SPECIAL HOMOGENEOUS LINEAR SYSTEMS ON HIRZEBRUCH SURFACES —
ALGORITHMIC ISSUES

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Abstract. We present algorithms used in the computational part of the article “Special homogeneous
linear systems on Hirzebruch surfaces”.

The main aim of this paper is to give the detailed description of algorithms used in proving the main
Theorem 6 in [Dum 09a]. We will not repeat the definitions presented in [Dum 09a].

All algorithms are presented in self-explaining pseudo-code. We will use indentation to make our
algorithms easier to read. The ← means an assignment, i.e. in the line

A ← B;

we force A to be equal to B. We use only integers, so m ≥ 2 means that m ∈ {2, 3, 4, . . .}. The command
return finishes our algorithm immediately, so further commands (if any) won’t be executed. The control
structures (if, for, repeat and so on) will be used in two versions. The first one, with only one command
executed:

if . . . then command;
and the second one, with possibility of more than one command to execute:

if . . . then

command A;
command B;
. . . ;
end if

1. Basic algorithms

The first algorithm is used to m-reduce a given diagram, according to [Dum 09a, Definition 20].

Algorithm REDUCE:

Input: m ≥ 2, a diagram D = diag(a1, . . . , ak).
Output: diag(b1, . . . , bk) = redm(D) or NOT REDUCIBLE if D is not m-reducible.

if k < m then return NOT REDUCIBLE;
for j = 1, . . . , k do b j ← a j;
U ← ∅;
for j = k, . . . , k − m + 1 do

if aj < m then r ← aj else r ← max({1, . . . , m} \ U);
bi ← aj − r;
U ← U ∪ {r};
end for
if U = {1, . . . , m} then return diag(b1, . . . , bk) else return NOT REDUCIBLE;

Example 1. Let us compute REDUCE(3, diag(5, 5, 4, 2)). First we put (b1, b2, b3, b4) = (5, 5, 4, 2) and take
U = ∅. Next, for j = 4, 3, 2 we will proceed as in the “for” loop. For j = 4 we have a4 = 2 < 3 = m,
so we take r = 2, b4 = 2 − 2 = 0, and we add {2} to the set U. The second step, for j = 3, gives
r = max({1, 2, 3} \ {2}) = 3, so r = 3 and b3 = 1. Before passing to the third step, we put U = {2, 3}.

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In the last step we will have $r = \max\{1, 2, 3\} \setminus \{2, 3\} = 1$, so $b_2 = 4$. At the end we have $U = \{1, 2, 3\}$, so REDUCE(3, diag(5, 5, 4, 2)) = diag(5, 4, 1).

We will apply a sequence of reductions to a diagram $D$, so we define an auxiliary algorithm SEQUENCE-REDUCE. By red$_m^{(k)}$ we denote 

\[ \text{red}_m^{(k)} = \underbrace{\text{red}_m \circ \cdots \circ \text{red}_m}_k. \]

**Algorithm SEQUENCE-REDUCE**

**Input:** $m \geq 2$, a number $k \geq 1$, a diagram $D$.

**Output:** red$_m^{(k)}(D)$ or NOT REDUCIBLE if reduction fails at some step.

**repeat** $k$ times

\[ D \leftarrow \text{REDUCE}(m, D); \]

\[ \text{if } D = \text{NOT REDUCIBLE then return NOT REDUCIBLE;} \]

**end repeat**

**return** $D$;

**Example 2.** Let us compute sequence-reduce(4, 3, diag([4]×3, [5]×5)). The reductions goes as follows:

\[ \text{diag}([4]×3, 5, 5, 5, 5) \rightarrow \text{diag}([4]×3, 5, 4, 3, 2, 1) \rightarrow \text{diag}(4, 4, 4, 5) \rightarrow \text{diag}(3, 2, 1, 1), \]

so

\[ \text{SEQUENCE-REDUCE}(4, 3, \text{diag}([4]×3, [5]×5)) = \text{diag}(3, 2, 1, 1), \]

while

\[ \text{SEQUENCE-REDUCE}(4, 4, \text{diag}([4]×3, [5]×5)) = \text{NOT REDUCIBLE}. \]

The following two algorithms will be used in the algorithm CH. The first one simply reduces all diagrams from a given set (omitting not reducible ones), the second finds all diagrams from a given set $D$ which cannot be reduced to some diagram from the second given set $G$.

**Algorithm RED**

**Input:** $m \geq 2$, a number $k \geq 1$, a set $D$ of diagrams.

**Output:** the set $\mathcal{G} = \{ \text{red}_m^{(k)}(D) : D \in D, D$ is $m$-reducible $k$ times$\}$. 

$\mathcal{G} \leftarrow \emptyset$;

**for each** $D \in \mathcal{D}$ **do**

\[ D \leftarrow \text{SEQUENCE-REDUCE}(m, k, D); \]

\[ \text{if } D \neq \text{NOT REDUCIBLE then } \mathcal{G} \leftarrow \mathcal{G} \cup \{D\}; \]

**end for each**

**return** $\mathcal{G}$;

**Algorithm REDOUT**

**Input:** $m \geq 2$, a number $k \geq 1$, a set $D$ of diagrams, a set $\mathcal{G}$ of diagrams.

**Output:** the set \{ $D \in \mathcal{D} : \text{red}_m^{(k)}(D) \notin \mathcal{G}$ \}.

**for each** $D \in \mathcal{D}$ **do**

\[ G \leftarrow \text{SEQUENCE-REDUCE}(m, k, D); \]

\[ \text{if } G \in \mathcal{G} \text{ then } \mathcal{D} \leftarrow \mathcal{D} \setminus \{D\}; \]

**end for each**

**return** $\mathcal{D}$;

We will often reduce a diagram $D$ as many times as possible, so we define an auxiliary algorithm TOP-REDUCE.

**Algorithm TOP-REDUCE**

**Input:** $m \geq 2$, a diagram $D = \text{diag}(a_1, \ldots, a_k)$.

**Output:** $G = \text{diag}(b_1, \ldots, b_k) = \text{red}_m (\text{red}_m (\ldots (D) \ldots ))$ such that $G$ is not $m$-reducible.
repeat
  \( G \leftarrow \text{REDUCE}(m, D); \)
  if \( G = \text{NOT REDUCIBLE} \) then return \( D; \)
  \( D \leftarrow G; \)
end repeat

Observe that if \( D \) is not \( m \)-reducible then \( \text{TOP-REDUCE}(D) = D. \)

**Example 3.** Let us compute \( \text{TOP-REDUCE}(3, \text{diag}(5, 5, 4, 2)) \). We have
\[
\text{REDUCE}(3, \text{diag}(5, 5, 4, 2)) = \text{diag}(5, 4, 1),
\]
\[
\text{REDUCE}(3, \text{diag}(5, 4, 1)) = \text{diag}(3, 1)
\]
\[
\text{REDUCE}(3, \text{diag}(3, 1)) = \text{NOT REDUCIBLE}.
\]

Hence \( \text{TOP-REDUCE}(3, \text{diag}(5, 5, 4, 2)) = \text{diag}(3, 1). \)

Now we present an algorithm to find all admissible \( h \)-\text{diag}(b_1, \ldots, b_{m-1})\)-tails for multiplicity \( m \), see [Dum 09a, Definition 35]. By the length of a diagram \( D = \text{diag}(a_1, \ldots, a_k) \) we denote the number of its non-zero layers,
\[
\text{length}(D) = \#\{j : a_j > 0\}.
\]

**Algorithm** \( h \)-tails

**Input:** \( m \geq 2, h > m, D = \text{diag}(b_1, \ldots, b_{m-1}). \)

**Output:** the set \( \mathcal{D} \) of all admissible \( h \)-\text{D}-tails for multiplicity \( m \), or \text{ERROR} if some reduction stops too early.

\[
\mathcal{D} \leftarrow \emptyset;
\]
repeat
  \( \mathcal{D} \leftarrow \mathcal{D} \cup \{D\}; \)
  \( D \leftarrow \text{diag}(h) + D; \)
  \( D \leftarrow \text{TOP-REDUCE}(m, D); \)
  if \( \text{length}(D) \geq m \) then return \text{ERROR};
  if \( D \in \mathcal{D} \) then return \( D; \)
end repeat

**Example 4.** We will find all admissible \( 4 \)-\text{diag}(0,0)-tails for multiplicity 3. The set \( \mathcal{D} \) is empty at the beginning, and we put \( D = \text{diag}(0, 0) = \emptyset \) into \( \mathcal{D} \). Now we take new \( D = \text{diag}(4) \), reduce it as many times as possible, but, in fact, \( \text{TOP-REDUCE}(3, \text{diag}(4)) = \text{diag}(4) \). So we go at the beginning of the “repeat” loop and add \( \text{diag}(4) \) to the set \( \mathcal{D} \). Now we take new \( D = \text{diag}(4) + \text{diag}(4) = \text{diag}(4, 4) \), still it cannot be 3-reduced. Thus we add it into \( \mathcal{D} \), which now is equal to \( \{\emptyset, \text{diag}(4), \text{diag}(4, 4)\} \). Taking \( D = \text{diag}(4) + \text{diag}(4, 4) = \text{diag}(4, 4, 4) \) we obtain \( \text{TOP-REDUCE}(3, \text{diag}(4, 4, 4)) = \emptyset \). Since the last diagram already belongs to \( \mathcal{D} \), the algorithm terminates.

Observe that the size of \( \mathcal{D} \) depends also on \( h \) and \( \text{diag}(b_1, \ldots, b_{m-1}). \) For example,
\[
\text{h-tails}(3, 5, \text{diag}(0, 0)) = \{\emptyset, \text{diag}(5), \text{diag}(5, 5), \text{diag}(3), \text{diag}(5, 3), \text{diag}(4, 3), \text{diag}(4, 2), \text{diag}(4, 1), \text{diag}(3, 1)\};
\]
while
\[
\#\text{h-tails}(3, 5, \text{diag}(1, 0)) = \#\text{h-tails}(3, 5, \text{diag}(0, 0)) + 4.
\]

**Example 5.** We will compute \( \text{h-tails}(5, 9, \text{diag}(0, 0, 0, 0)) \). The computations can be written in the following short form:
\[
\emptyset \rightarrow \text{diag}(9) \rightarrow \text{diag}(9, 9) \rightarrow \text{diag}(9, 9, 9) \rightarrow \text{diag}(9, 9, 9, 9) \rightarrow \text{diag}(8, 7, 6, 5, 4) \rightarrow \text{diag}(7, 5, 3) \rightarrow \text{diag}(9, 7, 5, 3) \rightarrow \text{diag}(8, 7, 3) \rightarrow \text{diag}(9, 8, 7, 3) \rightarrow \text{diag}(9, 9, 8, 7, 3) \rightarrow \text{diag}(8, 5, 2) \rightarrow \text{diag}(9, 8, 5, 2) \rightarrow \text{diag}(8, 6, 4) \rightarrow \text{diag}(9, 8, 6, 4) \rightarrow \text{diag}(9, 9, 8, 6, 4) \rightarrow \text{diag}(8, 7, 5, 1) \rightarrow \text{diag}(9, 8, 7, 5, 1) \rightarrow \text{diag}(7, 5, 3)
\]
and we finish, since the last diagram has been found earlier.

Since we are interested in collecting all admissible $h$-$D$-tails for all $D \in \mathcal{D}$, we present an auxiliary algorithm, called $\text{ltails}$.

**Algorithm $\text{ltails}$**

**Input:** $m \geq 2$, $h > m$, a set $\mathcal{D}$ of diagrams.

**Output:** the set $\mathcal{G}$ of all admissible $h$-$D$-tails for multiplicity $m$ and $D \in \mathcal{D}$, or ERROR if some reduction stops too early.

\[
\mathcal{G} \leftarrow \emptyset;
\]
for each $D \in \mathcal{D}$ do
\[
\mathcal{H} \leftarrow \text{h-tails}(m, h, D);
\]
if $\mathcal{H} = \text{ERROR}$ then return ERROR;
\[
\mathcal{G} \leftarrow \mathcal{G} \cup \mathcal{H};
\]
end for each

return $\mathcal{G}$;

Sometimes we also want to find all (top) reductions of diagrams of the form $\text{diag}(\lceil h \rceil \times n) + D$ for some fixed $n$ and a set $\mathcal{D}$ of diagrams.

**Algorithm $\text{atails}$**

**Input:** $m \geq 2$, $h > m$, $n > 0$, a set $\mathcal{D}$ of diagrams.

**Output:** the set $\mathcal{G}$ of all diagram from $\{ \text{diag}(\lceil h \rceil \times n) + D : D \in \mathcal{D} \}$ reduced as many times as possible, or ERROR if some reduction stops too early.

\[
\mathcal{G} \leftarrow \emptyset;
\]
for each $D \in \mathcal{D}$ do
\[
\mathcal{G} \leftarrow \text{diag}(\lceil h \rceil \times n) + D;
\]
\[
\mathcal{G} \leftarrow \text{top-reduce}(m, \mathcal{G});
\]
if $\text{length}(\mathcal{G}) \geq m$ then return ERROR;
\[
\mathcal{G} \leftarrow \mathcal{G} \cup \{ \mathcal{G} \};
\]
end for each

return $\mathcal{G}$;

**Example 6.** Let us compute $\text{atails}(3, 5, 2, \mathcal{D})$ for

\[
\mathcal{D} = \{ \emptyset, \text{diag}(5), \text{diag}(5, 5), \text{diag}(3) \}.
\]

For each $D \in \mathcal{D}$ we execute $\text{top-reduce}(3, \text{diag}(5, 5) + D)$. The result is

$\{ \text{diag}(5, 5), \text{diag}(3), \text{diag}(5, 3), \text{diag}(4, 3) \}$.

Observe that $\text{atails}(3, 4, 2, \mathcal{D}) = \text{ERROR}$, since

$\text{top-reduce}(3, \text{diag}(4, 4, 5)) = \text{diag}(4, 4, 2)$,

which is too long.

Now our aim is to enumerate all admissible $\text{diag}(a_1, \ldots, a_m)$-tails for multiplicity $m$, see [Dum 09a, Definition 39]. All admissible tails could be found by iterating symbolic reductions, see [Dum 09a, Definition 37] and the discussion after Example 40. Therefore we present an algorithm to produce all symbolic reductions of a given diagram of the form $\text{diag}(a_1, \ldots, a_m, [x]^{xk})$. This amounts to substitute $[x]^{xk}$ by all reasonable integers and reduce obtained diagrams. Observe that for $k = 0$, the symbolic reduction is equal to the $m$-reduction.

If $D = \text{diag}(a_1, \ldots, a_k)$ then by $\text{cut}(D, r)$ we denote the diagram given by

\[
\text{cut}(D, r) = \begin{cases} 
\text{diag}(a_1, \ldots, a_r) & r \leq k, \\
D & r \geq k.
\end{cases}
\]
Algorithm SYMB-REDUCE

Input: $m \geq 2$, $\text{diag}(a_1, \ldots, a_m, [x]^{\times k})$, $0 \leq k \leq m - 1$.
Output: the set $\mathcal{D}$ of all symbolic reductions of $\text{diag}(a_1, \ldots, a_m, [x]^{\times k})$.

$\mathcal{D} \leftarrow \emptyset$;
if $k = 0$ then
   $D \leftarrow \text{REDUCE}(m, \text{diag}(a_1, \ldots, a_m))$;
   if $D \neq \text{NOT REDUCIBLE}$ then $\mathcal{D} \leftarrow \{D\}$;
return $\mathcal{D}$;
end if
for each $(c_1, \ldots, c_k)$ satisfying $\min\{m + 1, a_m\} \geq c_1 \geq c_2 \geq \cdots \geq c_k$ do
   $D \leftarrow \text{diag}(a_1, \ldots, a_m, c_1, \ldots, c_k)$;
   $D \leftarrow \text{REDUCE}(m, D)$;
   if $D \neq \text{NOT REDUCIBLE}$ then
      $G \leftarrow \text{cut}(D, m)$;
      $\ell \leftarrow \text{length}(D) - \text{length}(G)$;
      $G \leftarrow G + \text{diag}([x]^{\times \ell})$;
      $\mathcal{D} \leftarrow \mathcal{D} \cup \{G\}$;
   end if
end for each
return $\mathcal{D}$;

Example 7. Let us compute SYMB-REDUCE$(3, \text{diag}(5, 5, 5, x, x))$. In the “foreach” loop we must consider all pairs $(c_1, c_2)$ satisfying $4 \geq c_1 \geq c_2$. For each such pair we take $\text{diag}(5, 5, 5, c_1, c_2)$, 3-reduce it, and change the fourth and fifth number into symbols $x$, if necessary. We present the computations in the following table.

| $(c_1, c_2)$ | $\text{diag}(5, 5, 5, c_1, c_2)$ | $\text{REDUCE}(\text{diag}(5, 5, 5, c_1, c_2))$ | the result |
|-------------|---------------------------------|---------------------------------|------------|
| (4, 4)      | $\text{diag}(5, 5, 5, 4, 4)$    | $\text{diag}(5, 5, 4, 2, 1)$    | $\text{diag}(5, 5, 4, x, x)$ |
| (4, 3)      | $\text{diag}(5, 5, 5, 4, 3)$    | $\text{diag}(5, 5, 4, 2)$       | $\text{diag}(5, 5, 4, x)$   |
| (4, 2)      | $\text{diag}(5, 5, 5, 4, 2)$    | $\text{diag}(5, 5, 4, 1)$       | $\text{diag}(5, 5, 4, x)$   |
| (4, 1)      | $\text{diag}(5, 5, 5, 4, 1)$    | $\text{diag}(5, 5, 3, 1)$       | $\text{diag}(5, 5, 3, x)$   |
| (4, 0)      | $\text{diag}(5, 5, 5, 4)$       | $\text{diag}(5, 4, 3, 1)$       | $\text{diag}(5, 4, 3, x)$   |
| (3, 3)      | $\text{diag}(5, 5, 5, 3, 3)$    | $\text{diag}(5, 5, 4, 1)$       | $\text{diag}(5, 5, 4, x)$   |
| (3, 2)      | $\text{diag}(5, 5, 5, 3, 2)$    | $\text{diag}(5, 5, 4)$          | $\text{diag}(5, 5, 4)$      |
| (3, 1)      | $\text{diag}(5, 5, 5, 3, 1)$    | $\text{diag}(5, 5, 3)$          | $\text{diag}(5, 5, 3)$      |
| (3, 0)      | $\text{diag}(5, 5, 5, 3)$       | $\text{diag}(5, 4, 3)$          | $\text{diag}(5, 4, 3)$      |
| (2, 2)      | $\text{diag}(5, 5, 5, 2, 2)$    | NOT REDUCIBLE                    |            |
| (2, 1)      | $\text{diag}(5, 5, 5, 2, 1)$    | $\text{diag}(5, 5, 2)$          | $\text{diag}(5, 5, 2)$      |
| (2, 0)      | $\text{diag}(5, 5, 5, 2)$       | $\text{diag}(5, 4, 2)$          | $\text{diag}(5, 4, 2)$      |
| (1, 1)      | $\text{diag}(5, 5, 5, 1, 1)$    | NOT REDUCIBLE                    |            |
| (1, 0)      | $\text{diag}(5, 5, 5, 1)$       | $\text{diag}(5, 3, 2)$          | $\text{diag}(5, 3, 2)$      |
| (0, 0)      | $\text{diag}(5, 5, 5)$          | $\text{diag}(4, 3, 2)$          | $\text{diag}(4, 3, 2)$      |

Now we can enumerate all admissible $\text{diag}(a_1, \ldots, a_m)$-tails for multiplicity $m$. To do this, we will consider all symbolic reductions of symbolic reductions of $\ldots$ and so on. Each diagram obtained in this way, which is short enough (i.e. without $x$ and with length at most $m - 1$), satisfies the desired property.

Algorithm TAILS

Input: $m \geq 2$, $\text{diag}(a_1, \ldots, a_m)$.
Output: the set $\mathcal{D}$ of all admissible $\text{diag}(a_1, \ldots, a_m)$-tails.

$\mathcal{D} \leftarrow \emptyset$;
$\mathcal{W} \leftarrow \{\text{diag}(a_1, \ldots, a_m, [x]^{\times (m - 1)})\}$;
repeat
   choose $W \in \mathcal{W}$;
   $\mathcal{W} \leftarrow \mathcal{W} \setminus \{W\}$;
   $\mathcal{R} \leftarrow \text{SYMB-REDUCE}(m, W)$;
repeat
\[ W \leftarrow W \cup \mathcal{R}; \]

for each \( D \in \mathcal{R} \) do

\[
\text{if } \text{length}(D) < m \text{ then } D \leftarrow D \cup \{D\};
\]

end for each

until \( W = \emptyset \)

To show that the above algorithm terminates after a finite number of steps, observe that in each step, after choosing \( W \in W \) and producing \( \mathcal{R} = \text{SYMB-REDUCE}(m, W) \), we have

\[
\#W > \max\{\#R : R \in \mathcal{R}\},
\]

where

\[
\#\text{diag}(a_1, \ldots, a_k, [x]^j) = a_1 + \cdots + a_k.
\]

Example 8. We will find \( \text{TAILS}(2, \text{diag}(5, 5)) \). The example for \( m = 3 \) would be a bit too long. The idea is to compute consecutive symbolic reductions of \( \text{diag}(5, 5, x) \). In the first step we proceed as in Example 7.



| \( D \) | \( c_1 \) | \( D(c_1) \) | \( \text{red}_2(D(c_1)) \) | the result |
|--------|--------|----------|-----------------|-----------|
| diag(5, 5, x) | 3 | diag(5, 5, 3) | diag(5, 4, 1) | diag(5, 4, x) |
| diag(5, 5, x) | 2 | diag(5, 5, 2) | diag(5, 4) | diag(5, 4) |
| diag(5, 5, x) | 1 | diag(5, 5, 1) | diag(5, 3) | diag(5, 3) |
| diag(5, 5, x) | 0 | diag(5, 5) | diag(4, 3) | diag(4, 3) |

In the next step we take all obtained diagrams and perform all possible symbolic reductions.



| \( D \) | \( c_1 \) | \( D(c_1) \) | \( \text{red}_2(D(c_1)) \) | the result |
|--------|--------|----------|-----------------|-----------|
| diag(5, 4, x) | 3 | diag(5, 4, 3) | diag(5, 3, 1) | diag(5, 3, x) |
| diag(5, 4, x) | 2 | diag(5, 4, 2) | diag(5, 3) | diag(5, 3) |
| diag(5, 4, x) | 1 | diag(5, 4, 1) | diag(5, 2) | diag(5, 2) |
| diag(5, 4, x) | 0 | diag(5, 4) | diag(4, 2) | diag(4, 2) |
| diag(5, 4) | diag(5, 4) | diag(4, 2) | diag(4, 2) | diag(4, 2) |
| diag(5, 4) | diag(5, 3) | diag(5, 3) | diag(4, 1) | diag(4, 1) |
| diag(4, 3) | diag(4, 3) | diag(3, 1) | diag(3, 1) | diag(3, 1) |

Again, in the third step:



| \( D \) | \( c_1 \) | \( D(c_1) \) | \( \text{red}_2(D(c_1)) \) | the result |
|--------|--------|----------|-----------------|-----------|
| diag(5, 3, x) | 3 | diag(5, 3, 3) | diag(5, 2, 1) | diag(5, 2, x) |
| diag(5, 3, x) | 2 | diag(5, 3, 2) | diag(5, 2) | diag(5, 2) |
| diag(5, 3, x) | 1 | diag(5, 3, 1) | diag(5, 1) | diag(5, 1) |
| diag(5, 3, x) | 0 | diag(5, 3) | diag(4, 1) | diag(4, 1) |
| diag(5, 3) | diag(5, 3) | diag(4, 1) | diag(4) | diag(4) |
| diag(4, 2) | diag(4, 2) | diag(3) | diag(3) | diag(3) |
| diag(4, 1) | diag(4, 1) | diag(2) | diag(2) | diag(2) |
| diag(3, 1) | diag(3, 1) | diag(1) | diag(1) | diag(1) |

The diagrams with length 1 are no more 2-reducible, so they won’t produce any additional admissible tail. We present the fourth step:



| \( D \) | \( c_1 \) | \( D(c_1) \) | \( \text{red}_2(D(c_1)) \) | the result |
|--------|--------|----------|-----------------|-----------|
| diag(5, 2, x) | 2 | diag(5, 2, 2) | diag(5, 1) | diag(5, 1) |
| diag(5, 2, x) | 1 | diag(5, 2, 1) | diag(5) | diag(5) |
| diag(5, 2, x) | 0 | diag(5, 2) | diag(4) | diag(4) |
| diag(5, 2) | diag(5, 2) | diag(4) | diag(4) | diag(4) |
| diag(5, 1) | diag(5, 1) | diag(3) | diag(3) | diag(3) |
| diag(4, 1) | diag(4, 1) | diag(2) | diag(2) | diag(2) |

In the final step we must reduce \( \text{diag}(5, 1) \) to obtain \( \text{diag}(3) \). We collect all obtained diagrams of length at most 1, thus

\[
\text{TAILS}(2, \text{diag}(5, 5)) = \{ \text{diag}(5), \text{diag}(4), \text{diag}(3), \text{diag}(2), \text{diag}(1) \}.
\]

Observe that we can skip some of the above reducing.
2. Algorithms to Compute Sets \( \mathcal{D} \)

In [Dum 09a, Section 7] we construct various sets of diagrams. Each set serves for showing that some given family of systems contains only non-special ones. To be more precise, we define a family \( \mathcal{S} \) of systems together with a finite set \( \mathcal{D} \) of diagrams such that if for each \( D \in \mathcal{D} \) and \( r = \lfloor \frac{\#D}{2} \rfloor \), the systems \( \mathcal{L}(D; m \times r) \) and \( \mathcal{L}(D; m \times (r+1)) \) are non-special then \( \mathcal{S} \) contains only non-special systems.

The first algorithm computes the set \( \mathcal{D} \) from [Dum 09a, Proposition 42]. For \( D = \text{diag}(a_1, \ldots, a_k) \) let \( \text{rev}(D) = \text{diag}(a_k, \ldots, a_1) \). Similarly, for a set \( \mathcal{D} \) of diagrams, let

\[
\text{rev}(\mathcal{D}) = \{ \text{rev}(D) : D \in \mathcal{D} \}.
\]

Algorithm `setDsign`

**Input:** \( m \geq 4, N \geq m \).

**Output:** the set \( \mathcal{D} \) from Proposition 42.

\[
\begin{align*}
\mathcal{D} & \leftarrow \emptyset; \\
\mathcal{L} & \leftarrow \text{h-tails}(m, m+1, \text{diag}([0]^{m(m-1)})); \\
& \text{for } j = m+2, \ldots, 2m-3 \text{ do} \\
& \quad \mathcal{L} \leftarrow \text{atails}(m, j, N, \mathcal{L}); \\
& \quad \mathcal{L} \leftarrow \text{ltails}(m, j, \mathcal{L}); \\
& \text{end for} \\
\mathcal{R} & \leftarrow \text{tails}(m, \text{diag}([2m-1]^{N})); \\
& \text{for each } (L, R) \in \mathcal{L} \times \mathcal{R} \text{ do} \\
& \quad \mathcal{D} \leftarrow \mathcal{D} \cup \{ \text{rev}(L) + \text{diag}([2m-2]^{N}) + R \}; \\
& \text{end for} \\
& \text{return } \mathcal{D};
\end{align*}
\]

**Example 9.** We will show the example for \( m = 5, N = 11 \) (which is a part of our computation to prove Theorem 6 in [Dum 09a]). We will not present all the details, since the output would be too big. In our case we do the following:

\[
\begin{align*}
\mathcal{L} & \leftarrow \text{h-tails}(5, 6, \text{diag}(0, 0, 0, 0)); \\
\mathcal{L} & \leftarrow \text{atails}(5, 7, 11, \mathcal{L}); \\
\mathcal{L} & \leftarrow \text{ltails}(5, 7, \mathcal{L}); \\
\mathcal{L} & \leftarrow \text{ltails}(5, 8, \mathcal{L}); \\
\mathcal{R} & \leftarrow \text{tails}(5, \text{diag}(9, 9, 9, 9));
\end{align*}
\]

In the first step we obtain

\[
\mathcal{L} = \{ \emptyset, \text{diag}(6), \text{diag}(6, 6), \text{diag}(6, 6, 6), \text{diag}(6, 6, 6, 6) \}.
\]

In the second step, for each \( D \in \mathcal{L} \), we take \text{top-reduce}(5, \text{diag}([7]^{11} + D) \). After reducing, we will have

\[
\mathcal{L} = \{ \text{diag}(7, 6, 4), \text{diag}(7, 7, 6, 3), \text{diag}(6, 4, 3, 1), \text{diag}(5), \text{diag}(7, 4) \}.
\]

In the third step we look for \text{top-reduce} of all diagrams of the form

\[
\text{diag}([7]^{k}) + D, \quad k \geq 0, \quad D \in \mathcal{L}.
\]

We will not enumerate all of them, since after this step, \( \#\mathcal{L} = 53 \). In the fourth step we look for

\[
\text{diag}([8]^{k}) + D, \quad k \geq 0, \quad D \in \mathcal{L},
\]

and the resulting set contains 119 diagrams. Now we look for admissible tails. After computations, we obtain \( \mathcal{R} \) of cardinality 147.

Now we must “glue” diagrams from \( \mathcal{L} \) and \( \mathcal{R} \) to produce \( 119 \cdot 147 = 17493 \) diagrams in \( \mathcal{D} \). One can check that, for example,

\[
\text{diag}(8, 6, 3, 1) \in \mathcal{L}, \quad \text{diag}(7, 6, 5, 4) \in \mathcal{R},
\]

so we have

\[
\text{diag}(1, 3, 6, 8, [8]^{11}, 7, 6, 5, 4) \in \mathcal{D}.
\]
Our next algorithm computes the set $\mathcal{D}$ from Proposition 44 in [Dum 09a].

**Algorithm setbign23**

**Input:** $2 \leq m \leq 3$, $N \geq m$.  
**Output:** the set $\mathcal{D}$ from Proposition 44.

$\mathcal{D} \leftarrow \emptyset$;  
$L \leftarrow \text{H-TAILS}(m, m + 1, \text{diag}([0]^{(m-1)}));$  
$L \leftarrow \text{LTAILS}(m, m + 2, L);$  
$\mathcal{R} \leftarrow \text{TAILS}(m, \text{diag}([m + 3 \times m]));$  
for each $(L, R) \in L \times R$ do  
\[ \mathcal{D} \leftarrow \mathcal{D} \cup \{\text{rev}(L) + \text{diag}([m + 2] \times N) + R\}; \]
end for each  
return $\mathcal{D};$

**Example 10.** The example for $m = 3$ would be very nice and illustrating, but also too long. So we will compute setbign23(2, 2). We have three steps:

$L \leftarrow \text{H-TAILS}(2, 3, \text{diag}(0));$  
$L \leftarrow \text{LTAILS}(2, 4, L);$  
$\mathcal{R} \leftarrow \text{TAILS}(2, \text{diag}(5, 5));$

In the first step we obtain  
$L = \{\emptyset, \text{diag}(3)\}.$

In the next step  
$L = \{\text{diag}(4), \text{diag}(3), \text{diag}(2), \text{diag}(1)\}.$

In Example 8 we have shown that  
$\mathcal{R} = \{\text{diag}(5), \text{diag}(4), \text{diag}(3), \text{diag}(2), \text{diag}(1)\}.$

So the final set is  
$\mathcal{D} = \{\text{diag}(a, [4] \times 2, b) : 1 \leq a \leq 4, 1 \leq b \leq 5\}.$

Our next algorithm computes the set $\mathcal{D}$ from [Dum 09a, Proposition 47].

**Algorithm setbignb**

**Input:** $m \geq 2$, $N \geq m$, $b \geq m + 2$.  
**Output:** the set $\mathcal{D}$ from Proposition 47.

$\mathcal{D} \leftarrow \emptyset;$  
$L \leftarrow \text{H-TAILS}(m, m + 1, \text{diag}([0]^{(m-1)}));$  
for $j = m + 2, \ldots, b - 1$ do  
\[ L \leftarrow \text{ATAILS}(m, j, N, L); \]
\[ L \leftarrow \text{LTAILS}(m, j, L); \]
end for  
$L \leftarrow \text{HTAILS}(m, b, L);$  
$\mathcal{R} \leftarrow \text{H-TAILS}(m, b + 1, \text{diag}([0]^{(m-1)}));$  
for each $(L, R) \in L \times \mathcal{R}$ do  
\[ \mathcal{D} \leftarrow \mathcal{D} \cup \{\text{rev}(L) + \text{diag}([b] \times N) + R\}; \]
end for each  
return $\mathcal{D};$

**Example 11.** We will show the example for $m = 5$, $N = 11$, $b = 8$ (which is a part of our computation to prove Theorem 6 in [Dum 09a]). In our case we do the following:

$L \leftarrow \text{H-TAILS}(5, 6, \text{diag}(0, 0, 0, 0));$  
$L \leftarrow \text{ATAILS}(5, 7, 11, L);$  
$L \leftarrow \text{LTAILS}(5, 7, L);$  
$L \leftarrow \text{LTAILS}(5, 8, L);$
This is very similar to what we did in Example 9, except for the last step. So we will have \( \#L = 119 \).
Now we must look for all possible top-reductions of \( \text{diag}([9]\times k) \). By Example 5 we have \( \#R = 15 \), so \( \#D = 15 \cdot 119 = 1785 \).

The next algorithm computes the set \( D \) from [Dum 09a, Proposition 50]. For a diagram \( D = \text{diag}(a_1, \ldots, a_k) \) let

\[
\text{cut}(D, \ell) = \begin{cases}
    \text{diag}(a_{k-\ell+1}, a_{k-\ell+2}, \ldots, a_k) & \ell \leq k, \\
    D & \ell > k.
\end{cases}
\]

**Algorithm setnb**

**Input:** \( m \geq 2, n \geq 2, B \geq 2m - 1 \).

**Output:** the set \( D \) from Proposition 50.

\[
D \leftarrow \emptyset;
G \leftarrow \text{diag}([m + 1]^{\times n}, \ldots, [B]^{\times n}, B + 1);
H \leftarrow \text{cut}(G, m);
K \leftarrow \text{cut}(G, n(B - m) - m + 1);
R \leftarrow \text{tails}(m, H);
\text{for each } R \in R \text{ do}
\quad D \leftarrow D \cup \{ K + R \};
\text{end for each}
\text{return } D;
\]

**Example 12.** Let us compute \( \text{setnb}(3, 2, 6) \). We will have

\[ G = \text{diag}(4, 4, 5, 6, 6, 7), \]
so we take

\[ H = \text{diag}(6, 6, 7), \quad K = \text{diag}(4, 4, 5, 5). \]

Now we must find \( \mathcal{R} = \text{tails}(m, H) \) by considering all symbolic reductions of \( \text{diag}(6, 6, 7, x, x) \), and take \( D = \{ \text{diag}(4, 4, 5, 6) + R : R \in \mathcal{R} \} \).

The next algorithm computes the set \( D \) from [Dum 09a, Proposition 52].

**Algorithm setnba**

**Input:** \( m \geq 2, n \geq 2, b \geq m + 1, A \geq 0 \).

**Output:** the set \( D \) from Proposition 52.

\[
D \leftarrow \emptyset;
R \leftarrow \text{h-tails}(m, b + 1, \text{diag}([0]^{\times (m - 1)})));
\text{for each } R \in R \text{ do}
\quad D \leftarrow D \cup \{ \text{diag}([m + 1]^{\times n}, \ldots, [b]^{\times n}, [b + 1]^{\times (A + 1)}) + R \};
\text{end for each}
\text{return } D;
\]

**Example 13.** For \((m, n, b, A) = (3, 2, 5, 0)\) we will have

\[
\mathcal{R} = \text{h-tails}(3, 6, \text{diag}(0, 0))
\quad = \{ \emptyset, \text{diag}(6), \text{diag}(6, 6), \text{diag}(4, 2), \text{diag}(5, 1) \},
\]
so

\[ D = \{ \text{diag}(4, 4, 5, 6) + R : R \in \mathcal{R} \}. \]

The next algorithm computes the set \( D \) from [Dum 09a, Proposition 54].

**Algorithm setpb**
Input: \( m \geq 2, B \geq 3(m-1). \)

Output: the set \( \mathcal{D} \) from Proposition 54.

\[
\mathcal{D} \leftarrow \emptyset; \\
\mathcal{R} \leftarrow \text{tails}(m, \text{diag}(B - m + 2, B - m + 3, \ldots, B + 1)); \\
\text{for each } R \in \mathcal{R} \text{ do} \\
\quad \mathcal{D} \leftarrow \mathcal{D} \cup \{ \text{diag}(1, 2, \ldots, B - m + 1) + R \}; \\
\text{end for each} \\
\text{return } \mathcal{D};
\]

Example 14. We will compute \( \text{setpb}(3, 9) \). Hence our computation starts with

\[
\mathcal{R} = \text{tails}(3, \text{diag}(8, 9, 10)).
\]

We obtain \( \# \mathcal{R} = 28 \) and take 28 diagrams of the form

\[
\text{diag}(1, 2, \ldots, 7) + R : R \in \mathcal{R}.
\]

For example, we will have \( \text{diag}(1, 2, 3, 4, 5, 6, 7, 5) \in \mathcal{D} \).

The next algorithm computes the set \( \mathcal{D} \) from [Dum 09a, Proposition 56].

Algorithm \text{setpba}

Input: \( m \geq 2, B \geq m, A \geq b. \)

Output: the set \( \mathcal{D} \) from Proposition 56.

\[
\mathcal{D} \leftarrow \emptyset; \\
\mathcal{R} \leftarrow \text{h-tails}(m, b+1, \text{diag}([0]^{(m-1)})); \\
\text{for each } R \in \mathcal{R} \text{ do} \\
\quad \mathcal{D} \leftarrow \mathcal{D} \cup \{ \text{diag}([b+1]^{(A+1)}) + R \}; \\
\text{end for each} \\
\text{return } \mathcal{D};
\]

Example 15. It is easy to check that

\[
\text{setpba}(3, 7, 7) = \{ \text{diag}([8]^{8}), \text{diag}([8]^{9}), \text{diag}([8]^{10}), \text{diag}([8]^{8}, 5, 1), \\
\text{diag}([8]^{8}, 6, 2), \text{diag}([8]^{8}, 7, 3), \text{diag}([8]^{8}, 7, 5) \}.
\]

3. Checking non-speciality

We begin with an algorithm to decide whether a given system \( \mathcal{L}(D; m^{x^r}) \) is special or not. The computations will be performed over \( \mathbb{F}_p \) and for some randomly chosen coordinates of points. Therefore, if our “specialized” system is non-special then obviously the general one is also non-special. In the opposite case we only know that our method does not work. We begin with preparing the matrix for our system (see [Dum 09c]).

Algorithm \text{i-matrix}

Input: a diagram \( D, m \geq 2, r \geq 1, p_1, \ldots, p_r \in \mathbb{F}_p^2 \).

Output: the matrix \( M \) associated to \( \mathcal{L}(D; mp_1, \ldots, mp_r) \).

Property: computations over \( \mathbb{F}_p \).

\[
\mathcal{M} \leftarrow \{ x^a y^b : (a, b) \in D \}; \\
f_M \leftarrow \text{a one-to-one correspondence from } \mathcal{M} \text{ to } \{1, \ldots, \#D\}; \\
\mathcal{C} \leftarrow \{ (k, d_x, d_y) : 1 \leq k \leq r, d_x + d_y < m \}; \\
f_C \leftarrow \text{a one-to-one correspondence from } \mathcal{C} \text{ to } \{1, \ldots, r \binom{m}{2} \}; \\
\text{for each } (x^a y^b, (k, d_x, d_y)) \in \mathcal{M} \times \mathcal{C} \text{ do} \\
\quad g \leftarrow \frac{\partial^k \partial^{d_x + d_y}}{\partial x^{d_x} \partial y^{d_y}} (x^a y^b); \\
\quad g \leftarrow g(p_k); \\
\quad M(f_M(x^a y^b), f_C(k, d_x, d_y)) \leftarrow g; \\
\text{end for each}
\]
return $M$;

**Example 16.** Let us compute $M$ for $D = \text{diag}(3, 2, 1)$, $m \times r = 2 \times 2$, $p_1 = (0, 0)$, $p_2 = (2, 1)$. The set of monomials

$\mathcal{M} = \{1, x, y, x^2, xy, y^2\}$

will be ordered by $f_\mathcal{M}$ as above, the set of conditions will also be ordered as indicated:

$\mathcal{C} = \{(1, 0, 0), (1, 1, 0), (1, 0, 1), (2, 0, 0), (2, 1, 0), (2, 0, 1)\}$.

Now, for example, take $x^2$ and $(1, 1, 0)$. The polynomial

$$g = \frac{\partial x^2}{\partial x}(0, 0) = 0$$

will be inserted into $M$ in the $f_\mathcal{M}(x^2) = 4$th row and the $f_\mathcal{C}(1, 1, 0) = 2$nd column. For the same monomial and condition $(2, 1, 0)$ we will have

$$g = \frac{\partial x^2}{\partial x}(2, 1) = 4$$

inserted into $M[4, 5]$. The entire matrix is equal to

$$M = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 4 & 4 & 0 \\
0 & 0 & 0 & 2 & 1 & 2 \\
0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}.$$

For a matrix $M$ over $F_p$, let $\text{rank}(M)$ denote the rank of $M$ computed, for example, by using the Gauss elimination.

**Algorithm** \textsc{ns}

**Input:** $m \geq 2$, $r \geq 1$, a diagram $D$, a number of tries $t \geq 1$.

**Output:** \textsc{non-special} or \textsc{not decided},

\textsc{non-special} implies that $\mathcal{L}(D; m \times r)$ is non-special.

**Property:** computations over $F_p$.

**repeat** $t$ times

$(p_1, \ldots, p_r) \leftarrow$ randomly chosen points in $F_p^2$;

$M \leftarrow$ \textsc{i-matrix}(\text{D}, m, r, p_1, \ldots, p_r);

$k \leftarrow$ \textsc{rank}(M);

if $k = \min\{\#D, r(m+1)\}$ then return \textsc{non-special};

**end repeat**

**return** \textsc{not decided};

**Example 17.** We will have \textsc{ns}(2, 2, \text{diag}(3, 2, 1), t) = \textsc{not decided}, since the system $\mathcal{L}(\text{diag}(3, 2, 1); 2, 2) = \mathcal{L}(2; 2, 2)$ is special.

Taking $D = \text{diag}(2, 1)$, $m = 1$ and $r = 3$ we will obtain \textsc{ns}(1, 3, \text{diag}(2, 1), t) = \textsc{non-special} if and only if the algorithm chooses $p_1, p_2, p_3$ not lying on a line. For a non-special system, the result \textsc{non-special} is much more probable if the number $t$ of tries is big. During computations, it appeared that “big” in our case means $t \geq 6$.

We will check non-speciality of all systems $\mathcal{L}(D; m \times r)$ for $D \in \mathcal{D}$, fixed $m$ and all $r \geq 1$.

**Algorithm** \textsc{check}

**Input:** $m \geq 2$, a set of diagrams $\mathcal{D}$, a number of tries $t \geq 1$.

**Output:** a set $\mathcal{G} \subset \mathcal{D}$ such that for every $G \in \mathcal{G}$, $r \geq 1$,

$\mathcal{L}(G; m \times r)$ is non-special.

$\mathcal{G} \leftarrow \varnothing$;

**for each** $D \in \mathcal{D}$ **do**
\[ r \leftarrow \left\lfloor \frac{\#D}{m+2} \right\rfloor; \]

if \( \text{NS}(m, r, D, t) = \text{NON-SPECIAL} \) then
  if \( \text{NS}(m, r+1, D, t) = \text{NON-SPECIAL} \) then
    \( \mathcal{G} \leftarrow \mathcal{G} \cup \{D\} \); 
  end if
end if
end for each

return \( \mathcal{G} \);

Observe that the above algorithm is sufficient to check whether all diagrams in \( D \) gives non-special systems for a fixed multiplicity. However, running it on the set \( D \) (of cardinality 17493) from Example 9 would consume too much time. Therefore we will reduce all diagrams from \( D \) several times (this should decrease the number of diagrams) and check whether they are non-special. If yes, we are done due to [Dum 09a, Theorem 27]. If no, we must deal with diagrams that reduces to special ones. This will be explained in more details after presenting the algorithm.

Algorithm \text{ch}

\textbf{Input:} \( m \geq 2 \), a set of diagrams \( D \), 
a number \( u \geq 0 \) of reductions, 
a number \( v \geq 0 \) of reductions performed on reversed diagrams.

\textbf{Output:} \( \text{OK or NOT DECIDED} \), 
\( \text{OK} \) implies that for every \( D \in D \), \( r \geq 1 \), \( \mathcal{L}(D; m^r) \) is non-special.

if \( u > 0 \) then
  \( \mathcal{R} \leftarrow \text{RED}(m, u, D); \)
  \( \mathcal{R} \leftarrow \text{CHECK}(m, \mathcal{R}, 6); \)
  \( D \leftarrow \text{REDOUT}(m, u, D, \mathcal{R}); \)
end if

if \( v > 0 \) then
  \( D \leftarrow \text{rev}(D); \)
  \( \mathcal{R} \leftarrow \text{RED}(m, v, D); \)
  \( \mathcal{R} \leftarrow \text{CHECK}(m, \mathcal{R}, 6); \)
  \( D \leftarrow \text{REDOUT}(m, v, D, \mathcal{R}); \)
end if

\( \mathcal{R} \leftarrow \text{CHECK}(m, D, 16); \)

if \( \mathcal{R} = D \) then return \text{OK} else return \text{NOT DECIDED};

\textbf{Example 18.} Let us deal with \text{ch}(3, D, 8, 0) for \( D \) from Example 15

\[ D = \{ \text{diag}([8]^{\times 8}), \text{diag}([8]^{\times 9}), \text{diag}([8]^{\times 10}), \text{diag}([8]^{\times 8}, 5, 1), \]
\[ \text{diag}([8]^{\times 8}, 6, 2), \text{diag}([8]^{\times 8}, 7, 3), \text{diag}([8]^{\times 8}, 7, 5) \}. \]

Each diagram must be 3-reduced 8 times.

| \( D \)       | \( \text{red}^8(D) \)     |
|----------------|---------------------------|
| \( \text{diag}([8]^{\times 8}) \) | \( \text{diag}([8]^{\times 8}) \) |
| \( \text{diag}([8]^{\times 9}) \) | \( \text{diag}([8]^{\times 6}, 2) \) |
| \( \text{diag}([8]^{\times 10}) \) | \( \text{diag}([8]^{\times 8}, 6, 2) \) |
| \( \text{diag}([8]^{\times 8}, 5, 1) \) | \( \text{diag}([8]^{\times 8}, 7, 5, 2) \) |
| \( \text{diag}([8]^{\times 8}, 6, 2) \) | \( \text{diag}([8]^{\times 8}, 8, 6, 2) \) |
| \( \text{diag}([8]^{\times 8}, 7, 3) \) | \( \text{diag}([8]^{\times 8}, 8, 7, 3) \) |
| \( \text{diag}([8]^{\times 8}, 7, 5) \) | \( \text{diag}([8]^{\times 8}, 8, 7, 5) \) |

We end up with the set \( \mathcal{R} \) containing 6 diagrams (reducing decreased the number of cases). We can check that \( \mathcal{R} = \text{CHECK}(3, \mathcal{R}, 6) \), so we are done and the result is \text{OK}. An additional advantage lies in the size of matrices, since each \( m \)-reduction decreases the number of rows and columns by \( \binom{m+1}{2} \).
Example 19. The set $\mathcal{D}$ from Example 18 contains 17493 diagrams. We will run \text{CH}(5, \mathcal{D}, 3, 0). So we must 5-reduce every diagram in $\mathcal{D}$ three times, which gives the set $\mathcal{R}$ of cardinality 6234. All of these diagrams appeared to be non-special, so

$$\mathcal{R} = \text{CHECK}(5, \mathcal{R}, 6)$$

and we are done.

Example 20. We will present the number of diagrams involved in computing \text{CH}(6, \mathcal{D}, 14, 13) for

$$\mathcal{D} = \text{SETBIGNB}(6, 51, 8).$$

The set $\mathcal{D}$ contains 5472 diagrams. 4617 of them can be 6-reduced 14 times and we obtain the set $\mathcal{R}$ of 2991 diagrams. By checking speciality we obtain that 2832 diagrams from $\mathcal{R}$ are non-special, while the rest is probably special. So in $\mathcal{D}$ we have 855 not-reducible diagrams together with 250 that reduces to special ones. Now we reverse 1105 diagrams, reduce them (all are reducible) 13 times to obtain the set with 562 diagrams. Again not all of them are non-special, we are left with 46 diagrams that reduces to 24 special ones.

In the next algorithm we deal with systems $\mathcal{L}_n(a, b; m^{x^r})$ for fixed $m$, $n$, $a$ and $b$. Our aim is to identify those $r$, for which the system is special.

**Algorithm** \text{FINALNBA}

**Input:** $m \geq 2$, $n \geq 0$, $a, b \geq 0$.

**Output:** the set $\mathcal{L} \subset \mathbb{N}$ such that

- if $r \notin \mathcal{L}$ then $\mathcal{L}(a, b; m^{x^r})$ is non-special.

1. \begin{align*}
\mathcal{L} & \leftarrow \emptyset; \\
\text{if } n = 0 \text{ then } D & \leftarrow \text{diag}([a + 1]^{x(b + 1)}); \\
\text{if } n \geq 2 \text{ then } D & \leftarrow \text{diag}([1]^{x_1}, \ldots, [b]^{x_n}, [b + 1]^{x(a + 1)});
\end{align*}

2. $r \leftarrow \left\lceil \frac{\#D}{n + 1} \right\rceil$;

3. repeat
   1. $A \leftarrow \text{NS}(m, r, D, 16)$;
   2. if $A = \text{NOT DECIDED}$ then $\mathcal{L} \leftarrow \mathcal{L} \cup \{r\}$;
   3. $r \leftarrow r - 1$;

4. until $A = \text{NON-SPECIAL}$;

5. $r \leftarrow \left\lceil \frac{\#D}{n + 1} \right\rceil + 1$;

6. repeat
   1. $A \leftarrow \text{NS}(m, r, D, 16)$;
   2. if $A = \text{NOT DECIDED}$ then $\mathcal{L} \leftarrow \mathcal{L} \cup \{r\}$;
   3. $r \leftarrow r + 1$;

7. until $A = \text{NON-SPECIAL}$;

8. return $\mathcal{L}$;

Example 21. Let us compute \text{FINALNBA}(3, 0, 5, 4). We have $D = \text{diag}([6]^{x^5})$, so $\#D = 30$ and, at the beginning, $r = 5$. In the first step we have

$$\text{NS}(3, 5, D, 16) = \text{NOT DECIDED},$$

since in fact $\mathcal{L}_0(5, 4; 3^{x^5})$ is special. So we take $\mathcal{L} = \{5\}$ and compute

$$\text{NS}(3, 6, D, 16) = \text{NON-SPECIAL}.$$

Since also

$$\text{NS}(3, 4, D, 16) = \text{NON-SPECIAL},$$

we finish with $\mathcal{L} = \{5\}$.

The last group of algorithms checks whether a given system is $-1$-special, see \cite{Dum09a, Definition 3}. We begin with auxiliary algorithms \textsc{take-line} and \textsc{cremona} (see \cite{Dum09c, Theorem 3}). We put

- \textsc{take-line}(\mathcal{L}(d; m_1, \ldots, m_r)) = \mathcal{L}(d - 1; m_1 - 1, m_2 - 1, m_3, \ldots, m_r),
- \textsc{cremona}(\mathcal{L}(d; m_1, \ldots, m_r)) = \mathcal{L}(d + k; m_1 + k, m_2 + k, m_3 + k, m_4, \ldots, m_r)
for \( k = d - m_1 - m_2 - m_3 \). We will also use \( \text{sort}(\mathcal{L}(d; m_1, \ldots, m_r)) \) to sort multiplicities in non-increasing order.

**Algorithm** SPEC

**Input:** \( m \geq 2, n \geq 0, a, b \geq 0, r \geq 1 \).

**Output:** 1-special if \( \mathcal{L}_n(a, b; m^r) \) is 1-special, ERROR otherwise.

**Remark:** for a system of curves \( L \) we define \( L_d \) to be the degree, \( L_{m_j} \) to be the \( j \)-th multiplicity.

\[
d \leftarrow (n+1)b + a;
\]

\[
m_0 \leftarrow nb + a;
\]

\[
e \leftarrow \text{edim} \mathcal{L}_n(a, b; m^r);
\]

**for** \( t = 0, \ldots, b \) **do**

\[
L \leftarrow \mathcal{L}(d-t; m_0, m^r, (b-t)^{(n+1)});
\]

**repeat**

if \( L_d - L_{m_1} - L_{m_2} < 0 \) then

\( L \leftarrow \text{take-line}(L) \);

**else**

if \( L_d - L_{m_1} - L_{m_2} - L_{m_3} < 0 \) then

\( L \leftarrow \text{cremona}(L) \);

**end if**

**end if**

\( L \leftarrow \text{sort}(L) \);

**until** \( L_d - L_{m_1} - L_{m_2} - L_{m_3} \geq 0 \);

**if** \( \text{edim} L > e \) **then return** 1-special;

**end for**

**return** ERROR;

**Example 22**. Let us compute SPEC(6, 8, 2, 8, 15), so we must consider \( L_8(2, 8; 6^{15}) \). We begin with planar system \( L = \mathcal{L}(74; 66, 6^{15}, 8^{15}) \) with edim \( L = -1 \). For \( t = 0 \) take the system \( \mathcal{L}(74; 66, 6^{15}, 8^{15}) \), sort multiplicities to obtain \( \mathcal{L}(42; 34, 8, 6^{15}) \). Then use CREMONA four times to obtain \( \mathcal{L}(36; 28, 6^{15}, 2) \). By the sequence of CREMONA we transform our system into \( \mathcal{L}(8; 2^{11}) \). Since edim \( L(8; 2^{11}) = -1 \), we pass to the case \( t = 1 \). Now we begin with \( \mathcal{L}(73; 66, 6^{15}, 7^{15}) \) and, by CREMONA, transform to \( \mathcal{L}(9; 6, 6, 2, 1^{15}) \). Now we use TAKE-LINE several times to produce \( \mathcal{L}(6; 3, 3, 2, 1^{15}) \). By CREMONA we finish with \( L(4; 1^{15}) \) of negative expected dimension. For \( t = 2 \) we begin with \( \mathcal{L}(72; 66, 6^{24}) \) and transform it to \( \mathcal{L}(6; 6, 6) \). Then, by TAKE-LINE, we obtain \( L(0; 0) \) of non-negative expected dimension, so the answer is 1-special.

4. IMPLEMENTATION AND RESULTS

All the presented algorithms have been implemented in Free Pascal and can be downloaded from [Dum 09b]. They are divided into two kinds, depending on method of working. The first kind simply works on given data, the second one prepares batch files with the list of instructions. For example, the implementation of RED (“red.pas”), of the first kind, performs sequence reductions on given diagrams (loaded from the specified file). The algorithm SETPB (of the second kind) prepares the batch file with the following instructions (for SETPB(3, 9))

```
tails 3 8,9,10,x,x rt
basediag 1 1 7 0 bt
gluedias inempty bt rt diag
```

The above instructions run tails, which produces all admissible diag(8, 9, 10)-tails and stores them in the file rt; basediag, which prepares diag(1, 2, 3, 4, 5, 6, 7) and stores it in bt; gluedias, which glues diagrams from inempty (by default, it contains only the empty diagram), bt and rt.

All algorithms with names beginning with SET are of the second kind, together with CH and FINALNBA. The others are of the first kind.

All algorithms produces log files, where the necessary information is stored. For example, the part of log file for multiplicity 2 contains:
setbign23 2 2
result: all systems $L_n(a,b)(2^r)$ are non-special
for $n>=2$, $a>=0$, $b>=5$, $r>=0$

ltails (all-h-D-admissible tails) 2 3
tails loaded:
1 tails loaded.
tails found:
3
4 entries used, 2 tails found.
job finished: 03:11:41:14
*************************************

ltails (all-h-D-admissible tails) 2 4
tails loaded:
3
2 tails loaded.
tails found:
3
4
1
2
11 entries used, 5 tails found.
job finished: 03:11:41:15
*************************************
basediag 4 0 2 0
base diagram:
4,4
job finished: 03:11:41:15
*************************************
tails (admissible tails) 2
diagram:
5,5,x
tails found:
4
3
2
1
5
17 entries used, 5 tails found.
job finished: 03:11:41:15
*************************************
gluediags (glue diagrams)
25 diagrams produced.
job finished: 03:11:41:15
*************************************
check
multiplicity: 2
diag(4,4,4) det <> 0 det <> 0
The information which set $D$ is considered, is stored in the preamble. It is then followed by names of programs together with additional detailed information. The shortlog files contain only preambles and names of programs, while infolog stores only preambles. The finitlog files contain informations on running spec. Each program informs when it has terminated (day:hour:min:second).

**References**

[Dum 09a] M. Dumnicki, *Special homogeneous linear systems on Hirzebruch surfaces*, [arXiv:0907.3818v1](https://arxiv.org/abs/0907.3818) (2009).

[Dum 09b] M. Dumnicki, [http://gamma.im.uj.edu.pl/dumnicki/interpol.htm](http://gamma.im.uj.edu.pl/dumnicki/interpol.htm), November, 2009.

[Dum 09c] *An algorithm to bound the regularity and nonemptiness of linear systems in $\mathbb{P}^n$*, J. Symb. Comp. 44, 1448–1462 (2009).