Some Considerations about the Bouchaud-Cates-Ravi-Edwards model for Granular Flow

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Abstract

In this paper we discuss some features of the BCRE model. We show that this model can be understood as a mapping from a two-dimensional to a one-dimensional problem, if some conditions are satisfied. We propose some modifications that (a) guarantee mass conservation in the model (what is not assured in its original form) and (b) correct undesired behaviors that appear when there are irregularities in the surface of the static phase. We also show that a similar model can be deduced both from the principle of mass conservation (first equation) and a simple thermodynamic model (from which the exchange equation can be obtained). Finally, we solve the model numerically, using different velocity profiles and studying the influence of the different parameters present in this model.

Key words: granular flow, grains, sandpile
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1 Introduction

Since the end of 1980s, when Bak et al. used a sandpile as a paradigm for self-organized criticality[6], the interest in granular materials experimented a revival. The topic, however, is not recent. The first studies with granular materials date back to 1773, when Coulomb first observed that this kind of matter could stand in equilibrium in piles at certain specific angles and, after

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that, other famous physicists studied the topic. Faraday discovered the convective instability in vibrating grains and Reynolds introduced the notion of dilatancy, just to cite some examples.

The study of granular materials is important since their applications in industry is wide enough to cover areas as distinct as civil construction and food transportation. The study of flows can uncover the behavior of dunes, sandstorms and avalanches. But those are just a few examples. Granular materials are present everywhere in nature, and in several branches of industry. Chemical industries, pharmaceutics, mining, geology are just some more examples of other areas where the study of grains can lead to important results.

In 1994, a paper by Bouchaud, Cates, Ravi and Edwards [1] presented an important model that became known as the BCRE model. This model was successful in describing the qualitative behavior of flowing grains, and had the additional advantage of being very simple. It assumes that a two-dimensional sandpile (figure 1), with rolling grains on its surface, can be divided in two “phases” (a static phase $h$ and a rolling phase $R$) and propose two coupled partial differential equations to model their behavior:

$$\frac{\partial R(x,t)}{\partial t} = -\frac{\partial}{\partial x} \left[ cR(x,t) \right] + \frac{\partial}{\partial x} \left[ D \frac{\partial R(x,t)}{\partial x} \right] + \Gamma(R,\tilde{h}), \quad (1)$$

$$\Gamma(R,\tilde{h}) = -R(x,t) \left[ \gamma \frac{\partial \tilde{h}(x,t)}{\partial x} + \kappa \frac{\partial^2 \tilde{h}(x,t)}{\partial x^2} \right] = -\frac{\partial \tilde{h}(x,t)}{\partial t}, \quad (2)$$

where $\gamma, \kappa > 0$ and $\tilde{h} = h + x \tan \theta_r$, $\theta_r$ is the angle of repose.

The first equation defines how the profile of the rolling phase evolves in time, and the second equation determines the profile of the static phase by setting the form of the exchange between rolling and static grains, depending on the local slope of the pile. The model is phenomenological, and is obtained from educated guesses about the properties observed in real sandpiles and desired for real grains.

This model gave important clues to how we could describe some interesting phenomena occurring in granular flow. A variety of papers utilized these equations to model the behavior of avalanches, stratification and flows in general [3–5,9,10].

The original model, however, is very simplified and need some improvements to describe more general situations. Some problems of consistency must also be solved if we want to have a model that do not violate some general principles, as mass conservation. The model, also, still lacks from a derivation from first principles or from a microscopic point of view.
In this paper we will address some of the points mentioned above. In section 2, we will show that an equation equivalent to equation (1) can be obtained from the principle of mass conservation, under the assumption that the densities of the static and rolling phases are constant in the vertical coordinate. In section 3, we discuss some limitations of equation (2) and present some possible alternatives to it. In section 4, we analyze the consequences of considering different velocity profiles for the rolling phase and discuss the relation of them with other models for granular materials present in literature, evaluating the role played by some of the parameters present in the model. In section 5, we present a new and simple model to describe the mechanism that underline the exchange of grains between static and rolling phases. From it we were able to deduce an equation of the type of equation (2). Finally, in the last section, we summarize our conclusions.

2 Mass Conservation

Consider, for instance, a two-dimensional sandpile with a rolling and a static phase. The rolling phase is located above the static phase, and slides over it (see figure 1). Let us assume that the sandpile can be treated as a continuous medium, \( \rho_r \) being the density in area of the rolling phase. The mass inside an interval \( x_o \) to \( x_o + \delta x \), where \( \delta x \) is small, is then given by

\[
\delta m_r = \int_{x_o}^{x_o+\delta x} \left[ \int_{h(x,t)}^{h(x,t)+r(x,t)} \rho_r (x, y, t) \, dy \right] \, dx,
\]

with

\[
h (x, t) \equiv \text{height of the static phase in a point } x \text{ at time } t, \]

\[
r (x, t) \equiv \text{height of the rolling phase in a point } x \text{ at a time } t.
\]

We define the linear density of the rolling phase by

\[
R(x_o, t) = \lim_{\delta x \to 0} \frac{\delta m_r}{\delta x} = \lim_{\delta x \to 0} \frac{1}{\delta x} \int_{x_o}^{x_o+\delta x} \left[ \int_{h(x,t)}^{h(x,t)+r(x,t)} \rho_r (x, y, t) \, dy \right] \, dx. \quad (3)
\]

Remembering that, in general,

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{x}^{x+\varepsilon} f (u) \, du = f (x), \quad (4)
\]
we can write (3) as
\[ R(x_0, t) = \int_{h(x_0, t)}^{h(x_0, t) + r(x_0, t)} \rho_r(x_0, y, t) \, dy. \] (5)

When \( \rho_r \) is independent of the vertical coordinate \( y \), equation (5) becomes
\[ R(x, t) = \rho_r(x, t) \, r(x, t). \] (6)

Repeating this procedure for the static phase, we obtain
\[ S(x_0, t) \equiv \lim_{\delta x \to 0} \frac{\delta m_s}{\delta x} = \int_0^{h(x_0, t)} \rho_s(x_0, y, t) \, dy, \] (7)
and
\[ S(x, t) = \rho_s(x, t) \, h(x, t), \] (8)
where \( m_s \) is the mass and \( \rho_s \) (again independent of \( y \)) is the density in area of the static phase.

Equations (6) and (8) define a one-to-one mapping from \( h \) and \( r \) to \( S \) and \( R \) respectively, mapping the two-dimensional sandpile in a one-dimensional problem in the case that the densities of the two phases do not vary with \( y \).

A particular case that follows in this category, used frequently in literature, is when \( \rho_r = \rho_s = \text{constant} \). The possibility of this mapping is not really unexpected, since, with this properties, the variables involved will only depend on the horizontal coordinate \( x \) and time.

Under the assumption of a continuous model, we can write, for the rolling phase, a mass conservation equation. For a one-dimensional fluid of density \( R \) that flows under a velocity field \( v \) (if \( v \) also does not vary in the \( y \) coordinate)
\[ \frac{\partial R}{\partial t} + \frac{\partial}{\partial x} (vR) = Q, \]
where \( Q \) represents the sources or sinks of this fluid. The BCRE model allows the exchange between rolling and static grains. So, when describing the rolling phase, the static phase acts like a source/sink at every point (the extra mass gained by the rolling phase equals the mass lost by the static phase). We can
then write
\[ \frac{\partial R}{\partial t} + \frac{\partial}{\partial x} (vR) = -\frac{\partial S}{\partial t}. \] (9)

Comparing (1) and (9), we can immediately see that the last is a slightly modified version of the first, where, instead of the height \( h \), we are working with the density \( S \) and the diffusion term (if there is one), will now depend on the specific shape of the velocity field \( v \).

Equation (9) is better than equation (1), since it guarantees mass conservation for any form of \( v \) (what does not happen with equation (1)). If we assume, for instance, a constant velocity field in (1) (as has already been done in some previous works \([9,3,4]\)), the diffusing term must be dropped out in order to ensure mass conservation.

In 1997 Makse \([9]\) applied the BCRE model to a mixture of two grains. He wrote equation (1) – with constant velocity and without diffusion – for each kind of grain
\[ \frac{\partial R_i}{\partial t} + v_i \frac{\partial R_i}{\partial x} = \Gamma_i, \quad i = 1, 2, \] (10)

and simulated this model with \( v_1 = v_2 \), showing that, depending on the value of some parameters, the grains would segregate or stratify. The assumption \( v_1 = v_2 \) was not justified in that paper. To exemplify the advantages of our equations, applying (9) to a mixture of two grains, and considering that the mixture is rolling with a constant average velocity \( v \), we obtain
\[ \frac{\partial (R_1 + R_2)}{\partial t} + v \frac{\partial}{\partial x} (R_1 + R_2) = \Gamma, \]

that is similar to the equations obtained by Markse (10). But now it is easy to see that \( v_1 = v_2 \) is not an \textit{ad hoc} assumption, but rather, an imposition of the model.

One more advantage of (9) is that now we can obtain a whole series of different granular flow regimes by varying the velocity profile of the rolling phase. We will discuss this point better in section 4.

3 Exchange of grains between static and rolling phases

We will now make some considerations about the second equation of the BCRE model, that we call exchange equation. In analogy with (9) we will first write
this equation (2) in terms of the new variables \( R \) and \( S \)

\[
\frac{\partial S}{\partial t} = R \left[ \gamma \left( \frac{\partial S}{\partial x} + \rho_s \tan \theta_r \right) + \kappa \frac{\partial^2 S}{\partial x^2} \right].
\]  

(11)

In [5], Boutreux proposed a modified version of this equation, to take into account a shielding effect, present on the upper grains of the rolling phase, due to the lower grains of this same layer. Adapted to the variables \( R \) and \( S \), this shielding effect can be introduced in equation (11) as

\[
\frac{\partial S}{\partial t} = \frac{R \xi'}{R + \xi'} \left[ \gamma \left( \frac{\partial S}{\partial x} + \rho_s \tan \theta_r \right) + \kappa \frac{\partial^2 S}{\partial x^2} \right].
\]  

(12)

where \( \xi' \) is a small constant related to the thickness of the layer of rolling grains that indeed interact with the static phase. Note that, if \( R \sim \xi' \) (a thin rolling phase), \( R \xi'/(R + \xi') \sim R \), but if \( R \gg \xi' \), then \( R \xi'/(R + \xi') \sim \xi' \).

Equation (12) is still not adequate to describe what happens close to the interface, if there are irregularities on it. As the grains flow, they can erode part of the static phase, creating a (sometimes big) crater with a positive slope in the right border (see figure 2). To see that, let us analyze a particular case, where the densities are constant.

The term \( \partial S/\partial x \) is directly related to the slope of the pile (given by \( \partial h/\partial x \)). If it is negative, the pile is inclined to the right (and if it is positive the pile is inclined to the left). There is no problem when the slope is negative. As expected, for a local slope above the repose angle, we have erosion (and for a slope below it we have acrision). But, when this term is positive, we always have acrision, what is not a reasonable behavior. To correct that we suggest a modification, where we consider the minus sign of the modulus of \( \partial S/\partial x \)

\[
\frac{\partial S}{\partial t} = \frac{R \xi'}{R + \xi'} \left[ \gamma \left( \rho_s \tan \theta_r - \frac{\partial S}{\partial x} \right) \right] + \kappa \frac{\partial^2 S}{\partial x^2}.
\]  

(13)

4 Diffusion and velocity profiles

One interesting, important and observed property of the BCRE model is that it includes a diffusion of the rolling grains. The presence of diffusion in this phase is observed and, in fact, very obvious. This diffusion, however, leads to some constraints on the velocity profile \( v \).

The correct expression for the velocity field \( v \), in equation (9), should, in principle, be derived from a momentum conservation equation. However, writing
such an equation is not an easy task, since it should take into account all the interactions between the grains and would depend on the stress tensor of the material. In general, we can only say that the velocity field is a function of $x$ and $t$ that depends on a variety of factors. Most of the works present in literature have assumed a constant velocity profile, for simplicity.

We will, in this section, analyze two possible functional forms for $v$, showing some results of computer simulations and studying its effects in the shape of the pile.

In all computer simulations we integrate the BCRE equations numerically by means of a finite difference scheme (see appendix) and constructed an online animation of the profile of both phases in real time. The figures presented below are screenshots of the animation generated by the program. We considered the following profiles:

(1) $v = v(R) = \alpha \partial_x R + \beta R + \delta$, where $\alpha, \beta, \delta$ are constants.

Here $v$ is considered as being a function of $R$ only, and a kind of gradient expansion was done. Note that, in this case, $v$ is an implicit function of $x$ and $t$ ($v = v(R)$ where $R = R(x, t)$). Substituting in (9) we obtain

$$\frac{\partial R}{\partial t} + \frac{\partial}{\partial x} (vR) = \frac{\partial R}{\partial t} + \alpha \left(\frac{\partial R}{\partial x}\right)^2 + (2\beta R + \delta) \frac{\partial R}{\partial x} + \alpha R \frac{\partial^2 R}{\partial x^2}. \quad (14)$$

Note that this functional form of $v$ includes a diffusion term. It can also be seen from (14) that:

i) The diffusion coefficient is proportional to $R$. This means that the higher the pile, the more it will diffuse. This is reasonable, and can be understood as a consequence of gravity.

ii) Equation (14) presents a non-linear term with a constant coefficient $\alpha$, ($\alpha \neq 0$). When diffusion is small and can be neglected (as in most cases studied in literature), the non-linear term is also unimportant, and the equation above is in agreement with previous works. Diffusion, however, affects the profile of the pile. Figure 3 shows the effects of this term. It compares the profile of a pile of grains, at the same time and for the same initial condition, for:

(a) the original model with constant velocity and (b), (c) equation (14) with $v = \alpha \partial_x R$, for two different values of $\alpha$. We can see that the final shape of the bump is quite different if $\alpha$ is not too small.

iii) This equation has also an advective term, i.e., a term on the first derivative of $R$ with respect to $x$. Its coefficient depends linearly on $R$ (for $\beta \neq 0$) \[3\].

\[3\] An advective term like that had already been proposed in other papers, as in [7].
Figure 4 shows how it affects the profile of the pile. We can see that there is a tendency to the formation of shock fronts in one of the sides of the bump.

(2) \( v = f(\partial_x h) \)

In a real pile of grains, there may be irregularities with positive slope, due to erosion. In this case, the velocity must depend on the sign of the slope, or else we will have avalanches climbing up the pile at the points with positive slope, with the same velocity as in the negative slope side. To correct this defect, we considered the following form for \( v \)

\[
v \begin{cases} 
\alpha \partial_x R + \beta R + \delta, & \text{if } \partial_x h \leq 0 \\
v_s, & \text{if } \partial_x h > 0,
\end{cases}
\]

(15)

where \( v_s \) is a constant.

For \( \partial_x h \leq 0 \) this expression is equivalent to the previous form of \( v \). But, for \( \partial_x h \geq 0 \), the rolling grains, now, meet a barrier of static grains, and move up this barrier with a constant velocity. There is no diffusion in this case, since the velocity is constant. But now this is a desired property: the grains slowly accumulate in the barrier (and do not diffuse). Figure 5 shows the profile of the pile at the same time when (a) the velocity is independent of the slope and (b), (c) \( v \) depends on the slope according to (15) for two different values of \( v_s \). The static phase is gray, and has been settled to an irregular shape to amplify the consequences of the modifications introduced.

When we were at the end of this work, we came to know about a work of Herrmann and Sauermann, on the behavior of the Barchan Dunes [2], that was also based on the BCRE model, and dealt with some of the points we present in this paper. They present a two-dimensional version of (9) but they do not deduce it. In particular, there is no mention of the need of independence of the densities with the \( y \) coordinate. Indeed, we believe that, (although it is not explicitly written) the simulations were performed under constant densities which is a particular case of \( y \) independence. They also used a slightly different version of (2), where the modulus of the slope of the static phase were also employed.

5 Simple Model for the exchange of grains between rolling and static phases

It is possible, on the bases of a naive model, to deduce an equation to describe the exchange of grains between rolling and static phases. Remember that
the densities of the rolling and static phases are different, the latter being higher than the former. As the rolling phase rolls over the static phase, the friction between them takes energy out of the rolling grains. Part of this energy becomes heat, but the rest of it is transferred to the grains of the static phase. This energy will agitate these grains and the density of the static phase right below the interface will increase until it attains the critical dilatancy and starts to move, becoming part of the rolling phase. This process is similar to a solid to liquid first order phase transition, the static phase playing the role of the solid (receives energy and starts to "melt"). Indeed, there are experimental evidences that, at least in the case when the transition is induced by tilting, it does have features of a first order phase transition [8]. We will assume that this analogy is valid. We can then calculate the mass of static grains that will "melt" (that is, receives energy and starts to roll) using an analogy with the latent heat equation

\[ \delta Q = L \delta m_s, \]

where \( \delta Q \) is the energy acquired from the rolling phase, \( \delta m_s \) is the mass of static grains that melts and \( L \) is a constant, analogue to the latent heat.

Let us now focus on what happens in a little interval \( \delta x \) of the horizontal coordinate of the pile (see figure 1). The mass that is melted is a portion of the total mass of static grains. However, not all the static grains receive energy from the rolling phase. The upper grains shield the lower grains from the contact with the rolling phase. We will consider that the mass that can actually receive energy (and, therefore, melt) is a fraction \( \xi h \) of the static phase (we will justify this assumption, now adopted for the sake of simplicity, ahead).

So, the mass that can be melted is given by

\[ m_s = \rho_s V = \xi h \delta x \rho_s = \xi h \delta x \frac{S}{h}, \]

where equation (8) were used. Then, \( \delta m_s = \delta S \xi \delta x \) and

\[ \delta Q = L \xi \delta S \delta x, \tag{16} \]

where \( \delta S \) is the variation in \( S \) caused by the melting of the static phase. Note that when the pile is too high, internal forces and gravity act to de-estabilize it, increasing the melting. This can justify the assumption, made above, that \( m_s \) is proportional to \( h \) (instead of being a constant layer).

If all the energy to melt the static phase comes from the rolling phase, and if it is a fraction of the kinetic energy lost due to friction in the interface, we
have

$$\delta Q = c \delta K,$$  \hspace{1cm} (17)

where $\delta K$ is the kinetic energy lost by the rolling phase. But $K = \frac{p^2}{2m_r}$, where $m_r$ is the mass of the rolling phase and $p$ its momentum. So we have

$$\delta K = \frac{2p \delta p}{2m_r} = \frac{(m_r v) \delta p}{m_r} = v \delta p.$$  \hspace{1cm} (18)

Remembering that $m_r = R \delta x$, supposing that the rolling phase transfers energy to the static phase only by friction, and that the friction is proportional to the weight of the rolling phase at $x$, we have

$$\frac{dp}{dt} = \mu m_r g \Rightarrow \delta p = \mu m_r g \delta t = \mu g R \delta x \delta t,$$  \hspace{1cm} (19)

and, from (16), (17), (18) and (19), we have

$$L \xi \delta S \delta x = c \mu g v R \delta t \delta x.$$

That, in the limit $\delta t, \delta S \to 0$, gives

$$\frac{\partial S}{\partial t} = \beta v R,$$  \hspace{1cm} (20)

where $\beta = \frac{c \mu g}{L \xi}$ is a constant.

This is a quite simple expression for the exchange equation. It can assume a variety of forms, depending on the velocity field $v$. It incorporates the velocity field explicitly, indicating that the exchange of grains between both phases depends on the exact shape of $v$, what is very reasonable, since the velocity of the rolling grains interfere directly with the energy lost in the collisions, that, ultimately, are responsible for the transformation of static grains in rolling grains.

Note also that if (20) is substituted in the mass conservation equation, we get

$$\frac{\partial R}{\partial t} + v \frac{\partial R}{\partial x} = q,$$

where

$$q = \left( \beta v - \frac{\partial v}{\partial x} \right) R.$$
When the velocity is a function of \( x, t \) and \( R \) only, \( q \) will be, in principle, also a function of these three variables (because \( R \) is a function of \( x \) and \( t \) only), and the resulting equation will be a well known quasi-linear partial differential equation of first order in \( R \), that can be solved by the method of characteristics given by the simple system

\[
\frac{dt}{1} = \frac{dx}{v(x,t,R)} = \frac{dR}{q(x,t,R)}.
\]

The only difficulty is that, if \( v \) has an explicit dependence on \( R \), the variable \( q \) will have a dependence in \( \partial R/\partial x \), what may make the system of characteristics difficult to be solved analytically in practice. However, a solution for \( R(x,t) \) will give consequently a solution for \( S(x,t) \) by means of (20). We intend to further explore this point in a following paper.

Furthermore, if equations (20) and (13) are equivalent, the velocity field must assume the form of

\[
v = \frac{\xi'/\beta}{R + \xi'} \left[ \gamma \left( \rho_s \tan \theta_r - \left| \frac{\partial S}{\partial x} \right| \right) + \kappa \frac{\partial^2 S}{\partial x^2} \right].
\]

This velocity field has some interesting features. First, note that \( v \) depends on the slope of the pile. The first term inside the square brackets becomes positive for a slope below the angle of repose and negative above it, what means that the velocity is higher where the pile is steeper. Second, the factor \( (R + \xi')^{-1} \) suggests that \( v \) is inversely proportional to the weight of the pile of rolling grains, what is reasonable.

6 Conclusions

In the first two sections of this paper, we pointed out some problems with the equations of the BCRE model, suggesting the following modification:

\[
\frac{\partial R}{\partial t} + \frac{\partial}{\partial x} (vR) = -\frac{\partial S}{\partial t}, \tag{21}
\]

\[
\frac{\partial S}{\partial t} = \frac{R \xi'}{R + \xi'} \left[ \gamma \left( \rho_s \tan \theta_r - \left| \frac{\partial S}{\partial x} \right| \right) + \kappa \frac{\partial^2 S}{\partial x^2} \right]. \tag{22}
\]

The first equation explicitly assures mass conservation. The second equation was modified to take into account the signal of the slope in the static phase. In addition, we introduced the new variables \( R \) and \( S \), giving a precision
definition for them, a point that was not totally clear in the literature until now.

We used the corrected equations (21) and (22) to simulate two possible velocity fields, and found acceptable results.

Finally, we have proposed a model for the exchange of grains between the rolling and static phases. From it, we obtained the alternative \[ \frac{\partial S}{\partial t} = \beta v R \] for the exchange equation, that is simpler, includes explicitly the velocity field, and lead to interesting results.

The model presented is very simple, and we are aware that many of the hypothesis made must be examined in more detail. We assumed that the friction is proportional to the weight of the rolling phase, and this is surely oversimplified. Probably there is a more complicated dependence on other parameters of the model as well. For instance, it is reasonable to suppose that the energy transfered to the static phase also depends on the shape of the grains, its density, the toughness of the material and a variety of other factors.

Also, we think that the fraction of static grains that receives energy from the rolling phase is probably a more general function of \( h \), not to speak of a dependence on the other variables of the model.

We have also neglected the inverse process, i.e., the transformation of rolling grains in static ones. Our model may be good to describe an avalanching process, where the inertia of the rolling grains is large, but may fail to describe a more general situation.

We hope to address this points in a following paper. However, we think it is amazing and interesting that a somewhat rich expression for \( v \), like (20), can be obtained from such a naive model.

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7 Appendix

The equations of the BCRE model were integrated with the operator splitting method [11]. Suppose a differential equation of the form

\[ \frac{\partial u}{\partial t} = L(u), \]
where $L(u) = \sum_{i=1}^{m} L_i(u)$ is a generic non-linear operator that can be written as a sum of $m$ other operators, and $u$ is a function of $x$ and $t$. If we have a good method to integrate each of the equations $\frac{\partial u}{\partial t} = L_i(u)$, then $u^{n+1}$ can be obtained through $m$ successive time steps:

$$
\begin{align*}
    u^{n+(1/m)} &= L_1(u^n, \delta t), \\
    u^{n+(2/m)} &= L_2(u^{n+(1/m)}, \delta t) \\
    \vdots &= \vdots \\
    u^{n+1} &= L_m(u^{n+(m-1)/m}, \delta t).
\end{align*}
$$

For the first equation of the model (and its variants), the $L_i$ operators are of the form

$$
L_1(r) = f(r) \frac{\partial r}{\partial x}, \quad L_2(r) = g(r) \frac{\partial^2 r}{\partial x^2}, \quad L_3(r) = q(r) \quad \text{and} \quad L_4(r) = k \left( \frac{\partial r}{\partial x} \right)^2,
$$

where $f$, $g$ and $q$ are arbitrary functions of $r$, and $k$ is a constant.

$L_1$ and $L_2$ where integrated with variants of the Crank-Nicholson method; the operator $L_3$ was integrated with a fourth order Runge-Kutta procedure and finally $L_4$ was integrated with a FTCS finite-difference scheme.

The second equation can be split into two operators of the form:

$$
L_1(h) = a \frac{\partial^2 h}{\partial x^2}, \quad L_2(h) = b \frac{\partial h}{\partial x},
$$

where $a$ and $b$ are constants. Now $L_1$ was integrated with the aid of Crank-Nicholson and the operator $L_2$ with the aid of a two-step Lax-Wendroff [11] procedure.

The program was run on line in a Intel Pentium III 400 Hz or in a AMD K6II 400 Hz (about 10 seconds for a single run in both cases).

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Figure Captions

**Figure 1** In the BCRE model, a two-dimensional pile with flowing grains is divided into two “phases” (static and rolling phases). They are described, respectively, by the variables $h$ and $R = \rho r$, where $h$ and $r$ are the heights of the static and rolling phases, respectively.

**Figure 2** Eventually the flow of rolling grains can erode the static phase creating a crater, and generating regions of positive slope.

**Figure 3** Profile, at equivalent time $t$ and same initial conditions, of the evolution of a pile in cases where: (a) the original model was considered with constant velocity ($v = 1$); (b) our suggestion mass conservation (equation (9)) with $\alpha = -0.01$ and (c) $\alpha = -0.04$. Gray area corresponds to the static phase and white area to the rolling phase.

**Figure 4** Effects of diffusion. Profile, at equivalent time $t$ and same initial conditions, of the evolution of a pile with a velocity profile given by $v = \alpha \frac{\partial}{\partial x} R + \beta R + \delta$, with $\delta = 1$, $\alpha = 0.01$ and (a) $\beta = 2$ and (b) $\beta = -1$ (A negative $\beta$ means that the thicker the rolling phase, the more difficult it is for grains to roll). Gray area corresponds to the static phase and white area to the rolling phase.

**Figure 5** Profile, at equivalent time $t$ and same initial conditions, of the evolution of a pile in the cases where (a) the velocity is independent of the sign of slope and (b) the velocity depends on the sign of the slope and $v_s = 0.9$ and (c) $v_s = 0.4$. Gray area corresponds to the static phase and white area to the rolling phase.
Figure 2

Static phase

positive slope

Rolling phase

Static phase
Figure 4
Figure 5