Hilbert space representation of maximal length and minimal momentum uncertainties

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Abstract

Perivolaropoulos has recently proposed a position-deformed Heisenberg algebra which includes a maximal length [Phys. Rev. D 95, 103523 (2017)]. He has shown that this length scale naturally emerges in the context of cosmological particle’s horizon or cosmic topology. Following this work, we propose a new deformed algebra and derive the maximal length uncertainty and its corresponding minimal momentum uncertainty from the generalized uncertainty principle. We construct a Hilbert space representation in the spectral representation of this length scale. We also construct the corresponding Fourier transform and its inverse representations. Finally, we propose \textit{n}-dimensional representation of this algebra.
Keywords: Generalized uncertainty principle; Position and momentum representation

1 Introduction

A minimum length scale of the order of Planck length is a feature of many models of quantum gravity that seek to unify quantum mechanics and gravitation. Recently, Perivolaropoulos [1] proposed a new version of a Generalized Uncertainty Principle (GUP) that yields simultaneously a maximal length and a minimal momentum uncertainties. An extension of this work has been carried out [2] and some applications have been performed to investigate the implications of such GUPs in thermodynamics of ideal gases [3, 4] and of Black hole [5]. In this work [1], Perivolaropoulos also predicted the simultaneous existence of minimal and maximal length measurements. More recently, one of us has shown that both measurable lengths can be obtained from position-dependent noncommutativity [6]. Its applications run from quantum optics [7], quantum thermodynamics [8] to non-Hermitian quantum mechanic scenarios [9].

In the present letter, we consider a generalized version of position-dependent deformed Heisenberg algebra [1] which includes a simultaneous presence of maximal position and minimal momentum uncertainties. Based on the GUP with minimal length scenarios [10, 11, 12, 13, 14, 15, 17], we study the functional analytic aspects of the maximal length uncertainty. We show that the spectral representation of this length scale forms a family of discrete eigenvalues that describes a lattice space. The corresponding set of eigenvectors exhibits properties similar to the standard Gaussian states which are consequences of quantum fluctuation at this scale. The rest of this letter is structured as follows: in the next section, we review the GUP with a maximal length and a minimal momentum. In the third section, we study the position representation of these uncertainty measurements, the corresponding Fourier transform and its inverse. Then finally, we generalise this representation in an arbitrary dimensional Hilbert space. In the last section we present our conclusion.

2 Position-deformed Heisenberg algebra

Let $\mathcal{H}_r = \mathcal{L}^2(\mathbb{R})$ be the Hilbert space of square integrable functions. $\mathcal{D}(\hat{X})$ and $\mathcal{D}(\hat{P})$ are respectively the domains of operators $\hat{X}$ and $\hat{P}$ maximally dense in $\mathcal{H}$. These operators $\hat{X}$ and $\hat{P}$ satisfy [6, 7, 8]

$$[\hat{X}, \hat{P}] = i\hbar(1 - \tau \hat{X} + \tau^2 \hat{X}^2),$$  \hspace{1cm} (1)

where $\tau \in \mathbb{R}_+^*$ is the GUP deformed parameter [10, 15, 17, 18]. By setting these operators

$$\hat{X} = \hat{x}, \quad \hat{P} = (1 - \tau \hat{x} + \tau^2 \hat{x}^2)\hat{p},$$  \hspace{1cm} (2)

in terms of the Hermitian operators $\hat{x}$ and $\hat{p}$ satisfy the ordinary Heisenberg algebra $[\hat{x}, \hat{p}] = i\hbar$, one recovers the relation [1]. Let $\phi(x) \in \mathcal{D}(\hat{X})$ and $\phi(p) \in \mathcal{D}(\hat{P})$ be
respectively the position and momentum representations. The action of the operators on these square integrable functions reads as follows

\[ \hat{X}\phi(x) = x\phi(x) \quad \text{and} \quad \hat{P}\phi(x) = -i\hbar(1 - \tau x + \tau^2 x^2)\frac{d}{dx}\phi(x), \quad x \in \mathbb{R}, \quad (3) \]

\[ \hat{X}\phi(p) = i\hbar\frac{d}{dp}\phi(p) \quad \text{and} \quad \hat{P}\phi(p) = \left(1 - i\hbar\tau\frac{d}{dp} - \tau^2\hbar^2\frac{d^2}{dp^2}\right)p\phi(p), \quad p \in \mathbb{R}. \quad (4) \]

For both representations, the corresponding completeness relations are given by

\[ \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2}|x\rangle\langle x| = \mathbb{I}, \quad (5) \]

\[ \int_{-\infty}^{+\infty} dp|p\rangle\langle p| = \mathbb{I}. \quad (6) \]

Consequently, the scalar product between two states \(|\Psi\rangle\) and \(|\Phi\rangle\) and the orthogonality of eigenstates become

\[ \langle\Psi|\Phi\rangle = \int_{-\infty}^{+\infty} dx \frac{d}{1 - \tau x + \tau^2 x^2}\Phi^*(x)\Phi(x), \quad \langle x|x'\rangle = (1 - \tau x + \tau^2 x^2)\delta(x - x'), \quad (7) \]

\[ \langle\Psi|\Phi\rangle = \int_{-\infty}^{+\infty} dp \Psi^*(p)\Phi(p), \quad \langle p|p'\rangle = \delta(p - p'). \quad (8) \]

For an operator \(\hat{A} = \{\hat{X}, \hat{P}\}\), its expectation value for both representations are given by

\[ \langle\hat{A}\rangle_{\phi(x)} = \langle\phi(x)|\hat{A}|\phi(x)\rangle = \int_{-\infty}^{+\infty} dx \frac{d}{1 - \tau x + \tau^2 x^2}\phi^*(x)\hat{A}\phi(x), \quad (9) \]

\[ \langle\hat{A}\rangle_{\phi(p)} = \langle\phi(p)|\hat{A}|\phi(p)\rangle = \int_{-\infty}^{+\infty} dp \phi^*(p)\hat{A}\phi(p), \quad (10) \]

and the corresponding dispersions are

\[ (\Delta_{\phi(x)}\hat{A})^2 = \langle\hat{A}^2\rangle_{\phi(x)} - \langle\hat{A}\rangle_{\phi(x)}^2 = \int_{-\infty}^{+\infty} dx \frac{d}{1 - \tau x + \tau^2 x^2}\phi^*(x)\left(\hat{A} - \langle\hat{A}\rangle_{\phi(x)}\right)^2\phi(x), \quad (11) \]

\[ (\Delta_{\phi(p)}\hat{A})^2 = \langle\hat{A}^2\rangle_{\phi(p)} - \langle\hat{A}\rangle_{\phi(p)}^2 = \int_{-\infty}^{+\infty} dp \phi^*(p)\left(\hat{A} - \langle\hat{A}\rangle_{\phi(p)}\right)^2\phi(p). \quad (12) \]

For any representation, an interesting feature can be observed from the commutation relation through the following uncertainty relation:

\[ \Delta X\Delta P \geq \frac{\hbar}{2} \left(1 - \tau \langle\hat{X}\rangle + \tau^2 \langle\hat{X}^2\rangle\right). \quad (13) \]

Using the relation \(\langle\hat{X}^2\rangle = \langle\Delta X\rangle^2 + \langle\hat{X}\rangle^2\), the equation (13) can be rewritten as a second order equation for \(\Delta X\) whose solution is

\[ \Delta X = \frac{\Delta P}{\hbar \tau^2} \pm \sqrt{\left(\frac{\Delta P}{\hbar \tau^2}\right)^2 - \frac{\langle\hat{X}\rangle}{\tau} \left(\tau \langle\hat{X}\rangle - 1\right) - \frac{1}{\tau^2}}. \quad (14) \]

3
This equation leads to the absolute minimal uncertainty \( \Delta P_{\text{min}} \) in \( P \)-direction and the absolute maximal uncertainty \( \Delta X_{\text{max}} \) in \( X \)-direction when \( \langle \hat{X} \rangle = 0 \), such that
\[
\Delta X_{\text{max}} = \frac{1}{\tau} \quad \text{and} \quad \Delta P_{\text{min}} = \hbar \tau.
\] (15)

It is well known that [10], the existence of minimal uncertainty raises the question of the loss of representation i.e the space is inevitably bounded by minimal quantity beyond which any further localization of particles is not possible. In the present situation, the minimal momentum \( \Delta P_{\text{min}} \) leads to the loss of representation in \( P \)-direction i.e the loss of \( \phi(p) \)-representation and a maximal measurement \( \Delta X_{\text{max}} \) conversely will be the physical space of wavefunction representations i.e all functions \( \phi(x) \in \mathcal{D}(\hat{X}) \subset \mathcal{H} = L^2(-\infty, +\infty) \) vanish at the boundary \( \phi(-\infty) = 0 = \phi(+\infty) \). Furthermore, the existence of the minimal momentum \( \Delta P_{\text{min}} \) induces also the loss of Hermicity of the operator \( \hat{P} \hat{P}^\dagger \neq \hat{P} \).

Using the relation (5) and by performing a partial integration [6], one can easily show that
\[
\langle \psi | \hat{P} \phi \rangle = \langle \hat{P}^\dagger \psi | \phi \rangle,
\] (17)
where
\[
\mathcal{D}(\hat{P}) = \{ \phi, \phi' \in L^2(-\infty, \infty); \phi(-\infty) = 0 = \phi'(+\infty) \}, \tag{18}
\]
\[
\mathcal{D}(\hat{P}^\dagger) = \{ \psi, \psi' \in L^2(-\infty, \infty) \}. \tag{19}
\]

3 Representation of the maximal length and the minimal momentum

In this section, we study the Hilbert space representation of the deformed Heisenberg algebra [11]. We define the position space representation that describes the maximal length uncertainty and its corresponding minimal momentum uncertainty. The quasi-representation of the position wavefunction is constructed through the Fourier transform and its inverse. Finally we extend the deformed Heisenberg algebra [11] to \( n \)-dimensional case.

3.1 Position space representation

The representation of the momentum operator \( \hat{P} \) reads as follows
\[
-i\hbar(1 - \tau x + \tau^2 x^2) \frac{d}{dx} \phi_\eta(x) = \eta \phi_\eta(x), \quad \eta \in \mathbb{R}.
\] (20)
By solving the differential equation (20), we obtain the position eigenvectors in the form
\[
\phi_\eta(x) = A \exp \left( \frac{2\eta}{\tau \hbar \sqrt{3}} \left[ \arctan \left( \frac{2\tau x - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right), \tag{21}
\]
where $A$ is an arbitrary constant. Then by normalization, $\langle \phi_\eta | \phi_\eta \rangle = 1$, we have

$$1 = \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2} \phi_\eta^*(x) \phi_\eta(x)$$

$$= A^2 \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2}. \quad (22)$$

so, we find

$$A = \sqrt{\frac{\tau \sqrt{3}}{2 \pi}}. \quad (23)$$

Substituting this equation (23) into the equation (21), we get

$$\phi_\eta(x) = \sqrt{\frac{\tau \sqrt{3}}{2 \pi}} \exp \left( i \frac{2\eta}{\tau \hbar \sqrt{3}} \left[ \arctan \left( \frac{2\tau x - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right). \quad (24)$$

This wave function describes the maximal length and the minimal momentum uncertainties. It is well known that [33] a compression of the wave function to a small volume leads to a sufficient increase of its kinetic energy, which leads to a divergence of the wave function. Since the wave function (24) describes a particle in a small region, we calculate the expectation value of its kinetic energy

$$E = \frac{1}{2m} \langle \phi_\eta | \hat{\mathbf{P}}^2 | \phi_\eta \rangle \quad (25)$$

$$= \frac{1}{2m} \int_{-\infty}^{+\infty} \phi_\eta^*(x) \left[ -ih(1 - \tau x + \tau^2 x^2) \frac{d}{dx} \right]^2 \phi_\eta(x). \quad (26)$$

By solving this integral, we find

$$E = \frac{\eta^2}{2m} < \infty. \quad (27)$$

This result is known as energetic stability or atomic stability [33]. In comparison with Kempf et al [10], the expectation value of the energy in our case does not diverge. According to this formalism [10], any states that have a well-defined minimal uncertainty measurement which is inside of a forbidden gap cannot have a finite energy. So that they cannot be accepted as physical states. Conversely to this formalism, here the energy is well defined therefore the states $\phi_\eta(x)$ are physically relevant although they are defined inside of the forbidden gap $0 < \Delta P < \Delta P_{\min}$. Thus, these states are sufficiently quantized with $\eta = \frac{\hbar \sqrt{3}}{2 \pi} n \ (n \in \mathbb{N})$ and they form a set of orthogonal basis.

The scalar product of the formal eigenstates is given by

$$\langle \phi_\eta | \phi_\eta \rangle = \frac{\tau \sqrt{3}}{2 \pi} \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2}$$
\[ \times \exp\left(\frac{i(\eta - \eta')}{\tau \sqrt[3]{3}} \left[ \arctan\left( \frac{2\tau x - 1}{\sqrt[3]{3}} \right) + \frac{\pi}{6} \right] \right) \] (28)

\[ = \frac{\tau h \sqrt[3]{3}}{2\pi(\eta - \eta')} \sin\left( \frac{2\pi(\eta - \eta')}{\tau h \sqrt[3]{3}} \right). \] (29)

This relation shows that, the normalized eigenstates (24) are no longer orthogonal. However, if one tends \((\eta - \eta') \to \infty\), these states become orthogonal

\[ \lim_{(\eta - \eta') \to \infty} \langle \phi_{\eta'} | \phi_{\eta} \rangle = 0. \] (30)

For \((\eta - \eta') \to 0\), we have

\[ \lim_{(\eta - \eta') \to 0} \langle \phi_{\eta'} | \phi_{\eta} \rangle = 1. \] (31)

These properties show that, the states \(|\phi_{\eta}\rangle\) are essentially Gaussians centered at \((\eta - \eta') \to 0\) (see Figure 1). This indicates quantum gravitational fluctuations at this scale and these fluctuations increase as one increases the gravitational effects.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Variation of \(\langle \phi_{\eta'} | \phi_{\eta} \rangle\) versus \(\eta - \eta'\).}
\end{figure}

Since at infinity \((\eta - \eta' \to \infty)\), the scalar product is zero, the states become orthogonal (30). Moreover, one has

\[ 2\pi \frac{\eta - \eta'}{\tau h \sqrt[3]{3}} = n\pi \implies \eta - \eta' = \frac{\tau h \sqrt[3]{3}}{2} n, \quad n \in \mathbb{Z}, \] (32)
which implies
\[ \langle \phi_{(\eta+\frac{\sqrt{3}}{2}n)\tau\hbar}|\phi_{(\eta+\frac{\sqrt{3}}{2}n')\tau\hbar} \rangle = \delta_{nn'} . \] (33)

The set of eigenvectors \( \{ |\phi_{(\eta+\frac{\sqrt{3}}{2}n')\tau\hbar} \rangle \} \) form a complete orthogonal basis which can diagonalize the operator \( \hat{P} \). Since the formal momentum eigenvectors \( |\phi_{\frac{n\tau}{\sqrt{3}}} \rangle \) are physically accepted and by referring to the work [37], one may tempted to interpret this result as if we are describing physics on a lattice in which each sites are spacing by the value \( \sqrt{3}\Delta P_{\text{min}} \).

3.2 Fourier transform and its inverse representations

Since the states \( \phi_{\eta}(x) \) are physically meaningful and are well localized, one can obtain the quasi-momentum representation by projecting an arbitrary state onto this localized states [10]

\[
\psi(\eta) = \langle \phi_{\eta}|\psi \rangle = \sqrt{\frac{\tau \sqrt{3}}{2\pi}} \int_{-\infty}^{+\infty} \psi(x)dx \frac{e^{-i\frac{2\eta}{\sqrt{3}} \left\{ \tan^{-1}\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right\}}}{1 - \tau x + \tau^2 x^2} .
\] (34)

The transformation that maps the quasi-position space wave functions into the momentum ones is the Fourier transformation. The inverse Fourier transform is given by

\[
\psi(x) = \frac{1}{\hbar \sqrt{2\pi \sqrt{3}}} \int_{-\infty}^{+\infty} d\eta \psi(\eta)e^{i\frac{2\eta}{\sqrt{3}} \left\{ \arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right\}} .
\] (35)

Moreover from equation (34), we can deduce

\[
\frac{d}{d\eta} e^{-i\frac{2\eta}{\sqrt{3}} \left\{ \arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right\}} = -i \frac{2}{\tau \sqrt{3}} \left\{ \arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right\} e^{-i\frac{2\eta}{\sqrt{3}} \left\{ \arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right\}} .
\] (36)

This equation is equivalent to

\[
i \frac{\tau \sqrt{3}}{2} \frac{d}{d\eta} = \left[ \arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right] = \left[ \arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \arctan\left(\frac{1}{\sqrt{3}}\right)\right] .
\] (37)

From the following relation [37]

\[
\arctan \alpha + \arctan \beta = \arctan\left(\frac{\alpha + \beta}{1 - \alpha \beta}\right) \quad \text{with} \quad \alpha \beta < 1,
\] (38)
we deduce that

\[
\tan \left[ \arctan \left( \frac{2\tau x - 1}{\sqrt{3}} \right) + \arctan \left( \frac{1}{\sqrt{3}} \right) \right] = \frac{\tau x \sqrt{3}}{2 - \tau x}. \tag{39}
\]

We notice from the equation (37) that the position operator is represented as

\[
\hat{X} = \frac{2}{\tau} \tan \left( \frac{i\tau \hbar}{\sqrt{3}} \frac{dx}{d\eta} \right) + \frac{\pi}{6}, \tag{40}
\]

\[
\hat{X} \psi(\eta) = \frac{2}{\tau} \tan \left( \frac{i\tau \hbar}{\sqrt{3}} \frac{dx}{d\eta} \right) \psi(\eta). \tag{41}
\]

From the action of \( \hat{P} \) on the quasi representation and using equation (3), we have

\[
\hat{P} \psi(\eta) = \eta \psi(\eta). \tag{42}
\]

Note that in the limit of \( \tau \), we recover the corresponding ordinary quantum mechanics results in momentum space

\[
\lim_{\tau \to 0} \hat{X} \psi(\eta) = i\hbar \frac{d}{d\eta} \psi(\eta), \quad \lim_{\tau \to 0} \hat{P} \psi(\eta) = \eta \psi(\eta). \tag{43}
\]

Now, based on the representation of the scalar product of two arbitrary vectors \(|\Psi\rangle\) and \(|\Phi\rangle\) reads as follows

\[
\langle \Psi | \Phi \rangle = \frac{1}{2\pi \hbar \sqrt{3}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{1 - \tau x + \tau^2 x^2} e^{i \frac{2(x - \eta)}{\sqrt{3}} \left[ \arctan \left( \frac{2x - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]} \Psi^*(\eta') \Phi(\eta) dx d\eta d\eta'. \tag{44}
\]

### 3.3 Generalization to n-dimensional position-deformed algebra

A generalization of the one-dimensional deformed-position Heisenberg algebra that preserves the rotational symmetry is

\[
[\hat{X}_i, \hat{P}_j] = i\hbar \delta_{ij} (1 - \tau \hat{X} + \tau^2 \hat{X}^2), \tag{45}
\]

where \( \hat{X} = \sum_{i=1}^{n} \sqrt{X_i X_i} \). This algebra induces a maximal length and a nonzero minimal momentum. Within this deformed algebra, the action of position and momentum operators on the position space representation can be written as

\[
\hat{X}_i \phi(x) = x_i \phi(x) \quad \text{and} \quad \hat{P}_j \phi(x) = -i\hbar (1 - \tau x + \tau^2 x^2) \partial_{x_j} \phi(x), \tag{46}
\]

where \( x = \sum_{i=1}^{n} \sqrt{x_i x_i} \). As in ordinary quantum mechanics, if we assume that the components of the position operator are commutative

\[
[\hat{X}_i, \hat{X}_j] = 0, \tag{47}
\]
then from the Jacobi identity
\[
[[\hat{P}_i, \hat{P}_j], \hat{X}_k] + [[\hat{P}_j, \hat{X}_k], \hat{P}_i] + [[\hat{X}_k, \hat{P}_i], \hat{P}_j] = 0
\] (48)
one obtains the commutation relations between the components of the momentum operator as \[11, 38, 39\]
\[
[[\hat{P}_i, \hat{P}_j]] = i\hbar \tau \left( 2\tau - \frac{1}{x} \right) \left( \hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i \right).
\] (49)
The identity operator can be expanded as
\[
\int_{-\infty}^{+\infty} \frac{d^nx}{1 - \tau x + \tau^2 x^2} |x\rangle \langle x| = \mathbb{I}.
\] (50)
The scalar product between two states \(|\Psi\rangle\) and \(|\Phi\rangle\) and the orthogonality of eigenstates become
\[
\langle\Psi|\Phi\rangle = \int_{-\infty}^{+\infty} \frac{d^nx}{1 - \tau x + \tau^2 x^2} \Psi^*(x) \Phi(x), \quad \langle x|x'\rangle = (1 - \tau x + \tau^2 x^2) \delta^n(x - x').
\] (51)
Now, if we consider the generalization of the previously defined as
\[
[[\hat{X}_i, \hat{P}_j]] = i\hbar \delta_{ij} (\mathbb{I} - f(\hat{X}) + g(\hat{X}^2)),
\] (52)
where \(f\) and \(g\) are deformed functions that we assume strictly positive. Then it is straightforward to show that
\[
\hat{X}_i \phi(x) = x_i \phi(x) \quad \text{and} \quad \hat{P}_j \phi(x) = -i\hbar \left( 1 - f(x) + g(x^2) \right) \partial_{x_j} \phi(x).
\] (53)
Therefore we find \[11\]
\[
[[\hat{P}_i, \hat{P}_j]] = i\hbar \left( -\frac{1}{x} f'(x) + g'(x^2) \right) \left( \hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i \right).
\] (54)
where by definition
\[
f'(x) = \frac{df}{dx} \quad \text{and} \quad g'(x^2) = \frac{dg}{dx^2}.
\] (55)
In our case \[45\], \(f(\hat{X}) = \tau \hat{X}\) and \(g(\hat{X}) = \tau^2 \hat{X}^2\). If we choose \(f(\hat{X}) = \mathbb{I} - \frac{i}{\tau - \alpha x^2}\) and \(g(\hat{X}^2) = 0\), we obtain the generalized form of Perivolaropoulos’s algebra \[11\] recently determined by Bensalem and Bouaziz \[4\].
4 Conclusion

In this paper we have studied a new generalized form of GUP introduced by Perivolaropoulos [1] which implies a maximal length uncertainty and a minimal observable momentum. Then, using the deformed algebra which generates this GUP, we studied the representation of the wave function that describes both uncertainty measurements. We have shown that with this maximal length uncertainty concept there is no divergency in energy spectrum of the particle. The state representations of this particle are physically meaningful and exhibit properties similar to the standard Gaussian states which are consequences of quantum fluctuations at that scale. We have also studied the quasi-representation of operators through the Fourier transform and its inverse. Finally, we have generalised the algebra into n-dimensional cases where a general representation of operators was studied.

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