PERIODS OF DOUBLE OCTIC CALABI–YAU MANIFOLDS

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Abstract. We compute numerical approximations of the period integrals for eleven rigid double octic Calabi–Yau threefolds and compare them with the periods of corresponding weight four cusp forms and find, as to be expected, commensurabilities. These give information on character of the correspondences of these varieties with the associated Kuga-Sato modular threefolds.

Introduction

Let $X$ be a rigid Calabi–Yau threefold and $\omega \in H^{3,0}(X)$ a regular 3–form on $X$. For a 3-cycle $\gamma \in H_3(X,\mathbb{Z})$ on $X$ we can form the period integral

$$\int_\gamma \omega \in \mathbb{C}.$$ 

The set of these period integrals form a lattice

$$\Lambda := \left\{ \int_\gamma \omega : \gamma \in H_3(X,\mathbb{Z}) \right\} \subset \mathbb{C}$$

and hence determine an elliptic curve

$$\mathbb{C}/\Lambda = H^{3,0}(X)^*/H_3(X,\mathbb{Z}) =: J^2(X)$$

which is just an example of the intermediate Jacobian of Griffiths [7].

It is now known that any rigid Calabi–Yau threefold defined over $\mathbb{Q}$ is modular [6], [5], in the sense that one has an equality of $L$-functions:

$$L(H^3(X), s) = L(f, s).$$

Here $f \in S_4(\Gamma_0(N))$ is a weight four cusp form for some level $N$. It is known more generally that a cusp-form $f \in S_{k+2}(\Gamma_0(N))$ can be interpreted as a $k$-form on the associated Kuga-Sato variety, which is (a desingularisation of) the $k$-fold fibre product of the universal elliptic curve over the modular curve $X_0(N)$, [4]. So in our case one expects the equality of $L$-functions to come from a correspondence between the rigid Calabi-Yau and the Kuga-Sato variety $Y$, which resolves the fibre
product $E \times_C E$ of the universal elliptic curve over the modular curve $C := X_0(N)$. Now the periods

$$\int_0^{i\infty} f(\tau) \tau^k d\tau$$

of the modular form determine the periods of the Kuga-Sato variety $Y$ and the correspondence between $X$ and $Y$ would imply that the period-lattice of $X$ is commensurable to the lattice derived from the modular form.

In this note we shall compute numerical approximations of period integrals for certain rigid double octic Calabi–Yau threefolds, i.e. Calabi–Yau threefolds constructed as a resolution of a double cover of projective space $\mathbb{P}^3$, branched along a surface of degree eight.

More specifically, we will look at the eleven arrangements of eight planes defined by linear forms with rational coefficients, described in the PhD thesis of C. Meyer [9]. These arrangements define eleven non–isomorphic rigid Calabi–Yau threefolds. He also determined the weight four cusp forms $f$ for these eleven rigid double octics using the counting of points in $\mathbb{F}_p$ for small primes $p$.

Each arrangement of real planes defines a partition of the real projective space $\mathbb{P}^3(\mathbb{R})$ into polyhedral cells and using these cells one can construct certain polyhedral 3-cycles on the desingularisation of the double octic. Using the explicit equations for the planes of the arrangement, one can write the period integral as an explicit sum of multiple integrals, which can be integrated numerically.

It turned out to be difficult to identify a complete basis of $H_3(X, \mathbb{Z})$ in terms of polyhedral cycles. But any two non–proportional periods of a rigid Calabi–Yau threefolds define a subgroup of finite index of $\Lambda$ and hence an elliptic curve isogeneous to the intermediate jacobian $J^2(X)$. From such a numerical lattice one can compute the lattice constants $g_2$ and $g_3$, and hence a Weierstrass equation and $j$–invariant of the curve defined by lattice spanned by the polyhedral 3–cycles.

We expect that a more refined topological analysis of the above situation will lead to more precise information on the nature of the correspondences between these varieties.
1. Double octic Calabi–Yau threefolds

By a double octic we understand a variety $X$ given as a double cover

$$\pi : X \longrightarrow \mathbb{P}^3$$

of $\mathbb{P}^3$, ramified over a surface $D \subset \mathbb{P}^3$ of degree eight. Such a double octic $X$ can be given by an equation in weighted projective space $\mathbb{P}(4,1,1,1,1)$ of the form

$$u^2 = F(x, y, z, t),$$

where the polynomial $F$ defines the ramification divisor $D$. If the surface $D$ is smooth, then $X$ is a smooth Calabi-Yau threefold, but we will be dealing here with the case that $D$ is a union of eight planes, so the polynomial acts as into a product of linear forms:

$$F = L_1L_2L_3L_4L_5L_6L_7L_8.$$ 

The associated double octic $X$ then is singular along the lines of intersection of the eight planes $D_i : = \{L_i = 0\}$. In case these planes have the property that

- no six intersect in a point,
- no four intersect along a line

one can construct a Calabi-Yau desingularisation of $X$. To do so, one first constructs a sequence of blow-ups with smooth centers

$$T : \tilde{\mathbb{P}}^3 \longrightarrow \mathbb{P}^3$$

and a divisor $\tilde{D}$ in $\tilde{\mathbb{P}}^3$ such that

- $\tilde{D}$ is non–singular (in particular reduced),
- $\tilde{D}$ is even as an element of the Picard group $\text{Pic}(\tilde{\mathbb{P}}^3)$,

by blowing-up the singularities of $D$ in the following order:

1. fivefold points,
2. triple lines,
3. fourfold points,
4. double lines.

In the first two cases we replace the branch divisor by its reduced inverse image: the strict transform plus the exceptional divisor. In the last two cases we replace the branch divisor by its strict transform.

The double cover

$$\tilde{\pi} : \tilde{X} \longrightarrow \tilde{\mathbb{P}}^3$$

of $\tilde{\mathbb{P}}^3$ branched along $\tilde{D}$ is now a smooth Calabi–Yau manifold, which we will call the double octic Calabi–Yau threefold of the arrangement.
We will also need to consider a particular *partial resolution* \( \hat{X} \) of \( X \), obtained as double cover of a space \( \hat{\mathbb{P}}^3 \) obtained by performing only the blow-ups in fivefold point, triple lines and double curves, so leaving out step (3) in the above procedure. Note that there are two types of fourfold points. The fourfold points on triple lines get removed in step (2), but the fourfold points that appear at the intersection of four generic planes produce an *ordinary double point* if we blow up consecutively the six curves of intersection of the strict transforms of these planes. After the blow-up of the first double line the strict transforms of the remaining two planes (not containing this line) intersect along a sum of two intersecting lines. So we have to blow-up four lines and a cross, the latter producing a node on the threefold \( \hat{\mathbb{P}}^3 \). The space doubly covering \( \mathbb{P}^3 \) and ramified over \( \hat{D} \) is a variety \( \hat{X} \) with twice as many nodes.

To summarise the situation, one can consider the following diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\sigma} & \hat{X} & \xrightarrow{\rho} & X \\
\downarrow{\bar{\eta}} & & \downarrow{\bar{\pi}} & & \downarrow{\pi} \\
\tilde{\mathbb{P}}^3 & \xrightarrow{S} & \hat{\mathbb{P}}^3 & \xrightarrow{R} & \mathbb{P}^3
\end{array}
\]

The vertical maps are two-fold covers, the map \( \sigma : \tilde{X} \rightarrow \hat{X} \) is a small resolution of the nodes of \( \hat{X} \) and \( \rho : \hat{X} \rightarrow X \) is a partial resolution of double octic variety \( X \). The composition \( RS \) is the map \( T : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) we started with, and \( \tau := \rho \sigma : \tilde{X} \rightarrow X \) is a resolution of singularities. In fact the resolutions \( \tilde{X} \) and \( \hat{X} \) depend on the choice of the order of blow-up of lines, in the above diagram we choose the same order of lines for both resolutions.

We will be concerned with 11 special arrangements that were studied by C. Meyer [9]. Their resolution \( \tilde{X} \) of the associated double octic lead to 11 different rigid Calabi-Yau varieties. For the convenience of the reader, we list here the arrangement numbers, second Betti-number
and the equations from [9].

| Number | $b_2(X)$ | Equation                                                                 | $\lambda$ |
|--------|----------|---------------------------------------------------------------------------|-----------|
| 1      | 70       | $xyz(x + y)(y + z)(z + t)(t + x)$                                         | $-1$      |
| 3      | 62       | $xyz(x + y)(y + z)(y - t)(x - y - z + t)$                                  | $1$       |
| 19     | 54       | $xyz(x + y)(y + z)(x - z - t)(x + y + z - t)$                              | $2$       |
| 32     | 50       | $xyz(x + y)(y + z)(x - y - z - t)(x + y - z + t)$                          | $-1$      |
| 69     | 50       | $xyz(x + y)(x - y + z)(x - y - t)(x + y - z - t)$                          | $-1$      |
| 93     | 46       | $xyz(x + y)(x - y + z)(y - z - t)(x + z - t)$                              | $2$       |
| 238    | 44       | $xyz(x + y + z - t)(x + y - z + t)(x - y + z + t)(-x + y + z + t)$          | $1$       |
| 239    | 40       | $xyz(x + y + z)(x + y + t)(x + z + t)(y + z + t)$                          | $1$       |
| 240    | 40       | $xyz(x + y + z)(x + y - z + t)(x - y + z + t)(x - y - z - t)$               | $-2$      |
| 241    | 40       | $xyz(x + y + z + t)(x + y - z - t)(y - z + t)(x + z - t)$                   | $1$       |
| 245    | 38       | $xyz(x + y + z)(y + z + t)(x - y - t)(x - y + z + t)$                       | $-2$      |

The meaning of the number $\lambda$ in the last column will be explained later.

2. 3–cycles on a double octic

In the above eleven examples of rigid double octic Calabi–Yau threefolds defined over $\mathbb{Q}$ are given. In all these examples the eight planes are given by equations with integral coefficients.

In general, an arrangement defined by real planes gives a decomposition of $\mathbb{P}^3(\mathbb{R})$ into a finite number of polyhedral cells. By combining these cells one can construct certain polyhedral cycles on the smooth model $\tilde{X}$. To explain this, let us fix one of these cells $C$ and consider its double covering $C$, that is, its preimage under the 2–fold covering map $\pi : X \longrightarrow \mathbb{P}^3$. Then $C$ is a 3–cycle in $X$ and determines an element in $H_3(X, \mathbb{Z})$, up to a sign determined by a choice of an orientation.

Question: Do the 3–cycles $C$ generate $H_3(X, \mathbb{Z})$?

However, $C$ will in general not be a 3–cycle on the desingularisation $\tilde{X}$, as the canonical map

$$\tau_* : H_3(\tilde{X}, \mathbb{Z}) \longrightarrow H_3(X, \mathbb{Z})$$

will not be surjective in general. To see this geometrically, we follow the fate of the cycle $C$ under the blow–up maps and see that its gets transformed into a chain on $\tilde{X}$ that we will still denote by $C$. Its boundary $\partial C$, as a chain on $\tilde{X}$, is a sum of 2–cycles contained in the
exceptional loci

\[ \partial C = \bigcup_i \Gamma_i. \]

Now observe that \( C \) is anti-symmetric with respect to the covering map \( \pi \), and as a consequence, these cycles \( \Gamma_i \) are anti-symmetric as well. On the other hand the exceptional divisor corresponding to a fivefold point or a triple line is fixed by the involution, while the exceptional locus corresponding to a double line is a blow-up of a conic bundle, so in all the three cases second cohomology group is symmetric. Hence each cycle \( \Gamma_i \) contained in the exceptional divisor corresponding to a double line, triple line or a fivefold point there is a boundary, i.e. there is a 3–chain \( C_i \) such that \( \delta C_i = \Gamma_i \). Hence, if we subtract from the chain \( C \) the chains \( C_i \) we get a chain with boundary contained in the exceptional divisors corresponding to the fourfold points. So we see that \( C \) can be lifted to a cycle \( \tilde{C} \) on the partial resolution \( \hat{X} \) and we have shown:

**Proposition:** The map

\[ \rho_* : H_3(\hat{X}, \mathbb{Z}) \longrightarrow H_3(X, \mathbb{Z}) \]

is surjective.

*Special role of the fourfold points*

So we see that we need to analyse the situation of a fourfold point in more detail. We only have to consider fourfold points \( p \) that are not on triple lines, so at which four planes intersect in general position. Near \( p \) the space \( \mathbb{R}^3 \) is decomposed into \( 2^4 = 16 \) cells, which are in one-to-one correspondence with the sign patterns

\[ (\text{sign}(L_1), \text{sign}(L_2), \text{sign}(L_3), \text{sign}(L_4)) \]

where the linear forms define the planes meeting at \( p \). Each cell has an opposite cell, obtained by a reversing all signs.
If we blow-up the point $p$, the exceptional divisor is a copy of $\mathbb{P}^2$, on which we find four lines in general position, corresponding to the four planes through $p$; these four lines decompose the real projective plane in seven regions, that can be colored into three 'black' and four 'white' regions.

On the double cover we find an exceptional divisor $E$ that is a double cover of this $\mathbb{P}^2$ ramified along these four lines, and each of the eight regions $R$ determine a 2-cycle $R$ in $E$.

These regions are in one–to–one correspondence the pairs of opposite cells. If $C$ is a cell corresponding to a region $R$, then the chain $C$ on the blow–up, then the boundary of this chain is precisely the 2-cycle corresponding to $R$:

$$\partial C = \pm R$$

What we learn from this is that we can cancel this boundary term of a cell by adding to it the boundary term of the opposite cell!

Hence, we can define a group of polyhedral cycles $PC^3$ consisting of elements

$$\sum_C n_C C, \quad n_C \in \mathbb{Z}$$
for which for all fourfold points $p$ one has:

$$
\sum_{p \in C} n_C = 0
$$

We can assume that the four planes we are considering have equations

$$
x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 0
$$

in appropriate coordinates. We will analyse what happens if we smooth out the fourfold point shifting the fourth plane to

$$
x + y + z = \epsilon,
$$

resolve this and then specialise back to $\epsilon = 0$.

Let us first blow–up the two disjoint lines

$$
x = y = 0, \quad z = x + y + z - \epsilon = 0.
$$

In one of the affine charts the blow–up of $\mathbb{P}^3$ is given by the equation

$$
x(y + 1) - z(v - 1) - \epsilon.
$$

The threefold is smooth unless $\epsilon = 0$ when it acquires a node at $x = 0, y = -1, z = 0, v = 1$. Since the surface $x = 0, z = 0$ is a Weil divisor on the threefold which is not Cartier (it is a component of the exceptional locus of the blow–up) the node admits a projective small resolution. Since the node does not lie on the branch divisor it gives two nodes on the double cover. Consequently, we get the partial resolution $\tilde{X}$ from the end of the section 1.

3. Determination of $H_3(\tilde{X}, \mathbb{Z})$

Denote by $X_t$ a smoothing of $\tilde{X}$. By [1], the deformations of $\tilde{X}$ correspond to deformations of the arrangements of eight planes that preserve the incidences between the planes in $D$. The deformations of the arrangement that preserves all the incidences except for the fourfold points correspond to smoothings of $\tilde{X}$.

By the work of J. Werner [12], the nodal variety $\tilde{X}$ is homotopy equivalent to its small resolution $\tilde{X}$ with 3–cells glued along the the exceptional lines (which topologically are 2–spheres). The nodal variety $\tilde{X}$ is also homotopy equivalent to its smoothing with 4–cells glued along the vanishing 3–cycles. As a consequence, one arrives at the following equations relating topological invariants of $X_t$, $\tilde{X}$ and $\hat{X}$:
\[b_4(X_t) + 2p_4 + b_3(\tilde{X}) = b_4(\tilde{X}) + b_3(X_t)\]
\[b_2(X_t) = b_2(\tilde{X})\]
\[b_3(\tilde{X}) + 2p_4 + b_2(\tilde{X}) = b_3(\tilde{X}) + b_2(X_t)\]
\[b_4(\tilde{X}) = b_4(\tilde{X})\]

and hence
\[b_3(\tilde{X}) = b_3(\tilde{X}) + b_2(X_t) - b_2(\tilde{X}) + 2p_4 = b_4(\tilde{X}) + b_3(X_t) - b_4(X_t) - 2p_4\]

where \(p_4\) is the number of (smoothed) fourfold points in \(D\) that do not lie on a triple line.

For the eleven rigid double octics from [9] we get

| No. | \(b_3(\tilde{X})\) | \(b_3(X_t)\) | \(p_4^0\) |
|-----|----------------|---------------|-----------|
| 1   | 3             | 4             | 1         |
| 3   | 5             | 8             | 3         |
| 19  | 6             | 10            | 4         |
| 32  | 7             | 12            | 5         |
| 69  | 7             | 12            | 5         |
| 93  | 8             | 14            | 6         |
| 238 | 11            | 20            | 12        |
| 239 | 11            | 20            | 10        |
| 240 | 11            | 20            | 10        |
| 241 | 11            | 20            | 10        |
| 245 | 11            | 20            | 9         |

4. An example

In order to find two independent cycles, we draw projections of intersections of all arrangement planes onto the \((x, y)\)-plane, and consider the equations of the planes not perpendicular to it as functions in \(z\). The easiest case is the arrangement No 1. which has a single \(p_4^0\) point. We will go through some details of this example.

The equation of this arrangement is
\[xyz(x + y)(y + z)(z + t) = 0\]

and the only \(p_4^0\) point is \((1, -1, 1, -1)\). The affine change of variables
\[t \mapsto t - x,\]
maps this point to the plane at infinity. The arrangement is then given in affine coordinates by the equation

$$xyz(1-x)(x+y)(y+z)(-x+z+1) = 0$$

while the $p_0^1$ point is the point at infinity $(1, -1, 1, 0)$. The planes defined the first, second, fourth and fifth factor of the above product are perpendicular to the $(x,y)$–plane and intersect the plane in lines $x = 0, y = 0, x = 1, x + y = 0$. The planes in the arrangement that are not perpendicular to the $(x,y)$–plane can be seen as graphs over the $(x,y)$–plane and are given by

$$z = f_3(x,y) := 0$$
$$z = f_6(x,y) := -y$$
$$z = f_7(x,y) := x - 1$$

The projections of the lines of intersection of these planes are given by

$$f_3 = f_6 : y = 0$$
$$f_3 = f_7 : x = 1$$
$$f_6 = f_7 : x + y = 1$$

In the $(x,y)$–plane we have two bounded domains

$\mathcal{I}$: \quad $x > 0, y > 0, x + y < 1$
$\mathcal{II}$: \quad $x < 1, y < 0, x + y > 0$

For points $(x, y)$ in these regions, the functions $f_3, f_6, f_7$ satisfy there the following inequalities

$\mathcal{I}$: \quad $f_7 < f_6 < f_3$
$\mathcal{II}$: \quad $f_7 < f_3 < f_6$
Consequently, the domains lying over triangle I are given by
\[ x > 0, y > 0, x + y < 1, z > x - 1, z < -y \]
\[ x > 0, y > 0, x + y < 1, z > -y, z < 0 \]
As the “right” (horizontal) edge of the triangle II is the projection
the intersection of planes no. 3 and 7, the only cycle lying over that
triangle is given by
\[ x < 1, y < 0, x + y > 0, z > x - 1, z < 0, \]
the other domain
\[ x < 1, y < 0, x + y > 0, z > 0, z < -y \]
is not bounded by arrangement planes, if we want to use it we would
have to add the unbounded domain “across the edge”
\[ x > 1, x + y > 0, x + y < 1. \]
Instead we can choose a domain over triangle II
\[ x > 0, y < 0, x + y > 0, z > x - 1, z < 0. \]

5. Period integrals
When we are given a degree eight polynomial \( F(x, y, z, t) \), then the
double octic \( X \subset \mathbb{P}^4(4, 1, 1, 1, 1) \) defined by the equation
\[ u^2 - F(x, y, z, t) = 0 \]
comes with a preferred section \( \omega \in \Gamma(X, \omega_X) \) of its sheaf of dualising
differentials. In the affine chart \( t \neq 0 \) it can be written as
\[ \omega := \frac{dxdydz}{u} = \frac{dxdydz}{\sqrt{F}}. \]
The period integrals of \( X \) are thus of the form
\[ \int_\gamma \omega = \int_\gamma \frac{dxdydz}{\sqrt{F}} \]
where \( \gamma \) is a three-cycle in \( X \).

If in particular \( F \) defines a real arrangement of eight planes and
we have a bounded cell \( C \) in \( \mathbb{R}^p \) yielding a 3–cycle \( \tilde{C} \) in Calabi-Yau
threefold \( \tilde{X} \), the period integral
\[ \int_{\tilde{C}} \omega \]
is just equal to three-fold integral

\[ 2 \int \int \int_C \frac{dx \, dy \, dz}{\sqrt{F}}, \]

In the case considered in previous section (arrangement No. 1), the two periods integral are given by

\[
\begin{align*}
\int_0^1 & \int_0^{1-x} \int_{x-1}^{-y} \frac{1}{\sqrt{xyz(1-x)(x+y)(y+z)(-x+z+1)}} \, dz \, dy \, dx \\
\int_0^1 & \int_{-x}^0 \int_{x-1}^0 \frac{1}{\sqrt{xyz(1-x)(x+y)(y+z)(-x+z+1)}} \, dz \, dy \, dx.
\end{align*}
\]

To compute such integrals numerically, we used Maple. However, the function \( F \) can have zeros of multiplicity 5 at a vertex of a polyhedron of integration and thus the integrand is unbounded. As a result, a direct numerical integration usually does not yield a satisfactory precision in reasonable time. We used the following simple trick which allows us to get 12 digits precision without much effort, which is sufficient for the our purposes. Using an affine coordinate change, we reduce computations to the case of a integration over a cube \( 0 \leq x, y, z \leq 1 \), with the function \( F \) vanishing only for \( xyz = 0 \). Then substituting \((x, y, z) \mapsto (x^k, y^k, z^k)\) in the triple integral transforms the integral to the integration of a bounded function.

Note that depending on the sign of the function \( F \) in a given polyhedral cell \( C \), we get either a real or a purely imaginary number. The computation time in the latter case can be reduced considerably by just using the function \(-F\)! It should be noted that if we multiply \( F \) by a constant factor \( \lambda \), the corresponding period integral changes by a factor \( \sqrt{\lambda} \). In particular, if we change the sign of \( F \) the real and imaginary periods are interchanged.

In the case of arrangement nr. 1 everything works nicely, but in the other cases the picture of the decomposition of \( \mathbb{P}^3 \) becomes much more complicated and more generic fourfold points to take into account. We wrote a simple Maple code to produce a linear–cylindric decomposition and form cycles from the polyhedral cells. Then we used several changes of variables moving each of the planes of the arrangement to infinity, which allowed us to compute the integrals for most of the cycles. In all cases ratios of any two real and any two complex integrals were rational numbers (with numerator and denominator \( \leq 6 \)). Below is a table that summarises all different period integrals that appeared
in our calculations.

It should be kept in mind that all period integrals get multiplied by a common factor if we change the polynomial $F$ defining the arrangement. For these calculations we used the equations $F$ as listed in [9] scaled by $\lambda$ from the last column of table at the end of section 1.

| Arr. No. | Real integrals | Imaginary integrals |
|----------|----------------|---------------------|
| 1 | 55.9805041334, 111.961008267 | 69.3694986501i |
| 3 | 80.3028893419, 160.60577868 | 41.4134587444i, 82.8269174889i |
| 19 | 72.1085316451, 144.217063291 | 124.240376233i, 289.89421121i |
| 216.325594935 | 216.325594935i |
| 32 | 55.9805041335, 111.961008267 | 34.6847493250i, 69.3694986501i |
| 196.240376233i, 289.89421121i |
| 69 | 55.9805041335, 111.9610083 | 34.6847493252i, 138.738997300i |
| 223.922016533 | 277.4779945i |
| 93 | 55.9805041334 | 17.3423746625i, 69.3694986502i |
| 238 | 55.9805041334, 111.961008267 | 34.6847493250i |
| 239 | 48.5252148713, 145.575644614 | 35.2275632784i, 105.682689835i |
| 240 | 43.7468074540, 131.240422363 | 28.8234453872i, 57.6468907743i |
| 241 | 223.922016533 | 69.3694986503i |
| 245 | 21.8734037270, 87.4936149079 | 28.8234453872i, 115.293781548i |
| 131.240422362 |

For each arrangement, the computed period integrals generate a lattice in $\mathbb{C}$, which in turn defines an elliptic curve. This lattice might be a proper sublattice of $H_3(\tilde{X}, \mathbb{Z})$, but in any case it defines a elliptic curve that is isogeneous with the intermediate Jacobian $J^2(\tilde{X})$ of the corresponding Calabi–Yau threefold. In the following table we list lattice generators, $j$–invariant and coefficients of the classical Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3$$

of the elliptic curve, which are easily computed numerically via

$$g_2 = 60 \sum_{0 \neq m \in \Lambda} \frac{1}{m^4}, \quad g_6 = 140 \sum_{0 \neq m \in \Lambda} \frac{1}{m^6},$$

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6. Comparison with modular periods

Recall that for an Hecke-eigenform $f \in S_k(\Gamma_0(N))$ with $q$-expansion

$$f = \sum_{n=1}^{\infty} a_n q^n$$

the $L$-function is defined by the series:

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$  

It converges for $Re(s) > 1 + k/2$ and the completed $L$-function

$$\Lambda(f, s) := (\sqrt{N}/2\pi)^s \Gamma(s)L(f, s)$$

satisfies the functional equation

$$\Lambda(f, s) = w i^k \Lambda(f, k - s)$$

where $w$ is the sign of $f$ under the Atkin-Lehner involution. We note that

$$\Lambda(f, s) = (\sqrt{N})^s \int_0^{\infty} f(it)t^s \frac{dt}{t}$$

In our case we $k = 4$ and $w = 1$, so that the functional equation just reads

$$\Lambda(f, s) = \Lambda(f, 4 - s)$$

This means in particular

$$\Lambda(f, 1) = \Lambda(f, 3)$$

from which we get the equality

$$L(f, 3) = \frac{(2\pi)^2 \Gamma(1)}{N \Gamma(3)} L(f, 1) = \frac{2\pi^2}{N} L(f, 1)$$
Furthermore, we see from the functional equation that $L(f, k) = 0$ for $k = 0, -1, -2, -3, -4, \ldots$.

By direct point counting (and correcting for the singularities of course), C. Meyer was able to determine cusp-forms

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_4(\Gamma_0(N))$$

such that

$$a_p = Tr(Fr_p : H^3(X) \longrightarrow H^3(X))$$

In other words, one has equality of $L$-functions

$$L(H^3(X), s) = L(f, s)$$

The result is summarised in the following table (we multiplied the equation of the octic arrangement by the factor $\lambda$ to obtain modular form of minimal level).

| Form | q-expansion | Arrangements |
|------|-------------|--------------|
| 6/1  | $q - 2q^4 - 3q^5 + 4q^6 + 6q^7 + 6q^8 - 16q^9 + O(q^{10})$ | 240, 245 |
| 8/1  | $q - 4q^2 - 2q^5 + 24q^7 - 11q^9 - 44q^{11} + O(q^{12})$ | 1, 32, 69, 93, 238, 241 |
| 12/1 | $q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + 36q^{11} + O(q^{12})$ | 239 |
| 32/1 | $q + 22q^4 - 27q^9 + O(q^{12})$ | 19 |
| 32/2 | $q + 8q^3 - 10q^5 + 16q^7 + 37q^9 - 40q^{11} + O(q^{12})$ | 3 |

It is a remarkable fact that only five different modular forms appear.

In cases where two varieties $X, X'$ have the same modular form, one expects there exists a correspondence $\phi$ between $X$ and $X'$ that explains it.

It is gratifying to see that the numerical evaluation of the period integrals lead to the very same grouping of our examples.

Here we summarise the calculations of the critical $L$-values

| $f$  | $L(f, 1)$ | $L(f, 2)$ |
|------|-----------|-----------|
| 6/1  | 0.2216239155906750824671004425 | 0.50971042361593998888737819140 |
| 8/1  | 0.35450068373096471876555989149 | 0.69003116312339752511910542021 |
| 12/1 | 0.61457902590673022954002802969 | 0.9344401381419444281042898230 |
| 32/1 | 1.8265304425089816105284840591 | 1.43455365630418076432000460798 |
| 32/2 | 2.0340959450627923591429024672 | 1.6477891674251259412768429683 |

If we express the real and imaginary periods of the double octics we get, at least at the numerical level, nice proportionals with

$$\pi L(f, 2), \quad \pi^2 L(f, 1)$$

for the corresponding modular form.
The above calculations show that it is possible to verify numerically the relation between the periods of a rigid Calabi-Yau and the corresponding $L$-values of the attached modular form. However, one would like to push these calculations to a further level. One important problem that was left untouched by our calculations is the complete determination of the group 3-cycles in terms of polyhedral cycles. We identified some polyhedral cycles, but there is no guaranty that these generate the whole third homology group $H_3(\tilde{X}, \mathbb{Z})$. It follows from Poincaré-duality, that this group is generated by any two cycles with intersection $\pm 1$. This leads to the following question to determine the intersection number $\langle \delta, \gamma \rangle$ between two polyhedral cycles $\delta, \gamma$ in purely combinatorial terms of the cell appearing in $\delta$ and $\gamma$, and their mutual position inside the arrangement.

Apart from the eleven double octics with rational coefficients there are two other with coefficients in $\mathbb{Q}(\sqrt{-3})$ and one in $\mathbb{Q}(\sqrt{5})$ (cf. [2, 3]). However, in these cases presented method did not yield any reasonable approximation of the period integrals.

### Outlook

| Form 6/1 | 240 | 43.7468074540... = 20\pi^2 L(f, 1) |
| Form 8/1 | 245 | 21.8734037270... = 10\pi^2 L(f, 1) |
| | 28 | 28.8234453871... = 18\pi L(f, 2) |
| | 28 | 28.8234453871... = 18\pi L(f, 2) |
| | 32 | 32.8234453871... = 18\pi L(f, 2) |
| | 69 | 69.3694986501... = 32\pi L(f, 2) |
| | 93 | 93.3694986501... = 32\pi L(f, 2) |
| | 238 | 238.3694986501... = 32\pi L(f, 2) |
| Form 12/1 | 239 | 48.5252148713... = 8\pi^2 L(f, 1) |
| Form 32/1 | 239 | 35.2275632785... = 12\pi L(f, 2) |
| Form 32/2 | 19 | 72.1085316452... = 4\pi^2 L(f, 1) |
| | 72 | 72.1085316452... = 16\pi L(f, 2) |
| | 3 | 80.3028893419... = 4\pi^2 L(f, 1) |
| | 41 | 41.4134587443... = 8\pi L(f, 2) |
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