Abstract. We study the combinatorics of free and simplicial line arrangements. After some preparation, we start by proving the Dirac Motzkin Conjecture for line arrangements whose characteristic polynomials split over $\mathbb{R}$. Then, we give a combinatorial analogue of the fact that a rank three hyperplane arrangement having isometric chambers is a Coxeter arrangement. We proceed by giving a classification of free simplicial line arrangements whose vertices have weight bounded by four. Finally, we show that any free line arrangement whose vertices have weight bounded by five consists of at most 185 lines.

1. Introduction

Arrangements of hyperplanes are classical objects of study in combinatorics and geometry (see for example [18]). In this note we consider the special case of rank three hyperplane arrangements. These may be identified with arrangements of lines in the projective plane. More precisely, we will be interested in the combinatorics of free and simplicial line arrangements in the real projective plane (see Section 2 for precise definitions).

Despite some major progress (see for instance the papers [4], [8], [9]), a complete classification of simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$ still remains an open problem. However, there is a catalogue published by Grünbaum (see [13]), listing all currently known isomorphism classes of such arrangements except for four arrangements later discovered by M. Cuntz (see [4]).

The current belief is that -up to finitely many corrections- the given catalogue is complete. In this paper we collect some more evidence for this belief. In particular, we show that Grünbaum’s catalogue contains all free simplicial line arrangements whose vertices have weight bounded by four (see Theorem 5). Similarly, we prove that a free simplicial line arrangement whose vertices have weight bounded by five consists of at most 40 lines (see Theorem 6). This implies that there is only a (rather small) finite number of such arrangements possibly missing in Grünbaum’s catalogue.

Motivated by the work of Green and Tao (see the paper [12]), we also study the Dirac Motzkin Conjecture in the special case of line arrangements whose characteristic polynomials split over $\mathbb{R}$. We are able to resolve the conjecture completely in this setup, including a classification of all extremal examples (see Theorem 2).
Moreover, we prove a combinatorial analogue of the fact that hyperplane arrangements in $\mathbb{R}^3$ having isometric chambers are Coxeter arrangements (see Theorem 3 and the paper [10]).

Most given arguments are purely combinatorial and focus on the $t$-vector of an arrangement (see Section 2 for precise definitions). Therefore, one obtains corresponding statements for arrangements of pseudolines.

Our techniques also allow us to prove finiteness results concerning line arrangements which are only free but not necessarily simplicial. In particular, we prove that there are only finitely many isomorphism classes of free line arrangements in $\mathbb{P}^2(\mathbb{R})$ having only vertices of weight bounded by five (see Theorem 8).

Acknowledgement. I wish to thank Michael Cuntz for many helpful discussions. I am also thankful for the financial support received from the Deutsche Forschungsgemeinschaft (DFG).

2. Definitions and notation

We start with the notions of rank three hyperplane arrangements and projective line arrangements.

**Definition 1.** a) Let $\mathcal{A}$ be a finite set of linear hyperplanes in $\mathbb{R}^3$. Then $\mathcal{A}$ is called a hyperplane arrangement. The arrangement is called trivial if $\bigcap_{H \in \mathcal{A}} H \neq \{0\}$. Otherwise, it is called nontrivial. The connected components of $\mathbb{R}^3 \setminus \left( \bigcup_{H \in \mathcal{A}} H \right)$ are called the chambers of $\mathcal{A}$. Let $\mathcal{V}_\mathcal{A}$ be the set of codimension two subspaces which are contained in at least two different hyperplanes of $\mathcal{A}$. The rays emanating from the origin which are contained in an element of $\mathcal{V}_\mathcal{A}$ are called the vertices of $\mathcal{A}$. These vertices partition each hyperplane into connected components which are called the edges of $\mathcal{A}$. If $K$ is a chamber of $\mathcal{A}$ then we associate to $K$ a weighted undirected graph $\Gamma^K$, called the Coxeter diagram at $K$, in the following way: the vertices of $\Gamma^K$ are given by the $s$ hyperplanes $H_1, \ldots, H_s$ bounding $K$. Two vertices $H_i, H_j$ are connected by an edge in $\Gamma^K$ if and only if there are at least three hyperplanes of $\mathcal{A}$ containing the subspace $H_i \cap H_j$. The weight of this edge is then given by the precise number of hyperplanes of $\mathcal{A}$ containing $H_i \cap H_j$. For a given chamber $K$, we define $w(K)$ to be the sum of all edgeweights of the Coxeter diagram $\Gamma^K$ and we call this quantity the weight of the chamber $K$. We write $f_0^\mathcal{A}, f_1^\mathcal{A}, f_2^\mathcal{A}$ for the number of vertices, edges, and chambers of $\mathcal{A}$ respectively. We say that two hyperplane arrangements $\mathcal{A}, \mathcal{A}'$ are isomorphic if the cell decompositions induced on the unit sphere $\mathbb{S}^2$ by $\mathcal{A}$ and $\mathcal{A}'$ are combinatorially isomorphic.

b) Let $\mathcal{A}$ be a finite set of projective lines in $\mathbb{P}^2(\mathbb{R})$. Then $\mathcal{A}$ is called a projective line arrangement. The arrangement is called trivial, if there is a point incident with all lines of $\mathcal{A}$. Otherwise, it is called nontrivial. The connected components of $\mathbb{P}^2(\mathbb{R}) \setminus \left( \bigcup_{\ell \in \mathcal{A}} \ell \right)$ are called the chambers of $\mathcal{A}$. The points in $\mathbb{P}^2(\mathbb{R})$ incident with at least two lines of $\mathcal{A}$ are called the vertices of $\mathcal{A}$. The vertices of $\mathcal{A}$ partition its lines into segments, which are called
the edges of \( A \). If \( K \) is a chamber of \( A \) then we associate to \( K \) a weighted undirected graph \( \Gamma^K \), called the Coxeter diagram at \( K \), in the following way: the vertices of \( \Gamma^K \) are given by the \( s \) lines \( \ell_1, \ldots, \ell_s \) bounding \( K \). Two vertices \( \ell_i, \ell_j \) are connected by an edge in \( \Gamma^K \) if and only if there are at least three lines of \( A \) passing through the point \( \ell_i \cap \ell_j \). The weight of this edge is then given by the precise number of lines of \( A \) passing through \( \ell_i \cap \ell_j \).

For a given chamber \( K \), we define \( w(K) \) to be the sum of all edgeweights of the Coxeter diagram \( \Gamma^K \) and we call this quantity the weight of the chamber \( K \). We write \( f_A^i, f_A^1, f_A^2 \) for the number of vertices, edges, and chambers of \( A \) respectively. We say that two line arrangements \( A, A' \) are isomorphic if the cell decompositions induced on \( \mathbb{P}^2(\mathbb{R}) \) by \( A \) and \( A' \) are combinatorially isomorphic.

Remark 1. a) Consider the natural projection \( \pi : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{P}^2(\mathbb{R}) \). Observe that for any given projective line arrangement \( A \) in \( \mathbb{P}^2(\mathbb{R}) \) we find a unique arrangement \( \bar{A} \) of linear hyperplanes in \( \mathbb{R}^3 \) such that under the projection \( \pi \) the hyperplanes (with the origin removed) of \( \bar{A} \) are mapped to the lines of \( A \). Clearly, we have \( f_1^{\bar{A}} = 2f_1^A \) for \( 0 \leq i \leq 2 \) and two projective line arrangements \( A, A' \) are isomorphic if and only if the corresponding hyperplane arrangements \( \bar{A}, \bar{A}' \) are isomorphic.

b) Of course one can also study hyperplane arrangements in \( \mathbb{K}^r \) for some positive \( r \in \mathbb{N} \) and some field \( \mathbb{K} \). However, some of the concepts in the above definition make sense only in case \( \mathbb{K} = \mathbb{R} \). Detailed definitions in the most general setup can be found in the classical book [18].

c) By the above definition it is clear that for every \( n \in \mathbb{N} \), there are only finitely many isomorphism classes corresponding to arrangements consisting of precisely \( n \) lines.

Now we can introduce the most important invariant for this paper.

Definition 2. Let \( A \) be a projective line arrangement. For each \( 2 \leq i \leq |A| \) we define \( t_i^A \) to be the number of vertices of \( A \) incident with precisely \( i \) lines of \( A \). We define \( t^A := (t_i^A)_{2 \leq i \leq |A|} \) and call it the \( t \)-vector of \( A \). Moreover, if \( v \) is a vertex of \( A \) incident with precisely \( i \) lines of \( A \), then we say that the weight or multiplicity of \( v \) is \( i \). A vertex of weight two is sometimes also called a double point. Similarly, a vertex of weight three is sometimes called a triple point.

Next we define the second most important invariant for this paper, the so called characteristic polynomial.

Definition 3. Let \( A \) be a projective line arrangement and denote by \( \bar{A} \) the corresponding arrangement of linear hyperplanes in \( V := \mathbb{R}^3 \). Define \( L_{\bar{A}} \) to be the collection of all subspaces of the form \( \bigcap_{H \in X} H \), where \( X \) ranges over subsets of \( \bar{A} \). Ordering \( L := L_{\bar{A}} \) by reverse inclusion makes it into a lattice, the so called intersection lattice of \( \bar{A} \). Denote its Möbius function by \( \mu \) and
define the characteristic polynomial of $A$ and $\tilde{A}$ by the formula
\[ \chi(A, t) := \chi(\tilde{A}, t) := \sum_{X \in L} \mu(V, X) t^{\dim(X)}. \]

Finally, we fix some notation and define the notions of simplicial and free line arrangements.

**Definition 4.** a) Let $A$ be a projective line arrangement in $\mathbb{P}^2(\mathbb{R})$. Let $Q \subset \mathbb{K} \subset \mathbb{C}$ be an extension field such that all roots of $\chi(A, t)$ are contained in $\mathbb{K}$. Then we say that $A$ splits over $\mathbb{K}$. If $k \in \mathbb{N}$ is maximal with the property $t^k \neq 0$, then we write $m(A) = k$ and we say that $A$ has multiplicity $k$. b) Let $A$ be a projective line arrangement in $\mathbb{P}^2(\mathbb{R})$. Then $A$ is called simplicial if every chamber of $A$ is bounded by precisely three lines of $A$. We say that $A$ is irreducible if $\Gamma_K$ is connected for any chamber $K$. Otherwise $A$ is called reducible. c) Let $A$ be a projective line arrangement in $\mathbb{P}^2(\mathbb{R})$ and let $\tilde{A}$ be the associated hyperplane arrangement in $\mathbb{R}^3$. By choosing coordinates we identify the symmetric algebra $\text{Sym}(\mathbb{R}^3)^*$ with the polynomial ring $S := \mathbb{R}[x, y, z]$. For every $H \in \tilde{A}$ choose a homogeneous linear form $l_H \in S$ such that $H = \ker(l_H)$ and let $D$ be the free $S$-module of derivations of $S$ over $\mathbb{R}$. We define the submodule $D_A := \{ \theta \in D \mid \forall H \in \tilde{A} : \theta(l_H) \in (l_H) \} \subset D$. Then $A$ is called free if the module $D_A$ is a free $S$-module.

**Remark 2.** a) Let $A$ be a reducible simplicial arrangement of $n$ lines. Then it is easy to see that $A$ is a so called near pencil arrangement: $A$ consists of $n - 1$ lines passing through some vertex $v$ and one further line not containing $v$ (see [7, Lemma 1]). In this case, one checks that $A = A_1 \times A_2$ is obtained as the product of two arrangements in lower dimension. This explains why such arrangements are called reducible. b) If $A, A'$ are simplicial line arrangements, then $A$ and $A'$ are isomorphic if and only if the graphs induced by the corresponding triangulations of $\mathbb{P}^2(\mathbb{R})$ are isomorphic: the existence of a bijection between the 2-cells which preserves the incidence structure is then automatic (see [3, Lemma 3.12]). c) The above definition of freeness may be generalized to hyperplane arrangements in $\mathbb{K}^r$ for some arbitrary field $\mathbb{K}$ and $r \in \mathbb{N}_{>0}$, for details on this see [18, Chapter 4]. Similarly, one may generalize part b) of the above definition to study simplicial arrangements in $\mathbb{R}^r$ for $r \in \mathbb{N}_{>0}$. See also the paper [5] for a notion of combinatorial simpliciality, which makes sense even in $\mathbb{K}^r$ for an arbitrary field $\mathbb{K}$ and which coincides with the classical notion in case $\mathbb{K} = \mathbb{R}$. d) It is well known that the roots of the characteristic polynomial of any free line arrangement are integral. More precisely, the multiset of roots coincides with the multiset of degrees of homogeneous generators in a basis for $D_A$ (this is essentially Terao’s factorization theorem, see [18, Theorem 4.137]). Thus, any free line arrangement splits over $\mathbb{Q}$ and therefore over $\mathbb{R}$. 
3. Combinatorics of Free and Simplicial Line Arrangements

In this section we study the combinatorics of free and simplicial line arrangements in detail. We start with some results on the $t$-vector of such an arrangement. We then proceed to prove the Dirac Motzkin Conjecture for line arrangements whose characteristic polynomials have only real roots. Afterwards, we prove a combinatorial analogue of the theorem which says that every hyperplane arrangement in $\mathbb{R}^3$ having isometric chambers is a Coxeter arrangement.

Then, a classification of simplicial line arrangements which split over $\mathbb{R}$ and which have multiplicity at most four is given. It turns out that such an arrangement is automatically crystallographic.

Moreover, we prove that there are only finitely many isomorphism classes of simplicial line arrangements which split over $\mathbb{R}$ and which have multiplicity at most five. More precisely, we show that such an arrangement consists of at most 40 lines. We are able to give classification results in this situation if we impose some further restriction.

We also obtain some finiteness results concerning free line arrangements which are not necessarily simplicial.

Throughout this section, we denote all isomorphism classes of simplicial line arrangements in the same way as in the papers [4], [13].

3.1. Basic relations involving $t^A$ and bounds for $t^A_2, t^A_3$. Let $\mathcal{A}$ be a simplicial projective line arrangement. In this subsection we first collect some known results on $t^A$. We then proceed to derive upper and lower bounds for the numbers $t^A_2, t^A_3$ (mostly in terms of $|\mathcal{A}|$). These results will then be used in the following subsections.

We start with the following basic lemma whose proof may be omitted.

**Lemma 1.** Let $\mathcal{A}$ be a nontrivial projective line arrangement in $\mathbb{P}^2(\mathbb{R})$. Write $n := |\mathcal{A}|$. Then for the $t$-vector $t^A$ the following relations hold:

1. $\sum_{i \geq 2} \binom{i}{2} t^A_i = \binom{n}{2}$,
2. $1 + \sum_{i \geq 2} (i - 1) t^A_i = f^A_2$,
3. $\sum_{i \geq 2} t^A_i = f^A_0$,
4. $\sum_{i \geq 2} i t^A_i = f^A_1$,
5. $3 + \sum_{i \geq 4} (i - 3) t^A_i \leq t^A_2$.

**Remark 3.** Inequality (5) is also known as Melchior’s inequality (see [17]). We observe that $\mathcal{A}$ is simplicial if and only if we have equality in (5).

Thus, for any simplicial arrangement $\mathcal{A}$ we have $f^A_2 = 2(f^A_0 - 1) = \frac{2}{3} f^A_1$. 

The following two lemmata are easy but very important results.

**Lemma 2.** a) Let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Suppose there is a line $\ell \in \mathcal{A}$ containing an edge bounded by two vertices of weight two. Then $\mathcal{A}$ is not irreducible.

b) Assume that $\mathcal{A}$ is an irreducible simplicial line arrangement. Then we have the estimate $4t_2^A \leq f_2^A$. Equality holds if and only if every chamber contains a vertex of weight two.

c) Let $\mathcal{A}$ be an irreducible simplicial line arrangement. Then we have the following analogue of Melchior’s inequality for the number $t_3^A$:

$$t_3^A \geq 4 + \sum_{i \geq 5} (i - 4)t_i^A.$$  

Equality holds if and only if we have equality in b), i.e. if every chamber of $\mathcal{A}$ contains a double point.

**Proof.**

a) By definition $\mathcal{A}$ is not irreducible if there exists a chamber $K$ such that the Coxeter diagram $\Gamma^K$ is not connected. Moreover, if $\ell \in \mathcal{A}$ contains an edge $e$ bounded by two vertices of weight two, then the Coxeter diagrams $\Gamma^K_i$ corresponding to the chambers $K_1, K_2$ containing $e$ are not connected.

b) By part a) every chamber has at most one vertex of weight two. Denote the set of chambers of $\mathcal{A}$ by $K$ and denote by $V_2$ the set of vertices of $\mathcal{A}$ which have weight two. Then clearly we have $\sum_{K \in K} K \cap V_2 \leq f_2^A$. On the other hand, every $v \in V_2$ is contained in exactly four chambers. Hence $4t_2^A = \sum_{K \in K} K \cap V_2 \leq f_2^A$. The statement about equality is now obvious.

c) By part b) and Remark 3 we have $-2 + 2 \sum_{i \geq 2} t_i^A = f_2^A \geq 4t_2^A$. Using the fact that we have equality in (5) and rearranging terms, we obtain the desired inequality. Now assume that $t_3^A = 4 + \sum_{i \geq 5} (i - 4)t_i^A$. This implies $t_2^A = -1 + \sum_{i \geq 3} t_i^A$, which yields $2t_2^A = f_2^A$.  

**Remark 4.** a) If $\mathcal{A}$ is simplicial but not irreducible, then part a) of Remark 2 tells that $\mathcal{A}$ is a near pencil arrangement with $t_2^A = |A| - 1$ and $f_2^A = 2(|A| - 1)$. So in this case one has $f_2^A < 4t_2^A$.

b) Lemma 2 shows that for irreducible simplicial line arrangements, the number of triple points is as small as possible if and only if the number of double points is as big as possible. Examples of arrangements with $4t_2^A = f_2^A$ are for instance given by the finite reflection arrangements in $\mathbb{R}^3$.

**Lemma 3.** Let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Assume that $\mathcal{A}$ has some vertex $v$ of weight two such that every neighbour of $v$ has weight three. Then $|A| \in \{6, 7\}$.

**Proof.** This follows from [7, Lemma 3].  

The following corollary has no direct applications in this paper. Nonetheless, it seems to be interesting in its own right.
Corollary 1. Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ with $|\mathcal{A}| \geq 8$. Then there exists a chamber containing at most one vertex of weight three.

Proof. Suppose that every chamber of $\mathcal{A}$ has at least two vertices of weight three. By [8, Corollary 3.18] we know that there exists a chamber containing a vertex of weight two and a vertex of weight three. Hence we have a vertex of weight two whose neighbours all have weight three. By the last lemma we conclude that $|\mathcal{A}| \leq 7$, a contradiction. □

We proceed to give a little lemma which tells us about the combinatorial consequences of the fact that an arrangement $\mathcal{A}$ splits over $\mathbb{R}$.

Lemma 4. Let $\mathcal{A}$ be an arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$ and consider the number $m := \frac{(n+1)^2}{2} - 4f_2^A$. Then for the characteristic polynomial $\chi(\mathcal{A},t)$ we have the following formula:

$$\chi(\mathcal{A},t) = t^3 - nt^2 + (f_2^A - 1)t + n - f_2^A.$$ 

In particular, the roots of $\chi(\mathcal{A},t)$ are given by $1$, $\frac{n-1+\sqrt{m}}{2}$, $\frac{n-1-\sqrt{m}}{2}$. Thus, $\mathcal{A}$ splits over $\mathbb{R}$ if and only if $m \geq 0$. In this case we have $f_2^A \leq \frac{(n+1)^2}{4}$.

Proof. Let $\bar{\mathcal{A}}$ be the hyperplane arrangement associated to $\mathcal{A}$ and let $L$ denote its intersection lattice. By abuse of notation we write $\mu(X) := \mu(V, X)$ for every $X \in L$. Then by definition we have $\chi(\mathcal{A},t) = \sum_{X \in L} \mu(X)t^{\dim(X)}$. If $X \in L$ has codimension 2 then $\mu(X) = |A_X| - 1$ and if $X \in L$ has codimension 1 then $\mu(X) = -1$. Further, $\mu(V) = 1$ and $\mu(\emptyset) = -\sum_{\emptyset \neq Y \in L} \mu(Y)$. Write $L_2$ for the subset of $L$ consisting of all elements of codimension 2. The claim then follows from the identity $\sum_{X \in L_2} |A_X| - 1 = \sum_{i \geq 2} i(i-1)t_i$ by straightforward calculation. For this observe that $\sum_{i \geq 2} i(i-1)t_i = f_2^A - 1$ as the Euler characteristic of $\mathbb{P}^2(\mathbb{R})$ equals one. □

Remark 5. It is known that every crystallographic simplicial line arrangement is inductively free and therefore the corresponding characteristic polynomial splits over $\mathbb{R}$ (see [1]). Moreover, among the 51 known examples of isomorphism classes of irreducible simplicial arrangements with up to 27 lines which are non-crystallographic, there are 31 whose characteristic polynomials have only real roots.

In order to obtain an upper bound for the number of double points in an arbitrary irreducible simplicial line arrangement $\mathcal{A}$, it suffices by part b) of Lemma 2 to bound the number of chambers of $\mathcal{A}$. This is done in the following Proposition.

Proposition 1. a) Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Then we have the estimate $t_2^A \leq \frac{(n+1)^2}{16}$, where $n := |\mathcal{A}|$. Moreover, this bound is tight.

b) If additionally $\mathcal{A}$ splits over $\mathbb{R}$ then we have $t_2^A \leq \frac{(n+1)^2}{16}$. 
Proof. a) By part b) of Lemma 2 we have $4t_2^A \leq f_2^A$. Further, using the relations given in Lemma 2 together with Remark 3 one can check that
\[ f_2^A = \binom{n}{2} + 1 - \sum_{i \geq 3} \binom{i-1}{2} t_i^A \leq \binom{n}{2} + 1 - t_3^A - 3t_4^A + 9. \]

Using $t_3^A \geq 4$, we may conclude that $4t_2^A \leq f_2^A \leq \frac{n^2-n+12}{2} - 3t_4^A$, proving the first claim. The reflection arrangement of type $B_3$ is an example for which the given bound is tight.

b) By Lemma 4 the characteristic polynomial $\chi(A, t)$ splits over $\mathbb{R}$ if and only if $\frac{(n+1)^2}{4} \geq f_2^A$. Using part b) of Lemma 2 it follows that $\frac{(n+1)^2}{4} \geq f_2^A \geq 4t_2^A$. This proves the claim. \qed

Remark 6. a) We can determine all simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$ which split over $\mathbb{R}$ and for which the bound in part a) of Proposition 1 is tight: in this case we have
\[ \frac{n^2-n}{14} + \frac{6}{7} = \frac{n^2}{16} + \frac{n}{8} + \frac{1}{16} = \frac{(n+1)^2}{16}. \]

This in turn leads to the inequality $\frac{n^2}{112} - \frac{11n}{56} + \frac{89}{112} \leq 0$, which holds only for $6 \leq n \leq 16$. Using the results in [4], it follows that the arrangements $A(6, 1), A(9, 1), A(13, 2)$ are the only examples.

b) Let $A$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ with $|A| = n$.

If $t_2^A = \frac{\binom{n}{2}+6}{7}$ then we have equality in the chain of inequalities
\[ 4t_2^A \leq f_2^A \leq \frac{n^2-n}{3} - \frac{2}{3}t_2^A + 4. \]

This in turn implies that $A$ has multiplicity at most four. Moreover, we observe that for any simplicial line arrangement $A$ one has
\[ t_3^A = 4 \iff t_4^A = \frac{\binom{n}{2}-15}{7} \iff t_2^A = \frac{\binom{n}{2}+6}{7}. \]

In the following, we want to establish an upper bound for $\min(t_2^A, t_3^A)$ and a lower bound for $\max(t_2^A, t_3^A)$. In order to do this we use a little lemma which may be interesting in its own right.

Lemma 5. Let $A$ be an irreducible simplicial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$. Then the following statements are true:

a) $2(t_2^A+2) \leq 4 + \frac{t_4^A}{3} + \sum_{i \geq 5}(i-4)t_i^A = 2t_2^A + t_3^A \leq 2t_2^A + t_3^A + \frac{t_4^A}{3} \leq \frac{\binom{n}{2}}{3} + 5$.

b) If $t_i^A = 0$ for $i > 3$ then we have $2t_2^A + t_3^A + \frac{t_4^A}{3} = \frac{\binom{n}{2}}{3} + 5$. In particular, we cannot have $t_4^A \equiv 2 \pmod{3}$ for such an arrangement.

Proof. a) Equation (1) from Lemma 1 gives $3t_3^A = \binom{n}{2} - t_2^A - \sum_{i \geq 4} (i-4)t_i^A$.

Moreover, for $i \geq 5$ we always have $\frac{\binom{n}{2}}{3} \geq 5(i-3)$ and so we conclude that
\[ 6t_4^A + \sum_{i \geq 5}(i-2)t_i^A \geq t_4^A + 5 \sum_{i \geq 4}(i-3)t_i^A = 5t_2^A + t_4^A - 15. \]

From this, it follows that $3t_3^A \leq \binom{n}{2} - 6t_2^A - t_4^A + 15$, proving the upper bound for $2t_2^A + t_3^A + \frac{t_4^A}{3}$. 


Next we show that $2t_2^A + t_3^A = 4 + \frac{f_3^A}{2} + \sum_{i \geq 5}(i-4)t_i^A$: observe that $t_2^A = 3 + \sum_{i \geq 4}(i-3)t_i^A = 4 + \sum_{i \geq 4} t_i^A - 1 + \sum_{i \geq 5}(i-4)t_i^A$. Adding $t_2^A + t_3^A$ on both sides of the last equation gives the desired equality.

Finally, the inequality $2(t_2^A + 2) \leq 4 + \frac{f_3^A}{2} + \sum_{i \geq 5}(i-4)t_i^A$ follows from part b) of Lemma 2.

b) If $t_i^A = 0$ for $i > 6$ then $t_2^A + 3t_3^A + 6t_4^A + 10t_5^A + 15t_6^A = (\binom{n}{2})$, using equation (1) from Lemma 1. By simpliciality of $\mathcal{A}$ we have $t_2^A = 3 + t_4^A + 2t_5^A + 3t_6^A$ (see Remark 3). We conclude that $6t_2^A + 3t_3^A + t_4^A - 15 = (\binom{n}{2})$. It follows $n^2 - n - 2t_4^A \equiv 0 \pmod{3}$. As the polynomial $X^2 - X + 2$ is irreducible over the finite field $\mathbb{F}_3$, it follows that $t_4^A \not\equiv 2 \pmod{3}$. This completes the proof.

\begin{corollary}
Let $\mathcal{A}$ be an irreducible simplicial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$. Then we have $\min(t_2^A, t_3^A) \leq \frac{n^2 - n + 30}{18}$ and $\max(t_2^A, t_3^A) > \frac{t_4^A}{6}$.
\end{corollary}

\begin{proof}
Assume that $\max_{i \geq 2} t_i^A = t_3^A$. Then by Lemma 3 we have

$$3t_2^A \leq 2t_2^A + t_3^A \leq \frac{(\binom{n}{2})}{3} + 5,$$

$$\frac{f_3^A}{2} < 2t_2^A + t_3^A \leq 3t_3^A.$$

This proves the claim in case $\max_{i \geq 2} t_i^A = t_3^A$. The case $\max_{i \geq 2} t_i^A = t_2^A$ is dealt with similarly.
\end{proof}

We close this subsection with a theorem which asserts that for a simplicial line arrangement $\mathcal{A}$, at least one of $t_2^A, t_3^A$ is quadratic in $|\mathcal{A}|$. For this we need one more result:

\begin{proposition}
Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ with $m := |\mathcal{A}|$. Then we have $t_i^A = 0$ for all $i > m$. Moreover, we always have $t_m^A \leq 1$.
\end{proposition}

\begin{proof}
Suppose that there was some $i > m$ such that $t_i^A > 0$. Pick a vertex $v$ of weight $i$ and denote the set of lines passing through $v$ by $L_v$. Then there are $2i$ chambers $K_1, \ldots, K_{2i}$ having $v$ as a vertex. Each of these chambers has precisely one wall supported by a line not contained in $L_v$. As $|\mathcal{A} \setminus L_v| < m$, we conclude that there must be some $\ell \in \mathcal{A} \setminus L_v$ such that $\ell$ is a wall in three neighbouring chambers $K_{j_1}, K_{j_2}, K_{j_3}$. But then $\ell$ contains a segment bounded by two vertices of weight two. It follows that $\mathcal{A}$ is not irreducible, contradicting our initial assumption. This proves the first claim. Next we show that $t_m \leq 1$. Suppose that $t_m > 1$. Then clearly $t_m = 2$ and we denote the two vertices of weight $m$ by $v_1$ and $v_2$. Then $v_1$ and $v_2$ may or may not be connected by a line of $\mathcal{A}$ and one checks that any line of $\mathcal{A}$ not passing through both $v_1$ and $v_2$ contains a segment bounded by two vertices of weight two. Therefore $\mathcal{A}$ is not irreducible. This completes the proof.
\end{proof}
Using what we have established so far together with a result from [16], we are now in a position to prove the following theorem:

**Theorem 1.** Let $\mathcal{A}$ be an irreducible simplicial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$. Then the following statements hold:

a) We have $\max (t_2^A, t_3^A) > \frac{n^2 + 3n}{27}$.

b) If $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$, then we have $t_2^A \geq 3 + \frac{(n-5)^2 - 4}{4m(\mathcal{A}) - 8}$. Equivalently, we have $\sum_{i \geq 4} (i-3)t_i^A \geq \frac{(n-5)^2 - 4}{4m(\mathcal{A}) - 8}$. Moreover, one has the following inequalities:

$$t_3^A + \frac{2t_4^A + 3t_5^A}{m(\mathcal{A})} \geq 4 + \frac{(n-5)^2 - 4}{4m(\mathcal{A})} + \sum_{i \geq 4} (i-6)t_i^A \geq 4 + \frac{(n-5)^2 - 4}{4m(\mathcal{A})}.$$ 

**Proof.** a) Observe that by Proposition 2 we know that $t_i^A = 0$ for $i > \frac{n}{2}$. Thus, we may apply [16, Proposition 11.3.1] to obtain the following estimate:

$$f_1^A = \sum_{i \geq 2} it_i^A \geq \frac{n^2 + 3n}{3}.$$ 

As $\mathcal{A}$ is simplicial we have $3f_2^A = 2f_1^A$ (see Remark 3), hence using Corollary 2 we obtain the inequality

$$\max (t_2^A, t_3^A) > \frac{f_2^A}{6} \geq \frac{n^2 + 3n}{27},$$

finishing the proof of part a).

b) By equation (1) from Lemma 1 we have $2t_3^A = \frac{n^2 - n}{3} - \frac{2n^2}{3} - \sum_{i \geq 4} i^2 - i t_i^A$. Combining this with equation (2) and inequality (5) from Lemma 1 and remembering Lemma 4 we obtain

$$\frac{(n + 1)^2}{4} \geq f_2^A = \frac{n^2 - n + 3}{3} + \frac{t_2^A}{3} - \sum_{i \geq 4} i^2 - 4i + 3 t_i^A$$

$$\geq \frac{n^2 - n + 3}{3} + 1 + \sum_{i \geq 4} i - 3 t_i^A - \sum_{i \geq 4} i^2 - 4i + 3 t_i^A$$

$$= \frac{n^2 - n + 6}{3} - \frac{1}{3} \sum_{i \geq 4} (i - 2)(i - 3) t_i^A.$$

We conclude that $\sum_{i \geq 4} (i-2)(i-3)t_i^A \geq \frac{n^2 - 10n + 21}{4} = \frac{(n-5)^2}{4} - 1$. Note that $m(\mathcal{A}) \geq 3$ because $\mathcal{A}$ is irreducible. By definition, we have $i - 2 \leq m(\mathcal{A}) - 2$ for every $i$ such that $t_i^A > 0$. Using this, we obtain the following chain of inequalities:

$$(m(\mathcal{A}) - 2)(t_2^A - 3) \geq \sum_{i \geq 4} (i-2)(i-3)t_i^A \geq \frac{(n-5)^2}{4} - 1.$$ 

We conclude that $t_2^A \geq 3 + \frac{(n-5)^2 - 4}{4m(\mathcal{A}) - 8}$. As $\mathcal{A}$ is simplicial, this allows us to write $3 + \sum_{i \geq 4} (i-3)t_i^A = t_2^A \geq 3 + \frac{(n-5)^2 - 4}{4m(\mathcal{A}) - 8}$, proving the first two statements.
For the last statement, we note that \( i(i - 4) - (i - 2)(i - 3) = i - 6 \) and we use part c) of Lemma 2 to obtain

\[
m(\mathcal{A})(t^A_3 - 4) \geq \sum_{i \geq 5} i(i - 4)t^A_i = \sum_{i \geq 5} (i - 2)(i - 3)t^A_i + \sum_{i \geq 5} (i - 6)t^A_i
\]
\[
= \sum_{i \geq 4} (i - 2)(i - 3)t^A_i + \sum_{i \geq 4} (i - 6)t^A_i
\]
\[
\geq \frac{(n - 5)^2 - 4}{4} + \sum_{i \geq 7} (i - 6)t^A_i - 2t^A_4 - t^A_5.
\]

This completes the proof of the theorem. □

**Remark 7.**

a) Note that [16, Proposition 11.3.1] is a result on arrangements in \( \mathbb{P}^2(\mathbb{C}) \). However, by complexification every arrangement \( \mathcal{A} \) in \( \mathbb{P}^2(\mathbb{R}) \) yields an arrangement \( \mathcal{A}_C \) in \( \mathbb{P}^2(\mathbb{C}) \) with the same intersection lattice.

b) If \( \mathcal{A} \) has multiplicity at most five, then the estimate given in part a) of Theorem 1 can be improved by taking advantage of [21, Theorem 1]: this result says that \( f^A_2 \geq \frac{2n^2 - 2n + 4m(\mathcal{A})}{m(\mathcal{A}) + 3} \).

We close this subsection with the observation that the first estimate in part b) of Theorem 1 stays valid for arrangements which are non-simplicial:

**Corollary 3.** Let \( \mathcal{A} \) be a nontrivial arrangement of \( n \) lines in \( \mathbb{P}^2(\mathbb{R}) \) whose characteristic polynomial splits over \( \mathbb{R} \). If \( m(\mathcal{A}) \geq 3 \), then \( t^A_2 \geq 3 + \frac{(n-5)^2-4}{4m(\mathcal{A})-8} \).

**Proof.** Basically the proof for part b) of Theorem 1 can be copied word by word: observe that in order to pass from line (6) to line (7) in said proof, one only needs relation (5) from Lemma 1 which holds for any nontrivial arrangement. The condition \( m(\mathcal{A}) \geq 3 \) is needed to divide by \( m(\mathcal{A}) - 2 \). □

### 3.2. The Dirac Motzkin Conjecture for free and simplicial line arrangements

In this subsection we study the so-called **Dirac Motzkin Conjecture**, which asserts that for a nontrivial arrangement \( \mathcal{A} \) of \( n \) lines in \( \mathbb{P}^2(\mathbb{R}) \) one always has

\[
t^A_2 \geq \left\lfloor \frac{n}{2} \right\rfloor.
\]

This has been a famous open problem for a long time until in the paper [12] said conjecture has been shown to be a theorem for *sufficiently large* arrangements. However, the lower bounds given in the paper concerning the “sufficiently large” part are of double exponential order.

We study this conjecture in the context of line arrangements whose characteristic polynomials have only real roots. This is motivated by the fact that there is an infinite family of such arrangements (denoted \( \mathcal{R}(1) \) in [13]) with \( t^A_2 = \frac{|\mathcal{A}|}{2} \) for every \( \mathcal{A} \) in the family.

In the paper [12], the dual point configurations corresponding to arrangements in the family \( \mathcal{R}(1) \) are the so-called “Böröcky examples”, denoted
by $X_{2m}$ for $m \in \mathbb{N}_{\geq 3}$. Moreover, we note that these arrangements are all (inductively) free and simplicial.

By Lemma 2, every chamber of an irreducible simplicial line arrangement contains at most one double point. Therefore, the simpliciality of the line arrangements corresponding to the Böröcky examples may not seem surprising: simplicial arrangements could in general be expected to yield “corner cases” of the Dirac Motzkin Conjecture.

Besides the points mentioned above, this point of view is supported by the observation that, apparently, the only known examples with $t^A_2 < \frac{|A|}{2}$ are the simplicial arrangements $A(7,1), A(13,4)$. We observe that the first arrangement splits over $\mathbb{R}$ while the second does not. Moreover, the first arrangement is known as the “Kelly-Moser configuration” while the second one is known as the “Crowe-McKee configuration” (see [15, 3]).

Remark 8. We note that the arrangements from the infinite family $\mathcal{R}(1)$ are completely characterized by their $t$-vector. Indeed, the smallest arrangement in the family $\mathcal{R}(1)$ is the arrangement $A' := A(6,1)$ which is characterized by the vector $t^{A'} = (3,4)$. Similarly, if $|A| \geq 8$ then $A$ belongs to $\mathcal{R}(1)$ if and only if there exists some $m \in \mathbb{N}_{\geq 4}$ such that $|A| = 2m$ and $t^A_2 = m, t^A_m = \frac{m^2 - m}{2}, t^A_m = 1$ while $t^A_i = 0$ for every $i \notin \{2,3,m\}$. In the papers [4], [13] the corresponding isomorphism class is denoted by $A(2m,1)$.

Note also that for $n \geq 2$, one may add a suitable line to the arrangement $A(4n,1)$ to obtain a new simplicial arrangement denoted by $A(4n+1,1)$. The arrangements of type $A(4n+1,1)$ with $n \geq 2$ constitute another infinite family, which is denoted by $\mathcal{R}(2)$. Again, we refer to [13] for more details.

Motivated by the above observations, we now state and prove the result of this subsection, which resolves the Dirac Motzkin Conjecture for arrangements whose characteristic polynomials have only real roots:

**Theorem 2.** Let $A$ be a nontrivial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$ whose characteristic polynomial splits over $\mathbb{R}$. Then the following statements hold:

a) We have $t^A_2 \geq \lfloor \frac{n}{2} \rfloor$.

b) If $t^A_2 = \lfloor \frac{n}{2} \rfloor$, then $A$ is simplicial. More precisely: if $n$ is even, then $A$ belongs to the infinite family $\mathcal{R}(1)$. If $n$ is odd, then $A$ is the Kelly-Moser example.

**Proof.** a) We consider three cases, corresponding to size and multiplicity of the given arrangement:

Case i): Assume that $3 \leq n \leq 7$.

Then by relation (5) from Lemma 2 we have $t^A_2 \geq 3 \geq \lfloor \frac{n}{2} \rfloor$.

Case ii): Assume that $n \geq 8$ and $m(A) \geq \frac{n}{2}$.

Then there exists $j \geq \frac{n}{2} \geq 4$ such that $t^A_j > 0$ and relation (5) from Lemma 2 yields $t^A_2 \geq 3 + \sum_{i \geq 4}(i-3)t^A_i \geq 3 + (j-3)t^A_j \geq 3 + \frac{n}{2} - 3 = \frac{n}{2} \geq \lfloor \frac{n}{2} \rfloor$.

Case iii): Assume that $n \geq 8$ and $m(A) < \frac{n}{2}$.
As \(1 + \binom{n}{2} > \frac{(n+1)^2}{4}\) for \(n \geq 8\), we conclude that \(m(A) \geq 3\) (remember equation (2) from Lemma 1 and Lemma 3). By Corollary 3, we have:

\[
t^2_A \geq 3 + \frac{(n-5)^2 - 4}{4m(A) - 8} \geq 3 + \frac{(n-5)^2 - 4}{4\frac{n-1}{2} - 8} = \frac{1}{2} \left(n + 1 - \frac{4}{n-5}\right).
\]

Clearly, for \(n \geq 9\) one has \(t^2_A \geq \frac{1}{2}(n + 1 - \frac{4}{n-5}) \geq \frac{n}{2} \geq \lfloor \frac{n}{2} \rfloor\). For \(n = 8\) we obtain \(\frac{23}{6} \leq t^2_A \in \mathbb{N}\), which implies \(t^2_A \geq 4 = \frac{n}{2} \geq \lfloor \frac{n}{2} \rfloor\).

b) Let \(A\) be an arrangement such that \(t^2_A = \lfloor \frac{n}{2} \rfloor\). We consider two cases, corresponding to the parity of \(n\):

Case i): Assume that \(n\) is odd.

Then we have \(t^2_A = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} < \frac{n}{2}\). The proof of part a) shows that we necessarily have \(3 \leq n \leq 7\). Indeed, for \(n \geq 8\) one always has \(t^2_A \geq \frac{n}{2}\).

Moreover, for \(3 \leq n \leq 6\) we also have \(t^2_A \geq 3 \geq \frac{n}{2}\). We conclude that \(n = 7\) and \(t^2_A = 3\). In particular, \(A\) is simplicial as we have equality in relation (5) from Lemma 1. Using [4], we conclude that \(A\) is the Kelly-Moser example.

Case ii): Assume that \(n\) is even.

Then we have \(t^2_A = \frac{n}{2}\). First assume that \(n \geq 8\). We show that the multiplicity of \(A\) is precisely \(\frac{n}{2}\). If not, then \(m(A) > \frac{n}{2}\) or \(m(A) < \frac{n}{2}\).

So assume that \(m(A) > \frac{n}{2}\). Then \(t^2_A \geq 3 + \sum_{i \geq 4}(i-3)t^A_i > 3 + \frac{n-6}{2} = \frac{n}{2}\), a contradiction.

Now assume that \(m(A) < \frac{n}{2}\). Then \(i - 2 \leq m(A) - 2 \leq \frac{n-2}{2} - 2 = \frac{n-6}{2}\) for every \(i\) such that \(t^A_i > 0\). Using that \(\frac{n-6}{2} = t^A_3 - 3\) and remembering the proof of part b) of Theorem 1, we conclude that:

\[
\frac{(n-6)^2}{4} = \frac{n-6}{2}(t^2_A - 3) \geq \sum_{i \geq 4}(i-2)(i-3)t^A_i \geq \frac{(n-5)^2}{4} - 1.
\]

Clearly, this is impossible for \(n \geq 8\). We obtain \(m(A) = \frac{n}{2}\). Using this, relation (5) from Lemma 1 yields \(t^A_3 = 1\) while \(t^A_i = 0\) for \(i \notin \{2, 3, \frac{n}{2}\}\).

Using equation (1) from Lemma 1, it follows that \(t^3_A = \frac{n^2 - 2n}{8}\). By Remark 8, we may conclude that \(A\) belongs to the infinite family \(\mathcal{R}(1)\).

It remains to consider the cases \(n = 4\) and \(n = 6\). For \(n = 4\), we obtain \(2 = \frac{n}{2} = t^2_A \geq 3\), which is impossible. If \(n = 6\), then \(t^2_A = \frac{n}{2} = 3\).

In particular, we have equality in relation (5) from Lemma 1. Using the enumeration in [4], it follows that \(A\) is the arrangement \(A(6, 1)\), which is the smallest arrangement from the family \(\mathcal{R}(1)\). This completes the proof. \(\square\)

**Remark 9.** We remark that there are non-simplicial arrangements for which we have equality in the Dirac Motzkin Conjecture: if one removes a suitable line from the (simplicial) arrangement \(A(13, 4)\), then one obtains a non-simplicial arrangement \(A\) with \(t^2_A = \frac{|A|}{2}\). Note also that the corresponding characteristic polynomial \(\chi(A, t)\) has a non-real root, in accordance with Theorem 2. However, as pointed out already, the only two known arrangements with \(t^2_A < \frac{|A|}{2}\) are both simplicial.
3.3. A combinatorial characterization of spherical rank three Coxeter arrangements. In this subsection we prove that (up to isomorphism) spherical rank three Coxeter arrangements are characterized as those line arrangements \( \mathcal{A} \) having the property that \( \Gamma^K \cong \Gamma \) for any chamber \( K \) of \( \mathcal{A} \) and a suitable fixed connected graph \( \Gamma \). This may be regarded as a combinatorial analogue of the theorem which asserts that spherical rank three Coxeter arrangements may be characterized as those arrangements in \( \mathbb{R}^3 \) which have isometric chambers (see [10]).

**Lemma 6.** Let \( \mathcal{A} \) be a nontrivial line arrangement in \( \mathbb{P}^2(\mathbb{R}) \). Assume that there is a connected graph \( \Gamma \) such that \( \Gamma^K \cong \Gamma \) for every chamber \( K \) of \( \mathcal{A} \). Then there exists \( x \in \mathbb{N} \) such that

\[
\Gamma = \bullet \cdots \bullet \xrightarrow{\bullet \quad \bullet \quad \bullet} \bullet
\]

**Proof.** Write \( n := |\mathcal{A}| \) and observe that \( \mathcal{A} \) is not a near pencil arrangement. By Shannon’s theorem (see [19]) we know that \( \mathcal{A} \) contains at least \( n \) chambers which are triangles. By assumption, this implies that every chamber of \( \mathcal{A} \) must be a triangle and hence the arrangement \( \mathcal{A} \) is necessarily simplicial. We have \( t_2^A \geq 3 \) by inequality (5) from Lemma 1. Moreover, we have \( t_3^A \geq 4 \) by Lemma 2. This proves the claim. \( \Box \)

**Proposition 3.** Fix \( x \in \mathbb{N} \) and let \( \mathcal{A} \) be a simplicial line arrangement in \( \mathbb{P}^2(\mathbb{R}) \) such that \( \Gamma^K \cong \Gamma = \bullet \cdots \bullet \xrightarrow{\bullet \quad \bullet \quad \bullet} \bullet \) for every chamber \( K \) of \( \mathcal{A} \). Then \( x \in \{3, 4, 5\} \). If \( x = 3 \) then \( \mathcal{A} \) is of type \( A(6,1) \), if \( x = 4 \) then \( \mathcal{A} \) is of type \( A(9,1) \) and if \( x = 5 \) then \( \mathcal{A} \) is of type \( A(15,1) \). In particular, \( \mathcal{A} \) is isomorphic to a spherical Coxeter arrangement.

**Proof.** Set \( n := |\mathcal{A}| \) and suppose that \( x = 3 \), so \( \mathcal{A} \) has only vertices of weight two or three. Then by Lemma 3 the arrangement \( \mathcal{A} \) is of type \( A(6,1) \) or \( A(7,1) \). Since the arrangement \( A(7,1) \) contains a chamber having only vertices of weight three, it follows that \( \mathcal{A} \) is of type \( A(6,1) \).

Now assume that \( x > 3 \). As \( \mathcal{A} \) is simplicial, we have

\[
t_2^A - (x - 3)t_x^A - 3 = 0.
\]

On the other hand, we have the following identities

\[
t_2^A + \left( \frac{3}{2} \right) t_3^A + \left( \frac{x}{2} \right) t_x^A - \left( \frac{n}{2} \right) = 0,
\]

\[
2t_2^A - 3t_3^A = 0,
\]

\[
3t_3^A - xt_x^A = 0.
\]

Regarding \( x \) as a variable, we consider the function field \( \mathbb{F} := \mathbb{Q}(x) \) and think of \( n, t_2^A, t_3^A, t_x^A \) as variables in a polynomial ring \( R := \mathbb{F}[n, t_2^A, t_3^A, t_x^A] \).
In $R$ we consider the ideal $I$ generated by the relations (9), (10), (11), (12) and we compute the following Gröbner basis for $I$:

$$I = \left(2 \binom{n}{2} + \frac{12x + 6x^2}{x - 6}, t_2^A + \frac{3x}{x - 6}, t_3^A + \frac{2x}{x - 6}, t_4^A + \frac{6}{x - 6}\right).$$

As $t_2^A > 0$ and $x \geq 4$ we infer that $x - 6 < 0$, hence $4 \leq x \leq 5$. For $x = 4$ we obtain $n = 9$ and $t^A = (6, 4, 3)$. If $x = 5$ then $n = 15$ and $t^A = (15, 10, 0, 6)$. Now we may use the enumeration given in [4] to obtain the full statement.

Lemma 3 and Proposition 3 now immediately give us the desired theorem:

**Theorem 3.** Let $A$ be a nontrivial line arrangement in $\mathbb{P}^2(\mathbb{R})$ with associated set of chambers $K$. Then the following statements are equivalent:

i) There exists a connected graph $\Gamma$ such that $\Gamma^K \cong \Gamma$ for every $K \in K$.

ii) $A$ is isomorphic to a spherical Coxeter arrangement.

3.4. Simplicial line arrangements having low multiplicities. In this subsection we study simplicial arrangements having multiplicity at most six. We prove that there are only finitely many (isomorphism classes of) irreducible simplicial arrangements which split over $\mathbb{R}$ and whose multiplicity is bounded by five. Among these arrangements we are able to give a classification of those having multiplicity bounded by four.

If $A$ has multiplicity bounded by six and if $t_2^A$ is not too large compared to $t_3^A$, then we can also prove that there are only finitely many possibilities for the isomorphism class of $A$, again provided that $A$ splits over $\mathbb{R}$.

Finally, we show that the validity of an old conjecture stated in [11] is related to Theorem 4, which gives asymptotically optimal estimates for $t_2^A, t_3^A, t_4^A$, and which is considered the main result of this subsection.

**Theorem 4.** Let $A$ be an irreducible simplicial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$. Suppose that $A$ splits over $\mathbb{R}$ and assume that $A$ has multiplicity at most six. Then we have the following estimates:

\begin{align}
(n - 5)^2 + 44 &\leq t_2^A \leq (n + 1)^2, \\
\frac{n^2 - 22n + 185}{24} &\leq t_3^A \leq \frac{n^2 + 116n - 597}{24}, \\
t_4^A + t_5^A &\leq \frac{3}{2}n - \frac{17}{2}, \\
\frac{n^2 - 46n + 225}{48} &\leq t_6^A \leq \frac{n^2 + 2n - 47}{48}.
\end{align}

In particular, all estimates hold true if $A$ is a free simplicial arrangement having multiplicity bounded by six.

**Proof.** By Remark 3, Lemma 4 and part b) of Lemma 2 we have $t_2^A = 3 + t_4^A + 2t_5^A + 3t_6^A \leq \frac{(n + 1)^2}{16}$. We conclude $t_6^A \leq \frac{n^2 + n}{24} - \frac{t_4^A}{3} - \frac{2t_5^A}{3} - \frac{47}{48}$. Using Corollary 3 we obtain $t_2^A \geq 3 + \frac{(n - 5)^2 - 4}{16}$, proving inequality (13).
As $3t_3^A = \binom{n}{2} - t_2^A - \sum_{i \geq 4} \binom{i}{2} t_i^A$ and $f_2^A = 2(f_0^A - 1)$, we may rewrite the condition that $A$ splits over $\mathbb{R}$ as $\frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{t_3^A}{2} - \frac{t_4^A}{6} \leq t_6^A$. For this observe that $(n+1)^2 \geq 4f_2^A$, by Lemma 4.

Now we combine the upper and lower bounds for $t_6^A$ and obtain the following estimate:

$$\frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{t_3^A}{2} - \frac{t_4^A}{6} \leq t_6^A \leq \frac{n^2}{48} + \frac{n}{24} - \frac{t_3^A}{3} - \frac{2t_5^A}{3} - \frac{16}{48}. \tag{17}$$

This implies inequality (15). Inequality (16) now follows from (17) using (15): we have $\frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{t_3^A}{2} - \frac{t_4^A}{6} \leq \frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{t_3^A}{2} - \frac{16}{48}$ and as $t_4^A + t_5^A \leq \frac{3n}{2} - \frac{17}{6}$ we conclude $\frac{n^2}{48} - \frac{46n + 225}{54} = \frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{3n}{4} + \frac{17}{4} \leq t_4^A$. Moreover, $t_6^A \leq \frac{n^2}{48} + \frac{n}{24} - \frac{t_3^A}{3} - \frac{2t_5^A}{3} - \frac{47}{48} \leq \frac{n^2}{48} + \frac{n}{24} - \frac{47}{48}$ because $t_i^A \geq 0$ for all $i \geq 2$.

Finally, the lower bound in (14) follows from part b) of Theorem 4 using (15). The upper bound follows from (13) and (16) using equation (1) from Lemma 1. This completes the proof. \hfill \Box

We now draw some conclusions from Theorem 4.

**Theorem 5.** Let $A$ be an irreducible simplicial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most four. Assume that $A$ splits over $\mathbb{R}$. Then $n \leq 16$. In particular, we have a complete list of such arrangements.

**Proof.** We have $t_i^A = 0$ for $i > 4$. Therefore, by equation (5) from Lemma 1 it follows $t_2^A = 3 + t_4^A$. Consequently, by Theorem 4 we obtain the upper bound $t_2^A \leq 3 + \frac{3}{2}n - \frac{17}{2}$. Using Corollary 3 we obtain $t_2^A \geq 3 + \frac{(n-5)^2-4}{8}$. It follows $\frac{(n-10)n+45}{8} \leq t_2^A \leq \frac{3}{2}n - \frac{17}{2}$ which implies $1 \leq n \leq 16$. This proves the first claim. The second claim follows from the first using the results in [4]. \hfill \Box

**Corollary 4.** Let $A$ be a line arrangement satisfying the conditions of Theorem 5. Then $A$ admits a crystallographic rootset. In particular, every free simplicial line arrangement having multiplicity bounded by four admits a crystallographic rootset.

**Proof.** Both claims can be verified by inspecting the enumeration given in [4]. For the second claim, observe that the roots of the characteristic polynomial of any free arrangement are given by its exponents. Thus, these roots are integral and in particular real. \hfill \Box

**Remark 10.** a) As every crystallographic arrangement is inductively free (see [4]), the last corollary shows that for a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ whose multiplicity is at most four, the notions of being free and being inductively free coincide.

b) We observe that every line arrangement satisfying the conditions of Theorem 5 is obtained as a subarrangement of the arrangement $A(13, 2)$ (see Figure 1). This can be verified by inspecting the Hasse diagram given in
The arrangement of type $A(13, 2)$. The line at infinity is contained in the arrangement, indicated by the symbol “∞”.

Figure 1

One should also observe that the latter arrangement is obtained as a restriction of the reflection arrangement of type $F_4$. Thus, all free simplicial line arrangements having multiplicity bounded by four originate from a reflection group.

c) Another interesting observation concerning the arrangement $A := A(13, 2)$ is the following: one computes that $t^A = (12, 4, 9)$, so that in particular we have $t^A_3 + t^A_4 = |A|$. Now let $R$ be the set consisting of all vertices of $A$ having weight at least three. By the above, we have $|R| = |A|$ and one checks that the arrangement $B := \{\ker(\alpha) \mid \alpha \in R\}$ in the dual space is isomorphic to $A$. Moreover, $A$ is the largest simplicial arrangement with this property. More on such duality results can be found in the paper [6].

The next theorem yields the announced finiteness result for simplicial arrangements splitting over $\mathbb{R}$ and having multiplicity bounded by five. We also obtain a classification result in this situation if the number of double points is not too large.

**Theorem 6.** Let $A$ be an irreducible simplicial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most five. Assume that $A$ splits over $\mathbb{R}$. Then the following statements hold:

a) We have $n \leq 40$.

b) If $t^A_2 \leq \frac{13}{16} t^A_3$, then $n \leq 27$. In particular, we have a complete list of such arrangements.
Proof. a) By assumption and Theorem 4 we have \( \frac{n^2 - 40n + 225}{48} \leq t_6^A = 0 \). This implies \( n \leq 40 \).

b) First, we observe that \( t_2^A \geq \frac{n^2 - 10n + 57}{12} \), by Corollary 3. As \( \frac{13}{16} t_2^A \leq t_3^A \) we may invoke Lemma 5 to arrive at the inequality \( \frac{n^2 - 10n + 57}{12} \leq t_2^A \leq \frac{8(n^2 - n + 30)}{135} \). It follows that we necessarily have \( n \leq 27 \). Using the results obtained in [4], this completes the proof. \( \square \)

For arrangements having multiplicity six, we can only prove finiteness results if the number of double points is not too large. More precisely, we will show that for \( \epsilon > 0 \) there are only finitely many isomorphism classes of simplicial arrangements \( A \) with \( t_2^A \leq \frac{24}{16 + \epsilon} t_3^A \) (of course provided that \( \chi(A, t) \) splits over \( \mathbb{R} \) and \( m(A) \leq 6 \)).

**Theorem 7.** Let \( A \) be an irreducible simplicial arrangement of \( n \) lines in \( \mathbb{P}^2(\mathbb{R}) \) having multiplicity bounded by six. Assume that \( A \) splits over \( \mathbb{R} \) and that \( t_2^A \leq \frac{24}{16 + \epsilon} t_3^A \) for some \( \epsilon > 0 \). Then \( n \leq \frac{2\sqrt{254016 - 11\epsilon^2 - 144\epsilon + 5\epsilon + 1008}}{14} \) and \( \epsilon \leq \frac{72}{11}(6\sqrt{15} - 1) \).

**Proof.** By Theorem 4 we have \( \frac{(n-5)^2 + 44}{16} \leq t_2^A \) and \( t_3^A \leq \frac{n^2 + 116n - 597}{24} \). By assumption we obtain \( \frac{n^2 + 116n - 597}{16 + \epsilon} \geq \frac{(n-5)^2 + 44}{16} \). This gives the result. \( \square \)

Unfortunately, we are not able to prove that the set \( M \) of isomorphism classes of arrangements considered in Theorem 4 is finite: the case \( t_2^A \geq t_3^A \) remains open in general. However, we can relate the finiteness of this set to the following conjecture stated in [11]:

**Conjecture 1.** Let \( 5 \leq k \in \mathbb{N} \) be a natural number and let \( \mathcal{A}_k \) denote the set of all isomorphism classes of line arrangements in \( \mathbb{P}^2(\mathbb{R}) \) having multiplicity at most \( k \). Then for any sequence of arrangements \( (A_\nu)_{\nu \in \mathbb{N}} \) such that \( A_\nu \in \mathcal{A}_k \) and \( \lim_{\nu \to \infty} |A_\nu| = \infty \) we have

\[
\lim_{\nu \to \infty} \frac{t_k^{A_\nu}}{|A_\nu|^2} = 0.
\]

Then Theorem 4 yields the following corollary, which closes this subsection.

**Corollary 5.** If \( |M| = \infty \) then Conjecture 1 is false for \( k = 6 \). Equivalently, if Conjecture 1 is true for \( k = 6 \), then \( |M| < \infty \).

**Proof.** By Theorem 4 we have \( |A| - 46 |A| + 225 \leq t_6^A \leq \frac{|A|^2 + 2|A| - 47}{48} \), if \( A \in M \). So if \( |M| = \infty \) we find a sequence \( (A_\nu)_{\nu \in \mathbb{N}} \) such that \( \lim_{\nu \to \infty} |A_\nu| = \infty \) and \( A_\nu \in M \) for every \( \nu \in \mathbb{N} \). But then \( \lim_{\nu \to \infty} \frac{t_6^{A_\nu}}{|A_\nu|^2} = \frac{1}{48} > 0 \) and Conjecture 1 is false for \( k = 6 \). \( \square \)
3.5. **Free line arrangements having only few vertices of high multiplicity.** In this subsection our main result is Theorem 8 which says that a free line arrangement having only vertices of weight bounded by five consists of at most 185 lines. In particular, there are only finitely many isomorphism classes of such arrangements. Moreover, it turns out that if the number of vertices of $A$ having high multiplicity is not too large, then $|A|$ may be bounded from above. This is made precise in Theorem 9.

We start with the following lemma which is similar to Theorem 4 of the paper [20].

**Lemma 7.** Let $A$ be an arrangement of $n \geq 8$ lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most five. Assume that $A$ splits over $\mathbb{R}$. Then the following estimates hold:

\begin{align}
(18) & \quad \frac{t_4^A}{3} + t_5^A \geq \frac{(n - 5)^2 - 4}{24}, \\
(19) & \quad t_2^A \geq \frac{n^2 - 46n + 233}{8} + 2t_4^A, \\
(20) & \quad \max(t_4^A, t_5^A) \geq \frac{n^2 - 10n + 21}{32}.
\end{align}

**Proof.** By part b) of [20, Theorem 1] we have $t_2^A + \frac{3t_2^A}{2} \geq 8 + \frac{t_4^A}{2} + \frac{5t_4^A}{2}$. As $3t_3^A = \binom{n}{2} - t_2^A - 6t_4^A - 10t_5^A$ we may rewrite this as $t_2^A + \frac{n^2 - n}{4} - 3t_4^A - 5t_5^A \geq 8 + \frac{t_4^A}{2} + \frac{5t_4^A}{2}$. It follows $t_5^A \leq \frac{n^2 - n}{30} + \frac{t_4^A}{15} - \frac{7t_4^A}{15} - \frac{16}{15}$. By Lemma 4 the splitting of $\chi(A, t)$ over $\mathbb{R}$ translates to $(n + 1)^2 \geq 4f_2^A$. Equation (2) in Lemma 1 yields $f_2^A = 1 + f_1^A - f_0^A = 1 + t_2^A + 2t_3^A + 3t_4^A + 4t_5^A = \frac{n^2 - n + 3}{3} + \frac{t_4^A}{3} - t_4^A - \frac{8t_5^A}{3}$. Now we use inequality (5) from Lemma 1 to conclude that the estimate

$$
\frac{n^2}{4} + \frac{n}{2} + \frac{1}{4} \geq f_2^A \geq \frac{n^2 - n + 6}{3} - \frac{2t_4^A}{3} - \frac{t_5^A}{3}
$$

holds. From this we deduce that $t_5^A \geq \frac{n^2}{24} - \frac{5n}{72} + \frac{7}{8} - \frac{t_4^A}{3}$, proving (18).

We have thus established the following chain of inequalities:

$$
\frac{n^2 - 10n}{24} + \frac{7}{8} - \frac{t_4^A}{3} \leq t_5^A \leq \frac{n^2 - n}{30} + \frac{t_4^A}{15} - \frac{7t_4^A}{15}.
$$

This implies (19). In order to prove (20) we consider two cases. First assume that $t_4^A \leq t_5^A$. Then by the above we know that $t_5^A \geq \frac{n^2 - 10n}{24} + \frac{7}{8} - \frac{t_4^A}{3} \geq \frac{n^2 - 10n}{24} + \frac{7}{8} - \frac{t_4^A}{3}$. From this we conclude that $t_5^A \geq \frac{n^2 - 10n + 21}{32}$. The case $t_4^A \geq t_5^A$ is dealt with similarly. This finishes the proof.

With the last lemma we are ready to prove the main result of this subsection.

**Theorem 8.** Let $A$ be a nontrivial line arrangement in $\mathbb{P}^2(\mathbb{R})$ having multiplicity bounded by five. Assume that $A$ splits over $\mathbb{R}$. Then $|A| \leq 185$. 

Proof. We may assume that \( n := |A| > 7 \). As \( A \) splits over \( \mathbb{R} \) we have \( \frac{(n+1)^2}{4} \geq f_2^A \). Together with the first estimate in Lemma 7 this yields
\[
\frac{(n+1)^2}{4} \geq f_2^A \geq 1 + t_2^A + 2t_3^A + 4\left(\frac{t_4^A}{3} + t_5^A\right) \geq 1 + t_2^A + \frac{n^2 - 10n + 21}{6}.
\]
We conclude \( \frac{n^2 + 26n - 51}{12} \geq t_2^A \). But now the second estimate in Lemma 7 gives us the following chain of inequalities:
\[
\frac{n^2 + 26n - 51}{12} \geq t_2^A \geq \frac{n^2}{12} - 46n + 233.
\]
This implies \( n \leq 19 \), finishing the proof. □

Remark 11. a) An important open problem in the theory of hyperplane arrangements is Terao’s Conjecture, which claims that for an arrangement \( A \) the property of being free is combinatorial, more precisely: if \( A, B \) are hyperplane arrangements in some finite dimensional vector space \( V \) over some field \( K \) such that \( L_A \cong L_B \), then \( B \) is also free.

Theorem 8 implies that for real projective line arrangements whose multiplicity is at most five, Terao’s Conjecture is “in principle” a finite problem.

b) If one additionally requires \( t_5^A = 0 \) in the above theorem, then we can conclude \( n \leq 19 \). This follows immediately from [21, Theorem 1].

We observe that Lemma 7 also yields the following classification results in the simplicial case:

Corollary 6. Let \( A \) be a simplicial line arrangement in \( \mathbb{P}^2(\mathbb{R}) \) such that \( \chi(A,t) \) has only real roots.

a) If \( t_i^A = 0 \) for \( i \notin \{2, 3, 5\} \) and if every chamber contains at most one vertex of weight five, then \( |A| \leq 26 \). In particular, we have a complete list of such arrangements.

b) If \( m(A) = 5 \) and \( t_2^A \geq 3t_3^A \), then \( A \) is the arrangement of type \( A(21, 2) \).

Proof. a) By Lemma 7 we have \( t_5^A \geq \frac{n^2 - 10n + 21}{24} \). On the other hand, as every chamber contains at most one vertex of weight five and because \( \chi(A,t) \) has only real roots, it follows that \( 10t_5^A \leq f_2^A \leq \frac{(n+1)^2}{4} \). Thus, \( n \) must satisfy the inequality \( \frac{n^2 - 10n + 21}{24} \leq \frac{(n+1)^2}{4} \). This implies \( n \leq 26 \).

b) By part b) of Theorem 1 we have \( t_3^A + \frac{2t_4^A + t_5^A}{5} \geq 4 + \frac{(n-5)^2 - 4}{20} \). By assumption and Theorem 4 we obtain
\[
\frac{(n+1)^2}{16} \geq t_2^A \geq 3\left(4 + \frac{(n-5)^2 - 4}{20} - \frac{2t_4^A + t_5^A}{5}\right) \geq 3\left(4 + \frac{(n-5)^2 - 4}{20} - \frac{2(3n^2 - 17)}{5}\right).
\]
This implies \( n \leq 29 \). In order to derive the result from the enumeration given in [1], it remains to exclude the possibilities \( n = 28, 29 \).

So assume that \( n = 29 \). But then \( 2t_4^A + t_5^A \geq \frac{277}{4} > 69, t_4^A + t_5^A \leq 35 \). This forces \( t_4^A = 35, t_5^A = 0 \), contradicting our assumption \( m(A) = 5 \).
Now assume that \( n = 28 \). Then we have \( 2t_4^A + t_5^A \geq \frac{3055}{38}, t_4^A + t_5^A \leq \frac{65}{7} \) and the only solutions are the pairs \((t_4^A, t_5^A) \in \{(31, 2), (32, 0), (31, 1), (33, 0)\}\). Because \( m(A) = 5 \), we are left with the two possibilities \( t_4^A = 31, t_5^A = 2 \) and \( t_4^A = 32, t_5^A = 1 \). However, both pairs violate relation (18) from Lemma 7. This completes the proof.

**Remark 12.** Let \( A \) be a simplicial line arrangement in \( \mathbb{P}^2(\mathbb{R}) \). Assume that \( \chi(A, t) \) splits over \( \mathbb{R} \) and that \( m(A) = 4 \). If \( t_4^A \geq 3t_5^A \), then \( A \) is the arrangement of type \( A(13, 2) \). This follows from Theorem 8.

Next we show that Theorem 8 is actually a special case of a more general technical statement. Observe that contrary to the preceding results, in the following we do not require all vertices of the considered arrangements to have weight bounded by five.

Moreover, in the proof of the next theorem we use the concept of pseudoline arrangements. For precise definitions and background on this, see [2]. For us it will be enough to know the following: an arrangement of pseudolines is a finite set \( B \) of \( n \geq 3 \) smooth closed curves in \( \mathbb{P}^2(\mathbb{R}) \) such that the following conditions are satisfied:

- Curves in \( B \) do not intersect themselves.
- Different curves in \( B \) intersect transversally at precisely one point.
- Not all curves in \( B \) pass through the same point.

Observe that one may define a \( t \)-vector \( t_B \) for a pseudoline arrangement \( B \) in the same way as for straight line arrangements. Moreover, the values \( f_i^B \) for \( 0 \leq i \leq 2 \) are defined in the same way as well. Using these definitions, all results (except part a) of Theorem 8 from Subsection 3.1 as well as Lemma 7 hold true for arrangements of pseudolines. Using this, we are ready to prove the following theorem.

**Theorem 9.** Let \( A \) be a line arrangement in \( \mathbb{P}^2(\mathbb{R}) \) splitting over \( \mathbb{R} \). Write \( \Delta_i := \sum_{j=4}^{i-2} j \) and assume \( 0 \leq t_i^A \leq \alpha_i \) for \( i \geq 6 \) and real numbers \( \alpha_i \). Then we have \( |A| \leq 95 + 2\sqrt{2056 + 63\sum_{i \geq 6} \Delta_i \alpha_i} \). In particular, if \( \alpha_i = 0 \) for all \( i \geq 6 \) then \( |A| \leq 185 \). If additionally \( t_5^A = 0 \) then \( |A| \leq 19 \).

**Proof.** We construct a pseudoline arrangement \( A' \) with \( n := |A| = |A'| \) in the following way: by a perturbation we transform any vertex \( v \) of \( A \) having weight \( x \geq 6 \) into one vertex \( v' \) of \( A' \) of weight five and \( \lambda(x) \) vertices \( w'_1, ..., w'_{\lambda(x)} \) of \( A' \) of weight two. All vertices of \( A \) of weight at most five remain unchanged. One checks that \( \lambda(x) = \sum_{j=5}^{x-1} j \). Therefore we obtain \( f_2^{A'} = f_2^A + \sum_{i \geq 6} \Delta_i t_i^A \). The assumptions imply \( f_2^{A'} \leq \frac{(n+1)^2}{4} + \sum_{i \geq 6} \Delta_i \alpha_i \) and by construction we have \( t_i^{A'} = 0 \) for \( i > 5 \). Hence we may write \( f_2^{A'} = 1 + f_1^{A'} - f_0^{A'} = 1 + t_2^{A'} + 2t_3^{A'} + 3t_4^{A'} + 4t_5^{A'} = \frac{n^2-n+3}{3} + \frac{t_4^{A'} + t_3^{A'} - 8t_5^{A'}}{3} \).
Now we use Melchior’s inequality to conclude that the estimate
\[
\frac{n^2}{4} - \frac{n}{2} + \frac{1}{4} + \sum_{i \geq 6} \Delta_i \alpha_i \geq f_2^{A'} \geq \frac{n^2 - n - 6}{3} - \frac{2t_4^{A'} - 3}{3} - 2t_5^{A'}
\]
holds. From this we deduce that \(\frac{t_4^{A'}}{3} + t_5^{A'} \geq \frac{n^2}{24} - \frac{5n}{12} + \frac{7}{8} - \frac{1}{3} \sum_{i \geq 6} \Delta_i \alpha_i\).
Moreover, by \([20, \text{Theorem 1}]\) we have \(t_2^{A'} + \frac{3t_3^{A'}}{2} \geq 8 + \frac{t_2^{A'}}{3} + \frac{5t_4^{A'}}{2}\). As
\(3t_3^{A'} = (\binom{n}{2} - t_2^{A'} - 6t_3^{A'} - 10t_4^{A'})\) we may rewrite the last inequality as \(\frac{t_2^{A'}}{2} + \frac{n^2 - n}{4} - 3t_4^{A'} - 5t_5^{A'} \geq 8 + \frac{t_2^{A'}}{2} + \frac{5t_4^{A'}}{2}\).
It follows \(t_5^{A'} \leq \frac{n^2 - n}{30} + \frac{t_2^{A'}}{15} - \frac{7t_4^{A'}}{15} - \frac{16}{15} \leq \frac{n^2 - n}{30} + \frac{t_2^{A'}}{3} - \frac{4t_4^{A'}}{3} - \frac{16}{15}\), yielding the following chain of inequalities:
\[
\frac{n^2 - n}{30} + \frac{t_2^{A'}}{15} - \frac{16}{15} \geq \frac{t_4^{A'}}{3} + \frac{t_5^{A'}}{3} \geq \frac{n^2}{24} - \frac{5n}{12} + \frac{7}{8} - \frac{1}{2} \sum_{i \geq 6} \Delta_i \alpha_i.
\]
We conclude that \(t_2^{A'} \geq \frac{n^2 - 46n + 233}{8} - \frac{15}{2} \sum_{i \geq 6} \Delta_i \alpha_i\). Moreover, \(f_2^{A'} = 1 + t_2^{A'} + 2t_3^{A'} + 4\left(\frac{t_4^{A'}}{3} + t_5^{A'}\right) + \frac{5t_4^{A'}}{3} \geq 1 + \frac{n^2 - 10n + 21}{24} - 2 \sum_{i \geq 6} \Delta_i \alpha_i\), implying the estimate \((\frac{n+1}{4})^2 - \frac{n^2 - 10n + 21}{6} - 3 \sum_{i \geq 6} \Delta_i \alpha_i - 1 \geq t_2^{A'}\).
Thus, we now have established the following estimates:
\[
\frac{n^2 + 26n - 51}{12} + 3 \sum_{i \geq 6} \Delta_i \alpha_i \geq t_2^{A'} \geq \frac{n^2}{4} - \frac{46n + 233}{8} - \frac{15}{2} \sum_{i \geq 6} \Delta_i \alpha_i.
\]
This is possible only for \(n \leq 95 + 2 \sqrt{2056 + 63 \sum_{i \geq 6} \Delta_i \alpha_i}\).
It remains to prove the last assertion of the theorem. So assume that \(t_4^{A} = 0\) for \(i \geq 5\). Then by \([21, \text{Theorem 1}]\) we have \(2 \frac{n^2 - n}{4} \leq f_2^{A} \leq \frac{(n+1)^2}{4}\), which implies \(n \leq 19\).

If \(A\) is a line arrangement in \(\mathbb{P}^2(\mathbb{R})\), then we write \(\overline{A}\) for the isomorphism class of \(A\). Using this notation, the last theorem immediately yields the following corollary, which closes this subsection.

**Corollary 7.** Let \(\epsilon > 0\) be a real parameter and let \(3 \leq x \in \mathbb{N}\). Define the set \(\mathfrak{A}_x^{\epsilon} := \{\overline{A} \mid \max_{i \geq 6} t_i^{A} \leq |A|^{2-\epsilon}, t_i^{A} = 0 \text{ for } i > x, \chi(A, t) \text{ splits over } \mathbb{R}\}\).
Then \(\mathfrak{A}_x^{\epsilon}\) is finite for any choice of \(x\) and \(\epsilon\).

### 4. Open problems and related questions

In this section, we point out two possibly interesting related problems which we have not been able to resolve yet. Motivated by Theorem 8 we start with the following question on free line arrangements:

**Problem 1.** For each \(x \geq 6\), prove or disprove that there are only finitely many isomorphism classes of free line arrangements in \(\mathbb{P}^2(\mathbb{R})\) whose vertices all have weight bounded by \(x\).
Observe that in order to obtain a positive answer for the last problem, it would be enough to prove a subquadratic upper bound for the values $t_i^A$, $6 \leq i \leq x$, for any sufficiently large (of course depending on $x$) free line arrangement $A$. This follows from Corollary 7.

Finally, by Theorem 2 and Remark 9 we are led to state the following problem concerning the Dirac Motzkin Conjecture:

**Problem 2.** Let $A$ be a nontrivial arrangement of $n$ lines in $\mathbb{P}^2(\mathbb{R})$. Assume that $t_2^A < \frac{n}{2}$. Is it true that $A$ is necessarily simplicial?

Note that the answer to Problem 2 is “yes” if the characteristic polynomial of the arrangement in question has only real roots: by part b) of Theorem 2, the Kelly-Moser example is the only arrangement having the required properties.

**References**

[1] M. Barakat and Michael Cuntz, *Coxeter and crystallographic arrangements are inductively free*, Adv. Math. **229** (2012), no. 1, 691–709.
[2] A. Björner and M. Las Vergnas and B. Sturmfels and N. White and G. Ziegler, *Oriented Matroids*, Cambridge University Press (1993).
[3] D. W. Crowe and T. A. McKee, *Sylvester’s problem on collinear points*, Math Mag. **41** (1968), 30–34.
[4] Michael Cuntz, *Simplicial arrangements with up to 27 lines*, Discrete Comput. Geom. **48** (2012), 682–701.
[5] Michael Cuntz and D. Geis, *Combinatorial simpliciality of arrangements of hyperplanes*, Beitr. Algebra Geom. **56** (2015), no. 2, 439–458.
[6] Michael Cuntz and D. Geis, *Simplicial arrangements and duality*, in preparation.
[7] Michael Cuntz and D. Geis, *Tits arrangements on cubic curves*, preprint, (2017), 15 pp., available at [arXiv:1711.02438](https://arxiv.org/abs/1711.02438).
[8] Michael Cuntz and I. Heckenberger, *Finite Weyl Groupoids Of Rank Three*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1369–1393.
[9] Michael Cuntz and I. Heckenberger, *Finite Weyl Groupoids*, J. Reine Angew. Math. **702** (2015), 77–108.
[10] R. Ehrenborg and C. Klivans and N. Reading, *Coxeter arrangements in three dimensions*, Beitr. Algebra Geom. **57** (2016), 1–7.
[11] P. Erdős and G. Purdy, *Some combinatorial problems in the plane*, Journal of Combinatorial Theory, Series A **25** (1978), no. 2, 205–210.
[12] B. Green and T. Tao, *On sets defining few ordinary lines*, Discrete Comput. Geom. **50** (2013), 409–468.
[13] B. Grünbaum, *A catalogue of simplicial arrangements in the real projective plane*, Ars Math. Contemp. **2** (2009), no. 1, 25 pp.
[14] B. Grünbaum, *Arrangements and spreads*, CBMS Regional conference series in mathematics **10** (1972), 114 pp.
[15] L. M. Kelly and W. O. J. Moser, *On the number of ordinary lines determined by $n$ points*, Canad. J. Math. **10** (1958), 210–219.
[16] A. Langer, *Logarithmic Orbifold Euler Numbers of Surfaces with Applications*, Proc. London Math. Soc. **86** (2003), no. 2, 358–396.
[17] E. Melchior, *Über Vielseite der projektiven Ebene*, Dtsch. Math. **5** (1940), 461–475.
[18] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag Berlin Heidelberg (1992).
[19] R.W. Shannon, *Simplicial cells in arrangements of hyperplanes*, Geometriae Dedicata 8 (1979), no. 2, 179–187.

[20] I.N. Shnurnikov, *A $t_k$ inequality for arrangements of pseudolines*, Discrete Comput. Geom. 55 (2016), 284–295.

[21] I.N. Shnurnikov, *Into how many regions do $n$ lines divide the plane if at most $n - k$ of them are concurrent?*, Moscow University Mathematics Bulletin 65 (2010), no. 5, 208–212.

David Geis, Leibniz Universität Hannover, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Fakultät für Mathematik und Physik, Welfengarten 1, D-30167 Hannover, Germany

E-mail address: geis@math.uni-hannover.de