Online Steiner Tree with Deletions

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Abstract

In the online Steiner tree problem, the input is a set of vertices that appear one-by-one, and we have to maintain a Steiner tree on the current set of vertices. The cost of the tree is the total length of edges in the tree, and we want this cost to be close to the cost of the optimal Steiner tree at all points in time. If we are allowed to only add edges, a tight bound of \(\Theta(\log n)\) on the competitiveness has been known for two decades. Recently it was shown that if we can add one new edge and make one edge swap upon every vertex arrival, we can still maintain a constant-competitive tree online.

But what if the set of vertices sees both additions and deletions? Again, we would like to obtain a low-cost Steiner tree with as few edge changes as possible. The original paper of Imase and Waxman (SIAM J. Disc. Math, 4(3):369–384, 1991) had also considered this model, and it gave an algorithm that made at most \(O(n^{3/2})\) edge changes for the first \(n\) requests, and maintained a constant-competitive tree online. In this paper we improve on these results:

- We give an online algorithm that maintains a Steiner tree under only deletions: we start off with a set of vertices, and at each time one of the vertices is removed from this set—our Steiner tree no longer has to span this vertex. We give an algorithm that changes only a constant number of edges upon each request, and maintains a constant-competitive tree at all times. Our algorithm uses the primal-dual framework and a global charging argument to carefully make these constant number of changes.
- We also give an algorithm that maintains a Steiner tree in the fully-dynamic model (where each request either adds or deletes a vertex). Our algorithm for this setting makes a constant number of changes per request in an amortized sense.

1 Introduction

The online Steiner tree problem needs little introduction: we have an underlying metric space, one-by-one some points in the metric space get designated as vertices, and we have to maintain a Steiner tree on the current set of vertices. The cost of the tree is the total length of edges in the tree, and we want that at each time \(t\) this cost stays close to the cost of the optimal Steiner tree on the first \(t\) vertices. If we are allowed only to add edges, a tight bound of \(\Theta(\log n)\) on the competitiveness is known [IW91]. In fact, this paper considered the “dynamic” version of the problem as well, and asked what would happen if we were allowed to change the Steiner tree along the way, swapping a small number of previously added edges for shorter non-tree edges in order to decrease the cost. The rationale for this problem was natural: much as in dynamic data structures, it seems natural to “rewire” the tree over time, as long as the overhead in terms of the number of rewirings is not too high, and there is considerable benefit in terms of cost.

Imase and Waxman [IW91] showed that one could maintain a constant-competitive Steiner tree while only making \(t^{3/2}\) changes in the first \(t\) time-steps; this is a considerable improvement over the naive bound of \(O(t^2)\) obtained by just recomputing the tree at each time-step. More recently, Megow et
al. [MSVW12] made a significant improvement, showing that $O(t)$ changes were enough to maintain a constant-competitive tree. Hence the number of changes required is an amortized constant, improving on the previous $t^{1/2}$-amortized bound. This was improved in [GGK13] to showing that, in fact, one could make a single edge swap upon every vertex arrival (in addition to the new edge being added) — i.e., a non-amortized bound — and still achieve constant-competitiveness.

But what if the set of vertices can undergo both additions and deletions? Again, we would like to maintain a low-cost Steiner tree with as few edge changes as possible. Imase and Waxman [IW91] considered this model too; their $t^{3/2}$-change algorithm really is presented for this more general case where both additions and deletions take place (the “fully-dynamic” case). Very recently [LOP+13] gave an $O(\log \Delta)$-change algorithm for the fully-dynamic case, where $\Delta$ is the ratio of the maximum to minimum distance in the metric.

1.1 Our Results

We first consider online algorithms that maintain a Steiner tree under deletions only. In this model, we start off with a set of vertices, and at each time one of the vertices is deleted from the this set by an adversary. This means our Steiner tree no longer has to span this vertex — though we are free to use this deleted vertex in our tree if we like. In fact we are allowed to maintain any subgraph of the original graph, as long as all the current vertices lie in a single connected component of our subgraph. We pay the cost of this subgraph, and want this cost to be comparable to that of the optimal Steiner tree on the vertex set. Getting an amortized bound showing a constant number of changes for this deletions-only case is not difficult (and is implicit in the [IW91] paper). Our first result is an algorithm that changes only a non-amortized constant number of edges upon each deletion request, and maintains a constant-competitive tree at all times.

Then we consider the fully-dynamic model, where each request either adds or deletes a vertex from the current set. For this setting, we give an algorithm that again maintains a constant-competitive tree, but now the algorithm makes an amortized constant number of changes per request; i.e., for all $t$, it makes $O(t)$ changes in the first $t$ steps. Getting an algorithm with a non-amortized constant bound for this setting remains a open problem.

1.1.1 Our Techniques

In order to get an algorithm that makes only a small number of changes in each step, one natural idea would be that if a vertex is deleted, and it happens to be a high degree vertex, we just keep it around. This might be fine because there cannot be too many high degree non-alive vertices (a vertex is called non-alive if it has been deleted). And if a low degree vertex gets deleted, then we remove all edges incident to it and update the tree (this can only change the tree by a constant number of edges, proportional to the degree). Of course, this still requires arguing that the extra cost to retain the high-degree vertices and their incident edges is small. But there is a bigger issue — the above description is inherently incomplete because deletion of a low degree vertex can cause other high degree non-alive vertices in the tree to become low degree vertices. Hence, we might not be able to argue that the tree has only a small number of non-alive low-degree vertices.

Our algorithm for the deletion-only case is essentially a primal-dual one, and though we state and analyze it purely combinatorially, the primal-dual viewpoint will be useful to state the intuition. We run a primal-dual algorithm for Steiner tree at each time-step. However if a vertex is deleted, we cannot afford to remove it from the primal-dual process, because this might change the moat-growing and edge-addition in very unpredictable ways. On the other hand, we cannot afford to keep it around either: growing a dual moat around it generates “fake” duals — the dual we generate in this way is not feasible for the cut-packing dual — and hence we have to be careful if we want to charge against this
dual. First, for an amortized bound: we use the fact that if a majority of the dual moats at each point in time are “real”, we can globally charge the fake dual to these real duals. If the majority of these dual moats are “fake”, then we can afford to make drastic changes in the Steiner tree – we can get rid of the vertices in these fake dual moats, and run the primal-dual process afresh from this point onwards. The new edges added can be charged to the deletions corresponding to the vertices in these fake dual moats. Thus, we shall ensure that at any point of time, at least half of the dual moats are real.

To get a non-amortized bound, we need some more ideas. We will not be able to make sure that at all times a constant fraction of the dual moats are real. Instead we will ensure this in a global manner. We will show that if such a property does not hold, there will be many fake moats which can be removed without requiring the tree to change in too many edges.

For the fully-dynamic case, we use the same greedy algorithm as in the work of [IW91]: if there is a tree edge \(e\) and a non-tree edge \(f\) such that \(T - e + f\) is still a spanning tree, we swap \((e, f)\). To handle deletions, if there is a deleted vertex that has degree one or two in the current tree, we remove such a low-degree vertex from the tree; we keep rest of the deleted vertices in our Steiner tree. We show via a potential function argument (which extends a potential function from [GGK13]) that the number of edge changes performed in \(t\) steps is only \(O(t)\). This gives the claimed amortized constant-change constant-competitive algorithm.

### 1.2 Other Related Work

The primal-dual method has been used extensively in offline algorithm design. Its use for Steiner forest was pioneered by Agrawal, Klein, and Ravi [AKR95], and extended by Goemans and Williamson [GW95]. In online algorithm design, the use of primal-dual techniques is more recent (see, e.g., [BN07]), though some early uses of primal-dual in online algorithms are for online Steiner forest [AAB04, BC97]. Our use of primal-dual in online algorithms is of a somewhat different flavor, we show how to implement a sequence of runs of an offline primal-dual algorithm in an online fashion. Although this high-level idea also drives the algorithm in Gu et al. [GGK13], the actual details differ in important ways. Unlike the algorithm in [GGK13], an adversary can now choose to delete a high degree vertex in the tree, which forces us to keep such deleted vertices in our tree, and accounting for their cost requires many of our new technical ideas. Moreover, we now give a global charging for the primal dual process, which is different from the techniques in that previous work.

The dynamic Steiner tree was recently also studied by Lacki et al. [LOP+13], who give a constant-competitive \(O(\log \Delta)\)-amortized algorithm for the fully-dynamic case (where \(\Delta\) is the ratio of the maximum to minimum distance in the metric). They also consider the dynamic algorithms problem of maintaining a competitive Steiner tree, counting not just the number of edges changed, but also the running time required to maintain such a Steiner tree. For this problem on planar graphs, they use an combination of online algorithms and dynamic \((1+\epsilon)\)-approximate near-neighbor to give \(O(\sqrt{n} \log^5 n \log \Delta)\)-time updates for dynamic Steiner tree. Improving and extending their results remains an interesting line of research.

Our work is morally related to the work of [KLS05, KLSvZ08] who consider algorithms for Steiner forest based on a timed analysis: they give a primal-dual algorithm which also grows fake (infeasible) duals, but still show that the Steiner forest they create via this dual process is a 2-approximation.

### 2 Steiner Tree with Deletions: Notation and Definitions

In the case of only deletions, we start off with the set of vertices \(V = \{1, \ldots, n\}\) with a metric \(d(\cdot, \cdot)\) on it, and we want to maintain a Steiner tree connecting them as vertices get deleted one at a time. At each time-step \(t\), one of the vertices gets deleted. Assuming that our algorithm is oblivious to the
names of the vertices, we assume that the vertex $t$ gets deleted at time-step $t$. Hence, after the $t^{th}$
time-step we need to ensure that the vertices in $[t+1 \ldots n]$ are connected. We say that a vertex is alive
at a time $t$ if it has not been deleted in the first $t$ time-steps. So, vertices $[t+1,n]$ are alive at time $t$.
For brevity, we denote the alive vertices after time $t$ by $A_t := [t+1 \ldots n]$, and the deleted vertices by
$D_t := [t] = V \setminus A_t$.

A valid solution at time $t$ is potentially a forest $F_t = (V, E_t)$, such that there is one tree in this forest that
contains all the alive vertices, i.e., the vertices in $A_t$ are in a connected component. Hence the initial
forest $F_0$ is a spanning tree on $V$. The cost of a forest $F$ is $\text{cost}(F) := \sum_{e \in F} d(e)$, and our algorithm is
$C$-competitive if at all times $t$ it maintains a forest $F_t$ such that $\text{cost}(F_t) \leq C \cdot \text{cost}($mst$(A_t))$. We want
to give a constant competitive algorithm such that the number of changes in the forest is small. The
number of changes at time $t$ is $|F_{t-1} \triangle F_t|$, and we want the number of changes to be constant either in
an amortized sense (i.e., $\sum_{t \leq T} |F_{t-1} \triangle F_t| = O(T)$ for all $T \in [n]$), or better still, in a non-amortized
sense (i.e., $|F_{t-1} \triangle F_t| = O(1)$ for all $T \in [n]$).

Observe that if we are giving an amortized bound for the deletions-only case, it is better to delete all
trees in $F_t$ except the one containing the vertices in $A_t$—this reduces the cost and the total number of
changes in the forest increase by an additive linear term only. However, this is not necessarily the case
in the non-amortized setting, where the restrictions on the number of changes means it might make
sense to keep around some components containing only deleted vertices.

3 Deletions: The Amortized Setting

We first give an amortized algorithm for the deletions-only case, and then we build on this to give the
non-amortized algorithm in Section 4. While the algorithm does not explicitly refer to LPs and duals,
there is a clear primal-dual intuition for the amortized algorithm: when a vertex is deleted, we keep it
around as a “zombie” node, and grow duals around it. We just ensure that the number of alive nodes
growing duals at any time is at least the number of zombies growing duals. If this condition is violated,
we show how to remove a set of zombies, and only change a comparable number of tree edges.

We now describe the process formally. For each vertex $v \in V$, we have a bit $b_v \in \{0,1\}$ which says
whether $v$ is not deleted ($b_v = 1$) or deleted ($b_v = 0$). Each vertex also has a threshold $\tau_v$, which is
initially set to $\tau_{\text{max}} := \max_{u,v \in V} \lfloor \log_2 d(u,v) \rfloor$. We maintain the invariant that $b_v = 1 \implies \tau_v = \tau_{\text{max}}$.
Moreover, the threshold for a vertex is non-increasing as more vertices get deleted.

A cluster $C = (V(C), T(C))$ is a set of vertices $V(C)$, along with a spanning tree $T(C)$ joining them.

Clusters come in three flavors:

- **Alive:** Cluster $C$ is alive if it contains an undeleted vertex, i.e., at least one $v \in V(C)$ has $b_v = 1$.

- **Zombie:** $C$ is a zombie cluster at level $\ell \in \mathbb{Z}_{\geq 0}$ if it is not alive, but there is at least one vertex
  $v \in V(C)$ which has threshold $\tau_v > \ell$.

- **Dead:** $C$ is a dead cluster at level $\ell$ if it is neither alive nor a zombie: i.e., all its vertices have
  been deleted, and all of them have thresholds $\tau_v \leq \ell$.

Single vertices are trivially clusters, and hence the same definitions apply for vertices too. Hence each
deleted vertex is either a zombie or it is dead; moreover, a deleted vertex is dead at or above level $\tau_v$
and a zombie at lower levels. Since being dead or a zombie are associated with some level $\ell$, we will
talk about being $\ell$-dead or $\ell$-zombie. We also use $\ell$-non-dead to mean (alive or $\ell$-zombie).
3.1 A Hierarchical Clustering Algorithm

A clustering is a set of clusters such that their vertex sets partition the set of vertices. A hierarchical clustering $\mathcal{C}$ is a collection of clusterings $\mathcal{C}_0, \mathcal{C}_1, \ldots$, with one clustering $\mathcal{C}_\ell = (V(C_\ell), T(C_\ell))$ for each level $\ell \in \mathbb{Z}_{\geq 0}$, such that each cluster $C \in \mathcal{C}_{\ell+1}$ is the union of some clusters $C_1, C_2, \ldots, C_p \in \mathcal{C}_\ell$ — i.e., $V(C) = \bigcup_{i=1}^{p} V(C_i)$, and the edges of the tree $T(C)$ are a super-set of the edges of the trees for each of the $T(C_i)$. In other words, when we combine some $p$ clusters in $\mathcal{C}_\ell$ into $C$, we obtain the tree $T(C)$ for $C$ by connecting up the trees for each of the clusters by $p - 1$ edges. For each clustering $\mathcal{C}_i$, there is a forest associated with it, namely $\bigcup_{C \in \mathcal{C}_i} T(C)$, the union of the trees for each of its clusters: we denote this forest by $E(\mathcal{C}_i)$.

Our algorithm maintains such a hierarchical clustering. This algorithm (FormCluster) takes the information $(b_v, \tau_v)$ for each $v \in V$, and outputs a hierarchical clustering $\mathcal{C} = \{\mathcal{C}_\ell\}$. Given two clusters $C = (V(C), T(C))$ and $C' = (V(C'), T(C'))$, define merge($C, C'$) as the cluster with vertex set $V(C) \cup V(C')$ and spanning tree $T(C) \cup T(C') \cup \{e\}$, where $e$ is the edge between the closest vertices in $V(C)$ and $V(C')$. In line 4 of the algorithm, we assume some consistent ways to break ties and ambiguity. E.g., when choosing clusters $C, C'$ to merge, we choose the closest pair $C, C'$, and break ties lexicographically. Also note that $d(C, C')$ refers to the distance between the closest pair of vertices $V(C)$ and $V(C')$ respectively.

**Algorithm 3.1 FormCluster**

Require: Values $(b_v, \tau_v)$ for all $v \in V$.

1: Initialize $\mathcal{C}_0$ to be the $n$ clusters with singleton vertex sets $\{1\}, \{2\}, \ldots, \{n\}$.
2: for $\ell = 0, 1, 2, \ldots$ do
3: $\mathcal{C}_{\ell+1} \leftarrow \mathcal{C}_{\ell}$
4: while there exist $C, C' \in \mathcal{C}_{\ell+1}$ which are $\ell$-non-dead with $d(C, C') \leq 2^{\ell+1}$ do
5: $\mathcal{C}_{\ell+1} \leftarrow \mathcal{C}_{\ell+1} \cup \{\text{merge}(C, C')\} \setminus \{C, C'\}$
6: end while
7: return the hierarchical clustering $\mathcal{C} = \{\mathcal{C}_\ell\}$

There are two special levels associated with this hierarchical clustering: the level $r(\mathcal{C})$ is the lowest level such that the clustering $\mathcal{C}_{r(\mathcal{C})}$ contains a single alive cluster (though there may be other zombie and dead clusters at level $r(\mathcal{C})$). The level $s(\mathcal{C})$ is the lowest level for which the clustering $\mathcal{C}_{s(\mathcal{C})}$ contains a single non-dead cluster, all other clusters are dead at this level. Note that for $\ell \geq s(\mathcal{C})$, the clustering $\mathcal{C}_{\ell}$ is the same as $\mathcal{C}_{s(\mathcal{C})}$, and hence we can stop the algorithm after this point.

3.2 The Amortized Algorithm

The main algorithm uses the above clustering procedure. We assume that all inter-point distances are at least, say, 2. At time 0, we start off with $b_v = 1$ and $\tau_v = \tau_{\max}$ for all $v$. Let $\mathcal{C}^{(0)} \leftarrow \text{FormCluster}(\{(b_v, \tau_v)\}_{v \in V})$, and $r_0 \leftarrow r(\mathcal{C}^{(0)})$. Output the tree containing the alive vertices in the forest $E(\mathcal{C}^{(0)})$; call this $T_0$. (Observe: this tree will be a spanning tree on $V$.) Suppose we have a clustering $\mathcal{C}^{(t-1)}$, and now vertex $t$ gets deleted. We first try the lazy thing: just change the bit $b_t$ to 0, and run FormCluster to get a hierarchical clustering $\widehat{\mathcal{C}}$. Observe that for each $\ell$, the clustering $\mathcal{C}_{\ell}$ is identical to $\mathcal{C}_{\ell}^{(t-1)}$, except that possibly one cluster in $\mathcal{C}_{\ell}$ may be zombie instead of alive. For level $\ell$, if the number of zombie clusters in $\widehat{\mathcal{C}}_{\ell}$ is strictly less than the number of alive clusters, call the level good, else call it bad.

- Case I: Suppose all levels in $\widehat{\mathcal{C}}$ are good, then set $\mathcal{C}^{(t)} \leftarrow \widehat{\mathcal{C}}$ and $T_t \leftarrow T_{t-1}$ and stop.
• Case II: there are some bad levels in \( \hat{G} \). Let \( \ell^{*} \) be the lowest bad level. Let \( Z_{\ell} \) be the set of the zombie clusters in \( \hat{G}_{\ell^{*}} \), and let \( Z_{t} \) be the set of vertices in these clusters. For each vertex \( v \in Z_{t} \), we set its threshold \( \tau_{v} \) to 0. Hence all nodes in \( Z_{t} \) will be dead at all levels and never again take part in cluster formation for future timesteps. Run the FormCluster algorithm, now with these new thresholds, to get hierarchical clustering \( \hat{C}(t) \). Again, \( r_{t} \leftarrow r(\hat{C}(t)) \), and output the tree containing the alive vertices in the forest \( E(\hat{C}(t)) \); call this tree \( T_{t} \).

This completes the description of our algorithm.

3.3 The Analysis

The following facts follows directly from the algorithm above.

Fact 3.1 Suppose we are in Case II, and consider \( \ell < \ell^{*} \):

- If a cluster \( C \in \hat{G}_{\ell} \) is such that \( V(C) \cap Z_{\ell} = \emptyset \), then \( C \) is a cluster in \( \hat{C}(t) \) as well with the same status (alive/dead/zombie). Similarly, a cluster \( C \in \hat{C}(t) \) with \( V(C) \cap Z_{\ell} = \emptyset \) is also a cluster in \( \hat{C}_{\ell} \) with the same status.
- All vertices in \( Z_{t} \) appear in \( \hat{C}_{\ell} \) as singleton dead clusters.
- If \( C \in \hat{G}_{\ell} \) satisfies \( V(C) \cap Z_{t} \neq \emptyset \), then \( V(C) \subseteq Z_{t} \).

Fact 3.2 In Case II, for levels \( \ell \geq \ell^{*} \), all clusters are alive or dead; there are no zombies.

Lemma 3.3 There are no bad levels in \( \hat{C}(t) \).

Proof. If we are in Case I and set \( \hat{C}(t) = \hat{G} \), then we know \( \hat{G} \) has no bad levels. Else we are in Case II, and decrease the thresholds of some deleted nodes, and run FormCluster again. For \( \ell < \ell^{*} \), Fact 3.1 says that each alive cluster in \( \hat{C}(t) \) corresponds to an alive cluster in \( \hat{G} \), whereas the number of zombie clusters in \( \hat{C}(t) \) is no more than the number of zombie clusters in \( \hat{G} \). Since \( \ell^{*} \) was the lowest numbered bad cluster, level \( \ell < \ell^{*} \) was good in \( \hat{G} \) and hence is good in \( \hat{C}(t) \). For \( \ell \geq \ell^{*} \), the clustering \( \hat{C}(t) \) contains only alive or dead clusters by Fact 3.2, so is trivially good.

Fact 3.4 In each clustering \( \hat{C}_{\ell}(t) \), all the dead clusters are singletons. Moreover, \( r(\hat{C}(t)) = s(\hat{C}(t)) \).

Proof. When we reduce the threshold for some node, we set it to zero, which gives the first statement. For the second statement, suppose when all the alive nodes belong to the same cluster at level \( r(\hat{C}(t)) \), there is another zombie cluster. Then this level would be bad, which would contradict Lemma 3.3.

Lemma 3.5 The number of edges that need to be added or dropped in going from \( T_{t-1} \) to \( T_{t} \) is at most \( 3|Z_{t}| \).

Proof. We must be in Case II, else \( T_{1} = T_{t-1} \) and there are no edge changes. Let \( \hat{C}_{\ell^{*}}(t) \) have \( p \) non-dead clusters, which by Fact 3.2 are all alive. Let \( |Z_{t}| = q \) (recall that \( Z_{t} \) is the set of zombie clusters in \( \hat{G}_{\ell^{*}} \)). By Fact 3.3 (and the fact that \( \hat{C}(t-1) \) and \( \hat{G} \) have the same clusters, modulo some being alive in the former and zombie in the latter), the non-dead clusters in \( \hat{G}_{\ell^{*}} \) and in \( \hat{C}_{\ell^{*}}(t-1) \) are precisely these \( p + q \) clusters. By the definition of \( \ell^{*} \) being a bad level, \( q \geq p \). The number of edges that have changed in going from \( T_{t-1} \) to \( T_{t} \) are:

(a) Those edges within clusters of \( Z_{t} \) are gone; there are exactly \( |Z_{t}| - |Z_{t}| = |Z_{t}| - q \) of these.
(b) The \( p + q - 1 \) edges that connect up the clusters in \( \hat{C}_{\ell^{*}}(t-1) \) have potentially been dropped.
(c) We add in \( p - 1 \) new edges to connect up the \( p \) alive clusters in \( \hat{C}_{\ell^{*}}(t) \).

So the total number of edge changes is

\[ |Z_{t}| - q + (p + q - 1) + p = |Z_{t}| + 2p - 1 < |Z_{t}| + 2q. \]

Finally, note that \( q = |Z_{t}| \leq |Z_{t}| \), so this is less than \( 3|Z_{t}| \).
The above lemma shows that our algorithm makes constant number of changes in the tree in an amortized sense. Indeed, the set of vertices in \( Z_t \) are disjoint for different values of \( t \) – once a node enters the set \( Z_t \), it cannot belong to a zombie cluster in subsequent timesteps.

**Fact 3.6** Any two non-dead clusters in \( \mathcal{C}_t^{(t)} \) are at distance more than \( 2^\ell \) from each other.

**Lemma 3.7** For any \( t \), the cost of \( T_t \) output by the algorithm is within \( O(1) \) of the optimal Steiner tree on the alive nodes \([t + 1, n]\).

**Proof.** Let \( \kappa_{t,\ell} \) be the number of non-dead clusters in \( \mathcal{C}_{t}^{(t)} \). Observe that \( \kappa_{t,\ell} > 1 \) for all \( \ell < r(\mathcal{C}_t^{(t)}) \) and \( \kappa_{t,\ell} = 1 \) for all other \( \ell \). Since all levels in \( \mathcal{C}_{t}^{(t)} \) are good (by Lemma 3.3), we know that \( \lceil \kappa_{t,\ell}/2 \rceil \) clusters at level \( \ell \) are alive clusters. And by Fact 3.6 all these are at distance at least \( 2^\ell \) from each other. A standard dual packing gives a lower bound on the cost of the optimal Steiner tree of

\[
\sum_{\ell \geq 0} \left( \left\lceil \frac{\kappa_{t,\ell}}{2} \right\rceil - 1 \right) \cdot 2^{\ell-2} \geq \frac{1}{2} \sum_{\ell=0}^{r(\mathcal{C}_t^{(t)})-1} \left( \frac{\kappa_{t,\ell}}{4} \right) \cdot 2^{\ell-2}. \tag{3.1}
\]

Let \( n_{t,\ell} \) be the number of edges added in forming \( \mathcal{C}_t^{(t)} \) from \( \mathcal{C}_{t-1}^{(t)} \). Hence the cost of the tree \( T_t \) is at most

\[
\sum_{\ell \geq 1} n_{t,\ell} \cdot 2^\ell.
\]

Moreover, \( n_{t,\ell} = \kappa_{t,\ell-1} - \kappa_{t,\ell} \), since the number of edges added is exactly the reduction in the number of components, so the cost of the tree \( T_t \) is at most

\[
\sum_{\ell \geq 1} \left( \kappa_{t,\ell-1} - \kappa_{t,\ell} \right) \cdot 2^\ell \leq 2\kappa_{t,0} + \sum_{\ell=1}^{r(\mathcal{C}_t^{(t)})} \kappa_{t,\ell} 2^\ell.
\]

This is at most a constant times the lower bound (3.1), which proves the result. \( \blacksquare \)

A constant-competitive constant-amortized algorithm for the deletions-only case can be inferred from the techniques of Imase and Waxman [IW91], so the result is not surprising. But the above algorithm can be extended to the non-amortized setting, as we show next.

### 4 Deletions: The Non-Amortized Setting

We now describe our algorithm in the non-amortized setting. Our algorithm is a direct extension of the one above, so let us think about why we get a large number of changes. This could happen because of two reasons. Firstly, if there were some deletion that caused a large zombie cluster to be marked dead, we would remove all the edges within the tree connecting this cluster and hence make a large number of changes. The main observation is that since we could pay for all the edges within this tree in the previous step, we should be also able to pay for most of them at the next step, and it should suffice to remove a constant number of edges. To do this, we will not just set the thresholds to \( \tau_{\text{max}} \) or 0, but will slowly lower them.

Secondly, we happened to mark a small zombie cluster dead, but it was being used to connect many other clusters. We get around this problem by marking only those zombie clusters dead which would be used for connecting a small number of clusters in subsequent steps – we show that it is always possible to find such zombie clusters.
### 4.1 A Modified Cluster-Formation Algorithm

The first change from the previous algorithm is that we want the tree $T_t$ to be similar to $T_{t-1}$. So we explicitly ensure this by being as similar to an “old” clustering $\mathcal{C}'$ given as input; the algorithm is otherwise very similar to Algorithm FormCluster, and we assume the reader is familiar with that section. Again, let $E(\mathcal{C}_t)$ be the edges contained in the forest corresponding to a clustering $\mathcal{C}_t$.

**Algorithm 4.1 FormClusterNew**

**Require:** Values $(b_v, \tau_v)$ for all $v \in V$, old hierarchical clustering $\mathcal{C}'$.

**Ensure:** A hierarchical clustering $\mathcal{C} = \{\mathcal{C}_t\}$.

1. Initialize $\mathcal{C}_0$ to be the $n$ clusters with singleton vertex sets $\{1\}, \{2\}, \ldots, \{n\}$.
2. for $j = 0, 1, 2, \ldots$ do
   3. $\mathcal{C}_{j+1} \leftarrow \mathcal{C}_j$
   4. while there exists edge $e \in E(\mathcal{C}'_{j+1})$ between
      5. $j$-non-dead clusters $C, C' \in \mathcal{C}'_{j+1}$ do
      6. Define a new cluster $C''$ with
         7. $V(C'') = V(C) \cup V(C')$
         8. and $T(C'') = T(C) \cup T(C') \cup \{e\}$.
      9. $\mathcal{C}_{j+1} \leftarrow \mathcal{C}_{j+1} \cup \{C, C'\}$ \(\triangleright\) the cluster $C''$ is $j$-non-dead
   10. while there exist $j$-non-dead clusters $C, C' \in \mathcal{C}_j$
      11. with $d(C, C') \leq 2^j$ do
         12. $\mathcal{C}_{j+1} \leftarrow \mathcal{C}_{j+1} \cup \{\text{merge}(C, C')\} \setminus \{C, C'\}$ \(\triangleright\) again, merge$(C, C')$ is $j$-non-dead
      13. return the new hierarchical clustering $\mathcal{C} = \{\mathcal{C}_t\}$

Again there are two special levels: level $r(\mathcal{C})$ is the lowest level such that $\mathcal{C}_{r(\mathcal{C})}$ contains a single alive cluster, and level $s(\mathcal{C})$ is the lowest level where $\mathcal{C}_{s(\mathcal{C})}$ contains a single non-dead cluster, all other clusters are dead at this level.

### 4.2 The Non-Amortized Algorithm

Again, assume that all inter-point distances are at least 2. At time 0, start off with $b_v = 1$ and $\tau_v = \tau_{\max}$ for all $v$. Let $\mathcal{C}^{(0)} \leftarrow \text{FormCluster}(\{(b_v, \tau_v)\}_{v \in V})$, and $r_0 \leftarrow r(\mathcal{C}^{(0)})$. Output the tree containing the alive vertices in the forest $E(\mathcal{C}_{r_0}^{(0)})$; call this $T_0$.

Consider the clustering $\mathcal{C}^{(t-1)}$ corresponding to thresholds $\tau^{(t-1)}$ and bits $b^{(t-1)}$. Now vertex $t$ is deleted, so set $b_t^{(t)} = 0$, and $b_j^{(t)} = b_j^{(t-1)}$ otherwise, and run FormClusterNew$(\tau^{(t-1)}, b^{(t)})$ to get a new hierarchical clustering $\mathcal{C}$. As before, $\mathcal{C}^{(t-1)}$ and $\hat{\mathcal{C}}$ will be identical at all levels, except for perhaps one cluster at each level being alive in the former and zombie in the latter. The algorithm FormClusterNew is described above.

The definition of a level being good is slightly different now. We first develop some more notation. Denote the edges added at level $\ell$ of any clustering $\mathcal{C}$ as $E_\ell(\mathcal{C})$; i.e., $E_\ell(\mathcal{C}) := E(\mathcal{C}_\ell) \setminus E(\mathcal{C}_{\ell-1})$. Let $m_\ell(\mathcal{C}) := |E_\ell(\mathcal{C})|$ be the cardinality of this set, and $m_{\ell+1}(\mathcal{C}) := \sum_{j > \ell} m_j(\mathcal{C})$ denote the edges added at levels (strictly) above level $\ell$, up to and including level $s(\mathcal{C})$. (Note this includes edges above level $r(\mathcal{C})$; of course no edges are added above level $s(\mathcal{C})$.) Finally, let $\#\text{alive}(\mathcal{C}_\ell)$ denote the number of alive clusters in $\mathcal{C}_\ell$. 


A level $\ell$ of a hierarchical clustering $\mathcal{C}$ is good if
\[
m_{>\ell}(\mathcal{C}) \leq 3 \# \text{alive}(\mathcal{C}_\ell);
\] (4.2)
i.e., if the number of edges added above level $\ell$ is at most three times the number of alive components at level $\ell$. Note that if the number of alive components were more than a third of the number of non-dead components, the level would be good. But since we will now use the thresholds in a more nuanced way, a level may be good even if almost all the non-dead components are zombies.

One final definition: given a hierarchical clustering $\mathcal{C}$ and a level $\ell$, the level-$\ell$ skeleton $\mathcal{G}_\ell(\mathcal{C})$ is an undirected graph defined as follows. The vertex set is the set of clusters of $\mathcal{C}_\ell$. There is an edge in $\mathcal{G}_\ell(\mathcal{C})$ connecting two clusters $C, C' \in \mathcal{C}_\ell$ precisely when there is an edge between $C, C'$ in the set $\cup_{\ell'<\ell} E_{\ell'}(\mathcal{C})$. In other words, if we were to take the clustering $\mathcal{C}_s(\mathcal{C})$ and collapse each cluster $C \in \mathcal{C}_\ell$ into a single node, we would get $\mathcal{G}_\ell(\mathcal{C})$. By our construction, the skeleton is always a forest, and contains $m_{>\ell}(\mathcal{C})$ edges. The degree of a cluster $C \in \mathcal{C}_\ell$ is the degree of the corresponding node in $\mathcal{G}_\ell(\mathcal{C})$.

Now we are ready to state the algorithm. Recall that we constructed $\hat{\mathcal{C}} \leftarrow \text{FormClusterNew}(\tau^{(t-1)}, b^{(t)})$.

Again, there are two cases:

- **Case I:** Suppose all levels in $\hat{\mathcal{C}}$ are good, or for every bad level $\ell$ we have $m_{>\ell}(\hat{\mathcal{C}}) < 36$. Then set $\mathcal{C}^{(t)} \leftarrow \hat{\mathcal{C}}$.
- **Case II:** There exists a bad level $\ell$ such that $m_{>\ell}(\hat{\mathcal{C}}) \geq 36$. Let $\ell^*$ be the highest such level, and consider the skeleton $\mathcal{G}_{\ell^*}(\hat{\mathcal{C}})$. Choose a set $Z_t$ of 6 zombie clusters from $\mathcal{G}_{\ell^*}$ that have degree 1 or 2 in $\mathcal{G}_{\ell^*}(\hat{\mathcal{C}})$ \footnote{Such a set exists by Fact 4.3, since $\mathcal{G}_{\ell^*}(\hat{\mathcal{C}})$ is a forest of at least 36 edges, if we choose $A$ to be the set of alive clusters, Fact 4.2 implies there must be 6 non-alive clusters of degree one or two. Since these have non-zero degree in $\mathcal{G}_{\ell^*}$, they cannot be dead at this level and must be zombies.}. Let $Z_t$ be the vertices in these clusters. For each $v \in Z_t$, set $\tau^{(t)}_v \leftarrow \min(\tau_v^{(t-1)}, \ell^*)$. Observe that these nodes now form singleton dead clusters at level $\ell^*$ according to the new thresholds $\tau^{(t)}$, whereas at least one of them in each cluster must have had threshold above $\ell^*$, i.e., $\forall C \in Z_t, \exists v \in V(C) : \tau_v^{(t-1)} > \ell^* = \tau_v^{(t)}$. This is true because these clusters are non-dead.

So we’re making progress in terms of strictly decreasing the threshold for some vertices. Now set $\mathcal{C}^{(t)} \leftarrow \text{FormClusterNew}(\tau^{(t)}, b^{(t)})$.

In either case, let $r_t \leftarrow r(\mathcal{C}^{(t)})$ be the lowest level with a single alive cluster, and return the forest corresponding to this level — i.e., $F_t \leftarrow E(\mathcal{G}_r^{(t)})$.

For future convenience, define $s_t \leftarrow s(\mathcal{C}^{(t)})$ to be the lowest level with a single non-dead cluster, so all the edges in $\mathcal{G}_s^{(t)}$ belong to $E(\mathcal{G}_s^{(t)})$. In Case I, $r_t$ may be much smaller than $r_{t-1}$ but $s_t = s_{t-1}$. Moreover, let $F'_t$ be the edges added at all levels of the algorithm — so $F'_t \setminus F_t$ are the edges added in levels $r_t + 1, \ldots, s_t$.

Before we begin the analysis, notice the differences between this algorithm and the previous amortized one: previously, badness meant the number of zombies was more than the alive clusters at that level, now badness means the number of edges being added above the level is much more than the number of alive clusters in that level. Previously, we chose all the zombie clusters at the lowest bad level and made their nodes dead right at level 0. Now we choose the highest bad level $\ell^*$ and carefully choose some six zombie clusters, and make their nodes dead only at level $\ell^*$ — this will ensure that only a small number of edges will change between timesteps $t$ and $t + 1$.

### 4.3 The Analysis

At a high level, the analysis will proceed analogously to Section 3.3, but the details are more interesting. If we are in Case I, things are simple. Indeed, for each $\ell$, the clusterings $\mathcal{C}_\ell^{(t)}$ and $\mathcal{C}_\ell^{(t-1)}$ have the same
clusters, apart from potentially one cluster being zombie in the former and alive in the latter. Since bad levels, if any, satisfy \(m_{>\ell}(C^{(t)}) < 36\), we get that for every level \(\ell\), we have \(m_{>\ell}(C^{(t)}) < 3\#\text{alive}(C^{(t)}) + 36\) if \(C^{(t)}\) was produced from Case I.

The bulk of the work will be to show an analogous inequality for Case II. Here, we first show that the structure of the clusterings at times \(t - 1\) and \(t\) differ only in a controlled fashion. In fact, we show that a large number of “safe” edges will be common to \(F_t\) and \(F_{t-1}\). This will allow us to show that all levels in the clustering we output may not be good, we still have \(m_{>\ell}(C^{(t)}) < 3\#\text{alive}(C^{(t)}) + O(1)\) in this case.

And why is such a bound useful? If there is a single living cluster at some level, only a constant number of edges are added above this level, and dropping them will change only a constant number of edges. On the other hand, if more than one cluster is alive, we are able to pay for the edges added at this level. Putting all this together will ensure that \(|F_t \Delta F_{t-1}|\) is bounded, and \(F_t\) is constant competitive.

### 4.3.1 The Structure of Clusters

**Lemma 4.1** The clusters in \(C^{(t)}\) are a refinement of the clusters in \(C^{(t-1)}\): for each cluster \(C \in C^{(t-1)}, V(C)\) is the union of vertex sets \(V(C_1), \ldots, V(C_p)\) corresponding to some clusters \(C_1, C_2, \ldots, C_p \in C^{(t)}\).

Suppose we are in Case II. Then each of the clusters in \(Z_t\) belong to \(C^{(t)}\) for all \(\ell \geq \ell^*\), and they are dead clusters at these levels. Moreover, if cluster \(C \in C^{(t-1)}\) is non-dead, then each of the corresponding \(C_i \in C^{(t)}\) are either in \(Z_t\) or are non-dead.

**Proof.** The lemma is immediate in Case I, so assume we are in Case II. We prove the lemma by induction on \(\ell\). If \(\ell \leq \ell^*\), then since we only reduced the thresholds of some nodes down to \(\ell^*\) and changed \(b_\ell = 0\), clusters that were non-dead according to \((\tau^{(t-1)}_\ell, b^{(t-1)}_\ell)\) are also non-dead according to \((\tau^{(t)}_\ell, b^{(t)}_\ell)\), and there is no change in the actions of \*FormClusterNew\* for these “low” levels. For such levels \(\ell \leq \ell^*\), there is a bijection between clusters in \(C^{(t-1)}\) and \(C^{(t)}\).

Now consider level \(\ell > \ell^*\), and assume the first (refinement) claim is true for level \(\ell - 1\). Now suppose we add an edge between the spanning trees \(T(C_1)\) and \(T(C_2)\) of two clusters \(C_1, C_2\) respectively in \(C^{(t-1)}\) (to form a cluster at level \(\ell\)), and hence \(d(C_1, C_2) \leq 2\ell\). By the inductive hypothesis, \(V(C_1)\) is contained in \(V(C'_1), C'_1 \in C^{(t-1)}\) and \(V(C_2)\) is contained in \(V(C'_2), C'_2 \in C^{(t-1)}\), and so \(d(C'_1, C'_2) \leq 2\ell\). Hence, \(C'_1\) and \(C'_2\) will become part of the same cluster in \(C^{(t-1)}\). This proves the refinement claim for level \(\ell\).

For the second statement, the clusters in \(Z_t \subseteq C^{(t-1)}\) all exist in \(C^{(t)}\) (by the above claim about a bijection for levels \(\ell \leq \ell^*\)), and they are all \(\ell^*\)-dead in \(C^{(t)}\) (by construction of \(\tau^{(t)}\)), so they will remain in all subsequent levels \(\ell \geq \ell^*\).

For the last statement, for \(\ell \geq \ell^*\), suppose \(C \in C^{(t-1)}\) corresponds to \(C_1, C_2, \ldots, C_p \in C^{(t)}\), and \(C_i\) is dead. The only possibilities for \(C_i\) are (a) it was already dead in \(C^{(t-1)}\), in which case \(C = C_i\) and \(C\) will be dead as well, which is a contradiction, or (b) \(C_i\) is one of the clusters in \(Z_t\).

### 4.3.2 Safe Edges

The results of this section are interesting only when we are in Case II, and \(\ell^*\) is defined. Consider the skeleton \(G^*_t(C^{(t-1)})\); recall that the clusters in \(Z_t\) correspond to degree-one or degree-two nodes in this graph. We define a set of \*boundary clusters* \(B_t\) to be those clusters in \(G^*_t(C^{(t-1)})\) that are not in \(Z_t\) but have at least one cluster in \(Z_t\) as a neighbor. Hence \(|B_t| \leq 2|Z_t|\).

An edge \(e \in E(G^{(t-1)}\_m)\) (which is the set of all edges added in the hierarchical clustering \(G^{(t-1)}\)) is called \*safe\* if either (a) \(e\) belongs to \(E(G^{(t-1)}\_m)\), i.e., it was added at level \(\ell \leq \ell^*\), or (b) at least one endpoint of \(e\) belongs to a cluster not in \(Z_t \cup B_t\). In other
words, an edge is unsafe if and only if it belongs to $\mathcal{G}_t^\star(\mathcal{C}^{(t-1)})$ and both endpoints fall in clusters in $Z_t \cup \mathcal{B}_t$.

**Fact 4.2** At most $3|Z_t|-1$ edges are unsafe.

**Proof.** The edges in $\mathcal{G}_t^\star(\mathcal{C}^{(t-1)})$ form a forest, and an unsafe edge is a subset of these edges that has both endpoints in $Z_t \cup \mathcal{B}_t$. So there are at most $|Z_t| + 2|Z_t| - 1$ unsafe edges. Moreover, each cluster in $Z_t$ has at most two neighbors, so $|\mathcal{B}_t| \leq 2|Z_t|$, which proves the claim. 

**Lemma 4.3** If $e \in E_\ell(\mathcal{C}^{(t-1)})$ is a safe edge, then $e \in E_\ell(\mathcal{C}^{(t)})$. In other words, every safe edge added at level $\ell$ at time $t-1$ is added at level $\ell$ at time $t$.

**Proof.** For $\ell \leq \ell^\star$, this follows because the algorithms at time $t-1$ and time $t$ behave the same until level $\ell^\star$: every edge is safe, and is added at the same time. For $\ell > \ell^\star$, consider a safe edge $e = (x, y) \in E_\ell(\mathcal{C}^{(t-1)})$ going between clusters $C_{t-1}, C'_{t-1} \in \mathcal{C}^{(t-1)}$. Let $C_t, C'_t$ be the clusters in $\mathcal{C}^{(t)}$ containing $x$ and $y$ respectively. Lemma 4.2 implies that $C_t \subseteq C_{t-1}$ and $C'_t \subseteq C'_{t-1}$.

First, observe that $x, y \notin Z_t$ (where $Z_t$ is the set of vertices lying in the clusters of $Z_t$). Indeed, if $x \in Z_t$, then the cluster in $\mathcal{C}^{(t-1)}$ containing $x$ at level $\ell^\star$ would belong to $Z_t$, and then $y \in Z_t \cup \mathcal{B}_t$, since $\mathcal{B}_t$ contains all the neighboring clusters of $Z_t$ in $\mathcal{C}^{(t-1)}$. This contradicts $(x, y)$ being safe.

We now claim that both $C_t$ and $C'_t$ are non-dead in $\mathcal{C}^{(t)}$. Suppose $C_t$ was dead. Note that $C_t \notin Z_t$ because $x \notin Z_t$. Therefore, the second part of Lemma 4.2 implies that $C_{t-1}$ would be dead in $\mathcal{C}^{(t-1)}$. But then the edge $(x, y)$ would not be added, a contradiction. A similar argument shows that $C'_t$ is not dead. Moreover, since the clustering at time $t$ is a refinement of that at time $t-1$ (again by Lemma 4.2), adding the edge $e$ to $E_\ell(\mathcal{C}^{(t-1)})$ will not create a cycle. Hence, we will add $e$ to $E_\ell(\mathcal{C}^{(t)})$. 

To summarize, there are very few edges that are unsafe (Fact 4.2), and safe edges are added at the same level at timestep $t$ as at timestep $t-1$. This will be useful to show that the edge set in consecutive steps remains pretty similar.

### 4.3.3 Bounding the Changes

Let us define some syntactic sugar. Let the number of alive clusters in $\mathcal{C}^{(t)}$ be denoted $a_{t, \ell}$ instead of $\#\text{alive}(\mathcal{C}^{(t)})$. Let the number of edges added at levels above $\ell$ in $\mathcal{C}^{(t)}$ be denoted by $m_{t, > \ell}$ instead of $m_{t, > \ell}(\mathcal{C}^{(t)})$.

**Lemma 4.4** For all levels $\ell$, $a_{t, \ell} \geq a_{t-1, \ell} - 1$.

**Proof.** The clustering $\mathcal{C}^{(t)}$ is a refinement of the clustering $\mathcal{C}^{(t-1)}$, so each alive cluster in $\mathcal{C}^{(t-1)}$ gives rise to at least one alive cluster in $\mathcal{C}^{(t)}$ — except for the cluster containing vertex $t$, which might become a zombie at time $t$, and accounts for the subtraction of one. 

**Lemma 4.5** The difference in the total number of edges added at timesteps $t-1$ and $t$ is

$$|F_{t-1}^{\ell'}| - |F_t^{\ell'}| \geq |Z_t|/2 = 3. \quad (4.3)$$

Moreover:

$$m_{t, > \ell} \leq m_{t-1, > \ell} - |Z_t|/2 \leq m_{t-1, > \ell} - 3 \quad \forall \ell \leq \ell^\star \quad (4.4)$$

$$m_{t, > \ell} \leq m_{t-1, > \ell} + 3|Z_t| \leq m_{t-1, > \ell} + 18 \quad \forall \ell > \ell^\star \quad (4.5)$$
Proof. Pick level \( \ell_M = \max(s_{t-1}, s_t) \). We claim the difference in the number of clusters at level \( \ell_M \) is
\[
|\mathcal{C}_{\ell_M}^{(t)}| - |\mathcal{C}_{\ell_M}^{(t-1)}| \geq |\mathcal{Z}_t|/2. \tag{4.6}
\]
To see this, observe that the vertex set in each cluster in \( \mathcal{C}_{\ell_M}^{(t)} \) is union of the vertex sets of some clusters in \( \mathcal{C}_{\ell_M}^{(t-1)} \) by Lemma 4.1, so the difference above is definitely non-negative. Moreover, each of the clusters in \( \mathcal{Z}_t \) forms an isolated cluster in \( \mathcal{C}_{\ell_M}^{(t-1)} \), but it used to have positive degree in \( \mathcal{C}_{\ell_M}^{(t-1)} \). The extreme case is when these clusters induce a matching, but that still increases the number of clusters by \( |\mathcal{Z}_t|/2 \). This proves (4.6).

For any level \( \ell \), the quantity \( m_{t-1, > \ell} = m_{ > \ell}(\mathcal{C}_t^{(t-1)}) \) is the number of edges added above level \( \ell \), which is equal to the reduction in the number of clusters above this level. Hence \( m_{t-1, > \ell} = |\mathcal{C}_t^{(t-1)}| - |\mathcal{C}_t^{(t-1)}| \). Similarly \( m_{t, > \ell} = |\mathcal{C}_t^{(t)}| - |\mathcal{C}_t^{(t)}| \). Since \( |\mathcal{C}_t^{(t-1)}| = |\mathcal{C}_t^{(t)}| \) for \( \ell \leq \ell^* \), we have
\[
m_{t-1, > \ell} - m_{t, > \ell} = (|\mathcal{C}_t^{(t-1)}| - |\mathcal{C}_t^{(t-1)}|) - (|\mathcal{C}_t^{(t-1)}| - |\mathcal{C}_t^{(t-1)}|) \geq |\mathcal{Z}_t|/2,
\]
the last from (4.6). This proves (4.4). Also, \( |F_{t-1}^{(t-1)}| - |F_t^{(t-1)}| = m_{t-1, \geq 0} - m_{t, > \ell} - m_{t, > \ell} \geq |\mathcal{Z}_t|/2 \), and so (4.3) also follows.

For a level \( \ell \geq \ell^* \), all the safe edges at \( \ell \) and lower levels in time \( t - 1 \) are added at the corresponding level in time \( t \) as well (Lemma 4.3). To maximize the difference, it can only be the case that all the unsafe edges (of which there are at most \( 3|\mathcal{Z}_t| \)) might not have been added yet. This proves (4.7).

Plugging in \( |\mathcal{Z}_t| = 6 \) gives the numerical values.

4.3.4 The Key Invariant

We now prove the key invariant. In the amortized case, we could prove that for each hierarchical clustering \( \mathcal{C}_t^{(t)} \), all levels were good. In the non-amortized case, this will not be true. However, we will show a slightly weaker invariant. Recall the notion of goodness (4.2): for any clustering \( \mathcal{C} \), level \( \ell \) is good if \( m_{ > \ell}(\mathcal{C}) \leq 3 \#\text{ alive}(\mathcal{C}_\ell) \). Using our shorthand, a good level \( \ell \) for timestep \( t \) means \( m_{t, > \ell} \leq 3a_{t, \ell} \).

What about bad levels?

Lemma 4.6 (Invariant) For all timesteps \( t \), if level \( \ell \) is bad for \( \mathcal{C}_t^{(t)} \), and \( m_{t, > \ell} \geq 36 \). Then
\[
m_{t, > \ell} \leq 3a_{t, \ell} + 54. \tag{4.7}
\]

Proof. We prove this by induction on \( t \). Initially, at time \( t = 0 \), all vertices are alive. For any level \( \ell \), the number of edges added above that level can be at most the number of components at that level. Thus \( m_{0, > \ell} \leq a_{0, \ell} - 1 \). This means all levels are good, and the invariant is vacuously true.

Suppose (4.7) holds true at some time \( t - 1 \) for all bad levels \( \ell \). We need to show that (4.7) holds at time \( t \) for all bad levels \( \ell \) as well. If we were in Case I, then we know that \( m_{t, > \ell} \leq 3a_{t, \ell} + 36 \) (since either all levels of \( \mathcal{C}_t \) were good, or they had \( m_{ > \ell}(\mathcal{C}_t) < 36 \).

Hence we need to consider when we get to time \( t \) using Case II. Let \( \ell^* \) be as defined by the algorithm — the highest bad level \( \ell \) in the intermediate hierarchical clustering \( \mathcal{C}_t \) with \( m_{ > \ell}(\mathcal{C}_t) = m_{t-1, > \ell} \geq 36 \).

Now take \( \mathcal{C}_t^{(t)} \), and first consider a bad level for some \( \ell \leq \ell^* \). There are several cases.

- Suppose \( \ell \) was a good level in \( \mathcal{C}_t^{(t-1)} \): by definition of goodness, \( m_{t-1, > \ell} \leq 3a_{t-1, \ell} \). Therefore,
\[
m_{t, > \ell} \leq m_{t-1, > \ell} - 3 \leq 3a_{t-1, \ell} - 3 \leq 3(a_{t, \ell} + 3) = 3a_{t, \ell} + 54.
\]

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• Suppose \( \ell \) was a bad level in \( C(t-1) \), but \( m_{t-1,>\ell} < 36 \): in this case,

\[
m_{t,>\ell} \leq m_{t-1,>\ell} - 3 < 36 - 3,
\]

and so the invariant holds trivially.

• Finally, suppose \( \ell \) was a bad level in time \( t-1 \), and \( m_{t-1,>\ell} \geq 36 \): we can now apply the invariant at time \( t-1 \) to this level \( \ell \). So, we get

\[
m_{t,>\ell} \leq m_{t-1,>\ell} - 3 \leq 3a_{t-1,\ell} + 54 - 3 \leq 3(a_{t,\ell} + 1) + 54 - 3 \leq 3a_{t,\ell} + 54.
\]

The other case to consider is when the bad level \( \ell \) at time \( t \) satisfies \( \ell > \ell^* \). We claim that such a level \( \ell \) at time \( t-1 \) must have either been good, or satisfies \( m_{t-1,>\ell} < 36 \). Indeed, by the choice of \( \ell^* \), if \( m_{t-1,>\ell} \geq 36 \) we must have had \( m_{t-1,>\ell} \leq 3\#\text{alive}(C_{\ell}) \leq 3a_{t-1,\ell} \), and hence would be good. Hence we just have to consider these two cases.

• Suppose \( \ell \) was good at time \( t-1 \), i.e., \( m_{t-1,>\ell} \leq 3a_{t-1,\ell} \). Then

\[
m_{t,>\ell} \leq m_{t-1,>\ell} + 18 \leq 3a_{t-1,\ell} + 18 \leq 3(a_{t,\ell} + 1) + 18 < 3a_{t,\ell} + 54.
\]

• \( m_{t-1,>\ell} < 36 \): in this case,

\[
m_{t,>\ell} \leq m_{t-1,>\ell} + 18 < 36 + 18 = 54 \leq 3a_{t,\ell} + 54.
\]

This completes the proof of the invariant.

To recap, the invariant says that for bad levels, the number of edges added to \( F_t' \) above that level is at most thrice the number of active components plus an additive constant. This is contrast to good levels, where the additive constant is missing.

### 4.3.5 The Final Accounting

**Lemma 4.7 (Lipschitz)** The number of edges in \( F_{t-1} \Delta F_t \) is at most O(1).

**Proof.** Recall the difference between \( F_t \) and \( F_t' \) is that the latter contains edges added after there is a single alive cluster, and until there is a single non-dead cluster. In particular, the difference \( |F_t' \setminus F_t| = m_{t,>rt} \). By the invariant, since \( a_{t,rt} = 1 \), this difference is at most 55. Moreover,

\[
|F_{t-1} \Delta F_t| \leq |F_{t-1}' \Delta F_t'| + |F_t' \setminus F_t| + |F_{t-1}' \setminus F_{t-1}| \leq |F_{t-1}' \Delta F_t'| + 110.
\]

By the refinement property (Lemma 4.1) we know that \( |F_t'| \leq |F_t'_{t-1}| \). And Lemma 4.3 and Fact 4.2 show that \( |F'_{t-1} \setminus F_t'| \) is at most \( 3|Z_t| - 1 = 17 \). Hence,

\[
|F_{t-1}' \Delta F_t'| = |F_{t-1}' \setminus F_t'| + |F_t' \setminus F_{t-1}'| \leq 2|F_{t-1}' \setminus F_t'| \leq 34.
\]

This proves the Lipschitz property.

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Theorem 4.8 For any time \( t \), the cost of \( F_t \) is at most \( O(1) \) times the optimal Steiner tree cost on the non-deleted nodes \([t + 1, n]\).

Proof. The proof is very similar to that of Lemma 3.7. The lower bound on the Steiner tree is again at least

\[
\sum_{\ell} 2^{\ell-2} \cdot a_{t,\ell} \cdot 1(a_{t,\ell} \geq 2) = \sum_{\ell=1}^{r_t-1} 2^{\ell-2} \cdot a_{t,\ell}.
\]

The edges of the forest \( F_t \) output by our algorithm are added in levels \( \ell \in \{1, \ldots, r_t\} \), and have total cost at most \( \sum_{\ell=1}^{r_t} 2^\ell \cdot m_{t,\ell} \). By Lemma 4.4, these quantities are within a constant factor of each other, which completes the proof.

4.4 Elementary Fact

Finally, one elementary fact, capturing that every forest must have a large number of low-degree vertices.

Fact 4.9 Suppose we are given a forest \( F \) with at least 36 edges on some set \( V \) of vertices, where \( V \) is partitioned into sets \( A \) and \( B \). If the number of edges in \( F \) is more than \( 3|A| \), then there must exist a set \( S \subseteq B \) of 6 nodes where the degrees of nodes in \( S \) are either one or two.

Proof. Let \( V(F) \) denote the nodes in \( F \) that have degree at least 1. Consider the set \( L \subseteq V(F) \) of the “low” degree nodes, i.e., the degree 1 or degree-2 nodes in \( F \). At least half the nodes in \( V(F) \) must lie in \( \ell \). (Indeed, all nodes in \( V(F) \) - \( L \) contribute degree at least 3, and the nodes in \( \ell \) contribute degree at least 1, and the average degree of nodes in \( V(F) \) is strictly less than 2 since it is a forest.) So \( |L| \geq |V(F)|/2 \geq |E(F)|/2 \). Since \( |A| \leq |E(F)|/3 \), we have that \( |L \setminus A| \geq |E(F)|/6 \geq 6 \); these are chosen to be in \( S \).

5 The Fully Dynamic Case

We now consider the fully-dynamic case, where the input sequence has both additions and deletions. Hence each request \( \sigma_t \) is either \((\text{add}, t)\), or \((\text{del}, t')\) for some \( t' < t \). We assume that each vertex that is added is a “new” vertex, and hence has a new index. Moreover, this means there is no point to deleting vertices multiple times, each vertex can be assumed to be deleted at most once.

In the fully-dynamic case, observe that the process can go on indefinitely and the metric can be arbitrarily large, so \( n \) will denote some arbitrary instance in time, instead of denoting the size of the metric as in the previous section. Let \( V_n = \{ t \in [n] \mid \sigma_t = (\text{add}, t) \} \) be the set of vertices that have appeared until time \( n \). Let \( D_n = \{ s \in [n] \mid \exists t \in [n] \text{ s.t. } \sigma_t = (\text{del}, s) \} \) be the vertices that have been deleted until time \( n \); since each vertex is deleted at most once, this is well-defined. Let \( A_n = V_n \setminus D_n \) be the “alive” vertices at time \( n \).

We will assume that the inter-point distances are specified in the following particular manner — this will be convenient for us in the following analysis. (In Section 3.1, we argue this does not change the problem by more than a constant factor.) Let \( d_t(\cdot, \cdot) \) be the distances between the vertices in \( V_t \). If we see \((\text{add}, t + 1)\), we are given the distances from \( t + 1 \) to all vertices in \( A_t \subseteq V_t \): i.e., to only the alive vertices. The guarantee we have is that the newly given distances form a metric along with the old distances, and hence do not violate the triangle inequality. The distances from \( t + 1 \) to vertices in \( D_t \) must be inferred using the triangle inequality: \( d_{t+1}(t + 1, s) = \min_{s' \in A_t} (d(t + 1, s') + d_t(s', s)) \). Note that the former summand is a new distance given as input, the second summand is inductively defined.

Let \( T_t \) be the tree at time \( t \), and \( V(T_t) \) denote the set of vertices in it. The following lemma is immediate:

Lemma 5.1 The distances \( d_t \) satisfy the following properties:
(a) The closest distance from \( t + 1 \) to vertices in \( V(T_t) \) is to some alive vertex; i.e. some vertex in \( A_t \).

(b) The metric \( d_{t+1} \) restricted to \( V_t \) is the same as \( d_t \).

The tree \( T_t \) at some time \( t \) is valid if it uses any vertices in \( V_t \), whether they are alive or dead, but it contains all the alive vertices \( A_t \). (Hence \( T_t \) is a Steiner tree on \( A_t \), with \( D_t \) being the Steiner vertices.) Consider the two cases for request \( \sigma_{t+1} \):

- Case I: \( \sigma_{t+1} = (\text{add}, t + 1) \). In this case we are now given the distances from \( t + 1 \) to all vertices in \( A_t \), and hence can infer the new distance metric \( d_{t+1}(\cdot, \cdot) \). We now must add at least one edge from \( t + 1 \) to \( V(T_t) \) to get connectivity, and then are allowed to make any edge swaps, and also potentially drop some deleted vertices from the tree to get tree \( T_{t+1} \).

- Case II: \( \sigma_{t+1} = (\text{del}, s) \). We mark the vertex \( s \in V(T_t) \) as deleted. We are allowed to make any edge swaps, and also potentially drop some deleted vertices from the tree to get tree \( T_{t+1} \).

Finally, the cost of tree \( T_t \) is \( \text{cost}(T_t) := \sum_{e \in E_t} d_t(e), \) the sum of lengths of the edges in \( T_t \). We call this tree \( \alpha \)-competitive if \( \text{cost}(T_t) \leq \alpha \cdot \text{opt}(A_t) \), i.e., it costs not much more than the minimum cost Steiner tree on the alive vertices. The algorithm is said to be \( \alpha \)-competitive in the fully-dynamic model if it maintains a tree that is \( \alpha \)-competitive at all times, when the input consists of both additions and deletions.

The main theorem of this section is the following:

**Theorem 5.2** There is a 4-competitive algorithm for Steiner tree in the fully-dynamic model that, for every \( t \), performs at most \( O(t) \) edge additions and deletions in the first \( t \) steps.

Before we give the algorithm, let us define \( c \)-swaps and \( c \)-stability. For some Steiner tree \( T \) on the terminals in \( A_t \), suppose there exist \( e \in E(T) \) and \( f \not\in E(T) \) such that (i) the graph \( T - e + f \) is also a Steiner tree on \( A_t \), and (ii) \( d_t(e) \geq c \cdot d_t(f) \). Then we say that \( (e, f) \) is a valid \( c \)-swap, and performing the valid \( (e, f) \) swap means changing the current tree from \( T \) to \( T - e + f \). A tree is \( c \)-stable if there do not exist any valid \( c \)-swaps.

Following Imase and Waxman [IW91], a tree \( T \) with vertex set \( V(T) \) is called an extension tree for a set of vertices \( S \) if (i) it is a Steiner tree on \( S \)—i.e., \( S \subseteq V(T) \), and (b) all Steiner vertices in \( T \), i.e., vertices in \( V(T) \), are of degree strictly greater than 2. Given a Steiner tree \( T \) that is not an extension tree (i.e., \( T \) has Steiner vertices of degree 1 or 2), the following operations produce an extension tree \( T' \).

For any degree-1 Steiner vertex (i.e., leaf Steiner vertex), delete the vertex and its incident edge. For any degree-2 Steiner vertex \( u \) with edges to \( v, w \), delete the vertex \( u \) and edges \((u, v), (u, w)\), and add the edge \((v, w)\). Note that such an operation might create more low-degree vertices: repeat the process on these vertices until the resulting tree is an extension tree for \( S \).

Our algorithm is the following:

- For an addition \( \sigma_t = (\text{add}, t) \), attach \( t \) to the closest vertex \( p_t \) from \( V(T_{t-1}) \). Call the edge \((t, p_t)\) the greedy edge for time \( t \). By [Lemma 5.1] the vertex \( p_t \) is alive. Now perform any valid 2-swaps until we get a 2-stable tree.

- For a deletion \( \sigma_t = (\text{del}, s) \), mark \( s \) as a Steiner vertex in \( T_{t-1} \). Convert this Steiner tree on \( A_t = A_{t-1} \setminus \{s\} \) to an extension tree as described above. Perform any valid 2-swaps until the tree is 2-stable. This might create low-degree vertices, so repeat these two steps iteratively until we get a 2-stable extension tree on the vertex set \( A_t \). Note that this process will terminate because during edge swaps, we are reducing the cost of the tree, and during conversion to an extension tree, we are removing some vertices which will not appear again.
Recall that for a set of vertices $S$, $\text{opt}(S)$ denotes the cost of the optimal Steiner tree on $S$. Let $\text{mst}(S)$ denote the cost of the minimum spanning tree on $S$. Let $\text{cost}(\text{mst}(S))$ denote the cost of this tree. The argument about the cost of the tree follows from known results [W91, Lemma 5]:

**Theorem 5.3** If $T = (V, E)$ is a $c$-stable extension tree for a set of vertices $S$, then

$$\text{cost}(T) \leq 2c \cdot \text{cost}(\text{mst}(S)) \leq 4c \cdot \text{opt}(S).$$

This shows that the tree maintained by the algorithm is 4-competitive. To prove Theorem 5.2, it suffices to now bound the number of edge additions and deletions performed during the algorithm. The following lemma follows from the fact that the closest vertex to a newly arriving vertex is one of the alive vertices at that time.

**Lemma 5.4** For any $n$, consider the algorithm after the first $n$ requests. The greedy edges added by the algorithm are the same edges that would be added by a greedy algorithm running on the subsequence of just the additions (and none of the deletions) in the sequence $\sigma_{1 \ldots n}$.

**Corollary 5.5** For any $n$, let $E_g$ be the set of greedy edges added by the algorithm on input sequence $\sigma_{1 \ldots n}$. Then

$$\prod_{(t,p_t) \in E_g} d_n(t,p_t) = \prod_{(t,p_t) \in E_G} d_t(t,p_t) \leq 4|V_n| \cdot \prod_{e \in \text{mst}(V_n)} d_n(e).$$

**Proof.** This follows from [GGKT, Theorem 5.1], which bounds the product of the greedy edge lengths added in the sequence $\sigma_{1 \ldots n}$ to the edge lengths of the minimum spanning tree of the added vertices $V_n$—this result assumes that we are inserting vertices only. But the lemma above shows that we can indeed make such an assumption without affecting the set of greedy edges which get added.

It is now convenient to define a slightly different process, where we maintain a spanning tree $\hat{T}_t$ on all vertices in $V_t$, instead of a Steiner vertex on terminals in $A_t$. Some of the edges in $\hat{T}_t$ will be colored red, and others black. One invariant will be that deleting all the red edges in $\hat{T}_t$ will leave exactly the tree $T_t$. Hence the red edges give us a forest, where each tree in this forest contains a single vertex from $T_t$.

Suppose we have inductively defined $\hat{T}_t$ thus far. There are four different operations:

(a) Any greedy edges added to $T_t$ are also added to $\hat{T}$ and colored black.
(b) Any swaps done in $T_t$ are also mimicked in $\hat{T}$—observe that these are swaps between black edges.
(c) If we delete some degree-1 vertex from $T_t$, we merely mark this edge as red in $\hat{T}_t$.
(d) If we delete a degree-2 vertex $u$ in $T_t$, and connect its neighbors $(v,w)$ by an edge, in $\hat{T}_t$ we also add the new black edge $(v,w)$, delete the black edges $(u,v), (u,w)$, and add a red edge from $u$ to the closer of $\{v,w\}$.

Note that all these moves maintain that $\hat{T}_t$ is a spanning tree on the vertices in $V_t$; all edges of $T_t$ are contained in it, and are colored black. Define the potential of any forest $F$ on the metric $d_n$ as

$$\Phi_n(F) := \prod_{e \in E(F)} d_n(e). \quad (5.8)$$

Hence, Corollary 5.5 says that $\Phi_n(E_g) \leq 4|V_n| \cdot \Phi_n(\text{mst}(V_n))$. (Notice that the greedy edges $E_g$ do form a forest—in fact, a spanning tree—on $V_n$.) Let us track how the potential of the tree $\hat{T}_t$ changes.

**Lemma 5.6**

$$\Phi_n(\hat{T}_n) \leq \Phi_n(E_g) \cdot \left(\frac{1}{2}\right)^{n_b} \cdot 2^{n_d}$$

where $n_b$ is the number of 2-swaps (i.e., number of invocations of operation (b)), and $n_d$ is the number of invocations of operation (d).
Proof. The change in the product due to operation (a) is captured by the product of the greedy edges. Operation (b) causes some edge to be replaced by an edge of at most half the length, which accounts for $(1/2)^{n_b}$. Operation (c) does not change the product, only the color of an edge. Operation (d) essentially replaces the longer of edges $(u, v), (u, w)$—say the longer one is $(u, w)$—by the edge $(v, w)$. By the triangle inequality, $(u, w)$ has length $d_n(v, w) \leq d_n(u, v) + d_n(u, w) \leq 2d(u, w)$. Hence the product of edge lengths increases by at most a factor of 2. This accounts for $2^{n_d}$.

Putting Corollary 5.3 and Lemma 5.6 together, and using that $\hat{T}_n$ is a spanning tree on $V_n$, we get

$$\frac{\Phi_n(\hat{T}_n)}{\Phi_n(\text{mst}(V_n))} \leq 4|V_n| \cdot 2^{-n_b+n_d}. \quad (5.9)$$

As observed in [3GK13, Lemma 5.2], the fact that $\hat{T}_n$ is a spanning tree on $V_n$ and mst$(V_n)$ is a minimum spanning tree on $V_n$ implies that $\Phi_n(\hat{T}_n) \geq \Phi_n(\text{mst}(V_n))$. Combining this with (5.9), we get

$$n_b \leq 2|V_n| + n_d.$$

Note that the algorithm performs at most $|V_n|$ edge deletions, since each execution of operations (c) and (d) causes one edge deletion. Also, each operation (d) also causes one edge swap (in addition to the edge deletion), as does an execution of operation (b). Hence the total number of swaps is at most

$$n_b + n_d \leq 2|V_n| + 2n_d.$$

Finally, $n_d = n - |V_n|$, because there are a total of $n$ requests and $|V_n|$ of them are vertex additions (so the rest must be deletions). This means the total number of edge swaps in the first $n$ requests is $2n$, which completes the proof of Theorem 5.2.

5.1 A Word about the Distance Specification

Recall that when a new point was added, we specified the distances from this new point to the old points in a particular fashion. Let us recall this again. Suppose $d_t(\cdot, \cdot)$ are the current distances between the vertices in $V_t$. If we see $(\text{add}, t+1)$, we are given the distances from $t+1$ to all vertices in $A_t \subseteq V_t$: i.e., to only the alive vertices. The guarantee we have is that the newly given distances form a metric along with the old distances, and hence do not violate the triangle inequality. The distances from $t+1$ to vertices in $D_t$ must be inferred using the triangle inequality: $d_{t+1}(t+1, s) = \min_{s' \in A_t} (d(t+1, s') + d_t(s', s))$. Note that the former summand is a new distance given as input, the second summand is inductively defined.

Perhaps a more natural model is where we are given distances to all the previous vertices (both alive and deleted), again subject to the triangle inequality. We now claim the two models are the same up to constant factors, and hence it is fine to work with the former model. Indeed, suppose when we see $(\text{add}, t+1)$, we are told distances $d'(t+1, x)$ for all $x \in V_t$, and this gives us a metric $d_{t+1}'(\cdot, \cdot)$ on $V_t$. We could then ignore the distances to the already-deleted vertices, define $d(t+1, y) := d'(t+1, y)$ for all alive vertices $y \in A_t$ and extend it by the triangle inequality as above to get the distances $d_{t+1}$ on all of $V_{t+1}$. Clearly the distances $d_{t+1}$ is at least the cost according to the actual distances $d_{t+1}'$. Moreover, this definition inductively maintains $d_{t+1}(x, y) = d_{t+1}'(x, y)$ for all $x, y \in A_{t+1}$, so the cost of the optimal Steiner tree on $A_{t+1}$ using the actual metric $d_{t+1}'$ is at least half the cost of the MST on $A_{t+1}$ with respect to $d_{t+1}'$ (and hence also with respect to $d_{t+1}$). This completes the proof that working in the distances-specified-to-alive-points only changes the competitive ratio by a factor of 2.
6 Discussion

Several interesting questions remain unanswered. We do not know how to get a non-amortized constant competitive algorithm for the fully-dynamic case which makes $O(1)$ swaps per insertion or deletion. Obtaining similar results for the Steiner forest problem (even in the amortized setting, even for insertions only) remains an interesting open problem.

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