Chern–Simons Hadronic Bag from Quenched Large-\(N\) QCD

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Abstract

SU(\(N\)) reduced, quenched, gauge theories have been shown to be related to string theories. We extend this result and show how a 4-dimensional, reduced, quenched, Yang–Mills theory, supplemented by the topological term, can be related through the Wigner–Weyl–Moyal correspondence to an open 3-brane model. The boundary of the 3-brane is described by a Chern–Simons 2-brane. We identify the bulk of the 3-brane with the interior of a hadronic bag and the world-volume of the Chern–Simons 2-brane with the dynamical boundary of the bag. We estimate the value of the induced bag constant to be a little less than 200 MeV.

Recent proposals for a non-perturbative formulation of String Theory [1] have renewed the interest for matrix models of non-Abelian gauge theories. Large-\(N\) Yang–Mills theories on a \(D\)-dimensional spacetime [2] have been shown to be equivalent to reduced matrix models, where the original \(N \times N\) matrix gauge field \(A_{\mu j}^i(x)\) is replaced by the same field at a single point, say \(x^\mu = 0\) [3] (for a recent review see [4]). Partial derivative operators are replaced by commutators with a fixed diagonal matrix \(p_{\mu j}^i\), playing the role of translation generator and called the quenched momentum [5]. Accordingly, the covariant derivative becomes \(iD_\mu = [p_\mu + A_\mu, \ldots]\). Thus, the reduced, quenched, Yang–Mills field strength is

\[
F_{\mu\nu}^{ij} \equiv [iD_\mu, iD_\nu]^{ij},
\]

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which leads to the matrix Yang–Mills action in four dimensions

\[ S_{\text{qYM}}^{\text{red.}} = -\frac{1}{4} \left( \frac{2\pi}{a} \right)^4 \frac{N}{g_{\text{YM}}^2} \text{Tr} F_{\mu\nu} F^{\mu\nu}, \]  

(1)

where \( g_{\text{YM}} \) is the strong coupling constant and \( a \equiv 2\pi/\Lambda \) is an inverse momentum cut-off, or lattice spacing. \( S_{\text{qYM}}^{\text{red.}} \) is the starting point to study large-\( N \) Yang–Mills theory [4]. For our purposes it is important to supplement \( S_{\text{qYM}}^{\text{red.}} \) with the contribution from topologically non-trivial field configurations, i.e. instantons, which is accounted for by the topological term \( \epsilon^{\lambda\mu\nu\rho} F_{\lambda\mu} F_{\nu\rho} \) (the reason for this choice will be evident later on). Thus

\[ S_{\text{qYM}}^{\text{red.}} \longrightarrow S_q \equiv S_{\text{qYM}}^{\text{red.}} + S_{q\theta}, \]

where

\[ S_{q\theta} = -\frac{\theta N g_{\text{YM}}^2}{16\pi^2} \left( \frac{2\pi}{a} \right)^4 \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} \]

and \( \theta \) is the vacuum angle.

The matrix model (1) in the limit \( N \to \infty \) has been shown to describe strings [6]. Here we would like to extend this result and show that the model encoded by \( S_{q}^{\text{red.}} \) includes, in the large-\( N \) limit, not only strings, but also an open 3-brane with a Chern–Simons 2-brane as its dynamical boundary. These higher dimensional objects would provide appropriate models for hadronic bags embedded in a 4-dimensional target spacetime. In this way we would hopefully fill the gap between QCD, as the fundamental quantum field theory of strong interactions, and more phenomenologically oriented models for strongly interacting, confined objects. To make evident the relationship between matrix gauge fields and extended objects we shall implement the effective and simple correspondence among unitary operators in Hilbert space and ordinary functions in a non-commutative phase space [7], which has been originally established by the Wigner–Weyl–Moyal (WWM) formulation of quantum mechanics (see [8] for recent applications of this quantization method).

The eigenvalues of the unitary operator \( D_{\mu i j} \) span a \( D \)-dimensional eigen-lattice [9] with \( 0 \leq D \leq 4 \). Thus \( D_{\mu} \) can be expressed in terms of \( 2D \) independent matrices, \( p_i, q_j, i, j = 1, \ldots, D \), through the WWM relation

\[ D_{\mu} \equiv \frac{1}{(2\pi)^D} \int d^D p d^D q A_\mu(k, z) \exp \left( iq^i p_i + ip^jq_j \right), \]  

(2)

\(^4\) The subscript “\( q \)” means *quenched*, not to be confused with “quantum”. 
where the operators $p_i, q_j$ satisfy the Heisenberg algebra

$$[p_i, q_j] = -i\hbar \delta_{ij}$$

and $(q^i, p^j)$ play the role of coordinates in a Fourier dual space. $\hbar$ is the deformation parameter, which for historical reason is often represented by the same symbol as the Planck constant.

The basic idea under this approach is to identify the Fourier space as the dual of a $(D + D)$-dimensional world-manifold of a $p = 2D - 1$ brane. The case $D = 1$, corresponding to strings, has been investigated in depth because of the relation between the $su(\infty)$ Lie algebra and $sdiff(\Sigma)$, the area preserving diffeomorphism algebra over a 2-dimensional manifold $\Sigma$. Much less attention has been given, in this framework, to higher dimensional objects (a remarkable exception is [10]).

In this letter we shall consider the case $D = 2$ and show that in the limit $N \rightarrow \infty$ the action $S_{red}^q$ becomes a bag action endowed with a dynamical boundary.

The WWM relation establishes a one-to-one correspondence between a linear operator, $D_\mu$ in our case, acting over a Hilbert space $H$ of square integrable functions on $\mathbb{R}^D$ and a smooth function $A_\mu(x, y)$, which is the anti-Fourier transform of $A_\mu(k, z)$ in (2):

$$A_\mu(q, p) = \frac{1}{N} \text{Tr}_H [D_\mu \exp (-ip_i q^i - iq_j p^j)]$$

$$A_\mu(x, y) = \int d^D q d^D p A_\mu(q, p) \exp \left(i q_i x^i + i p_j y^j\right),$$

where $\text{Tr}_H$ means the sum over diagonal elements with respect to an orthonormal basis in $H$. Under the WWM correspondence the matrix product turns into the Moyal product, or $\ast$-product, as follows:

$$UV \longrightarrow U(x, y) \ast V(x, y) \equiv \exp \left[\frac{i \hbar}{2} \left(\frac{\partial^2}{\partial x_i \partial y^i} - \frac{\partial^2}{\partial y^j \partial x^j}\right)\right] U(x, y) V(x', y') \bigg|_{x' = x}^{y' = y}$$

$$\equiv \exp \left[\frac{i \hbar}{2} \omega_{ab} \frac{\partial^2}{\partial \sigma^a \partial \xi^b}\right] U(\sigma) V(\xi) \bigg|_{\sigma = \xi},$$

where $\omega_{ab}$ is the symplectic form over the $2D$-dimensional manifold with $\sigma \equiv (x, y)$ as coordinates. By means of the non-commutative $\ast$-product it is possible to express the commutator between two matrices, $U, V$, as the Moyal Bracket.
between their corresponding Weyl symbols, $U(x, y)$, $V(x, y)$:

$$\frac{1}{\hbar} [U, V] \longrightarrow \{U, V\}_{MB} \equiv \frac{1}{\hbar} (U \ast V - V \ast U) \equiv \omega^{ij} \partial_i U \circ \partial_j V,$$

where we introduced the $\circ$-product, which corresponds to the “even” part of the $\ast$-product [11]. In the limit of vanishing deformation parameter the Moyal bracket reproduces the Poisson bracket

$$\lim_{\hbar \to 0} \{U, V\}_{MB} = \{U, V\}_{PB}.$$

The last step in the mapping of matrix theory into a “field model” is carried out through the identification of the “deformation parameter” $\hbar$ with the inverse of $N$:

$$\hbar \longrightarrow \frac{2\pi}{N} : \quad \lim_{N \to \infty} f(N) \longrightarrow \lim_{\hbar \to 0} f(\hbar).$$

Consistently, the large-$N$ limit of the SU($N$) matrix theory, where the $A_\mu$ quantum fluctuations freeze, corresponds to the quantum mechanical classical limit, $\hbar \to 0$, of the WWM corresponding field theory $^5$ (from now on, we shall refer to the “classical limit” without distinguishing between large-$N$ and small $\hbar$).

Finally, we can rewrite the trace operation as an integration over 2D coordinates

$$\frac{(2\pi)^4}{N^3} \text{Tr}_H \longrightarrow \int d^D x \, d^D y \equiv \int d^{2D} \sigma$$

and [11]

$$\mathcal{F}_{\mu \nu} \equiv \{A_\mu, A_\nu\}_{MB} = \omega^{ab} \partial_a A_\mu \circ \partial_b A_\nu.$$

After this technical detour, let us come back to the two terms in the reduced action (1), which are mapped by the WWM correspondence into

$$W_{qYM}^{\text{red}} = -\frac{1}{4} \left(\frac{2\pi}{a}\right)^4 \frac{N^4}{(2\pi)^3} \frac{2\pi}{N} \frac{1}{g_{YM}^2} \int d^{2D} \sigma \mathcal{F}_{\mu \nu} \ast \mathcal{F}^{\mu \nu}$$

$$= -\frac{1}{16} \left(\frac{2\pi}{a}\right)^4 \frac{N}{(2\pi)}^2 \frac{1}{g_{YM}^2}.$$  

$^5$ It is important to remark that we are not attempting a WWM quantization of the classical quenched field theory, but we are only investigating the effects of deforming the Lie algebraic structure.
\[
W_{\text{red}}^{\mu} = -\frac{g^2_{\text{YM}}}{16\pi^2} \left( \frac{2\pi}{a} \right)^4 \left( \frac{N^4}{2\pi} \right) \frac{g^2_{\text{YM}}}{G} \left( \frac{2\pi}{a} \right)^2 \epsilon^{\mu\nu\rho\sigma} \int d^{2D} \sigma \omega^{\mu} \partial_a A_\mu \partial_b \partial_a A_\nu \partial_{[m} A^{\mu} \partial_{n]} A^\nu \omega^{mn},
\]

We could also have written equation (3) without the \( \ast \)-product between the two Moyal brackets due to the following property of the integration over phase space in the absence of a boundary [12], \( \int d^4 \sigma U \ast V \equiv \int d^4 \sigma U V \), but extra terms will appear whenever boundaries are present [13]. Since these extra terms are the ones in which we will be interested in, we shall keep the \( \ast \)-product in (3).

Before discussing the case \( D = 2 \), it can be useful to show how the classical limit of \( S_1^{\text{red}} \) with \( D = 1 \) is related to the action of a bosonic string, which for simplicity we assume to be closed. In this case only the term \( W_{\text{red}}^{\mu} \) has to be taken into account and gives

\[
W_{\text{red}}^{\mu} \approx -\frac{1}{16} \left( \frac{2\pi}{a} \right)^4 \left( \frac{N^4}{2\pi} \right) \frac{g^2_{\text{YM}}}{G} \left( \frac{2\pi}{a} \right)^2 \int d^{2D} \sigma \omega^{\mu} \partial_a A_\mu \partial_b \partial_a A_\nu \partial_{[m} A^{\mu} \partial_{n]} A^\nu \omega^{mn}.
\]

where we took into account that both the \( \ast \) and \( \circ \) products collapse into the ordinary product in the classical limit [11]. Provided one appropriately rescales the gauge field, (5) reproduces the Schild action for the relativistic, bosonic string [14,15]. Let us remark that the Schild action is not invariant under reparametrization but only under the more restricted group of area preserving diffeomorphisms; this result establishes the known correspondence between SU(\( \infty \)) and \( \text{sdiff}(\Sigma) \) [6].

Moving to the case \( D = 2 \), it can be useful to recall that the action for a \( p \)-brane can be written in several different forms [16]. With hindsight, we need to recall the conformally invariant 4-dimensional \( \sigma \)-model action introduced in [17]

\[
S_{\text{DT}} = -\frac{\mu_p^4}{4} \int d^4 \sigma \sqrt{h} h^{am} h^{bn} \partial_{[a} X^\mu \partial_{b]} X^\nu \partial_{[m} X_\mu \partial_{n]} X_\nu.
\]
Table 1
This table summarizes the various actions for the bulk 3-brane and boundary 2-brane we used in the paper.

| $p$-brane Actions | Brane Fields | Symmetry |
|--------------------|--------------|----------|
| $S_{\text{DT}} \propto \int d^4 \sigma \sqrt{h} h^{am} h^{bn} \partial_a X^\mu \partial_b X^\nu \partial_m X_\mu \partial_n X_\nu$ | $X^\mu(\sigma)$ | Reparametrization + Conformal Invariance |
| $S_{\text{Schild}} \propto \int d^4 \sigma \{X^\mu, X^\nu\}_{\text{PB}} \{X_\mu, X_\nu\}_{\text{PB}}$ | $X^\mu(\sigma)$ | Volume Preserving Diffeomorphisms |
| $S_{\text{NG}} \propto \int d^4 \sigma \sqrt{-\det(\partial_m X_\mu \partial_n X^\mu)}$ | $X^\mu(\sigma)$ | Reparametrization Invariance |
| $S_{\text{CS}} \propto \epsilon_{\lambda \mu \nu \rho} \int d^3 \xi X^\lambda \{X^\mu, X^\nu, X^\rho\}_{\text{NPB}}$ | $X^\mu(\xi)$ | Reparametrization Invariance |

$$S_{\text{CS}} = -\frac{\kappa}{3! \times 4!} \epsilon_{\lambda \mu \nu \rho} \int d^3 \xi X^\lambda \epsilon^{abc} \partial_a X^\mu \partial_b X^\nu \partial_c X^\rho$$

$$= -\frac{\kappa}{4!} \epsilon_{\lambda \mu \nu \rho} \int d^3 \xi X^\lambda \{X^\mu, X^\nu, X^\rho\}_{\text{NPB}} . \ (7)$$

In equation (6) the indices inside square brackets are anti-symmetrized and target spacetime indices are saturated by a flat metric $\eta_{\mu \nu}$. In this $\sigma$-model approach $X^\mu$ would be the coordinates of a 3-brane in 4-dimensional target spacetime and the $\sigma^m$ are coordinates on the 4-dimensional world-volume. Moreover, $\eta_{\mu \rho}$ is the flat Minkowski metric tensor in target spacetime, while $h_{mn}(\sigma)$ is an independent, auxiliary, world-volume metric and provides the reparametrization invariance of the model. It can be worth to remind that, once the auxiliary metric is algebraically solved in terms of the induced metric, i.e. $h_{mn} \propto \partial_m X \cdot \partial_n X$, then $S_{\text{DT}}$ turns into a Nambu–Goto type action. But, the Nambu–Goto action for a 3-brane embedded in a 4-dimensional target spacetime is nothing but the world-volume of the brane itself. Accordingly, the constant $\mu_0^4$ in front of it can be identified with the (constant) pressure inside the bag. Despite its non-trivial look, $S_{\text{DT}}$ does not describe transverse propagating degrees of freedom, but only a constant energy density and constant pressure, non-dynamical spacetime domain. All the dynamics is carried by the boundary of the domain, in a way which seems to satisfy the holographic principle [18] in a very strict sense: all the non-trivial dynamical degrees of freedom are confined to the membrane enclosing the bag. Among various kind of relativistic membranes the Chern–Simons one is a very interesting object [19]. In the action $S_{\text{CS}}$ the 3-volume element of the membrane is represented by the Nambu–Poisson brackets, while $\kappa$ is a constant with dimension of energy per unit 3-volume. The presence of the Nambu–Poisson brackets suggests a new kind of formulation of both classical and quantum mechanics for such
an object, which is worth investigating by itself [20]. Moreover, the formal structures of (3, 4) and (6, 7) are so similar that one expects some kind of relationship among these actions. On the other hand, we notice that while the $W_{\text{qYM}}^{\text{red.}}$ action is defined over a flat phase space, $S_{\text{DT}}$ involves integration over a curved world-volume. In the latter case a Moyal deformation would be no longer valid, due to the lack of associativity of the $\ast$-product, and a Fedosov deformation quantization would be required [21]. However, the Weyl symmetry of the $S_{\text{DT}}$ also suggests to restrict the world metric to the conformally flat sector:

$$h_{mn} = e^{2\phi(\sigma)}\eta_{mn}. \quad (8)$$

We shall not consider in this paper the quantum conformal anomaly, which could spoil the choice (8), and concentrate our attention only over the equivalence between classical actions. Furthermore, we can implement the following relation between the Poisson bracket and the simplectic form in four dimensions:

$$\partial_{[a}X^\mu\partial_{b]}X^\nu = \frac{1}{4}\omega_{ab}\{X^\mu, X^\nu\}_{\text{PB}}.$$ 

Thus, we find that the $S_{\text{DT}}$ action in a conformally flat background geometry looks like a generalized Schild action [14] for a 3-brane:

$$S_{\text{DT}} = -\frac{\mu^4}{16}\int d^4 \sigma \{X^\mu, X^\nu\}_{\text{PB}} \{X^\mu, X^\nu\}_{\text{PB}} \equiv S_{\text{Schild}}. \quad (9)$$

The alleged correspondence between $W_{\text{qYM}}^{\text{red.}}$ and $S_{\text{DT}}$ can be seen as follows. By choosing $D = 2$ in (3), we find

$$W_{\text{qYM}}^{\text{red.}} \approx -\frac{1}{16}\left(\frac{2\pi}{a}\right)^4 N^2 \frac{1}{g_{\text{YM}}^2} \times \int d^4 \sigma \omega_{ab} \partial_{[a}A_{\mu}\partial_{b]}A_{\nu}\omega^{mn}\partial_{[m}A^\mu\partial_{n]}A^\nu$$

$$= -\frac{1}{16}\left(\frac{2\pi}{a}\right)^4 N^2 \frac{1}{g_{\text{YM}}^2} \int d^4 \sigma \{A_{\mu}, A_{\nu}\}_{\text{PB}} \{A^\mu, A^\nu\}_{\text{PB}};$$

$$W_{\text{qYM}}^{\text{red.}} = -\frac{\theta g_{\text{YM}}^2}{16\pi^2}\left(\frac{2\pi}{a}\right)^4 \frac{N^4}{(2\pi)^4} \frac{2\pi}{N} \epsilon^{\mu\nu\rho\sigma} \int \Sigma d^4 \sigma F_{\mu\nu} F_{\rho\sigma}$$

$$= -\frac{\theta g_{\text{YM}}^2}{64\pi^2}\left(\frac{2\pi}{a}\right)^4 \left(\frac{N}{2\pi}\right)^2 \epsilon^{\mu\nu\rho\sigma} \quad (10)$$

In two dimensions every metric is conformally flat. In our case, i.e. four dimensions, we need to require conformal flatness.
\[
\times \int \frac{d^4 \sigma \omega^{[a_b} \partial_a A_{\mu} \partial_b A_{\nu} \partial_m A_{\rho} \partial_n A_{\sigma} \omega^{m_n]}}{\Sigma} = -\frac{\theta g_{YM}^2}{32\pi} \left( \frac{2\pi}{a} \right)^4 \left( \frac{N}{2\pi} \right)^2 \epsilon^{\mu\nu\rho\sigma} \int d^3 s A_{\mu} \{ A_{\nu}, A_{\rho}, A_{\sigma} \}_{NPB}.
\]

By rescaling the fields according with \[7\]

\[
A_{\mu} \rightarrow \left( \frac{2\pi}{N} \right)^{1/4} X_{\mu}, \\
F_{\mu\nu} \rightarrow \left( \frac{2\pi}{N} \right)^{1/2} \{ X_{\mu}, X_{\nu} \}_{PB}
\]

we get

\[
W_{qYM}^{red} + W_{q\theta}^{red} = -\frac{1}{16} \left( \frac{2\pi}{a} \right)^4 \frac{1}{g_{YM}^2} \int d^4 \sigma \{ X_{\mu}, X_{\nu} \}_{PB} \{ X^{\mu}, X^{\nu} \}_{PB}
\]

\[
-\frac{\theta g_{YM}^2}{32\pi} \left( \frac{2\pi}{a} \right)^4 \epsilon^{\mu\nu\rho\sigma} \int d^3 s X_{\mu} \{ X_{\nu}, X_{\rho}, X_{\sigma} \}_{NPB},
\]

which matches the sum of the actions \( S_{DT} \) and \( S_{CS} \) provided we identify

\[
\mu_0^4 \longleftrightarrow \frac{1}{4\pi} \left( \frac{2\pi}{a} \right)^4 \frac{4\pi}{g_{YM}^2} \tag{10}
\]

and

\[
\kappa \longleftrightarrow \frac{\theta g_{YM}^2}{32\pi} \left( \frac{2\pi}{a} \right)^4.
\]

As a consistency check of our dynamically generated bag pressure consider equation (10) in the strong coupling regime, where it is conventionally assumed \[7\] It can be useful to list the dimensions of various quantities in natural units. The main reason is that quenched, dimensional reduced, gauge variables have not canonical dimensions:

\[
[D_{\mu}] = [A_{\mu}] = \text{length}, \quad [F_{\mu\nu}] = (\text{length})^2,
\]

\[
[X^{\mu}] = \text{length}, \quad [\Lambda] = (\text{length})^{-1},
\]

\[
[a] = \text{length}, \quad [\sigma^m] = 1, \quad [\{ X^{\mu}, X^{\nu} \}_{PB}] = (\text{length})^2,
\]

\[
[\mu_0] = (\text{length})^{-1}, \quad [\kappa] = (\text{length})^{-4}.
\]
Fig. 1. The figure shows the web of relationships among the various actions discussed in this note. It can be read as “map” to move from the Yang–Mills action, in the lower left corner, to the complete bag action in the upper right corner.

\[ g_{YM}^2/4\pi \approx 0.18. \]  

If we identify the inverse lattice spacing \(2\pi/a\) with the QCD scale \(\Lambda_{QCD} \approx 200\text{MeV}\), then equation (10) provides

\[ \mu_0^4 \equiv \frac{1}{4\pi} (200\text{MeV})^4 \frac{1}{0.18}. \]

The actual value of \(\mu_0\) is close to \(\Lambda_{QCD}\); this is not a bad result, compared with the phenomenological value \(\mu_0 \approx 110\text{MeV}\), if one takes into account the uncertainty on the value of \(\Lambda_{QCD}\), i.e. \(120\text{MeV} \leq \Lambda_{QCD} \leq 350\text{MeV}\).

In summary, we have shown that reduced, quenched SU(\(N\)) gauge theory can fit, in 4-dimensions and in the large-\(N\) limit, not only strings but 3-branes with a dynamical boundary as well. The WWM correspondence maps the original matrix action (1) into the phase space action (3, 4), where the gauge field strength is replaced by a Moyal bracket for the Weyl symbol of the matrix gauge field. The large-\(N\), or classical, limit of (3) reproduces the Dolan–Tchrakian action \(S_{DT}\) for a 3-brane in the conformally flat background geometry (8) and a Chern–Simons action for its boundary. In even dimensional target spacetime the \(S_{DT}\) functional matches the Schild action (9). In analogy with the string case, the action (9) is not invariant under reparametrization but it is only invariant under residual world-volume preserving diffeomorphisms. With hindsight, this is not a surprise: the reduced symmetry seems to be the memory of the constant inverse volume factor \((2\pi/a)^4\) in front of the original reduced quenched action (1). A similar conclusion was obtained in [22], where the light-cone gauge choice leads to a residual \(p\)-volume preserving diffeomorphisms invariance, while we have \((p + 1)\)-world-volume preserving diffeomorphisms. In the limiting case \(p = 3, D = 4\) the \(S_{DT}\) action degenerates into a pure volume term with no proper dynamics. All the physical degrees of freedom are carried by the Chern–Simons membrane enclosing the bag.
tracing back the bulk and boundary terms to the original Yang–Mills action
the following correspondence will show up:

\[
\text{“glue”: } S_{qYM}^{\text{red}} \longleftrightarrow S_{DT} \propto \text{Bulk Volume}
\]
\[
\text{instantons: } S_{q\theta}^{\text{red}} \longleftrightarrow S_{CS} \propto \text{Boundary Membrane}.
\]

Finally, an order of magnitude estimate of the induced bag constant results
to be in agreement with the phenomenological value, and suggests a model
for hadrons as QCD vacuum bubbles bounded by Chern–Simons membranes.
This new formulation of the hadronic bag model and its generalization to the
case of higher dimensional branes are currently under investigation [23].

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