Comments on Extended $t$-$J$ Models, Nodal Liquids and Supersymmetry

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Abstract
In the context of extended $t$ -- $J$ models, with intersite Coulomb interactions of the form $-V \sum_{\langle ij \rangle} n_i n_j$, with $n_i$ denoting the electron number operator at site $i$, nodal liquids are discussed.

We use the spin-charge separation ansatz as applied to the nodes of a d-wave superconducting gap. Such a situation may be of relevance to the physics of high-temperature superconductivity. We point out the possibility that at certain points of the parameter space supersymmetric points may occur, characterized by dynamical supersymmetries between the spinon and holon degrees of freedom, which are quite different from the symmetries in conventional supersymmetric $t$ -- $J$ models. Such symmetries pertain to the continuum effective field theory of the nodal liquid, and one’s hope is that the ancestor lattice model may differ from the continuum theory only by renormalization-group irrelevant operators in the infrared. We give plausible arguments that nodal liquids at such supersymmetric points are characterized by superconductivity of Kosterlitz-Thouless type. The fact that quantum fluctuations around such points can be studied in a controlled way, probably makes such systems of special importance for an eventual non-perturbative understanding of the complex phase diagram of the associated high-temperature superconducting materials.

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1 Introduction

The study of strongly correlated electron systems (SCES) is a major enterprise in modern condensed matter physics primarily due to high temperature (planar) superconductors, fractional Hall conductors and more recently in semiconductor quantum dots. Owing to various non-Fermi liquid features of SCES many believe that the low-energy excitations of these systems are influenced by the proximity of a critical Hamiltonian in a generalized coupling-constant space. In this scenario, known as spin-charge separation [1], these excitations are spinons, holons and gauge fields.

Important paradigm for SCES are the conventional Hubbard model, or its $t - t'$ extension, both of which have been conjectured to describe the physics of high-temperature superconducting doped antiferromagnets. Numerical simulations of such models [2], in the presence of very-low doping, have provided evidence for electron substructure (spin-charge separation) in such systems.

In ref. [3], an extension of the spin-charge separation ansatz, allowing for a particle-hole symmetric formulation away from half-filling, was introduced by writing:

$$\chi_{\alpha\beta} \equiv \begin{pmatrix} \psi_1 & \psi_2 \\ -\psi_2^\dagger & \psi_1^\dagger \end{pmatrix} \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix},$$

(1)

where the fields $z_{\alpha,i}$ obey canonical bosonic commutation relations, and are associated with the spin degrees of freedom (‘spinons’), whilst the fields $\psi$ are Grassmann variables, obeying Fermi statistics, and are associated with the electric charge degrees of freedom (‘holons’). There is a hidden non-abelian gauge symmetry $SU(2) \otimes U_S(1)$ in the ansatz, which becomes a dynamical symmetry of the pertinent planar Hubbard model, studied in ref. [3].

The ansatz (1) is different from that of refs. [4], where the holons are represented as charged bosons, and the spinons as fermions. That framework, unlike ours, is not a convenient starting point for making predictions such as the behaviour of the system under the influence of strong external fields. As argued in [5], a strong magnetic field induces the opening of a second superconducting gap at the nodes of the $d$-wave gap, in agreement with recent experimental findings on the behaviour of the thermal conductivity of high-temperature cuprates under the influence of strong external magnetic fields [6].

In [3] a single-band Hubbard model was used. Such a model should not be regarded as merely phenomenological for cuprate superconductors in the sense that it can be rigorously derived from chemically realistic multiband models with extra nearest-neighbour interactions of the form [4]:

$$H_{int} = -V \sum_{<ij>} \eta_i \eta_j \quad \eta_i \equiv \sum_{\alpha=1}^{2} c_{\alpha,i}^\dagger c_{\alpha,i},$$

(2)

What we shall argue below is that the presence of interactions of the form (2) is crucial for the appearance of supersymmetric points in the parameter space of the spin-charge separated model. Such points occur for particular doping concentrations. In this talk we shall only sketch the basic ideas. A more detailed account of the work will be
given in a future publication [3]. As we shall discuss, this supersymmetry is a dynamical symmetry of the spin-charge separation, and occurs between the spinon and holon degrees of freedom of the ansatz (1). Its appearance may indicate the onset of unconventional superconductivity of Kosterlitz-Thouless (KT) type [3, 10] in the liquid of excitations about the nodes of the d-wave superconducting gap (“nodal liquid”), to which we restrict our attention for the purposes of this work.

It should be stressed that the supersymmetry characterizes the continuum relativistic effective (gauge) field theory of the nodal liquid. The ancestor lattice model is of course not supersymmetric in general. What, however, one hopes is that at such supersymmetric points the universality class of the continuum low-energy theory is the same as that of the lattice model, in the sense that the latter differs from the continuum effective theory only by renormalization-group irrelevant operators (in the infrared). This remains to be checked by detailed studies, which do not constitute the topic of this talk.

Supersymmetry provides, in general, a much more controlled way for dealing with quantum fluctuations about the ground state of a field-theoretic system than a non-supersymmetric theory [11]. In this sense, one hopes that by working in such supersymmetric points in the parameter space of the nodal liquid she/he might obtain some exact results about the phase structure, which might be useful for a non-perturbative understanding of the complex phase diagrams that characterize the physics of the (superconducting) doped antiferromagnets.

Significant progress towards a non-perturbative understanding of Non-Abelian gauge field theories based on supersymmetry have been made by Seiberg and Witten [12]. The fact that the spin-charge separation ansatz (1) of the doped antiferromagnet is known to be characterized by such non-Abelian gauge structure is an encouraging sign. However, it should be noted that in the case of ref. [12] extended supersymmetries were necessary for yielding exact results. As we shall discuss below, in the case of doped antiferromagnets, and under special conditions, the supersymmetric points are characterized by $N = 1$ three-dimensional supersymmetries, although under certain circumstances the supersymmetry may be elevated to $N = 2$ [13], for which some exact results concerning the phase structure can be obtained [14]. However, in the realistic circumstances of a condensed-matter system such as a high-temperature superconductor, even the $N = 1$ supersymmetry of the supersymmetric points is expected to be broken at finite temperatures or under the influence of external electromagnetic fields. Nevertheless, one may hope that by viewing the case of broken supersymmetry as the result of some perturbation that takes one away from the supersymmetric point, valuable non-perturbative information may still be obtained. As we shall see, a possible example of this concerns the above-mentioned KT superconducting properties [1] that characterize such points.

The structure of the talk is as follows: In section 2 we describe briefly the statistical model which gives rise to the continuum relativistic effective (2+1)-dimensional field theory of the nodal liquid. In section 3 we discuss the properties and (non-abelian gauge) symmetries of the spin-charge separation ansatz that characterizes the model. In the next section we discuss the intersite Coulomb interactions, which are of crucial importance for the existence of supersymmetric points. In section 5 we state the conditions for $N=1$ supersymmetry at such points, and describe briefly their importance for yielding super-
conductivity of Kosterlitz-Thouless type. We conclude in section 6 with some prospects for future work.

2 The Model and its Parameters

In reference [7] it was argued that BCS-like scenarios for high $T_c$ superconductivity based on extended $t - J$ models yield reasonable predictions for the critical temperature $T_c^{\text{max}}$ at optimum doping. There it was argued that a pivotal role was played by next-to-nearest neighbour and third neighbour hoppings, $t'$ and $t'''$ respectively. In particular the combination $t_- \equiv t' - 2t'''$ determines the shape of the Fermi surface and the nature of the saddle points and the associated $T_c^{\text{max}}$.

Our aim is to use the extended $t - J$ model studied in [7] in order to discuss the appearance of relativistic charge liquids at the nodes of the associated d-wave superconducting gap. We will argue that the nodes characterize the model in a certain range of parameters. We will demonstrate that at a certain regime of the parameters and doping concentration the nodal liquid effective field theory of spin-charge separation exhibits supersymmetry. This supersymmetry is dynamical and should not be confused with the non-dynamical symmetry under a graded supersymmetry algebra that characterises the spectrum of doped antiferromagnets at two special points of the parameter space [15]. We shall also discuss unconventional mechanisms for superconductivity in the nodal liquid similar to the ones proposed in [9, 10].

To start with let us describe briefly the extended $t - J$ model used in Ref. [7]. The Hamiltonian is given by:

$$ H = P (H_{\text{hop}} + H_J + H_V) P + PH_\mu P ,$$

where:

(a) $$ H_{\text{hop}} = - \sum_{(ij)} t_{ij} c_{i\alpha}^+ c_{j\alpha} - \sum_{[ij]} t'_{ij} c_{i\alpha}^+ c_{j\alpha} - \sum_{(ij)} t''_{ij} c_{i\alpha}^+ c_{j\alpha} ,$$

and $\langle \ldots \rangle$ denotes nearest neighbour (NN) sites, $\{ \ldots \}$ next-to-nearest neighbour (NNN), and $\{ \}$ third nearest neighbour. Here repeated spin (or "colour") indices are summed over. The Latin indices $i, j$ denote lattice sites and the Greek indices $\alpha = 1, 2$ are spin components.

(b) $$ H_J = J \sum_{(ij)} T_{i,\alpha\beta} T_{j,\beta\alpha} + J' \sum_{[ij]} T_{i,\alpha\beta} T_{j,\beta\alpha} ,$$

with $T_{i,\alpha\beta} = c_{i\alpha}^+ c_{i\beta}$. The quantities $J, J'$ denote the couplings of the appropriate Heisenberg antiferromagnetic interactions. We shall be interested in the regime where $J' << J$.

(c) $$ H_\mu = \mu \sum_i c_{i\alpha}^+ c_{i\alpha} ,$$

and $\mu$ is the chemical potential.
(d) 

\[ H_V = -V \sum_{\langle ij \rangle} n_i n_j , \]  

(7)

and \( n_i = \sum_{\alpha=1}^2 c_{i\alpha}^+ c_{i\alpha} \). This is an effective static NN interaction which is provided in the bare \( t - J \) model by the exchange term, because of the extra magnetic bond in the system when two polarons are on neighbouring sites \[7\]. In ref. \[7\] the strength of the interaction is taken to be:

\[ V \approx 0.585 J , \]  

(8)

This is related to the regime of the parameters used in \[7\], for which the NN hoping elements \( t << J \). However, one may consider more general models \[8\], in which the above restrictions are not valid, and \( V \) is viewed as an independent parameter of the effective theory, e.g.

\[ V \approx b J , \]  

(9)

with \( b \) a constant to be determined phenomenologically. As we shall discuss below, this turns out to be useful for the existence of supersymmetric points in the parameter space of the model.

(e) The operator \( P \) projects out double occupancy at a site.

We define the doping parameter \( 0 < \delta < 1 \) by

\[ \sum_{\alpha=1}^2 \langle c_{i\alpha}^+ c_{i\alpha} \rangle = 1 - \delta , \]  

(10)

d-wave pairing, which seems to have been confirmed experimentally for high-\( T_c \) cuprates, was assumed in \[7\]. A d-wave gap is represented by an order parameter of the form

\[ \Delta \left( \vec{k} \right) = \Delta_0 (\cos k_x a - \cos k_y a) , \]  

(11)

where \( a \) is the lattice spacing. The relevant Fermi surface is characterised by the following four nodes where the gap vanishes:

\[ \left( \pm \frac{\pi}{2a}, \pm \frac{\pi}{2a} \right) , \]  

(12)

We now consider the generalized dispersion relation \[3, 16\] for the quasiparticles in the superconducting state:

\[ E \left( \vec{k} \right) = \sqrt{\left( \varepsilon \left( \vec{k} \right) - \mu \right)^2 + \Delta^2 \left( \vec{k} \right) } , \]  

(13)

In the vicinity of the nodes it is reasonable \[3, 16\] to assume that \( \mu \approx 0 \) or equivalently we may linearize about \( \mu \), i.e. write \( \varepsilon \left( \vec{k} \right) - \mu \approx v_D |\vec{q}| \) \[3\] where \( v_D \) is the effective velocity at the node and \( q \) is the wave-vector with respect to the nodal point.
Non-Abelian spin-charge separation in the t-J model

As already mentioned in the introduction, in ref. [3] it was proposed that for the large-U limit of the doped Hubbard model the following ‘particle-hole’ symmetric spin-charge separation ansatz occurs at each site $i$:

\[
\chi_{\alpha\beta,i} = \psi_{\alpha\gamma,i} z_{\gamma\beta,i} \equiv \left( \begin{array}{c}
\psi_1 & \psi_2 \\
-\psi_2 & \psi_1^\dagger 
\end{array} \right)_i \left( \begin{array}{c}
z_1 \\
-z_2
\end{array} \right)_i
\]

(14)

where the fields $z_{\alpha,i}$ obey canonical bosonic commutation relations, and are associated with the spin degrees of freedom (‘spinons’), whilst the fields $\psi_{\alpha,i}$, $a = 1, 2$ have fermionic statistics, and are assumed to create holes at the site $i$ with spin index $\alpha$ (‘holons’). The ansatz (14) has spin-electric-charge separation, since only the fields $\psi$ carry electric charge. Generalization to the non-Abelian model allows for inter-sublattice hopping of holes which is observed experimentally.

It is worth noticing that the anticommutation relations for the electron fields $c_\alpha, c_\beta^\dagger$, do not quite follow from the ansatz (14). Indeed, assuming the canonical (anti) commutation relations for the $\psi$ fields, one obtains from the ansatz (14)

\[
\{ c_{1,i}, c_{2,j} \} \sim 2\psi_{1,i}\psi_{2,i}\delta_{ij} \\
\{ c_{1,i}^\dagger, c_{2,j}^\dagger \} \sim 2\psi_{2,i}\psi_{1,i}\delta_{ij} \\
\{ c_{1,i}, c_{2,j}^\dagger \} \sim \{ c_{2,i}, c_{1,j}^\dagger \} \sim 0 \\
\{ c_{\alpha,i}, c_{\beta,j}^\dagger \} \sim \delta_{ij} \sum_{\beta=1,2} [z_{\beta,i} z_{\beta,j} + \psi_{\beta,i} \psi_{\beta,j}^\dagger], \quad \alpha = 1, 2 \quad \mathrm{no \ sum \ over \ } i, j
\]

(15)

To ensure canonical commutation relations for the $c$ operators therefore we must impose at each lattice site the (slave-fermion) constraints

\[
\psi_{1,i}\psi_{2,i} = \psi_{1,i}^\dagger \psi_{2,i}^\dagger = 0, \\
\sum_{\beta=1,2} [z_{\beta,i} z_{\beta,i} + \psi_{\beta,i} \psi_{\beta,i}^\dagger] = 1
\]

(16)

Such relations are understood to be satisfied when the holon and spinon operators act on physical states. Both of these relations are valid in the large-$U$ limit of the Hubbard model and encode the non-trivial physics of constraints behind the spin-charge separation ansatz (14). They express the constraint at most one electron or hole per site, which characterizes the large-$U$ Hubbard models we are considering here.

There is a local phase (gauge) non-Abelian symmetry hidden in the ansatz (14) \[3\] $G = SU(2) \times U_S(1)$, where $SU(2)$ stems from the spin degrees of freedom, $U_S(1)$ is a statistics changing group, which is exclusive to two spatial dimensions and is responsible for transforming bosons into fermions and vice versa. As remarked in [3], the $U_S(1)$ effective interaction is responsible for the equivalence between the slave-fermion ansatz (i.e. where the holons are viewed as charged bosons and the spinons as electrically neutral fermions [4]) and the slave boson ansatz (i.e. where the holons are viewed as charged
fermions and the spinons as neutral bosons [17, 3]. This is analogous (but not identical) to the bosonization approach of [18] for anyon systems.

The application of the ansatz [14] to the Hubbard (or t-j models) necessitates a 'particle-hole' symmetric formulation of the Hamiltonian (3), which as shown in [3], is expressible in terms of the operators \( \chi \). Upon appropriate linearizations of the various four-field operators involved using the Hubbard-Stratonovitch method, we obtain the effective spin-charge separated action for the doped-antiferromagnetic model of [3]:

\[
H_{HF} = \sum_{<ij>} \left( \text{tr} \left[ \frac{8}{J} \Delta_{ij} \Delta_{ji} + |A_1| (t_{ij} (1 + \sigma_3) + \Delta_{ij}) \psi_j V_{ji} U_{ji} \psi_i^\dagger \right] + \text{tr} \left[ K \chi_i V_{ij} U_{ij} z_j \right] + \text{h.c.} \right) + \ldots ,
\]

with the \( \ldots \) denoting chemical potential terms. This form of the action, describes low-energy excitations about the Fermi surface of the theory. \( \Delta_{ij} \) is a Hubbard-Stratonovich field that linearizes four-electron interaction terms in the original Hubbard model. The quantities \( V_{ij} \) and \( U_{ij} \) denote lattice link variables associated with elements of the \( SU(2) \) and \( U_S(1) \) groups respectively. They are associated [3] with phases of vacuum expectation values of bilinears \( < z_i z_j > \) and/or \( < \psi_i^\dagger (t_{ij} (1 + \sigma_3) + \Delta_{ij}) \psi_j > \). It is understood that, by integrating out in a path integral over \( z \) and \( \psi \) variables, fluctuations are incorporated, which go beyond a Hartree-Fock treatment. The quantity \( |A_1| \) is the amplitude of the bilinear \( < z_i z_j > \) assumed frozen [3]. By an appropriate normalization of the respective field variables, one may set \( |A_1| = 1 \), without loss of generality. In this normalization, one may then parametrize the quantity \( K \), which is the amplitude of the appropriate fermionic bilinears, as [3, 10]:

\[
K \equiv \left( J |\Delta_z|^2 \eta^2 \right)^{1/2} ; \quad \eta \equiv \sum_{\alpha=1}^2 \psi_\alpha \psi_\alpha^\dagger >= 1 - \delta ,
\]

with \( \delta \) the doping concentration in the sample. The quantity \( |\Delta_z| \) is considered as an arbitrary parameter of our effective theory, of dimensions \( [\text{energy}]^{1/2} \), whose magnitude is to be fixed by phenomenological or other considerations (see below). To a first approximation we assume that \( \Delta_z \) is doping independent [1]. The dependence on \( J \) and \( \delta \) in (18) is dictated [10] by the correspondence with the conventional antiferromagnetic \( CP^1 \) \( \sigma \)-model in the limit \( \delta \to 0 \).

The model of ref. [7] differs from that of [3] in the existence of NNN hopping \( t' \) and triplle neighbor hopping \( t'' \), which were ignored in the analysis of [3]. For the purposes of this work, which focuses on the low-energy (infrared) properties of the continuum field theory of (17), this can be taken into account by assuming that

\[
|t_{ij}| = t_{ij}' \equiv t + 2t_+ , \quad t_+ \equiv t' + 2t''
\]

in the notation of [7]. The relation stems from the observation that in the continuum low-energy field-theory limit such NNN and triple hopping terms can be Taylor expanded

\[1\]

However, from its definition, as a \( < \ldots > \) of a quantum model with complicated \( \delta \) dependences in its couplings, the quantity \( \Delta_z \) may indeed exhibit a doping dependence. For some consequences of this we refer the reader to the discussion in section 6, below, and in ref. [8].
(in derivatives). It is the terms linear in derivatives that yield the shift (19) of the NN neighbor hopping element \( t \). Higher derivatives terms, of the form \( \partial_x \partial_y \) are suppressed in the low-energy (infrared) limit.

It is important to note that the model of [3], as well as its extension (17), in contrast to that discussed in [9], involves only a single lattice structure, with nearest neighbor hopping \( <ij> \) being taken into account, \( t_{ij} \). The antiferromagnetic nature is then viewed as a ‘colour’ degree of freedom, being expressed via the non-Abelian gauge structure of the spin-charge separation ansatz (14). As we shall discuss later, this is very important in yielding the correct number of fermionic (holons \( \Psi \)) degrees of freedom in the continuum low-energy field theory to match the bosonic degrees of freedom (spinons \( z \)) at the supersymmetric point.

4 The Effective Low-Energy Gauge Theory

The conventional lattice gauge theory form of the action (17) is derived upon freezing the fluctuations of the \( \Delta_{ij} \) field, assuming, as usual, the flux phase for the gauge field \( U_S(1), \) with flux \( \pi \) per lattice plaquette, and assembling the fermionic degrees of freedom into two 2-component Dirac spinors [3]:

\[
\tilde{\Psi}_{1,i} = (\psi_1 - \psi_2^\dagger)_i, \quad \tilde{\Psi}_{2,i} = (\psi_2 \psi_1^\dagger)_i
\]  

(20)

The fermionic part of the long-wavelegth lattice lagrangian, then, reads:

\[
S = \frac{1}{2} K' \sum_{i,\mu} [\bar{\Psi}_i (-\gamma_\mu) U_{i,\mu} V_{i,\mu} \Psi_{i+\mu} + \\
\bar{\Psi}_{i+\mu} (\gamma_\mu) U_{i,\mu}^\dagger V_{i,\mu}^\dagger \Psi_i] + \text{Bosonic CP}^1 \text{ parts} 
\]  

(21)

where the Bosonic \( \text{CP}^1 \) parts denote magnon-field \( z \) dependent terms, and are given in (17). The coefficient \( K' \) is a constant which stems from the \( t_{ij} - \) and \( \Delta_{ij} - \) dependent coefficients in front of the fermion terms in (17). The fermions \( \Psi \) in (21) are two-component ‘coloured’ spinors, related to the spinors in (20) via a Kawamoto-Smit transformation [19]

\[
\Psi_c(r) = \gamma_0^c \cdots \gamma_2^c \tilde{\Psi}_c(r) \quad \bar{\Psi}_c(r) = \bar{\Psi}_c(r)(\gamma_2^c)^{r_2} \cdots (\gamma_0^c)^{r_0}
\]  

(22)

where \( r \) is a point on the spatial lattice, and \( c \) is a ‘colour’ index \( c = 1, 2 \) expressing the initial antiferromagnetic nature of the system; the \( \gamma \) matrices are \( 2 \times 2 \) antihermitean Dirac matrices on a Euclidean Lattice satisfying the algebra

\[
\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}
\]  

(23)

In terms of the Pauli matrices \( \sigma_i, i = 1, \ldots 3, \) the \( \gamma \) matrices are given by \( \gamma_\mu = i\sigma_\mu, \mu = 1, 2, 3. \) Notice that fermion bilinears of the form \( \bar{\Psi}_{i,c} \Psi_{i,c'} \) (\( i=\text{Lattice index} \)) are just

\[
\bar{\Psi}_{i,c} \Psi_{i,c'} = \bar{\Psi}_{i,c} \tilde{\Psi}_{i,c'}
\]  

(24)
due to the Clifford algebra (23), and (anti-) hermiticity properties of the $2 \times 2$ $\gamma$ matrices on the Euclidean lattice. On a lattice, in the path integral over the fermionic degrees of freedom in a quantum theory, the variables $\overline{\Psi}$ and $\Psi$ are viewed as independent. In view of this, the spinors $\Psi_\alpha$ in (20) may be replaced by $\overline{\Psi}_\alpha$, as being path integral variables on a Euclidean Lattice appropriate for the Hamiltonian system (8). This should be kept in mind when discussing the microscopic structure of the theory in terms of the holon creation and annihilation operators $\psi_\alpha^\dagger, \psi_\alpha, \alpha = 1, 2$.

An order of magnitude estimate of the modulus of $\Delta_{ij}$ then, which determines the strength of the coefficient $K'$ may be provided by its equations of motion. Assuming that the modulus of (the dimensionless) fermionic bilinears is of order unity, then, we have as an order of magnitude

$$K' \sim \left( t'_{+} + \frac{J}{8} \right)$$

(25)

Notice that in the regime of the parameters of $[7]$ $t << t_{+}$ and $t_{+} \simeq \frac{3}{2}J$ for a momentum regime close to a node in the fermi surface, of interest to us here. Thus

$$K' \simeq 25J/8$$

(26)

For reasons that will become clear below we may consider a regime of the parameters of the theory for which

$$K' >> K = \sqrt{J} |\Delta_z| \left( 1 - \delta \right), \quad 0 < \delta < 1$$

(27)

For the model of $[8]$, for instance, on account of (26), this condition implies that

$$\sqrt{J}/|\Delta_z| \gg 0.32 \left( 1 - \delta \right), \quad 0 < \delta < 1$$

(28)

By appropriately rescaling the fermion fields $\Psi$ to $\Psi'$, so that in the continuum they have a canonical Dirac term, we may effectively constrain the $z$ fields to satisfy the $CP^1$ constraint:

$$|z_\alpha|^2 + \frac{1}{K'} (\Psi' - \text{bilinear terms}) = 1$$

where now the fields $\Psi$ are dimensionful, with dimensions of $[\text{energy}]$. A natural order of magnitude of these dimensionful fermion bilinear terms is of the order of $K^2$, which plays the rôle of the characteristic scale in the theory, being related directly to the Heisenberg exchange energy $J$. In the limit $K' >> K$ (27) therefore the fermionic terms in the constraint can be ignored, and the constraint assumes the standard $CP^1$ form involving only the $z$ fields (this being also the case for the model of $[9, 10]$, in a specific regime of the microscopic parameters). As discussed in $[13, 8]$, such a form for the constraint is the one appropriate for supersymmetrization.

As we shall see later, however, the condition (28) alone, although necessary, is not sufficient to guarantee the existence of supersymmetric points. Supersymmetry imposes additional restrictions, which in fact rule out the existence of supersymmetric points for the model of $[8]$ compatible with superconductivity $[9]$. However, this does not prevent one

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from considering more general models in which $K'$ is viewed as a phenomenological parameter, not constrained by (26). In that case, supersymmetric points may occur for a certain regime of the respective parameters.

With the above in mind we consider from now on the standard $CP^1$ constraint involving only $z$ fields. By an appropriate normalization of $z$ to $z' = z / \sqrt{1 - \delta}$ the constraint then acquires the familiar normalized $CP^1$ form $|z_\alpha|^2 = 1$ form. This implies a rescaling of the normalization coefficient $K$ in (17):

$$K \to \frac{1}{\gamma} \equiv K(1 - \delta) \simeq \sqrt{J}|\Delta_z|(1 - \delta)^2$$  \hspace{1cm} (29)

In the naive continuum limit, then, the effective lagrangian of spin and charge degrees of freedom describing the low-energy dynamics of the Hubbard (or $t - j$) model (17) of $\chi$ is then:

$$L_2 \equiv \frac{1}{\gamma} \text{Tr} \left| \left( \partial_\mu + ig\sigma^a B^a_\mu + ig_\sigma^a A_\mu \right) z \right|^2 + \bar{\Psi} D_\mu \gamma_\mu \Psi$$  \hspace{1cm} (30)

with $z_\alpha$ a complex doublet satisfying the constraint

$$|z_\alpha|^2 = 1$$  \hspace{1cm} (31)

The Trace $\text{Tr}$ is over group indices, $D_\mu = \partial_\mu - ig_1 a_\mu^S - ig_2 \sigma^a B^a_\mu$, $B^a_\mu$ is the gauge potential of the local ('spin') $SU(2)$ group, and $a_\mu$ is the potential of the $US(1)$ group.

It should be remarked that, we are working in units of the Fermi velocity $v_F (= v_D)$ of holes, which plays the rôle of the limiting velocity for the nodal liquid. For the nodal liquid at the supersymmetric points we also assume that $v_F \simeq v_S$, where $v_S$ is the effective velocity of the spin degrees of freedom. The relativistic form of the fermionic and bosonic terms of the action (30) is valid only in this regime of velocities. This is sufficient for our purposes in this work. Indeed, at the supersymmetric points, where we shall restrict our analysis here, the mass gaps for spinons and holons, which may be generated dynamically, are equal by virtue of supersymmetry at zero temperatures and in the absence of any external fields. Hence it makes sense to assume the equality in the propagation velocities for spin and charge degrees of freedom, given that this situation is consistent with the respective dispersion relations. This is not true, of course, for excitations away from such points.

5 The NN interaction terms $H_V$

We will now discuss the terms

$$H_V = -V \sum_{\langle ij \rangle} n_i n_j$$  \hspace{1cm} (32)

introduced in ref. [4]. With the above discussion in mind for the spinors (20) we note that, under the ansatz (14), at a site $i$ the electron number operator $\eta_i$ is expressed, through the Determinant (Det) of the $\chi$ matrix in (14), in terms of the spin, $z_\alpha$, $\alpha = 1, 2,$ and
charge $\psi_\alpha, \alpha = 1, 2$, operators as:

$$\eta_i \equiv \sum_{\alpha=1}^2 c_{\alpha,i}^\dagger c_{\alpha,i} = \text{Det}\chi_{\alpha\beta,i} =$$

$$\text{Det}\hat{z}_{\alpha\beta,i} + \text{Det}\hat{\psi}_{\alpha\beta,i} = \sum_{\alpha=1}^2 (\psi_\alpha\psi_\alpha^\dagger + |z_\alpha|^2)$$

(33)

We may express the quantum fluctuations for the Grassmann fields $\psi_\alpha$ (which now carry a ‘colour’ index $\alpha = 1, 2$ in contrast to Abelian spin-charge separation models) via:

$$\psi_{\alpha,i}\psi_{\alpha,i}^\dagger = \langle \psi_{\alpha,i}\psi_{\alpha,i}^\dagger \rangle + :\psi_{\alpha,i}\psi_{\alpha,i}^\dagger :, \text{ no sum over } i$$

(34)

where : ... : denotes normal ordering of quantum operators, and from now on, unless explicitly stated, repeated indices are summed over. Since

$$\langle \psi_{\alpha,i}\psi_{\alpha,i}^\dagger \rangle \equiv 1 - \delta , \text{ no sum over } i$$

$\delta$ the doping concentration in the sample (34), we may rewrite $\eta_i$ as

$$\eta_i = |z_\alpha|^2 + (1 - \delta) + :\psi_\alpha\psi_\alpha^\dagger :$$

which in terms of the spinors $\Psi$ is given by (c.f. (20),(24)):

$$\eta_i = 2 - \delta + \frac{1}{2} \left( \Psi^\dagger_\alpha \sigma_3 \Psi_\alpha \right)_i$$

(35)

where $\sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ acts in (space-time) spinor space, and we took into account the $CP^1$ constraint (31).

Consider now the attractive interaction term $H_V$ (32), introduced in ref. [7]. We then observe than the terms linear in $(2 - \delta)$ in the expression for $H_V$ can be absorbed by an appropriate shift in the chemical potential, about which we linearize to obtain the low-energy theory. We can therefore ignore such terms from now on.

Next, we make use of the fact, mentioned earlier, that in a Lattice path integral the spinors $\Psi^\dagger_\alpha$ may be replaced by $\Psi_\alpha$. From the structure of the spinors (20), then, we observe that we may rewrite the $H_V$ term effectively as a Thirring vector-vector interaction among the spinors $\Psi$

$$H_V = +\frac{V}{4} \sum_{<ij>} \left( \Psi_\alpha \gamma_\mu \Psi_\alpha \right)_i \left( \Psi_\beta \gamma^\mu \Psi_\beta \right)_j$$

(36)

where summation over the repeated indices $\alpha, \beta (= 1, 2)$, and $\mu = 0, 1, 2$, with $\mu = 0$ a temporal index, is understood. To arrive at (36) we have expressed $\sigma_3$ as $-i\gamma_0$, and used the Clifford algebra (23), the off-diagonal nature of the $\gamma_{1,2} = i\sigma_{1,2}$ matrices, as well as the constraints (16). In particular the latter imply that any scalar product between Grassmann variables $\psi_\alpha$ (or $\psi_\beta^\dagger$) with different ‘colour’ indices vanish.
Taking the continuum limit of (36), and ignoring higher derivative terms involving four-fermion interactions, which by power counting are irrelevant operators in the infrared, we obtain after passing to a Lagrangian formalism

\[ \mathcal{L}_V = -\frac{V}{4K'^2} \left( \overline{\Psi}_\alpha \gamma_\mu \Psi_\alpha \right)^2 \]  

where we have used rescaled spinors, with the canonical Dirac kinetic term with unit coefficient, for which the canonical form of the $CP^1$ constraint (31) is satisfied. For notational convenience we use the same notation $\Psi$ for these spinors as the unscaled ones.

Although this is called the naive continuum limit, it captures correctly the leading infrared behaviour of the model.

We then use a Fierz rearrangement formula for the $\gamma$ matrices

\[ \gamma^\mu_{ab} \gamma_\mu_{cd} = 2 \delta_{ad} \delta_{bc} - \delta_{ab} \delta_{cd} \]

where Latin letters indicate spinor indices, and Greek Letters space time indices. The Thirring (four-fermion) interactions (36) then become:

\[ \left( \overline{\Psi}_\alpha \gamma_\mu \Psi_\alpha \right)^2 = -3 \left( \overline{\Psi}_\alpha \Psi_\alpha \right)^2 - 4 \sum_{\alpha<\beta} \left( \overline{\Psi}_\alpha \Psi_\beta \overline{\Psi}_\beta \Psi_\alpha \right) \]  

(38)

As mentioned above, in the model of [3], due to the first of the constraints (16), the mixed colour terms vanish, thereby leaving us with pure Gross-Neveu attractive interaction terms of the form:

\[ \mathcal{L}_V = +\frac{3V}{4K'^2} \left( \overline{\Psi}_\alpha \Psi_\alpha \right)^2 \]  

(39)

which describe the low-energy dynamics of the interaction (32) in the context of the non-Abelian spin-charge separation (14). It should be stressed that (39) is specific to our spin-charge separation model.

Moreover in the context of the spinors (20), a condensate of the form $< \overline{\Psi}_\alpha \Psi_\alpha >$ on the lattice vanishes because of the constraints (16). Such condensates would violate parity (reflection) operation on the planar spatial lattice, which on the spinors $\tilde{\Psi}$ is defined to act as follows:

\[ \tilde{\Psi}_1 (x) \rightarrow \sigma_1 \tilde{\Psi}_2 (x), \quad \tilde{\Psi}_2 (x) \rightarrow \sigma_1 \tilde{\Psi}_1 (x) \]

or equivalently, in terms of the (microscopic) holon operators $\psi_\alpha, \alpha = 1, 2,$

\[ \psi_1 (x) \rightarrow \psi_2^\dagger (x), \quad \psi_2 (x) \rightarrow -\psi_1^\dagger (x). \]

To capture correctly this fact in the context of our effective continuum Gross-Neveu interaction (39) the coupling strength must be subcritical, i.e. weaker than the critical coupling for mass generation. The critical coupling of the Gross-Neveu interaction is expressed in terms of a high-energy cut-off scale $\Lambda$ as [20]:

\[ 1 = 4g^2 \int_{S_\Lambda} \frac{d^3q}{8\pi^3q^2} = \frac{2g^2\Lambda}{\pi^2} \]  

(40)
where $q$ is a momentum variable and $S_\Lambda$ is a sphere of radius $\Lambda$. The divergent $q$-integral is cut-off at a momentum scale $\Lambda$ which defines the low-energy theory of interest. For the case of interest $g^2 = \frac{3V}{4K\pi}$: on using (29), then, the condition of sub-criticality requires that

$$\Lambda < 10^2 J.$$  

which is in agreement with the fact that in all effective models for doped antiferromagnets used in the literature the Heisenberg exchange energy serves as an upper bound for the energies of the excitations of the effective (continuum) theory.

6 Conditions for N=1 Supersymmetry and Potential Phenomenological Implications

We turn now to conditions for supersymmetrization of the above continuum theory. Below we shall sketch only the main results, which will be sufficient for the purposes of this talk. Details will appear in a forthcoming publication [8]. For simplicity we shall ignore the non-Abelian $SU(2)$ interactions, keeping only the Abelian $U_S(1)$ ones, which has been shown to be responsible for dynamical mass generation (and superconductivity) in the model of [3]. The extension to supersymmetrizing the full gauge multiplet $SU(2) \times U_S(1)$ will be the topic of a forthcoming work. However we shall still maintain the colour structure in the spinors, which is important for the ansatz (14).

As discussed in detail in [13, 21] the conditions for $N=1$ supersymmetric extensions of a $CP^1_\sigma$ model is that the constraint is of the standard $CP^1_\sigma$ form (31), supplemented by attractive four-fermion interactions of the Gross-Neveu type (39), whose coupling is related to the coupling constant of the kinetic $z$-magnon terms of the $\sigma$-model in a way so as to guarantee the balance between bosonic and fermionic degrees of freedom. Specifically, in terms of component fields, the pertinent lagrangian reads:

$$L = g_1^2 [D_\mu \bar{z}^\alpha D^\mu z^\alpha + i \nabla \cdot D \Psi + F^\alpha F^\alpha + 2i(\bar{\Psi} \gamma^\alpha \Psi - \bar{\Psi} \gamma^\alpha \eta z^\alpha)]$$ (42)

where $D_\mu$ denotes the gauge covariant derivative with respect to the $U_S(1)$ field. The analysis of [13, 21] shows that

$$F^\alpha F^\alpha = \sum_{\alpha=1}^{2} \frac{1}{4} \left( \nabla^\alpha \Psi_\alpha \right)^2$$ (43)

We thus observe that the $N=1$ supersymmetric extension of the $CP^1_\sigma$ model necessitates the presence of attractive Gross-Neveu type interactions among the Dirac fermions of each sublattice, in addition to the gauge interactions.

In the context of the effective theory (30), (37), discussed in this article, the $N = 1$ supersymmetric effective lagrangian (42) is obtained under the following restrictions

\(^3\)Ignoring the $SU(2)$ interactions implies, of course, that the ‘colour’ structure becomes a ‘flavour’ index; however, this is essential for keeping track of the correct degrees of freedom required by supersymmetry in the problem at hand [13].
among the coupling constants of the statistical model:

\[
\frac{3V}{K'^2} = \gamma = \frac{1}{\sqrt{J|\Delta_z|(1 - \delta)^2}}, \quad 0 < \delta < 1
\]

(44)

Note that in the context of the model of ref. [7], for which (8),(26) are valid, the relation (44) gives the supersymmetric point in the parameter space of the model at the particular doping concentration \(\delta = \delta_s\):

\[
(1 - \delta_s)^2 \simeq \frac{5.56\sqrt{J}}{|\Delta_z|}, \quad 0 < \delta_s < 1
\]

(45)

Then, compatibility with (27),(28) requires \(1 - \delta_s \gg 1.8\), which implies that the model of [7] does not have supersymmetric points. However, one may consider more general models [8] in which \(V\) and \(K' \sim t'_+ + J/8\) are treated as independent phenomenological parameters (c.f. [9]); in such a case one can obtain regions of parameters that characterize the supersymmetric points (27),(44), compatible with superconductivity.

Some comments are now in order:

First, it is quite important to remark that in the model of [3], where the antiferromagnetic structure of the theory is encoded in a colour (non-Abelian) degree of freedom of the spin-charge separated composite electron operator (1) on a single lattice geometry, there is a matching between the bosonic (\(z\) spinon fields) and fermionic (\(\Psi\) holon fields) physical degrees of freedom, as required by supersymmetry, without the need for duplicating them by introducing “unphysical” degrees of freedom [13].

The gauge multiplet of the \(CP^1\) \(\sigma\)-model also needs a supersymmetric partner which is a Majorana fermion called the gaugino. As shown in [13], such terms lead to an effective electric-charge violating interactions on the spatial planes, given that the Majorana gaugino is a real field, and as such cannot carry electric charge (which couples as a phase to a Dirac field). These terms can be interpreted as the removal or addition of electrons due to interlayer hopping.

Another important point we wish to make concerns the four-fermion attractive Gross-Neveu interactions in (42),(43). As discussed in detail in [22, 8], if the coupling of such terms is supercritical, then a parity-violating fermion (holon) mass would be generated in the model. However, the condition (41), which is valid in the statistical model of interest to us here, implies that the respective coupling is always subcritical, and thus there is no parity-violating dynamical mass gap for the holons, induced by the contact Gross-Neveu interactions. This leaves one with the possibility of parity conserving dynamical mass generation, due to the statistical gauge interactions in the model [3, 22].

A detailed analysis of such phenomena in the context of our \(CP^1\) model is left for future work. We note at present, however, that in \(N = 1\) supersymmetric gauge models, supersymmetry-preserving dynamical mass is possible [13, 23, 24]. In fact, as discussed in [24], although by supersymmetry the potential is zero, and thus there would naively seem that there is no obvious way of selecting the non-zero mass ground state over the zero mass one, however there appear to be instabilities in the quantum effective action in the massless phase, which manifest themselves through instabilities of the pertinent running coupling.
From a physical point of view, such a phenomenon would imply that, for sufficiently strong gauge couplings, the zero temperature liquid of excitations at the nodes of a $d$ wave superconducting gap would be characterized by the dynamical opening of mass gaps for the holons. At zero temperature, and for the specific doping concentrations corresponding to the supersymmetric points, as advocated above, the nodal gaps between spinon and holons would be equal, in agreement with the assumed equality of the respective propagation velocities $v_F = v_S$, which yielded the relativistic form of the effective continuum action (30) of the nodal excitations at the supersymmetric points. Moreover, it is known [4, 11, 3] that in the context of the gauge model, under the influence of an external electromagnetic field, the nodal gap may become superconducting, with a Kosterlitz-Thouless (KT) type superconductivity, not characterized by a local order parameter.

At finite temperatures, however, at which supersymmetry is explicitly broken, this equality of mass gaps would disappear. Moreover, as the crude analysis of [3] indicates, such gaps would disappear at temperatures which are much lower than the critical temperature of the (bulk) $d$-wave superconducting gap. For instance, for a typical set of the parameters of the $t – j$ model used in [3], the nodal critical temperature is of order of a few $mK$, which is much smaller than the 100 K bulk critical temperature of the high temperature superconductors. The application of an external magnetic field in the strongly type II, high-temperature superconducting oxides, which is another source for explicit breaking of the potential supersymmetry, enhances the critical temperature [4] up to 30 K, thereby providing a potential explanation for the recent findings of [3].

However, if such situations with broken supersymmetry are viewed as cases of perturbed supersymmetric points, then one might hope of obtaining non-perturbative information on the phase structure of the liquid of nodal excitations in spin-charge separating scenario of (gauge) high-temperature superconductors. This may also prove useful for a complete physical understanding of the entire phenomenon, including excitations away from the nodes. In fact, as discussed in detail in [22], the presence of supersymmetric points at certain doping concentrations, would favour superconductivity due to the suppression of potentially dangerous non-perturbative effects (instantons) of the compact statistical gauge field that would be responsible for giving the gauge field a mass, thereby destroying the superconducting nature of the gap. In the model of [3] such instanton configurations are unavoidable due to the non-Abelian nature of the gauge symmetry characterizing the spin-charge separation ansatz (1). In [22] a breakdown of superconductivity due to instanton effects has been interpreted as implying a “pseudogap” phase: a phase in which there is dynamical generation of a mass gap for the nodal holons, which, however, is not characterized by superconducting properties.

In this respect, the supersymmetric points [27], [14], for which such instanton effects are argued [22] to be strongly suppressed in favour of KT superconductivity, would constitute “superconducting stripes” in the temperature-doping phase diagram of the nodal liquid (see fig. [1] [3]). Theoretically, the stripes should have zero thickness, given that they occur for specific doping concentrations [44]. However, in practice, there may be uncer-

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4It should be stressed that the term “stripe” here is meant to denote a certain region of the temperature-doping phase diagram of the nodal liquid and should not be confused with the stripe structures in real space which characterizes the cuprates at special doping concentrations.
Figure 1: A possible scenario for the temperature-doping phase diagram of a charged, relativistic, nodal liquid in the context of spin-charge separation. At certain doping concentrations ($\delta_{SS}$) there are dynamical supersymmetries among the spinon and holon degrees of freedom, responsible for yielding thin “stripes” in the phase diagram (shaded region) characterized by Kosterlitz-Thouless (KT) superconductivity without a local order parameter. The diagram is conjectural at present. It pertains strictly to the nodal liquid excitations about the d-wave nodes of a superconducting gap, and hence, should not be confused with the phase diagram of the entire (high-temperature) superconductor.


tainties (due to doping dependences) in the precise value for the parameter $\Delta_z$ entering (44), which might be responsible for giving the superconducting stripe a certain (small) thickness. A detailed analysis of such important issues is still pending. It is hoped that due to supersymmetry one should be able to discuss some exact analytic results at least for zero temperatures.

7 Conclusions

From the above discussion it is clear that supersymmetry can be achieved in the effective continuum field theories of doped antiferromagnetic systems exhibiting spin-charge separation only for particular doping concentrations (cf. (27),(44)). One’s hope is that the ancestor lattice model will lie in the same universality class (in the infrared) as the continuum model, in the sense that it differs from it only by the action of renormalization-group irrelevant operators. This remains to be checked by explicit lattice calculations. We should note at this stage that this is a very difficult problem; in the context of four-dimensional particle-physics models it is still unresolved [23]. However, in view of the apparent simpler form of the three-dimensional lattice models at hand, one may hope that these models are easier to handle.
By varying the doping concentration in the sample, one goes away from the super-symmetric point and breaks supersymmetry explicitly at zero temperatures. At finite temperatures, or under the influence of external electromagnetic fields at the nodes of the d-wave gap, supersymmetry will also be broken explicitly. Therefore, realistic systems observed in nature will be characterized by explicitly broken supersymmetries even close to zero temperatures. However there is value in deriving such supersymmetric results in that at such points in the parameter space of the condensed-matter system it is possible to obtain analytically some exact results on the phase structure of the theory. Supersymmetry may allow for a study of the quantum fluctuations about some exact ground states of the spin-charge separated systems in a controlled way. Then one may consider perturbing around such exact solutions to get useful information about the non-supersymmetric models.

We have argued that such special points will yield KT superconducting “islands” in a temperature doping phase diagram of the nodal liquid, upon the dynamical generation of holon-spinon mass gaps (of equal size). This is due to special properties of the supersymmetry, associated with the suppression of non-perturbative effects of the (compact) gauge fields entering the spin-charge separation ansatz (\[1\]). This, of course, needs to be checked explicitly by carrying out the appropriate instanton calculations in the spirit of the non-perturbative modern framework of \[12\]. At present, such non-perturbative effects can only be checked explicitly in three dimensions for highly extended supersymmetric models \[26\]. It is, however, possible that some exact results could be obtained at least for the \( N = 2 \) supersymmetric models which may have some relevance for the effective theory of the nodal liquid at the supersymmetric points \[13\]. Then, one may get some useful information for the \( N = 1 \) models by viewing the latter as supersymmetry-breaking perturbations of the \( N = 2 \) models. Such issues remain for future investigations, but we hope that the speculations made in the present work provide sufficient motivation to carry out research along these directions.

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