Mode-matching technique for transmission calculations in electron waveguides at high magnetic fields

J. J. Palacios and C. Tejedor

Departamento de Física de la Materia Condensada. Universidad Autónoma de Madrid.

Cantoblanco, 28049, Madrid. Spain.

Abstract

In this paper we present a mode-matching technique to study the transmission coefficient of mesoscopic devices such as electron waveguides in the presence of high magnetic fields for different situations. A detailed study of the difficulties rising due to the presence of the magnetic field is given and the differences with the zero magnetic field case are stressed. We apply this technique to calculate the transmission at non-zero magnetic field of two completely different systems: a) a quantum box built up on a quantum wire (or electron waveguide) by means of two barriers and b) a meandering quantum wire, \textit{i.e.}, a wire with changes in the guiding direction. In the former case we analyze the so-called Coulomb Blockade and Aharanov-Bohm regimes and in the latter one we focus on the realistic case of soft, circular bends joining the different sections of the wire.

73.40.Cg, 73.50.Jt
I. INTRODUCTION

Landauer-Büttiker formula relates, in a simple manner, the linear response conductance with the transmission coefficient of an electronic system and it has been applied successfully in a number of transport problems in mesoscopic devices [1]. Only two ingredients are needed: a) reservoirs in which thermalization may occur and b) a region, whose transport properties we want to know, free of inelastic scattering. The transmission coefficient of this region is directly responsible for the conductance in such a way that elastic processes are the only ones to be taken into account.

In order to calculate the transmission, many methods have been developed over the past few years. On the one hand, tight-binding-like techniques [2], in which space is discretized into cells, have been applied satisfactorily in many problems, especially, in those including disorder [3]. On the other hand, mode-matching methods have been used in problems like those of disorder-free electron waveguides. Conductance calculations for a ballistic constriction in the absence [4] and in the presence of a magnetic field [5, 6] can be done with this method. Following with the so-called two-probes systems as the one just mentioned, Bagwell dealt with the effects on the conductance of a quantum wire of one and two δ-function-like scattering centers [7, 8]. Changes in the confining geometry of a quantum wire is also a suitable problem to be treated within the framework of the matching technique; for instance, cavities studied by Kasai et al. [9] and Wu et al. [10] and single and multiple bends [11, 12]. One-dimensional periodic structures have been also studied with this technique [13]. As for cross wires or junctions, encountered in three and four-probes devices, this method has revealed itself extraordinarily useful to understand phenomena such as negative resistances [14], resonant tunneling through bound states in open systems [15], quenching of the Quantum Hall Effect (QHE) [16], box resonators in crossed wires [17], etc.

Most of the work above mentioned has been done at zero magnetic field. This fact reduces the interest of it since many interesting phenomena in mesoscopic physics derive from the QHE [18] and are related with the possibility of applying a magnetic field perpendicular
to the two-dimensional electron gas (2DEG). The difficulties inherent to the magnetic field can be almost completely removed if waveguides with parabolic confinement potential are considered for the calculations [9,21]. However, screening properties due to the electron-electron interaction [22] show that, at not a very low density, the shape of the effective confinement potential is rather flat in the middle of the quantum wire and rises quickly at the edges of it (independently of the fact of having been defined either by split-gates [23], or by etching [24]). Additional step-like structure may appear in the classically calculated [25], or self-consistently calculated confining potential [26] but we will not consider this possibility in this paper in order to simplify our model. Bearing this fact in mind we consider appropriate to simulate the confinement by means of a square hard wall potential as done in Refs. [18,14].

Sec. II is devoted to the discussion of the numerical details for this case. In Sec. III we analyze the conductance properties of a quantum box in the presence of a magnetic field and in Sec. IV those of a twisting quantum wire with circular, soft bends. Finally, the conclusions of our work are presented in Sec. V.

II. MATCHING TECHNIQUE IN MAGNETIC FIELDS

The starting point of a matching method relies on obtaining the most general wave function for all the different regions involved in the problem according to an usual scattering-like problem. We consider a waveguide in such a way that electrons move freely in the $y$ direction and feel hard walls in the $x$ one. In Fig. I such a waveguide is shown including a region of arbitrary potential in the middle of it. The Schrödinger equation in the Landau gauge

$$\mathbf{A} = (0, Bx l_m, 0)$$  \hspace{1cm} (1)

becomes separable in the $x$ and $y$ variables for each different region $I, II, III$, being characterized respectively by different confinement potential profiles $V^{I,II,III}(x)$, and it looks like:
\[- \frac{\partial^2 \phi_{n,III}^I}{\partial x^2} + (k_{n,III}^I + x)^2 \phi_{n,III}^I(x) + v_{n,III}^I(x) \phi_{n,III}^I(x) = \epsilon \phi_{n,III}^I(x) \]  \tag{2}

with

\[ \Phi_{n,III}^I(x,y) = e^{ik_{n,III}^I y} \phi_{n,III}^I(x) \]  \tag{3}

being the total wave function for a given wave vector \(k_n\) in each region \(I, II, III\) where \(x\) and \(y\) are given in units of the magnetic length \(l_m = \sqrt{\hbar/eB}\), wave vectors \(k_n\) in \(l_m^{-1}\), \(\epsilon = 2E/\hbar\omega_c\), \(v_{n,III}^I(x) = 2V_{n,III}^I(x)/\hbar\omega_c\) (\(\omega_c\) is the cyclotron frequency \(eB/m^*\) for particles of charge \(e\) with effective mass \(m^*\) under a magnetic field \(B\) and \(E\) is the energy). Spin splitting effects due to the magnetic field are neglected for the GaAs-AlGaAs heterostructure considered below in all the calculations.

In the case depicted in Fig. 1 both regions I and III are identical and the potential profile in the \(x\) direction for both of them is depicted in Fig. 2a. The dispersion relations for real wave vectors for the lowest subbands have been depicted too. The energy was fixed in such a way that there are three subbands occupied or three current carrying channels. Now the problem consists of finding the wave vectors \(k_n\) (we drop labels \(I, II, III\)) and corresponding transversal modes \(\phi_n(x)\) for a given energy. Finding the complex band structure is a simple problem in the case of zero magnetic field (for any confinement potential in the \(x\) direction) since wave vectors and modes can be found analytically thanks to the usual parabolic dispersion relations (free electrons in the \(y\) direction). It is also simple the case of a parabolic confinement potential in the presence of a magnetic field \[9,21\]. However, if the dispersion relations are no longer parabolic, the problem must be solved numerically as it was already stressed in Ref. [18]. Wave vectors \(k_n\) are the solutions of the intersection between the complex dispersion relations (real part of it is shown in Fig. 2a) and the plane of constant energy. Details on this calculation are given in the Appendix and the result of this intersection is shown in Fig. 2b. Solid dots denote the wave vectors in the \(k\)-complex plane. Those lying on the real axis (with null imaginary part) correspond to extended modes and those with non-zero imaginary part correspond to evanescent or exploding modes. The latter ones can
be grouped into two types depending on whether its real part is null or not. The former ones belong to the subbands (four for each subband) with a dispersion relation having flat regions (bulk regions within the terminology of the QHE) and the latter ones belong to the rest of subbands with a minimum in their dispersion relation (two for each subband\[27\]).

This picture is modified as we change the magnetic field in the following way: If we increase the magnetic field dots with null real part split into two (arrows pointing outwards showing this fact) and, if we decrease it, double dots collapse into one (arrows pointing inwards).

Turning off the field completely, purely real wave vectors and purely imaginary ones appear in the \(k\)-complex plane corresponding to parabolic dispersion relations of subbands below or above the chosen energy respectively. The total number of subbands \(N\) is given by the sum of \(n_1\) (number of subbands below the chosen energy), \(n_2\) (number of subbands above that energy with flat regions in their dispersion relation) and \(n_3\) (the rest of them having a minimum in their dispersion relation). So, \(n_T = 2n_1 + 4n_2 + 2n_3\) is the total number of wave vectors involved in the \(N\)-subbands problem. Sometimes, this fact has not been duly appreciated in tight-binding-like schemes\[3\].

It is also shown in the Appendix how the \(\phi_n(x)\) functions (transversal modes) can be found once we know the \(k_n\) wave vectors. The above discussion has focused on regions I and III but is valid in general terms for region II provided that its potential is only dependent on \(x\) and not on \(y\), in order for the Schrödinger equation to be separable. In this way, the most general wave function at a given energy can be expressed like:

\[
\begin{align*}
\text{Region } I: \quad & \Phi^I(x, y) = \sum_{n=1}^{\infty} \alpha_n \Phi^I_n(x, y) \\
\text{Region } II: \quad & \Phi^{II}(x, y) = \sum_{n=1}^{\infty} \beta_n \Phi^{II}_n(x, y) \\
\text{Region } III: \quad & \Phi^{III}(x, y) = \sum_{n=1}^{\infty} \gamma_n \Phi^{III}_n(x, y)
\end{align*}
\]

(4)

The coefficients in the above expressions must be determined by matching on the delimiting interface but there are additional conditions inherent to the scattering problem and, as far as our problem is concerned, not all the solutions present in the \(k\)-complex plane can be considered. In region I the incident and reflecting modes must be taken into account (those on the left and right arms of the real axis respectively) so as the non-exploding ones (dots
in the lower part of the $k$-complex plane in Fig. 2b). In region $II$ there is no restriction and all the solutions must be taken into account and in region $III$ only out-going modes (dots on the real axis on the left arm) and those with dots in the upper half plane must be considered. So, the solution looks like this:

\[
\begin{align*}
\text{Region } I: & \quad \Phi^I(x, y) = \Phi^I_0(x, y) + \sum_{j=1}^{\infty} r_{ij} \Phi^I_j(x, y) \\
\text{Region } II: & \quad \Phi^I(x, y) = \sum_{j=1}^{\infty} \beta_{ij} \Phi^I_j(x, y) \\
\text{Region } III: & \quad \Phi^{II}(x, y) = \sum_{j=1}^{\infty} t_{ij} \Phi^{II}_j(x, y)
\end{align*}
\]

(5)

where a single incident mode $i$ has been chosen (the same must be done for all the incident modes) and the surviving coefficients from those $\alpha_n$, $\beta_n$ and $\gamma_n$ appearing initially in Eq. 4 have been relabeled as $r_{ij}$, $\beta_{ij}$ and $t_{ij}$.

Now the matching consists of the standard problem of invoking continuity of the wave function and its derivative across the two interfaces delimiting the three regions:

\[
\begin{align*}
\Phi^I_i(x, y) + \sum_{j=1}^{\infty} r_{ij} \Phi^I_j(x, y) = \sum_{j=1}^{\infty} \beta_{ij} \Phi^I_j(x, y) \\
\sum_{j=1}^{\infty} \beta_{ij} \Phi^I_j(x, L) = \sum_{j=1}^{\infty} t_{ij} \Phi^I_j(x, L)
\end{align*}
\]

(6) (7)

\[
\begin{align*}
\left| \frac{\partial}{\partial y} \left\{ \Phi^I_i(x, y) + \sum_{j=1}^{\infty} r_{ij} \Phi^I_j(x, y) \right\} \right|_{y=0} = \left| \frac{\partial}{\partial y} \left\{ \sum_{j=1}^{\infty} \beta_{ij} \Phi^I_j(x, y) \right\} \right|_{y=0} \\
\left| \frac{\partial}{\partial y} \left\{ \sum_{j=1}^{\infty} \beta_{ij} \Phi^I_j(x, y) \right\} \right|_{y=L} = \left| \frac{\partial}{\partial y} \left\{ \sum_{j=1}^{\infty} t_{ij} \Phi^I_j(x, y) \right\} \right|_{y=L}
\end{align*}
\]

(8) (9)

By projecting the obtained equations onto a basis of states in the $|\psi_m\rangle$ (for instance those of the problem without magnetic field, \textit{i.e.} sines and cosines in the $x$ direction) we reduce the problem to that of solving a non-homogeneous system of equations (by picking as many $|\psi_m\rangle$ states as necessary) from which we obtain the transmission coefficients:

\[
\begin{align*}
\langle \psi_m | \phi^I_i \rangle &= -\sum_{j=1}^{\infty} r_{ij} \langle \psi_m | \phi^I_j \rangle + \sum_{j=1}^{\infty} \beta_{ij} \langle \psi_m | \phi^{II}_j \rangle \\
0 &= -\sum_{j=1}^{\infty} \beta_{ij} e^{ik^L_j} \langle \psi_m | \phi^{II}_j \rangle + \sum_{j=1}^{\infty} t_{ij} e^{ik^{III}_j} \langle \psi_m | \phi^{III}_j \rangle \\
 ik^I_i \langle \psi_m | \phi^I_i \rangle &= -\sum_{j=1}^{\infty} ik^I_j r_{ij} \langle \psi_m | \phi^I_j \rangle + \sum_{j=1}^{\infty} ik^{II}_j \beta_{ij} \langle \psi_m | \phi^{II}_j \rangle
\end{align*}
\]

(10) (11) (12)
\[ 0 = -\sum_{j=1}^{\infty} ik_{j}^{II} e^{ik_{j}^{II}L} \beta_{ij} \langle \psi_{m} | \phi_{j}^{II} \rangle + \sum_{j=1}^{\infty} ik_{m}^{III} e^{ik_{j}^{III}L} t_{ij} \langle \psi_{m} | \phi_{j}^{III} \rangle \]  \tag{13}

Of course, the matching technique has its limitations and is not adequate if one has to separate the problem in too many different regions or regions too large compared with the magnetic length. Numerical errors due to the exponential behavior of the evanescent modes rise unavoidably in the linear system of equations to be solved for those situations.

III. QUANTUM BOXES

The case in which region II is a simple flat barrier can be solved easily with the expressions in Sec. II. In this section, we extend our scheme to the problem of a quantum box defined by means of two barriers crossing a wire of width \( W \) separated by a distance \( D \) (we restrict here to the symmetrical case of equal barriers). This is shown with the inset in Fig. 3a and it is intended to modelize real systems [23,24]. Once we know the scattering matrices \( t \equiv t_{ij} \) and \( r \equiv r_{ij} \) for a single barrier it is easy to obtain the total transmission \( (T) \) and reflection \( (R) \) matrices:

\[ T = t p (1 - w)^{-1} t \]  \tag{14}
\[ R = t p r p (1 - w)^{-1} t + r \]  \tag{15}

where \( w = r p r p \) and \( p \) is the diagonal propagation matrix whose diagonal elements are \( e^{ik_{i}D} \) with \( k_{i} \) being the wave vectors of all the transmitted modes, extended and evanescent, in the region between barriers. The evanescent ones are important if the barriers are nearby each other or the energy is below but close to the bottom of any subband as it was already pointed out for the case of a waveguide with two point-like scatterers [8]. The subspace of extended modes in the total transmission matrix at a given Fermi energy \( E_{F} \) gives us directly the linear response conductance of the system through the Landauer-Büttiker formula:

\[ G = 2e^{2}/h \sum_{i,j}^{n_{1}} \frac{v_{j}}{v_{i}} |T_{ij}(E_{F})|^{2} \]  \tag{16}
with $v_j$ and $v_i$ being the velocities of the outgoing and incoming modes respectively. A trivial expression for the velocity of these modes (in units of $l_m s^{-1}$) can be obtained from the Hellman-Feynman theorem:

$$v_i = \omega_c (k_i - \bar{x})$$

(17)

where $\bar{x}$ is the center of gravity of the mode.

Figs. 3a and 3b show the conductance as a function of the Fermi energy for two different values of the magnetic field. The values of $W$ and $D$ are 4 and 8 $l_m$ (for the case of 1 T) respectively, which roughly corresponds to a box of dimensions 0.1 and 0.2 $\mu$m respectively. In actual experiments the magnetic field is usually swept or, alternatively, a voltage applied to a bottom or top metal gate. In the latter cases, the effect of the sweeping can be simulated by changing the Fermi energy as we have done in our calculations. There clearly appear two different regions (hereafter labeled as region A and region B). The inset in Fig. 3a show the semiclassical or adiabatic picture of edge channels running along the boundaries. Although this is not the case in our geometry any more, it will serve as a visual help for clarifying purposes. Region A corresponds to the case of having only one Landau level occupied in the three regions (leads and box) with a kinetic energy of the edge state not high enough for the electron to pass over the barrier. That means that any non-zero transmission through a single barrier will be due to a tunneling process. So, the peaks in the conductance of the two barriers defining the box correspond to resonant tunneling through quasi-bound states in the box. These resonances appear as a consequence of the term $(1 - w)^{-1}$ accounting for the multiple reflections within the box. It can be equally seen as due to the poles of the Green’s function in the box within the framework of the Generalized Transfer Hamiltonian (GTH) [28]. These states do not correspond to the semiclassical idea of edge channels unless the magnetic field is high enough as it can be drawn from Fig. 3 in which, the evolution of one of these states is shown as the magnetic field is increased. At low fields they resemble to those of a two-dimensional, field-free square well but increasing the field they evolve to its edge state-like nature.
However, in this region A, Coulomb effects are extremely important since the number of electrons confined in the box is a well defined quantity and corresponds to an integer number (in our case of the order of 10). This fact gives rise to the so-called Coulomb blockade phenomenon \cite{29}. It is not the aim of this paper to discuss quantitatively these Coulomb effects on the conductance of our system although a few words can be said on the issue. It has already been discussed by Palacios et al. \cite{30} that the influence of Coulomb interaction on the conductance of this system (as to the region A is concerned) is summarized in the following facts: a) The position of the peaks is shifted by a charging or activation energy that varies in a non-trivial way with the number of electrons in the box and with the magnetic field and it can not be described in terms of classical capacitances or Anderson models; b) the height of the peaks is not only reduced from 2 to 1 (in units of $e^2/h$) as classical theories predict \cite{29,31} but it is reduced drastically even more due to the strong correlation suffered by the small number of electrons in this box. This regime of few electrons is being reached in recent experiments \cite{24,32} but a complete study of it is lacked.

Let us now pay attention to the region B in which two Landau levels are occupied. The first one has enough kinetic energy to run over the barrier but the second one can not overcome it. Again, the semiclassical picture is helpful although not completely correct as it will be seen below. In the cases depicted in both Fig. 3a and Fig. 3b there is an almost flat region at a fixed value of $2e^2/h$ corresponding to the first Landau running free over the barriers. In the case of 1 T (Fig. 3a) three peaks appear superimposed to this plateau and they correspond to resonant tunneling of the edge state belonging to the second Landau level. This phenomenon has been known as a particular *Aharanov-Bohm* effect in singly-connected systems \cite{33}. They do not reach the value of $4e^2/h$ because of the non-adiabatic behavior of the edge states in this geometry. There is a mixing between them which avoids perfect resonance. In Fig. 3b we show the results for a higher value of the magnetic field. The two peaks at lower $E_F$ on the plateau in Fig. 3a have become dips while the third one remains as a peak but narrower than before. That can be understood in the following way: the higher the magnetic field is, the more localized the states are, so that their coupling to
the leads is smaller. This fact narrows the peaks (as it can be equally seen in region A). Once these states have become confined enough, they are used as routes for the first Landau edge states to backscatter, so, reducing the conductance below the value of the plateau. Instead of a resonant transmission, this phenomenon is a resonant backscattering. The peaks on the conductance plateau are also characteristic of any dot with an adiabatic potential profile but the dips appearing in Fig. 3b are only due to the non-adiabatic behavior of the edge states in this geometry and have been reported in several experimental works \cite{23,34}. Recently \cite{35,36}, Coulomb effects have been observed in this regime although the charge within the dot is not an integer quantity any more due to the presence of traveling channels along the dot. These results have been discussed theoretically \cite{37} in terms of compressible and incompressible regions for the case of a circular geometry with soft walls but this is not the case presented here.

Figure 4 shows results of conductance, for the same box, in the case of sweeping the magnetic field for a given Fermi energy of $6\hbar\omega_c/2$. This is the most commonly found experimental situation and, from Fig. 4, we can see how the results are qualitatively the same as those presented in Figs. 3a and 3b. So, the above discussion remains valid in general terms. Notice, only, the appearance of additional structure in the conductance, for high values of the magnetic field, close to the region of resonant tunneling. This structure is due to backscattering of the only carrying-current mode remaining at those fields and only appear at high enough fields. As can be seen from Figs. 3a and 3b, at the fields shown there, no structure appear close to the region A of resonant tunneling.

As to the individual behavior of the transmission coefficients $T_{ij}$, it suffices to say that they remain diagonal for the whole range of values of $B$ except for the values at which peaks in the conductance appear. At those values, the four coefficients $T_{11}, T_{12}, T_{21}$ and $T_{22}$ (for the case of two Landau levels occupied) share equally the transmission. It is there that the nonadiabatic behavior of the edge states in this geometry becomes more notorious.
IV. MEANDERING WIRES

As another example of the matching technique we have chosen that of a meandering wire with soft bends. The case of one bend without magnetic field has already been studied by Sols et al. [11] and Sprung et al. [12]. We present here the general way of treating multiple bends in magnetic fields and we show results for the simplest case of a single one. In Fig. 6a it is illustrated what we understand for a soft bend. This shape is the adequate one for the matching and it can be expected to represent that of a real bend in a waveguide. Unlike the situation presented in Sec. II this time the matching is going to be done between two (instead of three) different regions (Fig. 6b). This fact will be justified later on. There is an additional difference: the interface between regions does not present a jump in the potential along the propagation direction as before but presents a change in the confining geometry. The Schrödinger equation in region $I$ is the same as Eq. 2 but in region $II$, due to the new boundary conditions, we must use polar coordinates with the symmetrical gauge

$$A = (0, Br l_m/2, 0)$$

(18)

in order to separate the Schrödinger equation by using the wave function

$$\Phi_{II}^m(r, \theta) = e^{im\theta} \chi_n(r)$$

(19)

for a given $m_n$. Defining $\chi_n(r) = \psi_n(r)/\sqrt{r}$ we obtain:

$$- \frac{\partial^2\psi_n(r)}{\partial r^2} + \left[ - \frac{1}{4r^2} + \frac{1}{r^2} \left( \frac{r^2}{2} - m_n \right)^2 \right] \psi_n(r) + v(r)\psi_n(r) = \epsilon\psi_n(r)$$

(20)

The units have been chosen in the same manner as those of Eq. 2 and the search of the complex $m_n$ quantum numbers is undertaken in exactly the same way as that of the $k_n$ wave vectors in Sec. II. All the facts concerning the matching in Sec. II remain valid but now everything is done for only one delimiting interface:

$$\Phi_i^I(x, 0) + \sum_{j=1}^{\infty} r_{ij}\Phi_j^I(x, 0) = \sum_{j=1}^{\infty} t_{ij}\Phi_j^{II}(r, \theta)$$

(21)

$$\left| \frac{\partial}{\partial y} \left\{ \Phi_i^I(x, y) + \sum_{j=1}^{\infty} r_{ij}\Phi_j^I(x, y) \right\} \right|_{y=0} = \left| \frac{1}{r \partial \theta} \left\{ e^{ir^2\sin\theta\cos\theta/2} \sum_{j=1}^{\infty} t_{ij}\Phi_j^{II}(r, \theta) \right\} \right|_{\theta=0}$$

(22)
These equations look similar to those in Sec. II with the exception of the term \( e^{ir^2 \sin \theta \cos \theta/2} \) accounting for the required unification of the gauge. The matching must be done in a unique gauge, i.e., wave function in region II must be expressed in the same gauge used for region I and vice versa. Again, we must reduce the expressions above to a non-homogeneous system of equations by projecting onto a given basis in order to obtain the scattering matrices \( t_{ij} \) and \( r_{ij} \). It must be stressed that now the problem does not present symmetry in the sense that the incident mode can be chosen in two different ways, any of them giving different resulting scattering matrices. For instance, the scattering matrices corresponding to incident modes from region I will be denoted by \( t^I_{ij} \) and \( r^I_{ij} \) and those corresponding to incident modes from region II by \( t^{II}_{ij} \) and \( r^{II}_{ij} \) (Fig. 3b). The superscripts \( i \) and \( o \) denote incident modes along the inner side of the waveguide or along the outer side. Reversing the magnetic field we obtain a different set of scattering matrices: \( t^o_{ij} \), \( r^o_{ij} \), \( t^{II}_{ij} \) and \( r^{II}_{ij} \).

All these matrices are necessary in order to calculate the transmission or reflection of the single bend (Fig. 3a). To achieve this, Eq. 14 and Eq. 15 must be generalized in the following way (dropping the subscripts \( ij \)):

\[
T^i = t^{II} p^i (1 - w)^{-1} t^I \\
R^i = t^{II} p^o r^{II} p^i (1 - w)^{-1} t^I + r^I
\]

(24) (25)

with \( w = r^{II} p^o r^{II} p^i \) where \( p^i \) and \( p^o \) are the diagonal propagation matrices for internal and external modes within the curved region respectively. Similar equations can be obtained if the incident modes come from the the other extreme of the bend (Fig. 3a):

\[
T^o = t^{II} p^i (1 - w)^{-1} t^I \\
R^o = t^{II} p^o r^{II} p^i (1 - w)^{-1} t^I + r^I
\]

(26) (27)

It was also possible to undertake the problem as a matching of three regions [14] as in Sec. II but numerical problems, even for bends of a small angle, leads us to prefer the method above. Besides, any number of bends can be joined in the same way by combining the latter four matrices through the diagonal propagation matrix \( p \) of straight regions (see Sec. III).
Fig. 7a shows results of the conductance for a single bend (of \( \theta \) angle equal to \( \pi/2 \)) as a function of the Fermi energy with the first subband occupied. The width of the wire is 1 \( l_m \) and four different inner radii \( R_i \) have been chosen. It can be seen the way in which the softness of this inner radius affects, crucially, the value of the conductance near the bottom of the subband but it is not important for larger values of the Fermi energy. Fig. 7b shows the behavior of the conductance for the Fermi energy near the second subband. An additional characteristic is the appearance of a dip when the second subband is about to enter. Again the important parameter is the inner radius which, given in magnetic lengths, shows how an increasing magnetic field reestablishes a perfect conductance even near the bottom of the subbands. In this way we can expect perfect conductance in these kind of bends for reasonably low values of the magnetic field.

V. CONCLUSIONS

The conclusions of the work can be summarized in the following points: a) The matching technique in the presence of a magnetic field is not a trivial problem and must be faced carefully in order to obtain correct results for actual situations. b) Although other techniques can be used for strips or straight waveguides, that presented here allows to face in the same manner changes, either in the potential or in the geometry of the waveguide, without changing the framework used. In addition to it, experimental results can be reproduced and easily understood.

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APPENDIX A: NUMERICAL SOLUTION OF SCHRÖDINGER EQUATION

Schrödinger equation \( \phi''(x) + f(x, k)\phi(x) - \epsilon\phi(x) = 0 \) (A1)

where \( f(x, k) = (x + k)^2 + v(x) \). If we discretized the \( x \) variable into \( N + 1 \) one-dimensional cells (with \( N \) of the order of 100 in the actual calculations) within the limits \( x_0 \) and \( x_{N+1} \) so that \( x_i = iq + x_0 \) with \( q = (x_{N+1} - x_0)/(N + 1) \) then, \( \phi(x_i) = \phi_i \) and the second derivative can be expressed (to the lowest order) as

\[
\phi_i'' = \alpha(\phi_{i-1} - 2\phi_i + \phi_{i+1})
\]

with \( \alpha = q^2 \). Eq. (A1) becomes

\[
- \alpha(\phi_{i-1} - 2\phi_i + \phi_{i+1}) + f(x, k)\phi_i - \epsilon\phi_i = 0
\]

(A3)

With the following boundary conditions for the \( \phi_i \) function

\[
\begin{align*}
\phi_0 &= 0 \\
\phi_{N+1} &= 0
\end{align*}
\]

(A4)

the Schrödinger equation \( \phi''(x) + f(x, k)\phi(x) - \epsilon\phi(x) = 0 \) can be transformed into the following homogeneous system of equations:

\[
\begin{pmatrix}
W_1(k) & -\alpha & 0 & \ldots & 0 \\
-\alpha & W_2(k) & -\alpha & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & -\alpha & W_{N-1}(k) & -\alpha & 0 \\
0 & -\alpha & W_N(k) & -\alpha & 0
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_{N-1} \\
\phi_N
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

(A5)

with \( W_i(k) = [f(iq + x_0 + k) - \epsilon] + 2\alpha \). Such equation system has a non-trivial solution if and only if
\[
\begin{pmatrix}
W_1(k) & -\alpha & 0 & \cdots \\
-\alpha & W_2(k) & -\alpha & \cdots \\
0 & \ddots & 0 \\
\vdots & & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \ddots \\
0 & -\alpha & W_{N-1}(k) & -\alpha \\
-\alpha & W_{N}(k) & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

where \( \text{det} \) denote the determinant which can be expressed by means of a recurrence relation as

\[
D_{-1}(k) = 0
\]
\[
D_0(k) = 1
\]
\[
D_1(k) = W_N'(k)
\]
\[
\vdots
\]
\[
D_i(k) = W_{N-i+1}'(k)D_{i-1}(k) - D_{i-2}(k)
\]
\[
\vdots
\]
\[
D_N(k) = W_1'(k)D_{N-1}(k) - D_{N-2}(k) = F(k)
\]

with \( W_i' = -W_i/\alpha \). The complex roots \( k_n \) of the so-defined \( F(k) \) function are the wave vectors we are looking for.

Once we know the wave vectors \( k_n \) we can calculate the corresponding \( \phi_i(k_n) \) complex functions to be used in the matching by means of

\[
\phi_{i+1}(k_n) = -\phi_{i-1}(k_n) + [(P_i(k_n) - \epsilon)/\alpha + 2]\phi_i(k_n)
\]

where \( P_i(k_n) = f(iq + x_0 + k_n) \).

In order to check the functions obtained above we can resort to standard Sturm-Liouville theory. Those functions must behave in a way that the following expression must be obeyed (we return to the usual continuous space \( \phi_i(k_n) \equiv \phi_n(x) \)):

\[
(k_i - k_j^*) \int_{x_0}^{x_{N+1}} (k_i + k_j^* - 2x)\phi_j^*(x)\phi_i(x) = 0
\]
Eq. 20 can be treated in exactly the same way just by substituting $f(x, k)$ by

$$f(r, m) = -\frac{1}{4r^2} + \frac{1}{r^2} \left( \frac{r^2}{2} - m \right)^2 + v(r)$$  \hspace{1cm} (A10)

and the $\psi_n(r)$ functions must obey the expression

$$(m_i - m_j^*) \int_{r_0}^{r_{N+1}} \left( \frac{(m_i + m_j^*)}{r^2} - 1 \right) \psi_j^*(r) \psi_i(r) = 0$$  \hspace{1cm} (A11)
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FIGURES

FIG. 1. Schematic view of a waveguide crossed by a barrier-like potential showing the three regions I, II, III involved in the matching problem. The thin lines give an idea of the trajectories followed by the edge states although they do not behave adiabatically in this geometry any more.

FIG. 2. a) Dispersion relation of the lowest subbands in a waveguide in the presence of a perpendicular magnetic field. It is also shown the chosen energy above the three lowest subbands. b) All the wave vectors found numerically for a given energy depicted in the $k$-complex plane. The effect of either increasing or decreasing the magnetic field is shown visually by the arrows pointing outwards or inwards respectively.

FIG. 3. a) Conductance versus Fermi energy for a magnetic field of 1 T. The length $L$ of the box is $8l_m$ and the width $W$ is $4l_m$. The height of the barriers is $1.5\frac{\hbar\omega_c}{2}$ and the width of them $0.5l_m$. The inset shows the semiclassical picture of trajectories followed by the edge states in both regions A and B. b) Conductance for the same box in the case of 1.56 T.

FIG. 4. Conductance versus magnetic field for a Fermi energy of $6\frac{\hbar\omega_c}{2}$ of a box of length $8l_m$, width $4l_m$, height of the barriers $1.5\frac{\hbar\omega_c}{2}$ and width of them $0.5l_m$.

FIG. 5. Evolution of the fifth state when increasing the magnetic field. The picture shows how the density tends to stick to the walls forming an edge state running all the way round the box.

FIG. 6. a) View of the total bend showing the two different possibilities for incoming modes as explained in the text. b) Schematic view of the matching between regions I and II of different geometry. The arrows show the four different possibilities for the incoming modes as explained in the text.

FIG. 7. a) Conductance of a waveguide of $1l_m$ wide with a bend of angle $\pi/2$ for different inner radii $R_i$ as a function of the Fermi energy. The first subband is the only one occupied. b) Conductance with the first and second subbands occupied for different inner radii in the same case than a).