On the étale site of marked schemes

Alexander Schmidt

When considering the étale site of a scheme it is often of interest to consider a variant which forces a given set of points to split in at least one member of a covering. Examples are the étale site of a marked curve used in [Scm], where a finite set of closed points is considered and the Nisnevich site [Nis], where all points are required to split. In this note we develop this approach in greater generality. Furthermore, we close a small gap in the literature by showing that any Nisnevich covering of a quasi-compact scheme has a finite subcovering.

1 Definition of the marked site

Let \( X \) be a scheme and let \( T \) be a set of points of \( X \). We will loosely write \( T \subset X \) and call the pair \( (X, T) \) a marked scheme. A morphism \( f : (Y, S) \to (X, T) \) of marked schemes is a scheme morphism \( f : Y \to X \) with \( f(S) \subset T \).

**Definition 1.1.** Let \( (X, T) \) be a marked scheme. The marked étale site \( (X, T)_{\text{et}} \) consists of the following data: The category \( \text{Cat}(X, T)_{\text{et}} \) is the category of morphisms \( f : (U, S) \to (X, T) \) such that

a) \( (f : U \to X) \) is étale, and

b) \( S = p^{-1}(T) \).

A family \( (p_i : (U_i, S_i) \to (U, S))_{i \in I} \) of morphisms in \( \text{Cat}(X, T)_{\text{et}} \) is a covering if it is surjective and any point \( s \in S \) splits, i.e., there exists an index \( i \) and a point \( u_i \in S_i \) mapping to \( s \) such that the induced field homomorphism \( k(s) \to k(u_i) \) is an isomorphism.

**Example 1.2.** For \( T = \emptyset \), we obtain the small étale site of \( X \), for \( T = X \) the Nisnevich site [Nis].

A morphism of marked schemes induces a morphism of the associated marked étale sites in the obvious way.

We consider the following “geometric points” of \( (X, T)_{\text{et}} \): we fix a separable closure \( k(x)^s \) of \( k(x) \) for every scheme-theoretic point \( x \in X \) and consider the following morphisms of marked schemes

1.) for \( x \notin T \), the natural morphism \( (\text{Spec} k(x)^s, \emptyset) \to (X, T) \).

2.) for \( x \in T \), the natural morphisms \( (\text{Spec} \kappa, \text{Spec} \kappa) \to (X, T) \) for every subextension \( \kappa/k(x) \) of \( k(x)^s/k(x) \).

If \( f : P \to X \) is any of the morphisms described in 1.) and 2.), the assignment \( F \mapsto \Gamma(P, f^* F) \) is a topos-theoretic point of \( (X, T)_{\text{et}} \) and one easily verifies that this family of points is conservative. In particular, exactness of sequences of abelian sheaves can be checked stalkwise.

We denote the cohomology of a sheaf \( F \in \text{Sh}_{\text{et}}(X, T) \) of abelian groups on \( (X, T)_{\text{et}} \) by \( H^*_\text{et}(X, T, F) \).
2 Excision

Let \((X,T)\) be a marked scheme, \(i : Z \to X\) a closed immersion and \(U = X \setminus Z\) the open complement. The right derivatives of the left exact functor “sections with support in \(Z\)"

\[ F \mapsto \ker(F(X,T) \to F(U,T \cap U)) \]

are called the cohomology groups with support in \(Z\). Notation: \(H^p_Z(X,T,F)\).

The proof of the next proposition is standard (cf. [Art, III, (2.11)] for the étale case without marking).

**Proposition 2.1.** There is a long exact sequence

\[ 0 \to H^0_Z(X,T,F) \to H^0_{\text{et}}(X,T,F) \to H^0_{\text{et}}(U,T \cap U,F) \to H^1_Z(X,T,F) \to H^1_{\text{et}}(X,T,F) \to H^1_{\text{et}}(U,T \cap U,F) \to \ldots \]

**Proposition 2.2** (Excision). Let \(\pi : (X',T') \to (X,T)\) be a morphism of marked schemes, \(Z \to X\), \(Z' \to X'\) closed immersions and \(U = X \setminus Z\), \(U' = X' \setminus Z'\) the open complements. Assume that

- \(\pi : X' \to X\) is étale,
- \(T' = \pi^{-1}(T)\),
- \(\pi\) induces an isomorphism \(Z'_{\text{red}} \iso Z_{\text{red}}\),
- \(\pi(U') \subset U\).

Then the induced homomorphism

\[ H^p_Z(X,T,F) \iso H^p_{\text{et}}(X',T',\pi^*F) \]

is an isomorphism for every sheaf \(F \in \text{Sh}_{\text{et}}(X,T)\) and all \(p \geq 0\).

**Proof.** The standard proof for étale topology applies: By the general theory, \(\pi^*\) is exact. Since \(\pi\) belongs to \(\text{Cat}(X,T)_{\text{et}}\), \(\pi^*\) has the exact left adjoint “extension by zero”, hence \(\pi^*\) sends injectives to injectives. Therefore it suffices to deal with the case \(p = 0\). Without changing the statement, we can replace all occurring schemes by their reductions. By assumption,

\[ (X',T') \coprod (U,T \cap U) \to (X,T) \]

is a covering. For \(\alpha \in H^0_Z(X,T,F)\) mapping to zero in \(H^0_{\text{et}}(X',T',\pi^*F)\) we therefore obtain \(\alpha = 0\).

Now let \(\alpha' \in H^0_Z(X',T',\pi^*F)\) be given. We show that \(\alpha'\) and \(0 \in H^0(U,T \cap U,F)\) glue to an element in \(H^0_{\text{et}}(X,T,F)\). The only nontrivial compatibility on intersections is \(p_1^*(\alpha') = p_2^*(\alpha')\) for \(p_1, p_2 : (X' \times_X X',T' \times_T T') \to (X',T')\). This can be checked on stalks noting that \(Z' \iso Z\) implies that the two projections \(Z' \times_Z Z' \to Z'\) are the same.

3 Continuity

**Proposition 3.1.** Let \(X\) be a quasi-compact scheme and let \(T \subset X\) be a closed subscheme. Then every étale covering of \((X,T)\) admits a finite subcovering.

**Proof.** Since \(X\) has a finite affine Zariski-open covering, we may assume that \(X\) is affine, in particular \(X\) is quasi-separated. Then also \(T\) is quasi-compact and quasi-separated. Let

\[ \coprod_{i \in I} (U_i, S_i) \to (X,T) \]
be an étale covering. Let, for \( i \in I \), \( K_i \subset T \) be the set of points in \( T \) which split in \( U_i \to X \). By [Src] Lemma 13.3, \( K_i \) is ind-constructible, i.e., open in the constructible topology of \( T \), which is compact by [EGA4] 1.9.15 (iii). Since \( T = \bigcup_i K_i \) by assumption, we find a finite subset \( J \subset I \) with \( T = \bigcup_{i \in J} K_i \). Furthermore, since \( X \) is quasi-compact and étale morphisms are open, we find a finite subset \( J' \subset I \) such that \( \bigcup_{i \in J'} U_i \to X \) is an étale covering. We conclude that \( \bigcup_{i \in J, j \in J'} (U_i, S_i) \to (X, T) \) is a finite subcovering of \( \bigcup_{i\in J} (U_i, S_i) \to (X, T) \).

As in [SGA4] VII, 3.2 for the unmarked étale site, we define the *restricted marked étale site*

\[(X, T)_{\text{et}}^{\text{res}}\]

as the restriction of \((X, T)_{\text{et}}\) to the subcategory of all \((U, S) \in (X, T)_{\text{et}}\) where \( U \to X \) is of finite presentation. Assume that \( X \) is quasi-compact and quasi-separated. Then the same is true for any such \( U \) and Proposition 3.1 shows that the restricted site is noetherian. Moreover, the categories of sheaves on \((X, T)_{\text{et}}\) and \((X, T)_{\text{et}}^{\text{res}}\) are naturally equivalent. Hence the same argument as in the unmarked étale case [SGA4] VII, Prop. 3.3 shows

**Theorem 3.2.** Let \( X \) be a quasi-compact and quasi-separated scheme and let \( T \subset X \) be a closed subscheme. Let \((F_i)\) be a filtered direct system of abelian sheaves on \((X, T)_{\text{et}}\). Then

\[\colim_i H^p_{\text{et}}(X, T, F_i) \cong H^p_{\text{et}}(X, T, \colim_i F_i)\]

for all \( p \geq 0 \).

Next we consider inverse limits of marked schemes.

**Theorem 3.3.** Let \((X, T)\) be a marked scheme with \( T \) closed in \( X \) and let \( X_i \to X \), \( i \in I \), be an inverse system of \( X \)-schemes. Assume that all \( X_i \) are quasi-separated and quasi-compact and that all transition morphisms are affine. Let \( T_i \) be the preimage of \( T \) in \( X_i \) and put \( X_\infty = \lim X_i \), \( T_\infty = \lim T_i \).

Then the restricted site \((X_\infty, T_\infty)_{\text{et}}^{\text{res}}\) is the limit site of the sites \((X_i, T_i)_{\text{et}}^{\text{res}}\).

**Corollary 3.4.** With the notation and assumptions of Theorem 3.3, let \( F \) be a sheaf of abelian groups on \((X, T)_{\text{et}}\). We denote its inverse image on \((X_i, T_i)_{\text{et}}\) and \((X_\infty, T_\infty)_{\text{et}}\) by \( F_i \) and \( F_\infty \). Then the natural map

\[\colim_i H^p_{\text{et}}(X_i, T_i, F_i) \longrightarrow H^p_{\text{et}}(X_\infty, T_\infty, F_\infty)\]

is an isomorphism for all \( p \geq 0 \).

**Proof of Theorem 3.3.** By [Art] III, Theorem 3.8, the site \((X_\infty)_{\text{et}}^{\text{res}}\) is naturally equivalent to the limit site of the \((X_i)_{\text{et}}^{\text{res}}\). In view of Proposition 3.1 it therefore suffices to show that for every quasi-compact étale surjection \( U_i \to X_i \), with the property that every point of \( T_\infty \) splits in \( U_\infty = U_i \times X_i \), \( X_\infty \to X_\infty \) there exist \( j \geq i \) such that every point of \( T_j \) splits in \( U_j = U_i \times X_j \). We follow the proof of [Src] Lemma 13.2 for Nisnevich coverings. By [Src] Lemma 13.3, the subset \( S_j \subset T_j \) of points that split in \( U_j \to X_j \) is ind-constructible for all \( j \geq i \). Denoting the projection by \( u_j : T_\infty \to T_j \), the assumption on \( U_\infty \to X_\infty \) implies \( T_\infty = \bigcup u_j^{-1}(S_j) \). Considering the \( T_j \subset X_j \) as reduced, closed subschemes, we may apply [EGA4] Cor. 8.3.4 to obtain \( S_j = T_j \) for some \( j \).

**Remark 3.5.** Let \( A \) be a ring and let \((A \to B_i)_{i=1}^\infty\) be an affine Nisnevich covering. We write \( A \) as the union of its finitely generated subrings. Then, by Proposition 3.1 and Theorem 3.3, there exists a finite subset \( J \subset I \), a subring \( A' \subset A \) which is
finitely generated over \( \mathbb{Z} \) and a finite Nisnevich covering \((A' \to B'_j)_{j \in J}\) such that \(B_j \cong A \otimes_{A'} B'_j\) for all \(j \in J\).

Hence the refined definition of Nisnevich coverings for general rings introduced by Lurie in [DAG, XI, Definition 1.1 and Remark 1.15] coincides with the naive definition.

**Corollary 3.6.** Let \((X, T)\) be a marked scheme with \(T\) closed in \(X\) and \(Z = \{z_1, \ldots, z_n\}\) a finite set of closed points of \(X\). Put \(X^h_i = \text{Spec}(O_{X,z_i})\). Then, for every sheaf \(F\) of abelian groups on \((X, T)\) and all \(p \geq 0\)

\[
H^p_Z(X, T, F) \cong \bigoplus_{i=1}^n H^p_{\{z_i\}}(X^h_i, T \cap X^h_i, F).
\]

**Proof.** Since \(H^p_Z(X, T, F) \cong \bigoplus_{i=1}^n H^p_{\{z_i\}}(X, T, F)\), we may assume that \(Z = \{z\}\) consists of a single closed point. Excision shows that \(H^p_{\{z\}}(X, T, F) = H^p_{\{z\}}(U, T \cap U, F)\) for every affine étale open neighbourhood \(U\) of \(z\). Since \(X^h_i\) is the limit over all these \(U\), the long exact sequences of Proposition 2.1 together with Corollary 3.4 show the result.

Using Corollary 3.4, it is easy to calculate the stalks of the higher direct images of the site morphism \((X, T)_{\text{et}} \to (X, X)_{\text{et}} = X_{\text{Nis}}\). The Leray spectral sequence together with the fact that the Nisnevich cohomological dimension of noetherian schemes is bounded by the Krull dimension [Nis, Theorem 1.32] yields:

**Corollary 3.7.** Let \(X\) be a noetherian scheme of finite Krull dimension \(d\), \(T \subset X\) closed and assume that there exists a nonnegative integer \(N\) such that

\[
cd(k(x)) \leq N
\]

for all points \(x \in X \setminus T\). Then for every abelian torsion sheaf \(F\) on \((X, T)_{\text{et}}\) we have

\[
H^p_{\text{et}}(X, T, F) = 0 \quad \text{for } p > N + d.
\]

## 4 Galois covers

**Definition 4.1.** A Galois cover of \(X\) with finite Galois group \(G\) in the site \((X, T)_{\text{et}}\) is a morphism \((Y, S) \to (X, T)\) in \((X, T)_{\text{et}}\) together with a right action of \(G\) on \(Y\) over \(X\) such that the following holds:

1. \((Y, S) \to (X, T)\) is a covering for the site \((X, T)_{\text{et}}\).

2. \(Y \to X\) is an étale Galois cover, i.e.,

\[
Y \times G \to Y \times_X Y, \quad (y, g) \mapsto (y, yg)
\]

is an isomorphism.

Since \(G\) acts transitively on the set of points in \(Y\) over a given point \(x \in X\), we see that every \(t \in T\) splits completely in \(Y/X\).

**Proposition 4.2** (Hochschild-Serre spectral sequence). Let \((Y, S) \to (X, T)\) be a Galois cover with finite group \(G\) and \(F \in Sh_{\text{et}}(X, T)\). Then there is a natural spectral sequence

\[
E_2^{pq} = H^p(G, H^q_{\text{et}}(Y, S, F)) \Rightarrow H^p_{\text{et}}(X, T, F).
\]
Proof. The proof is word-by-word the same as for the étale cohomology, see [AM, Theorem 2.20].

Remark 4.3. Assume that $X$ is quasi-compact and quasi-separated and $T \subset X$ closed. Let

$$(Y_i, S_i) \to (X, T)$$

be a directed inverse system of Galois covers with finite Galois groups $G_i$ and

$$(Y, S) = \lim(Y_i, S_i).$$

Then $(Y, S) \to (X, T)$ is a pro-Galois cover with profinite Galois group $G = \lim G_i$. By Theorem 3.3, for $F \in Sh_{et}(X, T)$, the groups $H^q_{et}(Y, S, F) = \text{colim} H^q_{et}(Y_i, S_i, F)$ are discrete $G$-modules and we obtain the profinite Hochschild-Serre sequence

$$E^p_2 = H^p(G, H^q_{et}(Y, S, F)) \Rightarrow H^{p+q}_{et}(X, T, F),$$

where $H^*(G, -)$ is the continuous cohomology of the profinite group $G$ with values in a discrete $G$-module (see [NSW, I, §2]).

5 Fundamental group

We recall some facts from Artin-Mazur [AM]. Let $\mathscr{C}$ be a pointed site and $HR(\mathscr{C})$ the category of pointed hypercovers of $\mathscr{C}$ [AM, §8]. If $\mathscr{C}$ is locally connected, then the “connected component functor” $\pi$ defines an object

$$\Pi \mathscr{C} = \{\pi(K)\}_{K \in HR(\mathscr{C})}$$

in the pro-category of the homotopy category of pointed simplicial sets. By definition, the fundamental group of $\mathscr{C}$ is the pro-group $\pi_1(\Pi \mathscr{C})$.

Let $X$ be a locally noetherian scheme. Then (cf. [AM, §9]) the site $X_{et}$, and hence also $(X, T)_{et}$ is locally connected. Pointing $(X, T)_{et}$ by choosing any “geometric” point $\bar{x}$ described at the end of Section 4 we obtain the étale fundamental group $\pi^et(X, T, \bar{x})$. It is independent of the choice of $\bar{x}$ up to isomorphism, which is canonical up to inner automorphisms. By [AM, Cor. 10.7], for any group $G$, the set $\text{Hom}(\pi^et(X, T, \bar{x}), G)$ is in bijection with the set of isomorphism classes of pointed (over $\bar{x}$) $G$-torsors in $(X, T)_{et}$. In particular, $\pi^et_1(X, \varnothing, \bar{x})$ is the enlarged étale fundamental group of [SGA3, X, §6] and its profinite completion is the usual étale fundamental group of $X$ defined in [SGA1]. If $\bar{x}$ is a geometric point of $X$, then $\pi^et_1(X, T, \bar{x})$ is a factor group of $\pi^et_1(X, \varnothing, \bar{x})$, which is profinite for normal $X$ by [AM, Thm. 11.1]. Hence we obtain the following result.

Proposition 5.1. Let $X$ be a noetherian, normal, connected scheme and $T \subset X$. Then (for any choice of base point) $\pi^et_1(X, T)$ is a profinite group. Its finite quotients are in bijection with the isomorphism classes of finite connected pointed étale Galois covers of $X$ in which every point $t \in T$ splits completely.

Example 5.2. For general $(X, T)$, the fundamental group need not be profinite. For example, let $k$ be a field and $N = \mathbb{P}^1_k/(0 - 1)$ the node over $k$. Then

$$\pi^et_1(N, T) \cong \begin{cases} \mathbb{Z} \times \text{Gal}_k, & T = \varnothing \\ \mathbb{Z}, & T = X. \end{cases}$$

We will use the notation $\hat{\pi}^et_1(X, T)$ for the profinite completion of $\pi^et_1(X, T)$, hence we have a completion map $\pi^et_1(X, T) \to \hat{\pi}^et_1(X, T)$ which is an isomorphism by Proposition 5.9 if $X$ is a noetherian, normal and connected scheme.

We end this section with the following observation concerning products.
Proposition 5.3. Let \( k \) be a field, \( X \) and \( Y \) geometrically connected schemes of finite type over \( k \) and \( S \subset X(k), T \subset Y(k) \) nonempty sets of \( k \)-rational points. Let \( a \) and \( b \) be geometric points of \( X \setminus S \) and \( Y \setminus T \) with values in a common separably closed extension field of \( k \). Assume that at least one of the schemes \( X \) and \( Y \) is proper over \( k \). Then the natural map
\[
\hat{\pi}_1^{et}(X \times_k Y, S \times T, (a, b)) \to \hat{\pi}_1^{et}(X, S, a) \times \hat{\pi}_1^{et}(Y, T, b)
\]
is an isomorphism of profinite groups.

Proof. We omit the base points from notation. For a connected scheme \( X \), let \( \hat{X} \) denote the profinite universal cover. For a subset \( S \subset X \), the kernel of \( \hat{\pi}_1^{et}(X) \to \hat{\pi}_1^{et}(X, S) \) is the (closed) normal subgroup of \( \hat{\pi}_1^{et}(X) = \text{Gal}(\hat{X}|X) \) generated by the decomposition groups of the points in \( S \), i.e., it is the (closed) subgroup of \( \text{Gal}(\hat{X}|X) \) generated by all automorphisms which fix a point \( \tilde{s} \in \hat{X} \) lying over some \( s \in S \). We denote this group by \( K(X, S) \).

Now assume we are in the situation of the proposition. By the topological invariance of the étale topology we may assume that \( k \) is perfect. Let \( \bar{k} \) be an algebraic closure of \( k \). We denote the base changes to \( \bar{k} \) by \( (X, S) \) and \( (Y, T) \). By [SGA1, X, 1.7], we have a natural isomorphism
\[
\hat{\pi}_1^{et}(\hat{X} \times_k \hat{Y}) \cong \hat{\pi}_1^{et}(\hat{X}) \times \hat{\pi}_1^{et}(\hat{Y}).
\]
Moreover, by [SGA1 IX, 6.1], we have a natural exact sequence
\[
1 \to \hat{\pi}_1^{et}(\hat{X}) \to \hat{\pi}_1^{et}(\hat{X}) \to \text{Gal}(\bar{k}|k) \to 1.
\]
This and the similar sequence for \( Y \) shows the isomorphism
\[
\hat{\pi}_1^{et}(X \times_k Y) \cong \hat{\pi}_1^{et}(X) \times_{\text{Gal}(\bar{k}|k)} \hat{\pi}_1^{et}(Y), \tag{\ast}
\]
where the term on the right hand side is a fibre product in the category of profinite groups. We consider the corresponding diagram of étale Galois covers.

\[
\begin{array}{ccc}
\hat{X} \times_k \hat{Y} & \xrightarrow{\pi_1^{et}(Y)} & \hat{X} \\
\downarrow & & \downarrow \\
X \times_k Y & \xrightarrow{\pi_1^{et}(X)} & \hat{X} \times_k \hat{Y}
\end{array}
\]

Let \( (s, t) \in S \times T \subset X \times_k Y \) and let \( (\tilde{s}, \tilde{t}) \in \hat{X} \times_k \hat{Y} = \hat{X} \times_k \hat{Y} \) be a point lying above \( (s, t) \). An element \( \sigma \in \hat{\pi}_1^{et}(X \times_k Y) = \text{Gal}(\hat{X} \times_k \hat{Y}|X \times_k Y) \) fixes \( (\tilde{s}, \tilde{t}) \) if and only if its image in \( \hat{\pi}_1^{et}(X) = \text{Gal}(\hat{X} \times_k \hat{Y}|X \times_k Y) \) fixes \( \tilde{s} \in \hat{X} \) and its image in \( \hat{\pi}_1^{et}(Y) = \text{Gal}(\hat{X} \times_k \hat{Y}|X \times_k Y) \) fixes \( \tilde{t} \in \hat{Y} \). Hence, the isomorphism \( \ast \) induces an isomorphism of subgroups
\[
K(X \times_k Y, S \times T) \cong K(X, S) \times_{\text{Gal}(\bar{k}|k)} K(Y, T). \tag{\ast\ast}
\]
The isomorphisms (⋆) and (⋆⋆) together induce an isomorphism
\[ \hat{\pi}_1^{\text{et}}(X \times_k Y, S \times T) \xrightarrow{\sim} C \]
with
\[ C = \coker(K(X, S) \times_{\text{Gal}(\overline{k}/k)} K(Y, T) \rightarrow \hat{\pi}_1^{\text{et}}(X) \times_{\text{Gal}(\overline{k}/k)} \hat{\pi}_1^{\text{et}}(Y)). \]
The natural homomorphism \( C \rightarrow \hat{\pi}_1^{\text{et}}(X, S) \times \hat{\pi}_1^{\text{et}}(Y, T) \) is injective. To conclude the proof of the proposition, it remains to show surjectivity, i.e., we have to show that every element in \( \hat{\pi}_1^{\text{et}}(X, S) \times \hat{\pi}_1^{\text{et}}(Y, T) \) has a preimage in \( \hat{\pi}_1^{\text{et}}(X) \times_{\text{Gal}(\overline{k}/k)} \hat{\pi}_1^{\text{et}}(Y) \subset \hat{\pi}_1^{\text{et}}(X) \times \hat{\pi}_1^{\text{et}}(Y) \). For this it suffices to show that the composite map \( K(X, S) \rightarrow \pi_1^{\text{et}}(X) \rightarrow \text{Gal}(\overline{k}/k) \) is surjective. This is true since \( K(X, S) \) contains the decomposition group of a \( k \)-rational point. \( \square \)

6 A modification

We consider a modification of the marked étale site which was used in \( \text{Sch} \) for one-dimensional, noetherian regular schemes.

**Definition 6.1.** The strict marked étale site \( (X, T)_{\text{et-s}} \) consists of the following data: \( \text{Cat}(X, T)_{\text{et-s}} \) is the category of morphisms \( f : (U, S) \rightarrow (X, T) \) such that
a) \( f : U \rightarrow X \) is étale,
b) \( S = p^{-1}(T) \), and
c) for every \( u \in S \) mapping to \( t \in T \) the induced field homomorphism \( k(s) \rightarrow k(u) \)
is an isomorphism.

Coverings are surjective families.

**Proposition 6.2.** (i) If \( T \subset X \) consists of a finite set of closed points, then the natural morphism of sites \( \varphi : (X, T)_{\text{et}} \rightarrow (X, T)_{\text{et-s}} \) induces isomorphisms
\[ H^p_{\text{et-s}}(X, T, F) \xrightarrow{\sim} H^p_{\text{et}}(X, T, \varphi^*F), \ H^p_{\text{et-s}}(X, T, \varphi_*G) \xrightarrow{\sim} H^p_{\text{et}}(X, T, G) \]
for any \( F \in \text{Sh}_{\text{et-s}}(X, T) \), \( G \in \text{Sh}_{\text{et-s}}(X, T) \) and \( p \geq 0 \).

(ii) For locally noetherian \( X \) (and any chosen base point), the natural map
\[ \pi_1^{\text{et}}(X, T) \rightarrow \pi_1^{\text{et-s}}(X, T) \]
is an isomorphism.

**Proof.** Let \( (U, S) \in \text{Cat}(X, T)_{\text{et-s}} \) and assume that \( (f_i : (U_i, S_i) \rightarrow (U, S)) \) is a covering in \( (X, T)_{\text{et}} \). Removing for all \( i \) the finitely many points \( s \in S_i \) such that \( k(f_i(s)) \) is not an isomorphism from \( U_i \), we obtain a strict covering \( (f_i : (U_i', S_i') \rightarrow (U, S)) \) which is a refinement of the original one. Hence \( \varphi_*\varphi^*F = F \) and \( R^q\varphi_*G = 0 \) for \( q \geq 1 \). In view of the Leray spectral sequence, this shows (i).

Assertion (ii) follows since both pro-groups represent the same functor: for any group \( G \), a \( G \)-torsor in \( (X, T)_{\text{et}} \) is the same as a \( G \)-torsor in \( (X, T)_{\text{et-s}} \). \( \square \)

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Alexander Schmidt, Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Deutschland
e-mail: schmidt@mathi.uni-heidelberg.de