Kaluza-Klein Cosmology: the bulk metric

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The Cosmological Principle is applied to a five-dimensional vacuum manifold. The general (non-trivial) solution is explicitly given. The result is a unique metric, parametrized with the sign of the space curvature ($k = 0, \pm 1$) and the signature of the fifth coordinate. Friedmann-Robertson-Walker (FRW) metrics can be obtained from this single 'mother' metric (M-metric), by projecting onto different space-homogeneous four-dimensional hypersurfaces. The expansion factor $R$ is used as time coordinate in order to get full control of the equation of state of the resulting projection. The embedding of a generic (equilibrium) mixture of matter, radiation and cosmological constant is given, modulo a quadrature, although some signature-dependent restrictions must be accounted for. In the 4+1 case, where the extra coordinate is spacelike, the condition ensuring that the projected hypersurface is of Lorentzian type is explicitly given. An example showing a smooth transition from an Euclidian to a Lorentzian 4D metric is provided. This dynamical signature change can be considered a classical counterpart of the Hartle-Hawking 'no-boundary' proposal, without resorting to the 'imaginary time' idea. The resulting FRW model shows an initial singularity at a finite value of the expansion factor $R$. It can be termed as a 'Big unfreeze', as it is produced just by the beginning of time, without affecting space geometry.

I. INTRODUCTION

Since the pioneering work of Kaluza-Klein, adding extra dimensions to the standard (4D) spacetime has shown to be a good strategy in the quest for unification. In modern particle physics, this has led to the brane-world models [1-3], where the basic physics scenario is a higher-dimensional 'bulk' spacetime, in which gravity acts, although ordinary matter and fields are supposed to be confined in 4-dimensional 'branes' (see ref. [4] for a purely classical description). This approach evokes Plato’s Cavern allegory, because the observed physical fields are seen just as projections (shadows) on the branes (the cavern walls), whereas the real objects are out of sight, living in some higher-dimensional reality (the bulk). The unifying power comes mainly from the fact that a single object can project different shadows in different walls. That is, different physics in the brane can be obtained just by considering different projections of a single bulk object.

In our case, this object is the 5D manifold geometry and the shadows depict the matter-energy content of the 4D spacetime. This has been termed as spacetime-matter (STM) approach (see [5] for a review). It has also been implemented in Cosmology, after the pioneering work of Ponce de Leon [6] (see [7] for a review). In some of these works, different projections of the same bulk metric are considered [8,9], so that different FRW metrics are obtained just by projecting onto different branes. A further step in that direction has been taken recently [10]: when embedding spatially flat FRW metrics in a bulk with 3+2 signature (timelike extra coordinate), a single bulk metric was obtained, termed as 'mother metric' or rather M-metric. The aim of this paper is to generalize this unifying result to all types of spatial curvature ($k = 0, \pm 1$) and bulk signature (either spacelike or timelike extra coordinate).

We will adopt a top-down approach, starting from the natural generalization of the Cosmological Principle to the five-dimensional bulk manifold. We will assume both isotropy and homogeneity with respect to three space coordinates (the ones that can be observed on the brane). Moreover, we will assume that the cosmological scale factor $R$ is a valid time coordinate. In the 3+2 case, this just amounts to a suitable coordinate choice in the time plane. In the 4+1 case, this amounts to require that the constant $R$ hypersurfaces be space-like. The cosmological expansion will affect then to all physical (timelike) observers.

With this only hypothesis, we write down in section 2 the general form of the 5D line element and the corresponding field equations in the evolution formalism. The general solution is obtained in section 3; it happens to be a single metric, which we call 'M-metric' because it is the generalization of the one previously obtained [10] in the 3+2 case for $k = 0$. Our result actually applies to all curvature cases ($k = 0, \pm 1$) and for both spacelike and timelike extra dimensions. We show that this metric has a Killing vector, whose integral lines are orthogonal to the $R$ coordinate (time) lines. This means that the coordinates in which we explicitly write down the M-metric have a clear geometrical and physical meaning.

We study in section 4 different brane projections of the M-metric, leading to different FRW models. Apart from the more conventional models (open universes starting from a Big Bang), we can generate in this way some other (more exotic) cases. In the 3+2 case, we can get regular universe models, where the Big Bang lies in the infinite past, or even emergent models, starting from a quasi-stationary state [11,12]. In the 4+1 case, we can generate smooth signature transitions from Euclidean to Lorentzian metrics in the brane [13,14], which can be considered as the classical counterpart of the Hartel-Hawking 'no boundary' proposal [15], without recourse to the imaginary time mechanism.
Finally, in section 5, we point out that using the expansion factor as a time coordinate provides a direct control of the matter-energy content in the brane, allowing to impose suitable equations of state. We study the case of an equilibrium combination of matter, radiation and cosmological constant. The explicit expression of the bulk metric embedding is provided, modulo a quadrature. In the process, we identify some signature restrictions: in the 3+2 case, one can get this generic mixture only in the closed universe case ($k = +1$). This confirms some previous results [10], in the sense that Campbell’s theorem [16] depends on some assumptions that can not be overlooked.

II. 5D COSMOLOGICAL FRAMEWORK: EVOLUTION FORMALISM

We will consider here 5-dimensional (5D) vacuum metrics, where the extra coordinate is labeled by $\psi$. In our case, where we assume homogeneity and isotropy of the physical 3-space, this ‘bulk’ metric would read

$$ds^2 = \epsilon A^2(\psi, t) d\psi^2 - N^2(\psi, t) dt^2 + R^2(t) \gamma_{ij} dx^i dx^j,$$

where the three-dimensional metric $\gamma_{ij}$ is of constant curvature, that is

$$R_{ij} = 2k \gamma_{ij} \quad (k = 0, \pm 1),$$

and $\epsilon = \pm 1$, so that the $\psi$ coordinate can be either space-like (4+1 case, $\epsilon = 1$) or time-like (3+2 case, $\epsilon = -1$).

Note that we have chosen the $t$-coordinate lines orthogonal to the constant $R$ hypersurfaces. This can always be done in the 3+2 case, where it would be just a coordinate choice in the $(\psi, t)$ time plane. But the 4+1 case requires assuming that the resulting coordinate lines are actually timelike. This amounts to discard the possibility of having a system of physical observers moving along constant $R$ hypersurfaces.

We will consider a slicing of the 5D manifold by the family of constant $\psi$ hypersurfaces (evolution formalism). On every slice we will recover a FRW line element, namely

$$-N^2(\psi, t) dt^2 + R^2(t) \gamma_{ij} dx^i dx^j \equiv g_{ab} dx^a dx^b,$$

where $a, b = 1, 2, 3, 4$. The corresponding extrinsic curvature $K_{ab}$ can be easily calculated in our case:

$$K_{ab} \equiv -\frac{1}{2A} \partial_\psi g_{ab} = -\frac{N'}{AN} u_a u_b,$$

where the primes stand for $\psi$ derivatives and $u^a$ is the future-pointing time unit vector (the FRW metric four-velocity)

$$u^a = \frac{1}{N} \delta^a_1,$$

which of course verifies

$$\nabla_a u_b = \frac{\dot{R}}{NR} (g_{ab} + u_a u_b),$$

where the dots stand for $t$ derivatives and $\nabla$ is the covariant derivative operator in the constant $\psi$ slices. The energy density $\rho$ can be obtained directly from the Friedmann equation:

$$\left(\frac{\dot{R}}{NR}\right)^2 = \frac{\rho}{3} - \frac{k}{R^2}.$$

In the evolution formalism, the vacuum field equations can be written as [17]:

- The scalar constraint:
  $$K_a^b K^a_b - (trK)^2 = (4)R.$$  

- The vector constraint:
  $$\nabla_b [K_a^b - trK \delta_a^b] = 0.$$  

- The evolution equations:
  $$\partial_\psi K_a^b = \epsilon \nabla_a \partial^b A + A \left[ -\epsilon (4)R_a^b + trK K_a^b \right],$$

where $(4)R_{ab}$ is the Ricci tensor of the FRW metric.

Allowing for the degenerate algebraic structure of the extrinsic curvature (4), the scalar constraint amounts to:

$$(4)R = \rho - 3\rho = 0,$$

which implies that the FRW metric obtained in every constant $\psi$ slice is a pure radiation metric.
III. GENERAL SOLUTION: THE M-METRIC

Let us now consider the vector constraint (9). Allowing for (4) and (6), one gets
\[ \dot{R} N' = 0. \] (12)
We will not consider here the trivial case \( R = \text{const} \), for which every projection of the bulk metric would lead to Minkowsky space on the brane. We will rather conclude that \( N = N(t) \) so that the extrinsic curvature \( K_{ab} = 0 \). (13)
The factor \( N(t) \) can be eliminated by a suitable redefinition of the \( t \) parameter in the 5D metric (1). We get then:
\[ ds^2 = \epsilon A^2(\psi, t) d\psi^2 - dt^2 + R^2(t) \gamma_{ij} dx^i dx^j. \] (14)
The evolution equation (10) reads now
\[ \frac{1}{A} \nabla_a \partial^b A = (4) R_a^b, \] (15)
where the \( A \) derivatives are computed on the constant \( \psi \) slices, that is
\[ \partial_a A = -A u_a. \] (16)
Allowing for (11), the Ricci tensor in the FRW slice corresponds to the pure radiation case:
\[ (4) R_a^b = \frac{C}{R^4} (\delta_a^b + 4 u_a u^b), \] (17)
where \( C \) is an arbitrary constant. According to the Friedmann equation (7), the expansion factor verifies
\[ \dot{R}^2 = \frac{C}{R^2} - k. \] (18)
The space components of (15) can now be written in a simple form:
\[ \frac{\dot{A}}{A} \dot{R} = -\frac{C}{R^3} = \dot{R}, \] (19)
which can be explicitly solved:
\[ A(\psi, t) = \lambda(\psi) \dot{R}. \] (20)
The integration factor \( \lambda \) can be easily removed by a suitable definition of the variable \( \psi \). The remaining components in (15) provide no additional restriction. As a result, the general vacuum solution for the cosmological case (1) can be written, in explicit form, as:
\[ ds^2 = \epsilon \left( \frac{C}{R^2} - k \right) d\psi^2 - \left( \frac{C}{R^2} - k \right)^{-1} dR^2 + R^2 \gamma_{ij} dx^i dx^j. \] (21)
We have chosen here the \((\psi, R)\) coordinate pair, because in this way all metric coefficients are fully specified. This shows that the general solution (21) is actually a single vacuum metric, which we will call ‘M-metric’ in what follows. A single ‘mother’ metric in the bulk for the full set of embedded FRW metrics, which can be recovered by projecting (21) onto different, infinitely-many, 4D hypersurfaces (branes). The particular solution for the \( \epsilon = -1, k = 0 \) parameter choice has been recently published (10).

Let us note that the M-metric (21) has a Killing vector
\[ \xi \equiv \partial_\psi. \] (22)
This implies that the \( \psi \)-coordinate lines have an intrinsic geometrical meaning. On the other hand, from the physical point of view, the time coordinate is precisely the cosmological expansion factor \( R \). This intrinsic meaning, both from the geometrical and the physical point of view, allows an straightforward comparison with other forms of the same metric, like the ones proposed in the pioneering work of Ponce de Leon (6).
IV. NON-TRIVIAL PROJECTIONS: REGULAR, EMERGENT AND SIGNATURE-CHANGING UNIVERSES

We have seen that the trivial projection on $\psi = \text{const}$ hypersurfaces leads to pure radiation FRW universe. Another option is to select instead a nontrivial hypersurface in order to get completely different FRW models. In order to visualize the behaviour of the resulting 4D projections, we will switch to a new time coordinate, namely

$$u = \int_0^R \frac{L^2 dL}{C - k L^2},$$

so that the $(\psi, u)$ sector in the M-metric (21) gets a explicit conformally-flat form:

$$ds^2 = \left( \frac{C}{R^2} - k \right) (\epsilon d\psi^2 - du^2) + R^2(u) \gamma_{ij} dx^i dx^j.$$

We can consider now projections defined by different choices of

$$\phi(\psi, u) = \text{constant},$$

which can be visualized as curves in the $(u, \psi)$ plane (see figure 1). Note that in the 4+1 case, we must fulfill the additional causality condition

$$|du/d\psi| > 1 \quad (\epsilon = +1).$$

Otherwise, the projected (brane) metric would be of Euclidean type, instead of the Lorentzian one.

We show in figure 1 some qualitatively different projections. The green line in the right-hand-side corresponds to a standard model, defined by

$$u = \sqrt{\psi^2 - \phi^2}$$

on the constant $\phi$ hypersurfaces. The resulting FRW universe shows a Big Bang singularity ($u = 0$) and verifies everywhere the causality condition (26); it will work for both choices of the signature parameter

![FIG. 1: Timelines in the $(\psi, u)$ plane of the bulk M-metric, each one leading to a different FRW brane projection. The big-bang singularity is here the $u = 0$ line. The dotted line marks the unit slope, as required from the causality condition arising in the $\epsilon = +1$ case. The green line on the right corresponds to a standard, open big bang model. The blue line in the center corresponds to an emergent universe ($\epsilon = -1$ case). The red line on the left corresponds either to a regular FRW model, without big bang ($\epsilon = -1$) or to a 4D metric showing a smooth transition, at the point marked with a dot, from Euclidean to Lorentzian signature ($\epsilon = +1$).](image-url)

A very different model is obtained by taking instead

$$u = \phi + 2 \tanh \psi, \quad \phi > 2$$

(blue line in the center). Note that this choice violates the causality condition everywhere, so it can only work for the $\epsilon = -1$ signature choice. This case corresponds to a regular universe, with no Big Bang singularity. It can be thought as an approximation to some ‘Emergent Universe’, as defined by Ellis and Maartens [11, 12]. The universe expansion starts here from a quasi-stationary state and makes a smooth transition to another quasi-stationary state with a higher value of the expansion factor.
Finally, the red line in the left-hand-side, given by

\[ u = -\dot{\phi}^2/\psi \]  

(29)
corresponds also to a regular universe, but it verifies the causality condition only in the upper part of the plot \((u > \phi)\). This is not a problem in the \(\epsilon = -1\) case (\(\psi\) timelike): there is no beginning, as the big bang singularity is just in the infinite past limit. Although the singularity is not actually reached, the physical conditions near there can be very close to those just after the big bang in standard models.

In the \(\epsilon = +1\) case (\(\psi\) spacelike), however, we get a completely different description. There is a dynamical signature change: a smooth transition from an Euclidian to a Lorentzian 4D metric in the brane. This change of signature can be considered \([13, 14]\) a classical counterpart of the Hartle-Hawking ‘no-boundary’ proposal \([15]\), without resorting to the ‘imaginary time’ idea. The resulting Lorentzian (FRW) model will then show an initial singularity (both density and the Hubble factor blow-up) at a finite value of the expansion parameter \((u = \phi)\).

V. MATTER-ENERGY CONTENT ON THE BRANE: REALISTIC EQUATIONS OF STATE

Let us consider now the general FRW line element. Using the scale factor \(R\) as the time coordinate, and allowing for the general Friedmann equation \([7]\), we get

\[ ds^2 = -\left(\frac{\rho}{\bar{3}} R^2 - \bar{k}\right)^{-\frac{1}{2}} dR^2 + R^2 \gamma_{ij} dx^i dx^j, \]  

(30)
where \(\rho(R)\) is the density, and the pressure \(p\) is given by

\[ \frac{d\rho}{dR} + 3 \frac{\rho + p}{R} = 0. \]  

(31)
We can see that the explicit form of the FRW metric is provided just by specifying the total density on the brane in terms of the scale factor. Allowing for \([31]\), this can be done by imposing any suitable equation of state: radiation \((\rho \propto R^{-4})\), incoherent matter \((\rho \propto R^{-3})\), cosmological constant (constant \(\rho = \Lambda\)), etc.

We can compare this form of FRW metrics with the projection of the bulk metric \([21]\) onto an arbitrary (brane) hypersurface, given by \([25]\). We get:

\[ ds^2 = \left[\epsilon\left(\frac{C}{R^2} - k\right) \left(\frac{d\psi}{dR}\right)^2 - \left(\frac{C}{R^2} - k\right)^{-1}\right] dR^2 + R^2 \gamma_{ij} dx^i dx^j. \]  

(32)
A simple calculation shows that the two metrics, \([30]\) and \([32]\), match if and only if \(d\psi/dR\) verifies

\[ \epsilon \left(\frac{C}{R^2} - k\right) \left(\frac{d\psi}{dR}\right)^2 = \frac{\rho R^4 - 3 C}{\rho R^4 - 3 k R^2}. \]  

(33)
At first sight, it seems that \([33]\) guarantees that any FRW metric can be embedded in the cosmological 5D bulk \([21]\). But one must pay attention to the signature restrictions contained in \([33]\), namely

\[ \epsilon = \text{sgn}(\rho R^4 - 3 C). \]  

(34)
We will explore the consequences of this restriction by considering a generic equilibrium combination of matter, radiation and cosmological constant terms (with positive density values), namely

\[ \rho = \rho_{\text{matter}} + \rho_{\text{rad}} + \Lambda. \]  

(35)
We can see that choosing the arbitrary constant \(C\) as

\[ 3 C \leq \rho_{\text{rad}} R^4 \]  

(36)
ensures that the embedding is possible in the \(\epsilon = +1\) case (4+1 bulk manifold). The specific expression for \(\psi(R)\) can be obtained from \([33]\), modulo a quadrature.

If we are rather interested in the \(\epsilon = -1\) case (3+2 bulk manifold), we see that open universes \((k = 0, -1)\) cannot be embedded in this way, as \(\rho R^4\) grows without bound (except in the pure radiation case). On the contrary, closed universes \((k = +1)\) can always be embedded by choosing a value of \(C\) greater than the maximum value of \(\rho R^4/3\). This limitation for the open case goes against the common belief that the Campbell theorem \([16]\) (see ref. \([18]\) for its extension to the pseudo-Riemannian case) ensures the embedding of any four-dimensional metric into a five-dimensional Ricci-flat manifold, where the extra dimension can be either space-like or time-like. Let us stress here that what the theorem really says is that any 4D metric can be embedded in a 5D Ricci-flat manifold \textit{provided that} the embedding equations \([21, 25]\) hold at least on a single 4D hypersurface. Our results show that this assumption is actually not fulfilled for the density choice \([33]\) with the signature combination \(\epsilon = -1, k = 0, -1\).
We have found the general solution for metrics verifying the natural extension of the Cosmological Principle to a five-dimensional (bulk) manifold. Apart from the trivial case, there is a unique solution for every curvature sign ($k = 0, \pm 1$) and every signature of the extra coordinate ($\epsilon = \pm 1$): the M-metric (21). This metric has a Killing vector (22), and we have used adapted orthogonal coordinates ($\psi, R$), where $\psi$ is the corresponding cyclic coordinate and $R$ is the cosmological expansion factor. In this way, our coordinates have a sound geometrical and physical meaning.

We have considered the projections of the M-metric onto space-homogeneous four-dimensional hypersurfaces (branes) in order to recover FRW metrics. The use of $R$ as the time coordinate in the brane allows one to express the generic FRW metric directly in terms of the matter-energy density (30). This is crucial for controlling the equation of state of the resulting projection. We have studied the case of a generic (equilibrium) mixture of matter, radiation and cosmological constant. We have provided, modulo a quadrature, the transformation $\psi(R)$ allowing to recover the corresponding metric for both signature combinations ($\epsilon = \pm 1$) in the closed universe case ($k = +1$). In the open case ($k = 0, -1$), we have shown that, apart from the pure radiation case, the generic combination can be obtained only when the extra coordinate $\psi$ is space-like.

The ($\psi, R$) plane in the bulk happens to be a powerful tool for devising the evolution properties of the resulting projected spacetimes: one only has to select a suitable time curve, which will be kept as the physical time coordinate in the brane. We shown how to obtain by this method some FRW regular solutions, evolving from the infinite past (no Big Bang), that could be useful to deal with the Cosmological Horizon problem. These are regular FRW models in standard General Relativity: there is no need to recur to alternative theories in order to get these appealing cosmological models. We have also seen how to obtain in this way models that approach some instances of the well-known 'Emergent Universe' inflationary models [11, 12].

In the 4+1 case, where the extra coordinate is spacelike, we have identified the causality condition (26) which ensures that the projected (brane) hypersurface is of Lorentzian type. When violated, the projection leads to an Euclidean 4D manifold, without any time dimension. We can take advantage of this fact in order to get universe models showing a dynamical signature change: a smooth transition from an Euclidian to a Lorentzian 4D metric (see fig. 1). This change of signature can be considered [13, 14] a classical counterpart of the Hartle-Hawking 'no-boundary' proposal, without resorting to the 'imaginary time' idea. The resulting Lorentzian (FRW) model will then show an initial singularity (both density and the Hubble factor blow-up) at a finite value of the scale factor $R$. This has been termed as 'Big freeze' singularity type [19] because it has been associated with the final stage of cosmological evolution, as an alternative to the 'Big rip' [20]. This term is misleading in our case, where we could rather use the term 'Big unfreeze' for this singularity, as it is produced just by the beginning of time, without affecting space geometry. As a final remark, let us point out that having the cosmological bulk metric (21) in fully explicit form can open the door to new advances in some related fields. For instance, a semiclassical Kaluza-Klein approach to Cosmology would clearly benefit from the uniqueness of the spacetime background: any result obtained from the vacuum line element (21) would allow to draw general conclusions, valid beyond any particular case. For more potential applications, see for instance ref. [7].

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