The existence and approximation of the periodic solutions for system of second order nonlinear differential equations by using Lebesgue integrable

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ABSTRACT

In this paper we study the existence and approximation of the periodic solutions for system of second order nonlinear differential equations according to Lebesgue integrable concept and by assuming that each of measurable at t and bounded by $f(t,x,y,x,y), g(t,x,y,x,y)$ the functions Lebesgue integrable functions numerical -- analytic method has been used to study the periodic The solutions of ordinary differential equations which were introduced by A.M. Samoilenko.
INTRODUCTION

There are many subjects in physics and technology use mathematical
methods that depends on the nonlinear differential equations, and it
became clear that the existence of the periodic solutions and its algorithm
structure from more important problems in the present time, because of
the great possibility for employment the electronic computers the
numerical analytic method [6] which suggested by Samoilenko to study
the periodic solutions for the linear and nonlinear differential equations
became the effective mean to find the periodic solutions and its algorithm
structure and this method include uniformly sequences of the periodic
functions and the result of that study is the using of the periodic solutions
on a wide range in the difference of the new processes in industry and
technology as in the studies [1,2,4,5].

We study in this paper the system of the nonlinear differential equations
with the form

\[
\begin{align*}
\frac{d^2 x(t)}{dt^2} &= (A + B(t))(x(t) + \dot{x}(t)) + f(t, x, y, \dot{x}, \dot{y}) \\
\frac{d^2 y(t)}{dt^2} &= (C + D(t))(y(t) + \dot{y}(t)) + g(t, x, y, \dot{x}, \dot{y})
\end{align*}
\] ..........(1)

represents a closed domain and bounded, the each of
the functions

\[
\begin{align*}
f(t, x, y, \dot{x}, \dot{y}) &= (f_1(t, x, y, \dot{x}, \dot{y}), f_2(t, x, y, \dot{x}, \dot{y}), \ldots, f_n(t, x, y, \dot{x}, \dot{y})) \\
g(t, x, y, \dot{x}, \dot{y}) &= (g_1(t, x, y, \dot{x}, \dot{y}), g_2(t, x, y, \dot{x}, \dot{y}), \ldots, g_n(t, x, y, \dot{x}, \dot{y}))
\end{align*}
\]

are defined, vector and continuous in the domain
\[(t, x, y, \dot{x}, \dot{y}) \in [0, T] \times D \times D_1 \times D_2 \times D_3 \] ..........(2)

and periodic in \(t\) of period \(T\) where \(D_1, D_2, D_3\) represents a closed
domain and bounded partly from the \(R^n\). Euclidean space and by
assumption that the two functions \(f(t, x, y, \dot{x}, \dot{y}), g(t, x, y, \dot{x}, \dot{y})\)
satisfy the following inequalities:

\[
\begin{align*}
\|f(t, x, y, \dot{x}, \dot{y})\| &\leq m_1(t) \quad , \quad \|g(t, x, y, \dot{x}, \dot{y})\| \leq m_2(t) \\
\|f(t, x_1, y_1, \dot{x}_1, \dot{y}_1) - f(t, x_2, y_2, \dot{x}_2, \dot{y}_2)\| &\leq K_1(t)\|x_1 - x_2\| + K_2(t)\|y_1 - y_2\| \\
&\quad + K_3(t)\|\dot{x}_1 - \dot{x}_2\| + K_4(t)\|\dot{y}_1 - \dot{y}_2\| \\
\|g(t, x_1, y_1, \dot{x}_1, \dot{y}_1) - g(t, x_2, y_2, \dot{x}_2, \dot{y}_2)\| &\leq L_1(t)\|x_1 - x_2\| + L_2(t)\|y_1 - y_2\| \\
&\quad + L_3(t)\|\dot{x}_1 - \dot{x}_2\| + L_4(t)\|\dot{y}_1 - \dot{y}_2\|
\end{align*}
\] ..........(3)
for all \( t \in [0, T] \), \( x, x_1, x_2 \in D, y, y_1, y_2 \in D_1, \dot{x}, \dot{x}_1, \dot{x}_2 \in D_2 \) and \( \dot{y}, \dot{y}_1, \dot{y}_2 \in D_3 \) where each of
\[
m_1(t), m_2(t), K_1(t), K_2(t), K_3(t), K_4(t), L_1(t), L_2(t), L_3(t), L_4(t)\]
\( 0 \leq t \leq T \) be Lebesgue integrable functions in the interval and by assuming that, \( A = [A_{ij}], B(t) = [B_{ij}(t)], C = [C_{ij}], D(t) = [D_{ij}(t)] \) positive matrices are defined in the domain \( -\infty, 0 \leq t \leq T < \infty \) continuous and periodic in \( t \) have size \((n \times n)\) and satisfy the following inequalities
\[
\left\| e^{A(t-s)} \right\| \leq Q, \quad \left\| e^{C(t-s)} \right\| \leq R \quad \text{.........(6)}
\]
\[
\left\| B(t) \right\| \leq H, \quad \left\| D(t) \right\| \leq J, \quad \left\| A \right\| = N_1, \quad \left\| C \right\| = N_2 \quad \text{.........(7)}
\]
\[
\left\| x_0 \right\| = \delta_o, \quad \left\| y_0 \right\| = \sigma_o \quad \text{.........(8)}
\]
where \( -\infty, 0 \leq t \leq T < \infty, Q, R, H, J, \delta_o, \sigma_o \) positive constants and let \( 0 \leq t \leq b_1 \leq T, 0 \leq t \leq b_2 \leq T \) where \( b_1, b_2 \) are two chosen points so that
\[
\int_{0}^{b_1} m_1(t) \, dt \leq c_1, \quad \int_{0}^{b_1} K_1(t) \, dt \leq c_2 \leq 1, \quad \int_{0}^{b_1} K_2(t) \, dt \leq c_3 \leq 1
\]
\[
, \quad \int_{0}^{b_1} K_3(t) \, dt \leq c_4 \leq 1, \quad \int_{0}^{b_1} K_4(t) \, dt \leq c_5 \leq 1 \quad \text{.........(9)}
\]
\[
\int_{0}^{b_2} m_2(t) \, dt \leq \delta_1, \quad \int_{0}^{b_2} L_1(t) \, dt \leq \delta_2 \leq 1, \quad \int_{0}^{b_2} L_2(t) \, dt \leq \delta_3 \leq 1
\]
\[
, \quad \int_{0}^{b_2} L_3(t) \, dt \leq \delta_4 \leq 1, \quad \int_{0}^{b_2} L_4(t) \, dt \leq \delta_5 \leq 1 \quad \text{.........(10)}
\]

**Definition 1 [6]:**

The system of nonlinear differential equations (1) where the right-hand side defined and continuous and periodic in \( t \) of period \( T \) in the domain (2)
is said to be system – \( T \) if
1. The two sets \( D_f, D_{1f}, D_g, D_{1g} \) are not empty
\[
\begin{align*}
D_f &= D - \frac{T}{2} \left( \frac{T}{2} N + QC_1 \right) \neq \phi \\
D_{1f} &= D_2 - \left( \frac{T}{2} N + QC_1 \right) \neq \phi \\
D_g &= D_1 - \frac{T}{2} \left( \frac{T}{2} N^* + R\delta_1 \right) \neq \phi \\
D_{1g} &= D_3 - \left( \frac{T}{2} N^* + R\delta_1 \right) \neq \phi
\end{align*}
\]

.........(11)
where \( N^* = R^2 J\sigma_o, N = Q^2 H\delta_o \), \( \| \| = \max_{t \in [0,T]} \| \)

2- The greatest eigen value of the matrix

\[
\Omega_2 = \begin{bmatrix}
\frac{T}{2} Q(N_1 + H) + QC_2 & \frac{T}{2} Q(H + QC_4) & \frac{T}{2} QC_3 & \frac{T}{2} QC_5 \\
\frac{T}{2} Q(N_1 + H) + QC_2 & \frac{T}{2} QH + QC_4 & QC_3 & QC_5 \\
\frac{T}{2} R\delta_2 & \frac{T}{2} R\delta_4 & \frac{T}{2} (R(N_2 + J) + R\delta_3) & \frac{T}{2} (RJ + R\delta_3) \\
R\delta_2 & R\delta_4 & \frac{T}{2} (R(N_2 + J) + R\delta_3) & \frac{T}{2} (RJ + R\delta_3)
\end{bmatrix}
\]

is not greater than 1 that is

\[
\lambda_{\max}(\Omega_2) = \frac{\tilde{\xi}_1 + \sqrt{\tilde{\xi}_1^2 + 4\tilde{\xi}_2}}{2} < 1
\]

......(12)

Where

\[
\tilde{\xi}_1 = \frac{T}{2} \rho_1 + \rho_2 + \frac{T}{2} \rho_3 + \rho_4 \quad , \quad \tilde{\xi}_2 = \frac{T^2}{4} v_3 - \frac{T^2}{4} \rho_1 \rho_3 - \frac{T}{2} \rho_1 \rho_4 - \frac{T}{2} \rho_2 \rho_3 - \frac{T}{2} v_1 v_4 + \frac{T}{2} v_2 v_3 - \rho_2 \rho_4 + v_2 v_4
\]

\[
\rho_1 = \frac{T}{2} Q(N_1 + H) + QC_2, \quad \rho_2 = \frac{T}{2} QH + QC_4, \quad \rho_3 = \frac{T}{2} R(N_2 + J) + R\delta_3, \\
\rho_4 = \frac{T}{2} RJ + R\delta_5, \quad v_1 = QC_3, \quad v_2 = QC_5, \quad v_3 = R\delta_2, \quad v_4 = R\delta_4
\]

Definition 2 [6]:-

The value of intermediary \( \mu^* = (\mu_1^*, \mu_2^*) \) in the point \((t, x_o, y_o)\) which be on it the solution of the system

\[
\frac{d^2 x(t)}{dt^2} = (A + B(t))(x(t) + \dot{x}(t)) + f(t, x, y, \dot{x}, \dot{y}) - \mu_1^* \\
\frac{d^2 y(t)}{dt^2} = (C + D(t))(y(t) + \dot{y}(t)) + g(t, x, y, \dot{x}, \dot{y}) - \mu_2^*
\]

......(13)

Periodic in \( t \) of period \( T \) is called constant - \( \Delta^* \) for the system (1) through the point \( t = 0, x = x_o, y = y_o \) if the intermediary is unique in that point.
Section One : The periodic approximate solution for the system (1)

Lemma 1 :-
Assume that each of \( f(t,x,y,\dot{x},\dot{y}) \) and \( g(t,x,y,\dot{x},\dot{y}) \) be vector function and continuous and defined in the interval \([0,T]\) then the inequality

\[
\begin{align*}
\left\| M_1^*(t,x_o,y_o,0,0) \right\| &\leq \left( \beta_1(t)N + QC_1 \right) \\
\left\| M_2^*(t,x_o,y_o,0,0) \right\| &\leq \left( \beta_2(t)N^* + R\delta \right)
\end{align*}
\]

holds for \( 0 \leq t \leq T \), \( \beta_1(t) \leq \frac{T}{2} \), \( \beta_2(t) \leq \frac{T}{2} \), where

\[
N^* = R^2 J\sigma_o, N = Q^2 H\delta_o
\]

\[
\begin{align*}
\beta_1(t) &= \frac{t(2e\|A\|\|T-t\| - e\|A\|\|T\| - E\| + T(e\|A\|\|T\| - e\|A\|\|T-t\|))}{e\|A\|\|T\| - E\|} \\
\beta_2(t) &= \frac{t(2e\|C\|\|T-t\| - e\|C\|\|T\| - E\| + T(e\|C\|\|T\| - e\|C\|\|T-t\|))}{e\|C\|\|T\| - E\|}
\end{align*}
\]

\[
M_1^*(t,x_o,y_o,0,0) = \int_0^t e^{A(t-s)} [B(s)(x_o e^{At} + 0) + f(s,x_o,y_o,0,0) -
\frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)} [B(s)(x_o e^{At} + 0) + f(s,x_o,y_o,0,0)] ds] ds
\]

\[
M_2^*(t,x_o,y_o,0,0) = \int_0^t e^{C(t-s)} [D(s)(y_o e^{Ct} + 0) + g(s,x_o,y_o,0,0) -
\frac{C}{e^{CT} - E} \int_0^T e^{C(T-s)} [D(s)(y_o e^{Ct} + 0) + g(s,x_o,y_o,0,0)] ds] ds
\]

Proof

\[
\| M_1(t,x_o,y_o,0,0) \| \leq \\
\leq \left\| E \right\| - \left( \frac{e\|A\|\|T\| - e\|A\|\|T-t\|}{e\|A\|\|T\| - E\|} \right) \left[ \int_0^t e^{A(t-s)} \|\|B(s)\|\| x_o \| e^{At} \| + \|f(s,x_o,y_o,0,0)\| \right] ds + \\
+ \frac{e\|A\|\|T\| - e\|A\|\|T-t\|}{e\|A\|\|T\| - E\|} \int_t^T e^{A(t-s)} \|\|B(s)\|\| x_o \| e^{At} \| + \|f(s,x_o,y_o,0,0)\| \right] ds
\]

\[
= \beta_1(t)N + QC_1
\]

\[
\text{......(15)}
\]
and also
\[ \|M_2(t,x_o,y_o,0,0)\| \leq \]
\[ \leq \left[\|E\| - \left(\frac{\|e\|c^T - \|e\|c^{(T-t)}}{\|e\|c^T - \|E\|}\right) \right] \int_0^t \|e^{C(t-s)}\| \left[\|D(s)\| \|y_o\| \|e^{Ct}\| + \|g(s,x_o,y_o)\|\right]ds + \]
\[ + \frac{\|e\|c^T - \|e\|c^{(T-t)}}{\|e\|c^T - \|E\|} \int_t^T \|e^{C(t-s)}\| \left[\|D(s)\| \|y_o\| \|e^{Ct}\| + \|g(s,x_o,y_o)\|\right]ds \]
\[ = \beta_2(t)N^* + R\delta_1 \]
(16)

from (15) and (16) we conclude that the inequality (14) holds for
\[ 0 \leq t \leq T, \quad \beta_1(t) \leq \frac{T}{2}, \quad \beta_2(t) \leq \frac{T}{2} \]

Now we define the two sequences of the functions
\[ \{x_m(t,x_o,y_o)\}_{m=0}^\infty, \]
\[ \{y_m(t,x_o,y_o)\}_{m=0}^\infty \]
on the given domain as follows
\[ x_{m+1}(t,x_o,y_o) = x_o e^{At} + L^2 \left( e^{A(t-s)} \left[ Ax_m(t,x_o,y_o) + B(t)(x_m(t,x_o,y_o) + \hat{x}_m(t,x_o,y_o)) + 
+ f(t,x_m(t,x_o,y_o),y_m(t,x_o,y_o),\hat{x}_m(t,x_o,y_o),\hat{y}_m(t,x_o,y_o)) \right] \right) \]
(17)
with \[ x_o(t,x_o,y_o) = x_o e^{At}, \quad m = 0,1,2, \]

\[ y_{m+1}(t,x_o,y_o) = y_o e^{Ct} + L^2 \left( e^{C(t-s)} \left[ Cy_m(t,x_o,y_o) + D(t)(y_m(t,x_o,y_o) + \hat{y}_m(t,x_o,y_o)) + 
+ g(t,x_m(t,x_o,y_o),y_m(t,x_o,y_o),\hat{x}_m(t,x_o,y_o),\hat{y}_m(t,x_o,y_o)) \right] \right) \]
(18)
with \[ y_o(t,x_o,y_o) = y_o e^{Ct}, \quad m = 0,1,2, \]

also now we define the two sequences of the functions \[ \{\hat{x}_m(t,x_o,y_o)\}_{m=0}^\infty, \]
\[ \{\hat{y}_m(t,x_o,y_o)\}_{m=0}^\infty \]
on the given domain as follows
\[ \hat{x}_{m+1}(t,x_o,y_o) = \int_0^t e^{A(t-s)} \left[ Ax_m(s,x_o,y_o) + B(s)(x_m(s,x_o,y_o) + \hat{x}_m(s,x_o,y_o)) + 
+ f(s,x_m(s,x_o,y_o),y_m(s,x_o,y_o),\hat{x}_m(s,x_o,y_o),\hat{y}_m(s,x_o,y_o)) \right] ds \]
\[ - \frac{A}{e^{AT} - E} \int_0^T e^{A(T-t)} \left[ Ax_m(s,x_o,y_o) + B(s)(x_m(s,x_o,y_o) + \hat{x}_m(s,x_o,y_o)) + 
+ f(s,x_m(s,x_o,y_o),y_m(s,x_o,y_o),\hat{x}_m(s,x_o,y_o),\hat{y}_m(s,x_o,y_o)) \right] ds \]
(19)

\[ \hat{y}_{m+1}(t,x_o,y_o) = \int_0^t e^{C(t-s)} \left[ Cy_m(s,x_o,y_o) + D(s)(y_m(s,x_o,y_o) + \hat{y}_m(s,x_o,y_o)) + 
+ g(s,x_m(s,x_o,y_o),y_m(s,x_o,y_o),\hat{x}_m(s,x_o,y_o),\hat{y}_m(s,x_o,y_o)) \right] ds \]
\[ - \frac{C}{e^{Ct} - E} \int_0^T e^{C(T-t)} \left[ Cy_m(s,x_o,y_o) + D(s)(y_m(s,x_o,y_o) + \hat{y}_m(s,x_o,y_o)) + 
+ g(s,x_m(s,x_o,y_o),y_m(s,x_o,y_o),\hat{x}_m(s,x_o,y_o),\hat{y}_m(s,x_o,y_o)) \right] ds \]
(20)
We define the two operators $L, L^2$ as follows:

\[
L(e^{A(t-s)}H(t,x_0,y_0)) = \int_0^t e^{A(t-s)}[H(s,x_0,y_0) - \Delta_1]ds,
\]

\[
L(e^{C(t-s)}Q(t,x_0,y_0)) = \int_0^t e^{C(t-s)}[Q(s,x_0,y_0) - \Delta_2]ds,
\]

\[
L^2(e^{A(t-s)}H(t,x_0,y_0)) = \int_0^T \left[ L(e^{A(t-s)}H(s,x_0,y_0)) - \frac{1}{T} \int_0^T L(e^{A(t-s)}H(s,x_0,y_0)) ds \right] ds,
\]

\[
L^2(e^{C(t-s)}Q(t,x_0,y_0)) = \int_0^T \left[ L(e^{C(t-s)}Q(s,x_0,y_0)) - \frac{1}{T} \int_0^T L(e^{C(t-s)}Q(s,x_0,y_0)) ds \right] ds,
\]

where

\[
\Delta_1(x_0,y_0) = \frac{A}{e^{4T} - E} \int_0^T e^{A(T-t)} \left[ Ax(t,x_0,y_0) + B(t)(x(t,x_0,y_0) + \dot{x}(t,x_0,y_0)) + f(t,x(t,x_0,y_0),y(t,x_0,y_0),\dot{x}(t,x_0,y_0),\dot{y}(t,x_0,y_0)) \right] dt
\]

\[
H(t,x_0,y_0) = Ax(t,x_0,y_0) + B(t)(x(t,x_0,y_0) + \dot{x}(t,x_0,y_0)) + f(t,x(t,x_0,y_0),y(t,x_0,y_0),\dot{x}(t,x_0,y_0),\dot{y}(t,x_0,y_0))
\]

\[
\Delta_2(x_0,y_0) = \frac{C}{e^{CT} - E} \int_0^T e^{C(T-t)} \left[ Cy(t,x_0,y_0) + D(t)(y(t,x_0,y_0) + \dot{y}(t,x_0,y_0)) + g(t,x(t,x_0,y_0),y(t,x_0,y_0),\dot{x}(t,x_0,y_0),\dot{y}(t,x_0,y_0)) \right] dt
\]

\[
Q(t,x_0,y_0) = Cy(t,x_0,y_0) + D(t)(y(t,x_0,y_0) + \dot{y}(t,x_0,y_0)) + g(t,x(t,x_0,y_0),y(t,x_0,y_0),\dot{x}(t,x_0,y_0),\dot{y}(t,x_0,y_0))
\]

It is clear that if each of $e^{A(t-s)}H(t,x_0,y_0), e^{C(t-s)}Q(t,x_0,y_0)$ be defined, continuous and periodic in $t$ on the interval $[0,T]$ then each of $\mathcal{L}(e^{A(t-s)}H(t,x_0,y_0)), \mathcal{L}(e^{C(t-s)}Q(t,x_0,y_0))$ and $\mathcal{L}(e^{C(t-s)}Q(t,x_0,y_0))$, $\mathcal{L}^2(e^{C(t-s)}Q(t,x_0,y_0))$ are also continuous and periodic in $t$ defined on the same interval. By lemma 1 we obtain

\[
\|\mathcal{L}(e^{A(t-s)}H(t,x_0,y_0))\| \leq \beta_1(t)\|e^{A(t-s)}\| \|f(s,x_0,y_0,0,0)\| \leq \frac{T}{2} N + QC_1, \quad \ldots (21)
\]

\[
\|\mathcal{L}(e^{C(t-s)}Q(t,x_0,y_0))\| \leq \beta_2(t)\|e^{C(t-s)}\| \|g(s,x_0,y_0,0,0)\| \leq \frac{T}{2} N^* + R\delta_1, \quad \ldots (22)
\]
The existence and approximation ....

\[ \| L^{2} (e^{A(t-s)} H_{c}(t,x_{0},y_{0})) \| \leq \alpha(t) \| L (e^{A(t-s)} H_{c}(t,x_{0},y_{0})) \| \leq \frac{T}{2} \left( \frac{T}{2} N + Q C_{1} \right) \quad \ldots \ldots (23) \]

\[ \| L^{2} (e^{C(t-s)} Q_{c}(t,x_{0},y_{0})) \| \leq \alpha(t) \| L (e^{C(t-s)} Q_{c}(t,x_{0},y_{0})) \| \leq \frac{T}{2} \left( \frac{T}{2} N^{*} + R \delta_{1} \right) \quad \ldots \ldots (24) \]

for all \( t \in [0,T] \), \( \beta_{1}(t) \leq \frac{T}{2} \), \( \beta_{2}(t) \leq \frac{T}{2} \), \( \alpha(t) \leq \frac{T}{2} \) where \( \alpha(t) = 2t(1 - \frac{t}{T}) \).

Theorem 1:-

Assume that the system (1) satisfy the inequalities (3),(4),(5) and the conditions (11), (12) and if each of \( f(t,x,y,\dot{x},\dot{y}) \) and \( g(t,x,y,\dot{x},\dot{y}) \) are measurable functions at \( t \) in the system (1) and defined in the domain (2) and satisfy the inequalities above and if the inequalities (9),(10) satisfies also and if for the system a periodic solution \( x = x(t,x_{0},y_{0}), y = y(t,x_{0},y_{0}) \) pass through the point \( (t,x_{0},y_{0}) \) then the two sequences of the functions

\[ x_{m+1}(t,x_{0},y_{0}) = x_{0} e^{A t} + \int_{0}^{t} e^{A(t-s)} \left[ A x_{m}(t,x_{0},y_{0}) + B(t)(x_{m}(t,x_{0},y_{0}) + \dot{x}_{m}(t,x_{0},y_{0})) + f(t,x_{m}(t,x_{0},y_{0}), y_{m}(t,x_{0},y_{0}), \dot{x}_{m}(t,x_{0},y_{0}), \dot{y}_{m}(t,x_{0},y_{0})) \right] ds \]

with \( x_{0}(t,x_{0},y_{0}) = x_{0} e^{A t} \), \( \frac{dx_{m}}{dt} = \dot{x}_{m}(t,x_{0},y_{0}) \), \( m = 0,1,2, \ldots \)

\[ y_{m+1}(t,x_{0},y_{0}) = y_{0} e^{C t} + \int_{0}^{t} e^{C(t-s)} \left[ C y_{m}(t,x_{0},y_{0}) + D(t)(y_{m}(t,x_{0},y_{0}) + \dot{y}_{m}(t,x_{0},y_{0})) + g(t,x_{m}(t,x_{0},y_{0}), y_{m}(t,x_{0},y_{0}), \dot{x}_{m}(t,x_{0},y_{0}), \dot{y}_{m}(t,x_{0},y_{0})) \right] ds \]

with \( y_{0}(t,x_{0},y_{0}) = y_{0} e^{C t} \), \( \frac{dy_{m}}{dt} = \dot{y}_{m}(t,x_{0},y_{0}) \), \( m = 0,1,2, \ldots \)

periodicity in \( t \) of period \( T \), its uniformly convergent when \( m \rightarrow \infty \) in the domain

\( (t,x_{0},y_{0}) \in [0,T] \times D_{f} \times D_{g} \) \quad \ldots \ldots (27)

from the two functions \( x_{\infty}(t,x_{0},y_{0}) \) and \( y_{\infty}(t,x_{0},y_{0}) \) are defined in the domain (27) continuous and periodic in \( t \) of period \( T \) and satisfies the system of integral equations.
\[ x(t, x_0, y_0) = x_0 e^{At} + L^2 (e^{A(t-s)}) [Ax(t, x_0, y_0) + B(t)(x(t, x_0, y_0) + \dot{x}(t, x_0, y_0)) + f(t, x(t, x_0, y_0), y(t, x_0, y_0), \dot{x}(t, x_0, y_0), \dot{y}(t, x_0, y_0))] \]
\[ y(t, x_0, y_0) = y_0 e^{Ct} + L^2 (e^{C(t-s)}) [Cy(t, x_0, y_0) + D(t)(y(t, x_0, y_0) + \dot{y}(t, x_0, y_0)) + g(t, x(t, x_0, y_0), y(t, x_0, y_0), \dot{x}(t, x_0, y_0), \dot{y}(t, x_0, y_0))] \]

which are a unique solution for the system (1) and satisfies the following inequality

\[
\begin{align*}
\left\| x_\infty(t, x_0, y_0) - x_m(t, x_0, y_0) \right\| + \left\| \dot{x}_\infty(t, x_0, y_0) - \dot{x}_m(t, x_0, y_0) \right\| + \left\| y_\infty(t, x_0, y_0) - y_m(t, x_0, y_0) \right\| + \left\| \dot{y}_\infty(t, x_0, y_0) - \dot{y}_m(t, x_0, y_0) \right\| & \leq \Omega_2^m \left( E - \Omega_2 \right)^{-1} V_o \\
\end{align*}
\]

where

\[
V_o = \left\{ \begin{array}{c}
\frac{T}{2} \left( \frac{T}{2} N + QC_1 \right) \\
\frac{T}{2} \left( \frac{T}{2} N + QC_1 \right) \\
\frac{T}{2} \left( \frac{T}{2} N^* + R\delta_1 \right) \\
\frac{T}{2} \left( \frac{T}{2} N^* + R\delta_1 \right)
\end{array} \right. \}
\]

E the unit matrix

Proof :-

When we look to the two sequences of the functions

\[ x_1(t, x_0, y_0), x_2(t, x_0, y_0), ..., x_m(t, x_0, y_0), ... \]

\[ y_1(t, x_0, y_0), y_2(t, x_0, y_0), ..., y_m(t, x_0, y_0), ... \]

that are defined in (25),(26) we find that each of the sequences of the functions are defined and continuous in the domain (2) and periodic in t of period T.

By lemma 1 and from (25) when m=0 we obtain

\[\left\| x_1(t, x_0, y_0) - x_0(t, x_0, y_0) \right\| = L^2 (e^{A(t-s)}) H_0(t, x_0, y_0)\]

By using (23) we get

\[\left\| x_1(t, x_0, y_0) - x_0(t, x_0, y_0) \right\| \leq \frac{T}{2} \left( \frac{T}{2} N + QC_1 \right)\]

Then \( x_1(t, x_0, y_0) \in D \) for all \( x_0 \in D_f, y_o \in D_g \)

also from the relation (19) and by lemma 1 when m=0 we obtain

\[\left\| \dot{x}_1(t, x_0, y_0) - 0 \right\| = \left\| L(e^{A(t-s)} H_0(t, x_0, y_0)) \right\|\]

By using (21) we get

\[\left\| \dot{x}_1(t, x_0, y_0) - 0 \right\| \leq \frac{T}{2} N + QC_1\]
The existence and approximation ....

Then \( \dot{x}_1(t, x_o, y_o) \in D \) for all \( x_o \in D_f, y_o \in D_g \)
also from the relation (26) when \( m=0 \) we obtain
\[
\left\| y_1(t, x_o, y_o) - y_o(t, x_o, y_o) \right\| = \left\| L^2 \left( e^{C(t-s)} Q_o(t, x_o, y_o) \right) \right\|
\]
By using (24) we get
\[
\left\| y_1(t, x_o, y_o) - y_o(t, x_o, y_o) \right\| \leq \frac{T}{2} \left( N^* + R \delta_1 \right)
\]
Then \( y_1(t, x_o, y_o) \in D_1 \) for all \( x_o \in D_f, y_o \in D_g \)
also from the relation (20) when \( m=0 \) we obtain
\[
\left\| \dot{y}_1(t, x_o, y_o) - 0 \right\| = \left\| L \left( e^{C(t-s)} Q_o(t, x_o, y_o) \right) \right\|
\]
By using (24) we get
\[
\left\| \dot{y}_1(t, x_o, y_o) - 0 \right\| \leq \frac{T}{2} N^* + R \delta_1
\]
Then \( \dot{y}_1(t, x_o, y_o) \in D_3 \) for all \( x_o \in D_f, y_o \in D_g \)
and by using the mathematical induction we can prove the truth of the following inequalities for \( m \geq 1 \)
\[
\begin{align*}
\left\| x_m(t, x_o, y_o) - x_o(t, x_o, y_o) \right\| & \leq \frac{T}{2} \left( N + QC_1 \right) \\
\left\| \dot{x}_m(t, x_o, y_o) - 0 \right\| & \leq \frac{T}{2} N + QC_1 \\
\left\| y_m(t, x_o, y_o) - y_o(t, x_o, y_o) \right\| & \leq \frac{T}{2} \left( N^* + R \delta_1 \right) \\
\left\| \dot{y}_m(t, x_o, y_o) - 0 \right\| & \leq \frac{T}{2} N^* + R \delta_1
\end{align*}
\]
that is \( x_m(t, x_o, y_o) \in D, \dot{x}_m(t, x_o, y_o) \in D_2, y_m(t, x_o, y_o) \in D_1, \dot{y}_m(t, x_o, y_o) \in D_3 \)
for all \( x_o \in D_f, y_o \in D_g \)
now we prove that the two sequences
\[
\{x_m(t, x_o, y_o)\}_{m=0}^{\infty}, \{y_m(t, x_o, y_o)\}_{m=0}^{\infty}
\]
are uniformly convergent in the domain (27) and thereupon each of the two ends for the two sequences are periodicity and continuous in the same domain
from the previous proof we find
\[ \|x_1(t, x_o, y_o) - x_o(t, x_o, y_o)\| \leq \frac{T}{2} (\frac{T}{2} N + QC_1) ,\]
\[ \|\dot{x}_1(t, x_o, y_o) - 0\| \leq \frac{T}{2} N + QC_1 ,\]
\[ \|y_1(t, x_o, y_o) - y_o(t, x_o, y_o)\| \leq \frac{T}{2} (\frac{T}{2} N^* + R\delta_1) ,\]
\[ \|\dot{y}_1(t, x_o, y_o) - 0\| \leq \frac{T}{2} N^* + R\delta_1 .\]

Now when \( m=1 \) in (25) and by using (23) we find
\[ \|x_2(t, x_o, y_o) - x_1(t, x_o, y_o)\| \leq \alpha(t) (\beta(t) Q(N_1 + H) + QC_2) \|x_1(t, x_o, y_o) - x_o(t, x_o, y_o)\| + \alpha(t) (\beta(t) QH + QC_4) \|\dot{x}_1(t, x_o, y_o) - 0\| + \alpha(t) QC_3 \|y_1(t, x_o, y_o) - y_o(t, x_o, y_o)\| + \alpha(t) QC_5 \|\dot{y}_1(t, x_o, y_o) - 0\| \]

also when \( m=1 \) in (19) and by using (21) we find
\[ \|\dot{x}_2(t, x_o, y_o) - \dot{x}_1(t, x_o, y_o)\| \leq (\beta_1(t) Q(N_1 + H) + QC_2) \|x_1(t, x_o, y_o) - x_o(t, x_o, y_o)\| + \beta_1(t) QH + QC_4 \|\dot{x}_1(t, x_o, y_o) - 0\| + QC_3 \|y_1(t, x_o, y_o) - y_o(t, x_o, y_o)\| + QC_5 \|\dot{y}_1(t, x_o, y_o) - 0\| \]

also when \( m=1 \) in (26) and by using (24) we find
\[ \|y_2(t, x_o, y_o) - y_1(t, x_o, y_o)\| \leq \alpha(t) R\delta_2 \|x_1(t, x_o, y_o) - x_o(t, x_o, y_o)\| + \alpha(t) R\delta_4 \|\dot{x}_1(t, x_o, y_o) - 0\| + \alpha(t) (\beta_2(t) Q(N_2 + J) + R\delta_3) \|y_1(t, x_o, y_o) - y_o(t, x_o, y_o)\| + \alpha(t) (\beta_2(t) RJ + R\delta_5) \|\dot{y}_1(t, x_o, y_o) - 0\| \]

also when \( m=1 \) in (22) and by using (20) we find
\[ \|\dot{y}_2(t, x_o, y_o) - \dot{y}_1(t, x_o, y_o)\| \leq R\delta_2 \|x_1(t, x_o, y_o) - x_o(t, x_o, y_o)\| + R\delta_4 \|\dot{x}_1(t, x_o, y_o) - 0\| + (\beta_2(t) Q(N_2 + J) + R\delta_3) \|y_1(t, x_o, y_o) - y_o(t, x_o, y_o)\| + (\beta_2(t) RJ + R\delta_5) \|\dot{y}_1(t, x_o, y_o) - 0\| \]

And so by using the mathematical induction we can prove the truth of the following two inequalities:
The existence and approximation ....

\[ \| \chi_{m+1}(t, x_0, y_0) - \chi_m(t, x_0, y_0) \| \leq \]
\[ \leq \alpha(t) (\beta_1(t) Q(N_1 + H) + QC_2) \| \chi_m(t, x_0, y_0) - \chi_{m-1}(t, x_0, y_0) \| + \]
\[ + \alpha(t) (\beta_1(t) QH + QC_4) \| \dot{\chi}_m(t, x_0, y_0) - \dot{\chi}_{m-1}(t, x_0, y_0) \| + \]
\[ + \alpha(t) QC_3 \| y_m(t, x_0, y_0) - y_{m-1}(t, x_0, y_0) \| + \]
\[ + \alpha(t) QC_5 \| \dot{y}_m(t, x_0, y_0) - \dot{y}_{m-1}(t, x_0, y_0) \| , \quad ........(32) \]

\[ \| \dot{\chi}_{m+1}(t, x_0, y_0) - \dot{\chi}_m(t, x_0, y_0) \| \leq \]
\[ \leq (\beta_1(t) Q(N_1 + H) + QC_2) \| \chi_m(t, x_0, y_0) - \chi_{m-1}(t, x_0, y_0) \| + \]
\[ + (\beta_1(t) QH + QC_4) \| \dot{\chi}_m(t, x_0, y_0) - \dot{\chi}_{m-1}(t, x_0, y_0) \| + \]
\[ + QC_3 \| y_m(t, x_0, y_0) - y_{m-1}(t, x_0, y_0) \| + \]
\[ + QC_5 \| \dot{y}_m(t, x_0, y_0) - \dot{y}_{m-1}(t, x_0, y_0) \| , \quad ........(33) \]

\[ \| y_{m+1}(t, x_0, y_0) - y_m(t, x_0, y_0) \| \leq \]
\[ \leq \alpha(t) R \delta_2 \| \chi_m(t, x_0, y_0) - \chi_{m-1}(t, x_0, y_0) \| + \]
\[ + \alpha(t) R \delta_4 \| \dot{\chi}_m(t, x_0, y_0) - \dot{\chi}_{m-1}(t, x_0, y_0) \| + \]
\[ + \alpha(t) (\beta_2(t) R(N_2 + J) + R \delta_3) \| y_m(t, x_0, y_0) - y_{m-1}(t, x_0, y_0) \| + \]
\[ + \alpha(t) (\beta_2(t) RJ + R \delta_5) \| \dot{y}_m(t, x_0, y_0) - \dot{y}_{m-1}(t, x_0, y_0) \| , \quad ........(34) \]

\[ \| \dot{y}_{m+1}(t, x_0, y_0) - \dot{y}_m(t, x_0, y_0) \| \leq \]
\[ \leq R \delta_2 \| \chi_m(t, x_0, y_0) - \chi_{m-1}(t, x_0, y_0) \| + \]
\[ + R \delta_4 \| \dot{\chi}_m(t, x_0, y_0) - \dot{\chi}_{m-1}(t, x_0, y_0) \| + \]
\[ + (\beta_2(t) R(N_2 + J) + R \delta_3) \| y_m(t, x_0, y_0) - y_{m-1}(t, x_0, y_0) \| + \]
\[ + (\beta_2(t) RJ + R \delta_5) \| \dot{y}_m(t, x_0, y_0) - \dot{y}_{m-1}(t, x_0, y_0) \| , \quad ........(35) \]

we rewrite (32),(33),(34),(35) with vectors form and on the following mode
\[ V_{m+1}(t) \leq \Omega_1(t) V_m(t) \]

where
Now we take the maximum value for the two sides of the inequality (36) for
\[0 \leq t \leq T\]
we find that
\[V_{m+1} \leq \Omega_2 V_m\]
\\[\text{where } \Omega_2 = \max_{t \in [0, T]} \Omega_1(t)\]
\\[\Omega_2 = \begin{bmatrix}
\frac{T}{2} \left[ N_1 + H \right] + QG_1 & \frac{T}{2} \left[ QH + QC_4 \right] & \frac{T}{2} QC_3 & \frac{T}{2} QC_5 \\
\frac{T}{2} \left[ Q(N_1 + H) + QG_1 \right] & \frac{T}{2} \left[ QH + QC_4 \right] & QC_3 & QC_5 \\
\frac{T}{2} R\delta_2 & \frac{T}{2} R\delta_4 & \frac{T}{2} (R(N_2 + J) + R\delta_3) & \frac{T}{2} (RJ + R\delta_5) \\
R\delta_2 & R\delta_4 & (\frac{T}{2} R(N_2 + J) + R\delta_3) & (\frac{T}{2} RJ + R\delta_5)
\end{bmatrix}\]
\\
and by repetition (37) we have
\[V_{m+1} \leq \Omega_2^n V_o\]
\\[\text{also we find}\]
\[\sum_{i=1}^{m} V_i \leq \sum_{i=1}^{m} \Omega_2^{i-1} V_o\]
\\[\text{since the matrix } \Omega_2 \text{ has the greatest eigen value as in (12) this lead to that the sequence (39) is uniformly convergent that is }\]
The existence and approximation ….

\[
\lim_{m \to \infty} \sum_{i=1}^{m} \Omega_{i}^{2-1} V = \sum_{i=1}^{\infty} \Omega_{i}^{2-1} V = (E - \Omega_{2})^{-1} V_0 \quad \text{..........}(40)
\]

and then the relation (40) ascertain on the convergence of sequences of the functions \( [x_m(t,x_o,y_o), \dot{x}_m(t,x_o,y_o), y_m(t,x_o,y_o), \dot{y}_m(t,x_o,y_o)] \) on the domain (27).

We assume that

\[
\lim_{m \to \infty} x_m(t,x_o,y_o) = x_\infty(t,x_o,y_o), \quad \lim_{m \to \infty} \dot{x}_m(t,x_o,y_o) = \dot{x}_\infty(t,x_o,y_o),
\]

\[
\lim_{m \to \infty} y_m(t,x_o,y_o) = y_\infty(t,x_o,y_o), \quad \lim_{m \to \infty} \dot{y}_m(t,x_o,y_o) = \dot{y}_\infty(t,x_o,y_o),
\]

since each of the sequences of the functions (25),(19),(26),(20) are continuous and periodic in t of period \( T \) then each of the ends for the sequences are continuous and periodic in t of period \( T \) and thereupon be

\[
x_\infty(t,x_o,y_o) = x(t,x_o,y_o), \quad y_\infty(t,x_o,y_o) = y(t,x_o,y_o),
\]

\[
\dot{x}_\infty(t,x_o,y_o) = \dot{x}(t,x_o,y_o), \quad \dot{y}_\infty(t,x_o,y_o) = \dot{y}(t,x_o,y_o)
\]

And in addition for that the using of lemma 1 and the relation (40) the inequality (30) satisfies for \( m \geq 0 \)

now we prove that \( x(t,x_o,y_o), y(t,x_o,y_o) \) are unique solution for the system (1) by contradiction.

we assume there at least another two different solutions 

\( x(t,x_o,y_o), y(t,x_o,y_o), u(t,x_o,y_o) \) for the system (1) are each of it defined and continuous and periodic in t of period \( T \).

from the form in below :

\[
r(t,x_o,y_o) = x_o e^{At} + L^2 (e^{A(t-s)} [A](t,x_o,y_o) + B(t)(r(t,x_o,y_o) + \dot{r}(t,x_o,y_o)) + f(t,r(t,x_o,y_o),u(t,x_o,y_o),\dot{r}(t,x_o,y_o),\dot{u}(t,x_o,y_o))])
\]

\[
H^*(t,x_o,y_o) = Ar(t,x_o,y_o) + B(t)(r(t,x_o,y_o) + \dot{r}(t,x_o,y_o)) + f(t,r(t,x_o,y_o),u(t,x_o,y_o),\dot{r}(t,x_o,y_o),\dot{u}(t,x_o,y_o)) \quad \text{..........}(42)
\]

\[
\dot{r}(t,x_o,y_o) = L(e^{A(t-s)} [A](t,x_o,y_o) + B(t)(r(t,x_o,y_o) + \dot{r}(t,x_o,y_o)) + f(t,r(t,x_o,y_o),u(t,x_o,y_o),\dot{r}(t,x_o,y_o),\dot{u}(t,x_o,y_o)))] \quad \text{..........}(43)
\]
\[ u(t,x_0,y_0) = y_0 e^{Ct} + L^2 (e^{C(t-s)}[Cu(t,x_0,y_0) + D(t)(u(t,x_0,y_0) + \dot{u}(t,x_0,y_0))] + g(t,r(t,x_0,y_0),u(t,x_0,y_0),\dot{r}(t,x_0,y_0),\dot{u}(t,x_0,y_0))]] \]

\[ Q^*(t,x_0,y_0) = Cu(t,x_0,y_0) + D(t)(u(t,x_0,y_0) + \dot{u}(t,x_0,y_0)) + g(t,r(t,x_0,y_0),u(t,x_0,y_0),\dot{r}(t,x_0,y_0),\dot{u}(t,x_0,y_0)) \] ........(44)

\[ \ddot{u}(t,x_0,y_0) = L(e^{C(t-s)}[Cu(t,x_0,y_0) + D(t)(u(t,x_0,y_0) + \dot{u}(t,x_0,y_0))] + g(t,r(t,x_0,y_0),u(t,x_0,y_0),\dot{r}(t,x_0,y_0),\dot{u}(t,x_0,y_0))]] \] ........(45)

We must prove that \( r(t,x_0,y_0) = x(t,x_0,y_0), \dot{r}(t,x_0,y_0) = \dot{x}(t,x_0,y_0), \)
\( y(t,x_0,y_0) = u(t,x_0,y_0), \dot{y}(t,x_0,y_0) = \dot{u}(t,x_0,y_0) \) and this from
\[ \|x(t,x_0,y_0) - r(t,x_0,y_0)\| \leq \frac{T}{2} \left[ Q(N_1 + H) + QC_2 \right] \|x(t,x_0,y_0) - r(t,x_0,y_0)\| + \]
\[ + \frac{T}{2} \left( QH + QC_4 \right) \|\ddot{x}(t,x_0,y_0) - \dot{r}(t,x_0,y_0)\| + \frac{T}{2} QC_3 \|y(t,x_0,y_0) - u(t,x_0,y_0)\| + \]
\[ + \frac{T}{2} QC_5 \|\ddot{y}(t,x_0,y_0) - \dot{u}(t,x_0,y_0)\| \] ........(46)

\[ \|\ddot{x}(t,x_0,y_0) - \dot{r}(t,x_0,y_0)\| \leq \left( \frac{T}{2} Q(N_1 + H) + QC_2 \right) \|x(t,x_0,y_0) - r(t,x_0,y_0)\| + \]
\[ + \left( \frac{T}{2} QH + QC_4 \right) \|\ddot{x}(t,x_0,y_0) - \dot{r}(t,x_0,y_0)\| + QC_3 \|y(t,x_0,y_0) - u(t,x_0,y_0)\| + \]
\[ + QC_5 \|\ddot{y}(t,x_0,y_0) - \dot{u}(t,x_0,y_0)\| \] ........(47)

\[ \|y(t,x_0,y_0) - u(t,x_0,y_0)\| \leq \frac{T}{2} R\delta_2 \|x(t,x_0,y_0) - r(t,x_0,y_0)\| + \]
\[ + \frac{T}{2} R\delta_4 \|\ddot{x}(t,x_0,y_0) - \dot{r}(t,x_0,y_0)\| + \frac{T}{2} \left( \frac{T}{2} R(N_2 + J) + R\delta_2 \right) \|y(t,x_0,y_0) - u(t,x_0,y_0)\| + \]
\[ + \frac{T}{2} \left( \frac{T}{2} R\delta_2 + R\delta_2 \right) \|\ddot{y}(t,x_0,y_0) - \dot{u}(t,x_0,y_0)\| \] ........(48)

\[ \|\ddot{y}(t,x_0,y_0) - \dot{u}(t,x_0,y_0)\| \leq R\delta_2 \|x(t,x_0,y_0) - r(t,x_0,y_0)\| + \]
\[ + R\delta_4 \|\ddot{x}(t,x_0,y_0) - \dot{r}(t,x_0,y_0)\| + \left( \frac{T}{2} R(N_2 + J) + R\delta_2 \right) \|y(t,x_0,y_0) - u(t,x_0,y_0)\| + \]
\[ + \left( \frac{T}{2} R\delta_2 + R\delta_2 \right) \|\ddot{y}(t,x_0,y_0) - \dot{u}(t,x_0,y_0)\| \] ........(49)

From (46),(47),(48),(49) we find
The existence and approximation ....

\[
\begin{pmatrix}
| x(t, x_o, y_o) - r(t, x_o, y_o) | \\
| \dot{x}(t, x_o, y_o) - \dot{r}(t, x_o, y_o) | \\
| y(t, x_o, y_o) - u(t, x_o, y_o) | \\
| \ddot{y}(t, x_o, y_o) - \ddot{u}(t, x_o, y_o) |
\end{pmatrix} \leq \Omega_2
\]

\[
\begin{pmatrix}
| x(t, x_o, y_o) - r(t, x_o, y_o) | \\
| \dot{x}(t, x_o, y_o) - \dot{r}(t, x_o, y_o) | \\
| y(t, x_o, y_o) - u(t, x_o, y_o) | \\
| \ddot{y}(t, x_o, y_o) - \ddot{u}(t, x_o, y_o) |
\end{pmatrix} \leq \Omega_2^m
\]

and by repetition (50) we obtain

\[
\begin{pmatrix}
| x(t, x_o, y_o) - r(t, x_o, y_o) | \\
| \dot{x}(t, x_o, y_o) - \dot{r}(t, x_o, y_o) | \\
| y(t, x_o, y_o) - u(t, x_o, y_o) | \\
| \ddot{y}(t, x_o, y_o) - \ddot{u}(t, x_o, y_o) |
\end{pmatrix} \leq \Omega_2^m
\]

but from the condition (12) we get \( \Omega_2^m \to 0 \) when \( m \to \infty \) and then we have from the last inequality that

\[
\begin{align*}
x(t, x_o, y_o) &= r(t, x_o, y_o), \\
\dot{x}(t, x_o, y_o) &= \dot{r}(t, x_o, y_o), \\
y(t, x_o, y_o) &= u(t, x_o, y_o), \\
\ddot{y}(t, x_o, y_o) &= \ddot{u}(t, x_o, y_o)
\end{align*}
\]

that is each of \( x(t, x_o, y_o), y(t, x_o, y_o) \) are unique solution for the system (1)

Section Two : The existence of the periodic solution for the system (1)

The problem of the existence of the periodic solution for the system (1) is connected with unique form with existence zero for all of the two functions

\[
\Delta_1^* = \Delta_1^*(x_o, y_o), \Delta_2^* = \Delta_2^*(x_o, y_o)
\]

in the form below :

\[
\Delta_1^* : D_f \times D_g \to R^n, \quad \Delta_1^*(x_o, y_o) = \frac{1}{T} \int_0^T L(e^{A(t-s)}) H(t, x_o, y_o) dt
\]

\[
\Delta_2^* : D_f \times D_g \to R^n, \quad \Delta_2^*(x_o, y_o) = \frac{1}{T} \int_0^T L(e^{C(t-s)}) Q(t, x_o, y_o) dt
\]

where \( x_o(t, x_o, y_o) \) it is the end of the sequence \( x_m(t, x_o, y_o) \) and \( y_o(t, x_o, y_o) \)

it is the end of the sequence \( y_m(t, x_o, y_o) \). The existence of these two functions are impossible to prove except by successive approximation from the following two sequences of the functions
\[ \Delta^*_{1m} : D_f \times D_g \rightarrow R^n, \]
\[ \Delta^*_{1m}(x_o, y_o) = \frac{1}{T} \int_0^T L(e^{A(t-s)} H_m(t, x_o, y_o))dt \], \hspace{1cm} ..........(53) 
\[ \Delta^*_{2m} : D_f \times D_f \rightarrow R^n, \]
\[ \Delta^*_{2m}(x_o, y_o) = \frac{1}{T} \int_0^T L(e^{C(t-s)} Q_m(t, x_o, y_o))dt \], \hspace{1cm} ..........(54) 

where \( m = 0, 1, 2, \ldots \).

**Theorem 2:**

If the assumptions and the conditions of the theorem 1 be given then the following inequalities:
\[ \| \Delta^*_{1}(x_o, y_o) - \Delta^*_{1m}(x_o, y_o) \| \leq \]
\[ \leq \left\{ \left( \frac{T}{2} Q(N_1 + H) + QC_2 \right) \left( \frac{T}{2} QH + QC_4 \right) QC_3 QC_5 \right\}, \Omega^m_2 (E - \Omega_2)^{-1} V_o \right\} = d^*_m \]
\[ \hspace{1cm} ..........(55) \]
\[ \| \Delta^*_{2}(x_o, y_o) - \Delta^*_{2m}(x_o, y_o) \| \leq \]
\[ \leq \left\{ \left( R \delta_2 R \delta_4 \left( \frac{T}{2} R(N_2 + J) + R \delta_3 \right) \left( \frac{T}{2} R J + R \delta_5 \right) \right), \Omega^m_2 (E - \Omega_2)^{-1} V_o \right\} = b^*_m \]
\[ \hspace{1cm} ..........(56) \]

satisfies for \( m \geq 0, x_o \in D_f, y_o \in D_g \) and \( d^*_m, b^*_m \) are positive numerical constants.

and \( \langle \rangle \) denote to the non cross product in the Euclidean space \( R^n \).

**Proof :-**

By the equations (51), (53) and the conditions (30) we have
The existence and approximation ....

\[ \| \Delta^*(x_0, y_0) - \Delta^*_{1m}(x_0, y_0) \| \leq \beta_1(t) e^{A(t-s)} \left[ \| A \| \| x_0(t, x_0, y_0) - x_m(t, x_0, y_0) \| + \right. \\
+ B(t) \left[ \| x_0(t, x_0, y_0) - x_m(t, x_0, y_0) \| + \| \dot{x}_0(t, x_0, y_0) - \dot{x}_m(t, x_0, y_0) \| \right] + \\
+ e^{A(t-s)} \int_0^t \left[ K_1(s) \| x_0(s, x_0, y_0) - x_m(s, x_0, y_0) \| + K_2(s) \| y_0(s, x_0, y_0) - y_m(s, x_0, y_0) \| + \\
+ K_3(s) \| \dot{x}_0(s, x_0, y_0) - \dot{x}_m(s, x_0, y_0) \| + K_4(s) \| \dot{y}_0(s, x_0, y_0) - \dot{y}_m(s, x_0, y_0) \| \right] ds \\
\leq \left( \frac{T}{2} Q(N_1 + H) + QC_2 \right) \left( \frac{T}{2} QH + QC_4 \right) QC_3 QC_5 \Omega^m_2(E - \Omega_2)^{-1} V_0 = d^*_m \\
\]

also by the equations (52), (54) and the conditions (30) we have

\[ \| \Delta^*_{2m}(x_0, y_0) \| \leq \beta_2(t) e^{C(t-s)} \left[ \| C \| \| x_0(t, x_0, y_0) - x_m(t, x_0, y_0) \| + \\
+ D(s) \left[ \| x_0(t, x_0, y_0) - x_m(t, x_0, y_0) \| + \| \dot{x}_0(t, x_0, y_0) - \dot{x}_m(t, x_0, y_0) \| \right] + \\
+ e^{C(t-s)} \int_0^t \left[ L_1(s) \| x_0(s, x_0, y_0) - x_m(t, x_0, y_0) \| + L_2(s) \| y_0(s, x_0, y_0) - y_m(t, x_0, y_0) \| + \\
+ L_3(s) \| \dot{x}_0(s, x_0, y_0) - \dot{x}_m(t, x_0, y_0) \| + L_4(s) \| \dot{y}_0(s, x_0, y_0) - \dot{y}_m(t, x_0, y_0) \| \right] ds \\
\leq \left( \frac{R\delta_2}{2} + R\delta_4 \left( \frac{T}{2} R(N_2 + J) + R\delta_3 \right) \right) \left( \frac{T}{2} RJ + R\delta_5 \right) \Omega^m_2(E - \Omega_2)^{-1} V_0 = b^*_m \\
\]

By the helping of the theorem 2 we introduce the following theorem and take into consideration the truth of the two inequalities (55) and (56) for

\[ m \geq 0 \].

**Theorem 3:-**

Let the right hand side from the system (1) are defined in the domain

\[ a \leq x, \dot{x} \leq b, c \leq y, \dot{y} \leq d, 0 \leq t \leq T \] on \( R^1 \) and assuming that the two sequences of the functions (53),(54) satisfies the following inequalities :
For all
\[ d_m^*(t) = \left( \begin{array}{c} \frac{T}{2} Q(N_1 + H) + QC_2 \\ \frac{T}{2} QH + QC_4 \end{array} \right) \begin{array}{c} QC_3 \\ QC_5 \end{array} \Omega_m^w(\epsilon - \Omega_2)^{-1} V_0, m \geq 0 \]
\[ b_m^*(t) = \left( \begin{array}{c} R \delta_2 \\ R \delta_4 \end{array} \right) \begin{array}{c} \frac{T}{2} R(N_2 + J) + R \delta_3 \\ \frac{T}{2} R J + R \delta_5 \end{array} \Omega_m^w(\epsilon - \Omega_2)^{-1} V_0, m \geq 0 \]
\[ N^m = \frac{T}{2} J \sigma_0, N = Q^2 H \delta_0, \quad h^m = \frac{T}{2} N^m + R \delta_1, \quad h = \frac{T}{2} N + QC \]

then the system (1) have periodic solutions
\[ x = x(t, x_0, y_0), y = y(t, x_0, y_0) \]
for \( x_0 \in [a + \frac{T}{2} h, b - \frac{T}{2} h], y_0 \in [c + \frac{T}{2} h^*, d - \frac{T}{2} h^*] \).

**Proof :-**

Let \( x_1, x_2 \) be any two points in the interval \( [a + \frac{T}{2} h, b - \frac{T}{2} h], y_2, y_1 \)
be any two points in the interval \( [c + \frac{T}{2} h^*, d - \frac{T}{2} h^*] \) so that
The existence and approximation ....

\[ \Delta_{lm}^* (x_1, y_1) = \min_{a+\frac{T}{2} h \leq x \leq b-\frac{T}{2} h} \min_{c+\frac{T}{2} h^* \leq y \leq d-\frac{T}{2} h^*} \Delta_{lm}^* (x_o, y_o) \]

\[ \Delta_{lm}^* (x_2, y_2) = \max_{a+\frac{T}{2} h \leq x \leq b-\frac{T}{2} h} \max_{c+\frac{T}{2} h^* \leq y \leq d-\frac{T}{2} h^*} \Delta_{lm}^* (x_o, y_o) \]  \[\text{........(59)}\]

\[ \Delta_{2m}^* (x_1, y_1) = \min_{a+\frac{T}{2} h \leq x \leq b-\frac{T}{2} h} \min_{c+\frac{T}{2} h^* \leq y \leq d-\frac{T}{2} h^*} \Delta_{2m}^* (x_o, y_o) \]

\[ \Delta_{2m}^* (x_2, y_2) = \max_{a+\frac{T}{2} h \leq x \leq b-\frac{T}{2} h} \max_{c+\frac{T}{2} h^* \leq y \leq d-\frac{T}{2} h^*} \Delta_{2m}^* (x_o, y_o) \]  \[\text{........(60)}\]

by using the inequalities (55),(56),(57),(58) we have

\[ \Delta_1^* (x_1, y_1) = \Delta_{lm}^* (x_1, y_1) + (\Delta_1^* (x_1, y_1) - \Delta_{lm}^* (x_1, y_1)) < 0 \]  \[\text{........(61)}\]

\[ \Delta_1^* (x_2, y_2) = \Delta_{lm}^* (x_2, y_2) + (\Delta_2^* (x_2, y_2) - \Delta_{lm}^* (x_2, y_2)) > 0 \]  \[\text{........(62)}\]

and from the continuity of the two functions \(\Delta_1^* (x_o, y_o), \Delta_2^* (x_o, y_o)\) and the two inequalities (61),(62) then there exist an isolated single point \((x_o, y_o) \neq (x_o, y_o)\) and

\(x_o \in [x_1, x_2], y_o \in [y_1, y_2]\) where \(\Delta_1^* (x_o, y_o) = 0, \Delta_2^* (x_o, y_o) = 0\)

this mean that for the system (1) periodic solution \(x = (t, x_o, y_o), y = (t, x_o, y_o)\) for \(x_o \in [a + \frac{T}{2} h, b - \frac{T}{2} h], y_o \in [c + \frac{T}{2} h^*, d - \frac{T}{2} h^*] \).

**Remark 1 :-**

We completed the formulation for the text of the theorem 3 with the proof when \(R^n = R^1\) that is when \(x_o, y_o\) are two non vectors quantities
**Theorem 4**

Let

\[ \Delta_1^* : D_f \times D_g \to R^n \]

\[ \Delta_1^*(x_o, y_o) = \frac{1}{T} \int_{T}^{0} L(e^{A(t-s)} H_\infty (t,x_o,y_o)) dt \]

\[ \Delta_2^* : D_f \times D_g \to R^n \]

\[ \Delta_2^*(x_o, y_o) = \frac{1}{T} \int_{T}^{0} L(e^{C(t-s)} Q_\infty (t,x_o,y_o)) dt \]

where \( x_\infty (t,x_o,y_o) \) it is the end of the periodic sequence (25) and the function

\( y_\infty (t,x_o,y_o) \) it is the end of the periodic sequence (26) then the following inequalities

\[ \| \Delta_1^*(x_o, y_o) \| \leq M_{13} \]

\[ \| \Delta_2^*(x_o, y_o) \| \leq M_{14} \]

where

\[ M_{13} = \frac{T}{2} Q(N_1 + H) M_9 [\delta_o Q + \frac{T^2}{4} Q^2 H C_1 M_3 + \frac{T}{2} Q C_1 + \frac{T}{2} Q H M_3 \delta_o Q + \]

\[ + \frac{T^3}{8} Q^3 H^2 M_3^2 C_1 + \frac{T^2}{4} Q^2 H C_1 M_3 ] + Q C_1 (\frac{T}{2} Q H M_3 + 1) \]

\[ M_3 = (1 - \frac{T^2}{2} Q H)^{-1} \]

\[ M_9 = (1 - \frac{T^2}{4} Q(N_1 + H)(1 + \frac{T}{2} Q H M_3))^{-1} \]

where

\[ M_{14} = \frac{T}{2} R(N_2 + J) M_{11} [\sigma_o R + \frac{T^2}{4} R^2 J \delta_1 M_5 + \frac{T}{2} R \delta_1 + \frac{T}{2} R J M_5 \sigma_o R + \]

\[ + \frac{T^3}{8} R^3 J^2 M_5^2 \delta_1 + \frac{T^2}{4} R^2 J M_5 \delta_1 ] + R \delta_1 (\frac{T}{2} R J M_5 + 1) \]

\[ M_5 = (1 - \frac{T}{2} R J)^{-1} \]

\[ M_{11} = (1 - \frac{T^2}{4} R(N_2 + J)(1 + \frac{T}{2} R J M_5))^{-1} \]
The existence and approximation ....

\[ \left\| \Delta^*_1(x^1, y^1) - \Delta_1^*(x^2, y^2) \right\| \leq \left( \frac{T}{2} \right) E_3 W_1 W_3 W_5 W_6 W_9 W_{10} W_{11} \gamma_1 + W_1 W_5 W_8 \gamma_2 \left\| x^1 - x^2 \right\| Q + \left( W_3 W_6 W_9 W_{10} W_{11} \right) \gamma_1 \left\| y^1 - y^2 \right\| R \] 

\[ \left\| \Delta^*_2(x^1, y^1) - \Delta_2^*(x^2, y^2) \right\| \leq \left( \frac{T}{2} \right) E_3 W_1 W_3 W_5 W_6 W_9 W_{10} W_{11} \gamma_3 + W_4 W_5 W_8 \gamma_4 \] 

satisfies for \( x_o, x^1_o, x^2_o \in D_f, y_o, y^1_o, y^2_o \in D_g \)

Where

\[ E_1 = \frac{T}{2} Q(N_1 + H) + QC_2 \]
\[ E_2 = \frac{T}{2} QH + QC_4 \]
\[ E_3 = Q C_3 \]
\[ E_4 = Q C_5 \]
\[ E_5 = \frac{T}{2} R \delta_2 \]
\[ E_6 = \frac{T}{2} R \delta_3 \]
\[ E_7 = \frac{T}{2} R(N_2 + J) \]
\[ E_8 = \frac{T}{2} R J + R \delta_5 \]

\[ \gamma_1 = E_1 W_5 W_8 a_1 + E_1 E_2 W_2 W_5 W_8 a_1 + E_2 E_3 W_2 + E_2 E_4 W_2 W_5 W_7 W_8 a_1 a_4 + E_2 E_4 W_2 W_7 a_5 + E_3 + E_4 W_2 W_7 W_8 a_1 a_4 + E_4 W_7 a_5 \]

\[ \gamma_2 = E_1 W_5 W_8 W_{10} W_{11} a_1 a_7 + E_1 + E_1 E_2 W_2 W_5 W_8 W_{10} W_{11} a_1 a_7 + E_1 E_2 W_2 + E_2 E_3 W_5 W_7 W_8 W_{10} W_{11} a_1 a_4 a_7 + E_2 E_4 W_2 W_7 a_4 + E_2 E_4 W_2 W_7 W_{10} W_{11} a_5 a_7 + E_3 W_{10} W_{11} a_7 + E_4 W_5 W_7 W_{10} W_{11} a_1 a_4 a_7 + E_4 W_7 a_4 + E_4 W_7 W_{10} W_{11} a_5 a_7 \]

\[ \gamma_3 = E_5 W_5 W_8 a_1 + E_1 E_6 W_2 W_5 W_8 a_1 + E_6 E_3 W_2 + E_4 E_6 W_2 W_5 W_7 W_8 a_1 a_4 + E_4 E_6 W_2 W_7 a_5 + E_7 + E_8 W_5 W_7 W_8 a_1 a_4 + E_8 W_7 a_5 \]

\[ \gamma_4 = E_5 W_5 W_8 W_{10} W_{11} a_1 a_7 + E_5 + E_1 E_6 W_2 W_5 W_8 W_{10} W_{11} a_1 a_7 + E_1 E_6 W_2 + E_3 E_6 W_2 W_{10} W_{11} a_7 + E_4 E_6 W_2 W_3 W_7 W_8 W_{10} W_{11} a_1 a_4 a_7 + E_4 E_6 W_2 W_7 a_4 + E_4 E_6 W_2 W_7 W_{10} W_{11} a_5 a_7 + E_7 W_{10} W_{11} a_7 + E_8 W_5 W_7 W_{10} W_{11} a_1 a_4 a_7 + E_8 W_7 a_4 + E_8 W_7 W_{10} W_{11} a_5 a_7 \]

\[ W_1 = \left( 1 - \frac{T}{2} E_1 \right)^{-1}, W_2 = \left( 1 - E_2 \right)^{-1}, W_3 = \left( 1 - \frac{T}{2} E_7 \right)^{-1}, W_4 = \left( 1 - E_8 \right)^{-1} \]
\[ a_1 = \left[ \frac{T}{2} E_2 E_3 W_1 W_2 + \frac{T}{2} E_3 W_1 \right], \quad W_5 = (1 - \frac{T}{2} E_1 E_2 W_1 W_2)^{-1} \]

\[ a_2 = \left[ \frac{T}{2} E_2 E_4 W_1 W_2 + \frac{T}{2} E_4 W_1 \right], \quad W_6 = (1 - \frac{T}{2} E_3 W_3 W_5 a_1)^{-1} \]

\[ a_3 = \left[ \frac{T}{2} E_5 W_3 W_5 a_2 + \frac{T}{2} E_8 W_3 \right], \quad W_7 = (1 - E_4 E_6 W_2 W_4)^{-1} \]

\[ a_4 = [E_5 W_4 + E_1 E_6 W_2 W_4] \quad , \quad W_8 = (1 - W_3 W_7 a_2 a_4)^{-1} \]

\[ a_5 = [E_3 E_6 W_2 W_4 + E_7 W_4] \quad , \quad W_9 = (1 - \frac{T}{2} E_3 E_6 W_2 W_3 E_6)^{-1} \]

\[ a_6 = \left[ \frac{T}{2} E_4 E_6 W_3 W_6 + a_3 W_6 \right], \quad W_{10} = (1 - W_7 W_9 a_6 a_5)^{-1} \]

\[ a_7 = \left[ \frac{T}{2} E_1 E_6 W_2 W_3 W_6 W_9 + W_7 W_9 a_6 a_4 \right], \quad W_{11} = (1 - W_5 W_8 W_{10} a_1 a_7)^{-1} \]

**Proof :-**

From properties of the function \( x_\infty(t, x, y) \) and the function \( y_\infty(t, x, y) \) that fixative by theorem 1 then each of the two functions \( \Delta_1^*(x, y), \Delta_2^*(x, y) \),

\( x_\infty \in D_f, y_\infty \in D_g \) continuous and bounded by non negative constants \( M_{13}, M_{14} \) on arrangement in the domain (2).

From the relation (63) we find that

\[ \| \Delta_1^*(x, y) \| \leq \| L(\epsilon A^{t-s}) H_\infty(t, x, y) \| \]

and by using lemma 1 we get

\[ \| \Delta_1^*(x, y) \| \leq \frac{T}{2} Q(N_1 + H) x_\infty(t, x, y) \| + \frac{T}{2} QH \| x_\infty(t, x, y) \| + QC_1 \quad ....(69) \]

Since that \( x_\infty(t, x, y) \) satisfy the integral equation (28) we find that

\[ \| x_\infty(t, x, y) \| \leq \delta_0 Q + \frac{T^2}{4} Q(N_1 + H) x_\infty(t, x, y) \| + \frac{T^2}{4} QH \| x_\infty(t, x, y) \| + \frac{T}{2} QC_1 \quad ....(70) \]

Also we have
The existence and approximation ....

\[ \| \dot{x}_\infty(t, x_0, y_0) \| \leq \frac{T}{2} Q(N_1 + H) M_3 \| x_\infty(t, x_0, y_0) \| + Q C_1 M_3 \] \quad \text{........(71)}

and by substitutions the inequality (71) in the inequality (70) we obtain

\[ \| x_\infty(t, x_0, y_0) \| \leq \delta_0 Q M_9 + \frac{T}{2} Q C_1 \left( \frac{T}{2} Q H M_3 + 1 \right) M_9 \] \quad \text{........(72)}

and by substitutions the inequality (72) in the inequality (71) we obtain

\[ \| \dot{x}_\infty(t, x_0, y_0) \| \leq \frac{T}{2} Q(N_1 + H) M_3 M_9 \delta_0 + \frac{T^2}{4} Q^2(N_1 + H) C_1 M_3 M_9 \left( \frac{T}{2} Q H M_3 + 1 \right) + Q C_1 M_3 \] \quad \text{........(73)}

by substitutions the two inequalities (72),(73) in the inequality (70) we obtain (65).

From the relation (64) we find that

\[ \| \Delta_2^* (x_0, y_0) \| \leq \| L \left( e^{C(t-s)} Q_\infty(t, x_0, y_0) \right) \| \]

and by using lemma 1 we get

\[ \| \Delta_2^* (x_0, y_0) \| \leq \frac{T}{2} R(N_2 + J) \| y_\infty(t, x_0, y_0) \| + \frac{T}{2} R J \| \dot{y}_\infty(t, x_0, y_0) \| + R \delta_1 \] \quad \text{........(74)}

since that \( y_\infty(t, x_0, y_0) \) satisfy the integral equation (29) we find

\[ \| y_\infty(t, x_0, y_0) \| \leq \sigma_0 R + \frac{T^2}{4} R(N_2 + J) \| y_\infty(t, x_0, y_0) \| + \frac{T^2}{4} R J \| \dot{y}_\infty(t, x_0, y_0) \| + \frac{T}{2} R \delta_1 \] \quad \text{........(75)}

also we have

\[ \| \dot{y}_\infty(t, x_0, y_0) \| \leq \frac{T}{2} R(N_2 + J) M_5 \| y_\infty(t, x_0, y_0) \| + R \delta_1 M_5 \] \quad \text{........(76)}

and by substitutions the inequality (76) in the inequality (75) we obtain

\[ \| y_\infty(t, x_0, y_0) \| \leq \sigma_0 R M_{11} + \frac{T}{2} R \delta_1 \left( \frac{T}{2} R J M_5 + 1 \right) M_{11} \] \quad \text{........(77)}

and by substitutions the inequality (77) in the inequality (76) we obtain

\[ \| \dot{y}_\infty(t, x_0, y_0) \| \leq \frac{T}{2} R(N_2 + J) M_5 M_{11} \sigma_0 R + \frac{T^2}{4} R^2(N_2 + J) \delta_1 M_5 M_{11} \left( \frac{T}{2} R J M_5 + 1 \right) + R \delta_1 M_5 \] \quad \text{........(78)}

by substitutions the two inequalities (77),(78) in the inequality (74) we obtain (66).

from the relation (63) we find that
\[ \| \Delta^*_1(x_0^1, y_0^1) - \Delta^*_1(x_0^2, y_0^2) \| \leq E_1 \| x_\infty(t, x_0^1, y_0^1) - x_\infty(t, x_0^2, y_0^2) \| + \\
+ E_2 \| \hat{x}_\infty(t, x_0^1, y_0^1) - \hat{x}_\infty(t, x_0^2, y_0^2) \| + E_3 \| y_\infty(t, x_0^1, y_0^1) - y_\infty(t, x_0^2, y_0^2) \| + \\
+ E_4 \| \hat{y}_\infty(t, x_0^1, y_0^1) - \hat{y}_\infty(t, x_0^2, y_0^2) \| \quad \text{...........(79)} \]

where
\[
x(t, x_0^k, y_0^k) = x_0^k e^{At} + L^2(e^{A(t-s)}H(t, x_0^k, y_0^k)) \quad \text{...........(80)}
\]
\[
y(t, x_0^k, y_0^k) = y_0^k e^{Ct} + L^2(e^{C(t-s)}Q(t, x_0^k, y_0^k)) \quad \text{...........(81)}
\]

where \( k = 1,2 \).

since \( x_\infty(t, x_0, y_0), y_\infty(t, x_0, y_0) \) satisfies the two equations (28),(29) on arrangement.

from the relation (80) we find that
\[
\| x_\infty(t, x_0^1, y_0^1) - x_\infty(t, x_0^2, y_0^2) \| \leq W_1 \| x_0^1 - x_0^2 \| + \frac{T}{2} E_2 W_1 \| \hat{x}_\infty(t, x_0^1, y_0^1) - \hat{x}_\infty(t, x_0^2, y_0^2) \| 
\]
\[
+ \frac{T}{2} E_3 W_1 \| y_\infty(t, x_0^1, y_0^1) - y_\infty(t, x_0^2, y_0^2) \| + \frac{T}{2} E_4 W_1 \| \hat{y}_\infty(t, x_0^1, y_0^1) - \hat{y}_\infty(t, x_0^2, y_0^2) \| \quad \text{...........(82)}
\]

also we find that
\[
\| \hat{x}_\infty(t, x_0^1, y_0^1) - \hat{x}_\infty(t, x_0^2, y_0^2) \| \leq E_1 W_2 \| x_\infty(t, x_0^1, y_0^1) - x_\infty(t, x_0^2, y_0^2) \| 
\]
\[
+ E_3 W_2 \| y_\infty(t, x_0^1, y_0^1) - y_\infty(t, x_0^2, y_0^2) \| + E_4 W_2 \| \hat{y}_\infty(t, x_0^1, y_0^1) - \hat{y}_\infty(t, x_0^2, y_0^2) \| 
\]
\[ \quad \text{...........(83)} \]

from the relation (81) we find that
\[
\| y_\infty(t, x_0^1, y_0^1) - y_\infty(t, x_0^2, y_0^2) \| \leq W_3 \| y_0^1 - y_0^2 \| + \frac{T}{2} E_3 W_3 \| x_\infty(t, x_0^1, y_0^1) - x_\infty(t, x_0^2, y_0^2) \| 
\]
\[
+ \frac{T}{2} E_6 W_3 \| \hat{x}_\infty(t, x_0^1, y_0^1) - \hat{x}_\infty(t, x_0^2, y_0^2) \| + \frac{T}{2} E_8 W_3 \| \hat{y}_\infty(t, x_0^1, y_0^1) - \hat{y}_\infty(t, x_0^2, y_0^2) \| 
\]
\[ \quad \text{...........(84)} \]

also we find that
\[
\| \hat{y}_\infty(t, x_0^1, y_0^1) - \hat{y}_\infty(t, x_0^2, y_0^2) \| \leq E_5 W_4 \| x_\infty(t, x_0^1, y_0^1) - x_\infty(t, x_0^2, y_0^2) \| + \\
+ E_6 W_4 \| \hat{x}_\infty(t, x_0^1, y_0^1) - \hat{x}_\infty(t, x_0^2, y_0^2) \| + E_7 W_4 \| y_\infty(t, x_0^1, y_0^1) - y_\infty(t, x_0^2, y_0^2) \| 
\]
\[ \quad \text{...........(85)} \]

after chain from substitutions in the inequalities (82),(83),(84),(85)we find
The existence and approximation ....

\[ \| x_\infty(t, x^1_v, y^1_v) - x_\infty(t, x^2_v, y^2_v) \| \leq [W_1 W_3 W_8 + \frac{T}{2} E_5 W_1 W_5 (W_5)^2 W_6 W_8 W_9 W_{10} W_{11} a_1 + \]
\[ + W_1 (W_5)^2 (W_8)^2 W_{10} W_{11} a_1 a_7 \| x^1_v - x^2_v \| Q + W_3 W_6 W_9 W_{10} W_{11} a_1 \| y^1_v - y^2_v \| R \]

\[ \| y_\infty(t, x^1_v, y^1_v) - y_\infty(t, x^2_v, y^2_v) \| \leq [\frac{T}{2} E_5 W_1 W_3 W_6 W_9 W_{10} W_{11} + \]
\[ + W_1 W_5 W_8 W_{10} W_{11} a_7 \| x^1_v - x^2_v \| Q + W_3 W_6 W_9 W_{10} W_{11} \| y^1_v - y^2_v \| R \]

\[ \| \dot{x}_\infty(t, x^1_v, y^1_v) - \dot{x}_\infty(t, x^2_v, y^2_v) \| \leq [\frac{T}{2} E_5 W_1 W_3 (W_5)^2 W_6 W_7 W_8 W_9 W_{10} W_{11} a_1 a_4 + \]
\[ + W_1 (W_5)^2 W_7 (W_8)^2 W_{10} W_{11} a_1 a_4 a_7 + W_1 W_5 W_7 W_8 a_4 + \]
\[ + \frac{T}{2} E_5 W_1 W_3 W_5 W_6 W_7 W_9 W_{10} W_{11} a_5 + W_1 W_5 W_7 W_8 W_{10} W_{11} a_5 a_7 \| x^1_v - x^2_v \| Q + \]
\[ + W_3 W_6 W_7 W_8 W_9 W_{10} W_{11} a_4 + W_3 W_6 W_7 W_9 W_{10} W_{11} a_5 \| y^1_v - y^2_v \| R \]

\[ \| \dot{x}_\infty(t, x^1_v, y^1_v) - \dot{x}_\infty(t, x^2_v, y^2_v) \| \leq \frac{T}{2} E_1 W_1 W_2 W_3 (W_5)^2 W_6 W_8 W_9 W_{10} W_{11} a_1 + \]
\[ + E_1 W_1 W_2 (W_5)^2 (W_8)^2 W_{10} W_{11} a_1 a_7 + E_1 W_1 W_5 W_8 + \]
\[ + \frac{T}{2} E_5 W_1 W_2 W_3 W_5 W_6 W_9 W_{10} W_{11} + E_3 W_1 W_2 W_5 W_8 W_{10} W_{11} a_7 + \]
\[ + \frac{T}{2} E_5 W_1 W_2 W_3 (W_5)^2 W_6 W_7 W_8 W_9 W_{10} W_{11} a_1 a_4 + \]
\[ + E_4 W_1 W_2 (W_5)^2 W_7 (W_8)^2 W_{10} W_{11} a_1 a_4 a_7 + E_4 W_1 W_2 W_7 W_8 a_4 + \]
\[ + \frac{T}{2} E_4 E_5 W_1 W_2 W_3 W_5 W_6 W_7 W_9 W_{10} W_{11} a_5 + E_4 W_1 W_2 W_5 W_7 W_8 W_{10} W_{11} a_5 a_7 \| x^1_v - x^2_v \| Q + \]
\[ + [E_1 W_1 W_2 W_3 W_6 W_8 W_9 W_{10} W_{11} a_1 + E_3 W_1 W_3 W_6 W_9 W_{10} W_{11} + \]
\[ + E_4 W_1 W_3 W_5 W_6 W_7 W_8 W_9 W_{10} W_{11} a_4 + E_4 W_2 W_3 W_6 W_7 W_9 W_{10} W_{11} a_5 \| y^1_v - y^2_v \| R \]

by substitutions the inequalities (86),(87),(88),(89) in the inequality (79) we obtain (67).
also by the relation (64) we find that
by substitutions the inequalities (86),(87),(88),(89) in the inequality (90) we obtain (68).

**Remark 2:-**

The theorem 4 confirm the stability of the solution for the system of non linear differential equations that is when a slight change happening in the point \((x_0, y_0)\) then a slight change will happen in the two functions

\[
\Delta_1^* = \Delta_1^*(x_0, y_0), \quad \Delta_2^* = \Delta_2^*(x_0, y_0)
\]

[for this remark return to [3].]

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