Research Article

Inequalities for Riemann–Liouville Fractional Integrals of Strongly \((s, m)\)-Convex Functions

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The results of this paper provide two Hadamard-type inequalities for strongly \((s, m)\)-convex functions via Riemann–Liouville fractional integrals and error estimations of well-known fractional Hadamard inequalities. Their special cases are given and connected with the results of some published papers.

1. Introduction

The most prominent inequality for convex functions is the well-known Hadamard inequality stated in the following.

**Theorem 1** (see [1]). Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on the interval \( I \), where \( x, y \in I \) with \( x < y \). Then, the following inequality holds:

\[
f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_{x}^{y} f(u) \, du \leq \frac{f(x) + f(y)}{2}.
\]  

(1)

Convex functions are extended, generalized, and refined in different ways to define new types of convex functions. For instance, \(s\)-convex, \(m\)-convex, \((s, m)\)-convex, strongly convex, and strongly \((s, m)\)-convex functions are extensions of convex functions. The aim of this paper is to establish integral inequalities by using the class of strongly \((s, m)\)-convex functions. We give definitions of \((s, m)\)-convex and strongly \((s, m)\)-convex functions as follows.

**Definition 1** (see [2]). A function \( f : [0, \infty) \rightarrow \mathbb{R} \) is called \((s, m)\)-convex in the second sense, if the following inequality holds:

\[
f(tx + m(1-t)y) \leq t^sf(x) + m(1-t)^sf(y),
\]

(2)

for every \( x, y \in [0, \infty) \), \( t \in [0, 1] \) and \( s, m \in (0, 1] \times [0, 1] \).

**Definition 2** (see [3]). A function \( f : [0, \infty) \rightarrow \mathbb{R} \) is called strongly \((s, m)\)-convex in the second sense with modulus \( C > 0 \), if the following inequality holds:

\[
f(tx + m(1-t)y) \leq t^sf(x) + m(1-t)^sf(y) - Cmt(1-t)(y-x)^2,
\]

(3)

for every \( x, y \in [0, \infty) \), \( t \in [0, 1] \) and \( s, m \in (0, 1] \times [0, 1] \).

By setting \((s, m) = (s, 1)\), \((s, m) = (1, m)\), and \((s, m) = (1, 1)\) in (2), we get \(s\)-convex [4], \(m\)-convex [5], and convex functions, respectively, while by setting \((s, m) = (s, 1)\), \((s, m) = (1, m)\), and \((s, m) = (1, 1)\) in (3), we get strongly \(s\)-convex [6], strongly \(m\)-convex [7], and strongly convex [6] functions, respectively.

Next we give definition of Riemann–Liouville fractional integrals \( I^\alpha_s f \) and \( I^\alpha_m f \) which are utilized to get the desired results of this paper.

**Definition 3** (see [8]). Let \( f \in L_1[x, y] \). Then, Riemann–Liouville fractional integral operators of order \( a > 0 \) are given by
\[ f^\alpha_x f (u) = \frac{1}{\Gamma (\alpha)} \int_x^u (u-t)^{\alpha-1} f (t) \, dt, \quad u > x, \]
\[ f^\alpha_y f (u) = \frac{1}{\Gamma (\alpha)} \int_u^y (t-u)^{\alpha-1} f (t) \, dt, \quad u < y, \]

where \( \Gamma (\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt \) is the gamma function and \( f^\alpha_y f (u) = f^\alpha_y f (u) = f (u) \).

The following special functions are also involved in the findings of this paper.

**Definition 4.** The beta function, also referred to as first type Euler integral, is defined by
\[ \beta (a, s) = \int_0^1 t^{a-1} (1-t)^{s-1} \, dt, \]
where \( \Re (a), \Re (s) > 0 \).

Close association of the beta function to the gamma function is an important factor of the beta function
\[ \beta (a, s) = \frac{\Gamma (a) \Gamma (s)}{\Gamma (a+s)} \]

The beta function is symmetric, i.e., \( \beta (a, s) = \beta (s, a) \). A generalization of the beta function, called the incomplete beta function, is defined by
\[ \beta (b; a, s) = \int_0^b t^{a-1} (1-t)^{s-1} \, dt, \]
where \( \Re (a), \Re (s) > 0 \) with \( 0 < b < 1 \). The incomplete beta function \( \beta (b; a, s) \) weakens to the ordinary \( \beta (a, s) \) (beta function) by setting \( b = 1 \).

In [8], the Hadamard inequality is studied for Riemann–Liouville fractional integrals which is stated in the following theorem.

**Theorem 2.** Let \( f : [x, y] \to \mathbb{R} \) be a positive function with \( 0 \leq x < y \) and \( f \in L_1 [x, y] \). If \( f \) is convex function on \([x, y] \), then the following inequality for fractional integrals holds:
\[ f \left( \frac{x+y}{2} \right) \leq \frac{f (x) + f (y)}{2} \]
with \( a > 0 \).

The inequality stated in the aforementioned theorem motivates the researchers to work in this direction by establishing other kinds of inequalities for Riemann–Liouville fractional integrals. In the past decade, several classical inequalities have been extended via different kinds of fractional integral operators. The Hadamard inequality is one of the most studied inequalities for fractional integral operators. For some recent work, we refer the readers to [3, 8–17].

This paper is organized as follows. In Section 2, two versions of the Hadamard inequality for strongly \((s,m)\)-convex functions via Riemann–Liouville fractional integrals are given. Their connection with the well-known results is established in the form of corollaries and remarks. In Section 3, the error estimations of Hadamard inequalities for Riemann–Liouville fractional integrals are obtained by using differentiable strongly \((s,m)\)-convex functions.

## 2. Main Results

**Theorem 3.** Let \( f \in L_1 [x, y] \) be a positive function with \( 0 \leq x < y \). If \( f \) is strongly \((s,m)\)-convex function on \([x, y]\) with modulus \( C \geq 0 \), \( m \neq 0 \), \( 0 < s \leq 1 \), then the following fractional integral inequality holds:

\[ f \left( \frac{x+y}{2} \right) \leq \frac{1}{2^s} f (xt + m (1-t)y) + \frac{m f (yt + (1-t) x/m)}{2^s} \]

where
\[ C = \frac{Cm (y-x)^2 + m (y-x/m)^2)^2}{2 (a+1) (a+2)} \]

By multiplying inequality (11) with \( t^{s-1} \) on both sides and then integrating over the interval \([0, 1]\), we get

\[ \int_0^1 f \left( \frac{x+y}{2} \right) \, dt \leq \frac{1}{2^s} f (xt + m (1-t)y) + \frac{m f (yt + (1-t) x/m)}{2^s} \]

By setting \( u = xt + m (1-t)y \) and \( v = yt + (1-t) (x/m) \), we have

\[ f \left( \frac{u+v}{2} \right) \leq \frac{f (u) + f (v)}{2} - \frac{Cm (u-v)^2}{4}. \]
\begin{equation}
\frac{1}{2} \int_{0}^{1} f(xt + m(1-t)y)x^{\alpha - 1}dt + \frac{m}{2} \int_{0}^{1} f(yt + (1-t)\frac{x}{m})y^{\alpha - 1}dt - \frac{Cm}{4} \int_{0}^{1} \left| t(x-y) + (1-t)\left( my - \frac{x}{m} \right) \right|^2 t^{\alpha - 1} dt.
\end{equation}

By change of variables, we will get

\begin{equation}
\frac{1}{\alpha} f\left( \frac{x + my}{2} \right) \leq \frac{\Gamma(\alpha)}{2^\alpha (my - x)^\alpha} \left[ \frac{1}{\Gamma(\alpha)} \int_{my}^{x} (my - u)x^{\alpha - 1}f(u)du + \frac{m^{\alpha + 1}}{\Gamma(\alpha)} \right]
\end{equation}

\begin{equation}
\cdot \int_{(x/m)}^{y} \left( v - \frac{x}{m} \right)^{\alpha - 1}f(v)dv - \frac{Cm}{4} \left( \frac{(x-y)^2}{\alpha + 2} + \frac{2(my - (x/m))^2}{\alpha(\alpha + 1)(\alpha + 2)} + \frac{2(x-y)(my - (x/m))}{(\alpha + 1)(\alpha + 2)} \right).
\end{equation}

Further, the above inequality takes the following form:

\begin{equation}
2^{\alpha - 1} f\left( \frac{x + my}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2^\alpha (my - x)^\alpha} \left[ f_x, f(my) + m^{\alpha + 1} f_y, f\left( \frac{x}{m} \right) \right]
\end{equation}

\begin{equation}
- \frac{2^{\alpha - 1} Cm}{4} \left( \frac{(x-y)^2}{\alpha + 2} + \frac{2(my - (x/m))^2}{\alpha(\alpha + 1)(\alpha + 2)} + \frac{2(x-y)(my - (x/m))}{(\alpha + 1)(\alpha + 2)} \right).
\end{equation}

From the definition of strongly \((s,m)-\)convex function with modulus \(C\), for \(t \in [0,1]\), we have the following inequality:

\begin{equation}
f(tx + m(1-t)y) + mf\left( yt + (1-t)\frac{x}{m} \right)
\end{equation}

\begin{equation}
\leq t^s (f(x) + mf(y)) + m(1-t)^s \left( f(y) + mf\left( \frac{x}{m^s} \right) \right)
\end{equation}

\begin{equation}
- Cmt(1-t) \left( (y-x)^2 + m\left( y - \frac{x}{m^s} \right)^2 \right).
\end{equation}

By multiplying inequality (15) with \(t^{\alpha - 1}\) on both sides and then integrating over the interval \([0,1]\), we get

\begin{equation}
\int_{0}^{1} f(tx + m(1-t)y)t^{\alpha - 1}dt + m \int_{0}^{1} f(yt + (1-t)\frac{x}{m})y^{\alpha - 1}dt
\end{equation}

\begin{equation}
\leq (f(x) + mf(y)) \int_{0}^{1} t^{s\alpha - 1}dt + m \left( f(y) + mf\left( \frac{x}{m^s} \right) \right) \int_{0}^{1} t^{\alpha - 1}(1-t)^s dt
\end{equation}

\begin{equation}
- Cm \left( (y-x)^2 + m\left( y - \frac{x}{m^s} \right)^2 \right) \int_{0}^{1} t^\alpha (1-t)dt.
\end{equation}
By change of variables, we will get

\[
\frac{\Gamma(\alpha)}{(my - x)^\beta} \left[ \frac{1}{\Gamma(\alpha)} \int_{my}^{x} (my - u)^{\alpha-1} f(u) \, du + \frac{m^{\alpha+1}}{\Gamma(\alpha)} \int_{(x/m)}^{y} \left( \frac{v - x}{m} \right)^{\alpha-1} f(v) \, dv \right] \\
\leq f(x) + mf(y) + m \left( f(y) + mf\left( \frac{x}{m^2} \right) \right) \beta(s + 1, \alpha) - \frac{C m \left( (y-x)^2 + m(y-(x/m)^2)^2 \right)}{(\alpha + 1)(\alpha + 2)}.
\]

Further, the above inequality takes the following form:

\[
\frac{\Gamma(\alpha+1)}{2(my - x)^\beta} \left[ J_{n,1}^e \left( f(my) + m^\alpha J_{n,1}^e \left( f\left( \frac{x}{m} \right) \right) \right) \right] \\
\leq \alpha f(x) + mf(y) + ma \left( f(y) + mf\left( \frac{x}{m^2} \right) \right) \beta(s + 1, \alpha) \\
\leq \frac{C m \left( (y-x)^2 + m(y-(x/m)^2)^2 \right)}{2(\alpha + 1)(\alpha + 2)}.
\]

From inequalities (14) and (18), one can get inequality (9).

Remark 1

(i) For \( s = 1 \) in (9), we have the result for strongly \( m \)-convex function [18].

(ii) For \( m = 1 \) and \( s = 1 \) in (9), we have the result for strongly convex function.

(iii) For \( m = 1, s = 1, \) and \( C = 0, \) we get \([16, \text{Theorem 2}]\).

(iv) For \( m = 1, s = 1, \alpha = 1, \) and \( C = 0, \) we get the classical Hadamard inequality.

(v) For \( m = 1 \) and \( C = 0, \) we get \([17, \text{Theorem 3}]\).
with \( a > 0 \).

**Proof.** Let \( t \in [0, 1] \). Using strong \((s, m)\)-convexity of function \( f \) for \( u = x(t/2) + m((2 - t)/2)y \) and \( v = ((2 - t)/2)(x/m) + y(t/2) \) in inequality (10), we have

\[
\begin{align*}
\int_0^1 f\left(\frac{x + my}{2}\right) t^{a - 1} dt &\leq \frac{1}{2} \int_0^1 f\left(\frac{x + m(2 - t)}{2} y\right) + m f\left(\frac{(2 - t)}{2} x + m t\right) \frac{1}{2} \int_0^1 f\left(\frac{x - t}{2} + m(2 - t) y\right) + m f\left(\frac{(2 - t)}{2} x + m t\right) \frac{1}{2} \left(\frac{1}{2} (x - y) + \frac{2 - t}{2} (my - \frac{x}{m})\right)^2 dt. \\
&= \frac{1}{2} \int_0^1 f\left(\frac{x + m(2 - t)}{2} y\right) + m f\left(\frac{(2 - t)}{2} x + m t\right) \frac{1}{2} \left(\frac{1}{2} (x - y) + \frac{2 - t}{2} (my - \frac{x}{m})\right)^2 dt.
\end{align*}
\]

By multiplying (22) with \( t^{a-1} \) on both sides and making integration over \([0, 1]\), we get

\[
\frac{2^a}{a} f\left(\frac{x + my}{2}\right) \leq \frac{2^a \Gamma(a)}{(my - x)^a} \left[ \frac{1}{\Gamma(a)} \int_0^1 (my - u)^{a-1} f(u) du + m \Gamma(a) \int_0^1 \left(\frac{v - \frac{x}{m}}{a} \right)^{a-1} f(v) dv \right]
\]

Further, it takes the following form:

\[
\begin{align*}
2^{a-1} f\left(\frac{x + my}{2}\right) &\leq \frac{2^a \Gamma(a + 1)}{(my - x)^a} \left[ f^{(\frac{a}{my + (x/m) + 2})}(ym) + m f^{(\frac{a}{my + (x/m) + 2})}(\frac{x}{m}) \right] \\
- \frac{2^{a-1} C m \Gamma(a + 1)}{4} \left[ \frac{(x - y)^2}{4(a + 2)} + \frac{(my - (x/m))^2(a^2 + 5a + 8)}{4(a + 1)(a + 2)} + \frac{(x - y)(my - (x/m))(a + 3)}{2(a + 1)(a + 2)} \right].
\end{align*}
\]

The first inequality of (21) can be seen in (25). Now we prove the second inequality of (21). Since \( f \) is strongly \((s, m)\)-convex function and \( t \in [0, 1] \), we have the following inequality:
\[ f\left(\frac{x}{2} + m\left(\frac{2-t}{2}\right)y\right) + mf\left(\frac{2-t}{2}\right)\left(\frac{x}{m} + y\frac{t}{2}\right) \leq \left(\frac{t}{2}\right)^* (f(x) + mf(b)) \]
\[ + m\left(\frac{2-t}{2}\right)\left(f(y) + mf\left(\frac{x}{m^2}\right)\right) - \frac{Cm(2-t)}{4} \left[(y-x)^2 + m\left(y - \frac{x}{m^2}\right)^2\right]. \] (26)

By multiplying inequality (26) with \(t^{a-1}\) on both sides and making integration over \([0, 1]\), we get
\[
\int_0^1 f\left(\frac{x}{2} + m\left(\frac{2-t}{2}\right)y\right) t^{a-1} dt + m \int_0^1 f\left(\frac{2-t}{2}\right)\left(\frac{x}{m} + y\frac{t}{2}\right) t^{a-1} dt
\leq \frac{1}{2^a} (f(x) + mf(y)) \int_0^1 t^{a-1} dt + m \int_0^1 \left(f(y) + mf\left(\frac{x}{m^2}\right)\right) t^{a-1} dt
\]
\[ - \frac{Cm}{4} \left[(y-x)^2 + m\left(y - \frac{x}{m^2}\right)^2\right] \int_0^1 t^a (2-t) dt. \] (27)

By using change of variables and computing the last integral, from (27), we get
\[
\frac{2^a \Gamma(a)}{(my - x)^a} \left[\int_0^{(x+my)/2m}(my - u)^{a-1} f(u) du + m^{a+1} \int_0^{((my+x)/2m)} (y - \frac{x}{m})^{a-1} f(v) dv\right]
\leq \frac{f(x) + mf(y)}{2^a (a + s)} + 2^a m \left(f(y) + mf\left(\frac{x}{m^2}\right)\right) \beta\left(\frac{1}{2}, s + 1, a\right)
\[ - \frac{Cm\left(y-x\right)^2 + m\left(y - \left(\frac{x}{m^2}\right)\right)^2}{4(a+1)(a+2)} (a + 3) \] (28)

Further, it takes the following form:
\[
\frac{2^{a-1} \Gamma(a+1)}{(my - x)^a} \left[\int_0^{(x+my)/2m} f(ym) + m^{a+1} \int_0^{((x+ym)/2m)} f\left(\frac{x}{m}\right)\right]
\leq \frac{f(x) + mf(y)}{2^{a+1} (a + s)} + 2^{a-1} ma \left(f(y) + mf\left(\frac{x}{m^2}\right)\right) \beta\left(\frac{1}{2}, s + 1, a\right)
\[ - \frac{Cma\left(y-x\right)^2 + m\left(y - \left(\frac{x}{m^2}\right)\right)^2}{8(a+1)(a+2)} (a + 3) \] (29)
From inequalities (25) and (29), we have inequality (21). □

**Remark 2**

(i) For $s = 1$ in (21), we get the result for strongly $m$-convex function [18].

(ii) For $m = 1, s = 1, C = 0$, we get [[16], Theorem 2]

\[
\frac{f(x + y)}{2} + \frac{C(y - x)^2}{2(\alpha + 1)(\alpha + 2)}
\]

\[
\leq \frac{2^{\alpha - 1}\Gamma(\alpha + 1)}{(y - x)^\alpha} \left[ (f^a_{\langle x+y\rangle/2})'(y) + (f^a_{\langle x+y\rangle/2})'(x) \right]
\]

\[
\leq \frac{f(x) + f(y)}{2} - C\alpha(y - x)^2(\alpha + 3) \frac{4(\alpha + 1)(\alpha + 2)}{4(\alpha + 1)(\alpha + 2)}
\]

**Corollary 3.** For $m = 1$ and $s = 1$ in (21), we have the result for Riemann–Liouville fractional integrals of strongly convex function:

\[
2^{t-1}f^a_{\langle x+y\rangle/2} \leq \frac{2^{\alpha - 1}\Gamma(\alpha + 1)}{(y - x)^\alpha} \left[ (f^a_{\langle x+y\rangle/2})'(y) + (f^a_{\langle x+y\rangle/2})'(x) \right]
\]

\[
\leq \alpha(f(x) + f(y)) \left( \frac{1}{2}\beta(1; s + 1, \alpha) + 2^{\alpha - 1}\beta(1; s + 1, \alpha) \right)
\]

**Corollary 4.** For $m = 1$ in (21), we get the result for Riemann–Liouville fractional integrals of strongly $s$-convex function:

\[
2^{t-1}f^a_{\langle x+y\rangle/2} \leq \frac{2^{\alpha - 1}\Gamma(\alpha + 1)}{(y - x)^\alpha} \left[ (f^a_{\langle x+y\rangle/2})'(y) + (f^a_{\langle x+y\rangle/2})'(x) \right]
\]

\[
\leq \alpha(f(x) + f(y)) \left( \frac{1}{2}\beta(1; s + 1, \alpha) + 2^{\alpha - 1}\beta(1; s + 1, \alpha) \right)
\]

**Corollary 5.** For $m = 1$ and $C = 0$ in (21), we get the result for Riemann–Liouville fractional integrals of $s$-convex function:

\[
2^{t-1}f^a_{\langle x+y\rangle/2} \leq \frac{2^{\alpha - 1}\Gamma(\alpha + 1)}{(y - x)^\alpha} \left[ (f^a_{\langle x+y\rangle/2})'(y) + (f^a_{\langle x+y\rangle/2})'(x) \right]
\]

\[
\leq \alpha(f(x) + f(y)) \left( \frac{1}{2}\beta(1; s + 1, \alpha) + 2^{\alpha - 1}\beta(1; s + 1, \alpha) \right)
\]

**Corollary 6.** For $m = 1$ and $\alpha = 1$ in (1), we have the Hadamard inequality for strongly $s$-convex function:

\[
2^{t-1}f^a_{\langle x+y\rangle/2} \leq \frac{2^{\alpha - 1}\Gamma(\alpha + 1)}{(y - x)^\alpha} \left[ (f^a_{\langle x+y\rangle/2})'(y) + (f^a_{\langle x+y\rangle/2})'(x) \right]
\]

\[
\leq \alpha(f(x) + f(y)) \left( \frac{1}{2}\beta(1; s + 1, \alpha) + 2^{\alpha - 1}\beta(1; s + 1, \alpha) \right)
\]

3. Error Estimations of Riemann–Liouville Fractional Integral Inequalities

The following two lemmas are very useful to obtain the results of this section.

**Lemma 1** (see [8]). Let $f: [x, y] \rightarrow \mathbb{R}$ be a differentiable mapping on $(x, y)$ with $x < y$. If $f' \in L[x, y]$, then the following fractional integral equality holds:
Then the following equality for fractional integrals holds:

\[
\frac{f(x) + f(y)}{2} - \frac{\Gamma(a + 1)}{2(y - x)^a} \left[ (f_x^a f) (y) + (f_y^a f) (x) \right] = \frac{y - x}{2} \int_0^1 (1 - t)^a t^a f'(tx + (1-t)y) dt.
\]  

(34)

**Lemma 2** (see [10]). Let \( f : [x, y] \rightarrow \mathbb{R} \) be a differentiable mapping on \((x, y)\) with \( x < y \). If \( f' \in [x, m y], m \in (0, 1] \), then the following equality for fractional integrals holds:

\[
\begin{align*}
2^{s-1} \frac{\Gamma(a + 1)}{(my - x)^a} & \left[ (f_x^a f) (my) + m^{a+1} (f_y^a f) \left( \frac{x}{m} \right) \right] \\
& - \frac{1}{2} \left[ f(x + my) + m f \left( \frac{x + my}{2m} \right) \right] \\
& = \frac{mb - a}{4} \left[ \int_0^1 t^a f' \left( t + \frac{2 - t}{2} \right) dt + \int_0^1 t^a f' \left( y + \frac{2 - t}{2} \right) dt \right].
\end{align*}
\]  

(35)

**Theorem 5.** Let \( f : [x, y] \rightarrow \mathbb{R} \) be a differentiable mapping on \((x, y)\) with \( x < y \). If \(|f'|\) is a strongly \((s, m)\)-convex function on \([x, my]\) with modulus \( C \geq 0, m \neq 0, \) and \( 0 < s \leq 1 \), then the following fractional integral inequality holds:

\[
\begin{align*}
\left| \frac{f(x) + f(y)}{2} - \frac{\Gamma(a + 1)}{2(y - x)^a} \left[ (f_x^a f) (y) + (f_y^a f) (x) \right] \right| \\
& \leq \frac{y - x}{2} \left[ f'(x) \left( \beta \left( \frac{1}{2}; a + 1, s + 1 \right) - \beta \left( \frac{1}{2}; s + 1, a + 1 \right) + \frac{1 - (1/2)^{a+s}}{a + s + 1} \right) \right. \\
& \left. + m \left| f' \left( \frac{y}{m} \right) \right| \left( \beta \left( \frac{1}{2}; a + 1, s + 1 \right) - \beta \left( \frac{1}{2}; s + 1, a + 1 \right) + \frac{1 - (1/2)^{a+s}}{a + s + 1} \right) \right] \\
& \leq 2Cm (\frac{((y/m) - x)}{(a + 2)(a + 3)} \left( 1 - \frac{a + 4}{2^{a+2}} \right),
\end{align*}
\]  

(36)

with \( a > 0 \).

**Proof.** Since \(|f'|\) is strongly \((s, m)\)-convex function on \([x, y]\) for \( t \in [0, 1] \), we have

\[
|f'(tx + (1-t)y)| \leq t^a |f'(x)| + m(1-t)^a \left| \frac{f'}{m} \left( \frac{y}{m} \right) \right| - Cmt (1-t) \left( \frac{y}{m} - x \right)^2.
\]  

(37)
By using Lemma 1 and (37), we have

\[
\left| f(x) + f(y) \right| - \frac{\Gamma(\alpha + 1)}{2(y - x)\alpha}\left[(J_{\alpha}^{\alpha} f)(y) + (J_{\alpha}^{\alpha} f)(x)\right]
\leq \frac{y - x}{2} \left[ |f'(x)| \left( \beta \left( \frac{1}{2}; \alpha + 1, s + 1 \right) \right) + \frac{1}{\alpha + s + 1} \right]
\leq \frac{y - x}{2} \left[ |f'(x)| \left( \beta \left( \frac{1}{2}; \alpha + 1, s + 1 \right) \right) + \frac{1}{\alpha + s + 1} \right] + \frac{2C(y - x)^2}{(\alpha + 2)(\alpha + 3)} \left( 1 - \frac{\alpha + 4}{2} \right).\]

After simplifying the last inequality of (38), we get \( \Box \) (ii) By setting \( s = 1 \) in inequality (36), we get [[18], Theorem 8].

**Remark 3**

(i) By setting \( C = 0 \) in inequality (36), one can get result for \((s, m)\)-convex function.

\[
\left| f(x) + f(y) \right| - \frac{\Gamma(\alpha + 1)}{2(y - x)\alpha}\left[(J_{\alpha}^{\alpha} f)(y) + (J_{\alpha}^{\alpha} f)(x)\right]
\leq \frac{y - x}{2} \left[ |f'(x)| \left( \beta \left( \frac{1}{2}; \alpha + 1, s + 1 \right) \right) + \frac{1}{\alpha + s + 1} \right] + \frac{2C(y - x)^2}{(\alpha + 2)(\alpha + 3)} \left( 1 - \frac{\alpha + 4}{2} \right).\]

**Corollary 7.** By taking \( m = 1 \) in (36), we have the result for Riemann–Liouville fractional integrals of strongly \( s \)-convex function:

\[
\left| f(x) + f(y) \right| - \frac{\Gamma(\alpha + 1)}{2(y - x)\alpha}\left[(J_{\alpha}^{\alpha} f)(y) + (J_{\alpha}^{\alpha} f)(x)\right]
\leq \frac{y - x}{2} \left[ \frac{1}{2} \left( \frac{1}{2}; \alpha + 1, s + 1 \right) \right] + \frac{1}{\alpha + s + 1} \right] + \frac{2C(y - x)^2}{(\alpha + 2)(\alpha + 3)} \left( 1 - \frac{\alpha + 4}{2} \right).\]

**Corollary 8.** By taking \( m = 1 \) and \( s = 1 \) in inequality (36), we have the result for Riemann–Liouville fractional integrals of strongly convex function:

\[
\left| f(x) + f(y) \right| - \frac{\Gamma(\alpha + 1)}{2(y - x)\alpha}\left[(J_{\alpha}^{\alpha} f)(y) + (J_{\alpha}^{\alpha} f)(x)\right]
\leq \frac{y - x}{2} \left[ \frac{1}{2} \left( \frac{1}{2}; \alpha + 1, s + 1 \right) \right] + \frac{1}{\alpha + s + 1} \right] + \frac{2C(y - x)^2}{(\alpha + 2)(\alpha + 3)} \left( 1 - \frac{\alpha + 4}{2} \right).\]
Corollary 9. By taking $m = s = 1$ and $\alpha = 1$ in inequality (36), we get the following inequality:

\[
\left| \frac{f(x) + f(y)}{2} - \frac{1}{y-x} \int_x^y f(u)\,du \right| \leq \frac{y-x}{8} \left( |f'(x)| + |f'(y)| \right) - \frac{C(y-x)^3}{32}. \tag{41}
\]

Inequality (41) provides the refinement of [[19], Theorem 2.2].

Theorem 6. Let $f: [x, y] \rightarrow \mathbb{R}$ be a differentiable mapping on $(x, y)$ with $x < y$. If $|f'|^q$ is strongly $(s, m)$-convex on $[x, my]$ with modulus $C \geq 0$, $(s, m) \in (0, 1]^2$ for $q \geq 1$, then the following inequality for fractional integrals holds:

\[
\left| \frac{2^{\alpha+1}(\alpha+1)}{(my-x)^2} \left( \int_0^{x+my} f(t)\,dt \right) + m^{\alpha+1} \int_0^{\alpha+1}(2x-my) f\left(\frac{x}{m}\right) \right| \leq \frac{my-x}{4\alpha+1} \left( |f'(x)| + |f'(y)| \right) - \frac{Cm(y-x)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)} \tag{42}
\]

Proof. By applying Lemma 2 and strong $(s, m)$-convexity of $|f'|$, we have

\[
\left| \frac{2^{\alpha+1}(\alpha+1)}{(my-x)^2} \left( \int_0^{x+my} f(t)\,dt \right) + m^{\alpha+1} \int_0^{\alpha+1}(2x-my) f\left(\frac{x}{m}\right) \right| \leq \frac{my-x}{4} \left[ t^a f' \left( \frac{t}{2} x + m \frac{2-t}{2} y \right) \right]_0^1 + \int_0^1 t^a f' \left( \frac{\frac{t}{2} x + m \frac{2-t}{2} y}{my-x} \right) \,dt - \frac{1}{2} \left( f' \left( \frac{x+my}{2} \right) \right.
\]

\[
+ m f\left( \frac{x+my}{2} \right) \right| \leq \frac{my-x}{4} \left[ \left( \frac{|f'(x)| + |f'(y)|}{2^a} \right) \right]_0^1 \int_0^1 t^a \,dt - \frac{Cm(y-x)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)} \tag{43}
\]

\[
\cdot \left( \beta \left( \frac{1}{2} s + 1, \alpha + 1 \right) - \frac{Cm(\alpha+4)}{4(\alpha+2)(\alpha+3)} \left( (y-x)^2 + \left( \frac{x}{m^2} - y \right)^2 \right) \right).
\]
Hence, we have inequality (42).

**Remark 4**

(i) For \( s = 1 \) in inequality (42), we have the result for strongly \( m \)-convex function [18].

(ii) For \( s = 1, m = 1, \) and \( C = 0 \) in inequality (42), we get [16, Theorem 5].

**Corollary 10.** For \( s = 1 \) and \( m = 1 \) in inequality (42), we have the result for Riemann–Liouville fractional integrals of strongly convex function:
Corollary 11. For $m = 1$ in inequality (42), we have the result for Riemann–Liouville fractional integrals of strongly $s$-convex function:

\[
\left| 2^{\alpha-1}(\alpha + 1) \left( \int_{(y-x)^{\alpha}} \left[ \left( J_{(x+y)^{\alpha}} f \right)(y) + \left( J_{(x-y)^{\alpha}} f \right)(x) \right] - f \left( \frac{x + y}{2} \right) \right) \right|
\]

\[
\leq \frac{y - x}{4(\alpha + 1)^{1/\rho}} \left[ \left( \int_{(x+y)^{\alpha}} f'(y)^{q} \beta_{1/2; s + 1, \alpha + 1} \left( \frac{C}{4(\alpha + 2)(\alpha + 3)} \right)^{(1/q)} \right) + \left( \int_{(x-y)^{\alpha}} f'(x)^{q} \beta_{1/2; s + 1, \alpha + 1} \left( \frac{C}{4(\alpha + 2)(\alpha + 3)} \right)^{(1/q)} \right) \right].
\]

Theorem 7. Let $f : [x, y] \rightarrow \mathbb{R}$ be a differentiable mapping on $(x, y)$ with $x < y$. If $f''$ is strongly $(s, m)$-convex function on $[x, my]$ with modulus $C \geq 0$, $(s, m) \in (0, 1)^{2}$ for $q > 1$, then the following fractional integral inequality holds:

\[
\left| 2^{\alpha-1}(\alpha + 1) \left( \int_{(my-x)^{\alpha}} \left[ \left( J_{(x+my)^{\alpha}} f \right)(my) + \left( J_{(x-my)^{\alpha}} f \right)(x) \right] - f \left( \frac{x + my}{2} \right) \right) \right|
\]

\[
\leq \frac{my - x}{4(\alpha + p + 1)^{(1/p)}} \left[ \left( \int_{(x+my)^{\alpha}} f'(y)^{q} \beta_{1/2; s + 1, \alpha + 1} \left( \frac{C}{4(\alpha + 2)(\alpha + 3)} \right)^{(1/q)} \right) + \left( \int_{(x-my)^{\alpha}} f'(x)^{q} \beta_{1/2; s + 1, \alpha + 1} \left( \frac{C}{4(\alpha + 2)(\alpha + 3)} \right)^{(1/q)} \right) \right].
\]

where $\alpha > 0$. Proof. By applying Lemma 2 and then using Hölder inequality and strong $(s, m)$-convexity of $|f''|$, we get
We have used \( A^q + B^q \leq (A + B)^q \), for \( A \geq 0, B \geq 0 \). This completes the proof. \( \square \)

Remark 5

(i) For \( s = 1 \) in inequality (47), we get [[18], Theorem 10].

(ii) For \( s = 1 \) and \( C = 0 \) in inequality (47), we get [[10], Theorem 2.7].

(iii) For \( s = 1, m = 1, \) and \( C = 0 \) in inequality (47), we get [[16], Theorem 6].

Corollary 12. For \( \alpha = 1 \) and \( m = 1 \), we have the result for \( s \)-convex function:

\[
\left| \frac{1}{y-x} \int_x^y f(u)du - f\left(\frac{x+y}{2}\right) \right| \leq \frac{(y-x)(2^s(s+1))^{1/p-1}}{4(p+1)^{1/p}}
\]

\[
\cdot \left[ \left( |f'(x)|^q + (2^{s+1} - 1)^q |f'(y)|^q - \frac{2^s C(s+1)(y-x)^2}{6} \right)^{1/q} \right. \\
\left. + \left( 2^{s+1} - 1 \right)^q |f'(x)|^q + |f'(y)|^q - \frac{2^s C(s+1)(y-x)^2}{6} \right]^{1/q} \\
\leq \frac{(y-x)(2^s(s+1))^{1/p-1} - \frac{2^s C(s+1)}{3}}{4(p+1)^{1/p}} \cdot \left[ 2^{s+1} \left( |f'(x)| + |f'(y)| \right) - \frac{2^s C(s+1)}{3} (y-x)^2 \right].
\]

Corollary 13. For \( \alpha = 1 \) and \( m = q = 1 \), we have

\[
\left| \frac{1}{y-x} \int_x^y f(u)du - f\left(\frac{x+y}{2}\right) \right| \\
\leq \frac{y-x}{4(s+1)} \left[ 2\left( |f'(x)| + |f'(y)| \right) - \frac{C(s+1)}{3} (y-x)^2 \right].
\]

Corollary 14. For \( \alpha = 1 \) and \( m = q = s = 1 \), we have

\[
\left| \frac{1}{y-x} \int_x^y f(u)du - f\left(\frac{x+y}{2}\right) \right| \\
\leq \frac{y-x}{4} \left[ (|f'(x)| + |f'(y)|) - \frac{C}{3} (y-x)^2 \right].
\]
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