Z., 1987, 194, 375-396.

[8] Dibenedetto E. Degenerate parabolic equations Springer - Verlag Publ., Berlin, 1993.

[9] Dziuk G., Elliott C.M. Finite element methods for surface PDEs, Acta, 2013, 22, 289-396. doi: http://dx.doi.org/10.1007/s0962492913000056.

[10] Zheng S N, Song X F, Jiang Z X. Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux. J Math Anal Appl, 2004, 298: 308-324.

[11] Fritelli Massimo, Madzvamuse Anotida, Sgura Ivanonne, Venkataraman Chandra Sekhar. Lumped finite elements for reaction-cross-diffusion systems on stationary surfaces Computers and Mathematics with Applications, 2017, 74, 3008-3023 doi: https://doi.org/10.1016/j.camwa.2017.07.04

[12] Francesco M.Di., Esposito A., Fagioli S. Nonlinear degenerate cross-diffusion systems with nonlocal interaction Nonlinear Analysis, 2018, 169, 94-117. doi: http://dx.doi.org/10.1016/j.na.2017.12.003.

[13] Galaktionov V. A., Kurdyumov S. P., Samarskii A. A. On the method of stationary states for quasilinear parabolic equations Math. USSR-Sb, (1990), 67(2), 449-471. doi: http://dx.doi.org/10.1070/SM1990v067n02ABEH002901.

[14] Gerisch A., Chaplain M.A.J. Robust numerical methods for taxis-diffusion-reaction systems, applications to biomedical problems, Math. Comp. Mod., 2006, 43, 49-75. doi: http://dx.doi.org/10.1016/j.mcm.2004.05.016.

[15] Gerstenmayer Anita, Jungel Ansgar. Analysis of a degenerate parabolic crossdiffusion system for ion transport, Journal of Mathematical Analysis and Applications, Nonlinear Analysis, 2018, 461(1), 523-543. doi: https://doi.org/10.1016/j.jmaa.2018.01.024

[16] Wang S, Xie C H., Wang M X. The blow-up rate for a system of heat equations completely coupled in the boundary conditions. Nonlinear Anal, 1999, 35: 389–398.

[17] Kalashnikov A.S. Some questions of the qualitative theory of nonlinear second order degenerate parabolic equations, UMN, 1987, V.42, Iss. 2 (254), 135-176.

[18] Lieberman G. M. Second order parabolic differential equations, River Edge, World Scientific Publ., 1996. 452. doi: http://dx.doi.org/10.1016/j.sdam.2004.05.016.

[19] Levine H. The role of critical exponents in blowup theorems SIAM Rev., 1990, 32(2), 262-288.

[20] Murray J.D. Mathematical Biology. Springer, Berlin, 2002–2003, 3rd, ed.

[21] Malchow H, Petrovskii S.V. Spatiotemporal patterns in ecology and epidemiology: theory, models, and simulations, Chapman and Hall/CRC Press, London, 2008.

[22] Nie Y-Y, Thomee V. A lumped mass finite-element method with quadrature for a non-linear parabolic problem,IMA J. Numer. Anal., 1985, 5, 371-396. doi: http://dx.doi.org/10.1093/imanum/5.4.371

[23] Mi, Y.S., Mu, C.L., and Chen, B.T., Critical exponents for a nonlinear degenerate parabolic system coupled via nonlinear boundary flux, J. Korean Math. Soc., 48(3), 2011, 513-527.

[24] Quiros F, Rossi J.D. Blow-up set and Fujita-type curves for a degenerate parabolic system with nonlinear conditions, Indiana Univ Math J, 2001, 50, 629-654.
[25] Rakhmonov Z. Estimates for solutions of a nonlinear system of heat conduction equations with variable density and with a non-local boundary condition, Bulletin NUU, 2016, no.1 (2), 145-154.

[26] Rakhmonov Z. On the properties of solutions of multidimensional nonlinear filtration problem with variable density and nonlocal boundary condition in the case of fast diffusion, Journal of Siberian Federal University. Mathematics & Physics, 2016, 9(2), P. 236-245. doi: https://doi.org/10.17516/1997-1397-2016-9-2-225-234

[27] Rakhmonov Z.R., Urunbayev J.E. On a Problem of Cross-Diffusion with Nonlocal Boundary Conditions, Journal of Siberian Federal University. Mathematics and Physics, 2019, no. 5, 614-620, doi: https://doi.org/10.17516/19971397-2019-12-5-614-620.

[28] Samarskii A.A., Galaktionov V.A., Kurdyumov S.P., Mikhailov A.P. Blow-up in Problems for Quasilinear Parabolic Equations, Gruyter Publ., 1995.

[29] Tsyganov M.A., Biktashev V.N., Brindley J., Holden A.V., Ivanitsky G.R. Waves in cross-diffusion systems - a special class of nonlinear waves, UFN, 2007, vol. 177, issue 3, P. 275-300, doi: http://dx.doi.org/10.1070/PU2007v050n03A BEH006114.

[30] Vanag V.K., Epstein I.R. Cross-diffusion and pattern formation in reaction-diffusion systems, Phys. Chem. Chem. Phys., 2009, 11, 897-912. doi: http://dx.doi.org/10.1039/B8 13825G.

[31] Wu Z.Q., Zhao J.N., Yin J.X. and Li H.L. Nonlinear Diffusion Equations, World Scientific, Singapore, 2001.

[32] L.Desvillettes, C. Soresina. Non-triangular cross-diffusion systems with predator-prey reaction terms. Ricerche mat., 68, 2019, 295-314. https://doi.org/10.1007/s11587-018-0403-y.

[33] Nayana Mukherjee, S. Ghora and Malay Banerjee. Cross-diffusion induced Turing and non-Turing patterns in Rosenzweig–MacArthur model. Letters in Biomathematics, 2019. DOI: 10.1080/23737867.2019.158598.

[34] Esther S. Daus, Ansgar Jüngel and Bao Quoc Tang. Exponential Time Decay of Solutions to Reaction-Cross-Diffusion Systems of Maxwell–Stefan Type. Arch. Rational Mech. Anal. 235, 2020, 1059–1104. https://doi.org/10.1007/s00205-019-01439-9.