Dual solutions of stagnation-point flow over an exponentially stretching/shrinking sheet in a porous medium with suction and velocity slip: A stability analysis

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Abstract. In this paper, we investigate the boundary layer flow near the stagnation-point region over an exponentially stretching/shrinking sheet with suction and velocity slip in a porous medium. The governing equations are first reduced to the ordinary differential equations with the help of similarity transformations. The obtained equations are then solved numerically using the bvp4c programme in MATLAB. It is observed that the ordinary differential equations have dual (first and second) solutions. The analysis of the results obtained reveals that as the porosity, suction and velocity slip parameters increase, the range in regards to the region of existence of dual solutions increases as well. Stability analysis is performed to determine the stability of the dual solutions obtained. The results indicate that the first solution is stable while the second solution is unstable.

1. Introduction

In the current boundary layer flow research, the study of the flows in porous media remains as an area of interest due to its various remarkable applications in different fields such as geothermal energy and oil recovery, material processing, fuel cell technologies as well as trickle bed chromatography [1]. Due to vast range of these applications, it has therefore attracted considerable attention of many researchers. Ishak et al. [2] studied the mixed convection stagnation-point flow on a vertical surface in a porous medium using the Darcy-Forchheimer model [3]. Pal and Mandal [4] discussed the combined effects of viscous dissipation, heat source/sink, chemical reaction and thermal radiation on the mixed convection heat and mass transfer stagnation-point flow over stretching/shrinking sheet in a porous medium. Recently, Abu Bakar et al. [5] examined the mixed convection flow in a porous medium saturated with a nanofluid along a vertical cylinder with suction and thermal radiation.

The aim of the present study is to extend the work of Bachok et al. [6], with consideration of the flow in a porous medium over an exponentially stretching/shrinking sheet with suction and velocity slip effects. The main result of the present study indicates the existence of dual (first and second) solutions. Thus, the stability of the dual solutions are determined by performing the stability analysis. Some tables and figures are presented and the effects of the porosity, suction and velocity slip parameter are discussed further.
2. Governing equations

This study considers a steady laminar boundary layer flow towards the stagnation-point over an exponentially stretching/shrinking sheet with suction and velocity slip embedded in a porous medium. We assume that the stretching/shrinking velocity is denoted by \( u_s(x) \), while \( u_r(x) \) stands for the free-stream velocity. Under these conditions, the two-dimensional governing equations of the present model are given as follows [2]:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} = u_r \frac{du}{dx} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{v}{K_i} (u_r - u), \tag{2}
\]

where \( u \) and \( v \) denote the velocity components along the \( x \)- and \( y \)-directions, respectively, \( v \) denotes the kinematic viscosity while \( K_i(x) \) refers to the permeability of the porous medium. The boundary conditions for the velocity are given by

\[
u = u(x) + N_i(x) \frac{\partial u}{\partial y}, \quad v = v_w(x) \quad \text{at} \quad y = 0,
\]

\[
u \to u_r \quad \text{as} \quad y \to \infty.
\]

Here \( N_i(x) \) denotes the velocity slip factor while \( v_w(x) \) denotes the mass flux velocity with \( v_w < 0 \) for suction. In order for the similarity solutions of the equations (1) and (2) along with the boundary conditions (3) to exist, \( u_r(x), u_w(x) \), \( v_w(x) \), \( K_i(x) \) and \( N_i(x) \) are respectively given by

\[
u = ae^{sL}, \quad u_r(x) = be^{sL}, \quad v_w(x) = -(av/2L)^{1/2} e^{sL} s, \quad K_i(x) = K_0 e^{-sL}, \quad N_i(x) = N_0 e^{-sL}, \tag{4}
\]

where \( a, b, K_0 \) and \( N_0 \) are constants. Furthermore \( s(>0) \) denotes the suction parameter. Now, we define the following similarity transformations [6]:

\[
u = ae^{sL} f' \eta, \quad v = -(av/2L)^{1/2} e^{sL} (f + \eta f'), \quad \eta = (a/2L)^{1/2} e^{sL} y, \tag{5}
\]

where \( \eta \) denotes the dimensionless similarity variable and a prime denotes differentiation with respect to \( \eta \). In view of these relations, equation (1) is identically satisfied while equation (2) reduces to

\[
u f'''' + ff' - 2f'^2 + 2 + K (1 - f') = 0, \tag{6}
\]

subject to the boundary conditions given by

\[
u f'(0) = s, \quad f'(0) = c + Nf''(0), \quad f'(\infty) = 1, \tag{7}
\]

where \( K = 2vL/aK_0 \) represents the porosity parameter, \( c = b/a \) denotes the stretching/shrinking parameter with \( c > 0 \) and \( c < 0 \) corresponding to stretching and shrinking sheet, respectively. In addition, \( N = N_0 (a/2L)^{1/2} \) represents the velocity slip parameter.

The physical quantities of interest is the skin friction coefficient \( C_f \) given by

\[
u = \frac{\tau_w}{\rho u_r^2} \tag{8}
\]

where wall shear stress \( \tau_w \) is given by

\[
u = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \tag{9}
\]

with \( \mu \) represents the dynamic viscosity. Substituting (5) into (8) and (9), yields

\[
u = 2 Re_s^{1/2} C_f = f''(0), \tag{10}
\]

where \( Re_s = u_r(x) L / \nu \) represents the local Reynolds number.
3. Stability analysis

For the intention of stability analysis (see Merkin [7]), the unsteady equation is considered. In this respect, equation (1) remains the same, while equation (2) is replaced by

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_e \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial y^2} + K_1 (u_e - u),
\]

(11)

where \( t \) is the time. Following Merkin [7] and Weidman et al. [8], a new similarity variable \( \tau \) is introduced. With the introduction of \( \tau \) and similarity transformation (5), we now apply the new similarity transformations given, as follows:

\[
u = a e^{\frac{v}{2L}} \frac{\partial f}{\partial \eta} (\eta, \tau), \quad v = -(av/2L)^{\frac{v}{2}} e^{\frac{v}{2L}} \left[f(\eta, \tau) + \eta \frac{\partial f}{\partial \eta} (\eta, \tau) + 2\tau \frac{\partial f}{\partial \tau} (\eta, \tau)\right],
\]

(12)

\[
\eta = (a/2vL)^{\frac{v}{2}} e^{\frac{v}{2L}} y, \quad \tau = (at/2L)e^{\frac{v}{2L}}.
\]

Using the new similarity transformations (12), equation (11) now can be written as

\[
\frac{\partial^2 f}{\partial \eta^2} + f \frac{\partial^2 f}{\partial \eta^2} - 2 \left( \frac{\partial f}{\partial \eta} \right)^2 + 2 + K \left(1 - \frac{\partial f}{\partial \eta}\right) + 2\tau \left( \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta \partial \tau} \right) - \frac{\partial^2 f}{\partial \eta \partial \tau} = 0,
\]

(13)

and the boundary conditions are as follows:

\[
f(0, \tau) = s, \quad \frac{\partial f}{\partial \eta}(0, \tau) = c + N \frac{\partial^2 f}{\partial \eta^2}(0, \tau), \quad \frac{\partial f}{\partial \eta}(\infty, \tau) = 1.
\]

(14)

Following the analysis proposed by Weidman et al. [8], in order to test the stability of solution \( f(\eta) = f_0(\eta) \) which satisfies the equation (6) subject to the boundary equations (7), we express

\[
f(\eta, \tau) = f_0(\eta) + e^{\gamma \tau} F(\eta, \tau),
\]

(15)

where \( F(\eta, \tau) \) is smaller relative to \( f_0(\eta) \) and \( \gamma \) is an unknown eigenvalue. Substituting (15) into equation (13) along with the help of equation (5), we obtain the following linear eigenvalue equation:

\[
F''_0 + f_0 F''_0 + f_0'' F_0 - (4f_0 + K - \gamma) F_0 = 0,
\]

(16)

subject to the new boundary conditions given by

\[
F_0(0) = 0, \quad F_0'(0) = NF_0''(0), \quad F_0'(\infty) = 0.
\]

(17)

The stability of the solution is determined by the smallest eigenvalue \( \gamma \) where \( \gamma > 0 \) reveals that there is an initial decay of disturbances leading to a stable solution. Conversely, if \( \gamma < 0 \), then there exists an initial growth of disturbances and hence corresponds to unstable solution. Harris et al. [9] proposed that the range of possible eigenvalues can be investigated by relaxing an appropriate boundary condition. It follows that, we decide to relax the condition \( F_0'(\infty) = 0 \) in (17), and solve the equation (16) subject to (17) considering the new condition \( F_0''(0) = 1 \).

4. Results and discussion

The ordinary differential equation (6) subject to (7) is solved numerically using the bvp4c programme in MATLAB for selected values of the porosity parameter \( K \), suction parameter \( s \), stretching/shrinking parameter \( c \) and velocity slip parameter \( N \). Table 1 compare the values of the reduced skin friction coefficient \( f''(0) \) with those reported in Bachok et al. [6] and Subhashini et al. [10]. An excellent agreement between the presented solutions are observed which ensures that the present results are correct and accurate.

The changes in the reduced skin friction coefficient \( f''(0) \) with \( c \) for selected values of parameters \( K, s \) and \( N \) are shown in Figure 1. Interestingly, we observe that dual (first and second) solutions exist in the range \( c > c_c \), while no solution exists for \( c < c_c \), where \( c_c \) is the critical value of \( c \). In precise, for
negative values of $c$ (shrinking sheet) there exists $c_c$. It is worth mentioning that the boundary layer separates from the surface when $c = c_c$. Thus, we are unable to obtain a solution for $c < c_c$ by using boundary layer approximations. The value of $f''(0)$ increases in the range $c_c < c < 1$ for rising values of $K$. Conversely, the opposite trend appears in the range of $c > 1$. Besides, it is observed from Table 2 that the values of $|c_c|$ increase as the parameters $K$, $s$ and $N$ increases, suggesting that these parameters effectively influence for uplifting the range of existence of dual solutions. It is therefore suggests that the combined effects of porosity, suction and velocity slip could decelerate the separation of boundary layer.

**Table 1.** Values of $f''(0)$ for different values of $c$ when $K = 0$, $s = 0$ and $N = 0$.

| $c$  | Bachok et al. [6] | Subhashini et al. [10] | Present study |
|------|-------------------|------------------------|--------------|
| $-0.5$ | 2.1182            | 2.1176                 | 2.118169     |
| $0$   | 1.6872            | 1.6863                 | 1.687218     |
| $0.5$ | 0.9604            | 0.9617                 | 0.960416     |

**Table 2.** The values of $|c_c|$ for different values of $K$, $s$ and $N$.

| $K$ | $s$ | $N$ | $|c_c|$ |
|-----|-----|-----|--------|
| 0   | 1   | 1   | 5.1102 |
| 1   |     |     | 6.5452 |
| 2   |     |     | 8.0513 |
| 1   | 0   |     | 4.2996 |
|     | 0.5 |     | 5.2867 |
| 0   |     | 0.5 | 2.6262 |
| 0.5 |     |     | 4.3480 |

**Figure 1.** Variation of $f''(0)$ with $c$ for different values of $K$ when $s = 1$ and $N = 1$. 
Figure 2. Velocity profiles $f'(\eta)$ for different values of $K$ when $c = -2$, $s = 1$ and $N = 1$.

Figure 3. Velocity profiles $f'(\eta)$ for different values of $s$ when $c = -2$, $K = 1$ and $N = 1$.

Figure 4. Velocity profiles $f'(\eta)$ for different values of $N$ when $c = -2$, $K = 1$ and $s = 1$. 
The velocity profiles $f'(\eta)$ are presented in Figures 2–4. It is analysed from these figures that the progressive values of parameters $K$, $s$ and $N$ causes an escalation in the dimensionless velocity profiles. From the physical point of view, it shows that the velocity of fluid increases when the porosity, suction and velocity slip parameters increase. Hence, the momentum boundary layer thickness becomes thinner. Apart from that, it can be seen that the boundary layer thickness for the first solution is always smaller in comparison with the second solution.

### Table 3. The values of $\gamma_1$ for different values of $c$ when $K = 1$, $s = 1$ and $N = 1$.  

| $c$  | $\gamma_1$ (first solution) | $\gamma_1$ (second solution) |
|-----|-----------------------------|------------------------------|
| -5  | 3.9264                      | -3.8624                      |
| -6  | 2.3828                      | -2.3640                      |
| -6.5| 0.6892                      | -0.6878                      |
| -6.54| 0.2248                      | -0.2247                      |

Due to the existence of dual (first and second) solutions as shown in Figures 1–4, the present study employs the stability analysis in order to test the stability of the dual solutions by obtaining the smallest eigenvalue $\gamma_1$. The selected values of the parameter $c$ together with $\gamma_1$ are listed in Table 3 for $K = 1$, $s = 1$ and $N = 1$. It is apparent from this table that $\gamma_1 > 0$ for the first solution while $\gamma_1 < 0$ for the second solution which implies that the first solution is stable, while the second solution is unstable. These results are similarly reported by other previous studies (see for example [7–9, 11–13]).

### 5. Conclusion

The problem of a stagnation-point flow over an exponentially stretching/shrinking sheet with porosity, suction and velocity slip effects is numerically studied. The governing equations describing the problem are solved numerically by utilizing the bvp4c programme in MATLAB. The results are obtained for the reduced skin friction coefficient and the velocity profiles in order to discover the effects of the stretching/shrinking parameter $c$, porosity parameter $K$, suction parameter $s$ as well as velocity slip parameter $N$. The presence of porosity, suction and velocity slip effect causes an increase in the dimensionless velocity profiles as well as possibility of delaying the separation of boundary layer. Dual (first and second) solutions are obtained in a specific range of the stretching/shrinking parameter. The stability analysis is implemented in order to analyse the stability of the dual solutions. The results established that the first solution is stable, while the second solution is unstable.

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### References

[1] Shit G C, Haldar R and Mandal S 2017 Adv. Powder Technol. 28 1519–30
[2] Ishak A, Nazar R and Pop I 2008 Int. J. Heat Mass Transfer 51 1150–55
[3] Vafai K and Tien C L 1981 Int. J. Heat Mass Transfer 24 195–203
[4] Pal D and Mandal G 2014 Nucl. Eng. Des. 273 644–52
[5] Abu Bakar S, Arifin N M, Md Ali F, Bachok N, Nazar R and Pop I 2018 Appl. Sci. 8 483
[6] Bachok N, Ishak A and Pop I 2012 Int. J. Heat Mass Transfer 55 8122–28
[7] Merkin J H 1985 J. Eng. Math. 20 171–79
[8] Weidman P D, Kubitschek D G and Davis A M J 2006 Int. J. Eng. Sci. 44 730–37
[9] Harris S D, Ingham D B and Pop I 2009 Trans. Porous Media 77 267–85
[10] Subhashini S V, Sumathi R and Momoniat E 2014 Meccanica 49 2467–78
[11] Hamid R A and Nazar R 2016 AIP Conference Proceedings 1750 030022
[12] Naganthran K and Nazar R 2016 AIP Conference Proceedings 1784 050004
[13] Jusoh R, Nazar R and Pop I 2017 Int. J. Mech. Sci. 124 166–73