DUALIZING COMPLEX OF THE FACE RING
OF A SIMPLICIAL POSET

KOHJI YANAGAWA

Abstract. A finite poset $P$ is called simplicial, if it has the smallest element $\hat{0}$, and every interval $[\hat{0}, x]$ is a boolean algebra. The face poset of a simplicial complex is a typical example. Generalizing the Stanley-Reisner ring of a simplicial complex, Stanley assigned the graded ring $A_P$ to $P$. This ring has been studied from both combinatorial and topological perspective. In this paper, we will give a concise description of a dualizing complex of $A_P$, which has many applications.

1. Introduction

All posets (partially ordered sets) in this paper will be assumed to be finite. By the order given by inclusion, the power set of a finite set becomes a poset called a boolean algebra. We say a poset $P$ is simplicial, if it admits the smallest element $\hat{0}$, and the interval $[\hat{0}, x] := \{ y \in P \mid y \leq x \}$ is isomorphic to a boolean algebra for all $x \in P$. For the simplicity, we denote rank$(x)$ of $x \in P$ just by $\rho(x)$. If $P$ is simplicial and $\rho(x) = m$, then $[\hat{0}, x]$ is isomorphic to the boolean algebra $2^{\{1, \ldots, m\}}$.

Let $\Delta$ be a finite simplicial complex (with $\emptyset \in \Delta$). Its face poset (i.e., the set of the faces of $\Delta$ with the order given by inclusion) is a simplicial poset. Any simplicial poset $P$ is the face (cell) poset of a regular cell complex, which we denote by $\Gamma(P)$. For $\hat{0} \neq x \in P$, $c(x) \in \Gamma(P)$ denotes the open cell corresponds to $x$. Clearly, $\dim c(x) = \rho(x) - 1$. While the closure $c(x)$ of $c(x)$ is always a simplex, the intersection $c(x) \cap c(y)$ for $x, y \in P$ is not necessarily a simplex. For example, if two $d$-simplices are glued along their boundaries, then it is not a simplicial complex, but gives a simplicial poset.

In the rest of the paper, $P$ is a simplicial complex. For $x, y \in P$, set

$$[x \lor y] := \text{the set of minimal elements of } \{ z \in P \mid z \geq x, y \}.$$ 

More generally, for $x_1, \ldots, x_m \in P$, $[x_1 \lor \ldots \lor x_m]$ denotes the set of minimal elements of the common upper bounds of $x_1, \ldots, x_m$.

Set $\{ y \in P \mid \rho(y) = 1 \} = \{ y_1, \ldots, y_n \}$. For $U \subset [n] := \{1, \ldots, n\}$, we simply denote $\bigvee_{i \in U} y_i$ by $[U]$. Here $\emptyset = \{ \hat{0} \}$. If $x \in [U]$, then $\rho(x) = \#U$. For each $x \in P$, there exists a unique $U$ such that $x \in [U]$. Let $x, x' \in P$ with $x \geq x'$ and $\rho(x) = \rho(x') + 1$, and take $U, U' \subset [n]$ such that $x \in [U]$ and $x' \in [U']$. Since $U = U' \bigcup \{i\}$ for some $i$ in this case, we can set

$$\alpha(i, U) := \# \{ j \in U \mid j < i \} \quad \text{and} \quad \epsilon(x, x') := (-1)^{\alpha(i, U)}.$$ 

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Then $\epsilon$ gives an incidence function of the cell complex $\Gamma(P)$, that is, for all $x, y \in P$ with $x > y$ and $\rho(x) = \rho(y) + 2$, we have

$$
\epsilon(x, z) \cdot \epsilon(z, y) + \epsilon(x, z') \cdot \epsilon(z', y) = 0,
$$
where $\{z, z'\} = \{w \in P \mid x > w > y\}$.

Stanley [9] assigned the commutative ring $A_P$ to a simplicial poset $P$. For the definition, we remark that if $[x \vee y] \neq \emptyset$ then $\{z \in P \mid z \leq x, y\}$ has the largest element $x \wedge y$. Let $\mathbb{k}$ be a field, and $S := \mathbb{k}[t_x \mid x \in P]$ the polynomial ring in the variables $t_x$. Consider the ideal

$$
I_P := \left( t_{x \wedge y} \sum_{z \in [x \vee y]} t_z \mid x, y \in P \right) + (t_0 - 1)
$$
of $S$ (if $[x \vee y] = \emptyset$, we interpret that $t_{x \wedge y} \sum_{z \in [x \vee y]} t_z = t_x t_y$), and set

$$
A_P := S/I_P.
$$

We denote $A_P$ just by $A$, if there is no danger of confusion. Clearly, $\dim A_P = \text{rank } P = \dim \Gamma(P) + 1$. For a rank 1 element $y_i \in P$, set $t_i := t_{y_i}$. If $\{x\} = [U]$ for some $U \subset [n]$ with $\#U \geq 2$, then $t_x = \prod_{i \in U} t_i$ in $A$, and $t_x$ is a “dummy variable”.

Since $I_P$ is a homogeneous ideal under the grading given by $\deg(t_x) = \rho(x)$, $A$ is a graded ring. If $\Gamma(P)$ is a simplicial complex, then $A_P$ is generated by degree 1 elements, and coincides with the Stanley-Reisner ring of $\Gamma(P)$.

Note that $A$ also has a $\mathbb{Z}^n$-grading such that $\deg t_i \in \mathbb{N}^n$ is the $i$th unit vector. For each $x \in P$, the ideal

$$
\mathfrak{p}_x := \left( t_z \mid z \not\leq x \right)
$$
of $A$ is a ($\mathbb{Z}^n$-graded) prime ideal with $\dim A/\mathfrak{p}_x = \rho(x)$, since $A/\mathfrak{p}_x \cong \mathbb{k}[t_i \mid y_i \leq x]$.

In [1], Duval adapted classical argument on Stanley-Reisner rings for $A_P$, and got basic results. Recently, M. Masuda and his coworkers studied $A_P$ with a view from toric topology, since the equivariant cohomology ring of a torus manifold is of the form $A_P$ (cf. [4, 5]). In this paper, we will introduce another approach.

Let $R$ be a noetherian commutative ring, $\text{Mod } R$ the category of $R$-modules, and mod $R$ its full subcategory consisting of finitely generated modules. The dualizing complex $D^*_R$ of $R$ gives the important duality $\text{RHom}_R(-, D^*_R)$ on the bounded derived category $\text{D}^b(\text{mod } R)$ (cf. [2]). If $R$ is a (graded) local ring with the maximal ideal $\mathfrak{m}$, then the (graded) Matlis dual of $H^{-i}(D^*_R)$ is the local cohomology $H^i_{\mathfrak{m}}(R)$.

We have a concise description of a dualizing complex $A_P$ as follows. This result refines Duval’s computation of $H^i_{\mathfrak{m}}(A)$ ([1, Theorem 5.9]).

**Theorem 1.1.** Let $P$ be a simplicial poset with $d = \text{rank } P$, and set $A := A_P$. The complex

$$
I^*_A : 0 \to I_{A}^{-d} \to I_{A}^{-d+1} \to \cdots \to I_{A}^{0} \to 0,
$$
given by

$$
I_{A}^{-i} := \bigoplus_{x \in P, \rho(x) = i} A/\mathfrak{p}_x,
$$
and

$$\partial_A^{-i} : I_A^{-i} \supset A/p_x \ni 1_{A/p_x} \mapsto \sum_{\rho(y) = i-1, \ y \leq x} \epsilon(x, y) \cdot 1_{A/p_y} \in \bigoplus_{\rho(y) = i-1, \ y \leq x} A/p_y \subset I_A^{-i+1}$$

is isomorphic to a dualizing complex $D_A^*$ of $A$ in $D^b(\text{Mod} A)$.

In [11], the author defined a squarefree module over a polynomial ring, and many applications have been found. This idea is also useful for our study. In fact, regarding $A$ as a squarefree module over the polynomial ring $\text{Sym} A_1$, Duval’s formula of $H_m^i(A)$ can be proved quickly (Remark 2.6). Moreover, we can show that a theorem of Murai and Terai ([7]) on the $h$-vectors of simplicial complexes also holds for simplicial posets (Theorem 5.6). In the present paper, we will define a squarefree module over $A$ to study the interaction between the topological properties of $\Gamma(P)$ and the homological properties of $A$.

The category $\text{Sq} A$ of squarefree $A$-modules is an abelian category with enough injectives, and $A/p_x$ is an injective object. Hence $I_A^* \in D^b(\text{Sq} A)$, and $\mathbb{D}(-) := \text{Hom}_A^*(-, I_A^*)$ gives a duality on $K^b(\text{Inj-Sq}) (\cong D^b(\text{Sq} A))$, where $\text{Inj-Sq}$ denotes the full subcategory of $\text{Sq} A$ consisting of all injective objects (i.e., finite direct sums of copies of $A/p_x$ for various $x \in P$). Via the forgetful functor $\text{Sq} A \to \text{mod} A$, $\mathbb{D}$ coincides with the usual duality $\text{RHom}_A(-, D_A^*)$ on $D^b(\text{mod} A)$.

By [13], we can assign a squarefree $A$-module $M$ the constructible sheaf $M^+$ on (the underlying space $X$ of) $\Gamma(P)$. In this context, the duality $\mathbb{D}$ corresponds to the Poincaré-Verdier duality on the derived category of the constructible sheaves on $X$ up to translation as in [8, 13]. In particular, the sheafification of the complex $I_A^*[-1]$ coincides with the Verdier dualizing complex of $X$ with the coefficients in $k$, where $[-1]$ represents the translation by $-1$. Using this argument, we can show the following. At least for the Cohen-Macaulay property, the next result has been shown in Duval [1]. However our proof gives new perspective.

**Corollary 1.2** (see, Theorem 4.4). The Cohen-Macaulay, Gorenstein* and Buchsbaum properties, and Serre’s condition $(S_i)$ of $A_P$ depend only on the topology of the underlying space of $\Gamma(P)$ and $\text{char}(k)$. Here we say $A_P$ is Gorenstein*, if $A_P$ is Gorenstein and the graded canonical module $\omega_{A_P}$ is generated by its degree 0 part.

While Theorem 1.1 and the results in §4 are very parallel to the corresponding ones for toric face rings ([8]), the construction of a toric face ring and that of $A_P$ are not so similar. Both of them are generalizations of the notion of Stanley-Reisner rings, but the directions of the generalizations are almost opposite (for example, Proposition 5.1 does not hold for toric face rings). The prototype of the results in [8] and the present paper is found in [13]. However, the subject there is “sheaves on a poset”, and the connection to our rings is not so straightforward.

2. The proof of Theorem 1.1

In the rest of the paper, $P$ is a simplicial poset with rank $P = d$. We use the same convention as the preceding section, in particular, $A = A_P$, $\{ y \in P \mid \rho(y) = 1 \} = \{ y_1, \ldots, y_n \}$, and $t_i := t_{y_i} \in A$. 

For a subset $U \subseteq [n] = \{1, \ldots, n\}$, $A_U$ denotes the localization of $A$ by the multiplicatively closed set $\{ \prod_{i \in U} t_i^{a_i} | a_i \geq 0 \}$.

**Lemma 2.1.** For $x \in [U]$, 
\[ u_x := \frac{t_x}{\prod_{i \in U} t_i} \in A_U \]
is an idempotent. Moreover, $u_x \cdot u_{x'} = 0$ for $x, x' \in [U]$ with $x \neq x'$, and 
\[ 1_{A_U} = \sum_{x \in [U]} u_x. \] 

Hence we have a $\mathbb{Z}^n$-graded direct sum decomposition 
\[ A_U = \bigoplus_{x \in [U]} A_U \cdot u_x \]
(if $[U] = \emptyset$, then $A_U = 0$).

**Proof.** Since $\prod_{i \in U} t_i = \sum_{x \in [U]} t_x$ in $A$, the equation (2.1) is clear. For $x, x' \in [U]$ with $x \neq x'$, we have $[x \vee x'] = \emptyset$ and $t_x \cdot t_{x'} = 0$. Hence $u_x \cdot u_{x'} = 0$ and 
\[ u_x = u_x \cdot 1_{A_U} = u_x \cdot \sum_{x'' \in [U]} u_{x''} = u_x \cdot u_x. \]

Now the last assertion is clear. \hfill \square

Let $\text{Gr} A$ be the category of $\mathbb{Z}^n$-graded $A$-modules, and $\text{gr} A$ its full subcategory consisting of finitely generated modules. Here a morphism $f : M \to N$ in $\text{Gr} A$ is an $A$-homomorphism with $f(M_a) \subset N_a$ for all $a \in \mathbb{Z}^n$. As usual, for $M$ and $a \in \mathbb{Z}^n$, $M(a)$ denotes the shifted module of $M$ with $M(a)_b = M_{a+b}$. For $M, N \in \text{Gr} A$, 
\[ \text{Hom}_A(M, N) := \bigoplus_{a \in \mathbb{Z}^n} \text{Hom}_{\text{Gr} A}(M, N(a)) \]
has a $\mathbb{Z}^n$-graded $A$-module structure. Similarly, $\text{Ext}_A^i(M, N) \in \text{Gr} A$ can be defined. If $M \in \text{gr} A$, the underlying module of $\text{Hom}_A(M, N)$ is isomorphic to $\text{Hom}_A(M, N)$, and the same is true for $\text{Ext}_A^i(M, N)$.

If $M \in \text{Gr} A$, then $M^\vee := \bigoplus_{a \in \mathbb{Z}^n} \text{Hom}_k(M_{-a}, k)$ can be regarded as a $\mathbb{Z}^n$-graded $A$-module, and $(-)^\vee$ gives an exact contravariant functor from $\text{Gr} A$ to itself, which is called the graded Matlis duality functor. For $M \in \text{Gr} A$, it is Matlis reflexive (i.e., $M^{\vee \vee} \cong M$) if and only if $\dim_k M_a < \infty$ for all $a \in \mathbb{Z}^n$.

**Lemma 2.2.** $A_U \cdot u_x$ is Matlis reflexive, and $E_A(x) := (A_U \cdot u_x)^\vee$ is injective in $\text{Gr} A$. Moreover, any indecomposable injective in $\text{Gr} A$ is isomorphic to $E_A(x)(a)$ for some $x \in P$ and $a \in \mathbb{Z}^n$.

**Proof.** Clearly, $A_U \cdot u_x$ is a $\mathbb{Z}^n$-graded $\mathbb{k}[t_i^{\pm 1} | i \in U]$-module. For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, let $a' \in \mathbb{N}^n$ be the vector whose $i$th component is 
\[ a'_i = \begin{cases} a_i & \text{if } i \notin U, \\ 1 & \text{otherwise.} \end{cases} \]
Then we have \( \dim_k(A_U \cdot u_x)_a = \dim_k(A \cdot t_x)_{a'} \leq \dim_k A_{a'} < \infty \), and \( A_U \cdot u_x \) is Matlis reflexive.

The injectivity of \( E_A(x) \) follows from the same argument as [6, Lemma 1.13]. In fact, we have a natural isomorphism

\[
\Hom_A(M, E_A(x)) \cong (M \otimes_A E_A(x))^\vee
\]

for \( M \in \text{Gr} \ A \) by [6, Lemma 11.16]. Since \( E_A(x)^\vee \cong A_U \cdot u_x \) is a flat \( A \)-module, \( \Hom_A(\cdot, E_A(x)) \) gives an exact functor.

Since \( E_A(x) \) is the injective envelope of \( A/p_x \) in \( \text{Gr} \ A \), and an associated prime of \( M \in \text{Gr} \ A \) is \( p_x \) for some \( x \in P \), the last assertion follows.

If \( \{A_U \cdot u_x\}_{a} \neq 0 \) for \( a \in \mathbb{N}^n \), then it is obvious that \( a \in \mathbb{N}^U \) (i.e., \( a_i = 0 \) for \( i \notin U \)). As shown in the above proof, we have \( \dim_k(A_U \cdot u_x)_{-a} = 1 \) with \( t^{-a} \cdot u_x := u_x/\prod_{i \in U} t_{ia}^a \in (A_U \cdot u_x)_{-a} \) in this case.

For \( M \in \text{Gr} \ A \), its “\( \mathbb{N}^n \)-graded part” \( M_{\geq 0} := \bigoplus_{a \in \mathbb{N}^n} M_a \) is a submodule of \( M \). Then we have a canonical injection

\[
\phi_x \colon A/p_x \longrightarrow E_A(x)
\]

defined as follows: The set of the monomials \( x^a := \prod_{i \in U} t_{ia}^a \in A/p_x \cong \mathbb{k}[t_i \mid i \in U] \) with \( a \in \mathbb{N}^U \) forms a \( \mathbb{k} \)-basis of \( A/p_x \) (\( \prod_{i \in U} t_i = t_x \) here), and \( \phi_x(x^a) = (E_A(x))_a = \Hom_k((A_U \cdot u_x)_{-a}, \mathbb{k}) \) for \( a \in \mathbb{N}^U \) is simply given by \( t^{-a} \cdot u_x \mapsto 1 \). Note that \( \phi_x \) induces the isomorphism

\[
A/p_x \cong E_A(x)_{\geq 0}.
\]

The Čech complex \( C^\bullet \) of \( A \) with respect to \( t_1, \ldots, t_n \) is of the form

\[
0 \to C^0 \to C^1 \to \cdots \to C^d \to 0 \quad \text{with} \quad C^i = \bigoplus_{U \subseteq [n]} A_U \quad (U \supseteq i)
\]

(note that if \( \#U > d = \dim A \) then \( A_U = 0 \)). The differential map is given by

\[
C^i \ni A_U \ni a \mapsto \sum_{U^\prime \supseteq U \atop \#U^\prime = i+1} (-1)^{a(U^\prime \setminus U)} f_{U^\prime, U}(a) \in \bigoplus_{U^\prime \supseteq U \atop \#U^\prime = i+1} A_{U^\prime} \subset C^{i+1},
\]

where \( f_{U^\prime, U} : A_U \to A_{U^\prime} \) is the natural map.

Since the radical of the ideal \( (t_1, \ldots, t_n) \) is the graded maximal ideal \( \mathfrak{m} := (t_x \mid \hat{0} \neq x \in P) \), the cohomology \( H^i(C^\bullet) \) of \( C^\bullet \) is isomorphic to the local cohomology \( H^i_{\mathfrak{m}}(A) \). Moreover, \( C^\bullet \) is isomorphic to \( \Gamma_{\mathfrak{m}}A \) in the bounded derived category \( D^b(\text{Mod} \ A) \). Here \( \Gamma_{\mathfrak{m}} : D^b(\text{Mod} \ A) \to D^b(\text{Mod} \ A) \) is the right derived functor of \( \Gamma_{\mathfrak{m}} : \text{Mod} \ A \to \text{Mod} \ A \) given by \( \Gamma_{\mathfrak{m}}(M) = \{ s \in M \mid \mathfrak{m}^i s = 0 \text{ for } i \gg 0 \} \).

The same is true in the \( \mathbb{Z}^n \)-graded context. We may regard \( \Gamma_{\mathfrak{m}} \) as a functor from \( \text{Gr} \ A \) to itself, and let \( \Gamma_{\mathfrak{m}} : D^b(\text{Gr} \ A) \to D^b(\text{Gr} \ A) \) be its right derived functor. Then \( C^\bullet \cong \Gamma_{\mathfrak{m}}(A) \) in \( D^b(\text{Gr} \ A) \).

Let \( D_A^\bullet \) be the \( \mathbb{Z}^n \)-graded normalized dualizing complex of \( A \). By the \( \mathbb{Z}^n \)-graded version of the local duality theorem [2, Theorem V.6.2], \( (D_A^\bullet) \cong \Gamma_{\mathfrak{m}}(A) \) in \( D^b(\text{Gr} \ A) \). Since \( D_A \in D^b_{\text{gr}} \), it is Matlis reflexive, and we have

\[
* D_A^\bullet \cong (D_A^\bullet)^\vee \cong \Gamma_{\mathfrak{m}}(A)^\vee \cong (C^\bullet)^\vee.
\]
Since each \((C^i)^\vee\) is isomorphic to the injective object
\[
\bigoplus_{x \in P} E_A(x)
\]
in Gr \(A\), \((C^i)^\vee\) actually coincides with \(*D_A^i\). Hence \(*D_A^i\) is of the form
\[
0 \to \bigoplus_{x \in P} E_A(x) \to \bigoplus_{\rho(x)=d-1} E_A(x) \to \cdots \to E_A(\emptyset) \to 0,
\]
where the cohomological degree is given by the same way as \(I_A^i\).

For each \(i \in \mathbb{Z}\), we have an injection \(\phi^i : I_A^i \to *D_A^i\) given by
\[
I_A^i = \bigoplus_{\rho(x)=i} A/p_x \subset A/p_x \to E_A(x) \subset \bigoplus_{\rho(x)=i} E_A(x) = *D_A^i.
\]

By the definition of \(\phi_x : A/p_x \to E_A(x) = (A_U \cdot u_x)^\vee\), we have a cochain map
\[
\phi^i : I_A^i \to *D_A^i.
\]

**Lemma 2.3.** For all \(i\), the cohomology \(H^i(*D_A^i)\) of \(*D_A^i\) is \(\mathbb{N}^n\)-graded.

This lemma immediately follows from Duval's description of \(H^i_m(A)\) ([11, Theorem 5.9]), but we give another proof using the notion of squarefree modules. This approach makes our proof more self-contained, and we will extend this idea in the following sections.

Let \(S = \mathbb{k}[x_1, \ldots, x_n]\) be a polynomial ring, and regard it as a \(\mathbb{Z}^n\)-graded ring. For \(a = (a_1, \ldots, a_n) \in \mathbb{N}^n\), let \(x^n\) denote the monomial \(\prod x_i^{a_i} \in S\).

**Definition 2.4 ([11]).** With the above notation, a \(\mathbb{Z}^n\)-graded \(S\)-module \(M\) is called squarefree, if it is finitely generated, \(\mathbb{N}^n\)-graded (i.e., \(M = \bigoplus_{a \in \mathbb{N}^n} M_a\)), and the multiplication map \(M_a \ni t \mapsto x_it \in M_{a+\mathbf{e}_i}\) is bijective for all \(a = (a_1, \ldots, a_n) \in \mathbb{N}^n\) and all \(i\) with \(a_i > 0\). Here \(\mathbf{e}_i \in \mathbb{N}^n\) is the \(i\)th unit vector.

The following lemma is easy, and we omit the proof.

**Lemma 2.5.** Consider the polynomial ring \(T := \text{Sym} A_1 = \mathbb{k}[t_1, \ldots, t_n]\) (note that \(T\) is not a subring of \(A\)). Then \(A\) is a squarefree \(T\)-module.

**Remark 2.6.** Since \(A\) is a squarefree \(T\)-module, Duval's formula on \(H^i_m(A)\) immediately follows from [11, Lemma 2.9]. However, since \(H^i_m(A)\) has a finer “grading” (see [11, Corollary 3.6] below), the formula will be mentioned in Corollary 4.3.

**The proof of Lemma 2.3.** Let \(T\) be as in Lemma 2.5. For \(\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n\), \(T(-1)\) is the \((\mathbb{Z}^n\)-graded) canonical module of \(T\). By the local duality theorem, we have
\[
H^i(*D_A^i) \cong \text{Ext}_1^i(A, *D_A^i) \cong \text{Ext}_{T}^{n+i}(A, T(-1)).
\]

By [11, Theorem 2.6], \(\text{Ext}_{T}^{n+i}(A, T(-1))\) is a squarefree module, in particular, \(\mathbb{N}^n\)-graded. \(\square\)

**The proof of Theorem 1.1.** Recall the cochain map \(\phi : I_A^i \to *D_A^i\) constructed before Lemma 2.3. By [2.2], \(\phi\) gives the isomorphism \(I_A^i \cong (*D_A^i)_{\geq 0}\). Hence \(\phi\) is a quasi-isomorphism by Lemma 2.3. Since \(*D_A^i \cong D_A^i\) in \(D(A)\), we are done. \(\square\)
3. Squarefree Modules over $A_P$

In this section, we will define a squarefree module over the face ring $A = A_P$ of a simplicial poset $P$. For this purpose, we equip $A$ with a finer “grading”, where the index set is no longer a monoid (similar idea has appeared in [11]).

Recall the convention that $\{ y \in P \mid \rho(y) = 1 \} = \{ y_1, \ldots, y_n \}$ and $t_i = t_{y_i} \in A$. For each $x \in P$, set

$$M(x) := \bigoplus_{y \leq x} N e^x_y,$$

where $e^x_y$ is a basis element. So $M(x) \cong N^{\rho(x)}$ as additive monoids. For $x, z$ with $x \leq z$, we have an injection $\iota_{x,z} : M(x) \ni e^x_y \mapsto e^z_y \in M(z)$ of monoids. Set

$$M := \lim_{x \in P} M(x),$$

where the direct limit is taken in the category of sets with respect to $\iota_{x,z} : M(x) \to M(z)$ for $x, z \in P$ with $x \leq z$. Note that $M$ is no longer a monoid. Since all $\iota_{x,z}$ is injective, we can regard $M(x)$ as a subset of $M$. For each $a \in M$, $\{ x \in P \mid a \in M(x) \}$ has the smallest element, which is denoted by $\sigma(a)$.

We say a monomial

$$m = \prod_{x \in P} t^m_x \in A \quad (n_x \in \mathbb{N})$$

is standard, if $\{ x \in P \mid n_x \neq 0 \}$ is a chain. In this case, set $\sigma(m) := \max \{ x \in P \mid n_x \neq 0 \}$. If $n_x = 0$ for all $x \neq 0$, then $m = 1$. Hence 1 is a standard monomial with $\sigma(1) = 0$. As shown in [9], the set of standard monomials forms a $k$-basis of $A$.

There is a one-to-one correspondence between the elements of $M$ and the standard monomials of $A$. For a standard monomial $m$, set $U := \{ i \in \pi \mid y_i \leq \sigma(m) \}$. Then we have $\sigma(m) \in [U]$. There is $a \in N^U$ such that the image of $m$ in $A/_{\pi(m)} \cong k[t_i \mid i \in U]$ is a monomial of the form $\prod_{i \in U} e^m_i$. So $m$ corresponds to $a \in M(\sigma(m)) = \bigoplus_{i \in U} N e^{\sigma(m)}_i \subset M$ whose $e^{\sigma(m)}_i$-component is $a_i$. We denote this $m$ by $\underline{a}$.

Let $a, b \in M$. If $[\sigma(a) \vee \sigma(b)] \neq \emptyset$, then we can take the sum $a + b \in M(x)$ for each $x \in [\sigma(a) \vee \sigma(b)]$. Unless $[\sigma(a) \vee \sigma(b)]$ consists of a single element, we cannot define $a + b \in M$. Hence we denote each $a + b \in M(x)$ by $(a + b)|x$.

**Definition 3.1.** $M \in \text{Mod } A$ is said to be $M$-graded if the following are satisfied:

1. $M = \bigoplus_{a \in k} M_a$ as $k$-vector spaces;
2. For $a, b \in M$, we have

$$t^a M_b \subset \bigoplus_{x \in [\sigma(a) \vee \sigma(b)]} M_{(a+b)}|x.$$

Hence, if $[\sigma(a) \vee \sigma(b)] = \emptyset$, then $t^a M_b = 0$.

Clearly, $A$ itself is an $M$-graded module with $A_a = k t^a$. Since there is a natural map $M \to N^n$, an $M$-graded module can be seen as an $N^n$-graded module.
If $M$ is an $\mathcal{M}$-graded $A$-module, then

$$M_{\mathcal{M}} := \bigoplus_{\underline{a} \in \mathcal{M}(x)} M_{\underline{a}}$$

is an $\mathcal{M}$-graded submodule for all $x \in P$, and

$$M_{\leq x} := M/M_{\mathcal{M}}$$

is a $\mathbb{Z}^{\mathcal{M}(x)}$-graded module over $A/p_x \cong k[t_i | y_i \leq x]$.

**Definition 3.2.** We say an $\mathcal{M}$-graded $A$-module $M$ is squarefree, if $M_{\leq x}$ is a squarefree module over the polynomial ring $A/p_x \cong k[t_i | y_i \leq x]$ for all $x \in P$.

Note that squarefree $A$-modules are automatically finitely generated, and can be seen as squarefree modules over $T = \text{Sym} A_1$.

Clearly, $A$ itself, $p_x$ and $A/p_x$ for $x \in P$, are squarefree. Let $\text{Sq} A$ be the category of squarefree $A$-modules and their $A$-homomorphisms $f : M \to M'$ with $f(M_{\underline{a}}) \subset M'_{\underline{a}}$ for all $\underline{a} \in \mathcal{M}$. For example, $I_\bullet A$ is a complex in $\text{Sq} A$. To see basic properties of $\text{Sq} A$, we introduce the incidence algebra of the poset $P$ as in [12] (so consult [12] for further information).

The incidence algebra $\Lambda$ of $P$ over $k$ is a finite dimensional associative $k$-algebra with basis $\{ e_{x,y} | x, y \in P, x \geq y \}$ whose multiplication is defined by

$$e_{x,y} \cdot e_{z,w} = \delta_{y,z} e_{x,w},$$

where $\delta_{y,z}$ denotes Kronecker’s delta.

Set $e_x := e_{x,x}$ for $x \in P$. Each $e_x$ is an idempotent, and $\Lambda e_x$ is indecomposable as a left $\Lambda$-module. Clearly, $e_x \cdot e_y = 0$ for $x \neq y$, and that $1_\Lambda = \sum_{x \in P} e_x$. Let $\text{mod} \Lambda$ be the category of finitely generated left $\Lambda$-modules. As a $k$-vector space, $N \in \text{mod} \Lambda$ has the decomposition $N = \bigoplus_{x \in P} e_x N$. Henceforth we set $N_x := e_x N$.

Clearly, $e_{x,y} N_y \subset N_x$, and $e_{x,y} N_z = 0$ if $y \neq z$.

For each $x \in P$, we can construct a left $\Lambda$-module as follows: Set

$$E_\Lambda(x) := \bigoplus_{y \in P, y \leq x} k \bar{e}_y,$$

where $\bar{e}_y$’s are basis elements. The module structure of $E_\Lambda(x)$ is defined by

$$e_{z,w} \cdot \bar{e}_y = \begin{cases} \bar{e}_z & \text{if } w = y \text{ and } z \leq x; \\ 0 & \text{otherwise.} \end{cases}$$

Then $E_\Lambda(x)$ is indecomposable and injective in $\text{mod} \Lambda$. Conversely, any indecomposable injective is of this form. Moreover, $\text{mod} \Lambda$ is an abelian category with enough injectives, and the injective dimension of each object is at most $d$.

**Proposition 3.3.** There is an equivalence between $\text{Sq} A$ and $\text{mod} \Lambda$. Hence $\text{Sq} A$ is an abelian category with enough injectives and the injective dimension of each object is at most $d$. An object $M \in \text{Sq} A$ is an indecomposable injective if and only if $M \cong A/p_x$ for some $x \in P$. 
Proof. Let $N \in \text{mod } \Lambda$. For each $a \in \mathbb{M}$, we assign a $k$-vector space $M_a$ with a bijection $\mu_a: N_{\sigma_a} \rightarrow M_a$. We put an $\mathbb{M}$-graded $A$-module structure on $M := \bigoplus_{a \in \mathbb{M}} M_a$ by

$$ t^a s = \sum \mu_{(a+b)|x}(e_{x,\sigma_b} \cdot \mu^{-1}_b(s)) \text{ for } s \in M_b. $$

To see that $M$ is actually an $A$-module, note that both $(t^a/b)s$ and $(t^a/t^b)s$ equal

$$ \sum \mu_{(a+b+c)|x}(e_{x,\sigma_c} \cdot \mu^{-1}_c(s)) \text{ for } s \in M_c. $$

We can also show that $M$ is squarefree.

To construct the inverse correspondence, for $x \in P$ with $r = \rho(x)$, set $a(x) := (r, r, \ldots, r) \in \mathbb{N}^r \cong \mathbb{M}(x) \subset \mathbb{M}$. If $x \geq y$, then there is a degree $a(x) - a(y) \in \mathbb{M}(x) \subset \mathbb{M}$ such that $a(x) - a(y) = e_{x,y}$. (One might think a simpler definition $a(x) := (1, 1, \ldots, 1) \in \mathbb{N}^r$ works. However this is not true. In this case, the candidate of $a(x) - a(y)$ belongs to $M(x)$ for some $z \in P$ with $z < x$. So $a(x) - a(y) + a(y)$ does not exist, unless $\#[y \cup z] = 1$.) Now we can construct $N \in \text{mod } \Lambda$ from $M \in \text{Sq} A$ as follows: Set $N_x := M_a(x)$, and define the multiplication map $N_y \ni s \mapsto e_{x,y} \cdot s \in N_x$ by $M_{a(y)} \ni s \mapsto t^{a(x) - a(y)} s \in M_{a(x)}$ for $x, y \in P$ with $x \geq y$.

By this correspondence, we have $\text{Sq} A \cong \text{mod } \Lambda$. For the last statement, note that $E_A(x) \in \text{mod } \Lambda$ corresponds to $A/p_x \in \text{Sq} A$.

Let $\text{Inj-Sq}^-$ be the full subcategory of $\text{Sq} A$ consisting of all injective objects, that is, finite direct sums of copies of $A/p_x$ for various $x \in P$. As is well-known, the bounded homotopy category $K^b(\text{Inj-Sq}^-)$ is equivalent to $D^b(\text{Sq} A)$. Since

$$ \text{Hom}_A(A/p_x, A/p_y) = \begin{cases} A/p_y & \text{if } x \geq y, \\ 0 & \text{otherwise}, \end{cases} $$

we have $\text{Hom}_A^*(J^*, I_A^*) \subset K^b(\text{Inj-Sq}^-)$ for all $J^* \subset K^b(\text{Inj-Sq}^-)$. Moreover, $\text{Hom}_A^*(-, I_A^*)$ gives a functor

$$ D: K^b(\text{Inj-Sq}^-) \rightarrow K^b(\text{Inj-Sq}^-)^{\text{op}}. $$

Proposition 3.4. Via the forgetful functor $U: \text{Inj-Sq} \rightarrow \text{gr } A$, $D$ coincides with $\text{RHom}_A(-, \ast D_A^*)$. More precisely, we have a natural isomorphism

$$ \Phi: U \circ D \cong \text{RHom}_A(-, \ast D_A^*) \circ U. $$

Here both $U \circ D$ and $\text{RHom}_A(-, \ast D_A^*) \circ U$ are functors from $K^b(\text{Inj-Sq}^-)$ to $D^b(\text{gr } A)$.

Proof. The cochain map $\phi^*: I_A^* \rightarrow \ast D_A^*$ induces the natural transformation $\Phi$. It remains to prove that $\Phi(J^*) : D(J^*) \rightarrow \text{RHom}_A(J^*, \ast D_A^*)$ is a quasi isomorphism for all $J^* \in K^b(\text{Inj-Sq}^-)$. For this fact, we use a similar argument to the final steps of the previous section (while the same argument as the proof of [8 Proposition 5.4] also works).

Note that $J^*$ is a complex of squarefree modules over the polynomial ring $T := \text{Sym } A_1$. Since $\text{RHom}_A(J^*, \ast D_A^*) \cong \text{RHom}_T(J^*, \ast D_T^*)$ by the local duality theorem, the cohomologies of $\text{RHom}_A(J^*, \ast D_A^*)$ are squarefree.
$T$-modules, in particular, $\mathbb{N}^n$-graded. On the other hand, through $\Phi$, $\mathbb{D}(J^\bullet)$ is isomorphic to the $\mathbb{N}^n$-graded part of $R\text{Hom}_A(J^\bullet, \mathcal{D}^\bullet_A)$. \hfill \Box

Remark 3.5. By the equivalence $K^b(\text{Inj-Sq}) \cong \mathbb{D}^b(\mathcal{S}q A)$, $\mathbb{D}$ can be regarded as a contravariant functor from $\mathbb{D}^b(\mathcal{S}q A)$ to itself. Then, through the equivalence $\mathcal{S}q R \cong \text{mod } \Lambda$, $\mathbb{D}$ coincides with the functor $\mathbb{D} : \mathbb{D}^b(\text{mod } \Lambda) \to \mathbb{D}^b(\text{mod } \Lambda)^{op}$ defined in \cite{13} up to translation. Hence, for $M^\bullet \in \mathbb{D}^b(\mathcal{S}q A)$, the complex $\mathbb{D}(M^\bullet)$ has the following description: The term of cohomological degree $p$ is

$$\mathbb{D}(M^\bullet)^p := \bigoplus_{i + p(x) = -p} (M^i_{\mathfrak{a}(x)})^* \otimes_k A/p_x,$$

where $(-)^*$ denotes the $k$-dual, and $\mathfrak{a}(x) \in \mathbb{M}(x) \subset \mathbb{M}$ is the one defined in the proof of Proposition 3.3. The differential is given by

$$(M^i_{\mathfrak{a}(x)})^* \otimes_k A/p_x \ni f \otimes 1_{A/p_x} \mapsto \sum_{\rho(x) = y, \rho(y) = p(x) - 1} \epsilon(x, y) \cdot f_y \otimes 1_{A/p_y} + (-1)^p \cdot f \circ \partial_{M^i}^{-1} \otimes 1_{A/p_x},$$

where $f_y \in (M^i_{\mathfrak{a}(y)})^*$ denotes $M^i_{\mathfrak{a}(y)} \ni s \mapsto f(\mathfrak{a}(x) - \mathfrak{a}(y) \cdot s) \in k$, and $\epsilon(x, y)$ is the incidence function. We also have $\mathbb{D} \circ \mathbb{D} \cong \text{id}_{\mathbb{D}^b(\mathcal{S}q A)}$.

Since $H^{-i}(\mathbb{D}(M)) \cong \text{Ext}^{-i}_{\mathcal{D}^\bullet_A}(M, \mathcal{D}^\bullet_A) \cong H^i_{\mathfrak{m}}(\mathcal{M})^\vee$ in $\text{Gr } A$, we have the following.

Corollary 3.6. If $M \in \mathcal{S}q A$, then the local cohomology $H^i_{\mathfrak{m}}(\mathcal{M})^\vee$ can be seen as a squarefree $T$-module.

4. Sheaves and Poincaré-Verdier Duality

The results in this section are parallel to those in \cite{18} Section 6 (or earlier work \cite{12}). Although the relation between the rings treated there and our $A_P$ is not so direct, the argument is very similar. So we omit the detail of some proofs here.

Recall that a simplicial poset $P$ gives a regular cell complex $\Gamma(P)$. Let $X$ be the underlying space of $\Gamma(P)$, and $c(x)$ the open cell corresponding to $0 \neq x \in P$. Hence, for each $x \in P$ with $\rho(x) \geq 2$, $c(x)$ is an open subset of $X$ homeomorphic to $\mathbb{R}^{n-1}$ (if $\rho(x) = 1$, then $c(x)$ is a single point), and $X$ is the disjoint union of the cells $c(x)$. Moreover, $x \geq y$ if and only if $c(x) \supset c(y)$.

As in the preceding section, let $\Lambda$ be the incidence algebra of $P$, and $\text{mod } \Lambda$ the category of finitely generated left $\Lambda$-modules. In \cite{13}, we assigned the constructible sheaf $N^1$ on $X$ to $N \in \text{mod } \Lambda$. By the equivalence $\mathcal{S}q A \cong \text{mod } \Lambda$, we have the constructible sheaf $M^+$ on $X$ corresponding to $M \in \mathcal{S}q A$. Here we give a precise construction for the reader’s convenience. For the sheaf theory, consult \cite{3}.

For $M \in \mathcal{S}q A$, set

$$\text{Sp}^\vee(M) := \bigcup_{0 \neq x \in P} c(x) \times M_{\mathfrak{a}(x)},$$

where $\mathfrak{a}(x) \in \mathbb{M}(x) \subset \mathbb{M}$ is the one defined in the proof of Proposition 3.3. Let $\pi : \text{Sp}^\vee(M) \to X$ be the projection map which sends $(p, m) \in c(x) \times M_{\mathfrak{a}(x)} \subset \text{Sp}^\vee(M)$ to $p \in c(x) \subset X$. For an open subset $U \subset X$ and a map $s : U \to \text{Sp}^\vee(M)$, we will consider the following conditions:
\[ (\ast) \] \( \pi \circ s = \text{id}_U \) and \( s_p = t^{\mathcal{A}(x) - \mathcal{A}(y)} \cdot s_q \) for all \( p \in c(x) \cap U, \ q \in c(y) \cap U \) with \( x \geq y \). Here \( s_p \in M_{\mathcal{A}(x)} \) (resp. \( s_q \in M_{\mathcal{A}(y)} \)) is an element with \( s(p) = (p, s_p) \) (resp. \( s(q) = (q, s_q) \)).

\[ (\ast\ast) \] There is an open covering \( U = \bigcup_{i \in I} U_i \) such that the restriction of \( s \) to \( U_i \) satisfies \((\ast)\) for all \( i \in I \).

Now we define a sheaf \( M^+ \) on \( X \) as follows: For an open set \( U \subset X \), set

\[ M^+(U) := \{ s \mid s : U \to \text{Spé}(M) \text{ is a map satisfying } (\ast\ast) \} \]

and the restriction map \( M^+(U) \to M^+(V) \) for \( U \supset V \) is the natural one. It is easy to see that \( M^+ \) is a constructible sheaf with respect to the cell decomposition \( \Gamma(P) \).

For example, \( A^+ \) is the \( \mathbb{k} \)-constant sheaf \( \mathbb{k}_X \) on \( X \), and \( (A/\mathfrak{m})^+ \) is (the extension to \( X \) of) the \( \mathbb{k} \)-constant sheaf on the closed cell \( c(x) \).

Let \( \text{Sh}(X) \) be the category of sheaves of finite dimensional \( \mathbb{k} \)-vector spaces on \( X \). The functor \( (\cdot)^+ : \text{Sq} \to \text{Sh}(X) \) is exact.

As mentioned in the previous section, \( \mathbb{D} : \mathbb{D}^b(\text{Sq} \to \mathbb{D}^b(\text{Sq}))^{\text{op}} \to \mathbb{T} \circ \mathbb{D} : \mathbb{D}^b(\text{mod} \ \Lambda) \to \mathbb{D}^b(\text{mod} \ \Lambda)^{\text{op}} \), where \( \mathbb{D} \) is the one defined in [13], and \( \mathbb{T} \) is the translation functor (i.e., \( \mathbb{T}(M^+)^{\cdot} = M^{+1} \)). Through \( (\cdot)^+ : \text{mod} \ \Lambda \to \text{Sh}(X) \), \( \mathbb{D} \) gives the Poincaré-Verdier duality on \( \mathbb{D}^b(\text{Sh}(X)) \), so we have the following.

**Theorem 4.1.** For \( M^\bullet \in \mathbb{D}^b(\text{Sq} \to \mathbb{D}^b(\text{Sq})) \), we have

\[ \mathbb{T}^{-1} \circ \mathbb{D}(M^\bullet)^+ \cong \mathcal{R} \text{Hom}((M^\bullet)^+, \mathcal{D}_X^\bullet) \]

in \( \mathbb{D}^b(\text{Sh}(X)) \). In particular, \( \mathbb{T}^{-1}((I^\bullet)^+) \cong \mathcal{D}_X^\bullet \), where \( I^\bullet \) is the complex constructed in Theorem 1.1, and \( \mathcal{D}_X^\bullet \) is the Verdier dualizing complex of \( X \) with the coefficients in \( \mathbb{k} \).

The next result follows from results in [13] (see also [8 Theorem 6.2]).

**Theorem 4.2.** For \( M \in \text{Sq} \), we have the decomposition \( H^i_m(M) = \bigoplus_{\mathfrak{a} \in \mathbb{M}} H^i_m(M)^{-\mathfrak{a}} \) by Corollary 3.6. Note that \( \mathbb{M} \) has the element \( \mathfrak{0} \). Then the following hold.

(a) There is an isomorphism

\[ H^i(X, M^+) \cong H^{i+1}_m(M)^{-\mathfrak{a}} \text{ for all } i \geq 1, \]

and an exact sequence

\[ 0 \to H^0_m(M)^{-\mathfrak{a}} \to M_0 \to H^0(X, M^+) \to H^1_m(M)^{-\mathfrak{a}} \to 0. \]

(b) If \( \mathfrak{0} \neq \mathfrak{a} \in \mathbb{M} \) with \( x = \sigma(\mathfrak{a}) \), then

\[ H^i_m(M)^{-\mathfrak{a}} \cong H^{i-1}_c(U_x, M^+|_{U_x}) \]

for all \( i \geq 0 \). Here \( U_x = \bigcup_{z \geq x} c(z) \) is an open set of \( X \), and \( H^*_c(\cdot) \) stands for the cohomology with compact support.

Let \( \tilde{H}^i(X; \mathbb{k}) \) denote the \( i \)th reduced cohomology of \( X \) with coefficients in \( \mathbb{k} \). That is, \( \tilde{H}^i(X; \mathbb{k}) \cong H^i(X; \mathbb{k}) \) for all \( i \geq 1 \), and \( \tilde{H}^0(X; \mathbb{k}) \oplus \mathbb{k} \cong H^0(X; \mathbb{k}) \), where \( H^i(X; \mathbb{k}) \) is the usual cohomology of \( X \).
Corollary 4.3 (Duval Theorem 5.9). We have
\[ [H^i_m(A)]_0 \cong \tilde{H}^{i-1}(X; k) \quad \text{and} \quad [H^i_m(A)]_{-\mathfrak{a}} \cong H^{i-1}_c(U_x; k) \]
for all \( i \geq 0 \) and all \( 0 \neq \mathfrak{a} \in \mathbb{M} \) with \( x = \sigma(\mathfrak{a}) \).
For this \( \mathfrak{a} \in \mathbb{M} \) (but \( \mathfrak{a} \) can be 0 here), \([H^i_m(A)]_{-\mathfrak{a}}\) is also isomorphic to the \( i \)th cohomology of the cochain complex
\[ K^*_x : 0 \to K^i_{x^{\rho(x)}} \to K^i_{x^{\rho(x)+1}} \to \cdots \to K^d_x \to 0 \quad \text{with} \quad K^i_x = \bigoplus_{z \geq x} k b_z \]
(bz is a basis element) whose differential map is given by
\[ b_z \mapsto \sum_{\rho(w) = \rho(z) + 1} e(w, z) b_w. \]
Duval uses the latter description, and he denotes \( H^i(K^*_x) \) by \( H^{i-\rho(x)-1}(\mathfrak{a} k x) \).

Proof. The former half follows from Theorem 1.2 by the same argument as Corollary 6.3. The latter part follows from that \( H^i_m(A) \cong H^i(\mathbb{D}(A))^\vee \) and that \( (\mathbb{D}(A)^\vee)_{-\mathfrak{a}} = K^*_x \) as complexes of \( k \)-vector spaces. \( \square \)

Theorem 4.4 (c.f. Duval 11). Set \( d := \text{rank} P = \dim X + 1 \). Then we have the following.

(a) \( A \) is Cohen-Macaulay if and only if \( \mathcal{H}^i(\mathbb{D}^*_X) = 0 \) for all \( i \neq -d + 1 \), and \( \tilde{H}^i(X; k) = 0 \) for all \( i \neq d - 1 \).

(b) Assume that \( A \) is Cohen-Macaulay and \( d \geq 2 \). Then \( A \) is Gorenstein*, if and only if \( \mathcal{H}^{-d+1}(\mathbb{D}^*_X) \cong k_X \). (When \( d = 1 \), \( A \) is Gorenstein* if and only if \( X \) consists of exactly two points.)

(c) \( A \) is Buchsbaum if and only if \( \mathcal{H}^i(\mathbb{D}^*_X) = 0 \) for all \( i \neq -d + 1 \).

(d) Set
\[ d_i := \begin{cases} 
\dim(\text{supp } \mathcal{H}^{-i}(\mathbb{D}^*_X)) & \text{if } \mathcal{H}^{-i}(\mathbb{D}^*_X) \neq 0, \\
-1 & \text{if } \mathcal{H}^{-i}(\mathbb{D}^*_X) = 0 \text{ and } \tilde{H}^i(X; k) \neq 0, \\
-\infty & \text{if } \mathcal{H}^{-i}(\mathbb{D}^*_X) = 0 \text{ and } \tilde{H}^i(X; k) = 0.
\end{cases} \]

Here \( \text{supp } \mathcal{F} = \{ p \in X \mid \mathcal{F}_p \neq 0 \} \) for a sheaf \( \mathcal{F} \) on \( X \). Then, for \( r \geq 2 \), \( A \) satisfies Serre’s condition \( (S_r) \) if and only if \( d_i \leq i - r \) for all \( i < d - 1 \).

Hence, Cohen-Macaulay (resp. Gorenstein*, Buchsbaum) property and Serre’s condition \( (S_r) \) of \( A \) are topological properties of \( X \), while they may depend on \( \text{char}(k) \).

As far as the author knows, even in the Stanley-Reisner ring case, (d) has not been mentioned in literature yet.

Recall that we say \( A \) satisfies Serre’s condition \( (S_r) \) if depth \( A_p \geq \min\{ r, \text{ht } p \} \) for all prime ideal \( p \) of \( A \). The next fact is well-known to the specialist, but we will sketch the proof here for the reader’s convenience.

Lemma 4.5. For \( r \geq 2 \), \( A \) satisfies the condition \( (S_r) \) if and only if \( \dim H^{-i}(I^*_A) \leq i - r \) for all \( i < d \). Here the dimension of the 0 module is \(-\infty\).
Proof. For a prime ideal \( p \), the normalized dualizing complex of \( A_p \) is quasi-isomorphic to \( T^{-\dim A/p} (I_A^* \otimes_A A_p) \). Hence we have
\[
\text{depth } A_p = \min \{ i \mid (H^{-i}(I_A^*)) \otimes_A A_p \neq 0 \} - \dim A/p.
\] (4.1)
Recall that if \( A \) satisfies Serre’s condition \((S_2)\) then \( A \) is pure (equivalently, \( \dim A/p = d \) for all minimal prime ideal \( p \)). Similarly, if \( \dim H^{-i}(I_A^*) < i \) for all \( i < d \), then \( A \) is pure. (In fact, if \( p \) is a minimal prime ideal of \( A \) with \( i := \dim A/p < d \), then depth \( A_p = 0 \) implies that \( p \) is a minimal prime of \( H^{-i}(I_A^*) \). It follows that \( \dim H^{-i}(I_A^*) = i \). This is a contradiction.) So we may assume that \( A \) is pure, and hence \( \dim A/p + \text{ht } p = d \) for all \( p \). Now the assertion follows from \([1,1]\). \(\Box\)

The proof of Theorem 4.4. We can prove (a)-(c) by the same way as \([8, Theorems 6.4 and 6.7]\). For (d), note that \( d_j = \dim H^{-j-1}(I_A^*) - 1 \). So the assertion follows from Lemma 4.5. \(\Box\)

5. Further discussion

This section is a collection of miscellaneous results related to the arguments in the previous sections.

For an integer \( i \leq d - 1 \), the poset
\[
P^{(i)} := \{ x \in P \mid \rho(x) \leq i + 1 \}
\]
is called the \( i \)-skeleton of \( P \). Clearly, \( P^{(i)} \) is simplicial again, and set \( A^{(i)} := A_{P^{(i)}} \). Then it is easy to see that the (Theorem 1.1 type) dualizing complex \( I_{A^{(i)}}^* \) of \( A^{(i)} \) coincides with the brutal truncation
\[
0 \rightarrow I_{A^{(i)}}^{i-1} \rightarrow I_{A^{(i)}}^{-i} \rightarrow \cdots \rightarrow I_{A}^0 \rightarrow 0
\]
of \( I_A^* \). Since depth \( A = \min \{ i \mid H^{-i}(I_A^*) \neq 0 \} \) and \( \dim A = \max \{ i \mid H^{-i}(I_A^*) \neq 0 \} \), we have the equation
\[
\text{depth } A_P = 1 + \max \{ i \mid A^{(i)} \text{ is Cohen-Macaulay} \},
\] (5.1)
which is \([1, Corollary 6.5]\).

Contrary to the Gorenstein* property, the Gorenstein property of \( A_P \) is not topological. This phenomena occurs even for Stanley-Reisner rings. But there is a characterization of \( P \) such that \( A_P \) is Gorenstein. For posets \( P_1, P_2 \), we regard \( P_1 \times P_2 = \{ (x_1, x_2) \mid x_1 \in P_1, x_2 \in P_2 \} \) as a poset by \( (x_1, x_2) \geq (y_1, y_2) \Leftrightarrow x_i \geq y_i \) in \( P_i \) for each \( i = 1, 2 \).

**Proposition 5.1.** \( A_P \) is Gorenstein if and only if \( P \cong 2^V \times Q \) as posets for a boolean algebra \( 2^V \) and a simplicial poset \( Q \) with \( A_Q \) is Gorenstein*.

**Proof.** The sufficiency is clear. In fact, if \( P \cong 2^V \times Q \) then \( A := A_P \) is a polynomial ring over \( A_Q \). So it remains to prove the necessity.

By Lemma 2.3, \( A \) is a squarefree module over the polynomial ring \( T := \text{Sym } A_1 \). We say \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) is squarefree, if \( a_i = 0, 1 \) for all \( i \). If this is the case, we identify \( a \) with its support \( \{ i \mid a_i = 1 \} \subset [n] \). Hence a subset \( F \subset [n] \) sometimes means the corresponding squarefree vector in \( \mathbb{N}^n \).
Since $A$ is Gorenstein (in particular, Cohen-Macaulay) now, a minimal $\mathbb{Z}^n$-graded $T$-free resolution of $A$ is of the form

$$L_\bullet : 0 \to L_{n-d} \to \cdots \to L_1 \to L_0 \to 0 \quad \text{with} \quad L_i = \bigoplus_{F \subset [n]} T(-F)^{\beta_i,F}$$

by \cite[Corollary 2.4]{11}.

Let $1 := (1, 1, \ldots, 1) \in \mathbb{N}^n$. Note that $\mathsf{Hom}_T^\bullet(L_\bullet, T(-1))$ is a minimal $\mathbb{Z}^n$-graded $T$-free resolution of the canonical module $\omega_A = \mathsf{Ext}_T^{n-d}(A, T(-1))$ of $A$ up to translation, and $\omega_A \cong A(-V)$ for some $V \subset [n]$. Set $W := [n] \setminus V$. Since $\mathsf{Hom}_T(T(-F), T(-1)) \cong T(-(n] \setminus F))$, we have the following:

(*) If $\beta_{i,F} \neq 0$ for some $i$, then $F \subset W$.

If $[V] = [\bigvee_{i \in V} y_i] = \emptyset$, then by the construction of $A$, there is some $F \subset V$ with $\beta_{i,F} \neq 0$, and it contradicts to the statement (*). Even if $\# [V] \geq 2$, the same contradiction occurs. Hence $[V] = \{x\}$ for some $x \in P$. We denote the closed interval $[0, x]$ by $2^V$.

Set

$$Q := \{ z \in P \mid z \not\geq y_i \text{ for all } i \in V \} = \prod_{U \subset W} [U].$$

If $\# [x' \vee z] \neq 1$ for some $x' \in 2^V$ and $z \in Q$, then $\beta_{i,F} \neq 0$ for some $F$ with $F \cap V \neq \emptyset$, and it contradicts to (*). Hence, for all $x' \in 2^V$ and $z \in Q$, we have $\# [x' \vee z] = 1$. Denoting the element of $[x' \vee z]$ by $x' \vee z$, we have an order preserving map

$$\psi : 2^V \times Q \ni (x', z) \mapsto x' \vee z \in P.$$  

Moreover, since $P = \bigsqcup_{U \subset [n]} [U]$, $\psi$ is an isomorphism of posets, and we have

$$P \cong 2^V \times Q.$$  

Clearly, $Q$ is a simplicial poset. Set $B := A_Q$. Since $A \cong B[t_i \mid i \in V]$ and

$$A(-V) \cong \omega_A \cong (\omega_B[t_i \mid i \in V])(-V),$$

$B$ is Gorenstein*.

Let $\Sigma$ be a finite regular cell complex with the underlying topological space $X(\Sigma)$, and $Y, Z \subset X(\Sigma)$ closed subsets with $Y \supset Z \neq \emptyset$. Set $U := Y \setminus Z$, and let $h : U \to Y$ be the embedding map. We can define the Cohen-Macaulay property of the pair $(Y, Z)$, which generalizes the Cohen-Macaulay property of a relative simplicial complex introduced in \cite[III. §7]{11}. See Lemma \ref{lem:5.3} below.

**Definition 5.2.** We say the pair $(Y, Z)$ is Cohen-Macaulay (over $k$), if $H^i_C(U; k) = 0$ for all $i \neq \dim U$ and $R^i h_* D_U^\bullet = 0$ for all $i \neq -\dim U$. Here $D_U^\bullet$ is the Verdier dualizing complex of $U$ with the coefficients in $k$.

We say an ideal $I \subset A$ is squarefree, if it is generated by a subset of $\{ t_x \mid x \in P \}$. Clearly, an ideal $I$ is a squarefree submodule of $A$ if and only if it is a squarefree ideal. For a squarefree ideal $I$, $\sigma(I) := \{ x \in P \mid t_x \in I \}$ is an order filter (i.e., $x \in \sigma(I)$ and $y \geq x$ imply $y \in \sigma(I)$), and $U_I := \bigcup_{x \in \sigma(I)} c(x)$ is an open set of $X$.

The sheaf $I^+$ is (the extension to $X$ of) the $k$-constant sheaf on $U_I$.  

\end{document}
Proposition 5.3. (1) A squarefree ideal $I$ with $I \subseteq \mathfrak{m}$ is a Cohen-Macaulay module if and only if $(\overline{U_1}, \overline{U_1} \setminus U_1)$ is Cohen-Macaulay in the sense of Definition [5,2].

(2) The sequentially Cohen-Macaulay (see [10, III. Definition 2.9]) property of $A$ depends only on $X$ (and char$(k)$).

Proof. (1) Set $U := U_1$, and let $h : U \to \overline{U}$ be the embedding map. The assertion follows from the fact that $T^{-1}(\mathbb{D}(I)^\vee)[c] \cong R \mathcal{H}_s \mathcal{D}_s^0$; and $[H^{-1}(\mathbb{D}(I))]_0 \cong H^{-1}_e(\overline{U}; k)$ by [13] (see also [12, Proposition 4.10]).

(2) Follows from (1) by the same argument as [14, Theorem 4.7].

Remark 5.4. While it is not stated in [8], the statements corresponding to Lemma [1.4] the equation (5.1) and Proposition 5.3 hold for a cone-wise normal toric face ring.

As in [13], we regard the finite regular cell complex $\Sigma$ as a poset by $\sigma \supseteq \tau$. Here we use the convention that $0 \in \Sigma$. We say $\Sigma$ is a meet-semilattice (or, satisfies the intersection property), if the largest common lower bound $\sigma \wedge \tau \in \Sigma$ exists for all $\sigma, \tau \in \Sigma$. It is easy to see that $\Sigma$ is a meet-semilattice if and only if $\#[\sigma \vee \tau] \leq 1$ for all $\sigma, \tau \in \Sigma$. The underlying cell complex of a toric face ring (especially, a simplicial complex) is a meet-semilattice.

For $\sigma \in \Sigma$, let $U_\sigma$ be the open subset $\bigcup_{\tau \geq \sigma} \tau$ of $X(\Sigma)$. As shown in [13], if $X(\Sigma)$ is Cohen-Macaulay and $\Sigma$ is a meet-semilattice, then $(\overline{U_\sigma}, \overline{U_\sigma} \setminus U_\sigma)$ is Cohen-Macaulay for all $\sigma$. (If $\Sigma$ is not a meet-semilattice, we have an easy counter example.) While a simplicial poset $P$ is not a meet-semilattice in general, the above fact remains true. We also remark that an indecomposable projective in $\text{Sq} \ A$ is isomorphic to the ideal $J_x := (t_x) \subset A$ for some $x \in P$.

Proposition 5.5. If $A$ is Cohen-Macaulay (resp. Buchsbaum) then the ideal $J_x := (t_x)$ is a Cohen-Macaulay module for all $x \in P$ (resp. for all $0 \neq x \in P$).

Proof. Let $a \in \mathcal{M}$ with $\sigma(a) = y$. With the notation of Proposition [1.3] recall that

$$R \Gamma_m A \cong (\mathbb{D}(A)^\vee)_{-a} \cong K_y^*.$$  

Similarly, we have

$$R \Gamma_m J_x \cong (\mathbb{D}(J_x)^\vee)_{-a} \cong \bigoplus_{z \in [x \vee y]} K_z^*.$$  

To see the second isomorphism, note that if $w \geq x, y$ then there exists a unique $z \in [x \vee y]$ such that $w \geq z$.

If $A$ is Cohen-Macaulay (resp. Buchsbaum) then $H^i_m(A)_{-a} \cong H^i(R \Gamma_m A)_{-a} = 0$ for all $i < d$ and all $a \in \mathcal{M}$ (resp. $0 \neq a \in \mathcal{M}$). Hence we are done.

Regarding $A = A_P$ as a $\mathbb{Z}$-graded ring, we have

$$\sum_{i \geq 0} (\dim_k A_i) \cdot \lambda^i = \frac{h_0 + h_1 \lambda + \cdots + h_d \lambda^d}{(1 - \lambda)^d},$$  

for some integers $h_0, h_1, \ldots, h_d$ by [10] Proposition 3.8. We call $(h_0, \ldots, h_d)$ the $h$-vector of $P$. The behavior of the $h$-vectors of simplicial complexes is an important subject of combinatorial commutative algebra. The $h$-vector of a simplicial poset has also been studied, and striking results are given in [10] [4].
Recently, Murai and Terai gave nice results on the $h$-vector of a simplicial complex $\Delta$ such that the Stanley-Reisner ring $k[\Delta]$ satisfies Serre’s condition ($S_r$). We show that one of them also holds for simplicial posets.

**Theorem 5.6** (c.f. Murai-Terai [7, Theorem 1.1]). Let $P$ be a simplicial poset with the $h$-vector $(h_0, h_1, \ldots, h_s)$. If $A$ satisfies Serre’s condition ($S_r$) then $h_i \geq 0$ for all $i \leq r$.

**Proof.** By virtue of Lemma 2.5, we can imitate the proof of [7, Theorem 1.1].

Let $\Delta$ be a simplicial complex with the vertex set $[n]$, $S = k[x_1, \ldots, x_n]$ the polynomial ring, and $k[\Delta] = S/I_\Delta$ the Stanley-Reisner ring of $\Delta$. To prove our theorem, we replace $k[\Delta]$ and $S$ in their argument by $A = A_P$ and $T = \text{Sym} A_1$ respectively. In the former half of the proof, they deal $k[\Delta]$ as just a finitely generated (graded) $S$-module, and the argument is clearly applicable to our $A$ and $T$. The latter half of their proof is based on the fact that $\text{Ext}_{I_\Delta}^i(k[\Delta], S(-1))$ is a squarefree $S$-module of dimension at most $n-i-r$. Hence if the following holds, the argument in [7] works in our case.

**Claim.** $\text{Ext}_{I_\Delta}^i(A, T(-1))$ is a squarefree $T$-module of dimension at most $n-i-r$.

The proof is easy. In fact, the squarefree-ness follows from Lemma 2.5 and [11, Theorem 2.6]. Moreover, since $\text{Ext}_{I_\Delta}^i(A, T(-1)) \cong \text{Ext}_A^{n+i}(A, D^*_A) \cong H^{-n+i}(I_A^*)$ by the local duality, we have $\text{Ext}_{I_\Delta}^i(A, T(-1)) \leq n-i-r$ by Lemma 4.5. $\Box$

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Department of Mathematics, Kansai University, Suita 564-8680, Japan
E-mail address: yanagawa@ipcku.kansai-u.ac.jp