QSSR ESTIMATE OF THE $B_B$ PARAMETER
AT NEXT-TO-LEADING ORDER

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ABSTRACT

We compute the leading $\alpha_s$ corrections to the two-point correlator of the $O_{\Delta B=2}$ operator which controls the $B^0\bar{B}^0$ mixing. Using this result within the QCD spectral sum rules approach and some phenomenologically reasonable assumptions in the parametrization of the spectral function, we conclude that the vacuum saturation values $B_B \simeq B_{B^*} \simeq 1$ are satisfied within 15%.

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The violation of the CP symmetry is still one of the most intriguing phenomena in particle physics. The standard model (SM) offers a pattern for an explanation of this violation through the complex phase of the unitary Cabibbo-Kobayashi-Maskawa quark mixing matrix for three generations. The rigorous experimental test of the pattern requires knowing the numerical values of some hadronic matrix elements at low energies with some non-perturbative QCD methods.

In this paper we discuss an estimate of the $B^0_d$-$\bar{B}^0_d$ mixing parameter within the QCD spectral sum rules (QSSR) approach [1,2] along the lines of Ref. [3], but by including the new $\alpha_s$ corrections and by taking care on the detailed contributions of different $B$-like states to the spectral function. In so doing, we estimate the two-point correlator of the $O_{\Delta B=2} = (\bar{b}_L\gamma_\mu d_L)^2$ operator, where $q_L$ are left-handed quark fields

$$T(x) = \langle 0 | TO_{\Delta B=2}(x)O_{\Delta B=2}(0) | 0 \rangle. \quad (1)$$

The leading term has a trivial expression in the configuration space. Indeed, to leading order in $\alpha_s$, it simply reads

$$T_0(x) = 2 N_c^2 \left( 1 + \frac{1}{N_c} \right) 16 S'(x, m) S(-x, 0) S'(x, m) S(-x, 0)$$

$$= 2 \left( 1 + \frac{1}{N_c} \right) \text{tr}[S(x, m) S(-x, 0)] \text{tr}[S(x, m) S(-x, 0)] = 2 \left( 1 + \frac{1}{N_c} \right) \Pi_5(x) \Pi_5(x) \quad (2)$$

where $S(x, m)$ is the free fermion propagator and $N_c$ stands for the number of quark colours. The prime means taking only the part of the propagator that is proportional to a $\gamma$ matrix. The function $\Pi_5(x) = \langle 0 | T j_5(x) j_5(0) | 0 \rangle$ is the two-point correlator associated to the current $j_5 = \bar{b}_i \gamma_\mu d_i$.

One can also rewrite Eq. (2) in the form

$$T_0(x) = 2 \left( 1 + \frac{1}{N_c} \right) \Pi_{\mu\nu}(x) \Pi^{\mu\nu}(x), \quad (3)$$

where $\Pi_{\mu\nu}(x) = \langle 0 | T j_L^\mu(x) j_L^\nu(0) | 0 \rangle$ and $j_L^\mu = \bar{b}_L \gamma_\mu d_L$, which has the following Lorentz decomposition in $x$-space

$$\Pi^{\mu\nu}(x) = (-\partial^\mu \partial^\nu + g^{\mu\nu} \partial^2) \Pi_T(x^2) - \partial^\mu \partial^\nu \Pi_L(x^2).$$

Formulae (2) and (3) demonstrate an explicit factorization of the correlator (1) in the configuration space to leading order in $\alpha_s$.

The dispersion representation in $x$-space for any two-point correlator $\Pi_j(x) (j = T, L, 5)$ has the form

$$i \Pi_j(x^2) = \int_{s_j}^\infty r_j(s) D(x, s) ds, \quad (4)$$

where $D(x, s)$ is a free boson propagator with the “mass” $\sqrt{s}$. The spectral functions $r_j$, read to leading order in $\alpha_s$:

$$r_L^{(0)}(s) = N_c \frac{1}{16\pi^2} z(1 - z)^2, \quad r_T^{(0)}(s) = N_c \frac{1}{16\pi^2} \frac{1}{3} (1 - z)^2 (2 + z),$$
\[ r_s^{(0)}(s) = m_b^2 N_c \frac{1}{16 \pi^2} \frac{2(1-z)^2}{z}, \]  
(5)

where \( z = m_b^2 / s \), \( m_b \) is the \( b \)-quark pole mass.

One can define the spectral function \( \rho(s) \) of the full correlator \( T(x) \) in the same way as in Eq. (4) and express it, to first order in \( \alpha_s \), in terms of the spectral functions \( r_j(s) \) associated to the two-line correlators. Therefore

\[ \rho(s) = \int r_1(s_1)r_2(s_2)\Phi(s; s_1, s_2)ds_1ds_2, \]  
(6)

where

\[ \Phi(s; s_1, s_2) = \frac{1}{16 \pi^2 s} \sqrt{(s - s_1 - s_2)^2 - 4s_1s_2} \]  
(7)

is the two-body phase-space factor, and the concrete form of the spectral functions \( r_j(s) \) entering Eq. (6) depends on the representation chosen for the correlator \( T(x) \) (as in Eqs. (2,3) for \( T_0(x) \)). The integration region in Eq. (6) is determined by the properties of the phase-space factor (7) for corresponding representation of the whole correlator; \( \Phi(s; s_1, s_2) \) is supposed to be equal to zero within kinematically forbidden regions.

To leading order in \( 1/N_c \), one can write the correlator \( T(x) \) as a product of two two-line correlators given in Eq. (4). It is worthwhile to notice that this decomposition is gauge-invariant and finite, i.e. it does not require any renormalization that is a reflection of the vanishing of the operator \( O_{\Delta B=2} \) anomalous dimension in leading order in \( 1/N_c \).

Including \( \alpha_s \) corrections, the spectral function in the factorization approximation reads:

\[ \rho_{\text{fact}}(s) = \rho_0(s)(1 + \Delta \rho_f(s)), \]  
(8)

where \( \rho_0(s) \) generates the leading-order term and \( \Delta \rho_f(s) \) is a (properly normalized) factorized correction in the \( \alpha_s \) order. The colour structure is the following

\[ \rho_0(s) = N_c^2 \left( 1 + \frac{1}{N_c} \right) \tilde{\rho}_0(s), \quad \Delta \rho_f(s) = C_F \frac{\alpha_s}{\pi} \Delta \tilde{\rho}_f(s), \]  

where \( C_F = (N_c^2 - 1)/2N_c \) for the \( SU(N_c) \) colour group and the quantities with the tilde contain no explicit \( N_c \) dependence.

The representation of the spectral density \( \rho_{\text{fact}}(s) \) through those of two-line correlators for the vector-like decomposition in Eq. (3) is:

\[ \rho_{\text{fact}}(s) = \int ds_1ds_2 \Phi(s; s_1, s_2) \left\{ \left( \frac{s_{12}^2}{4} + 2s_1s_2 \right) r_T(s_1) r_T(s_2) \right\} + \left( \frac{s_{12}^2}{4} - s_1s_2 \right) \left( r_T(s_1) r_L(s_2) + r_L(s_1) r_T(s_2) \right) + \frac{s_{12}^2}{4} r_L(s_1) r_L(s_2) \]  

where \( s_{12} = s - s_1 - s_2 \). The non-factorizable corrections are of the \( 1/N_c \) order and the full spectral density can be written in the form

\[ \rho(s) = \rho_0(s) \left( 1 + \Delta \rho_f(s) + \Delta \rho_{nf}(s) \right) \]  
(9a)
and

\[ \Delta \rho_{nf}(s) = \frac{C_F}{N_c + 1} \frac{\alpha_s}{\pi} \Delta \tilde{\rho}_{nf}(s) = \frac{1}{2} \left( 1 - \frac{1}{N_c} \right) \frac{\alpha_s}{\pi} \Delta \tilde{\rho}_{nf}(s). \]  

(9b)

One of the diagrams contributing to the non-factorizable part of the whole spectral density is given in Fig. 1.

Several comments are in order here. The decomposition (9) is gauge-invariant. The same is true for the anomalous dimension of the operator \( O_{\Delta B=2} \). We use four-dimensional algebra of Dirac’s \( \gamma \) matrices throughout the computation. This seems quite natural because it allows one to make Fierz rearrangements freely, which is crucial for establishing the validity of factorization. At the same time this approach simply implies the special choice of the renormalization scheme. So, this scheme is not the standard \( \overline{\text{MS}} \) one and it is not the real dimensional reduction as used in [4] either. To the considered order in \( \alpha_s \), however, the scheme dependence reduces to a certain choice of the normalization parameter \( \mu \). In principle, one can fix the scheme by direct comparison of our results with the massless limit for corresponding correlator [5] or, to put it another way, by comparing the non-logarithmic parts of corrections in the massless limit one can obtain the relation between our parameter \( \mu \) and the corresponding \( \mu_{\overline{\text{MS}}} \) or \( \mu_{\text{DimRed}} \). Thus, the answer in any desired scheme can be easily recovered by considering the massless limit. In the present paper we do not dwell on this point.

We have computed numerically the spectral density \( \rho(s) \) in the first non-leading order in \( \alpha_s \). We analyse our results, paying special attention to the presentation of the entire spectral density \( \rho(s) \) as a sum of factorizable and non-factorizable pieces, as in Eq. (9). This decomposition is useful from the theoretical point of view. It is also interesting from the \( 1/N_c \) analysis, and it is quite convenient technically. As for the factorizable part of the spectral density (Eq. 8) the analytical expressions for two-line spectral functions \( r_{T,L}(s) \) are well known to first order in \( \alpha_s \), and we use them as they have been given in Refs. [6,7]. For the non-factorizable part of the spectral density, we have to compute the gluon-exchange diagram both for unequal mass lines and for equal mass lines of a two-line correlator. We have done it using the REDUCE system of analytical computation [8] and the table of two-loop integrals given in [6,7]. After using a Fierz transformation, these non-factorizable diagrams can be represented in form analogous to that of the factorizable diagrams, i.e. the corresponding analytical expression is given by a product of two traces in Dirac’s indices. The transformed diagrams are given in Fig. 2.

For diagrams with a gluon connecting \( b \) or \( d \)-quarks in the loop (Fig. 2(a,b)), the resulting representation can be rendered into a product of two scalar spectral densities of two-line correlators. The relevant part of the spectral density reads:

\[
r_m(s) = \frac{1}{16\pi^2} s \left\{ (2(1 - 2z)v + C_F \frac{\alpha_s}{\pi} \left( -4v(1 - 2z)(1 + 2 \ln \frac{\mu^2}{m_b^2}) \right) \right.
\]

\[
+ 4(1 - 2z)(1 - v) \ln z + (1 + v)(4zv + (3 + v)(z - 1)) \ln \frac{1 - v}{1 + v}
\]

\[
(1 - 2z)^2 \phi(u) + v(18z - 13) + 16z - 8 \right\},
\]
\( \epsilon = 2 - D/2 \), \( D \) is the dimensional regularization parameter, \( v = \sqrt{1 - 4z} \equiv \sqrt{1 - 4z} \), \( \phi(u) = 8(Sp(u^2) - Sp(-u)) + 4(2 \ln(1 - u^2) - \ln(1 + u)) \ln u, u = (1 - v)/(1 + v) \),

\[
Sp(u) = -\int_0^u \frac{\ln(1 - t)}{t} dt.
\]

For a zero mass correlator (d-quark), the spectral density has the form

\[
0\rho(s) = \frac{1}{16\pi^2} s \left\{ \frac{1}{2} + 2C_F \frac{\alpha_s}{\pi} \left( -4\left( \frac{1}{\epsilon} + 2 \ln \frac{\mu^2}{s} \right) - 21 \right) \right\}.
\]

The corresponding representation for diagrams, with different quark flavours tied with a gluon in the loop, is of the vector type. The nonfactorizable first order corrections to the spectral densities are:

\[
r_{L,T}^{(0)}(s) = r_{L,T}^{(0)}(s) + C_F \frac{\alpha_s}{\pi} r_{L,T}^{(1)}(s),
\]

\[
r_{L}^{(1)}(s) = z(1 - z)^2 \left( \frac{1}{\epsilon} + 2 \ln \frac{\mu^2}{m_b^2} \right) - 2z(1 - z)^2 \ln(1 - z) \ln(\frac{z}{1 - z}) + z(1 - z)^2(4z^2 - 3z + \frac{1}{2}) \ln(\frac{z}{1 - z}) + 4z(1 - z)^2 Sp(-\frac{z}{1 - z}) + z(1 - z)(2z - 3) \ln(1 - z) + (z^2 + 3) \ln z + \frac{65}{12} z^3 - \frac{17}{2} z^2 + \frac{21}{4} z - \frac{2}{3},
\]

\[
3r_{T}^{(1)}(s) = (z + 2)(1 - z)^2 \left( \frac{1}{\epsilon} + 2 \ln \frac{\mu^2}{m_b^2} \right) - 2(z + 2)(1 - z)^2 \ln(1 - z) \ln(\frac{z}{1 - z}) + (1 - z)^2(2z^2 - 2z^2 + \frac{5}{2} z + 2) ln(\frac{z}{1 - z}) + 4(z + 2)(1 - z)^2 Sp(-\frac{z}{1 - z}) + z(1 - z)(2z + 1) \ln(1 - z) + (3z^3 + 2z^2 - \frac{9}{2} z + 2) \ln z + \frac{41}{12} z^3 + \frac{3}{2} z^2 - \frac{75}{4} z + \frac{46}{3}.
\]

The whole computation can be subjected to a powerful test consisting in the cancellation of divergences by the one-loop renormalization constant of the operator \( O_{\Delta B=2} \), which is known to be \( Z_{O_{\Delta B=2}} = 1 - 3(1 - 1/N_c) \frac{\alpha_s}{4\pi} \). We checked this cancellation explicitly.

The QCD results for the spectral density are given in Table 1.

At larger \( s \) (say, \( s \geq 5m_b^2 \)), where the perturbative QCD result makes sense, one can notice that the non-factorizable correction dies out faster than the factorizable one, so that the full correction comes mainly from that of the two-line correlator. Therefore, at sufficiently large \( s \), the factorization (in the sense of Eqs. (2) and (3)) becomes exact and the ‘formal’ \( 1/N_c \) suppression of non-factorizable corrections becomes numerically valid.

Using the QSSR approach for estimating \( B_B \), we can work with the moments:

\[
M_i(s_{th}) = \int_{4m_b^2}^{s_{th}} \rho(s)s^{-i}ds,
\]

(10)
which can be decomposed as:

\[ M_i(s_{th}) = M_i^0(s_{th}) \left( 1 + \Delta M_i^f(s_{th}) + \Delta M_i^{nf}(s_{th}) \right), \]

according to the corresponding decomposition of the spectral density, Eq. (9a).

We show in Table 2 the strength of the non-factorizable correction for \( s \geq 5m_b^2 \) for the following set of input parameters: \( \Lambda^{(5)}_{\overline{MS}} = 175 \) MeV, \( m_b = 4.6 \) GeV, and with the one-loop expression for the strong coupling constant \( \alpha_s(\mu) = \frac{6\pi}{23}\ln(\mu/\Lambda^{(5)}_{\overline{MS}}) \) at the point \( \mu = m_b \). For given regions of integration in Eq. (10), the moments of the factorizable spectral density are practically independent of the power \( i \) of the weight function \( s^{-i} \). They change by less than 5% for \( i=1-10 \). The non-factorizable correction changes its sign within the integration region for \( s \geq 5.5m_b^2 \), and its moments are more sensitive to the power \( i \). One can notice that the non-factorizable correction does not exceed a 15% level with respect to the full factorized spectral density at sufficiently large \( s \). The full order-\( \alpha_s \) corrections (factorizable+non-factorizable) can be quite large for both the spectral density itself and its moments depending on the energy \( s \) as they can reach a magnitude of 100% with respect to the leading term. Hopefully, the non-factorizable corrections, measured in terms of the fully factorized (lowest order + \( \alpha_s \) terms) spectral density are, nevertheless, still moderate. However, all the factorizable corrections to the correlator (1), no matter how large they are, can be absorbed into the calculation of the decay constant \( f_B \) from the two-point correlator with two quark lines, in such a way that the relevant corrections to the \( B_B \) parameter are only due to the non-factorizable ones.

Contrary to the QCD part, the estimate of the phenomenological contributions to the correlator is much more involved. For the analysis to be performed one might make the following phenomenological assumptions:

- the minimal choice of operator: \( O_{\Delta B=2} = g_{\Delta B=2} \partial_{\mu} B \partial^{\mu} B \), can give a good description of the spectral function.
- the contributions of the \( BB \) and \( B^*B^* \) pairs to the spectral function are equal due to their approximate degeneracy and to the equal values of their decay constants [9] (this feature is fully satisfied in the large \( m_b \) limit);
- we do not have exotic contributions due to a singlet mixture of colored states or due to some four-quark-like states which may restore the \( N_c \)-structure of the correlator in order to match the proper \( (1+1/N_c) \) construction entering the standard definition of the \( B_B \) parameter.

Within these phenomenological assumptions, we can conclude that the effect of inclusion of the perturbative corrections into the QCD part of the correlator on the \( B_B \) parameter is reasonably small and the change is not larger than 15%. The absolute value of \( B_B \) however cannot be unambiguously established using the perturbative part of the correlator only. Assuming that non-perturbative non-factorizable corrections are small and the factorizable corrections are properly taken into account through \( f_B \) we find after correcting for exotic contributions in theoretical part of the correlator

\[ B_B \simeq B_B^* \simeq (1.00 \pm 0.15) \]
where we have not explicitly written the small effect due to the anomalous dimension of the operator $O_{\Delta B=2}$.

An improvement of our result needs an explicit quantitative check of the previous phenomenological assumptions. Most probably, the QSSR approach based on the three-point function [10] may be more appropriate, as it does not need some of the assumptions which we have used for the two-point function. Unfortunately, one has to face, in this case, a highly involved computation of the non-recursive three-loop non-factorizable diagrams in order to obtain the $\alpha_s$ corrections.

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### Table 1: Normalized spectral densities

| $s/m_b^2$ | $\Delta \rho_f$ | $\Delta \rho_{nf}$ | $\Delta \rho_{nf}/(1 + \Delta \rho_f)$ |
|-----------|-----------------|-------------------|-------------------------------------|
| 5.5       | 1.03            | 0.02              | 0.01                                |
| 6.0       | 0.95            | 0.21              | 0.11                                |
| 6.5       | 0.89            | 0.29              | 0.15                                |
| 7.0       | 0.84            | 0.32              | 0.17                                |

### Table 2: Normalized moments of spectral densities

| $i$ | $s_{th}/m_b^2$ | $\Delta M_f$ | $\Delta M_{nf}$ | $\Delta M_{nf}/(1 + \Delta M_f)$ |
|-----|----------------|--------------|-----------------|-------------------------------------|
| 0   | 5.5            | 1.07         | -0.16           | -0.08                               |
|     | 6.0            | 0.99         | 0.11            | 0.06                                |
|     | 6.5            | 0.93         | 0.23            | 0.12                                |
|     | 7.0            | 0.88         | 0.29            | 0.15                                |
| 5   | 5.5            | 1.08         | -0.21           | -0.10                               |
|     | 6.0            | 1.00         | 0.08            | 0.04                                |
|     | 6.5            | 0.94         | 0.20            | 0.11                                |
|     | 7.0            | 0.89         | 0.27            | 0.14                                |
| 10  | 5.5            | 1.09         | -0.27           | -0.13                               |
|     | 6.0            | 1.01         | 0.02            | 0.01                                |
|     | 6.5            | 0.96         | 0.16            | 0.08                                |
|     | 7.0            | 0.91         | 0.23            | 0.12                                |
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