K–K excitations on $AdS_5 \times S^5$ as $N = 4$ “primary” superfields.

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Abstract

We show that the K–K spectrum of IIB string on $AdS_5 \times S^5$ is described by “twisted chiral” $N = 4$ superfields, naturally described in “harmonic superspace”, obtained by taking suitable gauge singlets polynomials of the $D3$-brane boundary $SU(n)$ superconformal field theory.

To each $p$-order polynomial is associated a massive K–K short representation with $256 \times \frac{1}{12} p^2 (p^2 - 1)$ states. The $p = 2$ quadratic polynomial corresponds to the “supercurrent multiplet” describing the “massless” bulk graviton multiplet.
1 Introduction

In recent time increasing evidence of a close connection between AdS\(^{p+2}\) supergravities and \(p+1\) dimensional superconformal field theories has emerged \([1]-[8]\).

The original duality between these theories was proposed in the context where the latter describe the dynamics of the degrees of freedom of \(p\)-branes and the former the near horizon geometry \([9]\). However it is appealing to investigate whether this correspondence is more general and valid in more general contexts.

Indeed the proposal of relating the correlators of superconformal field theories to the classical Anti de Sitter supergravity partition function \([6],[7]\) may be a suitable tool to investigate non-perturbative properties of scale invariant field theories \([10]\). In a more geometrical context these proposals are a realisation of the previous attempt \([11]\) to describe “elementary massless particles” on AdS in terms of “more elementary constituents”, the singletons \([12]\).

Indeed it was recently shown \([4],[5],[6],[7],[13]\) that massless particles, associated to diffeomorphisms, supersymmetry and Yang–Mills symmetry in the AdS bulk, correspond to particular “conserved currents” composite operators in the boundary singleton theory where \(\partial AdS_{p+2} = \mathcal{M}_{p+1}\) and \(\mathcal{M}_{p+1} \sim S^p \times R\) is a particular completion of Minkowski space. In particular it was shown that some of the K–K excitations of ten dimensional type IIB supergravity on \(AdS_5 \times S^5\) \([16]\) can be described taking suitable polynomials of \(N = 1,2\) chiral multiplets where the \(N = 4\) theory is analyzed in terms of superfields of lower supersymmetry \([7]\).

It was then realized that these K–K excitations are contained in \(N = 4\) superfield towers where apparent different towers do indeed correspond to different \(\theta\) components of the same superfield \([13]\).

It is the aim of this note to complete this analysis, by using the description of \(N = 4\) superfields through the use of extra suitable variables \(u_r^A (r = 1,2, A = 1, \cdots, 4)\) which are coordinates of the coset \(SU(4)/S(U(2)\times U(2))\) \([14],[15]\).

In particular we will show that the analysis in \([13]\) does indeed give the entire K–K spectrum of massless and massive states of IIB string compactified on \(AdS_5 \times S^5\) \([16]\).

The \(N = 4\) superfields obtained by taking the singlet of the \(p\)-order polynomial of the \(N = 4\) gauge singleton superfield, in the \(p\)-symmetric traceless
representation of $SU(4)$ (corresponding to Dynkin labels $(0, p, 0)$ [17]), does
indeed give a short representation containing $256 \times \frac{1}{12} p^2 (p^2 - 1)$ states whose
highest spin 2 state is in the $p - 2$ symmetric traceless representation of
$SU(4)$.

2 The K–K spectrum of IIB on $AdS_5 \times S^5$

The K–K spectrum of maximal supergravity on $AdS_5 \times S^5$ [16] was given,
in terms of unitary irreducible representations (U.I.R.) of the superalgebra
$SU(2, 2/4)$, in [18]. The masses of the individual spin states were given in
[19].

Let us briefly recall that the singleton representation corresponds to the
gauge-field multiplet on the boundary. It is the representation with $p = 1$
of [18]. To correctly count the states one must take into account the gauge
modes on the boundary which must be subtracted from the original spin
assignments. Indeed the singleton representation corresponds to a “massless”
representation of the same superalgebra acting as superconformal algebra on
the four dimensional boundary of $AdS_5$.

Therefore the singleton multiplet contains $2^4$ states, eight bosons with
helicities $\pm 2$ (1) and zero (6) and eight fermions with helicities $+\frac{1}{2}$ (4) and
$-\frac{1}{2}$ (4) where in brackets we denoted $SU(4)$ representations.

The massless representation on $AdS_5$, which corresponds to the graviton
multiplet of $AdS_5$, is given by the $p = 2$ towers of [18]. Again, due to gauge
modes in the bulk, to correctly count the states one must subtract them from
the spin 2, $\frac{3}{2}$ and 1 gauge fields.

The gauge modes in the bulk, which must be subtracted from the $(1, 1)$,
$(\frac{1}{2}, 1) + (1, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}) O(4)$ reps. describing the gauge fields, correspond
to $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0) + (0, \frac{1}{2})$ and $(0, 0) O(4)$ representations.

When this is done, the massless multiplet is seen to contain precisely 256
states, 128 bosons and 128 fermions.

These states are given in ref. [18] with the proviso that, due to the gauge
modes, the three above reps. should account for five, eight and three states
respectively (instead of nine, twelve and four).

The massive K–K towers are given from the table in ref. [18] for $p \geq 3$.
For massive states no gauge modes are present so states are classified
by $(J_0, J_1, J_2)$ labels with multiplicity $(2J_1 + 1)(2J_2 + 1)$ of a given $O(4)$
representation.
A $SU(4)$ representation, with Dynkin labels $(a_1, a_2, a_3)$ \cite{17}, has dimension

$$d(a_1, a_2, a_3) = \frac{1}{12} (a_1 + 1)(a_2 + 1)(a_3 + 1) \left( 1 + \frac{a_1 + a_2}{2} \right) \left( 1 + \frac{a_2 + a_3}{2} \right) \left( 1 + \frac{a_1 + a_2 + a_3}{3} \right)$$ \hspace{1cm} (1)

All the states with a given $p$ must be included. This gives $256 \times \frac{1}{12} p^2 (p^2 - 1)$ states, half bosons and half fermions.

Note in particular that such representation, uniquely denoted by $p$, contains a scalar state with conformal weight $E_0 = p$ in the $(0, p, 0)$ $SU(4)$ rep., a spin 2 (massive graviton) state with middle weight $E_0 = p + 2$ in the $(0, p - 2, 0)$ rep. of $SU(4)$ and a (real) scalar in the $(0, p - 4, 0)$ rep. of $SU(4)$ with maximal weight $E_0 = p + 4 \ (p \geq 4)$.

It also appears from the table of \cite{18} that the representations for $p = 1, 2, 3$ are not generic, while for $p \geq 4$ they are generic. We will understand in the next section why this is the case.

3 “Twisted chiral superfields” and K–K states as composite singleton excitations

In the present section we would like to complete the proof of the assertion of ref. \cite{13} that the entire K–K spectrum of IIB on $AdS_5 \times S^5$ is given by a single tower of $SU(n)$ singlets obtained by taking the traces of a $p$-order polynomial of a $SU(n)$ Lie algebra valued $N = 4$ singleton superfield, whose $\theta = 0$ term corresponds to the $(0, p, 0)$ $SU(4)$ irreducible representation.

We will first give the result and later show how this follows from the “twisted chiral superfields” of ref. \cite{13}.

The lowest order polynomial is the quadratic polynomial in the singletons. This superfield is the supercurrent multiplet of ref. \cite{20} and was discussed in \cite{14,15,13}.

Because this superfield satisfies some constraints which imply “current conservation” for spin 2, spin $\frac{3}{2}$ and spin 1 currents in the $1, 4$ (4) and 15 reps. of $SU(4)$, it is easily seen that it contains $256 = 128_B + 128_F$ states.

The $(E_0, J_1, J_2)$ assignment of these states was given in ref. \cite{13}.

\footnote{We thank Leonardo Castellani for a useful discussion on this point.}
Since $256 = 2^8$ it then follows that this is a representation of a Clifford algebra with 8 creation and 8 destruction operators or, equivalently, a superfield with 8 $\theta$’s (anticommuting) coordinates.

An $N=4$ chiral superfield would have such property, but its highest spin would be a spin 2 in the $(2,0)$ rep. of $SO(4)$. Instead, in order this superfield to contain spin 2 in the graviton representation $(1,1)$, one must have what we call a “twisted chiral” superfield, where out of the eight $\theta$’s four are 2-left-handed spinors and four are 2-right-handed spinors (instead of four 2-left-handed spinors as for a $N=4$ chiral superfield). We call such a superfield twisted chiral. It is easy to see that the independent components of such a superfield go up to $\theta^4 \overline{\theta}^4$. Moreover it contains a scalar of weight $E_0 = p$ in the $\theta = 0$ component which is in the $(0,p,0)$ rep. of $SU(4)$, a $(1,1)$ rep. of $SO(4)$ with $E_0 = p + 2$ in the middle $\theta^2 \overline{\theta}^2$ component and a real scalar, in $(0,p-4,0)$ $SU(4)$ rep. in its highest $\theta^4 \overline{\theta}^4$ component, with $E_0 = p + 4$.

This superfield is generic for $p \geq 4$, obtained by taking a $p$-order polynomial in the singleton superfield.

It corresponds to the $p \geq 4$ $SU(2,2/4)$ reps. discussed in the previous section where the $\theta$-component expansion contains $256 \times \frac{1}{12} p^2 (p^2 - 1)$ states. This number is obtained by multiplying the $\theta$ expansion degeneracy with the $SU(4)$ rep. of the spin 2 state. This state lies in the $(0,p-2,0)$ rep. with dimension $\frac{1}{12} p^2 (p^2 - 1)$.

It is obvious that for $p = 2, 3$ the superfield is not generic because its expansion goes up to $\theta^2 \overline{\theta}^2$ ($p = 2$) and $\theta^3 \overline{\theta}^3$ ($p = 3$). Still the degeneracy is given by the above formula for $p = 0, 1$. Note that in these cases the states of highest weights have $E_0 = p + 2$ and $E_0 = p + 3$ respectively.

In the second part of this section we recall the concept of “twisted chiral” superfield of the $N = 4$ four dimensional superconformal algebra [14]. The singleton superfield (Lie algebra valued in $SU(n)$) is $W_{AB}$ and it satisfies the constraints [20]

$$W_{[AB]} = \frac{1}{2} \epsilon_{ABCD} W^{(CD)}$$

$$\mathcal{D}_{\alpha A} W_{[BC]} = \mathcal{D}_{\alpha [A} W_{BC]}.$$  (3)

The $(x, \theta^\alpha, \theta^{\dot{\alpha}})$ superspace can be extended to “harmonic superspace” [14, 15] by introducing extra coordinates, elements of the coset $SU(4)/S(U(2) \times U(2))$.  

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Then the superfield carries an induced representation of $SU(4)$ with isotropy group $SU(2) \times SU(2) \times U(1)$.

Let us define $U^A_I$ as an element $U \in SU(4)$. 

$$U^A_I = (u^A_r, u^A_{r'}) \quad r, r' = 1, 2 \quad (A = 1, \cdots, 4)$$

Then we can introduce the superfield $W(x, \theta, \bar{\theta}, u)$:

$$W = \frac{1}{2} \epsilon^{rs} u^A_r u^B_s W_{[AB]}(x, \theta, \bar{\theta})$$

It satisfies the following properties:

$$D_{\alpha r} W = D_{\dot{\alpha} r'} W = 0$$
$$D_s W = 0$$
$$D_0 W = W$$

where:

$$D_{\alpha I} = U^A_I D_{\alpha A}$$
$$D^{I}_{\dot{\alpha}} = (U^{-1})_{A}^{I} D^{A}_{\dot{\alpha}}$$

$D_0$ denotes the derivative with respect to the $U(1)$ part of the isotropy group and:

$$D^J_I : D^J_I U^B_K = \delta^K_J U^B_I - \frac{1}{4} \delta^K_J U^B_I$$

Properties (3) and (7) were called Grassmann– and F–analiticity in [13].

Since (3) and (6) are linear it is obvious that

$$A_p = Tr W^p$$

satisfies (3) and (6) together with

$$D_0 W = p W$$

Moreover, eq. (3) means that $A_p$ depends only on $\theta_{\alpha r'}$ and $\theta^r_{\dot{\alpha}}$, i.e. on 2 left and 2 right $\theta$’s.

It then follows that the superfield (12) has a $\theta$ expansion which only contains $(J_1, J_2)$ representations with $J_1 \leq 1$, $J_2 \leq 1$ with highest spin 2 in the $(1, 1)$ representation as discussed in the previous section.
The superfields \([12]\), called “primary superfield” in ref. [15], corresponds to the K–K excitations previously discussed. In particular the singleton composite \(W^p\) precisely describes a short representation of the \(SU(2,2/4)\) algebra discussed in [13] where a unitary representation is denoted by a number \(p\) and a given state by the quantum numbers \((E_0, J_1, J_2, (a_1, a_2, a_3))\) where \((a_1, a_2, a_3)\) is the Dynkin label of the given \(SU(4)\) representation.

It is straightforward to see that the \(\theta = 0\) component of \(A^p\), denoted by \(\phi_{A_1B_1\ldots A_pB_p}\), defined by:

\[
\text{Tr} \epsilon^{r_1s_1} \ldots \epsilon^{r_ps_p} u_{r_1}^{A_1} u_{s_1}^{B_1} \ldots u_{r_p}^{A_p} u_{s_p}^{B_p} \Phi_{A_1B_1\ldots A_pB_p} \quad (14)
\]

precisely corresponds to a scalar in the \((0, p, 0)\) \(SU(4)\) representation with \(E_0 = p\).

The \(\theta^4\bar{\theta}^4\) component is a real scalar

\[
\phi_{A_1B_1\ldots A_pB_p}(\theta^4\bar{\theta}^4) = (\theta^4\bar{\theta}^4)_{(A_1B_1\ldots A_4B_4)} \Phi_{A_5B_5\ldots A_pB_p} - \text{Traces} \quad (15)
\]

which is in the \((0, p - 4, 0)\) \(SU(4)\) rep. with \(E_0 = p + 4\). This state exists for \(p \geq 4\).

The K–K graviton excitations lie in the \(\theta^2\bar{\theta}^2\) component, they belong to the \((0, p - 2, 0)\) \(SU(4)\) rep. and have \(E_0 = p + 2\).

For \(p = 2\), \(\text{Tr} W^2\), as shown in ref. [13], is the supercurrent multiplet of ref. [20, 21] and describes the massless graviton multiplet of the Anti-de Sitter bulk, it is a bilinear composite of singletons, as already shown in [3].

For a given \(p\), the total number of component fields is precisely \(2^8 \times \frac{1}{12} p^2 (p^2 - 1)(p \geq 2)\).

All superfields with \(p > 2\) are massive.

By expanding \(\text{Tr} W^p\) in powers of \(\theta\) one finds precisely the massless and massive spectrum, described in ref. [19], of type IIB string on \(AdS_5 \times S^5\), in terms of composite singleton excitations.

The K–K masses are simply obtained by using the formulae of ref. [13]:

- **Bosons:**

\[
m_{(0,0)}^2 = E_0(E_0 - 4) \\
m_{(1,0)}^2 = (E_0 - 2)^2 \\
m_{(1,1)}^2 = (E_0 - 1)(E_0 - 3) \\
m_{(1,1)}^2 = E_0(E_0 - 4) \quad (16)
\]
\[ |m_{(\frac{1}{2},0)}| = |m_{(0,\frac{1}{2})}| = E_0 - 2 \]  \hspace{1cm} (17)

We note in particular that, as a consequence of superfield multiplication, we can read out which specific singleton composite corresponds to a given state of the K–K spectrum, as given in [19]. As an example we note that the complex scalar \( F^2 + i F \tilde{F} \) (the axion–dilaton) corresponds to a \( m^2 = 0 \) state of the \( p = 2 \) (supercurrent) multiplet [4,13].

A real \( F^4 \) scalar \( \tilde{F}^4 \) corresponds to a \( m^2 = 32 \) state of the \( p = 4 \) massive composite [19].

No other \( F^p \) terms \( (p > 4) \) appear as singleton composites in the K–K tower.

4 Theories with lower supersymmetries

For \( N = 1, 2 \) four dimensional superconformal theories, corresponding to singleton field theories of \( SU(2,2/1), SU(2,2/2) \) superalgebras, the analysis of K–K excitations can be done in terms of a “chiral” singleton gauge field strength and additional “chiral” and/or “hypermultiplets”.

For \( N = 2 \) superconformal field theories, the \( N = 4 \) AdS\(_5\) supergravity has three kinds of bulk multiplets: the graviton, tensor and gauge multiplets.

For \( N = 1 \) superconformal field theories the bulk multiplets are four, the graviton, gauge, tensor and hyper multiplet [22], [23].

All these multiplets can be obtained as suitable \( N = 1 \) composites of singleton multiplets of chiral (antichiral) type, i.e. the \( N = 1 \) chiral field strength \( W_{\alpha} \) and additional chiral matter.

Due to the fact that the matter content in theories with lower supersymmetry is model dependent, the analysis of K–K excitations in terms of boundary composite fields is more complex in this case.

The analysis of the K–K spectrum for a class of these theories has just appeared in the literature [24].

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3Here \( F \) denotes the \( SU(n) \) gauge field strength.

4We benefited of a discussion with Juan Maldacena on this particular composite operator.
5 Conclusions

In this work we have shown that the towers of K–K excitations of IIB supergravity on \( AdS_5 \times S^5 \) is described by precisely the infinite sequence of “primary” analytic conformal \( N = 4 \) superfields discussed in [14]. There it was shown that their correlation functions in harmonic superspace are holomorphic sections of a line bundle. We prefer to call these superfields “twisted chiral”, in analogy with a similar structure in \( N = 2 \) 2\( D \) superconformal field theories [23].

From our correspondence and from the proposal in [6],[7] it emerges that such holomorphic correlators are precisely the boundary values of \( N = 8 \) supergravity amplitudes in \( AdS_5 \). It would be interesting to explore the consequences of this relation in the context of CFT/AdS duality.

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References

[1] J.M. Maldacena, “The Large \( N \) Limit of Superconformal Field Theories and Supergravity”, hep-th/9711200

[2] K. Sfetsos, K. Skenderis, “Microscopic derivation of the Bekenstein-Hawking entropy formula for non-extremal black holes”, hep-th/9711138; H.J. Boonstra, B. Peeters and K. Skenderis, “Branes and anti-de Sitter spacetimes”, hep-th/9801076; S.Hyun, “U-duality between Three and Higher Dimensional Black Holes”, hep-th/9704003

[3] P. Claus, R. Kallosh and A. Van Proeyen, “M5-brane and superconformal (0,2) tensor multiplet in 6 dimensions”, hep-th/9711161; R. Kallosh, J. Kumar and A. Rajaraman, “Special Conformal Symmetry of World-volume Actions”, hep-th/9712073; P. Claus, R. Kallosh, J. Kumar, P. Townsend and A. Van Proeyen, “Conformal theory of M2, D3, M5 and D1+D5 branes”, hep-th/9801200
[4] S. Ferrara and C. Fronsdal, “Conformal Maxwell theory as a singleton field theory on AdS$_5$, IIB three-branes and duality”, hep-th/9712239

[5] S. Ferrara and C. Fronsdal, “Gauge fields as composite boundary excitations”, hep-th/9802126

[6] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “Gauge Theory Correlators from Non-Critical String Theory”, hep-th/9802109

[7] E. Witten, “Anti-de Sitter Space And Holography”, hep-th/9802150

[8] M.J. Duff, H. Lu and C.N. Pope, “AdS$_5 \times S^5$ Untwisted”, hep-th/9803061

[9] C. W. Gibbons and P. K. Townsend, Phys. Rev. Lett. 71 (1993) 3754; M. P. Blencowe and M. J. Duff, Phys. Lett. B203 (1988) 229; Nucl. Phys B310 (1988), 389; M. J. Duff, Class. Quantum Grav. 5 (1988) 189; E. Bergshoeff, M. J. Duff, C. N. Pope and E. Sezgin, Phys. Lett. B199 (1988) 69; H. Nicolai, E. Sezgin and Y. Tanii, Nucl. Phys B305 (1988) 483.

[10] E. Witten, “Anti-de Sitter Space, Thermal Phase Transition, And Confinement in Gauge Theories”, hep-th/9803131; Soo-Jong Rey, S. Theisen and Jung-Tay Yee, “Wilson-Polyakov Loop at Finite Temperature in Large N Gauge Theory and Anti-de Sitter Supergravity”, hep-th/9803133; A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, “Wilson Loops in the Large N Limit at Finite Temperature”, hep-th/9803137

[11] M. Flato and C. Fronsdal, Lett. Math. Phys. 2 (1978) 421; Phys. Lett. 97B (1980) 236; “Interacting singletons”, hep-th/9803013

[12] M. Flato and C. Fronsdal, J. Math. Phys. 22 (1981) 1100; Phys. Lett. 172B (1986) 412; Lett. Math. Phys. 2 (1978) 421; Phys. Lett. 97B (1980) 236; A. Angelopoulos, M. Flato, C. Fronsdal and D. Sternheimer, Phys. Rev.D23 (1981) 1278

[13] S. Ferrara, C. Fronsdal and A. Zaffaroni, “On N=8 Supergravity on AdS$_5$ and N=4 Superconformal Yang-Mills theory”, hep-th/9802203

[14] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1 (1984) 469; Class. Quant. Grav. 2 (1985) 155
[15] P.S. Howe and P.C. West, “Non-perturbative Green’s functions in theories with extended superconformal symmetry”, [hep-th/9509140], “Operator product expansions in four-dimensional superconformal field theories”, [hep-th/9611073], Phys.Lett. B389 (1996) 273; “Is N=4 Yang-Mills Theory Soluble?”, [hep-th/9611074]; “Superconformal Invariants and Extended Supersymmetry”, [hep-th/9611075], Phys.Lett. B400 (1997) 307

[16] M. Gündaydin, L.J. Romans and N.P. Warner, “Gauged N = 8 supergravity in five dimensions”, Phys. Lett 154B (1985) 268; M. Pernici, K. Pilch and P. van Nieuwenhuizen, “Gauged N = 8 D = 5 supergravity”, Nucl. Phys. 259 (1985) 460

[17] R. Slansky, “Group theory for unified model building”, Phys. Rep. 79 (1981) 1

[18] M. Gündaydin and N. Marcus, Class. Quantum Grav. 2 (1985) L11

[19] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, “The mass spectrum of chiral N = 2 D = 10 supergravity on S^5”, Phys. Rev. D32 (1985) 389

[20] P. Howe, K.S. Stelle and P.K. Townsend, “Supercurrents”, Nucl. Phys. B192 (1981) 332

[21] E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B182 (1981) 173

[22] S. Ferrara and A. Zaffaroni, “N = 1, 2 4D Superconformal Field Theories and Supergravity in AdS_5”, [hep-th/9803060]

[23] M. Gündaydin, L.J. Romans and N.P. Warner, “Compact and noncompact gauged supergravity theories in five dimensions”, Nucl. Phys. B272 (1986) 598

[24] Y. Oz and J. Terning, “Orbifolds of AdS_5xS^5 and 4d Conformal Field Theories”, [hep-th/9803167]

[25] S. Gates, Jr., C. Hull and M. Rocek, “Twisted multiplets and new supersymmetric nonlinear sigma models” Nucl. Phys. B248 (1984) 157; E. Witten, “Phases of N=2 theories in two-dimensions”, Nucl. Phys. B403 (1993) 159, [hep-th/9301042]