Quark Propagator and Chiral Symmetry with String Tension

Ramesh Anishetty* and Santosh Kumar Kudtarkar†
The Institute of Mathematical Sciences, Chennai 600 113, India
February 26, 2019

Abstract

General properties of the light and heavy quark propagators have been investigated in the context of string tension interaction. Confinement, chiral symmetry breaking, spectral properties of the propagator are analytically studied and numerically validated. We show that the propagator is analytic in the infrared region even for massless quarks with a non zero radius of convergence. Emergence of more than one mass scale is exemplified. Massless limit of the quark propagator does exhibit critical behaviour.

Keywords: QCD, Quark masses, String Tension, Schwinger-Dyson Equation, Chiral Symmetry.

*ramesha@imsc.res.in
†sant@imsc.res.in
1 Introduction

Confinement of quarks and their transient behaviour as dictated by the quark propagator is being investigated in a theory [1] which shows many features of QCD. We have considered colored quarks in QCD with renormalisable string tension in the large $N$ expansion. We compute the behaviour of the quark propagator from first principles in this quantum field theory as a function of momentum, renormalised mass and coupling constant.

The presence of string tension results in the absence of the pole in the quark propagator thus manifesting the non-existence of quarks as asymptotic states in the quantum field theory. In addition for any renormalised mass parameter $m \geq 0$, we find that the analytic structure of the propagator is non-trivial. Firstly for space like momenta ($p^2 < 0$) the quark propagator is analytic. For any renormalised mass there exists a Taylor series solution in $p^2$ to the Schwinger-Dyson(SD) equation. This series is numerically estimated to converge for $p^2 < \tilde{m}^2 \neq 0$ where $\tilde{m}$ is called the threshold mass. In all, associated with a quark, there are three mass parameters namely, the renormalised mass $m$, the threshold mass $\tilde{m}$ and the asymptotic mass $M$ which capture the behaviour of the quark propagator for space like region. These three masses are very distinct and are estimated from phenomenology. We also discuss how the so called constituent mass in various bound states is related to $\tilde{m}$. We find for heavy quarks where the renormalised mass scale $m^2$ is larger than string tension $\sigma$, there is a qualitative difference in the analytic properties which can be physically interpreted as, in space-time light quarks have zero transient time existence while heavy quarks are almost about to be free and hence have relatively long time transient existence.

Massless quark theory does exhibit spontaneous chiral symmetry breaking in the presence of string tension and the theory is self-consistently defined with out the need for ultraviolet(UV) renormalisation unlike the massive case. Consequently the quark propagator function is not a continuous function of $m$ at $m = 0$. The spin independent quark propagator, due to Goldstone phenomenon, is also the massless pion wave function with the asymptotic mass of the quark in spacelike region being the inverse of the effective size of the pion. The SD equations also admit a chiral symmetric solution with confinement, this is understood to be an unstable vacuum. We also show that photon like behaviour of the gluon cannot cause chiral symmetry breaking for any value of the strong coupling constant $g$ in the rainbow graph approximation.

Much of this analysis is done analytically by exploiting certain computational simplifications that we state in the paper. This simplification procedure can be applied for any relativistic field theory. Numerical investigation was essential to establish the existence of solutions in the space like region and thus estimate the asymptotic masses of the quarks as a function of the parameters of the theory. A recent review on SD equations is given in [2].

2 Lagrangian with String Tension

The Minkowski Lagrangian of the theory is given by [1]
\[ Z = \int DADQDCD\chi D\chi \exp(iS_0 + i \int j(Q + A)) , \]
\[ S_0 = \int \left( \frac{-1}{4g^2} F^2 + C_\mu (-D^2) Q^\mu + \chi^- \chi^\mu - \frac{\sigma}{2} C_\mu^2 \right) + \int (\overline{Q} i \gamma_{\mu} D^\mu (A + Q) q - m\overline{Q}) \]  

where \( F = F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc} A_\mu^a A_\mu^b \) is the antisymmetric gauge field tensor, \( A_\mu^a \) are the gauge fields, \( g \) is the gauge coupling constant, \( f^{abc} \) are the structure constants of the corresponding non-abelian gauge group. \( Q_\mu \) and \( C_\mu \) are bosonic vector fields and \( \chi \) are Grassmann Lorentz vector fields. All of them are in adjoint representation of the gauge group and transform covariantly under local gauge transformations. The significance of these fields is discussed in [1]. \( D_\mu^{ab} = D_\mu (A) = \partial_\mu \delta^{ab} + f^{acb} A_\mu^c \) is the covariant derivative operator. \( q \) and \( \overline{q} \) are quark fields(fermions) in the fundamental representation of the gauge group with a mass \( m \). As can be seen from the Lagrangian the quark fields have a \( \overline{Q} Aq \) and a \( \overline{Q}Qq \) interaction vertices. \( \sigma \) is the string tension which is asymptotically free [1]. Gauge fixing for the gluons has to be done just as in QCD and in the Feynman gauge the \( A_\mu \) and the \( Q_\mu \) propagators are given by

\[ < Q_\mu^a Q_\nu^b > = i\sigma \frac{\eta_{\mu\nu}\delta^{ab}}{(q^2 + i\epsilon)^2} \quad \text{and} \quad < A_\mu^a A_\nu^b > = \frac{-i\eta_{\mu\nu}\delta^{ab}}{q^2 + i\epsilon} \]  

We consider a systematic expansion in the various coupling constants \( \sigma N, g^2 N \) and \( 1/N \), where \( \sigma \) is the string tension, \( g \) is the gauge coupling and \( N \) is the number of colours, of which the leading infrared(IR) contributions come from the \( \sigma N \) term alone. Consequently we perform a non-perturbative expansion in \( \sigma N \) but perturbative expansion in \( g^2 N \) and \( 1/N \) [3]. This yields for the quark propagator a summation of the rainbow graphs with the \( < Q_\mu^a Q_\nu^b > \) propagator alone. The SD equation can be considered with only this term alone and solved self-consistently. It turns out this equation has no UV infinite renormalisation. But we know that including the \( g^2 N \) term requires UV renormalisation. This raises the question whether non-perturbative IR physics is truly independent of the UV renormalisation procedure. To address this question in our analysis we include in the SD equation all the leading contributions in the various asymptotic IR and UV regions. The IR region is dominated by the \( < Q_\mu^a Q_\nu^b > \) propagator whereas the UV region is dominated by the \( < A_\mu^a A_\nu^b > \) propagator which gives the third term in the SD equation. It should be mentioned that to order \( g^2 N \) in addition to the above term there are some more graphs but these are less important in the UV region and are not expected to make a significant contribution, for simplicity we neglect them. In this work we do not use running \( \sigma, g \) or \( m \) in our calculations. The SD equation for the quark propagator with both the \( < Q_\mu^a Q_\nu^b > \) and \( < A_\mu^a A_\nu^b > \) propagator (Fig[1]) is,

\[ S^{-1}(p) = \frac{1}{i} (\gamma.p - m) + \int \frac{d^4 k}{(2\pi)^4} \left( \frac{\sigma N}{(k^2 + i\epsilon)^2} \gamma_\mu S(p - k) \gamma^\mu - \frac{g^2 N}{(k^2 + i\epsilon)} \gamma_\mu S(p - k) \gamma^\mu \right) \]  

where \( S(p) \) is the full quark propagator with momentum \( p_\mu \) and mass \( m \).
Figure 1: Schwinger Dyson Equation for Quark propagator

The dashed lines in the figure corresponds to $< Q^a_\mu Q^b_\nu >$ propagator and the springy line to the gluon. The solid line with the blob is the full quark propagator and the solid line without the blob is the bare quark propagator. Spin decomposing the quark propagator $S(p)$ as $S(p) = i(A(p^2)\gamma.p + B(p^2))$ and substituting it in (3), we can express the SD equation as a coupled nonlinear equation in terms of two scalar functions $A(p^2)$ and $B(p^2)$.

3 Causal Representation

The solution space of SD nonlinear equations in general can be very large to address. Relativistic Quantum Field Theory does impose certain qualifications and hence only those alone are of interest to us. Generally these restrictions are encoded in the Kallen-Lehmann representation. The physics consideration that go into this are that positive energies propagate forward in time and negative energies backward in time (as imposed by the Feynman $i\epsilon$ prescription) and all intermediate states have a positive norm. While these are essential for any physical propagators, it remains questionable whether unobservables should also have these properties. In particular the $\frac{1}{p^2}$ propagators necessarily generate negative norm intermediate states in perturbation theory [4]. Therefore we disregard the second assumption and make the minimal assumptions necessary namely Lorentz covariance and Feynman $i\epsilon$ prescription yielding the causal representation.

$$A(p^2) = \int_{-\infty}^{\infty} d\alpha \frac{\tilde{A}(\alpha)}{p^2 - \alpha + i\epsilon}$$  \hspace{1cm} (4)

$$B(p^2) = \int_{-\infty}^{\infty} d\alpha \frac{\tilde{B}(\alpha)}{p^2 - \alpha + i\epsilon}$$  \hspace{1cm} (5)

where $\tilde{A}(\alpha)$ and $\tilde{B}(\alpha)$ are functions that are not required to be positive. Infact the above are the Cauchy-Riemann integral representations for analytic functions. All we are demanding is that the quark propagator that we are interested in should be an analytic function of momentum. From the analyticity properties one can infer that the imaginary parts of $A(p^2)$ and $B(p^2)$ are the causal weight functions $\tilde{A}(\alpha)$ and $\tilde{B}(\alpha)$ themselves. In eq.(4) and eq.(5) out of generality we consider the entire range of $\alpha$. However as it turns out in the succeeding analysis $\tilde{A}(\alpha)$ and $\tilde{B}(\alpha)$ are non-vanishing only for $\alpha > 0$. 

4
4 Regularisations

The \(< Q_\mu^a Q_\nu^b >\) propagator behaviour $1/(k^2 + i\epsilon)^2$ is highly singular in the IR regime. In 3+1 dimensions the $+i\epsilon$ prescription is not sufficient to regulate it. Even after Wick rotation it has a logarithmic singularity which makes this propagator ill-defined. Consequently the theory demands a regularisation of this IR divergence. There are many equivalent ways of regularising this but we will mention the one which is convenient to implement in our relativistic quantum field theory. In all loop calculations we are convoluting this singular propagator with another quantity which has an integral representation which in its generic form is $\frac{1}{(k-p)^2 - \alpha + i\epsilon}$ and higher powers thereof. We illustrate the IR regularisation prescription by explicitly showing the steps in carrying out the $B(p^2)$ integration. Consider the following integral that occurs in the SD equation,

$$\int \frac{d^4k}{i(2\pi)^4} \frac{B((p-k)^2)}{k^4}$$

(6)

In eq.(6), we substitute the spectral representation eq.(5) for $B((p-k)^2)$ and using Feynman parametrisation, we do the 1-loop momentum integral explicitly by making the standard Wick rotation.

$$\int_{-\infty}^{\infty} d\alpha \int \frac{d^4k}{i(2\pi)^4} \frac{1}{(k^2 + i\epsilon)^2 (k-p)^2 - \alpha + i\epsilon} \tilde{B}(\alpha) = \int_{-\infty}^{\infty} d\alpha \int_0^1 \frac{dx}{(4\pi)^2} \frac{x \tilde{B}(\alpha)}{(xp^2 - \alpha + i\epsilon)}$$

(7)

$$= \frac{1}{(4\pi)^2} \int_0^1 \frac{dx}{(1-x)} x B(xp^2)$$

(8)

In the above expression we have done the $\alpha$ integration formally to get back $B(p^2)$. As is evident in eq.(8) the IR singularity manifests itself as the end-point singularity at $x = 1$ in the Feynman parameter $x$. The divergence is regulated by explicitly subtracting the $x = 1$ singularity in the integrand.

$$\int_0^1 \frac{dx}{(1-x)} x B(xp^2) \rightarrow \int_0^1 \frac{dx}{(1-x)} (xB(xp^2) - B(p^2))$$

(9)

$$= \int_0^1 dx \ln(1-x) \frac{d}{dx}(xB(xp^2))$$

This defines the regularisation which can be unambiguously implemented in the theory. The same procedure follows through for the integration involving $A(p^2)$.

The gluon term in the SD equation is UV divergent which needs to be regularised. The standard dimensional regularisation and subsequent multiplicative renormalisation of the quantum fields and parameters can be implemented. We find the same can also be obtained in a uniform way by implementing the following representation for the gluon propagator

$$\frac{1}{p^2 + i\epsilon} = -\int_0^\infty \frac{1}{(p^2 - \beta + i\epsilon)^2} d\beta$$

(10)
Using the causal representation we can do the loop momentum integration. The UV divergent terms are \( \int_0^\infty d\beta A(\beta) \) and \( \int_0^\infty d\beta B(\beta) \) which can be removed by renormalising the wave function and the quark mass parameter as in ordinary perturbation theory. The finite SD equation for renormalised mass \( m \) are,

\[
\frac{A(p^2)}{p^2A(p^2) - B(p^2)^2} = 1 + 2\bar{\sigma} \int_0^1 dx \frac{x^2 A(xp^2) - A(p^2)}{1-x} - \bar{\alpha} p^2 \int_0^1 dx A(xp^2)(x^2 - 1) \tag{11}
\]

\[
\frac{B(p^2)}{p^2A(p^2) - B(p^2)^2} = m + 4\bar{\sigma} \int_0^1 dx \frac{xB(xp^2) - B(p^2)}{1-x} - 4\bar{\alpha} p^2 \int_0^1 dx B(xp^2)(x - 1) \tag{12}
\]

where we have defined \( \bar{\sigma} = \frac{\sigma_N}{(4\pi)^2} \) and \( \bar{\alpha} = \frac{g^2 N}{(4\pi)^2} \). In the rest of the paper for convenience we will work in mass units of \( \bar{\sigma} = 1 \).

### 5 Confinement and Asymptotics

All states which exist for asymptotic times are physical and they manifest as poles in the corresponding Green’s functions. Physical quarks should be realised as poles in the quark propagators. Looking for poles in eq.\((11)\), if \( A(p^2) \) has a pole for some value of \( p^2 \), the left hand side of eq.\((11)\) vanishes and the right hand side using the last representation in eq.\((11)\), shows an edge singularity divergence. This contradiction in eq.\((11)\) implies absence of pole like solutions. Similar is the case for \( B(p^2) \). Finally if both \( A(p^2) \) and \( B(p^2) \) have poles at the same value of \( p^2 \) then by taking the ratio of eq.\((11)\) and eq.\((12)\) we note that again the equation cannot be matched. These observations make us conclude that \( A(p^2) \)and \( B(p^2) \) cannot have poles, furthermore we conclude by refining this observation that \( A(p^2) \) and \( B(p^2) \) cannot even have any divergent behaviour for any value of \( p^2 \).

To analyse the large space like \( p^2 \) \((p^2 \rightarrow -\infty)\) behaviour of \( A(p^2) \) and \( B(p^2) \) in eq.\((11)\) and eq.\((12)\), we rescale the integrand \( x \rightarrow \frac{x}{p^2} \) and obtain the approximate equation,

\[
\frac{1}{p^2A(p^2)} \sim 1 + \frac{1}{p^2} \int_0^{p^2} dx \frac{x^2 A(x) - A(p^2)}{(1 - \frac{x}{p^2})} - \frac{\bar{\alpha} p^2}{p^2} \int_0^{p^2} dx A(x)(\frac{x^2}{p^4} - 1) \tag{13}
\]

\[
\frac{B(p^2)}{p^2A(p^2)^2} \sim m + 4\frac{1}{p^2} \int_0^{p^2} dx \frac{x B(x) - B(p^2)}{(1 - \frac{x}{p^2})} - 4\frac{\bar{\alpha} p^2}{p^2} \int_0^{p^2} dx B(x)(\frac{x}{p^2} - 1) \tag{14}
\]

We further split the integral range to \((0, 1)\) and \((1, p^2)\). It is easy to see that the latter dominates and we can determine the asymptotic behaviour of \( A(p^2) \) and \( B(p^2) \) self-consistently in the UV regime. It is interesting to note that the UV behaviour of \( A(p^2) \) and \( B(p^2) \) is not influenced by the IR behaviour at all as expected in perturbation theory.

For \( p^2 \rightarrow -\infty \)

\[
A(p^2) \sim \frac{1}{p^2(1, \frac{1}{\sqrt{2\bar{\alpha} \ln(-p^2)}})} \tag{15}
\]
\[ B(p^2) \sim \frac{1}{p^2}(m, \ln(-p^2)) \] (16)

In the above equations the first term in the bracket refers to the asymptotic behaviour in the absence of the \( \bar{\alpha} \) term while the second term refers to the asymptotic behaviour in the presence of the \( \bar{\alpha} \) term. It has to be noted that except in the chiral limit, upto logarithms the large momentum behaviour of \( A(p^2) \) and \( B(p^2) \) with and without the gluons are the same.

Next we look at the small momentum \( (p^2 \to 0) \) limit. From the integral equation the small \( p^2 \) behaviour of \( A(p^2) \) and \( B(p^2) \) are also self-consistently determined without any contribution from the UV region. In fact at \( p^2 = 0 \), \( A(0) \) and \( B(0) \) are determined to be

\[ B(0) = \frac{m \pm \sqrt{m^2 + 16}}{8} \] (17)

\[ A(0) = \frac{B(0)}{m - B(0)} \]

Near \( p^2 = 0 \) we can have series expansion solution \( A(p^2) = \sum_{n=0}^{\infty} a_n p^{2n} \) and \( B(p^2) = \sum_{n=0}^{\infty} b_n p^{2n} \) for eq.(11) and eq.(12). Substituting the series and equating the coefficients of like powers of \( p^{2n} \), we get the following recursion relations for the coefficients \( b_n \) and \( a_n \),

\[ a_n = \frac{b_n + 2 \sum_{m=0}^{n-1} (2a_m b_{n-m} f_{n-m} - \bar{\alpha} (2a_m b_{n-m-1} h_{n-m-1} - a_{n-m-1} b_m g_{n-m-1}))}{m - 4b_0 f_0 + 2b_0 f_{n+1}} - \frac{2 \sum_{m=1}^{n} a_{n-m} b_m f_{n-m+1}}{m - 4b_0 f_0 + 2b_0 f_{n+1}} \] (18)

\[ b_n = \frac{a_{n-1} - 2 \sum_{m=0}^{n-1} (a_m a_{n-m-1} f_{n-m} + 2 \bar{\alpha} b_m b_{n-m-1} h_{n-m-1}) + 2 \bar{\alpha} \sum_{m=0}^{n-2} a_m a_{n-m-2} g_{n-m-2}}{m - 4b_0 f_0 - 4b_0 f_n} + \frac{4 \sum_{m=1}^{n-1} b_m b_{n-m} f_{n-m}}{m - 4b_0 f_0 - 4b_0 f_n} \] (19)

where,

\[ f_i = \sum_{t=0}^{i} \frac{1}{t + 1}; h_i = \frac{1}{(i + 1)(i + 2)}; g_i = \frac{1}{(i + 1)(i + 3)} \]

The radius of convergence of the series solution is difficult to ascertain analytically. Investigating it numerically, we find that for small masses \( m \), the series converges in a small interval around \( p^2 = 0 \). As we increase the mass \( m \), the interval of convergence also increases. We attribute the finite radius of convergence to the onset of the imaginary part in the SD equation at some value of the \( p^2 > 0 \) and this value is proportional to the mass \( m \). The
onset of this behaviour is perhaps due to a branch cut and consequently no Taylor expansion can converge. Numerical solution of the integral equations eq. (11) and eq. (12) support this assumption. The onset of non-analyticity in \( A(p^2) \) and \( B(p^2) \) in general can be different but we find that for small quark mass \( m \) they are about the same. We cannot ascertain precisely the nature of non-analyticity but it can be
\[
A(p^2) \sim \text{const} + \frac{1}{\ln(m^2 - p^2)} \quad \text{or even softer.}
\]
This follows because if we assume that \( A(p^2) \) or \( B(p^2) \) is singular at some value of \( p^2 \), the r.h.s of eq. (11) and eq. (12) will yield even more singular contribution for the same value of \( p^2 \) thus mismatching the equation. Similar is the case for \( B(p^2) \). The Taylor expansion suggests that for small quark masses \( m \), for both \( A(p^2) \) and \( B(p^2) \) functions the non-analyticity threshold is at \( \tilde{m}^2 \) and is numerically estimated to be
\[
\begin{align*}
\tilde{m}^2 &\sim 0.02 + m \times \text{const} \quad \text{for} \quad m \ll 1 \quad (20) \\
\tilde{m}^2 &\sim m^2 \quad \text{for} \quad m > 1
\end{align*}
\]
For large quark masses we estimate \( \tilde{m}^2 \) by doing the naive large \( m \) or equivalently the small \( \sigma \) and \( \alpha \) perturbation theory. In this perturbation theory where \( \tilde{\sigma} \) is also small, it is easy to recognise that \( A(p^2) \) and \( B(p^2) \) may not have strict poles but poles softened by logarithms such as \( \frac{\ln(p^2 - m^2)}{p^2 - m^2} \). Hence there is a net singularity softer than a pole. This is a remarkably different behaviour as compared to that of light quarks.

In eq. (14), if we pick \( B(0) \) negative solution, we are able to establish analytically that the imaginary part may exist only for \( p^2 > 0 \) and both \( A(p^2) \) and \( B(p^2) \) are negative for space like \( p^2 \). From these two observations it is self-consistent to infer that that the imaginary part of \( A(p^2) \) and \( B(p^2) \) are also positive. Consequently we can conclude that the quark propagator in our theory is consistent with the standard Kallen-Lehmann representation.

6 Numerical Solutions

The numerical solutions for \( A(p^2) \) and \( B(p^2) \) in the space-like \( (p^2 < 0) \) region can be obtained by iteration. Solutions exist with or without the gluon term. We ignore the gluon term for simplicity. The functions \( A(p^2) \) and \( B(p^2) \) are recast in terms of two new functions \( \tilde{A}(p^2) \) and \( \tilde{B}(p^2) \) through the following transformation
\[
\begin{align*}
A(p^2) &= \frac{\tilde{A}(p^2)}{p^2 A(p^2)^2 - B(p^2)^2} \\
B(p^2) &= \frac{\tilde{B}(p^2)}{p^2 A(p^2)^2 - B(p^2)^2}
\end{align*}
\]
This is substituted in eq. (11) and solved for \( \tilde{A}(p^2) \) and \( \tilde{B}(p^2) \) by giving a seed which is \( \tilde{A}(0) \) and \( \tilde{B}(0) \). Such an equation is numerically convenient to iterate. \( A(p^2) \) and \( B(p^2) \) can be obtained by solving eq. (21) in terms of \( \tilde{A}(p^2) \) and \( \tilde{B}(p^2) \). The nature of the integral equation (11) is such that to solve it at a particular value of momentum, one has to know the functions at all values of momentum below that particular value all the way upto \( p^2 = 0 \).
The momentum $p^2$ is discretised and the integration over $x$ is done numerically. We use an interpolation formula to fit the iterate and do the $x$ integration. For values of $m > 0.5$ the iteration method works very well. But for small values of the quark mass $m$, due to the onset of the imaginary part very close to $p^2 = 0$, one has to be very careful. The solutions for very light masses have been determined with about 5% inaccuracy.

The solution for light quark masses are shown in Fig[2a] and Fig[2b]. The string tension $\bar{\sigma}$ sets the scale for small $p^2$ and the dominant contribution in this region comes from this term. For $p^2 > 1$ the momentum $p^2$ sets the scale. The large momentum behaviour given by eq.(15) and eq.(16) sets in at around $p^2 \sim 3$. For small values of $m$, $A(p^2)$ has a minima. For masses $m > 0.82$ this minimum does not exist in the spacelike region.

![Figure 2: Numerical solutions for (a) $B(p^2)$ and (b) $A(p^2)$ for light masses.](image)

The solutions for heavy quark masses, are shown in Fig[3a] and Fig[3b]. Unlike in the case of light quarks, the mass $m$ dominates over the string tension and sets the scale for small value of momentum. For large momentum like in the light quark case, its the momentum that sets the scale.

It can be noted that all the solutions for $A(p^2)$ and $B(p^2)$ in the large momentum limit goes over smoothly to the asymptotic expressions in (15) and (16). Upto to logarithmic corrections, $A(p^2)$ and $B(p^2)$ have essentially the canonical $\frac{1}{p^2 - M^2}$ behaviour in the spacelike region for all masses $m$ where $M$ is called the asymptotic mass. A useful parametrisation for $A(p^2)$ and $B(p^2)$ is,

\[
A(p^2) \sim -\frac{A(0) M^2}{p^2 - M^2_a} \\
B(p^2) \sim -\frac{B(0) M^2}{p^2 - M^2_b}
\] (22)
Figure 3: Numerical solutions for (a) $B(p^2)$ and (b) $A(p^2)$ for Heavy masses.

The “asymptotic masses” $M^2_a$ and $M^2_b$ depend on the mass $m$. A convenient way of determining $M^2_a$ and $M^2_b$ is by defining them to be that value of $p^2$ where $A(p^2)$ and $B(p^2)$ are reduced by half from their values at $p^2 = 0$. Doing so we can infer the asymptotic masses $M_a$ and $M_b$ from the numerical data. In Fig[4] we have plotted $M_a$ and $M_b$ versus $m$ and for comparison we have also plotted $M_0 = \frac{B(0)}{A(0)}$ which is

$$M_0 = \frac{7m + \sqrt{m^2 + 16}}{8} \quad (23)$$

The asymptotic mass does increase almost linearly with $m$, all through from light to heavy quarks. There are two asymptotic masses for any quark, namely $M_a$, the spin-dependent asymptotic mass and $M_b$ the spin-independent asymptotic mass and the former is significantly heavier than the latter for light quarks.

We can extend the numerical ansatz to small $p^2 > 0$ as well. Beyond which as we understand from the Taylor expansion the numerical solutions cannot converge. At this point we have to work with the real and imaginary parts of $A(p^2)$ and $B(p^2)$ which is numerically cumbersome to handle. The asymptotic mass behaviour of the quark propagator given in eq.(22) is to be taken for spacelike $p^2$ only. For timelike $p^2 > 0$ this is qualitatively unacceptable due to confinement.

The above procedure of determining numerical solutions can be done for $B(0)$ positive also. We find that the numerical solution is not so stable. This is partly because in the the r.h.s of eq.(12), we can see that the integral contribution is of opposite sign to that of the constant term $m$. Therefore at some point $B(p^2)$ will vanish and change sign. The existence of this solution can be doubted numerically but it cannot be ascertained. In any case this
solution is not of physical interest as it disobeys Kallen-Lehmann positivity condition. From eq.(4) and eq.(5), knowing that $\tilde{A}(\alpha)$ and $\tilde{B}(\alpha)$ are non vanishing for $p^2 > \tilde{m}^2 > 0$, $A(p^2)$ and $B(p^2)$ will never vanish in the space like region if $\tilde{A}(\alpha)$ and $\tilde{B}(\alpha)$ are positive.

7 Chiral Limit

We look at the theory when the bare mass of the quark vanishes. Then the theory has a global continuous symmetry namely chiral symmetry. In our large $N$ and small $g^2N$ expansion, doing just the wave function renormalisation, we obtain the SD equations where we have eq.(11) and

$$\frac{B(p^2)}{p^2A(p^2)^2 - B(p^2)^2} = 4\bar{\sigma} \int_0^1 dx \frac{xB(xp^2) - B(p^2)}{(1 - x)} + 4\bar{\alpha} \int_0^1 dx \int_{-\infty}^{xp^2} d\beta B(\beta) \quad (24)$$

The last term has a constant perhaps finite or infinite. If it is infinite, then renormalisation is necessarily demanded. If it is finite renormalisation is an option. In either cases if we do any renormalisation, finite or infinite, it is evident that this implies we have formally included a bare mass term and then subsequently arranged the renormalised mass to vanish. This procedure cannot respect chiral symmetry. This is easy to see by noting that the massless Goldstone meson wave function satisfies the following Bethe-Salpeter equation [8],

$$\phi(p^2) = (p^2A(p^2)^2 - B(p^2)^2)(4\bar{\sigma} \int_0^1 dx \frac{x\phi(xp^2) - \phi(p^2)}{(1 - x)} + 4\bar{\alpha} \int_0^1 dx \int_{-\infty}^{xp^2} d\beta \phi(\beta)) \quad (25)$$

In a renormalisable theory, $\phi(p^2)$ does not undergo any other renormalisation. Consequently in eq.(25) the last term cannot be altered. It is clear that the solution to eq.(25)
is $\phi(p^2) = B(p^2)$ provided we do not make any finite or infinite renormalisation of eq. (24). 
$\phi(p^2) = B(p^2)$ is precisely the consequence of the Goldstone theorem. This property of SD 
and Bethe-Salpeter equations, first realised by Mandelstam [3], is true in theories with sponta-
neous symmetry breaking in the rainbow graph approximation. To reiterate, any mass 
renormalisation scheme if adopted such that the constant term in the last term in eq. (24) is 
effectively absent then chiral symmetry is explicitly broken [6, 7].

Alternatively we can investigate if eq. (24) as such has any solutions. The crucial property 
for the existence of the solution is its UV behaviour. Using exactly the same approximation 
as in eq. (13) and eq. (14) we find that behaviour of $A(p^2)$ for large spacelike momentum is 
unchanged from eq. (15), while $B(p^2)$ behaves as

$$B(p^2) \sim \frac{\text{const}}{(p^2)^2} \left(\frac{1}{p^2}, \frac{1}{\ln(-p^2)}\right)$$

(26)

The UV behaviour of $B(p^2)$ is indeed more convergent than the canonical behaviour. Con-
sequently the last term in eq. (24) is finite. Thus our system is self consistently determined 
with only wave function renormalisation and no mass renormalisation.

In the small $p^2$ regime not much of a difference occurs and indeed $B(0) = \pm \frac{1}{2}$ (neglecting 
the $\bar{\alpha}$ term) as expected from eq. (17). Asymptotic behaviour of $B(p^2)$ as in eq. (26) is 
determined up to a constant and this can have either sign. Now it is possible to interpolate 
both the solutions to their corresponding asymptotic behaviour without $B(p^2)$ vanishing. 
These solutions are chiral conjugate. This is because in eq. (24) there is an ambiguity in 
the sign of $B(p^2)$. Both $B(p^2)$ and $-B(p^2)$ are possible. Thus picking one solution breaks 
the chiral symmetry spontaneously. Note that $A(p^2)$ remains the same irrespective of the 
choice of $B(p^2)$, because by definition $A(p^2)$ is chirally symmetric. Both $A(p^2)$ and $B(p^2)$ 
have analytic expansions around $p^2 = 0$ as determined from the series solution, eq. (18) and 
eq. (19) with $\bar{\alpha} = 0$. The radius of convergence of this series is numerically inferred to be 
nonvanishing as shown in eq. (19) for $\bar{\alpha} = 0$.

Since the asymptotic behaviour of $B(p^2)$ is very different from the canonical, we should 
epect the asymptotic mass definition to be modified. Namely for space like $p^2$

$$B(p^2) \sim \frac{B(0)(M_b^2)^2}{(p^2 - M_b^2)^2} \left(\frac{-M_b^2}{(p^2 - M_b^2)}, \frac{1}{\ln(-p^2)}\right)$$

(27)

Numerically the asymptotic mass can be estimated and it is found that $M_b^2 \sim (0.7 - 0.9)$. 
The qualitatively different behaviour of $B(p^2)$ in the exact chiral limit cannot be obtained 
continuously by taking the renormalised mass $m \to 0$. Hence this theory exhibits critical 
behaviour at $m = 0$. This as we see is a consequence coming totally due to the absence of 
mass renormalisation in contrast to the massive case. The chiral limit exhibits spontaneous 
symmetry breaking but also shows that the spin independent propagation of a quark is non-
analytic as a function of the quark mass $m$ at $m = 0$. Inspite of this in the IR region as seen 
from the Taylor series we do not see any signature of the critical behaviour.

SD equations (11) and (24) also have chiral symmetric solutions wherein $B(p^2)$ vanishes 
for all $p^2$ and $A(p^2)$ is non-vanishing. We can show there exists a series solution for $A(p^2) =$
\[
\frac{1}{\sqrt{-p^2}} \sum_{n=0}^{\infty} a_n (-p^2)^n \]

for small \( p^2 \) and the asymptotic behaviour remains unchanged as in (14). This solution preserves chiral symmetry. Note that this is also a confining solution as there is no pole for the quark propagator. Indeed the propagator is finite at \( p^2 = 0 \) and has only a square root branch cut starting from the origin i.e., the threshold mass vanishes. SD equations can have many solutions. Each of them actually correspond to different possible stable or unstable vacua. The criterion to pick the minimum vacuum configuration cannot be inferred from the SD equations alone. This can be carried out by summing the vacuum graphs which in turn depend on the propagator functions. We have not attempted this analysis here. By our expectations either we have one solution to the SD equation in which case that will correspond to the vacuum or there are three (odd) solutions of which two are related by chiral symmetry and hence to be degenerate stable vacua and the third has to be unstable. Any choice of the stable vacua therefore will yield symmetry breakdown and the chirally symmetric choice is unstable.

We now address symmetry breakdown between two stable vacua in the absence of string tension. In eq.(24) we put \( \tilde{\sigma} = 0 \). This is a truncation of QCD in which we are keeping the rainbow graphs due to gluons alone. This is not a controlled approximation but it is interesting to know whether less singular interactions with sufficiently large coupling constant \( \tilde{\alpha} \) can cause spontaneous symmetry breakdown. The answer as we prove now is that it cannot cause symmetry breakdown. The asymptotic UV behaviour remains the same as before. In the absence of the \( \tilde{\sigma} \) term we do expect the quark propagator to obey the Kallen-Lehmann representation, consequently for space like \( p^2 \), \( A(p^2) \) and \( B(p^2) \) cannot have any other zeroes other than at asymptotic limit. Furthermore we also know if \( B(0) \) is positive(negative) and it remains positive(negative) in the entire space like regime, due to Kallen-Lehmann representation. The consequence in eq.(24) at \( p^2 = 0 \) is that the r.h.s is positive(negative) but the l.h.s is positive(negative) showing an inconsistency. Thus the only consistent alternative is \( B(0) = 0 \) and hence no chiral symmetry breakdown.

8 Discussion

We speculate some consequences of our semi-analytic analysis supported by numerical estimates for strong interaction phenomenology. In [8] by fitting the vector meson mass we estimate \( \tilde{m} \sim 0.5 \text{Gev} \). This threshold mass is not a physical mass accessible at asymptotic times, however in most dynamics of quarks it still should give similar effects in the transient existence. For example in stable bound states quarks would have to have energy less than \( \tilde{m} \). Consequently an electromagnetic probe will find the quark degrees of freedom to have a “constituent” mass depending upon the bound state but necessarily less than the threshold mass. Similarly a high energy quark will radiate and lose energy but it will cease to radiate when it reaches the energy of the threshold mass.

From [8] the light quark (\( u \) and \( d \)) renormalised mass is estimated from the pion mass to be about 6\( \text{Mev} \). This is consistent with other phenomenological considerations [9] and is substantially different from \( \tilde{m} \). In this theory string tension runs and it decreases at high energies. We define heavy quark mass in comparison to relevant string tension. If \( m^2 > \tilde{\sigma} \),
we find that all notions of masses $m$, $\tilde{m}$ and $M$ are the same. Furthermore heavy quarks in some sense are not severely effected by string tension. Apart from confinement, it is suggested that heavy quarks have a longer transient existence in contrast to light quarks.

String tension, naively can cause non-unitary behaviour in the theory. We find that with in this limited exercise of quark propagator and their bound states many of the consequences of unitarity still hold true although we do not have any formal proof. This issue should be understood more carefully. Infact we find numerically the solution of the SD equation which is consistent with the requirements of unitarity is unique and stable. While the other possible solutions are numerically unstable.

Chiral limit of the theory can be consistently defined only if there is no mass UV renormalisation. For this reason alone chiral limit is a critical point since for any other massive case the theory is defined only by mass renormalisation. Inspite of this the infrared behaviour of the quark propagator smoothly extrapolates from small mass to vanishing mass. Because of this PCAC(Partial Conservation of Axial Current) relations still hold. We find chiral symmetric solution of the SD equation and it is also confining due to the absence of a pole but it is expected to be an unstable vaccum.

The techniques of solving SD equations that we have enunciated here is fairly general. The massless particle exchange that is considered explicitly has some algebraic simplifications. Essentially eq.7 and the trick of eq.10 can be adopted for any generic relativistic field theory to get equations of the type 11 and 12.

All our considerations above are equally applicable to technicolour models 10. Indeed the existence of many mass scales $m$, $\tilde{m}$, and $M$ is the scenario suggested in the literature, here we have a concrete compatible model where it is realised.

References

[1] Ramesh Anishetty, *Perturbative QCD with String Tension*, hep-ph/9804204.
[2] R. Alkofer and L. Smekal, Phys.Rept. 353(2001) 281; C. D. Roberts and A. G. Williams, Prog.Part.Nucl.Phys. 33 (1994)477.
[3] G. ’t Hooft, Nucl.Phys. B72(1974)461.
[4] G. ’t Hooft and M. J. Veltman, *Diagrammar*, Particle Interactions At Very High Energies, NATO Adv. Study Inst. Series, Sect.B, vol.4B, 177; A.Pais and G. E. Uhlenbeck, Phys. Rev. 79,(1950)145.
[5] S. Mandelstam, Phys. Rev. D20(1979) 3223.
[6] H. Pagels, Phys. Rev. D19(1979) 3080.
[7] K. Lane, Phys. Rev. D10(1974) 2605.
[8] R.Anishetty and Santosh.K.Kudtarkar, *Mesons: Relativistic Bound states with String Tension*, hep-ph/0305...
[9] H. Leutwyler, Phys. Lett. B 378 (1996) 313.

[10] E. Farhi and L. Susskind, Phys. Rept. 74 (1981) 277; R. K. Kaul, Rev. Mod. Phys., 55 (1983) 449.