Anti-de Sitter Black Holes, Thermal Phase Transition and Holography in Higher Curvature Gravity

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We study anti-de Sitter black holes in the Einstein-Gauss-Bonnet and the generic \( R^2 \) gravity theories, evaluate different thermodynamic quantities, and also examine the possibilities of Hawking-Page type thermal phase transitions in these theories. In the Einstein theory, with a possible cosmological term, one observes a Hawking-Page phase transition only if the event horizon is a hypersurface of positive constant curvature \((k = 1)\). But, with the Gauss-Bonnet or and the \((\text{Riemann})^2\) interaction terms, there may occur a similar phase transition for a horizon of negative constant curvature \((k = -1)\). We examine the finite coupling effects, and find that \( N > 5 \) could trigger a Hawking-Page phase transition in the latter theory. For the Gauss-Bonnet black holes, one relates the entropy of the black hole to a variation of the geometric property of the horizon based on first law and Noether charge. With \((\text{Riemann})^2\) term, however, we can do this only approximately, and the two results agree when, \( r_H >> L \), the size of the horizon is much bigger than the AdS curvature scale. We establish some relations between bulk data associated with the AdS black hole and boundary data defined on the horizon of the AdS geometry. Following a heuristic approach, we estimate the difference between Hubble entropy \((S_H)\) and Bekenstein-Hawking entropy \((S_{BH})\) with \((\text{Riemann})^2\) term, which, for \( k = 0 \) and \( k = -1 \), would imply \( S_{BH} \leq S_H \).

PACS numbers: 04.70.Dy, 04.90.+e, 11.10.Kk, 98.80.Cq

I. INTRODUCTION

Anti-de Sitter black hole thermodynamics, which produces an aggregate of ideas from thermodynamics, quantum field theory and general relativity, is certainly one of the most remarkable tools to study quantum gravity in space-time containing a horizon \([1]\). In recent years a great deal of attention has been focused on such black holes. There are various familiar reasons for that. One of them, of course, is its role in the AdS/CFT duality \([2]\), and in particular, Witten’s interpretation \([3, 4]\) of the Hawking-Page phase transition between thermal AdS and AdS black hole \([5]\) as the confinement-deconfinement phases of the Yang-Mills (dual gauge) theory defined on the asymptotic boundaries of the AdS geometry. It is possible that our observable universe is a “brane” living on the boundary of a higher-dimensional black hole, and physics of such black hole is holographically related to that of early (brane) universe cosmology \([6]\). Most of the results in the literature are formulated in terms of the AdS/CFT conjecture and based on the Einstein’s theory with a negative cosmological constant, where one has the well known Bekenstein-Hawking area-entropy law

\[
S = \frac{k_B c^3 A}{h 4G},
\]

where \( A \) is volume of the horizon corresponding to the surface at \( r = r_H \). In the following, we adopt the standard convention of setting \( c = \hbar = k_B = 1 \). One of the impressive features of \([1]\) is its universality to all kinds of black holes \([6, 7]\) irrespectively of their charges, shapes and rotation. Nonetheless, it has been known that \([1]\) no longer applies to the higher curvature (HC) theories in \( D > 4 \) dimensions (see Ref. \([8]\) for review). Thus, the study of AdS black holes and boundary CFTs is well motivated both from the field theoretic and the cosmological points of view.

One notes that, on general grounds, any effective gravity action will involve, besides the usual Einstein term, the higher curvatures and also derivative terms corresponding to the low energy matter fields(see for example \([10]\)). In particular, when the effect of gravitational fluctuations are small compared to the large number of matter fluctuations, one can neglect graviton loops, and look for a stationary point of the combined gravitational action, and the effective action for the matter fields. This is implied by solving

\[
R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G \langle T_{ab} \rangle,
\]

where the source being the expectation value of the matter energy momentum tensor, which may include the contribution from HC terms. If one allows non-conformally invariant matter fields, one must take into account the...
non-conformally invariant local terms \[ \square R \], which in four dimensions read

\[ (T) = \alpha_1 \mathcal{F} - \alpha_2 \mathcal{G} + \alpha_3 \nabla^2 R, \]  

(3)

where the Gauss-Bonnet invariant \( \mathcal{G} \) and the square of Weyl tensor are

\[ \mathcal{G} = R^2 - 4 R_{ij} R^{ij} + R_{ijkl} R^{ijkl}, \]
\[ \mathcal{F} = \frac{1}{3} R^2 - 2 R_{ij} R^{ij} + R_{ijkl} R^{ijkl}. \]  

(4)

Here \( \alpha_1, \alpha_2, \alpha_3 \) are defined by certain combinations of the number of real scalars, Dirac fermions and vectors in the theory being considered, \( \nabla^2 R \) is a variation of the local term \( \int d^4 x \sqrt{|g|} R^2 \) in the effective action, which generally does not carry dynamical information. The authors in \[ 11, 13 \] have derived \( \mathcal{G} \) in \( d + 1 = 5 \) for conformal anomaly in the \( R^2 \) gravity using the AdS/CFT correspondence. This suggests that higher curvature terms in the bulk theory can arise as next-to-leading order corrections in the \( 1/N \) (large \( N \)) expansion of the boundary CFTs in the strong 't Hooft coupling limit \[ 4, 13, 14 \].

Paraphrasing, any effective stringy gravity action also contains higher curvature terms of different order as loop corrections to string amplitudes. The most suggestive combination of higher derivative terms is perhaps the GB invariant, which is attributed to the heterotic string effective action \[ 14, 17 \]. The theory with a Gauss-Bonnet term is free of ghost when expanding about the flat space \[ 16 \]. This is true \[ 18 \] also in the recently discovered warped brane-world model \[ 19 \] (see \[ 20 \] for discussion with GB term). It is noteworthy that the higher derivative correction terms with “small” coefficients will not just only produce some modifications of the solution of the unperturbed (Einstein) theory \[ 21 \], but importantly also contain whole classes of new solutions in an anti-de Sitter space \[ 22 \], see Ref. \[ 23 \] for brane-world solutions with the time varying gravitational constant and \( \Lambda \). Moreover, the Einstein-Gauss-Bonnet theory in \( D \geq 5 \) clearly exhibits some new black hole solutions \[ 24 \], which are unavailable to the Einstein theory.

AdS/CFT correspondence asserts that physics in the bulk of AdS spacetime is fully described by a CFT on the boundary, an intuitive notion of holography \[ 25 \]. As a result, in the Einstein theory with a bulk cosmological term, the thermodynamic quantities of the holographic dual CFTs defined on \( S^3 \times S^1 \) at high temperatures may be identified with those of the bulk AdS Schwarzschild black holes \[ 4 \]. Within the context of the AdS/CFT conjecture, the higher curvature terms in the bulk (string) theory correspond to finite coupling large \( N \) effects in the gauge theory, one may therefore allow HC terms, which are squares in Ricci scalar, Ricci tensor and Riemann tensor, into the effective bulk action. A ghost free combination of these terms is the Gauss-Bonnet invariant. The higher derivative supergravity action including a (Riemann)\(^2\) term may induce some extra degrees of freedom of ghost behavior around stable fixed points. It is not known what this leads to the finite temperature field theory. Any ghost field, if exists, is expected to decouple in the large \( N \) limit, so they do not project out on the conformal boundary. However, in the full string theory there is no such problem. We can therefore extract a bulk (Riemann)\(^2\) term directly from AdS/CFT, which is necessary if we wish to study the finite coupling effects.

If the AdS black hole horizon is a hypersurface with zero \( (k = 0) \) or negative \( (k = -1) \) curvature, the black hole is always stable and the corresponding boundary field theory at finite temperature is dominated by the black hole. While, for the horizon of a positive curvature \( (k = 1) \), one sees a Hawking-Page phase transition \[ 1 \], see also \[ 26 \]. This is indeed an observation one can make in the Einstein’s theory, where the qualitative features of Hawking-Page phase transitions are independent of the dimensions. However, in the higher derivative theory with a Gauss-Bonnet term or a (Riemann)\(^2\) term, the situation could be different, where phase structures actually depend on the number of spatial dimensions \( d \) and the horizon geometry \( k \). As we have no good reasons that a dual description should exist for the Gauss-Bonnet theory, it would be of more interest to study the AdS black hole solutions with a (Riemann)\(^2\) term and to examine the possibility of the Hawking-Page phase transition. Wald has shown \[ 27 \] that one can relate the variations in properties of the black hole as measured at horizon to the variations of the geometric property of the horizon based on the first law and evaluation of the Noether charge (see Refs. \[ 23, 24, 30 \] for a clear generalization). This has been realized for the Gauss-Bonnet black hole, in that case we also have the exact solutions. But with a (Riemann)\(^2\) term, this prediction is only a good approximation. In particular, the two entropies one finds with the (Riemann)\(^2\) term closely agree in the limit \( r_H >> L \), while they completely agree for \( k = 0 \).

The paper is organized as follows. In next section we shall begin with our effective action and present some curvature quantities for a general metric ansatz. Section III deals in detail with the Gauss-Bonnet black hole thermodynamics in AdS space, including thermal phase structures. In section IV we study AdS black holes with a trivial (Riemann)\(^2\) interaction \( (\gamma = 0) \). In section V we begin our discussion of the black hole thermodynamics with a non-zero \( \gamma \), and present formulas for free energy, entropy and energy. We also briefly discuss about the finite coupling effects. In section VI, we present certain realizations of the FRW-type brane equations, and also make a comparison between the Bekenstein-Hawking entropy and the Hubble (or holographic) entropy. Section VII contains conclusions.
II. ACTION, METRIC ANSATZ AND CURVATURE QUANTITIES

Perhaps, a natural tool to explore the AdS/CFT is to implement the general higher derivative terms to the effective action, and to study thermodynamics of the anti-de Sitter black holes. To begin with, we consider the following \((d+1)\)-dimensional gravitational action, containing terms up to quadratic in the curvatures,

\[
I = \int d^{d+1}x \sqrt{-g_{d+1}} \left[ \frac{R}{\kappa_{d+1}} - 2\Lambda + \alpha R^2 + \beta R_{ab}R^{ab} + \gamma R_{abcd}R^{abcd} + \cdots \right] + \frac{2}{\kappa} \int_{\partial B} d^d x \sqrt{|g_d|} \mathcal{K} + \cdots ,
\]

where \(\kappa_{d+1} = 16\pi G_{d+1}\), \(\mathcal{K} = \mathcal{K}_{ab}\) is the trace of the extrinsic curvature of the boundary, \(\mathcal{K}_{ab} = \nabla_a n_b\), where \(n^a\) is the unit normal vector on the boundary. The last term above is attributed to the Gibbon-Hawking boundary action. When working in \((d+1)\)-dimensional anti-de Sitter space \((\Lambda < 0)\), one may drop the surface terms including the Gibbon-Hawking action. However, the surface term, including the higher order, might be essential to evaluate the conserved quantities when the solutions are extended to de Sitter (dS) spaces \[31, 32\].

We define the metric ansatz in the following form

\[
ds^2 = -e^{2\phi(r)} dt^2 + e^{-2\phi(r)} dr^2 + r^2 \sum_{i,j=1}^{d-1} h_{ij} dx^i dx^j ,
\]

where \(h_{ij}\) is the horizon metric for a manifold \(M^{d-1}\) with the volume \(V_{d-1} = \int d^{d-1}x \sqrt{h}\). For \([3]\), the non-vanishing components of the Riemann tensor are

\[
\begin{align*}
R_{tttt} &= e^{2\phi(r)} \left( \phi'' + 2\phi' \frac{r}{2} \right) , \\
R_{ttij} &= e^{4\phi(r)} \phi' h_{ij} = -e^{4\phi(r)} R_{tij} , \\
R_{ijkl} &= r^2 R_{ijkl}(h) = r^{-2} e^{2\phi(r)} (h_{ik} h_{jl} - h_{il} h_{jk}) .
\end{align*}
\]

We readily obtain the following non-trivial components of the Ricci tensor

\[
\begin{align*}
R_{tt} &= -e^{4\phi(r)} R_{tt} = e^{4\phi(r)} \left( \phi'' + 2\phi' \frac{r}{2} + \frac{(d-1)\phi'}{r} \right) , \\
R_{ij} &= R_{ij}(h) = e^{2\phi(r)} ((d-2) + 2r \phi') h_{ij} ,
\end{align*}
\]

with \(k\) being the curvature constant, whose value determines the geometry of the horizon. The boundary topology of the Einstein space \((M^{d-1})\) looks like

\[
\begin{align*}
k = 1 &\rightarrow S^{d-1} : \text{Euclidean de Sitter space (sphere)} \\
k = 0 &\rightarrow B^{d-1} : \text{flat space} \\
k = -1 &\rightarrow H^{d-1} : \text{anti - de Sitter space (hyperbolic)} .
\end{align*}
\]

This means that event horizon of the black hole can be a hypersurface with positive, zero or negative curvature. For spherically symmetric black holes, the event horizon is generally a spherical surface with \(k = 1\). While, if the horizon is zero or negative constant hypersurface, the black holes are referred as topological black holes. The thermodynamics of topological or asymptotically anti-de Sitter black holes in the Einstein gravity was investigated, e.g., in Refs. \[33, 34, 35, 36, 37\].

Before going ahead, let us assume that the \((d+1)\)-dimensional spacetime is an Einstein space

\[
R_{abcd} = -\frac{1}{d} (g_{ac} g_{bd} - g_{ad} g_{bc}) , \quad R_{ab} = -\frac{d}{r^2} g_{ab} .
\]

This is always possible provided that the horizon geometry is also an Einstein space \[38\]

\[
R_{ijkti}(h) = k (h_{ik} h_{jt} - h_{it} h_{jk}) , \quad R_{ij}(h) = (d-2) k h_{ij} .
\]

Then one easily computes the Ricci scalar in \((d+1)\) spacetime dimensions

\[
R = \frac{(d-1)(d-2)k}{r^2} e^{2\phi(r)} \left( 2\phi'' + 4\phi' \frac{r}{2} + \frac{4(d-1)\phi'}{r} + \frac{(d-1)(d-2)}{r^2} \right) = \frac{(d-1)(d-2)k}{r^{d-1}} \left( \frac{1}{r^{d-1}} e^{2\phi(r)} \right)'' .
\]
III. GAUSS-BONNET BLACK HOLE IN ADS SPACE

Let us set at first $\alpha = -\beta/4 = \gamma$ in (13), and also drop the Hawking-Gibbon term. Then the equations of motion following from (13) simply read

$$\kappa_{d+1}^{-1} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) + \Lambda g_{ab} - \frac{\alpha}{2} g_{ab} R_{CB}^2 + 2\alpha \left( R R_{ab} - 2 R_{acbd} R^{cd} + R_{acde} R^{cde} - 2 R_{a} R^{bc} \right) = 0$$

(13)

with the Gauss-Bonnet invariant $R_{CB}^2 = R^2 - 4R_{ab} R^{ab} + R_{abcd} R^{abcd}$. The explicit form of the metric solution following from (13) is (see also the Refs. [16, 24, 38, 39])

$$e^{2\phi} = k + \frac{r^2}{2\alpha} + \epsilon \frac{r^2}{2\alpha} \left[ 1 - \frac{4\hat{\alpha}}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) + \frac{4\hat{\alpha} m}{r^d} \right]^{1/2},$$

(14)

where $\epsilon = \mp 1$, $\hat{\alpha} = (d-2)(d-3)\alpha \kappa_{d+1}$ and $m$ is an integration constant with dimensions of $(\text{length})^{d-2}$, which is related to the ADM mass $M$ of the black hole via

$$M = \frac{(d-1)V_{d-1}}{\kappa_{d+1}} m.$$

(15)

We should note that the AdS curvature squared $\ell^2$ ($= -\ell_{dS}^2$) is related to the cosmological constant $\Lambda$ via

$$\Lambda = -\frac{d(d-1)}{2\kappa_{d+1} \ell^2}, \quad \text{where} \quad \frac{1}{\ell^2} = 1 - \frac{\hat{\alpha}}{\ell^2},$$

(16)

so that $\ell^2 > 0$ for $\Lambda < 0$ (anti-de Sitter), while $\ell^2 < 0$ for $\Lambda > 0$ (de Sitter). Notice that there are two branches in the solution (13), because $e^{2\phi(r)}$ is determined by solving a quadratic equation, we denote by $r_+$ the $\epsilon = -1$ branch, and by $r_-$ the $\epsilon = +1$ branch. For large $r$

$$- g_{00}(r_+) \sim k - \frac{m}{r^{d-2}} + \frac{r^2}{\ell^2} + \hat{\alpha} O(r^{4-2d}),$$

(17)

$$- g_{00}(r_-) \sim k - \frac{m}{r^{d-2}} - \frac{r^2}{\ell^2} + \hat{\alpha} O(r^{4-2d}).$$

(18)

The $\epsilon = -1$ branch gives a $(d+1)$-dimensional Schwarzschild anti-de Sitter (SAdS) solution. While, $\epsilon = +1$ branch gives the SAdS solution, since $\ell^2 > \hat{\alpha}$, but with a negative gravitational mass [39]. However, $\epsilon = +1$ branch is unstable for certain parameter values of $\hat{\alpha}$ and $m$ [16]. In this paper we only consider (14), rather than its perturbative cousins [13, 18].

A. Thermodynamic quantities

From (14), the mass of the black hole $M$ can be expressed in terms of the horizon radius $r_+ = r_H$

$$M = \frac{(d-1)V_{d-1} r_+^{d-2}}{\kappa_{d+1}} \left[ k + \frac{r_+^2}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) + \frac{\hat{\alpha} k^2}{r_+^2} \right].$$

(19)

The positions of the horizons may be determined as the real roots of the polynomial $q(r = r_H) = 0$, where

$$q(r) = k r^{d-2} + \hat{\alpha} k^2 r^{d-4} + \frac{r^d}{\ell^2} - m.$$

(20)

Of course, in the limit $\ell^2 \to \infty$ (i.e. $\Lambda \to 0$) and $k = 1$, one recovers the results in [38]. One derives $q(r) = 0$ directly from Eq. (14) or from the field equations by setting $e^{2\phi(r)} = 0$ at the horizon $r = r_+$, but one satisfies $e^{2\phi(r)} > 0$ for $r > r_+$. When $d+1 = 5$, the black hole horizon is at

$$r_+^2 = \frac{\ell^2}{2} \left[ -k + \sqrt{k^2 + \frac{4(m-\hat{\alpha} k^2)}{\ell^2}} \right].$$

(21)
There is a mass gap at \( m = \dot{\alpha}k^2 \), so all black holes have a mass \( M \geq 3V_3\dot{\alpha}k^2/\kappa_5 \equiv M_0 \), the requirement \( M > M_0 \) is needed to have a black hole interpretation. From Eq. (14) and Eq. (20), we easily see that the roots of \( q(r) \) must also satisfy

\[
1 + \frac{4\dot{\alpha}^2 k^2}{r^4} + \frac{4\dot{\alpha}k}{r^2} \geq 0. \tag{22}
\]

When \( k \geq 0 \), this is trivially satisfied in any spaces \( (\ell^2 > 0 \text{ (AdS)}, \ell^2 < 0 \text{ (dS)}, \text{ and } \ell^2 = \infty \text{ (flat)}) \), since \( \dot{\alpha} > 0 \). If the horizon of the black hole is a hypersurface with a negative curvature \( (k = -1) \), the constraint (22) implies \( r^2 \geq 2\dot{\alpha} \) at \( r = r_+ \), giving the minimum size of the black hole horizon. For \( k = +1 \), Eq. (22) gives \( r_+^2 \geq -2\dot{\alpha} \), as first noticed in \([4]\). For \( k = 0 \), \( r_+ \) is not constrained in terms of \( \dot{\alpha} \).

To study the black-hole thermodynamics, it is customary to find first the Euclidean action by analytic continuation. After Wick-rotating the time variable \( t \to i\tau \), one regularizes Euclidean section \( \mathcal{E} \) by identifying the Killing time coordinate with a period \( \tau = \beta \). In the standard approach \([4, 5, 40]\), one subtracts energy of the reference geometry which is simply anti-de Sitter space produced by a setting \( m = 0 \) in \([14]\), i.e. \( M = 0 \). This process is equivalent to subtracting off the zero-temperature free energy in the field theory calculation. One may add the counter terms of holographic normalization instead of subtracting the energy of a reference geometry. In the latter approach the number of holographic terms grow with the spacetime dimensions, thus for simplicity we follow \([1]\). The Euclidean action \( \hat{I} \) therefore reads

\[
\hat{I} = -\frac{V_{d-1} r_+^{d-4}}{\kappa_{d+1} (d-3)} \left[ (d-1)\beta \left( k r_H^2 - \dot{\alpha} k^2 \right) - 8\pi r_H^3 + 3(d-1)\beta \frac{T_H^4}{\ell^2} \left( 1 - \frac{\dot{\alpha}}{\ell^2} \right) \right], \tag{23}
\]

where \( \beta = 1/T \) is the periodicity in Euclidean time. With \( \ell^2 = \infty \) (i.e. \( \Lambda = 0 \)) and \( k = 1 \), we correctly reproduce the result in \([12]\). Since the temperature of the black hole horizon is identified by the periodicity in imaginary time of the metric, \( T \) defines Hawking temperature \( (T_H) \) of the black hole given by \( (e^{2\phi(r)})' |_{r=r_+} = 4\pi T_H \). Hence

\[
\frac{1}{\beta} = T_H = \frac{d-2}{4\pi r_+} \frac{1}{(r_+^2 + 2\dot{\alpha} k)} \left[ k r_+^2 + \frac{d-4}{d-2} \dot{\alpha} k^2 + \frac{d}{d-2} \frac{r_+^4}{\ell^2} \left( 1 - \frac{\dot{\alpha}}{\ell^2} \right) \right]. \tag{24}
\]

We plot in Fig. (1) the inverse temperature \( (\beta) \) of the black hole versus the horizon radius \( r_+ \) for \( \dot{\alpha} = 0 \) and \( k = 0, \pm 1 \). Then, in Fig. (2) we plot \( \beta \) vs. \( r_+ \) for \( \dot{\alpha} \neq 0 \) and \( k = 1 \), when the number of spatial dimensions \( d \) is 3, 4, 5, and 9. In term of the black hole temperature, we clearly see that only in \( d+1 = 5 \) and for \( k = 1 \) there occurs a new phase of locally stable small black hole. In the Einstein gravity, one simply discards the region \( \beta \to 0 \) as \( r \to 0 \), because thermodynamically it is an unstable region \([4, 13]\). For the EGB gravity in \( d = 4 \), however, both the conditions: (i) \( \beta \to \infty \) as \( r \to 0 \) and (ii) \( \beta \to 0 \) as \( r \to \infty \) are physical.
We may identify the Euclidean action with the free energy times $\beta$. Therefore

$$F = -\frac{V_{d-1} r_+^{d-4}}{\kappa_{d+1} (d-3)} \left[(d-1) \left( k r_+^2 - \hat{\alpha} k^2 \right) - 8 \pi r_+^3 T_H + 3(d-1) \frac{r_+^4}{l^2} \right]$$

$$= \frac{V_{d-1} r_+^{d-4}}{\kappa_{d+1} (d-3) (r_+^2 + 2\hat{\alpha} k)} \left[(d-3)r_+^4 \left( k - \frac{r_+^2}{l^2} \right) - \frac{6(d-1) \hat{\alpha} k r_+^4}{l^2} \right.\left. + (d-7) \hat{\alpha} k^2 r_+^2 + 2(d-1) \hat{\alpha}^2 k^3 \right].$$

(25)

In the second line above we have substituted the value of $T_H$ from Eq. (24). Interestingly, the free energy (25) was obtained in Ref. [24] by using the thermodynamic relation $F = M - TS$, where entropy $S$ was evaluated there using $S = \int_0^{r_+} T^{-1} dM$. So these two apparently different prescriptions for calculating free energy ($F$ read from the Euclidean action and $F$ derived from the first law) give the same results. One therefore computes the energy

$$E = \frac{\partial \hat{I}}{\partial \beta_0} = M,$$

(26)

where $M$ is still given by (19), and the entropy

$$S = \beta_0 E - \hat{I} = \frac{4\pi V_{d-1} r_+^{d-4}}{\kappa_{d+1}} \left[1 + \frac{(d-1) \hat{\alpha} k}{(d-3) r_+^2} \right].$$

(27)

The minimum of the Hawking temperature is given by solving $\partial T_H/\partial r_+ = 0$, where

$$\frac{\partial T_H}{\partial r_+} = \frac{1}{4\pi} \frac{1}{(r_+^2 + 2\hat{\alpha} k)^2} \left[-(d-2) k r_+^2 + \frac{d r_+^4}{l^2} + \frac{6d \hat{\alpha} k r_+^2}{l^2} - (d-8) \hat{\alpha} k^2 - \frac{2(d-4) \hat{\alpha}^2 k^3}{r_+^4} \right].$$

(28)

If $\hat{\alpha} = 0$, for $k = 0$ and $k = -1$, one easily sees that there is no minimum of temperature, thus $k = 0$ and $k = -1$ black holes exist for all temperatures. But the situation could be different for $\hat{\alpha} \neq 0$. In next subsections we implement these results to investigate the thermal phase transition between AdS black hole and thermal AdS space.

B. Phase Transition in the Einstein Gravity

A black hole at high temperature is stable, while it is unstable at low temperature, and there can occur a phase transition between thermal AdS and AdS black hole at some critical temperature. In [3], existence of the first order
phase transition was interpreted in terms of quantum gravity rather than boundary conformal field theory. Only recently, Witten [4] has interpreted this behavior as the confinement-deconfinement transition in dual Yang-Mills (gauge) theory. In Witten’s interpretation the thermodynamics of the black hole corresponds to the thermodynamics of the strongly coupled super-Yang-Mills (SYM) theory in the unconfined phase, while the thermal anti-de Sitter space corresponds to the confined phase of the gauge theory. The results in Einstein’s theory are obtained by setting $\alpha = 0$. Therefore

$$T_H = \frac{(d - 2)}{4\pi r_+} \left( k + \frac{d}{d - 2} \frac{r_+^2}{\ell^2} \right) ,$$

$$F = \frac{V_3 r_+^{d-2}}{\kappa_5} \left( k - \frac{r_+^2}{\ell^2} \right) .$$

The minimum of $T_H$ occurs at $r_+ = \ell \sqrt{k/2}$, implying that $T_{min} = \sqrt{2k/(\pi \ell)}$. A thermal AdS phase is preferred for $T < T_{min}$. By definition, free energy of the thermal AdS space is zero, but from (30) $F = 0$ only at $r_+ = \ell \sqrt{k}$. The critical Hawking temperature is therefore $T_H = T_c = 3\sqrt{k/(2\pi \ell)}$. Thus, $k = 0$ and $k = -1$ cases are of less interest to explain the phase transition. But, for $k = 1$, $T_c$ defines a possible first order phase transition. When $T_H > T_c$ a stable AdS black hole exists, but a thermal AdS phase is preferred for $T_c > T_H$.

### C. Phase Transition in the Einstein-Gauss-Bonnet Gravity

We may allow a non-trivial GB coupling $\alpha$ to investigate the Hawking-Page transitions in gravity side. It is suggestive to consider the $d + 1 \geq 6$ case separately, since the properties of the solutions differ from the special case $d + 1 = 5$, for example, see Fig. (3). We consider here only the $d + 1 = 5$ case for two reasons: (i) from $AdS_5/CFT_4$ point of view, and (ii) other than in $d + 1 = 5$ there is no new phase transition of locally stable small black hole. Then

$$F = -\frac{V_3}{\kappa_5} \left[ 3k \left( r_H^2 - \hat{\alpha} k \right) - 8\pi r_H^3 T_H + \frac{9r_H^4}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) \right] ,$$

where $\hat{\alpha} = 2\alpha \kappa_5$. Thus, free energy can be zero at the critical Hawking temperature

$$T_H = T_c = \frac{1}{8\pi r_H} \left[ 3k \left( 1 - \frac{\hat{\alpha} k}{r_H^2} \right) + \frac{9r_H^2}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) \right] .$$

The minimum of $T_H$ is given by solving

$$r_+^2 \left( k - \frac{2r_+^2}{\ell^2} \right) + 2\hat{\alpha} k \left( k - \frac{6r_+^2}{\ell^2} \right) = 0 ,$$
FIG. 4: Gauss-Bonnet black hole: free energy vs horizon radius for \( k = 0 \) and \( d = 4 \). The curve with \( F > 0 \) corresponds to \( \hat{\alpha}/\ell^2 = (0.7)/(0.836)^2 \), which gives a dS solution, since \( \ell^2 > \hat{\alpha} \) and hence \( \Lambda > 0 \). The other three curves with \( F < 0 \) from up to down correspond respectively to \( \hat{\alpha}/\ell^2 = (0.7)/(0.84)^2 \), \( (0.7)/(0.85)^2 \) and \( 0.7/(0.9)^2 \).

FIG. 5: Gauss-Bonnet black hole: inverse temperature vs horizon radius for the case \( k = 1 \) in \( d = 4 \) and \( \hat{\alpha}/\ell^2 = (0.3)/64 \). The upper curve corresponds to \( T^{-1} < T_c^{-1} \) and the lower one to \( T^{-1} > T_c^{-1} \). The region where \( T_H \) exceeds \( T_c \) is shown.

provided \( (r_+^2 + 2\hat{\alpha}k) \neq 0 \). In the following, it would be suggestive to consider the \( k = 0, 1 \) and \( -1 \) cases separately.

\( k = 0 \) case: From Eq. (24) in \( d = 4 \) and Eq. (32), \( F = 0 \) only if \( \ell^2 = \hat{\alpha} \). For \( \ell^2 > \hat{\alpha} \), since one has \( F < 0 \), the black hole is always stable. For \( \ell^2 < \hat{\alpha} \), free energy (31) appears to be positive, but this limit is not allowed for AdS black hole, because \( \ell^2 < \hat{\alpha} \) necessarily implies \( \Lambda > 0 \) and the black hole mass in Eq. (19) is negative. Thus \( \ell^2 \geq \hat{\alpha} \) puts a bound for flat \( (k = 0) \) AdS black holes. We see no evidence of a phase transition for \( k = 0 \) even if \( \hat{\alpha} \neq 0 \). This behavior is not changed for \( (d + 1) > 5 \), unlike the case will be with \( k \neq 0 \).

\( k = \pm 1 \) case: This is the most interesting case, since the boundary of the bulk manifold will have an intrinsic geometry same as the background of the boundary field theory at finite temperature (i.e. \( S^1 \times S^3 \)). From Eq. (24) in \( d = 4 \) and Eq. (32), we have \( F \geq 0 \), only if

\[
\left( 1 - \frac{r_+^2}{\ell^2} \right) - \frac{3\hat{\alpha}}{r_+} \left( 1 - \frac{2\hat{\alpha}}{r_+^2} + \frac{6r_+^2}{\ell^2} \right) \geq 0.
\]  (34)

One has \( F > 0 \) for \( T < T_c \), while \( F < 0 \) for \( T > T_c \). As seen in Fig. (4), small Gauss-Bonnet black hole with a spherical horizon, which has a small positive free energy at start, evolves to thermal AdS phase, attains a maximum positive free energy at some \( r = r_+ \), and eventually goes to a stable black hole phase for large \( r \).
FIG. 6: Gauss-Bonnet black hole: free energy vs horizon radius for the case $k = 1$ in $d = 4$. The three curves from up to down correspond respectively to the cases $\hat{\alpha}/\ell^2 = (0.3)/100$, $(0.3)/64$ and $\hat{\alpha}/\ell^2 = (0.3)/36$. The $k = 1$ is the most plausible situation for a Hawking-Page phase transition even if $\hat{\alpha} \neq 0$.

At the critical radius $r_+^2 = 2\hat{\alpha}$, free energy is zero when $r_+^2 = \ell^2$, $F$ is always negative for $r_+^2 > 2\hat{\alpha}$ and $r_+^2 > \ell^2$, but $F$ can be positive for $2\hat{\alpha} < r_+^2 < \ell^2$, which may trigger a Hawking-Page type transition even if the horizon is hyperbolic (see Figs. 8 and 9). We note from Fig. 9 that under the limit $0.48 < \hat{\alpha}/\ell^2 < 0.52$, free energy is always positive. Thus $k = -1$ may allow one to study the boundary field theory at finite temperature with different background geometries $\mathbb{H} \times S^1$. Due to a possible phase transition for black holes with a hyperbolic horizon, one may find it particularly amusing that when the hypersurface is $AdS_3 \times S^1$ ($AdS_3$ may be obtained by analytic continuation of $dS_3$) quantum gravity in $AdS_5$ can be dual to a boundary conformal field theory on $AdS_4$ background. This possibly reflects that the geometry on the boundary is not dynamical, since there are no gravitational degrees of freedom in the dual CFT.

$k = -1$ case: In this case, the $F \leq 0$ condition reads

$$
\left(1 + \frac{r_+^2}{\ell^2}\right) + \frac{3\hat{\alpha}}{r_+^2} \left(1 + \frac{2\hat{\alpha}}{r_+^2} - \frac{6 r_+^2}{\ell^2}\right) \geq 0.
$$

(35)
FIG. 8: Gauss-Bonnet black hole: free energy vs horizon radius for the case \( k = -1 \) in \( d = 4 \). The four curves from up to down correspond respectively to the values \( \hat{\alpha}/\ell^2 = (0.3)/(0.8)^2 \), \( (0.3)/(0.65)^2 \), \( (0.3)/(0.6)^2 \) and \( (0.3)/4 \). For \( \hat{\alpha}/\ell^2 > (0.3)/(0.6)^2 \) and \( \hat{\alpha}/\ell^2 < (0.3)/(1.5)^2 \), the free energy is always negative.

FIG. 9: Gauss-Bonnet black hole: free energy (\( F \)) vs horizon radius (\( r_H \)) for the case \( k = -1 \) in \( d = 4 \). All the curves in between \( (0.3)/(0.7595)^2 \leq \hat{\alpha}/\ell^2 \leq (0.3)/(0.7905)^2 \) coincide each other. This is the region where a small topological black hole might prefer the thermal AdS phase, \( F > 0 \).

IV. THERMODYNAMIC QUANTITIES WITH \( \gamma = 0 \)

For \( \gamma = 0 \), the field equations following from (3), integrate to give the metric solution (5) with (4)

\[
e^{2\phi(r)} = k + \frac{r^2}{L^2} - \frac{\mu}{r^{d-2}},
\]

where \( L^2 \) is related to the cosmological term \( \Lambda \) via

\[
\Lambda = -\frac{d(d-1)}{2\kappa_{d+1}L^2} \left( 1 - \frac{(d-3)\varepsilon}{2(d-1)} \right),
\]

where

\[
\varepsilon = \frac{2d(\beta + (d+1)\alpha)\kappa_{d+1}}{L^2}.
\]

The integration constant \( \mu \) is related to the black mass \( M = (d-1) V_{d-1} \mu/\kappa_{d+1} \). When \( \mu = 0 \), the solution (5) locally corresponds to AdS metric, while \( \mu \neq 0 \) gives AdS black hole solutions. For the background metric (5), the
The classical action takes the form

$$I = - \frac{V_3}{2\pi G_5 T} \int_{r_H}^{r_\infty} dr \, r^3 (1 - \varepsilon) ,$$

(39)

where $V_3$ is the volume of the manifold $M^3$ and $\varepsilon = 8\kappa_5 (\beta + 5\alpha) / L^2$. After a proper regularization, the Euclidean action is identified with the free energy ($F$) times $1/T_H$. Following [42], the free energy ($F$) and the Hawking temperature $T_H$ evaluated for $d = 4$ read

$$F = \frac{V_3 r_H^2}{16\pi G_5} \left( k - \frac{r_H^2}{L^2} \right) (1 - \varepsilon) ,$$

(40)

$$T_H = \frac{1}{4\pi} \left[ e^{2\phi(r)} \right]^{\prime} |_{r=r_H} = \frac{1}{\pi L^2} \left( r_H + \frac{k L^2}{2r_H} \right) .$$

(41)

As in [42], if one defines $\tilde{G}_5 (1 - \varepsilon) = G_5$, entropy and energy will have the usual form

$$S = - \frac{dF}{dT_H} = \frac{V_3 r_H^3}{4G_5} ; \quad E = F + TS = \frac{3V_3}{16\pi G_5} \equiv M .$$

(42)

Thus energy of the AdS black hole can be identified simply by mass of the black hole, in terms of a renormalized Newton constant $\tilde{G}_5$. Moreover, these quantities may be identified, up to a conformal factor, with the same quantities defined on the boundary of $AdS_5$.

### A. The role of boundary terms

For a definiteness, we work in $d + 1 = 5$, and define $e^{-\sigma(y)} = r$. Then we can bring the metric [52] in the following Randall-Sundrum type five-dimensional warped metric ¹

$$ds^2 = e^{-2\sigma(y)} \left( -d\tau^2 + \sum_{i=1}^{3} \gamma_{ij} dx^i dx^j \right) + dy^2 ,$$

(43)

where $y$ denotes an extra (fifth) space transverse to the brane, which picks out a family of hypersurfaces, $y = \text{const}$. To make the role of the “brane” dynamic, one also adds to [52] the following boundary term, corresponding to the vacuum energy on the hypersurface,

$$I_{\partial B} = \int_{\partial B} d^d x \sqrt{|g_{(d)}|} (-T) + \cdots ,$$

(44)

where, the ellipsoids represent the higher order surface terms, and $\partial B$ denotes the hypersurface with a constant extrinsic curvature. To the leading order, one may drop the higher order surface terms by imposing certain restriction on their scalar invariants (see, for example, [53]), and this is the choice we adopt here. The extrinsic curvature $K_{ab}$ can be easily calculated from [53]. The variation $\delta I + \delta I_{\partial B}$ at the 4$d$ boundary would rise to give

$$\delta I + \delta I_{\partial B} = \int d^d x \sqrt{|g_{(d)}|} \left[ \left( \frac{8(1-\varepsilon)}{\kappa_5} - \frac{8}{\tilde{\kappa}} \right) \delta\sigma + 4 \left( \frac{8}{\kappa} \sigma' - \frac{2(1-\varepsilon)}{\kappa_5} \sigma' + \frac{T}{2} \right) \delta\sigma \right] .$$

(45)

Let us assume that the gravitational couplings $\kappa$ and $\tilde{\kappa}$ are related by [52]

$$\frac{1}{\tilde{\kappa}} - \frac{1-\varepsilon}{\kappa_5} = 0 \Rightarrow \tilde{G}_5 = \frac{G_5}{(1-\varepsilon)} ,$$

(46)

so that the first bracket in [45] vanishes. The dynamical equations on the brane then reduce to

$$T = - \frac{12(1-\varepsilon)}{\kappa_5} \sigma'|_{y=0+} = \frac{12(1-\varepsilon)}{\kappa_5 L} ; \quad \Lambda = - \frac{6}{\kappa_5 L^2} \left( \frac{1 - \varepsilon}{6} \right) ,$$

(47)

where $\varepsilon$ is given by [53], and in the second step $\sigma'|_{y=0+} = -1/L$ has been used. The condition $\sigma' = -1/L$ seems useful to recover the RS-type fine tunings.

¹ The related coordinate transformations are given in Ref. [52], which read:

$e^{2\phi(t)} - e^{-2\sigma}e^{-2\phi}g^2 = e^{-2\sigma}$; $e^{2\phi(t')} - e^{-2\sigma}e^{-2\phi}g'^2 = 1$, where $\phi = \partial_t \sigma$, $\sigma' = \partial_{t'} \sigma$. 
V. THERMODYNAMIC QUANTITIES WITH $\gamma \neq 0$

In AdS/CFT, the higher curvature terms in the bulk have coefficients that are uniquely determined [13, 14], thus it is suggestive to specify the coefficient $\gamma$. In particular, one computes the trace anomaly of a $N = 2$ SCFT in AdS$_{5}$ supergravity or conformal anomaly on $S^{4}$ in the $N = 4$ super YM theory. One finds there the order $N$ gravitational contribution to the anomaly from a (Riemann)$^{2}$ term. The string theory dual to $N = 2$ SCFTs with the gauge group $Sp(N)$ has been conjectured to be type IIB string theory on $AdS_{5} \times S^{5}$/Z$_{2}$ background [14, 44], whose low energy effective (bulk) action in five dimensions reads

$$S = \int d^{5}x \sqrt{-g} \left[ \frac{R}{\kappa_{5}} - 2\Lambda + \gamma R_{abcd}R^{abcd} \right] + \text{(boundary terms)}, \quad (48)$$

if we define

$$\frac{L^{3}}{\kappa_{5}} = \frac{N^{2}}{4\pi^{2}}, \quad \Lambda = -\frac{6}{\kappa_{5} L^{2}} = \frac{6N^{2}}{4\pi^{2}} \frac{1}{L^{3}}, \quad \gamma = \frac{6N}{24 \cdot 16\pi^{2}} \frac{1}{L}, \quad (49)$$

hence $\gamma > 0$. The third term in (48) is suppressed by $1/N$ with respect to the first two terms, thus taking $N \to \infty$ (i.e. $\alpha' = L^{2}/(4\pi g_{st} N)^{1/2} \to 0$, where $g_{st}$ is the string coupling) enables one to take classical limit of the string theory on $AdS_{5} \times S_{5}$ [5]. If we wish to study finite $N$ coupling effects, it is necessary to take into account the leading higher derivative corrections, including (Riemann)$^{2}$ term.

A string background $AdS_{5} \times S^{5}$ is an exact solution of Type IIB theory [2], where one knows that the no of $R^{2}$ term, and sub-leading corrections come only from (Weyl)$^{4}$ term. Therefore, the motivation for including (Riemann)$^{2}$ term in (48) is suppressed by $1/N$ with respect to the first two terms, thus taking $N \to \infty$ (i.e. $\alpha' = L^{2}/(4\pi g_{st} N)^{1/2} \to 0$, where $g_{st}$ is the string coupling) enables one to take classical limit of the string theory on $AdS_{5} \times S_{5}$ [5]. A Hawking-Page phase transition can occur only when the $F < 0$ (thermal AdS) and $F > 0$ (black hole). We will comment upon the finite coupling effects, but before that we evaluate the black hole parameters for the above theory.

A. Black hole solutions

When $\gamma \neq 0$, we can find only a perturbative metric solution, which reads, in $(d + 1) = 5$,

$$e^{2\phi(r)} = k - \frac{\mu}{\gamma} \left(1 + \frac{2\hat{\gamma}}{\ell^2}\right) + \frac{r^{2}}{\ell^{2}} \left(1 + \frac{2\hat{\gamma}}{3\ell^{2}}\right) + \frac{\hat{\gamma} \mu^{2}}{r^{6}} + \mathcal{O}(\hat{\gamma})^{2}, \quad (50)$$

where $\hat{\gamma} = 2\gamma \kappa_{5}$. Since $\mu$ is an integration constant with dimension of $(\text{length})^{d-2}$, we may rescale

$$\mu \left(1 + \frac{2\hat{\gamma}}{\ell^2}\right) \rightarrow \tilde{M}, \quad \frac{1}{\ell^{2}} \left(1 + \frac{2\hat{\gamma}}{3\ell^{2}}\right) \rightarrow \frac{1}{L^{2}}, \quad (51)$$

in order to bring the solution (50) in the usual form

$$e^{2\phi(r)} = k - \frac{\tilde{M}}{r^{2}} + \frac{r^{2}}{L^{2}} + \frac{\hat{\gamma} \tilde{M}^{2}}{r^{6}} + \mathcal{O}(\hat{\gamma})^{2}. \quad (52)$$

The integration constant $\tilde{M}$ at the singularity $e^{2\phi(r)} = 0$ reads, from Eq. (52), as

$$\tilde{M}_{+} = r_{+}^{2} \left[k + \frac{r_{+}^{2}}{L^{2}} + \frac{\hat{\gamma}}{r_{+}} \left(k^{2} + \frac{2k r_{+}^{2}}{L^{2}} + \frac{r_{+}^{4}}{L^{4}}\right)\right] + \mathcal{O}(\hat{\gamma})^{2}, \quad (53)$$

where $r_{+} = r_{H}$ is the black hole horizon. The corresponding Hawking temperature $T_{H}$ is given by

$$T = -\frac{1}{\beta} = \frac{1}{4\pi} \left[e^{2\phi(r)}\right]'|_{r=r_{+}} = \frac{1}{\pi L^{2}} \left[r_{+} + \frac{k L^{2}}{2r_{+}} - \frac{\hat{\gamma}}{L^{2}} \left(r_{+} + \frac{2k L^{2}}{r_{+}} + \frac{k^{2} L^{4}}{r_{+}^{3}}\right)\right] + \mathcal{O}(\hat{\gamma})^{2}. \quad (54)$$

The black hole solutions with $k = 0$ are qualitatively similar to those of Einstein’s theory, so only the $k = \pm 1$ cases are of interest. We plot inverse Hawking temperature versus the horizon radius in Fig. (10) with $k = +1$ at some fixed values of $\hat{\gamma}/L^{2}$, and in Fig. (11), but with $k = -1$. A Hawking-Page phase transition can occur only when the AdS-Schwarzschild solutions possess both phases: $F > 0$ (thermal AdS) and $F < 0$ (black hole). For $k = +1$, there
FIG. 10: Black hole in $\text{(Riemann)}^2$ gravity: inverse temperature vs horizon radius ($r_H$) for $k = +1$ in $d = 4$. The three curves from up to down correspond respectively to $\tilde{\gamma}/L^2 = (0.15)/(4.0)^2$, $(0.15)/(2.511)^2$ and $(0.15)/(2.0)^2$.

FIG. 11: Black hole in $\text{(Riemann)}^2$ gravity: inverse temperature vs horizon radius for $k = -1$ in $d = 4$. The three curves from up to down correspond respectively to $\tilde{\gamma}/L^2 = (0.15)/(0.3875)^2$, $(0.15)/(0.4)^2$ and $(0.15)/(0.41)^2$. Black holes with $\tilde{\gamma}/L^2 > (0.15)/(0.3875)^2$ (i.e. $\tilde{\gamma} > 3\ell^2$ or $\Lambda > 0$) are unstable. A line asymptotic to each curve shifts towards right as $\tilde{\gamma}/L^2$ is decreased, hence a bold asymptote appears.

can occur such a transition, driven by finite size effects, in particular, when $\tilde{\gamma}/L^2 < (0.15)/(2.511)^2$. This implies the existence of $F > 0$ region within a certain range of $r_H$, see Fig. (12).

From Fig. (11) we see that the thermodynamic properties of topological black holes ($k = -1$) are qualitatively similar to those of the five dimensional Gauss-Bonnet black holes with a spherical event horizon ($k = 1$). This behavior, however, disappears for a small $\tilde{\gamma}/L^2$, for example, when $\tilde{\gamma}/L^2 < (0.15)/(0.55)^2$, which actually corresponds to the $F < 0$ region in Fig. (13).

In the high temperature limit $T >> 1/L$ (i.e. $r_+ >> L$), the horizon radius may be expressed as

$$r_+ = \frac{\pi L^2 T}{2(1 - \delta)} \left[ 1 + \sqrt{1 - \frac{2(1 - \delta)(1 - 4\delta)k}{\pi^2 L^2 T^2}} + k^2 \delta O \left( \frac{1}{L^4 T^4} \right) \right], \quad (55)$$

where $\delta = \tilde{\gamma}/L^2 \equiv 1/(8N)$. For $\delta = 0$ (or $N \to \infty$), one has the known results of Einstein’s theory: $T_{\text{min}} = \sqrt{2k}/(\pi L)$ and $T_c = 3\sqrt{k}/(2\pi L)$. The (Riemann)$^2$ correction, therefore, makes $T_{\text{min}}$ smaller and $T_c$ larger.

We follow the Euclidean prescription [4, 5, 7] for regularizing the action and identify the Euclidean action with the
FIG. 12: Black hole in $(\text{Riemann})^2$ gravity: free energy vs horizon radius for $k = +1$ in $d = 4$. The three curves from up to down correspond respectively to $\hat{\gamma}/L^2 = (0.15)/2$, $(0.15)/(2.511)^2$, and $(0.15)/(2.0)^2$.

FIG. 13: Black hole in $(\text{Riemann})^2$ gravity: free energy vs horizon radius for $k = -1$ in $d = 4$. The three curves from up to down correspond respectively to $\hat{\gamma}/L^2 = (0.15)/2$, $(0.15)/(2.5475)^2$, and $(0.15)/(2.7)^2$.

free energy times $1/T_H$. For $(d + 1) = 5$, we find the following expression for free energy $^2$

$$F = \frac{V_3}{\kappa_5} \left[ \left( 1 - \frac{2\hat{\gamma}}{L^2} \right) \frac{\hat{M} - 2r_+^4}{L^2} - \frac{6\hat{\gamma}\hat{M}^2}{r_+^4} \right] + O(\hat{\gamma}^2)$$

$$= \frac{V_3r_+^2}{\kappa_5} \left[ k - \frac{r_+^2}{L^2} - \frac{12\hat{\gamma}k}{L^2} - \frac{3\hat{\gamma}r_+^2}{L^4} - \frac{5\hat{\gamma}^2k^2}{r_+^4} \right] + O(\hat{\gamma}^2)$$

(56)

where we have substituted Eq. (53) in the second step above. In Figs. (12) and (13), free energy of the black hole vs horizon radius is plotted respectively for $k = +1$ and $k = -1$. For $k = +1$, existence of a thermal AdS phase ($F > 0$) requires $\hat{\gamma}/L^2 < (0.15)/(2.511)^2(= 1/42)$, which implies that $N > 5$. There is no phase transition even for a closed geometry ($k = +1$) when $\hat{\gamma}/L^2 > (0.15)/(2.511)^2$, and hence $N \leq 5$, because in this limit free energy is always negative. The existence of a minimum $N$ for the possible Hawing-Page phase transition is not in any contradiction. There is no restriction for taking a large $N$, rather this limit more and more ensures existence of the both phases: $F > 0$ and $F < 0$. As $N$ is increased, $\hat{\gamma}/L^2 = 1/(8N)$ gets decreased, and hence an AdS phase exists for small $r_H$.

$^2$ This result differs in sign for the last term from that of [43]. Perturbative black hole solution with $R^2$ terms and its thermodynamic behavior was further discussed in [45], but we deserve some differences in the results so far reported.
while a stable black hole phase always exists for large \( r_H \). For \( k = -1 \), \( F > 0 \) requires that \( N \) be bounded from up, i.e. \( \tilde{\gamma}/L^2 > (0.15)/(0.5475)^2(= 1/2) \), which does not give a physical bound for \( N \).

In order to examine the finite coupling effects, we may express the free energy density, read from Eq. (56), in terms of \( T \) and \( N \). We can define the regularized action as \( \Delta I = V_3 \beta \Delta F \), where \( V_3 \) is the coordinate volume of \( \tau (\equiv -it) \) and \( d\Omega_3(k) \), and \( \Delta F \) is the free energy density. Then, since \( \tilde{\gamma}/L^2 = 1/(8N) \) from (49), using (57) and (56) we get

\[
\Delta F = -\frac{\pi^2 T^4}{4} \left[ \left( N^2 + \frac{7N}{8} \right) - \left( 3N^2 - \frac{3N}{4} \right) \frac{k\beta^2}{\pi^2 L^2} + \left( \frac{3N^2}{2} - \frac{23N}{16} \right) \frac{k^2\beta^4}{\pi^4 L^4} + O \left( \frac{\beta}{L} \right)^6 \right].
\]

(57)

It is not a priori clear what is the finite temperature field theory to which this corresponds. In view of the conjectured duality between a string theory on \( AdS_5 \times S^5/\mathbb{Z}_2 \) background and \( \mathcal{N} = 2 \) supersymmetric \( Sp(N) \) gauge theory, this result may be relevant to the \( \mathcal{N} = 2 \) \( Sp(N) \) gauge theory at finite temperature, which is not known. One may therefore wish to compare (57) with those of (58) with those of \( \mathcal{N} = 2 \) \( Sp(N) \) gauge theory (which has \( 4(N(2N + 1) + (2N^2 + 7N - 1)) \) boson-fermion pairs) at weak coupling (or free field limit \( g^2N \rightarrow 0 \)):

\[
\Delta F = -\frac{\pi^2 T^4}{3} \left( N^2 + 2N - \frac{1}{4} \right).
\]

(58)

To the leading order, a difference of factor \( 3/4 \) is seen between the weak and strong coupling limits. A possible explanation for the origin of this difference on \( AdS_5 \times S^5 \) background, including the (Weyl) \(^4 \) terms, was first given in (46), and further comments on this difference were made in the review (47) (see also references therein). It is expected that a similar relation would apply to the string theory on \( AdS_5 \times S^5/\mathbb{Z}_2 \) background. Note that even on the flat space \( k = 0 \), two results do not seem to match each other, the role is here being played by (Riemann)\(^2 \) term.

By considering a 5d AdS Schwarzschild background, but with a trivial (Riemann)\(^2 \) bulk term, and using AdS/CFT correspondence, we can calculate the free energy density, which is relevant to the \( N \rightarrow \infty \) (large \( \text{t ' Hooft coupling} \) \( g^2N \) limit of the \( \mathcal{N} = 2 \) \( Sp(N) \) gauge theory at finite temperature, which is not known. One may therefore wish to compare (57) with those of (58) with those of \( \mathcal{N} = 2 \) \( Sp(N) \) gauge theory (which has \( 4(N(2N + 1) + (2N^2 + 7N - 1)) \) boson-fermion pairs) at weak coupling (or free field limit \( g^2N \rightarrow 0 \)).

\[
\Delta F = -\frac{\pi^2 N^2 T^4}{8} \left[ 1 - \frac{3k\beta^2}{\pi^2 L^2} + \frac{3k^2\beta^4}{2\pi^4 L^4} + O \left( \frac{\beta}{L} \right)^6 \right].
\]

(59)

While, the high temperature limit of the free energy density for four dimensional \( \mathcal{N} = 4 \) supersymmetric SU(\( N \)) theory at weak coupling is given by (18)

\[
\Delta F = -\frac{\pi^2 N^2 T^4}{6} \left[ 1 - \frac{3k\beta^2}{2\pi^2 L^2} + O \left( \frac{\beta}{L} \right)^6 \right].
\]

(60)

There is already a mismatch in the sub-leading terms between (59) and (60). It would be not surprising if similar situation arises for \( \mathcal{N} = 2 \) \( Sp(N) \) gauge theory. There is also a temperature independent term in (59), which is interesting to study further. A difference of factor 1/2 in the leading term for free energy density between (57) and (59) simply arises from the dictionary of AdS/CFT: in SU(\( N \)) SYM theory one uses \( G_3 L^5 = 8\pi^3 g_\beta^2 \alpha'^4 = \pi L^8/(2N^2) \) which differs by 1/2 from (19) due to a volume factor Vol(\( S^5/\mathbb{Z}_2 \)) versus Vol(\( S^5 \)).

To evaluate thermodynamic quantities of the black holes, including entropy, we go back to the results (54) and (56). The entropy of the black hole can be determined by using the relation

\[
S = \frac{dF}{dT_H} = -\frac{dF}{dr_+} \frac{dr_+}{dT_H},
\]

where the derivative terms \( dF/dr_+ \) and \( dT_H/dr_+ \) take the following forms

\[
\frac{dF}{dr_+} = -\frac{4V_3\gamma^3}{\kappa_5 L^2} \left[ 1 - \frac{kL^2}{2r_+^2} + \frac{3\tilde{\gamma}}{L^2} \left( 1 + \frac{2kL^2}{r_+} \right) \right] + O(\tilde{\gamma}^2),
\]

\[
\frac{dT_H}{dr_+} = \frac{1}{\pi L^2} \left[ 1 - \frac{kL^2}{2r_+^2} - \frac{\tilde{\gamma}}{L^2} \left( 1 - \frac{2kL^2}{r_+} - \frac{3k^2L^4}{r_+^4} \right) \right] + O(\tilde{\gamma}^2).
\]

(62)

Hence we obtain (since \( \tilde{\gamma} = 2\gamma\kappa_5 \))

\[
S = \frac{4\pi V_3\gamma^3}{\kappa_5 L^2} \left[ 1 + \frac{8\gamma\kappa_5}{L^2} \left( 1 + \frac{kL^2}{r_+} - \frac{3k^2L^4}{4r_+^4} \right) \left( 1 - \frac{kL^2}{2r_+} \right)^{-1} \right] + O(\tilde{\gamma}^2).
\]

(63)
For a large size black hole $2r_+^2 >> kL^2$, one may approximate the entropy

$$S \simeq \frac{V_3 r_+^3}{4G_5} \left[ 1 + \frac{8\gamma \kappa_5}{L^2} \left( 1 + \frac{3kL^2}{2r_+^2} \right) \right] + \mathcal{O}(\gamma^2).$$ (64)

Thus entropy of the black hole with (Riemann)$^2$ term does not satisfy the area formula $S = A/4G$. Rather, just as the Einstein term in the action is corrected by (Riemann)$^2$ term, the Einstein contribution to black hole entropy ($S = A/4G$) receives (Riemann)$^2$ corrections. This has been expected in the literature (see Ref. [9] for review), but here we further show that entropy (64) agrees with Wald’s formula for entropy [27] for $r_+^2 >> L^2$. Moreover, we may implement these results to reproduce boundary data on the horizon of the AdS geometry.

We can also find the black hole energy, defined as $E \equiv T \mathcal{S} + F$,

$$E = \frac{3V_4}{\kappa_5} \left[ \hat{M}_o + 2\gamma \kappa_5 \left( \frac{3}{L^4} + \frac{5k r_+^2}{2L^2} - k^2 - \frac{k^3L^2}{2r_+^4} \right) \left( 1 - \frac{kL^2}{2r_+^2} \right)^{-1} \right] + \mathcal{O}(\gamma^2).$$ (65)

Here $\hat{M}_o = r_+^2 (k + r_+^2/L^2)$ is the value of $\hat{M}_o$ when $\gamma = 0$. For $k = 1$, one apparently sees a singularity at $2r_+^2 = L^2$, but there is no singularity in the formulas for energy Eq. (63) and entropy Eq. (64). One easily checks that at $r_+^2 = L^2/2$, for $k = 1$, the first round bracket in (63) also vanishes, thus the above formulas are applicable to all three possible values: $k = 0, \pm 1$.

It has been known that entropy of the black hole can be expressed as a local geometric (curvature) density integrated over a space-like cross section of the horizon. Notably, an entropy formula valid to any effective gravitational action including higher curvature interactions was first proposed in [27], and was nicely generalized in Refs. [28, 30]. One can infer from [30] that the black hole entropy for the action $F$ takes in five dimensions the following form

$$\tilde{S} = \frac{4\pi}{\kappa_5} \int_{\text{horizon}} d^3x \sqrt{h} \left\{ 1 + 2\alpha \kappa_5 R + \beta \kappa_5 \left( R - h^{ij} R_{ij} \right) \right\},$$ (66)

where $h$ is the induced metric on the horizon. For $d + 1 = 5$, since the curvatures are defined in the following form

$$R = \frac{-20}{L^2}, \quad h^{ij} R_{ij} = \frac{-12}{L^2}, \quad h^{ij} h^{kl} R_{ijkl} = \frac{6k}{r_+^2} + \mathcal{O}(\hat{\gamma}),$$ (67)

from the Eq. (66), when $\alpha = \beta = 0$, we read

$$\tilde{S} = \frac{V_3 r_+^3}{4G_5} \left[ 1 + \frac{8\kappa_5}{L^2} \left( 1 + \frac{3kL^2}{2r_+^2} \right) \right] + \mathcal{O}(\gamma^2).$$ (68)

Hence the two expressions for entropies, i.e. Eq. (64) and Eq. (68), are identical. Our results are suggestive, and clearly contradict the observations made in Ref. [13] in this regard.

From the $(d + 1)$ dimensional analogue of the formula (64), we may calculate entropy of the Gauss-Bonnet black hole, using the relation $\alpha = -\beta/4 = \gamma$,

$$\tilde{S} = \frac{4\pi}{\kappa_5} \int d^{d-1}x \sqrt{h} \left[ 1 + 2\alpha \kappa_{d+1} R(h) \right].$$ (69)

At the horizon, one sets $e^{2\phi(r_+)} = 0$, and reads the value of $R(h)$ from (12). Hence

$$S = \frac{V_{d-1} r_+^{d-1}}{4G_{d+1}} \left[ 1 + \frac{(d - 1)}{(d - 3)} \frac{2\alpha k}{r_+^2} \right],$$ (70)

which coincides with Eq. (27). Myers and Simon [3] have derived the result (70) for entropy of the black hole in asymptotically flat backgrounds ($L^2 \to \infty$ or $\Lambda = 0$) with $k = 1$, but here we see that this holds for arbitrary $\Lambda$. This mimics that the geometry on the boundary or the cosmological term in the bulk is not dynamical. This is plausible and possibly gives some insights of the holography.

### B. Quantities with non-trivial $\alpha, \beta, \gamma$

Furthermore, one could find a perturbative solution for arbitrary $\alpha, \beta$ and $\gamma$ at a time, but this calculation is complicated due to a perturbative expansion. Without loss of any generality, one can follow a different, but equivalent,
prescription, in which one combines the results obtained for (i) $\alpha, \beta \neq 0$ and $\gamma = 0$, to the results obtained for (ii) $\alpha = \beta = 0$ and $\gamma \neq 0$. Thus, by combining the results \cite{10, 12} and \cite{15, 16, 17}, we obtain the following expressions for free energy ($F$), entropy ($S$) and energy ($E$) of the 5d AdS black hole with quadratic curvature terms

$$F = \frac{V_3}{\kappa_5} \left[ r_H^2 \left( k - \frac{r_H^2}{L^2} \right) (1 - \varepsilon) - 10 \gamma \kappa_5 \left( k + \frac{r_H^2}{L^2} \right)^2 \right] + \mathcal{O}(\gamma^2),$$

$$S = \frac{4 \pi V_3 r_H^3}{\kappa_5} \left[ (1 - \varepsilon) + \frac{12 \alpha \kappa_5}{L^2} \left( \frac{k L^2}{2 r_H^2} - \frac{k^2 L^4}{2 r_H^4} \right) \left( 1 - \frac{k L^2}{2 r_H^2} \right)^{-1} \right] + \mathcal{O}(\gamma^2),$$

$$E = \frac{3 V_3}{\alpha} \left[ M_0 (1 - \varepsilon) + 2 \gamma \kappa_5 \left( \frac{5 r_H^4}{L^4} + \frac{7 k r_H^2}{2 L^2} - 2 k^2 - \frac{k^3 L^2}{2 r_H^2} \right) \frac{1}{1 - k L^2 / 2 r_H^2} \right] + \mathcal{O}(\gamma^2),$$

where $\varepsilon = 4 \kappa_5 (\gamma + 2 \beta + 10 \alpha) / L^2$, and $M_0$ is related to the mass parameter $\tilde{M}$ when $\alpha = \beta = \gamma = 0$. The consistency of the above results is reflected from the regularities of the expressions for entropy \cite{12} and energy \cite{13} at $k L^2 = 2 r_H^2$.

In the large $N$ limit, one has $L^3 >> \kappa_5$ (from the first expression of Eq. \cite{13}), and in AdS/CFT, the coefficients of the higher curvature terms in the bulk, in general, satisfy the limit $1 > \varepsilon > 0$. In Ref. \cite{14} the subleading contribution to the AdS$_5$/CFT$_4$ trace anomaly was considered. In terms of boundary metric $g_{ij}^{(0)}$, supergravity prediction of $\mathcal{O}(N)$ contribution to the trace anomaly was found to be \cite{14}

$$\frac{6 N}{24 \times 16 \pi^2} \left[ R_{ijkl}^{(0)} R^{(0)i j k l} - \frac{13}{4} R^{(0)} R^{(0)i j} + \frac{3}{4} R^{(0)2} \right],$$

from where we see that $1 < \varepsilon < 0$ holds in large $N$ limit. Since $\gamma > 0$, the free energy \cite{11} is always negative for $k = 0$ and $k = -1$, and such black holes are globally stable. This means that provided $(1 - \varepsilon) > 0$ and $\gamma > 0$, there may not occur a Hawking-Page phase transition for AdS black hole with a Ricci flat or hyperbolic horizon for the above theory. For the $k = 1$ case, $F$ can be negative for large black hole with $r_H^2 >> L^2$ since $\gamma > 0$. While, for a small size black hole $r_H^2 < L^2$, the free energy $F$ can be positive.

As a consistency check of the above formulas, we can express the curvature squared terms in the Gauss-Bonnet form: $\alpha = -\beta/4 = \gamma$. In this case, since $\varepsilon = 12 \alpha \kappa_5 / L^2$, the free energy \cite{11} reduces to

$$F = \frac{V_3}{\kappa_5} \left[ r_+^2 \left( k - \frac{r_+^2}{L^2} \right) + \hat{\alpha} \left( \frac{r_+^4}{L^4} - \frac{16 k r_+^2}{L^2} - 5 k^2 \right) \right] + \mathcal{O}(\alpha^2),$$

where $\hat{\alpha} = 2 \alpha \kappa_5$ and $r_+ = r_H$. This agrees with the exact expression of free energy \cite{16}, where one sets $d = 4$ and $(1 + 2 \hat{\alpha} k / r_+^2)^{-1} \equiv (1 - 2 \hat{\alpha} k / r_+^2)$, and $L \equiv \ell$. We may read entropy for the Gauss-Bonnet black hole from the formula Eq. \cite{12}, in the limit $r_+^2 >> L^2$,

$$S_{GB} = \frac{4 \pi V_3 r_+^3}{\kappa_5} \left[ 1 + \frac{12 \alpha \kappa_5 k}{r_+^2} \right].$$

This very nicely agrees in five dimension ($d = 4$) with the expression \cite{27} or \cite{23} (obtained using two different prescriptions). For $k = 0$, one has $S_{GB} = (4 \pi V_3 r_+^3 / \kappa_5) \equiv A/4G$. This special connection is the reminiscent of the topological behavior of the GB invariant, but as we have already seen $S = A/4G$ does not hold in the generic higher derivative theories.

VI. HOLOGRAPHY BEYOND ADS/CFT

Via holography \cite{23}, it has been known that thermodynamic quantities of a boundary CFT can be determined by those of the global AdS (supergravity) vacuum, the notion of the celebrated AdS/CFT correspondence \cite{2}. Witten \cite{4} further argued that such a correspondence may exist even if we give a finite temperature to the bulk AdS (so that pure AdS bulk is replaced by AdS-Schwarzschild black hole), and define a CFT on the boundary at finite temperature. Of course, finite temperature breaks both the supersymmetry and the conformal invariance. Still one can associate the mass (energy), temperature and entropy of the black hole with the corresponding quantities in the boundary field theory at finite temperature \cite{4}. Many checks that have been performed in the literature are either based on the Einstein’s theory or limited in the large $N$ limit, where one neglects any higher curvature corrections. However, when one includes a bulk (Riemann) term, and extracts boundary data (e.g. free energy, entropy, energy) from AdS-Schwarzschild solution, one finds that they do not agree to the sub-leading order. This we will exhibit below in a simple cosmological context.
A. Relating boundary and bulk parameters

To reproduce boundary data from bulk data and vice versa, one may consider the coordinate transformations given in the footnote [1], which are solved by [42]

$$\sigma^* = \sigma - e^{2\phi(r)} e^{2\sigma(y)} ,$$

(77)

where $d\eta = e^{-\sigma(y)} d\tau$, with $\eta$ being a new time parameter, $\sigma^* = \partial_\eta \sigma$ and $\sigma' = \partial_y \sigma$. One specifies the functions $r = r(\eta), \ t = t(\eta)$, so that $-\sigma^* \equiv \dot{r}/r$ defines the Hubble parameter $H$. Eq. (77) ensures that the induced metric on the brane takes the standard Robertson-Walker form

$$ds_2^2 = -d\eta^2 + r^2(\eta) d\Omega_{d-1}^2 ,$$

(78)

and the radial distance $r$ measures the size of $d$-dimensional universe from the center of the black hole [6]. When applying the holography in the above context, one could consider a $d$-dimensional brane with a constant tension in the background of an $(d+1)$-dimensional AdS black hole. One also regards the brane as the boundary of the AdS geometry, and further assumes that $\sigma'|_{y=0} = -L^{-1}$ at the horizon $r = r_H$, where $e^{2\phi(r)} = 0$. Then it is clear that Eq. (77) leads to $H = \pm 1/L$. Using Eq. (36), one finds the first Friedman equation in $(d+1)$-dimensions [43]

$$H^2 + \frac{k}{r^2} = \frac{\dot{M}}{r^d} \equiv \frac{\kappa_d}{(d-1)(d-2)} \rho ,$$

(79)

where $\kappa_d = 16\pi G_{(d)}$, $\rho = \dot{E}/V$ is the matter energy density on the boundary, $V = V_{d-1} r^{d-1}$, and

$$\dot{E} = \frac{(d-1)(d-2) \dot{M} V_{d-1}}{\kappa_d r} .$$

(80)

The energy $\dot{E}$ coincides with the gravitational energy $E$ (Eq. (42)), up to a conformal factor. Differentiation of Eq. (79) gives the second Friedman equation

$$\dot{H} - \frac{k}{r^2} = -\frac{\kappa_d}{2(d-2)} (\rho + p) , \quad \text{where} \quad p = \frac{(d-2) \dot{M}}{\kappa_d r^d}$$

(81)

is the matter pressure on the brane. Since $-\rho + (d-1) p = 0$, the induced CFT matter is radiation-like $T_{\mu}^\mu = 0$. For a non-zero $\gamma$ (but with $\alpha = \beta = 0$), however, the FRW equation reads

$$H^2 + \frac{k}{r^2} = \frac{\kappa_4}{6} \rho ,$$

(82)

where $\rho = \dot{E}/V$, $V = V_3 r^3$ and

$$\dot{E} = \frac{6V_3}{\kappa_4 r} \left[ \dot{M} - \frac{\dot{\gamma} M^2}{r^4} \right] .$$

(83)

Differentiation of Eq. (82) gives the second FRW equation

$$\dot{H} - \frac{k}{r^2} = -\frac{\kappa_4}{4} \left( \frac{\dot{E}}{V} + p \right) ,$$

(84)

with

$$p = \frac{2}{\kappa_4 r^4} \left[ \dot{M} - \frac{5\dot{\gamma} M^2}{r^4} \right] .$$

(85)

From Eqs. (83) and (85), one has

$$-\frac{\dot{E}}{V} + 3p = -\frac{24\dot{\gamma} M^2}{\kappa_4 r^8} + O(\dot{\gamma}^2) .$$

(86)
This simply means that matter fields on the boundary (brane) does not satisfy a radiation condition \((T^\mu_\mu = 0)\) for \(\gamma \neq 0\). This is not an unexpected behavior with a bulk (Riemann)\(^{2}\) term. To proceed further, it is desirable to define the relation between \(\kappa_4\) and \(\kappa_5\). They can be related by (see [49] for \(\varepsilon = 0\) case)

\[
\kappa_4 = \frac{2}{L} \frac{\kappa_5}{1 + \lambda \varepsilon}.
\]  

(87)

If \(\alpha = \beta = 0\) and \(\gamma \neq 0\), one has \(\varepsilon = 4\kappa_5 \gamma / L^2\). The magnitude of \(\lambda\) can be fixed from the propagator analysis. If one regards the brane as the boundary of the AdS geometry, then it is suggestive to consider a moment in the brane’s cosmological evolution at which the brane crosses the black hole horizon \(r = r_{\text{brane}} = r_H\), so that \(\tilde{M}(r_{\text{brane}}) = \tilde{M}(r_H)\). After implementing the relation (87), i.e.

\[
\kappa_4 = \frac{2}{L} \frac{\kappa_5}{1 + 2\hat{\gamma} / L^2} = \frac{2 \kappa_5}{L} \left(1 - \frac{2\hat{\gamma}}{L^2}\right),
\]  

(88)

from Eq. (83) we find

\[
\tilde{E} = \frac{3V_3}{\kappa_5} \frac{L}{r} \left[\tilde{M}_0 + \frac{4\gamma \kappa_5 r_+^2}{L^2} \left(k + \frac{r_+^2}{L^2}\right)\right] + \mathcal{O}(\hat{\gamma}^2).
\]  

(89)

One may assume that energy \(\tilde{E}\) is rescaled by

\[
\tilde{E} = \frac{L}{r} E_{AdS}.
\]  

(90)

The origin of the factor \(L/r\) is entirely “holographic” in spirit. It has been argued in Ref. [51] that for non-critical branes (i.e. \(k \neq 0\)), the AdS length scale \(L\) may be replaced by \(1/T\), where \(T\) is the brane tension. At any rate, we see that the two expressions for energy, Eq. (65) and \(E_{AdS}\) (which one reads from Eqs. (89) and (90)) do not match to the subleading order, but for \(\gamma = 0\) they do coincide.

Finally, we comment upon the size and location of the brane for a flat and static brane. Instead of \(\sigma' = -1/L\), let us assume that \(\sigma' = -1/\ell\) holds. If the brane (hypersurface) is flat (\(k = 0\)), the horizons defined by \(e^{2\phi(r)} = 0\) (Eq. (50)) read

\[
r_+^4 = \mu t^2 + \frac{\mu \hat{\gamma}}{3} + \mathcal{O}(\hat{\gamma}^2), \quad r_-^4 = \mu \hat{\gamma} \left(1 - \frac{\hat{\gamma}}{t^2} - \frac{16}{3} \frac{\hat{\gamma}^2}{t^4}\right) + \mathcal{O}(\hat{\gamma}^3).
\]  

(91)

If the brane is also static (i.e. \(H = 0\)), the Friedmann equation (84) would rise to give two horizons. However, the one which reduces to that of the Einstein gravity for \(\gamma = 0\) and would be of more physical interest is

\[
r_{\text{brane}}^4 = \mu \hat{\gamma} \left(1 - \frac{2\hat{\gamma}}{t^2} + \frac{2}{3} \frac{\hat{\gamma}^2}{t^4}\right) + \mathcal{O}(\hat{\gamma}^3).
\]  

(92)

Since \(\gamma > 0\), there is a critical point at

\[
1 = \frac{6\hat{\gamma}}{t^2} \equiv \frac{12 \gamma \kappa_5}{t^2},
\]  

(93)

where the brane coincides with the black hole inner horizon \(r_{\text{brane}}^4 = r_+^4\). The brane lies outside the (black hole) inner horizon if \(t^2 > 6\hat{\gamma}\), and it lies inside the black hole horizon if \(t^2 < 6\hat{\gamma}\).

B. Entropy bounds in holography

For completeness and comparison, we list some of the interesting proposals for entropy bounds in holography [52, 53]. Consider the Cardy formula [54] of two-dimensional conformal field theory

\[
S_H = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{ck}{24}\right)},
\]  

(94)
where the CFT generator (eigenvalue) $L_0$ represents the product $E r$ of the energy and radius, $k$ is the intrinsic curvature of the CFT boundary, $c/24$ is the shift in eigenvalues caused by the Casimir effect. In [13, 14], it was shown that the formula [14] can be generalized to arbitrary $d$-dimensions, which then corresponds to the FRW type brane equations of $d$ dimensions. One identifies there [33, 35]

$$L_0 = \frac{E r}{d-1}, \quad \frac{c}{6} = \frac{4(d-2)V_3 r^2}{\kappa_d}, \quad S_H = \frac{4\pi(d-2)HV}{\kappa_d},$$

where $S_H$ is the Hubble entropy [3]. The Cardy formula puts a bound $S_H < S_B$, where

$$S_B = \frac{2\pi r}{d-1} E$$

is the Bekenstein entropy. The “holographic bound” proposed by ’t Hooft and Susskind reads $S \leq S_B$, where

$$S_B = \frac{4\pi(d-2)}{\kappa_d} V.$$  

(96)

This mimics that entropy is smaller than Bekenstein-Hawking entropy of the largest black hole that fits in the given volume. Another entropy bound is the “Hubble bound” proposed in [32], which reads $S \leq S_H$. This heralds that entropy is bounded by total entropy of Hubble size black hole.

For our purpose, we find entropy formula defined in (95) interesting. Though initially derived for two-dimensional CFT, this formula may be applied to CFTd/AdS5 [1], then one regards $S_H$ as the holographic entropy. One may worry about when applying this formula to a theory with (Riemann)² term. We therefore follow here a heuristic approach to estimate a difference between $S_H$ and $S_{BH}$. Note that, for $\gamma \neq 0$, a non-conformal behavior in entropy is seen also from the scaling relation between $\kappa_4$ and $\kappa_5$, the former one involves a non-trivial $\gamma$ via [38]. Using the value $H_0 = 1/L$ and Eq. (68), the Hubble entropy $S_H$ in (95) takes the form, for $d = 4$

$$S_H = \frac{4\pi V_3 r_H^2}{\kappa_5} \left(1 + \frac{4\gamma \kappa_5}{L^2}\right) = \frac{V}{\pi L^3} \left(N^2 + \frac{N}{4}\right).$$

(98)

Of course, in the large $N$ limit the correction term is negligible. Nevertheless, we may calculate the difference in Bekenstein-Hawking entropy $S_{BH}$ (Eq. (63)) and Hubble entropy (Eq. (68)), which reads

$$S_H - S_{BH} = \frac{4\pi V_3 r_H^3}{\kappa_5} \left[-4\frac{\kappa_5}{L^2} \left(1 + \frac{5kL^2}{2r_H^2} - \frac{3k^2L^4}{2r_H^2}\right) \left(1 - \frac{kL^2}{2r_H^2}\right)^{-1}\right] = -48\pi V_3 \gamma r_H \left(k + \frac{r_H^2}{3L^2}\right) = -\frac{3N V_3 r_H}{4\pi L} \left(k + \frac{r_H^2}{3L^2}\right).$$

(99)

Note that the difference in entropies can arise only in the order of $N$, though each entropy is proportional to $N^2$. Moreover, a negative sign in the difference implies that $S_H \geq S_{BH}$. An obvious fact that $S_H$ does not coincide with $S_{BH}$ is not surprising with $\gamma \neq 0$, because in higher derivative gravity entropy is not directly related to the horizon. The entropy formulas for the boundary field theory and the FRW equations coincide only for a radiation dominated (brane) universe [4]. Of course, for $\gamma = 0$, one has $S_H = S_{BH}$. For $k = -1$, one has $S_H > S_{BH}$ if $r_H^2 > 3L^2$, but one has $S_H < S_{BH}$ for $L^2 < r_H^2 < 3L^2$. In the $k = 0$ and $k = 1$ cases, one finds $S_H \leq S_{BH}$. Thus for a flat and spherical AdS black holes, we propose a new entropy bound $S_H \leq S_{BH}$. These results could be useful to understand possible entropy bounds for the non-trivial (Riemann)² interaction.

VII. CONCLUSIONS

A possibility is that holography beyond the AdS/CFT persists in true quantum gravity and unified field theory, which may require inclusion of the higher order curvature derivative terms. With this motivation, we studied in details the thermodynamic properties of anti-de Sitter black holes for the Einstein-Gauss-Bonnet theory, and the Einstein term in the action corrected by the generic $R^2$ terms, and discussed their thermodynamic behavior including the thermal phase transitions in AdS space. We obtained in useful form the expression of black hole free energy for the theories with Guass-Bonnet and (Riemann)² interaction terms, and calculated entropy and energy. In the Einstein’s theory, only the $k = 1$ case is physical to explain the Hawking-Page phase transition. This is suggestive, because the AdS solution with $k = 1$ can be embedded in ten-dimensional IIB supergravity such that the supergravity
background is of the form $AdS_5 \times S^5$, and the boundaries of the bulk manifold will have the same intrinsic geometry as the background of the dual theory at finite temperature. We also noted that, unlike to the Einstein theory, in the EGB theory there may occur a Hawking-Page phase transition also for an event horizon of negative curvature ($k = -1$) hypersurface. The free energy of such a topological black hole starts from a negative value, reaches a positive maximum at some $r = r_+$, and then again goes to negative infinity as $r_+ \to \infty$. Thus, a hyperbolic AdS black hole, though can have a thermal AdS phase within a small range of $r_H$, globally prefers black hole phase for large $r_H$.

It is shown that the contribution from the squares of Ricci scalar and Ricci tensor can be absorbed into free energy, entropy, and energy via a redefinition of the five dimensional gravitational constant and the radius of curvature of AdS space. By introducing the Gibbon-Hawking surface term and a boundary action corresponding to the vacuum entropy, and energy via a redefinition of the five dimensional gravitational constant and the radius of curvature of AdS space. It is shown that the contribution from the squares of Ricci scalar and Ricci tensor can be absorbed into free energy, entropy, and energy via a redefinition of the five dimensional gravitational constant and the radius of curvature of AdS space. By introducing the Gibbon-Hawking surface term and a boundary action corresponding to the vacuum entropy, and energy via a redefinition of the five dimensional gravitational constant and the radius of curvature of AdS space. We recovered the RS type fine tunings as natural consequences of the variational principle. An interesting observation is that the thermodynamic properties of $k = -1$ AdS black holes, for $d + 1 = 5$, under a critical value of $\gamma/L^2$ in (Riemann)$^2$ gravity are qualitatively similar to those of Gauss Bonnet black hole with a spherical event horizon ($k = 1$). For a spherical horizon geometry, the (Riemann)$^2$ term (i.e. a finite coupling but $N > 5$) may induce a Hawking-Page phase transition, though a large $N$ is preferred. We discussed the corrections to the free energy density originated from the higher order curvature corrections to the supergravity action, which change only the $N$ dependence coefficients by sub-leading terms. Nonetheless, such corrections are necessary to look the behavior of AdS/CFT next to leading order, which we otherwise compare only to the CFTs leading terms, e.g., in the free energy and entropy. Moreover, in (Riemann)$^2$-gravity, the Bekenstein-Hawking entropy agrees with the value obtained using similar formula when the black hole horizon $r_H$ is much larger than the AdS length scale $L$. The Bekenstein-Hawking entropy in the Gauss-Bonnet theory coincides with the value directly evaluated using Wald’s formula. These are interesting and quite pleasing results. We also obtained formulas for free energy, entropy and energy of the AdS black holes with generic $R^2$ terms, which are applicable to all three geometries: $k = 0, \pm 1$.

We have also established some relations between the boundary field theory parameters defined on the brane and the bulk parameters associated with the Schwarzschild AdS black hole in five dimensions. We found the FRW type equations on the brane, which exhibit that matter on the AdS boundary does not satisfy the radiation condition ($T_{\mu\nu}^\tau = 0$) for a non-zero $\gamma$. Using a heuristic but viable approach, we calculated a difference between the holographic entropy and the Bekenstein-Hawking entropy, which generally implies $S_H \leq S_{BH}$. The essential new ingredient in our analysis is the role of the higher curvature terms in explaining the essential features of the black hole thermodynamics, including thermal phase transitions, finite coupling effects, and holography.

**Acknowledgements**

This work was supported in part by the BK21 program of Ministry of Education. One of us (IPN) also acknowledges partial support from the Seoul Foundation, Korea, and would like to thank ASICTP for hospitality where part of the work is done. Helpful discussions with M. Blau, Ed Gava and K. S. Narain are gratefully acknowledged.

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