On Free Baxter Algebras: Completions and the Internal Construction *

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1 Introduction

In a previous paper [4], we gave an explicit construction of a free Baxter algebra. This construction is called the shuffle Baxter algebra since it is described in terms of mixable shuffles. In this paper and its sequel [5], we will continue the study of free Baxter algebras.

There are two goals of this paper. The first goal is to extend the construction of shuffle Baxter algebras to completions of Baxter algebras. This process is motivated by a construction of Cartier [2] and is analogous to the process of completing a polynomial algebra to obtain a power series algebra. However, as we will see later, unlike the close similarity of properties of a polynomial algebra and a power series algebra, properties of a shuffle Baxter algebra and its completion can be quite different.

The second goal is to establish a connection between the shuffle Baxter algebra we have constructed to the standard Baxter algebra constructed by Rota [10]. The shuffle Baxter algebra is an external construction in the sense that it is a free Baxter algebra obtained without reference to any other Baxter algebra. On the other hand, the standard Baxter algebra is an internal construction, obtained as a Baxter subalgebra inside a naturally defined Baxter algebra constructed originally by Baxter [1]. There are several restrictions on Rota’s original construction of standard Baxter algebras. By modifying Rota’s method and making use of the shuffle Baxter algebra construction, we are able to construct the standard Baxter algebra in full generality. The shuffle product

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construction of a free Baxter algebra has the advantage that its module structure and Baxter operator can be easily described. The description of a free Baxter algebra as a standard Baxter algebra has the advantage that its multiplication is very simple. We will give an explicit description of the isomorphism between the shuffle Baxter algebra and the standard Baxter algebra. This description will enable us to make use of properties of both the shuffle product description and Rota’s description of free Baxter algebras. Some applications will be given in [5].

We will start with a brief summary of definitions and basic properties of the shuffle Baxter algebra in section 2. We also take the opportunity to extend the construction of shuffle Baxter algebras to the category of Baxter algebras not necessarily having an identity. In section 3 we define the completion of a Baxter algebra by making use of the filtration given by the Baxter operator, and give a description of a free complete Baxter algebra in terms of mixable shuffles. In section 4 we construct the standard Baxter algebras, generalizing Rota. Variations of the construction for complete Baxter algebras and for Baxter algebras not necessarily having an identity are also considered.

2 Shuffle Baxter algebras

We write $\mathbb{N}$ for the additive monoid of natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{n \in \mathbb{N} \mid n > 0\}$ for the positive integers.

Let $\text{Rings}$ denote the category of commutative rings with identity. For any $C \in \text{Rings}$, let $\text{Alg}_C$ denote the category of $C$-algebras with identity. For $C \in \text{Rings}$ and for any $C$-modules $M$ and $N$, the tensor product $M \otimes N$ is taken over $C$ unless otherwise indicated. Let $M$ be a $C$-module. For $n \in \mathbb{N}$, denote

$$M^{\otimes n} = M \otimes \ldots \otimes M$$

with the convention that $M^{\otimes 0} = C$.

2.1 Baxter algebras

Baxter algebras were first studied by Baxter [11] and the category of Baxter algebras was first studied by Rota [10]. We recall basic definitions and properties of Baxter algebras. See [11] [10] for details.

**Definition 2.1** Let $C$ be a ring, $\lambda \in C$, and let $R$ be a $C$-algebra.

- A **Baxter operator of weight $\lambda$ on $R$ over $C$** is a $C$-module endomorphism $P$ of $R$ satisfying

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy), \quad x, \ y \in R. \quad (1)$$
• A Baxter C-algebra of weight $\lambda$ is a pair $(R, P)$ where $R$ is a $C$-algebra and $P$ is a Baxter operator of weight $\lambda$ on $R$ over $C$.

• Let $(R, P)$ and $(S, Q)$ be two Baxter C-algebras of weight $\lambda$. A homomorphism of Baxter C-algebras $f : (R, P) \to (S, Q)$ is a homomorphism $f : R \to S$ of C-algebras with the property that $f(P(x)) = Q(f(x))$ for all $x \in R$.

If the meaning of $\lambda$ is clear, we will suppress $\lambda$ from the notation. Note that our $\lambda$ is $-q$ in the notation of Rota [11]. Let $\text{Bax}_{C, \lambda}$ denote the category of Baxter C-algebras of weight $\lambda$. A Baxter ideal of $(R, P)$ is an ideal $I$ of $R$ such that $P(I) \subseteq I$. Other concepts of $C$-algebras, such as subalgebra and quotient algebra, can also be defined for Baxter algebras [4].

### 2.2 Shuffle Baxter algebras with an identity

Let $A \in \text{Alg}_C$. In a previous work [4], we used mixable shuffles to construct a mixable shuffle algebra $\text{III}_C(A)$ and proved that it is a free Baxter C-algebra on $A$.

For $m, n \in \mathbb{N}_+$, define the set of $(m, n)$-shuffles by

$$S(m, n) = \left\{ \sigma \in S_{m+n} \mid \sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(m), \sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \ldots < \sigma^{-1}(m+n) \right\}.$$

Given an $(m, n)$-shuffle $\sigma \in S(m, n)$, a pair of indices $(k, k+1)$, $1 \leq k < m+n$ is called an admissible pair for $\sigma$ if $\sigma(k) \leq m < \sigma(k+1)$. Denote $\mathcal{T}^\sigma$ for the set of admissible pairs for $\sigma$. For a subset $T$ of $\mathcal{T}^\sigma$, call the pair $(\sigma, T)$ a mixable $(m, n)$-shuffle, where $(\sigma, T)$ is identified with $\sigma$ if $T$ is the empty set. Denote

$$\bar{S}(m, n) = \{(\sigma, T) \mid \sigma \in S(m, n), T \subset \mathcal{T}^\sigma\}$$

for the set of $(m, n)$-mixable shuffles.

For $m, n \in \mathbb{N}_+$, denote $x = x_1 \otimes \ldots \otimes x_m \in A^\otimes m$, and $y = y_1 \otimes \ldots \otimes y_n \in A^\otimes n$. For $\sigma \in S_m$, denote

$$\sigma(x) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \ldots \otimes x_{\sigma(m)}.$$

Denote $x \otimes y = x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n \in A^\otimes (m+n)$, and, for $\sigma \in S_{m+n}$, denote

$$\sigma(x \otimes y) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(m+n)},$$

where

$$u_k = \begin{cases} x_k, & 1 \leq k \leq m, \\ y_{k-m}, & m+1 \leq k \leq m+n. \end{cases}$$

**Definition 2.2** Let $x \in A^\otimes m$, $y \in A^\otimes n$ and $\sigma \in S(m, n)$. 

1. \( \sigma(x \otimes y) \in A^\otimes(m+n) \) is called a shuffle of \( x \) and \( y \).

2. Let \( T \) be a subset of \( T_\sigma \). The element
\[
\sigma(x \otimes y; T) = u_{\sigma(1)} \hat{} u_{\sigma(2)} \hat{} \ldots \hat{} u_{\sigma(m+n)},
\]
where for each pair \((k, k+1), 1 \leq k < m + n,\)
\[
u_{\sigma(k)} \hat{} u_{\sigma(k+1)} = \left\{ \begin{array}{ll}
u_{\sigma(k)} u_{\sigma(k+1)}, & \text{if } (k, k+1) \in T \\
u_{\sigma(k)} \otimes u_{\sigma(k+1)}, & \text{if } (k, k+1) \notin T
\end{array} \right.
\]
is called a mixable shuffle of \( x \) and \( y \).

Fix a \( \lambda \in C \). For \( x = x_0 \otimes x_1 \otimes \ldots \otimes x_m \in A^\otimes(m+1) \) and \( y = y_0 \otimes y_1 \otimes \ldots \otimes y_n \in A^\otimes(n+1) \), define
\[
x \diamond y = \sum_{(\sigma, T) \in S(m,n)} \lambda^{|T|} x_0 y_0 \otimes \sigma(x \otimes y; T) \in \bigoplus_{k \leq m+n+1} A^\otimes k.
\]
Then \( \diamond \) extends to a mapping
\[
\diamond : A^\otimes(m+1) \times A^\otimes(n+1) \to \bigoplus_{k \leq m+n+1} A^\otimes k, m, n \in \mathbb{N}
\]
by \( C \)-linearity. Let
\[
\Pi_C(A) = \Pi_C(A, \lambda) = \bigoplus_{k \in \mathbb{N}} A^\otimes(k+1) = A \oplus A^\otimes 2 \oplus \ldots.
\]
Extending by additivity, the map \( \diamond \) gives a \( C \)-bilinear map
\[
\diamond : \Pi_C(A) \times \Pi_C(A) \to \Pi_C(A)
\]
with the convention that
\[
A \times A^\otimes(m+1) \to A^\otimes(m+1)
\]
is the scalar multiplication on the left \( A \)-module \( A^\otimes(m+1) \). Define a \( C \)-linear endomorphism \( P_A \) on \( \Pi_C(A) \) by assigning
\[
P_A(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = 1_A \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n,
\]
for all \( x_0 \otimes x_1 \otimes \ldots \otimes x_n \in A^\otimes(n+1) \) and extending by additivity. Let \( j_A : A \to \Pi_C(A) \) be the canonical inclusion map. We proved the following theorem in [4].

**Theorem 2.3** 1. The \( C \)-module \( \Pi_C(A) \), together with the multiplication \( \diamond \), is a commutative \( C \)-algebra with an identity.
2. \((\mathcal{III}_C(A), P_A)\), together with the natural embedding \(j_A : A \rightarrow \mathcal{III}_C(A)\), is a free Baxter \(C\)-algebra on \(A\) (of weight \(\lambda\)). In other words, for any Baxter \(C\)-algebra \((R, P)\) and any \(C\)-algebra map \(\varphi : A \rightarrow R\), there exists a unique Baxter \(C\)-algebra homomorphism \(\tilde{\varphi} : (\mathcal{III}_C(A), P_A) \rightarrow (R, P)\) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{j_A} & \mathcal{III}_C(A) \\
\downarrow{\varphi} & & \downarrow{\tilde{\varphi}} \\
R & & 
\end{array}
\]

commutes.

The Baxter \(C\)-algebra \((\mathcal{III}_C(A), P_A)\) will be called the **shuffle Baxter \(C\)-algebra (of weight \(\lambda\)) on \(A\)**. When there is no danger of confusion, we will often suppress the symbol \(\diamond\) and simply denote \(xy\) for \(x \diamond y\) in \(\mathcal{III}_C(A)\).

For a given set \(X\), let \(C[X]\) be the polynomial \(C\)-algebra on \(X\) with the natural embedding \(X \hookrightarrow C[X]\). Let \((\mathcal{III}_C(X), P_X)\) be the Baxter \(C\)-algebra \((\mathcal{III}_C(C[X]), P_{C[X]})). (\mathcal{III}_C(X), P_X)\) will be called the **shuffle Baxter \(C\)-algebra (of weight \(\lambda\)) on \(X\)**.

**Proposition 2.4** \((\mathcal{III}_C(X), P_X)\), together with the set embedding

\[
j_X : X \hookrightarrow C[X] \xrightarrow{j_C[X]} \mathcal{III}_C(C[X]),
\]

is a free Baxter \(C\)-algebra on the set \(X\), described by the following universal property: For any Baxter \(C\)-algebra \((R, P)\) over \(C\) and any set map \(\varphi : X \rightarrow R\), there exists a unique Baxter \(C\)-algebra homomorphism \(\tilde{\varphi} : (\mathcal{III}_C(X), P_X) \rightarrow (R, P)\) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j_X} & \mathcal{III}_C(X) \\
\downarrow{\varphi} & & \downarrow{\tilde{\varphi}} \\
R & & 
\end{array}
\]

commutes.

If we choose \(A = C\) in the construction of free Baxter \(C\)-algebras, then we get

\[
\mathcal{III}_C(C) = \bigoplus_{n=0}^{\infty} C^{\otimes (n+1)} = \bigoplus_{n=0}^{\infty} C^1^{\otimes (n+1)},
\]

where \(1^{\otimes (n+1)} = 1_C \otimes \ldots \otimes 1_C\). Thus \(\mathcal{III}_C(C)\) is a free \(C\)-module on the basis \(1^{\otimes n}, n \geq 1\).

**Proposition 2.5** For any \(m, n \in \mathbb{N}\),

\[
1^{\otimes (m+1)} \otimes 1^{\otimes (n+1)} = \sum_{k=0}^{m} \binom{m+n-k}{k} \lambda^k 1^{\otimes (m+n+1-k)}.
\]
2.3 Shuffle Baxter algebras without an identity

We now construct a shuffle Baxter algebra in the category of Baxter algebras not necessarily having an identity.

For $C \in \text{Rings}$, let $\text{Alg}^0_C$ be the category of $C$-algebras not necessarily having an identity and let $\text{Bax}^0_C$ be the category of Baxter $C$-algebras not necessarily having an identity. For $A \in \text{Alg}^0_C$, we will use mixable shuffles to construct a free Baxter algebra on $A$ in $\text{Bax}^0_C$. This construction was given in a special case in [4]. A similar construction can be carried out if $C$ is assumed to be a commutative ring not necessarily having an identity, but we will not give details here.

Let $C \in \text{Rings}$ and $A \in \text{Alg}^0_C$ be given. We use a well-known construction [2,6] to embed $A$ in an element $A^+ \in \text{Alg}_C$. Let $A^+ = C \oplus A$ with the addition defined componentwise and the multiplication defined by

$$(c, a)(d, b) = (cd, cb + da + ab), c, d \in C, a, b \in A.$$  

Then $A^+$ is in $\text{Alg}_C$ with $(1_C, 0)$ as the identity and $a \mapsto (0, a)$ embeds $A$ in $A^+$ as a subobject in $\text{Alg}^0_C$ (in fact, as an ideal).

Define

$$\Pi_C(A)^0 = \oplus_{n \in \mathbb{N}} ((A^+)^{\otimes n} \otimes A)$$

with the convention that $(A^+)^{\otimes 0} = C$. Thus $\Pi_C(A)^0$ is the $C$-submodule of $\Pi_C(A^+)$ generated by tensors of the form

$$x_0 \otimes \ldots \otimes x_n, x_i \in A^+, 0 \leq i \leq n - 1, x_n \in A.$$  

Since any mixable shuffle of $x_0 \otimes \ldots \otimes x_m$ and $y_0 \otimes \ldots \otimes y_n$ has either $x_m$ or $y_n$ or $x_my_n$ as the last tensor factor, we see that $\Pi_C(A)^0$ is a $C$-subalgebra of $\Pi_C(A^+)$. It is also clearly closed under the Baxter operator $P_{A^+}$. So $\Pi_C(A)^0$, with the restriction of $P_{A^+}$, denoted by $P_A$, is a subobject of $\Pi_C(A^+)$ in $\text{Bax}^0_C$. It is called the shuffle Baxter algebra on $A$ (of weight $\lambda$) in the category $\text{Bax}^0_C$.

**Proposition 2.6** $(\Pi_C(A)^0, P_A)$, together with the natural embedding $j_A : A \to \Pi_C(A)^0$, is a free Baxter $C$-algebra on $A$ (of weight $\lambda$).

**Proof:** For any $(R, P) \in \text{Bax}^0_C$ and any morphism $\varphi : A \to R$ in $\text{Alg}^0_C$, we will display a unique morphism $\bar{\varphi} : (\Pi_C(A)^0, P_A) \to (R, P)$ in $\text{Bax}^0_C$ that extends $\varphi$. For each $n \in \mathbb{N}$, we will define a $C$-linear map

$$\bar{\varphi}_n : (A^+)^{\otimes n} \otimes A \to R.$$  

If $n = 0$, we define

$$\bar{\varphi}_0 : (A^+)^{\otimes 0} \otimes A = A \to R$$

by $\bar{\varphi}_0(x_0) = \varphi(x_0), x_0 \in A$. Assuming $\bar{\varphi}_n$ is defined, we define

$$\bar{\varphi}_{n+1} : (A^+)^{n+1} \times A \to R$$
by
\[ \varphi_{n+1}(x_0, \ldots, x_{n+1}) = cP(\tilde{\varphi}_n(x_1 \otimes \ldots \otimes x_{n+1})) + \varphi(x_0') P(\tilde{\varphi}_n(x_1 \otimes \ldots \otimes x_{n+1})), \]
if \( x_0 = (c, x_0') \in A^+ = C \oplus A \). Using the induction hypothesis, we see that this map is
\( C \)-multilinear, and so induces
\[ \tilde{\varphi}_{n+1} : (A^+) \otimes (n+1) \otimes A \rightarrow R. \]
We then use \( \tilde{\varphi}_n, n \in \mathbb{N} \) to define
\[ \hat{\varphi} = \sum_{n=0}^{\infty} \tilde{\varphi}_n : \mathfrak{III}_C(A)^0 = \oplus_{n=0}^{\infty} ((A^+) \otimes^n \otimes A) \rightarrow R. \] (2)
Since the products of \( \mathfrak{III}_C(A)^0 \subseteq \mathfrak{III}_C(A^+) \) and \( A \) both satisfy the mixable shuffle
product identity (see [4, Proposition 4.2]), \( \tilde{\varphi} \) is a morphism in \( \text{Bax}^0_C \). On the other
hand, for \( x_0 \otimes \ldots \otimes x_n \in (A^+) \otimes^n \otimes A \) with \( x_0 = (c, x_0') \), we have
\[ x_0 \otimes \ldots \otimes x_n = cP_A(x_1 \otimes \ldots \otimes x_n) + \varphi(x_0') P_A(x_1 \otimes \ldots \otimes x_n). \]
Thus equation (2) is the only possible way to define a morphism in \( \text{Bax}^0_C \) from \( \mathfrak{III}_C(A)^0 \) to \( R \) that extends \( \varphi \). This verifies the required universal property of \( \mathfrak{III}_C(A)^0 \). ■

3 Complete Baxter algebras

We will define a natural decreasing filtration on Baxter algebras and study the associated completion. We will show that the completion of a shuffle Baxter algebra is a free object in the category of complete Baxter algebras. In this section, we retain the assumption that all algebras have an identity.

3.1 Filtrations and completions

Let \((R, P)\) be a Baxter algebra. For any subset \( U \) of \( R \), denote \( < U >_B \) for the Baxter
ideal of \( R \) generated by \( U \). We will define a decreasing filtration \( \text{Fil}^n R \) of ideals of
\((R, P)\) as follows. Define \( \text{Fil}^0 R = R \). For any \( n \in \mathbb{N} \), assume that \( \text{Fil}^n R \) is defined,
and inductively define
\[ \text{Fil}^{n+1} R = < P(\text{Fil}^n R) >_B. \]
Thus, for example,
\[ \text{Fil}^1 R = < P(R) >_B \quad \text{and} \quad \text{Fil}^2 R = < P(< P(R) >_B) >_B. \]
Since each \( \text{Fil}^n R \) is a Baxter ideal of \( R \), we have \( P(\text{Fil}^n R) \subseteq \text{Fil}^n R \). Therefore,
\[ \text{Fil}^{n+1} R = < P(\text{Fil}^n R) >_B \subseteq \text{Fil}^n R. \]
So \( \{\text{Fil}_n R\}_{n \in \mathbb{N}} \) defines a decreasing filtration of Baxter ideals on \((R, P)\). Assuming \(\text{Fil}^1 R \neq R\), then each \(R/\text{Fil}_n^n R, n \in \mathbb{N}_+\), is a Baxter \(C\)-algebra. Since each of the projections \(R/\text{Fil}_n^{n+1} R \to R/\text{Fil}_n^n R, n \in \mathbb{N}_+\), is a Baxter \(C\)-algebra homomorphism, the inverse limit \(\lim_{\leftarrow} (R/\text{Fil}_n^n R)\) is also a Baxter \(C\)-algebra.

**Definition 3.1** Let \((R, P)\) be a Baxter algebra.

1. The decreasing filtration \(\text{Fil}_n^n R\) on \(R\) is called the **Baxter filtration on \(R\)**.

2. The Baxter algebra \((R, P)\) is called **proper** if \(\text{Fil}^1 R\) is a proper Baxter ideal of \(R\).

3. Denote \(\text{Bax}_C^\prime\) for the subcategory of \(\text{Bax}_C\) consisting of proper Baxter \(C\)-algebras.

4. For \((R, P) \in \text{Bax}_C^\prime\), the inverse limit \(\hat{R} = \lim_{\leftarrow} (R/\text{Fil}_n^n R)\) with the induced Baxter operator \(\hat{P}\) is called the **(Baxter) completion of \((R, P)\)**.

5. \((R, P) \in \text{Bax}_C^\prime\) is called **(Baxter) complete** if the natural Baxter \(C\)-algebra homomorphism
   
   \[\pi_R : R \to \lim_{\leftarrow} (R/\text{Fil}_n^n R)\]

   is an isomorphism.

Let \(\lambda \in C\) and \(A \in \text{Alg}_C\). It is easy to see (Proposition 3.2) that the shuffle Baxter algebra \(\text{III}_C(A)\) of weight \(\lambda\) is proper. On the other hand, for \(\lambda \in C\), define \(P_\lambda\) on \(A\) by \(P_\lambda(a) = -\lambda a\). Then \((A, P_\lambda)\) is a Baxter algebra of weight \(\lambda\). If \(\lambda\) is invertible in \(A\), then \(P_\lambda(A) = -\lambda A = A\). So \((A, P_\lambda)\) is proper. If \(\lambda\) is not invertible in \(A\), then \((A, P_\lambda)\) is proper. In fact, the Baxter completion of \((A, P_\lambda)\) is the same as \(\lim_{\leftarrow} A/\lambda^n A\), the \(\lambda\)-adic completion of \(A\).

**Proposition 3.2** For each \(f : (R, P) \to (S, Q)\) in \(\text{Bax}_C^\prime\), there is a unique \(\hat{f} : (\hat{R}, \hat{P}) \to (\hat{S}, \hat{Q})\) in \(\text{Bax}_C\) making the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow \pi_R & & \downarrow \pi_S \\
\hat{R} & \xrightarrow{f} & \hat{S}
\end{array}
\]

commute.

Before proving the proposition, we first give an elementary fact on Baxter algebras.

**Lemma 3.3** Let \(f : (R, P) \to (S, Q)\) be a morphism in \(\text{Bax}_C^\prime\). For any subset \(U\) of \(R\), we have \(f(< U >_B) \subseteq < f(U) >_B\).
Proof: For a given subset $U$ of $R$, consider the morphism

$$R \xrightarrow{f} S \to S/\langle f(U) \rangle_B$$

in $\text{Bax}_C$. Since $U$ is in the kernel of the morphism, it follows that $\langle U \rangle_B$ is in the kernel of the morphism. Thus $f(\langle U \rangle_B) \subseteq \langle f(U) \rangle_B$. ■

Proof of Proposition 3.2: We first apply induction on $k$ to show that $f(\text{Fil}^k R) \subseteq \text{Fil}^k S$. For $k = 0$, this just says $f(R) \subseteq S$. Assume that $f(\text{Fil}^k R) \subseteq \text{Fil}^k R$. Applying Lemma 3.3 to $P(\text{Fil}^k R)$, we obtain

$$f(\text{Fil}^{k+1} R) = f(\langle P(\text{Fil}^k R) \rangle_B) \subseteq \langle f(P(\text{Fil}^k R)) \rangle_B$$

$$= \langle Q(f(\text{Fil}^k R)) \rangle_B$$

$$\subseteq \langle Q(\text{Fil}^k S) \rangle_B$$

$$= \text{Fil}^{k+1} S.$$

This completes the induction. Then the proposition follows from general results on completions [9, p.57]. ■

3.2 An alternative description

We now give an interpretation of a complete free Baxter algebra in terms of infinite sequences. We consider the following situation. Let $R$ be a $C$-algebra and let $R_i$, $i \in \mathbb{N}$, be $C$-submodules of $R$ such that

1. $R = \oplus_{k \in \mathbb{N}} R_k$ as a $C$-module, and

2. $F^n R \overset{\text{def}}{=} \oplus_{k>n} R_k$ is an ideal of $R$, $n \in \mathbb{N}$.

We will define a multiplication on $\prod_{k \in \mathbb{N}} R_k$. The definition is similar to the case when $R$ is graded $C$-algebra. For lack of a suitable reference, we give details below.

Fix a $k \in \mathbb{N}$. Let $(x^{(n)})_n$ be a sequence of elements in $R_k$. If there is an $n_0 \in \mathbb{N}$ such that $x^{(n)} = x^{(n_0)}$ for $n \geq n_0$, then define $\lim_{n \to \infty} x^{(n)} = x^{(n_0)} \in R_k$. Further, let $(x^{(n)}) = (x^{(n)}_k)_k$ be a sequence of elements in $\prod_{k \in \mathbb{N}} R_k$. If $\lim_{n \to \infty} x^{(n)}_k$ exists for each $k$, then define

$$\lim_{n \to \infty} x^{(n)} = (\lim_{n \to \infty} x^{(n)}_k)_k \in \prod_{k \in \mathbb{N}} R_k.$$

For any $x = (x_k)_k \in \prod_k R_k$ and any $n \in \mathbb{N}$, define $x^{[n]} = (x^{[n]}_k)_k \in \prod_{k \in \mathbb{N}} R_k$ by

$$x^{[n]}_k = \begin{cases} x_k, & k \leq n, \\ 0, & k > n \end{cases}$$
Now let \( x = (x_k)_k \) and \( y = (y_k)_k \) be two elements of \( \prod_{k \in \mathbb{N}} R_k \). For given \( n \in \mathbb{N} \), we have \( x^{[n]}_k, y^{[n]}_k \in R = \oplus_k R_k \). So \( x^{[n]}_k y^{[n]}_k \) can be uniquely expressed as \( (z_k^{(n)})_k \), \( z_k^{(n)} \in R_k \) and \( z_k^{(n)} = 0 \) for \( k >> 0 \). For each fixed \( k \), we obtain a sequence \( (z_k^{(n)})_n \) in \( R_k \). When \( n \geq k \), we have \( x^{[n]}_k, y^{[n]}_k \in R = \oplus_k R_k \). So \( x^{[n]}_k y^{[n]}_k \) can be uniquely expressed as \( (z_k^{(n)})_k \), \( z_k^{(n)} \in R_k \) and \( z_k^{(n)} = 0 \) for \( k >> 0 \). For each fixed \( k \), we obtain a sequence \( (z_k^{(n)})_n \) in \( R_k \). When \( n \geq k \), we have \( x^{[n]}_k = x^{[k]}_k + x' \) and \( y^{[n]}_k = y^{[k]}_k + y' \) with \( x', y' \in F^{k+1} R \). Since \( F^{k+1} R \) is an ideal of \( R \), we have \( x^{[n]}_k y^{[n]}_k \equiv x^{[k]}_k y^{[k]} \pmod{F^{k+1} R} \).

Therefore \( z_k^{(n)} = z_k^{(k)} \) for \( n \geq k \) and \( \lim_{n \to \infty} z_k^{(n)} \in R_k \) is well-defined. Define

\[
(x_k)_k (y_k)_k = ( \lim_{n \to \infty} z_k^{(n)} )_k \in \prod_k R_k.
\]

In other words,

\[
(x_k)_k (y_k)_k = \lim_{n \to \infty} x^{[n]}_k y^{[n]}.
\]

It can be easily verified that this defines an associative, commutative multiplication on \( \prod_{k \in \mathbb{N}} R_k \), making it into a commutative \( C \)-algebra.

**Proposition 3.4** Let \( R \) be a \( C \)-algebra and let \( R_i, \ i \in \mathbb{N} \), be \( C \)-submodules of \( R \) such that

1. \( R = \oplus_{k \in \mathbb{N}} R_k \) as a \( C \)-module, and
2. \( F^n R \defeq \oplus_{k>n} R_k \) is an ideal of \( R \), \( n \in \mathbb{N} \).

There is a unique \( C \)-algebra isomorphism

\[
\psi_R : \varprojlim (R/F^k R) \cong \prod_{k \in \mathbb{N}} R_k
\]

that makes the diagram in \( \text{Alg}_C \)

\[
\begin{array}{ccc}
R & \longrightarrow & \prod_{k \in \mathbb{N}} R_k \\
\downarrow \pi_R & & \bigg\uparrow \psi_R \\
\varprojlim (R/F^k R) & & \\
\end{array}
\]

commute.

**Proof:** By definition, \( \varprojlim (R/F^k R) \) is the inverse limit of the inverse system \( p_{n+1,n} : R/F^{n+1} R \to R/F^n R \), \( n \geq 1 \). An element \( ((x_k^{(n)})_k + F^n R)_n \in \prod_{n \in \mathbb{N}} R/F^n R \) is an element of \( \varprojlim (R/F^k R) \) if and only if, for any \( n \geq 1 \),

\[
p_{n+1,n}((x_k^{(n+1)})_k + F^{n+1} R) = (x_k^{(n)})_k + F^n R.
\]
Since \( R/F^nR \cong \bigoplus_{k \leq n} R_k \), this is so if and only if \( x_k^{(n+1)} = x_k^{(n)} \) for \( k \leq n \). Therefore \( (x_k^{(n)})_k + F^nR_n = \lim_{\leftarrow} (R/F^kR) \) if and only if there is \( (y_n)_n \in \prod_{n \in \mathbb{N}} R_n \) such that, for any \( n \), \( x_k^{(n)} = y_k \) for \( k \leq n \). In fact, we can take \( y_k = x_k^{(k)} \). This gives the desired map \( \psi_R : \lim_{\leftarrow} (R/F^kR) \to \prod_{k \in \mathbb{N}} R_k \). More precisely, we have
\[
\psi_R((x_k^{(n)})_k + F^nR_n) = (x_k^{(k)})_k.
\]
(3)
For \( x = (x_k)_k \in R = \bigoplus_{k \in \mathbb{N}} R_k \), \( \pi_R(x) \in \lim_{\leftarrow} (R/F^kR) \) is defined to be the sequence \( (x_k)_k + F^nR_n \) which corresponds under \( \psi_R \) to the element \( (x_k)_k \in \prod_{k \in \mathbb{N}} R_k \). This proves the commutativity of the diagram. 

### 3.3 Complete shuffle Baxter algebras

We now consider the completion of \( \Pi_C(A) \). Recall that we denote \( \Pi_C^k(A) \) for the \( C \)-submodule \( A^{\otimes (k+1)} \) of \( \Pi_C(A) \). We denote \( \lim_{\leftarrow} \Pi_C^k(A) / \text{Fil}^k \Pi_C(A) \) by \( \hat{\Pi}_C(A) \).

**Proposition 3.5** Given \( k \in \mathbb{N}_+ \),

1. \( \text{Fil}^k \Pi_C(A) = \bigoplus_{n \geq k} \Pi_C^n(A) \),
2. \( \text{Fil}^k \Pi_C(A) \) is a Baxter homogeneous ideal of \( \Pi_C(A) \), and
3. The quotient Baxter \( C \)-algebra \( \Pi_C(A) / \text{Fil}^k \Pi_C(A) \) is isomorphic to \( \bigoplus_{n=0}^{k-1} \Pi_C^n(A) \) as a \( C \)-module.

**Proof:** 1. It follows from the definition of \( \diamond \) that, for any \( C \)-algebra \( A \),
\[
\Pi_C^m(A) \diamond \Pi_C^n(A) \subseteq \bigoplus_{k=\max\{m,n\}}^{m+n} \Pi_C^k(A).
\]
This shows that \( \bigoplus_{n \geq k} \Pi_C^n(A) \) is a Baxter ideal. Next we prove
\[
\text{Fil}^k \Pi_C(A) = \bigoplus_{n \geq k} \Pi_C^n(A)
\]
(4)
by induction on \( k \). By definition, \( \text{Fil}^1 \Pi_C(A) \) is the Baxter ideal generated by \( P_A(\Pi_C(A)) = 1_A \otimes \Pi_C(A) \). On the other hand, \( \bigoplus_{n \geq 1} \Pi_C^n(A) \) equals \( A \diamond (1_A \otimes \Pi_C(A)) \), hence is also generated by \( 1_A \otimes \Pi_C(A) \). This verifies equation (4) for \( k = 1 \). Assume that \( \text{Fil}^k \Pi_C(A) = \bigoplus_{n \geq k} \Pi_C^n(A) \) for a \( k \in \mathbb{N}_+ \). From this we obtain that \( \text{Fil}^{k+1} \Pi_C(A) \) is the Baxter ideal generated by
\[
P_A(\text{Fil}^{k+1} \Pi_C(A)) = P_A(\bigoplus_{n \geq k} \Pi_C^n(A)) = 1_A \otimes (\bigoplus_{n \geq k} \Pi_C^n(A)).
\]
On the other hand, \( \bigoplus_{n \geq k+1} \Pi^n_C(A) \) equals \( R \circ (1_A \otimes \bigoplus_{n \geq k} \Pi^n_C(A)) \), hence is also generated by \( 1_A \otimes \bigoplus_{n \geq k} \Pi^n_C(A) \). This verifies equation (1) for \( k+1 \).

Other statements in the proposition follow immediately from the first one. ■

It follows from Proposition 3.5 that \( R = \Pi_C(A) = \bigoplus_{n \in \mathbb{N}} \Pi^n_C(A) \) satisfies the two conditions for \( R \) in Proposition 3.4. Thus the product \( \prod_{k \in \mathbb{N}} \Pi^n_C(A) \) is a \( C \)-algebra. Define an operator \( \hat{P} \) on this product by \( \hat{P}((x_k)) = (1_A \otimes x_{k-1}) \) with the convention that \( 1_A \otimes x_{k-1} = 0 \) for \( k = 0 \).

**Theorem 3.6**

1. \( \hat{P} \) is a Baxter operator on \( \prod_{k \in \mathbb{N}} \Pi^n_C(A) \), and \( \psi_A \stackrel{\text{def}}{=} \psi_{\Pi_C(A)} : \hat{\Pi}_C(A) \rightarrow \prod_{k \in \mathbb{N}} \Pi^n_C(A) \) is an isomorphism of \( C \)-Baxter algebras.

2. Given a morphism \( f : A \rightarrow B \) in \( \text{Alg}_C \), we have the following commutative diagram in \( \text{Bax}_C \)

\[
\begin{array}{ccc}
\hat{\Pi}_C(A) & \xrightarrow{\psi_A} & \prod_{k \in \mathbb{N}} \Pi^n_C(A) \\
\downarrow \hat{\Pi}_C(f) & & \downarrow \prod_{k \in \mathbb{N}} \Pi^n_C(f) \\
\hat{\Pi}_C(B) & \xrightarrow{\psi_B} & \prod_{k \in \mathbb{N}} \Pi^n_C(B)
\end{array}
\]

where \( \hat{\Pi}_C(f) \) is induced by \( \Pi_C(f) \) which is in turn induced by \( f \), and \( f_k : \Pi^n_C(A) \rightarrow \Pi^n_C(B) \) is the tensor power morphism of \( C \)-modules \( f^{\otimes(k+1)} : A^{\otimes(k+1)} \rightarrow B^{\otimes(k+1)} \) induced from \( f \).

**Proof:** 1. Let \( ((x_k^{(n)})_k + \text{Fil}^n \Pi_C(A))_n \in \hat{\Pi}_C(A) \) be given. Using the formula (3) for the map \( \psi_A \), we have

\[
(\hat{P} \circ \psi_A)(((x_k^{(n)})_k + \text{Fil}^n \Pi_C(A))_n)
\]

\[
= (1_A \otimes x_{k-1}^{(k)})
\]

and

\[
(\psi_A \circ \hat{P})(((x_k^{(n)})_k + \text{Fil}^n \Pi_C(A))_n)
\]

\[
= \psi_A(((1_A \otimes x_{k-1}^{(n)})_k + \text{Fil}^n \Pi_C(A))_n)
\]

\[
= (1_A \otimes x_{k-1}^{(k)})
\]

Thus \( \hat{P} \circ \psi_A = \psi_A \circ \hat{P} \). Since \( \psi_A \) is an isomorphism of \( C \)-algebras and since \( \hat{P} \) is known to satisfy the identity defining a Baxter operator, the above equation implies that the same identity is satisfied by \( \hat{P} \).

2. Given \( ((x_k^{(n)})_k + \text{Fil}^n \Pi_C(A))_n \in \hat{\Pi}_C(A) \), using the formula (3) we have
\[
\left( \prod_k f_k \circ \psi_A \right) \left( ((x_k^{(n)})_k + \text{Fil}^n \Pi_C(A))_n \right) \\
= \prod_k f_k ((x_k^{(k)})_k) \\
= (f_k(x_k^{(k)}))_k
\]

and

\[
\left( \psi_A \circ \hat{\Pi}_C(A) \right) \left( ((x_k^{(n)})_k + \text{Fil}^n \Pi_C(A))_n \right) \\
= \psi_A ((f_k(x_k^{(n)}))_k + \text{Fil}^n \Pi_C(B)_n) \\
= (f_k(x_k^{(k)}))_k.
\]

This proves that the diagram commutes. ■

As an example of Theorem 3.6 consider the case when \( A = C \) and \( \lambda = 0 \). Let \( HC \) be the ring of Hurwitz series over \( C \) [7], defined to be the set of sequences
\[
\{(a_n) \mid a_n \in C, n \in \mathbb{N}\}
\]

in which the addition is defined componentwise and the multiplication is defined by
\[
(a_n)(b_n) = (c_n)
\]

with
\[
c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.
\]

Denote \( e_n \) for the sequence \( (a_k) \) in which \( a_n = 1_C \) and \( a_k = 0 \) for \( k \neq n \). Since \( e_n e_m = \binom{m+n}{n} e_{m+n} \), the following corollary follows from Proposition 2.5.

**Corollary 3.7** The assignment
\[
1 \otimes (n+1) \mapsto e_n, \ n \geq 0
\]
defines an isomorphism
\[
\hat{\Pi}_C(C) \to HC.
\]

By Theorem 3.6 and part three of Proposition 3.5 we have the isomorphism of inverse systems
\[
\hat{\Pi}_C(A)/\text{Fil}^k \hat{\Pi}_C(A) \cong \Pi_C(A)/\text{Fil}^k \Pi_C(A).
\]

Thus the completion of \( \hat{\Pi}_C(A) \) is itself, so it is complete. We next verify the free universal property of \( \hat{\Pi}_C(A) \) in the category \( \text{Bax}_C \) of complete Baxter \( C \)-algebras. Abbreviate \( \pi_A \) for \( \pi_{\Pi_C(A)} \).
Theorem 3.8 \((\hat{\mathbb{M}}_C(A), \hat{P}_A)\), together with the natural embedding \(j_A : A \xrightarrow{j_A} \mathbb{M}_C(A) \xrightarrow{\pi_A} \hat{\mathbb{M}}_C(A)\), is a free complete Baxter \(C\)-algebra on \(A\) (of weight \(\lambda\)). In other words, for any complete Baxter \(C\)-algebra \((R, P)\) and any \(C\)-algebra map \(\varphi : A \to R\), there exists a unique Baxter \(C\)-algebra homomorphism \(\hat{\varphi} : (\hat{\mathbb{M}}_C(A), \hat{P}_A) \to (R, P)\) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{j_A} & \hat{\mathbb{M}}_C(A) \\
\downarrow{\varphi} & & \downarrow{\hat{\varphi}} \\
R & \end{array}
\]

commutes.

**Proof:** Given a \((R, P)\) and \(\varphi : A \to R\) as in the statement of the theorem, by the universal property of \((\mathbb{M}_C(A), j_A)\) in \(\mathbf{Bax}_C\), there is a unique \(\hat{\varphi} : \mathbb{M}_C(A) \to R\) in \(\mathbf{Bax}_C\) such that \(\hat{\varphi} \circ j_A = \varphi\). Since \(R\) is complete, by Proposition 3.2 there is \(\hat{\varphi} \defeq \hat{\varphi} : \hat{\mathbb{M}}_C(A) \to \hat{R} \cong R\) such that \(\hat{\varphi} \circ \pi_A = \varphi\). Then we have

\[
\hat{\varphi} \circ j_A = \hat{\varphi} \circ \pi_A \circ j_A = \hat{\varphi} \circ j_A = \varphi.
\]

This proves the existence of \(\hat{\varphi}\). The uniqueness follows from the uniqueness of \(\hat{\varphi}\) and the uniqueness of the completion. \(\blacksquare\)

### 3.4 Completions of Baxter algebras

It is clear that complete Baxter \(C\)-algebras, together with the Baxter algebra homomorphisms between them, form a full subcategory \(\hat{\mathbf{Bax}}_C\) of \(\mathbf{Bax}_C\). Recall that we denote \(\mathbf{Bax}'_C\) for the full subcategory of \(\mathbf{Bax}_C\) consisting of proper Baxter algebras. Denoting \(I_C : \mathbf{Bax}_C \to \mathbf{Bax}'_C\) for the natural inclusion of categories, we then have

**Proposition 3.9**

1. For any \(R \in \mathbf{Bax}'_C\), the Baxter completion \(\hat{R}\) of \(R\) is complete.

2. The assignments \((R, P) \mapsto (\hat{R}, \hat{P})\) and \(f \mapsto \hat{f}\) define a functor \(F_C\) from \(\mathbf{Bax}'_C\) to \(\hat{\mathbf{Bax}}_C\), and the morphisms \(\pi_R : R \to \hat{R}\), \(R \in \mathbf{Bax}'_C\), define a natural transformation between the identity functor on \(\mathbf{Bax}'_C\) and the functor \(I_C \circ F_C : \mathbf{Bax}'_C \to \mathbf{Bax}'_C\).

**Proof:** We only need to prove that \(\hat{R}\) is complete. The rest of the proof is clear.

Note that the filtration \(\text{Fil}^k\) on \(\hat{R}\) is not defined to be the natural filtration induced by the filtration \(\text{Fil}^k\) on \(R\). So the general results on completions do not apply. Instead, we will use the facts that any Baxter algebra is a quotient of a shuffle Baxter algebra,
and that the completion of a shuffle Baxter algebra is complete, which follows from Proposition 3.3 and Theorem 3.6.

For a given \((R, P) \in \text{Bax}_C\), by the universal property of free Baxter algebras, there is an \(A \in \text{Alg}_C\) and a surjective morphism \(\psi : \text{III}_C(A) \to R\) in \(\text{Bax}_C\). We will prove by induction on \(n \in \mathbb{N}^+\) that

\[
\psi(\text{Fil}^n \text{III}_C(A)) = \text{Fil}^n R. \tag{5}
\]

For \(n = 1\) we have

\[
\psi(\text{Fil}^1 \text{III}_C(A)) = \psi(\text{III}_C(A)P_A(\text{III}_C(A)))
= \psi(\text{III}_C(A))P(\psi(\text{III}_C(A)))
= RP(R).
\]

Note that \(RP(R)\) is the ideal of \(R\) generated by \(P(R)\). Since \(P(RP(R)) \subseteq P(R)\), it is in fact the Baxter ideal of \(R\) generated by \(P(R)\). Thus \(RP(R) = \text{Fil}^1 R\). So equation (5) holds for \(n = 1\).

Assume that the equation holds for \(n\). Part one of Proposition 3.5 shows that \(\text{Fil}^{n+1} \text{III}_C(A)\) is the ideal of \(\text{III}_C(A)\) generated by \(P_A(\text{Fil}^n \text{III}_C(A))\). Then by induction we have

\[
\psi(\text{Fil}^{n+1} \text{III}_C(A)) = \psi(\text{III}_C(A)P_A(\text{Fil}^n \text{III}_C(A)))
= \psi(\text{III}_C(A))P(\psi(\text{III}_C(A)))
= RP(\text{Fil}^n R).
\]

Since

\[
P(RP(\text{Fil}^n R)) = P(\psi(\text{III}_C(A))P(\psi(\text{Fil}^n \text{III}_C(A))))
= \psi(P_A(\text{III}_C(A))P_A(\text{Fil}^n \text{III}_C(A))))
= \psi(P_A(\text{Fil}^{n+1} \text{III}_C(A))))
\subseteq \psi(\text{III}_C(A)P_A(\text{Fil}^n \text{III}_C(A))))
= RP(\text{Fil}^n R),
\]

\(RP(\text{Fil}^n R)\) is the Baxter ideal of \(R\) generated by \(P(\text{Fil}^n R)\), so is equal to \(\text{Fil}^{n+1} R\). This completes the induction.

Because of equation (5), the morphism \(\psi : \text{III}_C(A) \to R\) induces a morphism

\[
\psi_n : \text{III}_C(A)/\text{Fil}^n \text{III}_C(A) \to R/\text{Fil}^n R
\]

for each \(n \in \mathbb{N}_+\) and the kernel of \(\psi_n\) is \((\ker \psi + \text{Fil}^n \text{III}_C(A))/\text{Fil}^n \text{III}_C(A)\) which is isomorphic to \(\ker \psi/(\ker \phi \cap \text{Fil}^n \text{III}_C(A))\). Thus we have the exact sequence of inverse systems

\[
0 \to (\ker \psi + \text{Fil}^n \text{III}_C(A))/\text{Fil}^n \text{III}_C(A) \to \text{III}_C(A)/\text{Fil}^n \text{III}_C(A) \to R/\text{Fil}^n R \to 0
\]
and the transition map of the left inverse system is identified with the natural map
\[ \ker \psi/(\ker \psi \cap \Fil^{n+1} \Pi_C(A)) \to \ker \psi/(\ker \psi \cap \Fil^n \Pi_C(A)) \]
so is surjective. By [13, Lemma 3.5.3], for the first derived functor \( R^1 \lim \) of the inverse limit,
\[ R^1 \lim (\ker \psi + \Fil^n \Pi_C(A))/\Fil^n \Pi_C(A) = 0. \]
Therefore the above exact sequence of inverse systems gives the surjective morphism
\[ \hat{\psi} : \hat{\Pi}_C(A) \to \hat{R}. \]

Because of Theorem 3.6, \( \Fil^{n+1} \hat{\Pi}_C(A) \) is the ideal of \( \hat{\Pi}_C(A) \) generated by \( \hat{P}_A(\Fil^n \hat{\Pi}_C(A)) \).
Then the same argument for \( \psi : \Pi_C(A) \to R \) in the previous part of the proof can be repeated for the morphism \( \hat{\psi} : \hat{\Pi}_C(A) \to \hat{R} \). In particular, we have, for any \( n \in \mathbb{N}^+ \),
\[ \psi(\Fil^n \hat{\Pi}_C(A)) = \Fil^n \hat{R}. \quad (6) \]
We then obtain a surjective morphism
\[ \hat{\psi} : \hat{\Pi}_C(A) \to \hat{R}. \]
Since \( \hat{\Pi}_C(A) \) is its own completion, we have the commutative diagram
\[
\begin{array}{ccc}
\hat{\Pi}_C(A) & \cong & \hat{\Pi}_C(A) \\
\downarrow \hat{\psi} & & \downarrow \hat{\psi} \\
\hat{R} & \xrightarrow{\pi_{\hat{R}}} & \hat{R}
\end{array}
\]
in which both of the vertical maps are surjective. By the commutativity of the diagram, \( \pi_{\hat{R}} \) is surjective. Since \( \cap_n \Fil^n \Pi_C(A) = 0 \), by equation (6), we have \( \cap_n \Fil^n \hat{R} = 0 \). Thus \( \pi_{\hat{R}} \) is injective. Therefore, \( \hat{R} \) is complete. □

4 The standard Baxter algebra

The standard Baxter algebra constructed by Rota in [10] is a free object in the category \( \text{Bax}^0_C \) of Baxter algebras not necessarily having an identity. It is described as a Baxter subalgebra of another Baxter algebra whose construction goes back to Baxter [1]. In Rota’s construction, there are further restrictions that \( C \) is a field of characteristic zero, the free Baxter algebra obtained is on a finite set \( X \), and the weight \( \lambda \) is 1. By making use of shuffle Baxter algebras, we will show that Rota’s description can be modified to yield a free Baxter algebra on an algebra in the category \( \text{Bax}_C \) of Baxter algebras with an identity, with a mild restriction on the weight \( \lambda \). We will also provide a similar construction for algebras not necessarily having an identity, and for complete Baxter algebras.
4.1 The standard Baxter algebra of Rota

We first briefly recall the construction of Rota of a standard Baxter algebra $\mathcal{S}(X)$ on a set $X$. For details, see [10] [12].

As before, let $C$ be a commutative ring with an identity, and fix $\lambda \in C$. Let $X$ be a given set. For each $x \in X$, let $t^{(x)}$ be a sequence $t^{(x)} = (t^{(x)}_1, \ldots, t^{(x)}_n, \ldots)$ of distinct symbols $t^{(x)}_n$. We also require that the sets $\{t^{(x)}_n\}_n$ and $\{t^{(x)}_n\}_n$ are disjoint for $x_1 \neq x_2$ in $X$. Denote

$$
\overline{X} = \bigcup_{x \in X} \{t^{(x)}_n \mid n \in \mathbb{N}_+\}
$$

and denote $\mathfrak{A}(X)$ for the ring of sequences with entries in $C[\overline{X}]$, the $C$-algebra of polynomials with variables in $\overline{X}$. Thus the addition, multiplication and scalar multiplication by $C[\overline{X}]$ in $\mathfrak{A}(X)$ are defined componentwise. It will useful to have the following description of $\mathfrak{A}(X)$. For $k \in \mathbb{N}_+$, denote $\gamma_k$ for the sequence $(\delta_{n,k})_n$, where $\delta_{n,k}$ is the Kronecker delta. Then we can identify a sequence $(a_n)_n$ in $\mathfrak{A}(X)$ with a series

$$
\sum_{n=1}^{\infty} a_n \gamma_n = a_1 \gamma_1 + a_2 \gamma_2 + \ldots.
$$

Then the addition, multiplication and scalar multiplication by $C[\overline{X}]$ are given termwise.

Define

$$
P'_X = P'_{X,\lambda} : \mathfrak{A}(X) \to \mathfrak{A}(X)
$$

by

$$
P'_X(a_1, a_2, a_3, \ldots) = \lambda (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots).
$$

In other words, each entry of $P'_X(a)$, $a = (a_1, a_2, \ldots)$, is $\lambda$ times the sum of the previous entries of $a$. If elements in $\mathfrak{A}(X)$ are described by series $\sum_{n=1}^{\infty} a_n \gamma_n$ given above, then we simply have

$$
P'_X \left( \sum_{n=1}^{\infty} a_n \gamma_n \right) = \lambda \sum_{n=1}^{\infty} (\sum_{i=1}^{n-1} a_i) \gamma_n.
$$

It is well-known [1] [10] that, for $\lambda = 1$, $P'_X$ defines a Baxter operator of weight 1 on $\mathfrak{A}(X)$. It follows that, for any $\lambda \in C$, $P'_X$ defines a Baxter operator of weight $\lambda$ on $\mathfrak{A}(X)$, since it can be easily verified that for any Baxter operator $P$ of weight 1, the operator $\lambda P$ is a Baxter operator of weight $\lambda$. Hence $(\mathfrak{A}(X), P'_X)$ is in $\text{Bax}_C$.

**Definition 4.1** Let $\mathcal{S}(X)^0$ be the Baxter subalgebra in $\text{Bax}_C^0$ of $\mathfrak{A}(X)$ generated by the sequences $t^{(x)} = (t^{(x)}_1, \ldots, t^{(x)}_n, \ldots)$, $x \in X$. $\mathcal{S}(X)^0$ is called the standard Baxter algebra on $X$.

Note that $\mathcal{S}(X)^0$ is denoted by $\mathcal{S}(X)$ in Rota’s notation. We reserve $\mathcal{S}(X)$ for the free Baxter algebra on $X$ with an identity that will be defined below.

**Theorem 4.2 (Rota)** [11] $\mathcal{S}(X)^0, P'_X$ is a free Baxter algebra on $X$ in the category $\text{Bax}_C^0$. 

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4.2 The standard Baxter algebra in general

Given \( A \in \text{Alg}_C \), we now give an alternative construction of a free Baxter algebra on \( A \) in the category \( \text{Bax}_C \) of Baxter algebras with an identity.

For each \( n \in \mathbb{N}_+ \), denote \( A \otimes n \) for the tensor power algebra. Denote the direct limit algebra

\[
\overline{A} = \lim_{\rightarrow} A \otimes n
\]

where the transition map is given by

\[
A \otimes n \to A \otimes (n+1), \ x \mapsto x \otimes 1_A.
\]

Note that the multiplication on \( A \otimes n \) here is different from the multiplication on \( A \otimes n \) when it is regarded as the \( C \)-submodule \( \mathfrak{I}_{C}^{n-1}(A) \) of \( \mathfrak{I}_{C}(A) \). To distinguish between the two contexts, we will use the notation \( \mathfrak{I}_{C}^{n}(A) \) for \( A \otimes (n+1) \subseteq \mathfrak{I}_{C}(A) \).

Let \( A(A) \) be the set of sequences with entries in \( A \). Thus we have

\[
A(A) = \prod_{n=1}^{\infty} \overline{A} \gamma_n = \left\{ \sum_{n=1}^{\infty} a_n \gamma_n, a_n \in \overline{A} \right\}.
\]

Define addition, multiplication and scalar multiplication on \( A(A) \) componentwise, making \( A(A) \) into a \( \overline{A} \)-algebra, with the sequence \((1, 1, \ldots)\) as the identity. Define

\[
P'_A = P'_{A,\lambda} : A(A) \to A(A)
\]

by

\[
P'_A(a_1, a_2, a_3, \ldots) = \lambda(0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots).
\]

Then \( (A(A), P'_A) \) is in \( \text{Bax}_C \). For each \( a \in A \), define \( t^{(a)} = (t^{(a)}_k) \) in \( A(A) \) by

\[
t^{(a)}_k = \otimes_{i=1}^k a_i (= \otimes_{i=1}^{\infty} a_i), \ a_i = \left\{ \begin{array}{ll} a, & i = k, \\ 1, & i \neq k. \end{array} \right.
\]

**Definition 4.3** Let \( \mathcal{G}(A) \) be the Baxter subalgebra in \( \text{Bax}_C \) of \( A(A) \) generated by the sequences \( t^{(a)} = (t^{(a)}_1, \ldots, t^{(a)}_n, \ldots), \ a \in A \). \( \mathcal{G}(A) \) is called the **standard Baxter algebra** on \( A \).

Since \( \mathfrak{I}_{C}(A) \) is a free Baxter algebra on \( A \), the morphism in \( \text{Alg}_C \)

\[
A \to A(A), \ a \mapsto t^{(a)}
\]

extends uniquely to a morphism in \( \text{Bax}_C \)

\[
\Phi : \mathfrak{I}_{C}(A) \to \mathfrak{A}(A).
\]
We will prove in Theorem 4.6 that, when $\lambda$ is not a zero divisor in $\overline{A}$, $\Phi$ is an isomorphism. Hence $(\mathfrak{G}(A), P'_A)$ is a free Baxter algebra on $A$ in the category $\text{Bax}_C$. Before proving the theorem, we will first give some notations and preliminary results.

For $k \in \mathbb{N}_+$, denote

$$F^k \mathfrak{A}(A) = \{(a_i) \in \mathfrak{A}(A) \mid a_i = 0, \ i \leq k\} = \{\sum_{n=k+1}^{\infty} a_n \gamma_n \mid a_n \in \overline{A}\}.$$ 

Also denote $F^0 \mathfrak{A}(A) = \mathfrak{A}(A)$. Clearly each $F^k \mathfrak{A}(A)$ is an ideal of $\mathfrak{A}(A)$. Define

$$F^k \mathfrak{G}(A) = F^k \mathfrak{A}(A) \cap \mathfrak{G}(A).$$

Then we have

$$F^k \mathfrak{G}(A) = \{(a_i) \in \mathfrak{G}(A) \mid a_i = 0, \ i \leq k\}.$$ 

$F^k \mathfrak{G}(A)$ are ideals of $\mathfrak{G}(A)$. Recall from Section 3 that there is a canonical (Baxter) filtration $\text{Fil}^k$ on $\mathfrak{A}(A)$ and $\mathfrak{G}(A)$ defined by $P'_A$. We will explain the relation between them in Lemma 4.10.

**Lemma 4.4** For any $k \in \mathbb{N}$, we have

1. $\mathfrak{A}(A)P'_A(F^k \mathfrak{A}(A)) \subseteq F^{k+1} \mathfrak{A}(A)$.
2. $\Phi(\text{Fil}^k \mathfrak{G}(A)) \subseteq F^k \mathfrak{A}(A)$.

Similar inclusions hold for $\mathfrak{G}(A)$.

**Proof:** We only need to verify the inclusions for $\mathfrak{A}(A)$. The inclusions for $\mathfrak{G}(A)$ follows immediately. By the definition of $P'_A(F^k \mathfrak{A}(A))$ we have $P'_A(F^k \mathfrak{A}(A)) \subseteq F^{k+1} \mathfrak{A}(A)$. Since $\mathfrak{A}(A)P'_A(F^k \mathfrak{A}(A))$ equals to the ideal of $\mathfrak{A}(A)$ generated by $P'_A(F^k \mathfrak{A}(A))$, we get the first inclusion.

The second inclusion is clear for $k = 0$. By induction on $k$, for $k > 0$, we have

$$\Phi(\text{Fil}^k \mathfrak{G}(A)) \subseteq \Phi(\mathfrak{G}(A)P'_A(\text{Fil}^{k-1} \mathfrak{G}(A))))$$
$$\subseteq \Phi(\mathfrak{G}(A)P'_A(\Phi(\text{Fil}^{k-1} \mathfrak{G}(A))))$$
$$\subseteq \Phi(\mathfrak{G}(A)P'_A(F^{k-1} \mathfrak{A}(A)))$$
$$\subseteq F^k \mathfrak{A}(A).$$

**Lemma 4.5** For $n \in \mathbb{N}_+$ and $a_1 \otimes \ldots \otimes a_n \in \mathfrak{G}(A)$, we have

$$\Phi(a_1 \otimes \ldots \otimes a_n) = \lambda^{n-1}(a_n \otimes \ldots \otimes a_1)\gamma_n + F^n \mathfrak{G}(A).$$
Proof: By definition, for $a_1 \in A \subseteq \mathcal{W}_C(A)$,

$$\Phi(a_1) = \sum_{k=1}^{\infty} t_k^{(a_1)} \gamma_k = a_1 \gamma_1 + (1 \otimes a_1) \gamma_2 + \ldots.$$ 

So the lemma is proved for $n = 1$. Assume that the lemma is proved for $n$, and consider $a_1 \otimes \ldots \otimes a_{n+1} \in \mathcal{W}_C^n(A)$. Applying Lemma 4.4 we have

$$\Phi(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = \Phi(a_1 P_A(a_2 \otimes \ldots \otimes a_{n+1}))$$

$$= \Phi(a_1) \Phi(P_A(a_{n+1} \otimes \ldots \otimes a_2))$$

$$= \Phi(a_1) P'_A(\Phi(a_{n+1} \otimes \ldots \otimes a_2))$$

$$= \left( \sum_{k=1}^{\infty} t_k^{(a_1)} \gamma_k \right) P'_A(\lambda^{n-1}(a_{n+1} \otimes \ldots \otimes a_2) \gamma_n + \text{a term in } F^n \mathcal{S}(A))$$

$$= \left( \sum_{k=1}^{\infty} t_k^{(a_1)} \gamma_k \right) (\lambda^n(a_{n+1} \otimes \ldots \otimes a_2) \gamma_{n+1} + \text{a term in } F^{n+1} \mathcal{S}(A))$$

$$= \lambda^n(a_{n+1} \otimes a_n \otimes \ldots \otimes a_1) \gamma_{n+1} + \text{a term in } F^{n+1} \mathcal{S}(A).$$

This completes the induction. ■

Now we are ready to prove the main theorem of this section.

**Theorem 4.6** Assume that $\lambda \in C$ is not a zero divisor in $\overline{A}$. The morphism in $\mathbf{Bax}_C$

$$\Phi : \mathcal{W}_C(A) \to \mathcal{S}(A)$$

induced by sending $a \in A$ to $t^{(a)} = (t_1^{(a)}, \ldots, t_n^{(a)}, \ldots)$ is an isomorphism.

**Corollary 4.7** When $\lambda$ is not a zero divisor in $\overline{A}$, $(\mathcal{S}(A), P'_A)$ is a free Baxter algebra on $A$ in the category $\mathbf{Bax}_C$. ■

**Corollary 4.8** Assume that $\lambda$ is not a zero divisor in $C$. Let $X$ be a set. The morphism in $\mathbf{Bax}_C$

$$\Phi : \mathcal{W}_C(X) \to \mathcal{S}(X)$$

induced by sending $x \in X$ to $t^{(x)} = (t_1^{(x)}, \ldots, t_n^{(x)}, \ldots)$ is an isomorphism. The restriction of $\Phi$ to $\mathcal{W}_C(X)^0$ is an isomorphism in $\mathbf{Bax}_C^0$ from $\mathcal{W}_C(X)^0$ to $\mathcal{S}(X)^0$.

**Proof:** Applying Theorem 4.6 to the case when $A = C[X]$, we obtain $\Phi : \mathcal{W}_C(X) \cong \mathcal{S}(X)$. Since $\mathcal{W}_C(X)^0 \subseteq \mathcal{W}_C(X)$ is generated by $X$ in $\mathbf{Bax}_C^0$ and $\mathcal{S}(X)^0 \subseteq \mathcal{S}(X) = \Phi(\mathcal{W}_C(X))$ is generated by $\Phi(X)$ in $\mathbf{Bax}_C^0$, the corollary follows. ■

**Remarks:**
1. The proof of Theorem 4.6 specialized to the setting of Corollary 4.8 also gives another proof of Theorem 4.2.

2. The above construction of $\mathcal{S}(A)$ for $A \in \text{Alg}_C$ can be modified to give the construction of an internal free Baxter algebra $\mathcal{S}(A)^0$ in $\text{Bax}_C^0$ on $A$ for $A \in \text{Alg}_C^0$. The situation is similar to the construction of shuffle Baxter algebras not necessarily having an identity, discussed in section 2.

**Proof of Theorem 4.6** Since $(\mathcal{W}_C(A), P_A)$ is a free Baxter algebra on $A$ in $\text{Bax}_C$, the assignment

$$\Phi: A \to \mathcal{S}(A), \quad a \mapsto t^{(a)} = (t_1^{(a)}, \ldots, t_n^{(a)}, \ldots)$$

induces a morphism $\Phi: \mathcal{W}_C(A) \to \mathcal{S}(A)$ in $\text{Bax}_C$. Since $\mathcal{S}(A)$ is the Baxter subalgebra of $\mathfrak{A}(A)$ generated by $A$, the morphism $\Phi$ is onto. So we only need to verify that $\Phi$ is injective.

For any $G \in \mathcal{W}_C(X)$, we can uniquely write $G = \sum_{n \in \mathbb{N}} G_n$ with $G_n \in \mathcal{W}_C^n(A)$. Suppose $\Phi(G) = 0$; we will show by induction on $n \in \mathbb{N}$ that $G_n = 0$. For $n = 0$ we have $\mathcal{W}_C^0(A) = A$. So $G_0$ is in $A$. By Lemma 4.2, we have

$$\Phi(\sum_{k \geq 1} G_k) \in F^1\mathfrak{A}(A).$$

Thus the first component of $\Phi(F)$ in $\mathfrak{A}(A)$ is from

$$\Phi(G_0) = G_0 \gamma_1 + \text{a term in } F^1\mathfrak{A}(A).$$

Thus $\Phi(G) = 0$ implies $G_0 \gamma_1 = 0$. Therefore, $G_0 = 0$.

Now assume that $G_k = 0$ for $k \leq n$ and consider $G_{n+1} \in \mathcal{W}_C^{n+1}(A) = A^{\otimes(n+2)}$. Thus $G_{n+1}$ can be expressed as

$$G_{n+1} = \sum_{i=1}^k a_1^{(i)} \otimes \ldots \otimes a_{n+2}^{(i)}, \quad k \in \mathbb{N}_+, \quad a_j^{(i)} \in A.$$

Since $G_k = 0$ for $k \leq n$, and by Lemma 4.4 for $k \geq n + 2$,

$$\Phi(G_k) \in \Phi(\mathcal{W}_C(A)) \subseteq \Phi(\text{Fil}^k \mathcal{W}_C(A)) \subseteq \Phi(\text{Fil}^{n+2} \mathcal{W}_C(A)) \subseteq F^{n+2} \mathcal{S}(A),$$

the only contribution of $\Phi(G)$ to the coefficient of $\gamma_{n+2}$ is from $\Phi(G_{n+1})$. By Lemma 4.6, this coefficient is

$$\lambda^{n+1} \sum_{i=1}^k a_{n+2}^{(i)} \otimes \ldots \otimes a_1^{(i)}.$$

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Thus $\Phi(F) = 0$ implies that

$$\lambda^{n+1} \sum_{i=1}^{k} a_{n+2}^{(i)} \otimes \ldots \otimes a_{1}^{(i)} = 0.$$ 

Since $\lambda$ is not a zero divisor in $\tilde{A}$, we further have

$$\sum_{i=1}^{k} a_{n+2}^{(i)} \otimes \ldots \otimes a_{1}^{(i)} = 0$$

as an element in the tensor power algebra $A^{\otimes (n+2)}$. But this element being zero or not depends only on the $C$-module structure of $A^{\otimes (n+2)}$ and the $C$-module map

$$A^{\otimes (n+2)} \to A^{\otimes (n+2)}, a_{n+2} \otimes \ldots \otimes a_{1} \mapsto a_{1} \otimes \ldots \otimes a_{n+2}$$

is an isomorphism. Thus we also have

$$G_{n+1} = \sum_{i=1}^{k} a_{1}^{(i)} \otimes \ldots \otimes a_{n+2}^{(i)} = 0.$$ 

This completes the induction. Thus $\Phi$ is injective and hence an isomorphism. □

4.3 Completions

We now give an internal construction of free complete Baxter algebras by showing that the complete free Baxter algebra $\hat{\Pi}C(A)$ can be embedded into $\mathfrak{A}(A)$.

Let $\mathfrak{A}(A)'$ be the subgroup of $\mathfrak{A}(A)$ consisting of sequences with finitely many non-zero entries. Then we have

$$\mathfrak{A}(A)' = \bigoplus_{n=1}^{\infty} A\gamma_n.$$ 

Define a filtration on $\mathfrak{A}(A)'$ by taking

$$F^{k}\mathfrak{A}(A)' = \mathfrak{A}(X) \cap F^{k}\mathfrak{A}(A)' = \bigoplus_{n=k+1}^{\infty} A\gamma_n.$$ 

By Proposition 3.4 we have

$$\lim_{\leftarrow} (\mathfrak{A}(A)' / \text{Fil}^{k}\mathfrak{A}(A)') \cong \prod_{k \in \mathbb{N}_+} \tilde{A} \gamma_n$$

with the addition, multiplication and scalar multiplication defined componentwise. Therefore,

$$\lim_{\leftarrow} (\mathfrak{A}(A)' / \text{Fil}^{k}\mathfrak{A}(A)') \cong \mathfrak{A}(A).$$
From the definition of $F^k \mathcal{A}(A)$ and $F^k \mathcal{A}(A)'$,

$$\mathcal{A}(A)'/\text{Fil}^n \mathcal{A}(A)' \cong \mathcal{A}(A)/\text{Fil}^n \mathcal{A}(A).$$

So $\mathcal{A}(A)$ is the completion of itself with respect to the filtration $F^k \mathcal{A}(X)$. Since $F^k \mathcal{G}(A) = \mathcal{G}(A) \cap F^k \mathcal{A}(A)$, we have the injective map of inverse systems

$$\mathcal{G}(A)/F^k \mathcal{G}(A) \rightarrow \mathcal{A}(A)/F^k \mathcal{A}(A), k \in \mathbb{N}_+.\label{eq:1}$$

So

$$\lim_{\leftarrow} \mathcal{G}(A)/F^k \mathcal{G}(A) \hookrightarrow \lim_{\leftarrow} \mathcal{A}(A)/F^k \mathcal{A}(A) \cong \mathcal{A}(A). \quad (7)$$

We can easily describe the image of $\lim_{\leftarrow} \mathcal{G}(A)/F^k \mathcal{G}(A)$ in $\mathcal{A}(A)$. It consists of sequences $(b^{(n)})_n$, $b^{(n)} \in \mathcal{A}$, that can be expressed as an infinite sum of the form

$$\sum_{k=1}^{\infty} (b^{(n)}_k)_n,$$

where $(b^{(n)}_k)_n \in \mathcal{G}(A)$ for each $k$. This means that, for any fixed $n \in \mathbb{N}_+$, all but finitely many $b^{(n)}_k$, $k \in \mathbb{N}_+$, are non-zero, and $\sum_{k=1}^{\infty} b^{(n)}_k = b^{(n)}$. We denote this image by $\tilde{\mathcal{G}}(A)$ with the induced Baxter algebra structure.

On the other hand, we also have the Baxter filtration $\text{Fil}^k$ on $\mathcal{A}(A)$ and $\mathcal{G}(A)$ (section 3).

**Theorem 4.9**

1. The Baxter algebra $\mathcal{A}(A)$ is complete.

2. Assume that $\lambda \in C$ is not a zero divisor in $\mathcal{A}$. The isomorphism $\Phi : Π_1C(A) \rightarrow \mathcal{G}(A)$ extends to an isomorphism of complete Baxter algebras

$$\tilde{\Phi} : Π_1C(A) \rightarrow \tilde{\mathcal{G}}(A).$$

We first prove a lemma.

**Lemma 4.10** For any $k \in \mathbb{N}_+$, we have

1. $\text{Fil}^k \mathcal{A}(A) = \lambda^k F^k \mathcal{A}(A)$.

2. Assume that $\lambda \in C$ is not a zero divisor in $\mathcal{A}$. $\text{Fil}^k \mathcal{G}(A) = F^k \mathcal{G}(A)$.

**Proof:** 1. We prove that, for any $n \in \mathbb{N}_+$,

$$P'_A(F^k \mathcal{A}(A)) = \lambda F^{k+1} \mathcal{A}(A).$$
By the definition of $P'_A : \mathfrak{A}(A) \to \overline{\mathfrak{A}}(A)$, we have $P'_A(F^k\mathfrak{A}(A)) \subseteq \lambda F^{k+1}\mathfrak{A}(A)$. On the other hand, any element in $\lambda F^{k+1}\mathfrak{A}(A)$ is of the form
\[
\lambda \sum_{i=k+2}^{\infty} a_i \gamma_i, \ a_i \in \overline{A}.
\]
Then we have
\[
P'_A\left( \sum_{i=k+1}^{\infty} (a_{i+1} - a_i) \gamma_i \right) = \lambda \sum_{i=k+2}^{\infty} a_i \gamma_i.
\]
Here we take $a_{k+1} = 0$. This proves the equation.

When $k = 1$, we have $P'_A(\mathfrak{A}(A)) = \lambda F^1\mathfrak{A}(A)$. Since $F^1\mathfrak{A}(A)$ is already a Baxter ideal, we have Fil$^1\mathfrak{A}(A) = \lambda F^1\mathfrak{A}(A)$. Inductively, assuming that Fil$^k\mathfrak{A}(A) = \lambda^k F^k\mathfrak{A}(A)$, then we have
\[
P'_A(\text{Fil}^k\mathfrak{A}(A)) = P'_A(\lambda^k F^k\mathfrak{A}(A)) = \lambda^{k+1} F^{k+1}\mathfrak{A}(A).
\]
Since $F^{k+1}\mathfrak{A}(A)$ is a Baxter ideal of $\mathfrak{A}(A)$, it is Fil$^{k+1}\mathfrak{A}(A)$, the Baxter ideal generated by $P'_A(\text{Fil}^k\mathfrak{A}(A))$.

2. By Lemma 4.4, Lemma 4.5 and Theorem 4.6 we have, for any $a \in \Pi_C(A)$
\[
\Phi(a) \in \text{Fil}^k \mathfrak{S}(A)
\iff a \in \text{Fil}^k \Pi_C(A)
\iff \Phi(a) \in \mathfrak{S}(A) \cap F^k \mathfrak{A}(A)
\iff \Phi(a) \in F^k \mathfrak{S}(A).
\]
This proves the second equation. □

**Proof of Theorem 4.9:** From the first equation of Lemma 4.10, we have the exact sequence of inverse systems
\[
0 \to F^k \mathfrak{A}(A/\lambda k A) \to \mathfrak{A}(A)/\text{Fil}^k \mathfrak{A}(A) \to \mathfrak{A}(A)/F^k \mathfrak{A}(A) \to 0.
\]
Clearly $\lim_{\leftarrow} F^k \mathfrak{A}(A/\lambda k A) = 0$. Thus we have
\[
\hat{\mathfrak{A}}(A) \leftarrow \lim_{\leftarrow} (\mathfrak{A}(A)/F^k \mathfrak{A}(A)) = \mathfrak{A}(A).
\]
This proves the first statement.

Next assume that $\lambda \in C$ is not a zero divisor in $\overline{A}$. Then by the second statement of Lemma 4.10
\[
\hat{\mathfrak{S}}(A) \cong \lim_{\leftarrow} (\mathfrak{S}(A)/F^k \mathfrak{S}(A)).
\]
Then by Theorem 4.6 and equation (7) we obtain
\[
\hat{\Pi}_C(A) \cong \hat{\mathfrak{S}}(A) \to \mathfrak{A}(A).
\]
□
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