Signed Hultman Numbers and Signed Generalized Commuting Probability in Finite Groups

Robert Shwartz
Department of Mathematics
Ariel University, Israel
robertsh@ariel.ac.il

Vadim E. Levit
Department of Computer Science
Ariel University, Israel
levitv@ariel.ac.il

Abstract
Let $G$ be a finite group. Let $\pi$ be a permutation from $S_n$. We study the distribution of probabilities of equality
\[ a_1 a_2 \cdots a_{n-1} a_n = a_{\pi_1}^{\epsilon_1} a_{\pi_2}^{\epsilon_2} \cdots a_{\pi_{n-1}}^{\epsilon_{n-1}} a_{\pi_n}^{\epsilon_n}, \]
when $\pi$ varies over all the permutations in $S_n$, and $\epsilon_i$ varies over the set $\{+1, -1\}$. By [6], the case where all $\epsilon_i$ are $+1$ led to a close connection to Hultman numbers.

In this paper we generalize the results, permitting $\epsilon_i$ to be $-1$. We describe the spectrum of the probabilities of signed permutation equalities in a finite group $G$. This spectrum turns out to be closely related to the partition of $2^n \cdot n!$ into a sum of the corresponding signed Hultman numbers.

Keywords: commuting probability, signed permutation, signed Hultman number, breakpoint graph, finite group.
MSC 2010 classification: 20P05, 20B05, 05A05, 20D60.

1 Introduction

The study of the probability that two random elements in a finite group $G$ commute (commuting probability) has received considerable research attention. In 1968, Erdős and Turan proved that
\[ \Pr(a_1 a_2 = a_2 a_1) > \frac{\log(\log |G|)}{|G|}. \]

In early 1970s, Dixon observed that the commuting probability is $\leq \frac{1}{12}$ for every finite non-Abelian simple group (this was submitted as a problem in Canadian Mathematical Bulletin 13 (1970), with a solution appearing in 1973). In 1973, Gustafson proved that the commuting probability is equal to $\frac{k(G)}{|G|}$, where $k(G)$ is the number of conjugacy classes in $G$ [9]. Based on this observation, Gustafson then obtained the upper bound of the commuting probability in any finite non-Abelian group to be $\frac{5}{8}$ [9]. This upper bound is actually attained in many finite groups, including the two non-abelian groups of order 8. Since then, there has been significant research concerning probabilistic aspects
of finite groups. Many of these studies can be regarded as variations of the commuting probability problem. For example, Das and Nath [13], [7] study the probability \( \Pr(\pi)(g) \) of the equality
\[
a_1a_2...a_{n-1}a_n a_\pi^{-1}_1 a_\pi^{-1}_2 ... a_\pi^{-1}_{n-1} a_\pi^{-1}_n = g
\]
in a finite group \( G \). The word
\[
a_1a_2...a_{n-1}a_n a_\pi^{-1}_1 a_\pi^{-1}_2 ... a_\pi^{-1}_{n-1} a_\pi^{-1}_n,
\]
in which \( a_1a_2...a_{n-1}a_n \) vary over all the elements of \( G \), is denoted by \( \omega \). Thus this is a generalization of the classical study of the commuting probability, in which case \( \omega = a_1a_2a_1^{-1}a_1^{-1} \) and \( g = 1 \) [14]. In the same direction of generalization of the commuting probability, Cherniavsky, Goldstein, Levit, and Shwartz [6] have introduced an interesting connection between the distribution of the probabilities
\[
\Pr_\pi(G) = \Pr(a_1a_2 ... a_n = a_\pi_1 a_\pi_2 ... a_\pi_n),
\]
for a finite group \( G \), when \( \pi \in S_n \), and the number of alternating cycles in the Hultman decomposition of \( \pi \) [11], [8]. It was shown in [6], that this probability for a generic group \( G \) depends only on the Hultman decomposition of the permutation \( \pi \) and that the spectrum of probabilities, as \( \pi \) runs over all the permutations in \( S_n \), is the decomposition of \( n! \) to the Hultman numbers. In this paper, we generalize the results of [6], where we consider the distribution of the probabilities
\[
\Pr_\pi(G) = \Pr(a_1a_2 ... a_n = a_\pi_1 a_\pi_2 ... a_\pi_n),
\]
for a finite group \( G \), where \( \pi \in S_n \), and \( \epsilon = \{+1,-1\}^n \). Notice, the paper [6] deals with the case where \( \epsilon = (1,1,...,1) \) (i.e., \( \epsilon_i \) for every \( 1 \leq i \leq n \)).

In this paper, we show that \( \Pr_\pi(G) \) for a generic group \( G \) depends only on the signed Hultman decomposition [2] (which is a generalization of the Hultman decomposition) of the signed-permutation \( \pi_\epsilon \), (where \( \pi_\epsilon(i) = \epsilon_i \cdot \pi(i) \), for every \( \pi \in S_n \) and every \( 1 \leq i \leq n \)), and that the spectrum of probabilities, as \( \pi \) runs over all the permutations in \( S_n \), and \( \epsilon \) runs over all the \( n \)-tuples of \( \{+1,-1\}^n \), is the decomposition of \( 2^n \cdot n! \) to signed Hultman numbers. Similarly to the permutation group \( S_n \), the signed-permutation group is a Coxeter group as well, which is denoted by \( B_n \). The idea to generalize theorems concerning permutation groups to signed-permutation groups has a rich history. For instance, there is a theorem about Coxeter covers of symmetric groups [13], where characterizing special quotients of Coxeter groups are defined by a line Coxeter graph as extensions of a symmetric group \( S_n \). The theorem was generalized into a theorem about Coxeter covers of classical Coxeter groups [3], where we consider characterization of special quotients of particular Coxeter groups as extensions of a classical Coxeter group \( B_n \) or \( D_n \) (a subgroup of \( B_n \), which can be considered as a specific signed-permutation group. For more details see [3]), by using signed-graphs as a generalization of the dual line Coxeter graphs used in [15]. By the theorem of MacMahon [12], the major-index of the permutations in \( S_n \) is equi-distributed with the Coxeter length. Roichman and Adin [4] proposed a generalization of the theorem of MacMahon, where they defined the flag-major-index for signed-permutations, which is a generalization of the major-index for permutations, and they proved that the flag-major-index is equi-distributed with the Coxeter length. A few
years later, Shwartz, Adin and Roichman [16] generalized further the flag-major-index for $D_n$ (even-signed permutations [4]), where they proved that the flag-major-index is equi-distributed with the Coxeter length of $D_n$ as well.

For Hultman numbers, signed Hultman numbers, and the related definitions and notations we refer to [8], [2]. Recall, that the Hultman number $S_{H(n,k)}$ counts the number of permutations $\pi$ in $S_n$ whose cycle graph $G(\pi)$ decomposes into $k$ alternating cycles. For a permutation $\pi$ in $S_n$, let $H(\pi)$ be the Hultman decomposition of the cycle graph $G(\pi)$ of $\pi$ into alternating cycles, and let $|H(\pi)|$ be the number of alternating cycles in $H(\pi)$.

The paper is organized as follows. In Section 2, we give basic definitions about groups, permutations, signed-permutations, generalized commuting probabilities, and signed generalized commuting probabilities, which we use in the paper. In Section 3, we recall the basic definitions about signed Hultman decompositions, breakpoint graphs (generalization of the cycle graphs [8]), and the related alternating cycles for the signed-permutation groups $B_n$ as defined in [2]. In Section 4, we define $(|x| − |y|)$-exchange and $(|x| − |y|)$-cyclic operations on the breakpoint graph $Gr(\pi)$ for $\pi \in B_n$, as a generalization of the definitions of $(x − y)$-exchange and $(x − y)$-cyclic operations on the cycle graph $G(\pi)$ for $\pi \in S_n$, as defined in [6]. We also define a new operation which has not been defined or used yet in [6], namely $([|x|] − [|x| + 1])$-sign-change operation. We show some important properties of the mentioned operations on $Gr(\pi)$, which we need in order describe the connections between the signed Hultman decompositions of $B_n$ and the signed generalized commuting probability, which is induced by a signed-permutation $\pi \in B_n$. In Section 5, we prove the main theorem of the paper, which draws a connection between the number of the alternating cycle of the breakpoint graph $Gr(\pi)$ for an element $\pi$ in the signed-permutation group $B_n$, and the signed generalized commuting probability, which is induced by a signed-permutation $\pi$. We show that the main theorem is a generalization of the theorem about the connection between the number of the alternating cycles of the cycle graph $G(\pi)$ for an element $\pi$ in the symmetric group $S_n$, and the generalized commuting probability, which is induced by a signed-permutation $\pi$, as described in [6]. Moreover, we will observe two special families of finite groups $G$, where the signed generalized commuting probability has interesting properties. In Section 6, we give our conclusions about the results of the paper, which we compare to the results of [6], since this paper is a generalization of it. Finally, we offer ideas for further generalizations of the generalized commuting probability.

2 Preliminaries

We start with some important definitions about finite groups, which we use throughout the paper.

**Definition 2.1** Let $G$ be a finite group.

- For every $g, h \in G$, denote by $g^h$ the element $h^{-1}gh$, which is the conjugate of $g$ by $h$.
- For every $g \in G$, denote by $\Omega_g(G)$ the conjugacy class of $g$ in $G$, i.e. the set of all the elements of $G$ the form $h^{-1}gh$, where $h \in G$. 

For every $g \in G$, denote by $C_G(g)$ the centralizer of the element $g \in G$, i.e. the subgroup of $G$, consisting of all the elements $h \in G$, such that $gh = hg$.

For every $g, h \in G$, denote by $[g, h]$ the commutator $g^{-1}h^{-1}gh$ of $g$ and $h$.

Denote by $c(G)$ the number of conjugacy classes in $G$.

Denote by $\Omega(G, 1), \ldots, \Omega(G, c(G))$ the conjugacy classes of $G$, ordered in some fixed order, where $\Omega(G, 1)$ is $\Omega_1(G)$ (The conjugacy class of 1).

For every $1 \leq j \leq c(G)$, denote by $\Omega(G, j^{-1})$ the conjugacy class, which contains the inverses of the elements of $\Omega(G, j)$.

For every $1 \leq j \leq c(G)$, denote by $\Omega(G, j^2)$ the conjugacy class, which contains the squares of the elements of $\Omega(G, j)$.

For every $g \in G$, denote by $ic_g$ the integer such that $g \in \Omega(G, ic_g)$.

For a sequence $(g_1, g_2, \ldots, g_n)$ of $n$ elements of $G$, denote by $\text{Stab.Prod}_n(g_1, g_2, \ldots, g_n)$ the set of all the sequences $(a_1, a_2, \ldots, a_n)$ of $n$ elements of $G$ such that

$$a_1^{-1}g_1a_1 \cdot a_2^{-1}g_2\cdots a_n^{-1}g_na_n = g_1 \cdot g_2 \cdots g_n.$$
Definition 2.3 Let $G$ be a finite group.

- An element $g \in G$ is considered to be 'real', if $g$ is conjugate to its inverse, $g^{-1}$.
- $G$ is called an ambivalent group, if every element of $G$ is 'real'.
- A conjugacy class $\Omega$ of $G$ is called a 'real conjugacy class', if every element of $g \in \Omega$ is a 'real element'.
- Denote by $rc(G)$ the number of 'real conjugacy classes' of a finite group $G$.

By Definition 2.3 it can be easily concluded that a finite group $G$ is an ambivalent group if and only if every conjugacy class of $G$ is a 'real conjugacy class'. Therefore, $c_{i_1, \ldots, i_n} = c_{i_1, \ldots, i_n}(G)$, for every conjugacy class of $G$, if and only if $G$ is an ambivalent group.

Definition 2.4 Let $inv(G)$ be the number of involutions in a finite group $G$; i.e, the number of elements $b \in G$, such that $b^2 = 1$.

Now, we recall the definition of signed-permutation group, which is the Coxeter group $B_n$ (for details see [4]).

Definition 2.5 For every $n \in \mathbb{N}$,

- Let $S_n$ be the permutation group of the elements of the set $\{1, 2, \ldots, n\}$;
- Let $B_n$ be the permutation group of the elements of the set $\{+1, +2, \ldots, +n, -1, -2, \ldots, -n\}$ such that every permutation $\pi$ satisfies $\pi_{-i} = -\pi_{+i}$, for every $1 \leq i \leq n$, where $-(i)$ is defined to be $+i$.

Remark 2.6 Definition 2.5 implies that every $\pi \in B_n$ is uniquely determined by $\{\pi_{+1}, \pi_{+2}, \ldots, \pi_{+n}\}$. Therefore, every $\pi \in B_n$ can be considered as a signed permutation, where we denote $\pi = \langle \pi_{+1} \pi_{+2} \ldots \pi_{+n}\rangle$.

Definition 2.7 For every element $i \in \{+0, +1, +2, \ldots, +n\}$, let it be sign $(i) = (+)$, and for every element $i \in \{-0, -1, -2, \ldots, -n\}$, let it be sign $(i) = (-)$

Definition 2.8 Let $\pi$ be a sign-permutation in $B_n$. Then $|\pi|$ is the corresponding permutation of $S_n$, which satisfies $|\pi|_i = |\pi_i|$.

Now, we recall some definition concerning generalized commuting probability.

Definition 2.9 $Pr^n(G) = Pr(a_1a_2 \cdots a_m = a_ma_{m-1} \cdots a_1)$

Proposition 2.10 $Pr(G) = Pr^{n-|H(\pi)|+1}(G)$. Where,
Now, we define signed generalized commuting probability, $\Pr_{\pi}(G)$ for $\pi \in B_n$ as a generalization of the definition of the generalized commuting probability $\Pr_{\pi}(G)$ for $\pi \in S_n$ which was defined in [6].

**Definition 2.11** Let $G$ be a finite group, and $\pi$ be a signed-permutation in $B_n$. Then,

$$\Pr_{\pi}(G) := \Pr(a_1a_2\cdots a_n = a_{|\pi|_1}\epsilon_1(\pi)a_{|\pi|_2}\epsilon_2(\pi)\cdots a_{|\pi|_n}\epsilon_n(\pi))$$

where for every $1 \leq i \leq n$, either $\epsilon_i(\pi) = 1$ or $\epsilon_i(\pi) = -1$ such that:

- In case $\text{sign}(\pi_i) = (+)$, $\epsilon_i(\pi) = 1$;
- In case $\text{sign}(\pi_i) = (-)$, $\epsilon_i(\pi) = -1$.

**Definition 2.12** Let $m(\pi)$ be

$$\frac{n - \sum_{i=1}^{n} \epsilon_i}{2}.$$

**Corollary 2.13** $m(\pi)$ is the number of indices $1 \leq i \leq n$, such that $\epsilon_i = -1$.

**Definition 2.14** A signed permutation $\pi$ considered as a positive element in case $m(\pi) = 0$, which means $\epsilon_i(\pi) = 1$ for every $1 \leq i \leq n$, otherwise $\pi$ is considered as a non-positive element.

Now, we generalize the definition of $\Pr^k(G)$, for negative $k$ as well.

**Definition 2.15** For every group $G$, and every $k \in \mathbb{N}$, define $\Pr^{-k}(G)$ to be

$$\Pr^{-k}(G) = \Pr(a_1a_2\cdots a_n = a_1^{-1}a_2^{-1}\cdots a_k^{-1}a_{k+1}a_{k+2}\cdots a_n)$$

**Definition 2.16** Let $I^{(-k)} \in B_n$ the following signed permutation:

- $I^{(-k)}_j = -j$ for every $j$ such that $|j| \leq k$;
- $I^{(-k)}_j = j$ for every $j$ such that $|j| > k$.

i.e., $I^{(-k)} = (-1 \ldots -k + (k+1) + n)$.

**Remark 2.17** $\Pr^{-k}(G) = \Pr_{\pi}(G)$ for $\pi = I^{(-k)}$.

**Proposition 2.18**

$$\Pr^{-1}(G) = \frac{\text{inv}(G)}{|G|}.$$ 

**Proof.** Let $G$ be a finite group, then

$$\Pr^{-1}(G) = \Pr(a = a^{-1}) = \Pr(a^2 = 1) = \frac{\text{inv}(G)}{|G|}.$$ 

$\blacksquare$
Proposition 2.19

\[ \Pr^{-2}(G) = \frac{rc(G)}{|G|}. \]

Proof.

\[ \Pr^{-2}(G) = \Pr(ab = a^{-1}b^{-1}) = \]
\[ = \Pr(ab = (ba)^{-1}) = \Pr(b^{-1}(ba)b = (ba)^{-1}) = \]
\[ = \Pr(b^{-1}ab = a^{-1}) = \frac{rc(G)}{|G|} \]



Proposition 2.20

\[ \Pr^{-2n}(G) = \Pr(a_1a_2 \cdots a_{2n-1}a_{2n} = a_1^{-1}a_2^{-1} \cdots a_{2n-1}^{-1}a_{2n}) = \]
\[ = \sum_{i_1,i_2,\ldots,i_n} c_{i_1,i_2,\ldots,i_n,1}^{(G)} \frac{c_{i_1,i_2,\ldots,i_n,1}^{(G)}}{|\Omega_{i_1}| \cdot |\Omega_{i_2}| \cdots |\Omega_{i_n}|} \cdot |G|^n. \]

\[ \Pr^{-2n+1}(G) = \Pr(a_1a_2 \cdots a_{2n-1}a_{2n}a_{2n+1} = a_1^{-1}a_2^{-1} \cdots a_{2n-1}^{-1}a_{2n}^{-1}a_{2n+1}) = \]
\[ = \sum_{i_1,i_2,\ldots,i_n,j} c_{i_1,i_2,\ldots,i_n,1}^{(G)} \frac{c_{i_1,i_2,\ldots,i_n,1}^{(G)}}{|\Omega_{i_1}| \cdot |\Omega_{i_2}| \cdots |\Omega_{i_n}|} \cdot |G|^n. \]

Remark 2.21 Since the proof of Proposition 2.20 is based on very similar arguments as the proof of the formula for \( \Pr^{2k}(G) \), in Section 5 of [6], we leave the proof of the proposition for the reader.

Remark 2.22 Since \( \Pr^{2}(G) = \frac{c(G)}{|G|} \), Proposition 2.19 implies that \( \Pr^{-2}(G) \) equals to \( \Pr^{2}(G) \) if and only if every conjugacy class of a finite group \( G \) is a 'real conjugacy class', which holds if and only if \( G \) is a finite ambivalent group. Therefore, \( \Pr^{k}(G) \) is not necessarily equal to \( \Pr^{-k}(G) \) in general.

From Proposition 2.20 we conclude the following corollary, which classifies the cases where \( \Pr^{k}(G) = \Pr^{-k}(G) \).

Corollary 2.23 For every integer \( k \geq 1 \),

\[ \Pr^{2k}(G) = \Pr^{-2k}(G) \]

if and only if \( G \) is an ambivalent group, which means that every element \( g \in G \) is conjugate to its inverse \( g^{-1} \in G \).

7
Proof. By \[6\],

\[
\Pr^{2n}(G) = \Pr(a_1a_2 \cdots a_{2n-1}a_{2n} = a_2a_1 \cdots a_{2n}) = \\
\sum_{i_1, i_2, \ldots, i_n, j=1}^{c(G)} \frac{|\Omega_j| \cdot c_{i_1,i_2,\ldots,i_n;j}^2(G)}{|\Omega_1| \cdot |\Omega_2| \cdot \cdots \cdot |\Omega_n| \cdot |G|^n}.
\]

and by Propositions 2.2, 2.20

\[
\Pr^{-2n}(G) = \Pr(a_1a_2 \cdots a_{2n-1}a_{2n} = a_1^{-1}a_2^{-1} \cdots a_{2n-1}^{-1}a_{2n}^{-1}) = \\
\sum_{i_1, i_2, \ldots, i_n, j=1}^{c(G)} \frac{|\Omega_j| \cdot c_{i_1,i_2,\ldots,i_n;j}^{-1}(G)}{|\Omega_1| \cdot |\Omega_2| \cdot \cdots \cdot |\Omega_n| \cdot |G|^n}.
\]

The case where, \(\Pr^{-2k}(G) = \Pr^{2k}(G)\) for every positive integer \(k\), is equivalent to \(c_{i_1,i_2,\ldots,i_n;j}(G) = c_{i_1,i_2,\ldots,i_n;j}^{-1}(G)\) for every conjugacy class of \(G\), which holds if and only if the group \(G\) is ambivalent. ■

Lemma 2.24

\[
\Pr^{-k}(G) = \Pr(a_1^2a_2^2 \cdots a_k^2 = 1)
\]

Proof.

\[
\Pr(a_1a_2 \cdots a_k = a_1^{-1}a_2^{-1} \cdots a_k^{-1})
\]

is equivalent to

\[
\Pr(a_k^2, a_{k-1}^{-1}a_k, a_k^{-1}a_{k-1}^2a_k \cdots a_k^{-1}a_{k-1}^{-1}a_2^{-1}a_2 \cdots a_1^{-1}a_1a_k = 1)
\]

Then, by substituting \(b_i = a_k\) and \(b_i = \prod_{j=i+1}^{k} a_j\), for every \(1 \leq i \leq k\), we have:

\[
\Pr(a_1a_2 \cdots a_k = a_1^{-1}a_2^{-1} \cdots a_k^{-1}) = \Pr(b_1^2 \cdot b_2^2 \cdots b_k^2 = 1).
\]

From Lemma 2.24, we conclude the following corollaries:

Corollary 2.25 A finite group \(G\) has an odd order, if and only if

\[
\Pr^{-k}(G) = \frac{1}{|G|}
\]

for every \(k \in \mathbb{N}\).
Proof. Let $G$ be a finite group. Then, for every $g \in G$ there exists only one $g' \in G$ such that $g'^2 = g$, if and only if the order of $G$ is odd. By Lemma 2.24, $\Pr^{-k}(G) = \Pr(a_1^2 a_2^2 \cdots a_k^2 = 1)$. Therefore, in case of group $G$ the order of which an odd integer, $\Pr^{-k}(G) = \Pr(a_1' a_2' \cdots a_k' = 1)$. Hence, obviously, $\Pr^{-k}(G) = \frac{1}{|G|}$ in case of odd order $G$. In case of $G$ the order of which is an even integer, by Sylow Theorem, the group $G$ contains at least one non-trivial involution (i.e., an element of order 2 in $G$). Thus, by Proposition 2.18,

$$\Pr^{-1}(G) = \frac{\text{inv}(G)}{|G|} > \frac{1}{|G|}.$$  

Corollary 2.26 If $G$ is a finite abelian group, then

$$\Pr^{-k}(G) = \Pr^{-1}(G) = \frac{\text{inv}(G)}{|G|},$$

for every $k \in \mathbb{N}$.

Proof. By Lemma 2.24, $\Pr^{-k}(G) = \Pr(a_1^2 a_2^2 \cdots a_k^2 = 1)$. Since $G$ is an abelian group, $a_1^2 a_2^2 \cdots a_k^2 = (a_1 \cdot a_2 \cdots a_k)^2$. Thus,

$$\Pr^{-k}(G) = \Pr((a_1 \cdot a_2 \cdots a_k)^2 = 1) = \Pr(a^2 = 1) = \Pr^{-1}(G) = \frac{\text{inv}(G)}{|G|}.$$  

Corollary 2.27 Let $G = \bigoplus_{i=1}^{m} G_i$ (i.e., $G$ is a direct sum of $m$ groups $G_i$, for $1 \leq i \leq m$), then

$$\Pr^{-k}(G) = \prod_{i=1}^{m} \Pr^{-k}(G_i),$$

for every $k \in \mathbb{N}$.

Proof. By Lemma 2.24, $\Pr^{-k}(G) = \Pr(a_1^2 a_2^2 \cdots a_k^2 = 1)$. Since $G = \bigoplus_{i=1}^{m} G_i$, it is satisfied that $a_j = \prod_{i=1}^{m} b_{j,i}$, such that $b_{j,i} \in G_i$, for every $1 \leq j \leq k$. Thus we conclude:

$$\Pr^{-k}(G) = \Pr(a_1^2 a_2^2 \cdots a_k^2 = 1) = \Pr(\prod_{i=1}^{m} b_{j,i} = 1) = \prod_{i=1}^{m} \Pr^{-k}(G_i).$$

3 Signed Hultman decomposition

Now, we recall the definition of signed Hultman decomposition of the signed-permutations of $B_n$, as defined in [2]. The signed Hultman decomposition is a generalization of the Hultman decomposition of permutations of $S_n$, which is defined in [11], [8].
Definition 3.1 Let $\pi \in B_n$. Consider the set $H_n$ of $2n + 2$ vertices named by

$$H_n = \{+0, +1, \ldots, +n, -0, -1, \ldots, -n\}.$$ 

Defining two types of edges.

- There are gray-edges connecting $[i] \leftrightarrow [-(i + 1)]$ for every $i \in H_n$;
- There are black-edges connecting $[\pi_i] \leftrightarrow [\pi_{-(i+1)}]$ for every $i \in H_n$,

such that in the specific case, where $i$ or $i + 1$ equals to $(+0)$ or $(-0)$, $i + 1$ is defined as follows:

- $(−1) + 1$ is considered to be $(-0)$;
- $(−0) + 1$ is considered to be $(-n)$;
- $(+n) + 1$ is considered to be $(+0)$.

Then, considering the breakpoint graph $\text{Gr}(\pi)$, the edges of which are two-colored containing the black and the gray non-oriented edges.

Remark 3.2 Notice, both the gray and the black edges are non-oriented in the breakpoint graph $\text{Gr}(\pi)$ for $\pi \in B_n$, in contrast to the edges in the cycle graph $\text{Gr}(\pi)$ for $\pi \in S_n$ as described in [5], [8], and [6].

Definition 3.3 Let $s(\pi)$ be the number of the alternating cycles of $\text{Gr}(\pi)$, where in every cycle black edge is followed by gray edge, and gray edge is followed by black edge.

Example 3.4 Let $\pi = (3~−1~2~4)$. Then $\pi$ contains two alternating cycles, in the following way:

$$[+0] \leftrightarrow [−1] \leftrightarrow [+2] \leftrightarrow [−4] \leftrightarrow [−3] \leftrightarrow [+0]$$

$$[+4] \leftrightarrow [−0] \leftrightarrow [+4]$$

Therefore, $s(\pi) = 2$.

Remark 3.5 Our notation of the vertices of $\text{Gr}(\pi)$ is slightly different from the notation in [2], where for every integer $0 \leq i \leq n$,

- we denote by $+i$ the vertex denoted by $i^h$ in [2];
• we denote by \(-i\) the vertex denoted by \(i\) in \([2]\).

**Proposition 3.6** Let \(I^{(k)}\) be a sign permutation of \(B_n\) as defined in Definition 2.16. Then, \(s(I^{(-k)}) = n - k + 1\) (i.e., \(Gr(I^{(-k)})\) contains \(n - k + 1\) alternating cycles).

**Proof.** By the definition of \(I^{(k)}\), and the definition of the alternating cycle, we get \(n - k\) alternating cycles where each one contains two vertices \(+j\) and \(-j - 1\), for every \(k + 1 \leq j \leq n\), and one more alternating cycle which contains the remaining \(2k + 1\) vertices which are \(j\) such that \(+0 \leq j \leq +k\), and \(-k \leq j \leq -1\).

**Definition 3.7** Denote by \(\pi^\circ\) the element

\[
\pi^\circ = (\pi_+, \pi_{+(n-1)}, \pi_{+(n-2)}, \ldots, \pi_0) \cdot (\pi_{-0}, \pi_{-1}, \pi_{-2}, \ldots, \pi_{-n})
\]

of \(S_{H_n}\).

**Proposition 3.8** Let \(\pi \in B_n\), then there are \(m(\pi)\) elements \(k\) such that \(\text{sign}(k) = (-)\) in the same cycle to \(+0\) of \(\pi^\circ\).

**Proof.** The cycle of \(\pi^\circ\), where \(+0\) is located has the form \((\pi_+, \pi_{+(n-1)}, \pi_{+(n-2)}, \ldots, \pi_0)\). Hence, by Corollary 2.13, the number of elements with sign \((-)\) in that cycle equals to \(m(\pi)\).

**Definition 3.9**

\[
\pi^\circ = \pi^\bullet \cdot (+0, +1, \ldots, +n) \cdot (-n, \ldots, -1, -0).
\]

**Corollary 3.10** Let \(\pi \in B_n\). Then, we have the following connections between \(\pi^\circ\) and the breakpoint graph \(Gr(\pi)\):

**Conclusion 3.11**

- For \(x, y \in H_n\), \(\pi^\circ_x = y\) if and only if the cycle graph \(Gr(\pi)\) contains \([x] \leftrightarrow [-(x+1)] \leftrightarrow [y];\)

- The number of cycles of \(\pi^\circ\) as a permutation of the \(2n + 2\) elements of \(H_n\) equals to \(2s(\pi)\).

### 4 Exchange, cyclic, and sign-change operations

First, we recall and generalize the definitions of \([(x) - (y)]\)-exchange, and \([(x) - (y+1)]\)-cyclic operations on \(Gr(\pi)\), which were defined in [6]. In addition, we define \([(|x|) - (|x+1|)]\)-sign-exchange operation. We use the operations for the proof of the theorems about the connections between the signed generalized commuting probability \(Pr_\pi(G)\) and the number of alternating cycles \(s(\pi)\) in \(Gr(\pi)\), similar to its use in [6].

**Definition 4.1** Let \(\pi \in B_n\) a signed permutation. Let \(x, y, w, z \in H_n\) such that \(y \neq x, y \neq x + 1,\) and

\[
[-z] \leftrightarrow [x] \leftrightarrow [-(x+1)] \leftrightarrow [y], \quad [-y] \leftrightarrow [w]
\]
is satisfied in $Gr(\pi)$. Then by ($[x] - [y]$)-exchange operation on $\pi$ we obtain $\theta$ such that in $Gr(\theta)$ the following holds:

$$[-z] \leftrightarrow [y], \quad [-y] \leftrightarrow [x] \leftrightarrow [(x+1)] \leftrightarrow [w]$$

and all the other arrows of $Gr(\theta)$ are the same as in $Gr(\pi)$.

**Observation 4.2** Let $\pi \in B_n$, $x, y, z, w \in H_n$, which satisfy the conditions of Definition 4.7, and $\theta \in B_n$ which is obtainable from $\pi$ by an $([x] - [y])$-exchange operation. Then, the following properties hold:

- If $x = w$ or $y = z$ then $\theta = \pi$ (i.e., the $([x] - [y])$-exchange operation does not do anything to $\pi$);
- In case $m(\pi) = 0$ (i.e., $\pi(i) > 0$ for every $1 \leq i \leq n$), then $m(\theta) = 0$ as well, such that $|\theta|$ is obtained from $|\pi|$ by $(x - y)$-exchange operation as defined in [6], by the same $x$ and $y$;
- $\theta^* = (x, y, w)(-y, -(x+1), -z) \cdot \pi^*$ (i.e., obtaining $\theta$ by an $([x] - [y])$-exchange operation on $\pi$ changes just the location of $y$ from being between $x + 1$ and $w$ in a cycle of $\pi^*$, to being located between $z$ and $x$ in a cycle of $\theta^*$, and the location of $-y$ from being between $-w$ and $-(x+1)$ in a cycle of $\pi^*$ to being located between $-x$ and $-z$ in a cycle of $\theta^*$);
- If $\text{sign} (\pi^{-1}_x) \neq \text{sign} (\pi^{-1}_{x+1})$ then either $m(\theta) = m(\pi) - 1$ or $m(\theta) = m(\pi) + 1$;
- If $\text{sign} (\pi^{-1}_x) = \text{sign} (\pi^{-1}_{x+1})$ then $m(\theta) = m(\pi)$;
- $s(\pi) = s(\theta)$;
- $Pr_\pi(G) = Pr_\theta(G)$;
- Obtaining $\theta$ by performing a $([x] - [y])$-exchange operation on $\pi$ if and only if obtaining $\pi$ by performing a $([-x-1] - [y])$-exchange operation on $\theta$.

**Proof.** The proof of most of the parts of the observation is a direct consequence of the definition. The proof of the part $Pr_\pi(G) = Pr_\theta(G)$, in case $\theta$ is obtainable from $\pi$ by a $([x] - [y])$-exchange operation, can be proved by the same argument as in the case of $S_n$, which has been proved in Section 4.1. of [6].

**Example 4.3** Consider the following signed-permutation $\pi \in B_n$

- $\pi = (6 \ 2 \ 3 \ 4 \ 5 \ 1)$. Then

$$\pi^* = (0+1, -5, +4, -3, +2, +6) \cdot (-6, -2, +3, -4, +5, -1, -0).$$

Thus,

$$\pi^o = \pi^* \cdot (0+1, -5, +2, +3, +4, +5, +6) \cdot (-6, -5, -4, -3, -2, -1, -0)
\quad = (0+1, -5, +2, +3, +4, +5, +6) \cdot (-6, -5, -4, -3, -2, -1, -0)$$

$$\quad = (0+1, -5, +2, +3, +4, +5, +6).$$
Thus, \( \theta = (+2, -4, +5) \cdot (+4, -3, +3) \cdot \pi \)
\begin{align*}
\theta^* &= (+2, -4, +5) \cdot (+4, -3, +3) \cdot \pi^* \\
&= (+0, +1, -5, -3, -4, +2, +6) \cdot (-6, -2, +4, +3, +5, -1, -0).
\end{align*}

Thus,
\begin{align*}
\theta^* &= \theta^* \cdot (+0, +1, +2, +3, +4, +5, +6) \cdot (-6, -5, -4, -3, -2, -1, -0) \\
&= (+0, -5, +2, +5) \cdot (-6, -3, +4, -1) \cdot (+1, +6) \cdot (-0, -2) \cdot (-4) \cdot (+3).
\end{align*}

By performing a \((+2) \leftrightarrow [-4]\)-exchange operation on \(\pi\)
(we have: \(x = (+2), \ y = (-4), \ w = (+5), \ z = (-3)\)) we obtain \(\theta\) such that:
\[\theta^* = (+2, -4, +5) \cdot (+4, -3, +3) \cdot \pi^* \]
\[= (+0, +1, -5, -3, -4, +2, +6) \cdot (-6, -2, +4, +3, +5, -1, -0).\]

Thus,
\begin{align*}
\theta^* &= \theta^* \cdot (+0, +1, +2, +3, +4, +5, +6) \cdot (-6, -5, -4, -3, -2, -1, -0) \\
&= (+0, -5, +2, +5) \cdot (-6, -3, +4, -1) \cdot (+1, +6) \cdot (-0, -2) \cdot (-4) \cdot (+3).
\end{align*}

By performing a \((-3) \leftrightarrow [-4]\)-exchange operation on \(\theta\) we obtain \(\pi\):

\[\pi = (+5, +3, +2, +6, -4, -1).\]

Then
\[\pi^* = (+0, -1, -4, +6, +2, +3, +5) \cdot (-5, -3, -2, -6, +4, +1, -0).\]

Thus,
\begin{align*}
\pi^* &= \pi^* \cdot (+0, +1, +2, +3, +4, +5, +6) \cdot (-6, -5, -4, -3, -2, -1, -0) \\
&= (+0, -0, +4) \cdot (-5, +6, -1) \cdot (+1, +3) \cdot (-4, -2) \cdot (+2, +5) \cdot (-6, -3),
\end{align*}

By performing a \((-3) \leftrightarrow [-4]\)-exchange operation on \(\pi\)
(we have: \(x = (+3), \ y = (+1), \ w = (-0), \ z = (+2)\)) we obtain \(\theta\) such that:
\[\theta^* = (+3, +1, -0) \cdot (-1, -4, -2) \cdot \pi^* \]
\[= (+0, -4, +6, +2, +1, +3, +5) \cdot (-5, -3, -1, -2, -6, +4, -0).\]

Thus,
\begin{align*}
\theta^* &= \theta^* \cdot (+0, +1, +2, +3, +4, +5, +6) \cdot (-6, -5, -4, -3, -2, -1, -0) \\
&= (+0, +3, -0, +4) \cdot (-5, +6, -4, -1) \cdot (+1) \cdot (-2) \cdot (+2, +5) \cdot (-6, -3).
\end{align*}
For Definition 4.4
Then we define

and each

with

If

with

If

sign

then by performing a ([−4] − [−1])-exchange operation on θ, we obtain π.

Definition 4.4 For π ∈ Bn, such that π∗ ∈ S(2 + 2n) contains one of the following:

• \( x + 1 \to y \to x \), (If \( x = +n \), becomes \( +0 \to y \to +n \)), and assume \( \text{sign} (x) = (+) \);

• \( -x - 1 \to y \to x \), (If \( x = +n \), becomes \( -0 \to y \to +n \)), and assume \( \text{sign} (x) = (+) \);

• \( -x + 1 \to y \to x \), (If \( x = -n \), becomes \( +0 \to y \to -n \)), and assume \( \text{sign} (x) = (-) \).

Then we define \( (|x| - |y|) \)-cyclic operation as follows (In case π∗ contains \( x + 1 \to y \to x \) and \( \text{sign} (x) = \text{sign} (y) (+) \), the definition is the same as in [4]):

• If \( \text{sign} (y) = (+) \), \( x + 1 \to y \) in π∗, and \( y > x + 1 \), then in π∗ we replace \( x + 1 \) with \( y - 1 \) and each \( t \), for \( t = x + 2, ..., y - 1 \), we replace with \( t - 1 \). We also replace \( -x - 1 \) with \( -y - 1 \) and each \( t \), where \( t = -x - 2, ..., -y + 1 \), we replace with \( t + 1 \);

• If \( \text{sign} (y) = (+) \), \( x + 1 \to y \) in π∗, and \( y < x \), then in π∗ we replace \( x \) with \( y + 1 \) and each \( t \), for \( t = y + 1, ..., x - 1 \), we replace with \( t + 1 \). We also replace \( -x \) with \( -y - 1 \) and each \( t \), for \( t = -y - 1, ..., -x + 1 \), we replace with \( t - 1 \);

• If \( \text{sign} (y) = (-) \), \( x + 1 \to y \) in π∗, and \( |y| > x + 1 \), then in π∗ we replace \( x + 1 \) with \( y \) and each \( t \), where \( t = x + 2, ..., |y| \), we replace with \( t + 1 \). We also replace \( -x - 1 \) with \( -y \) and each \( t \), where \( t = -x - 2, ..., -|y| \), we replace with \( t + 1 \);

• If \( \text{sign} (y) = (-) \), \( x + 1 \to y \) in π∗, and \( |y| < x \), then in π∗ we replace \( x \) with \( y \) and each \( t \), for \( t = |y| + 1, ..., x - 1 \), we replace with \( t + 1 \). We also replace \( -x \) with \( -y - 1 \) and each \( t \), for \( t = -|y| - 1, ..., -x + 1 \), we replace with \( t - 1 \);

• If \( \text{sign} (y) = (+) \), \( -x - 1 \to y \) in π∗, and \( y > x + 1 \), then in π∗ we replace \( x + 1 \) with \( y - 1 \) and each \( t \), where \( t = x + 2, ..., y - 1 \), we replace with \( t - 1 \). We also replace \( -x - 1 \) with \( -y - 1 \) and each \( t \), where \( t = -x - 2, ..., -|y| \), we replace with \( t + 1 \);

• If \( \text{sign} (y) = (+) \), \( -x - 1 \to y \) in π∗, and \( y < x \), then in π∗ we replace \( x + 1 \) with \( y \) and each \( t \), for \( t = y, ..., x \), we replace with \( t + 1 \). We also replace \( -x - 1 \) with \( -y - 1 \) and each \( t \), for \( t = -y, ..., -x \), we replace with \( t - 1 \);
Example 4.7 Consider the following signed-permutation

Notice the following observations concerning arguments.

The proof of the first three parts of the observation comes directly from the definition of \(|x|\) and \(|y|\)-cyclic operation as defined in Definition 3.7. The last part has been proved partially in Section 4.2. of [6] (The case where \(\pi^*\) contains \(x + 1 \rightarrow y \rightarrow x\), and where \(\text{sign}(y) = (+)\)), and for the other cases, it can be proved by very similar arguments.

Example 4.7 Consider the following signed-permutation \(\pi \in B_n\)
• \(\pi = (-2 \ -6 \ +3 \ +1 \ +5 \ +4).\) Then

\[\pi^* = (+0, +4, +5, +1, +3, -6, -2) \cdot (+2, +6, -3, -1, -5, -4, -0).\]

We can perform \([-2] - [-6]\)-cyclic operation on \(\pi\) and obtain a new permutation \(\theta\). We have

\[\theta^* = (+0, +3, +4, +1, +2, -6, +5) \cdot (-5, +6, -2, -1, -4, -3, -0).\]

Hence, \(\theta = (+5 \ -6 \ +2 \ +1 \ +4 \ +3);\)

• \(\pi = (-5 \ +3 \ +6 \ +1 \ +4 \ +2).\) Then

\[\pi^* = (+0, +2, +4, +1, +6, +3, -5) \cdot (+5, -3, -6, -1, -4, -2, -0).\]

We can perform \([-5] - [-3]\)-cyclic operation on \(\pi\) and obtain a new permutation \(\theta\). We have

\[\theta^* = (+0, +2, +5, +1, +6, +3, -4) \cdot (+4, -3, -6, -1, -5, -2, -0).\]

Hence, \(\theta = (+4 \ +3 \ +6 \ +1 \ +5 \ +2);\)

• \(\pi = (+2 \ -6 \ -3 \ +1 \ +5 \ +4).\) Then

\[\pi^* = (+0, +4, +5, +1, -3, -6, +2) \cdot (-2, +6, +3, -1, -5, -4, -0).\]

We can perform \([+2] - [-6]\)-cyclic operation on \(\pi\) and obtain a new permutation \(\theta\). We have

\[\theta^* = (+0, +3, +4, +1, +6, -5, +2) \cdot (-2, +5, -6, -1, -4, -3, -0).\]

Hence, \(\theta = (+2 \ -5 \ +6 \ +1 \ +4 \ +3);\)

• \(\pi = (+5 \ +3 \ -6 \ +1 \ +4 \ +2).\) Then

\[\pi^* = (+0, +2, +4, +1, -6, +3, +5) \cdot (-5, -3, +6, -1, -4, -2, -0).\]

We can perform \([+5] - [-3]\)-cyclic operation on \(\pi\) and obtain a new permutation \(\theta\). We have

\[\theta^* = (+0, +2, +5, +1, -3, +4, +6) \cdot (-6, -4, +3, -1, -5, -2, -0).\]

Hence, \(\theta = (+6 \ +4 \ -3 \ +1 \ +5 \ +2).\)

**Definition 4.8** For \(\pi \in B_n\), such that \(\pi^* \in S(2 + 2n)\) contains either

\[|x| \to -|x| + 1, \quad |x| + 1 \to -|x|,\]

or

\[-|x| \to |x| + 1, \quad -|x| + 1 \to |x|,\]

we define \(([|x|] - [|x| + 1])\)-sign-change operation by replacing in \(\pi^*\) the following replacements:
• Replace $x$ by $-x$ and $-x$ by $x$;
• Replace $x+1$ by $-x-1$ and $-x-1$ by $x+1$.

i.e., If $\pi^*$ contains:
\[
w \to |x| \to -(|x|+1) \to z, \quad -z \to |x|+1 \to -|x| \to -w,
\]
then by performing a $([|x|] - ([|x|+1])$-sign-change operation on $\pi$ we obtain $\theta$ such that $\theta^*$ contains:
\[
w \to -|x| \to |x|+1 \to z, \quad -z \to -(|x|+1) \to |x| \to -w.
\]
By performing a $([|x|] - ([|x|+1])$-sign-change operation on $\theta$ we obtain back $\pi$.

**Observation 4.9** Let $\pi$ and $\theta$ be two signed-permutations of $B_n$ such that $\theta$ is obtainable from $\pi$ by a $([|x|] - ([|x|+1])$-sign-change operation. Then the following properties hold:

- If $|x| \neq 0$ or $|x| \neq n$, then $m(\theta) = m(\pi)$;
- $s(\theta) = s(\pi)$;
- $Pr(\pi(G)) = Pr(\theta(G))$.

**Proof.** The proof of the first part comes directly from Definition 1.8. Thus, we turn to the second part of the observation. Assume $\pi^*$ contains:
\[
w \to |x| \to -(|x|+1) \to z, \quad -z \to |x|+1 \to -|x| \to -w,
\]
Therefore, $Gr(\pi)$ has the form:
\[
\ldots \leftrightarrow [w] \leftrightarrow [-|x|] \leftrightarrow [|x|-1] \leftrightarrow \ldots,
\ldots \leftrightarrow [-z] \leftrightarrow [-(|x|+1)] \leftrightarrow [|x|+1] \leftrightarrow [-(|x|+2)] \leftrightarrow \ldots
\]
Now, we obtain $\theta$ by performing a $([|x|] - ([|x|+1])$-sign-change operation, such that $\theta^*$ contains:
\[
w \to -|x| \to |x|+1 \to z, \quad -z \to -(|x|+1) \to |x| \to -w,
\]
Therefore, $Gr(\theta)$ has the form:
\[
\ldots \leftrightarrow [w] \leftrightarrow [|x|] \leftrightarrow [-(|x|+1)] \leftrightarrow [-|x|] \leftrightarrow [|x|-1] \leftrightarrow \ldots,
\ldots \leftrightarrow [-z] \leftrightarrow [|x|+1] \leftrightarrow [-(|x|+2)] \leftrightarrow \ldots
\]
Hence, $s(\pi) = s(\theta)$. Now, we prove the last part of the observation, i.e., $Pr(\pi(G)) = Pr(\theta(G))$, in case $\theta$ is obtainable from $\pi$ by a $([|x|] - ([|x|+1])$-sign-change operation.
First, assume $|x| \neq 0$ and $|x| \neq n$. Assume also, $\pi_k = |x|+1$ and $\pi_{k+1} = -|x|$, for some $+1 \leq k \leq n-1$. Then:
\[
Pr(\pi(G)) = Pr(a_1 \cdots a_{x-1}a_x(a_{x+1}a_{x-1})a_xa_{x+2} \cdots a_n = a_{\pi_k}^1 \cdots a_{\pi_{k-1}}^{\epsilon_{\pi_{k-1}}} (a_{x+1}a_{x-1})a_{\pi_{k+1}}^{\epsilon_{\pi_{k+1}}} \cdots a_{\pi_n}^{\epsilon_{\pi_n}})
\]
\[
= Pr(a_1 \cdots a_{x-1}a_x(a_{x+1}a_{x-1})a_xa_{x+2} \cdots a_n = a_{\pi_k}^1 \cdots a_{\pi_{k-1}}^{\epsilon_{\pi_{k-1}}} (a_{x+1}a_{x-1})a_{\pi_{k+1}}^{\epsilon_{\pi_{k+1}}} \cdots a_{\pi_n}^{\epsilon_{\pi_n}})
\]
\[
= Pr(a_1 \cdots a_n = a_{\theta_{\pi_k}}^1 \cdots a_{\theta_{\pi_n}})
\]
\[
= Pr(\theta(G)).
\]
The case, $|x| \neq 0$, $|x| \neq n$, $\pi_k = |x|$ and $\pi_{k+1} = -(|x| + 1)$, can be proved by the same argument, while showing

$$\Pr([a_1 \cdots a_n]^{-1} = [a_{\pi_1}^{\epsilon_{\pi_1}} \cdots a_{\pi_n}^{\epsilon_{\pi_n}}]^{-1}) = \Pr([a_1 \cdots a_n]^{-1} = [a_{\theta_1}^{\epsilon_{\theta_1}} \cdots a_{\theta_n}^{\epsilon_{\theta_n}}]^{-1}).$$

Now, it remains to show $\Pr_\pi(G) = \Pr_\theta(G)$, in case $\theta$ is obtainable from $\pi$ by a $([-|x|] - [n] - [-|x| + 1])$-sign-change operation, where $|x| = 0$ or $|x| = n$. Assume, $|x| = n$ (The case of $|x| = 0$ is proved by the same argument). Consider the following notations:

1. $\alpha = a_1 a_2 \cdots a_{n-1};$
2. $\beta = a_{\pi_1}^{\epsilon_{\pi_1}} \cdot a_{\pi_2}^{\epsilon_{\pi_2}} \cdots a_{\pi_{n-1}}^{\epsilon_{\pi_{n-1}}}$.

Then:

$$\Pr_\pi(G) = \Pr(\alpha \cdot a_n = \beta \cdot a_n^{-1}) = \Pr(a_n^2 = \alpha^{-1} \cdot \beta),$$

where $a_n$ is independent on $\alpha$ and on $\beta$. Since we obtain $\theta$ from $\pi$ by a $([n] - [-0])$-sign-change operation, we have:

$$\Pr_\theta(G) = \Pr(\alpha \cdot a_n = \beta^{-1} \cdot a_n^{-1}) = \Pr([a_n \cdot a_n]^{-2} = \beta \cdot \alpha^{-1}) = \Pr(\gamma^2 = \alpha^{-1} \cdot \beta),$$

where $\gamma = \beta^{-1} \cdot a_n \cdot \alpha \cdot \beta$. Since, $a_n$ is independent on $\alpha$ and on $\beta$, we have $\gamma$ is independent on $\alpha$ and on $\beta$ as well. Hence, $\Pr_\pi(G) = \Pr_\theta(G)$.  

**Example 4.10** Consider the following signed-permutation $\pi \in B_n$

- $\pi = \langle -6 - 2 + 3 + 1 + 5 + 4 \rangle$. Then
  
  $$\pi* = (+0,+4,+5,+1,+3,-2,-6) \cdot (+6,+2,-3,-1,-5,-4,-0).$$
  
  We can perform $([-2] - [-3])$-sign-change operation on $\pi$ and obtain a new permutation $\theta$. We have
  
  $$\theta* = (+0,+4,+5,+1,+3,+2,-6) \cdot (+6,-2,+3,+1,-5,-4,-0).$$
  
  Hence, $\theta = \langle -6 + 2 - 3 + 1 + 5 + 4 \rangle$;

- $\pi = \langle +3 - 4 + 5 + 2 - 1 - 6 \rangle$. Then
  
  $$\pi* = (+0,-6,-1,+2,+5,-4,+3) \cdot (-3,+4,-5,-2,+1,+6,-0).$$
  
  We can perform $([-6] - [-0])$-sign-change operation on $\pi$ and obtain a new permutation $\theta$. We have
  
  $$\theta* = (-0,+6,-1,+2,+5,-4,+3) \cdot (-3,+4,-5,-2,+1,-6,+0).$$
  
  Hence, $\theta = \langle -6 + 1 - 2 - 5 + 4 - 3 \rangle$

**Definition 4.11** Two signed permutations $\pi$ and $\theta$ in $B_n$ are considered $([|x|] - [|y|])$-equivalent, if it is possible to obtain either $\pi$ from $\theta$ or $\theta$ from $\pi$ by a finite sequence of $([|x|] - [|y|])$-exchange, $([|x|] - [|y|])$-cyclic, or $([|x|] - [0])$-sign-change operations.
Proposition 4.12 Let \( \theta \) and \( \pi \) be two sign permutations in \( B_n \) which are \((|x| - |y|)\)-equivalent, then \( m(\pi) = 0 \) if and only if \( m(\theta) = 0 \).

Proof. Let \( \theta \) and \( \pi \) be two sign permutations in \( B_n \) which are \((|x| - |y|)\)-equivalent. Then either \( \theta \) is obtainable from \( \pi \) or \( \pi \) is obtainable from \( \theta \) by a sequence of \((|x| - |y|)\)-exchange, \((|x| - |y|)\)-cyclic, and \((||x|| - ||x| + 1||)\)-sign-change operations. Therefore, it is enough to prove the following: The case where \( \theta \) is obtainable from \( \pi \) either by one \((|x| - |y|)\)-exchange, or by one \((|x| - |y|)\)-cyclic, or by one \((||x|| - ||x| + 1||)\)-sign-change operation. Then, by Observations 4.2, 4.6, 4.9, we conclude \( m(\theta) = 0 \) if and only if \( m(\pi) = 0 \). \( \blacksquare \)

Lemma 4.13 Let \( \pi \) and \( \theta \) in \( B_n \) which are \((|x| - |y|)\)-equivalent, then

\[ s(\pi) = s(\theta) \]

\[ \Pr_\pi(G) = \Pr_\theta(G) \]

in every finite group \( G \).

Proof. By Observations 4.2, 4.6, and 4.9 \( s(\pi) = s(\theta) \) and \( \Pr_\pi(G) = \Pr_\theta(G) \) if \( \theta \) is obtainable from \( \pi \) either by one \((|x| - |y|)\)-exchange, or by one \((|x| - |y|)\)-cyclic, or by one \((||x|| - ||x| + 1||)\)-sign-change operation. Therefore, the results hold in case where \( \theta \) is obtainable from \( \pi \) by any finite number of \((|x| - |y|)\)-exchange, or \((|x| - |y|)\)-cyclic or \((||x|| - ||x| + 1||)\)-sign-change operations. \( \blacksquare \)

5 The main result

In this section, we prove the main result of the paper, whereby Theorem 5.6 we show a strong connection between \( \Pr_\pi(G) \) and \( s(\pi) \), the number of the alternating cycles of \( \pi \) in the breakpoint graph \( Gr(\pi) \), for every \( \pi \in B_n \). Theorem 5.6 is a generalization of the main theorem of [6], where it have been proved that for \( \pi \in S_n \), the generalized commuting probability, \( \Pr_\pi(G) \) depends on the number of the alternating cycles on the cycle graph of \( \pi \). The proof of Theorem 5.6 makes use of several technical lemmas.

Lemma 5.1 Let \( \pi' \in B_{n-1} \) be a signed permutation, and let \( \pi \) be a signed permutation in \( B_n \), such that:

\[ \pi_i = \pi'_i \quad \text{for every } i \text{ such that } |i| \leq n - 1; \]

\[ \pi_{+n} = -n. \]

Then \( s(\pi) = s(\pi') \).

Proof. Look at the arrows of \( Gr(\pi) \) and \( Gr(\pi') \). Since \( \pi_i = \pi'_i \) for every \( i \) such that \( |i| \leq n - 1 \), all the arrows of \( Gr(\pi) \) and \( Gr(\pi') \) are the same, apart from the following segment:

\[ \pi_{+n} \leftrightarrow [-0] \leftrightarrow [+n] \leftrightarrow \ldots \]
• In $Gr(\pi)$ the following holds:

$$\ldots \Leftrightarrow [\pi_{+(n-1)}] \Leftrightarrow [+n] \Leftrightarrow [-0] \Leftrightarrow [-n] \Leftrightarrow [+\pi(n-1)] \Leftrightarrow \ldots$$

Hence, $s(\pi) = s(\pi')$. ■

Lemma 5.2 Let $\pi'$ and $\theta'$ be two $([x] - [y])$-equivalent signed permutations in $B_{n-1}$. Let $\pi$ and $\theta$ be two signed permutations in $B_n$ such that:

- $\pi'_i = \pi_i$ and $\theta'_i = \theta_i$ for every $i$ such that $|i| \leq n - 1$;
- $\pi_{+n} = \theta_{+n} = -n$.

Then $\pi$ and $\theta$ are $([x] - [y])$-equivalent as well.

Proof. It is enough to show that $\pi$ and $\theta$ are $([x] - [y])$-equivalent in case $\theta'$ is obtainable by either performing a single $([x] - [y])$-exchange, or a single $([x] - [y])$-cyclic, or a single $([|x|] - [|x| + 1])$-sign-change operation on $\pi'$. If $\theta'$ is obtainable by performing a $([x] - [y])$-exchange operation on $\pi'$ for $x \neq +(n-1)$, then we obtain $\theta$ by performing a $([x] - [y])$-exchange operation on $\pi$ for the same $x$ and $y$. Therefore, assume $\theta'$ is obtainable by performing a $([+\pi(n-1)] - [y])$-exchange operation on $\pi'$. Then by Observation 4.2 $y = \pi'_{+(n-1)}$ and either $\pi'_i = +(n-1)$ and then

$$\theta' = \langle \pi_{+1} \cdots + (n-1) \pi_{+(n-1)} \pi_{i+1} \cdots \pi_{+(n-2)} \rangle$$

or $\pi_i = -(n-1)$ and then

$$\theta' = \langle \pi_{+1} \cdots \pi_{-(n-1)} - (n-1) \pi_{i+1} \cdots \pi_{+(n-2)} \rangle.$$ 

Now, by performing a $([-0] - [-\pi_{+(n-1)}])$-exchange operation on $\pi$, we obtain

$$\tau = \langle \pi_{+1} \cdots \pi_{+(n-2)} - n \pi_{-(n-1)} \rangle.$$ 

Then by performing a $([+\pi(n-1)] - [-\pi_{+(n-1)}])$-exchange operation on $\tau$, we obtain $\theta$. Now assume, $\theta'$ is obtainable by performing a $([x] - [y])$-cyclic operation on $\pi'$. If $x \neq n - 1$ or $x \neq -(n-1)$, then one can obtain $\theta$ by performing a $([x] - [y])$-cyclic operation on $\pi$, for the same $x$ and $y$. Therefore, assume either $x = n - 1$ or $x = -(n-1)$. If $x = n - 1$ and $\pi^*_{0} = y$, then

$$\pi^* = (+0, -n, y, n-1, \ldots) \cdot (\ldots, -n + 1, -y, +n, -0).$$

In case $\text{sign}(y) = (+)$, we obtain $\theta$ by performing first a $([+\pi(n-1)] - [y])$-cyclic, then a $([y] - [+n])$-exchange, and finally a $([+0] - [-y])$-exchange operation. In case $\text{sign}(y) = (-)$, we obtain $\theta$ by performing first $([+\pi(n-1)] - [y])$-cyclic, then $([y] - [-n])$-cyclic, and finally $([+n] - [-y + 1])$-exchange operation. By similar arguments, $\theta$ is obtainable from $\pi$ in all the rest of the cases, where $\theta'$ is obtainable from $\pi'$ by performing a $([x] - [y])$-cyclic operation. Now assume, $\theta'$ is obtainable by a $([|x|] - [|x| + 1])$-sign-change operation on $\pi'$. If $|x| \neq n - 1$, then one can obtain $\theta$ by
performing a \([|x|] - [x + 1]\)-sign-change operation on \(\pi\), for the same \(|x|\). Therefore, assume \(|x| = n - 1\). Then we have
\[
\pi^* = (+0, -n, -(n - 1), \pi_{+}(n - 2), \ldots, \pi_{+1}) \cdot (\pi_{-1}, \ldots, \pi_{-(n - 2)}, + (n - 1), + n, -0).
\]
By performing a \([n] - [0]\)-sign-change operation on \(\pi\), we obtain \(\mu\) such that
\[
\mu^* = (+0, \pi_{-1}, \ldots, \pi_{-(n - 2)}, + (n - 1), -n) \cdot (+ n, -(n - 1), \pi_{+}(n - 2), \ldots, \pi_{+1}, -0).
\]
By performing a \([n - 1] - [n]\)-sign-change operation on \(\mu\), we obtain \(\eta\) such that
\[
\eta^* = (+0, \pi_{-1}, \ldots, \pi_{-(n - 2)}, -(n - 1), + n) \cdot (-n, + (n - 1), \pi_{+}(n - 2), \ldots, \pi_{+1}, -0).
\]
Finally, by performing a \([+ (n - 1)] - [-0]\)-cyclic operation on \(\eta\), we obtain the desired \(\theta\), where:
\[
\theta^* = (+0, -n, \pi_{-1}, \ldots, \pi_{-(n - 2)}, -(n - 1)) \cdot (+ (n - 1), \pi_{+}(n - 2), \ldots, \pi_{+1}, + n, -0).
\]

\[\blacksquare\]

**Lemma 5.3** Let \(\pi \in B_n\) be a signed permutation such that \(\pi_i = -n\), for some \(1 \leq i \leq n\), then there exists \(\theta \in B_n\) such that the following holds:
\begin{itemize}
  \item \(\theta + n = -n\);
  \item \(\pi\) is obtainable by a sequence of \(n - i\) times \([-0] - [-y]\)-exchange operations repeatedly starting on \(\theta\), such that \(\theta_j = \pi_j\) for every \(+1 \leq j < i\);
  \item \(\theta\) and \(\pi\) are \([|x|] - [y]\)-equivalent;
  \item \(s(\theta) = s(\pi)\).
\end{itemize}

**Proof.** Let \(\pi \in B_n\) be a signed permutation such that \(\pi_i = -n\) for some \(+2 \leq i \leq +n\). Then,
\[
\pi^* = (+0, \pi_{+n}, \ldots, \pi_{i + 1}, -n, \pi_{i - 1}, \ldots, \pi_{+1}) \cdot (\pi_{-1}, \ldots, \pi_{-(i - 1)}, + n, \pi_{-(i + 1)}, \ldots, \pi_{-n}, -0).
\]
Now, by performing a \([-0] - [-[\pi_{i-1}]]\)-exchange operation, we obtain \(\mu\) such that
\[
\mu^* = (+0, \pi_{-(i - 1)}, \pi_{+n}, \ldots, \pi_{i + 1}, - n, \pi_{i - 2}, \ldots, \pi_{+1}) \cdot (\pi_{-1}, \ldots, \pi_{-(i - 2)}, + n, \pi_{-(i + 1)}, \ldots, \pi_{-n}, \pi_{i - 1}, - 0).
\]
Therefore, we obtain \(\mu\) such that:
\begin{itemize}
  \item \(\mu_{i - 1} = \pi_i = -n\);
  \item \(\mu_j = \pi_j\), for \(+1 \leq j < i - 1\);
  \item \(\mu_j = \pi_j + 1\), for \(i - 1 \leq j \leq +(n - 1)\);
  \item \(\theta + n = \pi_{-(i - 1)}\).
\end{itemize}
Therefore, we conclude that every \( \pi \in B_n \) such that \( \pi_i = -n \) for some \(+1 \leq i \leq +n\), is obtainable from \( \theta \) such that \( \theta_{+n} = -n \) by performing \( ([0] - [y])\)-exchange operations \( n - i \) times repeatedly, starting on \( \theta \), such that \( \pi_j = \theta_j \) for every \(+1 \leq j < i\). Hence, we get the rest of the results of the lemma.

Lemma 5.4 Let \( \pi \in B_n \) be a signed permutation such that \( \text{sign}(\pi_i) = (+) \) and \( \pi_{i-1} = \pi_{-i} + 1 \) for some \(+2 \leq i \leq +n\), and there exists \(+1 \leq j < i\), such that \( \pi_j = +n\). Then there exists a signed permutation \( \theta \in B_n \) such that the following holds:

- \( \theta_{+n} = -n \);
- \( \theta_k = \pi_{i-1} \) for some \(+1 \leq k \leq +(n - 1)\);
- \( \theta \) and \( \pi \) are \(([x] - [y])\)-equivalent;
- \( s(\theta) = s(\pi) \).

Proof. Assume \( \pi \in B_n \) is a signed permutation such that \( \text{sign}(\pi_i) = (+) \) and \( \pi_{i-1} = \pi_{-i} + 1 \) for some \(+2 \leq i \leq +n\), and there exists \(+1 \leq j < i - 1\), such that \( \pi_j = +n\). Then, by performing \( ([\pi_i] - [y])\)-exchange operations \( i - 1 - j \) times repeatedly, where starting on \( \pi \), we obtain a signed permutation \( \mu \) such that:

- \( \mu_i = \pi_i \);
- \( \mu_{i-1} = -n \);
- \( \mu_j = \pi_{i-1} = \pi_{-i} + 1 \), which implies \( \text{sign}(\mu_j) = (-) \), since \( \text{sign}(\pi_{i-1}) = (-) \).

Now, by using Lemma 5.3, we conclude that \( \mu \) is obtainable from \( \theta \) by a sequence of \( ([0] - [y])\)-exchange operations \( n - i - 1 \) times repeatedly, such that:

- \( \theta_{+n} = -n \);
- \( \theta_k = \mu_k \) for every \(+1 \leq k < i - 1\).

Since \( j < i - 1 \), we have \( \theta_j = \mu_j = \pi_{-i} + 1 \). Therefore, \( \text{sign}(\theta_j) = (-) \). Now, since \( \theta \) and \( \mu \) are \([x] - [y]\)-equivalent and \( \mu \) and \( \pi \) are \([x] - [y]\)-equivalent too, we have \( \theta \) and \( \pi \) are \([x] - [y]\)-equivalent as well. Hence, the lemma holds.

Lemma 5.5 Let \( \pi \in B_n \) be a signed permutation such that \( m(\pi) \geq 1 \), then there exists \( \theta \in B_n \) such that:

- \( \theta_{+n} = -n \);
- \( \theta \) and \( \pi \) are \([x] - [y]\)-equivalent;
- \( s(\theta) = s(\pi) \).
Proof. The proof is in induction on $n$. Notice, $\pi = (-1)$ is the only element in $B_1$ with $m(\pi) \geq 1$, therefore the lemma holds trivially in case of $n = 1$. Assume by induction that the lemma holds for $n_0 < n$, and we prove it for $n_0 = n$. If $\pi_i = -n$ for some $+0 \leq i \leq +n$, then by Lemma 5.3, $\pi$ is $(x - [y])$-equivalent to a signed permutation $\theta \in B_n$, such that $\theta_{+n} = -n$ and $s(\theta) = s(\pi)$. Moreover, by Lemma 5.2, $s(\theta) = s(\theta')$, where $\theta' \in B_{n-1}$, such that $\theta'_i = \theta_i$ for every $-(n-1) \leq i \leq +(n-1)$. If $\text{sign}(\theta'_i) = (-)$ for some $+1 \leq j \leq +(n-1)$, then by the induction hypothesis, $\theta'$ is $(x - [y])$-equivalent to every $\tau'$ such that $s(\tau') = s(\theta')$ and $\text{sign}(\tau'_k) = (-)$ for some $+1 \leq k \leq +(n-1)$. Then by Lemma 5.2, $s(\tau) = s(\tau')$, where $\tau \in B_n$ such that $\tau_i = \tau_i'(i)$ for every $-(n-1) \leq i \leq +(n-1)$ and $\tau_{+n} = -n$. Since $\theta'$ is $(x - [y])$-equivalent to $\tau'$, by Lemma 5.1, $\theta$ is $(x - [y])$-equivalent to $\tau$ as well. Thus we conclude, $\theta$ is $(x - [y])$-equivalent to $\tau$ if the following holds:

- $\theta_{+n} = \tau_{+n} = -n$;
- $s(\theta) = s(\tau)$;
- $\text{sign}(\theta_i) = \text{sign}(\tau_j) = (-)$ for some $+1 \leq i, j \leq +(n-1)$.

Now, consider $\tau \in B_n$ such that $\tau_{+n} = -n$ but $\text{sign}(\tau_j) = (+)$ for all $+1 \leq j \leq +(n-1)$. Then

$$\tau^* = (+0, -n, \tau_{+(n-1)}, \ldots, \tau_{+1}) \cdot (\tau_{-1}, \ldots, \tau_{-(n-1)}, +n, -0).$$

Now, by performing a $([-n] - [\tau_{+(n-1)}])$-exchange operation on $\tau$ we obtain $\mu$ such that

$$\mu^* = (+0, \tau_{-(n-1)}, -n, \ldots, \tau_{+1}) \cdot (\tau_{-1}, \ldots, +n, \tau_{+(n-1)}, -0).$$

Now, by performing a $([-n] - [\tau_{-(n-1)}])$-cyclic operation on $\mu$, we obtain $\zeta$ such that

$$\zeta^* = (+0, \tau_{-(n-1)} - 1, \tau_{+(n-1)}, \ldots, +n, \ldots) \cdot (\ldots, -n, \ldots, \tau_{-(n-1)}, \tau_{+(n-1)} + 1, -0).$$

Now, by performing a $([\tau_{+(n-1)}] - [\tau_{+(n-1)} + 1])$-sign-change operation on $\zeta$, we obtain $\eta$ such that

$$\eta^* = (+0, \tau_{+(n-1)} + 1, \tau_{-(n-1)}, \ldots, +n, \ldots) \cdot (\ldots, -n, \ldots, \tau_{+(n-1)}, \tau_{-(n-1)} - 1, -0).$$

Since $\text{sign}(\eta_{+n}) = (+)$, $\eta_{n-1} = \eta_{-n-1} + 1$, and $\eta_j = +n$ for some $+1 \leq j < +n$, by Lemma 5.4, there exists a signed permutation $\theta \in B_n$, such that:

- $\theta_{+n} = -n$;
- $\theta_k = \eta_{+(n-1)}$ for some $+1 \leq k \leq +(n-1)$;
- $\theta$ and $\eta$ are $(x - [y])$-equivalent;
- $s(\theta) = s(\eta)$.

Now, since $\eta$ and $\pi$ are $(x - [y])$-equivalent and $s(\eta) = s(\pi)$, we conclude $\pi$ and $\theta$ are $(x - [y])$-equivalent and $s(\pi) = s(\theta)$ as well.
Theorem 5.6 Let $\pi$ and $\theta$ be two sign permutations in $B_n$ such that $s(\pi) = s(\theta) = k$ then

- In case $m(\pi) = m(\theta) = 0$, then $\pi$ and $\theta$ are $([x] - [y])$-equivalent, and
  $$\Pr_{\pi}(G) = \Pr_{\theta}(G) = \Pr^{n-k+1}(G).$$

- In case $m(\pi) > 0$ and $m(\theta) > 0$, then $\pi$ and $\theta$ are $([x] - [y])$-equivalent, and
  $$\Pr_{\pi}(G) = \Pr_{\theta}(G) = \Pr^{-(n-k+1)}(G).$$

**Proof.** If $m(\pi) = 0$ then $\pi = |\pi|$, and therefore $\Pr_{\pi}(G) = \Pr_{|\pi|}(G)$. Then, we deal with the special case which have been proved in [6]. Therefore, assume $m(\pi) > 0$. Then by Lemma 5.5 $\pi$ is $([x] - [y])$-equivalent to a signed permutation $\theta$ such that $s(\theta) = s(\pi)$ and $\theta_{+n} = -n$. By Proposition 5.6, $s(I^{(k)}) = s(\pi) = s(\theta)$ for $k = n - s(\pi) + 1$. Therefore, by using Lemma 5.6 again, $I^{(k)}$ is $([x] - [y])$-equivalent to $\theta$ as well. Hence, we conclude that $I^{(k)}$ and $\pi$ are $([x] - [y])$-equivalent for $k = n - s(\pi) + 1$. Hence, we get
  $$\Pr_{\pi}(G) = \Pr_{\theta}(G) = \Pr^{-(n-k+1)}(G).$$

We notice that there are two extreme cases of finite group $G$ for applying Theorem 5.6.

1. The case of finite ambivalent group $G$, where every element $g \in G$ is conjugate to its inverse $g^{-1} \in G$.
2. The case of group of odd order $G$, where no element $g \neq 1$ is conjugate to its inverse $g^{-1} \in G$.

Hence, we get the following two corollaries.

**Corollary 5.7** Let $G$ be a finite group, then
  $$\Pr_{\theta}(G) = \Pr_{\pi}(G),$$
for every $\theta$ and $\pi$ in $B_n$ such that $s(\theta) = s(\pi)$ if and only if $G$ is an ambivalent finite group. Which means, $\Pr_{\pi}(G)$ depends only on $s(\pi)$ (The number of alternating cycles in $Gr(\pi)$), regardless of whether $\pi$ is a positive or a non-positive signed permutation.

**Proof.** The result holds by applying Corollary 2.23 in Theorem 5.6.

**Corollary 5.8** Let $G$ be a finite group, then
  $$\Pr_{\theta}(G) = \Pr_{\pi}(G) = \frac{1}{|G|},$$
for every every $\theta$ and $\pi$ in $B_n$ such that $\theta$ and $\pi$ are non-positive (which means $m(\theta) > 0$ and $m(\pi) > 0$), without any dependance on the values of $s(\theta)$ and $s(\pi)$, if and only if $G$ is an odd order group. Which means, every non-positive $\pi$ satisfies $\Pr_{\pi}(G) = \frac{1}{|G|}$, regardless of the number of alternating cycles in $Gr(\pi)$. 

24
Proof. The result holds by applying Corollary 2.25 in Theorem 5.6.

Finally, we have the following corollary, which generalizes Corollary 5.8.

**Corollary 5.9** Let $G = G_1 \bigoplus G_2$ such that $G_1$ is an abelian 2-group, and $G_2$ is a group which has an odd order (i.e., $G_1$ is the 2-sylow subgroup of $G$). Then

$$\Pr_{\theta}(G) = \Pr_{\pi}(G) = \Pr^{-1}(G) = \frac{\text{inv}(G)}{|G|},$$

for every $\theta$ and $\pi$ in $B_n$ such that $\theta$ and $\pi$ are non-positive (which means $m(\theta) > 0$ and $m(\pi) > 0$), without any dependance on the values of $s(\theta)$ and $s(\pi)$.

Proof. Now, since $G_1$ is an abelian group, by Corollary 2.26, $\Pr^{-k}(G_1) = \frac{\text{inv}(G_1)}{|G_1|}$, for every $k \in \mathbb{N}$. Similarly, since the order of $G_2$ is odd, by Corollary 2.25, $\Pr^{-k}(G_2) = \frac{1}{|G_2|}$, for every $k \in \mathbb{N}$. Hence, by using $G = G_1 \bigoplus G_2$ and Corollary 2.27, we conclude

$$\Pr^{-k}(G) = \Pr^{-k}(G_1) \cdot \Pr^{-k}(G_2) = \frac{\text{inv}(G_1)}{|G_1| \cdot |G_2|} = \frac{\text{inv}(G)}{|G|},$$

for every $k \in \mathbb{N}$. Now, by applying Theorem 5.6, we get the desired result of the corollary.

\[\blacksquare\]

**6 Conclusion and future plans**

In this paper, we generalize the results of [6], where we find an interesting connection between the signed-Hultman decomposition of a signed-permutation $\pi \in B_n$ and the signed generalized commuting probability $\Pr_{\pi}(G)$, which is a generalization of the generalized commuting probability, which was defined in [6]. In contrast to the results of [6], in the case of $\pi \in B_n$, the following changes occur:

- For $\pi \in B_n$, the parity of $n - s(\pi) + 1$ is not necessarily even, it can be any integer;

- If $G$ is a non-ambivalent group, then $s(\pi) = 2k$ induces two equivalence classes of $\Pr_{\pi}(G)$, namely:
  - $\Pr^{2k}(G) = \Pr_{\pi}(G)$ for a positive $\pi \in B_n$;
  - $\Pr^{-2k}(G) = \Pr_{\pi}(G)$ for a non-positive $\pi \in B_n$,

  such that, $\Pr^{2k}(G) \neq \Pr^{-2k}(G)$;

- If $G$ has an odd order, then $\Pr_{\pi}(G) = \Pr^{-k}(G) = \frac{1}{|G|}$ for every non-positive $\pi \in B_n$ and every $k \in \mathbb{N}$.

- If $G$ has an abelian 2-sylow subgroup, and $G$ is a direct sum of its 2-sylow subgroup with an odd order group, then $\Pr_{\pi}(G) = \Pr^{-k}(G) = \frac{\text{inv}(G)}{|G|}$ for every non-positive $\pi \in B_n$ and every $k \in \mathbb{N}$.

25
It might be interesting to find further generalizations of the generalized commuting probability. For instance, classifying the probabilities of

\[ \Pr(a_1a_2 \cdots a_n = a'_1a'_2 \cdots a'_{n}), \]

where \( \pi \) is a permutation of \( S_n \) and \( a'_i \) is a specific automorphic or anti-automorphic image of \( a_i \), under a defined automorphism of the group \( G \).

References

[1] R. M. Adin, Y. Roichman, The Flag Major Index and Group Actions on Polynomial Rings, *Europ. J. Combinatorics* 22, (2001), 431-446.

[2] N. Alexeev, A. Pologova, M. A. Alekseyev, Generalized Hultman Numbers and the Distribution of Multi-break Distances, *Algorithms for Computational Biology*, Lecture Notes in Bioinformatics 9199, (2015) 3–12.

[3] M. Amram, R. Shwartz, M. Teicher, Coxeter covers of the classical Coxeter groups, *International Journal of Algebra and Computation*, 20 (2010) 1041-1062.

[4] A. Bjorner, F. Brenti, Combinatorics of Coxeter groups, in: GTM, vol. 231, Springer, 2004.

[5] B. Bafna and P. A. Pevzner, Sorting by transpositions, *SIAM J. Discrete Math.* 11 (1998) 224-240.

[6] Y. Cherniavsky, A. Goldstein, V. E. Levit, R. Shwartz, Hultman Numbers and Generalized Commuting Probability in Finite Groups, *Journal of Integer Sequences* 20 (2017), Article 17.10.7.

[7] A. K. Das and R. K. Nath, A generalization of commutativity degree of finite groups, *Communications in Algebra* 40 (2012) 1974-1981.

[8] J. P. Doignon, A. Labarre, On Hultman Numbers, *Journal of Integer Sequences* 10 (2007), Article 07.6.2.

[9] W. H. Gupta, What is the probability that two group elements are commute?, *American Mathematical Monthly* 80 (1973) 1031-1034.

[10] S. Grusea, A. Labarre, The distribution of cycles in breakpoint graphs of signed permutations, *Discrete Applied Mathematics* 161 (2013) 1448–1466.

[11] A. Hultman, Toric permutations, Master’s thesis, Department of Mathematics, KTH, Stockholm, Sweden (1999).

[12] P. A. MacMahon, Combinatory Analysis I-II, *Cambridge University Press, London/New-York* (1916) (Reprinted by Chelsea, New-York 1960.).

[13] R. K. Nath and A.K. Das, On generalized commutativity degree of a finite group, *Rocky Mountain Journal of Mathematics* 41 (2011) 1987-2000.
[14] R. K. Nath, Some new directions in commutativity degree of finite groups: Recent developments, *Lambert Academic Publishing* (2011).

[15] L. Rowen, M. Teicher, U. Vishne, Coxeter covers of the symmetric groups, *Journal of Group Theory*, 8, (2005) 139-169.

[16] R. Shwartz, R. M. Adin, and Y. Roichman, Major Indices and Perfect Bases for Complex Reflection Groups, *The Electronic Journal of Combinatorics* 15 (2008) Research paper 61.