Shafarevich-Tate groups of abelian varieties

Igor V. Nikolaev

Abstract. The Shafarevich-Tate group \( \Sha(A) \) measures the failure of the Hasse principle for an abelian variety \( A \). Using a correspondence between the abelian varieties and the higher dimensional non-commutative tori, we prove that \( \Sha(A) \cong \Cl(\Lambda) \oplus \Cl(\Lambda) \) or \( \Sha(A) \cong \left( \mathbb{Z}/2^k \mathbb{Z} \right) \oplus \Cl_{odd}(\Lambda) \oplus \Cl_{odd}(\Lambda) \), where \( \Cl(\Lambda) \) is the ideal class group of a ring \( \Lambda \) associated to the K-theory of the non-commutative tori and \( 2^k \) divides the order of \( \Cl(\Lambda) \). The case of elliptic curves with complex multiplication is considered in detail.

1. Introduction

The study of Diophantine equations is the oldest part of mathematics. If such an equation has an integer solution, then using the reduction modulo any prime \( p \), one gets a solution of the equation lying in the finite field \( \mathbb{F}_p \) and a solution in the field of real numbers \( \mathbb{R} \). The equation is said to satisfy the Hasse principle, if the converse is true. For instance, the quadratic equations satisfy the Hasse principle, while the equation \( x^4 - 17 = 2y^2 \) has a solution over \( \mathbb{R} \) and each \( \mathbb{F}_p \), but no rational solutions. Measuring the failure of the Hasse principle is a difficult open problem of number theory.

Let \( A_K \) be an abelian variety over the number field \( K \) which we assume to be simple, i.e. the \( A_K \) has no proper sub-abelian varieties over \( K \). Denote by \( K_v \) the completion of \( K \) at the (finite or infinite) place \( v \). Consider the Weil-Châtelet group \( WC(A_K) \) of the abelian variety \( A_K \) and the group homomorphism:

\[
WC(A_K) \to \prod_v WC(A_{K_v}).
\]

The Shafarevich-Tate group \( \Sha(A_K) \) of \( A_K \) is defined as the kernel of homomorphism (1.1). The variety \( A_K \) satisfies the Hasse principle, if and only if, the group \( \Sha(A_K) \) is trivial. Little is known about the \( \Sha(A_K) \) in general. The existing methods include an evaluation of the analytic order of \( \Sha(A_K) \) based on the second part of the Birch and

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Swinnerton-Dyer Conjecture and an exact calculation of the p-part of \( \text{III}(\mathcal{A}_K) \), see e.g. [Rubin 1989] [R].

The aim of our note is calculation of the group \( \text{III}(\mathcal{A}_K) \) based on a functor \( F \) between the \( n \)-dimensional abelian varieties \( \mathcal{A}_K \) and the \( 2n \)-dimensional non-commutative tori \( \mathcal{A}_\Theta \), i.e. the \( C^* \)-algebras generated by the unitary operators \( U_1, \ldots, U_{2n} \) satisfying the commutation relations \( \{ U_i U_j = e^{2\pi i \theta_{ij}} U_i U_j \mid 1 \leq i, j \leq 2n \} \) described by a skew-symmetric matrix

\[
\Theta = \begin{pmatrix}
0 & \theta_{12} & \cdots & \theta_{1,2n} \\
-\theta_{12} & 0 & \cdots & \theta_{2,2n} \\
\vdots & \vdots & \ddots & \vdots \\
-\theta_{1,2n} & -\theta_{2,2n} & \cdots & 0
\end{pmatrix} \in GL_{2n}(\mathbb{R}).
\]

The functor maps isomorphic abelian varieties \( \mathcal{A} \) and \( \mathcal{A}' \) to the Morita equivalent algebras \( \mathcal{A}_\Theta = F(\mathcal{A}) \) and \( \mathcal{A}'_\Theta = F(\mathcal{A}') \), see e.g. [N, Section 1.3]. Restricting \( F \) to the simple abelian varieties \( \mathcal{A}_K \), one gets the non-commutative tori \( \mathcal{A}_\Theta(k) = F(\mathcal{A}_K) \), where \( \Theta(k) \) is the matrix (1.2) defined over a number field \( k \subset \mathbb{R} \).

Roughly speaking, the idea is this. To calculate the \( \text{III}(\mathcal{A}_K) \), we recast (1.1) in terms of the K-theory of algebra \( \mathcal{A}_\Theta(k) = F(\mathcal{A}_K) \). Recall that an isomorphism class of \( \mathcal{A}_\Theta(k) \) is defined by the triple \( (K_0(\mathcal{A}_\Theta(k)), K_0^+(\mathcal{A}_\Theta(k)), \Sigma(\mathcal{A}_\Theta(k))) \) consisting of the \( K_0 \)-group, the positive cone \( K_0^+ \) and the scale \( \Sigma \) of the algebra \( \mathcal{A}_\Theta(k) \) [Blackadar 1986] [B, Section 6], see also Section 2.3. It is proved, that \( \Sigma(\mathcal{A}_\Theta(k)) \) is a torsion group, such that \( WC(\mathcal{A}_K) \cong \Sigma(\mathcal{A}_\Theta(k)) \) (lemma 3.1). The RHS of (1.1) corresponds to the crossed product \( C^* \)-algebra \( \mathcal{A}_\Theta(k) \rtimes_{L_v} \mathbb{Z} \), where the notation is explained in Section 2.4.2. It is proved, that

\[
\prod_v WC(\mathcal{A}_K) \cong \prod_v K_0 \left( \mathcal{A}_\Theta(k) \rtimes_{L_v} \mathbb{Z} \right) \quad \text{(corollary 3.4)}.
\]

Thus (1.1) can be written in the form:

\[
\Sigma(\mathcal{A}_\Theta(k)) \rightarrow \prod_v K_0 \left( \mathcal{A}_\Theta(k) \rtimes_{L_v} \mathbb{Z} \right). \tag{1.3}
\]

Both sides of (1.3) are functions of a single positive matrix \( B \in GL(2n, \mathbb{Z}) \), see definition 2.1. However, the LHS of (1.3) depends on the similarity class of \( B \), while the RHS of (1.3) depends on the characteristic polynomial of \( B \). This observation is critical, since it puts the elements of \( \text{III}(\mathcal{A}_K) \) into a one-to-one correspondence with the similarity classes of matrices having the same characteristic polynomial. The latter is an old problem of linear algebra and it is known, that the number of such classes is finite. They correspond to the ideal classes \( Cl(\Lambda) \) of an order \( \Lambda \) in the field \( k = \mathbb{Q}(\lambda_B) \), where \( \lambda_B \) is the Perron-Frobenius eigenvalue of matrix \( B \) [Latimer & MacDuffee 1933] [LM]. Let \( Cl(\Lambda) \cong (\mathbb{Z}/2^k\mathbb{Z}) \oplus Cl_{\text{odd}}(\Lambda) \) for \( k \geq 0 \). Using the Atiyah pairing between the K-theory and the K-homology, one gets the following result.
Theorem 1.1. The Shafarevich-Tate group of an abelian variety $A_K$ is a finite group given by the formulas:

\[(1.4) \quad \text{III}(\mathcal{A}_K) \cong \begin{cases} \text{Cl} (\Lambda) \oplus \text{Cl} (\Lambda), & \text{if } k \text{ is even}, \\ \left(\mathbb{Z}/2^k\mathbb{Z}\right) \oplus \text{Cl odd} (\Lambda) \oplus \text{Cl odd} (\Lambda), & \text{if } k \text{ is odd}. \end{cases} \]

The article is organized as follows. Preliminary facts can be found in Section 2. The proof of theorem 1.1 is given in Section 3. As an illustration, we consider the abelian varieties with complex multiplication in Section 4.

2. Preliminaries

We briefly review the abelian varieties, the Shafarevich-Tate groups and the $n$-dimensional non-commutative tori. For a detailed exposition, we refer the reader to [Mumford 1983] [M2, Chapter 2, §4], [Cassels 1966] [C], [Blackadar 1986] [B, Section 6] and [N, Section 3.4.1], respectively.

2.1. Abelian varieties. Let $C^n$ be the $n$-dimensional complex space and $\Lambda$ be a lattice in the underlying $2n$-dimensional real space $\mathbb{R}^{2n}$. The complex torus $C^n/\Lambda$ is said to be an $n$-dimensional (principally polarized) abelian variety $\mathcal{A}$, if it admits an embedding into a projective space. In other words, the set $\mathcal{A}$ is both an additive abelian group and a complex projective variety. Recall that the Siegel upper half-space $\mathbb{H}_n$ consists of the symmetric $n \times n$ matrices with complex entries $\tau_i$, such that:

$$\mathbb{H}_n := \left\{ \tau = (\tau_i) \in C^{n(n+1)/2} \mid \Im(\tau_i) > 0 \right\}.$$  

The points of $\mathbb{H}_n$ parametrize the $n$-dimensional abelian varieties, i.e. $\mathcal{A} \cong \mathcal{A}_{\tau}$, for some $\tau \in \mathbb{H}_n$. Denote by $Sp(2n, \mathbb{R})$ the $2n$-dimensional symplectic group, i.e. a subgroup of the linear group defined by the equation:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where $A, B, C, D$ are the $n \times n$ matrices with real entries and $I$ is the identity matrix. It is well known, that the abelian varieties $\mathcal{A}_{\tau}$ and $\mathcal{A}_{\tau'}$ are isomorphic, whenever

$$\tau' = \frac{A\tau + B}{C\tau + D}, \quad \text{where} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z}).$$

2.2. Shafarevich-Tate group. Let $K \subset C$ be a number field and let $\mathcal{A}_K$ be an abelian variety over $K$. The principal homogeneous space of $\mathcal{A}_K$ is a variety $\mathcal{G}$ over $K$ with a map $\mu : \mathcal{G} \times \mathcal{A}_K \to \mathcal{G}$ satisfying (i) $\mu(x, 0) = x$, (ii) $\mu(x, a + b) = \mu(\mu(x, a), b)$ for all $x \in \mathcal{G}$ and $a, b \in \mathcal{A}_K$ and (iii) for all $x \in \mathcal{G}$ the map $a \mapsto \mu(x, a)$ is a $K$-isomorphism between $\mathcal{A}_K$ and $\mathcal{G}$. The homogeneous spaces $(\mathcal{G}, \mu)$ and $(\mathcal{G}', \mu')$ are equivalent, if there is a $K$-isomorphism $\mathcal{G} \to \mathcal{G}'$ compatible with the maps $\mu$ and
\(\mu\) \(\mu\) \(\mu\). If \(\mathcal{C}\) has \(K\)-points, then the equivalence class \((\mathcal{C}, \mu)\) is said to be trivial. The non-trivial classes \((\mathcal{C}, \mu)\) correspond to the varieties \(\mathcal{C}\) without \(K\)-points, but with the \(L\)-points, where \(L\) is a Galois extension of \(K\). The Weil-Châtelet group \(WC(\mathcal{A}_K)\) is an additive abelian group of the equivalence classes \((\mathcal{C}, \mu)\), where the trivial class corresponds to the zero element of the group, see [Cassels 1966] [C] for the details. The \(WC(\mathcal{A}_K)\) is a torsion group, i.e. each element of the group has finite order. It is not hard to see, that if \(K \subset K_v\) is an extension of \(K\) by completion at the place \(v\), then there exists a natural group homomorphism:

\[
(2.3) \quad WC(\mathcal{A}_K) \rightarrow WC(\mathcal{A}_{K_v}).
\]

The kernel of \((2.3)\) consists of those homogeneous spaces having the \(K_v\)-points, but no \(K\)-points. The Shafarevich-Tate group \(\text{III}(\mathcal{A}_K)\) of \(\mathcal{A}_K\) is defined as the kernel of homomorphism \(WC(\mathcal{A}_K) \rightarrow \prod_v WC(\mathcal{A}_{K_v})\). Thus the abelian variety \(\mathcal{A}_K\) satisfies the Hasse principle, if and only if, the group \(\text{III}(\mathcal{A}_K)\) is trivial.

2.3. \(K\)-theory of \(C^*\)-algebras. The \(C^*\)-algebra \(\mathcal{A}\) is an algebra over \(\mathbb{C}\) with a norm \(a \mapsto ||a||\) and an involution \(a \mapsto a^*\) such that it is complete with respect to the norm and \(||ab|| \leq ||a|| ||b||\) and \(||a^*a|| = ||a||^2\) for all \(a, b \in \mathcal{A}\). Any commutative \(C^*\)-algebra is isomorphic to the algebra \(C_0(X)\) of continuous complex-valued functions on some locally compact Hausdorff space \(X\). Any other algebra \(\mathcal{A}\) can be thought of as a non-commutative topological space. By \(M_\infty(\mathcal{A})\) one understands the algebraic direct limit of the \(C^*\)-algebras \(M_n(\mathcal{A})\) under the embeddings \(a \mapsto \text{diag}(a, 0)\). The direct limit \(M_\infty(\mathcal{A})\) can be thought of as the \(C^*\)-algebra of infinite-dimensional matrices whose entries are all zero except for a finite number of the non-zero entries taken from the \(C^*\)-algebra \(\mathcal{A}\). Two projections \(p, q \in M_\infty(\mathcal{A})\) are equivalent, if there exists an element \(v \in M_\infty(\mathcal{A})\), such that \(p = v^*v\) and \(q = vv^*\). The equivalence class of projection \(p\) is denoted by \([p]\). We write \(V(\mathcal{A})\) to denote all equivalence classes of projections in the \(C^*\)-algebra \(M_\infty(\mathcal{A})\), i.e. \(V(\mathcal{A}) := \{[p] : p = p^* = p^2 \in M_\infty(\mathcal{A})\}\). The set \(V(\mathcal{A})\) has the natural structure of an abelian semi-group with the addition operation defined by the formula \([p] + [q] := \text{diag}(p, q) = [p' \oplus q']\), where \(p' \sim p, \ q' \sim q\) and \(p' \perp q'\). The identity of the semi-group \(V(\mathcal{A})\) is given by \([0]\), where \(0\) is the zero projection. By the \(K_0\)-group \(K_0(\mathcal{A})\) of the unital \(C^*\)-algebra \(\mathcal{A}\) one understands the Grothendieck group of the abelian semi-group \(V(\mathcal{A})\), i.e. a completion of \(V(\mathcal{A})\) by the formal elements \([p] - [q]\). The image of \(V(\mathcal{A})\) in \(K_0(\mathcal{A})\) is a positive cone \(K_0^+(\mathcal{A})\) defining the order structure \(\leq\) on the abelian group \(K_0(\mathcal{A})\). The pair \((K_0(\mathcal{A}), K_0^+(\mathcal{A}))\) is known as a dimension group of the \(C^*\)-algebra \(\mathcal{A}\). The scale \(\Sigma(\mathcal{A})\) is the image in \(K_0^+(\mathcal{A})\) of the equivalence classes of projections in the \(C^*\)-algebra \(\mathcal{A}\). The \(\Sigma(\mathcal{A})\) is a generating, hereditary and directed subset of \(K_0^+(\mathcal{A})\), i.e. (i) for each \(a \in K_0^+(\mathcal{A})\)
there exist \( a_1, \ldots, a_r \in \Sigma(A) \) such that \( a = a_1 + \cdots + a_r \); (ii) if \( 0 \leq a \leq b \in \Sigma(A) \), then \( a \in \Sigma(A) \) and (iii) given \( a, b \in \Sigma(A) \) there exists \( c \in \Sigma(A) \), such that \( a, b \leq c \). Each scale can always be written as \( \Sigma(A) = \{ a \in K_0^+(A) \mid 0 \leq a \leq u \} \), where \( u \) is an order unit of \( K_0^+(A) \). The pair \( (K_0(A), K_0^+(A)) \) and the triple \( (K_0(A), K_0^+(A), \Sigma(A)) \) are invariants of the Morita equivalence and isomorphism class of the \( C^* \)-algebra \( A \), respectively.

### 2.4. Higher dimensional non-commutative tori.

#### 2.4.1. Definition and properties.

The even-dimensional non-commutative torus \( A_\Theta \) is the universal \( C^* \)-algebra generated by the unitary operators \( U_1, \ldots, U_{2n} \) and satisfying the relations \( \{ U_j U_i = e^{2\pi i \Theta_{ij}} U_i U_j \mid 1 \leq i, j \leq 2n \} \), where \( \Theta = (\Theta_{ij}) \) is a skew-symmetric real matrix \((1.2)\). Let \( SO(n, n; \mathbb{Z}) \) be a subgroup of the group \( GL(2n, \mathbb{Z}) \) preserving the quadratic form \( x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n} \). The matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(n, n; \mathbb{Z}) \), if and only if,

\[
\begin{align*}
\Theta^T D + C^T B &= I, \\
A^T C + C^T A &= 0 = B^T D + D^T B,
\end{align*}
\]

where \( A, B, C, D \in GL(n, \mathbb{Z}) \) and \( I \) is the identity matrix. The non-commutative tori \( A_\Theta \) and \( A_{\Theta'} \) are Morita equivalent, whenever

\[
\Theta' = A \Theta + B \\
C \Theta + D,
\]

where \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(n, n; \mathbb{Z}) \).

The reader can verify, that conditions \((2.4)\) follow from the matrix equation \((2.1)\), but not vice versa. In other words, one gets an inclusion \( Sp(2n, \mathbb{Z}) \subseteq SO(n, n; \mathbb{Z}) \). This observation prompts the construction of a functor \( F \) from the \( n \)-dimensional abelian varieties \( \mathcal{A} \) to the \( 2n \)-dimensional non-commutative tori \( A_\Theta \), such that if \( \mathcal{A} \) and \( \mathcal{A}' \) are isomorphic abelian varieties, the \( A_\Theta = F(\mathcal{A}) \) and \( A_{\Theta'} = F(\mathcal{A}') \) will be the Morita equivalent non-commutative tori, see \((2.2)\) and \((2.5)\). The restriction of \( F \) to the simple abelian varieties \( \mathcal{A}_K \) corresponds to the non-commutative tori \( A_{\Theta(k)} = F(\mathcal{A}_K) \), where \( \Theta(k) \) is the matrix over a number field \( k \) of degree \( \deg(k|\mathbb{Q}) = 2n \).

#### 2.4.2. Crossed product \( A_{\Theta(k)} \rtimes \Lambda \mathbb{Z} \).

Let \( A_\Theta \) be a \( 2n \)-dimensional non-commutative torus endowed with the canonical trace \( \tau : A_\Theta \to \mathbb{C} \). Since \( K_0(A_\Theta) \cong \mathbb{Z}^{2n-1} \) and \( \tau_* : K_0(A_\Theta) \to \mathbb{R} \) is a homomorphism induced by \( \tau \), one gets a \( \mathbb{Z} \)-module \( \Lambda := \tau_* (K_0(A_\Theta)) \subseteq \mathbb{R} \) of rank \( 2^{n-1} \). The generators of \( \Lambda \) belong to the ring \( \mathbb{Z}[\theta_{ij}] \). In what follows, we assume that \( \theta_{ij} \in \mathbb{K} \). In this case one gets the algebraic constraints and the rank of \( \Lambda \) is equal to \( 2n \) [N, Remark 6.6.1]. In other words, \( \Lambda \cong \mathbb{Z} + \mathbb{Z} \theta_1 + \cdots + \mathbb{Z} \theta_{2^{n-1}} \), where \( \theta_i \in \mathbb{K} \).

**Definition 2.1.** We shall write \( B \in GL(2n, \mathbb{Z}) \) to denote a positive matrix, such that \( (1, \theta_1, \ldots, \theta_{2n-1}) \) is the normalized Perron-Frobenius eigenvector of \( B \).
Let \( \pi(n) \) be an integer-valued function defined in \([N, \text{Section 6.5.3}]\). Consider the characteristic polynomial \( \text{Char} \left(B^{x(p)}\right) = x^{2n} - a_1 x^{2n-1} - \cdots - a_{2n-1} x - 1 \) and a matrix

\[
L_v = \begin{pmatrix}
a_1 & 1 & \ldots & 0 & 0 \\
a_2 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{2n-1} & 0 & \ldots & 0 & 1 \\
p & 0 & \ldots & 0 & 0
\end{pmatrix},
\]

where \( p \) is the prime underlying \( v \). The \( L_v \) defines an endomorphism of the algebra \( A_{\Theta(k)} \) by its action on the generators \( U_1, \ldots, U_{2n} \). We consider the crossed product \( C^*\)-algebra \( A_{\Theta(k)} \rtimes_{L_v} Z \) associated to such an action.

**Remark 2.2.** There exists an isomorphism:

\[
K_0 \left(A_{\Theta(k)} \rtimes_{L_v} Z\right) \cong A_{F_p},
\]

where \( A_{F_p} \) is a localization of the \( A_K \) at the prime ideal \( \mathcal{P} \subset K \) over \( p \) \([N, \text{Section 6.6}]\). Formula (2.8) hints that the crossed product \( A_{\Theta(k)} \rtimes_{L_v} Z \) is a recast of the variety \( A_{K_v} \).

### 3. Proof of theorem 1.1

We refer the reader to Section 1 for an outline of the proof. The detailed argument is given below. We split the proof in a series of lemmas.

**Lemma 3.1.** The scale \( \Sigma(A_{\Theta(k)}) \) has the structure of a torsion group, so that \( WC(A_K) \cong \Sigma(A_{\Theta(k)}) \), where \( A_{\Theta(k)} = F(A_K) \).

**Proof.** (i) For the sake of clarity, we restrict to the case \( n = 1 \), i.e. when the variety \( \mathcal{A} \) is an elliptic curve \( \mathcal{E} \). In this case matrix (1.2) can be written as:

\[
(3.1) \quad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.
\]

We shall denote the corresponding non-commutative torus by \( A_\phi \). One gets

\( \deg (k|Q) = 2 \), i.e. \( k \) is a real quadratic field. Therefore \( F(\mathcal{E}_K) = A_\phi \), where \( \theta \in k \) is a quadratic irrationality.

To prove that the scale \( \Sigma(A_\phi) \) has the structure of a torsion group, we shall use the Minkowski question-mark function \( ?(x) \) \([\text{Minkowski 1904}] \) \([M1, \text{p. 172}]\). Such a function is known to map quadratic irrational numbers of the unit interval to the rational numbers of the same interval preserving their natural order. This observation implies \( \Sigma(A_\phi) \subset Q/Z \), i.e. the scale \( \Sigma(A_\phi) \) is a subgroup of the torsion group \( Q/Z \). Let us pass to a detailed argument.
Let \( \tau \) be the canonical trace on the algebra \( \mathcal{A}_\theta \). Since \( K_0(\mathcal{A}_\theta) \cong \mathbb{Z}^2 \), one gets \( \tau_\ast(K_0(\mathcal{A}_\theta)) = \mathbb{Z} + \mathbb{Z}\theta \) and \( \tau_\ast(K_0^+(\mathcal{A}_\theta)) = \{ m + n\theta \geq 0 \mid m, n \in \mathbb{Z} \} \). It is known that \( \tau_\ast(u) = 1 \), where \( u \in K_0^+(\mathcal{A}_\theta) \) is an order unit. Therefore the traces on the scale \( \Sigma(\mathcal{A}_\theta) \) are given by the formula:

\[
(3.2) \quad \tau_\ast(\Sigma(\mathcal{A}_\theta)) = [0, 1] \cap \mathbb{Z} + \mathbb{Z}\theta.
\]

Recall that the Minkowski question-mark function is defined by the convergent series

\[
(3.3) \quad ?(x) := a_0 + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2a_1 + \cdots + a_k},
\]

where \( x = [a_0, a_1, a_2, \ldots] \) is the continued fraction of the irrational number \( x \). The \( ?(x) : [0, 1] \rightarrow [0, 1] \) is a monotone continuous function with the following properties: (i) \( ?(0) = 0 \) and \( ?(1) = 1 \); (ii) \( ?(\mathbb{Q}) = \mathbb{Z}[\frac{1}{2}] \) are dyadic rationals and (iii) \( ?(\mathcal{Q}) = \mathbb{Q} - \mathbb{Z}[\frac{1}{2}] \), where \( \mathcal{Q} \) are quadratic irrational numbers [Minkowski 1904] [M1, p. 172].

Consider subset \( (3.2) \) of the interval \([0, 1]\). Since \( \theta \) is a quadratic irrationality, we conclude that the set \( \tau_\ast(\Sigma(\mathcal{A}_\theta)) \subset \mathcal{Q} \). By property (iii) of the Minkowski question-mark function, one gets an inclusion \( ?(\tau_\ast(\Sigma(\mathcal{A}_\theta))) \subset \mathcal{Q}/\mathbb{Z} \). Thus we constructed a one-to-one map:

\[
(3.4) \quad \Sigma(\mathcal{A}_\theta) \longrightarrow \mathcal{Y} \subset \mathcal{Q}/\mathbb{Z},
\]

where \( \mathcal{Y} := ?(\tau_\ast(\Sigma(\mathcal{A}_\theta))) \). It follows from \( (3.4) \), that \( \Sigma(\mathcal{A}_\theta) \) is a torsion group.

Remark 3.2. Formula \( (3.4) \) is part of a duality between the \( K \)-theory of non-commutative tori and the Galois cohomology of abelian varieties. Such a study is beyond the scope of present paper.

(ii) Let us show, that \( WC(\mathcal{E}_K) \cong \Sigma(\mathcal{A}_\theta) \), where \( \cong \) is an isomorphism of the torsion groups. Indeed, let \( \mathcal{C} \) be the principal homogeneous space of the elliptic curve \( \mathcal{E}_K \). It is known, that \( \mathcal{C} \cong \mathcal{E}_K' \) is the twist of \( \mathcal{E}_K \), i.e. an isomorphic but not \( K \)-isomorphic elliptic curve \( \mathcal{E}_K' \). It follows, that the \( \mathcal{A}_\theta \) and \( \mathcal{A}_\theta' = F(\mathcal{E}_K') \) are Morita equivalent, but not isomorphic \( C^* \)-algebras. This would imply \( (K_0(\mathcal{A}_\theta), K_0^+(\mathcal{A}_\theta)) \cong (K_0(\mathcal{A}_\theta'), K_0^+(\mathcal{A}_\theta')) \), but \( (K_0(\mathcal{A}_\theta), K_0^+(\mathcal{A}_\theta), \Sigma(\mathcal{A}_\theta)) \not\cong (K_0(\mathcal{A}_\theta), K_0^+(\mathcal{A}_\theta), \Sigma(\mathcal{A}_\theta')) \). In other words, the principal homogeneous spaces of elliptic curve \( \mathcal{E}_K' \) are classified by the scales of the algebra \( \mathcal{A}_\theta \).

On the other hand, each element of \( K_0^+(\mathcal{A}_\theta) \) can be taken for an order unit \( u \) of the dimension group \( (K_0(\mathcal{A}_\theta), K_0^+(\mathcal{A}_\theta)) \). Since \( \Sigma(\mathcal{A}_\theta) = \{ a \in K_0^+(\mathcal{A}_\theta) \mid 0 \leq a \leq u \} \), we conclude that the elements of \( K_0^+(\mathcal{A}_\theta) \) classify all scales of the algebra \( \mathcal{A}_\theta \). We can always restrict to the generating subset \( \Sigma(\mathcal{A}_\theta) \subset K_0^+(\mathcal{A}_\theta) \), since all other elements of the positive cone \( K_0^+(\mathcal{A}_\theta) \) correspond to the finite unions \( \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k \) of the generating homogeneous spaces \( \mathcal{C}_i \). It remains to notice, that
equivalence classes of the principal homogeneous spaces \((E, \mu)\) coincide with the isomorphism classes of the algebra \(A_\theta\). In other words, one gets an isomorphism of the torsion groups:

\[
WC(E_K) \cong \Sigma(A_\theta).
\]

(iii) The general case \(n > 1\) is proved by an adaption of the argument for the case \(n = 1\). Notice that one must use the Perron-Frobenius \(n\)-dimensional continued fractions and a higher-dimensional analog of the Minkowski question-mark function. The details are left to the reader.

Lemma 3.1 is proved. □

**Lemma 3.3.** \(WC(\mathfrak{A}_K) \cong K_0 \left( A_{\Theta(k)} \times L_\nu \mathbb{Z} \right)\), where \(A_{\Theta(k)} = F(\mathfrak{A}_K)\).

**Proof.** The abelian group \(WC(\mathfrak{A}_K)\) is known to be finite, see e.g. [Borel & Serre 1964] [BS]. Specifically,

\[
WC(\mathfrak{A}_K) \cong (\widehat{\mathfrak{A}}_K)^*,
\]

where \(\widehat{\mathfrak{A}}_K\) are the rational points of \(K\) on the dual abelian variety \(\mathfrak{A}_K\) and \((\widehat{\mathfrak{A}}_K)^*\) is the character group of \(\widehat{\mathfrak{A}}_K\) [Tate 1958] [T].

Recall that \(K_0 \left( A_{\Theta(k)} \times L_\nu \mathbb{Z} \right) \cong \mathcal{A}(F_p)\), where \(A_{\Theta(k)} = F(\mathfrak{A}_K)\) and \(\mathcal{A}(F_p)\) is a localization of the abelian variety \(\mathfrak{A}_K\) at the prime ideal \(\mathfrak{P}\) over prime \(p\), see remark 2.2. In view of the Hensel Lemma, the points of \(\mathcal{A}(F_p)\) are the rational points of \(\mathfrak{A}_K\), where \(v\) is the place over prime \(p\). Since the torsion points of \(\widehat{\mathfrak{A}}_K\) and the torsion points of \(\widehat{\mathfrak{A}}_K\) coincide, we conclude that \(\widehat{\mathfrak{A}}_K \cong \mathcal{A}(F_p)\). On the other hand, since finite abelian groups are Pontryagin self-dual, one gets \((\widehat{\mathfrak{A}}_K)^* \cong \mathcal{A}(F_p)\). Therefore isomorphism (3.6) can be written as:

\[
WC(\mathfrak{A}_K) \cong K_0 \left( A_{\Theta(k)} \times L_\nu \mathbb{Z} \right) .
\]

Lemma 3.3 is proved. □

**Corollary 3.4.** \(\prod_v WC(\mathfrak{A}_K) \cong \prod_v K_0 \left( A_{\Theta(k)} \times L_\nu \mathbb{Z} \right)\).

**Proof.** One takes the direct sum of finite groups \(WC(\mathfrak{A}_K)\) over all non-archimedean places \(v\) of the field \(K\). The corollary follows from isomorphism (3.7).

**Lemma 3.5.**

\[
\Pi(\mathfrak{A}_K) = \begin{cases} 
\text{Cl} (\Lambda) \oplus Cl (\Lambda), & \text{if } k \text{ is even}, \\
\left( \mathbb{Z}/2^k\mathbb{Z} \right) \oplus Cl_{\text{odd}} (\Lambda) \oplus Cl_{\text{odd}} (\Lambda), & \text{if } k \text{ is odd}.
\end{cases}
\]

**Proof.** For the sake of clarity, let us outline the main idea. In view of lemma 3.1 and corollary 3.4, one can replace (1.1) by a group homomorphism:

\[
\Sigma (A_{\Theta(k)}) \to \prod_v K_0 \left( A_{\Theta(k)} \times L_\nu \mathbb{Z} \right) .
\]
To calculate the kernel $\mathcal{X}_\Theta(A)$ of (3.8), recall that the algebra $\mathcal{A}_\Theta(k)$ at the LHS of (3.8) depends on the similarity class of a single matrix $B \in GL(2n, \mathbb{Z})$, see definition 2.1. On the other hand, the RHS of (3.8) depends solely on the characteristic polynomial of $B$. Roughly speaking, the problem of kernel of (3.8) reduces to an old question of the linear algebra: how many non-similar matrices with the same characteristic polynomial are there in $GL(2n, \mathbb{Z})$? This problem was solved by [Latimer & MacDuffee 1933] [LM]. Their theorem says that the similarity classes of matrices $B \in GL(2n, \mathbb{Z})$ with the fixed polynomial $\text{Char}(B)$ are in one-to-one correspondence with the ideal classes $\text{Cl}(\Lambda)$ of an order $\Lambda$ of a number field $k$ generated by the eigenvalues of $B$. Coupled with the fact, that the scale $\Sigma(\mathcal{A}_\Theta(k))$ contains the scales corresponding to all similarity classes, the Latimer-MacDuffee Theorem implies that the (half of) kernel of (3.8) is isomorphic to the group $\text{Cl}(\Lambda)$. The rest of the proof follows from the Atiyah pairing between the K-theory and the K-homology. We pass to a step by step argument.

(i) We use notation of Sections 2.3 and 2.4.2. Recall that the positive cone $K^+_0(\mathcal{A}_\Theta(k))$ and the scale $\Sigma(\mathcal{A}_\Theta(k))$ can be recovered from (2.6) by solving the inequality $\Lambda \geq 0$ and $0 \leq \Lambda \leq 1$, respectively. Since $\theta_i \in k$ are components of the normalized Perron-Frobenius eigenvector of a matrix $B \in GL(2n, \mathbb{Z})$, we conclude that the similarity class of $B$ defines the algebra $\mathcal{A}_\Theta(k)$ up to an isomorphism, and vice versa. The set of all pairwise non-similar matrices with the characteristic polynomial $\text{Char}(B)$ will be denoted by $B_1, \ldots, B_h$.

(ii) Let us show, that $\Lambda_1 \subset \cdots \subset \Lambda_h$, where $\Lambda_i$ are $\mathbb{Z}$-modules corresponding to the matrices $B_i$. Indeed, since $B_i$ has the same characteristic polynomial as $B$, we conclude that the components of the normalized Perron-Frobenius eigenvector of $B_i$ must lie in the number field $k$. Therefore $\Lambda_i$ is a full $\mathbb{Z}$-module in the field $k$. It is well known, that set $\{\Lambda_i\}_{i=1}^h$ of such modules can be ordered by inclusion $\Lambda_1 \subset \cdots \subset \Lambda_h$, where $\Lambda_h$ is the maximal $\mathbb{Z}$-module.

REMARK 3.6. The inclusion of scales $\Sigma_1 \subset \cdots \subset \Sigma_h$ follows from the inclusion $\Lambda_1 \subset \cdots \subset \Lambda_h$ and the inequality $0 \leq \Lambda_i \leq 1$.

(iii) Let us prove that the matrix $L_v$ given by formula (2.7) does not depend on $\{B_i\}_{1 \leq i \leq h}$. Indeed, the $a_k$ in (2.7) are coefficients of the characteristic polynomial of the matrix $B^{\pi(p)}$, where $\pi(p)$ is a positive integer. Notice that the spectrum $\text{Spec}(B_i) = \{\lambda_{i_1}, \ldots, \lambda_{i_{2n}}\}$ does not depend on the matrix $B_i$, since $\text{Char}(B_i)$ does not vary. Because $\text{Spec}(B_i^{\pi(p)}) = \{\lambda_{i_1}^{\pi(p)}, \ldots, \lambda_{i_{2n}}^{\pi(p)}\}$, we conclude that the spectrum of the matrix $B_i^{\pi(p)}$ is independent of $B_i$. Therefore the
polynomial \( \text{Char} \left( B_i^{\pi(p)} \right) \) and the coefficients \( a_k \) in (2.7) are the same for all matrices \( B_i \). It follows from (2.7), that \( L_{\nu} \) is independent of the choice of matrix \( \{ B_i \mid 1 \leq i \leq h \} \).

(iv) Let us calculate the kernel of homomorphism (3.8). Without loss of generality, we assume \( B \cong B_h \). By remark 3.6, we have the inclusion of torsion groups:

\[
\Sigma_1 \subset \cdots \subset \Sigma_{h-1} \subset \Sigma \left( A_{\Theta(k)} \right) .
\]

Let \( e \in \prod v K_0 \left( A_{\Theta(k)} \times L, \mathbb{Z} \right) \) be the trivial element of torsion group at the RHS of (3.8). From item (iii), it is known that the preimage of \( e \) under the homomorphism (3.8) consists of \( h \) distinct elements of the group \( \Sigma \left( A_{\Theta(k)} \right) \). Indeed, each \( \Sigma_i \) in (3.9) has a unique such element lying in \( \Sigma_i \setminus \Sigma_{i-1} \), since otherwise the corresponding abelian variety would have two different reductions modulo \( p \). We conclude therefore, that the kernel of homomorphism (3.8) is an abelian group of order \( h \).

(v) Let us calculate the Shafarevich-Tate group \( \text{III}(\mathcal{A}_K) \). It follows from item (iv), that \( h = |\text{Cl} (\Lambda)| \), where \( \text{Cl} (\Lambda) \) is the ideal class group of \( \Lambda \), see [Latimer & MacDuffee 1933] [LM]. It is easy to see, that the kernel of (3.8) is isomorphic to the group \( \text{Cl} (\Lambda) \). To express \( \text{III}(\mathcal{A}_K) \) in terms of the group \( \text{Cl} (\Lambda) \), recall that the K-homology is dual to the K-theory [Blackadar 1986] [B, Section 16.3]. Roughly speaking, the cocycles in K-theory are represented by the vector bundles. Atiyah proposed using elliptic operators to represent the K-homology cycles. An elliptic operator can be twisted by a vector bundle, and the Fredholm index of the twisted operator defines a pairing between the K-homology and the K-theory. In particular, \( K^0(\mathcal{A}_{\Theta(k)}) \cong K_0(\mathcal{A}_{\Theta(k)}) \), where \( K^0(\mathcal{A}_{\Theta(k)}) \) is the zero K-homology group of the the algebra \( \mathcal{A}_{\Theta(k)} \). By analogy with (3.4), one gets \( \Sigma_0(\mathcal{A}_{\Theta(k)}), \Sigma^0(\mathcal{A}_{\Theta(k)}) \hookrightarrow \mathbb{Q}/\mathbb{Z} \) and, therefore, a pair of embeddings:

\[
\text{Cl} (\Lambda_0), \text{Cl} (\Lambda^0) \hookrightarrow \mathbb{Q}/\mathbb{Z}.
\]

Since \( \text{Cl} (\Lambda_0) \cong \text{Cl} (\Lambda^0) \) are finite abelian groups, the Atiyah pairing gives rise to a bilinear form \( Q_{F_p}(x, y) \) over the finite field \( F_p \) for each prime \( p \) dividing \( |\text{Cl} (\Lambda)| \). The (3.10) is a group homomorphism, if and only if, the \( Q_{F_p}(x, y) \) is an alternating form, i.e. \( \{ Q_{F_p}(x, x) = 0 \mid \forall x \in \text{Cl} (\Lambda_0) \oplus \text{Cl} (\Lambda^0) \} \). Recall that the alternating bilinear forms \( Q_{F_2}(x, y) \) exist, if and only if, \( p \neq 2 \). Namely, the form \( Q_{F_2}(x, y) \) is always symmetric, i.e. \( Q_{F_2}(x, x) \neq 0 \) unless \( x = 0 \). Thus there are no Atiyah pairing in the case \( p = 2 \). The rest of the proof follows the Basis Theorem for finite abelian groups. Lemma 3.5 is proved. \( \Box \)
Remark 3.7. Both cases of lemma 3.5 are realized by the concrete abelian varieties, see [Rubin 1989] [R, Examples (B) and (C)] and [Poonen & Stoll 1999] [PS, Proposition 27], respectively.

Theorem 1.1 follows from lemma 3.5.

4. Complex multiplication

The \(\Sha(\mathcal{A}_K)\) can be calculated using theorem 1.1, if the functor \(F\) is given explicitly. To illustrate the idea, we consider the abelian varieties with complex multiplication.

4.1. Abelian varieties. Denote by \(K\) a CM-field, i.e. the totally imaginary quadratic extension of a totally real number field \(k\). The abelian variety \(\mathcal{A}_CM\) is said to have complex multiplication, if the endomorphism ring of \(\mathcal{A}_CM\) contains the field \(K\), i.e. \(K \subset \text{End } \mathcal{A}_CM \otimes \mathbb{Q}\).

The \(\deg (K | \mathbb{Q}) = 2n\), where \(n\) is the complex dimension of the \(\mathcal{A}_CM\).

For a canonical basis in \(K\) the lifting of the Frobenius endomorphism has the matrix form:

\[
F_{r_v} = \begin{pmatrix}
a_1 & 1 & \ldots & 0 & 0 \\
-a_2 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
a_{2n-1} & 0 & \ldots & 0 & 1 \\
-p & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

The functor \(F\) is acting by the formula \(F_{r_v} \mapsto L_v\), where \(L_v\) is given by (2.7), see [N, p. 179] for the details. Since \(L_v \in \text{End } \Lambda\), one can recover the ring \(\Lambda\) from the eigenvalues of matrix \(L_v\). Let us consider the simplest case \(n = 1\), i.e. the elliptic curves with complex multiplication.

4.2. Elliptic curves. Denote by \(\mathcal{E}_{CM}\) an elliptic curve with complex multiplication by the ring \(R = \mathbb{Z} + fO_K\), where \(K = \mathbb{Q}(\sqrt{-D})\) is an imaginary quadratic field with the square free discriminant \(D > 1\) and \(f \geq 1\) is the conductor of \(R\). The ring \(\Lambda = F(R)\) is given by the formula \(\Lambda = \mathbb{Z} + f'\mathbb{O}_K\), where \(k = \mathbb{Q}(\sqrt{D})\) is the real quadratic field and the conductor \(f' \geq 1\) is the least integer solution of the equation \(|\text{Cl} (\mathbb{Z} + f'\mathbb{O}_K)| = |\text{Cl} (R)|\), see [N, Theorem 1.4.1]. Moreover, there exists a group isomorphism \(\text{Cl} (\Lambda) \cong \text{Cl} (R)\). Using theorem 1.1, one gets the following corollary.

Corollary 4.1. \(\Sha(\mathcal{E}_{CM}) \cong \text{Cl} (R) \oplus \text{Cl} (R)\).

Example 4.2. ([Rubin 1989] [R, Example (B)]) Let \(\mathcal{E}_{CM}\) be the Fermat cubic \(x^3 + y^3 = z^3\). The \(\mathcal{E}_{CM}\) has complex multiplication by the ring \(R \cong \mathbb{Z} + O_K\), where \(K \cong \mathbb{Q}(\sqrt{-3})\). It is well known, that in this case \(\text{Cl} (R)\) is trivial. We conclude from corollary 4.1, that the \(\Sha(\mathcal{E}_{CM})\) is a trivial group. An alternative proof of this fact is based
on the exact calculation of the $p$-part of $\Sha(E_{\text{CM}})$ and can be found in [Rubin 1989] [R, Theorem 1 and Example (B)].

**Example 4.3.** ([Rubin 1989] [R, Example (C)]) Let $E_{\text{CM}}$ be the modular curve $X_0(49)$ given by the equation $y^2 + xy = x^3 - x^2 - 2x - 1$. The $E_{\text{CM}}$ has complex multiplication by the ring $R \cong \mathbb{Z} + \mathcal{O}_K$, where $K \cong \mathbb{Q}(\sqrt{-7})$. It is well known, that the group $Cl(R)$ is trivial. Using corollary 4.1, one concludes that the $\Sha(E_{\text{CM}})$ is a trivial group. A different proof of this result can be found in [Rubin 1989] [R, Theorem 1 and Example (C)].

**Remark 4.4.** Theorem 1.1 is true for the simple abelian varieties $A_K$. If $A_K$ is not simple, then the functor $F$ splits and one gets different formulas for the group $\Sha(A_K)$, see the corresponding calculations in the excellent paper [Stein 2004] [S, Theorem 3.1].

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