Essential enhancements in Abelian networks: 
continuity and uniform strict monotonicity

Lorenzo Taggi 
Sapienza Università di Roma 
Dipartimento di Matematica ‘Guido Castelnuovo’

July 28, 2023

Abstract

We prove that in wide generality the critical curve of the activated random walk model is a continuous function of the deactivation rate, and we provide a bound on its slope which is uniform with respect to the choice of the graph. Moreover, we derive strict monotonicity properties for the probability of a wide class of ‘increasing’ events, extending previous results of Rolla and Sidoravicius (2012). Our proof method is of independent interest and can be viewed as a reformulation of the ‘essential enhancements’ technique – which was introduced for percolation – in the framework of Abelian networks.

1 Introduction

The activated random walk model (ARW) is a particle system with conserved number of particles. It is a special case of a class of models introduced by Spitzer in the ’70s and it is not only of great mathematical interest but also physically relevant due to its connections to self-organised criticality [8]. The informal definition of the model is as follows. Let $G = (V, E)$ be an infinite undirected unimodular graph (for example $\mathbb{Z}^d$ or a regular tree). Each particle can either be of type A (active) or of type S (sleeping, or inactive). At time zero, the number of particles is sampled according to a Poisson distribution with parameter $\mu \in [0, \infty)$ independently at every vertex, where $\mu$ is the particle density, and every particle is of type A. An independent exponential clock with rate $\lambda \in [0, \infty)$, the deactivation rate, is associated to every active particle. Every A-particle performs a continuous time simple random walk independently until its own clock rings. When this happens, the A-particle turns into the S-state. Every S-particle is at rest. Moreover, whenever a S-particle shares the vertex with an A-particle, the S-particle is instantaneously activated, i.e, it becomes an A-particle. It follows from this definition that, almost surely, a particle of type S can be observed only if it does not share the vertex with other particles.

Let $P_{\lambda, \mu}$ be the probability measure of the interacting particle system defined informally above, whose existence on unimodular graphs was proved in [14]. A central and natural question is whether the dynamics dies out with time or whether it is sustained at all times. More precisely, we say that the system fixes if for every finite set $A \subset V$ there exists a time $t_A < \infty$ such that for any time $t > t_A$ no active particle jumps from a vertex of $A$, and that it is active if it does not fixate. The critical density is defined as,

$$\forall \lambda \in [0, \infty), \quad \mu_c(\lambda) := \inf \left\{ \mu \in \mathbb{R}_0^+ : P_{\lambda, \mu}(\text{ARW is active}) > 0 \right\}.$$ (1.1)

It was proved in [12] that the probability that the model is active is either zero or one, that it does not decrease with $\mu$ and does not increase with $\lambda$. This ensures the existence of a unique transition point between the regime of a.s. local fixation and the regime of a.s. activity. In recent
years significant effort has been made for proving basic properties of the critical curve, \( \mu = \mu_c(\lambda) \). It was proved in [20] that \( \mu_c(\lambda) \geq \frac{1}{1+\lambda} \) in any vertex-transitive graph, generalising and extending previous results from [12, 19]. It is known from [18] that \( \mu_c(\lambda) \leq 1 \) for any \( \lambda \in [0, \infty) \) in wide generality. It was proved in [3, 14, 20, 21, 22] that on various graphs \( \mu_c(\lambda) < 1 \) for any \( \lambda \in (0, \infty) \) and that \( \mu_c(\lambda) \to 0 \) as \( \lambda \to 0 \). Moreover, it was proved in [11, 13] that, on \( \mathbb{Z} \), \( \mu_c(\lambda) = O(\sqrt{\lambda}) \) in the limit as \( \lambda \to 0 \). It was proved in [13] that the critical density is universal. Our first main theorem states a new general property of the critical curve, namely that it is a continuous function of the deactivation parameter \( \lambda \).

**Theorem 1.1.** On any unimodular graph the two following properties hold:

1. \( \mu_c(\lambda) \) is a continuous function of \( \lambda \) in \((0, \infty)\),
2. for any \( \lambda \in (0, \infty) \), \( \limsup_{\delta \to 0} \frac{\mu_c(\lambda+\delta)-\mu_c(\lambda)}{\delta} \leq \frac{1}{\lambda(1+\lambda)} \).

Continuity of the critical curve at \( \lambda = 0 \) (more precisely, right-continuity, namely \( \lim_{\lambda \to 0^+} \mu_c(\lambda) = \mu_c(0) = 0 \)) in \( \mathbb{Z} \) was proved in [4, 9, 20]. Our Theorem 1.1 generalizes such a continuity property to all positive values of \( \lambda \) and holds for any unimodular graph. Even though the critical curve is expected to strongly depend on the graph, the second claim of Theorem 1.1 provides a bound on its slope which is uniform with respect to the choice of the graph. Moreover, the assumption that the graph is unimodular is only required to give sense to the continuous time dynamics and then to (1.1). For a more general notion of critical density (see equation (6.1) and Remark 6.2 below) our theorem holds on any locally-finite infinite connected graph.

### 1.1 Strict monotonicity properties

Our first main theorem is a consequence of our second theorem, which derives new general monotonicity properties for the probability of a wide class of events, which will be referred to as ‘relevant’. This class includes all the events which are increasing and which depend on how many times the vertices are visited by the particles (we refer to Section 3 for a precise definition). For example, the event \( A = \{ \forall x \in K, M(x) > H(x) \} \), where \( K \subset \mathbb{V} \) is finite, \( M(x) \) is the number of times the active particles jump from \( x \) and \( (H(x))_{x \in K} \) is any integer-valued vector, is relevant. This is an important class of events, since one can deduce whether the system fixates or is active by determining the limiting probability of appropriately defined sequences of these events.

The derivation of monotonicity properties is very useful and allows a deeper understanding of the model. From the definition of the activated random walk dynamics it is reasonable to expect that the probability of any relevant event is non-increasing with respect to \( \lambda \) and non-decreasing with respect to \( \mu \). The proof of this claim is non-trivial and was derived in [12] by employing a graphical representation.

Here we address a related question, namely do monotonicity properties hold if we increase the deactivation rate and the particle density at the same time? This question is challenging, since the increase of the deactivation rate and of the particle density play against each other. Indeed, higher deactivation rate implies that the model is ‘less active’, while higher particle density implies that the model is ‘more active’. Our Theorem 1.2 below studies a regime where a positive increase in \( \mu \) compensates for a small enough increase in \( \lambda \). More precisely, if take an arbitrary point of the phase diagram, \( (\lambda, \mu) \in \mathbb{R}_+^2 \), and we move up-right along a semi-line which starts from \( (\lambda, \mu) \) and whose slope, \( s \), satisfies \( s \geq \frac{1}{\lambda(1+\lambda)} \), then the probability of the event does not decrease. Remarkably, our estimate on the minimal slope is uniform not only with respect to the choice of the graph, but also with respect to the choice of the event, provided that it is relevant. The monotonicity result of Rolla and Sidoravicius [12] can thus be viewed as corresponding to the special case \( s = \infty \) of our theorem.
Theorem 1.2. Consider any unimodular graph, let \( A \) be any relevant event. Let \((\lambda, \mu) \in \mathbb{R}_+^2\) be an arbitrary point of the phase diagram, let \( C_{\lambda, \mu} \) be the region above the semi-line with slope \( \frac{1}{\lambda(1+\lambda)} \) which starts from \((\lambda, \mu)\),

\[
C_{\lambda, \mu} := \left\{ (x, y) \in \mathbb{R}^2 : y \geq \frac{1}{\lambda(1+\lambda)} (x - \lambda) + \mu, x \geq \lambda \right\}.
\] (1.2)

Then, for any pair \((\lambda', \mu') \in C_{\lambda, \mu}\),

\[
P_{\lambda, \mu}(A) \leq P_{\lambda', \mu'}(A).
\] (1.3)

As we show in Section 3 below, relevant events can be defined in the framework of the Diaconis-Fulton representation, which is well-defined on any locally-finite graph. Hence our theorem can be stated in wider generality, see Remark 6.2 below.

1.2 Proof method: Essential enhancements

Our proof method can be viewed as a reformulation of the ‘Essential enhancements’ technique – which was mostly used in Percolation [2, 3] – in the framework of Abelian networks. Our method may find applications in the study of other Abelian models, for example the frog model [10], oil and water [6, 7], the stochastic sandpile model [12], the Abelian sandpiles [11], see also [5]. Our proof uses the setting of the Diaconis-Fulton graphical representation [12], where some random instructions – operators which act on the particle configuration moving active particles to their neighbours or trying to let the A-particle turn into a S-particle – are used to mimic the dynamics without employing the variable ‘time’. Such a graphical representation fulfils the fundamental Abelian property which, informally, states that the relevant quantities – for example the number of times an active particle jumps from each vertex – do not depend on the order according to which such instructions are used.

Our proof is divided into three main steps. The first step of the proof is the derivation of a Russo’s formula [17] – which is a classical formula in percolation – for activated random walks, Theorem 4.4 below. This formula relates the partial derivative with respect to \( \lambda \) of the probability of any relevant event to the expected number of instructions which are ‘sleeping essential’ for the event. Such instructions will be defined later and, informally, are those instructions whose removal would cause the occurrence of the event. Similarly, such a formula relates the partial derivative with respect to \( \mu \) of the probability of any relevant event to the expected number of vertices which are ‘particle essential’ for the event, namely vertices such that the addition of one more particle there would cause the occurrence of the event. In the second step of the proof we derive the following differential inequality, which holds for any relevant event \( A \),

\[
- \frac{\partial}{\partial \lambda} P_{\lambda, \mu}(A) \leq \frac{1}{\lambda(1+\lambda)} \frac{\partial}{\partial \mu} P_{\lambda, \mu}(A),
\] (1.4)

where \( P_{\lambda, \mu} \) is the law of the initial particle configuration and of the random instructions. The two following properties of the odometer – a fundamental quantity which counts how many times an active particle jumps from each vertex – are derived and used for the proof of (1.4). The first property is that the removal of a ‘sleep’ instruction does not affect the value of the odometer, unless such a removed instruction occupies a very specific location in the array of instructions. Such a property allows us to the deduce that, on each vertex, at most one instruction is ‘sleeping essential’. The second property states that if the removal of a sleep instruction lets the event \( A \) occur, then also the addition of a particle at the same vertex lets \( A \) occur, provided that \( A \) is relevant. This leads to the conclusion that if on a vertex we have a sleeping-essential instruction, then the vertex is also particle-essential. Such two properties combined allow the comparison between the partial derivatives and lead to (1.4). In the third step we derive our monotonicity theorem by using the differential inequality, (1.4), and we derive our main continuity theorem by using our monotonicity theorem.
We conclude with some natural questions which might be answered by further developing our framework. To begin, the derivation of the inverse of the inequality \(1.4\) (with some other positive and bounded constant uniformly in \(A\) in place of \(\frac{1}{\lambda (1+\lambda)}\)) would allow us to answer the following open question.

**Open Problem 1.** Prove that \(\mu_c(\lambda)\) is strictly increasing with respect to \(\lambda\).

Our proof shows that the critical density is a continuous function of the deactivation rate and provides a bound for its right and left derivatives, but unfortunately it does not show that the right and left derivatives coincide. This considerations lead to the following natural question, to which we expect the answer to be positive.

**Open Problem 2.** Prove that \(\mu_c(\lambda)\) is differentiable with respect to \(\lambda\).

In the framework of percolation differentiability properties of several quantities of interest have been studied for example in [16].

**Organisation of the paper.** This paper is organised as follows. In Section 2 we recall the properties of the Diaconis-Fulton representation. In Section 3 we introduce the main definitions and discuss the properties of the jump odometer. In Section 4 we present the equivalent of Russo’s formula for activated random walk. In Section 5 we present the proof of (1.4). In Section 6 we present the proof of our main theorems, Theorem 1.1 and 1.2.

**Notation**

We use the notation \(\mathbb{N} = \{1, 2, 3, \ldots\}\), \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\), \(\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}\), and \(\mathbb{R}_+^0 = \{x \in \mathbb{R} : x \geq 0\}\) and the convention \(\inf\{\emptyset\} = \infty\). The following table presents part of the notation which is introduced in Sections 2 and 3 below.

| Symbol | Description |
|--------|-------------|
| \(\eta\) | \(\eta(x)\) for \(x \in V\) particle configuration |
| \(\tau\) | \(\tau(x, j)\) for \(x \in V, j \in \mathbb{N}_0\) array of instructions |
| \(\mathcal{H} \times \mathcal{I}\) | set of realisations, with \(\eta \in \mathcal{H}\) and \(\tau \in \mathcal{I}\) |
| \(\mathcal{S} \subset \mathcal{H} \times \mathcal{I}\) | smallest \(\sigma\)-algebra generated by open subsets of \(\mathcal{H} \times \mathcal{I}\) |
| \(\mathcal{P}_{\lambda, \mu}\) | probability measure on \(\mathcal{H} \times \mathcal{I}\) |
| \(\mathcal{H}_a = \mathbb{N}_0^V\) | set of particle configurations with only active particles |
| \(\tau_{xy}\), with \(\ell \geq 1\) | \(\ell\)th jump instruction of \(\tau\) at \(x\), sleep instruction at \(x\), instruction ‘jump from \(x\) to \(y\)’ |
| \(J_{\rho}^{x,\ell}\) | \(J_{\rho}^{x,\ell}(\tau)\) \(\ell\)th jump instruction of \(\tau\) at \(x\), |
| \(m_{K, \eta, \tau}, M_{K, \eta, \tau}\) | odometer, jump odometer |
| \(\mathcal{W} \subset \mathcal{H} \times \mathcal{I}\) | \(\{\eta, \tau \in \mathcal{H} : m_{K, \eta, \tau}(x) < \infty\text{ for any finite } K \subset V \text{ and } x \in K\}\) number of s. instr. between the \(\ell - 1\)th and the \(\ell\)th j. instr. |
| \(\mathcal{S}_{\rho}^{x, \ell}\) | array with no s. instr. between the \(k - 1\)th and the \(k\)th j. instr. |
| \(\Gamma_{\rho}^{x, k}(\tau)\) | array with one s. instr. between the \(k - 1\)th and the \(k\)th j. instr. |
| \(\nu_j, \nu_{>j}\) | probability that a vertex hosts \(j\) (resp. \(>j\)) particles |
| \(\nu_j'\) | derivative of \(\nu_{>j} = \nu_{>j}(\mu)\) |

2 Definitions and graphical representation

In this section we introduce the Diaconis-Fulton graphical representation for the dynamics of ARW, partially following [12].
2.1 Particle configuration and array of instructions

To begin, we fix a graph $G = (V, E)$, which is always assumed to be undirected, connected, infinite, and locally finite. For any $x \in V$, we denote by $d_x$ the degree of the vertex $x$, which corresponds to the number of vertices which are connected to $x$ by an edge. We refer to the arbitrary chosen vertex $o \in V$ as root. We write $x \sim y$ when $x$ and $y$ are neighbours, i.e., $\{x, y\} \in E$. The set of particle configurations is denoted by $H = \{0, \rho, 1, 2, 3, \ldots\}^V$, where a vertex being in state $\rho$ denotes that the vertex has one $S$-particle, while being in state $i \in \{0, 1, 2, \ldots\}$ denotes that the vertex contains $i$ A-particles. We employ the following order on the states of a vertex: $0 < \rho < 1 < 2 < \cdots$. In a configuration $\eta \in H$, a vertex $x \in V$ is called stable if $\eta(x) \in \{0, \rho\}$, and it is called unstable if $\eta(x) \geq 1$. We denote by $I$ the set of arrays of instructions, i.e., each element of $I$ is an array of instructions $\tau = (\tau_{x,y})_{x,y \in \mathbb{N}_0}$, where for each $x \in V$ and $j \in \mathbb{N}_0$,

$$\tau_{x,j} \in \{\tau_{x,y} \cup \{\tau_{x,y} : y \sim x\},$$

where $\tau_{x,y}$ and $\tau_{x,y}$, called jump and sleep instruction respectively, are operators acting on the particle configuration which are defined as follows. Given any configuration $\eta$ such that $x$ is unstable, performing the instruction $\tau_{x,y}$ in $\eta$ yields another configuration $\eta'$ such that $\eta'(z) = \eta(z)$ for all $z \in V \setminus \{x, y\}$, $\eta'(x) = \eta(x) - 1 \eta\{x \geq 1\}$, and $\eta'(y) = \eta(y) + 1 \{\eta(x) \geq 1\}$. We use the convention that $1 + \rho = 2$, while $k - 1$ is defined only if $k \geq 1$. Similarly, performing the instruction $\tau_{x,y}$ to $\eta$ yields a configuration $\eta'$ such that $\eta'(z) = \eta(z)$ for all $z \in V \setminus \{x\}$, and if $\eta(x) = 1$ we have $\eta'(x) = \rho$, otherwise $\eta'(x) = \eta(x)$. Note that $\tau_{x,y}$ and $\tau_{x,y}$ cannot be applied if $\eta(x) \in \{0, \rho\}$.

2.2 Use of the instructions and stabilisation of a set

Fix a particle configuration $\eta \in H$ and an instruction array $\tau \in I$. We say that the instruction $\tau_{x,y}$ is legal for $x$ if $x$ is unstable in $\eta$, otherwise it is illegal. We say that we use the instruction $(x, j)$, $x \in V$, $j \in \mathbb{N}$, of the array $\tau$ for $\eta$, or that we use the instruction $\tau_{x,y}$ for $\eta$, when we act on the current particle configuration $\eta$ through the operator $\tau_{x,y}$. When we use an instruction $(x, j)$ for some $j \in \mathbb{N}$, sometimes we may simply say that ‘we topple $x$’. Let $\alpha$ be a sequence

$$\alpha = ((x_1, n_1), (x_2, n_2), \ldots, (x_k, n_k)),$$

define the operator $\Phi_{\alpha,\tau}$ as

$$\Phi_{\alpha,\tau} := \tau_{x_{\ell},n_{\ell}} \cdots \tau_{x_2,n_2} \tau_{x_1,n_1},$$

and for $1 \leq \ell \leq k$ define the subsequence $\alpha^{(\ell)} := ((x_1, n_1), (x_2, n_2), \ldots, (x_\ell, n_\ell))$. We say that $\alpha$ is a legal sequence for $\eta$ if the three following properties hold at the same time:

(i) For any $x \in V$, let $u_x := \inf\{\ell \in \{1, \ldots, k\} : x_\ell = x\}$. If $u_x < \infty$, then $n_{u_x} = 0$. In other words, the first instruction which is used at any vertex $x$ is $\tau_{x,0}$.

(ii) For any $i \in \{1, \ldots, k - 1\}$, let $j(i) := \inf\{\ell > i : x_\ell = x_i\}$. If $j(i) < \infty$, then $n_{j(i)} = n_i + 1$. In other words, every time we use an instruction, we use the one which is located ‘right above’ the one which was used right before at the same vertex.

(iii) For any $i \in \{1, \ldots, k\}$, $\tau_{x_i,n_i}$ is legal for $\eta_{i-1} := \Phi_{\alpha^{(i-1)},\tau} \eta$.

Let $m_\alpha = (m_\alpha(x) : x \in V)$ be given by $m_\alpha(x) = \sum_{i \in \{1, \ldots, k\}} \mathbb{1}\{x_i = x\}$, the number of times the vertex $x$ appears in $\alpha$. Let $M_{\alpha,\tau} = (M_{\alpha,\tau}(x) : x \in V)$ be given by,

$$M_{\alpha,\tau}(x) = \sum_{i=1}^{k} \mathbb{1}\{x_i=x, \tau_{x_i,n_i} \neq \tau_{x_i,0}\},$$

the number of jump instructions of $\alpha$. Let $K$ be a finite subset of $V$. A configuration $\eta$ is said to be stable in $K$ if all the vertices $x \in K$ are stable. We say that $\alpha$ is contained in $K$ if $x_i \in K$ for any $i \in \{1, \ldots, k\}$. We say that $\alpha$ stabilizes $\eta$ in $K$ if every $x \in K$ is stable in $\Phi_{\alpha,\eta}$. 

5
Figure 1: An array $\tau \in I$ with $S^{y,1}_\tau = 1$, $S^{y,2}_\tau = 0$, $t^{y,1}_\tau = 1$ and $t^{y,3}_\tau = 3$. In the figure we assume that the instructions below the bold profile are those which have been used for the stabilisation of $\eta \in H_a$ in $[0,L]$, where the particles of $\eta$ correspond to the black circles. The array of instructions $\Gamma^{y,n}_\tau(\tau)$, with $n = M_{K,\eta,\tau}(y) + 1 = 5$, is obtained from $\tau$ by ‘removing’ the two dark sleep instructions above the vertex $y$.

2.3 Odometers and Abelian property

For any subset $K \subset V$, any $x \in V$, any particle configuration $\eta$, and any array of instructions $\tau$, we define

$$m_{K,\eta,\tau}(x) := \sup_{\alpha \subset K} m_{\alpha}, \quad M_{K,\eta,\tau}(x) := \sup_{\alpha \subset K} M_{\alpha,\tau},$$

where the sup is taken over the legal sequences of instructions which are contained in $K$. We refer to $m_{K,\eta,\tau}$ as the odometer function, or simply odometer, and to $M_{K,\eta,\tau}$ as jump odometer. The following lemma gives a fundamental property of the Diaconis-Fulton representation. For the proof we refer to [15].

Lemma 2.1 (Abelian Property). Let $(\eta,\tau) \in H \times I$, fix any finite set $K \subset V$. If $\alpha$ and $\beta$ are both legal sequences for $\eta$ that are contained in $K$ and stabilize $\eta$ in $K$, then $m_{K,\eta,\tau} = m_\alpha = m_\beta$, and $M_{K,\eta,\tau} = M_\alpha = M_\beta$. In particular, $\Phi_\alpha \eta = \Phi_\beta \eta$.

2.4 Counters

It will be useful to identify the jump or sleep instructions placed at specific locations of the array. For this reason we introduce some very useful variables, which depend on the instruction array and on some indices. Let $\tau \in I$ be an array of instructions, fix a vertex $x \in V$ and an integer $m \in \mathbb{N}$. We let $J^{x,m}_\tau$ be the $m$-th jump instruction of $\tau$ at $x$ and $t^{x,m}_\tau$ be its corresponding index. More precisely, we set $t^{x,0}_\tau := -1$, and, for any $m \in \mathbb{N}$, we define

$$t^{x,m}_\tau := \min\{n > t^{x,m-1}_\tau \ : \ \tau^{x,n} \neq \tau_{x,\rho}\}, \quad J^{x,m}_\tau := \tau^{x,t^{x,m}_\tau}. \quad (2.1)$$

Moreover, for any $m \in \mathbb{N}$ we let $S^{x,m}_\tau$ be the number of sleep instructions of $\tau$ at $x$ between the $m - 1$ th and the $m$ th jump instruction,

$$S^{x,m}_\tau := t^{x,m}_\tau - t^{x,m-1}_\tau - 1. \quad (2.2)$$

For example, the variable $S^{y,1}_\tau$ represents the number of sleep instructions which are located before the first jump instruction at $x$, and this variable equals zero if the first instruction at $x$ is a jump instruction. See Figure 1 for an example.
2.5 Partial orders and monotonicity properties

We now introduce a partial order between particle configurations and arrays of instructions. Given two particle configurations $\eta, \eta' \in \mathcal{H}$, we write $\eta' \geq \eta$ if $\eta'(x) \geq \eta(x)$ for all $x \in V$. Given two arrays $\tau, \tau'$, we write $\tau' \geq \tau$ if

$$\forall x \in V, \forall m \in \mathbb{N}, \quad J^{x,m}_\tau = J^{x,m}_{\tau'} \quad S^{x,m}_\tau \geq S^{x,m}_{\tau'}.$$  

In other words, either $\tau' = \tau$ or $\tau'$ is obtained from $\tau$ by removing some sleep instructions. The next lemma presents the monotonicity properties of the Diaconis-Fulton representation, which is a straightforward adaptation of [12, Lemmas 3 and 5].

Lemma 2.2 (Monotonicity). If $K_1 \subset K_2 \subset V$, $\eta \leq \eta'$, $\tau \leq \tau'$, then $M_{K_1,\eta,\tau} \leq M_{K_2,\eta',\tau'}$

By monotonicity, given any growing sequence of subsets $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq V$ such that $\lim_{m \to \infty} V_m = V$, the limits

$$m_{\eta,\tau} := \lim_{m \to \infty} m_{V_m,\eta,\tau}, \quad M_{\eta,\tau} := \lim_{m \to \infty} M_{V_m,\eta,\tau},$$

exist and do not depend on the particular sequence $\{V_m\}_m$.

2.6 Probability measure and initial particle distribution

We now introduce a probability measure on the space of particle configurations and arrays of instructions. The distribution of the initial particle configuration is supported in $H \times I$. Parameter $\mu$ is denoted by $\nu$. The distribution of the initial particle configuration is supported in $H$. We now introduce a probability measure on the space of particle configurations and arrays of instructions. Given two particle configurations $\eta, \eta' \in \mathcal{H}$, we write $\eta' \geq \eta$ if $\eta'(x) \geq \eta(x)$ for all $x \in V$. Given two arrays $\tau, \tau'$, we write $\tau' \geq \tau$ if

$$\forall x \in V, \forall m \in \mathbb{N}, \quad J^{x,m}_\tau = J^{x,m}_{\tau'} \quad S^{x,m}_\tau \geq S^{x,m}_{\tau'}.$$  

In other words, either $\tau' = \tau$ or $\tau'$ is obtained from $\tau$ by removing some sleep instructions. The next lemma presents the monotonicity properties of the Diaconis-Fulton representation, which is a straightforward adaptation of [12, Lemmas 3 and 5].

Lemma 2.2 (Monotonicity). If $K_1 \subset K_2 \subset V$, $\eta \leq \eta'$, $\tau \leq \tau'$, then $M_{K_1,\eta,\tau} \leq M_{K_2,\eta',\tau'}$

By monotonicity, given any growing sequence of subsets $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq V$ such that $\lim_{m \to \infty} V_m = V$, the limits

$$m_{\eta,\tau} := \lim_{m \to \infty} m_{V_m,\eta,\tau}, \quad M_{\eta,\tau} := \lim_{m \to \infty} M_{V_m,\eta,\tau},$$

exist and do not depend on the particular sequence $\{V_m\}_m$.

3 Relevant events and properties of the jump odometer

The goal of this section is to introduce the definition of increasing and relevant events and discuss some properties of the jump odometer. Recall that $\mathcal{H}$ denotes the set of particle configurations and that $I$ denotes the set of arrays of instructions. Let $S$ be the smallest sigma-algebra generated by all the open subsets of $\mathcal{H} \times I$ with respect to the natural product topology.
Definition 3.1. We say that an event $A \in S$ is **increasing** if

$$\eta, \tau \in A, \quad \eta \geq \tilde{\eta}, \quad \tilde{\tau} \geq \tau \implies (\tilde{\eta}, \tilde{\tau}) \in A.$$  

Definition 3.2. An event $A \in S$ is said to be **relevant** if it can be written as,

$$A = \{\forall x \in K \ M_{K,\eta,\tau}(x) \geq H(x)\},$$

for some finite $K \subset V$ and for some function $H \in \mathbb{N}_0^K$. We refer to the set $K$ as **domain** of $A$.

For example, the event $\{M_{K,\eta,\tau}(\alpha) \geq L\}$, for $L \in \mathbb{N}$ and $\alpha \in K \subset V$, is relevant and has domain $K$, since it can be written as $\{M_{K,\eta,\tau}(x) \geq H(x) \ \forall x \in K\}$ for the function $H \in \mathbb{N}_0^K$ which is such that $H(x) = L \delta_o(x)$ for any $x \in K$. Note that, by Lemma 2.2, any relevant event is increasing.

We now discuss some properties of the jump odometer and of the relevant events. For any arbitrary pair $(y, m) \in V \times \mathbb{N}$ we introduce two operators, $\Gamma^{y,m}_1, \Gamma^{y,m}_0 : \mathcal{I} \to \mathcal{I}$ acting on the instruction array as follows. For an arbitrary array $\tau \in \mathcal{I}$, we let $\Gamma^{y,m}_0(\tau) \in \mathcal{I}$ be the new array of instructions which is obtained from $\tau$ by removing all the sleep instruction between the $m$th and the $m+1$th jump instruction at $y$. More precisely, $\Gamma^{y,m}_0(\tau)$ is defined as the unique array such that for any $x \in V$ and $k \in \mathbb{N},$

$$J^{x,k}_{\Gamma^{y,m}_0(\tau)} = J^{x,k}_\tau \quad \text{and} \quad S^{x,k}_{\Gamma^{y,m}_0(\tau)} = \begin{cases} 0 & \text{if } x = y \text{ and } k = m \\ S^{x,k}_\tau & \text{otherwise}, \end{cases}$$

where we recall that the variables $J^{x,m}_\tau$ and $S^{x,m}_\tau$ were defined in Section 2.4. See Figure 1 for an example. Moreover, we define a new instruction array $\Gamma^{y,m}_1(\tau) \in \mathcal{I}$, which is is obtained from $\tau$ by setting to one the number of sleep instructions between the $m$th and the $(m+1)$th jump instruction at $y$. More precisely, $\Gamma^{y,m}_1(\tau)$ is defined as the unique array such that for any $x \in V$ and $k \in \mathbb{N},$

$$J^{x,k}_{\Gamma^{y,m}_1(\tau)} = J^{x,k}_\tau \quad \text{and} \quad S^{x,k}_{\Gamma^{y,m}_1(\tau)} = \begin{cases} 1 & \text{if } x = y \text{ and } k = m \\ S^{x,k}_\tau & \text{otherwise}. \end{cases}$$

We now discuss some properties of the jump odometer. To begin, introduce the set $W$ of pairs $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ such that, for any finite $K \subset V$ and any $x \in K$, $m_{K,\eta,\tau}(x) < \infty$. Any $(\eta, \tau) \in W$ is such that the stabilisation of any finite set of sites uses a finite number of instructions and, clearly, $P_{\lambda,\mu}(W) = 1$. The first simple lemma states that only the removal of the sleep instructions which have been used at last at sites during the stabilisation might affect the jump odometer.

**Lemma 3.3.** Consider a pair $(\eta, \tau) \in W$, let $K \subset V$ be a finite set, fix an arbitrary vertex $y \in V$. For any $n \in \mathbb{N}$ such that $n \neq M_{K,\eta,\tau}(y) + 1$, we have that

$$M_{K,\eta,\tau} = M_{K,\eta,\Gamma^{y,n}_0(\tau)}.$$  

**Proof.** Fix $(\eta, \tau) \in W$, $y \in V$ and $n \in \mathbb{N}$. The claim is obvious if $n > M_{K,\eta,\tau}(y) + 1$, if $y \notin K$ or if $\Gamma^{y,n}_0 = 0$, hence we can assume that $n \leq M_{K,\eta,\tau}(y)$, $y \in K$ and that $\Gamma^{y,n}_0 > 0$. We set for brevity $\tau' = \Gamma^{y,n}_0(\tau)$. We first perform a stabilisation of $K$ under the constraint that no instruction after the $n - 1$th jump instruction at $y$ is used. More precisely, we perform a legal sequence of topplings as follows, namely we use any instruction at $x \in K$ with $x \neq y$ as long as $x$ is unstable, and we use any instruction at $y$ as long as $y$ is unstable and no instruction after the $n - 1$th jump instruction at $y$ is used. We iterate this procedure until no further instruction can be, unless violating such constraints. When we have done, we obtain a particle configuration $\eta'$ which is such that,

$$\eta'(x) = \begin{cases} \geq 2 & \text{if } x = y, \\ \in \{0, \rho\} & \text{if } x \neq y, \end{cases}$$

(3.2)

The fact that $\eta'(x) \in \{0, \rho\}$ for any $x \neq y$ follows from the definition of our toppling procedure. The fact that $\eta'(y) \geq 2$ will be now proved by contradiction. Indeed, suppose that this was not
true, namely that either (a) \( \eta'(y) \in \{0, \rho\} \) or (b) \( \eta'(y) = 1 \). If (a) was true, then we would have stabilized \( \eta \) in \( K \) using strictly less than \( M_{K,\eta,\tau}(y) \) jump instructions at \( y \), contradicting our assumption. Similarly, if (b) was true, then by using one more instruction at \( y \) for \( \eta \) (which is a sleep instruction by assumption) we would have stabilized \( \eta \) in \( K \) using strictly less than \( M_{K,\eta,\tau}(y) \) jump instructions, contradicting again our assumptions. This leads to (3.2) as desired. We now complete the stabilisation of \( \eta \) in \( K \) starting from the particle configuration \( \eta' \) and following an arbitrary stabilisation procedure and denote by \( \alpha \) the sequence of instructions which were used for the stabilisation of the initial particle configuration. Since \( \eta'(y) \geq 2 \), the next \( S_{\eta,\tau}^{0,n} \) instructions at \( y \), which are of type sleep by assumption, do not affect the particle configuration. Hence, they can be removed from the sequence \( \alpha \) and from the array \( \tau \) without the jump odometer being affected from this removal. This leads to the new array \( \tau' \) and to a new legal sequence of instructions of \( \tau', \alpha' \), which stabilises \( \eta \) in \( K \) and satisfies \( M_{\alpha,\tau} = M_{\alpha',\tau'} \). By the Abelian property, the proof is concluded. 

\[\Box\]

The next simple lemma states that the jump odometer does not depend on the precise number of sleep instructions between any two consecutive jump instructions, but only on whether such a number is zero or strictly positive.

**Lemma 3.4.** Consider a pair \( (\eta, \tau) \in W \), let \( K \subset V \) be a finite set, fix an arbitrary vertex \( y \in V \). For any integer \( n \in \mathbb{N} \) such that \( S_{\eta,\tau}^{0,n} > 0 \) we have, 

\[ M_{K,\eta,\tau} = M_{K,\eta,1^{\alpha,n}(\tau)} \]

**Proof.** The proof of the claim is similar to the one of Lemma 3.3. We assume that \( S_{\eta,\tau}^{0,n} > 1 \), the claim is trivial otherwise. If \( n > M_{K,\eta,\tau}(y) + 1 \) or \( y \notin K \), then the proof is trivial. If \( n < M_{K,\eta,\tau}(y) + 1 \), then the proof is analogous to the proof of Lemma 3.3. Suppose then that \( n = M_{K,\eta,\tau}(y) + 1 \) (a glance at Figure [ ] may help). We first perform a stabilisation of \( K \) under the constraint that no instruction after the \( n - 1 \)th jump instruction at \( y \) is used, as defined in the proof of Lemma 3.3.

We call \( \eta' \) the particle configuration we obtain. We claim that, 

\[ \eta'(x) = \begin{cases} 0, & \text{if } x = y, \\ \rho, & \text{if } x \neq y. \end{cases} \]  

(3.3)

The fact that \( \eta'(x) \in \{0, \rho\} \) for any \( x \neq y \) follows from the definition of our toppling procedure. The fact that \( \eta'(y) \in \{0,1\} \) will be now proved by contradiction. Indeed, if \( \eta'(y) > 1 \), then in order to conclude the stabilisation of \( K \) it would be necessary to use at least one additional jump instruction at \( x \), and this would imply that more than \( M_{K,\eta,\tau}(y) \) jump instructions are used at \( y \) for the stabilisation of \( K \), thus contradicting our assumptions. Moreover, if \( \eta'(y) = \rho \), then this would mean that the last instruction which was used at \( y \) was a sleep instruction and, since this sleep instruction must be located before the \( n - 1 \)th jump instruction at \( y \) (unless violating the definition of our toppling procedure), this implies that the initial particle configuration has been stabilised using strictly less than \( M_{K,\eta,\tau}(y) \) jump instruction at \( y \), leading again to a contradiction. 

This leads to (3.3). We now complete the stabilisation. In case \( \eta'(y) = 1 \), the use of the next instruction at \( y \), which is sleep by assumption, stabilises the particle configuration \( \eta' \) in \( K \). Hence, the removal of the next \( S_{\eta,\tau}^{0,n} - 1 > 0 \) sleep instructions at \( y \) does not affect the jump odometer, since these instructions are not used for the stabilisation. Similarly, in case \( \eta'(y) = 0 \), we conclude that \( \eta' \) is already stable and that no sleep instruction between the \( n - 1 \)th and the \( n \)th jump instruction at \( y \) was used for the stabilisation of \( \eta \) in \( K \). Hence, by the Abelian property, also in this case the removal of the next \( S_{\eta,\tau}^{0,n} - 1 \) sleep instructions at \( y \) from the array does not affect the jump odometer. This concludes the proof. 

\[\Box\]

The next lemma is an immediate application of the previous one and states that any relevant event does not depend on the precise number of sleep instructions which are located between
two consecutive jump instructions, but only on whether this number is zero or strictly positive. The lemma also states the obvious fact that the relevant event \([3.1]\) does not depend on the instructions outside its domain and on the instructions at sites in the domain which are not used for the stabilisation of the domain.

**Lemma 3.5.** Consider an arbitrary finite set \(K \subset V\), let \(A\) be any relevant event with domain \(K\). For every \((\eta_1, \tau_1), (\eta_2, \tau_2) \in W\) satisfying \(\eta_1(x) = \eta_2(x)\) and

\[
\forall j \in \{1, \ldots, M_{K,\eta_1,\tau_1}(x) + 1\} \quad S_{\tau_1}^{x,j} > 0 \iff S_{\tau_2}^{x,j} > 0 \quad \text{and} \quad J_{\tau_1}^{x,j} = J_{\tau_2}^{x,j},
\]

for every \(x \in K\), we have that

\[M_{K,\eta_2,\tau_2} = M_{K,\eta_1,\tau_1}.\]  \((3.4)\)

This in turn implies that,

\[(\eta_1, \tau_1) \in A \iff (\eta_2, \tau_2) \in A.\]  \((3.5)\)

**Proof.** Let \((\eta_1, \tau_1), (\eta_2, \tau_2)\) be as in the assumptions of the lemma. Let \(U\) be the set of pairs \((x, n) \in K \times \mathbb{N}\) such that \(1 \leq n \leq M_{K,\eta_1,\tau_1}(x) + 1\) and \(S_{\tau_1}^{x,n} > 0\). Let \((x_1, n_1), \ldots, (x_k, n_k)\) be a sequence of elements in \(U\) such that each element of \(U\) appears one time in the sequence. Define the arrays \(\tau_1' := \Gamma_{1,1}^{x_1,n_1} \Gamma_{1,2}^{x_2,n_2} \cdots \Gamma_{1,k}^{x_k,n_k} \tau_1\) and \(\tau_2' := \Gamma_{1,1}^{x_1,n_1} \Gamma_{1,2}^{x_2,n_2} \cdots \Gamma_{1,k}^{x_k,n_k} \tau_2\). By Lemma 3.4 we deduce that \(M_{K,\eta_1,\tau_1} = M_{K,\eta_1,\tau_1}'\) and that \(M_{K,\eta_2,\tau_2} = M_{K,\eta_2,\tau_2}'\). Moreover, note that if \((y, j)\) belongs to a sequence of instructions of \(\tau'_1\) which stabilises \(\eta_1\) in \(K\), by our assumptions on \(\tau_1, \tau_2\) and by the fact that \(j \leq M_{K,\eta_1,\tau_1}(y) + 1\) we then have that \(\tau_1'^{y,j} = \tau_2'^{y,j}\). Hence, by the Abelian property and by the fact that \(\eta_1\) and \(\eta_2\) are identical in \(K\) we have that \(M_{K,\eta_1,\tau_1}' = M_{K,\eta_2,\tau_2}'\). Combining the identities we derived so far we obtain \((3.4)\), as desired. The claim \((3.5)\) now follows immediately from the definition of relevant event.

\(\square\)

### 4 Partial derivatives

In this section we introduce the notion of particle and sleeping essential pairs and present our formula for the partial derivatives of the probability of relevant events. For an arbitrary particle configuration \(\eta \in \mathcal{H}\), vertex \(x \in V\) and integer \(k \in \mathbb{N}_0\), we denote by \(\eta^{(x,k)} \in \mathcal{H}\) the particle configuration which is obtained from \(\eta\) by setting \(k\) (active) particles at \(x\), i.e,

\[\eta^{(x,k)}(y) := \begin{cases} k & \text{if } y = x, \\ \eta(y) & \text{if } y \neq x. \end{cases}\]

Let now \(A \in S\) be an arbitrary event.

**Definition 4.1.** For every vertex \(x \in V\) and integer \(k \in \mathbb{N}_0\), we define the event \{the pair \((x, k)\) is particle-essential\} for the event \(A\) as the set of realisations \((\eta, \tau) \in \mathcal{H} \times \mathcal{I}\) such that,

\[(\eta^{(x,k)}, \tau) \notin A \quad \text{and} \quad (\eta^{(x,k+1)}, \tau) \in A.\]

Sometimes, we will write \(p\)-essential in place of particle-essential.

**Definition 4.2.** For every vertex \(x \in V\) and integer \(k \in \mathbb{N}\), we define the event \{the pair \((x, k)\) is sleeping-essential for \(A\)\} as the set of realisations \((\eta, \tau) \in \mathcal{H} \times \mathcal{I}\) such that

\[(\eta, \Gamma_{1}^{x,k}(\tau)) \notin A \quad \text{and} \quad (\eta, \Gamma_{-}^{x,k}(\tau)) \in A.\]

Sometimes, we will write \(s\)-essential in place of sleeping-essential.
we denote by \( \nu \) define \( k \) correspond to the jump instructions. We start with the construction of the For every \( m, \nu \in \mathbb{N} \) independent random variables in \( \Sigma \) corresponding to different values of \( \nu \) we define \( \nu_{\nu,j} = \nu_{\nu}(j) := \sum_{\ell \geq j} \nu_{\ell} \), the probability that a vertex hosts more than \( j \) particles and let \( \nu'_{\nu,j} := \frac{d}{d\mu} \nu_{\nu,j}(\mu) \) be its derivative with respect to \( \mu \), noting that \( \nu'_{\nu,j} = \nu_{\nu} \).

**Theorem 4.4.** Let \( \mathcal{A} \) be any relevant event. The function \( \mathcal{P}_{\lambda,\mu}(\mathcal{A}) \) is differentiable in \( \mathbb{R}_+^2 \) and, for every \( (\lambda', \mu') \in \mathbb{R}_+^2 \), we have that,

\[
\frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda,\mu}(\mathcal{A})|_{\lambda', \mu'} = -\left(\frac{1}{1 + \lambda'}\right)^2 \sum_{y \in \mathcal{V}, j \in \mathbb{N}} \mathcal{P}_{\lambda',\mu'}((y, j) \text{ is sleeping-essential for } \mathcal{A}),
\]

\[
\frac{\partial}{\partial \mu} \mathcal{P}_{\lambda,\mu}(\mathcal{A})|_{\lambda', \mu'} = \sum_{y \in \mathcal{V}, j \in \mathbb{N}} \mathcal{P}_{\lambda',\mu'}((y, j) \text{ is particle-essential for } \mathcal{A}) \nu'_{\nu,j}(\mu').
\]

The remainder of this section is devoted to the proof of Theorem 4.4, which is divided into three subsections. In Section 4.1 we introduce a coupling which allows the comparison of ARW-systems with different values of \( \mu \) and \( \lambda \). In Sections 4.2 and 4.3 we will use such a coupling to present the proof of (4.1) and (4.2), respectively. From now on, we will write \( \frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda,\mu}(\mathcal{A}) \) in place of \( \frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda,\mu}(\mathcal{A})|_{\lambda', \mu'} \), sometimes we will write \( \partial_\lambda \) for \( \frac{\partial}{\partial \lambda} \), and we will do the same for the partial derivative with respect to \( \mu \).

### 4.1 Probability space for coupled activated random walk models

We now introduce a new probability space which allows us to couple activated random walk systems corresponding to different values of \( \mu \geq 0 \) and \( \lambda \geq 0 \). This new probability space will be denoted by \( (\Sigma, \mathcal{F}, \mathcal{P}) \). To begin, let \( (X_x)_{x \in \mathcal{V}}, (Y_{x,m})_{x \in \mathcal{V}, m \in \mathbb{N}}, \) and \( (A_{x,m})_{x \in \mathcal{V}, m \in \mathbb{N}} \) be three sequences of independent random variables in \( (\Sigma, \mathcal{F}, \mathcal{P}) \) which are distributed as follows. The variables \( (X_x)_{x \in \mathcal{V}} \) and \( (Y_{x,m})_{x \in \mathcal{V}, m \in \mathbb{N}} \) have uniform distribution in \([0,1] \), while the variables \( (A_{x,m})_{x \in \mathcal{V}, m \in \mathbb{N}} \) are such that, for each \( x \in \mathcal{V} \), and \( m \in \mathbb{N}, A_{x,m} \) takes values in \( \{ \tau_{xy} : x \in \mathcal{V}, y \sim x \} \), and has distribution

\[
\mathcal{P}(A_{x,m} = \tau_{xy}) = \frac{1}{d_x}.
\]

The variables \( (X_x)_{x \in \mathcal{V}} \) will be used to sample the initial particle configurations, the variables \( (Y_{x,m})_{x \in \mathcal{V}, m \in \mathbb{N}} \) will be used to sample the sleep instructions, and the variables \( (A_{x,m})_{x \in \mathcal{V}, m \in \mathbb{N}} \) will correspond to the jump instructions. We start with the construction of the initial particle configuration. Define the function \( \eta_{\mu} : \Sigma \rightarrow \mathcal{H} \), which depends on the parameter \( \mu \geq 0 \). For every \( x \in \mathcal{V} \), let \( k \in \mathbb{N}_0 \) be the unique integer such that \( X_x \in [\nu_{<k}(\mu), \nu_{<k+1}(\mu)) \), where \( \nu_{<k}(\mu) := 1 - \nu_{>k-1}(\mu) \) and \( \nu_{<0}(\mu) := 0 \). Then, \( \eta_{\mu}(x) := k \). Note that, it follows by construction that,

\[
\forall \mu \geq 0, \quad \forall x \in \mathcal{V}, \quad \mathcal{P}(\eta_{\mu}(x) = k) = \nu_k(\mu),
\]

and that the variables \( (\eta_{\mu}(x))_{x \in \mathcal{V}} \) are independent. We now construct the array of instructions. For every \( m \in \mathbb{N} \) and \( x \in \mathcal{V} \), we define the functions \( R_{\lambda,m}^x : \Sigma \rightarrow \mathbb{N} \), which represent the number of sleep instructions between the \( m - 1 \)-th and the \( m \)-th jump instruction at \( x \) and depend on the parameter \( \lambda \in [0, \infty) \),

\[
R_{\lambda,m}^x := \begin{cases} \ell & \text{if } Y_{x,m} \in \left( \left( \frac{\lambda}{1+\lambda} \right)^{\ell+1}, \left( \frac{\lambda}{1+\lambda} \right)^\ell \right] \\ 0 & \text{otherwise.} \end{cases}
\]
Note that, by construction,

\[ \forall \lambda \in \mathbb{R}_+^+, \; \forall x \in V, \; \forall m \in \mathbb{N}, \; \forall \ell \in \mathbb{N}_0, \; \mathcal{P}(R_{x,m}^\lambda = \ell) = \frac{1}{1 + \lambda} \left( \frac{\lambda}{1 + \lambda} \right)^\ell, \]  

(4.4)

and that the variables \((R_{x,m}^\lambda)_{x \in V, m \in \mathbb{N}}\) are independent. Moreover, we define the function \(\tau_\lambda : \Sigma \to \mathcal{I}\), which represents the instruction array for the coupled activated random walk systems as the unique array of instructions such that

\[ \forall x \in V, \; \forall m \in \mathbb{N}, \; f_{x,m}^\lambda := A_{x,m} \quad \text{and} \quad S_{\tau_\lambda} := R_{\lambda}^x. \]

By construction, we proved the following proposition.

**Proposition 4.5.** Let \(\lambda \in [0, \infty)\) and \(\mu \in [0, \infty)\). Sample the pair \((\eta, \tau) \in \mathcal{H} \times \mathcal{I}\) according to \(\mathcal{P}_{\lambda,\mu}^\nu\) and let \(\eta_\lambda : \Sigma \to \mathcal{H}\) and \(\tau_\lambda : \Sigma \to \mathcal{I}\) be the random variables in the probability space \((\Sigma, \mathcal{F}, \mathcal{P})\) which have been defined above. We have that,

\[ (\eta, \tau) \overset{d}{=} (\eta_\mu, \tau_\lambda), \]

where \(\overset{d}{=}\) denotes equality in distribution. From this, we deduce that, for every event \(A \in \mathcal{S}\),

\[ \mathcal{P}( (\eta_\mu, \tau_\lambda) \in A) = \mathcal{P}_{\lambda,\mu}( (\eta, \tau) \in A), \]

(4.5)

In the next two subsections we will use this coupling to prove equations (4.1) and (4.2).

### 4.2 Proof of equation (4.1)

Let \(A\) be any relevant event with (finite) domain \(K \subset V\). To begin, we deduce from Proposition 4.5 that, since \(A\) is increasing, then for any \(\delta > 0\),

\[ \mathcal{P}_{\lambda,\mu}(A) - \mathcal{P}_{\lambda+\delta,\mu}(A) = \sum_{\tilde{M} \in \mathbb{N}_0^K} \mathcal{P}(M_{\tilde{K},\tilde{\eta},\tau_0} = \tilde{M}, (\eta_\mu, \tau_\lambda) \in A, (\eta_\mu, \tau_{\lambda+\delta}) \notin A), \]

(4.6)

where, since \(M_{\tilde{K},\tilde{\eta},\tau}(y) = 0\) for \(y \notin K\) a.s, by a slight abuse of notation we consider the function \(M_{\tilde{K},\tilde{\eta},\tau}\) as taking values in \(\mathbb{N}_0^K\) rather than in \(\mathbb{N}_0^V\). Recall the coupling construction which was defined in Section 4.1. For arbitrary \((y, j) \in K \times \mathbb{N}\), we define the events in the probability space \(\mathcal{P}\),

\[ B_{y,j} := \{ R_{x,m}^\lambda > 0 \} \cap \{ R_{x,m}^\lambda = 0 \}, \]

\[ B_{y,j}^- := \bigcap_{(x,k) \in K \times \mathbb{N}; \; k \leq \tilde{M}(x) + 1} \left\{ R_{x,k}^\lambda > 0 \iff R_{x,k}^\lambda > 0 \right\}, \]

\[ B_{\tilde{M}} := \bigcup_{(v,n),(x,k) \in K \times \mathbb{N}; \; (v,n) \neq (x,k), n \leq \tilde{M}(v) + 1, k \leq \tilde{M}(x) + 1} \left\{ R_{x,n}^\lambda > 0, R_{x,k}^\lambda > 0, R_{x,n}^\lambda = 0, R_{x,k}^\lambda = 0 \right\}. \]

Using the fact that \(A\) is relevant, applying Lemma 3.5 and the conditional probability formula, we obtain that

\[ \mathcal{P}(M_{\tilde{K},\tilde{\eta},\tau_0} = \tilde{M}, (\eta_\mu, \tau_\lambda) \in A, (\eta_\mu, \tau_{\lambda+\delta}) \notin A) = \]

\[ \sum_{(y, j) \in K \times \mathbb{N}; \; j \leq \tilde{M}(y) + 1} \mathcal{P}(M_{\tilde{K},\tilde{\eta},\tau_0} = \tilde{M}, (\eta_\mu, \tau_\lambda) \in A, (\eta_\mu, \tau_{\lambda+\delta}) \notin A, B_{y,j}^- \left| B_{y,j}^+ \right. \mathcal{P}(B_{y,j}^+) \]

\[ + \mathcal{P}(M_{\tilde{K},\tilde{\eta},\tau_0} = \tilde{M}, (\eta_\mu, \tau_\lambda) \in A, (\eta_\mu, \tau_{\lambda+\delta}) \notin A, B_{\tilde{M}}^2). \]  

(4.7)
We now let $f_δ(\tilde{M})$ and $u_δ(\tilde{M})$ be the respectively the first and second term in the right-hand side (RHS) of the previous expression. Thus we deduce from (4.5) and (4.7) that,

$$
\lim_{\delta \to 0^+} \frac{1}{\delta} \left[ P_{\lambda,\mu}(A) - P_{\lambda+\delta,\mu}(A) \right] = \lim_{\delta \to 0^+} \sum_{M \in \mathbb{N}_0^K} \frac{1}{\delta} f_δ(M) + \lim_{\delta \to 0^+} \sum_{M \in \mathbb{N}_0^K} \frac{1}{\delta} u_δ(M).
$$

(4.8)

We now consider the two terms in the right-hand side (RHS) of the previous identity separately.

### 4.2.1 First term in the RHS of (4.8)

To begin, note that in the limit as $\delta \rightarrow 0^+$, uniformly in $y \in V$ and $j \in \mathbb{N}$,

$$
P(B_{\delta,j,+}) = P\left( y, j \in \left( \frac{\lambda}{1+\lambda}, 1 \right] \setminus \left( \frac{\lambda+\delta}{1+\lambda+\delta}, 1 \right] \right) = \delta \left( \frac{1}{1+\lambda} \right)^2 + O(\delta^2).
$$

(4.9)

Consider now an arbitrary $\tilde{M} \in \mathbb{N}_0^K$. Using independence, the definition of sleeping-essential pair, the important Remark 4.3 for the second identity, and the fact that for each given $\tilde{M} \in \mathbb{N}_0^K$, $\lim_{\delta \to 0^+} P(B_{\delta,j,-}) = 1$ for the third identity, we obtain that,

$$
\lim_{\delta \to 0^+} \frac{1}{\delta} f_δ(\tilde{M}) = \lim_{\delta \to 0^+} P\left( M_K, \eta_\mu, \tau_\lambda = M, (\eta_\mu, \tau_\lambda) \in A, (\eta_\mu, \tau_\lambda+\delta) \notin A, B_{\delta,j,-}^y \right) \left( \frac{1}{1+\lambda} \right)^2

= \lim_{\delta \to 0^+} P\left( M_K, \eta_\mu, \tau_\lambda = \tilde{M}, \left( \eta_\mu, \tau_\lambda+\delta \right) \notin A, B_{\delta,j,-}^y \right) \left( \frac{1}{1+\lambda} \right)^2

= P_{\lambda,\mu}\left( \{ (y,j) \text{ is s-essential for } A \} \cap \{ M_K, \eta_\mu, \tau_\lambda = \tilde{M} \} \right) \left( \frac{1}{1+\lambda} \right)^2.
$$

(4.10)

Moreover, from (4.9) we deduce that for any $\tilde{M} \in \mathbb{N}_0^K$,

$$
\frac{1}{\delta} f_δ(\tilde{M}) \leq \sum_{y \in K} (\tilde{M}(y) + 1) P(M_K, \eta_\mu, \tau_\lambda = \tilde{M}) \left( \frac{1}{1+\lambda} \right)^2 + O(\delta).
$$

Since the quantity in the RHS of the previous expression is summable in $\tilde{M}$ and the sum is uniformly bounded in $\delta \in (0, 1)$, we can use (4.10) and apply the dominated convergence theorem to conclude that,

$$
\lim_{\delta \to 0^+} \sum_{\tilde{M} \in \mathbb{N}_0^K} \frac{1}{\delta} f_δ(\tilde{M}) = \sum_{\tilde{M} \in \mathbb{N}_0^K} \lim_{\delta \to 0^+} \frac{1}{\delta} f_δ(\tilde{M})

= \left( \frac{1}{1+\lambda} \right)^2 \sum_{\tilde{M} \in \mathbb{N}_0^K} \left( \sum_{y,j \in K \times \mathbb{N} : y < \tilde{M}(y)+1} P_{\lambda,\mu}\left( \{ (y,j) \text{ is s-essential for } A \} \cap \{ M_K, \eta_\mu, \tau_\lambda = \tilde{M} \} \right) \right)

= \left( \frac{1}{1+\lambda} \right)^2 \sum_{y,j \in K \times \mathbb{N}} P_{\lambda,\mu}\left( \{ (y,j) \text{ is s-essential for } A \} \right).
$$

To conclude the proof it remains then to show that the second term in the RHS of (4.8) equals zero.
4.2.2 Second term in the RHS of (4.8)

We now prove that the second term in the RHS of (4.8) equals zero. For this, note that in the limit as \( \delta \to 0^+ \),

\[
\sum_{M \in \mathbb{N}_0^K} u_\delta(\tilde{M}) \leq \sum_{M \in \mathbb{N}_0^K} \mathcal{P}(M_{K, \eta_\mu, \tau_0} = \tilde{M}, B_M^2)
\]

\[
\leq \sum_{M \in \mathbb{N}_0^K} \sum_{(x,k),(y,j) \in K \times \mathbb{N} : (x,k) \neq (y,j)} \mathcal{P}(Y_{y,j}, Y_{x,k} \in \left( \frac{\lambda}{1+\lambda}, 1 \right] \setminus \left( \frac{\lambda + \delta}{1+\lambda + \delta}, 1 \right], M_{K, \eta_\mu, \tau_0} = \tilde{M}),
\]

\[
= (\delta^2\left(\frac{1}{1+\lambda}\right)^4 + o(\delta^2)) \sum_{(x,k),(y,j) \in K \times \mathbb{N}_0 : (x,k) \neq (y,j)} \mathcal{P}(M_{K, \eta_\mu, \tau_0}(x) \geq k, M_{K, \eta_\mu, \tau_0}(y) \geq j). \quad (4.11)
\]

where for the first inequality we used the union bound, for the first identity we used the independence between the function \( M_{K, \eta_\mu, \tau_0} \) and the functions \( Y_{y,j} \). Now note that, since \( K \) is finite, then the sum in the last expression is finite and depends only on \( \mu \) and \( K \). This implies that the second term in the RHS of (4.8) equals zero and concludes the proof of the right partial derivative.

4.2.3 The left partial partial derivative

To see that (4.11) holds with ‘\( \delta \to 0^- \)’ in place of ‘\( \delta \to 0^+ \)’ we observe that, since the event \( \mathcal{A} \) is increasing, then for any \( \delta > 0 \) we have that,

\[
\mathcal{P}_{\lambda-\delta, \mu}(\mathcal{A}) - \mathcal{P}_{\lambda, \mu}(\mathcal{A}) = \sum_{M \in \mathbb{N}_0^K} \mathcal{P}(M_{K, \eta_\mu, \tau_0} = \tilde{M}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, (\eta_\mu, \tau_{\lambda-\delta}) \in \mathcal{A}).
\]

Now the proof follows the same steps as the proof of the right partial derivative, with the following events

\[
C_{\bar{M}}^{y,j,+} := \{ R_{\lambda, \delta}^{y,j} = 0 \} \cap \{ R_{\lambda}^{y,j} > 0 \},
\]

\[
C_{\bar{M}}^{y,j,-} := \bigcap_{(x,k) \in K \times \mathbb{N} : (x,k) \neq (y,j)} \left\{ R_{\lambda, \delta}^{x,k} > 0 \iff R_{\lambda}^{x,k} > 0 \right\},
\]

\[
C_{\bar{M}}^{2} := \bigcup_{(v,n),(x,k) \in K \times \mathbb{N} : (v,n) \neq (x,k), n \leq \tilde{M}(v)+1, k \leq \tilde{M}(x)+1} \left\{ R_{\lambda, \delta}^{v,n} = 0, R_{\lambda}^{x,k} > 0, R_{\lambda}^{v,n} > 0, R_{\lambda}^{x,k} > 0 \right\}.
\]

playing the role of those defined in (4.6) for every \( (y,j) \in K \times \mathbb{N} \) and \( \tilde{M} \in \mathbb{N}_0^K \), and

\[
\lim_{\delta \to 0^+} \mathcal{P}(M_{K, \eta_\mu, \tau_0} = \tilde{M}, (\eta_\mu, \tau_{\lambda-\delta}) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, C_{\bar{M}}^{y,j,-}) = \mathcal{P}_{\lambda, \mu}(\{(y,j) \text{ is s-essential for } \mathcal{A}\} \cap \{M_{K, \eta_\mu, \tau_0} = \tilde{M}\}).
\]

playing the role of (4.10). Since we obtain the same formula for the right and left partial derivative, the proof is concluded.
4.3 Proof of equation (4.2)

We now turn to the proof of the second partial derivative. Let \( A \) be an arbitrary relevant event. From Proposition 4.5, by the fact that \( A \) is increasing we obtain that for any \( \delta > 0 \),

\[
\mathcal{P}_{\lambda, \mu + \delta}(A) - \mathcal{P}_{\lambda, \mu}(A) = \sum_{\eta \in \mathcal{H}_a} \mathcal{P}\left( (\eta_{\mu + \delta}, \tau_\lambda) \in A, (\eta_{\mu}, \tau_\lambda) \notin A, \eta_{\mu} = \tilde{\eta} \right). \tag{4.12}
\]

To begin, we define the sets,

\[
\begin{align*}
\mathcal{E}^{x,+} &:= \{ \eta_{\mu + \delta}(x) > \eta_{\mu}(x) \}, \\
\mathcal{E}^{x,-} &:= \{ \forall y \in K \setminus \{ x \}, \eta_{\mu + \delta}(y) = \eta_{\mu}(y) \}, \\
\mathcal{E}^2 &:= \{ \exists x_1, x_2 \in K : x_1 \neq x_2 \text{ and } \eta_{\mu + \delta}(x_1) > \eta_{\mu}(x_1), \eta_{\mu + \delta}(x_2) > \eta_{\mu}(x_2) \}, \\
\mathcal{E}^2_{\tilde{\eta}} &:= \{ \eta \in \mathcal{H} : \forall y \in K \setminus \{ x \} \eta_{\mu}(y) = \tilde{\eta}(y) \},
\end{align*}
\tag{4.13}
\]

where the first three sets are elements of the sigma-algebra \( \mathcal{F} \), the last set is defined as a subset of \( \mathcal{H} \), and \( \tilde{\eta} \in \mathcal{H} \). Now note that, for each \( \tilde{\eta} \in \mathcal{H}_a \), we have that,

\[
\mathcal{P}\left( (\eta_{\mu + \delta}, \tau_\lambda) \in A, (\eta_{\mu}, \tau_\lambda) \notin A, \eta_{\mu} = \tilde{\eta} \right) = \\
\sum_{x \in K} \mathcal{P}\left( (\eta_{\mu + \delta}, \tau_\lambda) \in A, (\eta_{\mu}, \tau_\lambda) \notin A, \mathcal{E}^{x,+}, \eta_{\mu}(x) = \tilde{\eta}(x) \right) \\
\times \mathcal{P}\left( \mathcal{E}^{x,+}, \eta_{\mu}(x) = \tilde{\eta}(x) \right) + \mathcal{P}\left( (\eta_{\mu + \delta}, \tau_\lambda) \in A, (\eta_{\mu}, \tau_\lambda) \notin A, \mathcal{E}^2, \eta_{\mu} = \tilde{\eta} \right). \tag{4.14}
\]

We now let \( g_\delta(\tilde{\eta}) \) and \( h_\delta(\tilde{\eta}) \) respectively be the first and second term in the RHS of the previous expression. Using (4.14) in (4.12) we then deduce that

\[
\lim_{\delta \to 0^+} \frac{\mathcal{P}_{\lambda, \mu + \delta}(A) - \mathcal{P}_{\lambda, \mu}(A)}{\delta} = \lim_{\delta \to 0^+} \sum_{\tilde{\eta} \in \mathcal{H}_a} \frac{1}{\delta} g_\delta(\tilde{\eta}) + \lim_{\delta \to 0^+} \sum_{\tilde{\eta} \in \mathcal{H}_a} \frac{1}{\delta} h_\delta(\tilde{\eta}). \tag{4.15}
\]

We now consider the two terms in the RHS of the previous expression separately.

4.3.1 First term in the RHS of (4.15)

To begin, from Proposition 4.5 and from a simple computation we deduce that, in the limit as \( \delta \to 0^+ \),

\[
\begin{align*}
\mathcal{P}(\eta_{\mu + \delta}(x) > \eta_{\mu}(x), \eta_{\mu}(x) = k) &= \mathcal{P}\left( X_x \in [\nu_{\leq k-1}(\mu), \nu_{\leq k}(\mu)] \cap [\nu_{\leq k}(\mu + \delta), 1] \right) \\
&= \nu_{>k}(\mu + \delta) - \nu_{>k}(\mu) = \delta \nu'_{>k}(\mu) + E_{k,\delta},
\end{align*}
\tag{4.16}
\]

where the last identity defines \( E_{k,\delta} \), which satisfies \( |E_{k,\delta}| \leq \frac{\delta^2}{\min(\nu_1,1)} k \nu_{>k-1}(\mu + \delta) \) for any \( \delta > 0 \). Fix now an arbitrary particle configuration \( \tilde{\eta} \in \mathcal{H}_a \). Since \( \lim_{\delta \to 0^+} \mathcal{P}(\mathcal{E}^{x,-}) = 1 \), we deduce from the definition of particle-essential pair, from Remark 4.3 and from (4.16) that,

\[
\begin{align*}
\lim_{\delta \to 0^+} \frac{1}{\delta} g_\delta(\tilde{\eta}) &= \lim_{\delta \to 0^+} \sum_{x \in K} \mathcal{P}\left( (\eta_{\mu + \delta}, \tau_\lambda) \in A, (\eta_{\mu}, \tau_\lambda) \notin A, \mathcal{E}^{x,-}, \eta_{\mu}(x) = \tilde{\eta}(x) \right) \nu'_{>\tilde{\eta}(x)} \\
&= \sum_{x \in K} \mathcal{P}\left( \{ (\eta_{\mu}, \tau_\lambda) \in \{ (x, \tilde{\eta}(x)) \text{ is p-essential for } A \} \right) \cap \{ \eta_{\mu} \in \mathcal{E}^{x,-}_\tilde{\eta} \} \nu'_{>\tilde{\eta}(x)}. \tag{4.17}
\end{align*}
\]
Moreover, note that from \((4.16)\), from the fact that \(\nu'_{>k} = \nu_k\) for any \(k \in \mathbb{N}_0\), and from Remark \(4.3\) we deduce that for any \(\delta > 0\),
\[
\frac{1}{\delta} g_\delta(\tilde{\eta}) \leq \sum_{x \in K} \mathcal{P}(\eta_{\mu} \in \mathcal{E}_\tilde{\eta}^\nu) \left(\frac{\nu_{\tilde{\eta}}(x)}{\nu_{\tilde{\eta}}(x)} + \frac{E_{k,\delta}}{\delta}\right)
\leq |K| \mathcal{P}(\eta_{\mu} = \tilde{\eta}) + \delta \frac{1}{\min\{\mu, 1\}} \mathcal{P}(\eta_{\mu} = \tilde{\eta}) \sum_{x \in K} \tilde{\eta}(x) \frac{\nu_{\tilde{\eta}}(x) - (\mu + \delta)}{\nu_{\tilde{\eta}}(x) \mu}.
\]
Since the quantity in the RHS is summable in \(\tilde{\eta}\) and the sum is uniformly bounded for \(\delta \in (0, 1)\), we can use the dominated convergence theorem and deduce from \((4.17)\) that,
\[
\lim_{\delta \to 0^+} \sum_{\tilde{\eta} \in \mathcal{H}_a} \frac{1}{\delta} g_\delta(\tilde{\eta}) = \sum_{\tilde{\eta} \in \mathcal{H}_a} \lim_{\delta \to 0^+} \frac{1}{\delta} g_\delta(\tilde{\eta})
= \sum_{\tilde{\eta} \in \mathcal{H}_a} \sum_{x \in K} \mathcal{P}\left(\left\{\eta_{\mu}, \tau_\lambda \mid (x, \tilde{\eta}(x)) \text{ is p-essential for } A \right\} \cap \{\eta_{\mu} \in \mathcal{E}_\tilde{\eta}^\nu\}\right) \nu_{\tilde{\eta}}(x)
= \sum_{x \in K, j \in \mathbb{N}_0} \mathcal{P}_{\lambda, \mu}(x, j) \text{ is particle-essential for } A \nu_{\tilde{\eta}}(x).
\]
To conclude the proof it remains then to show that the second term in the RHS of \((4.15)\) equals zero.

4.3.2 Second term in the RHS of \((4.15)\)

We now show that the second term in the RHS of \((4.15)\) equals zero. For this, we apply the union bound and obtain that,
\[
\mathcal{P}\left(\left\{\eta_{\mu+\delta}, \tau_\lambda \mid (\eta_{\mu}, \tau_\lambda) \notin A, \mathcal{E}^2\right\}\right)
\leq \sum_{x_1, x_2 \in K} \sum_{x_1 \neq x_2} \mathcal{P}\left(\eta_{\mu}(x_1) = k_1, \eta_{\mu}(x_2) = k_2, \eta_{\mu+\delta}(x_1) > k_1, \eta_{\mu+\delta}(x_2) > k_2\right)
\leq |K|^2 \left(\sum_{k \geq 0} \mathcal{P}\left(\eta_{\mu}(\circ) = k, \eta_{\mu+\delta}(\circ) > k\right)\right)^2
\leq |K|^2 \left(\sum_{k \geq 0} (\nu_{>k}(\mu + \delta) - \nu_{>k}(\mu))\right)^2 = \delta^2 |K|^2, \quad (4.18)
\]
where we used the fact that for the Poisson distribution \(\sum_{k \geq 0} \nu_{>k}(\mu) = \mu\). The previous inequality implies that the second term in the RHS of \((4.15)\) equals zero. This concludes the proof of the right partial derivative.

4.3.3 The left partial derivative

We now show that \((4.2)\) holds with \(\delta \to 0^\text{−}\) in place of \(\delta \to 0^\text{+}\). Since \(A\) is increasing, we obtain that, for positive \(\delta > 0\),
\[
\mathcal{P}_{\lambda, \mu-\delta}(A) - \mathcal{P}_{\lambda, \mu}(A) = -\sum_{\tilde{\eta} \in \mathcal{H}_a} \mathcal{P}\left(\left\{\eta_{\mu-\delta}, \tau_\lambda \mid (\eta_{\mu}, \tau_\lambda) \notin A, (\eta_{\mu}, \tau_\lambda) \in A, \eta_{\mu} = \tilde{\eta}\right\}\right).
\]
The proof is now analogous to that of the right partial derivative, with the sets
\[
D^{x,+} := \left\{\eta_{\mu-\delta}(x) < \eta_{\mu}(x)\right\},
D^{x,−} := \{\forall y \in K \setminus \{x\}, \eta_{\mu-\delta}(y) = \eta_{\mu}(y)\},
D^2 := \{|x_1, x_2 \in K : x_1 \neq x_2 \text{ and } \eta_{\mu-\delta}(x_1) < \eta_{\mu}(x_1), \eta_{\mu-\delta}(x_2) < \eta_{\mu}(x_2)\},
\]

16
playing the role of those defined in (4.13), with
\[ P(\eta_{\mu-\delta}(x) < \eta_\mu(x), \eta_\mu(x) = k) = P\left( X_\mu \in [\nu_{\leq k-1}(\mu), \nu_{\leq k}(\mu)] \cap [0, \nu_{\leq k-1}(\mu - \delta)] \right), \]
(4.19)
playing the role of (4.16) for every \( k \geq 1 \), and
\[ \lim_{\delta \to 0^+} P\left( (\eta_{\mu-\delta}, \tau_\lambda) \not\in A, (\eta_\mu, \tau_\lambda) \in A, \mathcal{D}^{x,-}, \eta_\mu \in \mathcal{E}_\eta^x | \mathcal{D}^{x,+}, \eta_\mu(x) = \tilde{\eta}(x) \right) =
\]
\[ P\left( \{ (\eta_\mu, \tau_\lambda) \in \{(x, \tilde{\eta}(x) - 1) \text{ is } p\text{-essential for } A \} \cap \{ \eta_\mu \in \mathcal{E}_\eta \} \right). \]
being used for any \( \tilde{\eta} \in \mathcal{H}_a \) and \( x \in K \) such that \( \tilde{\eta}(x) \geq 1 \) in the step which is analogous to (4.17). This concludes the proof.

5 The key differential inequality

The goal of this section is to state and prove Theorem 5.4 below, which provides a precise formulation of the differential inequality (1.4). This section is divided into three subsections. In the first subsection we provide an alternative formula for (4.1), corresponding to Proposition 5.2 below. In the second subsection we state an important comparison lemma. In the last subsection we state and prove our differential inequality.

5.1 Alternative formula for (4.1)

Our first step is to provide an alternative formula for the partial derivative with respect to \( \lambda \) which appears in Theorem 4.4. This is a consequence of Lemma 3.3 and is important for the comparison with the partial derivative with respect to \( \mu \). We start with a technical lemma, which is a consequence of Lemma 3.3.

Lemma 5.1. Let \( A \) be a relevant event with domain \( K \). For every \( y \in K, n \in \mathbb{N} \),
\[ \{(y, n) \text{ is } s\text{-essential for } A\} \cap \{(\eta, \tau) \in \mathcal{W} : M_{K,\eta,\tau}(y) \neq n - 1 \text{ and } S^{\eta, n}_y > 0\} = \emptyset. \]
(5.1)

Proof. Suppose that \( (\eta, \tau) \in \{(y, n) \text{ is } s\text{-essential for } A\} \cap \mathcal{W} \) and that \( S^{\eta, n}_y > 0 \). We will show that it is necessarily the case that \( M_{K,\eta,\tau}(y) = n - 1 \), thus implying (5.1). To begin, we deduce by definition of sleeping-essential pair and by the fact that \( S^{\eta, n}_y > 0 \) that,
\[ (\eta, \tau) \not\in A \quad \text{and} \quad (\eta, \Gamma^{\eta, n}(\tau)) \in A. \]
(5.2)

Suppose that \( M_{K,\eta,\tau}(y) \neq n - 1 \). This will lead to a contradiction. Indeed, by Lemma 3.3 we deduce that \( M_{K,\eta,\tau} = M_{K,\eta,\Gamma^{\eta, n}(\tau)} \). From this, from the fact that \( A \) is relevant and from the fact that \( (\eta, \tau) \not\in A \), we deduce that \( (\eta, \Gamma^{\eta, n}(\tau)) \not\in A \). This, however, contradicts (5.2), and we obtained the desired contradiction.

We now present our alternative formula for the partial derivative with respect to \( \lambda \) which appears in Theorem 4.4.

Proposition 5.2. Let \( A \) be a relevant event. for every \( \lambda \in (0, \infty) \) we have that,
\[ \frac{\partial}{\partial \lambda} P_{\lambda, \mu}(A) = -\frac{1}{\lambda (1 + \lambda)} \sum_{y \in K} \sum_{n=1}^{\infty} P_{\lambda, \mu}\left( \{(y, n) \text{ is } s\text{-essential for } A\} \cap \{M_{K,\eta,\tau}(y) = n - 1\} \cap \{S^{\eta, n}_y > 0\} \right), \]
(5.3)
Proof. Below we use Remark 4.3 for the first identity and Lemma 5.1 for the third identity, obtaining that, for every \( y \in K \),
\[
\sum_{j=1}^{\infty} \mathcal{P}_{\lambda,\mu}((y, j) \text{ is s-essential}) = \frac{1 + \lambda}{\lambda} \sum_{j=1}^{\infty} \mathcal{P}_{\lambda,\mu}((y, j) \text{ is s-essential} \cap \{S^y_j > 0\})
\]
\[
= \frac{1 + \lambda}{\lambda} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \mathcal{P}_{\lambda,\mu}((y, j) \text{ is s-essential} \cap \{S^y_j > 0\} \cap \{M_K(y) = n\})
\]
\[
= \frac{1 + \lambda}{\lambda} \sum_{j=1}^{\infty} \mathcal{P}_{\lambda,\mu}((y, j) \text{ is s-essential} \cap \{S^y_j > 0\} \cap \{M_K(y) = j - 1\}).
\]

By using the previous formula and Theorem 4.4, we conclude the proof.

5.2 Comparison lemma

We now state and prove our comparison lemma, Lemma 5.3 below, which is the core of the proof of our differential inequality.

Lemma 5.3 (Comparison lemma). Consider any relevant event \( A \) with domain \( K \subset V \). For any \( y \in K \), \( n,j \in \mathbb{N}_0 \), with \( n > 0 \), we have that,
\[
W \cap \{(y, n) \text{ is s-essential for } A \} \cap \{\eta(y) = j\} \cap \{S^{y,n}_\tau > 0\} \cap \{M_{K,\eta}(y) = n-1\} \subset W \cap \{(y, j) \text{ is p-essential for } A \} \cap \{\eta(y) = j\} \cap \{S^{y,j}_\tau > 0\} \cap \{M_{K,\eta}(y) = n-1\}.
\]

Proof. Let \((\eta, \tau)\) be a realisation which belongs to the event in the left-hand side (LHS) of (5.4), we will show that it also belongs to the event in the RHS. Set \( n = M_{K,\eta,\tau}(y) + 1 \). By definition of sleeping-essential pair and by the fact that \( S^{y,n}_\tau > 0 \) we deduce that, \((\eta, \tau) \not\in A\) and \((\eta, \Gamma_y^{y,n}(\tau)) \in A\).

By monotonicity, Lemma 2.2, we have that,
\[
M_{K,\eta,\tau} \leq M_{K,\eta,\Gamma_y^{y,n}(\tau)} \leq M_{K,\eta^y,\Gamma_y^{y,n}(\tau)}.
\]

By the fact that, by the Abelian property, \( n = M_{K,\eta,\tau}(y) + 1 < M_{K,\eta^y,\tau}(y) + 1 \), we deduce from Lemma 3.3 that
\[
M_{K,\eta^y,\Gamma_y^{y,n}(\tau)} = M_{K,\eta^y,\tau}.
\]

From (5.5), (5.6), (5.7), and from the fact that \( A \) is relevant we deduce that, \((\eta^y, \tau) \in A\). Summarising, \((\eta, \tau)\) is such that (1) \((\eta, \tau) \not\in A\) by (5.5), (2) \(\eta(y) = j\) by assumption, and (3) \((\eta^y, \tau) = (\eta^{y,j+1}, \tau) \in A\) by (5.5), and (5.7). These three facts imply that the pair \((\eta, \tau)\) belongs to the event ‘\((y, j)\) is p-essential for \( A \)’. From this we deduce that \((\eta, \tau)\) belongs to the event in the RHS of (5.4). This concludes the proof.

5.3 Differential inequality

We are now ready to state the main result of this section. Its proof will employ our comparison lemma.

Theorem 5.4 (Differential inequality). Let \( G = (V, E) \) be an arbitrary undirected locally-finite connected graph, let \( A \) be a relevant event. Then, for any \((\lambda, \mu) \in \mathbb{R}_+^2\), we have that,
\[
- \frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda,\mu}(A) \leq \frac{1}{\lambda(1 + \lambda)} \frac{\partial}{\partial \mu} \mathcal{P}_{\lambda,\mu}(A).
\]
Proof. Let \( K \subset V \) be the domain of the relevant event \( A \). Using Proposition 5.2 for the first step, our Comparison Lemma for the second step, Remark 4.3 the fact that for Poisson distributions \( \nu_k = \nu_k \) and Theorem 4.4 for the last step, we obtain that,

\[
- \frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda, \mu}(A) = \frac{1}{\lambda(1+\lambda)} \sum_{y \in K} \sum_{j \geq 0} \mathcal{P}_{\lambda, \mu} \left( \{ (y, n) \text{ is s-ess.} \} \cap \{ \eta(y) = j \} \cap \{ S_{y,n}^\mu > 0 \} \cap \{ M_{K, \eta, \tau}(y) = n - 1 \} \right)
\leq \frac{1}{\lambda(1+\lambda)} \sum_{y \in K} \sum_{j \geq 0} \mathcal{P}_{\lambda, \mu} \left( \{ (y, j) \text{ is p-ess.} \} \cap \{ \eta(y) = j \} \cap \{ S_{y,n}^\mu > 0 \} \cap \{ M_{K, \eta, \tau}(y) = n - 1 \} \right)
\leq \frac{1}{\lambda(1+\lambda)} \sum_{y \in K} \sum_{j \geq 0} \mathcal{P}_{\lambda, \mu} \left( \{ (y, j) \text{ is p-essential} \} \cap \{ \eta(y) = j \} \right)
= \frac{1}{\lambda(1+\lambda)} \frac{\partial}{\partial \mu} \mathcal{P}_{\lambda, \mu}(A).
\]

This concludes the proof. \( \square \)

6 Proof of Theorems 1.1 and 1.2

Recall the definition of the curve \( C_{\lambda, \mu} \) which was provided in (1.2). We will start with the proof of Theorem 1.2.

Proof of Theorem 1.2. Consider any relevant event \( A \). Let \( (\lambda, \mu) \in \mathbb{R}_+^2 \) be an arbitrary point in the phase diagram, take any arbitrary point \( (x, y) \in C_{\lambda, \mu} \), we assume that \( x > \lambda \) (when \( x = \lambda \), the result we are going to prove is already known from [12]). Let \( (X(t), Y(t))_{t \in [\lambda, \infty)} \) be a curve such that \( X(0) := \lambda \), \( Y(0) := \mu \), and for any \( t \in [\lambda, \infty) \),

\[
\begin{cases}
X(t) := t, \\
Y(t) := s(t - \lambda) + \mu,
\end{cases}
\]

where \( s \geq \frac{1}{\lambda(1+\lambda)} \) is such that there exists a positive \( T \in \mathbb{R} \) such that \( X(T) = x \), \( Y(T) = y \). From the fundamental theorem of calculus we deduce that,

\[
\mathcal{P}_{x,y}(A) = \mathcal{P}_{\lambda, \mu}(A) + \int_{\lambda}^{x} \partial_t \mathcal{P}_{X(t), Y(t)}(A) : \left( \partial_t X(t), \partial_t Y(t) \right) = \mathcal{P}_{\lambda, \mu}(A) + \int_{\lambda}^{x} \left( \partial_t \mathcal{P}_{\lambda, \mu}(A) \right)_{\lambda = t, \mu = Y(t)} + s \partial_\mu \mathcal{P}_{\lambda, \mu}(A)_{\lambda = t, \mu = Y(t)} \\
\geq \mathcal{P}_{\lambda, \mu}(A) + \int_{\lambda}^{x} 0 = \mathcal{P}_{\lambda, \mu}(A),
\]

where for the first inequality we used Theorem 5.4 and the fact that \( s \geq \frac{1}{\lambda(1+\lambda)} \geq \frac{1}{t(1+t)} \) for every \( t \geq 0 \). This concludes the proof. \( \square \)

We now provide a more general definition of critical density,

\[
\forall \lambda \in [0, \infty) \, \zeta_c(\lambda) := \inf \left\{ \mu \in \mathbb{R}_0^+ : \mathcal{P}_{\lambda, \mu}(m(o) = \infty) > 0 \right\}, \quad (6.1)
\]

By monotonicity, the random variable \( m_{\eta, \tau}(o) \) is well defined for every locally-finite connected infinite graph \( G \). From now on we will say that ARW \textit{fixates} if \( m(o) < \infty \) and that it is active.
otherwise. Such a notion of activity and fixation reduces to the one which has been introduced in
Section 1, thus implying the identity $\zeta_c(\lambda) = \mu_c(\lambda)$, whenever Lemma 2.3 holds. The next theorem
is an immediate consequence of Theorem 1.2.

**Theorem 6.1.** Let $G$ be a locally-finite connected graph. For any $(\lambda, \mu) \in \mathbb{R}_+^2$ and $(\lambda', \mu') \in \mathcal{C}_{\lambda, \mu}$
we have that,

$$
P_{\lambda, \mu}(\text{ARW active}) \leq P_{\lambda', \mu'}(\text{ARW active}).$$

**Proof.** Define $B_L := \{x \in V : d(x, o) \leq L\}$. For any $L, H \in \mathbb{N}$ consider the relevant event,

$$A_{L, H} := \{M_{B_L}(o) > H\}.$$ 

From our monotonicity theorem, Theorem 1.2, we deduce that for any $L, H \in \mathbb{N}$,

$$P_{\lambda, \mu}(A_{L, H}) \leq P_{\lambda', \mu'}(A_{L, H}).$$

From this and from Lemma 2.3 we then deduce that,

$$P_{\lambda, \mu}(\text{ARW active}) = \lim_{H \to \infty} \lim_{L \to \infty} P_{\lambda, \mu}(A_{L, H}) \leq \lim_{H \to \infty} \lim_{L \to \infty} P_{\lambda', \mu'}(A_{L, H}) = P_{\lambda', \mu'}(\text{ARW active}).$$

This concludes the proof. □

### 6.1 Proof of Theorem 1.1

We now present the proof of our continuity theorem.

**Proof.** It suffices to prove the second claim of Theorem 1.1 since it implies the first claim. In
the whole proof we use the fact that it is known from [18] that on any vertex-transitive graph
the critical density is finite for every $\lambda \in [0, \infty)$. Consider an arbitrary $\lambda \in (0, \infty)$ and, for any $\epsilon > 0$
and $t \in (-\infty, \infty)$ define the function,

$$Y_\epsilon(t) := \zeta_c(\lambda) + \epsilon + \frac{1}{\lambda(1 + \lambda)} (t - \lambda).$$

Suppose that,

$$\limsup_{\delta \to 0^+} \frac{\zeta_c(\lambda + \delta) - \zeta_c(\lambda)}{\delta} > \frac{1}{\lambda(1 + \lambda)}, \quad \text{(6.2)}$$

we look for a contradiction with this claim. From (6.2) it follows that we can find a small enough
$\Delta > 0$ such that there exists an infinite positive sequence $(\delta_n)_{n \in \mathbb{N}}$ converging to zero with $n$
such that, for any large enough $n$,

$$\zeta_c(\lambda + \delta_n) \geq \zeta_c(\lambda) + \delta_n \left(\frac{1}{\lambda(1 + \lambda)} + \Delta\right). \quad \text{(6.3)}$$

From Theorem 6.1 and from the definition of the critical density we deduce that, $\zeta_c(t) \leq Y_\epsilon(t)$. From
this we deduce that,

$$\forall \epsilon > 0, \quad \forall n \in \mathbb{N}, \quad \zeta_c(\lambda + \delta_n) \leq Y_\epsilon(\lambda + \delta_n) = \zeta_c(\lambda) + \epsilon + \delta_n \frac{1}{\lambda(1 + \lambda)}.$$

We can then find $n \in \mathbb{N}$ large and $\epsilon > 0$ small such that the previous inequality contradicts (6.3)
and thus also (6.2). We then found the desired contradiction and concluded the proof of the second
claim of the theorem for $\delta \to 0^+$. The proof for $\delta \to 0^-$ is analogous, hence the proof is concluded. □

We conclude with a remark about the generality of our results, which hold on graphs more general
than unimodular.
Remark 6.2. Relevant events do not depend on the clock realisations of the continuous time dynamics and have been defined in the framework of the Diaconis-Fulton representation, which is well-defined on any locally-finite infinite connected graph. Hence, our Theorem 1.2 can be stated in wider generality, namely for any infinite connected locally-finite graph, provided that the probability measure $P_{\lambda,\mu}$ is replaced by $P_{\lambda,\mu}$ in (1.3).

Moreover, contrary to (1.1), the critical density (6.1) is well-defined on any locally-finite infinite connected graph. Our proof of Theorem 1.1 then implies that $\zeta_c(\lambda)$ is a continuous function of $\lambda$ in $(0, \infty)$ on any locally-finite infinite connected graph, provided that it is known that $\zeta_c(\lambda) < \infty$ for some $\lambda > 0$.

Acknowledgements

This work started as the author was affiliated to Technische Universität Darmstadt, it has been carried on while the author was affiliated to the University of Bath, it was concluded as the author was affiliated to Sapienza Università di Roma. The author acknowledges support from DFG German Research Foundation BE/5267/1 and from EPSRC Early Career Fellowship EP/N004566/1. The author thanks the two anonymous referees for carefully reviewing the paper and their important and useful comments.

References

[1] A. Asselah, L. Rolla, B. Shapira: Diffusive bounds for the critical density of activated random walks. ALEA, 19, 457–465 (2022).
[2] M. Aizenman and G. Grimmett: Strict monotonicity for critical points in percolation and ferromagnetic models. Journ. Stat. Phys. 63 (1991), pp. 817–835.
[3] P. Balister, B. Bollobás, O. Riordan: Essential enhancements revisited. Preprint: arXiv 1402.0834 (2014).
[4] R. Basu, S. Ganguly, C. Hoffman: Non-fixation of symmetric Activated Random Walk on the line for small sleep rate. Commun. Math. Phys. 358 (2018), No 3.
[5] B. Bond and L. Levine Abelian Networks I. Foundations and Examples SIAM J. Discrete Math. Vol. 30 (2016), No. 2
[6] E. Candellero, S. Ganguly, C. Hoffman, L. Levine: Oil and water: a two-type internal aggregation model. Ann. Probab., 45 (2017), No 6A.
[7] E. Candellero, A. Stauffer, L. Taggi: Abelian oil and water dynamics does not have an absorbing-state phase transition. Transactions of the A.M.S. , Vol 347, No 4, pp. 2733-2752 (2021).
[8] R. Dickman, L.T. Rolla, V. Sidoravicius: Activated Random Walkers: Facts, Conjectures and Challenges. J. Stat. Phys., 138 (2010), pp. 126-142.
[9] N. Forien, A. Gaudillaire: Active Phase for Activated Random Walks on the Lattice in all Dimensions. ArXiv: 2203.02476.
[10] C. Hoffman, T. Johnson, and M. Junge: Recurrence and transience for the frog model on trees. Ann. Probab. 45 (2017), No 5, pp. 2826-2854.
[11] A.A. Járai: Abelian sandpiles: an overview and results on certain transitive graphs. Markov Process. Relat. Fields, 18 (2012) 111-156

[12] L. T. Rolla and V. Sidoravicius: Absorbing-State Phase Transition for Driven-Dissipative Stochastic Dynamics on $\mathbb{Z}$. Invent. Math., 188 (2012), No 1.

[13] L. T. Rolla, V. Sidoravicius, O. Zindy: Universality and sharpness in activated random walks. Ann. Instit. Henri Poincaré (A), 20 (2018), No 6.

[14] L. T. Rolla and L. Tournier: Sustained Activity for Biased Activated Random Walks at Arbitrarily Low Density. Ann. Instit. Henri Poincaré (B), 54 (2018), No 2.

[15] L. T. Rolla: Activated Random Walks on $\mathbb{Z}^d$. Probability Surveys, Vol. 17 (2020).

[16] L. Russo: A note on percolation. Z. Wahrscheinlichkeitstheorie Verw. Gebiete 32 (1978), pp. 39-48.

[17] L. Russo: On the critical percolation probabilities, Z. Wahrscheinlichkeitstheorie Verw. Gebiete 56 (1981), pp. 229–237.

[18] E. Shellef: Nonfixation for activated random walk. ALEA, 7, (2010).

[19] V. Sidoravicius and A. Teixeira: Absorbing-state transitions for Stochastic Sandpiles and Activated Random Walk. Electr. J. Probab., 22 (2017), No 33.

[20] A. Stauffer and L. Taggi: Critical density of activated random walks on transitive graphs. Ann. Probab., 46, (2018), No 4.

[21] L. Taggi: Absorbing-state phase transition in biased activated random walk. Electr. J. Probab., 21 (2016), No 13.

[22] L. Taggi: Active phase for activated random walks on $\mathbb{Z}^d$, $d \geq 3$ with density less than one and arbitrary sleeping rate. Ann. Instit. Henri Poincaré (B), 55 (2019), No 3.