Global geometry of the supersymmetric $AdS_3/CFT_2$ correspondence in M-theory

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Abstract

We study the global geometry of a general class of spacetimes of relevance to the supersymmetric $AdS_3/CFT_2$ correspondence in eleven-dimensional supergravity. Specifically, we study spacetimes admitting a globally-defined $\mathbb{R}^{1,1}$ frame, a globally-defined frame bundle with structure group contained in $\text{Spin}(7)$, and an $AdS_3$ event horizon or conformal boundary. We show how the global frame bundle may be canonically realised by globally-defined null sections of the spin bundle, which we use to truncate eleven-dimensional supergravity to a gravitational theory of a frame with structure group $\text{Spin}(7)$, $SU(4)$ or $Sp(2)$. By imposing an $AdS_3$ boundary condition on the truncated supergravity equations, we define the geometry of all $AdS_3$ horizons or boundaries which can be obtained from solutions of these truncations. In the most generic case we study, we reproduce the most general conditions for an $AdS_3$ manifold in M-theory to admit a Killing spinor. As a consistency check on our definitions of $AdS$ geometries we verify that they are satisfied by known gauged supergravity $AdS_3$ solutions. We discuss future applications of our results.
1 Introduction

The formulation of the AdS/CFT correspondence [1] has stimulated intense and ongoing interest in the geometry of Anti-de Sitter manifolds, and their quantum field theoretic description, in string and M-theory. By now there exists an extensive list of explicit supersymmetric AdS solutions of the field equations of ten- and eleven-dimensional supergravity, and an extensive range of solution generating techniques - for example, by taking the near-horizon limit of an elementary or wrapped brane configuration [2]-[6], or by applying the gravity dual of a marginal field theory deformation to a known solution [7], [8]. More generally, there are many known Minkowski solutions which asymptote to AdS, either at a horizon or a conformal boundary. The elementary brane solutions describe interpolations from a conical special holonomy manifold at a spacelike infinity to an internal AdS spacelike infinity associated to an event horizon; and there are many known globally Minkowski and asymptotically AdS solutions admitting an interpretation as the dual of an RG flow to a superconformal fixed point, for example [11]-[13]. More generally still, there are Minkowski solutions without an AdS region which may be interpreted as dual to confining gauge theories, such as the warped deformed conifold [14].

Our primary goal in this paper is to define the general global features of the geometry of supersymmetric spacetimes in eleven dimensions which are globally or locally AdS\(_3\). The globally AdS\(_3\) spacetimes arise as the horizon manifolds of branes, or the fixed point manifolds of RG flows; the locally AdS\(_3\) spacetimes can be interpreted as the full brane or RG flow solutions. Our approach is a direct continuation of that of [9], [10]. For our basic set-up, we require the global existence of a warped \(\mathbb{R}^{1,1}\) frame, with a global reduction of the frame bundle on the transverse space; the metric is given by

\[
ds^2 = 2e^+ \otimes e^- + ds^2(\mathcal{M}_8) + e^9 \otimes e^9,\tag{1.1}
\]

where we impose that \(e^+ = L^{-1} dx^+, e^- = dx^-,\) and \(L < \infty\) globally; that \(L\), the metric on \(\mathcal{M}_8\), and the basis one-form \(e^9\) are everywhere independent of the coordinates \(x^\pm\); and that \(e^9 \neq 0\) is everywhere non-vanishing. We demand that the flux respects the Minkowski isometries; in other words, that it is given by

\[
F = e^{+-} \wedge H + G,\tag{1.2}
\]

with \(H\) and \(G\) independent of the Minkowski coordinates, globally. Our final assumption is that \(\mathcal{M}_8\) admits a globally-defined \(G\)-structure. We will study globally defined Spin(7), SU(4) and Sp(2) structures on \(\mathcal{M}_8\). The existence of a globally defined Spin(7)
structure on $\mathcal{M}_8$ is equivalent to the existence of a no-where vanishing $\text{Spin}(7)$ invariant Cayley four-form $\phi$ on $\mathcal{M}_8$. For $SU(4)$, the globally-defined forms are the almost complex structure $J$ and the $(4,0)$ form $\Omega$. For $Sp(2)$, the existence of the global structure is equivalent to the existence of a triplet of everywhere non-zero almost complex structures $J^A, \ A = 1, 2, 3$.

Our assumption of the existence of a global frame bundle is a stronger one than the more traditional assumption of the existence of a generic section of the spin bundle - a globally non-vanishing Killing spinor. All sorts of complications can potentially occur in the global behaviour of generic sections of the spin bundle - timelike spinors becoming null, spinors becoming parallel, and so forth - that seriously restrict their usefulness as a global tool. Part of our motivation for assuming the existence of a frame bundle is that it provides significant global control over the geometry, and these issues do not arise. Heuristically, a second motivation is that the workings of AdS/CFT appear to be reflected in the very special global properties of the relevant supergravity solutions, and we believe that all known $AdS$, brane or RG flow supergravity solutions satisfy this assumption. A third, more concrete motivation for this assumption is that it has played an important rôlle in the recent beautiful work on $\mathcal{N} = 1$ superconformal field theories in four dimensions and interpolations from Calabi-Yau cones to $AdS_5 \times Y^{p,q}$ manifolds in IIB [15]-[21]. The metric and flux for these supergravity solutions are given by

$$d\mathbf{s}^2 = \left[1 + \frac{1}{R^4}\right]^{-1/2} d\mathbf{s}^2(\mathbb{R}^{1,3}) + \left[1 + \frac{1}{R^4}\right]^{1/2} [dR^2 + R^2 d\mathbf{s}^2(\mathcal{M}_5)],$$

$$F = (1 + \ast) \text{Vol}_{\mathbb{R}^{1,4}} \wedge d\left[1 + \frac{1}{R^4}\right]^{-1}, \quad (1.3)$$

where $d\mathbf{s}^2(\mathcal{M}_5)$ is a Sasaki-Einstein metric on $Y^{p,q}$. As $R \to \infty$, the metric asymptotes to a singular Calabi-Yau cone:

$$d\mathbf{s}^2 \to d\mathbf{s}^2(\mathbb{R}^{1,3}) + dR^2 + R^2 d\mathbf{s}^2(\mathcal{M}_5). \quad (1.4)$$

A global geometry of this form would be singular at $R = 0$. However, in the interpolating solution, this singularity is excised, and removed to infinity. The apex of the cone is thereby rendered non-compact, and opens up into an internal, asymptotically $AdS_5$ region, at infinite proper distance. The Penrose diagram, in the $t - R$ plane, for the maximal analytic extension of this manifold [22] is shown in Figure 1. An important global assumption in identifying the geometric dual of a-maximisation [21] is that the Calabi-Yau singularity is Gorenstein. This means that the incomplete special holonomy manifold obtained upon excising the singularity is globally Calabi-Yau; it admits an
Figure 1: Penrose diagram for the maximal analytic extension of an interpolation from a Calabi-Yau cone to $AdS_5$ in IIB.

everywhere non-vanishing complex structure and holomorphic three-form. An equivalent statement of this assumption is that the interpolating solution (where the singularity is indeed excised, and removed to infinity) admits a global reduction of the frame bundle to a principal $SU(3)$ sub-bundle, on an incomplete region of spacetime bounded by the special holonomy asymptotics and the $AdS$ horizon - a causal diamond of the Penrose diagram. Analytic extension of the frame bundle across an event horizon appears to be facilitated by the doubling of supersymmetry on the $AdS$ horizon manifold. However we do not explore the issue of analytic extension across a horizon any further here, and we henceforth restrict attention to regions of spacetime bounded by asymptopia and $AdS$ horizons, admitting a global reduction of the frame bundle. This restriction to a causal diamond of a Penrose diagram is in any event in keeping with more general ideas about holography, and also plays an important rôle in the quantum gravity of de Sitter space [23], [24].
We will now begin to explore what information about the geometry of (1.1), (1.2) we can extract, from eleven-dimensional supergravity, given our global assumptions. Eleven dimensional supergravity is not designed to manipulate frame bundles directly - the Killing spinor equation is instead an equation for sections of the spin bundle. In demanding the existence of a globally defined frame bundle, we have not assumed any a priori realisation of the frame bundle by sections of the spin bundle. Therefore, in order to use eleven-dimensional supergravity, we must find a way of associating globally defined sections of the spin bundle to a globally-defined frame bundle. By this we mean finding the Killing spinors whose bilinears produce the structure forms. Clearly, they should be singlets of the structure group. They may be selected in a natural way, by using the Clifford action of the structure forms on the eleven dimensional spin bundle. This Clifford action is defined for an \( n \)-form \( A \) on \( M_8 \) by

\[
A \cdot \eta = \frac{1}{n!} A_{i_1 \ldots i_n} \Gamma^{i_1 \ldots i_n} \eta,
\]

where \( \eta \) is a Majorana spinor in eleven dimensions and the \( \Gamma_i \) are eleven-dimensional gamma-matrices, with \( i = 1, \ldots, 8 \). Taking the example of \( Spin(7) \), the Clifford action \( \phi \cdot \eta \) decomposes an arbitrary spinor \( \eta \) into modules of the structure group of \( \phi \); each module is an eigenmodule of the Clifford action of the structure form, with a different eigenvalue. What distinguishes the singlets, in general, is that they are the modules of highest norm eigenvalue. Thus, normalising appropriately, we may obtain the singlets of the structure group \( Spin(7) \) as solutions of

\[
\frac{1}{14} \phi \cdot \eta = -\eta.
\]

Any globally-defined Killing spinors which give a realisation of the globally defined \( Spin(7) \) structure must lie in the two-dimensional solution space of (1.6). We will impose one further condition on all Killing spinors throughout this paper: we demand that all Killing spinors have a definite \( \mathbb{R}^{1,1} \) chirality; in other words, that they are eigenspinors of \( \Gamma^{+-} \). Combining this restriction with (1.6), we get the global definition of the Killing spinors realising a \( Spin(7) \) structure:

\[
\Gamma^{+-} \frac{1}{14} \phi \cdot \eta = \pm \eta.
\]

On a special holonomy manifold, in less geometrical language, this would be called the kappa-symmetry projection for a probe M5 brane wrapped on a Cayley four-cycle.

For our global \( Spin(7) \) structure, there are two distinct Killing spinor realisations; one where only one solution of (1.7) exists globally, and one where both solutions exist.
globally. When given a wrapped-brane interpretation, the first case can be associated to interpolating solutions involving deformations of the normal bundle of a Cayley four-cycle of a Spin(7) manifold. The second case is associated to interpolations from a Spin(7) cone to an AdS$_4$ horizon (foliated by AdS$_3$ leaves). For SU(4) and Sp(2), finding the spinorial realisations of the global frame bundle is very similar, and is discussed in detail in section 3.

Having found the spinorial realisation of the frame bundle, we may truncate eleven-dimensional supergravity, globally, to a gravitational theory in eleven dimensions for a frame bundle which is not Spin(1,10), but rather Spin(7), in the generic case we consider. One may re-interpret the BPS conditions for the globally-defined Killing spinor(s) realising the Spin(7)-structure - together with such components of the field equations and Bianchi identity as are not implied by their existence - as being instead the truncation of the field equations of eleven-dimensional supergravity to a frame bundle with structure group Spin(7); in effect, a classical theory of Spin(7) gravity in eleven dimensions.

Let us illustrate this truncation for the most generic case we consider in this paper; a global Spin(7) structure realised by a single solution of (1.7). We will refer to this as a Cayley structure, a Cayley frame bundle, or simply Cayley geometry, henceforth. All other cases we study may be regarded as particular cases of this one, with more restrictive global conditions. The BPS conditions with our frame and a single globally defined Killing solution of (1.7) may easily be obtained from the results of [25]. The conditions on the intrinsic torsion of the globally-defined Spin(7)-structure are

$$e^9 \wedge \left[ -L^3 e^9 \, d(L^{-3} e^9) + \frac{1}{2} \phi \, d\phi \right] = 0, \quad (1.8)$$

$$\langle e^9 \wedge + \star_9 \rangle [e^9 \, d(L^{-1} \phi)] = 0. \quad (1.9)$$

Here $\star_9$ denotes the Hodge dual on the space transverse to the Minkowski factor. The operation $\wedge$ is defined, for an $n$-form $A$ and an $m$-form $B$, $m > n$, by

$$A \wedge B_{\mu_{n+1} \cdots \mu_m} = \frac{1}{n!} A^{\mu_1 \cdots \mu_n} B_{\mu_1 \cdots \mu_n \mu_{n+1} \cdots \mu_m}. \quad (1.10)$$

Then the flux is given by

$$F = d(e^{+9}) - \star d(e^{+9} \wedge \phi) - \frac{L^{10/7}}{2} e^9 \wedge d(L^{-10/7} \phi) + \frac{1}{4} \phi \wedge [e^9 \wedge (e^9 \wedge d^9)] + F^{27}. \quad (1.11)$$

We have defined the operation $\wedge$ for an $n$-form $A$ and a two-form $B$ on $\mathcal{M}_S$ according to

$$A \wedge B = n A_{[i_1 \ldots i_{n-1}] B_{i_n}]i_n}. \quad (1.12)$$
Observe that $\phi \circ \phi : \Lambda^2(\mathcal{M}_8) \to \Lambda^4(\mathcal{M}_8)$. The $F^{27}$ term in the flux is a four-form on $\mathcal{M}_8$ in the $27$ of $Spin(7)$ which is unfixed by the truncation. Thus, the general equations for the truncation of eleven dimensional supergravity to Cayley geometry are the torsion conditions (1.8) and (1.9), coupled to the Bianchi identity

$$dF = 0$$

(1.13)

and, as it turns out (all other field equation components being implied), the $+ - 9$ component of the four-form field equation

$$\star \left( d \star F + \frac{1}{2} F \wedge F \right) = 0.$$  

(1.14)

Having obtained the truncated supergravity equations, we must also specify the boundary conditions of interest to us. We will impose the existence of an $AdS_3$ region, which we view as being associated either to a horizon or a conformal boundary of a globally Minkowski solution. It will be very interesting to explore more sophisticated boundary conditions in the future. Because of the global structure, topological considerations will be important in doing this. Generically, one would expect a solution with an $AdS_3$ region to go to some flux geometry at other asymptopia. But one could easily imagine imposing more specialised boundary conditions, such as the existence of more than one $AdS$ region - as relevant for the dual of an RG flow between fixed points. From a mathematical point of view, perhaps the most interesting additional boundary condition would be asymptotic fall-off of the flux. This is because far from a gravitating source, the spacelike asymptotics necessarily, and automatically, have $Spin(7)$ holonomy; they must be Ricci-flat by Einstein, and special holonomy by the frame bundle. Solutions of the truncated supergravity equations with these boundary conditions describe interpolations from special holonomy spacelike asymptopia to $AdS$ horizons. Because of the global structure, the $AdS$ horizon geometry of an interpolating solution will be intimately related to that of the asymptotic special holonomy manifold. We will return to a discussion of these boundary conditions in the conclusions.

In this paper, we will impose the most general $AdS_3$ boundary condition on the supergravity truncations we study, leaving additional specialisations for the future. As we shall explain in detail, we do this by inserting the most general locally $AdS_3$ frame into the globally-defined $\mathbb{R}^{1,1}$ frame, and converting the equations for the globally-defined Minkowski structure into a set of equations for the locally-defined $AdS_3$ structure. For the generic case of Cayley geometry, the local $AdS$ structure is $G_2$, with associative

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1This equation can of course receive quantum corrections, as can the Killing spinor and Einstein equations, but we will ignore them.
three-form $\Phi$ and co-associative four-form $\Upsilon$. We shall see that locally, the metric may always be cast in the form

$$d{s^2} = \frac{1}{{\lambda m^2 }}\left[ {d{s^2}(AdS_3 ) + \frac{{\lambda^3 }}{{4\sin^2 \theta }}d\rho \otimes d\rho } \right] + d{s^2}(N_7 ), \quad (1.15)$$

where the $G_2$ structure is defined on $N_7$, and $\lambda$, $\theta$ and the frame on $N_7$ are independent of the $AdS_3$ coordinates. The restrictions on the intrinsic torsion of the locally defined $G_2$ structure may be expressed as

$$\hat{\rho} \wedge d(\lambda^{-1} \Upsilon) = 0, \quad (1.16)$$

$$\lambda^{5/2} d (\lambda^{-5/2} \sin \theta \text{Vol}_7 ) = -4m\lambda^{1/2} \cos \theta \hat{\rho} \wedge \text{Vol}_7, \quad (1.17)$$

$$d\Phi \wedge \Phi = \frac{4m\lambda^{1/2}}{\sin \theta} (4 - \sin^2 \theta) \text{Vol}_7 - 2\cos \theta \ast_8 d\log \left( \frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta} \right); \quad (1.18)$$

the flux is given by

$$F = \frac{1}{{m^2 }}\text{Vol}_{AdS_3} \wedge d[\rho - \lambda^{-3/2} \cos \theta]$$

$$+ \frac{\lambda^{3/2}}{\sin^2 \theta} \left( \cos \theta + \ast_8 \right) \left( d[\lambda^{-3/2} \sin \theta \Phi] - 4m\lambda^{-1} \Upsilon \right) + 2m\lambda^{1/2} \Phi \wedge \hat{\rho}, \quad (1.19)$$

and the definitions of $\ast_8$ and the basis one-form $\hat{\rho}$ hopefully are obvious. In [26], Martelli and Sparks gave a classification of all minimally supersymmetric $AdS_3$ spacetimes in M-theory; the conditions we have obtained on the local $G_2$ structure of an $AdS_3$ region in Cayley geometry are identical to theirs. We regard these conditions as being valid locally on the horizon or conformal boundary of a globally Minkowski solution, or globally for a globally $AdS$ solution of Cayley geometry.

As we have said, we study truncations of eleven-dimensional supergravity to several different frame bundles, with different spinorial realisations. For a $Spin(7)$ bundle, we study spinorial realisations by either one or two globally defined Killing spinors. We refer to the resulting truncations of eleven-dimensional supergravity as Cayley or $Spin(7)$ geometry, respectively. The $AdS_3$ conditions we derive from Cayley geometry define the geometry of all M-theory duals of $N = (1, 0)$ two-dimensional CFTs. The $AdS_3$ conditions we derive from $Spin(7)$ geometry reproduce the $AdS_4 \times \text{Weak } G_2$ Freund-Rubin solutions, with the $AdS_4$ foliated by $AdS_3$ leaves. For an $SU(4)$ frame bundle we

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2The orientations for the various Hodge stars will be specified when they next appear.

3Up to a minor discrepancy in (3.14) of [26] which we have corrected.

4A subtlety in the global validity of these conditions for globally $AdS$ manifolds is discussed in section 5.
study three distinct spinorial realisations. The first, with the maximal number (four) of globally-defined Killing spinors, produces a truncation we refer to as $SU(4)$ geometry. The $AdS_3$ conditions we derive from $SU(4)$ geometry produce the Freund-Rubin $AdS_4 \times SE_7$ solutions. The other two spinorial realisations of an $SU(4)$ frame bundle we study have two globally defined Killing spinors. We refer to the associated truncations as Kähler-4 and Special Lagrangian-4 (SLAG) geometry. Given a wrapped brane interpretation, one would say that a solution of these truncations described an M5-brane wrapped on, respectively, a Kähler or SLAG four-cycle of a Calabi-Yau four-fold, with a membrane extended in the directions transverse to the Calabi-Yau and intersecting the fivebrane in a string. We believe that the $AdS_3$ conditions we derive from SLAG geometry define all M-theory duals of $N = (1, 1)$ CFTs; and similarly that the $AdS_3$ conditions we derive from Kähler-4 geometry (together with the $AdS_3$ conditions of [9] from co-associative geometry with a global Minkowski $G_2$ frame bundle) define all M-theory duals of $N = (2, 0)$ CFTs. For an $Sp(2)$ frame bundle, we again study three distinct spinorial realisations. The first, with the maximal number (six) of globally-defined Killing spinors, produces a truncation we call $Sp(2)$ geometry. Again, the $AdS_3$ conditions we derive from $Sp(2)$ geometry just give the appropriate Freund-Rubin solutions, this time the direct product of $AdS_4$ with a Tri-Sasaki-Einstein manifold. The other two spinorial realisations of an $Sp(2)$ frame bundle we study have three globally defined Killing spinors. We refer to the associated truncations of eleven dimensional supergravity as Quaternionic Kähler (QK) and Complex Lagrangian (CLAG) geometry. We believe that the $AdS_3$ conditions we derive from these truncations define all M-theory duals of $N = (3, 0)$ and $N = (2, 1)$ CFTs respectively.

The remainder of this paper is organised as follows. For the convenience of the reader who is not interested in their derivation, in section 2 we summarise our main technical results: the truncation of eleven-dimensional supergravity to Cayley, Kähler-4, SLAG, QK or CLAG geometry, together with the associated conditions for an $AdS_3$ region. These equations are the result of involved calculations. As a consistency check, we have verified that explicit Kähler-4 and SLAG $AdS_3$ solutions, known from gauged supergravity, satisfy our definitions of $AdS$ geometry in the appropriate truncations, by explicitly elucidating their structure. Since our results for Kähler-4 and SLAG geometry are derived directly from Cayley geometry, and our results for CLAG and QK in turn are derived from those for Kähler-4 and SLAG, this serves as a rigid overall consistency check. The remainder of the paper (with the exception of the conclusions) is concerned with the derivation of the results of section 2. In section 3, we discuss the globally-defined G-structures and spinorial realisations thereof which are of interest to us. In section 4,
we explain in more detail how to obtain the supergravity truncation in each case. Section 5 is concerned with the derivation of the local conditions for an $AdS_3$ region in each truncation. Section 6 discusses the verification of the globalised $AdS$ torsion conditions for known solutions. Section 7 concludes with some more observations, speculations and suggestions for future directions.

## 2 Summary of results

In this section, we will summarise our technical results for Cayley, Kähler-4, SLAG, QK and CLAG geometry. In each case, we will give the globally defined spinorial realisation of the frame bundle, the associated truncation of eleven-dimensional supergravity, and a definition of the geometry of an arbitrary $AdS_3$ region in the truncation. For the Kähler-4 and SLAG geometries, as an overall consistency check, we present a known exact solution of the $AdS$ equations, with its structure made manifest, that we have verified satisfies our $AdS$ conditions.

We take positive orientation in eleven dimensions to be defined by

$$\text{Vol}_{11} = e^- \wedge e^+ \wedge \frac{1}{14} \phi \wedge \phi \wedge e^9.$$  \hfill (2.1)

In every case, the globally-defined Minkowski frame is given by (1.1) of the introduction; positive orientation for $\star_9$, the Hodge dual on the space transverse to the Minkowski factor, is defined by

$$\text{Vol}_9 = \frac{1}{14} \phi \wedge \phi \wedge e^9.$$  \hfill (2.2)

In every case, the truncation of eleven-dimensional supergravity to the global frame bundle consists of the quoted torsion conditions for the globally-defined Minkowski structure coupled to the Bianchi identity and the $+ - 9$ component of the four-form field equation. The $AdS_3$ geometries automatically solve the four-form field equation, and for them it is in every case sufficient to impose the Bianchi identity in addition to the torsion conditions to ensure that they are solutions of eleven dimensional supergravity. For the $AdS$ geometries, the warp factor, the frame on the transverse space, and the flux, are independent of the $AdS$ coordinates. We define the basis one-form $\hat{\rho}$ in the local $AdS$ frame in every case according to

$$\hat{\rho} = \frac{\lambda}{2m \sin \theta} d\rho.$$  \hfill (2.3)

The electric flux for the $R^{1,1}$ geometries, in every case, takes the form

$$F_{\text{elec}} = d(e^{+ - 9}).$$  \hfill (2.4)
while for the \textit{AdS} geometries, in every case, it takes the form

$$F_{\text{elec}} = \frac{1}{m^2} \text{Vol}_{\text{AdS}} \wedge d[\rho - \lambda^{-3/2} \cos \theta]. \quad (2.5)$$

Now we will state our results.

2.1 \textbf{Cayley geometry}

In this case, \(\mathcal{M}_8\) admits a globally-defined \textit{Spin}(7) structure which is realised by a single Killing solution of

$$\Gamma^+ \cdot \frac{1}{14} \phi = -\eta. \quad (2.6)$$

\textbf{Global truncation} The truncation of eleven-dimensional supergravity to this geometry is defined by

$$e^9 \wedge \left[-L^3 e^9 \cdot d(L^{-3} e^9) + \frac{1}{2} \phi \cdot d\phi\right] = 0, \quad (2.7)$$

$$e^9 \wedge [e^9 \cdot d(L^{-1} \phi)] = 0, \quad (2.8)$$

$$F = d(e^{+9}) - \star d(e^{-} \wedge \phi) - \frac{L^{10/7}}{2} e^9 \cdot d(L^{-10/7} \phi) + \frac{1}{4} \phi \cdot [e^9 \cdot (e^9 \wedge de^9)] + F^{27}. \quad (2.9)$$

\textbf{AdS geometry} The geometry of an \textit{AdS} region in this truncation is as follows.

Locally the metric may be cast in the form

$$d\textbf{s}^2 = \frac{1}{\lambda m^2} \left[ds^2(\text{AdS}_3) + \frac{\lambda^3}{4 \sin^2 \theta} d\rho \otimes d\rho\right] + ds^2(\mathcal{N}_7), \quad (2.10)$$

where \(\mathcal{N}_7\) admits a \textit{G}_2 structure, with associative three-form \(\Phi\) and co-associative four-form \(\Upsilon\). The torsion conditions are

$$\hat{\rho} \wedge d(\lambda^{-1} \Upsilon) = 0, \quad (2.11)$$

$$\lambda^{5/2} d(\lambda^{-5/2} \sin \theta \text{Vol}_7) = -4m \lambda^{1/2} \cos \theta \hat{\rho} \wedge \text{Vol}_7, \quad (2.12)$$

$$d\Phi \wedge \Phi = \frac{4m \lambda^{1/2}}{\sin \theta} (4 - \sin^2 \theta) \text{Vol}_7 - 2 \cos \theta \star_8 d \log \left(\frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta}\right); \quad (2.13)$$

and the magnetic flux is given by

$$F_{\text{mag}} = \frac{\lambda^{3/2}}{\sin^2 \theta} \left(\cos \theta + \star_8 \right) \left(d[\lambda^{-3/2} \sin \theta \Phi] - 4m \lambda^{-1} \Upsilon\right) + 2m \lambda^{1/2} \Phi \wedge \hat{\rho}. \quad (2.14)$$

Positive orientation on the space transverse to the \textit{AdS} factor is defined by \(\frac{1}{7} \Phi \wedge \Upsilon \wedge \hat{\rho}\).
2.2 Kähler-4 geometry

In this case, $\mathcal{M}_8$ admits a globally-defined $SU(4)$ structure. The structure is realised by two globally defined null Killing spinors, which are solutions of

$$\frac{1}{12} \Gamma^{+-}(J \wedge J) \cdot \eta = -\eta.$$  \hfill (2.15)

**Global truncation** The truncation of eleven-dimensional supergravity to this geometry is defined by the torsion conditions for the global $SU(4)$ structure

$$J_{\perp} \, \text{d}e^9 = 0,$$
$$\text{d}(L^{-1}\text{Re}\Omega) = 0,$$
$$e^9 \wedge [J_{\perp} \, \text{d}J - Le^9_{\perp} \, \text{d}(L^{-1}e^9)] = 0,$$ \hfill (2.16)

and the four-form

$$F = \text{d}(e^{+9}) + \frac{1}{2} \ast \text{d} (e^{+-} \wedge J \wedge J) + \frac{1}{4} L^2 e^9_{\perp} \, \text{d} (L^{-2}J \wedge J)$$
$$- \frac{1}{4} (J \wedge J) \diamond [e^9_{\perp} (e^9 \wedge \text{d}e^9)] + F^{20}. \hfill (2.17)$$

Here $F^{20}$ is a four-form on $\mathcal{M}_8$ in the 20 of $SU(4)$ (a primitive (2,2) form) which is not fixed by the truncation.

**AdS geometry** The local metric for an $AdS_3$ region in this geometry is

$$\text{d}s^2 = \frac{1}{\lambda m^2} \left[ \text{d}s^2(AdS_3) + \frac{\lambda^3}{4 \sin^2 \theta} \, \text{d}\rho \otimes \text{d}\rho \right] + e^7 \otimes e^7 + \text{d}s^2(\mathcal{N}_6),$$  \hfill (2.18)

where $\mathcal{N}_6$ admits an $SU(3)$ structure. Using $J$ and $\Omega$ to denote the structure forms of this local $SU(3)$ structure (hopefully without risk of confusion with the structure forms of the global $SU(4)$ structure), the local $AdS_3$ torsion conditions are

$$\hat{\rho} \wedge \text{d}(\lambda^{-1}J \wedge J) = 0,$$
$$\text{d}(\lambda^{-3/2} \sin \theta \text{Im}\Omega) = 2m\lambda^{-1}(e^7 \wedge \text{Re}\Omega - \cos \theta \hat{\rho} \wedge \text{Im}\Omega),$$
$$J_{\perp} \, \text{d}e^7 = \frac{2m\lambda^{1/2}}{\sin \theta} (2 - \sin^2 \theta) - \cos \theta \hat{\rho}_{\perp} \, \text{d} \log \left( \frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta} \right). \hfill (2.21)$$

The magnetic flux is

$$F_{\text{mag}} = \frac{\lambda^{3/2}}{\sin^2 \theta} (\cos \theta + \ast_8) (\text{d}[\lambda^{-3/2} \sin \theta J \wedge e^7] - 2m\lambda^{-1}J \wedge J) + 2m\lambda^{1/2} J \wedge e^7 \wedge \hat{\rho}.$$  \hfill (2.22)

Positive orientation on the space transverse to the $AdS$ factor is defined by $\frac{1}{6}J \wedge J \wedge J \wedge e^7 \wedge \hat{\rho}$. 

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Exact solution We have verified that the following is an exact solution of the AdS torsion conditions and Bianchi identity in this truncation. Topologically, the space transverse to the AdS factor is an $S^4$ bundle over a negatively curved Kähler-Einstein manifold. This solution was first constructed in gauged supergravity, as the near-horizon limit of an M5 brane wrapped on a Kähler four-cycle in a Calabi-Yau four-fold, in [4]. The metric is given by

$$d s^2 = \frac{1}{\lambda m^2} \left[ ds^2(AdS_3) + \frac{3}{4} ds^2(KE_4) + (1 - \lambda^3 f^2)dy^a \otimes dy^a + \frac{\lambda^3}{4(1 - \lambda^3 f^2)} d\rho \otimes d\rho \right],$$

where

$$\lambda^3 = \frac{3}{4(1 + \rho^2/12)}, \quad f^2 = \frac{4}{9}\rho^2.$$  \hspace{1cm} (2.23)

The $Y^a, a = 1, \ldots, 4$ are constrained coordinates on an $S^3$, $Y^a Y^a = 1$. We define $K^A, A = 1, 2, 3, K^A K^B = -\delta^{AB} - e^{ABC} K^C$, to be a triplet of self-dual two-forms on $KE_4$, and we choose $K^3$ to label the Kähler form. We define

$$dY^a = dY^a - \frac{1}{4} K^{\alpha \beta \gamma a} \omega_{\alpha \beta} K^{a b} Y^b, \hspace{1cm} (2.24)$$

where $\omega_{\alpha \beta}$ are the spin connection one-forms of $KE_4$. Finally $ds^2(KE_4)$ is normalised such that the Ricci form is given by $R = -K^3$. Defining the functions

$$g = \sqrt{3}/2\lambda^{1/2} m, \quad h = \sqrt{1 - \lambda^3 f^2}/\lambda^{1/2} m,$$

the $SU(3)$ structure forms are given by

$$e^7 = hK^3_{ab} Y^a dY^b,$$

$$J = g^2 K^3 + h^2 \frac{1}{2} K^3_{ab} dY^a \wedge dY^b,$$

$$\text{Re}\Omega = -g^2 h \left[ K^2 \wedge K^1_{ab} Y^a dY^b + K^1 \wedge K^2_{ab} Y^a dY^b \right],$$

$$\text{Im}\Omega = -g^2 h \left[ K^2 \wedge K^2_{ab} Y^a dY^b - K^1 \wedge K^1_{ab} Y^a dY^b \right].$$  \hspace{1cm} (2.25)

In [5], [6], many infinite families of AdS$_3$ solutions, generalising this one, were constructed. All these families will satisfy our AdS equations for Kähler geometry.

2.3 Special Lagrangian geometry

Again in this case $M_8$ admits a globally defined $SU(4)$ structure. It is realised by two globally defined null Killing solutions of

$$\Gamma^+ \eta = \pm \eta,$$

$$\frac{1}{8} \Gamma^+ \text{Re}\Omega \cdot \eta = -\eta.$$  \hspace{1cm} (2.26)
Global truncation  In this case, the torsion conditions for the global SU(4) structure are

\[ d(L^{-1/2}J) = 0, \]
\[ \text{Im}\Omega \wedge d\text{Re}\Omega = 0, \]
\[ e^9 \wedge [\text{Re}\Omega \lrcorner d\text{Re}\Omega - 2L^{3/2}e^9 \lrcorner d(L^{-3/2}e^9)] = 0. \] (2.29)

The flux is given by

\[ F = d(e^{+9}) + \star d(e^{+9} \wedge \text{Re}\Omega) + \frac{1}{2}L^{7/4}e^9 \lrcorner d(L^{-7/4}\text{Re}\Omega) \]
\[ - \frac{1}{2}\text{Re}\Omega \diamond [e^9 \lrcorner (e^9 \wedge de^9)] + F^{20}. \] (2.30)

**AdS geometry**  The local AdS$_3$ frame and orientation are as for Kähler-4 geometry, and again $N_6$ admits an SU(3) structure. The AdS torsion conditions, for the local SU(3) structure forms $J$, $\Omega$, are

\[ e^7 \wedge \hat{\rho} \wedge d\left(\frac{\text{Re}\Omega}{\sin \theta}\right) = 0, \] (2.31)
\[ d(\lambda^{-1} \sin \theta e^7) = m\lambda^{-1/2}(J + \cos \theta e^7 \wedge \hat{\rho}), \] (2.32)
\[ \text{Im}\Omega \wedge d\text{Im}\Omega = \frac{m\lambda^{1/2}}{\sin \theta}(6 + 4 \cos^2 \theta)\text{Vol}_6 \wedge e^7 - 2 \cos \theta \star_8 d\log \left(\frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta}\right). \] (2.33)

The magnetic flux is given by

\[ F_{\text{mag}} = - \frac{\lambda^{3/2}}{\sin^2 \theta}(\cos \theta + \star_8)(d[\lambda^{-3/2} \sin \theta \text{Im}\Omega] + 4m\lambda^{-1}\text{Re}\Omega \wedge e^7) - 2m\lambda^{1/2}\text{Im}\Omega \wedge \hat{\rho}. \] (2.34)

**Exact solution**  We have verified that the following is an exact solution of the AdS torsion conditions and Bianchi identity in this truncation. Topologically, the eight-manifold transverse to the AdS factor is an $S^4$ bundle over $H^4$. Again this solution was first constructed (in seven-dimensional gauged supergravity) in [4], as the near-horizon limit of an M5 brane wrapped on a SLAG four-cycle of a Calabi-Yau four-fold. The metric is given by

\[ ds^2 = \frac{1}{\lambda m^2} \left[ ds^2(AdS_3) + \frac{2}{3} ds^2(H^4) + (1 - \lambda^3 f^2)DY^a \otimes DY^a \right. \]
\[ + \frac{\lambda^3}{4(1 - \lambda^3 f^2)} d\rho \otimes d\rho \right]. \] (2.35)
where
\[ \lambda^3 = \frac{2}{3(1 + \rho^2/8)}, \quad f^2 = \frac{9}{16} \rho^2; \] (2.36)
the \( Y^a, a = 1, \ldots, 4 \) are constrained coordinates on an \( S^3 \), \( Y^a Y^a = 1 \), and we define
\[ D Y^a = dY^a + \omega^a_b Y^b, \] (2.37)
where \( \omega_{ab} \) are the spin connection one-forms of \( \mathcal{H}^4 \). Finally \( ds^2(\mathcal{H}^4) \) is normalised such that the curvature two-form is given by \( R_{ab} = -\frac{1}{3} \epsilon^{a} e_{b} \wedge e_{c} \). Defining the functions
\[ g = \sqrt{\frac{2}{3}} \frac{1}{\lambda^{1/2} m}, \quad h = \sqrt{1 - \lambda^3 f^2} \frac{1}{\lambda^{1/2} m}, \] (2.38)
and with \( ds^2(\mathcal{H}^4) = \delta_{ab} e^a \otimes e^b \), the \( SU(3) \) structure forms are given by
\[ e^7 = -gY^a e^a, \]
\[ J = g h e^a \wedge D Y^a, \]
\[ \text{Re}\Omega = g^2 \frac{1}{3!} \epsilon^{abcd} Y^a e^b \wedge e^c \wedge e^d - gh \frac{1}{2} \epsilon^{abcd} Y^a D Y^b \wedge D Y^c \wedge e^d, \]
\[ \text{Im}\Omega = g^2 h \frac{1}{2} \epsilon^{abcd} Y^a D Y^b \wedge e^c \wedge e^d - h^3 \frac{1}{3!} \epsilon^{abcd} Y^a D Y^b \wedge D Y^c \wedge D Y^d. \] (2.39)

### 2.4 Quaternionic Kähler geometry

In this case \( \mathcal{M}_8 \) admits a globally defined \( Sp(2) \) structure. Defining the form \( \Xi_1 \) in terms of the three almost complex structures according to
\[ \Xi_1 = \frac{1}{2} J^A \wedge J^A, \] (2.40)
the \( Sp(2) \) structure is realised by three globally defined null Killing solutions of
\[ \frac{1}{10} \Gamma^+ \Xi_1 \cdot \eta = -\eta. \] (2.41)

**Global truncation**  The torsion conditions of the global truncation are
\[ J^A \wedge d e^g = 0, \]
\[ d (L^{-1} \text{Re}\Omega^A) = 0, \]
\[ e^g \wedge [ J^A \wedge d J^A - L e^g \wedge d (L^{-1} e^g) ] = 0, \] (2.42)
where there is no sum on \( A \) in the third equation. Here \( \Omega^A \) are the \((4, 0)\) forms associated to the almost complex structures \( J^A \). More details of their definition are given in the
The flux is
\[ F = d(e^+ \wedge e^-) + \frac{1}{3} \ast d(e^+ \wedge \Xi_1) + \frac{1}{6} L^{14/5} e^9 \wedge d(L^{-14/5} \Xi_1) \]
\[ - \frac{1}{4} \Xi_1 \wedge [e^9 \wedge (e^9 \wedge de^9)] + F^{14}, \] (2.43)
where \( F^{14} \) is a four-form on \( M_8 \) in the \( 14 \) of \( Sp(2) \) which is unfixed by the truncation.

**AdS geometry** The local metric for an \( AdS_3 \) region in this geometry is
\[ ds^2 = \frac{1}{\lambda m^2} ds^2(AdS_3) + e^A \otimes e^A + \hat{\rho} \otimes \hat{\rho} + ds^2(N_4), \] (2.44)
where \( N_4 \) admits a local \( SU(2) \) structure, specified by a triplet of self-dual \( SU(2) \) forms \( K^A \). The local \( AdS_3 \) torsion conditions are
\[ \hat{\rho} \wedge d\left[ \lambda^{-1} \left( \text{Vol}_4 + \frac{1}{6} e^{ABC} K^A \wedge e^{BC} \right) \right] = 0, \] (2.45)
\[ \frac{1}{3} \left( K^A + \frac{1}{2} e^{ABC} e^{BC} \right) \wedge de^A = \frac{2m\lambda^{1/2}}{\sin \theta} (2 - \sin^2 \theta) \]
\[ - \cos \theta \hat{\rho} \wedge d \log \left( \frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta} \right), \] (2.46)
\[ d[\lambda^{-3/2} \sin \theta (K^2 \wedge e^2 - K^1 \wedge e^1)] = 2m\lambda^{-1} [K^2 \wedge e^{123} - K^1 \wedge e^{23}] \]
\[ + 2m\lambda^{-1} \cos \theta [K^2 \wedge e^2 - K^1 \wedge e^1] \wedge \hat{\rho}, \] (2.47)
together with permutations of the last equation. The magnetic flux is given by
\[ F_{\text{mag}} = - \frac{\lambda^{3/2}}{\sin \theta} (\cos \theta + 8) \left[ d\left( \lambda^{-3/2} \sin \theta \left[ \frac{1}{3} K^A \wedge e^A + e^{123} \right] \right) \right] \]
\[ - 4m\lambda^{-1} \left( \text{Vol}_4 + \frac{1}{6} e^{ABC} K^A \wedge e^{BC} \right) - 2m\lambda^{1/2} \left[ \frac{1}{3} K^A \wedge e^A + e^{123} \right] \wedge \hat{\rho}. \] (2.48)
Positive orientation on the space transverse to the \( AdS \) factor is defined by \( \frac{1}{6} K^A \wedge K^A \wedge e^{123} \wedge \hat{\rho} \).

**2.5 Complex Lagrangian geometry**

In this case, \( M_8 \) again admits a global \( Sp(2) \) structure. Defining
\[ \Xi_2 = \frac{1}{2} (J^1 \wedge J^1 - \text{Re}\Omega^2 + \text{Re}\Omega^3), \] (2.49)
it is realised by three globally defined null Killing solutions of
\[ \Gamma^{+-} \eta = \pm \eta, \] (2.50)
\[ \frac{1}{10} \Gamma^{+-} \Xi_2 \cdot \eta = -\eta. \] (2.51)
Global truncation  In this case, effecting the global truncation is technically more difficult. We have performed it under the assumption that $e^9 \wedge de^9 = 0$. Then the torsion conditions are given by

\[ d(L^{-1/2}J^2) = d(L^{-1/2}J^3) = 0, \]

\[ e^9 \wedge [J^1 \wedge dJ^1 - Le^9 \wedge d(L^{-1}e^9)] = 0. \]  

(2.52)

The flux is

\[ F = d(e^{−9}) + \frac{1}{2} \, d(e^{−} \wedge \Xi_2) + \frac{1}{4} L^{11/5} e^9 \wedge d(L^{-11/5} \Xi_2) + F^{14}. \]

(2.53)

AdS geometry  The local frame, structure and orientation for an AdS$_3$ region in this geometry are identical to those in Quaternionic Kähler geometry. We have derived the AdS$_3$ torsion conditions by decomposing those in SLAG and Kähler-4 geometry (exactly how we do this is discussed in section 5) rather than from the equations for the global truncation of the previous paragraph. This means that our AdS equations are independent of the assumption $e^9 \wedge de^9 = 0$ that we made for the global Minkowski frame above. The torsion conditions we find are

\[ \bar{\rho} \wedge d[\lambda^{-1}(\text{Vol}_4 + K^3 \wedge e^{12})] = 0, \]

(2.54)

\[ (K^3 + e^{12}) \wedge de^3 = \frac{2m\lambda^{1/2}}{\sin \theta}(2 - \sin^2 \theta) - \cos \theta \bar{\rho} \wedge d \log \left( \frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta} \right), \]

(2.55)

\[ d(\lambda^{-1} \sin \theta e^1) = m\lambda^{-1/2}(K^1 + e^{23} + \cos \theta e^1 \wedge \bar{\rho}), \]

(2.56)

\[ d(\lambda^{-1} \sin \theta e^2) = m\lambda^{-1/2}(K^2 + e^{31} + \cos \theta e^2 \wedge \bar{\rho}). \]

(2.57)

The magnetic flux is

\[ F_{\text{mag}} = \frac{\lambda^{3/2}}{\sin^2 \theta}(\cos \theta + *s)(d(\lambda^{-3/2} \sin \theta[K^3 \wedge e^3 + e^{123}]) - 4m\lambda^{-1}(\text{Vol}_4 + K^3 \wedge e^{12})) \]

\[ + \frac{2m\lambda^{1/2}}{K^3 \wedge e^3 + e^{123}} \wedge \bar{\rho}. \]

(2.58)

3  Spinorial realisation of the frame bundles

In this section, we will discuss the spinorial realisations of the globally defined frame bundles we study. A global reduction of the frame bundle to a sub-bundle is by definition equivalent to the existence of a globally-defined G-structure. The existence of a globally
defined $G$-structure, for our purposes, is equivalent to the existence of a set of globally-defined forms, invariant under the action of the structure group $G$. We will use the action of the structure forms on the spin bundle to define the spinorial realisation of the $G$-structure. When the flux vanishes asymptotically, the structure forms asymptote to the calibrations of the asymptotic special holonomy manifold.

To start, we will specify the $Spin(1,10)$ structure we use for eleven dimensional supergravity. We use all the supergravity and spinorial conventions of [25], which are employed consistently throughout [25], [27]-[32], the papers we will use in the next section for truncating supergravity to the frame bundles of this section. We work in the null frame of the introduction,

$$ds^2 = 2e^+ \otimes e^- + ds^2(M_8) + e^9 \otimes e^9.$$  \hspace{1cm} (3.1)

We recall that we impose, globally, that $e^+ = L^{-1}dx^+, e^- = dx^-, L < \infty, e^9 \neq 0$; and that $L$ and the frame on the space transverse to the Minkowski factor are independent of the coordinates $x^\pm$. The orientations we use are defined in section 2.

3.1 $Spin(7)$ and associated local $AdS_3$ structures

A global $Spin(7)$ structure in eleven dimensions is defined by the nowhere vanishing one-forms $e^\pm, e^9$, and the nowhere vanishing Cayley four-form $\phi$. We choose the components of $\phi$ to be

$$\begin{align*}
-\phi &= e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + e^{1357} \\
&+ e^{2468} - e^{1368} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457}.
\end{align*}$$  \hspace{1cm} (3.2)

On a special holonomy manifold, $\phi$ calibrates Cayley four-cycles. The embedding of our $Spin(7)$ structure group in $Spin(1,10)$ (which is entirely at our discretion) is defined by this choice of $\phi$, together with the globally-defined forms $e^\pm, e^9$.

The most general geometry we study is Cayley geometry, where the $Spin(7)$ structure is realised by a single globally defined null Killing spinor. This may be chosen to satisfy the projection

$$\frac{1}{14} \Gamma^{+-} \phi \cdot \epsilon = -\epsilon.$$  \hspace{1cm} (3.3)

With our choice of the components of $\phi$, this projection is equivalent to

$$\Gamma^{1234} \epsilon = \Gamma^{3456} \epsilon = \Gamma^{5678} \epsilon = \Gamma^{1357} \epsilon = -\Gamma^{+-} \epsilon = -\epsilon.$$  \hspace{1cm} (3.4)

We will reserve the notation $\epsilon$ for a globally-defined Killing spinor satisfying this projection in the frame (3.1). The statements regarding the eigenvalues of the $Spin(7)$
modules of the spin bundle may be verified by evaluating the Clifford action of \( \phi \) in a specific basis for the spin bundle; a useful choice (which we will have used for all the spinor algebra described in this section) is that constructed in [29]. The spinor \( \epsilon \) is the spinorial realisation of the frame bundle for the geometric dual of an \( N = (1, 0) \) CFT.

Having found the spinorial realisation of the Cayley structure, the structure forms may be obtained as bilinears of the Killing spinor (apart from \( e^- \), which is put in by hand in our frame definition). As discussed in detail in [25], [27], the only non-zero bilinears are the one-, two-, and five-forms, which are

\[
K = e^+, \\
\Theta = e^{+9}, \\
\Sigma = e^+ \wedge \phi.
\]

(3.5)

An essential point in our construction is the patching of the G-structures of the global \( \mathbb{R}^{1,1} \) and local \( AdS_3 \) regions. We will now examine this in detail, by imposing the most general local warped product \( AdS_3 \) frame ansatz on our globally-defined frame. Globally, we have

\[
ds^2 = L^{-1} ds^2(\mathbb{R}^{1,1}) + ds^2(\mathcal{M}_8) + e^9 \otimes e^9.\]

(3.6)

Observe, that in Poincaré coordinates, every \( AdS_3 \) space is foliated by \( \mathbb{R}^{1,1} \) leaves:

\[
\frac{1}{m^2} ds^2(AdS_3) = e^{-2mr} ds^2(\mathbb{R}^{1,1}) + dr^2.
\]

(3.7)

Therefore we demand that for a local \( AdS \) region, \( L \) in (3.6) is given by

\[
L = e^{2mr} \lambda,
\]

(3.8)

for some function \( \lambda \) which is independent of the \( AdS \) coordinates. For a general \( \mathbb{R}^{1,1} \) solution with an \( AdS \) horizon, this expression for \( L \) is local and valid for large positive \( r \). For an \( AdS_3 \) conformal boundary, it is valid for large negative \( r \). For a globally \( AdS \) solution, it is valid for all \( r \). To get an \( AdS \) metric, we must also pick out the \( AdS \) radial one-form \( \hat{r} = \lambda^{-1/2} dr \) from the space transverse to the \( \mathbb{R}^{1,1} \) factor. In an \( AdS \) region, this one-form will in general be a linear combination of \( e^9 \), and a one-form lying entirely in \( \mathcal{M}_8 \). Using the transitive action of \( Spin(7) \) on \( \mathcal{M}_8 \) (an action which, by definition, leaves the Killing spinor and Cayley form invariant) we may choose the part of \( \hat{r} \) lying in \( \mathcal{M}_8 \) to lie entirely along the basis one-form \( e^8 \). Then we may write the locally-defined \( AdS_3 \) frame as a rotation of the globally-defined \( \mathbb{R}^{1,1} \) frame, as

\[
\hat{r} = \sin \theta e^8 + \cos \theta e^9, \\
\hat{\rho} = \cos \theta e^8 - \sin \theta e^9.
\]

(3.9)
with $0 < \theta \leq \pi/2$. We demand that $\hat{\rho}$, together with the remaining basis one-forms transverse to the AdS factor, are locally independent of the AdS coordinates. The local metric becomes

$$ds^2 = \frac{1}{\lambda m^2} ds^2(AdS_3) + \hat{\rho} \otimes \hat{\rho} + ds^2(N_7).$$

(3.10)

This frame-rotation technique was first employed in [33]. Because we have locally picked out a preferred vector on $\mathcal{M}_8$, the eleven-dimensional structure group is reduced, locally, from $Spin(7)$ to a $G_2$ which acts on $N_7$. This $G_2$ structure is specified by the local decomposition of the globally-defined $\phi$, into an associative three-form $\Phi$ and a co-associative four-form $\Upsilon$ according to

$$-\phi = \Upsilon + \Phi \wedge e^8,$$

(3.11)

so that

$$\Phi = e^{127} + e^{347} + e^{567} + e^{246} - e^{136} - e^{145} - e^{235},$$

$$\Upsilon = e^{1234} + e^{1256} + e^{3456} + e^{1357} - e^{1467} - e^{2367} - e^{2457}.$$  

(3.12)

Having defined the most general global Minkowski and associated local AdS structures of interest to us, we now describe how they may be further reduced, by imposing the existence of even more exceptional global structures. The first exceptional case we consider is where both spinorial singlets of the global $Spin(7)$ structure are Killing. They are defined by the projection

$$\frac{1}{14} \phi \cdot \eta = -\eta.$$  

(3.13)

The local decomposition of the Cayley four-form under $G_2$ in an $AdS_3$ patch will be exactly as above; however, the local supersymmetry in the $AdS_3$ patch will double, so now there will be four locally-defined $Spin(7)$ structures, whose common subgroup is the locally-defined $G_2$. The second linearly independent globally defined Killing spinor realising the maximal $Spin(7)$ structure is proportional to the basis spinor [29]

$$\Gamma^- \epsilon.$$  

(3.14)

Now we will look at the spinorial realisations of frame bundles with a reduced structure group.

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\(^5\)Obviously, $\theta = 0$ is a special case; it will be discussed separately in section 5.
3.2  \textit{SU(4) and associated local AdS}_3 \textit{ structures}

In this subsection we will define the spinorial realisations of an \textit{SU(4)} frame bundle of interest to us. What we call \textit{SU(4)} geometry is when all four spinorial singlets of the structure group of the frame bundle are Killing and globally defined. The other spinorial realisations we consider - defining what we call Kähler-4 and SLAG geometry - are when two of the singlets of the structure group are Killing and globally defined. Again, the Killing spinors may be naturally selected by the action of the structure forms on the spin bundle. With asymptotically vanishing flux, the interpolating solutions of Kähler-4 and SLAG geometry will involve deformations of the normal bundles of respectively Kähler-4 and SLAG-4 cycles of the special holonomy manifolds to which they asymptote.

In the globally defined frame (3.1), we demand that \( \mathcal{M}_8 \) admits an everywhere non-zero almost complex structure two-form \( J \) and a \((4,0)\) form \( \Omega \). We may always take the \textit{SU(4)} structure group to be embedded in \( \text{Spin}(1,10) \) such that their components are given by

\begin{align*}
J &= e^{12} + e^{34} + e^{56} + e^{78}, \\
\Omega &= (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6)(e^7 + ie^8).
\end{align*}

\textit{SU(4) geometry}  The \textit{SU(4)} singlets are defined globally by the action of the structure forms on the spin bundle; they are the four solutions of

\begin{align*}
\Gamma^+ \eta &= \pm \eta, \\
\frac{1}{12} (J \wedge J) \cdot \eta &= -\eta,
\end{align*}

or equivalently

\begin{align*}
\Gamma^- \eta &= \pm \eta, \\
\frac{1}{8} \text{Re}\Omega \cdot \eta &= \pm \eta.
\end{align*}

A third equivalent form of these conditions is

\begin{align*}
\Gamma^{1234} \eta = \Gamma^{3456} \eta = \Gamma^{5678} \eta = \pm \Gamma^{+-} \eta = -\eta.
\end{align*}

\textit{SU(4)} geometry is defined by requiring that all four solutions of these projections are Killing. Explicitly, the Killing solutions of these equations are proportional to

\begin{align*}
e, \quad \Gamma^- \epsilon, \quad \frac{1}{4} J \cdot \epsilon, \quad \frac{1}{4} \Gamma^- J \cdot \epsilon.
\end{align*}

This maximal realisation of an \textit{SU(4)} structure is relevant for interpolations from Calabi-Yau cones to Sasaki-Einstein manifolds.
Kähler-4 geometry  The spinorial realisation of a Kähler-4 structure is given by two globally defined Killing spinors which satisfy the projection
\[
\frac{1}{12} \Gamma^{+-} (J \wedge J) \cdot \eta = - \eta. \quad (3.23)
\]
This is equivalent to the maximal \(SU(4)\) projections supplemented by \(\Gamma^{+-} \eta = \eta\). The basis spinors of (3.22) which survive this projection are
\[
\epsilon, \quad \frac{1}{4} J \cdot \epsilon. \quad (3.24)
\]
Both of these spinors have positive chirality under \(\Gamma^{+-}\). The bilinears associated to these basis spinors are
\[
K = e^+, \\
\Theta = e^{+9}, \\
\Sigma_{\pm} = e^+ \wedge \phi_{\pm}, \quad (3.25)
\]
where
\[
\phi_{\pm} = - \left( \frac{1}{2} J \wedge J \pm \text{Re} \Omega \right), \quad (3.26)
\]
with \(\phi_+\) coming from \(\epsilon\) and \(\phi_-\) from \(\frac{1}{4} J \cdot \epsilon\). This is the spinorial realisation of the \(SU(4)\) frame bundle of relevance for the geometric duals of \(N = (2, 0)\) conformal field theories.

SLAG geometry  The spinorial realisation of the \(SU(4)\) frame bundle defining SLAG geometry is specified by two globally defined Killing spinors, which satisfy the projections
\[
\Gamma^{+-} \eta = \pm \eta, \\
\frac{1}{8} \Gamma^{+-} \text{Re} \Omega \cdot \eta = - \eta. \quad (3.27)
\]
This is equivalent to the maximal \(SU(4)\) projections, supplemented by
\[
\Gamma^{+-1357} \eta = - \eta. \quad (3.28)
\]
The pair of basis spinors surviving this projection are
\[
\epsilon, \quad \frac{1}{4} \Gamma^- J \cdot \epsilon. \quad (3.29)
\]
The bilinears associated to the basis spinor \(\frac{1}{4} \Gamma^- J \cdot \epsilon\) are
\[
K = -e^-, \\
\Theta = e^{-9}, \\
\Sigma = e^- \wedge \left( \frac{1}{2} J \wedge J - \text{Re} \Omega \right). \quad (3.30)
\]
Observe that these Killing spinors have opposite $\mathbb{R}^{3,1}$ chirality, so this is the spinorial realisation of the $SU(4)$ frame bundle of relevance for the geometric duals of $N = (1, 1)$ CFTs.

**Local $AdS_3$ structures** Now we will give the local $AdS_3$ structures which arise from each spinorial realisation of the globally-defined $SU(4)$ structures. In this case, picking out a local $AdS_3$ radial direction with a component on $\mathcal{M}_8$ reduces the structure group, locally, to $SU(3)$. The metric is given by

$$ds^2 = \frac{1}{\lambda m^2} ds^2(AdS_3) + e^7 \otimes e^7 + \hat{\rho} \otimes \hat{\rho} + ds^2(\mathcal{N}_6),$$

(3.31)

where the locally defined $\mathcal{N}_6$ admits the local $SU(3)$ structure. For each of $Spin(7)$ structures which collectively define a Kähler-4 structure, $-\phi_\pm = \Upsilon_\pm + \Phi_\pm \wedge e^8$, we find that the associated local $AdS_3 G_2$ structures are

$$\Phi_\pm = J_{SU(3)} \wedge e^7 \mp \text{Im}\Omega_{SU(3)},$$

$$\Upsilon_\pm = \frac{1}{2} J_{SU(3)} \wedge J_{SU(3)} \pm \text{Re}\Omega_{SU(3)} \wedge e^7.$$  

(3.32)

We see how the two local $G_2$ structures in turn define a local $SU(3)$ structure. For the globally-defined SLAG structures, the $G_2$ structures of a local $AdS_3$ patch are

$$\Phi_\pm = \pm J_{SU(3)} \wedge e^7 - \text{Im}\Omega_{SU(3)},$$

$$\Upsilon_\pm = \pm \frac{1}{2} J_{SU(3)} \wedge J_{SU(3)} + \text{Re}\Omega_{SU(3)} \wedge e^7,$$

(3.33)

and collectively they define a different embedding of the local $SU(3)$ in the local $G_2$. In both cases

$$J_{SU(3)} = e^{12} + e^{34} + e^{56},$$

$$\Omega_{SU(3)} = (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6).$$

(3.34)

The local $AdS_3$ structure for $SU(4)$ geometry is obvious.

### 3.3 $Sp(2)$ and associated local $AdS_3$ Structures

Finally we will discuss the spinorial realisations of an $Sp(2)$ frame bundle of interest to us. The discussion closely follows that of the $SU(4)$ case.

We obtain a globally defined $Sp(2)$ structure by demanding that $\mathcal{M}_8$ admits a triplet of everywhere non-zero almost complex structures $J^A$, $A = 1, 2, 3$. These obey the algebra

$$J^A J^B = -\delta^{AB} + \epsilon^{ABC} J^C.$$  

(3.35)
We can always choose a basis such that the components of the three almost complex
are given by
\[ J^1 = e^{12} + e^{34} + e^{56} + e^{78}, \quad (3.36a) \]
\[ J^2 = -e^{13} + e^{24} - e^{57} + e^{68}, \quad (3.36b) \]
\[ J^3 = e^{14} + e^{23} + e^{67} + e^{58}. \quad (3.36c) \]

Note that \( J^1 = J \), with \( J \) given in (3.15). Each almost complex structure has a corresponding \((4, 0)\) form given by
\[
\Omega^1 = \frac{1}{2} J^2 \wedge J^2 - \frac{1}{2} J^3 \wedge J^3 + i J^2 \wedge J^3,
\]
\[
\Omega^2 = \frac{1}{2} J^3 \wedge J^3 - \frac{1}{2} J^1 \wedge J^1 + i J^3 \wedge J^1,
\]
\[
\Omega^3 = \frac{1}{2} J^1 \wedge J^1 - \frac{1}{2} J^2 \wedge J^2 + i J^1 \wedge J^2. \quad (3.37)
\]

**Sp(2) geometry** What we call \( Sp(2) \) geometry is defined by the existence of six Killing singlets of the structure group of the global \( Sp(2) \) frame bundle, which satisfy the projections
\[
\Gamma^{+-} \eta = \pm \eta,
\]
\[
\frac{1}{10} \tilde{\Xi}_1 \cdot \eta = -\eta, \quad (3.38)
\]
or equivalently
\[
\Gamma^{+-} \eta = \pm \eta,
\]
\[
\frac{1}{10} \tilde{\Xi}_2 \cdot \eta = \pm \eta \quad (3.39)
\]
The Killing solutions of these projections are proportional to the basis spinors
\[
\epsilon, \frac{1}{4} J^1 \cdot \epsilon, \frac{1}{4} J^2 \cdot \epsilon, \Gamma^{-} \epsilon, \frac{1}{4} \Gamma^{-} J^1 \cdot \epsilon, \frac{1}{4} \Gamma^{-} J^2 \cdot \epsilon. \quad (3.40)
\]
This realisation of an \( Sp(2) \) frame bundle is of relevance for interpolations from Hyperkähler cones to Tri-Sasaki-Einstein manifolds.

**QK geometry** QK geometry is defined by the existence of three Killing spinorial realisations of the frame bundle which satisfy the projection
\[
\frac{1}{10} \Gamma^{+-} \tilde{\Xi}_1 \cdot \eta = -\eta. \quad (3.41)
\]
This projects out half of the Killing spinors of the maximal structure; the Killing solutions of this projection are proportional to the basis spinors
\[ \epsilon, \frac{1}{4} J^1 \cdot \epsilon, \frac{1}{4} J^2 \cdot \epsilon. \] (3.42)

The bilinears associated to these basis spinors are
\[ K^A = e^+, \]
\[ \Theta^A = e^{+9}, \]
\[ \Sigma^A = e^+ \wedge \phi^A, \] (3.43)

where
\[ \phi^A = -\frac{1}{2} J^A \wedge J^A - \text{Re} \Omega^A, \] (3.44)

with no sum on \( A \). This is the spinorial realisation of an \( Sp(2) \) frame bundle of relevance to the geometric duals of \( N = (3, 0) \) CFTs. On a special holonomy manifold, the supersymmetric cycle calibrated by \( \Xi_1 \) is Kähler-4 with respect to all three complex structures.

**CLAG geometry**  CLAG geometry is defined by the existence of three Killing spinors which satisfy the projections
\[ \Gamma^{+-} \eta = \pm \eta, \]
\[ \frac{1}{10} \Gamma^{+-} \Xi_2 \cdot \eta = -\eta. \] (3.45)

Again, this projects out half of the Killing spinors of the maximal structure; the subspace defined by this projection is spanned by the basis spinors
\[ \epsilon, \frac{1}{4} J^1 \cdot \epsilon, \frac{1}{4} \Gamma^- J^2 \cdot \epsilon. \] (3.46)

The bilinears associated to the first two of these basis spinors are the same as in (3.43) with \( A = 1, 3 \); the bilinears associated to \( \frac{1}{4} \Gamma^- J^2 \cdot \epsilon \) are
\[ K = -e^-, \]
\[ \Theta = e^{-9}, \]
\[ \Sigma = e^- \wedge \left( \frac{1}{2} J^3 \wedge J^3 - \text{Re} \Omega^3 \right). \] (3.47)

This is the spinorial realisation of the frame bundle of relevance to the geometric duals of \( N = (2, 1) \) CFTs. On a special holonomy manifold, the supersymmetric cycles calibrated by \( \Xi_2 \) are Kähler-4 with respect to \( J^1 \), and SLAG with respect to \(-\text{Re} \Omega^2\) and \(\text{Re} \Omega^3\).
Local $AdS_3$ structures In this case, picking out a local $AdS_3$ radial direction with a component on $\mathcal{M}_8$ reduces the structure group near an $AdS$ horizon to $SU(2)$. The local metric is given by

$$ds^2 = \frac{1}{\lambda m^2} ds^2(AdS_3) + e^5 \otimes e^5 + e^6 \otimes e^6 + e^7 \otimes e^7 + \hat{\rho} \otimes \hat{\rho} + ds^2(\mathcal{N}_4),$$

(3.48)

where the locally-defined $\mathcal{N}_4$ admits a local $SU(2)$ structure. For each of the three $Spin(7)$ structures which are collectively equivalent to a QK structure, $-\phi^A = \Upsilon^A + \Phi^A \wedge e^8$, we find the associated local $AdS_3$ structures

$$\Phi^1 = e^{567} + K^3 \wedge e^7 + K^2 \wedge e^6 - K^1 \wedge e^5,$$

$$\Upsilon^1 = \text{Vol}_{\mathcal{N}_4} + K^3 \wedge e^{56} - K^2 \wedge e^{57} - K^1 \wedge e^{67},$$

(3.49)

$$\Phi^2 = e^{567} - K^3 \wedge e^7 + K^2 \wedge e^6 + K^1 \wedge e^5,$$

$$\Upsilon^2 = \text{Vol}_{\mathcal{N}_4} - K^3 \wedge e^{56} - K^2 \wedge e^{57} - K^1 \wedge e^{67},$$

(3.50)

$$\Phi^3 = e^{567} + K^3 \wedge e^7 - K^2 \wedge e^6 - K^1 \wedge e^5,$$

$$\Upsilon^3 = \text{Vol}_{\mathcal{N}_4} + K^3 \wedge e^{56} + K^2 \wedge e^{57} + K^1 \wedge e^{67},$$

(3.51)

where the $K^A$ are the self-dual $SU(2)$ invariant two-forms on $\mathcal{N}_4$, given by

$$K^1 = e^{14} + e^{23},$$

$$K^2 = -e^{13} + e^{24},$$

$$K^3 = e^{12} + e^{34}.$$  

(3.52)

They satisfy the algebra $K^A K^B = -\delta^{AB} - e^{ABC} K^C$. For CLAG geometry, two of the local $G_2$ structures $\{\Phi^1, \Upsilon^1\}, \{\Phi^3, \Upsilon^3\}$, have exactly the same form as for QK geometry, while the third structure is now given by

$$\Phi^2 = -e^{567} + K^1 \wedge e^7 - K^2 \wedge e^6 - K^3 \wedge e^5,$$

$$\Upsilon^2 = -\text{Vol}_{\mathcal{N}_4} + K^1 \wedge e^{56} + K^2 \wedge e^{57} - K^3 \wedge e^{67}.$$  

(3.53)

4 Truncating eleven-dimensional supergravity

In this section, we will truncate eleven dimensional supergravity to the global frame bundles of interest to us. The different ways in which we do this are parameterised by the different spinorial realisations of the frame bundles we defined in the previous section.
The papers [25], [27]-[32] essentially provide a machine for doing this. The input is the global Minkowski frame, and in each case, the most general Killing spinors satisfying the appropriate global projection conditions. The output (with human intervention) is the most general BPS conditions in each case. These, coupled to the Bianchi identity and outstanding component of the field equations, define the truncation of eleven dimensional supergravity to the frame bundles.

Since they are qualitatively similar to one another (and qualitatively different to the other cases), and have also already received much attention, we will first briefly discuss the maximal structures, before moving on to the remaining cases.

4.1 Spin(7), SU(4) and Sp(2) geometry

The BPS conditions for the maximal structures may be obtained by a trivial restriction and globalisation of the local conditions of [30] (for Spin(7)) and [31] (for SU(4) and Sp(2)) to our global $\mathbb{R}^{1,1}$ frame. To state them, it is convenient to make some frame redefinitions (for this subsection only) so let us define $L = H^{2/3}$, $e^9 = H^{-1/3}\hat{e}^9$, and conformally rescale the frame on $\mathcal{M}_8$ so that the metric becomes

$$ds^2 = H^{-2/3}[ds^2(\mathbb{R}^{1,1}) + \hat{e}^9 \otimes \hat{e}^9] + H^{1/3}d\tilde{s}^2(\mathcal{M}_8).$$

The Killing spinors for Spin(7) geometry are given by

$$\epsilon, \quad H^{-1/3}\Gamma^-\epsilon.$$ (4.2)

The torsion conditions in this case may be succinctly summarised by saying that $\hat{e}^9$ is Killing, $d\hat{e}^9$ is a two-form on $\mathcal{M}_8$ in the 21 of Spin(7), and $d\tilde{s}^2(\mathcal{M}_8)$ is globally a metric of Spin(7) holonomy. The flux is given by

$$F = d(e^{+\cdot-9}) + F^{27},$$ (4.3)

where $F^{27}$ is a four-form on $\mathcal{M}_8$ in the 27 of Spin(7). The Bianchi identity and field equation reduce to

$$dF^{27} = 0,$$ (4.4)

$$\nabla^2 H = -de^9 \wedge de^9 - \frac{1}{2}F^{27} \wedge F^{27},$$ (4.5)

where in the second equation all operations are defined in the conformally rescaled metric. For SU(4) geometry, $d\tilde{s}^2(\mathcal{M}_8)$ is globally restricted further to a metric of SU(4) holonomy, $de^9$ to the 15, and $F_{\text{mag}}$ to the 20. For Sp(2) geometry, $d\tilde{s}^2(\mathcal{M}_8)$ has global Sp(2) holonomy, $de^9$ belongs to the 10, and $F_{\text{mag}}$ to the 14. The most general
AdS$_3$ horizons in these truncations are the Freund-Rubin solutions, the direct products $AdS_4 \times \mathcal{M}_7$. In $Spin(7)$ geometry, $\mathcal{M}_7$ has weak $G_2$ holonomy; for $SU(4)$ geometry, $\mathcal{M}_7$ is Sasaki-Einstein; and for $Sp(2)$ geometry, it is Tri-Sasaki-Einstein. These geometries were discussed in detail in [34]. We have nothing else to say about maximal structures, and will henceforth focus on the remaining cases, where the global frame bundles are realised by half the number of Killing spinors as the maximal structures.

4.2 Cayley geometry

Cayley geometry is given by a globally defined $Spin(7)$ frame bundle realised by a single globally defined global null Killing spinor. In [25], the most general local BPS conditions implied by the existence of a single locally defined null Killing spinor were derived. We may thus obtain the truncation of eleven dimensional supergravity to Cayley geometry simply by restricting the conditions of [25] to a Minkowski frame and globalising them. The single null Killing spinor is $\epsilon$, and the torsion conditions and flux are as given in the introduction.

4.3 Kähler-4, SLAG, QK and CLAG geometry

Now we move to the remaining cases, where the derivation of the torsion conditions is considerably more involved. We have used a combination of techniques. The Kähler-4 and QK torsion conditions may be extracted, with considerable effort, by restricting and globalising the appropriate local classifications of [31]. We derive the SLAG and CLAG conditions from scratch, using the machinery of [32]. In the Kähler-4 case, the general solution for the Killing spinors, given our frame, is

$$\epsilon; \frac{1}{4}J \cdot \epsilon.$$ (4.6)

The Killing spinors have constant components in this spinorial basis. The same is true of the QK Killing spinors; in general they are

$$\epsilon; \frac{1}{4}J^1 \cdot \epsilon; \frac{1}{4}J^2 \cdot \epsilon.$$ (4.7)

For SLAG geometry, the general solution for the Killing spinors is

$$\epsilon; L^{-1/2} \frac{1}{4} \Gamma^- J \cdot \epsilon.$$ (4.8)

Finally for CLAG, the Killing spinors are

$$\epsilon; \frac{1}{4}J^1 \cdot \epsilon; L^{-1/2} \frac{1}{4} \Gamma^- J^2 \cdot \epsilon.$$ (4.9)
In all cases, a useful consistency check on our torsion conditions and expressions for the flux is provided by the generalised calibration conditions of [27]:

\[
\begin{align*}
    dK &= \frac{2}{3} \Theta \mathcal{J} F + \frac{1}{3} \Sigma \mathcal{J} \star F, \\
    d\Theta &= K \mathcal{J} F, \\
    d\Sigma &= K \mathcal{J} \star F - \Theta \wedge F.
\end{align*}
\]

(4.10)

These are conditions on the exterior derivatives of the bilinears of the Killing spinors, in eleven dimensions. For any of the Killing spinors we look at, with a flux of the form of (1.2), these are equivalent to

\[
\begin{align*}
    H &= L d(L^{-1} e^9), \\
    d \log L &= \frac{2}{3} L e^9 \mathcal{J} d(L^{-1} e^9) + \frac{1}{3} \phi \mathcal{J} \star_9 G, \\
    L d(L^{-1} \phi) &= - \star_9 G + e^9 \wedge G.
\end{align*}
\]

(4.11)

The particular choice of \( \phi \) depends on the particular choice of Killing spinor, and is as given in the previous section. A module of the flux which is in fact fixed by supersymmetry drops out of these equations, so they are not sufficient conditions for supersymmetry in general. However, for the modules which they contain, they provide us with a useful consistency check. The results we obtain are as stated in section two. The details of the calculations are uninstructive, how to do them is explained in [31], [32], and so we have suppressed them.

5 The \( AdS_3 \) geometries

In this section, we will explain in more detail how we derive the conditions on the geometry of \( AdS \) boundary regions of solutions of Cayley, Kähler-4, SLAG, QK and CLAG geometry. By boundary, we mean either event horizon or conformal boundary. For Cayley geometry, we derive the conditions by inserting the local \( AdS_3 \) frame of 3.1 into the torsion conditions for the global truncation, and also by demanding that the flux at an \( AdS \) boundary respects the \( AdS \) isometries. This requirement on the flux means that it takes the form

\[
F = \text{Vol}_{AdS_3} \wedge g + F_{\text{mag}},
\]

(5.1) with

\[
\partial_a g = \partial_a F_{\text{mag}} = \dot{e}^a \mathcal{J} F_{\text{mag}} = 0,
\]

(5.2)
where $\hat{e}^a$ are the AdS basis one-forms. As explained in section three, the global structure group is reduced locally at an AdS boundary. We define the AdS geometry by the conditions on the intrinsic torsion of the locally defined structure, together with the flux, in each case.

An interesting question is that of the global structure of a manifold with global AdS$_3$ isometry. Recall the frame rotation of 3.1:

\begin{align}
\hat{r} &= \sin \theta e^8 + \cos \theta e^9, \\
\hat{\rho} &= \cos \theta e^8 - \sin \theta e^9,
\end{align}

with $\hat{r} = \lambda^{-1/2} dr$ and $L = e^{2mr} \lambda$. In the generic case of a Cayley bundle, provided that $\theta \neq 0$ globally, this frame rotation reduces the global $Spin(7)$ structure associated to the $\mathbb{R}^{1,1}$ isometry to a global $G_2$ structure associated to the AdS$_3$ isometry. Thus for a manifold with global AdS$_3$ isometries, the $G_2$ structure will only fail to be globally defined if there exist points where $\theta = 0$. It may be readily verified that the torsion conditions and flux for a Cayley bundle imply that $\theta = 0$ in an open neighbourhood is inconsistent with AdS$_3$ isometry of the neighbourhood - an AdS$_3$ frame and flux with $\hat{r} = e^9$ does not solve the supergravity equations for a Cayley frame bundle. A much more subtle issue - which we have not attempted to resolve - is what happens at isolated points of a global AdS manifold where $\theta = 0$, and what the existence of such points implies for the geometry or topology.

In the remainder of this section, we will first discuss how, by imposing AdS$_3$ isometry on the electric flux (which is universally given by $F_{\text{elec}} = d(e^{+9})$ in all cases we study) one may introduce local coordinates for the local AdS frame. We will then show how the general necessary and sufficient minimally supersymmetric AdS$_3$ conditions of Martelli and Sparks may be derived by imposing an AdS$_3$ boundary condition on an arbitrary solution of Cayley geometry. Finally we will derive the AdS boundary geometry of a solution with a Kähler-4, SLAG, QK or CLAG bundle.

### 5.1 Coordinates for the AdS frame

Imposing AdS$_3$ isometries on the electric flux, we demand that

\[ Ld(L^{-1}e^9) = \hat{r} \wedge g, \]

with $g$ independent of $r$. Then using (5.3), we get

\[ e^{2mr} \partial_r (e^{-2mr} \sin \theta) \lambda^{1/2} \hat{\rho} + \lambda^{3/2} \tilde{d}(\lambda^{-3/2} \cos \theta) = -g, \]
\[ \tilde{d}(\lambda^{-1} \sin \theta \hat{\rho}) = 0, \]
where $\tilde{d}$ denotes the exterior derivative on the space transverse to the $AdS$ factor. Since $g$, $\hat{\rho}$ and $\lambda$ are independent of $r$, the first of these equations implies that the rotation angle $\theta$ is also independent of $r$. Then the second equation implies that locally there exists a coordinate $\rho$ such that

$$\hat{\rho} = \frac{\lambda}{2m \sin \theta} d\rho. \quad (5.6)$$

Therefore in general, we may write the metric near an $AdS_3$ boundary as

$$\text{d}s^2 = \frac{1}{\lambda m^2} \left[ \text{d}s^2(AdS_3) + \frac{\lambda^3}{4\sin^2 \theta} d\rho \otimes d\rho \right] + \text{d}s^2(N_7), \quad (5.7)$$

where $N_7$ is defined by

$$\text{d}s^2(M_8) = \text{d}s^2(N_7) + e^8 \otimes e^8. \quad (5.8)$$

There is a special case in which we can do more, and integrate the frame rotation completely. In general, in the Minkowski frame, there exists a coordinate $z$ such that the one-form $e^9$ is given by

$$e^9 = C(dz + \sigma), \quad (5.9)$$

for a function $C$ and a one-form $\sigma$ on $M_8$ which are independent of the Minkowski coordinates. When $\sigma = 0$ (equivalently, when $e^9 \wedge de^9 = 0$) both the Minkowski and associated $AdS$ BPS conditions simplify considerably, and we can integrate the frame rotation.

To see how to do this, we first solve for $dz$ in (5.3), and then take the exterior derivative. We get

$$d(\lambda^{-1/2} C^{-1} \cos \theta) \wedge dr - \frac{1}{2m} d(\lambda C^{-1}) \wedge d\rho = 0. \quad (5.10)$$

We may immediately deduce that $C = \hat{C}(r, \rho) \lambda$. Then

$$d(\lambda^{-3/2} \hat{C}^{-1} \cos \theta) \wedge dr = \frac{1}{2m} \partial_r \hat{C}^{-1} dr \wedge d\rho \quad (5.11)$$

implies that

$$\lambda^{-3/2} \cos \theta = f(\rho), \quad (5.12)$$

for some arbitrary function $f$. Therefore, when $\sigma = 0$ in the Minkowski frame, we may write the metric in the $AdS$ frame as

$$\text{d}s^2 = \frac{1}{\lambda m^2} \left[ \text{d}s^2(AdS_3) + \frac{\lambda^3}{4(1 - \lambda^3 f^2)} d\rho \otimes d\rho \right] + \text{d}s^2(N_7). \quad (5.13)$$
The electric flux is then given by
\[ g = (1 - \partial_\rho f)\lambda^{3/2}d\rho. \] (5.14)

It is instructive to compare this expression with the results of [9], where the supersymmetry conditions for \( AdS \) boundaries with global Minkowski frames, frame bundles with structure group contained in \( G_2 \), and purely magnetic fluxes, were derived. For the \( AdS \) geometries, an expression for the metric of the form of (5.13) was found, in every case with \( f = \rho \). In the present context, we see that when \( \sigma = 0 \), \( f \) essentially sources the electric flux, and that as in [9], \( f = \rho \) implies that the fluxes are purely magnetic. To conclude this subsection, we record the expression for \( e^8 \) in the \( AdS \) frame, when \( \sigma = 0 \) in the Minkowski frame:
\[ e^8 = \lambda^{-1/2} \sqrt{1 - \lambda^3 f^2}dr + \frac{\lambda^{5/2}f}{2m\sqrt{1 - \lambda^3 f^2}}d\rho. \] (5.15)

5.2 \( AdS \) boundaries in Cayley geometry

In this subsection, we will derive the minimal \( AdS_3 \) BPS conditions quoted in the introduction by imposing an \( AdS_3 \) boundary condition on Cayley geometry. First we will derive the \( AdS \) torsion conditions, and then the relationship between the flux and the torsion in the \( AdS \) limit. We start with the torsion conditions
\[ e^9 \wedge \left[ -L^3 e^9 \wedge d(L^{-3}e^9) + \frac{1}{2} \phi \wedge d\phi \right] = 0, \] (5.16)
\[ (e^9 \wedge +*g)[e^9 \wedge d(L^{-1}\phi)] = 0. \] (5.17)

One of the reasons why we have written the torsion conditions in this coordinate independent form is that it makes it much easier to perform the frame rotation. Now we do this, using
\[ \phi = -\Upsilon - \Phi \wedge e^8, \]
\[ e^8 = \sin \theta \dot{r} + \cos \theta \dot{\rho}, \]
\[ e^9 = \cos \theta \dot{r} - \sin \theta \dot{\rho}, \] (5.18)

to evaluate (5.16) and (5.17) in the \( \dot{r}, \dot{\rho} \) frame. We have seen that \( \theta \) must be independent of the \( AdS \) radial coordinate, and we demand that the only \( r \) dependence in the rotated frame enters in the warping of the \( \mathbb{R}^{1,1} \) factor. We split all exterior derivative as \( d = \dot{r} \wedge \partial_r + \dot{\rho} \wedge \partial_\rho + d_7 \). Separating out the \( \dot{r} \wedge \dot{\rho} \) and \( \cos \theta \dot{r} - \sin \theta \dot{\rho} \) components, (5.16)
contains the two independent equations

\[
\frac{1}{2} \Upsilon_{,d_7} \Phi + \frac{1}{2} \cos \theta \Upsilon_{,d_\rho} \Upsilon + \lambda^{7/2} \partial_\rho (\lambda^{-7/2} \cos \theta) - 6 m \lambda^{1/2} \sin \theta = 0, \tag{5.19}
\]

\[
\Upsilon_{,d_7} \Upsilon - \frac{1}{2} \cos \theta \Phi_{,d_\rho} \Upsilon - 3 d_7 \log \lambda + \cos^2 \theta \partial_\hat{\rho} \left( \frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta} \right) = 0, \tag{5.20}
\]

where in deriving the second of these equations we have used the $G_2$ identities $\Phi_{,d_7} \Phi = -\Upsilon_{,d_7} \Upsilon$, $\Phi_{,d_\rho} (A \wedge \Phi) = -4 \Phi_{,d_\rho}$. Applying the same procedure to (5.17), we find the single condition

\[
0 = 4 m \lambda^{1/2} \cos \theta \Upsilon + \lambda^{3/2} \star_7 \partial_\rho (\lambda^{-3/2} \sin \theta \Phi) + \lambda \sin \theta \partial_\rho (\lambda^{-1} \Upsilon)
\]

\[
+ \sin \theta \cos \theta \partial_\hat{\rho} \log \left( \frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta} \right) \wedge \Phi. \tag{5.21}
\]

To proceed, we decompose (5.19), (5.20) and (5.21) into modules of $G_2$, to extract out the independent conditions. We will then show that these can be repackaged in the form of [26]. First consider (5.21). This is an equation for four-forms of $G_2$, and hence a priori contains $1, 7$ and $27$ parts. To treat the $\partial_\rho$ terms, it is useful to introduce

\[
Q_{ij} = \delta_{ik} (\partial_\rho e^k)_j, \tag{5.22}
\]

where indices run from 1 to 7. Since we have chosen the frame so that $\Phi$ and $\Upsilon$ have constant components, we may then write

\[
(\partial_\rho \Phi)_{i_1 i_2 i_3} = 3 \Phi_{k[i_1 i_2} Q_{i_3]}^k,
\]

\[
(\partial_\rho \Upsilon)_{i_1 i_2 i_3 i_4} = -4 \Upsilon_{k[i_1 i_2 i_3} Q_{i_4]}^k. \tag{5.23}
\]

Since $Q$ is an a priori arbitrary rank 2 tensor of $G_2$, it contains $1, 7, 14$ and $27$ parts, and encodes the intrinsic torsion modules of the eight-dimensional $G_2$ structure contained in the $\hat{\rho}$ derivatives of $\Phi$ and $\Upsilon$. Acting on these $G_2$ invariant forms, the $14$ part of $Q$ drops out of (5.23). We can separate out the remaining parts of $Q$ according to

\[
Q_{ij} = \frac{1}{i} \gamma \delta_{ij} + \Phi_{ijk} \beta^k + Q_{ij}^{27}, \tag{5.24}
\]

where $Q_{ij}^{27}$ is a symmetric traceless tensor. Now we insert this expression for $Q_{ij}$, together with (5.23), into (5.21). The $Q_{ij}^{27}$ drops out; this is most easily seen by choosing any particular element of the $27$ and verifying that its contribution to (5.21) vanishes. The remaining terms are given by

\[
0 = \left[ 4 m \lambda^{1/2} \cos \theta + \lambda^{5/2} \partial_\rho (\lambda^{-5/2} \sin \theta) + \sin \theta \gamma \right] \Upsilon
\]

\[
+ \left[ \sin \theta \cos \theta \partial_\hat{\rho} \log \left( \frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta} \right) - 6 \sin \theta \beta \right] \wedge \Phi. \tag{5.25}
\]
where in evaluating the 7 terms we have used the $G_2$ identities given in the appendix of [35]. Therefore we must have

\[
\gamma = -4m\lambda^{1/2}\frac{\cos \theta}{\sin \theta} - \partial_\rho \log(\lambda^{-5/2} \sin \theta), \quad (5.26)
\]

\[
\beta = \frac{1}{6} \cos \theta \partial_7 \log \left(\frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta}\right). \quad (5.27)
\]

This exhausts the content of (5.21). Now, using our expression for $\gamma$, (5.19) gives the singlet part of $d_7 \Phi$. We find

\[
\Upsilon \cdot d_7 \Phi = \frac{4m\lambda^{1/2}}{\sin \theta} (4 - \sin^2 \theta) - 2 \cos \theta \partial_\rho \log \left(\frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta}\right). \quad (5.28)
\]

Next, using our expression for $\beta$ in (5.20), we obtain

\[
\Upsilon \cdot d_7 \Upsilon = 3d_7 \log \lambda. \quad (5.29)
\]

This exhausts all the torsion conditions. Finally, it may be verified that the four conditions (5.26)-(5.29) are equivalent to

\[
\hat{\rho} \wedge d(\lambda^{-1} \Upsilon)^7 = 0, \quad (5.30)
\]

\[
\lambda^{5/2} d \left(\lambda^{-5/2} \sin \theta \text{Vol}_7\right) = -4m\lambda^{1/2} \cos \theta \hat{\rho} \wedge \text{Vol}_7, \quad (5.31)
\]

\[
d\Phi \wedge \Phi = \frac{4m\lambda^{1/2}}{\sin \theta} (4 - \sin^2 \theta) \text{Vol}_7 - 2 \cos \theta \ast_8 d \log \left(\frac{\lambda^{3/2} \cos \theta}{\sin^2 \theta}\right). \quad (5.32)
\]

Equation (5.30) is equivalent to (5.29), (5.31) to (5.26), and (5.32) is equivalent to (5.27) and (5.28).

Next we must impose the AdS boundary condition on the magnetic flux, which in the Minkowski frame is

\[
F_{\text{mag}} = -\ast d(e^+ \wedge \phi) - \frac{L^{10/7}}{2} e^9 \wedge d(L^{-10/7} \phi) + \frac{1}{4} \phi \circ [e^9 \wedge (e^9 \wedge de^9)] + F^{27}. \quad (5.33)
\]

We define

\[
F^{27} = G^{27} + H^{27} \wedge e^8; \quad (5.34)
\]

eight-dimensional self-duality of $F^{27}$ then implies that $G^{27} = \ast_7 H^{27}$ (recall that the 27 of $\text{Spin}(7)$ is irreducible under $G_2$). Now we perform the frame rotation, and impose vanishing of the components of $F_{\text{mag}}$ along the AdS radial direction. We find the
conditions
\[ \hat{\rho} \wedge d(\lambda^{-1} \Upsilon) = 0, \]  
\[ H^{27} = \frac{1}{\sin \theta} \star_7 \lambda \partial_\mu (\lambda^{-1} \Upsilon) + \star_7 d_7 \left( \frac{\cos \theta}{\sin \theta} \Phi \right) + \frac{10m}{7} \lambda^{1/2} \cos \theta \Phi \]
\[ + \frac{1}{2} \lambda^{27/14} \partial_\mu (\lambda^{-27/14} \sin \theta \Phi) + \frac{1}{4} \sin \theta \cos \theta d_7 \log(\lambda^{-3/2} \cos \theta) \wedge \Upsilon. \]
\[ (5.35) \]

The first of these equations implies (5.30), and together with (5.31) and (5.32), comprises the torsion conditions given in the introduction. The left-hand side of (5.36) only contains a term in the $27$ of $G_2$, and hence the $1$ and $7$ parts of the right-hand side must vanish. It may be verified that they do, using (5.30), (5.31) and (5.32). Using (5.36), the magnetic flux may be expressed as
\[ F_{\text{mag}} = \hat{\rho} \wedge \left[ -\frac{\cos \theta}{\sin \theta} \star_7 \lambda \partial_\mu (\lambda^{-1} \Upsilon) - \cos \theta \star_7 d_7 \left( \frac{\cos \theta}{\sin \theta} \Phi \right) \right] + 2m \lambda^{1/2} \Phi - \star_7 \lambda^{3/2} d_7 (\lambda^{-3/2} \sin \theta \Phi) \]
\[ + \left[ \star_7 H^{27} - \frac{24m}{7} \lambda^{1/2} \cos \theta \Upsilon - \star_7 \lambda^{3/2} \partial_\mu (\lambda^{-3/2} \sin \theta \Phi) \right] - \frac{1}{2} \sin \theta \lambda^{10/7} \partial_\mu (\lambda^{-10/7} \Upsilon) + \frac{1}{4} \sin \theta \cos \theta d_7 \log \left( \frac{\sin^4 \theta}{\lambda^{9/2} \cos \theta} \right) \wedge \Phi. \]
\[ (5.36) \]

After some manipulation, this expression may be shown to be equivalent to
\[ F_{\text{mag}} = \frac{\lambda^{3/2}}{\sin^2 \theta} \left( \cos \theta + \star_8 \left( d(\lambda^{-3/2} \sin \theta \Phi) - 4m \lambda^{-1} \Upsilon \right) + 2m \lambda^{1/2} \Phi \wedge \hat{\rho}, \right) \]
\[ (5.37) \]

which is in turn equivalent to
\[ \lambda^{3/2} d(\lambda^{-3/2} \sin \theta \Phi) = \star_8 F_{\text{mag}} - \cos \theta F_{\text{mag}} + 2m \lambda^{1/2} (\Upsilon + \cos \theta \Phi \wedge \hat{\rho}). \]
\[ (5.38) \]

This exhausts all conditions.

5.3 AdS boundaries in Kähler-4 and SLAG geometry

There are two ways in which we can derive the BPS conditions for an AdS boundary in Kähler-4 or SLAG geometry. The first is to impose the frame rotation on the appropriate global truncation of eleven dimensional supergravity, just as for Cayley geometry. The second and technically simpler way is to use the local AdS structures of section 3. This is what we have done. An AdS region in Kähler-4 geometry admits a pair of local $G_2$ structures, which are equivalent to a local $SU(3)$ structure, according to\[
\Phi_\pm = J_{SU(3)} \wedge e^7 \mp \Im \Omega_{SU(3)}, \]
\[ \Upsilon_\pm = \frac{1}{2} J_{SU(3)} \wedge J_{SU(3)} \pm \Re \Omega_{SU(3)} \wedge e^7. \]
\[ (5.40) \]
Both these \(G_2\) structures must satisfy the local Cayley \(AdS_3\) conditions. Similarly for SLAG geometry, where the local \(G_2\) structures of an \(AdS\) region are

\[
\Phi_\pm = \pm J_{SU(3)} \wedge e^7 - \Im \Omega_{SU(3)},
\]

\[
\Upsilon_\pm = \pm \frac{1}{2} J_{SU(3)} \wedge J_{SU(3)} + \Re \Omega_{SU(3)} \wedge e^7,
\]  

(5.41)

In each case, the local \(AdS\) metric is

\[
ds^2 = \frac{1}{\lambda m^2} \left[ ds^2(AdS_3) + \frac{\lambda^3}{4 \sin^2 \theta} d\rho \otimes d\rho \right] + e^7 \otimes e^7 + ds^2(N_6).
\]  

(5.42)

For the remainder of this subsection it is understood that structure forms are of \(SU(3)\), and we will suppress their subscripts. This doubling of the structures provides a very convenient way of arriving at the BPS conditions in each case. For example, the Cayley condition \(\hat{\rho} \wedge d(\lambda^{-1} \Upsilon) = 0\) decomposes for both Kähler-4 and SLAG into the pair of equations

\[
\hat{\rho} \wedge d(\lambda^{-1} J \wedge J) = 0,
\]

\[
\hat{\rho} \wedge d(\lambda^{-1} \Re \Omega \wedge e^7) = 0.
\]  

(5.43)

Similarly, decomposing (5.31) and (5.32) leads to identical equations for \(AdS_3\) regions in both Kähler-4 and SLAG geometries. What distinguishes these geometries is the decomposition of the flux. Requiring a magnetic flux of the form (5.38) for both the local \(G_2\) structures in Kähler-4 geometry, we find the conditions

\[
F_{mag} = \frac{\lambda^{3/2}}{\sin^2 \theta} (\cos \theta + *_8)(d[\lambda^{-3/2} \sin \theta J \wedge e^7] - 2m \lambda^{-1} J \wedge J) + 2m \lambda^{1/2} J \wedge e^7 \wedge \hat{\rho},
\]

(5.44)

\[
0 = -\frac{\lambda^{3/2}}{\sin^2 \theta} (\cos \theta + *_8)(d[\lambda^{-3/2} \sin \theta \Im \Omega] + 4m \lambda^{-1} \Re \Omega \wedge e^7) - 2m \lambda^{1/2} \Im \Omega \wedge \hat{\rho}.
\]  

(5.45)

For the local \(AdS\) structures in SLAG geometry, we instead get

\[
0 = \frac{\lambda^{3/2}}{\sin^2 \theta} (\cos \theta + *_8)(d[\lambda^{-3/2} \sin \theta J \wedge e^7] - 2m \lambda^{-1} J \wedge J) + 2m \lambda^{1/2} J \wedge e^7 \wedge \hat{\rho},
\]  

(5.46)

\[
F_{mag} = -\frac{\lambda^{3/2}}{\sin^2 \theta} (\cos \theta + *_8)(d[\lambda^{-3/2} \sin \theta \Im \Omega] + 4m \lambda^{-1} \Re \Omega \wedge e^7) - 2m \lambda^{1/2} \Im \Omega \wedge \hat{\rho}.
\]  

(5.47)

It is this formal difference in the decomposition of the flux that endows \(AdS\) regions in the two geometries with such different properties.
In each case, not all the equations obtained by performing this decomposition of the Cayley AdS conditions are independent. We have reduced them to a minimal set of necessary and sufficient independent conditions, which are quoted in section 2.

5.4 AdS boundaries in QK and CLAG geometry

To derive the BPS conditions for an AdS boundary in QK or CLAG geometry, we may either impose the frame rotation on the torsion conditions of section 4, or (which is again technically more convenient) we can use the local AdS$_3$ structures of section 3 to further decompose the torsion conditions for an AdS region in Kähler-4 or SLAG geometry. The derivation proceeds in a very similar way to that of the derivation of the geometry of AdS regions in Kähler-4 or SLAG from Cayley, and we have suppressed the details. In QK and CLAG geometry, the local AdS metric is

$$\frac{ds^2}{\lambda m^2} = \left[ ds^2(AdS_3) + \frac{\lambda^3}{4 \sin^2 \theta} d\rho \otimes d\rho \right] + e^5 \otimes e^5 + e^6 \otimes e^6 + e^7 \otimes e^7 + ds^2(N_4).$$

(5.48)

Relabelling $(e^5, e^6, e^7) \rightarrow (e^1, e^2, e^3)$, from section 3, we find that an AdS region in QK or CLAG geometry admits three local $SU(3)$ structures; in terms of the $SU(2)$ forms $K^A$ on $N_4$, these are given by

$$e^A = e^7,$$

$$J^A = K^A + \frac{1}{2} \epsilon^{ABC} e^B \wedge e^C,$$

$$\text{Re}\Omega^1 = K^3 \wedge e^2 + K^2 \wedge e^3,$$

$$\text{Im}\Omega^1 = K^3 \wedge e^3 - K^2 \wedge e^2,$$

(5.49) (5.50) (5.51) (5.52)

together with permutations of the last two equations. In QK geometry, each of these local structures must individually satisfy the conditions for a local AdS$_3$ SU(3) structure in Kähler-4 geometry. In CLAG geometry, the structure forms $e^3$, $J^3$ and $\Omega^3$ must together satisfy the AdS conditions in Kähler-4 geometry, while the $e^A$, $J^A$, $\Omega^A$, $A = 1, 2$, must satisfy the AdS conditions in SLAG geometry. In each case, reducing these conditions to a minimal necessary and sufficient independent set, we get the results quoted in section 2.

6 Explicit solutions

In this section we show that the known solutions with AdS$_3$ factors fit nicely in the framework developed in the previous sections. To do so we consider solutions describing
Table 1: Known solutions of wrapped M5-branes on 4-cycles in a CY$_4$.

|                  | $p$ | $q$ | $a_1$  | $a_2$  | $e^{10\Lambda}$ | $c_1$ | $c_2$ |
|------------------|-----|-----|--------|--------|------------------|-------|-------|
| SLAG 4-cycle in CY$_4$ | 4   | 1   | $e^{4\Lambda}$ | $e^{-6\Lambda}$ | $\frac{2}{3}$ | 1     | $\frac{3}{2}$ |
| Kähler 4-cycle in CY$_4$ | 4   | 1   | $e^{4\Lambda}$ | $e^{-6\Lambda}$ | $\frac{2}{3}$ | 1     | $\frac{3}{2}$ |

the near-horizon limit of M5-branes wrapping SLAG 4-cycles and Kähler 4-cycles in a CY$_4$. These were first found in [4] in seven-dimensional gauged supergravity. Throughout this section we will follow the notation and conventions of [9].

For the known solutions, the eleven-dimensional metric can be put into the form

$$m^2 ds^2 = \Delta^{-2/5} \left[ \frac{a_1}{u^2} ds^2(\mathbb{R}^{1,5-d}) + a_2 ds^2(\Sigma_d) \right] + \Delta^{4/5} \left[ e^{2q\Lambda} u^{2c_1} DX^a DX^a + e^{-2p\Lambda} u^{2c_2} dX^\alpha dX^\alpha \right],$$  

(6.1)

with $a = 1\ldots p$ and $\alpha = 1\ldots q$ and $p + q = 5$. The constants $a_1$ and $a_2$ specify the relative size of the AdS factor and the $d$-cycle $\Sigma_d$. For the cases corresponding to M5-branes wrapping 4-cycles, we have $d = 4$, $p = 4$ and $q = 1$. The relevant values of the remaining constants for the two cases that we will be discussing are summarized in Table 1. Following [9], we have defined

$$DX^a = dX^a + B^a_b X^b$$  

(6.2)

where $B^a_b$ is determined by the spin connection on $\Sigma_d$. Comparing this form of the metric with (1.1), we identify

$$L = \frac{\Delta^{2/5} u^2}{a_1}, \quad C = \frac{\Delta^{2/5} u^{c_2}}{a_1},$$  

(6.3)

where $e^\theta = C dz$ for the case at hand. Introducing new coordinates

$$X^a = u^{-c_1} \cos \tau Y^a, \quad X^\alpha = u^{-c_2} \sin \tau Y^\alpha, \quad c_1 = e^{-2q\Lambda} \sqrt{a_1}, \quad c_2 = e^{2p\Lambda} \sqrt{a_2},$$  

(6.4)

where $Y^a$ and $Y^\alpha$ parametrise an $(p-1)$-sphere and an $(q-1)$-sphere respectively, the metric becomes

$$m^2 ds^2 = \Delta^{-2/5} \left[ \frac{a_1}{u^2} ds^2(\mathbb{R}^{1,5-d}) + du^2 + a_2 ds^2(\Sigma_d) + e^{2(q-p)\Lambda} d\tau^2 \right] + \Delta^{4/5} \left[ e^{2q\Lambda} \cos \tau^2 dY^a dY^a + e^{-2p\Lambda} \sin \tau^2 dY^\alpha dY^\alpha \right],$$  

(6.5)

From the definition of $\Delta$ in [4], we have that

$$(a_1 e^{\lambda})^{-3} = \Delta^{-6/5} = e^{-2q\Lambda} \cos \tau + e^{2p\Lambda} \sin \tau.$$  

(6.6)

This form of the metric corrects some errors in eq. 9.6 of [9].
Specialising to our cases of interest \( p = 4 \) and \( q = 1 \) from now on, the metric can be put into the generic form (5.13) by introducing a new coordinate \( \rho \) as follows. Define

\[
f(\rho) = a_1 c_2 e^{-p \Lambda} \sin \tau ,
\]

where \( f(\rho) \) is the same function as in section 5.1. Then, the metric becomes

\[
ds^2 = \frac{1}{\lambda m^2} \left[ ds^2(AdS_3) + \frac{\lambda^3}{4(1 - \lambda^3 f^2)} d\rho \otimes d\rho + \frac{a_2}{a_1} ds^2(\Sigma_4) + \frac{1}{c_1} (1 - \lambda^3 f^2) D Y^a \otimes D Y^a \right],
\]

where, in order to get the form (5.13), we have set

\[
f(\rho) = \frac{c_2}{2} \rho .
\]

From the analysis in section 5.1, we conclude that this solution carries electric flux, which is indeed the case for the solutions presented in [4].

To identify the \( AdS \) radial coordinate in \( \mathcal{M}_8 \), one defines a one-form [9]

\[
e^8 = \frac{\Delta^{2/5} e^{\Lambda}}{m} \left( c_1 \cos \tau \frac{du}{u} + \sin \tau d\tau \right)
= \lambda^{-1/2} \sqrt{1 - \lambda^3 f^2} dr + \frac{\lambda^{5/2} f}{2m \sqrt{1 - \lambda^3 f^2}} d\rho ,
\]

with \( u = e^{mr} \). This expression matches our previous one in (5.15).

Now we are ready to check that the solutions of [4] satisfy our equations. We discuss the SLAG-4 and the Kähler-4 cases separately.

**SLAG-4**

For the SLAG-4 case, \( B_{ab} = \tilde{\omega}^a_b \), where \( \tilde{\omega}^a_b \) is the spin connection on \( \Sigma_4 \), which is just the four-hyperboloid \( \mathcal{H}^4 \) of unit curvature. As described in [9], the \( SU(4) \) structure is given by

\[
J = e^a \wedge f^a ,
\]

\[
\Omega = \frac{1}{4!} e^{abcd}(e^a + if^a)(e^b + if^b)(e^c + if^c)(e^d + if^d) ,
\]

where \( e^a = \Delta^{-1/5} \sqrt{a_2 m^{-1}} \tilde{e}^a \) and \( f^a = \Delta^{2/5} e^{\Lambda} c_1 m^{-1} DX^a \). Here \( \{e^a\} \) denote a basis of 1-forms on \( \mathcal{H}^4 \). In the \( AdS \) limit, the \( SU(4) \) structure decomposes under \( SU(3) \) in the following way:

\[
J = e^a \wedge (\tilde{f}^a - Y^a e^8)
= J_{SU(3)} + e^7 \wedge e^8 ,
\]
where
\[ e^7 = -Y^a e^a , \quad \text{and} \quad \tilde{f}^a = \frac{\Delta^{2/5} e^{qA}}{m} \cos \tau Y^a . \] (6.14)

Similarly, the holomorphic 4-form \( \Omega \) decomposes as
\[ \Omega = (\text{Re} \Omega_{SU(3)} + i \text{Im} \Omega_{SU(3)}) \wedge (e^7 + ie^8) , \] (6.15)
from which we find
\[ \text{Re} \Omega_{SU(3)} = \frac{1}{3!} \epsilon^{abcd} Y^a e^{bcd} - \frac{1}{2} \epsilon^{abcd} \tilde{f}^{bc} \wedge e^d , \] (6.16)
\[ \text{Im} \Omega_{SU(3)} = -\frac{1}{3!} \epsilon^{abcd} \tilde{f}^{bcd} + \frac{1}{2} \epsilon^{abcd} \tilde{f}^b \wedge e^{cd} . \] (6.17)

Using these expressions, it is straightforward to show that the SL AG-4 solution of [4] satisfies (2.31)-(2.32). Showing that (2.33) also holds requires some more work. Consider first the LHS of (2.33). One should first note that
\[ D^2 Y^a = \frac{1}{2} \tilde{R}^a_{bcd} Y^b e^c \wedge e^d , \] (6.18)
where \( \tilde{R}^a_{bcd} \) is the Riemann tensor on \( \mathcal{H}^4 \) and hence \( \tilde{R}_{abcd} = 2k \delta_{[a} \delta_{d]} \), with \( k = -\frac{1}{3} \).

Then, we compute
\[ \tilde{d} \text{Im} \Omega = \frac{m \Delta^{-2/5} e^{-qA}}{\cos \tau} \frac{1}{2!} \epsilon^{abcd} \tilde{f}^{ab} e^{cd} + k \Delta^{6/5} e^{2qA} \cos^2 \tau Y^a e^b \left( \tilde{f}^{cd} - e^{cd} \right) e^7 , \] (6.19)
with \( q = 1 \) for SLAG-4 [9]. Taking the wedge product of the expression above with \( \text{Im} \Omega \) we obtain
\[ \text{Im} \Omega \wedge \tilde{d} \text{Im} \Omega = 6m \left( \frac{\Delta^{-2/5} e^{-qA}}{\cos \tau} + \frac{k \Delta^{4/5} e^{qA}}{a_2} \cos \tau \right) \text{Vol}_6 \wedge e^7 , \] (6.20)
where on a manifold with \( SU(3) \) structure, the volume 6-form may be defined by
\[ \text{Vol}_6 = \frac{1}{3!} J \wedge J \wedge J = - \left( e^{123} \tilde{f}^{123} + e^{124} \tilde{f}^{124} + e^{134} \tilde{f}^{134} + e^{234} \tilde{f}^{234} \right) . \] (6.21)

To put the result above into a more familiar form, we perform the following change of coordinates:
\[ \lambda = \frac{\Delta^{2/5}}{a_1} , \quad \sin \tau = \frac{e^{pA}}{a_1 c_2} f(\rho) , \quad \cos \tau = \frac{e^{qA}}{(a_1 \lambda)^{3/2}} \left( 1 - \lambda^2 f^2 \right)^{1/2} . \] (6.22)

Finally, introducing the values of \( a_1, a_2, c_1, c_2 \), for the SLAG-4 solutions of [4], and using that \( \cos \theta = \lambda^{3/2} f(\rho) \), (6.20) can be cast in the following form:
\[ \text{Im} \Omega \wedge \tilde{d} \text{Im} \Omega = \frac{3m \lambda^{1/2}}{\sin \theta} (1 + \cos^2 \theta) \text{ Vol}_6 \wedge e^7 . \] (6.23)
One can then show that this matches the RHS of (2.33). Furthermore, we have also checked that the Bianchi identity for the four-form field strength is satisfied. To do this, the following identities are useful:

\[ d \left[ \epsilon^{abcd} Y^a D Y^b \wedge D Y^c \wedge D Y^d \right] = -\epsilon^{abcd} Y^a \wedge D Y^b \wedge D Y^c \wedge e^d \wedge (Y^e e^e), \]

\[ d \left[ \epsilon^{abcd} Y^a \wedge e^b \wedge e^c \wedge e^d \right] = -2 \epsilon^{abcd} Y^a D Y^b \wedge e^c \wedge e^d \wedge (Y^e e^e), \]

\[ \epsilon^{abcd} D Y^a \wedge D Y^b \wedge e^c \wedge e^d = -2 \epsilon^{abcd} Y^a D Y^b \wedge e^c \wedge e^d \wedge (Y^e e^e). \] (6.24)

Kähler-4

For the Kähler-4 solutions presented in [4], one can take \( B^{12} = B^{34} \) with all other components vanishing. Then one has

\[ B^{12} + B^{34} = -\frac{1}{2} \omega_{ab} K^{3ab}, \] (6.25)

where \( K^3 \) is defined in (3.52). Making the following ansatz for the \( SU(4) \) structure,

\[ J = e^1 \wedge e^2 + e^3 \wedge e^4 + f^1 \wedge f^2 + f^3 \wedge f^4, \] (6.26)

\[ \Omega = (e^1 + ie^2)(e^3 + ie^4)(f^1 + if^2)(f^3 + if^4), \] (6.27)

we find, in the \( AdS \) limit, that the \( SU(3) \) structure is given by

\[ e^7 = K^{3ab} Y^a \bar{f}^b, \]

\[ J_{SU(3)} = e^1 \wedge e^2 + e^3 \wedge e^4 + \bar{f}^1 \wedge \bar{f}^2 + \bar{f}^3 \wedge \bar{f}^4, \]

\[ \text{Re} \Omega_{SU(3)} = -K^2 \wedge K^{1ab} Y^a \bar{f}^b - K^1 \wedge K_{ab}^2 Y^a \bar{f}^b, \]

\[ \text{Im} \Omega_{SU(3)} = -K^2 \wedge K_{ab}^3 Y^a \bar{f}^b + K^1 \wedge K_{ab}^3 Y^a \bar{f}^b. \] (6.28)

where the \( K^A, A = 1, 2, 3 \), have been defined in (3.52). These expressions are just the same as those in eq.(2.27). Now it is a straightforward exercise to check that our equations (2.19)-(2.21) are satisfied and so is the Bianchi identity.

7 Conclusions

In this paper, we have formalised a proposal for a universal feature of the global geometry of supergravity solutions of relevance to the supersymmetric \( AdS_3/CFT_2 \) correspondence in M-theory. A supergravity solution associated to a CFT - a region of
spacetime containing a local $AdS$ region - should admit a globally-defined $\mathbb{R}^{1,1}$ frame, and a global reduction of its frame bundle, to one with structure group contained in $Spin(7)$. From this starting assumption, we have seen how many individual features of AdS/CFT geometry may be assembled into a coherent overall picture. Probe brane kappa-symmetry projections arise from the global definition of the spinorial realisation of the frame bundle. Solutions with asymptotically vanishing flux automatically asymptote to special holonomy manifolds. The existence of a globally-defined frame bundle allows for the global truncation of the field equations of eleven-dimensional supergravity. The general necessary and sufficient conditions for minimally supersymmetric $AdS_3$ geometry in M-theory may be derived by imposing an $AdS_3$ boundary condition on the truncation of supergravity to a Cayley frame bundle. The same applies for $AdS_2$ with an $SU(5)$ frame bundle [10], $AdS_4$ (with magnetic fluxes) and a $G_2$ frame bundle; and $AdS_5$ with an $SU(3)$ frame bundle [9]. The minimal truncations, and associated $AdS$ conditions, may be refined by further reducing the structure group of the frame bundle and/or by demanding additional Killing spinor realisations. Freund-Rubin or gauged supergravity $AdS$ solutions satisfy the general equations for $AdS$ horizons in the appropriate geometries.

One of the original motivations for this work, and that of [9], [10], was to map out the supersymmetric $AdS$ landscape of M-theory. At this point, it worth summarising what has been achieved. The strategy in each of these papers is first to impose the existence of a global Minkowski frame bundle, realised by Killing spinors of a definite Minkowski chirality, and then to impose a general $AdS$ boundary condition on the global truncation of eleven-dimensional supergravity to the Minkowski frame bundle. Modulo quotients, this approach covers all supersymmetric $AdS$ spacetimes which may be obtained from solutions with globally defined Minkowski frame bundles, and definite chirality Minkowski Killing spinors. For $AdS_2$, with the exception of near-horizon limits of M5 branes wrapped on the direct product of a SLAG-3 and a Kähler-2 cycle in a manifold of $SU(3) \times SU(2)$ holonomy, and modulo some technical caveats, the results of [10] are complete. For $AdS_3$ with less than sixteen supersymmetries, we believe that the combined results of this paper and [9] are complete. We have certainly covered all cases which admit a wrapped brane interpretation, and in full generality. There exist half-BPS $AdS_3$ solutions of M-theory; we have not performed a general investigation of this interesting case here and we leave it for the future. For $AdS_4$ with electric fluxes, the Freund-Rubin solutions are exhaustive. For $AdS_4$ with purely magnetic fluxes, admitting a wrapped brane interpretation and modulo some technical caveats explained therein, the results of [36] and [9] are complete. The existence or otherwise of supersymmetric
AdS\textsubscript{4} solutions with dyonic fluxes is an open problem. For AdS\textsubscript{5} spacetimes admitting a wrapped brane interpretation, and again modulo some technical caveats detailed in [9], the results of [33] and [37] are complete. This, then, is the status of the classification; the most interesting cases that have not been covered are half-BPS AdS\textsubscript{3} and dyonic AdS\textsubscript{4}. If there exist any other AdS solutions of M-theory which are not covered by the classification, they will necessarily have very complicated and unusual geometry.

This global framework, and the results of the classification, open the way for many future applications. The most obvious is to use the geometrical insight provided by the AdS torsion conditions to construct new explicit AdS solutions. A more important question is the development of a theory of boundary conditions for solutions of the truncated supergravity equations. With our general AdS\textsubscript{3} boundary condition, we have taken a first step in this direction, but there are many other possibilities to be explored.

As we mentioned before, a class of boundary conditions which is particularly interesting mathematically is special holonomy spacelike asymptotics with vanishing fluxes, and spacelike AdS asymptotics associated to event horizons. To our knowledge, the only known solutions of this form are the elementary brane solutions, associated to interpolations from conical special holonomy manifolds to Freund-Rubin AdS horizons. These interpolating solutions are intimately associated to the resolution of singularities of the asymptotic special holonomy manifolds; and in these cases, the AdS/CFT correspondence provides a definition of how the singularities may be resolved quantum gravitationally. This has been made manifest in the work on the $Y^{p,q}$/quiver gauge theory correspondence in IIB. Open string theory on the Calabi-Yau - with Dirichlet boundary conditions restricting the open strings to the vicinity of the singularity - reduces at low energies to a conformally invariant quiver gauge theory. The AdS/CFT correspondence states that this field theory is dual to the low-energy limit of closed IIB strings on a large-volume $AdS_5 \times Y^{p,q}$ manifold. The field theory at weak 't Hooft coupling encodes the toric data of the Calabi-Yau singularity. It also encodes, at strong 't Hooft coupling, the Sasaki-Einstein data of the AdS manifold. This means that, at low energies, the physical content of open string theory near a conical Calabi-Yau singularity and closed string theory on a large-volume AdS blow-up of the singularity are contained in the same quantum field theory. Going from weak to strong 't Hooft coupling in the field theory gives a quantum definition of the singularity-resolving geometrical interpolation. It would be very interesting if other interpolating solutions, associated to the resolution of other types of special holonomy singularities, could be constructed. Extending the intuition obtained from conical interpolations, it seems likely that such solutions will be associated to the resolution of singularities of collapsing supersymmetric cycles of the
asymptotic special holonomy manifolds. Understanding how to construct such solutions - and how to associate a given $AdS$ near-horizon geometry to a special holonomy infinity, and vice versa - will require a detailed understanding of how to match the boundary data at each spacelike infinity. It will certainly require the use of more sophisticated geometrical techniques than those employed here. A complementary approach to finding explicit interpolating solutions would be to try to establish existence or obstruction theorems. An example of such a result - an obstruction theorem for Sasaki-Einstein metrics - has recently appeared in the context of conical interpolations in IIB [38]. It will be interesting to explore a more general extension of this kind of approach.

Another extension will be to consider more general boundary conditions for solutions of the global supergravity truncations - solutions with an $AdS$ region which asymptote to flux geometries, or solutions with multiple $AdS$ regions. Known solutions are likely to provide useful insights into the general form of boundary conditions one should impose in these cases.

Though we have focussed on its classical, geometrical limit throughout this paper, the AdS/CFT correspondence is of course a quantum phenomenon. We have a much less satisfactory understanding of the quantum aspects of AdS/CFT in M-theory than we do in IIB, and improving this situation is a major outstanding problem. Perhaps the most promising line of attack is to exploit the duality between M-theory and IIB, where things are much better understood. This can work in two directions. By imposing $T^2$ isometries on the eleven-dimensional supergravity truncations - or their $AdS$ limits - one could obtain the subset of M-theory solutions which can be reduced and T-dualised to IIB. This is how the $Y^{p,q}$ were discovered. Conversely, where a IIB AdS/CFT dual is known, if the geometry admits a lift to M-theory one could do so, in the hope of gaining an understanding of how the known quantum field theory encodes the eleven-dimensional geometrical data. Ultimately it might be possible to extend the intuition thus obtained to M-theory geometries not admitting a IIB reduction, though this will certainly require significant new insights.

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