Abstract. A finite connected 2-complex $K$ whose fundamental group $G$ is of cohomological dimension 2 is aspherical iff the subgroup $\Sigma_K \subset H_2(K)$ of spherical 2-cycles is zero.

A finite connected subcomplex of an aspherical 2-complex is aspherical iff its fundamental group is of cohomological dimension 2.

§0. Introduction

Two famous unsolved conjectures of low dimensional topology are the Eilenberg-Ganea conjecture [EG] and the Whitehead conjecture [Wh]. The Eilenberg-Ganea conjecture states that a group of cohomological dimension (cd) 2 is of geometric dimension (geomdim) 2 (the standard reference for these notions is K. Brown’s book [Br] Chapter VIII §2). The Whitehead conjecture states that a connected subcomplex of an aspherical 2-complex is aspherical. It is equivalent to saying that a connected subcomplex of a contractible 2-complex is aspherical. There is an extensive literature on it including, notably, work of J. F. Adams [Ad] and James Howie [Ho1]. A result of Bestvina and Brady [BB] now 17 years old, states that not both of these conjectures can be true. If, for example, the Eilenberg-Ganea conjecture is true, then the Whitehead conjecture must be false. The proof gives no indication about the truth or falsity of the individual conjectures. There is also an article of Howie’s [Ho2] which gives a new argument for this result.

I view the Whitehead conjecture [Wh] as the combination of two distinct conjectures. First, that a connected subcomplex $K$ of an aspherical 2-complex has fundamental group $G$ of cohomological dimension 2, and, second, the Eilenberg-Ganea conjecture, that a group of cohomological dimension 2 has geometric dimension 2, so is the fundamental group of an aspherical 2-complex. This article deals with the
second conjecture in the case $K$ is finite. I have nothing to say about the first of these conjectures, since my methods apply to geometry only when the fundamental group is already assumed to be of cohomological dimension 2.

In this note I shall establish the following result.\footnote{A stronger result is proved in §3 below after I became aware of results of B. Eckmann [Ec]. I have not made any changes in the first two sections, but merely added on the improvements at the end in §3.}

**Theorem.** Let $K$ be a finite connected 2-complex whose fundamental group $G$ is of cohomological dimension 2. Then $K$ is aspherical iff

1. the relation module $Z_1(\tilde{K})$ is stably free, and
2. the subgroup $\Sigma_K \subset H_2(K)$ of spherical 2-cycles is 0.

The connection with the Whitehead conjecture is that every subcomplex $K$ of a contractible 2-complex has $H_2(K) = 0$ (and hence, a fortiori, $\Sigma_K = 0$), as one sees from the exact homology sequence for the pair. So, with the additional assumptions that the subcomplex is finite and connected and its fundamental group is of cd 2, and the necessary condition (from the Theorem) that the relation module be stably free, it follows that $K$ is aspherical.

**Remark.** (due to Jim Howie:) Suppose $K$ is a connected subcomplex of an aspherical 2-complex $L$ (so we are replacing contractible by the wider context of aspherical). Let $\hat{K}$ be the covering of $K$ contained in the universal cover $\hat{L}$ of $L$, $\hat{K} \subset \hat{L}$, and let $\hat{K}$ be the universal cover of $K$. So we have coverings $\hat{K} \to \hat{K} \to K$, all with the same $\pi_2$. So the map $\pi_2(K) \to H_2(K)$ factors through $H_2(\hat{K}) = 0$. It follows that $\Sigma_K = 0$ in this situation, whether or not $H_2(K) = 0$. So with the further assumptions that $K$ is finite, that $\pi_1(K)$ is of cohomological dimension 2, and that the relation module $Z_1(\hat{K})$ is stably free, it follows that $K$ is aspherical.

\section*{§1. Proof of Theorem and some Corollaries\footnote{See §3 for the strengthened version of the Theorem of §0.}}

In this section $K$ is a finite connected 2-complex whose fundamental group $G$ is of cohomological dimension 2, and $\hat{K}$ is the universal covering complex of $K$. We have the fibration $G \to \hat{K} \to K$ with discrete fibre $G$. This is classified\footnote{See any good book on algebraic topology, like [Sp].} by a map $K \to BG$ with fibre $\hat{K}$, where $BG$ is an Eilenberg-MacLane space of type $K(G, 1)$. We can apply the Serre spectral sequence to this fibration to get the $E^2_{p,q}$ term $H_p(G, H_q(\hat{K}, \mathbb{Z}))$ converging to the $E^\infty$ term, which is the successive quotients of a filtration of $H_{p+q}(K, \mathbb{Z})$.

The spectral sequence has only two non-zero rows, rows 0 and 2, since $\hat{K}$ is simply connected (so row 1 is zero) and since $\hat{K}$ is a 2-complex (so rows $n$ with $n \geq 3$ are zero). So we have an edge-term exact sequence

\[0 \to H_3(G, \mathbb{Z}) \to H_0(G, H_2(\hat{K}, \mathbb{Z})) \to H_2(K, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \to 0.\]
The 0 on the left comes about since \( K \) is a 2-complex so \( H_3(K) = 0 \). Since \( \text{cd}(G)=2 \), it follows that \( H_3(G, \mathbb{Z}) = 0 \), so the exact sequence simplifies to

\[
0 \to H_0(G, H_2(\tilde{K}, \mathbb{Z})) \to H_2(K, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \to 0.
\]

**Lemma 1.2.** If \( \Sigma_K = 0 \), then \( H_0(G, H_2(\tilde{K}, \mathbb{Z})) = 0 \).

For the proof, note that the image of the map \( H_0(G, H_2(\tilde{K}, \mathbb{Z})) \to H_2(K, \mathbb{Z}) \) is \( \Sigma_K \), the group of spherical 2-cycles of \( K \).

Now we examine the hypothesis that \( \text{cd}(G)=2 \) more closely. Let \( P =: H_2(\tilde{K}, \mathbb{Z}) \).

**Lemma 1.3.** \( P \) is a finitely generated projective \( \mathbb{Z}[G] \)-module and a direct summand of \( C_2(\tilde{K}, \mathbb{Z}) \).

**Proof.** Consider the exact sequence

\[
0 \to Z_1(\tilde{K}) \to C_1(\tilde{K}) \to C_0(\tilde{K}) \to \mathbb{Z} \to 0.
\]

But \( C_1 \) and \( C_0 \) are free \( \mathbb{Z}[G] \)-modules and hence projective. So it follows from [Br] Chapter VIII §2 Lemma 2.1 (iv) (using \( n = 2 \) there and the fact that \( \text{cd}(G)=2 \)) that \( Z_1(\tilde{K}) \) is projective.

Now \( B_1(\tilde{K}) = Z_1(\tilde{K}) \) since \( \tilde{K} \) is simply connected. It follows that the exact sequence

\[
0 \to H_2(\tilde{K}) \to C_2(\tilde{K}) \to B_1(\tilde{K}) \to 0
\]

splits. Thus \( P \) is projective and a direct summand of \( C_2(\tilde{K}) \) with complementary direct summand \( Z_1(\tilde{K}) = B_1(\tilde{K}) \), and the lemma is proved.

Now assume that \( K \) is aspherical. Then \( P = H_2(\tilde{K}) = 0 \), so \( C_2(\tilde{K}) \cong B_1(\tilde{K}) = Z_1(\tilde{K}) \), whence \( Z_1(\tilde{K}) \) is free (and hence a fortiori stably free).

Summarizing, if \( K \) is aspherical, it follows that \( \Sigma_K = 0 \) and \( Z_1(\tilde{K}) \) is stably free, which proves the necessity of conditions (1) and (2) in the Theorem.

Next we proceed to the sufficiency of conditions (1) and (2). Since \( \Sigma_K = 0 \), it follows from Lemma 1.2 that \( H_0(G, H_2(\tilde{K}, \mathbb{Z})) = 0 \). But this is the same as \( P/IP \), where \( P = H_2(\tilde{K}, \mathbb{Z}) \) and \( I \) is the augmentation ideal of \( \mathbb{Z}[G] \). Thus \( P/IP = 0 \).

Furthermore we are assuming that \( Z_1(\tilde{K}) \) is stably free. We do elementary expansions on \( K \), attaching a finite number of 2-discs trivially at the base point. The effect on the chains of the universal cover is to take the direct sum of \( C_2 \) and \( C_1 \) with the same finite number of copies of \( \mathbb{Z}[G] \), and map them by the identity map. So \( Z_1 = B_1 \) is increased by the direct sum of the same number of copies of \( \mathbb{Z}[G] \). This has the effect of stabilizing \( Z_1 \), which we may assume now is free. In addition there is no change in \( H_2 \) of \( \tilde{K} \), which is still \( P \). So we may, without loss of generality, adjust the notation and assume that \( K \) is such that \( Z_1(\tilde{K}) \) is free and \( P/IP = 0 \), with \( P = H_2(\tilde{K}) \). Our goal is to prove that \( P = 0 \).
Now take the split exact sequence 1.4 and apply the functor $\mathbb{Z} \otimes_{\mathbb{Z}[G]}$ to it. The left term is $P/IP$ and hence vanishes, so the result is that $\mathbb{Z} \otimes_{\mathbb{Z}[G]} C_2(\tilde{K}) \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} Z_1(\tilde{K})$. But both $C_2$ and $Z_1$ are free $\mathbb{Z}[G]$ modules, so it follows that they are of the same rank. Thus the boundary map $\partial : C_2(\tilde{K}) \to Z_1(\tilde{K})$ is a split epimorphism of free modules of the same rank. Let $h : Z_1(\tilde{K}) \to C_2(\tilde{K})$ be a right inverse for $\partial$, so $\partial \circ h = Id$, the identity map of $Z_1$. Since $C_2$ and $Z_1$ are free of the same rank and equipped with free bases, the maps $\partial$ and $h$ are given by $n$-by-$n$ matrices over $\mathbb{Z}[G]$ with respect to these bases, so we are dealing with $M_n(\mathbb{Z}[G])$.

Let $E = h \circ \partial : C_2(\tilde{K}) \to C_2(\tilde{K})$. Computation shows that $E^2 = E$, so the image of $E$ is a direct summand of $C_2(\tilde{K})$ which is isomorphic to $Z_1(\tilde{K})$. It follows that $Id - E$ projects $C_2(\tilde{K})$ onto an isomorphic copy of $P$, where here $Id$ is the identity map of $C_2$.

Let $\Lambda$ be the ring of $\mathbb{Z}[G]$-linear endomorphisms of $C_2(\tilde{K})$, so $\Lambda \cong M_n(\mathbb{Z}[G])$ where $n$ is the rank of $C_2(\tilde{K})$ as a free $\mathbb{Z}[G]$-module. We consider $M_n(\mathbb{Z}[G]) = M_n(\mathbb{Z})[G]$, the group ring of $G$ with coefficients in $M_n(\mathbb{Z})$, and extend scalars to $\mathbb{C}$ to get $M_n(\mathbb{C})[G]$. A typical element of this ring may be written as $z =: \sum g m_g g$, where $m_g \in M_n(\mathbb{C})$ and almost all $m_g$ are zero.

Define a map $t : M_n(\mathbb{C})[G] \to \mathbb{C}$ by $t(z) =: \text{trace}(m_1)$, where $m_1$ is the coefficient of 1, the unit element of $G$, in $z$. In addition $M_n(\mathbb{C})[G]$ has an involution $x \to x^*$ given by $x^* = \sum_{g \in G} x_g T g^{-1}$ if $x = \sum_{g \in G} x_g g$; here the superscript $T$ indicates the transposed matrix and “bar” is complex conjugation. The following facts are valid for $t$.

1. $t(Id) = n$, where $Id$ is the identity matrix.
2. $t$ is $\mathbb{C}$-linear.
3. $t(xy) = t(yx)$ for all $x, y$ in $M_n(\mathbb{C})[G]$.
4. $t(xx^*) \geq 0$ and $t(xx^*) = 0$ iff $x = 0$.

Only the last item requires comment. Let $x = \sum_{g \in G} x_g g$ and let $r_i(x_g)$ denote the $i^{th}$ row of the matrix $x_g$. Then $t(xx^*) = \sum_{g \in G} \sum_{i=1}^n ||r_i(x_g)||^2$, where, for a row vector $v$, $||v||$ is the $L^2$-norm of the vector. Now $M_n(\mathbb{C})[G]$ is equipped with a Hermitian inner product given by $< \sum_{g \in G} x_g, \sum_{g \in G} y_g > = t(\sum_{g \in G} x_g y_g T)$. The non-degeneracy follows from the identity $< x, x > = t(xx^*) \geq 0$ and equal to 0 iff $x = 0$. It follows that $M_n(\mathbb{C})[G]$ may be completed to a Hilbert space $H$. Elements of $\Lambda$ act as bounded operators $B(H)$ on $H$. Let $A$ denote the closure of the set of these operators in the norm topology of $B(H)$. Then $A$ is a $C^*$ algebra, so $1 + f f^*$ is invertible for all $f \in A$. Recall that an element $x$ is self-adjoint if $x = x^*$ and a projection if it is both idempotent and self-adjoint. The next two results are due to Kaplansky. I am following [Ma] in the exposition.

4I am following almost verbatim the exposition given in [Mo], with appropriate changes for the wider context of $n$-by-$n$ matrices rather than 1-by-1.

5So here $r_i(x_g)$ is the vector over $\mathbb{C}$ with components $(x_g, i_1, x_g, i_2, \ldots, x_g, i_n)$ and $||r_i(x_g)||^2 = \sum_k |x_g, i_k|^2$.

6See W. Rudin, Functional Analysis, Second Edition, McGraw-Hill, Inc., 1991, p. 295, Theorem 11.28 (f).
1.5 Let $R$ be a unital ring with involution $\ast$ such that for any $x \in R$ the element $1 + xx^\ast$ is invertible in $R$. For every idempotent $f \in R$ we can find a projection $e \in R$ so that $Rf = Re$. (For the proof see [Ma] Prop. 2.4. Note that we are using left modules rather the right modules in [Ma].)

1.6 Let $R$ be a unital ring and $e, f \in R$ two idempotents such that $Re = Rf$. Then $e$ and $f$ are similar, so there exists an invertible element $s \in R$ so that $s^{-1} fs = e$. (See [Ma] Prop. 2.5.)

It follows that every idempotent in $A$ is similar to a projection. So the idempotent $E \in M_n(\mathbb{Z}[G])$ is similar in the larger ring to a projection $\Pi$. Also the function $t$ is continuous so extends a function also called $t$ on $A$. I also recall Lemma 1 of [Mo] which asserts that if $t(xx^\ast) = 0$ for $x \in A$, then $x = 0$. Although the context is more restricted there, the same argument given there works here (it is, basically, a continuity argument).

**Lemma 1.7.** For the idempotent $E$ in $M_n(\mathbb{C})[G]$, $t(E)$ is real and $0 \leq t(E) \leq n$.

**Proof.** $E$ is similar in $A$ to the projection $\Pi$, so $t(E) = t(\Pi)$, using property 3. But $\Pi = \Pi \cdot \Pi^\ast$, so $t(\Pi) > 0$ if $\Pi \neq 0$, by the analog of Lemma 1 of [Mo]. Similarly $t(Id - \Pi) > 0$ if $\Pi \neq Id$.

Now we can complete the proof of the Theorem. We have $\partial \cdot h$ is the identity of $Z_1(\tilde{K})$, a free module of rank $n$, so $t(\partial \cdot h) = n$. But $t(\partial \cdot h) = t(h \cdot \partial)$, so $t(E) = t(h \cdot \partial) = n$. Thus $t(Id - E) = 0$, and it follows that $Id - E = 0$. Hence $E$ is the identity, the maps $\partial$ and $h$ are mutual inverses, $P = 0$, $K$ is aspherical, and the proof is complete. □

Let us deduce some corollaries of the Theorem.

**Corollary 1.** If $K$ is a finite connected 2-complex whose fundamental group is of cohomological dimension 2 and if $H_2(K) = 0$ and the relation module $Z_1(\tilde{K})$ is stably free, then $K$ is aspherical. In particular, if $K$ is a finite connected subcomplex of a contractible 2-complex, and if the cohomological dimension of $\pi_1(K)$ is 2 and if the relation module is stably free, then $K$ is aspherical. □

**Corollary 2.** Suppose $G$ is a finitely presented perfect group\footnote{Or more generally if its first betti number is 0.} of cohomological dimension 2 which admits a balanced\footnote{A finite presentation is balanced if it has the same number of generators as defining relators.} presentation $\mathcal{P}$. If $K = K(\mathcal{P})$ is the associated 2-complex to the presentation\footnote{$K(\mathcal{P})$ has a single vertex, one 1-cell for each generator, and one 2-cell for each defining relator whose attaching map spells out the relator in the 1-skeleton, with appropriate orientations.} and if the relation module is stably free, then $K$ is aspherical.

**Proof.** An Euler characteristic computation shows that the second betti number of $K$ is 0. Since $K$ is a 2-complex, this implies $H_2(K) = 0$, so Corollary 1 applies.

**Remark.** Perfect groups are in a sense opposite to locally indicable groups. Both Howie’s and Adams’s methods [Ho1],[Ad] fail for perfect groups, and Howie’s apply to locally indicable groups. So Corollary 2 above reaches into territory not
Corollary 3. Let $K$ be a finite connected 2-complex whose fundamental group $G$ is of cohomological dimension 2 and such that $G$ is of type FL.\footnote{A group $G$ is of type FL if the trivial $G$-module $\mathbb{Z}$ has a finite resolution by finitely generated free $\mathbb{Z}[G]$-modules.} Suppose in addition that $H_2(K) = 0$. Then $K$ is aspherical.

Proof. One checks that conditions 1 and 2 in the Theorem are satisfied, with 2 following since there are no nonzero 2-cycles. Since $G$ is of type FL, there is a finite resolution

$$0 \to L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \to \mathbb{Z},$$

with the $L_i$ finitely generated free $\mathbb{Z}[G]$-modules. Let $Z_i$ be the kernel of the map $L_i \to L_{i-1}$. Since $G$ is assumed of cohomological dimension 2, $Z_1$ is finitely generated and projective. It follows that all the short exact sequences

$$0 \to Z_i \to L_i \to Z_{i-1} \to 0$$

split for $2 \leq i \leq n$, where $Z_n = L_n$. Now it follows by downward induction that $Z_n, Z_{n-1}, \ldots, Z_2, Z_1$ are stably free. In particular $Z_1$ is stably free. But $Z_1$ is stably equivalent to the relation module for $K$, namely, $Z_1(\tilde{K})$, and it follows that $Z_1(\tilde{K})$ is stably free.

Hence both conditions of the Theorem are satisfied, and it follows that $K$ is aspherical.

There is a version of the last corollary which is analogous to Howie’s observation in §0, as follows.

Corollary 4. Let $K$ be a finite connected subcomplex of an aspherical 2-complex $L$. Assume that the fundamental group $G$ of $K$ is of cohomological dimension 2 and of type FL. Then $K$ is aspherical. □

Remark. Corollary 4 is in fact a necessary and sufficient condition for a finite connected subcomplex $K$ of an aspherical 2-complex $L$ to be aspherical, namely, that $G = \pi_1(K)$ be of cohomological dimension (at most) 2 and that it be of type FL. The sufficiency is the content of Corollary 4. For necessity, assume that such $K$ is aspherical. Then the chain complex of $\tilde{K}$ gives a finite free resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}[G]$-modules, so $G$ is of type FL. The fact that $G$ is of cohomological dimension (at most) 2 follows from [Br] p. 184, Lemma 2.1(iv).

§2. THE $t$-RANK OF A PROJECTIVE MODULE AND SOME OPEN QUESTIONS

Definition. The $t$-rank of $P$, a finitely generated projective module over $\mathbb{Z}[G]$, is defined to be $t(E)$, where $E$ is an idempotent matrix in $M_n(\mathbb{Z})[G]$ with image $P$. For simplicity denote the $t$-rank of $P$ by $t(P)$. 

A group $G$ is of type FL if the trivial $G$-module $\mathbb{Z}$ has a finite resolution by finitely generated free $\mathbb{Z}[G]$-modules.
Here is a collection of properties of the t-rank, all easily proved.

1. $t(P)$ is well-defined and independent of the ambient matrix algebra used to define it.
2. $t(P + Q) = t(P) + t(Q)$.
3. If $F$ is a free module over $\mathbb{Z}[G]$, of rank $n < \infty$, then $t(F) = n$.
4. $t(P)$ is a non-negative integer with $t(P) = 0$ iff $P = 0$.

There is another notion of rank of a f.g. projective module $P$ over $\mathbb{Z}[G]$, namely the rank over $\mathbb{Z}$ of $\mathbb{Z} \otimes_{\mathbb{Z}[G]} P$. It is easy to see that the two notions of rank agree on free modules, and more generally they agree on stably free modules.

**Remark.** Let $E$ be an idempotent matrix in $M_n(\mathbb{Z}[G])$ with image the projective module $P$. Then the two notions of rank are easily computed as follows. If $E = \sum_{g \in G} e_g g$ where $e_g \in M_n(\mathbb{Z})$, and if $e_g$ is the matrix $(e_g,ij)$, then the $t$-rank is given by $t(P) = \sum_{1 \leq i \leq n} e_{1,ii}$ whereas the other notion of rank is calculated as $\epsilon(\text{trace}(E)) = \sum_{g,i} e_{g,ii}$. If $P$ is stably free, the two calculations must give the same answer.

**Question.** Do the two notions of rank agree on all finitely generated projective modules over $\mathbb{Z}[G]$? If there is a projective module $P$ for which the two notions disagree, that module cannot be stably free. In particular, if $P/IP = 0$ but $P \neq 0$, then $P$ cannot be stably free.

**Question.** If $P$ is a finitely generated projective module over an integral group ring such that $P/IP = 0$, is $P = 0$?

**Question.** Is every relation module of a finite presentation of a group $G$ of cohomological dimension 2 stably free? It is equivalent to ask whether such $G$ is of type FL.

**Question.** If $P$ is a finitely generated projective module over $\mathbb{Z}[G]$, is $\ell_2(G) \otimes_{\mathbb{Z}[G]} P$ stably free? An affirmative answer to this question would imply affirmative answers to the preceding two questions.

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11. This makes sense since $\mathbb{Z}[G]$ has invariant basis number.
12. Over $\mathbb{Z}$, the target of $\epsilon$, any idempotent matrix is similar to a diagonal matrix with 1’s and 0’s on the main diagonal, and the rank is the number of 1’s, i.e. the trace. Over the commutative ring $\mathbb{Z}$ trace is a similarity invariant, from which the rank formula can be derived.
13. I was motivated originally to ask this question by Nakayama’s lemma, but it has turned out to be much deeper. In a sense it says that the augmentation ideal plays the role of the Jacobson radical as far as projective modules over group rings are concerned. I had originally approached the question, whether $P/IP = 0$ implies $P = 0$, as a question of ring theory. Howie pointed out to me that there is a canonical isomorphism $I/I^2 \cong G_{ab}$, where $G_{ab}$ is the abelianization of the group $G$ and $I$ is the augmentation ideal of $\mathbb{Z}[G]$. Hence if $G$ is perfect, $I = I^2 = I^3 = \cdots = I^\infty$, and the ring theoretic approach fails. But for abelian groups and potentially a larger class admitting no perfect subgroups, this approach may conceivably succeed.
14. Putatively it should be free of rank $t(P)$.
15. I have asked the experts in the field of operator algebras and come up with either no response or that they don’t know the answer.
**Remark.** I want to pose a vague question to bring the Eilenberg-Ganea conjecture into the discussion: what in addition to cohomological dimension 2 and type FL (both necessary conditions) does one need to characterize when a finitely presented group has a finite aspherical presentation?\textsuperscript{16}

**Remark.** My results have no bearing on the result of [BB], that one of the two conjectures, Whitehead’s or Eilenberg-Ganea’s, must be false. The reason is I am considering only finite presentations. The Whitehead conjecture considers arbitrary not necessarily finite subcomplexes of a contractible 2-complex and the Eilenberg-Ganea conjecture considers arbitrary groups, not necessarily finitely presented, of cohomological dimension 2. In this generality, one of the two conjectures must be false.

§3. Update and further results

Since sections 0–2 were written, I have become aware of an article by B. Eckmann [Ec] whose results enable me to strengthen the Theorem of §1 and to settle some of the questions asked at the end of §2, at least for groups of cohomological dimension 2. Eckmann’s results for our purposes can be summarized as follows.

**Theorem of Eckmann.** \textsuperscript{18} If $G$ is a countable group of cohomological dimension 2 over $\mathbb{Z}$ and $P$ is any finitely generated projective $\mathbb{Z}[G]$-module, then $\ell_2(G) \otimes_{\mathbb{Z}[G]} P$ is a free $\ell_2(G)$-module of rank the $\mathbb{Z}$-rank of $\mathbb{Z} \otimes_{\mathbb{Z}[G]} P$.

To apply this result we need a lemma, which makes use of the central point of the proof of the Theorem of §1.

**Lemma 3.1.** Let $G$ be a group with the property that for any finitely generated projective module $P$ over $\mathbb{Z}[G]$ one has $\ell_2(G) \otimes_{\mathbb{Z}[G]} P$ is a free $\ell_2(G)$-module of rank the $\mathbb{Z}$-rank of $\mathbb{Z} \otimes_{\mathbb{Z}[G]} P$. If $Q$ is a finitely generated projective $\mathbb{Z}[G]$-module such that $Q/IQ = 0$, then $Q = 0$.

**Proof.** Let $P$ be a finitely generated projective such that $Q \oplus P = \mathbb{Z}[G]^n$. Since $Q/IQ = 0$, it follows that the $\mathbb{Z}$-rank of $\mathbb{Z} \otimes_{\mathbb{Z}[G]} P = n$. Now tensor with $\ell_2(G)$ over $\mathbb{Z}[G]$ to get $\ell_2(G) \otimes_{\mathbb{Z}[G]} P$ is free of rank $n$ as an $\ell_2(G)$-module. It follows that we have homomorphisms of free $\ell_2(G)$-modules $\partial : \ell_2(G)^n \rightarrow \ell_2(G) \otimes_{\mathbb{Z}[G]} P$ and $h : \ell_2(G) \otimes_{\mathbb{Z}[G]} P \rightarrow \ell_2(G)^n$ of the same rank $n$ such that $\partial \circ h = 1 \ell_2(G) \otimes_{\mathbb{Z}[G]} P$. Since these maps are given by $n$-by-$n$ matrices over $\ell_2(G)$, we can apply the function $t$ (which extends by continuity to $M_n(\ell_2(G))$) to get $n = t(1_P) = t(\partial \circ h) = t(h \circ \partial)$. But $h \circ \partial$ is an idempotent endomorphism of $\ell_2(G)^n$. It follows that

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\textsuperscript{16}I have tried and failed so far to prove that just these two conditions sufficed. I am grateful to Howie for having found the gap in my argument.

\textsuperscript{17}The statement that follows is a combination of several of his results, where I have omitted references to various forms of the Bass conjecture to limit demands on the reader. Also I only need the results for groups of cd 2 over $\mathbb{Z}$, whereas he considers other fields.

\textsuperscript{18}I am thankful to Stefan Friedl for catching an omission in the statement in an earlier version of this paper.
t(ℓ₂(G) ⊗ℤ[G] Q) = t(Id − h ◦ ∂) = n − n = 0, where Id is the identity map of ℓ₂(G). Hence ℓ₂(G) ⊗ℤ[G] Q = 0, whence Q = 0, and the proof is complete.

**Corollary 3.2.** If G is a countable group of cohomological dimension 2 over ℤ and P is a finitely generated projective module over ℤ[G] such that P/IP = 0, then P = 0.

**Proof.** This is immediate from Eckmann’s theorem and the lemma.

We can now strengthen the Theorem of §1.

**Main Theorem 3.3.** Let K be a finite connected 2-complex whose fundamental group is of cohomological dimension 2. Assume that the subgroup Σₖ ⊂ H₂(K) of spherical 2-cycles is 0. Then it follows that

1. K is aspherical, and
2. the relation module Z₁(˜K) is stably free, and
3. the fundamental group G of K is of type FL.

**Proof.** From the exact sequence 1.1 we deduce from the hypothesis that Σₖ = 0 that H₀(G, H₂(˜K, ℤ)) = 0. That is, if we let P = H₂(˜K, ℤ), P/IP = 0. Since P is finitely generated and projective over ℤ[G], it follows from Corollary 3.2 that P = 0. Thus K is aspherical. Also from the split exact sequence

0 → H₂(˜K) → C₂(˜K) → Z₁(˜K) → 0

we deduce that Z₁(˜K) is stably free. It follows easily that G is of type FL. This completes the proof.

**Corollary 3.4.** Let K be a finite connected subcomplex of an aspherical 2-complex such that the fundamental group G of K is of cohomological dimension 2. Then K is aspherical.

**Proof.** This is immediate from Corollary 4 of §2 and Theorem 3.3.

**Remark.** The question remains open whether P/IP = 0 implies P = 0 for a finitely generated projective module over the group ring ℤ[G] of a general group G. There are many classes of groups for which this holds by our argument and by the results of the appendix to [Ec]; the question is intimately related to the various forms of the Bass conjecture, which we have chosen not to discuss here.

§4. Some decision problems for finitely presented groups of cohomological dimension 2

The Main Theorem of §3 reduces the question of asphericity of K = K(P) to a question about Σₖ ⊂ H₂(K), when P is a finite presentation of a group G with cd(G) = 2. Thus there is the question about how effective are calculations involving Σₖ ⊂ H₂(K) and H₂(G) = H₂(K)/Σₖ.

For general finitely presented groups G there is no effective procedure for calculating H₂(G) nor even for finding its rank nor whether or not it is zero [Go].
splits. Thus there is a section \( h : Z_1(\tilde{K}) \to C_2(\tilde{K}) \) so that \( \partial \circ h = 1 \), the identity map of \( Z_1(\tilde{K}) \). It follows that \( E = h \circ \partial \) is an idempotent endomorphism of \( C_2(\tilde{K}) \); the kernel of \( E \) is isomorphic to \( H_2(\tilde{K}) \) and the image of \( E \) is \( Z_1(\tilde{K}) \). To avoid questions about the effectiveness of choosing \( h \), we shall assume that \( E \) is given, say, as an \( n \)-by-\( n \) matrix with entries in \( \mathbb{Z}[G] \), where \( n \) is the rank of \( C_2(\tilde{K}) \) as a free \( \mathbb{Z}[G] \)-module.\(^{19}\)

**Proposition 4.1.** In the situation just described, assume \( E \) is given as an idempotent matrix over \( \mathbb{Z}[G] \) whose image is isomorphic to \( Z_1(\tilde{K}) \) and such that the kernel of \( E \) is isomorphic to \( H_2(\tilde{K}) \). Then the following are true.

1. \( \Sigma_K \) is effectively computable,
2. \( H_2(G) \) is effectively computable,
3. \( \beta_2(G) \) is effectively computable, and
4. one can effectively decide whether or not \( H_2(G) \) is zero.
5. \( \Sigma_K \) is a direct summand of \( H_2(K) \) and \( H_2(G) \) is a torsion-free abelian group.

**Remark.** It is convenient to factor \( \partial : C_2(\tilde{K}) \to C_1(\tilde{K}) \) as \( \partial = \iota \circ p \), where \( p : C_2(\tilde{K}) \to B_1(\tilde{K}) = Z_1(\tilde{K}) \) is a split epimorphism of finitely generated projective \( \mathbb{Z}[G] \)-modules and \( \iota : Z_1(\tilde{K}) \to C_1(\tilde{K}) \) is injective. Then \( H_2(K) = \ker(C_2(\tilde{K}) \to C_1(\tilde{K})) \) and \( \Sigma_K = \mathbb{Z} \otimes_{\mathbb{Z}[G]} \ker(E) \). Since \( E = h \circ p \), one has \((\iota \circ p) \circ E = \iota \circ p \circ (h \circ p) = \iota \circ p \), and one sees that \( \Sigma_K \subset H_2(K) \), as a check on the notation.

**Proof of 4.1.** One has \( \Sigma_K = \ker(\epsilon(E)) \), where we recall that \( \epsilon : \mathbb{Z}[G] \to \mathbb{Z} \) is the augmentation. Similarly \( H_2(K) \) is the kernel of a square matrix over \( \mathbb{Z} \). It follows that conclusions 1–4 are consequences of the theorem of elementary divisors for matrices over \( \mathbb{Z} \).

For the final conclusion I show first that \( H_2(G) \) is torsion-free.\(^{20}\) We have for any prime number \( p \), \( 0 = H^3(G, \mathbb{Z}/p\mathbb{Z}) = \text{Hom}(H_3(G, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \oplus \text{Ext}(H_2(G, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) = \text{Ext}(H_2(G, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \), by the universal coefficient theorem and the assumption that \( \text{cd}(G) = 2 \). But if there were any non-zero torsion in \( H_2(G) \), it would show up in the Ext terms. It follows that \( H_2(G) \) is torsion-free.

\(^{19}\)The various definitions of projective module are all equivalent in classical mathematics, but some are more effective than others. We have chosen to formulate the proposition in terms of the most effective definition by idempotent matrices.

\(^{20}\)I thank Jim Howie for this argument.
Finally, since \( H_2(G) = H_2(K)/\Sigma_K \) is torsion-free, it follows that all of the elementary divisors for the pair \( (H_2(K), \Sigma_K) \) are either 0 or 1. Hence \( \Sigma_K \) is a direct summand of \( H_2(K) \), and the proof is complete.

**Problem.** Can one by applying a sequence Tietze transformations to \( P \) arrange that \( \Sigma_K = 0 \)? This is just the special case of the Eilenberg-Ganea conjecture for finitely presented groups of cohomological dimension 2.

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\[21\] Conclusion 5 of the proposition can be interpreted as saying there are no obvious obstructions to this case of the Eilenberg-Ganea conjecture being true.