The vector form of a Lorentz transformation which is separated with time and space parts is studied. It is necessary to introduce a new definition of the relative velocity in this transformation, which plays an important role for the calculations of various invariant physical quantities. The Lorentz transformation expressed with this vector form is geometrically well interpreted in a hyperbolic space. It is shown that the Lorentz transformation can be interpreted as the law of cosines and sines for a hyperbolic triangle in hyperbolic trigonometry. So the triangle made by the two origins of inertial frames and a moving particle has the angles whose sum is less than 180°.

PACS numbers: 02.40Yy, 03.30+p, 11.30.Cp, 45.20.-d, 98.80Hw

I. INTRODUCTION

The special theory of relativity starts with the fundamental two postulates which are for relativity and the constancy of the speed of light in all inertial frames [1, 2]. The transformation between inertial frames is known as the Lorentz transformation which satisfies the two postulates. Therefore physical quantities and formulas are the same forms in all inertial frames. This is achieved by using the four vector notation which is combined with the space and time components, such as \( v^\mu = dx^\mu/d\tau \), \( F^\mu = dp^\mu/d\tau \) and so on [4]. However the separate notation of four vectors does not show the covariance manifestly in the transformation equations. The transformed coordinate system used in the standard texts [1, 2] has a little complicated transformation formulas, such as,

\[
\begin{align*}
t' &= \gamma(t - v \cdot r), \\
r' &= r + \frac{\gamma - 1}{v^2}(v \cdot r)v - \gamma vt,
\end{align*}
\]

thus, it is often doubtful that the transformed coordinate system has the same physical formulas as the original coordinate system has. These transformation equations are not true vector forms. They are merely components of the vector equation, because the basis vectors are used in common in the two frames. Physical quantities are measured in an inertial frame with its own units of time and length. Physical formulas composed of these quantities which are scalars, vectors or tensors are formed in a frame with its own unit vectors. Therefore it is difficult to say relativity in the two frames with the above equations, because the transformed equations often suffer from unwanted factors, such as, the \( \gamma \) factor. Due to this trouble maker, the Biot-Savart law, for an example in electromagnetism, is explained with sophisticated terms, such as, the duration time of measurement for the magnetic field come from the transformed electric field of a moving charged particle [2].

A rotating coordinate system is a good example for the above arguments, of which transformation is

\[
x'_i = [e^{iJ\theta}]_{ij}x_j, \tag{2}
\]

where \( J_k \) is a generator of the rotation group \( O(3) \). Its time derivative is the velocity, which can be calculated with the transformation equation as follows

\[
v'_i = \frac{dx'_i}{dt} = [e^{iJ\theta}]_{ij}\left(\frac{dx_j}{dt} + i[J]_{ij}\frac{d\theta_k}{dt}x_l\right) \\
= [e^{iJ\theta}]_{ij}(v_j + \epsilon_{jkl}\omega_kx_l) \tag{3}
\]

where \( [J]_{ij} = i\epsilon_{jkl} \) in the regular representation of the rotation group [3]. Then the vector form of the velocity is

\[
v' = v'_i\hat{e}_i = [e^{iJ\theta}]_{ij}(v_j + \epsilon_{jkl}\omega_kx_l)\hat{e}_i \\
= (v_j + \epsilon_{jkl}\omega_kx_l)\hat{e}_j \\
= v + \omega \times r, \tag{4}
\]
where the unprimed basis vectors are transformed as \( \hat{e}_j = \hat{e}'_i[e^{iJ_k\theta_k}]_{ij} \). The acceleration is

\[
a'_i = \frac{dv'_i}{dt} = [e^{iJ_k\theta_k}]_{ij}(a_j + \epsilon_{jkl}\omega_k\epsilon_{lmn}\omega_m x_n + 2\epsilon_{jkl}\omega_k v_l),
\]

where we consider a constant angular velocity. The vector form of the above equation is

\[
a' = a + \omega \times (\omega \times r) + 2\omega \times v,
\]

and, if a mass is multiplied to the equation, the Newton’s law is

\[
F' = ma' = ma + m\omega \times (\omega \times r) + 2m\omega \times v,
\]

where the second term is the centrifugal force and the last term is the Coriolis force as shown in the standard texts. Therefore the position vectors in the two frames are written by

\[
r' = x'_i\hat{e}'_i = [e^{iJ_k\theta_k}]_{ij}x_j\hat{e}'_i = x_j\hat{e}_j = r,
\]

irrespective of the rotation angle which is constant or varying with time. Since the mass times acceleration is expressed as the same forms in the two frames, the Newton’s law can be said to be expressed as covariant manner in the vector representation rather than in the component representation of the vector. There is no rotation matrix, namely, an unwanted factor, in Eq. (7) compared to Eq. (5).

Therefore the correct vector form of the Lorentz transformation should be expressed with its own components and unit vectors. It is more natural that relativity can be expected after obtaining the correct vector form of the Lorentz transformation. As learned from non-relativistic kinematics, if a physical phenomenon is described with a geometrical picture, it is much easier to comprehend it. The correct vector forms of the Lorentz transformation are well interpreted with geometrical pictures, which are the law of cosines and sines in hyperbolic trigonometry. It is of course already known that the velocity space of the special theory of relativity is a Lobatchewsky space, that is, a hyperbolic space, but our interpretation gives more clear extended explanation for the hyperbolic space. So the Thomas precession can be interpreted as the time rate of the angle defect.

The outline of this paper is that a rotational transformation is adapted to a Lorentz transformation in section II in order to obtain the correct vector form of the Lorentz transformation, the method of calculation for invariant quantities under the Lorentz transformation and the geometrical interpretations of the Lorentz transformation. In section III, most of basic physical quantities are investigated under the Lorentz transformation according to the guidelines of section II, and developed further. In section IV, the Lorentz transformation for an energy and a momentum is shown to be the law of cosines and sines in hyperbolic trigonometry. Finally some conclusions are given. Since spherical and hyperbolic trigonometries are not seen in the standard texts for physics, they are introduced in Appendices A and B. Spherical trigonometry is very useful to introduce an angle vector as shown in Appendix A and to comprehend hyperbolic trigonometry and its space.

II. ROTATIONAL TRANSFORMATION

A Lorentz transformation is similar to a rotational transformation, if the relative velocity \( v \) between two frames is regarded as tangent of a rotational angle \( \phi \), that is, \( \tan \phi = -iv \). So the converse of the similarity is also true and gives us a good insight for the special theory of relativity. Moreover they are expressed as the same transformation formulas in the Euclidean space time. A rotated coordinate system \( K' \) with an angle \( \theta \) around the z-axis of a coordinate system \( K \) as shown in Fig. 1 has the components as follows

\[
x' = x \cos \theta + y \sin \theta,
\]

\[
y' = -x \sin \theta + y \cos \theta,
\]

\[
z' = z.
\]

Then let \( \cos \theta \) and \( \sin \theta \) be assigned to \( 1/\sqrt{1+v^2} \) and \( v/\sqrt{1+v^2} \), respectively. The above transformation becomes similar to a Lorentz transformation as follows

\[
x' = \frac{1}{\sqrt{1+v^2}}(x + vy) = \gamma(x + vy),
\]

\[
y' = \frac{1}{\sqrt{1+v^2}}(y - vx) = \gamma(y - vx),
\]

\[
z' = z,
\]

(10)
FIG. 1: The rotational transformation (a) around $z$-axis can be analogized to the Lorentz transformation as shown in the figure (b) by regarding as $\tan \theta = v$. The new relative velocity vector can be decomposed into the two vectors depicted with unfilled arrows in the primed coordinate system and unprimed coordinate system, respectively.

where $\gamma$ is used as $1/\sqrt{1 + v^2}$ only in this section. We shall see in the next section that the difference between the rotational and the Lorentz transformations comes from that between the metrics in the two transformations, if the coordinate $x$ is regarded as time axis and the coordinates $y$ and $z$ are done as space axes. Hence the signs before the terms involving $v$ are different from those of the Lorentz transformation.

The vector forms of the Lorentz transformation used in the standard texts\cite{1, 2} are again written in the version of this rotational transformation by

$$x' = \gamma(x + vy) = \gamma(x + v \cdot r),$$
$$r' = y'y' + z'k'$$
$$= y'y' + z'k' + (\gamma - 1)y'y' - \gamma vx'y'$$
$$= r + \gamma - \frac{1}{v^2}(r \cdot v)v - \gamma vx,$$  \hspace{1cm} (11)

where the basis vectors are used in common in the two coordinate systems. These equations are not, therefore, true vector forms, but tell us only components of the transformation equation. The correct definition of the vector in the unprimed coordinate system should be

$$r = y'j + z'k.$$  \hspace{1cm} (12)

Hence the following transformation for the basis vectors should be applied to the above equation (11):

$$\hat{i}' = i \cos \theta + j \sin \theta,$$
$$\hat{j}' = -i \sin \theta + j \cos \theta,$$
$$\hat{k}' = \hat{k}.$$  \hspace{1cm} (13)

Then the vector form of the transformation equation is calculated as

$$r' = r + y'y' - y'j$$
$$= r - \frac{y'j - y'y'}{x}x$$
$$= r - vx,$$  \hspace{1cm} (14)
where $\mathbf{v}$ is defined as a new relative velocity between the two frames at the point indicated by the coordinates in the space time, and used a different type letter to distinguish the new relative velocity vector from usual vectors. We will call this type velocity the new relative velocity from now on because we have no name to call it suitably. The vectors in the two frames are connected via this new relative velocity. The definition of the new relative velocity is exactly correct in a sense that, if we consider it in general situation, namely, curvilinear coordinate systems, the variation of a vector should take into account both its component and unit vector, such as, $\Delta y_\hat{j} + y' \Delta \hat{j}' = \Delta \hat{j} + \Delta \hat{j}'$. Therefore the definition of a velocity should be

$$\mathbf{v} = \frac{dx}{dt} = \frac{d\hat{x} + r d\hat{x}}{dt}, \quad (15)$$

even in the rectangular coordinate system, because the basis vectors are defined only in an inertial frame. The $x$ coordinate has the following transformation:

$$x' = x + x'\hat{i} - x\hat{i} = x + \frac{y\hat{j} - y'\hat{j}'}{x} \hat{i} = x + \mathbf{v}_x, \quad (16)$$

because the vectors $\mathbf{s}$ and $\mathbf{s}'$ in the two coordinate systems are identical as follows

$$\mathbf{s}' = x'\hat{i} + y'\hat{j}' + z'\hat{k}' = x\hat{i} + y\hat{j} + z\hat{k} = \mathbf{s}, \quad (17)$$

and we get the following equation:

$$x'\hat{i} - x\hat{i} = y\hat{j} - y'\hat{j}'. \quad (18)$$

The component forms of the Lorentz transformation to which we are accustomed can be calculated by multiplying the basis vectors to Eqs. (14) and (16) as

$$x' = x' \cdot \hat{i} = x\hat{i} + \mathbf{v} \cdot \hat{i} \cdot x = \gamma x + \mathbf{v} \cdot \mathbf{r} = x \cos \theta + y \sin \theta, \quad (19)$$

$$y' = y' \cdot \hat{j} = y\hat{j} + \mathbf{v} \cdot \hat{j} \cdot x = \gamma y - \mathbf{v} \cdot \mathbf{r} = y \cos \theta - x \sin \theta,$n

$$z' = z' \cdot \hat{k} = z\hat{k} + \mathbf{v} \cdot \hat{k} \cdot x = z;.$$

where the scalar products of the basis vectors are the direction cosines of each axis and the scalar products of the new velocity vector to the basis vectors are calculated as

$$\mathbf{v} \cdot \hat{i}' = \frac{(y\hat{j} - y'\hat{j}')}{x} \cdot \hat{i}' = \frac{y\hat{j} \cdot \hat{i}'}{x} = \frac{y \sin \theta}{x},$$

$$\mathbf{v} \cdot \hat{j}' = \frac{(y\hat{j} - y'\hat{j}')}{x} \cdot \hat{j}' = \frac{y\hat{j} \cdot \hat{j}'}{x} = \frac{y \cos \theta}{x},$$

$$\mathbf{v} \cdot \hat{k}' = \frac{(y\hat{j} - y'\hat{j}')}{x} \cdot \hat{k}' = 0. \quad (20)$$

These equations tell us the direction of the new velocity vector as shown in Fig. 4, the new velocity vector is represented in the primed and unprimed coordinate systems as

$$\mathbf{v}x = y' \sin \theta \hat{i} + x' \sin \theta \hat{j} = y \sin \theta \hat{i} + x \sin \theta \hat{j} = \mathbf{v}' x'. \quad (21)$$

If the new relative velocity in the primed coordinate system is defined as this equation, the vector forms of the transformation equations (14) and (16) are expressed as covariant forms in the two frames. The scalar product of the new relative velocity to the position vector is calculated as

$$\mathbf{v} \cdot \mathbf{r} = \frac{1}{x} (y\hat{j} - y'\hat{j}') \cdot (y\hat{j} + z\hat{k})$$

$$= \frac{1}{x} (y^2 - y' \cos \theta) = \frac{1}{x} (y^2 \sin^2 \theta + xy \sin \theta \cos \theta)$$

$$= \frac{x' y \sin \theta}{x} = x' \mathbf{v} \cdot \hat{i}'. \quad (22)$$
The new relative velocity in the primed coordinate system can be defined as

$$v' = \frac{y' j - y j'}{x'},$$

and the scalar product of it to the position vector is related with the above relevant quantities as follows

$$v' \cdot r = x v \cdot j' = y \sin \theta = \frac{x}{x'} v \cdot r.$$  \tag{24}

The new relative velocities $v$ and $v'$ are defined different from each other in general in the two coordinate systems. However the new relative velocities of the origins of each other frames are reduced to the usual vectors

$$v = \frac{y j}{x} = v_j = \mathbf{v}, \quad v' = -\frac{y' j'}{x'} = -v_j' = v',$$

from Eqs. (14) and (23), where the magnitude of the usual relative velocities are the same in both frames. This is assumed without mention as a postulate in the special theory of relativity. The usual relative velocity vectors are different from each other in their direction because their unit vectors are different. In general all moving particles have their own basis vectors of the inertial frame so that they should have the form of the velocity like Eq. (15) or the new relative velocities. However their velocity vectors can be expressed as usual vectors in an observing frame, because the particles are assumed to be at the origins of their own frames as shown in above equations. The vector form of the Lorentz transformation requires the different new relative velocity from point to point so that the magnitude of it is also varying according to points. The magnitude of the new relative velocity is calculated as

$$v \cdot v = \frac{1}{x^2} (y^2 + y'^2 - 2y y' \cos \theta) = 1 \left( x^2 + y^2 \right) \sin^2 \theta = \frac{\sin^2 \theta}{\cos^2 \alpha} = v^2,$$  \tag{26}

where $\alpha$ is the angle of the vector $s$ making with x-axis as shown in Fig. 1. The magnitude of the new relative velocity observed in the primed coordinate system is written by

$$v' \cdot v' = \frac{1}{x'^2} (x'^2 + y'^2) \sin^2 \theta = \frac{\sin^2 \theta}{\cos^2 \beta} = v'^2,$$  \tag{27}

where $\beta$ is the angle of the vector $s$ making with $x'$-axis as shown in Fig. 1. Therefore the relation between the two new relative velocities has the identity:

$$v^2 \cos^2 \alpha = v'^2 \cos \beta = \sin^2 \theta = \frac{v^2}{1 + v^2},$$  \tag{28}

Now the vector $s$ is invariant under the rotational transformation so that the transformed vector $s'$ is calculated as

$$s'^2 = x'^2 + r' \cdot r' = \gamma^2 x^2 + 2 \gamma x v \cdot r + (v' \cdot r)^2 + r \cdot r - 2 x v \cdot r + x^2 v \cdot v = \gamma^2 x^2 + r \cdot r = s^2,$$  \tag{29}

where Eqs. (22), (24), and (26) are inserted in the second line of the equation. This calculation is a little different from the calculation in the component form of the Lorentz transformation, but leads to the same result.

### III. LORENTZ TRANSFORMATION

The vector form of a Lorentz transformation for a coordinate system moving with a velocity along the x-axis of a rest coordinate system is trivially achieved with the transformation of the previous section by considering the difference of the metrics. This can be also obtained from the general transformation below, let us pay attention to the general case. A Lorentz transformation for an arbitrary direction of the relative velocity with respect to the origin of the rest frame is just Eq. (15) whose matrix form is

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v x & -\gamma v y & -\gamma v z \\ -\gamma v x & 1 + (\gamma - 1) n_x n_x & (\gamma - 1) n_x n_y & (\gamma - 1) n_x n_z \\ -\gamma v y & (\gamma - 1) n_x n_y & 1 + (\gamma - 1) n_y n_y & (\gamma - 1) n_y n_z \\ -\gamma v z & (\gamma - 1) n_x n_z & (\gamma - 1) n_y n_z & 1 + (\gamma - 1) n_z n_z \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}.$$  \tag{30}
which are the array of the components of Eq. (11), and where \( v \) is the usual relative velocity between the origins of the two inertial frames. The \( \gamma, \gamma v \) and \( v \) are assigned to \( \cosh \theta, \sin \theta \) and \( \tanh \theta \), respectively, and the direction cosines of the relative velocity vector can be thought of as \( n_x = \sin \theta \cos \phi, n_y = \sin \theta \sin \phi, \) and \( n_z = \cos \theta \). The basis vectors of the coordinate system are also transformed as

\[
\begin{pmatrix}
\hat{h}' \\
\hat{i}' \\
\hat{j}' \\
\hat{k}'
\end{pmatrix} = \begin{pmatrix}
\gamma v_{n_x} & \gamma v_{n_y} & \gamma v_{n_z} \\
\gamma v_{n_x} & (\gamma - 1)n_x n_y & (\gamma - 1)n_x n_z \\
\gamma v_{n_y} & (\gamma - 1)n_x n_y & (\gamma - 1)n_y n_z \\
\gamma v_{n_z} & (\gamma - 1)n_x n_z & (\gamma - 1)n_y n_z
\end{pmatrix}
\begin{pmatrix}
\hat{h} \\
\hat{i} \\
\hat{j} \\
\hat{k}
\end{pmatrix},
\]

where the transformation matrix is not only inverse to the matrix of the coordinate transformation, but also a transformation matrix for the corresponding covariant vector to the coordinate.

From the calculations according to the previous section, a position and a time vectors in the moving frame are written by

\[
\begin{align*}
\mathbf{r}' &= \mathbf{r} - \mathbf{vt}, \\
\mathbf{t}' &= \mathbf{t} + \mathbf{vt},
\end{align*}
\]

where the transformation for the position vector is ironically similar to the Galilean transformation, but an exact Lorentz transformation due to the new relative velocity. The new relative velocities are defined as

\[
\mathbf{v} = \frac{x' \hat{i} + y' \hat{j} + z' \hat{k} - x \hat{i} - y \hat{j} - z \hat{k}}{t} = \frac{t' \hat{h}' - \hat{t} h}{t},
\]

\[
\mathbf{v}' = \frac{x' \hat{i} + y' \hat{j} + z' \hat{k} - x' \hat{i}' - y' \hat{j}' - z' \hat{k}'}{t'} = \frac{t' \hat{h}' - \hat{t} h}{t'},
\]

since an event four vector \( \mathbf{s} \) in the rest frame is identical to \( \mathbf{s}' \) in the moving frame as follows

\[
\mathbf{s} = x' \hat{e}_x = \hat{t} h + x' \hat{i} + y' \hat{j} + z' \hat{k} = t' \hat{h}' + x' \hat{i}' + y' \hat{j}' + z' \hat{k}' = x'^\mu \hat{e}_\mu = \mathbf{s}'.
\]

It should be noted that \( \hat{h} \cdot \hat{h}' = 1 \) and \( \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = -1 \), because the metric is represented in terms of its basis vectors as \( g_{\mu \nu} = \hat{e}_\mu \cdot \hat{e}_\nu \). The time component of the transformation equations is calculated from the time vector equation as

\[
\mathbf{t}' = \mathbf{t} \cdot \hat{h}' = (\mathbf{t} + \mathbf{vt}) \cdot \hat{h}' = \gamma t + \mathbf{v} \cdot \mathbf{r}
\]

where the scalar product of the time component basis vector to the new relative velocity vector is calculated as

\[
\mathbf{t} \mathbf{v} \cdot \hat{h}' = \hat{u} \cdot \mathbf{r} \sinh \theta = \frac{t}{t'} \mathbf{v} \cdot \mathbf{r} = \mathbf{v}' \cdot \mathbf{r}
\]

The scalar products of the new relative velocity to the other basis vectors are

\[
\mathbf{v} \cdot \hat{i}' = -n_x \sinh \theta, \quad \mathbf{v} \cdot \hat{j}' = -n_y \sinh \theta, \quad \mathbf{v} \cdot \hat{k}' = -n_z \sinh \theta.
\]

From the above knowledge, the component equations of the Lorentz transformation can be obtained from the scalar product of every basis vector to Eq. (12), which agree to Eq. (30) as shown in the previous section.

The square of the new relative velocity vector is calculated from the above definition as

\[
\mathbf{v} \cdot \mathbf{v} = (\frac{t' \hat{h}' - \hat{t} h}{t})^2 = (1 + \frac{(\hat{u} \cdot \mathbf{r})^2}{t^2}) \sinh^2 \theta,
\]

which is the same form as shown in the previous section. Now we can check the following invariant quantity with using the above calculations according to the procedure of the previous section:

\[
\mathbf{s}^2 = t'^2 + \mathbf{r}' \cdot \mathbf{r}' = t'^2 - r'^2 = t^2 - r^2 = t^2 + \mathbf{r} \cdot \mathbf{r} = \mathbf{s}^2.
\]
This equation may be useful to explain the time dilation phenomenon with the muon decay in elementary course of relativity. The origins of the two inertial frames which are the frame of a muon rest frame and the frame of an observer on the earth are coincided at the position and time that a muon is produced in the atmosphere by a cosmic ray. We observe the muon with a velocity $v$, and it travels a distance $r = vt$. Since the muon does not travel in the muon rest frame, that is, $r' = 0$, the time interval which the muon travels is observed as $t = t'/\sqrt{1 - v^2}$ on the earth.

An inverse Lorentz transformation can be obtained by applying the inverse matrix of the transformation to the transformation matrix equation Eq. (39), so the direction cosines of the relative velocity between the origins of the two frames are used in common in both frames. The unit vector of the relative velocity in the primed coordinate system is, therefore, calculated as

$$\hat{n}' = n_x\hat{i}' + n_y\hat{j}' + n_z\hat{k}' = \hat{n}\cosh{\vartheta} + \hat{h}\sinh{\vartheta},$$

which joins the first line of the transformation matrix equation for the basis vectors and forms a pair of the Lorentz transformation for the unit vectors as follows

$$\hat{h}' = \hat{h}\cosh{\vartheta} + \hat{n}\sinh{\vartheta},$$

$$\hat{n}' = \hat{n}\cosh{\vartheta} + \hat{h}\sinh{\vartheta},$$

$$\hat{m}' = \hat{m},$$

where two unit vectors of $\hat{m}$ which are orthogonal to $\hat{n}$ can be chosen as the remaining space basis vectors. Using the transformation matrix equations for the coordinate and the unit vectors, the scalar and vector products of the unit vector of the relative velocity to the position vector in the primed coordinate system are directly calculated as

$$\hat{n}' \cdot \hat{r}' = -n_xx' - n_yy' - n_zz' = t\sinh{\vartheta} + \hat{n} \cdot \hat{r} \cosh{\vartheta},$$

$$\hat{n}' \times \hat{r}' = (n_yz' - n_zy')\hat{i}' + (n_zx' - n_xz')\hat{j}' + (n_xx' - n_yx')\hat{k}' = \hat{n} \times \hat{r},$$

$$\hat{n}' \times (\hat{n}' \times \hat{r}') = -\hat{n}' (\hat{n}' \cdot \hat{r}') - \hat{r}' = \hat{n} \times (\hat{n} \times \hat{r}),$$

where the vector product is not well defined in relativity. Such an unsatisfactory definition may be due to a non-commutative property between the operation of a vector product and the Lorentz transformation, or insufficient definitions for the extended indices, 0, 1, 2, 3 in the four dimension for a vector product. Anyway the operation after the transformation needs a transformation rule for the vector product itself, that is, the symbol $\times$, but the transformation after the operation does not need to do so. Therefore the latter is regarded as the prescription for a vector product in this works. Moreover scalar products are well defined in the four dimension. Vector products which can be expressed with scalar products, such as triple vector products, does not matter in the Lorentz transformation after the operation of vector products.

Now the corresponding coordinate transformation to the above transformation for the unit vectors is

$$t' = t\cosh{\vartheta} + \hat{n} \cdot \hat{r}\sinh{\vartheta},$$

$$\hat{n}' \cdot \hat{r}' = t\sinh{\vartheta} + \hat{n} \cdot \hat{r}\cosh{\vartheta},$$

$$\hat{n}' \times (\hat{n}' \times \hat{r}') = \hat{n} \times (\hat{n} \times \hat{r}),$$

where the minus sign in the matrix of the Lorentz transformation is hidden here in the scalar product due to the metric. Since a vector is decomposed into a transverse and a normal components with respect to a specified unit vector, such as $\hat{r} = - (\hat{n} \cdot \hat{r})\hat{n} - \hat{n} \times (\hat{n} \times \hat{r})$, the transformation for $\hat{n}' \times \hat{r}'$ is discarded because it is irrelevant to the vector $\hat{r}$. This transformation for an arbitrary direction of a relative velocity agrees to that for a relative velocity along $x$-axis which can be obtained by substituting $\hat{n} = \hat{i}$. The position vectors in the transformation equations are expressed exactly because they have their own components and unit vectors in their initial frames. The new relative velocity is written by

$$\mathbf{vt} = \mathbf{v}'t' = \hat{n} \cdot \hat{r}\sinh{\vartheta}\hat{h}' + t\sinh{\vartheta}\vartheta\hat{n}'$$

$$= \hat{n}' \cdot \hat{r}'\sinh{\vartheta}\hat{h} + t'\sinh{\vartheta}\vartheta\hat{n},$$

(44)

which gives the following identity:

$$(\hat{n} \cdot \hat{r})\hat{h}' + t\hat{n}' = (\hat{n}' \cdot \hat{r}')\hat{h} + t'\hat{n},$$

(45)

by eliminating $\sinh{\vartheta}$ in the above equation. The scalar products of the unit vectors $\hat{h}$ and $\hat{n}$ to the identity give again the above Lorentz transformation with the correct vectors, that is, Eq. (43). Therefore the new relative velocity has
the Lorentz transformation properties between the two vectors \( \mathbf{r} \) and \( \mathbf{r}' \) in the two frames in Eq. (32). The above transformation Eq. (13) can be also obtained from the scalar and vector products of the primed unit vector of the relative velocity to the transformed position vector as follows

\[
\hat{n}' \cdot \mathbf{r}' = \hat{n}' \cdot (\mathbf{r} - \mathbf{v}t) = t \sinh \vartheta + \hat{n} \cdot \mathbf{r} \cosh \vartheta,
\]

\[
\hat{n}' \times (\hat{n}' \times \mathbf{r}') = -\hat{n}'(\hat{n}' \cdot \mathbf{r}') - (\mathbf{r} - \mathbf{v}t) = \hat{n} \times (\hat{n} \times \mathbf{r}),
\]  

(46)

by using the above new relative velocity and the transformation for the unit vector.

Since the position vector is a sort of a displacement vector, an infinitesimal displacement vector can be investigated in the same way as follows

\[
d\mathbf{r}' = d\mathbf{r} - \mathbf{v}dt,
\]

\[
dt' = dt + \mathbf{v}dt,
\]  

(47)

where the new relative velocity is defined in the same way, but different from Eq. (44) as follows

\[
\mathbf{v}dt = \mathbf{v}'dt' = \hat{n} \cdot d\mathbf{r} \sinh \vartheta' + dt \sinh \vartheta \hat{n}'
\]

\[
= \hat{n}' \cdot d\mathbf{r}' \sinh \vartheta + dt' \sinh \vartheta \hat{n}.
\]  

(48)

This transformation for the infinitesimal displacement vector is not the derivative of the transformation equations for the position and time vectors, because the new relative velocities between the two transformations are different from each other. A new relative velocity is always defined in the way of Eq. (32). A new relative velocity is defined different not only from point to point, but also from vector to vector. The invariant quantity for the infinitesimal displacement vector is also calculated in the same way of the above event vectors as follows

\[
ds'^2 = dt'^2 + \mathbf{dr}' \cdot \mathbf{dr}' = dt'^2 - dr'^2 = dt^2 + dr \cdot dr = ds^2.
\]  

(49)

The transformation with usual vectors for the infinitesimal displacement vector is represented as

\[
dt' = dt \cosh \vartheta + \hat{n} \cdot d\mathbf{r} \sinh \vartheta,
\]

\[
\hat{n}' \cdot d\mathbf{r}' = dt \sinh \vartheta + \hat{n} \cdot d\mathbf{r} \cosh \vartheta,
\]

\[
\hat{n}' \times (\hat{n}' \times d\mathbf{r}') = \hat{n} \times (\hat{n} \times d\mathbf{r}),
\]  

(50)

which can be used to calculate the transformation for the velocities of a moving particle observed in both frames. The components of the velocity of the particle which are parallel and perpendicular to the relative velocity are written by

\[
\hat{n}' \cdot u' = \frac{\hat{n} \cdot (u - v)}{1 + v \cdot u},
\]

\[
\hat{n}' \times (\hat{n}' \times u') = \frac{\hat{n} \times (\hat{n} \times u)}{\gamma(1 + v \cdot u)},
\]  

(51)

by dividing the last two equations by \( dt' \). Using the above vector equations for the infinitesimal displacement vector, the velocity addition rule in vector form is written by

\[
u' = \frac{d\mathbf{r}'}{dt'} = \frac{d\mathbf{r} - \mathbf{v}dt}{\gamma dt + \mathbf{v}' \cdot d\mathbf{r}}
\]

\[
= \frac{\mathbf{u} - \mathbf{v}}{\gamma + \mathbf{v}' \cdot \mathbf{u}} = \frac{\mathbf{u} - \mathbf{v}}{\gamma(1 + v \cdot u)},
\]  

(52)

which is consistent with the above component equations by doing the scalar and vector products of the primed unit vector of the relative velocity to it.

Since an energy and a momentum are transformed as coordinates under the Lorentz transformation, the transformed momentum and energy vectors are written by

\[
p' = p - \mathbf{v}E,
\]

\[
E' = E + \mathbf{v}E,
\]  

(53)

where the new relative velocities in momentum space are also defined as

\[
\mathbf{v} = \frac{E \hat{\mathbf{h}} - E\hat{\mathbf{h}}}{E}, \quad \mathbf{v}' = \frac{E' \hat{\mathbf{h}} - E\hat{\mathbf{h}}}{E'},
\]  

(54)
These velocities are calculated as

\[ \mathbf{v} E = \mathbf{v}' E' = \hat{n} \cdot \mathbf{p} \sinh \theta \hat{n}' + E \sinh \theta \hat{\mathbf{n}}. \]

These new relative velocities \( \mathbf{v} \) and \( \mathbf{v}' \) defined in momentum space are equal to those defined in the transformation for the infinitesimal displacement vectors, if the velocity of the particle involved in the transformation is expressed as \( \mathbf{u} = \frac{dr}{dt} = \mathbf{p}/E \). So the velocity addition rule is again written in vector form by

\[
\mathbf{u}' = \frac{\mathbf{p}'}{E'} = \frac{\mathbf{p} - \mathbf{v}E}{\gamma \mathbf{E} - \mathbf{v}' \cdot \mathbf{p}} = \frac{\mathbf{u} - \mathbf{v}}{\gamma + \mathbf{v}' \cdot \mathbf{u}}
\]

which agrees to Eq. (52), because the momentum is defined in relativity as \( \mathbf{p} = E \mathbf{u} \) and the new relative velocities are equal to each other in the two transformations. The energy in the primed coordinate system is calculated from the energy vector as

\[
E' = E' \cdot \hat{l}' = E \cosh \vartheta + \mathbf{v}' \cdot \mathbf{p} = E \cosh \vartheta + \hat{n} \cdot \mathbf{p} \sinh \vartheta.
\]

The energy-momentum relation which is an invariant quantity in momentum space is also calculated as

\[ m^2 = E'^2 + \mathbf{p}' \cdot \mathbf{p}' = E^2 + \mathbf{p} \cdot \mathbf{p}, \]

which is equal to its mass squared. Using the new relative velocity the transformation of the energy and the momentum with usual vectors are calculated as

\[
E' = E \cosh \vartheta + \hat{n} \cdot \mathbf{p} \sinh \vartheta,
\hat{n}' \cdot \mathbf{p}' = E \sinh \vartheta + \hat{n} \cdot \mathbf{p} \cosh \vartheta,
\hat{n}' \times (\hat{n}' \times \mathbf{p}') = \hat{n} \times (\hat{n} \times \mathbf{p}),
\]

where the last two equations give again the transverse and normal components of the velocity addition rule which are Eq. (51), if they are divided by the primed energy, that is, the first equation.

In the unprimed coordinate system, the energy and the momentum of a moving particle are represented with a hyperbolic angle as \( E = m \cosh \alpha = m/\sqrt{1-u^2} \) and \( \mathbf{p} = m \hat{n} \sinh \alpha = m \mathbf{u}/\sqrt{1-u^2} \), respectively, where \( u \) is the magnitude of the velocity of the particle. From the above transformations for the energy and the momentum, it is also possible to do so in the primed coordinate system by using the hyperbolic trigonometric identities in Appendix B. Using Eq. (B2) in Appendix B, the energy of the particle in the primed coordinate system is calculated as

\[
E' = E \cosh \vartheta + \hat{n} \cdot \mathbf{p} \sinh \vartheta
= m \cosh \alpha \cosh \vartheta + m \hat{n} \cdot \hat{m} \sinh \alpha \sinh \vartheta
= m \cos(\alpha \hat{n} \cdot \vartheta) = m \cosh \beta = \frac{m}{\sqrt{1-u'^2}}.
\]

where \( \vartheta' \) is the magnitude of the velocity of the particle which is observed in the primed coordinate system. From Eqs. (58) and (60), the momentum of the particle in the primed coordinate system is calculated as

\[
\mathbf{p}' = \sqrt{m^2 - E'^2} = \frac{mu'}{\sqrt{1-u'^2}} = m \hat{l}' \sinh \beta,
\]

where \( \hat{l}' \cdot \hat{l}' = -1 \). This equation agrees to the following calculation by using Eqs. (B3) and (B5) in Appendix B

\[
\hat{n}' \cdot \mathbf{p}' = \hat{n}' \cdot \hat{l}' \mathbf{p}' = E \sinh \vartheta + \hat{n} \cdot \mathbf{p} \cosh \vartheta
= m \cosh \alpha \sinh \vartheta + m \hat{n} \cdot \hat{m} \cosh \vartheta \sinh \alpha
= m \hat{n} \cdot [\hat{m} \sinh \alpha \cosh \vartheta - \hat{n} \sinh \vartheta \cosh \alpha + \frac{\gamma}{\gamma - 1} \hat{n} \times (\hat{n} \times \hat{m}) \sinh \alpha]
= m \hat{n} \cdot \sin(\alpha \hat{n} \cdot \vartheta) = m \hat{n} \cdot \hat{l} \sinh \beta = \hat{n}' \cdot (m \hat{l}' \sinh \beta),
\]

where the identities \( \hat{n} \cdot \{\hat{n} \times (\hat{n} \times \hat{m})\} = 0 \), and \( \hat{n} \cdot \hat{n} = -1 \) are used. As previously mentioned, the direction cosines of the unit vector of the relative velocity are invariant under the Lorentz transformation, such as, \( \hat{n} \cdot \hat{i} = \hat{n}' \cdot \hat{i}' = -n_x \),
\[ \hat{n} \cdot \hat{j} = \hat{n}' \cdot \hat{j}' = -n_y, \] and \( \hat{n} \cdot \hat{k} = \hat{n}' \cdot \hat{k}' = -n_z. \] This can be generalized that the angle between two unit vectors is invariant under the Lorentz transformation, that is, \( \hat{n}' \cdot \hat{l}' = \hat{n} \cdot \hat{l}. \) Hence the magnitude of the primed momentum is \( p' = m \sinh \beta. \) Therefore the momentum vector of the particle in the primed coordinate system agrees to Eq. (61). The difference between the two unit vectors \( \hat{l}' \) and \( \hat{l} \) is similar to that between the unit vectors \( \hat{n}' \) and \( \hat{n} \) of the relative velocities.

Since the direction cosines of an unit vector is invariant under the Lorentz transformation, this can be regarded as

\[ \hat{l}' = l_x \hat{l}' + l_y \hat{j}' + l_z \hat{k}' = \hat{l} + (\hat{n} \cdot \hat{l})(\hat{n} \cdot \hat{n}'), \]

and thus, the angle between two unit vectors is calculated to be invariant under the Lorentz transformation as follows

\[ \hat{l}' \cdot \hat{n}' = [\hat{l} - (\hat{n} \cdot \hat{l})(\hat{n} \cdot \hat{n})] \cdot (\hat{n} \cosh \vartheta + \hat{n} \sinh \vartheta) = \hat{l} \cdot \hat{n}. \]

More elegant form of the transformation of the unit vector can be written by

\[ \hat{l}' + (\hat{n}' \cdot \hat{l}') \hat{n}' = \hat{l} + (\hat{n} \cdot \hat{l}) \hat{n}, \]

or \( \hat{n}' \times (\hat{n}' \times \hat{l}') = \hat{n} \times (\hat{n} \times \hat{l}), \]

which means that the normal component of the unit vector \( \hat{l} \) with respect to the unit vector of the relative velocity is the same in both frames. Therefore Eqs. (64) and (65) can be regarded as the transformation rules for an unit vector. The invariant quantity for an unit vector is

\[ \hat{l}' \cdot \hat{l}' = \hat{l} \cdot \hat{l} = -1, \]

under the Lorentz transformation. This is an inevitable condition for relativity, because all the other inertial frames should have the same measures, namely, units of time and length as the inertial frame where we live.

So using the transformation rules for an unit vector, the transverse and the normal components of the momentum in the primed frame are written in terms of the variables in the unprimed frame by

\[ \hat{n}' \cdot \rho' = \cosh \vartheta \{ \hat{n} \cdot \rho - \frac{1}{\gamma} \hat{n} \times (\hat{n} \times \rho) \}, \]

\[ \hat{n}' \times (\hat{n}' \times \rho') = \cosh \vartheta \{ \hat{n} \times \{ \hat{n} \times (\hat{n} \rho + \frac{1}{\gamma} \hat{n} \times (\hat{n} \times \rho)) \} \}, \]

where the following equation obtained from Eq. (62):

\[ \rho' \hat{l}' = \cosh \vartheta \{ \rho \rho - \frac{1}{\gamma} \hat{n} \times (\hat{n} \times \rho) \} \]

\[ = \rho - (\gamma - 1)(\hat{n} \cdot \rho) \hat{n} - \gamma E, \]

is used, which is nothing but the component equation and agrees to the form of Eq. (60). If these equations are divided by the primed energy, another form of the velocity addition rule is obtained as follows

\[ \hat{n}' \cdot \mathbf{u}' = \frac{\hat{n} \cdot \{ \mathbf{u} - v + \frac{1}{\gamma} \hat{n} \times (\hat{n} \times \mathbf{u}) \} \cdot 1 + v \cdot \mathbf{u}}{1 + v \cdot \mathbf{u}} = \frac{\hat{n} \cdot (\mathbf{u} - v)}{1 + v \cdot \mathbf{u}}, \]

\[ \hat{n}' \times (\hat{n}' \times \mathbf{u}') = \frac{\hat{n} \times \{ \mathbf{u} - v + \frac{1}{\gamma} \hat{n} \times (\hat{n} \times \mathbf{u}) \} \cdot 1 + v \cdot \mathbf{u}}{1 + v \cdot \mathbf{u}} = \frac{\hat{n} \times (\hat{n} \times \mathbf{u})}{\gamma(1 + v \cdot \mathbf{u})}, \]

which should be compared with Eq. (61). These equations agree to Eq. (61), if the irrelevant terms are dropped out. Since the two components of the velocity in the primed coordinate system are described as the same form in terms of the unprimed physical quantities, the magnitude of the velocity of the particle in the primed coordinate system can be regarded as

\[ u' = \sqrt{1 - \frac{(1 - u^2)(1 - u^2)}{(1 + v \cdot \mathbf{u})^2}} \leq 1, \]
where the velocity vector of the particle in the primed coordinate systems can be expressed as this equation without the symbol of the absolute value. Of course, this velocity vector is also not the real primed velocity of the particle, but the vector which has unprimed unit vector like Eq. (68).

The scalar product of two different physical quantities is invariant under the Lorentz transformation, though their new relative velocities are different from each other. The new relative velocity in the coordinate transformation is different from that in the momentum transformation; it is not difficult to calculate the invariant quantity like followings

\[ p'_\mu s'^\mu = E't' + p' \cdot r' = Et + p \cdot r = p_\mu s'^\mu, \]  

where the following calculations are used:

\[ \mathbf{v}_r \cdot \mathbf{v}_p t E = (\hat{n} \cdot \mathbf{r})(\hat{n} \cdot \mathbf{p}) \sinh^2 \vartheta - tE \sinh^2 \vartheta, \]

\[ \mathbf{v}_r \cdot \mathbf{p} t = t(\hat{n} \cdot \mathbf{p}) \cosh \vartheta \sinh \vartheta + (\hat{n} \cdot \mathbf{r})(\hat{n} \cdot \mathbf{p}) \sinh^2 \vartheta, \]

\[ \mathbf{v}_p \cdot \mathbf{r} E = E(\hat{n} \cdot \mathbf{r}) \cosh \vartheta \sinh \vartheta + (\hat{n} \cdot \mathbf{r})(\hat{n} \cdot \mathbf{p}) \sinh^2 \vartheta, \]  

where subscripts mean the kind of new relative velocities.

Like infinitesimal displacement vectors, infinitesimal displacement of the momentum vectors are transformed as

\[ dp' = dp - v dE, \]

\[ dE' = dE + v dE, \]  

where the new relative velocity is calculated as

\[ vdE = \mathbf{v}' dE' = \hat{n} \cdot \mathbf{d}p \sinh \vartheta \hat{h}' + dE \sinh \vartheta \hat{n}', \]

\[ = \hat{n}' \cdot \mathbf{d}p' \sinh \vartheta \hat{h} + dE' \sinh \vartheta \hat{n}. \]  

The scalar products of \( \hat{h} \) and \( \hat{n} \) to the new relative velocity generate the following two Lorentz transformation equations and the double vector products of the primed unit vector of the relative velocity to the infinitesimal momentum displacement vector can be calculated as

\[ dE' = dE \cosh \vartheta + \hat{n} \cdot \mathbf{d}p \sinh \vartheta, \]

\[ \hat{n}' \cdot \mathbf{d}p' = dE \sinh \vartheta + \hat{n} \cdot \mathbf{d}p \cosh \vartheta, \]

\[ \hat{n}' \times (\hat{n}' \times \mathbf{d}p') = \hat{n} \times (\hat{n} \times \mathbf{d}p). \]  

Since a force is defined as the time derivative of a momentum, the transformation of the force in vector form is calculated as

\[ F' = \frac{dp'}{dt'} = \frac{dp - v dE}{\gamma dt - \mathbf{v}' \cdot d\mathbf{r}}, \]

\[ = \frac{F - v \frac{dE}{dt}}{\gamma (1 + v \cdot \mathbf{u})} = \frac{F - v \frac{dE}{dt}}{\gamma (1 + v \cdot \mathbf{p}/E)}. \]  

The power is transformed as

\[ P' = \frac{dE'}{dt'} = \frac{dE + \mathbf{v} \cdot F}{1 + v \cdot \mathbf{u}} = \frac{P + \mathbf{v} \cdot F}{1 + v \cdot \mathbf{u}}. \]  

The parallel and perpendicular components of the force to the relative velocity are transformed as

\[ \hat{n}' \cdot F' = \frac{\hat{n} \cdot F - v \frac{dE}{dt}}{1 + v \cdot \mathbf{u}}, \]

\[ \hat{n}' \times (\hat{n}' \times F') = \frac{\hat{n} \times (\hat{n} \times F)}{\gamma (1 + v \cdot \mathbf{u})}. \]  

Since derivatives of coordinates are transformed like a four vector, the separate vector forms of them can be written by

\[ \square = \frac{\partial}{\partial t} \hat{h} + \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} = \frac{\partial}{\partial t} \hat{h} + \nabla \]

\[ = \frac{\partial}{\partial t} \hat{h} - \frac{\partial}{\partial x} \hat{i} - \frac{\partial}{\partial y} \hat{j} - \frac{\partial}{\partial z} \hat{k} = \frac{\partial}{\partial t} \hat{h} - \nabla, \]  

where the following calculations are used:

\[ \mathbf{v}_r \cdot \mathbf{v}_p t E = (\hat{n} \cdot \mathbf{r})(\hat{n} \cdot \mathbf{p}) \sinh^2 \vartheta - tE \sinh^2 \vartheta, \]

\[ \mathbf{v}_r \cdot \mathbf{p} t = t(\hat{n} \cdot \mathbf{p}) \cosh \vartheta \sinh \vartheta + (\hat{n} \cdot \mathbf{r})(\hat{n} \cdot \mathbf{p}) \sinh^2 \vartheta, \]

\[ \mathbf{v}_p \cdot \mathbf{r} E = E(\hat{n} \cdot \mathbf{r}) \cosh \vartheta \sinh \vartheta + (\hat{n} \cdot \mathbf{r})(\hat{n} \cdot \mathbf{p}) \sinh^2 \vartheta, \]  

where subscripts mean the kind of new relative velocities.
where the checked unit vectors mean contravariant vectors. Since we are more familiar with a gradient operator with covariant unit vectors than contravariant ones, the minus sign should appear before the gradient operator. Using chain rule the vector forms of the derivatives transformation are obtained as

$$\nabla' = \nabla + \frac{\partial}{\partial t} \hat{h}',$$

$$\frac{\partial}{\partial t} \hat{h}' = \frac{\partial}{\partial t} \hat{h} + \nu \frac{\partial}{\partial t},$$  \hspace{1cm} (80)

where all the unit vectors are used with covariant vectors, because it is easy to calculate scalar and vector products with other physical quantities which almost have covariant unit vectors. The new relative velocity in the transformation of derivatives is calculated as

$$\rho' = \rho' \nabla' = \hat{n} \cdot \nabla' = \hat{n} \cdot \nabla + \rho \sinh \theta,$$

$$\hat{n}' \cdot \nabla' = \rho' \nabla' = \hat{n} \cdot \nabla + \rho \sinh \theta,$$

$$\hat{n} \times (\hat{n} \times \nabla') = \hat{n} \times (\hat{n} \times \nabla).$$ \hspace{1cm} (82)

The invariant quantity for the derivatives is the D’Alembertian:

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 = \Box,$$  \hspace{1cm} (83)

where the D’Alembertian is used with the squared quantity, because it is necessary to distinguish it from the four dimensional gradient operator.

If a charge and a current densities together transform like a four vector under the Lorentz transformation, they can be written in vector form by

$$j' = j - \rho \nu,$$

$$\rho' = \rho + \rho \nu,$$  \hspace{1cm} (84)

where the new relative velocity for the charge and current densities can be calculated as

$$\rho' = \rho' \nabla' = \hat{n} \cdot \nabla' = \hat{n} \cdot \nabla + \rho \sinh \theta,$$

$$\hat{n}' \cdot \nabla' = \rho' \nabla' = \hat{n} \cdot \nabla + \rho \sinh \theta,$$

$$\hat{n} \times (\hat{n} \times \nabla') = \hat{n} \times (\hat{n} \times \nabla).$$\hspace{1cm} (85)

The scalar products of the unit vectors $\hat{h}$ and $\hat{n}$ to the new relative velocity and the double vector products of $\hat{n}'$ to the primed current vector give

$$\rho' = \rho' \nabla' = \hat{n} \cdot \nabla' = \hat{n} \cdot \nabla + \rho \sinh \theta,$$

$$\hat{n}' \cdot \nabla' = \rho' \nabla' = \hat{n} \cdot \nabla + \rho \sinh \theta,$$

$$\hat{n} \times (\hat{n} \times \nabla') = \hat{n} \times (\hat{n} \times \nabla).$$ \hspace{1cm} (86)

The invariant quantity for the charge and current densities can be calculated as

$$J^\mu J'_\mu = \rho^2 + j' \cdot j' = \rho^2 (1 - u'^2) = \rho^2 (1 - u^2) = \rho^2 + j \cdot j = J^\mu J_\mu,$$ \hspace{1cm} (87)

where the velocities in the parentheses mean $u' = j'/\rho'$ and $u = j/\rho$. Using the two kinds of the transformation equations for the derivatives and the charge-current density the divergence of a current can be calculated to be an invariant quantity as

$$\partial_\mu J'^\mu = \partial_\mu \rho' - \nabla' \cdot j' = \partial_\mu \rho - \nabla \cdot j = \partial_\mu J^\mu,$$ \hspace{1cm} (88)
where this continuity equation also can be obtained from the total derivative of a charge density with respect to time as
\[
\frac{d}{dt} \rho = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \frac{dx}{dt} + \frac{\partial \rho}{\partial y} \frac{dy}{dt} + \frac{\partial \rho}{\partial z} \frac{dz}{dt} = \partial_0 \rho - \nabla \cdot \mathbf{j}. \tag{89}
\]

Therefore if the charge density is time independent in an inertial frame, we know from the above two equations that the charge is conserved in all the other inertial frames.

As shown above, if a scalar and a vector potentials together transform like a four vector under the Lorentz transformation, which can be thought to be come from the following covariant propagator
\[
A^\mu(r) = \int d^4r' D(r - r')J^\mu(r'), \tag{90}
\]
where the propagator \(D(r - r')\) should satisfy the wave equation for the electromagnetic field, then its transformation property is the same as the charge-current density. The vector forms of the scalar and the vector potentials are written by
\[
A' = A - \phi \mathbf{v},
\phi' = \phi + \phi \mathbf{v}, \tag{91}
\]
where the new relative velocity is
\[
\phi \mathbf{v} = \phi' \mathbf{v}' = \mathbf{\hat{n}} \cdot \mathbf{A} \sinh \vartheta \mathbf{\hat{n}}' + \phi \sinh \vartheta \mathbf{\hat{n}}', \nonumber
= \mathbf{\hat{n}}' \cdot \mathbf{A}' \sinh \vartheta \mathbf{\hat{n}} + \phi' \sinh \vartheta \mathbf{\hat{n}}. \tag{92}
\]

The scalar products of the unit vectors \(\mathbf{\hat{n}}\) and \(\mathbf{\hat{n}}'\) to the new relative velocity and the double vector products of \(\mathbf{\hat{n}}'\) to the primed vector potential give
\[
\phi' = \phi \cosh \vartheta + \mathbf{\hat{n}} \cdot \mathbf{A} \sinh \vartheta, \nonumber
\mathbf{\hat{n}}' \cdot \mathbf{A}' = \phi \sinh \vartheta + \mathbf{\hat{n}} \cdot \mathbf{A} \cosh \vartheta, \nonumber
\mathbf{\hat{n}}' \times (\mathbf{\hat{n}}' \times \mathbf{A}') = \mathbf{\hat{n}} \times (\mathbf{\hat{n}} \times \mathbf{A}). \tag{93}
\]

The divergence of the scalar and vector potentials is an invariant quantity under the Lorentz transformation which is nothing but the Lorentz gauge:
\[
\partial_\mu A'^\mu = \partial_0 \phi' - \nabla' \cdot \mathbf{A}' = \partial_0 \phi - \nabla \cdot \mathbf{A} = \partial_\mu A'^\mu. \tag{94}
\]

The square of the sum or subtraction of two different physical quantities is invariant under the Lorentz transformation, if the dimensions and the transformation properties of the two physical quantities are equal to each other. As shown above the energy-momentum and the scalar-vector potential have the same transformation properties. So the multiplication of the electromagnetic coupling constant, which is a scalar under the Lorentz transformation, to the scalar and vector potentials have the same dimensions as the energy-momentum relation. The sums of the two physical quantities transform as follows
\[
E' - e\phi' = (E - e\phi) \cosh \vartheta + (\mathbf{\hat{n}} \cdot \mathbf{p} - e\mathbf{\hat{n}} \cdot \mathbf{A}) \sinh \vartheta, \nonumber
\mathbf{p}' - e\mathbf{A}' = \mathbf{p} - e\mathbf{A} - \mathbf{v}_p E + e\mathbf{v}_A \phi, \tag{95}
\]
where the sum of the two new relative velocities is
\[
E\mathbf{v}_p - e\phi \mathbf{v}_A = \{(\mathbf{\hat{n}} \cdot \mathbf{p} - e\mathbf{\hat{n}} \cdot \mathbf{A}) \mathbf{\hat{h}}' + (E - e\phi) \mathbf{\hat{n}}'\} \sinh \vartheta \nonumber
= \{(\mathbf{\hat{n}}' \cdot \mathbf{p}' - e\mathbf{\hat{n}}' \cdot \mathbf{A}') \mathbf{\hat{h}} + (E' - e\phi') \mathbf{\hat{n}}\} \sinh \vartheta. \tag{96}
\]

Using the new relative velocity the following quantity is calculated to be invariant under the Lorentz transformation as
\[
(E' - e\phi')^2 + (\mathbf{p}' - e\mathbf{A}') \cdot (\mathbf{p}' - e\mathbf{A}') = (E - e\phi)^2 + (\mathbf{p} - e\mathbf{A}) \cdot (\mathbf{p} - e\mathbf{A}) = M^2, \tag{97}
\]
which corresponds that an equation of motion for a particle in the electromagnetic fields is invariant under the Lorentz transformation, if the invariant quantity is set to be the mass squared of the particle as shown in gauge theories.
The γ matrices used in the Dirac equation are transformed like the basis vectors under the Lorentz transformation, because \((\gamma^0)^2 = 1\) and \((\gamma^1)^2 = -1\). Using the direction cosines of the relative velocity the γ matrix for the direction of the relative velocity can be defined as
\[
\gamma_n = n_x \gamma_1 + n_y \gamma_2 + n_z \gamma_3.
\] (98)

The Lorentz transformation for the γ matrices is similar to the basis vectors as follows
\[
\gamma'_0 = \gamma_0 \cosh \theta + \gamma_n \sinh \theta,
\gamma'_n = \gamma_n \cosh \theta + \gamma_0 \sinh \theta.
\] (99)

The γ matrices for the space parts are transformed like an unit vector as follows
\[
\gamma'_i \cdot \gamma'_n = \gamma_i \cdot \gamma_n,
\gamma'_i = \gamma_i + (\gamma_i \cdot \gamma_n)(\gamma_n - \gamma'_n).
\] (100)

The scalar product of the γ matrices to the above sum of the two different physical quantities is invariant as follows
\[
\gamma^0(E' - e\phi') + \gamma' \cdot (p' - eA') = \gamma^0(E - e\phi) + \gamma \cdot (p - eA).
\] (101)

This equation is not proved, but rather defined as all the other polar vectors in this paper in a sense that the basis vectors are chosen in the primed coordinate system in order to have the same direction cosines of the primed relative velocity as those of the unprimed relative velocity. Therefore there is no new relative velocity for the basis vectors. Like the above calculations the invariance of the square of sum of several energy-momentum relations is useful in the energy momentum conservation for many physical reaction processes.

Hitherto polar vectors are investigated under the Lorentz transformation, they all have the similar transformation properties. It is an interesting question whether an axial vector has the same transformation property or not. An angular momentum vector is the very axial vector. Since the definition of an angular momentum vector consists of a position vector and a momentum vector for which we know well the Lorentz transformation properties, by using the prescription for a vector product and the transformation matrices for the coordinate and the momentum, an angular momentum in the primed frame is calculated as
\[
L' = r' \times p'
= (y'p'_z - z'p'_y)i' + (z'p'_x - x'p'_z)j' + (x'p'_y - y'p'_x)k'
= L + (\hat{n} \cdot L)(\hat{n} - \hat{n}') + (\hat{n} \times p)(\hat{n} \cdot r) - (\hat{n} \times r)(\hat{n} \cdot p) - (\hat{n} \times p)(\hat{n}' \cdot r') + (\hat{n} \times r)(\hat{n}' \cdot p'),
\] (102)

where it is easy to calculate the transformation for \(\hat{n}' \cdot L' = \hat{n} \cdot L\) by replacing the basis vectors in the above equation with the direction cosines of \(\hat{n}'\). At first view the angular momentum seems to be transformed like an unit vector under the Lorentz transformation, but does not so because it is difficult to confirm that \(\hat{n}' \times (\hat{n}' \times L') = \hat{n} \times (\hat{n} \times L)\). The best way in this situation is to calculate the transformation property of the magnitude of the angular momentum:
\[
L'^2 = -L' \cdot L' = -L \cdot L + (Er - tp)^2 - (E'r' - t'p')^2,
\] (103)

where the following invariant relation:
\[
\hat{n}' \cdot (E'r' - t'p') = \hat{n} \cdot (Er - tp),
\] (104)

is used, which is similar to the equation \(\hat{n}' \cdot L' = \hat{n} \cdot L\). The magnitude of an angular momentum is not an invariant quantity in relativity, but invariant with \((Er - tp)^2\). So it is necessary to calculate the transformation property of this physical quantity as follows
\[
E'r' - t'p' = (Er - tp) \cosh \vartheta - \hat{n} \cdot (Er - tp) \sinh \vartheta \hat{n} + \hat{n} \times L \sinh \vartheta.
\] (105)

This calculation shows some hints on the transformation property for the angular momentum, but not satisfactory. Since a double cross product physical quantity can be represented as scalar product physical quantities, the following quantity for an angular momentum can be transformed properly under the Lorentz transformation as
\[
\hat{n}' \times L' = \hat{n}' \times (r' \times p') = -r'(\hat{n}' \cdot p') + p'(\hat{n}' \cdot r')
= \hat{n} \times L \cosh \vartheta + \hat{n} \times [\hat{n} \times (Er - tp)] \sinh \vartheta,
\] (106)
where the last term has the following transformation property:
\[
\hat{n}' \times [\hat{n}' \times (E'r' - t'p')] = \hat{n} \times [\hat{n} \times (Er - tp)] \cosh \vartheta + \hat{n} \times \mathbf{L} \sinh \vartheta.
\] (107)

It is interesting that this transformation should be compared with Eq. (105). Thus the above two transformation equations and the invariance of the transverse components of the angular momentum and \(Er - tp\) are regarded as the transformation rules for the angular momentum. Therefore the invariant quantity for the angular momentum is
\[
\begin{align*}
(n' \times n')^2 - [n' \times (n' \times (E' r' - t' p'))]^2 &= (n \times n \times (E - t p))^2 - [n \times (n \times (E - t p))]^2 \\
&= \mathbf{L} \cdot \mathbf{L} + (n \times n \times (E r - t p)) - (E r - t p) \cdot (E r - t p) - \{n \times (E r - t p)\}^2
\end{align*}
\] (108)

where the squared quantities mean the scalar product of the vectors in the first two lines. This agrees to Eq. (103), if the transverse components are dropped out.

From the transformation property of the angular momentum, the transformation for a torque can be calculated as
\[
\begin{align*}
\hat{n}' \cdot \mathbf{\tau}' &= \hat{n}' \cdot \frac{d \mathbf{L}'}{d t'} = \frac{\hat{n}' \cdot \mathbf{\tau}}{\gamma (1 + \mathbf{v} \cdot \mathbf{u})}, \\
\hat{n}' \times \mathbf{\tau}' &= \hat{n}' \times \frac{d \mathbf{L}'}{d t'} = \frac{\hat{n} \times \mathbf{\tau} + \hat{n} \times \{P(v \times r) - t(v \times \mathbf{F})\}}{1 + \mathbf{v} \cdot \mathbf{u}}.
\end{align*}
\] (109)

It is inevitable to include the counterpart of the angular momentum in the transformation rules for the torque:
\[
\begin{align*}
\hat{n}' \cdot (P'r' - t'\mathbf{F}') &= \frac{\hat{n}' \cdot (P r - t \mathbf{F})}{\gamma (1 + \mathbf{v} \cdot \mathbf{u})}, \\
\hat{n}' \times \{\hat{n}' \times (P'r' - t'\mathbf{F}')\} &= \frac{\hat{n} \times \{P r - t \mathbf{F} + v \times \mathbf{L}\}}{1 + \mathbf{v} \cdot \mathbf{u}}.
\end{align*}
\] (110)

The electric and magnetic fields are the very axial vectors like an angular momentum in electromagnetism, which are written by
\[
\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t},
\] (111)

where the difference between the electromagnetic fields and the angular momentum is only the minus sign before the gradient operator due to its covariant vector nature. So they transform like an axial vector under the Lorentz transformation as follows
\[
\begin{align*}
\hat{n}' \cdot \mathbf{B}' &= \hat{n} \cdot \mathbf{B}, \\
\hat{n}' \cdot \mathbf{E}' &= \hat{n} \cdot \mathbf{E}, \\
\hat{n}' \times \mathbf{B}' &= \hat{n} \times \mathbf{B} \cosh \vartheta - \hat{n} \times (\hat{n} \times \mathbf{E}) \sinh \vartheta, \\
\hat{n}' \times (\hat{n}' \times \mathbf{E}') &= \hat{n} \times (\hat{n} \times \mathbf{E}) \cosh \vartheta - \hat{n} \times \mathbf{B} \sinh \vartheta.
\end{align*}
\] (112)

This transformation shows that the origin of a magnetic field is due to the Lorentz transformation [3], and is consistent with the Biot-Savart law. From this transformation property, the invariant quantity for the electromagnetic field can be obtained as
\[
(\hat{n}' \times \mathbf{B}')^2 = [\hat{n}' \times (\hat{n}' \times \mathbf{E}')]^2 = (\hat{n} \times \mathbf{B})^2 - \{\hat{n} \times (\hat{n} \times \mathbf{E})\}^2
\] (113)

Since the transverse components of the electric and magnetic fields are invariant under the Lorentz transformation, if they are dropped out, the remaining parts are also invariant as follows
\[
B^2 - E^2 = B^2 - E^2 = \frac{1}{2} F_{\mu \nu} F^{\mu \nu},
\] (114)

which is known as the norm of electromagnetic 2-form or Faraday [4]. This appears also in the Lagrangian for an electromagnetic interaction as a photon field.
IV. GEOMETRICAL ASPECTS OF THE LORENTZ TRANSFORMATION

The three angles $\alpha$, $\beta$, and $\vartheta$ mentioned for the transformation in momentum space in the previous section make a triangle in a hyperbolic space like Fig. 2, whose vertices are the origins of the observing two frames $O$, and $O'$, and the position of an observed particle $P$. While the three angles $\alpha$, $\beta$, and $\vartheta$ are global parameters, the unit vectors $\hat{l}$, $\hat{m}$ and $\hat{n}$ are defined locally at each vertex which are tangential directions on a hyperbolic sphere. So these unit vectors are designated by unprimed vectors in the rest frame, primed vectors in the moving frame and double primed vectors in the reference frame of the particle.

![Diagram](image)

FIG. 2: The three vertices of the hyperbolic triangle are the positions of the origin of the primed frame, the origin of the unprimed frame and the particle according to the angle $A$, $B$ and $C$. The unit vector $\hat{l}_r$ can be regarded as the complete rotated vector of $\hat{l}$ around the triangle.

An observer in the rest frame sees the origin of the moving frame with the velocity $v = \hat{n} = \hat{n} \tanh \vartheta$ and the moving particle with the momentum $p = p\hat{n} = m\hat{n} \sinh \alpha$ whose velocity is $u = p/E = \hat{m} \tanh \alpha$. The angle between the two velocities $v$, and $u$ can be defined as $\cos B = -\hat{n} \cdot \hat{m}$. An observer in the moving frame sees the origin of the rest frame with the velocity $v' = -v\hat{n}' = -\hat{n}' \tanh \vartheta$ and the moving particle with the momentum $p' = p\hat{n}' = m\hat{n}' \sinh \beta$ whose velocity is $u' = p'/E' = \hat{l}' \tanh \beta$. The angle between the two velocities $v'$, and $u'$ can be defined as $\cos A = \hat{n}' \cdot \hat{l}'$. Since the magnitude of the relative velocity between two origins of reference frames is observed to be the same in both reference frames, an observer in the frame of the moving particle sees the origin of the rest frame with the velocity $u''_r = -\hat{m}'' \tanh \alpha$ and the origin of the moving frame with the velocity $u''_m = -\hat{l}'' \tanh \beta$, respectively. The angle between the two velocities $u''_r$ and $u''_m$ can be defined as $\cos C = -\hat{m}'' \cdot \hat{l}''$.

The law of cosines in hyperbolic trigonometry shown in Appendix B can be applied to this triangle as

\[
\begin{align*}
\cosh \alpha &= \cosh \beta \cosh \vartheta - \hat{n}' \cdot \hat{l}' \sinh \beta \sinh \vartheta, \\
\cosh \beta &= \cosh \alpha \cosh \vartheta + \hat{n} \cdot \hat{m} \sinh \alpha \sinh \vartheta, \\
\cosh \vartheta &= \cosh \alpha \cosh \beta + \hat{m}'' \cdot \hat{l}'' \sinh \alpha \sinh \beta.
\end{align*}
\]

(115)

If the mass of the observed particle is multiplied to the above equations, these equations are nothing but the Lorentz transformation as follows

\[
E = E' \cosh \vartheta - \hat{n}' \cdot p' \sinh \vartheta,
\]
where the last equation is obtained from the calculation of \( \cos C = -\hat{m}'' \cdot \hat{l}'' \) by using the transformation equations:

\[
\begin{align*}
\hat{m}'' &= \hat{m} \cos \alpha + \hat{h} \sinh \alpha, \\
\hat{l}'' &= \hat{l} \cosh \beta + \hat{h'} \sin \beta.
\end{align*}
\]

These transformations are explained again later. Since the unit vector \( \hat{l}'' \) is the transformed vector from the primed coordinate system, it is worth to note that \( \hat{m}'' \cdot \hat{l}'' = \hat{m} \cdot \hat{l} \neq \hat{m} \cdot \hat{l} \) compared to \( \hat{m} \cdot \hat{l} = \hat{m} \cdot \hat{l} \) in the previous section. The reason is shown in Fig. If the cosines of the angles are replaced by the unit vectors defined above, then the following relations are obtained

\[
\begin{align*}
(E - E')^2 + (p - p')^2 + 2EE'(1 - \cosh \vartheta) &= 0, \\
or \quad p \cdot p' &= -p^2 - \sinh \vartheta E' \hat{n} \cdot \hat{p} = p \cdot (p - vE),
\end{align*}
\]

where the last equation agrees to Eq. (53) and is of importance to check whether transformed formulas are correct or not. The first two equations in the above Lorentz transformation give the remaining partners of the Lorentz transformation by inserting each of them into the other equation as follows

\[
\begin{align*}
\hat{n} \cdot \hat{p} &= \hat{n}' \cdot \hat{p}' \cosh \vartheta - E' \sinh \vartheta, \\
\hat{n}' \cdot \hat{p}' &= \hat{n} \cdot \hat{p} \cosh \vartheta + E \sinh \vartheta.
\end{align*}
\]

Therefore the Lorentz transformation for an energy-momentum can be interpreted geometrically as the law of cosines for the hyperbolic triangle. The law of cosines for the corresponding polar triangle is applied to the hyperbolic triangle as

\[
\begin{align*}
\cos A &= -\cos B \cos C + \sin B \sin C \cosh \alpha, \\
\cos B &= -\cos A \cos C + \sin A \sin C \cosh \beta, \\
\cos C &= -\cos A \cos B + \sin A \sin B \cosh \vartheta.
\end{align*}
\]

If the cosines of the angles are replaced by the unit vectors defined above, then the following relations are obtained

\[
\begin{align*}
\cos A &= \hat{n}' \cdot \hat{l}' = \hat{n}'' \cdot \hat{l}'' - (\hat{m}'' \times \hat{n}'') \cdot (\hat{m}'' \times \hat{l}'') + \sin B \sin C \cosh \alpha, \\
\cos B &= -\hat{n} \cdot \hat{m} = -\hat{n}'' \cdot \hat{m}'' + (\hat{l}'' \times \hat{n}'' \cdot \hat{l}'' \times \hat{n}'') + \sin A \sin C \cosh \beta, \\
\cos C &= -\hat{m}'' \cdot \hat{l}'' = -\hat{m} \cdot \hat{l} + (\hat{n} \times \hat{l}) \cdot (\hat{n} \times \hat{m}) + \sin A \sin B \cosh \vartheta = -\hat{m} \cdot \hat{l}' + (\hat{n} \times \hat{l}') \cdot (\hat{n} \times \hat{m}') + \sin A \sin B \cosh \vartheta.
\end{align*}
\]

These equations are of importance for the relations among the three angles \( A, B \) and \( C \) and play a crucial role for the derivation of the law of sine just later.

There are three pairs of the Lorentz transformation among the three inertial frames. In order to obtain another form of the law of sine for the hyperbolic triangle, it is necessary to remind the Lorentz transformation for the unit vectors among the three inertial frames. The Lorentz transformation for the unit vectors of the moving frames is again written as shown in previous section by

\[
\begin{align*}
\hat{n}' &= \hat{n} \cosh \vartheta + \hat{h} \sinh \vartheta, \\
\hat{l}' &= \hat{l} \cosh \vartheta + \hat{h'} \sin \vartheta.
\end{align*}
\]

The Lorentz transformation for the unit vectors of the reference frame of the particle can be written in terms of the unit vectors of the moving frame by

\[
\begin{align*}
\hat{l}'' &= \hat{l} \cosh \beta + \hat{h'} \sin \beta, \\
\hat{n}'' &= \hat{n} \cosh \beta + \hat{l} \sinh \beta,
\end{align*}
\]

because the relative velocity between the two frames is the velocity of the particle observed in the moving frame. The Lorentz transformation for the unit vectors of the reference frame of the particle can be written in terms of the unit vectors of the rest frame by

\[
\begin{align*}
\hat{m}'' &= \hat{m} \cosh \alpha + \hat{h} \sinh \alpha, \\
\hat{h}'' &= \hat{h} \cosh \alpha + \hat{m} \sinh \alpha,
\end{align*}
\]
multiplied to the law of sine, the result is the primed momentum vector described in unprimed coordinate system

\[ \hat{h} \cosh \alpha + \hat{m} \sinh \alpha = \hat{h}' \cosh \beta + \hat{l}' \sinh \beta, \]  

(125)

which means that every time axis of all inertial frames has the same nature as that of the inertial frame where we live, though it is rotated through the Lorentz transformation, as if the heavens look the same everywhere on the earth in regards to a spherical space. The reason can be thought that, while the time axis is only one, the space axes are three so that the Lorentz transformation causes mutual rotations among the space axes like the unit vectors \( l_r \) and \( l \) as shown in Fig. [2].

The scalar product of \( \hat{h} \) to the identity yields

\[ \cosh \alpha = \hat{h} \cdot \hat{h}' \cosh \beta + \hat{h} \cdot \hat{l}' \sinh \beta = \cosh \beta \cosh \vartheta - \cos A \sinh \beta \sinh \vartheta, \]

(126)

which is the law of cosine for the angle \( \alpha \). The scalar product of \( \hat{h}' \) to the identity yields

\[ \cosh \beta = \hat{h}' \cdot \hat{h} \cosh \alpha + \hat{h}' \cdot \hat{m} \sinh \alpha = \cosh \alpha \cosh \vartheta - \cos B \sinh \alpha \sinh \vartheta, \]

(127)

which is the law of cosine for the angle \( \beta \). These two laws of cosine agree to Eq. (113). The scalar product of \( \hat{l}' \) to the identity gives

\[ \hat{l}' \cdot \hat{l}' \sinh \beta = \hat{l}' \cdot \hat{h} \cosh \alpha + \hat{l}' \cdot \hat{m} \sinh \alpha = -\hat{l}' \cdot \hat{n}' \cosh \alpha \sinh \vartheta + \hat{l}' \cdot \hat{m} \sinh \alpha. \]

(128)

If a mass is multiplied to the equation, the resulting equation is written by

\[ \hat{l}' \cdot \hat{p}' = \hat{l}' \cdot (\hat{p} - \hat{n}' E \sinh \vartheta) = \hat{l}' \cdot (\hat{p} - \hat{v} E), \]

(129)

where \(-\hat{h}'(\hat{n} \cdot \hat{p}) \sinh \vartheta\) is added because the space unit vector is orthogonal to the unit vector of time. The law of sine for the hyperbolic triangle means the Lorentz transformation for a momentum vector, because the magnitude of a momentum can be expressed as a hyperbolic sine. This equation is further calculated as

\[ \hat{l}' \cdot \hat{l}' \sinh \beta = \hat{l} \cdot \hat{l} \sinh \beta = \hat{l} \cdot \hat{m} \cosh \vartheta \sinh \alpha - \hat{l} \cdot \hat{n} \cosh \alpha \sinh \vartheta - (\hat{n} \times \hat{m}) \cdot (\hat{n} \times \hat{l}) \cosh \vartheta - 1) \sinh \alpha, \]

(130)

where the law of cosines for the polar triangle and the following calculation:

\[ \hat{l}' \cdot \hat{m} = (\hat{n} \times \hat{m}) \cdot (\hat{n} \times \hat{l}) - (\hat{n} \cdot \hat{m}) (\hat{n} \cdot \hat{m}) \cosh \vartheta, \]

(131)

are used. Then the hyperbolic sine is written again in vector form by

\[ \hat{l} \sinh \beta = \hat{m} \cosh \vartheta \sinh \alpha - \hat{n} \cosh \alpha \sinh \vartheta + \hat{n} \times (\hat{n} \times \hat{m}) \cosh \vartheta - 1) \sinh \alpha. \]

(132)

This can be regarded as another form of the law of sine in hyperbolic trigonometry compared to Eq. (A4) in spherical trigonometry. So the sum of the squares of the hyperbolic cosine and sine is equal to unity as follows

\[ \cosh^2 \beta + \hat{l} \cdot \hat{l} \sinh^2 \beta = 1. \]

(133)

This also means the energy momentum relation, if a square of mass is multiplied to the equation. If a mass is multiplied to the law of sine, the result is the primed momentum vector described in unprimed coordinate system

\[ p' \hat{l} = \cosh \vartheta (\hat{p} - \hat{E} \hat{v}) + \hat{n} \times (\hat{n} \times \hat{p}) \cosh \vartheta - 1), \]

(134)
can be represented as the unit vectors transformations reduce to a pure Lorentz boost and a rotational transformation \[1, 2\]. This rotational transformation if we consider that the space time is not flat. This is related to the Thomas precession, because successive Lorentz which is not a contradiction to the invariance of angle under the Lorentz transformation as mentioned previously, the angle defect of the hyperbolic triangle in the hyperbolic space are also varying. Therefore the angular velocity for the angle defect can be obtained. This may be the Thomas precession in this geometrical interpretation.

A simple example is very illustrative for the hyperbolic triangle. Let us calculate all the angles involved in the triangle. A moving particle with a momentum \( \mathbf{p} = p \hat{\mathbf{j}} \) whose mass is detected as 1 MeV is observed in the rest frame. Another inertial frame is moving with a relative velocity \( \mathbf{v} = \hat{\mathbf{i}} \). The energy and the momentum of the particle are calculated to be observed in the moving frame as \( E' = \sqrt{E^2 + m^2} \) and \( \mathbf{p}' = p' \hat{\mathbf{l}}' = -E\sinh \vartheta \hat{\mathbf{l}}' + p \hat{\mathbf{j}} \) by using the Lorentz transformation. The three hyperbolic angles of the triangle are calculated as

\[
\begin{align*}
\tanh \alpha &= u = \frac{p}{E} = \frac{p}{\sqrt{1 + p^2}}, \\
\tanh \beta &= u' = \frac{p'}{E'} = \frac{\sqrt{p^2 + E^2 \sinh^2 \vartheta}}{E \cosh \vartheta} = \frac{1}{\sqrt{u^2 + v^2 - u^2 v^2}}, \\
\tanh \vartheta &= v.
\end{align*}
\]

The three real angles of the hyperbolic triangle are

\[
\begin{align*}
\cos A &= -\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0, \\
\cos B &= \hat{\mathbf{l}}' \cdot \hat{\mathbf{j}}' = \frac{E \sinh \vartheta}{\sqrt{p^2 + E^2 \sinh^2 \vartheta}} = \frac{v}{\sqrt{u^2 + v^2 - u^2 v^2}}, \\
\cos C &= -\hat{\mathbf{l}}'' \cdot \hat{\mathbf{j}}'' = -\cos A \cos B + \sin A \sin B \cosh \vartheta = \frac{p \cosh \vartheta}{\sqrt{p^2 + E^2 \sinh^2 \vartheta}} = \frac{u}{\sqrt{u^2 + v^2 - u^2 v^2}}.
\end{align*}
\]

If \( u = v \), the hyperbolic triangle is an equilateral right triangle so that \( \cos B = \cos C = 1/\sqrt{2 - u^2} \). Table shows some numerical calculations for this triangle from non-relativistic to relativistic regions.
| tanh α | tanh β | A   | B   | C   | 180 − A − B − C |
|-------|-------|-----|-----|-----|-----------------|
| 0.1   | 0.1   | 0.141067 | 90  | 44.856 | 44.856 | 0.28792 |
| 0.3   | 0.3   | 0.414608 | 90  | 43.6496 | 43.6496 | 2.7008  |
| 0.5   | 0.5   | 0.661438 | 90  | 40.8934 | 40.8934 | 8.21321 |
| 0.7   | 0.7   | 0.860174 | 90  | 35.5323 | 35.5323 | 18.9355 |
| 0.9   | 0.9   | 0.981784 | 90  | 23.5519 | 23.5519 | 42.8962 |
| 0.99  | 0.99  | 0.999802 | 90  | 8.02958 | 8.02958 | 73.9408 |
| 0.999 | 0.999 | 0.999998 | 90  | 2.56   | 2.56   | 84.88    |

TABLE I: The three sides and the three angles of the hyperbolic triangle are calculated from non-relativistic to relativistic regions. The angle defect runs from zero to 90 ° according to the relativistic motion of the observed particle and the relative velocity between two inertial frames.

The triangle of Fig. 2 is quite different from a triangle in a plane whose sum of the three angles is 180 °. The sum of the three angles for a hyperbolic triangle is less than 180 °, while the sum of the three angles for a spherical triangle is more than 180 °. The general equation for the sum of the three angles for a triangle is

\[ A + B + C = \pi \pm \frac{\text{area}}{R^2}, \]  

where the area means the area of the triangle and \( R \) means the radius of a sphere (\( + \)) or a hyperbolic sphere (\( - \)). The minus sign is due to the imaginary radius of the hyperbolic sphere. The curvature \( R \) is infinite in plane trigonometry. The more relativistic are the motion of an observed particle and the relative velocity, the larger is the area of the hyperbolic triangle as shown in table I. The maximum value of the angle defect is approaching to 90 ° in this example. For an equilateral triangle the maximum value of the angle defect is approaching to 180 °, which can be easily shown with the law of cosine because all the three angles and the hyperbolic angles, respectively, are equal. The angle between the relative velocities is expressed with the velocity as \( \cos A = \frac{1}{1 + \sqrt{1 - v^2}} \).

V. CONCLUSIONS

In this paper we have studied the separate notation of the vector forms of the Lorentz transformation and its geometrical aspects. The four vector notation have indeed powerful merits not only in the special theory of relativity but also in the general theory of relativity for the covariance of physical laws. However the detailed structure of space time is not seen in the four vector notation. The separate notation of four vectors can show some clues for the geometry of our universe, besides it can express the vector form of the Lorentz transformation as covariant forms. Three inertial frames whose relative velocities among them are all non-zero make a hyperbolic triangle. The sum of its three angles is less than 180 °. While the space of an inertial frame is flat which is a tangential space on a hyperbolic sphere, the space of all inertial frames is a hyperbolic space, namely, a Lobatchewsky space which are on the hyperbolic sphere. It is more easy to imagine this picture on an unit sphere rather than a hyperbolic sphere, if some results are taken into account carefully, such as, the imaginary radius of the hyperbolic sphere, metric tensor, and so on.

Naturally two inertial frames are rotated mutually with a hyperbolic angle with respect to each other. Therefore the vector equations of the Lorentz transformation between the inertial frames in the separate notation of four vectors can not be represented without mixing time and space components. Such mixed quantities are expressed in the new relative velocity. The \( \gamma \) factor appeared in the component equations of the Lorentz transformation between two frames does not appears in the vector equations, because their own unit vectors are used. This factor comes from the scalar product between the unit vectors of the two frames in the calculation of components from the vector transformation equation. It is shown that we can do with these vector equations what we can do with the components of the Lorentz transformation, such as, calculations of invariant quantities. It is possible to transform an axial vector in the four dimension under the Lorentz transformation which are represented as a tensor in the component form of the Lorentz transformation.

No matter how this paper is worthless, at least very adventurous astronauts who want to journey around the universe should know this work, if they want to return home where their grandsons or grand-grandsons will welcome them due to the time dilation of their space ship. Navigators on the sea should know that the earth is round so that their direction for the destination should be determined by spherical trigonometry according to their sailed distance.
FIG. 3: An angle vector can be defined as an arc on an unit sphere like this figure. The angle vectors $a\hat{m}, b\hat{n}$ and $c\hat{l}$ have the arcs $a, b$ and $c$ as their magnitudes and the unit vectors $\hat{m}, \hat{n}$ and $\hat{l}$ as their directions, respectively. The figure shows how to add angle vectors, that is, $c\hat{l} = a\hat{m} \oplus b\hat{n}$. The length $pq$ is equal to $\cos c$, and the length $pr$ is $\sin c$.

However navigators in the universe should know that the space is not flat so that their direction for the destination should be determined by hyperbolic trigonometry according to the velocity of their space ship.

APPENDIX A: ANGLE VECTOR

An angle has its magnitude and unique direction to be able to define. However it is not treated as a proper vector in the standard texts, on the contrary infinitesimal angles, angular velocities and angular accelerations are regarded as proper vectors. This treatment for an angle encounters a contradiction, when an angular velocity summed with two vectors or decomposed into several vectors is integrated with respect to time. This integration gives a finite angle vector. Therefore a finite angle should be defined also as a proper vector as the angular velocity and acceleration are done. The problems to make it difficult to define an angle as a vector may be the definition of the sum of two angle vectors or confusion with a rotational operation. Using spherical trigonometry, such problems are resolved.

To begin with, let us define an angle vector as a least arc arrow on an unit sphere which is naturally lying on the great circle of the sphere as shown in Fig. 3. The magnitude of the angle vector is the length of the arc and its direction is represented as an arrow on the sphere, or equivalently an unit vector at the center of the sphere perpendicular to the plane which is made of the center of the sphere and the arc arrow according to the right handed rule.

If two angle vectors are equivalent, then the magnitudes of the two angle vectors are equal to each other and they lie on the same great circle on the sphere with the same direction of an arrow. The negative angle vector is represented as the arc arrow with opposite direction of the angle vector.

The sine and cosine of an angle vector have the following relations:

$$\cos \alpha \hat{n} = \cos \alpha, \quad \sin \alpha \hat{n} = \hat{n} \sin \alpha,$$

which can be checked by expanding them. If an angle vector is the sum of two angle vectors as shown in Fig. 3:

$$c\hat{l} = a\hat{m} \oplus b\hat{n},$$

(A2)
where it is necessary to distinguish the addition of angle vectors from that of usual vectors so that the different type of an addition symbol should be employed, then the sum of the angle vectors satisfies the following relations:

\[
\cos c = \cos(a\hat{m} + b\hat{n}) = \cos a \cos b - \hat{m} \cdot \hat{n} \sin a \sin b, \tag{A3}
\]

\[
\hat{l} \sin c = \sin(a\hat{m} + b\hat{n}) = \hat{m} \sin a \cos b + \hat{n} \cos a \sin b - \hat{m} \times \hat{n} \sin a \sin b. \tag{A4}
\]

Therefore the magnitude and the direction of the angle vector \(\hat{c}\) are calculated as

\[
c = \cos^{-1}[\cos a \sin b + \hat{m} \cdot \hat{n} \sin a \sin b],
\]

\[
\hat{l} = \frac{1}{\sin c}[\hat{m} \sin a \cos b + \hat{n} \cos a \sin b - \hat{m} \times \hat{n} \sin a \sin b], \tag{A5}
\]

and thus the angle vector addition rule is neither commutative nor anti-commutative. It is very important to keep the order of operations for angle vectors. The calculation for other angle vector, as an example, should be

\[
\hat{m} \cdot \hat{n} = -a\hat{m} \oplus \hat{l}. \tag{A6}
\]

The proof of the above relations is just referring to spherical trigonometry [10]. A spherical triangle on an unit sphere has three sides, \(a, b\) and \(c\), and the corresponding opposite angles \(A, B\) and \(C\), respectively, as shown in Fig. 3. All the three sides lie on their great circles. Each angle included by two sides is the angle between the sectional planes of the great circles on which the sides lie. The laws of sines and cosines for the spherical triangle are

\[
\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}, \tag{A6}
\]

\[
\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{A7}
\]

\[
\cos b = \cos a \cos c + \sin a \sin c \cos B, \tag{A8}
\]

\[
\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{A9}
\]

If we take \(\hat{m} \cdot \hat{n} = -\cos C\) in Eq. (A9), then Eq. (A3) is proved.

If we calculate the laws of cosine as \([\text{Eq. (A9)} \times \cos a + \text{Eq. (A8)}]/\sin a\) and \([\text{Eq. (A9)} \times \cos b + \text{Eq. (A7)}]/\sin b\), then we get the following relations:

\[
\cos A \sin c = \cos a \sin b - \cos C \sin a \cos b, \tag{A10}
\]

\[
\cos B \sin c = \sin a \cos b - \cos C \cos a \sin b. \tag{A11}
\]

These equations agree to the scalar products of unit vectors \(\hat{m}\) and \(\hat{n}\) to Eq. (A4), respectively, if we take \(\hat{l} \cdot \hat{n} = \cos A\) and \(\hat{l} \cdot \hat{m} = \cos B\). The last term of Eq. (A4) can be put in by hand considering the unity of sine squared and cosine squared in the right and left handed sides, but it can also be proved. The sum of the above equations is calculated as

\[
\sin c = \frac{1 - \cos C}{\cos A + \cos B} \cos a \sin b + \frac{1 - \cos C}{\cos A + \cos B} \sin a \cos b
\]

\[
= \cos A \cos a \sin b + \cos B \sin a \cos b + \sin b \sin C \sin^2 a \sin b, \tag{A12}
\]

where the law of cosines for the polar triangle is used, which are the same as that for the spherical triangle except the sign. The polar triangle corresponding to the above spherical triangle has the three angles \(A', B'\) and \(C'\) and the corresponding sides \(a', b'\) and \(c'\) which are related to that of the spherical triangle as follows

\[
A' = \pi - a, \quad B' = \pi - b, \quad C' = \pi - c,
\]

\[
a' = \pi - A, \quad b' = \pi - B, \quad c' = \pi - C, \tag{A13}
\]

then the following relations are satisfied

\[
\cos A = -\cos B \cos C + \sin B \sin C \cos a, \tag{A14}
\]

\[
\cos B = -\cos A \cos C + \sin A \sin C \cos b, \tag{A15}
\]

\[
\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{A16}
\]

This polar triangle can be imagined in Fig. 3 as the triangle made by the three end points of the unit vectors \(-\hat{l}, \hat{m}\) and \(\hat{n}\), where the minus sign before \(\hat{l}\) is due to the angle \(c\) which is not cyclic here. The above equation (A12) is nothing but the scalar product of \(\hat{l}\) to Eq. (A4), so it is understood that \(\hat{l} \cdot \hat{m} \times \hat{n} = -\sin a \sin B \sin C\), where \(-\sin C\) comes from the cross product of the unit vectors \(\hat{m}\) and \(\hat{n}\), and \(\sin a \sin B\) which is equal to \(\sin b \sin A\) comes from the dot product of the two vectors \(\hat{l}\) and \(\hat{m} \times \hat{n}\) by using the Napier’s rule for the right spherical triangle, which is
Therefore, in principle, it may be possible to derive another form of law of sines in vector form as follows

\[
\cos\left(\frac{\pi}{2} - h\right) = \sin h = \sin a \sin B = \sin b \sin A.
\] (A17)

Since the angular velocities of the three angles are defined as

\[
\omega_a = \frac{d(a\hat{n})}{dt}, \quad \omega_b = \frac{d(b\hat{n})}{dt}, \quad \omega_c = \frac{d(c\hat{l})}{dt},
\] (A18)

the time derivative of \(\cos c\) is calculated as

\[
- \sin c \frac{dc}{dt} = - (\sin a \cos b + \hat{n} \cdot \hat{m} \cos a \sin b) \frac{da}{dt} - (\hat{n} \cdot \hat{m} \sin a \cos b + \cos a \sin b) \frac{db}{dt},
\] (A19)

This equation can be rewritten by

\[
\hat{l} \sin c \frac{d(c\hat{l})}{dt} = \left(\hat{m} \sin a \cos b + \hat{n} \cos a \sin b - \hat{m} \times \hat{n} \sin a \sin b\right) \frac{d(a\hat{n})}{dt} - \left(\hat{m} \sin a \cos b + \hat{n} \cos a \sin b - \hat{m} \times \hat{n} \sin a \sin b\right) \frac{d(b\hat{n})}{dt},
\] (A20)

which is just the sum of angular velocities

\[
\omega_c = \omega_a \oplus \omega_b.
\] (A21)

This agrees to the direct time derivative of \(c\hat{l} = a\hat{n} \oplus b\hat{n}\). This equation shows that the angular velocity regarded as a proper vector in the standard texts has the same properties as the angle vector defined above. The angular acceleration is also calculated in the same way as the angular velocity is done. Considering that the angular velocities of the rotation and the precession for a symmetrical top cannot be interchanged, non-commutative properties of the rotations as mentioned in the standard texts \[2, 3\] may not be a proper reason for the insufficient definition of an angle vector. As the same way, we can not rotate a symmetrical top after precessing by using an angular acceleration, that is, a non-rotating symmetrical top does not precess. This angle vector addition rule also appears partly in the matrix for the Eulerian angles in the rigid body motion.

**APPENDIX B: HYPERBOLIC TRIGONOMETRY**

It is difficult to imagine a hyperbolic space directly which is known as a Lobatchewsky space \[1\]. So the hyperbolic space is supposed to be the space on an unit sphere as shown in the previous section with an imaginary radius which can be called a hyperbolic sphere here. If the metric for the space part is taken as \(\hat{n} \cdot \hat{n} = -1\), then the sine and cosine of the angle vector are written by hyperbolic sine and cosine as follows

\[
\cos \alpha \hat{n} = \cosh \alpha, \quad \sin \alpha \hat{n} = \hat{n} \sinh \alpha.
\] (B1)

Due to the metric the sum of two hyperbolic angle vectors is expressed with hyperbolic sines and cosines as

\[
\cosh c = \cosh (a\hat{n} \oplus b\hat{n}) = \cosh a \cosh b + \hat{m} \cdot \hat{n} \sinh a \sinh b,
\] (B2)

\[
\hat{l} \sinh c = \sin (a\hat{n} \oplus b\hat{n}) = \hat{m} \sinh a \cosh b + \hat{n} \cosh a \sinh b - i \hat{m} \times \hat{n} \sinh a \sinh b,
\] (B3)

where the last equation should have an imaginary term in order that the cosine squared and sine squared should be unity. Since the last term is not imaginary in the scalar equation as

\[
\sinh c = \cos A \cosh a \sinh b + \cos B \sinh a \cosh b - \sin B \sin C \sinh^2 a \sinh b,
\] (B4)

which is also derived from the laws of sines and cosines for a hyperbolic triangle and its polar triangle as shown below. Therefore, in principle, it may be possible to derive another form of law of sine in vector form as follows

\[
\hat{l} \sinh c = \sin (a\hat{n} \oplus b\hat{n}) = \hat{m} \sinh a \cosh b + \hat{n} \sinh b \cosh a + \hat{n} \times (\hat{n} \times \hat{m})(\cosh b - 1) \sinh a,
\] (B5)
which is obtained in section IV in a different way as shown in Appendix A.

The unit vectors for the directions of the hyperbolic angle vectors are defined globally, as shown in the spherical space, in the hyperbolic space where the triangle is. It is difficult to calculate the operations between such unit vectors because we don’t know how to draw the unit vectors in the hyperbolic space. Instead of treating the unit vectors in such a way, it is much more proper and easy to treat the unit vectors as local variables than global ones at the starting and end points of the hyperbolic angle vectors. Since a velocity in the Lorentz transformation can be represented as a hyperbolic angle, the directions of the relative velocities in two inertial frames are defined as the unit vectors at the starting point and the end point of the hyperbolic angle on the hyperbolic sphere. Therefore the unit vectors defined locally in the hyperbolic space are well interpreted as physical quantities and treated easily.

The law of sines and cosines for a hyperbolic triangle are

\[
\begin{align*}
\sinh a &= \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}, \\
\cosh a &= \cosh b \cosh c - \sinh b \sinh c \cos A, \\
\cosh b &= \cosh a \cosh c - \sinh a \sinh c \cos B, \\
\cosh c &= \cosh a \cosh b - \sinh a \sinh b \cos C.
\end{align*}
\] (B6, B7, B8, B9)

The law of cosines for the polar triangle corresponding to the hyperbolic triangle are

\[
\begin{align*}
\cos A &= -\cos B \cos C + \sin B \sin C \cosh a, \\
\cos B &= -\cos A \cos C + \sin A \sin C \cosh b, \\
\cos C &= -\cos A \cos B + \sin A \sin B \cosh c.
\end{align*}
\] (B10, B11, B12)

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