Thermodynamics of extended bodies in special relativity

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February 2010

Abstract – Relativistic thermodynamics is generalized to accommodate four-dimensional rotation in a flat spacetime. An extended body can be in equilibrium when each of its elements moves along a Killing flow. There are three types of basic Killing flows in a flat spacetime, each of which corresponds to translational motion, spatial rotation, and constant linear acceleration; spatial rotation and constant linear acceleration are regarded as four-dimensional rotation. Translational motion has been mainly investigated in the past literature of relativistic thermodynamics. Thermodynamics of the other two is derived in the present paper.

Introduction. – Numerous papers have been published on relativistic equilibrium thermodynamics of extended bodies (bodies with finite volumes) in a flat spacetime. Most of the papers focused on the equilibrium of a body with translational motion, i.e., each element of the body has the same velocity. There has been a heated controversy on the relativistic temperature with the translational motion (see, e.g., [1] and references therein) in the 1960s, and papers are still published to this date (e.g., [2–5]).

The equilibrium temperature of an extended body in translational motion is uniform within the body, in other words, the local temperature at each point is the same. In contrast, there can be equilibrium states in which the local temperature is not uniform. Equilibrium with varying local temperature was first examined by Tolman and Ehrenfest [6] in the context of general relativity. With some assumptions they concluded the local temperature should be inversely proportional to the square root of the temporal component of the metric tensor. This result can be derived from a more general approach [7,8], which shows equilibrium can take place only when the local inverse temperature four-vector satisfies the Killing equation.

This result in general relativity is of course applicable to flat spacetimes. The local inverse temperature four-vector depends on the local velocities, and consequently the condition of equilibrium determines the velocity distribution by the Killing equation. There are three types of basic Killing vectors in a flat spacetime, each of which corresponds to translational motion, spatial rotation, and constant linear acceleration. Correspondingly there are three types of equilibrium in a flat spacetime. The case of translational motion is trivial: the uniform temperature.

In the case of spatial rotation, the equilibrium state is determined by the well-known effect which is often expressed as “the rim of a rotating wheel is hotter than the axis.” This effect can be interpreted as the result of centrifugal force acting on the energy. The energy is equivalent to the mass in relativity and subject to the centrifugal force, resulting in higher energy density on the outer side of a wheel. The same effect takes place in the case of constant acceleration, resulting in non-uniform local temperatures in the direction of acceleration.

The equilibrium with spatial rotation or constant acceleration is a well-known fact, however, the past literature focused on its microscopic aspect. In other words, the distribution of the local temperatures has been mainly investigated, and little attention has been paid for the global thermodynamical properties of extended bodies. The basic strategy of thermodynamics is based on the fact that the macroscopic properties of a body in equilibrium can be well represented by a small number of thermodynamical parameters, such as temperature, pressure, etc. It is trivial in non-relativistic thermodynamics that the temperature can be expressed with one single value, since the local temperatures are all equal in equilibrium. In contrast, there can be equilibrium with non-uniform local temperatures as discussed above, and it is not well known how to represent its global temperature so far.

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40007-p1
The purpose of the present paper is to generalize the concept of global temperature, or global inverse temperature more precisely, to be applicable to the relativistic equilibrium with non-uniform local temperature. Even when the local temperature varies, the condition of equilibrium is a stringent constraint, which can determine thermodynamical state uniquely by a small number of parameters, namely the generalized inverse temperature.

The second law of thermodynamics will be extended to accommodate four-dimensional rotation (spatial rotation and constant acceleration) with the generalized inverse temperature. This second law can tell how the energy-momentum or angular momentum are transferred between extended bodies thermodynamically. Further, by introducing the generalized inverse temperature we can obtain a clear insight on the relativistic thermal equilibrium. We can tell the similarity and difference of the equilibrium states in translational motion, spatial rotation, and constant acceleration.

Non-relativistic thermodynamics is basically the theory of energetics. The temperature is defined based on the conservation of the energy. When we generalize it to relativity, energy must be treated as one component of the energy momentum which is expressed by a contravariant vector (1-vector). Consequently the inverse temperature becomes a covariant four vector (1-form); this formulation was first proposed by van Kampen [9] and later refined by Israel [10].

The energy-momentum is a conserved quantity resulting from the translational symmetry. There can be another conserved quantity, four-dimensional angular momentum namely, in a Minkowski spacetime resulting from the rotational symmetry. As we will see in the present paper, there exists another inverse temperature corresponding to the angular momentum. In general, rotation can have six independent directions in a four-dimensional space. Consequently the four-dimensional angular momentum is a contravariant bivector (2-vector), with six components, and the corresponding inverse temperature becomes a covariant bivector (2-form) with six components.

After deriving general expression for the inverse temperature, two specific cases of four-dimensional rotation will be examined in the present paper. In a three-dimensional space, any successive two rotations can be combined into one rotation. This is not true for a four-dimensional space because there can be two independent rotations. For example, a rotation in the $x_0x_1$-plane is independent of the rotation in the $x_2x_3$-plane and the two cannot be combined into one single rotation. In the present paper we treat the motion of a single rotation; this is not general, but can clarify the essential thermodynamical properties of four-dimensional rotational motion with simplicity. We examine two basic single rotations in a Minkowski spacetime: spatial rotation and constant acceleration. With respect to the constant acceleration, we further focus on the Rindler motion, which is one limited case but important in applications.

**Temperature and velocity in equilibrium.** Let us suppose an extended body with a three-volume $\Sigma$ is in the equilibrium state, and examine its thermodynamical properties. The local inverse temperature $\xi(\bar{x})$ at each point within the body is defined as

$$\xi_\mu(\bar{x}) = \frac{u_\mu(\bar{x})}{T(\bar{x})},$$

where $T(\bar{x})$ and $\bar{u}(\bar{x})$ are the local proper temperature and the four-velocity of the matter at a point $\bar{x}$ [7,8]; we denote a vector or a tensor as a whole by a bar, e.g., $\bar{\xi}$, and each of its components by a subscript or a superscript, e.g., $\xi_\mu$, in the present paper (precisely speaking, the position $\bar{x}$ may not be a vector, however, this expression does not cause confusion in a flat spacetime). We use natural units $\hbar = c = G = k_B = 1$ throughout this paper unless otherwise stated.

When the body is in equilibrium, $\xi$ satisfies the following Killing equation [7,8]:

$$\nabla_\nu \xi_\mu - \nabla_\mu \xi_\nu = 0.$$  
(2)

The general solution to the above equation in a flat spacetime is given as (see appendix)

$$\xi_\mu = \beta_\mu + \lambda_{\mu\nu}(x^\nu - x_0^\nu),$$  
(3)

where $\beta$ is a constant vector, $x_0$ is a certain fixed point that corresponds to the center of four-dimensional rotation, and $\lambda$ is an anti-symmetric tensor that satisfies $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ [7]. Both $\beta$ and $\lambda$ do not depend on the position $\bar{x}$. The solution to (2) has ambiguity in its amplitude since the Killing equation is linear; the amplitude is determined so as to make the appropriate temperature by (1). When $\beta$, $\lambda$, and $x_0$ are given, $T(\bar{x})$ and $\bar{u}(\bar{x})$ can be calculated from (1) as

$$T(\bar{x}) = |\beta_\mu + \lambda_{\mu\nu}(x^\nu - x_0^\nu)|,$$

$$u_\mu(\bar{x}) = T^{-1}(\bar{x})[\beta_\mu + \lambda_{\mu\nu}(x^\nu - x_0^\nu)].$$  
(4)

Now that we obtain the equilibrium state, let us examine its thermodynamical properties. Suppose there is an adiabatic energy-momentum supply to the body, and the local energy momentum density increases by $\Delta T_\mu^\mu(\bar{x})$; this is not uniform within the body in general. The local change of the entropy four vector at a point $\bar{x}$ is given as $\Delta s_\mu = \xi_\mu \Delta T_\mu^\mu$ (see, e.g., [10,11]), thus we can write the total entropy change in the three-volume $\Sigma$ as

$$\Delta S = \int_\Sigma \beta_\mu \lambda_{\mu\nu}(x^\nu - x_0^\nu) \Delta T_\mu^\mu \ d\Sigma^\nu = \beta_\mu \Delta G^\mu + \lambda_{\mu\nu}\Delta M^{\mu\nu}.$$  
(5)

In the above expression $\Delta G$ and $\Delta M$ are the changes in energy-momentum and four-dimensional angular momentum defined by

$$\Delta G^\mu = \int_\Sigma \Delta T_\mu^\nu \ d\Sigma^\nu, \quad \Delta M^{\mu\nu} = \int_\Sigma (x^\nu - x_0^\nu) \Delta T_\mu^\nu \ d\Sigma^\nu.$$  
(6)
Precisely speaking, both $\Delta \hat{G}$ and $\Delta \hat{M}$ are frame dependent when the body is not isolated [12]. However, the dependence is canceled out when we calculate the entropy and thus we do not pay attention to this point in the following.

Most of the papers on relativistic thermodynamics of extended bodies assume the case with $\lambda = 0$, i.e., for the translational motion without rotation or acceleration. In this case van Kampen [9] and Israel [10] suggested the concept of temperature should be extended by defining a four-vector $\beta_\mu$ as an inverse temperature.

When $\lambda \neq 0$, we find the parameter $\lambda$ in (5) plays the same role to $\Delta \hat{M}$ as $\beta$ does to $\Delta \hat{G}$. Therefore, $\lambda$ can be regarded as a thermodynamical parameter like the inverse-temperature four-vector $\beta$. Since $\lambda$ is a bivector (2-form) with six independent components, the inverse temperature has ten independent components in total: four of $\beta_\mu$ and six of $\lambda_{\mu\nu}$. This number corresponds to the number of independent Killing vector fields in a Minkowski spacetime, or equivalently, the number of conserved quantities resulting from the symmetry of spacetime.

The ten components of inverse temperature is the generalization of the inverse-temperature four-vector; we can treat not only energy-momentum but also four-dimensional angular momentum. In a Minkowski spacetime, the single rotation can be further divided into two categories: spatial rotation and double rotation. In the following we concentrate on the single rotation for simplicity. In the case of the Minkowski spacetime, the single rotation can be further divided into two categories: spatial rotation and constant acceleration. These two have similar mathematical structures, however, their physical properties are considerably different.

The direction of a single rotation can be specified by a two-dimensional plane in which the rotation takes place. In a three-dimensional space, specifying a two-dimensional plane is equivalent to specifying an axis orthogonal to the plane. This does not work in a space with dimension higher than four because the orthogonal direction to a plane is not unique, therefore, we need to specify the direction by a two-dimensional plane.

Two-dimensional planes in a Minkowski spacetime are categorized into two groups in general. One consists of the planes spanned by two spacelike vectors, and planes in the other group are spanned by one timelike and one spacelike vectors; the former defines the spatial rotation and the latter defines the constant acceleration. We will examine these two in the following.

Spatial rotation. For spatial rotation, we can choose the $xy$-plane as the plane of rotation without loss of generality, then we have $\lambda_{xy} \neq 0$, $\lambda_{ti} = \lambda_{yz} = \lambda_{xz} = 0$ in (3). We still have two degrees of freedom in the choice of axis directions and can set $\beta_y = \beta_z = 0$ with them. Then the Killing vector $\xi$ in (3) can be written as

$$\xi = (\beta_t, \beta_x + \lambda_{xy}(y - y_0) - \lambda_{xy}(x - x_0), 0).$$

The above expression can be simplified further by choosing the origin as $x_0 = 0$ and $y_0 = \beta_x/\lambda_{xy}$, which gives $\xi = (\beta_t, \lambda_{xy}y, -\lambda_{xy}x, 0)$. Then (4) is reduced to

$$\bar{u} = \frac{1}{\sqrt{1 - \Omega^2 r^2}} (1, \Omega y, -\Omega x, 0, 0),$$

where $\Omega = \lambda_{xy}/\beta_t$ is the angular velocity and $r = \sqrt{x^2 + y^2}$ is the three-dimensional distance from the rotating axis. As well known, the motion must be restricted within the light cylinder, i.e., $r < \Omega^{-1}$ to keep the causality.

Then the inverse temperature is written as

$$\begin{aligned}
\beta_\mu &= \begin{cases}
T_0^{-1} & (\mu = t), \\
0 & \text{(otherwise),}
\end{cases} \\
\lambda_{\mu\nu} &= \begin{cases}
\Omega T_0^{-1} & (\mu, \nu = x, y), \\
-\Omega T_0^{-1} & (\mu, \nu = y, x), \\
0 & \text{(otherwise),}
\end{cases}
\end{aligned}$$

where $T_0 = T(\bar{x}_0)$ is the temperature at the axis. The local temperature becomes $T = T_0/\sqrt{1 - \Omega^2 r^2}$.

We understand from the above expression that the thermodynamical state of a rotating body is essentially determined by two parameters $\Omega$ and $T_0$, and the ten components of the inverse temperature are derived from the Lorentz transform. The general expression of the four-velocity becomes

$$\bar{u} = \frac{1}{\sqrt{1 - \Omega^2 r^2}} [U_\mu + \Omega_{\mu\nu}(x^\nu - x_0^\nu)],$$

40007-p3
where \( U_{\mu} = \beta_{\mu}/(\beta_{\nu} \beta^{\nu}) \), \( \Omega_{\mu\nu} = \lambda_{\mu\nu}/(\beta_{\nu} \beta^{\nu}) \), and \( r \) is the three-dimensional distance from the rotating axis. The vector \( \vec{U} \) represents the four-velocity of the rotation center, and it is perpendicular to the rotation plane, i.e., \( U^\mu \Omega_{\mu\nu} q^\nu = 0 \) for any four-vector \( \vec{q} \). Note that the origin \( x_0 \) is not identical to \( x_0 \) in (3) because of the shift \( \beta_x/\lambda_{xy} \) to obtain (9).

**Constant acceleration.** Constant acceleration is characterized by the rotating plane spanned by one timelike and one spacelike vector, and we can choose the coordinate such that the latter becomes the \( tx \)-plane, resulting \( \lambda_{tx} \neq 0, \lambda_{ty} = \lambda_{tz} = \lambda_{ij} = 0 \). Further we can simplify the expression with \( \beta_t = \beta_x = 0 \) by choice of the origin and \( \beta_z = 0 \) by choice of the axis direction in the same way as in the above subsection.

Then the four-velocity becomes

\[
\vec{u} = \frac{1}{\sqrt{\Omega^{\rho^2 - 1}}} (\Omega x, \Omega t, 1, 0),
\]

where \( \rho^2 = x^2 - t^2 \) and \( \Omega = \lambda_{tx}/\beta_y \). The motion must be in the region of \( |\Omega\rho - 1| > 1 \) to keep the causality. The inverse temperature becomes

\[
\beta_{\mu} = \begin{cases} 
T_0^{-1} & (\mu = y), \\
0 & \text{(otherwise)}, 
\end{cases}
\]

\[
\lambda_{\mu\nu} = \begin{cases} 
\Omega T_0^{-1} & (\mu, \nu = t, x), \\
-\Omega T_0^{-1} & (\mu, \nu = x, t), \\
0 & \text{(otherwise)}. 
\end{cases}
\]

The above expressions are similar to (9): here the spatial coordinate \( y \) takes the place of the temporal coordinate \( t \) in (9). One difference to be noted is that a case with \( \beta_y = 0 \) \( (T_0, \Omega \to \infty \) with finite \( \Omega T_0^{-1} \) \) is allowed here. The four-vector \( \vec{u} \) must be timelike, therefore, \( \beta_t \) must be nonzero in the spatial rotation. The four-velocity of the acceleration, in contrast, can be timelike even when \( \beta_y = 0 \).

The motion of constant acceleration is usually investigated assuming \( \beta_y = 0 \) in the past literature. In this case (12) becomes the well-known Rindler motion:

\[
\vec{u} = \frac{1}{\rho} (x, t, 0, 0).
\]

It should be noted that the parameter \( \Omega \) vanishes in the above expression. This means the equilibrium velocity distribution of the constant acceleration is uniquely determined in general without tuning parameters. The acceleration at each trajectory is given as \( a = 1/\rho \), in other words, the magnitude of the acceleration is determined by the distance from the origin.

In this case the local temperature is inversely proportional to \( \rho \), i.e., \( T(\rho) \propto 1/\rho \), therefore it diverges at the origin. This means the definition of the global inverse temperature based on \( T_0 \), which was done in (10) or (13), is inappropriate. The inverse temperature in this case may be written using a new parameter \( \Lambda = \Omega T_0^{-1} \) as

\[
\beta_{\mu} = 0,
\lambda_{\mu\nu} = \begin{cases} 
\Lambda & (\mu, \nu = t, x), \\
-\Lambda & (\mu, \nu = x, t), \\
0 & \text{(otherwise)}. 
\end{cases}
\]

Consequently the thermal property of the system is derived from only one parameter \( \Lambda \), in contrast to the two independent parameters \( \Omega \) and \( T_0 \) for the spatial rotation.

**Concluding remarks.** Theory of relativistic thermodynamics is generalized to accommodate four-dimensional rotation in the present paper. The equilibrium state of an extended body (body with finite volume) has been mainly investigated in the past literature assuming the translational motion, i.e., the velocity of each element vanishes in the comoving frame. There can be other types of motion, spatial rotation and constant acceleration namely, with which an equilibrium state is possible. These types of motion can be regarded as four-dimensional rotation, and equilibrium is possible because of the rotational symmetry of the Minkowski spacetime.

From the non-relativistic theory of statistical mechanics we understand the conventional temperature is the result of energy conservation law. For relativistic translational motion, momentum also obeys its conservation law and should be treated in the same way as energy. This approach was adopted by van Kampen [9] and Israel [10], who proposed to treat the inverse temperature as a four-vector. The four components of the vector come from the conservation laws of energy and three components of momentum. When we generalize their theory to accommodate four-dimensional rotation, four-dimensional angular momentum must be treated in the same way as the energy-momentum because it is also a conserved quantity.

Consequently there arises another kind of inverse temperature for the angular momentum; this inverse temperature becomes covariant bivector (2-form) since the angular momentum is contravariant bivector (2-vector). Then the thermodynamical behavior of the body is completely determined by two inverse temperatures with ten components in total.

In the present paper the above formulation is applied to the motion of single rotation. There are two types of single rotation in the Minkowski spacetime; one is spatial rotation and the other is constant acceleration. The result shows that the ten components are derived from the two parameters, the global temperature \( T_0 \) and the angular momentum \( \Lambda \) namely, by the Lorentz transform.

In the case of spatial rotation, local temperature at each point of the body is uniquely determined by these two parameters, in agreement with the past literature: the rim of a rotating disk has higher temperature than the center. Constant acceleration has the similar properties
in general. However, the case of Rindler motion, which is one special case but especially important in applications, has a singular property. The ten components of the inverse temperature is derived from only one parameter $\Lambda = \Omega/T_0$, since we can eliminate the inverse temperature of energy-momentum ($\beta = 0$).

We make a brief comment on the relation of the present result to thermodynamics of quantum vacua before closing this section. It is believed that an observer with relativistic constant acceleration (Rindler motion) finds a quantum vacuum thermalized with a certain temperature (see, e.g., [13]). This effect, which is called Unruh effect, seem to contradict the result obtained in the present paper, e.g. vacuum thermalized with a certain temperature (see, Appendix). Therefore, the rotation at constant rate means the motion with constant acceleration.

To see the above point with actual calculation, let us examine a spacial case with $\lambda_{\mu \nu} = 0$ as an example. We can set $\lambda_{ty} = \lambda_{tz} = 0$ with an appropriate choice of the spatial axis. Suppose a “test particle” is moving along the Killing flow in (3) whose four-velocity is given by the second equation of (4). Then its equation of motion may be written as

$$V_x(t) = \frac{dX(t)}{dt} = \frac{\lambda_{tz} t}{1 + \lambda_{tz} X(t)}.$$  \hspace{1cm} (A.3)

The above equation gives the following velocity with the initial value $V_x = 0$ and $X = X_0$ at $t = 0$:

$$V_x(t) = \frac{\lambda_{tz} t}{\sqrt{(1 + \lambda_{tz} X_0)^2 + \lambda_{tz}^2 t^2}}.$$  \hspace{1cm} (A.4)

When $\lambda_{tz} X_0 \ll 1$, the three-velocity $V_x$ is much smaller than unity, which means the non-relativistic limit.

The above expression gives the non-relativistic constant acceleration $V_x = \lambda_{tz} t$ in this limit. Thus we understand the motion corresponds to $\lambda(t)$ is the relativistic generalization of constant acceleration.

The motion defined by (3) (or (4) equivalently) is the superposition of the translational motion, spatial rotation, and constant acceleration in general. It should be noted the decomposition in (A.1) is not unique but dependent on the choice of reference frame. We have examined the consequences of $\lambda^{(s)}$ and $\lambda^{(t)}$ separately in this appendix for simplicity, however, the general case may be complicated and cannot be understood as a simple superposition of elements.

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