New parametrization of cosmic neutrino oscillations

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Abstract

A new parameterization of neutrinos oscillations in vacuum, assuming the conventional three flavor scheme, is proposed and discussed. Applications of this parameterization are considered, that concern the study of the flavor ratios, of their uncertainties, the expectations for the signal due to Glashow resonance. It is shown that a Gaussian treatment describes to a good level of approximation the effects of the uncertainties on the mixing angles and on the CP violating phase. The recently obtained agreement of observations with the hypothesis of cosmic neutrino oscillations is confirmed.

1 Introduction

After IceCube results, see e.g. [1], the importance of a precise description of oscillations has increased greatly. In the present paper, we propose a new choice of the relevant parameters that quantify the effect of oscillations, discussing their usefulness.

Let us begin by recalling the main achievements in the discussion of cosmic neutrinos oscillations. The general formula for the vacuum averaged oscillations of several neutrinos was given in [2]. Ref. [3] studied for the first time the implications of the observed oscillation phenomena on cosmic neutrinos. Since then, various authors remarked the possibility to measure flavor ratios, possibly aiming to constrain the parameters of oscillations, e.g. [4] [5] [6]. In [7] the single parameter that rules cosmic neutrino oscillations and that depends linearly upon unknown quantities was identified; then, it was remarked [8] that this leads to a strong correlation between the effect of the oscillation parameters on the probabilities of oscillation, reducing the chances of measuring the oscillation parameters. The non-linear effects have been studied in various subsequent papers including [9] [10] [11]. The expression for all relevant parameters was given in [12] in the context of a specific model for neutrino masses. The study of the oscillations of IceCube neutrinos was performed in [13], [14], [15], compare with [16]. In [14] the impact on the flavor ratio of the uncertainties on oscillation parameters was analyzed, and the present work develops the discussion. An alternative
| Parameter | Mean value | Standard deviation |
|-----------|------------|--------------------|
| $P_0$     | 0.109      | 0.005              |
| $P_1$     | 0.000      | 0.029              |
| $P_2$     | 0.010      | 0.007              |

Table 1: Table of present values and errors of the new parameters.

choice of the parameterization that gives different insight on the allowed ranges is discussed in [17].

We will introduce new parameters useful to discuss oscillations of cosmic neutrinos and obtain their analytical expressions in terms of the known mixing parameters. We give their numerical values and illustrate the usefulness of these parameters by discussing three applications: 1) we compare the predicted flavor fractions and those that are allowed by the present observations; 2) we quantify the uncertainties in the prediction of the fraction of muon neutrinos due to oscillations; 3) we prove that the intensity of the Glashow resonance [18] differs greatly in the alternative cases of $pp$- and $p\gamma$-production, even accounting for the uncertainties of oscillations. Throughout this work, we argue that a Gaussian treatment of the errors of these parameters is quite adequate for the present precision.

2 Definition of new parameters

Here, we give a new parameterization of the vacuum oscillation probabilities of cosmic neutrinos. First of all, it should be noted that we have to describe only three relevant parameters rather than the four standard parameters $\theta_{12}, \theta_{23}, \theta_{13}, \delta$. The new parameters of oscillations are $P_0, P_1, P_2$ defined as follow,

$$
\begin{align*}
    P_0 &= \frac{P_{ee} - \frac{1}{3}}{2}, \\
    P_1 &= \frac{P_{e\mu} - P_{e\tau}}{2}, \\
    P_2 &= \frac{P_{\mu\mu} + P_{\tau\tau} - 2P_{\mu\tau}}{4}
\end{align*}
$$

They agree with the parameters $A, B, C$ introduced in [12] in a more specific context. In terms of mixing angles and CP phase violation, their expressions are,

$$
\begin{align*}
    P_0 &= \frac{1}{2} \left\{ (1 - \epsilon^2) \left[ 1 - \frac{\sin^2 \theta_{12}}{2} \right] + \epsilon^2 - \frac{1}{3} \right\} \\
    P_1 &= \frac{1 - \epsilon}{2} \left\{ \gamma \cos 2\theta_{12} + \beta \frac{1 - 3\epsilon}{2} \right\} \\
    P_2 &= \frac{1}{2} \left\{ \gamma^2 + \frac{3}{4} \beta^2 (1 - \epsilon)^2 \right\}
\end{align*}
$$

1F.V. thanks Klas Hultqvist for raising this issue at the Neutrino Telescope conference at Venice (March 2015) and for useful discussions. For previous work on the subject see [19, 20, 21].
where we introduce for convenience the following 4 small parameters,

\[ \epsilon = \sin^2 \theta_{13} \]
\[ \alpha = \sin \theta_{13} \cos \delta \sin 2\theta_{12} \sin 2\theta_{23} \]
\[ \beta = \cos 2\theta_{23} \]
\[ \gamma = \alpha - \frac{\beta}{2} \cos 2\theta_{12}(1 + \epsilon) \]

The parameters \( \alpha, \beta, \gamma \) are small, first order, and not known precisely, whereas \( \epsilon \) is second order (= very small) and precisely known. In this sense, \( P_0 \) is a zeroth-order parameter; \( P_1 \) is first order in \( \alpha \) and \( \beta \) and agrees with the leading order expression given in \([7]\); \( P_2 \) is second order in \( \alpha \) and \( \beta \). Note moreover that \( P_2 \) is bound to be positive.

Using the present knowledge of mixing angle and phase CP violation \([22]\), we obtain the values and the errors of new parameters. We show the results in the Table 1, assuming normal mass hierarchy. It is easy to repeat the same steps with inverted hierarchy, but the differences are not large. From this Table we notice that with present data the average values obey \( \langle P_0 \rangle \gg \langle P_1 \rangle \simeq \langle P_2 \rangle \) whereas their variances obey \( \delta P_1 \gg \delta P_0 \simeq \delta P_2 \). \( P_0 \) is well known, because it is related to survival probability of solar low energy neutrinos and \( \theta_{13} \)–or \( \epsilon \)–is well measured by reactor experiments. As we see from figure 1, \( P_0 \) and \( P_1 \) are well represented by Gaussian functions; \( P_2 \) is not Gaussian but it is very small parameter. For these reasons, as we show in the following, we can use a Gaussian approximation without introducing severe inaccuracies in the numerical analysis of the oscillations.

At this point, it is useful to write in terms of \( P_0, P_1, P_2 \) the matrix that contains the probabilities of oscillations of cosmic neutrinos, whose elements are given by \( P_{\ell \ell'} = \sum_1^3 |U_{\ell i}^2 U_{\ell' i}^2| \) \([2]\) with \( \ell, \ell' = e, \mu, \tau \). We obtain the symmetric matrix \( P \) that acts on the vector of fluxes \( F^0 = (F_e^0, F_\mu^0, F_\tau^0) \) before oscillations and gives the observable vector of fluxes.
\( F = (F_e, F_\mu, F_\tau) \) simply as \( F = P \ F^0 \) where
\[
P = \begin{pmatrix}
\frac{1}{3} + 2P_0 & \frac{1}{3} - P_0 + P_1 & \frac{1}{3} - P_0 - P_1 \\
\frac{1}{3} + \frac{P_0}{2} - P_1 + P_2 & \frac{1}{3} + \frac{P_0}{2} - P_2 \\
\frac{1}{3} + \frac{P_0}{2} + P_1 + P_2 & \frac{1}{3} + \frac{P_0}{2} + P_2
\end{pmatrix}
\] (4)

The probabilities of oscillation assume a very simple form, in fact they depend linearly upon the new parameters. Moreover, in first approximation, they could be expressed only in terms of \( P_0 \), because \( P_1 \) and \( P_2 \) give small corrections. Using the value of Table 1 and the new parameterization of oscillation matrix, we obtain the probabilities of oscillations,
\[
P_{ee} = 0.552 \pm 0.010, \quad P_{e\mu} = P_{e\tau} = 0.224 \pm 0.029 \\
P_{\mu\tau} = 0.378 \pm 0.008, \quad P_{\mu\mu} = P_{\tau\tau} = 0.398 \pm 0.029
\] (5)

Two couples of probabilities have (almost) the same values, because with the present best fit value \( \langle P_1 \rangle = 0 \) and the numerical differences between these expressions are small.

## 3 Applications

We will consider two quantities, that are affected by oscillations; the flavor ratios and the number of events due to Glashow resonance.

Let us denote the initial fraction of \( \nu_\ell \) with \( \xi_\ell^0 = \frac{F_\ell^0}{\sum_\ell F_\ell^0} \) and the fraction of \( \nu_\ell \) after oscillations with \( \xi_\ell = \frac{F_\ell}{\sum_\ell F_\ell} \), where of course \( \sum_\ell F_\ell^0 = \sum_\ell F_\ell \). Suppose that the initial flavor ratio is given by \( (\xi_e^0, \xi_\mu^0, \xi_\tau^0) = (1 - g - h, g, h) \). After propagation the flavor ratio is modifying in that way:
\[
\xi_e = \frac{1}{3} + (2 - 3g - 3h)P_0 + (g - h)P_1 \quad \text{(6)}
\]
\[
\xi_\mu = \frac{1}{3} + \frac{1}{2}(-2 + 3g + 3h)P_0 + (1 - 2g - h)P_1 + (g - h)P_2 \quad \text{(7)}
\]
\[
\xi_\tau = \frac{1}{3} + \frac{1}{2}(-2 + 3g + 3h)P_0 + (-1 + g + 2h)P_1 - (g - h)P_2 \quad \text{(8)}
\]

Below, we will emphasize \( \xi_\mu \) since it is quite directly connected to an observable quantity, namely, the fraction of track-type events.

### 3.1 Flavor ratio

A first application of the new parameterization is the study of the flavor ratio of neutrinos, considering different mechanisms of production\(^2\). In particular, we consider:

\(^2\)Intermediate possibilities have been also considered in \cite{19, 23}. 

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4
Figure 2: Flavor triangle. The present observational information on the flavor composition of cosmic neutrinos at 1, 2 and 3σ is compared with the expectations derived for four different hypotheses on the mechanism of production of the neutrinos. The ellipses derive from a Gaussian treatment of the error based on Table I, see also Appendix A.

1. pions decay \((h = 0, g = 2/3; \text{blue})\);
2. neutrons decay \((h = 0, g = 0; \text{green})\);
3. damped muons \((h = 0, g = 1; \text{red})\);
4. charm mesons \((h = 0, g = 1/2; \text{orange})\).

Using the equation (6), (7) and (8), we represent the allowed regions by propagating the errors on the predictions by a Monte Carlo simulation; this gives the 4 dotted regions of the flavor triangle, figure 2.

These regions can be compared with those obtained with a Gaussian treatment of the errors on \(P_0\), \(P_1\) and \(P_2\); this gives the 4 ellipses, enclosing the 99% CL region. We see that the differences between these two treatments are not very important. The Gaussian approach seems appropriate for the present needs; this is significantly easier to implement, the details are given in Appendix A.

The theoretical regions on the flavor triangle of Fig. 2 (i.e. the dotted areas obtained by Montecarlo and the elliptic curves corresponding to Gaussian approximation) depend only by initial flavor ratio and are not affected by the energy spectrum of the neutrinos, that is assumed to be universal for all neutrinos.

On the contrary, the 1σ, 2σ and 3σ zones depend on energy distribution of neutrinos. The confidence levels, indicated in the flavor triangle, correspond to the result of the data.
analysis of IceCube events (the high energy starting events, whose initial vertex is in the detectors and the passing muons, i.e. the throughgoing muons) discussed in [15] and [14] and considering a power law distribution given by,

$$\frac{d\phi}{dE_\nu} = \phi_0 E_\nu^{-\alpha}$$

Thus, the flavor ratio detected is a function of spectral index $\alpha$. Here, we have used $\alpha = 2$, but the actual value is not crucial. In fact the most important conclusion is just that the small number of events presently available, does not allow us yet to exclude any mechanism of production [14]. This remains true also with $\alpha = 2.3$ (the actual best fit of IceCube’s HESE events) or $\alpha = 2.6$ (the best fit including also low energy events).

The confidence levels are in reasonable accordance with those of IceCube data analysis [15]. A novelty of the present analysis is that we show, for the first time, the impact of uncertainty on oscillation parameters in the triangle of flavor, both with Montecarlo approach and Gaussian approach: See again figure 2. These uncertainties agree well with those discussed in [14].

### 3.2 Errors on flavor ratio

Let us discuss further the point of the errors. The flavor ratios given in Eq. 6, 7 and 8 depend linearly by $P_0$, $P_1$ and $P_2$. Therefore, it is straightforward and quite easy to evaluate the Gaussian errors on these quantities. Let us focus on $\xi_\mu$. 

Figure 3: Gaussian errors on fraction of $\nu_\mu$ on Earth as a function of the neutrino fractions at the source.
Figure 4: Fraction of $\nu_e$ due to $pp$ or $p\gamma$ interaction. The distribution obtained by a Montecarlo extraction compares well with the Gaussian distributions obtained from Table 1 (continuous lines).

From the formula for $\xi_\mu$, equation (7), we see that the term linear in $P_1$ becomes very small when $\nu_e$ and $\nu_\mu$ are about equal (e.g. charm mesons), and the figure 3 confirms that this type of mechanisms gives very small errors on the flavor ratio measured at Earth (indeed, an initial composition of 1:1:1 would not be modified, or in other words, the error would be just 0.)

On the contrary, the mechanisms that produce only $\nu_e$ (neutrons decay) or only $\nu_\mu$ (damped muons) give the biggest error, about 10% on final flavor ratio. The pion decay, that is the most plausible mechanism, is between the two extreme situations; the error on the muon fraction $\xi_\mu$ is about 5%.

We note incidentally that the expectations from the pion decay mechanism agree quite well with the results of the analysis of the existing data; see again Fig. 2.

3.3 Glashow resonance

With the formalism of this paper it is easy to write analytical expressions for some interesting signal including the effect of three flavor oscillations. We analyze the case Glashow resonance [18, 19, 20, 21], i.e. the production of $W^-$ starting from electron antineutrino due to the process,

$$\bar{\nu}_e + e^- \rightarrow W^-$$  \hspace{1cm} (10)

The process is possible when the antineutrino with energy greater than 6.3 PeV collides with an electron at rest.

An interesting point for us is that different astrophysical mechanisms produce different fraction of $\bar{\nu}_e$. From the $pp$ interaction, e.g. [24], we can obtain all the type of pions,
instead from $p\gamma$ interaction, e.g. [23], we obtain only $\pi^+$ and $\pi^0$. After the decays the flavor ratio for $pp$ are approximatively equal for neutrinos and antineutrinos,

$$\xi^0 = (1, 2, 0)/6\,, \xi^0 = (1, 2, 0)/6$$

(11)

while for $p\gamma$, the neutrino and antineutrino channels contain a different number of particles and lead to different flavors,

$$\xi^0 = (1, 1, 0)/3\,, \xi^0 = (0, 1, 0)/3$$

(12)

where we have normalized the two fluxes to a single particle. Thus the fraction of electron antineutrinos at Earth are given by a linear expression in the parameters $P_0$ and $P_1$.

$$\xi_{e\bar{\nu}}^{pp} = \frac{1}{6} + \frac{1}{3}P_1$$

(13)

$$\xi_{e\bar{\nu}}^{p\gamma} = \frac{1 - 3P_0}{9} + \frac{1}{3}P_1$$

(14)

These two distribution can be obtained with Monte Carlo simulation, using the distributions of mixing angle and CP phase violation. The results are illustrated in the figure 4. They show that the two cases are well distinguished despite the uncertainties on the oscillation parameters; more specifically, the signal is about two times stronger in the $pp$ case than in the $p\gamma$ case. Note that the very small differences with the result of the Gaussian approximation, evident from Fig. 4, justify the use of a linear approach.

4 Summary and discussion

The results of IceCube have greatly increased the interest in an accurate description of propagation of cosmic neutrinos, accounting in particular for the minimal hypothesis of three flavor oscillation in vacuum. In this work, we have discussed a natural choice of the parameters that allows a very direct and simple description of these oscillation phenomena.

We have illustrated the usefulness of the new parameters $P_0, P_1, P_2$ by discussing the expectations on the neutrino flavor ratios and their errors. We have also analyzed the expectations on the intensity of the signal due Glashow resonance, that depends on the mechanism of neutrino production.

The expectations that we have obtained by using a Gaussian treatment of the new parameters are very similar to those obtained in more complex descriptions of three flavor oscillations [14]. The results of the analysis of the flavor of cosmic neutrinos compare well with those obtained by IceCube collaboration [15] and [14], as we have shown displaying them on the triangle of flavor, and including in this triangle, for the first time, the effect of the uncertainties due to the oscillation parameters.
A Allowed regions in the Gaussian approximation

Let us consider the two dimensional Gaussian likelihood,

\[
\mathcal{L} = \exp \left[ -\frac{1}{2} (\vec{v} - \langle \vec{v} \rangle)^t \Sigma^{-2} (\vec{v} - \langle \vec{v} \rangle) \right] \quad \text{where} \quad \vec{v} = \left( \begin{array}{c} x \\ y \end{array} \right), \quad \Sigma^2 = \left( \begin{array}{cc} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{array} \right)
\]

(15)

A confidence level \((0 < \text{C.L.} < 1)\) defines the allowed region \(\mathcal{L} > \mathcal{L}_{\text{max}} (1 - \text{C.L.})\). Its contour is an ellipse that can be obtained from the following parametric expression,

\[
\left( \begin{array}{c} x(\varphi) \\ y(\varphi) \end{array} \right) = \left( \begin{array}{c} \langle x \rangle \\ \langle y \rangle \end{array} \right) + \sqrt{-2 \log(1 - \text{C.L.})} \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \sigma_+ \cos \varphi \\ \sigma_- \sin \varphi \end{array} \right)
\]

(16)

where \(\varphi = [0, 2\pi]\) and where we defined,

\[
\theta = \frac{1}{2} \arctan \left[ \frac{2\sigma_y^2}{\sigma_y^2 - \sigma_x^2} \right], \quad \sigma^2 = \frac{2(\sigma_x^2 \sigma_y^2 - \sigma^4)}{\sigma_x^2 + \sigma_y^2 + (\sigma_x^2 - \sigma_y^2) \sqrt{1 + \tan^2 2\theta}}
\]

(17)

In the flavor triangle, we have known linear combinations of \(P_0, P_1, P_2\),

\[
x = (\xi_\mu - \xi_\tau) / \sqrt{3} \equiv x_0 + x_i P_i \quad \text{and} \quad y = \xi_\tau \equiv y_0 + y_i P_i
\]

(18)

From Tab. \([1]\) one evaluates \(\langle x \rangle = x_0 + x_i \langle P_i \rangle\), \(\sigma^2 = x_i^2 \delta P^2_i\), \(\sigma^2 = x_i y_i \delta P^2_i\), etc.

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