Partially integrable generalizations of classical integrable models by combination of characteristics method and Hopf-Cole transformation

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August 9, 2018

Abstract

We represent an integration algorithm combining the characteristics method and Hopf-Cole transformation. This algorithm allows one to partially integrate a large class of multidimensional systems of nonlinear Partial Differential Equations (PDEs). A specific generalization of the equation describing the dynamics of two-dimensional viscous fluid and a generalization of the Korteweg-de Vries equation are examples of such systems. The richness of available solution space for derived nonlinear PDEs is discussed.

1 Introduction

A possible way to construct new classes of (partially) integrable nonlinear multidimensional Partial Differential Equations (PDEs) is combining features of different classical integration algorithms. Thus, the multidimensional nonlinear PDEs having features of PDEs linearizable (C-integrable [1]) by the Hopf-Cole transformation [2, 3] and integrable by the Inverse Spectral Transform Method (ISTM) [4, 5, 6, 7] (S-integrable) have been derived in refs. [8, 9]. A combination of the characteristics method [10] with the Hopf-Cole transformation is proposed in refs. [11, 9]. The characteristics method was also combined with the ISTM in refs. [12, 9] and with the commuting vector fields [13, 14] in ref. [15]. There are also several generalizations of the characteristics method [16, 17, 18, 19, 20] integrating certain classes of systems of multidimensional nonlinear PDEs. In addition, there is a relation [21] between the nonlinear PDEs linearizable by the Hopf-Cole transformation and integrable by the ISTM.

The algorithm proposed in this paper allows one to partially integrate a new class of multidimensional nonlinear PDEs using the features of the characteristics method and Hopf-Cole transformation. Unlike the nonlinear PDEs derived in refs. [11, 9], a new type of nonlinear PDEs involves additional arbitrary functions of fields and their derivatives. Note that the term ”partially integrable PDEs” has two different meanings. First, the PDE is partially integrable if the available solution space is not full [22]. Second, the PDE is partially integrable if its solution space is described in terms of the lower dimensional PDEs [24]. Our nonlinear PDEs are partially integrable in both senses.

The general result is covered by the following theorem.

Theorem. Let \( W(x, t) \) be a solution to the algebraic system

\[
W(x, t)F(u(x, t), x, t) = \eta(u(x, t)),
\]

\[
W(x, t)F_{u^{(n)}}(u(x, t), x, t) - \eta_{u^{(n)}}(u(x, t)) = \eta^{(n)}(x, t), \quad n = 1, \ldots, K,
\]

\[
\det W \neq 0,
\]

\[
t = \{t_1, \ldots, t_M\}, \quad x = \{x_1, \ldots, x_N\}, \quad u = \{u^{(1)}, \ldots, u^{(K)}\},
\]
where \( W \) is the \( N_1 \times N_1 \) matrix function of independent variables \( x \) and \( t \); \( F, \eta \) and \( \eta^{(n)} \) are the \( N_1 \times N_2 \) \((N_1 > N_2)\) matrix functions of arguments with

\[
N_1 = N_2(K + 1).
\] (5)

Let \( \eta \) and \( \eta^{(n)} \) be arbitrary functions, while \( F \) satisfy the following system of compatible linear PDEs:

\[
F_{t_m} - \sum_{j=1}^{M} F_{x_j} V^{(mj)}(u) = F \Gamma^{(m)}(u), \ m = 1, \ldots, M,
\] (6)

where \( V^{(mj)} \) and \( \Gamma^{(m)} \), are arbitrary \( N_2 \times N_2 \) diagonal matrix functions of \( u \). Then elements of the matrix function \( W \) satisfy the following system of nonlinear equations:

\[
W_{t_m} W^{-1} \eta(u) - \sum_{j=1}^{N} W_{x_j} W^{-1} \eta(u) V^{(mj)}(u) + \sum_{i=1}^{K} \left( u_{t_m}^{(i)} - \sum_{j=1}^{N} u_{x_j}^{(i)} V^{(mj)}(u) \right) \eta^{(i)}(x, t) + \eta(u) \Gamma^{(m)}(u) = 0,
\] (7)

where \( u^{(i)} \) are arbitrary functions of \( x \) and \( t \).

**Proof.** First, we differentiate eq.(1) with respect to \( t_m \) and \( x_j \):

\[
E^{(t_m)} := W_{t_m} F + W F_{t_m} + W \sum_{i=1}^{K} F_{u^{(i)}} u_{t_m}^{(i)} = \sum_{i=1}^{K} \eta_{u^{(i)}} u_{t_m}^{(i)},
\] (8)

\[
E^{(x_n)} := W_{x_n} F + W F_{x_n} + W \sum_{i=1}^{K} F_{u^{(i)}} u_{x_n}^{(i)} = \sum_{i=1}^{K} \eta_{u^{(i)}} u_{x_n}^{(i)}.
\] (9)

Consider the combination \( E^{(t_m)} - \sum_{j=1}^{N} E^{(x_j) V^{(mj)}} \) and take into account eqs.(2) and (6). We obtain:

\[
W_{t_m} F - \sum_{j=1}^{N} W_{x_j} F V^{(mj)} + \sum_{i=1}^{K} \eta^{(i)} \left( u_{t_m}^{(i)} - \sum_{j=1}^{N} u_{x_j}^{(i)} V^{(mj)} \right) + W F \Gamma^{(m)}(u) = 0.
\] (10)

Since \( W \) is invertible, from eq.(1) we have: \( F = W^{-1} \eta \). Substituting it into eq.(10) we obtain eq.(7). \( \Box \)

**Remark 1.** If \( W = 1, \eta = 0, N_1 = 1, N_2 = K \) and \( \text{det}\{F_{i,j}^{(u^{(j)}): i, j = 1, \ldots K}\} \neq 0 \), then eq.(2) may be disregarded. Herewith eq.(1) yields the usual characteristics method [10] and eq.(7) must be replaced with the system

\[
u_{t_m}^{(i)} - \sum_{j=1}^{N} u_{x_j}^{(i)} V^{(mj)} = 0, \ \ i = 1, \ldots, K, \ \ m = 1, 2, \ldots.
\] (11)

If \( N_1 = N_2, \eta = F_{x_1} \) and \( \text{det} F \neq 0 \), then \( W = F_{x_1} F^{-1} \) so that we have the matrix Hopf-Cole transformation. To derive the nonlinear PDE for the matrix \( W \) in this case we have to replace eqs.(6) by another system for \( F \). For instance, if

\[
F_{t_m} = \partial_x^{m} F + F B^{(m)} - B^{(m)} F, \ m = 1, 2, \ldots,
\] (12)
(where $B^{(m)}$ are diagonal constant matrices) then $W$ satisfies the matrix Bürgers hierarchy \[23\].

**Remark 2.** If $N_2 = 1$, i.e. $K = N_1 - 1$, then eqs. (11) and (2) may be written as the following single matrix equation:

$$W \hat{F} = \hat{\eta},$$

(13)

where $\hat{F}$ and $\hat{\eta}$ are block-row matrices

$$\hat{F} = [F \ F_{u(1)} \cdots F_{u(N_1-1)}], \quad \hat{\eta} = [\eta \ \eta^{(1)} + \eta_{u(1)} \cdots \eta^{(N_1-1)}].$$

(14)

To satisfy the condition (3) we require det $\hat{F} \neq 0$ and det $\hat{\eta} \neq 0$. Then

$$W = \hat{\eta} \hat{F}^{-1}.$$  

(15)

Emphasize, that $\eta(u)$, $\eta^{(i)}(x, t)$ and $V^{(mj)}(u)$ are arbitrary functions of arguments with appropriate matrix dimensions.

Note that functions $u^{(i)}$ and $\eta^{(i)} (i = 1, \ldots, K)$ are arbitrary functions of $x$ and $t$ in eq. (17). Consequently, we have to add relations among $u^{(j)}$, $\eta^{(i)}$ and elements of $W$. Further, each matrix equation with fixed $m$ in the system (7) involves $N_1 N_2$ scalar equations for $N_1^2$ scalar fields $W_{ij}$ ($i, j = 1, \ldots, N_1$). Thus, the complete system consists of several equations (7) with different $m = 1, \ldots, T$, so that $N_1 N_2 T = N_1^2$. Consequently, the nonlinear system is $N + T$-dimensional.

### 2 Nonlinear PDEs corresponding to $N_1 = 2, N_2 = 1$

Consider the case $N_1 = 2$, $N_2 = 1$, $T = 2$ and denote $u^{(1)} \equiv u$. Then the $(N + 2)$-dimensional system of nonlinear PDEs (7) reads:

$$E_{a}^{(m)} := A_{w}^{w:1} \left( (W_{a1})_{tm} - \sum_{j=1}^{N} (W_{a1})_{xj} V_{1}^{(mj)}(u) \right) +$$

$$A_{w}^{w:2} \left( (W_{a2})_{tm} - \sum_{j=1}^{N} (W_{a2})_{xj} V_{1}^{(mj)}(u) \right) +$$

$$A^{v} \eta_{a1}^{(1)}(W, u) \left( u_{tm} - \sum_{j=1}^{N} u_{xj} V_{1}^{(mj)}(u) \right) + A^{v} \eta_{a1} \Gamma_{1}^{(m)} = 0, \quad \alpha = 1, 2, \quad m = 1, 2.$$

(16)

where

$$A_{w}^{w:1} = W_{22} \eta_{11}(u) - W_{12} \eta_{21}(u), \quad A_{w}^{w:2} = W_{11} \eta_{21}(u) - W_{21} \eta_{11}(u),$$

$$A^{v} = (W_{11} W_{22} - W_{12} W_{21}).$$

(17)

Let $\eta$ be independent on $u$ for the sake of simplicity,

$$u = W_{11}, \quad v = W_{12}, \quad p = W_{21}, \quad q = W_{22}.$$  

(18)

Then, introducing the operators $L^{(m)}$ acting on an arbitrary function $h(x, t)$ by the formula

$$L^{(m)}(h) = h_{tm} - \sum_{j=1}^{N} h_{xj} V_{1}^{(mj)}(u),$$

(19)
we write eq.(16) as

\begin{align}
L^{(m)}(u) + G_{11} L^{(m)}(v) + G_{12} \Gamma_1^{(m)} &= 0, \; m = 1, 2, \\
L^{(m)}(p) + G_{21} L^{(m)}(q) + G_{22} L^{(m)}(v) + G_{23} \Gamma_1^{(m)} &= 0, \; m = 1, 2
\end{align}

(20)

\begin{align}
G_{11} &= \frac{A^{(w;2)} f_1}{A^{(w;1)} \eta_{11}}, \\
G_{12} &= \frac{A^{(v)} \eta_{11}}{A^{(w;1)} \eta_{11}}, \\
G_{21} &= \frac{A^{(w;2)}}{A^{(w;1)}}, \\
G_{22} &= -\frac{A^{(v)} A^{(w;2)} \eta_{21}}{A^{(w;1)} (A^{(w;1)} + A^{(v)} \eta_{11})}, \\
G_{23} &= \frac{\eta_{21} A^{(v)}}{A^{(w;1)}} - \frac{\eta_{11} \eta_{11} (A^{(v)})^2}{A^{(w;1)} (A^{(w;1)} + A^{(v)} \eta_{11})}
\end{align}

(22)

(23)

(24)

(25)

(26)

where

\begin{align}
A^{(w;1)} &= q \eta_{11} - v \eta_{21}, \quad A^{(w;2)} = u \eta_{21} - p \eta_{11}, \quad A^{(v)} = u q - p v.
\end{align}

(27)

Owing to the arbitrary functions \(\eta_{11}^{(1)}(x, t)\) \((i = 1, 2)\), coefficients \(G_{12}\) and \(G_{23}\) may be arbitrary functions of \(x\) and \(t\). Consequently, we may impose the following relations

\begin{align}
G_{12} = f_1(U), \quad G_{23} = f_2(U)
\end{align}

(28)

where list of arguments \(U\) in arbitrary functions \(f_i\) \((i = 1, 2)\) involves the fields \(u, v, p, q\) and their derivatives: \(U = \{u, v, p, q, u_x, v_x, p_x, q_x, u_t, v_t, p_t, q_t, \ldots\}\). Substituting \(G_{12}\) and \(G_{23}\) from eqs.\((23)\) and \((26)\) into eqs. \((28)\) and solving them for \(\eta_{11}^{(1)}\), \(i = 1, 2\), we obtain:

\begin{align}
\eta_{i1}^{(1)} = \frac{A^{(v)} \eta_{11} - A^{(w;1)} f_i}{A^{(v)} f_1}, \quad i = 1, 2.
\end{align}

(29)

Now eqs.\((20)\) and \((21)\) read as

\begin{align}
L^{(m)}(u) + G_{11} L^{(m)}(v) + f_1 \Gamma_1^{(m)} &= 0, \; m = 1, 2, \\
L^{(m)}(p) + G_{21} L^{(m)}(q) + G_{22} L^{(m)}(v) + f_2 \Gamma_1^{(m)} &= 0, \; m = 1, 2,
\end{align}

(30)

(31)

where, in virtue of eqs.\((29)\), \(G_{ij}\) read

\begin{align}
G_{11} &= \frac{A^{(w;2)} f_1}{A^{(v)} \eta_{11}} = \frac{f_1 (u \eta_{21} - p \eta_{11})}{\eta_{11} (u q - p v)}, \\
G_{21} &= \frac{A^{(w;2)}}{A^{(w;1)}} = \frac{u \eta_{21} - p \eta_{11}}{q \eta_{11} - v \eta_{21}}, \\
G_{22} &= \frac{A^{(w;2)} (A^{(w;1)} f_2 - A^{(v)} \eta_{21})}{A^{(v)} A^{(w;1)} \eta_{11}} = \frac{(u \eta_{21} - p \eta_{11}) (v \eta_{21} (p - f_2) - q (u \eta_{21} - \eta_{11} f_2))}{\eta_{11} (u q - p v) (q \eta_{11} - v \eta_{21})}
\end{align}

(32)

Eqs.\((30)\) and \((31)\) must be considered as the ultimate general \((N + 2)\)-dimensional system of nonlinear PDEs in this example.
2.1 Reduction of the system \((30, 31)\) to the single PDE

There is a particular reduction of the system \((30, 31)\) leading to the single nonlinear PDE for the function \(u\). Namely, if the second term in the system \((30)\) is negligible, then this system reads

\[
u_{tm} - \sum_{j=1}^{N} u_{x_j} V^{(mj)}_1(u) + f_1 \Gamma_1^{(m)} = 0, \quad m = 1, 2, \tag{33}
\]

i.e. \(u\) satisfies the system of two commuting flows. In this case, it is reasonable to take \(f_1\) as an arbitrary function of \(u\) and its derivatives, while eq.\((31)\) may be disregarded. Of course, commutativity condition imposes additional differential relation on the field \(u\). To avoid this relation we have to disregard eqs.\((33)\) with \(m = 2\), keeping the only equation

\[
u_{t_1} - \sum_{j=1}^{N} u_{x_j} V^{(1j)}_1(u) + f_1 \Gamma_1^{(1)} = 0. \tag{34}
\]

This reduction may be realized by means of the multi-scale expansion with

\[
u \sim v \sim p \sim q \sim \varepsilon(u), \quad V^{(mk)} \sim \varepsilon(V), \quad \partial_{x_k} \sim \varepsilon(x), \quad \partial_{tm} \sim \varepsilon(x) \varepsilon(V), \quad f_1 \sim \varepsilon(u) \varepsilon(x) \varepsilon(V), \tag{35}\]

where all \(\varepsilon\)'s are small positive parameters \((\ll 1)\). In this case \(G_{11}^{(1)} \sim \varepsilon(x) \varepsilon(V)\). Consequently, the second term in eq.\((30)\) is of the order \(\varepsilon(u) \varepsilon(x) \varepsilon(V))^2\), while all other terms are of the order \(\varepsilon(u) \varepsilon(x) \varepsilon(V)\). Thus, the leading \(\varepsilon(u) \varepsilon(x) \varepsilon(V)\)-order terms yield eq.\((33)\).

Example 1: three-dimensional viscous fluid flow. Let \(N = 3\), \(V^{(1)}(u) = u\), \(f = -\nu \Delta u\), \((\Delta = \sum_{i=1}^{3} \partial_{x_i}^2\) is the three-dimensional Laplasian). Then eq.\((34)\) yields a special case of three-dimensional viscous fluid flow with the constant kinematic viscosity \(\nu\)

\[
u_{t_1} - \sum_{j=1}^{3} u_{x_j} u - \nu \Delta u = 0. \tag{36}
\]

A two-dimensional \((M = 2)\) version of this equation was derived in [24] using a different algorithm.

Example 2: multidimensional generalization of the Korteweg-de Vries equation \((KdV)\). Let \(V^{(1)} = u\), \(V^{(1j)} = 0 \quad (j > 1)\), \(f = \sum_{i,j,k=1}^{N} a_{ijk} u_{x_i x_j x_k}\), \(a_{ijk} = \text{const}\). Then eq.\((34)\) yields an \((N + 1)\)-dimensional version of KdV:

\[
u_{t_1} - u_{x_1} u + \sum_{i,j,k=1}^{N} a_{ijk} u_{x_i x_j x_k} = 0. \tag{37}
\]

2.2 Solution space to the nonlinear system \((30, 31)\)

To obtain solutions to the derived system of nonlinear PDEs \((30, 31)\) we, first, write expressions for the elements of the matrix function \(W\) given by the system \((15)\) with \(N_1 = 2\):

\[
W_{i1} = \frac{\eta^{(1)}_{i1} F_{21} - \eta_{i1} (F_{21})_u}{F_{21}(F_{11})_u - F_{11}(F_{21})_u}, \quad W_{i2} = -\frac{\eta^{(1)}_{i1} F_{11} - \eta_{i1} (F_{11})_u}{F_{21}(F_{11})_u - F_{11}(F_{21})_u}, \tag{38}
\]
where $F_{a1}$ ($\alpha = 1, 2$) are solutions to eq. (39):

$$F_{a1} = e^{\sum_{m=1}^{M} \Gamma_i^{(m)} t_m} \mathcal{F}_{a1} \left( x_1 + \sum_{m=1}^{M} t_m V_1^{(m)}(u), \ldots, x_N + \sum_{m=1}^{M} t_m V_1^{(mN)}(u) \right).$$

Substituting $W_{ij}$ from eqs. (38) and $\eta^{(i)}_1$ ($i = 1, 2$) from eq. (29) into the system (18) we obtain

$$u = \frac{\eta_{11} F_{21} - f_1 F_{11} F_{21} - f_1 \eta_{11} (F_{21})_u}{f_1 (F_{21})_u - F_{11} (F_{21})_u},$$

$$p = \frac{\eta_{21} F_{21} - f_2 F_{11} F_{21} - f_1 \eta_{21} (F_{21})_u}{f_1 (F_{21})_u - F_{11} (F_{21})_u},$$

$$v = \frac{-\eta_{11} F_{11} + f_1 F_{11}^2 + f_1 \eta_{11} (F_{11})_u}{f_1 (F_{21})_u - F_{11} (F_{21})_u},$$

$$q = \frac{-\eta_{21} F_{11} + f_2 F_{11}^2 + f_1 \eta_{21} (F_{11})_u}{f_1 (F_{21})_u - F_{11} (F_{21})_u}.$$

If $f_i$ depend on all fields and their derivatives, then the system (40) is a lower dimensional system of PDEs for the fields $u, v, p, q$ (dimensionality of this PDE must be less then the dimensionality of the system (30, 31)). Derivatives in eqs. (40, 43) appear due to the functions $f_i$ ($i = 1, 2$), which may depend on the partial derivatives of the fields $u, v, p$ and $q$. In particular, if $f_i$ ($i = 1, 2$) are the function of single field $u$ and its derivatives, then only eq. (40) is a PDE for the function $u$, while eqs. (41, 43) are the non-differential equations for the fields $v, p$ and $q$.

Now we briefly characterize the solutions $u, v, p$ and $q$ to the system (30, 31), which are implicitly given by eqs. (40, 43). As was noted above, the system (40, 43) is a system of PDEs for the fields $u, v, p, q$ whose dimensionality is less then $N + 2$, i.e. we do not completely integrate the original system of PDEs (30, 31), but reduce its dimensionality. Further, eqs. (40, 43) involve two arbitrary functions $\mathcal{F}_{i1}, i = 1, 2, \ldots, N$ variables. For this reason, the available solution space is not full (for the fullness, the solution space to the $(N + 2)$-dimensional system of four PDEs must involve four arbitrary functions of $N + 1$ variables). Thus the system (30, 31) is partially integrable by our algorithm. Since all fields are given implicitly by eqs. (40, 43), they describe the wave breaking in general.

## 3 Conclusions

The algorithm proposed in this paper allows one to solve a new class of multi-dimensional nonlinear PDEs using the features of the characteristics method and Hopf-Cole transformation. The principal novelty of this algorithm is the presence of arbitrary functions of fields in eq. (7), i.e. functions $\eta, \eta^{(i)}$ and $\Gamma^{(m)}$. Thus, this algorithm possesses new features which do not appear in both method of characteristics and method of direct linearization by the Hopf-Cole transformation. The derived nonlinear PDEs are partially integrable since, first, one has to solve a lower dimensional PDE in order to construct solutions and, second, the available solution space is not full. The important question is whether the multi-scale expansion (35) may be properly implemented in the proposed algorithm so that eq. (34) would be exactly solvable.

It is important that, among the derived nonlinear PDEs, there are such equations that may be considered as multidimensional generalizations of known integrable models. The physical applications of these generalizations become obvious provided that some terms are negligible, see eq. (34) reduced from the system (30, 31).
This work is supported by the RFBR grant 10-01-00787 and by the Program for Support of Leading Scientific Schools (grant No.6170.2012.2).

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