A COMPARISON THEOREM FOR SEMI-ABELIAN SCHEMES OVER A SMOOTH CURVE.

FABIEN TRIHAN AND DAVID VAUCLAIR

Abstract. We compare flat cohomology with crystalline syntomic complexes in two cases: 1) p-divisible groups over a separated $\mathbb{F}_p$-scheme with local finite $p$-bases, 2) semi-abelian schemes over a separated irreducible smooth curve.

Contents

1. Introduction 2
2. Preliminaries (part I) 11
  2.1. Diagrams and (co)fibered categories 11
  2.2. Fibered topoi and derived categories 14
  2.3. Usual sites for log schemes 22
  2.4. Crystalline sites for log schemes 23
  2.5. 1-motives and $p$-divisible groups 26
  2.6. Technical remarks on certain sheaves 30
3. Dévissage of flat cohomology 33
  3.1. Functorial mapping fibers in derived categories 33
  3.2. Some useful diagrams and functors 35
  3.3. Vanishing cohomology 41
  3.4. Complete Mayer-Vietoris for semi-abelian schemes over $C$. 45
  3.5. Rigid uniformization around semi-stable fibers 48
  3.6. Dévissage in $p$-divisible groups 53
4. Preliminaries (part II) 54
  4.1. Projective systems and $p$-divisibility over a $\mathbb{Z}/p$-algebra 54
  4.2. Limits and quasi-coherence on $p$-adic formal schemes 57
  4.3. Limits and quasi-coherence on crystalline sites 62
  4.4. Local $p$-bases and local embeddings 65
  4.5. Exactness for crystals 69
  4.6. Twists by effective logarithmic divisors 73
5. Twisted syntomic complexes for Dieudonné crystals 75
  5.1. The category of $(1, \phi)$-modules 76
  5.2. Local embeddings with Frobenius and cohomological descent 77
  5.3. The relative Frobenius and Cartier’s descent for crystals 85
  5.4. The mod $p$ Hodge Filtration on the small crystalline étale site 95
  5.5. Twisted syntomic complexes on the étale site 104
  5.6. Syntomic complexes on the syntomic site 109
6. Dévissage of twisted syntomic complexes 116
  6.1. Complete Mayer Vietoris 116

Date: May 12, 2015.
1. Introduction

Consider an odd prime number $p$. To a scheme $X$ over $\mathbb{F}_p$ is naturally associated two kinds of integral $p$-adic cohomology groups. On the one hand we have the $p$-adic étale cohomology groups $H^q_{\text{et}}(X, \mathbb{Z}_p)$. On the other hand we have the crystalline cohomology groups $H^q_{\text{crys}}(X/\mathbb{Z}_p, \mathcal{O}_{X/\mathbb{Z}_p})$. If $X$ is smooth (or more generally if $X$ is syntomic over a scheme with local finite $p$-bases) then it is well known (using e.g. [FM]) that both are related by a long exact sequence

\[ H^q_{\text{et}}(X, \mathbb{Z}_p) \to H^q_{\text{crys}}(X/\mathbb{Z}_p, \mathcal{O}_{X/\mathbb{Z}_p}) \to H^{q+1}_{\text{crys}}(X/\mathbb{Z}_p, \mathcal{O}_{X/\mathbb{Z}_p}) \to \]

It is natural to expect generalizations of this result when $\mathbb{Z}_p$ is replaced by more general coefficients. A first (easy) step is to replace $\mathbb{Z}_p$ with an arbitrary lisse $\mathbb{Z}_p$-sheaf. A possible way of investigation to treat more general coefficients is to replace the étale topology with finer ones such as the flat or syntomic topology (note that such a change of topology does not affect cohomology for lisse $\mathbb{Z}_p$-sheaves). Using the techniques of Fontaine-Messing again one could presumably treat positive Tate twists $\mathbb{Z}_p(i)$ (at least for $i \leq p-1$). The Tate module of abelian schemes have been treated in [Ba]. The purpose of this paper to establish a suitable comparison theorem for $p$-divisible groups in the first place and then for semi-abelian schemes following the ideas of [KT].

Assume $X$ is separated and has local finite $p$-bases. Consider the covariant Dieudonné functor of [BBM], $D$, from the category $p\text{-div}(X)$ of $p$-divisible groups over $X$ to the category $\mathcal{D}C(X)$ of Dieudonné crystals over $X$. It is known that $D$ is fully faithful [BM] and is in fact an equivalence [dJ]. However the proofs given in loc. cit. do not produce explicitly a quasi-inverse functor. Instead they rely on deformation techniques in order to reduce to the case where $X$ is the spectrum of a field. In this paper we construct the syntomic complex functor

\[ S_{\text{syn},X} : \mathcal{D}C(X) \to D^b(X_{\text{syn}}^N, \mathbb{Z}/p^k) \]

whose target is the bounded derived category of $\mathbb{Z}/p^k$-modules on the small syntomic site of $X$ (here $\mathbb{Z}/p^k$ denotes the pro-ring $(\mathbb{Z}/p^k)_{k \leq 1}$). By construction we have a functorial...
distinguished triangle

$$\sum_{\text{syn}, X}(D) \xrightarrow{\mathbb{Z}/p \otimes_{\mathbb{Z}/p+1} u_{X/(\mathbb{Z}/p+1)}}^{1-\phi} \text{Fil}^1 D_{++1} \xrightarrow{u_{X/(\mathbb{Z}/p+1)}} X.$$ 

where $D_+$ is the restriction of $D$ to the ind crystalline topos $(X/(\mathbb{Z}/p))^\text{crys, syn}, \text{Fil}^1$ denotes the natural $mod p$ Hodge filtration on $D$, $u_{X/(\mathbb{Z}/p)}$ is the projection to $X^\text{crys, syn}$, and $1$ denotes the obvious morphism and $\phi$ is the unique $\mathcal{O}^\text{crys}$ semi-linear morphism such that $p\phi$ is induced by the Frobenius of $D$. In this setting our first main result is the following.

**Theorem 1.1.** (8.1) Let $G_p$ denote the projective system of $p$ power torsion subgroups of $G$ viewed as sheaves on the small syntomic site of $X$. There is a canonical isomorphism

$$G_p \simeq \sum_{\text{syn}, X}(D(G))$$

This isomorphism is functorial with respect to $G$ and $X$.

In particular, applying $R\text{limproj}$ and taking cohomology in 3 yields a long exact sequence

$$H^0_{\text{syn}}(X, T_p(G)) \xrightarrow{H^0_{\text{crys}}(X/\mathbb{Z}_p, \text{Fil}^1 D(G))}^{1-\phi} H^0_{\text{crys}}(X/\mathbb{Z}_p, D(G)) +1$$

which boils down to (1) for $G = \mathbb{Q}_p/\mathbb{Z}_p$.

Consider now a separated irreducible smooth curve $C$ over $\mathbb{F}_p$. Our second main result is a comparison theorem for semi-abelian schemes over $C$. Using rigid uniformization around bad fibers we construct a functor $D$ from the category of semi-abelian schemes over $C$ with good reduction on a dense open $U$ to the category $\mathcal{D}(C^2)$ of Dieudonné crystals over the logarithmic curve $(C, Z)$ where $Z$ is the reduced divisor complementary to $U$. By construction the restriction of this functor to abelian schemes coincides with the one of [BBM]. Next we construct the twisted syntomic complex functor

$$\sum_{\text{et}, C^1}(-Z) : \mathcal{D}(C^2) \rightarrow \mathcal{D}^0(C^1, Z/p)$$

whose restriction to $\mathcal{D}(C)$ can be recovered from the functors (2) on $X = C$ and $X = Z$ by projection from the small syntomic topos to the small étale one (more precisely a version of (2) is needed above the diagram $X = (C \leftarrow Z)$, see below for explanations). Let $z : Z \rightarrow C$ denote the inclusion morphism and $\Gamma^Z : \text{Mod}(C^\text{et}, Z/p) \rightarrow \text{Mod}(C^N, Z/p)$ denote the functor taking $M$ to the kernel of the specialization morphism $M \rightarrow z_*z^{-1}M$.

We prove the following.

**Theorem 1.2.** (8.1) Consider a semi-abelian scheme $A$ over $C$ whose restriction to $U$ is abelian. Let $\epsilon : C_F \rightarrow C_{\text{et}}$ denote the projection of the big flat topos to the small étale one. There is a canonical isomorphism

$$R\epsilon_*R\Gamma^Z A \simeq \sum_{\text{et}, C^1}(-Z)(D(A))$$

This isomorphism is functorial with respect to $A$ and $Z$.

Applying $R\text{limproj}$ and passing to cohomology yields in particular a long exact sequence

$$H^q_{\text{FL}}(C, T_p(A)) \rightarrow H^q_{\text{crys}}(C^2/\mathbb{Z}_p, \text{Fil}^1 D(A)(-Z)) \rightarrow H^q_{\text{crys}}(C^2/\mathbb{Z}_p, D(A)(-Z)) +1$$

where $H^q_{\text{FL}}$ means flat cohomology vanishing at $Z$. If $A$ is in fact an abelian scheme and $Z = \emptyset$ this result follows from [1].
1.0.1. The results of this paper certainly won’t come as a surprise to experts. In fact theorems 1.1 and 1.2 are nothing more than sheafified versions of [KT] 5.10 and 5.13. These refinements will be the main ingredient for a proof of a non commutative Iwasawa main conjecture in [TV]. Our original motivation was thus to check whether or not the proofs of loc. cit. could be sheafified as well. While doing so, we discovered several problems in loc. cit. and it finally became easier to rewrite everything than trying to fix the mistakes and/or missing arguments. It might be worth however to explain the length of this paper with regards to theirs. We encountered mainly two sources of difficulties.

The first one concerns the proof of the comparison theorem for p-divisible groups. The strategy of [KT] 5.10 is to use the equivalence of categories between p-divisible groups and Dieudonné crystals and to interpret cohomology as higher extension groups. Unfortunately some confusions regarding continuous cohomology create a serious gap in the argumentation and we do not know how to fix it (some detailed explanations are given in [Va to KT]). We circumvent this difficulty by following the method of [Br] instead. Namely we show that the vanishing results for Ext’s of Br together with the techniques of [FM] are sufficient to treat not only abelian schemes but p-divisible groups as well.

The second source of difficulty concerns the proof of the comparison theorem for semi-abelian schemes by reduction to the case of p-divisible groups. Roughly speaking the strategy of [KT] 5.13 is to perform a parallel devissage in flat and crystalline cohomology. Unfortunately this devissage is merely sketched in loc. cit. and some delicate issues are left to the reader. On the side of flat cohomology the main ingredient is to replace A with a diagram of p-divisible groups using Raynaud’s rigid uniformization around bad fibers. However a precise sheaf theoretic interpretation is not given and the cohomological consequences are left to the reader to find. Here these tasks will be achieved at the end of section 3.6. On the side of crystalline cohomology their first step is to define $D(A)$ by gluing $BBM$’s $D(A|U)$ with an ad hoc (logarithmic) Dieudonné crystal on the complete neighborhood of bad points. We will check that this definition is functorial and is the only one possible by studying the structure of the category of log 1-motives introduced in [KT]. Their second step is to relate the twisted syntomic complex of $D(A)$ to the one attached to the diagram of p-divisible groups introduced before. In loc. cit. explanations are only given in the local situation and globalization is left to the reader. Unfortunately the local computations explained in loc. cit. 5.14 in presence of a lifting are not enough to perform the cohomological descent hinted in loc. cit. 5.8. We solve this issue by using hypercoverings with divisors in the spirit of [SN] (see 5.7).

An additional new difficulty is that we have to compare the syntomic complex on the étale site (5) (which are useful for dévissage) with the ones on the syntomic site (2) (which appear in [Li]). We achieve this comparison by introducing an intermediate variant (using linearized de Rham complexes). A significant part of this work is thus devoted to the construction of several variants of the syntomic complex functor and to comparing them.

1.0.2. The organization of the paper is the following.

2 Preliminaries (part I). Constructions like gluing functors, projective limits etc. are often needed at the level of derived categories. A convenient way to get rid of technical complications is to work systematically above an arbitrary diagram of (log) schemes rather than above a single (log) scheme. The relevant basic facts about (co)fibered categories and (weakly) fibered topoi are recalled in [21, 22]. The language recalled there will also be useful when studying various categories of hypercoverings and the syntomic complex
functors defined above them. Our conventions concerning the usual (pre)topologies and crystalline topoi are explained in 2.3 2.4 Some terminology and well known results about 1-motives and $p$-divisible groups are recalled in 2.5.

3 Dévissage of flat cohomology. Let $A/C$ be a semi-abelian scheme with good reduction over $U = C - Z$. If $v$ is a point of $Z$ with residue field $k_v$ we denote $Z_v = \text{Spec}(k_v)$, $C_v = \text{Spec}(O_v)$ the complete neighborhood of $v$ in $C$ and $U_v = \text{Spec}(K_v)$ the generic point of $C_v$. Consider the diagram $J^+$ with the following subschemes of $C$ as vertices: $U$ and for $v$ running in $Z$, the $Z_v$’s, the $C_v$’s and the $U_v$’s, and the inclusion morphisms between them as edges. The purpose of this chapter is to define a $p$-divisible group $H$ over $J^+$ (ie. a $p$-divisible group over each one of the subschemes in question together with base change morphisms between them) from which one can rebuild the projection of the vanishing local sections $R^\Gamma Z A_p \in D^b(C^B_E, \mathbb{Z}/p)$ to the small étale site (see 3.26 or (9) below for explanations).

Let us explain the logical steps leading to this result. The first step is to replace the big flat site with the small étale one. In 3.3 we begin with an elementary study of the vanishing (local) sections functor $R^\Gamma Z$ in the general setting of fibered topoi. This study relies notably on 3.11 (where the functorial properties of mapping fibers are discussed) and on the notions of acyclicity in fibered topoi which are discussed in 2.11. A significant drawback of the functor $R^\Gamma Z$ is its lack of functoriality with respect to the chosen topos. For instance if $\epsilon$ denotes the projection morphism from the big (say flat) to the small (say étale) topos then $R^\Gamma Z$ does not commute to $R\epsilon^*$. This is mainly due to the following fact: if $G$ is a group scheme over $C$ then the small étale sheaf over $C$ represented by $G$ has no memory of the group scheme $Z \times_C G$ over $Z$. To circumvent this issue, we enrich the picture by replacing $C$ with the diagram $C^+ := (Z \to C)$. We define a functor $R^\Gamma Z(C, -)$ going from the topos of $C^+$ to that of $C$. As it turns out this new variant of the vanishing sections functor commutes to $R\epsilon_*$. In the setting of big topoi we have moreover $R^\Gamma Z(C, (\cdot)|_{C^+}) \simeq R^\Gamma Z(-)$. Applying this to $A_p$ gives (see 3.14)

\[
R\epsilon_* R^\Gamma Z A_p \simeq R^\Gamma Z(C, R\epsilon_* A_p, |_{C^+})
\]

In 3.3 we achieve the next step which is to replace $C$ (resp. $C^+$) with the diagram $J$ formed by $U$, the $U_v$’s and the $C_v$’s for $v$ in $Z$ (resp. $J^+$, obtained from $J$ by adding the $Z_v$’s). Consider the natural morphism $m : J \to C$. The functor $Rm_*$ fits into a familiar distinguished triangle of Mayer Vietoris type (see 3.4) and is called the (complete) Mayer Vietoris functor. We prove that it allows to recover $R\epsilon_* A_p$ from the restriction of $A_p$ to $J$ and similarly for vanishing sections. More precisely (see 3.17):

\[
R^\Gamma Z(C, R\epsilon_* A_p, |_{C^+}) \simeq Rm_* R^\Gamma Z (J, R\epsilon_* A_p, |_{J^+})
\]

In addition to abstract nonsense the proof notably relies on the fact that the direct image functor of $C_{v, \text{et}} \to C_{\text{et}}$ is exact (lemma 3.6) together with the following property of Abelian varieties which was found by Greenberg and Milne: if $K$ denotes the function field of $C$, $v$ a point of $C$ and $K_v$ (resp. $K^b_v$) the completion (resp. henselization) of $K$ at $v$ then $H^q(K_v, A_{[K_v]})$ coincides with $H^q(K^b_v, A_{[K^b_v]})$ for $q \geq 1$.

Let us now explain the construction of the $p$-divisible group $H$ on $J^+$ which will serve as a substitute for $A_{p, |_{J^+}}$. First we have to define a $p$-divisible group $H_U$ (resp. $H_{U_v}$, $H_{C_v}$, $H_{Z_v}$ for each $v$ in $Z$). When restricted to the open of good reduction $A$ gives rise directly to a $p$-divisible group. Thus we simply set $H_U := A_{U, p^\infty}$ and $H_{U_v} := A_{U_v, p^\infty}$. Since
A has semi-stable reduction at \( v \) we get a \( p \)-divisible group on \( \mathbb{Z}_v \) simply by replacing \( A|_{\mathbb{Z}_v} \) with its connected component \( A^0|_{\mathbb{Z}_v} \) (which is the extension of a abelian variety \( B_{k_v} \) by a torus \( T_{k_v} \)): we set \( H_{\mathbb{Z}_v} = A^0|_{\mathbb{Z}_v} \). We know from [SGA7-I] IX that \( A^0|_{\mathbb{Z}_v} \) admits a canonical lifting to a group scheme \( G_v \) (the Raynaud group) which is the extension of an abelian scheme lifting \( B_{k_v} \) by a torus \( T_{k_v} \) and whose completion along the special fiber is isomorphic to the completion of \( A^0_{C_v} \). We get a \( p \)-divisible group on \( C_v \) by setting \( H_{C_v} = G_v|_{p^{-\infty}} \). We make the following key observation: since \( G_v|_{p^{-\infty}} \) is finite over \( C_v \) the previous formal isomorphism induces a morphism of group schemes \( e_v : G_v|_{p^{-\infty}} \to A^0|_{C_v} \). This remarks allow us to define the base change morphisms giving rise to the \( p \)-divisible \( H \) over \( J^+ \) as well as a morphism \( H \to A^0|_{J^+} \) over \( J^+ \). This morphism is certainly not an isomorphism in general since \( A^0|_{C_v} \) might not be finite over \( C_v \). In 3.5 we show that it is nevertheless possible to recover \( A^0|_{J^+} \) from \( H \) using Raynaud’s result concerning the generic fiber (in the sense of rigid geometry) of the morphism \( e_v \). In 3.6 we use this to prove (see 3.21 and the proof of 3.26) that

\[
R_{\Gamma}^{Z|J}(J^+, R\epsilon_* A_{p,|J^+}) \simeq \text{Sma} R_{\Gamma}^{Z|J}(J^+, R\epsilon_* H_p)
\]

where \( \text{Sma} \) is an exact functor defined over \( J_{et} \), designed to neglect the generic fiber of the \( C_v \) components (see 3.7 for more details). A key ingredient in the proof is the introduction of the small quasi-finite flat site (see 3.20) which is fine enough to compute cohomology and small enough to express sheaf theoretic consequences of Raynaud’s rigid uniformization. The final dévissage result 3.26 is obtained by putting (6), (7) and (8) together:

\[
R\epsilon_* R_{\Gamma}^{Z|J} A_p \simeq Rm_* \text{Sma} R_{\Gamma}^{Z|J}(J, R\epsilon_* H_p)
\]

4. Preliminaries (part II). The purpose of this chapter is to review a number of technical results regarding projective systems and exactness which will be useful in the various constructions of the syntomic complex coming next.

In 4.1, we review the basic lemma designed to divide Frobenius by \( p \) (lem. 4.3). This involves the notion of \( (L-) \)-normalized modules or complexes (def. 4.2) and the normalizing functor \( \langle 1 \rangle^* \) (def. 4.1). In 4.2, 4.3 we design a convenient framework regarding \( p \)-adic formal schemes limits and quasi-coherence. A standard feature of crystalline cohomology is that one is often led to pass from finite to \( p \)-adic coefficients and conversely. Inside the crystalline topos this causes no difficulty (see 4.20). When passing to realizations on the other hand (e.g. for de Rham complexes) one has to add some quasi-coherence assumptions in order to get (partial) analogous results. The relevant notions are adapted from [Be5] 3.2 and discussed in 4.2.4 - 4.2.5.

In 4.4 we review a notion of finite \( p \)-bases for \( (p \)-adic formal) log schemes which is stronger than the one defined in 1.5 but slightly easier to handle. We also introduce the corresponding categories of (global and local) embeddings, which will be enriched later.

When dealing with crystals one has to be careful with the notion of subobjects because the inclusion of the category of crystals inside the category of modules on the crystalline site is not left exact in general. The relevant phenomenons are discussed in 4.5 using the previously introduced categories of embeddings (rather than a restricted crystalline site in the spirit of [Be1]). In 4.6 we discuss the relation between effective logarithmic and
Cartier divisors. Putting everything together we study to what extent a twist of a module or a crystal can be regarded as a subobject of the latter (see \[4.48\], \[4.49\]).

5 Twisted syntomic complexes for Dieudonné crystals. In this chapter we define functors

\[
S_{syn,.,X}^{1,\phi} : DC(X) \to Mod^1,\phi(X_{syn}^{\mathbb{N}}, \mathcal{O}^{crys})
\]

\[
S_{et,.,X}^{1,\phi}(-h) : DC(X^\sharp) \to D^+(Mod^1,\phi(X_{et}^{\mathbb{N}}, \mathcal{O}^{crys}))
\]

The first (resp. second) one is defined when \(X \in \mathcal{B}_0\) (resp. \((X^\sharp, h) \in \mathcal{B}_0^\prime\)) i.e. is a diagram of separated schemes (resp. a diagram of separated log-schemes and logarithmic divisors) whose type \(\Delta\) satisfies a certain finiteness condition (namely, \(\delta/\Delta\) must be finite for each \(\delta\)) and whose vertices \(X_\delta\) (resp. \(X^\sharp_\delta\)) have local \(p\)-bases over \(\mathbb{F}_p\), see \[5.1\]. The target of these functors is the (derived) category of \((1, \phi)\)-modules, defined and briefly studied in \[5.1\]. The functors \[2\], \[5\] are respectively obtained from \[10\], \[11\] by forming the mapping fiber of \(1 - \phi\) (see \[5.2\] (iv)). Aside from the definitions themselves the main purpose of this chapter is to establish a canonical isomorphism

\[
Re_\ast S_{sym,.,X}^{1,\phi} D \simeq S_{et,.,X}^{1,\phi} D
\]

in the case \(X^\sharp = X\) (trivial log structure). Here \(\epsilon : X_{syn} \to X_{et}\) denotes the canonical morphism. This task requires several intermediate variants of the syntomic complex on the small étale and syntomic sites.

Let us briefly enumerate these constructions and explain the strategy leading to \[12\]. In \[5.2\] we begin with some observations regarding applications of cohomological descent in crystalline cohomology and we define some categories of semi-simplicial embeddings with additional structures (Frobenius lifts, logarithmic divisors) which are adapted to the various constructions that we have in mind. Let us explain this roughly. A semi-simplicial global embedding \(\iota : U^\sharp[_\cdot] \to Y^\sharp[_\cdot]\) together with a Frobenius lift is in the category \(HR_{F,et}^{\sharp,\sharp}\) if \(U^\sharp[_\cdot]\) is a hypercovering in the topos \(X_{et}\). The full subcategory \(HR_{F,et}^{\sharp,\sharp, crys} \subset HR_{F,et}^{\sharp,\sharp}\) is defined by the additional condition that the logarithmic divided power envelope \(T^\sharp[_\cdot]\) of \(\iota\) is a hypercovering in the topos \((X^\sharp/\mathbb{Z}_p)_{crys, et}\). The latter is adapted to the computation of crystalline cohomology à la Čech whereas the former is adapted to the computation using de Rham complexes. This leads naturally to the construction of two functors with value in a category of complexes of \((1, \phi)\)-modules:

\[
S_{et,.,T^\sharp[_\cdot]}^{1,\phi}(-h), S_{et,.,T^\sharp[_\cdot]}^{\ast,1,\phi}(-h) : DC(X^\sharp) \to Kom^1,\phi(T^\sharp[_\cdot], et, \mathcal{O}^{crys})
\]

The first one (Čech) is not necessary in this text. It is nevertheless the most direct way of defining the syntomic complex and will serve as a warming up for the other constructions. The second one (de Rham) is well adapted to dévissage results and will be used in the next chapter.

In the case of a trivial log structure, we give a construction of the functor \[10\] which is global (i.e. does not involve semi-simplicial embeddings). This construction has the advantage of being close to the context of \([FM]\), \([Ba]\) and will be used in the comparison theorem for \(p\)-divisible groups. We also define linearized de Rham versions of the syntomic complex \(SL\Omega^{1,\phi}_{et,.,T^\sharp[_\cdot]}\), \(SL\Omega^{\ast,1,\phi}_{sym,.,T^\sharp[_\cdot]}\) on the small étale and syntomic site respectively.

The main steps leading to \[12\] may be summarized as follows. Section \[5.3\] is concerned with the versions on the étale site. Using the gluing lemma \[5.9\] we prove (lem.
that the projection of \( S_{et, T}^{1, \phi} (-h) \) (resp. of \( S_{et, T}^{1, \phi} (-h) \)) to \( D^+ (Mod^{1, \phi} (X_{et, \mathcal{O}^{crys})) \) is essentially independent of the semi-simplicial embedding as long as the latter is chosen in \( H_{et}^{2} \) (resp. \( H_{et}^{2, crys}, H_{et}^{2, crys} \)). We prove furthermore (lem. 5.51) that the resulting three functors \( D C (X^2) \to D^+ (Mod^{1, \phi} (X_{et, \mathcal{O}^{crys})) \) are canonically isomorphic and we define (11) as anyone of them (prop. 5.52, def. 5.53). Section 5.6 is concerned with the versions on the syntomic site. We prove (prop. 5.65) that the globally defined functor (10) coincides with the projection of \( S\Omega_{et, T}^{1, \phi} \) to \( D^+ (Mod^{1, \phi} (X_{et, \mathcal{O}^{crys})) \). The isomorphism (12) is finally obtained by proving (prop. 5.66):

\[
R_{\phi} S\Omega_{et, T}^{1, \phi} \simeq S\Omega_{et, T}^{1, \phi}
\]

Let us now give some explanations regarding the constructions enumerated above. In section 5.3 we review some basics concerning the relative Frobenius and the Cartier operator. Then we establish the crystalline version of the Cartier equivalence over \( X \) (prop. 5.25) and deduce some elementary consequences for crystals over \( X^2 \). These results will be the main ingredient of section 5.4 where a crystalline sub sheaf \( Fil^1 D \) is defined for an arbitrary Dieudonné crystal \( D \) and shown to satisfy a canonical isomorphism \( D/Fil^1 D \simeq i_* Lie(D) \) (prop. 5.35), the tangent sheaf \( Lie(D) \) being defined in terms of the Verschiebung operator modulo \( p \). This filtration and a similar isomorphism are then extended to the linearized semi-simplicial crystal \( L(D_{T^2}) \) and its twisted versions (prop. 5.37). In section 5.5 we check that in each three cases the Frobenius vanishes modulo \( p \) on the subcomplex defined by using \( Fil^1 D \). Using this and the normalization functor \( \langle 1 \rangle^* \) (see 4.1) we are able to divide Frobenius by \( p \). This yields the desired definition for the (Cech, de Rham and linearized de Rham) versions of the syntomic complex on the small étale site (prop. 5.49). In section 5.6 the construction and elementary properties of the (global and linearized de Rham) versions of the syntomic complex on the syntomic site of \( X \) are given using the same ingredients than in 5.5 together with some well known properties of crystalline cohomology with respect to the syntomic topology.

6 Dévissage of twisted syntomic complexes. Our general strategy to obtain the comparison result in the case of a semi-abelian scheme \( A/C \) is to perform a dévissage for the twisted syntomic complex of \( D_{C^2} (A) \) which is roughly parallel to the one of chapter 3 in order to reduce to the case of \( p \)-divisible groups. This won’t be achieved before section 8.2 for the following reasons. On the one hand the semi-stable Dieudonné functor \( D_{C^2} \) has yet to be defined (this will be done in the next chapter). On the other hand the desired reduction involves switching from \( D_{C^2} (A) \) to \( D_{T^2} (H) \) where \( H \) is the diagram of \( p \)-divisible groups defined in chapter 3. This will be done using a trick relying on the comparison result for \( p \)-divisible groups established in section 8.1. The purpose of the present chapter is nevertheless to establish two key dévissage results.

In section 6.1 we consider the diagram of log schemes \( J^2 \) which is \( J \) together with the log structure coming from \( C^2 \). Letting \( m : J^2 \to C^2 \) denote the natural morphism, we prove (prop. 6.1(i)) that \( m^* \) induces an equivalence between Dieudonné crystals on \( C^2 \) and diagrams of Dieudonné crystals over \( J^2 \) whose base change morphisms are invertible (we call those cartesian). By Zariski localization on \( C^2 \) we reduce to the case where a lifting of \( C^2 \) by a \( p \)-adic formal log scheme with \( p \)-bases over \( Spf(\mathbb{Z}_p) \) is given and then the result essentially boils down to gluing locally free modules from \( J \) to \( C \). Looking at
de Rham complexes yields furthermore a canonical isomorphism (prop. 6.1 (iii))

\begin{equation}
S_{et,-,C^t}^{1,\phi}(-h)(D) \simeq Rm_* S_{et,-,J^+}^{1,\phi}(-m^{-1}h)(m^*D)
\end{equation}

In section 6.2 we use the second variant of the local vanishing sections functor to describe the twisted syntomic complex in a specific situation. Namely consider a Dieudonné crystal $D$ over $J$ and denote respectively $o^* D$ and $\rho^* D$ its pullback to $J^t$ and $J^+$. Consider furthermore the smooth divisor $Z_J$ as an effective logarithmic divisor of $J$. Then we establish a canonical isomorphism

\begin{equation}
S_{et,-,J^+}^{1,\phi}(-Z_J)(o^* D) \simeq R\prod_{Z_J}(J, S_{et,-,J^+}^{1,\phi}(\rho^* D))
\end{equation}

Here the key idea is to work with semi-simplicial embeddings $\iota : U^t_i \to Y^t_i$ of $H^{\text{log}}$-modules. This approach amounts to extending the Dieudonné functor on the category of $\text{log}$-divisible groups over $C_v$ and $U_v$ (here we use the exact structure inherited from the category of abelian sheaves on the big flat site (see 2.33, 7.2).

We do this in two steps. The first one is achieved in section 7.1. By studying the structure of the category $\mathcal{M}_{\text{log}}(C'_v)$ for $C'_v$ varying among the finite étale $C_v$-schemes we show (prop. 7.8) that given a stack $\mathcal{C}$ of exact categories on the small finite étale site $\mathcal{M}_{\text{log}}$ amounts to extending the homomorphism

\begin{equation}
\text{Ext}_{\mathcal{M}(C_v)}^1(\mathbb{Z}, \mathbb{Z}(1)) \to \text{Ext}_{\mathcal{C}(C_v)}^1(F(\mathbb{Z}), F(\mathbb{Z}(1)))
\end{equation}

induced by $F$ to $\text{Ext}_{\mathcal{M}_{\text{log}}(C_v)}^1(\mathbb{Z}, \mathbb{Z}(1))$. The second step will be achieved in section 7.2. It consists in proving that if $\mathcal{C}$ denotes the category of Dieudonné crystals over (finite étale extensions of) $C'_v$ such an extension of (17) exists and is moreover unique if one imposes the desired compatibility with the analogous homomorphism over $U_v$ (prop. 7.18). This verification could be achieved by brute force and explicit calculations since it is nothing more than the investigation of Kummer extensions of Dieudonné crystals over $C_v$, $C'_v$ and $U_v$. Here we have chosen a homological approach by embedding the exact category of Dieudonné crystals into the larger abelian category of $(f,v)$-modules. This approach relies mainly on the compatibility of the Dieudonné functor with Cartier duality ([BBM] chap. 5).

8 The comparison theorem. The purpose of this chapter is to prove theorems 1.1 and 1.2. The first one is proven in section 8.1 and is rather independant from the rest of the
text (it only uses the global construction of the syntomic complex in section 5.6). The
second one is then deduced by dévissage using the main results of chapters 3, 5, 6, 7.

In section 8.1 we consider the case of $p$-divisible groups. We first consider the case of
a single base scheme $X$ with local finite $p$-bases over $\mathbb{F}_p$. The basic idea is to reduce to
the case of $\mu_{p^\infty}$ by using Cartier bi-duality. Following the arguments reproduced in [Br]
we begin by checking that Fontaine-Messing’s exact sequence

$$
0 \longrightarrow \mu_{p^k} \longrightarrow \mathcal{L} \longrightarrow I_k \longrightarrow O_{k^{crys}} \longrightarrow 0
$$

on $\text{Spec}(\mathbb{F}_p)_{\text{syn}}$ remains valid on $X_{\text{syn}}$ (8.9 (iii)). Now if $G$ is an arbitrary $p$-divisible
group on $X$ whose Cartier dual is denoted $G^*$ we show that (18) gives rise to an exact sequence

$$
0 \longrightarrow \mathcal{E}xt^1_{X_{\text{syn}}}(G^*, \mu_{p^k}) \longrightarrow \mathcal{E}xt^1_{X_{\text{syn}}}(G^*, I_k) \longrightarrow \mathcal{E}xt^1_{X_{\text{syn}}}(G^*, O_{k^{crys}}) \longrightarrow 0
$$

thanks to the vanishing theorem of [BR75]. Next we observe that the first term in (19) is
nothing but $G^*_{p^k}$, i.e. $G_{p^k}$. It thus remains only to compare the second and third term to
the corresponding ones in the distinguished triangle (3) describing $S_{\text{syn}, k, X}(D_X(G))$. For
the third one this relies directly on the fact that for certain coefficient, $\mathcal{E}xt^1$ commutes to
the projection of the crystalline topos on the small syntomic topos. Next we need to
to check that $\mathcal{E}xt^1_{X_{\text{syn}}}(G^*, \mathbb{G}_a)$ and the morphism induced by $\mathcal{O}_{X/(\mathbb{Z}/p^k)} \to i_\ast \mathbb{G}_a$ respectively coincide with $\text{Lie}(D(G))$ and the canonical morphism $D(G) \to i_\ast \text{Lie}(D(G))$. This is done
in lemma 8.11 using the Cartier isomorphism on the syntomic site. Then we conclude
easily from the fact that modulo the previous identifications the map induced by the Frobenius of $O_{k^{crys}}$ coincides with the one induced by the Frobenius of $D_X(G)$.

Our arguments do not extend directly to the case where $X$ is a diagram (of schemes with local $p$-bases). Fortunately the result extends nevertheless thanks to the fact (proven
by the above arguments) that $S_{\text{syn}, k, X}(D_X(G))$ is concentrated in degree 0.

In section 8.2 we put everything together in order to treat the case of a semi-abelian
scheme $A/C$. Namely (9), theorem 1.2 (applied $H \in p\text{div}(J)$), (12) and (16) give an
isomorphism

$$
R\text{et}_s R\Gamma\mathbb{Z} A_p \simeq Rm_\ast S\text{ma}_{\text{et}, \ast J}(\omega Z J)(D_J(H)|_{\text{et}})
$$
on the one hand while (15) together with a basic property of the functor $\text{Sma}$ give

$$
S_{\text{et}, \ast C}(\omega Z (D_{C\ast}(A))) \simeq Rm_\ast S\text{ma}_{\text{et}, \ast J}(\omega Z J)(D_{C\ast}(A)|_{\text{et}})
$$
on the other hand. We conclude by showing that the natural morphism $D_J(H)|_{\text{et}} \to
D_{C\ast}(A)|_{\text{et}}$ becomes an isomorphism after applying $S_{\text{et}, \ast J}$ and the functor $\text{Sma}$.

Appendix (crystals on log schemes with local finite $p$-bases). In this chapter we offer a
quick review of top crystalline sites for fine log schemes (top being a pretopology be-
tween $et$ and $fl$) and cohomology of their crystals in the context of local finite $p$-bases. In
9.1 we check the pseudo-functoriality of the small crystalline top topos and of the usual
(quasi-) morphisms $p$, $\varepsilon$, $i$, $u$. In 9.2 we review the basic definitions and properties of
crystals. In 9.3 and 9.4 we assume given a closed immersion of the base log scheme into
a $p$-adic formal log scheme with local finite $p$-bases and we restrict our attention to the
étale topology. In this context we review the classical theory of integrable quasi-nilpotent
connections, hyper dp-stratifications, hyper dp-differential operators, linearization functors and de Rham complexes. While doing so we pay special attention to the functoriality of the theory with respect to the base log scheme and the chosen closed immersion. Let us point out that this chapter is not independent from chapters 2 and 4. More precisely we have the following logical relations: 2.4 $\Rightarrow$ 9.1 $\Rightarrow$ 9.2 $\Rightarrow$ 4.3 and 4.4 $\Rightarrow$ 9.3 $\Rightarrow$ 4.5 $\Rightarrow$ 9.4.

1.0.3. A few further generalizations of such comparison results can easily be imagined. The case of finite locally free groups can be treated roughly in the same way [Pe to Va], or alternatively as a consequence of the case of $p$-divisible groups [Va to Pe]. Other possible generalizations are log $p$-divisible groups in the sense of [Tr] or alternatively abelian varieties with general reduction. We hope to come back to these cases in future works.

1.0.4. Needless to say this paper relies entirely on the previous work of K. Kato and F. Trihan. The references to their original text [KT] are so numerous that we have given up listing them all. We hope that the self contained approach taken here will serve as a useful reference in the future.

We are grateful to T. Tsuji for his help and suggestions, especially concerning the version of the syntomic complex involving linearized de Rham complexes and also for the localization triangles in compactly supported log crystalline cohomology using blowing ups.

2. Preliminaries (part I)

2.1. Diagrams and (co)fibered categories.

We review basic facts concerning (co)fibered categories over a base category $B$ and their natural extension to the category of diagrams in $B$.

2.1.1. Recall the following definition from [Gr].

Definition 2.1. Let $C$ denote a category.

(i) A diagram $X$ of $C$ is a functor $X : \Delta \to C$, $\delta \mapsto X_\delta$, $(\delta \to \delta') \mapsto (X_\delta \to X_{\delta'})$ for some small category $\Delta$ which is called the type of $X$. We sometimes use simplified notations as $X/\Delta$ or $(X_\delta)$.

(ii) Given diagrams $X, X'$ of respective type $\Delta, \Delta'$ a morphism of diagrams $f : X \to X'$ is a couple $f = (F, \alpha)$ where $F : \Delta \to \Delta'$ is a functor and $\alpha : X \to X' \circ F$ is a natural transformation. Using abusively the letter $f$ to designate either $f$, $F$ or $\alpha$ we sometimes write simply $f_\delta : X_\delta \to X'_{f(\delta)}$.

(iii) Morphisms of diagrams are composed in an obvious way and the resulting category is denoted $\text{Diag}(C)$.

The category $\text{Diag}(C)$ is strictly 2-functorial with respect to the category $C$, meaning that a functor $F : C \to C'$ canonically induces a functor $\text{Diag}(C) \to \text{Diag}(C')$ and similarly for natural transformations between such functors everything being strictly compatible with composition.

We always identify $C$ with a full subcategory of $\text{Diag}(C)$ by sending an object to the corresponding punctual diagram. Let us emphasize that even though diagrams of type $\Delta$ in $C$ are in bijection with (and are often identified to) diagrams of type $\Delta^{op}$ in $C^{op}$ there is no obvious relation between the categories $\text{Diag}(C^{op})$ and $\text{Diag}(C)$ or $\text{Diag}(C)^{op}$. 

We are grateful to T. Tsuji for his help and suggestions, especially concerning the version of the syntomic complex involving linearized de Rham complexes and also for the localization triangles in compactly supported log crystalline cohomology using blowing ups.
2.1.2. Let $\mathcal{C}at$ denote the 2-category of categories and let $\mathcal{B}$ denote a category. The following result is well known.

**Lemma 2.2.** The following 2-categories are strictly equivalent in a natural way.

(i) The 2-category of contravariant pseudo-functors $\mathcal{B} \to \mathcal{C}at$, pseudo morphisms and natural transformations between them.

(ii) The 2-category of fibered categories above $\mathcal{B}$ endowed with a normalized cleavage, cartesian $\mathcal{B}$-functors and $\mathcal{B}$-natural transformations.

(iii) The 2-category of cofibered categories above $\mathcal{B}^{op}$ endowed with a normalized co-cleavage, cocartesian $\mathcal{B}^{op}$-functors and $\mathcal{B}^{op}$-natural transformations.

Proof. This follows easily from [SGA1] VI. Let us only hint the strict 2-functors $(\text{ii}) \to (\text{iii})$ and $(\text{i}) \to (\text{ii})$ in order to fix the ideas regarding directions of arrows. We write morphisms in $\mathcal{B}$ rather than in $\mathcal{B}^{op}$. Start with a contravariant pseudo-functor $\mathcal{F} : \mathcal{B} \to \mathcal{C}at$, $X \mapsto \mathcal{F}(X)$, $f \mapsto f^*$, $(fg)^* \simeq g^*f^*$. Then:

- The corresponding fibered category $p : \mathcal{F}_{fib} \to \mathcal{B}$ is constructed in such a way that the fiber category $p^{-1}(X)$ above an object $X$ of $\mathcal{B}$ is $\mathcal{F}(X)$ while $\text{Hom}_f(\xi, \eta) = \text{Hom}_{\mathcal{F}}(\xi, f^*\eta)$ if $p(\xi) = X$, $p(\eta) = Y$ and $f : X \to Y$ is a morphism in $\mathcal{B}$. The cleavage is given by the collection of tautological morphisms $f^*\eta \to \eta$.

- The corresponding cofibered category $q : \mathcal{F}_{cof} \to \mathcal{B}^{op}$ is constructed in such a way that the fiber category $q^{-1}(X)$ above an object $X$ of $\mathcal{B}$ is $\mathcal{F}(X)$ while and $\text{Hom}_f(\xi, \eta) = \text{Hom}_{\mathcal{F}}(f^*\xi, \eta)$ if $q(\xi) = X$, $q(\eta) = Y$ and $f : X \leftarrow Y$ is a morphism in $\mathcal{B}$. The co-cleavage is given by the collection of tautological morphisms $\xi \to f^*\xi$.

It is sometimes useful to notice that one can switch between the point of views $(\text{ii})$ and $(\text{iii})$ by changing the directions of the arrows in fibers. Namely let $\mathcal{F} : \mathcal{B} \to \mathcal{C}at$ contravariant as in $(\text{i})$ and consider the contravariant pseudo-functor $\mathcal{F}^{\circ} : \mathcal{B} \to \mathcal{C}at$, defined as $X \mapsto \mathcal{F}(X)^{op}$, $f \mapsto f^*$, $g^*f^* \simeq (fg)^*$. Then we have natural isomorphisms

$$
(\mathcal{F}^{\circ})_{fib}/\mathcal{B} \simeq (\mathcal{F}_{cof})^{op}/\mathcal{B} \quad \text{and} \quad (\mathcal{F}^{\circ})_{cof}/\mathcal{B}^{op} \simeq (\mathcal{F}_{fib})^{op}/\mathcal{B}^{op}.
$$

Summarizing we found four relative categories naturally attached to a contravariant pseudo-functor $\mathcal{F} : \mathcal{B} \to \mathcal{C}at$: two fibrations $(\mathcal{F}_{fib}/\mathcal{B}$ and $(\mathcal{F}^{\circ})_{fib}/\mathcal{B})$ and two cofibrations $(\mathcal{F}_{cof}/\mathcal{B}^{op}$ and $(\mathcal{F}^{\circ})_{cof}/\mathcal{B}^{op})$. To pick one of these four it is enough to specify the fiber categories and wether this is a fibration or a cofibration. In the body of the paper we will thus often drop the subscripts $(\text{-})_{fib}$ and $(\text{-})_{cof}$ from the notations without any danger of confusion.

**Remark 2.3.**

(i) In practice one sometimes has to consider $\mathcal{B}$-functors $A : \mathcal{F}_{fib} \to \mathcal{G}_{fib}$ which are not necessarily cartesian or alternatively $\mathcal{B}^{op}$-functors $B : \mathcal{F}_{cof} \to \mathcal{G}_{cof}$ which are not necessarily cocartesian. These data are no longer equivalent. More precisely:

- the data of $A$ corresponds bijectively to a lax morphism $\mathcal{F} \to \mathcal{G}$, ie. a functor $A_X : \mathcal{F}(X) \to \mathcal{G}(X)$ for each $X$ in $\mathcal{B}$ and a natural transformation $\alpha_f : A_X f^* \to f^* A_Y$ for each $f : X \to Y$ in $\mathcal{B}$, these data being submitted to the composition constraint.

- the data of $B$ corresponds bijectively to a colax morphism $\mathcal{F} \to \mathcal{G}$, ie. a functor $B_X : \mathcal{F}(X) \to \mathcal{G}(X)$ for each $X$ in $\mathcal{B}$ and a natural transformation
2.1.3. If \( X/\Delta \) is a diagram of \( \mathcal{B} \) we use the simplified notations

\[
\begin{align*}
\mathcal{F}_{\text{fib}}(X) & := \Gamma(\Delta \times_{X,\mathcal{B},p} \mathcal{F}_{\text{fib}}/\Delta) \\
\mathcal{F}_{\text{cof}}(X) & := \Gamma(\Delta^{\text{op}} \times_{X,\mathcal{B}^{\text{op}},q} \mathcal{F}_{\text{cof}}/\Delta^{\text{op}})
\end{align*}
\]

where \( \Gamma(-) \) denotes the category of sections. These categories identify respectively with the category of diagrams of type \( \Delta \) above \( X \) in \( \mathcal{F}_{\text{fib}} \) (recall that \( X : \Delta \rightarrow \mathcal{B} \)) and with the category of diagrams of type \( \Delta^{\text{op}} \) above \( X \) in \( \mathcal{F}_{\text{cof}} \) (here we think of \( X : \Delta^{\text{op}} \rightarrow \mathcal{B}^{\text{op}} \)). Note that for a punctual diagram \( X \), we have trivially \( \mathcal{F}_{\text{fib}}(X) \simeq \mathcal{F}_{\text{cof}}(X) \).

**Lemma 2.4.** The contravariant pseudo-functor \( \mathcal{F} : \mathcal{B} \rightarrow \mathcal{C} \) admits two natural extensions to \( \text{Diag}(\mathcal{B}) \), namely:

(i) A contravariant pseudo-functor \( \mathcal{F}^{\text{diag}} : \text{Diag}(\mathcal{B}) \rightarrow \mathcal{C} \) satisfying \( \mathcal{F}^{\text{diag}}(X) \simeq \mathcal{F}_{\text{fib}}(X) \) for any \( X \) in \( \text{Diag}(\mathcal{B}) \).

(ii) A contravariant pseudo-functor \( \mathcal{F}^{\text{codiag}} : \text{Diag}(\mathcal{B}) \rightarrow \mathcal{C} \) satisfying \( \mathcal{F}^{\text{codiag}}(X) \simeq \mathcal{F}_{\text{cof}}(X) \) for any \( X \) in \( \text{Diag}(\mathcal{B}) \).

**Proof.** We notice that \( \text{Diag}(p) : \text{Diag}(\mathcal{F}_{\text{fib}}) \rightarrow \text{Diag}(\mathcal{B}) \) is automatically a fibered category and inherits a canonical normalized cleavage from \( \mathcal{F} \). Let us describe the cleavage explicitly. If \( f : X/\Delta \rightarrow X'/\Delta' \) is a morphism in \( \text{Diag}(\mathcal{B}) \) and \( \xi'/\Delta' \) is a diagram of \( \mathcal{F}_{\text{fib}} \) above \( X' \) then \( f^*\xi' \) is the diagram of type \( \Delta \) with vertices given by the formula \( (f^*\xi')_\delta := f^*_{\delta}(\xi'_{f(\delta)}) \) and edges deduced from the edges of \( \xi' \) using the universal property of the tautological morphisms \( f^*_{\delta}(\xi'_{f(\delta)}) \rightarrow \xi'_{f(\delta)} \) in \( \mathcal{F}_{\text{fib}} \). We define \( \mathcal{F}^{\text{diag}} : \text{Diag}(\mathcal{B}) \rightarrow \mathcal{C} \) as the corresponding contravariant pseudo-functor obtained by \( \mathcal{F}^{\text{diag}}(X) \) for any \( X \) in \( \text{Diag}(\mathcal{B}) \) as desired.
Applying the previous construction to $\mathcal{F}^\circ$ instead of $\mathcal{F}$ and then reversing arrows again we find a pseudo-functor $\mathcal{F}^{\text{codiag}} := \mathcal{F}^{\text{codiag},0} : \text{Diag}(\mathcal{B}) \to \mathcal{E}\text{at}$ verifying

$$\mathcal{F}^{\text{codiag}}(X) = \mathcal{F}^{\text{codiag}}(X)^{\text{op}} \simeq (\mathcal{F}^\circ)_{\text{fib}}(X)^{\text{op}} \simeq \mathcal{F}_{\text{cof}}(X)$$

for $X$ in $\text{Diag}(\mathcal{B})$ and we are done.

Let us describe the cocleavage of the corresponding cofibered category $(\mathcal{F}^{\text{codiag}})_{\text{cof}}/\mathcal{B}^{\text{op}}$ for the sake of clarity. If $X/\Delta \leftarrow X'/\Delta' : f$ is a morphism in $\text{Diag}(\mathcal{B})$ and $\xi/\Delta^{\text{op}}$ is a diagram of $\mathcal{F}_{\text{cof}}$ above $X : \Delta^{\text{op}} \to \mathcal{B}^{\text{op}}$ then $(f^*\xi)$ is the diagram of type $\Delta^{\text{op}}$ in $\mathcal{F}_{\text{cof}}$ with vertices given by the formula

$$\xi_{\delta'} = f^*_\delta(\xi_{f(\delta')})$$

and edges defined in the following way: if $\delta' \leftarrow \delta'' : g'$ in $\Delta'$ then the morphism

$$f^*\xi_{\delta'} : (f^*\xi)_{\delta'} \to (f^*\xi)_{\delta''} \text{ in } \mathcal{F}_{\text{cof}}$$

corresponds to the composed morphism

$$g'^*f^*_\delta(\xi_{f(\delta')}) \sim f^*\delta(\xi_{f(\delta)}) \xrightarrow{f^*\delta(\xi_{f(\delta)})} f^*\delta(\xi_{f(\delta')}) \text{ in } \mathcal{F}(X'_{\delta''})$$

where the first isomorphism results from the commutativity of the following square

\[
\begin{array}{ccc}
X_{\delta'} & \xrightarrow{g'} & X'_{\delta''} \\
\downarrow f_{\delta'} & & \downarrow f_{\delta''} \\
X_{f(\delta')} & \xrightarrow{f(g')} & X_{f(\delta'')}
\end{array}
\]

\[\square\]

**Remark 2.5.** Consider another contravariant pseudo-functor $\mathcal{G} : \mathcal{B} \to \mathcal{E}\text{at}$.

(i) Every lax (resp. pseudo) morphism $\alpha : \mathcal{F} \to \mathcal{G}$ admits a canonical (resp. and essentially unique) extension to a lax (resp. pseudo) morphism $\alpha : \mathcal{F}^{\text{diag}} \to \mathcal{G}^{\text{diag}}$ (use the explicit description of the cleavages on $\text{Diag}(\mathcal{B})$).

(ii) Similarly colax (resp. pseudo) morphisms $\mathcal{F} \to \mathcal{G}$ extend canonically (resp. and essentially uniquely) to colax (resp. pseudo) morphisms $\mathcal{F}^{\text{codiag}} \to \mathcal{G}^{\text{codiag}}$.

2.1.4. In the text we will use the following simplified terminology and notations. Every fibered (resp. cofibered) category will implicitly be endowed with a normalized cleavage (resp. cocleavage). In view of the lemmas 2.2 and 2.3 we will often designate a fibered category $\mathcal{G}/\mathcal{B}$ or $\mathcal{G}/\text{Diag}(\mathcal{B})$ (resp. a cofibered category $\mathcal{G}/\mathcal{B}^{\text{op}}$ or $\mathcal{G}/\text{Diag}(\mathcal{B})^{\text{op}}$) simply by describing the contravariant pseudo-functor $\mathcal{F} : \mathcal{B} \to \mathcal{E}\text{at}$ satisfying $\mathcal{F}_{\text{fib}} = \mathcal{G}$ or $(\mathcal{F}^{\text{diag}})_{\text{fib}} = \mathcal{G}$ (resp. $\mathcal{F}_{\text{cof}} = \mathcal{G}$ or $(\mathcal{F}^{\text{codiag}})_{\text{cof}} = \mathcal{G}$).

2.2. Fibered topoi and derived categories.

2.2.1. We review basic facts about fibered topoi in a slightly more general context than [SGA4-II] VI. Let us begin by introducing some convenient terminology concerning the functoriality of sites and topos.

**Definition 2.6.**

(i) We use the term pretopology to designate a couple $(\mathcal{C}, \text{Cov})$ where $\mathcal{C}$ is a category and $\text{Cov}$ denotes a set of families of arrows in $\mathcal{C}$ (the coverings) defining a pretopology in the sense of [SGA4-II] II, 1.3.
(ii) Consider pretopologies \((\mathcal{C}_i, \text{Cov}_i), i = 1, 2\). A premorphism: \(v : (\mathcal{C}_1, \text{Cov}_1) \rightarrow (\mathcal{C}_2, \text{Cov}_2)\) is a functor \(V : \mathcal{C}_1 \leftarrow \mathcal{C}_2\) which sends \text{Cov}_2 into \text{Cov}_1 and commutes to base change by arrows belonging to coverings of \text{Cov}_2.

(iii) Consider the couple of adjoint functors \((v^{-1}, v_*): (\mathcal{C}_1, \text{Cov}_1) \rightarrow (\mathcal{C}_2, \text{Cov}_2)\) induced by a premorphism \(v : (\mathcal{C}_1, \text{Cov}_1) \rightarrow (\mathcal{C}_2, \text{Cov}_2)\) by passing to the associated topos. We say that \(v\) is a quasi-morphism if \(v^{-1}\) preserves finite products (and thus in particular final objects). We say that \(v\) is a is a morphism if \(v^{-1}\) is exact.

(iv) Consider topoi \(E_i, i = 1, 2\). A weak morphism of topoi: \(v : E_1 \rightarrow E_2\) is a couple of adjoint functors \((v^{-1}, v_*): E_1 \rightarrow E_2\).

(v) Consider a weak morphism of topoi \(v : E_1 \rightarrow E_1\). We say that \(v\) is a premorphism (resp. quasi-morphism, morphism) if there exist pretopologies \((\mathcal{C}_i, \text{Cov}_i), i = 1, 2, \text{equivalences } (\mathcal{C}_i, \text{Cov}_i) \simeq E_i, i = 1, 2\) and a premorphism (resp. quasi-morphism, morphism) \((\mathcal{C}_1, \text{Cov}_1) \rightarrow (\mathcal{C}_2, \text{Cov}_2)\) inducing \(v\).

We do not know whether or not premorphisms or quasi-morphisms of topoi are stable under composition. Premorphisms of pretopologies are called morphisms of topologies in \[\text{At} 2.4.2.\] Morphisms of topoi are the usual notion by \[\text{SGA4-I} \text{ IV, 4.9.2.}\]

In practice the following fact is useful to recognize quasi-morphisms (resp. morphisms) of pretopologies. Assume that \(\mathcal{C}_2\) has finite non empty products (resp. fiber products). Then \(v^{-1}\) commutes to them (resp. is exact) if and only if \(V\) does (resp. commutes to fiber products). This may be checked by using that an arbitrary sheaf can be written as an inductive limit of representable sheaves (resp. using \[\text{SGA4-I} 1, 5.2 \text{ and III, 1.3 or } \text{At} 2.4.8.\])

Remark 2.7. Each notion of morphism defined in \[2.6\] admits an obvious ringed variant (by adding the data of a morphism of rings \(v, A_1 \leftarrow A_2\)).

(i) A ringed premorphism of ringed pretopologies \(v\) gives rise to a couple of adjoint functors \((v^*, v_*): \text{Mod}(\mathcal{C}_1, A_1) \rightarrow \text{Mod}(\mathcal{C}_2, A_2)\). Here \(v^*M\) can be described as the sheaf associated to the presheaf of modules \(U_1 \mapsto \limind A_1(U_1) \otimes_{A_2(U_2)} M(U_2)\) the limit being taken in the category of \(A_1(U_1)\)-modules with respect to \((U_2, U_1 \rightarrow VU_2)\) running in the category \(U_1/C_2\).

(ii) If \(v\) is a quasi-morphism of pretopologies then \(v^{-1}\) preserves abelian groups, rings, modules and we have the familiar relation \(v^*M = A_1 \otimes_{A_2} v^{-1}M\). In this setting the local adjunction isomorphism \(v_*\text{Hom}_{A_1}(v^*M_2, M_1) \simeq \text{Hom}_{A_2}(M_2, v_*M_1)\) holds as well.

2.2.2. Let us introduce several variants of the notion of fibered topoi using \[2.6\]. We will then use \[2.4(ii)\] to extend a fibered topos to the category of diagrams of the base.

Definition 2.8. Let \(\mathcal{B}\) denote a category, \(\text{Cat}\) the 2-category of categories, \(\text{Pre}\) the 2-category of pretopologies (with premorphisms as 1-morphisms).

(i) A prevariable pretopology \(\mathcal{P}\) on \(\mathcal{B}\) is a covariant pseudo-functor \(\mathcal{P} : \mathcal{B} \rightarrow \text{Pre}\). We say that \(\mathcal{P}\) is a quasi-variable (resp. variable) pretopology if the premorphism of pretopologies \(\mathcal{P}(f)\) is in fact a quasi-morphism (resp. morphism) of pretopologies for any morphism \(f\) in \(\mathcal{B}\).

(ii) A weakly variable topos \(\mathcal{T}\) on \(\mathcal{B}\) is a contravariant pseudo-functor \(\mathcal{T} : \mathcal{B} \rightarrow \text{Cat}\) such that \(\mathcal{T}(X)\) is a topos for any object \(X\) in \(\mathcal{B}\), and the functor \(f^{-1} = \mathcal{T}(f)\) admits a right adjoint for any morphism \(f\) in \(\mathcal{B}\). We say that \(\mathcal{T}\) is a prevariable
Lemma 2.9. Let \( \mathcal{T} : B \to \mathcal{E} \text{at} \) be a weakly variable topos and consider the associated cofibered categories let \( \mathcal{T}_{cof}/B^{op}, (\mathcal{T}^{cof})^{op}/\text{Diag}(B)^{op} \) (2.2, 2.4).

(i) If \( X \) is a diagram of \( B \) then \( \mathcal{T}(X) := \mathcal{T}_{cof}(X) \) is a topos. Whence a weakly variable topos \( \mathcal{T}^{cof} \) on \( \text{Diag}(B) \). If \( \mathcal{T} \) is in fact variable then \( \mathcal{T}^{cof} \) is variable as well.

(ii) The cofibered categories \( \mathcal{T}_{cof}/B^{op} \) and \( (\mathcal{T}^{cof})^{op}/\text{Diag}(B)^{op} \) are in fact bifibered.

Proof. [i] The first statement results easily from Giraud’s criterion [\text{EGAIV}] IV, 1.2. The second statement results from the explicit description of \( f^{-1} \) given in (25).

[ii] The fact that \( \mathcal{T}_{cof}/B^{op} \) is bifibered follows from the definition of a weakly variable topos. Let us choose a right adjoint (together with adjunction maps) \( f_{*} \) of \( f^{-1} \) for each arrow \( f \) in \( B \), i.e. a bicleavage above \( B^{op} \). Let us check that \( (\mathcal{T}^{cof})^{op}/\text{Diag}(B)^{op} \) is bifibered. It is sufficient to define a right adjoint to the pullback functors by morphisms of diagrams. Given \( f : X/\Delta \to X'/\Delta' \) in \( \text{Diag}(B) \) and \( \xi \) in \( \mathcal{T}(X) \) we define \( f_{*}\xi \) as follows. The vertices of \( f_{*}\xi \) are given by the projective limit formula:

\[
(f_{*}\xi)_{\delta'} = \lim_{\leftarrow} \xi_{\delta} \quad \quad \quad (\text{27})
\]

where the projective limit is indexed by the category \( \Delta/\delta' \) (where an object is a couple \( (\delta, \alpha : f(\delta) \to \delta') \)). The edges of \( f_{*}\xi \) are deduced from those of \( \xi \) using the covariance of the category \( \Delta/\delta' \) with respect to \( \delta' \). The reader can check that \( f_{*} \) is naturally right adjoint to the functor \( f^{-1} \) described in (25).

In order to make the connection with the terminology of [\text{EGAIV}] Vbis, VI, let us mention that for a variable topos \( \mathcal{T} : B \to \mathcal{E} \text{at} \) the category \( \mathcal{T}_{cof}/B^{op} \) would be called a \( B^{op} \)-topos in [\text{EGAIV}] Vbis, 1.2.1, \( \mathcal{T}_{fib}/B \) would be called a fibered topos in [\text{EGAIV}] VI, 7.1.1 and \( \mathcal{T}_{cof}(X) \) is equivalent to the total topos associated to the fibered topos \( \Delta \times_{X,B,P} \mathcal{T}_{fib}/\Delta \) [\text{EGAIV}] VI, 7.4.7.

2.2.3. We discuss the functoriality of derived categories of modules. We say that \( (\mathcal{T}, A) \) is a ringed weakly variable topos on \( B \) (resp. that \( (\mathcal{P}, A) \) is a ringed prevariable pretopology on \( B \)) if \( \mathcal{T} \) is as in 2.8 [i] (resp. if \( \mathcal{P} \) is as in 2.8 [i]) and \( A \) is a ring of the total topos \( \Gamma(B^{op}, \mathcal{T}) \) (resp. \( \Gamma(B^{op}, \mathcal{P}) \)).

We make use of the traditional boundedness conditions since it will be enough for our purpose. The reader may probably get rid of them by using the theory of unbounded complexes [\text{Sp}], [\text{KS}].

Lemma 2.10. Consider a ringed weakly variable topos \( (\mathcal{T}, A) \) on \( B \).

(i) There is a canonical covariant pseudo-functor \( \text{Mod}(\mathcal{T}(-), A) : B \to \mathcal{E} \text{at}, X \mapsto \text{Mod}(\mathcal{T}(X), A_{X}), (f : X \to Y) \mapsto (f_{*} : \text{Mod}(\mathcal{T}(X), A_{X}) \to \text{Mod}(\mathcal{T}(Y), A_{Y})), f_{*}g_{*} \simeq (fg)_{*} \). If \( \mathcal{T} \) is in fact prevariable then the associated fibered category \( \text{Mod}(\mathcal{T}(-), A)_{fib}/B^{op} \) is in fact bifibered, a bicleavage being given by “the” natural adjunctions \( (f^{*}, f_{*}) \).

(ii) Right derivation gives rise to a covariant lax functor \( D^{+}(\mathcal{T}(-), A) : B \to \mathcal{E} \text{at}, X \mapsto D^{+}(\mathcal{T}(X), A_{X}), f \mapsto Rf_{*}, Rf_{*}Rg_{*} \to R(fg)_{*} \). If \( \mathcal{T} \) is prevariable then \( D^{+}(\mathcal{T}(-), A) \) is in fact a pseudo-functor, i.e. \( Rf_{*}Rg_{*} \simeq R(fg)_{*} \). If the \( Rf_{*} \)'s are
of finite cohomological dimension then similar statements hold with $D$ instead of $D^+$.

(iii) If $\mathcal{T}$ is variable then left derivation gives rise to a contravariant pseudo-functor $D^-(\mathcal{T}(\cdot), A) : X \mapsto D^-(\mathcal{T}(X), A_X)$, $f \mapsto Lf^*$, $L(fg)^* \simeq Lg^*Lf^*$. If the $Lf^*$s are of finite homological dimension then a similar statement holds with $D$ instead of $D^-$.

(iv) Assume that $\mathcal{T}$ is variable and that the $Rf^*$s are of finite cohomological dimension (resp. that the $Lf^*$s are of finite homological dimension). Then the cofibered category $D^-(\mathcal{T}(\cdot), A)_{cof}/\mathcal{B}^{op}$ (resp. the fibered category $D^+(\mathcal{T}(\cdot), A)_{fib}/\mathcal{B}^{op}$) is in fact bifibered, a bicleavage being given by natural adjunctions $(Lf^*, Rf^*)$.

Proof. This is standard. (i) is SGA4-II IV, 13. The first statement in (ii) (resp. (iii) is a consequence of the universal property of right (resp. left) derived functors. The second statement in (ii) is proved using Čech cohomology ([Ar] 2.4.5 or the proof of SGA4-II, V, 4.9). The last one in (ii) and (iii) is proven using an easy truncation trick. Finally (iv) is a consequence of the trivial duality theorem of SGA4-III XVII, 4.1.3.

In virtue of lemma 2.9, the parts (i), (iii), (iv) of lemma 2.10 extend to $\text{Diag}(\mathcal{B})$. It seems moreover plausible that $\mathcal{T}_{cof}^{\text{codag}}$ is prevariable as soon as $\mathcal{T}$ is prevariable and this would imply that (ii) extends as well. Instead of checking this we will now discuss some acyclicity conditions allowing us to compute derived functors over diagrams.

**Lemma + definition 2.11.** Consider a ringed weakly variable topos $(\mathcal{T}, A)$ on $\mathcal{B}$ and consider $(\mathcal{T}_{cof}^{\text{codag}}, A)$ denote the associated ringed weakly variable topos on $\text{Diag}(\mathcal{B})$ (2.4). Let $X/\Delta$ in $\text{Diag}(\mathcal{B})$.

(i) A module $M \in \text{Mod}(\mathcal{T}_{cof}(X), A_X)$ is flat if and only its components $M_\delta$ are flat.

(ii) Assume that $\mathcal{T}$ is variable. If $f : X/\Delta \to X'/\Delta'$ then $Lf^*$ can be computed componentwise, ie. $(Lf^*M)_\delta \simeq Lf^*_\delta M(f_\delta)$. In particular if $f' : X'/\Delta' \to X''/\Delta''$ then $L(ff')^* \simeq Lf'^*Lf^*$. In particular if $f' : X'/\Delta' \to X''/\Delta''$ then $L(ff')^* \simeq Lf'^*Lf^*$.

(iii) We say that $M \in \text{Mod}(\mathcal{T}_{cof}(X), A_X)$ has injective (resp. flasque) components, or that it is componentwise injective (resp. flasque) if $M_\delta$ is injective (resp. flasque in the sense of SGA4-II V, 4.1) for each $\delta \in \Delta$. If now $\mathcal{T} \simeq \mathcal{P}$ with $\mathcal{P}$ some prevariable pretopology we say that $M$ has $\mathcal{P}$-acyclic components, or that it is componentwise $\mathcal{P}$-acyclic if each $M_\delta$ is $\mathcal{P}(X_\delta)$-acyclic in the sense of loc. cit..

(iv) We say that $M \in \text{Mod}(\mathcal{T}_{cof}(X), A_X)$ is $d$-injective (resp. $d$-flasque, $d$-$\mathcal{P}$-acyclic, assuming $\mathcal{T} \simeq \mathcal{P}$ with $\mathcal{P}$ a prevariable pretopology) if it is isomorphic to $f_*M_1$ for some morphism $f : X_1/\Delta_1 \to X/\Delta$ with $\Delta_1$ discrete and some module $M_1$ over $(\mathcal{T}_{cof}(X_1), A_{X_1})$ which is componentwise injective (resp. flasque, $\mathcal{P}$-acyclic). The category of $d$-injective (resp. $d$-flasque, $d$-$\mathcal{P}$-acyclic) modules is cogenerating and stable by direct images of arbitrary morphisms of diagrams.

(v) Assume that $\mathcal{T}$ is variable (resp. that $\mathcal{T}$ is equivalent to $\mathcal{P}$ for some prevariable pretopology $\mathcal{P}$). If $M \in \text{Mod}(\mathcal{T}_{cof}(X), A_X)$ is $d$-flasque (resp. $d$-$\mathcal{P}$-acyclic) then it is componentwise flasque (resp. $\mathcal{P}$-acyclic). Injectives are direct factors of $d$-injectives and are thus componentwise flasque (resp. $\mathcal{P}$-acyclic) as well.

(vi) Assume that $\mathcal{T}$ is variable (resp. that $\mathcal{T}$ is equivalent to $\mathcal{P}$ for some prevariable pretopology $\mathcal{P}$). If $f : X/\Delta \to X'/\Delta'$ then $Rf_* : D^+(\mathcal{T}_{cof}(X), A_X) \to$
$D^+(\mathcal{T}_{\operatorname{cof}}(X'), A_{X'})$ can be computed using resolutions by $d$-flasques (resp. $d$-$\mathcal{P}$-acyclics). In particular if $f' : X'/\Delta' \to X''/\Delta''$ then $Rf'_* Rf_* \simeq R(f'f)_*$. In the particular case where $\Delta' = \Delta$ and $f$ induces the identity on $\Delta$ then $Rf_*$ can in fact be computed using resolutions by componentwise flasques (resp. componentwise $\mathcal{P}$-acyclics).

Proof. \((\text{i})\) and \((\text{ii})\) are obvious from the fact that tensor products can be computed componentwise.

\((\text{iv})\) Stability by direct images is tautological. The cogeneration statement needs only to be proven for $d$-injectives. If $\Delta$ itself is discrete this results from the fact that injectives are cogenerating. In the general case let $f : X_1/\Delta_1 \to X/\Delta$ denote the inclusion of the discrete diagram underlying $X$. Then $f_*$ preserves injectives (because $f$ is a flat morphism of ringed topoi, even though $\mathcal{T}$ is only assumed weakly variable), $M \to f_* f^* M$ is monomorphic (because $f^*$ is faithful) for any $M$ and the result follows.

\((\text{v})\) Let $f : X_1/\Delta_1 \to X/\Delta$ with $\Delta_1$ discrete. We have to prove that componentwise $\mathcal{P}$-acyclicity and componentwise flasqueness is preserved by $f_*$. It suffices to treat the case of $\mathcal{P}$-acyclicity. Let us thus consider $M_1$ with $\mathcal{P}$-acyclic components on $(\mathcal{T}_{\operatorname{cof}}(X_1), A_{X_1})$. Since $\Delta_1$ is discrete the projective limit formula (27) describing the $\delta$'s component of $f_* M_1$ boils down to a product of $f_{\delta_1,*} M_{\delta_1}$'s. Now each term of this product is $\mathcal{P}$-acyclic (1.2.4.5 or the proof of [SGA4-II] V, 4.9) and one may easily conclude by using the fact that for any set $I$ the derived functors $R^q \prod_{f_1} , q \geq 1$ vanish on families of $\mathcal{P}$-acyclics indexed by $I$ (see [SGA4-II] Vbis, 1.3.10 for details in the flasque case; the $\mathcal{P}$-acyclic case is similar). The statement about injectives is clear since $d$-injectives are cogenerating.

\((\text{vi})\) It suffices to treat the prevariable case. Consider a diagram $X : \Delta \to \mathcal{B}$ and a functor $f_1 : \Delta_1 \to \Delta$. Let us denote $X_{\mid \Delta_1}$ the diagram $X \circ f_1$. Then we have a a natural morphism $f_1 : X_{\mid \Delta_1} \to X$ induced by the identity of the vertices $X_{f_1(\delta_1)}$'s. A morphism of this form (for some functor $f_1$) will be called a change of indices. Note that for such $f_1$, the induced weak morphism $(\mathcal{T}(X_1), A_{X_1}) \to (\mathcal{T}(X), A_X)$ is in fact a flat morphism of ringed topoi.

Let us now turn to the statement for $f$ arbitrary. We begin with two facts.

Fact 1. Consider a change of indices $f_1 : X_1 \to X$ where the type $\Delta_1$ of $X_1$ is discrete. If $M_1$ is a componentwise $\mathcal{P}$-acyclic module on $(\mathcal{T}_{\operatorname{cof}}(X_1), A_{X_1})$ then $f_{1,*} M_1$ is $f_{*,*}$-acyclic.

Since $f_1 : \mathcal{T}_{\operatorname{cof}}(X_1) \to \mathcal{T}_{\operatorname{cof}}(X)$ is a morphism of topos, we have $Rf_{1,*} Rf_1_* M_1 \simeq R(f_1 f)_* M_1$. Now the arguments of (v) show that $R^q f_{1,*} M_1$ and $R^q (f_1 f)_* M_1$ vanish for $q \geq 1$ and the fact follows.

Fact 2. If $X \to X_{\mid \Delta} \to X'$ is the natural factorization of the morphism $f : X/\Delta \to X'/\Delta'$ then $Rf_* \simeq Rb_* Rf_*$. 

18
Let \( \Delta_1 \) be the discrete category underlying \( \Delta \) and consider the following commutative diagram

\[
\begin{array}{ccc}
X_{|\Delta_1} & \xrightarrow{a_1} & (X'_\Delta)_{|\Delta_1} \\
\downarrow f_1 & & \downarrow f'_1 \\
X & \xrightarrow{a} & X'_{|\Delta} & \xrightarrow{b} & X'
\end{array}
\]

Since the category of modules of the form \( f_{1,*}M_1 \) with \( M_1 \) injective is cogenerating in \( \text{Mod}(\mathcal{T}_{cof}(X), A_X) \) (see the proof of [iv]), it is sufficient to check that \( Rf_*f_{1,*}M_1 \simeq Rb_*Ra_*f_{1,*}M_1 \) for \( M_1 \) injective. This in turn is proven by the following series of isomorphisms:

\[
\begin{align*}
Rb_*Ra_*f_{1,*}M_1 & \simeq Rb_*a_*f_{1,*}M_1 \\
& \simeq Rb_*f'_{1,*}a'_{1,*}M_1 \\
& \simeq b_*f'_{1,*}a'_{1,*}M_1 \\
& \simeq f_*f_{1,*}M_1 \\
& \simeq Rf_*f_{1,*}M_1
\end{align*}
\]

The isomorphisms (29) and (33) are due to the fact that \( f_{1,*} \) preserves injectives (because \( f_1 : (\mathcal{T}(X_1), A_{X_1}) \to (\mathcal{T}(X), A_X) \) is a flat morphism of topos). The isomorphisms (30) and (32) are by commutativity of the diagram (28). The isomorphism (31) proceeds from fact 1 since \( f'_1 \) is a change of indices while \( a'_{1,*}M_1 \) has flasque components.

Let us now prove that \( \text{d-}\mathcal{P} \)-acyclics are \( f_* \)-acyclic. Consider a diagram \( X_1 \) whose type \( \Delta_1 \) is discrete and let \( f_1 : X_1 \to X \) denote an arbitrary morphism. As before we have a natural diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{a_1} & X'_\Delta \\
\downarrow f_1 & & \downarrow f'_1 \\
X & \xrightarrow{a} & X'_{|\Delta} & \xrightarrow{b} & X'
\end{array}
\]

Let \( M_1 \) be a componentwise \( \mathcal{P} \)-acyclic module on \( (\mathcal{T}_{cof}(X_1), A_{X_1}) \). The following series of isomorphisms prove the \( f_* \)-acyclicity of \( f_{1,*}M_1 \):

\[
\begin{align*}
Rf_*f_{1,*}M_1 & \simeq Rb_*Ra_*f_{1,*}M_1 \\
& \simeq Rb_*a_*f_{1,*}M_1 \\
& \simeq Rb_*f'_{1,*}a'_{1,*}M_1 \\
& \simeq b_*f'_{1,*}a'_{1,*}M_1
\end{align*}
\]

The isomorphism (35) follows from fact 2. The isomorphism (36) follows from the fact that \( f_{1,*}M_1 \) has \( \mathcal{P} \)-acyclic components and \( a \) induces the identity on \( \Delta \). The isomorphism (37) is obvious and (38) follows from fact 1 since \( f'_1 \) is a change of indices while \( a'_{1,*}M_1 \) has \( \mathcal{P} \)-acyclic components.

\[
\square
\]

2.2.4. Let us now discuss the functoriality of \( \text{Mod}(\mathcal{T}^{\text{codiag}}(-), A) \) and derived categories with respect to \( (\mathcal{T}, A) \). To state things precisely one should begin by observing that (pre,
quasi-) variable pretopologies and (weakly, pre, quasi-) variable topoi can be made into a 2-category in several manners. Let us review several possibilities for the definition of 1-morphisms.

**Definition 2.12.**  
(i) A premorphism \( g : \mathcal{P} \to \mathcal{P}' \) between prevariable pretopologies on \( \mathcal{B} \) is a collection of premorphisms \( g_X : \mathcal{P}(X) \to \mathcal{P}'(X) \) together with compatibility isomorphisms \( g_X : \mathcal{P}(f) \sim \mathcal{P}'(f) g_X \) satisfying the composition constraint. If the \( g_X \)'s are quasi-morphisms (resp. morphisms) then we say that \( g \) is a quasi-morphism (resp. morphism).

(ii) A weak morphism \( g : \mathcal{T} \to \mathcal{T}' \) between weakly variable topoi on \( \mathcal{B} \) is a collection of functors \( g^{-1}_X : \mathcal{T}(X) \leftarrow \mathcal{T}'(X) \) having right adjoints together with compatibility isomorphisms \( f^{-1} g^{-1}_X \simeq g^{-1}_X f^{-1} \) satisfying the composition constraint. If the \( g^{-1}_X \)'s are exact we say that \( g \) is a morphism.

If \( (\mathcal{P}, A) \) and \( (\mathcal{P}', A) \) are ringed prevariable pretopologies we define a premorphism \( (\mathcal{P}, A) \to (\mathcal{P}', A) \) as the data of a premorphism \( g : \mathcal{P} \to \mathcal{P}' \) together with a ring morphism \( A' \to g_* A \). We use a similar terminology for weak morphisms between weakly ringed topoi.

**Lemma 2.13.**  
(i) Any weak morphism (resp. morphism) \( g : \mathcal{T} \to \mathcal{T}' \) of weakly variable topoi on \( \mathcal{B} \) naturally extends to a weak morphism (resp. morphism) \( \mathcal{T}^{\text{codiag}} \to \mathcal{T}'^{\text{codiag}} \) of weakly variable topoi on \( \text{Diag}(\mathcal{B}) \).

(ii) Consider a weak morphism \( g : (\mathcal{T}, A) \to (\mathcal{T}', A') \) of ringed weakly variable topoi on \( \mathcal{B} \). Then:
- For any \( X/\Delta \) in \( \text{Diag}(\mathcal{B}) \) one has a natural functor \( g_{X,*} : \text{Mod}(\mathcal{T}_{\text{cof}}(X), A_X) \to \text{Mod}(\mathcal{T}_{\text{cof}}'(X), A'_X) \) satisfying \( (g_{X,*} M)_{\delta} = g_{X,*} M_{\delta} \).
- If \( f : X/\Delta \to X'/\Delta' \) there is a canonical functoriality isomorphism \( g_{X,*} f_* \simeq f_* g_{X,*} \). These isomorphisms for varying \( f \) satisfy the composition constraint, ie. give rise to a pseudo-morphism \( g_* : \text{Mod}(\mathcal{T}^{\text{codiag}}(-), A) \to \text{Mod}(\mathcal{T}'^{\text{codiag}}(-), A') \) of covariant pseudo-functors on \( \text{Diag}(\mathcal{B}) \).

(iii) Consider a premorphism of ringed prevariable pretopologies \( (\mathcal{P}, A) \to (\mathcal{P}', A') \) on \( \mathcal{B} \) and denote \( (\mathcal{T}, A) \to (\mathcal{T}', A') \) the associated weak morphism of ringed weakly variable topoi on \( \mathcal{B} \). Then:
- For any \( X/\Delta \) in \( \text{Diag}(\mathcal{B}) \) the functor \( g_{X,*} \) has a left adjoint \( g_{X,*}^{-1} \) satisfying \( (g_{X,*}^{-1} M')_{\delta} = g_{X,*}^{-1} M'_{\delta} \).
- If \( f : X/\Delta \to X'/\Delta' \) the functoriality isomorphisms of \( g_{X,*}^{-1} \) induce \( f* g_{X,*}^{-1} \simeq g_{X,*}^{-1} f \) by transposition. This defines a pseudo-morphism \( g^* : \text{Mod}(\mathcal{T}^{\text{codiag}}(-), A') \to \text{Mod}(\mathcal{T}'^{\text{codiag}}(-), A) \) of contravariant pseudo-functors on \( \text{Diag}(\mathcal{B}) \).

Proof. \( (i) \) is clear (define \( g_{X,*} \) and \( g_{X,*}^{-1} \) componentwise) and \( (ii) \) follows. In \( (iii) \) we define \( g_{X,*} \) componentwise in the obvious way.

\( \square \)

Let us turn to derived categories.

**Lemma 2.14.** Consider a morphism \( g : (\mathcal{T}, A) \to (\mathcal{T}', A') \) between ringed variable topoi on \( \mathcal{B} \).

(i) Let \( X/\Delta \) in \( \text{Diag}(\mathcal{B}) \).
- The functor \( R g_{X,*} : D^+(\mathcal{T}_{\text{cof}}(X), A_X) \to D^+(\mathcal{T}_{\text{cof}}'(X), A'_X) \) can be computed componentwise, ie. \( (R g_{X,*} M)_{\delta} \simeq R g_{X,*} M_{\delta} \).
Assume that Remark 2.15.

(i) The statement about \(Lg_X^*: D^-(\mathcal{T}_{cof}(X), A_X') \to D^-(\mathcal{T}_{cof}(X), A_X)\) can be computed componentwise i.e. \((Lg_X^* M)_\delta \simeq Lg_{X_\delta}^* M_\delta\).

(ii) Let \(f: X/\Delta \to X'/\Delta'\) be a morphism in \(\text{Diag}(\mathcal{B})\).

- There is a canonical functoriality isomorphism \(Rg_{X'}^* Rf_* \simeq Rf_* Rg_X^*\). When \(f\) varies, the resulting collection of isomorphisms satisfies the composition constraint, i.e. give rise to a pseudo-morphism \(Rg_*: D^+(\mathcal{T}^{\text{coding}}(-), A) \to D^+(\mathcal{T}^{\text{coding}}(-), A')\) between covariant pseudo-functors on \(\text{Diag}(\mathcal{B})\).

- There is a canonical isomorphism \(Lg_X^* Lf^* \simeq Lf^* Lg_{X'}^*\). When \(f\) varies the resulting collection of isomorphisms satisfies the composition constraint, i.e. give rise to a pseudo-morphism \(Lg_*: D^-(\mathcal{T}^{\text{coding}}(-), A') \to D^-(\mathcal{T}^{\text{coding}}(-), A)\) between contravariant pseudo-functors on \(\text{Diag}(\mathcal{B})\).

(iii) Consider another morphism \(g': (\mathcal{T}', A') \to (\mathcal{T}'', A'')\) between ringed variable topoi on \(\mathcal{B}\).

- There is a canonical composition isomorphism \(Rg_* Rg'_* \simeq R(g'g)_*\). When \(g, g'\) vary the resulting collection of isomorphisms satisfy the associativity constraint.

- There is a canonical composition isomorphism \(L(g'g)^* \simeq Lg_* Lg'^*\). When \(g, g'\) vary the resulting collection of isomorphisms satisfy the associativity constraint.

Proof. \(\boxed{\text{[i]}}\) The statement about \(Lg_X^*\) follows from \(\boxed{2.11(i)}\). The statement about \(Rg_X^*\) follows from the fact that injectives are componentwise flasque \(\boxed{2.11(v)}\).

In \(\boxed{\text{[ii]}}\) the statement about derived direct (resp. inverse) images can be checked using d-flasque (resp. flat) resolutions, see \(\boxed{2.11(iv), (v)}\) (resp. \(\boxed{2.11(i), (ii)}\)). Finally \(\boxed{\text{[iii]}}\) follows from \(\boxed{\text{[i]}}\) \(\square\)

**Remark 2.15.** Assume that \(\mathcal{T}, \mathcal{T}'\) are only prevariable and that \(g\) is induced by a pre-morphism (not necessarily a morphism) of prevariable pretopologies \(\mathcal{P} \to \mathcal{P}'\) where \(\mathcal{T} \simeq \mathcal{P}\) and \(\mathcal{T}' \simeq \mathcal{P}'^r\).

(i) The statement about \(Rg_X^*\) in \(\boxed{2.14(i)}\) still holds since injectives are componentwise \(\mathcal{P}\)-acyclic \(\boxed{2.11(v)}\).

(ii) The statement about \(Rg_X^*\) in \(\boxed{2.14(ii)}\) still holds as well since d-\(\mathcal{P}\)-acyclics are preserved by \(f_*\) and \(g_{X,*}\).

**2.2.5.** Let us discuss projective systems.

**Definition 2.16.** Let \(\mathbb{N}\) denote the set of positive integers viewed as a category with exactly one arrow \(k \to k'\) if \(k \geq k'\). Given a weakly fibered topos \(\mathcal{T}\) over \(\mathcal{B}\) we define \(\mathcal{T}^\mathbb{N}\) as follows. For \(X\) in \(\mathcal{B}\), \(\mathcal{T}^\mathbb{N}(X)\) is the category of projective systems \(k \mapsto \xi_k\) (we also use the notation \((\xi_k)\) or \(\xi\)) indexed by \(\mathbb{N}\). For \(f: X \to Y\), \(f^{-1}: \mathcal{T}^\mathbb{N}(Y) \to \mathcal{T}^\mathbb{N}(X)\) is defined componentwise: \((f^{-1}\xi)_k = f^{-1}\xi_k\).

Note that direct images can be computed componentwise as well: \((f_*\xi)_k = f_*\xi_k\). Here are some immediate properties of \(\mathcal{T}^\mathbb{N}\).

**Lemma 2.17.** Consider a weakly variable topos \(\mathcal{T}\) on \(\mathcal{B}\).

(i) There is a natural isomorphism \((\mathcal{T}^\mathbb{N})^\text{codiag} \simeq (\mathcal{T}^\text{codiag})^\mathbb{N}\) of weakly variable topos on \(\text{Diag}(\mathcal{B})\).

(ii) If \(\mathcal{T}\) is prevariable (resp. quasi-variable, variable) then so is \(\mathcal{T}^\mathbb{N}\).

(iii) The functor \(\xi \mapsto \xi_k\) defines a morphism of weakly variable topos \(i_k: \mathcal{T} \to \mathcal{T}^\mathbb{N}\).
(iv) The functor $\xi \mapsto (k \mapsto \xi)$ defines a morphism of weakly variable topoi $l : \mathcal{T}^N \to \mathcal{T}$.

Proof. Only (i) deserves an explanation. If $\mathcal{T}(X) \simeq \mathcal{P}(X)$ then $\mathcal{T}(X)^N$ is naturally equivalent to the category of sheaves on $\mathcal{P}(X) \times \mathbb{N}^{op}$ endowed with the pretopology for which $\text{Cov}(U, k)$ is the set of families $((U, k) \to (U, k))_i$ with $(U_i \to U)_i$ in $\text{Cov}(U)$. The statement results from this.

The following lemma will often be used implicitly.

Lemma 2.18. Let $g : \mathcal{T} \to \mathcal{T}'$ denote a weak morphism of weakly variable topoi on $\mathcal{B}$. Consider rings $A$, $A'$ in the respective total topos of $\mathcal{T}^N$, $\mathcal{T}'^N$ and a morphism of rings $A' \to g_* A$.

(i) Consider a morphism $f : X/\Delta \to X'/\Delta'$ in $\text{Diag}(\mathcal{B})$. The functor $Rf_* : D^+(\mathcal{T}_{cof}^N(X), A_{.,x}) \to D^+(\mathcal{T}_{cof}^N(X'), A'_{.,x'})$ can be computed $k$ by $k$:

$$i_k^{-1}Rf_* M \simeq Rf_* M_k.$$

(ii) Consider $X/\Delta$ in $\text{Diag}(\mathcal{B})$. The functor $Rg_{X,*} : D^+(\mathcal{T}_{cof}^N(X), A_{.,x}) \to D^+(\mathcal{T}_{cof}^N(X), A'_{.,x'})$ can be computed $k$ by $k$:

$$i_k^{-1}Rg_{X,*} M \simeq Rg_{X,*} M_k.$$

Proof. We may view $\mathcal{T}(X)^N$ as the total topos of the constant variable topos $k \mapsto \mathcal{T}(X)$ on $\mathbb{N}^{op}$. With this in mind (i) (resp. (ii)) is a formal consequence of 2.14 (i) (resp. 2.15 (i)) under the additional assumption that $\mathcal{T}$ is variable (resp. that $\mathcal{T}$, $\mathcal{T}'$ are prevariable and that $g$ is induced by a premorphism of pretopologies). The result still holds without this assumption however because the category $\mathcal{I}$ of modules $M$ with each $M_k$ injective has the following properties: 1) it is cogenerating (use the monomorphism $M \mapsto (k \mapsto \prod_{k' \leq k} M_{k'})$) and 2) bounded below complexes with objects in $\mathcal{I}$ which are acyclic are sent by $f_*$ and $g_{X,*}$ to acyclic complexes.

2.3. Usual sites for log schemes.

2.3.1. Let $\text{top}$ denote a property for morphisms of schemes which is a stable under isomorphism, base change and composition. We say that $\text{top}$ has fiber products if $X \times_Y Z$ is $\text{top}$ over $S$ whenever $X/S$, $Y/S$, $Z/S$ are $\text{top}$.

We use the notation $\mathcal{S}ch^2$ for the category of fine log schemes as defined in [Ka2] and we look at $\mathcal{S}ch$ as a full subcategory of $\mathcal{S}ch^2$ by endowing any scheme with the trivial log structure. For $X^2$ in $\mathcal{S}ch^2$, we let $X$ denote the underlying scheme. Finally we say that a morphism of $\mathcal{S}ch^2$ is $\text{top}$ if the underlying morphism of $\mathcal{S}ch$ is $\text{top}$.

Definition 2.19. Consider $X^2$ in $\mathcal{S}ch^2$.

(i) The $\sharp$-big top pretopology $\mathcal{TOP}^\sharp(X^2)$ is the category $\mathcal{S}ch^2/X^2$ together with surjective families of strict top morphisms as coverings. The associated topos $X^2_{\mathcal{TOP}^\sharp}$ is called the $\sharp$-big topos of $X^2$.

(ii) The big top pretopology $\mathcal{TOP}(X^2)$ is the full subcategory of $\mathcal{S}ch^2/X^2$ formed by the strict $Y^2/X^2$'s together with surjective families of strict top morphisms as coverings. The associated topos $X^2_{\mathcal{TOP}}$ is called the big top topos of $X^2$.

(iii) The small top pretopology $\text{top}(X^2)$ is the full subcategory of $\mathcal{S}ch^2/X^2$ formed by the strict top $Y^2/X^2$'s together with surjective families of strict top morphisms as coverings. The associated topos $X^2_{\text{top}}$ is called the small top topos of $X^2$.
The $\sharp$-big and big (resp. small) top pretopologies are pseudo-functorial in the sense of morphisms (resp. quasi-morphisms) with respect to $X^\sharp$ and thus give rise to variable topoi (resp. a quasi-variable topos) $T = (-)_{TOP^\sharp}$, $(-)_{TOP}$ (resp. $(-)_{top}$ which is variable if top has fiber products) on $B = \mathcal{S}ch^\sharp$.

The inclusions of sites viewed as continuous functors moreover give rise to

\[ (-)_{TOP^\sharp} \xrightarrow{p} (-)_{TOP} \xrightarrow{p} (-)_{top} \]

where the first $p$ is a morphism (2.12 (ii)) and the second $p$ is associated to a quasi-morphism (2.12 (i)). For a fixed $X^\sharp$, both $p_{X^\sharp}$'s admit a right section (ie. $r_{X^\sharp}^{-1} = p_{X^\sharp}^*$).

It follows in particular that the functors $p^{-1}$ are fully faithful. The morphisms $r_{X^\sharp}$ are not pseudo-functorial with respect to $X^\sharp$ however.

If $top_1$ is finer than $top_2$ then we have the natural morphism

\[ (-)_{TOP^1} \xrightarrow{\epsilon} (-)_{TOP^2} \]

The same holds for $TOP$ and $top$ except that in the latter case $\epsilon$ might not be a morphism if $top_2$ does not have fiber products (it is nevertheless associated to a quasi-morphism).

The weak morphisms $\epsilon$ and $p$ canonically pseudo-commute to each other.

2.3.2. The properties used most frequently as top in this paper are the following:

- fl means flat, locally of finite presentation.
- syn means of complete intersection [EGA4-IV] 19.3.1, 19.3.6, 19.3.7,
- et means étale,
- zar means open immersion,

Note that $X^\sharp_{FL}$ is thus equivalent to the usual fppf topos of the underlying scheme as defined e.g. in [SGA3-I] IV, 6.3.

We have morphisms of topoi

\[ (-)_{FL} \xrightarrow{\epsilon} (-)_{SYN} \xrightarrow{\epsilon} (-)_{ET} \xrightarrow{\epsilon} (-)_{ZAR} \]

and similarly for big strict or small topos (except that $(-)_{fl} \to (-)_{syn}$ is only a quasi-morphism).

2.3.3. If fl $\preceq$ top then $U^\sharp \mapsto \Gamma(U, \mathcal{O}_U)$ naturally defines a sheaf of rings on $TOP^\sharp(Spec(\mathbb{Z}))$ (here $\mathcal{O}_U$ denotes the structural ring of the scheme $U$). We use the following notation.

**Definition 2.20.** Let $X^\sharp$ in $\mathcal{S}ch^\sharp$. The ring of $X^\sharp_{TOP^\sharp}$, $X^\sharp_{TOP}$ or $X^\sharp_{top}$ obtained by restriction of the above sheaf of rings is called the structural ring and is denoted $\mathcal{O}$.

The quasi-variable topos defined in 2.3.1 are naturally ringed by $\mathcal{O}$ and the quasi-morphisms $p$, $\epsilon$ are naturally ringed as well.

2.4. Crystalline sites for log schemes.

2.4.1. We review crystalline sites in the absolute case. Let $\Sigma_k$ denote $Spec(\mathbb{Z}/p^k)$ if $1 \leq k < \infty$ and $Spf(\mathbb{Z}_p)$ if $k = \infty$. We denote $\mathcal{S}ch^\sharp/\Sigma_\infty$ or $\mathcal{S}ch^\sharp_{p, nil}$ the full subcategory of $\mathcal{S}ch^\sharp$ formed by the fine log schemes on which $p$ is locally nilpotent. All divided powers and divided power envelopes considered are implicitly compatible with the divided power structure of $(\mathbb{Z}_p,(p))$.
Definition 2.21. Let $X^\sharp$ in $\text{Sch}^\sharp/\Sigma_1$, $\Sigma = \Sigma_k$ with $1 \leq k \leq \infty$. Consider $\text{top}$ as in \[2.3.1\] and assume $fl \preceq \text{top} \preceq \text{zar}$.

(i) A dp-thickening in $\text{Sch}^\sharp/\Sigma$ is a quadruple $(U^\sharp, T^\sharp, \iota, \gamma)$ where $U^\sharp, T^\sharp$ are in $\text{Sch}^\sharp/\Sigma$, $\iota : U^\sharp \to T^\sharp$ is an exact closed immersion and $\gamma$ is a divided power structure on the closed immersion $U \to T$ underlying $\iota$. We often use simplified notations such as $(U^\sharp, T^\sharp)$ or even $T^\sharp$. A morphism of dp-thickenings is a couple $f = (f_U, f_T)$ where $f_U$, $f_T$ are compatible morphisms of log schemes and $f_T$ is compatible with divided power structures.

(ii) We say that a morphism of dp-thickenings $f : (U^\sharp, T^\sharp) \to (U'^\sharp, T'^\sharp)$ is cartesian if the underlying commutative square

$$
\begin{array}{ccc}
U'^\sharp & \longrightarrow & T'^\sharp \\
\downarrow f_U & & \downarrow f_T \\
U^\sharp & \longrightarrow & T^\sharp
\end{array}
$$

is cartesian in $\text{Sch}^\sharp$. We say that $f$ is top cartesian if $f_T$ is moreover strict and top.

(iii) The $\sharp$-big crystalline pretopology $\text{CRY} S^\sharp_{\text{top}}(X^\sharp/\Sigma)$ is defined as follows. An object $(U^\sharp/X^\sharp, T^\sharp, \gamma, \iota)$ of the underlying category is a dp-thickening in $\text{Sch}^\sharp/\Sigma$ together with a morphism $U^\sharp \to X^\sharp$. A morphism is a morphism $(f_U, f_T)$ of dp-thickenings such that $f_U$ is an $X^\sharp$-morphism. A covering is a surjective family of top cartesian morphisms. The associated topos $(X^\sharp/\Sigma)_{\text{CRY} S^\sharp_{\text{top}}}$ is called the $\sharp$-big crystalline top topos.

(iv) The big crystalline pretopology $\text{CRY} S_{\text{top}}(X^\sharp/\Sigma)$ is the full subcategory of $\text{CRY} S^\sharp_{\text{top}}(X^\sharp/\Sigma)$ formed by the $(U^\sharp/X^\sharp, T^\sharp, \gamma, \iota)$‘s with $U^\sharp/X^\sharp$ strict together with surjective families of top cartesian morphisms as coverings. The associated topos $(X^\sharp/\Sigma)_{\text{CRY} S_{\text{top}}}$ is called the big crystalline top topos.

(v) The small crystalline pretopology $\text{cryst}_{\text{top}}(X^\sharp/\Sigma)$ is the full subcategory of $\text{CRY} S^\sharp_{\text{top}}(X^\sharp/\Sigma)$ formed by the $(U^\sharp/X^\sharp, T^\sharp, \gamma, \iota)$‘s with $U^\sharp/X^\sharp$ strict top together with surjective families of top cartesian morphisms as coverings. The associated topos $(X^\sharp/\Sigma)_{\text{cryst}_{\text{top}}}$ is called the small crystalline top topos.

As in the case of usual topoi we have

$$
(X^\sharp/\Sigma)_{\text{CRY} S^\sharp_{\text{top}}} \xrightarrow{p} (X^\sharp/\Sigma)_{\text{CRY} S_{\text{top}}} \xrightarrow{p} (X^\sharp/\Sigma)_{\text{cryst}_{\text{top}}}
$$

where the first $p$ is a morphism, the second one is a quasi-morphism (exercise) and both admit a right section $r$.

For $1 \leq k \leq k' \leq \infty$ the inclusion of $\text{CRY} S^\sharp_{\text{top}}(X^\sharp/\Sigma_k)$ into $\text{CRY} S^\sharp_{\text{top}}(X^\sharp/\Sigma_{k'})$ is cocontinuous and thus induces a morphism

$$
(X^\sharp/\Sigma_k)_{\text{CRY} S^\sharp_{\text{top}}} \xrightarrow{\iota_{k,k'}} (X^\sharp/\Sigma_{k'})_{\text{CRY} S^\sharp_{\text{top}}}
$$

We will also use the simplified notation $\iota_k := \iota_{k,\infty}$. The same holds for $\text{CRY} S$ or $\text{cryst}$.

If $\text{top}_1$ is finer than $\text{top}_2$ then we have a natural morphism

$$
(X/\Sigma)_{\text{CRY} S^\sharp_{\text{top}_1}} \xrightarrow{\epsilon} (X/\Sigma)_{\text{CRY} S^\sharp_{\text{top}_2}}
$$
The same holds for \( CRY S \) or \( crys \) except that in the latter case \( \epsilon \) might not be a morphism if \( top_2 \) does not have fiber products (it is nevertheless always a quasi-morphism). The weak morphisms \( \epsilon \) and \( p \) canonically pseudo-commute to each other.

### 2.4.2

The forgetful functor \( CRY S^2_{TOP}(X^2/\Sigma) \to TOP^2(X^2), (U^2/X^2, T^2, \iota, \gamma) \mapsto U^2/X^2 \) is a morphism of pretopologies and thus induces a morphism of topoi \( i \) as follows:

\[
\begin{array}{ccc}
X^2_{TOP} & \xrightarrow{i} & (X^2/\Sigma)_{CRY S_{top}} \\
\downarrow & & \downarrow \\
\end{array}
\]

We do not know whether or not the forgetful functor is cocontinuous as well for an arbitrary \( top \). When this is the case it also induces a morphism \( u \) as above which is a right retraction for \( i \) (ie. \( i_* = u^{-1} \), \( u\iota \simeq i\iota \)). The situation is similar for big and small topoi.

**Lemma 2.22.** The forgetful functor is cocontinuous (and thus induces a morphism \( u \)) in the cases \( top = z ar, et \) or \( syn \).

**Proof.** We only treat the case of \( z ar \)-big sites. The other cases are similar. Consider the following property:

\((lift)\) For every \((U^2, T^2)\) in \( CRY S^2_{top}(X^2/\Sigma) \) and every strict \( top \) morphism \( V^2 \to U^2 \) there exists a surjective family of \( top \) morphisms \((V_i \to V)\) such that each \( V_i/U \) admits a lifting to a \( top \) morphisms \( T_i/T \).

Remark that if we set \( T^2_i := T^2 \times_T T_i \) and \( V^2_i := V^2 \times_V V_i \) and if we endow the closed immersions \( \iota_i : V_i \to T_i \) with the divided powers extending those of \( V \to T \) (using [BO] 3.22 and flatness of \( T_i/T \)) then the family \( ((V^2_i, T^2_i) \to (U^2, T^2)) \) is a covering in the sense of [2.21](iii). The property \((lift)\) would thus imply the desired cocontinuity: any covering of \( TOP(X^2) \) can be refined by the image of a covering in \( CRY S^2_{top}(X^2/\Sigma) \) under the forgetful functor.

Let us check \((lift)\) in each case. If \( top = z ar \) then \( V/U \) itself admits a lifting to \( T \) namely the open subscheme of \( T \) which has the same underlying topological space as \( V \). The case \( top = et \) is similar using the fact that the categories of \( et \)ale schemes over \( U \) and \( T \) are naturally equivalent [SGA4-II] VIII, 1.1. Turn to the case \( top = syn \). Recall that syntomic \( T \)-schemes are characterized by the property that any (one is enough) closed embedding into a smooth \( T \)-scheme is transversally regular relatively to \( T \) [EGA4-IV] 19.3.7. Replacing \( T, U \) and \( V \) by open coverings we may assume that these are affine schemes and that \( V \) is a closed subscheme into an open of \( A^d_{T/U} \) defined by a sequence \( x = (x_1, \ldots, x_n) \) which is transversally regular relatively to \( U \). Then any lift of the sequence \( x \) to the corresponding open of \( A^d_{T} \) is transversally regular relatively to \( T \) (use [EGA4-IV] 19.8.2 to replace \( U \) and \( T \) with a noetherian affine schemes and then apply [Mi1] I, 2.5, 2.6 (d)) and thus gives rise to the desired syntomic lift of \( V/U \).

\[\Box\]

**Lemma 2.23.** Let \( top = z ar, et \) or \( syn \) then the functor \( \iota_{k, k'} \) in (43) admits a right adjoint and is thus in particular exact.

**Proof.** It suffices to prove that the continuous functor \( T^2 \mapsto T^2_k \) inducing \( \iota_{k, k'} \) is cocontinuous as well. The proof is similar to 2.22.
2.4.3. Keep the notations and assumptions of [2.22]. If $T^\sharp = (U^\sharp, T^\sharp, t, \gamma)$ is an object of $\mathcal{C}RYS^S_{\mathrm{top}}(X^\sharp/\Sigma)$, we have natural quasi-morphisms of topoi

\[(X^\sharp/\Sigma)_{\mathcal{C}RYS^S, \mathrm{top}} \xrightarrow{f_{T^\sharp}} (X^\sharp/\Sigma)_{\mathcal{C}RYS^S, \mathrm{top}}/T^\sharp \xrightarrow{\lambda_{T^\sharp}} T^\sharp_{\mathrm{top}}\]

The left morphism is sometimes denoted $f_{T^\sharp}/X^\sharp$ if a reference to $X^\sharp$ is useful. It is the usual localization morphism. Recall that the topos on the middle naturally identifies to the topos of sheaves on $\mathcal{C}RYS^S_{\mathrm{top}}(X^\sharp/\Sigma)/T^\sharp$ (induced topology). Modulo this identification, the right quasi-morphism is induced by the continuous functor $T^\sharp/T^\sharp \mapsto (U^\sharp \times_{T^\sharp} T^\sharp, T^\sharp, t', \gamma')$ where $t'$ is the base change of the $t$ by the flat morphism $T^\sharp/T^\sharp$ and $\gamma'$ is the unique divided power structure on $t'$ extending $\gamma$ [BO] 3.21. Both $f_{T^\sharp}$ and $\lambda_{T^\sharp}$ are pseudo-functorial with respect to $T^\sharp$.

In the case $\mathcal{C}RYS$ or $crys$ we have similarly a morphism $f_{T^\sharp}$ and a quasi-morphism $\lambda_{T^\sharp}$. Both are pseudo-compatible with $p$ in the obvious way.

**Definition 2.24.** For $T^\sharp$ as above and $F$ in $(X^\sharp/\Sigma)_{\mathcal{C}RYS^S, \mathrm{top}}$ (resp. $(X^\sharp/\Sigma)_{\mathcal{C}RYS^S, \mathrm{top}}$, $(X^\sharp/\Sigma)_{\mathrm{crys}, \mathrm{top}}$) we use the following notations:

- $F_{|T^\sharp} := f_{T^\sharp}^{-1}F$ is the restriction of $F$ to $T^\sharp$.
- $F_{|T^\sharp} := \lambda_{T^\sharp}^{-1}F_{|T^\sharp}$ is the realization of $F$ on $T^\sharp$.

Some properties of these functors and their relation to the morphisms ([15]) will be explained in appendix [9.2]

2.4.4. The rules $(U^\sharp, T^\sharp) \mapsto \Gamma(T, O_T)$ and $(U^\sharp, T^\sharp) \mapsto \Gamma(U, O_U)$ naturally define two sheaves of rings on $\mathcal{C}RYS^S(\Sigma_1/\Sigma_\infty)$.

**Definition 2.25.** Let $X^\sharp$ in $\mathcal{S}ch^s$, $1 \leq k \leq \infty$.

(i) The ring of $(X^\sharp/\Sigma_k)_{\mathcal{C}RYS^S, \mathrm{top}}$, $(X^\sharp/\Sigma_k)_{\mathcal{C}RYS^S, \mathrm{top}}$, $(X^\sharp/\Sigma_k)_{\mathrm{crys}, \mathrm{top}}$ obtained by restriction of the first sheaf of rings above is called the structural ring and is denoted $O$, or $O_{X^\sharp/\Sigma_k}$ depending on the context.

(ii) The ring of $(X^\sharp/\Sigma_k)_{\mathcal{C}RYS^S, \mathrm{top}}$, $(X^\sharp/\Sigma_k)_{\mathcal{C}RYS^S, \mathrm{top}}$, $(X^\sharp/\Sigma_k)_{\mathrm{crys}, \mathrm{top}}$ obtained by restriction of the second sheaf of rings above is denoted $G_a$.

The weakly variable topoi $(-/\Sigma_k)_{\mathcal{C}RYS^S, \mathrm{top}}$, $(-/\Sigma_k)_{\mathcal{C}RYS^S, \mathrm{top}}$, $(-/\Sigma_k)_{\mathrm{crys}, \mathrm{top}}$ (prop. [9.2]) and the weak morphisms $p, e, i$ (prop. [9.3][1]) are naturally ringed by $O$. The morphisms ([43] and [16]) and their big or small analogues are naturally ringed by $O$ as well.

**Definition 2.26.** Assume that the morphism $u : (X^\sharp/\Sigma_k)_{\mathcal{C}RYS^S, \mathrm{top}} \rightarrow X^\sharp_{\mathrm{TOP}_5}$ exists. We set

$$O^\mathrm{crys}_k := u_\ast O$$

We use similar notations in the setting of big or small topoi. In the case $k = \infty$ we also use the notation $O^\mathrm{crys}$ instead of $O^\mathrm{crys}_\infty$.

2.5. 1-motives and $p$-divisible groups.

We review some well known definitions and properties concerning finite locally free groups, $p$-divisible groups, abelian schemes and 1-motives.
2.5.1. Part of this reminder is devoted to put an exact structure on the relevant categories. The reader may consult [Bu] for a reminder of basic facts concerning exact categories. Let us introduce a terminology which is adapted both for our present purpose and for later ones as well ([4.5]). Consider a functor $F$ between exact categories $C_1$, $C_2$ whose exact structures are both denoted $e$ for simplicity. We say that $F$ is $e$-exact if it sends short $e$-exact sequences of $C_1$ to short $e$-exact sequences of $C_2$. We say that it reflects $e$-exactness if a short sequence of $C_1$ whose image by $F$ is $e$-exact in $C_2$ is already $e$-exact. A fully $e$-exact subcategory of $C_1$ is a full additive subcategory which is closed by extensions and equipped with the exact structure induced by $e$.

If $C_1$ and $C_2$ are abelian and $e$ denotes their natural exact structure then $F$ is $e$-exact if and only if it is exact in the usual sense (ie. if it commutes to finite limits). It moreover reflects $e$-exactness if and only if it is moreover faithful. A fully abelian subcategory of an abelian category is a full subcategory which is abelian, closed by extensions and which is such that the inclusion functor is exact. In other words it is a fully $e$-exact subcategory which is stable by kernels and cokernels.

2.5.2. Let us fix some terminology.

**Definition 2.27.** Let $X$ be a scheme.

(i) A finite locally free group over $X$ is an abelian group of $X_{FL}$ which is represented by a finite locally free group scheme.

(ii) A $p$-divisible group $G$ over $X$ is an abelian group of $X_{FL}$ which is such that $p : G \to G$ is epimorphic, $\lim\text{ind} G_{p^k} \simeq G$ and each $G_{p^k}$ is a finite locally free group.

**Remark 2.28.**

(i) Finite locally free is equivalent to finite flat and locally of finite presentation.

(ii) Finite implies affine [EGA2] 6.1.4.

The Cartier dual of a finite locally free (resp. $p$-divisible) group $G$ is defined as follows:

\[(47) \quad G^* := \text{Hom}(G, \mathbb{G}_m)\]

\[(48) \quad (\text{resp. } G^* := \lim_k (G_{p^k})^*)\]

Using that $\text{Ext}^1(G, \mathbb{G}_m)$ vanishes for any finite locally free group $G$ [SGA7-I] VIII, 3.3.1 we find an exact sequence

\[0 \to (G_{p^k})^* \overset{(p^l)^*}{\to} (G_{p^{k+1}})^* \overset{(p^k)^*}{\to} (G_{p^{k+1}}^*)^* \to (G_{p^k})^* \to 0\]

for $G$ a $p$-divisible group. It follows in particular that $(G_{p^k})^*$ is naturally isomorphic to $(G^*)_{p^k}$ and that $G^*$ is a $p$-divisible group. This implies that the obvious biduality isomorphism $G \simeq G^{**}$ for finite locally free groups naturally extends to $p$-divisible groups.

**Lemma 2.29.**

(i) Finite locally free groups form a fully $e$-exact subcategory of the category of abelian groups of $X_{FL}$. This is in fact a fully abelian subcategory if $X = \text{Spec}(K)$ for some field $K$.

(ii) $p$-divisible groups form a fully $e$-exact subcategory of the category of abelian groups of $X_{FL}$.

Proof. (i) Let us prove the first assertion. Consider an exact sequence $0 \to G \to G' \to G'' \to 0$ where $G$ and $G''$ are finite locally free groups. Looking at $G'$ as an object of
we find that it is a torsor under the pullback of $G$ to $G''$. It is thus represented by a finite locally free group scheme over $G''$ in virtue of [SGA1] VIII, 7.9. We conclude by composition with the finite locally free morphism $G'' \to X$. The second assertion follows from [Gr3] 212-17, cor. 7.3. Statement (ii) follows from (i).

Lemma 2.30. Finite locally free groups are syntomic.

Proof. Since finite locally free groups are flat and locally of finite presentation we may assume [EGA4-IV], 19.3.6 that $X = Spec(K)$ for some field $K$. By descent we may furthermore assume that $K$ is algebraically closed [EGA4-IV] 19.3.9 (ii). In that case the category of finite locally group is fully abelian in $Ab(X_{FL})$ (2.29 (i)). Each one of its objects admits moreover a composition series with simple quotients, those being isomorphic to either one of the groups $\mu_p$, $\alpha_p$, $\mathbb{Z}/l$ ($l$ any prime number) [Oo] I, 2. We may thus conclude by dévissage since the property of being syntomic is preserved by forming extensions (use the same arguments than in the proof of 2.29).

2.5.3. The following definitions are standard.

Definition 2.31. Let $X$ be a scheme.

(i) An abelian scheme (or abelian variety if $X$ is the spectrum of a field) is an abelian group of $X_{FL}$ which is representable by a proper smooth $X$-group scheme whose fibers are geometrically connected.

(ii) A torus is an abelian group of $X_{FL}$ which is representable and locally isomorphic to a finite direct sum of $\mathbb{G}_m$’s.

(iii) A semi-abelian scheme is an abelian group of $X_{FL}$ which is representable by a smooth $X$-group scheme whose fibers are successive extensions of an étale finite locally free group, an abelian variety and a torus.

(iv) A twisted constant group is an abelian group of $X_{FL}$ which is representable and locally isomorphic to a finite direct sum of $\mathbb{Z}$’s.

(v) A 1-motive is a complex $[\Gamma \to G]$ of abelian groups of $X_{FL}$ placed in degrees $[0, 1]$ where $\Gamma$ is a twisted constant group and $G$ is the extension of an abelian scheme by a torus.

Remark 2.32. (i) In virtue of [SGA3-II] X, 4.5 (resp. [SGA3-II] X, 5.3) a torus (resp. twisted constant groups) in fact becomes isomorphic to a finite direct sum of $\mathbb{G}_m$’s (resp. $\mathbb{Z}$’s) over a suitable étale covering.

(ii) If $T$ is a torus and $A$ is an abelian scheme then every morphism $T \to A$ or $A \to T$ is trivial (use [SGA3-II] XV, 8.3, [Mi3] 3.1, 3.8 in the first case and [SGA7-I] VIII, 3.2 in the second case).

(iii) If $G$ is an extension of an abelian scheme $B$ by a torus $T$ then $G$ is automatically representable by a smooth group scheme [SGA1] VIII, 7.9. We know from [SGA3-II] XV, 8.17], [SGA3-II] XII, 7.1, b, that $G$ admits a unique maximal subtorus in the sense of [SGA3-II] XII, def. 1.3 (see also rem. 1.4, a’ of loc. cit.). Using (ii) we find that this maximal subtorus is nothing but $T$ and that it varies functorially with respect to $G$.

Lemma 2.33. Assume that $X$ is regular. The category $\mathcal{M}(X)$ of 1-motives is fully e-exact in the category of complexes of abelian groups of $X_{FL}$.
Proof. We have to show that \( \mathcal{M}(X) \) is stable by extensions. The category of twisted constant groups is stable by extensions thanks to [SGA3-II] X, 5.4. The only point deserving explanations is thus the following

Claim. Consider an exact sequence 0 \( \rightarrow \) \( G' \rightarrow G \rightarrow G'' \rightarrow 0 \) of abelian groups in \( X_{FL} \). If \( G', G'' \) are respectively an extension of an abelian scheme \( B', B'' \) by a torus \( T', T'' \) then \( G \) is an extension of an abelian scheme \( B \) by a torus \( T \) as well. We have moreover \( \dim_XT = \dim_XT' + \dim_XT'' \) and \( \dim_XB = \dim_XB' + \dim_XB'' \).

First we note that \( G \) is representable by a smooth group scheme (use [SGA1] VIII, 7.9 to reduce to the case where \( G' \) is an abelian scheme and then conclude by [Ra1] XIII, 2.6).

Let us now prove the claim assuming that \( G \) has a maximal subtorus, say \( T \). In virtue of [SGA3-II] XV, lem. 8.3 the kernel \( H' \) of \( T \rightarrow G'' \) is a multiplicative subgroup of finite type in \( G' \) [SGA3-II] IX, 1.1, 1.2. It clearly contains \( T' \) and the quotient \( H'/T' \) is a multiplicative subgroup of finite type in \( B' \) [SGA3-II] IX, 2.7. It follows easily from 2.32 (ii) that it is in fact finite locally free. The image of \( T \rightarrow G'' \) on the other hand is a subtorus of \( G'' \) [SGA3-II] XV, lem. 8.3 and is in fact equal to \( T'' \) [SGA3-II] XII, 7.1, e). Forming cohomology of the vertical complexes in the following commutative diagram with exact lines:

\[
\begin{array}{ccc}
0 & \rightarrow & T' \rightarrow G' \rightarrow B' \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T \rightarrow G \rightarrow G/T \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T'' \rightarrow G'' \rightarrow B'' \rightarrow 0
\end{array}
\]

produces an exact sequence

\[
0 \rightarrow H'/T' \rightarrow B' \rightarrow G/T \rightarrow B'' \rightarrow 0
\]

Since \( X \) is normal a result of Grothendieck [Ra1] XI, 1 ensures that \( B' \) is projective. It thus follows from [Ra2] 5. thm 1, applications (a) (iii) that the quotient of \( B' \) by the finite locally free group \( H'/T' \) is an abelian scheme. As a result \( B := G/T \) is the extension of two abelian schemes and is thus an abelian scheme as well (using [Ra1] XIII, 2.6 again). The assertion about dimensions is clear from the exact sequence (50) since \( \dim_XH'/T' = 0 \).

The above proof applies in particular to the case where \( X \) is the spectrum of a field. If now \( X \) is regular we have to show that \( G \) has a maximal subtorus. This follows from [SGA3-II] XV, 8.17 thanks to the dimension formulae in the fibers.

\[\square\]

Lemma 2.34. (i) Consider abelian schemes \( A, A' \) and an isogeny \( f : A \rightarrow A' \) (ie. \( f \) is a finite faithfully flat morphism of group schemes [SGA3-III] XXII, 4.2.9). The kernel of \( f \) is a finite locally free group.

(ii) If \( M = [\Gamma \rightarrow G] \) is a \( 1 \)-motive then \( M_{p^\infty} := Q_p/Z_p \otimes^L M \) is concentrated in degree 0 and is a \( p \)-divisible group. If \( X \) is regular the resulting functor

\[
(-)_{p^\infty} : \mathcal{M}(X) \rightarrow pdiv(X)
\]
is e-exact in the sense of 2.33.

(iii) Consider a semi-abelian scheme $A$. Its connected component $A^0$ (see \textit{SGA3-I} VIB def. 3.1) is semi-abelian as well. Multiplication by $p$ on $A$ is flat locally of finite presentation and locally quasi-finite. It is moreover surjective if $A = A^0$.

Proof. (i) Since $f$ is finite and flat the same is true for $\ker f$. The latter group scheme is furthermore locally of finite presentation since the unit section $X \to A'$ is a closed immersion (thanks to $A'/X$ being separated) whose ideal is finitely generated.

(ii) The case where $M = [0 \to A]$ with $A$ an abelian scheme follows from (i) thanks to the well known fact that $p : A \to A$ is an isogeny [Gr2]. The general case proceeds from there by dévissage using 2.29.

(iii) According to \textit{SGA3-I} VIB, thm. 3.10 $A^0$ is representable by an open subgroup scheme. The first statement follows from this and the definition. The other statements are follow easily from (ii) applied to the connected components of the fibers (use \textit{EGA4-III} 11.3.10 for flatness).

\[\square\]

Remark 2.35. In fact a result of Raynaud ensures that every finite locally free group arises as the kernel of an isogeny between abelian schemes after suitable Zariski localization on $X$ (\textit{BBM} 3.1.1).

Let us finally discuss the group of components when the base is a curve.

Lemma 2.36. Assume that $X$ is locally noetherian of dimension 1. Let $z : Z \to X$ denote the inclusion morphism of a reduced closed subscheme and let $U$ denote its complement. Consider a semi-abelian scheme $A/X$ whose restriction to $U$ is abelian. Its component group $\Phi := A/A^0$ verifies the following.

(i) The sheaf $z^{-1}\Phi$ is representable by an étale group scheme.

(ii) The adjunction morphism $\Phi \to z_*z^{-1}\Phi$ is invertible.

Proof. Under our assumptions $Z$ is a discrete scheme. Statement (i) thus follows simply from the fact that the formation of $A^0$ is compatible to base change (see \textit{SGA3-I} VIB prop. 3.3).

(ii) Consider the following diagram commutative diagram with exact lines in $Ab(X_{FL})$:

\[
\begin{array}{ccccccccc}
0 & \to & A^0 & \to & A & \to & \Phi & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & z_*z^{-1}A^0 & \to & z_*z^{-1}A & \to & z_*z^{-1}\Phi & \\
\end{array}
\]

It follows immediately from the definition of $A^0$ (see \textit{SGA3-I} VIB def. 3.1) that the kernel of the middle vertical morphism is contained in $A^0$. It thus remains only to observe that the right vertical morphism is epimorphic thanks to (i) and \textit{SGA1}, I, prop. 8.1.

\[\square\]

2.6. Technical remarks on certain sheaves.

We review basic facts about constant sheaves and $p$-divisible groups viewed as top sheaves on usual or crystalline sites.

2.6.1. Let us begin with constant sheaves.
Lemma 2.37. Assume that $\text{fl} \preceq \text{top} \preceq \text{zar}$.

(i) Let $X^\sharp$ in $\text{Sch}^\sharp$ and let us (temporarily) denote $M$ in $X^\sharp_{\text{TOP}}$ the constant sheaf associated to a set $M$. Then $M$ is representable by the $X^\sharp_{\text{TOP}}$-scheme $X^\sharp_{\text{TOP}}(M)$ consisting in a direct sum of copies of $X^\sharp$ indexed by $M$. The same is true in the setting of big or small topos.

(ii) Consider the morphism $i : X^\sharp_{\text{TOP}} \to (X^\sharp/\Sigma)_{\text{Crys}, \text{top}}$. Then $i_* X^\sharp(M)$ is the constant sheaf (rep. sheaf of groups) associated to $M$. The same is true in the setting of big or small topos.

(iii) The formation of constant sheaves is compatible with:

- the restriction functors $p_*$ between the $\sharp$-big, big and small crystalline or usual top topos (note that compatibility with $p^{-1}$ is tautological).
- the change of topology functors $\epsilon_*$ (compatibility with $\epsilon^{-1}$ is tautological as well).
- the functor $i_*$ from usual to crystalline topoi (compatibility with $i^{-1}$ is tautological as well).

Proof. We only prove [i] since [ii] is similar and [iii] follows from [i] and [ii]. Let $\tilde{H}^0(U,-)$ denote the Cech functor associated to a covering $U = (U^\sharp_\lambda \to U^\sharp)_{\lambda \in \Lambda}$ of some $U^\sharp$ in $\text{Sch}^\sharp$ and let $\tilde{H}^0(U,-)$ denote the direct limit of the previous functors for $U$ running in the category of coverings of $U^\sharp$. If $M_0$ denotes the constant presheaf associated to $M$ then $M_1 := U^\sharp \mapsto \tilde{H}^0(U^\sharp, M_0)$ is the separated presheaf associated to $M$ and $U^\sharp \mapsto \tilde{H}^0(U^\sharp, M_1)$ is the sheaf $M$ associated to $M$. These may be computed as follows. The value of $M_1$ on $U^\sharp$ is 1 if $U$ is the empty scheme and $M$ otherwise. Next $\tilde{H}^0(U, M_1)$ is a product of copies of $M$ indexed by the quotient $\overline{\Lambda}$ of $\Lambda$ by the equivalence relation $\lambda \simeq \lambda' \iff \exists n \geq 1, \exists \lambda_0, \ldots, \lambda_n, \lambda_0 = \lambda, \lambda_n = \lambda', \forall 0 \leq k \leq n - 1, U_{\lambda_k} \times_U U_{\lambda_{k+1}} \neq \emptyset$. Consider $\lambda \in \Lambda$ with equivalence class $\overline{\lambda}$ in $\overline{\Lambda}$. Define $U^\sharp_{\lambda}$ as the open sub log scheme of $U^\sharp$ whose underlying topological space is the union of the images of the $U_{\lambda'}$'s with $\lambda' \simeq \lambda$. This defines an open partition $\mathcal{P}_U$ of $U^\sharp$. A morphism $f : U^\sharp \to X^\sharp_{\text{TOP}}(M)$ on the other hand defines an open partition $\mathcal{P}_f$ of $U^\sharp$ as well. With this in mind we find that the natural morphism $\tilde{H}^0(U, M_1) \to \text{Hom}(U^\sharp, X^\sharp_{\text{TOP}}(M))$ is injective and that $f$ is in its image if and only if $\mathcal{P}_f$ is refined by $\mathcal{P}_U$. The result follows since every open partition is of the form $\mathcal{P}_U$ for an adequate covering $U$.

\[\square\]

In view of this lemma the constant sheaf on a usual or crystalline site represented by a set or group $M$ will simply be denoted $M$ from now on.

2.6.2. It is sometimes useful to view locally free groups or $p$-divisible groups as sheaves on big or small top (crystalline or usual) sites rather than on the big fl-usual site. The following lemma will be helpful.

Lemma 2.38. Assume that $\text{fl} \preceq \text{top} \preceq \text{top}' \preceq \text{zar}$ and let $X^\sharp$ in $\text{Sch}^\sharp$. The functors $\epsilon_* : X^\sharp_{\text{TOP}} \to X^\sharp_{\text{TOP}}$ and $i_* : X^\sharp_{\text{TOP}} \to (X^\sharp/\Sigma)_{\text{Crys}, \text{top}}$ commute to filtrant inductive limits. The same holds in the context of big or small sites.

Proof. Consider a filtrant inductive system $(F_i)$ in $X^\sharp_{\text{TOP}}$ (resp. $(X^\sharp/\Sigma)_{\text{Crys}, \text{top}}$). Let $F$ denote the direct limit computed in the category of presheaves and let $G$ denote the associated sheaf for the zar topology.
Fact. The presheaf $G$ on $\text{TOP}^\sharp(X^\sharp)$ (resp. $\text{CRYS}^\sharp(X^\sharp/\Sigma)$) is in fact a sheaf (for $\text{top}$)
It is in particular the direct limit computed in $X^\sharp_{\text{TOP}}$ (resp. $(X^\sharp/\Sigma)_{\text{CRYS,top}}$).

We may use the characterization of $\text{top}$ sheaves given in [SGA3-1] IV, 6.2.1, 6.2.3 in
the following situation (with the notations of loc. cit.): $\mathcal{C}$ is the category underlying $\text{TOP}^\sharp(X^\sharp)$ (resp. $\text{CRYS}^\sharp(X^\sharp/\Sigma)$), $\mathcal{C}'$ is the full subcategory of $\mathcal{C}$ formed by the $U^\sharp/X^\sharp$'s
where $U$ is an affine scheme (resp. the $(U^\sharp/X^\sharp, T^\sharp, \iota, \gamma)$'s where $T$ is an affine scheme), $P$
is the set of open coverings, $P'$ is the set of finite surjective families of $\text{top}$ strict morphisms
(resp. of cartesian $\text{top}$ strict morphisms) of finite presentation in $\mathcal{C}'$. We must check that
$G$ satisfies the descent condition for the families which belong to $P$ or $P'$. To start with
we notice that $F$ verifies the descent condition for the families which are elements of $P'$.
Indeed in the category of sets, filtrant inductive limits commute to finite projective limits.
The fact will thus be proven if we check that $F$ and $G$ coincide on the objects of $\mathcal{C}'$. Now
$F$ is clearly a separated presheaf for $\text{zar}$ and $G$ can thus be computed from $F$ by applying
the Cech functor only once. The Cech computation in question can be done using only
finite families thanks to quasi-compactness and the fact follows.

It follows immediately from this fact that $\epsilon_*$ commutes to filtrant inductive limits. The
case of $i_*$ follows easily as well using moreover that the continuous functor inducing $i_*$
is cocontinuous in the case $\text{top} = \text{zar}$.

The same proof holds verbatim if $\text{TOP}^\sharp$ and $\text{CRYS}^\sharp$ are replaced respectively with
$\text{TOP}$ and $\text{CRYS}$ or $\text{top}$ and $\text{crys}$.

\[\square\]

Corollary 2.39. Assume that $\text{fl} \leq \text{top} \leq \text{syn}$. Let $\text{flf}(X)$ and $\text{pdiv}(X)$ respectively
denote the full subcategory of $\text{Ab}(X_{\text{FL}})$ formed by finite locally free groups and $p$-divisible
groups. Consider the functor $\epsilon_*$ (resp. $p_\epsilon \epsilon_*, i_* \epsilon_*, i_* p_\epsilon \epsilon_*$) from $\text{Ab}(X_{\text{FL}})$ to $\text{Ab}(X_{\text{TOP}})$
(resp. $\text{Ab}(X_{\text{top}}), \text{Ab}((X/\Sigma)_{\text{CRYS,top}}), \text{Ab}((X/\Sigma)_{\text{crys,top}})$). We denote
$\text{flf}_{\text{TOP}}(X)$ and $\text{pdiv}_{\text{TOP}}(X)$ (resp. $\text{flf}_{\text{top}}(X)$ and $\text{pdiv}_{\text{top}}(X)$, $\text{flf}_{\text{CRYS,top}}(X/\Sigma)$ and $\text{pdiv}_{\text{CRYS,top}}(X/\Sigma)$, $\text{flf}_{\text{crys,top}}(X/\Sigma)$ and $\text{pdiv}_{\text{crys,top}}(X/\Sigma)$) the essential image of $\text{flf}(X)$ and $\text{pdiv}(X)$.

(i) Let $G$ in $\text{pdiv}(X)$. Multiplication by $p$ is epimorphic on $\epsilon_* G$ (resp. $p_\epsilon \epsilon_* G$, $i_* \epsilon_* G$, $i_* p_\epsilon \epsilon_* G$).

(ii) The restriction of the functor $\epsilon_*$ (resp. $p_\epsilon \epsilon_*, i_* \epsilon_*, i_* p_\epsilon \epsilon_*$) to $\text{flf}(X)$ or $\text{pdiv}(X)$
is fully faithful and $\epsilon$-exact.

(iii) Assume now that $\text{fl} \leq \text{top}' \leq \text{top} \leq \text{syn}$. The adjunctions $(\epsilon^{-1}, \epsilon_*)$, $(p^{-1}, p_*)$,
$(i^{-1}, i_*)$ induce equivalences as follows:

\[\text{flf}_{\text{top}}(X) \xrightarrow{(i^{-1}, i_*)} \text{flf}_{\text{crys,top}}(X/\Sigma)\]
\[\text{flf}_{\text{TOP}}(X) \xrightarrow{(i^{-1}, i_*)} \text{flf}_{\text{CRYS,top}}(X/\Sigma)\]
\[\text{flf}_{\text{TOP}}(X) \xrightarrow{(i^{-1}, i_*)} \text{flf}_{\text{crys,top}}(X/\Sigma)\]
and similarly for \( \text{pdiv} \).

Proof. \([i]\) Since \( p_* \) preserves epimorphisms it is sufficient to consider the case of \( \epsilon_*G \) and \( i_*\epsilon_*G \). We may furthermore assume that \( \text{top} = \text{syn} \). In that case \( i_* \) preserves epimorphisms as well and it suffices to look at the case of \( \epsilon_*G \). Now the result follows immediately from the fact that the morphism of group schemes representing \( p : \epsilon_*G_{p^{k+1}} \to \epsilon_*G_{p^k} \) is surjective and syntomic (indeed \( G_{p^{k+1}} \to G_{p^k} \) is a torsor of \( (G_{p^k})_{\text{FL}} \) under the syntomic group scheme \( G_p \) (2.30)).

\([ii]\) Let \( v \) denote either one of the quasi-morphisms of topoi \( \epsilon, p\epsilon, i\epsilon, i\epsilon^p \). Let us explain full faithfulness. It is sufficient to show that the adjunction morphism \( v^{-1}v_*G \to G \) is invertible whenever \( G \) is in \( \text{flf}(X) \) or \( \text{pdiv}(X) \). Thanks to (2.38) it suffices to consider the case of \( \text{flf}(X) \). Now the functors \( \epsilon_* : \text{Ab}(X_{FL}) \to \text{Ab}(X_{\text{TOP}}) \), \( i_* : \text{Ab}(X_{\text{TOP}}) \to \text{Ab}((X/\Sigma)_{\text{CRYS,top}}) \) and \( i_* : \text{Ab}(X_{\text{top}}) \to \text{Ab}((X/\Sigma)_{\text{crys,top}}) \) being fully faithful themselves we are reduced to the case of \( p_* : \text{flf}(X) \to \text{flf}(X) \). The result then follows from (2.30).

Let us explain \( e \)-exactness. Consider a short \( e \)-exact sequence \( 0 \to G' \to G \to G'' \to 0 \) of \( \text{flf}(X) \) or \( \text{pdiv}(X) \). We have to show that the image of \( G \to G'' \) by \( v_* \) is epimorphic. Passing to \( p^k \)-torsion points in the case of \( \text{pdiv}(X) \) we are reduced to the case of \( \text{flf}(X) \). Since \( p_* : \text{Ab}(X_{\text{TOP}}) \to \text{Ab}(X_{\text{top}}) \) and \( p_* : \text{Ab}((X/\Sigma)_{\text{CRYS,top}}) \to \text{Ab}((X/\Sigma)_{\text{crys,top}}) \) are exact we may restrict our attention to big topoi. We may then assume that \( \text{top} = \text{syn} \). The case \( v = \epsilon \) results from (2.30) and the case \( v = i\epsilon \) follows by exactness of \( i_* : \text{Ab}(X_{\text{SYN}}) \to \text{Ab}((X/\Sigma)_{\text{CRYS,syn}}) \).

\([iii]\) It remains to show that the functors \( e^{-1} : \text{Ab}(X_{\text{TOP}}) \to \text{Ab}(X_{FL}), p^{-1} : \text{Ab}(X_{\text{top}}) \to \text{Ab}(X_{\text{TOP}}), p^{-1} : \text{Ab}((X/\Sigma)_{\text{crys,top}}) \to \text{Ab}((X/\Sigma)_{\text{CRYS,top}}), i^{-1} : \text{Ab}((X/\Sigma)_{\text{CRYS,top}}) \to \text{Ab}(X_{\text{TOP}}) \) and \( i^{-1} : \text{Ab}((X/\Sigma)_{\text{crys,top}}) \to \text{Ab}(X_{\text{top}}) \) preserve the corresponding categories of finite locally free groups and \( p \)-divisible groups. Let \( v \) denote one of the five quasi-morphisms \( \epsilon, p, i \) involved. We have to check that the adjunction morphism \( v^{-1}v_*G \to G \) is invertible when \( G \) is a locally free group or a \( p \)-divisible group. By (2.38) we may assume that \( G \) is a finite locally free group. Aside from the cases already treated in (ii) it remains to explain the case \( v = p : (X/\Sigma)_{\text{CRYS,top}} \to (X/\Sigma)_{\text{crys,top}} \). For \( G \) in \( \text{flf}(X) \) we will show that \( p^{-1}p_*i_*G \cong i_*G \). We may always assume that \( \Sigma = \Sigma_k \) with \( 1 \leq k < \infty \) and that \( X \) (hence \( G \)) is an affine scheme. Choose a closed immersion \( G \to Y \) where \( Y = \text{Spec} (\mathbb{Z}/p^k[[x_i]_{i\in I}]) \) (\( I \) possibly infinite) and let \( (G, T), \ (G, T^{(1)}) \) respectively denote the divided power envelope of \( G \) into \( Y, Y \times Y \). The result follows from the observation that \( i_*G \) (resp. \( p_*i_*G \)) may be described as the cokernel of the couple of morphisms \( (G, T^{(1)}) \to (G, T) \) (induced by projections) in the topos \( (X/\Sigma)_{\text{CRYS,top}} \) (resp. \( (X/\Sigma)_{\text{crys,top}} \)). \( \square \)

3. Dévissage of flat cohomology

The purpose of this section is to prove that the flat cohomology of the Néron model of an abelian variety can be recovered from a suitable diagram of \( p \)-divisible groups. This will be achieved in 3.26.

3.1. Functorial mapping fibers in derived categories.
We explain a basic construction designed to avoid the difficulties caused by the lack of functoriality of mapping cones and mapping fibers with respect to morphisms in the derived category.

3.1.1. Let \([1]\) denote the category \(\{0 \rightarrow 1\}\). If \(\mathcal{C}\) is a category \(\mathcal{C}^{[1]}\) is thus the category of arrows of \(\mathcal{C}\). Given an abelian category \(\mathcal{A}\) we have an obvious functor

\[
D(\mathcal{A}^{[1]}) \longrightarrow D(\mathcal{A})^{[1]}
\]

This functor is not an equivalence in general. The objects of the source category will be called true arrows. Consider the mapping fiber functor

\[
Kom(\mathcal{A})^{[1]} \longrightarrow Kom(\mathcal{A})
\]

taking a morphism of complexes \(f: A \rightarrow B\) to the simple complex associated to \(f\) viewed as a double complex placed in degrees \([0, 1] \times -\infty, +\infty\]. The functor (52) does not induce a functor \(D(\mathcal{A})^{[1]} \rightarrow D(\mathcal{A})\). We can nevertheless make the following definition.

Definition 3.1. The functorial mapping fiber

\[
MF: D(\mathcal{A}^{[1]}) \rightarrow D(\mathcal{A})
\]
is the functor deduced from (52) by the universal property of derived categories.

By definition we thus have a tautological distinguished triangle

\[
MF(f) \longrightarrow A \longrightarrow B \xrightarrow{+1}
\]

for any true arrow \(f: A \rightarrow B\). Assume now that \(\mathcal{A}\) has enough injectives. Consider left exact functors \(F_i: \mathcal{A} \rightarrow \mathcal{A}', i = 1, 2\) and a natural transformation \(t: F_1 \rightarrow F_2\). The triple \(F = (F_1, F_2, t)\) can be viewed as a left exact functor \(A \rightarrow A^{[1]}\). Deriving, we get \(RF: D^+(\mathcal{A}) \rightarrow D^+(A^{[1]})\). Now (53) yields a distinguished triangle

\[
MF(RF(M)) \longrightarrow RF_1(M) \longrightarrow RF_2(M) \xrightarrow{+1}
\]

which is functorial with respect to \(M\) in \(D^+(\mathcal{A})\).

Lemma 3.2. Let \(\mathcal{A}, F\) as above and consider moreover left exact functors \(F_3: \mathcal{A}' \rightarrow \mathcal{A}'',\) \(F_4: \mathcal{A}'' \rightarrow \mathcal{A}\) where \(\mathcal{A}'\) and \(\mathcal{A}''\) have enough injectives.

(i) The functor \(F_3^{[1]}: \mathcal{A}^{[1]} \rightarrow \mathcal{A}'^{[1]}\) induced by \(F_3\) is right derivable. The derived functor \(RF_3^{[1]}\) can be computed componentwise. There is a canonical isomorphism

\[
RF_3(MF(f')) \simeq MF(RF_3^{[1]}(f'))
\]

which is functorial with respect to the true arrow \(f'\).

(ii) Assume that \(F_1\) and \(F_2\) send injectives of \(\mathcal{A}\) to \(F_3\)-acyclic objects. There is a canonical isomorphism

\[
RF_3^{[1]} \circ RF \simeq R(F_3^{[1]} \circ F)
\]

(iii) Assume that \(F_4\) sends injectives of \(\mathcal{A}''\) to \(F_1\)-acyclic objects \(i = 1, 2\). There is a canonical isomorphism

\[
RF \circ RF_4 \simeq R(F \circ F_4)
\]

34
Proof. [i] The first statement follows from the fact that there are enough injectives in $\mathcal{A}^{[1]}$. The second and third statements follows from the fact that if $f' : A' \to B'$ is an injective object of the category $\mathcal{A}^{[1]}$ then both $A'$ and $B'$ are injective as well. 

[ii] This follows from the fact that $RF_3^{[1]}$ can be computed using true arrows whose source and target are acyclic for $RF_3$.

[iii] This follows from the fact that $F_i$-acyclicity for $i = 1, 2$ is equivalent to $F$-acyclicity.

3.1.2. Let us discuss an alternative point of view on the distinguished triangle \((54)\).

Lemma 3.3. Let $\mathcal{A}, F = (F_1, F_2, t)$ as in \((53)\) and set $F_0(M) := Ker(F_1(M) \to F_2(M))$. The functor $F_0$ is right derivable and there is a natural transformation

$$RF_0 \to MF \circ R(F_1 \to F_2)$$

This natural transformation is an isomorphism if the following holds:

(Condition) If $I$ is an injective object of $\mathcal{A}$ then $F_1(I) \to F_2(I)$ is epimorphic.

Proof. The first statement is clear since $F_0$ is left exact and $\mathcal{A}$ has enough injectives. The claimed arrow is induced by the obvious isomorphisms $RF_0 \simeq MF \circ R(F_0 \to 0)$ and the natural transformation $R(F_0 \to 0) \to R(F_1 \to F_2)$. If the condition holds then $0 \to F_0(I) \to F_1(I) \to F_2(I) \to 0$ is exact for $I$ injective and we may conclude.

3.2. Some useful diagrams and functors.

3.2.1. The following notations will be used from now on.

\[
\begin{array}{cccccc}
Z_v & \xrightarrow{z_v} & C_v & \leftarrow & C^*_v & \xleftarrow{j_v} & U_v \\
Z & \xrightarrow{z} & C & \leftarrow & C^* & \xleftarrow{j} & U
\end{array}
\]

(55)

- $C$ is an irreducible smooth curve over $\Sigma_1 = Spec(\mathbb{F}_p)$.
- $Z$ is a smooth effective divisor on $C$. In other terms $Z$ is a disjoint union of $Spec(k_v)$’s with $v$ running through a finite number of closed points of $C$.
- $C^*$ is the log scheme $(C, Z)$ and $U$ is the complementary open subscheme of $Z$ in $C$.
- For each point $v$ of $Z$ we let $Z_v = Spec(k_v)$ denote the corresponding reduced closed subscheme, as well as $C_v = Spec(O_v)$, $C^*_v = (C_v, Z_v)$ and $U_v = Spec(K_v)$ where $O_v$ is the completion of the local ring $O_{C,v}$ and $K_v$ is the fraction field of $O_v$.
- The arrows $i, j, o, i_v, o_v, i_v, z, c_v, c^*_v, u_v$ are the obvious ones.

3.2.2. Let us introduce notations for some variants of the diagram \((55)\). We consider star shaped diagrams $J, J^\sharp$ in which $U$ is the central vertex while the branches are indexed by $v$ running through the points of $Z$ and have the following form (we only draw one branch for convenience):

\[
J := ( C_v \xrightarrow{j_v} U_v \xleftarrow{u_v} U )
\]

\[
J^\sharp := ( C^*_v \xrightarrow{j_v} U_v \xleftarrow{u_v} U )
\]
Let us consider moreover the discrete diagram

(57) \[ Z_J := (Z_v) \]

whose vertices are indexed by the points of \( Z \) (this discrete diagram should not be confused with the punctual diagram \( Z \) itself). With these notations we have a natural commutative diagram as follows in the category \( \text{Diag}(\text{Sch}^2/\Sigma_1) \).

(58) \[
\begin{array}{ccc}
Z_J & \xrightarrow{z_J} & J \\
m_Z & & \circ & \circ \\
Z & \xrightarrow{z} & C & \circ \circ \\
\end{array}
\]

3.2.3. Let us discuss Mayer Vietoris functors and triangles which are relevant to completion at points of \( Z \). Recall that an object of the topos \( J_{et} \) is a diagram of the form

\[
F_{C_v} \xrightarrow{F_{j_v}} F_{U_v} \xrightarrow{F_{i_{U_v}}} F_U
\]

in \((-)_{et, cof} \) above \( \Delta^p \). In other terms \( F_U \) is an object of \( U_{et} \) and for each \( v \) in \( Z, F_{C_v} \) (resp. \( F_{U_v} \)) is an object of \( C_{v,et} \) (resp. \( U_{v,et} \)) while \( F_{j_v} : j_v^{-1}F_{C_v} \to F_{U_v} \) and \( F_{i_{U_v}} : i_{U_v}^{-1}F_U \to F_{U_v} \) are morphisms of \( U_{v,et} \).

Lemma 3.4. (complete Mayer Vietoris functors) Consider the natural morphism

\[
m : ((J^N_{et}, \mathbb{Z}/p)) \to (C^N_{et}, \mathbb{Z}/p)
\]

(i) For \( M = (M_{C_v} \to M_{U_v} \leftarrow M_U) \) in \( \text{Mod}(J^N_{et}, \mathbb{Z}/p) \) we have canonically

\[
m_* M \simeq \text{Ker} \left( \left( \prod_{v \in Z} \iota_{C_v,*} M_{C_v} \right) \times j_* M_U \to \left( \prod_{v \in Z} j_{v,*} \iota_{U_v,*} M_{U_v} \right) \right)
\]

(ii) For \( M = (M_{C_v} \to M_{U_v} \leftarrow M_U) \) in \( D^+(J^N_{et}, \mathbb{Z}/p) \) we have a canonical isomorphism

\[
Rm_* M \simeq MF(R \left( \left( \prod_{v \in Z} \iota_{C_v,*}(-)_{C_v} \right) \times j_* (-)_{U_v} \to \left( \prod_{v \in Z} j_{v,*} \iota_{U_v,*}(-)_{U_v} \right) \right)(M))
\]

where \( MF \) is the functorial mapping fiber \([33]\). Whence in particular a canonical distinguished triangle

\[
Rm_* M \longrightarrow \left( \prod_{v \in Z} R\iota_{C_v,*} M_{C_v} \right) \times Rj_* M_U \longrightarrow \left( \prod_{v \in Z} Rj_{v,*} R\iota_{U_v,*} M_{U_v} \right)^{+1}
\]

Proof. (i) This is nothing but the projective limit formula \([27]\).

(ii) The claimed isomorphism will follow from lemma \([33]\) once proven that

\[
(\prod_{v \in Z} \iota_{C_v,*} M_{C_v}) \times j_* M_U \xrightarrow{(\prod_{v \in Z} \iota_{C_v,*} M'(j_v))_{op1} - (j_{v,*} M'(\iota_{U_v}))_{op2}} \prod_{v \in Z} j_{v,*} \iota_{U_v,*} M_{U_v}
\]

is epimorphic when \( M \) is injective. Here \( M'(j_v) : M_{C_v} \to j_{v,*} M_{U_v} \) denotes the morphism deduced from \( M(j_v) : j_v^{-1} M_{C_v} \to M_{U_v} \) by adjunction and similarly for \( M'(\iota_{U_v}) \). Let \( |J| \) denote the discrete diagram underlying \( J \) and \( f : |J| \to J \) denote the obvious morphism. By injectivity of \( M \) the natural monomorphism \( M \to f_* f^{-1} M \) splits and it is thus sufficient to prove that \((\prod_{v \in Z} \iota_{C_v,*} N'(j_v)) \circ p_1 - (N'(\iota_{U_v}))_{v} \circ p_2 \) is epimorphic for \( N = f_* f^{-1} M \).
Now we are done since $N'(j_v)$ (or alternatively $N'(\iota_{U_v})$) is a split epimorphism as shown by the explicit formula

\[ f_* f^{-1} M = (M_{C_v} \times j_{v,*} M_{U_v}) \xrightarrow{p_2} M_{U_v} \xleftarrow{\bigcup_{w \in \mathbb{Z}} \iota_{U_v,*} M_{U_v}} M_U \]

The distinguished triangle follows by \([\mathbb{A}]\) since the component functors $(-)_{C_v}, (-)_{U_v}$, $(-)_{U_v}$ send injectives to flasque modules by \([\mathbb{A}]\).

\[ \square \]

**Remark 3.5.**

(i) The lemma \([\mathbb{A}]\) has been stated for $\mathbb{Z}/p$-modules for convenience of later reference. It would clearly work for arbitrary rings.

(ii) Restricting \([\mathbb{A}]\) to the $k$-th component yields a similar isomorphism and a distinguished triangle in $D^+ (C_{et}, \mathbb{Z}/p^k)$.

### 3.2.4

In practice the following lemma will be useful to compute $Rm_s$.

**Lemma 3.6.** The following functors are exact:

\[
\begin{align*}
t_{U_v,*} & : Ab(U_{v,et}) \to Ab(U_{et}) \\
t_{C_v,*} & : Ab(C_{v,et}) \to Ab(C_{et})
\end{align*}
\]

Proof. Let us explain the case of $t_{C_v,*}$ since the case of $t_{U_v,*}$ is easier (or alternatively, follows formally). We will check that for any geometric point $\pi$ of $C$ the functor $F \mapsto (t_{C_v,*} F)_\pi$ is exact. Obviously we can assume that $C$ is affine, say $C = Spec(A)$. With our usual notations $K$ is thus the fraction field of $A$.

Consider a closed point $w$ of $Spec(A)$. Let $A_w$ denote the completion of $A$ at $w$ and $K_w$ its fraction field. Let us fix a separable closure $K_w^{sep}$ (resp. $K_w^{sep}$) of $K$ (resp. $K_w$) as well as an embedding $i_w^{sep} : K_w^{sep} \to K_w^{sep}$ for each $w$. We denote $G$ the galois group of $K^{sep}/K$ and $D_{w^{sep}}$ (resp. $I_{w^{sep}}$) the decomposition (resp. inertia) group of $w^{sep}$ inside $G$.

By Krasner’s lemma \([\mathbb{L}_a]\) II, 2 prop. 4 $D_{w^{sep}}$ thus identifies to $Gal(K_w^{sep}/K_w)$.

Let $\mathcal{C}$ denote the ordered set of subalgebras $A'$ of $K^{sep}$ of finite type over $A$ such that the ring $A'$ is integrally closed. For $A'$ in $\mathcal{C}$ we identify the set of closed points of $Spec(A')$ with a subset of the set of places of the fraction field $K'$ of $A'$. We use the notation $w'$ to designate either the place $K'$ induced by $i_{w}^{sep}$ or, when it exists, the corresponding point of $Spec(A')$. The completion of $A'$ (resp. $K'$) with respect to $w'$ is denoted $A_{w'}'$ (resp. $K_{w'}'$). We use respectively the notation $G'$ (resp. $D'_{w'}, I_{w'}$) for the galois group of $K'/K$ (resp. the image of $D_{w^{sep}}, I_{w^{sep}}$ inside it). If $H$ is a subgroup of $G'$ we have a pseudo-commutative diagram of topoi

\[
\begin{align*}
& Spec(K'_v)_{et} \\
B_{K_v,D'_{v'}}/Spec(K'_v) & \xrightarrow{*} B_{K_v,G'}/Spec(K_v \otimes_K K') & B_{K_v,H}/Spec(K_v \otimes_K K') & \xrightarrow{*} Spec(K_v \otimes_K K')_{et} \\
B_{D'_{v'}} & \downarrow & B_{G'} & \downarrow & B_{H} & \downarrow & Set
\end{align*}
\]

which is moreover pseudo-functorial with respect to $Spec(K')$ (or $Spec(A')$) in the obvious way. Here we have used the notation $B_H$ (resp. $B_{R,H}$) for the topos of left $H$-sets (resp. left $H$-objects in $Spec(R)_{et}$) and the schemes $Spec(K_v \otimes_K K')$ are endowed with the left action of $G$ obtained by inverting the natural action of $G$ on $K'$. All the arrows of the
diagram are simply obtained by functoriality of classifying topos [SGA4-2] IV, 4.5 and localization. Let us only mention that in the bottom line the direct images functors are respectively induction from $D'_{v'}$ to $G'$, from $H$ to $G'$ as described e.g. in [Se2] and fixed points $(-)^H$. The squares of this diagram are subject to obvious base change isomorphisms expressing that the inverse image functors of the horizontal arrows pseudo-commute to the direct image functors of the vertical arrows. If now $K'/K$ is galois then $\text{Spec}(K_v \otimes_K K')$ is a $G'$-torsor in $\text{Spec}(K_v)_{et}$. In this situation the arrows indicated with a $\star$ are equivalences and $B_{K_v,C'/\text{Spec}(K_v \otimes_K K'_v)}$ is naturally equivalent to $\text{Spec}(K_v)_{et}$. Via these equivalences the direct image functor of the first (resp. second) vertical arrow simply sends $F \in \text{Spec}(K_v)_{et}$ to the set $\Gamma(K'_v, F)$ (resp. $\Gamma(K_v \otimes_K K', F)$) endowed with the left action of $H$ (resp. $G'$) induced by the natural action of $G'$ on $K'$ (resp. $D'_{v'}$ on $K'_v$).

We are now in position of proving the exactness of the functor $F \mapsto (\iota_{C_v, \star} F)_{\overline{x}}$. We consider the following cases.

Case 1. The geometric point $\overline{x}$ lies above the generic point $\eta$ of $C$: $\overline{x} : \text{Spec}(K^\text{sep}) \to C$. Consider the following filtrant subsets: $C_{\overline{x}} \subset C$ contains only the $A'$'s which are étale over $A$, $C'_{\overline{x}} \subset C_{\overline{x}}$ contains only the $A'$'s whose fractions field $K'$ is galois over $K$ and whose spectrum has no point above $v$. We have a series of functorial isomorphisms

\begin{align*}
\left(\iota_{C_v, \star} F\right)_{\overline{x}} & \simeq \lim_{\rightarrow A' \in C'_{\overline{x}}} \Gamma(A_v \otimes_A A', F) \\
\simeq & \lim_{\rightarrow A' \in C'_{\overline{x}}} \Gamma(K_v \otimes_K K', F) \\
\simeq & \lim_{\rightarrow A' \in C'_{\overline{x}}} \Gamma(\text{Ind}^{D'_{v'}} G', K'_v, F) \\
\simeq & \lim_{\rightarrow A' \in C'_{\overline{x}}} \text{Ind}^{D'_{v'}} G'_v \Gamma(K'_v, F) \\
& \simeq \text{Ind}^{D'_{v'}} G'_v F_{\overline{x}}
\end{align*}

where $\overline{x}_v : \text{Spec}(K^\text{sep}) \to \text{Spec}(A_v)$ is the geometric point satisfying $\iota^\text{sep}\overline{x}_v = \overline{x}$ and $\text{Ind}^{D'_{v'}} G'_v$ means discrete induction. The first isomorphism comes from the fact that $C'_{\overline{x}}$ is cofinal inside $C_{\overline{x}}$. The second one occurs since $\text{Spec}(A')$ has no points above $v$. The third one is induced by the isomorphism

\begin{align*}
K_v \otimes_K K' & \simeq \text{Ind}^{D'_{v'}} G' K'_v \\
a \otimes b & \mapsto (g \mapsto a g^\text{sep}(gb))
\end{align*}

The fourth one is by pseudo-commutativity of the first square in the above diagram of classifying topos. Since discrete induction commutes to filtrant inductive limits the fifth one comes from the fact that the functor $C'_{\overline{x}} \to C_{\overline{x}}$, $A' \mapsto K'_v$ is cofinal if $C_{\overline{x}}$ denotes the set of étale sub algebras of $K^\text{sep}/K_v$. The desired exactness follows from the fact that $\text{Ind}^{D'_{v'}} G'_v$ is exact.

Case 2. The geometric point $\overline{x}$ lies over a closed point $w \neq v$. Consider the following filtrant sets: $C'_{\overline{x}} \subset C$ contains only the étale $A'/A'$'s whose spectrum contain $w'$, $C_{\overline{x}}' \subset C$ contains only the $A'$'s satisfying 1) $A'/A$ is étale outside $w$, 2) $K'/K$ is galois and 3) $A'$ is the integral closure of $A[1/s]$ in $K'$ for some $s$ satisfying $w(s) = 0, v(s) > 0$. We define a functor $(-)^{w} : C'_{\overline{x}} \to C_{\overline{x}}$, $A' \mapsto A' := A'^{D'_{v'}}[1/t]$ where the valuation of $t$ is zero at $w'$.
and places which are not above \( w \) and is non zero at the other places above \( w \). We have a series of functorial isomorphisms

\[
(t_{C_v}, F)_{\overline{\pi}} \simeq \lim_{\rightarrow \mathcal{A} \in C_{\overline{\pi}}} \Gamma(A_v \otimes_A A', F)
\]

\[
\simeq \lim_{\rightarrow \mathcal{A} \in C_{\overline{\pi}}} \Gamma((\text{Ind}_{G_v}^{D_v} K_v')^{\prime}, F)
\]

\[
\simeq \lim_{\rightarrow \mathcal{A} \in C_{\overline{\pi}}} (\text{Ind}_{G_v}^{D_v} \Gamma(K_v', F))^{\prime}
\]

\[
\simeq (\text{Ind}_{G_v}^{D_v} \text{F}_{\overline{\pi}})_{\overline{\pi}}^{\prime}
\]

where \( \overline{\pi} \) is as in Case 1. The first isomorphism holds because the functor \((-)^{\prime}\) is cofinal. The second one follows from \([61]\) since \( A_v \otimes_A A' = A_v \otimes_A A'^{\prime} \) and \( \text{Spec}(A') \) has no points above \( v \). The third isomorphism follows from the diagram of classifying topoi by pseudo-commutativity of the first and third squares together with the base change isomorphism in the second square. The last isomorphism is similar to case 1. Exactness follows using that \( D_{v, \text{sep}} \) intersects trivially the conjugacy class of \( I_{v, \text{sep}} \) (by Krasner’s lemma and weak approximation \([\text{St} I, 3]\)).

Case 3. The geometric point \( \overline{\pi} \) lies above \( v \). Consider the following filtrant sets: \( \mathcal{C}_{\overline{\pi}} \subset \mathcal{C} \) contains only the étale \( A'/A' \)'s whose spectrum contain \( v' \), \( \mathcal{C}_v \subset \mathcal{C} \) contains only the \( A' \)'s satisfying 1) \( A' \) is étale outside \( v \), 2) \( K'/K \) is galois and 3) \( A' \) is the integral closure of \( A[1/s] \) in \( K' \) for some \( s \) satisfying \( v(s) = 0 \). We define a functor \((-)^{\prime} : \mathcal{C}_{\overline{\pi}} \to \mathcal{C}_v \), \( A' \mapsto A'' := A'^{\prime}[1/t] \) where the valuation of \( t \) is zero at \( v' \) and places which are not above \( v \) and is non zero at the other places above \( v \). We have the following series of functorial isomorphisms

\[
(t_{C_v}, F)_{\overline{\pi}} \simeq \lim_{\rightarrow \mathcal{A} \in C_{\overline{\pi}}} \Gamma(A_v \otimes_A A', F)
\]

\[
\simeq \lim_{\rightarrow \mathcal{A} \in C_{\overline{\pi}}} \Gamma((\text{Ind}_{G_v}^{D_v} K_v')^{\prime}, F)
\]

\[
\simeq \lim_{\rightarrow \mathcal{A} \in C_{\overline{\pi}}} (\text{Ind}_{G_v}^{D_v} \Gamma(K_v', F))^{\prime}
\]

\[
\simeq (\text{Ind}_{G_v}^{D_v} \text{F}_{\overline{\pi}})_{\overline{\pi}}^{\prime}
\]

where \( \overline{\pi}_v \) is as in Case 1 and \( \overline{\pi}_v \) is the geometric point of \( \text{Spec}(A_v) \) inducing \( \overline{\pi} \). The first isomorphism holds because \((-)^{\prime}\) is cofinal. The second isomorphism is deduced from \([64]\) by taking fixed points under \( I_{v'}^{\prime} \) and imposing integrality at \( v' \) (note that the fiber product is taken with respect to the adjunction map, explicitly described as \( f \mapsto f(1) \) on the left and the inclusion map on the right). The third isomorphism is obtained by descent along the surjective étale morphism

\[
\text{Spec}(\text{Ind}_{G_v}^{D_v} K_v') \sqcup \text{Spec}(A_v^{I_{v'}}) \to \text{Spec}((\text{Ind}_{G_v}^{D_v} K_v')^{\prime}, A_v^{I_{v'}})
\]

The last isomorphism holds because \( \mathcal{C}_{\overline{\pi}} \to \mathcal{C}_{\overline{\pi}_v}, A' \mapsto A'_v \) and \( \mathcal{C}_v \to \mathcal{C}_{\overline{\pi}_v}, A' \mapsto A'_v \) are cofinal if \( v'_1 \neq v' \) is a place of \( K' \) above \( v \) and \( \mathcal{C}_{\overline{\pi}_v} \subset \mathcal{C} \) contains only the étale \( A'/A' \)'s whose spectrum contains \( v' \). Exactness then follows using the isomorphism

\[
(\text{Ind}_{G_v}^{D_v} \text{F}_{\overline{\pi}} F_{\overline{\pi}_v}) \times F_{\overline{\pi}_v} \times_{F_{\overline{\pi}_v}} \simeq \text{Cont}(D_{v, \text{sep}} \backslash (G - D_{v, \text{sep}}), F_{\overline{\pi}_v})\times F_{\overline{\pi}_v}
\]

\[
(f, a) \mapsto (f \circ s, a)
\]

39
provided by a continuous section \( s \) of \( G \to D_{v,sep} \setminus G \) (Sez \( \text{[2], prop. 1.} \))

\[ \text{□} \]

**3.2.5.** We now introduce some *ad hoc* functors designed to neglect the generic fiber of \( C_v \).
Recall the well known equivalence between \( C_v \text{et} \) and triples \((F_1, F_2, f)\) where \( F_1 \in \mathcal{Z}_{v, \text{et}} \), \( F_2 \in U_{v, \text{et}}\) and \( f : F_1 \to z_{v}^{-1} j_{v, *}, F_2 \) \([\text{Mil}]\) II, 3.10.

**Definition 3.7. (Smashing functors)**

(i) The functor

\[ \text{Sma}_v : (C_v \leftarrow U_v)_{\text{et}} \to (C_v \leftarrow U_v)_{\text{et}} \]

is defined by sending \( F_1 \to F_2 \) to \( F_3 \to F_2 \) where \( F_3 \) corresponds to the triple \( (z_{v}^{-1} F_1, F_2, z_{v}^{-1} F_1 \to z_{v}^{-1} j_{v, *} F_2) \).

(ii) The functor

\[ \text{Sma} : J_{\text{et}} \to J_{\text{et}} \]

is defined by sending \( (F_{C_v} \to F_{U_v} \leftarrow F_U) \) to \( (G_{C_v} \to F_{U_v} \leftarrow F_U) \) where \( (G_{C_v} \to F_{U_v}) = \text{Sma}_v(F_{C_v} \to F_{U_v}) \).

These functors just defined have the following properties.

**Lemma 3.8.**

(i) The functor \( \text{Sma}_v \) (resp. \( \text{Sma} \)) is exact. It induces in particular an endofunctor of the categories \( \text{Mod}((C_v \leftarrow U_v)_{\text{et}}^{\text{N}}, \mathbb{Z}/p) \) and \( \text{D}((C_v \leftarrow U_v)_{\text{et}}^{\text{N}}, \mathbb{Z}/p) \) (resp. \( \text{Mod}(J_{\text{et}}^{\text{N}}, \mathbb{Z}/p) \) and \( \text{D}(J_{\text{et}}^{\text{N}}, \mathbb{Z}/p) \).

(ii) The morphisms \( F(j_v) : j_{v}^{-1} F_{C_v} \to F_{U_v} \) for variable \( F \) induces natural transformations \( \text{Sma}_v \to \text{Id} \) (resp. \( \text{Sma} \to \text{Id} \)). If \( F(j_v) \) (resp. each \( F(j_v) \)) is invertible the previous natural transformation induces an isomorphism:

\[ \text{Sma}_v(F) \simeq F \]

(resp. \( \text{Sma}(F) \simeq F \))

(iii) Consider an object \( M = (M_{C_v} \to M_{U_v}) \) (resp. \( M = (M_{C_v} \to M_{U_v} \leftarrow M_U) \)) of \( \text{D}((C_v \leftarrow U_v)_{\text{et}}^{\text{N}}, \mathbb{Z}/p) \) (resp. \( \text{D}(J_{\text{et}}^{\text{N}}, \mathbb{Z}/p) \)). Then

\[ \text{Sma}_v(M) = 0 \]

(resp. \( \text{Sma}(M) = 0 \))

if and only if \( M_{U_v} = 0 \) and \( z_{v}^{-1} M_{C_v} = 0 \) (resp. \( M_U = 0, M_{U_v} = 0 \) and \( z_{v}^{-1} M_{C_v} = 0 \))

Proof. Everything follows directly from the definition of \( \text{Sma} \) and \( \text{Sma}_v \).

\[ \text{□} \]

**Corollary 3.9.** Consider a morphism \( f : M \to N \) in \( \text{D}(J_{\text{et}}^{\text{N}}, \mathbb{Z}/p) \). The induced morphism \( \text{Sma}(M) \to \text{Sma}(N) \) is invertible if and only if the following morphisms are invertible:

\[ f_U : M_U \to N_U, f_{U_v} : M_{U_v} \to N_{U_v} \text{ and } z_{v}^{-1} f_{C_v} : z_{v}^{-1} M_{C_v} \to z_{v}^{-1} N_{C_v}. \]

Proof. Apply \( \text{[3.8(iii)]} \) to a cone of the morphism \( f \).

\[ \text{□} \]
3.3. Vanishing cohomology.

We define vanishing cohomology in a general setting and study its behaviour when changing the topos.

3.3.1. We begin with simple diagram theoretic constructions regarding a morphism of diagrams $i : Y \to X$. In the applications $i$ will be taken either as the closed immersion $z : Z \to C$, the closed immersion $z_J : Z_J \to J$ or slight variants of these.

**Definition 3.10.**
(i) We say that a functor $i : \Delta' \to \Delta$ is extremal if it is injective on objects, fully faithful, and if $\Delta'/\delta$ is empty for $\delta \notin i(\Delta')$.
(ii) Let $i : \Delta' \to \Delta$ be an extremal functor. We denote $\Delta^+$ the full subcategory of $\Delta \times [1]$ formed by the $(\delta,0)$'s with $\delta \in i(\Delta')$ and all $(\delta,1)$'s. Here $[1]$ denotes the category $\{0 \to 1\}$. We denote $i^+ : \Delta' \to \Delta^+$, $\delta' \mapsto (i(\delta'),0)$, $\sigma : \Delta \to \Delta^+$, $\delta \mapsto (\delta,1)$ and $\rho : \Delta^+ \to \Delta$, $(\delta,n) \mapsto \delta$.
(iii) We say that a morphism of diagrams $i : Y/\Delta' \to X/\Delta$ is of extremal type if the underlying functor $\Delta' \to \Delta$ is extremal. For $i$ of extremal type we denote

$$X^+ = (Y \to X)$$

the diagram of type $\Delta^+$ having $X^+_{i(\delta'),0} = Y_{\delta'}$ ($\delta'$ in $\Delta'$) and $X^+_{\delta,1} = X_\delta$ ($\delta$ in $\Delta$) as vertices and whose edges are the following:

- $X^+(i(f'),id_0) : X^+_{i(\delta'),0} \to X^+_{i(\delta'),0}$ is $Y(f')$ ($f : \delta_1' \to \delta'$ in $\Delta'$),
- $X^+(f,id_1) : X^+_{\delta,1} \to X^+_{\delta,1}$ is $X(f)$ ($f : \delta \to \delta$ in $\Delta$) and
- $X^+(i(f'),0 \to 1) : X^+_{i(\delta'),0} \to X^+_{i(\delta'),1}$ is $i_{\delta'} \circ Y(f')$ ($f' : \delta_1' \to \delta'$ in $\Delta'$). It is also equal to $X(i(f')) \circ i_{\delta'}$.

The injections $\Delta \to \Delta^+$ and $\Delta' \to \Delta^+$ extend naturally to morphisms of diagrams: $\sigma : X \to X^+$, $X_\delta \to X^+_{\delta,1}$ and $i^+ : Y \to X^+$, $Y_{\delta'} \to X^+_{i(\delta'),0}$. We define furthermore $\rho : X^+ \to X$ as the unique retraction of $\sigma$ factorizing $i$ as follows:

$$Y \xrightarrow{i^+} X^+ \xrightarrow{\sigma} X \xrightarrow{\rho} X$$

**Lemma 3.11.** Consider a ringed variable topos $(\mathcal{T}, A)$ on $\mathcal{B}$ (2.8 (ii)). Let $i$ of extremal type as in 3.10 (iii).

(i) The direct image functors induced by $i : Y \to X$ and $i^+ : Y \to X^+$ have the following description:

- $i_* : \text{Mod}(\mathcal{T}(Y), A_Y) \to \text{Mod}(\mathcal{T}(X), A_X)$ satisfies $(i_* M)_\delta = i_{\delta'}*M_{\delta'}$ if $\delta = i(\delta')$ and $(i_* M)_\delta = 0$ if $\delta \notin i(\Delta')$. A module $M$ is acyclic for $i_*$ if and only if each $M_{\delta'}$ is acyclic for $i_{\delta'}*$. 
- $i^+_* : \text{Mod}(\mathcal{T}(Y), A_Y) \to \text{Mod}(\mathcal{T}(X^+), A_{X^+})$ satisfies $(i^+_* M)_{i(\delta'),0} = M_{\delta'}$, $(i^+_* M)_{i(\delta),1} = i_{\delta'}*M_{\delta'}$ and $(i^+_* M)_{\delta,1} = 0$ if $\delta \notin i(\Delta')$. A module $M$ is acyclic for $i^+_*$ if and only if each $M_{\delta'}$ is acyclic for $i_{\delta'}^*$. 

(ii) There is a natural isomorphism $\sigma^{-1}A_{X^+} \simeq A_X$. The functors $\sigma^{-1}, \rho_* : \text{Mod}(\mathcal{T}(X^+), A_{X^+}) \to \text{Mod}(\mathcal{T}(X), A_X)$ are naturally isomorphic. In particular they commute to arbitrary limits. The functors $\sigma_*$ and $\rho^*$ are fully faithful.
Proof. [i] The calculation of \( i_* \) and \( i_*^+ \) is immediate from the projective limit formula \((27)\). Since injective modules of \((T(Y), A_Y)\) have flasque components by \((2,11)(v)\), a similar formula holds for \( R i_* \) and \( R i_*^+ \). The acyclicity statements follow.

[ii] Here again the calculation of \( \rho_* \) in \((ii)\) is straightforward from the projective limit formula. The last statement of \((ii)\) follows from the isomorphism \( \sigma^{-1} \cong \rho_* \) given that \( \rho \sigma = id_X \).

\[ \square \]

In \((3.11)\) we have used the simplified notation \( T(X) \) instead of \( T^{\text{cofib}}(X) \) or \( T_{\text{cof}}(X) \). We will continue to do so from now on. We will sometimes use the suggestive notation \((-)_{|Y}, (-)_{|X} \) (resp. \((-)_{|X}, (-)_{|X^+} \)) instead of \( \sigma^* \) (resp. \( i_*^* \) or \( i_*^{-1}, \rho^* \)) when no danger of confusion arises.

3.3.2. We are now ready to define the functors of vanishing sections and study their elementary properties in the general context of fibered topoi.

**Definition 3.12.** Let \( T, B, A, i : Y \to X, X^+ \) as in \((3.11)\).

(i) We define the following functors for modules on \((T(X), A_X)\):
- The functor of global sections vanishing at \( Y \),
  \[ \Gamma^Y(X, -) : Mod(T(X), A_X) \to Mod(T(X), A_X) \]
  is defined as \( \Gamma^Y(X, M) := \text{Ker}(\Gamma(X, M) \to \Gamma(Y, i_*^* M)) \).
- The functor of local sections vanishing at \( Y \),
  \[ \Gamma^Y(-) : Mod(T(X), A_X) \to Mod(T(X), A_X) \]
  is defined as \( \Gamma^Y(M) := \text{Ker}(M \to i_*^* M) \).

(ii) We define the following functors for modules on \((T(X^+), A_{X^+})\):
- The functor of sections on \( X \) vanishing at \( Y \),
  \[ \Gamma^Y(X, -) : Mod(T(X^+), A_{X^+}) \to Mod(T(X), A_X) \]
  is defined as \( \Gamma^Y(X, M) := \text{Ker}(\Gamma(X, \sigma^{-1} M) \to \Gamma(Y, i_*^{+1} M)) \).
- The functor of sections on \( X \) vanishing at \( Y \),
  \[ \Gamma^Y(-) : Mod(T(X^+), A_{X^+}) \to Mod(T(X), A_X) \]
  is defined as \( \Gamma^Y(X, M) := \text{Ker}(\sigma^{-1} M \to i_*^+ i_*^{+1} M) \) where the arrow is defined using the adjunction morphism \( id \to i_*^+ i_*^{+1} \) and the isomorphisms \( \sigma^{-1} \cong \rho_* \), \( \rho_* i_*^+ \cong i_* \) (explicitly the arrow is thus described on component \( \delta \) as \( M_{\delta,1} \to i_{\delta^*}, M_{\delta,0} \) if \( \delta = i(\delta') \) and \( M_{\delta,1} \to 0 \) if \( \delta \notin i(\Delta') \)).

Let us emphasize that even in the case of punctual diagrams the functor \( \Gamma^Y \) should not be mistaken for a functor of the type “sections with support in \( Y \)” whose usual notation is \( \Gamma_Y \). It will be useful to consider the following condition for a morphism of diagrams \( f : X_1/\Delta_1 \to X_2/\Delta_2 \):

\( \text{opimm}(f, T, A) \): the morphisms of ringed topoi \((T(X_1/\delta), A_{X_1/\delta}) \to (T(X_2/f(\delta)), A_{X_2/f(\delta)}) \) induced by \( f_\delta \) are open immersions for all \( \delta \) in \( \Delta_1 \).

Recall that by definition an open immersion of ringed topoi is the morphism of localization attached to an open of the target topos. When this condition is realized one finds easily that \( i : T(Y) \to T(X) \) and \( i^+ : T(Y) \to T(X^+) \) are open immersions as well. In that case:
- The morphism $\sigma : \mathcal{T}(X) \to \mathcal{T}(X^+)$ is a closed immersion complementary to $i^+$ and the functors \[(\text{sections vanishing at } Y)\] can be interpreted as follows: $\prod^Y(X, -) = \prod_{X}(-), \quad \Gamma^Y(X, -) = \Gamma_{X}(X^+, -)$.

- If there exists $Y^c \to X$ in $\mathcal{B}$ inducing a closed immersion $\mathcal{T}(Y^c) \to \mathcal{T}(X)$ complementary to $i$ (this is not always the case) then the functors of \[(\text{iii})\] can be interpreted as follows: $(\prod^Y_{Y^c}) = \prod_{Y^c}, \quad \Gamma^Y(X, -) = \Gamma_{Y^c}(X, -)$.

Let us gather some properties of the derived functors $R\prod^Y(-)$ and $R\Gamma^Y(X, -)$.

**Lemma 3.13.** Keep the notations and assumptions of 3.12

(i) There are canonical isomorphism and distinguished triangle

$$ R\prod^Y(X, M) \simeq MF(R(\sigma^{-1} \to i_*i_{-1})(M)) $$

$$ R\prod^Y(X, M) \longrightarrow M \longrightarrow Ri_*M_Y \longrightarrow$$

which are functorial with respect to $M$ in $D^+(\mathcal{T}(X), A_X)$. If $M$ is a complex such that the objects of $M_Y$ are acyclic for $i_*$ (e.g. if the objects of $M$ have flasque components, 3.11(iii)) then the right hand side in the above isomorphism is canonically isomorphic to the mapping fiber of $M_{|X} \to i_*M_Y$.

(ii) Assume that $\text{opimm}(i, \mathcal{T}, A)$ holds. There are canonical isomorphism and distinguished triangle

$$ R\prod^Y(M) \simeq MF(R(id \to i_*i_{-1})(M)) $$

$$ R\prod^Y(M) \longrightarrow M \longrightarrow Ri_*M_Y \longrightarrow$$

which are functorial with respect to $M$ in $D^+(\mathcal{T}(X), A_X)$. If $M$ is a complex such that the objects of $M_Y$ are acyclic for $i_*$ (e.g. if the objects of $M$ have flasque components, 3.11(iii)) then the right hand side in the above isomorphism is canonically isomorphic to the mapping fiber of $M \to i_*M_Y$.

(iii) Assume that $\text{opimm}(i, \mathcal{T}, A)$ holds. There is a canonical isomorphism

$$ R\prod^Y(X, \rho^{-1}M) \simeq R\prod^Y(M) $$

which is functorial with respect to $M$ in $D^+(X, A_X)$.

(iv) Consider another morphism $i_1 : Y_1/\Delta_1 \to X_1/\Delta_1$ of extremal type. Assume given morphisms of diagrams $f : X_1 \to X, \quad f_Y : Y_1 \to Y$ satisfying $fi_1 = if_Y$ and denote $f^+ : X^+_1 \to X^+$ the induced morphism. There is a canonical isomorphism

$$ R\prod^Y(X, Rf^+_*M) \simeq Rf_*R\prod^Y(X_1, M) $$

which is functorial with respect to $M$ in $D^+(\mathcal{T}(X^+_1), A_X^+)$.

(v) Let $g : (\mathcal{T}, A) \to (\mathcal{T}', A')$ be a morphism of variable topoi. There is a canonical isomorphism

$$ R\prod^Y(X, Rg_*M) \simeq Rg_*R\prod^Y(X, M) $$

which is functorial with respect to $M$ in $D^+(\mathcal{T}(X^+), A_X^+)$.

Proof. (i) Let $f : |X^+| \to X^+$ denote the inclusion of the discrete diagram underlying $X^+$. As in the proof of 3.1(i) the isomorphism will follow once checked that $M_{|X} \to i_*M_Y$ is epimorphic for $M = f_*N$. If $\delta = i(\delta')$ we have natural isomorphisms

$$ (f_n, N)._{\delta, 1} \simeq \prod_{(g', \delta_{1}' \to \delta')} (i(\delta'), 0 \to 1).N_{i(\delta_{1}')}.0 \times (i(\delta'), id_1).N_{i(\delta_{1}')}.1 $$

$$ (f_n, N)._{\delta, 0} \simeq \prod_{(g', \delta_{1}' \to \delta')} (i(\delta'), id_0).N_{i(\delta_{1}')}.0 $$
We see on this description that \((f_*N)_{δ,1} \rightarrow i_{\nu,*}(f_*N)_{δ,0}\) is split epimorphic. The distinguished triangle and the last assertion follow from the isomorphism since \(R(i^+_1+i^{-1}) \simeq (Ri^+_1)i^{-1}\) (note that \(i^{-1}\) sends injectives to componentwise flasques by \([2.11](v)\) hence \(i^+_1\)-acyclic by \([3.11](i)\).

\[\text{Let } f : [X] \rightarrow X\text{ denote the inclusion of the discrete diagram underlying } X.\text{ Once again it suffices to check that } M \rightarrow i_*i^{-1}M\text{ is epimorphic for } M = f_*N\text{ with } N\text{ injective. Using } [SGA4-II] V, 4.7\text{ this follows from } opimm(i, T, A)\text{ thanks to the fact that each } (f_*N)_{δ} \text{ is flasque } \text{[2.11](v)}.\text{ The distinguished triangle and the last assertion follow as in } \text{[i]}(\text{here } i^{-1}\text{ preserves injectives thanks to } opimm(i, T, A)).\]

\[\text{[ii]}\text{ Since } id \rightarrow i_*i^{-1}\text{ identifies to } σ^{-1}\rho^{-1} \rightarrow i_*i^{-1}\rho^{-1}\text{ we have a natural isomorphism } \Gamma^X(X, ρ^{-1}(−)) \simeq \Gamma^Y(−).\text{ In order to pass to derived functors we need to know that } ρ^{-1}\text{ sends injectives to } \Gamma^Y(X, −)\text{-acyclics. Consider an injective module } M\text{ over } (X, A_X)\text{. Then } ρ^{-1}M\text{ has flasque components and } σ^{-1}\rho^{-1}M \rightarrow i_*i^{-1}ρ^{-1}M\text{ (i.e. } M \rightarrow i_*i^{-1}M)\text{ is epimorphic as already observed in the proof of } \text{[i]}.\text{ It thus follows from } \text{[i]}\text{ that } ρ^{-1}M\text{ is } \Gamma^Y(X, −)\text{-acyclic as desired.}\]

\[\text{[iv]}\text{ For any module } M\text{ over } (T(X^+_1), A_{X^+_1})\text{ the left commutative diagram below}\]

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{i_1^+} & X_1^+ \\
\downarrow{f_Y} & & \downarrow{f} \\
Y & \xrightarrow{i} & X
\end{array}
\]

\[\begin{array}{c}
i_1^+ \\
\rho_1 \\
\rho
\end{array}
\]

\[\begin{array}{c}
i_1^+i^{-1} \\
f \\
i
\end{array}
\]

\[(74)\]

induces the right commutative square in \(Mod(T(X), A_X)\) (use formula \((27)\) and the fact that \(i_1\) is extremal to check the base change isomorphism \(f_Yi_1^+i^{-1} \simeq i^+_1f^+_1\) underlying the first vertical isomorphism). Deriving the bottom arrow with respect to \(M\) gives \(R(i^+_1+i^{-1}) \rightarrow ρ_*\) \(\circ \ Rf^+_1(M)\) since \(f^+_1\) sends injectives to componentwise flasques (hence \(i^+_1\)-acyclics thanks to \(R(i^+_1+i^{-1}) \simeq (Ri^+_1)i^{-1}\) and \([3.11](i)\) by \([2.11](iv)\) \([v]\)). Deriving the top arrow on the other hand gives \(Rf_*\circ R(ρ_{1,*} \rightarrow i_1^+i^{-1})(M)\) since \(ρ_{1,*}\) and \(i_1^+i^{-1}\) send injectives to \(f_*\)-acyclics. The latter claim deserves an explanation. The functors \(ρ_{1,*}\) and \(i_1^+\) both preserve \(d\)-injectives. Since \(i_1\) is extremal type it may be checked easily that \(i^+_1\)-preserves \(d\)-injectives as well. In particular both \(ρ_{1,*}\) and \(i_1^+\) send injectives to direct factors of \(d\)-injectives (hence \(f_*\)-acyclics \([2.11](v)\)). The desired isomorphism comes by functorial mapping fibers.

\[\text{[v]}\text{ The construction of the morphism is similar to the first part of } \text{[iv]}\text{ (replace } f \text{ with } g)\text{ using only that } i_*i^+_1\text{ sends injectives to componentwise flasques (hence } g_*\text{-acyclic } \text{[2.11](i)}).\]

\[\square\]

Let us emphasize the following result which justifies the introduction of \(X^+\) and definition \([3.12]\text{(ii)}\) In practice it will be applied when \(g\) is the morphism \(ε : (−)_FL \rightarrow (−)_{et}\).

**Corollary 3.14.** Keep the notations and assumptions of \([3.12]\). Consider \(g : (T, A) \rightarrow (T', A')\) as in \([3.13]\text{(v)}\). If \(opimm(i, T, A)\) holds there is a canonical isomorphism

\[Rg_*Ri^+_1(M) \simeq R\Gamma^Y(X, Rg_*M_{|X^+})\]

which is functorial with respect to \(M\) in \(D^+(T(X), A_X)\).
regards to the morphism Lemma 3.16.

Consider a morphism of extremal type scheme. Let \( \mathcal{M} \) denote the natural morphism (so that \( \iota_{\mathcal{M}} = M_k \)).

(i) There is a canonical isomorphism

\[
\iota_k^{-1} R\Gamma Y (X, M) \simeq R\Gamma Y (X, M_k)
\]

which is functorial with respect to \( M \), in \( D^+ (\mathcal{T}_N(X^+), A_{\cdot, X^+}) \). If \( \text{opimm}(i, \mathcal{T}, A) \) holds there is also a canonical isomorphism

\[
\iota_k^{-1} R\Gamma Y (N) \simeq R\Gamma Y (N_k)
\]

which is functorial with respect \( N \), in \( D^+ (\mathcal{T}_N(X), A_{\cdot, X}) \).

(ii) Consider a subcategory \( \Delta_1 \) of \( \Delta \) and set \( \Delta'_1 = \Delta' \times_\Delta \Delta_1 \). Let \( X_1 / \Delta_1 \) (resp. \( Y_1 / \Delta'_1 \)) denote the restriction of \( X \) (resp. \( Y \)) to \( \Delta_1 \) (resp. \( \Delta'_1 \)) and denote \( f : X_1 \to X \), \( f^+ : X_1^+ \to X^+ \) denote the inclusion morphisms. There is a canonical isomorphism

\[
f^{-1} R\Gamma Y (X, M) \to R\Gamma Y_1 (X_1, f^{-1} M)
\]

which is functorial with respect to \( M \), in \( D^+ (\mathcal{T}_N(X^+), A_{\cdot, X^+}) \). If \( \text{opimm}(i, \mathcal{T}, A) \) holds there is also a canonical isomorphism

\[
f^{-1} R\Gamma Y (M) \to R\Gamma Y_1 (f^{-1} M)
\]

which is functorial with respect \( N \), in \( D^+ (\mathcal{T}_N(X), A_{\cdot, X}) \).

Proof. Thanks to \( \text{Lemma 3.13 (iii)} \) it suffices to establish the first isomorphism in \( (i) \) and \( \text{Lemma 3.13 (i)} \). Now both cases follow from \( \text{Lemma 3.13 (i)} \) using the following remark. If \( M \) is injective in \( \text{Mod}(\mathcal{T}_N(Y), A_{\cdot, Y}) \) then each \( M_{k, \gamma} \) is flasque (use \( \text{Lemma 2.11 (v)} \) applied to the variable topos \( \mathcal{T}^- (\cdot) \) on \( \mathcal{B} \times \mathcal{N}^{op} \)). In particular \( \iota_k^{-1} M \) is acyclic for \( \iota_* : \text{Mod}(\mathcal{T}(Y), A_{\cdot, Y}) \to \text{Mod}(\mathcal{T}(X), A_{\cdot, X}) \) and \( f^{-1} M \) is acyclic for \( i_* : \text{Mod}(\mathcal{T}(Y^+_1), A_{\cdot, Y^+_1}) \to \text{Mod}(\mathcal{T}(X^+_1), A_{\cdot, X^+_1}) \).

\( \square \)

### 3.4. Complete Mayer-Vietoris for semi-abelian schemes over \( C \).

We establish a Mayer Vietoris isomorphism (and triangle) relatively to the complete neighborhoods of the points of \( Z \subset C \). We begin by recalling an acyclicity result with regards to the morphism

\[
\epsilon : (-)^{FL} \to (-)^{et}
\]

**Lemma 3.16.** Consider a morphism of extremal type \( i : Y / \Delta' \to X / \Delta \) between diagrams of schemes and assume that each \( i_{\gamma} : Y_{\gamma} \to X_{\delta} \) is a closed immersion. Consider an abelian group \( M \) in \( X_{FL} \) (resp. a module \( M \), on \( (X_{FL}^N, \mathbb{Z}/p) \)) and assume that \( M = i_* N \) (resp. \( M = i_* N_\gamma \)) where each \( N_{\gamma} \) (resp. each \( N_{\gamma, k} \)) is representable by a smooth group scheme.

(i) The group \( M \) (resp. the module \( M \)) is acyclic for \( \epsilon_* \). The group \( N \) (resp. \( N \)) is acyclic for \( \epsilon_* \) and \( i_* \).

\( 45 \)
(ii) Both $RΓ^Y(M)$ and $RΓ^Y(X, ε_*M|_{X^+})$ (resp. $RΓ^Y(M)$ and $RΓ^Y(X, ε_*M,|_{X^+})$) are zero.

Proof. Using [2.14(i), 3.11(i), and 3.15(i)] it is sufficient to consider the case where $X$ and $Y$ are schemes (i.e., punctual diagrams) and $M$ is an abelian group.

(i) We know from [Mi1] III, 3.9 or [Gr1] Brauer III, 11.7 that $N$ is acyclic for $ε_*$. Since moreover the functor $i_* : Y_{et} → X_{et}$ is exact we find that $N$ is acyclic for the functor $i_* : Y_{FL} → X_{FL}$ as well (sheafify the isomorphism $H^q(X'_{FL}, Ri_*N') \cong H^q(X'_{et}, i'_*ε_*N')$ for $X'$ varying in the big flat site of $X$, $i' : Y' → X'$ the base change of $i$ to $X'$ and $N' = N|_{Y'}$). Finally the acyclicity of $M = i_*N$ for $ε_*$ results from the isomorphisms $Rε_*M \cong Ri_*Ri_*N ≃ Ri_*Rε_*N ≃ i_*ε_*N$.

(ii) We remark that the assumption $opimm(i, (−)^N_{FL}, ℤ/p)$ holds since arbitrary subschemes give rise to open subtopoi in the big top topos for as long top is coarser than or equal to $fl$. It thus follows from [3.13(ii)] that $RΓ^Y(M)$ is the mapping fiber of $M → i_*i^{-1}M$. Now the latter arrow is invertible since $M$ is in the essential image of the fully faithful functor $i_*$. We have thus proven that $RΓ^Y(M) = 0$. The case of $RΓ^Y(X, ε_*M|_{X^+})$ follows by [3.14(i)]

We can now establish the following result which states that the diagram $Rε_*A_{J^+, p}$ is sufficient to retrieve the projection of $RΓ^Z(A_p)$ to the small étale topos of $C$.

**Proposition 3.17.** Consider a semi-abelian scheme $A/C$ whose restriction to $U$ is abelian. Recall the closed immersions of extremal type $z : Z → C$ and $z_J : Z_J → J$ occurring in (58). There is a canonical isomorphism

$$Rε_*RΓ^Z(A_p) \cong Rm_*RΓ^Z(J, Rε_*A_{J^+, p})$$

in $D^+(C^N_{et}, ℤ/p)$. Moreover either side of the isomorphism remains unchanged if $A_p$ is replaced by $A^p$ or $ℤ/p \otimes ℤ A$.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
Z_J & \xrightarrow{z^+} & J^+ \\
m_J \downarrow & & \downarrow \rho_J \\
Z & \xrightarrow{z} & C^+ \\
& \sigma_J & \downarrow m \\
& \sigma & \downarrow \\
& C & \xrightarrow{m} & J
\end{array}
$$

For any $M$ in $D^+(C^N_{et}, ℤ/p)$, $D^+(C_{FL}, ℤ/p^k)$ or $D^+(C_{FL})$ we have canonically

$$Rε_*RΓ^Z(M) \cong Rε_*RΓ^Z(C, ρ^{-1}M)$$

$$\cong RΓ^Z(C, Rε_*ρ^{-1}M)$$

$$\cong Rm_*RΓ^Z(J, Rε_*Rm_*m^+, ρ^{-1}M)$$

$$\cong Rm_*RΓ^Z(J, Rε_*Rm_*m^+ m^{-1}, ρ^{-1}M)$$

where (76) is 3.13(iii), (77) is 3.13(v), (78) is the adjunction morphism for $m^+$, (79) is 3.13(iv), (80) is clear and (81) is 3.13(v). It thus remains to prove that (78) is invertible for $M = A_p$ or equivalently for $M = A_{p^k}$ (3.13(i)).
Claim. Consider $N = (N_Z \leftarrow N_C)$ in $\text{Mod}(C_{FL}^+, \mathbb{Z})$. If $N_C = A$ or $N_C \simeq z_*z^{-1}N_C$ then the following natural morphism is invertible:

\begin{equation}
R\epsilon_*N \rightarrow R\epsilon_*Rm^+_zM^+N^{-1}
\end{equation}

Let us first explain why the claim implies the proposition. The first case of the claim applied to $N = \rho^{-1}M$ shows that $(78)$ is invertible for $M = A \in D(C_{FL})$, and thus also (by scalar extension via $\mathbb{Z} \rightarrow \mathbb{Z}/p^k$) for $M = \mathbb{Z}/p^k \otimes^L_\mathbb{Z} A \in D^+(C_{FL}, \mathbb{Z}/p^k)$. Since multiplication by $p^k$ is epimorphic on $A^0$ (see 2.34(iii)) we have a long exact sequence in the first line below where $\Phi = A/A^0$ denote the group of components of $A$. The second line is the tautological distinguished triangle of truncation in $D(C_{FL}, \mathbb{Z}/p^k)$:

\begin{equation}
0 \rightarrow A^0 \rightarrow A \rightarrow A/p^k \rightarrow 0
\end{equation}

\begin{equation}
A/p^k \rightarrow \mathbb{Z}/p^k \otimes_\mathbb{Z} A[-1] \rightarrow A/p^k[-1]^{-1}
\end{equation}

Since $\Phi \simeq z_*z^{-1}\Phi$ (2.36(ii)) the second case of the claim implies that $(78)$ is invertible for $M = \Phi/p^k \simeq A/p^k$ or $\Phi/p^k$. Using the distinguished triangle we conclude that $(78)$ is invertible for $M = A/p^k$ and finally for $A^0/p^k$ as well using the exact sequence. The last statement of the proposition follows from the above proof together with the vanishing of $R\Gamma^Z(\Phi/p^k)$ and $R\Gamma^Z(\Phi/p^k)$ (2.36(iii), 3.16(ii)).

Let us now prove the claim using the conservative couple of functors $z^{+,-1} : D^+(C_{et}^+) \rightarrow D^+(Z_{et}), \sigma^{-1} : D^+(C_{et}^+) \rightarrow D^+(C_{et})$. Some preliminary observations are in order. First we note that $z^{+,-1}$ and $\sigma^{-1}$ both commute to $R\epsilon_*$ (because $z^+$ and $\sigma$ are inclusions of diagrams, see 2.14(i)). Next we observe that the natural base change morphisms $z^{+,-1}Rm^+_j \rightarrow m_{Z,*}z^{+,-1}$ and $\sigma^{-1}Rm^+_j \rightarrow Rm_\sigma^{-1}$ are both invertible (use the projective limit formula for $m_{Z,*}$, note that $m_{Z,*}$ is exact and that $\sigma^{-1} = \rho_*$ preserves injectives). With these observations in mind one may check without difficulty that the image of the morphism $(82)$ under $z^{+,-1}$ and $\sigma^{-1}$ respectively identifies with

\begin{equation}
R\epsilon_*z^{+,-1}N \rightarrow R\epsilon_*m_{Z,*}m^Z_{Z,-}z^{+,-1}N
\end{equation}

\begin{equation}
\text{and } R\epsilon_*\sigma^{-1}N \rightarrow R\epsilon_*Rm_*m^{-1}_Z\sigma^{-1}N
\end{equation}

On the one hand (83) is clearly an isomorphism since $m_{Z,*}$ is in fact an equivalence of topoi. It thus remains to prove that the morphism

\begin{equation}
R\epsilon_*N_C \rightarrow R\epsilon_*Rm^+_zm^{-1}_zs^{+,-1}N_C
\end{equation}

is invertible. Let us investigate the target of $(85)$. By 3.4(ii) and 3.6 we have a distinguished triangle

\begin{equation}
R\epsilon_*Rm^+_zm^{-1}_zs^{+,-1}N_C \rightarrow (\prod_{v \in Z} \iota_{C_v,*}R\epsilon_*(N_{C_v|C_v})) \oplus \sum_{v \in Z} Rj_*\iota_{U_v,*}R\epsilon_*(N_{C_v|U_v}) \rightarrow \prod_{v \in Z} Rj_*\iota_{U_v,*}R\epsilon_*(N_{C_v|U_v}) \rightarrow 1
\end{equation}

In the case $N_Z \simeq z_*z^{-1}N_C$ the third term and the second summand of the middle term vanish and the claim follows immediately. Let us now explain the case $N_C = A$. By [Ra4] and the previous case we may (and will) assume that $A$ is the Néron model of its generic fiber (note that the quotient $Q$ of the component group of the Néron model by $\Phi$ satisfies $Q \simeq z_*z^{-1}Q$ thanks to 3.16(i)). Recall from 3.16 that $A$ is acyclic for $\epsilon_*$ as well as its restriction to $U$, $C_v$ or $U_v$. Thus we only have to show that the above distinguished
triangle induces an exact sequence
\[
0 \longrightarrow \epsilon_* A \longrightarrow (\prod_{v \in Z} \epsilon_* A_{|C_v}) \oplus j_* \epsilon_* A_{|U} \longrightarrow \prod_{v \in Z} \epsilon_* A_{|U_v} \longrightarrow 0
\]
and isomorphisms
\[
R^q j_* \epsilon_* A_{|U} \longrightarrow \prod_{v \in Z} \epsilon_* A_{|U_v}^{\text{rig}}
\]
for \(q \geq 1\). The exactness of \((86)\) follows from the Néron extension property which implies that \(\epsilon_* A \rightarrow j_* \epsilon_* A_{|U}\) and \(\epsilon_* A_{|C_v} \rightarrow j_* \epsilon_* A_{|U_v}\) are invertible. Let us now investigate the stalks of the morphism \((87)\) at a geometric point \(\overline{v}\) of \(C\). Both sides give zero unless \(\overline{v}\) lies above a point \(v \in Z\). In the latter case we have on the one hand
\[
(R^q j_* \epsilon_* A_{|U})_{\overline{v}} \cong H^q(K^{sh}, \epsilon_* A_{|K^{sh}}) \cong \lim_{\rightarrow K'} H^q(K'^{h}, \epsilon_* A_{|K'^{h}})
\]
where both isomorphisms follow from [SGA4-II] VI, 5.7. Here \(K^{sh}\) denotes the fraction field of the strictly henselian ring \(\mathcal{O}_{C, \overline{v}}\) (with the notations of 3.6 \(K^{sh} = (K^{sep})_{\text{sep}}\), \(K'/K\) runs through finite subextensions of \(K^{sh}/K\) and \(K'^{h} = (K^{sep})_{\text{h}}\) is the fraction field of the corresponding henselian ring. On the other hand we have
\[
(l_{C_v, *} R^q j_{\overline{v}, * \epsilon_* A_{|U_v}})_{\overline{v}} \cong H^q(K'^{ur}, \epsilon_* A_{|K'^{ur}}) \cong \lim_{\rightarrow K'} H^q(K'^{ur}, \epsilon_* A_{|K'^{ur}})
\]
where \(K'^{ur}/K_v\) denotes the maximal unramified subextension of \(K^{sep}/K_v\). Here the first isomorphism follows from case 2 in 3.6 (note that \((R^q j_{\overline{v}, * \epsilon_* A_{|U_v}})_{\overline{v}} = 0\)) and the other ones follow from [SGA4-II] VI, 5.7. The morphism \((87)\) is thus the direct limit of the morphisms
\[
H^q(K'^{h}, \epsilon_* A_{|K'^{h}}) \rightarrow H^q(K'^{ur}, \epsilon_* A_{|K'^{ur}})
\]
for \(K'\) as above. We may now conclude using [Mi2] I, 3.10 (a) if \(q = 1\) and loc. cit. III, 6.10, 6.13 if \(q \geq 2\) (in that case both sides are in fact trivial), recalling that the étale cohomology of \(A_{|K'^{h}}\) (resp. \(A_{|K'^{ur}}\)) in degree \(\geq 1\) identifies with the direct limit of the flat cohomology of its torsion points.

\[\square\]

3.5. Rigid uniformization around semi-stable fibers.

3.5.1. For the purpose of this paragraph, let \(R\) denote a complete discrete valuation ring, \(t\) a uniformizer and \(k = R/(t)\) (resp. \(K = R^1(t)\)) its residual (resp. fraction) field. As explained in [Be2] 0, one has a diagram of functors:
\[
\begin{array}{ccc}
Sch_{\text{ft}}/R & 
\xrightarrow{(-)^{\text{for}}}_K

\xrightarrow{(-)^{\text{an}}}_K

Sch_{\text{ft}}/K & 
\end{array}
\]
where we have used the following notations:
- \(Sch_{\text{ft}}/R\) (resp. \(Sch_{\text{ft}}/K\)) denotes the category of \(\text{Spec}(R)\)-schemes (resp. \(\text{Spec}(K)\)-schemes) which are locally of finite type,
- $\text{For}/R$ denotes the category of $t$-adic formal schemes over $\text{Spf}(R)$ which are locally of finite type over $\text{Spf}(R)$, and $(-)^{\text{for}}$ is the functor of ($t$-adic) completion defined in [EGA1] 10.8.

- $\text{Rig}/K$ denotes the category of rigid analytic spaces over the $t$-adic field $K$ and the functors $(-)|_K$, $(-)^{\text{an}}$ are respectively defined in [Be2] 0.2, 0.3.

This diagram is not commutative. Instead for $X/R$ in $\text{Sch}_{\text{f}l}/R$, there is a functorial morphism $(X^{\text{for}})|_K \to (X^\text{an})|_K$ which is an open immersion if $X/R$ is separated and an isomorphism if $X/R$ is proper [loc. cit., 0.3.5].

3.5.2. Start with a semi-stable abelian variety $A/K$. Let $A/R$ denote the Néron model of $A/K$ and $A^0/R$ its connected component. There is an essentially unique pair $(G, e)$ where $G/R$ is a smooth algebraic group scheme which is the extension of a torus $T/R$ by an abelian scheme $B/R$ and

$$e : G^{\text{for}} \simeq (A^0)^{\text{for}}$$

is an isomorphism of groups in $\text{For}/R$ (note that $e$ is not algebraic in general). We refer to [SGA7-I] IX, 7 for the construction of $(G, e)$ and simply refer to $G/R$ as the Raynaud group attached to $A/K$.

The next result is originally due to [Ra3]. Here we slightly reformulate the statement given [BL] where the reader is referred for a proof.

**Proposition 3.18.** (Raynaud’s uniformization) [BL] thm. 1.2.

(i) There exists a unique arrow $e'$ making the following diagram commutative:

$$
\begin{array}{ccc}
(G^{\text{for}})|_K & \longrightarrow & (A^0)^{\text{for}}|_K \\
\downarrow & & \downarrow \\
(G)|_K^{\text{an}} & \longrightarrow & A^\text{an}
\end{array}
$$

(ii) The kernel of $e'$ computed in the category of groups in $\text{Rig}/K$ is of the form $\Gamma^\text{an}|_K$ for some group $\Gamma$ in $\text{Sch}_{\text{f}l}/R$ which is étale locally isomorphic to a constant free abelian group of rank $\dim_K T|_K$.

(iii) If the torus $T$ is split then $\Gamma$ is constant and $e'$ induces a surjection

$$
(G)|_K^{\text{an}}(S) \longrightarrow A^\text{an}(S)
$$

for any $S$ in $\text{Rig}/K$ verifying $H^1(S, \mathbb{Z}) = 0$.

Proof. This is [BL] 1.2 modulo the following remark. In loc. cit. (b) and (c) it is stated (using their notations) that $\text{Ker} p$ is a lattice and that the rigid analytic quotient of $E$ by $\text{Ker} p$ is isomorphic to $A$. Now looking into the construction of the quotient by glueing affinoids étale locally one finds easily [i] and [ii]. An example of this construction is given in [FV] 6.4.

□

**Corollary 3.19.** (i) Let $\text{Fin}/R$ denote the category of finite $\text{Spec}(R)$-schemes. The isomorphism $e$ of (89) induces an isomorphism of abelian presheaves on $\text{Fin}/R$:

$$G \stackrel{e}{\sim} A^0$$
(ii) Let $\text{Fin}/K$ denote the category of finite $\text{Spec}(K)$-schemes endowed with any topology which is finer than (or equal to) the étale topology. The morphism $e'$ of 3.18 (i) induces an exact sequence of abelian sheaves on $\text{Fin}/K$:

$$0 \to \Gamma_{|K} \to G_{|K} \to A \to 0$$

(iii) The morphisms (i) and (ii) are compatible, i.e. the following square

$$\begin{array}{ccc}
G(S) & \xrightarrow{e} & A^0(S) \\
\downarrow & & \downarrow \\
G_{|K}(S_{|K}) & \xrightarrow{e'} & A(S_{|K})
\end{array}$$

commutes for any $S$ in $\text{Fin}/R$.

Proof. (i) it suffices to notice that for any $H$ in $\text{Sch}_{lft}/R$ and $S$ in $\text{Fin}/R$ the functor $(-)_{\text{for}}$ identifies $H(S)$ with $H_{\text{for}}(S_{\text{for}})$ (just because finite $R$-algebras are t-adically complete).

(ii) Similarly for any $H$ in $\text{Sch}_{lft}/K$ and $S$ in $\text{Fin}/K$ the functor $(-)^{\text{an}}$ identifies $H(S)$ with $H^{\text{an}}(S^{\text{an}})$ (because finite $K$-algebras are complete for the $t$-adic topology).

(iii) is a straightforward consequence of the definition of the analytification functor. □

3.5.3. In order to collect the sheaf theoretic consequences of 3.19 it will be useful to introduce some intermediary sites. We use the general conventions of 2.3.1 for big and small pretopologies.

Definition 3.20. Let $X$ be any scheme. We will consider the small (resp. big) top pretopology of $X$ $\text{top}(X)$ (resp. $\text{TOP}(X)$) and the associated topos $X_{\text{top}}$ (resp. $X_{\text{TOP}}$) for $\text{top} = \text{qff, ff, or fet}$ defined as follows:

- qff stands for locally of finite presentation locally quasi-finite and flat,
- ff stands for finite and flat,
- fet stands for finite and étale.

The various inclusions of sites viewed as continuous functors give rise to quasi-morphisms of topos fitting in the following pseudo-commutative diagram.

It might be useful to point out that the topologies generated by $Q\text{FF}(X)$ and $F\text{L}(X)$ coincide, i.e. $X_{Q\text{FF}} = X_{F\text{L}}$. The morphism $\alpha$ is thus nothing but the projection morphism $p : X_{F\text{L}} \to X_{fl}$. In particular $\alpha_*$ commutes to arbitrary limits. Here is an alternative description of the topos $X_{qff}$ which will be useful for our purpose. Let $\text{top}$ stand for separated, of finite presentation, quasi-finite and flat. Then it follows from [AÃ] 3.1.3 and [EGA1] chap.1, 5.5.1, [EGA4-I] chap 4, 1.6.2, [EGA2] 6.2.4 that $X_{qff}$ is equivalent to $X_{\text{top}}$. 

50
Let us finally point out the equivalences $X_{qff} \simeq X_{ff}$ and $X_{fet} \simeq X_{et}$ in the particular case where $X$ is the spectrum of a field.

Let us go back to our situation, using the notation $X = \text{Spec}(R)$ and

$$ x = \text{Spec}(k) \xrightarrow{s} X \xleftarrow{j} \eta = \text{Spec}(K) $$

Lemma 3.21. (i) The continuous functor $ff(X) \to qff(X)$ (resp. $fet(X) \to et(X)$) inducing the left (resp. right) vertical premorphism in (92) is cocontinuous as well. Whence a morphism of topoi in the opposite direction

$$ \iota : X_{ff} \to X_{qff} \quad (\text{resp. } \iota : X_{fet} \to X_{et}) $$

(ii) The morphism $\iota$ defined in (i) is a closed immersions of topoi. The complementary open immersion is the morphism denoted $j$ in the top (resp. bottom) line of the following pseudo-commutative diagram of topoi

$$
\begin{array}{ccc}
X_{ff} & \xrightarrow{\iota} & X_{qff} \\
\downarrow{\beta} & & \downarrow{\beta} \\
x_{et} & \sim & X_{fet} \\
\end{array}
$$

(iii) Smooth group schemes over $X$ are acyclic for the functor $\beta_\ast$.

Proof. (i) In view of the above comments it is sufficient to prove that the inclusion of $\text{fin}(X)$ into the category of separated quasi-finite flat $X$-schemes admits a right adjoint which is continuous (recall that quasi-finite implies of finite type, hence of finite presentation since $X$ is noetherian). If $U/X$ is separated and quasi-finite there is a disjoint decomposition $U = U^{\text{fin}} \sqcup U_2$ where $U^{\text{fin}}/X$ is finite and $x \times_X U_2 = \emptyset$ [M1] I, 4.2 (c). Note that if $U'/X$ is finite then $\text{Hom}_X(U', U_2) = \emptyset$ (since $\text{Hom}_X(U', \eta) = \emptyset$). It follows in particular that $U^{\text{fin}}$ varies functorially with respect to $U/X$. Whence the desired functor $qff(X) \to ff(X)$. This functor is easily seen to be continuous (use that the finite part is a direct sum of spectrums of local rings [M1] I, 4.2 (c)) as well as right adjoint to the inclusion.

(ii) Let us explain the first line of (94) since the second is similar (and well known). The fact that $\iota$ is an immersion follows from the fact that the inclusion $ff(X) \to qff(X)$ is fully faithful. According to [SGA4-I] IV, 9.3.4 we have to show that the following properties (a), (b) are equivalent for any sheaf $F$ on $qff(X)$: (a) $F \to \iota_\ast \iota^{-1}F$ is an isomorphism, (b) $j^{-1}F = 1$. Now this equivalence is clear from the fact that $\iota_\ast \iota^{-1}F(U) = F(U^{\text{fin}})$.

(iii) We already know that smooth group schemes are acyclic for $\epsilon_\ast$ (3.16) and that $\alpha_\ast$ is exact. The result follows formally.

\[\square\]

In view of the next proposition let us emphasize that 3.21 (ii) implies an equivalence between $\text{Ab}(X_{qff})$ (resp. $\text{Ab}(X_{et})$) and the category of triples $(F_1, F_2, f)$ where $F_1 \in \text{Ab}(X_{ff})$ (resp. $\text{Ab}(X_{fet})$), $F_2 \in \text{Ab}(\eta_{qff})$ (resp. $\text{Ab}(\eta_{et})$) and $f : F_1 \to \iota^{-1}j_\ast F_2$ [SGA4-I] IV, 9.5.4. These equivalences (and thus also the functors $j_!$ of extension by zero) are clearly compatible via the restriction functor $\beta_\ast$.

3.5.4. Let us come back to the devissage of $A^0_{p\infty}$. We begin on the $qff$ topos.
Corollary 3.22.  
(i) The projections of 3.19 (i) and (ii) to $X_{ff}$ and $\eta_{ff} \simeq \eta_{aff}$ glue into an exact sequence as follows in $\text{Ab}(X_{aff})$:

$$
0 \longrightarrow j_! \alpha_* \Gamma |_{\eta} \longrightarrow \alpha_* G \longrightarrow \alpha_* A^0 \longrightarrow 0
$$

(ii) Applying $\mathbb{Z}/p^k \otimes_{\mathbb{Z}} L$ to (i) gives an exact sequence in $\text{Mod}(X_{aff}, \mathbb{Z}/p^k)$:

$$
0 \longrightarrow \alpha_* G_{p^k} \longrightarrow \alpha_* A^0_{p^k} \longrightarrow j_! \alpha_* (\Gamma |_{\eta}/p^k) \longrightarrow 0
$$

Proof. (i) We use the interpretation of $\text{Ab}(X_{qff})$ in terms of triples recalled after 3.21. The sheaf $\alpha_* G$ corresponds to the triple $(\iota^{-1} \alpha_* G, j^{-1} \alpha^* G, \text{nat} : \iota^{-1} \alpha_* G \to \iota^{-1} j_* j^{-1} \alpha^* G)$.

Explicitly for $S/\mathbb{Z}$ in $\text{fin}(X)$ (resp. $S/\eta$ in $\text{fin}(\eta)$) we have $\iota^{-1} \alpha_* G(S) = G(S)$ (resp. $j^{-1} \alpha_* G(S) = G_{|K}(S_K)$). A similar interpretation holds for $A^0$. The desired exact sequence is thus nothing but a reformulation of 3.19 (iii).

(ii) Exactness on the right follows from the fact that multiplication by $p^k$ is epimorphic on $G^1$ and thus on $\alpha_* G$ too.

Remark 3.23. From 3.19 and 3.22 one can go back to the big flat site as follows.

(i) Since $\Gamma |_{\eta}$ is representable by a group of $qff(\eta)$ (i.e. a locally finite group scheme) the adjunction morphism $\alpha^{-1} \alpha_* \Gamma |_{\eta} \to \Gamma |_{\eta}$ is invertible. In particular the first arrow of 3.22 (i) (or equivalently 3.19 (ii)) defines a morphism

$$(95) \Gamma |_{\eta} \to G |_{\eta} \quad \text{in} \quad \text{Ab}(\eta_{FL})$$

(ii) Similarly it follows from the fact that $G_{p^k}$ is representable by a group of $qff(X)$ (2.34 (ii)) that the first arrow of 3.22 (ii) defines a morphism

$$(96) G_{p^k} \to A^0_{p^k} \quad \text{in} \quad \text{Ab}(X_{FL}^N)$$

(iii) We may view the exact sequence of 3.19 (ii) as a quasi-isomorphism $[\Gamma |_{\eta} \to G |_{\eta}] \to A |_{\eta}$ in the category of complexes of abelian groups of $\eta_{aff}$. Applying $\mathbb{Z}/p^k \otimes \mathcal{L} (\cdot)$ and pulling back to the big flat topos yields an isomorphism of exact sequences in $\text{Ab}(\eta_{FL}^N)$:

$$(97) \begin{array}{c}
0 \longrightarrow G |_{\eta,p} \longrightarrow \mathbb{Z}/p^k \otimes_{\mathbb{Z}} L \Gamma |_{\eta} \to G |_{\eta} \longrightarrow \Gamma |_{\eta}/p \longrightarrow 0 \\
\downarrow \iota \quad \downarrow \iota \quad \downarrow \iota \\
0 \longrightarrow G |_{\eta,p} \longrightarrow A |_{\eta,p} \longrightarrow \Gamma |_{\eta}/p \longrightarrow 0
\end{array}$$

where the bottom left arrow is compatible with (96).

Proposition 3.24. The image of the morphism (95) under the functor

$$R\Gamma^x(X, R\epsilon_* \rho^{-1}) : D^+(X_{FL}, \mathbb{Z}/p^k) \to D^+(X_{et}, \mathbb{Z}/p^k)$$

fits into a canonical distinguished triangle

$$R\Gamma^x(X, R\epsilon_* G |_{X^{+},p}) \longrightarrow R\Gamma^x(X, R\epsilon_* A^0_{X^{+},p}) \longrightarrow j_! \epsilon_* (\Gamma |_{\eta}/p) \longrightarrow +1$$
Proof. The restriction of the morphism defined in 3.23(ii) to FL(x) (resp. qff(x)) clearly coincides with the natural isomorphism of group schemes $G_{|x,p^k} \simeq A_{|x,p^k}^0$ induced by (59) (resp. coincides with the first arrow of 3.22(ii)). If $X^+$ denote the diagram $(x \to X)$ as in 3.10 the exact sequence of 3.22(i) can thus be completed into an exact sequence

$$0 \to (\alpha_s A_{|x,p^k}^0) \to (\alpha_s A_{|x,p}^0) \to (0 \to j_!\alpha_s \Gamma|/p^k) \to 0$$

over $(X_{qff}^+, \mathbb{Z}/p^k)$. Apply now the functor $R\beta_* : D^+(X_{qff}^+, \mathbb{Z}/p) \to D^+(X_{et}^+, \mathbb{Z}/p)$ and then $R\Gamma^x(X, -) : D^+(X_{et}^+, \mathbb{Z}/p) \to D^+(X_{et}^+, \mathbb{Z}/p)$ to get a distinguished triangle

$$R\Gamma^x(X, R\beta_s \alpha_s G|_{X^+}) \to R\Gamma^x(X, R\beta_s \alpha_s A_{|X^+}^0) \to R\beta_* j_!\alpha_s \Gamma|/p^{k+1}$$

(use e.g. 3.13(i) for the third term). The announced distinguished triangle follows since $\alpha_s$ is exact and $j_!\alpha_s \Gamma|/p$ is $\beta_*$ acyclic (use 3.16 to check this).

\[\square\]

3.6. Dévissage in $p$-divisible groups.

3.6.1. We are now in position to define a $p$-divisible group $H$ over the diagram $J^+$ which will serve as a replacement for $A_{|J^+, p^\infty}$. By a $p$-divisible group $H$ over $J^+$ we mean an object of $pdiv(J^+)$ where $pdiv$ denotes the cofibered category of $p$-divisible groups over $\mathcal{S}ch^{op}$ (here we use the conventions of (2.1.4); in other terms, $H$ is a group in $J^+_{FL}$ whose components are $p$-divisible groups.

Definition 3.25. Let $A/C$ be a semi-abelian scheme whose restriction to $U$ is abelian. We define a $p$-divisible group $H$ over $J^+ = (Z_v \to C_v \leftarrow U_v \to U)$:

$$H := (G_v|Z_v, p^\infty \leftarrow G_v, p^\infty \to A_{|U_v, p^\infty} \leftarrow A_{|U_v}$$

where $G_v/C_v$ is the Raynaud group attached to $A_{|U_v}/U_v$ as in 3.5.2 and $j_v^{-1}G_v, p^\infty \to A_{|U_v}$ is the arrow induced by 3.23(ii).

The following result is the final stage of our dévissage. It states that $R\epsilon_\ast H_p$ is sufficient to retrieve the projection of $R\Gamma^Z(A_p)$ to the small étale topoi of $C$.

Proposition 3.26. Recall the functor $Sma : D(J_{et}^N, \mathbb{Z}/p) \to D(J_{et}^N, \mathbb{Z}/p)$ from 3.7. There is a canonical isomorphism

$$R\epsilon_\ast R\Gamma^Z A_p \simeq Rm_\ast Sm \Gamma^Z(J, R\epsilon_\ast H_p)$$

Proof. According to 3.17 it suffices to build an isomorphism between the right hand side and $Rm_\ast R\Gamma^Z(J, R\epsilon_\ast A_{|J^+, p}^0)$ in $D(C_{et}^N, \mathbb{Z}/p)$. Consider the morphism $H_p \to A_{|J^+, p}^0$ in $Mod(J_{et}^N, \mathbb{Z}/p)$ which is induced by 3.23(ii) on the vertices $Z_v, C_v$ and which is the identity on the vertices $U, U_v$. This and the natural transformation $Sma \to id$ (3.8(ii)) induce morphisms:

$$(98) \quad Sma R\Gamma^Z(J, R\epsilon_\ast H_p) \to Sma R\Gamma^Z(J, R\epsilon_\ast A_{|J^+, p}^0)$$

$$(99) \quad \to \Gamma^Z(J, R\epsilon_\ast A_{|J^+, p}^0)$$

in $D(J_{et}^N, \mathbb{Z}/p)$. We will prove that (98) and (99) are invertible.
In order to prove that (98) is an isomorphism it suffices to check the conditions of 3.9 for the morphism
\[ RΓ_{Z,j}(J, Rǫ_*H_p) \rightarrow RΓ_{Z,j}(J, Rǫ_*A^0_{j+1, p}) \]
Let us examine the component $U$. By [3.13][ii] there is a functorial isomorphism
\[ (RΓ_{Z,j}(J, M))_U \simeq RΓ^0(U, M_{U+}) \]
for $M$ in $D^+(J^{+\infty}, \mathbb{Z}/p)$. Here $\emptyset$ denote the diagram of empty type and the right hand side is thus naturally isomorphic to $M_U$. It follows that the morphism (100)$_U$ identifies to $(Rǫ_*H_p)_{U} \rightarrow (Rǫ_*A^0_{p, j+1})_{U}$. It is an isomorphism since $Rǫ_*$ can be computed componentwise on $J^+$. The same arguments show that (100)$_U$ is an isomorphism. Let us now examine the component $C_v$. By [3.15][ii] again we have
\[ (RΓ_{Z,j}(J, M))_{C_v} \simeq RΓ_{Z,v}(C_v, M_{C_v^+}) \]
The image of the morphism (100)$_{C_v}$ by $z_v^{-1}$ thus identifies to
\[ z_v^{-1}RΓ_{Z,v}(C_v, Rǫ_*H_{p, j}^{C_v^+}) \rightarrow z_v^{-1}RΓ_{Z,v}(C_v, Rǫ_*A^0_{p, j}^{C_v^+}) \]
and is an isomorphism by [3.24].

In order to prove that (99) is an isomorphism it suffice to check the conditions of 3.8 [ii] ie. that
\[ j_v^{-1}(RΓ_{Z,j}(J, Rǫ_*A^0_{j, p}))_{C_v} \rightarrow (RΓ_{Z,j}(J, Rǫ_*A^0_{j+1, p}))_{U_v} \]
is an isomorphism. This in turn follows immediately from [3.13][ii] and the fact that $j_v^{-1}Rǫ_*A^0_{C_v, p} \rightarrow Rǫ_*A^0_{U_v, p}$ is invertible (because $j_v$ is étale).

\[ \square \]

4. Preliminaries (part II)

4.1. Projective systems and $p$-divisibility over a $\mathbb{Z}/p$-algebra.

4.1.1. Consider a ringed variable topos on $\mathbb{N}^op$: $k \mapsto E_k$, $(k \leq k') \mapsto (\imath_{k, k'} : (E_k, A_k) \rightarrow (E_{k'}, A_{k'}))$ and let $(E, A)$ denote the associated ringed total topos. A module $M$ on the latter ringed topos is thus a projective system (in the sense of the appropriate cofibered category) whose $k^{th}$ term $M_k$ is in $Mod(E_k, A_k)$ and whose transition morphisms are $M_k \leftarrow \imath_{k, k'}^* M_{k'}$, $k \leq k'$. In what follows transition morphisms will often be implicit.

**Definition 4.1.** If $j \geq 1$ we define an endomorphism $\langle j \rangle$ of the ringed topos $(E, A)$ by the formulae $((\langle j \rangle)^{-1} F)_k = \imath_{k, k+j}^* F_{k+j}$ ($k \geq 1$) and $((j)_* F)_k = \imath_{k, k+j} F_{k-j}$ ($k \geq j + 1$), $((j)_* F)_k = 1$ ($1 \leq k \leq j$) together with the natural morphism $\langle j \rangle^{-1} A \rightarrow A$.

We note that there is a natural isomorphism $\langle j \rangle \simeq \langle 1 \rangle^j$ as well as a natural morphism $id \rightarrow \langle j \rangle$ (given by $((\langle j \rangle)^{-1} F)_k = \imath_{k, k+j}^{-1} F_{k+j} \rightarrow F_k$). In the following definition we make use of the derived functor $L(\langle j \rangle)^*$ for unbounded complexes [KS] 18.9.6.

**Definition 4.2.** (i) We say that $M$ in $Mod(E, A)$ is normalized if $\langle 1 \rangle^* M \simeq M$ (ie. if $\imath_{k, k+1}^* M_{k+1} \simeq M_k$ for all $k \geq 1$).

(ii) We say that $M$ in $D(E, A)$ is $L$-normalized if $L(\langle 1 \rangle)^* M \simeq M$ (ie. if $L\imath_{k, k+1}^* M_{k+1} \simeq M_k$ for all $k \geq 1$).
The following lemma will be used to “divide the Frobenius” by \( p \) in the definition of syntomic complexes. For \( i \geq 0 \), we use the notation \((E_{+i}, A_{+i})\) for the ringed total topos associated to \( k \mapsto (E_{+i+k}, A_{+i+k}) \), \((k \leq k') \mapsto \iota_{k+i,k'+i} \). The morphisms \((E_k, A_k) \to (E_{k+i}, A_{k+i})\) for varying \( k \) and \( i \) give rise to natural morphisms denoted as follows:

\[
\begin{array}{ccc}
(E_k, A_k) & \xrightarrow{i_{k+i}} & (E_{+i}, A_{+i}) \\
& \searrow & \downarrow \iota_{j, j+i} \\
& & (E_{+i+j}, A_{+i+j})
\end{array}
\]

**Lemma 4.3.** Assume that each one of the morphisms of topos \( \iota_{k,k+1} : E_k \to E_{k+1} \) is an equivalence and that \( A_{+i} \) is a flat normalized \( \mathbb{Z}/p \)-algebra. Consider modules \( L, M, N \), over \((E, A)\).

(i) The module \( M \) is \( L \)-normalized if and only it is normalized and \( \mathbb{Z}/p \)-flat.

(ii) For \( j \geq 1 \), there is a functorial exact sequence

\[
\iota_{j-1,j+*}\langle j \rangle^* M \xrightarrow{\tau_{j-1,j}} M_j \xrightarrow{\iota_{j-1,j}^*\iota_{j,j+1}^*} M_{j+1} \to 0
\]

of modules on \((E_{j+1}, A_{j+1})\). If \( M \) is \( \mathbb{Z}/p \)-flat the exact sequence can be extended with a 0 on the left. If \( M \) is normalized then the last term can be replaced with \( \iota_{j-1,j}^* M_j \).

(iii) If \( N \) is normalized and \( M \) is \( L \)-normalized then the group \( \text{Hom}(N, M) \) of \( A \)-linear morphisms is \( p \)-torsion free.

(iv) Consider an exact sequence \( 0 \to N \to M \to L \to 0 \) of modules on \((E, A)\) where \( L \) is normalized and \( M \) is \( L \)-normalized. If \( L \) is killed by \( p^j \), then \( (j)^* N \) and \( \tau_{j-1} L (j)^* L \) are \( L \)-normalized and fit into a canonical distinguished triangle of \( D(E, A) \):

\[
\begin{align*}
\langle j \rangle^* N & \to \langle j \rangle^* M & \tau_{j-1} L \langle j \rangle^* L & \to \langle j \rangle^* N[1] \\
0 & \to L' & \langle j \rangle^* N & \to M & \to L & \to 0
\end{align*}
\]

The associated long exact sequence of cohomology is

\[
0 \to L' \to \langle j \rangle^* N \to M \to L \to 0
\]

where \( L' \) denote a projective system where \( L'_k = L_k \) if \( k \geq j \) and \( \iota_{k,k}^* L'_k \to L_k \) is zero if \( k' \geq k + j \).

Proof. The morphisms \( \iota_{1,k} : E_1 \to E_k \) for varying \( k \) induce an equivalence \( E_1^\mathbb{N} \simeq E \). We may thus assume that \((E, A)\) is of the form \((E^\mathbb{N}, A)\).

[i] Since \( A \) is flat and normalized over \( \mathbb{Z}/p \), \( \mathbb{T}_{\mathbb{Z}/p^k}(A_k, -) \) and \( \mathbb{T}_{\mathbb{Z}/p^{k+j}}(\mathbb{Z}/p^k, -) \) coincide. Using this we find that \( M \) is \( L \)-normalized if it is \( \mathbb{Z}/p \)-flat and normalized. The other implication follows from the fact that \( M_k \) is \( \mathbb{Z}/p^k \)-flat if and only if \( \mathbb{T}_{\mathbb{Z}/p^k}(\mathbb{Z}/p, M_k) \) vanishes for \( q \geq 1 \).

[ii] The claimed exact sequence in \((E^\mathbb{N}, A_{+j})\) boils down to

\[
M_{+j}/p' \xrightarrow{p'} M_{+j} \xrightarrow{p} M_{+j}/p' \to 0
\]

which is straightforward from \( 0 \to \mathbb{Z}/p \to \mathbb{Z}/p^j \to \mathbb{Z}/p^j \to 0 \).

[iii] Recall that \( \langle j \rangle^* \langle j \rangle M_{+j} \) if \( k \geq j + 1 \) and 0 otherwise. Multiplication by \( p^j \) on \( M \) thus factors through \( \langle j \rangle^* M \). Since \( M \) is \( \mathbb{Z}/p \)-flat the resulting morphism...
\( \langle j \rangle^* \langle j \rangle_* M \to M \) is a monomorphism. We conclude by the following commutative square where the top arrow is an isomorphism because \( M \) is normalized:

\[
\begin{array}{ccc}
\text{Hom}(N, M.) & \xrightarrow{\langle j \rangle^*} & \text{Hom}(\langle j \rangle^* N, \langle j \rangle^* M.) \\
\downarrow \quad \rho^j & & \downarrow \quad \tau \\
\text{Hom}(N, M.) & \xrightarrow{\text{Hom}(N, \rho^j)} & \text{Hom}(N, \langle j \rangle_*(\langle j \rangle^* M.)
\end{array}
\]

\[\text{iv}\] Applying \( L\langle j \rangle^* \) to the given exact sequence gives a distinguished triangle

\[
L\langle j \rangle^* N \to L\langle j \rangle^* M \to L\langle j \rangle^* L \to \langle j \rangle^* N[1]
\]

The claimed distinguished triangle follows by truncation since the second term is concentrated in degree 0. Let us prove that \( (103) \) becomes

\[
\tau_{\geq -1} L\langle j \rangle^* L \simeq \tau_{\geq -1} \mathbb{Z}/p \hat{\otimes}_{\mathbb{Z}/p^+} \mathbb{Z}/p^{+j} \hat{\otimes}_{\mathbb{Z}} L
\]

\[
\tau_{\geq -1} \mathbb{Z}/p \hat{\otimes}_{\mathbb{Z}/p^+} \mathbb{Z}/p^{+j} \hat{\otimes}_{\mathbb{Z}} L
\]

in \( D(E^N, \mathbb{Z}/p) \). Consider the natural morphism

\[
\mathbb{Z}/p \hat{\otimes}_{\mathbb{Z}} L \to \mathbb{Z}/p \hat{\otimes}_{\mathbb{Z}/p^+} L
\]

in \( D(E^N, \mathbb{Z}/p^+) \). We claim that this morphism becomes invertible after \( \tau_{\geq -1} \). To check this it is sufficient to look at the \( k \)-th component and restrict scalars to \( \mathbb{Z} \). The morphism \( (105) \) becomes

\[
[L \xrightarrow{p^k} L] \to [\cdots \to L \xrightarrow{p^1} L \xrightarrow{p^k} L]
\]

and our claim results from the fact that \( p^1 \) is zero on \( L \). The right hand side in \( (104) \) is thus isomorphic to the left hand side of \( (105) \). It is in particular \( L \)-normalized. The description of \( L' \) in the long exact sequence of cohomology is obtained by letting \( k \) vary in the righthand side of \( (106) \): in \( D(E_+, \mathbb{Z}) \) we have an isomorphism

\[
\tau_{\geq -1} L\langle j \rangle^* L \simeq [L' \xrightarrow{p^1} L]
\]

Let us emphasize however that such an isomorphism certainly does not hold in general at the level of \( D(E_+, \mathbb{Z}/p) \).

Assume now given a ringed topos \((E, A)\) together with a morphism of variable ringed topoi on \( \mathbb{N}^{op} \): \( i_k : (E_k, A_k) \to (E, A), k \geq 1 \). In this situation we have a morphism

\[
l : (E, A) \to (E, A)
\]

and the typical example of normalized module (resp. an \( L \)-normalized complex) is \( l^* M \) (resp. \( Ll^* M \)).

**Lemma 4.4.** Assume that each \( i_k : E_k \to E \) is an equivalence. Assume that \( A \) is a flat \( \mathbb{Z}_p \)-algebra and \( A \simeq \mathbb{Z}/p \otimes l^{-1} A \). The restriction of the functor

\[
Rl_* : D^+(E_+, A) \to D^+(E, A)
\]
Definition 4.6. Let \( S \) be the full subcategory of \( L \)-normalized complexes is fully faithful.

Proof. Since \( Ll^* \) preserves \( D^+ \) it suffices to prove that the adjunction morphism \( Ll^*\text{Rl}_*M \rightarrow M \) is invertible for \( M \) \( L \)-normalized. As in the proof of \( \text{[EGA]}4.3 \) we may assume that \( E = E^N \) and \( A = \mathbb{Z}_p \). Then we invoke \( \text{[EGA]}2.2 \) (i).

\[ \square \]

Example 4.5. Let \( E = X^{\text{syn}}_\text{syn}, \; E = X^{\text{syn}}, \; A = \mathbb{Z}_p \) and \( G \in \text{pdiv}(X) \).

(i) The projective system \( G_p \) is \( L \)-normalized.

(ii) The isomorphism \( Ll^*\text{Rl}_*G_p \simeq G_p \) means that the sheaves \( R^q\text{l}_*G_p \) are uniquely \( p \)-divisible (but not necessarily trivial) for \( q \geq 1 \). For instance if \( X = \Sigma_1 \) then the restriction of \( R^q\text{l}_*\mathbb{Z}/p^q \) to the small \( \acute{e} \text{tale site} \) is \( \mathbb{Q}_p \).

4.2. Limits and quasi-coherence on \( p \)-adic formal schemes.

4.2.1. We adopt the following conventions regarding \( p \)-adic schemes, \( p \)-adic fine log schemes and their associated sites.

Definition 4.6. (i) A \( p \)-adic scheme \( X \) is an ind object of the category of schemes which is isomorphic to the inductive system of schemes \( X = (X_k)_{k \geq 1} \) arising from a \( p \)-adic formal scheme in the sense of \( \text{[EGA]}1 \). The category of \( p \)-adic schemes is denoted \( \text{Sch}_p \). The category \( \text{Sch}_{p,\text{nil}} \) of schemes where \( p \) is locally nilpotent is identified to a full subcategory of \( \text{Sch}_p \) in the natural way.

(ii) A \( p \)-adic log scheme \( X^\natural \) is an ind object of the category of log schemes which is isomorphic to an inductive system of log schemes \( X^\natural = (X^\natural_k)_{k \geq 1} \) satisfying \( X^\natural_k \simeq \Sigma_k \times X^\natural_{k+1} \) and such that the underlying ind-object of the category of schemes is in \( \text{Sch}_p \). The category of \( p \)-adic log schemes is denoted \( \text{Sch}^\natural_p \). The category \( \text{Sch}^\natural \) of fine log schemes and \( \text{Sch}_p \) are identified to full subcategories of \( \text{Sch}^\natural_p \) in the natural way.

We note that no information is lost by passing from the inductive system \( X^\natural \) to the associated ind-object \( X^\natural \) since \( X^\natural_k \simeq \Sigma_k \times X^\natural \) (product computed in \( \text{Sch}^\natural_p \)).

In virtue of \( \text{[EGA]}1 \) chap. 0, 10.6.2, 10.6.4 the category \( \text{Sch}_p \) is equivalent to the usual category of \( p \)-adic formal schemes (without any noetherian assumption). The category \( \text{Sch}^\natural_p \) on the other hand is probably too large to be a good category (see the remark after \( \text{[EGA]}4.8 \) below).

Definition 4.7. Let \( \text{top} \) be as in \( \text{[EGA]}2.3.1 \) and \( X^\natural \) in \( \text{Sch}^\natural_p \).

(i) The pretopology \( \text{TOP}^\natural(X^\natural) = \text{Sch}^\natural_p/X^\natural \) endowed with the pretopology for which a covering \( (U^\natural_f \rightarrow U) \) is a family whose reduction mod \( p^k \) is a covering in \( \text{TOP}^\natural(X^\natural) \) for all \( k \geq 1 \). The associated topos is denoted \( X^\natural_{\text{TOP}} \). The pretopology \( \text{TOP}(X^\natural) \) (resp. \( \text{top}(X^\natural) \)) is the full subcategory of \( \text{TOP}^\natural(X^\natural) \) formed by the \( U^\natural/X^\natural \)’s whose reduction mod \( p^k \) is in \( \text{TOP}(X^\natural_k) \) (resp. \( \text{top}(X^\natural_k) \)) for all \( k \geq 1 \) and where coverings are defined as in \( \text{TOP}^\natural(X^\natural) \). The associated topos is denoted \( X^\natural_{\text{top}} \) (resp. \( X^\natural_{\text{top}} \)).

(ii) The topos \( X^\natural_{\text{TOP}} \) is the \( \# \)-big top topos of \( X \) viewed as a diagram of type \( \mathbb{N}^\text{opp} \). Explicitly an object of \( X^\natural_{\text{TOP}} \) is thus a collection \( (F_k) \), \( F_k \in X^\natural_{k,\text{TOP}} \) together with transition morphisms \( F_{k+1} \rightarrow \iota_{k,k+1}^*F_k \) (here \( \iota_{k,k+1} \) denotes the inclusion \( X^\natural_k \rightarrow X^\natural_{k+1} \)). The topos \( X^\natural_{\text{TOP}} \) and \( X^\natural_{\top} \) are defined similarly.
The functoriality properties of these topoi with respect to \( X^\sharp \) and \( \text{top} \) are similar to the case of fine log schemes. We have moreover a pseudo-commutative diagram of topoi

\[
\begin{array}{ccc}
X^\sharp_{k,\text{TOP}} & \xrightarrow{i_k} & X^\sharp_{l,\text{TOP}} \\
\downarrow{l} & & \downarrow{l} \\
X^\sharp_{\text{TOP}} & & X^\sharp_{\text{TOP}}
\end{array}
\]

(107)

where the morphisms \( i_k \) and \( l \) have the following explicit description (as usual the description of transition morphisms are implicitly left to the reader).

- The functor \( i_k^{-1} \) takes \( F \) to \( F_k \). The functor \( i_k,* \) takes \( F \) to \( F' \) where \( F'_k = i_k,k,F \) if \( k' \geq k \) and 1 otherwise.
- The functor \( l^{-1} \) takes \( F \) to \( F' \) where \( F'_k = i_k^{-1}F \). The functor \( l_* \) takes \( F \) to \( \limproj_{j,k} F_k \).

When \( X^\sharp \) varies, (107) is a diagram of morphisms between variable topoi on \( \mathcal{S}ch^\sharp_p \). In the case of big and small topoi it has a weak analogue. These diagrams are pseudo-compatible with the premorphisms of projection between \( \sharp \)-big, big and small topoi and with the quasi-morphisms of changing the topology.

**Definition 4.8.** Let \( \text{top} \) be as in 2.3.1 and \( X^\sharp \) in \( \mathcal{S}ch^\sharp_p \).

(i) The structural \( \mathcal{O} \) ring of \( X^\sharp_{\text{TOP}} \) is defined as \( \mathcal{O} := \mathcal{O}_k \) where \( \mathcal{O}_k \) denotes the structural ring of \( X^\sharp_{k,\text{TOP}} \). The structural ring \( \mathcal{O} \) of \( X^\sharp_{l,\text{TOP}} \) is defined as \( \mathcal{O} := l_*\mathcal{O} \). The structural ring of the small topoi are defined by restriction.

(ii) Consider the morphism of monoids \( M_{X,*} \xrightarrow{} \mathcal{O}_k \) of \( X_{k,\text{et}} \) defining the log structure of \( X^\sharp_k \). Letting \( k \) vary defines a monoid \( M_{X,*} \) in \( X_{*,\text{et}} \) and a morphism \( M_{X,*} \xrightarrow{} \mathcal{O} \). The monoid \( M_X \) of \( X^\sharp_{*,\text{et}} \) is defined as \( M_X := l_*M_{X,*} \).

It does not seem clear to us wether or not \( M_X \to \mathcal{O} \) is always a fine log structure in the sense of [Sh]. This won’t be a problem for us since in practice our \( p \)-adic log schemes will always have nice charts (see 4.4.1).

**4.2.2.** Before discussing quasi-coherence over \( p \)-adic schemes, let us recall a possible definition for quasi-coherent modules on the usual sites of a fine log scheme. Log structure play essentially no role in the following discussion. In particular \( \sharp \)-big topoi could be replaced by big topoi in 4.9, 4.10 and 4.12 below.

**Definition 4.9.** Consider \( X^\sharp \) in \( \mathcal{S}ch^\sharp \). Let \( \text{prop} \in \{\text{qcoh}, \text{lf}, \text{lf ft} \} \). Let \( \text{Mod}_{\text{prop}}(X^\sharp_{\text{zar}}, \mathcal{O}) \) denote the category of quasi-coherent modules (resp. locally free, locally free of finite type) on the scheme \( X \) if \( \text{prop} = \text{qcoh} \) (resp. \( \text{lf}, \text{lf ft} \)). Consider the projection morphism \( p \) from the \( \sharp \)-big to small topoi. We define \( \text{Mod}_{\text{prop}}(X^\sharp_{\text{ZAR}}, \mathcal{O}) \) as the essential image of the functor

\[ p^* : \text{Mod}_{\text{prop}}(X^\sharp_{\text{zar}}, \mathcal{O}) \to \text{Mod}(X^\sharp_{\text{ZAR}}, \mathcal{O}) \]

A reasonable definition of quasi-coherence on the other usual sites requires the following lemma.

**Lemma 4.10.** Let \( \text{top} \) be as in 2.3.1 and let \( \text{prop} \) be as in 4.9.

(i) If \( M \) is in \( \text{Mod}_{\text{qcoh}}(X^\sharp_{\text{ZAR}}, \mathcal{O}) \) then it is in fact a sheaf on \( \text{TOP}(X^\sharp) \). We may thus define

\[ \text{Mod}_{\text{prop}}(X^\sharp_{\text{TOP}}, \mathcal{O}) := \text{Mod}_{\text{prop}}(X^\sharp_{\text{ZAR}}, \mathcal{O}) \]
(ii) Let $\epsilon$ denote the morphism of change of topology. The essential images of the following functors coincide:

\[ p_* : \text{Mod}_{\text{prop}}(X^*_\text{TOP}, \mathcal{O}) \to \text{Mod}(X^*_\text{top}, \mathcal{O}) \]

\[ \epsilon^* : \text{Mod}_{\text{prop}}(X^*_\text{zar}, \mathcal{O}) \to \text{Mod}(X^*_\text{top}, \mathcal{O}) \]

We define $\text{Mod}_{\text{prop}}(X^*_\text{top}, \mathcal{O})$ as the essential image in question.

(iii) Consider the following pairs of adjoint functors

\[ (p^*, p_) : \text{Mod}(X^*_\text{TOP}, \mathcal{O}) \to \text{Mod}(X^*_\text{top}, \mathcal{O}) \]

\[ (\epsilon^*, \epsilon_*) : \text{Mod}(X^*_\text{top}, \mathcal{O}) \to \text{Mod}(X^*_\text{zar}, \mathcal{O}) \]

Each one of these four functors preserves prop. The induced adjunctions on $\text{Mod}_{\text{prop}}$ are equivalences.

(iv) The contravariant pseudo-functors $\text{Mod}_{\text{prop}}((-)_{\text{TOP}}, \mathcal{O})$ and $\text{Mod}_{\text{prop}}((-)_{\text{top}}, \mathcal{O})$ are stacks for fl (i.e. verifies fl descent).

(v) A module $M$ of $(X^*_\text{TOP}, \mathcal{O})$ or $(X^*_\text{top}, \mathcal{O})$ verifies prop if and only if its restrictions to the elements of a surjective family of top morphisms do.

Proof. Recall the following fact from sheaf theory. Let $\mathcal{C}$ denote a full subcategory of $\text{Sch}^\sharp/\mathcal{X}^\sharp$ such that $U^\sharp \in \mathcal{C}$ and $V^\sharp/U^\sharp \in \text{top}(U^\sharp)$ imply $V^\sharp \in \mathcal{C}$. Consider on the one hand the ringed site $((\mathcal{C}, \text{zar}), \mathcal{O})$ (resp. $((\mathcal{C}, \text{top}), \mathcal{O})$) obtained by endowing $\mathcal{C}$ with the zar (resp. top) topology and the structural ring. Consider on the other hand $\text{Mod}((-)_{\text{zar}}, \mathcal{O})$ as a bifibered category over $\mathcal{C}^{\text{op}}$. Sending a module $F$ on $((\mathcal{C}, \text{zar}), \mathcal{O})$ to the collection $\xi$ of its restrictions $\xi(U) := F|_{U_{\text{zar}}(U)}$ together with the natural base change morphisms $\xi(f) : f^* F|_{U_{\text{zar}}(U)} \to F|_{U_{\text{zar}}(U')}$, $f : U' \to U$ realizes an equivalence of categories between the modules over $((\mathcal{C}, \text{zar}), \mathcal{O})$ and a full subcategory, say $\mathcal{F}_{\text{zar}}$, of the category of sections of the cofibered category $\text{Mod}((-)_{\text{zar}}, \mathcal{O})/\mathcal{C}^{\text{op}}$. Explicitly a section $\xi$ of the latter cofibered category is in $\mathcal{F}_{\text{zar}}$ if and only if $\xi(f) : f^* \xi(U) \to \xi(U')$ is invertible for every open immersion $f$. Under this equivalence the category of sheaves of modules over $((\mathcal{C}, \text{top}), \mathcal{O})$ corresponds to a full category, say $\mathcal{F}_{\text{zar}, \text{top}}$, of $\mathcal{F}_{\text{zar}}$. Explicitly a section $\xi$ in $\mathcal{F}_{\text{zar}}$ is in fact in $\mathcal{F}_{\text{zar}, \text{top}}$ if and only if it satisfies the following property (top descent): if $(f_i : U_i^\sharp \to U^\sharp)$ is a top surjective family then

\[ \xi(U)^\sharp \simeq \ker\left( \prod_i \xi(f_i)^* \xi(U_i^\sharp) \to \prod_i (f_j \times_{U^\sharp} f_k)^* \xi(U_j^\sharp \times_{U^\sharp} U_k^\sharp) \right) \]

Apply this to the case where $\mathcal{C}$ is either $\text{Sch}^\sharp/\mathcal{X}^\sharp$ or the category underlying $\text{top}(\mathcal{X}^\sharp)$. Starting from a module $M$ of $(X^*_\text{zar}, \mathcal{O})$ the pullback of $M$ to $((\mathcal{C}, \text{zar}), \mathcal{O})$ corresponds to the collection $\xi$ of pullbacks $\xi(U^\sharp) = f^* M$, $f : U^\sharp \to X^\sharp$ endowed with the obvious base change (iso)morphisms. If $M$ is quasi-coherent then it follows from descent theory that $\xi$ automatically verifies the property of top descent. In other terms the pullback of $M$ to $((\mathcal{C}, \text{zar}), \mathcal{O})$ is already a sheaf for top, and thus coincides with the pullback of $M$ to $((\mathcal{C}, \text{top}), \mathcal{O})$. All statements are straightforward from this remark. We take the opportunity to mention that the claim of [SGA4-II] VII, 2.1, c (which would imply full faithfulness of the functor $\epsilon^*$ on the whole category of modules) is incorrect (counter examples already exist for top = et).

Remark 4.11. (i) The category $\text{Mod}_{\text{qcoh}}(X^*_\text{top}, \mathcal{O})$ is abelian and the inclusion functor into $\text{Mod}(X^*_\text{top}, \mathcal{O})$ is exact. In the case top = zar this follows from [EGA1]
In the general case this follows from the flatness of the morphism \( \epsilon : (X^\sharp_{\text{top}}, \mathcal{O}) \to (X^\sharp_{\text{zar}}, \mathcal{O}) \) (which in turn results from the interpretation of \( \epsilon^* \) in terms of collections of small Zariski sheaves as in the above proof).

(ii) Consider a subsheaf \( F \) of a module \( M \) in \((X^\sharp_{\text{top}}, \mathcal{O})\). We may form the submodule \(< F >\) of \( M \) generated by \( F \) (ie. the image in \( M \) of the free module with basis \( F \)). It is not always the case that \(< F >\) is quasi-coherent even if \( M \) is. This is the case however if \( M \) is quasi-coherent and \( F \) is a constant sheaf.

**Lemma 4.12.** Consider a quasi-coherent module \( M \) on \((X^\sharp_{\text{TOP}}, \mathcal{O})\) (resp. \((X^\sharp_{\text{top}}, \mathcal{O})\)).

(i) Consider \( \text{top}' \) between \( \text{zar} \) and \( \text{top} \) and let \( \epsilon \) denote the morphism \( X^\sharp_{\text{TOP}} \to X^\sharp_{\text{TOP}} \) (resp. \( X^\sharp_{\text{top}} \to X^\sharp_{\text{top}} \)). Then \( M \) is acyclic for \( \epsilon_* \).

(ii) Let \( f : X^\sharp \to X^\sharp \) denote a morphism in \( \text{Sch}^\sharp \) whose underlying morphism of schemes is affine. Then \( M \) is acyclic for \( f_* \). The module \( f_\ast M \) is moreover quasi-coherent.

Proof. We only explain the case of small sites since the case of \( \sharp \)-big sites is similar (or alternatively follows formally). Let \( \mathcal{C}(X^\sharp, \text{top}) \) denote the full subcategory of \( \text{top}(X^\sharp) \) formed by the \( U^\sharp/U^\sharp \)'s with \( U \) affine. Endow \( \mathcal{C}(X^\sharp, \text{top}) \) with the pretopology of surjective families of \( \text{top} \) morphisms. The associated topos is thus equivalent to \( X^\sharp_{\text{top}} \) (see [A1] 3.1.3). It follows from faithfully flat descent theory that the Cech cohomology of a quasi-coherent module for an arbitrary covering in \( \mathcal{C}(X^\sharp, \text{top}) \) is trivial in degree \( \geq 1 \). In other terms quasi-coherent modules are \( \mathcal{C}(X^\sharp, \text{top}) \)-acyclic. Statement [i] (resp. the first statement of [ii]) thus follows from the fact that \( \epsilon \) (resp. \( f \)) arises from a premorphism \( \mathcal{C}(X^\sharp, \text{top}) \to \mathcal{C}(X^\sharp, \text{top}') \) (resp. \( \mathcal{C}(X^\sharp, \text{top}) \to \mathcal{C}(X^\sharp, \text{top}') \)). The second statement of [ii] follows from [EGA2] 1.2.6, 1.5.2 together with the characterization explained in the proof of [L10] for quasi-coherent modules as cocartesian sections of the cofibered category of quasi-coherent modules on the small Zariski site of a variable base.

\( \square \)

4.2.3. We discuss briefly a first candidate for the notions of quasi-coherence on a \( p \)-adic log scheme \( X^\sharp \). Logarithmic structures are left aside from the discussion since \( X^\sharp_{\text{top}} \simeq X_{\text{top}} \) and \( X^\sharp_{\text{top}} \simeq X_{\text{top}} \). We restrict furthermore to the étale topology which has the advantage that the morphisms \( t_{k,k+1} : X_{k,\text{et}} \to X_{k+1,\text{et}} \) and \( t_k : X_{k,\text{et}} \to X_{\text{et}} \) are equivalences.

**Definition 4.13.** Consider \( X \) in \( \text{Sch}_p \).

(i) We say that a module \( M \) of \((X_{\text{et}}, \mathcal{O})\) is quasi-coherent in the sense of [4.10] if each \( M_k \) is quasi-coherent.

(ii) Let \( M \) denote a module on \((X_{\text{et}}, \mathcal{O})\). We say that \( M \) is quasi-coherent if \( l^\ast M \) is quasi-coherent in the sense of [ii] and \( M \to l_\ast l^\ast M \) is an isomorphism.

**Remark 4.14.** By definition a \( p \)-adic scheme admits a covering by open sub \( p \)-adic schemes which are affine. Since quasi-coherent modules are acyclic on affine schemes the Mittag-Leffler criterion of [BO] 7.20 ensures the following:

(i) The natural morphism \( \psi_k^{-1} \mathcal{O}/p^k \to \mathcal{O}_k \) is invertible.

(ii) More generally if \( M \) is a normalized quasi-coherent module of \((X_{\text{et}}, \mathcal{O})\) then \( M_k \simeq \psi_k^{-1} l_* M \simeq \psi_k^{-1} l_* M/p^k \).
(iii) The functor \( l_\ast \) is in particular fully faithful on the category of normalized quasi-coherent modules. Whence a tautological equivalence

\[
\text{Mod}_{\text{norm,qcoh}}(X,\text{et},\mathcal{O}) \xrightarrow{(l^\ast,l_\ast)} \text{Mod}_{\text{coh}}(X,\text{et},\mathcal{O})
\]

We say that \( X \) is flat over \( \Sigma_\infty \) if each \( X_k \) is flat over \( \Sigma_k \). We say that \( X \to Y \) is a closed immersion if each \( X_k \to Y_k \) is a closed immersion.

**Lemma 4.15.** Consider \( X \) in \( \text{Sch}_p \).

(i) The structural ring \( \mathcal{O} \) of \( X_{\text{et}} \) is \( \mathbb{Z}_p \)-flat if and only if \( X \) is flat over \( \Sigma_\infty \).

(ii) Assume that \( X \) is flat over \( \Sigma_\infty \) and consider a closed immersion \( Y \to X \). Then \( Y \) is flat over \( \Sigma_\infty \) if and only if the corresponding quasi-coherent ideal \( I \) of \( (X,\text{et},\mathcal{O}) \) is normalized.

**Proof.** (i) follows from \([MW]\) lem. 2.1 and (ii) is straightforward.

\[\square\]

**4.2.4.** A drawback of the category of quasi-coherent modules in \( (X_{\text{et}},\mathcal{O}) \) as defined in \[4.13\] is that it fails to be abelian. This may justify the following alternative definition.

**Definition 4.16.** Consider a flat \( p \)-adic scheme \( X \) over \( \Sigma_\infty \).

(i) Consider \( M \) in \( D^+(X_{\text{et}},\mathcal{O}) \). We say that \( M \) is quasi-coherent if its cohomology modules are quasi-coherent in the sense of \[4.13\](i).

(ii) Consider \( M \) in \( D^+(X_{\text{et}},\mathcal{O}) \). We say that \( M \) is \( L \)-quasi-coherent if \( Ll^\ast M \) is quasi-coherent in the sense of \([ii]\) and \( M \to Rl_\ast Ll^\ast M \) is an isomorphism.

**Remark 4.17.** In virtue of \[4.14\](i) the morphism \( l : (X,\text{et},\mathcal{O}) \to (X,\text{et},\mathcal{O}) \) verifies the assumptions of \[4.4\]. Whence a tautological equivalence

\[
D^+_{L\text{norm,qcoh}}(X,\text{et},\mathcal{O}) \xrightarrow{(Ll^\ast,RL_\ast)} D^+_\text{Lqcoh}(X,\text{et},\mathcal{O})
\]

A drawback of the category \( D^+_\text{Lqcoh}(X,\text{et},\mathcal{O}) \) is that it is not stable by truncation. This issue might probably be resolved as in \([EK]\). We do not pursue this however since in practice the result \[4.3\] will be enough for our purpose.

**4.2.5.** The notions of quasi-coherence \([4.13]\) and \( L \)-quasi-coherence \([4.16]\) do not seem to be simply related. We will content ourselves with the following lemma.

**Lemma 4.18.** Assume that \( X \) is flat over \( \Sigma_\infty \). Consider a module \( M \) of \((X_{\text{et}},\mathcal{O})\) which is such that each \( i_k^\ast M \) is quasi-coherent.

(i) Assume that \( M \) is \( \mathbb{Z}_p \)-flat. Then \( M \) is quasi-coherent if and only it is \( L \)-quasi-coherent.

(ii) If \( M \) is killed by \( p^k \) for some \( k \geq 0 \) then it is quasi-coherent and \( L \)-quasi-coherent.

**Proof.** Let us denote \( M = l^\ast M \).

(i) Since \( M \) is \( \mathbb{Z}_p \)-flat the module \( M \) is both normalized and \( L \)-normalized. Thus it suffices to observe that \( l_\ast M \simeq RL_\ast M \) by Mittag-Leffler (\([BO]\) 7.20).

(ii) We have to prove that the morphisms \( M \to l_\ast l^\ast M \) and \( M \to RL_\ast Ll^\ast M \) are invertible. This is clear for the first one. To deduce it for the second one it suffices to check that \( l_\ast M \simeq RL_\ast M \) and \( RL_\ast l^\ast M = 0 \). The first condition follows from the fact that \( M \) is constant from rank \( k \) (ie. \( \mu_{k,k^\prime},_s M_k \simeq M_{k^\prime} \) for \( k^\prime \geq k \)). The second condition follows
by [Ek] 1.1 thanks to the fact that $\tau_{\leq -1} Ll^* M = N[1]$ where $N$ is essentially zero (ie. $\iota_{k,k'}^* N_{k'} \to N_k'$ is zero for $k' >> k'$) as seen using the natural $\mathbb{Z}_p$-flat resolution of $\mathbb{Z}/p$.

4.3. Limits and quasi-coherence on crystalline sites.

4.3.1. Consider $X^z$ in $\mathcal{S}ch^z/\Sigma_1$. Letting $k$ vary in definition 2.21 defines a variable topos on $\mathbb{N}^{op}$: $k \mapsto (X^z/\Sigma_k)^{CRYS^z,\text{top}}$, $(k \leq k') \mapsto \iota_{k,k'} : (X^z/\Sigma_k)^{CRYS^z,\text{top}} \to (X^z/\Sigma_{k'})^{CRYS^z,\text{top}}$. The same is true for big and small crystalline topoi.

\textbf{Definition 4.19.} The topos $(X^z/\Sigma_.)^{CRYS^z,\text{top}}$ is the total topos associated to the above variable topos. Explicitly an object is thus a collection $(F_k)_{k \in \mathbb{N}}$, $F_k \in (X^z/\Sigma_k)^{CRYS^z,\text{top}}$, together with transition morphisms $F_{k+1} \to \iota_{k,k+1,*} F_k$. The topos $(X^z/\Sigma.)^{CRYS,\text{top}}$ and $(X^z/\Sigma.)^{cryst,\text{top}}$ are defined similarly.

The functoriality properties of this topos with respect to $X^z$ and $\text{top}$ are similar to the case where $k$ is fixed. We have moreover a pseudo-commutative diagram of topos

\begin{equation}
\begin{array}{ccc}
(X^z/\Sigma_k)^{CRYS^z,\text{top}} & \xrightarrow{\iota_{k,k'}} & (X^z/\Sigma_{k'})^{CRYS^z,\text{top}} \\
\downarrow{\iota_k} & & \downarrow{\iota_{k',k}} \\
(X^z/\Sigma_\infty)^{CRYS^{z,\text{top}}} & \xrightarrow{\iota} & (X^z/\Sigma_\infty)^{CRYS^{z,\text{top}}}
\end{array}
\end{equation}

The formulae describing (107) describe (108) as well. When $X^z$ varies, (108) is a diagram of morphisms between variable topos on $\mathcal{S}ch^z$. In the case of big and small topoi it has a weak analogue. These diagrams are pseudo-compatible with the weak morphisms of projection between $z$-big, big and small topoi and with the weak morphisms of changing the topology.

The morphisms $i$, $u$ of (45) naturally induce morphisms

\begin{equation}
\begin{array}{ccc}
(X^z/\Sigma_{\text{TOP}^0}) & \xleftarrow{i} & (X^z/\Sigma.)^{CRYS^z,\text{top}}
\end{array}
\end{equation}

and similarly for big and small topoi. The functoriality properties of these morphisms are similar to the case where $k$ is fixed.

Assume now given $T^z$ in $\mathcal{S}ch^z$ as well as an exact divided power immersion $(U^z, T^z, t, \gamma)$. In $(X^z/\Sigma_k)^{CRYS^z,\text{top}}$ we have natural isomorphisms $(U^z, T^z_k) \to \iota_{k,k+1}(U^z, T^z_{k+1})$. Let us simply denote $T^z_k$ the object $(X^z/\Sigma_k)^{CRYS^z,\text{top}}$ obtained by inverting these isomorphisms. Then (46) naturally induces a morphism and a weak morphism as follow: morphisms of topos

\begin{equation}
\begin{array}{ccc}
(X^z/\Sigma.).^{CRYS^z,\text{top}} & \xleftarrow{f_{T^z}} & (X^z/\Sigma.).^{CRYS^z,\text{top}}/T^z \\
& \xrightarrow{\lambda_{T^z}} & T^z_{\text{top}}
\end{array}
\end{equation}

Generalizing 2.24 we define the restriction $F_{|T^z} := f_{T^z}^{-1} F$ of $F$ to $T^z$ and the realization $F_{T^z} := \lambda_{T^z,*} F_{|T^z}$ of $F$ on $T^z$. Similar observations and notations hold with $\text{CRYS}$ or $\text{cryst}$ instead of $\text{CRYS}^z$.

4.3.2. We discuss limits in crystalline topos. The relevant morphism of ringed topos is

\begin{equation}
l : ((X^z/\Sigma.).^{CRYS^z,\text{top}}, \mathcal{O}) \to ((X^z/\Sigma_\infty)^{CRYS^z,\text{top}}, \mathcal{O})
\end{equation}
Remark 4.21. are invertible for $k$. observing that the transition morphisms

we are thus finally reduced to check that $R\tilde{l}_M$, $P\tilde{l}_M$ $(\text{Lemma } 4.20.)$

coincides with the derived functor of $(-)$ $M$. This formula immediately implies that $M \rightarrow l_*l^{-1}M$ (resp. $l^{-1}l_*M \rightarrow M$) is invertible for any $M$ (resp. any normalized $M$).

(ii) It is sufficient to show that $(R\tilde{l}_M)_T = 0$, $q \geq 1$, if $T^q = (U^q, T^q)$ is an object of $\text{CRY S}^\infty(\mathbb{X}/\mathbb{Z}p)$ where $p$ is nilpotent, say $p^{k_0} = 0$. First we notice that $(-)_T \circ R\tilde{l}_*$ coincides with the derived functor of $(-)_T \circ l_*$. Now $(-)_T \circ l_* \simeq l_* \circ (-)_T$ where $(-)_T = \lambda_{T^q, l} f_{T^{-1}}$ as in (110). Next we claim that $R(l_* \circ (-)_T) \simeq Rl_* \circ R(-)_T$. This claim simply results from (2.15(ii)) applied to the following variable pretopologies on $\mathbb{N}^{op} \cup \{\infty\}$: $\mathcal{P} : k \mapsto \text{CRY S}^\infty(\mathbb{X}/\mathbb{S}_k)$, $\mathcal{P}' : k \mapsto \text{top}(T^q_k)$, the premorphism of pretopologies $g : \mathcal{P} \rightarrow \mathcal{P}'$ given by $T^q_k / T^q_k \mapsto (U^q_k \times \tau^q_k, T^q_k, T^q_k)$ (note that $g_{|\mathbb{N}^{op}, l} = (-)_T$) and the morphism of diagrams $f : \mathbb{N}^{op} \rightarrow \{\infty\}$ (note that $f_* = l_*$). Now $(-)_T$ is exact (see appendix) and we are thus finally reduced to check that $R\tilde{l}_M M_T$ vanishes for $q \geq 1$. We conclude by observing that the transition morphisms

$M_{T_{k+1}} \rightarrow l_{k,k+1,M} M_{T_k}$

are invertible for $k \geq k_0$. (iii) The proofs of (i) and (ii) work as well for $\text{CRY S}$ or $\text{cry s}$.

Remark 4.21. (i) The statements of (4.20) would hold verbatim for modules over an arbitrary ring.

(ii) The functors $l_*$ and $l^{-1}$ clearly preserve the crystal condition. The equivalence (4.20(i)) thus induces an equivalence (with obvious notations)

$\text{CRY S}_{\text{norm}}((\mathbb{X}/\mathbb{S})_{\text{CRY S}^\infty, \top}, \mathcal{O}) \overset{(l^{-1}, l_*)}{\sim} \text{CRY S}((\mathbb{X}/\mathbb{S}_\infty)_{\text{CRY S}^\infty, \top}, \mathcal{O})$

(iii) Let $\text{prop}$ as in (4.10). We say that a module $M$ has $\text{prop}$ realizations if each $M_T$ is $\text{prop}$. This condition is clearly preserved by $l_*$ and $l^{-1}$. Whence (with obvious notations):

$\text{Mod}_{\text{norm}, \text{prop}}((\mathbb{X}/\mathbb{S})_{\text{CRY S}^\infty, \top}, \mathcal{O}) \overset{(l^{-1}, l_*)}{\sim} \text{Mod}_{\text{prop}}((\mathbb{X}/\mathbb{S}_\infty)_{\text{CRY S}^\infty, \top}, \mathcal{O})$
(iv) The above remarks [(ii), (iii)] hold as well for CRY S or crys. Let $\Sigma = \Sigma$, or $\Sigma_k$, $k \leq \infty$ and let $p$ denote the premorphism of projection from the $\sharp$-big crystalline topos. The adjunction $(p^*, p_*)$ for $\mathcal{O}$-modules induces an equivalence

$$
\text{Crys}((X^*/\Sigma)_{\text{CRY S}_\ast, \text{top}}, \mathcal{O}) \xrightarrow{(p^*, p_*)} \text{Crys}((X^*/\Sigma)_{\text{crys}_\ast, \text{top}}, \mathcal{O})
$$

This equivalence clearly preserves the property of prop realizations. In the case $\Sigma = \Sigma$, the property norm is preserved as well. Similar remarks hold with $\text{CRY S} \sharp$ or crys instead of CRY S.

Lemma 4.22. Let $\Sigma$ denote either $\Sigma$, or $\Sigma_k$, $k \leq \infty$ and consider

$$
\epsilon : ((X^*/\Sigma)_{\text{CRY S}_\ast, \text{top}'}), \mathcal{O}) \rightarrow ((X^*/\Sigma)_{\text{CRY S}_\ast, \text{top}'), \mathcal{O})
$$

(i) The adjunction $(\epsilon^*, \epsilon_*)$ for $\mathcal{O}$-modules induces an equivalence

$$
\text{Crys}_{\text{qcoh}}((X^*/\Sigma)_{\text{CRY S}_\ast, \text{top}'}), \mathcal{O}) \xrightarrow{(\epsilon^*, \epsilon_*)} \text{Crys}_{\text{qcoh}}((X^*/\Sigma)_{\text{CRY S}_\ast, \text{top}'), \mathcal{O})
$$

(ii) Modules with quasi-coherent realizations are acyclic for $\epsilon_*$. 
(iii) Statements [(i)] and [(ii)] hold verbatim if $\text{CRY S} \sharp$ is replaced with CRY S or crys.

Proof. (i) We may assume that $\Sigma = \Sigma_k$. Recall that $\epsilon_*$ is fully faithful. We need to check the following: a. (resp. b.) the functor $\epsilon_*$ (resp. $\epsilon^*$) preserves the condition of being a crystal with quasi-coherent realizations and c. if $M$ is such a crystal then $M \simeq \epsilon_*, \epsilon^* M$. By abstract nonsense it is in fact sufficient to prove a. for arbitrary top and top' and b., c. for top = zar.

Let us prove a. Consider $M$ in $\text{Crys}_{\text{qcoh}}((X^*/\Sigma)_{\text{CRY S}_\ast, \text{top}'}), \mathcal{O})$ and let us check that $\epsilon_* M$ is a crystal with quasi-coherent realizations. Since $\epsilon_*$ pseudo-commutes to the realization functors, we only have to check the crystal condition, i.e. that for all $h : T_1^\ast \rightarrow T_2^\ast$, $h^*(\epsilon_* M)_T^\ast \rightarrow (\epsilon_* M)_T^\ast$ is invertible. Since both the source and the target are quasi-coherent this is equivalent to $\epsilon^* h^*(\epsilon_* M)_T^\ast \rightarrow \epsilon^* (\epsilon_* M)_T^\ast$ being invertible. Now the latter morphism identifies to $h^* M_{T_1^\ast} \rightarrow M_{T_2^\ast}$ (recall that by quasi-coherence $\epsilon^* \epsilon_* M_{T_1^\ast} \simeq M_{T_1^\ast}$). It is thus invertible indeed by the crystal condition for $M$.

Let us prove c. in the case top = zar. Consider $M$ in $\text{Crys}_{\text{qcoh}}((X^*/\Sigma)_{\text{CRY S}_\ast, \text{zar}}), \mathcal{O})$. The corresponding section $\xi$ of $\text{Mod}((-)_{\text{top}'}, \mathcal{O})$ over $\text{CRY S}^\sharp_{\text{zar}}(X^*/\Sigma)$ verifies that $\xi(f)$ is invertible for any top cartesian $f$ (and in fact for any $f$). Now it follows easily from this and [SGA1] VIII, 1.1.16 that the descent condition of 9.4(iiv) holds, i.e. that $M$ is in fact a sheaf for top'. In other terms $M \rightarrow \epsilon_* \epsilon^* M$ is invertible.

Let us now prove b. in the case top = zar. Consider $M$ in $\text{Crys}_{\text{qcoh}}((X^*/\Sigma)_{\text{CRY S}_\ast, \text{zar}})$ and let us check that $h^*(\epsilon^* M)_T^\ast \rightarrow (\epsilon^* M)_T^\ast$ is invertible. This will result from the crystal condition for $M$ if we show that $\epsilon^* (M_{T_1^\ast}) \rightarrow (\epsilon^* M)_{T_2^\ast}$ is invertible. Let us interpret the latter morphism in the category $F_{\text{zar, top}'}$ defined during the proof of 4.10. Let $\mathcal{C} = \text{top}'(T_1^\ast)$.

By c. above the source of the latter morphism identifies to $(f : T^\ast \rightarrow T_1^\ast) \mapsto M_{T^\ast}$ whereas the target identifies to $(f : T^\ast \rightarrow T_2^\ast) \mapsto f^* M_{T_2^\ast}$ by quasi-coherence of $M_{T_1^\ast}$. We may thus conclude by the crystal condition for $M$.

The cases CRY S and crys follow formally from the case CRY S by 4.21(iv).
(ii) Thanks to 2.13(i) it suffices to treat the case $\Sigma = \Sigma_k$ here as well. Now since $(-)_{T^1}$ and $\epsilon_*$ are induced by commuting continuous functors we have the following isomorphisms:

\begin{align*}
(111) & \quad (R\epsilon_* M)_{T^2} \cong R(((-)_{T^2} \circ \epsilon_*)M \\
(112) & \quad \cong R(\epsilon_* ((-))_{T^1})M \\
(113) & \quad \cong R\epsilon_* M_{T^2}
\end{align*}

We conclude by 4.12(ii) The case $CRYS$ or $crys$ is similar (or alternatively follows from the case $CRYS^\sharp$).

\[ \square \]

4.4. Local $p$-bases and local embeddings.

4.4.1. We recall the notion of $p$-bases for morphisms of schemes introduced by Kato in [Ka3] as well as a simplified notion of $p$-bases for morphisms of (p-adic) log schemes.

**Definition 4.23.** (i) A morphism $f : X \to Y$ between schemes of characteristic $p$ is relatively perfect if the relative Frobenius morphism

$$F^{(X/Y)} : X \to X^{(p/Y)}$$

is invertible.

(ii) A morphism $f : X \to Y$ between p-adic schemes is relatively perfect if each $f_k : X_k \to Y_k$ is formally étale and if $f_1 : X_1 \to Y_1$ is relatively perfect in the sense of (i).

(iii) A finite $p$-basis (of cardinal $d$) for a morphism $f : X \to Y$ of p-adic schemes is a $d$-uple $\underline{s} \in \Gamma(X, \mathcal{O}_X)^d$ such that the induced morphism

$$(\underline{s}, f) : X_k \to A^d_{\Sigma_k} \times Y_k$$

to the affine space (of dimension $d$) over $Y$ is relatively perfect for each $k$.

(iv) A finite $p$-basis (of cardinal $(d, e)$) for a morphism of p-adic log schemes $f : X^\sharp \to Y^\sharp$ is a couple $(\underline{s}, \underline{t})$ where $\underline{s} \in \Gamma(X, \mathcal{O}_X)^d$, $\underline{t} \in \Gamma(X, M_X)^e$ such that the induced morphism

$$(\underline{s}, \underline{t}, f) : X^\sharp_k \to A^d_{\Sigma_k} \times (A^e_{\Sigma_k}, \mathbb{N}^e) \times Y^\sharp_k$$

to the affine space of dimension $d + e$ over $Y_k$ with log structure induced by $M_Y$ and the canonical one on $A^e_{\Sigma_k}$ is strict and relatively perfect for each $k$.

(v) A morphism of p-adic log schemes $f : X^\sharp \to Y^\sharp$ has local finite $p$-bases if there exist strict étale coverings $(Y^\sharp_i \to Y^\sharp)$, $(X^\sharp_{ij} \to X^\sharp \times_{X^\sharp} Y^\sharp_i)$ such that each $X^\sharp_{ij}$ has finite $p$-bases over $Y^\sharp_i$.

**Remark 4.24.** In [Ts2] 1.4, the author defines a $p$-basis for a morphism $f : X^\sharp \to Y^\sharp$ of fine $\Sigma_k$-log schemes as a set of elements $(b_\lambda)_{\lambda}$ in $\Gamma(X, \mathcal{O}_X)$ together with a chart $\mathbf{ch}$ for $f$ having certain properties. If $(\underline{s}, \underline{t})$ is a finite $p$-basis in the sense of 4.23(iv) then $\underline{t}$ can be viewed as chart $\mathbf{ch}$ for the morphism $f$ and the couple $((s_\lambda), \mathbf{ch})$ is a $p$-basis in the sense of loc. cit. Our definition is more restrictive however since we only consider log structures of the type $\mathbb{N}^e$.

Let us gather some known useful properties which will be used freely in the text. We begin with facts about relative perfectness.

**Lemma 4.25.** (i) If a morphism of p-adic schemes is étale (ie. if its reductions mod $p^k$ are étale) then it is relatively perfect.
(ii) Relative perfectness as defined in 4.23 (i), (ii) is a Zariski local notion on the source. It is moreover stable by base change and composition.

(iii) Consider a relatively perfect morphism $X \to Y$ over $\Sigma_1$. If $\text{Spec}(A)$ is an affine open subscheme of $X$, $I$ is a finitely generated ideal in $A$ and $\hat{A}$ is the $I$-adic completion of $A$ then $\text{Spec}(\hat{A})$ is relatively perfect over $Y$.

(iv) Consider a commutative square in $\text{Sch}_p$:

$$
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow i & & \downarrow f \\
T & \longrightarrow & Y
\end{array}
$$

If $f$ is relatively perfect and if $i$ is a nilimmersion of order $n$ (ie. if each $U_k \to T_k$ is a closed immersion defined by an ideal whose sections on every open are nilpotent of order $n$) then there exists a unique arrow $h$ making the full diagram commutative.

(v) Let $X$ in $\text{Sch}_p$. The category of relatively perfect $p$-adic schemes over $X$ and $X_1$ are naturally equivalent. If $Y/X$ is relatively perfect then $Y_k/X_k$ is flat if and only if $Y_1/X_1$ is flat.

(vi) Consider a relatively perfect morphism $f : X \to Y$ over $\Sigma_1$. If $Y$ is regular then $f$ is flat. If $Y$ is regular and $X$ is locally noetherian then $X$ is regular.

Proof. Statement (i) is proven in [Ka1] 1.3 and (ii) is elementary. Statement (iii) is a particular case of [dJ1] 1.1.3. Let us briefly indicate the proof of (iv). By formal étaleness of $X_k/Y_k$ we are easily reduced to the case where $Y = Y_1$. But then the result follows from the fact that for $p^r \geq n$, $U^{(p^r/T)} \simeq T$ while $X \simeq X^{(p^r/Y)}$. In (v) both statements follow from [Ka] lem. 1 together with (iv). The first statement of (vi) is a particular case of [Ka4] 1.5 (whose proof refers to [Ka1] 5.2). The second statement is a consequence of formal smoothness over $\Sigma_1$ [EGA4-I] chap. 0, 22.5.8.

Now some facts about finite $p$-bases.

**Lemma 4.26.**

(i) Consider a log scheme of the form $X^\sharp = (X, Z)$ where $X$ is smooth over $\Sigma_1$ and $Z$ is a normal crossing divisor [Ka2] (1.5) (1), (2.5). Then both $X$ and $X^\sharp$ have local finite $p$-bases over $\Sigma_1$.

(ii) Morphisms of $p$-adic log schemes having finite $p$-bases or local finite $p$-bases are stable by composition and base change.

(iii) Consider a commutative square in $\text{Sch}_p^\sharp$

$$
\begin{array}{ccc}
U^\sharp & \longrightarrow & X^\sharp \\
\downarrow i & & \downarrow f \\
T^\sharp & \longrightarrow & Y^\sharp
\end{array}
$$

If $f$ has local finite $p$-bases and if $i$ is an exact nilimmersion of order $n$ (ie. its reductions are exact nilimmersions of order $n$) then after replacing $T^\sharp$ by a strict étale covering one can find $h$ rendering the diagram commutative.

(iv) Consider a $p$-adic log scheme $Y^\sharp$ and a morphism $X^\sharp \to Y^\sharp$ with $p$-basis $(s, t)$. Up to canonical isomorphism there exists a unique triple $(\tilde{X}^\sharp, \tilde{f} : \tilde{X}^\sharp \to Y^\sharp, (\tilde{s}, \tilde{t}))$ where $\tilde{f} : \tilde{X}^\sharp \to Y^\sharp$ is a lifting of $f$ and $(\tilde{s}, \tilde{t})$ is a $p$-basis for $\tilde{f}$ lifting $(s, t)$.
(v) If \((s, t)\) is a \(p\)-basis for a morphism of \(p\)-adic log schemes \(f : X^\sharp \to Y^\sharp\) then the module of logarithmic differentials is free:

\[
\Omega_{X^\sharp/Y^\sharp} := t_1^* \Omega_{X^\sharp/Y^\sharp} = \left( \bigoplus_{i=1}^e OX_{t_i} \right) \oplus \left( \bigoplus_{i=1}^e Od\log t_i \right)
\]

Proof. \([i]\) Up to étale localization we may assume that \(X\) is noetherian and that \(Z\) is a strict normal crossing divisor ([19J2] 2.4), i.e. \(Z = \bigcup_{i=1}^e Z_i\) (reduced scheme structure) with each \(Z_i := \cap_{j \neq i} Z_j\) smooth over \(\Sigma_1\) and of codimension \#\(J\) in \(X\) for each \(J \subset \{1, \ldots, e\}\). By Zariski localization and induction on \(e\) it suffices to assume that \(Z(1, \ldots, e)\) is non empty and to find \(p\)-bases at the neighborhood of each point of \(Z(1, \ldots, e)\) (by convention \(Z(1, \ldots, e) = X\) if \(e = 0\)). Consider a closed point \(x\) of \(Z(1, \ldots, d)\) and choose for each \(i\) a generator \(t_i\) of the ideal of \(\text{Spec}(\mathcal{O}_{X,x}) \times_X Z_i\) in \(\text{Spec}(\mathcal{O}_{X,x})\). Since \(\text{Spec}(\mathcal{O}_{X,x}) \times_X Z(1, \ldots, e)\) is regular of codimension \(e\) in \(\text{Spec}(\mathcal{O}_{X,x})\) it follows from [Se] chap. III prop. 15, chap. IV prop. 22] that \((t_1, \ldots, t_e)\) can be completed into a system of parameters generating \(\mathcal{m}_{X,x}\), say \((t_1, \ldots, t_e, s_1, \ldots, s_d)\). The resulting morphism \((s, t) : \text{Spec}(\mathcal{O}_{X,x}) \to A_{\Sigma_1}^{d+e}\) extends to a morphism \(U \to A_{\Sigma_1}^{d+e}\) which is étale at \(x\) [EGA-IV 17.5.3 for some open neighborhood \(U\) of \(x\). Shrinking \(U\) if necessary we may assume that this morphism is étale everywhere [EGA-IV 17.11.4] and that \(Z \cap U = \bigcup_{i=1}^e V(t_i)\). Then \((s, t)\) is a \(p\)-basis for \(U^\sharp := U \times_X X^\sharp\) and the proof is finished.

Statement \([ii]\) is immediate from the definition and \([iii]\) \([iv]\) respectively follow from \([12,iv]\) \([iv]\) Statement \([v]\) follows from [EGA-IV] chap. 0, 20.7.7 and [Ka2] 3.12. In \([vi]\) the equivalence follows from [Ka4] 1.4 (whose proof rely on [Ka1] 5.2 and [Ka] lem. 1) together with the fact that \(\iota_{1,k,*}(M_{X_1, k}/G_m) \simeq M_{X_k, k}/G_m\).

\[\square\]

Remark 4.27. Consider a perfect field \(k\). Here are some basic examples of a \(p\)-adic log scheme \(X^\sharp = X^\sharp_k\) with finite \(p\)-bases over \(\Sigma_\infty\).

(i) \(Y^\sharp = \text{Spec}(W(k)[S_1, \ldots, S_d, T_1, \ldots, T_e])\) or \(\text{Spec}(W(k)[[S_1, \ldots, S_d, T_1, \ldots, T_e]])\) and \(Y^\sharp = (Y, (\mathbb{N}^e \to \mathcal{O}, e_i \mapsto T_i))\). A \(p\)-basis is given by \((s, t)\) where \(s_i = S_i\) and \(t_i = e_i\).

(ii) \(Y = \text{Spec}(W(k)[S_1, \ldots, S_d, T_1, \ldots, T_e][(S_1 - \alpha_1)^{-1}, \ldots, (S_d - \alpha_d)^{-1}])\) with \(\alpha_i \in W(k)^\times\) or \(\text{Spec}(W(k)[[S_1, \ldots, S_d, T_1, \ldots, T_e]])\), \(Y^\sharp = (Y, (\mathbb{N}^e \to \mathcal{O}, e_i \mapsto T_i))\). A \(p\)-basis is given by \((\emptyset, t)\) where \(t_i = S_i - \alpha_i\), \(i \leq d\) and \(t_d = e_{d-1}, i \geq d + 1\).

(iii) Example \([ii]\) can be covered étale locally by example \([ii]\] (enlarge \(k\) if necessary and choose \(\alpha_1 = \ldots = \alpha_d = \alpha\) for \(d + 1\) values of \(\alpha\) whose reduction mod \(p\) are distinct and non zero). As a result if \(Y^\sharp\) is an arbitrary \(p\)-adic log scheme with local finite \(p\)-bases then they can in fact always be chosen of the form \((\emptyset, t)\).

4.4.2. We define categories of embeddings using the notion of finite \(p\)-bases introduced in \([4.4.1]\)

Definition 4.28. (i) We define the category \(\text{Emb}^\sharp\) and a full subcategory \(\text{Emb}^\sharp_{glob} \subset \text{Emb}^\sharp\) as follows:

- An object of \(\text{Emb}^\sharp\) (resp. \(\text{Emb}^\sharp_{glob}\)), called a local (resp. global) embedding, is a triple \((U^\sharp / X^\sharp, Y^\sharp, \iota)\) where \(U^\sharp\) and \(X^\sharp\) are fine separated \(\Sigma_1\)-log schemes, \(Y^\sharp\)
is a $p$-adic log scheme with local finite $p$-bases over $\Sigma_\infty$, $U^\sharp/X^\sharp$ is strict étale surjective (resp. $U^\sharp = X^\sharp$) and $\iota : U^\sharp \to Y^\sharp$ is a closed immersion (ie. $U^\sharp \to Y^\sharp_k$ is a closed immersion in the sense of log schemes).

- A morphism $f : (U^\sharp/X^\sharp,Y^\sharp,\iota) \to (U'^\sharp/X'^\sharp,Y'^\sharp,\iota')$ consists in a triple of compatible morphisms $(f_X,f_U,f_Y)$.

(ii) The category $\Emb^\sharp$ (resp. $\Emb^\sharp,\glob$) is viewed as a $\Sch^\sharp/\Sigma_1$-category via the forgetful functor $(U^\sharp/X^\sharp,Y^\sharp,\iota) \to X^\sharp$. The fiber above $X^\sharp$ is denoted $\Emb^\sharp(X^\sharp)$ (resp. $\Emb^\sharp,\glob(X^\sharp)$) and its objects are called local (resp. global) embeddings for $X^\sharp$. We say that $X^\sharp$ is locally (resp. globally) embeddable if $\Emb^\sharp(X^\sharp)$ (resp. $\Emb^\sharp,\glob(X^\sharp)$) is non empty.

(iii) A morphism $f = (f_X,f_U,f_Y)$ in $\Emb^\sharp$ (resp. $\Emb^\sharp,\glob$) is said to be above $f_X$, or to extend $f_X$ locally (resp. globally). We say that $f$ lifts $f_X$ locally (resp. globally) if the commutative square

$$
\begin{array}{ccc}
U^\sharp & \to & Y^\sharp \\
\downarrow f_U & & \downarrow f_Y \\
U'^\sharp & \to & Y'^\sharp
\end{array}
$$

is moreover cartesian.

We often use simplified notations such as $(U^\sharp,Y^\sharp)$ or even $Y^\sharp$ to designate an object of $\Emb^\sharp$ or $\Emb^\sharp,\glob$. In practice it will also be useful to consider the following smaller categories.

**Definition 4.29.**

(i) Let $\Sch^{\sharp,\lfpb}/\Sigma_1$ (resp. $\Sch^{\sharp,\lfpb}/\Sigma_1$) denote the category of separated log schemes with local finite $p$-bases over $\Sigma_1$ (resp. and whose log structure is trivial).

(ii) Let $\Emb^\sharp,\lfpb$ and $\Emb^{\sharp,\lfpb}$ denote the respective full subcategories of $\Emb^\sharp,\glob$ and $\Emb^\sharp$ defined by the condition that $X^\sharp$ (hence $U^\sharp$) has local finite $p$-bases over $\Sigma_1$.

(iii) Let $\Emb^{\glob}$, $\Emb^{\glob,\lfpb}$, $\Emb$, $\Emb^{\lfpb}$ denote the respective full subcategories of $\Emb^\sharp,\glob$, $\Emb^\sharp,\glob,\lfpb$, $\Emb^\sharp$, $\Emb^{\sharp,\lfpb}$ defined by the condition that $X^\sharp$ (hence $U^\sharp$) and $Y^\sharp$ have trivial log structures.

**Remark 4.30.** If $X = X^\sharp$ then $X$ is locally (resp. globally) embeddable if and only if $\Emb(X)$ (resp. $\Emb^{\glob}(X)$) is non empty (just forget log structures on $Y^\sharp$).

**Lemma 4.31.** Let $X^\sharp$ be a separated fine log scheme over $\Sigma_1$. Let $* \subset \{\sharp,\lfpb\}$.

(i) The categories $\Emb^{*,\glob}(X^\sharp)$ and $\Emb^{*}(X^\sharp)$ have products.

(ii) Consider a morphism $f_X : X'^\sharp \to X^\sharp$ and assume that $\Emb^{*}(X'^\sharp)$ is non empty. For any $Y'^\sharp \in \Emb^{*}(X'^\sharp)$ there exists $Y'^\sharp \in \Emb^{*}(X'^\sharp)$ and $f : Y'^\sharp \to Y^\sharp$ in $\Emb^{*}$ above $f_X$.

(iii) If $X^\sharp/\Sigma_1$ has finite $p$-bases then the category $\Emb^{*,\glob}(X^\sharp)$ contains liftings (ie. objects satisfying $X^\sharp \simeq Y^\sharp_1$). Closed subschemes of $X^\sharp$ are in particular globally embeddable.

(iv) The category $\Emb^{*}(X^\sharp)$ is non empty in the following cases:

1) $X^\sharp/\Sigma_1$ has local finite $p$-bases.
2) $X/\Sigma_1$ is of finite type.
3) $X = \Spec(A)$ where $A$ is the completion of a $\mathbb{F}_p$-algebra of finite type.
In the latter two cases, the category $Emb^\sharp(X^\sharp)$ contains objects $(U^\sharp/X^\sharp, Y^\sharp)$ with $U$ affine and $Y^\sharp = (Spf(\mathbb{Z}_p\{N^d\}))/N^e$ for some $e \geq 0$.

(v) If $X^\sharp$ has local finite $p$-bases and $Y^\sharp \in Emb^\sharp(X^\sharp)$ then its logarithmic divided power envelope $T^\sharp := D(X^\sharp, Y^\sharp)$ is flat over $\Sigma_\infty$.

Proof. (i) The product $(U^\sharp/X^\sharp, Y^\sharp, \iota) \times (U'^\sharp/X^\sharp, Y'^\sharp, \iota')$ is represented by

$$(U^\sharp \times_X U'^\sharp/X^\sharp, Y^\sharp \times Y'^\sharp, \iota'' : U^\sharp \times_X U'^\sharp \to Y^\sharp \times Y'^\sharp)$$

where $\iota''$ is natural morphism deduced from $\iota \times \iota'$. It is indeed a closed immersion thanks to $X^\sharp$ being separated. (EGA1) chap. 1, 5.4.2.

(ii) Let us begin by choosing an arbitrary $(U^\sharp, Y^\sharp, \iota')$ in $Emb^\sharp(X^\sharp)$. Observe that $U'^\sharp \times_X U^\sharp \cong U''^\sharp \times_X (X^\sharp \times_X U^\sharp)$ is a strict étale surjective $X^\sharp$-scheme via the first projection and that the morphism $\iota' : U'^\sharp \times_X U^\sharp \to Y'^\sharp \times Y^\sharp$ is a closed immersion. Projecting on second factors, we get the desired morphism above $f_X$:

$$f : (U'^\sharp \times_X U^\sharp, Y'^\sharp \times Y^\sharp, \iota'') \to (U^\sharp, Y^\sharp, \iota)$$

(iii) The first statement follows from (1.20(iv)) and the second statement follows from the first one.

(iv) Case 1 follows from (iii). Case 2: Note that if a local embedding of the claimed type exists for each $X_i^\sharp$ in a finite family then it also exists for the disjoint union of the $X_i^\sharp$. Replacing $X^\sharp$ with a strict étale covering if necessary we may thus assume given a fine chart $P \to M_X$. Choosing a surjection $\mathbb{N}^e \to P$ and a closed immersion $X \to Spec(\mathbb{F}_p[\mathbb{N}^d])$ gives a closed immersion $\iota : X^\sharp \to Spec(\mathbb{F}_p[\mathbb{N}^{d+e}], \mathbb{N}^e)$. Replacing $X^\sharp$ with an open covering and composing $\iota$ with an appropriate translation we may assume that this closed immersion factors through $Spec(S, \mathbb{N}^{d+e})$ and the result follows. Case 3 is deduced from case 2 by completion (4.26). (v) Since the question is étale local on $T$ and since the étale sites of $T$ and $X$ are naturally equivalent we may assume that $X^\sharp$ is separated and has finite $p$-bases. But then we can find a lifting $\tilde{X}^\sharp$ in $Emb^\sharp_{\text{glob}}(X^\sharp)$ and it follows from (1.32) 1.8 that $T^\sharp_k := D(X^\sharp, Y_k \times X^\sharp_k)$ is simultaneously an algebra of divided power polynomials above $X^\sharp_k$ and $T^\sharp_k$. Since $\tilde{X}^\sharp_k$ is flat over $\Sigma_k$, the result follows by faithfully flat descent along $T^\sharp_k/T^\sharp_k$. □

4.5. Exactness for crystals.

The purpose of this paragraph is to discuss some exactness properties of the categories of crystals over $(X^\sharp/\Sigma)_\text{crys,et}$ or $(X^\sharp/\Sigma_\infty)_\text{crys,et}$ and of the realization functors attached to local embeddings.

4.5.1. We use the terminology of [2541]

Lemma 4.32. Let $X^\sharp$ in $\text{Sch}^\sharp/\Sigma_1$, $\Sigma = \Sigma_\infty$ and prop $\subset \{\text{qcoh, lf, lft, norm}\}$. The full subcategory $\text{Crys}_{\text{prop}}((X^\sharp/\Sigma)_\text{crys,et}, \mathcal{O})$ of $\text{Mod}((X^\sharp/\Sigma)_\text{crys,et}, \mathcal{O})$ is closed by extensions. The induced exact structure will be denoted $e_M$ (exactness of modules).

Proof. This is clear from the exactness of the realization functors (9.4(iii)). □

In practice the exact structure $e_M$ is not very useful. In the subsequent paragraphs we will review another (weaker) canonical exact structure.
4.5.2. We begin with a preliminary result concerning the exactness of the pullback functor for modules with connections.

**Definition 4.33.** Consider a morphism \( f_X : X^\sharp \to X'^\sharp \) between fine separated \( \Sigma_1 \)-log schemes. We say that \( f_X \) admits flat liftings to local embeddings if there exist \((U^\sharp_1/X^\sharp, Y^\sharp_1, \iota_1)\) in \( \text{Emb}^{\sharp}(X^\sharp) \), \((U'^\sharp_1/X'^\sharp, Y'^\sharp_1, \iota'_1)\) in \( \text{Emb}^{\sharp}(X'^\sharp) \) and a morphism \( f = (f_X, f_U, f_Y) \) between them which lifts \( f_X \) locally \([4.28 (iii)]\) and which is such that each one of the morphisms \( Y_k \to Y'_k \) underlying \( f_Y \) is flat.

**Lemma 4.34.** If \( X'^\sharp \) is locally embeddable and \( f_X : X^\sharp \to X'^\sharp \) is a strict étale morphism between separated \( \Sigma_1 \)-log schemes then \( f_X \) admits flat liftings to local embeddings.

Proof. Replacing \( X'^\sharp \) by an étale covering if necessary we may assume given a global embedding \((X^\sharp, Y'^\sharp, \iota')\). We conclude using [SGA1] I, I.8.1.

Consider a global embedding \( Y^\sharp \) of \( X^\sharp \) with logarithmic divided power envelope denoted \( T^\sharp \). In this situation we may define the category of module with quasi-nilpotent integrable connection over \( T^\sharp_k \) as in [9.19]. This category will sometimes be denoted \( \nabla-\text{Mod}^{pd}(X^\sharp, Y^\sharp) \) instead of \( \nabla-\text{Mod}(T^\sharp_k) \) in order to emphasize the dependance in \( Y^\sharp \). We also denote \( \nabla-\text{Mod}^{pd}(X^\sharp, Y^\sharp) \) the category obtained by letting \( k \) vary. Note that these categories are clearly abelian.

**Lemma 4.35.** Consider a morphism \( f = (f_X, f_Y) : (X^\sharp, Y^\sharp, \iota) \to (X'^\sharp, Y'^\sharp, \iota') \) in \( \text{Emb}^{\sharp, \text{glob}} \). If \( f_X \) admits flat liftings to local embeddings then the pullback functor

\[
f^* : \nabla-\text{Mod}^{pd}(X^\sharp, Y^\sharp) \to \nabla-\text{Mod}^{pd}(X'^\sharp, Y'^\sharp)
\]

described in \([9.21]\) is exact.

Proof. Denote \( T^\sharp \) and \( T'^\sharp \) the respective logarithmic divided power envelope of \( \iota \) and \( \iota' \). Since et\( (X^\sharp) \simeq \text{et}(T^\sharp) \) and et\( (X'^\sharp) \simeq \text{et}(T'^\sharp) \) we find that the problem is étale local on \( X' \) and \( X \). We may thus assume given a morphism \( f_1 = (f_X, f_{1,Y}) : (X^\sharp, Y^\sharp_1) \to (X'^\sharp, Y'^\sharp_1) \) in \( \text{Emb}^{\sharp, \text{glob}} \) such that the \( f_{1,Y,k} \)'s are flat. Letting \( Y^\sharp_2 \) (resp. \( Y'^\sharp_2 \)) denote the product \( Y^\sharp \times Y^\sharp_1 \) (resp. \( Y'^\sharp \times Y'^\sharp_1 \)) computed in \( \text{Emb}^{\sharp, \text{glob}}(X^\sharp) \) (resp. \( \text{Emb}^{\sharp, \text{glob}}(X'^\sharp) \)) and \( f_2 = f \times f_1 \) we get from \([9.20]\) a pseudo commutative diagram

\[
\begin{array}{cccc}
\nabla-\text{Mod}^{pd}(Y^\sharp) & \xrightarrow{p_1} & \nabla-\text{Mod}^{pd}(Y^\sharp_2) & \xrightarrow{p_2} & \nabla-\text{Mod}^{pd}(Y^\sharp_1) \\
\nabla-\text{Mod}^{pd}(Y'^\sharp) & \xrightarrow{p_1'} & \nabla-\text{Mod}^{pd}(Y'^\sharp_2) & \xrightarrow{p_2'} & \nabla-\text{Mod}^{pd}(Y'^\sharp_1) \\
\end{array}
\]

where horizontal arrows are equivalence. We are thus reduced to the case where \( f = f_1 \) but then the statement follows from \([9.21]\) since the logarithmic divided power envelope \( T^\sharp \) of \( Y^\sharp \) is flat over the logarithmic divided power envelope \( T'^\sharp \) of \( Y'^\sharp \).

\[\square\]

4.5.3. We are now in position of discussing the case of crystals over \((X^\sharp/\Sigma)_{\text{cris-et}}\).

**Proposition 4.36.** Consider \( X^\sharp \) in \( \text{Sch}^\sharp/\Sigma_1 \) which is locally embeddable and let \( T^\sharp \) denote the logarithmic divided power envelope of some local embedding \( Y^\sharp \) for \( X^\sharp \).
(i) The category $\text{Crys}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$ is abelian and

$(-)_{\mathcal{T}^\sharp} : \text{Crys}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O}) \to \text{Mod}(\mathcal{T}^\sharp_{\text{crys,et}}, \mathcal{O})$

is an exact faithful functor.

(ii) Crystals with quasi-coherent realizations form a fully abelian subcategory $\text{Crys}_{\text{qcoh}}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$ of $\text{Crys}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$.

(iii) If $\text{prop} \subset \{\text{norm, qcoh, lf, lff} \}$ then $\text{Crys}_{\text{prop}}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$ is closed by extensions in the abelian category $\text{Crys}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$. The induced exact structure will be denoted $e$ (exactness of crystals).

Proof. (i) Let $\Delta$ denote the category whose objects are $[\nu]$, $0 \leq \nu \leq 2$ and whose arrows are the strictly increasing maps. Consider the diagram of type $\Delta^{\text{op}}$ whose vertex $Y^\sharp_{[\nu]} = (U^\sharp_{[\nu]}, Y^\sharp_{[\nu]})$ is the $\nu + 1$-th fold product of $Y^\sharp$ computed in the category $\text{Emb}^\sharp(X^\sharp)$. It follows from 9.20 and 9.10 (iii) that $\text{Crys}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$ is equivalent to the category $\Gamma_{\text{cort}}(\mathcal{F}/\Delta)$ of cartesian sections of the cofibered category over $\mathcal{F}/\Delta$ given by $[\nu] \mapsto \nabla\text{-}\text{Mod}^\sharp(Y^\sharp_{[\nu]}), (f : [\nu] \to [\nu']) \mapsto f^*$. The result now follows by exactness of the functors $f^*$. (4.34, 4.35).

(ii) Using the crystal condition we find easily that the conditions of having quasi-coherent realizations can be tested by realization on the given $\mathcal{T}^\sharp$. The result follows by exactness of the realization functor $(-)_{\mathcal{T}^\sharp}$ of (i). The proof of (iii) is similar.

\[\square\]

Remark 4.37. (i) Concretely, a short sequence of crystals $\mathcal{E} : 0 \to M_1 \to M_2 \to M_3 \to 0$ is $e_\mathcal{M}$-exact (resp. $e$-exact) if all realizations of the $\mathcal{E}_k$'s are exact (resp. if the realization of $\mathcal{E}_k$ at $T^\sharp_k$ is exact for each $k$, $T^\sharp$ being fixed as in 4.30). The exact structure $e_\mathcal{M}$ is thus stronger than $e$. This may be reformulated by saying that $e$ is not induced by the inclusion of the category of crystals into the category of modules over $((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$. Instead it is induced by the inclusion into the category of modules over a suitable variant of the restricted crystalline site of [Be1 IV, 2].

(ii) If a cofibered category has a cleavage by functors commuting to direct limits then its category of sections has the same exactness properties than its fibers. It thus follows from 4.30 (ii) that the category $\text{Crys}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$ is abelian as well when $X^\sharp$ is a diagram whose vertices are locally embeddable (no condition is needed on the edges here).

4.5.4. Let us turn to the case $k = \infty$. As a preliminary observation let us notice that the equivalence 4.14 (iii) endows $\text{Mod}_{\text{qcoh}}(\mathcal{T}_\text{et}^\sharp, \mathcal{O})$ with a canonical exact structure which will be denoted $e$.

Proposition 4.38. Let $X^\sharp$, $Y^\sharp$, $T^\sharp$ as in 4.30.

(i) The fully faithful functor

$l^{-1} : \text{Crys}((X^\sharp/\Sigma_\infty)_{\text{crys,et}}, \mathcal{O}) \to \text{Crys}((X^\sharp/\Sigma)_{\text{crys,et}}, \mathcal{O})$

induces an exact structure on its source which will be denoted $e$ (exactness of crystals).

(ii) If $\text{prop} \subset \{\text{qcoh, lf, lff} \}$ then $\text{Crys}_{\text{prop}}((X^\sharp/\Sigma_\infty)_{\text{crys,et}}, \mathcal{O})$ is a fully $e$-exact subcategory of $\text{Crys}((X^\sharp/\Sigma_\infty)_{\text{crys,et}}, \mathcal{O})$. 

71
(iii) Consider the realization functor \((-)_{T^2} : \text{Mod}( (X^2/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O}) \to \text{Mod}(T^2_{et}, \mathcal{O})\)
defined by the formula \(M_{T^2} = l_{*}(l^{-1}M)_{T^2}\). This induces a functor
\[\text{Crys}_{qcoh}((X^2/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O}) \to \text{Mod}_{qcoh}(T^2_{et}, \mathcal{O})\]
which is faithful e-exact and reflects e-exactness.

Proof. (i) This follows from the \textit{crys} variant of the equivalence 4.21 (ii) together with
the case \(\text{prop} = \text{norm}\) in 4.36 (iii).

(ii) This follows from 4.36 (i) thanks to the fact that the functor
\[l^{-1} : \text{Crys}_{\text{prop}}((X^2/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O}) \to \text{Crys}_{\text{prop,norm}}((X^2/\Sigma)_{\text{crys,et}}, \mathcal{O})\]
is an equivalence (4.21 (ii), (iii), (iv)).

(iii) We have a pseudo-commutative square as follows:

\[
\begin{array}{ccc}
\text{Crys}_{qcoh}((X^2/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O}) & \xrightarrow{l^{-1}} & \text{Crys}_{qcoh}((X^2/\Sigma)_{\text{crys,et}}, \mathcal{O}) \\
\downarrow (-)_{T^2} & & \downarrow (-)_{T^2} \\
\text{Mod}_{qcoh}(T^2_{et}, \mathcal{O}) & \xrightarrow{l^*} & \text{Mod}(T^2_{et}, \mathcal{O})
\end{array}
\]

Indeed if \(M\) is a crystal with quasi-coherent realizations in \((X^2/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O})\) then
\((l^{-1}M)_{T^2}\) is normalized and quasi-coherent, hence verifies \(l^*l_*(l^{-1}M)_{T^2} \simeq (l^{-1}M)_{T^2}\). Now
the statement follows formally from the fact that the horizontal arrows and the right
vertical one are faithful e-exact and reflect e-exactness (4.36 (i)).

\(\square\)

4.5.5. If one is interested only in crystals with locally free realizations the situation is
simpler.

**Proposition 4.39.** The exact structures \(e_M\) and \(e\) coincide on \(\text{Crys}_{lf}((X^2/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O})\)
and \(\text{Crys}_{lf}((X^2/\Sigma)_{\text{crys,et}}, \mathcal{O})\).

Proof. In both cases one is reduced to prove that a sequence of locally free crystals
\(0 \to M' \to M \to M'' \to 0\) in \((X^2/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O})\) which is e-exact is \(e_M\)-exact as well. Let
\(T^2\) be as in 4.36 and consider an arbitrary \(T^2_{et}\) in \(\text{crys}_{et}(X^2/\Sigma^\infty)\). Replacing \(T^2\) with a
covering if necessary we may always assume that there exists a morphism \(g : T^2 \to T^2_k\).
Consider the following commutative diagram

\[
\begin{array}{c}
0 \longrightarrow M'_{T^2} \longrightarrow M_{T^2} \longrightarrow M''_{T^2} \longrightarrow 0 \\
\uparrow \downarrow \uparrow \downarrow \downarrow \downarrow \\
0 \longrightarrow g^*M'_{T^2_k} \longrightarrow g^*M_{T^2_k} \longrightarrow g^*M''_{T^2_k} \longrightarrow 0
\end{array}
\]

The vertical arrows are invertible by the crystal condition and the bottom line is exact by
local freeness of \(M''\). The top horizontal line is thus exact as well and we are done (see
4.37 (i)).

\(\square\)

**Remark 4.40.** Statements 4.35, 4.36 and 4.39 have obvious counterparts over \(\Sigma^\infty\) if
\(k < \infty\).
4.6. **Twists by effective logarithmic divisors.**

4.6.1. We recall the basics of twisting in a general setting. Let \((E, A)\) be a ringed topos and assume given an integral log structure \(\alpha : M_E \to A\) (ie. \(M_E\) is a monoid of \(E\), \(\alpha\) is a morphism of monoids, \(M_E^\times := \alpha^{-1}A^\times\) is sent isomorphically to \(A^\times\) by \(\alpha\) and the natural morphism of \(M_E\) into its fraction group \(\text{Fr}_E\) is monomorphic).

**Definition 4.41.** Let \((E, A, M_E, \alpha)\) as above.

(i) An effective log divisor \(h\) is a section of the monoid \(M_E/M_E^\times\).

(ii) If \(h\) is an effective log divisor, the associated line bundle

\[
A(-h) := P_h \land_{M_E^\times} A
\]

is defined as follows: \(P_h\) is the \(M_E^\times\)-torsor determined by \(h\) (ie. \(P_h := q^{-1}\{h\}\) where \(q : M_E \to M_E/M_E^\times\)) and \(A(-h)\) is the quotient of \(P_h \times A\) by the action of \(M_E^\times\) given as \(m.(p,a) = (pm^{-1}, \alpha(m)a)\) endowed with the addition \((p,a) + (pm,b) = (p + \alpha(m)b)\) and the action of \(A\) given as \(a.(p,b) = (p,ab)\).

(iii) If \(M\) is a module of \((E, A)\) and \(h\) an effective log divisor we set

\[
M(-h) := M \otimes_A A(-h)
\]

We make some remarks about these definitions:
- if \(h \in \Gamma(U, M_E)\) is a preimage of \(h\) for some \(U \subset E\) then it induces a trivialization \(A(-h)|_U \simeq A|_U, (mh, a) \mapsto ma\).
- the twisting functor \((-h) : \text{Mod}(E, A) \to \text{Mod}(E, A)\) is exact since \(A(-h)\) is locally isomorphic to \(A\).
- the twisted module \(M(-h)\) can as be described as a contracted product as well: \(M(-h) \simeq P_h \land_{M_E^\times} M\).

**Lemma 4.42.**

(i) Let \(h \in \Gamma(E, M_E/M_E^\times)\) and assume given a factorization \(h = h_1 + h_2\). Then we have canonical isomorphism and morphism of \(A\)-modules as follow:

\[
A(-h_1) \otimes_A A(-h_2) \to A(-h)\quad\text{nat} : A(-h) \to A(-h_1)
\]

(ii) If \(f : (E', M_{E'}) \to (E, M_E)\) is a morphism of integral log ringed topoi, \(h \in \Gamma(E, M_E/M_E^\times)\) and \(h' = f^{-1}h\) there are canonical isomorphisms

\[
f^*(M(-h)) \simeq (f^*M)(-h') \quad\text{and}\quad (f_*M')(-h) \simeq f_*(M'(-h'))
\]

which are functorial with respect to \(M \in \text{Mod}(E, A)\) and \(M' \in \text{Mod}(E', A')\).

Proof. [i] The isomorphism is given by the formula \((m_1, a_1) \otimes (m_2, a_2) \mapsto (m_1m_2, a_1a_2)\). The second morphism is induced by the morphism

\[
A(-h_2) = P_{h_2} \land_{M_E^\times} A \to A \land A^\times A \to A
\]

where the first arrow is induced by \(\alpha\) and the second is multiplication in \(A\).

[ii] Since \(A(-h)\) is locally isomorphic to \(A\) both isomorphisms follow formally from the more obvious isomorphism \(f^*(A(-h)) \simeq A'(-h')\).

\[\square\]

4.6.2. We explain briefly the relation between log divisors and effective Cartier divisors.

**Definition 4.43.** Consider a scheme \(X\) and let top between zar and fl.

73
Lemma 4.44. Let $\epsilon : X_{\text{top}} \to X_{\text{zar}}$ denote the natural morphism. There is a natural isomorphism $\epsilon_* (S_{\text{top}}/\mathbb{G}_m) \simeq S_{\text{zar}}/\mathbb{G}_m$. The notion of top effective Cartier divisors is in particular independent of top.

Proof. Note first that by flatness of the morphism $\epsilon : (X_{\text{top}}, \mathcal{O}) \to (X_{\text{zar}}, \mathcal{O})$ we have $\epsilon_* S_{\text{top}} = S_{\text{zar}}$. The point is thus to prove that $\epsilon_* S_{\text{top}}/\epsilon_* \mathbb{G}_m \simeq \epsilon_* (S_{\text{top}}/\mathbb{G}_m)$. If $S_{\text{top}}$ was a group then we could conclude directly from the fact that $\mathbb{G}_m$ is $\epsilon_*$-acyclic. We may still conclude however thanks to the following fact (whose proof is straightforward and left to the reader).

Claim. Consider a pretopology $(\mathcal{C}, \text{Cov})$. If $G$ is a sheaf of groups acting freely on a sheaf of sets $F$ then for any $U$ in $\mathcal{C}$ there is a monomorphism $F(U)/G(U) \to F/G(U)$ whose image coincides with the preimage of zero under the boundary map $F/G(U) \to H^1(U, G)$.

Definition 4.45. Consider a fine log scheme $X^\sharp$ over $\Sigma_k$.

(i) A flat fine chart for $X^\sharp$ over $\Sigma_k$ is a strict flat morphism $X^\sharp \to (\text{Spec}(\mathbb{Z}/p^k[P]), P)$ with $P$ a finitely generated integral monoid.

(ii) We say that $X^\sharp/\Sigma_k$ has local flat fine charts if there is a surjective strict étale family $U^\sharp_{\lambda} \to X^\sharp$ where each $U^\sharp_{\lambda}/\Sigma_k$ has a flat fine chart.

Let us point out that in virtue of [4.25 (vi)] a finite $p$-basis over $\Sigma_k$ automatically induces a flat fine chart over $\Sigma_k$.

Lemma 4.46. Assume that $X^\sharp$ has local flat fine charts over $\Sigma_k$.

(i) The logarithmic structure $\alpha : M_X \to \mathcal{O}$ on $(X_{\text{et}}, \mathcal{O})$ induces monomorphisms $M_X \hookrightarrow S_{\text{et}}$ and $M_X/\mathbb{G}_m \hookrightarrow S_{\text{et}}/\mathbb{G}_m$.

(ii) Assume that $X^\sharp$ has a $p$-basis (4.4). Then $M_X$ identifies to the submonoid sheaf of $S_{\text{et}}$ generated by $\mathbb{G}_m$ and the $t_i$’s. In particular if $z_i : V_X(t_i) \to X$ denote the inclusion morphism of the divisor defined by $t_i$ then $M_X/\mathbb{G}_m \simeq \prod_i z_{i,*}\mathbb{N}$.

Proof. The proof of (i) and the first statement of (ii) is given in [Ka2] 1.5.3, 1.10. For the proof of the last statement we may assume that $X$ is an affine space as in [K2] 1.27 (i). We have to describe the sub monoid sheaf of $S_{\text{et}}/\mathbb{G}_m$ generated by the $t_i$’s. Consider an étale $X$-scheme $U$ and denote $z_{i,U} : V_U(t_i) \to U$. Since $U$ is regular the Zariski sheaf of its Cartier divisors identifies to its sheaf of Weil divisors and we find that the sub monoid Zariski sheaf generated by the $t_i$’s identifies to $\prod_i z_{i,U,*}\mathbb{N}$. The resulting collection of Zariski sheaves for variable $U$ clearly verifies the étale descent condition and corresponds to the étale sheaf $\prod_i z_{i,*}\mathbb{N}$ as claimed.

Remark 4.47. (i) Consider an arbitrary fine log scheme $X^\sharp$. We know that the set of points where the log structure is non trivial is closed (see e.g. [Sh] 2.3.1). The
corresponding reduced closed subscheme will be denoted \( \text{Supp}(M_X) \). Using \textbf{4.16} (iv) and \textbf{4.17} (ii) we find that the ideal generated by the image of \( M_X \) into \( \mathcal{O} \) is quasi-coherent. It thus defines a closed subscheme which we denote \( \text{Center}(M_X) \). Note that if \( Y^\natural \to X^\natural \) is a strict morphism then \( \text{Supp}(M_Y) = Y \times_X \text{Supp}(M_X) \) and \( \text{Center}(M_Y) = Y \times_X \text{Center}(M_X) \).

(ii) If \( X^\natural \) has a \( p \)-basis \( (\Sigma, t) \) then \textbf{4.40} implies that the closed subschemes defined in (i) have the following explicit description: \( \text{Supp}(M_X) = V_X(\prod t_i) \) and \( \text{Center}(M_X) = V_X(t_1, \ldots, t_e) \). Note in particular that the latter is reduced too.

\textbf{4.6.3.} Let \( X^\natural \) in \( \mathcal{S}ch_p \). The collection of log structures \( M_{X_k} \to \mathcal{O} \) defines an integral log structure \( M_X \to \mathcal{O} \) on the ringed topos \((X^\natural_{,\text{et}}, \mathcal{O})\). Note that by exactness of the \( \iota_{1,k} : X_1^\natural \to X_k^\natural \) the logarithmic divisors on \( X_k \) are in bijection with those of \( X_1 \). In particular \( \Gamma(X^\natural_{,\text{et}}, M_X / M_X^\natural) \cong \Gamma(X_1, M_{X_1} / \mathbb{G}_m) \).

\textbf{Lemma 4.48.} Let \( h \in \Gamma(X^\natural_{,\text{et}}, M_X / M_X^\natural) \).

(i) The twisting functor can be computed componentwise: \( \iota_k^{-1}(M_k(-h_k)) \cong M_k(-h_k) \).

(ii) The image of the canonical morphism \( \text{nat}_k : \mathcal{O}(-h_k) \to \mathcal{O} \) is the ideal generated by \( \alpha(h_k) \). If \( X_k^\natural \) has flat fine charts over \( \Sigma_k \) then \( \text{nat}_k \) is a monomorphism in the category \( \text{Mod}(X_k^\natural_{,\text{et}}, \mathcal{O}) \).

(iii) The twisting functor \( -h : \text{Mod}(X^\natural_{,\text{et}}, \mathcal{O}) \to \text{Mod}(X^\natural_{,\text{et}}, \mathcal{O}) \) is exact and preserves the properties norm, \( \text{qcoh}, \text{lf}, \text{lfft} \).

Proof. Statement (i) is a special case of \textbf{4.42} (ii). In (ii) the first assertion follows from the fact that \( \iota_1,k \) can be described as multiplication on \( \mathcal{O} \) by \( \tilde{h}_k \) a preimage of \( h \) giving a trivialization as explained after \textbf{4.41}. The second assertion follows from \textbf{4.46} (i). Statement (iii) is clear from the fact that the properties in question are \( \text{étale} \) local.

Consider \( X^\natural \) in \( \mathcal{S}ch/\Sigma_1 \). The collection of log structures \( M_T \to \mathcal{O} \) for \( T^\natural = (U^\natural, T^\natural, \iota) \) in \( \text{Crys}_{\text{et}}((X^\natural / \Sigma)_k) \) defines an integral log structure \( M_X / \Sigma \to \mathcal{O}_{X/\Sigma} \) on the ringed topos \( ((X^\natural / \Sigma)_{\text{crys},\text{et}}, \mathcal{O}) \) (\( \Sigma = \Sigma \) or \( \Sigma_k \), \( 1 \leq k \leq \infty \)). Note that a logarithmic divisor \( h \) of \( X^\natural \) induces canonically a logarithmic divisor \( h_T^\natural \) of \( T^\natural \) by exactness of \( \iota \). Whence in particular a bijection \( \Gamma((X^\natural / \Sigma)_{\text{crys},\text{et}}, M_X / \Sigma / \mathcal{O}_{X/\Sigma}^\natural) \cong \Gamma(X, M_X / \mathbb{G}_m) \).

\textbf{Lemma 4.49.} Let \( X^\natural, T^\natural, h \) as above.

(i) The twisting functor \( -h : \text{Mod}((X/\Sigma)_{\text{crys},\text{et}}, \mathcal{O}) \to \text{Mod}((X/\Sigma)_{\text{crys},\text{et}}, \mathcal{O}) \) is pseudo-compatible with the realization functor \( -\iota_T^\natural \), i.e. \( M(-h_T) \cong M_{T^\natural}(-h_T) \). In particular it preserves crystals.

(ii) If \( X^\natural \) is locally embeddable then \( \text{nat} : \mathcal{O}(-h) \to \mathcal{O} \) is a monomorphism in the category of crystals.

Proof. Statement (i) follows from \textbf{4.42} (ii). Statement (ii) is a consequence of \textbf{4.36} (i) 4.48 (ii).

\textbf{5. Twisted syntomic complexes for Dieudonné crystals}

75
The purpose of this chapter is to define and relate several variants of the syntomic complex functors announced in (2), (5). On the étale site we carry on three parallel constructions of a twisted syntomic complex which are shown to be isomorphic (5.38, 5.40, 5.51). Then we give two constructions on the syntomic site in the case of trivial log structures and we relate them to the previous ones (5.59, 5.63, 5.65).

5.1. The category of \((1, \phi)\)-modules.

Roughly speaking (each variant of) the syntomic functor will be defined as the mapping fiber of \(1 - \phi\) where \(1, \phi\) are morphisms to be defined. The purpose of this section is to summarize some basic facts about the category keeping track of \(1, \phi\) themselves (rather than their mapping fiber).

5.1.1. It will be convenient to define the category of \((1, \phi)\)-modules as a category of modules on a suitable ringed topos. The following constructions will be used e.g. when \(E = X^N \times T^{2}_{[\cdot], \text{et}}, X^N \), etc. in sections 5.5, 5.6

Lemma 5.1. Let \(E\) denote a topos and consider the category \(E^{1, \phi}\) of \(E\)-valued presheaves on the category \(1 \equiv 0\). Explicitly:

- an object is a quadruple \(X = (X_1, X_0, 1_X, \phi_X)\) where \(X_1, X_0\) are objects of \(E\) and \(1_X, \phi_X\) are morphisms \(X_1 \to X_0\) in \(E\).
- a morphism \(X \to Y\) is a couple of morphisms \(X_1 \to Y_1, X_0 \to Y_0\) which are compatible with \(1_X, 1_Y\) as well as \(\phi_X, \phi_Y\).

(i) The category \(E^{1, \phi}\) is a topos.

(ii) There are two morphisms of topoi \(0, 1 : E \to E^{1, \phi}\) such that \(1^{-1}(X_1, X_0, 1, \phi) = X_1, \phi = (X, 1, 1, 1), 0^{-1}(X_1, X_0, 1, \phi) = X_0, 0_*(X) = (X \times X, X, p_1, p_2)\)

(iii) There is a morphism of topoi \(\pi : E^{1, \phi} \to E\) such that \(\pi^{-1}X = (X, X, id_X, id_X), \pi_*1\) is \(1\) and \(\pi_01 \simeq id_E\) canonically.

(iv) The topos \(E^{1, \phi}\) and the morphisms \(1, 0, \pi\) are canonically pseudo-functorial with respect to \(E\).

(v) The functors \(1^{-1}, 0^{-1}, \pi^{-1}\) have left adjoints, \(1_0, 0_0, \pi_0\) satisfying \(1_0(X) = (X, X \sqcup X, i_1, i_2), 0_0(X) = (0, X, 0, 0), \pi_0(X_1, X_0, 1, \phi) = Coker(1, \phi)\).

Proof. (i) is clear from Giraud’s criterion and the remaining statements are immediate.

The following lemma explains how the topos \(E^{1, \phi}\) gives rise to \((1, \phi)\)-modules.

Lemma 5.2. Assume now that \((E, A)\) is a ringed topos and that \(A\) is equipped with a given endomorphism \(F\). Let \((F)\) denote the endomorphism of the ringed topos \((E, A)\) which is the identity of \(E\) together with \(F\). Let us consider the category \(\text{Mod}^{1, \phi}(E, (A, F))\) (or simply \(\text{Mod}^{1, \phi}(E, A)\) if there is no ambiguity about \(F\)) where:

- an object is a quadruple \(M = (M_1, M_0, 1_M, \phi_M)\) where \(M_1, M_0\) are modules of \((E, A)\) and \(1_M : M_1 \to M_0, \phi_M : M_1 \to (F), M_0\) are morphisms in \(\text{Mod}(E, A)\).
- a morphism \(M \to N\) is a couple of morphisms \(M_1 \to N_1, M_0 \to N_0\) in \(\text{Mod}(E, A)\) which are compatible with \(1_M, 1_N\) as well as with \(\phi_M, \phi_N\).

(i) The category \(\text{Mod}^{1, \phi}(E, A)\) is canonically isomorphic to the category of modules of the ringed topos \((E^{1, \phi}, (A, A, id_A, F))\).
(ii) The morphisms $1$, $0$, $\pi$ induce tautological morphisms of ringed topoi

\[
(E, A) \xrightarrow{1} (E^1, (A, A, id_A, F)) \xrightarrow{\pi} (E, A^{F=1})
\]

where $A^{F=1} := \pi_*A$ denotes the subring of fixed points of the endomorphism $F$.

(iii) The pullback functors for modules $1^*$, $0^*$, $\pi^*$ have left adjoints, $1_!$, $0_!$, $\pi_!$, satisfying $1_!(M) = (M, M \oplus (F)^*M, i_1, i_2)$, $0_!(M) = (0, M, 0, 0)$, $\pi_!(M_1, M_0, 1, \phi) = \text{Coker}(1 - \phi)$. The second one is exact and the other two ones are left derivable. There is moreover an exact sequence in $\text{Mod}^{1,\phi}(E, A)$

\[
0 \longrightarrow 0, M_0 \longrightarrow M \longrightarrow 1, M_1 \longrightarrow 0
\]

which is functorial with respect to $M = (M_1, M_0, 1_M, \phi_M)$ in $\text{Mod}^{1,\phi}(E, A)$.

(iv) In $D^+(E, A^{F=1})$ one has a canonical distinguished triangle

\[
R\pi_*M \longrightarrow M_1 \xrightarrow{1_M - \phi_M} M_0 \longrightarrow R\pi_*M[1]
\]

which is functorial with respect to $M = (M_1, M_0, 1_M, \phi_M)$ in $D^+(\text{Mod}^{1,\phi}(E, A))$.

Proof. In (ii) it suffices to observe that a module action

\[(A, A, id_A, F) \times (M_1, M_0, 1_M, \phi_M) \rightarrow (M_1, M_0, 1_M, \phi_M)\]

is equivalent to the data of module actions $A \times M_i \rightarrow M_i$ for which $1_M$ is linear and $\phi_M$ is $F$-linear. The proof of (ii), (iii) is routine. Let us explain (iv). By scalar restriction to $(A^{F=1}, A^{F=1}, id_{A^{F=1}}, id_{A^{F=1}})$ we can assume that $A^{F=1} = A$ (recall that $R\pi_*$ commutes to restricting scalars [SGA4-II V, 5.1, 2]). But then the result will follow from 3.3 applied to the case where the functors $F_1$, $F_2$, $F$ and $F_0$ respectively send $M = (M_1, M_0, 1_M, \phi_M)$ to $M_1$, $M_0$, $1_M - \phi_M$ and $\pi_*M$ once checked that $1_M - \phi_M$ is surjective as soon as $M$ is an injective object of $\text{Mod}^{1,\phi}(E, A)$. This in turn results from the fact that $1_{M'} - \phi_{M'}$ is surjective for $M' = (M_1 \times M_0 \times M_0, M_0, p_2, p_3)$ together with the fact that the natural embedding $f : M \hookrightarrow M'$ (built from the adjunction map between the forgetful functor $M \mapsto (M_1, M_0)$ and its right adjoint) splits (by injectivity of $M$).

\[\square\]

5.2. Local embeddings with Frobenius and cohomological descent.

In this section we define several categories of semi-simplicial local embeddings with additional structures which are adapted to the constructions we have in mind. We also prove that they satisfy the assumptions of a categorical lemma providing a canonical way of gluing local constructions.

5.2.1. As a motivation let us begin with two applications of cohomological descent in the small étale crystalline topos. Let $X^\sharp$ in $\text{Sch}^\sharp/\Sigma_1$ and consider a semi-simplicial $p$-adic $dp$-thickening $T^\sharp_{[1]} = (U^\sharp_{[1]} X^\sharp, T^\sharp_{[1]})$. Denoting $T^\sharp_{[1]} = \text{limind}_k T^\sharp_{[1,k]}$ the associated semi-simplicial crystalline sheaf we have the following pseudo-commutative diagram (9.6(ii)):
Proposition 5.3. Let $X^\sharp$ in $\text{Sch}/\Sigma_1$.

(i) Assume that $T^\sharp_{[\cdot]}$ is a hypercovering of $(X^\sharp/\Sigma_\infty)_{\text{crys,et}}$. For any $M$ in $D^+((X^\sharp/\Sigma_\infty)_{\text{crys,et}},\mathcal{O})$ the adjunction morphism $M \to Rf_{T^\sharp_{[\cdot]},*}f_{T^\sharp_{[\cdot]}1}^{-1}M$ is invertible and induces an isomorphism

$$Ru_*M \simeq Rf_{T^\sharp_{[\cdot]},*}\lambda_{T^\sharp_{[\cdot]},*}f_{T^\sharp_{[\cdot]}1}^{-1}M \quad \text{in } D^+(X^\sharp_{et,N},\mathcal{O}_{\text{crys}})$$

(ii) Assume that $U^\sharp_{[\cdot]}$ is a hypercovering of $X^\sharp_{et}$ and that $T^\sharp_{[\cdot]}$ is the logarithmic divided power envelope of some semi-simplicial object $(U^\sharp_{[\cdot]},Y^\sharp_{[\cdot]})$ of $\text{Emb}^\sharp_{\text{glob}}$. For any module $M$ on $((X^\sharp/\Sigma_\cdot)_{\text{crys,et}},\mathcal{O})$ the adjunction morphism $M \to Rf_{U^\sharp_{[\cdot]},*}f_{U^\sharp_{[\cdot]}1}^{-1}M$ is invertible. If $M$ is a crystalline this isomorphism together with the Poincaré-resolvent (see [9.32] and [9.34]) induce

$$Ru_*M \simeq Rf_{U^\sharp_{[\cdot]},*}\Omega^\bullet_{T^\sharp_{[\cdot]}*}(M) \quad \text{in } D^+(X^\sharp_{et,N},\mathcal{O}_{\text{crys}})$$

Proof. The first statement of (i) (resp. (ii)) is an easy consequence of [SGA4-II] V, 7.3.2 applied to the hypercovering $T^\sharp_{[\cdot]}$ (resp. $U^\sharp_{[\cdot]}$) of the topos $(X^\sharp/\Sigma_{\cdot})_{\text{crys,et}}$. The second statement of (i) follows by pseudo-commutativity of (115). Let us now explain the second statement of (ii). Using [9.34] we get a semi-simplicial version of the Poincaré lemma [9.32] (i)

$$f_{U^\sharp_{[\cdot]}1}^{-1}M \simeq L(\Omega^\bullet_{T^\sharp_{[\cdot]}*}(M_{,[U^\sharp_{[\cdot]}]})) \text{ in } D^+((U^\sharp_{[\cdot]}/\Sigma_{\cdot})_{\text{crys,et}},\mathcal{O})$$

Using [9.31] (ii) as in the proof of [9.32] (i) we get

$$Ru_*f_{U^\sharp_{[\cdot]}1}^{-1}M \simeq \iota^{-1}\Omega^\bullet_{T^\sharp_{[\cdot]}*}(M_{,[U^\sharp_{[\cdot]}]}) \text{ in } D^+((U^\sharp_{[\cdot],N},\mathcal{O}_{\text{crys}}))$$

and the result follows by (115). In the statement of the proposition we have used the simplified notation

$$\Omega^\bullet_{T^\sharp_{[\cdot]}*}(M) := \Omega^\bullet_{T^\sharp_{[\cdot]}*}(M_{,[U^\sharp_{[\cdot]}]})$$

and we will continue to do so in the rest of the text.
Remark 5.4. Assume that \( T^{\sharp}_{[\lambda]} \) satisfies simultaneously the conditions of \( \| 5.3 (i) \| (ii) \) and let \( M \) denote a crystal. Consider the following morphisms in \( D^+(T^{\sharp}_{[\lambda]}, \mathcal{O}^{\text{crys}}) \):

\[
\begin{array}{ccc}
(M)_{T^{\sharp}_{[\lambda]}} & \sim & (L\Omega_{T^{\sharp}_{[\lambda]} - }^{-1} (M))_{T^{\sharp}_{[\lambda]}} \\
\rightarrow & & \leftarrow \\
t_* u_* & t_* u_* f_{T^{\sharp}_{[\lambda]} - }^- \rightarrow & \lambda_{T^{\sharp}_{[\lambda]} - }^- f_{T^{\sharp}_{[\lambda]} - }^-
\end{array}
\]

where the middle arrow is deduced from \( 9.6 (ii) \) as follows:

\[
t_* u_* \rightarrow t_* u_* f_{T^{\sharp}_{[\lambda]} - }^- \rightarrow \lambda_{T^{\sharp}_{[\lambda]} - }^- f_{T^{\sharp}_{[\lambda]} - }^-
\]

On the one hand it follows easily from \( (115) \) and the proof of \( 5.3 (i), (ii) \) that the middle arrow of \( (118) \) in question becomes invertible after applying

\[
Rf_{T^{\sharp}_{[\lambda]} - } : D^+(T^{\sharp}_{[\lambda]}, \mathcal{O}^{\text{crys}}) \rightarrow D^+((X^{\sharp}_{\text{et}}, \mathcal{O}^{\text{crys}})).
\]

Stupid truncation on the other hand provides a morphism from the last term of \( (118) \) to the first one. As explained in \( [BD] \), one can prove that the image of the latter morphism under \( Rf_{T^{\sharp}_{[\lambda]} - } \) is also invertible. We have not tried to compare these two isomorphisms.

In view of constructing twisted syntomic complexes above fine log schemes or diagrams of such it will be convenient to introduce the following variants of the categories of embeddings defined in \( \| 4.28 \| 4.29 \).

**Definition 5.5.** Consider \( * \subset \{ \#, \lf pb \} \).

(i) Let \( \text{Emb}^*_F \) (resp. \( \text{Emb}^*_F^{\text{glob}} \)) denote the following category. An object \( (U^{\sharp}_{\div}/X^{\sharp}, Y^{\sharp}, \nu, \tilde{F}) \) is an object of \( \text{Emb}^{\#} \) (resp. \( \text{Emb}^{\#}^{\text{glob}} \)) together with a Frobenius lift \( \tilde{F} \) on \( Y^{\sharp} \). A morphism is a morphism of \( \text{Emb}^{\#} \) which is compatible with Frobenius lifts.

(ii) Let \( H\text{R}^{\#}_{F_{\text{et}}} \) (resp. \( H\text{R}^{\#}_{F_{\text{crys}}} \)) denote the category of simple-simplicial objects of the category \( \text{Emb}^{\#} \) of the form \( Y^{\sharp}_{\div} = (U^{\sharp}_{\div}/X^{\sharp}, Y^{\sharp}, \cup, \tilde{F}_{\div}) \) for some constant semi-simplicial object \( X^{\sharp} \) of \( \text{Sch}^I/\Sigma_1 \) and such that \( U^{\sharp}_{\div} \) (resp. the logarithmic divided power envelope \( T^{\sharp}_{\div} \) of \( U^{\sharp}_{\div} \)) is a hypercovering of the topos \( X^{\sharp}_{\text{et}} \) (resp. \( (X^{\sharp}/\Sigma_{\infty})_{\text{crys, et}} \)).

(iii) Let \( \text{Sch}^{\#}_{\div}/\Sigma_1 \) denote the category of objects \( (X^{\sharp}, h) \) where \( X^{\sharp} \) is in \( \text{Sch}^I/\Sigma_1 \) and \( h \in \Gamma(X, M_X/\mathcal{G}_m) \) is an effective log divisor on \( X^{\sharp} \). A morphism \( (X^{\sharp}, h) \rightarrow (X^{\sharp}, h') \) is a morphism \( f : X^{\sharp} \rightarrow X^{\sharp} \) which is compatible with log divisors in the sense that \( h \) divides \( f^* h' \) in \( \Gamma(X, M_X/\mathcal{G}_m) \). If \( C = \text{Emb}^*_F, \text{Emb}^*_F^{\text{glob}} \) (resp. \( H\text{R}^{\#}_{F_{\text{et}}}, H\text{R}^{\#}_{F_{\text{crys}}} \)) we view it as a category above \( \text{Sch}^I/\Sigma_1 \) via the forgetful functor \( Y^{\sharp} \rightarrow X^{\sharp} \) (resp. \( Y^{\sharp}_{\div} \rightarrow X^{\sharp} \)). We further denote \( C^{\#}_{\div} \) the category of objects \( (Y^{\sharp}, h) \) (resp. \( (Y^{\sharp}_{\div}, h) \)) where \( h \) is an effective log divisor on \( X^{\sharp} \) and we view it as category above \( \text{Sch}^{\#}_{\div}/\Sigma_1 \) via \( (Y^{\sharp}, h) \rightarrow (X^{\sharp}, h) \) (resp. \( (Y^{\sharp}_{\div}, h) \rightarrow (X^{\sharp}, h) \)).

(iv) Let \( X^{\sharp} \) (resp. \( (X^{\sharp}, h) \)) denote a diagram in the category \( \text{Sch}^I/\Sigma_1 \) (resp. \( \text{Sch}^{\#}_{\div}/\Sigma_1 \)). If \( C = \text{Emb}^*_F, \text{Emb}^*_F^{\text{glob}}, H\text{R}^{\#}_{F_{\text{et}}} \) or \( H\text{R}^{\#}_{F_{\text{crys}}} \) we denote simply \( C(X^{\sharp}) \) the fiber at \( X^{\sharp} \) of \( \text{Diag}(C)/\text{Diag}(\text{Sch}^I/\Sigma_1) \) and \( C(X^{\sharp}, h) \simeq C(X^{\sharp}) \) the fiber at \( (X^{\sharp}, h) \) of \( \text{Diag}(C^{\#}_{\div})/\text{Diag}(\text{Sch}^{\#}_{\div}/\Sigma_1) \).

The following lemma is a complement to \( \| 4.31 \| \).

**Lemma 5.6.** Consider \( * \subset \{ \#, \lf pb \} \) and let \( X^{\sharp}, X^{\sharp}_{\div} \) denote diagrams of \( \text{Sch}^I/\Sigma_1 \) whose vertices are separated, have local finite p-bases if \( \lf pb \in * \) and have trivial log structure if \( \# \notin * \).
(i) The categories $\text{Emb}_F^*(X^\sharp)$, $\text{Emb}_F^{*,\text{glob}}(X^\sharp)$, $\text{HR}_F^{*,\text{et}}(X^\sharp)$ and $\text{HR}_F^{*,\text{crys}}(X^\sharp)$ have finite non empty products.

(ii) The category $\text{HR}_F^{*,\text{et}}$ contains $\text{HR}_F^{*,\text{crys}}$. These categories have a non empty fiber above $X^\sharp$ if and only if $\text{Emb}_F^*(X^\sharp)$ is non empty.

(iii) Consider a morphism $f_X : X^\sharp \to X'^\sharp$ and assume that $\text{Emb}_F^*(X^\sharp)$ is non empty. Any given object above $X'^\sharp$ in the category $\text{Diag}(\text{Emb}_F^*)$ (resp. $\text{Diag}(\text{HR}_F^{*,\text{et}})$, $\text{Diag}(\text{HR}_F^{*,\text{crys}})$) is the target of a morphism above $f_X$.

(iv) Assume that the type $\Delta$ of $X^\sharp$ is such that $\delta/\Delta$ is finite for any $\delta$ in $\Delta$ (this is e.g. the case if $\Delta$ is finite or if $\Delta = \mathbb{N}$). The category $\text{Emb}_F^*(X^\sharp)$ is non empty if $\text{Emb}_F^*(X^\sharp)$ is non empty for each $\delta$ in $\Delta$. This is in particular the case if lfpb $\in *$.

Proof. (i) As in 4.31 we can form products componentwise (note that the property of being a hypercovering is stable by products [SGA4-II] V, 7.3.4).

(ii) Since the property of being a hypercovering is expressed in terms of fiber products it is preserved by the inverse image functor of any morphism of topoi. The first claim thus follows from the fact that $U^\sharp_\delta = i^{-1}Y^\sharp_\delta$. To get the second one it suffices to observe that the coskeleton construction makes sense in $\text{Emb}_F^*(X^\sharp)$ thanks to (i) and gives a right adjoint of the functor $\text{HR}_F^{*,\text{crys}}(X^\sharp) \to \text{Emb}_F^*(X^\sharp)$, $Y^\sharp_{[\nu]} \mapsto Y^\sharp_{[\nu]}$.

(iii) The proof of 4.31 (ii) works here as well.

(iv) Assume given local embeddings $(U^\sharp_\delta/X^\sharp_\delta, Y^\sharp_\delta, \iota_\delta)$ for each $\delta$ in $\Delta$. Replacing $Y^\sharp_\delta$ by a strict étale $Y^\sharp_{\delta'}$-p-adic log scheme if necessary and similarly with $U^\sharp_\delta$ we may assume that each $Y^\sharp_\delta$ has $p$-bases and also that $Y^\sharp_\delta$ (and thus $U^\sharp_\delta$) is separated. Choosing $p$-bases induces Frobenius lifts $\tilde{F}_\delta$. The collection $(U^\sharp_\delta/X^\sharp_\delta, Y^\sharp_\delta, \tilde{F}_\delta)$ for $\delta$ in $\Delta$ can be seen as a diagram of type $|\Delta|$ (the discrete subcategory underlying $\Delta$). Now we observe that the forgetful functor $\text{for} : (\text{Sch}_p^\Delta)^\Delta \to (\text{Sch}_p^\Delta)^{|\Delta|}$ has a right adjoint, say $\text{cofor}$. Explicitly $\text{cofor}$ sends a collection $(Z^\sharp_\delta)$ to the diagram $\delta \mapsto \prod \overline{Z^\sharp_{\delta'}}$ where the product is indexed on the finite set of objects $\delta \to \delta'$ in $\delta/\Delta$. Note that thanks to the finiteness assumption on the $\delta/\Delta$'s the vertices of $\text{cofor}(Z^\sharp_\delta)$ have local finite $p$-bases if and only each $Z^\sharp_\delta$ has. Let $|X|$ denote the diagram of type $|\Delta|$ underlying $X$. The forgetful functor $\text{for}_X : (\text{Sch}_p^\Delta)^X \to (\text{Sch}_p^\Delta)^{|\Delta|}/|X|$ admits similarly a right adjoint $\text{cofor}_X$ sending $(Z^\sharp_\delta)$ to $\delta \mapsto \prod X^\sharp_\delta \times_{X^\sharp_{\delta'}} Z^\sharp_{\delta'}$. By separatedness of the $X^\sharp_\delta$'s, we find that the natural morphism $\text{cofor}_X(Z^\sharp_\delta) \to \text{cofor}(Z^\sharp_\delta)$ is a closed immersion [EGA] chap. 1, 5.4.2. The vertices of $\text{cofor}_X(Z^\sharp_\delta)$ are moreover separated over $\Sigma_1$ (resp. have local finite $p$-bases over $\Sigma_1$, resp. are étale above the vertices of $X$) if and only the same is true for each $Z^\sharp_\delta$. Since the Frobenius lifts $\tilde{F}_\delta$ induce a Frobenius lift $\tilde{F}$ on $\text{cofor}Y^\sharp$ by functoriality we have finally obtained an object $(U^\sharp/X^\sharp, Y^\sharp, \iota, \tilde{F})$ of $\text{Emb}_F^*(X^\sharp)$.

□

For some purpose it will be useful to work with exact closed immersions rather than closed immersions. The reader may consult [SN] for more elaborate variants of the following lemma.

**Lemma 5.7.** Let $* = \emptyset$ or $\text{lfpb}$ and denote $\text{Emb}_F^{*,\text{ex},*}$ the full subcategory of $\text{Emb}_F^{*,*}$ formed by the objects $(U^\sharp/X^\sharp, Y^\sharp, \iota, \tilde{F})$ where $Y^\sharp$ has local finite $p$-bases of the form $(\underline{a}, t)$ (and thus has local charts of type $\mathbb{N}$) and $\iota$ is an exact closed immersion. Also let $\text{HR}_F^{*,\text{et},*,*}$ and $\text{HR}_F^{*,\text{crys},*,*}$ denote the respective full subcategories of $\text{HR}_F^{*,*,*}$ and $\text{HR}_F^{*,*,*}$ formed by the $Y^\sharp_{[\nu]}$'s for which each $Y^\sharp_{[\nu]}$ is in $\text{Emb}_F^{*,\text{ex},*}$. Consider a diagram $X^\sharp$ of $\text{Sch}_p^\sharp/\Sigma_1$ whose vertices are separated and have furthermore local finite $p$-bases in the case $* = \text{lfpb}$.

80
(i) The categories $Emb^\ast_{et,ex}(X^2)$, $HR^\ast_{et,ex}(X^2)$ and $HR^\ast_{crys,ex}(X^2)$ have finite non empty products.

(ii) Assume that $X^2_\delta$ has local charts of type $\mathbb{N}$ and that $\delta/\Delta$ is finite for all $\delta$ in $\Delta$. The categories $Emb^\ast_{et,ex}(X^2)$, $HR^\ast_{et,ex}(X^2)$ and $HR^\ast_{crys,ex}(X^2)$ are non empty if and only if each vertex $X^2_\delta$ is locally embeddable (this condition is empty if $\ast = lfpb$).

Proof. (i) Consider the full subcategory $Emb^\ast_{diagonal,ex}(X^2)$ of $Emb^\ast_{et,ex}(X^2)$ formed by the objects $(U^2/X^2, Y^2, t, F)$ such that étale locally $Y^2$ admits $p$-bases $(s, t)$ satisfying that each one of the $t^i$’s is a chart for $U^2$ (ie. $N \to U^2$, $1 \to i$ $t^i$ is a chart). On the other hand $Emb^\ast_{diagonal,ex}(X^2)$ clearly contains $Emb^\ast_{et,ex}(X^2)$. On the other hand the category $Emb^\ast_{diagonal,ex}(X^2)$ is a full subcategory of $Emb^\ast_{et,ex}(X^2)$ which is stable by products as the reader will check easily. To prove that $Emb^\ast_{et,ex}(X^2)$ has products as claimed it is thus enough to show that the inclusion functor $inc : Emb^\ast_{et,ex}(X^2) \to Emb^\ast_{diagonal,ex}(X^2)$ has a right adjoint $ex$ which is a $\mathcal{S}ch^\infty/\Sigma_1$-functor. For $(U^2/X^2, Y^2, t, F_{Y^2})$ in $Emb^\ast_{diagonal,ex}(X^2)$ we set $ex(U^2/X^2, Y^2, t, F_{Y^2}) := (U^2/X^2, Y^2, i, F_{Y^2})$ where $i : X^2 \to Y^2$ and $F_{Y^2}$ are to be described now.

Let $Z_k = \text{Supp}(M_{Y_k})$ and $z_k = \text{Center}(M_{Y_k})$ (see 4.17). Consider the blowing up $\overline{Y}_k$ of $Y_k$ centered at $z$. We define $\tilde{Y}_k$ as the open complement of the strict transform of $Z_k$ in $\overline{Y}_k$. Etale locally on $Y_k$ we thus have explicitly

$$(120) \quad \tilde{Y}_k \simeq Spec_{Y_k} \mathcal{O}_{Y_k} \otimes \mathbb{Z}[t_1, \ldots, t_s] \mathbb{Z}[y_1, \ldots, y_{12}]$$

where the tensor product is taken with respect to $t_1 \mapsto t_1$ and $t_i \mapsto y_{1i}t_1$, $i \geq 2$. The universal property of blowing ups gives a morphism $\overline{\tau} : U \to \overline{Y}$ above $\tau : U \to Y$. Since each $t^i$ is a chart for $U^2$ there exists for each $i \geq 2$ a unique $y_{ij} \in \mathbb{G}_m(U_i)$ satisfying $t^i = u_{1i}t^i$. In particular $\tau$ factors through $\overline{Y}$. If we endow $\tilde{Y}_k$ with the log structure induced by $Y_k$ then we find an exact closed immersion $\tilde{i} : U \to \tilde{Y}_k$ where $\tilde{Y}_k$ admits a local finite $p$-basis of the form $((s_1, \ldots, s_d, y_1, \ldots, y_{12}), t_1)$. The construction of $(U^2/X^2, \tilde{Y}_k, \tilde{i})$ is clearly functorial with respect to $(U^2/X^2, Y^2, \tau)$. The required Frobenius lift $F_{Y^2}$ may thus be defined by functoriality and this ends the construction of the functor $ex$. We leave it to the reader to check that it is canonically right adjoint to $inc$ (one adjunction morphism is given by the structural morphism $Y^2 \to Y^2$ of the blowing up and the other one is the identity).

Let us emphasize that the functor $ex$ does not affect logarithmic divided power envelopes since the blowing up is log étale [Ka2]. 5.3, 5.4. Forming logarithmic divided power envelopes commutes in particular to products computed in the category $Emb^\ast_{et,ex}(X^2)$. It follows immediately that $HR^\ast_{et,ex}(X^2)$ and $HR^\ast_{crys,ex}(X^2)$ also have products.

(ii) Since $Emb^\ast_{et,ex}(X^2)$ has finite non empty products which are compatible with logarithmic divided power envelopes the coskeleton construction provides a right adjoint to $HR^\ast_{et,ex}(X^2) \to Emb^\ast_{et,ex}(X^2)$, $Y^2 \mapsto Y^2$. It is thus sufficient to prove that $Emb^\ast_{et,ex}(X^2)$ is non empty if the $X^2_\delta$’s are locally embeddable. Start from a local embedding $(U^2/X^2, Y^2, t, s)$ for each $\delta$. Applying e.g. 5.5 (iii) to an appropriate strict étale surjective $X^2_\delta \to X^2_\delta$ we may always assume that $U^2_\delta$ has charts of type $\mathbb{N}$, say $f : U^2_\delta \to (\text{Spec}(\mathbb{Z}[t]), (0))$. We may assume furthermore given a family $(Y^2_{\delta,\lambda})\lambda$ and compatible isomorphisms $Y^2_{\delta,\lambda} \simeq \sqcup_{\lambda} Y^2_{\delta,\lambda}$ such that each $Y^2_{\delta,\lambda}$ has $p$-bases of the form $(s, t) = ((s_1, \ldots, s_d), (t_1, \ldots, t_e))$. Let $U^2_{\delta,\lambda} := U^2 \times_{Y^2_\delta} Y^2_{\delta,\lambda}$. There is at least one index $i$ such that the image of $t_i$ in $\Gamma(U^2_{\delta,\lambda}, M_{U^2_{\delta,\lambda}}/\mathbb{G}_m)$ coincides with the image of $f^it$. 

81
Consider the log structure on $Y_s$ such that $((s_1, \ldots, s_d, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_e), t_i)$ is a $p$-basis. The resulting log structure on $Y_s$ defines an object of $\text{Emb}_F^{\text{ex}+}(X_s^\sharp)$. We may now apply the cofor construction as in the proof of 5.6 (iv) in order to produce an object in $\text{Emb}_F^{\text{diagonal}+}(X^\sharp)$. We conclude using the functor $\text{ex} : \text{Emb}_F^{\text{diagonal}+}(X^\sharp) \to \text{Emb}_T^{\text{ex}+}(X^\sharp)$ defined in the proof of (i).

\[ \square \]

**Remark 5.8.** As already observed in 5.6 (ii), a semi-simplicial local embedding $Y^\sharp$ which is in $HR_{F,\text{crys}}^\sharp$ is automatically in $HR_F^\sharp$. This implication is obviously strict since the property of being in $HR_F^\sharp$ only depends on $U_1^\sharp$. In particular it does not see log structures. To illustrate this let us start with $X^\sharp$ as in 5.7 (ii) and some $Y^\sharp$ in $\text{Emb}_F^{\text{diagonal}}(X^\sharp)$. The coskeleton construction provides some $Y^\sharp_s$ in $HR_{F,\text{crys}}(X^\sharp)$. The underlying $Y^\sharp_s$ obtained by forgetting log structures is clearly in $HR_F^\sharp$. But certainly not in $HR_{F,\text{crys}}(X)$ in general (unless $X = X^\sharp$). Indeed let $\tilde{T}^\sharp_s$ denote the logarithmic divided power envelope of $Y^\sharp_s$. Since the closed immersions $U^\sharp_{\nu} \to Y^\sharp_{\nu}$ are exact the underlying $\tilde{T}^\sharp_s$ coincides with the divided power envelope of $Y^\sharp_s$. Now $Rf_{\tilde{T}^\sharp_s,\ast} \mathcal{O}$ computes the crystalline cohomology of $(X^\sharp/\Sigma)$ but certainly not that of $(\tilde{T}^\sharp_s/\Sigma)$. 

5.2.2. We give a preliminary result which will reduce the construction of the syntomic complex to the case where a global embedding is given. Let us first fix some conventions.

If $Y^\sharp = (U^\sharp/X^\sharp, Y^\sharp, \iota, F_Y, h)$ is an object of $\text{Emb}_F^{\text{et}}(X^\sharp) \subset \text{Emb}_F^{\text{et}}$, we denote $Y^\sharp,\text{glob} = (U^\sharp, Y^\sharp, \iota, F_Y, h, \nu)$ the underlying object of $\text{Emb}_F^{\text{et}}(U^\sharp) \subset \text{Emb}_F^{\text{et}}$. The resulting functor $(-)_{\text{glob}} : \text{Emb}_F^{\text{et}} \to \text{Emb}_F^{\text{et}}$ extends to

\[ (-)_{\text{glob}} : \text{Diag}(HR_{F,\text{div}}^\sharp) \to \text{Diag}(\text{Emb}_{F,\text{div}}^\sharp) \]

by sending a diagram of semi-simplicial embeddings of type $\Delta$ to the underlying diagram of global embeddings of type $\Delta \times \text{Simp}$ where $\text{Simp}$ denotes the semi-simplicial category. Let us emphasize that $(-)_{\text{glob}}$ is not a $\text{Diag}(\text{Sch}^\sharp/\Sigma_1)$-functor.

Consider cofibered categories $\mathcal{F}$, $\mathcal{G}$ over $B^{\text{op}}$, a $B$-category $\mathcal{C}$ and a $C^{\text{op}}$-functor $A : \mathcal{F}_{|\text{op}} \to \mathcal{G}_{|\text{op}}$. We say that $A$ descends to $B^{\text{op}}$ if there exists a $B^{\text{op}}$-functor $\overline{A} : \mathcal{F} \to \mathcal{G}$ and an isomorphism $\alpha : A \simeq \overline{A}_{|\text{op}}$. In that case the couple $(\overline{A}, \alpha)$ is unique up to a unique isomorphism.

**Proposition 5.9.** Let $*_1 \in \{\text{crys}, \text{et} \}$ and $*_2 \in \{\sharp, \text{lfpb} \}$.

(i) Let $\mathcal{F}$ and $\mathcal{G}$ denote cofibered categories above $\text{Emb}_{F,\text{div}}^{*_{2} \text{glob},\text{op}}$. Any functor $S : \mathcal{F} \to \mathcal{G}$ above $\text{Emb}_{F,\text{div}}^{*_{2} \text{glob},\text{op}}$ canonically extends to a functor

\[ S : (\mathcal{F}^{\text{co}})_{\text{Diag}(HR_{F,\text{div}}^{*_{2} \text{et}})\text{op}} \to (\mathcal{G}^{\text{co}})_{\text{Diag}(HR_{F,\text{div}}^{*_{2} \text{et}})\text{op}} \]

above $\text{Diag}(HR_{F,\text{div}}^{*_{2} \text{et}})\text{op}$ in the sense that

\[ S(M)_{\delta, [\nu]} = S(M_{\delta, [\nu]}) \text{ in } \mathcal{G}(Y^{*_{2}\text{glob}}, [\nu]) \]

for any object $Y^{*_{2}\text{glob}}, M$ in $\mathcal{F}(Y^{*_{2}\text{glob}}, [\nu])$ in $\Delta \times \text{Simp}$. The extension of $S$ is moreover naturally functorial.

(ii) Let $B$ denote the essential image of the functor $\text{Diag}(HR_{F,\text{div}}^{*_{1} \sharp}) \to \text{Diag}(\text{Sch}_{\text{div}}^{\sharp}/\Sigma_1)$. Consider cofibered categories $\mathcal{F}$, $\mathcal{G}$ above $B^{\text{op}}$ and a $\text{Diag}(HR_{F,\text{div}}^{*_{1} \sharp})^{\text{op}}$-functor

\[ S : \mathcal{F}_{\text{Diag}(HR_{F,\text{div}}^{*_{1} \sharp})^{\text{op}}} \to \mathcal{G}_{\text{Diag}(HR_{F,\text{div}}^{*_{1} \sharp})^{\text{op}}} \]
The functor \( S \) descends to a \( \mathcal{B}^{op} \)-functor if and only if the base change morphism \( f^* S(M) \to S(f^* M) \) is invertible for any object \((X^\sharp, h)\) in \( \mathcal{B} \) and any morphism \( f \) in \( \text{Diag}(HR_{F,\text{div}}^{s_1,s_2}) \) above \((X^\sharp, h)\).

Proof. \( (i) \) Passing from \( \text{Emb}_{F,\text{div}}^{s_2,\text{glob},op} \) to \( \text{Diag}(\text{Emb}_{F,\text{div}}^{s_2,\text{glob}})^{op} \) relies on \( 2.5 (ii) \) It only remains to base change via \( (121) \).

The proof of \( (ii) \) relies on the following lemma.

**Lemma 5.10.**

(i) Consider categories \( \mathcal{C}, \mathcal{D} \) and assume that the category \( \mathcal{C} \) has products. If \( S : \mathcal{C}^{op} \to \mathcal{D} \) is a functor which sends every arrow of \( \mathcal{C} \) to an isomorphism in \( \mathcal{D} \) then it is isomorphic to a constant functor.

(ii) Consider a \( \mathcal{B} \)-category \( \pi_C : \mathcal{C} \to \mathcal{B} \) satisfying the following conditions:

a. For all \( X \) in \( \mathcal{B} \) the product of two objects is representable in \( \mathcal{C}(X) \). Moreover if \( p_i : \mathcal{C} \times_B \mathcal{C} \to \mathcal{C}, i = 1, 2 \) denote the projection functors there exists a functor \( P : \mathcal{C} \times B \to \mathcal{C} \) and natural transformations \( P \to p_i \) extending the product functor \( P_X : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X) \) and the natural transformations \( P_X \to p_i(X) \).

b. For all \( X \) in \( \mathcal{B} \) the fiber category \( \mathcal{C}(X) \) is non empty. Moreover for every \( x : X' \to X \) in \( \mathcal{B} \) and \( Y \) in \( \mathcal{C}(X) \) there exists a morphism \( y : Y' \to Y \) above \( x \) in \( \mathcal{C} \).

Consider a \( \mathcal{B}^{op} \)-category \( \pi_D : \mathcal{D} \to \mathcal{B}^{op} \). If \( S : \mathcal{C}^{op} \to \mathcal{D} \) is a \( \mathcal{B}^{op} \)-functor then the following conditions are equivalent:

c. For all \( X \) in \( \mathcal{B} \) and all morphism \( y : Y' \to Y \) in \( \mathcal{C}(X) \) the morphism \( S(y) : S(Y') \to S(Y) \) is invertible in \( \mathcal{D}(X) \).

c'. There exists a section \( S \) of \( \pi_D \) and an isomorphism \( \alpha : S \simeq S \circ \pi_C \).

Moreover in \( c' \) the couple \((\overline{S}, \alpha)\) is unique up to a unique isomorphism.

(iii) Let \( \mathcal{C} \) be a \( \mathcal{B} \)-category satisfying a., b. as in \( (ii) \). Consider \( \mathcal{B}^{op} \)-categories \( \mathcal{F}, \mathcal{G} \) and a \( \mathcal{C}^{op} \)-functor

\[
(122) \quad S : \mathcal{F}_{\mathcal{C}^{op}} \to \mathcal{G}_{\mathcal{C}^{op}}
\]

The following conditions are equivalent.

d. If \( y : Y \to Y' \) is a morphism in \( \mathcal{C}(X) \) and \( f : F' \to F \) is a morphism of \( \mathcal{F}_{\mathcal{C}^{op}} \) which is cocartesian above \( y \) (i.e. such that \( f \circ y : F' \to F \) is invertible) then \( S(f) \) is also cocartesian above \( y \).

d'. The functor \( S \) descends to a \( \mathcal{B}^{op} \)-functor.

The proof of \( 5.10 \) will be given below. Let us first finish the proof of \( 5.9 (ii) \) by checking that \( 5.10 (iii) \) can be applied with \( \mathcal{C} = \text{Diag}(HR_{F,\text{div}}^{s_1,s_2}) \). By \( 5.6 (i) \) we have the product functor \( \mathcal{C}(X^\sharp) \times \mathcal{C}(X^\sharp) \to \mathcal{C}(X^\sharp) \) for each \( X^\sharp \) in \( \mathcal{B} \). Consider now a morphism \( f : X^\sharp \to X'^\sharp \) in \( \mathcal{B} \) and a couple of morphisms \( Y^\sharp_{1,[\cdot]} \to Y'^\sharp_{1,[\cdot]}, Y^\sharp_{2,[\cdot]} \to Y'^\sharp_{2,[\cdot]} \) in \( \text{Diag}(HR_{F,\text{div}}^{s_1,s_2}) \) above \( f \). Recalling that products are computed componentwise we find an obvious morphism \( Y^\sharp_{1,[\cdot]} \times Y^\sharp_{2,[\cdot]} \to Y'^\sharp_{1,[\cdot]} \times Y'^\sharp_{2,[\cdot]} \) and condition a. follows. Condition b. is ensured by the definition of \( \mathcal{B} \) and \( 5.6 (iii) \). We conclude by noticing that condition d. is a reformulation of the condition that \( f^* S(M) \simeq S(f^* M) \) for all \( f \)'s in the fibers of \( \text{Diag}(HR_{F,\text{div}}^{s_1,s_2}) \to \text{Diag}(\mathcal{Sch}_{\text{div}}/\Sigma_1) \).
Proof of 5.10. (i) We have to show that up to isomorphism, the functor $S$ factors through the punctual category. This is equivalent to producing a factorization (up to isomorphism) $S \simeq \overline{S} \circ \pi$ where $\pi : C \to C^{rig}$ is the canonical functor, $C^{rig}$ denoting the category which has the same set of objects as $C$ but where there is exactly one arrow $Y \to Y'$ for any couple of objects $(Y, Y')$. Given $Y$ in $C$ we set $\overline{S}(Y) := S(Y)$. If $Y'$ is another object in $C$ and $y$ denotes the only arrow $Y \to Y'$ in $C^{rig}$ we set $\overline{S}(y) := S(p_2)^{-1}S(p_1)$ where denote the projection morphisms. Let us check that $\overline{S}$ is a functor. For $y' : Y' \to Y''$ in $C^{rig}$ the equality $\overline{S}(y'y) = \overline{S}(y)\overline{S}(y')$ is ensured by the following commutative diagram of $C$:

Next the equality $\overline{S}(id) = id$ is ensured by the following commutative diagram of $C$:

It remains to check that $\overline{S} \circ \pi = S$, i.e. that $S(p_2)^{-1}S(p_1) = S(y)$ if $y : Y \to Y'$ is any arrow of $C$. Now the following commutative diagram

of $C$ reduces us to the case $Y' = Y$ and $y = id$ and thus we are done since this case already has been checked.

(ii) The unicity part is easy (using b.) and left to the reader. Let us prove that c. implies c'. Thanks to (i) applied to the functors $S_X : C(X)^{op} \to D(X)$ we already know that there is a constant functor $\overline{S}_X$ and an isomorphism $\alpha_X : S_X \simeq \overline{S}_X$ for each object $X$ of $B$. Let us denote $\overline{S}(X)$ the unique value of the functor $\overline{S}_X$. Let us enrich the collection of the $\overline{S}(X)$'s into a functor $\overline{S} : B^{op} \to D$. Let $x : X' \to X$ in $B$ and choose (by b.)
a morphism \( y : Y' \to Y \) in \( \mathcal{C} \) above \( x \). The following commutative diagram shows that \( S(x) := \alpha_X'(Y') \circ S(y) \circ \alpha_X(Y)^{-1} \) is well defined, i.e. does not depend on the choice of \( y \):

\[
\begin{array}{c}
S(Y_1) \xrightarrow{\alpha_X(Y_1)} S(P(Y_1, Y_2)) \xrightarrow{S(P(y_1, y_2))^{-1}} S(P(Y'_1, Y'_2)) \xrightarrow{\alpha_X'(Y'_1)} S(Y'_1) \\
\alpha_X(Y_2) \xrightarrow{\alpha_X(P(y_1, y_2))} \alpha_X'(P(y_1, y_2)) \xrightarrow{\alpha_X'(P'(y'_1, y'_2))^{-1}} \alpha_X'(Y'_2) \\
\alpha_X'(Y'_2) \xrightarrow{\alpha_X'(P'(y'_1, y'_2))^{-1}} S(Y'_2)
\end{array}
\]

Thanks to assumption \( b \), a couple of composable arrows \( x : X' \to X, x' : X'' \to X' \) in \( \mathcal{B} \) can always be lifted to a couple of composable arrows \( y : Y' \to Y, y' : Y'' \to Y' \) in \( \mathcal{C} \) and it follows immediately that \( S(xx') = S(x')S(x) \). The functor \( S \) just constructed verifies \( \pi_D \circ S = id \) since \( \alpha_X \) is above the identity and \( S \) is a \( \mathcal{B}^{op} \)-functor. It remains to notice that by construction of the \( S(x) \)'s the collection of isomorphisms \( \alpha_{\pi_C(Y)} : S(Y) \simeq S(\pi_C(Y)) \) is functorial with respect to \( Y \), i.e. gives rise to an isomorphism \( S \simeq S \circ \pi_C \).

\[\text{[iii]}\] follows from \( \text{[ii]} \) applied to the following situation \( C^{top} := \mathcal{F}_{C^{op}}, B^{op} := \mathcal{F}, \mathcal{D}' := F \times_{B^{op}} \mathcal{G} \) (note that for \( F \) in \( \mathcal{F}(X) \), the categories \( C'(F) \) and \( C(X) \) are isomorphic).

\[\square\]

In view of the applications we make the following definition.

**Definition 5.11.** Let \( \mathcal{B}_0 \) (resp. \( \mathcal{B}_0 \)) denote the full subcategory of \( \text{Diag}(\mathcal{S}ch^{\ast}_{\text{div}}) \) (resp. \( \text{Diag}(\mathcal{S}ch_{\text{div}}) = \text{Diag}(\mathcal{S}ch) \)) formed by the diagrams of log schemes with effective logarithmic divisor (resp. the diagrams of schemes) whose type verifies the finiteness condition of \( 5.6[iv] \) and whose vertices are separated with local finite \( p \)-bases.

**Lemma 5.12.** The category \( \mathcal{B}_0 \) (resp. \( \mathcal{B}_0 \)) is contained in the essential image of the functor \( \text{Diag}(HR_{F,\text{div}}^{crys,lfpb}) \to \text{Diag}(\mathcal{S}ch^{\ast}_{\text{div}}) \) (resp. \( \text{Diag}(HR_{F}^{crys,lfpb}) \to \text{Diag}(\mathcal{S}ch) \)).

Proof. This results from \( 4.31[iv] \) and \( 5.6[iv] \)

\[\square\]

5.3. The relative Frobenius and Cartier’s descent for crystals.

The purpose of this section is to review the crystalline interpretation of Cartier’s descent for crystals with trivial \( p \)-curvature in the case of local finite \( p \)-bases (see \( 5.25 \)) and to gather the exactness results needed for the definition of the mod \( p \) Hodge filtration of Dieudonné crystals. Logarithmic structures play essentially no role until \( 5.3.4 \) where we discuss elementary consequences of the Cartier equivalence on \( X \) for the category of crystals on \( (X^2/\Sigma_1) \).

**5.3.1.** Let us begin with a review of the relative Frobenius and of the inverse Cartier operator in the language of topoi.

We denote \( F \) or \( F_{X^2} \) the absolute Frobenius endomorphism of \( X^2 \) in \( \mathcal{S}ch^{\ast}/\Sigma_1 \). If \( U^2 \) is in \( TOP^{\ast}(X^2) \) we denote \( F_0U^2 \) the object deduced from \( U^2 \) by composing the structural morphism \( U^2 \to X^2 \) with \( F_{X^2} \). The absolute Frobenius of \( U^2 \) is thus a morphism

\[ F_{U^2} : F_0U^2 \to U^2 \text{ in } TOP^{\ast}(X^2). \]
Similarly if \((U^\sharp, T^\sharp)\) is an object of \(\text{CRYST}_\text{top}(\mathcal{X}^\sharp/\Sigma_1)\) we denote \(F_0(U^\sharp, T^\sharp)\) the object deduced from \((U^\sharp, T^\sharp)\) by composing the structural morphism \(U^\sharp \to \mathcal{X}^\sharp\) with \(F_{\mathcal{X}^\sharp}\). The absolute Frobenius of \(U^\sharp\) and \(T^\sharp\) is then a morphism

\[
F_{(U^\sharp, T^\sharp)}: F_0(U^\sharp, T^\sharp) \to (U^\sharp, T^\sharp) \text{ in } \text{CRYST}_\text{top}(\mathcal{X}^\sharp/\Sigma_1).
\]

In both case we have obtained a natural transformation of functors

\[
F: F_0 \to \text{id}
\]

**Definition 5.13.**

(i) Let \(F : \mathcal{X}_\text{TOP}^\sharp \to \mathcal{X}_\text{TOP}^\sharp\) or \((\mathcal{X}^\sharp/\Sigma_1)\text{CRYST}_\text{top} \to (\mathcal{X}^\sharp/\Sigma_1)\text{CRYST}_\text{top}\) denote the morphism induced by \(F : \mathcal{X}^\sharp \to \mathcal{X}^\sharp\). We define the relative Frobenius

\[
F(-/\mathcal{X}^\sharp) : F \to \text{id}
\]

as the morphism induced by (123) viewed a natural transformation between cocontinuous functors.

(ii) In the case \(\text{TOP}\) or \(\text{top}\) (resp. \(\text{CRYST}\) or \(\text{crys}\)) we deduce relative Frobenius morphism from (i) as follows

\[
F(-/\mathcal{X}^\sharp) : F = pFr \to pr \simeq \text{id}
\]

Note that \(F(-/\mathcal{X}^\sharp)\) can equivalently be seen as a natural transformation \(F_* \to \text{id}\), or \(\text{id} \to F^{-1}\). The relative Frobenius on the usual and \(\text{top}\) crystalline topoi are compatible with each other via the functoriality isomorphism \(Fi \simeq iF\) and \(Fu \simeq uF\) (when \(u\) exists).

Recall that \(ui \simeq \text{id}\) canonically. It is of course not true that \(iu \simeq \text{id}\); there is however a natural morphism \(\text{nat} : iu \to \text{id}\) (since \(i_* \simeq u^{-1}\).

**Lemma + definition 5.14.** Let \(\text{top} = \text{et}\) or \(\text{syn}\).

(i) Consider the Frobenius morphism \(F : (\mathcal{X}^\sharp/\Sigma_1)\text{CRYST}_\text{top} \to (\mathcal{X}^\sharp/\Sigma_1)\text{CRYST}_\text{top}\). There exists a unique morphism \(C^{-1}\), called the inverse Cartier morphism, factorizing \(F(-/\mathcal{X}^\sharp)\) as follows:

\[
\begin{array}{c}
\text{id} \\
\downarrow \\
\text{id}
\end{array}
\]

(ii) In the case \(\text{CRYST}\) or \(\text{crys}\) the inverse Cartier morphism \(C^{-1}\) is defined from 5.14 as follows:

\[
\begin{array}{c}
\text{id} \\
\downarrow \\
\text{id}
\end{array}
\]

Proof. (i) Recall that the morphisms \(F\), \(\text{id}\) and \(i\) are respectively induced by the cocontinuous functor sending \((U^\sharp, T^\sharp)\) to \(F_0(U^\sharp, T^\sharp)\), \((U^\sharp, T^\sharp)\) and \((U^\sharp, U^\sharp)\). Since these functors are continuous as well it is not difficult to check that the claimed factorization
of $F(-/X^1)$ is equivalent to a functorial factorization

$$
\begin{array}{ccc}
F_0(U^2, T^2) & \xrightarrow{F} & (U^2, T^2) \\
\downarrow{C^{-1}} & & \downarrow{\text{nat}} \\
(U^2, U^2) & \xrightarrow{i} & (U^2, T^2)
\end{array}
$$

Uniqueness is clear and existence results from the divided power structure on the ideal $I_{U/T}$ of the closed immersion $U \to T$, using $x^p = p! x^{[p]} = 0$. □

As in the case of the relative Frobenius morphism, the inverse Cartier morphism $C^{-1}$ can equivalently be seen as a natural transformation $F_* \to i_* u_*$ or $u^{-1}i^{-1} \to F^{-1}$. Of particular importance in section 5.4 will be the morphism

$$
C^{-1} : i^{-1} \to u_* F^{-1}
$$

which is deduced from the latter by adjunction.

The following compatibilities will be used in section 8.1.

**Lemma 5.15.** Let $\text{top} = \text{et}$ or $\text{syn}$.

(i) Consider abelian groups $A, B$ in $(X^2/\Sigma_1)_{\text{CRYST,top}}$ or $X^2_{\text{TOP}}$. The following square is commutative:

$$
\begin{array}{ccc}
\text{RHom}(A, B) & \xrightarrow{F(B/X^2)} & \text{RHom}(A, F^{-1}B) \\
\downarrow{F(\text{RHom}(A,B)/X^2)} & & \downarrow{F(A/X^2)} \\
\text{RHom}(A, i_! Ru_* F^{-1}B) & \xrightarrow{\text{adj}_0} & i_* Ru_* \text{RHom}(u^{-1}i^{-1}A, F^{-1}B)
\end{array}
$$

(ii) Let $\text{top}$ be as in 5.14 and consider abelian groups $A, B$ in $(X^2/\Sigma_1)_{\text{CRYST,top}}$. The following diagram is commutative:

$$
\begin{array}{ccc}
\text{RHom}(A, B) & \xrightarrow{C^{-1}} & \text{RHom}(A, i_* Ru_* F^{-1}B) & \xrightarrow{\text{adj}_0} & i_* Ru_* \text{RHom}(u^{-1}i^{-1}A, F^{-1}B) \\
\downarrow{C^{-1}} & & \downarrow{\sim} & & \downarrow{\text{nat}} \\
i_* Ru_* F^{-1} \text{RHom}(A, B) & \xrightarrow{\sim} & i_* Ru_* \text{RHom}(F^{-1}A, F^{-1}B) & \xrightarrow{F(A/X)} & i_* Ru_* \text{RHom}(A, F^{-1}B)
\end{array}
$$

(iii) If $X = X^2$ (resp. and if the absolute Frobenius morphism of $X$ is $\text{top}$) then (i) and (ii) hold verbatim with big (resp. small) sites instead of $\#$-big sites.

Proof. It is sufficient to prove analogous compatibilities before deriving. We begin with a general fact whose proof is left to the reader. If $f, g : E \to E'$ are any two morphisms of topoi and $\alpha : f \to g$ is a morphism between them, then the following natural square of $\text{Ab}(E')$ is commutative for any $A$ in $\text{Ab}(E')$, $B$ in $\text{Ab}(E)$:

$$
\begin{array}{ccc}
\text{Hom}(A, f_* B) & \xrightarrow{\alpha_*} & \text{Hom}(A, g_* B) \\
\downarrow{i} & & \downarrow{i} \\
\text{f}_* \text{Hom}(f^{-1}A, B) & \xrightarrow{\sim} & \text{g}_* \text{Hom}(g^{-1}A, B)
\end{array}
$$

Let us now prove the lemma.
Recall that the natural transformation $F^{-1} \to id$ can be deduced from $F_* \to id$ by composition as follows:

$$id \xrightarrow{\cdot} F_* F^{-1} \xrightarrow{\cdot} F^{-1}$$

We thus have to prove that the exterior square of the following diagram is commutative:

$$\begin{array}{ccc}
\text{Hom}(A, B) & \rightarrow & \text{Hom}(A, F_* F^{-1} B) \\
\downarrow & & \downarrow \\
F_* F^{-1} \text{Hom}(A, B) & \sim & F_* \text{Hom}(F^{-1} A, F^{-1} B) \\
\downarrow & & \downarrow \\
F^{-1} \text{Hom}(A, B) & \sim & \text{Hom}(F^{-1} A, F^{-1} B)
\end{array}$$

It suffices to prove that the interior squares are commutative. The left ones cause no difficulty and the top right one follows from (127) applied to the morphism $F(-/X^2) : F \to id$ (note that the bottom right triangle is tautologically commutative by definition of the bottom horizontal arrow in (127)).

Here the arrows denoted $C^{-1}$ are meant as the ones induced by the morphism $C^{-1} : id \to i_* u_* F^{-1}$. The latter can be decomposed as

$$id \xrightarrow{\cdot} F_* F^{-1} \xrightarrow{\cdot} i_* u_* F^{-1}$$

We thus have to prove that the exterior square of the following diagram is commutative:

$$\begin{array}{ccc}
\text{Hom}(A, B) & \rightarrow & \text{Hom}(A, F_* F^{-1} B) \\
\downarrow & & \downarrow \\
F_* F^{-1} \text{Hom}(A, B) & \sim & F_* \text{Hom}(F^{-1} A, F^{-1} B) \\
\downarrow & & \downarrow \\
i_* u_* F^{-1} \text{Hom}(A, B) & \sim & i_* u_* \text{Hom}(A, F^{-1} B)
\end{array}$$

It suffices to prove that the interior squares are commutative. The left ones cause no difficulty and the top right one is then (127) applied to the morphism $C^{-1} : F \to iu$. Commutativity of the bottom right square then results from the commutativity of the triangle

$$\begin{array}{c}
u^{-1}i^{-1}A \\
\xrightarrow{\text{nat}} \\
F^{-1}A
\end{array}$$

The previous proofs remain valid as long as $F$ is a localization morphism (so that $\text{Hom}$ commutes to $F^{-1}$).

\[\Box\]

5.3.2. Let us now discuss the linear variants of $C^{-1}$ and $F(-/X^2)$. The following definitions are taken from [BBM] 4.3.4.2.
Lemma 5.16. Let $\text{top} = \text{et}$ or $\text{syn}$. We use the following notations.

(i) We denote respectively $(C^{-1})$, $(\text{nat})$, $(F)$, $(\phi)$, the morphism of ringed topoi which is the identity of $(X^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}$ (or $X^\sharp_{\text{TOP}}$) together with the morphism of rings $C^{-1}: \mathbb{G}_a = u^{-1} i^{-1} O \to F^{-1} O \simeq O$, nat : $O \to i_* i^{-1} O = \mathbb{G}_a$, $F : O \to O$, $x \mapsto x^p$, $F : \mathbb{G}_a \to \mathbb{G}_a$, $x \mapsto x^p$ $(C^{-1} : O = i^{-1} O \to u_* F^{-1} O \simeq O_{\text{CRYS}}$, nat : $O_{\text{CRYS}} = u_* O \to u_* i_* i^{-1} O = O$, $F : O_{\text{CRYS}} \to O_{\text{CRYS}}$, $x \mapsto x^p$, $F : O \to O$, $x \mapsto x^p$).

(ii) We denote

$$
((X^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}, O) \xrightarrow{\phi} (X^\sharp_{\text{TOP}}, O)
$$

the morphism of ringed topoi defined by the morphism of topoi $u$ together with the

(iii) We use similar notations in the context of big or small topoi (replace the isomorphisms $F^{-1} O \simeq O$ by the functoriality morphisms $F^{-1} O \to O$).

Let us gather some compatibilities into a lemma.

Lemma 5.17. (i) There is a canonically pseudo-commutative diagram of ringed topoi

$$
\begin{array}{ccc}
((X^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}, \mathbb{G}_a) & \xrightarrow{(\text{nat})} & ((X^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}, O) \\
\downarrow i & & \downarrow (C^{-1}) \\
(X^\sharp_{\text{TOP}}, O) & \xrightarrow{(\phi)} & (X^\sharp_{\text{TOP}}, O_{\text{CRYS}}) \\
\end{array}
$$

This diagram is pseudo-functorial with respect to $X^\sharp$. The same is true in the context of big or small topoi and the resulting diagrams are pseudo-compatible via the projection weak morphisms $p$.

(ii) We have canonical isomorphisms

$$
i_* \simeq u^{-1} : \text{Mod}(S^\sharp_{\text{TOP}}, O) \to \text{Mod}((S^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}, \mathbb{G}_a)
$$

$$
(C^{-1})^* \simeq (F)^*(\text{nat})_* : \text{Mod}((S^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}, \mathbb{G}_a) \to ((S^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}, O)
$$

$$
\phi^* \simeq (F)^* i_* : \text{Mod}(S^\sharp_{\text{TOP}}, O) \to \text{Mod}((S^\sharp/\Sigma_1)_{\text{CRYS},\text{top}}, O)
$$

and similarly in the context of big or small topoi.

Proof. (i) results easily from the definitions.

(ii) The first isomorphism is clear. The second one results from the fact that $\text{nat} : O \to \mathbb{G}_a$ is epimorphic and the third one follows formally.

Lemma 5.18. (i) The morphism $F(-/X^\sharp)$ of 5.13 (in either the $\sharp$-big, big, small, crystalline or usual topoi setting) induces a morphism between the following (weak) endomorphisms of ringed topoi:

$$
F \xrightarrow{F(-/X^\sharp)} (F)
$$

This morphism is compatible with $p$, $\epsilon$, $f$, $\lambda$, $i$, $\phi$. 89
(ii) The morphism $C^{-1}$ of $\text{(5.14)}$ (in either the $\mathbb{Z}$-big, big or small crystalline topos setting) induces a morphism between the following (weak) endomorphisms of ringed topoi:

$$\begin{CD}
F @>C^{-1}>> i\phi
\end{CD}$$

This morphism is compatible with $p$, $\epsilon$, $f$.

Proof. Left to the reader. \hfill $\square$

**Lemma 5.19.** Assume that top is coarser or equal than et.

(i) On $(X^\sharp_{\text{top}}, \mathcal{O})$ the relative Frobenius $F(\mathbb{Z}/X^\sharp) : F \rightarrow (F)$ is an isomorphism.

(ii) On $\text{Crys}((X^\sharp/\Sigma_1)_{\text{CRY S}, \text{top}}, \mathcal{O})$ the relative Frobenius $F(\mathbb{Z}/X^\sharp) : (F)^* \rightarrow F^*$ is an isomorphism. In particular the functor $(F)^*$ preserves crystals of $((X^\sharp/\Sigma_1)_{\text{CRY S}, \text{top}}, \mathcal{O})$.

The same is true with $\text{CRY S}$ or crys instead of $\text{CRY S}^\sharp$.

Proof. (i) Since $F(U^\sharp/\mathbb{Z}) : U^\sharp \rightarrow F^{-1}U^\sharp$ is an isomorphism for any $U^\sharp$ in $\text{top}(X^\sharp)$ we find that $F_*M \rightarrow (F)_*M$ is invertible for any module $M$.

(ii) It suffices to prove that $(F)^*M|_{T^1} \rightarrow F^*M|_{T^1}$ is invertible for any $T^\sharp$ in $\text{CRY S}_{\text{top}}^\sharp(X^\sharp/\Sigma_1)$. This follows from (i) by compatibility of the relative Frobenius with $\lambda_{T^1}$ since $M|_{T^1} \simeq \lambda_{T^1}^*M|_{T^1}$.

\hfill $\square$

**Remark 5.20.** The statements of $\text{(5.19)}$ have the following consequences for arbitrary top (using $\text{(4.10)}$, $\text{(4.22)}$).

(i) If $M$ is a quasi-coherent module of the $\mathbb{Z}$-big, big or small top topos of $X^\sharp$ then $F(M/X^\sharp) : (F)^*M \rightarrow F^*M$ is an isomorphism.

(ii) If $M$ is a quasi-coherent crystal of the $\mathbb{Z}$-big, big or small crystalline top topos of $(X^\sharp/\Sigma_1)$ then $F(M/X^\sharp) : (F)^*M \rightarrow F^*M$ is an isomorphism.

**5.3.3.** Consider a separated log schemes with local finite $p$-bases over $\Sigma_1$ and let $Y^\sharp = T^\sharp = X^\sharp$. Let $X_\lambda$ and $(\emptyset, \underline{1})$ as in section 9.3 so that the $\mathcal{O}$-algebras $\mathcal{P}_X^{\sharp(1)}$ and $\mathcal{D}^\sharp$ of $X^\sharp_{\text{et}}$ have the following local description:

$$(\mathcal{P}_X^{\sharp(1)})|_{X_\lambda} \simeq \mathcal{O}|_{X_\lambda} \langle \tau^\sharp > \rangle \quad \text{and} \quad (\mathcal{D}^\sharp)|_{X_\lambda} \simeq \mathcal{O}|_{X_\lambda} [\partial^\sharp]$$

Let us furthermore denote $\mathcal{P}_X^{\sharp(1)}$, $\mathcal{D}$ the rings of $X_{\text{et}}$ corresponding to the underlying scheme without log structures $X = Y = T$, so that explicitly

$$(\mathcal{P}_X^{\sharp(1)})|_{X_\lambda} \simeq \mathcal{O}|_{X_\lambda} \langle \tau > \rangle \quad \text{and} \quad (\mathcal{D})|_{X_\lambda} \simeq \mathcal{O}|_{X_\lambda} [\partial]$$

where $\tau_a \mapsto t_a \tau^\sharp_a$ (resp. $\partial_a^\sharp \mapsto t_a \partial_a$) under the natural ring homomorphism

$$\mathcal{P}_X^{\sharp(1)} \rightarrow \mathcal{P}_X^{\sharp(1)} \quad \text{and} \quad \mathcal{D}^\sharp \rightarrow \mathcal{D}$$

evaluated at $X_\lambda$.

**Definition 5.21.** We use the following notations.

(i) Let $\mathcal{P}_X^{\sharp(1)}$ denote the structural ring of $(X \times_{F,X,F} X)_{\text{et}}$ viewed as an object $X_{\text{et}}$ via the diagonal immersion $X \rightarrow X \times_{F,X,F} X$ and endowed with the structure of $\mathcal{O}$-algebra coming from the first projection.
(ii) Let $D^F$ denote the module $Hom(P_\Delta^{(1),F}(X),O)$ of $(X_{et},O)$.

Lemma 5.22. (i) The $O$-module $P_\Delta^{(1),F}$ naturally identifies to a direct summand of the $O$-module $P_\Delta^{(1)}$ which is moreover stable by multiplication and comultiplication. Explicitly:
\[(P_\Delta^{(1),F})|_{X_\Delta} \simeq O_{|X_\Delta}[\tau_1, \ldots, \tau_d]/(\tau_1^p, \ldots, \tau_d^p).
\]
(ii) The $O$-module $D^F$ has a natural $O$-algebra structure for which it is a quotient of $D$. Explicitly:
\[(D^F)|_{X_\Delta} \simeq O_{|X_\Delta}[\partial_1, \ldots, \partial_d]/(\partial_1^p, \ldots, \partial_d^p).
\]

Proof. Since the diagonal closed immersion $X \to X \times_{F,X,F} X$ has divided powers we have a canonical morphism $inc : X \times_{F,X,F} X \to T^{(1)}$. Since on the other hand the ideal of the closed immersion $X \times_{F,X,F} X \to X \times X$ is generated by $p^th$ powers the morphism $inc$ has a canonical retraction $ret : T^{(1)} \to X \times_{F,X,F} X$. All statements follow easily from the fact that $inc$ and $ret$ are compatible with the projection morphisms $p_0, p_1$.

Recall the following equivalences of categories (9.20)
\[\text{Crys}((X/\Sigma_1)_{crys,et},O) \simeq \nabla - \text{Mod}(X^{\circ})\]
and
\[\text{Crys}((X/\Sigma_1)_{crys,et},O) \simeq \nabla - \text{Mod}(X^{\circ})\]

Lemma + definition 5.23. Consider $M$ in $\text{Crys}((X/\Sigma_1)_{crys,et},O)$ corresponding to $(M_X, \nabla)$ in $\nabla - \text{Mod}(X)$. The following conditions are equivalent

(i) The morphism $\theta : M_X \to M_X \otimes P_\Delta^{(1)}$ takes its values in $M_X \otimes P_\Delta^{(1),F}$ viewed as a submodule of $M_X \otimes P_\Delta^{(1)}$.

(ii) The action of $D$ on $M_X$ factors through the quotient ring $D^F$.

When they hold we say that $M$ (or equivalently $(M_X, \nabla)$) has trivial $p$-curvature. The category $\text{Crys}^F((X/\Sigma_1)_{crys,et},O)$ of crystals with trivial p-curvature is a full subcategory of $\text{Crys}((X/\Sigma_1)_{crys,et},O)$ which is stable by subobjects and quotient objects. In particular it is abelian and the inclusion functor is exact.

Proof. This follows from (9.15)

Remark 5.24. Consider $M$ in $\text{Crys}^F((X/\Sigma_1)_{crys,et},O)$ and let $(M_X, \epsilon : P_\Delta^{(1)} \otimes M_X \simeq M_X \otimes P_\Delta^{(1)})$ denote the corresponding hyper $dp$-stratification (9.15, 9.20). The crystal $M$ has trivial $p$-curvature if and only if $\epsilon(P_\Delta^{(1),F} \otimes M_X) \subset (M_X \otimes P_\Delta^{(1),F})$. In that case the resulting
\[\epsilon^F : P_\Delta^{(1),F} \otimes M_X \simeq M_X \otimes P_\Delta^{(1),F}\]
verifies the cocycle condition since $\epsilon$ does. Note that $\epsilon$ can be recovered from $\epsilon^F$ by scalar extension via $P_\Delta^{(1),F} \to P_\Delta^{(1)}$.

The following result gives the interpretation of Cartier equivalence using the morphism
\[\phi : ((X/\Sigma_1)_{crys,et},O) \to (X_{et},O)\]
using mainly the arguments of [Be4] 2.3.6. This interpretation is peculiar to small étale sites and the reader is referred to [FM] II, 1.6 for a good understanding of the situation in the setting of small syntomic sites.
Proposition 5.25. Recall that $X$ has local finite $p$-bases over $\Sigma_1$. The functor $\phi^* : \text{Mod}(X_{et}, \mathcal{O}) \to \text{Mod}((X/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$ induces an equivalence of categories

$$\text{Mod}(X_{et}, \mathcal{O}) \simeq \text{Crys}^F((X/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$$

The properties $\text{qcoh}, \text{lf}, \text{lfft}$ are moreover preserved under this equivalence.

Proof. We begin with a lemma.

Lemma 5.26. (i) Consider a cartesian square of schemes

$$\begin{array}{ccc}
U_1 & \xrightarrow{\iota} & T_1 \\
\downarrow h & & \downarrow h \\
U_2 & \xrightarrow{\iota} & T_2
\end{array}$$

where the horizontal arrows are finite morphisms. The base change morphism $h^*\iota_* M \to \iota_* h^* M$ is invertible for any $M$ in $\text{Mod}(U_{2,et}, \mathcal{O})$.

(ii) Let $f : Y \to X$ denote a finite surjective morphism and set $Y' := Y \times_X Y$, $Y'' := Y \times_Y Y \times_Y Y$. Let $\text{prop} \in \{\emptyset, \text{qcoh}, \text{lf}, \text{lfft}\}$. The category $\text{Mod}_{\text{prop}}(X_{et}, \mathcal{O})$ is equivalent to that of $\mathcal{O}$-modules on $Y_{et}$ satisfying $\text{prop}$ with descent datum relatively to $f$.

(iii) Consider a commutative triangle

$$\begin{array}{ccc}
U' & \xrightarrow{\iota'} & T \\
\downarrow h & & \downarrow h \\
U & \xrightarrow{\iota} & T
\end{array}$$

in $\text{Sch}/\Sigma_1$ and assume that $\iota$ and $\iota'$ are nilimmersions of order $p$. The adjunction morphism $id \to h_* h^*$ induces an isomorphism $(F)^* \iota_* M \simeq (F)^* \iota'_* h^* M$ for any $M$ in $\text{Mod}(U_{et}, \mathcal{O})$.

Proof. (i) and (ii) are respective variants of [SGA4-II] VIII, prop. 5.5 and th. 9.4. The proofs given there for abelian sheaves can easily be adapted.

(iii) Since $\iota$ and $\iota'$ are nilimmersions of order $p$ the absolute Frobenius of $T$ admits the following compatible factorizations:

$$\begin{array}{ccc}
T & \xrightarrow{\iota} & T \\
\downarrow h & & \downarrow h \\
U & \xrightarrow{\iota'} & U'
\end{array}$$

Now $\iota_* : \text{Mod}(U_{et}, \mathcal{O}) \to \text{Mod}(T_{et}, \mathcal{O})$ is a fully faithful functor hence

$$F^* \iota_* M \simeq \overline{F}^* \iota_* M \simeq \overline{F}^* M$$

Similarly $F^* \iota'_* h^* M \simeq \overline{F}^* h^* M$. The natural morphism

$$F^* \iota_* M \to F^* \iota'_* h^* M$$

thus identifies to the transitivity isomorphism

$$\overline{F}^* M \simeq \overline{F}^* h^* M$$

The result follows by 5.19 (i).
We can now proceed with the proof of 5.25. The first thing to check is that for any \( M \) in \( \text{Mod}(X_{et}, \mathcal{O}) \), \( \phi^* M \) is a crystal. Consider \((h_T, h_U) : (U_1, T_1, \iota_1, \gamma_1) \to (U_2, T_2, \iota_2, \gamma_2)\) in \( \text{Crys}_{et}(X/\Sigma_1) \). We have the following compatible isomorphisms in \( \text{Mod}(T_{1,et}, \mathcal{O}) \):

\[
\begin{align*}
(\phi^* M)_{T_1} \overset{5.17 (ii)}{\simeq} ((F)^* i_* M)_{T_1} \overset{9.2 (ii)}{\simeq} (F)^* i_* M_{T_1} \overset{9.4 (i)}{\simeq} (F)^* i_{1,*} M_{U_1} \\
\uparrow \quad \uparrow \quad \uparrow \quad \quad \quad \uparrow \\
(h_T^*(\phi^* M)_{T_2} \overset{5.17 (ii)}{\simeq} h_T^*((F)^* i_* M)_{T_2} \overset{9.2 (ii)}{\simeq} (F)^* h_T^*(i_* M)_{T_2} \overset{9.4 (i)}{\simeq} (F)^* h_T^* i_{2,*} M_{U_2}
\end{align*}
\]

It thus suffices to prove that the arrow denoted \( ch \) is invertible. This in turn follows from 5.26 (i) and (iii) applied respectively to the square and triangle of the commutative diagram:

\[
\begin{array}{ccc}
U_1 & \xrightarrow{h_U} & U_1 \times T_2 \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
T_1 & \xrightarrow{h_T} & T_2
\end{array}
\]

We have thus checked that \( \phi^* M \) is a crystal. Let \((M_X, \epsilon)\) denote the corresponding hyper \( dp \)-stratification. Using the isomorphism \( \phi^* M \simeq (F)^* i_* M \overset{5.17 (ii)}{\simeq} (F)^* i_* M \) one checks easily that \( \epsilon \) verifies the condition of 5.24. Using the diagonal equivalence \( \iota : X_{et} \simeq (X \times_{F,X,F} X)_{et} \) and 5.19 (i) we may translate the corresponding \( \epsilon^F \) on \((\phi^* M)_X \simeq (F)^* M \) as a descent datum on \( F^* M \) along \( F \). This descent datum is the canonical one and we may thus conclude by 5.26 (ii) applied to \( f = F \) (note that \( F \) is finite since \( X^2 \) has local finite \( p \)-bases).

\[
\square
\]

5.3.4. We explain some exactness properties of the category of crystals with respect to the following morphisms:

\[
\phi^\sharp : ((X^2/\Sigma_1)_{\text{crys},et}, \mathcal{O}) \xrightarrow{o} ((X/\Sigma_1)_{\text{crys},et}, \mathcal{O}) \xrightarrow{\phi} (X_{et}, \mathcal{O})
\]

(here as in the previous paragraphs we have identified \((X_{et}^2, \mathcal{O})\) and \((X_{et}, \mathcal{O})\)). For simplicity we assume that \( X^2 \) has a fixed global finite \( p \)-basis of the form \((\emptyset, \underline{t})\).

**Definition 5.27.**

(i) We say that a module \( M \) on \((X_{et}, \mathcal{O})\) is \( t \)-torsion free if multiplication by \( t_i \) on \( M \) is monomorphic for all \( i \). The fully \( e \)-exact subcategory of \( \text{Mod}(X_{et}, \mathcal{O}) \) formed by such modules is denoted \( \text{Mod}_{t-f}^e(X_{et}, \mathcal{O}) \).

(ii) We say that a crystal on \(((X/\Sigma_1)_{\text{crys},et}, \mathcal{O})\) is \( t \)-torsion free if its realization at \( X \) is \( t \)-torsion free. The fully \( e \)-exact subcategory of \( \text{Crys}((X/\Sigma_1)_{\text{crys},et}, \mathcal{O}) \) formed by such crystals is denoted \( \text{Crys}_{t-f}^e((X/\Sigma_1)_{\text{crys},et}, \mathcal{O}) \). We use similar notations with \( X^2 \) instead of \( X \).

**Remark 5.28.**

(i) Definition 5.27 (ii) does not depend on the chosen \( p \)-basis \((\emptyset, \underline{t})\). Indeed if \( M \) is \( t \)-torsion free then multiplication on \( M \) by any local section of the monoid of \( X^2 \) is monomorphic as well.
Lemma 5.29. (i) The functors

\[
(131) \quad \sigma^* : \text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O}) \to \text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O})
\]
\[
(132) \quad \phi^* : \text{Mod}(\mathcal{X}, \mathcal{O}) \to \text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O})
\]
\[
(133) \quad \phi^* : \text{Mod}(\mathcal{X}, \mathcal{O}) \to \text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O})
\]

preserve $\mathcal{L}$-torsion freeness and are conservative for this property. The respective right adjoints $\phi_*$, $\phi^*$ of the latter two ones preserve $\mathcal{L}$-torsion freeness as well.

(ii) The functors (131), (132), (133) are exact. The functor (132) is fully faithful. The functor (131) (resp. (133)) is faithful and its restriction to $\mathcal{L}$-torsion free crystals (resp. modules) is fully faithful.

(iii) Consider a short exact sequence $E : 0 \to M_1 \to M_2 \to M_3 \to 0$ of the category $\text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O})$ or $\text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O})$ accordingly. Let $\text{prop} \in \{\mathcal{L}_{fr}, \text{qcoh}, \text{lf}, \text{lf}\}$.

- If $M_2 \simeq \phi_* M_2'$ for some $M_2'$ in $\text{Mod}(\mathcal{X}, \mathcal{O})$ then $E \simeq \phi_* E'$ for some short exact sequence $E' : 0 \to M_1' \to M_2' \to M_3' \to 0$ of $\text{Mod}(\mathcal{X}, \mathcal{O})$. Moreover $M_i'$ satisfies $\text{prop}$ if and only if $M_i$ does.

- If $M_1$, $M_3$ are in $\text{Crys}_{\mathcal{L}_{fr}}(\mathcal{X}/\Sigma_1, \mathcal{O})$ and $M_2 \simeq \sigma^* M_2'$ for some $M_2'$ in $\text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O})$ (resp. $M_2 \simeq \phi^* M_2'$ for some $M_2'$ in $\text{Mod}(\mathcal{X}, \mathcal{O})$) then $E \simeq \sigma^* E'$ for some short exact sequence $E' : 0 \to M_1' \to M_2' \to M_3' \to 0$ of $\text{Crys}(\mathcal{X}/\Sigma_1, \mathcal{O})$ (resp. $\text{Mod}(\mathcal{X}, \mathcal{O})$). Moreover $M_i'$ satisfies $\text{prop}$ if and only if $M_i$ does.

Proof. (i) That $\phi^*$ (resp. $\phi_*$, $\phi^*$) preserves $\mathcal{L}$-torsion freeness follows easily from the third isomorphism in 5.17(ii) using the flatness of $F : X \to X$ (resp. follows from the fact that $\phi_* M$ is a submodule of $(F)_* M_X$, resp. similarly). For a crystal $M$ the isomorphism $(\sigma^* M)_X \simeq M_X$ shows that $\sigma^*$ does not affect the property of being $\mathcal{L}$-torsion free. The remaining statements follow formally from these facts together with full faithfulness of $\phi^*$ (5.25).

(ii) In terms of modules with connection, $\sigma^*$ sends $(M, \nabla)$ to $(M, \nabla')$ where $\nabla'$ is deduced from $\nabla$ via $\Omega_{X/\Sigma_1} \to \Omega_{X/\Sigma_1}$. This shows that (131) is exact and faithful. Recall that $\Omega_{X/\Sigma_1} \simeq \bigoplus_i \mathcal{O}dt_i$, $\Omega_{X_1/\Sigma_1} \simeq \bigoplus_i \mathcal{O}dlog(t_i)$ and $dt_i \mapsto t_i dlog(t_i)$; in particular $M \otimes \Omega_{X/\Sigma_1} \to M \otimes \Omega_{X/\Sigma_1}$ is monomorphic as long as $M$ is $\mathcal{L}$-torsion free. This remark shows that the restriction of $\sigma^*$ to $\mathcal{L}$-torsion free crystals is fully faithful. The functor (132)
is exact and fully faithful in virtue of 5.23 and 5.25. The statement about (133) follows by composition.

(iii) The case of \( \phi^* \) is clear by 5.25 and (i). In the case of \( \phi \) we have to check that the connection of \( M_{1,X^i} \) and \( M_{3,X^i} \) have no logarithmic poles, or equivalently that \( M_{1,X^i} \) is stable by the \( \partial_i \)'s. This is true since \( \partial_i^2 = t_i \partial_i \) and \( M_{3,X^i} \) is \( \xi \)-torsion free. The case of \( \phi_{\xi}^* \) follows by the isomorphism \( \phi_{\xi}^* \simeq \phi^* \).

\[ \square \]

Lemma 5.30. The essential image of the functor

\[ (F)^* : \text{Crys}_{\mathbb{L}fr}(\mathbb{X}^z/\Sigma_1,\text{crys,et},\mathcal{O}) \rightarrow \text{Crys}_{\mathbb{L}fr}(\mathbb{X}^z/\Sigma_1,\text{crys,et},\mathcal{O}) \]

(see 5.19 (ii)) is contained in the essential image of

\[ (134) \quad \phi_{\xi}^* : \text{Mod}_{\mathbb{L}fr}(X_{et},\mathcal{O}) \rightarrow \text{Crys}_{\mathbb{L}fr}(\mathbb{X}^z/\Sigma_1,\text{crys,et},\mathcal{O}) \]

Proof. It suffices to prove that the following diagram is pseudo-commutative:

\[
\begin{array}{ccc}
\text{Crys}(\mathbb{X}^z/\Sigma_1,\text{crys,et},\mathcal{O}) & \xrightarrow{(F)^*} & \text{Crys}(\mathbb{X}^z/\Sigma_1,\text{crys,et},\mathcal{O}) \\
\downarrow & & \downarrow \\
\text{Mod}(X_{et},\mathcal{O}) & \xrightarrow{i-1} & \text{Mod}(X_{et},\mathcal{O}) \\
\end{array}
\]

Let \( M' \) in \( \text{Crys}(\mathbb{X}^z/\Sigma_1,\text{crys,et},\mathcal{O}) \) and let \((M_X', \nabla')\) denote the corresponding connection. Using 5.19 (ii) and the description of \( F^* \) recalled in 9.21 we find that \( M := (F)^* M' \) corresponds to \((M_X, \nabla)\) where \( M_X = (F)^* M_X' \) and \( \nabla \) is the trivial connection \( \lambda \otimes x \mapsto (1 \otimes x) \otimes d\lambda \). The claimed pseudo-commutativity follows easily using the isomorphism \( \phi_{\xi}^* \simeq (F)^* i_* \) (5.17 (ii)).

\[ \square \]

5.4. The mod \( p \) Hodge Filtration on the small crystalline étale site.

In this section we define the tangent sheaf and the mod \( p \) Hodge filtration of a twisted Dieudonné crystal over \( X^z \). Then we deduce a filtration on the corresponding linearized crystal and de Rham complexes. Finally we use the functor \((1)^* \) to normalize \( \text{Fil}^1 \) and get the morphism \( \phi \) occurring in the definition of syntomic complexes. Here we have to assume that \( X^z \) has local finite \( p \)-bases over \( \Sigma_1 \) since the construction relies on Cartier’s descent.

5.4.1. Recall the following definitions (see e.g. [dJ1] rem. 2.4.10).

Definition 5.31. Let \( X^z \) in \( \text{Sch}^z/\Sigma_1 \).

(i) A Dieudonné crystal on \( X^z \) is a triple \((D, f, v)\) where \( D \) a crystal of locally free modules of finite type on \((\mathbb{X}^z/\Sigma_\infty,\text{crys,et}) \) and \( f : F^* D \rightarrow D, v : D \rightarrow F^* D \) satisfy \( fv = p \) and \( vf = p \). The category of Dieudonné crystals on \( X^z \) is denoted \( DC(X^z) \).

(ii) Assume that \( X^z \) is locally embeddable. A truncated Dieudonné crystal of level 1 on \( X^z \) is a triple \((D, f, v)\) where \( D \) is a crystal of locally free modules of finite type on \((\mathbb{X}^z/\Sigma_1,\text{crys,et}) \), and \( f : F^* D \rightarrow D, v : D \rightarrow F^* D \) fit into an exact sequence \( D \rightarrow F^* D \rightarrow D \rightarrow F^* D \) of \( \text{Crys}(\mathbb{X}^z/\Sigma_1,\text{crys,et},\mathcal{O}) \). The category of truncated Dieudonné crystals of level 1 on \( X^z \) is denoted \( DC_1(X^z) \).
**Lemma 5.32.** Assume that $X^{\sharp}$ is locally embeddable.

(i) There is an exact structure $e$ on $\mathcal{D}C(X^{\sharp})$ (resp. $\mathcal{D}C_1(X^{\sharp})$) such that the forgetful functor to $\text{Crys}_{\text{ff}}((X^{\sharp}/\Sigma_{\infty})_{\text{crys,et}}, \mathcal{O})$ (resp. $\text{Crys}_{\text{ff}}((X^{\sharp}/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$) is $e$-exact and reflects exactness.

(ii) The natural morphism $\iota_1 : ((X^{\sharp}/\Sigma_1)_{\text{crys,et}}, \mathcal{O}) \to ((X^{\sharp}/\Sigma_{\infty})_{\text{crys,et}}, \mathcal{O})$ induces an $e$-exact functor

$$\iota_1^{-1} : \mathcal{D}C(X^{\sharp}) \to \mathcal{D}C_1(X^{\sharp})$$

We use the notation $\mathcal{D}(\mathcal{D}, \mathcal{D}, \pi) := \iota_1^{-1}(D, f, v)$.

Proof. (i) We say that $0 \to (D, f, v) \to (D', f', v') \to (D'', f'', v'') \to 0$ of $\mathcal{D}C(X^{\sharp})$ (resp. $\mathcal{D}C_1(X^{\sharp})$) is exact if the underlying sequence $0 \to D \to D' \to D'' \to 0$ of $\text{Crys}((X^{\sharp}/\Sigma_{\infty})_{\text{crys,et}}, \mathcal{O})$ (resp. $\text{Crys}((X^{\sharp}/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$) is exact. The reader may check directly that this defines an exact structure, i.e., verifies the axioms of [Bui] def. 2.1 using that $F^*$ is $e$-exact on $\text{Crys}_{\text{ff}}((X^{\sharp}/\Sigma_{\infty})_{\text{crys,et}}, \mathcal{O})$ (resp. $\text{Crys}_{\text{ff}}((X^{\sharp}/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$).

(ii) Let us check that the claimed functor is well defined. Consider $(D, f, v)$ in $\mathcal{D}C(X^{\sharp})$ and let $D_k, f_k : F^*D_k \to D_k, v_k : D_k \to F^*D_k$ denote the image $D, f, v$ under the functor $\iota_1^{-1} : \text{Crys}(X^{\sharp}/\Sigma_{\infty}, \mathcal{O}) \to \text{Crys}(X^{\sharp}/\Sigma_k, \mathcal{O})$. We need to show that the sequence

$$D_1 \xrightarrow{v_1} F^*D_1 \xrightarrow{f_1} D_1 \xrightarrow{v_1} F^*D_1$$

is exact in the category of crystals over $(X^{\sharp}/\Sigma_1)$. Since $\iota_{1,2,*} : \text{Crys}(X^{\sharp}/\Sigma_1, \mathcal{O}) \to \text{Crys}(X^{\sharp}/\Sigma_k, \mathcal{O})$ is exact faithful and $\iota_{1,2,*}D_1 \simeq D_2/p$ it is equivalent to show the exactness of

$$D_2/p \xrightarrow{v_2} F^*D_2/p \xrightarrow{f_2} D_2/p \xrightarrow{v_2} F^*D_2/p$$

Since $p = f_2v_2$ we have

$$F^*D_2/(\text{Ker } f_2 + \text{Im } v_2) \xrightarrow{f_2} D_2/\text{Im } p$$

Now $\text{Ker } f_2$ contains $\text{Im } f_2$ and coincides with $\text{Im } p$ (check this using e.g. (iv) or (vii)). Exactness at the second term follows. Exactness at the third term is similar. The functor $\iota_1^{-1}$ thus well defined. Its exactness is clear by local freeness.

$$\square$$

**Remark 5.33.** Replacing the small étale crystalline site with the $\sharp$-big, big or small top crystalline site, $\mathfrak{f} \mathfrak{l} \preceq \text{top} \preceq \text{et}$, we may define similarly categories of Dieudonné crystals $\mathcal{D}C_{\text{CRYS, top}}(X^{\sharp}), \mathcal{D}C_{\text{CRYS, top}}(X^{\sharp}), \mathcal{D}C_{\text{CRYS, top}}(X^{\sharp})$.

(i) The various categories of Dieudonné crystals obtained in that way are all naturally equivalent by 4.27 and 4.22 (note that for a quasi-coherent crystal $M$, we have $F^*\epsilon_* M \simeq \epsilon_* F^* M$ and $F^*p_* M \simeq p_* F^* M$) and thus naturally endowed with an exact structure $e$.

(ii) Using 4.39 we find that the forgetful functor from the category of Dieudonné crystals on the $\sharp$-big, big or small top crystalline site to the corresponding category of $\mathcal{O}$-modules is $e$-exact (because locally free crystals are flat, hence acyclic for $\epsilon^*$ and $p^*$) and reflects exactness (because quasi-coherent crystals are acyclic for $\epsilon_*$ and $p_*$, see 4.22 (ii)).

**Definition 5.34.** Assume that $X^{\sharp}/\Sigma_1$ has local finite $p$-bases.
(i) Let $D = (D, f, v)$ in $\mathcal{D}C_1(X^2)$.
- We define $\text{Lie}(D)$ as the following module on $(X_{et}, \mathcal{O})$:
\[
\text{Lie}(D) := \phi_*(\text{Coker } v)
\]
- The Hodge filtration on the module of $((X^2/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$ underlying $D$ is defined as follows: $\text{Fil}^2 D = 0$, $\text{Fil}^0 D = D$ and $\text{Fil}^1 D$ is the kernel of the composed arrow
\[
\text{Fil}^1 D := \text{Ker}(\text{can}_D : D \to i_*i^{-1}D \to i_*\text{Lie}(D))
\]
where the first one is the adjunction morphism and the second one is the inverse Cartier operator $[129]$: $i^*D \to \phi_*F^*D$.

(ii) Let $D = (D, f, v)$ in $\mathcal{D}C(X^2)$.
- We define $\text{Lie}(D)$ as $\text{Lie}(\overline{D})$.
- The mod $p$ Hodge filtration on the module of $((X^2/\Sigma_\infty)_{\text{crys,et}}, \mathcal{O})$ underlying $D$ is defined as follows: $\text{Fil}^p D$ is the inverse image of $t_1*\text{Fil}^p\overline{D}$ by the adjunction morphism $D \to t_1*\overline{D}$.

Proposition 5.35. Assume that $X^2/\Sigma_1$ has local finite $p$-bases. Consider $D$ in $\mathcal{D}C(X^2)$.

(i) The morphism $\text{can}_\overline{D} : \overline{D} \to i_*\text{Lie}(\overline{D})$ occurring in the definition of $\text{Fil}^1\overline{D}$ is an epimorphism. It induces an exact sequence in $\text{Mod}((X^2/\Sigma_\infty)_{\text{crys,et}}, \mathcal{O})$:
\[
0 \longrightarrow \text{Fil}^1 D \longrightarrow D \overset{\text{can}_D}{\longrightarrow} i_*\text{Lie}(D) \longrightarrow 0
\]

(ii) Consider a morphism $f : X^2 \to X^2$ where $X^2$ has local finite $p$-bases as well. There is a natural base change isomorphism $ch_f$ rendering the following square commutative:
\[
\begin{array}{ccc}
\text{can}_f & \longrightarrow & \text{Lie}(f^*D) \\
\downarrow & & \downarrow \text{ch}_f \\
\text{can}_{f^*i^{-1}D} & \longrightarrow & f^*\text{Lie}(D)
\end{array}
\]

The family of the $\text{ch}_f$’s satisfies the composition constraint.

(iii) The functor $\text{Lie} : \mathcal{D}C(X^2) \to \text{Mod}(X_{et}^2, \mathcal{O})$ is $e$-exact.

Proof. \[\square\] Up to étale localization we may assume given a $p$-basis of the form $(\emptyset, \emptyset)$ as in paragraphs 5.3.3 and 5.3.4 (see 4.27 (iii)). Since $\overline{D}$ is a crystal $\overline{D} \to i_*i^{-1}\overline{D}$ is epimorphic in $\text{Mod}((X^2/\Sigma_1)_{\text{crys}}, \mathcal{O})$ (by [9,6 (i)] and the crystal condition). We want to prove that $i_*i^{-1}\overline{D} \to i_*\text{Lie}(\overline{D})$ is epimorphic, ie. that
\[
i^{-1}\overline{D} \overset{\text{can}_1}{\longrightarrow} \phi_*F^*\overline{D} \overset{\phi_*\text{nat}}{\longrightarrow} \phi_*\text{Coker } \overline{v}
\]
is epimorphic in $\text{Mod}(X_{et}, \mathcal{O})$. Here $\text{nat}$ denotes the tautological morphism $F^*\overline{D} \to \text{Coker } \overline{v}$. Let us begin with the following

Claim: The following adjunction morphisms are isomorphisms:
\[
i^{-1}\overline{D} \to \phi_*\phi^*i^{-1}\overline{D}
\]
\[
\phi^*\phi_*F^*\overline{D} \to F^*\overline{D}
\]
\[
\phi^*\phi_*\text{Coker } \overline{v} \to \text{Coker } \overline{v}.
\]
Let us prove the claim. The isomorphism \(i^{-1}\overline{D} \simeq \overline{D}_{X_t}\) shows that \(i^{-1}\overline{D}\) is \(t\)-torsion free. The first isomorphism of the claim follows by full faithfulness of the restriction of \(\phi^*\) to \(t\)-torsion free modules (see \[5.29\](ii) where \(\phi^*\) was denoted \(\phi^{\sharp\ast}\)). Thanks to \[5.29\](i) the second and third isomorphism will follow if we show that \(F^*\overline{D}\) and \(\text{Coker } \overline{\varphi}\) are in the essential image of the fully faithful functor \[134\]. In the case of \(F^*\overline{D}\) this results directly from \[5.30\] and \[5.19\](ii). Next we notice that the crystal \(\text{Coker } \overline{\varphi}\) is \(t\)-torsion free since it is a subcrystal of \(\overline{D}\). We may thus conclude that \(\text{Coker } \overline{\varphi}\) is in the essential image of \(\phi^*\) as well by \[5.29\](iii) and this ends the proof of the claim.

We will now prove that the composed arrow \[135\] is epimorphic. As the reader may check we have a commutative diagram

\[
\begin{array}{ccc}
\phi_\ast \phi^* i^{-1}\overline{D} & \xrightarrow{\sim} & \phi_\ast F^*\overline{D} \\
\downarrow \phi_\ast \phi^* & & \downarrow \phi_\ast F^*\overline{D}/X^2 \\
\phi_\ast \phi^* i^{-1}\overline{D} & \xrightarrow{\sim} & \phi_\ast (F)^*\overline{D}
\end{array}
\]

where the lower isomorphism is by pseudo-commutativity of the triangle in the proof of \[5.30\]. The left (resp. right) vertical arrow is invertible as well by the above claim (resp. \[5.19\](ii)). The first arrow in \[135\] is thus invertible.

Let us now investigate the second arrow in \[135\]. Consider the following commutative square of \(\text{Crys}(X^\sharp/\Sigma_1, \mathcal{O})\):

\[
\begin{array}{ccc}
F^*\overline{D} & \xrightarrow{\text{nat}} & \text{Coker } \overline{\varphi} \\
\downarrow \phi^* \phi_\ast & & \downarrow \phi^* \phi_\ast \text{Coker } \overline{\varphi} \\
\phi^* \phi_\ast F^*\overline{D} & \xrightarrow{\phi^* \phi_\ast (\text{nat})} & \phi^* \phi_\ast \text{Coker } \overline{\varphi}
\end{array}
\]

Vertical arrows are isomorphisms by the claim. The arrow \(\text{nat}\) is epimorphic and thus \(\phi^* \phi_\ast (\text{nat})\) also. We conclude that \(\phi_\ast (\text{nat})\) is epimorphic as well since

\[\phi^*: \text{Mod}(X_{et}, \mathcal{O}) \to \text{Crys}((X^\sharp/\Sigma_1)_{\text{crys,et}}, \mathcal{O})\]

is conservative for epimorphisms (it is a faithful exact functor between abelian categories by \[5.29\](i) (ii) and \[4.36\](i)). We have thus proven that the composed arrow \[135\] is epimorphic as desired.

We already know that the bottom sequence of the following tautologically commutative diagram of \(\text{Mod}((X^\sharp/\Sigma_{\infty})_{\text{crys}}, \mathcal{O})\) is exact (recall that \(t_{1,*}\) is exact by \[2.23\]):

\[
\begin{array}{cccccc}
0 & \to & \text{Fil}^1 D & \xrightarrow{\rightarrow} & \overline{D} & \xrightarrow{\rightarrow} & i_\ast \text{Lie}(D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{t_{1,*}} & \text{Fil}^1 \overline{D} & \xrightarrow{\rightarrow} & t_{1,*} \overline{D} & \xrightarrow{\rightarrow} & t_{1,*} i_\ast \text{Lie}(\overline{D}) & \to & 0
\end{array}
\]

Exactness of the top sequence follows formally since the left square is cartesian (by definition of \(\text{Fil}^1 D\)) and the middle vertical arrow is epimorphic (since \(\overline{D}\) is a crystal).

The definition of the morphism \(ch_f\) is purely formal from the canonical isomorphisms \(t_1 f \simeq f t_1\), \(F f \simeq F f\), \(\phi f \simeq f \phi\), \(i f \simeq f i\). The composition constraint for the \(ch_f\)'s moreover follows from the compatibility of these isomorphisms with respect to composition in \(f\). It remains to check that \(ch_f\) is in fact an isomorphism. We can always assume
that both $X^\sharp$ and $X'^\sharp$ have global $p$-bases as in the proof of [1]. There is the following commutative diagram on $(X'_\log, \mathcal{O})$:

\[
\begin{array}{c}
\text{Lie}(f^*\mathcal{D}) & \xrightarrow{\sim} & \phi_* f^* \text{Coker } \mathcal{V} \\
\text{ch}_{\text{Lie}} & \uparrow & \phi_* f^* \phi^* \text{Lie}(\mathcal{D}) \\
\end{array}
\]

where the isomorphism denoted 1 is defined by identifying $\phi^* \text{Coker } \mathcal{V}$ and $\text{Coker } f^* \mathcal{V}$. Let us finally check that the arrows denoted 2 are isomorphisms. It suffices to check that $\text{Lie}(\mathcal{D})$ are $t$-torsion free (5.29 (i), (ii)). This is indeed the case by 5.29 (i) since $\phi^* \text{Lie}(\mathcal{D}) \simeq \text{Coker } \mathcal{V}$ and $\phi^* f^* \text{Lie}(\mathcal{D}) \simeq f^* \text{Coker } \mathcal{V}$ are $t$-torsion free as already explained.

(iii) The question is local so we may again assume given a $p$-basis as in [1]. The functor

\[
\mathcal{D}C(X^\sharp) \rightarrow \text{Crys}((X^\sharp/\Sigma_1)_{\text{crys,et}}, \mathcal{O})
\]

is $e$-exact since $\iota_1^{-1} : \mathcal{D}C(X^\sharp) \rightarrow \mathcal{D}C_1(X^\sharp)$ is $e$-exact (5.32 (ii)) hence $e_{M}$-exact (4.40) and there is an isomorphism $\text{Coker } \mathcal{V} \simeq \text{Ker } \mathcal{V}$ (induced by $f : F^* \mathcal{D} \rightarrow \mathcal{D}$). Next we observe that $\phi_* : \text{Crys}((X^\sharp/\Sigma_1)_{\text{crys,et}}, \mathcal{O}) \rightarrow \text{Mod}(X_{\log, \mathcal{O}})$ has the following property thanks to 5.29 (ii) (iii) if $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is $e$-exact and the $M_i$’s are $t$-torsion free as well as in the essential image of $\phi^*$ then $0 \rightarrow \phi_* M_1 \rightarrow \phi_* M_2 \rightarrow \phi_* M_3 \rightarrow 0$ is exact.

\[\square\]

**Remark 5.36.** The realization of $\text{Coker } \mathcal{V}$ at $X^\sharp$ has a right resolution as follows

\[
(C\text{oker } \mathcal{V})_{X^\sharp} \xrightarrow{\eta} [D_{X^\sharp} \xrightarrow{\eta} F^* D_{X^\sharp} \xrightarrow{\eta} D_{X^\sharp} \xrightarrow{\eta} \ldots]
\]

It is thus quasi-coherent and even locally free of finite type under the additional assumption that $X$ is locally noetherian (hence regular by 4.25 (vi)). The same is true for $\text{Lie}(\mathcal{D})$ by 5.29 (ii), (iii).

5.4.2. Let us consider a separated log scheme $X^\sharp/\Sigma_1$ with local finite $p$-bases and assume moreover given a global embedding $X^\sharp \rightarrow Y^\sharp$ whose logarithmic divided power envelope is denoted $\iota : X^\sharp \rightarrow T^\sharp$. We will use the following notations.

- Letting $k$ vary in (223) defines a functor $L_{\log}$ below. We define furthermore a functor $L$ by commutativity of the following triangle:

\[
(140) \quad H\text{dp}_{\text{norm}}(T^\sharp, \mathcal{O}) \xrightarrow{L} C\text{rys}_{\text{norm}}((X^\sharp/\Sigma_1)_{\text{crys,et}}, \mathcal{O})
\]

- If $M$ is a module on $((X^\sharp/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$ we write $M := L^{-1} M$ (resp. $M_{T^\sharp} := M_{T^\sharp}$) the associated normalized module of $((X^\sharp/\Sigma_1)_{\text{crys,et}}, \mathcal{O})$ (resp. its realization on $(T^\sharp_{\text{et}}, \mathcal{O})$).

- If $M$ is a crystal of $((X^\sharp/\Sigma_\infty)_{\text{crys}}, \mathcal{O})$, we write $\Omega^\sharp_{T^\sharp}(M) := \Omega^\sharp_{T^\sharp}(M)$ the associated de Rham complex in $(T^\sharp_{\text{et}}, \mathcal{O}_{\text{crys}})$.

**Proposition 5.37.** Consider $D \in \mathcal{D}C(X^\sharp)$.
(i) There exists a canonical morphism \(\text{can}_L\) rendering the right square below commutative. We let \(\text{Fil}^1(L(D_{T^*_h}))\) denote its kernel. Whence a tautological morphism of exact sequences on \((X^2/\Sigma^\infty)_{\text{crys},\text{et}}, \mathcal{O})\):

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Fil}^1 D \\
\downarrow & & \downarrow \text{can}_D \\
D & \longrightarrow & i_*\text{Lie}(D) \\
\downarrow & \text{aug} & \downarrow \| \\
0 & \longrightarrow & \text{Fil}^1(L(D_{T^*_h})) \\
\downarrow & & \downarrow \text{can}_L \\
L(D_{T^*_h}) & \longrightarrow & i_*\text{Lie}(D) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\]

from 5.34 (i), (ii) that are thus reduced to noticing the commutativity of the following diagram where 1:

\[
\begin{array}{ccc}
0 & \longrightarrow & (\text{Fil}^1 D)(-h)_{T^*_h} \\
\downarrow & \text{can}_L & \downarrow \| \\
0 & \longrightarrow & D(-h)_{T^*_h} \\
\downarrow & \text{can}_D & \downarrow \\
0 & \longrightarrow & (i_*\text{Lie}(D)(-h))_{T^*_h} \\
\end{array}
\]

where the top (resp. bottom) one is deduced from the bottom (resp. top) one in (i) by twisting and applying \(\iota_*\iota^{-1}\) (resp. \((l^{-1}(-))_{T^*_h}\)).

(ii) There is a natural isomorphism of exact sequences on \((T^*_h, \mathcal{O}_{\text{crys}})\) for any \(h \in \Gamma(X, M_X/\mathbb{G}_m)\).

\[
\begin{array}{ccc}
0 & \longrightarrow & \iota_*u_*l^{-1}(\text{Fil}^1(L(D_{T^*_h}))(-h)) \\
\downarrow & \iota_*u_*l^{-1}(L(D_{T^*_h})(-h)) & \text{can}_L \\
0 & \longrightarrow & \iota_*l^{-1}(\text{Lie}(D)(-h)) \\
\downarrow & \iota_* & \downarrow \\
0 & \longrightarrow & (\text{Fil}^1 D)(-h)_{T^*_h} \\
\downarrow & \iota_* & \downarrow \| \\
0 & \longrightarrow & D(-h)_{T^*_h} \\
\downarrow & \iota_* & \downarrow \\
0 & \longrightarrow & (i_*\text{Lie}(D)(-h))_{T^*_h} \\
\end{array}
\]

Proof. (i) The only point requiring explanations is the definition of \(\text{can}_L\). Recall from 5.34 (i), (ii) that \(\text{can}_D\) factorizes via the adjunction morphisms \(D \rightarrow i_*i^{-1}D\). By 4.20 (i) it is thus sufficient to provide a canonical factorization \(\alpha\) of \(D \rightarrow i_*i^*D\) via \(D \rightarrow f_{T^*_h}f_{T^*_h}^{-1}D \simeq L(D_{T^*_h})\). We do this by observing that the right vertical arrow is invertible in the following commutative diagram of adjunction morphisms:

\[
\begin{array}{ccc}
D & \longrightarrow & i_*i^{-1}D \\
\downarrow & & \downarrow \alpha \\
\iota_*\iota^{-1}D & \longrightarrow & \iota_*\iota^{-1}D \\
\end{array}
\]

(ii) It suffices to check that the right hand square is commutative. Using 4.20 (i) this will follow from the commutativity of the following diagram

\[
\begin{array}{ccc}
\iota_*u_*((f_{T^*_h}f_{T^*_h}^{-1}D)(-h)) & \overset{\alpha}{\longrightarrow} & \iota_*u_*((i_*i^{-1}D)(-h)) \\
\downarrow & \iota_* & \downarrow \sim \\
\lambda_{T^*_h}f_{T^*_h}^{-1}(D(-h)) & \longrightarrow & \lambda_{T^*_h}f_{T^*_h}^{-1}i_*i^{-1}(D(-h)) \\
\end{array}
\]

where the definition of each arrow uses 4.42 (ii) except the bottom horizontal one. We are thus reduced to noticing the commutativity of the following diagram where 1 : \(\iota_*u_* \rightarrow 100\)
\( \lambda_{T^1, s} f^{-1}_{T^1} \) is the natural morphism deduced from \( \text{9.6 (ii)} \). 2 expresses the isomorphism \( \mathcal{B} = \text{id} \) and the unlabeled arrows are adjunction morphisms.

\[
\begin{array}{cccc}
\lambda_{T^1, s} f^{-1}_{T^1} & \lambda_{T^1, s} f^{-1}_{T^1} & \lambda_{T^1, s} f^{-1}_{T^1} & \lambda_{T^1, s} f^{-1}_{T^1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\lambda_{T^1, s} f^{-1}_{T^1} & \lambda_{T^1, s} f^{-1}_{T^1} & \lambda_{T^1, s} f^{-1}_{T^1} & \lambda_{T^1, s} f^{-1}_{T^1} \\
\end{array}
\]

\[ \text{9.1(iii)} \]

Let us explain the morphism \( f^1(\overline{\psi}) \to \overline{\psi} \). It is sufficient to define compatible morphisms for the vertices of the right hand square. Using obvious notations we define:
- \( \mathcal{B}_f : f^* D \to f^* D \) as the identity;
- \( \mathcal{B}_f : f^* i_* \text{Lie}(D) \to \text{Lie}(f^* D) \) using \( i f \simeq f i \) and the base change morphism for \( \text{Lie} \) \( \text{9.35 (ii)} \);
- \( \mathcal{B}_f : f^* L(D_{T^1}) \to L'((f^* D)_{T^1}) \) as the natural morphism \( f^* l_* f_{T^1, s} f_{T^1} \to l_* f_{T^1, s} f_{T^1} \) resulting from \( l f \simeq f l \) and \( f f_{T^1} \simeq f_T f \).

In each case the composition constraint for the \( \mathcal{B}_f \)'s results from the fact that each one of the morphisms used to build \( \mathcal{B}_f \) verifies the composition constraint. Compatibility with \( \text{can}_D \) (resp. \( \text{can}_L \)) follows from \( \text{5.35 (ii)} \) (resp. and \( \text{141} \)). Compatibility with \( \text{aug} \) follows from \( \text{224} \).

Let us explain the morphisms \( f^1(\overline{\psi}) \to \overline{\psi} \). We define:
- \( \mathcal{B}_f : (f^* D(-h')_{T^2})_{T^2} \to (f^* D(-h'))_{T^2} \) using \( \text{9.10 (ii)} \) and \( \text{4.42 (ii)} \) (i)
- \( \mathcal{B}_f : f^* L(D_{T^2})(-h) \to f^* L'((f^* D)'_{T^2})(-h') \) using the morphisms \( l f \simeq f l \) and \( f^* u_* \text{Lie}(D) \simeq f^* i_*(f^* D) \) using the morphisms \( l f \simeq f l \) and \( f^* u_* \text{Lie}(D) \simeq f^* i_*(f^* D) \) (see above), \( f^* D(-h) \to f^* D(-h') \) \( \text{4.42 (ii) \ (ii)} \), \( L'((f^* D)'_{T^2})(-h') \) \( \text{4.42 (ii)} \).
- \( \mathcal{B}_f : f^* i_* l_* (-h)_{T^1} \to f^* i_* l_* (-h)_{T^1} \) using \( l f \simeq f l \) and \( l f \simeq f l \) and the base change morphism for \( \text{Lie} \);
- \( \mathcal{B}_f : f^* i_* \text{Lie}(D)(-h)_{T^1} \to i_* \text{Lie}(f^* D)(-h)_{T^2} \) using \( \text{9.10 (ii)} \). If \( l f \simeq f l \) and \( i f \simeq f i \).

Here again the composition constraint causes no difficulty. Let us explain the compatibility with respect to the boundaries of the right hand square in \( \text{5.37 (ii)} \). Compatibility with the horizontal arrows denoted \( \text{can}_L \) and \( \text{can}_D \) just follows from the corresponding compatibility in \( \text{i} \). Compatibility with the left vertical arrow (denoted \( \text{9.31 (ii)} \) easily reduces to the commutativity of the following diagram

\[
\begin{array}{cccc}
\iota u f_{T^2} & \iota u f_{T^2} & \iota u f_{T^2} & \iota u f_{T^2} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\lambda_{T^2} f & \lambda_{T^2} f & \lambda_{T^2} f & \lambda_{T^2} f \\
\end{array}
\]

\[ \text{9.1(ii)} \]

\[ \text{9.1(ii)} \]

which in turn causes no difficulty. Compatibility with the right vertical arrow follows formally (alternatively, it would follows from the functoriality of \( \text{9.6 (ii)} \) in a sense which the reader can imagine). 

\[ \square \]
**Definition 5.38.** Fix an effective log divisor \( h \in \Gamma(X, M_X / \mathbb{G}_m) \). We define three functors with value in the category of complexes of modules of \((T^\sharp_{,et}, \mathcal{O}_{\text{cris}})\) endowed with a one step filtration \( \text{Fil}^1 \subset \text{Fil}^0 \):

\[
\text{Fil}^0 \Omega^\cdot_{,T^2}(-h), \text{Fil}^1 \Omega^\cdot_{,T^2}(-h), \text{Fil}^0 L\Omega^\cdot_{,T^2}(-h) : \mathcal{D}C(X^\sharp) \to \text{Fil}^{0,1} \text{Kom}(T^\sharp_{,et}, \mathcal{O}_{\text{cris}})
\]

Their description is the following (in iii), \( \text{Fil}^1 \) is viewed as a subcomplex of \( \text{Fil}^0 \) via the isomorphism \((L(D_{T^2}))(-h) \simeq L(D(-h))\) resulting from Definition 5.41:

(i) \( \text{Fil}^0 \Omega_{,T^2}(-h)(D) \) is \( (L(D_{T^2}))(-h) \)

and \( \text{Fil}^1 \Omega_{,T^2}(-h)(D) := \text{Fil}^0 \Omega_{,T^2}(-h)(D) \) for \( q \geq 1 \)

(ii) \( \text{Fil}^0 L\Omega^\cdot_{,T^2}(-h)(D) \) is \( (L(\Omega^\cdot_{,T^2}(D(-h))))_{T^2} \)

and \( \text{Fil}^1 L\Omega^\cdot_{,T^2}(-h)(D) := \text{Fil}^0 L\Omega^\cdot_{,T^2}(-h)(D) \) for \( q \geq 1 \)

(iii) \( \text{Fil}^0 \Omega^\cdot_{,T^2}(-h)(D) = (L(D_{T^2}))(-h) \)

and \( \text{Fil}^1 \Omega^\cdot_{,T^2}(-h)(D) := \text{Fil}^0 \Omega^\cdot_{,T^2}(-h)(D) \) for \( q \geq 1 \)


**Remark 5.39.** Using [2,3,7] (iii) and [9,34] we find that the functors of [2,3,7] are subject to natural base change morphisms turning them into colax morphisms between contravariant pseudo functors on \( \text{Emb}_{d_{\text{div}}}^{\text{glob}, \text{lfph}} \):

\[
\text{Fil}^\cdot(-)\#(-), \text{Fil}^\cdot L\Omega^\cdot(-)\#(-), \text{Fil}^0 L\Omega^\cdot(-)\#(-) : \mathcal{D}C(-) \to \text{Fil}^{0,1} \text{Kom}((-)_{,et}^{\#}, \mathcal{O}_{\text{cris}})
\]

They extend in particular to diagrams [2,3,7] (i).

**5.4.3.** We will now use the functor \( (1)^* \) defined in 4.1 in order to get a normalized version of the filtered complexes defined in 5.38. To begin with, we replace the ring \( \mathcal{O}_{\text{cris}} \) (which is not \( \mathbb{Z}/p \)-normalized for the étale topology) with the following.

**Definition 5.40.** Recall the morphism of ringed topoi \( l : T^\sharp_{,et} \to T^\sharp_{,et} \). We set

\[
\tilde{\mathcal{O}}_{\text{cris}} := \mathbb{Z}/p \otimes_l \mathcal{O}_{\text{cris}}
\]

**Definition 5.41.** Consider the endomorphism \( (1)^* \) of the ringed topos \((T^\sharp_{,et}; \tilde{\mathcal{O}}_{\text{cris}})\) defined as in 4.4. Let \( \text{Fun}^i \) denote one of the three functors defined in 5.38. We define a functor with values in the category of arrows of complexes of modules of \((T^\sharp_{,et}; \tilde{\mathcal{O}}_{\text{cris}})\),

\[
\tilde{\text{Fun}}^i : \mathcal{D}C(X^\sharp) \to \text{Kom}(T^\sharp_{,et}, \tilde{\mathcal{O}}_{\text{cris}})[1]
\]

by setting \( \tilde{\text{Fun}}^i(D) := (1)^* \text{Fun}^i(D) \).

**Remark 5.42.** The morphism \((1)^* \) is functorial with respect to \( T^\sharp \) in the obvious way. The three functors just defined thus naturally extend to colax morphisms between contravariant pseudo functors on \( \text{Emb}_{d_{\text{div}}}^{\text{glob}, \text{lfph}} \):

\[
\tilde{\text{Fil}}^\cdot(-)\#(-), \tilde{\text{Fil}}^\cdot L\Omega^\cdot(-)\#(-), \tilde{\text{Fil}}^0 L\Omega^\cdot(-)\#(-) : \mathcal{D}C(-) \to \text{Kom}((-)_{,et}^{\#}, \mathcal{O}_{\text{cris}})[1]
\]

as in 5.39.

The following lemma explains the difference between \( \tilde{\text{Fun}}^i \) and \( \text{Fun}^i \).
Lemma 5.43. Let \((T^\sharp, h)\) denote the logarithmic divided power envelope of some \((Y^\sharp, h)\) in \(\text{Em}_{\Sigma_\infty}^{\text{glob}, \text{fpb}}\) and let \(\text{Fun}^i\) (resp. \(\text{Fun}^\flat\)), denote one of the three functors defined in 5.38, 5.39 (resp. 5.41, 5.42).

(i) There is a natural morphism of functorial distinguished triangles in \(D^b(T^\sharp_{\text{et}}, \mathcal{O}_{\text{crys}}^\flat)\)

\[
\begin{align*}
\tilde{\text{Fun}}^1(D) & \xrightarrow{1} \tilde{\text{Fun}}^0(D) \xrightarrow{\text{can}} \text{Lie}_{T^\sharp}(-h)(D) \\
\text{Fun}^1(D) & \xrightarrow{1} \text{Fun}^0(D) \xrightarrow{\text{can}} \text{Lie}_{T^\sharp}(-h)(D)
\end{align*}
\]

where \(\text{Lie}_{T^\sharp}(-h)(D)\) is the direct image via \((X^N_{\text{et}}, \mathcal{O}) \to (T^\sharp_{\text{et}}, \mathcal{O}_{\text{crys}}^\flat)\) of the constant projective system \(\text{Lie}(D)(-h)\) and \(\text{Lie}_{T^\sharp}(-h)(D) := \tau_{\geq -1}L(1)^*\text{Lie}_{T^\sharp}(-h)(D)\). The cohomology of the latter complex is described explicitly as follows: \(H^0\) is \(\text{Lie}_{T^\sharp}(-h)(D)\); the \(k\)-th component of \(H^{-1}\) is \(\text{Lie}_{k,T^\sharp}(-h)(D)\) as well but the transition morphisms of the projective system are zero.

(ii) The objects of the complexes \(\tilde{\text{Fun}}^i(D)\) (resp. \(\text{Fun}^i(D)\)) come from normalized (resp. \(L\)-normalized) quasi-coherent modules on \(T_{\text{et}}\) by scalar restriction via \(\mathcal{O}_{\text{crys}}^\flat \to \mathcal{O}\) (resp. \(\mathcal{O}_{\text{crys}}^\flat \to \mathcal{O}\)). They are in particular normalized (resp. \(L\)-normalized) and \(L\)-acyclic. The complexes \(\tilde{\text{Fun}}^i(D)\) are \(L\)-normalized and \(\text{Lie}_{T^\sharp}(-h)(D)\) as well.

Proof. Everything is straightforward from 4.3 (iv) once observed that \(T\) is flat over \(\Sigma_\infty\) and that the tautologically normalized \(\mathbb{Z}/p\)-algebra \(\mathcal{O}_{\text{crys}}^\flat\) is consequently flat as well (note that by 4.15 (i) the structural ring of \(T_{\text{et}}\) is \(\mathbb{Z}_p\)-flat, hence \(\mathcal{O}_{\text{crys}}^\flat\) as well).

\(\square\)

5.4.4. The difference between \(\tilde{\text{Fun}}^i\) and \(\text{Fun}^i\) disappears when taking limits. Let us discuss this briefly.

Definition 5.44. Let \(\text{Fun}^i\) denote one of the three functors defined in 5.38. We define a functor

\[\text{Fun}^i : \mathcal{D}(X^\sharp) \to \text{Fil}^0,1 \text{Kom}(T^\sharp_{\text{et}}, \mathcal{O}_{\text{crys}}^\flat)\]

by setting \(\text{Fun}^i(D) := \text{Fil}^i\text{Fun}^\flat(D)\).

Lemma 5.45. (i) Consider the natural morphism \(l : (T^\sharp_{\text{et}}, \mathcal{O}_{\text{crys}}^\flat) \to (T^\sharp_{\text{et}}, \mathcal{O}_{\text{crys}}^\flat)\).

The images of the vertical arrows in 5.43 (i) by \(\text{R}l^\sharp\) are isomorphisms. The distinguished triangle obtained by applying \(\text{R}l^\sharp\) to either one of the horizontal lines boils down to an exact sequence

\[
\begin{array}{cccc}
0 & \text{Fun}^1(D) & \text{Fun}^0(D) & \text{Lie}_{T^\sharp}(-h)(D) & 0 \\
\end{array}
\]

where \(\text{Lie}_{T^\sharp}(-h)(D)\) is the direct image of \(\text{Lie}(D)(-h)\) via \((X^N_{\text{et}}, \mathcal{O}) \to (T^\sharp_{\text{et}}, \mathcal{O}_{\text{crys}}^\flat)\). Conversely the top line of 5.43 (i) can be obtained by applying \(\text{L}l^\sharp\) to this exact sequence.

(ii) The objects of the complexes \(\text{Fil}^i\text{Lie}_{T^\sharp}(-h)(D), \text{Fil}^i\text{Lie}_{T^\sharp}^\flat (-h)(D), \text{Fil}^i\text{L}l^\sharp\text{Lie}_{T^\sharp}(-h)(D)\) and \(\text{Lie}_{T^\sharp}(-h)(D)\) come from quasi-coherent and \(L\)-quasi-coherent modules on \(T^\sharp_{\text{et}}\) by scalar restriction via \(\mathcal{O}_{\text{crys}}^\flat \to \mathcal{O}\).

\(\square\)

To prove the first assertion it suffices to notice that the image of \(L^1(1)^*\text{Lie}_{T^\sharp}(-h)(D)\) by \(\text{R}l^\sharp\) vanishes (the transition morphisms are zero). The remaining assertions of (i) (ii) are straightforward from 5.43 (ii).
5.5. Twisted syntomic complexes on the étale site.

In this section we complete the construction of the twisted syntomic complexes (on the étale site) for twisted Dieudonné crystals over an $X$ having local finite $p$-bases over $\Sigma_1$. We first define the morphism $\phi$ in presence of a global embedding with Frobenius lift (5.49) and we conclude using the gluing lemma (5.9).

5.5.1. We begin with some basic facts about liftings of the relative Frobenius on small étale site.

**Lemma 5.46.** Consider a $p$-adic log scheme $T^\flat$ endowed with a Frobenius lift $\tilde{F}_{T^\sharp}$.

(i) The relative Frobenius $F^{{(-/T^\sharp)}}: id \to F^{-1}$ on $T^\sharp_{1,et}$ uniquely extends to a natural transformation of endofunctors of $T^\sharp_{*,et}$:

$$\tilde{F}^{{(-/T^\sharp)}}: id \to \tilde{F}^{-1}_{T^\sharp}$$

called the lifted relative Frobenius (attached to $\tilde{F}_{T^\sharp}$).

(ii) Consider the ringed topos $(T^\sharp_{*,et}, \mathcal{O})$. There exists a unique endomorphism $\tilde{F}$ of the ring $\mathcal{O}$ which simultaneously extends:

- the endomorphism of the structural ring of $T^\sharp_{*,zar}$ defining $F_{T^\sharp}$, and
- the Frobenius endomorphism $F: f \mapsto f^p$ of the structural ring of $T^\sharp_{1,et}$.

Explicitly this endomorphism can be obtained by composing the lifted relative Frobenius $\tilde{F}(\mathcal{O}/T^\sharp) : \mathcal{O} \to \tilde{F}^{-1}_{T^\sharp} \mathcal{O}$ with the functoriality morphism along $\tilde{F}_{T^\sharp}$, $\tilde{F}_{T^\sharp}^*: \tilde{F}^{-1}_{T^\sharp} \mathcal{O} \to \mathcal{O}$.

Proof. Everything follows easily from the fact that $T^\sharp_{1,et} \to T^\sharp_{k,et}$ is an equivalence.

5.5.2. We will now show that “Frobenius is uniquely divisible by $p$ on $\tilde{Fil}^1$”.

**Definition 5.48.** Let $(T^\sharp, \tilde{F}_{T^\sharp})$ denote the logarithmic divided power envelope of some $(X^\sharp, Y^\sharp, \tilde{F}_{Y^\sharp})$ in $Emb^{\sharp, glob}_{\Sigma_F}$. If $h \in \Gamma(X, M_X/\mathbb{G}_m)$ and $(D, f, v)$ in $\mathcal{D}_C(X^\sharp)$ we use the following notations.
- We let \( f : F^*(D(-h)) \to D(-h) \) in \( \mathcal{C} \text{rys}(X^\sharp/\Sigma_{\infty})_{\text{crys,et}, \mathcal{O}} \) denote

\[
F^*(D(-h)) \xrightarrow{\mathrm{proj}} (F^*D)(-ph) \to (F^*D)(-h) \xrightarrow{f} D(-h)
\]

- We let \( Fr : D(-h)_{T^\sharp} \to (\tilde{F})_*D(-h)_{T^\sharp} \) in \( \text{Mod}(T^\sharp_{\text{crys},et}, \mathcal{O}) \) (or \( \text{Mod}(T^\sharp_{\text{crys},et}, \mathcal{O}_{\text{crys}}) \)) denote

\[
D(-h)_{T^\sharp} \xrightarrow{\tilde{F}(-/T^\sharp)} (\tilde{F})_*F^*_{T^\sharp}(D(-h))_{T^\sharp} \xrightarrow{\mathfrak{9}(\mathfrak{1}(\mathfrak{1})))} (\tilde{F})_*F^*(D(-h))_{T^\sharp} \xrightarrow{f} (\tilde{F})_*D(-h)_{T^\sharp}
\]

- We let \( Fr : \Omega^*_{T^\sharp}(D(-h)) \to (\tilde{F})_*\Omega^*_{T^\sharp}(D(-h)) \) in \( \text{Kom}(T^\sharp_{\text{crys},et}, \mathcal{O}_{\text{crys}}) \) denote

\[
\Omega^*_{T^\sharp}(D(-h)) \xrightarrow{\tilde{F}(-/T^\sharp)} (F)_*\tilde{F}^*_{T^\sharp}\Omega^*_{T^\sharp}(D(-h)) \xrightarrow{\mathfrak{22}(\mathfrak{7})} (F)_*\Omega^*_{T^\sharp}(F^*(D(-h))) \xrightarrow{f} (F)_*(\tilde{F})_*\Omega^*_{T^\sharp}(D(-h))
\]

- We let \( Fr : (L(\tilde{F})_*\Omega^*_{T^\sharp}(D(-h))))_{T^\sharp} \to (\tilde{F})_*((L(\tilde{F})_*\Omega^*_{T^\sharp}(D(-h))))_{T^\sharp} \) in \( \text{Kom}(T^\sharp_{\text{crys},et}, \mathcal{O}_{\text{crys}}) \) denote

\[
(L(\tilde{F})_*\Omega^*_{T^\sharp}(D(-h))))_{T^\sharp} \xrightarrow{\tilde{F}(-/T^\sharp)} (F)_*\tilde{F}^*_{T^\sharp}(L(\tilde{F})_*\Omega^*_{T^\sharp}(D(-h))))_{T^\sharp} \xrightarrow{\mathfrak{22}(\mathfrak{4})} (F)_*(L(\tilde{F})_*\Omega^*_{T^\sharp}(F^*(D(-h))))_{T^\sharp} \xrightarrow{f} (F)_*(L(\tilde{F})_*\Omega^*_{T^\sharp}(D(-h))))_{T^\sharp}
\]

\[\]

**Proposition 5.49.** Let \( (X^\sharp, Y^\sharp, \iota, \tilde{F}, h) \) in \( \text{Emb}_{F,\text{div}}^{\text{glob,lfpb}} \) and denote \( T^\sharp \) the logarithmic divided power of \( \iota \). Let \( \text{Fun}^i \) denote one of the three functors defined in 5.42 and let \( D \in \mathcal{D}C(X^\sharp) \). There exists a unique morphism \( \phi \) in \( \text{Kom}(T^\sharp_{\text{crys},et}, \mathcal{O}_{\text{crys}}) \) rendering the following square commutative:

\[
\text{Fun}^1(D) \xrightarrow{\phi} (\tilde{F})_*\text{Fun}^0(D) \\
\downarrow 1 \hspace{2cm} \downarrow p \\
\text{Fun}^0(D) \xrightarrow{Fr} (\tilde{F})_*\text{Fun}^0(D)
\]

The morphism \( \phi \) is functorial with respect to \( D \) and compatible with the base change morphisms arising from morphisms in \( \text{Emb}_{F,\text{div}}^{\text{glob,lfpb}} \) (5.42).

Proof. Unicity, functoriality and compatibility to base change follows from 4.3 (iii) 5.43 (ii). Let us prove existence. By 4.3 (ii) we have the bottom exact sequence in the following commutative diagram:

\[
\text{Fun}^1_{t,1}(D) \xrightarrow{t_{t,1+1}} \text{Fun}^1_{t,1+1}(D) \\
\downarrow \text{Fr}_{t,1} \hspace{2cm} \downarrow \text{Fr}_{t,1} \\
0 \xrightarrow{\text{Fr}_{t,1}} (\tilde{F})_*\text{Fun}^0_{t,1+1}(D) \xrightarrow{p} (\tilde{F})_*\text{Fun}^0_{t,1+1}(D) \xrightarrow{p} (\tilde{F})_*\text{Fun}^0_{t,1+1}(D) \xrightarrow{p} 0
\]

We want to prove the existence of the dotted arrow marked \( ? \), since the desired \( \phi \) will follow by applying \( t_{t,1+1} \) to it (note that \( t_{t,1+1} \) is fully faithful and commutes to \( (\tilde{F})_* \)). It is thus sufficient to prove that the morphism

\[
Fr \circ 1 : \text{Fun}^1_{t,1}(D) \to (\tilde{F})_*\text{Fun}^0_{t,1}(D) \text{ in Kom}(T^\sharp_{t,1,et}, \mathcal{O}^\vee_{1,\text{crys}})
\]

105
vanishes. Let us examine the three cases of definition 5.48. Using the definition of the base change morphisms (227) and (224) it is clear that (142) vanishes in degree $\geq 1$ in all cases. It thus remains to prove that

\[ Fr_1 : D(-h)_{T'_1} \to (F)_* D(-h)_{T'_1} \]  
\[ Fr_1 : (L(D(-h)_{T'_1}))_{T'_1} \to (F)_* (L(D(-h)_{T'_1}))_{T'_1} \]

respectively vanish on $(Fil^1 D)(-h)_{T'_1}$ and $(Fil^1 L(D_{T'_1}))(-h)_{T'_1}$. Thanks to 4.49(ii) we see that

\[ M(-h)_{T'_1} \hookrightarrow M_{T'_1} \]

if $M$ is either one of the locally free crystals $D$ or $L(D_{T'_1})$. This monomorphism is compatible with $Fr_1$ and we may thus assume that $h = 1$. 

Consider the following commutative diagram on $((X^2/\Sigma_1)_{\text{crys}}, \mathcal{O})$:

```
\[ \begin{array}{ccc}
\mathcal{F}_r \\
\downarrow \text{adj} \\
D \\
\downarrow \text{can}_D \end{array} \]
```

where $\text{nat} : i_\ast \phi_* \to (F)_*$ is deduced from the isomorphism $\phi^* \simeq (F)^*i_\ast (5.17(\text{ii})).$ Since $Fil^1 \mathcal{D} = \text{Ker} \text{ can}_D$, it must be in the kernel of $\mathcal{F}_r$. This ends the proof in the case $M = D$ since $Fr_1$ coincides with the realization of $\mathcal{F}_r$ at $T'_1$.

The case $M = L(D_{T'_1})$ is proven similarly using the following commutative diagram of $\text{Mod}((X^2/\Sigma_1)_{\text{crys, et}}, \mathcal{O})$:

```
\[ \begin{array}{ccc}
\mathcal{F}_r \\
\downarrow \text{adj} \\
\mathcal{D} \\
\downarrow \text{can}_L \end{array} \]
```

**Definition 5.50.** As in 5.49 let $T^2$ denote the logarithmic divided power envelope of some $(X^2, Y^2, \iota, \tilde{F}, h)$ in $Emb_{F, \text{div}}^{\text{glob, div}}$. For $D \in \mathcal{D}C(X^2)$ we define three complexes of

\[ \square \]
(1, φ)-modules of \((T^z_\text{et}, \tilde{O}^{\text{crys}})\) as follows.

\[
\begin{align*}
S_{\text{et}, T^z}(h)(D) & := \langle \text{Fil}^1_{\text{et}, T^z}(h)(D), \text{Fil}^0_{\text{et}, T^z}(h)(D), 1, \phi \rangle \\
\Omega^*_{\text{et}, T^z}(h)(D) & := \langle \text{Fil}^1 \Omega^*_{\text{et}, T^z}(h)(D), \text{Fil}^0 \Omega^*_{\text{et}, T^z}(h)(D), 1, \phi \rangle \\
SL\Omega^*_{\text{et}, T^z}(h)(D) & := \langle \text{Fil}^1 L\Omega^*_{\text{et}, T^z}(h)(D), \text{Fil}^0 L\Omega^*_{\text{et}, T^z}(h)(D), 1, \phi \rangle
\end{align*}
\]

By functoriality with respect to \(D\) and \((X^z, Y^z, i, \tilde{F}, h)\) this defines three functors

\[
S_{\text{et}}, \Omega^*_{\text{et}}, SL\Omega^*_{\text{et}, T^z} : \text{DC}(-) \to \text{Kom}^{1, \phi}((-)_{\text{et}, T^z}, \tilde{O}^{\text{crys}})
\]

above \(\text{Emb}_{F, \text{div}}^{\text{glob}, \text{lfpb}, \text{op}}\) and thus above \(\text{Diag}(HR_{F, \text{div}}^{x, \text{et}, \text{lfpb}})\) also, using \ref{5.51}[i].

It might be worth to spell out the meaning of the functor \([145]\) above \(\text{Diag}(HR_{F, \text{div}}^{x, \text{et}, \text{lfpb}})\) in terms of a collection of functors together with base change morphisms. Let \(S\) denote either \(S_{\text{et}, T^z}\), \(\Omega^*_{\text{et}, T^z}\) or \(SL\Omega^*_{\text{et}, T^z}\). For each diagram \(Y^z_{[1]} = (U^z_{[1]} / X^z, Y^z_{[1]}, i_{[1]}, \tilde{F}_{[1]})(h)\) of \(HR_{F, \text{et}}^x\) such that each vertex of \(X^z\) (hence of \(U^z_{[1]}\) has local finite \(p\)-bases and for each \(h \in \Gamma(X, M_X / \mathbb{G}_m)\) \([145]\) yields a functor

\[
S(-h|_{U^z_{[1]}}) : \text{DC}(U^z_{[1]}) \to \text{Kom}^{1, \phi}(T^z_{[1], \text{et}}, \tilde{O}^{\text{crys}})
\]

where \(T^z_{[1]}\) denotes the logarithmic divided power envelope of \(u^z_{[1]}\). The collection of these functors for \((Y^z, h)\) varying in \(\text{Diag}(HR_{F, \text{div}}^{x, \text{et}, \text{lfpb}})\) is moreover endowed with a canonical family of base change morphisms satisfying the usual cocycle condition.

**Lemma 5.51.**  
(i) The functors \([145]\) are canonically related by natural transformations above \(\text{Diag}(HR_{F, \text{div}}^{x, \text{et}, \text{lfpb}})\) as follows:

\[
S_{\text{et}}^{1, \phi} \xrightarrow{\alpha'} SL\Omega^*_{\text{et}}^{1, \phi} \xrightarrow{\alpha''} S\Omega^*_{\text{et}}^{1, \phi}
\]

Fix a diagram \((X^z, h)\) of \(\text{Sch}_{\text{div}}^{x, \text{lfpb}} / \Sigma_1\). Let \(T^z_{[1]}\) denotes the logarithmic divided power envelope of some object \(Y^z_{[1]}\) in \(HR_{F, \text{div}}^{x, \text{lfpb}}\) \((X^z, h)\) \ref{5.50}[ii][iv].

- If \(* = \text{et}\) and \(D_{[1]} \in \text{DC}(U^z_{[1]})\) the morphism \(\alpha'\) induces an isomorphism

\[
S_{\text{et}, T^z_{[1]}}^{1, \phi}(-h|_{U^z_{[1]}})(D_{[1]}) \simeq SL\Omega^*_{\text{et}, T^z_{[1]}}^{1, \phi}(-h|_{U^z_{[1]}})(D_{[1]}) \text{ in } D(\text{Mod}^{1, \phi}(T^z_{[1], \text{et}}, \tilde{O}^{\text{crys}}))
\]

- If \(* = \text{crys}\) and \(D \in \text{DC}(X^z)\) the morphism \(\alpha''\) induces an isomorphism

\[
Rf_{T^z_{[1], \text{et}}} S\Omega^*_{\text{et}, T^z_{[1]}}^{1, \phi}(-h|_{U^z_{[1]}})(D_{[1]}) \simeq Rf_{T^z_{[1], \text{et}}} SL\Omega^*_{\text{et}, T^z_{[1]}}^{1, \phi}(-h|_{U^z_{[1]}})(D_{[1]}) \text{ in } D(\text{Mod}^{1, \phi}(X^z, \mathbb{N}, \tilde{O}^{\text{crys}}))
\]

(ii) Let \(S\) denote one of the three functors \([145]\) and fix a diagram \((X^z, h)\) of \(\text{Sch}_{\text{div}}^{x, \text{lfpb}} / \Sigma_1\) as well as \(D \in \text{DC}(X^z)\). If \(T^z_{2[1], \text{et}} \to T^z_{1[1], \text{et}}\) is the logarithmic divided power envelope of some morphism \(Y^z_{2[1]} \to Y^z_{1[1]}\) in \(HR_{F, \text{div}}^{x, \text{lfpb}}(X^z, h)\) the base change morphism for \(S\) induces

\[
Rf_{T^z_{2[1], \text{et}}} S_{T^z_{2[1], \text{et}}}^{1, \phi}(-h|_{U^z_{2[1]}})(D_{[1]}) \to Rf_{T^z_{1[1], \text{et}}} S_{T^z_{1[1], \text{et}}}^{1, \phi}(-h|_{U^z_{1[1]}})(D_{[1]}) \text{ in } D(\text{Mod}^{1, \phi}(X^z, \mathbb{N}, \tilde{O}^{\text{crys}}))
\]

This morphism is invertible in the following cases:

- if \(S\) is either \(S_{\text{et}}^{1, \phi}\) or \(SL\Omega^*_{\text{et}}^{1, \phi}\) and \(* = \text{crys}\),
- if \(S\) is \(S\Omega^*_{\text{et}}^{1, \phi}\) and \(* = \text{et}\)
Proof. (i) Putting together 5.43 and 5.3 gives morphisms

\[ \tilde{\text{Fil}}^i_{T,1}(h_{U[1]}) \to \tilde{\text{Fil}}^i L\Omega^\bullet_{T,1}(h_{U[1]}) \to \tilde{\text{Fil}}^i L\Omega^\bullet_{T,1}(h_{U[1]}) \]

which are compatible with 1 and \( Fr \). They are compatible with \( \phi \) as well by 4.3 (iii). Whence the claimed morphisms \( \alpha' \) and \( \alpha'' \). An immediate dévissage (using 5.2 (iii) 5.45 (i)) together with 5.3 5.4 show that \( \alpha' \) is always an isomorphism and that \( \alpha'' \) becomes an isomorphism once we apply \( Rf_{\Gamma[1,m]} \) to it. Similar arguments apply in (ii).

The above lemma shows that the assumptions of 5.9 (ii) are satisfied. From this and 5.12 we get immediately the following stronger result which roughly says that the base change morphism 5.51 (ii) is in fact independent of the given morphism \( g : Y^\sharp_{1,[1]} \to Y^\sharp_{2,[1]} \) and that it still exists even if \( g \) does not.

**Proposition 5.52.** (i) Up to a unique isomorphism there exists a unique couple \((\mathcal{S}^\sharp_{et,x^\sharp}, \alpha)\) such that the following conditions are satisfied:

- \( \mathcal{S}^{1,\phi}_{et,x^\sharp} \) is a functor \( DC(-) \to D(\text{Mod}^{1,\phi}(\mathcal{D}^{\text{crys}})) \) above the category \( \mathcal{B}^\sharp_0 \) defined in 5.11. In other terms it is a collection of functors indexed by the \( (X^\sharp, h) \)'s of \( \mathcal{B}^\sharp_0 \) together with a canonical family of base change morphisms satisfying the usual cocycle condition.

\[ \mathcal{S}^{1,\phi}_{et,x^\sharp}(h) : DC(X^\sharp) \to D(\text{Mod}^{1,\phi}(X^{\sharp,N}, \tilde{\mathcal{O}}^{\text{crys}})) \]

- \( \alpha \) is a collection of functorial isomorphisms indexed by the diagrams of \( H\mathcal{R}^{\sharp,\text{et,fppf}}_F \):

\[ \alpha_Y : \mathcal{S}^{1,\phi}_{et,x^\sharp}(h)(D) \to Rf_{\Gamma[1,m]}(\mathcal{S}^{1,\phi}_{et,x^\sharp}(h)(D)) \]

A morphism \( (X^\sharp_1, h_1) \to (X^\sharp_2, h_2) \) induces a base change morphism on the left side. If \( Y^\sharp_{1,[1]} \to Y^\sharp_{2,[1]} \) is furthermore a morphism of \( H\mathcal{R}^{\sharp,\text{et}}_F \) above \( X^\sharp_1 \to X^\sharp_2 \) then it induces a base change morphism on the right side. Both base change morphisms are compatible via \( \alpha \).

(ii) If \( (X^\sharp, h) \) is in \( \mathcal{B}^\sharp_0 \) and \( Y^\sharp_{[1]} \) is in \( H\mathcal{R}^{\sharp,\text{crys}}_F(X^\sharp) \) then (i) and 5.51 (ii) induce canonical isomorphisms as follows

\[ \alpha' \alpha : \mathcal{S}^{1,\phi}_{et,x^\sharp}(h)(D) \to Rf_{\Gamma[1,m]}(\mathcal{S}^{1,\phi}_{et,x^\sharp}(h)(D)) \]

\[ \alpha''^{-1} \alpha' \alpha : \mathcal{S}^{1,\phi}_{et,x^\sharp}(h)(D) \to Rf_{\Gamma[1,m]}(\mathcal{S}^{1,\phi}_{et,x^\sharp}(h)(D)) \]

Those morphisms are compatible with the base change morphism attached to morphisms of \( H\mathcal{R}^{\sharp,\text{crys,fppf}}_F \) in the same sense as above.

**Definition 5.53.** If \( (X^\sharp, h) \) is in \( \mathcal{B}^\sharp_0 \) and \( D \) is in \( DC(X^\sharp) \) we define the syntomic complex of \( D \) twisted by \( (h) \) on the étale site as follows:

\[ \mathcal{S}_{et,x^\sharp}(h)(D) := R\pi_* \mathcal{S}^{1,\phi}_{et,x^\sharp}(h)(D) \in D(X^\sharp,N, \tilde{\mathcal{O}}^{\text{crys},F=1}) \]

where \( \mathcal{S}^{1,\phi}_{et,x^\sharp}(h) \) is the functor (147) and \( \pi : (X^{\sharp,N,1,\phi}, \tilde{\mathcal{O}}^{\text{crys}}) \to (X^{\sharp,N,\text{crys},F=1}) \) denotes the projection morphism 5.2 (ii). Finally we set

\[ \mathcal{S}_{et,x^\sharp}(h)(D) := Rl_* \mathcal{S}_{et,x^\sharp}(h)(D) \in D(X^\sharp, \tilde{\mathcal{O}}^{\text{crys},F=1}) \]
Remark 5.54. The functor \((\mathcal{F}_1)\) sends short exact sequences of Dieudonné crystals to distinguished triangles (this follows immediately from \(5.53\) (iii)).

Proposition 5.55. Consider \((X^2, h)\) in \(B_0^\ast\).

(i) If \(\mathcal{T}_\ast\) is the logarithmic divided power envelope of some \(Y^2\) in \(HR^e_F(X^2)\) there are canonical distinguished triangles in \(D(X^2, N_1, \Omega^{crys,F=1})\)

\[
\begin{align*}
Rf_{\mathcal{T}^\ast_{\ast,\ast}} \mathcal{F}^{\mathcal{F}^1} \Omega^\ast_{\mathcal{T}^\ast_{\ast,\ast}} (h_{U^{\ast}_{\mathcal{T}^\ast_{\ast,\ast}}}) & \to Rf_{\mathcal{T}^\ast_{\ast,\ast}} \mathcal{F}^{\mathcal{F}^0} \Omega_{\mathcal{T}^\ast_{\ast,\ast}} (h_{U^{\ast}_{\mathcal{T}^\ast_{\ast,\ast}}}) \to Rf_{\mathcal{T}^\ast_{\ast,\ast}} \Omega_{\mathcal{T}^\ast_{\ast,\ast}} (h_{U^{\ast}_{\mathcal{T}^\ast_{\ast,\ast}}}) \\
S_{et, X^2} (-h)(D) & \to Rf_{\mathcal{T}^\ast_{\ast,\ast}} \mathcal{F}^{\mathcal{F}^1} \Omega^\ast_{\mathcal{T}^\ast_{\ast,\ast}} (h_{U^{\ast}_{\mathcal{T}^\ast_{\ast,\ast}}}) (D_{U^{\ast}_{\mathcal{T}^\ast_{\ast,\ast}}}) \\
\mathcal{S}_{et, X^2} (-h)(D) & \to Rf_{\mathcal{T}^\ast_{\ast,\ast}} \Omega_{\mathcal{T}^\ast_{\ast,\ast}} (h_{U^{\ast}_{\mathcal{T}^\ast_{\ast,\ast}}}) (D_{U^{\ast}_{\mathcal{T}^\ast_{\ast,\ast}}})
\end{align*}
\]

(ii) If \(\mathcal{T}_\ast\) is in fact in \(HR^e_F^{crys}\) there are analogous distinguished triangles with \(\mathcal{F}^{\mathcal{F}^1} \Omega^\ast_{\mathcal{T}^\ast_{\ast,\ast}}\) instead of \(\mathcal{F}^{\mathcal{F}^0} \Omega^\ast_{\mathcal{T}^\ast_{\ast,\ast}}\). The resulting couples of triangles are moreover canonically isomorphic.

Proof. The first distinguished triangle is provided by \(5.43\) (i) and the second by \(5.2\) (iv) \(\Box\)

Remark 5.56. Here again the situation becomes clearer after passing to the limit. Indeed from \(5.3\) (i) we find that the distinguished triangles of \(5.55\) (ii) in the case \(\mathcal{F}^{\mathcal{F}^0}\) become

\[
\begin{align*}
Ru_\ast((\mathcal{F}^{\mathcal{F}^1}D)(-h)) & \to Ru_\ast(D(-h)) \to \text{Lie}(D)(-h) +1 \\
S_{et, X^2} (-h)(D) & \to Ru_\ast((\mathcal{F}^{\mathcal{F}^1}D)(-h)) \to Ru_\ast(D(-h)) +1
\end{align*}
\]

in \(D(X_{et}, \Omega^{crys,F=1})\). Conversely it follows from \(5.43\) (ii) that the triangles of \(5.55\) can be retrieved from these ones by applying \(L^\ast\) (note that \(\mathcal{O}^{crys,F=1} \simeq \mathbb{Z}/p \otimes \mathcal{O}^{crys,F=1} \simeq \mathbb{Z}/p\), as will be recalled in \(5.4\) (iii)).

5.6. Syntomic complexes on the syntomic site.

We explain two constructions of the syntomic complex using the syntomic topology for schemes without log structures. The first one is global, i.e. does not involve local embeddings or cohomological descent. The second one is local and involves the linearized de Rham complex. It will serve as a bridge to the previous constructions. We will consider the following situations in parallel until \(5.63\):

- The global situation where \(X\) is a separated scheme with local finite \(p\)-bases over \(\Sigma_1\).
- The local situation where \((i : X \to Y, F)\) is an object of \(\text{Emb}^\text{glob,lf} \mathcal{F}\) and \(T\) denotes the divided power envelope of \(i\).

Let us gather some technical facts in a lemma.

Lemma 5.57. Consider the global situation.

(i) Let \(\epsilon : (X^N_{sym}, \mathcal{O}) \to (X^N_{et}, \mathcal{O})\) denote the natural morphism. If \(M\) is a module on \((X^N_{et}, \mathcal{O})\) (resp. a quasi-coherent local crystal on \(((X/\Sigma_1)_{crys,et}/T, \mathcal{O})\)) then the following morphisms are invertible

\[
\begin{align*}
\epsilon^i_* M & \to i_* \epsilon^* M \\
\epsilon^* f_{T, *} M & \to f_{T, *} \epsilon^* M
\end{align*}
\]

109
(ii) Let \(((X/\Sigma_\cdot)_\text{crys, syn}, \mathcal{O}) \to ((X/\Sigma_\cdot)_\text{crys, et}, \mathcal{O})\) denote the natural morphism. If \(M\) is a crystal of \(((X/\Sigma_\cdot)_\text{crys, et}, \mathcal{O})\) then \(\epsilon^* M\) is a crystal of \(((X/\Sigma_\cdot)_\text{crys, syn}, \mathcal{O})\). If \(M\) is furthermore quasi-coherent, locally free or locally free of finite type then the same is true for \(\epsilon^* M\). In that case \(M \to R\epsilon_* \epsilon^* M\) is moreover an isomorphism.

(iii) If \(M\) is a quasi-coherent crystal of \(((X/\Sigma_\cdot)_\text{crys, syn}, \mathcal{O})\) then it is acyclic for the functor \(u_*\). The following natural morphism (induced by \(id \to i_* i^*\)) is moreover an epimorphism in \(\text{Mod}(X_{\text{syn}}, \mathcal{O})\):

\[ u_* M \to i^* M. \]

Proof. Let us explain the first isomorphism. First we note that we have an isomorphism \(\epsilon^1 i_* \cong i_* \epsilon^{-1}\) for abelian sheaves expressing the compatibility of \(u\) and \(\epsilon\). The claimed analogous morphism for modules will follow formally if we prove that \(\epsilon^* \mathbb{G}_a \to \mathbb{G}_a\) is an isomorphism, i.e. that \(\epsilon^* I \to I\) (here \(I\) denotes the canonical ideal in the structural ring of the crystalline sites). This in turn can be checked easily using that each affine object \((U', T')\) of \(\text{crys, syn}(X/\Sigma_{\infty})\) admits a morphism to some affine \((U, T)\) in \(\text{crys, et}(X/\Sigma_{\infty})\) where \(T\) is the divided power envelope of \(U\) inside a polynomial algebra of the form \(\mathbb{Z}/p^k[x_\alpha, y_\beta]\) where the \(x_\alpha\) (resp. \(y_\beta\)) are sent to generators of the algebra of \(U\) (resp. the ideal of \(U'\) inside \(T'\)).

The second isomorphism is a formal consequence of \([1.20]\) since \(\epsilon_*\) is fully faithful on the category of quasi-coherent crystals \([1.22] (i)\) and of quasi-coherent local crystals (easy variant of \(\text{loc. cit.}\)).

(ii) This a repetition of \([1.22] (i)\) (ii) (iii)

To prove the first statement it is sufficient to show that the syntomic sheaf associated to the presheaf \(U \mapsto H^q((U/\Sigma_k)_{\text{crys, syn}}, M_k|U)\) vanishes for all \(k \geq 1\) and \(q \geq 1\). We have isomorphisms

\[ H^q((U/\Sigma_k)_{\text{crys, syn}}, M_k|U) \cong H^q((U/\Sigma_k)_{\text{crys, et}}, \epsilon_*(M_k|U)) \]

\[ \cong H^q(U_{et}, \mathcal{O}_{D(U,Y_k)}/\mathcal{S}_k(\epsilon_*(M_k|U))) \]

if \(Y \in E\text{mb}_{\text{glob}}(U)\) \([1.32] (i)\)

and the second isomorphism is functorial with respect to morphisms in \(E\text{mb}_{\text{glob}}\). Now if \(U\) and \(Y\) are affine the latter is a subquotient of

\[ \Gamma(U_{et}, \mathcal{O}_{D(U,Y_k)}/\mathcal{S}_k(\epsilon_*(M_k|U))) \cong M_k(U, D(U,Y_k)) \otimes_{\mathcal{O}(Y_k)} \mathcal{O}_{Y_k}/\mathcal{S}_k(Y_k) \]

It is thus sufficient to notice that any global section \(\omega = f dg\) of \(\mathcal{O}_{Y_k}/\mathcal{S}_k\) vanishes when restricted to \(Y_k := \text{Spec}_{\mathbb{Z}/p^k}(\mathcal{O}[x]/(x^p - g))\) which is a syntomic covering of \(Y_k\). The second statement is proven similarly.

(iv) Let us prove that \(u_* M\) is flat over \(\mathbb{Z}/p^k\) as soon as \(M\) is a locally free crystal (not necessarily of finite type) on \(((X/\Sigma_k)_{\text{crys, syn}, \mathcal{O}})\). We will show that \(u_* M \otimes_{\mathbb{Z}/p^k} (-)\) preserves monomorphisms \(N \hookrightarrow N'\) of \(\mathbb{Z}/p^k\) modules on \(X_{\text{crys, syn}}\). Since the question is local we can always assume that \(X\) affine and choose a lifting \(X\) with \(p\)-bases over \(\mathbb{Z}/p^k\). Let \(N'' : U \mapsto N'(U)/N(U)\) denote the cokernel presheaf. It is sufficient to show that the
sheaf associated to the following presheaf vanishes:

\[ K : U \mapsto \text{Tor}_1^{\mathbb{Z}/p^k}(u_*M(U), N''(U)) \]

We will prove that for any \( U \) affine and \( s \in K(U) \) there is a syntomic covering \( U' \to U \) killing \( s \). Let \( U/X \) syntomic with \( U \) affine and choose a transversally regular immersion \( U \to \mathbb{A}_X^n \) above \( X \). Let us choose a transversally regular sequence \((x)\) above \( X \) defining this immersion. Then any lift \( \tilde{x} \) of \( x \) to \( Y := \mathbb{A}_X^n \) is transversally regular (reduce to the case where \( Y \) is noetherian by [EGA4-IV] 19.8.2 and then use [Mi] I, 2.6 (d)). Let us choose one such lift and denote \( \tilde{U} \) the corresponding syntomic \( X \)-scheme. By [Be3] 1.5.3 (i) with \( m = 0 \), the divided power envelope \( D(\tilde{U}, Y) \) is flat over \( \tilde{X} \). Let us emphasize that \( D(\tilde{U}, Y) \) coincides with \( D(U,Y) \) since all divided powers are intended to be compatible with \( p \) (see loc. cit. 1.3.1). Applying (9.32) forming derived global sections and truncating we get a distinguished triangle of \( \mathbb{Z}/p^k \)-modules:

\[
\xymatrix{ u_*M(U) \ar[r] & \Gamma(\mathcal{O}_U, \Omega_{D(U,Y)}/\Sigma_k(\epsilon_*(M_U))) \ar[r] & \tau_{\geq 1} \Gamma(\mathcal{O}_U, \Omega_{D(U,Y)}/\Sigma_k(\epsilon_*(M_U))) \ar[r]^+ & }
\]

Since the non zero objects of the middle term are flat over \( \mathbb{Z}/p^k \) and placed in positive degrees this induces

\[
\text{Tor}_1^{\mathbb{Z}/p^k}(u_*M(U), N''(U)) \cong \text{Tor}_2^{\mathbb{Z}/p^k}(\tau_{\geq 1} \Gamma(\mathcal{O}_U, \Omega_{D(U,Y)}/\Sigma_k(\epsilon_*(M_U))), N''(U))
\]

This abelian group thus admits a finite (here we use that \( Y \) has a finite \( p \)-basis) filtration which graduations are subquotients of

\[
\text{Tor}_2^{\mathbb{Z}/p^k}(H^q(\mathcal{O}_U, \Omega_{D(U,Y)}/\Sigma_k(\epsilon_*(M_U))), N''(U))
\]

with \( q \geq 1 \). Applying repeatedly the argument of (iii) produces a syntomic covering \( Y'/Y \) such that the image of \( s \) vanishes in

\[
\text{Tor}_2^{\mathbb{Z}/p^k}(\tau_{\geq 1} \Gamma(\mathcal{O}_U, \Omega_{D(U,Y')}\Sigma_k(\epsilon_*(M_U'))), N''(U'))
\]

where \( U' := U \times_Y Y' \). The result follows by functoriality of the above distinguished triangle with respect to \( (U \to Y) \).

\[ \square \]

**Definition 5.58.** Consider the global (resp. local) situation and let \( D \in \mathcal{D}(X) \).

(i) We define a quasi-coherent module on \((X_{\text{syn}}, \mathcal{O})\) as follows

\[ \text{Lie}_{\text{syn}}(D) := \text{e}^* \text{Lie}(D) \]

(ii) We define a quasi-coherent crystal on \((X/\Sigma_{\infty})_{\text{crys,syn}}, \mathcal{O})\) as follows

\[ \text{Fil}^0 D_{\text{syn}} = D_{\text{syn}} := \text{e}^* D \]

(respectively \( \text{Fil}^0 L_{\text{syn}} \Omega_T^* (D) = L_{\text{syn}} \Omega_T^* (D) := \text{e}^* L \Omega_T^* (D) \))

(iii) We define a sub-module of the crystal defined in 5.58 as follows

\[ \text{Fil}^1 D_{\text{syn}} := \text{Ker}(\text{can}_{D}^{\text{syn}} : D_{\text{syn}} \to i_* \text{Lie}^{\text{syn}}(D)) \]

(respectively \( \text{Fil}^1 L_{\text{syn}} \Omega_T^* (D) := \text{Ker}(\text{can}_{L}^{\text{syn}} : L_{\text{syn}} \Omega_T^* (D) \to i_* \text{Lie}^{\text{syn}}(D)) \))

where \( \text{can}_{D}^{\text{syn}} \) (respectively \( \text{can}_{L}^{\text{syn}} \)) is the arrow deduced from the corresponding one in 5.37 (iii) using the base change morphism of 5.57 (i).
Next we project to $X_{\text{syn}}^N$ and modify the resulting $\mathcal{O}^{\text{crys}}$-modules (resp. complexes) exactly as in 5.41. Here we use the functor $\langle 1 \rangle^*$ for $(E, A) = (X_{\text{syn}}^N, \mathcal{O}^{\text{crys}})$. Concretely the functor $\langle 1 \rangle^*$ is thus simply $M \mapsto \mathcal{O}^{\text{crys}} \otimes_{\mathcal{O}^{\text{crys}}_{+1}} M$. A pleasant feature of the present setting is that the ring $\mathcal{O}^{\text{crys}}$ need not to be modified, thanks to Proposition 5.60.

**Definition 5.59.** Consider the global (resp. local) situation and let $D \in \mathcal{D}C(X)$.

(i) We define a filtered module (resp. complex) as follows

$$\text{Fil}_i^{\text{crys}} D := u_i^{-1} \text{Fil}^i D_{\text{syn}} \quad \text{in } \text{Mod}(X_{\text{syn}}^N, \mathcal{O}^{\text{crys}}), \; i = 0, 1$$

(resp. $\text{Fil}^i \Omega_{\cdot, T}^\bullet(D) := u_i^{-1} \text{Fil}^i \Omega_{\cdot, T}^\bullet(D) \quad \text{in } \text{Kom}(X_{\text{syn}}^N, \mathcal{O}^{\text{crys}}), \; i = 0, 1$)

(ii) We define a couple of modules (resp. complexes) as follows

$$\text{Fil}^i \Omega_{\cdot, T}^\bullet(D) := \langle 1 \rangle^* \text{Fil}^i \Omega_{\cdot, T}^\bullet(D) \quad \text{in } \text{Kom}(X_{\text{syn}}^N, \mathcal{O}^{\text{crys}}), \; i = 0, 1$$

Note that in virtue of Proposition 5.60(iv) $\text{Fil}^i \Omega_{\cdot, T}^\bullet$ and $\text{Fil}^i \Omega_{\cdot, T}^\bullet$ coincide for $i = 0$.

**Proposition 5.60.** Consider the global (resp. local) situation and consider $D \in \mathcal{D}C(X)$. Let $\text{Fun}^*$ (resp. $\text{Fun}^*$) denote one of the two couples of functors defined in Proposition 5.59(i) (resp. (ii)).

(i) There is a natural morphism of functorial distinguished triangles in $D(X_{\text{syn}}^N, \mathcal{O}^{\text{crys}})$:

$$\begin{array}{ccc}
\text{Fun}^1(D) & \xrightarrow{1} & \text{Fun}^0(D) \\
\downarrow & & \downarrow \\
\text{Lie}_i(-h)(D) & \xrightarrow{\text{can}} & \text{Lie}_i(D) \\
\downarrow & & \downarrow \\
\text{Fun}^1(D) & \xrightarrow{1} & \text{Fun}^0(D) \\
\end{array}$$

where $\text{Lie}_i(D)$ is the constant projective system $\text{Lie}(D)$ and $\text{Lie}_i(D) := \tau_{i-1} L(\langle 1 \rangle^* \text{Lie}(D))$.

(ii) The objects of the complexes $\text{Fun}^*$ are $L$-normalized $\mathcal{O}^{\text{crys}}$-modules. The objects of the complexes $\text{Fil}^i \Omega_{\cdot, T}^\bullet(D)$ are moreover $\epsilon_\ast$-acyclic.

(iii) In the case $\text{Fun}^* = \text{Fil}^i \Omega_{\cdot, T}^\bullet$ the image of the diagram in (i) by $\text{Re}_\ast$ is naturally isomorphic to the image by $\epsilon_\ast$ of the diagram in 5.41(i) in the case $\text{Fun}^* = \text{Fil}^i \Omega_{\cdot, T}^\bullet$. More precisely there is a canonical commutative cube as follows in $\text{Kom}(X_{\text{syn}}^N, \mathcal{O}^{\text{crys}})$:

\[
\begin{array}{ccc}
\text{Fil}^i \Omega_{\cdot, T}^\bullet(D) & \xrightarrow{\sim} & \epsilon_\ast \text{Fil}^i \Omega_{\cdot, T}^\bullet(D) \\
\downarrow & & \downarrow \\
\text{Fil}^0 \Omega_{\cdot, T}^\bullet(D) & \xrightarrow{\sim} & \epsilon_\ast \text{Fil}^0 \Omega_{\cdot, T}^\bullet(D) \\
\downarrow & & \downarrow \\
\text{Fil}^1 \Omega_{\cdot, T}^\bullet(D) & \xrightarrow{\sim} & \epsilon_\ast \text{Fil}^1 \Omega_{\cdot, T}^\bullet(D) \\
\downarrow & & \downarrow \\
\text{Fil}^0 \Omega_{\cdot, T}^\bullet(D) & \xrightarrow{\sim} & \epsilon_\ast \text{Fil}^0 \Omega_{\cdot, T}^\bullet(D) \\
\end{array}
\]
Proof. \(\text{(i)}\) Using \ref{exist-score} (i), \ref{exist-score} (iii), \ref{exist-score} (i) and the isomorphism \(l^{-1}i_* \cong i_* l^{-1}\) we find the following exact sequence on \(((X/\Sigma), \mathcal{O})\)

\[
\begin{array}{cccccc}
0 & l^{-1}Fil^1D^{syn} & l^{-1}Fil^0D^{syn} & i_* l^{-1}\text{Lie}^{syn}(D) & 0 \\
0 & l^{-1}Fil^1L^{syn}\Omega^q_{\mathcal{T}}(D) & l^{-1}Fil^0L^{syn}\Omega^q_{\mathcal{T}}(D) & i_* l^{-1}\text{Lie}^{syn}(D) & 0 \\
\end{array}
\]

These sequences stay exact after applying \(u_*\) thanks to the second statement of \ref{exist-score} (i) Whence the bottom distinguished triangle. The top one follows by \ref{4.3} (iv).

\(\text{(ii)}\) The first statement is part of \ref{4.3} (iv) as well. Let us prove the acyclicity statements. First we note that \(Fil^0L^{\text{crys}}\Omega^q_{\mathcal{T}}(D)\) is \(\epsilon_*/\text{acyclic}\) by the conjunction of \ref{exist-score} (ii) and (iii). The same is true for \(\text{Lie}(D)\) and \(L^1\langle 1\rangle \text{Lie}(D)\) by \ref{4.12} (i). Given the isomorphisms

\[
\tilde{F}il^1L^{\text{crys}}\Omega^q_{\mathcal{T}} \cong Fil^0L^{\text{crys}}\Omega^q_{\mathcal{T}}
\]

for \(q \geq 1\) and the exact sequence

\[
0 \longrightarrow L^1\langle 1\rangle \text{Lie}(D) \longrightarrow \tilde{F}il^1L^{\text{crys}}\Omega^0_{\mathcal{T}}(D) \longrightarrow Fil^0L^{\text{crys}}\Omega^0_{\mathcal{T}}(D) \xrightarrow{\text{can}} \text{Lie}(D) \longrightarrow 0
\]

it is thus sufficient to check that the image of \(\text{can}\) by \(\epsilon_*\) remains epimorphic. This is indeed the case by \ref{exist-score} (i) thanks to the following compatible isomorphisms

\[
\begin{array}{ccc}
\epsilon_*Fil^0L^{\text{crys}}\Omega^0_{\mathcal{T}}(D) & \longrightarrow & \epsilon_*\text{Lie}^{syn}(D) \\
\downarrow \quad \downarrow & & \downarrow \\
l^{-1}Fil^0\Omega^0_{\mathcal{T}}(D) & \longrightarrow & l^{-1}\text{Lie}_{\mathcal{T}}(D)
\end{array}
\]

resulting from \ref{9.6} (ii) together with the full faithfulness of \(\epsilon^*\) on quasi-coherent modules and quasi-coherent crystals.

\(\text{(iii)}\) The arguments giving \eqref{eps-syn} provide the bottom commutative square and the rest follows formally.

\[\square\]

\textbf{5.6.1.} From here the construction is similar to the case of the étale topology.

\textbf{Definition 5.61.} Let \((D, f, v) \in DC(X)\). We use the following notations.

- In the global situation we let \(Fr : Fil^0L^{\text{crys}}D \to (F)_*Fil^0L^{\text{crys}}D\) in \(\text{Mod}(X^N_{\text{syn}}, \mathcal{O}^{\text{crys}})\) denote

\[
\begin{array}{cccccc}
u_* l^* \epsilon^* D & \xrightarrow{F(-/X^N)} & (F)_* F^* u_* l^* \epsilon^* D & \longrightarrow & (F)_* u_* l^* \epsilon^* F^* D & \xrightarrow{f} & (F)_* u_* l^* \epsilon^* D \\
\end{array}
\]

- In the local situation we let \(Fr : Fil^0L^{\text{crys}}\Omega^q_{\mathcal{T}}(D) \to (F)_*Fil^0L^{\text{crys}}\Omega^q_{\mathcal{T}}(D)\) in \(\text{Kom}(X^N_{\text{syn}}, \mathcal{O}^{\text{crys}})\) denote

\[
\begin{array}{cccccc}
u_* l^* \epsilon^* L(\Omega^q_{\mathcal{T}}(D)) & \xrightarrow{F(-/X^N)} & (F)_* F^* u_* l^* \epsilon^* L(\Omega^q_{\mathcal{T}}(D)) & \longrightarrow & (F)_* u_* l^* \epsilon^* F^* L(\Omega^q_{\mathcal{T}}(D)) & \xrightarrow{f} & (F)_* u_* l^* \epsilon^* L(\Omega^q_{\mathcal{T}}(F^*(D))) \\
\end{array}
\]

113
Proposition 5.62. Consider the global (resp. local situation) and let $D \in \mathcal{D}C(X)$. There exist unique morphisms $\phi$ in $\text{Kom}(X^{\mathbb{N}}_{\text{sym}}, \mathcal{O}^{\text{crys}})$ rendering the following squares commutative:

$$
\begin{array}{ccc}
\tilde{\text{Fil}}^{\text{crys},1}D & \xrightarrow{\phi} & (F)_*\tilde{\text{Fil}}^{\text{crys},0}D \\
\downarrow 1 & & \downarrow p \\
\tilde{\text{Fil}}^{\text{crys},0}D & \xrightarrow{Fr} & (F)_*\tilde{\text{Fil}}^{\text{crys},0}D
\end{array}
\begin{array}{ccc}
\text{Fil}^1L^{\text{crys}}\Omega^*_{,T}(D) & \xrightarrow{\phi} & (F)_*\text{Fil}^0L^{\text{crys}}\Omega^*_{,T}(D) \\
\downarrow 1 & & \downarrow p \\
\text{Fil}^0L^{\text{crys}}\Omega^*_{,T}(D) & \xrightarrow{Fr} & (F)_*\text{Fil}^0L^{\text{crys}}\Omega^*_{,T}(D)
\end{array}
$$

These morphisms are functorial with respect to $D$ and $X$ or $Y$ accordingly.

Proof. Using [4.3 (ii), (iii)] as in the proof of 5.49 we are reduced to prove that the first component

\begin{align*}
(149) & \qquad Fr_1 : \text{Fil}^0_{1,\text{crys}}D \to (F)_*\text{Fil}^0_{1,\text{crys}}D \\
(150) & \qquad Fr_1 : \text{Fil}^0L^{\text{crys}}\Omega^*_{1,\text{T}}(D) \to (F)_*\text{Fil}^0L^{\text{crys}}\Omega^*_{1,\text{T}}(D)
\end{align*}

of the morphisms $Fr$ vanish respectively on $\text{Fil}^0_{1,\text{crys}}D$ and $\text{Fil}^1L^{\text{crys}}\Omega^*_{1,\text{T}}(D)$. For $q \geq 1$, the arrow (150) is zero since the base change morphism (224) occurring in its definition (see 5.61) is zero. Next let $M$ denote either the module $D$ or the complex $L(\Omega^q_{\text{T}}(D))$ on $((X/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O})$ and $f : F^*M \to M$ the morphism given by the Frobenius of the Dieudonné crystal (using (221) in the second case). The reader may check that $Fr_1 : u_*\iota_1^{-1}e^*M \to (F)_*u_*\iota_1^{-1}e^*M$ identifies with the composed morphism

\begin{align*}
(151) & \quad u_*e^*\iota_1^{-1}M \xrightarrow{u_*e^*Fr} u_*e^*(F)_*M \xrightarrow{(F)_*u_*e^*\iota_1^{-1}M}
\end{align*}

where $\iota_1 : ((X/\Sigma)_{\text{crys,et}}, \mathcal{O}) \to ((X/\Sigma^\infty)_{\text{crys,et}}, \mathcal{O})$ and $Fr$ is the morphism introduced during the proof of 5.49. Consider now the commutative diagram

\begin{align*}
(152) & \quad e^*\iota_1^{-1}\text{Fil}^1D \xrightarrow{e^*\iota_1^{-1}D} \text{Fil}^1D^{\text{sym}} \xrightarrow{\iota_1^{-1}\text{Lie}(D)} 0 \\
& \quad \downarrow \downarrow \downarrow \\
& \quad 0 \xrightarrow{\iota_1^{-1}\text{Fil}^1D^{\text{sym}}} \text{Fil}^1D^{\text{sym}} \xrightarrow{\iota_1^{-1}e^*\iota_1^{-1}\text{Lie}(D)} 0
\end{align*}

where the first line is exact and is obtained from [5.37 (i)] by applying the right exact functor $e^*\iota_1^{-1}$ while the second one is exact as well and is deduced from [5.58 (iii)] by the exact functor $\iota_1^{-1}$. It has been shown in 5.49 that $Fr$ vanishes on the kernel of $\iota_1^{-1}\text{can}_D$. It follows in particular that $e^*Fr$ vanishes on the image of the left vertical arrow in (152). Applying $u_*$ we find that (150) vanishes on $\text{Fil}^0_{1,\text{crys}}D := u_*\iota_1^{-1}\text{Fil}^1D^{\text{sym}}$ as desired. The case of (150) is similar.

\[\square\]

Definition 5.63. (i) In the global situation we define a functor $\mathcal{S}^1_{\text{sym...X}} : \mathcal{D}C(X) \to \text{Mod}^{1,\phi}(X^{\mathbb{N}}_{\text{sym}}, \mathcal{O}^{\text{crys}})$ by setting

$$
\mathcal{S}^1_{\text{sym...X}}(D) := (\tilde{\text{Fil}}^1_{1,\text{crys}}(D), \tilde{\text{Fil}}^0_{0,\text{crys}}(D), 1, \phi)
$$

By functoriality with respect to $D$ and $X$ this defines a functor

$$
\mathcal{S}^1_{\text{sym...}} : \mathcal{D}C(-) \to \text{Mod}^{1,\phi}((-)^{\mathbb{N}}_{\text{sym}}, \mathcal{O}^{\text{crys}})
$$
Proposition 5.64. Let \( \text{Emb} \) These isomorphisms are functorial with respect to proof. Recall that there is a natural quasi-isomorphism in \( \text{Fil} \) to get a couple of morphisms which are compatible with the morphism 1. These morphisms are clearly compatible with indeed the case by 5.57 (iii) (note also that local freeness implies \( \varepsilon = 0 \), ie. that the image of (153) under \( \text{Kom} \) in \( u \), \( \phi \) and thus above \( \text{Emb}^{\text{glob,lfpb},op} \) and thus above \( \text{Diag}(\text{Fil}^{\text{et,lfpb}}) \) as well well (5.58 (ii)).

(ii) In the local situation we define a functor \( SL\Omega^{1,\phi}_{\text{syn},T} : DC(X) \rightarrow \text{Kom}^{1,\phi}(X^N_{\text{syn}}, \mathcal{O}^{\text{crys}}) \) by setting

\[
SL\Omega^{1,\phi}_{\text{syn},T}(D) := (\text{Fil}^1L^{\text{crys}}\Omega^1_{\text{crys},T}(D), \text{Fil}^0L^{\text{crys}}\Omega^0_{\text{crys},T}(D), 1, \phi)
\]

By functoriality with respect to \( D \) and \( (X,Y,\iota,\tilde{F}) \) this defines a functor \( SL\Omega^{1,\phi}_{\text{syn},-} : DC(-) \rightarrow \text{Kom}^{1,\phi}((-)^N_{\text{syn}}, \mathcal{O}^{\text{crys}}) \) above \( \text{Emb}_{\Phi}^{\text{glob,lfpb},op} \) and thus above \( \text{Diag}(\text{Fil}^{\text{et,lfpb}}) \) as well (5.9 (ii)).

Proposition 5.64. Let \( X \) denote a diagram of \( \text{Sch}^{slf,\Sigma_1} \) and assume given \( Y \) in \( \text{Emb}_{\Phi}^{\text{glob}}(X) \). For \( D \in DC(X) \), there are canonical isomorphisms

\[
\begin{align*}
S^1_{\text{syn},X}(D) & \simeq SL\Omega^{1,\phi}_{\text{syn},T}(D) \quad \text{in } D(\text{Mod}^{1,\phi}(X^N_{\text{syn}}, \mathcal{O}^{\text{crys}})) \\
i^{-1}S\Omega^{1,\phi}_{\text{syn},T}(D) & \simeq \text{Re}_{*}SL\Omega^{1,\phi}_{\text{syn},T}(D) \quad \text{in } D(\text{Mod}^{1,\phi}(X^N_{\text{et}}, \mathcal{O}^{\text{crys}})).
\end{align*}
\]

These isomorphisms are functorial with respect to \( D \) and \( Y \) in the obvious sense.

Proof. Recall that there is a natural quasi-isomorphism

\[(153) \quad D \rightarrow L\Omega^*_T(D)\]

in \( \text{Kom}((X/\Sigma_\infty)_{\text{crys,et}}, \mathcal{O}) \) (9.32 (i)). Applying \( \varepsilon^* \) and taking 5.58 into account we get compatible morphisms \( \text{Fil}^iD^\text{syn} \rightarrow \text{Fil}^iL^\text{syn}\Omega^i_T(D), i = 0, 1 \). Apply then \( l^*, u_* \) and \( (1)^* \) to get a couple of morphisms

\[(154) \quad \widetilde{\text{Fil}}^i,\text{crys}D \rightarrow \widetilde{\text{Fil}}^iL^{\text{crys}}\Omega^i_{\text{crys},T}, i = 0, 1 \]

which are compatible with the morphism 1. These morphisms are clearly compatible with \( Fr \) and thus with \( \phi \) also, thanks to 4.3 (ii). Up to now we have thus obtained a canonical morphism

\[
S^1_{\text{syn},X}(D) \rightarrow SL\Omega^{1,\phi}_{\text{syn},T}(D)
\]

in \( \text{Kom}^{1,\phi}(X^N_{\text{syn}}, \mathcal{O}^{\text{crys}}) \). It thus remains to check that this is a quasi-isomorphism. By 5.2 (iii) and 5.60 (i) we see that it suffices to check that (154) is a quasi-isomorphism for \( i = 0 \), ie. that the image of (153) under \( u_*l^{-1}\varepsilon^* \) remains a quasi-isomorphism. This is indeed the case by 5.57 (iii) (note also that local freeness implies \( \varepsilon^* \) acyclicity). Similarly the second isomorphism easily reduces to 9.0 (ii) using 5.2 (iii) 5.60 (iii) and 5.57 (i) (ii). 

Let us turn to the global situation. For a semi-simplicial \( T_{[\cdot]} \) as in 5.2.1 we have the following partial counterpart for (115):

\[
\begin{array}{ccc}
(X/\Sigma)_{\text{crys,et}}, \mathcal{O} & \xrightarrow{f_{U_{[\cdot]}}} & ((U/\Sigma)_{\text{crys,et}}, \mathcal{O})/T_{[\cdot]} \\
\downarrow u & & \downarrow u \\
X_{\text{syn}}, \mathcal{O}^{\text{crys}} & \xrightarrow{f_{U_{[\cdot]}}} & (U_{\Sigma}, \mathcal{O}^{\text{crys}})
\end{array}
\]
This diagram and (i) are compatible with each other in the obvious way via the morphisms $\epsilon : (\mathcal{X}/\Sigma,\mathcal{O}) \to (\mathcal{X}/\Sigma,\mathcal{O})$, $\epsilon : (\mathcal{X}_{\text{syn}},\mathcal{O}_{\text{syn}}) \to (\mathcal{X}_{\text{et}},\mathcal{O}_{\text{et}})$ and their localizations.

**Proposition 5.65.** Consider $X$ in $\mathcal{B}_0$ and $Y_{[\cdot]}$ in $H_{\mathcal{F}}^B(X)$ with divided power envelope $T_{[\cdot]}$.

(i) The adjunction morphisms induce the following isomorphisms:

$$
S_{\text{syn},X}^1(D) \simeq Rf_{U_{[\cdot]}}^* S_{\text{syn},U_{[\cdot]}}^1(D) \quad \text{in } D(\text{Mod}^{1,\phi}(\mathcal{X}_{\text{syn}},\mathcal{O}_{\text{crys}}))
$$

$$
S_{\text{et},X}^1(D) \simeq Rf_{U_{[\cdot]}}^* S_{\text{et},U_{[\cdot]}}^1(D) \quad \text{in } D(\text{Mod}^{1,\phi}(\mathcal{X}_{\text{et}},\mathcal{O}_{\text{crys}}))
$$

(ii) The isomorphisms of (i) and 5.64 induce

$$
Re_\epsilon S_{\text{syn},X}^1(D) \simeq S_{\text{et},X}^1(D) \quad \text{in } D(\text{Mod}^{1,\phi}(\mathcal{X}_{\text{et}},\mathcal{O}_{\text{crys}}))
$$

This isomorphism is functorial with respect to $D$ and independant of the choice of $Y_{[\cdot]}$.

Proof. [i] The first (resp. second) morphism is invertible by cohomological descent in $X_{\text{syn}}$ (resp. $X_{\text{et}}$) and devissage using 5.52 [ii] and 5.60 [i] (resp. and 5.43 [i]).

[ii] The claimed isomorphism follows from [i] by 5.64 applied to $U_{[\cdot]}$ viewed as a diagram. The independance with respect to the choice of $Y_{[\cdot]}$ follows from the connectedness of the category $HR_{\mathcal{F}}^B(X)$.

6. DÉVISSAGE OF TWISTED SYNTOMIC COMPLEXES

In this section we prove two dévissage properties of the twisted syntomic complexes defined in 5.50, 5.52 (i).

6.1. Complete Mayer Vietoris.

Recall the natural morphism of diagrams $m^2 : J^2 \to C^2$ from (58). We have the following diagram of ringed topoi

$$
\begin{array}{ccc}
((J^2/\Sigma,\mathcal{O})) & \xrightarrow{m^2} & ((C^2/\Sigma,\mathcal{O})) \\
\downarrow u & & \downarrow u \\
(J_{\text{et}}^{\Sigma},\mathcal{O}_{\text{crys}}) & \xrightarrow{m} & (C_{\text{et}}^{\Sigma},\mathcal{O}_{\text{crys}})
\end{array}
$$

Here (as often) we have identified the small étale site of a log scheme with the small étale site of the underlying scheme. The following proposition expresses that Dieudonné crystals and their syntomic complexes glue from $J^2$ to $C^2$.

**Proposition 6.1.** Let us denote $\Delta_J$ the type of the diagram $J^2$.

(i) The pullback functor

$$
m^2_* : \text{Mod}((C^2/\Sigma_{\infty},\mathcal{O})) \to \text{Mod}((J^2/\Sigma_{\infty},\mathcal{O}))
$$

induces equivalences of categories:

$$
\begin{aligned}
\text{Crys}_{\text{ff}}((C^2/\Sigma_{\infty},\mathcal{O})) & \xrightarrow{m^2_*} \Gamma_{\text{cart}}(\Delta_J^{\text{op}} \times_{J^2}(\mathcal{S}_{\text{ch}}/\Sigma_{\infty})) \circ \text{Crys}_{\text{ff}}((/\Sigma_{\infty},\mathcal{O})/\Delta_J^{\text{op}}) \\
\text{DC}(C^2) & \xrightarrow{m^2_*} \Gamma_{\text{cart}}(\Delta_J^{\text{op}} \times_{J^2}(\mathcal{S}_{\text{ch}}/\Sigma_{\infty})) \circ \text{DC}(\Delta_J^{\text{op}})
\end{aligned}
$$

116
where $\Gamma_{\text{cart}}$ denotes the full subcategory of $\mathcal{D}$ formed by cartesian sections.

(ii) For $M \in \text{Mod}((C^\times/S)^{\text{crys},\text{et}}, \mathcal{O})$ there is a canonical morphism

$$R_{\text{et}} M \rightarrow Rm_* R_{\text{et}} M^{\sharp\ast} \quad \text{in } D(C^N_{\text{et}}, \Omega^{\text{crys}})$$

Assume now that $M$ is a crystal. Let us choose a morphism $m^\sharp_{\Delta} : \Delta_1 \rightarrow H R^F_{\text{et}}$ above $m^\sharp = \dot{\Delta}_1, \Delta_2, (\text{ii}), (\text{iii}), (\text{iv})$ and denote $m^\sharp = m^\sharp_{\Delta_1} : T_{\text{et}} \rightarrow T_{\text{et}}$ its logarithmic divided power envelope. The previous morphism of $D(C^N_{\text{et}}, \Omega^{\text{crys}})$ identifies to the natural morphism

$$Rf_{T^\ast(\text{et})} : \Omega^\bullet_{T^\ast(\text{et})} (M) \rightarrow Rm_* Rf_{T^\ast(\text{et})} \Omega^\bullet_{T^\ast(\text{et})}(m^\sharp M)$$

It is an isomorphism if $M$ is a locally free crystal of finite type.

(iii) For $D \in D(C^\times)$ and $h \in \Gamma(C, M_{\text{c}} / \mathcal{G}_m)$ the canonical base change morphism induces an isomorphism

$$S^1_{\text{et}, v, \Delta}(D) \cong Rm_* S^1_{\text{et}, v, \Delta}(D) \quad \text{in } D(\text{Mod}^1, \Omega^{\text{crys}, F=1}(\mathcal{O})$$

as well as a similar isomorphism without the superscripts $1, \phi$ in $D(C^N_{\text{et}}, \Omega^{\text{crys}, F=1})$.

Proof. [i] Thanks to $m^\ast$ being compatible with $F^\ast$, it suffices to prove that the first functor is fully faithful and essentially surjective. We will do this by Zariski descent along the coskeleton of an appropriate covering. Let $(C^\times_{\Delta} \rightarrow C^\times)$ be an open covering such that each $C^\times_{\Delta}$ is affine, has $p$-bases of the form $(0, t)$ and such that each point $v$ of $Z$ is contained in exactly one of the $C^\times_{\Delta}$s. Let $v \mapsto \lambda_v$ be the injection of $|Z|$ into the set of indices defined by the condition $v \in C_{\lambda_v}$. Consider $U_{J^\Delta} := C_{\lambda_v} \times C^\times U$, $C^\times_{v, \Delta} := C^\times_{\lambda_v} \times C^\times U$, $J^\Delta_{v, \Delta} := C^\times_{v, \Delta} \times C^\times U$ and form $C_{[v]} := \sqcup C_{v, \Delta} U_{[v]} := \sqcup U_{[v]}$, $C^\times_{v, [v]} := \sqcup C^\times_{v, \Delta} U_{[v]} := \sqcup U_{[v]}$ as well as the diagram of type $\Delta_f$ represented by $(C^\times_{v, [v]} \rightarrow U_{[v]} \rightarrow U_{[v]}))$. Let us furthermore denote $C^\times_{v, [v]} U_{[v]}$, $C^\times_{v, [v]} U_{[v]}$ and $J^\Delta_{[v]}$ the respective $\nu + 1$-th fibered power over $C^\times, U$, $C^\times_{v, [v]} U_{[v]}$ and $J^\Delta_{[v]}$ of $C^\times_{v, [v]} U_{[v]}$, $C^\times_{v, [v]} U_{[v]}$ and $J^\Delta_{[v]}$. By Zariski descent along $C^\times_{v, [v]} \rightarrow C^\times$ and $J^\Delta_{[v]} \rightarrow J^\Delta$ it is enough to prove that the induced functor

$$\text{Crys}^\ast((C^\times_{[v]}/\Sigma_\infty)^{\text{crys},\text{et}}, \mathcal{O}) \xrightarrow{m^\ast_{[v]}} \Gamma_{\text{cart}}(\Delta_f \times_{\text{et}} (\text{Sch}/\Sigma_\infty)^{\text{crys},\text{et}}, \mathcal{O})$$

is an equivalence, ie that for each $\nu \geq 0$ and multi-index $\Delta = (\lambda_0, \ldots, \lambda_\nu)$ the induced functor

$$\text{Crys}^\ast((C^\times_{[v]/\Delta}/\Sigma_\infty)^{\text{crys},\text{et}}, \mathcal{O}) \xrightarrow{m^\ast_{[v], \Delta}} \Gamma_{\text{cart}}(\Delta_f \times_{\text{et}} (\text{Sch}/\Sigma_\infty)^{\text{crys},\text{et}}, \mathcal{O})$$

is an equivalence where $C^\times_{[v]/\Delta} := C^\times_{\lambda_v} \times C^\times \times C^\times_{\lambda_v}$ and $J^\Delta_{[v]/\Delta} := J^\Delta_{[v]} \times_{\text{et}} \times_{\text{et}} \times_{\text{et}} J^\Delta_{\lambda_\nu}$.

Since $U_{\lambda_v} = C_{\lambda_v}$ for $\lambda_v \neq \lambda_v$, the equivalence is trivial if $\lambda_v \neq \lambda_v$ for some $i \neq j$ (in that case $C^\times_{[v]/\Delta} := U_{[v]/\Delta}$ and $C^\times_{[v]/\Delta} := U_{[v]/\Delta}$ for all $v$). On the other hand, if $\Delta = (\lambda_0, \ldots, \lambda_\nu)$ then $C^\times_{[v]/\Delta}$ and $J^\Delta_{[v]/\Delta}$ respectively coincide with $C^\times_{[v]}$ and $J^\Delta_{[v]}$. We are thus reduced to the case $\nu = 0$ and it suffices to treat the case $\lambda = \lambda_v$ for some $v$ in $Z$ since the other cases are trivial as well. In that case let us choose a $p$-basis $(0, t)$ for $C_{\lambda_v}$ and denote $R = \Gamma(C_{\lambda_v}, \mathcal{O})$. The ring $R$ is thus an étale $\mathbb{F}_p[T]$ algebra via $T \mapsto t$ and that $t$ is a uniformizer at $v$. Denote furthermore $R_v$ the $t$-adic completion of $R$. Hence with these notations we have $C_{\lambda_v} = (\text{Spec} R, 1 \mapsto t)$, $U_{\lambda_v} = \text{Spec} R[t^{-1}])$, $C^\times_{\lambda_v} = (\text{Spec} R_v, 1 \mapsto t)$ $U_{\lambda_v} = \text{Spec} R_v[t^{-1}]$). Let $\tilde{R}_k$, (resp. $\tilde{R}_{v, k}$) denote the essentially unique étale $\mathbb{Z}/p^k[t]-$
Consider the functor sending an object

\[(M_1, \nabla_1), (M_2, \nabla_2), \alpha : \tilde{R}_v, k[t^{-1}] \otimes \tilde{R}_v, k[t^{-1}] (M_1, \nabla_1) \simeq \tilde{R}_v, k[t^{-1}] \otimes \tilde{R}_v, k (M_2, \nabla_2)\]

in the target of category of \(m^{\ast \ast}_{0, \lambda_v}\) to the \(\tilde{R}_k\) module

\[M := \text{Ker}(M_1 \oplus M_2 \xrightarrow{\alpha \cdot \underline{-1}} M_2[t^{-1}])\]

endowed with the induced connection (which turns out to be well defined by exactness of

\[0 \longrightarrow \Omega_{\tilde{R}_k} \longrightarrow \Omega_{\tilde{R}_k, k[t^{-1}] \oplus \Omega_{\tilde{R}_v, k, (1, -1)} \longrightarrow \Omega_{\tilde{R}_v, k[t^{-1}]}\].

This functor is clearly right adjoint to \(m^{\ast \ast}_{0, \lambda_v}\). Note that this functor preserves the property of being normalized when \(k\) varies thanks to the fact that the arrow \(M_1 \oplus M_2 \rightarrow M_2[t^{-1}]\) is in fact onto and its target is flat over \(\mathbb{Z}/p^k\). Let us now check that it is in fact a quasi-inverse to \(m^{\ast \ast}_{0, \lambda_v}\).

- full faithfulness of \(m^{\ast \ast}_{0, \lambda_v}\). It suffices to check that for a locally \(\tilde{R}_k\)-module \(M\) the sequence

\[0 \longrightarrow M \longrightarrow M[t^{-1}] \oplus M_v \xrightarrow{(1, -1)} M_v[t^{-1}]\]

is exact. This is indeed the case thanks to (156).

- full faithfulness of the right adjoint. If \(((M_1, \nabla_1), (M_2, \nabla_2), \alpha)\) is in the target category of \(m^{\ast \ast}_{0, \lambda_v}\), the following natural morphisms are invertible:

\[\tilde{R}_k[t^{-1}] \otimes_{\tilde{R}_k} \text{Ker}(M_1 \oplus M_2 \xrightarrow{\alpha \cdot \underline{-1}} M_2[t^{-1}]) \xrightarrow{(1, \text{pr}_1)} M_1 \text{ in Mod}(\tilde{R}_k[t^{-1}])\]

\[\tilde{R}_v, k \otimes_{\tilde{R}_k} \text{Ker}(M_1 \oplus M_2 \xrightarrow{\alpha \cdot \underline{-1}} M_2[t^{-1}]) \xrightarrow{(1, \text{pr}_2)} M_2 \text{ in Mod}(\tilde{R}_v, k)\]

Indeed Nakayama’s lemma reduces us to the case \(k = 1\) and then we may use the classification of finitely generated modules over the Dedekind rings \(R\) and \(R_v\) to show that \((M_1, M_2, \alpha)\) is in fact necessarily of the form \((M[t^{-1}], M_v, \text{can})\) for some \(R\)-module \(M\). Localizing if necessary we conclude using (157).

The claimed morphism is simply induced by the natural morphism \(M \rightarrow Rm^{\ast \ast}_{0, \lambda_v} M\).

If \(M\) is a crystal the claimed interpretation of the image of this morphism by \(R_{u, v}\) in terms of de Rham complexes using hypercoverings follows easily from the proof of (5.3)(ii)

Assume now that \(M\) is a crystal of locally free modules of finite type. In order to prove the claimed isomorphism we may use the de Rham interpretation via global embeddings. In fact we may even choose \(\tilde{m} : \tilde{J}^\sharp \rightarrow \tilde{C}^\sharp\) in \(Emb^{\text{glob}}\) above \(m\) such that \(\tilde{J}^\sharp\) and \(\tilde{C}^\sharp\) are respectively liftings of \(C^\sharp\) and \(J^\sharp\). Let us briefly explain this. On the one hand since \(C\) is
smooth of dimension 1 the obstruction to the existence of a formally smooth $p$-adic lifting $\tilde{C}$ furnished by deformation theory vanishes (note that in general it not possible to lift the Frobenius however). Next we may define $\tilde{C}^2$ by choosing an arbitrary lifting of the log structure of $C^2$. The lifting $\tilde{J}^2$ and the arrow $\tilde{m}$ are then obtained by relative perfectness of the edges of $J^2$ over $C^2$. It remains to check that the natural morphism

$$\Omega_{C^1_v}^\bullet (M) \to R\tilde{m}_*\Omega_{J^2_v}^\bullet (m^*sM)$$

is invertible in $D(\tilde{C}^{\text{et,N}}, \mathcal{O}_{\text{crys}}')$. Since this question is étale local on $\tilde{C}$ we may replace $C^2$ with $C^1$ as in [i]. Since quasi-coherent modules are acyclic for the direct image of affine morphisms we find (using 3.4 (ii)) that the statement boils down to the exactness of the sequence

$$0 \to \Omega_{C_\lambda}^\bullet (M) \to \iota_{C_\lambda,*}\Omega_{C_\lambda}^\bullet (j^\lambda_M) \oplus j_{\lambda,*}\iota_{U_\lambda,*}\Omega_{U_\lambda}^\bullet (j^\lambda_M) \to j_{\lambda,*}\iota_{U_\lambda,*}\Omega_{U_\lambda}^\bullet (j^\lambda_M) \to 0$$

where the notations $j_\lambda$, $j_{v,\lambda}$, $\iota_{U_\lambda}$, $\iota_{C_\lambda}$ are used abusively to denote either the morphisms induced by $j$, $j_v$, $\iota_{U_\lambda}$, $\iota_{C_\lambda}$ in the diagram $J_\lambda$ or its lifting $\tilde{J}_\lambda$. Now the exactness in question is trivial if $|C_\lambda| \cap |Z| = \emptyset$ and results from (156) if $\lambda = \lambda_v$.

[iii] It suffices to prove the isomorphism in $D(\text{Mod}^l,\phi(C^{\text{et},\tilde{O}_{\text{crys}}}))$. Consider a morphism $Y_{J^1} \to Y_{C^1}$ above $m^2$ in $HR^{\text{et},\phi}$ and denote $T_{C^1,\lambda} \to T_{C^1,\lambda}$ its logarithmic divided power envelope. We have to show that the induced morphism

$$Rf_{C^1,\lambda,*}S\Omega_{et,T_{C^1,\lambda}}^{1,\phi}(-h_{|U_{C^1}})(D_{|U_{C^1}}) \to Rm_*Rf_{J^1,\lambda,*}S\Omega_{et,T_{J^1,\lambda}}^{1,\phi}(-h_{|U_{J^1}})(D_{|U_{J^1}})$$

is invertible. We have the following distinguished triangles in $D(\text{Mod}^l,\phi(T_{C^1,\lambda},\text{et},\tilde{O}_{\text{crys}}))$ (see 5.2 (iii) and 5.43 (i)):

$$(0, \text{Fil}^0 \Omega_{T_{C^1,\lambda}}^\bullet (-h_{|U_{C^1}})(D_{|U_{C^1}}), 0, 0, 0) \quad \text{and} \quad (\text{Fil}^1 \Omega_{T_{C^1,\lambda}}^\bullet (-h_{|U_{C^1}})(D_{|U_{C^1}}), 0, 0, 0)$$

These distinguished triangles are compatible with the analogous ones where $C^2$ is replaced with $J^2$. Thanks to [iii] we are thus reduced to prove that

$$Rf_{C^1,\lambda,*}\text{Lie}_{et,T_{C^1,\lambda}}(D_{|U_{C^1}})(-h_{|U_{C^1}}) \to Rm_*Rf_{J^1,\lambda,*}\text{Lie}_{et,T_{J^1,\lambda}}(D_{|U_{J^1}})(-h_{|U_{J^1}})$$

is invertible. We know from 5.35 (ii) that

$$\text{ch}_{\text{Lie}} : m^*\text{Lie}(D) \to \text{Lie}(m^*sD)$$

is invertible. As before it follows easily from (156) and 3.4 (ii) that

$$\text{Lie}(D)(-h) \to Rm_*\text{Lie}(m^*sD)(-h_{|J})$$
is invertible as desired.

Let us write down the complete Mayer Vietoris distinguished triangles encoded in the above proposition.

**Corollary 6.2.** (i) If \( M \) is a locally free crystal of finite type of \((C^\# / \Sigma_{\infty})_{\text{crys,et}}, \mathcal{O})\) there is a canonical distinguished triangle in \(D(C^{\# \mathbb{N}}, \mathcal{O}_{\text{crys}})\):

\[
Ru_* M \xrightarrow{\oplus_{v \in |Z|} \iota_{C_{v},*} Ru_* t_{C_{v}}^* M} Ru_* \iota_{C_{v}}^* M \xrightarrow{+1} \oplus_{v \in |Z|} Ru_* \iota_{U_{v}}^* M.
\]

(ii) For \( D \in DC(C^\#) \) and \( h \in \Gamma(C, M_C / \mathbb{G}_m) \) there is a canonical distinguished triangle in \(D(M \text{od}^{1,\phi} (C^\# \mathbb{N}, \tilde{\mathcal{O}}_{\text{crys}}))\):

\[
S^{1,\phi}_{\text{et},.,C^\#} (-h)(D) \xrightarrow{\oplus_{v \in |Z|} \iota_{C_{v},*} S^{1,\phi}_{\text{et},.,C_{v}} (-h|_{C_{v}})(\iota_{C_{v}}^* D)} \oplus Ru_* S^{1,\phi}_{\text{et},.,U} (j^* D) \xrightarrow{+1} \oplus_{v \in |Z|} Ru_* S^{1,\phi}_{\text{et},.,U_{v}} (\iota_{U_{v}}^* j^* D)
\]

and similarly without the superscripts \((1,\phi)\) in \(D(C^{\# \mathbb{N}}, \mathbb{Z}/p)\).

Proof. This follows from the previous proposition thanks to \[3.6\] and the natural distinguished triangle describing \(Rm_* (3.4 (ii))\).

\[\square\]

### 6.2. A localization triangle.

We show that for smooth divisors, the twisted syntomic complex of a logarithmic Dieudonné crystal with trivial residue can be recovered from a diagram of syntomic complexes without twists. This can be viewed as a refined version of the expected localization triangle for compactly supported log crystalline cohomology as defined in \[131\].

In order to make a precise statement we will need the functor

\[
R \Gamma^{Z_J} (J_{-}, -) : D^+ (J^+ \mathbb{N}_{\text{et}}, \mathbb{Z}/p) \rightarrow D^+ (J^+ \mathbb{N}_{\text{et}}, \mathbb{Z}/p)
\]

defined in \[3.12 (ii)\]. In order to treat simultaneously the case of \( J^\# \) and \( C^\# \) it will be convenient to modify slightly the notations introduced in \(58\) as follows:

\[
(158)
\]

\[
\begin{align*}
\xymatrix{& J \ar[d]^m \ar[r]^o_J & J^\# \\
Z_C \ar[u]^{m_J} \ar[r]^{z_C} & C \ar[r]^{\rho_C} & C^\#}
\end{align*}
\]

**Proposition 6.3.** Consider the morphism of extremal type \( z_J : Z_J \rightarrow J \) and let us furthermore denote \((-)|_{J^+} := \rho^* \) the pullback via the following morphism \[3.10 (iii)\]

\[
\rho : ((J^+ / \Sigma)_{\text{crys,et}}, \mathcal{O}) \rightarrow ((J / \Sigma)_{\text{crys,et}}, \mathcal{O}).
\]

(i) If \( M \) is a crystal on \((J / \Sigma)_{\text{crys,et}}\) there is a canonical morphism

\[
o_{J,*} Ru_*(o_J^* M(-Z_J)) \rightarrow R \Gamma^{Z_J} (J, Ru_*(M_{-J^+}))
\]

in \(D(J^+ \mathbb{N}_{\text{et}}, \mathcal{O}_{\text{crys}})\). It is an isomorphism if \( M \) is locally free.
(ii) If $D$ is in $DC(J)$ there is canonical isomorphism

$$o_J^*S_{\ell et, J^+}^{1, \phi}(-Z_J)(o_J^*D) \simeq R^\infty_{Z_J}(J, S_{\ell et, J^+}^{1, \phi}(D_{J^+}))$$

in $D(\text{Mod}^{1, \phi}(J_{et}^\phi, \tilde{O}^{crys}))$. A similar isomorphism holds in $D(J_{et}^\phi, \tilde{O}^{crys})$ without the superscripts $1, \phi$.

(iii) The statements (i), (ii) are functorial with respect to $D, C$ and $Z$. Moreover they hold verbatim if the letter $J$ is replaced with the letter $C$.

Proof. We only prove (iii) since (i) is easier and will be essentially proven along the way. Using 5.7 we choose an object $Y_{[\cdot], J^2} = (U_{[\cdot], J^2}, Y_{[\cdot], J^2}, t, F)$ in $HR_{F}^{\ell et, ex}(J^2)$. Let us denote $U_{[\cdot], J}$ (resp. $Y_{[\cdot], J}$) the diagram obtained from $U_{[\cdot], J^2}$ (resp. $Y_{[\cdot], J^2}$) by forgetting log structures. Let us furthermore denote $U_{[\cdot], Z_J}$ (resp. $Y_{[\cdot], Z_J}$) the diagram obtained from $U_{[\cdot], X_J}$ (resp. $Y_{[\cdot], X_J}$) as follows. For each $\nu$ and $\delta$ the vertex $U_{[\nu], \delta, Z_J}$ (resp. $Y_{[\nu], \delta, Z_J}$) is the support of the log structure of $U_{[\nu], \delta, J^2}$ (resp. $Y_{[\nu], \delta, J^2}$) as defined in 4.47 (1) (it is thus a reduced closed subscheme of $U_{[\nu], \delta, J}$ (resp. $Y_{[\nu], \delta, J}$). If $S = J$, $J^2$, or $Z_J$ we let furthermore $U_{[\cdot], S} \to T_{[\cdot], S}$ denote the logarithmic divided power envelope of $U_{[\cdot], S} \to Y_{[\cdot], S}$.

Summarizing we have thus obtained the following commutative diagram of diagrams of simplicial $p$-adic log schemes:

\[
\begin{array}{ccc}
Y_{[\cdot], Z_J} & \xrightarrow{z^Y} & Y_{[\cdot], J} \\
\downarrow & & \downarrow \\
T_{[\cdot], Z_J} & \xrightarrow{z^T} & T_{[\cdot], J} \\
\downarrow & & \downarrow \\
U_{[\cdot], Z_J} & \xrightarrow{z_U} & U_{[\cdot], J} \\
\end{array}
\]

We claim that each square of this diagram is cartesian. Let us explain this using that $Y_{[\cdot], J^2}$ is in $HR_{F}^{\ell et, ex}(J^2)$. On the right side this follows from the fact that each $U_{[\nu], \delta, J} \to Y_{[\nu], \delta, J}$ is an exact closed immersion. Note that this implies in particular that the morphisms $T^2_{[\cdot], J^2, \nu, \delta} \to T_{[\cdot], J^2, \nu, \delta}$ is an equivalence. Next let us choose a $p$-basis of the form $(s, t)$ for $Y_{[\nu], J^2, \delta}$. On the left side the exterior square is cartesian since the ideal of the closed immersions $z_{Y_{[\nu], \delta, k}}$ and $z_{U_{[\nu], \delta}}$ are both generated by the image of $t$. Since the image of $t$ does not divide zero in the structure sheaf of $U$ it follows from [BO] 3.5 that the top square is cartesian as well.

It is also true that $U_{[\cdot], J^2} \to J^2$ (resp. $U_{[\cdot], Z_J}/Z_J$) is the base change of $U_{[\cdot], J}/J$ by $o_J : J^2 \to J$ (resp. $z_J : Z_J \to J$). It is thus a hypercovering for $\ell et$ topology and the previous constructions in particular gives rise to a couple of arrows

\[
\begin{array}{c}
Y_{[\cdot], Z_J} \xrightarrow{z} Y_{[\cdot], J} \xleftarrow{o_J} Y_{[\cdot], J^2}
\end{array}
\]

in $\text{Diag}(HR_{F}^{\ell et})$ above the top line in (159). It will be useful to notice that the formation of (160) is functorial with respect to the object $Y_{[\cdot], J^2} \in HR_{F}^{\ell et, ex}(J^2)$ which has been chosen at the beginning of the proof.

Let us denote $h \in \Gamma(J, M_J / G_m)$ the diagram of effective log Cartier divisors corresponding to $-Z_J$. In virtue of 1.49(ii) and cartesianity properties explained before we
have canonical exact sequences

\[
\begin{array}{c}
0 \longrightarrow \mathcal{O}(\mathfrak{h}|_{U,[1],j}) \longrightarrow \mathcal{O}_{T,[1],j} \longrightarrow z_{T,*}(\mathcal{O}_{T,[1],z,j}) \longrightarrow 0 \quad \text{in} \ \text{Mod}(T,[1],z,t,\mathcal{O})
\end{array}
\]

and

\[
\begin{array}{c}
0 \longrightarrow \mathcal{O}(\mathfrak{h}|_{U,[1],j}) \longrightarrow \mathcal{O} \longrightarrow z_{U,*}\mathcal{O} \longrightarrow 0 \quad \text{in} \ \text{Mod}(U^{\text{N}}_{[1],t,\text{ct}},\mathcal{O})
\end{array}
\]

which are compatible with each other in the obvious way. Tensoring the first (resp. second) one over \(\mathcal{O}\) with \(D_{T,[1],j}\) (resp. with \(\text{Lie}(D_{U,[1],j})\) then pushing forward to \(T,[1],j\)) and restricting scalars to \(\mathcal{O}^{\text{crys}}\) gives an exact sequence which may be identified to the first (resp. second) line of the following commutative diagram of \(\text{Mod}(T,[1],J_{s,t},\mathcal{O}^{\text{crys}})\) by using that for \(M\) locally free \(M \otimes f^*N \simeq f_*(f^*M \otimes N)\) (resp. by using \(\text{(ii)}\) and flatness of the logarithmic differentials on \(T,[1],J_{s,t}\)).

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{O}_{T,[1],j} & \longrightarrow & D_{T,[1],j} \\
go_T((\mathfrak{o}^*_j D)_{T,[1],j}(-h|_{T,[1],j})) & \longrightarrow & z_{T,*}((\mathfrak{z}^*_j D)_{T,[1],j}) \\
\end{array}
\end{array}
\]

Next let \(q \geq 1\) and consider the functoriality morphisms induced by \(\mathfrak{z}^*_j\):

\[
\begin{array}{c}
\begin{array}{ccc}
o_T^q \Omega^q_{T,[1],j} (\mathfrak{o}^*_j D(-h)|_{U,[1],j}) & \longrightarrow & \mathcal{O}^{\text{crys}}_{T,[1],j} (D) \\
& \longrightarrow & z_{T,*}\mathcal{O}^{\text{crys}}_{T,[1],z,j} (\mathfrak{z}^*_j D)
\end{array}
\end{array}
\]

as well as the natural morphism

\[
\begin{array}{c}
o_T^* \Omega^q_{T,[1],j} (\mathfrak{o}^*_j D(-h)|_{U,[1],j}) \xrightarrow{\text{nat}} o_T^* \Omega^q_{T,[1],j} (\mathfrak{o}^*_j D)|_{U,[1],j}
\end{array}
\]

Using flatness of the realization of \(\mathfrak{o}^*_j D(-h)\) over \((T,[1],J_{s,t},\mathcal{O})\) and comparing the local description of logarithmic differentials on \(T,[1],j\) and \(T,[1],t\), we find that \(o_T^*\) is monomorphic. The arrow \(\text{nat}\) is monomorphic as well thanks to \(\text{(ii)}\) and flatness of the logarithmic differentials on \(T,[1],J_{s,t}\).

Claim: The image of \(\text{nat}\) coincides with \(o_T^* \text{(Ker} z_T^*)\).

Let us prove this. We can assume \(D = \mathcal{O}\), fix \(\nu, \delta\) and choose a \(p\)-basis \((s_1, \ldots, s_d)\) for \(Y_{[1],J_{s,t},\delta}\). Then \((s_1, \ldots, s_d, t, \emptyset)\) is a \(p\)-basis for \(Y_{[1],J_{s,t},\delta}\) and \((s_1, \ldots, s_d, \emptyset)\) is a \(p\)-basis for \(Y_{[1],Z_{s,t},\delta}\). We find in particular the following bases for the modules of differential forms of degree \(q\):

\[
\begin{array}{c}
\Omega^q_{T,[1],J_{s,t},\delta,k} (\mathfrak{h}|_{T,[1],j}) : \\
\Omega^q_{T,[1],J_{s,t},\delta,k} : \\
\Omega^q_{T,[1],J_{s,t},\delta,k}
\end{array}
\]

The claim follows immediately thanks to the following isomorphisms of rings:

\[
\begin{array}{c}
o_T^* \mathcal{O}_{T,[1],J_{s,t},\delta,k} \simeq o_T^* \mathcal{O}_{T,[1],J_{s,t},\delta,k} \\
\mathcal{O}_{T,[1],J_{s,t},\delta,k}/(t) \simeq z_{T,*}\mathcal{O}^{\text{crys}}_{T,[1],J_{s,t},\delta,k}
\end{array}
\]

Putting everything together we find compatible exact sequences

\[
\begin{array}{c}
o_T^* \text{Fil}^i \Omega^*_{T,[1],J_{s,t}} (\mathfrak{o}^*_j D)|_{U,[1],j} \longrightarrow \text{Fil}^i \Omega^*_{T,[1],J} (D|_{T,[1],j}) \longrightarrow z_{T,*}\text{Fil}^i \Omega^*_{T,[1],J_{s,t},\delta,k} (\mathfrak{z}^*_j D)|_{U,[1],j}
\end{array}
\]

for \(i = 0, 1\) on \((T,[1],J_{s,t},\mathcal{O}^{\text{crys}})\). These sequences are compatible with each other via \(1\) and \(Fr\). Let us now restrict scalars to \(\mathcal{O}^{\text{crys}}\) and apply the functor \((1)^*\). Since \(o_T^*\) and \(z_{T,*}\)
are exact functors they commute to \( \langle 1 \rangle^* \) and this yields exact sequences

\[
\sigma_T^* \widetilde{\text{Fil}}^n \Omega^*_{T_{[1],J}} (-h_{[1],J})( (o^*_J D)_{[1],J} ) \longrightarrow \widetilde{\text{Fil}}^n \Omega^*_{T_{[1],J}} (D_{[1],J}) \longrightarrow z_{T,1}^* \widetilde{\text{Fil}}^n \Omega^*_{T_{1},J} ( (z^*_J D)_{[1],J} )
\]

(injectivity of the left arrow is checked easily by diagram chasing using that \( T Z_J \) is flat over \( \Sigma_{\infty} \) for the case \( i = 0 \) and \( 5.43 \) together with the fact that that

\[
\sigma_T^* H^{-1} \widetilde{\text{Lie}}_{.,T_{[1],J}} ( (o^*_J D)_{[1],J} ) (-h_{[1],J}) \longrightarrow H^{-1} \widetilde{\text{Lie}}_{.,T_{[1],J}} (D_{[1],J})
\]

for the case \( i = 1 \). These exact sequences are compatible with each other via \( 1 \) and \( \phi \) (use \( 4.3 \) (ii)).

We may thus interpret them as exact sequences in the category of \((1,\phi)\)-modules over \( (T_{[1],J,et}, \widetilde{O}_{\text{crys}}) \) or equivalently as an isomorphism

\[
\sigma_T^* S^L_{\eta.,T_{[1],J}} (-h_{[1],J})( (o^*_J D)_{[1],J} ) \longrightarrow MF (S^+_J \rightarrow z_J^* S^+_J) \]

where

\[
S^+ := S^L_{\eta.,T_{[1],J}} (D_{J^+})
\]

is the syntomic complex on \( J^+ \) associated to the object of \( \text{HR}_F^\text{et}(J^+) \) which is defined by the arrow \( Y_{[1],J} \rightarrow Y_{[1],J} \). The isomorphism of the proposition follows by applying \( RF_{T_{[1],J,\ast}} \) (use \( 3.13 \) (i), (iv)). Finally we note that the independence of choices is a formal consequence of the connectedness of the category \( \text{HR}_F^\text{et,ex}(J^+) \) (which in turn is ensured by the existence of products, see \( 5.7 \)).

\[
\Box
\]

Let us write down the localization triangles encoded in the above proposition.

**Corollary 6.4.** (i) If \( M \) is a locally free crystal of \( (X/\Sigma,)_\text{crys,et}, \mathcal{O} \) there is a canonical distinguished triangle in \( D(X^N, \widetilde{\mathcal{O}}_{\text{crys}}) \):

\[
Ru_*(o^*_X M_{(-Z_{\Sigma}))} \longrightarrow Ru_* M \longrightarrow z_{X,\ast} Ru_*(z^*_X M) +1
\]

(ii) If \( D \) is in \( DC(X) \) there is a canonical distinguished triangle in \( D(X^N, \widetilde{\mathcal{O}}_{\text{crys}}) \):

\[
S^L_{X^\ast,\eta,et} (o^*_X D) \longrightarrow S^L_{X^\ast,et} (D) \longrightarrow z_{X,\ast} S^L_{X^\ast,et} (z^*_X D) +1
\]

and similarly without the superscripts \((1,\phi)\) in \( D(X^N, \mathbb{Z}/p) \)

Proof. Assertion \( \Box \) (resp. \( \Box \)) follows from \( 6.3 \) (i) (resp. \( 6.3 \)) by \( 3.13 \) (i)

\[
\Box
\]

7. **Dieudonné crystals for semi-stable abelian varieties**

The purpose of this chapter is to define a Dieudonné crystal over \( (\mathcal{O}_r^d/\Sigma_{\infty})_{\text{crys,et}} \) associated to semi-abelian scheme \( A/C \) whose restriction to \( U \) is abelian. We begin with a result concerning the structure of log 1-motives on a complete discrete valuation ring.
7.1. Dévissage of log 1-motives.

Let $X$ denote the spectrum of a complete discrete valuation ring $R$, and let $s : \text{Spec}(k) \to X$ (resp. $j : \text{Spec}(K) \to X$) the inclusion of its special (resp. generic) point. For any $X'/X$, we let $s'$, $j'$ denote the morphisms deduced from $s$, $j$ by base change.

7.1.1. The following definition is [KT] 4.6.1.

**Definition 7.1.**

(i) The category $\mathcal{M}_{\log}(X)$ of log 1-motives over $X$ is defined as follows:
- an object is a triple $(\Gamma, G, f)$ where $\Gamma$ is a twisted constant group (2.31 (iv)), while $G$ is the extension of an abelian scheme $B$ by a torus $T$ and and $f : \Gamma \to j_*j^{-1}G$ is a morphism in $\text{Ab}(X_{FL})$.
- a morphism from $(\Gamma, G, f)$ to $(\Gamma', G', f')$ is a couple of morphisms $\Gamma \to \Gamma'$, $G \to G'$, compatible with $f$ and $f'$.

(ii) Denote $B$ the category $\text{fet}(X)$. Define $\mathcal{M}_{\log}/B$ as the fibered category corresponding to the contravariant pseudo-functor $(X'/X) \mapsto \mathcal{M}_{\log}(X')$, $f \mapsto f^{-1}$ where $f^{-1}$ is the pullback functor deduced from $(\cdot)_{FL}$.

The functor sending a usual 1-motive $f : \Gamma \to G$ over $X'$ to the log 1-motive $f : \Gamma \to j_*j^{-1}G$ deduced from it is fully faithful. We may thus identify the category of usual 1-motives over variable bases with a full subcategory of $\mathcal{M}_{\log}$.

Just as usual 1-motives, any log 1-motive $(\Gamma, G, f)$ over $X$ comes with a functorial increasing weight filtration. It is defined as follows:

$\text{Fil}^0 = (\Gamma, G, f)$
$\text{Fil}^{-1} = (0, G, 0)$
$\text{Fil}^{-2} = (0, T, 0)$
$\text{Fil}^{-3} = (0, 0, 0)$

$T$ being the maximal subtorus of $G$ (see 2.32 (iii) for the functoriality).

We consider the following full $B$- subcategories of $\mathcal{M}_{\log}$:
- $\mathcal{M}$ : 1-motives viewed as log 1-motives, as explained above.
- $\mathcal{M}_{\log}$ : those $(\Gamma, G, f)$ with $Gr^{-1} = 0$, ie. such that $G = T$ is a torus.
- $\mathcal{MT} = \mathcal{M} \cap \mathcal{M}_{\log}$.

**Remark 7.2.** The $B$-categories $\mathcal{M}$, $\mathcal{M}_{\log}$, $\mathcal{MT}$ and $\mathcal{MT}_{\log}$ are naturally $B$-e-exact categories (i.e. the fibers are naturally exact and the pullback functors are $e$-exact) and the inclusion functors are $B$-e-exact. We leave this to the reader using only the exactness of $j^{-1}$ together with the stability under extensions of the categories of twisted constant groups, tori and extensions of abelian schemes by tori (see the proof of 2.33).

7.1.2. Any log 1-motive $(\Gamma, G, f)$ over $X$ can be seen canonically as object of the category of extensions $\text{EXT}^1_{\mathcal{M}_{\log}(X)}((\Gamma, 0, 0), (0, G, 0))$. A nice feature of the latter is that it is endowed with the Baer sum $\boxplus$ which is an exact bifunctor underlying the usual addition of the group $\text{Ext}^1_{\mathcal{M}_{\log}(X)}$. We will need to extend the bifunctor $\boxplus$ as follows.

**Definition 7.3.** Consider an exact category $\mathcal{C}$. 

124
(i) We define \( \mathcal{C} \Box \mathcal{C} \) as the category of diagrams of the form

\[
\begin{array}{ccccccc}
0 & \longrightarrow & A_1 & \overset{i_1}{\longrightarrow} & B_1 & \overset{p_1}{\longrightarrow} & C_1 & \longrightarrow & 0 \\
\alpha \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \overset{i_2}{\longrightarrow} & B_2 & \overset{p_2}{\longrightarrow} & C_2 & \longrightarrow & 0
\end{array}
\]

in which both lines are short e-exact sequences and both following morphisms are admissible (ie. are part of a short e-exact sequence of \( \mathcal{C} \)): \( A_2 \hookrightarrow B_1 \times B_2 \), \( a \mapsto (i_1 \alpha(a), -i_2(a)) \), \( B_1 \times B_2 \rightarrow C_1 \), \( (b_1, b_2) \mapsto p_1(b_1) - \gamma p_2(b_2) \). For simplicity, such an object is usually denoted simply \( (B_1, B_2) \).

(ii) We define \( \Box : \mathcal{C} \Box \mathcal{C} \rightarrow \mathcal{C} \) as the functor sending \( (B_1, B_2) \) as above to its Baer sum \( B_1 \boxplus B_2 := \text{Ker}(B_3 \rightarrow C_1 \), \( (b_1, b_2) \mapsto p_1(b_1) - \gamma p_2(b_2) \) where \( B_3 := \text{Coker}(A_2 \rightarrow B_1 \times B_2 \), \( a \mapsto (i_1 \alpha(a), -i_2(a)) \)).

The reader is invited to check that this definition makes sense, ie. that the involved kernels and cokernels exist indeed using the axioms of an exact category (see e.g. [Bu] def. 2.1). Note that \( \Box \) is an e-exact functor if \( \mathcal{C} \Box \mathcal{C} \) is endowed with its obvious exact structure. The formation of the category \( \mathcal{C} \Box \mathcal{C} \) and of the functor \( \Box \) is moreover canonically pseudo-functorial with respect to \( \mathcal{C} \) in the obvious sense.

In our situation, we thus have a \( B \)-e-exact functor

\[
(161) \quad \Box : \mathcal{M}_{log} \Box \mathcal{M}_{log} \rightarrow \mathcal{M}_{log}
\]

where \( \mathcal{M}_{log} \Box \mathcal{M}_{log} \) denote the natural \( B \)-e-exact fibered category whose fiber at \( X'/X \) is \( \mathcal{M}_{log}(X') \Box \mathcal{M}_{log}(X') \). Let us indicate a useful computation in a special case. If an object of \( \mathcal{M}_{log}(X) \Box \mathcal{M}_{log}(X) \) is of the particular form

\[
(162) \quad \begin{array}{ccccccc}
0 & \longrightarrow & (0, G_1, 0) & \overset{(0, id)}{\longrightarrow} & (\Gamma_1, G_1, f_1) & \overset{(id, 0)}{\longrightarrow} & (\Gamma_1, 0, 0) & \longrightarrow & 0 \\
\alpha \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (0, G_2, 0) & \overset{(0, id)}{\longrightarrow} & (\Gamma_2, G_2, f_2) & \overset{(id, 0)}{\longrightarrow} & (\Gamma_2, 0, 0) & \longrightarrow & 0
\end{array}
\]

then its Baer sum is naturally isomorphic to \( (\Gamma_2, G_1, f_1 \circ \gamma + j, j^{-1}(\alpha) \circ f_2) \).

7.1.3. As recalled in \ref{2.32}(i) tori and twisted constant groups are already locally trivial for the étale topology. Here \( X \) is the spectrum of a complete discrete valuation ring and the finite étale topology is in fact sufficient (use [Mi1] I., 4.2 c)). This and \ref{2.37}(iii) implies in particular that for a twisted constant group \( \Gamma \), we have \( \epsilon^{-1} \epsilon_* \Gamma \simeq \Gamma \) if

\[
\epsilon : X_{FL} \rightarrow X_{fet}
\]
denotes the quasi-morphism induced by the inclusion \( f_{et}(X) \subset FL(X) \).

Consider a fixed uniformizer \( t \) of \( R \). In \( \text{Ab}(X_{fet}) \) this choice provides a splitting of the valuation exact sequence

\[
0 \longrightarrow \epsilon_* \mathbb{G}_m \longrightarrow \epsilon_* j_* j^{-1} \mathbb{G}_m \overset{\epsilon_* \mathbb{Z}}{\longrightarrow} 0
\]

\[125\]
More generally consider an extension $G$ of an abelian scheme $B$ by a torus $T$ and denote $\Gamma^*: = \text{Hom}(T,G_m)$ (resp. $\Gamma^{\vee}: = \text{Hom}(\Gamma^*,\mathbb{Z})$) the character (resp. co character) group of $T$. Then $t$ induces a split monomorphism $t_T: \epsilon_*\Gamma^{\vee} \simeq \text{Hom}(\epsilon_*\Gamma^*,\epsilon_*\mathbb{Z}) \to \text{Hom}(\epsilon_*\Gamma^*,\epsilon_*j_*j^{-1}G_m) = \epsilon_*j_*j^{-1}T$. Let us furthermore denote $t_G: \epsilon_*\Gamma^{\vee} \to \epsilon_*j_*j^{-1}G$ the morphism given by $t_T$ and the inclusion $T \subset G$.

**Lemma 7.4.** The morphism $t_G$ induces the following isomorphisms:

(163) $\epsilon_*G \times \epsilon_*\Gamma^{\vee} \xrightarrow{(j^*, t_G)} \epsilon_*j_*j^{-1}G$

(164) $\text{Hom}(\Gamma, G \times \Gamma^{\vee}) \xrightarrow{\sim} \text{Hom}(\Gamma, j_*j^{-1}G)$

Proof. The second isomorphism follows immediately from the first one. Let us thus explain (163). First we note that smooth group schemes are acyclic for $\epsilon_*$ in virtue of (3.16) and (3.21)(i). Applying this to $T$ and using that $\epsilon_*B \to \epsilon_*j_*j^{-1}B$ is an isomorphism (this follows from the Néron extension property) we get the following commutative diagram with exact lines:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \epsilon_*T & \longrightarrow & \epsilon_*G & \longrightarrow & \epsilon_*B & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \epsilon_*j_*j^{-1}T & \longrightarrow & \epsilon_*j_*j^{-1}G & \longrightarrow & \epsilon_*j_*j^{-1}B & \longrightarrow & 0 \\
& & & & & & & & \\
\end{array}
$$

Diagram chasing reduces us to the case $T = G$ but then we are done by the construction of $t_T$ explained above.

Note that $T$ being functorial with respect to $G$, the isomorphisms of the lemma are functorial as well.

**Definition 7.5.**

(i) Let $(\Gamma, G, f)$ in $\mathcal{M}_{\log}(X)$ and denote $(f_0, v(f))$ the preimage of $f$ under the isomorphism (164). The resulting couple of log 1-motives

$$(\Gamma, G, f_0), (\Gamma, T, t_T \circ v(f))$$

in $\mathcal{M}(X) \times \mathcal{MT}_{\log}(X)$ coming from this decomposition is called the $t$-decomposition of $(\Gamma, G, f)$ (here we abusively denote $t_T$ the morphism $\Gamma^{\vee} \to j_*j^{-1}T$ obtained using that $\epsilon^{-1}\epsilon_*\Gamma^{\vee} \simeq \Gamma^{\vee}$).

(ii) We say that $(\Gamma, G, f)$ in $\mathcal{M}_{\log}(X)$ is a multiple of $t$ if $f_0 = 0$ (or equivalently if $f = t_G \circ v(f)$). The full subcategory of $\mathcal{M}_{\log}(X)$ formed by the multiples of $t$ is denoted $\mathcal{M}'_{\log}(X)$.

Let us emphasize that the $t$-decomposition introduced in (i) is functorial with respect to $(\Gamma, G, f)$ in $\mathcal{M}_{\log}(X)$. Let us also notice that a log 1-motive $(\Gamma, G, f)$ is in fact in $\mathcal{M}(X)$ if and only if $v(f) = 0$.

**Lemma 7.6.**

(i) Let $\mathcal{M}(X) \Box \mathcal{MT}'_{\log}(X)$ denote the full subcategory of $\mathcal{M}_{\log}(X) \Box \mathcal{M}_{\log}(X)$ whose objects are of the form

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (0, G, 0) & \xrightarrow{(0, id)} & (\Gamma, G, f) & \xrightarrow{(id, 0)} & (\Gamma, 0, 0) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (0, T, 0) & \xrightarrow{(0, id)} & (\Gamma, T, g) & \xrightarrow{(id, 0)} & (\Gamma, 0, 0) & \longrightarrow & 0 \\
& & & & & & & & \\
\end{array}
$$

\[126\]
where \((\Gamma, G, f)\) is in \(\mathcal{M}(X)\) and \((\Gamma, T, g)\) is in \(\mathcal{M}_\log^t(X)\) (ie. \(g = t_T \circ v(g)\)).

The functor \((161)\) induces an equivalence of categories

\[
\mathcal{M}(X) \boxtimes \mathcal{M}_\log^t(X) \overset{\cong}{\longrightarrow} \mathcal{M}_\log(X).
\]

(ii) Let \(\mathcal{M}(X) \boxtimes \mathcal{M}_\log^0(X)\) denote the full subcategory of \(\mathcal{M}(X) \boxtimes \mathcal{M}_\log^t(X)\) whose objects are those of the above form satisfying furthermore \(g = 0\). Then we have an equivalence of categories

\[
\mathcal{M}(X) \boxtimes \mathcal{M}_\log^0(X) \overset{\cong}{\longrightarrow} \mathcal{M}(X).
\]

Proof. (ii) The assertion follows from the functoriality of the weight filtration together with the computation of the Baer sum on \((161)\) (alternatively, it is also a consequence of \((i)\).

(i) An object of the indicated form is sent by \(\boxplus\) to \((\Gamma, G, f + g)\), where \(g\) is seen as a morphism \(\Gamma \to G\). Essential surjectivity thus results from \(7.4\). Full faithfulness on the other hand boils down to the fact that the \(t\)-decomposition is functorial with respect to the log 1-motive.

\(\square\)

Let \(\mathbb{Z}, \mathbb{Z}(1)\) respectively denote the objects \((\mathbb{Z}, 0, 0), (0, \mathbb{G}_m, 0)\) of \(\mathcal{M}(X)\). If \(x \in R^\times\) (resp. \(K^\times\)), we denote \(\text{Kum}(x) := (\mathbb{Z}, \mathbb{G}_m, x : \mathbb{Z} \to \mathbb{G}_m)\) (resp. \(\text{Kum}(x) := (\mathbb{Z}, \mathbb{G}_m, \mathbb{Z} \to j_* j^{-1} \mathbb{G}_m)\) the corresponding Kummer 1-motive (resp. log 1-motive). Note that sending \(x\) to the class of \(\text{Kum}(x)\) realizes isomorphisms of groups \(R^\times \simeq \text{Ext}^1_{\mathcal{M}(X)}(\mathbb{Z}, \mathbb{Z}(1))\) and \(K^\times \simeq \text{Ext}^1_{\mathcal{M}_\log(X)}(\mathbb{Z}, \mathbb{Z}(1))\) (compatibility with group laws relies on the computation of \(\boxplus\) on \((161)\)). If \(M\) is a twisted constant group, we denote furthermore \(M \otimes \text{Kum}(x)\) the 1-motive (resp. log 1-motive) \((M, M \otimes \mathbb{G}_m, id_M \otimes x)\).

**Lemma 7.7.** The Baer sum functor induces an equivalence between \(\mathcal{M}_\log^t(X)\) and the full subcategory of \(\mathcal{M}_\log(X) \boxtimes \mathcal{M}_\log(X)\) whose objects are of the form

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (0, M \otimes \mathbb{G}_m, 0) & \overset{(0, id)}{\longrightarrow} & M \otimes \text{Kum}(t) & \overset{(id, 0)}{\longrightarrow} & (M, 0, 0) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \rightarrow & (0, M \otimes \mathbb{G}_m, 0) & \overset{(0, id)}{\longrightarrow} & (\Gamma, M \otimes \mathbb{G}_m, 0) & \overset{(id, 0)}{\longrightarrow} & (\Gamma, 0, 0) & \rightarrow & 0
\end{array}
\]

with \(M\) a twisted constant group.

Proof. An object of the above form is sent to \((\Gamma, M \otimes \mathbb{G}_m, g)\) where \(g\) is the composition of the morphism \(f : \Gamma \to M\) with the morphism \(id_M \otimes t : M \to M \otimes j_* j^{-1} \mathbb{G}_m\). Essential surjectivity results from the definition of \(\mathcal{M}_\log^t(X)\) (take \(M = \Gamma^{*v}\)) while full faithfulness results from the fact that \(id_M \otimes t\) is monomorphic.

\(\square\)

**Proposition 7.8.** Let \(\mathcal{C}/\mathcal{B}\) be a stack of exact categories and assume given a \(\mathcal{B}\)-exact cartesian functor \(F : \mathcal{M} \to \mathcal{C}\) (ie. a collection of \(e\)-exact functors \(F_X' : \mathcal{M}(X') \to \mathcal{C}(X')\) together with isomorphisms \(f^* F_{X'} \simeq F_{X''} f^{-1}\) for all \(f : X'' \to X'\), such that the obvious composition constraint holds). The category of extensions of \(F\) to a \(\mathcal{B}\)-exact functor

\[
F_\log : \mathcal{M}_\log \to \mathcal{C}
\]
is canonically equivalent to the discrete category whose underlying set is the set of homomorphisms

\[ |F_{\log}| : K^\times \simeq Ext^1_{\mathcal{M}_{\log}(X)}(\mathbb{Z}, \mathbb{Z}(1)) \rightarrow Ext^1_{\mathcal{C}(X)}(F(\mathbb{Z}), F(\mathbb{Z}(1))) \]

extending the homomorphism

\[ |F| : R^\times \simeq Ext^1_{\mathcal{M}(X)}(\mathbb{Z}, \mathbb{Z}(1)) \rightarrow Ext^1_{\mathcal{C}(X)}(F(\mathbb{Z}), F(\mathbb{Z}(1))) \]

induced by \( F \). This equivalence is independent on the choice of \( t \) made earlier.

Proof. Consider the category \( \text{Cart}Ext^F_B(\mathcal{M}_{\log}, \mathcal{C}) \) whose objects (resp. morphisms) are the cartesian \( B \)-exact functors extending \( F \) (resp. \( B \)-morphisms between them) and the discrete category \( \text{Hom}^{|F|}(K^\times, Ext^1_{\mathcal{C}(X)}(F(\mathbb{Z}), F(\mathbb{Z}(1)))) \) whose objects are the homomorphisms extending \( |F| \). We need to show that the natural functor

\[ \frac{|-| : \text{Cart}Ext^F_B(\mathcal{M}_{\log}, \mathcal{C})}{\text{Cart}Ext^F_B(\mathcal{M}_{\log}, \mathcal{C})} \rightarrow \frac{\text{Hom}^{|F|}(K^\times, Ext^1_{\mathcal{C}(X)}(F(\mathbb{Z}), F(\mathbb{Z}(1))))}{\text{Hom}^{|F|}(K^\times, Ext^1_{\mathcal{C}(X)}(F(\mathbb{Z}), F(\mathbb{Z}(1))))} \]

is an equivalence. Let us fix a uniformizer \( t \) and consider the following natural factorization (via restriction to \( \mathcal{M}_{\log}^t \) of the functor \( |-| \):

\[ \text{Cart}^F_B(\mathcal{M}_{\log}, \mathcal{C}) \xrightarrow{\text{res}^t} \text{Cart}^t_F(\mathcal{M}_{\log}^t, \mathcal{C}) \xrightarrow{|-|^t} \text{Hom}^{|F|}(K^\times, Ext^1_{\mathcal{C}(X)}(F(\mathbb{Z}), F(\mathbb{Z}(1)))) \]

We are going to show that both functors \( \text{res}^t \) and \( |-|^t \) are equivalences.

The given \( B \)-exact functor \( F \) induces a pseudo-commutative diagram which corresponds to the solid part of the following diagram where the vertical equivalences are given by [7.6]

\[ \begin{array}{ccc}
\mathcal{M} & \xrightarrow{\square F} & \mathcal{C} \\
\mathcal{M} & \xleftarrow{\square B} & \mathcal{C} \\
\mathcal{M} & \xleftarrow{\square B \mathcal{M}^0} & \mathcal{M} \\
\mathcal{M} & \xrightarrow{\square B \mathcal{M}^t_{\log}} & \mathcal{M} \\
\mathcal{M}_{\log} & \xrightarrow{\square B \mathcal{M}^t_{\log}} & \mathcal{M}_{\log} \\
\end{array} \]

This diagram together with pseudo-functoriality of \( \square \) and \( \boxdot \) shows that it is equivalent to extend either one of the \( B \)-exact functors \( F \), \( F \square F \) or \( \tilde{F} \) respectively into a \( B \)-exact functor \( F_{\log}, F_{\log} \square F_{\log} \) or \( \tilde{F}_{\log} \). Now extending \( \tilde{F} \) into \( \tilde{F}_{\log} \) is also equivalent to extending \( \text{res}^t(F) : \mathcal{M}^t \rightarrow \mathcal{C} \) into a functor \( \mathcal{M}^t_{\log} \rightarrow \mathcal{C} \). This shows that \( \text{res}^t \) is an equivalence.

Using [7.7] together with étale descent in \( \mathcal{C} \) we are reduced to the case \( \Gamma \simeq \mathbb{Z}^a \) and \( M \simeq \mathbb{Z}^b \). Thus we see that extending \( \text{res}^t(F) \) to \( \mathcal{M}^t_{\log} \) simply boils down to the choice of the image of the Kummer log 1-motive \( \text{Kum}(t) \). Since the group \( K^\times / R^\times \) is generated by \( t \), this shows that \( |-|^t \) is an equivalence.

\[ \square \]
7.2. The semi-stable Dieudonné functor.

The goal of this section is to construct the Dieudonné crystal of a semi-abelian scheme over $C_v$ and then over $C$ by gluing the local constructions for $v$ in $Z$ with the usual Dieudonné crystal of $[BBM]$ over $U$.

7.2.1. The Dieudonné functor to be constructed will take its values into the exact category of Dieudonné crystals. In view of cohomological methods it will be convenient to consider the larger category of $(f,v)$-modules defined just below.

**Definition 7.9.** Consider a ringed topos $(E,A)$ (with $A$ commutative) and an object $X$ of $E$ together with an endomorphism $F : X \to X$. We still denote $F$ the associated localization morphism $F : (E_X, A|_X) \to (E_X, A|_X)$.

(i) We define the category $\text{Mod}^F(E_X, A|_X)$ of $(f,v)$-modules as follows:

- An object is a triple $(M, f_M, v_M)$ where $M$ is a module of $(E_X, A|_X)$ and $f_M : F^{-1}M \to M$, $v_M : M \to F^{-1}M$ are morphisms in $\text{Mod}(E_X, A|_X)$.
- A morphism $(M, f_M, v_M) \to (N, f_N, v_N)$ is a morphism $a : M \to N$ in $\text{Mod}(E_X, A|_X)$ which is compatible with the $f'$s and $v$'s.

(ii) We define a bifunctor

$$\text{Hom}^F_{A|_X} : \text{Mod}^F(E_X, A|_X)^{op} \times \text{Mod}^F(E_X, A|_X) \to \text{Mod}^F(E_X, A|_X)$$

by the formula

$$\text{Hom}^F_{A|_X}((M, f_M, v_M), (N, f_N, v_N)) := (\text{Hom}_{A|_X}(M, N), \text{Hom}_{A|_X}(v_M, f_N), \text{Hom}_{A|_X}(f_M, v_N))$$

where $\text{Hom}_{A|_X}$ denotes inner homomorphisms over $(E_X, A|_X)$.

Consider the category $E^F_X$ of triples $(Y, f_Y, v_Y)$ where $Y, f_Y : F^{-1}Y \to Y$ and $v_Y : Y \to F^{-1}Y$ are in $E_X$. The ring $(A|_X, f_A, v_A)$ where $f_A$ is the canonical identification $F^{-1}A|_X \simeq A|_X$ and $v_A = f_A^{-1}$ will simply be denoted $A|_X$.

**Lemma 7.10.** Let $(E, A, X, F)$, $E^F_X$ and $A|_X$ as above.

(i) The pair $(E^F_X, A|_X)$ is a ringed topos. It is pseudo functorial with respect to $(E, A, X, F)$ if a morphism $(E, A, X, F) \to (E', X', A', F')$ means a morphism of ringed topoi $g : (E, A) \to (E', A')$ together with an isomorphism $g^{-1}X' \simeq X$ which is compatible with $F$ and $F'$.

(ii) There is a canonical isomorphism $\text{Mod}^F(E_X, A|_X) \simeq \text{Mod}(E^F_X, A|_X)$. Via this identification the adjunction $(g^*, g_*) : \text{Mod}(E^F_X, A|_X) \to \text{Mod}(E^F_X, A|_X)$ attached to $(E, A, X, F) \to (E', X', A', F')$ translates as follows:

$$g^*(M', f', v') = (g^*M', f, v)$$

where $f : F^{-1}g^*M' \simeq g^*F^{-1}M' \xrightarrow{g^*(f')} g^*M'$

and $v : g^*M' \xrightarrow{g^*(v')} g^*F^{-1}M' \simeq F^{-1}g^*M'$

$$g_*(M, f, v) = (g_*M, f', v')$$

where $f' : F^{-1}g_*M \simeq g_*F^{-1}M \xrightarrow{g_*(f)} g_*M$

and $v' : g_*M \xrightarrow{g_*(v)} g_*F^{-1}X \simeq F^{-1}g_*M$

(here the base change isomorphisms are due to the fact that $F'$ is a localization morphism whose pullback by $g$ is $F'$).

(iii) The forgetful functor $\chi^{-1} : \text{Mod}^F(E_X, A|_X) \to \text{Mod}(E_X, A|_X)$ is the pullback of morphism of ringed topoi. It has moreover a left adjoint $\chi_!$ which is exact.
Proof. (i) and (ii) are routine.

(iii) Let us give an explicit description of $\chi_1$ in order to check that it is exact. Let $[F^{-1}]^{N} \ast [F]^{N}$ denote the free associative monoid with unit on the labels $[F^{-1}], [F]$. Sending $[F^{-1}]$ to $F^{-1}$ and $[F]$ to $F$ defines an action of $[F^{-1}]^{N} \ast [F]^{N}$ on the category $\text{Mod}(E/X, A/X)$ in the strict sense. Explicitly given a word $w = [F^{-1}]^{a_1}[F]^{b_1} \cdots [F^{-1}]^{a_l}[F]^{b_l}$ of length $2l$ ($l \geq 0$, $a_i \geq 1$, $b_i \geq 1$) and an $A$-module $M$, $w.M = (F^{-1})^{a_1}(F)^{b_1}\cdots (F^{-1})^{a_l}(F)^{b_l}M$. Set $\chi_1 M = (N, f_N, v_N)$ where

$$N = \bigoplus_{w \in [F^{-1}]^{N} \ast [F]^{N}} w.M,$$

$f_M : F^{-1}M \to M$ is induced by multiplying the indices on the left by $[F^{-1}]$ together with the identity $F^{-1}(w.M) = ([F^{-1}]w)M$ in component $w$, and $v_M : M \to F^{-1}M$ is induced by multiplying the indices on the left by $[F]$ together with the adjunction morphism $w.M \to F^{-1}F(w.M) = F^{-1}([F]w.M)$ in component $[F]w$. The adjunction morphisms $M \to \chi^{-1}_1 \chi_1 M$ and $\chi_1 \chi^{-1}_1 (M, f_M, v_M) \to (M, f_M, v_M)$ are respectively defined by sending $M$ into the component 1 and sending the component $[F^{-1}]^{a_1}[F]^{b_1} \cdots [F^{-1}]^{a_l}[F]^{b_l}$ into $M$ by applying successively: the morphism $F \chi_1 M \to M$ deduced from $v_M$ by adjunction $\beta_1$ times, the morphism $f_M \alpha_1$ times and so on. We leave it to the reader to check the adjunction property. This description makes it clear that $\chi_1$ is exact since it is built from the exact functors $F$ and $F^{-1}$ using direct sums. Let us mention that the functor $\chi_*$ has an analogous description with $[F^{-1}]^{N} \ast [F]^{N}$ replaced by $[F_*]^{N} \ast [F^{-1}]^{N}$.

\[\square\]

Remark 7.11. The bifunctor $\mathcal{H}om^{f_v}_{A/X}$ should not be confused with the bifunctor $\mathcal{H}om^{f_v}_{E/f_v, A/X}$ of inner homomorphisms in the ringed topos $(E/X, A/X)$.

The bifunctor $\mathcal{H}om^{f_v}_{A/X}$ is right derivable and thus induces:

$$R\mathcal{H}om^{f_v}_{A/X}(-, -) : D^{-}(E^f_{/X}, A_{/X})^{op} \times D^{+}(E^f_{/X}, A_{/X}) \to D^{+}(E^f_{/X}, A_{/X})$$

We will also need a variant $\mathcal{H}om^{f_v}_{A/X}$ where the first argument is in the category of abelian groups instead of $A$-modules. Deriving gives:

$$R\mathcal{H}om^{f_v}_{A/X}(-, -) : D^{-}(E^f_{/X})^{op} \times D^{+}(E^f_{/X}, A_{/X}) \to D^{+}(E^f_{/X}, A_{/X})$$

Finally we will use furthermore variants where the first argument (resp. the target) takes its values in the derived category of inductive systems of $(f, v)$-abelian groups (resp. projective systems of $(f, v)$-modules) and conversely. Derived functors in this setting can be computed by taking components of the projective or inductive systems accordingly.

Lemma 7.12. (i) One has a canonical bifunctorial isomorphism

$$\mathcal{E}xt^{f_i}_{A/X}((M, f_M, v_M), (N, f_N, v_N)) = (\mathcal{E}xt^{i}_{A/X}(M, N), f, v)$$

where $f$ is induced by $v_M$ and $f_N$ while $v$ is induced by $f_M$ and $v_N$.

(ii) Let $A/X \otimes (-) : \text{Mod}^{f_v}(E/X, \mathbb{Z}) \to \text{Mod}^{f_v}(E/X, A_{/X})$ denote the pullback functor induced by $(E, A, X, F) \to (E, \mathbb{Z}, X, F)$. There is a canonical isomorphism of bifunctors $D^{-}(E^f_{/X})^{op} \times D^{+}(E^f_{/X}, A_{/X}) \to D^{+}(E^f_{/X}, A_{/X})$:

$$R\mathcal{H}om^{f_v}_{A/X}(A/X \otimes (M, f_M, v_M), (N, f_N, v_N)) \overset{adj}{\cong} R\mathcal{H}om^{f_v}((M, f_M, v_M), (N, f_N, v_N))$$
(iii) Let $l_* : \text{Mod}^{f_v}(E/{X}, A|_X)^{\mathbb{N}} \to \text{Mod}^{f_v}(E/{X}, A|_X)$ denote the functor taking an projective system of $(f, v)$-modules to its inverse limit. There is a canonical isomorphism of bifunctors $D^-(E^{f_v,\text{opp}}/{X}) \times D^+(E^{f_v}_{/X}, A|_X) \to D^+(E^{f_v}_{/X}, A|_X)$. 

\[ \text{RHom}^{f_v}(\text{lim}(M, f_M, v_M), (N, f_N, v_N)) \approx \text{Rl}_* \text{RHom}^{f_v}((M, f_M, v_M), (N, f_N, v_N)) \]

A similar isomorphism holds with $\text{RHom}^{f_v}_{A|_X}$ instead of $\text{RHom}^{f_v}$. 

(iv) For each $n \geq 0$ there is a canonical morphism of functors $D^-(E^{f_v}_{/X}, A|_X) \to D(E^{f_v}_{/X}, A|_X)$ 

\[ \text{id} \to \text{RHom}^{f_v}_{A|_X}(\tau \leq_n \text{RHom}^{f_v}_{A|_X}(-, A|_X), A|_X) \]

The morphisms obtained for different values of $n$ are compatible in the obvious way.

Proof. (i) This follows from the fact that the functor $\chi^{-1}$ preserve injectives (7.10 (iii)) as well as $F^{-1}$ (localization).

(ii) The isomorphism is clear if $(M, f_M, v_M)$ is flat over $(E^{f_v}_{/X}, \mathbb{Z})$ and $(N, f_N, v_N)$ is injective over $(E^{f_v}_{/X}, A|_X)$. The general case follows by taking resolutions.

(iii) Start with the obvious isomorphism 

\[ \text{Hom}^{f_v}(\text{lim}(M, f_M, v_M), (N, f_N, v_N)) \approx l_* \text{Hom}^{f_v}((M, f_M, v_M), (N, f_N, v_N)) \]

Let us explain why the announced isomorphism can be deduced by right derivation. Say that an object $M = (M, f_M, v_M)$ in $\text{Mod}^{f_v}(E^{\text{opp}}/{X}, \mathbb{Z})$ is good if $M_k \to M_{k'}$ for $k \leq k'$. Every inductive system $M$ is a quotient of a good one e.g. $g(d(M)) \to M$, where $g(d(M)) = \oplus_{j \leq k} M_j$. The result will follow if we can prove that the projective system $\text{Hom}^{f_v}(M, N)$ is $l_*$-acyclic as soon as $M$ is good and $N$ is injective. Since $\chi^{-1} : \text{Mod}^{f_v}(E/{X}, A|_X)^{\mathbb{N}} \to \text{Mod}(E/{X}, A)^{\mathbb{N}}$ preserves injectives (7.10 (iii)) we are thus reduced to prove that for $M$ good and $N$ injective $L := \text{Hom}(M, N)$ is acyclic for the functor $l_*$. In order to check that $l_* L \cong \text{Rl}_* L$, it is sufficient to check that $\Gamma(U, l_* L) \cong \text{Rl}_* \Gamma(U, L)$ for every $U$ in $E$ or equivalently $\text{lim}_{\text{proj}} \text{Hom}(M|_U, N|_U) \cong \text{Rlim}_{\text{proj}} \Gamma(U, L)$. Since $N$ is injective we find that $\Gamma(U, L) \cong \text{Hom}(M|_U, N|_U)$ (SGA4-II V) and that the transition maps of the latter projective system are surjective. The result follows (using e.g. the Mittag Leffler criterion for $A(U)$-modules).

(iv) The desired morphism may be obtained by using any injective resolution of $A|_X$ in $(E^{f_v}_{/X}, A|_X)$.

\[ \square \]

7.2.2. In the context of big or $\mathbf{2}$-big crystalline topoi, we will use the following notations:

- For $Y^{\mathbf{2}}$ in $\text{Sch}^\mathbf{2}/\Sigma_1$ we denote $(Y^{\mathbf{2}}/\Sigma_\infty)^{f_v}$ the topos denoted $E^{f_v}_{/X}$ in 7.10 with $E = (\Sigma_1/\Sigma_{\infty})_{\text{CRYS}, fl}$, $X = i_* Y^{\mathbf{2}}$ and $F$ the endomorphism of $X$ induced by the absolute Frobenius of $Y^{\mathbf{2}}$.

- For $Y$ in $\text{Sch}/\Sigma_1$ we denote $(Y/\Sigma_{\infty})^{f_v}$ the topos denoted $E^{f_v}_{/X}$ in 7.10 with $E = (\Sigma_1/\Sigma_{\infty})_{\text{CRYS}, fl}$, $X = i_* Y$ and $F$ the endomorphism of $X$ induced by the absolute Frobenius of $Y$.

**Lemma 7.13.** Assume that $X^{\mathbf{2}}/\Sigma_1$ is locally embeddable.

(i) The inclusion functor $\text{DC}_{\text{CRYS}, fl}(X^{\mathbf{2}}) \to \text{Mod}((X^{\mathbf{2}}/\Sigma_\infty)^{f_v}, O)$ is $e$-exact and reflects $e$-exactness.
(ii) For $D_1, D_2$ in $\mathcal{DC}(X^\sharp)$ the functor of $[\dagger]$ induces an injection

$$\text{Ext}^1_{\mathcal{DC}_{\text{CRYS}^\sharp, f_1}(X^\sharp)}(D_1, D_2) \xrightarrow{} \text{Ext}^1_{(X^\sharp/\Sigma^\infty)^{f^u, \mathcal{O}}}(D_1, D_2)$$

(iii) If $X^\sharp = X$ then $[\ddagger]$ and $[\dagger]$ hold as well with $(X/\Sigma^\infty)^{f^u}$ and $\text{CRYS}^\sharp$ respectively replaced by $(X/\Sigma^\infty)^{f^u}$ and $\text{CRYS}$.

Proof. $[\dagger]$ This follows from 5.33 $[\ddagger]$ given that the forgetful functor $\chi^{-1}$ is exact. Statement $[\ddagger]$ follows formally. The case of $[\ddagger]$ is similar. □

7.2.3. Let us introduce some simplified notations for certain $(f, v)$-modules in crystalline toposi. In what follows we implicitly use the obvious isomorphism $F^{-1}i_* \simeq i_* F^{-1}$.

- If $G$ is an abelian group of $(\Sigma^1/\Sigma^\infty)_{\text{CRYS}^\sharp, f_1}$ we use the notation

$$(167)\quad G = (G, 1, 1)$$

for the corresponding trivial $(f, v)$-module of $((X^\sharp/\Sigma^\infty)_{\text{CRYS}^\sharp, f_1}, \mathbb{Z})$, i.e. the abelian group $G_{(X, f, v)}$ of $(X^\sharp/\Sigma^\infty)^{f^u}$ where $f$ is the trivial identification $G \simeq F^{-1}G$ (not to be confused with the relative Frobenius) and $v$ is the inverse of $f$. This convention applies for instance to $G_m, G^\log_m, G_a, \mathcal{O}, \mathcal{I}$.

- If $G$ is a finite locally free group or a $p$-divisible group over $X$ then we denote

$$(168)\quad G^{f^u} = (i_* G, f_G, v_G)$$

the abelian group of $(X^\sharp/\Sigma^\infty)^{f^u}$ where $v_G$ is induced by the relative Frobenius $F^{(G/X^\sharp)} : G \to F^{-1}G$ and $f_G$ is induced by $(F^{(G^*/X^\sharp)})^* : F^{-1}G \to G$ where $(-)^*$ denotes the Cartier dual (see section 2.5).

We use similar notations in the setting of big (instead of $\sharp$-big) toposi in the case $X^\sharp = X$.

7.2.4. Our starting point for the construction of Dieudonné crystals of semi-abelian schemes is a result from [BBM] for the Dieudonné crystal of a $p$-divisible group.

If $M$ is an abelian group (resp. $\mathcal{O}$-module) of $(X/\Sigma^\infty)^{f^u}$ we use the following notation for bidualizing functors

$$B(M) = \mathcal{R}\text{Hom}_{\mathcal{O}(X/\Sigma^\infty)}^{f^u}(\tau_{\leq 1} \mathcal{R}\text{Hom}_{(X/\Sigma^\infty)}^{f^u}(M, \mathcal{O}), \mathcal{O}) \text{ in } D((X/\Sigma^\infty)^{f^u}, \mathcal{O})$$

(resp. $B_{\mathcal{O}}(M) = \mathcal{R}\text{Hom}_{\mathcal{O}(X/\Sigma^\infty)}^{f^u}(\tau_{\leq 1} \mathcal{R}\text{Hom}_{\mathcal{O}(X/\Sigma^\infty)}^{f^u}(M, \mathcal{O}), \mathcal{O}) \text{ in } D((X/\Sigma^\infty)^{f^u}, \mathcal{O})$).

Here and in the following we denote respectively $\mathcal{R}\text{Hom}_{\mathcal{O}(X/\Sigma^\infty)}^{f^u}$ and $\mathcal{R}\text{Hom}_{(X/\Sigma^\infty)}^{f^u}$ the functors $[165]$ and $[166]$.

Remark 7.14. Because the definition of the functor $B$ involves truncation it does not commute to shifting. Instead we have natural distinguished triangles:

$$B(M[-1]) \xrightarrow{} B(M)[-1] \xrightarrow{} \mathcal{R}\text{Hom}_{\mathcal{O}(X/\Sigma^\infty)}^{f^u}(\mathcal{E}\text{xt}^{f^u, 2}_{(X/\Sigma^\infty)}(M, \mathcal{O}), \mathcal{O})[2] \xrightarrow{}$$

$$B(M)[1] \xrightarrow{} B(M[1]) \xrightarrow{} \mathcal{R}\text{Hom}_{\mathcal{O}(X/\Sigma^\infty)}^{f^u}(\mathcal{E}\text{xt}^{f^u, 1}_{(X/\Sigma^\infty)}(M, \mathcal{O}), \mathcal{O})[3] \xrightarrow{}$$

A similar remark holds for $B_{\mathcal{O}}$.

Proposition 7.15. Let $X$ in $\mathcal{S}ch/\Sigma_1$ and $G$ in $p\text{div}(X)$. 

(i) If $G^*$ denotes the dual $p$-divisible group then $\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(G^*,f_v, \mathcal{O}) = 0$ and

$$D(G) := \mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^*,f_v, \mathcal{O}) \text{ in } \text{Mod}((X/\Sigma_{\infty})^{f_v}, \mathcal{O})$$

is a Dieudonné crystal. The resulting functor $p\text{div}(X) \to D\mathcal{C}_{\mathcal{R}YS, fl}(X)$ is exact.

(ii) There are canonical isomorphisms in $\text{Mod}((X/\Sigma_{\infty})^{f_v}, \mathcal{O})$:

$$D(G) \simeq B(G^{f_v})[-1] \simeq \mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(\mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^{f_v}, \mathcal{O}), \mathcal{O})$$

Proof. [i] follows immediately from 7.12 [i] and BBM 3.3.3. 

This is proven in BBM 5.3.6 by reduction to the case of finite locally free groups. Let us only recall the definition of the arrow $B(G^{f_v})[-1] \to D(G)$ in question since it will be needed below. Consider the exact sequence

$$0 \to 1 + \mathcal{I} \to \mathcal{O}^\times \to \mathbb{G}_m \to 0$$

of $(\Sigma_1/\Sigma_{\infty})_{\mathcal{R}YS, fl}$. Combining with the logarithm $1 + \mathcal{I} \to \mathcal{O}$ and passing to trivial $(f,v)$-modules of $(X/\Sigma_{\infty})$ we get a morphism

$$\log : \mathbb{G}_m[-1] \to \mathcal{O} \text{ in } D((X/\Sigma_{\infty})^{f_v})$$

According to 7.12 [i] and BBM 5.2.7 the morphism (169) induces an isomorphism

$$R\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(\tau \leq 1 R\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(G^{f_v}_{p^k}, \mathcal{O}), \mathcal{O}) \simeq \tau \leq 0 R\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(\tau \leq 1 R\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(G^{f_v}_{p^k}, \mathcal{O}), \mathcal{O})$$

in $D((X/\Sigma_{\infty})^{f_v,N}, \mathcal{O})$. The claimed isomorphisms will follow by computing the effect of $R\mathcal{L}_*$ on both side. We need the following

Fact. There is a natural isomorphism $\limind_k \mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(G^{f_v}_{p^k}, \mathcal{O}) \simeq \mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^{f_v}, \mathcal{O})$ and $\limind_k \mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^{f_v}_{p^k}, \mathcal{O})$ vanishes.

The proof of this fact is left to the reader (look at realizations of the exact sequence of $\text{Mod}((X/\Sigma_{\infty})^{f_v}, \mathcal{O})$

\[
\begin{array}{cccc}
\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(G^{f_v}_{p^k}, \mathcal{O}) & \mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^{f_v}, \mathcal{O}) & \mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^{f_v}_{p^k}, \mathcal{O})
\end{array}
\]

resulting from BBM 3.3.3 and let $k$ run in $\mathbb{N}^{op}$.

Using this fact we obtain the following series of isomorphisms:

$$R\mathcal{L}_*(\text{LHS}) \simeq R\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(\lim\tau \leq 1 R\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(G^{f_v}_{p^k}, \mathcal{O}), \mathcal{O})$$

(171)

$$\simeq R\mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(\mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^{f_v}, \mathcal{O}), \mathcal{O})$$

(172)

$$\simeq \mathcal{H}om_{(X/\Sigma_{\infty})}^{f_v}(\mathcal{E}xt_{(X/\Sigma_{\infty})}^{f_v,1}(G^{f_v}, \mathcal{O}), \mathcal{O})$$
where (170) is by 7.12 (iii) and (172) is by local freeness of $\mathcal{E}_x^{1}(X/\Sigma_{\infty})(G, \mathcal{O})$ (see (i)). Having noticed that $R_{l}(LHS)$ is concentrated in degree 0, we find that

\begin{align*}
(173) \quad Rl_*(RHS) & \simeq Rl^0_*(RHS) \\
(174) & \simeq Rl^0_*(\tau_{\leq 0} RHom_{(X/\Sigma_{\infty})}^{fv}(\tau_{\leq 1} RHom_{(X/\Sigma_{\infty})}^{fv}(G_{p}^{fv}, G_{m}[-1]), \mathcal{O})) \\
(175) & \simeq Rl^0_*(\tau_{\leq 0} RHom_{(X/\Sigma_{\infty})}^{fv}(G_{p}^{*,fv}[-1], \mathcal{O})) \\
(176) & \simeq Rl^1_*(\tau_{\leq 1} RHom_{(X/\Sigma_{\infty})}^{fv}(G_{p}^{*,fv}, \mathcal{O})) \\
(177) & \simeq Rl^1_*(RHom_{(X/\Sigma_{\infty})}^{fv}(G_{p}^{*,fv}, \mathcal{O})) \\
(178) & \simeq Ext_{(X/\Sigma_{\infty})}^{1}(G^{*,fv}, \mathcal{O})
\end{align*}

where (173) - (177) are trivial and (178) is by 7.12 (iii)

\[\square\]

**Definition 7.16.** Let $X$ in $\text{Sch}/\Sigma_{1}$ and $M$ in $\mathcal{M}(X)$. We set

\[D_X(M) := D(M_{p^{\infty}})\]

where $M_{p^{\infty}}$ is the $p$-divisible group associated to $M$ (2.34 (ii)). This defines a functor $D_X : \mathcal{M}(X) \to \mathcal{D}_{CRYS, fl}(X)$ which is canonically pseudo functorial with respect to $X$ and $e$-exact if $X$ is regular (2.33).

Another basic input is the following calculation from [BM].

**Proposition 7.17.** Let $X$ in $\text{Sch}/\Sigma_{1}$.

(i) There are canonical isomorphisms in $\mathcal{D}_{CRYS, fl}(X)$:

\[D_X(\mathbb{Z}) := D(\mathbb{Q}_{p}/\mathbb{Z}_{p}) \simeq (\mathcal{O}, p, 1)\]

\[D_X(\mathbb{Z}(1)) := D(\mu_{p^{\infty}}) \simeq (\mathcal{O}, 1, p)\]

(ii) Assume that $X$ is regular. There is a commutative diagram of abelian groups

\[
\begin{array}{ccc}
\Ext^1_{\mathcal{M}(X)}(\mathbb{Z}, \mathbb{Z}(1)) & \xrightarrow{\delta} & \mathbb{G}_m(X) \\
|D_X| & & \text{nat} \\
\Ext^1_{\mathcal{D}_{CRYS, fl}(X)}(D_X(\mathbb{Z}), D_X(\mathbb{Z}(1))) & \xrightarrow{(6)} & \Ext^1_{\mathcal{X}^{1}(\Sigma_{\infty})^{t, v}, \mathcal{O}}((\mathcal{O}, p, 1), (\mathcal{O}, 1, p)) \xrightarrow{adj} \Ext^1_{\mathcal{X}^{1}(\Sigma_{\infty})^{t, v}, ((\mathbb{Z}, p, 1), (\mathcal{O}, 1, p))} \\
\end{array}
\]

where $\delta$ denotes the obvious isomorphism $x \mapsto [Kum(x)]$ and $|D_X|$ denotes the map induced by the $e$-exact functor $D_X$ defined in 7.16.

Proof. [1] Since the structural ring $\mathcal{O}$ of $(X/\Sigma_{\infty})_{CRYS, fl}$ is $p$-torsion the natural morphism $(\mathbb{Q}_{p}/\mathbb{Z}_{p})^{fv} = (\mathbb{Q}_{p}/\mathbb{Z}_{p}, p, 1) \to (\mathbb{Z}, p, 1)[1]$ of $D((X/\Sigma_{\infty})^{fv})$ induces isomorphisms

\[\mathcal{O}^{L}(\mathbb{Q}_{p}/\mathbb{Z}_{p})^{fv} \simeq (\mathcal{O}, p, 1)[1]\]

\[RHom_{(X/\Sigma_{\infty})}^{fv}((\mathbb{Q}_{p}/\mathbb{Z}_{p})^{fv}, \mathcal{O}) \simeq (\mathcal{O}, 1, p)[-1]\]

in $D((X/\Sigma_{\infty})^{fv}, \mathcal{O})$. The announced computation of $D_X(\mathbb{Z})$ and $D_X(\mathbb{Z}(1))$ follows immediately (using 7.15 (ii) for the first one):

134
\[ D_X(\mathbb{Z}) := D(\mathbb{Q}_p/\mathbb{Z}_p) \]
\[ B((\mathbb{Q}_p/\mathbb{Z}_p)^{f_v})[-1] \]
\[ R\text{Hom}^{f_v}_{\mathcal{O}(X/\Sigma_{\infty})}((\mathcal{O}, 1, p)[-1], \mathcal{O})[-1] \]
\[ (\mathcal{O}, p, 1) \]

\[ D_X(\mathbb{Z}(1)) := D(\mu_{p\infty}) \]
\[ \approx \mathcal{E}\text{xt}^{f_v, 1}_{(X/\Sigma_{\infty})}((\mathbb{Q}_p/\mathbb{Z}_p)^{f_v}, \mathcal{O}) \]
\[ \approx (\mathcal{O}, p, 1) \]

The following claim will be needed for the proof of \([\text{iii}]\).

**Claim.** The above isomorphism \( D(\mu_{p\infty}) \simeq (\mathcal{O}, 1, p) \) coincides with

\[ D(\mu_{p\infty}) \xrightarrow{[7.12\text{[ii]}]} B((\mathbb{Q}_p/\mathbb{Z}_p)^{f_v})[-1] \xrightarrow{\log} B_\mathcal{O}((\mathcal{O}, 1, p)) \xrightarrow{\sim} (\mathcal{O}, 1, p) \]

where \( \log \) is obtained by applying \( B_\mathcal{O} \) to the morphism \( \mathcal{O} \otimes^L (\mu_{p\infty})^{f_v} \to (\mathcal{O}, 1, p)[1] \) of \( D((X/\Sigma_{\infty})^{f_v}, \mathcal{O}) \) deduced from \((169)\) (apply \([7.12]\) with \( M = (\mathcal{O}, 1, p) \)).

Let us prove the claim. To begin with we notice that for any module or abelian group \( M \) we have a natural morphism

\[ \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\tau \leq i} R\text{Hom}^{f_v}_{(X/\Sigma_{\infty})}((\mu_{p\infty})^{f_v}, M) \to \lim_{\tau \leq i} R\text{Hom}^{f_v}_{(X/\Sigma_{\infty})}((\mu_{p})^{f_v}, M) \]

(since \( \mu_{p\infty} \simeq \mathbb{Z}/p \otimes^L \mu_{p\infty}[-1] \)) which is an isomorphism if and only if

\[ \mathcal{T}\text{or}_1(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{E}\text{xt}^{f_v, i+1}_{(X/\Sigma_{\infty})}((\mu_{p\infty})^{f_v}, M)) = 0 \]

This is in particular the case if \( M = \mathcal{O}, i = 1 \) (\([\text{BBM} 3.3.3\text{ (iii)}]\)) or \( M = \mathbb{G}_m, i = 0 \) (use \([7.12]\text{[ii]}\) together with the vanishing of \( \mathcal{E}\text{xt}^{f_v, 1}_{(X/\Sigma_{\infty})}(\mu_{p}, \mathbb{G}_m) \) (\([\text{BBM} 1.1.8, \text{SGA7-II VIII, 3.3.1}]\) and the vanishing of \( \mathbb{Z}/p^k \otimes^L R^1\mu_{\infty, X}(X/\Sigma_{\infty})(\mu_{p}, \mathbb{G}_m) \) (\([4.5]\))).

This observation explains the second commutative square in the following diagram

Let us explain the other commutative squares. The first commutative square is clear from the construction of the isomorphism \([7.13\text{[ii]}]\). The fourth one is obvious. To get the third one we notice that each corner is in fact concentrated in degree 0 so that we may
drop the superscripts $fv$ to check commutativity. Then we are done using the following commutative square of abelian groups in $(X/\Sigma_{\infty})_{CRYS, fl}$:

$$
\begin{array}{ccc}
\Ext^1_{(X/\Sigma_{\infty})}(\mu_p, \mathcal{O}) & \xrightarrow{\log: \mathbb{G}_m \to \mathcal{O}[1]} & \Hom_{(X/\Sigma_{\infty})}(\mu_p, \mathbb{G}_m) \\
\Hom_{\mathcal{O}((X/\Sigma_{\infty}))}(\mathcal{O}, \mathcal{O}) & \xrightarrow{\log: \mu_p \to \mathcal{O}} & \mathbb{Z}
\end{array}
$$

The claim is now proven.

In the diagram below we let [i] denote the arrow induced by the isomorphisms established in [i]

The claim established in [i] implies that the pentagon on the left is commutative. The commutativity of the rest of the diagram causes no difficulty and will be left to the reader. Let us only give some explanations concerning the definition of some arrows. The arrow denoted $(-)_{p\infty}$ is well defined thanks to 2.34(ii) and 2.39(ii). The arrow denoted $\text{nat}$ follows from the following observation: for $x: \mathbb{Z} \to \mathbb{G}_m$ the morphism $F^{-1}x : F^{-1}\mathbb{Z} \to F^{-1}\mathbb{G}_m$ translates as $1 \mapsto x(1)^p$ if one makes the obvious identifications $F^{-1}\mathbb{Z} = \mathbb{Z}$ and $F^{-1}\mathbb{G}_m = \mathbb{G}_m$. The arrows denoted $\log$ are deduced from $\mathcal{O} \otimes^L (\mathbb{G}_m)^{fv} \to (\mathcal{O}, 1, p)[1]$ by applying $B_\mathcal{O}$ and 7.14. The first and third arrow denoted $|B|$ are defined using respectively the isomorphisms $B((\mathbb{Q}_p/\mathbb{Z}_p)[-1]) \simeq B((\mathbb{Q}_p/\mathbb{Z}_p)[-1]$ (by 7.14 and BBM 3.3.3, (iii)) and $B((\mathbb{Z}, p, 1)[1]) \sim B((\mathbb{Z}, p, 1)[1]$ (by 7.14 again).

\[\square\]

**Proposition 7.18.** Let $X$, $R$, $K$, $k$ as in 7.1. Consider furthermore $X^2 = (X, \Spec(k))$ and the natural morphism $o : X^2 \to X$. Denote $|D_X|$ and $|D_K|$ the homomorphisms on $\Ext^1$'s induced by the functors defined by 7.13(i), 5.33(ii) respectively for $X$ and $\Spec(K)$. There exists a canonical homomorphism $|D_X^2|$ rendering the following diagram...
We define the desired dotted arrows as follows.

- The morphisms $\log$ are deduced from (180) and the logarithm $1 + \mathcal{I}_{X^\flat/\Sigma_\infty} \to \mathcal{O}_{X^\flat/\Sigma_\infty}$.

Here $M^{gp}_{X^\flat}$ denotes the sheaf of groups on $FL^\sharp(X^\flat)$ defined by the collection of sheaves $M^{gp}_{U^\flat}$ (the fraction group of the structural monoid on $et(U^\sharp)$) together with the natural functoriality morphisms for $U^\sharp$ varying in $Sch^\sharp/X^\sharp$. That this is indeed a sheaf is explained in the proof of [Ka3] thm 3.2 and relies essentially on the fact that small étale sheaves (and thus in particular morphisms of such) satisfy fl descent ([SGA4-II] VIII, 9.4). As usual $M^{gp}_{X^\sharp}$ is viewed as a sheaf on $CRYST\sharp(X^\sharp/\Sigma_\infty)$ via $i_*$. It should not be confused with $M^{gp}_{X^\sharp/\Sigma_\infty} : (U^\sharp, T^\sharp) \mapsto \Gamma(T^\sharp, M^{gp}_{T^\sharp})$ (which is also sheaf for similar reasons). Both are related by the following exact sequence:

$0 \to 1 + \mathcal{I}_{X^\sharp/\Sigma_\infty} \to M^{gp}_{X^\sharp/\Sigma_\infty} \to M^{gp}_{X^\sharp} \to 0$
It now remains to check that if an extension class \([E]\) is produced by the dotted vertical arrow from an element of \(\Gamma(X^\sharp, M_{X^\sharp})\) then \(E\) is in fact in \(\mathcal{D}C_{\text{CRY S}^\sharp, fl}(X^\sharp)\). First we note that \(E\) is a locally free crystal since it is an extension of two copies of \(\mathcal{O}\). Finally we notice that the condition \(fv = p, vf = p\) can be checked after pulling back to \(((\text{Spec}(K)/\Sigma_\infty)_{\text{CRY S}^\sharp, fl}, \mathcal{O})\) (observe realizations at a lifting).

\[\square\]

### 7.2.5.

We are now in position of defining the Dieudonné functor for log 1-motives and semi-abelian schemes over \(X\). We use the following notations

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{j} & X^\sharp \\
\downarrow j^\sharp & & \downarrow \circ \\
X & & X
\end{array}
\]

for the obvious morphisms.

**Corollary 7.19.** There exists a canonical \(e\)-exact functor \(D_{X^\sharp}\) fitting into a canonically pseudo-commutative diagram as follows:

\[
\begin{array}{cccccc}
\mathcal{M}(X) & \xrightarrow{\subset} & \mathcal{M}_{\text{log}}(X) & \xrightarrow{j^{-1}} & \mathcal{M}(K) \\
\mathcal{D}_X & \mathcal{D}_{X^\sharp} & \mathcal{D}_K \\
\mathcal{D}C(X) & \xrightarrow{\circ^*} & \mathcal{D}C(X^\sharp) & \xrightarrow{j^{!*}} & \mathcal{D}C(K)
\end{array}
\]

This diagram is moreover canonically pseudo-functorial with respect to finite étale base change.

Proof. This directly results from 7.8 and 7.18.

\[\square\]

**Definition 7.20.** Consider a semi-abelian scheme \(A/X\).

1. The log 1-motive of \(A\) and the 1-motive of \(A_{\eta}\) are defined respectively as

\[
\begin{align*}
M_{\text{log}}(A) & := (\Gamma, G, f) \quad \text{in } \mathcal{M}_{\text{log}}(X) \\
M(A_{\eta}) & := (\Gamma_{\eta}, G_{\eta}, f) \quad \text{in } \mathcal{M}(\eta)
\end{align*}
\]

where \(\Gamma, G\) (resp. \(f\)) are as in 3.19 (resp. 3.25 (i)).

2. The Dieudonné crystal of \(A/X\) is defined as follows:

\[
D_{X^\sharp}(A) := D_{X^\sharp} (M_{\text{log}}(A/X))
\]

Let us write down explicitly how the Dieudonné crystals of \(A, G\) and \(\Gamma\) are related.

**Proposition 7.21.**

1. The weight filtration on \(M_{\text{log}}(A)\) induces a canonical exact sequence as follows in \(\mathcal{D}C(X^\sharp)\):

\[
\begin{array}{cccccc}
0 & \longrightarrow & D_{X^\sharp}(0, G, 0) & \longrightarrow & D_{X^\sharp}(M_{\text{log}}(A)) & \longrightarrow & D_{X^\sharp}(\Gamma, 0, 0) & \longrightarrow & 0
\end{array}
\]

2. The weight filtration of \(M(A_{\eta})\) and the bottom exact sequence of 3.25 (iii) induce isomorphic exact sequences as follows in \(\mathcal{D}C(\eta)\):

\[
\begin{array}{cccccccc}
0 & \longrightarrow & D_{\eta}(0, G_{\eta}, 0) & \longrightarrow & D_{\eta}(M(A_{\eta})) & \longrightarrow & D_{\eta}(\Gamma_{\eta}, 0, 0) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_{\eta}(G_{\eta, p^\infty}) & \longrightarrow & D_{\eta}(A_{\eta, p^\infty}) & \longrightarrow & D_{\eta}(\mathbb{Q}_p/\mathbb{Z}_p \otimes \Gamma_{\eta}) & \longrightarrow & 0
\end{array}
\]

138
The top exact sequence is moreover naturally isomorphic to the one deduced from (i) by $j^\ast : DC(X^\delta) \to DC(\eta)$.

Proof. (i) This follows from the fact that $D_{X^\delta} : \mathcal{M}_{log}(X) \to DC(X^\delta)$ is exact.
(ii) The compatible exact sequences are deduced from 3.23 (ii) by applying the $\mathcal{e}$-exact functor $D_\eta : pdiv(\eta) \to DC(\eta)$. The last assertion is by compatibility of the functors $D_{X^\delta}$ and $D_\eta$ (7.19).

7.2.6. We are now in position to define the Dieudonné crystal of a semi-abelian scheme over $C$.

Definition 7.22. Let $\text{SAS}(C, Z)$ denote the category of semi-abelian schemes over $C$ which are abelian above $U$. We define a functor $D_{C^\delta} : \text{SAS}(C, Z) \to DC(C^\delta)$ by sending $A/C$ to the Dieudonné crystal over $(C^\delta/Z_p)$ corresponding to $m^\ast D_{C^\delta}(A/C) := (D_{C^\delta}(A|_{C^\delta}) \to D_{U^\delta}(A|_{U^\delta}) \leftarrow D_{U}(A|_{U}))_{v \in \mathbb{Z}}$ in $DC(J^\delta)$ under the equivalence 6.1 (i).

Note that this definition uses the isomorphism $j^\ast v D_{C^\delta}(A|_{C^\delta}) \simeq D_{U^\delta}(A|_{U^\delta})$ of 7.21 (ii).

The following result explains the relation with the Dieudonné crystal arising from the diagram of $p$-divisible groups introduced in 3.25.

Lemma 7.23. Consider the restriction $H|_J$ to $J$ of the diagram of $p$-divisible groups $H$ defined in 3.25. We have a canonical exact sequence as follows in $DC(J^\delta)$:

$$0 \to o^\ast J(H|_J) \to m^\ast D_{C^\delta}(A/C) \to (o^\ast J((Q_p/Z_p \otimes \Gamma_v \to 0 \leftarrow 0)_{v \in \mathbb{Z}}) \to 0$$

Proof. This results directly from 7.21.

8. The comparison theorem

8.1. Diagrams of $p$-divisible groups.

The purpose of this paragraph is to establish the comparison theorem for $p$-divisible groups over a diagram of schemes whose vertices have local finite $p$-bases. This will be achieved in 8.9 for $Q_p/Z_p$ and $\mu_{p^\infty}$ and in 8.14 for arbitrary $p$-divisible groups using Cartier biduality.

8.1.1. Let us begin with technical observations regarding the computation of $\mathcal{E}xt^q$'s on diagrams.

Lemma 8.1. Consider a ringed variable topos $(\mathcal{T}, A)$ on a category $\mathcal{B}$ and let $X/\Delta$ in $\text{Diag}(\mathcal{B})$.

(i) Consider an abelian group $M$ of $\mathcal{T}(X)$ and a module $N$ over $(\mathcal{T}(X), A_X)$. If $M$ is cocartesian (i.e. $f^{-1}M_\delta \simeq M_{f^\delta}$ for all $f : \delta \to \delta$ in $\Delta$) then the following natural morphism of $D^+(\mathcal{T}(X_\delta), A_{X_\delta})$ is invertible for all $\delta \in \Delta$:

$$R\mathcal{H}om_{\mathcal{T}(X)}(M, N)_\delta \to R\mathcal{H}om_{\mathcal{T}(X_\delta)}(M_\delta, N_\delta)$$
(ii) Consider $\mathcal{H}om_{\mathcal{T}(X)}$ as a bifunctor $\text{Ab}(\mathcal{T}(X))^{\text{op}} \times \text{Mod}(\mathcal{T}(X), A_X) \to \text{Mod}(\mathcal{T}(X), A_X)^{\text{top}}$. This bifunctor is right derivable into $D^-(\mathcal{T}(X))^{\text{op}} \times D^+(\mathcal{T}(X), A_X) \to D^+(\mathcal{T}(X), A_X)^{\text{top}}$ and there is a bifunctorial isomorphism

$$R\mathcal{H}om_{\mathcal{T}(X)}(\lim M_k, N) \simeq Rl_* \mathcal{H}om_{\mathcal{T}(X)}(M, N)$$

Proof. (i) Let us compute the $A_X$-module $\mathcal{H}om_{\mathcal{T}(X)}(M, N)$. Consider $\delta$ in $\Delta$, $U$ in $\mathcal{T}(X)$ and let $\mathcal{T}(X/U)$ denote the topos of sections of the cofibered category over $\Delta/\delta$ whose fiber at $g : \delta' \to \delta$ is $\mathcal{T}(X_{\delta'})/g^{-1}U$. Let $h : \mathcal{T}(X/U) \to \mathcal{T}(X)$ denote the morphism induced by $\Delta/\delta \to \Delta$, $g \mapsto \delta'$ and the localization morphisms $\mathcal{T}(X_{\delta'})/g^{-1}U \to \mathcal{T}(X_{\delta'})$. Use [SGA4-II] VI, 7.4.7 we find a natural isomorphism

$$\mathcal{H}om_{\mathcal{T}(X)}(M, N)_\delta(U) \simeq \mathcal{H}om_{\mathcal{T}(X/U)}(h^{-1}M, h^{-1}N)$$

Via this identification the image by $H^0$ of the natural morphism in question translates into

$$\mathcal{H}om_{\mathcal{T}(X/U)}(h^{-1}M, h^{-1}N) \longrightarrow \mathcal{H}om_{\mathcal{T}(X)}(M_\delta|U, N_\delta|U)$$

and is thus an isomorphism since $M$ is cocartesian. The case of derived functors follows since $N_\delta$ is flasque if $N$ is injective [2.11(v)].

(ii) Right derivability causes no difficulty (use injective resolutions of the second argument). The proof of the claimed isomorphism is similar to (and easier than) the proof of 7.12 (iii). □

Let us now gather some results about the behaviour of $\mathcal{E}xt^i$’s while traveling through the various topoi which will be involved in our proof. The next statement and later ones implicitly use 2.39 to switch between the various incarnations of a $p$-divisible group.

Lemma 8.2. Let $X$ in $\text{Sch}/\Sigma_1$ and $\Sigma = \Sigma_k$, $1 \leq k \leq \infty$. Let $\mathfrak{f}l \leq \top \leq \text{zar}$ (e.g. top $= \text{syn}$) and let $\epsilon$ denote the quasi-morphism from $\mathfrak{f}l$ to top.

(i) Let $G/X$ denote a quasi-compact quasi-separated group (resp. a $p$-divisible group viewed as a group in $(X/\Sigma)_{\text{CRY S}, \text{fl}}$). Consider a quasi-coherent crystal $M$ of $((X/\Sigma)_{\text{CRY S}, \text{fl}}, \mathcal{O})$ (resp. and assume that top $\leq \text{syn}$). Then for $0 \leq i \leq 2$ and $n \geq 0$:

$$\epsilon_* \mathcal{E}xt^i_{(X/\Sigma)_{\text{CRY S}, \text{fl}}}(G, \mathcal{T}^{[n]}M) \simeq \mathcal{E}xt^i_{(X/\Sigma)_{\text{CRY S}, \text{top}}}(\epsilon_* G, \epsilon_* \mathcal{T}^{[n]}M)$$

(ii) Assume $k < \infty$. Let $G/X$ denote a $p$-divisible group viewed as a group in $X_{\text{syn}}$. If $M$ is a module of $((X/\Sigma)_{\text{CRY S}, \text{syn}}, \mathcal{O})$ then for $0 \leq i \leq 1$:

$$u_* \mathcal{E}xt^i_{(X/\Sigma)_{\text{CRY S}, \text{syn}}}(u^{-1}G, M) \simeq \mathcal{E}xt^i_{X_{\text{SYN}}}(G, u_* M)$$

The same is true for syn and crys instead of $\text{SYN}$ and $\text{CRY S}$.

(iii) Consider a syntomic group scheme (resp. a $p$-divisible group) $G/X$ and let $0 \leq i \leq 2$ (resp. $0 \leq i \leq 1$).

- If we view $G$ as a group in $X_{\text{SYN}}$ and if $M$ is any abelian group of $X_{\text{SYN}}$ (resp. which is killed by a power of $p$) then:

$$p_* \mathcal{E}xt^i_{X_{\text{SYN}}}(G, M) \simeq \mathcal{E}xt^i_{X_{\text{syn}}}(p_* G, p_* M)$$

- If we view $G$ as a group of $((X/\Sigma)_{\text{CRY S}, \text{syn}}$ and if $M$ is an abelian group of $(X/\Sigma)_{\text{CRY S}, \text{syn}}$, satisfying $M \simeq i_* i^{-1}M$ (resp. and which is killed by a power of $p$)
Lemma 8.4. Consider $X$ in $\mathcal{S}ch/\Sigma_1$ and a $p$-divisible group $G/X$.

(i) Let $1 \leq k < \infty$. If we view $G$ as an abelian group in $X_{\text{SYN}}$ then:

$$\mathcal{E}xt^k_{\mathcal{S}ch/\Sigma_1}(G, \mu_p) = 0$$

and

$$\mathcal{E}xt^k_{X_{\text{SYN}}}(G, \mathcal{O}) = 0$$

The same is true with syn instead of SYN.

(ii) Let $k \leq 1 \leq \infty$, then:

$$\mathcal{E}xt^2_{(X/\Sigma_k)_{\text{CRY,S,syn}}}(G, \mathcal{O}) = 0$$

Remark 8.3. Thanks to 8.1, the isomorphisms of 8.3 have obvious variants in the case $\Sigma = \Sigma_1$ (replace $G$ with $t^{-1}G$ in the formulae).

The following lemma relies on the vanishing theorem of [Br] and will be a cornerstone in our proof.

Lemma 8.4. Consider $X$ in $\mathcal{S}ch/\Sigma_1$ and a $p$-divisible group $G/X$.

(i) Let $1 \leq k < \infty$. If we view $G$ as an abelian group in $X_{\text{SYN}}$ then:

$$(iv) \text{ Let } G/X \text{ denote a } p\text{-divisible group viewed as an abelian group of } (X/\Sigma)_{\text{CRY,S,top}}. \text{ If } M \text{ is a quasi-coherent crystal of } ((X/\Sigma)_{\text{CRY,S,top}}, \mathcal{O}) \text{ then for } 0 \leq i \leq 2, n \geq 0:\n
\mathcal{E}xt^i_{(X/\Sigma)_{\text{CRY,S,top}}}(G, \mathcal{I}^n M) \simeq \lim_{\leftarrow j} \mathcal{E}xt^i_{(X/\Sigma)_{\text{CRY,S,top}}}(G_{p^j}, \mathcal{I}^n M)$$

(v) Let $G/X$ denote a $p$-divisible group viewed as an abelian group of $X_{\text{TOP}}$. If top is finer or equal than syn and if $M$ is an abelian group of $X_{\text{TOP}}$ killed by a power of $p$ then for $0 \leq i \leq 1$:

$$\mathcal{E}xt^i_{X_{\text{TOP}}}(G, M) \simeq \lim_{\leftarrow j} \mathcal{E}xt^i_{X_{\text{TOP}}}(G_{p^j}, M)$$

The same is true with top instead of TOP.

Proof. (i) In the first case the proof of [BBM] 2.3.11 works here as well (just notice that the $\text{CRY,S,top}$ analogue of loc. cit. 1.1.19 and 1.3.6 is true and then use directly loc. cit. 2.3.1, 2.3.9). Similarly (iv) is a straightforward adaptation of [BBM] 2.4.5. The second case of (i) follows from the first one using (iv) and 2.38.

(v) This follows from 8.1(ii) using that $Ri_!\mathcal{H}om_{X_{\text{TOP}}}(G_{p^i}, M) = 0$ (thanks to the fact that the transition morphisms $\mathcal{H}om_{X_{\text{TOP}}}(G_{p^{i+1}}, M) \to \mathcal{H}om_{X_{\text{TOP}}}(G_{p^i}, M)$ are zero if $M$ is killed by $p^i$).

The first assertion is [Ba] 1.18 if $G$ is a syntomic group scheme. The case of $p$-divisible groups follows by 2.30 (v) and $\mathcal{E}xt^i$. The second assertion follows from the first one since $i_*$ commutes to $p_*$ and $\mathcal{E}xt^i$.

(ii) The claimed isomorphism is obtained from

$$Ru_*R\mathcal{H}om_{(X/\Sigma)_{\text{CRY,S,syn}}}(u^{-1}G, M) \simeq R\mathcal{H}om_{X_{\text{SYN}}}(G, Ru_*M)$$

by forming $H^1$. The computation of the left hand term uses the vanishing of $\mathcal{H}om_{(X/\Sigma)_{\text{CRY,S,syn}}}(u^{-1}G, M)$ (which follows from the fact that $p$ is epimorphic on $u^{-1}G$ while $M$ is killed by $p^k$) and the computation of the right hand term uses the vanishing of $\mathcal{H}om_{X_{\text{SYN}}}(G, R^1u_*M) = 0$ (which follows from the fact that $p$ is epimorphic on $G$ while $R^1u_*M$ is killed by $p^k$). The same arguments apply in the case of small topoi.

□
Proof. [i] Let us examine the first case. The starting point is the vanishing of \( \mathcal{E}xt^i(G_{p^l}, \mathcal{G}_m) \) for \( i = 1, 2 \) which is an easy consequence of SGA7-II VIII, 3.3.1 for \( i = 1 \) (use that \( G_{p^l}^* \) is acyclic for \( \epsilon_*, \epsilon : X_{FL} \to X_{SYN} \)) and of the main result of [Br] for \( i = 2 \) (here we use that \( p \neq 2 \)). In the case of \( X_{syn} \) use furthermore [8.2(iii)]. Using the Kummer exact sequence \( 0 \to \mu_{p^k} \to \mathbb{G}_m \to \mathbb{G}_m \to 0 \) we obtain the following facts:
- \( \mathcal{E}xt^2(G_{p^l}, \mu_{p^k}) = 0; \)
- for \( l' \geq l \) the morphism
\[
\mathcal{E}xt^1(G_{p^{l'}}, \mu_{p^k}) \to \mathcal{E}xt^1(G_{p^l}, \mu_{p^k})
\]
identifies to the morphism \( G_{p^{l'}}/p^k \to G_{p^l}/p^k \) induced by \( p^{l' - l} : G_{p^{l'}} \to G_{p^l}^* \). It is thus an isomorphism for \( l \geq k \).

The vanishing of \( \mathcal{E}xt^2(G, \mathcal{O}) \) will follow from the spectral sequence
\[
R^j \mathcal{E}xt^i(G_p, \mu_{p^k}) \Rightarrow \mathcal{E}xt^{i+j}(G, \mathcal{O})
\]
\([8.1(ii)]\) once checked that \( R^{2-j} \mathcal{E}xt^i(G_p, \mu_{p^k}) \) vanishes for \( j = 0, 1, 2 \). Recall that \( R^j \mathcal{M} \) can be computed by sheafifying \( U \mapsto R\limproj_k R\Gamma(U, \mathcal{M}) \). For \( j = 1, 2 \) the desired vanishing follows directly from the above facts. For \( j = 0 \), it results from the fact that
\[
\mathcal{H}om(G_{p^{l'}}, \mu_{p^k}) \to \mathcal{H}om(G_{p^l}, \mu_{p^k})
\]
is zero for \( l' \geq l + k \).

The vanishing of \( \mathcal{E}xt^2(G, \mathcal{O}) \) is proved in the same way using the following facts which result from [BBM] 3.3.2 and its proof (thanks to the isomorphisms \( \epsilon_* \mathcal{E}xt^i_{X_{FL}}(G_{p^l}, \mathcal{O}) \simeq \mathcal{E}xt^i_{X_{SYN}}(\epsilon_* G_{p^l}, \epsilon_* \mathcal{O}) \)
\([1.2(i)]\) and \( p_* \mathcal{E}xt^i_{X_{SYN}}(G_{p^l}, \mathcal{O}) \simeq \mathcal{E}xt^i_{X_{syn}}(p_* G_{p^l}, p_* \mathcal{O}) \)
\([8.2(iii)]\):
- \( \mathcal{E}xt^2(G_{p^l}, \mathcal{O}) = 0; \)
- for \( l' \geq l \) the following morphism is invertible:
\[
\mathcal{E}xt^1(G_{p^{l'}}, \mathcal{O}) \to \mathcal{E}xt^1(G_{p^l}, \mathcal{O})
\]
- for \( l' \geq l + 1 \) the following morphism is zero:
\[
\mathcal{H}om(G_{p^{l'}}, \mathcal{O}) \to \mathcal{H}om(G_{p^l}, \mathcal{O})
\]

\([ii]\) This is [BBM] 3.3.3, 3.3.4 modulo [8.2(i)].

\[\Box\]

Remark 8.5. (i) In [8.2(i)] \( X_{SYN} \) or \( X_{syn} \) (resp. \( G \)) can be replaced by \( X_{SYN}^N \) or \( X_{syn}^N \) (resp. the constant projective system \( l^{-1} G \)) thanks to [8.2(ii)].
(ii) We do not know whether or not the analogue of [8.2(ii)] holds on the small crystalline topos. Unpleasantly this will force us to switch frequently between small and big topos.

8.1.2. Let us come to the techniques of [FM] which are needed for our purpose. We begin with the description of \( \mathcal{O}_{k, c}^{cys} \) using divided power envelopes of Witt vectors and syntomic sheafification.

Lemma 8.6. Consider the presheaf \( W_{dp}^{dp} \) on \( SYN(\Sigma_1) \) sending \( U \) to the divided power envelope of \( W_k(U) \) with respect to the kernel \( I_k(U) \) of \( W_k(U) \to \Omega(U) \), \( (x_0, \ldots, x_k) \mapsto x_0^k \). Let \( I_k^{dp}(U) \) denote the tautological dp-ideal of \( W_{dp}^{dp}(U) \).

(i) The sheaf associated to \( W_{dp}^{dp} \) is canonically isomorphic to \( \mathcal{O}_{k, c}^{cys} \) in \( \Sigma_1, SYN \). The ring \( \mathcal{O}_{c}^{cys} \) is in particular normalized.
(ii) If \( U \) is an affine semi-perfect \( \Sigma_1 \)-scheme (i.e. \( U = \text{Spec}(A) \) and the Frobenius of \( A \) is surjective) the isomorphism of (i) induces 
\[
W_k^\text{dp}(U) \simeq \mathcal{O}_k^\text{crys}(U)
\]

(iii) If \((U_i)_{i \in I}\) is a filtrant projective system of affine \( \Sigma_1 \)-schemes and \( U_\infty = \limproj U_i \) then
\[
\lim \mathcal{O}_k^\text{crys}(U_i) \simeq \mathcal{O}_k^\text{crys}(U_\infty)
\]

(iv) If \( U \) is an affine \( \Sigma_1 \)-scheme, there exists a filtrant projective system \((U_i/U)_{i \in I}\) of affine syntomic surjective \( U \)-schemes whose transition morphisms are syntomic surjective as well and such that \( U_\infty \) semi-perfect. If there exists a closed immersion \( U \hookrightarrow Y \) where \( Y \) has finite \( p \)-bases over \( \Sigma_1 \) then we may chose \( I = \mathbb{N} \) and \( U_i = U^{(p^i/Y)} \). If the ideal of the closed immersion is moreover generated by a regular sequence, then \( W_k^\text{dp}(U_\infty) \) is flat over \( \mathbb{Z}/p^k \).

Proof. Recall that for any \( U/\Sigma_1 \) we have \( \mathcal{O}_k^\text{crys}(U) = \limproj \mathcal{O}(T') \) where the projective limit is indexed by \((U', T', \gamma')\) in \( \text{CRYS}_{\text{syn}}(U/\Sigma_k) \) and does not change if only affine \( T' \)s are considered. Thanks to the existence of \( \gamma' \) the ring homomorphism
\[
\begin{align*}
W_k(U) &\to \mathcal{O}(T') \\
(x_0, x_1, \ldots, x_{k-1}) &\mapsto (\tilde{x}_0)^{p^k} + p(\tilde{x}_1)^{p^{k-1}} + \cdots + p^{k-1}(\tilde{x}_k)^p
\end{align*}
\]
is well defined if \( \tilde{x}_i \in \mathcal{O}(T') \) denotes an arbitrary lift of the image of \( x_i \) in \( \mathcal{O}(U') \). Passing to divided powers and letting \((U', T', \gamma')\) and \( U \) vary this defines a morphism of presheaves of rings on \( \text{SYN}(\Sigma_1) \):
\[
\Theta_k : W_k^\text{dp} \to \mathcal{O}_k^\text{crys}
\]
We will prove that \( \Theta_k \) induces the isomorphism announced in (i) after sheafification. The other statements will be proved along the way.

Clearly Since \( F_A : x \mapsto x^p \) is surjective, the ring homomorphism \((x_0, \ldots, x_k) \mapsto x_0^{p^k} \) is surjective and defines a \( pd \)-thickening \((U, T) := (\text{Spec}(A), \text{Spec}(W_k^\text{dp}(A))) \). Let us check that \((U, T) \) is final in \( \text{CRYS}_{\text{syn}}(U/\Sigma_k) \). Consider a \( pd \)-ideal \( J \) of some \( \mathbb{Z}/p^k \)-algebra \( B \). Any ring homomorphism \( f : A \to B/J \) lifts uniquely to \( \tilde{f} : W_k(A) \to B \) by the formula
\[
(181) \quad \tilde{f}(x_0, x_1, \ldots, x_{k-1}) = (\tilde{x}_0)^{p^k} + p(\tilde{x}_1)^{p^{k-1}} + \cdots + p^{k-1}(\tilde{x}_k)^p
\]
To see this we first note that thanks to divided powers on \( J, \tilde{f}([x_0^{p^k}]) = (\tilde{f}([x_0]))^{p^k} \) is uniquely determined by the image \( f(x_0^{p^k}) \) of \( \tilde{f}([x_0]) \) in \( B/J \) and we conclude using the formula \( (x_0, x_1^p, \ldots, x_{k-1}^{p^{k-1}}) = [x_0] + p[x_1] + \cdots + p^{k-1}[x_{k-1}] \) in \( W_k(A) \).

Using a construction similar to the proof of 5.6 (iv) (here the finiteness of \( \delta/\Delta \) is not required) we find easily a projective system \((U_i, Y_i)_{i \in I}\) where \( Y_i = \text{Spec}(B_i) \), with \( B_i \) is a polynomial algebra \( B_i \) (in possibly infinitely many variables) and \( B_\infty := \limind B_i \) as well (it is in fact an infinite tensor product indexed by \( I \) of polynomial algebras). For \( i \in I \cup \{ \infty \} \) let us denote respectively \( T_i \) and \( T_i^{(1)} \) the divided power envelope of \( U_i \) inside \( Y_i = \text{Spec}(B_i) \) and \( Y_i \times Y_i \). Then \( \mathcal{O}_k^\text{crys}(U_i) \simeq \text{Ker}(\mathcal{O}(T_i) \rightarrow \mathcal{O}(T_i^{(1)})) \) and the claim follows.

Let \( A = \mathcal{O}(U) \) and consider a set \( I := \bigsqcup_{n \geq 1} I_n \) where \( I_1 \) is the set of finite subsets of \( A, A_{1,i} := \otimes_{a \in I} A[X]/(X^p - a), A_1 = \limind A_{1,i}, \) and where for \( n \geq 2, I_n, A_{n,i} \) and \( A_n \) are inductively defined respectively as follows: \( I_n \) is the set of finite subsets in \( A_{n-1}, A_{n,i} := \otimes_{a \in I} A_{n-1}[X]/(X^p - a), A_n = limind A_{n,i} \). The set \( I \) is naturally ordered and the inductive system \( i \in I \mapsto U_i := \text{Spec}(A_i) \) satisfies the required conditions. In the
second case, the choice of a \( p \)-basis \( s = (s_1, \ldots, s_d) \) for \( Y \) induces the right commutative square below

\[
\begin{array}{ccc}
U(p^{i+1}/Y) & \xrightarrow{\delta} & Y \\
\downarrow & & \downarrow F \\
U(p^i/Y) & \xrightarrow{\delta} & Y \\
\end{array}
\]

The right vertical arrow is clearly syntomic surjective. The other vertical ones are thus syntomic surjective as well since both squares are cartesian. Let us explain the last statement. Set \( B = \mathcal{O}(Y) \) and chose a regular sequence \( f = (f_1, \ldots, f_r) \) generating the ideal of \( U \). Then \( U_\infty = \text{Spec}(A_\infty) \) where \( A_\infty = B_\infty/(f_1, \ldots, f_r) \) and \( B_\infty = \lim \text{ind} B \).

Also \( W_k^{dp}(U_\infty) \) is the divided power envelope (compatible with \( (p) \), as always) of the ideal of \( W_k(B_\infty) \) generated by the \([f_i^{p-r}]\)'s. According to [Be3] 1.5.3 (i), it is sufficient to prove that \( W_k(B_\infty)/([f_1^{p-r}], \ldots, [f_r^{p-r}]) \) is flat over \( \mathbb{Z}/p^k \). We do this by induction on \( r \). For \( r = 0 \) this follows from the fact that \( B_\infty \) is perfect. Let us denote \( J_i = ([f_1^{p-r}], \ldots, [f_i^{p-r}]), \) if \( 0 \leq i \leq r \). For \( r \geq 1 \) it is sufficient to prove that \( Tor_1^{\mathbb{Z}/p^k}(\mathbb{Z}/p, W_k(B_\infty)/J_r) \) vanishes, ie. that the following sequence is exact:

\[
0 \rightarrow \mathbb{Z}/p \otimes (J_r/J_{r-1}) \rightarrow \mathbb{Z}/p \otimes (W_k(B_\infty)/J_{r-1}) \rightarrow \mathbb{Z}/p \otimes (W_k(B_\infty)/J_r) \rightarrow 0
\]

Now the image of \([f_i^{p-r}]\) in \( \mathbb{Z}/p \otimes (W_k(B_\infty)/J_{r-1}) \simeq B_\infty/(f_1^{p-r}, \ldots, f_{r-1}^{p-r}) \) is a non zero divisor and thus \( J_r \cap (pW_k(B_\infty) + J_{r-1}) = pJ_r + J_{r-1} \) as desired.

\[\text{(i)}\] Let \( F \) denote either the kernel or cokernel presheaf of the morphism \( \Theta_k \) and let \( U \in \text{SYN}(\Sigma_1) \). Thanks to \[\text{(ii)}\] \[\text{(iii)}\] and \[\text{(iv)}\] we find that for each \( x \in F(U) \) there exists a syntomic surjective family \((U_i \rightarrow U)\) such that the restriction of \( x \) to each \( U_i \) vanishes. It follows in particular that the sheaf associated to \( F \) is zero.

\[\square\]

Let us continue with some technical complements in the setting of small syntomic topoi.

We use the notation

\[\tilde{\mathcal{I}}_{\text{crys}} := \langle 1 \rangle^* \mathcal{I}_{\text{crys}}\]

where \( \langle 1 \rangle \) has the same meaning as in \[\text{5.59}\] (ie. \( \tilde{\mathcal{I}}_{\text{crys}} = \mathcal{I}_{k+1}/p^k \), see \[\text{4.1}\]).

**Lemma 8.7.** Consider a regular closed immersion of affine schemes \( U \rightarrow Y \) where \( Y \) has finite \( p \)-bases over \( \Sigma_1 \).

(i) The modules \( \mathcal{O}_{\text{crys}} \) and \( \tilde{\mathcal{I}}_{\text{crys}} \) of \((U^{|\Sigma_1|}, \mathbb{Z}/p)\) are flat and normalized.

(ii) If \( Y \) is semi-perfect then \( \Theta_k \) induces

\[J_{k+1}^{dp}(U)/p^k \simeq \tilde{\mathcal{I}}_{k}^{\text{crys}}(U)\]

(iii) If \( U_\infty := \lim proj(U^{|\Sigma_1|}/p^i) \) then

\[\lim \tilde{\mathcal{I}}_{k}^{\text{crys}}(U_i) \simeq \tilde{\mathcal{I}}_{k}^{\text{crys}}(U_\infty)\]

\[\text{(i)}\] The case of \( \mathcal{O}_{\text{crys}} \) follows easily from the flatness statement in \[\text{8.6(\text{iv})}\] as in the proof of \[\text{8.6(\text{i})}\]. Alternatively one could refer to \[\text{5.57(\text{iv})}\]. The case of \( \tilde{\mathcal{I}}_{\text{crys}} \) follows by \[\text{4.3(\text{iv})}\].

\[\text{\[\text{iii}\]}\] Starting with the exact sequence defining \( J_{k+1}^{dp}(U) \) (resp. \( \tilde{\mathcal{I}}_{k+1}^{\text{crys}} \)) and applying \[\text{4.3(\text{iv})}\] (resp. and evaluating at \( U \)) gives the top (resp. bottom) exact sequence in the following
commutative diagram:

\[(183) \quad 0 \rightarrow \mathcal{O}(U) \rightarrow I^d_{k+1}(U)/p^k \rightarrow I^d_k(U) \rightarrow 0 \]

\[0 \rightarrow \mathcal{O}(U) \rightarrow \mathcal{I}^{\text{crys}}_k(U) \rightarrow \mathcal{I}^{\text{crys}}_k(U) \rightarrow 0 \]

The result follows since the right vertical arrow is invertible thanks to 8.6 (ii). 

(iii) Because \( \mathcal{O} \) is a cyclic on the \( U_i \)'s we have an exact sequence \( 0 \rightarrow \mathcal{O}(U_i) \rightarrow \mathcal{I}^{\text{crys}}_k(U_i) \rightarrow \mathcal{I}^{\text{crys}}_k(U_i) \rightarrow 0 \). Thanks to 8.7 (iii) we may conclude by taking \( \lim \) and comparing with the bottom line of (183).

\[\square\]

Remark 8.8. (i) The Cartier morphism \( c_1 := C^{-1} : \mathcal{O} \rightarrow \mathcal{O}^{\text{crys}}_1 \) of 5.14 (ii) generalizes as follows. For any \( i \geq 1 \) there is a unique \( c_i \) fitting into a commutative triangle of rings

\[\begin{array}{ccc}
\mathcal{O} & \xrightarrow{\text{nat}} & \mathcal{O} \\
\mathcal{O}^{\text{crys}}_i & \xrightarrow{F} & \mathcal{O}^{\text{crys}}_i \\
\mathcal{O}^{\text{crys}}_i & \xrightarrow{i} & \mathcal{O}^{\text{crys}}_i
\end{array}\]

(ii) It follows e.g. from 8.4 that \( \mathcal{O}^{\text{crys}} \) is normalized in \( (\Sigma_{1,SYN}, \mathbb{Z}/p) \). A result we have a well defined morphism: \( p^{k-i} : \mathcal{O}^{\text{crys}}_i \simeq \mathcal{O}^{\text{crys}}_k/p^i \rightarrow \mathcal{O}^{\text{crys}}_k \). Using this we find that \( \Theta_k \) can be described globally as the morphism deduced from

\[W_k(U) \rightarrow \mathcal{O}^{\text{crys}}_k(U) \]

\[(x_0, x_1, \ldots, x_{k-1}) \mapsto c_k(x_0) + pc_{k-1}(x_1) + \cdots + p^{k-1}c_1(x_{k-1}) \]

by the universal property of divided power envelopes.

(iii) If \( U \) is any scheme of characteristic \( p \) we have a commutative diagram or rings:

\[(184) \quad W^d_{k}(U) \xrightarrow{\Theta_k} \mathcal{O}^{\text{crys}}_k(U) \]

\[F_k \downarrow \quad \text{nat} \quad \text{nat} \]

\[W^d_{1}(U) \xrightarrow{\Theta_1} \mathcal{O}^{\text{crys}}_1(U) \]

\[\mathcal{O}(U) \xrightarrow{\text{nat}} \mathcal{O}(U) \]

This diagram is functorial with respect to \( k \) and \( U \) in the obvious way.

Lemma 8.9. Assume that \( X \) admits locally a closed regular immersion into a \( \Sigma_1 \)-scheme \( Y \) with finite \( p \)-bases.

(i) The following sequence of \( (X^N, \mathcal{O}^{\text{crys}}) \) is exact:

\[0 \rightarrow \mathcal{I}^{\text{crys}} \rightarrow \mathcal{O}^{\text{crys}} \rightarrow \mathcal{O} \rightarrow 0 \]

The modules \( \mathcal{O} \) and \( \mathcal{I} \) of \( (X/\Sigma, \mathcal{O})^{\text{crys, syn}} \) are acyclic for \( u_* \).
(ii) There exists a unique $\phi$ making the following square commutative over $(X^N_{\text{syn}}, \mathcal{O}^{\text{crys}})$

$$
\begin{array}{ccc}
\mathcal{I}^{\text{crys}} & \xrightarrow{\phi} & \mathcal{O}^{\text{crys}} \\
1 & \downarrow & \downarrow p \\
\mathcal{O}^{\text{crys}} & \xrightarrow{F} & \mathcal{O}^{\text{cris}}
\end{array}
$$

(iii) There are canonical exact sequences as follows over $(X^N_{\text{syn}}, \mathbb{Z}/p)$:

$$
0 \rightarrow \mathbb{Z}/p \rightarrow \mathcal{O}^{\text{crys}} \xrightarrow{1-F} \mathcal{O}^{\text{crys}} \rightarrow 0
$$

$$
0 \rightarrow \mu_p \rightarrow \mathcal{I}^{\text{crys}} \xrightarrow{1-\phi} \mathcal{O}^{\text{crys}} \rightarrow 0
$$

(iv) The morphism $\phi$ and the exact sequences of (i), (iii) are functorial with respect to $X$. All statements thus generalize verbatim if $X$ is replaced by a diagram of $\text{Sch}/\Sigma_1$ whose vertices have local finite $p$-bases.

Proof. (i) The exact sequence follows either from 5.57 (iii) or 8.6 (i). The acyclicity of $\mathcal{O}$ (resp. $\mathbb{G}_m$) is by 5.57 (iii) (resp. is immediate) and that of $\mathcal{I}$ follows from the exact sequence.

(ii) Thanks to 8.7 (i) existence and unicity follow from 4.3 (ii), (iii) as in the proof of 5.49, once noticed that the Frobenius endomorphism of $\mathcal{O}^{\text{crys}}_k$ vanishes on $\mathcal{I}^{\text{crys}}_k$ ($x^p = p!x^p = 0$).

(iii) The first exact sequence is left to the reader. Let us explain the second one. Starting with the exact sequence of abelian groups

$$
0 \rightarrow 1 + \mathcal{I} \rightarrow \mathcal{O}^X \xrightarrow{\mu_p} \mathbb{G}_m \rightarrow 0 \quad \text{in} \quad (\Sigma_1/\Sigma_k)_{\text{crys}, \text{syn}}
$$

then combining with $\log : 1 + \mathcal{I} \rightarrow \mathcal{I}$, applying $\mathbb{Z}/p^k \otimes^{\mathbb{L}} (-)$ and $u_*$ we find a morphism $\mu_p^k \rightarrow \mathcal{I}^{\text{crys}}_k$ in $\Sigma_1, \text{SYN}$. Since the Frobenius acts like $p$ on $\mu_p^k$ this morphism factors via the kernel of $p - F : \mathcal{I}^{\text{crys}}_k \rightarrow \mathcal{O}^{\text{crys}}_k$. Next we let $k$ vary, restrict to the small syntomic site of $X$ and apply the functor $1^*$ in order to get a morphism $1^*\mu_p^k \rightarrow 1^*\mathcal{I}^{\text{crys}}_k$ whose composition with $p - F : \mathcal{I}^{\text{crys}}_k \rightarrow \mathcal{O}^{\text{crys}}_k$ is zero. Next we note that $(1)^*\mu_p^k \simeq \mu_p$. Since $p \circ (1 - \phi) = p - F$ and $\mathcal{O}^{\text{crys}}_k$ is flat and normalized over $\mathbb{Z}/p$, this yields by 4.3 (iii) a complex

$$
0 \rightarrow \mu_p \xrightarrow{\mathcal{L}} \mathcal{I}^{\text{crys}} \xrightarrow{1-\phi} \mathcal{O}^{\text{crys}} \rightarrow 0
$$

over $(X^N_{\text{syn}}, \mathbb{Z}/p)$. It remains to show that this complex is exact. Since $\mu_p^\infty$, $\mathcal{I}^{\text{crys}}$ and $\mathcal{O}^{\text{crys}}$ are $L$-normalized it is sufficient to check exactness on component $k = 1$ (use 4.3 (ii)). Using 8.6 (ii), (iii), (iv) and 8.7 (ii), (iii) we are reduced to check the exactness of the sequence

$$
0 \rightarrow \mu_p(A_\infty) \xrightarrow{\mathcal{L}} I^p_2(A_\infty)/p \xrightarrow{1-\phi} W^{dp}(A_\infty) \rightarrow 0
$$

with $A_\infty = B_\infty/(f_1, \ldots, f_r)$ as in the proof of the last statement in 8.6 (iv). In order to do this our first task is to give an explicit description of $W^{dp}(A_\infty)/pI^{dp}_2(A_\infty)$, $W^{dp}(A_\infty)$ and of the maps $\mathcal{L}$ and $1 - \phi$. We will use the following facts.

(Fact 1): The natural map $D(W_k(B_\infty), ([f_i^{p^{-k}}], \ldots, [f_r^{p^{-k}}])) \rightarrow W^{dp}(A_\infty)$ is an isomorphism.

(Fact 2): The $[f_i^{p^{-k}}]$’s form a regular sequence in $W_2(B_\infty)$. 

146
(Fact 3): Consider the ideal $J$ generated by a regular sequence $(g_1, \ldots, g_r)$ in a $\mathbb{Z}/p^k$-algebra $C$. For $d \geq r$, let $< Y >$ denote the tautological pd-ideal of $C < Y > = C < Y_1, \ldots, Y_d >$ and let $I$ denote the ideal generated by the $Y_i - f_i$'s. Then $I \cap < Y >$ coincides with $I < Y >$. It is in particular a sub pd-ideal of $< Y >$ and $D(C, I) = C < Y > / I$.

Let us explain this briefly. Fact 1: this simply follows from the fact that the image of the $p^d[f_i^{p^r}]$'s vanish in $D(W_k(B_{\infty}), ([f_1^{p^r}], \ldots, [f_r^{p^r}]))$, turning the latter into an algebra over $W_k(A_{\infty})$. Fact 2: During the proof of [8.6 (iv)] we have established that $W_k(B_{\infty})/([f_1^{p^r}], \ldots, [f_r^{p^r}])$ is $\mathbb{Z}/p^k$-flat for $0 \leq i \leq r$. If $0 \leq i \leq r - 1$ this implies that $\mathbb{Z}/p^k \simeq K = 0$ where $K$ denotes the kernel of the multiplication by $[f_{i+1}]$ on $W_k(A_{\infty})/([f_1^{p^r}], \ldots, [f_r^{p^r}]).$ It follows that $K = pK = p^kK = 0$ as desired. Fact 3: the equality $I \cap < Y > = I < Y >$ can easily be proven by induction on $r$ and the remaining statements follow immediately.

Using these facts we find the following decomposition into cyclic $A_{\infty}$-modules (resp. $W_2(A_{\infty})$-modules):

$$W_1^d(A_{\infty}) \simeq D(B_{\infty}, (f_1^{p^r}, \ldots, f_r^{p^r}))$$

$$\simeq B_{\infty} < Y_1, \ldots, Y_r > / (Y_1 - f_1^{p^r}, \ldots, Y_r - f_r^{p^r})$$

$$\simeq \bigoplus_n A_{\infty}([f_i^{p^r}])^{|\nu|}$$

$$W_2^d(A_{\infty})/pI_2^d(A_{\infty}) \simeq D(W_2(B_{\infty}), ([f_1^{p^r}], \ldots, [f_r^{p^r}])/pI_2^d(B_{\infty})$$

$$\simeq W_2(B_{\infty}) < Y_1, \ldots, Y_r > / p < Y > = (Y_1 - [f_1^{p^r}], \ldots, Y_r - [f_r^{p^r}])$$

$$\simeq W_2(A_{\infty}/(f_1^{p^r}, \ldots, f_r^{p^r}))$$

$$\simeq \bigoplus_{n \geq 1} W_2(A_{\infty}/(f_1^{p^r}, \ldots, f_r^{p^r}))/p.[f_i^{p^r}]^{|\nu|}$$

Next we observe that the following square is commutative:

$$\begin{array}{ccc}
(0, a) [f_i^{p^r}]^{|\nu|} & \xrightarrow{\Theta_2} & W_2^d(A_{\infty}) \xrightarrow{\Theta_2} \mathcal{O}_{crys}^d(A_{\infty}) \\
\downarrow V & & \downarrow \Theta_1 \\
(0, a) [f_i^{p^r}]^{|\nu|} & \xrightarrow{\Theta_2} & W_1^d(A_{\infty}) \xrightarrow{\Theta_2} \mathcal{O}_{crys}^d(A_{\infty})
\end{array}$$

where the vertical arrow denoted $V$ is defined by the given formula. With this in mind we find that the morphism $\phi: I_2^d(A_{\infty}) \rightarrow W_1^d(A_{\infty})$ is described by the formulae (which are valid only since $p \geq 3$)

$$\phi(0, a) = a^p$$

$$\phi((a_0, a_1)[f_i^{p^r}]) = (p-1)!a_0^p(f_i^{p^r})^{|\nu|}$$

$$\phi((a_0, a_1)[f_i^{p^r}]) = 0 \text{ if } \nu \neq 0$$

An arbitrary element of $I_2^d(A_{\infty})/p$ may be written $x = (0, a) + \sum_i a_i[f_i^{p^r}]/p^2\left(\sum_{\nu \neq 0} a_i(f_i^{p^r})^{|\nu|}\right)$. Then we have:

$$(1 - \phi)(x) = \left(\sum_i a_i^p(f_i^{p^r}) + \sum_{\nu \neq 0} (a_i^p f_i^{p^r})^{|\nu|}\right) - \left(a^p - \sum(a_i^p f_i^{p^r})^{|\nu|}\right)$$
This formula shows that $1 - \phi : I^{dp}_2(A_\infty)/p \rightarrow W^{dp}_1(A_\infty)$ is surjective and that $x$ is in its kernel if and only if the following are satisfied in $A_\infty/(f^{p-1})$: 

$$\begin{align*}
& a = \sum a_i f_i^{p-2} \\
& a_i p = -a_i (i = 1, \ldots, r) \\
& a_i^{np} = 0 \quad (|i| \geq 2)
\end{align*}$$

ie. if $x$ is of the form

$$x = (0, \sum a_i f_i^{p-2}) + \sum [a_i f_i^{p-2}] - \sum [a_i f_i^{p-2}] [p]$$

In other terms $\text{Ker}(1 - \phi)$ is generated by elements of the form $(z, z) - [z][p]$ with $z$ running in one of the ideals $(f_i^{p-2})$ of $A_\infty/(f^{p-1})$.

Let us now investigate the map $L : \mu_p(A_\infty) \rightarrow I^{dp}_2(A_\infty)$. Let $\zeta = 1 + z^{p^2}$ denote an element of $\mu_p(A_\infty)$ with $z \in (f^{p-1})$. By definition of $L$, we have $L(\zeta) = \log((1 + [z])^{p^2})$.

We have the following equalities in $I^{dp}_2(A_\infty)/p$:

$$\begin{align*}
L(1 + z^{p^2}) &= \log(1 + p(\sum_{k=1}^{p-1} (\frac{1}{p} k)) k) + [z^{p^2}] \\
&= p \sum_{k=1}^{p-1} (\frac{1}{p} k - [z^{p^2}]) + \sum_{k=1}^{p-1} [z^{p^2}] - [z^{p^2}] [p] \\
&= (\sum_{k=1}^{p-1} (\frac{1}{p} k - [z^{p^2}]) - \sum_{k=1}^{p-1} [z^{p^2}] [p])
\end{align*}$$

Now $x^p \mapsto \sum_{k=1}^{p-1} (\frac{1}{p} k - [z^{p^2}] - \sum_{k=1}^{p-1} [z^{p^2}] [p])$ is a bijection of the ideal $(f^{p-1})$ of $A_\infty/(f^{p-2})$. In particular the map $L$ is injective. Since its image is moreover a priori contained in $\text{Ker}(1 - \phi)$ the previous description of the latter yields the desired equality.

(iv) Consider $X'$ with local finite $p$-bases and a morphism $f : X' \rightarrow X$. We have to check that the following squares of abelian groups of $X'_{\text{syn}}$ are commutative:

$$\begin{array}{ccc}
\mathcal{O}_{\text{crys}} & \xrightarrow{F} & \mathcal{O}_{\text{crys}} \\
\downarrow & & \downarrow \\
\tilde{\mathcal{O}}_{\text{crys}} & \xrightarrow{\phi} & \mathcal{O}_{\text{crys}} \\
\end{array} \quad \begin{array}{ccc}
\mathcal{T}_{\text{crys}} & \xrightarrow{\phi} & \mathcal{O}_{\text{crys}} \\
\downarrow & & \downarrow \\
\tilde{\mathcal{T}}_{\text{crys}} & \xrightarrow{\phi} & \mathcal{O}_{\text{crys}} \\
\end{array} \quad \begin{array}{ccc}
\mu_p & \xrightarrow{L} & \tilde{\mathcal{T}}_{\text{crys}} \\
\downarrow & & \downarrow \\
\mu_p & \xrightarrow{L} & \tilde{\mathcal{T}}_{\text{crys}} \\
\end{array}$$

The first square is clear. The second follows from the first one using Lemma 8.10(iii). The third one is obvious since both horizontal arrows come from a morphism in $\Sigma_{1,SYN}^N$ by restriction.

Lemma 8.10. Assume $X/\Sigma_1$ has local finite $p$-bases.

(i) The Cartier morphism $C^{-1} : \mathcal{O} \rightarrow \mathcal{O}^{\text{crys}}_\Sigma$ is monomorphic in $X_{\text{syn}}$. It induces an exact sequence as follows over $((X/\Sigma_1)_{\text{crys, syn}}, \mathcal{O})$:

$$0 \rightarrow \mathcal{T} \rightarrow \mathcal{O} \rightarrow i_* \phi_* \mathcal{O}$$

(ii) Let us use the notation $\epsilon$ for either one of the morphisms $(X_{\text{syn}}, \mathcal{O}) \rightarrow (X_{\text{et}}, \mathcal{O})$ or

$$((X/\Sigma_1)_{\text{crys, syn}}, \mathcal{O}) \rightarrow ((X/\Sigma_1)_{\text{crys, et}}, \mathcal{O})$$. If $M$ is a locally free crystal with trivial $p$-curvature on $((X/\Sigma_1)_{\text{crys, et}}, \mathcal{O})$ then the canonical base change morphism

$$\epsilon^* \phi_* M \rightarrow \phi_* \epsilon^* M$$

is monomorphic in $\text{Mod}(X_{\text{syn}}, \mathcal{O})$
8.1.3. The following result explains a posteriori the ad hoc definition given in 5.34.

**Lemma 8.11.** Assume that $X/\Sigma_1$ has local finite $p$-bases and consider $G$ in $p$-$\text{div}(X)$. Let $D$ denote the associated Dieudonné crystal over $((X/\Sigma_1)_{\text{crys},et}, \mathcal{O})$. Using the definitions of 5.38 we have the following.

(i) There is a canonical isomorphism of exact sequences over $((X/\Sigma_1)_{\text{crys,syn}}, \mathcal{O})$:

\[
\begin{array}{c}
0 \rightarrow p_\ast \mathcal{E}xt^1_{\Sigma_1}(G^*, \mathcal{I}) \rightarrow p_\ast \mathcal{E}xt^1_{\Sigma_1}(G^*, \mathcal{O}) \rightarrow \mathcal{E}xt^1_{\Sigma_1}(G^*, \mathcal{G}_a) \rightarrow 0 \\
0 \rightarrow \text{Fil}^1 D^{\text{syn}} \rightarrow \text{Fil}^0 D^{\text{syn}} \rightarrow i_\ast \text{Lie}^{\text{syn}}(D) \rightarrow 0
\end{array}
\]

(ii) There is a canonical isomorphism of exact sequences over $(X_{\Sigma_1}^N, \mathcal{O}^{\text{crys}})$:

\[
\begin{array}{c}
0 \rightarrow \mathcal{E}xt^1_{\Sigma_1}(l^{-1}G^*, \mathcal{I}^{\text{crys}}) \rightarrow \mathcal{E}xt^1_{\Sigma_1}(l^{-1}G^*, \mathcal{O}^{\text{crys}}) \rightarrow \mathcal{E}xt^1_{\Sigma_1}(l^{-1}G^*, \mathcal{O}) \rightarrow 0 \\
0 \rightarrow \text{Fil}^1 D^{\text{crys}} \rightarrow \text{Fil}^0 D^{\text{crys}} \rightarrow \text{Lie}^{\text{crys}}(D) \rightarrow 0
\end{array}
\]

(iii) The diagrams of (i) and (ii) are naturally functorial with respect to $X$.

Proof. (i) Recall that by definition $D^{\text{syn}} = e^* D$ and $\overline{D} = \iota_1^{-1} D$. Let us furthermore denote $\overline{D}^{\text{syn}} := \iota_1^{-1} D^{\text{syn}} \simeq e^* \overline{D}$, $D^{\text{SYN}} := p^* D^{\text{syn}}$ and $\overline{D}^{\text{SYN}} := \iota_1^{-1} D^{\text{SYN}} \simeq p^* \overline{D}^{\text{syn}}$. By 5.13(ii) we have a commutative square as follows in $((X/\Sigma_1)_{\text{CRY},\Sigma_1,\text{syn}}, \mathcal{O})$: (note that $\iota_1^{-1} \mathcal{E}xt^1_{\Sigma_1}(\cdot, -) \simeq \mathcal{E}xt^1_{\Sigma_1}(\cdot, -)$) since $\iota_1^{-1}$ has an exact left adjoint.)
Claim. The image of the bottom (resp. top) horizontal arrow in (186) identifies canonically with $i_*\text{Lie}^{\text{syn}}(D)$ (resp. $\mathcal{E}xt^1_{(X/\Sigma_1)_{\text{crys,syn}}}(G^*, \mathcal{G}_a)$).

Let us prove the first part of the claim. To begin with, we notice that the monomorphism $\text{Coker} \xrightarrow{\varepsilon_*} \overline{\text{D}}^{\text{syn}}$ in $\text{Crys}((X/\Sigma_1)_{\text{crys,syn}}, \mathcal{O})$ induces a monomorphism

\begin{equation}
\phi_* \text{Coker} \xrightarrow{\phi_*(\overline{\mathcal{J}}^{\text{syn}})} \phi_* \overline{\text{D}}^{\text{syn}} \text{ in } \text{Mod}(X_{\text{syn}}, \mathcal{O})
\end{equation}

of $\mathcal{O}$-modules on $X_{\text{syn}}$. The claim concerning the image of the top arrow then follows from the commutative diagram

\begin{equation}
\xymatrix{
\overline{\text{D}}^{\text{syn}} \ar[r]^{C^{-1}} & i_* \phi_* \text{Coker} \ar[d]_{\varepsilon_*} \ar[r]^{\phi_*(\overline{\mathcal{J}}^{\text{syn}})} & i_* \phi_* \overline{\text{D}}^{\text{syn}} \\
\epsilon^* \overline{\text{D}} \ar[r]^{C^{-1}} & i_* \epsilon_* \phi_* \text{Coker} \ar[d]_{\overline{\text{ch}}} & i_* \text{Lie}^{\text{syn}}(D) \\
}
\end{equation}

since the arrow denoted $\overline{\text{ch}}$ is monomorphic by 8.10 (ii). Note that the bottom arrow denoted $C^{-1}$ uses the isomorphism $\epsilon^* i_* \simeq i_* \epsilon^*$ of 5.57 (i).

Let us now prove the second part of the claim. Consider the exact sequences

\begin{equation}
0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O} \xrightarrow{\text{nat}} \mathcal{G}_a \longrightarrow 0 \quad \text{over } (X/\Sigma_1)_{\text{CRYS,syn}}
\end{equation}

\begin{equation}
0 \longrightarrow \mathcal{G}_a \xrightarrow{C^{-1}} i_* \phi_* \mathcal{O} \longrightarrow \text{Coker } C^{-1} \longrightarrow 0 \quad \text{over } (X/\Sigma_1)_{\text{crys,syn}} \text{ (see 8.10 (i))}
\end{equation}

It suffices to prove that the left (resp. right) vertical arrow in the following commutative square is epimorphic (resp. monomorphic)

\begin{equation}
\xymatrix{
p_* \mathcal{E}xt^1_{(X/\Sigma_1)_{\text{CRYS,syn}}}(G^*, \mathcal{O}) \ar[r]^{C^{-1}} & \mathcal{E}xt^1_{(X/\Sigma_1)_{\text{crys,syn}}}(G^*, i_* \phi_* \mathcal{O}) \\
p_* \mathcal{E}xt^1_{(X/\Sigma_1)_{\text{cly}^{\text{syn}}}}(G^*, \mathcal{G}_a) \ar[r]^{\sim} & \mathcal{E}xt^1_{(X/\Sigma_1)_{\text{crys,syn}}}(G^*, \mathcal{G}_a) \\
}
\end{equation}
Now this follows from the vanishing of $\mathcal{E}xt^2_{(X/\Sigma_1)}(G^*, \mathcal{I})$, 8.4 (ii) (resp. of $\mathcal{H}om_{(X/\Sigma_1)}(G^*, \text{Coker } C^{-1})$).

Using the claim and applying $i_{1,*}$ to (186) we find compatible epimorphisms as follows over $((X/\Sigma_\infty)_{\text{crys},\text{syn}}, \mathcal{O})$:

\[
p_*\mathcal{E}xt^1_{(X/\Sigma_1)_{\text{crys},\text{syn}}}(G^*, \mathcal{O}) \overset{\cong}{\longrightarrow} i_{1,*}p_*\mathcal{E}xt^1_{(X/\Sigma_1)_{\text{crys},\text{syn}}}(G^*, \mathcal{O}) \overset{\text{nat}}{\longrightarrow} \mathcal{E}xt^1_{(X/\Sigma_\infty)_{\text{crys},\text{syn}}}(G^*, \mathcal{O}_a)
\]

The result now follows immediately from the vanishing of $\mathcal{H}om_{(X/\Sigma_\infty)_{\text{crys},\text{syn}}}(G^*, \mathcal{O}_a)$ and the definition of $\text{Fil}^1 D^\text{syn}$.

By definition the second line is obtained from the second line of (i) by the functor $u_*\mathcal{L}^{-1}$ (5.59 (i)). It is exact by 5.60 (i). It is thus sufficient to observe that the first line is also obtained from the first one by $u_*\mathcal{L}^{-1}$ (recall that $l^{-1}p_* \simeq p_*\mathcal{L}^{-1}$, $u_*p_* \simeq p_*u_*$ and use 8.1 (ii), 8.2 (ii)).

We have to check that there are naturally defined base change morphisms for each vertex of the diagrams in question and that those are compatible to each other. Consider a morphism $f : X' \to X$ where $X'$ has local finite $p$-bases as well. Let $G'$ denote the base change of $G$ to $X'$. For the purpose of this proof we view $G$ (resp. $G'$) as an abelian group in $X_{\text{SYN}}$ (resp. $X'_{\text{SYN}}$). We thus have $f^{-1}G \simeq G'$. Using the natural isomorphisms $f^{-1}i_* \simeq i_*f^{-1}$, $pf \simeq fp$ and the compatibility of $\mathcal{E}xt^1$ with localization on the big syntomic crystalline site we find compatible base change morphisms (these are in fact isomorphisms but we don’t need that here) over $((X'/\Sigma_\infty)_{\text{crys},\text{syn}}, \mathcal{O})$:

\[
\begin{align*}
p_*\mathcal{E}xt^1_{(X'/\mathcal{L}_p)_{\text{crys},\text{syn}}}(i_*G'^*, \mathcal{I}) & \overset{\cong}{\longrightarrow} p_*\mathcal{E}xt^1_{(X'/\mathcal{L}_p)_{\text{crys},\text{syn}}}(i_*G'^*, \mathcal{O}) \quad p_*\mathcal{E}xt^1_{(X'/\mathcal{L}_p)_{\text{crys},\text{syn}}}(i_*G'^*, \mathcal{O}_a) \\
f_*p_*\mathcal{E}xt^1_{(X/\mathcal{L}_p)_{\text{crys},\text{syn}}}(i_*G^*, \mathcal{I}) & \overset{\cong}{\longrightarrow} f_*p_*\mathcal{E}xt^1_{(X/\mathcal{L}_p)_{\text{crys},\text{syn}}}(i_*G^*, \mathcal{O}) \quad f_*p_*\mathcal{E}xt^1_{(X/\mathcal{L}_p)_{\text{crys},\text{syn}}}(i_*G^*, \mathcal{O}_a)
\end{align*}
\]

Let us now explain the base change morphisms for the second line of (i). As before we use the following notations:

\[
\begin{align*}
D^{\text{SYN}} & := \mathcal{E}xt^1_{(X/\Sigma_\infty)_{\text{SYN}}}(i_*G, \mathcal{O}) & \text{in } \mathcal{C}rys((X/\Sigma_\infty)_{\text{crys,syn}}, \mathcal{O}) \\
\overline{D}^{\text{SYN}} & := i_1^{-1}D^{\text{SYN}} & \text{in } \mathcal{C}rys((X/\Sigma_1)_{\text{crys,syn}}, \mathcal{O}) \\
D^{\text{syn}} & := p_*D^{\text{syn}} & \text{in } \mathcal{C}rys((X/\Sigma_\infty)_{\text{crys,syn}}, \mathcal{O}) \\
\overline{D}^{\text{syn}} & := i_1^{-1}\overline{D}^{\text{syn}} & \text{in } \mathcal{C}rys((X/\Sigma_1)_{\text{crys,syn}}, \mathcal{O}) \\
D & := \epsilon_*D^{\text{syn}} & \text{in } \mathcal{C}rys((X/\Sigma_\infty)_{\text{crys,et}}, \mathcal{O}) \\
\overline{D} & := i_1^{-1}D & \text{in } \mathcal{C}rys((X/\Sigma_1)_{\text{crys,et}}, \mathcal{O})
\end{align*}
\]

We define similarly crystals $D^{\text{SYN}}$, $\overline{D}^{\text{SYN}}$, $D^{\text{syn}}$, $\overline{D}^{\text{syn}}$, $D'$, $\overline{D}'$ starting from $G'$ instead of $G$. A base change morphism $f^*D^{\text{syn}} \to D^{\text{syn}}$ has been described just above. Note that this also induces $f^*\overline{D}^{\text{syn}} \to \overline{D}^{\text{syn}}$ (using $i_1f \simeq f i_1$), $f^*D \to D'$ (using $\epsilon f \to f \epsilon$) and $f^*\overline{D} \to \overline{D}'$ (using $i_1f \simeq f i_1$). From the latter and $\phi f \to f \phi$ we deduce the base change morphism $f^*\text{Lie}(D) \to \text{Lie}(D')$ considered in 5.33 (ii). Recall now that $\text{Lie}^{\text{syn}}(D) := \epsilon^*\text{Lie}(D)$. Using $f \epsilon \to \epsilon f$ we next deduce the base change morphism $f^*\text{Lie}^{\text{syn}}(D) \to \text{Lie}^{\text{syn}}(D')$ implicitly hinted in 5.60 (i). Finally a base change morphism $f^*i_*\text{Lie}^{\text{syn}}(D) \to i_*\text{Lie}^{\text{syn}}(D')$ for the right term in the second line of (i) follows using
if $\rightarrow fi$. In order to get the expected base change morphisms for the second line of [i] we have to check that the following natural square (where the horizontal arrows are defined using $can$ and $i\epsilon \rightarrow \epsilon i$) is commutative:

\begin{equation}
\begin{array}{c}
D'^{syn} \rightarrow \epsilon^* i_* Lie^{syn}(D') \\
\downarrow \quad \downarrow \\
f^* D^{syn} \rightarrow f^* i_* Lie^{syn}(D)
\end{array}
\end{equation}

Even though this was also already implicit in 5.63 [i] we give some details here. We have to prove that the exterior square of the following diagram (where the arrows are the obvious ones) commutes:

\begin{equation}
\begin{array}{c}
\epsilon^* D' \rightarrow \epsilon^* i_* Lie(D') \rightarrow i_* \epsilon^* Lie(D') \\
\downarrow \quad \downarrow \quad \downarrow \\
\epsilon^* f^* D \rightarrow \epsilon^* f^* i_* Lie(D) \rightarrow i_* \epsilon^* f^* Lie(D) \\
\downarrow \quad \downarrow \quad \downarrow \\
f^* \epsilon^* D \rightarrow f^* \epsilon^* i_* Lie(D) \rightarrow f^* i_* \epsilon^* Lie(D)
\end{array}
\end{equation}

Commutativity of the bottom left (resp. top right) square is tautological (resp. is immediate if one introduces $\epsilon^* i_* f^* Lie(D)$). Commutativity of the top left square is deduced from 5.35 [ii] using only pseudo-functoriality of the morphism $i$. Commutativity of the bottom right hand square is a formal consequence of the pseudo-functoriality of the isomorphism $i\epsilon \simeq \epsilon i$.

We have now checked that (189) commutes. As a result $f^* D^{sym} \rightarrow D'^{sym}$ also induces a base change morphism $f^* Fil^1 D^{sym} \rightarrow Fil^1 D'^{sym}$ so that we have described base change morphisms for each vertex of the commutative diagram [i]. It remains to check their compatibility with respect to the vertical arrows of [i]. In the case of the middle one there is nothing to prove. The case of the left and right ones follow formally. The tenacious reader may check by himself that the base change morphisms just described verify the composition constraint with respect to morphisms $X'' \rightarrow X' \rightarrow X$.

Functoriality of the diagram in [ii] with respect to $X$ follows immediately since it is obtained from [i] by $u_* l^{-1}$ while both $u$ and $l$ are pseudo-functorial with respect to $X$.

\[\square\]

**Lemma 8.12.** Assume that $X/\Sigma_1$ has local finite $p$-bases. Consider $G$ in $pdiv(X)$ and view it as group in $X_{syn}$. We use the simplified notation $\mathcal{E}xt^n$ for $\mathcal{E}xt^n_{X_{syn}}$ or $\mathcal{E}xt^n_{X_{sym}^{\Sigma_1}}$.

(i) The modules $\mathcal{E}xt^1(l^{-1}G^*, \tilde{I}_{crys})$ and $\mathcal{E}xt^1(l^{-1}G^*, O_{crys})$ are normalized and $\mathbb{Z}/p$-flat. Moreover 8.11 [ii] induces compatible isomorphisms as follows over $(X_{sym}^{\Sigma_1}, O_{crys})$:

\[
\begin{array}{c}
Fil^1_{crys} D \rightarrow 1 \rightarrow Fil^0_{crys} D \\
\downarrow l \quad \quad \downarrow l \\
\mathcal{E}xt^1(l^{-1}G^*, \tilde{I}_{crys}) \rightarrow \mathcal{E}xt^1(l^{-1}G^*, 1) \rightarrow \mathcal{E}xt^1_{X_{sym}^{\Sigma_1}}(l^{-1}G^*, O_{crys})
\end{array}
\]
(ii) The following square of \((X_{syn}^N, \mathcal{O}^{crys})\) is commutative:

\[
\begin{array}{ccc}
\tilde{Fil}^{1, \text{crys}} D & \xrightarrow{\phi} & (F)_* \tilde{Fil}^{0, \text{crys}} D \\
\downarrow l & & \downarrow l \\
\mathcal{E}xt^1(l^{-1}G^*, \tilde{I}^{\text{crys}}) & \xrightarrow{\mathcal{E}xt^1(l^{-1}G^*, \phi)} & (F)_* \mathcal{E}xt^1(l^{-1}G^*, \mathcal{O}^{\text{crys}})
\end{array}
\]

Proof. (i) By \(5.57\) we already know that \(D^{\text{crys}} \simeq \mathcal{E}xt^1(l^{-1}G^*, \mathcal{O}^{\text{crys}})\) is \(\mathbb{Z}/p\)-flat and normalized. The same holds for \((1)^* \mathcal{E}xt^1(l^{-1}G^*, \tilde{I}^{\text{crys}})\) thanks to \(4.3\) applied to the top exact sequence of \(8.1\) (ii). In order to get the claimed compatible isomorphisms it suffices to apply the functor \((1)^*\) to the left square in \(8.1\) (ii) and to prove that the natural arrow

\[
(1)^* \mathcal{E}xt^1(l^{-1}G^*, \tilde{I}^{\text{crys}}) \rightarrow \mathcal{E}xt^1(l^{-1}G^*, (1)\tilde{I}^{\text{crys}})
\]

is an isomorphism. Now this in turn will follow from the following commutative diagram by the five lemma once proven that its lines are exact:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{E}xt^1(G^*, \mathcal{O}) & \rightarrow & \mathcal{E}xt^1(G^*, \mathcal{O}^{\text{crys}}) & \rightarrow & 0 \\
\downarrow \| & & \downarrow \| & & \downarrow \| & & \\
0 & \rightarrow & \mathcal{E}xt^1(G^*, \mathcal{O}) & \rightarrow & \mathcal{E}xt^1(G^*, \mathcal{O}^{\text{crys}}) & \rightarrow & 0
\end{array}
\]

The first exact line is deduced from the first exact sequence of \(8.1\) (ii) using \(4.3\) (iv). The second line on the other hand is deduced from the exact sequence

\[(190)\]

\[
0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}^{\text{crys}} \rightarrow \mathcal{O} \rightarrow 0
\]

(apply \(4.3\) (iv) to \(8.9\) (i)) (note that \(190\) remains exact after applying \(\mathcal{E}xt^1(G^*, -)\) since \(\text{Hom}(G^*, \mathcal{O}^{\text{crys}}), \mathcal{E}xt^2(G^*, \mathcal{O})\) and \(\text{Hom}(G^*, \mathcal{O})\) vanish by \(2.39\) (i) and \(8.4\) (i)).

(ii) Using \(4.3\) (iii) and the definition of \(\phi\) \((5.62)\) it is sufficient to prove that the following square commutes:

\[
\begin{array}{ccc}
\tilde{Fil}^{1, \text{crys}} D & \xrightarrow{Fr} & (F)_* \tilde{Fil}^{0, \text{crys}} D \\
\downarrow l & & \downarrow l \\
\mathcal{E}xt^1(l^{-1}G^*, \tilde{I}^{\text{crys}}) & \xrightarrow{\mathcal{E}xt^1(l^{-1}G^*, Fr)} & (F)_* \mathcal{E}xt^1(l^{-1}G^*, \mathcal{O}^{\text{crys}})
\end{array}
\]

where \(Fr\) is as in \(5.61\) and \(F\) denotes the Frobenius endomorphism of \(\mathcal{O}^{\text{crys}}\). We conclude by \(5.15\) (i) (iii).

\[\square\]

Remark 8.13. In view of \(8.1\) (ii) we may reformulate \(8.12\) (i), (ii) as an isomorphism

\[
\mathcal{S}^{\phi}_{syn, X}(D(G)) \simeq \mathcal{E}xt^1(\pi^{-1}l^{-1}G^*, (\tilde{I}^{\text{crys}}, \mathcal{O}^{\text{crys}}, 1, \phi)) \text{ in } \text{Mod}^{1, \phi}(X_{syn}^N, \mathcal{O}^{\text{crys}})
\]

Where \(\pi\) denotes the canonical morphism \(X_{syn}^N, 1, \phi \to X_{syn}^N\) \((5.1)\).

We are now in position of proving the expected comparison theorem for \(p\)-divisible groups.
**Theorem 8.14.** Let $X$ denote a diagram of $\text{Sch}/\Sigma_1$ whose vertices have local finite $p$-bases and consider $G \in p\text{-}\text{div}(X)$. There is a canonical isomorphism

$$G_p \simeq S_{\text{syn},X}(D(G)) \quad \text{in } D(X^N_{\text{syn}}, \mathbb{Z}/p)$$

Proof. Since $G_p$ is concentrated in degree 0 it suffices to construct the desired isomorphism in the case where $X$ is a single scheme and to check its compatibility with respect to the natural base change morphisms relative to a morphism $X' \to X$. Let thus $X$ denote a scheme with local finite $p$-bases over $\Sigma_1$. In virtue of 8.13 and of the tautological isomorphism

$$S_{\text{syn},X}(D(G)) \simeq R\pi_* S_{\text{syn},X}^1 \phi(D(G))$$

it suffices to establish a canonical isomorphism in $X^N_{\text{syn}}$:

$$G_p \simeq R\pi_* \mathcal{E}xt^1(\pi^{-1}l^{-1}G^*, (\overline{\mathcal{I}}_{\text{crys}}, \mathcal{O}_{\text{crys}}, 1, \phi))$$

Let us compute the right hand side. Recall from 5.2 (iv) that one has a canonical distinguished triangle

$$R\pi_* \mathcal{E}xt^1(\pi^{-1}l^{-1}G^*, (\overline{\mathcal{I}}_{\text{crys}}, \mathcal{O}_{\text{crys}}, 1, \phi)) \longrightarrow \mathcal{E}xt^1(l^{-1}G^*, \overline{\mathcal{I}}_{\text{crys}}) \xrightarrow{1-\phi} \mathcal{E}xt^1(l^{-1}G^*, \mathcal{O}_{\text{crys}})$$

Since $\mathcal{E}xt^q(l^{-1}G^*, \mu_p)$ vanishes for $q = 0, 2$ (8.1 (i), 2.39 (i), 8.4 (i)) the second exact sequence in 8.9 (iii) produces an exact sequence

$$0 \longrightarrow \mathcal{E}xt^1(l^{-1}G^*, \mu_p) \longrightarrow \mathcal{E}xt^1(l^{-1}G^*, \overline{\mathcal{I}}_{\text{crys}}) \xrightarrow{1-\phi} \mathcal{E}xt^1(l^{-1}G^*, \mathcal{O}_{\text{crys}}) \longrightarrow 0$$

Comparing with the above we get

$$(191) \quad R\pi_* \mathcal{E}xt^1(\pi^{-1}l^{-1}G^*, (\overline{\mathcal{I}}_{\text{crys}}, \mathcal{O}_{\text{crys}}, 1, \phi)) \simeq \mathcal{E}xt^1(l^{-1}G^*, \mu_p) \quad \text{in } \text{Mod}(X^N_{\text{syn}}, \mathbb{Z}/p)$$

Let us compute the right hand side. Thanks to 2.39 (i) we find that for $k$ fixed, the exact sequence

$$0 \longrightarrow G_{p^k} \longrightarrow G^* \xrightarrow{p^k} G^* \longrightarrow 0$$

gives rise to an isomorphism between $G_{p^k} \simeq \text{Hom}(G_{p^k}, \mu_p)$ and $\mathcal{E}xt^1(G^*, \mu_p)$. Letting $k$ vary we get

$$(192) \quad G_p \simeq \mathcal{E}xt^1(l^{-1}G^*, \mu_p) \quad \text{in } \text{Mod}(X^N_{\text{syn}}, \mathbb{Z}/p)$$

This ends the proof in the case of a single scheme $X$. In order to pass to the case of a diagram $X/\Delta$ of arbitrary type it suffices to check that the isomorphisms (191) and (192) are both naturally functorial with respect to $X$. This causes no difficulty using the first isomorphism of 8.2 (iii) (note that (191) and (192) both come from a morphism in the big topos). 

\[ \square \]

**Remark 8.15.** The isomorphism of the theorem may be reformulated as exact sequence over $(X^N_{\text{syn}}, \mathbb{Z}/p)$. over $(X^N_{\text{syn}}, \mathcal{O}_{\text{crys}})$:

$$0 \longrightarrow G_p \longrightarrow \hat{\mathcal{F}}il^{1,\text{crys}}(D(G)) \xrightarrow{1-\phi} \hat{\mathcal{F}}il^{0,\text{crys}}(D(G)) \longrightarrow 0$$

154
Corollary 8.16. Keep the assumptions of 8.14 and denote $\epsilon : X_{FL} \to X_{et}$. There is a canonical isomorphism:

$$\text{Res}_* G_p \simeq \mathcal{S}_{et\ldots X}(D(G)) \quad \text{in} \quad D(X_{et}^N, \mathbb{Z}/p)$$

Proof. This follows immediately from 8.14 and 5.65 (ii) given that $G_p$ is acyclic for the projection functor $X_{FL} \to X_{syn}$ (this follows easily from [dix exp, BR III, 11.7] using [BBM] 3.1.1).

\[\square\]

8.2. Semi-abelian schemes over $C$.

We can finally prove our main result.

Theorem 8.17. Assume $A$ is a semi-abelian scheme over $C$ whose restriction to $U$ is abelian. Let $\epsilon : C_{FL} \to C_{et}$ denote the canonical morphism. There is a canonical isomorphism

$$\text{Res}_* \text{R}_{\pi}^Z A_p \simeq \mathcal{S}_{et\ldots C^*(C)}(-Z)(D_{C^*}(A)) \quad \text{in} \quad D(C_{et}^N, \mathbb{Z}/p)$$

Proof. Recall the diagram of $p$-divisible groups $H \in p\text{div}(J^+)$ defined in 3.25. Denoting $D := D_{J^+}(H)$ the associated diagram of Dieudonné crystals we have the following series of canonical isomorphisms in $D(C_{et}^N, \mathbb{Z}/p)$:

$$\text{Res}_* \text{R}_{\pi}^Z A_p \simeq \text{Rm}_* \text{Sma} \text{R}_{\pi}^Z (J, \text{Res}_* H_p) \quad \text{(3.26)}$$

$$\simeq \text{Rm}_* \text{Sma} \text{R}_{\pi}^Z (J, \mathcal{S}_{et\ldots J^+}(D)) \quad \text{(8.16)}$$

$$\simeq \text{Rm}_* \text{Sma} \mathcal{S}_{et\ldots J^+}(-Z_J)(D_{J^+}\mid_{J^+}) \quad \text{(6.3 (ii) applied to $D\mid_{J^+}$)}$$

$$\simeq \text{Rm}_* \text{Sma} \mathcal{S}_{et\ldots J^+}(-Z_J)(D_{C^*}(A)\mid_{J^+}) \quad \text{(see below)}$$

$$\simeq \mathcal{S}_{et\ldots C^*}(-Z)(D_{C^*}(A)) \quad \text{(8.16 (iii))}$$

Let us explain the fourth isomorphism in details. According to 7.23 and 5.54 we have a canonical distinguished triangle

$$\text{(193)} \quad \mathcal{S}_{et\ldots J^+}(-Z_J)(D_{J^+}\mid_{J^+}) \longrightarrow \mathcal{S}_{et\ldots J^+}(-Z_J)(D_{C^*}(A)\mid_{J^+}) \longrightarrow \mathcal{S}_{et\ldots J^+}(-Z_J)(D'_{J^+}\mid_{J^+}) +1$$

in $D(J_{et}^N, \mathbb{Z}/p)$. Here $D' := D(H')$ is the Dieudonné crystal associated to the following $p$-divisible group on $J^+ = (Z_v \to C_v \leftarrow U_v \to U)$:

$$H' = (\mathbb{Q}_p/\mathbb{Z}_p \otimes \Gamma_v/\mathbb{Z}_v \leftarrow \mathbb{Q}_p/\mathbb{Z}_p \otimes \Gamma_v \to 0 \leftarrow 0)$$

We claim that the third term in (193) vanishes after applying the functor $\text{Sma}$ (3.7 (ii)). Let us explain this. From 6.3 (iii) and 8.16 we have

$$\mathcal{S}_{et\ldots J^+}(-Z_J)(D'_{J^+}\mid_{J^+}) \simeq \text{R}_{\pi}^Z (J, \mathcal{S}_{et\ldots J^+}(D'))$$

By 3.13 (ii) (iii) we have a distinguished triangle

$$\text{R}_{\pi}^Z (J, \epsilon_* H'_p) \longrightarrow \epsilon_* H'_{J^+\mid_{J^+}} \longrightarrow z_{J,\epsilon_* H'_p} +1$$

and thus an isomorphism

$$\text{R}_{\pi}^Z (J, \epsilon_* H'_p) \simeq (j_v! \epsilon_* \Gamma_{v\mid U_v}/\mathbb{Z}/p \to 0 \leftarrow 0)$$

in $D(J_{et}^N, \mathbb{Z}/p)$ (recall that $J = (C_v \leftarrow U_v \to U)$). The claim then follows from 3.13 (iii)
It might be worth to state a down to earth (weaker) version of this result.

**Corollary 8.18.** There is a canonical distinguished triangle

\[
\begin{array}{c}
R\Gamma^Z(C, T_p(A)) \longrightarrow R\Gamma(C^Z/\Sigma_\infty, \text{Fil}^1 D_{C^Z}(A)(-Z)) \overset{1-\phi}{\longrightarrow} R\Gamma(C^Z/\Sigma_\infty, D_{C^Z}(A)(-Z)) \overset{+1}{\longrightarrow}
\end{array}
\]

where \(p.\phi\) is induced by the Frobenius endomorphism of the third term.

Proof. Apply \(Rl_*\), \(R\Gamma(C_{et}, -)\) and 5.2(iv) to 8.17.\(\square\)

9. **Appendix (crystals on log schemes with local finite p-bases)**

9.1. **Small crystalline sites, functoriality.**

This section contains some technical results about crystalline sites which could not find their place in the text. We use the definitions and notations introduced in 2.4. In particular \(\Sigma = \Sigma_k\) with \(1 \leq k \leq \infty\) and \(fl \preceq \text{top} \preceq \text{zar}\).

9.1.1. We explain pseudo-functoriality of the \(\sharp\)-big and small crystalline topoi \((X^\sharp/\Sigma)\) with respect to \(X^\sharp\) in \(Sch^\sharp/\Sigma_1\). A similar discussion holds when either the \(\sharp\)-big or small topos is replaced with the big one but we omit it to avoid lengthy statements.

**Definition 9.1.** Let \(f : X'^\sharp \to X^\sharp\).

(i) We define \(f_{CRYS^\sharp} : (X'^\sharp/\Sigma)_{CRYS^\sharp, top} \to (X^\sharp/\Sigma)_{CRYS^\sharp, top}\) as the morphism induced by the forgetful cocontinuous functor \(f_0 : CRYS^\sharp_{top}(X'^\sharp/\Sigma) \to CRYS^\sharp_{top}(X^\sharp/\Sigma)\).

(ii) We define a weak morphism \(f_{crys} : (X'^\sharp/\Sigma)_{crys, top} \to (X^\sharp/\Sigma)_{crys, top}\) by the formula

\[
f_{crys} := pf_{CRYS^\sharp}r
\]

where \(p : (X'^\sharp/\Sigma)_{CRYS^\sharp, top} \to (X'^\sharp/\Sigma)_{crys, top}\), and \(r : (X^\sharp/\Sigma)_{crys, top} \to (X^\sharp/\Sigma)_{CRYS^\sharp, top}\) are as in 4.2.

In the body of the text we simply write \(f\) instead of \(f_{CRYS^\sharp}\) or \(f_{crys}\). For the following discussion however it is convenient to keep the indices.

It follows immediately from the definition that there are natural isomorphisms

\((ff')_{CRYS^\sharp} \simeq f_{CRYS^\sharp}f'_{CRYS^\sharp}\)

satisfying the composition constraint (look at cocontinuous functors). In other words, \((-/\Sigma)_{CRYS^\sharp, top}\) is a variable topos on \(Sch^\sharp/\Sigma_1\). The case of small crystalline topoi requires more work.

**Proposition 9.2.** Assume that \(top\) is finer or equal than \(ct\).

(i) There are natural isomorphism \((fg)_{crys} \to f_{crys}g_{crys}\) satisfying the composition constraint. In other words \((-/\Sigma)_{crys, top}\) is a weakly variable topos on \(Sch^\sharp/\Sigma_1\).

(ii) If \(top\) has fiber products then it is in fact a variable topos.

Proof. (i) Since \(r\) is a right section for \(p\) there is a natural morphism \(id \to rp\). Using this for each fine \(\Sigma_1\)-log scheme we find a family of morphisms

\(194\)

\((ff')_{crys} \to f_{crys}f'_{crys}\)

156
satisfying the composition constraint. We will now prove that (194) is an isomorphism in three steps.

**Step 1.** Consider \( f : X'^{\sharp} \to X^{\sharp} \) and \( f' : X'^{\sharp} \to X'^{\sharp} \). The morphism (194) is invertible in the following cases:

(i) \( f' \) is strict top, or
(ii) \( f \) is strict étale.

In case (i) a straightforward verification shows that \( f'^{\text{crys}} \) is in fact induced by the cocontinuous functor \( f'^{0} : \text{crys}_{\text{top}}(X'^{\sharp}/\Sigma_{1}) \to \text{crys}_{\text{top}}(X^{\sharp}/\Sigma_{1}) \) (compute inverse images) and we may conclude from the resulting isomorphism \( r f'^{\text{crys}} \simeq f'^{\text{crys}}_{\text{top}} r \). In case (ii) we need the following fact which is a consequence of [SGA4-II] VIII, 1.1 together with [BO] 3.22.

**Fact.** The category of étale cartesian dp-thickenings over a given dp-thickening \((U^{\sharp}, T^{\sharp})\) is naturally equivalent to the category of strict étale log schemes over \( U^{\sharp} \).

Using this fact we find that the cocontinuous functors underlying \( f^{\text{crys}} \) and \( f^{\text{crys}}_{\text{top}} \) have compatible right adjoints. We may then conclude from the resulting isomorphism \( f^{\text{crys}} \simeq p f^{\text{crys}}_{\text{top}} \).

**Step 2.** We may always assume that the schemes \( X, X', X'' \) are affine and that \( f' : X'^{\sharp} \to X^{\sharp} \) has a chart

\[
\begin{array}{ccc}
X'^{\sharp} & \xrightarrow{c'} & X^{\sharp} \\
\downarrow & & \downarrow \\
(Spec(\mathbb{Z}[P'']), P'') & \xrightarrow{\text{ch}} & (Spec(\mathbb{Z}[P']), P')
\end{array}
\]

where \( P' \) and \( P'' \) are finitely generated integral monoids.

Let us explain this. Given arbitrary \( f \) and \( f' \) we can always find a family of commutative diagrams

\[
\begin{array}{ccc}
X''^\sharp & \xrightarrow{f'''} & X'^\sharp \\
\downarrow & & \downarrow \\
X'^\sharp & \xrightarrow{f'} & X^\sharp
\end{array}
\]

where \( h, h', h'' \) are strict étale, \( X, X', X'' \), \( f' \) satisfy the assumptions of Step 2 and the family of the \( h'' \)'s is surjective. These diagrams induce squares

\[
\begin{array}{ccc}
h'' - 1, h' - 1, f' - 1 \xrightarrow{f'' - 1} & \xrightarrow{f'' - 1 (f f') - 1} & h'' - 1, h - 1 \\
\downarrow & & \downarrow \\
(f f')_{\text{crys}} \xrightarrow{h_{\lambda, \text{crys}} f_{\text{crys}} f_{\text{crys}}} & \xrightarrow{h_{\lambda, \text{crys}} (f f')_{\text{crys}}} & h_{\lambda, \text{crys}}
\end{array}
\]

where the vertical isomorphisms are by Step 1 and which is commutative thanks to the composition constraint. It remains to notice that the family of the \( h'' - 1, h' - 1, f' - 1 \)'s is conservative (indeed the essential images of the underlying cocontinuous functors are generating thanks to the Fact used in the proof of Step 1).

**Step 3.** The morphism (194) is invertible under the assumptions of Step 2.
The reader may easily establish the formula

\begin{equation}
(196) \quad f_{cryst,*}(U_T, T^\sharp) \simeq \lim_{\leftarrow C_f(U_T, T^\sharp)} F(U_T, T^\sharp)
\end{equation}

where the projective limit is indexed on the category \( C_f(U_T, T^\sharp) \) of couples \((U_T / X_T, T_T^\sharp)\), \( f : f_0(U_T / X_T, T_T^\sharp) \rightarrow (U_T / X_T^\sharp, T_T^\sharp) \) where \((U_T / X_T, T_T^\sharp)\) is an object of \( \text{cryst}_{top}((X_T^\sharp / \Sigma)) \), \( f_0(U_T / X_T, T_T^\sharp) \) is the object of \( \text{CRY} S^2_{top}(X_T^\sharp / \Sigma) \) obtained by composition of \( U_T \rightarrow X_T \) with \( f : X_T^\sharp \rightarrow X_T^\sharp \) and \( \tilde{f} \) is a morphism in \( \text{CRY} S^2_{top}(X_T^\sharp / \Sigma) \). The projective limit remains the same if the category \( C_f(U_T, T^\sharp) \) is replaced by the full subcategory \( C^*_f(U_T, T^\sharp) \) defined by the conditions that the scheme underlying \( T_T^\sharp \) is affine and that the composed morphism \( U_T \rightarrow X_T \xrightarrow{\sim} \text{spec}(\mathbb{Z}[P]), P' \) extends to \( T_T^\sharp \) (this happens étale locally on \( T_T^\sharp \) thanks to [Ka2] 2.10). Let us denote \( h = f f' \). The reader may check that the evaluation of the direct image functors underlying \( (194) \) at a given \( F \) and \( (U_T, T^\sharp) \) is

\begin{equation}
(197) \quad \lim_{\leftarrow C^*_f(U_T, T^\sharp)} F(U_T, T^\sharp) \rightarrow \lim_{\leftarrow C^*_f(U_T, T^\sharp)} F(U_T, T^\sharp)
\end{equation}

where \( C^*_h(U_T, T^\sharp) \) and \( C^*_f(U_T, T^\sharp) \) are defined in the same way than \( C^*_f(U_T, T^\sharp) \), while \( C^*_f(U_T, T^\sharp) \) is the category where an object is a couple \(((U_T^\sharp / X_T^\sharp, T_T^\sharp), \tilde{f}), ((U_T / X_T, T_T^\sharp), f)\) whose first (resp. second) argument is in \( C^*_f(U_T, T^\sharp) \) (resp. \( C^*_f(U_T, T^\sharp) \)) and where morphisms are defined in the natural way.

In order to prove that \( (194) \) is an isomorphism it is sufficient to prove that this is the case when \( T \) is affine. In that case we will prove that the natural “composition” functor

\begin{equation}
(198) \quad C^*_f(U_T, T^\sharp) \rightarrow C^*_h(U_T, T^\sharp)
\end{equation}

is cofinal, ie. that the category \( C := (U_T^\sharp / X_T^\sharp, T_T^\sharp), h / C^*_h(U_T, T^\sharp) \) is connected for any \((U_T^\sharp / X_T^\sharp, T_T^\sharp), h \) in \( C^*_h(U_T, T^\sharp) \).

Let us first prove that \( C \) is non empty, ie. that there exist \(((U_T^1 / X_T^1, T_T^1), \tilde{f}_1) \) in \( C^*_f(U_T, T^\sharp), ((U_T^1 / X_T^1, T_T^1), f_1) \) in \( C^*_f(U_T, T^\sharp) \) and a morphism \( e_1 \) in \( \text{cryst}_{top}(X_T^\sharp / \Sigma) \) rendering the following diagram commutative in \( \text{CRY} S^2_{top}(X_T^\sharp / \Sigma) \):

\begin{equation}
(199) \quad h_0(U_T^\sharp / X_T^\sharp, T_T^\sharp) \xrightarrow{h_0(e_1)} f_0 f'_0(U_T^\sharp / X_T^\sharp, T_T^\sharp) \xrightarrow{f_0(f'_1)} f_0(U_T^\sharp / X_T^\sharp, T_T^\sharp)
\end{equation}

Consider the affine schemes \( U' := X' \times_X U \) and choose polynomial schemes \( Y'' = \text{spec}(\mathbb{Z}[P]), Y' = \text{spec}(\mathbb{Z}[x]) \) as well as closed immersions \( i'' \), \( i' \) and morphisms \( y'', y' \) fitting into a commutative diagram of the form

\begin{equation}
\begin{array}{ccc}
U & \xrightarrow{U''} & U' \\
\downarrow \quad \quad \quad \gamma'' & & \quad \downarrow \quad \quad \quad \gamma' \\
Y & \xrightarrow{y'} & Y'
\end{array}
\end{equation}

Putting this together with a chart for \( f' \) as in \( (195) \) together with the choice of a lifting to \( T_T^\sharp \) of the morphism \( U_T^\sharp \rightarrow X_T^\sharp \rightarrow \text{spec}(\mathbb{Z}[P]), P' \) easily produces a commutative
The desired object of \( C \) is obtained by forming logarithmic divided power envelopes of the bottom vertical closed immersions with respect to the divided power of \((U^2, T^2)\).

It remains to prove that two objects in \( C \) are always related by a chain of arrows. Let us simply denote \(((\tilde{f}_1, \tilde{f}_1'), e_1)\) the object of \( C \) just constructed and consider an other arbitrary object \(((\tilde{f}_2, \tilde{f}_2'), e_2)\). We will now hint the construction of a chain

\[ ((\tilde{f}_1, \tilde{f}_1'), e_1) \leftarrow ((\tilde{f}_3, \tilde{f}_3'), e_3) \rightarrow ((\tilde{f}_4, \tilde{f}_4'), e_4) \leftarrow ((\tilde{f}_1, \tilde{f}_1'), e_1) \]

and the proof will be finished. The reader may easily guess how to define a commutative diagram of closed immersions

\[
\begin{array}{c}
(U''', Y'' \times \text{Spec}(\mathbb{Z}[P'', P']) \times T^2) \longrightarrow (U'', Y' \times \text{Spec}(\mathbb{Z}[P', P'] \times T^2) \\
(U'', Y'' \times \text{Spec}(\mathbb{Z}[P'', P']) \times T''_2) \longrightarrow (U'_2, Y' \times \text{Spec}(\mathbb{Z}[P', P'] \times T'_2) \\
(U''_2, T''_2) \longrightarrow (U''_2, T''_2)
\end{array}
\]

whose first terms are above

\[
\begin{array}{c}
X''' \longrightarrow X'' \longrightarrow X' \longrightarrow X
\end{array}
\]

and whose bottom (resp. top) line is obtained from \(((\tilde{f}_2, \tilde{f}_2'), e_2)\) by forgetting divided powers (resp. was used in the definition of \(((\tilde{f}_1, \tilde{f}_1'), e_1)\)). We define \(((\tilde{f}_3, \tilde{f}_3'), e_3)\) (resp. \(((\tilde{f}_4, \tilde{f}_4'), e_4)\)) as the object of \( C \) obtained from the second line by forming logarithmic divided powers envelopes of the middle (resp. bottom) line with respect to the divided powers of \((U^2, T^2)\). This gives a chain \(((\tilde{f}_1, \tilde{f}_1'), e_1) \leftarrow ((\tilde{f}_3, \tilde{f}_3'), e_3) \rightarrow ((\tilde{f}_4, \tilde{f}_4'), e_4)\). We conclude by noticing that the universal property of logarithmic divided power envelopes with respect to \((U^2, T^2)\) gives a morphism \(((\tilde{f}_2, \tilde{f}_2'), e_2) \rightarrow ((\tilde{f}_4, \tilde{f}_4'), e_4)\).

\[\boxed{\text{ii}}\] A straightforward adaptation of [Bel] chap. 3, 2.1.4 shows that the crystalline sites under consideration have fiber products as soon as \( \text{top} \) does. The result follows.

\[\square\]

9.1.2. Let us now investigate the behaviour of some usual (weak) morphisms with respect to crystalline functoriality.

**Proposition 9.3.** Let \( fl \preceq \text{top}' \preceq \text{top} \preceq \text{et} \).
(i) The morphisms (42), (44), (45) naturally define weak morphisms of weakly variable topoi on $\mathcal{S}ch^\Sigma/\Sigma_1$ fitting into a canonically pseudo-commutative cube

$$\begin{array}{ccc}
(-)_{\text{top}} & \xrightarrow{i} & (-/\Sigma)_{\text{crys,top}} \\
p & ^{\epsilon}\downarrow & ^{\epsilon}\downarrow \\
(-)_{\text{TOP}} & \xrightarrow{i} & (-/\Sigma)_{\text{CYRS}^2}\text{top} \\
p & ^{\epsilon}\downarrow & ^{\epsilon}\downarrow \\
(-)_{\text{TOP}^2} & \xrightarrow{i} & (-/\Sigma)_{\text{CYRS}^2}\text{top}
\end{array}$$

(ii) If $\text{top}$ and $\text{top}'$ satisfy the property (lift) considered in 2.22 then (44) naturally defines a morphism of weakly variable topoi as well and there is a canonically pseudo-commutative cube

$$\begin{array}{ccc}
(-)_{\text{top}} & \xrightarrow{i} & (-/\Sigma)_{\text{crys,top}} \\
p & ^{\epsilon}\downarrow & ^{\epsilon}\downarrow \\
(-)_{\text{TOP}} & \xrightarrow{i} & (-/\Sigma)_{\text{CYRS}^2}\text{top} \\
p & ^{\epsilon}\downarrow & ^{\epsilon}\downarrow \\
(-)_{\text{TOP}^2} & \xrightarrow{i} & (-/\Sigma)_{\text{CYRS}^2}\text{top}
\end{array}$$

The isomorphism $ui \simeq id$ is moreover functorial and compatible with $p$ and $\epsilon$.

Proof. (i) A similar pseudo-commutative cube above a fixed $X^\Sigma$ in $\mathcal{S}ch^\Sigma/\Sigma_1$ is obvious by looking at continuous functors. Let $(v_{X^\Sigma})_{X^\Sigma \in \mathcal{S}ch^\Sigma/\Sigma_1}$ denote one of the twelve collections of weak morphisms involved. We need to achieve the following tasks:

(A) Enrich $(v_{X^\Sigma})$ into a pseudo-morphism $v$, i.e. define a functoriality isomorphism $fv_{X^\Sigma} \simeq v_{X^\Sigma}f$ for each $f : X^\Sigma \to X', this family of isomorphisms being submitted to the composition constraint.

(B) Check that the isomorphisms expressing the pseudo-commutativity of the cube above a fixed $X^\Sigma$ are functorial with respect to $X^\Sigma$, i.e. compatible with the functoriality isomorphisms of (A).

We first explain how to achieve task (A). Let us begin with $v = p$. The case of $p_{X^\Sigma}^{-1} : X_{\text{top}}^\Sigma \to X_{\text{TOP}}^\Sigma$ is obvious by looking at continuous functors. Let us thus consider the case of $p_{X^\Sigma}^{-1} : (X^\Sigma/\Sigma)_{\text{crys,top}} \to (X^\Sigma/\Sigma)_{\text{CYRS}^2\text{top}}$. The morphisms $p_{X^\Sigma}^{-1}r_{X^\Sigma}^{-1} \to id$ for $f : X' \to X^\Sigma$ give a family of morphisms

$$p_{X^\Sigma}^{-1}f_{\text{crys}}^{-1} \to f_{\text{CYRS}^2}^{-1}p_{X^\Sigma}^{-1}$$

(200)
satisfying the composition constraint. If now \( f'' : X'' \to X'' \) then (200) becomes an
isomorphism after applying \( r_{X''}^{-1} f'_{Crys^2} \) to it\(^{[2]}(i)\). It follows that (200) is in fact an
isomorphism and task (A) is achieved in the case \( v = p \).

Next we consider \( v = \epsilon \). Here again the case of usual topoi is obvious by looking at
continuous functors. The case of big crystalline topoi is clear as well (sheafification
pseudo-commutes to restriction). The case of small crystalline topoi can be deduced as
follows. We define an isomorphism \( \epsilon_{X^\sharp, s} f_{Crys^2, s} \simeq f_{Crys^2, s} \epsilon_{X^\sharp, s} \) by composition:

\[
\begin{align*}
(201) \quad f_{Crys^2, s} \epsilon_{X^\sharp, s} & \simeq f_{Crys^2, s} \epsilon_{X^\sharp, s} p_{X^\sharp, s} p_{X^\sharp}^{-1} \\
(202) \quad & \simeq f_{Crys^2, s} p_{X^\sharp, s} \epsilon_{X^\sharp, s} p_{X^\sharp}^{-1} \\
(203) \quad & \simeq p_{X^\sharp, s} f_{Crys^2, s} \epsilon_{X^\sharp, s} p_{X^\sharp}^{-1} \\
(204) \quad & \simeq p_{X^\sharp, s} \epsilon_{X^\sharp, s} f_{Crys^2, s} p_{X^\sharp}^{-1} \\
(205) \quad & \simeq \epsilon_{X^\sharp, s} p_{X^\sharp, s} f_{Crys^2, s} p_{X^\sharp}^{-1} \\
(206) \quad & \simeq \epsilon_{X^\sharp, s} f_{Crys^2, s} p_{X^\sharp, s} p_{X^\sharp}^{-1} \\
(207) \quad & \simeq \epsilon_{X^\sharp, s} f_{Crys^2, s}
\end{align*}
\]

The composition constraint for this family of isomorphism is a formal consequence (ex-
ercise!) of the previously established composition constraint for the following weak mor-
phisms of variable topoi:

\[
p : (-/\Sigma)_{Crys^2, top} \to (-/\Sigma)_{Crys, top}
\]

\[
p : (-/\Sigma)_{Crys^2, top} \to (-/\Sigma)_{Crys, top}
\]

\[
\epsilon : (-/\Sigma)_{Crys^2, top} \to (-/\Sigma)_{Crys^2, top}
\]

Let us finally consider \( v = i \). The case of big topoi is obvious by looking at cocontinuous
functors. The case of small topoi follows formally exactly as for \( v = \epsilon \).

Let us now explain task (B). The left hand face is clear by looking at continuous func-
tors. The right, top and bottom faces are tautological from the definition of functoriality
isomorphisms for \( \epsilon \) and \( i \) in the setting of small crystalline topoi. Using that \( p \) admits a
right retraction we see that it only remains to consider the front face ie. to check that
the following diagram commutes:

\[
\begin{align*}
\begin{array}{ccc}
f_{Crys^2} i_{X^\sharp} \epsilon_{X^\sharp} & \simeq & i_{X^\sharp} f_{Crys^2} \epsilon_{X^\sharp} \\
\downarrow & & \downarrow \\
& & \\
f_{Crys^2} \epsilon_{X^\sharp} i_{X^\sharp} & \simeq & \epsilon_{X^\sharp} f_{Crys^2} i_{X^\sharp}
\end{array}
\end{align*}
\]

This causes no difficulty when looking at inverse image functors.

\[\text{[ii]}\] Task (A) for \( v = u \) in the setting of big topoi is obvious by looking at cocontinuous
functors. The case of small topoi follows formally exactly as for \( v = \epsilon \). Regarding
\( \epsilon \) (B) it is sufficient to consider the front face. By cocontinuity of the functor
\( Crys^2_{top}(X^\sharp/\Sigma) \to TOP^\sharp(U^\sharp) \), \( (U^\sharp, T^\sharp) \to U^\sharp \) we find that the \( top' \) sheafification of
\( (U^\sharp, T^\sharp) \mapsto F(U) \) and \( U^\sharp \mapsto F(U^\sharp) \) coincide. This defines an isomorphism

\[
\epsilon_{X^\sharp, i} u_{X^\sharp, i}^{-1} \simeq u_{X^\sharp, i}^{-1} \epsilon_{X^\sharp, i}
\]

whose compatibility with \( f_{Crys^2}^{-1} \) is immediate.

The functoriality of the isomorphism \( id \simeq u_{X^\sharp, i}^{-1} \) is clear by looking at cocontinuous
functors. Compatibility with \( p_{X^\sharp} \) and \( \epsilon_{X^\sharp} \) causes no difficulty either looking at direct or
inverse images.

\( \square \)
9.2. Crystals on crystalline sites.

We review the definition and basic properties of crystals.

9.2.1. Let us begin with some properties of the realization functors \((-)^\flat_T\) defined in 2.24.

**Lemma 9.4.**

(i) Consider \((U^e, T^e)\) in \(\text{CRY}S^e_{\text{top}}(X/\Sigma)\). The functor \(\text{top}(T^e) \to \text{CRY}S^e_{\text{top}}(X/\Sigma)/(U^e, T^e), \ T^e/T^e \mapsto (U^e \times_{T^e} T^e, T^e)/(U^e, T^e)\) is fully faithful and continuous. The topology of \(\text{top}(T^e)\) is induced by the topology of \(\text{CRY}S^e_{\text{top}}(X/\Sigma)/(U^e, T^e)\) via this functor.

(ii) The realization functor \((-)^\flat_T : (\text{CRY}S^e_{\text{top}}(X/\Sigma)) \to T^e_{\text{top}}\) is the inverse image functor of a morphism of topoi. The corresponding functor between categories of \(\mathcal{O}\)-modules commutes in particular to arbitrary limits and commutes to tensor products.

(iii) Sending a sheaf \(F\) on \(\text{CRY}S^e_{\text{top}}(X/\Sigma)\) to \(\xi = ((U^e, T^e) \mapsto F_{T^e}, (f : (U'^e, T'^e) \to (U^e, T^e)) \mapsto (f^{-1} F_{T^e} \to F_{T'^e}))\) induces a natural equivalence

\[
(X^e/\Sigma)_{\text{CRY}S^e_{\text{top}}} \simeq \text{Real}(\text{CRY}S^e_{\text{top}}(X/\Sigma))
\]

where the right category denotes the full subcategory of \(\Gamma(\text{CRY}S^e_{\text{top}}(X/\Sigma), (-))_{\text{top}}\) whose objects are the sections \(\xi : (U^e, T^e) \mapsto \xi(U^e, T^e), (f : (U'^e, T'^e) \to (U^e, T^e)) \mapsto (\xi(f) : f^{-1} \xi(U^e, T^e) \to \xi(U'^e, T'^e))\) satisfying the condition that \(\xi(f)\) is an isomorphism if \(f\) is top cartesian.

(iv) Consider \(\text{top'}\) finer than \(\text{top}\). A sheaf \(F\) on \(\text{CRY}S^e_{\text{top}}(X/\Sigma)\) is a sheaf for \(\text{top'}\) if and only if the corresponding \(\xi\) satisfies the following descent condition: if \(f_i : (U^e_i, T^e_i) \to (U^e, T^e)\) is a covering in \(\text{CRY}S^e_{\text{top}}(X/\Sigma)\) then

\[
\xi(U^e, T^e) \simeq \ker \left( \prod_i f_i \ast \xi(U^e_i, T^e_i) \Rightarrow \prod_{j,k} f_{jk} \ast \xi(U^e_{jk}, T^e_{jk}) \right)
\]

where \(f_{jk} : (U^e_{jk}, T^e_{jk}) \to (U^e, T^e)\) is the product of \(f_i\) and \(f_k\) computed in the category \(\text{CRY}S^e_{\text{top}}(X^e/\Sigma)/(U^e, T^e)\). The full subcategory of \(\text{Real}(\text{CRY}S^e_{\text{top}}(X^e/\Sigma))\) formed by such sections will be denoted \(\text{Real}_{\text{top'}}(\text{CRY}S^e_{\text{top}}(X^e/\Sigma))\).

Proof. Everything is straightforward from definition 2.21.

Let us emphasize that the morphism of topoi mentioned in (ii) is not pseudo-functorial with respect to \((U^e, T^e)\).

**Remark 9.5.** Here are some complementary remarks.

(i) Let \(T^e = (U^e, T^e)\) in \(\text{CRY}S^e_{\text{top}}(X^e/\Sigma)\). An obvious local variant of 9.4 (iii), (iv) provides equivalences

\[
(X^e/\Sigma)_{\text{CRY}S^e_{\text{top}}} / T^e \simeq \text{Real}(\text{CRY}S^e_{\text{top}}(X^e/\Sigma)/T^e)
\]

(208)

\[
(X^e/\Sigma)_{\text{CRY}S^e_{\text{top'}}} / T^e \simeq \text{Real}_{\text{top'}}(\text{CRY}S^e_{\text{top'}}(X^e/\Sigma)/T^e)
\]

(209)
(ii) Consider $T^x, T'^x$, a morphism $h : T'^x \to T^x$ in $\text{CRYS}_{\text{top}}^x (X^x/\Sigma)$, $f_{T^x}$, the structural morphism of $T^x$ viewed as an object of the topos and $f : X'^x \to X^x$, a morphism in $\text{Sch}^x/\Sigma_1$. The functors

\[
\begin{align*}
\lambda_{T^1}^{-1} : T^x_{\text{top}} & \to (X^x/\Sigma)_{\text{CRYS}_{\text{top}}^x}/T^x \\
\eta_{T^x} : (X^x/\Sigma)_{\text{CRYS}_{\text{top}}^x} & \to (X^x/\Sigma)_{\text{CRYS}_{\text{top}}^x}/T^x \\
h^{-1} : (X^x/\Sigma)_{\text{CRYS}_{\text{top}}^x}/T^x & \to (X^x/\Sigma)_{\text{CRYS}_{\text{top}}^x}/T'^x \\
f^{-1} : (X^x/\Sigma)_{\text{CRYS}_{\text{top}}^x} & \to (X'^x/\Sigma)_{\text{CRYS}_{\text{top}}^x}
\end{align*}
\]

have the following convenient translation in terms of the corresponding categories of realizations: $(\lambda_{T^1}^{-1} F) : (h_1 : T^x_1 \to T^x) \mapsto h_1^{-1}F$, $f_{T^x}^{-1} \xi : (T^x_1/T^x) \mapsto \xi(T^x_1)$, $h^{-1} \xi : (T^x_1/T'^x) \mapsto \xi(T^x_1)$.

(iii) In [9.4 (iv)] the descent condition needs only to be checked for families $f_i : T^x_i \to T^x$ where the morphisms of schemes underlying the $f_i$'s are affine (see [EGA4-I] II, 2.3, [EGA2] 1.5.1).

(iv) Statements 9.4 (iii), (iv), 9.5 (i), (ii), (iii) have obvious counterparts for the categories of modules.

(v) Statements 9.4 (i), (ii), (iii) and the remarks 9.5 (i) (208), (iii), as well as their counterparts for modules hold verbatim if CRYS is replaced with crys. The same is true for 9.5 (ii) if one assumes that $f$ is strict and top.

Let us write down some compatibilities which are used in the body of the text.

**Lemma 9.6.** Consider $X^x$ in $\text{Sch}^x/\Sigma_1$ and let $T^x = (U^x/X^x, T^x, \iota, \gamma)$ in $\text{CRYS}_{\text{top}}^x (X^x/\Sigma)$ (resp. crys$_{\text{top}}^x (X^x/\Sigma)$). Let $f_{T^x}$ and $\lambda_{T^x}$ as in [46] and let $p$ denote the quasi-morphism of projection from $\Sigma^x$-big to small crystalline or usual top topos.

(i) There are canonical isomorphisms

\[
\begin{align*}
(i_* F)_{(U^x, T^x)} & \simeq p_* t_*(F_{U^x}) \quad (\text{resp. } (i_* F)_{(U^x, T^x)} \simeq t_* F_{U^x}) \\
p_*(i^{-1}G)_{(U^x)} & \simeq G_{(U^x, U^x)} \quad (\text{resp. } (i^{-1}G)_{U^x} \simeq G_{U^x, U^x})
\end{align*}
\]

which are functorial with respect to $F$ in the $\Sigma^x$-big (resp. small) top topos and $G$ in the $\Sigma^x$-big (resp. small) crystalline topos. topos.

(ii) If $U^x = X^x$ then we have a canonical morphism

\[
p_{U^x} f_{T^x} \to \lambda_{T^x} \quad (\text{resp. } u_{U^x} f_{T^x} \to \lambda_{T^x})
\]

This is in fact an isomorphism if top = zar or et.

Proof. The isomorphisms of (i) are immediate from the definitions. Let us prove (ii) in the case $\text{CRY} S^x$ using the local realization functors

\[
(-)^{T'^x/T^x} : (X^x/\Sigma)_{\text{CRYS}_{\text{top}}^x}/T^x \to T'^x_{\text{top}}
\]

underlying the equivalence (208). Given $F$ in $T^x_{\text{top}}$ and $h : (U'^x, T'^x) \to (U^x, T^x)$ in $\text{CRYS}_{\text{top}}^x (X^x/\Sigma)/T^x$ we have (using (i) 9.5 and full faithfulness of $p^{-1}$)

\[
\begin{align*}
(\lambda_{T^1}^{-1} F)_{T'^x/T^x} & \simeq h^{-1}F \\
(f_{T^x}^{-1} u^{-1} t^{-1} p^{-1} F)_{T'^x/T^x} & \simeq (i_* t_* (l^{-1}p^{-1} F))_{T'^x} \\
& \simeq p_* t_* (l^{-1}p^{-1} F)_{U^x} \\
& \simeq p_* t_* l^{-1} h^{-1} F \\
& \simeq i_* t_* l^{-1} F
\end{align*}
\]
The adjunction morphism \( id \to \iota_\ast \iota^{-1} \) for varying \( T^\varphi/T^\varphi \)'s and \( F \)'s gives rise to the desired morphism \( puu_f T_2 \to \lambda_T \). If now \( top = et \) (resp. \( zar \)) then \( \iota : U^\varphi_{top} \to T^\varphi_{top} \) is an equivalence (resp. an isomorphism) and this morphism is thus invertible as claimed. The case of small crystalline topoi is similar.

\[ \square \]

9.2.2. Let us come to the definition of crystals.

**Lemma 9.7.** Consider \( M \) in \( \text{Mod}((X^\varphi/\Sigma)_{\text{CRY S}^2, top}, \mathcal{O}) \) and \( T^\varphi \) in \( \text{CRY S}^2_{top}(X^\varphi/\Sigma) \). The following conditions are equivalent:

(i) The adjunction morphism \( \lambda^\varphi_T M_{T^\varphi} \to M_{\mid T^\varphi} \) is invertible.

(ii) The base change morphism \( h^\ast M_{T^\varphi} \to M_{T^\varphi} \) is invertible for all \( h : T^\varphi \to T^\varphi \) in \( \text{CRY S}^2_{top}(X^\varphi/\Sigma)/T^\varphi \).

The same holds verbatim with \( \text{crys} \) instead of \( \text{CRY S}^2 \).

**Proof.** This results from the description of \( \lambda^\varphi_T \) in terms of realizations \( (\emptyset, 5, (i), (iv)). \)

\[ \square \]

**Definition 9.8.** We say that \( M \) is a crystal of \( ((X^\varphi/\Sigma)_{\text{CRY S}^2, top}, \mathcal{O}) \) if the equivalent conditions of 9.7 are verified for all \( T^\varphi \)'s. The full subcategory of \( \text{Mod}((X^\varphi/\Sigma)_{\text{CRY S}^2, top}, \mathcal{O}) \) formed by crystals is denoted \( \text{Crys}((X^\varphi/\Sigma)_{\text{CRY S}^2, top}, \mathcal{O}) \). We use similar definitions and notations in the context of small crystalline topoi.

**Remark 9.9.** Let \( (T_i^\varphi) \) denote a family of in \( \text{CRY S}^2_{top}(X^\varphi/\Sigma) \) which covers the final object of the topos. If one of the conditions of 9.7 hold with \( T^\varphi = T_i^\varphi \) for each \( i \) then \( M \) is a crystal. Indeed if \( T^\varphi \) is arbitrary the conditions 9.7 for \( T^\varphi \) are top local on \( T^\varphi \) and it is thus sufficient to check them under the assumption that \( \text{Hom}(T^\varphi, T_i^\varphi) \neq \emptyset \). In that case the verification is straightforward. The same remark holds with \( \text{crys} \) instead of \( \text{CRY S}^2 \).

The category of crystals enjoys the following useful properties.

**Lemma 9.10.** Assume that \( top \) is finer or equal than \( et \).

(i) The adjunction \( (p_\ast, p^\ast) \) for \( \mathcal{O} \)-modules on crystalline sites induces an equivalence

\[
\text{Crys}((X^\varphi/\Sigma)_{\text{CRY S}^2, top}, \mathcal{O}) \overset{(p^\ast,p_\ast)}{\longrightarrow} \text{Crys}((X^\varphi/\Sigma)_{\text{Crys}, top}, \mathcal{O})
\]

(ii) Consider a morphism \( g : X^\varphi \to X^\varphi \) in \( \text{Sch}/\Sigma_1 \). Let \( T^\varphi \) in \( \text{CRY S}^2_{top}(X^\varphi/\Sigma) \), \( T^\varphi \) in \( \text{CRY S}^2_{top}(X^\varphi/\Sigma) \) and \( h : T^\varphi \to T^\varphi \) a morphism of dp-thickenings which is compatible with \( g \). There is a natural morphism

\[
h^\ast(M_{T^\varphi}) \to (g^\ast M)_{T^\varphi}
\]

which is functorial with respect to \( M \) in \( \text{Mod}((X^\varphi/\Sigma)_{\text{CRY S}^2, top}, \mathcal{O}) \). This is an isomorphism if \( M \) is a crystal. The category of crystals is in particular stable by \( g^\ast \). The same is true with \( \text{crys} \) instead of \( \text{CRY S}^2 \) with no restriction on \( g \).

(iii) If \( top \) is moreover coarser or equal than \( syn \) then the contravariant pseudo-functor \( \text{Crys}((-/\Sigma)_{\text{CRY S}^2, top}, \mathcal{O}) : \text{Sch}/\Sigma_1 \to \mathcal{E} \text{at} \) is a stack for \( top \) (ie. verifies top descent). The same statement holds verbatim with \( \text{crys} \) instead of \( \text{CRY S}^2 \).

**Proof.** Let us make a preliminary observation. Assume given a surjective family of \( top \) morphisms \( (f_i : U^\varphi_i \to X^\varphi) \) where the \( U^\varphi_i \)'s are affine and possess fine charts. Let \( T_{i,j}^\varphi \) denote the logarithmic divided power envelope of \( U^\varphi_i \) inside a log scheme of the form
A module $M$ of $(X, \mathcal{O})$ in mind one sees immediately that the condition of being a crystal is

and the following functors are fully faithful:

and the following functors are fully faithful:

According to definition 9.8 (resp. and 9.9 together with the above preliminary observation) a module $M$ of $(X, \mathcal{O})$ in the essential image of $\lambda_{T_z}$ for all $T_z$'s in the small top crystalline site. With the above diagram in mind one sees immediately that the condition of being a crystal is preserved by the functors $p^*$ and $p_*$. It remains to check that $p_*$ is fully faithful on the category $\text{Crys}(X, \mathcal{O})$. If $M$ is in the latter category and $T_z$ is in the small crystalline site then $f_{T_z}^* M$ is in the essential image of $\lambda_{T_z}$, hence of $p_{T_z}^*$, i.e.

Using the preliminary observation again it follows that $p^* p_* M \simeq M$ as desired.

Let us use the notations $f_{T_z/X_1} = f_{T_z}$ and $\lambda_{T_z/X_1} = \lambda_{T_z}$ in order to emphasize the dependence in $X_1$. We have the following pseudo-commutative diagram of ringed topoi and quasi-morphisms:

\[
\begin{array}{ccc}
((X_1, \mathcal{O})_{\text{Crys, top}}, \mathcal{O}) & \xrightarrow{f_{T_z/X_1}} & ((X_1, \mathcal{O})_{\text{Crys, top}}, T_1, \mathcal{O}) \\
\downarrow \phi & & \downarrow \phi \\
((X_1, \mathcal{O})_{\text{Crys, top}}, \mathcal{O}) & \xrightarrow{f_{T_z/X_1}} & ((X_1, \mathcal{O})_{\text{Crys, top}}, T_1, \mathcal{O}) \\
\downarrow h & & \downarrow \lambda_{T_z/X_1} \\
((X_1, \mathcal{O})_{\text{Crys, top}}, \mathcal{O}) & \xrightarrow{f_{T_z/X_1}} & ((X_1, \mathcal{O})_{\text{Crys, top}}, T_1, \mathcal{O}) & \xrightarrow{\lambda_{T_z/X_1}} & T_1, \mathcal{O} \\
\downarrow h & & & & \downarrow \lambda_{T_z/X_1} \\
\end{array}
\]
If $M$ is a module of $((X^\sharp/\Sigma)_{\text{CRYS}^*\text{et,top}}, \mathcal{O})$ the desired morphism is defined by composition as follows:

$$h^*\lambda_{T^\sharp/X^\sharp_1} s f^*_{T^\sharp/X^\sharp_1} M \to \lambda_{T^\sharp/X^\sharp_1} h^* f^*_{T^\sharp/X^\sharp_1} M \simeq \lambda_{T^\sharp/X^\sharp_1} s f^*_{T^\sharp/X^\sharp_1} g^* M$$

If $M$ is a crystal then $f^*_{T^\sharp,\Sigma} M$ is in the essential image of $\lambda_{T^\sharp/X^\sharp_1}$ and the first arrow is thus invertible (by full faithfulness of $\lambda$ as follows:

$$(\text{iv})$$ and 4.27 (iii); this condition could be relaxed as in [CV]). For the purpose of local

$$(\text{iv})$$ and 4.27 (iii); this condition could be relaxed as in [CV]). For the purpose of local

9.3. Modules with logarithmic connection.

9.3.1. We review the classical relation between crystals and modules with connection on the small étale site). Throughout all this section we let $1 \leq k < \infty$ and we consider an immersion (i.e., a closed immersion followed by a strict open immersion) $X^\sharp \to Y^\sharp$ where $X^\sharp$ is over $\Sigma_1$ and $Y^\sharp$ has local finite finite $p$-bases over $\Sigma_k$. This means in particular that $Y^\sharp$ has a strict étale covering by $Y^\sharp_{i,j}$’s having $p$-bases over $\Sigma_k$ of the form $(\emptyset, \ell)$ (see 4.23 (iv) and 4.27 (iii) this condition could be relaxed as in [CV]). For the purpose of local descriptions we fix such a strict covering and we use furthermore the following notations: $X^\sharp_{i,j}/X^\sharp$ denotes the base change of $Y^\sharp_{i,j}/Y^\sharp$ and $T_{\lambda} := D(X^\sharp_{i,j}, Y^\sharp_{i,j})$. We also fix a $p$-basis $(\emptyset, \ell)$ for each $Y^\sharp_i$ (e and $\ell = (t_1, \ldots, t_\ell)$ thus depend on $\lambda$ even though we do not write it in order to keep notations reasonable).

All tensor products and inner homomorphisms are taken with respect to the ring $\mathcal{O}$ unless mentioned otherwise.

9.3.2. Let us denote $Y^\sharp(i)$ the $(i+1)$-th fold product of $Y^\sharp$ with itself over $\Sigma_k$ and $(X^\sharp, T^\sharp(i))$ (resp. $(X^\sharp, T^\sharp, n(i))$) the object of $Crys(X^\sharp/\Sigma_k)$ which is the logarithmic divided power envelope (resp. of order $n$) of the diagonal morphism $X^\sharp \to Y^\sharp(i)$. Let

$$\mathcal{P}^\sharp(i)_{T,Y} \text{ (resp. } \mathcal{P}^\sharp(i,n)_{T,Y})$$

denote the ring of $Y_{zar}$ which is obtained from the structural ring of $T^\sharp(i)$ (resp. $T^\sharp(i,n)$) by pullback and pushforward along the obvious morphisms

$$Y_{zar} \leftarrow T_{zar}^{(i)} \simeq T_{zar}^{(i)} \text{ (resp. } Y_{zar} \leftarrow T_{zar}^{(i,n)} \simeq T_{zar}^{(i,n)})$$

Let $d_0, \ldots, d_i : \mathcal{P}^{(i)}_{T,Y} \to \mathcal{P}^{(i)}_{T,Y}$ or $\mathcal{P}^{(i,n)}_{T,Y} \to \mathcal{P}^{(i,n)}_{T,Y}$ denote the ring morphisms corresponding to the projections. Unless explicitly mentioned, these rings will be viewed as $\mathcal{O}_Y$-algebras via $d_0$ and the natural $\mathcal{O}_Y$-algebra structure of $\mathcal{P}^{(0)}_{T,Y}$ (resp. $\mathcal{P}^{(0,n)}_{T,Y}$).

Consider now $f : Y^\sharp \to Y^\sharp$ and denote $X^\prime := X^\sharp \times_{Y^\sharp} Y^\sharp$. Consider the logarithmic divided power envelope (resp. of order $n$) $T^\sharp(i)$ (resp. $T^\sharp, n(i)$) of $X^\sharp \to Y^\sharp, i+1$ and denote similarly $\mathcal{P}^\sharp(i)_{T^\prime,Y}$ and $\mathcal{P}^{(i,n)}_{T^\prime,Y}$ the corresponding $\mathcal{O}_{Y^\prime}$-algebras of $Y_{zar}$.

**Lemma 9.12.** If $f$ is strict étale then the natural morphisms

$$f^* \mathcal{P}^\sharp(i)_{T,Y} \to \mathcal{P}^\sharp(i)_{T^\prime,Y} \text{ (resp. } f^* \mathcal{P}^{(i,n)}_{T,Y} \to \mathcal{P}^{(i,n)}_{T^\prime,Y})$$


of \( \text{Mod}(Y'_{zar}, \mathcal{O}_{Y'}) \) are invertible. Here \( f^* \) denotes the module pullback functor of the morphism \( f : (Y'_{zar}, \mathcal{O}_{Y'}) \to (Y_{zar}, \mathcal{O}_Y) \).

Proof. We have to show that the squares of the following commutative diagram are cartesian

\[
\begin{array}{ccc}
X'^i & \to & T'^i_{(i), n} \\
\downarrow & & \downarrow \\
X^i & \to & T^i_{(i), n}
\end{array}
\]

Here the arrows denoted \( p_0 \) are induced by the first projection \( Y'^{(i)} \to Y^i \) and \( Y'^{(i)} \to Y^i \). Since logarithmic divided power envelopes and logarithmic divided power envelopes of order \( n \) commute to strict étale base change this is turn is equivalent to the natural morphism \( T'^{(i), n} \to D(X'^i, Y'^i \times Y^{(i-1)}) \) being invertible. It suffices to prove that the latter morphism is strict étale and that it lifts the identity of \( X'^i \). The result thus follows from the fact that it admits a natural decomposition as

\[
D(X'^i, Y'^{(i)}) \to D(X'^{i+1/X^i}, Y'^{(i)}) \to D(X'^i, Y'^i \times Y^{(i-1)})
\]

where the first arrow is the open immersion induced by the diagonal open immersion \( X'^i \to X'^{i+1/X^i} \) and the second is the base change of the strict étale morphism \( Y'^{(i)} \to Y^i \times Y^{(i-1)} \) (note that \( X'^i \times_{Y'^i \times Y^{(i-1)}} Y'^{(i)} \cong X'^{i+1/X^i} \)).

\[\square\]

This implies in particular that the quasi-coherent modules \( \mathcal{P}^{(i)}_{T/Y} \) and \( \mathcal{P}^{(i), n}_{T/Y} \), for varying strict étale \( Y^i \)-log schemes \( Y'^i \) satisfy descent and thus come from algebras of \((Y_{et}, \mathcal{O})\) (see the proof of [1,10]). The latter will simply be denoted \( \mathcal{P}^{(i)}_{T,Y} \) and \( \mathcal{P}^{(i), n}_{T,Y} \) as well.

**Lemma 9.13.** Let \( \iota : (T^i_{et}, \mathcal{O}) \to (Y^i_{et}, \mathcal{O}) \) denote the tautological morphism.

(i) There are canonical algebras \( \mathcal{P}_T^{(i)}, \mathcal{P}_T^{(i), n} \) in \((T^i_{et}, \mathcal{O})\) such that

\[\iota_* \mathcal{P}_T^{(i)} \cong \mathcal{P}_T^{(i)} \quad \text{and} \quad \iota_* \mathcal{P}_T^{(i), n} \cong \mathcal{P}_T^{(i), n}\]

(ii) The algebra \( \mathcal{P}_T^{(0)} \) is \( \mathcal{O} \) itself. The algebra \( \mathcal{P}_T^{(i)} \) has the following explicit local description:

\[\mathcal{P}_T^{(i)}_{|T_{\lambda}^i} \cong \mathcal{P}_T^{(0)}_{|T_{\lambda}^i} \quad \text{with} \quad u_\lambda^{a, \mu} \text{the unique section of} \mathcal{P}_T^{(i)} \text{over} T_{\lambda}^i \text{satisfying} \ u_\lambda^{a, \mu} d_\lambda t_a = d_\mu t_a. \]

We use the following standard notations in the case \( i = 1 \): \( \tau_a^i := u_\lambda^{0,1} - 1 \) and \( \text{dlog} t_a \) is its image in \( \mathcal{P}^{(1)}_{1,1}(T_{\lambda}^i) \).

(iii) Let \( \mathcal{P}_Y^{(i)}, \mathcal{P}_Y^{(i), n} \) denote the algebras of \((Y_{et}, \mathcal{O})\) whose construction is similar to \( \mathcal{P}_T^{(i)}, \mathcal{P}_T^{(i), n} \) but where the closed immersion \( X^i \to Y^i \) is replaced from the start by \( Y^i_1 \to Y^i \) (recall that \( Y_1 := \Sigma_1 \times \Sigma h \ Y \) and that \( Y = D(Y_1, Y) \)). Then we have canonical isomorphisms

\[\iota^* \mathcal{P}_Y^{(i)} \cong \mathcal{P}_T^{(i)} \quad \text{and} \quad \iota^* \mathcal{P}_Y^{(i), n} \cong \mathcal{P}_T^{(i), n}\]

Proof. (i) It is clear from the construction that \( \mathcal{P}_T^{(i)} \) and \( \mathcal{P}_T^{(i), n} \) are supported on the closed subtopos \( X_{et} \) of \( Y_{et} \). The result follows. The local description \( \text{[iii]} \) is a straightforward consequence of the one given in [1s2 1.8]. This description shows that the natural morphisms \( \iota^* \mathcal{P}_Y^{(i)} \to \mathcal{P}_T^{(i)} \) and \( \iota^* \mathcal{P}_Y^{(i), n} \to \mathcal{P}_T^{(i), n} \) are invertible as claimed in \( \text{[iii]} \).
It follows from (iii) that one has a canonical exact sequence of $P_T^{s(1)}$-modules of $T^s_{et}$. 

$$\begin{align*}
0 & \to \Omega_{T^s} \longrightarrow P_T^{s(1),1} \longrightarrow s \to \mathcal{O} \to 0
\end{align*}$$

where we denote abusively

$$\Omega_{T^s} := \iota^* \Omega_{Y^s}$$

and $s$ is the morphism induced by the diagonal immersion. After scalar restriction to $\mathcal{O}$ via $d_0$ (resp. $d_1$) the morphism $s$ has a canonical splitting given by $d_0$ (resp. $d_1$). We will further denote

$$d := d_1 - d_0 : \mathcal{O} \to \Omega_{T^s} \subset P_T^{s(1),1}$$

the canonical $\mathcal{O}$-derivation (it is not universal in general because $\Omega_{T^s}$ is not the module of Kähler differentials, see (211)).

**9.3.3.** Since for any strict étale $Y^s/Y^s$ as above, the diagonal closed immersion $X^s \to T^{s(1)} \times_{T^s} T^{s(1)}$ is exact (because $p_0 : T^{s(1)} \to T^s$ is strict) and has divided powers (as in [Be63] 2.1.3 use flatness of the projections $p_0, p_1$ to extend the divided powers in two different compatible ways) the morphism $Y^{s2} \times_Y Y^{s2} \to Y^{s2}$, $(a, b, c) \mapsto (a, c)$ induces $T^{s(1)} \times_{T^s} T^{s(1)} \to T^{s(1)}$. The resulting morphisms for varying $Y^s$ are compatible with each other (use that $T^{s(1)} \cong Y^{s2} \times_{Y^s} T^{s(1)}$) and gives rise to morphisms as follows in $\text{Mod}(T^s_{et}, \mathcal{O})$:

$$\begin{align*}
\delta & : P_T^{s(1)} \to P_T^{s(1)} \otimes d_1 \otimes d_0 P_T^{s(1)} \\
\delta^{n,n'} & : P_T^{s(1),n+n'} \to P_T^{s(1),n} \otimes d_1 \otimes d_0 P_T^{s(1),n'}.
\end{align*}$$

These morphisms are compatible with their counterparts for $P_Y^{s(1)}$. Let us write an explicit local formula. Over $T^s_\lambda$ we have

$$\delta(\tau^s_a + 1) = (\tau^s_a + 1) \otimes (\tau^s_a + 1).$$

**Definition 9.14.** We define the $\mathcal{O}$-algebra $D^s$ of $dp$-differential operator on $T^s$ (abuse of language) as the following module of $(T^s_{et}, \mathcal{O})$

$$D^s := \lim_{\longrightarrow n} D^{s^n} \quad \text{where} \quad D^{s^n} := \text{Hom}_\mathcal{O}(P_T^{s(1),n}, \mathcal{O})$$

**9.3.4.** Following closely [BO] we discuss briefly integrable quasi-nilpotent connections and hyper $dp$-stratifications.
Lemma 9.15. Let $M$ in $\text{Mod}(T^\sharp_{et}, \mathcal{O})$. The following data $[i]$, $[ii]$, $[iii]$ are equivalent.

(i) A morphism $\nabla : M \to M \otimes \Omega^2_T$ of abelian groups in $T^\sharp_{et}$ satisfying $\nabla(fx) = f\nabla(x) + x \otimes d(f)$ for any strict étale $T^\sharp/T$, $f \in \Gamma(T^\sharp, \mathcal{O})$, $x \in \Gamma(T^\sharp, M)$.

(ii) A morphism $\theta_1 : M \to M \otimes_{d_0} \mathcal{P}^{(1),1}_T$ in $\text{Mod}(T^\sharp_{et}, \mathcal{O})$ where the target is endowed with its right $\mathcal{O}$-module structure (i.e. the one coming from the $\mathcal{O}$-algebra structure of $\mathcal{P}^{(1),1}_T$ given by $d_1$) which gives back the identity of $M$ when composed with $M \otimes_{d_0} \mathcal{P}^{(1),1}_T \to M_T$.

(iii) A morphism $\nabla : \mathcal{D}^{(1),1} \otimes_{d_1} M \to M$ in $\text{Mod}(T^\sharp_{et}, \mathcal{O})$ where the source is endowed with its left $\mathcal{O}$-module structure (i.e. the one coming from the $\mathcal{O}$-algebra structure of $\mathcal{P}^{(1),1}_T$ given by $d_0$) which gives back the identity of $M$ when composed with $M \to \mathcal{D}^{(1),1} \otimes_{d_1} M$.

Such data is called a (logarithmic) connection on $M$.

Proof. $[i] \Leftrightarrow [ii]$ is by setting $\theta_1(x) = (x \otimes 1) + \nabla(x)$ and $[ii] \Leftrightarrow [iii]$ is by adjunction using that $\mathcal{P}^{(1),1}_T$ is locally free of finite type.

Lemma 9.16. Let $(M, \nabla)$ be a module with connection as in 9.15. The following conditions $[i]$, $[ii]$, $[iii]$ are equivalent.

(i) The curvature morphism

$$K : M \to M \otimes \wedge^2 \Omega^2_T$$

is zero. Here $K$ is the morphism of abelian groups of $T^\sharp_{et}$ defined as $K(x) = \nabla^1(\nabla(x))$ where $\nabla^1(x \otimes \omega) = \nabla(x) \wedge \omega + x \otimes d^1(\omega)$ and $d^1(y \, d\log z) = dy \wedge d\log z$.

(ii) There exists a morphism $\theta_2$ rendering the following square of $\text{Mod}(T^\sharp_{et}, \mathcal{O})$ commutative:

$$
\begin{array}{ccc}
M & \xrightarrow{\theta_1} & M \otimes_{d_0} \mathcal{P}^{(1),1}_T \\
\downarrow{\theta_2} & & \downarrow{\theta_1 \otimes 1} \\
M \otimes_{d_0} \mathcal{P}^{(1),2}_T & \xrightarrow{1 \otimes d^1} & M \otimes_{d_0} \mathcal{P}^{(1),1}_T \otimes_{d_1} \mathcal{P}^{(1),1}_T
\end{array}
$$

Here the tensor products modules are viewed as $\mathcal{O}$-modules via $d_1$ on the last factor.

(iii) There exists a morphism $\nabla_2$ rendering the following square of $\text{Mod}(T^\sharp_{et}, \mathcal{O})$ commutative:

$$
\begin{array}{ccc}
\mathcal{D}^{11} \otimes_{d_0} \mathcal{D}^{11} \otimes M & \xrightarrow{1 \otimes \nabla} & \mathcal{D}^{21} \otimes M \\
\downarrow{\nabla_2} & & \downarrow{\nabla} \\
\mathcal{D}^{22} \otimes M & \to & M
\end{array}
$$

Here the bottom arrow is induced by the composition in $\mathcal{D}^\sharp$ and the tensor product are viewed as $\mathcal{O}$-modules via left multiplication on the first factor.

When these conditions are verified we say that the connection $\nabla$ is integrable.
Proof. A straightforward computation with our chosen $p$-basis of $Y^x_\ast$ shows that
\[
(\theta_1 \otimes 1)(\theta_1(x)) = x \otimes 1 \otimes 1 + \sum_a (\nabla(\partial^x_a \otimes x)) \otimes \delta^{-1}(\tau^x_1) \\
+ \sum_a (\nabla(\partial^x_a \otimes (\nabla(\partial^x_a - 1) \otimes x)) \otimes \delta^{-1}(\tau^x_2[2]) + \sum_{a \leq b}(\nabla \partial^x_a) \nabla \partial^x_b(x)(\delta^{-1}(\tau^x_{a+b})) \\
+ K(x)
\]
where $K(x) = \sum_{a < b}((\nabla(\partial^x_a \otimes (\nabla(\partial^x_a \otimes x)))) - (\nabla(\partial^x_a \otimes (\nabla(\partial^x_a \otimes x)))) \otimes (\tau^x_1 \otimes \tau^x_2)$
and that $K(x)$ is sent to $K(x)$ via the canonical morphism
\[
M \otimes_{d_0} \mathcal{P}_{T}^{(1),1} \to M \otimes_{d_0} \text{Coker}(\delta^{1,1}) \simeq M \otimes \wedge^2 \mathcal{P}_{T}^{(1),1}
\]
This shows that conditions (i) and (ii) are equivalent. Conditions (ii) and (iii) on the other hand are clearly equivalent by adjunction.

If $\nabla$ is an integrable connection on $M$ then it follows from the explicit description of $\mathcal{D}^1$ that there is a unique structure of $\mathcal{D}^1$-module on $M$ extending $\nabla$ (ie. such that $\partial x := \nabla(\partial \otimes x)$ for any $\partial$ in $\mathcal{D}^1$ and $x$ in $M$). By adjunction, one deduces a right $\mathcal{O}$-linear morphism
\[
\theta : M \to \limproj_n (M \otimes_{d_0} \mathcal{P}_{T}^{(1),n})
\]
lifting $\theta_1$ and satisfying the cocycle condition $(\theta \otimes 1)(\theta(x)) = \delta(\theta(x))$. The explicit local description of $\theta$ is given by the following Taylor formula:
\[
(216) \quad \theta(x) = (\sum_{|\alpha| \leq n} \partial_{[\lambda]}^x \otimes \tau^{[\lambda]}_n)_n
\]

Lemma 9.17. Consider a module with integrable connection $(M, \nabla)$. The following conditions (i), (ii) are equivalent.

(i) For all strict étale $T^u$ over $T^x$ and $x \in \Gamma(T^u, M)$ all but a finite number of the $\partial_{[\lambda]}^x$’s vanish in $\Gamma(T^u, M)$.

(ii) The morphism (215) factors through a morphism
\[
\theta : M \to M \otimes_{d_0} \mathcal{P}_{T}^{(1)}
\]
When these conditions are verified we say that the integrable connection $\nabla$ is quasi-nilpotent.

Proof. This is straightforward from the formula (216).

\[
\square
\]

Proposition 9.18. Let $M$ in $\text{Mod}(T^x_{\text{et}}, \mathcal{O})$. The data of an integrable quasi-nilpotent connection $\nabla$ on $M$ is equivalent to a hyper $dp$-stratification i.e. a $\mathcal{P}_{T}^{(1)}$-linear isomorphism
\[
\epsilon : \mathcal{P}_{T}^{(2)} \otimes_{d_0} M \simeq M \otimes_{d_0} \mathcal{P}_{T}^{(1)}
\]
satisfying the cocycle condition
\[
(\mathcal{P}_{T}^{(2)} \otimes_{d_0,1} \mathcal{P}_{T}^{(1)}) \circ (\mathcal{P}_{T}^{(2)} \otimes_{d_0,2} \mathcal{P}_{T}^{(1)} \epsilon) = (\mathcal{P}_{T}^{(2)} \otimes_{d_0,2} \mathcal{P}_{T}^{(1)} \epsilon)
\]

Proof. On deduces $\epsilon$ from $\theta$ by scalar extension via $d_1$. Then condition (216) (ii) may be translated into the cocycle condition using that $\mathcal{P}_{T}^{(2)} \simeq \mathcal{P}_{T}^{(1)} \otimes_{d_0} \mathcal{P}_{T}^{(1)}$ and that via this identification $\delta : \mathcal{P}_{T}^{(1)} \rightarrow \mathcal{P}_{T}^{(1)} \otimes_{d_0} \mathcal{P}_{T}^{(1)}$ translates into $d_{0,2}$.

\[
\square
\]
9.3.5. We use the following notation.

**Definition 9.19.** The category \( \nabla \text{-Mod}(T^\sharp) \) is defined as follows. An object is a module with quasi-nilpotent integrable connection. A morphism \((M, \nabla) \to (M', \nabla')\) is an \( \mathcal{O} \)-linear morphism which is compatible with the given connections. If \(*\) is either qcOH, lf or lff then we denote furthermore \( \nabla \text{-Mod}_*(T^\sharp) \) the full subcategory formed by the \((M, \nabla)\) with \( M \) satisfying \(*\).

Note that as in the case of the algebra of \( dp \)-differential operators the reference to \( T^\sharp \) only is abusive since the category depends in fact a priori on the closed immersion \( X^\sharp \to Y^\sharp \) rather then \( T^\sharp \) itself.

We have the following analogue of [BO] 6.6.

**Proposition 9.20.** There is a canonical equivalence

\[
\text{Crys}_*((X^\sharp/\Sigma_k)_{\text{crys,et}}, \mathcal{O}) \simeq \nabla \text{-Mod}_*(T^\sharp).
\]

Proof. The local lifting property for log schemes with local finite \( p \)-bases together with the universal property of logarithmic divided power envelopes ensure that \( T \) covers the final object of \((X^\sharp/\Sigma_k)_{\text{crys,et}}\). As a result, the category of modules of \((X^\sharp/\Sigma_k)_{\text{crys,et}}, \mathcal{O}\) is equivalent to the category of modules of \((X^\sharp/\Sigma_k)_{\text{crys,et}}/T^\sharp, \mathcal{O}\) endowed with a descent datum satisfying the cocycle condition. It is thus equivalent to the category of modules on \((T_{et}, \mathcal{O})\) endowed with the extra data coming from the descent datum \((\text{9.7}(i), \text{9.9})\). It only remains to notice that a descent datum \( \epsilon : p^*_1 M \simeq p^*_0 M \) on a module \( M \) over \((X^\sharp/\Sigma_k)_{\text{crys,et}}/T^\sharp\) exactly translates into a descent datum \( \epsilon : p^*_1 M_{T^\sharp} \simeq p^*_0 M_{T^\sharp} \) on the realization \( M_{T^\sharp} \) i.e. (using the diagonal equivalence \( T_{et} \to T_{et}(1) \)) into a \( \mathcal{P}_{T}(1) \)-linear isomorphism \( \epsilon : \mathcal{P}_{T}(1) \otimes M \simeq M \otimes_{\mathcal{D}_T(1)} \mathcal{P}_{T}(1) \). We may conclude by \[\text{9.18}\] in the case \(*\) = \( \emptyset \). The other cases follow (see \[\text{1.21(3)}\] for the meaning of the category on the left).

9.3.6. We explain inverse images for modules with connection.

**Lemma 9.21.** Let \( X \to Y \) and \( X' \to Y' \) as in \[\text{9.3.1}\] and assume given a commutative square

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}
\]

Let \( f_T : T^\sharp \to T'^\sharp \) denote the morphism obtained by forming logarithmic divided power envelopes. Let \( M' \) in \( \text{Crys}((X'^\sharp/\Sigma_k,\text{crys,et}), \mathcal{O}) \) and consider its pullback \( M := f^*_X M' \) in \( \text{Crys}((X^\sharp/\Sigma_k,\text{crys,et}), \mathcal{O}) \). If \( M' \) corresponds to \((N', \nabla')\) in \( \nabla' \text{-Mod}(T'^\sharp) \) then \( M \) corresponds to \((N, \nabla)\) in \( \nabla \text{-Mod}(T^\sharp) \) where \( N \simeq f^*_X N' \) and \( \nabla \) has the following alternative characterizations:

(i) if \( \nabla' \) corresponds to \( \epsilon' : \mathcal{P}_{T'}(1) \otimes N' \simeq N' \otimes \mathcal{P}_{T'}(1) \) on \((T_{et}', \mathcal{P}_{T'}(1))\) then \( \nabla \) corresponds to the morphism \( \epsilon : \mathcal{P}_{T}(1) \otimes N \simeq N \otimes \mathcal{P}_{T}(1) \) deduced from \( \epsilon' \) by pullback via \((T_{et}, \mathcal{P}_{T}(1)) \to (T_{et}', \mathcal{P}_{T'}(1))\).

(ii) if \( \nabla' \) corresponds to \( \theta' : N' \to N' \otimes \mathcal{P}_{T'}(1) \) on \((T_{et}', \mathcal{O})\) (the \( \mathcal{O} \)-module structure on the target is via \( d_1 \)) then \( \nabla \) corresponds to the morphism \( \theta : N \to N \otimes \mathcal{P}_{T}(1) \) deduced from \( \theta' \) by pullback via \( f_T : (T_{et}, \mathcal{O}) \to (T_{et}', \mathcal{O}) \) and the natural “base change morphism” \( f_T^*(N' \otimes \mathcal{P}_{T'}(1)) \to N \otimes \mathcal{P}_{T}(1) \).
(iii) $\nabla$ is the unique connection on $M$ rendering the following diagram of abelian groups of $T_{et}$

$$
\begin{array}{ccc}
N & \xrightarrow{\nabla} & N \otimes \Omega_{T^2} \\
\downarrow & & \downarrow \\
\text{f}_{T}^{-1}N' & \xrightarrow{f_{T}^{-1}\nabla'} & \text{f}_{T}^{-1}(N' \otimes \Omega_{T^2})
\end{array}
$$

commutative, the right vertical arrow being induced by $\text{f}_{T}^{-1}N' \to N$, $\text{f}_{T}^{-1}\mathcal{O} \to \mathcal{O}$ and $\text{f}_{T}^{-1}\Omega_{T^2} \to \Omega_{T^2}$.

Proof. The first statement is 9.10 (ii) since $N' = M'_{et}$ and $N = M_{et}$. Characterization (i) is also an easy consequence of 9.10 (ii) by looking at the commutative following commutative diagram of $CRY S^g_{et}(X^g/\Sigma_1)$

$$
\begin{array}{ccc}
(X, T^g) & \xrightarrow{p_0} & (X^g, T^{g(1)}) & \xrightarrow{p_1} & (X, T^g) \\
\downarrow & & \downarrow & & \downarrow \\
(X' \to T^{g(1)}) & \xrightarrow{p_0} & (X'^{g(1)}, T^{g(1)}) & \xrightarrow{p_1} & (X'^{g}, T^{g})
\end{array}
$$

Characterization (ii) follows immediately and (iii) as well (note that by linearity, $\theta$ is characterized by its composition with the $\text{f}_{T}^{-1}\mathcal{O}$-linear map $\text{f}_{T}^{-1}N' \to N$).

$\square$

9.4. Crystalline and de Rham cohomology of crystals (linearization functors).

We discuss the de Rham interpretation of the étale crystalline cohomology of a crystal in the local case following the exposition of [BO]. We keep the notations and assumptions of 9.3. Unless mentioned otherwise tensor products and inner homomorphisms are taken with respect to the ring $\mathcal{O}$.

9.4.1. Let us begin with the definition of the category of hyper $dp$-differential operators.

**Definition 9.22.** The category $Hdp(T^g)$ is defined as follows:
- an object is a module of $(T^g_{et}, \mathcal{O})$,
- the set $Hdp(M, N)$ of morphisms from $M$ to $N$ is $\text{Hom}(\mathcal{P}_T^{g(1)}, M, N)$ (the tensor product is viewed as an $\mathcal{O}$-module via $d_0$ on $\mathcal{P}_T^{g(1)}$) and composition is defined by the formula $f \circ Hdp g := f \circ (1 \otimes g) \circ \delta$.

We note that if $\mathcal{O}^{d=0}$ denotes the kernel of the canonical derivation $d: \mathcal{O} \to \Omega_{T^2}$ (217) we have the forgetful functor

$$
Hdp(T^g) \to \text{Mod}(T^g_{et}, \mathcal{O}^{d=0})
$$

The notation $Hdp(T^g)$ is abusive since this category $a priori$ depends on $X^g \to Y^g$ rather than $T^g$ itself. When emphasis on this dependance is needed we will also use the more correct notation $Hdp^{dp}(X^g \to Y^g)$.

**Lemma 9.23.** There is a functor

$$
\otimes: \nabla-\text{Mod}(T^g) \times Hdp(T^g) \to Hdp(T^g)
$$

which sends:
- a couple of objects $((M, \nabla), N)$ to $M \otimes N$,
- a couple of morphisms \( (f : (M_1, \nabla_1) \to (M_2, \nabla_2), g : \mathcal{P}_T^{(1)} \otimes N_1 \to N_2) \) to the composed morphism \( (f \otimes g) \circ (e_1 \otimes 1) : \mathcal{P}_T^{(1)} \otimes M_1 \otimes N_1 \to M_2 \otimes N_2 \) where \( e_1 \) is the hyper \( dp \)-stratification corresponding to \( \nabla_1 \).

Proof. Use the cocycle condition for \( e_1 \) in order to check the compatibility with respect to composition in \( Hdp(T^2) \).

\[ \square \]

9.4.2. We review the de Rham complex of a module with logarithmic connection.

**Lemma 9.24.** (i) There exists a unique de Rham complex of \( \text{Mod}(T_{et}, \mathcal{O}^{d=0}) \)

\[
\Omega_{T^2}^\bullet := \left[ \mathcal{O} \xrightarrow{d^0} \Omega_{T^1} \xrightarrow{d^1} \ldots \xrightarrow{d^{q-1}} M \otimes \Lambda^q \Omega_{T^1} \right]
\]

whose differentials are characterized by the following formulae:
- \( d^0 = d := d_1 - d_0 : \mathcal{O} \to \Omega_{T^2} \subseteq \mathcal{P}_{T^1}^{(1)}, \)
- \( d^{q+1}(\omega \wedge dlog m) = (d^q \omega) \wedge dlog m \) (or equivalently \( d^{q+q'} \omega \wedge \omega' = d^q \omega \wedge \omega' + (-1)^q \omega \wedge d^{q'} \omega' \))

(ii) There is a functor \( Hdp(T^2) \) which is sent to \( \Omega_{T^2}^\bullet \), under the forgetful functor \( [217] \). Its differentials \( \tilde{d}^q \) are given by the formulae:
- \( \tilde{d}^0 := 1 - d_1 s \) (with \( s \) as in \( [210] \)),
- \( \tilde{d}^q(\lambda \otimes \omega) := (d^0 \lambda) \wedge \omega + s(\lambda)d^q \omega \).

Proof. A simplicial construction of the differentials complex can be carried out by adapting \( [11] \) chap. VIII, 1.2.8 to our case (denote \( I = Ker s : \mathcal{P}_T^{(1)} \to \mathcal{O} \) and check directly that the well defined maps \( d^0 = d_1 - d_0 : \mathcal{O} \to \Omega_{T^2} \), \( d^{q+1} : I \otimes I^{q+2} \to I^{q+2} \), \( a \otimes b \mapsto (1 \otimes a - \delta a + a \otimes 1) - a \otimes d^q b \) induce \( d^q : \Lambda^q(I/I[2]) \to \Lambda^{q+1}(I/I[2]) \) satisfying the desired formulae).

\[ \square \]

**Definition 9.25.** For \( M = (M, \nabla) \) in \( \nabla\text{-Mod}(T^2) \) we define

\[
\Omega_{T^2}^\bullet(M) := (M, \nabla) \otimes \Omega_{T^1}^\bullet
\]

in \( Hdp(T^2) \). Here the de Rham complex on the right is meant in the sense of \( [9.24](ii) \) and the tensor product is \( \otimes_{T^2} \).

The differential of degree \( q \) of this complex is denoted \( \nabla^q \) and can be described explicitly by the following formula:

\[
\nabla^q(\lambda \otimes m \otimes \omega) = (m \otimes d^0(\lambda) + s(\lambda)\nabla(m)) \wedge \omega + s(\lambda)m \otimes d^q \omega
\]

9.4.3. We discuss the linearization functor. If \( M \) is an \( \mathcal{O} \)-module we usually view \( \mathcal{P}_T^{(1)} \otimes M \) as an \( \mathcal{O} \)-module via \( d_0 \) and \( M \otimes \mathcal{P}_T^{(1)} \) as an \( \mathcal{O} \)-module via \( d_1 \). In order to keep notations simple we will use the obvious isomorphism of \( \mathcal{O} \)-modules

\[
(218) \quad \text{exch} : \mathcal{P}_T^{(1)} \otimes M \simeq M \otimes \mathcal{P}_T^{(1)}
\]

(this is not a hyper \( dp \)-stratification, only a triviality) obtained by “exchanging factors in \( \mathcal{P}_T^{(1)} \)” (in terms of local coordinates \( \text{exch}(\tau_0^2 \otimes m) = m \otimes (1 + \tau_0^2)^{-1} - 1 \).

**Lemma 9.26.** There is a functor

\[
L_{T^2} : Hdp(T^2) \to \nabla\text{-Mod}(T^2)
\]
which sends:
- an $\mathcal{O}$-module $M$ to $L_T! (M) := (\mathcal{P}_T^{(1)} \otimes M, \nabla)$ where $\nabla$ is “derivation on the first factor”, i.e. corresponds to the $d_1$-linear morphism

$$M \otimes \mathcal{P}_T^{(1)} \overset{\sim}{\underset{\text{id} \otimes \delta}{\longrightarrow}} M \otimes \mathcal{P}_T^{(1)} \otimes \mathcal{P}_T^{(1)} \overset{\sim}{\underset{\text{exch}^{-1}}{\longrightarrow}} (\mathcal{P}_T^{(1)} \otimes M) \otimes \mathcal{P}_T^{(1)}$$

- an $\mathcal{O}$-linear morphism $f : \mathcal{P}_T^{(1)} \otimes M \to N$ to the composed $\mathcal{O}$-linear morphism

$$\mathcal{P}_T^{(1)} \otimes M \overset{(\delta \otimes \text{id})}{\longrightarrow} \mathcal{P}_T^{(1)} \otimes \mathcal{P}_T^{(1)} \otimes M \overset{\text{id} \otimes f}{\longrightarrow} \mathcal{P}_T^{(1)} \otimes N$$

Proof. Once noticed that $L_T! (f) = \text{exch}^{-1} \circ (\text{id} \otimes \delta) \circ (f \otimes \text{id}) \circ \text{exch}$ one may check that everything boils down to the coassociativity of $\delta$.

Let us write a local formula for the connection $\nabla$ of $L_T! (M)$:

$$\nabla \left( \sum_a \prod_b ((1 + \tau_a^z)^{-1} - 1)^{[n_a]} \otimes x_a \right) = \sum_a \sum_b (1 + \tau_a^z)^{-1} \prod_b ((1 + \tau_b^z)^{-1} - 1)^{[n_a]} \otimes x_{a+b} \otimes d \log t_b$$

Lemma 9.27. There is a canonical isomorphism in $\nabla$-$\text{Mod}(T^z)$

$$M \otimes L_T! (N) \simeq L_T! (M \otimes N)$$

which is functorial with respect to $(M, N)$ in $\nabla$-$\text{Mod}(T^z) \times \text{Hdp}(T^z)$. Here the first tensor product is meant in the sense of 9.23 while the second is the usual one in $\nabla$-$\text{Mod}(T^z)$ (the connection on $M'' = M \otimes M'$ is the one corresponding to $\epsilon'' = (\epsilon' \otimes 1) \circ (1 \otimes \epsilon)$ i.e. $\nabla'' = \nabla \otimes 1 + 1 \otimes \nabla'$)

Proof. The claimed isomorphism is

$$M \otimes \mathcal{P}_T^{(1)} \otimes N \overset{\text{exch}}{\longrightarrow} N \otimes \mathcal{P}_T^{(1)} \otimes M \overset{\text{id} \otimes \epsilon_M}{\longrightarrow} N \otimes M \otimes \mathcal{P}_T^{(1)} \overset{\text{exch} \otimes N}{\longrightarrow} \mathcal{P}_T^{(1)} \otimes M \otimes N$$

Compatibility with connections and bifunctoriality is easily checked using the cocycle condition for $\epsilon_M$.

9.4.4. We have now all the ingredients needed to state and prove the logarithmic version of the Poincaré lemma.

Lemma 9.28. The morphism $d_0 : \mathcal{O} \to \mathcal{P}_T^{(1)}$ induces a quasi-isomorphism

$$\mathcal{O} \overset{\text{aug}}{\longrightarrow} L_T! (\Omega^*_{T^z})$$

of complexes in the abelian category $\nabla$-$\text{Mod}(T^z)$.

Proof. The differentials of the complex of $\mathcal{O}$-modules underlying $L_T! (\Omega^*_{T^z})$ are given by the following formulae, which are the logarithmic variant of [BO] 6.11. Recall that
We may thus compute the source of (220) as follows

\[ h^*(f_{T^q,M})_{T^q} \Rightarrow (f_{T^q,M})_{T^q} \]

is invertible for each \( h : (U'^q, T'^q, \gamma') \rightarrow (X^q, T^q, \gamma) \) in \( \text{CRY} S^q_{et}(X^q/\Sigma_k) \). (see 9.5(ii)). As in the proof of [Be1] IV 3.1.6 we may use loc. cit. I 1.7.2 to check that there is a unique divided power structure \( \gamma'_1 \) producing the upper left hand corner in the following cartesian square of \( \text{CRY} S^q_{et}(X^q/\Sigma_k) \).

\[
\begin{array}{ccc}
(U'^q, T'^q) \times_{T^q, p_0} T^{q(1)} & \xrightarrow{h} & (X^q, T^{q(1)}, \gamma_1) \\
p_0 & & \downarrow p_0 \\
(U'^q, T'^q, \gamma') & \xrightarrow{h} & (X^q, T^q, \gamma)
\end{array}
\]

We may thus compute the source of (220) as follows

\[
h^*(f_{T^q,M})_{T^q} = h^* \lambda_{T^q}\lambda_{T^1}^{-1} f_{T^q,M}
\]

\[
\simeq h^* \lambda_{T^q} p_{0,s} p_1^q M
\]

\[
\simeq h^* p_{0,s} \lambda_{T^q(1),s} p_1^q M
\]

\[
\simeq h^* p_{0,s} M_{T^{q(1)}}
\]

\[
\simeq p_{0,s} h^* M_{T^{q(1)}}
\]

(see 9.5(ii) (iv))

Here the last base change isomorphism is due to the fact that the vertical arrows denoted \( p_0 \) are affine on the underlying schemes and induce equivalences \( (T'^q \times_{T^q} T^{q(1)})_{et} \simeq T'^q_{et}, T^{q(1)}_{et} \simeq T^{q(1)}_{et} \) (the main point is the isomorphism \( h^* p_{0,s} O \simeq p_{0,s} h^* O \) which follows from
Lemma 9.31. \(\text{(i)}\) Consider the morphism of topoi
\[ \lambda_{T^\sharp} \times T_{p^\sharp}^\ast \to T_{p^\sharp}^\ast M \]
and we may conclude using the assumption on \(M\) that the resulting isomorphism is compatible with connections. The proof of \(\text{(ii)}\) is left to the reader.

\(\square\)

Lemma 9.30. \(\text{(i)}\) The following diagram is canonically pseudo-commutative:
\[
\begin{array}{ccc}
\text{Mod}(T^\sharp_{et}, \mathcal{O}) & \xrightarrow{f^\ast_{T^\sharp}} & \text{Crys}((X^\sharp/\Sigma_k)_{CRYS,et}, \mathcal{O}) \\
\downarrow \text{nat} & & \downarrow \text{(-)}_{T^\sharp} \\
Hdp(T^\sharp) & \xrightarrow{L_{T^\sharp}} & \nabla\text{-Mod}(T^\sharp)
\end{array}
\]
\(\text{(ii)}\) If \(M\) is a crystal on \((X^\sharp/\Sigma_k)_{CRYS,et}, \mathcal{O})\) then \(f^\ast_{T^\sharp} M\) satisfies the condition of \(\text{9.29}\). Consider the adjunction morphism
\[
(221) \quad M \xrightarrow{f^\ast_{T^\sharp}} f^\ast_{T^\sharp} M
\]
as a morphism in \(\text{Crys}((X^\sharp/\Sigma_k)_{CRYS,et}, \mathcal{O})\). Via the equivalence \(\text{9.20} \ (221)\) translates into a morphism \(\nabla\text{-Mod}(T^\sharp)\)
\[
(222) \quad M_{T^\sharp} \xrightarrow{\text{aug}} L_{T^\sharp}(M_{T^\sharp})
\]
which may be described as the following composed morphism in \(\text{Mod}(T^\sharp_{et}, \mathcal{O})\):
\[
M_{T^\sharp} \xrightarrow{id \otimes d_0} M_{T^\sharp} \otimes \mathcal{P}_T^{(1)} \xrightarrow{\epsilon_{T^\sharp}^{-1} \otimes M_{T^\sharp}} \mathcal{P}_T^{(1)} \otimes M_{T^\sharp}
\]

Proof. \(\text{(i)}\) One checks that for a module \(M\) over \((T^\sharp_{et}, \mathcal{O})\) the \(\mathcal{O}\)-modules \(L_{T^\sharp}(M)\) and \(\lambda_{T^\sharp} \times f^\ast_{T^\sharp} f^\ast_{T^\sharp} M\) both identify naturally with \(p_{0^\ast} M\) and that the resulting isomorphism is compatible with connections. The proof of \(\text{(ii)}\) is left to the reader.

\(\square\)
9.4.6. We are now in position of proving the expected analogue of [BO] 6.12 and 7.1. Here we restrict ourselves to the small étale crystalline site. Let \( L \) denote the following composed functor:

\[
Hdp(T^\natural) \xrightarrow{\mathbf{L} T^\natural} \nabla-\text{Mod}(T^\natural) \xrightarrow{\mathbf{L} T^\natural} \text{Crys}((X^\natural/\Sigma_k)_{\text{cryst,et}}, \mathcal{O})
\]

(223)

Proposition 9.32. Consider a crystal \( M \) on \((X^\natural/\Sigma_k)_{\text{cryst,et}}\).

(i) The morphism (222) induces a quasi-isomorphism

\[
M \xrightarrow{\text{aug}} L(\Omega^\bullet_{T^\natural}(M_{T^\natural}))
\]

in the category of complexes of modules of \((X^\natural/\Sigma_k)_{\text{cryst,et}}, \mathcal{O})\).

(ii) The quasi-isomorphism (i) induces an isomorphism

\[
Ru_* M \xrightarrow{\sim} \Omega^\bullet_{T^\natural}(M_{T^\natural})
\]

in the derived category of modules on \((X^\natural_{et}, \mathcal{O}^\text{crys}_k)\).

Proof. (i) Begin with the case \( M = \mathcal{O} \). We already know from 9.28 that \( L(\Omega_{T^\natural}) \) is a resolution of \( \mathcal{O} \) via \( d_0 \) in the abelian category \( \text{Crys}((X^\natural/\Sigma_k)_{\text{cryst,et}}, \mathcal{O}) \). To see that it is in fact a resolution in \( \text{Mod}((X^\natural/\Sigma_k)_{\text{cryst,et}}, \mathcal{O}) \) it suffices to notice that

\[
\mathcal{O}_{T^\natural} \xrightarrow{\sim} L(\Omega^\bullet_{T^\natural}(\mathcal{O}_{T^\natural}))
\]

remains a quasi-isomorphism when pulled back to \((T^\prime_{et}, \mathcal{O})\) via an arbitrary morphism \( g : T^\prime \rightarrow T^\natural \) of the crystalline site (indeed the modules \( \mathcal{O} \) and \( L(\Omega^\bullet_{T^\natural}(\mathcal{O})) \) are flat). The case of an arbitrary crystal \( M \) follows from the isomorphism

\[
M \otimes L(\Omega^\bullet_{T^\natural}(M_{T^\natural})) \simeq L(\Omega^\bullet_{T^\natural}(M_{T^\natural})) \text{ in } \text{Mod}((X^\natural/\Sigma_k)_{\text{cryst,et}}, \mathcal{O})
\]

(use 9.27) by flatness of the modules \( L(\Omega^\bullet_{T^\natural}) \).

(ii) According to 9.31 (i) we have a canonical isomorphism

\[
u_\natural L(\Omega^\bullet_{T^\natural}(M_{T^\natural})) \simeq \Omega^\bullet_{T^\natural}(M_{T^\natural})
\]

in the category of complexes of modules on \((X^\natural_{et}, \mathcal{O}^\text{crys}_k)\). The result will thus follow from the following

Claim. The module \( L(N) \) of \((X^\natural/\Sigma_k)_{\text{cryst,et}}, \mathcal{O}) \) is \( u_* \)-acyclic for any \( N \) in \( \text{Mod}(T^\natural_{et}, \mathcal{O}) \).

Let us prove the claim. Let \( T^\natural \) denote an arbitrary object in \( \text{crys}_{et}(X^\natural/\Sigma_k) \) and form the product \( T^{\prime \natural} = T^\natural \times_{\text{cryst}} T^\natural \) computed in the crystalline site. If \( p_0 : T^{\prime \natural} \rightarrow T^\natural \) and \( p_1 : T^{\prime \natural} \rightarrow T^\natural \) denote the canonical projections, we have the following natural isomorphisms:

\[
(-)_{T^\natural} \circ f_{T^\natural} \simeq \lambda_{T^\natural} \circ f_{T^\natural} \circ f_{T^\natural} \simeq \lambda_{T^\natural} \circ p_0 \circ p_1 \simeq p_0 \circ \lambda_{T^\natural} \circ p_1.
\]
Since $p_0 : T_{et}^\eta \to T_{et}^\eta$ is an equivalence these isomorphisms show that the functor $f_{T^*,*}$ is exact. Whence isomorphisms

$$Ru_*(\lambda_T^*)N \cong Ru_*(f_{T^*,*}L_T\lambda_T^*)N$$

$$\cong R(u_*f_{T^*,*})\lambda_T^*N$$

and the claim follows since $u_*f_{T^*,*}$ is isomorphic to $\iota^{-1}\lambda_T^*$, hence exact.

\[ \square \]

9.4.7. Letting $k$ vary in 9.3, 9.4 causes no difficulty. Functoriality with respect to the closed immersion $X^2 \to Y^2$ fixed at the beginning of 9.3 is less straightforward and so we briefly give some explanations.

Recall from 9.21 that the category $\nabla\text{-}\text{Mod}(\text{--})$ is naturally a contravariant pseudo-functor in the argument $X^2 \to Y^2$. The category $Hdp(\text{--})$ on the other hand is naturally a covariant pseudo-functor with respect to $X^2 \to Y^2$ as explained in the following lemma.

Lemma 9.33. Consider a morphism $f = (f_X, f_Y) : (X^2 \to Y^2) \to (X'^2 \to Y'^2)$ between closed immersions as in 9.3, and let $f_T : T^2 \to T'^2$ denote the morphism obtained by forming logarithmic divided power envelopes. There is a natural functor

$$f_* : Hdp(T^2) \to Hdp(T'^2)$$

sending a module $N$ to $f_{T^*,*}N$.

Proof. We have to specify the effect of $f_*$ on morphisms. We define the image of a morphism $g : N_1 \to N_2$ in $Hdp(T^2)$ (ie. $g : P_T^{(1)} \otimes N_1 \to N_2$ in $\text{Mod}(T_{et}^2, O)$) as the composed morphism $P_T^{(1)} \otimes f_{T^*,*}N_1 \to f_{T^*,*}(P_T^{(1)} \otimes N_1) \to f_{T^*,*}N_2$ (the first arrow is the base change morphism). One checks without difficulty that this defines indeed a functor.

\[ \square \]

Lemma 9.34. Let $f = (f_X, f_Y)$ as in 9.33.

(i) There is a canonical base change morphism

$$ch_f : f^*L_{T^2}(\Omega_{T^2}(M')) \to L_{T'^2}(\Omega_{T'^2}(f^*M))$$

in $\nabla\text{-}\text{Mod}(T^2)$ which is functorial with respect to $M'$ in $\nabla\text{-}\text{Mod}(T'^2)$. The collection of the $ch_f$'s satisfies the composition constraint $ch_g \circ g^*ch_f = ch_{fg}$.

(ii) The base change morphism renders the following square of $\nabla\text{-}\text{Mod}(T^2)$ commutative:

\[ (224) \]

\[ \begin{array}{ccc}
    f^*M' & \xrightarrow{\text{aug}} & L_{T^2}(\Omega_{T^2}(f^*M)) \\
    \downarrow f^* & \xrightarrow{\text{aug}} & \downarrow ch_f \\
    f^*L_{T'^2}(\Omega_{T'^2}(M')) & \xrightarrow{ch_f} & L_{T'^2}(\Omega_{T'^2}(f^*M))
\end{array} \]

Proof. Let us only explain the construction of $ch_f$. First we notice that there is a natural base change morphism

\[ (225) \]

$$f^*L_{T^2}(f_*N) \to L_{T^2}(N)$$

in $\nabla\text{-}\text{Mod}(T^2)$ for $N$ in $Hdp(T^2)$. Next it follows from the description of $f^*$ in 9.21 that there is a natural base change morphism

\[ (226) \]

$$f^*\Omega_{T^2}(M') \to \Omega_{T^2}(f^*M')$$
for \( M' \) in \( \nabla\text{-}Mod(T'^\#) \). From there one easily deduces a morphism

\[
\Omega_{T'^\#}^\bullet(M') \to f_*\Omega_{T^\#}^\bullet(f^*M')
\]

of complexes in \( \text{Hdp}(T'^\#) \). The morphism \( ch_f \) is obtained by combining (225) and (227).

\[\square\]

References

[Ar] M. Artin, Grothendieck topologies. Notes on a seminar (1962).

[Ba] W. Bauer, On the conjecture of Birch and Swinnerton-Dyer for abelian varieties over function fields in characteristic \( p > 0 \). Invent. Math. 108, no. 2 (1992), p. 263-287.

[Be1] P. Berthelot, Cohomologie cristalline des schémas de caractèreistique \( p > 0 \). (French) Lecture Notes in Mathematics 407. Springer-Verlag, Berlin-New York (1974). 604 p.

[Be2] P. Berthelot, Cohomologie rigide et cohomologie rigide à support propre I, preprint (1991).

[Be3] P. Berthelot, D-modules arithmétiques. I. Opérateurs différentiels de niveau fini. (French) [Arithmetical D-modules. I. Differential operators of finite level] Ann. Sci. Ecole Norm. Sup. (4) 29, no. 2 (1996), p. 185-272.

[Be4] P. Berthelot, D-modules arithmétiques. II. Descente par Frobenius. (French) [Arithmetic D-modules. II. Frobenius descent] Mém. Soc. Math. Fr. (N.S.) 81 (2000), vi+136 p.

[Be5] P. Berthelot, Introduction à la théorie arithmétique des \( \mathcal{D} \)-modules. Astérisque 279 (2002), p. 1-80.

[BBM] P. Berthelot, L. Breen, W. Messing, Théorie de Dieudonné cristalline. II. Lecture Notes in Mathematics 930. Springer-Verlag (1982).

[BD] B. Bhatt, A. J. de Jong, Crystalline cohomology and de Rham cohomology, arXiv:1110.5001v1 [math.AG].

[BM] P. Berthelot, W. Messing, Théorie de Dieudonné cristalline III. The Grothendieck Festschrift I, Progress in Mathematics 86, Birkhäuser (1990), p. 171-247.

[BO] P. Berthelot, A. Ogus, Notes on crystalline cohomology. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1978).

[BL] B. Siegfried, W. Lütkebohmert, Degenerating abelian varieties. Topology 30, no. 4 (1991), p. 653-698.

[Br] B. Lawrence, Un théorème d’annulation pour certains \( \text{Ext}^1 \) de faisceaux abéliens. Ann. Sci. Ecole Norm. Sup. (4) 8 no. 3 (1975), p. 339-352.

[Bu] Bühler, Theo, Exact categories. Expo. Math. 28, no. 1 (2010), p. 1-69.

[dJ1] A. J. de Jong, Crystalline Dieudonné module theory via formal and rigid geometry. Publications Mathématiques de l’IHES, 82 (1995), p. 5-96.
[dJ2] A. J. de Jong, *Smoothness, semi-stability and alterations*. Inst. Hautes tudes Sci. Publ. Math. No. 83 (1996), p. 51-93.

[EGA1] A. Grothendieck, J.A. Dieudonné, *Eléments de géométrie algébrique*. I. Grundlehren der Mathematischen Wissenschaften, 166, Springer-Verlag, Berlin, (1971), ix+466 p.

[EGA2] A. Grothendieck, *Eléments de géométrie algébrique. II. Etude globale élémentaire de quelques classes de morphismes*. Inst. Hautes Etudes Sci. Publ. Math. 8 (1961), 222 p.

[EGA4-I] A. Grothendieck, *Eléments de géométrie algébrique. IV. Etude locale des schémas et des morphismes de schémas I*. (French) Inst. Hautes Etudes Sci. Publ. Math. 20 (1964), 259 p.

[EGA4-III] A. Grothendieck, *Eléments de géométrie algébrique. III*. Inst. Hautes Etudes Sci. Publ. Math. 28 (1966), 255 p.

[EGA4-IV] A. Grothendieck, *Eléments de géométrie algébrique. IV. Etude locale des schémas et des morphismes de schémas IV*. Inst. Hautes Etudes Sci. Publ. Math. 32 (1967), 361 p.

[Ek] T. Ekedahl, *On the adic formalism*. The Grothendieck Festschrift, vol. II, Prog. Math., 87. Birkhäuser Boston, Boston, MA (1990), p. 197-218.

[FM] J.-M. Fontaine, W. Messing, *p-adic periods and p-adic étale cohomology*. Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math. 67, Amer. Math. Soc., Providence, RI (1987), p. 179-207.

[FV] J. Fresnel, M. van der Put, *Rigid analytic geometry and its applications*. Progress in Math. 218. Birkhäuser Boston, Inc., Boston (2004), xii+296 p.

[Gr] A. Grothendieck, *Sur quelques points d’algèbre homologique*. Tohoku Math. J. 9, no. 2 (1957), p. 119-221.

[Gr1] A. Grothendieck, *Le groupe de Brauer. III. Exemples et compléments*. Dix Exposés sur la Cohomologie des Schémas. North-Holland, Amsterdam; Masson, Paris (1968), p. 88-188.

[Gr2] A. Grothendieck, *Un théorème sur les homomorphismes de schémas abéliens*. Inv. Math. 2 (1966), p. 59-78.

[Gr3] A. Grothendieck, *Technique de descente et théorèmes d’existence en géométrie algébrique. III. Préschémas quotients*. Séminaire Bourbaki 13, no 212 (1960/61).

[Ill] L. Illusie, *Complexe cotangent et déformations. II*. Lecture Notes in Mathematics, bf 283, Springer-Verlag, Berlin-New York (1972). vii+304 p.

[KS] M.Kashiwara, P. Schapira, *Categories and sheaves*. Grundlehren der Mathematischen Wissenschaften 332, Springer-Verlag (2006).

[Ka] K. Kato, *A generalization of local class field theory by using K-groups III*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29, no. 1 (1982), p. 31-43.

[Ka1] K. Kato, *Duality theories for the p-primary étale cohomology. I. Algebraic and topological theories*. (Kinosaki, 1984), Kinokuniya, Tokyo (1986), p. 127-148.
[Ka2] K. Kato, *Logarithmic structures of Fontaine-Illusie*. Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD (1989), p. 191-224.

[Ka3] K. Kato, *Logarithmic structures of Fontaine-Illusie II, logarithmic flat topology*. Incomplete version.

[Ka4] K. Kato, *The explicit reciprocity law and the cohomology of Fontaine-Messing*. Bull. Soc. Math. France 119 no. 4 (1991), p. 397-441.

[KT] K. Kato, F. Trihan, *On the conjecture of Birch and Swinnerton-Dyer in characteristic p > 0*. Invent. Math. 153 (2003), p. 537-592.

[Kl] S. L. Kleiman, *Misconceptions about K_X*. Enseign. Math. (2) 25, no. 3-4 (1980), p. 203-206.

[La] S. Lang, *Algebraic number theory*. Second edition. Graduate Texts in Mathematics, 110. Springer-Verlag, New York (1994). p. xiv+357.

[Mi1] J. S. Milne, *étale cohomology*. Princeton Mathematical Series, 33, Princeton University Press (1980).

[Mi2] J. S. Milne, *Arithmetic duality theorems*. Second edition. BookSurge, LLC, Charleston, SC (2006).

[Mi3] J. S. Milne, *Abelian varieties*. Arithmetic geometry (Storrs, Conn., 1984), Springer, New York (1986), p. 103-150.

[Mo] C. Montagnon, *Généralisation de la théorie arithmétique des D-modules la géométrie logarithmique*, thèse, Université de Rennes (2002), 84 p.

[MW] P. Monsky, G. Washnitzer, *Formal cohomology I*. Ann. of Math. (2) 88 (1968), p. 181-217.

[Oo] F. Oort, *Commutative group schemes*, Lecture Notes in Mathematics, 15, Springer Verlag, Berlin-New York (1966), vi+133 p.

[Pe to Va] C. Pépin, *Lettre à D. Vauclair*. (April 10th, 2013).

[Ra1] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics, 119. Springer Verlag, Berlin-New York (1970), ii+218 p.

[Ra2] M. Raynaud, *Passage au quotient par une relation d’équivalence plate*. Proc. Conf. Local Fields (Driebergen, 1966). Springer, Berlin (1967), p. 78-85.

[Ra3] M. Raynaud, *Variétés abéliennes et géométrie rigide*. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome I. Gauthiers-Villars, Paris (1971), p. 473-477.

[Ra4] M. Raynaud, *Néron models*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag (1989).

[Se] J.-P. Serre, *Local Algebra*. Springer-Verlag (2000).

[Se1] J.-P. Serre, *Corps locaux*. (French) Deuxième édition. Publications de l’Université de Nancago, No. VIII. Hermann, Paris (1968). 245 p.
[Se2] J.-P. Serre, Cohomologie Galoissienne. Lecture Notes in Mathematics, Springer-Verlag, Berlin (1994).

[SGA1] A. Grothendieck, Revetements etales et groupe fondamental. Fasc. II: Exposés 6, 8, 11. Séminaire de Géométrie Algébrique, 1960/61. Troisième édition, corrigée, Institut des Hautes Études Scientifiques, Paris (1963), i+163 p.

[SGA3-I] M. Demazure, A. Grothendieck, eds. Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA3) - vol. 1. Lecture notes in mathematics 151 (1970). Springer-Verlag, Berlin, New York, xv+564 p.

[SGA3-II] M. Demazure, A. Grothendieck, eds. Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA3) - vol. 2. Lecture notes in mathematics 152 (1970). Springer-Verlag, Berlin, New York, ix+654 p.

[SGA3-III] M. Demazure, A. Grothendieck, eds. Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA3) - vol. 3. Lecture notes in mathematics 153 (1970). Springer-Verlag, Berlin, New York, vii+529 p.

[SGA4-I] Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos. Séminaire de Géométrie Algébrique du Bois-Marie 1963/1964 (SGA4). Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics 269 (1972). Springer-Verlag, Berlin-New York, xix+525 p.

[SGA4-II] Théorie des topos et cohomologie étale des schémas. Tome 2 Séminaire de Géométrie Algébrique du Bois-Marie 1963/1964 (SGA4). Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics 270 (1972). Springer-Verlag, Berlin-New York, iv+418 p.

[SGA4-III] Théorie des topos et cohomologie étale des schémas. Tome 3 Séminaire de Géométrie Algébrique du Bois-Marie 1963/1964 (SGA4). Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics 305 (1973). Springer-Verlag, Berlin-New York, vi+640 p.

[SGA7-I] Groupes de monodromie en géométrie algébrique I. Séminaire de géométrie Algébrique du Bois-Marie 1967/1969 (SGA7 I). Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Lecture Notes in Mathematics, 288 (1972). Springer-Verlag, Berlin-New York, viii+523 p.

[Sh] A. Shiho, Crystalline fundamental groups. I. Isocrystals on log crystalline site and log convergent site. J. Math. Sci. Univ. Tokyo, 7, no. 4 (2000), p. 509-656.

[SN] Y. Nakakjima, A. Shiho, Weight filtrations on log crystalline cohomologies of families of open smooth varieties. Lecture Notes in Mathematics, 1959. Springer-Verlag, Berlin, (2008), x+266 p.

[Sp] N. Spaltenstein, Resolutions of unbounded complexes. Compositio Math. 65, no. 2 (1988), p. 121-154.

[TV] F. Trihan, D. Vauclair, non-commutative Iwasawa main conjecture for abelian varieties over function fields. In preparation.

[Ts1] T. Tsuji, Poincaré duality for logarithmic crystalline cohomology. Compositio Math. 118, no. 1 (1999), p. 11-41.
[Ts2] T. Tsuji, *On nearby cycles and D-modules of log schemes in characteristic p > 0*. Compos. Math. 146 no. 6 (2010), p. 1552-1616.

[Tr] F. Trihan, *A note on semistable Barsotti-Tate groups*. J. Math. Sci. Univ. Tokyo 15, no. 3 (2008), p. 411-425.

[Va to KT] D. Vauclair, *Letter to K. Kato and F. Trihan* (July 29th, 2011).

[Va to Pe] D. Vauclair, *Lettre à C. Pépin* (March 25th, 2013).

DEPARTMENT OF INFORMATION AND COMMUNICATION SCIENCES, FACULTY OF SCIENCE AND TECHNOLOGY, SOPHIA UNIVERSITY, 4 YONBANCHO, CHIYODA-ku, TOKYO 102-0081 JAPAN.

E-mail address: f-trihan-52m@sophia.ac.jp

LABORATOIRE DE MATHEMATIQUES NICOLAS ORESME, UNIVERSITÉ DE CAEN, CAMPUS 2, 14032 CAEN CEDEX, FRANCE.

E-mail address: david.vauclair@unicaen.fr