Two-Loop Quark and Gluon Form Factors in Dimensional Regularisation

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Abstract

We compute the two-loop corrections to the massless quark form factor $\gamma^* \rightarrow q \bar{q}$ and gluon form factor $H \rightarrow gg$ to all orders in the dimensional regularisation parameter $\epsilon = (4 - d)/2$. The two-loop contributions to the form factors are reduced to linear combinations of master integrals, which are computed in a closed form, expressed as $\Gamma$-functions and generalised hypergeometric functions of unit argument. Using the newly developed HypExp-package, these can be expanded to any desired order, yielding Laurent expansions in $\epsilon$. We provide expansions of the form factors to order $\epsilon^2$, as required for ultraviolet renormalisation and infrared factorisation of the three-loop form factors.
1 Introduction

The infrared pole structure of renormalised multi-loop amplitudes in dimensional regularisation with $d = 4 - 2\epsilon$ space-time dimensions can be predicted from an infrared factorisation formula, which was first conjectured in [1], where it was formulated up to two loops. A proof of the formula, together with an explicit formulation up to three loops was derived later in [2]. The simplest multi-loop amplitudes where the infrared factorisation formula can be applied are three-point functions, involving two partons coupled to an external current: the quark form factor $\gamma^* \to q\bar{q}$ and the gluon form factor $H \to gg$. The QCD corrections to these form factors can in particular be used to fix a priori unknown constants in the infrared factorisation formula, thus enabling an unambiguous prediction for multi-loop amplitudes involving more than two external partons.

In the infrared factorisation formula for a given form factor (or more generally for a given multi-leg amplitude) at a certain number of loops, infrared singularity operators act on the form factor evaluated with a lower number of loops. The infrared singularity operators contain explicit infrared poles $1/\epsilon^2$ and $1/\epsilon$. They do therefore project subleading terms in $\epsilon$ from the lower order form factors.

At present, two-loop corrections to the massless quark [3] and gluon [4] form factors are known to order $\epsilon^0$. Two-loop corrections to this order were also obtained for massive quarks [5]. The infrared structure of the massless form factors and infrared cancellations with real radiation contributions are described in detail in [6]. Very recently, results to order $\epsilon^2$ were obtained for the quark form factor [7].

The calculation of these corrections proceeds through a reduction [8–11] of all two-loop Feynman integrals appearing in the form factors to a small set of master integrals. The reduction is exact in $\epsilon$, such that the evaluation of the form factors is limited only by the order to which the master integrals can be computed. The massless two-loop form factors contain three two-loop master integrals, which can be computed either using various analytical methods [12] or numerically order-by-order in their Laurent expansion using the sector decomposition algorithm [13]. Up to now, exact expressions were known only for two of these master integrals, while the third (the so-called two-loop crossed triangle graph) was known only as a Laurent expansion up to finite terms [14].

In this letter, we derive an exact expression for the two-loop crossed triangle graph in terms of generalised hypergeometric functions of unit argument in Section 2. Using the HypExp-package [15] for the Laurent expansion of generalised hypergeometric functions, this can be expanded to any desired order in $\epsilon$. Together with the exact expressions for the one- and two-loop quark and gluon form factors in Section 3, this allows the expansion of these form factors to higher orders in $\epsilon$. For illustration, we list the one-loop form factors to order $\epsilon^4$ and the two-loop form factors to order $\epsilon^2$ in Section 4; these orders appear for example in the infrared factorisation of the corresponding three-loop from factors. Finally, Section 5 contains conclusions and an outlook.
2 Two-loop master integrals

The virtual two-loop vertex master integrals were first derived to order $\epsilon^0$ in [14] in the context of the calculation of the two-loop quark form factor [3]. All but the crossed triangle graph $A_6$ can be expressed in terms of $\Gamma$-functions to all orders in $\epsilon$.

Factoring out a common

$$S_{\Gamma} = \left( \frac{(4\pi^\epsilon)}{16\pi^2 \Gamma(1-\epsilon)} \right),$$

and introducing $q^2 = (p_1 + p_2)^2$, they read

$$A_{2,LO} = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2(k - p_1 - p_2)^2} = S_{\Gamma} (-q^2)^{-\epsilon} \Gamma(1 + \epsilon) \Gamma^3(1 - \epsilon)i \frac{\Gamma(2 - 2\epsilon)}{\epsilon},$$

$$A_3 = \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{k^2l^2(k - l - p_1 - p_2)^2} = S_{\Gamma}^2 (-q^2)^{1-2\epsilon} \frac{\Gamma(1 + 2\epsilon) \Gamma^5(1 - \epsilon) - 1}{\Gamma(3 - 3\epsilon) 2(1 - 2\epsilon) \epsilon},$$

$$A_4 = \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{k^2l^2(k - p_1 - p_2)^2(k - l - p_2)^2} = S_{\Gamma}^2 (-q^2)^{-2\epsilon} \frac{\Gamma(1 - 2\epsilon) \Gamma(1 + \epsilon) \Gamma^4(1 - \epsilon) \Gamma(1 + 2\epsilon) - 1}{\Gamma(2 - 3\epsilon) 2(1 - 2\epsilon) \epsilon^2}. $$

No exact expression for $A_6$ was known up to now. Following the steps outlined in [14], we obtain

$$A_6 = \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{k^2l^2(k - p_1 - p_2)^2(l - l - p_2)^2(l - p_1)^2} = S_{\Gamma}^2 (-q^2)^{-2-2\epsilon} \left[ - \frac{\Gamma^3(1 - \epsilon) \Gamma(1 + \epsilon) \Gamma^4(1 - 2\epsilon) \Gamma^3(1 + 2\epsilon)}{\epsilon^3 \Gamma^2(1 - 4\epsilon) \Gamma(1 + 4\epsilon)} \right] + \frac{\Gamma^4(1 - \epsilon) \Gamma(1 + \epsilon) \Gamma(1 - 2\epsilon) \Gamma(1 + 2\epsilon)}{2 \epsilon^4 \Gamma(1 - 3\epsilon)} _3F_2(1, -4\epsilon, -2\epsilon; 1 - 3\epsilon, 1 - 2\epsilon; 1) \frac{1}{1 - 3\epsilon}. $$

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\[- \frac{4 \Gamma^4(1 - \epsilon) \Gamma(1 - 2\epsilon) \Gamma(1 + 2\epsilon)}{\epsilon^2 (1 + \epsilon) (1 + 2\epsilon) \Gamma(1 - 4\epsilon)} \, 3F_2(1, 1, 1 + 2\epsilon; 2 + \epsilon, 2 + 2\epsilon; 1) \\
- \frac{\Gamma^5(1 - \epsilon) \Gamma(1 + 2\epsilon)}{2 \epsilon^3 \Gamma(1 - 3\epsilon)} \, 4F_3(1, 1 - \epsilon, -4\epsilon, -2\epsilon; 1 - 3\epsilon, 1 - 2\epsilon, 1 - 2\epsilon; 1) \right]. \tag{5}\]

While $A_{2,LO}$, $A_3$ and $A_4$ can be expanded using any standard computer algebra programme, the expansion of $A_6$ requires the expansion of generalised hypergeometric functions in their parameters. For this purpose, a dedicated package, \texttt{HypExp} [15], was developed recently. Using this, we obtain the eighth-order expansion:

$$A_6 = S_1^2 \left( -q^2 \right)^{-2 - 2\epsilon} \left[ -\frac{1}{\epsilon^4} + \frac{5\pi^2}{6\epsilon^2} + \frac{27}{\epsilon} \zeta_3 + \frac{23\pi^4}{36} + \left( 117\zeta_5 - 8\pi^2\zeta_3 \right) \epsilon \\
+ \left( \frac{19\pi^6}{315} - 267\zeta_3^2 \right) \epsilon^2 + \left( -\frac{109\pi^4}{10} \zeta_3 - 40\pi^2\zeta_5 - 6\zeta_7 \right) \epsilon^3 \\
+ \left( 44\pi^2\zeta_3^2 - \frac{1073\pi^8}{3024} - 2466\zeta_3\zeta_5 + 264\zeta_{5,3} \right) \epsilon^4 + \mathcal{O}(\epsilon^5) \right], \tag{6}\]$$

where we encountered a multiple zeta value in the last term.

### 3 Quark and gluon form factors at two loops

The tree-level quark and gluon form factors are obtained by normalising the corresponding tree-level vertex functions to unity:

$$F_q^{(0)} = 1, \quad F_g^{(0)} = 1. \tag{7}$$

The unrenormalised one-loop and two-loop form factors are calculated from the relevant Feynman diagrams. Using integration-by-parts [8] and Lorentz invariance [10] identities (which can be solved symbolically for massless two-loop vertex integrals, see the appendix of [16]), these can be reduced [9–11] to the master integrals listed in Section 2.

The unrenormalised one-loop quark and gluon form factors read:

$$F_q^{(1,B)} = -ig^2 \frac{N^2 - 1}{N} \frac{d^2 - 7d + 16}{2(d - 4)} A_{2,LO}, \tag{8}$$

$$F_g^{(1,B)} = ig^2 N \frac{d^3 - 16d^2 + 68d - 88}{(d - 4)(d - 2)} A_{2,LO}, \tag{9}$$

where $N = 3$ is the number of colours and $g$ is the bare QCD coupling parameter.

The unrenormalised two-loop quark and gluon form factors for $N_F$ massless quark flavours are:

$$F_q^{(2,B)} = g^4 \frac{N^2 - 1}{N} \left\{ - \frac{N^2 - 1}{N} \frac{(d^2 - 7d + 16)^2}{4(d - 4)^2} A_{2,LO}^2 \\
+ N \frac{(d^5 - 18d^4 + 138d^3 - 552d^2 + 1144d - 980)(3d - 8)}{2(d - 3)(d - 4)^3} A_{3} \right\} \frac{q^2}{A_{3}^2}.$$
\[ + \frac{1}{N} \frac{(9d^6 - 358d^5 + 4309d^4 - 24466d^3 + 72896d^2 - 110064d + 66080)(3d - 8)}{16(d - 3)(d - 4)^3 (2d - 7)} A_3 \]
\[ + N \frac{3d^6 - 82d^5 + 819d^4 - 4030d^3 + 10344d^2 - 12824d + 5632}{4(d - 1)(d - 4)^2 (3d - 8)} A_4 \]
\[ - \frac{1}{N} \frac{(21d^5 - 789d^4 + 9422d^3 - 53864d^2 + 163200d^2 - 253472d + 159232)(2d - 7)}{(3d - 8)(d - 4)^2} A_4 \]
\[ + N_F \frac{(3d^3 - 31d^2 + 110d - 128)(d - 2)}{2(d - 1)(d - 4)(3d - 8)} A_4 \]
\[ - \frac{1}{N} \frac{d^3 - 20d^2 + 104d - 176}{32(2d - 7)} (g^2)^2 A_6 \] 

The renormalised form factors are obtained by introducing the renormalised QCD coupling constant and the renormalised effective coupling of \( H \) to the gluon field strength [4], and subsequent expansion in powers of the renormalised coupling.
4 Expansion of two-loop form factors

The renormalised form factors are expanded in the renormalised coupling constant. In the $\overline{\text{MS}}$ scheme, the bare coupling $\alpha_0 = g^2/(4\pi)$ is related to the renormalised coupling $\alpha_s \equiv \alpha_s(\mu^2)$, evaluated at the renormalisation scale $\mu^2$ by

$$\alpha_0 \mu_0^{2\epsilon} S_\epsilon = \alpha_s \mu^{2\epsilon} \left[ 1 - \frac{11N - 2NF}{6\epsilon} \left( \frac{\alpha_s}{2\pi} \right) + \mathcal{O}(\alpha_s^2) \right], \quad (12)$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma} \quad \text{with the Euler constant } \gamma = 0.5772\ldots$$

and $\mu_0^2$ is the mass parameter introduced in dimensional regularisation to maintain a dimensionless coupling in the bare QCD Lagrangian density. For simplicity, we set $\mu^2 = q^2$. If the squared momentum transfer $q^2$ is space-like ($q^2 < 0$), the form factors are real, while they acquire imaginary parts for time-like $q^2$. These imaginary parts (and corresponding real parts) arise from the $\epsilon$-expansion of

$$\Delta(q^2) = (-\text{sgn}(q^2) - i0)^{-\epsilon}. \quad (13)$$

The renormalised form factors can then be written as

$$F_{q,g}(q^2) = 1 + \left( \frac{\alpha_s}{2\pi} \Delta(q^2) \right) F_{q,g}^{(1)} + \left( \frac{\alpha_s}{2\pi} \Delta(q^2) \right)^2 F_{q,g}^{(2)} + \mathcal{O}(\alpha_s^3). \quad (14)$$

Expanding the first and second order coefficients of the form factors to $\epsilon^4$ and $\epsilon^2$ respectively, we obtain:

$$F_q^{(1)} = \left( N - \frac{1}{N} \right) \left[ \frac{1}{2\epsilon^2} - \frac{3}{4\epsilon} - 2 + \frac{\pi^2}{24} \right] \epsilon^2 + \left( -8 + \frac{\pi^2}{6} + \frac{7}{4} \zeta_3 + \frac{47\pi^4}{2880} \right) \epsilon^2 + \left( -16 - \frac{7\pi^2}{72} \zeta_3 + \frac{14}{3} \zeta_5 + \frac{31}{10} \zeta_5 + \frac{\pi^2}{3} + \frac{47\pi^4}{1920} \right) \epsilon^3 + \left( -32 - \frac{7\pi^2}{48} \zeta_3 + \frac{28}{3} \zeta_5 - \frac{49}{36} \zeta_5 + \frac{93}{20} \zeta_5 + \frac{2\pi^2}{3} + \frac{47\pi^4}{720} + \frac{949\pi^6}{241920} \right) \epsilon^4 + \mathcal{O}(\epsilon^5), \quad (15)$$

$$F_q^{(2)} = \left( N - \frac{1}{N} \right) \left[ -\frac{1}{\epsilon^2} - \frac{11}{6\epsilon} + \frac{\pi^2}{12} \right] \epsilon + \left( -1 + \frac{7}{3} \zeta_3 \right) \epsilon + \left( -3 + \frac{47\pi^4}{1440} \right) \epsilon^2 + \left( -7 - \frac{7\pi^2}{36} \zeta_3 + \frac{31}{5} \zeta_5 + \frac{\pi^2}{12} \right) \epsilon^3 + \left( -15 + \frac{7}{3} \zeta_3 - \frac{49}{18} \zeta_5 + \frac{\pi^2}{4} + \frac{949\pi^6}{120960} \right) \epsilon^4 + \frac{NF}{3\epsilon} + \mathcal{O}(\epsilon^5), \quad (16)$$

$$F_q^{(2)} = \left( N - \frac{1}{N} \right) \left\{ N \left[ \frac{1}{8\epsilon^4} + \frac{17}{16\epsilon^3} + \frac{433}{288\epsilon^2} \right] + \mathcal{O}(\epsilon^5) \right\}.$$
\[ F^{(2)}_{g} = N^2 \left[ \frac{1}{2\epsilon^4} + \frac{77}{24\epsilon^3} + \frac{1}{\epsilon^2} \left( \frac{175}{72} - \frac{\pi^2}{24} \right) + \frac{1}{\epsilon} \left( \frac{119}{54} - \frac{11\pi^2}{144} \right) + \frac{8237}{648} - \frac{33}{4} \zeta_3 + \frac{67\pi^2}{144} - \frac{7\pi^4}{240} \right) \right. \\
\left. \frac{200969}{3888} + \frac{23\pi^2}{72} \zeta_3 - \frac{1139}{108} \zeta_3 + \frac{71}{20} \zeta_5 + \frac{53\pi^2}{108} - \frac{1111\pi^4}{8640} \right) \epsilon + \frac{4082945}{23328} - \frac{11\pi^2}{216} \zeta_3 - \frac{13109}{162} \zeta_3 + \frac{901\pi^2}{36} - \frac{341}{20} \zeta_5 + \frac{85\pi^2}{1296} - \frac{1943\pi^4}{8640} + \frac{257\pi^6}{6720} \epsilon^2 \right] + \mathcal{O}(\epsilon^3) \]
\[
+ \left( -\frac{3663205}{46656} + \frac{31\pi^2}{108}\zeta_3 + \frac{274}{81}\zeta_3 - \frac{9}{10}\zeta_5 + \frac{883\pi^2}{2592} - \frac{73\pi^4}{2592}\right)\epsilon^2 \right]
\]
\[
+ \frac{N_F}{N} \left[ -\frac{1}{8\epsilon} + \left( \frac{67}{48} - \zeta_3 \right) + \left( \frac{2027}{288} - \frac{23}{6}\zeta_3 - \frac{7\pi^2}{144} - \frac{\pi^4}{54}\right)\epsilon \right]
\]
\[
+ \left( \frac{47491}{1728} + \frac{5\pi^2}{18}\zeta_3 - \frac{281}{18}\zeta_3 - 4\zeta_5 - \frac{209\pi^2}{864} - \frac{23\pi^4}{324}\right)\epsilon^2 \right] + N_F^2 \frac{1}{9\epsilon^2} + \mathcal{O}(\epsilon^3). \tag{18}
\]

5 Conclusions and outlook

In this letter, we computed the two-loop quark and gluon form factors to all orders in the dimensional regularisation parameter \(\epsilon\). The principal ingredient to this calculation is the two-loop crossed triangle graph \(A_6\), for which we computed an exact expression in terms of generalised hypergeometric functions of unit argument, which can be expanded to any desired order in \(\epsilon\) using the HypExp-package.

A potential application of the form factors derived here is the extraction of the complete set of infrared pole terms of the genuine three-loop quark form factor from the recently derived three-loop splitting and coefficient functions in deep inelastic scattering [17]. In turn, these allow to fix the yet unknown hard radiation constants in the infrared factorisation formula at three loops. Parts of these constants were derived previously from \(\mathcal{N} = 4\) supersymmetry relations [18, 19].

The two-loop vertex master integrals feature as subtopologies in the reduction of the three-loop form factor contributions, appearing if one of the three loops is disconnected from the others by pinching the connecting propagators. In this case, their terms to \(\epsilon^2\) are required.

The calculation presented here illustrates the applicability of the HypExp-package in the calculation of multi-loop corrections in quantum field theory. Functions similar to those which were expanded here appear also in multi-particle phase space integrals in massless [20] and massive decay processes [21]. Since all these integrals correspond to particular cuts of multi-loop two-point functions, one might expect that three-loop and four-loop two-point functions could also be expanded using HypExp to high orders in \(\epsilon\), as required for multi-loop calculations of fully inclusive observables [22].

Note added: While finalising this letter, an independent paper addressing very similar issues appeared. In hep-ph/0507039 [7], Moch, Vermaseren and Vogt compute the two-loop quark form factor to order \(\epsilon^2\) and apply it in the extraction of the pole parts of the three-loop quark form factor from deep inelastic coefficient functions. In this paper, the hard radiation constants for infrared factorisation at three-loops and related resummation coefficients are extracted for processes involving quarks only. Expanding our unrenormalised quark form factor (10) to order \(\epsilon^2\), we confirm the result (B.1) of [7].

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