Wiener and non-Wiener dynamics of open systems

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Abstract. The paper discusses mathematical and physical differences in describing open systems by means of the algebraic perturbation theory of different order.

1. Introduction
Mathematical models of open optical systems allow describing a large class of real physical objects which interact with the surrounding electromagnetic fields, whose typical frequencies lie in the optical spectral region. One specific feature of optical and near-frequency ranges of electromagnetic fields is the small-sized elementary electromagnetic emitters compared with the typical wavelengths involved in interactive processes. The other key feature of the given systems is the engagement in describing dynamics of these systems’ values which have different typical time-change scales, i.e. fast and slowly changing time functions.

Being engaged in describing dynamics of open optical systems of different time-change scale functions leads to use averaging methods of rapidly varying values over characteristic time intervals of the problem. Similar methods were developed as early as in papers [1,2], they appear to be rather cumbersome [3] in their conventional formulation.

The Markov approximation is true of open systems, in which dynamic equations, e.g. the Schrödinger equation, become mathematically uncertain. Thus, for further analysis, either the kinetic (master) equations, which are be derived, are used, or a rigorous status is assigned to the Schrödinger equation, which afterwards emerges as a stochastic differential equation (SDE) [4]. The use of SDEs allows us to obtain the kinetic equation. This leads to a transparency in calculation and analysis of the model, and facilitates derivation of the kinetic equation.

In terms of the averaging method, the author has developed its algebraic formulation (algebraic perturbation theory) for constructing an effective Hamiltonian [5–7] and obtaining a kinetic equation for open optical systems. Used for the first time in [8], the derivation/formulation of quantum SDEs for the Schrödinger equation with the effective Hamiltonian was obtained within the algebraic theory. This approach has proven particularly fruitful in allowing us to analyze a lot of problems in resonant optics and predicting a variety of fundamental physical effects, such as entanglement of quantum states and stabilization of excited states of the atomic ensemble relative to collective decay [9,10]. The notions of Wiener and non-Wiener dynamics of open quantum systems were introduced as well. Mathematical differences are apparent in the SDEs – non-Wiener SDEs [10] differ from Wiener ones [4] due to the quantum counting process that arises in constructing an effective Hamiltonian when the second order algebraic perturbation theory is taken into account.
2. Algebraic perturbation theory and the effective Hamiltonian

Use of algebraic perturbation theory in quantum optics lies in the transition from a state vector $|\Psi\rangle$ to a new state vector by means of unitary transformation $|\Psi\rangle \rightarrow U|\Psi\rangle$. For the Schrödinger equation determined by the Hamiltonian $H$, $i\hbar \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle$, the given transition from the vector should be followed by the Hamiltonian’s change,

$$\widetilde{H} = UHU^* - i\hbar U \frac{d}{dt} U^*, \tag{1}$$

for the quantum system to be described by the Schrödinger equation with the transformed Hamiltonian

$$i\hbar \frac{d}{dt} \widetilde{|\Psi\rangle} = \widetilde{H} |\widetilde{\Psi}\rangle. \tag{2}$$

Such transformations are widely known. In case of $U = \exp(ith/h)$, $\widetilde{H} = 0$, then we have the Heisenberg representation. In case of $U = \exp(iH_{eff}/h)$, $\widetilde{H} = \exp(iH_{eff}/h)V \exp(-iH_{eff}/h)$, then we come to the Dirac picture, where the system’s Hamiltonian $H = H_0 + V$, is expressed as an Hamiltonian $H_0$ of isolated subsystems and interaction operator $V$.

For convenience, principles of the algebraic perturbation theory will be formulated in the Dirac picture. If a unitary operator $U$ is expressed by a Hermitian operator $U = e^{-iS}$, $S^* = S$, then the transformed Hamiltonian (1) can be expanded in series over $S$ (the Baker-Hausdorff formula)

$$\widetilde{H} = e^{-iS}H_0 e^{iS} + e^{-iS}V e^{iS} - i\hbar e^{-iS} \frac{d}{dt} e^{iS} =$$

$$= H_0 - i[S, H_0] - \frac{i}{2}[S, [S, H_0]] - ... + V - i[S, V] - \frac{i}{2}[S, [S, V]] - ... - i\hbar e^{-iS} \frac{d}{dt} e^{iS}. \tag{3}$$

When the interaction operator $V$ is small in a certain sense, then the operator $S$ and the transformed Hamiltonian (3) can be expressed as series over the interaction constant

$$S(t) = S^{(0)}(t) + S^{(2)}(t) + ... \quad \text{and} \quad \widetilde{H}(t) = \widetilde{H}^{(0)}(t) + \widetilde{H}^{(2)}(t) + ... \tag{4}$$

From now on $S^{(n)}$ and $\widetilde{H}^{(n)}$ denote the terms which have $n$-th order over the interaction constant. An explicit expression for the time argument $t$ indicates the operators in the Dirac picture. Assuming that the interaction operator has the first order over the interaction constant, substituting (4) into the expansion (3) for terms of the same order gives the basic formulas of algebraic perturbation theory:

$$\widetilde{H}^{(0)}(t) = \hbar \sum_i \{b_i^+ + b_i\} \sum_{i,j} d_{ij} [E_i, E_j]^{(i)}$$

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\[ H^{\text{Eff}}(t) = \tilde{H}^{(1)}(t) + \tilde{H}^{(2)}(t), \quad \tilde{H}^{(3)}(t) = \sum_q \gamma_q b_q^* d_{q1} e^{i(\omega_q - \omega_{21})t} R_e + \sum_q \gamma_q b_q d_{q2} e^{-i(\omega_q - \omega_{21})t} R_e, \]
\[ \tilde{H}^{(2)}(t) = H^{S0}(t) + V^{D,D}, \quad V^{D,D} = -\sum_q |d_{q2}|^2 \hbar^{-1} (\omega_q + \omega_{21})^{-1} (R_R + R_R - N_a), \]
\[ H^{S0}(t) = \sum_q \gamma_q b_q^* b_q e^{-i(\omega_q - \omega_{21})t} \{ \Pi_1(\omega_q, \omega_q) \pm N_a + \Pi_1(\omega_q, \omega_q) R_e \}, \]
\[ \Pi_1(\omega) = \sum_q |d_{q2}|^2 \hbar^{-1} (\omega_{q} + \omega) - (\omega_{q} - \omega)^{-1}, \quad \Pi_1(\omega, \omega') = \frac{1}{2} \{ \Pi_2(\omega) + \Pi_2(\omega') \pm (\Pi_1(\omega) + \Pi_1(\omega')) \}, \]
\[ R_e = \frac{1}{2} \sum_i (|E_2 >^i < E_2|^i - |E_i >^i < E_1|^i), \quad R_1 = \sum_i |E_1 >^i < E_2|^i, \quad R_2 = \sum_i |E_2 >^i < E_1|^i, \]
\[ [R_3, R_1] = \pm R_3, \quad [R_3, R_2] = 2R_3. \]

Here \(|E_i >^i\) is the state vector of \(i\)-th atom with energy \(E_i\), \(d_{q2}\) is the dipole moment matrix element, \(\omega_{21}\) is the transition frequency between ground and excited levels, \(\omega_q = (E_q - E_e)\hbar^{-1}\), \(\omega_q\) is the photon frequency with a wave vector \(\vec{q}\), \(b_q\) is the boson annihilation operator, \(\gamma_q\) is the characteristics of electromagnetic field geometry.

For a three-dimensional, unidirectional, and one-dimensional models of a quantized electromagnetic field to be developed, we proceed to new boson operators for the creation and annihilation of quanta \(b^*_a\) and \(b_a\) that depend only on the effective projection of the photon momentum, which is to be compared to its frequency. Then we can go to the integration from summation:

\[ \tilde{H}^{(1)}(t) = \int_0^\infty y^{(1)}(\omega) b^*_a b_a e^{-i(\omega - \omega_{21})t} R_e d\omega + \int_0^\infty y^{(1)*}(\omega) b_a^* b_a e^{-i(\omega - \omega_{21})t} R_e d\omega, \quad (6a) \]
\[ H^{S0}(t) = \int_0^\infty \int_0^\infty \left\{ y^{(2)}(\omega, \omega') \pm N_a + y^{(2)*}(\omega, \omega') R_e \right\} b_a^* b_a e^{-i(\omega - \omega')t} d\omega d\omega'. \quad (6b) \]

Thus, the expression \(H^{\text{Eff}}(t) = \tilde{H}^{(1)}(t) + H^{S0}(t) + V^{D,D}\) is the effective Hamiltonian of an open optical system made up of a localized atomic ensemble of \(N_a\) identical atoms and its electromagnetic environment.

3. Markov approximation and stochastic differential equation

The Markov approximation lies in:
1. the delta-correlated condition of averaged operators \(<b^*_a b_a > = 0\), \(<b_a^* b_a^* > = \delta(\omega - \omega')\) (a vacuum field with a zero-photon density is considered);
2. neglect of frequency dependences of the effective Hamiltonian parameters \(y^{(1)}(\omega) = \text{const}\) and \(y^{(2)}(\omega, \omega') = \text{const}\);
3. the formal extension of the lower integration limit to \(-\infty\), since, due to the requirement for a slow change of functions, the characteristic frequencies \(\omega\) lie in the vicinity of the resonant transition frequency \(\omega_{21}\).

When meeting the Markov conditions, the effective Hamiltonian terms, that are responsible for the interaction with broadband vacuum fields, are expressed in terms of the basic quantum stochastic processes: annihilating process \(B(t) = \int_0^t dt' b(t')\), \(b(t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} d\omega e^{-i(\omega - \omega_{21})t} b_a\) and counting process \(\Lambda(t) = \int_0^t dt' b^*(t') b(t')\). From now on all values are considered to be dimensionless, and for the sake of
clarity, they are designated using the same symbols (see, the introduction of dimensionless parameters, e.g. in work [10]).

Ito differentials of quantum stochastic processes \(dB(t) = B(t + dt) - B(t)\) and \(d\Lambda(t) = \Lambda(t + dt) - \Lambda(t)\) satisfy the Hudson - Parthasarathy algebra [11]

\[
d\Lambda(t) dB(t) = dB(t) d\Lambda(t) = d\Lambda(t) dB(t) = dB(t) d\Lambda(t) = dB^+(t)dt = dB(t) dt = 0, \quad dtdt = 0,
\]

\[
< dB(t) >= < dB^+(t) > = < d\Lambda(t) >= 0.
\]

In terms of quantum stochastic processes, the dimensionless effective Hamiltonian \(H^{\text{Eff}}(t)\) can no longer be written, since the differential equation (2) with \(\bar{H} = H^{\text{Eff}}(t)\) was not defined mathematically - the introduced stochastic processes are not differentiable in the usual sense. Then the evolution equation is written, for example, in the form of \(dU(t, t_0) = \exp(iH^{\text{Eff}}(t) dt) - 1\) where the integral formal solution (1) is taken into account, and \(dU(t, t_0)\) is the Ito differential. The expression \(H^{\text{Eff}}(t) dt\) is a single symbol which means the following

\[
H^{\text{Eff}}(t) dt = H^{\text{Eff} - S}(t) dt + Y^* dB(t) + YdB^+(t) + Y_\lambda d\Lambda(t).
\]

Here, the term \(H^{\text{Eff} - S}(t) = Y^{D - D}\), representing the effective Hamiltonian of the atomic ensemble, is singled out separately. In addition, new relevant operators \(Y\) and \(Y_\lambda\) were introduced, so that the resulting equations could describe a general situation, and concrete expressions for the introduced operators would be easily obtained from a comparison of the expressions presented above.

The representation of the effective Hamiltonian by quantum stationary stochastic processes serves as the basis for further derivation of both a quantum SDE for the evolution operator and the kinetic equation of an open system. In case of algebra (7), we regard Wiener-type SDEs as being quantum ones which were obtained by means of quantum Wiener processes with the account of only the first-order algebraic perturbation theory \(H^{\text{Eff}}(t)dt = H^{\text{Eff} - S}(t)dt + Y^* dB(t) + YdB^+(t)\), \(Y_\lambda \neq 0\). In this context it would be reasonable to refer to the kinetic equations obtained by means of these SDEs as kinetic equations governed by quantum Wiener processes. Written with a quantum counting process and based on the effective Hamiltonian (8), \(Y_\lambda \neq 0\), a quantum SDE could be generally referred to as the quantum SDE of a generalized (with the account of the counting process) Langevin-type or non-Winner-type ones in the sense that SDEs are not determined by quantum Wiener processes only. It is obvious that at \(Y_\lambda \equiv 0\) use of non-Winner dynamics gives rise to the results characterizing the Wiener dynamics of an open optical system.

The operator represented by the counting process is generally small, since it is determined by the magnitude of the second order over the interaction constant. However, its value increases in the ensemble of identical atoms as the number of atoms \(N_\lambda\) grows [10]. Due to Hudson - Parthasarathy algebra relations, the counting process with a noise source of a zero-photon density becomes meaningful, regardless of a similar growth in an ensemble of operators during Wiener processes (of the first order over the interaction constant). Thus the “accumulative” property \(d\Lambda(t)dt = d\lambda(t)\) of the counting process manifests itself.

4. Markov approximation and stochastic differential equation

The specific feature of differential relations involving stochastic process differentials (Ito differentials) is non-compliance with the Leibniz rule. In case of the evolution operator, it takes the following form

\[
d(U(t)U^+(t)) = (dU(t))U^+(t) + U(t)dU^+(t) + (dU(t))(dU^+(t)).
\]

In this case \(d(U(t)U^+(t)) = 0\), which shows the unitary dynamics of an open system together with its electromagnetic environment.
A quantum SDE for the evolution operator is derived from calculating the Ito differential \( dU(t) \) with the account of the Hudson-Parthasarathy algebra (7).

Since the given effective Hamiltonian could be written in a rather general form (8), the resulting non-Wiener SDE would be widely applied. We consider operators \( Y_\lambda \) to be diagonal, which is the case in the given example, since \( Y_\lambda = c R_\lambda + c N_s / 2 \). Thus we obtain the following result:

\[
dU(t) = -iH^{\text{Eff}-S}(t)dtU(t) + \left[ Y^+(Y_\lambda^e + iY_\lambda)(Y_\lambda^*)^{-2}Ydt + Y^+Y_\lambda^ey^\lambda t + Y_\lambda^eY_\lambda^{-1}YdB^+(t) + Y_\lambda^ey_\lambda d\Lambda(t)\right]U(t),
\]

(9)

Non-Wiener operator factors \( Y_\lambda^e, Y_\lambda^ey_\lambda^{-1}, (Y_\lambda^* + iY_\lambda)(Y_\lambda^*)^{-2} \) differ a quantum non-Wiener type SDE from the Wiener-type SDE. In the absence of a counting process \( Y_\lambda = 0 \), non-Wiener operator factors are equal \( Y_\lambda^e = 0, Y_\lambda^ey_\lambda^{-1} = -i, (Y_\lambda^* + iY_\lambda)(Y_\lambda^*)^{-2} = -1/2 \) and the Wiener SDE [4], determined only by the Ito differential of the Wiener process, follows from the non-Wiener SDE (9):

\[
dU(t) = -iH^{\text{Eff}-S}(t)dtU(t) - (\lambda Y'Ydt + iy^\lambda dB^+(t) + iydB^+(t))U(t).
\]

(10)

5. Kinetic equations. Wiener and non-Wiener dynamics
The outlined method of representing the dynamic Schrödinger equation with the SDE-type and effective Hamiltonian (in the Markov approximation), the kinetic equation for the density matrix \( \rho^{S+\text{Env}}(t) = |\Psi(t)\rangle \langle \Psi(t) | \) of an open system and its environment is directly calculated by the Ito differential \( dp^{S+\text{Env}}(t) \), using (9) and the algebra of Ito differentials (7). The kinetic equation for the density matrix \( \rho^S(t) = Tr_{\text{Env}}(\rho^{S+\text{Env}}(t)) \) of an open quantum system follows from the equation for \( \rho^{S+\text{Env}}(t) \) when it is averaged over the environment states with the account of relations

\[
Tr_{\text{Env}}(\rho^{S+\text{Env}}(t)dB(t)) = Tr_{\text{Env}}(\rho^{S+\text{Env}}(t)dB^+(t)) = Tr_{\text{Env}}(\rho^{S+\text{Env}}(t)d\Lambda(t)) = 0.
\]

We have

\[
\frac{d\rho^S(t)}{dt} = -i[H^{\text{Eff}-S}(t), \rho^S(t)] + \frac{Y^e}{Y_\lambda^e}Yp^S(t)Y^+Y^e + Y^* + iY_\lambda^eYp^S(t)Y^* - iY_\lambda^e - Y^e - Y^e - Y_\lambda^e - Y_\lambda^e.
\]

This kinetic equation can be easily rewritten as the Lindblad equation

\[
\dot{\rho}_S = i[H^{\text{Eff}-S}, \rho_S] - \Gamma_\rho \rho_S, \quad \Gamma_\rho = -i[p^S, H^{\text{Eff}-S}] + \frac{Y^e}{Y_\lambda^e}Yp^S(t)Y^+Y^e + Y^* + iY_\lambda^eYp^S(t)Y^* - iY_\lambda^e - Y^e - Y^e - Y_\lambda^e - Y_\lambda^e
\]

with Lindblad operators \( L^S = Y_\lambda^eY_\lambda^e - Y^*, H^{\text{Shift}-S} = Y^*(\sin Y_\lambda - Y_\lambda)Y^2 Y^*, L^S = 2Y^*(1 - \cos Y_\lambda)Y^2 Y^* \). In the case of Wiener dynamics, Lindblad operators take a simple form \( L^S = -iY^*, L^S = 2Y^* Y^*, H^{\text{Shift}-S} = 0 \).

Thus, in the proposed approach [6-10], the Lindblad operators have a different form depending on the terms included in the effective Hamiltonian.

If we consider only the atomic ensemble states that are symmetric with respect to permutations of atoms, then it is convenient to introduce eigenvectors \( |m\rangle \) of the operator \( R : R_m |m\rangle = m |m\rangle \), \( -r \leq m \leq r \). Here \( r \) is the parameter characterizing the irreducible representation of the angular momentum algebra \( su(2) \) for describing symmetric states of an ensemble of \( 2r = N_i \) identical atoms. The state of an ensemble of fully excited quantum particles is given by a vector \( |r\rangle \), and the state of an ensemble, whose particles are in the ground state, is given by a vector \( | -r \rangle \). The equation for elements in the density matrix of the ensemble has the simplest form in the case when \( \Pi (\omega, \omega') = y^{(2)}(\omega, \omega') = c = 0 \):

\[
\dot{\rho}_{mn} = -2Y^*(1 - \cos(\eta_0 N_s))(\eta_0 N_s)^2 [g_{m,m}p_{nn} - g_{m,m}p_{m+1,1}].
\]

(11)
Here, the dimensionless parameters are explicitly written \( \tau = \omega_0 t \), \( \chi = \sqrt{2} \omega_2 d \mu^{-1} e^{\omega_2/2 \hbar} \), \( \eta_0 = 4\omega_2^3 \mu^{-2} c^{-3} \Pi(\omega_2) \), \( \Pi_1(\omega) \approx \Pi_2(\omega) \equiv \Pi(\omega) \). The parameter \( \mu \) is introduced to describe various models of the quantized electromagnetic field [10].

The non-Wiener dynamics differs from the Wiener one by a factor \( 2[1 - \cos(\eta_0 N_s)](\eta_0 N_s)^{-2} \). It has been “embedded” in the formula for the Wiener dynamics, since, at the limiting value \( Y_\alpha = 0 \), \( 2[1 - \cos(\eta_0 N_s)](\eta_0 N_s)^{-2} = 1 \), when the non-Wiener dynamics is not accounted, we have equations for the usual Dicke superradiance theory for a localized atomic ensemble.

If \( Y_\alpha \neq 0 \), then a non-Wiener factor \( 2[1 - \cos(\eta_0 N_s)](\eta_0 N_s)^{-2} \) is an oscillating function which tends to vanish at \( \eta_0 N_s = 2\pi \). This means that \( \frac{1}{\pi} \rho_{nm} = 0 \) and collective excited states of an atomic ensemble do not decay. In other words, we observe the suppression of collective spontaneous emission, which is maintained in case of insignificant deviations from the condition \( \eta_0 N_s = 2\pi \). It has been estimated that the number of atoms in an ensemble must be at least a hundred [10] in order to fulfill the condition of full suppression of collective spontaneous emission.

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References

[1] Krylov N M and Bogoliubov N N 1947 *Introduction to Non-Linear Mechanics* (Princeton: Princeton University Press)
[2] Bogoliubov N and Mitropolsky Y 1961 *Asymptotic Methods in the Theory of Non-linear Oscillations* (New York: Gordon and Breach)
[3] Butylkin V S, Kaplan A E, Khronopulo Yu G and Yakubovich E I 1989 *Resonant Nonlinear Interactions of Light with Matter* (Berlin: Springer-Verlag)
[4] Gardiner C W and Collet M J 1985 *Phys.Rev.A* 31 3761
[5] Basharov A M, Maimistov A I and Manykin E A 1983 *Sov.Phys.JETP* 57 282
[6] Maimistov A I and Basharov A M 1999 *Nonlinear optical waves* (Dordrecht: Kluwer Academic)
[7] Basharov A M 2012 *JETP* 115 371
[8] Basharov A M 1992 *Sov. Phys. JETP* 75 611
[9] Basharov A M 2002 *JETP* 84 1070
[10] Basharov A M 2011 *Phys.Rev. A* 84 013801
[11] Hudson R L and Parthasarathy K R 1984 *Comm.Math.Phys.* 93 301