Approximation via toy Fock space – the vacuum-adapted viewpoint

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Dedicated to Slava Belavkin on the occasion of his 60th birthday

Abstract

After a review of how Boson Fock space (of arbitrary multiplicity) may be approximated by a countable Hilbert-space tensor product (known as toy Fock space) it is shown that vacuum-adapted multiple quantum Wiener integrals of bounded operators may be expressed as limits of sums of operators defined on this toy space, with strong convergence on the exponential domain. The vacuum-adapted quantum Itô product formula is derived with the aid of this approximation and a brief pointer is given towards the unbounded case.

1 Introduction

The idea of using discrete approximations in quantum stochastic calculus goes back at least as far as Meyer’s notes [16], where he gave credit to Jean-Lin Journé. Around the same time, articles by Parthasarathy [18] and Lindsay and Parthasarathy [15] showed that certain quantum flows (which are generalisations of classical diffusions) may be approximated by so-called spin random walks, while Accardi and Bach produced (in an unpublished preprint — see [1, 17]) a central-limit theorem which may be viewed as a result on toy-Fock-space approximation. These ideas have recently been the subject of renewed interest.

Attal revisited and extended the Journé–Meyer ideas in [2], giving a heuristic derivation of the quantum Itô product formula using the approximation, and this was followed by further work of Attal and Pautrat [3] and of Pautrat [19]. Their point of view may be considered as physical, rather than probabilistic; in [9], Gough examined the physics of this set-up and explained its connexion with Holevo’s time-ordered exponentials.

Meanwhile, Sinha [21] revived the ideas of [15], emphasising that, in many cases, sufficiently strong convergence holds to enable one to deduce that the limit flows are *-homomorphic. Further work in this direction has done by Sahu [20], who moved away from the spin approach of Lindsay–Parthasarathy–Sinha to adopt the same type of coupling between system and noise as Attal–Pautrat; it is this direct (as opposed to spin) coupling which is used below.

Many other people have worked with these concepts, including Bouten, van Handel and James [6] (in quantum filtering), Brun [7] and Gough and Sobolev [10] (who view the situation as physicists), Franz and Skalski [8] (for
constructing random walks on quantum groups), Kümmerer ([12] gives a detailed physical interpretation of discrete models and is an excellent introduction to his earlier work in this area) and Leitz-Martini [13] (who expressed many of these approximation ideas rigorously using non-standard analysis; for example, the discrete Itô table of Attal [2] Section VII agrees with the continuous-time version only in the limit, but in the non-standard setting the anomalous terms are infinitesimal [13] (2.2.8)).

Here, a vacuum-adapted approach to approximation is adopted and, as might be expected, a very straightforward theory results. After revising the embedding of toy Fock space into Boson Fock space in Section 2, modified versions of the vacuum-adapted Wiener integral are defined in Section 3. The natural 'discrete integral' (which is, of course, a sum) is examined in Section 4 and is shown to be given, up to an error term, by the modified integral previously defined. Section 5 extends this working to the case of multiple integrals, Section 6 shows how the quantum Itô product formula arises naturally from the discrete approximation and Section 7 points the way to further developments involving unbounded operators. Applications of these results will appear elsewhere [5].

1.1 Conventions and Notation

All sesquilinear inner products are conjugate linear in the first variable. We follow [14] for the most part, although the ordering of certain objects is changed: for us, the initial space always appears first (the ‘usual’ convention, to quote Lindsay [14, p.183]).

The vector space of linear operators between vector spaces $U$ and $V$ is denoted by $\mathcal{L}(U; V)$, or $\mathcal{L}(U)$ if $U$ and $V$ are equal; the identity operator on a vector space $V$ is denoted by $I_V$. The operator space of bounded operators between Hilbert spaces $H_1$ and $H_2$ is denoted by $\mathcal{B}(H_1; H_2)$, abbreviated to $\mathcal{B}(H_1)$ if $H_1$ equals $H_2$. The tensor product of Hilbert spaces and bounded operators is denoted by $\otimes$; the algebraic tensor product is denoted by $\odot$. The restriction of a function $f$ to a set $A$ is denoted by $f|_A$; the indicator function of $A$ is denoted by $1_A$. If $P$ is a proposition then the expression $1_P$ has the value 1 if $P$ is true and 0 if $P$ is false.

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2 Toy and Boson Fock spaces

Men more frequently require to be reminded than informed.
— Samuel Johnson, The Rambler, No. 2 (1749–50).

Notation 2.1. Let $k$ be a complex Hilbert space (called the multiplicity space) and let $\hat{k} := \mathbb{C} \oplus k$ be its one-dimensional extension. Elements of $\hat{k}$ will be written as column vectors, with the first entry a complex number and the second a vector in $k$; if $x \in k$ then $\vec{x} := \left( \frac{1}{x} \right)$.

Definition 2.1. Toy Fock space is the countable tensor product

$$\Gamma := \bigotimes_{n=0}^{\infty} \hat{k}(n)$$

with respect to the stabilising sequence $(\omega(n) := \left( \frac{1}{n} \right))_{n=0}^{\infty}$, where $\hat{k}(n) := \hat{k}$ for each $n$; the subscript $(n)$ is used here and below to indicate the relevant copy. (For information on infinite tensor products of Hilbert spaces, see, for example [11, Exercise 11.5.29].)

For all $n \in \mathbb{Z}_+ := \{0, 1, 2, \ldots \}$, let

$$\Gamma_{n+1} := \bigotimes_{m=0}^{n} \hat{k}(m) \text{ and } \Gamma_{n} := \bigotimes_{m=n}^{\infty} \hat{k}(m),$$

where $\Gamma_{0} := \mathbb{C}$. The identity $\Gamma = \Gamma_{n} \otimes \Gamma_{n}$ is the analogue of the continuous tensor-product structure of Boson Fock space.

Notation 2.2. For any interval $A \subseteq \mathbb{R}_+$, let $\mathcal{F}_A$ denote Boson Fock space over $L^2(A; k)$ and let $\mathcal{F} := \mathcal{F}_{\mathbb{R}_+}$. For further brevity, let $K = L^2(\mathbb{R}_+; k)$.

If $\tau := \{0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots \}$ is a partition of $\mathbb{R}_+$ (so that $\tau_n \to \infty$ as $n \to \infty$) then there exists an isometric isomorphism

$$\Pi_{\tau} : \mathcal{F} \overset{\sim}{\longrightarrow} \mathcal{F}_{\tau} := \bigotimes_{n=0}^{\infty} \mathcal{F}_{[\tau_n, \tau_{n+1}]} : \varepsilon(f) \mapsto \bigotimes_{n=0}^{\infty} \varepsilon(f|_{[\tau_n, \tau_{n+1}]}),$$

where the tensor product is taken with respect to the stabilising sequence $(\Omega_{[\tau_n, \tau_{n+1}]} := \varepsilon(0|_{[\tau_n, \tau_{n+1}]}))_{n=0}^{\infty}$ and $\varepsilon(g)$ denotes the exponential vector in $\mathcal{F}_A$ corresponding to the function $g \in L^2(A; k)$. The set of all such partitions of $\mathbb{R}_+$ is denoted by $T$.

Definition 2.2. For all $\tau \in T$ and $n \in \mathbb{Z}_+$, define the natural isometry

$$j[\tau]_n : \hat{k} \to \mathcal{F}_{[\tau_n, \tau_{n+1}]} : \left( \frac{1}{\tau} \right) \mapsto \lambda \Omega_{[\tau_n, \tau_{n+1}]} + x \Omega_{[\tau_n, \tau_{n+1}]},$$

where $\Omega_{[\tau_n, \tau_{n+1}]} := 1_A/\|1_A\|_{L^2(\mathbb{R}_+)}$ is the normalised indicator function of the interval $A \subseteq \mathbb{R}_+$, viewed as an element of the one-particle subspace of $\mathcal{F}_A$. These give an isometric embedding

$$J_{\tau} : \Gamma \to \mathcal{F}_{\tau} : \bigotimes_{n=0}^{\infty} \theta_n \mapsto \bigotimes_{n=0}^{\infty} j[\tau]_n(\theta_n).$$
Note that $Q_\tau := \Pi_\tau^* J_\tau J_\tau^* \Pi_\tau$ is an orthogonal projection on $F$ and
\[ J_\tau^* \Pi_\tau \varepsilon(f) = \bigotimes_{n=0}^{\infty} f_\tau(n) \quad \forall f \in K, \tag{6} \]
where
\[ f_\tau(n) := \frac{1}{\sqrt{\tau_{n+1} - \tau_n}} \int_{\tau_n}^{\tau_{n+1}} f(t) \, dt \quad \forall n \in \mathbb{Z}_+. \tag{7} \]

**Notation 2.3.** Let $T$ be the directed set of all partitions of $\mathbb{R}_+$, ordered by inclusion; the expression '$f_\tau \to f$ as $|\tau| \to 0$' means that the net $(f_\tau)_{\tau \in T}$ converges to $f$. For all $\tau \in T$, let $P_\tau \in B(K)$ be the orthogonal projection given by
\[ P_\tau f := \sum_{n=0}^{\infty} \frac{1}{\tau_{n+1} - \tau_n} \int_{\tau_n}^{\tau_{n+1}} f(t) \, dt \, 1_{[\tau_n, \tau_{n+1}]} \quad \forall f \in K. \tag{8} \]

**Lemma 2.1.** The projection $P_\tau$ converges strongly to $I_K$ as $|\tau| \to 0$.

**Proof.** If $f \in K$ is continuous and compactly supported, a uniform-continuity argument may be used to show that $P_\tau f \to f$ uniformly; the density of such functions in $K$ completes the proof. \( \square \)

**Theorem 2.1.** As $|\tau| \to 0$, the projection $Q_\tau$ converges strongly to $I_F$.

**Proof.** By (6) and (8), if $f, g \in K$ then (compare [18, (2.10)])
\[ \langle \varepsilon(f), Q_\tau \varepsilon(g) \rangle = \prod_{n=0}^{\infty} \left( 1 + \langle f_\tau(n), g_\tau(n) \rangle \right) \]
\[ = \exp \left( \sum_{n=0}^{\infty} \log \left( 1 + \int_{\tau_n}^{\tau_{n+1}} \langle f(t), P_\tau g(t) \rangle \, dt \right) \right) \]
\[ \sim \exp \left( \sum_{n=0}^{\infty} \int_{\tau_n}^{\tau_{n+1}} \langle f(t), P_\tau g(t) \rangle \, dt \right) \]
\[ \to \exp \left( \int_{0}^{\infty} \langle f(t), g(t) \rangle \, dt \right) \quad \text{as } |\tau| \to 0, \tag{9} \]
by Lemma 2.1, so $Q_\tau \to I_F$ weakly on $\mathcal{E}$, the linear span of the set of exponential vectors; the asymptotic identity (9) holds because $\log(1 + z) = z + O(z^2)$ as $z \to 0$. Since each $Q_\tau$ is an orthogonal projection, strong convergence on $\mathcal{E}$, so on $F$, follows. \( \square \)

### 3 Modified QS integrals

*Natura abhorret vacuum.*

— François Rabelais, *Gargantua et Pantagruel*, Bk. 1, Ch. 5 (1534).

To examine the behaviour of the discrete approximations which will be constructed in the following sections, it is useful first to introduce a slight extension
of the iterated QS integral (QS being, of course, an abbreviation for quantum stochastic).

**Notation 3.1.** Let $h$ be a fixed complex Hilbert space (the initial space) and let $\tilde{\mathcal{F}} := h \otimes \mathcal{F}$, $\tilde{\Gamma} := h \otimes \Gamma$ and $\tilde{\mathcal{E}} := h \otimes \mathcal{E}$. As is customary, the tensor sign will be omitted before exponential vectors: $u \varepsilon(f) := u \otimes \varepsilon(f)$.

**Definition 3.1.** Given a Hilbert space $H$, an $H$-process $X = (X_t)_{t \in \mathbb{R}^+}$ is a weakly measurable family of linear operators with common domain $H \otimes \mathcal{E}$, i.e.,

$$X_t \in \mathcal{L}(H \otimes \mathcal{E}; H \otimes \mathcal{F}) \quad \forall t \in \mathbb{R}^+$$

and $t \mapsto \langle u \varepsilon(f), X_t v \varepsilon(g) \rangle$ is measurable for all $u, v \in H$ and $f, g \in K$.

An $H$-process $X$ is vacuum adapted if

$$\langle u \varepsilon(f), X_t v \varepsilon(g) \rangle = \langle u \varepsilon(1_{[0,t]}f), X_t v \varepsilon(1_{[0,t]}g) \rangle$$

for all $t \in \mathbb{R}^+$, $u, v \in H$ and $f, g \in K$. Equivalently, the identity $X_t = (I_H \otimes \mathbb{E}_t)X_t(I_H \otimes \mathbb{E}_t)$ holds for all $t \in \mathbb{R}^+$, where $\mathbb{E}_t \in \mathcal{B}(\mathcal{F})$ is the second quantisation of the multiplication operator $f \mapsto 1_{[0,t]}f$ on $K$.

An $H$-process $X$ is semi-vacuum-adapted if $(I_H \otimes \mathbb{E}_t)X_t = X_t$ for all $t \in \mathbb{R}^+$; clearly every vacuum-adapted process is semi-vacuum-adapted.

If $M \in \mathcal{B}(K)$ then an $h \otimes k$-process $X$ is $M$-integrable if

$$\|X\hat{\nabla}^M \theta\|_{L^2([0,t]; h \otimes k \otimes \mathcal{F})}^2 = \int_0^t \|X_s\hat{\nabla}^M \theta\|^2 ds < \infty \quad \forall \theta \in \tilde{\mathcal{E}}, t \in \mathbb{R}^+, \tag{12}$$

where the modified gradient $\hat{\nabla}^M : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{F}} \oplus (h \otimes k \otimes \mathcal{F})$ is the linear operator such that

$$u \varepsilon(f) \mapsto [u \otimes \hat{M}f] \varepsilon(f) = \left(\frac{u \varepsilon(f)}{|u \otimes Mf| \varepsilon(f)}\right)$$

and the definition $\hat{\nabla}^M u \varepsilon(f) := [u \otimes \hat{M}f(s)] \varepsilon(f)$ is extended by linearity.

**Notation 3.2.** Let $\Delta \in \mathcal{B}(h \otimes k \otimes \mathcal{F})$ be the orthogonal projection onto $h \otimes k \otimes \mathcal{F}$ and $\Delta^\perp := I_{h \otimes k \otimes \mathcal{F}} - \Delta$ the projection onto its complement, $\tilde{\mathcal{F}}$.

**Theorem 3.1.** Let $M \in \mathcal{B}(K)$. If $X$ is an $M$-integrable, semi-vacuum-adapted $h \otimes k$-process, there exists a unique semi-vacuum-adapted $h$-process $\Lambda_\Omega(X; M)$, the modified QS integral of $X$, such that, for all $t \in \mathbb{R}^+$,

$$\|\Lambda_\Omega(X; M)_t \theta\| \leq c_t \|X\hat{\nabla}^M \theta\|_{L^2([0,t]; h \otimes k \otimes \mathcal{F})} \quad \forall \theta \in \tilde{\mathcal{E}}, \tag{14}$$

where $c_t := \sqrt{2 \max\{t,1\}}$, and

$$\langle u \varepsilon(f), \Lambda_\Omega(X; M)_t v \varepsilon(g) \rangle = \int_0^t \langle [u \otimes \hat{f}(s)] \varepsilon(f), [v \otimes \hat{M}g(s)] \varepsilon(g) \rangle ds \tag{15}$$

for all $u, v \in h$ and $f, g \in K$. 

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Proof. This follows from the behaviour of the Bochner integral and the abstract Itô integral: for all $t \in \mathbb{R}_+$ and $\theta \in \mathcal{E}$ let

$$
\Lambda_\Omega(X; M)_t \theta := \int_0^t \Delta X_s \hat{\nabla}^M \theta \, ds + \mathcal{I}_t(\Delta X \hat{\nabla}^M \theta),
$$

(16)

where $\mathcal{I}_t$ is the Itô integral on $[0, t]$ (the adjoint of the adapted gradient). As $s \mapsto \Delta X_s \hat{\nabla}^M \theta$ is an adapted vector process, i.e., $\Delta X_s \hat{\nabla}^M \theta \in \mathfrak{h} \otimes \mathfrak{k} \otimes \mathcal{F}_{[0, s]}$ for (almost) all $s \in \mathbb{R}_+$, this is a good definition, and the isometric nature of the Itô integral [4, Proposition 3.28] implies that

$$
\|\Lambda_\Omega(X; M)_t \theta\|^2 \leq 2t \int_0^t \|\Delta X_s \hat{\nabla}^M \theta\|^2 \, ds + 2 \int_0^t \|\Delta X_s \hat{\nabla}^M \theta\|^2 \, ds
$$

\[ \leq c^2_t \int_0^t \|X_s \hat{\nabla}^M \theta\|^2 \, ds, \tag{17} \]

as claimed. The identity (15) follows immediately and yields semi-vacuum-adaptedness. 

\[ \Box \]

Remark 3.1. As may be seen from (15), the modified integral preserves semi-vacuum-adaptedness but need not preserve vacuum-adaptedness. This identity also shows that if $A \in \mathcal{B}(\mathfrak{h})$ commutes with $X$, in the sense that

$$
X_t(A \otimes I_{\mathfrak{k} \otimes \mathcal{F}}) = (A \otimes I_{\mathfrak{k} \otimes \mathcal{F}})X_t \quad \forall t \in \mathbb{R}_+, \tag{18} \]

then $A$ commutes with $\Lambda_\Omega(X; M)_t$ in the same sense: $\Lambda_\Omega(X; M)_t(A \otimes I_{\mathcal{F}}) = (A \otimes I_{\mathcal{F}})\Lambda_\Omega(X; M)_t$ for all $t \in \mathbb{R}_+$.

Notation 3.3. For all $n \geq 1$ and $t \in \mathbb{R}_+$, let

$$
\Delta_n(t) := \{ t := (t_1, \ldots, t_n) \in [0, t]^n : t_1 < \cdots < t_n \} \subseteq \mathbb{R}^n_+ \tag{19} \]

and, given $M \in \mathcal{B}(\mathcal{K})$, define $(\hat{\nabla}^M)^n \in \mathcal{L}(\hat{\mathcal{E}}; L^2(\Delta_n(t); \mathfrak{h} \otimes \hat{\mathcal{K}}^\otimes_n \otimes \mathcal{F})$ such that

$$
(\hat{\nabla}^M)^n u \varepsilon(f) := ((\hat{\nabla}^M)^n u \varepsilon(f))(t) := [u \otimes \overline{f}^\otimes_n(t)] \varepsilon(f) \tag{20} \]

for all $u \in \mathfrak{h}$, $f \in \mathcal{K}$ and $t \in \Delta_n(t)$, where $\widehat{g}^\otimes_n(t) := g(t_1) \otimes \cdots \otimes g(t_n)$ for all $g \in \mathcal{K}$ and $t \in \mathbb{R}^n_+$.

Theorem 3.2. Let $M \in \mathcal{B}(\mathcal{K})$. If $n \geq 1$, $X \in \mathcal{B}(\mathfrak{h} \otimes \hat{\mathcal{K}}^\otimes_n)$ and $Y$ is a locally uniformly bounded, semi-vacuum-adapted $\mathbb{C}$-process then there exists a unique semi-vacuum-adapted $\mathfrak{h}$-process $\Lambda^n_{\Omega}(X \otimes Y; M)$, the modified $n$-fold QS integral, such that, for all $t \in \mathbb{R}_+$,

$$
\|\Lambda^n_{\Omega}(X \otimes Y; M)_t \theta\|^2 \leq c^2_{tn} \int_{\Delta_n(t)} \|(X \otimes Y)_t) \hat{\nabla}^M \theta\|^2 \, dt \tag{21} \]

\]
for all \( \theta \in \mathcal{E} \) and
\[
\langle u \varepsilon(f), \Lambda_\Omega^n(X \otimes Y; M)_t v \varepsilon(g) \rangle = \int_{\Delta_n(t)} \langle u \otimes \hat{f}^{\otimes n}(t), X[v \otimes \hat{M}g^{\otimes n}(t)] \rangle \langle \varepsilon(f), Y_t \varepsilon(g) \rangle \, dt
\]
for all \( u, v \in \mathfrak{h} \) and \( f, g \in K \).

**Proof.** If \( n = 1 \) then the result follows by applying Theorem [3.1] to the process \( X \otimes Y : t \mapsto X \otimes Y_t \). Now suppose the theorem holds for a particular \( n \geq 1 \), let \( X \in \mathcal{B}(\mathfrak{h} \otimes \hat{k}^{\otimes n+1}) \) and define \( X' := \hat{R}_{n+1}^* X \hat{R}_{n+1} \), where the unitary map \( \hat{R}_{n+1} : \mathfrak{h} \otimes \hat{k}^{\otimes n+1} \to \mathfrak{h} \otimes \hat{k}^{\otimes n+1} \) implements the permutation
\[
\begin{align*}
u \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} &\mapsto u \otimes x_2 \otimes \cdots \otimes x_{n+1} \otimes x_1.
\end{align*}
\]

By replacing \( \mathfrak{h} \) with \( \mathfrak{h} \otimes \hat{k} \), this assumption yields a semi-vacuum-adapted \( \mathfrak{h} \otimes \hat{k} \)-process \( \Lambda_\Omega^n(X' \otimes Y; M) \) such that, for all \( t \in \mathbb{R}_+ \), \( u, v \in \mathfrak{h} \), \( x, y \in \hat{k} \) and \( f, g \in K \),
\[
\langle (u \otimes x) \varepsilon(f), \Lambda_\Omega^n(X' \otimes Y; M)_t (v \otimes y) \varepsilon(g) \rangle = \int_{\Delta_n(t)} \langle u \otimes x \otimes \hat{f}^{\otimes n}(t), X'[v \otimes y \otimes \hat{M}g^{\otimes n}(t)] \rangle \langle \varepsilon(f), Y_t \varepsilon(g) \rangle \, dt
\]
\[
= \int_{\Delta_n(t)} \langle u \otimes \hat{f}^{\otimes n}(t) \otimes x, X[v \otimes \hat{M}g^{\otimes n}(t) \otimes y] \rangle \langle \varepsilon(f), Y_t \varepsilon(g) \rangle \, dt.
\]

Letting
\[
\Lambda_\Omega^{n+1}(X \otimes Y; M) := \Lambda_\Omega(\Lambda_\Omega^n(X' \otimes Y; M); M)
\]
gives the result: if \( t \in \mathbb{R}_+ \), \( u, v \in \mathfrak{h} \) and \( f, g \in K \) then
\[
\langle u \varepsilon(f), \Lambda_\Omega^{n+1}(X \otimes Y; M)_t v \varepsilon(g) \rangle = \int_0^t \langle [u \otimes \hat{f}(s)] \varepsilon(f), \Lambda_\Omega^n(X' \otimes Y; M)_s (v \otimes \hat{M}g(s)) \varepsilon(g) \rangle \, ds
\]
\[
= \int_0^t \int_{\Delta_n(s)} \langle u \otimes \hat{f}^{\otimes n+1}(t, s), X[v \otimes \hat{M}g^{\otimes n+1}(t, s)] \rangle \langle \varepsilon(f), Y_t \varepsilon(g) \rangle \, dt \, ds
\]
\[
= \int_{\Delta_{n+1}(t)} \langle u \otimes \hat{f}^{\otimes n+1}(t), X[v \otimes \hat{M}g^{\otimes n+1}(t)] \rangle \langle \varepsilon(f), Y_t \varepsilon(g) \rangle \, dt;
\]
the norm estimate (21) (and \( M \)-integrability of \( \Lambda_\Omega^n(X' \otimes Y; M) \)) may be shown similarly.

**Proposition 3.1.** If \( n \geq 1 \), \( X \in \mathcal{B}(\mathfrak{h} \otimes \hat{k}^{\otimes n}) \) and \( X \otimes E \) is the vacuum-adapted \( \mathfrak{h} \otimes \hat{k}^{\otimes n} \)-process given by setting \( (X \otimes E)_t := X \otimes E_t \) then
\[
\Lambda_\Omega^n(X) := \Lambda_\Omega^n(X \otimes E; I_K)
\]
is a vacuum-adapted, bounded process: each \( \Lambda_\Omega^n(X)_t \) extends uniquely to an element of \( \mathcal{B}(\hat{F}) \), the vacuum-adapted \( n \)-fold quantum Wiener integral of \( X \).

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Proof. Note first that if $Z$ is a locally uniformly bounded, vacuum-adapted $h \otimes k$-process and $\theta \in \hat{E}$ then

$$\|Z\tilde{\nabla}\theta\|_{L^2([0,t];(h \otimes k) \otimes \mathcal{F})}^2 \leq \|Z\|_{L^2}\int_0^t \|(I_{h \otimes k} \otimes \mathcal{E}_s)\tilde{\nabla}_s\theta\|^2 \, ds$$

$$= \|Z\|_{L^2}\int_0^t (\|\mathcal{E}_s\theta\|^2 + \|\mathcal{D}_s\theta\|^2) \, ds$$

$$\leq \|Z\|_{L^2}(t+1)\|\theta\|^2,$$  \quad (25)

where $\| \cdot \|_{\infty,t}$ is the essential-supremum norm on $[0,t]$, $\tilde{\nabla} := \tilde{\nabla}^{hk}$ and $\mathcal{D}$ is the adapted gradient on $\hat{F}$ \cite{1} Proposition 3.27. Hence $\Lambda_\Omega(Z;I_{k})_t$ extends to a unique element of $\mathcal{B}(\hat{F})$ for all $t \in \mathbb{R}_+$ and $\Lambda_\Omega(Z;I_{k})$ is a locally uniformly bounded, vacuum-adapted $h$-process. The result now follows from the inductive construction of $\Lambda_{\Omega_1}^n(X)$ given in the proof of Theorem 3.2. \qed

Proposition 3.2. Let $M, N \in \mathcal{B}(\mathcal{K})$. If $n \geq 1$, $X \in \mathcal{B}(h \otimes \hat{k}^\otimes n)$, $Y$ is a locally uniformly bounded, semi-vacuum-adapted $C$-process and $t \in \mathbb{R}_+$ then

$$\left\| \left( \Lambda_{\Omega}^n(X \otimes Y; M)_t - \Lambda_{\Omega}^n(X \otimes Y; N)_t \right) u \varepsilon(f) \right\|^2$$

$$\leq 2^{n-1}c_2n \|Y\|_{\infty,t}^2\|\varepsilon(f)\|^2 \sum_{m=1}^n L_m,$$  \quad (26)

for all $u \in h$ and $f \in \mathcal{K}$, where $\| \cdot \|_{\infty,t}$ is the essential-supremum norm on the interval $[0,t]$,

$$L_m := \int_{\Delta_n(t)} \|X(u \otimes \hat{M}^{\otimes m-1}(t_m)) \otimes [(M-N)f](t_m) \otimes \hat{N}^{\otimes n-m}(t_m))\|^2 \, dt$$

$$t_m := (t_1, \ldots, t_{m-1}) \text{ and } t_m := (t_{m+1}, \ldots, t_n).$$  \quad (27)

Proof. Note that, with notation as in the proof of Theorem 3.2

$$\Lambda_{\Omega}^{n+1}(X \otimes Y; M) - \Lambda_{\Omega}^{n+1}(X \otimes Y; N)$$

$$= \Lambda_{\Omega}(\Lambda_{\Omega}^n(X' \otimes Y; M); M) - \Lambda_{\Omega}(\Lambda_{\Omega}^n(X' \otimes Y; M); N)$$

$$+ \Lambda_{\Omega}(\Lambda_{\Omega}^n(X' \otimes Y; M) - \Lambda_{\Omega}^n(X' \otimes Y; N); N).$$

Now use induction, together with \cite{13}, \cite{21} and the fact that

$$\left\| \left( \Lambda_{\Omega}(Z; M)_t - \Lambda_{\Omega}(Z; N)_t \right) u \varepsilon(f) \right\|^2$$

$$\leq c_2^2 \int_0^t \|Z_s([u \otimes [(M-N)f](s)]\varepsilon(f))\|^2 \, ds,$$  \quad (28)

by \cite{10}. \qed
**Definition 3.2.** If \( A \) is an ordered set and \( n \geq 1 \) then \( A^{n,\uparrow} \) is the collection of strictly increasing \( n \)-tuples of elements of \( A \). Given \( \tau \in T \), let
\[
[\tau_p, \tau_{p+1}] := \{ t \in \mathbb{R}_+^n : \tau_p \leq t_i < \tau_{p+1} \ (i = 1, \ldots, n) \}
\tag{29}
\]
for all \( p = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n \) and, for all \( t \in \mathbb{R}_+ \), let
\[
\Delta^n_\tau(t) := \bigcup_{p \in \{0, \ldots, m-1\}^n} [\tau_p, \tau_{p+1}] \quad \text{if } t \in [\tau_m, \tau_{m+1}].
\tag{30}
\]

**Theorem 3.3.** Let \( M \in \mathcal{B}(K) \). If \( n \geq 1 \), \( \tau \in T \), \( X \in \mathcal{B}(\mathfrak{h} \otimes \mathfrak{k}^\otimes n) \) and \( Y \) is a locally uniformly bounded, semi-vacuum-adapted \( \mathbb{C} \)-process then there exists a unique semi-vacuum-adapted \( h \)-process \( \Lambda^n_\Omega(X \otimes Y; M)_T \), the modified \( n \)-fold QS integral subordinate to \( \tau \), such that, for all \( t \in \mathbb{R}_+ \),
\[
\| \Lambda^n_\Omega(X \otimes Y; M)_T \| \theta^2 \leq c' \int_{\Delta^n_\tau(t)} \| (X \otimes Y_t)(\overline{\nabla} M)_t \| \theta^2 \, dt \tag{31}
\]
for all \( \theta \in \overline{E} \) and
\[
\langle u \varepsilon(f), \Lambda^n_\Omega(X \otimes Y; M)_T v \varepsilon(g) \rangle
= \int_{\Delta^n_\tau(t)} \langle u \otimes \tilde{f}^\otimes n(t), X[v \otimes \overline{M}^\otimes n(t)] \rangle \langle \varepsilon(f), Y_t \varepsilon(g) \rangle \, dt \quad \tag{32}
\]
for all \( u, v \in h \) and \( f, g \in K \).

**Proof.** When \( n = 1 \), apply Theorem 3.1 to the process \( X \otimes Y : t \mapsto X \otimes Y_t \) and let
\[
\Lambda^1_\Omega(X \otimes Y; M)_T := \bigoplus_{m=0}^{\infty} 4_{t \in [\tau_m, \tau_{m+1}]} \Lambda_\Omega(X \otimes Y; M)_{\tau_m}. \tag{33}
\]

Now suppose the theorem holds for a particular \( n \geq 1 \), let \( X \in \mathcal{B}(\mathfrak{h} \otimes \mathfrak{k}^\otimes n+1) \) and define \( X' := \widehat{R}_n \otimes X \widehat{R}_{n+1} \) as in the proof of Theorem 3.2. The semi-vacuum-adapted \( h \otimes k \)-process \( \Lambda^n_\Omega(X' \otimes Y; M)_T \) is such that, for all \( t \in \mathbb{R}_+ \), \( u, v \in h \), \( x, y \in k \) and \( f, g \in K \),
\[
\langle (u \otimes x) \varepsilon(f), \Lambda^n_\Omega(X' \otimes Y; M)_T (v \otimes y) \varepsilon(g) \rangle
= \int_{\Delta^n_\tau(t)} \langle u \otimes \tilde{f}^\otimes n(t) \otimes x, X[v \otimes \overline{M}^\otimes n(t) \otimes y] \rangle \langle \varepsilon(f), Y_t \varepsilon(g) \rangle \, dt,
\]
so, as
\[
\{0, \ldots, m-1\}^{n+1,\uparrow} = \bigcup_{k=0}^{m-1} \{(p_1, \ldots, p_n, k) : p \in \{0, \ldots, k-1\}^n \}. \tag{33}
\]
and therefore \( \Delta_{n+1}^\tau(\tau_m) = \bigcup_{k=0}^{m-1} (\Delta_n^\tau(\tau_k) \times \lbrack \tau_k, \tau_{k+1} \rbrack) \), letting

\[
\Lambda_n^{m+1}(X \otimes Y; M)^\tau := \sum_{m=0}^{\infty} \mathbb{1}_{t \in \lbrack \tau_m, \tau_{m+1} \rbrack} \Lambda\Omega(\Lambda_n^\tau(X' \otimes Y; M)^\tau; M)_{\tau_m}
\]

(34) gives the result. \( \square \)

**Proposition 3.3.** Let \( M \in \mathcal{B}(K) \). If \( n \geq 1, \tau \in T, X \in \mathcal{B}(h \otimes \hat{k}^\otimes n) \) and \( Y \) is a locally uniformly bounded, semi-vacuum-adapted \( \mathbb{C} \)-process then

\[
\| (\Lambda_n^\tau(X \otimes Y; M)_t - \Lambda_n^\tau(X \otimes Y; M)_t^\tau) \theta \|^2 \\
\leq 2^{n-1} \epsilon_f^2 \int_{\Delta_n(t) \setminus \Delta_n^\tau(t)} \| (X \otimes Y_\tau)(\nabla^M)^\tau_t \theta \|^2 \, dt
\]

(35) for all \( t \in \mathbb{R}_+ \) and \( \theta \in \tilde{E} \).

**Proof.** This follows by induction and the fact that if \( t \in \lbrack \tau_m, \tau_{m+1} \rbrack \) then the set

\[
\{(t, s) : s \in [0, \tau_m], t \in \Delta_n(s) \setminus \Delta_n^\tau(s) \} \cup \{(t, s) : s \in [\tau_m, t], t \in \Delta_n(s) \}
\]

is contained in \( \Delta_{n+1}(t) \setminus \Delta_{n+1}^\tau(t) \). \( \square \)

4 The toy integral

I see salvation in discrete individuals
– Anton Chekhov, Letter to I.I. Orlov (22nd February, 1899).

**Definition 4.1.** For all \( n \in \mathbb{Z}_+ \) let \( \lambda_n : \mathcal{B}(h \otimes \hat{k}) \to \mathcal{B}(\tilde{\Gamma}) \) be the normal \( * \)-homomorphism such that \( B \otimes C \mapsto B \otimes I_{\Gamma_n} \otimes C \otimes P^\omega_{\Gamma_{n+1}} \), where

\[
P^\omega_{\Gamma_{n+1}} : \Gamma_{n+1} \to \Gamma_{n+1} ; \quad \bigotimes_{m=n+1}^{\infty} x_n \mapsto \bigotimes_{m=n+1}^{\infty} \langle \omega(m), x_n \rangle \omega(m)
\]

(36) is the orthogonal projection onto the one-dimensional subspace of \( \Gamma_{n+1} \) spanned by the vector \( \otimes_{m=n+1}^{\infty} \omega(m) \).

**Notation 4.1.** For all \( \tau \in T \), let

\[
D_\tau := I_h \otimes J_\tau^* \Pi_\tau : \bar{\mathcal{F}} \to \bar{\Gamma}; \quad u(\varepsilon(f)) \mapsto u \otimes \bigotimes_{n=0}^{\infty} f_\tau(n)
\]

(37) and note that, as \( |\tau| \to 0 \),

\[
D_\tau^* D_\tau = I_h \otimes \Pi_\tau^* J_\tau^* \Pi_\tau = I_h \otimes Q_\tau \to I_{\bar{\mathcal{F}}}
\]

(38) in the strong operator topology and

\[
D_\tau D_\tau^* = I_h \otimes J_\tau^* \Pi_\tau \Pi_\tau^* J_\tau = I_{\bar{\mathcal{F}}},
\]

(39) since \( \Pi_\tau \) is an isometric isomorphism and \( J_\tau \) an isometry.
the right-hand side of (42) becomes

\[ \text{Remark 4.1.} \quad \text{Let } X \in \mathcal{B}(h \otimes \hat{k}) \text{ and } t \in \mathbb{R}_+ \text{ be fixed. For all } \tau \in T, \text{ let } n = n(\tau) \in \mathbb{Z}_+ \text{ be such that } t \in [\tau_n, \tau_{n+1}[ \text{ and note that} \]

\[ \langle u \varepsilon(f), D_T^*\tilde{\mathbb{S}}_n(X)D_{\tau}v\varepsilon(g) \rangle = \prod_{m=0}^{n-1} \left(1 + \langle f_\tau(m), g_\tau(m)\rangle\right)\langle u \otimes \tilde{f}_\tau(n), X[v \otimes \tilde{g}_\tau(n)]\rangle \quad (40) \]

for all \( u, v \in h \) and \( f, g \in K \). As \( |\tau| \to 0, \tau_n \not\to t \) and

\[ \prod_{m=0}^{n-1} \left(1 + \langle f_\tau(m), g_\tau(m)\rangle\right) = \langle J^*_\tau \Pi_\tau \varepsilon(f), J^*_\tau \Pi_\tau \varepsilon(1_{[0,\tau_n]}(g)) \rangle = \langle Q_\tau \varepsilon(f), E_{\tau_n} \varepsilon(g) \rangle \to \langle \varepsilon(f), E_\varepsilon(g) \rangle. \quad (41) \]

To analyse the second term in the right-hand side of (40), let \( X = (E F, G H) \), where \( E \in \mathcal{B}(h) \), \( F \in \mathcal{B}(h \otimes k; h) \), \( G \in \mathcal{B}(h_0 \otimes k) \) and \( H \in \mathcal{B}(h \otimes k) \). Then

\[ \langle u \otimes \tilde{f}_\tau(n), X[v \otimes \tilde{g}_\tau(n)]\rangle = \langle u, Ev \rangle \langle u, F[v \otimes \tilde{g}_\tau(n)]\rangle \]

\[ + \langle u \otimes \tilde{f}_\tau(n), H[v \otimes \tilde{g}_\tau(n)]\rangle; \quad (42) \]

this equation shows the necessity of scaling the components of \( X \) in order to obtain non-trivial limits. Replacing \( X \) by \( X_{\tau,n} \), where

\[ \left( \begin{array}{c} E \\ F \\ \hline G \\ H \end{array} \right)_{\tau,n} := \left( \begin{array}{c} (\tau_{n+1} - \tau_n)E \\ (\tau_{n+1} - \tau_n)^{1/2} F \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{cc} (\tau_{n+1} - \tau_n)^{1/2} G & \langle E F \rangle \\ 0 & 1 \end{array} \right) \quad (43) \]

the right-hand side of (42) becomes

\[ \int_{\tau_n}^{\tau_{n+1}} \left( \langle u, Ev \rangle \right) + \langle u, F[v \otimes g(t)]\rangle + \langle u \otimes f(t), Gv \rangle + \langle u \otimes f(t), H[v \otimes P_\tau g(t)]\rangle \right) dt \]

\[ = \int_{\tau_n}^{\tau_{n+1}} \left( \langle u \otimes \tilde{f}(t), X[v \otimes \tilde{g}(t)]\rangle + \langle u \otimes f(t), H[v \otimes (P_\tau g - g)(t)]\rangle \right) dt. \quad (45) \]

\[ \text{Theorem 4.1.} \quad \text{For all } X \in \mathcal{B}(h \otimes \hat{k}) \text{ and } t \in \mathbb{R}_+, \]

\[ \Sigma_{\tau}(X)_t := \sum_{n=0}^{\infty} 1_{\tau_{n+1} \in [0,t]} D_T^* \tilde{\mathbb{S}}_n(X_{\tau,n})D_{\tau} \to \Lambda_\Omega(X)_t \quad \text{as } |\tau| \to 0 \quad (46) \]

strongly on \( \tilde{\mathcal{E}} \).
Proof. It follows from \[45\] that $\Sigma_t(X)_t$ can be written as the sum of a semi-vacuum-adapted QS integral and an Itô-integral remainder term. Let

$$E^\tau_t := \sum_{n=0}^{\infty} \mathbb{1}_{t \in [\tau_n, \tau_{n+1}]} \mathbb{E}_{\tau_n} \quad \forall t \in \mathbb{R}_+$$ \hspace{1cm} (47)

and note that $t \mapsto X \otimes Q^\tau E^\tau_t$ is vacuum-adapted. If $t \in [\tau_m, \tau_{m+1}]$ then

$$\langle u \varepsilon(f), \Sigma_t(X)_t \varepsilon(g) \rangle = \sum_{n=0}^{m-1} \langle Q^\tau \varepsilon(f), \mathbb{E}_{\tau_n} \varepsilon(g) \rangle \langle u \otimes f^\tau(n), X_{\tau,n}[v \otimes g^\tau(n)] \rangle$$

$$= \int_0^{\tau_m} \langle [u \otimes f^\tau(s)] \varepsilon(f), (X \otimes Q^\tau E^\tau_t)([v \otimes g^\tau(s)] \varepsilon(g)) \rangle \, ds$$

$$+ \int_0^{\tau_m} \langle [u \otimes f^\tau(s)] \varepsilon(f), H[v \otimes (P_t g - g)(s)] \otimes Q^\tau E^\tau_t \varepsilon(g) \rangle \, ds. \hspace{1cm} (48)$$

If $I_s$ denotes the abstract Itô integral on $[0, s]$ then this shows that

$$\langle \Sigma_t(X)_t - \Lambda \Omega(1_{[0, \tau_m]} X \otimes Q^\tau E^\tau_t; I_K)_t \varepsilon(g) \rangle = I_{\tau_m} (H[v \otimes (P_t g - g)(\cdot)] \otimes Q^\tau E^\tau_t \varepsilon(g)); \hspace{1cm} (49)$$

as the Itô integral is an isometry, the norm of this quantity is bounded above by

$$\|H\| \|v\| \|P_t g - g\|_{L^2([0, t]; \mathbb{E}_s)} \varepsilon(g) \| \rightarrow 0 \quad \text{as } |\tau| \rightarrow 0.$$

Finally, since $Q^\tau E^\tau_s \rightarrow \mathbb{E}_s$ strongly as $|\tau| \rightarrow 0$ for all $s \in \mathbb{R}_+$, Theorem 3.1 and the dominated-convergence theorem imply that

$$\Lambda \Omega(X)_t - \Lambda \Omega(1_{[0, \tau_m]} X \otimes Q^\tau E^\tau_t; I_K)_t = \Lambda \Omega(X \otimes (E \otimes 1_{[0, \tau_m]} Q^\tau E^\tau_t); I_K)_t \hspace{1cm} (50)$$

tends to 0 strongly on $\tilde{\mathbb{E}}$ as $|\tau| \rightarrow 0$, as required. \hfill \Box

5 Multiple integrals

O, thou hast damnable iteration
– William Shakespeare, Henry IV, Part 1, Act I, Scene ii (1596).

Remark 5.1. For all $X \in B(h \otimes k^\otimes 2)$, $t \in \mathbb{R}_+$ and $\tau \in T$, let

$$\Sigma^\tau_t(X)_t := \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \mathbb{1}_{[\tau_{m+1}, \tau]} \mathbb{E}_{\tau_{n+1}}(X_{\tau,m,n} \otimes D^\tau_t) \hspace{1cm} (51)$$

where $\mathbb{E}_{\tau_{m,n}} : B(h \otimes k^\otimes 2) \rightarrow B(\tilde{T})$ is the normal $*$-homomorphism such that

$$B \otimes C_1 \otimes C_2 \mapsto B \otimes I_{\tau_{m,n}} \otimes C_1 \otimes P_{\tau_{m+1},n}^\tau \otimes C_2 \otimes P_{\tau_{n+1}}, \hspace{1cm} (52)$$
with \( P^\omega_{[m+1,n]} \) and \( P^\omega_{[m+1]} \) the orthogonal projections onto \( \otimes_{k=m+1}^{n-1} \mathbb{C} \omega(k) \) and \( \otimes_{k=n+1}^{\infty} \mathbb{C} \omega(k) \), respectively.

To find the correct scaling for \( X_{\tau,m,n} \), note that if

\[
\Psi[\tau]_n := \begin{pmatrix} (\tau_{n+1} - \tau_n)^{1/2} & 0 \\ 0 & I_n \end{pmatrix} \in \mathcal{B}(\widehat{k}) \quad \forall n \in \mathbb{Z}_+
\tag{53}
\]

and \( Y \in \mathcal{B}(h \otimes \widehat{k}) \) then \( Y_{\tau,n} = (I_h \otimes \Psi[\tau]_n)Y(I_h \otimes \Psi[\tau]_n) \), so let

\[
X_{\tau,m,n} := (I_h \otimes \Psi[\tau]_m \otimes \Psi[\tau]_n)X(I_h \otimes \Psi[\tau]_m \otimes \Psi[\tau]_n)
\tag{54}
\]

for all \((m, n) \in \mathbb{Z}^2_+ \). Having examined the case of multiplicity two, the general case is now clear.

**Definition 5.1.** For all \( n \geq 1, X \in \mathcal{B}(h \otimes \widehat{k}^{\otimes n}) \), \( t \in \mathbb{R}_+ \) and \( \tau \in \mathbb{T} \), let

\[
\Sigma_n^\tau(X)_t := \sum_{p \in \mathbb{Z}^n_+} \mathbb{I}_{T_{\tau,n+1} \in [0,t]}D^*_t \tilde{\sigma}_p(X_{\tau,p})D^*_t,
\tag{55}
\]

where \( \tilde{\sigma}_p : \mathcal{B}(h \otimes \widehat{k}^{\otimes n}) \to \mathcal{B}(\widehat{\mathcal{B}}) \) is the normal *-homomorphism such that

\[
B \otimes C_1 \otimes \cdots \otimes C_n \mapsto B \otimes I_{\tau_{p_1}} \otimes C_1 \otimes P^\omega_{[p_1+1,p_2]} \otimes \cdots \otimes C_n \otimes P^\omega_{[p_n+1]},
\tag{56}
\]

in which \( C_m \) acts on \( \widehat{k}_{(p_m)} \) for \( m = 1, \ldots, n \) and \( P^\omega : x \mapsto \langle x, \omega \rangle \omega \) acts on \( \widehat{k}_{(q)} \) for all \( q \geq p_1 \) such that \( q \notin \{p_1, \ldots, p_n\} \), and

\[
X_{\tau,p} := (I_h \otimes \Psi[\tau]_{p_1} \otimes \cdots \otimes \Psi[\tau]_{p_n})X(I_h \otimes \Psi[\tau]_{p_1} \otimes \cdots \otimes \Psi[\tau]_{p_n}).
\tag{57}
\]

This is the discrete analogue of the vacuum-adapted \( n \)-fold quantum Wiener integral of \( X \).

**Theorem 5.1.** If \( n \geq 1, X \in \mathcal{B}(h \otimes \widehat{k}^{\otimes n}) \) and \( t \in \mathbb{R}_+ \) then

\[
\Sigma_n^\tau(X)_t = \Lambda_n^\tau(X \otimes Q_{\tau}E^\tau; P_{\tau})^t \to \Lambda_n^\tau(X)_t
\tag{58}
\]

strongly on \( \widehat{\mathcal{E}} \) as \( |\tau| \to 0 \).

**Proof.** If \( p \in \mathbb{Z}^n_+ \) then, with the obvious extension of notation,

\[
\langle u \in (f), D^*_t \tilde{\sigma}_p(X_{\tau,p})D^*_t v \in (g) \rangle = \prod_{m=0}^{p_1-1} (f_{\tau}(m), g_{\tau}(m)) \langle u \otimes \bigotimes_{k \in p} f_{\tau}(k), X_{\tau,p} \otimes \bigotimes_{k \in p} g_{\tau}(k) \rangle
\tag{59}
\]

for all \( u, v \in h \) and \( f, g \in K \). Furthermore, as

\[
\Psi[\tau]_k f_{\tau}(k) = \frac{1}{\sqrt{\tau_{k+1} - \tau_k}} \int_{\tau_k}^{\tau_{k+1}} f(t) \, dt,
\tag{60}
\]

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it follows that
\[
\langle u \otimes \bigotimes_{k \in p} f_r(k), X_{\tau \cdot p} [v \otimes \bigotimes_{k \in p} g_r(k)] \rangle \\
= \int_{[\tau \cdot p, \tau \cdot p + 1]} \langle u \otimes \hat{f}^{\otimes n}(t), X[v \otimes \hat{g}^{\otimes n}(t)] \rangle \, dt, \tag{61}
\]
which gives the identity. That the limit is as claimed may be established by writing the difference \(\Sigma^n_\tau(X) - \Lambda^n_\Omega(X)\) as
\[
\Lambda^n_\Omega(X \otimes Q_{\tau E^*}; P_{\tau}) - \Lambda^n_\Omega(X \otimes Q_{\tau E^*}; P_{\tau}) \\
+ \Lambda^n_\Omega(X \otimes (Q_{\tau E^*} - E); P_{\tau}) + \Lambda^n_\Omega(X \otimes E; P_{\tau}) - \Lambda^n_\Omega(X \otimes E; I_K)
\]
and employing Proposition 3.3, Theorem 3.2 and Proposition 3.2.

6 Product formulae

\textit{Entia non sunt multiplicanda praeter necessitatem.}

– William of Ockham.

**Remark 6.1.** Given \(p = (p_1, \ldots, p_m) \in \mathbb{Z}^n_+\) and \(q = (q_1, \ldots, q_n) \in \mathbb{Z}^n_+\) with \(p_m < q_1\), let
\[
p \cup q := (p_1, \ldots, p_m, q_1, \ldots, q_n) \in \mathbb{Z}^{m+n}_+.
\tag{62}
\]
If the normal \(\ast\)-homomorphism
\[
\tilde{s}_{n,m} : B(h \otimes \hat{k}^\otimes n \otimes \hat{k}^\otimes m) \to B(h \otimes \hat{k}^\otimes m \otimes \hat{k}^\otimes n)
\]
is determined by the transposition \(B \otimes C \otimes D \to B \otimes D \otimes C\) then, letting
\[
Y \triangleright X := [\tilde{s}_{n,m}(Y \otimes I_{\hat{k}^\otimes m})](X \otimes (P^\omega)^\otimes n)
\tag{63}
\]
and
\[
X \triangleleft Y := (X \otimes (P^\omega)^\otimes n)[\tilde{s}_{n,m}(Y \otimes I_{\hat{k}^\otimes m})],
\tag{64}
\]
it is readily verified that
\[
\tilde{s}_q(Y) \tilde{s}_p(X) = \tilde{s}_{p \cup q}(Y \triangleright X) \quad \text{and} \quad \tilde{s}_p(X) \tilde{s}_q(Y) = \tilde{s}_{p \cup q}(X \triangleleft Y)
\tag{65}
\]
for all \(X \in B(h \otimes \hat{k}^\otimes m)\) and \(Y \in B(h \otimes \hat{k}^\otimes n)\). Furthermore, for any \(\tau \in T\),
\[
(Y \triangleright X)_{\tau \cdot p \cup q} = Y_{\tau \cdot q} \triangleright X_{\tau \cdot p} \quad \text{and} \quad (X \triangleleft Y)_{\tau \cdot p \cup q} = X_{\tau \cdot p} \triangleleft Y_{\tau \cdot q},
\tag{66}
\]
since \(\Psi[\tau]_\rho C \Psi[\tau]_\rho P^\omega = \Psi[\tau]_\rho CP^\omega \Psi[\tau]_\rho\) for all \(p \in \mathbb{Z}^n_+\) and \(C \in B(\hat{k})\). The following Proposition is an immediate consequence of these observations.

**Proposition 6.1.** (Fubini) If \(X \in B(h \otimes \hat{k}^\otimes m)\) and \(Y \in B(h \otimes \hat{k}^\otimes n)\) then
\[
\Sigma_{\tau}^{m+n}(Y \triangleright X)_{t} = \sum_{q \in \mathbb{Z}^{n}_+} \mathbbm{1}_{\tau \cdot q \in \mathbb{Z}^{n}_+} D^*_\tau \tilde{s}_q(Y_{\tau \cdot q})D^*_\tau \Sigma_{\tau}^m(X)_{\tau \cdot q},
\tag{67}
\]
and
\[
\Sigma_{\tau}^{m+n}(X \triangleleft Y)_{t} = \sum_{q \in \mathbb{Z}^{n}_+} \mathbbm{1}_{\tau \cdot q \in \mathbb{Z}^{n}_+} \Sigma_{\tau}^m(X)_{\tau \cdot q} D^*_\tau \tilde{s}_q(Y_{\tau \cdot q})D^*_\tau
\tag{68}
\]
for all \(\tau \in T\) and \(t \in \mathbb{R}^n_+\).
Theorem 6.1. (Quantum Itô product formula) If \( X, Y \in \mathcal{B}(h \otimes \hat{k}) \) then
\[
\Lambda^1_\Omega(Y)\Lambda^1_\Omega(X) = \Lambda^2_\Omega(Y \rhd X) + \Lambda^2_\Omega(Y \lhd X) + \Lambda^1_\Omega(Y \Delta X),
\tag{69}
\]
where \( \Delta \in \mathcal{B}(h \otimes \hat{k}) \) denotes the orthogonal projection onto \( h \otimes k \).

Proof. Note first that if \( \alpha_{\tau,n} := (\tau_{n+1} - \tau_n)^{1/2} \) for all \( \tau \in T \) and \( n \in \mathbb{Z}_+ \) then
\[
(I_h \otimes \Psi[\tau])^2 = I_h \otimes \Psi[\tau]^2 = \Delta + \alpha_{\tau,n}^2 \Delta^\perp,
\tag{70}
\]
whence
\[
Y_{\tau,n}X_{\tau,n} = (X\Delta Y)_{\tau,n} + \alpha_{\tau,n}^2(X\Delta^\perp Y)_{\tau,n}.
\tag{71}
\]
This working, the fact that \( \tilde{s}_m \) is a homomorphism for all \( m \in \mathbb{Z}_+ \) and the identities (67–68) with \( m = n = 1 \) imply that
\[
\Sigma_\tau(Y)_{\tau} \Sigma_\tau(X)_{\tau} = \Sigma_\tau^2(Y \rhd X)_{\tau} + \Sigma_\tau^2(Y \lhd X)_{\tau} + \Sigma_\tau(Y \Delta X)_{\tau} + Z^*_\tau
\tag{72}
\]
for all \( \tau \in T \) and \( t \in \mathbb{R}_+ \), where
\[
Z^*_\tau := \sum_{m=0}^{\infty} \mathbb{1}_{[\tau_{m+1}, \tau_m]} \alpha_{\tau,m}^2 D^*_\tau \tilde{s}_m ((Y \Delta^\perp X)_{\tau,m}) D_\tau.
\tag{73}
\]
Working as in the proof of Theorem 4.1 (compare (49)) shows that
\[
Z^*_\tau u(f) = \Lambda_\Omega(Y \Delta^\perp X \otimes W^\tau; I_K)_{\tau} u(f)
\tag{74}
\]
for all \( t \in [\tau_n, \tau_{n+1}] \), \( u \in h \) and \( f \in K \), where \( V := \Delta Y \Delta^\perp X \Delta \) and
\[
W^*_\tau := Q_\tau \sum_{n=0}^{\infty} \mathbb{1}_{[\tau_n, \tau_{n+1}]} \alpha_{\tau,n}^2 E_{\tau_n} \forall t \in \mathbb{R}_+.
\tag{75}
\]
Now \( W^*_\tau \to 0 \) in norm as \( |\tau| \to 0 \), since \( \|W^*_\tau\| \leq \sup_{n \geq 1} \alpha_{\tau,n}^2 \) for all \( t \in \mathbb{R}_+ \), so \( Z_{\tau,n} \to 0 \) strongly on \( \tilde{E} \), by Theorem 5.1 and Itô isometry. Combining this with Theorem 5.1 it follows that
\[
\Sigma_\tau(Y)_{\tau} \Sigma_\tau(X)_{\tau} \to \Lambda^2_\Omega(Y \rhd X)_{\tau} + \Lambda^2_\Omega(Y \lhd X)_{\tau} + \Lambda^1_\Omega(Y \Delta X)_{\tau}
\tag{76}
\]
strongly on \( \tilde{E} \) as \( |\tau| \to 0 \) and this gives the result.

Remark 6.2. The quantum Itô formula (69) may be compared to that valid for the usual form of adaptedness [14, Exercise after Proposition 3.20].
7 Further development

Unbounded hopes were placed on each successive extension
– George Bernard Shaw, Socialism: Principles and Outlook, Shavian Tract No. 4, The Illusions of Socialism and Socialism: Principles and Outlook (1956).

This section contains little analysis, but sets out the basic situation once one moves beyond bounded integrands.

Definition 7.1. An admissible triple \((h_0, k_0, S)\) is a dense subspace \(h_0 \subseteq h\), a dense subspace \(k_0 \subseteq k\) and a subset \(S \subseteq K\) such that
\begin{enumerate}
  \item each \(f \in S\) has compact support,
  \item \(f(t) \in k_0\) for all \(t \in \mathbb{R}_+\) and \(f \in S\)
\end{enumerate}
and (iii) \(\mathcal{E}_S := \text{lin}\{\varepsilon(f) : f \in S\}\) is dense in \(\mathcal{F}\).

Definition 7.2. If \(X \in \mathcal{L}(h_0 \odot \hat{k}_0; h \odot \hat{k})\), where \(h_0\) is a subspace of \(h\), \(k_0\) is a subspace of \(k\) and \(\hat{k}_0 := \mathbb{C} \odot k_0 \subseteq \hat{k}\), then
\[\tilde{s}_n(X) := U_n^*(X \odot I_{\Gamma_n}) \odot P_n^* U_n \in \mathcal{L}(h_0 \odot \bigoplus_{m=0}^{\infty} k_0; \tilde{\Gamma}) \tag{77}\]
for all \(n \in \mathbb{Z}_+\), where the unitary operator \(U_n : \tilde{\Gamma} \rightarrow \tilde{\Gamma}\) is such that
\[u \odot \bigotimes_{m=0}^{\infty} x_m \mapsto u \odot \bigotimes_{m=0}^{n-1} x_m \odot \bigotimes_{m=n+1}^{\infty} x_m\] \tag{78}
and
\[\bigotimes_{m=0}^{\infty} k_0 := \text{lin}\left\{ \bigotimes_{m=0}^{\infty} x_m \left| x_m \in k_0 \forall m \geq 0, \exists l \in \mathbb{Z}_+ : x_l = x_{l+1} = \cdots = \omega \right\}. \tag{79}\]

Proposition 7.1. Let \((h_0, k_0, S)\) be admissible. If \(X \in \mathcal{L}(h_0 \odot \hat{k}_0; h \odot \hat{k})\) and \(t \in \mathbb{R}_+\) are such that \(f_0^t \|X[u \odot \hat{f}(s)]\|^2 ds < \infty\) for all \(u \in h_0\) and \(f \in S\) then
\[\Sigma_\tau(X)_t := \sum_{n=0}^{\infty} \delta_{\tau_n+1 \in [0, t]} D_\tau \tilde{s}_n(X_{\tau_n}) D_\tau \rightarrow \Lambda_\Omega(X)_t \tag{80}\]
weakly on \(\tilde{E}_S := h_0 \odot \mathcal{E}_S\) as \(|\tau| \rightarrow 0\), where \(\Lambda_\Omega(X) := \Lambda_\Omega(X \odot \mathbb{E}; I_K)\).

Proof. Note that
\[\langle u \varepsilon(f), D_\tau \tilde{s}_n(X) D_\tau v \varepsilon(g) \rangle = \prod_{m=0}^{n-1} \langle f_\tau(m), g_\tau(m) \rangle \int_{\tau_n}^{\tau_{n+1}} \langle u \odot \check{P}_\tau f(s), X[v \odot \hat{g}(s)] \rangle ds \tag{81}\]
for all $n \in \mathbb{Z}_+$, $u, v \in h$ and $f, g \in S$, so if $t \in [\tau_n, \tau_{n+1}]$ then, as $|\tau| \to 0$,
\[
\langle u \varepsilon(f), \Sigma_t^\tau X(v \varepsilon(g)) \rangle = \int_0^{\tau_n} \langle u \otimes \hat{P}_t f(s), X[v \otimes g(s)] \rangle \langle \varepsilon(f), Q_t \varepsilon_\tau g(s) \rangle ds \\
\rightarrow \int_0^t \langle u \otimes f(s), X[v \otimes g(s)] \rangle \langle \varepsilon(f), \varepsilon(1_{[0,s]} g(s)) \rangle ds.
\]

Remark 7.1. Similarly, if $X \in L(h_0 \otimes \hat{\kappa}^n_0; h \otimes \hat{\kappa}^n_0)$ and $t \in \mathbb{R}_+$ are such that
\[
\int_{\Delta_n(t)} \|X[u \otimes \hat{f}^n(t)]\|^2 dt < \infty \quad \forall u \in h_0, f \in S
\]
then $\Sigma^n_t(X) \to \Lambda^n_t(X)$ weakly on $\tilde{E}_S$ as $|\tau| \to 0$; the proper definitions of $\Sigma^n_t(X)$ and $\Lambda^n_t(X)$ should be clear from the above.

Bibliography

[1] L. ACCARDI and A. BACH, Central limits of squeezing operators, in: Quantum probability and applications IV (Rome, 1987), L. Accardi and W. von Waldenfels (eds.), Lecture Notes in Mathematics 1396, Springer, Berlin, 1989, 7–19.

[2] S. ATTAL, Approximating the Fock space with the toy Fock space, Séminaire de Probabilités XXXVI, J. Azéma, M. Émery, M. Ledoux and M. Yor (eds.), Lecture Notes in Mathematics 1801, Springer, Berlin, 2003, 477–491.

[3] S. ATTAL and Y. PAUTRAT, From repeated to continuous quantum interactions, Ann. Henri Poincaré 7 (2006), 59–104.

[4] A.C.R. BELTON, Some self-adjoint quantum semimartingales, Proc. London Math. Soc. (3) 92 (2006), 791–816.

[5] A.C.R. BELTON, Random-walk approximation to vacuum cocycles, arXiv:math.OA/0702700, version 2, 2007.

[6] L. BOUTEN, R. VAN HANDEL and M.R. JAMES, A discrete invitation to quantum filtering and feedback control, arXiv:math.PR/0606118, version 4, 2006.

[7] T.A. BRUN, A simple model of quantum trajectories, Amer. J. Phys. 70 (2002), 719–737.

[8] U. FRANZ and A. SKALSKI, Approximation of quantum Lévy processes by quantum random walks, arXiv:math.FA/0703339, version 1, 2007.

[9] J. GOUGH, Holevo-ordering and the continuous-time limit for open Floquet dynamics, Lett. Math. Phys. 67 (2004), 207–221.
[10] J. Gough and A. Sobolev, *Stochastic Schrödinger equations as limit of discrete filtering*, Open Sys. Inf. Dyn. 11 (2004), 235–255.

[11] R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras. Volume II: Advanced theory*, Graduate Studies in Mathematics 16, American Mathematical Society, Providence, Rhode Island, 1997.

[12] B. Kümmerer, *Quantum Markov processes and applications in physics*, in: Quantum independent increment processes II, M. Schürmann and U. Franz (eds.), Lecture Notes in Mathematics 1866, Springer, Berlin, 2006, 259–330.

[13] M. Leitz-Martini, *Quantum stochastic calculus using infinitesimals*, doctoral thesis, Eberhard Karls Universität Tübingen, 2001. [http://w210.ub.uni-tuebingen.de/dbt/volltexte/2002/458/](http://w210.ub.uni-tuebingen.de/dbt/volltexte/2002/458/)

[14] J.M. Lindsay, *Quantum stochastic analysis — an introduction*, in: Quantum independent increment processes I, M. Schürmann and U. Franz (eds.), Lecture Notes in Mathematics 1865, Springer, Berlin, 2005, 181–271.

[15] J.M. Lindsay and K.R. Parthasarathy, *The passage from random walk to diffusion in quantum probability. II*, Sankhyā Ser. A 50 (1988), 151–170.

[16] P.-A. Meyer, *Éléments de probabilités quantiques. I–V*, in: Séminaire de Probabilités XX, J. Azéma and M. Yor (eds.), Lecture Notes in Mathematics 1204, Springer, Berlin, 1986, 186–312.

[17] P.-A. Meyer, *Éléments de probabilités quantiques. X. Approximation de l’oscillateur harmonique (d’après L. Accardi et A. Bach)*, in: Séminaire de Probabilités XXIII, J. Azéma, P.-A. Meyer and M. Yor (eds.), Lecture Notes in Mathematics 1372, Springer, Berlin, 1989, 175–182.

[18] K.R. Parthasarathy, *The passage from random walk to diffusion in quantum probability*, J. Appl. Probab. 25A (1988), 151–166.

[19] Y. Pautrat, *From Pauli matrices to quantum Itô formula*, Math. Phys. Anal. Geom. 8 (2005), 121–155.

[20] L. Sahu, *Quantum random walks and their convergence*, [arXiv:math.0505438](http://arxiv.org/abs/math.0505438) version 1, 2005.

[21] K.B. Sinha, *Quantum random walk revisited*, in: Quantum Probability, M. Bożejko, W. Młotkowski and J. Wysoczański (eds.), Banach Center Publications 73, Polish Academy of Sciences, Warsaw, 2006, 377–390.