RISK-MINIMIZATION AND HEDGING CLAIMS ON A JUMP-DIFFUSION MARKET MODEL, FEYNMAN-KAC THEOREM AND PIDE.

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Abstract. At first, we solve a problem of finding a risk-minimizing hedging strategy on a general market with ratings. Next, we find a solution to this problem on Markovian market with ratings on which prices are influenced by additional factors and rating, and behavior of this system is described by SDE driven by Wiener process and compensated Poisson random measure and claims depend on rating. To find a tool to calculate hedging strategy we prove a Feynman-Kac type theorem. This result is of independent interest and has many applications, since it enables to calculate some conditional expectations using related PIDE’s. We illustrate our theory on two examples of market. The first is a general exponential Lévy model with stochastic volatility, and the second is a generalization of exponential Lévy model with regime-switching.

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1. Introduction

Fölmer and Sondermann [12] introduced the concept of risk-minimization as a tool for hedging and pricing in the incomplete markets. They considered the problem of hedging a payment at maturity which is non-attainable, that is a claim \( H \) for which there is no self-financing strategy that replicates \( H \). The idea of Fölmer and Sondermann was to drop the self-financing condition and look for strategies that hedges \( H \) perfectly and minimize conditional variance of the remaining cost at each time \( t \). In [11] they proved that there exists a unique risk-minimizing hedging strategy for an arbitrary square integrable payoff at fixed maturity \( T < \infty \) provided that the process of discounted price of risky asset is a martingale. The problem of finding the explicit form of risk-minimization strategy for an European contingent claim being a function of asset price at time \( T \) were solved by Elliott and Föllmer [9] in the one-dimensional Markovian case as well as in general case by using orthogonal martingale representation. Møller [24] generalized results of Fölmer and Sondermann to the case in which liabilities of the hedger are described by an arbitrary square integrable and càdlàg payment process. Subsequently, Schweizer [28] proved that if a discounted price process is not a martingale then one cannot, in general, find a risk-minimizing strategy. To overcome this problem Schweizer [28] introduced the new concept of the local-risk-minimization which generalizes risk-minimization and it is suitable for general special semimartingales. However, to apply a local-risk-minimization strategy it is necessary to know drift coefficient, and usually it is hard to estimate it (see discussion in Tankov [31]). In this paper, we consider the risk-minimization problem.

In Section 2 we present the state of knowledge, so we recall main definitions and results on existence and uniqueness of risk-minimizing strategy for discounted cumulated payments.

In Section 3 we solve the risk-minimization problem on a general market with ratings. We consider a market with time horizon \( T^* < \infty \) and with \( d \) primary risky assets with the price process denoted by \( S \) and a money account with the price process \( B \). The dynamics of discounted price process \( S_t := S_t/B_t \) is given by SDE driven by a Wiener process, a compensated integer-valued random measure, and a martingales \( M^{ij} \) influenced by a credit rating of some corporate or a state of economy. Theorem 3.1 states that the risk-minimizing strategy, under appropriate
conditions, solves some linear system of equations. Moreover, using the Galtchouk-Kunita-Watanabe decomposition of discounted cumulated payments, we give an explicit form of this strategy in terms of coefficients which appear in the dynamics describing prices and in terms of processes which appear in a martingale representation of discounted cumulated payments. Existence of this representation is assured by a weak property of predictable representation. There are many models in which this property holds. In Theorem 3.4 we formulate the necessary and sufficient conditions for attainability of a dividend process. Since these conditions are very restrictive this theorem underline the necessity of using another methods of hedging the risk for non-attainable dividend streams, e.g., finding of the risk-minimizing strategies as we do here.

In Section 4 we will consider very flexible Markovian market model with ratings, which is driven by a standard Wiener process, a compensated Poisson random measure, and counting point processes with intensities. As we can see in Section 5 this allows us to derive more explicit formulae for risk-minimizing strategies by means of solutions to some partial integro-differential equations (PIDE’s). We assume that information available to the market participants is modeled by a multidimensional process \((Y, C)\) given as a solution to SDE, where \(Y = (S, R)\) with \(S\) being a process of prices of tradable risky assets and \(R\) represent some factors of economic environment such as interest rates, inflation or stochastic volatility. The evolution of price depends on economic conditions of market which are described by rating system \(C\) and additional non-tradable risk factor process \(R\). So we assume that dynamics of the process \((Y, C)\) is described by SDE \((\text{1.8})\) in which the credit rating of corporate has influence on asset prices \(S\) and other non-tradable factor \(R\) by changing drift and volatility and moreover a change in credit rating causes a jump. The money account on this market is given by \((\text{4.1})\). In Theorem \((\text{4.1})\) we establish a 0-achieving risk-minimizing strategy for a general payment stream \(D\) given by \((\text{4.12})\) which corresponds to rating sensitive claims considered in Jakubowski and Niewegłowski \([19]\). We connect this problem with finding the ex-dividend price function \(v\), and for a sufficiently regular function \(v\) we can find a 0-achieving risk-minimizing strategy for \(D\) written explicitly in the terms of this function \(v\).

In Section 5 we formulate and prove a Feynman-Kac type theorem for components of a weak solution to SDE \((\text{1.8})\). So, we connect a problem of calculation of conditional expectation of some natural functionals of these components with solution to corresponding PIDE with given boundary conditions. Note also that if there is no component \(C\) we have obtained the classical Feynman-Kac theorem. It is worth to notice that we solve a general problem which is of independent interest and has many applications, among others in finding the risk-minimizing strategies.

In Section 6 we discuss possibly extensions of our results to risk-minimization problems. Moreover, we present examples of applications of our results considering a general exponential Lévy model with stochastic volatility, a model which generalizes a regime switching model with jumps and semi-Markovian regime switching models. In appendix we prove the result which connects the existence of replication strategy for a payment stream and the form of Galtchouk-Kunita-Watanabe decomposition of some random variable.

Our models are very general, and can be applied to the most models which appear in finance, e.g. stochastic volatility models, regime switching Black Scholes type models considered among others by Elliott, Chan and Siu \([10]\), Siu and Yang \([30]\), Yao, Zhang and Zhou \([32]\), Yuen and Yang \([33]\) and also regime switching Lévy models studied by Chourdakis \([9]\), Mijatovic and Pistorius \([23]\), Kim, Fabozzi, Lin and Rachey \([21]\). In our models we consider the additional factors which influence the prices, so known the results can be generalized. This justifies the assertion that our models well described a real market.

Looking at our model from point of view of insurance (so changing interpretation of abstract objects in our model) we obtain new results in insurance, e.g. on risk minimizing hedging strategies of insurance payment processes studied by Møller \([24]\).
2. Risk-minimization

In this section we briefly recall main definitions and theorems which allows us to define precisely notion of risk-minimality and present results on existence and uniqueness of risk-minimizing hedging strategies. We will consider processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the "usual conditions" and $\mathbb{F} = \mathcal{F}_{T^+}$. By $\mathbb{P}$ we denote the predictable sigma algebra on $\Omega \times \mathbb{R}_+$. We consider a market with a risky assets price process denoted by $S$ and a bank account price process $B$. By $\varphi = (\phi, \eta)$ we denote a strategy that describes number of assets hold in portfolio at time $t$, i.e. number of assets invested in risky assets and in bank account, respectively. We specify later the measurability requirements imposed on $\varphi$. The process $V_t(\varphi) := \phi^T_t S_t + \eta_t B_t$ denotes the wealth of portfolio $\varphi$ at time $t$. By $V^*_t(\varphi)$ we denote the discounted wealth of the portfolio $\varphi$, so $V^*_t(\varphi) = \phi^T_t S^*_t + \eta_t$. We make an assumption that $S^*_t := S_t/B_t$ is a square integrable martingale under the measure $\mathbb{P}$. By $\langle S^*_t \rangle$ we denote the matrix of predictable covariations of $S^*$, i.e.,

$$\langle S^*_t \rangle_t = \begin{bmatrix} \langle S^*_t, S^*_t \rangle_t \end{bmatrix}_{i,j},$$

where $\langle S^*_t, S^*_t \rangle_t$ is the unique predictable process such that $S^*_t S^*_t - \langle S^*_t, S^*_t \rangle_t$ is a martingale. In the sequel we use convention from stochastic integration theory writing $\int_t^u$ instead of $\int_{[t,u]}$ in stochastic and Lebesgue integrals.

**Definition 2.1.** We say that $\varphi = (\phi, \eta)$ is an $L^2$-strategy if $\phi$ is a predictable process such that

$$\mathbb{E} \left( \int_0^T \phi^T_t d\langle S^* \rangle_t \phi_t \right) < \infty,$$

$\eta$ is an $\mathbb{F}$-adapted process, $V^*_t(\varphi) := \phi^T_t S^*_t - \eta_t$ is a right continuous and square integrable. An $L^2$-strategy $\varphi$ is called 0-achieving strategy, if $V_T(\varphi) = 0$.

We consider a contract between two parties, a seller (also called hedger) and a buyer, which specifies precisely the cash-flows between these two parties. These cash-flows are described by a càdlàg processes $D$, i.e. $D_t$ represents accumulated payments (both outflows and injections of cash from the buyer) up to time $t$. We assume additionally that $D$ matures at some finite non-random time $T < \infty$, that is we have $D_t = D_T$ for $t \geq T$. This process $D$ is called a dividend process or a payments stream process. Seller of this claim can actively trade in the market according to strategy $\varphi$. Since we do not restricted to self-financing strategies this strategy can generate during period $[0,T]$ some cost which is defined below.

**Definition 2.2.** The cost process of an $L^2$-strategy $\varphi = (\phi, \eta)$ associated with a dividend process $D$ is given by

$$C^D_t(\varphi) := \int_0^t \frac{1}{B_u} dD_u + V^*_t(\varphi) - \int_0^t \phi^T_u dS^*_u$$

for $0 \leq t \leq T$, provided that $\int_{[0,t]} \frac{1}{B_u} dD_u$ is square integrable. If the process $C^D$ is a constant, then we say that $\varphi$ finances dividend process $D$ (or $\varphi$ is a self-financing for $D$). If $C^D$ is a martingale, then we say that $\varphi$ is a mean-self-financing for $D$.

The cost is the sum of three components, the first describes the cumulated discounted dividends, the second is equal to the discounted wealth of the portfolio and the third is equal to the minus one multiply by the discounted gains from trading using $\varphi$. Obviously, the result of this simple arithmetics deserves, from the hedger point of view, for the name accumulated cost. In this paper we take a point of view of the hedger, i.e., a person who is obligated to deliver payments according to a dividend process $D$. Therefore the gains from trading strategy are subtracted because a negative cost is a hedger’s income. Therefore $V_t(\phi)$ could be interpreted as the wealth of portfolio after delivering payments described by $D$ at time $t$. Since $D$ matures at $T$, it is quite natural to restrict our considerations to the 0-achieving strategies since the hedger of $D$ should stop trading after $T$. For more detailed motivation of above definition we refer to Moller [24]. We stress that
we don’t assume that $D$ is positive or has positive jumps. With a strategy $\varphi$ we connect another quantity.

**Definition 2.3.** The risk process $R^D(\varphi)$ of a strategy $\varphi$ associated with $D$ is defined by

$$R^D_t(\varphi) = E \left( (C^D_T(\varphi) - C^D_t(\varphi))^2 | \mathcal{F}_t \right)$$

for $0 \leq t \leq T$.

Föllmer and Sonderman [11] proposed the following definition of risk-minimality:

**Definition 2.4.** We say that $\varphi = (\phi, \eta)$ is a risk-minimizing strategy (for a dividend process $D$) if for any $t \in [0, T]$ and any strategy $\hat{\varphi} = (\hat{\phi}, \hat{\eta})$ satisfying

\begin{align*}
V^*_t(\varphi) &= V^*_t(\hat{\varphi}) \quad \mathbb{P} - \text{a.s.}, \\
\phi_s = \hat{\phi}_s &\text{ for } s \leq t, \text{ and } \eta_s = \hat{\eta}_s \text{ for } s \leq t,
\end{align*}

we have $R^D_t(\hat{\varphi}) \geq R^D_t(\varphi)$.

Sometimes we call $\varphi$ a $D$-risk-minimizing strategy. A strategy $\hat{\varphi}$ satisfying (2.3) and (2.4) is called the admissible continuation of $\varphi$ at time $t$.

Föllmer and Sonderman [11] noticed that the problem of finding risk-minimizing hedging strategy can be solved by using the Galtchouk-Kunita-Watanabe decomposition (GKW decomposition). However, in [11] only the case of a single payment at maturity is considered. The general case of payment streams was considered by Møller [24] who noticed that the GKW decomposition of martingale defined by

$$V^*_t := E \left( \int_0^T \frac{1}{B_u} dD_u | \mathcal{F}_t \right), \quad 0 \leq t \leq T,$$

is a very useful in deriving $D$-risk-minimizing hedging strategies. We recall that if $S^*$ and $V^*$ are square integrable martingales, then $V^*$ can be uniquely decomposed (Galtchouk-Kunita-Watanabe) as

$$V^*_t = V^*_0 + \int_0^t \phi^D_u dS^*_u + L^D_t,$$

where $L^D$ is a zero mean martingale orthogonal to $S^*$ (i.e., $S^*L^D$ is a martingale) and $\phi^D$ is a predictable process satisfying integrability assumption [24]. Møller [24] obtained formula for a risk-minimizing hedging strategy for a dividend process $D$ using the GKW decomposition of $V^*$:

**Theorem 2.5.** [24] There exist a unique 0-achieving risk-minimizing strategy $\varphi = (\phi, \eta)$ for the dividend process $D$. It is of the following form

$$(\phi_t, \eta_t) = \left( \phi^D_t, V^*_t - \int_0^t \frac{1}{B_u} dD_u - \phi^D_t S^*_t \right),$$

where $\phi^D$ is from GKW decomposition of $V^*$. The risk process of risk-minimizing strategy $\varphi$ is given by $R^D_t(\varphi) = E \left( (L^D_t - L^D_t)^2 | \mathcal{F}_t \right)$.

Subsequently, we consider only the 0-achieving risk-minimizing strategies for dividend processes. To shorten notation, we sometimes omit description "0-achieving" in a risk-minimizing strategy.

**Remark 2.6.** Föllmer and Sonderman [11] restrict themselves to finding risk-minimizing strategies such that $V^*_T(\varphi) = X$. This is a consequence, as explained by Møller [24], of considering only single payments at maturity and slightly different definition of the cost of strategy which is independent of the claim $X$. Note that this modification yields that for a 0-achieving risk-minimizing strategy $\varphi$ we have

$$V_t(\varphi) = B_t E \left( \int_0^T \frac{1}{B_u} dD_u | \mathcal{F}_t \right),$$

i.e., the wealth of the 0-achieving risk-minimizing strategy is equal to the ex-dividend price of the claim with dividend process $D$. 

3. Risk-minimizing hedging strategy on a general market with ratings

In this section we apply results described in the previous section to solving risk-minimization problem on the general market with ratings. We consider a market with time horizon $T^* < \infty$ and with $d$ primary risky assets with the price process denoted by $S$ and a money account with the price process $B$.

The dynamics of discounted price process $S^*_t := S_t/b_t$ is given by

$$dS^*_t = \sigma_t dW_t + \int_{\mathbb{R}^n} F_i(x) \tilde{\Pi}(dx, dt) + \sum_{i,j \in \mathcal{K}, j \neq i} \rho^{i,j}_t dM^{i,j}_t, \quad S^*_0 = s. \tag{3.1}$$

Here, $W$ is an $n$ dimensional Wiener process, by $\tilde{\Pi}$ we denote a compensated random measure

$$\tilde{\Pi}(dx, du) := \Pi(dx, du) - Q(dx, du),$$

where $\Pi(dx, dt)$ is assumed to be an integer-valued random measure on $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}_+)$ with the unique compensator $Q(dx, dt)$ which has density $\nu = (\nu_u)$ with respect to time, i.e.,

$$Q(dx, du) = \nu_u(dx) du,$$

where $\nu_u$ is a Lévy measure. Processes $M^{i,j}$ are driven by an additional source of uncertainty, i.e., by a càdlàg process $C$ taking values in a finite set $\mathcal{K} = \{1, \ldots, K\}$ which can be interpreted as a credit rating of some corporate or as a state of economy. For each $i \in \mathcal{K}$ we define the process

$$H^i_t := 1_{\{i\}}(C_t)$$

which is a càdlàg process, taking values in $\{0, 1\}$, indicating in which state the process $C$ is at given time $t$. In such models $\sigma$ has often a form

$$\sigma_t = \sum_{i \in \mathcal{K}} H^i_t \tilde{\sigma}_t^i,$$

where $\tilde{\sigma}^i$ depends only on $S$. The same remark concerns $F$. Moreover, we define processes $H^{i,j}$ by letting

$$H^{i,j}_t := \sum_{0 \leq u \leq t} H^i_u - H^j_u.$$

This process counts the number of transition from state $i$ to $j$ during time interval $[0, t]$. Therefore $H^{i,j}$ are mutually orthogonal point processes with jumps equal to 1.

In this paper we make the standing assumptions:

**Assumption EI.** There exist nonnegative bounded processes $\lambda^{i,j}$, $i, j \in \mathcal{K}$, $j \neq i$, such that processes $M^{i,j}$ defined by

$$M^{i,j}_t = H^{i,j}_t - \int_0^t \lambda^{i,j}_u du \tag{3.2}$$

are martingales.

**Assumption NCJ.** We require that the random measure $\Pi$ and the processes $H^{i,j}$, for $i, j \in \mathcal{K}$, $i \neq j$, have no common jumps, i.e., for every $t > 0$ and every $b > 0$,

$$\int_0^t \int_{\|x\| > b} \Delta H^{i,j}_u \Pi(dx, du) = 0 \quad \mathbb{P} - a.s. \tag{3.3}$$

**Assumption INT.** The process $\sigma$ is predictable with values in the space of matrices of dimension $d \times n$, $F$ is a mapping from $\Omega \times \mathbb{R}_+ \times \mathbb{R}^n$ with values in $\mathbb{R}^d$ which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)$ measurable, and $\rho^{i,j}$ are predictable processes with values in the space of vectors of dimension $d$ satisfying

$$\mathbb{E} \left( \int_0^{T^*} \left( \|\sigma_t\|^2 + \int_{\mathbb{R}^n} \|F_i(x)\|^2 \nu_t(dx) + \sum_{i,j \in \mathcal{K}, j \neq i} \left\| \rho^{i,j}_t \right\|_{\lambda^{i,j}_t}^2 \right) dt \right) < \infty. \tag{3.4}$$
One of consequences of Assumption EI is that $H^{i,j}$ are counting processes with rating dependent intensities $\lambda^{i,j}$. It is worth to note that

$$\lambda^{i,j}_t = H^{i,j}_t \cdot \lambda^{i,j}_t, \quad dt \times d\mathbb{P} \text{ a.e.,}$$

so $\lambda^{i,j}_t$ depends on rating at time $t$ but in general it may also depend on the trajectory of rating process $C$ on the interval $[0, t]$. Assumption INT implies that $S^*$ is a square integrable martingale, so $\mathbb{P}$ is a martingale measure. We denote by $G$ the matrix-valued stochastic process

$$G_t := \sigma_t (\sigma_t)^\top + \int_{\mathbb{R}^n} F_t(x)(F_t(x))^\top \nu_t(dx) + \left( \sum_{i,j \in K : j \neq i} \rho^{i,j}_t \left( \rho^{i,j}_t \right)^\top \lambda^{i,j}_t \right).$$

The process $G$ exists and is finite by Assumption INT. One could notice that $G$ represents the density with respect to time of predictable covariations of $S^*$, i.e., the matrix of predictable covariations of $S^*$ under $\mathbb{P}$ is given by

$$d\langle S^* \rangle_t = G_t dt.$$

The theorem below is the main result of this section and states that, under appropriate conditions, the risk-minimizing strategy can be calculated by solving a linear system of equations. We find an explicit form of risk-minimizing strategy in terms of coefficients which appear in formulas describing dynamics of prices and in terms of processes which appear in a martingale representation of discounted cumulated payments.

**Theorem 3.1.** Fix a dividend process $D$. Assume that the square integrable random variable $X := \int_0^T \frac{1}{B_u} dD_u$ representing discounted cumulated payments up to maturity time $T$ has the representation

$$X = \mathbb{E}(X) + \int_0^T \delta_t^\top dW_t + \int_0^T \int_{\mathbb{R}^n} J_t(x) \tilde{\Pi}(dt, dx) + \sum_{i,j \in K : j \neq i} \int_0^T \gamma^{i,j}_t dM^{i,j}_t.$$

Then there exists a $\theta$-achieving risk-minimization strategy $\varphi = (\phi, \eta)$ for the dividend process $D$. The component $\phi$ of this strategy is the predictable version of the solution to the linear system

$$G_t \phi_t = A_t,$$

where a predictable process $A$ is given by the formula

$$A_t := \sigma_t \delta_t + \int_{\mathbb{R}^n} F_t(x) J_t(x) \nu_t(dx) + \sum_{i,j \in K : j \neq i} \rho^{i,j}_t \gamma^{i,j}_t \lambda^{i,j}_t.$$

The second component $\eta$ of strategy $\varphi$ is given by

$$\eta_t = V^{\ast}_t - \int_0^t \frac{1}{B_u} dD_u - \phi^\top_t S^*_t.$$

Moreover, the dynamics of $L^X$ has the form

$$dL^X_t = \left( \delta_t^\top - \phi^\top_t \sigma_t \right) dW_t + \int_{\mathbb{R}^n} (J_t(x) - \phi^\top_t F_t(x)) \tilde{\Pi}(dt, dx) + \sum_{i,j \in K : j \neq i} \left( \gamma^{i,j}_t - \phi^\top_t \rho^{i,j}_t \right) dM^{i,j}_t.$$

**Proof of Theorem 3.1.** By Theorem 2.5, we only need to find the explicit form of GKW decomposition of $V^*$ defined by (2.5). The GKW decomposition of $V^*$ always exists since $V^*$ is a square integrable martingale. Thus, we are looking for $\phi$ and $L^X$ such that

$$dV^{\ast}_t = \phi^\top_t dS^*_t + dL^X_t.$$
where \( L^X \) is a martingale orthogonal to the martingale \( S^* \), that is \( \langle L^X, S^{*(k)} \rangle = 0 \) for each \( k = 1, \ldots, d \). Suppose that \( \phi \) satisfies (5.10), then using representations (3.6) and (3.11) we have

\[
dL^X_t = \delta_t^\top dW_t + \int_{\mathbb{R}^n} J_t(x) \Pi(dt, dx) + \sum_{i,j \in K : j \neq i} \gamma^{i,j}_t dM^{i,j}_t
- \phi_t^\top \left( \sigma_t dW_t + \int_{\mathbb{R}^n} F_t(x) \Pi(dt, dx) + \sum_{i,j \in K : j \neq i} \rho^{i,j}_t dM^{i,j}_t \right).
\]

Rearranging yields

\[
dL^X_t = (\delta_t^\top - \phi_t^\top \sigma_t) dW_t + \int_{\mathbb{R}^n} (J_t(x) - \phi_t^\top F_t(x)) \Pi(dt, dx)
+ \sum_{i,j \in K : j \neq i} \left( \gamma^{i,j}_t - \phi_t^\top \rho^{i,j}_t \right) dM^{i,j}_t.
\]

Now we want to find \( \phi \) such that \( L^X \) is orthogonal to each \( S^{*(k)} \). First notice that the angle bracket of the continuous martingale parts of \( S^{*(k)} \) and \( L^{X,c} \) has the form

\[
d\langle S^{*(k)}, L^{X,c} \rangle_t = (\sigma_t)_k \left( \delta_t^\top - \phi_t^\top \sigma_t \right)^\top dt,
\]

where for a matrix \( E \) by \( (E)_k \) we denote the operation of taking \( k \)-th row from the matrix \( E \). By assumption NCJ

\[
\Delta S^{*(k)}_t \Delta L^X_t = \int_{\mathbb{R}^n} (F_t(x)_k (J_t(x) - \phi_t^\top F_t(x))^\top \Pi(dt, dx)
+ \sum_{i,j \in K : j \neq i} (\rho^{i,j}_t)_k \left( \gamma^{i,j}_t - \phi_t^\top \rho^{i,j}_t \right)^\top \Delta H^{i,j}_t.
\]

Therefore, since \( X \in L^2(\mathbb{P}) \), the following condition is equivalent to the strong orthogonality

\[
0 = (\sigma_t) \left( \delta_t^\top - \phi_t^\top \sigma_t \right)^\top
+ \int_{\mathbb{R}^n} (F_t(x)) (J_t(x) - \phi_t^\top F_t(x))^\top \nu_t(dx) + \sum_{i,j \in K : j \neq i} \rho^{i,j}_t \left( \gamma^{i,j}_t - \phi_t^\top \rho^{i,j}_t \right)^\top \lambda^{i,j}_t
\]

where \( 0 = (0, \ldots, 0)^\top \). This equality simplifies to

\[
\left( \sigma_t (\sigma_t)^\top + \int_{\mathbb{R}^d} F_t(x)(F_t(x))^\top \nu_t(dx) + \left( \sum_{i,j \in K : j \neq i} \rho^{i,j}_t (\rho^{i,j}_t)^\top \lambda^{i,j}_t \right) \right) \phi_t
\]

(3.11)

Using definition of \( G \) and \( A \) (i.e., (3.5) and (3.8)) we see that (3.11) is, in fact, (3.7). Therefore, the existence of GKW decomposition implies that \( A_t \in \text{Im}G_t \). Taking \( \phi \) as a minimum norm solution to (3.7) we have an explicit form of \( \phi \) in GKW decomposition. Now, the assertion of theorem follows from Theorem 2.5.

\[\square\]

**Corollary 3.2.** Assume that \( G_t \) is invertible for every \( t \in [0, T] \). Then the component \( \phi \) of 0-achieving D-risk-minimizing strategy is given by the following formula

\[
\phi_t = (G_t)^{-1} \left( \sigma_t \delta_t + \int_{\mathbb{R}^n} F_t(x) J_t(x) \nu_t(dx) + \sum_{i,j \in K : j \neq i} \rho^{i,j}_t \gamma^{i,j}_t \lambda^{i,j}_t \right).
\]

\[\square\]

**Proof.** It is an immediate consequence of invertibility of \( G \) and (3.7).
Corollary 3.3. The risk process of D-risk-minimizing strategy has the form

\[
R^D_t = E \left( \int_t^T \| \delta_u - \phi_u^T \sigma_u \|^2 du + \int_t^T \int_{\mathbb{R}^n} (J_u(x) - \phi_u^T F_u(x))^2 \nu_u(dx) du \right) \\
+ \sum_{i,j \in \mathcal{K}, j \neq i} \int_t^T (\gamma^{i,j}_u - \phi_u^T \rho^{i,j}_u)^2 \lambda^{i,j}_u du \bigg| \mathcal{F}_t ) .
\]

We can also formulate conditions ensuring attainability of dividend processes. For definition of \( \mathbb{P} \)-admissibility and attainability of dividend process \( D \) we refer to [19, Def. 16.13 and 16.9]. In [19, Prop. 16.14] it is proved that \( \varphi \) is \( \mathbb{P} \)-admissible and replicates \( D \) if and only if \( V_t(\varphi) = B_t \mathbb{E} \left( \int_t^T \frac{1}{n^2} dD_u | \mathcal{F}_t \right) \).

Theorem 3.4. Let \( D \) be a dividend process with representation \( (3.10) \). There exists a \( \mathbb{P} \)-admissible strategy \( \varphi = (\phi, \eta) \) which replicates \( D \) if and only if there exists a predictable process \( \phi \) satisfying:

\[
(\sigma_i)^T \phi_t = \delta_i \quad dt \times d\mathbb{P} \ a.e., \\
(F_t(x))^T \phi_t = J_t(x) \quad dt \times \nu(x) dx \times d\mathbb{P} \ a.e., \\
(\rho^{i,j}_t)^T \phi_t = \gamma^{i,j}_t, \quad \lambda^{i,j}_t dt \times d\mathbb{P} \ a.e. \ \forall i, j \in \mathcal{K}, i \neq j.
\]

Proof. We prove in Lemma 6.1 in Appendix that the existence of \( \mathbb{P} \)-admissible strategy \( \varphi = (\phi, \eta) \) which replicates \( D \) is equivalent to the fact that the process \( L^X \) is equal to zero. This fact and (3.9) give the assertion. \( \square \)

If the compensator of \( \Pi \) is a random measure such that all \( \nu_t \) has the same finite support (e.g., if \( \Pi \) is a Poisson random measure with Lévy measure \( \nu \) with finite support), then we can give sufficient conditions for attainability of an arbitrary square integrable dividend process \( D \). To formulate these conditions we introduce convenient notation: for vectors \( a_1, \ldots, a_q \) by \( [a_i]_{i=1}^q \) we denote the matrix created from theses vectors by setting them in columns, i.e.,

\[
[a_i]_{i=1}^q = [a_1 a_2 \ldots a_q].
\]

Proposition 3.5. Assume that for every \( t \in [0, T] \) the measure \( \nu_t \) has the same finite support, i.e., \( \supp \nu_t = \{ x_1, \ldots, x_q \}, \quad q \in \mathbb{N} \).

Let

\[
L^i_t := [\sigma_t, \nu_{t}(x_k)]_{k=1}^q, [\rho^{i,j}_t]_{j \in \mathcal{K}, j \neq i}.
\]

If, for every \( i \in \mathcal{K} \),

\[
\text{rank } (L^i_t) = n + q + K - 1 \quad \text{for every } t \in [0, T] \quad \mathbb{P} \ a.s.,
\]

then every square integrable dividend process \( D \) is attainable. Necessary condition for (3.14) is that number of assets is sufficiently large, i.e. \( d \geq n + q + K - 1 \).

Proof. Under the above assumptions, conditions formulated in Theorem 3.4 transform, for every \( i \in \mathcal{K} \), to the following system of finite number of equations:

\[
(\sigma_i)^T \phi_t = \delta_i \quad dt \times d\mathbb{P} \ a.e., \\
(F_{t}(x_k))^T \phi_t = J(t, x_k) \quad dt \times d\mathbb{P} \ a.e. \ \forall k = 1, \ldots, q, \\
(\rho^{i,j}_t)^T \phi_t = \gamma^{i,j}_t, \quad \lambda^{i,j}_t dt \times d\mathbb{P} \ a.e. \ \forall j \in \mathcal{K}, j \neq i,
\]

which can be written in the form

\[
(L^i_t)^T \phi_t = \left[ \delta^T_t, (J^T(t, x_k))_{k=1}^q, ((\gamma^{i,j}_t)^T)_{j \in \mathcal{K}, j \neq i} \right]^T
\]

Since (3.14) holds \( \mathbb{P} \) a.s., the minimum norm solution \( \phi \) to the above system of linear equations exists and is a predictable process. Hence \( \varphi = (\phi, \eta) \) is an admissible portfolio which replicates \( D \). \( \square \)
Remark 3.6. a) It is also worth to mention that Theorem 3.4 underlines the necessity of using other methods of hedging risk for non-attainable dividend streams. Thus, we consider in this paper the risk-minimizing strategies.

b) Theorem 3.4 also implies that condition (3.13) is to some extent necessary for attainability of an arbitrary square integrable dividend process D.

c) As we have seen, the replication of arbitrary dividend process requires sufficiently large number of assets which are in some sense linearly independent. In many cases this might be considered as a very unrealistic situation. Thus, a hedger has to choose the risk sources he/she wants to hedge against. For example, if a hedger will choose to hedge against Brownian-risk (continuous-risk), then he/she will be looking for \( \phi \) such that only the first condition is satisfied, i.e.,

\[
(\sigma_t)^T \phi_t = \delta_t.
\]

Usually this corresponds to the classical delta-hedging. Alternatively, a hedger might choose to hedge against credit-risk and then he/she will be interested in solving the following system of linear equations

\[
\left[ \rho_{i,j}^t \right]_{j \in K, j \neq i} \phi_t = \left[ \gamma_{i,j}^t \right]_{j \in K, j \neq i}.
\]

This kind of strategy is called delta-hedging of credit risk. Note that corresponding rank conditions for solving above systems is less demanding than (3.14).

In Theorem 3.1 we assume that the square integrable random variable \( X = \int_0^T \frac{1}{\Pi^u} dD_u \) representing discounted cumulated payments up to maturity time has certain martingale representation. This might be considered as a very restricted assumption, however this representation is ensured by a weak property of predictable representation of the triple \((W, \Pi, M)\) (see He, Wang and Yan [14 Def. 13.13]).

Definition 3.7. We say that \((W, \Pi, M)\) has a weak property of predictable representation with respect to \((\mathcal{F}, \mathbb{P})\) if every square integrable \((\mathcal{F}, \mathbb{P})\)-martingale \(N\) has the representation

\[
N_t = N_0 + \int_0^t \phi_u dW_u + \int_0^t \int_{\mathbb{R}^n} \psi_u(x)\Pi(dx, du) + \int_0^t \sum_{i,j \in K, j \neq i} \xi_u^{i,j} dM_u^{i,j},
\]

where \(\phi_u, \psi_u(x), \xi_u^{i,j}\) are predictable processes such that

\[
\mathbb{E} \left( \int_0^T |\phi_u|^2 du \right) < \infty, \quad \mathbb{E} \left( \int_0^T \int_{\mathbb{R}^n} |\psi_u(x)|^2 \nu_u(dx) du \right) < \infty,
\]

\[
\mathbb{E} \left( \int_0^T |\xi_u^{i,j}|^2 \lambda_u^{i,j} du \right) < \infty \quad i,j \in K, i \neq j.
\]

This property holds in many models, here we give two examples.

Example 1. Assume that \(C\) is a time homogenous Markov chain with an intensity matrix \(\Lambda\) which is independent of \(W\) and \(\Pi\) is an independent Poisson random measure. Then a weak property of predictable representation holds for \((W, \Pi, M)\) with respect to \((\mathcal{F}, \mathbb{P})\), where \(\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^\Pi \vee \mathcal{F}^C\). This follows by analogous arguments as in Becherer [3]. For the convenience of the reader we repeat these arguments. First, note that each of processes \(W, \Pi, M\) has the weak property of predictable representation with respect to their own filtration and therefore by independence and strong orthogonality of \(W, \Pi, M\) we have martingale representation (3.15) for martingales \(N\) such that \(N_T = 1_E 1_F 1_G\) where \(E \in \mathcal{F}^W_T, F \in \mathcal{F}^\Pi_T, G \in \mathcal{F}^C_T\). Finally, by standard approximation techniques, we obtain the asserted weak property of a predictable representation of \((W, \Pi, M)\) with respect to \((\mathcal{F}^W \vee \mathcal{F}^\Pi \vee \mathcal{F}^C, \mathbb{P})\).
3.1. Semimartingale dividend process. Assume that the square integrable random variable \( D \) constitute processes \( H^{i,j} \) with \( \tilde{P} \)-intensity processes denoted by \( \tilde{\lambda}^{i,j} \), so

\[
\hat{M}_{t}^{i,j} := H_{t}^{i,j} - \int_{0}^{t} \tilde{\lambda}_{u}^{i,j} du
\]

is a \( \tilde{P} \)-martingale. Assume that \((\tilde{W}, \tilde{P}, \hat{M})\) has the weak property of predictable representation with respect to \((\mathbb{F}, \tilde{P})\) and that \( \Pi \) is absolutely continuous with respect to \( \tilde{P} \) with the density process

\[
dm = \eta(- \begin{pmatrix} \beta_{t} d\tilde{W}_{t} + \int_{\mathbb{R}^{n}} (Y(t, x) - 1) \tilde{\Pi}(dt, dx) + \sum_{ij \neq i} (\kappa_{t}^{i,j} - 1) d\hat{M}_{t}^{i,j} \end{pmatrix}),
\]

\( \eta_{0} = 1. \)

Then \((W, \Pi, M)\) has the weak property of predictable representation with respect to \((\mathbb{F}, P)\) (see Jacod and Shiryaev [17, Thm. III.5.24]), where

\[
W_{t} := \tilde{W}_{t} - \int_{0}^{t} \beta_{u} du, \quad \Pi(dt, dx) := \Pi(dt, dx) - Y(t, x) \tilde{\nu}(dx) dt,
\]

\[
M_{t}^{i,j} := H_{t}^{i,j} - \int_{0}^{t} \kappa_{t}^{i,j} \tilde{\lambda}_{u}^{i,j} du, \quad i, j \in K, i \neq j.
\]

Example 2 shows, in principle, that we can construct a model having weak property of predictable representation with mutual dependence between \( W, \Pi \) and \( C \) starting from an independent ones.

3.1. Semimartingale dividend process. One of most the important class of dividend processes constitute processes \( D \) which are semimartingales with the following decomposition

\[
D_{t} = \xi^{D} \mathbb{1}_{t \geq T} + \int_{0}^{t} g^{D}_{u} du + \int_{0}^{t} \sum_{i \in K} (\delta^{D}_{u})^{i} dW_{u} + \int_{0}^{t} \sum_{i,j \in K} J^{D}_{u}(x) \tilde{\nu}(dx) du + \int_{0}^{t} \sum_{i,j \in K} \gamma^{D, i,j}_{u} dM_{t}^{i,j}
\]

with appropriately integrable quintuple of processes \((\xi^{D}, g^{D}, \delta^{D}, J^{D}, \gamma^{D})\). In this case finding risk minimization strategy boils down to finding martingale representation of

\[
\hat{X} := \frac{\xi^{D}}{B_{T}} + \int_{0}^{T} \frac{g^{D}}{B_{u}} du
\]

instead of \( X = \int_{0}^{T} \frac{B_{u}}{B_{T}} dD_{u} \). Indeed, applying the previous results for \( D \) yield

**Theorem 3.8.** Fix a dividend process \( D \) of the form (3.16), where \((\xi^{D}, g^{D}, \delta^{D}, J^{D}, \gamma^{D})\) satisfies

\[
\mathbb{E}\left( |\xi^{D} B_{T}^{-1}|^{2} + \int_{0}^{T} |B_{u}^{-1}(g^{D}_{u} + \delta^{D}_{u})|^{2} dt \right) < \infty, \quad \mathbb{E}\left( \int_{0}^{T} \int_{\mathbb{R}^{n}} |B_{u}^{-1} J^{D}_{u}(x)|^{2} \nu_{u}(dx) du \right) < \infty,
\]

\[
\mathbb{E}\left( \int_{0}^{T} |B_{u}^{-1} \gamma^{D, i,j}_{u}|^{2} \lambda_{u}^{i,j} du \right) < \infty \quad \text{for} \quad i, j \in K, i \neq j.
\]

Assume that the square integrable random variable \( \hat{X} := \frac{\xi^{D}}{B_{T}} + \int_{0}^{T} \frac{g^{D}}{B_{u}} du \) has the representation

\[
\hat{X} = \mathbb{E}(\hat{X}) + \int_{0}^{T} \tilde{\delta}_{t}^{T} dW_{t} + \int_{0}^{T} \tilde{J}_{t}(x) \tilde{\nu}(dx) dt + \sum_{i,j \in K, i \neq j} \int_{0}^{T} \tilde{\gamma}_{t}^{i,j} dM_{t}^{i,j}.
\]

Then there exists a \( \theta \)-achieving risk-minimization strategy \( \varphi = (\phi, \eta) \) for the dividend process \( D \). The component \( \phi \) of this strategy is the predictable version of the solution to the linear system

\[
G_{t} \phi_{t} = \hat{A}_{t},
\]
where a predictable process $\hat{X}$ is given by the formula

$$\hat{X}_t := \sigma_t (\hat{\delta}_t + B_t^{-1} \delta_t^D) + \int_{\mathbb{R}^n} F_t(x)(\hat{J}_t(x) + B_t^{-1} J_t^D(x)) \nu_t(dx)$$

$$+ \sum_{i,j \in K;j \neq i} \bar{\mu}_{i,j}(\gamma_t^{i,j} + B_t^{-1} \gamma_t^{D,i,j}) \lambda_t^{i,j}.$$ 

The second component $\eta$ of strategy $\varphi$ is given by

$$\eta_t = V_t^\ast - \int_0^t \frac{1}{B_u} dD_u - \phi_t^\top S_t^\ast.$$ 

Moreover, the dynamics of $L^X$ has the form

$$dL_t^X = (\hat{\delta}_t + B_t^{-1} \delta_t^D)^\top \phi_t + \int_{\mathbb{R}^n} \left( \hat{J}_t(x) + B_t^{-1} J_t^D(x) - \phi_t^\top F_t(x) \right) \tilde{\Pi}(dt, dx)$$

$$+ \sum_{i,j \in K;j \neq i} \left( \gamma_t^{i,j} + B_t^{-1} \gamma_t^{D,i,j} - \phi_t^\top \rho_t^{i,j} \right) dM_t^{i,j}.$$

Proof. Straightforward calculations show that if $\hat{X}$ has representation (3.18) then $X = \int_0^T B_u^{-1} dD_u$ has the representation

$$X = \mathbb{E}(X) + \int_0^T \left( \hat{\delta}_t + \delta_t^D \right)^\top dW_t + \int_0^T \int_{\mathbb{R}^n} \left( \hat{J}_t(x) + \frac{J_t^D(x)}{B_t} \right) \tilde{\Pi}(dt, dx)$$

$$+ \sum_{i,j \in K;j \neq i} \int_0^T \left( \gamma_t^{i,j} + \frac{\gamma_t^{D,i,j}}{B_t} \right) dM_t^{i,j}.$$ 

Applying Theorem 3.4 we obtain the assertion of theorem. 

Note that $\hat{X}$ can be connected with the ex-dividend price of $D$ in the following way

$$\pi_t(D) := B_t \mathbb{E} \left( \int_t^T \frac{1}{B_u} dD_u | F_t \right) = B_t \mathbb{E} \left( \xi^D \delta_t^D \right) + \int_t^T \frac{\xi_u^D}{B_u} d\xi_u^D | F_t \right) = B_t \left( \mathbb{E}(\hat{X} | F_t) - \int_0^t \frac{\xi_u^D}{B_u} d\xi_u^D \right)$$

The second conditional expectation resembles the one from Feynman-Kac type theorems. We will investigate this similarity in the next section where we will consider a very flexible Markovian framework. This allows us to derive explicit formulae for risk-minimizing strategies by means of solutions to some PIDE’s as we will see in Section 5. Using (3.19) we see that the previous results from this section could be rewritten for the semimartingale dividend processes with

$$\delta_t = \hat{\delta}_t + B_t^{-1} \delta_t^D,$$

$$J_t(x) = \hat{J}_t(x) + B_t^{-1} J_t^D(x),$$

$$\gamma_t^{i,j} = \hat{\gamma}_t^{i,j} + B_t^{-1} \gamma_t^{D,i,j}.$$ 

For instance, Theorem 3.4 concerning attainability can be reformulated as follows

**Theorem 3.9.** Let $D$ be a dividend process given by (3.10) for which $\hat{X}$ has representation (3.18). There exists a $\mathbb{P}$-admissible strategy $\varphi = (\phi, \eta)$ which replicates $D$ if and only if there exists a predictable process $\phi$ satisfying:

$$\left( \sigma_t \right)^\top \phi_t = \hat{\delta}_t + B_t^{-1} \delta_t^D \quad dt \times d\mathbb{P} \ a.e.,$$

$$\left( F_t(x) \right)^\top \phi_t = \hat{J}_t(x) + B_t^{-1} J_t^D(x) \quad dt \times \nu_t(dx) \times d\mathbb{P} \ a.e.,$$

$$\left( \rho_t^{i,j} \right)^\top \phi_t = \hat{\gamma}_t^{i,j} + B_t^{-1} \gamma_t^{D,i,j} \quad \lambda_t^{i,j} dt \times d\mathbb{P} \ a.e. \ \forall i, j \in K, i \neq j.$$
4. Risk-minimization in the Markovian market models with ratings

In this section we specify a Markovian market model. We assume that information available to the market participants is modeled by a multidimensional process $(Y, C)$ given as a solution to some SDE in $\mathbb{R}^{d+p} \times \mathcal{K}$. The first $d$ components of $Y$ are assumed to be a process $S$ of prices of tradable risky assets, and the remaining $p$ components, denoted by $R$, represent economic environment such as interest rates, inflation or stochastic volatility. Thus

$$S = (Y^i)_{i=1}^d, \quad R = (Y^i)_{i=d+1}^{d+p}. $$

$C$ is a stochastic process which has finite state space $\mathcal{K} = \{1, \ldots, K\}$ and that could be interpreted as a credit rating of corporate, so $C_u$ represents credit rating of corporate at time $u \leq T$. On the market there is a money account with the price process, denoted by $B$, depending on on economic conditions of market, i.e. $R$ and $C$. So $B$ is given as a unique solution to

$$dB_u = r(u, R_{u-}, C_{u-})B_u du, \quad B_0 = 1,$$

where $r$ is a measurable, deterministic and bounded function. Also the evolution of price depends on economic conditions of market which are described by rating system and additional non-tradable risk factor process $R$. So, we assume that our model is described by SDE in which the credit rating of corporate has influence on asset prices $S$ and other non-tradable factor $R$ by changing drift and volatility. Moreover, a change in credit rating from $i$ to $j$ at time $u$ causes a jump of size $\rho_{ij}^C(u, Y_{u-})$. Therefore, the evolution of $(S, R, C)$ is given as solution of the following SDE in $\mathbb{R}^{d+p} \times \mathcal{K}$ :

$$dS_u = \left( \begin{array}{c} \mu_S(u, S_{u-}, R_{u-}, C_{u-}) \\ \mu_R(u, R_{u-}, C_{u-}) & \sigma_S(u, S_{u-}, R_{u-}, C_{u-}) \\ \sigma_R(u, R_{u-}, C_{u-}) \end{array} \right) du + \left( \begin{array}{c} \sigma_S(u, S_{u-}, R_{u-}, C_{u-}) \\ \sigma_R(u, R_{u-}, C_{u-}) \\ \sigma_R(u, R_{u-}, C_{u-}) \end{array} \right) dW_u$$

$$+ \int_{\mathbb{R}^n} \left( \begin{array}{c} F_S(u, S_{u-}, R_{u-}, C_{u-}, x) \\ F_R(u, R_{u-}, C_{u-}, x) \end{array} \right) \bar{\Pi}(dx, du)$$

$$+ \sum_{i,j=1}^K \rho_{ij}^C(u, S_{u-}, R_{u-}) \rho_{ij}^R(u, R_{u-}) 1\{i\}(C_{u-})(dN_{i,j}^u - \lambda_{i,j}^C(u, S_{u-}, R_{u-})du),$$

$$dC_u = \sum_{i,j=1}^K (j - i) 1\{i\}(C_{u-})dN_{i,j}^u,$$

$$S_0 = s, \quad R_0 = r, \quad C_0 = c \in \mathcal{K},$$

where $W$ is a standard $r$-dimensional Wiener process, $\bar{\Pi}(dx, du)$ is a compensated Poisson random measure on $\mathbb{R}^n \times [0, T]$ with the intensity measure $\nu(dx)du$ where $\nu$ is a Lévy measure, and $N_{i,j}^u$ are counting point processes with intensities determined by $\lambda_{i,j}$, bounded continuous functions in $(u, y)$, i.e., for the fixed $(i, j)$, the process

$$\tilde{M}_{i,j}^u := N_{i,j}^u - \int_0^u \lambda_{i,j}(v, Y_{v-})dv$$

is a martingale. Note that the martingales defined by (4.2) take the following form

$$M_{i,j}^u = H_{i,j}^u - \int_0^u H_{i,j}^v \lambda_{i,j}(v, Y_{v-})dv,$$

since

$$H_{i,j}^u = \int_0^u 1\{i\}(C_{v-})dN_{i,j}^v.$$ 

Moreover, we require that the Poisson random measure $\Pi$ and the processes $N_{i,j}^u$, $i,j \in \mathcal{K}$, $i \neq j$, have no common jumps, i.e., for every $t > 0$ and every $b > 0$,

$$\int_0^t \int_{\|x\|>b} \Delta N_{i,j}^v \Pi(dx, dv) = 0 \quad \mathbb{P} - a.s.,$$

where $\Delta N_{i,j}^v$ denotes the jump of $N_{i,j}^v$ at the time $v$. This condition ensures that the martingale property holds for the processes $M_{i,j}^u$. The martingale property of $M_{i,j}^u$ is important because it allows us to use the theory of martingales to study the properties of the market model.
and for all \((i_1, j_1) \neq (i_2, j_2)\),
\begin{equation}
\Delta N^{i_1, j_1}_u \Delta N^{i_2, j_2}_u = 0 \quad \mathbb{P} - a.s.
\end{equation}

We also impose the following condition
\[\mu_S(u, s, r, c) = \sigma(u, s, r, c),\]
which implies that discounted prices of tradable assets \(S\) are local martingales under the probability \(\mathbb{P}\). The presence of \(\rho^{i,j}\) in the above SDE adds extra flexibility to model, so it is very important from the point of view of possible applications. For example, it enables us to introduce into a model a dependence of intensity of jumps \(C\) at time \(t\) on behavior of trajectory of process \(C\) up to time \(t\). This is the case of semi-Markov processes where \(\lambda^{i,j}\) at time \(t\) depends on time which the process \(C\) spends in the current state after the last jump. The process (say \(R^1\)) corresponding to this semi-Markovian dependence can be introduced in our framework by setting
\[dR^1_t = dt - \sum_{i,j \in K} R^1_t \mathbb{1}_i(C_{t-})dN^{i,j}_u.\]

So if we allow \(\lambda^{i,j}\) to be a (non-constant) function of \(R^1\) we obtain a semi-Markov model (see Section 6).

For convenience and brevity we introduce the following notation:
\[
Y_u := (S_u, R_u), \\
z := (s, r), \quad \mu_Y(u, z, c) := \begin{pmatrix} \mu_S(u, s, r, c) \\
\mu_R(u, r, c) \end{pmatrix}, \quad \sigma_Y(u, z, c) := \begin{pmatrix} \sigma_S(u, s, r, c) \\
\sigma_R(u, r, c) \end{pmatrix},
\]
\[F_Y(u, z, c, x) := \begin{pmatrix} F_S(u, s, r, c, x) \\
F_R(u, r, c, x) \end{pmatrix}, \quad \rho^{i,j}_{Y}(u, y) := \begin{pmatrix} \rho^{i,j}_{S}(u, s, r) \\
\rho^{i,j}_{R}(u, r) \end{pmatrix},
\]
\[a_Y(u, z, c) := \begin{pmatrix} a_{SS}(u, z, c) & a_{SR}(u, z, c) \\
a_{RS}(u, z, c) & a_{RR}(u, z, c) \end{pmatrix} := \begin{pmatrix} \sigma_S \sigma_{SS} & \sigma_S \sigma_{SR} \\
\sigma_R \sigma_{RS} & \sigma_R \sigma_{RR} \end{pmatrix}.
\]

Using this notation we can write SDE (4.2) as the following multidimensional SDE in \(\mathbb{R}^{d+p} \times \mathcal{K}\)
\begin{equation}
dY_u = \mu_Y(u, Y_u, C_u)du + \sigma_Y(u, Y_u, C_u)dw_u + \int_{\mathbb{R}^n} F_Y(u, Y_u, C_u, x)\Pi(dx, du)
+ \sum_{i,j=1}^K \rho^{i,j}_{Y}(u, Y_{u-})\mathbb{1}_i(C_{u-}-(dN^{i,j}_u - \lambda^{i,j}(u, Y_{u-}))du),
\end{equation}
\[dC_u = \sum_{i,j=1}^K (j - i)\mathbb{1}_i(C_u-)dN^{i,j}_u,
\]
\[Y_0 = y, \quad C_0 = c \in \mathcal{K},\]

with the coefficients being measurable functions \(\sigma_Y(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^{d+p} \times \mathcal{K} \to \mathbb{R}^{(d+p) \times r}, F_Y(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^{d+p} \times \mathcal{K} \times \mathbb{R}^n \to \mathbb{R}^{d+p}\) and \(\rho^{i,j}(\cdot, \cdot) : [0, T] \times \mathbb{R}^{d+p} \to \mathbb{R}^{d+p}\). This SDE is a non standard one since a driving noise depends on the solution itself (as in Jacod and Protter [16], Becherer [14]), that is, the noise \((N^{i,j})_{i,j \in K \setminus \{i\}}\) is not given a priori, it is also constructed. The solution is a quintuple \((Y, C, (N^{i,j})_{i,j \in K \setminus \{i\}}, W, \mathbb{P})\) together with a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\). This corresponds exactly to the concept of weak solutions of SDE’s (see Rogers and Williams [24]) and uniqueness in law is suitable concept of uniqueness for weak SDE’s. On the other hand the existence and uniqueness of weak solutions to SDE (4.8) (started at 0) is also required to model a market in the proper way. Indeed, if there is no uniqueness in law, then there is an ambiguity on a law of prices, so we do not have a market model specified correctly. Furthermore to establish connection between SDE’s and related PIDE’s (Markov property) we are also interested in existence and uniqueness of solutions to the SDE (4.8) started at every \(t \in [0, T]\) from every \((y, c) \in \mathbb{R}^{d+p} \times \mathcal{K}\) and defined on interval \([t, T]\). Note that these weak solutions
\((Y^{t,y,c}, C^{t,y,c}, (\{N^{i,j}\}^{t,y,c}),_{i,j \in K, j \neq i}, W^{t,y,c}, \Pi^{t,y,c})_{u \in [t,T]}\) can be constructed on possibly different filtered probability spaces \((\Omega^{t,y,c}, \mathbb{F}^{t,y,c}, \mathbb{P}^{t,y,c})\).

The expectation with respect to \(\mathbb{P}^{t,y,c}\) will be denoted by \(\mathbb{E}^{t,y,c}\). By \((H^{i,j})^{t,y,c}\) and \((M^{i,j})^{t,y,c}\) we denote processes defined by

\[
\begin{align*}
(H^{i,j})^{t,y,c}_u &:= \int_t^u I^1_i(C^{t,y,c}_v)d(N^{i,j})^{t,y,c}_v, \\
(M^{i,j})^{t,y,c}_u &:= (H^{i,j})^{t,y,c}_u - \int_t^u I^1_i(C^{t,y,c}_v)\lambda^{i,j}(v, Y^{t,y,c}_v)dv.
\end{align*}
\]

(4.9)

Sometimes, for brevity of notation, we will write \((4.9)\) considered as the law on an appropriate Skorochod space, solves the time dependent martingale problem. Let \(I\) be a class of functions defined by

\[
I = \{v : [0, T] \times \mathbb{R}^{d+p} \times K \to \mathbb{R} : v \text{ is measurable, and } \forall s \in \mathbb{R}^d \forall i \in K
\]

\[
\int_{\mathbb{R}^d} \left( |v'(u,z) + F_Y(u,z,i,x)| - v'(u,z) - \nabla v(u,z) F_Y(u,z,i,x) \right) |dx| < \infty \}.
\]

Let \(C^{1,2} = C^{1,2}([0, T] \times \mathbb{R}^{d+p} \times K)\) be the space of all measurable functions \(v : [0, T] \times \mathbb{R}^{d+p} \times K \to \mathbb{R}\) such that \(v(\cdot, \cdot, k) \in C^{1,2}([0, T] \times \mathbb{R}^{d+p})\) for every \(k \in K\), and let \(C^{1,2}_c\) be a set of functions \(f \in C^{1,2}\) with compact support. Define the family of operators \((A_t)_{t \in [0,T]}\) acting on \(C^{1,2}_c \cap I\) by setting

\[
A_t v(z, c) := \nabla v(u(z, c)) \mu_Y(u(z, c), c) + \frac{1}{2} \text{Tr} \left( a_Y(u(z, c), \nabla^2 v(u(z, c)) \right)
\]

\[
+ \int_{\mathbb{R}^d} \left( v(t,y) + F_Y(u(z, c), x, c) - v(t,y,c) - \nabla v(u(z, c)) F_Y(u(z, c), x, c) \right) \nu(dx)
\]

\[
+ \sum_{c' \in K \setminus c} \left( v(u(z, c') - v(u(z, c) c', z) - \nabla v(u(z, c)) \rho^{c,c'}_Y(u(z, c) c', z) \right) \lambda^{c,c'}(u, z).
\]

Here, by \(Tr\) we denote the trace operator, \(a_Y(u(z, c)) := \sigma_Y(u(z, c)) \sigma_Y(u(z, c)) \sigma_Y(u(z, c)) \sigma_Y(u(z, c))\), and \(\nabla^2 v\) is the matrix of second derivatives of \(v\) with respect to the components of \(y\). By \(\nabla v(\nabla s v, \nabla rv\text{ respectively})\) we denote the row vector of partial derivatives of function \(v\) with respect to \(y (s, r\text{ respectively})\).

We make the following assumption on coefficients of (4.8).

**Assumption LG.** Functions \(\mu_Y, \sigma_Y, F_Y, \rho_Y^{i,j}\) satisfy the following linear growth condition

\[ |\mu_Y(u(z, c))|^2 + |\sigma_Y(u(z, c))|^2 + \int_{\mathbb{R}^n} |F_Y(u(z, c, x)|^2 |dx| + \sum_{i,j \neq i} |\rho_Y^{i,j}(u(z, c)|^2 \leq K (1 + |z|^2), \]

Assuming (LG), among others, we can prove very important facts (see [20]) used in the sequel, such as:

1. The component \(Y^{t,y,c}\) of any weak solution to SDE (4.8), \(Y^{t,y,c}\) started from \((y, c)\) at \(t \in [0,T]\), and constructed on some filtered probability space \((\Omega^{t,y,c}, \mathbb{F}^{t,y,c}, \mathbb{P}^{t,y,c})\) satisfies

\[
\mathbb{E}^{t,y,c} \left( \sup_{u \in [t,T]} |Y^{t,y,c}_u|^2 \right) < \infty.
\]

2. \(C^{1,2}_c \subset I\).

3. For \(v \in C^{1,2}_c\) the mapping \((u, z, c) \mapsto A_u v(y, c)\) has quadratic growth.

By applying the Itô lemma and exploiting properties (1), (2) and (3) we obtain that for every \(v \in C^{1,2}_c\) the process \(M^v\) defined for \(u \in [t,T]\) by

\[
M^v_u := v(Y^{t,y,c}_u, C^{t,y,c}_u) - \int_t^u A_u v(Y^{t,y,c}_u, C^{t,y,c}_u)du
\]

is a \((\mathbb{P}^{t,y,c}, \mathbb{P}^{t,y,c})\) martingale (see [20] Prop. 2.5 and Thm. 5.1). We have, by construction, that \(\mathbb{P}^{t,y,c}(Y^{t,y,c}_t = y, C^{t,y,c}_t = c) = 1\), and this means that \(\mathbb{P}^{t,y,c}\) the law of \((Y^{t,y,c}_u, C^{t,y,c}_u)_{u \in [t,T]}\) considered as the law on an appropriate Skorochod space, solves the time dependent martingale problem for \((A_u)_{u \in [t,T]}\) started at \(t \) from \((y, c)\).
Of course, in general, assumption (LG) is to weak to guarantee existence and uniqueness of solutions to SDE (4.8), therefore, to present results as general as possible, we assume

**Assumption EUWS** (existence of unique weak solution). For each \((y, c) \in \mathbb{R}^{d+p} \times \mathcal{K}\) and \(t \in [0, T]\), the SDE (4.8) on the interval \([t, T]\) has a unique weak solution on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}^{t,y,c})\).

For some sufficient conditions ensuring that EUWS is satisfied we refer to Kurtz [22] and Jakubowski and Niewegowski [20] where the existence and weak uniqueness of SDE’s is related with the existence and well-possedness of corresponding martingale problem. In particular assumption EUWS yields that martingale problem for \((\mathcal{L}_u)_{u \in [t, T]}\) is well posed, and therefore \(\{\mathbb{P}^{t,y,c} : (s, c) \in \mathbb{R}^d \times \mathcal{K}\}\) is a Markov family (see [20] Prop. 4.5)). So, for a bounded functional \(h\), we have

\[
E^{t,y,c}(h((Y_u^{0,y,c}, C_u^{0,y,c})_{u \in [t,T]}) | \mathcal{F}_t) = E^{t,y',c'}(h((Y_u^{t,y',c'}, C_u^{t,y',c'})_{u \in [t,T]}) | (y', c') = (Y_t^{0,y,c}, C_t^{0,y,c}))
\]

where equality holds \(\mathbb{P}^{0,y,c}\) a.s.

Now we establish a 0-achieving risk-minimizing strategy for a rating sensitive claim (see [19]), i.e., for a payment stream \(D\)

\[
D_t := h(Y_T, C_T)1_{(t \geq T)} + \int_0^T g(u, Y_u, C_u)du + \sum_{i,j \neq i} \int_0^T \delta^{i,j}(u, Y_u) dH_u^{i,j},
\]

\(0 \leq t \leq T\), where \(h, g, \delta^{i,j}\) are measurable real-valued function such that

\[
\forall t \leq T \quad E \left( \int_t^T \frac{1}{B_u} dD_u \right) < \infty.
\]

We connect this with a problem of finding the *ex-dividend price function*, i.e. a function \(v\) such that \(v(t, Y_t, C_t) = V_t\), with the process \(V\) defined by

\[
V_t := B_t E \left( \frac{h(Y_T, C_T)}{B_T} + \int_t^T g(u, Y_u, C_u) B_u \frac{dD_u}{B_u} \right) + \sum_{i,j \neq i} \int_t^T \delta^{i,j}(u, Y_u) dH_u^{i,j} | \mathcal{F}_t).
\]

Note that the existence of such function \(v\) follows immediately from the Markov property of \((Y, C)\) with respect to \(\mathcal{F}\). Under assumption EUWS, by (4.11), we have

\[V_t = v(t, Y_t, C_t),\]

where

\[
v(t, y, c) := E^{t,y,c}(h(Y_T, C_T) B_T^{t,y,c} + \int_t^T g(u, Y_u, C_u) B_u^{t,y,c} du + \sum_{i,j \neq i} \int_t^T \delta^{i,j}(u, Y_u) dH_u^{i,j} | \mathcal{F}_t).
\]

with \(B_u^{t,y,c}\) defined by

\[
B_u^{t,y,c} = \exp \left( \int_0^u v(v, R_v^{t,y,c}, C_v^{t,y,c}) dv \right).
\]

So \(B_u^{t,y,c}\) is the unique solution, on the interval \([t, T]\), of ODE

\[
dB_u^{t,y,c} = v(u, R_u^{t,y,c}, C_u^{t,y,c}) B_u^{t,y,c} du, \quad B_0^{t,y,c} = 1,
\]

\(\forall (y', c')\).

Let us notice that the law of \(B_u^{t,y,c}\) is the same as the law of \(B_0^{t,y,c}\) on the set \(\{Y_t^{0,y,c} = y, C_t^{0,y,c} = c\}\) for arbitrary \((y', c')\).

In general, we do not know anything about regularity of \(v\). The next theorem states that for a sufficiently regular function \(v\) we can find a 0-achieving risk-minimizing strategy for \(D\) written explicitly in terms of this function \(v\).
Theorem 4.1. Consider the dividend process $D$ given by \eqref{eq:dividend-process}. Assume EUWS and that:

i) There exists $m \geq 1$ such that

\begin{equation}
\mathbb{E}\left(\sup_{t \in [0,T]} |Y_t|^{2m}\right) < \infty,
\end{equation}

and moreover the following growth condition for this $m$ holds

\begin{equation}
|h(z, i)|^2 + |g(u, z, i)|^2 + \sum_{j \in K : j \neq i} |\delta^{i,j}(u, z)|^2 \leq K(1 + |z|^{2m}).
\end{equation}

ii) There exist an ex-dividend price function $v$ which belongs to $C^{1,2} \cap \mathcal{I}$.

Then there exists a $\theta$-achieving risk-minimization strategy $\varphi = (\phi, \eta)$ for the dividend process $D$. The component $\phi$ of this strategy is the predictable version of solution to the linear system

\begin{equation*}
G(t, Y_{t-}, C_{t-}) \phi_t = A(t, Y_{t-}, C_{t-}),
\end{equation*}

where

\begin{align*}
G(u, z, i) := & a_{SS}(u, z, i) + \int_{\mathbb{R}^n} (F_S F_S^\top)(u, z, i, x)\nu(dx) + \sum_{j \in K : j \neq i} (\rho_{S}^{i,j} \rho_{S}^{j,i})^{\top}(u, z)\lambda^{i,j}(u, z), \\
A(u, z, i) := & a_{SS}(u, z, i)[(\nabla_S \nu(u, z, i))]^\top + a_{SR}(u, z, i)(\nabla_R \nu(u, z, i)) + \int_{\mathbb{R}^n} (F_S(u, z, i, x)(\nu(u + F_Y(u, z, i, x), i) - \nu(u, z))\nu(dx) \\
& + \sum_{j \in K : j \neq i} \rho_{S}^{i,j}(u, z)(\nu(u + \rho_{S}^{i,j}(u, z), j) - \nu(u, z)) + \delta^{i,j}(u, z))^\top \lambda^{i,j}(u, z) \\
\end{align*}

and

\begin{equation*}
\eta_t = \frac{v(t, Y_t, C_t)1_{\{t < T\}} - \phi_t^\top S_t}{B_t}.
\end{equation*}

Proof. Let $X = \int_0^T \frac{1}{B_t} dD_u$. We find the martingale representation of $X$ of the form \eqref{eq:condition-4.6}, and then apply Theorem 3.1 to obtain the asserted formulae. Using i), the Cauchy-Schwarz inequality and the isometry formula for stochastic integrals we see that $X \in L^2$. Thus, $M_t := \mathbb{E}(X|\mathcal{F}_t)$ is a martingale in $\mathcal{H}^2$ - the space of square integrable martingales. By definition of $V$ (i.e. \eqref{eq:theta-1}), $M$ can be represented as

\begin{align*}
M_t = & \frac{V_t}{B_t} + \int_0^t \frac{g(u, Y_{u-}, C_{u-})}{B_u} du + \sum_{i,j \neq i} \int_0^t \frac{\delta^{i,j}(u, Y_{u-})}{B_u} dH_{u}^{i,j} \\
& = \frac{v(t, Y_t, C_t)}{B_t} + \int_0^t \frac{g(u, Y_{u-}, C_{u-}) + \sum_{i,j \neq i} \delta^{i,j}(u, Y_{u-})H_{u}^{i,j}}{B_u} du \\
& + \sum_{i,j \neq i} \int_0^t \frac{\delta^{i,j}(u, Y_{u-})}{B_u} dM_{u}^{i,j}.
\end{align*}

Using ii), the Itô formula, \eqref{eq:asserted-formula} and \eqref{eq:condition-4.10} we obtain

\begin{align*}
d\frac{v(t, Y_t, C_t)}{B_t} = & \frac{1}{B_t} (dv(t, Y_t, C_t) - v(t, Y_{t-}, C_{t-})v(t, Y_t, C_t) dt) \\
& = \frac{1}{B_t} (dM_t^v + [(\partial_t + \mathcal{A}_t - v)(t, Y_{t-}, C_{t-})] dt),
\end{align*}
where $M^v$ is given by

\begin{equation}
M^v_t := \int_0^t \sum_i H^v_{u-} \nabla v(u, Y_u, i) \sigma_Y(u, Y_u) dW_u
+ \int_0^t \sum_i H^v_{u-} \int_{\mathbb{R}^n} (v(u, Y_u + F_Y(u, Y_u, i, x), i) - v(u, Y_u, i)) \tilde{\Pi}(du, dx)
+ \int_0^t \sum_{i,j \neq i} (v(u, Y_u + \rho^i_j(u, Y_u), j) - v(u, Y_u, i)) H^i_{u-} dM^{i,j}_u.
\end{equation}

Hence we obtain

\begin{equation}
M_t = v(0, Y_0, C_0) + \int_0^t \frac{1}{B_u} (dM^v_u + \sum_{i,j \neq i} \delta^{i,j}(u, Y_u) dM^{i,j}_u) + A_t,
\end{equation}

where

\begin{equation}
A_t = \int_0^t \frac{1}{B_u} \left( \sum_i H^v_{u-} \left[ (\partial_i + A_u - \tau) v(u, Y_u, i) + g(u, Y_u, i) + \sum_{j \neq i} \delta^{i,j}(u, Y_u) \lambda^{i,j}(u, Y_u) \right] \right) du.
\end{equation}

We see that RHS of (4.20) is a special semimartingale with the unique predictable finite variation part given by $A$. On the other hand, this special semimartingale is equal to $M$ which is a martingale and therefore $A$ is equal to zero (by uniqueness of canonical decomposition of special semimartingales). This implies that

\[ X = M_T = v(0, Y_0, C_0) + \int_0^T \frac{1}{B_u} (dM^v_u + \sum_{i,j \neq i} \delta^{i,j}(u, Y_u) dM^{i,j}_u). \]

So, we obtain that the predictable processes $\delta$, $J(\cdot, x)$, $\gamma_{i,j}$ in the representation (3.6) are given by

\begin{equation}
\delta_t = \sum_{i \in \mathcal{K}} \frac{H^i_{t-}}{B_t} \left[ (\sigma_S(t, Y_t, i))^T (\nabla_S v(t, Y_t, i))^T + (\sigma_R(t, Y_t, i))^T (\nabla_R v(t, Y_t, i))^T \right],
\end{equation}

\begin{equation}
J_t(x) = \sum_{i \in \mathcal{K}} \frac{H^i_{t-}}{B_t} (v(t, Y_t + F_Y(t, Y_t, i, x), i) - v(t, Y_t, i)),
\end{equation}

\[ \gamma^{i,j}_t = B_t^{-1} \left( v(t, Y_t + \rho^i_j(t, Y_t), j) - v(t, Y_t, i) + \delta^{i,j}(t, Y_t) \right). \]

Thus the assertion follows from Theorem 3.4. □

Remark 4.2. If in SDE (4.2) there is no component $R$ (i.e., $Y = S$) then, by Remark 3.4c and (4.22), the Brownian-risk can be eliminated by a usual delta-hedging strategy which is given by

\[ \phi_t = (\nabla_S v(t, S_{t-}, C_{t-}))^T. \]

If component $R$ appears in diffusion model, i.e., in (4.2) with $F_S = 0$, $F_R = 0$, $\rho_S = 0$ and $\rho_R = 0$, then the Brownian-risk cannot be eliminated completely. The risk-minimizing strategy is given by

\[ \phi_t = (\nabla_S v(t, Y_{t-}, C_{t-}))^T + (\sigma_{SS}^{-1} a_{SR})(t, Y_{t-}, C_{t-}) (\nabla_R v(t, Y_{t-}, C_{t-}))^T, \]

where $a_{SS}^{-1}$ denotes the Moore-Penrose pseudo-inverse of $a_{SS}$ (see Albert [1]). Hence, if $a_{SR} = 0$, i.e., if continuous martingale parts of $R$ and $S$ are orthogonal, then $\phi_t = (\nabla_S v(t, Y_{t-}, C_{t-}))^T$, and the risk-minimizing strategy is a usual delta hedging strategy.
Delta-hedging of credit risk strategy is \( \phi(t) = \sum_{i \in K} \mathbb{1}_i(C_{t-}) \phi_i' \), where \( \{ \phi_i' \}_{i \in K} \) are appropriately measurable solution to the system of linear equations:

\[
\left( \begin{array}{cccc}
\rho_{S,1,1}^i & \rho_{S,1,2}^i & \cdots & \rho_{S,1,d}^i \\
\rho_{S,2,1}^i & \rho_{S,2,2}^i & \cdots & \rho_{S,2,d}^i \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{S,i-1,1}^i & \rho_{S,i-1,2}^i & \cdots & \rho_{S,i-1,d}^i \\
\rho_{S,i,1}^i & \rho_{S,i,2}^i & \cdots & \rho_{S,i,d}^i \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{S,i,K,1}^i & \rho_{S,i,K,2}^i & \cdots & \rho_{S,i,K,d}^i
\end{array} \right) \phi_i' = \left( \begin{array}{c}
\Delta_{i,1} v + \delta_{i,1} \\
\Delta_{i,2} v + \delta_{i,2} \\
\vdots \\
\Delta_{i,i-1} v + \delta_{i,i-1} \\
\Delta_{i,i} v + \delta_{i,i} \\
\vdots \\
\Delta_{i,K} v + \delta_{i,K}
\end{array} \right),
\]

where

\[ \Delta_{i,j} v(t, Y_{t-}) := v(t, Y_{t-} + \rho_{ij}^i(t, Y_{t-}), j) - v(t, Y_{t-}, i), \]

and \( \rho_{i,j}^k(t, Y_{t-}) \) denotes the \( k \)-th coordinate of \( \rho_{i,j}^i(t, Y_{t-}) \). For convenience we drop the dependence on \( t \) and \( Y_{t-} \) in the above system of equations.

**Remark 4.3.** a) The conditions ensuring that (4.17) holds are (LG) together with (LGm), i.e.

(4.23)

\[ \int_{\mathbb{R}^n} \left| F_Y(u, z, i, x) \right|^{2m} \nu(dx) \leq K(1 + |z|^{2m}), \]

for some \( m \geq 1 \), see [20] Thm. 3.2. Note that if there exists a function \( K \) with

\[ \int_{\mathbb{R}^n} (|K(x)|^2 + |K(x)|^{2m}) \nu(dx) < \infty \]

for some \( m \geq 1 \), such that

\[ \frac{|F_Y(u, z, i, x)|}{1 + |z|} \leq K(x) \quad \forall z \in \mathbb{R}^{d+p}, \]

then \( F \) satisfies (LG) and (LGm).

b) Note that if (LG) and (4.23) hold, and \( v \in C^{1,2} \) is such that \( |\nabla^2 v(u, z, i)| \leq K(1 + |z|^{2m-2}) \) for each \( i \in K \), then \( v \in C^{1,2} \cap L \).

5. FEYNMAN-KAC THEOREM AND PIDE

In this section we formulate and prove Feynman-Kac Theorem for components \( (Y^{t,y,c}_u, C^{t,y,c}_u)_{u \in [t,T]} \) of a weak solution to SDE (4.18). So, we connect calculation of conditional expectation of some interesting and important functionals of \((Y^{t,y,c}_u, C^{t,y,c}_u)_{u \in [t,T]}\) with solution of corresponding PIDE with given boundary conditions. We consider functionals of the form (4.12) with \( h, g, \delta^{i,j} \) satisfying (4.13). With such functional we connect a process \( V \) given by (4.14), and for \( V \) we prove a Feynman-Kac type theorem. In general context our results concerns functionals of the type

\[
e^{-\int_t^T r(u, Y^{t,y,c}_u, C^{t,y,c}_u)du} h(Y^{t,y,c}_T, C^{t,y,c}_T) + \int_t^T e^{-\int_t^s r(u, Y^{t,y,c}_u, C^{t,y,c}_u)du} g(u, Y^{t,y,c}_u, C^{t,y,c}_u)du \]

\[+ \sum_{i,j \neq i} \int_t^T e^{-\int_t^s r(u, Y^{t,y,c}_u, C^{t,y,c}_u)du} \delta^{i,j}(u, Y^{t,y,c}_u) dH^{i,j}_u. \]

Note also that if there is no component \( C \) we have classical Feynman-Kac Theorem for jump-diffusion with killing at rate \( r \). Thus, we solve a general problem which is of independent interest and has many applications, among others in finding risk-minimizing strategies, which will be presented in this section.

The process \( A \) given by (4.20) is equal to zero as we see in the proof of Theorem 4.11. By right continuity of paths of \((Y, C)\) we obtain

\[ L_t := \int_0^t \frac{1}{B_u} \left( (\partial_t A_u - r)v(u, Y_u, C_u) + g(u, Y_u, C_u) + \sum_{j \in K \setminus C_u} \delta^{C_u,j}(u, Y_u) \lambda^{C_u,j}(u, Y_u) \right) du = 0. \]
The mean value theorem for right continuous functions yields

$$(\partial_t + A_t - \mathbf{r})v(0, y, c) + g(0, y, c) + \sum_{j\in\mathcal{K}\setminus c} \delta^{c,j}(0, y)\lambda^{c,j}(0, y) = 0.$$ 

Using similar arguments we obtain

**Theorem 5.1.** Assume EUWS and that (4.17) and (4.18) hold for some $m \geq 1$. Let $v$ be a function defined by (4.18). If $v \in C^{1,2} \cap \mathcal{I}$, then $v$ solves the following PIDE

$$v(t, y, c) = h(y, c).$$

This gives us a way to finding the ex-dividend price function $v$ and to show connection with so called Feynman-Kac type theorems. Thus to obtain $v$ we solve PIDE (5.1) and if solution is a sufficiently regular function, then $v$ has the stochastic representation

$$v(t, Y_t, C_t) = B_t \mathbb{E} \left[ \left( \frac{h(Y_T, C_T)}{B_T} + \int_t^T g(u, Y_u, C_u) \frac{du}{B_u} + \sum_{i,j \neq i} \int_t^T \delta^{i,j}(u, Y_u) \frac{dM^i_u}{B_u} \right) \bigg| F_t \right].$$

So $v$ is indeed the ex-dividend price function. The next theorem gives precise formulation of this idea.

**Theorem 5.2.** Let $(Y_{u_t}^{t,y,c}, C_{u_t}^{t,y,c})_{u \in [t, T]}$ be the components of a weak solution to SDE (4.18) started at $t$ from $(y, c) \in \mathbb{R}^{d+p} \times \mathcal{K}$. Assume that a function $v \in C^{1,2} \cap \mathcal{I}$ solves the PIDE (5.1) with the boundary condition (5.2) and the local martingale $\tilde{M}$ on $[t, T]$ with the dynamic

$$d\tilde{M}_t = \sum_i H_{-u}^i \nabla v(u, Y_{u-}^{t,y,c}, i) \sigma_Y(u, Y_{u-}^{t,y,c}, i) dW_{u}^{t,y,c}$$

$$(5.3) + \sum_i H_{-u}^i \int_{\mathbb{R}^n} (v(u, Y_{u-}^{t,y,c} + F_Y(u, Y_{u-}^{t,y,c}, i, x), i) - v(u, Y_{u-}^{t,y,c}, i)) \Pi^{i,y,c}(dx, du)$$

$$(5.3) + \sum_{i,j \neq i} (v(u, Y_{u-}^{t,y,c} + \rho_{ij}^{i}(u, Y_{u-}^{t,y,c}, j) - v(u, Y_{u-}^{t,y,c}, i) + \delta^{i,j}(u, Y_{u-}^{t,y,c})) d(M^{i,j})_{u}$$

$$(5.3) \tilde{M}_t = v(t, y, c)$$

is a martingale on $[t, T]$. Then

$$(5.4) v(t, y, c) = \mathbb{E}^{t,y,c} \left[ \frac{h(Y_T^{t,y,c}, C_T^{t,y,c})}{B_T^{t,y,c}} + \int_t^T g(u, Y_u^{t,y,c}, C_u^{t,y,c}) \frac{du}{B_u^{t,y,c}} \right]$$

$$+ \sum_{i,j \neq i} \int_t^T \frac{\delta^{i,j}(u, Y_u^{t,y,c})}{B_u^{t,y,c}} dH_u^{i,j}.$$ 

**Proof.** Let $v$ be a function satisfying PIDE (5.1) with the boundary condition (5.2). For $v \in C^{1,2} \cap \mathcal{I}$ using the Itô lemma (in a similar way as in [20 Thm. 2.4]), we have

$$v(T, Y_T^{t,y,c}, C_T^{t,y,c}) = v(t, y, c) + \int_t^T \frac{1}{B_u^{t,y,c}} (\partial_t + A_u - \mathbf{r}) v(u, Y_u^{t,y,c}, C_u^{t,y,c}) du + \int_t^T \frac{1}{B_u^{t,y,c}} dM_u,$$
where $M^u$ is given, for $w \geq t$, by

\[
M^u_w := \int_t^w \sum_i H^i_u \nabla v(u, Y^{t,y,c}_u, i) \sigma_y(u, Y^{t,y,c}_u, i) dW^{t,y,c}_u \\
+ \int_t^w \sum_i H^i_u \int_{\mathbb{R}^n} \left( v(u, Y^{t,y,c}_u + F_Y(u, Y^{t,y,c}_u, i, x), i) - v(u, Y^{t,y,c}_u, i) \right) \Pi^{t,y,c}(du, dx) \\
+ \int_t^w \sum_{i,j \neq i} (v(u, Y^{t,y,c}_u + \rho^{i,j}_Y(u, Y^{t,y,c}_u), j) - v(u, Y^{t,y,c}_u, i)) d(M^{i,j})^{t,y,c}_u.
\]

We have $v(T, Y^{t,y,c}_T, C^{t,y,c}_T) = v(T, Y^{t,y,c}_u, C^{t,y,c}_u)$, since $\Delta N^{t,j}_u = 0$ for every $i, j \in \mathcal{K}$, $i \neq j$ and $\Pi(A \times \{T\}) = 0$ for every $A \in \mathcal{B} \left( \mathbb{R}^n \right)$. Hence, by (4.9) and assumption that $v$ solves PIDE (5.1) with boundary condition (5.2), we get

\[
h(Y^{t,y,c}_T, C^{t,y,c}_T) + \int_t^T \frac{1}{B^{t,y,c}_u} \left( g(u, Y^{t,y,c}_u, C^{t,y,c}_u) + \sum_{i,j \in \mathcal{K}} H^i_u \delta^{i,j}(u, Y^{t,y,c}_u) \lambda^{i,j}(u, Y^{t,y,c}_u) \right) du \\
= v(t, y, c) + \int_t^T \frac{1}{B^{t,y,c}_u} dM^u_u,
\]

Thus, using (4.9), after rearranging, we obtain that this equality takes the form

\[
h(Y^{t,y,c}_T, C^{t,y,c}_T) + \int_t^T \frac{1}{B^{t,y,c}_u} \left( g(u, Y^{t,y,c}_u, C^{t,y,c}_u) + \sum_{i,j \in \mathcal{K}} H^i_u \delta^{i,j}(u, Y^{t,y,c}_u) \lambda^{i,j}(u, Y^{t,y,c}_u) \right) du \\
= v(t, y, c) + \int_t^T \frac{1}{B^{t,y,c}_u} dM^u_u + \sum_{i,j \neq i} \delta^{i,j}(u, Y^{t,y,c}_u) d(M^{i,j})^{t,y,c}_u.
\]

By assumption,

\[
\tilde{M}_u = M^u_u + \sum_{i,j \neq i} \int_t^w \delta^{i,j}(u, Y^{t,y,c}_u) d(M^{i,j})^{t,y,c}_u
\]

is a martingale, so taking the expectation in (5.6) we obtain

\[
\mathbb{E}^{t,y,c} \left( \frac{h(Y^{t,y,c}_T, C^{t,y,c}_T)}{B^{t,y,c}_T} + \int_t^T \frac{1}{B^{t,y,c}_u} g(u, Y^{t,y,c}_u, C^{t,y,c}_u) du + \sum_{i,j \neq i} \int_t^T \frac{1}{B^{t,y,c}_u} \delta^{i,j}(u, Y^{t,y,c}_u) d(M^{i,j})^{t,y,c}_u \right) = v(t, y, c).
\]

This finishes the proof.

In Theorem 5.2 above we assume that corresponding PIDE has classical solutions. In a very special case of our setting we can find sufficient conditions for this assumption to hold, e.g. see Gichman Skorochod [13] Theorem II.2.6 and Corollary II.2.2] (see also our example 6.2.2). The question about the form of sufficient conditions for existence of classical solutions to our PIDE is very interesting but it requires deeper study and therefore it is postponed to the following paper.

Assumption that the local martingale is a martingale is the weakest that one can formulate to obtain stochastic representation of classical solution of PIDE. In the theorem below we formulate some sufficient conditions for the corresponding martingale property. Let $C_m$ be a class of functions having polynomial growth of order $m$ defined by

\[
C_m = \{ v : [0, T] \times \mathbb{R}^{d+p} \times \mathcal{K} \to \mathbb{R} : v(\cdot, i) \text{ is continuous and} \quad |v(u, z, i)| \leq K(1 + |z|^m) \quad \forall i \in \mathcal{K} \},
\]

and $C^{1,2,m}$ be the space of all measurable functions $v : [0, T] \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}$ such that $v(\cdot, i) \in C^{1,2}([0, T] \times \mathbb{R}^d)$ for every $i \in \mathcal{K}$ and with the derivative with respect to $z$ having a polynomial growth of degree $m$, i.e., $|\nabla v(u, z, i)| \leq K(1 + |z|^m)$. 

Theorem 5.3. Let \( (Y_{t_u}^{t,v,c}, C_{t_u}^{t,v,c})_{u \in [t,T]} \) be the components of a solution to SDE (5.8) started at \( t \) from \( (y,c) \in \mathbb{R}^{d+p} \times \mathcal{K} \). Assume (LG), (LGm) for some natural number \( m \geq 1 \), and

\[
|\delta_{i,j}(u,z)|^2 \leq K(1 + |z|^{2m}), \quad \forall i,j \in \mathcal{K}, i \neq j.
\]

If \( v \in C^{1,2,m-1} \cap \mathcal{C}_m \) solves PIDE (5.1) with boundary condition (5.2), then (5.4) holds.

Proof. Fix \( t, y, c \). From Theorem 5.2 follows that it is sufficient to show that each stochastic integral in formula (5.3), defining the local martingale \( \hat{M} \), belongs to \( H^2 \) - the space of square integrable martingales. We start from the integral with respect to a Brownian motion, i.e.,

\[
\hat{M}_w := \int_t^w \sum_i H_{u-}^i \nabla v(u, Y_{u-}^{t,y,c}, i) \sigma_Y(u, Y_{u-}^{t,y,c}, i) dW_u.
\]

Since \( v \in C^{1,2,m-1} \) we infer, using (LG), that

\[
|\nabla v(u, z, i) \sigma_Y(u, z, i) \sigma_Y(u, z, i) \nabla v(u, z, i)^\top|^2 \leq |\nabla v(u, z, i)^2| \| \sigma_Y(u, z, i) \sigma_Y(u, z, i)^\top \|^2 \\
\leq L_1(1 + |z|^{2m-2})(1 + |z|^2) \\
\leq L_2(1 + |z|^{2m}).
\]

This and (5.3) imply, by Remark 4.3, that

\[
E^{t,y,c} \int_t^T \left| \nabla v(u, Y_{u-}^{t,y,c}, i) \sigma_Y(u, Y_{u-}^{t,y,c}, i) \nabla v(u, Y_{u-}^{t,y,c}, i)^\top \right|^2 du \\
\leq L_2 T E^{t,y,c} \left( 1 + \sup_{u \in [t,T]} |Y_{u-}^{t,y,c}|^{2m} \right) < \infty.
\]

Hence, by the Doob \( L^2 \)-inequality for local martingales applied to \( M^c \) we obtain that \( M^c \in H^2 \).

Now, we consider the second integral in (5.3), i.e.,

\[
\hat{M}^{d1}_w := \int_t^w \sum_i H_{u-}^i \int_{\mathbb{R}^n} (v(u, Y_{u-}^{t,y,c} + F_Y(u, Y_{u-}^{t,y,c}, i, x), i) - v(u, Y_{u-}^{t,y,c}, i)) \Pi(du, dx).
\]

Obviously \( \hat{M}^{d1} \in M_{loc} \). Let \( \hat{M}^{d1} \) be a process of quadratic variation of \( \hat{M}^{d1} \). We have

\[
\left[ \hat{M}^{d1} \right]_T = \int_t^T \sum_i H_{u-}^i \int_{\mathbb{R}^n} |v(u, Y_{u-}^{t,y,c} + F_Y(u, Y_{u-}^{t,y,c}, i, x), i) - v(u, Y_{u-}^{t,y,c}, i)|^2 \Pi(du, dx)
\]

and, as before, we use the Doob inequality to show that \( \hat{M}^{d1} \in H^2 \). So, it is suffices to show that \( E^{t,y,c}[\hat{M}^{d1}]_T < \infty \). Since \( v \in C^{1,2,m-1} \), we have

\[
|v(u, z + F_Y(u, z, i, x), i) - v(u, z, i)| \leq |\nabla v(u, z + \theta F_Y(u, z, i, x), i)| |F_Y(u, z, i, x)| \\
\leq L_3(1 + |z + \theta F_Y(u, z, i, x)|)^{m-1} |F_Y(u, z, i, x)| \\
\leq L_3(1 + |z|^{m-1} |F_Y(u, z, i, x)| + L_4 |F_Y(u, z, i, x)|^m).
\]

Hence, by (LG) and (LGm)

\[
\int_{\mathbb{R}^n} |v(u, z + F_Y(u, z, i, x), i) - v(u, z, i)|^2 \nu(dx) \\
\leq L_4(1 + |z|^{2m-2}) \int_{\mathbb{R}^n} |F_Y(u, z, i, x)|^2 \nu(dx) + L_4 \int_{\mathbb{R}^n} |F_Y(u, z, i, x)|^2 m \nu(dx) \\
\leq L_5 K(1 + |z|^{2m}).
\]
Using this estimation we obtain
\[
\mathbb{E}^{t,y,c} \int_t^T \int_{\mathbb{R}^n} |v(u, Y^{t,y,c}_{u^-} + F(u, Y^{t,y,c}_{u^-}, i, x), i) - v(u, Y^{t,y,c}_{u^-}, i)|^2 \Pi(du, dx)
\]
\[
= \mathbb{E}^{t,y,c} \int_t^T \int_{\mathbb{R}^n} |v(u, Y^{t,y,c}_{u^-} + F(u, Y^{t,y,c}_{u^-}, i, x), i) - v(u, Y^{t,y,c}_{u^-}, i)|^2 \nu(dx) du
\]
\[
\leq L_\nu \mathbb{E}^{t,y,c} \left( 1 + \sup_{u \in [t,T]} |Y^{t,y,c}_{u^-}|^2 \right) < \infty.
\]
Therefore \( \bar{M}^{d1} \in \mathcal{H}^2 \). Now we consider the third part, i.e.,
\[
\bar{M}^{d2}_u := \int_t^u \sum_{i,j \neq i} (v(u, Y^{t,y,c}_{u^-} + \rho_{i,j} Y^{i,j}(u, Y^{t,y,c}_{u^-}), j) - v(u, Y^{t,y,c}_{u^-}, i) + \delta^{i,j}(u, Y^{t,y,c}_{u^-})) dM^{i,j}_u.
\]
We obviously have
\[
\left| v(u, z + \rho_Y^{i,j}(u, z), j) - v(u, z, i) + \delta^{i,j}(u, z) \right|^2 
\]
\[
\leq L_7 \left( \left| v(u, z + \rho_Y^{i,j}(u, z), j) - v(u, z, j) \right|^2 + \left| v(u, z, j) - v(u, z, i) \right|^2 + \left| \delta^{i,j}(u, z) \right|^2 \right).
\]
The similar arguments as above, assumption (LG) and \( v \in C^{1,2,m-1} \) yield
\[
\left| v(u, z + \rho_Y^{i,j}(u, z), j) - v(u, z, j) \right|^2 \leq L_8 (1 + |z|^{2m}).
\]
This, \((5.7)\) and the fact that \( v \in C^{1,2,m-1} \cap C_m \), give
\[
\left| v(u, z + \rho_Y^{i,j}(u, z), j) - v(u, z, i) + \delta^{i,j}(u, z) \right|^2 \leq L_9 (1 + |z|^{2m}).
\]
Using boundedness of \( \lambda^{i,j} \) and repeating arguments we get
\[
\mathbb{E}^{t,y,c} \int_t^T \left| v(u, Y^{t,y,c}_{u^-} + \rho_Y^{i,j}(u, Y^{t,y,c}_{u^-}), j) - v(u, Y^{t,y,c}_{u^-}, i) + \delta^{i,j}(u, Y^{t,y,c}_{u^-}) \right|^2 du < \infty,
\]
so \( \bar{M}^{d2} \) is in \( \mathcal{H}^2 \), see \cite{17} Thm. I.4.40. This finishes the proof. 

\[ \square \]

6. Further generalizations and examples

6.1. Further generalizations. Results of the previous sections can be extended in several directions. One possibility is to assume that \( \bar{\Pi}(dx, du) \) is a compensated integer-valued random measure on \( \mathbb{R}^n \times [0,T] \) with the intensity measure given by
\[
Q(dx, du) = K(u, Y_{u^-}, C_{u^-}, x) \bar{\nu}(dx) du
\]
where \( \nu \) is a Lévy measure, and \( K \) is nonnegative real-valued measurable function. For such compensator we impose the following growth type conditions:

**Assumption GC-K.** Functions \( \sigma, F, \rho^{i,j} \) satisfy the linear growth condition
\[
(LG-K) \quad |\sigma(t, y, c)|^2 + \int_{\mathbb{R}^n} |F_Y(u, z, i, x)|^2 K(u, z, i, x) \bar{\nu}(dx) + \sum_{j:j \neq i} |\rho_Y^{i,j}(u, z)|^2 \leq L (1 + |z|^2),
\]
and the polynomial growth condition
\[
(LGm-K) \quad \int_{\mathbb{R}^n} |F_Y(u, z, i, x)|^{2m} K(u, z, i, x) \bar{\nu}(dx) \leq L (1 + |z|^{2m}),
\]
for some constant \( L > 0 \) and \( m \geq 1 \).

Under these conditions we have moment estimate \((4.17)\) which can be proved analogously as in \cite{20} (c.f., Remark \( \text{(13)} \)). Thus assuming EUWS we can easily generalize results of previous section. For example, family of generators \( A_t \) should be replaced by \( A^K_t \) defined by \((4.10)\) with
are interested in models with nonnegative price processes. In majority of financial applications this
first
dir
simplicity , that the coefficient of diffusion of $W \in \mathbb{R}$. This is provided if we assume that
$n$
where $\Sigma$ is a positive definite matrix.

We assume EWUS. This is provided if we assume that $\Sigma \in \mathbb{R}$.

$6.2$. Examples. The setup considered in this paper is very general, so it contains many models well
known in finance such as local volatility models and regime switching Lévy models. In this subsection
we present examples which demonstrate how our theory generalize known results in existing models.

$6.2.1$. General exponential Lévy model with stochastic volatility. In many practical applications we
are interested in models with nonnegative price processes. In majority of financial applications this
requirement is achieved by imposing a linear structure on the coefficients in dynamic of $S$ (i.e. the
first $d$ components of $Y$ in our model), by taking

$$dS^k_t = S^k_t \left( r(t, R_{t-}, C_{t-}) dt + \Sigma_k(t, R_{t-}, C_{t-})^\top dW_t + \int_{\mathbb{R}^d} \left( e^{\Sigma_k(t, R_{t-}, C_{t-})^\top x} - 1 \right) \Pi(dx, dt) \right) + \sum_{i,j \in \mathcal{K}; j \neq i} \left( e^{P^{i,j}(t, R_{t-})} - 1 \right) \lambda^{i,j}_t,$$

$$dR_t = \mu_R(t, R_{t-}, C_{t-}) dt + \sigma_R(t, R_{t-}, C_{t-})^\top dW_t,$$

where $W, \Pi, \lambda^{i,j}$ are as before with $\lambda^{i,j}$ being deterministic function. In this model we assume, for
simplicity, that the coefficient of diffusion of $R$ is modulated by $C$ - the process given in advance.

We assume EWUS. This is provided if we assume that $\Sigma_k : [0, T] \times \mathbb{R}^p \times \mathcal{K} \rightarrow \mathbb{R}^n$ and $P^{i,j}_k : [0, T] \times \mathbb{R}^p \rightarrow \mathbb{R}$ are continuous and bounded functions for all $k \in \{1, \ldots, d\}, \mu_R, \sigma_R$ satisfy linear
growth and local Lipschitz conditions and

$$\Sigma_k(t, r, c) + \Sigma_l(t, r, c) \in B \quad \forall k, l \in \{1, \ldots, n\},$$

for all

$$B = \left\{ v \in \mathbb{R}^n : \int_{|x| > 1} e^{v^\top x} \nu(dx) < \infty \right\}.$$

Let us denote by $\text{Exp}(a)$ a component-wise exponential function of vector $a$, by $1_d$ a $d$-dimensional
column vector of $1$. Assume that the dividend process $D$ is given by $1_d$ and the value function is
appropriately smooth. Using Theorem 4.1 we obtain that the component $\phi$ of a risk-minimizing
hedging strategy has, on the set $\{C_{t-} = i\}$, the form

$$\phi_t = \text{diag}(S_{t-})^{-1}(\tilde{G}_t(R_{t-}, i))^{-1}$$

$$\left( \Sigma^\top \Sigma(t, R_{t-}, i) \text{diag}(S_{t-}) \nabla_S v(t, Y_{t-}, i) + \Sigma^\top \sigma_R(t, Y_{t-}, i) \nabla_R v(t, Y_{t-}, i) \right)$$

$$+ \int_{\mathbb{R}^n} \left( \exp(\Sigma(t, R_{t-}, i)^\top x) - 1 \right) \left( v(t, S_{t-} \circ \exp(\Sigma(t, R_{t-}, i)^\top x), R_{t-}, i) - v(t, S_{t-}, R_{t-}, i) \right) \nu(dx)$$

$$+ \sum_{j \in \mathcal{K}; j \neq i} \left( \exp(P^{i,j}(t, R_{t-})) - 1 \right) \left( v(t, S_{t-} \circ \exp(P^{i,j}(t, R_{t-}, j), R_{t-}, j) - v(t, S_{t-}, R_{t-}, i) \right)$$

$$+ \delta^{i,j}(t, Y_{t-}) \lambda^{i,j}(t),$$

where

$$\tilde{G}_t(r, i) = \left( \Sigma^\top \Sigma(t, r, i) + \int_{\mathbb{R}^n} \left( \exp(\Sigma(t, r, i)^\top x) - 1_d \right) \left( \exp(\Sigma(t, r, i)^\top x) - 1_d \right)^\top \nu(dx) \right)$$

$$+ \sum_{j \in \mathcal{K}; j \neq i} \left( \exp(P^{i,j}(t, r)) - 1_d \right) \left( \exp(P^{i,j}(t, r)) - 1_d \right)^\top \lambda^{i,j}(t),$$
and \((\hat{G}_i(R_{t-}, i))^{-1}\) is the Moore-Penrose pseudo-inverse, and \(\circ\) denotes the Hadamard product (i.e. the componentwise product of matrices).

6.2.2. Generalization of exponential \(\text{L}\text{évy} \) model with regime-switching. Now we consider particular model of asset prices and present how our theory works. We consider market with money account and one risky asset. The dynamic of asset price process is given by

\[
\begin{align*}
& (6.1) \quad dS_t = S_{t-} \left( r(C_{t-}) dt + \sigma(C_{t-}) dW_t + \int_{\mathbb{R}^n} (\sigma^\top(C_{t-}) x - 1) \tilde{\Pi}(dx, dt) + \sum_{i,j \in \mathcal{K}, j \neq i} (e^{\rho^{i,j}} - 1)dM_{t}^{i,j} \right), \\
& dC_t = \sum_{i,j \in \mathcal{K}, j \neq i} (j - i) \mathbb{I}_{\{i\}}(C_{t-})dN_{t}^{i,j},
\end{align*}
\]

where \(\sigma: \mathcal{K} \to \mathbb{R}^n\), \(\rho^{i,j} \in \mathbb{R}\), \(N_{t}^{i,j}\) are independent Poisson processes with constant intensities \(\lambda^{i,j} > 0\), and \(\tilde{\Pi}(dx, dt)\) is a Poisson random measure with intensity measure \(\nu(dx)dt\) satisfying, for some \(m \geq 1\),

\[
(6.2) \quad \int_{|x|>1} e^{2m\sigma(i)\top x} \nu(dx) < \infty \quad \forall i \in \mathcal{K}.
\]

Our model generalize a regime switching model with jumps, extensively studied among others by Chourdakis \[5\], Mijatovic and Pistorius \[23\] or Kim et.al. \[21\] for which \(\rho^{i,j} = 0\) in \((6.1)\). Note that the coefficients of SDE \((6.1)\) satisfy standard assumptions for existence of a unique strong solution, and the coordinate \(C\) of solution \((S, C)\) is a Markov chain with the state space \(\mathcal{K}\). Theorem \[3.1\] yields that the component \(\phi\) of the risk-minimizing strategy for a dividend process \(D\) with representation \[(3.6)\] is given, on the set \(\{C_{t-} = i\}\), by

\[
\phi_t = \frac{(\sigma(i))\delta_t + \int_{\mathbb{R}^n} (e^{\sigma(i)\top x} - 1) J_t(x) \nu(dx) + \sum_{j \in \mathcal{K}, j \neq i} (e^{\rho^{i,j}} - 1) \gamma^{i,j}}{S_{t-} \hat{G}_t^{i}},
\]

where

\[
\hat{G}_t^{i} := \left( |(\sigma(i))|^2 + \int_{\mathbb{R}^n} (e^{\sigma(i)\top x} - 1)^2 \nu(dx) + \left( \sum_{j \in \mathcal{K}, j \neq i} (e^{\rho^{i,j}} - 1)^2 \lambda^{i,j} \right) \right).
\]

If \(D\) is given by \[(1.12)\], the growth condition \[(4.18)\] holds, and the value function \(v\) is sufficiently smooth, then by Theorem \[4.1\] on the set \(\{C_{t-} = i\}\), the component \(\phi\) has the representation

\[
\phi_t = (\hat{G}_t)\quad \left( |\sigma(i)|^2 \nabla v(t, S_{t-}, i) + \int_{\mathbb{R}^n} (e^{\sigma(i)\top x} - 1) v(t, S_{t-} e^{\sigma(i)\top x}, i) - v(t, S_{t-}, i) \right) \frac{1}{S_{t-}} \nu(dx)
\]

\[
+ \sum_{j \in \mathcal{K}, j \neq i} (e^{\rho^{i,j}} - 1) \left( v(t, S_{t-} e^{\rho^{i,j}}, j) - v(t, S_{t-}, i) + \delta^{i,j}(t, S_{t-}) \lambda^{i,j} \right),
\]

We can use Theorem \[4.1\] since \[(6.2)\] implies \[(4.17)\]. In the case of generalized exponential \(\text{L}\text{évy} \) model described above, to find the value function \(v\) we may try to solve the system of PIDE with the corresponding terminal conditions (see Theorem \[5.3\]), which in this model takes the following form:

\[
(6.3) \quad \partial_t v(t, s, c) + \nabla v(t, s, c) r(c) s + \frac{1}{2} s |\sigma(c)|^2 \nabla^2 v(t, s, c)
+ \int_{\mathbb{R}^n} \left( v(t, se^{\sigma(c)\top x}, c) - v(t, s, c) - \nabla v(t, s, c)(e^{\sigma(c)\top x} - 1) \right) \nu(dx)
+ \sum_{c' \in \mathcal{K} \setminus c} \left( v(t, se^{\rho^{c,c'}}, c') - v(t, s, c) - \nabla v(t, s, c)(e^{\rho^{c,c'}} - 1) + \delta^{c,c'}(t, s) \right) \lambda^{c,c'}
+ g(t, s, c) = 0,
\]

\(v(T, s, c) = h(s, c)\).
In the case of $\rho^{c,e} \equiv 0$ and $\nu \equiv 0$, the above system of PIDE reduces to that considered by Becherer and Schweizer \[4\], who gave sufficient conditions for existence of bounded classical solution to this system of PIDE. Generally, the problem of existence of classical solutions to this PIDE is complex and it is well known that in some cases classical solutions to this kind of PDE’s may not exist (see Norberg \[23\]). However, under some strong assumptions on functions $h, g, \delta^{i,j}, \lambda^{i,j}$, we can find a classical solution to PIDE system \[6.3\]. To see this we assume, for simplicity, that $\tau \equiv 0$. Let us consider the function

\[(6.4)\]

\[v(t, s, c) := \mathbb{E} \left( h(S_{T-t}^{0,s,c}, C_{T-t}^{0,s,c}) + \int_0^{T-t} n(S_{u-}^{0,s,c}, C_{u-}^{0,s,c}) \, du \right),\]

where $n$ is defined by

\[n(s, c) := g(s, c) + \sum_{c' \neq c} \delta^{c,c'}(s) \lambda^{c,c'},\]

and $(S_t^{0,s,c}, C_t^{0,s,c})$ is the solution to SDE \[6.1\] starting from $(s, c)$ at time $0$. We claim that if $h$ and $n$ are $C^2$-function in $s$ with bounded first and second derivatives, then $v$ given by \[6.4\] is a classical solution to PIDE system \[6.3\]. First let us note using Markov property of solution to our SDE that

\[v(t, S_t^{0,s,c}, C_t^{0,s,c}) = \mathbb{E} \left( h(S_T^{0,s,c}, C_T^{0,s,c}) + \int_t^T g(S_u^{0,s,c}, C_u^{0,s,c}) \, du + \sum_{i,j \neq i} \int_t^T \delta^{i,j}(S_u^{0,s,c}) dH_u^{i,j} \right| F_t \).

By Theorem \[5.1\] we know that $v$ solves PIDE \[6.3\] provided that the function $v$ is in $C^{1,2} \cap \mathcal{I}$, \[4.17\] and \[4.18\] hold. Assumption \[6.2\] implies, by Remark \[4.3a\], that \[4.17\] holds for $m = 1$. Moreover our assumptions on functions $h$ and $n$ implies \[4.18\]. So it is enough to show required smoothness of $v$. Let us notice that $S_t^{0,s,c}$ is continuously differentiable with respect to $s$, and the first two derivatives are given by

\[D_s^e := \partial_s S_t^{0,s,c} = \frac{S_t^{0,s,c}}{s},\]

\[= \exp \left( \int_0^t J(\sigma(C_u^{0,s,c})) \, du + \int_0^t \sigma(C_u^{0,s,c}) \, dZ_u + \sum_{i,j \in K : j \neq i} \int_0^t \rho^{i,j} dH_u^{i,j} \right),\]

\[\partial_s^2 S_t^{0,s,c} = 0.
\]

Therefore, $D_s^e$ is square integrable for every $u$. By assumption, $h$ and $n$ are function of $C^2$ class in $s$ with the bounded first and second derivatives, so by the dominated convergence theorem we have

\[\partial_s v(t, s, c) = \mathbb{E} \left( \partial_s h(S_{T-t}^{0,s,c}, C_{T-t}^{0,s,c}) D_{T-t}^e + \int_0^{T-t} \partial_s n(S_{u-}^{0,s,c}, C_{u-}^{0,s,c}) D_u^e \, du \right),\]

\[\partial_s^2 v(t, s, c) = \mathbb{E} \left( \partial_s^2 h(S_{T-t}^{0,s,c}, C_{T-t}^{0,s,c}) (D_{T-t}^e)^2 + \int_0^{T-t} \partial_s^2 n(S_{u-}^{0,s,c}, C_{u-}^{0,s,c}) (D_u^e)^2 \, du \right).
\]

Boundedness of derivatives of $h$ and $n$, together with independence of $D_u^e$ of $s$, yield that these derivatives are bounded functions in $s$. This implies, by Remark \[6.3\], that $v \in \mathcal{I}$. To get that $v$ is $C^1$ class in $t$ we first apply the Itô lemma to function $h$ and obtain

\[v(t, s, c) = h(s, c) + \mathbb{E} \left( \int_0^{T-t} (A_c h + n(S_u^{0,s,c}, C_u^{0,s,c}) \, du \right),\]

where $A_c$ denotes the operator

\[A_c h(s, c) := \frac{1}{2} s^2 |\sigma(c)|^2 \nabla^2 h(s, c) + \sum_{c' \in K \setminus c} \left( h(se^{\sigma^{c,c'}}, c') - h(s, c) \right) \lambda^{c,c'} + \int_{\mathbb{R}^n} \left( h(se^{\sigma(c)x}, c) - h(s, c) - \nabla h(s, c)(e^{\sigma(c)x} - 1) \right) \nu(dx).
\]
Since the process
\[ f(u) = (A, h + n)(S^0_{u}, C^0_{u}, C^d_{u}) \]
is continuous in probability, using the Pratt theorem we see that
\[ v(t, s, c) = h(s, c) + \int_t^{T-t} \mathbb{E} \left( (A, h + n)(S^0_{u}, C^0_{u}, C^d_{u}) \right) du, \]
so \( v \in C^{1,2} \cap \mathcal{I} \), which implies our claim.

6.2.3. Semi-Markovian regime switching models. Now we present how in our framework a feedback mechanism in jumps of \( Y \) and intensity of jumps of \( C \) gives extra flexibility in modeling. As an example we present how semi-Markov switching processes can be embedded in our framework. Let us recall that a semi-Markov nature of \( C \) is reflected in the fact that the compensator \( \lambda_{i,j} \) of jumps from \( i \) to \( j \) depends on time that process \( C \) spends in a current state after the last jump. A semi-Markov regime switching process can be embedded in our framework by considering the following SDE
\[
\begin{align*}
    dS_t &= S_{t-} \left( rd\tau + \sigma(C_{t-})dW_t + \int_{\mathbb{R}} (e^{\sigma(C_{t-})x} - 1) \tilde{\Pi}(dx, dt) \right) \\
    dR_t &= dt - \sum_{i,j \in \mathcal{K}, j \neq i} R_{t-} \mathbb{I}(i)(C_{t-})dN_{t}^{i,j} \\
    dC_t &= \sum_{i,j \in \mathcal{K}, j \neq i} (j - i) \mathbb{I}(i)(C_{t-})dN_{t}^{i,j},
\end{align*}
\]
where \( W \) is a Wiener process, \( \sigma(i) \geq 0 \), \( N^{i,j} \) are the point processes with intensities \( \lambda^{i,j} \) being functions of \( R_{t-} \), \( \Pi(dx, dt) \) is a Poisson random measure with intensity measure \( \nu(dx)dt \) satisfying
\[
\int_{|x| > 1} e^{2\sigma(i)x} \nu(dx) < \infty \quad \forall i \in \mathcal{K}.
\]

Note that the coefficients of this SDE satisfy assumptions of Theorem 5.3 [20], and if functions \( \lambda^{i,j} \) satisfy assumptions of this theorem then there exists a solution unique in law. Moreover, the coordinate \( C \) of solution \( (S, R, C) \), is a semi-Markov chain with the state space \( \mathcal{K} \). Theorem 3.1 yields that the component \( \phi \) of the risk-minimizing strategy for a dividend process \( D \) with representation (3.6) is given, on the set \( \{ C_{t-} = i \} \), by
\[
\phi_t = \left( \sigma(i)^\top \delta_t + \int_{\mathbb{R}^n} (e^{\sigma(i)^\top x} - 1) J_t(x) \nu(dx) \right) S_{t-} \tilde{G}_t^i,
\]
where
\[
\tilde{G}_t^i := \left( |\sigma(i)|^2 + \int_{\mathbb{R}^n} (e^{\sigma(i)^\top x} - 1)^2 \nu(dx) \right).
\]
If \( D \) is given by (4.12), the growth condition (4.18) holds, and the value function \( v \) is sufficiently smooth, then we can use Theorem 4.11 since (4.9) implies (4.17). So, we have, on the set \( \{ C_{t-} = i \} \),
\[
\phi_t = (\tilde{G}_t^i)^{-1} \left( |\sigma(i)|^2 \nabla v(t, S_{t-}, R_{t-}, i) \right)
+ \int_{\mathbb{R}^n} (e^{\sigma(i)^\top x} - 1) \frac{v(t, S_{t-}e^{\sigma(i)^\top x}, R_{t-}, i) - v(t, S_{t-}, R_{t-}, i)}{S_{t-}} \nu(dx).
\]

In the case of semi-Markovian regime switching exponential Lévy model described above, one of the way of finding the value function \( v \) is to solve the corresponding system of PIDE with the terminal conditions (see Theorem 5.3).
7. Appendix

In this appendix we prove the results which is outside the main scope of our paper, but we need it in the proof of Theorem 3.4. This result connects the form of GKW decomposition of \( \int_0^T \frac{1}{B_u} dD_u \) with the existence of replication strategy for a payment stream \( D \).

Lemma 7.1. The following conditions are equivalent:

1. There exist a \( \mathbb{P} \)-admissible strategy \( \varphi \) which replicates a payment stream \( D \).

2. In the GKW decomposition (2.6) of \( V^* \) we have \( L^X = 0 \).

Proof. \( \Leftarrow \Rightarrow \) Follows immediately from [19, Lem. 16.17].

\( \Rightarrow \) Assume that \( \varphi \) is a \( \mathbb{P} \)-admissible strategy which replicates \( D \). By [19, Prop. 16.14] we have that

\[
V_t(\varphi) = B_t \mathbb{E} \left( \int_t^T \frac{1}{B_u} dD_u | F_t \right).
\]

Hence

\[
V_t(\varphi) - \int_0^t \frac{1}{B_u} dD_u = \mathbb{E} \left( \int_0^T \frac{1}{B_u} dD_u | F_t \right).
\]

Using [19, Lem. 16.7] we obtain

\[
V_0(\varphi) + \int_0^T \phi_u^T dS_u^* = V^*_t.
\]

\[ \Box \]

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