Towards Uncovering Generic Effects Of Matter Sources In Anisotropic Quantum Cosmologies Via Bianchi II and VII_{h=0} Models

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Abstract

We solve the Bianchi II Wheeler DeWitt equation when a cosmological constant, an electromagnetic field and stiff matter are present using the Euclidean-signature semi classical method. Additionally we study the non-commutative quantum Bianchi II models with an electromagnetic field and apply our modified semi-classical method to the Bianchi VII_{h=0} models with and without the aforementioned matter sources. By applying this method we are able to study the ‘excited’ states of the Bianchi II models when a cosmological constant is present and the vacuum Bianchi VII_{h=0} models. In doing so we shed light on how matter sources could have influenced the evolution of the very early universe and expand upon the limited number of known non-trivial closed form solutions to the Wheeler DeWitt equation.

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This work is in memory of my parents, Susan Orchan Berkowitz, and Jonathan Mark Berkowitz
I. INTRODUCTION

Since the early days of studying the symmetry reduced Wheeler DeWitt equation, obtaining closed solutions for it of any kind has proven to be difficult. The "symmetry reduced" Wheeler Dewitt equation is called such because it is derived by first reducing the infinite number of degrees of freedom present in general relativity to a finite number by utilizing the symmetries allowed in homogeneous space-times, such as Bianchi A models. When a Bianchi A metric of the following form
\[
ds^2 = -N(t)^2 dt^2 + L^2 e^{2\alpha(t)} \left( e^{2\beta(t)} \right)_{ab} \omega^a \omega^b
\]
is inserted into the Einstein-Hilbert action expressed in terms of the ADM formalism of general relativity
\[
S = \frac{c^3}{16\pi G} \int dt d^3x \sqrt{h} \left( K_{ab} K^{ab} - K^2 + R - 2\Lambda \right) + S_{\text{matter}}; \tag{2}
\]
the resulting ADM action has a finite number of degrees of freedom. In the above metric \( L \) has units of length and sets a scale for the spatial size of our cosmology. From here one can construct a finite dimensional Hamiltonian which can be quantized as follows
\[
\begin{align*}
-e^{-3\alpha}p_{\alpha}^2 & \longrightarrow \frac{\hbar^2}{e^{(3-B)\alpha}} \frac{\partial}{\partial \alpha} \left( e^{-B\alpha} \frac{\partial}{\partial \alpha} \right) \\
e^{-3\alpha}p_{\beta_+}^2 & \longrightarrow -\frac{\hbar^2}{e^{3\alpha}} \frac{\partial^2}{\partial \beta_+^2} \\
e^{-3\alpha}p_{\beta_-}^2 & \longrightarrow -\frac{\hbar^2}{e^{3\alpha}} \frac{\partial^2}{\partial \beta_-^2}
\end{align*}
\tag{3}
\]
where \( B \) can be any real number and is the Hartle Hawking ordering parameter.

In the notation introduced by Misner\cite{1, 6} \( e^{\alpha(t)} \) is a measure of the local scale factor of the spatial surface and \( \left( e^{2\beta(t)} \right)_{ij} \) is diag \( \left( e^{2\beta(t)_+} + 2\sqrt{3}\beta(t)_-, e^{2\beta(t)_+} - 2\sqrt{3}\beta(t)_-, e^{-4\beta(t)_+} \right) \), where \( \beta_+ \) and \( \beta_- \) measures the amount of anisotropy present on the spatial hypersurface. The \( \omega^i \) factors are one forms defined on the spatial hypersurface of each Bianchi cosmology and obey \( d\omega^i = \frac{1}{2} C_{ijk} \omega^j \wedge \omega^k \) where the \( C_{ijk} \) are the structure constants of the invariance Lie group associated with each particular class of Bianchi models.

The Bianchi II and VII\(_h=0\) models which we will focus on in this paper have the following one forms respectively
\[
\begin{align*}
\omega^1 &= dy + xdz \\
\omega^2 &= dz \\
\omega^3 &= dx.
\end{align*}
\tag{4}
\]
\[ \omega^1 = \cos(z)dx + \sin(z)dy \]
\[ \omega^2 = -\sin(z)dx + \cos(z)dy \]
\[ \omega^3 = dz. \]  

Using the methodology presented in [7, 8] for writing out cosmological potentials for Bianchi A models with certain matter sources such as a cosmological constant and a primordial electromagnetic field, we will use the Euclidean-signature semi classical method[9, 10] to study the following Lorentzian signature Wheeler DeWitt (WDW) equations[11]

\[ \Box \psi - B \frac{\partial \psi}{\partial \alpha} + U_i \psi = 0 \]
\[ U_{II} = \frac{1}{12} e^{4\alpha + 4\beta_+ + 4\sqrt{3}\beta_-} + 24\Lambda e^{6\alpha} + 2b^2 e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} + \rho \]
\[ U_{VII} = \frac{4}{3} e^{4(\alpha + \beta_+)} \sinh^2 \left(2\sqrt{3}\beta_-\right) + 24\Lambda e^{6\alpha} + 2b^2 e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} + \rho; \]

where \( \Box \) is the three-dimensional d’Alembertian in the minisuperspace with signature, \((+--))\), \(\rho\) is the stiff matter term, and \(b^2\) represents how strong the electromagnetic field is.

In comparison to scalar fields and other matter sources, very little attention has been given to the study[12–15] of primordial electromagnetic fields within the context of quantum cosmology. Considering that new evidence[16, 17] for the existence of a femto Gauss strength intergalactic magnetic field has been uncovered by observing gamma rays there is now an additional incentive to study what effects electromagnetic fields have on classical/quantum cosmological evolution. Through studying the effects of electromagnetic fields on quantum universes we can better understand nucleogenesis and how seeds of anisotropy developed in our early universe. Even though on large scales our universe is incredibly isotropic and homogeneous it is possible that our early universe possessed a considerable amount of anisotropy. Thus it is useful to study anistropic classical/quantum cosmologies so we can better understand what our universe could have been like when it was extremely young. To accomplish this we will study the quantum Bianchi II and VII\(_{h=0}\) models with matter sources in order to determine what possible effects an electromagnetic field and cosmological constant could have induced in our early universe.

Equations (6) can be seen as the analogue of the Schrödinger equation for these quantum cosmologies. They however possess many fundamental differences from the Schrödinger equation which obscures the meaning behind \(\psi\). Two notable differences are the absence of any first order time derivative, and the requirement that physically meaningful \(\psi\)’s must be
annihilated by the quantized Hamiltonian constraint $\hat{H}$, which leads to the problem of time manifesting itself as
\[ i\hbar \frac{\partial \Psi}{\partial t} = N\hat{H}\Psi \]
\[ \frac{\partial \Psi}{\partial t} = 0. \]
(7)

A way around this for our purposes is to denote one of the Misner variables to be our clock. A good clock increases monotonically. Out of the variables we can choose from, $\alpha$ which is related to the spatial size of our Bianchi II universe is the best candidate for our clock and will be for practical purposes our ”time” [2] parameter.

The motivation for finding solutions to the WDW equation for a variety of cosmological models lies in obtaining a glimpse of how quantum gravity could have affected the evolution of our universe. The word ”glimpse” is used because employing the procedure of symmetry reduction, which we outlined, comes at a cost. Formally by applying symmetry reduction in the manner we just described we are freezing out inhomogeneous modes which would be present if one were to first quantize general relativity with its infinitely many degrees of freedom, and then find solutions to the full functional Wheeler Dewitt equation [2](this particular WDW equation is expressed using a specific operator ordering)
\[ N\left( \frac{16\pi G\hbar^2}{c^4\sqrt{h}} - G_{abcd} \frac{\delta}{\delta h(x)_{ab}} \frac{\delta}{\delta h(x)_{cd}} + \frac{(3)R\sqrt{hc^4}}{16\pi G} \right) \Psi = 0 \]
(8)

which in some limit corresponds to classical homogeneous space-times. Because inhomogeneous modes are not present in the solutions to the symmetry reduced WDW equations we are unable to take into account the full spectrum of quantum effects that dictate cosmic evolution. Despite these shortcomings, due to the immense difficulties in solving the full functional WDW equation, and the lack of a complete theory of quantum gravity, symmetry reduction remains one of our best tools for uncovering what effects a theory of quantum gravity can have on cosmological evolution. By including multiple matter sources in our WDW equation as we have done, we can even better understand what effects the manifestation of quantum gravity had on the very early universe, and its subsequent development.

Closed form solutions [11, 18–21] to the symmetry reduced WDW equations, including those of the Bianchi II and VII$_h=0$ models have been found using a plethora of methods, such as traditional semi-classical methods and via elementary separation of variables after
performing a suitable coordinate change. Recently though a Euclidean-signature semi classical method has been developed [10] which has some interesting applications to certain field theories in addition to quantum cosmology[9] and ordinary quantum mechanics[22]. It was used by Bae and Moncrief [9] to compute asymptotic solutions for the Bianchi IX WDW equation for any arbitrary Hartle-Hawking ordering parameter. Furthermore, it was also used by the author to compute for two particular values of the Hartle-Hawking ordering parameters new closed form solutions to the Bianchi IX and VIII [23] WDW equations and find new leading order solutions when a cosmological constant, and a primordial electromagnetic field are present. In this work we will use the Euclidean-signature semi classical method to find a plethora of new solutions associated with the quantum Bianchi II and VII$ _{h=0} $ models. As it will be seen this method possesses advantages over other traditional methods that have been used to solve the WDW equation in the past.

This paper will be organized as follows. In the next section we will explain what the Euclidean-signature semi classical method is and how it applies to quantum Bianchi A models. Then we will showcase our method by obtaining closed form solutions to the Bianchi II WDW equation that are similar to the ones first derived by [18]. Afterwards we will show how all of the equations that this method provides can be solved which will give us a family of asymptotic and closed form solutions. From there we will derive the electromagnetic potential term which appears in (6). Then we will obtain new solutions to the Bianchi II WDW equation when a cosmological constant, primordial electromagnetic field and stiff matter are present and discuss its ‘excited’ states. Next we will take a detour from applying our modified semi-classical method and turn our attention to the non-commutative quantum Bianchi II models with a primordial electromagnetic field, and stiff matter.

Moving on from Bianchi II we will turn our attention to the quantum Bianchi VII$ _{h=0} $ models. Using the Euclidean-signature semi classical method we will first study its vacuum ‘ground’ and ‘excited’ states. Afterwards we will study its ‘ground’ states when matter sources are present. Finally, we will discuss the fascinating qualitative properties that our wave functions of the universe possess and give some concluding remarks.
II. THE EUCLIDEAN-SIGNATURE SEMI CLASSICAL METHOD

Our outline of this method will follow closely [9]. The method described in this section and its resultant equations can in principle be used to find solutions (closed and asymptotic) to a wide class of quantum cosmological models such as all of the Bianchi A, Kantowski Sachs models, and the FLRW models.

The first step we will take in solving the Wheeler DeWitt equation is to introduce the ansatz
\[ \Psi_h^{(0)} = e^{-S_h/h} \]
where \( S_h \) is a function of \( (\alpha, \beta_+, \beta_-) \). We will rescale \( S_h \) in the following way
\[ S_h := \frac{G}{c^3 L^2} S_h \]
where \( S_h \) is dimensionless and admits the following power series in terms of this dimensionless parameter
\[ X := \frac{L_{\text{Planck}}^2}{L^2} = \frac{G \hbar}{c^3 L^2}. \]

The series is given by
\[ S_h = S_{(0)} + X S_{(1)} + \frac{X^2}{2!} S_{(2)} + \cdots + \frac{X^k}{k!} S_{(k)} + \cdots \]

and as a result our initial ansatz now takes the following form
\[ \Psi_h^{(0)} = e^{-\frac{X}{2} S_{(0)} - S_{(1)} - \frac{X}{2} S_{(2)} - \cdots} \]

Substituting this ansatz into the Wheeler-DeWitt equation and requiring satisfaction, order-by-order in powers of \( X \) leads immediately to the sequence of equations
\[ \left( \frac{\partial S_{(0)}}{\partial \alpha} \right)^2 - \left( \frac{\partial S_{(0)}}{\partial \beta_+} \right)^2 - \left( \frac{\partial S_{(0)}}{\partial \beta_-} \right)^2 + U = 0 \]
\[ 2 \left[ \frac{\partial S_{(0)}}{\partial \alpha} \frac{\partial S_{(1)}}{\partial \alpha} - \frac{\partial S_{(0)}}{\partial \beta_+} \frac{\partial S_{(1)}}{\partial \beta_+} - \frac{\partial S_{(0)}}{\partial \beta_-} \frac{\partial S_{(1)}}{\partial \beta_-} \right] + B \frac{\partial^2 S_{(0)}}{\partial \alpha^2} - \frac{\partial^2 S_{(0)}}{\partial \beta_+^2} + \frac{\partial^2 S_{(0)}}{\partial \beta_-^2} = 0, \]
\[
2 \left[ \frac{\partial S(0)}{\partial \alpha} \frac{\partial S(k)}{\partial \alpha} - \frac{\partial S(0)}{\partial \beta_+} \frac{\partial S(k)}{\partial \beta_+} - \frac{\partial S(0)}{\partial \beta_-} \frac{\partial S(k)}{\partial \beta_-} \right] + k \left[ B \frac{\partial S(k-1)}{\partial \alpha} - \frac{\partial^2 S(k-1)}{\partial \alpha^2} + \frac{\partial^2 S(k-1)}{\partial \beta_+^2} + \frac{\partial^2 S(k-1)}{\partial \beta_-^2} \right] + \sum_{\ell=1}^{k-1} \frac{k!}{\ell! (k-\ell)!} \left( \frac{\partial S(\ell)}{\partial \alpha} \frac{\partial S(k-\ell)}{\partial \alpha} - \frac{\partial S(\ell)}{\partial \beta_+} \frac{\partial S(k-\ell)}{\partial \beta_+} - \frac{\partial S(\ell)}{\partial \beta_-} \frac{\partial S(k-\ell)}{\partial \beta_-} \right) = 0
\]

We will refer to \( S(0) \) in our WDW wave functions as the leading order term, which can be used to construct a semi-classical approximate solution to the Lorentzian signature WDW equation, and call \( S(1) \) the first order term. The \( S(1) \) term can also be viewed as our first quantum correction, with the other \( S(k) \) terms being the additional higher order quantum corrections, assuming that they are smooth and globally defined. This is reflected in the fact that the higher order transport equations depend on the operator ordering used in defining the Wheeler Dewitt equation, which is an artifact of quantization. Additionally in some cases one can find a solution to the \( S(1) \) equation which allows the \( S(2) \) equation to be satisfied by zero. Then one can write down the following as a solution to the WDW equation for either a particular value of the Hartle-Hawking ordering parameter, or for an arbitrary ordering parameter depending on the \( S(1) \) which is found.

\[
\Psi_0 = e^{-\frac{1}{2} S(0) - S(1)}
\]

This can be easily shown. Let’s take \( S(0) \) and \( S(1) \) as arbitrary known functions which allow the \( S(2) \) transport equation to be satisfied by zero, then the \( k = 3 \) transport equation can be expressed as

\[
2 \left[ \frac{\partial S(0)}{\partial \alpha} \frac{\partial S(3)}{\partial \alpha} - \frac{\partial S(0)}{\partial \beta_+} \frac{\partial S(3)}{\partial \beta_+} - \frac{\partial S(0)}{\partial \beta_-} \frac{\partial S(3)}{\partial \beta_-} \right] = 0
\]

which is clearly satisfied by \( S(3) = 0 \). The \( S(4) \) equation can be written in the same form as \( (18) \) and one of its solution is 0 as well, thus resulting in the \( S(5) \) equation possessing the same form as \( (18) \). One can easily convince oneself that this pattern continues for all of the \( k \geq 3 \) \( S(k) \) transport equations as long as the solution of the \( S(k-1) \) transport equation is chosen to be 0. Thus if an \( S(1) \) exists which allows one to set the solutions to all of the higher order transport equations to zero the infinite sequence of transport equations generated by our ansatz truncates to a finite sequence of equations which allows us to construct a closed
form wave function satisfying the WDW equation. Not all solutions to the \( S_{(1)} \) transport equation will allow the \( S_{(2)} \) transport equation to be satisfied by zero; however in our case, we were able to find \( S_{(1)} \)'s which cause the \( S_{(2)} \) transport equation to be satisfied by zero, thus allowing one to set all of the solutions to the higher order transport equations to zero as shown above. This will enable us to construct new ‘ground’ state closed form solutions to the Lorentzian signature Bianchi II Wheeler Dewitt equation for arbitrary ordering parameter. It should be noted that using an alternate form of operator ordering for the WDW equation which we will introduce later one can construct solutions to it using just the \( S_{(0)} \) term.

Under certain conditions which we cannot rigorously articulate yet, wave functions that behave as ‘excited’ states can be calculated by introducing the following ansatz.

\[
\Psi_h = \phi_h e^{-S_h/h} \tag{19}
\]

where

\[
S_h = \frac{c^3 L^2}{G} S_h = \frac{c^3 L^2}{G} \left( S_{(0)} + X S_{(1)} + \frac{X^2}{2!} S_{(2)} + \cdots \right)
\]

is the same series expansion as before and \( \phi_h \) can be expressed as the following series

\[
\phi_h = \phi_{(0)} + X \phi_{(1)} + \frac{X^2}{2!} \phi_{(2)} + \cdots + \frac{X^{k(\star)}}{k!} \phi_{(k)} + \cdots \tag{20}
\]

with \( X \) being the same dimensionless quantity as before. Inserting (19) with the expansions given by (12) and (20) into the Wheeler DeWitt equation (6) and by matching equations in powers of \( X \) leads to the following sequence of equations.

\[
\begin{align*}
- \frac{\partial \phi_{(0)}}{\partial \alpha} \frac{\partial S_{(0)}}{\partial \alpha} + \frac{\partial \phi_{(0)}}{\partial \beta_+} \frac{\partial S_{(0)}}{\partial \beta_+} + \frac{\partial \phi_{(0)}}{\partial \beta_-} \frac{\partial S_{(0)}}{\partial \beta_-} &= 0, \tag{21} \\
- \frac{\partial \phi_{(1)}}{\partial \alpha} \frac{\partial S_{(0)}}{\partial \alpha} + \frac{\partial \phi_{(1)}}{\partial \beta_+} \frac{\partial S_{(0)}}{\partial \beta_+} + \frac{\partial \phi_{(1)}}{\partial \beta_-} \frac{\partial S_{(0)}}{\partial \beta_-} \\
&+ \left( - \frac{\partial \phi_{(0)}}{\partial \alpha} \frac{\partial S_{(1)}}{\partial \alpha} + \frac{\partial \phi_{(0)}}{\partial \beta_+} \frac{\partial S_{(1)}}{\partial \beta_+} + \frac{\partial \phi_{(0)}}{\partial \beta_-} \frac{\partial S_{(1)}}{\partial \beta_-} \right) \\
&+ \left( \frac{1}{2} \left( -B \frac{\partial \phi_{(0)}}{\partial \alpha} \frac{\partial^2 \phi_{(0)}}{\partial \alpha^2} - \frac{\partial^2 \phi_{(0)}}{\partial \beta_+^2} - \frac{\partial^2 \phi_{(0)}}{\partial \beta_-^2} \right) \right) &= 0,
\end{align*}
\]

(22)
must restrict the quantities satisfying equation (21) which are raised to the power \((\alpha, \beta_+, \beta_-)\), and inserting (4.9, 4.18 – 4.20) from [9] that \(\phi_{(0)}\) is a conserved quantity under the flow of \(S_0\). This means that any function \(F(\phi_{(0)})\) is also a solution of equation (21). Wave functions constructed from these functions of \(\phi_0\) are only physical if they are smooth and globally defined. Beyond the semi-classical limit, if smooth globally defined solutions can be proven to exist for the higher order \(\phi\) transport equations, then one may be able to construct a family of \('excited'\) closed form or asymptotic solutions to the WDW equation.

If our \('excited'\) states \(\Psi_h = \phi_n e^{-S_h/\hbar}\) behave as bound states then in the sense of their mathematical structure they qualitatively possess the same form as excited states for the quantum harmonic oscillator \(\psi_n(x) = H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}},\) where \(H_n\) are the Hermite polynomials in which \(n\) is a positive integer which specifies its form. Because the solutions of the \(\phi_{(0)}\) equation are quantities conserved along the flow generated by \(S_{(0)}\), any multiple \(\phi_{(0)}^n\) also satisfies equation (21). On purely physical grounds the amount of numbers required to specify an \('excited'\) state equals the number of excitable degrees of freedom present. For Bianchi A models with non dynamical matter sources that amounts to two numbers corresponding to the two anistropic degrees of freedoms. As a result \(\phi_{(0)}\) has the following form \(\prod_{i=1}^n f^{m_i}(\alpha, \beta_+, \beta_-)_i\); where \(f(\alpha, \beta_+, \beta_-)_i\) are independent conserved quantities satisfying equation (21) which are raised to the power \(m_i\), and \(n\) is the number of excitable degrees of freedom. If all of the \(f(\alpha, \beta_+, \beta_-)_i\)'s vanish at some point or points in minisuperspace then to ensure that our wave function is smooth and globally defined we must restrict the \(m_i\) to be positive integers which results in our \('excited'\) states being \('bound'\) states just like for the quantum harmonic oscillator. This discretization of the quantities that are used to denote our \('excited'\) states is the mathematical manifestation of quantization one would expect excited states to possess. If none of our conserved quantities \(f(\alpha, \beta_+, \beta_-)_i\)
vanish in minisuperspace then our 'excited' states within the context of our method as is currently formulated can be considered 'scattering' states akin to the quantum free particle and \( m_i \) can be any real number. It is also possible that only some \( f(\alpha, \beta_+, \beta_-) \)'s vanish while the other do not, in this case our 'excited' states are a hybrid of 'bound' and 'scattering' states which means some degrees of freedoms are 'bound' while others are 'scattering'.

In some cases though solutions to the aforementioned equations which don’t vanish contribute terms which do not impart qualitative features of 'excited' states onto the wave functions. We examine such a case for the vacuum quantum Bianchi II models. As we will discuss soon this points to the fact that we don’t have yet a fully adequate definition of what constitutes an 'excited' state in quantum cosmology. Beyond the list of requirements we gave above there might be additional restrictions on the forms that \( S(0) \) and \( \phi_0 \) are permitted to have in order for proper 'excited' states to be constructed from them. Also it may prove useful for establishing the existence of 'excited' states for a particular class of Bianchi A models to quantize its first order perturbations around an LRS background and seeing if the quantum states of the perturbations exhibit any discreteness as was done in [24]. Additional information about 'excited' states can be found in [9]. In what follows we will set \( X = 1 \).

III. 'GROUND' STATES OF THE VACUUM BIANCHI II WHEELER DEWITT EQUATION

As was reported in [18], there are three solutions to the Euclidean-signature Hamilton Jacobi equation (14) that corresponds to the Bianchi II WDW equation (6) when \( \Lambda = 0, b = 0, \) and \( \rho = 0 \) given by

\[
S_{(0)}^1 := \frac{1}{12} e^{2(\alpha + \sqrt{3}\beta_- + \beta_+)}.
\]

\[
S_{(0)}^2 := \left( \frac{1}{12} e^{2\alpha + 2\sqrt{3}\beta_- + 2\beta_+} + f(2\alpha + 2\beta_+) \right)
\]

\[
S_{(0)}^3 := \left( \frac{1}{12} e^{2\alpha + 2\sqrt{3}\beta_- + 2\beta_+} + g(2\alpha + \sqrt{3}\beta_- - \beta_+) \right)
\]

where \( f(x) \) and \( g(x) \) are arbitrary single variable functions. In this section we will study 'ground' state solutions to the Bianchi II Wheeler DeWitt equation obtained using the
Euclidean-signature semi classical method. The superscripts for the $S_{(k)}$ terms in this paper will play the role of an index unless stated otherwise.

Starting with (24) if we insert it into our first 'ground' state transport equation (15) we obtain the following simple differential equation

$$2 \frac{\partial S_{(1)}}{\partial \alpha} - 2 \sqrt{3} \frac{\partial S_{(1)}}{\partial \beta_-} - 2 \frac{\partial S_{(1)}}{\partial \beta_+} + B + 6 = 0 \quad (27)$$

This transport equation in principle has infinitely many solutions. However for the purposes of trying to find a $S_{(1)}$ which will allow the solutions to the higher order $k \geq 2$ transport equations to be satisfied by zero, we will choose the following to be our $S_{(1)}$

$$S_{(1)}^1 := \frac{1}{2} \left( -B + 2x_2 + 2\sqrt{3}x_3 - 6 \right) \alpha + x_2 \beta_+ + x_3 \beta_- \quad (28)$$

where $x_2$ and $x_3$ are free parameters, which can be real or complex. We choose this linear form because when we insert it into the $S_{(2)}$ transport equation (16), its source term can be made to vanish by adjusting our free parameters. To clearly illustrate this if we insert (28) into the source term of the $S_{(2)}$ transport equation we obtain the following expression

$$f(x_2, x_3) = B^2 - 4 \left( 2 \sqrt{3}(x_2 - 3)x_3 - 6x_2 + 2x_3^2 + 9 \right), \quad (29)$$

which can be easily made to vanish by solving for one of its free parameters $(x_2, x_3)$ such that $f(x_2, x_3) = 0$. The constraint of $f(x_2, x_3) = 0$ was first derived in [18]. If we solve for $x_2$, the following $S_{(1)}$ will allow all of the higher order transport equations to be satisfied by zero and will allow us to easily write out a closed form solution for any arbitrary ordering parameter $B$ to the Bianchi II Wheeler DeWitt equation

$$S_{(1)}^1 := \frac{1}{2} \left( -B + 2x_2 + 2\sqrt{3}x_3 - 6 \right) \alpha + x_2 \beta_+ + x_3 \beta_-$$

$$x_2 = \frac{B^2 - 8x_3^2 + 24\sqrt{3}x_3 - 36}{8(\sqrt{3}x_3 - 3)} \quad (30)$$

This results in us obtaining one of the solutions compatible with the constraint that was first reported in [18] to the Bianchi II WDW equation for any arbitrary ordering parameter $B$

$$\psi = e^{\left( -\frac{x_3(2\alpha + \sqrt{3}\beta_- - \beta_+)}{\sqrt{3}} \right)} \frac{1}{12} e^{2(\alpha + \sqrt{3}\beta_- + \beta_+)} \left( \frac{B^2 + 12}{8(\sqrt{3}x_3 - 3)} \right) + \frac{1}{2} \alpha(B+2) - 2\beta_+ \quad (31)$$
A nice feature of this solution is that it still possesses the free parameter \( x_3 \), which allows one to form a wide variety of wave functions from it using superposition. The above is a general method to find closed form ‘ground’ state solutions to the Wheeler DeWitt equation using the Euclidean-signature semi classical method.

Moving on to (25) if we choose \( f = x_1 e^{2\alpha + 2\beta} \) where \( x_1 \) is a free parameter and apply the same procedure we obtain an \( S_{(1)} \) which possesses one free parameter

\[
S_{(1)}^2 := x_2 \alpha + \sqrt{3} \beta_+ + \left( \frac{B}{2} + x_2 \right) \beta_+,
\]

and when inserted into (16) results in a source term \(-\frac{1}{2}B^2 - 6\) which does not vanish for any real values of the ordering parameter but does vanish when \( B = 2\sqrt{3}i \). However, if we want a solution involving a real value of the ordering parameter we can use the Euclidean-signature semi classical method to construct an asymptotic solution to the Bianchi II Wheeler Dewitt equation for any arbitrary ordering parameter for this \( S_{(0)}(25) \).

If we choose the following ansatz for our higher order \( k \geq 2 \) quantum corrections

\[
S_{(k)}^2 := g(B)k e^{\left( -2\alpha(k-1) - 2(k-1)(\sqrt{3}\beta_- + \beta_+) \right)};
\]

and insert it into (16) we can prove that the problem of solving the higher order transport partial differential equations reduces to solving a recurrence equation, where we are solving for some function \( g(B)_k \) of the Hartle-Hawking ordering parameter. The first step in our proof is to insert (33) into the homogeneous portion of equation (16) which results in the following expression

\[
2(k - 1)g(B)_k e^{\left( -2(k-2)(\alpha + \sqrt{3}\beta_- + \beta_+) \right)}.
\]

The next step is to rewrite the source terms of equation (16) as follows

For \( k = 2 \)

\[
2 \left[ B \frac{\partial S_{(1)}}{\partial \alpha} - \frac{\partial^2 S_{(1)}}{\partial \alpha^2} + \frac{\partial^2 S_{(1)}}{\partial \beta_+^2} + \frac{\partial^2 S_{(1)}}{\partial \beta_-^2} \right] + 2 \left( \frac{\partial S_{(1)}}{\partial \alpha} \frac{\partial S_{(1)}}{\partial \alpha} - \frac{\partial S_{(1)}}{\partial \beta_+} \frac{\partial S_{(1)}}{\partial \beta_+} - \frac{\partial S_{(1)}}{\partial \beta_-} \frac{\partial S_{(1)}}{\partial \beta_-} \right)
\]
For $k = 3$

$$
3 \left[ B \frac{\partial S_{(2)}}{\partial \alpha} - \frac{\partial^2 S_{(2)}}{\partial \alpha^2} + \frac{\partial^2 S_{(2)}}{\partial \beta_+^2} + \frac{\partial^2 S_{(2)}}{\partial \beta_-^2} \right] + 6 \left( \frac{\partial S_{(1)}}{\partial \alpha} \frac{\partial S_{(2)}}{\partial \alpha} - \frac{\partial S_{(1)}}{\partial \beta_+} \frac{\partial S_{(2)}}{\partial \beta_+} - \frac{\partial S_{(1)}}{\partial \beta_-} \frac{\partial S_{(2)}}{\partial \beta_-} \right)
$$

(36)

For $k > 3$

$$
k \left[ B \frac{\partial S_{(k-1)}}{\partial \alpha} - \frac{\partial^2 S_{(k-1)}}{\partial \alpha^2} + \frac{\partial^2 S_{(k-1)}}{\partial \beta_+^2} + \frac{\partial^2 S_{(k-1)}}{\partial \beta_-^2} \right]
$$

$$
+ \sum_{\ell=2}^{k-2} \frac{k!}{\ell! (k-\ell)!} \left( \frac{\partial S_{(\ell)}}{\partial \alpha} \frac{\partial S_{(k-\ell)}}{\partial \alpha} - \frac{\partial S_{(\ell)}}{\partial \beta_+} \frac{\partial S_{(k-\ell)}}{\partial \beta_+} - \frac{\partial S_{(\ell)}}{\partial \beta_-} \frac{\partial S_{(k-\ell)}}{\partial \beta_-} \right)
$$

$$
+ 2k \left( \frac{\partial S_{(1)}}{\partial \alpha} \frac{\partial S_{(k-1)}}{\partial \alpha} - \frac{\partial S_{(1)}}{\partial \beta_+} \frac{\partial S_{(k-1)}}{\partial \beta_+} - \frac{\partial S_{(1)}}{\partial \beta_-} \frac{\partial S_{(k-1)}}{\partial \beta_-} \right)
$$

(37)

As the reader can easily verify if we were to insert (33) into the source terms (35) and (36), the resulting expressions would be some constants which are proportional to (33), and thus would allow one to calculate the $k=2$ and $k=3$ quantum corrections by simply solving for $g(B)_k$ and inserting it back into (33). To prove that this is the case for the higher order $k > 3$ quantum corrections all we need to do is insert our $S_{(k)}^2$ and our linear $S_{(1)}^2$ into (37).

Doing so yields the following amazing simplification

$$
- 2(k - 2)k(B - 6k + 12)g(B)_{k-1}e^{-2(k-2)(\alpha + \sqrt{3}\beta_- + \beta_+)}
$$

$$
+ \sum_{\ell=2}^{k-2} \frac{k!}{\ell! (k-\ell)!} \left( -12(\ell - 1)(k - \ell - 1)g(B)_\ell e^{-2(k-2)(\alpha + \sqrt{3}\beta_- + \beta_+)}g(B)_{k-\ell} \right)
$$

$$
+ 2(B + 6)(k - 2)kg(B)_{k-1}e^{-2(k-2)(\alpha + \sqrt{3}\beta_- + \beta_+)}.
$$

(38)

Putting this all together, and solving for $g(B)_k$ results in

$$
g(B)_k = \frac{\sum_{\ell=2}^{k-2} \frac{-12(\ell - 1)k!(k-\ell-1)g(B)_\ell g(B)_{k-\ell}}{\ell!(k-\ell)!}}{2 - 2k} - 6k(k - 2)g(B)_{k-1}.
$$

(39)

As the reader can see our infinite sequence of linear partial differential equations has become a recurrence relation for our higher order quantum corrections. A computer algebra system like Mathematica can easily compute the terms of this recurrence relation and as a result the $S_{(k)}^2$ quantum corrections can in principle be obtained to any order $k$. The above calculation presents an alternative to [25] for obtaining asymptotic solutions to the Wheeler DeWitt equation using the Euclidean-signature semi classical method.
We have constructed a method to obtain all of the $S_2^{(k)}$ quantum corrections to the semi-classical wave function $\Psi_h = e^{-\frac{i}{\hbar} S_2^{(0)}}$, and as a result are able to construct a wide variety of asymptotic solutions to the Bianchi II Wheeler DeWitt equation for any Hartle-Hawking ordering parameter $S_2^{(k, k>} = \frac{1}{2} \sum_{k=l}^{l-2} \frac{12(l-1)!l(l-1)!g(B)g(B)_{k-1}}{l(l-1)!} - 6k(2k-2)g(B)_{k-1} \left( e^{-2\alpha(k-1)-2(k-1)(\sqrt{3}\beta_-+\beta_+)} \right)$

\[ S_2^{(k, k>} := \frac{1}{2} \sum_{k=4}^{\infty} \frac{12(k-1)!g(B)g(B)_{k-1}}{l(l-1)!} - 6k(2k-2)g(B)_{k-1} \left( e^{-2\alpha(k-1)-2(k-1)(\sqrt{3}\beta_-+\beta_+)} \right) \]

\[ S_2^{(2)} := \frac{1}{4} (B^2 + 12) \left( e^{-2\alpha-2(\sqrt{3}\beta_-+\beta_+)} \right) \]

\[ S_2^{(3)} := -\frac{9}{2} (B^2 + 12) \left( e^{-4\alpha-4(\sqrt{3}\beta_-+\beta_+)} \right) \]

\[ (0) \Psi_h = e^{-\frac{i}{\hbar} S_2^{(0)}} e^{-S_2^{(1)}} - \frac{x}{\hbar} e^{S_2^{(2)}} - \frac{x^2}{\hbar^2} e^{S_2^{(3)}} - \sum_{k=4}^{\infty} \frac{x^{k-1}}{\hbar^k} S_2^{(k)} \]

It would be instructive to see if through some manner of non trivial summation such as a Borel sum if the resulting asymptotic terms converge to some wave function which behaves in an interesting fashion. Regardless of the convergence properties of these terms, the fact that such an asymptotic solution can be found in the first place is remarkable. There are other problems in physics where such a technique for computing an asymptotic expansion can prove to be very useful.

The explicit forms of the $S_2^{(2)}$ and $S_2^{(3)}$ quantum corrections shown above can be easily computed by the reader using (34), (35), and (36). Our quantum corrections possess the important property that they decay as $\alpha$ grows. Because $\alpha$ is related to the spatial size of our Bianchi II universe, physically it makes sense that our quantum corrections become increasingly important the smaller our universe becomes, while conversely becoming negligible in the classical limit of $\alpha >> 0$. Because our solutions are asymptotic we only need to sum up a finite number of terms to get a good approximation for the full wave function. As a result of our solutions being asymptotic we will qualitatively analyze the properties of the following wave functions which are composed from $S_2^{(0)}$, $S_2^{(1)}$, and $S_2^{(2)}$

\[ \psi = e^{\left( \frac{1}{\hbar} \left( -3(B^2+12)e^{-2(\alpha+\sqrt{3}\beta_-+\beta_+)} - 2e^{2(\alpha+\beta_+)}(e^{2\sqrt{3}\beta_-+12}\chi)-24x2(\alpha+\beta_+)-24\sqrt{3}\beta_-+12\beta_+e^{B} \right) \right)} \]

where we made $x1$ an imaginary number. Because both $x1$ and $x2$ are free parameters there are infinitely many different wave functions we can choose to analyze. To narrow things
down for our purposes we will set $x_2$ and the ordering parameter $B$ equal to zero, and then form the following wave function $\Psi = \int_{-\infty}^{\infty} e^{-x^2/2} \psi dx$ based on the linearity of the WDW equation resulting in

$$\Psi = \sqrt{\pi} e^{\frac{1}{12} \left( -18 e^{-2(\alpha + \sqrt{3} \beta_+ + \beta_-)} - e^{2(\alpha + \sqrt{3} \beta_+ + \beta_-)} - 3 e^{4(\alpha + \beta_+ + \beta_-)} - 12 \sqrt{3} \beta_- \right)}.$$  \hfill (42)

We will display three plots(figures 1, 2, and 3) of this wave function for different values of $\alpha$ below and discuss them qualitatively towards the end of this paper.

![Figure 1: $\alpha = -1.5$](image1)

![Figure 2: $\alpha = 0$](image2)

![Figure 3: $\alpha = 1.5$](image3)

**IV. SOLVING THE ‘EXCITED’ STATE TRANSPORT EQUATIONS IN CLOSED FORM**

Even though the quantum Bianchi II models without any matter sources can be solved in closed form via separation of variables, applying this method has allowed us to obtain
explicit quantum corrections to semi classical wave functions of the form (13). For some cases such as the Taub models\[21\] superpositions of separable solutions can be constructed using integration which yield wave functions having the form of (13). Computing an exact expression which looks like $\Psi_0 = e^{-\frac{1}{2}S_0 - \frac{1}{3}S_1 - \frac{1}{2}S_2 - ...}$ though is contingent upon knowing how to integrate usually some Bessel function times a kernel such as $e^{-w^2}$ in closed form.

As we have shown in the previous section our modified semi classical method allows us to bypass those mathematical difficulties and obtain solutions which are more mathematically transparent. In addition the wave functions that this method obtains for us possess non-trivial characteristics such as their behavior being highly dependent upon $\alpha$, which as was previously mentioned is our internal clock and also dictates the scale factor of our Bianchi II universes. Another non-trivial feature is the manifestation of discreteness in our wave functions which will be showcased in our ’excited states later on.

We will now go over how all of the ’excited’ state transport equations can be solved for the case when (24) is our $S_0$. As it can be seen from our ’excited’ state transport equations(21-23), in order to solve them we first need solutions to their ground state counterparts (14-16). If we insert (24) into (21) we obtain

$$\left( \frac{\partial \phi_0}{\partial \alpha} - \sqrt{3} \frac{\partial \phi_0}{\partial \beta_-} - \frac{\partial \phi_0}{\partial \beta_+} \right) = 0,$$

which is an elementary linear transport equation which has the following solutions

$$\phi_{10} := f_1 \left( (3\alpha + \sqrt{3}\beta_-), (3\beta_+ - \sqrt{3}\beta_-) \right),$$

where $f_1$ is function of both the expressions $3\alpha + \sqrt{3}\beta_-$ and $3\beta_+ - \sqrt{3}\beta_-$. As a result we have infinitely many choices for our $\phi_{0}$. We can exploit the properties of the ’excited’ state transport equations and our solutions to the ’ground’ state equations to pick an ansatz which will give us the forms for all of our $\phi_{(k)}$ terms. Using the same reasoning presented in [9] for the Bianchi IX models we will pick the following to be our ansatz for the higher order $\phi_{(k)}$ terms

$$\phi_{1k} := j(B)_k e^{\left( (m_1-2k)\alpha + \frac{1}{\sqrt{3}}(-6k+m_1-m_2)\beta_- + (m_2-2k)\beta_+ \right)}.$$  

The parameters $(m_1, m_2)$ in certain circumstances can plausibly be interpreted as graviton excitation numbers for the ultra long wavelength gravitational wave modes embodied in the $(\beta_+, \beta_-)$ anisotropic degrees of freedom [24]. If we assume $(m_1, m_2)$ represent physical
quantities then they must be real numbers, despite the fact that states with complex \((m_1, m_2)\) can also satisfy the Wheeler DeWitt equation as will be shown below. Because our \(\phi_k\) terms do not vanish anywhere, \(m_1\) and \(m_2\) can be any real numbers, and if they lead to 'excited' states they would be scattering states. Before we solve for the explicit form of \(j(B)_k\) we will pick a different ansatz to showcase the versatility of this method.

If we choose our \(\phi_0^1\) to be \((\alpha + \sqrt{3}\beta_-)^m_1 (3\beta_+ - \sqrt{3}\beta_-)^m_2\), we can obtain leading order bound states because both of our expressions vanish for real finite values of the Misner variables. Going beyond leading order we can actually find a closed form solution to the Bianchi II Wheeler Dewitt equation by simply inserting \((\alpha + \sqrt{3}\beta_-)^m_1 (3\beta_+ - \sqrt{3}\beta_-)^m_2 e^{-S_0^{(1)} - S_1^{(1)}}\) into it and noticing that for \(m_2 = 1, m_1 = 0,\) and \(x = 3 = \frac{1}{4} (4\sqrt{3} - \sqrt{B^2 + 12})\) that it is satisfied by

\[
\psi = \frac{1}{3} (3\beta_+ - \sqrt{3}\beta_-) e^{\left(\frac{1}{4} (-e^{2(\alpha + \sqrt{3}\beta_- + \beta_+)} + 2\alpha (2\sqrt{3}\sqrt{B^2+12+3B-6} + \sqrt{B^2+12(3\beta_+ + \sqrt{3}\beta_-) - 12(\sqrt{3}\beta_- + \beta_+)}))\right)}.
\]

(46)

This solution shows that one doesn’t have to stick to the type of ansatz used in [9] to find solutions to the Wheeler Dewitt equation using this method. As a matter of fact it may be more advantageous for the sake of finding 'excited' states for the vacuum Bianchi II models after choosing (24) to be our \(S_0\) to use a different ansatz than (45) as we will discuss soon.

Moving on, if we pick (30) to be the \(S_1\) for our 'excited' state transport equations, a significant simplification occurs. Because our closed form solution (31) is constructed solely from an \(S_0\) and an \(S_1\) term all of the higher order \(S_{(k)}\) terms can be set to zero as was explained earlier. This significantly simplifies our 'excited' state transport equations because they depend on those higher order \(S_{(k)}\) terms which we can set to zero. The same is true for any Bianch A model which has a closed form solution where its \(S_{(k>1)}\) terms vanish (an even greater simplification occurs if (6) is satisfied by \(e^{-S_0}\)). As a result our sequence of transport equations becomes

\[
- \frac{\partial \phi_{(k)} \partial S_{(0)}}{\partial \alpha} + \frac{\partial \phi_{(k)} \partial S_{(0)}}{\partial \beta_+} + \frac{\partial \phi_{(k)} \partial S_{(0)}}{\partial \beta_-} + k \left( - \frac{\partial \phi_{(k-1)} \partial S_{(1)}}{\partial \alpha} + \frac{\partial S_{(1)} \partial S_{(1)}}{\partial \beta_+} + \frac{\partial \phi_{(k-1)} \partial S_{(1)}}{\partial \beta_-} \right) + k \frac{1}{2} \left( - B \frac{\partial \phi_{(k-1)}}{\partial \alpha} + \frac{\partial^2 \phi_{(k-1)}^{(*)}}{\partial \alpha^2} - \frac{\partial^2 \phi_{(k-1)}}{\partial \beta_+^2} - \frac{\partial^2 \phi_{(k-1)}}{\partial \beta_-^2} \right).
\]

(47)

Our situation supremely simplifies further if we can find a \(\phi_k\) which is able to satisfy its
associated transport equation when it equals zero. If \( \phi_k = 0 \) satisfies the kth order 'excited' state transport equation then the k+1th order transport equation will reduce to

\[
- \frac{\partial \phi_{(k+1)}}{\partial \alpha} \frac{\partial S_0}{\partial \alpha} + \frac{\partial \phi_{(k+1)}}{\partial \beta_+} \frac{\partial S_0}{\partial \beta_+} + \frac{\partial \phi_{(k+1)}}{\partial \beta_-} \frac{\partial S_0}{\partial \beta_-} = 0, \tag{48}
\]

which is satisfied by \( \phi_{k+1} = 0 \). Thus the k+2th order transport equations reduces to

\[
- \frac{\partial \phi_{(k+2)}}{\partial \alpha} \frac{\partial S_0}{\partial \alpha} + \frac{\partial \phi_{(k+2)}}{\partial \beta_+} \frac{\partial S_0}{\partial \beta_+} + \frac{\partial \phi_{(k+2)}}{\partial \beta_-} \frac{\partial S_0}{\partial \beta_-} = 0 \tag{49}
\]

and it is also satisfied by \( \phi_{k+2} = 0 \). When a kth order \( \phi_k \) equation is satisfied by zero, all of the higher order \( \phi_{k+n} \) transport equations can also be satisfied by zero as well. This results in a truncation of the infinite sequence of 'excited' state transport equations to a finite sequence and allows one to find closed form solutions to the Wheeler DeWitt equation for any model to which the above applies to. Inserting our ansatz (45) into (47) yields

\[
\begin{align*}
  j(B)_{k-1} & \left(3m1 \left(B^2 + 8m2 + 36\right) - 3m2 \left(B^2 + 16m2 - 36\right) + 144k^2 \left(\sqrt{3}x3 - 3\right) \\
  - 144k \left(\sqrt{3}x3 - 3\right) + 24m1^2 - 8x3(m1 + 2m2) \left(\sqrt{3}m1 - \sqrt{3}m2 - 3x3 + 6\sqrt{3}\right) \right) \\
  + 24 \left(\sqrt{3}x3 - 3\right) j(B)_k &= 0,
\end{align*} \tag{50}
\]

which allows us to easily find a simple recurrence relation for \( j(B)_k \)

\[
\begin{align*}
  j(B)_k &= \frac{1}{24 \left(\sqrt{3}x3 - 3\right)} j(B)_{k-1} \left(3m1 \left(B^2 + 8m2 + 36\right) - 3m2 \left(B^2 + 16m2 - 36\right) + 144k^2 \left(\sqrt{3}x3 - 3\right) \\
  - 144k \left(\sqrt{3}x3 - 3\right) + 24m1^2 - 8x3(m1 + 2m2) \left(\sqrt{3}m1 - \sqrt{3}m2 - 3x3 + 6\sqrt{3}\right) \right).
\end{align*} \tag{51}
\]

This recursion relation can be solved in closed form using Mathematica in terms of Pochhammer functions. The full expression is too long and cumbersome to express in this paper. However we will display the explicit form for \( j(B)_k \) and all of our \( \phi_k \)'s when our closed form solution (31) has \( x3=0 \)

\[
\begin{align*}
  j(B)_k &= \frac{1}{\pi} \left(-6\right)^k \cos \left(\frac{1}{12} \pi \sqrt{m1 \left(B^2 + 8m2 + 36\right) - m2 \left(B^2 + 16m2 - 36\right) + 8m1^2 + 36} \right) \\
  \Gamma \left(k - \frac{1}{12} \sqrt{8m1^2 + (BB^2 + 8m2 + 36) m1 - m2 \left(B^2 + 16m2 - 36\right) + 36 + \frac{1}{2}} \right) \\
  \Gamma \left(k + \frac{1}{12} \left(\sqrt{8m1^2 + \left(B^2 + 8m2 + 36\right) m1 - m2 \left(B^2 + 16m2 - 36\right) + 36 + 6}\right) \right) \\
  \phi_k &= j(B)_k e^{(m1-2k)\alpha + \frac{1}{\sqrt{3}} (-6k+m1-m2)\beta_- + (m2-2k)\beta_+}.
\end{align*} \tag{52}
\]
Going back to (51) and (45) we see that we possess the freedom to pick the values of any of our free parameters $m_1$, $m_2$, and $x_3$, for any value of $k$ so that $\phi_k=0$. When $\phi_k=0$, the solutions to the subsequent transport equations can be satisfied by zero as well, thus truncating the infinite sequence of transport equations to a finite one; enabling us to construct closed form solutions using $S_{(0)}^{1}$, $S_{(1)}^{1}$, and $\phi_0$......$\phi_{k-1}$. Because for every value of $k$ we can set our free parameters so that $\phi_k=0$, this enables us to construct a closed form solution using $\phi_0$......$\phi_{k-1}$, and because $k$ can take on every possible positive integer value, we have found an infinite family of solutions to the Bianchi II Wheeler DeWitt equation for arbitrary ordering parameter. In addition because we only need to adjust one of our three free parameters so that $j(B)_k=0$, each one of our solutions has two free parameters which we can vary. If we choose $x_3$ to be the parameter which we adjust so that $j(B)_k=0$, each one of our closed form solutions at a value of $k$ can have their two 'excitation' numbers $m_1$ and $m_2$ be any real number. Naturally, this choice makes the most physical sense in terms of forming quantum Bianchi II scattering states. However, if we decide that $m_1$ and $m_2$ are not physical quantities, they can be complex numbers and still satisfy the Bianchi II Wheeler DeWitt equation.

Despite in principle solving all of the 'excited' state transport equations a quick look at our solutions reveals that they don’t behave as ‘excited’ states. Our $\phi_k$s (45) impart a linear term composed of Misner variables $(\alpha, \beta_+, \beta_-)$ to the exponent of (31), thus resulting in our solutions being a sum of functions which have the form of our ‘ground’ states (31).

Now this does not mean no ‘excited’ states exist for the vacuum Bianchi II models. In this section we chose to use (45) as our $\phi_k$s because it allowed us to easily solve for all of the 'excited’ state transport equations. Because our $\phi_0$ is a conserved quantity we could have used any function of it to construct our 'excited’ states. There very well could exist a $\phi_0$ which results in wave functions that have (24) as their semi classical term that qualitatively behave like 'excited’ states. Furthermore if we choose a different $S_{(0)}$ such as the infinitely many choices for (25) and (26) we may have obtained a simple form for our $\phi_0$ which could immediately lead to wave functions that behave as ‘excited’ states. Studying the perturbations of the LRS Bianchi II models as was done for the Taub models in [24] would also be very useful in establishing the existence of vacuum Bianchi II 'excited’ states. As we will see though when matter sources are included our solutions to the $\phi_0$ transport equation does result in wave functions which do behave as 'excited' states.
V. ELECTROMAGNETIC POTENTIALS FOR BIANCHI II AND VII

In this section we will compare two methods for obtaining the WDW equation (6). The first method will be based on directly quantizing the class of classical Hamiltonians for Bianchi A models that was developed in [8]. This will lead to a semi-classical treatment of our electromagnetic degree of freedom and will be what we use in the following sections to analyze how matter sources affect our wave functions. However we will also do a full quantum treatment of the electromagnetic degree of freedom and compare the two approaches. We will assume all of our electric and magnetic fields are parallel to each other as is justified in [26].

With this in mind our first task is to obtain solutions for Maxwell’s equations in the space-time (1) in terms of the Misner variables. In our calculations we will set \( L = 1 \) which has units of length. Starting from

\[
A = A_0 dt + A_1 \omega^1 + A_2 \omega^3 + A_3 \omega^3
\]  

(53)

and using the fact that \( d\omega^i = \frac{1}{2} C_{jk}^i \omega^j \wedge \omega^k \) to aide us in computing \( F = dA = \frac{1}{2} F_{\mu\nu} \omega^\mu \wedge \omega^\nu \) results in the following expression for \( F_{\mu\nu} \)

\[
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} + A_\alpha C_{\mu\nu}^\alpha.
\]  

(54)

In (54) differentiation is done through a vector dual to our one forms \( \omega^\mu \) which we denote as \( X_\mu \). Thus \( A_{\nu,\mu} = X_\mu A_\nu \). The electromagnetic portion of \( S_{\text{matter}} \) in (2) is

\[
S_{\text{matter}} = \int dtdx^3 N \sqrt{h} \left( -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \right) ,
\]  

(55)

where

\[
h_{ab} = e^{2\alpha(t)} \text{ diag} \left( e^{2\beta_+(t)} + 2\sqrt{3}\beta_-(t), e^{2\beta_+(t)} - 2\sqrt{3}\beta_-(t), e^{-4\beta_+(t)} \right).
\]  

(56)

Writing the action (55) in terms of its vector potential \( A \) and our structure constants results in the Lagrangian density which is derived in [8]

\[
\mathcal{L} = \Pi^s A_{0,s} - NH
\]

\[
\mathcal{L} = \Pi^s A_{0,s} - \Pi^s A_{s,0} - N \frac{2\pi}{\sqrt{h}} \Pi^p h_{sp}
\]

\[
- \frac{N\sqrt{h}}{16\pi} h^{ik} h^{sl} \left( 2A_{[i,s]} + A_m C_{is}^m \right) \left( 2A_{[k,l]} + A_m C_{kl}^m \right),
\]  

(57)
where
\[ \Pi^s = \frac{\partial \mathcal{L}}{\partial (X_0 A_s)} = \frac{\hbar^s \sqrt{\hbar}}{4N \pi} (-A_{0,j} + A_{j,0} + A_\alpha C^\alpha_{0j}) , \] (58)
and we allow the shift \( N^k \) to vanish. If we invoke the homogeneity of (1) then we can say that \( A_{i,j} = 0 \), and \( A_{0,j} = 0 \) which results in (57) simplifying to
\[ \mathcal{L} = \Pi^s A_{s,0} - N \left[ \frac{2\pi}{\sqrt{\hbar}} \Pi^s \Pi^p h_{ps} + \frac{\sqrt{\hbar}}{16 \pi} h^{ik} h^{sl} C^m_{kl} C^m_{is} A_m A_n \right] . \] (59)
The non zero Bianchi II and VII\(_{h=0}\) structure constants are the following respectively
\[
\begin{align*}
C^1_{23} &= -1 \quad (60) \\
C^1_{32} &= 1 \\
C^1_{13} &= -1 \\
C^1_{32} &= 1 \\
C^2_{13} &= 1 \\
C^2_{31} &= -1
\end{align*}
\]
We will now set \( A_2, A_3, \Pi^2, \) and \( \Pi^3 \) to zero as is justified in [8] and [26] and only consider the electromagnetic field produced by \( A_1 \) and \( \Pi^1 \); doing so results in the following Lagrangian density
\[ \mathcal{L} = \Pi^1 A_{1,0} - N \left[ \frac{2\pi}{\sqrt{\hbar}} \Pi^1 \Pi^1 h_{11} + \frac{\sqrt{\hbar}}{16 \pi} h^{ik} h^{sl} C^1_{kl} C^1_{is} A_1 A_1 \right] . \] (62)
As the reader can easily verify \( h^{ik} h^{sl} C^1_{kl} C^1_{is} = \frac{2h_{11}}{\hbar} \), which allows us to obtain the following set of Maxwell’s equations when \( A_1 \) and \( \Pi^1 \) are varied
\[ \dot{A}_1 - 4\pi \frac{1}{\sqrt{\hbar}} \Pi^1 h_{11} = 0 \] (63)
and
\[ \dot{\Pi}^1 + \frac{1}{4\pi \sqrt{\hbar}} h_{11} A_1 = 0. \] (64)
For the last equation we applied an integration by parts to the term \( \Pi^1 A_{1,0} \) and dropped the total derivative term which vanishes at the spatial boundary. The solutions for (63) and (64) are
\[ A_1 = \sqrt{2} B_0 \cos(\theta(t)) \] (65)
\[ \Pi^1 = \frac{1}{2\sqrt{2\pi}} B_0 \sin(\theta(t)) \] (66)
where $\theta(t)$ is an integral which is immaterial for our purposes and $B_0$ is an integration constant. Inserting (65) and (66) back into (62) results in

$$L = \Pi^1 A_{1,0} - N \frac{B_0^2}{4\pi \sqrt{\hbar}} \hbar_{11} (\sin(\theta(t))^2 + \cos(\theta(t))^2) = \Pi^1 A_{1,0} - N \frac{B_0^2}{4\pi} e^{-\alpha(t) + 2\beta(t)_+ + 2\sqrt{3}\beta(t)_-},$$

(67)

From (59) we can easily identify the electromagnetic Hamiltonian as

$$H_{em} = \frac{B_0^2}{4\pi} e^{-\alpha(t) + 2\beta(t)_+ + 2\sqrt{3}\beta(t)_-}$$

(68)

which can be added to the Hamiltonian constraint which is derivable from our action (2) and (3)

$$e^{-3\alpha(t)} (-p_\alpha^2 + p_+^2 + p_-^2) + U_g + \frac{B_0^2}{4\pi} e^{-\alpha(t) + 2\beta(t)_+ + 2\sqrt{3}\beta(t)_-} + \rho = 0$$

(69)

Quantizing (69) using the factor ordering we chose before, multiplying each side by $e^{3\alpha(t)}$, and rescaling $B_0$ results in the Wheeler DeWitt equations (6).

If we start with (62) and directly quantize our component of the total Hamiltonian constraint which is proportional to the lapse $N$ we obtain a similar, but slightly different contribution to the potential. Simplifying the term in brackets of (62) results in the following contribution to the Hamiltonian constraint derived with (2)

$$H_{em} = N \left[ e^{-\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \left( 16\pi^2 \Pi^1 \Pi^1 + A_1^2 \right) \right] / 8\pi.$$  

(70)

The term $(16\pi^2 \Pi^1 \Pi^1 + A_1^2)$ commutes with our total Hamiltonian constraint $H_{gravity} + H_{em}$. Thus we can solve the following WDW equation constructed by directly quantizing (70) with the rest of our constraint.

$$\Box \Psi - B \frac{\partial \Psi}{\partial \alpha} - e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \left( 2\pi \frac{\partial^2 \Psi}{\partial A_1^2} + \frac{1}{8\pi} A_1^2 \Psi \right) + U_{II \lor III} \Psi = 0$$  

(71)

by first solving this eigenvalue problem

$$-2\pi \frac{\partial^2 \Psi}{\partial A_1^2} + \frac{1}{8\pi} A_1^2 \Psi = b_n \Psi.$$  

(72)

This is simply the Schrödinger equation for a harmonic oscillator whose solutions are well known

$$\Psi = \psi(\alpha, \beta_+, \beta_-) e^{-\frac{A_1^2}{8\pi} H_{bn}} \left( \frac{A_1}{2\sqrt{\pi}} \right)$$

$$b_n = \frac{1}{2} (1 + 2n).$$  

(73)
Inserting our $\Psi$ from (73) into (71) yields

$$\Box \psi - B \frac{\partial \psi}{\partial \alpha} - b_n e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} + U_{II\nu\nu\nu} \psi = 0.$$  \hspace{1cm} (74)

This WDW equation is similar to what we had before except for the fact that the strength $b_n$ of the electromagnetic field is now quantized thanks to (73). By first solving the classical $A_i$ equations (63) in terms of the Misner variables we eliminate the electromagnetic field degree of freedom. By keeping it we can in theory study much more general quantum cosmologies for anisotropic models that involve electromagnetic fields. For now though working with just (69) is sufficient for what will follow.

VI. CLOSED FORM BIANCHI II 'GROUND' STATES WITH MATTER SOURCES

Using the following ansatz

$$S^4_{(0)} = -6\Lambda e^{4\alpha - 2\sqrt{3}\beta_- - 2\beta_+} + \frac{1}{12} e^{2\alpha + 2\sqrt{3}\beta_- + 2\beta_+} + \alpha x_1 + \beta_- x_3 + \beta_+ x_2$$  \hspace{1cm} (75)

the author found the following solutions to the Bianchi II Euclidean-signature Hamilton Jacobi equation corresponding to (6)

$$\left( \frac{\partial S^4_{(0)}}{\partial \alpha} \right)^2 - \left( \frac{\partial S^4_{(0)}}{\partial \beta_+} \right)^2 - \left( \frac{\partial S^4_{(0)}}{\partial \beta_-} \right)^2 + \frac{1}{12} e^{4\alpha + 4\beta_+ + 4\sqrt{3}\beta_-} + 24\Lambda e^{6\alpha} + 2b^2 e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} + \rho = 0$$

$$S^4_{(0)} = -6\Lambda e^{4\alpha - 2\sqrt{3}\beta_- - 2\beta_+} + \frac{1}{12} e^{2\alpha + 2\sqrt{3}\beta_- + 2\beta_+} - 2\alpha b^2 + \frac{1}{2} \beta_- \left( 2\sqrt{3}b^2 + \sqrt{\rho} \right) + \frac{1}{2} \beta_+ \left( 2b^2 - \sqrt{3}\sqrt{\rho} \right).$$  \hspace{1cm} (76)

If we insert (76) into (15) we obtain

$$-24\Lambda e^{4\alpha - 2(\sqrt{3}\beta_- + \beta_+)} \left( 2 \frac{\partial S_{(1)}}{\partial \alpha} + \sqrt{3} \frac{\partial S_{(1)}}{\partial \beta_-} + \frac{\partial S_{(1)}}{\partial \beta_+} + B \right)$$

$$+ \frac{1}{6} e^{\alpha + \sqrt{3}\beta_- + \beta_+} \left( 2 \frac{\partial S_{(1)}}{\partial \alpha} - 2\sqrt{3} \frac{\partial S_{(1)}}{\partial \beta_-} - 2 \frac{\partial S_{(1)}}{\partial \beta_+} + B + 6 \right)$$

$$- 4b^2 \frac{\partial S_{(1)}}{\partial \alpha} - 2b^2 \left( \sqrt{3} \frac{\partial S_{(1)}}{\partial \beta_-} + \frac{\partial S_{(1)}}{\partial \beta_+} \right)$$

$$+ \sqrt{\rho} \left( \sqrt{3} \frac{\partial S_{(1)}}{\partial \beta_+} - \frac{\partial S_{(1)}}{\partial \beta_-} \right) - 2b^2 B = 0;$$  \hspace{1cm} (77)

which in accordance with our previous reasoning can be satisfied by the following simple solution

$$S^{4}_{(1)} := \frac{1}{2} \alpha (-B - 2) + \frac{\sqrt{3} \beta_-}{2} + \frac{\beta_+}{2}.$$  \hspace{1cm} (78)
Inserting this into the source term of (16) yields $-\frac{B^2}{2}$ which vanishes when $B = 0$. Thus we have the following solution to the Bianchi II WDW equation when a cosmological constant, electromagnetic field and stiff matter are present

$$\Psi = e^{6\Lambda e^{4\alpha - 2\sqrt{3}\beta_+-\beta_+} - \frac{1}{12} e^{2(\alpha-\sqrt{3}\beta_++\beta_+)} + 2ab^2 + \alpha - \frac{1}{2}\beta_- (2\sqrt{3}b^2 + \sqrt{\rho} + \sqrt{3}) + \frac{1}{2}\beta_+ (2b^2 + \sqrt{3}\sqrt{\rho} - 1)}.$$ (79)

To understand what effects the electromagnetic field ($b^2$) has on our wave function (79) we shall display four plots (figures 4-7), and discuss them at the end of this paper.

| FIG. 4 | $\alpha = -\frac{1}{4}$ | $\Lambda = -1$ | $b=0$ | $\rho = 0$ |
| FIG. 5 | $\alpha = -\frac{1}{4}$ | $\Lambda = -1$ | $b=7$ | $\rho = 0$ |
| FIG. 6 | $\alpha = \frac{1}{4}$ | $\Lambda = -1$ | $b=0$ | $\rho = 0$ |
| FIG. 7 | $\alpha = \frac{1}{4}$ | $\Lambda = -1$ | $b=7$ | $\rho = 0$ |

We can obtain a solution for any Hartle-Hawking ordering parameter if we consider the case when only a cosmological constant is present. If we start with the following semi-classical term

$$S^5_{(0)} = -6\Lambda e^{4\alpha - 2\sqrt{3}\beta_- - 2\beta_+} + \frac{1}{12} e^{2\alpha + 2\sqrt{3}\beta_- + 2\beta_+}.$$ (80)
and insert it into (15) we obtain the following $S_{(1)}^5$

$$S_{(1)}^5 := \frac{1}{2} \alpha(-B - 2) + \beta_- x_1 + \beta_+ \left(2 - \sqrt{3}x_1\right). \quad (81)$$

When (81) is inserted into (16) we obtain this source term $-3B^2 - 4(1 - 2x_1)^2$ which vanishes when $x_1 = \frac{1}{4} \left(2 \pm \sqrt{3}B\right)$. This allows us to construct two independent solutions to the Bianchi II WDW equation, one for each of the two possible values of $x_1$ and sum them up to obtain

$$\Psi = \left(e^{i\frac{\beta_- B}{2}} + e^{\frac{1}{2}i\sqrt{3}\beta_+ B}\right) e^{\left(\frac{1}{12} \left(72\Lambda e^{4\alpha - 2(\sqrt{3}\beta_- + \beta_+)} - e^{2(\alpha + \sqrt{3}\beta_- + \beta_+)} + 6\alpha(B+2) - 3\Lambda(\beta_- + \sqrt{3}\beta_+) - 6(\sqrt{3}\beta_- + \beta_+)\right)\right)}, \quad (82)$$

These solutions non-trivially depend on the ordering parameter. When $B = 0$ these solutions are real, otherwise they are complex. We will plot (figures 8-10) $|\Psi|^2$ for three different values of $\alpha$. We will discuss in detail the plots towards the end of this paper.

VII. CLOSED FORM ‘EXCITED’ STATES OF THE $\Lambda \neq 0$ BIANCHI II WHEELER DEWITT EQUATION

The author was able to find the following $\phi_0$ for the case when only a cosmological constant is present.
\[
\phi^5_0 := \left( e^{\frac{1}{2} (3\beta_+ - \sqrt{3}\beta_-)} \right)^{m_1} \left( 48\Lambda e^{6\alpha - 4\sqrt{3}\beta_-} + e^{4(\alpha + \beta_+)} \right)^{m_2}.
\] (83)

This \( \phi_0 \) suggests that the 'excited' states of the quantum Bianchi II models when a cosmological constant is present have some interesting properties. When \( \Lambda > 0 \) our 'excited' states are scattering states because none of the terms exponentiated by our graviton excitation numbers \( m_1 \) and \( m_2 \) vanish for any real values of the Misner variables. However, when \( \Lambda < 0 \) the terms associated with \( m_1 \) don't vanish, while the term exponentiated by \( m_2 \) does vanish for real values of the Misner variables. Thus for \( \Lambda < 0 \) our \( m_1 \) term can be any real number, while \( m_2 \) is restricted to being either zero or a positive integer. The excited' states for the quantum Bianchi II models when \( \Lambda < 0 \) are hybrid scattering/bound states. This property is shared with the quantum Bianchi VIII models, which the author studied as well. The higher order \( \phi^k_5 \) terms in principle can be found by solving the rest of the transport equations. However, due to \( S^{5}_{(0)} \) possessing two terms with different \( \alpha \) dependence, it is more difficult to solve these transport equations as opposed to the ones we encountered earlier. The author in trying to solve the \( \phi^5_1 \) terms computed an unenlightening integral expression which we will omit. The plots we will show will be leading order in \( \phi \) 'excited' states.

To construct our graphs we will set \( m_1 = -3 \) and \( m_2 = 1 \) and graph the modulus squared of the following wave function

\[
\Psi_{Excited} = \sum_{m=1}^{10} (\phi^5_0)^m \Psi,
\] (84)

where \( \Psi \) is our exact wave function (82).

\[ \text{FIG. 11: } \alpha = -1 \quad \Lambda = -1 \quad B = 0 \]

\[ \text{FIG. 12: } \alpha = 0 \quad \Lambda = -1 \quad B = 0 \]
VIII. NON-COMMUTATIVE QUANTUM BIANCHI II WITH MATTER SOURCES

In this section we shall study the quantum non-commutative Bianchi II models when an electromagnetic field and stiff matter are present. To do so we shall use the following deformation of the ordinary commutation relations between the minisuperspace variables

\[
\begin{align*}
[\alpha_{\text{nc}}, \beta_{+\text{nc}}] &= i\theta_1, \\
[\alpha_{\text{nc}}, \beta_{-\text{nc}}] &= i\theta_2, \\
[\beta_{-\text{nc}}, \beta_{+\text{nc}}] &= i\theta_3.
\end{align*}
\]

This type of deformation of the configuration or phase space of a finite dimensional theory is employed in non-commutative quantum mechanics\[27–29\]. The non-commutative quantum Bianchi II models with stiff matter were thoroughly investigated in \[11\]. In this section we will follow their methodology and extend their results by including an electromagnetic field.

The purpose of imposing these non-commutative relations on the minisuperspace of our Bianchi II models is to obtain a better understanding of how non-commutative space-time could have affected cosmological evolution in the early universe. Many theories of quantum gravity predict that space-time itself manifests some form of discretization. One way for this supposed discretization to manifest mathematically is in the coordinates\((t, x_i)\) of space-time possessing non-vanishing commutation relations. For example in String Theory/M-Theory a non-commutative gauge theory emerges when describing the low energy excitations of open strings in the presence of a Neveu-Schwarz constant background B field\[30\][31]. As a result there has been a renewed interest in the study of non-commutative space-times.

A non-commutative space-time version of general relativity has been proposed\[32\] and in theory one can use it to directly study the full impact that non-commutative space-time has on classical cosmological evolution. However formulating general relativity in a non-commutative space-time results in a theory that is incredibly non-linear and very difficult to work with mathematically. A way to obtain some understanding of how non-commutative space-times can affect cosmology was proposed by \[33\]. Instead of directly studying the
cosmology of a theory of gravity with a non-commutative space-time one can study a cosmology with deformed minisuperspace commutative relations as presented in (85). The justification for this can be summed up by saying that it is reasonable to expect that a full non-commutative space-time theory of gravity would result in some effects which can be captured by introducing non-commutativity in the minisuperspace of its homogeneous cosmologies. Thus by studying non-commutative minisuperspace homogeneous cosmologies we are studying an effective toy model of a non-commutative theory of gravity with its degrees of freedom reduced by imposing the symmetries present in homogeneous space-times.

To begin the process of solving the non-commutative Bianchi II WDW equation we will implement the following Seiberg-Witten map[31]

\[ \begin{align*}
\alpha_{nc} & \rightarrow \alpha - \frac{\theta_1}{2} p_{\beta_+} - \frac{\theta_2}{2} p_{\beta_-}, \\
\beta_{-nc} & \rightarrow \beta_+ + \frac{\theta_2}{2} p_{\alpha} - \frac{\theta_3}{2} p_{\beta_+}, \\
\beta_{+nc} & \rightarrow \beta_+ + \frac{\theta_1}{2} p_{\alpha} + \frac{\theta_3}{2} p_{\beta_-}.
\end{align*} \tag{86} \]

Doing so results in the following modified potential term for (6) when \( \Lambda = 0 \)

\[ U(\Omega, \beta_{\pm}) = \frac{1}{12} e^{4\left[\alpha + \beta_+ \sqrt{3} \beta_- \frac{\partial}{\partial \Omega} + \frac{\partial}{\partial \beta_+} \left(\sqrt{3} \frac{\partial}{\partial \Omega} - \frac{\partial}{\partial \beta_-}\right) \right] + \frac{\partial}{\partial \lambda} \left(\sqrt{3} \frac{\partial}{\partial \Omega} - \frac{\partial}{\partial \lambda}\right) + \rho. \] \tag{87} \]

After applying the following coordinate transformation

\[ \xi = \Omega + \beta_+ + \sqrt{3} \beta_-, \quad \kappa = \Omega + \sqrt{3} \beta_-, \quad \lambda = \Omega - 2 \beta_+ + \sqrt{3} \beta_- \] \tag{88} \]

and applying the generalized Baker-Campbell-Hausdorff formula

\[ e^{\eta(\hat{A} + \hat{B})} = e^{-\eta^2[\hat{A}, \hat{B}]} e^{\eta \hat{A}} e^{\eta \hat{B}} \] \tag{89} \]

we obtain the following WDW equation

\[ \begin{align*}
- B & \frac{\partial \Psi}{\partial \xi} - 3 \frac{\partial^2 \Psi}{\partial \xi^2} + 2 \frac{\partial \Psi}{\partial \kappa} - B \frac{\partial \Psi}{\partial \kappa} - 6 \frac{\partial^2 \Psi}{\partial \lambda^2} + \hat{B} \frac{\partial \Psi}{\partial \lambda} \\
+ & \frac{1}{12} e^{4\xi} e^{-2i\theta_1 \frac{\partial}{\partial \xi}} e^{6i\theta_2 \frac{\partial}{\partial \kappa}} e^{-\frac{4\sqrt{3} i \theta_3 \partial}{3 \partial \lambda}} e^{-2 \frac{\sqrt{3} i \theta_3 \partial}{3 \partial \lambda}} e^{-6 \sqrt{3} i \theta_3 \frac{\partial}{3 \partial \lambda}} \Psi \\
+ & 2b^2 e^{2\xi} e^{-i\theta_1 \frac{\partial}{\partial \xi}} e^{-3i\theta_2 \frac{\partial}{\partial \kappa}} e^{-\frac{4\sqrt{3} i \theta_3 \partial}{3 \partial \lambda}} e^{-2 \frac{\sqrt{3} i \theta_3 \partial}{3 \partial \lambda}} e^{-3 \sqrt{3} i \theta_3 \frac{\partial}{3 \partial \lambda}} \Psi + \rho \Psi = 0. \tag{90} \end{align*} \]

To solve this equation we will insert this ansatz into it, \( \Psi = f(\xi) e^{i c_1 \kappa} e^{i c_2 \lambda} \) where \( c_1 \) and \( c_2 \) are constants while keeping in mind that \( e^{i \theta \frac{\partial}{\partial \xi}} e^{\eta} \equiv e^{i \eta} e^{\eta} \), resulting in
\[ f(\xi) \left( 24b^2 e^{2c+2w} + e^{4c+4w} + g \right) - 12 \left( B \frac{\partial f}{\partial \xi} + 3 \frac{\partial^2 f}{\partial \xi^2} \right) = 0 \]

\[ w = \frac{1}{2} \left( \theta_1(c1 + 3c2) + \frac{2\theta_3 c1 + \theta_3 c1 + 9\theta_3 c2}{\sqrt{3}} \right) \]

\[ g = -12iBc1 - 12iBc2 + 12 \rho - 8c1^2 + 72c2^2. \]

The solution to this equation (91) is the following

\[ f(\xi) = e^{\frac{1}{12}(-2B(\xi+w) - e^{2(\xi+w)})} \left( e^{2(\xi+w)} \right)^{\frac{\sqrt{B^2 + g}}{12}} U \left( b^2 + \frac{1}{12} \left( \sqrt{B^2 + g + 6} \right), \frac{1}{6} \left( \sqrt{B^2 + g + 6} \right), \frac{1}{6} e^{2(\xi+w)} \right) \]

where U is the hypergeometric U function. There is a generalized Laguerre polynomial which also satisfies (91) but it yields solutions which do not appear to be physical. Using our ansatz, (88), and (92) one can express the solutions for the non-commutative Bianchi II WDW equation (6, 87). In what follows we will set \( \theta_2 = \theta_1, \quad \theta_3 = \theta_1, \quad \rho = 0. \) We will present a series of plots for our non-commutative wave \( \left| \int_{-\infty}^{\infty} e^{-1.5(c1-1.3)^2} e^{ic\kappa} f(\xi) dc1 \right|^2 \) function and discuss them at the end of this paper.

FIG. 15: \( \alpha = -2 \quad \theta_1 = 0 \quad B = 0 \quad b = 0 \)

FIG. 16: \( \alpha = -2 \quad \theta_1 = 1.5 \quad B = 0 \quad b = 0 \)
FIG. 17: $\alpha = -2 \quad \theta_1 = 1.5 \quad B = 0 \quad b = 4$

IX. QUANTUM VACUUM BIANCHI VII$_{h=0}$ ‘GROUND’ AND ‘EXCITED’ STATES

Moving on to the Bianchi VII$_{h=0}$ models, these two solutions were found for its Euclidean-Signature Hamilton-Jacobi equation (6, 14) by [18]

\[ S_6^{(0)} = \frac{1}{3} e^{2(\alpha + \beta_+)} \cosh \left( 2\sqrt{3}\beta_- \right) \]  \hspace{1cm} (93)

\[ S_7^{(0)} = \frac{1}{3} e^{2(\alpha + \beta_+)} \cosh \left( 2\sqrt{3}\beta_- \right) + x_1 e^{2\alpha + 2\beta_+} \]  \hspace{1cm} (94)

where $x_1$ is an arbitrary constant.

Starting with (93) if we insert it into (15) and employ the methodology that we used to solve the Bianchi II $S_{(1)}$ equation (27) we obtain

\[ S_{(1)}^{6} = \alpha x_1 + \frac{1}{12} \log \left( \sinh \left( 2\sqrt{3}\beta_- \right) \right) \left( B + 2x_1 - 2x_2 + 6 \right) + \beta_+ x_2 \]  \hspace{1cm} (95)

where both $x_1$ and $x_2$ are arbitrary constants. Inserting this term into the source term of equation (16) results in an expression which vanishes when our arbitrary constants equals

\[ \left\{ x_1 = \frac{1}{24} \left( -B^2 - 12B - 36 \right) , x_2 = \frac{1}{24} \left( 36 - B^2 \right) \right\} . \]  \hspace{1cm} (96)

This allow us to write down the following solution to the Bianchi VII$_{h=0}$ WDW equation

\[ \Psi = e^{\left( \frac{1}{24} (B+6)(\alpha(B+6)+\beta_+(B-6)) - \frac{1}{2} e^{2(\alpha + \beta_+)} \cosh \left( 2\sqrt{3}\beta_- \right) \right)} . \]  \hspace{1cm} (97)

This solution was first reported by [18].
For the Bianchi VII\(_{h=0}\) 'excited' states the author found the following solutions to the \(\phi_0\) equations for (93)

\[
\phi_0^6 := e^{6m2\alpha-6m1\beta_+} \sinh \left( 2\sqrt{3}\beta_- \right)^{m1+m2}.
\] (98)

Due to \(\sinh \left( 2\sqrt{3}\beta_- \right)\) vanishing we must restrict the values of \(m1\) and \(m2\) so that their sum \(m1 + m2\) always equals a positive integer or zero. Using (98) we can construct semi-classical 'excited' states. We can find closed form solutions to the Bianchi VII\(_{h=0}\) WDW equation by inserting \(\phi_0^6\Psi\) into it, where we used (97) for our \(\Psi\). By doing so we will will find that it is satisfied when \(m1 = \frac{1}{216} (B^2 + 84)\) and \(m2 = \frac{1}{216} (132 - B^2)\). This leads to the following closed form solution

\[
\Psi_{excited} = \sinh \left( 2\sqrt{3}\beta_- \right) e^{\left( \frac{1}{17} (-24 e^{2(\alpha+\beta_+)} \cosh(2\sqrt{3}\beta_-) + \alpha (B(B+36)+372) + \beta_+ (B^2-276)) \right)}.
\] (99)

This solution has a strange property though. The potential for the Bianchi VII\(_{h=0}\) models is invariant under the reflection of \(\beta_- \rightarrow -\beta_-\) while our 'excited state isn’t. This is another instance[23] in which the WDW equation admits solutions which do not respect the symmetry of its potential. However \(|\Psi_{excited}|^2\) does preserve the symmetry of the potential. This is important because if all of our observables are dependent upon \(|\Psi_{excited}|^2\), then the symmetry which is broken by (99) may not bear any practical consequences. Nonetheless the physical implications of symmetry breaking solutions of the Wheeler DeWitt equation is a topic which deserves to be investigated more.
X. QUANTUM VACUUM BIANCHI VII$_{h=0}$ 'GROUND' STATES WITH MATTER SOURCES

For the case when only an electromagnetic field and stiff matter are present the author found the following solution to the Euclidean-signature Hamilton Jacobi equation corresponding to (6)

$$S^7_{(0)} = \frac{1}{6} \left( 3b^2 \left( -2\alpha + \sqrt{3}\beta_- + \beta_+ \right) + 2e^{2(\alpha + \beta_+)} \cosh \left( 2\sqrt{3}\beta_- \right) + \frac{2\rho(\alpha + \beta_+)}{b^2} \right).$$

(100)

Inserting this expression in (15) and seeking an $S_{(1)}$ which is linear in the Misner variables results in

$$S^7_{(1)} = \frac{2\beta_+ (3b^4 - \rho)}{3b^4} - \frac{\alpha (3b^4B + 6b^4 + 4\rho)}{6b^4}.$$ 

(101)

If we insert (101) into the source term of (16) we obtain $\frac{8\rho}{b^4} - \frac{B^2}{2} - 6 = 0$ which vanishes when our ordering parameter equals $B = \frac{2\sqrt{4\rho - 3b^4}}{b^2}$. This allow us to write down the following solution which satisfies (6) when $\Lambda = 0$ and $B = \frac{2\sqrt{4\rho - 3b^4}}{b^2}$

$$\Psi = e^{\left( \frac{1}{2} \left( -3b^2(-2\alpha + \sqrt{3}\beta_- + \beta_+) - 2e^{2(\alpha + \beta_+)} \cosh \left( 2\sqrt{3}\beta_- \right) - \frac{2\rho(\alpha + \beta_+)}{b^2} \right) + \frac{2\rho(\alpha + \beta_+)}{3b^4} + \frac{\alpha \sqrt{4\rho - 3b^4} + \alpha - 2\beta_+}{b^2} \right)}.$$ 

(102)

In order for our ordering parameter to be real we require that $\rho \geq \frac{3}{4}b^4$. Nonetheless if one were interested in studying an asymptotic Bianchi VII$_{h=0}$ wave function, then they may be content to only include the first two terms of the expansion which we have calculated, or they can solve for higher order terms using our transport equations.

To accommodate a cosmological constant we have to make our stiff matter term $\rho = 3b^4$, which results in the following $S_{(0)}$

$$S^8_{(0)} = -3\Lambda e^{4\alpha + 2\sqrt{3}\beta_- - 2\beta_+} + \frac{1}{3} e^{2(\alpha + \beta_+)} \cosh \left( 2\sqrt{3}\beta_- \right) + \frac{1}{2} b^2 \left( \sqrt{3}\beta_- + 3\beta_+ \right).$$

(103)

Unfortunately as of the writing of this paper the author has been unable to find a solution to its corresponding $S_{(1)}$ equation. Hopefully in the future a solution to it can be found. However if we alter the operator ordering of our WDW equation we can satisfy it using just (103).
As pointed out by Moncrief and Ryan in [21] and shown explicitly in [34] using a different operator ordering then the Hartle-Hawking semi general ordering for the Wheeler DeWitt equation allows one to construct wave functions which satisfy it if one possesses pure imaginary solutions to its corresponding Lorentzian signature Hamilton Jacobi equation. We will review the derivation presented in [34] which allows us to construct solutions using just an \( S_0 \). The Hamiltonian constraint for the Bianchi A models can be expressed as

\[
H = G^{AB} p_A p_B + U = 0
\]  
(104)

where \( G^{AB} \) is the DeWitt supermetric and \( p_i \) are the canonical momenta. Likewise the regular Lorentzian signature Hamilton Jacobi is expressed as

\[
G^{AB} \frac{\partial J}{\partial x^A} \frac{\partial J}{\partial x^B} + U = 0.
\]  
(105)

Because (105) is the Lorentzian signature Hamilton Jacobi equation the signs in its derivatives are the opposite of those for the Euclidean case. That means for the Bianchi VII\(_{h=0}\) models with a cosmological constant, a primordial electromagnetic field and stiff matter it is satisfied by our previous solutions multiplied by \( \sqrt{-1} \) such that \( J = \sqrt{-1} S_0^i \). This allows us to rewrite (105) as

\[
G^{AB} p_A p_B + G^{AB} \frac{\partial S^i}{\partial x^A} \frac{\partial S^i}{\partial x^B} = G^{AB}(x) \pi_A^* \pi_B = 0
\]  
(106)

where

\[
\pi_A = p_A - \sqrt{-1} \frac{\partial S^i}{\partial x^A}
\]  
(107)

and is quantized as follows

\[
\hat{\pi}_A = -\sqrt{-1} \frac{\partial}{\partial x^A} - \sqrt{-1} \frac{\partial S^i}{\partial x^A},
\]  
(108)

Due to this quantization, wave functions of the form \( \Psi = e^{-S_0^i} \) mathematically behave in the following way

\[
\hat{\pi}_B \Psi = 0.
\]  
(109)

Thus we can satisfy the Bianchi VII\(_{h=0}\) WDW equation when a cosmological constant, primordial electromagnetic field and stiff matter are present if we order the WDW as follows

\[
\frac{1}{\sqrt{|G|}} \left[ \pi_A^* \left( \sqrt{|G|} G^{AB} \pi_B \right) \right] \Psi = 0.
\]  
(110)
If we only consider the Bianchi VII\(_{h=0}\) models with a cosmological constant our \(S(0)\) term simplifies to

\[
S^9_{(0)} = \frac{1}{3} e^{2(\alpha+\beta_+)} \cosh \left( 2\sqrt{3}\beta_- \right) - 3\Lambda e^{4\alpha \pm 2\sqrt{3}\beta_- - 2\beta_+}.
\]  

(111)

The \(\pm\) operator appropriate to our \(\beta_-\) term in (111) is a result of the fact that the Bianchi \(\text{VII}_{h=0}\) potential when only a cosmological constant is present is invariant under reflection of \(\beta_- \rightarrow \beta_-\). Via (15) and (111) we find the following \(S(1)\) term

\[
S^9_{(1)} = \frac{1}{2} \alpha (-B - 2) + 2\beta_+.
\]  

(112)

Using the well by now established methodology that we have presented throughout this paper we can show that a closed form solution to the WDW equation exists when our ordering parameter equals \(B = \pm 2\sqrt{3}i\) using just (16), (111), and (112).

As previously shown for an alternative operator ordering we can construct a closed form solution using just (111). The following solution to the Bianchi \(\text{VII}_{h=0}\) WDW equation with a cosmological constant which respects the operator ordering of (110) was generated by group averaging over the reflection symmetry \(\beta_- \rightarrow -\beta_-\) present in its potential

\[
\Psi = \left( e^{3\Lambda e^{4\alpha - 2(\sqrt{3}\beta_- + \beta_+)}} + e^{3\Lambda e^{4\alpha - 2(\sqrt{3}\beta_- + \beta_+)}} \right) e^{-\frac{1}{2} e^{2(\alpha + \beta_+)} \cosh(2\sqrt{3}\beta_-)}.
\]  

(113)

In order to see what effects an electromagnetic field has on our Bianchi \(\text{VII}_{h=0}\) wave function we will group average the wave function which satisfies (110) and is constructed using (103) despite the fact that the addition of an electromagnetic field breaks the reflection invariance in \(\beta_-\) of the potential. We are doing this so we can make a clear comparison between what happens when we just have a cosmological constant vs when we have both a cosmological constant and an electromagnetic field. Specifically we want to highlight how the electromagnetic field effects the anisotropy present in the universes represented by our wave functions. The actual wave function which has an electromagnetic field would only have one branch present.

\[
\Psi = \left( e^{3\Lambda e^{4\alpha - 2(\sqrt{3}\beta_- + \beta_+)}} + e^{3\Lambda e^{4\alpha + 2\sqrt{3}\beta_- - 2\beta_+}} \right) e^{\frac{1}{2} \left( -2e^{2(\alpha + \beta_+)} \cosh(2\sqrt{3}\beta_-) - 3b^2(\sqrt{3}\beta_- + 3\beta_+) \right)}
\]  

(114)
FIG. 20: Due to the electromagnetic field breaking the reflection symmetry under $\beta_-$ of the Bianchi VII$_h=0$ potential the actual wave function which satisfies (110) only has one ridge present as opposed to two, as is shown in figure 20(b). Both ridges are plotted so we can better compare the two wave functions.

XI. DISCUSSION

To begin analyzing our results we first need to adopt an interpretation for the wave functions we computed. Two interpretations of quantum mechanics which in the past have been used to extrapolate physics from Wheeler DeWitt wave functions within the context of quantum cosmology are the consistent histories approach [35] and pilot wave theory [36]. However, for our purposes, we will use the following admittedly naive interpretation which we will briefly outline. Even though we cannot interpret $|\psi|^2$ as a probability density due to the lack of a known dynamical unitary operator for the symmetry reduced Wheeler DeWitt equation, if we fix $\alpha$, and only consider wave functions which do not approach $\infty$ when their Misner variables approach $\pm \infty$, our wave functions are reminiscent of normalizable probability densities as can be seen from our plots. Each point in those plots at a fixed $\alpha$ represents a potential geometric configuration the universe can possess which is specified by the values of the Misner variables $(\alpha, \beta_+, \beta_-)$. Associated with each of those points in $\beta$ space at a fixed $\alpha$ is a value of $|\psi|^2$; it is not unreasonable to conjecture that the greater the value of $|\psi|^2$ is, the more likely a Bianchi II universe will possess the geometry given by the $\beta_+$ and $\beta_-$ Misner variables. For example if $|\psi(\alpha, \beta_1+, \beta_1-)|^2 > |\psi(\alpha, \beta_2+, \beta_2-)|^2$ we would interpret this to mean that a Bianchi II universe described by $\psi$ when it reaches a size
dictated by $\alpha$ is more likely to have a spatial geometry which possesses a level of anisotropy described by the values of $(\beta_{1+}, \beta_{1-})$ as opposed to $(\beta_{2+}, \beta_{2-})$.

A shortcoming of our interpretation is that it cannot assign numerical values of probability to a micro ensemble of Bianchi II universes with different values of $\alpha$, because $\int_{-\infty}^{\infty} |\psi|^2 d\beta_+ d\beta_-$ is not conserved in $\alpha$. Nonetheless we are picking this interpretation because it is intuitive for the solutions we are dealing with and facilitates the elucidation of the points the author wishes to make. In essence the author was inspired to pick this interpretation because he would like to let the bare solutions speak for themselves. The author strongly encourages future work to be done in extrapolating physics for the results presented in this paper using both the Bohemian approach and consistent histories, in conjunction to other quantitative approaches. What follows will be a rough qualitative description of what our wave functions are trying to tell us.

The plots for our 'ground' state asymptotic wave functions in figures (1-3) depict a wave travelling down the $\beta_+$ axis in the negative direction while the scale factor $e^{\alpha}$ grows. As our clock $\alpha$ ”ticks” forward in ”time”, this universe's geometric probability density will be peaked at an ever increasing negative value of $\beta_+$ and a roughly constant value of $\beta_-$. A variety of other asymptotic wave functions which behave differently could also be constructed using the methods outlined in III.

Moving on to our Bianchi II solutions with matter sources we notice in figures (4-5) that the electromagnetic field $(b^2)$ causes our wave function to travel in the negative $\beta_-$ direction. It was noted in [37] that magnetic fields within the context of LRS Bianchi I quantum cosmology induces anisotropy. This is clearly being mirrored for our particular case of the quantum Bianchi II models. However, the opposite can happen as well if our wave functions are centered on the positive portion of the $\beta_-$ axis because increasing the strength of the electromagnetic field would initially decrease anistropy by making our $\beta_-$ approach the origin for certain values of $\alpha$ which we take to be our internal clock. There is mathematical evidence that this duel nature of the electromagnetic field is a generic feature possessed by wave functions computed using this method as can be seen in the Bianchi IX models[23] and in results for the Taub models which will be released shortly. If an electromagnetic field was capable of doing something similar in our early universe then this duel behavior could have played a paramount role in nucleogensis and in the formation of seeds of anistropy.
In addition to inducing anistropy the electromangetic field also makes our wave functions thinner and thus more sharply peaked. This effect of making the wave function of the universe more sharply peaked could have played an important role in the early universe by causing quantum states which otherwise would be geometrically fuzzy, such as those whose wave function possesses multiple peaks to condense to a far more sharply defined state with one narrow central peak. In other words a primordial electromagnetic field might have played an important role in the early universe by facilitating a phase transition from a quantum universe to one that could be adequately described using classical mechanics.

One last feature to point out about the electromagnetic field is that its effects rapidly diminish as $\alpha$ grows. As can be seen in figures 6 and 7 increasing the strength of the electromagnetic field has far more milder effects on the wave function then it did in figures 4 and 5. This provides a quantum explanation for why an electromagnetic field might have played a large role in the early universe, but played a diminishing role as it grew in size.

For figure (10), in order for it to be of sufficient quality the author had to multiply the wave function by $10^{20}$, in actuality its magnitude is far less than the wave functions in figures (8) and (9). This is to be expected because a negative cosmological constant should act as a powerful force of attraction which resists the tendency of a universe to grow in size. Thus our wave function (82) ($\Lambda < 0$) decaying as $\alpha$ grows indicates that the likelihood that this universe will reach a state when $\alpha >> 0$ is low. Another feature of our figures (8-10) is that as $\alpha$ grows they become thinner. This shows that as $\alpha$ grows larger the universe that these wave function are supposed to represent become somewhat less fuzzy. Geometric fuzziness is a feature associated with the uncertainty relation of our minisuperspace as a result of quantum mechanics. As a result the larger a universe becomes the less we expect those quantum features of fuzziness to be present.

Our leading order 'excited' states when $\Lambda \neq 0$, figures (11-14), can be interpreted similarly to our other states. One noticeable difference is that their geometry is more "fuzzy" because the wave functions which describe this universe have multiple ridges/peaks. This is most clearly illustrated in the $\alpha = 0$ case where we can see three distinct ridges, one large ridge and two smaller ridges. Figure (14) further shows this. The potential geometries this Bianchi II universe can take on are located on one of these three visible ridges. As $\alpha$ grows those ridges appear to fuse and the geometries become slightly more "sharp", while also becoming more unlikely to occur due to the magnitude of the wave function decaying.
For the non-commutative quantum Bianchi II models our wave function behaves similarly to other non-commutative quantum cosmological models \cite{33,19} in the sense that multiple peaks are present when $\theta_i \neq 0$, and only one defined peak is present when $\theta_i = 0$. These multiple peaks indicate that non-commutativity in the early universe could facilitate the creation of many possible states which a quantum universe can tunnel into. Our universe might have been one of those states; and if it wasn’t for non-commutativity it might have been exceedingly unlikely for our universe to tunnel into the state which allowed it to evolve in the way it has.

When we include our electromagnetic field in figure (17) it doesn’t have much of an effect. However because we simplified our wave function by setting all three non-commutative parameters $\theta_i$ equal to each other we only looked at a small subset of possible quantum non-commutative universes. There very well could exist a region in $\theta$ space such that a primordial electromagnetic field has a profound effect on what states the early universe can tunnel into. Studying the the full effects of both non-communaitivty and primordial electromagnetic fields in the early universe could shed much light on how our homogeneous and isotropic universe came to be.

Our vacuum Bianchi VII$_{h=0}$ ‘excited’ states(figures 18-19) are two Gaussian like peaks which travel in the negative $\beta_+$ direction. Thus for a specific value of $\alpha$ our wave functions represent a universe which can tunnel in between two relatively defined anisotropic states. For our quantum Bianchi VII$_{h=0}$ models with matter sources(figures 20(a)-20(b)) it can be seen that when the electromagnetic field $b^2$ is zero that our wave function is peaked around $\beta_+ = \frac{1}{2}$ and $\beta_- = 0$. However when our electromagnetic field increases in strength the ridges overtake the aforementioned peak and our wave function represents a fuzzier and more anisotropic universe. Making a wave function of the universe less geometrically defined is another potential effect that an electromagnetic field can have. Within the context of non-commutative quantum cosmology this could mean creating additional peaks which otherwise wouldn’t be present.

XII. CONCLUDING REMARKS

In this paper we found many new solutions to the Wheeler DeWitt equations for the quantum Bianchi II and VII$_{h=0}$ models when a cosmological constant, an electromagnetic
field and a stiff matter term were present. By doing so we greatly expanded upon the known number of closed form solutions to the Wheeler Dewitt equation and provided results which can shed light on how matter sources affects the evolution of a quantum universe. Notably they further point[23] to a whole host of generic behaviors that are imparted on to the wave function of the universe from a primordial electromagnetic field. In order for the results in this paper to be potentially more relevant to our own universe in its early infancy it is crucial that the analysis that we carried out here be duplicated for other anisotropic quantum cosmologies models[38, 39] to establish how typical these effects are for these types of solutions.

In addition our results further facilitate the development of a rigorous understanding of 'excited' states in quantum cosmology. We have also shown that the Euclidean-signature semi classical method is apt for finding solution to the Wheeler DeWitt equation when matter sources are present. These results further show the immense utility of the Euclidean-signature semi classical method in solving Lorentzian signature problems. Thus the author very much looks forward to seeing what future applications of this method will produce.

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