Model-Checking Branching-Time Properties of Stateless Quantum Pushdown Systems

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Abstract

In this work, we define the quantum analogues of the probabilistic pushdown systems and Markov chains and investigate whether it is necessary to define a quantum analogue of probabilistic computational tree logic to describe the branching-time properties of the quantum Markov chain. We also study its model-checking problem and show that the model-checking of stateless quantum pushdown systems (qBPA) against probabilistic computational tree logic (PCTL) is generally undecidable, too.

The immediate corollaries of the above results are summarized in the work.

Keywords: Probabilistic pushdown systems, undecidability, probabilistic computational tree logic, quantum pushdown systems, model-checking

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1. Introduction

Logic is the origin and ongoing topic of theoretical computer science. Dating back to 1936, the goal of Alan Turing in defining the Turing machine [Tur37] was to investigate the logic issue of the Entscheidungsproblem. In the modern day, logic plays a fundamental role in computer science. Some of the key areas of logic that are particularly significant are computability theory, modal logic, and category theory. More significantly, the theory of computation is based on concepts defined by logicians and mathematicians such as Alonzo Church [Chu36a, Chu36b], and Alan Turing [Tur37].

Model checking [CGP99] is an essential tool for formal verification, which is an interesting branch of logic in computer science, in which one describes the system to be verified as a model of some logic, expresses the property to be verified as a formula in that logic, and then checks by using automated algorithms that the formula holds or not in that model [BK08]. Traditionally, model checking has been applied to finite-state systems and non-probabilistic programs. To the best of our knowledge, the verification of probabilistic programs was first considered in the 1980s [Var85]. During the last two decades, researchers have paid attention to model-checking of probabilistic infinite-state systems, for instance, [EKM06].

One of these probabilistic infinite-state systems is the probabilistic pushdown systems, which were dubbed “probabilistic pushdown automata” \(^2\) in [BBF+14, Bra07, EKM06]. Roughly, probabilistic pushdown systems can be seen as probabilistic pushdown automata with only an input symbol, which means that they can be considered as restricted versions of probabilistic pushdown automata. Their model-checking problem, which was initiated in [EKM06], has gotten a lot of attention, for example, in [Bra07, BBF+14], in which the model-checking of stateless probabilistic pushdown systems (pBPA) against PCTL*, as well as that of probabilistic pushdown systems (pPDS) against PCTL were resolved. However, the problem of model-checking of stateless probabilistic pushdown systems (pBPA) against PCTL still remains open in [Bra07, BBF+14] and later has been resolved by the author in [Lin14], which was first proposed in [EKM06].

The theory of applying quantum mechanisms to classical computation, i.e., what we call quantum computing, is unquestionably one of the hottest topics in modern times (see [Hir04]), which is based on information processing in physical systems that are so tiny that the dynamics of classical physics can not be applied

\(^2\)Throughout this work, we call it a “probabilistic pushdown system” rather than a “probabilistic pushdown automata,” which is different from [Bra07, BBF+14].
anymore. For more details on (see [Hir04]), we refer the reader to [Hir04], which contains the main topics of quantum computing from the viewpoint of theoretical computer science, and we just summarize some basic principles of it in the introduction Section. According to the quantum principle, a physical system may be in a superposition of various states, each having its observation a probability (i.e., quantum probability) assigned by quantum mechanics, which offers a way of using a kind of parallelism for computational purposes. But quantum parallelism differs essentially from the probability distributions since the quantum amplitudes can also be negative, which gives an option of utilizing interference in quantum algorithms.

Intrinsically, quantum algorithms are quantum Turing machines defined in [Deu85, BV97], and quantum software can be seen as a composition of various quantum algorithms. To the largest extent, quantum systems are devices which are equivalent to a universal quantum Turing machine [Deu85, BV97], while to the minimal extent, they are equivalent to very limited quantum automata, such as [MC00, KW97]. Regarding quantum automata, or variations of the quantum Turing machine, more and more questions about them have been answered. For example, the emptiness problem of quantum cut-point languages [Hir07], the equivalence problem regarding measure-many one-way quantum finite automata [Lin12], and the computation power question with small space bounds in the unbounded error setting [YC11], and so forth. However, there is seldom work concerning model-checking such quantum systems, to the best of our knowledge.

Naturally, one big question is raised: how to extend the classical model-checking method [CGP99] to such quantum systems? This issue has attracted much attention in the academic community, see the most recent survey [YF18]. Just as Markov chains are widely used as a formal model for discrete systems with random behavior, the notion of quantum Markov chains is also widely used as a formal model for discrete quantum systems with quantum probability behavior [Gud08, YF18]. There are various definitions of a quantum Markov chain, for example, the one defined in [Gud08] and others such as in [YF18]. For our purposes in this paper, we will extend the classical Markov chain to the quantum Markov chain for our purposes, based on the viewpoint on how to define the quantum Turing machine in [BV97].

To describe branching-time properties of quantum Markov chains, [YF18, BCM08], for instance, defined a quantum variant of probabilistic computational tree logic (QCTL/QCTL∗). Even so, [XFM+22] extends quantum computational tree logic to quantum computation tree logic plus (QCTL+). Although the probabilistic computational tree logic is a special case of the quantum variation of the probabilistic computational tree logic, we will not adopt the definition mentioned above. We believe that probabilistic computational tree logic is adequate for describing the branching-time properties of quantum Markov chains. Note that in [BV97], the definition of a quantum Turing machine accepting a language $L$ with probability $p$ if the

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3That is, the transition amplitudes preserve Euclidean length, which is equivalent to unitary of its time evolution operator; see Theorem A.5 in [BV97].
quantum Turing machine accepts every string \( w \in \mathcal{L} \) with probability at least \( p \) and rejects every string \( \tilde{w} \not\in \mathcal{L} \) with probability at least \( p \).\(^{4}\) That is to say, the transition amplitude has been finally translated to a probabilistic value. So, we assert that the probabilistic computational tree logic is enough to specify the branching-time properties of quantum Markov chains, if we translate the transition amplitudes in quantum Markov chains to probabilistic values.

### 1.1. Main Outcomes

As our main contribution to this work, we extend the quantum mechanisms to the pushdown systems and also extend the Markov chains to quantum Markov chains. We do not intend to define the quantum analog of probabilistic computational tree logic, but rather to specify the branching-time properties of quantum Markov chains with probabilistic computational tree logic. The aim of doing so is to see whether we have a theorem for quantum extension of pushdown systems that is similar to [Lin14]. We show the following:

**Theorem 1.** The model-checking of stateless quantum pushdown system (qBPA) against probabilistic computational tree logic PCTL is generally undecidable.

As will be obvious, the stateless quantum pushdown systems (qBPA) are a sub-class of quantum pushdown systems (qPDS). The above Theorem 1 has the following immediate corollary:

**Corollary 2.** The model-checking of quantum pushdown system (qPDS) against probabilistic computational tree logic PCTL is generally undecidable.

Furthermore, because the logic PCTL is a sub-logic of the more general probabilistic computational tree logic PCTL*, Corollary 2 implies:

**Corollary 3.** The model-checking of quantum pushdown system (qPDS) against probabilistic computational tree logic PCTL* is generally undecidable.

### 1.2. Related Work

Studying the model-checking question for probabilistic pushdown systems first appeared in [EKM06]. To the best of our knowledge, [EKM06] is the first work in the literature on model-checking of probabilistic infinite-state systems.

Most tightly related to our result in [Lin14] are papers on model-checking for probabilistic pushdown systems (pPDS) and basic probabilistic pushdown systems (pBPA) against PCTL/PCTL* [BBF+14, Bra07], where the results of undecidability of pPDS against PCTL and pBPA against PCTL* are obtained. However,

\(^{4}\)When \( p = \frac{2}{3} \), all such languages, accepted by some polynomial-time QTMs, form the complexity class “BQP”.

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as said by the authors of [BBF+14], they have not managed to extend the technique presented in [BBF+14] to pBPA and PCTL, which leaves open the model-checking question for pBPA against PCTL.

There are a few papers on model-checking of quantum systems; see [XFM+22] and references in [YF18]. In particular, quantum Markov chains and the closely related notions of quantum Markov processes and quantum random walks have been studied for many years [Acc76]. The one whose transition is given by completely positive maps was first defined in [Gud08], which is more general than that defined in this work. But, in view of [BV97], unitary operations are enough to construct universal quantum Turing machine. Then the general one defined in [Gud08], seems no more powerful than ours. In the references [YF18] and [XFM+22], they also defined the notion of quantum computational tree logic and its extension to describe properties of quantum Markov chains whose transitions are ruled by completely positive maps. In this work, since our motivation is to extend the notion of a probabilistic pushdown system to its quantum analogue and to study the corresponding model-checking questions in our own framework, the methods and ideas are unrelated to the aforementioned results.

1.3. Overview

The rest of this paper is structured as follows: in the next Section, some basic definitions will be reviewed and useful notation will be fixed. We define the notions of quantum Markov chain and quantum pushdown systems, which are the quantum analogues of Markov chain and probabilistic pushdown systems, respectively. The proof of our Theorem 1 is put into Section 4, and the last Section is for conclusions, in which some possible research directions are given.

2. Preliminaries

For convenience, we make the work self-contained, and most notation in probabilistic verification will follow [Bra07, BBF+14]. For elementary probability theory, the reader is referred to [Shi95] by Shiryaev or [Loe78a, Loe78b] by Loève. Let $|A|$ denote the cardinality of any finite set $A$. Let $\Sigma$ and $\Gamma$ denote non-empty finite alphabets, respectively. Then $\Sigma^*$ is the set of all finite words (including empty word $\epsilon$) over $\Sigma$, and $\Sigma^+ = \Sigma^*\setminus\{\epsilon\}$. For any word $w \in \Sigma^*$, $|w|$ represents its length. For example, let $\Sigma = \{0, 1\}$, then $|\epsilon| = 0$ and $|001101| = 6$.

2.1. Markov Chains

Roughly, Markov chains are probabilistic transition systems, which are accepted as the most popular operational model for the evaluation of the performance and dependability of information-processing systems [BK08].
Definition 1. A (discrete) Markov chain is a triple $\mathcal{M} = (S, \delta, P)$ where $S$ is a finite or countably infinite set of states, $\delta \subseteq S \times S$ is a transition relation such that for each $s \in S$ there exits $t \in S$ such that $(s,t) \in \delta$, and $P$ is a function from domain $\delta$ to range $(0,1]$ which to each transition $(s,t) \in \delta$ assigns its probability $P(s,t)$ such that $\sum P(s,t) = 1$ for each $s \in S$.

Remark 1. $\sum P(s,t)$ means $P(s,t_1) + P(s,t_2) + \cdots + P(s,t_i)$ where $\{(s,t_1), (s,t_2), \ldots, (s,t_i)\} \subseteq \delta$ is the set of all transition relations whose current state is $s$.

A path in $\mathcal{M}$ is a finite or infinite sequence of states of $S$: $\pi = s_0s_1 \cdots \in S^*$ (or $\in S^\omega$) such that $(s_i, s_{i+1}) \in \delta$ for each $i$. A run of $\mathcal{M}$ is an infinite path. We denote the set of all runs in $\mathcal{M}$ by $\text{Run}_\mathcal{M}$, and $\text{Run}_\mathcal{M}(\pi')$ to denote the set of all runs starting with a given finite path $\pi'$. If a run $\pi$ starts with a given finite path $\pi'$, then we denote this case as $\pi' \in \text{prefix}(\pi)$. Let $\pi$ be a run, then $\pi(i)$ denotes the state $s_i$ of $\pi$, and $\pi_i$ the run $s_is_{i+1}\cdots$. In this way, it is clear that $\pi_0 = \pi$. Further, a state $s'$ is reachable from a state $s$ if there is a finite path starting in $s$ and ending at $s'$.

For each $s \in S$, $(\text{Run}(s), F, P)$ is a probability space, where $F$ is the $\sigma$-field generated by all basic cylinders $\text{Cyl}(\pi)$ and $\pi$ is a finite path initiating from $s$,

$$\text{Cyl}(\pi) = \{ \tilde{\pi} \in \text{Run}(s) : \pi \in \text{prefix}(\tilde{\pi}) \},$$

and $P : F \to [0,1]$ is the unique probability measure such that

$$P(\text{Cyl}(\pi)) = \prod_{1 \leq i \leq |\pi| - 1} P(s_i, s_{i+1})$$

where $\pi = s_1s_2\cdots s_{|\pi|}$ and $s_1 = s$.

2.2. Probabilistic Computational Tree Logic

The logic PCTL was originally introduced in [HJ94], where the corresponding model-checking problem has been focused mainly on finite-state Markov chains.

Let $AP$ be a fixed set of atomic propositions. Formally, the syntax of probabilistic computational tree logic PCTL is given by

$$\Phi ::= p \mid \neg \Phi \mid \Phi_1 \land \Phi_2 \mid \text{P}_{\text{core}}(\varphi)$$

$$\varphi ::= X\Phi \mid \Phi_1 U \Phi_2$$

where $\Phi$ and $\varphi$ denote the state formula and path formula respectively; $p \in AP$ is an atomic proposition, $\in \in \{>,=\}$, $r$ is an rational with $0 \leq r \leq 1$.

We do not include other relations of comparison such as $\geq$, $\leq$, and $<$, because $\geq$ and $=$ are sufficient enough for our purpose.
Let $\mathcal{M} = (S, \delta, \mathcal{P})$ be a Markov chain and $\nu : AP \to 2^S$ an assignment and the symbol true the abbreviation of always true. Then the semantics of PCTL, over $\mathcal{M}$, is given by the following rules

$$
\mathcal{M}, s \models ^{\nu} \text{true} \quad \text{for any } s \in S,
$$

$$
\mathcal{M}, s \models ^{\nu} p \quad \text{iff } s \in \nu(p),
$$

$$
\mathcal{M}, s \models ^{\nu} \neg \Phi \quad \text{iff } \mathcal{M}, s \not\models ^{\nu} \Phi,
$$

$$
\mathcal{M}, s \models ^{\nu} \Phi_1 \land \Phi_2 \quad \text{iff } \mathcal{M}, s \models ^{\nu} \Phi_1 \text{ and } \mathcal{M}, s \models ^{\nu} \Phi_2,
$$

$$
\mathcal{M}, s \models ^{\nu} \mathcal{P}_{\text{bor}}(\varphi) \quad \text{iff } \mathcal{P}(\{\pi \in \text{Run}(s) : \mathcal{M}, \pi \models ^{\nu} \varphi\}) \bowtie r,
$$

$$
\mathcal{M}, \pi \models ^{\nu} X\Phi \quad \mathcal{M}, \pi(1) \models ^{\nu} \Phi
$$

$$
\mathcal{M}, \pi \models ^{\nu} \Phi_1 \mathcal{U} \Phi_2 \quad \exists k \geq 0 \text{ s.t. } \mathcal{M}, \pi(k) \models ^{\nu} \Phi_2 \text{ and } \forall j, 0 \leq j < k : \mathcal{M}, \pi(j) \models ^{\nu} \Phi_1
$$

**Remark 2.** The logic PCTL* contains the logic PCTL defined above as a sublogic, of which the path formula is generated by the following syntax:

$$
\varphi ::= \Phi | \neg \varphi | \varphi_1 \land \varphi_2 | X\varphi | \varphi_1 \mathcal{U} \varphi_2.
$$

The difference in formulas between PCTL and PCTL* is very clear: a well-defined PCTL formula is definitely a well-defined PCTL* formula. However, the inverse is not necessarily true. The semantics of PCTL* path formulas are defined as follows, over $\mathcal{M}$:

$$
\mathcal{M}, \pi \models ^{\nu} \Phi \quad \text{iff } \mathcal{M}, \pi(0) \models ^{\nu} \Phi,
$$

$$
\mathcal{M}, \pi \models ^{\nu} \neg \varphi \quad \text{iff } \mathcal{M}, \pi \not\models ^{\nu} \varphi,
$$

$$
\mathcal{M}, \pi \models ^{\nu} \varphi_1 \land \varphi_2 \quad \text{iff } \mathcal{M}, \pi \models ^{\nu} \varphi_1 \text{ and } \mathcal{M}, \pi \models ^{\nu} \varphi_2,
$$

$$
\mathcal{M}, \pi \models ^{\nu} X\varphi \quad \text{iff } \mathcal{M}, \pi_1 \models ^{\nu} \varphi,
$$

$$
\mathcal{M}, \pi \models ^{\nu} \varphi_1 \mathcal{U} \varphi_2 \quad \text{iff } \exists k \geq 0 \text{ s.t. } \mathcal{M}, \pi(k) \models ^{\nu} \varphi_2 \text{ and } \forall j, 0 \leq j < k : \mathcal{M}, \pi(j) \models ^{\nu} \varphi_1.
$$

**Remark 3.** The abbreviation of “s.t.” is “such that”. The logic PCTL or PCTL* can be interpreted over an MDP (Markov decision process) $\mathcal{M}$ in the similar way that we did with the Markov chain. But it is outside of our topic here.

### 2.3. Probabilistic Pushdown Systems

Let us recall the definitions of the probabilistic pushdown systems, being as follows:

$$
\text{7}
$$
**Definition 2.** A probabilistic pushdown system (pPDS) is a tuple \( \triangle = (Q, \Gamma, \delta, \mathcal{P}) \) where \( Q \) is a finite set of control states, \( \Gamma \) a finite stack alphabet, \( \delta \subseteq (Q \times \Gamma) \times (Q \times \Gamma^*) \) a finite set of rules satisfying

- for each \((p, X) \in Q \times \Gamma\) there is at least one rule of the form \((p, X, (q, \alpha)) \in \delta\); In the following we will write \((p, X) \to (q, \alpha)\) instead of \(((p, X), (q, \alpha)) \in \delta\).
- \( \mathcal{P} \) is a function from \( \delta \) to \([0, 1]\) which to each rule \((p, X) \to (q, \alpha)\) in \( \delta \) assigns its probability \( \mathcal{P}((p, X) \to (q, \alpha)) \in [0, 1] \).

Furthermore, without loss of generality, we assume \(|\alpha| \leq 2\). The configurations of \( \triangle \) are elements in \( Q \times \Gamma^* \).

The stateless probabilistic pushdown system (pPBA) is a probabilistic pushdown system (pPDS) whose state set \( Q \) is a singleton (or, we can even omit \( Q \) without any influence).

**Definition 3.** A stateless probabilistic pushdown system (pBPA) is a triple \( \triangle = (\Gamma, \delta, \mathcal{P}) \), whose configurations are elements \( \in \Gamma^* \), where \( \Gamma \) is a finite stack alphabet, \( \delta \) a finite set of rules satisfying

- for each \( X \in \Gamma \) there is at least one rule \((X, \alpha) \in \delta\) where \( \alpha \in \Gamma^* \). In the following, we write \( X \to \alpha \) instead of \((X, \alpha) \in \delta\); We assume, w.l.o.g., that \(|\alpha| \leq 2\).
- \( \mathcal{P} \) is a function from \( \delta \) to \([0, 1]\) which to every rule \( X \to \alpha \) in \( \delta \) assigns its probability \( \mathcal{P}(X \to \alpha) \in [0, 1] \).

Given a pPDS or pBPA \( \triangle \), all of its configurations with all of its transition rules and corresponding probabilities induce an infinite-state Markov chain \( M_\triangle \). The model-checking problem for properties expressed by the PCTL formula is defined as determining whether \( M_\triangle \vDash \psi \).

As shown in [EKS03], if there are no effective valuation assumptions, undecidable properties can be easily encoded to pushdown configurations. Thus, we consider the same assignment as in [EKS03, EKM06, BBF+14, Bra07], which was called a regular assignment. More precisely, let \( \triangle = (Q, \Gamma, \delta, \mathcal{P}) \) be a probabilistic pushdown system, an assignment \( \nu : AP \to 2^{Q \times \Gamma^*} \) (2\( \Gamma^* \) for a pPBA) is regular if \( \nu(p) \) is a regular set for each \( p \in AP \). In other words, finite automata \( A_p \) recognizes \( \nu(p) \) over the alphabet \( Q \cup \Gamma \), and \( A_p \) reads the stack of \( \triangle \) from bottom to top. Furthermore, the regular assignment \( \nu \) is simple if for each \( p \in AP \) there is a subset of heads \( H_p \subseteq Q \cup (Q \times \Gamma) \) s.t. \((q, \gamma \alpha) \in \nu(p) \Leftrightarrow (q, \gamma) \in H_p \) [BBF+14].

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2.4. Post Correspondence Problem

The Post Correspondence Problem (PCP), originally introduced by and shown to be undecidable by Post [Pos46], has been used to show that many problems arising from formal languages are undecidable.

Formally, a PCP instance consists of a finite alphabet \( \Sigma \) and a finite set \( \{ (u_i, v_i) \mid 1 \leq i \leq n \} \subseteq \Sigma^* \times \Sigma^* \) of \( n \) pairs of strings over \( \Sigma \), determining whether there is a word \( j_1 j_2 \cdots j_k \in \{1, 2, \cdots, n\}^+ \) such that

\[
\begin{align*}
  u_{j_1} u_{j_2} \cdots u_{j_k} &= v_{j_1} v_{j_2} \cdots v_{j_k},
\end{align*}
\]

There are numerous variants of the PCP definition, such as the 2-Marked PCP [HHW99], and the modified PCP [BBF14, Bra07]. The latter of which is the most convenient for our discussion here. Since the word \( w \in \Sigma^* \) is of finite length,\(^6\) we can suppose that \( m = \max\{|u_i|, |v_i|\}_{1 \leq i \leq n} \). If we put \( * \) into clearance between two letters of \( u_i \) or \( v_i \), to form the \( u'_i \) or \( v'_i \), such that \( |u'_i| = |v'_i| = m \), then the modified PCP problem is to ask whether there exists \( j_1 \cdots j_k \in \{1, \cdots, n\}^+ \) such that the equation \( u'_{j_1} \cdots u'_{j_k} = v'_{j_1} \cdots v'_{j_k} \) holds after erasing all \( \bullet \) in \( u'_i \) and \( v'_i \).

**Remark 4.** Essentially, the modified PCP problem is equivalent to the original PCP problem. That we stuff the \( n \)-pair strings \( u_i \) and \( v_i \) with \( \bullet \) to make them the same length is useful in Sections ?? and 4.

2.5. Quantum Principles

Let us introduce the quantum principle, which will be needed to define the quantum analogues of the probabilistic pushdown systems and Markov chains. For more details, we refer the reader to the excellent monograph [Hir04]; see p. 105 in [Hir04].

In quantum theory, for any isolated physical system, it is associated with a (finite dimensional) **Hilbert space**, denoted as \( \mathcal{H} \), which is called the state space of the system. In Dirac notation, the row vector (resp. column vector) \( \varphi \) is denoted as \( \langle \varphi \rangle \) (resp. \( |\varphi\rangle \)). Furthermore, \( \langle \varphi \rangle \) is the conjugate-transpose of \( |\varphi\rangle \), i.e., \( \langle \varphi \rangle = |\varphi\rangle^\dagger \). The inner product of two vector \( |\varphi\rangle \) and \( |\eta\rangle \) is denoted as \( \langle \varphi | \eta \rangle \). The norm (or length) of the vector \( |\varphi\rangle \), denoted by \( |||\varphi||| \), is defined to be \( |||\varphi||| = \sqrt{\langle \varphi | \varphi \rangle} \). A vector \( |\varphi\rangle \) is said to be unit if \( |||\varphi||| = 1 \).

Suppose that \( Q = \{ q_1, q_2, \cdots, q_m \} \) is the basic state set of a quantum system. Then the corresponding Hilbert space is \( \mathcal{H}_m = \text{span}\{ |q_i\rangle \mid q_i \in Q, 1 \leq i \leq m \} \) where \( |q_i\rangle \) is an \( m \) dimensional column vector having only 1 at the \( (i, 1) \) entry,\(^7\) together with the inner product \( \langle \cdot | \cdot \rangle \), defined to be \( \langle \alpha | \beta \rangle = \sum_{i=1}^{m} x_i^* y_i \) where \( x_i^* \) stands for the conjugate of \( x_i \) for any complex number \( x_i \in \mathbb{C} \). At any time, the state of this system is a **superposition** of \( |q_i\rangle, 1 \leq i \leq m \), and can be represented by a unit vector \( |\rho\rangle = \sum_{i=1}^{m} c_i |q_i\rangle \) with \( c_i \in \mathbb{C} \) such that \( \sum_{i=1}^{m} |c_i|^2 = 1 \) where \( |x| \) denotes the modulus of complex number \( x \).\(^8\) One can perform a measure

\(^6\)Thank goes to Dr. Forejt for reminding us of that \( |w| \in \mathbb{N} \) for any \( w \in \Sigma^* \) [For13].

\(^7\)\( |q_i\rangle \) is an \( m \) dimensional row vector having only 1 at the \( (1, i) \) entry and 0 elsewhere else.

\(^8\)That is, if \( x = a + b \sqrt{-1} \) then \( |x| = |a + b \sqrt{-1}| = \sqrt{a^2 + b^2} \).
on $\mathcal{H}_m$ to extract some information about the system. A measurement can be described by an observable, i.e., a Hermitian matrix $O = \lambda_1 P_1 + \cdots + \lambda_s P_s$ where $\lambda_i$ is its eigenvalue and $P_i$ is the projector onto the eigenspace corresponding to $\lambda_i$.

To summarize, the quantum principle stated above can be summarized by four postulates, which provide the way to describe and analyse quantum systems and their evolution.

**Postulate 1.** (State space) The state space of any isolated physical system is a complex vector space with inner product, i.e., a Hilbert space $\mathcal{H}$.

**Postulate 2.** (Evolution) The evolution of a closed quantum system is described by a unitary transformation.

**Postulate 3.** (Measurement) Quantum measurements are described by a collection $\{M_m\}$ of measurement operators such that $\sum_m M_m^\dagger M_m = I$, where $m$ refers to the possible measurement outcomes and $I$ the identity matrix. If the state is $|\psi\rangle$ immediately before the measurement then the probability that the result $m$ occurs is $P(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$ and the post-measurement state is $\frac{M_m |\psi\rangle}{\sqrt{P(m)}}$.

**Postulate 4.** (Composite) The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Specifically, if system $i \in \{1, 2, \cdots, n\}$ is prepared in $|\psi_i\rangle$, then the joint state of the total system is $\bigotimes_{i=1}^{n} |\psi_i\rangle = |\psi_1\rangle \otimes |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$.

Based on the above-introduced quantum principle, we will extend the notion of probabilistic pushdown processes and Markov chains to their quantum counterparts in Section 3.

### 3. Quantum Analogues of Markov Chain and Quantum Pushdown Systems

#### 3.1. Quantum Markov Chains

Mathematically, the quantum Markov chain is a reformulation of the ideas of a classical Markov chain, replacing the classical definitions of probability with quantum probability. And the transition rules also satisfy the unitary condition.

**Definition 4.** A (discrete) quantum Markov chain (QMC) is a triple $\tilde{M} = (S, \delta, Q)$ where $S$ is a finite or countably infinite set of basic states, $\delta \subseteq S \times S$ is a transition relation such that for each $s \in S$ there exits $t \in S$ such that $(s, t) \in \delta$, and $Q$ is a function from domain $\delta$ to $\mathbb{C}$ which to each transition $(s, t) \in \delta$ assigns its quantum amplitude $Q(s, t) \in \mathbb{C}$ such that $\sum_t |Q(s, t)|^2 = 1$ for each $s \in S$. 
A path in $\hat{M}$ is a finite or infinite sequence of states of $S: w = s_0s_1 \cdots$ such that $(s_i, s_{i+1}) \in \delta$ for each $i$. A run of $\hat{M}$ is an infinite path. We denote the set of all runs in $\hat{M}$ by $Run$, and $Run(w')$ to denote the set of all runs starting with a given finite path $w'$. Let $w$ be a run, then $w(i)$ denotes the basic state $s_i$ of $w$, and $w_i$ the run $s_is_{i+1} \cdots$. In this way, it is clear that $w_0 = w$. Further, a basic state $s'$ is reachable from a basic state $s$ if there is a finite path starting in $s$ and ending at $s'$.

For each $s \in S$, $(Run(s), \mathcal{F}, \mathcal{P})$ is a probability space, where $\mathcal{F}$ is the $\sigma$-field generated by all basic cylinders $Run(w)$ and $w$ is a finite path initiating from $s$,

$$Cyl(w) = \{ \tilde{w} \in Run(s) : w \in \text{prefix}(\tilde{w}) \},$$

and $\mathcal{P}: \mathcal{F} \to [0, 1]$ is the unique probability measure such that

$$\mathcal{P}(Cyl(w)) = \left| \prod_{1 \leq i \leq |w|-1} Q(s_i, s_{i+1}) \right|^2. \quad (1)$$

where $w = s_1s_2 \cdots s_{|w|}$ and $s_1 = s$.

**Remark 5.** Why is the $\mathcal{P}(Cyl(w))$ defined to (1) for the quantum Markov chain $\hat{M}$? First note that the amplitude matrix $Q_{\hat{M}}$ can be represented by

$$Q_{\hat{M}} \doteq \begin{pmatrix}
Q(s_1, s_1) & Q(s_1, s_2) & \cdots & Q(s_1, s_j) & \cdots \\
\vdots & \ddots & \cdots & \vdots & \cdots \\
Q(s_j, s_1) & Q(s_j, s_2) & \cdots & Q(s_j, s_j) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots
\end{pmatrix}$$

where $Q(s_i, s_j)^9$ means the quantum amplitude of $\hat{M}$ going to basic state $s_j$, when it being in basic state $s_i$.

---

$^9$For simplicity, we denote $Q(s_i, s_j)$ by $\delta_{ij}$ in the sequel; the meaning of $Q(s_i, s_j)$ is similar to probabilistic automaton [Hir08], i.e., from basic state $s_i$ to $s_j$ with quantum amplitude $Q(s_i, s_j)$.
Consider the finite path \( w = s_1 s_2 \cdots s_n \):

\[
\langle \varphi_1 \rangle = \langle s_1 \rangle = \begin{pmatrix}
Q(s_1, s_1) & Q(s_1, s_2) & \cdots & Q(s_1, s_j) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
Q(s_n, s_1) & Q(s_n, s_2) & \cdots & Q(s_n, s_j) & \cdots \\
\end{pmatrix}
\]

\[
= (1, 0, \ldots) \begin{pmatrix}
Q(s_1, s_1) & Q(s_1, s_2) & \cdots & Q(s_1, s_j) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
Q(s_{n+1}, s_1) & Q(s_{n+1}, s_2) & \cdots & Q(s_{n+1}, s_j) & \cdots \\
\end{pmatrix}
\]

\[
= (\delta_{11}, \delta_{12}, \ldots, \delta_{1j}, \ldots);
\]

\[
\langle \varphi_2 \rangle = (0, \delta_{12}, 0, \ldots) \begin{pmatrix}
Q(s_1, s_1) & Q(s_1, s_2) & \cdots & Q(s_1, s_j) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
Q(s_{n+1}, s_1) & Q(s_{n+1}, s_2) & \cdots & Q(s_{n+1}, s_j) & \cdots \\
\end{pmatrix}
\]

\[
= (\delta_{12} \delta_{21}, \delta_{12} \delta_{22}, \ldots, \delta_{12} \delta_{2j}, \ldots).
\]

Proceeding in this way, we have the following:

\[
\langle \varphi_{n-1} \rangle = (0, \ldots, 0, \delta_{12} \delta_{23} \cdots \delta_{n-2n-1}, \ldots) \begin{pmatrix}
Q(s_1, s_1) & Q(s_1, s_2) & \cdots & Q(s_1, s_j) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
Q(s_{n+1}, s_1) & Q(s_{n+1}, s_2) & \cdots & Q(s_{n+1}, s_j) & \cdots \\
\end{pmatrix}
\]

\[
= (\prod_{i=1}^{n-2} \delta_{ii+1}) \delta_{n-11, \ldots, (\prod_{i=1}^{n-2} \delta_{ii+1}) \delta_{n-1n}, \ldots}.
\]

Taking the \( n^{th} \) weight of \( \langle \varphi_{n-1} \rangle \) we get that the quantum amplitude is \( \prod_{i=1}^{n-1} \delta_{ii+1} \). That is, the final quantum amplitude from basic state \( s_1 \) to basic state \( s_n \) along the path \( w \) is \( \prod_{i=1}^{n-1} \delta_{ii+1} \). So the probability of \( P(Cyl(w)) \) is the the square of the modulus of complex number \( \prod_{i=1}^{n-1} \delta_{ii+1} \).
3.2. PCTL∗ is Enough to Describe Branching-Time Properties of Quantum Markov Chains

This sub-section is devoted to illustrating that to specify branching-time properties of QMCs, the probabilistic computational tree logic is enough. So there is no need to extend the quantum principle to probabilistic computational tree logic at all.

Observe the definition of quantum computational tree logic presented in [YF18]. Only a state formula is different from the probabilistic computational tree logic defined in [HJ94]. That is, the probability quantifier in the state formula $P_{\text{dep}}(\varphi)$ has been replaced with a super-operators quantifier, yielding $Q_{\text{dep}}(\varphi)$.$^{10}$ But here we follow the approach in [BV97], viewing the quantum Markov chain as a system driven by quantum principles and showing probabilistic behavior, so to describe the branching-time properties of QMCs, the probabilistic computational tree logic is much more suitable for our purpose.

3.3. Quantum Pushdown Systems

Now let us extend the notion of probabilistic pushdown systems to their quantum analogue as follows:

**Definition 5.** A quantum pushdown system (qPDS) is a tuple $\Delta = (Q, \Gamma, \delta, Q)$ where $Q$ is a finite set of control states, $\Gamma$ a finite stack alphabet, $\delta \subset (Q \times \Gamma) \times (Q \times \Gamma^*)$ a finite set of rules satisfying

- for each $(p, X) \in Q \times \Gamma$ there is at least one rule of the form $((p, X), (q, \alpha)) \in \delta$; In the following we will write $(p, X) \to (q, \alpha)$ instead of $((p, X), (q, \alpha)) \in \delta$. 

- $Q$ is a function from $\delta$ to $\mathbb{C}$ which to each rule $(p, X) \to (q, \alpha) \in \delta$ assigns its quantum amplitude $Q((p, X) \to (q, \alpha)) \in \mathbb{C}$ s.t. for each $(p, X) \in Q \times \Gamma$ satisfying the following

$$\sum_{(q, \alpha)} |Q((p, X) \to (q, \alpha))|^2 = 1$$

Further, without loss of generality, we assume $|\alpha| \leq 2$. The configurations of $\Delta$ are elements in $Q \times \Gamma^*$.

The stateless quantum pushdown system (qPBA) is a quantum pushdown system (qPDS) whose state set $Q$ is a singleton (or, we can even omit $Q$ without any influence).

**Definition 6.** A stateless quantum pushdown system (qPBA) is a triple $\Delta = (\Gamma, \delta, Q)$, whose configurations are elements $\in \Gamma^*$, where $\Gamma$ is a finite stack alphabet, $\delta$ a finite set of rules satisfying

- for each $X \in \Gamma$ there is at least one rule $(X, \alpha) \in \delta$ where $\alpha \in \Gamma^*$. In the following, we write $X \to \alpha$ instead of $(X, \alpha) \in \delta$; We assume, w.l.o.g., that $|\alpha| \leq 2$.

$^{10}$E is a super-operators.
• $Q$ is a function from $\delta$ to $\mathbb{C}$ which to every rule $X \rightarrow \alpha$ in $\delta$ assigns its amplitude $Q(X \rightarrow \alpha) \in \mathbb{C}$ s.t. for each $X \in \Gamma$, it meets the condition that

$$\sum_{\alpha} |Q(X \rightarrow \alpha)|^2 = 1.$$  

Given a qPDS or qBPA $\triangle$, all of its configurations with all of its transition rules and corresponding amplitudes induce an infinite-state quantum Markov chain $\widehat{M}_\triangle$. The model-checking problem for properties expressed by the PCTL formula is defined as determining whether $\widehat{M}_\triangle \models \nu \Psi$.  

4. Proof of Theorem 1

We are going to prove Theorem 1. We still fix $\Sigma = \{A, B, \bullet\}$ and the stack alphabet $\Gamma$ of a qBPA as follows:

$$\Gamma = \{Z, Z', C, F, S, N, (x, y), X(x, y), G^i_1 | (x, y) \in \Sigma \times \Sigma, 1 \leq i \leq n, 1 \leq j \leq m + 1\}$$

The elements in $\Gamma$ also serve as symbols of atomic propositions similar to [Lin14]. We will detail how to build the desirable stateless quantum pushdown system $\triangle = (\Gamma, \delta, Q)$.

Similar to [Lin14], our qBPA $\triangle$ also works by two steps, the first of which is to guess a possible solution to a modified PCP instance by storing pairs of words $(u_i, v_i)$ in the stack, which is done by the following transition rules:

\[\begin{align*}
Z & \rightarrow G^i_1 Z' \cdots |G^i_n Z'; \\
G^i_1 & \rightarrow G^{i+1}_1 (u_i(j), v_i(j)); \\
G^{m+1}_i & \rightarrow C|G^i_1| \cdots |G^i_n. 
\end{align*}\]

(2)

In the above rules, $(u_i(j), v_i(j))$ means that selecting the $i$-th $(u_i, v_i)$ and further selecting the $j$-th symbol in $u_i$ (say $x'$) and the $j$-th symbol in $v_i$ (say $y'$) form the $(x', y')$. Obviously, we should let the symbol $Z$ serve as the initial stack symbol. It begins by pushing $G^1_1 Z'$ ($\in \Gamma^*$) into the stack with quantum amplitude $\frac{\sqrt{n}}{n!} e^{\frac{i\pi \sqrt{n}}{n}}$. Then, the symbol at the top of the stack is $G^1_1$ (we read the stack from left to right). The rules in (2) state that $G^1_1$ is replaced with quantum amplitude $e^{0.1 \sqrt{n+1}}$ by $G^2_1 (u_1(1), v_1(1))$. The process will be repeated until $G^{m+1}_i (u_i(m), v_i(m))$ is stored at the top of the stack, indicating that the first pair of $(u_i, v_i)$ has been stored.

Then, with the amplitude $\frac{\sqrt{n}}{n+1} e^{\frac{i\pi \sqrt{n}}{n+1}}$ or $\frac{\sqrt{n}}{n+1} e^{\frac{i\pi \sqrt{n}}{n+1}}$ for $2 \leq l \leq n + 1$, the $\triangle$ will go to push symbol $C$ or $G^1_1$ into the stack, depending on whether the guessing procedure is at the end or not. When

\[\begin{align*}
\text{We still consider the case of that $\nu$ is a simple assignment.}
\end{align*}\]

\[\begin{align*}
\text{Note that $m$ is the common length of $u^i_1$ and $v^i_1$, and $n$ is the number of pairs of strings over $\Sigma$, see sub section 2.4.}
\end{align*}\]
the rule $G_i \rightarrow C$ is applied, the $\triangle$ goes to check whether the pairs of words stored in the stack are a solution of a modified PCP instance. It is clear that the above guess procedure will lead to a word

\[ j_1j_2\cdots j_k \in \{1,2,\cdots,n\}^+ \]

corresponding to the sequence of the words \((u_{j_1},v_{j_1}),(u_{j_2},v_{j_2}),\cdots,(u_{j_k},v_{j_k})\) pushed orderly into the stack. In addition, there are no other transition rules in the guessing-step for $\triangle$ except those illustrated by (2).

Lemma 1. A configuration of the form $C\alpha Z'$ is reachable from $Z$ if and only if $\alpha \equiv (x_1,y_1)\cdots(x_i,y_i)$ where $x_j,y_j \in \Sigma$, and there is a word $j_1j_2\cdots j_k \in \{1,2,\cdots,n\}^+$ such that $x_1\cdots x_1 = u_{j_1}\cdots u_{j_k}$ and $y_i\cdots y_i = v_{j_1}\cdots v_{j_k}$. And the probability from $Z$ to $C\alpha Z'$ is greater than 0.

The next step is for $\triangle$ to verify a stored pair of words, and the transition rules are given as follows:

\[
\begin{align*}
C & \rightarrow N, \\
N & \rightarrow F|S, \\
F & \rightarrow \epsilon, \\
S & \rightarrow \epsilon, \\
(x,y) & \rightarrow X_{(x,y)}|\epsilon, \\
Z' & \rightarrow X_{(A,B)}|X_{(B,A)}, \\
X_{(x,y)} & \rightarrow \epsilon.
\end{align*}
\]

Remark 6. We still emphasize again, there are no other rules in verifying-step besides those described by (3).

When the stack symbol $C$ is at the top of the stack, the $\triangle$ will check to see if the previous guess is a solution to the modified PCP instance. It first replaces $C$ with $N$ at the top of the stack, with an amplitude of $e^{\theta_1\sqrt{-1}}$, and then continues to push $F$ or $S$ into the stack, with a quantum amplitudes $\sqrt{\frac{\theta}{2}}e^{\frac{2\pi}{7}v^2}$ or $\sqrt{\frac{\theta}{2}}e^{\frac{2\pi}{7}v^2}$, depending on whether the $\triangle$ wants to check $u$'s or $v$'s.

Lemma 2 (Cf. [Lin14]). Let $\vartheta$ and $\overline{\vartheta}$ be two functions from $\{A,B,Z'\}$ to $\{0,1\}$, given by

\[
\vartheta(x) = \begin{cases} 
1, & x = Z' \\
1, & x = A \\
0, & x = B. 
\end{cases} \quad \overline{\vartheta}(x) = \begin{cases} 
1, & X = Z' \\
0, & X = A; \\
1, & X = B. 
\end{cases}
\]

Then, we have $\vartheta(x) + \overline{\vartheta}(x) = 1$ for $x \in \{A,B\}$. Furthermore, let $\rho$ and $\overline{\rho}$ be two functions from $\{A,B\}^+ Z'$ to $[0,1]$, given by

\[
\rho(x_1x_2\cdots x_n) \triangleq \sum_{i=1}^{n} \vartheta(x_i) \frac{1}{2^n}, \\
\overline{\rho}(x_1x_2\cdots x_n) \triangleq \sum_{i=1}^{n} \overline{\vartheta}(x_i) \frac{1}{2^n}.
\]
Then, for any \((u'_{j_1}, v'_{j_1}), (u'_{j_2}, v'_{j_2}),\ldots, (u'_{j_k}, v'_{j_k})\) ∈ \(A, B\)^+ × \(A, B\)^+,

\[u'_{j_1} u'_{j_2} \cdots u'_{j_k} = v'_{j_1} v'_{j_2} \cdots v'_{j_k}\] (4)

if and only if

\[\rho(u'_{j_1} \cdots u'_{j_k} Z') + \overline{\rho}(v'_{j_1} v'_{j_2} \cdots v'_{j_k} Z') = 1.\] (5)

**Proof.** It is very clear that the first half for \(\vartheta(x) + \overline{\vartheta}(x) = 1\) for \(x \in \{ A, B\}\). We show the second half.

The “only if” part is clear. Suppose that (4) holds true and that \(u'_{j_1} \cdots u'_{j_k} = y_1 \cdots y_l = v'_{j_1} \cdots v'_{j_k}\). Then we have

\[
\rho(y_1 \cdots y_l Z') + \overline{\rho}(y_1 \cdots y_l Z') = \sum_{i=1}^{l} (\vartheta(y_i) + \overline{\vartheta}(y_i)) \frac{1}{2^i} + (\vartheta(Z') + \overline{\vartheta}(Z')) \frac{1}{2^{l+1}} = 1
\]

The “if” part. If (5) is false, suppose that \(u'_{j_1} \cdots u'_{j_k} = x_1 x_2 \cdots x_l\) and \(v'_{j_1} \cdots v'_{j_k} = y_1 y_2 \cdots y_m\), then

\[
\rho(u'_{j_1} \cdots u'_{j_k} Z') + \overline{\rho}(v'_{j_1} \cdots v'_{j_k} Z') \neq 1.
\]

This case must involve the following two cases:

1. \(l \neq m\);
2. \(l = m\) and there is at least a \(k: 1 \leq k \leq l\) such that \(x_k \neq y_k\).

Then, either of the above two cases shows that \(u'_{j_1} u'_{j_2} \cdots u'_{j_k} \neq v'_{j_1} v'_{j_2} \cdots v'_{j_k}\). This complete the proof. □

Because of Lemma 2, the path formulas \(\varphi_1\) and \(\varphi_2\) defined in (6) and (7) are still useful. That is, these two path formulas are in connection with \(\rho(u'_{j_1} \cdots u'_{j_k} Z')\) and \(\overline{\rho}(v'_{j_1} \cdots v'_{j_k} Z')\), respectively. To do so, we should first prove the following:

**Lemma 3.** Let \(w = s_1 s_2 \cdots s_n\) be a path in quantum Markov chain \(\hat{M}\) and \(\mathcal{P}(s_i, s_j)\) the probability of \(\hat{M}\) from basic state \(s_i\) to \(s_j\). Then

\[
\mathcal{P}(\text{Cyl}(w)) = \prod_{i=1}^{n-1} \mathcal{P}(s_i, s_{i+1}).
\]

**Proof.** Suppose that \(\delta_{i(i+1)} = r_i e^{i \sqrt{-1}}\) where \(0 < r_i \leq 1\) and \(0 < \eta_i \leq 2\pi\) for \(1 \leq i \leq n - 1\). Then
\(P(s_i, s_{i+1}) = |\delta_{i(i+1)}|^2 = r_i^2\). So by (1) we get

\[
P(Cyl(w)) = \left| \prod_{i=1}^{n-1} \delta_{i(i+1)} \right|^2
\]

\[
= \left| \left( \prod_{i=1}^{n-1} r_i \right) e^{\left( \sum_{i=1}^{n-1} \eta_i \right) \nu^{-1}} \right|^2
\]

\[
= \prod_{i=1}^{n-1} r_i^2
\]

\[
= \prod_{i=1}^{n-1} P(s_i, s_{i+1}). \quad \square
\]

We summarize the above analysis in the following Lemma, which establishes the connection between \(P(\{\pi \in \text{Run}(\alpha Z') \mid \pi \models \varphi_1\})\) and the function \(\rho\), and the connection between \(P(\{\pi \in \text{Run}(\alpha Z') \mid \pi \models \varphi_2\})\) and the function \(\pi\), respectively. To prove it, we need to fix an additional notation: Let \(\text{trim}(b_1 b_2 \cdots b_n)\) denote the resultant word \(\in \{A, B\}^*\) in which all the \(\bullet\) in \(b_1 b_2 \cdots b_n\) are erased. Then \(\text{trim}(b_2 b_3 \cdots b_n)\) means the resultant word \(\in \{A, B\}^*\) in which all the \(\bullet\) in \(b_2 b_3 \cdots b_n\) are erased.

**Lemma 4 (Cf. [Bra07]).** Let \(\alpha = (x_1, y_1)(x_2, y_2) \cdots (x_i, y_i) \in \Sigma^* \times \Sigma^*\) be the pair of words pushed into the stack by \(\Delta\), where \(x_i, y_i \in \Sigma\), and \((u_{i_1}', v_{i_1}'), \ldots, (u_{i_k}', v_{i_k}')\), \(1 \leq i \leq k\), the pair of words after erasing all \(\bullet\) in \(x_1 x_2 \cdots x_i\) and \(y_1 y_2 \cdots y_i\). Then

\[
P(\{\pi \in \text{Run}(\alpha Z') \mid \pi \models \varphi_1\}) = \rho(u_{i_1}', u_{i_2}', \ldots, u_{i_k}', Z')
\]

and

\[
P(\{\pi \in \text{Run}(\alpha Z') \mid \pi \models \varphi_2\}) = \pi(v_{i_1}', v_{i_2}', \ldots, v_{i_k}', Z').
\]

**Proof.** We will show by induction on \(i\) that

\[
P(\alpha Z', \varphi_1) \triangleq P(\{\pi \in \text{Run}(\alpha Z') \mid \pi \models \varphi_1\}) = \rho(\text{trim}(x_1 x_2 \cdots x_i) Z');
\]

Similar arguments apply for

\[
P(\alpha Z', \varphi_2) \triangleq P(\{\pi \in \text{Run}(\alpha Z') \mid \pi \models \varphi_2\}) = \pi(\text{trim}(y_1 y_2 \cdots y_i) Z').
\]

Note that by (3), \(\alpha Z' \rightarrow \alpha Z'\) with transition probability 1, i.e., \(P(\alpha Z', \varphi_1) = P(\alpha Z', \varphi_1)\). Thus, to prove the lemma, we need only to show

\[
P(\alpha Z', \varphi_1) = \rho(\text{trim}(x_1 x_2 \cdots x_i) Z').
\]

We give a proof by induction on \(i\).

**Base case:** The case of \(i = 1\):
1. if \( x_1 = \bullet \), then \( \mathcal{P}((\bullet, z)Z', \varphi_1) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{1}{2} = \rho(\text{trim}(\bullet)Z') \);
2. if \( x_1 = B \), then \( \mathcal{P}((B, z)Z', \varphi_1) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} = \rho(\text{trim}(B)Z') \);
3. if \( x_1 = A \), then \( \mathcal{P}((A, z)Z', \varphi_1) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \rho(\text{trim}(A)Z') \).

Induction step: Assume the induction hypothesis for \( l = n - 1 \) is true, i.e.,

\[
\mathcal{P}((x_2, y_2)(x_3, y_3) \cdots (x_n, y_n)Z', \varphi_1) = \rho(\text{trim}(x_2x_3 \cdots x_n)Z').
\]

Now we consider the case of \( l = n \), i.e., \( \mathcal{P}((x_1, y_1)\alpha'Z', \varphi_1) \) where \( \alpha' = (x_2, y_2) \cdots (x_n, y_n) \). Note that \( (x_1, y_1)\alpha'Z \rightarrow \frac{1}{2} X \alpha'Z' \rightarrow \alpha'Z' \) and \( (x_1, y_1)\alpha'Z \rightarrow \frac{1}{2} \alpha'Z' \):

1. if \( x_1 = \bullet \), then \( \mathcal{P}((x_1, y_1)\alpha'Z', \varphi_1) = \frac{1}{2} \rho(\text{trim}(x_2 \cdots x)n)Z') + \frac{1}{2} \rho(\text{trim}(x_2 \cdots x_n)Z') = \rho(\text{trim}(x_1x_2 \cdots x_n)Z') \);
2. if \( x_1 = B \), then \( \mathcal{P}((x_1, y_1)\alpha'Z', \varphi_1) = \frac{1}{2} \rho(\alpha'Z', \varphi_1) = \frac{1}{2} \rho(\text{trim}(x_2 \cdots x_n)Z') = \rho(\text{trim}(x_1x_2 \cdots x_n)Z') \);
3. if \( x_1 = A \), then \( \mathcal{P}((x_1, y_1)\alpha'Z', \varphi_1) = \frac{1}{2} + \frac{1}{2} \rho(\text{trim}(x_2 \cdots x_n)Z') = \rho(\text{trim}(x_1x_2 \cdots x_n)Z') \).

From which it immediate follows that

\[
\mathcal{P}(\{ \pi \in \text{Run}(FaZ') \mid \pi \models \varphi_1 \}) = \rho(u'_{j_1} u'_{j_2} \cdots u'_{j_k} Z').
\]

The similar arguments apply for \( \mathcal{P}(\{ \pi \in \text{Run}(SaZ') \mid \pi \models \varphi_1 \}) = \overline{\rho}(u'_{j_1} u'_{j_2} \cdots u'_{j_k} Z'). \)

Let us still take \( \alpha = (A, A)(A, \bullet)(\bullet, A)(B, B) \) as an example:

\[
\begin{array}{c}
\text{CaZ'} \\
\text{NaZ'}
\end{array}
\]

\[
\begin{array}{c}
\text{FaZ'} \\
\text{SaZ'}
\end{array}
\]

Figure 1: CaZ’’s unfolding tree.

The above Figure 2 shows the evolutionary process of FaZ’ (with quantum amplitude), where \( \alpha_1 = (A, \bullet)(\bullet, A)(B, B) \), \( \alpha_2 = (\bullet, A)(B, B) \) and \( \alpha_3 = (B, B) \).

There are 4 paths from basic state \( FaZ' \) to basic states begin with \( X_{(A, z)} \) where \( z \in \Sigma \)

\[
\begin{align*}
\text{FaZ'} & \rightarrow \alpha Z' \rightarrow X_{(A, A)} \alpha_1 Z' & \text{(with amplitude } \frac{\sqrt{2}}{8} e^{(\theta_3 + \frac{1}{2} \theta_5) \sqrt{-1}}) \\
\text{FaZ'} & \rightarrow \alpha Z' \rightarrow \alpha_1 Z' \rightarrow X_{(A, \bullet)} \alpha_2 Z' & \text{(with amplitude } \frac{1}{8} e^{(\theta_3 + 3 \theta_5) \sqrt{-1}}) \\
\text{FaZ'} & \rightarrow \alpha Z \rightarrow \alpha_1 Z' \rightarrow \alpha_2 Z' \rightarrow X_{(\bullet, A)} \alpha_3 Z' \rightarrow \alpha_3 Z' \rightarrow Z' \rightarrow X_{(A, B)} & \text{(with amplitude } \frac{\sqrt{2}}{8} e^{(\theta_3 + \frac{1}{2} \theta_7 + \theta_6 + \theta_5) \sqrt{-1}}) \\
\text{FaZ'} & \rightarrow \alpha Z \rightarrow \alpha_1 Z' \rightarrow \alpha_2 Z' \rightarrow \alpha_3 Z \rightarrow Z' \rightarrow X_{(A, B)} & \text{(with amplitude } \frac{\sqrt{2}}{8} e^{(\theta_3 + 4 \theta_7 + \theta_6) \sqrt{-1}})
\end{align*}
\]
Figure 2: $\alpha Z'$'s unfolding tree in quantum transitions

So the total probability is \(\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{8}\right)^2 + \left(\frac{\sqrt{2}}{8}\right)^2\) which still matches the value

\[
\rho(AABZ') = \vartheta(A)\frac{1}{2} + \vartheta(A)\frac{1}{2^2} + \vartheta(B)\frac{1}{2^3} + \vartheta(Z')\frac{1}{2^4}
\]

Remark 7. In fact, the above two path formulas, $\varphi_1$ and $\varphi_2$, were used in [Bra07] to specify the same paths illustrated above. Note that the atomic propositions $F$, $S$ and $X_{(z,z')}$ ($z, z' \in \Sigma$) are valid in exactly all
By (9) and Lemma 2, we conclude that (8) holds true.

\[
\pi(k) \models X(A, z) \quad (k \geq 0)
\]

\[
\pi(i) \models \neg S \land \bigwedge_{z \in \Sigma} (\neg X(A, z) \land \neg X(B, z)) \quad (0 \leq i < k).
\]

So, by Lemma 4, we can still successfully reduce the modified PCP problem to the model-checking problem of whether \(CaZ \models \nu X(P_{=t_1}(\varphi_1) \land P_{=t_2}(\varphi_2))\) s.t. \(t_1 + t_2 = 1\), which can be formalized as:

**Lemma 5.** Let \(\alpha = (u_{j_1}, v_{j_1})(u_{j_2}, v_{j_2})\ldots(u_{j_k}, v_{j_k}) \in \Sigma^* \times \Sigma^*\) be the pair of words pushed into the stack by \(\Delta\). Let \((u'_i, v'_i)\), \(1 \leq i \leq j_k\), be the pair of words after erasing all \(\bullet\) in \(u_i\) and \(v_i\). Then

\[
u u'_{j_1} \ldots u'_{j_k} = v'_{j_1} \ldots v'_{j_k}
\]

if and only if

\[
\overline{M_\Delta, NaZ'} \models \nu P_{=\frac{t}{2}}(\varphi_1) \land P_{=\frac{1-t}{2}}(\varphi_2)
\]

where \(t\): \(0 < t < 1\) is a rational constant.

**Proof.** It is obvious that when \(\alpha\) is pushed into the stack of \(\Delta\), the stack’s content is \(CaZ'\) (read from left to right). Note that there is only one rule, \(C \rightarrow N\) which is applicable. Thus, with probability 1, the content of the stack changes to \(NaZ'\).

The “if” part. Suppose that

\[
\mathcal{M}_\Delta, NaZ' \models \nu P_{=\frac{t}{2}}(\varphi_1) \land P_{=\frac{1-t}{2}}(\varphi_2).
\]

The probability of paths from \(N\) that satisfy \(\varphi_1\) is then \(\frac{t}{2}\), and the probability of paths from \(N\) that satisfy \(\varphi_2\) is \(\frac{1-t}{2}\). As a result, the probability of paths from \(F\) satisfying \(\varphi_1\) is \(t\), while the probability of paths from \(S\) satisfying \(\varphi_2\) is \(1-t\). Because \(P(N \rightarrow F) = P(N \rightarrow S) = \frac{1}{2}\), we have the following:

\[
\rho(u'_{j_1} \ldots u'_{j_k} Z') + \overline{P}(v'_{j_1} \ldots v'_{j_k} Z') = t + (1 - t) = 1.
\]

By (9) and Lemma 2, we conclude that (8) holds true.

The “only if” part. Obviously, that (8) is true leads to

\[
\rho(u'_{j_1} \ldots u'_{j_k} Z') + \overline{P}(v'_{j_1} \ldots v'_{j_k} Z') = 1 \Rightarrow \rho(u'_{j_1} \ldots u'_{j_k} Z') = 1 - \overline{P}(v'_{j_1} \ldots v'_{j_k} Z').
\]

Namely, \(P(FaZ' \models \nu \varphi_1) = 1 - P(SaZ' \models \nu \varphi_2)\) which, together with \(P(N \rightarrow F) = P(N \rightarrow S) = \frac{1}{2}\), further implies that

\[
\mathcal{M}_\Delta, NaZ' \models \nu P_{=\frac{t}{2}}(\varphi_1) \land P_{=\frac{1-t}{2}}(\varphi_2).
\]

The lemma follows. □
Now, Theorem 1 can be proved as follows:

**Proof of Theorem 1.** Let \( \pi \) be a path of qBPA \( \triangle \), starting at \( C \), induced by \( CaZ' \), where \( \alpha \) is guessed by \( \triangle \) as a solution of the modified PCP instance. Then, we get that

\[
(8) \text{ is true } \iff \hat{M}_{\triangle}, N \alpha Z' \models^{e} \mathcal{P} = \frac{2}{t} (\varphi_1) \wedge \mathcal{P} = \frac{1}{2} (\varphi_2) \quad \text{(by Lemma 5)}
\]

\[
\iff \hat{M}_{\triangle}, CaZ \models^{e} \mathcal{X}[\mathcal{P} = \frac{2}{t} (\varphi_1) \wedge \mathcal{P} = \frac{1}{2} (\varphi_2)] \quad \text{(by } C \rightarrow N\text{)}
\]

\[
\iff \hat{M}_{\triangle}, C \models^{e} \mathcal{P} = \frac{2}{t} (\varphi_1) \wedge \mathcal{P} = \frac{1}{2} (\varphi_2) \quad \text{(by } Q(C \rightarrow N) = e^{\theta_3 \sqrt{-1}} \text{ where } 0 < \theta_3 \leq \frac{\pi}{2}\text{)}
\]

\[
\iff \hat{M}_{\triangle}, Z \models^{e} \mathcal{P} > 0 (\text{trueU}[C \wedge \mathcal{P} = 1 (\mathcal{X} [\mathcal{P} = \frac{2}{t} (\varphi_1) \wedge \mathcal{P} = \frac{1}{2} (\varphi_2) ])] \quad \text{(by Lemma 1)}.
\]

Thus

\[
\hat{M}_{\triangle}, Z \models^{e} \mathcal{P} > 0 (\text{trueU}[C \wedge \mathcal{P} = 1 (\mathcal{X} [\mathcal{P} = \frac{2}{t} (\varphi_1) \wedge \mathcal{P} = \frac{1}{2} (\varphi_2) ])])) \quad \text{(10)}
\]

if and only if \( \alpha \) is a solution of the modified PCP instance. As a result, an algorithm for determining whether (10) is true leads to an algorithm for solving the modified Post Correspondence Problem. ■

**Remark 8.** At a glance, our definition of quantum pushdown systems is somewhat similar to the quantum context-free grammars defined in [MC00]. In fact, there are two different notions because each production \( \beta \rightarrow \gamma \) of quantum context-free grammars is with a set of complex amplitudes. And our quantum transition rules defined in Definition 5 and Definition 6 only have a complex amplitude. So, the results regarding quantum context-free grammars in [MC00] is irrelevant to this work.

5. Conclusions

We have extended the notions of probabilistic pushdown systems and Markov chains to their quantum analogues. Moreover, the corresponding model-checking question for stateless quantum pushdown systems (qPBA) against PCTL has been investigated. We obtain a similar conclusion that this question is generally undecidable, which has rich implications. For example, it implies that both the model-checking questions for qPBA against PCTL* and for qPDS against PCTL are generally undecidable as well. Other questions concerning quantum pushdown systems are outside the scope of our interests.

Although the result in this work indicates that there exists no exact algorithm for model-checking pBPA against PCTL, as well as for model-checking qPBA (qPDS) against PCTL(PCTL*), the research method presented in [AAG+15] may be applicable, which is a possible direction.

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