COMPLEXITY OF NILPOTENT ORBITS AND THE
KOSTANT-SEKIGUCHI CORRESPONDENCE

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Abstract. Let $G$ be a connected linear semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $K_{\mathbb{C}} \to \text{Aut}(\mathfrak{p}_{\mathbb{C}})$ be the complexified isotropy representation at the identity coset of the corresponding symmetric space. Suppose that $\Omega$ is a nilpotent $G$-orbit in $\mathfrak{g}$ and $\mathcal{O}$ is the nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ associated to $\Omega$ by the Kostant-Sekiguchi correspondence. We show that the complexity of $\mathcal{O}$ as a $K_{\mathbb{C}}$ variety measures the failure of the Poisson algebra of smooth $K$-invariant functions on $\Omega$ to be commutative.

1. Introduction

The Kostant-Sekiguchi correspondence is a vital tool in the study of infinite dimensional representations of semisimple Lie groups. Let us recall some facts about this correspondence in case $G$ is a connected real, linear semisimple Lie group with maximal compact subgroup $K$. We obtain the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of $G$ (resp., $K$). The vector spaces $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ are then complexified to give a vector space decomposition of $\mathfrak{g}_{\mathbb{C}}$, the Lie algebra of $G_{\mathbb{C}}$ (the complexification of $G$), as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$. The Kostant-Sekiguchi correspondence is a bijection between the nilpotent $G$-orbits in $\mathfrak{g}$ and the nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$. (For the precise definition of the correspondence we refer the reader to [2].)

If $\Omega$ is a nilpotent $G$-orbit in $\mathfrak{g}$ and $\mathcal{O}$ is the nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ associated to $\Omega$ by the Kostant-Sekiguchi correspondence, then $(\Omega, \mathcal{O})$ is said to be a Kostant-Sekiguchi pair. Among the nice elementary properties of such a pair are: (1) $\Omega$ and $\mathcal{O}$ lie in the same $G_{\mathbb{C}}$ orbit which we denote by $\mathcal{O}_{\mathbb{C}}$ and (2) $\mathcal{O}$ is a Lagrangian submanifold of $\mathcal{O}_{\mathbb{C}}$ (relative to the Kostant-Souriau symplectic form on $\mathcal{O}_{\mathbb{C}}$.) Moreover, Vergne [9] has established a much deeper relationship between $\Omega$ and $\mathcal{O}$, namely that there is a $K$-equivariant diffeomorphism which maps $\Omega$ onto $\mathcal{O}$.

Recently, the author proved that if $(\Omega, \mathcal{O})$ form a Kostant-Sekiguchi pair then $\Omega$ is multiplicity free as a Hamiltonian $K$-space if and only if $\mathcal{O}$ is a spherical $K_{\mathbb{C}}$ variety [2]. (The definition of multiplicity free is given below in Remark 2.1. Spherical $K_{\mathbb{C}}$-varieties are defined below in Remark 2.3.) The goal of this paper is to prove a generalization of that result to all Kostant-Sekiguchi pairs. That generalization is contained in Theorem 3.1. In essence our theorem shows that the complexity of the $K_{\mathbb{C}}$ action on $\mathcal{O}$ measures the failure of the Poisson algebra of smooth $K$-invariant functions on $\Omega$ to be commutative.

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2. Notation and key definitions

We now introduce some further concepts and notations. Unless otherwise indicated, in this section $K$ will denote an arbitrary compact group. We assume that $K$ is contained in its complexification $K_{\mathbb{C}}$. Our basic reference for symplectic manifolds with Hamiltonian $K$-actions is [4].

If $A \subset G_{\mathbb{C}}$ is a Lie subgroup and $S \subset g_{\mathbb{C}}$ then $A^S$ denotes the subgroup of $A$ that fixes each element of $S$ under the adjoint action of $G_{\mathbb{C}}$ on $g_{\mathbb{C}}$. If $a \subset g_{\mathbb{C}}$ is a Lie subalgebra, then $a^S$ is defined similarly.

For the remainder of this section $X$ will denote a connected symplectic manifold which is a Hamiltonian $K$-space. Let the $K$-invariant symplectic form be $\omega_X$ and the moment map be $\Phi : X \to \mathfrak{k}^\ast$.

**Definition 2.1.** Let $\mathfrak{A} = C^\infty(X)^K$ be the algebra (with respect to Poisson bracket) of $K$-invariant smooth functions on $X$. $\mathfrak{z}$ denotes the center of $\mathfrak{A}$.

**Remark 2.1.** $X$ is said to be multiplicity free if $\mathfrak{A} = \mathfrak{z}$ i.e., $\mathfrak{A}$ is a commutative Poisson algebra.

In addition, we need to recall some facts about the symplectic structure of $\Omega$ and the action of $K$ on $\Omega$. ($K$ is a maximal compact subgroup of $G$.) Suppose that $\Omega = G \cdot E$. For each $E' \in \Omega$, we identify $T_{E'}(\Omega)$, the tangent space of $\Omega$ at $E'$, with the quotient $\mathfrak{g}/\mathfrak{g}^{E'}$. The Koszul-Souriau form $\omega_{\Omega}$ on $\Omega$ is defined by setting

$$w_{\Omega}|_{E'}(\bar{Y}, \bar{Z}) = \kappa(E', [Y, Z])$$

for all $Y, Z \in \mathfrak{g}$ where $\kappa$ denotes the Killing form of $\mathfrak{g}$. ($\bar{Y}$ and $\bar{Z}$ are the cosets of $Y$ and $Z$ in $\mathfrak{g}/\mathfrak{g}^{E'}$.) $w_{\Omega}$ is a symplectic form on $\Omega$.

The action of $K$ on $\Omega$ is the left action. If $\xi \in \mathfrak{k}$, then $\xi$ determines a global vector field $\xi_{\Omega}$ on $\Omega$ according to the following definition:

$$\xi_{\Omega} f(E') = \frac{d}{dt} \bigg|_{t=0} \left( f(\exp(-t\xi) \cdot E') \right)$$

where $f$ is any smooth function on $\Omega$ and $E' \in \Omega$. If $Y$ is any smooth vector field on $\Omega$, $Y(E')$ is the vector in $T_{E'}(\Omega)$ obtained by evaluating $Y$ at $E'$.

Since $w_{\Omega}$ is invariant under $G$, it is invariant under $K$. The $K$ action on $\Omega$ is Hamiltonian in the following sense. For each $\xi \in \mathfrak{k}$, there is a function $\phi^\xi \in C^\infty(\Omega)$ such that $i(\xi_{\Omega}) w_{\Omega} = -d\phi^\xi$, where $i(\xi_{\Omega})$ denotes interior multiplication by $\xi_{\Omega}$. We may take $\phi^\xi = \kappa(\xi, \cdot)$. We obtain the moment mapping $\Phi_{\Omega} : \Omega \to \mathfrak{k}^\ast$ by setting $\Phi_{\Omega}(E')(\xi) = \phi^\xi(E') = \kappa(E', \xi) = \kappa(E'_k, \xi)$ for all $E' \in \Omega$ and $\xi \in \mathfrak{k}$. $E'_k$ denotes the component of $E'$ in $\mathfrak{k}$.

Thus, $\Omega$ is a Hamiltonian $K$-space, with moment mapping $\Phi_{\Omega} : \Omega \to \mathfrak{k}$ defined by sending an element $E' \in \Omega$ to its component in $\mathfrak{k}$. (We have identified $\mathfrak{k}$ with its real dual space $\mathfrak{k}^\ast$ using the restriction of $\kappa$ to $\mathfrak{k}$.) Each function $f$ in $C^\infty(\Omega)$ gives rise to a smooth vector field $X_f$ satisfying $df(Y) = w_{\Omega}(Y, X_f)$ for all smooth vector fields $Y$ on $\Omega$. $X_f$ is said to be the Hamiltonian vector field associated to $f$. (If $\xi \in \mathfrak{k}$, then $X_{\phi^\xi} = \xi_{\Omega}$.) $C^\infty(\Omega)$ is a Poisson algebra under the Poisson bracket $\{\cdot, \cdot\}$ defined as follows: for $f, g \in C^\infty(\Omega)$, $\{f, g\} = w_{\Omega}(X_f, X_g)$. One checks that the linear mapping $\xi \mapsto \phi^\xi$ is a Lie algebra homomorphism $\mathfrak{k} \to C^\infty(\Omega)$ when $\{\cdot, \cdot\}$ is taken as the Lie bracket on $C^\infty(\Omega)$. 

Lemma 2.1. Let $E' \in X$. Set $W = W_{E'} = T_{E'}(K \cdot E')$ and $W^\perp$ equal to the orthogonal complement (with respect to $\zeta = w_{X|E'}$) of $W$ inside $T_{E'}(X)$, then we have the following orthogonal decomposition of $T_{E'}(X)$ into (real) symplectic vector spaces:

$$\frac{W}{W \cap W^\perp} \bigoplus \left((W \cap W^\perp) \oplus (W \cap W^\perp)^* \right) \bigoplus \frac{W^\perp}{W \cap W^\perp}. $$

The restriction of $\zeta$ to $(W \cap W^\perp) \oplus (W \cap W^\perp)^*$ is given by $\zeta((Y_1, \lambda_1), (Y_2, \lambda_2)) = \lambda_1(Y_2) - \lambda_2(Y_1)$ for all $Y_i \in (W \cap W^\perp)$, $\lambda_i \in (W \cap W^\perp)^*$. Moreover, (1) $\frac{W}{W \cap W^\perp}$ and $W \cap W^\perp$ are isomorphic (as $\mathfrak{t}^E$ modules) and (2) $W/\mathfrak{t}^E$ and $\frac{W}{W \cap W^\perp}$ are isomorphic (as $\mathfrak{t}^{E'}$ modules).

Proof. See Corollary 9.10 and Lemma 9.11 of [6]. These results are steps in the proof of the normal slice theorem of Guillemin, Sternberg and Marle. □

Definition 2.2. If $E'' \in X$ (or $\mathfrak{t}$), set $d(E'') = \dim K \cdot E''$. Let $d = d_X$ be the maximum dimension of a $K$-orbit in $X$ and let $m = m_X$ denote the minimal codimension of a $K$-orbit in $X$. $d_\Phi$ will denote the maximum dimension of a $K$ orbit in $\Phi(X)$. We define three important subsets of $X$.

(1) $X_d = \{E' \in X| \dim K \cdot E' = d\}$

(2) $X_0 = \{E' \in X| \exists$ an open set $U \subset X$ such that $E' \in U$ and $d(\cdot)$ is constant on $U\}$

(3) $X_\Phi = \{E' \in X| \dim K \cdot \Phi(E') = d_\Phi\} = \{E' \in X| \dim (\mathfrak{t}^E) \text{ is minimum}\}$

Remark 2.2. From Proposition 27.1 in [3] we know that $X_0$ is open and dense in $X$. In addition, $X_0 \subset X_d$. Otherwise there is a point in $X_0$ whose $K$ orbit has dimension $d' < d$. But, then there must be an open subset of points whose $K$ orbits have dimension $d'$ which is impossible.

Proposition 2.1. $X_\Phi$ is open and dense in $X$.

Proof. The fact that $X_\Phi$ is open is Lemma 1 in section 3 of [3]. The following proof (unpublished) that $X_\Phi$ is dense in $X$ is due to A. T. Huckleberry.

It suffices to show that $X_\Phi^c$, the complement of $X_\Phi$, has codimension at least 2 in $X$. We use induction on the dimension of $K$. If $\dim K = 0$, then $K$ is finite. Then $X_\Phi = X$ since all $K$ orbits in $\Phi(X)$ are zero dimensional. Suppose $\dim K > 0$. If $x \in X_\Phi^c$, we need to show that there is an open neighborhood of $x$ whose intersection with $X_\Phi^c$ has codimension at least 2. Let $K_0$ denote the identity component of $K$. First consider $X^{K_0}$, the set of fixed points of $K_0$ on $X$, and its intersection with $X_\Phi$. If $K_0 \cdot x = x$, we use the slice theorem to construct an open neighborhood $U = K \times_{K^*} \Sigma$ of $x$. Define $\Sigma_\Phi$ relative to the action of $(K^*)_0$ in the obvious way. The argument for Proposition 27.3 in [3] shows that either (a) $K_0$ (which equals $(K^*)_0$) acts trivially on $\Sigma$, or (b) $U^{K_0}$ has codimension $\geq 2$ in $U$. In case (a) $K$ acts as a finite group on $U$ so that $U = U_\Phi$. In case (b) $U^{K_0} \cap (U_\Phi)^c$ has codimension $\geq 2$ in $U$. This argument takes care of the points in $X_\Phi^c \cap X^{K_0}$. If $x \notin X^{K_0}$, then the slice neighborhood $U = K \times_{K^*} \Sigma$ has the property that $\dim K^* < \dim K$. By induction $(\Sigma_\Phi)^c$ has codimension at least 2 in $\Sigma$. Thus $(U_\Phi)^c = K \times_{K^*} (\Sigma_\Phi)^c$ has codimension at least 2 in $U$. □

Since $X_d$ and $X_\Phi$ are each open and dense in $X$, we have the following result.
Corollary 2.1. $X_\Phi \cap X_\delta$ is open and dense in $X$.

We know recall the notion of a coisotropic submanifold of $X$.

**Definition 2.3.** The orbit $K \cdot E'$ in $X$ is said to be coisotropic if $W_{E'}^\perp \subset W_{E'}$.

By results of Guillemin and Sternberg [7], $X$ is multiplicity free as a Hamiltonian $K$-space if and only if there is an open dense subset $U$ of $X$ such that for $E' \in U$ the orbit $K \cdot E'$ is coisotropic in $X$. Therefore, it is reasonable to use the size of the quotient $rac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp}$ for generic $K$ orbits to measure the failure of $X$ to be multiplicity free.

**Proposition 2.2.** There is an open dense subset $U$ of $X$ such that for all $E' \in U$, the non-negative integer

$$\dim \frac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp},$$

has a constant value. We denote this value by $2\epsilon(X)$.

**Proof.** Choose $U = X_\delta \cap X_\Phi$. Then for all $E' \in U$, the decomposition of $T_{E'}(X)$ in Lemma 2.1 implies that

$$\dim \frac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp} = \dim X - 2d + d_\Phi$$

Note that if there is another dense open set $U'$ on which the dimension function in 2 is constant, then $U'$ must have non-empty intersection with $X_\delta \cap X_\Phi$. Thus, $\epsilon(X)$ is well defined. \qed

As a corollary of the preceding proof we have

**Corollary 2.2.** Suppose that $E' \in X_\delta \cap X_\Phi$, then

$$\dim \frac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp} = 2\epsilon(X).$$

We recall from 3 or 2 the definition of the rank of the action of $K$ on $X$.

**Definition 2.4.** The rank of the $K$-action on $X$, denoted by $r_K(X)$, is equal to $\text{rank } K \cdot E'$ where $E' \in X$ and the orbit $K \cdot E'$ has maximum dimension among the $K$ orbits in $X$.

Finally, we recall from 4, the notions of rank and complexity of algebraic $K_C$ actions.

**Definition 2.5.** Suppose that $Y$ is a variety with $K_C$ action and $B_k$ is a Borel subgroup of $K_C$. The complex codimension of a generic $B_k$ orbit is called the complexity of $Y$, denoted $c_{K_C}(Y)$ or $c(Y)$ (when the reductive group $K_C$ is understood). If $U_k$ is the nilpotent radical of $B_k$ and $B_k \cdot z$ is a generic $B_k$ orbit, then the codimension of $U_k \cdot z$ in $B_k \cdot z$ is called the rank of $Y$. It is denoted $r_{K_C}(Y)$.

**Remark 2.3.** $c(Y)$ is also the transcendence degree (over $C$) of the $B_k$ invariant functions in the field of rational functions (with complex coefficients) on $Y$. $Y$ is spherical for $K_C$ if and only if $c(Y) = 0$. The rank of $Y$ is also the transcendence degree of the $U_k$ invariants in the field of rational functions on $Y$. 
3. Main Theorem

Our main result is:

**Theorem 3.1.** Let \((\mathcal{O}, \Omega)\) be a Kostant-Sekiguchi pair, then

(a) \(r_K(\mathcal{O}) = r_K(\Omega)\);
(b) \(c(\mathcal{O}) = c(\Omega)\).

This result was inspired by the main result of [1].

We assume from now on that \((\Omega, \mathcal{O})\) is a Kostant–Sekiguchi pair. The proof of Theorem 3.1 requires two important facts about \(\Omega\) and \(\mathcal{O}\). The first is the existence of a \(K\) invariant diffeomorphism

\[
\mathcal{M}_\mathcal{O} : \mathcal{O} \to K \times_{K^s} V_\mathcal{O}(\mathfrak{s})
\]

established in the proof of Proposition 5.2 of [2]. To describe the vector bundle \(K \times_{K^s} V_\mathcal{O}(\mathfrak{s})\) in [1], we recall the notation of [2].

There is a Kostant-Sekiguchi \(sl(2)\)-triple \(\{x, e, f\}\) such that \(\mathcal{O} = K_c \cdot e\). The Kostant-Sekiguchi property means that (1) \(x \in \mathfrak{k}\), \(e, f \in \mathfrak{p}_c\), (2) \(e = \sigma(f)\), where \(\sigma\) is conjugation on \(\mathfrak{g}_c\) relative to the real form \(\mathfrak{g}\), and (3) the following Lie bracket relations hold: \([x, e] = 2e, [x, f] = -2f\), and \([e, f] = x\). It follows that the Lie algebra \(C x \oplus C e \oplus C f\) is the complexification of a Lie subalgebra \(s\) of \(\mathfrak{g}\), where \(s\) is isomorphic to \(sl(2, \mathbb{R})\). \(V_\mathcal{O}(\mathfrak{s})\) is the quotient \([\mathfrak{t}_c, e]/[\mathfrak{t}, e]\). In [2] it is shown that \(K \times_{K^s} V_\mathcal{O}(\mathfrak{s})\) is diffeomorphic to the conormal bundle of \(K \cdot e\) inside the cotangent bundle of \(\mathcal{O}\). In addition, there is an isomorphism of \(K^s\) modules over \(\mathbb{R}\):

\[
V_\mathcal{O}(\mathfrak{s}) \simeq \mathfrak{t}^s/\mathfrak{t}^s + Z.
\]

where \(Z\) is the sum (over \(C\)) of the positive eigenspaces of \(ad(x)\) on \(\mathfrak{t}_c\) that do not lie in \(\mathfrak{t}_c^s\).

We now establish the main result, Theorem 3.1.

**Proof.** We first establish part (a).

By composing the \(K\)-invariant diffeomorphisms in [1] and [1], we have the assignment: \(E' \mapsto \mathcal{M} \circ \mathcal{V}(E')\). So we can identify \(E'\) with an equivalence class \([k_0, y + z]\) in \(K \times_{K^s} V_\mathcal{O}(\mathfrak{s})\) where \(y \in \mathfrak{t}^s/\mathfrak{t}^s\) and \(z \in Z\). (See equation [1].) It is more convenient to consider the equivalence class \(\mathcal{M} \circ \mathcal{V}(k_0^{-1} \cdot E') = [1, y + z]\) where \(1\) denotes the identity in \(K\). Assume that \(\dim K \cdot E' = d\). Since \(\dim K \cdot E'\) is maximum, (1) \((\mathfrak{t}^s)^{y+1}\) has minimum dimension among the subalgebras \((\mathfrak{t}^s)^{v}\) for \(v \in V_\mathcal{O}(\mathfrak{s})\) and (2) \((\mathfrak{t}^s)^{y}\) has minimum dimension among the subalgebras \((\mathfrak{t}^s)^{y}\) for \(y' \in \mathfrak{t}^s/\mathfrak{t}^s\). (See Lemma 6.2 of [2].)

Set \(\mathfrak{s}_R = (\mathfrak{t}^s)^{y}\) and \(\mathfrak{s}_C = (\mathfrak{t}^s)^{y}\). Set \(S\) equal to the connected subgroup of \(K^s\) with Lie algebra \(\mathfrak{s}_C\) and \(\mathfrak{s}_R\) equal to the connected subgroup of \(S\) with Lie algebra \(\mathfrak{s}_R\). Then \(S\) (resp., \(\mathfrak{s}_R\)) is a stabilizer of general position for the action of \(K^s\) on \(\mathfrak{t}^s/\mathfrak{t}_C^s\) (resp., \(\mathfrak{t}^s/\mathfrak{t}^s\)). Also, \(\mathfrak{t}^{[1, y+z]}\), the centralizer of \([1, y + z]\) in \(\mathfrak{t}\) is equal to \(\mathfrak{s}_R\). Since \(\mathcal{M} \circ \mathcal{V}\) is \(K\)-equivariant, \(\mathfrak{t}^{k_0^{-1} \cdot E'} = \mathfrak{t}^{[1, y+z]}\). Hence, since \(\mathfrak{t}^{E'} = Ad(k_0)(\mathfrak{t}^{k_0^{-1} \cdot E'})\), \(\mathfrak{t}^{E'}\) and \(\mathfrak{s}_R\) are isomorphic Lie algebras. Therefore,

\[
r_K(\Omega) := \text{rank } \mathfrak{t}^{\Phi(E')} - \text{rank } \mathfrak{t}^{E'} = \text{rank } K - \text{rank } \mathfrak{s}_R.
\]
From Panyushev (Theorem 2.3 in [3]) \( r_{K_C}(O) = r_{K_C}(K_C^x/K_C^z) + r_{S}(Z) \). By Corollary 2(i) of Theorem 1 in [4], \( r_{K_C}(K_C^x/K_C^z) = \text{rank } K_C^x - \text{rank } S_C^r \). By the observation following equation (6.7) in [2], \( r_{S}(Z) = \text{rank } S_C^r - \text{rank } S_C^s \). Hence, by equation (7),

(8) \( r_{K_C}(O) = \text{rank } K - \text{rank } S_C^s \),

which is the same as \( r_K(\Omega) \).

The argument for part (b) of the theorem starts by noting that for all \( E' \in \Omega \), Lemma 2.1 implies that if \( W = T_{E'}(K \cdot E') \), then

(9) \( \dim \Omega = \dim T_{E'}(\Omega) = 2 \dim \left( \mathfrak{t}^{\Phi(E')}/\mathfrak{k}^{E'} \right) + \dim \left( \mathfrak{k}/\mathfrak{t}^{\Phi(E')} \right) + \dim \frac{W^\perp}{W \cap W^\perp} \).

Therefore,

(10) \( \dim \Omega = 2(\dim \mathfrak{t}^{\Phi(E')}) - \dim \mathfrak{E}' + \dim \mathfrak{k} - \dim \mathfrak{t}^{\Phi(E')} + \dim \mathfrak{t}/\mathfrak{t}^{\Phi(E')} + \dim \frac{W^\perp}{W \cap W^\perp} \).

Choose \( E' \in \Omega_d \cap \Omega_b \). By Lemma 2 in section 3 of [3], we have \( [\mathfrak{t}^{\Phi(E')}, \mathfrak{t}^{\Phi(E')}] \subseteq \mathfrak{E}' \). Since \( \mathfrak{t}^{\Phi(E')} \subseteq \mathfrak{t}^{\Phi(E')} \), \( [\mathfrak{t}^{\Phi(E')}, \mathfrak{t}^{\Phi(E')}]=[\mathfrak{t}^{\Phi(E')}, \mathfrak{t}^{\Phi(E')}] \).

Since \( \mathfrak{t}^{\Phi(E')} \) and \( \mathfrak{t}^{E'} \) have the same maximum semisimple ideal,

(11) \( \dim \mathfrak{t}^{\Phi(E')} - \dim \mathfrak{t}^{E'} = \text{rank } \mathfrak{t}^{\Phi(E')} - \text{rank } \mathfrak{t}^{E'} \).

Applying Corollary 2.2, equation (10) becomes

(12) \( \dim \Omega = \dim \mathfrak{t}^{\Phi(E')} - \dim \mathfrak{t}^{E'} + \dim \mathfrak{k} - \dim \mathfrak{t}^{E'} + 2\epsilon(\Omega) \).

Since \( E' \in \Omega_d \cap \Omega_b \), equation (11) implies

\( \dim \Omega = \text{rank } \mathfrak{t}^{\Phi(E')} - \text{rank } \mathfrak{t}^{E'} + \dim \mathfrak{k} - \dim \mathfrak{t}^{E'} + 2\epsilon(\Omega) \),

where \( \dim \mathfrak{k} - \dim \mathfrak{t}^{E'} \) is the maximal dimension of a \( K \) orbit in \( \Omega \).

We have just shown that the codimension of the largest \( K \) orbit in \( \Omega \) is given by the expression

(13) \( \text{rank } \mathfrak{t}^{\Phi(E')} - \text{rank } \mathfrak{t}^{E'} + 2\epsilon(\Omega) = r_{K_C}(O) + 2\epsilon(\Omega) \).

The last assertion follows from part (a). On the other hand, by equation (6.8) in [2], taking into account equation (5.3) and the fact that \( c(\Omega) = c_{K_C}(K_C^x/K_C^z) + c_S(Z) \) (Theorem 2.3 in [3]), the codimension of the largest \( K \) orbit in \( \Omega \) is also given by the expression:

(14) \( r_{K_C}(O) + 2c(\Omega) \).

Part (b) of the theorem follows from equations (13) and (14).

\[ \square \]

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