Stationary Lifshitz Black Hole of New Massive Gravity

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(Dated: December 13, 2018)

Abstract

I present the stationary Lifshitz black hole solution of three-dimensional New Massive Gravity theory and study its elementary geometric and thermodynamical properties.

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I. INTRODUCTION

As well known, the celebrated AdS/CFT correspondence has also been utilized in non-relativistic condensed matter systems \([1, 2]\) (and the references therein). In this regard, the anisotropic scaling symmetry (also called the \textit{Lifshitz symmetry})

\[
t \mapsto \lambda^z t, \quad \rho \mapsto \frac{\rho}{\lambda}, \quad \vec{x} \mapsto \lambda \vec{x},
\]

where \(z > 1\) is called the \textit{dynamical exponent}, has been successfully imposed on boundary field theories with a corresponding bulk described by the static Lifshitz spacetime metric \([3]\)

\[
ds^2 = -\frac{\rho^{2z}}{\ell^2 z} dt^2 + \frac{\ell^2}{\rho^2} d\rho^2 + \frac{\ell^2}{\rho^2} d\vec{x}^2.
\] (1)

On the gravity side, one either needs (various types of) matter couplings \([3–5]\) and/or higher curvature models \([6, 7]\) to support the Lifshitz spacetime \((1)\) and/or analytic or numerical Lifshitz black hole and black brane solutions. In this regard, black hole solutions are special since they describe the finite temperature behavior of the dual non-relativistic field theories. However, there are only a few \textit{exact} static and, even less number of, stationary Lifshitz black holes known \([6, 7]\).

This work provides an important addition to this modest list of exact Lifshitz black holes: I present the stationary Lifshitz black hole of three-dimensional New Massive Gravity (NMG) theory, and study its basic geometric and thermodynamical properties. It is worth emphasizing that this solution can be used as a test case for investigating discrepancies between various methods for calculating gravitational charges of spacetimes with non-standard (in particular, non-AdS, and in general, anisotropic) asymptotics relevant for generalized (especially, non-relativistic) holography.

Briefly stated, NMG, the gravitational model of interest in this work, was originally introduced \([8]\) as a \textit{parity-preserving} and \textit{unitary} solution to the problem of consistently extending the Fierz-Pauli field theory for a massive spin-2 particle to include interactions. To this end, the source-free NMG action was obtained by adding a specially-chosen quadratic term to the cosmological Einstein-Hilbert piece and reads \([8]\)

\[
I_{\text{NMG}} = \int d^3x \sqrt{-g} \left( R - 2\Lambda_0 + \frac{1}{m^2} (S^{\mu\nu} S_{\mu\nu} - S^2) \right).
\] (2)

Here \(\Lambda_0\) is the cosmological constant (with dimensions \(1/\text{Length}^2\)), \(m\) is a mass parameter (with
dimensions $1/\text{Length}$, and the Schouten tensor $S_{\mu\nu}$ and its trace $S$ are given by

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}, \quad S \equiv g^{\mu\nu} S_{\mu\nu} = \frac{R}{4}.$$ (3)

The organization of the paper is as follows: In section II, I show how the stationary Lifshitz black hole can be obtained from the static Lifshitz black hole by a simple boost, and discuss its basic geometric properties. Section III is devoted to the calculation of the thermodynamical quantities of the stationary Lifshitz black hole, and on how the first law of black hole thermodynamics can be utilized to conjecture the energy and the angular momentum of this black hole. I then finish up with a discussion of possible future work. I give the technical details on the properties of the cubic polynomial that is essential for the derivation of the stationary Lifshitz black hole in appendix A, and examine the geodesics of the stationary Lifshitz black hole and compare them to the geodesics of the static one in appendix B.

II. LIFSHITZ SPACETIMES AND LIFSHITZ BLACK HOLES OF NMG

The field equation that follows from the variation of the action (2) is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \frac{1}{m^2} K_{\mu\nu} = 0,$$ (4)

where $K_{\mu\nu} \equiv \Box S_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} S + S S_{\mu\nu} - 4 S_{\mu\rho} S_{\nu}^\rho + \frac{1}{2} g_{\mu\nu} (3 S^{\rho\sigma} S_{\rho\sigma} - S^2)$. It was shown in [6] that, for the special choice

$$\Lambda_0 = -\frac{13}{2\ell^2}, \quad m^2 = \frac{1}{2\ell^2}$$ (5)

of the parameters, the static Lifshitz black hole

$$ds^2 = -\rho^6 \left( 1 - \frac{M \ell^2}{\rho^2} \right) dt^2 + \frac{d\rho^2}{(\rho^2 - M)} + \rho^2 d\theta^2$$ (6)

is a solution of the NMG field equations (4). Note that when the parameter $M$ is set to zero in (6), one is led to the static Lifshitz spacetime (with dynamical exponent $z = 3$) [3]

$$ds^2 = -\rho^6 \frac{\ell^6}{\ell^6} dt^2 + \frac{\ell^2}{\rho^2} d\rho^2 + \rho^2 d\theta^2.$$ (7)
Now let me rewrite (6) by making the coordinate transformation $\rho^2 = x$ as

$$ds^2 = -\frac{x^3}{\ell^6} \left( 1 - \frac{M \ell^2}{x} \right) dt^2 + \frac{dx^2}{4 x \left( \frac{x^2}{\ell^4} - M \right)} + x d\theta^2,$$

(8)

and simply boost (8) via

$$\begin{pmatrix} dt \\ d\theta \end{pmatrix} \rightarrow \frac{1}{\sqrt{1 - \omega^2}} \begin{pmatrix} 1 & -\omega \ell \\ -\omega/\ell & 1 \end{pmatrix} \begin{pmatrix} dt \\ d\theta \end{pmatrix},$$

(9)

where the “rotation parameter” $\omega$ is a real constant with $|\omega| < 1$, to arrive at the stationary metric

$$ds^2 = -\frac{dt^2}{(1 - \omega^2)} \left( \frac{x^3}{\ell^6} - \frac{M x^2}{\ell^4} \right) + \frac{2 \omega \ell dt d\theta}{(1 - \omega^2)} \left( \frac{x^3}{\ell^6} - \frac{M x^2}{\ell^4} - \frac{x}{x^2/\ell^4} \right) + \frac{\ell^2 d\theta^2}{(1 - \omega^2)} \left( \frac{x}{\ell^2} - \frac{\omega^2 x^3}{\ell^6} + \frac{M \omega^2 x^2}{\ell^4} \right) + \frac{dx^2}{4 x \left( \frac{x^2}{\ell^4} - M \right)}.$$

(10)

Let me identify the coefficient of the $d\theta^2$ term in (10) by introducing the coordinate transformation $x = x(r)$ that is described by the cubic polynomial\(^1\)

$$x + \frac{M \omega^2}{\ell^2} x - \frac{\omega^2}{\ell^4} x^3 - (1 - \omega^2) r^2 = 0,$$

(11)

such that (10) can be written as

$$ds^2 = -\frac{dt^2}{\omega^2 \ell^2} \left( (1 + \omega^2) x(r) - r^2 \right) - \frac{2 dt d\theta}{\omega \ell} (r^2 - x(r)) + r^2 d\theta^2 + \frac{(x')^2 dr^2}{4 x(r) \left( \frac{x'(r)}{\ell^2} - M \right)}.$$

(12)

Here prime denotes differentiation with respect to the coordinate $r$. Since the metric (12) and the polynomial (11) are both left invariant under $r \mapsto \omega r$ (and independently under $\omega \mapsto \omega$), one can think of the variables $(r, \theta)$ as the polar coordinates on the Euclidean plane with the ranges $r \geq 0$ and $\theta \in [0,2\pi)$, and assume $0 \leq \omega < 1$ without loss of generality. Here the temporal coordinate $t$ takes any real value $t \in \mathbb{R}$, and the metric is circularly-symmetric with Killing vectors $(\partial/\partial t)^\mu$ and $(\partial/\partial \theta)^\mu$.

Note that even though the static Lifshitz black hole (6) and, of course, the static Lifshitz spacetime (7) enjoy the Lifshitz scaling symmetry $(t \mapsto \lambda^3 t, \rho \mapsto \rho/\lambda, \theta \mapsto \lambda \theta)$ provided $M \mapsto M/\lambda^2$ as well [6], this is no longer so for the stationary metric (10) (with the understanding that $x \mapsto x/\lambda^2$).

\(^1\)The properties of the cubic polynomial (11) are studied in detail in appendix A. The existence of at least one real root $x(r)$ is guaranteed of course.
As a side remark, one can also keep $\omega$ "on", i.e. $\omega \neq 0$, but switch-off $M$ in (10), that is boost the static Lifshitz spacetime (7) (written in terms of the variable $x$) by (9), to arrive at what I will call as stationary Lifshitz spacetime\(^2\). Going backwards, this results merely in setting $M = 0$ in (12) and (11), of course.

The stationary metric (12) (or (10)) has a curvature singularity at $r = 0$ (or $x = 0$), which can also be seen from the curvature invariants

\[
R = \frac{26}{\ell^2} + \frac{8M}{x(r)}, \quad R_{\mu\nu}R^{\mu\nu} = 4 \left( \frac{65}{\ell^4} - \frac{38M}{\ell^2 x(r)} + \frac{6M^2}{x^2(r)} \right).
\]

As shown explicitly in appendix A, the cubic polynomial (11) is guaranteed to have at least one real root, so the function $x(r)$ in (12) is indeed well-defined. The $rr$-component of (12) diverges when $x(r) = \ell^2 M$ and for $M > 0$ this leads to the coordinate singularity at $r_+ \equiv \ell \sqrt{M/(1 - \omega^2)} > 0$, describing the event horizon. Thus I call the metric (12) (or equivalently (10)\(^3\)) as the stationary Lifshitz black hole of NMG, even though it is neither left invariant under the ‘proper’ Lifshitz scalings as pointed out earlier nor asymptotically Lifshitz as one formally takes $x$ (or $r$) $\to \infty$\(^4\), since it derives from the static Lifshitz black hole (6).

For the sake of convenience, I summarize the four metrics described in this section, and the relations between them, in Fig. 1, and present the stationary Lifshitz black hole explicitly using

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\(^2\)Here the token ‘stationary Lifshitz spacetime’ is an obvious misnomer since switching-off $M$ does not help in restoring the Lifshitz scaling symmetry lost with the turning-on of $\omega$. Please refer to the last sentence of the next paragraph for the rationale behind this choice.

\(^3\)For the interpretation of (10) still as a black hole, one must implicitly assume that the coordinate $x$ in (10) is in the range $0 < x < x_+$ (A3). One must not take (10) on its own and assume wrongly that $x \in \mathbb{R}$ here.

\(^4\)See the previous footnote on the range of the variable $x$. 

FIG. 1. The static and stationary Lifshitz spacetimes and Lifshitz black holes of NMG
the \( \rho \)-coordinate as well:

\[
\begin{align*}
\text{ds}^2 &= -\frac{dt^2}{(1 - \omega^2)} \left( \frac{\rho^6}{\ell^6} - \frac{M \rho^4}{\ell^4} - \frac{\omega^2 \rho^2}{\ell^2} \right) + 2 \frac{\omega \ell^2 dt \, d\theta}{(1 - \omega^2)} \left( \frac{\rho^6}{\ell^6} - \frac{M \rho^4}{\ell^4} - \frac{\rho^2}{\ell^2} \right) \\
&+ \frac{\ell^2 d\theta^2}{(1 - \omega^2)} \left( \frac{\rho^2}{\ell^2} - \frac{\omega^2 \rho^6}{\ell^6} + \frac{M \omega^2 \rho^4}{\ell^4} \right) + \frac{d\rho^2}{(\frac{\rho^2}{\ell^2} - M)}.
\end{align*}
\]

(13)

Finally, for ‘geometers at heart’ I briefly discuss the geodesics of the stationary Lifshitz black hole (13) and compare these with the geodesics of the static Lifshitz black hole (6) in appendix B.

III. THERMODYNAMICS OF THE STATIONARY LIFSHITZ BLACK HOLE

I now turn to the question/challenge of examining the thermodynamics of the stationary Lifshitz black hole. For that purpose, it is useful to review the analogous properties of the static Lifshitz black hole first.

It was in [9] that the energy of the static Lifshitz black hole was calculated first. The authors of [9] employed the so-called boundary stress tensor method, but did so with a non-trivial caveat: The counterterm they obtained was not uniquely determined; one could put, at best, two physical conditions to determine the three free parameters that needed to be fixed. The authors of [9] had to resort to the validity of the first law of thermodynamics, \( dE = TdS \), to get over this ambiguity. In hindsight, it is easy to see that one could in fact do away with the boundary stress tensor method altogether, calculate the temperature \( T \) and the entropy \( S \) through standard methods (e.g. using the Wald entropy [10]) and arrive at the energy of the static Lifshitz black hole with relative ease.

As an alternative, the authors of [11] have instead performed a dimensional reduction (by exploiting the circular symmetry of the static Lifshitz black hole (6)) of the NMG theory to arrive at a complicated two-dimensional dilaton gravity, studied the properties of the analogous black hole obtained so in two dimensions, and showed that the thermodynamics of the original black hole in three dimensions could be consistently derived from there. The upshot of both of these calculations is that (in the units that I adopt in this work) the relevant thermodynamic quantities of the static Lifshitz black hole read

\[
T = \frac{M^{3/2}}{2\pi \ell}, \quad S = \frac{2\pi \ell \sqrt{M}}{G}, \quad E = \frac{M^2}{4G},
\]

(14)

where \( G \) denotes the three-dimensional Newton’s constant, and indeed the first law of thermodynamics \( dE = TdS \) holds.
Recognizing the need for a more direct approach for the computation of conserved charges (such as energy and angular momentum) of spacetimes that do not asymptote to spaces of maximal symmetry (such as Minkowski or AdS spaces) but instead to exotic ones such as Lifshitz spaces, it was in [12] that an extension of the conserved Killing charge definition of the ADT procedure [13] was given and developed for quadratic curvature gravity models in generic dimensions. There it was also shown that this extension is background gauge invariant and reduces to the one in [13] when the background is a space of constant curvature. To cut a long story short, application of this hands-on approach to the static Lifshitz black hole [12, 14] led to the energy
\[ E = \frac{7M^2}{8G}, \] (15)
which is clearly different from (14) and not in accord with the first law of thermodynamics. This is quite discouraging, to say the least, but as discussed in [14], may stem from a number of reasons. Instead of going over that discussion once more, let me point out to the most obvious one here: Any number of hypotheses, especially “the assumption that the deviations vanish sufficiently fast as one asymptotically approaches to the boundary of spacetime described by the background” and/or “the applicability of the Stokes’ theorem on the relevant hypersurfaces and/or boundaries”, which were crucial in the derivation of the extended definition of conserved charges [12] at the first place, may be violated by exotic spacetimes such as Lifshitz black holes.

Despite the disappointment in the discrepancy between (14) and (15), the extended definition of the Killing charge was quite successful in directly determining the energy of the warped AdS black hole solution of NMG [15], but unfortunately led to a slightly different expression for the angular momentum (see [14] for details). One of the main motivations of the present work has been to find a new concrete and stationary example where the extended Killing charge definition [12] could be applied to, apart from the warped AdS black hole.

One may question why I am “insisting on” using the theoretical approach advanced in [12] when it has already failed in a number of cases as explained above. The plain reason is that it gives the “correct” structural form of the conserved quantities and falls flat only at the numerical factors in front of the charges. In what follows, I want to extract, at least, the forms of the energy \( E \) and the angular momentum \( L \) for the stationary Lifshitz black hole, see the effect of turning on the parameter \( \omega \) and speculate, if necessary, on the “correct” \( E \) and \( L \) from there on.

After this informative digression, let me turn back to the problem of studying the thermodynamics of the stationary Lifshitz black hole now. Using the definition for the angular velocity of
TABLE I. The energy and the angular momentum of the stationary Lifshitz black hole

| background                          | Energy $E$                  | Angular Momentum $L$ |
|-------------------------------------|-----------------------------|----------------------|
| static Lifshitz spacetime           | divergent                    | divergent            |
| stationary Lifshitz spacetime       | $\frac{M^2(7+11\omega^2)}{8G(1-\omega^2)}$ | $\frac{18M^2\ell\omega}{4G(1-\omega^2)}$ |
| static Lifshitz black hole          | divergent                    | $\frac{4M^2\ell\omega}{8G(1-\omega^2)}$ |

the horizon $\Omega_H$, the surface gravity $\kappa$, thus the temperature $T$, and Wald entropy $S$ [10], I find the following

$$\Omega_H = \frac{\omega}{\ell}, \quad T = \frac{\kappa}{2\pi} = \frac{M^{3/2}}{2\pi \ell} \sqrt{1-\omega^2}, \quad S = \frac{2\pi \ell \sqrt{M}}{G \sqrt{1-\omega^2}}$$ (16)

for elementary thermodynamical quantities. Here, I have suitably adapted the general discussion given in [14] to the conventions used throughout, i.e. I have set

$$\kappa = 16\pi G, \quad \Omega_1 = 2\pi, \quad \beta = \frac{1}{m^2\kappa} = \frac{2\ell^2}{\kappa}, \quad \alpha = -\frac{3}{8}\beta, \quad \gamma = 0$$

in [14]. As for the calculation of the energy $E$ and the angular momentum $L$, I again refer the reader to the detailed discussion given in Section IV of [14] (especially to the parts on the warped AdS black hole solution of NMG). A naive calculation using the static Lifshitz spacetime as background immediately leads to divergent $E$ and $L$. However, a close scrutiny makes it apparent that “the fall-off conditions that need to be satisfied by the deviations” (as alluded to earlier) are severely violated in this case. Keeping this observation in mind, it turns out that the most reasonable thing to do is to work with the stationary Lifshitz spacetime as background\(^5\). Sparing the reader form the gory details of rather long calculations, I have summarized the outcome of the energy and the angular momentum calculations of the stationary Lifshitz black hole with respect to one of three physically sensible background choices in Table I\(^6\).

Clearly one cannot have the first law of thermodynamics (in the form $dE = T\,dS + \Omega_H\,dL$) hold even by using the only reasonable pair

$$E = \frac{M^2(7+11\omega^2)}{8G(1-\omega^2)}, \quad L = \frac{18M^2\ell\omega}{8G(1-\omega^2)}$$ (17)

\(^{5}\)One may naively call for the background itself, i.e. the stationary Lifshitz spacetime in this case, to satisfy the very same fall-off conditions as the deviations, but this is against the whole essence of this procedure. In a sense, this background choice is the most natural one that “renormalizes” the divergences encountered.

\(^{6}\)I can briefly explain the steps taken though: One first calculates the “extended Killing charge density” at finite $r$ for which $r > r_+$, the outcome of which is a rather long expression, better not displayed here, but an even, rational function of finite $r$. Then I assumed $r \gg 1$, which is plausible, and ignoring the lesser powers of $r$ both in the numerator and the denominator led me to the results in Table 1. The relevant ones also reduced to what was earlier found in the $\omega \to 0$ limit.
Demand that i) \(dE = T \, dS + \Omega_H \, dL\) holds with (16), and that ii) all respective quantities approach their counterparts for the static Lifshitz black hole when one takes \(\omega \to 0\). Keeping the general features of both \(E\) and \(L\) intact, the cheapest way to do so is by tweaking the coefficients in (17), i.e. to take

\[
E = \frac{M^2 (2 + a \omega^2)}{8G(1 - \omega^2)}, \quad L = \frac{b M^2 \ell \omega}{8G(1 - \omega^2)},
\]

and later to determine the coefficients \((a, b)\) using the first law of thermodynamics. Doing so, one finds that there is only a unique nontrivial pair: \((a, b) = (6, 8)\). For what it is worth, I thus conjecture that the “correct” energy and the “correct” angular momentum of the stationary Lifshitz black hole is given by

\[
E = \frac{M^2 (1 + 3 \omega^2)}{4G(1 - \omega^2)}, \quad L = \frac{M^2 \ell \omega}{G(1 - \omega^2)}.
\]

IV. DISCUSSION

In this note I have presented the stationary Lifshitz black hole of NMG, and studied its elementary geometric and thermodynamical properties. Even though the charges calculated using the extended conserved Killing charge definition [12] were not in accord with the first law of thermodynamics, assuming the validity of the first law (and of course taking the Wald entropy for granted) I predicted the energy and the angular momentum of the stationary Lifshitz black hole.

It is a separate but, of course, a legitimate question to understand the “physical meanings” of the conserved quantities, since the asymptotic behavior, if any, of the metric (12) is far from clear. (12) must certainly be studied further, perhaps using numerical methods as well to alleviate the difficulties arising from the cubic polynomial (11). A more detailed examination of the geodesics, initiated in appendix B here, would undoubtedly be of help in this endeavor.

The stationary Lifshitz black hole should allow for getting rid of the ambiguity encountered in uniquely determining the three free parameters in the counterterm [9] that emerged when trying to use the boundary stress tensor method. It is also worth trying to generalize the dimensional reduction developed in [11] and to work out the conserved charges of the stationary Lifshitz black hole from that side. One immediate calculation that should be worth the effort is to compute the energy and the angular momentum by using the “quasilocal generalization” [16] of the conserved Killing charges method employed here.

As stated in the text, one obvious source of error causing the theoretical procedure developed
in [12] to fail is the violation of the fall-off conditions demanded from the deviations. Pending a computation, e.g. via the method of [16], to check the conjecture advanced in this work, one may contemplate devising a convenient cut-off mechanism and scrutinizing the effects, if any, of the boundary terms that are thrown away in the derivation of the field equations to the application of the procedure given in [12]. It certainly is worth the effort to understand the cause of the inconsistency between the conserved charge computation and the first law of black hole thermodynamics, and to find a solid solution to fix this problem.

To recapitulate, this work has mainly focused on the presentation of the stationary Lifshitz black hole and understanding the energy and the angular momentum through a rather conventional manner. It must also be worth studying other geometric features as well as physical properties and their consequences in the context of condensed matter physics via the AdS/CFT correspondence and NMG holography [18].

ACKNOWLEDGMENTS

I would like to thank Deniz Olgu Devecioğlu for a critical reading of the manuscript, and to Gaston Giribet for a useful correspondence after the first draft of this manuscript appeared in the arXiv.

Appendix A:

Here I want to examine the cubic polynomial (11) in more detail. In what follows, I will take $0 < \omega < 1$ and $0 < M$ to simplify the discussion on the black-hole interpretation of (12). Let me start by writing (11) in the canonical form

$$x^3 + a_2 x^2 + a_1 x + a_0 = 0,$$

where I have defined the coefficients

$$a_2 \equiv -M \ell^2 < 0, \quad a_1 \equiv -\frac{\ell^4}{\omega^2} < 0, \quad a_0 \equiv \frac{\ell^4}{\omega^2} (1 - \omega^2) r^2 > 0.$$
Using these, let me also introduce [17]

\[ Q \equiv \frac{a_1}{3} - \frac{a_2^2}{9} = -\frac{\ell^4}{9\omega^2}(3 + M^2\omega^2) < 0, \]

\[ P \equiv \frac{a_0 a_1}{6} - \frac{a_0 a_2}{2} - \frac{a_2^3}{27} = \frac{\ell^4}{2\omega^2} \left( \frac{M\ell^2}{27}(9 + 2M^2\omega^2) - (1 - \omega^2)r^2 \right), \]

\[ \Delta \equiv Q^3 + P^2 = \frac{\ell^8}{108\omega^6} \left( 27r^4 - \ell^4(4 + M^2\omega^2) - 2Mr^2\ell^2\omega^2(1 - \omega^2)(9 + 2M^2\omega^2) \right). \]

\[ \Xi \equiv \frac{3}{2} \sqrt{P + \sqrt{\Delta}}, \quad \Upsilon \equiv \frac{3}{2} \sqrt{P - \sqrt{\Delta}}. \]

Finally, the formal roots of the polynomial (11) are given by

\[ x_1(r) = \frac{M\ell^2}{3} + (\Xi + \Upsilon), \]

\[ x_2(r) = \frac{M\ell^2}{3} - \frac{1}{2}(\Xi + \Upsilon) + i\frac{\sqrt{3}}{2}(\Xi - \Upsilon), \quad (A1) \]

\[ x_3(r) = \frac{M\ell^2}{3} - \frac{1}{2}(\Xi + \Upsilon) - i\frac{\sqrt{3}}{2}(\Xi - \Upsilon). \]

Defining the critical value \( \tilde{r} \) as

\[ \tilde{r}^2 \equiv \frac{\ell^2}{27\omega(1 - \omega^2)} \left( M\omega(9 + 2M^2\omega^2) + 2(3 + M^2\omega^2)^{3/2} \right) > 0, \]

I find that [17] i) \( \Delta > 0 \) when \( r^2 > \tilde{r}^2 \), which implies that there exist one real and two complex conjugate roots of (11); ii) \( \Delta = 0 \) when \( r = \tilde{r} \), so that all roots of (11) are real and at least two of them are equal; iii) \( \Delta < 0 \) when \( 0 < r^2 < \tilde{r}^2 \), so that all roots of (11) are real and unequal. Note that since (11) can be cast as

\[ x \left( 1 + \frac{M\omega^2}{\ell^2} - \frac{\omega^2}{\ell^4} x^2 \right) = (1 - \omega^2)r^2 > 0, \quad (A2) \]

this, though crudely, further constrains the metric function \( x(r) \) to satisfy either

\[ x(r) < x_- \equiv \frac{M\ell^2}{2} \left( 1 - \sqrt{1 + \frac{4}{M^2\omega^2}} \right) < 0 \quad \text{or} \quad 0 < x(r) < x_+ \equiv \frac{M\ell^2}{2} \left( 1 + \sqrt{1 + \frac{4}{M^2\omega^2}} \right), \quad (A3) \]

where \( x_\pm \) denote the roots of the quadratic factor on the left hand side of (A2), which are different from the formal roots (A1) of the polynomial (11). The branch \( 0 < x(r) < x_+ \) allows for an event horizon since \( M\ell^2 < x_+ \) and is the one I use for the interpretation of (12) as a stationary Lifshitz black hole. Note that this is all the more plausible especially if one takes both \( \omega \) and \( M \) as small...
but positive, i.e. $\omega \gtrsim 0$ and $M \gtrsim 0$, since then it is easy to see that the upper bound of the inequality $x_+ \simeq \ell^2/\omega$ can be made arbitrarily large with $\ell \gg 1$.

Appendix B:

Here I briefly discuss the geodesics of the stationary Lifshitz black hole (13) and compare them to the geodesics of the static Lifshitz black hole (6). Denoting the derivatives with respect to an affine parameter with a dot, it immediately follows from (13) that the geodesics satisfy

$$L = -\frac{\ell^2}{(1 - \omega^2)} \left( \frac{\rho^6}{\ell^6} - \frac{M \rho^4}{\ell^4} - \frac{\omega^2 \rho^2}{\ell^2} \right) + \frac{2 \omega \ell \dot{t} \dot{\theta}}{(1 - \omega^2)} \left( \frac{\rho^6}{\ell^6} - \frac{M \rho^4}{\ell^4} - \frac{\rho^2}{\ell^2} \right)$$

$$+ \frac{\ell^2 \dot{\theta}^2}{(1 - \omega^2)} \left( \frac{\rho^2}{\ell^6} - \frac{\omega^2 \rho^6}{\ell^6} + \frac{M \omega^4 \rho^4}{\ell^4} \right) + \frac{\rho^2}{(\ell^2 - M)},$$

where $L = -1$ for timelike and $L = 0$ for null geodesics. For a physical particle which has energy $E = \partial L/\partial \dot{t} = \text{const.}$ and orbital angular momentum $J = \partial L/\partial \dot{\theta} = \text{const.}$, the elimination of $\dot{t}$ and $\dot{\theta}$ in terms of $E$ and $J$ in the obvious way leads to

$$L = \frac{1}{4\rho^2(1 - \omega^2)} \left[ J^2 \left( 1 + \frac{\ell^4 \omega^2}{\rho^2(M\ell^2 - \rho^2)} \right) + \mathcal{E}^2 \ell^2 \left( \omega^2 + \frac{\ell^4}{\rho^2(M\ell^2 - \rho^2)} \right) \right]$$

$$+ 2\ell \omega \mathcal{E} \left( 1 + \frac{\ell^4}{\rho^2(M\ell^2 - \rho^2)} \right) - \frac{\ell^2 \rho^2}{(M\ell^2 - \rho^2)}.$$  \hspace{1cm} (B1)

In principle from this one can study the solutions of the radial geodesics, but obviously the resultant expression for $\dot{\rho}$ is quite complicated in the generic case. However, when $\mathcal{E}$ and $\mathcal{J}$ are precisely related by $\mathcal{J} + \ell \omega \mathcal{E} = 0$, this simplifies considerably and allows one to arrive at

$$\dot{\rho}^2 - \left( \frac{\rho^2}{\ell^2} - M \right) L = \frac{\mathcal{E}^2 \ell^4 (1 - \omega^2)}{4\rho^4}. \hspace{1cm} (B2)$$

For a lightlike particle $L = 0$, and this simplifies further obviously.

If one is to repeat the analogous calculation for the static Lifshitz black hole (6) and to use $\bar{\mathcal{E}}$ and $\bar{\mathcal{J}}$ for the analogous physical quantities, one finds

$$L = \frac{\bar{\mathcal{J}}^2}{4\rho^2} - \frac{\ell^2 \rho^2}{(M\ell^2 - \rho^2)} + \frac{\bar{\mathcal{E}}^2 \ell^6}{4\rho^4(M\ell^2 - \rho^2)}, \hspace{1cm} (B3)$$

which unsurprisingly amounts to setting $\omega = 0$ and barring the relevant bits in (B1). In this case
when the orbital angular momentum $\mathcal{J} = 0$, the analog of (B2) becomes

$$\dot{\rho}^2 - (\frac{\rho^2}{\ell^2} - M) \mathcal{L} = \frac{\mathcal{E}^2 \ell^4}{4\rho^4},$$  \hspace{1cm} (B4)$$

which again amounts to setting $\omega = 0$ and barring $\mathcal{E}$ in (B2). Once again, for lightlike particles this becomes very simple.

By now, the upshot of all this discussion should be clear. The rotation introduced by $\omega$ indeed complicates the behavior of the geodesics, but not by a great margin. One can easily conclude, at least for the simple geodesics discussed above, that whatever attributes the geodesics of the static Lifshitz black hole (6) possess the same attributes are, more or less, also shared by the stationary Lifshitz black hole (13). In particular, one may be tempted to conclude that a timelike particle with a large enough $\mathcal{E}$ can go from the horizon $\rho^2 = \ell^2 M$ to the timelike surface $\rho_1 \equiv x_1^2$ (A3) within a finite affine parameter interval and proceed further to larger values of $\rho = x^2$ before being reflected back at some turning point $x_\mathcal{E} > x_+$, but such a thing is impossible as per what happens for the analogous case of the static Lifshitz black hole, and all the more so for the choices $\omega \gtrsim 0$, $M \gtrsim 0$ and $\ell \gg 1$.

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