1. Introduction

Branched structures, such as blood vessels in a human body, arrangements of stems in leaves or branches of trees belong to nature's impressive mechanical constructs. All of them have a common hierarchical structure associated with their multi-functionality. From a mechanical point of view, branches of trees are particularly intriguing: although being slender structures, trees repeatedly cope with large oscillations caused by different types of excitations (gusty winds, water waves, impacts, or even earthquakes), exhibiting a bearable level of damage in a high majority of cases. On the other hand, man-made lightweight and slender structures are required to have highly resistant characteristics and performance, but are still sensitive to high levels of vibration, which might result in their dysfunctionality. This parallel [1] provides a strong motivation for a biomimetic strategy to design hierarchical structures that perform rapid localization of externally applied energy to themselves, where this energy can be dissipated, opening some possibilities for a wide range of engineering applications such as vibration and shock isolation, aeroelastic instability suppression or seismic mitigation.

Introducing a bio-inspired hierarchy and targeted energy transfer into adaptive structures has promising potential, but is faced with several major scientific and engineering challenges. A first step in realization of highly-promising potentials of adaptive biomimetic structures is a deep and detailed insight and understanding of how trees damp vibrations. This insight is related to the creation of appropriate mechanical and mathematical models and involves multiscale modelling: the smaller scale structure should mirror the corresponding physical mechanism as well as the parameters of a tree as a structure at the higher scale. Mechanical models of branched trees created and examined so far have been either discrete (lumped) having a finite number of degrees of freedom, or continuous/elastic—having an infinite number of degrees of freedom. Multi-degree-of-freedom mod-
els of branched trees have been assumed to contain particles/discrete masses or rigid bodies (physical or mathematical pendula) and include: a Y-shaped system of pendula [1], a system of 13 rigid, massless rods hinged together [2], symmetric or asymmetric two- and three-branch physical pendula [3], and a coupled spring-mass-damper system [4, 5]. A branched tree as an elastic structure has also been investigated by using finite element method (FEM) as it conveniently gives vibration characteristics, such as mode frequencies, mode shapes and vibration amplitudes, but also local stresses [1, 6–11].

Potential biomimetic benefits of tree-like branching have been recognised in a number of studies. The review paper [12] of damped trees’ vibrations emphasized several beneficial aspects. First, the importance of structural damping is pointed out. It is realized through the movements of branches relative to the trunk and the associated coupling and tuning. Then, the energy transfer in a branched tree is such that mechanical energy is distributed more effectively than in a tree with stiff branches. A theoretical study carried out by Rodriguez et al [9] supported these views. They investigated a fractal-type sympodial and monopodial structure by FEM, and illustrated that the modal frequencies are divided into groups related to the parts of the structure they affect, so that the corresponding modal shapes can be localized in branches. Another interesting mechanism associated with large-amplitude vibrations and, thus, geometric nonlinearity, is the so-called, damping-by-branching. Theckes et al [1] considered a simple pendulum-type Y-shape two-degree-of-freedom model of a branched dynamical system and also a beam-type Y-shape finite element model, based on which they proposed two designs of bioinspired slender structures exhibiting an efficient damping-by-branching mechanism.

The additional parameters introduced in this study are the stiffness ratio \( \kappa \), and the damping ratio \( \beta \). The stiffness ratio defines the ratio between the stiffness coefficients of the higher order with respect to the previous one. It is assumed that the stiffness between the elements of higher order is smaller than the one between those of lower order, i.e. the range of the values of \( \kappa \) is assumed as \( \kappa \in (0, 1) \), which is accordance with this characteristic in real trees. The damping ratio defines the ratio between the damping coefficients of
the dampers of higher order with respect to the lower-order one. In real in-leaf trees, the damping increases from the trunk above, but the structure considered herein does not mimic in-leaf trees and this fact is not followed, but it is assumed that $\beta = 1/2$. It should be noted that the considerations related to the the stiffness ratio $\kappa$, and the damping ratio $\beta$ have not been examined so far. However, given the way they are introduced (as visco-elastic elements between the trunk and the base, the first-order branches and the trunk, and between the first- and second-order branches), they can be treated as suitable parameters through which one can achieve certain suitable/desired design of slender engineering structures as they can be chosen and practically realised in accordance with their values obtained from the vibration analyses and the detection
of potentially beneficial mechanical behaviour. The following analyses are structured to yield such values that can be used, together with a suitable determined branching angle, for bioinspired slender engineering structures with a view to achieving trunk’s amplitude that is either minimal possible or equal to zero.

3. Structure with first-order branching \((N = 1)\)

The model shown in figure 1(a) has three degrees of freedom and three generalized coordinates are chosen to be the absolute coordinates—the angles \(\psi_1\) and \(\psi_2\) between each pendulum and its position in the static equilibrium (figure 1(c)). Equations of motion will be formed by using Lagrange’s formalism and Lagrange’s equations of the second kind. For this purpose, tree scalar functions—the potential energy, the kinetic energy and the dissipative function, are formed first.

The gravitational potential energy is neglected, so that the system potential energy stems from the deflection of the springs and is given by:

\[
V_1 = \frac{1}{2} k_1 \dot{\psi}_1^2 + \frac{1}{2} k_1 (\psi - \psi_1)^2 + \frac{1}{2} k_1 (\psi_2 - \varphi)^2.
\]

(1)

As the trunk is in rotational motion and the branches are in general plane motion, the kinetic energy is given by:

\[
T_1 = \frac{1}{2} J_0 \dot{\varphi}^2 + \frac{1}{2} m_1 \dot{v}_{C_1}^2 + \frac{1}{2} J_1 \dot{\psi}_1^2 + \frac{1}{2} m_1 \dot{v}_{C_2}^2 + \frac{1}{2} J_2 \dot{\psi}_2^2.
\]

(2)

3.1. Small free vibrations: natural frequencies and mode shapes

Calculating the kinetic energy (2)–(4) while the system passes through the equilibrium position, i.e. by putting \(\varphi = 0, \psi_1 = 0\) and \(\psi_2 = 0\) into equations (3) and (4), and using Lagrange’s equation of the second kind for small vibrations in the form \(d (\partial T / \partial \dot{q}_i) / dt + \partial V / \partial q_i = 0\) [16], where \(\dot{q}_i \in \{ \varphi, \psi_1, \psi_2 \}\), the following differential equations for small vibrations are derived:

\[
\left( \frac{1}{3} m + 2m_1 \right) \ddot{\varphi} + \frac{m_1 l_1 \cos \alpha}{2} \left( \ddot{\psi}_1 + \ddot{\psi}_2 \right) + (k + 2k_1) \varphi - k_1 (\dot{\psi}_1 + \dot{\psi}_2) = 0,
\]

(6)

\[
\frac{m_1 l_1 \cos \alpha}{3} \ddot{\psi}_1 - \frac{m_1 l_1 \cos \alpha}{3} \dot{\psi}_1 - k_1 \varphi + k_1 \psi_1 = 0,
\]

(7)

\[
\frac{m_1 l_1 \cos \alpha}{3} \ddot{\psi}_2 - \frac{m_1 l_1 \cos \alpha}{3} \dot{\psi}_2 - k_1 \varphi + k_1 \psi_2 = 0.
\]

(8)

In order to find the solution of the system of homogeneous linear differential equations (6)–(8), one can assume them in the form: \(\varphi = A \cos (\omega t + \vartheta)\), \(\psi_1 = B_1 \cos (\omega t + \vartheta)\), \(\psi_2 = B_2 \cos (\omega t + \vartheta)\), where \(A, B_1\) and \(B_2\) are the amplitudes of vibrations of the trunk and the branches, while \(\omega\) in an unknown natural (modal) frequency.

Non-trivial solutions for the amplitudes \(A, B_1\) and \(B_2\) of the resulting system of algebraic equations exist provided that its characteristic determinant \(\Delta_1\) is equal to zero:

\[
\Delta_1 = \begin{vmatrix} k + 2k_1 - \frac{1}{3} (m + 2m_1) \omega^2 & -k_1 & - \frac{m_1 l_1 \cos \alpha}{2} \omega^2 \\ -k_1 & -k_1 - \frac{m_1 l_1 \cos \alpha}{2} \omega^2 & 0 \\ -k_1 & 0 & -k_1 - \frac{m_1 l_1 \cos \alpha}{2} \omega^2 \end{vmatrix} = 0.
\]

(9)

This yields the characteristic (frequency) equation, which can be written down as the following product:

\[
\Delta_1 = \Delta_{1,1} \Delta_{1,2} = 0,
\]

(10)

where

\[
\Delta_{1,1} = k_1 - \frac{m_1 l_1^2}{3} \omega^2,
\]

(11)

\[
\Delta_{1,2} = \left[ k + 2k_1 - \frac{1}{3} (m + 2m_1) \omega^2 \right] \left( k_1 - \frac{m_1 l_1^2}{3} \omega^2 \right) - 2 \left( k_1 + \frac{m_1 l_1 \cos \alpha}{2} \omega^2 \right)^2.
\]

(12)

Solutions of equations (10)–(12) give three natural frequencies. By comparing them and using the values from table 1, one can conclude that the solution of equations (11) is between two solutions defined by equation (12). The second one, labelled by the index II, can be written down as \(\omega^* = \sqrt{3k_1 / (m l_1^2)}\) stands for the square of the natural frequency of the first pendulum mimicking the trunk.
attached to the ground via the spring of stiffness \( k \). So, this second natural frequency is proportional to the natural frequency of the trunk and the coefficient of proportionality is the double square root of the stiffness ratio. By making equation (12) equal to zero, one finds that \( \omega_2^2 / (\omega^*)^2 \) and \( \omega_{II}^2 / \omega^* \) are the functions of \( \kappa \) and \( \alpha \). Thus, when normalized with the trunk’s natural frequency, the first and the third natural frequency are found to depend both on the stiffness ratio and the branching angle: the higher the stiffness ratio (for the fixed branching angle \( \alpha = 20^\circ \)), the higher these natural frequencies (figure 2(a)). However, the higher the branching angle (for the fixed stiffness ratio \( \kappa = 0.3 \)), the first natural frequency slightly increases with the increase of the branching angle, the second one is independent of it, while the third one first decreases, then it is equal to \( \omega_2 \) (this special case is labelled by a square in figure 2(b)), and then increases. This special case when two frequencies are equal is interesting for practical applications as it reduces the number of possible modes. Equating \( \omega_{II} \) and \( \omega_{III} \), one calculates that this special case appears when \( \cos \alpha = -2\lambda^{1/3} / 3 \). Note that this value does not depend on the stiffness ratio, but depends only on the lateral branching ratio. For the lateral branching ratio used in this study (see table 1), one has \( \alpha \approx 121.947^\circ \).

A mode shape ratio that corresponds to \( \omega_{II} \) is calculated to be \( A = 0 \), while the mode shape ratio defining the ratio between the amplitudes of two branches is \( b_{II,II} = B_2 / B_1 = -1 \). This implies that the trunk does not move, while the branches rotate out-of-phase (this is shown as Mode II, figure 3). This means that the mode is localized in the branches of first order.

The mode shape ratios that corresponds to \( \omega_1 \) and \( \omega_{III} \) are found to be characterized by \( b_{III,III} = B_3 / B_1 = 1 \). In addition, it is derived that \( b_{III,II} = B_3 / B_1 = \frac{\kappa - \kappa^2}{\kappa + \kappa^2} \). For \( \omega_3 \), this ratio is positive, which means that the trunk and the branches are in phase (see Mode I, figure 3). For \( \omega_{III} \), this ratio is negative, i.e. the trunk and the branches are out-of-phase (see Mode III, figure 3). Note also that this ratio \( A / B_1 \) increases with the the stiffness ratio (for the fixed branching angle). The same happens when the stiffness ratio is fixed and the branching angle increases. For example, when \( \alpha = 20^\circ \) and \( \kappa = 0.3 \), the mode shape ratios are: (0.700 731; 1 1), (0; −1; 1), (−0.344 535; 1; 1).

Thus, the analytical solutions are found to be:

\[
\varphi(t) = b_{III,II}^3 \cos(\omega_{III}t) + b_{III,III}B_{III,III}^3 \cos(\omega_{III}t), \\
\psi_1(t) = B_{III,II} \cos(\omega_{III}t) + B_{III,III} \cos(\omega_{III}t) + B_{III,III} \cos(\omega_{III}t), \\
\psi_2(t) = B_{III,II} \cos(\omega_{III}t) - B_{III,III} \cos(\omega_{III}t) + B_{III,III} \cos(\omega_{III}t),
\]

where the constant \( B_{II,II} \), \( B_{III,II} \) and \( B_{III,III} \) are defined by the initial conditions \( \varphi(0), \psi_1(0) \) and \( \psi_2(0) \). Note that the solutions given by equation (13) correspond to zero-valued initial angular velocities. In case of non-zero-valued initial angular velocities, either phases

**Figure 2.** Natural frequencies for \( N = 1 \) as a function of: (a) the stiffness ratio for \( \alpha = 20^\circ \); (b) the branching angle for \( \kappa = 0.3 \).
or Sine terms need to be added to the expressions in equation (13).

Going back to the original equations and analysing the case when two natural frequencies coincide \( (\cos \bar{\alpha} = -2\lambda^{1/3}/3) \), one can calculate \( A = 0 \), but \( B_1 \) and \( B_2 \) are arbitrary. Thus, the first mode is \( (1; 1; 1) \), while the second one is \( (0; \text{arbitrary}; \text{arbitrary}) \), where these arbitrary values are defined by the initial amplitudes. This is a very interesting case where the trunk does not move, while the branches behave as two physical pendula uncoupled from each other, but they oscillate with the same frequency. Both modes are schematically presented in figure 4.

The general solutions corresponding to this branching angle and zero-valued initial angular velocities are:

\[
\begin{align*}
\psi(t) &= A \cos (\omega_1 t) = A \cos (0.589t), \\
\psi_1(t) &= A \cos (\omega_1 t) + B_1 \cos (\omega_2 t) = A \cos (0.589t) + B_1 \cos (2\sqrt{\kappa} t), \\
\psi_2(t) &= A \cos (\omega_1 t) + B_2 \cos (\omega_2 t) = A \cos (0.589t) + B_2 \cos (2\sqrt{\kappa} t),
\end{align*}
\]

(14)

where again the constant \( A, B_1 \) and \( B_2 \) are defined by the initial conditions \( \psi(0), \psi_1(0) \) and \( \psi_2(0) \). Generally speaking, the branches can oscillate periodically or aperiodically, depending on the relationship between two frequencies. The first interesting special case occurs when \( 0.589 = 2\sqrt{\kappa} \), i.e. when \( \kappa = 0.08677 \). The corresponding response for all three angles in equation (14) is periodic then with the angular frequency 0.589, and the responses are given by:

\[
\begin{align*}
\psi(t) &= A \cos (0.589t), \\
\psi_1(t) &= (A + B_1) \cos (0.589t), \\
\psi_2(t) &= (A + B_2) \cos (0.589t).
\end{align*}
\]

So, despite having three degrees of freedom, these bioinspired structures designed with \( \bar{\alpha} = 121.947^\circ \) and \( \kappa = 0.08677 \) will have all three elements oscillating with the same frequency, while their amplitudes will be defined by the initial amplitudes, which can be practically beneficial as a completely predefined single-frequency response. Besides this case, the response will be periodic when two existing natural frequencies are commensurable, i.e. when \( \kappa = 0.08677(n_1/n_2)^2 \), where \( n_1 \) and \( n_2 \) are integers.
3.2. Small undamped forced vibrations: dynamic absorbers

Given the fact that engineering structures are often exposed to different kinds of excitation, the next aim is to investigate biomimetic benefits of the branched structure from figures 1(a) and (c) when harmonic excitation acts on the trunk. This excitation is assumed to have the form of a torque \( M_0 \cos \Omega t \), where \( M_0 \) is its constant magnitude, while \( \Omega \) is the excitation frequency.

The differential equation for motion can again be derived from Lagrange’s equation for small vibrations

\[
\frac{d}{dt} \left( \frac{\partial T(q_i = 0)}{\partial \dot{q}_i} \right) + \frac{\partial V}{\partial q_i} = Q_i \quad [16],
\]

where \( Q_i \) represents the ‘generalized force’ [16]. The generalized force exists only for \( \varphi \), i.e. \( Q_\varphi = M_0 \cos \Omega t \).

Thus, equation (6), will not be homogenous anymore, but will get the term \( M_0 \cos \Omega t \) on the right-hand side, while equations (7) and (8) will stay in the same form. The solutions for motion can now be assumed as \( \varphi = A \cos \Omega t \), \( \psi_1 = B_1 \cos \Omega t \), \( \psi_2 = B_2 \cos \Omega t \), which yields a system of algebraic equations in the forced amplitudes \( A, B_1 \) and \( B_2 \). The solutions derived for them are:

\[
A = \frac{M_0 \left( k_1 - \frac{m_{ll} \Omega^2}{k_1} \right)}{[k + 2k_1 - \left( \frac{1}{4}m + 2m_1 \right) \Omega^2] \left( k_1 - \frac{m_{ll} \Omega^2}{k_1} \right) - 2(k_1 + \frac{m_{ll} \cos \alpha_2 \Omega^2}{k_1})^2},
\]

(15)

\[
B_1 = B_2 = \frac{M_0 \left( k_1 + \frac{m_{ll} \cos \alpha_2 \Omega^2}{k_1} \right)}{[k + 2k_1 - \left( \frac{1}{4}m + 2m_1 \right) \Omega^2] \left( k_1 - \frac{m_{ll} \Omega^2}{k_1} \right) - 2(k_1 + \frac{m_{ll} \cos \alpha_2 \Omega^2}{k_1})^2}.
\]

(16)

Analysing the nominator in equation (15), one can see that the amplitude of the trunk is equal to zero if \( \Omega^2 / \omega^* = \omega_{II}^2 / \omega_{II}^* = 4 \kappa \). So, if the excitation frequency is tuned to the second natural frequency of this tree-like structure, the trunk will not oscillate, i.e. its amplitude will be zero. This corresponds to the so-called frequency of dynamic absorber [16, 17], and is labelled by the
black dot in figure 5(a), where the amplitude-frequency diagrams for the trunk (the red thick solid line) and the branches (the blue thin solid line) are plotted. The resonances at \( \Omega = \omega_1 \) and \( \Omega = \omega_{12} \) are also labelled. It is interesting to note that at the frequency of dynamic absorber, the amplitude of the branches is negative, i.e. of the same sign and they move in-phase, while the trunk does not oscillate. This behaviour is different from the one achieved in free oscillations, Mode II (figure 3). However, the fact that two branches oscillate with the same amplitude at the same frequency as the attachments to the harmonically excited trunk (a main part of the structure), makes the whole concept similar to Den Hartog’s concept of a specially designed attachment added to the harmonically excited main structure in-line to eliminate its vibrations. The branches here play the role of attachments, but there are two of them and they are symmetrically placed with respect to the trunk (not in-line as in Den Hartog’s case). Thus, they behave as branched absorbers, which can be useful when an in-line placement of the attachment is not possible but the branched one is.

In the special case corresponding to the branching angle \( \alpha = 121.947^\circ \), all three amplitude-frequency relationships are the same

\[
A = B_1 = B_2 = \frac{3M_0}{1 - \left( \frac{1}{2} + \frac{3}{2\pi^2} \right) \left( \frac{\omega}{\omega^*} \right)^2}.
\]

The corresponding amplitude-frequency curve is plotted in figure 5(b), indicating the behavior similar to an externally excited undamped one-degree-of-freedom simple harmonic oscillator, with the resonance occurring for \( \Omega/\omega^* = 0.5892 \) and no condition for the existence of dynamic absorbers. So, in this case, the branches cannot be a dynamic absorber for the trunk. However, the fact that this structure having three degrees of freedom can experience only one resonance can be beneficial from the viewpoint of the avoidance of the resonance phenomenon.

### 3.3. Small damped forced vibrations

The results presented in the previous section have been obtained for an undamped case. This section represents its extension to the case when the dampers from figure 1(c) exist. By taking into account the dissipative function given by equation (5) and by using Lagrange’s equation of the second kind for small damped vibrations

\[
d(\partial T/\partial q_i)/\partial t + \partial D(q_i = 0)/\partial q_i + DV/\partial q_i = Q_0 \]

the equations of motion can be derived. They are not given here for brevity as they just respectively contain the terms

\[
b_2 (\dot{\psi} - \dot{\psi}_1) = b_1 (\ddot{\psi}_2 - \ddot{\psi}_1),
\]

\[
\dot{b}_1 (\dot{\psi}_2 - \dot{\psi}_1)
\]

with respect to those analysed previously. The solutions for damped externally excited motion can now be assumed as

\[
\ddot{\psi} = A \cos \Omega t + A \sin \Omega t, \\
\psi_1 = B_1 \cos \Omega t + B_1 \sin \Omega t, \\
\psi_2 = B_1 \cos \Omega t + B_1 \sin \Omega t, \\
\]

which can be solved to find the forced amplitudes

\[
A = \sqrt{A_1^2 + A_2^2}, \\
B_1 = \sqrt{B_1^2 + B_1^2}, \\
B_2 = \sqrt{B_2^2 + B_2^2}.
\]

As being fairly complicated, these solutions are not shown here, but the corresponding amplitude-frequency diagrams are plotted in figure 6 for a different non-dimensional damping coefficient

\[
\zeta = \beta \omega^*/(2k) \text{ and } \alpha = 20^\circ \text{ (figures 6(a)–(c)) and } \alpha = 121.947^\circ \text{ (figures 6(d)–(f)).}
\]

They are shown as thick lines, while the undamped case is plotted in thin lines. When the diagrams from figures 6(a)–(c) are compared to the undamped amplitude-frequency diagrams from figure 5(a), one can notice several facts regarding the increase of the non-dimensional damping coefficient: the resonant peaks are finite, but their number changes depending on the value of \( \zeta \) (for certain damping values, the second resonant peak disappears); the amplitude of the trunk that corresponded to the dynamic absorber is non-zero now. When the diagrams from figures 6(d)–(f) are compared to the undamped amplitude-frequency diagram from figure 5(b), one can notice the existence of a finite and smaller resonant peak for a larger value of \( \zeta \), as in the damped externally excited simple harmonic oscillator [16].

To further explore the influence of the non-dimensional damping coefficient on the number of resonant peaks and the disappearance of the second resonant peak, the amplitude of the trunk is analysed numerically and is presented in figure 7: two distinctive regions can be recognized, related to the existence of only one resonant peak (thus is labelled by ‘1’), or two of them (thus is labelled by ‘2’). It is seen that when the branching angle is increased from zero, the higher this angle, the smaller the critical value of the non-dimensional damping coefficient yielding one resonant peak. Thus, from the biomimetic point of view, branching is beneficial until this special branching angle as it decreases the damping needed for the existence of only one resonance peak. After the special value \( \alpha = 121.947^\circ \), the opposite situation occurs. However, this enables one to formulate another design recommendation that reflects potentially useful practical applications of these bioinspired structures with respect to the desired outcome of having only one resonant peak (not two): if there is a possibility to have a higher damping ratio in the system, the branching angle should be smaller.

Figure 7 and the boundary between the green region (one resonant peak) and the red region (two resonant peaks) gives a possibility to define a critical combination of the damping ratio and the branching angle for which the transition from two to one resonant peaks exist. Therefore, it can be used to make an appropriate choice of the set of these two parameters, or only one parameter when the other one is predefined.

Besides giving useful insights into the response from the viewpoint of the number of resonance peaks, this analysis also raises the question of an optimal damping value for the dynamic absorber and its relationship with the stiffness ratio and the branching
angle to realize a minimum amplitude of the trunk. This question is the focus of current investigations and will be reported in due course. The current focus is also on the comparison of the benefits of having these two branches acting as dynamic absorbers for the trunk with respect to Den Hartog’s one-mass absorber set in-line with the main structure [17].

3.4. Freely large-amplitude vibrations

3.4.1. Internal resonances

Given the fact that trees reasonably well cope with large-amplitude vibrations, the next aim is to investigate the behaviour of the tree-inspired structure from figures 1(a) and (c) when performing large oscillations. This section is, thus, concerned with the exact solutions of motion formed based on the exact expression for the potential and kinetic energy (1)–(4), which will cover the case of large-amplitude vibrations. The equations of motion are formed by using Lagrange’s equation d(∂T/∂q)/dt – ∂T/∂q + ∂V/∂q = 0 [16], where q ∈ {ϕ, ψ1, ψ2}. They can be represented in a non-dimensional form

\[ \varphi'' + \frac{1 + 2\kappa}{\gamma_0 + 6\lambda_0} \varphi - \frac{\omega_0^2}{\gamma_0 + 6\lambda_0} \psi_1 + \frac{\omega_0^2}{\gamma_0 + 6\lambda_0} \psi_2 + \frac{3\lambda_0^{1/3}}{2(1 + 6\lambda_0^{1/3})} \sin(\alpha + \varphi - \psi_1) \psi_1^2 = 0, \]

\[ \psi_1'' + \frac{\kappa}{\lambda_1} \psi_1 - \frac{\kappa}{\lambda_1} \varphi = \frac{3}{2} \lambda_1^{1/3} \sin(\alpha + \varphi - \psi_1) \varphi'' + \frac{3}{2\lambda_1^{1/3}} \cos(\alpha + \varphi - \psi_1) \varphi'' = 0, \]

\[ \psi_2'' + \frac{\kappa}{\lambda_2} \psi_2 - \frac{\kappa}{\lambda_2} \varphi = \frac{3}{2} \lambda_2^{1/3} \sin(\alpha + \varphi - \psi_2) \varphi'' + \frac{3}{2\lambda_2^{1/3}} \cos(\alpha + \varphi - \psi_2) \varphi'' = 0, \]

where the primes stand for the derivation with respect to non-dimensional time \( \tau = t \omega_1^* \). It is seen that the equations of motion are nonlinear now, and these nonlinearities appear due to the fact that the velocities in equations (3) and (4) are displacement-dependent, thus yielding displacement-dependent inertial terms and additional terms quadratic in generalised coordinates and velocities with displacement-dependent coefficients. It can also be recognised that the coefficient in front of \( \psi_1 \) and \( \psi_2 \) in equations (18) and (20) corresponds to \( \omega_1^* \).

It is easy to check that Mode II identified in section 3.1 and shown in figure 3, when the trunk does not oscillate but the branches do, satisfies identically these equations of motion. So, this mode exists also when the system performs large-amplitude oscillations. It represents the so-called similar normal modes [18] in this system of coupled nonlinear oscillators. Modes I and III do not satisfy the equations of motion (18)–(20), but one can expect that new modes will be born due to the existing nonlinearities. However, one can conclude that from the biomimetic point of view, the sympodial tree-like structures from figures 1(a) and (c) are characterised by additional useful characteristics: when perturbed in a way that only their branches oscillate but not the trunk as defined by these modes, such behaviour will persist both when branches perform small vibrations (see section 3.1) and large vibrations (as explained at the beginning of this paragraph).

Given the fact that the equations of motion are nonlinear and coupled, the possibilities for internal resonance and the related energy transfer between the trunk and branches do exist. In order to determine analytically conditions for internal resonances, equations of motion (18)–(20) are to be treated by the method of multiple scales. Thus, time scales are introduced as \( T_N = \varepsilon^N t, \ N = 0, 1, 2, \ldots \) where \( \varepsilon \) stands for a small parameter, and where the abbreviation \( D_N() = \partial()/\partial D_N \) exist in the derivatives:

\[ (\cdot)' = D_0 + \varepsilon D_1 + \varepsilon^2 D_2, \]

\[ (\cdot)'' = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 D_0 D_2 + \varepsilon^2 D_3^2. \]

The solutions for the generalised coordinates are assumed as the following series expansions:

\[ \varphi = \varepsilon \varphi_0 + \varepsilon^2 \varphi_1 + \ldots, \]

\[ \psi_1 = \varepsilon \psi_{10} + \varepsilon^2 \psi_{11} + \ldots, \]

\[ \psi_2 = \varepsilon \psi_{20} + \varepsilon^2 \psi_{21} + \ldots. \]

Equations (21) and (22) are substituted into equations (18)–(20), developed into series with respect to the small parameter and, then, the terms of the same perturbation order are collected. Equations of the \( \varepsilon \)-order correspond to those of a system performing small oscillations considered in section 3.1. Their solution can be assumed in the form given by equation (13), but with \( t \) replaced by \( T_0. \) Further, by combining the equations of the \( \varepsilon^2 \)-order for \( \psi_1 \) and \( \psi_2 \), introducing \( \psi_{10} = \psi_{21} - \psi_{11} \) and using \( \lambda = 1/2 \), one can derive

\[ D_1 \psi_{10} + \omega_1^2 \psi_{10} = P_1 \sin[2\omega T_0 - \alpha] \]

\[ + P_2 \sin[\omega_0 T_0 - \omega_1 T_0 - \alpha] \]

\[ + P_3 \sin[\omega T_0 + \omega_1 T_0 - \alpha] \]

\[ + P_4 \sin[2\omega_0 T_0 - \alpha] \]

\[ + P_5 \sin[2\omega T_0 + \omega_1 T_0 + \alpha] \]

\[ + P_6 \sin[\omega T_0 + \omega_1 T_0 + \alpha] \]

\[ + P_7 \sin[2\omega_0 T_0 + \alpha] + P_8, \]

where \( P_1 - P_8 \) stand for expressions that depend on the first-order initial amplitudes, mode shape ratios, \( \omega_1, \) \( \omega_1^* \) and the branching angle. Their expressions are not given here as we are interested in the condition for the internal resonance only. Noting that \( \omega_1 < \omega_1^* < \omega_1 \) the internal resonances are recognised to exist when:

\[ \omega_1 = 2\omega_1^*, \]
also when $\omega_{II} = \omega_{III} - \omega_1$, which is a combination resonance. Substituting the expressions for $\omega_I, \omega_{II}, \omega_{III}$ determined in section 3.1 in these two conditions, figure 8 is plotted—it gives the combination of the stiffness ratio $\kappa$ and the branching angle $\alpha$ for which these two internal resonances occur. The 1:2 resonance exists for all $\alpha$ considered, but only for a range of $\kappa$, while the combination resonance exists for all $\kappa$, but for a range of $\alpha$ considered. It should be pointed out that in [1], only the 1:2 resonance was detected and examined. Although the model considered therein is the same as the model considered herein with a trunk and first-order branches (figures 1(a) and (c)), the assumption that $-\psi_1 = \psi_2$ limited the considerations performed therein and led to the omission of the combination resonance. The combination resonance detected here opens up the possibility to consider the role it has in energy transfer in trees and its benefits for tree-inspired structures. In addition, the fact that the 1:2 resonance does not occur for all combinations of the branching angle and the stiffness ratio is a novel finding that needs further investigations to determine how the energy transfer is carried out and when the combination resonance is the most efficient.

The previously described procedure is applied now to the structure with the branching angle $\bar{\alpha} = 121.947^\circ$. The first-order solution can be assumed in the form given by equation (14), where $t$ should be replaced by $T_0$. The differential equation for $\psi_{d1}$ is derived to be:

$$D_0^2 \psi_{d1} + \bar{\omega}_{II}^2 \psi_{d1} = Q_1 \sin \left(2\omega_1 T_0 - \alpha\right) + Q_2 \sin \left[\omega_1 T_0 + \omega_{II} T_0 - \alpha\right] + Q_3 \sin \left[\omega_1 T_0 + \omega_{II} T_0 + \alpha\right] + Q_4 \sin \left[2\omega_1 T_0 - \alpha\right] + Q_5 \sin \left[\omega_1 T_0 - \omega_{II} T_0 - \alpha\right] + Q_6 \sin \left[2\omega_1 T_0 + \alpha\right] + Q_7 \sin \left[\omega_1 T_0 - \omega_{II} T_0 + \alpha\right] + Q_8 \sin \left[\omega_1 T_0 + \omega_{II} T_0 - \alpha\right] + Q_9,$$

(24)

where $Q_1 - Q_9$ stand for expressions that depend on the initial amplitudes, $\omega_1$ and the branching angle. The internal resonances is identified for $\omega_{II} = 2\omega_1$. So, for this special branching angle, only the 1:2 internal resonance is found to occur and this is labelled in figure 8.
In the next part of the analyses of large-amplitude oscillations, we investigate the behaviour corresponding to the case when the whole tree-like structure is pulled-and-released as a rigid body structure, i.e. the initial amplitudes of the absolute generalised coordinates are equal, while the initial angular velocities are zero. To that end, the equations of motion (18)–(20) are solved numerically by introducing pulling for large initial angles equal to 0.5 rad = 28.65°. Spectral analyses are carried out on such numerically obtained responses for the whole range of values of the branching angle \( \alpha \in (0, 180^\circ) \) to show how the frequency (f) content changes for all three generalised coordinates (figure 9).

When viewed from above, these graphs can be compared with figure 2(b) to recognise the changes with respect to the frequencies appearing in the linear systems, i.e. the system performing small oscillations. For the trunk, the first (lowest) frequency is absolutely dominant. It is apparent that others frequencies also occur especially in the first-order branches and that they become dense for certain branching angle. Note that this conclusion about dense frequencies obtained herein based for the first time on the analytically derived exact equations of motion is in accordance with the conclusion presented in [9] obtained purely numerically based on FEM. According to the statement made in [9], dense multimodal dynamics can be interpreted as a beneficial strategy to prevent the trunk of real trees from bending excessively until the rupture. Given the rigid-body model with a finite number of degrees of freedom considered herein, one can use this statement from [9] (where a system with an infinite number of
degrees of freedom was considered) but to reformulate it by replacing ‘bending’ by ‘performing large-amplitude vibrations’ to recognise this beneficial strategy.

4. Structure with second-order branching
\((N = 2)\)

This section is the extension of the previous one in a way that the hierarchy is enriched with one more order that mimics the existence of second-order branches. The aim is to analyse the behaviour and characteristics that are the consequence of this new hierarchy, but the focus is still on the influence of the branching angle and the stiffness ratio.

The mechanical model of a sympodial tree-like structure of second order is shown in figure 1(b). It has four more physical pendula of mass \(m_2\), four torsional springs of stiffness \(k_2\), and four viscous dampers of the damping coefficient \(b_2\), whose characteristics are given in table 1. Four additional generalised coordinates \(\theta_1 - \theta_4\) (absolute angles) have been added as shown in figure 1(d).

With the gravitational potential energy neglected, the potential of the system can be expressed as:

\[
V_2 = V_1 + \frac{1}{2} k_2 (\theta_1 - \psi_1)^2 + \frac{1}{2} k_2 (\theta_2 - \psi_1)^2 \\
+ \frac{1}{2} k_2 (\theta_3 - \psi_2)^2 + \frac{1}{2} k_2 (\theta_4 - \psi_2)^2,
\]

where \(V_1\) is defined by equation (1).

The kinetic energy can also be related to the previously considered model and written down as

\[
T_2 = T_1 + \sum_{i=1}^{4} \left( \frac{1}{2} m_2 v_{\psi_i}^2 + \frac{1}{2} f_2 \dot{\psi}_i^2 \right),
\]

where \(T_1\) is defined by equation (2) and where the velocity of the centres of mass \(S_1 - S_4\) of the four new physical are:

\[
\begin{align*}
\dot{\mathbf{v}}_S^2 &= \mathbf{v}_S, \\
\mathbf{v}_{S_1} &= \mathbf{v}_A + \mathbf{v}_{B/A} + \mathbf{v}_{S_1/B}, \\
\mathbf{v}_{S_2} &= \mathbf{v}_A + \mathbf{v}_{B/A} + \mathbf{v}_{S_2/B}, \\
\mathbf{v}_{S_3} &= \mathbf{v}_A + \mathbf{v}_{C/A} + \mathbf{v}_{S_3/C}, \\
\mathbf{v}_{S_4} &= \mathbf{v}_A + \mathbf{v}_{C/A} + \mathbf{v}_{S_4/C}.
\end{align*}
\]

where:

\[
\begin{align*}
\mathbf{v}_A &= l \dot{\phi}, \\
\mathbf{v}_{B/A} &= l_1 \dot{\psi}_1, \\
\mathbf{v}_{C/A} &= l_1 \dot{\psi}_2, \\
\mathbf{v}_{S_1/B} &= l_2 \dot{\theta}_1, \\
\mathbf{v}_{S_2/B} &= l_2 \dot{\theta}_2, \\
\mathbf{v}_{S_3/C} &= l_2 \dot{\theta}_3, \\
\mathbf{v}_{S_4/C} &= l_2 \dot{\theta}_4.
\end{align*}
\]

The mass moments of inertia \(J_{S_i}\) in equation (26) are

\[
J_{S_1} = J_{S_2} = J_{S_3} = J_{S_4} = m_2 l_2^2 / 12.
\]

The dissipative function due to the new viscous dampers is

\[
D_2 = D_1 + \frac{1}{2} b_2 (\dot{\theta}_1 - \dot{\psi}_1)^2 + \frac{1}{2} b_2 (\dot{\theta}_2 - \dot{\psi}_1)^2 \\
+ \frac{1}{2} b_2 (\dot{\theta}_3 - \dot{\psi}_2)^2 + \frac{1}{2} b_2 (\dot{\theta}_4 - \dot{\psi}_2)^2.
\]

These scalar quantities are used subsequently to derive the equations of motion in different cases, but based on Lagrange’s equations of the second kind as in the previous section.
4.1. Small free vibrations: natural frequencies and mode shapes

Lagrange’s equations of the second kind yield the following equations of motion:

\[
\left(\frac{m_1}{2} + 2m_1 + 4m_2\right) \ddot{\tilde{\varphi}} + \left(\frac{m_1}{2} + 2m_2\right) l_1 \cos \alpha (\ddot{\tilde{\psi}}_1 + \ddot{\tilde{\psi}}_2)
+ \frac{m_2}{2} l_2 (\ddot{\tilde{\theta}}_2 + \ddot{\tilde{\theta}}_3) + \frac{m_2}{2} l_2 \cos 2\alpha (\ddot{\tilde{\psi}}_3 + \ddot{\tilde{\psi}}_4)
+ \left(k + 2k_1\right) \varphi - k_1 \psi_1 - k_1 \psi_2 = 0,
\]

\[
\left(m_1 + 2m_1\right) l_1 \cos \alpha \ddot{\varphi} + \left(m_1 + 2m_2\right) \ddot{\psi}_1 + \frac{m_2}{2} l_2 \cos \alpha (\ddot{\theta}_1 + \ddot{\theta}_4)
- k_1 \psi_1 - (k_1 + 2k_2) \psi_2 - k_2 \theta_3 - k_2 \theta_4 = 0,
\]

\[
\left(m_1 + 2m_2\right) l_1 \cos \alpha \ddot{\varphi} + \left(m_1 + 2m_2\right) \ddot{\psi}_2 + \frac{m_2}{2} l_2 \cos \alpha (\ddot{\theta}_1 + \ddot{\theta}_4)
- k_1 \psi_2 - (k_1 + 2k_2) \psi_1 - k_2 \theta_3 - k_2 \theta_4 = 0,
\]

\[
\Delta_{2,i} = \begin{vmatrix}
-\left(\frac{4\alpha_1 + 2\alpha_2}{2}\right) \omega^4 - \left(\frac{1}{2}\right) k_1 \omega^2 & \omega_1 \omega_2 \omega^3 - k_1 & \omega_1 \omega_3 \omega^3 - k_1 & \omega_1 \omega_4 \omega^3 - k_1 & \omega_2 \omega_3 \omega^3 - k_2 & \omega_2 \omega_4 \omega^3 - k_2 & \omega_3 \omega_4 \omega^3 - k_2 \\
0 & -\left(\frac{4\alpha_1 + 2\alpha_2}{2}\right) \omega^4 - \left(\frac{1}{2}\right) k_1 \omega^2 & \omega_1 \omega_2 \omega^3 - k_1 & \omega_1 \omega_3 \omega^3 - k_1 & \omega_1 \omega_4 \omega^3 - k_1 & \omega_2 \omega_3 \omega^3 - k_2 & \omega_2 \omega_4 \omega^3 - k_2 \\
0 & 0 & -\left(\frac{4\alpha_1 + 2\alpha_2}{2}\right) \omega^4 - \left(\frac{1}{2}\right) k_1 \omega^2 & \omega_1 \omega_2 \omega^3 - k_1 & \omega_1 \omega_3 \omega^3 - k_1 & \omega_1 \omega_4 \omega^3 - k_1 & \omega_2 \omega_3 \omega^3 - k_2 \\
0 & 0 & 0 & -\left(\frac{4\alpha_1 + 2\alpha_2}{2}\right) \omega^4 - \left(\frac{1}{2}\right) k_1 \omega^2 & \omega_1 \omega_2 \omega^3 - k_1 & \omega_1 \omega_3 \omega^3 - k_1 & \omega_1 \omega_4 \omega^3 - k_1 \\
0 & 0 & 0 & 0 & -\left(\frac{4\alpha_1 + 2\alpha_2}{2}\right) \omega^4 - \left(\frac{1}{2}\right) k_1 \omega^2 & \omega_1 \omega_2 \omega^3 - k_1 & \omega_1 \omega_3 \omega^3 - k_1 \\
0 & 0 & 0 & 0 & 0 & -\left(\frac{4\alpha_1 + 2\alpha_2}{2}\right) \omega^4 - \left(\frac{1}{2}\right) k_1 \omega^2 & \omega_1 \omega_2 \omega^3 - k_1 \\
0 & 0 & 0 & 0 & 0 & 0 & -\left(\frac{4\alpha_1 + 2\alpha_2}{2}\right) \omega^4 - \left(\frac{1}{2}\right) k_1 \omega^2
\end{vmatrix} = 0.
\]

Although being cumbersome, this characteristic determinant is intentionally given here for the sake of comparison with the characteristic determinant (9) for \(N = 1\). The symmetry and additional terms are easily noticed, which gives possibilities to extend the procedure to fractal-type structures of higher branching order.

The corresponding characteristic equation can be written down as a product:

\[
\Delta_2 = \Delta_{2,1} \Delta_{2,2} \Delta_{2,3} = 0,
\]

where:

\[
\Delta_{2,1} = 3k_1 - m_2 \omega^2,
\]

\[
\Delta_{2,2} = -18k_1 k_2 + \omega^2 (6k_2 l_1 m_1 + 36k_1 l_2 m_2 + 6k_1 l_2 m_2 \\
+ 12k_1 l_2 m_2 + 36k_1 l_1 l_2 m_2 \cos \alpha)
+ \omega^4 (-2l_1 l_2 m_2 - 12l_1 l_2 m_2^2 + 9l_1 l_2^2 m_2^2 \cos^2 \alpha),
\]

while \(\Delta_{2,3}\) is not shown here for brevity as being very long (it is a polynomial of the 8th order in \(\omega\) with the coefficients that depend on all the masses, lengths, stiffness coefficients and the branching angle).

By using the values from table 1, seven real solutions of equations (38)–(40) are obtained for natural frequencies and are plotted as a function of the stiffness ratio (figure 10(a)). These natural frequencies are again normalized with respect to the natural frequency of the trunk attached to the ground with the spring of stiffness \(k\), i.e. with the expression \(\sqrt{3k/\bar{m}^2}\). All the natural frequencies increase with the stiffness ratio, but it is seen that III to IV and V to IV are very close to each other for certain values of the stiffness ratio. Note that \(\omega_{IV}\) is the solution of equation (39), \(\omega_{VI}\) and \(\omega_{VII}\) are the solution of equation (40), while \(\Delta_{2,3} = 0\) gives the rest of four natural frequencies \(\omega_{I}, \omega_{II}, \omega_{III}, \omega_{IV}\).

In addition, the natural frequencies are plotted for a fixed stiffness ratio \(\kappa = 0.3\) as a function of the branching ratio (figure 10(b)). Several regions can be distinguished in which some of them are close to each other or coinciding, but the striking fact is that three of them coincide at the already recognised special branching value \(\bar{a} = 121.947^\circ\) and these natural frequencies are then equal to \(4\kappa\), which is also the value existing in the model with first-order branches only (see section 3.1).

The corresponding mode shape ratios are calculated for \(\kappa = 0.3\) and listed in table 2. Note that all of them are given with respect to the amplitude \(C_4\), which is the reason why the last one is always unity.

Figure 11 presents qualitatively all these mode shapes. It is seen that the majority of modes affects the whole structure (I, III, V, VII), while in Modes II, IV and VI the trunk does not oscillate (these modes are plotted below each other to emphasize that the
trunk is motionless in all of them). In Mode I, all the elements oscillate in phase. Model III is different from it in the way that the trunk oscillates out-of-phase with respect to the upper part. Modes V and VII represent further divisions as there is a change in the oscillation direction between the trunk, the first and second-order branches. Mode II and VI are localised in branches of both orders: in Mode II, the left and right symmetric part oscillate out-of-phase, while in Mode VI, the first-order branch is out-of-phase with respect to the second-order branches. Mode IV is localised in the second-order branches only, where both groups of these branches are synchronised in phase with respect to each other, but external and internal branches oscillate out-of-phase. Potential benefits of these structures lie particularly in the behaviour associated with the motionless trunk:

mode shape ratio given in Table 2 for Modes II, IV and VI define the initial displacement that will yield such motion in which the branches as the attachments might oscillate, but not the main part of the structure—the trunk. Note that the number of modes with the motionless trunk in the model with $N = 2$ is three, while in the model with $N = 1$ (section 1), there is only one such mode.

The mode shape ratios corresponding to the special branching angle $\alpha = 121.947^\circ$ are presented in Table 3, and the mode shapes are shown in Figure 12. Modes I, III, and V affect the whole structure, while in Modes II and IV the trunk does not oscillate. In Mode II, both the branching of the first and second order oscillate, while in Mode IV, only the second order branches oscillate. It can be seen that the number of modes with the motionless trunk is again higher than

| Mode | $b_A = A/C_4$ | $b_{B_1} = B_1/C_4$ | $b_{B_2} = B_2/C_4$ | $b_{C_1} = C_1/C_4$ | $b_{C_2} = C_2/C_4$ | $b_{C_3} = C_3/C_4$ | $b_{C_4} = C_4/C_4$ |
|------|---------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| I    | 0.503         | 0.868               | 0.868               | 1                   | 1.011               | 1.011               | 1                   |
| II   | 0             | −0.701              | 0.701               | −1                  | −1                  | 1                   | 1                   |
| III  | −0.259        | 0.236               | 0.236               | 1                   | 0.395               | 0.395               | 1                   |
| IV   | 0             | 0                   | 0                   | −1                  | 1                   | −1                  | 1                   |
| V    | 0.133         | −0.098              | −0.098              | 1                   | −1.752              | −1.752              | 1                   |
| VI   | 0             | 0.345               | −0.345              | −1                  | −1                  | 1                   | 1                   |
| VII  | 0.163         | −0.550              | −0.550              | 1                   | 0.886               | 0.886               | 1                   |

Figure 10. Natural frequencies for $N = 2$ as a function of: (a) the stiffness ratio for $\alpha = 20^\circ$; (b) the branching angle for $\kappa = 0.3$. 

Table 2. Mode shape ratios for $\kappa = 0.3$ and $\alpha = 20^\circ$. 

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![Figure 10. Natural frequencies for $N = 2$ as a function of: (a) the stiffness ratio for $\alpha = 20^\circ$; (b) the branching angle for $\kappa = 0.3$.](image-url)
for the case when $N = 1$. It is, thus, confirmed that the increase of hierarchy is beneficial for the increase of number of such modes for all branching angles.

4.2. Small undamped forced vibrations: dynamic absorbers
In the next part of the analysis, the tree-like structure with $N = 2$ considered previously is excited by the torque $M_0 \cos \Omega t$ that acts on the trunk. Consequently, equation (30) gets this term on the right-hand side. The solutions for the forced response can be assumed as in section 3.2 with $\theta_i = C_i \sin (\Omega t)$, $i = 1, \ldots, 4$. The amplitude of the trunk is found to be defined by

$$ A = \frac{M_0 (3k_2 - m_2 \bar{\Omega}^2) \Delta_{1,2} (\omega \to \Omega)}{\Delta_{2,3} (\omega \to \Omega)}. $$

Table 3. Mode shape ratios for $\kappa = 0.3$ and $\bar{\alpha} = 121.947^\circ$.

| Mode | $b_A = A/C_4$ | $b_B = B_1/C_4$ | $b_B = B_2/C_4$ | $b_C = C_1/C_4$ | $b_C = C_2/C_4$ | $b_C = C_3/C_4$ |
|------|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| I    | 0.955         | 1.1745          | 1.1745          | 1               | 1.571           | 1.571           |
| II   | 0             | $-1$            | 1               | $-1$            | $1$             | $1$             |
| III  | $-0.136$      | 0.936           | 0.936           | 1               | 0.803           | 0.803           |
| IV   | 0             | 0               | 0               |                 |                 |                 |
| V    | 0.183         | 0.132           | 0.132           | 1               | $-1.842$        | $-1.842$        |

$C_1 = -C_4$, $C_2 = -C_3$, where $C_3$ and $C_4$ are arbitrary.
It is of interest again to find the condition under which the trunk, although being excited, does not oscillate. Equation (41) gives that the amplitude of the trunk is equal to zero when
\[ \Delta_{2,2} (\omega \rightarrow \Omega) = 0. \]
This equation has two solutions and the dynamic absorber can appear if the excitation is tuned to the natural frequencies \( \omega_{II} \) and \( \omega_{VI} \). Note that these frequencies of the dynamic absorber depend on the masses and lengths of all branches and the stiffness coefficients of springs to which they are connected and also on the branching angle. The amplitude of the trunk is also equal to zero when
\[ \Omega = \sqrt{\frac{3k_2}{(m_1l_2^2)}} = \omega^* 4\kappa = \omega^* \omega_{IV}, \]
which corresponds to the natural frequency \( \omega_{IV} \) and has the value that appears in the structure with \( N = 1 \) as well. It should be pointed out that the amplitudes of the first-order branches are also equal to zero then. Thus, if the excitation frequency is tuned to the natural frequency of the pendulum from the second-order branching with a torsional spring, neither the trunk nor the first-order branches oscillate, which can be beneficial for practical applications as a way for eliminating oscillations of the elements from different order of hierarchy. All these cases are labelled in figure 13, where the amplitude-frequency diagrams for the trunk (the red thick solid line), the first-order branches (the blue thin solid line) and the external and internal second-order branches (the green thin solid and dashed line) are plotted.

The cases of zero amplitudes of the trunk are depicted by the black dot. The additional frequencies at which the first-order branches have a zero-valued amplitude are labelled by a triangle. Thus, for \( \alpha = 20^\circ \) (figure 13(a)) there are four resonances labelled by vertical dashed lines at the frequencies I, III, V and VII labelled by stars; the amplitude of the trunk can be equal to zero at three frequencies, and the amplitude of the first-order branches can be equal to zero at two frequencies.

For \( \alpha = 121.947^\circ \) (figure 13(b)) there are three resonances labelled by vertical dashed lines at the frequencies I, III, and V labelled by stars; the amplitude of the trunk can be equal to zero at two frequencies, and at the latter of these two, the amplitude of the first-order branches is equal to zero as well. So, it is seen again that the branches from the last order can be beneficially used to eliminate oscillations of all the elements preceding them: the first-order branches and the trunk.

4.3. Small damped forced vibrations

It is of interest now to determine how the response considered previously will change when damping exists. The damping terms have been added in accordance with the dissipative function given by equation (29) and the corresponding amplitude-frequency curves are plotted in figure 14. The amplitude-frequency diagrams are shown in parallel for \( \alpha = 20^\circ \) and \( \bar{\alpha} = 121.947^\circ \) and for the same non-dimensional damping coefficients. They illustrate the decrease in the resonant amplitudes and the reduction of the number of resonance zones with the increase of the non-dimensional damping coefficient.

In addition, it is also seen that the amplitudes that were equal to zero in the undamped case now have non-zero values and represent local minima. However, for certain values of the non-dimensional damping coefficient, these local minima do not exist anymore.

To detect the influence of the non-dimensional damping coefficient on the number of local extrema, the amplitude of the trunk is analysed numerically and is presented in figure 15: four distinctive regions can be recognizing, related to the existence of only one extremum, i.e. one resonant peak (this is labelled by ‘1’), three extrema (two maxima and one minimum), which is the case labelled by ‘3’. Considerably smaller are the regions of the combination of the branching angle and the non-dimensional damping coefficient
that yield five or seven extrema. Note that the shape of regions 1 and 3 and the ‘line’ that separate them is similar to those shown in the analogous graph plotted in figure 7 for the system with \( N = 1 \).

### 4.4. Free large-amplitude vibrations

The previous consideration of the system with \( N = 2 \) have been related to small vibrations, and are extended now to large-amplitude vibrations based on exacts equations of motion. The non-dimensional equations of motion can be derived by using the quantities (25)–(28) introducing the same non-dimensional time as in section 3. Analysing these seven equations, one can find out that Mode II, IV and VI identified in section 4.1 and shown in figure 11 satisfy them identically. Thus, they exist also when the system performs large-amplitude oscillations, and represent similar normal modes [18]. This characteristic determined in the model with first-order branching holds for second-order branching as well: when initial conditions are such to correspond some of these odes, the structure will oscillate in the same mode both when performing small vibrations and large-amplitude vibrations. This characteristic can be utilised for practical applications when large-amplitude vibrations of the corresponding structures are needed—although being geometrically nonlinear, the structure will behave in a similar way as the linear ones.

Analogously to the case with three degrees of freedom and figure 9, the behaviour of the whole tree-like seven-degree-of-freedom structure is analysed when pulled-and-released as a rigid body structure, i.e. for the equal initial amplitudes of the absolute generalised coordinates and zero-valued initial angular velocities. Thus, the corresponding non-dimensional equations of motion derived based on the quantities (25)–(28) are solved numerically by introducing pulling for large initial angles equal to \( 0.5 \text{ rad} = 28.65^\circ \). Spectral analyses of the responses obtained are carried out for the branching angle \( \alpha \in (0, 180^\circ) \) and are shown in figure 16 for the trunk and the left-hand side branches of the first and second order.

![Figure 13](image-url)
When viewed from above, these graphs can be compared to those presented in figure 10(b). Additional higher frequencies are seen for all the elements of the structure. The second-order branches have more dense spectra than the first-order ones and the spectrum for external and internal second-order branches slightly differ, which have not been noted in the literature so far. Besides being beneficial for understanding trees’ behaviour in the context of dense multimodal dynamics (see section 3.4), these characteristics will be of interest for studying the response of trees and tree-like structures to multi-frequency excitations and to winds in particular.

5. Discussion and conclusions

This study has been concerned with systematic and detailed investigations of tree-like structures of
Figure 15. Combinations of the branching angle and the non-dimensional damping coefficient $\zeta$ for which different number of extrema appear in the amplitude-frequency diagram of the trunk when $N = 2, \alpha = 20^\circ$ and $\kappa = 0.3$.

Figure 16. Spectral analyses of large-amplitude vibrations obtained numerically for $N = 2$ and $\kappa = 0.3$: (a) amplitude of $\varphi_1$; (b) amplitude of $\psi_1$; (c) amplitude of $\theta_1$; (d) amplitude of $\theta_2$. 
different branching hierarchy $N$. These structures contain either a trunk with first-order branches ($N = 1$), or a trunk with both first and second-order branches ($N = 2$). These structures are symposidal, i.e. they have only lateral segments in subsequent branching order. The corresponding mechanical models considered have comprised physical pendula coupled with torsional springs and viscous dampers. In the first case, the mechanical model has been with three degrees of freedom and in the second one, with seven degrees of freedom.

First, natural frequencies and modal shapes have been found analytically for the case of small undamped vibrations and for $N = 1$. It has been shown how the natural frequencies change with the stiffness ratio and the branching angle. While analysing the latter, it has been recognised that two higher frequencies coincide for the special branching angle $\bar{\alpha} = 121.947^\circ$. In both cases, one mode has been found to be localized in branches. This mode has been found to exist also in free large-amplitude vibrations, which can be beneficial in practice as it implies the same pattern of behaviour for small and large-amplitude vibrations. When harmonically exciting the trunk, the first-order branches can act as a dynamic absorber for the trunk at the second natural frequency of the structure. Unlike Den Hartog’s concept of a single-mass dynamic absorber specifically tuned to the main structure and placed in-line with it [17], the branches actually represent a different type of physical realisation of a dynamic absorber: they have the form of a two-mass attachment symmetrically arranged with respect to the main structure (trunk). This finding opens up the question related to the role of branches as dynamic absorbers and the relationship/advantages/disadvantages of these branched absorbers with respect to Den Hartog’s coupled spring-mass-dampers, which have widely been investigated in the literature. This issue is currently under investigation.

It has been found out that for the special value of the branching angle $\bar{\alpha} = 121.947^\circ$, the dynamic absorber does not exist and that the structure has only one resonance. Thus, although designed as a system with three degrees of freedom, this structure actually behaves as an externally excited undamped mass on a spring, i.e. as an externally excited system with one degree-of-freedom.

The influence of damping has been analysed as well. It has been found that certain values of the non-dimensional damping coefficient change the number of resonance peaks: for the branching angle smaller than the special one $\alpha = 121.947^\circ$, smaller values of the non-dimensional damping coefficient are needed to reduce the number of resonance peaks, which can be considered as a useful criterion for the design of tree-like structures with respect to the choice of the branching angle and the damping ratio. This criterion can be formulated in two ways, depending on the parameter that is defined. So, if higher values of the damping ratio are achievable, smaller values of the branching angle yields the behaviour with one resonance peak; if only smaller values of the damping ratio can be used, the branching angle should be chosen to be around the critical value $\bar{\alpha} = 121.947^\circ$ to yield one resonant real for the smallest possible damping ratio.

When considering free large-amplitude vibrations, the existence of internal resonances have been determined analytically, including a 1:2 resonance and also a combined resonance, which has not been identified in previous investigations. Additional new findings are associated with the fact that the 1:2 resonance has been found to exist for all values of the branching angle $\alpha$ considered, but only for a certain range of the stiffness ratio $\kappa$, while the combination resonance exists for all the values of the stiffness ratio, but only for a certain range of the branching angle. These findings require further investigations to determine how the related energy transfer is carried out and when it is the most efficient. The combination resonance detected in this study opens up the possibility to consider its role in the energy transfer in trees and its benefits for tree-inspired structures.

Analogous analyses have been carried out considering the structure with $N = 2$. The bioinspired branched model has seven degrees of freedom, but the number of modal frequencies can be smaller than seven and this depends on the branching angle. It has been obtained that three higher modal frequencies coincide for the special branching angle $\bar{\alpha} = 121.947^\circ$. In a general case, three out of seven modes are localized in branches, while in the special case two modes out of five have such characteristics. It should be pointed out that this possibility for the existence of modes with the motionless trunk and oscillating branches has been detected in the structure with $N = 1$ as well, but the analyses conducted for $N = 2$ have shown that the number of such modes increases with the increase of $N$. Thus, the hierarchy can be seen as being beneficial for the appearance of these modes.

When harmonically exciting the trunk, the hierarchy of the structure yields the cases in which the amplitude of the trunk is equal to zero, but also the cases when the amplitude of first-order branches is equal to zero, which can be beneficial for practical applications. Thus, the upper part of the structure acts as a dynamic absorber for the whole lower part and multiple amplitude absorption can exist. The influence of damping and the branching angle has been analysed to determine how the number of extrema changes with the combination of these two parameters.

The analyses conducted have also included the considerations of the frequency spectra for large-amplitude vibration for both $N = 1$ and $N = 2$ and their comparison with the modal frequencies appearing in small-amplitude vibrations. Additional higher frequencies have been noted for all the elements of the structure, while for the case $N = 1$, the first frequency is absolutely dominant in the trunk. The second-order branches have more dense spectra than the first-order
branches and the spectrum for external and internal second-order branches slightly differ, which have not been noted in the literature so far. However, these dense spectra are in accordance with the existing knowledge related to the fact that dense multimodal dynamics is seen as beneficial for prevent real trees from bending excessively until the rupture. In this respect, it would be useful to investigate further the analogous large-amplitude response of the tree-like structures to multi-frequency excitations and in particular to winds to determine when this multimodal dynamics is efficient in terms of the design parameters.

The analyses presented have been carefully methodologically organised and carried out, especially for small free and forced vibrations, but they have also showed fundamental beneficial properties of free large-amplitude vibrations (the existence of modes with a motionless trunk) and beneficial properties of forced small-amplitude vibrations (the existence of multiple dynamic absorbers). These analyses have opened up several questions, some of which have been stated above. The additional one is to examine a forced large-amplitude response of tree-inspired hierarchical structures, which could also be of interest to get insights into potential benefits of branching to minimise the amplitude of the trunk, and to avoid certain nonlinear phenomena or to utilise them for energy harvesting, for instance.

The models considered herein perform in-plane vibrations. Extending them to 3D tree-like structures would increase considerably the number of degrees of freedom. The motion of the trunk, for example, with the exclusion of intrinsic rotation (spin), would be described by the precession and nutation angles, i.e. it would have two degrees of freedom. First-order branching would require the existence of four branches symmetrically placed on the top of the trunk in 3D, and each of the branches will have two additional Euler angles (the precession angle and the nutation angles) introduced to describe its motion. Thus, the structure with first-order branching will have 10 degrees of freedom, and the structure with second-order branching with four second-order branches stemming from each first-order branch will have 44 degrees of freedom. Consequently, their analytical considerations would be cumbersome, and it would be easier to turn to simulation software or FEM, as done in [9], for example. However, even when focusing only on 3D structures with first-order branching, one can visualise (and prove theoretically as well) the existence of modes with the motionless trunk and four oscillating branches (two groups of branches oscillating in two orthogonal planes, and each of these two planes can be considered analogously as presented in this study). This corresponds to the localisation phenomenon, which is noted in 2D structures studied herein as well. Thus, although 3D structures would certainly give more details about possible behaviour, the fact that this localisation exists also in simpler 2D structures provides an additional argument to focus on simpler structures first as done herein and then to introduce more complex cases in future studies especially related to their experimental realisation and verification of the localisation phenomenon.

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