Roughening and inclination of competition interfaces

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We study the competition interface between two clusters growing over a random vacant sector of the plane in a simple set-up which allows us to perform formal computations and obtain analytical solutions. We demonstrate that a phase transition occurs for the asymptotic inclination of this interface when the final macroscopic shape goes from curved to non curved. In the first case it is random while in the second one it is deterministic. We also show that the flat case (stationary growth) is a critical point for the fluctuations: for curved and flat final profiles the fluctuations are in the KPZ scale (2/3); for non curve final profile the fluctuations are in the same scale of the fluctuations of the initial conditions, which in our model are Gaussian (1/2).

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Introduction  The behavior of the interface of a growing material has been investigated using the Eden model [1], ballistic deposition and other random systems. Typically, the growing region converges to an asymptotic deterministic shape and its fluctuations depend on the geometry of the initial condition [11-13]. A less well studied phenomenon is the competing growth of two materials. The interface between two growing clusters (competition interface) presents a random direction on the same scale as the deterministic shape [3-7]. In this letter we describe quite explicitly this phenomenon in a simple model. On grounds of universality, this will provide a guide to understand the interplay between the asymptotics of the competition interface and the final macroscopic shape in models with different growth and competition mechanisms.

We determine the inclination of the competition interface for a growth model called “last passage percolation” in a random sector of the plane of angle \( \theta \). The growth interfaces are mapped into particle configurations of the totally asymmetric simple exclusion process in one dimension (TASEP) [10]. Under Euler space-time rescaling, the particle density of the TASEP converges to a solution of the Burgers equation. This equation has travelling wave solutions (shocks) corresponding to the case \( \theta > 180^\circ \), and rarefaction fronts corresponding to \( \theta < 180^\circ \). A perturbation at one site of the initial particle configuration (called a second class particle) follows a characteristic of the equation or the path of a shock. To establish our results, we map the competition interface linearly onto the path of the second class particle.

The growth model  The random sector is parametrized by the asymptotic slope of its sides. Let \( \lambda \in (0,1] \) and \( \rho \in [0,1) \) and define a random path \( \gamma_0 = (\gamma_0(j))_{j \in \mathbb{Z}} \subseteq \mathbb{Z}^2 \) with \( \gamma_1(0) = (1,0) \), \( \gamma_0(0) = (1,1) \), \( \gamma_{-1}(0) = (0,1) \) as follows. Starting from \( (0,1) \), walk one unit up with probability \( \lambda \) and one unit left with probability \( 1 - \lambda \), repeatedly, to obtain \( \gamma_0^1 = (\gamma_0(j))_{j < 0} \). Then, starting from \( (1,0) \) walk down with probability \( \rho \) and right with probability \( 1 - \rho \) to get \( \gamma_0^2 = (\gamma_0(j))_{j > 0} \). \( \gamma_0^1 \) has asymptotic orientation \( (\lambda - 1, \lambda) \) while \( \gamma_0^2 \) has asymptotic orientation \( (1 - \rho, -\rho) \). Let \( C_0 \) be the sector with boundary \( \gamma_0 \), containing the first quadrant; its asymptotic angle \( \theta = \theta_{\lambda, \rho} \in [90^\circ, 270^\circ] \) is the angle between \( (\lambda - 1, \lambda) \) and \( (1 - \rho, -\rho) \). Notice that \( \theta \in [90^\circ, 180^\circ] \) if and only if \( \rho < \lambda \).

The path \( \gamma_0 \) is the growth interface at time 0. The dynamics are then defined as follows. For each \( z \in C_0 \) and each \( t \geq 0 \), we have a label \( \sigma(z) \in \{0,1,2\} \). The label is 0 if \( z \) is unoccupied at time \( t \), and is 1 or 2 if \( z \) belongs to cluster 1 or 2 respectively. Once occupied, a site remains occupied and keeps the same value forever. Initially, set \( \gamma_0(z) = 1 \) for all \( z \in \gamma_0^1 \), \( \sigma_0(z) = 2 \) for all \( z \in \gamma_0^2 \) and \( \sigma_0(z) = 0 \) for all \( z \in C_0 \setminus \gamma_0 \). Independently each vacant site \( z \in C_0 \setminus \gamma_0 \) becomes occupied with rate 1 provided \( z - (1,0) \) and \( z - (0,1) \) are occupied. Let \( G(z) \) be the time at which site \( z \) becomes occupied. At this time \( \sigma(z) \) assumes the value \( \sigma_0(z) \) where \( z \) is the argument that maximizes \( G(z - (1,0)) \) or \( G(z - (0,1)) \). Thus when a site becomes occupied it joins the cluster of whichever of its two neighbours (below and to the left) became occupied more recently. The label of the site (1,1) may be left ambiguous, but we stipulate that site 1,2 always joins cluster 1, and site 2,1 always joins cluster 2.

The process \( (G^1_t,G^2_t) \), where \( G^k_t \) is the set of sites \( z \in C_0 \) such that \( \sigma_t(z) = k \), describes the competing spatial growth model. The growth interface at time \( t \) is the polygonal path \( \gamma_t \) composed of sites \( z \in C_0 \) such that \( G(z) \leq t \) and \( G(z + (1,1)) > t \). The competition inter-
is a Markov process in the state space $\eta$ and $U$.

Simple exclusion and second-class particles

The totally asymmetric simple exclusion process $(\eta_t, t \geq 0)$ is a Markov process in the state space $\{0, 1\}^Z$ whose elements are particle configurations. $\eta(j) = 1$ indicates a particle at site $j$ at time $t$, otherwise $\eta(j) = 0$ (a hole is at site $j$ at time $t$). With rate 1, if there is a particle at site $j$, it attempts to jump to site $j+1$; if there is a hole at $j+1$ the jump occurs, otherwise nothing happens. The basic coupling between the two exclusion processes with initial configurations $\eta_0$ and $\eta'_0$ is the joint realization $(\eta, \eta')$ obtained by using the same potential jump times at each site for the two different initial conditions. Let $\eta_0$ and $\eta'_0$ be configurations of particles differing only at site $X(0) = 0$. With the basic coupling, the configurations at time $t$ differ only at a single site $X(t)$, the position of a so-called second-class particle. Such a particle jumps one step to its right to an empty site with rate 1, and jumps backwards one step with rate 1 when a (first class) particle jumps over it.

If $\eta_0$ is distributed according to the Bernoulli product measure with density $\lambda$ for $j \leq 0$ and $\rho$ for $j > 0$, then the asymptotic behavior of $X(t)$ shows a phase transition in the line $\lambda = \rho$: with probability one,

$$\lim_{t \to \infty} \frac{X(t)}{t} = \begin{cases} 1 - \rho - \lambda & \text{if } \lambda \leq \rho \\ U & \text{if } \lambda > \rho \end{cases} \quad (3)$$

where $U$ is a random variable uniformly distributed in $[1 - 2\lambda, 1 - 2\rho]$ ($[1, 1]$ for the deterministic case and $[\lambda, \rho]$ for the random case).

The limits are based on the following hydrodynamic limits. If $\eta_0$ is distributed with the product measure with densities $\lambda$ and $\rho$ as before, then the macroscopic density evolution is governed by the Burgers equation:

$$\lim_{\epsilon \to 0} \sum_{x \in \mathbb{Z}} f(x) \eta_{\epsilon}(x) = \int_{\mathbb{R}} f(r) u(r, t) dr \quad (4)$$

with probability one for all $f : \mathbb{R} \to \mathbb{R}$ with compact support, where $u(r, t)$ is the solution of the Burgers equation

$$\frac{\partial u(r, t)}{\partial t} + \frac{\partial}{\partial r} (u(r, t)(1 - u(r, t))) = 0, \quad r \in \mathbb{R}, \ t \geq 0$$

with initial condition $u(r, 0) = \lambda$ for $r \leq 0$ and $\rho$ for $r > 0$. If $\lambda = \rho$ the solution is constant, if $\lambda < \rho$ it is a shock:

$$u(r, t) = \begin{cases} \lambda & \text{if } r \leq (1 - \lambda - \rho)t \\ \rho & \text{if } r > (1 - \lambda - \rho)t \end{cases} \quad (5)$$

and it is a rarefaction front if $\lambda > \rho$:

$$u(r, t) = \begin{cases} \lambda & \text{if } r \leq (1 - 2\lambda)t \\ 1 - \frac{r}{2(1 - \rho)t} & \text{if } (1 - 2\lambda)t < r \leq (1 - 2\rho)t \\ \rho & \text{if } r > (1 - 2\rho)t \end{cases} \quad (6)$$

for initial product measures and $\eta'_0$ for initial measures satisfying $\eta'_0$ with $t = 0$; also $\eta'_0$)

The characteristics $v(a, t)$, corresponding to the Burger's equation and emanating from $a$, are the solutions of $dv/dt = 1 - 2u(v, t)$ with $v(0) = a$. The solutions are constant along the characteristics. When two characteristics carrying a different solution meet, they give rise to a shock. There is only one characteristic emanating from locations where the initial data is locally constant and there are infinitely many characteristics when there is a decreasing discontinuity. In particular, if the initial condition is $u(r, 0) = \lambda$ for $r < 0$ and $u(r, 0) = \rho$ for $r \geq 0$, then the characteristics $v_r(t)$ emanating from the point $v_r(0) = r$ are given by $v_r(t) = r + (1 - 2\lambda)t$ if $r < 0$ and $v_r(t) = r + (1 - 2\rho)t$ if $r > 0$. For $r = 0$ there are two cases. When $\lambda \leq \rho$, the characteristics emanating from positive sites are slower than those emanating from negative sites. They collide, giving rise to a shock traveling at speed $1 - \lambda - \rho$. When $\lambda > \rho$ there are infinitely many characteristics emanating from the origin.
for each $s \in [1 - 2\lambda, 1 - 2\rho]$ the line $v_0(t) = st$ is a characteristic emanating from 0. The limits show that the second-class particle follows the characteristic when there is only one (that is, when $\lambda = \rho$), that it follows the shock when the initial condition has an increasing discontinuity and that it chooses uniformly one of the characteristics emanating from a decreasing discontinuity.

**Growth and simple exclusion** Rost relates the simple exclusion process to the growth model as follows. Consider initial configurations $\eta_0$ for the exclusion process in which $\eta_0(0) = 0$ and $\eta_0(1) = 1$. Elsewhere let $\eta_0$ be distributed according to Bernoulli product measure with density $\lambda$ for $j < 0$ and $\rho$ for $j > 1$. Define the initial growth interface $\gamma_0$ by $\gamma_0(0) = (1, 1)$ and $\gamma_0(j) - \gamma_0(j - 1) = (1 - \eta_0(j), -\eta_0(j))$; then $\gamma_0$ has the same distribution as before. Label the particles sequentially from right to left and the holes from left to right, with the convention that the particle at site 1 and the hole at site 0 are both labeled 1. Let $P_j(0)$ and $H_j(0)$, $j \in \mathbb{Z}$ be the positions of the particles and holes respectively at time 0. The position at time $t$ of the $j$th particle $P_j(t)$ and the $i$th hole $H_i(t)$ are functions of the variables $G(z)$: $\eta \in \mathbb{R}^2 : s = h(r)$ is related to the hydrodynamic limit by $h'(r) = u(r, 1)/(1 - u(r, 1))$.

**Second class particles and competition interfaces** A key tool in proving is the observation that the process given by the difference of the coordinates of the competition interface $I(t) - J(t)$ behaves exactly as the second class particle initially put at the origin. To see this call the particle at site 1 *particle and the hole at site 0 *hole, and call this couple *pair. The dynamics of the *pair is the following: it jumps to the right when the *particle jumps to the right, and it jumps to the left when a particle jumps from the left onto the *hole. The *pair then behaves as a second class particle. The only difference is that it occupies two sites while the second class particle occupies only one. The labels of the *particle and *hole change with time. At time 0 they both have label 1 and the labels of the *pair are represented by the point $\varphi_0 = (1, 1)$, the initial value of the competition interface. If, say, $G(2, 1) < G(1, 2)$, then the *particle jumps over the second hole before the second particle jumps over the *hole (see Figure). In this case, the labels of the *pair at time $G(2, 1)$ are $(2, 1)$, which is exactly the argument that minimizes $\{G(2, 1), G(1, 2)\}$; thus, after the first jump of the *pair, its labels are given by $\varphi_1$. By recurrence, $\varphi_n$ gives exactly the labels of the *pair after its $n$th jump. Therefore the labels of the *particle and *hole are $J(t)$ and $I(t)$, respectively. In addition, $J(t) - 1$ is exactly the number of jumps that the *pair has made backwards up to time $t$, and $I(t) - 1$ is the number of its jumps forwards. This shows that if the

![FIG. 2: Pair representation of second class particle.](image)
let $w_z = G(z) - \max\{G(z-1,0), G(z - (0,1))\}$. Then $(w_z, z \in C_0(\gamma_0))$ is a sequence of i.i.d random variables with an exponential distribution of mean 1. Let $\Pi(z, z')$ be the set of all directed polymers (or up-right paths) $(z_1, \ldots, z_n)$ connecting $z$ to $z'$, and let $G(z', z)$ be the maximum over all $\pi \in \Pi(z', z)$ of $t(\pi)$, the sum of $w_z$ along the polymer $\pi$. Each site $z$ has energy $-w_z$, and the polymer $\pi$ has energy $-t(\pi)$. Thus $-G(z', z)$ is the minimal energy, or ground state, between $z'$ and $z$. There exists a unique polymer $M(z', z) \in \Pi(z', z)$ that attains the maximum. We say that the semi-infinite polymer $(\gamma_0, z)_n$ is maximizing if for all $n < m$ we have $(z_0, \ldots, z_m) = M(z_n, z_m)$. Every semi-infinite maximizing polymer $(\gamma_0, z)_n$ has an asymptotic inclination $e^{\alpha_k}$.

In the competition model, $G(z) = G(\gamma_0, z)$ and for $k = 1, 2, G^k_{\infty}$ is the set of sites $z$ such that $M(\gamma_0, z)$ originates from $\gamma^0_0$. Denote by $\xi$ the roughening exponent of semi-infinite maximizing polymers.

For $\theta = 180^\circ (\lambda = \rho)$ the process is stationary and the connection is explicit. Running the process forward and backward we extend $G(z)$ to all $z \in \mathbb{Z}^2$; $G^+ = (G(z), z \in \mathbb{Z}^2)$ and $G^- = (-G(-z), z \in \mathbb{Z}^2)$ are identically distributed. We define the forward competition interface starting at $z$, $\varphi^z = (\varphi^z_n)_n$, by setting $\varphi^z_0 = z$ and putting $\varphi^z_{n+1}$ equal to the argument of the minimum between $G(\varphi^z_n + (1,0))$ and $G(\varphi^z_n + (0,1))$, and the backward semi-infinite polymer starting at $z$, $M^z = (M^z_n)_n$, by setting $M^z_0 = z$ and putting $M^z_{n+1}$ equal to the argument of the maximum between $G(M^z_n - (1,0))$ and $G(M^z_n - (0,1))$. Note that $\varphi = \varphi(1,1)$ and that $M^z$ is a semi-infinite maximizing polymer. Together with the duality relation $\varphi^z(G^+) = M^z(G^-)$, this shows that the forward competition interface has the same law as the backward semi-infinite maximizing polymer and, in particular, they have the same fluctuations, so that $\chi = \xi$.

For $\theta < 180^\circ (\lambda > \rho)$ the competition interface $\varphi$ is enclosed by two semi-infinite maximizing polymers $M^1$ and $M^2$ starting from $\gamma^0_0$ and $\gamma^0_0$, respectively, and with the same inclination $\xi$. Therefore $\chi \leq \xi$ in this case.

**Conclusions**

The connections studied above between the competition interface, the second class particle and maximal polymers fit into the interplay between the fluctuation statistics and the global geometry of the growth interface developed by Prahofer and Spohn. If the final macroscopic profile is curved then the competition interface follows a random direction (characteristic) intersecting the final surface at a point with non-zero curvature. In this case we have the KPZ scaling and the competition interface gets the transversal fluctuations, indicating the exponent $\chi = 2/3$. If the macroscopic profile is not curved we have two different situations. In the flat case (stationary growth) the competition interface also follows the characteristics of the associated hydrodynamic PDE and we still have the KPZ scaling. In the shock case the competition interface gets the longitudinal fluctuations which, in this case, are produced by the Gaussian fluctuations ($\chi = 1/2$) of the initial profile. On microscopic grounds one might suggest different rules for growth and competition. By universality we expect that from the knowledge of the curvature of the final macroscopic shape one can infer the asymptotics of the competition interface. This fits with the exponents founded by Derrida and Dickman in the Eden context since, in this case, the macroscopic profile is curved for angles $\theta > 180^\circ$ (we notice that in their simulations they have considered periodic initial conditions and so the longitudinal fluctuations in the shock direction are governed by the exponent $1/3$).

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