On a closed form of rational generating functions for polynomials

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Abstract. Our goal in this work is to found a closed form for rational generating functions, these generate a various families of polynomials and generalized polynomials, in order to get the general recursive formula satisfied by these polynomials.

1. Introduction

First we remember the notion of ordinary generating function [1] of a family of polynomials in one variable.

Definition 1. \( f(x, t) \) is an ordinary generating function if and only if there exist a sequence \( P_k(x) \) of polynomials in \( \mathbb{Z}[x] \) and \( \delta > 0 \) a positive real number such that

\[
(1.1) \quad f(x, t) = \sum_{k \geq 0} P_k(x) t^k, \quad |t| < \delta.
\]

In this paper we consider a family of generating functions for these the polynomials associated obey to the same general recursive formula. Let the family \( \{A_0(x), A_1(x), A_2(x), \cdots, A_m(x), B_0(x) \neq 0, B_1(x), B_2(x), \cdots, B_n(x)\} \) of polynomials such that \( B_0(x), B_1(x), \cdots, B_n(x) \) are coprime and the rational function \( f(x, t) \) defined by \( f(x, t) = A(x, t)/B(x, t) \) where \( A(x, t) = \sum_{j=0}^{m} A_j(x) t^j \) and

\[
B(x) = \sum_{l=0}^{n} B_l(x) t^j.
\]

Taking \( h(x, t) = -\sum_{l=1}^{n} B_l(x)/B_0(x) t^l \) then \( h(x, 0) = 0 \). Since \( h(x, t) \) as a function of \( t \) is continuous on \( \mathbb{R} \) then there exist a constant \( \delta > 0 \) such that \( |h(x, t)| < 1 \)

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for \( |t| < \delta \). Furthermore

\[
\frac{1}{1 - h(x, t)} = \sum_{k \geq 0} h^k(x, t), |t| < \delta
\]

this result is deduced from the well known identity

\[
\frac{1}{1 - t} = \sum_{k \geq 0} t^k, |t| < 1.
\]

Without lost generality \( h^k(x, t) \) can be written in the following form

\[
h^k(x, t) = \sum_{j_1+j_2+\cdots+j_n=k} \binom{k}{j_1 \cdots j_n} \frac{B_{1}^{j_1}(x) \cdots B_{n}^{j_n}(x)}{B_{0}^{k+1}(x)} t^{j_1+j_2+\cdots+j_n}
\]

where \( \binom{k}{j_1 \cdots j_n} \) is the multinomial of order \( n \)

\[
\binom{k}{j_1 \cdots j_n} = \frac{k!}{j_1!j_2!\cdots j_n!}.
\]

Finally

\[
f(x, t) = A(x, t) \sum_{k \geq 0} \sum_{j_1+j_2+\cdots+j_n=k} \binom{k}{j_1 \cdots j_n} \frac{B_{1}^{j_1}(x) \cdots B_{n}^{j_n}(x)}{B_{0}^{k+1}(x)} t^{j_1+j_2+\cdots+j_n}
\]

to be a generating function; \( B_{0}^{k+1}(x) \) must divides \( A_{l}(x) \) for every \( k \geq 0 \) and \( 0 \leq l \leq m \) then \( B_{0}(x) \) must be 1. We conclude that all rational generating functions are of the form

\[
f(x, t) = \sum_{l=0}^{m} \frac{A_l(x)t^l}{B_l(x)t^l}
\]

with \( B_0(x) = 1 \).

2. Statement of main results

Let \( f(x, t) \) the rational function of two variables \( x \) and \( t \) considered bellow. Denoting \( \chi_m \) the characteristic function of the set \( \{0, 1, \cdots, m\} \). It means that

\[
\chi_m(k) = \begin{cases} 1, & \text{if } 0 \leq k \leq m, \\ 0, & \text{otherwise}. \end{cases}
\]

The recursive formula of polynomials generated by \( f(x, t) \) is given in the following theorem.

**Theorem 2.1.** If \( B_0(x) = 1 \) then \( f(x, t) \) generates the family \( \{P_k(x), k \geq 0\} \) of polynomials such that \( P_0(x) = A_0(x) \) and

\[
P_k(x) = \chi_m(k)A_k(x) - \sum_{j=1}^{\min(n,k)} B_j(x)P_{k-j}(x), k \geq 1
\]
Corollary 2.1. If $B_0(x) = 1$ then

$$g(x, t) = \frac{1}{B(x, t)}$$

generates the family $\{Q_k(x), \ k \geq 0\}$ of polynomials such that $Q_0(x) = 1$ and

$$Q_k(x) = -\sum_{j=1}^{\min\{n, k\}} B_j(x)Q_{k-j}(x), \ k \geq 1.$$  \hfill (2.2)

Proof. Just taking $A_0(x) = 1$ and $A_k(x) = 0$ for $k \geq 1$, then $m = 0$. After substitution in the formula (2.1) Theorem 2.1 we get a new family of polynomial $Q_k(x)$ satisfying the identity (2.2) Corollary 2.1. \hfill □

The family $\{P_k(x), \ k \geq 0\}$ depend only on the family $\{Q_k(x), \ k \geq 0\}$, it results from the convolution product of the two families $\{A_j(x), \ 0 \leq j \leq m\}$ and $\{Q_k(x), \ k \geq 0\}$. Explicit formula is given in the following proposition.

Proposition 2.1.

$$P_k(x) = \sum_{j=0}^{\min\{m, k\}} A_j(x)Q_{k-j}(x).$$  \hfill (2.3)

Proof. 

$$f(x, t) = \sum_{j=0}^{m} A_j(x)g(x, t) t^j$$

then

$$f(x, t) = \sum_{k \geq 0} \sum_{j=0}^{m} A_j(x)Q_k(x) t^{k+j}$$

and

$$f(x, t) = \sum_{j=0}^{m} \sum_{k \geq j} A_j(x)Q_{k-j}(x, t) t^k$$

which means that

$$f(x, t) = \sum_{k \geq 0} \min\{m, k\} \sum_{j=0}^{\min\{m, k\}} A_j(x)Q_{k-j}(x) t^k.$$ 

Furthermore

$$\sum_{k \geq 0} P_k(x) t^k = \sum_{k \geq 0} \sum_{j=0}^{\min\{m, k\}} A_j(x)Q_{k-j}(x) t^k.$$ 

Finally after comparison between the coefficients of $t^k$ in both sides of the equality we get

$$P_k(x) = \sum_{j=0}^{\min\{m, k\}} A_j(x)Q_{k-j}(x, t).$$ 

□
Corollary 2.2.

\[(2.4) \quad \chi_m(k)A_k(x) - P_k(x) = \sum_{j=1}^{\min\{n,k\}} \sum_{l=0}^{\min\{m,k-j\}} B_j(x)A_l(x)Q_{k-j-l}(x)\]

Proof. From (2.1) Theorem 2.1;

\[\chi_m(k)A_k(x) = \sum_{j=0}^{\min\{n,k\}} B_j(x)P_{k-j}(x)\]

But in means of the identity (2.3) Proposition 2.1;

\[P_{k-j}(x) = \sum_{l=0}^{\min\{m,k-j\}} A_l(x)Q_{k-j-l}(x)\]

then

\[\sum_{j=0}^{\min\{n,k\}} \sum_{l=0}^{\min\{m,k-j\}} B_j(x)A_l(x)Q_{k-j-l}(x) = \chi_m(k)A_k(x)\]

and the result (2.4) Corollary 2.2 follows. □

Now if \(0 \leq k \leq m\) we get

\[\sum_{j=0}^{k} \sum_{i=0}^{k-j} B_j(x)A_i(x)Q_{k-j-l}(x) = A_k(x)\]

and

\[\sum_{j=1}^{k} \sum_{i=0}^{k-j} B_j(x)A_i(x)Q_{k-j-l}(x) + \sum_{i=0}^{k} A_i(x)Q_{k-i}(x) = A_k(x)\]

then

\[A_k(x) - P_k(x) = \sum_{j=1}^{k} \sum_{i=0}^{k-j} B_j(x)A_i(x)Q_{k-j-l}(x)\]

2.1. Proof of Theorem 2.1. First we must remember the Cauchy product of a polynomial \(\sum_{k=0}^{n} R_k(x)t^k\) with an entire series \(\sum_{k=0}^{\infty} S_k(x)t^k\) to be

\[\left( \sum_{k=0}^{n} R_k(x)t^k \right) \left( \sum_{k=0}^{\infty} S_k(x)t^k \right) = \sum_{k=0}^{\min\{n,n\}} \left( \sum_{j=0}^{\min\{n,k\}} R_j(x)S_{k-j}(x) \right) t^k\]

which is an entire series too, for more details about the procedure we refer to [2].

Now writing

\[f(x, t) = \sum_{k=0}^{n} P_k(x)t^k\]

then

\[\left( \sum_{k=0}^{n} B_k(x)t^k \right) \left( \sum_{k=0}^{\infty} P_k(x)t^k \right) = \sum_{k=0}^{m} A_k(x)t^k\]
and

\[
\sum_{k \geq 0} \left( \sum_{j=0}^{\min\{n,k\}} B_j(x)P_{k-j}(x) \right) t^k = \sum_{k=0}^{m} A_k(x)t^k,
\]

furthermore

\[
\sum_{k \geq 0} \left( \sum_{j=0}^{\min\{n,k\}} B_j(x)P_{k-j}(x) \right) t^k = \sum_{k \geq 0} \chi_m(k)A_k(x)t^k.
\]

After identification we obtain

\[
\chi_m(k)A_k(x) = \sum_{j=0}^{\min\{n,k\}} B_j(x)P_{k-j}(x), \quad k \geq 0
\]

and the recursive formula (2.1) in Theorem 2.1 follows.

This identity states the recursive formula of a large families of polynomials. Including the polynomials generated by functions of the form

\[
\theta(x, t) = \frac{\sum_{j=0}^{m} A_j(x)t^j}{\left(\sum_{j=0}^{n} B_j(x)t^j\right)^h}
\]

where \(B_0(x) = 1\) and \(h > 1\) a positive integer. The reason is that \(\left(\sum_{j=0}^{n} B_j(x)t^j\right)^h\)

is only polynomial in \(\mathbb{Z}[x, t]\), it takes the following form

\[
\left(\sum_{j=0}^{n} B_j(x)t^j\right)^h = \sum_{j=0}^{hn} D_j(x)t^j.
\]

with \(D_0(x) = 1\).

The following table gives a few families of polynomials obeying the general recursive formula (2.1) Theorem 2.1.

| Table 1. few families \(P_k(x)\) of polynomials |
|-----------------------------------------------|
| \(A(x, t)\) | \(B(x, t)\) | Polynomial | Recursive formula |
|-----|-----|--------|-----------------|
| \(t\) | \(1 - xt - t^2\) | Fibonacci | \(F_k(x) - xF_{k-1}(x) - F_{k-2}(x) = 0, F_0(x) = 0, F_1(x) = 1\) |
| 1 | \(1 - t + xt^2\) | Catalan [3] | \(C_k(x) - C_{k-1}(x) + xC_{k-2}(x) = 0, C_0(x) = C_1(x) = 1\) |
| \(t\) | \(1 - xt - t^m\) | G. Fibonacci [1] | \(U_{n,m}(x) - xU_{n-1,m}(x) - U_{n-m,m}(x) = 0, n \geq m\) |
| \(t\) | \(1 - t - xt^2\) | Jacobsthal | \(J_k(x) - J_{k-1} - xJ_{k-2}(x) = 0, J_0(x) = J_1(x) = j_2(x) = 1\) |
| \(1\) | \(1 - pxt - qt^2\) | Horadam [1] | \(A_k(x) - pxA_{k-1} - qA_{k-2}(x) = 0, A_0(x) = 0, A_1(x) = 1\) |
| \(1 + gt^2\) | \(1 - pxt - qt^2\) | Horadam [1] | \(B_k(x) - pxB_{k-1} - qB_{k-2}(x) = 0, B_0(x) = 2, B_1(x) = x\) |
| \(2x + 2t\) | \(1 - 2xt - t^2\) | Pell [1] | \(P_k(x) - 2xP_{k-1} - P_{k-2}(x) = 0, P_0(x) = 0, P_1(x) = 1\) |
| \(2x + 2t\) | \(1 - 2xt - t^2\) | Pell-Lucas [1] | \(Q_k(x) - 2xQ_{k-1} - Q_{k-2}(x) = 0, Q_0(x) = 2, Q_1(x) = 2x\) |
| \(2 - xt\) | \(1 - xt - t^m\) | G. Lucas [1] | \(V_{n,m}(x) - xV_{n-1,m}(x) - V_{n-m,m}(x) = 0, n \geq m\) |
3. Application to generalized Catalan and Fibonacci polynomials

In the literature a large families of polynomials obey to the general recursive formula given in Theorem 2.1. In this section, we revisit the works [1, 2] and get, with a new method the recursive formulas satisfied by generalized Catalan polynomials and Fibonacci polynomials in the case $λ = 1$ for the first and $h = 1$ for the second.

Firstly we began by generalized Catalan polynomials, these generated by the function

$$
\frac{1 + A(x)t}{1 - mt + xt^m} = \sum_{k \geq 0} P_{k,m}^{1,A}(x)t^k
$$

According to Theorem 2.1; the following proposition states exactly the same recursive formula of the family $P_{k,m}^{1,A}(x)$ as in Corollary 2.1 [3] p. 166 by taking $λ = 1$.

**Proposition 3.1.** The family $\{P_{k,m}^{1,A}(x), k \geq 0\}$ is defined by

$$
P_{0,m}^{1,A}(x) = 1, \quad P_{1,m}^{1,A}(x) = A(x) + m,
$$

(3.1)

$$
P_{k,m}^{1,A}(x) = mP_{k-1,m}^{1,A}(x), \quad 2 \leq k < m,
$$

and

(3.2)

$$
P_{k,m}^{1,A}(x) = mP_{k-1,m}^{1,A}(x) - xP_{k-m,m}^{1,A}(x), \quad k \geq m.
$$

**Proof.** In means of the identity (2.1) Theorem 2.1 we deduce that

$$
P_{0,m}^{1,A}(x) = 1, \quad P_{1,m}^{1,A}(x) = A(x) + m
$$

and

$$
P_{k,m}^{1,A}(x) = -\sum_{j=1}^{\min\{m,k\}} B_j(x)P_{k-j,m}^{1,A}(x), \quad k \geq 2.
$$

with $B_1(x) = -m, B_m(x) = x$ and the others are zero. Explicitly

$$
P_{k,m}^{1,A}(x) = mP_{k-1,m}^{1,A}(x), \quad 2 \leq k < m
$$

and

$$
P_{k,m}^{1,A}(x) = mP_{k-1,m}^{1,A}(x) - xP_{k-m,m}^{1,A}(x), \quad k \geq m
$$

□

Without lost generality the identity (2.1) Theorem 2.1 can be adapted to polynomials of two variables in the following way

$$
f(x, y, t) = \frac{A(x, y, t)}{B(x, y, t)}
$$
with \( A(x, y, t) = \sum_{j=0}^{m} A_j(x, y)t^j \) and \( B(x, y, t) = \sum_{j=0}^{m} B_j(x, y)t^j \). With the same
demarche we conclude that \( f(x, y, t) \) is a generating function if and only if \( B_0(x, y) = 1 \). In this case
\[
f(x, y, t) = \sum_{k \geq 0} P_k(x, y)t^k
\]
and the corresponding recursive formula is
\[
(3.3) \quad P_k(x, y) = \chi_m(k)A_k(x, y) - \sum_{j=1}^{\min\{n, k\}} B_j(x, y)P_{k-j}(x, y)
\]
Secondly, the function
\[
f(x, y, t) = \frac{1 + A(x, y)t}{1 - xkt - ymtm+n} = \sum_{k \geq 0} G_{k}^{1,A}(x, y, k, m, n) t^\nu
\]
generates the generalized two variables Fibonacci polynomials \( G_{\nu}^{1,A}(x, y, k, m, n) \) and the same result as in Proposition 4.2 [2] is deduced where
\[
G_{0}^{1,A}(x, y, k, m, n) = 1, \quad G_{1}^{1,A}(x, y, k, m, n) = A(x, y) + x^k,
\]
\[
G_{\nu}^{1,A}(x, y, k, m, n) = x^kG_{\nu-1}^{1,A}(x, y, k, m, n), \quad 2 \leq \nu < n + m
\]
and
\[
G_{\nu}^{1,A}(x, y, k, m, n) = x^kG_{\nu-1}^{1,A}(x, y, k, m, n) + y^mG_{\nu-n-m}^{1,A}(x, y, k, m, n), \quad \nu \geq n + m.
\]
These two kinds of polynomials admit a natural generalization to the forms
\[
\sum_{j=0}^{m} A_j(x)t^j \quad (1 - mt + xt^m)^h = \sum_{k \geq 0} P_{k,m}(x)t^k
\]
and
\[
\sum_{j=0}^{m} A_j(x)t^j \quad (1 - xkt - ymtm+n)^h = \sum_{k \geq 0} G_{k}^{h,A}(x, y, k, m, n) t^k.
\]
The recursive formula satisfied by these polynomials is left as an exercise.

4. Conclusion

This method is efficient, it gives directly the recursive formula of infinitely many families of polynomials generated by rational functions. An open question is: can we found a general explicit formula for these polynomials?

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