Non-hermitean hamiltonians with unitary and antiunitary symmetry

Francisco M. Fernández and Javier García

INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica, Blvd. 113 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

Abstract

We analyse several non-Hermitian Hamiltonians with antiunitary symmetry from the point of view of their point-group symmetry. It enables one to predict the degeneracy of the energy levels and to reduce the dimension of the matrices necessary for the diagonalization of the Hamiltonian in a given basis set. We can also classify the solutions according to the irreducible representations of the point group. One of the main results of this paper is that PT-symmetric Hamiltonians with point-group symmetry $C_{2v}$ exhibit complex eigenvalues for all values of a potential parameter. In such cases the PT phase transition takes place at the trivial Hermitian limit and suggests that the phenomenon is not robust.

Key words: PT-symmetry, multidimensional oscillators, point-group symmetry, PT phase transition, broken PT symmetry

1 Introduction

It was shown some time ago that some complex non-Hermitian Hamiltonians may exhibit real eigenvalues [1][2]. The conjecture that such intriguing feature

1 e–mail: fernande@quimica.unlp.edu.ar
may be due to unbroken PT-symmetry [3] gave rise to a very active field of research [4] (and references therein). The first studied PT-symmetric models were mainly one-dimensional anharmonic oscillators [3–6] and lately the focus shifted towards multidimensional problems [7–15]. Among the most widely studied multidimensional PT-symmetric models we mention the complex versions of the Barbanis [7,8,10,12,14,15] and Hénon-Heiles [7,12] Hamiltonians. Several methods have been applied to the calculation of their spectra: the diagonalization method [7–10,12,14], perturbation theory [7,9,10,12], classical and semiclassical approaches [7,8], among others [12,15]. Typically, those models depend on a potential parameter $g$ so that the Hamiltonian is Hermitian when $g = 0$ and non-Hermitian when $g \neq 0$. Bender and Weir [14] conjectured that some of those models may exhibit PT phase transitions so that their spectra are entirely real for sufficiently small but nonzero values of $|g|$. Such phase transition appears to be a high-energy phenomenon.

Multidimensional oscillators exhibit point-group symmetry (PGS) [16,17]. As far as we know such a property has not been taken into consideration in those earlier studies of the PT-symmetric models, except for the occasional parity in one of the variables. It is to be expected that PGS may be relevant to the study of the spectra of multidimensional PT-symmetric anharmonic oscillators. One of the purposes of this paper is to start such research.

The main interest in the study of PT-symmetric oscillators has been to enlarge the class of such models that exhibit real spectra, at least for some values of the potential parameter. In such cases PT-symmetry is broken at particular values $g = g_c$ of the parameter that are known as exceptional points [18,21] and can be easily calculated as critical parameters by means of the diagonalization method [22]. The PT phase transition is determined by the smallest $|g_c|$. Another goal of this paper is to find PT-symmetric models that do not exhibit real spectra, except at the trivial Hermitian limit $g = 0$. 

2
In section 2 we outline the main ideas of unitary (point-group) and antiunitary symmetry. In section 3 we show that two exactly solvable PT-symmetric oscillators with different PGS exhibit quite different spectra. In sections 4, 5, and 6 we discuss some non-Hermitian operators, already studied earlier by other authors, from the point of view of PGS. All of them have been shown to exhibit high-energy phase transitions. In section 7 we show a PT-symmetric anharmonic oscillator with complex eigenvalues for all values of the potential parameter. Finally, in section 8 we summarize the main results of the paper and draw conclusions.

2 Unitary and antiunitary symmetry

We assume that there is a group of unitary transformations $G = \{U_1, U_2, \ldots, U_n\}$ and a set of antiunitary transformations $S = \{A_1, A_2, \ldots, A_m\}$ that leave the non-Hermitian Hamiltonian operator invariant

$$U_j H U_j^{-1} = H, \quad A_j H A_j^{-1} = H.$$  \hfill (1)

Therefore, if $\psi$ is an eigenvector of $H$ with eigenvalue $E$ we have

$$H U_j \psi = E U_j \psi,$$  \hfill (2)

and

$$H A_j \psi = E^* A_j \psi.$$  \hfill (3)

It follows from the latter equation that the eigenvalues of $H$ are either real or appear as pairs of conjugate complex numbers.

It is well known that a product of antiunitary operators is a unitary one \cite{23}. Therefore, since $A_i A_j$ leaves the Hamiltonian invariant then $A_i A_j = U_k \in G$, 

3
provided that $G$ is the actual symmetry point group for $H$ \cite{24,25}.

If $A_j \psi = \lambda \psi$ then the antiunitary symmetry is said to be unbroken and $E = E^*$. For some non-Hermitian Hamiltonians with degenerate states the eigenvalue can be real even though $A_j \psi \neq \lambda \psi$ \cite{22}.

3 Exactly solvable examples

In this section we discuss exactly solvable PT-symmetric models similar to those studied earlier by Nanayakkara \cite{9} and Cannata et al \cite{13}. In the present case we focus on the PGS of the Hamiltonian operators that was not considered by those authors. As a first simple model we consider the Hamiltonian operator

$$H = p_x^2 + p_y^2 + x^2 + y^2 + i a x y$$

where $a$ is a real parameter. It is exactly solvable and invariant under the operations of the symmetry point group $C_{2v} \colon \{E, C_2, \sigma_{v1}, \sigma_{v2}\}$ that transform the variables according to

$$E : (x, y) \rightarrow (x, y),$$
$$C_2 : (x, y) \rightarrow (-x, -y),$$
$$\sigma_{v1} : (x, y) \rightarrow (y, x),$$
$$\sigma_{v2} : (x, y) \rightarrow (-y, -x).$$

Note that $C_2$ is a rotation by an angle $\pi$ around the $z$ axis and $\sigma_v$ are vertical reflection planes \cite{24,25}. Unless otherwise stated, from now on it is assumed that the same transformations apply to the momenta $(p_x, p_y)$. In the case of a two-dimensional model the effect of the symmetry operations on the $z$ variable is irrelevant and for this reason there may be more than one point group suitable for the description of the problem. For example, here we can also choose the symmetry point groups $C_{2v}$ or $D_2$ \cite{24,25}. For concreteness
we restrict ourselves to the $C_{2v}$ point group with irreducible representations \{\(A_1\), \(B_1\), \(A_2\), \(B_2\)\}.

To the PGS discussed above we can also add the antiunitary operations

\[
A(x) = C_2(x)T, \quad A(y) = C_2(y)T,
\]

where \(T\) is the time reversal operation \[26\] and

\[
C_2(x) : (x, y) \mapsto (x, -y), \\
C_2(y) : (x, y) \mapsto (-x, y),
\]

are rotations by \(\pi\) about the \(x\) and \(y\) axis, respectively. Note that \(A(x)A(y) = C_2\) is an example of the product of two antiunitary operators that results in one of the elements of the symmetry point group for \(H\).

This model is separable into two harmonic oscillators by means of the change of variables

\[
x = \frac{1}{\sqrt{2}}(s + t), \\
y = \frac{1}{\sqrt{2}}(s - t),
\]

that leads to

\[
H = p_s^2 + p_t^2 + ks^2 + k^*t^2, \\
k = 1 + i\frac{a}{2},
\]

If we write \(\omega = \sqrt{k} = \omega_R + i\omega_I\) then the eigenvalues are given by

\[
E_{mn} = 2(m + n + 1)\omega_R + 2(m - n)i\omega_I,
\]

where \(m, n = 0, 1, \ldots \) and

\[
\omega_R = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{a^2}{4}}}, \quad \omega_I = \frac{a}{4\omega_R}.
\]
We see that all the eigenvalues with $m = n$ are real and those with $m \neq n$ are complex when $a \neq 0$ (more precisely: $E_{mn} = E_{nm}^*$. In this case the PT phase transition \cite{14} takes place at the trivial Hermitian limit $a = 0$. It is also obvious that the perturbation series for this model exhibits only powers of $a^2$ when $m = n$ and powers of $a$ when $m \neq n$.

The eigenfunctions can be written as

$$\psi_{mn}(s, t) = \phi_m(k, s) \phi_n(k^*, t), \quad (12)$$

where $\phi_m(k, s)$ is an eigenfunction of $p_s^2 + ks^2$. Therefore

$$A(x) \psi_{mn}(s, t) = \psi_{mn}^*(t, s) = \psi_{mn}(s, t),$$
$$A(y) \psi_{mn}(s, t) = \psi_{mn}^*(-t, -s) = (-1)^{m+n} \psi_{mn}(s, t) \quad (13)$$

that are consistent with equation (3).

The states $\psi_{2m, 2n}$, $\psi_{2m+1, 2n+1}$, $\psi_{2m+1, 2n}$ and $\psi_{2m, 2n+1}$ are bases for the irreducible representations $A_1$, $A_2$, $B_1$ and $B_2$, respectively. It is clear that only some of the states with symmetry $A_1$ and $A_2$ have real eigenvalues and that those with symmetry $B_1$ and $B_2$ exhibit only complex ones. Moreover, the antiunitary operators $A(x)$ and $A(y)$ transform functions of symmetry $B_1$ into functions of symmetry $B_2$ and viceversa, which shows that PT symmetry is broken for all $a \neq 0$. More precisely, the eigenvalue of $\psi_{2m+1, 2n} (B_1)$ is the complex conjugate of $\psi_{2m, 2n+1} (B_2)$.

We also appreciate that the eigenfunctions of the non-Hermitian Hamiltonian retain their symmetry in the Hermitian limit: $\lim_{a \to 0} \psi_{mn}(s, t) = \phi_m(1, s) \phi_n(1, t)$.

In order to test the effect of symmetry on the spectra of the non-Hermitian Hamiltonians we next consider the less symmetric operator

$$H = p_x^2 + p_y^2 + 2x^2 + y^2 + iaxy, \quad (14)$$
that is invariant under the operations of the point group $C_2$: \{E, C_2\}. In this case all the eigenvalues

$$E_{mn} = (2m + 1)\omega_1 + (2n + 1)\omega_2,$$

(15)

where

$$\omega_1 = \sqrt{\frac{3}{2} + \frac{\sqrt{1 - a^2}}{2}}, \quad \omega_2 = \sqrt{\frac{3}{2} - \frac{\sqrt{1 - a^2}}{2}},$$

(16)

are real provided that $|a| < 1$. In this less symmetric example we find a PT phase transition at the exceptional point $a = 1$. The eigenfunctions are bases for the irreducible representations \{A, B\}.

From the results of this section we may argue that PGS determines whether the PT symmetry is broken or unbroken. In order to confirm such conjecture we should find other examples (preferably non exactly solvable) with PT phase transitions at the trivial Hermitian limit. Before doing it we first discuss the non-Hermitian Hamiltonians studied so far from the point of view of PGS.

4 The Barbanis Hamiltonian

The PT-symmetric version of the Barbanis Hamiltonian

$$H = \frac{1}{2} \left(p_x^2 + p_y^2\right) + \frac{1}{2} \left(x^2 + y^2\right) + iaxy^2,$$

(17)

is one of the simplest nontrivial two-dimensional models chosen by several authors as a suitable illustrative example [7,8,10,12,14,15]. Most of them have in fact exploited the fact that it is invariant under $y$ parity: $P_y : (x, y) \rightarrow (x, -y)$. If we take into account that the effect of $P_y$ is equivalent to a rotation by an angle $\pi$ about the $x$ axis then we realize that the appropriate symmetry point...
group for this model is $C_2$ with elements \{$E, C_2(x)\}$ and irreducible representations \{$A, B$\} \cite{24,25}. This model with a rather low symmetry appears to exhibit a PT phase transition at $a \approx 0.1$ \cite{14}.

The slightly modified Hamiltonian \cite{14}

$$H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + \frac{1}{2} x^2 + y^2 + i a x^2 y,$$  \hspace{1cm} (18)$$

exhibits the same symmetry and in this case the phase transition occurs approximately at $a \approx 0.08$.

5 The Hénon-Heiles Hamiltonian

A more interesting non-Hermitian anharmonic oscillator is the PT-symmetric version of the Hénon-Heiles one \cite{7,12}

$$H = p_x^2 + p_y^2 + x^2 + y^2 + i a \left( x y^2 - \frac{1}{3} x^3 \right).$$  \hspace{1cm} (19)$$

Earlier treatments of this problem have taken into account the $y$ parity already discussed in the preceding section. This symmetry is insufficient to account for the existence of two-fold degenerate eigenvalues already mentioned by Wang \cite{12}. The fact is that this Hamiltonian is invariant under rotations around the $z$ axis by angles $2\pi/3$ and $4\pi/3$ as well as under three vertical and equivalent reflection planes $\sigma_v$ \cite{17}. The appropriate symmetry point group is thus $C_{3v}$ and the eigenfunctions are bases for the irreducible representations \{$A_1, A_2, E$\} \cite{24,25}. The latter is two-dimensional and accounts for the degeneracy just mentioned.

If instead of the three vertical planes $\sigma_v$ we choose three equivalent axes $C_2$ perpendicular to the principal $C_3$ one the suitable point group results to be $D_3$. The results coming from any of these choices are equivalent. In section 5
we already explained why we can choose more than one symmetry point group for the two-dimensional models discussed here.

The eigenvalues and eigenfunctions of the Hermitian operator $H(a = 0)$ are

$$E_{mn}(a = 0) = 2(m + n + 1), \ m, n = 0, 1, \ldots, \quad (20)$$

and

$$\varphi_{mn}(x, y) = \phi_m(x)\phi_n(y), \quad (21)$$

respectively, where $\phi_j(q)$ is a normalized eigenfunction of the harmonic oscillator $H = p_q^2 + q^2$. It is convenient for the discussion below to label the eigenfunctions as $\psi_{M,j}(x, y)$, where $M = m + n$, $j = 0, 1, \ldots, M$ and $E_{M,0} \leq E_{M,1} \leq \ldots \leq E_{M,M}$ so that (as outlined in section 3)

$$\lim_{a \to 0} \psi_{M,j}(x, y) = \sum_{i=0}^{M} c_{M-i,i,j} \varphi_{M-i,i}(x, y), \quad (22)$$

where the coefficients $c_{ij}$ are determined by the symmetry of the eigenfunction.

For example, the first eigenfunctions in this limit and their corresponding symmetries are

$$M = 0 : \{\varphi_{00}\}, \ A_1, \quad M = 1 : \{\varphi_{10}, \varphi_{01}\}, \ E, \quad M = 2 : \begin{cases} \{\frac{1}{\sqrt{2}}(\varphi_{20} + \varphi_{02})\}, \ A_1 \\ \{\frac{1}{\sqrt{2}}(\varphi_{20} - \varphi_{02}), \varphi_{11}\}, \ E \end{cases}. \quad (23)$$

The projection operators $P_S$ are suitable for a systematic construction of symmetry-adapted functions [24][25]. For example, for $M = 3$ we have

$$P^{A_1}\varphi_{30} = \frac{1}{4}\varphi_{30} - \frac{\sqrt{3}}{4}\varphi_{12},$$
$$P^{A_2} \varphi_{21} = \frac{3}{4} \varphi_{21} - \frac{\sqrt{3}}{4} \varphi_{03},$$

$$P^E \varphi_{30} = \frac{3}{4} \varphi_{30} - \frac{\sqrt{3}}{4} \varphi_{12},$$

$$P^E \varphi_{21} = \frac{1}{4} \varphi_{21} + \frac{\sqrt{3}}{4} \varphi_{03}. \quad (24)$$

These functions are not normalized to unity because \( \langle P^S \varphi | P^S \varphi \rangle \leq \langle \varphi | \varphi \rangle \) for any projection operator \( P^S \). Note that the functions with symmetry \( A_1 \) and \( A_2 \) exhibit even and odd parity, respectively, with respect to the operation \( P_y \) discussed above. On the other hand, one of the functions of the basis for the irreducible representation \( E \) is even and the other odd.
The order of the energy levels for this model when \( a \) is sufficiently small (say, \( a = 0.1 \)) is

\[
2(M + 1) \text{ Symmetry Ref. } [12]
\]

\[
\begin{array}{c|c|c}
2 & A_1 & E_{00} \\
4 & E & E_{10} = E_{11} \\
6 & A_1 & E_{20} \\
 & A_1 & E_{32} \\
8 & A_2 & E_{33} \\
 & E & E_{41} = E_{42} \\
 & E & E_{43} = E_{44} \\
10 & E & E_{41} = E_{42} \\
 & A_1 & E_{40} \\
 & E & E_{54} = E_{55} \\
 & A_1 & E_{52} \\
12 & A_2 & E_{53} \\
 & E & E_{50} = E_{51}
\end{array}
\]

where the last column shows the energy levels as labelled by Wang [12] who derived the perturbation expansions

\[
E_{00} = 2 + \frac{a^2}{18} - \frac{11a^4}{864} + \frac{6089a^6}{933120} - \frac{2221951a^8}{447897600} + \ldots
\]
$E_{10} = E_{11} = 4 + \frac{7}{18}a^2 - \frac{133}{864}a^4 + \frac{30191}{23328}a^6 - \frac{67779467}{447897600}a^8 + \ldots
$

$E_{20} = 6 + \frac{31}{18}a^2 - \frac{145}{288}a^4 + \frac{200923}{186624}a^6 - \frac{40752209}{29859840}a^8 + \ldots
$

$E_{21} = E_{22} = 6 + \frac{5}{9}a^2 - \frac{83}{144}a^4 + \frac{432493}{466560}a^6 - \frac{133188257}{74649600}a^8 + \ldots
$

$E_{30} = E_{31} = 8 + \frac{26}{9}a^2 - \frac{535}{432}a^4 + \frac{180037}{46656}a^6 - \frac{296084959}{223948800}a^8 + \ldots
$

$E_{32} = 8 + \frac{5}{9}a^2 - \frac{1123}{432}a^4 + \frac{1416869}{46656}a^6 - \frac{3963323843}{44789760}a^8 + \ldots
$

$E_{33} = 8 + \frac{5}{9}a^2 - \frac{115}{432}a^4 + \frac{12121}{46656}a^6 - \frac{15676999}{44789760}a^8 + \ldots
$

$E_{40} = 10 + \frac{91}{18}a^2 - \frac{2065}{864}a^4 + \frac{1208431}{186624}a^6 - \frac{1731827209}{89579520}a^8 + \ldots
$

$E_{41} = E_{42} = 10 + \frac{35}{9}a^2 - \frac{1085}{432}a^4 + \frac{1285829}{93312}a^6 - \frac{1478346167}{44789760}a^8 + \ldots
$

$E_{43} = E_{44} = 10 + \frac{7}{18}a^2 - \frac{2485}{864}a^4 + \frac{1063615}{186624}a^6 - \frac{181958169}{89579520}a^8 + \ldots
$

$E_{50} = E_{51} = 12 + \frac{127}{18}a^2 - \frac{1205}{288}a^4 + \frac{814129}{46656}a^6 - \frac{195820799}{29859840}a^8 + \ldots
$

$E_{52} = 12 + \frac{85}{18}a^2 - \frac{2633}{288}a^4 + \frac{1370563}{29160}a^6 - \frac{20818356203}{149299200}a^8 + \ldots
$

$E_{54} = 12 + \frac{85}{18}a^2 + \frac{55}{288}a^4 + \frac{70673}{5832}a^6 + \frac{354058961}{29859840}a^8 + \ldots
$

$E_{54} = E_{55} = 12 + \frac{1}{18}a^2 - \frac{1457}{288}a^4 + \frac{329257}{29160}a^6 - \frac{9599275547}{149299200}a^8 + \ldots
$

(26)

6 Hamiltonian operator in three dimensions

Bender et al [7] and Bender and Weir [14] also discussed some PT-symmetric Hamiltonians in three dimensions. One of them is

$$H = p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2 + iaxyz.$$  (27)

The eigenfunctions of $H_0 = H(a = 0)$ are given by

$$\varphi_{m,n,k}(x, y, z) = \phi_m(x)\phi_n(y)\phi_k(z), \ m, n, k = 0, 1, \ldots, \quad (28)$$
and the corresponding eigenvalues $E_{mnk}^{(0)} = 2M + 3$, $M = m + n + k$, are $(M + 1)(M + 2)/2$-fold degenerate. The perturbation $H' = iaxy$ splits the states of the three-dimensional Harmonic oscillator $H_0$ in the following way:

\[
\begin{align*}
\{2n, 2n, 2n\} &\rightarrow A_1 \\
\{2n + 1, 2m, 2m\} &\rightarrow T_2 \\
\{2n + 1, 2n + 1, 2m\} &\rightarrow T_2 \\
\{2n, 2m, 2m\} &\rightarrow A_1, E \\
\{2n + 1, 2n + 1, 2n + 1\} &\rightarrow A_1 \\
\{2n, 2m, 2k + 1\} &\rightarrow T_1, T_2 \\
\{2n, 2m + 1, 2k + 1\} &\rightarrow T_1, T_2 \\
\{2n, 2m, 2k\} &\rightarrow A_1, A_2, E, E \\
\{2n + 1, 2m + 1, 2k + 1\} &\rightarrow A_1, A_2, E, E
\end{align*}
\]

(29)

where $\{m, n, k\}_P$ stands for all the distinct permutations of the three positive integers. In this case the eigenvalues of (27) appear to be real for all $0 \leq a < a_c$ [14].

7 Non-Hermitian oscillator with $C_{2v}$ point-group symmetry

In section 3 we saw that the phase transition for the exactly solvable example with symmetry point group $C_{2v}$ occurs at $a = 0$. The purpose of this section is to verify whether another PT-symmetric anharmonic oscillator with that symmetry exhibits the same behaviour. A suitable example is the non-Hermitian modification of the Pullen-Edmonds Hamiltonian [16]

\[
H = p_x^2 + p_y^2 + \alpha \left(x^2 + y^2\right) + \beta x^2 y^2 + iaxy.
\]

(30)

Note that both the unitary and antiunitary transformations that leave this Hamiltonian invariant are exactly those already introduced in section 3. In fact, when $\alpha = 1$ and $\beta = 0$ we obtain the first exactly solvable example discussed there. When $a = 0$ we recover the Pullen-Edmonds Hamiltonian with $C_{4v}$ PGS [16].
In order to discuss the results from the point of view of PGS we apply the diagonalization method with symmetry-adapted products $\varphi_{mn}(x, y)$ of eigenfunctions $\phi_n(q)$ of the harmonic oscillator $H = p_q^2 + q^2$. We thus obtain basis sets with the following functions

\[
\begin{align*}
\varphi_{2m \, 2n}^+ & = \begin{cases} 
\varphi_{2n \, 2n}(x, y), & m = n \\
\frac{1}{\sqrt{2}} [\varphi_{2m \, 2n}(x, y) + \varphi_{2n \, 2m}(x, y)], & m \neq n
\end{cases}, \\
\varphi_{2m \, 2n}^- & = \frac{1}{\sqrt{2}} [\varphi_{2m \, 2n}(x, y) - \varphi_{2n \, 2m}(x, y)], & m \neq n,
\end{align*}
\]

\[
\begin{align*}
\varphi_{2m+1 \, 2n+1}^+ & = \begin{cases} 
\varphi_{2n+1 \, 2n+1}(x, y), & m = n \\
\frac{1}{\sqrt{2}} [\varphi_{2m+1 \, 2n+1}(x, y) + \varphi_{2n+1 \, 2m+1}(x, y)], & m \neq n
\end{cases}, \\
\varphi_{2m+1 \, 2n+1}^- & = \frac{1}{\sqrt{2}} [\varphi_{2m+1 \, 2n+1}(x, y) - \varphi_{2n+1 \, 2m+1}(x, y)],
\end{align*}
\]

(31)

with symmetry

\[
\begin{align*}
\varphi_{2m \, 2n}^+, \varphi_{2m+1 \, 2n+1}^+: & \ A_1, \\
\varphi_{2m \, 2n}^-, \varphi_{2m+1 \, 2n+1}^-: & \ A_2, \\
\varphi_{2m \, 2n}^+: & \ B_1, \\
\varphi_{2m \, 2n+1}^+: & \ B_2.
\end{align*}
\]

(32)

Since basis functions of different symmetry do not mix then we can carry out four independent diagonalizations, one for each irreducible representation.

Because $A(x)\varphi_{2m \, 2n+1}^+ = -\varphi_{2m \, 2n+1}^+$ then $A(x)\psi_{B_1} = \lambda_{B_1B_2}\psi_{B_2}$ and $E_{B_1} = E_{B_2}^*$ according to equation (3). Therefore, the eigenvalues for $B$ eigenfunctions are expected to be complex for any $a > 0$ as in the case of the exactly solvable model discussed in section 3. We have verified this conclusion by numerical calculation (see below).
Straightforward application of the diagonalization method with those symmetry-adapted basis sets shows that there are no real eigenvalues with eigenfunctions of symmetry $B$. More precisely, the characteristic polynomials for the bases with symmetry $B_1$ and $B_2$ exhibit odd powers of $g = ia$ which do not appear in those for the other two irreducible representations $A_1$ and $A_2$. The characteristic polynomials for the entire basis set $\{\psi_{mn}\}$ are only functions of $g^2$ and the complex eigenvalues appear as pairs of complex conjugate numbers. In other words, the coefficients of the characteristic polynomials are real for the full basis set as argued elsewhere [27]. On the other hand, the coefficients of the characteristic polynomials for $B_1$ and $B_2$ are complex and every complex root $E_{B_1}$ of the former has its counterpart $E_{B_2}^*$ as a root of the latter.

Doubts have been arisen about the existence of a discrete spectrum for the Hamiltonian (30) with $\alpha = 0$ and $a = 0$ [7]. Our numerical results suggest that it already exhibits positive discrete spectrum. However, we have decided to choose the less controversial model with $\alpha = 1$ which enables us to obtain more accurate eigenvalues with smaller matrix dimension. Figure 1 shows results for $\alpha = 1$, $\beta = 0.1$ and $0 \leq a \leq 1$. We appreciate that the $A$ states exhibit phase transitions at nonzero values of $a$ but the eigenvalues of symmetry $B$ are complex for all $a > 0$ as argued above. We clearly see that in this case the PT-symmetry is broken at $a = 0$ and the phase transition takes place at the trivial Hermitian limit. In other words, the PT phase transition is not such a robust phenomenon as it is believed [14].

8 Conclusions

Throughout this paper we have discussed Hamiltonians that are Hermitian when a potential parameter $a$ is zero and non-Hermitian but PT symmetric when $a \neq 0$. Those in sections 4, 5 and 6 discussed earlier by several authors exhibit different kinds of PGS but they share the property of having phase
transitions at nonzero values of $a$ \cite{14}. On the other hand, the exactly solvable PT-symmetric harmonic oscillator of section 3 exhibits a phase transition at $a = 0$; that is to say, some of its eigenvalues are complex for all values of $a > 0$. This operator exhibits $C_{2v}$ PGS and the eigenvalues for the $B$ eigenfunctions are complex. For such eigenfunctions the PT symmetry is broken for all values of $a$ and the phase transition occurs at the Hermitian limit.

In order to verify if the broken PT symmetry was due to PGS and not to the particular form of the Hamiltonian (an exactly solvable two-dimensional harmonic oscillator) we constructed other simple but nontrivial examples with the same PGS and found exactly the same behaviour: the eigenvalues with eigenfunctions of symmetry $B$ are complex for all nonzero values of the model parameter $a$. It is likely that broken PT symmetry may also be associated to other PGS. The most important conclusion of this paper is that the existence of a phase transition as a high-energy phenomenon \cite{14} is not a general property of PT-symmetric multidimensional oscillators. It does not appear to be a robust phenomenon.

**References**

[1] E. Caliceti, S. Graffi, and M. Maioli, Perturbation theory of odd anharmonic oscillators, Commun. Math. Phys. 75 (1980) 51-66.

[2] G. Alvarez, Bender-Wu branch points in the cubic oscillator, J. Phys. A 28 (1995) 4589-4598.

[3] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, Phys. Rev. Lett. 80 (1998) 5243-5246.

[4] C. M. Bender, Making sense of non-Hermitian Hamiltonians, Rep. Prog. Phys. 70 (2007) 947-1018.

[5] F. M. Fernández, R. Guardiola, and M. Znojil, Strong-coupling expansions for
the PT-symmetric oscillators \( V(x) = a(ix) + b(ix)^2 + c(ix)^3 \), J. Phys. A 31 (1998) 10105-10112.

[6] F. M. Fernández, R. Guardiola, J. Ros, and M. Znojil, A family of complex potentials with real spectrum, J. Phys. A 32 (1999) 3105-3116.

[7] C. M. Bender, G. V. Dunne, P. N. Meisinger, and M. Simsek, Quantum complex Hénon-Heiles potentials, Phys. Lett. A 281 (2001) 311-316.

[8] A. Nanayakkara and C. Abayaratne, Semiclassical quantization of complex Henon-Heiles systems, Phys. Lett. A 303 (2002) 243-248.

[9] A. Nanayakkara, Real eigenspectra in non-Hermitian multidimensional Hamiltonians, Phys. Lett. A 304 (2002) 67-72.

[10] A. Nanayakkara, Comparison of quantal and classical behavior of PT-symmetric systems at avoided crossings, Phys. Lett. A 334 (2005) 144-153.

[11] H. Bila, M. Tater, and M. Znojil, Comment on: "Comparison of quantal and classical behavior of PT-symmetric systems at avoided crossings" [Phys. Lett. A 334 (2005) 144], Phys. Lett. A 351 (2006) 452-456.

[12] Q-H Wang, Level crossings in complex two-dimensional potentials, Pramana J. Phys. 73 (2009) 315-322.

[13] F. Cannata, M. V. Ioffe, and D. N. Nishnianidze, Exactly solvable nonseparable and nondiagonalizable two-dimensional model with quadratic complex interaction, J. Math. Phys. 51 (2010) 022108.

[14] C. M. Bender and D. J. Weir, PT phase transition in multidimensional quantum systems, J. Phys. A 45 (2012) 425303.

[15] C. R. Handy and D. Vrincenau, Orthogonal polynomial projection quantization: a new Hill determinant method, J. Phys. A 46 (2013) 135202.

[16] R. A. Pullen and A. R. Edmonds, Comparison of classical and quantal spectra for a totally bound potential, J. Phys. A 14 (1981) L477-L484.

[17] R. A. Pullen and A. R. Edmonds, Comparison of classical and quantal spectra for the Hénon-Heiles potential, J. Phys. A 14 (1981) L319-L327.
[18] W. D. Heiss and A. L. Sannino, Avoided level crossing and exceptional points, J. Phys. A 23 (1990) 1167-1178.

[19] W. D. Heiss, Repulsion of resonance states and exceptional points, Phys. Rev. E 61 (2000) 929-932.

[20] W. D. Heiss and H. L. Harney, The chirality of exceptional points, Eur. Phys. J. D 17 (2001) 149-151.

[21] W. D. Heiss, Exceptional points - their universal occurrence and their physical significance, Czech. J. Phys. 54 (2004) 1091-1099.

[22] F. M. Fernández and J. Garcia, Critical parameters for non-hermitian Hamiltonians, arXiv:1305.5164 [math-ph]

[23] E. Wigner, Normal Form of Antiunitary Operators, J. Math. Phys. 1 (1960) 409-413.

[24] M. Tinkham, Group Theory and Quantum Mechanics, (McGraw-Hill Book Company, New York, 1964).

[25] F. A. Cotton, Chemical Applications of Group Theory, (John Wiley & Sons, New York, 1990).

[26] C. E. Porter, Fluctuations of quantal spectra, in: C. E. Porter (Ed.), Statistical theories of spectra: fluctuations, Vol. Academic Press Inc., New York and London, 1965.

[27] F. M. Fernández, On the real matrix representation of PT-symmetric operators, arXiv:1301.7639v3 [quant-ph]
Fig. 1. Lowest eigenvalues with symmetry $A_1$, $A_2$, $B_1$ and $B_2$ (top to bottom) of the Hamiltonian operator (30) with $\alpha = 0$ and $\beta = 0.1$