Structural stochastic responses determination via a sample-based stochastic finite element method

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\textbf{Abstract}

This paper presents a new stochastic finite element method for computing structural stochastic responses. The method provides a new expansion of stochastic response and decouples the stochastic response into a combination of a series of deterministic responses with random variable coefficients. A dedicated iterative algorithm is proposed to determine the deterministic responses and corresponding random variable coefficients one by one. The algorithm computes the deterministic responses and corresponding random variable coefficients in their individual space and is insensitive to stochastic dimensions, thus it can be applied to high dimensional stochastic problems readily without extra difficulties. More importantly, the deterministic responses can be computed efficiently by use of existing Finite Element Method (FEM) solvers, thus the proposed method can be easy to embed into existing FEM structural analysis softwares. Three practical examples, including low-dimensional and high-dimensional stochastic problems, are given to

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demonstrate the accuracy and effectiveness of the proposed method.

*Keywords:* Stochastic finite element method; High dimensions; Large scale; Stochastic responses;

1. Introduction

Ordinary or partial differential equations (PDEs) are powerful tools to describe many real life engineering and scientific processes. A wide body of numerical methods based on finite differences, finite elements, and boundary elements are available to approximately solve the governing equations for the response quantities of interest. In particular, Finite element method (FEM) has become state-of-the-art since it offers a simple way to solve very high resolution models in various computational physics problems, ranging from structural mechanics, thermodynamics to nano-bio mechanics [1]. Nevertheless, the FEM is deterministic in nature and is therefore limited to describe the general characteristics of a real life system. The considerable influence of inherent uncertainties on system behavior has led the scientific community to recognize the importance of a stochastic approach to realistic engineering systems [2]. More than ever, the goal becomes to represent and propagate uncertainties from the available data to the desired results through PDEs within the framework of stochastic equations [3, 4].

The modelling of uncertainties consists in defining a suitable probability space \((\Theta, \Sigma, \mathcal{P})\), where \(\Theta\) denotes the space of elementary events, \(\Sigma\) is a \(\sigma\)-algebra defined on \(\Theta\) and \(\mathcal{P}\) is a probability measure. In this paper, we consider the structural stochastic response \(u(\theta)\) of the problem is a stochastic function, with value in a certain function space, which has to verify almost
surely the stochastic partial differential equations (SPDEs) discretized by stochastic finite element equations \([5]\) as

\[
K(\theta)u(\theta) = F(\theta)
\]  

(1)

where \(K(\theta)\) is an operator representing properties of the physical model under investigation, which can be considered the stochastic stiffness matrix and \(F(\theta)\) is a stochastic load vector. Randomness on the model can be formalized as a dependency of the operator and loads on the elementary event \(\theta \in \Theta\) and it’s a great challenge to solve Equation (1) in the high dimensional stochastic space \(\Theta\).

As an extension of deterministic FEM, stochastic finite element method (SFEM) \([6, 7]\) has become a common tool for the solution of Equation (1). Given the representation of uncertain system parameters and environmental source in terms of random fields, it becomes possible to integrate discretization methods for the response and random fields to arrive at a system of random algebraic equations. Two prominent variants of the SFEM are the non-intrusive methods and the Galerkin-type methods. Although various non-intrusive methods, e.g., Monte Carlo simulation \([8]\), or regression and projection methods \([9]\), can be readily applied to compute the response statistics to an arbitrary degree of accuracy, this is the method of last resort since the attendant computational cost can be prohibitive for real life problems.

The Galerkin-type spectral methods \([5, 6, 10]\), which are developed for linear SFEM, provide an explicit functional relationship between the random input and output, hence allow easy evaluation of the statistics of the stochastic system response. These methods transform Equation (1) arising from spatial discretization of SPDEs into a deterministic finite element
equation by stochastic Galerkin projection, but the size of the deterministic finite element equation is significantly higher than that of the original SPDEs. Although several iterative solvers have been developed to decrease the substantial computational requisite \cite{11, 12}, the difficulty to build efficient preconditioners and memory requirements still limit their use to small-scale and low-dimensional stochastic problems. Also, the Curse of Dimensionality in stochastic spaces makes these methods more inefficient. For this line of approach to be successful in practice, it is crucial to have general-purpose and highly efficient numerical schemes for the solution of stochastic finite element equation \cite{11}.

In this article, we develop a highly efficient numerical method for the explicit and high precision solution of Equation (1) with application to structural responses that involve uncertainties. An universal construct of solution to stochastic finite element (SFE) equations is firstly developed, which is independent on the types of SFE problems. Based on the construct of this solution, we further develop a numerical algorithm for solving SFE equations. The representations of the stochastic solutions are applicable for high-dimensional stochastic problems, and more importantly, the stochastic analysis and deterministic analysis in the solution procedure can thus be implemented in individual space. In this way, the proposed algorithm for the solution of SFE equation integrate the advantages of the non-instrusive methods and the Galerkin-type methods simultaneously, and thus have great potential for uncertainty quantifications in structural analysis.

The paper is organized as follows: Section 2 briefly introduces the series expansion methods of random fields simulation and the derivation of stochas-
tic finite element equations. A new method outlines for solving stochastic
finite element equations is described in Section 3. Following this, the al-
gorithm implementation of the proposed method is elaborated in Section 4.
Three practical problems are used to demonstrate the accuracy and effective-
ess of the proposed method in Section 5. Some conclusions and prospects
are discussed in Sections 6.

2. Stochastic finite element method

2.1. Random fields expansion

In the framework of SFEM for structural analysis, uncertain physical pa-
rameters usually consists of the Young modulus, Poisson’s ratio, yield stress,
cross section geometry of physical systems, earthquake loading, wind loads,
etc. In most cases, due to the lack of relevant experimental data, assumptions
are made regarding probabilistic characteristics of random fields, such as Gaussian or non-Gaussian, stationary or non-stationary, etc [13, 14]. The
first step in applying the FEM to problems involving one or more of the
random parameters is to model random fields based on the assumptions of
probabilistic characteristics, thus random fields discretization is a key step
in the numerical solutions of stochastic finite element equations. In order to
derive stochastic finite element equations effectively, explicit expressions of
random fields are also crucial. In general, we represent random fields by an
enumerable set of random variables, and the series expansion of a second-
order random field $\omega(x, \theta)$, which is indexed on a bounded domain $D$, can be expressed as

$$\omega(x, \theta) = \sum_{i=0}^{M} \xi_i(\theta) \omega_i(x)$$  \hspace{1cm} (2)
where \( \{ \xi_i(\theta) \}_{i=0}^{M} \) and \( \{ \omega_i(x) \}_{i=0}^{M} \) are random variables and deterministic functions, respectively, and \( M \) is the number of retained items. Equation (2) can be obtained by some methods for discretization of random fields. Various discretization techniques are available in the literature for approximating random fields including shape function methods, optimal linear estimation, weighted integral methods, orthogonal series expansion \[15, 16\].

As a special case of the orthogonal series expansion, Karhunen-Loéve expansion is the most commonly used method in SFEM, and it has a form as

\[
\omega(x, \theta) = \omega_0(x) + \sum_{i=1}^{M} \xi_i(\theta) \sqrt{\lambda_i} \omega_i(x)
\]  

where \( \omega_0(x) \) is the mean function of the random field \( \omega(x, \theta) \), \( \{\lambda_i\}_{i=1}^{M} \) and \( \{\omega_i(x)\}_{i=1}^{M} \) are eigenvalues and eigenfunctions of the covariance function \( C_{\omega\omega}(x_1, x_2) \) of the random field \( \omega(x, \theta) \), and they are solutions of the homogeneous Fredholm integral equation of the second kind \[6, 17\],

\[
\int_{D} C_{\omega\omega}(x_1, x_2) \omega_i(x_1) \, dx_1 = \lambda_i \omega_i(x_2)
\]  

Due to the symmetry and the positive definiteness of covariance kernel \( C_{\omega\omega}(x_1, x_2) \), the eigenfunctions \( \{\omega_i(x)\}_{i=1}^{M} \) form a complete orthogonal set satisfying the equation

\[
\int_{D} \omega_i(x) \omega_j(x) \, dx = \delta_{ij}
\]  

where \( \delta_{ij} \) is the Kronecker delta function. An explicit expression for the random variables \( \{\xi_i(\theta)\}_{i=1}^{M} \) in Equation (2) can be obtained by

\[
\xi_i(\theta) = \frac{1}{\sqrt{\lambda_i}} \int_{D} [\omega(x, \theta) - \omega_0(x)] \omega_i(x) \, dx
\]
which is a set of uncorrelated standardized random variables and satisfy

$$E \{ \xi_i (\theta) \} = 0, \ E \{ \xi_i (\theta) \xi_j (\theta) \} = \delta_{ij} \quad (7)$$

where $E \{ \cdot \}$ is the expectation operator.

The Karhunen-Loève expansion (3) offers a unified and powerful tool [17, 18] for representing stationary and nonstationary, Gaussian and non-Gaussian random fields with explicitly known covariance functions. Karhunen-Loève expansion is optimal among series expansion methods in the global mean square error with respect to the number of random variables in the representation, which means that only a few terms $M$ are required in order to capture most of randomness, thus it has received much attentions in many disciplines. In stochastic finite element analysis, it has been widely used to discretize the random fields representing the randomness of structures and excitations. It is worth mentioning that the implementation of Karhunen-Loève expansion requires solutions of the integral equation (4) with the covariance function as the integral kernel. Although only a limited number of analytical eigen-solutions are available [6], the solution of the integral equation can be numerically approximated for random fields with arbitrary covariance functions. For random fields that are defined on two- and three-dimensional domains, the finite element method becomes the only available method for the discretization of the multi-dimensional integral eigenvalue problems [19, 20]. In this paper, the generation of the finite element mesh for random fields is same to that for responses.
2.2. Stochastic finite element equations

We simply recall the deterministic finite element method of the relevant formulation before dealing with stochastic problems. The deterministic finite element method in linear elasticity defined on $\Omega$ eventually derive a $N \times N$ linear system

$$Ku = F$$ (8)

where $N$ is the number of degrees of freedom, $K$, $u$, $F$ are global stiffness matrix, displacement vector and load vector, respectively. By assembling the element stiffness matrices $k^e$, the global stiffness matrix $K$ can be obtained as

$$k^e = \int_{\Omega_e} B^T DBd\Omega_e$$ (9)

where $B$ and $D$ stand for the strain matrix and the elasticity matrix, respectively.

We suppose that the material Young’s modulus is a random field and can be written as the form in Equation (3),

$$D (x, \theta) = D_0 \left[ \omega_0 (x) + \sum_{i=1}^{M} \xi_i (\theta) \sqrt{\lambda_i} \omega_i (x) \right]$$ (10)

where $D_0$ is a constant matrix, and random variables $\{\xi_i (\theta)\}_{i=1}^{M}$ construct a $M$-dimensional stochastic space. By substituting Equation (10) into Equation (9), the element stiffness matrix thus becomes as,

$$k^e (\theta) = k^e_0 + \sum_{i=1}^{M} \xi_i (\theta) k^e_i$$ (11)

where $k^e_0$ is the mean element stiffness matrix given by

$$k^e_0 = \int_{\Omega_e} \omega_0 (x) B^T D_0 Bd\Omega_e$$ (12)
and $k^e_i$ are deterministic matrices given by

$$k^e_i = \int_{\Omega_e} \sqrt{\lambda_i \omega_i(x)} B^T D_0 B \Omega_e$$  \hspace{1cm} (13)

The stochastic global stiffness matrix $K(\theta)$ in the stochastic finite element equation (1) is obtained by assembling the stochastic element stiffness matrices $k^e(\theta)$,

$$K(\theta) = \sum_{i=0}^{M} \xi_i(\theta) K_i$$  \hspace{1cm} (14)

where $\xi_0(\theta) \equiv 1$ and global matrices $K_i$ are obtained by assembling element matrices $k^e_i$ in the way similar to the deterministic case. In a similar way, we can get the stochastic global load vector as

$$F(\theta) = \sum_{l=0}^{Q} \eta_l(\theta) F_l$$  \hspace{1cm} (15)

After assembling the stochastic global stiffness matrix $K(\theta)$ and the stochastic global load vector $F(\theta)$, the stochastic finite element equation (1) can be rewritten as

$$\left( \sum_{i=0}^{M} \xi_i(\theta) K_i \right) u(\theta) = \sum_{l=0}^{Q} \eta_l(\theta) F_l$$  \hspace{1cm} (16)

The high precision solution of Equation (16) is one of the most important problems of the stochastic finite element method. Spectral stochastic finite element method (SSFEM) [21, 6] is a popular method in the past few decades. SSFEM represents the stochastic response $u(\theta)$ through polynomial chaos expansion (PCE) and transform Equation (16) into a deterministic finite element equation by stochastic Galerkin projection. The size of the deterministic finite element equation depends directly on the number of terms retained in the PCE and the number of degrees of freedom $N$, and the
computational cost for the solution of this system is much larger than that of the original problem. Although several improved methods [11, 22] have been developed to decrease computational costs, the Curse of Dimensionality still limit SSFEM to low-dimensional stochastic problems, thus it is crucial to develop a new method for the solution of Equation (16).

3. A new method for solving stochastic finite element equations

In order to avoid the difficulties of SSFEM, in this section, we propose a new method for solving the stochastic finite element equation (16) defined in low- and high-dimensional stochastic spaces. A natural idea is to represent the stochastic solution $u(\theta)$ of Equation (16) by use of random field expansions, however common methods are inactive since we almost know nothing about $u(\theta)$ except the governing equation (16). Inspired by Karhunen-Loève expansion (3) and the general spectral decomposition [23], we construct the $u(\theta)$ as

$$u(\theta) = \sum_{i=1}^{\infty} \lambda_i(\theta) d_i$$

(17)

where $\{\lambda_i(\theta)\}_{i=1}^{\infty}$ are random variables and $\{d_i\}_{i=1}^{\infty}$ are deterministic discretized basis vectors. Similar to the orthogonal conditions Equation (5) and (7) of Karhunen-Loève expansion, the following bi-orthogonal condition is introduced

$$d^T_i d_j = \delta_{ij}, \quad E\{\lambda_i(\theta) \lambda_j(\theta)\} = \kappa_i \delta_{ij}$$

(18)

where $E\{\cdot\}$ is the expectation operator and $\kappa_i = E\{\lambda_i^2(\theta)\}$.

It is shown in expansion (17) that the solution space of $u(\theta)$ is decoupled into a stochastic space and a deterministic space and it allows to compute
\{\lambda_i(\theta)\}_{i=1}^{\infty} in the stochastic space and \{d_i\}_{i=1}^{\infty} in the deterministic space, respectively. In this way, the difficulties in expanding the unknown solution random field of Equation (1) can be overcome. One only requires to seek a set of deterministic orthogonal vectors \{d_i\}_{i=1}^{\infty} and the corresponding uncorrelated random variables \{\lambda_i(\theta)\}_{i=1}^{\infty} such that the expanded solution in Equation (17) satisfies the Equation (1). In practical, we truncate Equation (17) at the \(k\)-th term as,

\[ u_k(\theta) = \sum_{j=1}^{k} \lambda_j(\theta) d_j \]  

(19)

As mentioned above, neither \{d_i\}_{i=1}^{k} nor \{\lambda_i(\theta)\}_{i=1}^{k} is known a priori, a natural choice is to successively determine these unknown couples \{\lambda_i(\theta), d_i\} one after another via iterative methods. In order to compute the couple \((\lambda_k(\theta), d_k)\), we suppose that the approximate solution \(u_{k-1}(\theta)\) has been obtained, then substituting Equation (19) into Equation (1) yields,

\[ K(\theta) \left[ \sum_{j=1}^{k-1} \lambda_j(\theta) d_j + \lambda_k(\theta) d_k \right] = F(\theta) \]  

(20)

If random variable \(\lambda_k(\theta)\) has been determined (or given an initial value), the \(d_k\) can be determined using stochastic Galerkin method and a dedicated iteration [23], this corresponds

\[ E \left\{ \lambda_k(\theta) K(\theta) \left[ \sum_{j=1}^{k-1} \lambda_j(\theta) d_j + \lambda_k(\theta) d_k \right] \right\} = E \{\lambda_k(\theta) F(\theta)\} \]  

(21)

Considering Equation (14) and (15), the Equation (21) about \(d_k\) can be simplified as,

\[ \left( \sum_{i=0}^{M} c_{ikk} K_i \right) d_k = \sum_{i=0}^{Q} b_{kl} F_i - \sum_{i=0}^{M} \sum_{j=1}^{k-1} c_{ijk} K_i d_j \]  

(22)
where
\[ c_{ijk} = E \{ \xi_i (\theta) \lambda_j (\theta) \lambda_k (\theta) \}, \quad b_{kl} = E \{ \eta_l (\theta) \lambda_k (\theta) \} \] (23)

Once \( d_k \) has been determined in Equation (22), the random variable \( \lambda_k (\theta) \) can be subsequently updated via the similar procedure. This requires to multiply \( d_k \) on both sides of Equation (20) to yield
\[
d_k^T K (\theta) \left[ k-1 \sum_{j=1}^{k-1} \lambda_j (\theta) d_j + \lambda_k (\theta) d_k \right] = d_k^T F (\theta) \] (24)

Considering Equation (14) and (15), the Equation (24) about \( \lambda_k (\theta) \) can be simplified as,
\[
\left( \sum_{i=0}^{M} g_{i,k,k} \xi_i (\theta) \right) \lambda_k (\theta) = \sum_{l=0}^{Q} h_{kl} \eta_l (\theta) - \sum_{i=0}^{M} \sum_{j=1}^{k-1} g_{ij,k} \xi_i (\theta) \lambda_j (\theta) \] (25)

where
\[
g_{ijk} = d_k^T K_i d_j, \quad h_{kl} = d_k^T F_l \] (26)

The classical SSFEM is to represent the stochastic solution of nodes \( \{ u_i (\theta) \}_{i=1}^{N} \) in terms of a set of polynomial chaos and transforms the original stochastic finite element equation into a deterministic finite element equation with size \( N \times \left( \frac{(M+p)!}{M!p!} \right) \), where \((\cdot)!\) represents the factorial operator, \( N, M \) and \( p \) are the number of system degrees of freedom, the number of random variables and the order of polynomial chaos expansion, respectively. The size of the deterministic finite element equation is significantly higher than that of the original stochastic finite element equation. For instance, the size is \( 1 \times 10^6 \) when \( N = 1000, M = 10 \) and \( p = 4 \), which leads to the Curse of Dimensionality, and is prohibitive for problems with high stochastic dimensions and large scales.
The method in this paper decouples the original stochastic finite element equation into a deterministic finite element equation (22) with size $N$ and one-dimensional stochastic algebraic equation (25). The iteration process of Equation (21) and (24) was used in the paper [23] to solve linear stochastic partial differential equations and subsequently applied to time-dependent and nonlinear problems [24, 25]. The key of this method is to transform the original SPDE to a deterministic PDE and a stochastic algebraic equation like Equation (25). The method for solving Equation (25)-like is to represent the random variable $\lambda_k(\theta)$ in terms of a set of polynomial chaos and transforms Equation (25)-like into a deterministic equation with size $\frac{(M+p)!}{M!p!}$, which greatly alleviates the Curse of Dimensionality, but is still prohibitive for problems with high stochastic dimensions. In order to avoid this difficulty, we develop a simulation method to determine $\lambda_k(\theta)$. For each realization of $\{\theta^{(r)}\}_{r=1}^{R}$, the $\lambda_k(\theta^{(r)})$ can be obtained by solving (25) as,

$$
\lambda_k(\theta^{(r)}) = \frac{\sum_{i=0}^{Q} h_{kl} \eta_{l}(\theta^{(r)}) - \sum_{i=0}^{M} \sum_{j=1}^{k-1} g_{ijk} \xi_{i}(\theta^{(r)}) \lambda_{j}(\theta^{(r)})}{\sum_{i=0}^{M} g_{ikk} \xi_{i}(\theta^{(r)})} \quad (27)
$$

It is important to note that Equation (27) has become a one-dimensional linear algebraic equation about $\lambda_k(\theta^{(r)})$. Compared to classic methods, we do not need to choose the type and order of polynomial chaos. The total computational cost for determining $\{\lambda_k(\theta^{(r)})\}_{r=1}^{R}$ is very low even for high stochastic dimensions, which hopefully avoid the Curse of Dimensionality. Then statistical methods are readily introduced to obtain $\lambda_k(\theta)$ from samples $\{\lambda_k(\theta^{(r)})\}_{r=1}^{R}$. Hence, this method will be particularly appropriate for a wide class of high-dimensional stochastic problems in practice.
4. Algorithm implementation

**Algorithm 1** Algorithm for solving linear stochastic finite element equations

1: while $\varepsilon_{global} > \varepsilon_1$ do
2:    initialize $\lambda_k^{(0)}(\theta)$
3:    while $\varepsilon_{local} > \varepsilon_2$ do
4:        compute $d_k^{(j)}$ by solving Equation (22)
5:        orthogonalization $d_k^{(j)} \perp d_i$, $i = 1, \cdots, k - 1$ by Equation (28) and normalization $d_k^{(j)} = \frac{d_k^{(j)}}{\|d_k^{(j)}\|}$
6:        compute $\lambda_k^{(j)}(\theta)$ by Equation (27)
7:        orthogonalization $\lambda_k^{(j)}(\theta) \perp \lambda_i(\theta), i = 1, \cdots, k - 1$ by Equation (28)
8:        compute local error $\varepsilon_{local}$, $j = j + 1$
9:    update $u(\theta)$ as $u_k(\theta) = \sum_{i=1}^{k-1} \lambda_i(\theta)d_i + \lambda_k(\theta)d_k$
10:   compute global error $\varepsilon_{global}$, $k = k + 1$

The above procedure for solving the stochastic finite element equation is summarized in Algorithm 1, which consists of an outer loop procedure and an inner loop procedure. The inner loop, which is from step 3 to 8, is used to determine the couple of $(\lambda_k(\theta), d_k)$. With an initial random variable $\lambda_k^{(0)}(\theta)$ given in step 2, $d_k^{(j)}$ can be determined in step 4 and 5 where superscript $j$ represents the $j$-th round of iteration. With the obtained $d_k^{(j)}$, the random variable $\lambda_k^{(j)}(\theta)$ is then updated in step 6 and 7. While the outer loop, which is from step 1 to 10, corresponds to recursively building the set of couples and then generates a set of couples such that the approximate solution in step 9 satisfies Equation (1).
Note that both $d_k^{(j)}$ and $\lambda_k^{(j)}(\theta)$ require orthogonalizations such that the bi-orthogonal condition in Equation (18) holds along the whole process, here we use the Gram-Schmidt Orthogonalization method in step 5 and 7. It is written as,

$$
\begin{align*}
    d_k^{(j)} &= d_k^{(j)} - \sum_{i=1}^{k-1} (d_k^{(j)T} d_i) d_i \\
    \lambda_k^{(j)}(\theta) &= \lambda_k^{(j)}(\theta) - \sum_{i=1}^{k-1} \frac{E\{\lambda_k^{(j)}(\theta)\lambda_i(\theta)\}}{E\{\lambda_i^2(\theta)\}} \lambda_i(\theta), \quad k \geq 2
\end{align*}
$$

For practical purposes, a certain number of truncated items are retained of the solution $u(\theta)$. The truncation criterion in step 1 is considered as a global error, which is defined as,

$$
\varepsilon_{global} = \frac{E\{\Delta u_k^2(\theta)\}}{E\{u_k^2(\theta)\}} = \frac{E\{\lambda_k^2(\theta)\}}{\sum_{i=1}^{k} \sum_{j=1}^{k} E\{\lambda_i(\theta)\lambda_j(\theta)\} d_i^T d_j} = \frac{E\{\lambda_k^2(\theta)\}}{\sum_{i=1}^{k} E\{\lambda_i^2(\theta)\}}
$$

which measures the contribution of the $k$-th couple $(\lambda_k(\theta), d_k)$ to the stochastic solution $u(\theta)$ and converges to the final solution when it achieves the required precision. Further, the stop criterion for computing each couple $(\lambda_k(\theta), d_k)$ is considered as a local error and defined as,

$$
\varepsilon_{local} = \frac{\|d_k^{(j)} - d_k^{(j-1)}\|}{\|d_k^{(j-1)}\|} = \frac{\|d_k^{(j)} - d_k^{(j-1)}\|}{\|d_k^{(j-1)}\|}
$$

which measures the difference between $d_k^{(j)}$ and $d_k^{(j-1)}$ and the calculation is stopped when $d_k^{(j)}$ is almost the same as $d_k^{(j-1)}$. 

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5. Applications

The numerical implementation of the proposed method is illustrated with the aid of three practical applications. The first application consists of an electric pylon frame with stochastic material properties and a stochastic load. The second application is a roof truss under stochastic wind loads defined in low-dimensional and high-dimensional stochastic spaces, which is to illustrate the efficiency of applying the proposed method to high-dimensional stochastic problems. The third example tests the ability of the proposed method for dealing with a large-scale engineering problem given by computing the deformation of a tunnel under the action of self-weight. These three examples serve to verify the validity and accuracy of the proposed method and demonstrate that there is the same solution construct for different problems.

5.1. Response of electric pylon frame with stochastic material property

In this problem, we consider a frame system as shown in Figure 1, which is an electric pylon frame consisting of 91 elements with square cross-sections. A load \( P \) is applied vertically downward at the far right tip of the arm of the pylon. Clamped boundary conditions are applied at the base of the frame model. Spatial nodes of the electric pylon frame model are defined in Table 1. All elements of the electric pylon frame are constructed of 300M steel and have identical cross-sectional areas. Deterministic material properties are given as, mass density \( \rho = 7.8 \text{g/cm}^3 \), cross-sectional area \( \bar{A} = 4 \text{cm}^2 \), Young’s modulus \( \bar{E} = 200 \text{GPa} \).

The response of the electric pylon forced under a load \( P \) deeply depends on these parameters. In order to better reflect the structural response influenced
Figure 1: 91-element electric pylon frame

Table 1: Nodal definitions of the electric pylon frame

| node | x     | y     | node | x     | y     | node | x     | y     | node | x     | y     |
|------|-------|-------|------|-------|-------|------|-------|-------|------|-------|-------|
| 1    | 12.11 | 0.00  | 13   | 29.96 | 29.73 | 14   | 33.76 | 17.44 | 15   | 21.16 | 33.76 |
| 2    | 36.58 | 0.00  | 14   | 17.44 | 33.76 | 15   | 21.16 | 48.45 | 26   | 17.93 | 48.45 |
| 3    | 15.18 | 9.53  | 15   | 21.16 | 33.76 | 16   | 27.78 | 33.76 | 27   | 21.88 | 48.45 |
| 4    | 24.47 | 9.53  | 16   | 27.78 | 33.76 | 26   | 17.93 | 48.45 | 27   | 21.88 | 48.45 |
| 5    | 33.67 | 9.53  | 17   | 31.41 | 33.76 | 25   | 14.70 | 48.45 | 27   | 21.88 | 48.45 |
| 6    | 17.77 | 9.53  | 18   | 19.06 | 36.34 | 25   | 14.70 | 48.45 | 26   | 17.93 | 48.45 |
| 7    | 33.67 | 9.53  | 19   | 29.88 | 36.34 | 26   | 17.93 | 48.45 | 27   | 21.88 | 48.45 |
| 8    | 24.47 | 9.53  | 20   | 14.45 | 41.77 | 27   | 21.88 | 48.45 | 28   | 26.97 | 48.45 |
| 9    | 33.67 | 9.53  | 21   | 29.88 | 36.34 | 28   | 26.97 | 48.45 | 27   | 21.88 | 48.45 |
| 10   | 24.47 | 9.53  | 22   | 0.00  | 48.45 | 29   | 31.01 | 48.45 | 27   | 21.88 | 48.45 |
| 11   | 33.67 | 9.53  | 23   | 8.56  | 48.45 | 30   | 34.16 | 48.45 | 29   | 31.01 | 48.45 |
| 12   | 12.11 | 0.00  | 24   | 12.11 | 48.45 | 31   | 36.74 | 48.45 | 30   | 34.16 | 48.45 |
| 13   | 29.96 | 29.73 | 25   | 14.70 | 48.45 | 32   | 40.38 | 48.45 | 31   | 36.74 | 48.45 |
| 14   | 17.44 | 33.76 | 26   | 17.93 | 48.45 | 33   | 48.86 | 48.45 | 32   | 40.38 | 48.45 |
| 15   | 21.16 | 33.76 | 27   | 21.88 | 48.45 | 34   | 6.46  | 50.31 | 33   | 48.86 | 48.45 |
| 16   | 27.78 | 33.76 | 28   | 26.97 | 48.45 | 35   | 42.40 | 50.31 | 34   | 6.46  | 50.31 |
| 17   | 31.41 | 33.76 | 29   | 31.01 | 48.45 | 36   | 10.01 | 51.32 | 35   | 42.40 | 50.31 |
| 18   | 19.06 | 36.34 | 30   | 34.16 | 48.45 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 |
| 19   | 29.88 | 36.34 | 31   | 36.74 | 48.45 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 |
| 20   | 14.54 | 41.77 | 32   | 40.38 | 48.45 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 |
| 21   | 29.88 | 36.34 | 33   | 48.86 | 48.45 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 |
| 22   | 0.00  | 48.45 | 34   | 6.46  | 50.31 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 |
| 23   | 8.56  | 48.45 | 35   | 42.40 | 50.31 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 |
| 24   | 12.11 | 48.45 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 | 36   | 10.01 | 51.32 |
by material and load variabilities, we consider the stochastic tensile stiffness and the stochastic bending stiffness as,

\[ EA = (\xi_1(\theta) + 0.2\xi_2(\theta)) \overline{EA}, \quad EI = (\xi_3(\theta) + 0.2\xi_4(\theta)) \overline{EI} \]  

(31)

and consider a stochastic load \( P(\theta) \) as

\[ P(\theta) = (1 + \xi_5(\theta) + \xi_6(\theta)) \overline{P} \]  

(32)

where \( \overline{P} = 1000N \). Independent random variables \( \{\xi_i(\theta)\}_{i=1}^6 \) in Equation (31) and (32) satisfy

\[ \log \{\xi_i(\theta)\}_{i=1}^4 \sim N(0, 0.3), \quad \xi_5(\theta), \xi_6(\theta) \sim N(0, 0.1) \]  

(33)

Similar to the derivation of Equation (16), a stochastic finite element equation for this problem can be obtained as,

\[ \left( \sum_{i=1}^{4} \xi_i(\theta) K_i \right) u(\theta) = \left( 1 + \sum_{i=5}^{6} \xi_i(\theta) \right) F \]  

(34)

In order to solve Equation (34) by use of Algorithm 1, the convergence criterias are set as \( \varepsilon_1 = \varepsilon_2 = 10^{-6}, \quad R = 1 \times 10^4 \) random samples, i.e. \( \{\xi_i(\theta^{(r)})\}_{r=1}^{1 \times 10^4}, \quad i = 1, \cdots, 2 \), are given in Equation (34) and \( \{\lambda_k^{(0)}(\theta^{(r)})\}_{r=1}^{1 \times 10^4} \) are adopt in step 2 in Algorithm 1. In this example, only two retained terms in Equation (19), the displacement components \( \{d_i(x, y)\}_{i=1}^2 \) and probability density functions (PDFs) of corresponding random variables \( \{\lambda_i(\theta)\}_{i=1}^2 \) shown in Figure 2 can achieve the required precision, which demonstrates the high efficiency of the proposed method. It is seen from Figure 2 that, the second random variable \( \lambda_2(\theta) \) is very small and makes almost no contributions to the approximate solution \( u_2(\theta) \). In practical, we determine whether
Figure 2: Displacement components \( \{d_i\}_{i=1}^2 \) (top) and PDFs of corresponding random variables \( \{\lambda_i(\theta)\}_{i=1}^2 \) (bottom)

Figure 3: Comparison of PDFs between the Monte Carlo simulation and the proposed method
step 2 in Algorithm 1 converges through computing the second term, thus the displacement component $d_2$ and the random variable $\lambda_2(\theta)$ are necessary.

Further we compare the proposed method with existing methods, including Monte Carlo simulation [8] and spectral stochastic finite element method (SSFEM) [21, 6]. Here Hermite Polynomial Chaos (PC) of 6 standard Gaussian random variables are adopted in the SSFEM, and the order of PC is set as $p = 3$ and $p = 4$. We test the computational efficiency of these methods by use of a personal laptop (dual-core, Intel i7, 2.40GHz) and the computational times of the proposed method, PC ($p = 3$), PC ($p = 4$) and $1 \times 10^6$ standard Monte Carlo simulations are 3.4s, 71.9s, 474.9s and 1412.7s, respectively, which demonstrates the high efficiency of the proposed method. Based on above methods, the resulted approximate PDFs of the response of the far right tip of the arm of the electric pylon are seen from Figure 3. The result of the two-term approximation of the proposed method is in very good accordance with that from the Monte Carlo simulation, while the PC method requires fourth order ($p = 4$) to achieve a similar accuracy. In addition, our method is based on random samples, thus can avoid choosing the order $p$ of PC basis. We observe in practice that the number of random samples has less influence on the computational cost. In the general case, the sample size is enough when it is sufficient to describe the statistical characteristics of the random variables.

5.2. Response of roof truss under stochastic wind loads

In this example, we consider the stochastic response of a roof truss under a stochastic wind load acting vertically downward on the roof. The roof truss, as shown in Figure 4, includes 185 spatial nodes and 664 elements, where ma-
The material properties of all members are set as Young’s modulus $E = 209\text{GPa}$ and cross-sectional areas $A = 16\text{cm}^2$. The stochastic wind load is a random field

$$C_{FF}(x_1, y_1; x_2, y_2) = \sigma_F^2 e^{-|x_1-x_2|/l_x} e^{-|y_1-y_2|/l_y},$$

where the variance function $\sigma_F^2 = 0.15$, the correlation lengths $l_x = l_y = 24$, and it can be expanded by use of Karhunen-Loève expansion Equation (3) with a $M$-term truncated as

$$f(x, y, \theta) = \sum_{i=0}^{M} \xi_i(\theta) f_i(x, y)$$

where $\xi_0(\theta) \equiv 1$ and the mean function $f_0(x, y) = 10\text{kN}$. \{f_i(x, y)\}_{i=1}^{M}$ is obtained by solving Equation (3). Based on the expansion Equation (35) of the stochastic wind load, the following stochastic finite element equation is obtained,

$$K u(\theta) = \sum_{i=0}^{M} \xi_i(\theta) F_i$$

In this example, the initializations give the random samples \{\xi_i(\theta^{(r)})\}_{i=1}^{M}$, \(i = 1, \cdots, M\) and the initial random variable samples \{\lambda_k^{(0)}(\theta^{(r)})\}_{i=1}^{M}$, and set
the convergence criterias as $\varepsilon_1 = \varepsilon_2 = 10^{-6}$. We first consider a low-dimensional case by choosing $M = 10$. It is seen from Figure 5: that the displacement components $\{d_i\}$ and corresponding random variables $\{\lambda_i(\theta)\}$ can be determined after 8 iterations, which demonstrates the fast convergence rate of the proposed method. Correspondingly, the number of couples $(\lambda_k(\theta), d_k)$ that constitute the stochastic response is adopted as $k = 8$. As shown in Figure 5a and Figure 5b, with the increasing of the number of
couples, the ranges of corresponding random variables are more closely approaching to zero, indicating that the contribution of the higher order random variables to the approximate solution decays dramatically.

For the maximum displacement of the whole roof truss, the resulted approximate PDF compared with $1 \times 10^6$ standard Monte Carlo simulations (MCS) is seen in Figure 6, which indicates that the result of eight-term approximation is in very good accordance with that from the Monte Carlo simulation. According to our experience, further increasing the number of couples will not significantly improve the accuracy since the series in Equation (19) has converged and thus the first few couples dominate the solution of the problem. This example demonstrates the success of our proposed construct of the stochastic solution and Algorithm 1 for the solution of practical problems.

![Figure 6: Comparison of PDFs between the MCS and Algorithm 1](image1)

![Figure 7: Time costs of different stochastic dimensions $M = 10 \sim 1000$](image2)

One of the main purposes the proposed method is to solve high-dimensional stochastic problems. Here we introduce the high-dimensional stochastic problems by choosing $M = 100 \sim 1000$, and test the computational efficiency of
different stochastic dimensions by use of a personal laptop (dual-core, Intel i7, 2.40GHz). Computational costs for solving Equation (36) of different stochastic dimensions are shown in Figure 7, which indicates that our proposed algorithm is efficient for high stochastic dimensions. The computational costs do not increase dramatically as the dimensions increase and is almost linear with the stochastic dimensions, which demonstrates the success of the proposed method for avoiding the Curse of Dimensionality.

5.3. Deformation of tunnel under the action of self-weight

This example is to compute the deformation of a tunnel under the action of self-weight [26]. In order to reduce the size of the stochastic finite element equation while ensuring the accuracy, triangle elements with gradients are used to generate a fine mesh for the tunnel structure and a coarse mesh for the rock, totally including 2729 nodes and 5145 triangle elements, as shown in Figure 8. Material properties and thicknesses of all components are seen from Table 2, here we consider the Young’s modulus of components as a random field with the mean value shown in Table 2 and the covariance function

$$C_{EE}(x_1, y_1; x_2, y_2) = \sigma^2_E e^{-|x_1-x_2|/l_x-|y_1-y_2|/l_y},$$

where variance function $\sigma_E = 0.1$, correlation lengths $l_x = 10$, $l_y = 20$. Similar to Example 5.2, we model the Young’s modulus random field by use of Karhunen-Loève expansion with 10 terms, and derive a stochastic finite element equation.

Given the random samples $\{\xi_i(\theta^{(r)})\}_{r=1}^{1 \times 10^4}$, $i = 1, \ldots, 10$, the initial random variable samples $\{\lambda_k^{(0)}(\theta^{(r)})\}_{r=1}^{1 \times 10^4}$ and set the convergence criteria as $\varepsilon_1 = 10^{-8}$, $\varepsilon_2 = 10^{-6}$, displacements $\{d_i\}$ and corresponding random variables $\{\lambda_i(\theta)\}$ can be determined after 6 iterations, as shown in Figure 9, which indicates the high efficiency of the proposed method. Figure 9a–
Figure 8: Model of the tunnel (top) and the finite element mesh (bottom)

Table 2: Descriptions of materials properties

| Material                        | Young’s modulus (GPa) | Poisson’s ratio | Mass density (kg/m³) | Thickness (m) |
|---------------------------------|-----------------------|----------------|-----------------------|---------------|
| rock                            | 2.0                   | 0.25           | 2200                  |               |
| rock reinforcement              | 2.6                   | 0.20           | 2300                  | 2.80          |
| concrete lining                 | 28.5                  | 0.20           | 2500                  | 0.20          |
| backfilling concrete            | 18.5                  | 0.20           | 2300                  | 0.50          |
| concrete spray                  | 28.5                  | 0.20           | 2200                  | 0.95          |
|      |       |       |       |       |       |       |       |       |       |       |       |       |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|      |       |       |       |       |       |       |       |       |       |       |       |       |
|      |       |       |       |       |       |       |       |       |       |       |       |       |
|      |       |       |       |       |       |       |       |       |       |       |       |       |
|      |       |       |       |       |       |       |       |       |       |       |       |       |

(a). Displacement \( \{d_i\}_{i=1}^6 \) in \( x \) direction (b). Displacement \( \{d_i\}_{i=1}^6 \) in \( y \) direction

(c). PDFs of \( \{\lambda_i(\theta)\}_{i=1}^6 \)

(d). Iterative errors of \( k \) retained items

Figure 9: Solutions of the couples \( \{\lambda_i(\theta), d_i\}_{i=1}^6 \) and iterative errors of the solving process
c) shows the displacement components \( \{d_i\}_{i=1}^6 \) and PDFs of corresponding random variables \( \{\lambda_i(\theta)\}_{i=1}^6 \), where Figure 9a and b are the displacement components in the \( x \) direction (horizontal direction) and \( y \) direction (vertical direction), respectively. Mean values and variances of the displacement are shown in Figure 10a and b, and as a part of the whole displacement (shown in Figure 8 bottom), mean values and variances of the tunnel displacement are seen from Figure 10a0 and b0. Both tunnel displacements and rock displacements can be captured efficiently, which once again demonstrates the effectiveness of the proposed method. Comparing with Figure 9(a–c) and
Figure 10, we observe that the first retained item, i.e. $E \{ \lambda_1 (\theta) \} d_{x1}$ and $E \{ \lambda_1 (\theta) \} d_{y1}$, can roughly approximate the mean displacements in the $x$ direction and $y$ direction. For most cases, the mean displacement considering uncertainties is very close to the displacement obtained from the deterministic case, but considering uncertainties can be better to reflect the variabilities of displacements, which is of great significance for structure design and evaluation, such as reliability analysis and sensitivity analysis [27]. It is seen from Figure 10b that, randomness in this example has more influence on the variance of the tunnel displacement in the $x$ direction and less influence on that in the $y$ direction, which provides a potential way for the design and evaluation of tunnel structures considering uncertainties.

Figure 10: Means and variances of the displacement in the $x$ and $y$ direction

6. Conclusions

In this paper, we develop a method for solving stochastic finite element equations and illustrate its accuracy and efficiency on three practical examples. The proposed method solves stochastic problems by use of a universal
solution construct and a dedicated iterative algorithm. It allows to solve high-dimensional stochastic problems with very low computational costs, which has been illustrated on numerical examples. Thus it appears as a powerful way to avoid the Curse of Dimensionality. In addition, since the stochastic analysis and deterministic analysis in the solving procedure are implemented in their individual spaces, the existing FEM and ODE codes can be readily incorporated into the computational procedure. In these senses, this method is particularly appropriate for large-scale and high-dimensional stochastic problems of practical interests and has great potential in uncertainty quantification of practical problems in science and engineering. In the follow-up research, it hopefully further applies the proposed method to a wider range of uncertainty quantification, such as reliability analysis, sensitivity analysis, etc.

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