Quantum Group Symmetric Bargmann Fock Construction*

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Abstract

Usually in quantum mechanics the Heisenberg algebra is generated by operators of position and momentum. The algebra is then represented on an Hilbert space of square integrable functions. Alternatively one generates the Heisenberg algebra by the raising and lowering operators. It is then natural to represent it on the Bargmann Fock space of holomorphic functions. In the following I show that the Bargmann Fock construction can also be done in the quantum group symmetric case. This leads to a 'q- deformed quantum mechanics' in which the basic concepts, Hilbert space of states and unitarity of time evolution, are preserved.

1 Introduction

There are already several approaches to q- deformed algebras of raising and lowering operators in the literature [see e.g. 6,7,8,9,11]. Let us define a 'q- deformed' Heisenberg algebra generated by the operators \( a^\dagger i \) and \( a^i \) by imposing the following commutation relations which were shown to be preserved under the action of the quantum group \( SU_q(n) \) [4,6]:

\[
a^r a^s + c R^{rs}_{ji} a^j a^i = 0, \quad a^r a^\dagger_s + 1/c R^{rs}_{ji} a^\dagger_j a^i = \delta^r_s, \quad a^\dagger_r a^\dagger_s + c R^{ij}_{sr} a^\dagger_j a^\dagger_i = 0 \quad (1)
\]

The R- matrix reads [for the foundations see e.g. 1,2]:

\[
R = q \sum_i e^i_i \otimes e^i_i + \sum_{i \neq j} e^i_i \otimes e^j_j + (q - 1/q) \sum_{i > j} e^i_j \otimes e^j_i \quad (2)
\]

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with $c_j \in M_n(C)$ matrix units. In the bosonic case we have $c = -1/q$ and in the fermionic case $c = q$, with $q$ a real number. Other commutation relations are possible but the above relations are natural here in the sense that the resulting algebra deviates the least possible from the undeformed case [see 4]. Using the generalized twisting method introduced in [10] it is easy to see, that no further continuous parameters can be introduced.

Since the $a_i^\dagger$ and $a_i$ will be represented as adjoint operators we can recover hermitean position and momentum operators:

$$x^i := (2m\omega)^{-1/2}(a_i^\dagger + a_i) \quad , \quad p^i := i(2m\omega)^{1/2}(a_i^\dagger - a_i) \quad (3)$$

In the fermionic case one may consider a $q$-deformed system of $n$ spin $1/2$'s in a constant magnetic field in $z$-direction. With similar linear combinations one then gets the $x$- and $y$- components of the $i$th spin.

We define the ground state as usual $<0|0>=1$ and $a_i^\dagger |0>=0$ for $i = 1, ..., n$. The scalar product $<,>$ can then be shown to be still positive definite.

For the simplest Hamiltonian $H := \omega N$ with $\omega \in \mathbb{R}^+$, which is obviously scalar transforming and hermitean, the energy spectrum is:

$$E_p = \omega(1 + c^{-2} + c^{-4} + \ldots + c^{-2(p-1)}) = \frac{\omega c^{-2p} - 1}{c^{-2} - 1} \quad \text{with} \quad p = 0, 1, 2, ... \quad (4)$$

2 Bargmann Fock representation

The operators $a_i$ and $a_i^\dagger$ can be represented as differentiation and multiplication operators on a space of (deformed) holomorphic functions [4,5]:

$$\rho : a_i \to \partial_{\bar{\eta}}^i \quad , \quad \rho : a_i^\dagger \to \eta_i$$

Wave functions are the polynomials exclusively in $\bar{\eta}$'s, which shall also be denoted holomorphic functions. The bar operation is an antialgebra mapping i.e. we have for example $\partial_{\bar{\eta}}^i \bar{\eta}^j = \eta^j \partial_{\eta}^i$. The commutation relations among the $\eta^i, \bar{\eta}^i, \partial_{\eta}^i$, and $\partial_{\bar{\eta}}^i$ are not unique. I use a choice [4] that is natural in the above mentioned sense, and which is similar but not identical to the differential calculus of Wess and Zumino [3].

The evaluation of differentiation is defined as follows: All $\partial_{\bar{\eta}}^i$'s are to be commuted to the right, the $\partial_{\eta}^i$'s to the left. When they arrive, the corresponding terms are to be set equal zero. What remains is the value of the differentiation.

Let us define the '$c$-deformed' exponential function

$$e_c^{(\partial_{\eta} \partial_{\bar{\eta}})} := \sum_r \frac{(\partial_{\eta} \partial_{\bar{\eta}})^r}{[r]_c!} \quad \text{with} \quad [r]_c := \frac{c^{2r} - 1}{c^2 - 1}. \quad (5)$$

The scalar product $(,)$ of the wave functions $\phi$ and $\psi$ (which are polynomials in $\bar{\eta}$) is then given [4] by the following "integral":

$$(\phi, \psi) := (\bar{\phi} e_c^{(\partial_{\eta} \partial_{\bar{\eta}})} \psi) \text{evaluated at } \eta = 0 = \bar{\eta} \quad (6)$$
The evaluation procedure is as follows: At first the differentiations are to be evaluated. Then all terms containing \( \eta \)'s and \( \bar{\eta} \)'s are to be set equal zero. Thus the result is a number. The new integral, which can of course also be used in the undeformed theory, has the same evaluation procedure for the bosonic as for the fermionic case.

It can be shown, that the operators \( a_k \) and \( a_k^{\dagger} \) are adjoint in respect to \((,\) i.e.
\[
(\rho(a_k^{\dagger}) \phi, \psi) = (\phi, \rho(a_k) \psi)
\]
and that the wave function of the ground state, 1, is normalized. Thus the scalar product \((,)\) coincides with the bracket \(<,>\) of the Fock space. The Hilbert space of all wave functions is defined to be the set of power series in \( \bar{\eta} \)'s that are square integrable in respect to \((,)\).

Let us introduce a more familiar notation for the scalar product:
\[
\int d\bar{\eta}d\eta \bar{\phi}e_{c}^{(\partial_{\eta}^{i}\partial_{\bar{\eta}}^{i})} \psi := (\bar{\phi}e_{c}^{(\partial_{\eta}^{i}\partial_{\bar{\eta}}^{i})} \psi)_{\text{evaluated at } \eta=0=\bar{\eta}}
\]
(7)

Like in the undeformed case it is now possible to represent every operator \( P \) on the Fock space also as an integral kernel \( G_{P}(\bar{\eta}', \eta) \):
\[
\int d\bar{\eta}d\eta G_{P}(\bar{\eta}', \eta)e_{c}^{(\partial_{\eta}^{i}\partial_{\bar{\eta}}^{i})} \psi(\bar{\eta}) = P\psi(\bar{\eta}')
\]
(8)

With natural commutation relations between different copies of the function space (e.g. primed and unprimed) the general rule for getting the integral kernel of an arbitrary normal ordered operator \( P(a^{\dagger}, a) \) turns out to be as follows: Starting with the integral kernel of the identity operator \( e_{c}^{(\partial_{\eta}^{i}\partial_{\bar{\eta}}^{i})} \) one writes for each \( a^{\dagger} \) a \( \bar{\eta}^{i} \) to the left of \( e_{c}^{(\partial_{\eta}^{i}\partial_{\bar{\eta}}^{i})} \) and for all \( a^{i} \) one writes \( \eta^{i} \) to the right of the \( e_{c}^{(\partial_{\eta}^{i}\partial_{\bar{\eta}}^{i})} \).

The green function i.e. the integral kernel of the time evolution operator \( U = e^{-i(t_{f}-t_{i})H} \) for the simple Hamiltonian of section 1 is found to be
\[
G_{U} = \sum_{r=0}^{\infty} \frac{(\bar{\eta}^{i}\eta^{i})^{r}}{[r]!} e^{-i\omega(t_{f}-t_{i})[r]_{1/c}}
\]
(9)

3 Introduction of driving forces

Let us introduce a 'classical' but \( g \)-deformed driving force \( f(t)g^{i} \) where \( f(t) \) denotes a complex-valued function describing the time dependence of the driving force and \( g \) shall be a constant unit vector: \( \bar{g}_{i}g^{i} = 1 \). The Hamiltonian now reads:
\[
H = \omega \bar{\eta}_{i}\partial^{i}_{\eta} - \bar{f}(t)\bar{g}_{i}\partial^{i}_{\bar{\eta}} - f(t)\bar{\eta}_{i}g^{i}
\]
(10)

Considering quantum mechanics as quantum field theory with zero space dimensions, our driving forces are \( g \)-deformed Schwinger sources. The algebra of the \( \bar{g} \)'s and \( g \)'s is now noncommutative, even in the bosonic case [5]. Nevertheless, results that do only depend on the length of the force vector are still ordinary complex numbers for
that the usual probability interpretation applies. E.g. the vacuum-vacuum transition amplitude for the switching on of a constant driving force can be calculated to be

\[ <0|0(t_f, t_i) = 1 + \frac{\bar{f}f}{\omega^2} \sum_{z=2}^{\infty} \frac{(-i\omega)^z}{z!} (t_f - t_i)^z \]

\[ + \frac{(\bar{f}f)^2}{\omega^4} \sum_{z=4}^{\infty} \left\{ z - 3 + \sum_{r=0}^{z-4} (z - 3 - r)([2]_{1/c})^{r+1} \right\} \frac{(-i\omega)^z}{z!} (t_f - t_i)^z \]

\[ + \ldots \]  

One recognizes that deviations from the undeformed case do not occur before the second order in \((\bar{f}f)\). This is plausible because the energy level of the first excitation is not deformed, while the levels (see Eq.4) deviate the more, the higher the excitation is. If it is possible to extend this formalism to a quantum field theory it can be expected that the ultraviolet behaviour is strongly influenced by the deformation parameter.

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4 References

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