DEMAND-INDEPENDENT TOLLS

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Abstract. Wardrop equilibria in nonatomic congestion games are in general inefficient as they do not induce an optimal flow that minimizes the total travel time. Tolls can be used to induce an optimum flow in equilibrium. The classical approach to find such tolls is marginal cost pricing. This requires the exact knowledge of the demand on the network. In this paper, we investigate under which conditions tolls exist that are independent of the demand in the network. We call them demand-independent optimum tolls. We show that such tolls exist when the cost functions are shifted monomials. Moreover non-negative demand-independent optimum tolls exist when the network is a directed acyclic multi-graph. Finally, we show that any network with a single origin-destination pair admits demand-independent optimum tolls that, although not necessarily non-negative, satisfy a budget constraint.

1. Introduction

The impact of selfish behavior on the efficiency of traffic networks is a longstanding topic in the algorithmic game theory and operations research literature. Already more than half a century ago, Wardrop (1952) stipulated a main principle of a traffic equilibrium that—in light of the omnipresence of modern route guidance systems—is as relevant as ever: “The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.” This principle can be formalized by modeling traffic as a flow in a directed network where edges correspond to road segments and vertices correspond to crossings. Each edge is endowed with a cost function that maps the total amount of traffic on it to a congestion cost that each flow particle traversing the edge has to pay. Further, we are given a set of commodities, each specified by an origin, a destination, and a flow demand. In this setting, a Wardrop equilibrium is a multi-commodity flow such that for each commodity the total cost of any used path is not larger than the total cost of any other path linking the commodity’s origin and destination.

While Wardrop equilibria are guaranteed to exist under mild assumptions on the cost functions (Beckmann et al., 1956), it is well known that they are inefficient in the sense that they do not minimize the overall travel time of all commodities (Pigou, 1920). A popular mechanism to decrease the inefficiency of selfish routing are congestion tolls. A toll is a payment that the system designer defines for each edge of the graph and that has to be paid by each flow particle traversing the edge. By carefully choosing the edges’ tolls, the system designer can steer the Wardrop
equilibrium in a favorable direction. A classic approach first due to Pigou (1920) is marginal cost pricing where the toll of each edge is equal to difference between the marginal social cost and the marginal private cost of the system optimum flow on that edge. Marginal cost pricing induces the system optimal flow—the one that minimizes the overall travel time—as a Wardrop equilibrium (Beckmann et al., 1956). Congestion pricing is not only an interesting theoretical issue that links system optimal flows and traffic equilibria, but also a highly relevant tool in practice, as various cities of the world, including Stockholm, Singapore, Bergen, and London, implement congestion pricing schemes to mitigate congestion (Gomez-Ibanez and Small, 1994; Small and Gómez-Ibáñez, 1997).

1.1. The problem. Marginal cost pricing is an elegant way to induce the system optimum flow as a Wardrop equilibrium, but the concept crucially relies on the exact knowledge of the travel demand. As an example consider the Pigou network in Fig. 1a for an arbitrary flow demand of $\mu > 0$ going from $o$ to $d$. The optimal flow only uses the lower edge with cost function $c(x) = x$ as long as $\mu \leq 1/2$. For demands $\mu > 1/2$, a flow of $1/2$ is sent along the lower edge and the remaining flow of $\mu - 1/2$ is sent along the upper edge. Using marginal cost pricing the toll of the lower edge is $\min\{\mu, 1/2\}$ and no toll is to be payed for the upper edge, see Fig. 1b. This toll scheme is problematic as it depends on the flow demand $\mu$ in the network. In particular, when the demand is estimated incorrectly, the resulting tolls will be sub-optimal. Assume the network designer expects a flow of $\mu = 1/4$ and, thus, sets a (marginal cost) toll of $\tau_2 = 1/4$ on the lower link. When the actual total flow demand is higher than expected and equal to $\mu = 1$, a fraction $1/4$ of
the flow uses the upper edge and 3/4 of the flow uses the lower edge resulting in a cost of $1/4 + 9/16$. However, it is optimal to induce an equal split between the edge which can be achieved by a toll of $1/2$ on the lower edge. Quite strikingly, the toll of $\tau_2 = 1/2$ is optimal for all possible demand values as it always induces the optimal flow: For demands less than $1/2$, a toll of $1/2$ on the lower edge does not hinder any flow particle from using the lower edge, which is optimal. On the other hand, for any demand larger $1/2$, the toll of $1/2$ on the lower edge forces the flow on the lower edge to not exceed $1/2$, which is optimal as well.

The situation is even more severe for the Braess network in Fig. 2. When the system designer expects a traffic demand of 1 going from $o$ to $d$, marginal cost pricing fixes a toll of 1 on both the upper left and the lower right edge (both with cost function $c(x) = x$). When the demand is lower than expected, say, $\mu = 1/2$, under marginal cost pricing, the flow is split equally between the lower and the upper path leading to a total cost of $5/4$. The optimal flow with flow value 1/2, however, only uses the zig-zag-path $o \rightarrow u \rightarrow v \rightarrow d$ with cost 1. It is interesting to note that this flow is actually equal to the Wardrop equilibrium without any tolls. To conclude this example, marginal cost pricing may actually increase the total cost of the Wardrop equilibrium when the travel demand is estimated incorrectly. We note that also in the Braess graph, there is a distinct toll vector that enforces the optimum flow as a Wardrop equilibrium for any demand. We will see that by setting a toll of 1 on the central edge from $u$ to $v$, the Wardrop equilibrium for any flow demand is equal to the respective optimum flow.

We conclude that for both the Pigou network and the Braess network, marginal cost pricing is not robust with respect to changes in the travel demand since wrong estimates about the travel demand lead to sub-optimal tolls. Since such changes may occur frequently in road networks (e.g., due to sudden weather changes, accidents, or other unforeseen events), marginal cost pricing does not use the full potential of congestion pricing and may even be harmful for the traffic. On the other hand, for both networks, there exist tolls that enforce the optimum flow as an equilibrium for any flow demand. In this paper, we systematically study conditions for such demand-independent tolls to exist.

1.2. Our results. In this paper we aim at finding tolls that induce the optimum flow as an equilibrium. We want these tolls to be independent of the demand in the network. When they exist, we call them demand-independent optimum tolls (DIOTs). First we prove that, when the cost functions are the sum of a monomial and a constant, DIOTs exist, regardless of the topology of the network and on the number of origin-destination (O/D) pairs. This class of cost functions includes affine costs, and functions similar to the monomials of degree 4 used by the U.S. Bureau of Public Roads (1964).

In general DIOTs are not non-negative. We provide an example of a network with shifted monomial costs where non-negative DIOTs do not exist. Besides conditions on the costs, conditions on the network are needed to guarantee the existence of non-negative DIOTs, like the Pigou network and the Braess network. In particular they exist for directed acyclic multi-graphs (DAMGs). This condition on the network is sufficient, but not necessary for the existence of non-negative DIOTs.

Under a weaker condition than DAMG, we prove the existence of DIOTs that follow a budget constraint of non-negativity of the total amount of tolls. This condition is satisfied by networks with a single O/D pair.
1.3. Related work. Marginal cost pricing as a means to reduce the inefficiency of selfish resource allocation was first proposed by Pigou (1920) and subsequently discussed by Knight (1924). Wardrop (1952) introduced the notion of a traffic equilibrium where each flow particle only uses shortest paths. Beckmann et al. (1956) showed that marginal cost pricing always induces the system optimal flow as a Wardrop equilibrium. The set of feasible tolls that induce optimal flows was explored by Bergendorff et al. (1997); Hearn and Ramana (1998); Larsson and Patriksson (1999). They showed that the set of optimal tolls can be described by a set of linear equations and inequalities.

This characterization led to various developments regarding the optimization of secondary objectives of the edge tolls, such as the minimization of the tolls collected from the users (Bai et al., 2004; Dial, 1999, 2000), or the minimization of the number of edges that have positive tolls (Bai and Rubin, 2009; Bai et al., 2010). A problem closely related to the latter is to compute tolls for a given subset of edges with the objective to minimize the total travel time of the resulting equilibrium. Hoefer et al. (2008) showed that this problem is NP-hard for general networks, and gave an efficient algorithm for parallel edges graphs with affine cost functions. Harks et al. (2015) generalized their result to arbitrary cost functions satisfying a technical condition. Bonifaci et al. (2011) studied generalizations of this problem with further restrictions on the set of feasible edge tolls.

For heterogenous flow particles that trade off money and time differently, marginal cost pricing cannot be applied to find tolls that induce the system optimum flow. In this setting, Cole et al. (2003) showed the existence of a set of tolls enforcing the system optimal flow, when there is a single commodity in the network. Similarly, Yang and Huang (2004) studied how it is possible to design toll structure when there are users with different toll sensitivity. Fleischer (2005) showed that in single source series-parallel networks the tolls have to be linear in the latency of the maximum latency path. Karakostas and Kolliopoulos (2006) and Fleischer et al. (2004) independently generalized this result to arbitrary networks. Han et al. (2008) extended the previous results to different classes of cost functions.

Most of the literature assumes that the charged tolls cause no disutility to the network users. For the case where tolls contribute to the cost, Cole et al. (2006) showed that marginal cost tolls do not improve the equilibrium flow for a large class of instances, including all instances with affine costs. They further showed that for these networks it NP-hard to approximate the minimal total cost that can be achieved as a Wardrop equilibrium with tolls. Karakostas and Kolliopoulos (2005) proved that the total disutility due to taxation is bounded with respect to the social optimum for large classes of latency functions. Moreover, they showed that, if both the tolls and the latency are part of the social cost, then for some latency functions the coordination ratio improves when taxation is used. For networks of parallel edges, Christodoulou et al. (2014) studied a generalization of edge tolls where cost functions are allowed to increase in an arbitrary way. They showed that for affine cost functions, the price of anarchy is strictly better than in the original network, even when the demand is not known.

Brown and Marden (2016, 2017) studied how marginal tolls can create perverse incentives when users have different sensitivity to the tolls and how it possible to circumvent this problem.
Caragiannis et al. (2006) studied the optimal toll problem for atomic congestion games. They proved that in the atomic case the optimal system performance cannot be achieved even in very simple networks. On the positive side they shown that there is a way to assign tolls to edges such that the induced social cost is within a factor of 2 to the optimal social cost. Singh (2008) observed that marginal tolls weakly enforce optimal flows. Fotakis and Spirakis (2008) showed that in series-parallel networks with increasing cost functions the optimal social cost can be induced with tolls. Meir and Parkes (2016) discussed how in atomic congestion games with marginal tolls multiple equilibria are near-optimal when there is a large number of players.

In Sandholm (2002, 2007) the problem is studied from a mechanism design perspective where the social planner has no information over the preferences of the users and has limited ability to observer the users’ behavior.

2. Model and preliminaries

In this section, we present some notation and basic definitions that are used in the sequel. Specifically, in Section 2.1, we describe nonatomic congestion games, whereas, in Section 2.2, we give a precise definition of the price of anarchy and of network tolls.

2.1. Network model. We consider a finite directed multi-graph \( G = (V, E) \) with vertex set \( V \) and edge set \( E \). We call \((v \to v')\) the set of all edges \( e \) whose tail is \( v \) and whose head is \( v' \). We assume that there is a finite set of origin-destination (O/D) pairs \( i \in I \) each with an individual traffic demand \( \mu^i \geq 0 \) that has to be routed from an origin \( o^i \in V \) to a destination \( d^i \in V \) via \( G \). Denote \( \mu = (\mu^i)_{i \in I} \). We call \( P^i \) the set of (simple) paths joining \( o^i \) to \( d^i \), where each path \( p \in P^i \) is a finite sequence of edges such that the head of each edge meets the tail of the subsequent edge. For as long as all pairs \((o^i, d^i)\) are different, the sets \( P^i \) are disjoint. Call \( P := \bigcup_{i \in I} P^i \) the union of all such paths.

Each path \( p \) is traversed by a flow \( f_p \in \mathbb{R}_+ \). Call \( f = (f_p)_{p \in P} \) the vector of flows in the network. The set of feasible flows is defined as

\[
F = \left\{ f \in \mathbb{R}^P : \sum_{p \in P} f_p = \mu^i \text{ for all } i \in I \right\}. \tag{2.1}
\]

In turn, a routing flow \( f \in F \) induces a load on each edge \( e \in E \) as

\[
x_e = \sum_{p \ni e} f_p. \tag{2.2}
\]

We call \( x = (x_e)_{e \in E} \) the corresponding load profile on the network.

For each \( e \in E \) consider a nondecreasing, continuous cost function \( c_e : [0, \infty) \to [0, \infty) \). Denote \( c = (c_e)_{e \in E} \). If \( x \) is the load profile induced by a feasible routing flow \( f \), then the incurred delay on edge \( e \in E \) is given by \( c_e(x_e) \); hence, with a slight abuse of notation, the associated cost of path \( p \in P \) is given by the expression

\[
c_p(f) \equiv \sum_{e \in P} c_e(x_e). \tag{2.3}
\]

We call the tuple \( \Gamma = (G, I, \mu, c) \) a (nonatomic) routing game.
2.2. Equilibrium, optimality, and the price of anarchy. A routing flow \( f^* \) is a Wardrop equilibrium (WE) of \( \Gamma \) if, for all \( i \in \mathcal{I} \), we have:
\[
c_p(f^*) \leq c_p(f^*) \text{ for all } p, p' \in \mathcal{P}^i \text{ such that } f_p^* > 0.
\]
(2.4)
This concept was introduced by Wardrop (1952).

A socially optimum (SO) flow is defined as a solution to the total cost minimization problem:
\[
\text{minimize } L(f) = \sum_{p \in \mathcal{P}} f_pe_p(f),
\]
subject to \( f \in \mathcal{F} \).

We write \( \text{Eq}(\Gamma) = L(f^*) \) and \( \text{Opt}(\Gamma) = \min_{f \in \mathcal{F}} L(f) \).
(2.5)

We write
\[
\text{PoA}(\Gamma) = \frac{\text{Eq}(\Gamma)}{\text{Opt}(\Gamma)}.
\]
(2.6)

Obviously, \( \text{PoA}(\Gamma) \geq 1 \).

2.3. Tolls. We want to explore the possibility of imposing tolls on the edges of the network in such a way that the equilibrium flow of the game with tolls produces a flow that is a solution of the original minimization problem (SO). In other words, we want to see whether it is possible to achieve an optimum flow as an equilibrium of a modified game.

We call \( \tau = (\tau_e)_{e \in \mathcal{E}} \) the toll vector. We call \( c^*_e \) the cost of edge \( e \) under the toll \( \tau \):
\[
c^*_e(x_e) := c_e(x_e) + \tau_e.
\]
(2.7)
Similarly
\[
c^*_p(f) := \sum_{e \in p} c^*_e(x_e).
\]
(2.8)
Define \( \Gamma^\tau := (\mathcal{G}, \mathcal{I}, \mu, c^\tau) \). A toll \( \tau \) such that the equilibrium \( f^\tau \) of the game \( \Gamma^\tau \) is an optimum of the game \( \Gamma \) is called a demand-independent optimum toll (DIOT) for \( \Gamma \).

3. Monomial costs

In this section, we assume that cost functions are shifted monomials of degree \( d \), i.e., there is an integer \( d \in \mathbb{N} \) and, for all edges \( e \in \mathcal{E} \), constants \( b_e, a_e \geq 0 \) such that
\[
c_e(x_e) = a_ex_e^d + b_e.
\]
(3.1)

The following theorem establishes a sufficient condition for the existence of DIOTs in games with shifted monomial costs.

**Theorem 3.1.** Consider the game \( \Gamma = (\mathcal{G}, \mathcal{I}, \mu, c) \), where for each \( e \in \mathcal{E} \) the cost \( c_e \) is a shifted monomial as in (3.1). If the toll vector \( \tau \) is such that for all \( i \in \mathcal{I} \) and all \( p, p' \in \mathcal{P}^i \) we have
\[
\sum_{e \in p}(d+1)\tau_e + db_e = \sum_{e \in p'}(d+1)\tau_e + db_e,
\]
(3.2)

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\(^1\)That all Wardrop equilibria have the same total cost is a standard result due to Beckmann et al. (1956).
then, for every non-negative traffic demand \( \mu \), the vector \( \tau \) is a DIOT for \( \Gamma \).

**Proof.** Fix a demand \( \mu \in \mathbb{R}^I \). Let \( \tilde{f} \) be an optimum flow in \( \Gamma \). The local optimality conditions imply that there exists \( \alpha(p, p') \geq 0 \) such that for all \( i \in I \) and all \( p, p' \in \mathcal{P} \) with \( \tilde{f}_{p} > 0 \), we have

\[
\alpha(p, p') + \sum_{e \in p} c_e(\tilde{x}_e) + c'_e(\tilde{x}_e)\tilde{x}_e = \sum_{e \in p'} c_e(\tilde{x}_e) + c'_e(\tilde{x}_e)\tilde{x}_e
\]

which gives

\[
\alpha(p, p') + \sum_{e \in p} ((d + 1)a_e\tilde{x}^d_e + b_e) = \sum_{e \in p'} ((d + 1)a_e\tilde{x}^d_e + b_e).
\] (3.3)

Now we want to show that, if (3.2) is satisfied, then \( \tilde{f} \) is a Wardop equilibrium of \( \Gamma^r \). Consider \( p, p' \in \mathcal{P} \) such that \( \tilde{f}_{p} > 0 \). We have

\[
\sum_{e \in p}(c_e(\tilde{x}_e) + \tau_e) = \sum_{e \in p}a_e\tilde{x}^d_e + b_e + \tau_e
\]

\[
= \sum_{e \in p}((d + 1)a_e\tilde{x}^d_e + b_e) - \sum_{e \in p} d a_e\tilde{x}^d_e - \tau_e
\]

\[
= \sum_{e \in p}((d + 1)a_e\tilde{x}^d_e + b_e) - \sum_{e \in p} d a_e\tilde{x}^d_e - \tau_e - \alpha(p, p'),
\]

\[
= \sum_{e \in p} c_e(\tilde{x}_e) + \tau_e + \sum_{e \in p'} ((d + 1)a_e\tilde{x}^d_e - \tau_e) - \sum_{e \in p} (d a_e\tilde{x}^d_e - \tau_e) - \alpha(p, p'),
\]

where the third equality comes from (3.3). By assumption, the toll vector \( \tau \) satisfies equation (3.2) which implies

\[
\sum_{e \in p} \tau_e - \sum_{e \in p'} \tau_e = -\frac{d}{d + 1} \left( \sum_{e \in p} b_e - \sum_{e \in p'} b_e \right).
\]

Therefore

\[
\sum_{e \in p} c_e(\tilde{x}_e) + \tau_e = \sum_{e \in p'} c_e(\tilde{x}_e) + \tau_e
\]

\[
+ \sum_{e \in p'} \left( d a_e\tilde{x}_e + \frac{d}{d + 1} b_e \right) - \sum_{e \in p} \left( d a_e\tilde{x}_e + \frac{d}{d + 1} b_e \right) - \alpha(p, p')
\]

\[
= \sum_{e \in p'} \left( c_e(\tilde{x}_e) + \tau_e \right) + \frac{d}{d + 1} \alpha(p, p') - \alpha(p, p')
\]

\[
= \sum_{e \in p'} \left( c_e(\tilde{x}_e) + \tau_e \right) - \frac{1}{d + 1} \alpha(p, p'),
\]

where the last equality stems from (3.3). The result follows from the fact that \( \alpha(p, p') \geq 0 \). \( \square \)

**Remark 3.1.** A solution to equation (3.2) always exists. For instance

\[
\hat{\tau} = (\hat{\tau}_e)_{e \in E}, \text{ with } \hat{\tau}_e = -\frac{d}{d + 1} b_e,
\] (3.4)
satisfies (3.2). We call \( \hat{\tau} \) the trivial DIOT.

**Proposition 3.2.** Consider a game \( \Gamma = (G, I, \mu, c) \), where for each \( e \in E \) the cost \( c_e \) is monomial as in (3.1). If \( \tau \) is a DIOT for \( \Gamma \) for any demand \( \mu \), then (3.2) holds for all \( i \in I \) and \( \tau_c \) satisfies (4.1). We first prove that the toll vector \( \mu^* \) of the corresponding social optimum \( \tilde{\tau} \).

**Proof.** Assume that \( \hat{\mu}^*, \hat{\mu}' > 0 \) for some demand \( \mu \). If \( \tau \) is a DIOT for \( \Gamma \), then

\[
\sum_{e \in p}(c_e(x) + \tau_e) = \sum_{e \in p'}(c_e(x) + \tau_e) \text{ which implies}
\]

\[
\sum_{e \in p}(a_e x^d + b_e + \tau_e) = \sum_{e \in p'}(a_e x^d + b_e + \tau_e). \tag{3.5}
\]

By the local optimality conditions, we further have

\[
\sum_{e \in p} c_e(x) + \sum_{e \in p} c_e(x) \tilde{x}_e = \sum_{e \in p'} c_e(x) + \sum_{e \in p'} c_e(x) \tilde{x}_e, \tag{3.6}
\]

which implies

\[
\sum_{e \in p} (d + 1) a_e \tilde{x}_e + b_e = \sum_{e \in p'} (d + 1) a_e \tilde{x}_e + b_e. \tag{3.7}
\]

Subtracting (3.6) from \( (d + 1) \) times (3.5), we obtain

\[
\sum_{e \in p} (db_e + (d + 1) \tau_e) = \sum_{e \in p'} (db_e + (d + 1) \tau_e). \tag{4.1}
\]

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4. **Nonnegative Tolls**

In general, the trivial DIOT toll \( \hat{\tau} \) is negative. Our next result shows that, for games played on a directed acyclic multi-graph (DAMG), a non-negative DIOT can always be found.

Given a DAMG there exists a topological sort, namely a linear ordering \( < \) of its vertices such that, if \( v < v' \), there is no path from \( v' \) to \( v \) in the DAMG. Notice that, in general the topological sort of a DAMG is not unique. Let \(|V| = n \) and call \( v^< = (v_1, \ldots, v_n) \) the vector of ordered vertices. For each edge \( e \in (v_i \rightarrow v_j) \), define

\[
\delta_e := j - i. \tag{4.2}
\]

Let

\[
\xi := \min_{e \in E} \frac{\tau_e}{\delta_e} \quad \text{and} \quad \chi := -\xi, \tag{4.3}
\]

where \( \xi = \max \{-\xi, 0\} \) is the negative part of \( \xi \). Define now

\[
\tau_e = \hat{\tau}_e + \delta_e \chi. \tag{4.4}
\]

**Theorem 4.1.** Consider the congestion game \( \Gamma = (G, I, \mu, c) \), where \( G \) is a DAMG and for each \( e \in E \) the cost \( c_e \) is monomial as in (3.1). Then there exists a non-negative DIOT for \( \Gamma \).

**Proof.** Let \( \hat{\tau} \) be the trivial DIOT of the game \( \Gamma \), and let \( \tau \) be defined as in (4.3). We first prove that the toll vector \( \tau \) is non-negative. Notice that \( \chi \), defined as in (4.2) is non-negative and \( \chi = 0 \) only if \( \hat{\tau}_e \geq 0 \) for all \( e \in E \). Assume that there exists a \( \hat{\tau}_e < 0 \) and let \( e^* \in \arg \min_{e \in E} \frac{\hat{\tau}_e}{\delta_e} \). Then

\[
\tau_{e^*} = \hat{\tau}_{e^*} + \delta_{e^*} \chi = \hat{\tau}_{e^*} - \delta_{e^*} \frac{\tau_{e^*}}{\delta_{e^*}} = 0.
\]
In general, whenever $\tau_e < 0$, we have
\[ \tau_e = \hat{\tau}_e + \delta_e \chi = \hat{\tau}_e - \delta_e \frac{\tau_e^*}{\delta_e^*} \geq \hat{\tau}_e - \delta_e \hat{\tau}_e = 0. \]

Now we prove that the toll vector $\tau$ is a DIOT. By Theorem 3.1, this means that it satisfies equation (3.2). First, notice that, by construction of the $\delta_e$, for any $i \in \mathcal{I}$, we have
\[ \sum_{e \in p} \delta_e = \sum_{e \in p'} \delta_e \quad \text{for all } p, p' \in \mathcal{P}^i. \] (4.4)

By (3.2) we have
\[ \sum_{e \in p} ((d+1)\hat{\tau}_e + db_e) = \sum_{e \in p'} ((d+1)\hat{\tau}_e + db_e), \]
hence, by (4.4),
\[ \sum_{e \in p} ((d+1)(\hat{\tau}_e + \delta_e \chi) + db_e) = \sum_{e \in p'} ((d+1)(\hat{\tau}_e + \delta_e \chi) + db_e), \]
that is
\[ \sum_{e \in p} ((d+1)\tau_e + db_e) = \sum_{e \in p'} ((d+1)\tau_e + db_e). \]

\[ \square \quad \square \]

Remark 4.1. The condition that the graph $\mathcal{G}$ is a DAMG is sufficient for the existence of a non-negative DIOT. It is not necessary, as the following counterexample shows. Let $\Gamma = (\mathcal{G}, \mathcal{I}, \mu, c)$ with $\mathcal{I} = \{1, 2\}$, $\mathcal{V} = \{v, v'\}$, $o_1 = d_2 = v$, $o_2 = d_1 = v'$, $e_1, e_2 \in (o_1 \rightarrow d_1)$, $e_3, e_4 \in (o_2 \rightarrow d_2)$, and the costs are as in Fig. 3.

The graph $\mathcal{G}$ is not a DAMG, but the following non-negative toll is a DIOT:
\[ \tau_1 = 1/2, \quad \tau_2 = 0, \quad \tau_3 = 1/2, \quad \tau_4 = 0. \]

We proceed to show that for graphs that contain a directed cycle, non-negative DIOTs need not exist, even in networks with affine costs.

Remark 4.2. There are networks that do not admit a non-negative DIOT. Consider the network in Fig. 4. Assume that it has a single O/D pair and
\[ o = v_1, \quad d = v_4. \]

Call $e_{jh}$ the edge that connects the tail $v_j$ to the head $v_h$ and, for $d = 1$, denote $c_{jh}$ the corresponding cost function:
\[ c_{jh}(x) = a_{jh}x + b_{jh}. \]
Let $a_{jh} = 1$ for all $j, h \in \{1, 2, 3, 4\}$. Moreover, assume that

$$b_{12} = b_{23} = b_{34} = 4, \quad b_{13} = b_{24} = 0, \quad b_{32} = 8.$$  

Under these assumptions all edges are used at the optimum and in equilibrium when the demand is large enough. By (3.4) the trivial optimal toll $\hat{\tau}$ is

$$\hat{\tau}_{12} = \hat{\tau}_{23} = \hat{\tau}_{34} = -2,$$  

$$\hat{\tau}_{13} = \hat{\tau}_{24} = 0,$$  

$$\hat{\tau}_{32} = -4.$$  

By Proposition 3.2, any optimum toll $\tau$ satisfies (3.2) and we obtain the equations

$$\tau_{12} = -2 + \chi_{12}, \quad \tau_{23} = -2 + \chi_{23}, \quad \tau_{34} = -2 + \chi_{34},$$  

$$\tau_{13} = \chi_{13}, \quad \tau_{24} = \chi_{24}, \quad \tau_{32} = -4 + \chi_{32},$$  

with

$$\chi_{12} + \chi_{23} + \chi_{34} = \chi_{13} + \chi_{34} = \chi_{12} + \chi_{24} = \chi_{13} + \chi_{32} + \chi_{24},$$  

which implies $\chi_{23} = -\chi_{32}$. For the tolls to be non-negative, we should have $\chi_{23} \geq 2$ and $\chi_{32} \geq 4$, which is clearly impossible. Hence no non-negative DIOT exist.

### 5. General case

In general there is no guarantee that a nonnegative DIOT exists. This, by itself, is not a big problem. It is conceivable that a social planner may sometime want to use negative tolls in order to achieve her goal. Nevertheless, the planner may be subject to budget constraints and not be able to afford a toll system that implies a global loss. Therefore it is interesting to study the existence of conditions for a DIOT $\tau$ such that the following budget constraint is satisfied:

$$\sum_{e \in E} \tau_e x_e \geq 0, \quad \text{for any feasible flow } f.$$  

If the network has a single O/D pair, then the budget constraint (5.1) is satisfied for some DIOT.

**Theorem 5.1.** Assume that the game $\Gamma$ has monomial costs as in (3.1). If there exists an order $\prec$ on $V$ such that for all $i \in I$ we have $o^i \prec d^i$, then there exists a DIOT $\tau$ that satisfies (5.1).

**Proof.** Consider the trivial DIOT $\hat{\tau}$. Define now

$$\tau_e(\gamma) = \hat{\tau}_e + \delta_e \gamma,$$  

where $\delta$ is defined as in (4.1). Notice that $\delta_e$ can be negative, but, for each path $p \in \mathcal{P}^i$ we have

$$\sum_{e \in p} \delta_e = d^i - o^i =: \Delta^i.$$  

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Hence
\[ \sum_{e \in E} \tau_e(\gamma)x_e = \sum_{e \in E} \hat{\tau}_e x_e + \gamma \sum_{e \in E} \delta_e x_e \]
\[ = \sum_{e \in E} \hat{\tau}_e x_e + \gamma \sum_{e \in E} \delta_e \sum_{p \ni e} f_p \]
\[ = \sum_{e \in E} \hat{\tau}_e x_e + \gamma \sum_{i \in I} \sum_{p \in P^i} f_p \sum_{e \in p} \delta_e \]
\[ = \sum_{e \in E} \hat{\tau}_e x_e + \gamma \sum_{i \in I} \sum_{p \in P^i} f_p \Delta^i \]
\[ = \sum_{e \in E} \hat{\tau}_e x_e + \gamma \sum_{i \in I} \mu^i \Delta^i. \]

Since \( \mu^i \) and \( \Delta^i \) are positive for all \( i \in I \), by choosing \( \gamma \) big enough, the quantity \( \sum_{e \in E} \tau_e(\gamma)x_e \) can be made positive. □ □

**Corollary 5.2.** Assume that the game \( \Gamma \) is played on a network with a single O/D pair with monomial costs as in (3.1). Then there exists a DIOT \( \tau \) that satisfies (5.1).

**Remark 5.1.** Using the counterexample of Remark 4.1, we can show that the condition of Theorem 5.1 is only sufficient for the existence of a DIOT that satisfies (5.1).

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