Non-Standard Analysis, Multiplication of
Schwartz Distributions, and Delta-Like
Solution of Hopf’s Equation

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Abstract

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We construct an algebra of generalized functions $\mathcal{E}(\mathbb{R}^d)$. We also construct an embedding of the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^d)$ into $\mathcal{E}(\mathbb{R}^d)$ and thus present a solution of the problem of multiplication of Schwartz distributions which improves J.F. Colombeau’s solution. As an application we prove the existence of a weak delta-like solution in $\mathcal{E}(\mathbb{R}^d)$ of the Hopf equation. This solution does not have a counterpart in the classical theory of partial differential equations. Our result improves a similar result by M. Radyna obtained in the framework of perturbation theory.

Key words and phrases: Schwartz distributions, multiplication of Schwartz distributions, Colombeau’s algebra of generalized functions, non-standard analysis, saturation principle, conservation law, Hopf equation, weak solution, shock wave.

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1 Introduction

In what follows $\mathcal{E}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$ denotes the class of $C^\infty$-functions on $\mathbb{R}^d$. Also $\mathcal{D}(\mathbb{R}^d) = C^\infty_0(\mathbb{R}^d)$ denotes the class of test functions on $\mathbb{R}^d$ and $\mathcal{D}'(\mathbb{R}^d)$ stands for the space of Schwartz distributions (Schwartz generalized functions) on $\mathbb{R}^d$ (H. Bremermann [1]).

The algebra of generalized functions $^*\mathcal{E}(\mathbb{R}^d)$ is a particular non-standard extension of the class $\mathcal{E}(\mathbb{R}^d)$. The field of the scalars $^*\mathbb{C}$ of the algebra $^*\mathcal{E}(\mathbb{R}^d)$ is a particular non-standard extension of the field of complex numbers $\mathbb{C}$ and the field of the real scalars $^*\mathbb{R}$ is a non-standard extension of $\mathbb{R}$. That means that both $^*\mathbb{C}$ and $^*\mathbb{R}$ are non-Archimedean fields containing non-zero infinitesimals, i.e. generalized numbers $h$ such that $0 < |h| < 1/n$ for all $n \in \mathbb{N}$. Since the involvement of non-Archimedean fields in applied mathematics is somewhat unusual, we start with a summary of the relevant definitions and results in the theory of ordered fields and non-Archimedean fields (Section 2).

In Sections 3-4 we present the basic facts of the theory of free filters and ultrafilters (C. C. Chang and H. J. Keisler [2]). We construct a particular ultrafilter on the space of test functions $\mathcal{D}(\mathbb{R}^d)$ which is important for the embedding of Schwartz distributions in the algebra $^*\mathcal{E}(\mathbb{R}^d)$.

In Sections 5-6 we present the construction of the fields of the complex and real non-standard numbers $^*\mathbb{C}$ and $^*\mathbb{R}$. In Section 7 we prove the Saturation Principle in $^*\mathbb{C}$ which plays a role in non-standard analysis similar to the role of the completeness of $\mathbb{R}$ and $\mathbb{C}$ in usual (standard) analysis. These sections might be viewed as an introduction to non-standard analysis (A. Robinson [12]). We should note that our exposition of non-standard analysis does not require any background in mathematical logic or model theory.

The construction of the algebra $^*\mathcal{E}(\mathbb{R}^d)$ is presented in Section 8; in short, $^*\mathcal{E}(\mathbb{R}^d)$ is a differential associative commutative algebra of general-
ized functions similar to (but much larger than) the class $\mathcal{E}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$. In Section 9 we state the Saturation Principle for $^\ast\mathcal{E}(\mathbb{R}^d)$, playing the role of the completeness property.

In Section 12 we construct the chain of embeddings $\mathcal{E}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d) \subset ^\ast\mathcal{E}(\mathbb{R}^d)$. These embeddings presents a solution of the problem of multiplication of Schwartz distributions similar to but different from Colombeau’s solution of the same problem (J.F. Colombeau [2]). The problem of multiplication of Schwartz distributions has an interesting and dramatic history. Soon after the distribution theory was invented by L. Schwartz, he proved that the space of distributions $\mathcal{D}'(\mathbb{R}^d)$ cannot be supplied with an associative and commutative product that reproduces the usual product in the spaces $\mathcal{C}^k(\mathbb{R}^d), k = 0, 1, 2, \ldots$. This negative result, known as Schwartz Impossibilities Result (L. Schwartz [14]), was the reason this problem was considered for a long time as unsolvable. In the late 1980’s Jean F. Colombeau offered a solution of the problem of multiplication of distributions by constructing an algebra of generalized functions $\mathcal{G}(\mathbb{R}^d)$ with the chain of algebraic embeddings $\mathcal{E}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d) \subset \mathcal{G}(\mathbb{R}^d)$ thus avoiding Schwartz Impossibilities Result (since $k = \infty$). One (slightly disturbing) feature of Colombeau’s solution is that the set of scalars $\overline{\mathbb{C}}$ of the algebra $\mathcal{G}(\mathbb{R}^d)$ is a ring with zero divisors, not a field as any set of scalars should be. In this respect our solution of the problem of multiplication of Schwartz distributions presents an important improvement of Colombeau’s theory: the set of scalars $^\ast\mathbb{C}$ of the algebra $^\ast\mathcal{E}(\mathbb{R}^d)$ is an algebraically complete $c^+$-saturated field (Section 7). As a consequence, the set of the real scalars $^\ast\mathbb{R}$ is a real closed Cantor complete field. We should notice that the fact that $^\ast\mathcal{E}(\mathbb{R}^d)$ is a differential algebra (not merely a linear space) is important for our goals in applied mathematics, in particular, for studying generalized solutions of non-linear partial differential equations such as shock-wave and delta-like solutions. Notice that these are the solutions after the formation of the shock
in many conservation law type equations.

In Section 14 we prove the existence of a weak delta-like solution of the Hopf equation $u_t(x, t) + u(x, t)u_x(x, t) = 0$ in the framework of $\mathcal{E}(\mathbb{R}^d)$. This solution has counterparts neither in the spaces of classical functions such as $C^k(\mathbb{R}^d), k = 1, 2, \ldots, \infty$, nor in the spaces of Schwartz distributions such as $\mathcal{D}'(\mathbb{R}^d)$. Our result improves a similar result by M. Radyna [11] obtained in the spirit of perturbation theory.
2 Ordered Fields

We will begin by defining ordered fields and giving examples of some well known and some lesser known orderings on the (non-archimedean) fields of rational functions and Laurent series.

Definition 2.1 Let \( \mathbb{K} \) be a field (ring). \( \mathbb{K} \) is called orderable if there exists a nonempty subset \( \mathbb{K}_+ \subset \mathbb{K} \) such that

1. \( 0 \notin \mathbb{K}_+ \)
2. \( x, y \in \mathbb{K}_+ \implies x + y, xy \in \mathbb{K}_+ \)
3. For every non-zero \( x \in \mathbb{K} \), either \( x \in \mathbb{K}_+ \) or \( -x \in \mathbb{K}_+ \)

\( \mathbb{K}_+ \) generates an order relation \( <_\mathbb{K} \) on \( \mathbb{K} \) as follows: \( x <_\mathbb{K} y \) iff \( y - x \in \mathbb{K}_+ \).

(\( \mathbb{K}, <_\mathbb{K} \)) is called a totally ordered field or simply an ordered field.

We will write \( < \) instead of \( <_{\mathbb{K}} \) when it is clear from context which field’s order relation we are referring to.

In addition to \( (\mathbb{R}, <) \) (where \( < \) is the usual order on \( \mathbb{R} \)) there are many (more interesting) examples of ordered fields. But first let us make a short detour:

Example 2.1 \( \mathbb{C} \), the set of complex numbers, is not orderable.

Proof Suppose there exists a subset \( \mathbb{C}_+ \) satisfying Definition 2.1. Then consider:

Case 1 Suppose \( i \in \mathbb{C}_+ \). Then \( i \cdot i = -1 \in \mathbb{C}_+ \), implying \( (-1) \cdot (-1) = 1 \in \mathbb{C}_+ \). This is impossible since \( -1 + 1 = 0 \notin \mathbb{C}_+ \).
Case 2 Suppose $i \notin \mathbb{C}_+$. Then $-i \in \mathbb{C}_+$, implying $(-i) \cdot (-i) = -1 \in \mathbb{C}_+$, leading to the same contradiction as in Case 1. ▲

The previous example can be generalized as follows:

**Theorem 2.1** A field $K$ is orderable if and only if it is formally real. This means that for every $n \in \mathbb{N}$ and every $x_k \in K$

$$\sum_{k=1}^{n} x_k^2 = 0 \text{ implies } x_k = 0 \text{ for all } k.$$  

For details on the subject of formally real fields and the proof of this theorem, see (Van Der Waerden [18], Chapter 11).

**Definition 2.2** Let $K$ and $L$ be ordered fields. If $\varphi : K \to L$ is a field homomorphism that preserves order, i.e. $x <_K y$ implies $\varphi(x) <_L \varphi(y)$, then $\varphi$ is said to be an **ordered field homomorphism**.

*Ordered field isomorphisms* and *ordered field embeddings* are defined similarly.

**Remark 2.1** There exists an ordered field embedding from $\mathbb{Q}$ into any ordered field $K$. We call it the **canonical embedding** of $\mathbb{Q}$ into $K$ and it is defined by: $\sigma(0) = 0$, $\sigma(n) = n \cdot 1$ and $\sigma(-n) = -\sigma(n)$ for $n \in \mathbb{N}$, and $\sigma(p/q) = \sigma(p)/\sigma(q)$ for $p, q \in \mathbb{Z}$, $q \neq 0$.

From now on, if $x \in \mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$, we will refer to $x$ and $\sigma(x) \in K$ interchangeably.

**Example 2.2** Let $(\mathbb{R}, <)$ be the field of real numbers with the usual order, and let $\mathbb{R}(x)$ be the set of rational functions in the variable $x$ with coefficients in $\mathbb{R}$. Note that we may think of $\mathbb{R}$ as a subfield of $\mathbb{R}(x)$, as represented by the constant functions. Then define $\mathbb{R}(x)_+ = \{R(x) : R(x) \in \mathbb{R}(x) \text{ and there}$
exists \( x_0 \in \mathbb{R} \) such that \( R(x) > 0 \) whenever \( x > x_0 \).

The ordered field generated by \( \mathbb{R}(x)_+ \), which we will refer to simply as \( \mathbb{R}(x) \), has some surprising properties. Namely:

(i) \( \mathbb{R}(x) \) contains \textbf{infinitely large elements} like \( f(x) = x \). This means that \( f(x) > n \) for all \( n \in \mathbb{N} \) (let \( x_0 = n \)).

(ii) \( \mathbb{R}(x) \) contains \textbf{positive infinitesimals} like \( g(x) = \frac{1}{x} \). This means that \( 0 < g(x) < \frac{1}{n} \) for all \( n \in \mathbb{N} \) (let \( x_0 = n \)).

\textbf{Remark 2.2} Let \( \mathbb{L} \) be an ordered integral domain (an ordered ring without zero divisors) and \( \mathbb{K} \) be the field of fractions of \( \mathbb{L} \). Define

\[
\mathbb{K}_+ = \{ \frac{x}{y} : x, y \in \mathbb{L}_+ \text{ or } -x, -y \in \mathbb{L}_+ \}
\]

The order generated by \( \mathbb{K}_+ \) is the only one which extends the order in \( \mathbb{L} \). It is said to be the order \textbf{inherited} from \( \mathbb{L} \).

\textbf{Example 2.3} With Remark (2.2) in mind, we may revisit Example (2.2).

If \( \mathbb{R}[x] \) is the ring of polynomials over \( \mathbb{R} \), we may define \( \mathbb{R}[x]_+ = \{ P(x) \in \mathbb{R}[x] : \text{lead}(P) > 0 \} \), where \( \text{lead}(P) \) is the leading coefficient of \( P(x) \).

The order generated on \( \mathbb{R}[x] \) by \( \mathbb{R}[x]_+ \) can be extended to \( \mathbb{R}(x) \) since \( \mathbb{R}(x) \) is the field of fractions of \( \mathbb{R}[x] \). That is, we may redefine \( \mathbb{R}(x)_+ = \{ \frac{P(x)}{Q(x)} : P(x), Q(x) \in \mathbb{R}[x] \text{ and lead}(P), \text{lead}(Q) > 0 \text{ or lead}(P), \text{lead}(Q) < 0 \} \)

This definition is equivalent to that given previously and the orders generated by the two are in fact one and the same. Now we may plainly see that \( f(x) = x \) is indeed an infinitely large element since \( \text{lead}(x - n) = 1 > 0 \) for all \( n \in \mathbb{N} \).

Similarly, \( g(x) = \frac{1}{x} \) is a positive infinitesimal because \( \frac{1}{n} - \frac{1}{x} = \frac{x - n}{nx} \) and \( \text{lead}(x - n), \text{lead}(nx) > 0 \) for all \( n \in \mathbb{N} \).
Example 2.4 The set
\[ \mathbb{R}(x^{\mathbb{Z}}) = \left\{ \sum_{n=m}^{\infty} a_n x^n : \ a_n \in \mathbb{R}, \ m \in \mathbb{Z}, \ \text{and} \ a_m \neq 0 \right\} \]
of Laurent series with coefficients in \( \mathbb{R} \) is a field under normal polynomial addition and multiplication.
We may define an order on \( \mathbb{R}(x^{\mathbb{Z}}) \) by
\[ \mathbb{R}_+(x^{\mathbb{Z}}) = \left\{ \sum_{n=m}^{\infty} a_n x^n \in \mathbb{R}(x^{\mathbb{Z}}) : \ a_m > 0 \right\} \]
Here, an element such as
\[ \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots \]
is infinitesimal.

Example 2.5 The field of (formal) Laurent Series may also be defined as follows:
\[ \mathbb{R}(x^{\mathbb{Z}}) = \left\{ \sum_{n=m}^{\infty} a_n x^n : \ a_n \in \mathbb{R}, \ m \in \mathbb{Z}, \ \text{and} \ a_m \neq 0 \right\} \]
If we now let
\[ \mathbb{R}_+(x^{\mathbb{Z}}) = \left\{ \sum_{n=m}^{\infty} a_n x^n \in \mathbb{R}(x^{\mathbb{Z}}) : \ a_m > 0 \right\} \]
then even a series that is divergent for all \( x \), such as
\[ x + 2x^2 + 6x^3 + \ldots + n!x^n + \ldots \]
is an infinitesimal in this field.
3 Filters and Ultrafilters

In this section we define and give examples of filters on an arbitrary infinite set. Having done so, we will prove the existence of an ultrafilter using the axiom of choice.

**Definition 3.1** Let $I$ be an infinite set and let $\mathcal{F} \subset \mathcal{P}(I)$, $\mathcal{F} \neq \emptyset$. If $\mathcal{F}$ satisfies:

(F1) If $A \in \mathcal{F}$ and $A \subset B \subset I$, then $B \in \mathcal{F}$.

(F2) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$

(F3) $\emptyset \notin \mathcal{F}$

then $\mathcal{F}$ is a filter on $I$. If it is also true that

(F4) $\bigcap_{A \in \mathcal{F}} A = \emptyset$

then $\mathcal{F}$ is called a free filter on $I$. A filter $\mathcal{F}$ is called countably incomplete if:

(F5) There exists a sequence of decreasing sets $I = I_0 \supset I_1 \supset I_2 \supset \ldots$ in $\mathcal{F}$ such that $\bigcap_{n=0}^{\infty} I_n = \emptyset$.

If $\mathcal{F}$ is a filter on $I$, it follows immediately from the definition that:

(i) $I \in \mathcal{F}$

(ii) $\mathcal{F}$ is closed under finite intersections.

(iii) If $A \in \mathcal{F}$ then $I \setminus A \notin \mathcal{F}$

(iv) If $\mathcal{F}$ is countably incomplete, then $\mathcal{F}$ is free.
Definition 3.2 A filter $U$ on a set $I$ is called an \textbf{ultrafilter} if for every filter $F$ on $I$, $U$ is not a proper subset of $F$. That is, there is no filter $F$ on $I$ that properly contains $U$.

Theorem 3.1 Let $U$ be a filter on $I$. Then $U$ is an ultrafilter on $I$ if and only if for every $A \subset I$ either $A \in U$ or $I \setminus A \in U$.

Proof Suppose $U$ is an ultrafilter on $I$ and $A, I \setminus A \notin U$. Let $\hat{U} = \{X : A \cup X \in U\}$. It is not hard to check that $\hat{U}$ is a filter that properly contains $U$, since $I \setminus A \in \hat{U}$. Thus $U$ cannot be an ultrafilter. To prove the other direction, suppose, to the contrary, that $U$ is not an ultrafilter. Then there exists a filter $V$ which is a proper extension of $U$. Let $A \in V \setminus U$. Then, since $U \subset V$, we have that $A \cap X \neq \emptyset$ for all $X \in U$. But $I \setminus A \in U$ by assumption, so $A \cap (I \setminus A) \neq \emptyset$, a contradiction. $\blacksquare$.

Example 3.1 Let $I = \mathbb{N}$, and fix $n \in \mathbb{N}$. Then $U = \{X : X \subset \mathbb{N}, n \in X\}$ is an ultrafilter on $\mathbb{N}$. However, $U$ is clearly not free.

Example 3.2 The filter $\mathcal{F}_r(\mathbb{N})$ consisting of all cofinite sets of natural numbers is called the Fréchet filter on $\mathbb{N}$. It is free and countably incomplete since $\bigcap_{n=1}^{\infty} (\mathbb{N} \setminus \{n\}) = \emptyset$ However, since neither the set of even numbers nor the set of odd numbers is in $\mathcal{F}_r(\mathbb{N})$, by Theorem 3.1, it is not an ultrafilter.

Theorem 3.2 Let $I$ be an infinite set and $F$ a free filter on $I$. Then there exists a free ultrafilter $U$ on $I$ such that $F \subseteq U$.

Proof Let $\hat{F}$ be the set of all filters on $I$ that contain $F$. $\hat{F} \neq \emptyset$ since $F \in \hat{F}$. Let $\hat{F}$ be ordered by set inclusion, and consider a linearly ordered subset $\mathcal{M} \subseteq \hat{F}$. Define $\hat{\mathcal{M}} = \bigcup_{M \in \mathcal{M}} M$. Note that if $A \in \hat{\mathcal{M}}$ then $A \in M$ for some $M \in \mathcal{M}$. Thus, if $A \subset B \subset I$, it follows that $B \in M$, implying
Also, if \(A, B \in \widehat{\mathcal{M}}\), then we must have that \(A \in M_1\) and \(B \in M_2\) for some \(M_1, M_2 \in \mathcal{M}\). Since \(\mathcal{M}\) is linearly ordered, we may assume without loss of generality that \(M_1 \subset M_2\). Thus \(A, B \in M_2\). Hence \(A \cap B \in M_2\), implying \(A \cap B \in \widehat{\mathcal{M}}\). Finally, we have that \(\emptyset \notin \widehat{\mathcal{M}}\) because otherwise \(\emptyset\) would be an element of some filter \(M \in \mathcal{M}\), which is impossible. We have just shown that \(\widehat{\mathcal{M}}\) is itself a filter. But since the choice of \(\mathcal{M}\) was arbitrary, we can conclude that every linearly ordered subset of \(\widehat{\mathcal{F}}\) has an upper bound in \(\widehat{\mathcal{F}}\). Thus, by Zorn’s Lemma, \(\widehat{\mathcal{F}}\) has a maximal element, \(\mathcal{U}\), which is an ultrafilter on \(I\) containing \(\mathcal{F}\). Also, since \(\mathcal{F} \subset \mathcal{U}\) and \(\mathcal{F}\) is free, we have that \(\bigcap_{A \in \mathcal{U}} A \subseteq \bigcap_{A \in \mathcal{F}} A = \emptyset\). Therefore \(\mathcal{U}\) is free. ▲.

4 Ultrafilter on \(\mathcal{D}(\mathbb{R}^d)\)

Here we define an ultrafilter on \(\mathcal{D}(\mathbb{R}^d)\), the set of test functions, in order to construct ordered, non-archimedean fields of non-standard real and complex numbers, \(*\mathbb{R}\) and \(*\mathbb{C}\), respectively.

**Definition 4.1** Let \(\mathcal{D}(\mathbb{R}^d)\) be the set of test functions on \(\mathbb{R}^d\). That is, \(\mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)\). For every \(n \in \mathbb{N}\), define the basic set \(\mathcal{B}_n\) by

\[
\mathcal{B}_n = \{ \varphi \in \mathcal{D}(\mathbb{R}^d) : \varphi \text{ is real-valued and symmetric,} \\
\varphi(x) = 0 \text{ for all } x \in \mathbb{R}^d, \|x\| \geq 1/n, \\
\int \varphi = 1 \\
\int x^\alpha \varphi = 0 \text{ for all } \alpha \in \mathbb{N}_0^d, 1 \leq |\alpha| \leq n, \\
1 \leq \int |\varphi| < 1 + \frac{1}{n} \}
\]

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and $\mathcal{B}_0 = \mathcal{D}(\mathbb{R}^d)$.

**Theorem 4.1**  
(i) $\mathcal{B}_n \neq \emptyset$ for all $n$.  
(ii) $\mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \ldots$  
(iii) $\bigcap_n \mathcal{B}_n = \emptyset$

**Proof** For the proof of (i), see (Oberguggenberger and Todorov [10]). (ii) follows from Definition (4.1). For (iii), suppose there were a function $\varphi$ such that $\varphi \in \bigcap_n \mathcal{B}_n$ for all $n$. Then consider $\hat{\varphi}(\xi) = \int \varphi(x)e^{i\xi x}dx$, the Fourier transform of $\varphi$. Since $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\hat{\varphi}$ is entire (Bremermann [1] Lemma 8.11, p.85). Therefore, we can write $\hat{\varphi}(\xi) = \sum_{\alpha \in \mathbb{N}^d_0} (\partial^\alpha \hat{\varphi})(0) \xi^\alpha$. But $0 = i|\alpha| \int x^\alpha \varphi(x)dx = i|\alpha| \int x^\alpha \varphi(x)e^{i\xi x}dx|_{\xi=0} = (\partial^\alpha \hat{\varphi})(0)$ for all $\alpha \neq 0$. It follows that $\hat{\varphi}$ is constant. However, by the same lemma as before, we also have that $\lim_{|\xi| \to \infty} \hat{\varphi}(\xi) = 0$. Thus $\hat{\varphi} = 0$, implying $\varphi(x) = 0$. This contradicts the property that $\int \varphi = 1$. Hence $\bigcap_n \mathcal{B}_n = \emptyset$. ▲

**Definition 4.2** Define the basic filter $\mathcal{F}_B$ on $\mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_B = \{ \Phi \subseteq \mathcal{D}(\mathbb{R}^d) : \mathcal{B}_n \subseteq \Phi \text{ for some } n \in \mathbb{N} \}.$$ 

Since each $\mathcal{B}_n$ is itself an element of $\mathcal{F}_B$, it follows from Theorem (4.1) that $\mathcal{F}_B$ is countably incomplete, and therefore free. Thus, by Theorem (4.2), there exists an ultrafilter $\mathcal{U}$ on $\mathcal{D}(\mathbb{R}^d)$ containing $\mathcal{F}_B$. We shall keep $\mathcal{U}$ fixed in what follows.

**5 Non-Standard Numbers**

We will now use the ultrafilter defined in the previous section to construct fields of non-standard real and complex numbers.
Definition 5.1 Let $U$ be as before, and let $\mathbb{C}^{D(\mathbb{R}^d)}$ be the ring of functionals from $D(\mathbb{R}^d)$ to $\mathbb{C}$ supplied with pointwise addition and multiplication. We shall denote these functionals as “families” $(A_\varphi)$ and treat the domain $D(\mathbb{R}^d)$ as an “index set”.

We may define the operations of absolute value, real part mapping, imaginary part mapping, and complex conjugation on the elements of $\mathbb{C}^{D(\mathbb{R}^d)}$ by:

\[
|\left(\begin{matrix} A_\varphi \end{matrix}\right)| = (A_\varphi) \\
\Re(\left(\begin{matrix} A_\varphi \end{matrix}\right)) = (\Re A_\varphi) \\
\Im(\left(\begin{matrix} A_\varphi \end{matrix}\right)) = (\Im A_\varphi) \\
\left(\begin{matrix} A_\varphi \end{matrix}\right) = (A_\varphi)
\]

Also, we may define an embedding of $\mathbb{C}$ into $\mathbb{C}^{D(\mathbb{R}^d)}$ by $c \mapsto (C_\varphi)$ where $C_\varphi = c$ for all $\varphi \in D(\mathbb{R}^d)$.

Define an equivalence relation $\sim_U$ on $\mathbb{C}^{D(\mathbb{R}^d)}$ by

\[
(A_\varphi) \sim_U (B_\varphi) \; \text{if} \; \{\varphi \in D(\mathbb{R}^d) : A_\varphi = B_\varphi\} \in U
\]

Finally, let $\ast \mathbb{C} = \mathbb{C}^{D(\mathbb{R}^d)}/\sim_U$. That is, $\ast \mathbb{C}$ consists of equivalence classes of functionals in $\mathbb{C}^{D(\mathbb{R}^d)}$. We may write $\langle (A_\varphi) \rangle$ to represent these classes, but to simplify notation we will denote by $\langle A_\varphi \rangle \in \ast \mathbb{C}$ the non-standard number (equivalence class of functionals) with representative $(A_\varphi)$. $\ast \mathbb{C}$ is called a field of complex non-standard numbers.

$\ast \mathbb{C}$ inherits the operations and embedding mentioned above from $\mathbb{C}^{D(\mathbb{R}^d)}$. With the embedding in mind, we shall treat elements of $\mathbb{C}$ as their images in $\ast \mathbb{C}$.

A non-standard number $\langle A_\varphi \rangle$ is called real if

\[
\{\varphi \in D(\mathbb{R}^d) : A_\varphi \in \mathbb{R}\} \in U
\]

We denote the set of all real non-standard numbers by $\ast \mathbb{R}$ and supply it
with an order relation as follows:

\[ \langle A_\varphi \rangle >_R 0 \text{ if } \{ \varphi \in D(\mathbb{R}^d) : A_\varphi > 0 \} \in \mathcal{U} \]

**Theorem 5.1**  
(i) Every number \( \gamma \in \ast \mathbb{C} \) can be uniquely represented in the form \( \gamma = \alpha + \beta i \) where \( \alpha, \beta \in \ast \mathbb{R} \) and \( \alpha = \Re \gamma, \beta = \Im \gamma, \) and \( |\gamma| = \sqrt{\alpha^2 + \beta^2} \).

(ii) \( \ast \mathbb{C} \) is an algebraically closed non-Archimedean field of characteristic zero. \( \mathbb{C} \) is a subfield of \( \ast \mathbb{C} \).

(iii) \( \ast \mathbb{R} \) is a totally ordered non-Archimedean real closed field. Moreover, \( \alpha > 0 \) in \( \ast \mathbb{R} \) iff \( \alpha = \beta^2 \) for some \( \beta \in \ast \mathbb{R}, \beta \neq 0. \) \( \mathbb{R} \) is an ordered subfield of \( \ast \mathbb{R} \).

*Proof* (i) Let \( \gamma \in \ast \mathbb{C}. \) Then \( \gamma = \langle C_\varphi \rangle \) for some \( (C_\varphi) \in \mathbb{C}^{D(\mathbb{R}^d)}. \) But for each \( \varphi, \)

\[ C_\varphi = A_\varphi + B_\varphi i, \]

where \( A_\varphi = \Re C_\varphi \) and \( B_\varphi = \Im C_\varphi. \) Thus \( \langle C_\varphi \rangle = \langle A_\varphi \rangle + \langle B_\varphi \rangle i. \)

To prove uniqueness, suppose that \( \langle C_\varphi \rangle = \langle D_\varphi \rangle + \langle E_\varphi \rangle i \) also. Then

\[ \{ \varphi : \Re C_\varphi = D_\varphi \} \cap \{ \varphi : \Re C_\varphi = A_\varphi \} = \{ \varphi : A_\varphi = D_\varphi \} \in \mathcal{U} \]

because \( \mathcal{U} \) is closed under intersections. Therefore \( \langle A_\varphi \rangle = \langle D_\varphi \rangle. \)

The same argument can be applied to show that \( \langle B_\varphi \rangle = \langle E_\varphi \rangle. \)

The proof for \( |\gamma| \) is similar.

(ii) It is not hard to check that \( \mathbb{C}^{D(\mathbb{R}^d)} \) really is a ring, and that \( \sim_\mathcal{U} \) really is an equivalence relation. It follows that \( \ast \mathbb{C} \) is a (commutative) ring.

To prove that \( \ast \mathbb{C} \) is a field, we must show that each non-zero element has a multiplicative inverse. For any non-zero \( \gamma \in \ast \mathbb{C}, \) we may choose a representative \( (C_\varphi) \) such that \( C_\varphi \neq 0 \) for all \( \varphi. \) Let \( D_\varphi = 1/C_\varphi \) and \( \delta = \langle D_\varphi \rangle. \) Then \( \delta \gamma = \langle 1 \rangle. \)

Let

\[ P(x) = \sum_{k=0}^{n} \alpha_k x^k, \quad \alpha_k \in \ast \mathbb{C} \text{ for all } k \]
be a polynomial in \( *\mathbb{C}[x] \). Define
\[
P_\varphi(x) = \sum_{k=0}^{n} A_{k, \varphi} x^k
\]
where \( \alpha_k = \langle A_{k, \varphi} \rangle \) for each \( k \). Since each \( P_\varphi(x) \) is a polynomial over \( \mathbb{C} \), there exists a number \( C_\varphi \in \mathbb{C} \) such that \( P_\varphi(C_\varphi) = 0 \). If we let \( \gamma = \langle C_\varphi \rangle \), it follows that \( P(\gamma) = 0 \) in \( *\mathbb{C} \).

That \( \mathbb{C} \) is a subfield of \( *\mathbb{C} \) is clear from the embedding.

(iii) The trichotomy of the order relation on \( *\mathbb{R} \) follows from the trichotomy of the order relation on \( \mathbb{R} \). For suppose \( A = \{ \varphi : A_\varphi < B_\varphi \} \), \( B = \{ \varphi : A_\varphi = B_\varphi \} \), and \( C = \{ \varphi : A_\varphi > B_\varphi \} \), for some non-standard real numbers \( \langle A_\varphi \rangle \), \( \langle B_\varphi \rangle \). Note that \( A \), \( B \), and \( C \) are mutually disjoint. Therefore, at most one of \( A \), \( B \), or \( C \) can be in \( U \). Also, \( A \cup B \cup C = \mathcal{D}(\mathbb{R}^d) \in U \). We can use this to prove that one of \( A \), \( B \), or \( C \) must be in \( U \). For suppose that none of \( A \), \( B \), or \( C \) is in \( U \). Then by Theorem (3.1), \( B \cup C \in U \) and \( A \cup C \in U \). Taking the intersection of these two sets, we would have \( C \in U \), a contradiction. ▲

**Definition 5.2** Define the sets of **infinitesimal**, **finite**, and **infinitely large** numbers as follows:

\[
\mathcal{I}(*\mathbb{C}) = \{ x \in *\mathbb{C} : |x| < 1/n \text{ for all } n \in \mathbb{N} \}
\]
\[
\mathcal{F}(*\mathbb{C}) = \{ x \in *\mathbb{C} : |x| < n \text{ for some } n \in \mathbb{N} \}
\]
\[
\mathcal{L}(*\mathbb{C}) = \{ x \in *\mathbb{C} : |x| > n \text{ for all } n \in \mathbb{N} \}
\]

It is not hard to prove that \( \mathcal{F}(*\mathbb{C}) \) is a subring of \( *\mathbb{C} \) and \( \mathcal{I}(*\mathbb{C}) \) is a maximal ideal in \( \mathcal{F}(*\mathbb{C}) \).

**Example 5.1** Define \((R_\varphi) \in \mathbb{C}^{\mathcal{D}(\mathbb{R}^d)} \) by
\[
R_\varphi = \sup\{\|x\| : x \in \text{supp}\varphi\}
\]
where \( \text{supp}\varphi = \{x \in \mathbb{R}^d : \varphi(x) \neq 0\} \) is the support of \( \varphi \). The non-standard number \( \rho = \langle R\varphi \rangle \) is a (positive) infinitesimal. For let \( \mathcal{A} = \{\varphi : 0 < R\varphi < 1/n\} \). Then for any \( \varphi \in \mathcal{B}_{n+1} \), we have \( \varphi \in \mathcal{A} \), by the definition of \( \mathcal{B}_{n+1} \). Thus \( \mathcal{B}_{n+1} \subset \mathcal{A} \), implying \( \mathcal{A} \in \mathcal{U} \). \( \rho \) is called the canonical infinitesimal in \( ^\ast\mathbb{C} \).

**Definition 5.3** Define the **standard part mapping** \( st : ^\ast\mathbb{R} \to \mathbb{R} \cup \{\pm\infty\} \) by

\[
st(x) = \begin{cases} 
\sup\{r \in \mathbb{R} : r < x\} & \text{if } x \in \mathcal{F}(^\ast\mathbb{R}) \\
\infty & \text{if } x \in \mathcal{L}(^\ast\mathbb{R}_+) \\
-\infty & \text{if } x \in \mathcal{L}(^\ast\mathbb{R}_-)
\end{cases}
\]

We may extend this definition to \( ^\ast\mathbb{C} \) by \( st(x + yi) = st(x) + st(y)i \).

**Theorem 5.2** If \( x \in \mathcal{F}(^\ast\mathbb{C}) \) then \( x \) has a unique asymptotic expansion: \( x = r + dx \) where \( r \in \mathbb{C} \) and \( dx \in \mathcal{I}(^\ast\mathbb{C}) \). In fact, \( r = st(x) \).

**Proof** We will prove the case for \( x \in \mathcal{F}(^\ast\mathbb{R}) \). The general result will follow. Let \( x \in \mathcal{F}(^\ast\mathbb{R}) \). First note that \( x - st(x) \in \mathcal{I}(^\ast\mathbb{R}) \), for otherwise we would have \( |x - st(x)| > 1/n \) for some \( n \), implying either that \( st(x) > x \) or that \( st(x) + 1/2n < x \). In either case, this is a contradiction to Definition (5.3). To prove uniqueness, suppose that \( x = r + dx \) and \( x = s + dy \) are two expansions of \( x \). Then we would have \( r - s = dx - dy \), implying that \( r - s \in \mathcal{I}(^\ast\mathbb{R}) \). But since \( r - s \in \mathbb{R} \), \( r - s = 0 \). Hence \( r = s \). Therefore \( r + dx = r + dy \), implying \( dx = dy \). \( \square \)

## 6 Internal Sets

In non-standard analysis, internal sets play the role of the “good” sets, in a similar way to the measurable sets in Lebesgue theory.
In what follows we will use the abbreviation \textbf{a.e.} to mean that the set of functions for which some statement is true is in $U$.

**Definition 6.1** Let $A \subseteq \mathbb{C}$. The \textbf{non-standard extension} of $A$ is

\[ *A = \{ (A_\varphi) \in *\mathbb{C} : A_\varphi \in A \text{ a.e.} \} \]

A set $A$ of non-standard numbers is called \textbf{internal standard} if it is the non-standard extension of some subset of $\mathbb{C}$. The set of all internal standard sets is denoted by $*\mathcal{P}(\mathbb{C})$.

**Example 6.1** The non-standard extensions of the intervals $(a, b)$, $[a, b]$, $(a, \infty)$, etc. are

\[ *(a, b) = \{ x \in *\mathbb{R} : a < x < b \} \]
\[ *[a, b] = \{ x \in *\mathbb{R} : a \leq x \leq b \} \]
\[ *(a, \infty) = \{ x \in *\mathbb{R} : a < x \} , \text{ etc.} \]

**Definition 6.2** Let $(A_\varphi) \in \mathcal{P}(\mathbb{C})^{\mathcal{P}(\mathbb{R}^d)}$ be a family of subsets of $\mathbb{C}$. We define the \textbf{internal set} generated by $(A_\varphi)$ by

\[ \langle A_\varphi \rangle = \{ (A_\varphi) \in *\mathbb{C} : A_\varphi \in A_\varphi \text{ a.e.} \} \]

A set is called \textbf{external} if it is not internal.

**Example 6.2** Let $A_\varphi = (0, R_\varphi)$, where $R_\varphi$ is as in Example 5.1. Then the internal set $\langle A_\varphi \rangle$ generated by $(A_\varphi)$ is the internal interval $(0, \rho)$. It is important to note (and easy to check) that this coincides with the more natural definition for $(0, \rho)$ given by

\[ (0, \rho) = \{ x \in *\mathbb{R} : 0 < x < \rho \} \]

A set $S \subset \mathbb{R}^d$ is called \textbf{relatively compact} if its closure $\overline{S}$ is compact in $\mathbb{R}^d$. Unless it is specified otherwise, we shall call \textit{Lebesgue measurable sets} of $\mathbb{R}^d$ simply \textit{measurable} sets.
Definition 6.3 An internal set \( \langle A_\varphi \rangle \) of \( \ast \mathbb{R}^d \) is called \( \ast \)-measurable (\( \ast \)-compact, \( \ast \)-relatively-compact, \( \ast \)-closed, \( \ast \)-open, etc.) if 
\( A_\varphi \) is measurable (compact, relatively compact, closed, etc.) in \( \mathbb{R}^d \) for a.e. \( \varphi \).

Let \( \rho \in \ast \mathbb{R} \) denote a positive infinitesimal in \( \ast \mathbb{R} \) (for example, \( \rho \) might be the positive infinitesimal defined in (Example 5.1)). We shall keep \( \rho \) fixed in what follows.

Definition 6.4 Let \( \rho \) be a positive infinitesimal in \( \ast \mathbb{R} \). We define the following (external) sets of non-standard numbers:
\[
\mathcal{M}_\rho(\ast \mathbb{C}) = \{ x \in \ast \mathbb{C} \mid |x| \leq \rho^{-n} \text{ for some } n \in \mathbb{N} \}
\]
\[
\mathcal{N}_\rho(\ast \mathbb{C}) = \{ x \in \ast \mathbb{C} \mid |x| < \rho^n \text{ for all } n \in \mathbb{N} \}
\]
\[
\mathcal{F}_\rho(\ast \mathbb{C}) = \{ x \in \ast \mathbb{C} \mid |x| < 1/\sqrt[2n]{\rho} \text{ for all } n \in \mathbb{N} \},
\]
\[
\mathcal{I}_\rho(\ast \mathbb{C}) = \{ x \in \ast \mathbb{C} \mid |x| \leq \sqrt[2n]{\rho} \text{ for some } n \in \mathbb{N} \},
\]
\[
\mathcal{C}_\rho(\ast \mathbb{C}) = \{ x \in \ast \mathbb{C} \mid \sqrt[2n]{\rho} < |x| < 1/\sqrt[2n]{\rho} \text{ for all } n \in \mathbb{N} \}.
\]

The numbers in \( \mathcal{M}_\rho(\ast \mathbb{C}) \) and \( \mathcal{N}_\rho(\ast \mathbb{C}) \) are called \( \rho \)-moderate and \( \rho \)-null non-standard numbers, respectively. Similarly, the numbers in \( \mathcal{F}_\rho(\ast \mathbb{C}) \), \( \mathcal{I}_\rho(\ast \mathbb{C}) \) and \( \mathcal{C}_\rho(\ast \mathbb{C}) \) are called \( \rho \)-finite, \( \rho \)-infinitesimal and \( \rho \)-constant, respectively.

7 Saturation Principle in \( \ast \mathbb{C} \)

Theorem 7.1 Let \( \{ A_n \} \) be a sequence of internal sets in \( \ast \mathbb{C} \) such that
\[
\bigcap_{n=0}^m A_n \neq \emptyset
\]
for all \( m \in \mathbb{N} \). (The sequence \( \{ A_n \} \) satisfies the finite intersection property.) Then
\[
\bigcap_{n=0}^\infty A_n \neq \emptyset.
\]
Proof Since each $A_n$ is internal,

$$A_n = \langle A_{n,\varphi} \rangle, \ A_{n,\varphi} \subseteq C$$

Also, since for each $m$ it is given that $\bigcap_{n=0}^{m} A_n \neq \emptyset$, this implies that for each $m$ there exists a non-standard number $\langle C_{m,\varphi} \rangle \in {}^\ast C$ such that

$$\langle C_{m,\varphi} \rangle \in \bigcap_{n=0}^{m} \langle A_{n,\varphi} \rangle$$

or, in other words,

$$\langle C_{m,\varphi} \rangle \in \langle A_{n,\varphi} \rangle \text{ for } 0 \leq n \leq m$$

This means that for a.e. $\varphi$ and $0 \leq n \leq m$,

$$C_{m,\varphi} \subseteq A_{n,\varphi}$$

Remembering that $U$ is closed under finite intersections, we see that for a.e. $\varphi$,

$$C_{m,\varphi} \subseteq \bigcap_{n=0}^{m} A_{n,\varphi}$$

Hence for each $m$,

$$\bigcap_{n=0}^{m} A_{n,\varphi} \neq \emptyset \text{ a.e.}$$

We may assume without loss of generality that $A_{0,\varphi}$ is non-empty for all $\varphi$. (Else define $A'_{0,\varphi} = A_{0,\varphi}$ if $A_{0,\varphi} \neq \emptyset$ and $A'_{0,\varphi} = C$ otherwise. Then it will still be true that $A_0 = \langle A'_{0,\varphi} \rangle$.)

Next define a function $\mu : \mathcal{D}(\mathbb{R}^d) \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$\mu(\varphi) = \max\{m \in \mathbb{N}_0 \cup \{\infty\} \mid \bigcap_{n=0}^{m} A_{n,\varphi} \neq \emptyset\}$$

Notice that $\mu$ is defined for all $\varphi$ due to our assumption for $A_{0,\varphi}$.

Thus we have

$$\bigcap_{n=0}^{\mu(\varphi)} A_{n,\varphi} \neq \emptyset \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d)$$
Hence for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$ there exists (by Axiom of Choice) $A_\varphi$ such that $A_\varphi \in \bigcap_{n=0}^{\mu(\varphi)} \mathbb{A}_{n,\varphi}$.

We intend to show that

$$\langle A_\varphi \rangle \in \bigcap_{n=0}^{\infty} \mathbb{A}_n$$

or, equivalently, that for every $m$, $A_\varphi \in \mathbb{A}_{m,\varphi}$ for a.e. $\varphi$.

If $\varphi$ is such that $\bigcap_{n=0}^{m} \mathbb{A}_{n,\varphi} \neq \emptyset$, this implies that $0 \leq m \leq \mu(\varphi)$. Thus $A_\varphi \in \mathbb{A}_{m,\varphi}$, by the choice of $A_\varphi$. Therefore,

$$\{ \varphi \mid \bigcap_{n=0}^{m} \mathbb{A}_{n,\varphi} \neq \emptyset \} \subseteq \{ \varphi \mid A_\varphi \in \mathbb{A}_{m,\varphi} \}$$

But the set on the left is in $\mathcal{U}$, and so the set on the right is also, as required. ▲

8 Non-Standard Smooth Functions

Having constructed the fields $^*\mathbb{R}$ and $^*\mathbb{C}$, the natural next step is to look at functions on these fields. However, for our purposes we will focus on a certain class of function contained in $^*\mathbb{C}$-$^*\mathbb{R}$. In what follows $\mathcal{E}(\mathbb{R})$ is the set of $\mathcal{C}^\infty$-functions from $\mathbb{R}$ into $\mathbb{C}$.

**Definition 8.1** A function $f \in ^*\mathbb{C}$-$^*\mathbb{R}$ is called *internal smooth* if there exists a family $(f_\varphi) \in \mathcal{E}(\mathbb{R})^{\mathcal{D}(\mathbb{R}^d)}$ such that for every $x = \langle X_\varphi \rangle \in ^*\mathbb{R}$

$$f(x) = \langle f_\varphi(X_\varphi) \rangle$$

The set of all internal smooth functions will be denoted by $^*\mathcal{E}(\mathbb{R})$.

**Remark 8.1** $^*\mathcal{E}(\mathbb{R})$ may equivalently be defined as the set of equivalence classes $(f_\varphi)$ of families of functions in $\mathcal{E}(\mathbb{R})^{\mathcal{D}(\mathbb{R}^d)}$, where the equivalence relation is as usual:

$$(f_\varphi) \sim_U (g_\varphi) \text{ if } f_\varphi = g_\varphi \text{ for a.e. } \varphi$$
It is not hard to prove that the value of an internal function does not depend on the choice of representatives. If $\langle X_\varphi \rangle = \langle Y_\varphi \rangle \in \ast \mathbb{R}$ and $\langle f_\varphi \rangle = \langle g_\varphi \rangle \in \ast \mathcal{E}(\mathbb{R})$, then

$$\{ \varphi \mid f_\varphi(X_\varphi) = g_\varphi(X_\varphi) \} \cap \{ \varphi \mid g_\varphi(X_\varphi) = g_\varphi(Y_\varphi) \} \subseteq \{ \varphi \mid f_\varphi(X_\varphi) = g_\varphi(Y_\varphi) \}$$

Since $\mathcal{U}$ is closed under intersections, $\langle f_\varphi(X_\varphi) \rangle = \langle g_\varphi(Y_\varphi) \rangle$.

The operations of addition, multiplication, and partial differentiation in $\ast \mathcal{E}(\mathbb{R}^d)$ are inherited from $\mathcal{E}(\mathbb{R}^d)$. Also, $\mathcal{E}(\mathbb{R}^d)$ is embedded in $\ast \mathcal{E}(\mathbb{R}^d)$ by $f \mapsto \ast f$ where $\ast f = \langle f_\varphi \rangle$, $f_\varphi = f$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

In what follows integrable means Lebesgue integrable.

**Definition 8.2** Let $\langle X_\varphi \rangle \subseteq \ast \mathbb{R}^d$ be a $\ast$-measurable internal set and let $\langle f_\varphi \rangle \in \ast \mathcal{E}(\mathbb{R}^d)$ be an internal function. We say that $\langle f_\varphi \rangle$ is $\ast$-integrable over $\langle X_\varphi \rangle$ if

$f_\varphi$ is integrable over $X_\varphi$ for a.e. $\varphi$

If $\langle f_\varphi \rangle$ is $\ast$-integrable over $\langle X_\varphi \rangle$, we define the integral:

\[
\left(1\right) \int_{\langle X_\varphi \rangle} \langle f_\varphi \rangle(x) \, dx = \left\langle \int_{X_\varphi} f_\varphi(x) \, dx \right\rangle.
\]

We also say that the integral converges in $\ast \mathbb{C}$ (since it is a number in $\ast \mathbb{C}$). Notice that as long as the integral converges for a.e. $\varphi$, we include this object in the equivalence class, even if the integral diverges for other $\varphi$.

**9 Internal Sets and Saturation Principle in $\ast \mathcal{E}(\mathbb{R}^d)$**

We define internal sets in $\ast \mathcal{E}(\mathbb{R}^d)$ similarly to those of $\ast \mathbb{C}$.
Definition 9.1  (i) Let \((\mathcal{F}_\varphi) \in \mathcal{P}(\mathbf{E}(\mathbb{R}^d))^{\mathbb{D}(\mathbb{R}^d)}\) be a family of subsets of \(\mathbf{E}(\mathbb{R}^d)\). We define the **internal set** generated by \((\mathcal{F}_\varphi)\) by

\[
\langle \mathcal{F}_\varphi \rangle = \{ \{ f_\varphi \} \in \mathbf{E}(\mathbb{R}^d) : f_\varphi \in \mathcal{F}_\varphi \text{ a.e.} \}
\]

A set is called **external** if it is not internal.

(ii) An internal set \(\mathcal{F}\) is called **standard** if there exists \(\mathcal{F} \subseteq \mathbf{E}(\mathbb{R}^d)\) such that \(\mathcal{F} = \langle \mathcal{F} \rangle\). In this case we may also write \(\mathcal{F} = \mathbf{E}\).

**Theorem 9.1** Let \(\{\mathcal{F}_n\}\) be a sequence of internal sets in \(\mathbf{E}(\mathbb{R}^d)\) such that

\[
\bigcap_{n=0}^{m} \mathcal{F}_n \neq \emptyset
\]

for all \(m \in \mathbb{N}\). (The sequence \(\{\mathcal{F}_n\}\) satisfies the finite intersection property.) Then

\[
\bigcap_{n=0}^{\infty} \mathcal{F}_n \neq \emptyset.
\]

**Proof** The proof is almost identical to that of (Theorem 7.1).

**Definition 9.2** We define the following (external) subsets of \(\mathbf{E}(\mathbb{R}^d)\):

\[
\mathcal{F}(\mathbf{E}(\mathbb{R}^d)) = \{ f \in \mathbf{E}(\mathbb{R}^d) | (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mathcal{F}(\mathbf{E}(\mathbb{R}^d))[\partial^\alpha f(x) \in \mathbf{E}(\mathbb{C})] \},
\]

\[
\mathcal{I}(\mathbf{E}(\mathbb{R}^d)) = \{ f \in \mathbf{E}(\mathbb{R}^d) | (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mathcal{F}(\mathbf{E}(\mathbb{R}^d))[\partial^\alpha f(x) \in \mathcal{I}(\mathbb{C})] \},
\]

\[
\mathcal{M}_\rho(\mathbf{E}(\mathbb{R}^d)) = \{ f \in \mathbf{E}(\mathbb{R}^d) | (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mathcal{F}(\mathbf{E}(\mathbb{R}^d))[\partial^\alpha f(x) \in \mathcal{M}_\rho(\mathbb{C})] \},
\]

\[
\mathcal{N}_\rho(\mathbf{E}(\mathbb{R}^d)) = \{ f \in \mathbf{E}(\mathbb{R}^d) | (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mathcal{F}(\mathbf{E}(\mathbb{R}^d))[\partial^\alpha f(x) \in \mathcal{N}_\rho(\mathbb{C})] \},
\]

\[
\mathcal{F}_\rho(\mathbf{E}(\mathbb{R}^d)) = \{ f \in \mathbf{E}(\mathbb{R}^d) | (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mathcal{F}(\mathbf{E}(\mathbb{R}^d))[\partial^\alpha f(x) \in \mathcal{F}_\rho(\mathbb{C})] \},
\]

\[
\mathcal{I}_\rho(\mathbf{E}(\mathbb{R}^d)) = \{ f \in \mathbf{E}(\mathbb{R}^d) | (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mathcal{F}(\mathbf{E}(\mathbb{R}^d))[\partial^\alpha f(x) \in \mathcal{I}_\rho(\mathbb{C})] \},
\]

\[
\mathcal{C}_\rho(\mathbf{E}(\mathbb{R}^d)) = \{ f \in \mathbf{E}(\mathbb{R}^d) | (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mathcal{F}(\mathbf{E}(\mathbb{R}^d))[\partial^\alpha f(x) \in \mathcal{C}_\rho(\mathbb{C})] \}.
\]

The functions in \(\mathcal{F}(\mathbf{E}(\mathbb{R}^d)), \mathcal{I}(\mathbf{E}(\mathbb{R}^d)), \mathcal{M}_\rho(\mathbf{E}(\mathbb{R}^d)), \mathcal{N}_\rho(\mathbf{E}(\mathbb{R}^d)), \mathcal{F}_\rho(\mathbf{E}(\mathbb{R}^d)), \mathcal{I}_\rho(\mathbf{E}(\mathbb{R}^d)), \text{ and } \mathcal{C}_\rho(\mathbf{E}(\mathbb{R}^d))\) are called **finite, infinitesimal, \(\rho\)-moderate,**
\(\rho\)-null, \(\rho\)-finite, \(\rho\)-infinitesimal and \(\rho\)-constant functions, respectively. For more details we refer to (Lightstone and Robinson [7]) and (Wolf and Todorov [17]).

10 Weak Equality

Definition 10.1 Let \(x, y \in \ast \mathbb{C}, \ f, g \in \ast \mathcal{E}(\mathbb{R}^d)\)

(i) \(x \approx y\) if \(x - y \in \mathcal{I}(\ast \mathbb{C})\)

(ii) \(x \overset{\rho}{=} y\) if \(x - y \in \mathcal{N}_\rho(\ast \mathbb{C})\)

(iii) \(f \approx g\) if \(f - g \in \mathcal{I}(\ast \mathcal{E}(\mathbb{R}^d))\)

(iv) \(f \overset{\rho}{=} g\) if \(f - g \in \mathcal{N}_\rho(\ast \mathcal{E}(\mathbb{R}^d))\)

(v) \(f \cong g\) if \(\int f(x)\tau(x)\, dx = \int g(x)\tau(x)\, dx\) for every \(\tau \in \mathcal{D}(\mathbb{R}^d)\)

(vi) \(f \overset{\rho}{\cong} g\) if \(\int f(x)\tau(x)\, dx \overset{\rho}{=} \int g(x)\tau(x)\, dx\) for every \(\tau \in \mathcal{D}(\mathbb{R}^d)\)

(vii) \(f \cong g\) if \(\int f(x)\tau(x)\, dx \approx \int g(x)\tau(x)\, dx\) for every \(\tau \in \mathcal{D}(\mathbb{R}^d)\)

It is not hard to prove that each of these weak equalities forms an equivalence relation in its respective space. Many results in non-standard analysis hold weakly in the sense of one of these weak equalities.
11 Schwartz Distributions

At this point, we must take a short detour to present some basic definitions and results from the Schwartz theory.

**Definition 11.1** A **distribution** is a mapping $F: \mathcal{D}(\mathbb{R}^d) \rightarrow \mathbb{C}$ that satisfies the following conditions:

(i) **Linearity:**

$$F[c_1 \tau_1 + c_2 \tau_2] = c_1 F[\tau_1] + c_2 F[\tau_2] \text{ for all } c_1, c_2 \in \mathbb{C} \text{ and } \tau_1, \tau_2 \in \mathcal{D}(\mathbb{R}^d).$$

(ii) **Continuity:** Let $\{\tau_k\}$ be a sequence in $\mathcal{D}(\mathbb{R}^d)$. Suppose there exists $R$ such that $\text{supp} \tau_k \subseteq \{x : |x| < R\}$ for all $k$. Also, suppose there exists $\tau \in \mathcal{D}(\mathbb{R}^d)$ such that $\partial^\alpha \tau_k \rightarrow \partial^\alpha \tau$ for all $\alpha \in \mathbb{N}_0^d$ uniformly as $k \rightarrow \infty$. Then $F[\tau_k] \rightarrow F[\tau]$.

We will denote by $\mathcal{D}'(\mathbb{R}^d)$ the set of all such distributions.

We supply $\mathcal{D}'(\mathbb{R}^d)$ with the usual pointwise addition and scalar multiplication. In addition, we define partial differentiation by

$$(\partial^\alpha F)[\tau] = (-1)^{|\alpha|} F[\partial^\alpha \tau]$$

and multiplication by a smooth function $g \in \mathcal{E}(\mathbb{R}^d)$ by

$$(gF)[\tau] = F[g\tau]$$

Both of these operations are well-defined since $\partial^\alpha \tau, g\tau \in \mathcal{D}(\mathbb{R}^d)$.

$\mathcal{L}_{\text{loc}}(\mathbb{R}^d)$, the set of locally integrable functions, is embedded in $\mathcal{D}'(\mathbb{R}^d)$ by the mapping

$$S(f) = \int f(t)\tau(t)dt$$
It is not hard to show that this embedding preserves the operations mentioned above.

Finally, we define the convolution of a distribution with a test function by

$$(F * \tau)(x) = F[\tau(x - t)]$$

**Theorem 11.1** If $F \in \mathcal{D}'(\mathbb{R}^d)$ and $\tau \in \mathcal{D}(\mathbb{R}^d)$ then $(F * \tau)(x) \in \mathcal{E}(\mathbb{R}^d)$ and $\partial^\alpha (F * \tau) = F * \partial^\alpha \tau$.

Before proving this theorem, we will state (without proof) a result from analysis. See (Rudin [13] p.148):

**Lemma 11.1** Suppose

$$\lim_{n \to \infty} f_n(x) = f(x) \quad (x \in E).$$

Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$ 

Then $f_n \to f$ uniformly on $E$ if and only if $M_n \to 0$ as $n \to \infty$.

**Proof of the theorem** We will prove the theorem for the case $d=1$. The general result will follow.

Let $f(x) = (F * \tau)(x)$. Fixing $x$, we wish to show that

$$f(x + h) - f(x) \to 0 \quad \text{as} \quad h \to 0$$

Note that

$$f(x + h) - f(x) = (F * \tau)(x + h) - (F * \tau)(x)$$

$$= F[\tau(x + h - t)] - F[\tau(x - t)]$$

$$= F[\tau(x + h - t) - \tau(x - t)]$$
by the linearity of $F$. Let

$$\psi(t) = \tau(x + h - t) - \tau(x - t)$$

$\psi$ is itself a test function, and if we restrict $|h| < 1$, then the support of $\psi$ and all its derivatives is contained in $E = \{y \mid y \leq r + |x| + 1\}$, where $r$ is the radius of the support of $\tau$. It is clear that any sequence $\{\psi_{h_n}\}$ where $h_n \to 0$ as $n \to \infty$ converges pointwise to 0 for all $x$ and $t$ (by the uniform continuity of $\tau$). Also, since one compact set, $E$, contains the support of $\psi_{h_n}$ for all $n$, and since each $\psi_{h_n}$ is continuous, $M_n = \sup_{t \in E} |\psi_{h_n}(t)|$ is achieved by $\psi_{h_n}$ for each $n$. Thus $M_n \to 0$, implying that $\{\psi_{h_n}\} \to 0$ uniformly, by the Lemma. Therefore, since $F$ is continuous in the sense of (Definition 11.1),

$$F[\psi_{h_n}] \to F[0] = 0$$

Since $x$ was chosen arbitrarily, this proves that $f = F \ast \tau$ is continuous.

To prove that $f'$ exists and that $f'(x) = (F \ast \tau')(x)$, we must show that

$$\frac{f(x + h) - f(x)}{h} - (F \ast \tau')(x) \to 0 \text{ as } h \to 0$$

Note that

$$\frac{f(x + h) - f(x)}{h} - (F \ast \tau')(x) = \frac{(F \ast \tau)(x + h) - (F \ast \tau)(x)}{h} - (F \ast \tau')(x)$$

$$= F \left[ \frac{\tau(x + h - t) - \tau(x - t)}{h} - \tau'(x - t) \right]$$

Now, if we let

$$\chi(t) = \frac{\tau(x + h - t) - \tau(x - t)}{h} - \tau'(x - t)$$

we can use the same argument as before to show that

$$F[\chi_{h_n}(t)] \to F[0] = 0 \text{ as } h \to 0$$
This proves that \((F \ast \tau)' = F \ast \tau'\). Since \(\tau'\) is itself a test function, the same proof works to show that \((F \ast \tau)'' = F \ast \tau''\) and so on. For functions of several variables, the same argument can be applied in each variable to show the general result. ▲

Before we can prove the embedding of the distributions in \(*\mathcal{E}(\mathbb{R}^d)\), we need a result showing that distributions can be “approximated” in a way by a certain sequence of test functions.

**Theorem 11.2** Let \(\{\delta_n\}\) be a sequence in \(\mathcal{D}(\mathbb{R}^d)\) such that \(\delta_n \in \mathcal{B}_n\) for every \(n\). Then for any distribution \(T \in \mathcal{D}'(\mathbb{R}^d)\), \(T \ast \delta_n \rightarrow T\) weakly. (A sequence of distributions \(\{F_k\}\) converges weakly to a distribution \(F\) if \(F_k[\tau] \rightarrow F[\tau]\) for all test functions \(\tau\).)

Before proving this theorem, we need two lemmas:

**Lemma 11.2** Let \(\{\delta_n\}\) be as above and let \(\tau\) be any test function. Then there exists \(R\) such that \(\text{supp}(\delta_n \ast \tau) \subset \{x : |x|\leq R\}\). Also, \(\partial^\alpha(\delta_n \ast \tau) \rightarrow \tau\) uniformly for every \(\alpha \in \mathbb{N}_0^d\).

**Proof** For each \(n\), \(\text{supp}\delta_n \subset \{x : |x| \leq 1/n\}\). In particular, \(\text{supp}\delta_n \subset \{x : |x| \leq 1\}\). If we let \(R_\tau\) be the radius of the support of \(\tau\) and set \(R = R_\tau + 1\), then it is not hard to see that \(\text{supp}(\delta_n \ast \tau) \subset \{x : |x| \leq R\}\).

As before, to show the uniform convergence it is enough to prove that

\[
\sup_{|x| \leq R} |(\delta_n \ast \tau)(x) - \tau(x)| \rightarrow 0
\]
Recalling that \( \int \delta_n = 1 \), we see that

\[
\sup_{|x| \leq R} |(\delta_n * \tau)(x) - \tau(x)| = \sup_{|x| \leq R} \left| \int \delta_n(t) \tau(x-t) dt - \tau(x) \int \delta_n(t) dt \right|
\]

\[
= \sup_{|x| \leq R} \left| \int_{|t| \leq 1/n} \delta_n(t)[\tau(x-t) - \tau(x)] dt \right|
\]

By the mean value theorem for integrals, there exists \( |t_n| \leq 1/n \) such that

\[
= \sup_{|x| \leq R} \left| \tau(x-t_n) - \tau(x) \right| \int \delta_n(t) dt
\]

and by the extreme value theorem there exists \( |x_n| \leq R \) such that

\[
= |\tau(x_n - t_n) - \tau(x_n)|
\]

This last expression vanishes as \( n \to \infty \) since \( \tau \) is uniformly continuous. The case \( \alpha \neq 0 \) is similar. \( \triangledown \)

For the proof of the next lemma see (Folland [5] p.318):

**Lemma 11.3** Suppose \( F \) is a distribution and \( \phi \) and \( \psi \) are test functions. Then \( (F * \phi)[\psi] = F[\tilde{\phi} * \psi] \), where \( \tilde{\phi}(x) = \phi(-x) \).

**Proof of the theorem** We must show that for any distribution \( T \) and any test function \( \tau \),

\[
(T * \delta_n)[\tau(x)] \to T[\tau(x)]
\]

Using (Lemma 11.3) and remembering that \( \delta_n \) is symmetric for all \( n \),

\[
(T * \delta_n)[\tau(x)] = T[(\delta_n * \tau)(x)] \to T[\tau(x)]
\]

by (Lemma 11.2) and the continuity of \( T \). \( \triangledown \)
12 Embedding of Schwartz Distributions in $\ast \mathcal{E}(\mathbb{R}^d)$

Finally, we are ready to define the embedding $\Sigma$ of $\mathcal{D}'(\mathbb{R}^d)$ into $\ast \mathcal{E}(\mathbb{R}^d)$ as follows:

$$\Sigma(T) = \langle T \ast \varphi \rangle$$

By (Theorem 11.1), $\Sigma(T) \in \ast \mathcal{E}(\mathbb{R}^d)$. From the definition of the convolution, it is clear that $\Sigma$ is linear. It remains to prove that $\Sigma$ is injective.

**Lemma 12.1** $\Sigma$ is injective.

**Proof** Since $\Sigma$ is linear, it is enough to show that $\Sigma(T) = 0$ implies $T = 0$.

If $\Sigma(T) = 0$, we have that $T \ast \varphi = 0$ a.e. That is, $\Phi = \{ \varphi \mid T \ast \varphi = 0 \} \in \mathcal{U}$. Thus $\emptyset \neq \Phi \cap \mathcal{B}_n \in \mathcal{U}$ for each $n$, where $\mathcal{B}_n$ are the basic sets. Therefore we can construct a sequence $\{ \varphi_n \}$ such that $\varphi_n \in \Phi \cap \mathcal{B}_n$ for each $n$. Then by (Theorem 11.2), we have that $T = 0$ since $T \ast \varphi_n = 0$ for every $n$. ▲

**Theorem 12.1**

(i) $\langle P \ast \varphi \rangle = \ast P$ for every polynomial $P \in \mathbb{C}[x_1, \ldots, x_d]$

(ii) $\langle f \ast \varphi \rangle \overset{\rho}{=} \ast f$ for all $f \in \mathcal{E}(\mathbb{R}^d)$

**Proof**

(i) Let $P \in \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial of degree $p$. By the Taylor formula,

$$P(x - t) = P(x) + \sum_{|\alpha| = 1}^{p} \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha P(x) t^\alpha$$

It follows that for every test function $\varphi$ and $x \in \mathbb{R}^d$

$$(P \ast \varphi)(x) = \int P(x-t) \varphi(t) dt = P(x) \int \varphi(t) dt + \sum_{|\alpha| = 1}^{p} \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha P(x) \int t^\alpha \varphi(t) dt$$

Notice that if $\varphi \in \mathcal{B}_n$ for some $n \geq p$, then $\int \varphi(t) dt = 1$ and $\int t^\alpha \varphi(t) dt = 0$, $|\alpha| = 1, 2, \ldots, p$. Thus we have

$$\mathcal{B}_n \subseteq \{ \varphi \mid P \ast \varphi = P \}$$

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implying that \( P * \varphi = P \) a.e. as required.

(ii) Let \( \xi \in \mathcal{F}(^\ast \mathbb{R}^d) \), \( n \in \mathbb{N} \), and \( \alpha \) be a multi-index. We have to show that \( |\partial^\alpha (f * \varphi)(\xi) - \partial^\alpha f(\xi)| < \rho^n \). We will show this for the case \( \alpha = 0 \), the general result will follow.

Since \( st(\xi) \in \mathbb{R}^d \) we can find an open relatively compact set \( \mathcal{O} \subset \mathbb{R}^d \) such that \( st(\xi) \in \mathcal{O} \) and by (Robinson [12], p.90 Theorem 4.1.4) \( \xi \in ^\ast \mathcal{O} \) and hence \( \xi \in ^\ast ^\ast \mathcal{O} \).

As before, the Taylor formula gives

\[
f(x - t) = f(x) + \sum_{|\alpha| = 1}^{n} \frac{(-1)^{|\alpha|} \partial^\alpha f(x)}{\alpha!} t^\alpha + \sum_{|\alpha| = n+1} \frac{(-1)^{|\alpha|} \partial^\alpha f(\eta(x, t))}{\alpha!} t^\alpha
\]

where \( \eta(x, t) \) is a point in \( \mathbb{R}^d \) “between \( x \) and \( t \)”. It follows that for every \( \varphi \in \mathcal{D}(\mathbb{R}^d) \)

\[
(f * \varphi)(x) = f(x) \int \varphi(t) dt + \sum_{|\alpha| = 1}^{n} \frac{(-1)^{|\alpha|} \partial^\alpha f(x)}{\alpha!} \int t^\alpha \varphi(t) dt +
\]

\[
\sum_{|\alpha| = n+1} \int \frac{(-1)^{|\alpha|} \partial^\alpha f(\eta(x, t))}{\alpha!} t^\alpha \varphi(t) dt
\]

Letting

\[
M \overset{\text{def}}{=} 2 \sum_{|\alpha| = n+1} \sup_{x \in K} \sup_{t \in \text{supp}(\varphi)} \left| \frac{\partial^\alpha f(\eta(x, t))}{\alpha!} \right|
\]

we have that \( M \rho^{n+1} < \rho^n \) since \( M \in \mathbb{R} \) and \( \rho \) is a positive infinitesimal. In other words, if \( \rho = \langle R_\varphi \rangle \), then \( MR_\varphi^{n+1} < R_\varphi^n \) a.e.
By the properties of $B_n$, it follows that for a.e. $\varphi, x \in K$,
\[
| (f \ast \varphi)(x) - f(x) | < \sup_{x \in K} | (f \ast \varphi)(x) - f(x) |
\]
\[
< \sum_{|\alpha|=n+1} \sup_{x \in K} \sup_{t \in \text{supp}(\varphi)} \left| \partial^{\alpha} f(\eta(x,t)) \right| \left( \sup_{t \in \text{supp}(\varphi)} |t|^{n-1} \right) \int |\varphi(t)| dt
\]
\[
\leq MR_{\varphi}^{n+1} < R_{\varphi}^n
\]
Finally, since $\xi \in \ast K$, we have that $|(f \ast \varphi)(\xi) - f(\xi)| < \rho^n$, as required.

The general result follows from the case $\alpha = 0$ and the fact that $\partial^{\alpha}(f \ast \varphi) = (\partial^{\alpha} f) \ast \varphi$. ▲

13 Conservation Laws in $\ast \mathcal{E}(\Omega)$ and the Hopf Equation

The embedding in the previous section is done deliberately, with the intent of showing that $\ast \mathcal{E}$ is a natural extension of $\mathcal{D}'$ and an appropriate setting for the study of weak solutions to non-linear partial differential equations, an abundance of which arise from the conservation law of physics.

**Theorem 13.1 (Conservation Laws in $\ast \mathcal{E}(\Omega)$)** Let $L \in \mathbb{R}_+ \cup \{\infty\}$, $F \in C^\infty(\mathbb{C})$ and let $\ast F$ be the non-standard extension of $F$. Let $u \in \ast \mathcal{E}(\Omega)$, where $\Omega = (0, L) \times (0, \infty)$. Then the following are equivalent:

(i) $u_t(x, t) + [\ast F(u(x, t))]_x = 0$ for all $x, t \in \ast \mathbb{R}$, $0 < x < L$, $t > 0$.

(ii) $u_t(x, t) + [\ast F'(u(x, t))] u_x = 0$ for all $x, t \in \ast \mathbb{R}$, $0 < x < L$, $t > 0$.

(iii) $\frac{d}{dt} \int_a^b u(x, t) dx = \ast F(u(a, t)) - \ast F(u(b, t))$ for every $a, b, t \in \ast \mathbb{R}$, $0 < a < b < L$, $t > 0$.

**Remark 13.1** The term “conservation law” is due to (iii) which in a classical setting is given by
\[
\frac{d}{dt} \int_a^b u(x, t) dx = F(u(a, t)) - F(u(b, t)).
\]
Here $u(x,t)$ stands for the density of a physical quantity (the density of the mass of a fluid, the density of the heat energy, etc.) in a rod of length $L$ and $F(u(x,t))$ stands for the flux of the quantity from left to right through the $x$-cross section. Then the above equality expresses the conservation of this quantity in any $(a,b)$-segment of the rod. Recall that, according to the classical theory, (i)-(iii) are equivalent for solutions $u$ in the class $C^2(\Omega)$ and for all $x,t,a,b \in \mathbb{R}$ in the corresponding intervals. The proof which follows can be generalized (without new complications) in the case of more complicated flux $F(u,u_x)$ or even $F(u,u_x,u_{xx})$.

Proof (i) $\Leftrightarrow$ (ii): The equivalency between (i) and (ii) follows immediately from the fact that the partial differentiation and extension mapping $\ast$ commute in $\ast E(\Omega)$ and the fact that $\ast E(\Omega)$ is a differential algebra (with Leibniz rule for differentiation of products and chain rule). So, we have

$$[\ast F(u(x,t))]_x = (\ast F)'(u(x,t)) u_x(x,t) = \ast F'(u(x,t)) u_x(x,t),$$

as required.

(i) $\Rightarrow$ (iii): We have $a = \langle a_\varphi \rangle, b = \langle b_\varphi \rangle$ and $u = \langle u_\varphi \rangle$ for some families of real numbers $(a_\varphi),(b_\varphi) \in \mathbb{R}^D(\mathbb{R}^2)$ and some family of smooth functions $(u_\varphi) \in E(\Omega)^D(\mathbb{R}^2)$. We have $\Phi = \{ \varphi \mid a_\varphi < b_\varphi \} \subseteq U$ since $a < b$ in $\ast \mathbb{R}$, by assumption. Thus (involving the classical arguments in the framework of $E(\Omega)$) we have $\Phi \subseteq \Phi_1$, where

$$\Phi_1 = \{ \varphi \mid \frac{d}{dt} \int_{a_\varphi}^{b_\varphi} u_\varphi(x,t) \, dx = F(u_\varphi(a_\varphi,t)) - F(u_\varphi(b_\varphi,t)) \text{ for all } t \in \mathbb{R}_+ \}.$$

The latter implies $\Phi_1 \subseteq U$ which implies (iii), as required, after transferring the result from representatives to the corresponding equivalence classes.

(i) $\Leftrightarrow$ (iii): Suppose (on the contrary) that there exist $\xi, \tau \in \ast \mathbb{R}$, $0 < \xi < L$, $\tau > 0$, such that $u_t(\xi,\tau) + [\ast F(u(\xi,\tau))]_x \neq 0$ in $\ast \mathbb{C}$. We have $\xi = \langle \xi_\varphi \rangle$ and $\tau = \langle \tau_\varphi \rangle$ for some $(\xi_\varphi),(\tau_\varphi) \in \mathbb{R}^D(\mathbb{R}^2)$. We denote $\Phi = \{ \varphi \mid (u_\varphi)_t(\xi_\varphi,\tau_\varphi) +$
\[ [F(u_{\varphi}(\xi_{\varphi}, \tau_{\varphi}))]_x \neq 0 \} \] and observe that \( \Phi \in \mathcal{U} \) (by our assumption). Also, we let

\[
\Phi_{\varphi} = \{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \frac{d}{dt} \int_{\alpha}^{\beta} u_{\varphi}(x, \gamma) \, dx \neq F(u_{\varphi}(\alpha, \gamma)) - F(u_{\varphi}(\beta, \gamma)), \\
0 < \alpha < \beta < L, \gamma > 0 \},
\]

and observe that \( \Phi_{\varphi} \neq \emptyset \) for all \( \varphi \in \Phi \) (by the classical theory in the framework of \( \mathcal{E}(\Omega) \)). By axiom of choice, there exist families \( (a_{\varphi}), (b_{\varphi}), (\gamma_{\varphi}) \in \mathbb{R}^\Phi \) such that \( (a_{\varphi}, b_{\varphi}, \gamma_{\varphi}) \in \Phi_{\varphi} \) for all \( \varphi \in \Phi \). If \( \varphi \in \mathcal{D}(\mathbb{R}^2) \setminus \Phi \), we define \( (a_{\varphi}, b_{\varphi}, \gamma_{\varphi}) \) anyhow (say, by \( a_{\varphi} = b_{\varphi} = \gamma_{\varphi} = 1 \)). These families (of real numbers) determine the non-standard real numbers \( a = \langle a_{\varphi} \rangle, b = \langle b_{\varphi} \rangle \) and \( t = \langle \gamma_{\varphi} \rangle \). Next, we observe that \( \Phi \subseteq \Psi \) (by the definition of \( \Phi_{\varphi} \)), where

\[
\Psi = \{ \varphi \mid \frac{d}{dt} \int_{a_{\varphi}}^{b_{\varphi}} u_{\varphi}(x, \gamma_{\varphi}) \, dx \neq F(u_{\varphi}(a_{\varphi}, \gamma_{\varphi})) - F(u_{\varphi}(b_{\varphi}, \gamma_{\varphi})), \\
0 < a_{\varphi} < b_{\varphi} < L, \gamma_{\varphi} > 0 \}.
\]

Next, \( \Phi \in \mathcal{U} \) implies \( \Psi \in \mathcal{U} \) which implies

\[
\frac{d}{dt} \int_{a}^{b} u(x, t) \, dx \neq F(u(a, t)) - F(u(b, t)), 0 < a < b < L, t > 0,
\]

in the framework of \( \ast \mathbb{C} \), contradicting (iii). The proof is complete. ▲

**Example 13.1 (Hopf Equation)** To appreciate the result of Theorem 13.1 we recall that (i)-(iii) might or might not be equivalent in classes of classical functions and Schwartz distributions larger than \( \mathcal{C}^2(\Omega) \), where the (important for the theory and applications) **fundamental solutions and shock wave solutions** belong. For a discussion we refer to (J. David Logan [9], p. 309-310). Here is an example: Let \( F(u) = \frac{1}{2}u^2 \) and \( L = \infty \), so we have \( \Omega = \mathbb{R}^2_+ \). In this case (i)-(iii) become:

(i) \( u_t(x, t) + [\frac{1}{2}u(x, t)^2]_x = 0 \) for all \( x, t \in \mathbb{R}, x > 0, t > 0 \).
(ii) \( u_t(x, t) + u(x, t)u_x(x, t) = 0 \) (Hopf equation) for all \( x, t \in \mathbb{R}, \ x > 0, \ t > 0. \)

(iii) \( \frac{d}{dx} \int_a^b u(x, t) \, dx = \frac{1}{2} [u^2(a, t) - u^2(b, t)] \) for every \( a, b, t \in \mathbb{R}, \ 0 < a < b, \ t > 0, \)

respectively. Let \( v \in \mathbb{R}_+ \), \( H \) be the Heaviside step function and let \( u(x, t) = 2vH(x - vt) \) be a shock wave. The next analysis shows that (i), (ii) and (iii) are not equivalent in the spaces of classical functions and Schwartz distributions:

(i) Since \( u = 2vH(x - vt) \notin \mathcal{C}^2(\mathbb{R}^2_+) \) this function can not be a classical solution of (i). However, \( u = 2vH(x - vt) \) is a (generalized) solution of (i) in the framework of the class of Schwartz distributions \( \mathcal{D}'(\mathbb{R}^2_+) \). Indeed, for the first term of (i) we have \( u_t(x, t) = -2v^2 \delta(x - vt) \), where \( \delta(x) \) is the Dirac delta function. For the second term we have \([\frac{1}{2}u(x, t)^2]_x = [\frac{1}{2}u_x^2H(x - vt)^2]_x = [2v^2H(x - vt)]_x = 2v^2 \delta(x - vt)\). Thus \( u(x, t) = 2vH(x - vt) \) is a (generalized) solution of (i). We should notice that \( u(x, t) = 2vH(x - vt) \) is also a weak solution of (i) in the framework of \( \mathcal{L}_{loc}(\mathbb{R}^2_+) \) (see the remark below).

(ii) \( u(x, t) = 2vH(x - vt) \) is clearly not a solution of (ii) in classical sense. Neither is it a (generalized) solution of (ii) in the class \( \mathcal{D}'(\mathbb{R}^2_+) \) because the term \( uu_x = 4v^2H(x - vt)\delta(x - vt) \) does not make sense within \( \mathcal{D}'(\mathbb{R}^2_+) \) (recall that there is no multiplication in \( \mathcal{D}'(\mathbb{R}^2_+) \)).

(iii) For the LHS of (iii) we have \( \frac{d}{dt} \int_a^b u(x, t) \, dx = \frac{d}{dt} \int_a^b 2vH(x - vt) \, dx = 2v \frac{d}{dt} \int_{a - vt}^{b - vt} H(x) \, dx = -2v^2 H(vt - a) \). For the RHS of (iii) we have
\[
\frac{1}{2} [u^2(a, t) - u^2(b, t)] = 2v^2 [H(a - vt) - H(b - vt)] = -2v^2 H(vt - a).
\]
Thus \( u(x, t) = 2vH(x - vt) \) is a solution of (iii).

Remark 13.2 (Weak Solution) Suppose the \( u \in \mathcal{L}_{loc}(\Omega) \) is a solution of \( u_t(x, t) + [F(u(x, t))]_x = 0 \) in the framework of \( \mathcal{D}'(\Omega) \) (that means that both \( u_t(x, t) \) and \([F(u(x, t))]_x \) are in \( \mathcal{D}'(\Omega) \)). These solutions are often called weak
solutions because they satisfy the weak equality:

$$
\iint_\Omega [u(x,t)\tau_t(x,t) + F(u(x,t))\tau_x(x,t)] \, dx \, dt = 0,
$$

for all test functions $\tau \in \mathcal{D}(\Omega)$. In $\mathcal{D}'(\Omega)$ we have:

$$
\iint_\Omega [u(x,t)\tau_t(x,t) + F(u(x,t))\tau_x(x,t)] \, dx \, dt = \langle u_t(x,t) + [F(u(x,t))]_x, \tau(x,t) \rangle,
$$

where $\langle \, , \, \rangle$ stands for the pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Thus every (generalized) solution in $\mathcal{D}'(\Omega)$ is also weak solution in $\mathcal{L}_{loc}(\Omega)$. In particular the function $u(x,t) = 2vH(x-vt)$ in the example above is both a generalized and a weak solution of (i).

14 Generalized Delta-like Solution of the Hopf Equation

In this section we will prove the existence of a delta-like weak solution to the Hopf equation of the type $^\rho \hat{\simeq}$. That is, we are looking for a function of the form

$$
u(x,t) = u_0 + A\frac{\Theta}{\rho} \left( \frac{x-vt}{\rho} \right)
$$

where $\Theta \in ^*\mathcal{S}(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$ is the class of rapidly decreasing functions, such as $e^{-x^2}$, $^*\mathcal{S}$ is its non-standard extension, defined similarly to $^*\mathcal{E}$), $\int \Theta(x)dx = 1$, $u_0, A, v \in \mathcal{M}_\rho(^*\mathbb{R})$ (we consider $A$ to be the amplitude of the soliton and $v$ its velocity), and for all $t > 0$,

$$
u_t + \nu\nu_x \overset{\rho}{\simeq} 0
$$

That is, for all $t > 0$, $\tau \in \mathcal{D}'(\mathbb{R})$,

$$
\int [\nu_t + \nu\nu_x] \tau(x) dx \overset{\rho}{\simeq} 0
$$
In addition, we would like this function to satisfy the conservation law, so that for all \( a, b \in \mathbb{R}, t > 0 \),
\[
\frac{d}{dt} \int_a^b u(x,t)\,dx = \frac{1}{2} \left[ u^2(a,t) - u^2(b,t) \right]
\]
Calculating \( u_t + uu_x \) we get
\[
u_t + uu_x = -\frac{Av}{\rho^2} \Theta' \left( \frac{x - vt}{\rho} \right) + \frac{u_0A}{\rho^2} \Theta' \left( \frac{x - vt}{\rho} \right) + \frac{A^2}{\rho^3} \Theta \left( \frac{x - vt}{\rho} \right) \Theta' \left( \frac{x - vt}{\rho} \right)
\]
Simplifying and letting \( \Theta \left( \frac{x - vt}{\rho} \right) \Theta' \left( \frac{x - vt}{\rho} \right) = \frac{\rho}{2} \left( \Theta^2 \left( \frac{x - vt}{\rho} \right) \right) \) gives us
\[
\int [u_t + uu_x] \tau(x)\,dx
\]
\[
= \frac{(u_0 - v)A}{\rho^2} \int \Theta' \left( \frac{x - vt}{\rho} \right) \tau(x)\,dx + \frac{A^2}{2\rho^2} \int \left( \Theta^2 \left( \frac{x - vt}{\rho} \right) \right) \tau(x)\,dx
\]
Integrating by parts and making the substitution \( y = \frac{x - vt}{\rho} \) gives
\[
= \int \left[ (v - u_0)A\Theta(y) - \frac{A^2}{2\rho} \Theta^2(y) \right] \tau'(vt + \rho y)\,dy
\]
Finally, using the Taylor formula for \( \tau'(vt + \rho y) \), we have that for each \( m \in \mathbb{N} \),
\[
= \sum_{n=0}^{m} \int \left[ (v - u_0)A\Theta(y) - \frac{A^2}{2\rho} \Theta^2(y) \right] y^n \frac{\tau^{(n+1)}(vt)}{n!} \rho^n \,dy + R_m(\tau)
\]
where the remainder term is
\[
R_m(\tau) = \rho^{m+1} \int \left[ (v - u_0)A\Theta(y) - \frac{A^2}{2\rho} \Theta^2(y) \right] y^{m+1} \frac{\tau^{(m+2)}(\eta(y,t))}{(m+1)!} \,dy
\]
We would like to find a function \( \Theta \) such that for every \( m \),
\[
\int \left[ (v - u_0)A\Theta(y) - \frac{A^2}{2\rho} \Theta^2(y) \right] y^n \,dy = 0, \quad 0 \leq n \leq m
\]
\[ |R_m(\tau)| < \rho^{m+k} \] (for some fixed k).

When \( m = 0 \), we have that
\[
A = \frac{2\rho(v - u_0)}{\int \Theta^2(y)dy}
\]
(remembering that \( \int \Theta(y)dy = 1 \)). Replacing this value of A, we have that for every \( m \),
\[
\int \Theta^2(y)dy \int \Theta(y)y^n dy = \int \Theta^2(y)y^n dy, \quad 0 \leq n \leq m
\]
Define
\[
S_m = \{ f \in S : \int f(x)x^n dx = \frac{\int f^2(x)x^n dx}{\int f^2(x)dx}, \quad 0 \leq n \leq m \}
\]
For each \( m \), \( S_m \) is non-empty by (M. Radyna [11] p. 275).

Now let
\[
\overline{S}_m = \{ f \in *S_m : f(0) = 0 \\
|\ln \rho|^{-1} \int |f(x)x^n| < 1 \\
|\ln \rho|^{-1} \int |f^2(x)x^n|dx < 1, \quad 0 \leq n \leq m \}
\]

The standard functions in \( *S_m \) certainly satisfy the second and third conditions, since their integrals will be standard (and therefore finite) and \( |\ln \rho|^{-1} \) is infinitely small. As for the first condition, we can say with certainty that if a function \( f \in *S_m \) has at least one zero, say \( f(-k) = 0 \), then \( g(x) \overset{\text{def}}{=} f(x - k) \in *S_m \) and \( g(0) = 0 \) by the following lemma:

**Lemma 14.1** Suppose \( f(x) \) satisfies
\[
\int f^2(x)dx \int f(x)x^n dx = \int f^2(x)x^n dx
\]
Then \( g(x) = f(x - k) \) also satisfies
\[
\int g^2(x) dx \int g(x) x^n dx = \int g^2(x) x^n dx
\]

**Proof** \( \text{Substituting } y = x - k, \text{ we get} \)
\[
\int g^2(x) dx \int g(x) x^n dx = \int f^2(y) dy \int f(y)(y + k)^n dy
\]
\[
= \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int f^2(y) dy \int f(y)y^j dy
\]
\[
= \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int f^2(y)y^j dy
\]
\[
= \int f^2(y) \sum_{j=0}^{n} \binom{n}{j} y^j k^{n-j} dy
\]
\[
= \int f^2(y)(y + k)^n dy
\]
\[
= \int f^2(x - k) x^n dx
\]
\[
= \int g^2(x) x^n dx \quad \▲
\]

Thus, if at least one function in \( *S_m \) has at least one zero, then \( \overline{S}_m \) will be non-empty. In addition, the sets \( \overline{S}_m \) are internal and \( \overline{S}_0 \supset \overline{S}_1 \supset \ldots \). Therefore, by the saturation principle, there exists a function \( \Theta(x) \in \bigcap_{n=0}^{\infty} \overline{S}_n \).

Noting that \( \int |\Theta(x)x^n| dx \) and \( \int |\Theta^2(x)x^n| dx \) are at most \( C_{\rho}(^\ast \mathbb{C}) \), we have that for this \( \Theta \),
\[
|R_m(\tau)| < \frac{\rho^{n+1}}{(m+1)!} \sup_{x \in \mathbb{R}} |\tau^{(m+2)}(x)| \int \left| \left( v - u_0 \right) A \Theta(y) - \frac{A^2}{2\rho} \Theta^2(y) \right| y^n dy < \rho^{n+k}
\]
where \( k \) is some real constant that depends on \( v, u_0, \) and \( A \). Therefore \( u(x, t) = u_0 + \frac{A}{\rho} \Theta \left( \frac{\rho - V}{\rho} \right) \) satisfies the Hopf equation weakly, in the sense of \( \sim \).

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If, in addition, it is true that $\Theta(0) = 0$, then $u(x,t)$ will also satisfy the conservation law:

for all $a, b \in \mathbb{R}$, $t > 0$,

$$\frac{d}{dt} \int_a^b u(x,t) \, dx = \frac{1}{2} [u^2(a,t) - u^2(b,t)]$$

Let us prove this by first calculating the left side:

$$\frac{d}{dt} \int_a^b \left[ u_0 + \frac{A}{\rho} \Theta \left( \frac{x - vt}{\rho} \right) \right] \, dx = \frac{d}{dt} \left[ u_0(b - a) + \frac{A}{\rho} \int_a^b \Theta \left( \frac{x - vt}{\rho} \right) \, dx \right]$$

$$= \frac{d}{dt} \left[ A \int_{\frac{a}{\rho}}^{\frac{b}{\rho}} \Theta(y) \, dy \right]$$

$$= A \left[ \Theta \left( \frac{b - vt}{\rho} \right) \left( \frac{-v}{\rho} \right) - \Theta \left( \frac{a - vt}{\rho} \right) \left( \frac{-v}{\rho} \right) \right]$$

$$= A \frac{v}{\rho} \left[ \Theta \left( \frac{a - vt}{\rho} \right) - \Theta \left( \frac{b - vt}{\rho} \right) \right]$$

Since $\Theta \in *\mathcal{S}$, if $a \neq vt$ and $b \neq vt$ then this quantity vanishes. However, if $a = vt$ or $b = vt$ (but not both) then we have that the left side equals $\pm \frac{A}{\rho} \Theta(0)$, respectively. If $\Theta(0) = 0$, this also vanishes.

Calculating the right side, we have:

$$\frac{1}{2} \left[ \left( u_0 + \frac{A}{\rho} \Theta \left( \frac{a - vt}{\rho} \right) \right)^2 - \left( u_0 + \frac{A}{\rho} \Theta \left( \frac{b - vt}{\rho} \right) \right)^2 \right]$$

$$= \frac{1}{2} u_0^2 + \frac{u_0 A}{\rho} \Theta \left( \frac{a - vt}{\rho} \right) + \frac{A^2}{2\rho^2} \Theta^2 \left( \frac{a - vt}{\rho} \right)$$

$$- \frac{1}{2} u_0^2 - \frac{u_0 A}{\rho} \Theta \left( \frac{b - vt}{\rho} \right) - \frac{A^2}{2\rho^2} \Theta^2 \left( \frac{b - vt}{\rho} \right)$$

$$= \frac{u_0 A}{\rho} \left[ \Theta \left( \frac{a - vt}{\rho} \right) - \Theta \left( \frac{b - vt}{\rho} \right) \right] + \frac{A^2}{2\rho^2} \left[ \Theta^2 \left( \frac{a - vt}{\rho} \right) - \Theta^2 \left( \frac{b - vt}{\rho} \right) \right]$$

Again, we have that if $a \neq vt$ and $b \neq vt$ then this quantity vanishes. If $a = vt$ or $b = vt$ (but not both) then the right side equals $\pm \left( \frac{u_0 A}{\rho} \Theta(0) + \frac{A^2}{2\rho^2} \Theta^2(0) \right)$, respectively. Here also, if $\Theta(0) = 0$ the right side vanishes, and so the conservation law holds.
In conclusion, we may make some conjectures based on the relation

$$A = \frac{2\rho(v - u_0)}{\int \Theta^2(y) dy}$$

There are many possibilities here, but if we assume (for simplicity) that $\int \Theta^2(y) dy$ is finite (and not infinitesimal) and $u_0 = 0$, then there are at least the following two particular cases:

(i) $u$ has infinitesimal amplitude with finite or infinitely large velocity, resembling a small signal, or

(ii) $u$ has non-infinitesimal, finitely large amplitude, and infinitely large velocity, resembling an explosion.

**Remark 14.1 (Connection with Perturbation Theory)** The closest to our result is the work by M. Radyna [11] in the framework of perturbation theory. M. Radyna proves the following result: For every $n \in \mathbb{N}$ there exists a function $\Theta_n \in \mathcal{S}(\mathbb{R})$ such that the function $u(x, t) = \frac{\Delta}{\epsilon} \Theta_n \left( \frac{x - vt}{\epsilon} \right)$ satisfies:

$$\left| \int_{\mathbb{R}} [u_t(x, t) + u(x, t)u_x(x, t)] \tau(x) dx \right| < \epsilon^n,$$

for every test function $\tau \in \mathcal{D}(\mathbb{R})$, every $t \in \mathbb{R}$ and all sufficiently small $\epsilon \in \mathbb{R}$.

For comparison we mention the following:

(a) Instead of a small real parameter $\epsilon$ we use a proper positive infinitesimal $\rho$. Our framework is, of course, quite different from M. Radyna’s theory.

(b) In contrast to M. Radyna’s result, we have proved the existence of a function $\Theta \in \ast \mathcal{S}(\mathbb{R})$ (not depending on $n$) such that the function $u(x, t) = \frac{\Delta}{\rho} \Theta \left( \frac{x - vt}{\rho} \right)$ satisfies:

$$\left| \int_{\mathbb{R}} [u_t(x, t) + u(x, t)u_x(x, t)] \tau(x) dx \right| < \rho^n,$$

for every test function $\tau \in \mathcal{D}(\mathbb{R})$, every $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$.  

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