Diffusive Synchrotron Radiation from Pulsar Wind Nebulae

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ABSTRACT

Diffusive Synchrotron Radiation (DSR) is produced by charged particles as they random walk in a stochastic magnetic field. The spectrum of the radiation produced by particles in such fields differs substantially from those of standard synchrotron emission because the corresponding particle trajectories deviate significantly from gyration in a regular field. The Larmor radius, therefore, is no longer a good measure of the particle trajectory. In this paper we analyze a special DSR regime which arises as highly relativistic electrons move through magnetic fields which have only random structure on a wide range of spatial scales. Such stochastic fields arise in turbulent processes, and are likely present in pulsar wind nebulae (PWNe). We show that DSR generated by a single population of electrons can reproduce the observed broad-band spectra of PWNe from the radio to the X-ray, in particular producing relatively flat spectrum radio emission as is usually observed in PWNe. DSR can explain the existence of several break frequencies in the broad-band emission spectrum without recourse to breaks in the energy spectrum of the relativistic particles. The shape of the radiation spectrum depends on the spatial spectrum of the stochastic magnetic field. The implications of the presented DSR regime for PWN physics are discussed.

Key words: acceleration of particles—shock waves—turbulence—supernova remnants—radiation mechanisms: non-thermal—magnetic fields

1 INTRODUCTION

In many astrophysical objects, radiation is produced by relativistic charged particles moving in magnetic fields. This radiation is called magneto-bremsstrahlung or synchrotron radiation. The theory of synchrotron radiation in the case of regular magnetic fields is well established (e.g., Ginzburg & Syrovatskii 1965; Pacholczyk 1970). However, in many astrophysical fields there is a stochastic, turbulent component, which can often dominate the regular field. Astrophysical magnetic fields, then, will often be variable over a wide range of spatial and temporal scales.

In order to calculate spectrum of synchrotron emission from a volume encompassing regions of varying field strength and orientation, it is necessary to average the microscopic emission intensity over the different field strengths and orientations (as well as over the range of particle energies which may be present). A common approach to this problem is to simply average the standard synchrotron formulae for a regular field over the varying magnetic field. However, in general, this approach is only correct if the field can be described as regular over the volume large compared to a typical particle orbit.

In the case of a field which has structure in a volume small compared to the average particle orbit, the particle paths deviate significantly from regular gyration around the field lines, and the standard formulae can lead to incorrect results, particularly for the emitted spectrum. For example, if a turbulent magnetic field is composed of random waves, then the ensemble of these waves will result in an incoherent superposition of random forces which will produce a stochastic electron trajectory quite different from the circular orbit in the regular field. Therefore, the standard synchrotron formulae will be in error in this case (Nikolaev & Tsytovich 1979; Toptygin & Fleishman 1987; Fleishman 2005). To correctly calculate the
emission spectrum in such a case requires not only averaging over the regions of different field strength and orientation but also over the many possible particle paths, since the microscopic nature of the particle path strongly influences the spectrum. This problem is highly non-linear: in addition to the electron path affecting the nature of the emission, electrons are efficiently mirrored from regions of high magnetic field, and thus spend a disproportionate time in the regions of lower field. As we will elaborate below, it is quite possible that the fields in astrophysical sources, in particular in pulsar wind nebulae (PWNe), will in fact have structure on very small scales.

To calculate the radiation from electrons moving in stochastic fields, and which do not necessarily have circular orbits, we turn to the theory of Diffusive Synchrotron Radiation (DSR), which attempts to calculate the average emission over the many possible particle paths. The theory was established some time ago [Toptygin & Fleishman (1987a; Toptygin et al. 1987)], and was reviewed recently by Fleishman (2005). However, despite the many astrophysical situations where it would be applicable, from the sun and geospace to extragalactic objects (Fleishman 2005), it is not yet in widespread use.

So far, two special cases of DSR as applied to astrophysical sources have been particularly considered. The first one is DSR generated in the presence of only small-scale random inhomogeneities and its application to prompt gamma-ray burst spectra (Fleishman 2006a). Note that a limiting special case of the DSR, namely one calculated in perturbative 1-d approximation, is also referred to as “jitter” radiation in the gamma-ray burst literature (for a review see, e.g., Piran 2005), which is essentially the same physical process as DSR. The second one is DSR generated in a superposition of regular and small-scale random magnetic fields, which is possibly relevant for the interpretation of spatially resolved wide-band spectra observed from some extragalactic jets (Fleishman 2006b).

In this paper, we focus on pulsar wind nebulae (PWNe). A magnetized and highly relativistic pulsar wind, which is shocked near the pulsar, is thought to be the primary source of both the nebular field and relativistic particle population. Plasma instabilities in relativistic shocks are thought to produce structure in the magnetic field down at very small scales, down to the skin depth (e.g., Kazimura et al. 1998; Piran 2003). It is therefore quite possible that the magnetic field in PWNe have significant structure on very small spatial scales, and consequently that the standard synchrotron formulae may not allow accurate calculations of the emission spectrum. In this paper, we calculate the broad-band emission spectra from PWNe, which are often difficult to understand using only standard synchrotron theory, using DSR.

We consider here a special case of DSR where the magnetic field represents an incoherent superposition of waves with different scales and random phases and orientations, and having a power-law spectrum. We note here that such a field, consisting of only random waves and having no regular component, is almost certainly an oversimplification. Real astrophysical fields, including those in PWNe, will often contain a combination of stochastic and regular magnetic fields. The case considered here, however, is an interesting limiting case, with the opposite limiting case being that of only regular field which is well known from the application of the standard synchrotron formulae. We note however, that the problem of calculating the emission in a more general case is non-linear (as mentioned above) and the emission spectrum cannot necessarily be easily determined from the two limiting cases of regular field only and random field only.

Nonetheless, as we will show, the very simple model of field we consider here produces broadband spectra for pulsar wind nebulae which in remarkable agreement with observations, with a minimum of additional parameters. Thus, even if a complete description of the synchrotron emission will require more elaborate modelling than is possible in this paper, the present success strongly suggests that DSR is a promising avenue for future, more detailed work.

We finally note that alternatives to the standard synchrotron models for PWNe were already suggested some time ago. For example, Arons (1972) considered a nonlinear inverse Compton radiation on a strong coherent electro-magnetic wave created by the rotation of the magnetized pulsar, and found that such a model predicts high levels of circular polarization in the diffuse radio emission, in contradiction with observations. In contrast, the DSR process assumes the presence of many stochastic incoherent static magnetic inhomogeneities in the nebula volume, which will give rise to very little (if any) circular polarization in agreement with observations.

The plan of the remainder of this paper is as follows: We give a brief general formulation of DSR in §2 and then discuss in particular DSR as produced by a distribution of relativistic electrons with a power-law distribution in energy in §3. We discuss the implications for PWNe in §4 and then compare DSR spectra to the observed ones for different PWNe in §5 and finally, give our conclusions in §6.

2 GENERAL FORMULATION

We give here a summary of full DSR theory for the radiation produced by particles moving in stochastic magnetic fields. As mentioned, it was first introduced by Toptygin & Fleishman (1987a), but reviewed and slightly reformulated to use more convenient notation by Fleishman (2006), which formulation we follow here. For a truly random field, which represents a superposition

\[ B(r, t) = \sum B(k, \omega)e^{ikr - \omega t + \phi_k} \]  

(1)
of the waves with random phases $\varphi_k$, where the sum $\sum$ means the summation and/or integration over all available Fourier components with various $\omega$ and $k$, we can write

$$\langle B_\alpha(k, \omega) B^*_\beta(k', \omega') \rangle = K_{\alpha \beta}(k, \omega) \delta(k - k') \delta(\omega - \omega')$$

(2)

where the brackets $\langle \cdot \rangle$ denote averaging over the random phases, so only the waves with exactly the same pairs of $k$ and $\omega$ are correlated. Stated another way, the presence of the delta-functions in equation (2) indicates that any waves with $k$ and $k + dk$ are entirely uncorrelated. As was pointed out in Fleishman (2005), the approximation of the standard synchrotron radiation formulae is not applicable in the case of a magnetic field composed of uncorrelated waves, independent of their wave length.

Although we do not know the detailed statistical properties of the magnetic turbulence in astrophysical objects, we consider here the DSR spectrum generated in the presence of the truly random magnetic field whose structure can be described by a power-law spectrum. We discuss possible implications of this regime for the interpretation of wide-band spectra of PWNe.

According to Toptygin & Fleishman (1987a) and Fleishman (2005), if the deviation of the particle trajectory from the straight line (due to effect of the regular magnetic field) is small compared to the correlation length of the random field, then the radiation spectrum takes the form

$$I_\omega = \frac{8Q^2 q(\omega)}{3\pi c} \gamma^2 \left(1 + \frac{\omega_{pe}^2 \gamma^2}{\omega^2}\right)^{-1} \Phi_1(s, r) + \frac{Q^2 \omega}{4\pi c^2} \left(1 + \frac{\omega_{pe}^2 \gamma^2}{\omega^2}\right) \Phi_2(s, r),$$

(3)

where $\gamma = E/Mc^2$ is the Lorentz-factor of emitting particle, $Q$ and $M$ is its charge and the mass ($Q = e$ and $M = m$ are the charge and the mass of the electron in practice), $\omega_{pe}$ is the plasma frequency, which enters via the dielectric permeability $\varepsilon(\omega) \approx 1 - \omega_{pe}^2/\omega^2$, $c$ is the speed of light, and $\Phi_1(s, r)$ and $\Phi_2(s, r)$ stand for the integrals:

$$\Phi_1(s, r) = 24s^2 \text{Im} \int_0^\infty dt \exp(-2sdt) \times \left[ \coth t \exp\left(-2r s_0^3 \text{coth} t - \sinh^{-1} t - t/2\right) - \frac{1}{t} \right],$$

(4)

$$\Phi_2(s, r) = 4rs^2 \text{Re} \int_0^\infty dt \frac{\cosh t - 1}{\sinh t} \times \exp\left(-2sdt - 2r s_0^3 \text{coth} t - \sinh^{-1} t - t/2\right),$$

(5)

which depend on the dimensionless parameters $s_0$, $s$, $r$:

$$s_0 = (1 - i)s = \frac{1 - i}{8\gamma^2} \left(\frac{\omega}{q(\omega)}\right)^{1/2} \left(1 + \frac{\omega_{pe}^2 \gamma^2}{\omega^2}\right),$$

(6)

$$r = 32\gamma^4 \left(\frac{\omega B_\perp}{\omega}\right)^2 \left(1 + \frac{\omega_{pe}^2 \gamma^2}{\omega^2}\right)^{-3},$$

(7)

$q(\omega)$ is the rate of scattering of the particle by magnetic inhomogeneities, which will be specified below, and $\omega_{B\perp} = QB_\perp/(Mc)$, where $B_\perp$ is the regular magnetic field component transverse to the line of sight.

In essence, it would be highly desirable to consider the joint effect of the random and the regular magnetic fields on the emitted radiation since both components can coexist in the nebula volume. Unfortunately, the full theory which allows for strong electron path deflections due to both random and regular field components is currently unavailable. Therefore, we considered here a limiting special case opposite to widely accepted case of synchrotron emission in the regular field. In particular, we consider the case when there is only random magnetic field and no regular field, i.e., $B_\perp \to 0$, so equation (3) is applicable and can be simplified further since $r \to 0$ and

$$\Phi_1(s, r = 0) \equiv \Phi(s) = 24s^2 \int_0^\infty dt \exp(-2sdt) \sin(2st) \times \left[ \coth t - \frac{1}{t} \right], \quad \Phi_2(s, r = 0) = 0,$$

(8)

thus

$$I_\omega = \frac{8Q^2 q(\omega)}{3\pi c} \gamma^2 \left(1 + \frac{\omega_{pe}^2 \gamma^2}{\omega^2}\right)^{-1} \Phi(s).$$

(9)

Although the integration in equation (3) cannot be performed analytically, the Migdal function $\Phi(s)$ has simple asymptotes for small and large values of $s$ (Migdal 1954, 1956):

$$\Phi(s) \simeq 1, \text{ if } s \gg 1, \quad \Phi(s) \simeq 6s, \text{ if } s \ll 1.$$
where \( \Gamma(z) \) is the Euler gamma-function, \( k_0 = 2\pi/L_0 \), \( L_0 \) is the largest scale of the magnetic turbulence, \( \nu \) is the spectral index of the turbulence, \( \langle B^2_\perp \rangle \) is the mean square of the random magnetic field. Here, in equation (11), we assume that the magnetic fluctuations are composed of some propagating eigen-modes with a dispersion relation \( \omega = \omega(\mathbf{k}) \), which results in the \( \delta \)-function \( \delta(\omega - \omega(\mathbf{k})) \) in equation (11). Therefore, the spatial dependence on \( \mathbf{k} \) and temporal dependence on \( \omega \) are strictly correlated for magnetic turbulence. Frequently, e.g., for normal MHD waves, the group velocity \( |v| = |\partial \omega/\partial k| \) is much less than the speed of light, \( c \). For such cases we can adopt \( \omega(\mathbf{k}) = 0 \), and, accordingly, \( \delta(\omega - \omega(\mathbf{k})) \approx \delta(\omega) \), which is the approximation of a quasi-static magnetic field, and therefore the turbulence spectral index \( \nu \) describes the distribution of the magnetic energy over different spatial scales.

A key parameter in the DSR theory, the rate of the particle scattering by magnetic inhomogeneities, \( q(\omega) \), can be approximated by (see Eq. 35 in Fleishman 2005):

\[
q(\omega) = \frac{\sqrt{\pi} \Gamma(\nu/2) \omega_0^{\nu-1}}{3 \Gamma(\nu/2 - 1/2) \gamma^2 \left[ (\pi^2/2)^2 (\gamma^{-2} + \omega_{pe}^2/\omega_0^2) + \omega_0^2 \right]^{\nu/2}},
\]

(12)

in the case of magnetic field spectrum given by equation (11), and where \( a \) is a number of the order of unity (Fleishman 2005), and \( \omega_0 = k_0c, \omega_{st}^2 = Q^2 \langle B^2_\perp \rangle / (Mc)^2 \) is the mean square of the cyclotron frequency in the random magnetic field.

In Figure 1 we show the DSR spectra calculated by the numerical integration of equation (9) in the case of strong random magnetic field, i.e., under condition

\[
\omega_{st} \gg \omega_0
\]

(13)

along with the DSR spectrum in a small-scale field. Although equation (9) in general must be integrated numerically, we can analytically obtain several characteristic asymptotes, which give a good qualitative idea of the general shape of the DSR spectrum.

As we will see, this regime contains a characteristic frequency

\[
\omega_{st} \equiv \left( \frac{2\pi c}{L_0} \right) \left( \frac{e^2}{m^2 c^2} \right)^{1/2} = \left( \frac{\omega_{st}}{\omega_0} \right)^{2\nu/\nu+1} = \left( \frac{\omega_{st}}{\omega_0} \right)^{2\nu/\nu+1} \omega_0 = \omega_{st},
\]

(14)

which plays a role similar to some extent to the role of the frequency \( \omega_B \) in the standard synchrotron theory. At high frequencies

\[
\omega \gg \omega_{st} \gamma^2 \quad \text{or} \quad \omega_{st} \gamma^2 \gg \omega_0 \gamma^2
\]

we have \( s \gg 1 \), so the radiation spectrum has the standard high-frequency form

\[
I_\omega = \frac{8Q^2 q(\omega)}{3\pi c} \gamma^2,
\]

(15)

with the spectral asymptote \( I_\omega \propto \omega^{-\nu} \), typical for the high-frequency perturbative regime of DSR. Note that if the magnetic field were regular with the same strength, then the bounding frequency would be \( \omega_{st} \gamma^2 \) in place of \( \omega_{st} \gamma^2 \) (equation (15), and the radiation intensity would decrease exponentially rather than as a power-law (equation (16)). The decrease of the bounding frequency (compared with the regular field regime) happens because the deviation of the particle trajectory from the straight line occurs more slowly in the random than in regular field, thus the region of applicability of the perturbation theory (asymptote (16)) broadens towards lower frequencies.

Accordingly, the amount of radiated energy in the random magnetic field is less than in the regular magnetic field with the same energy density. Stated another way, larger random (than regular) magnetic field is required to provide the same radiative losses.

The parameter \( s \) decreases with frequency, and when it falls below unity at

\[
\omega < \omega_{st} \gamma^2,
\]

the perturbation theory is no longer valid. For \( s \ll 1 \) we have \( \Phi(s) \approx 6s \), therefore

\[
I_\omega = \frac{2Q^2}{\pi c} \left( \omega q(\omega) \right)^{1/2}.
\]

(18)

This expression is valid down to relatively low frequencies, where parameter \( s \) increases again under the influence of the effect of density (term \( \omega_{pe}^2/\omega_0^2 \)) and again reaches the unity at a sufficiently low frequency.

The asymptotic regime \( \Phi(s) \approx 6s \), at \( s \ll 1 \), is due to multiple scattering of the fast particle by the uncorrelated long waves composing the random magnetic field at the scales \( l > c/\omega_{st} \). Even though the perturbation of the particle trajectory due to any single Fourier-component of the random field is small, their cumulative effect results in significant angular diffusion of the charged particle. Accordingly, the direction of the particle’s motion changes by a value exceeding the characteristic
Let us proceed now to the DSR spectra produced by a power-law distribution of relativistic electrons $dN_e/d\gamma = \xi^{-1} \gamma^{-\xi} d\gamma$, $\gamma_1 \leq \gamma \leq \gamma_2$, where $N_e$ is the number density of electrons with energies $E \geq mc^2 \gamma_1$, $\xi$ is the power-law index of the distribution, and $\gamma_1 \gg 1$.

Although several different regimes of DSR are possible, depending on the value of $\xi$, we consider the case when $\nu < \xi < 2\nu + 1$, (22)

which is probably of the most practical importance. Indeed, the turbulence spectral index is typically $\nu < 2$ (e.g., $\nu \approx 1.7$ for the Kolmogorov turbulence), while the particle index $\xi > 2$ (e.g., $\xi \approx 2.7$ for the galactic cosmic rays). In this regime, the standard non-thermal spectrum (typical also for synchrotron radiation), $P_\nu \propto \omega^{-\alpha_{\text{nth}}}$, where $\alpha_{\text{nth}} = (\xi - 1)/2$, produced by the inner part $\gamma_1 \ll \gamma \ll \gamma_2$ of the distribution (equation 21) is steeper than the non-perturbative DSR spectrum, $P_\nu \propto \omega^{-(\nu+1)/2}$, but shallower than the high-frequency perturbative spectrum, $P_\nu \propto \omega^{-\nu}$.

The bulk of the DSR energy produced by a single particle is emitted at frequencies $\omega \sim \omega_{\text{loc}} \gamma_1^2$ (if $\nu < 3$). Accordingly, it is easy to estimate that in the region of “intermediate” frequencies

$$\omega_{\text{loc}} \gamma_1^2 \ll \omega \ll \omega_{\text{loc}} \gamma_2^2,$$

(23)

the standard well-known non-thermal (synchrotron-like) spectrum $P_\nu \propto \omega^{-\alpha_{\text{nth}}}$ is formed. However, the DSR spectrum will deviate significantly from the standard synchrotron spectrum at high ($\omega > \omega_{\text{loc}} \gamma_2^2$) and low ($\omega < \omega_{\text{loc}} \gamma_1^2$) frequencies, where it will reproduce the single-particle spectra, $P_\nu \propto \omega^{-\nu}$ (in place of the exponential synchrotron cut-off) and $P_\nu \propto \omega^{-(\nu-1)/2}$ (in place of $P_\nu \propto \omega^{1/3}$) respectively. An example of the DSR spectrum generated by the power-law electron distribution is given in Figure 2. We note that at sufficiently low frequencies the spectrum decreases as $P_\nu \propto \omega^{(\nu+1)/2}$ and then as $P_\nu \propto \omega^{\nu+2}$, but these regions may or may not be observable from the Earth depending on the source parameters. Below we assume that this happens beyond the observed spectral range and, thus, do not consider further the effect of plasma density on the DSR spectra.
4 IMPLICATIONS FOR PULSAR WIND NEBULAE

What are the implications of DSR for PWNe? As has been shown, differences in the shape of the radiation spectrum between DSR and standard synchrotron theory will occur when a particular spectral region is formed primarily by electrons from near either end of the electron distribution ($\gamma_1$ or $\gamma_2$). It is currently believed that a pulsar driving a PWN supplies it with a wind of highly relativistic electrons, which may have $\gamma_1$ as large as $10^6$ (e.g., Wilson & Rees 1978; Kennel & Coroniti 1984a,b; Melatos & Melrose 1996; Chevalier 2000; Arons 2001). If this is the case, and the PWN emission is produced by synchrotron mechanism, then the broad-band spectrum in the radio, and possibly up to the infrared, would have the form $P_\nu \propto \nu^{1/3}$. (see Fig. 3), which is not observed. It has been suggested that there are in fact two distributions of electrons: the one mentioned above, called the nebular electrons, and a second distribution to fit the radio observations, called radio electrons. The postulated radio electrons have a relatively flat spectrum at $\gamma \ll 10^6$.

In particular in the case of the Crab Nebula, which is the best studied PWN, a standing shock forms in the pulsar wind at a distance of $\sim 0.1$ pc from the pulsar. This shock is thought to form a power-law energy spectrum of nebular electrons which produce the bulk of the nebula’s synchrotron emission (see e.g., Kennel & Coroniti 1984a,b; Chevalier 2000; Arons 2002). Highly mobile features, called wisps, which have been associated with this shock are observed in optical and X-ray emission, (e.g., Hester et al. 1996; Mori et al. 2002).

The observation of the wisps also in the radio (Bietenholz et al. 2001, 2004) suggests that the electrons responsible for the radio emission are accelerated in the same region as those responsible for the higher-energy emission. However, if the radio emission is produced by low ($\gamma < 10^3$) energy “radio” electrons, then their origin is unclear, since it is difficult to produce them in sufficient quantity from a pulsar wind with high $\gamma$ (e.g., Arons 2002; Atoyan 1999). It would, therefore, be highly desirable to interpret the whole PWN spectrum with the single electron population, rather than requiring a separate population of radio electrons.

The DSR mechanism indeed suggests a simple and straightforward interpretation of the observed wide-band PWN spectra with only a single population of electrons. Within the DSR model, the flat radio spectrum (with spectral indices $\alpha_r = 0.2 \pm 0.2$) should be associated with the low-frequency (non-perturbative) DSR asymptote $P_\nu \propto \nu^{-(\nu-1)/2}$. The spectral index required of the random magnetic field, therefore, is in the range $\nu = 1 - 1.8$, which is in remarkable agreement with current turbulence models (Vainshtein et al. 1993; Schekochihin & Cowley 2003).

The optical emission (and possibly also the millimetre or infrared and/or to X-ray emission) then, is produced by the inner part of the electron distribution (equation 24) and has the standard form $P_\nu \propto \nu^{-\alpha_{\text{opt}}}$, which can occur in the X-ray or gamma-ray range depending on the actual value of $\gamma_2$, which is also a decreasing function of the distance from the pulsar due to significant radiative losses at these high energies (see, e.g., Bocchino & Bykov 2001), the DSR model predicts a spectrum $P_\nu \propto \nu^{-\alpha_{\text{opt}}}$, resembling the spectrum of relatively small-scale magnetic inhomogeneities. Interestingly, within the simplest model which assumes a single power-law spectrum of the random magnetic field with the index $\nu$, we expect a specific correlation between the radio spectral index $\alpha_r = (\nu - 1)/2$ and X-ray (or gamma-ray) spectral index $\alpha_x = \nu$:

$$\alpha_r = (\alpha_x - 1)/2. \quad (24)$$

Note that this equality may not hold exactly if the spectrum of the magnetic irregularities deviates from a single power-law with index $\nu$.

Can DSR reproduce the other characteristics of PWNe emission? The most important such characteristic is probably the significant polarization often seen in PWNe (e.g., Bietenholz & Kronberg 1991). In our simplified case of a completely random field, the average degree of polarization will evidently be zero if the turbulence is isotropic. However, modern models of the astrophysical turbulence (Goldreich & Sridhar 1995; Beresnyak et al. 2003) suggest the turbulence is highly anisotropic. In particular, the field downstream from a strong shock, such as that in a pulsar wind, can be almost two-dimensional, with virtually all the power in the plane of the shock. Such highly anisotropic field seems common in relativistic shocks, as it has been seen in simulations of a variety of shocks (see, e.g., recent reviews by Piran 2003; Silva 2006).

The radiation produced in the presence of a field largely confined to a plane will be highly polarized in the direction normal to the plane. For example, if the $k$-vectors of the random waves composing the turbulent magnetic fields line in a plane, the degree of polarization can be as large as 50% in the high-frequency range of the DSR spectrum (Toptygin & Fleishman 1987b). Furthermore, the presence of a regular component of the field will also lead to polarized radiation. A detailed analysis of the polarization patterns produced in the case of a combination of regular and random fields applicable to PWNe is beyond the scope of the current paper. However, it is clear that a combination of regular and random fields with anisotropic turbulence can potentially produce rather strongly polarized radiation.
5 COMPARISON TO OBSERVED PULSAR WIND NEBULAE SPECTRA

5.1 The Crab Nebula

The Crab Nebula has been well observed at all wave-bands from the radio to the gamma-ray. We compare the broad-band spectrum of the Crab, taken from Atoman & Aharonian (1996) and Aharonian & Atoman (1998), with those obtained using DSR. As was pointed out in §4 above, within conventional synchrotron theory, the combination of a relatively flat radio spectrum but steeper emission spectra at higher frequencies is impossible to reconcile with a single power-law energy spectrum of electrons.

DSR theory, however, can reproduce the entire broad-band spectrum of the Crab nebula from a single population of nebular relativistic electrons without introducing a second population of the radio electrons. A DSR spectrum providing a good fit to the observed broad-band spectrum of the Crab Nebula is shown in Figure 2. Remarkably, this DSR spectrum was calculated by using the parameters commonly accepted for the Crab, i.e., \( B_{\text{rms}} = 3 \times 10^{-4} \) G, \( \gamma_1 = 3 \times 10^7 \), and \( \xi = 2.6 \). Compared with the case of the regular field, there are two new parameters specifying the properties of the random magnetic field: the largest spatial scale in the random field, \( L_0 \), and the corresponding spectral index, \( \nu \). These values adopted for these parameters were \( L_0 = 10^{14} \) cm, which corresponds to the regime of large-scale random magnetic field, and \( \nu = 1.54 \) to match the observed radio-to-infrared spectral index. We should note that the DSR spectrum depends on \( \omega_{\text{loc}} \), which is the combination of \( \langle B^2 \rangle \) and \( L_0 \), equation (14), rather than on \( \langle B^2 \rangle \) and \( L_0 \) separately (as far as \( \omega_0 < \omega_{\text{loc}}/\gamma_1 \)). Therefore, the spectrum will remain unchanged if the rms magnetic field value scales with \( L_0 \) as \( B_{\text{rms}} \propto L_0^{\frac{1}{\gamma-1}} \).

DSR is therefore capable of reproducing the radio spectrum of the Crab without recourse to a large population of “radio-emitting” electrons with \( \gamma \lesssim 10^4 \). As the electrons responsible for the radio emission are more energetic than in the standard synchrotron theory, a smaller number of them is required. The Crab’s spectral luminosities at 1 GHz and \( 10^4 \) GHz are \( \sim 5 \times 10^{34} \) erg s\(^{-1}\) Hz\(^{-1}\) and \( \sim 10^{34} \) erg s\(^{-1}\) Hz\(^{-1}\) respectively (assuming a distance of 2 kpc). The total number of relativistic electrons required to produce these luminosities can then be calculated from Equation 19 and is \( 2 \times 10^{39} \). Over the lifetime of the nebula, this corresponds to an average injection rate of \( 10^{38.5} \) electrons per second, or, assuming \( \gamma \sim 10^6 \), an average energy injection rate of \( \sim 10^{38.5} \) erg s\(^{-1}\). This injection energy is reasonable in light of the pulsar’s spindown luminosity of \( \sim 5 \times 10^{38} \) erg s\(^{-1}\), and consistent with other estimates of the current injection rate (e.g., Arons 2002).

5.2 3C 58

Perhaps the second most intensely studied PWN is 3C 58. We have compiled measurements of its broadband spectrum extending from a few MHz in the radio to X-ray, and show the resulting spectrum in Figure 3. There is a significant difference between the spectrum of 3C 58 and the Crab: In the Crab, the spectrum is an unbroken power-law from a below 100 MHz near 100 GHz, where a break with a change in slope of \( \Delta \alpha = 0.5 \) occurs. This break is interpreted within standard synchrotron theory as the synchrotron ageing break (e.g., Chevalier 2000). In 3C 58, by contrast, a spectral break must occur near ~100 GHz, with a change in slope which is > 0.5 (Green & Scheuer 1992). With standard synchrotron theory, a break with \( \Delta \alpha > 0.5 \) cannot be produced, and furthermore the low frequency of the break in 3C 58 implies an unreasonably large magnetic field (these difficulties have previously been pointed out by Green & Scheuer 1992, Woltjer et al. 1997, Salvati et al. 1998).

DSR however, naturally produces both spectra with several breaks and spectral breaks with \( \Delta \alpha > 0.5 \). In particular, in Figure 4 the spectrum of 3C 58 is reproduced with the parameters \( B_{\text{rms}} = 10^{-5} \) G, \( \gamma_1 = 0.8 \times 10^6 \), \( \xi = 3.2 \), \( L_0 = 10^{14} \) cm, \( \nu = 1.2 \), not very different from the parameters adopted for the Crab above. Remarkably, two quite differently looking spectra, those of Crab and 3C 58, can easily be reproduced within the same regime of DSR, although with appropriately adjusted parameter values.

5.3 Other PWNe

Broad-band spectral measurements as detailed as those for the Crab Nebula and 3C 58 are not available for other PWNe. Nevertheless, the radio and the X-ray spectral indices have been measured for a good number of systems. Equation (24) suggests that within the DSR model there should be a tight correlation between radio and high-energy (X-ray or gamma-ray) spectral indices of the emissivity of PWNe. It is known, however, that the size of a nebula observed in the X-rays is typically smaller than the corresponding radio and optical size (although a weak X-ray corona can extend outside the radio emitting region, see, e.g., Bocchino & Bykov 2001). This means that radiative losses of high-energy electrons are important and the overall PWN spectrum represents a convolution of the local emissivity spectrum with actual inhomogeneous spatial distribution of the emitting material.

As a result, the X-ray spectrum will represent an appropriately weighted average of two DSR spectral asymptotes, namely \( P_\nu \propto \omega^{-\alpha_{\text{loc}}} \) and \( P_\nu \propto \omega^{-\nu} \), and the X-ray spectral index of the whole nebula will lie somewhere in between these two values,
with the softest possible X-ray spectral index being $\alpha_x = \nu$, while the hardest one being $\alpha_x = \alpha_{nth}$. Stated another way, equation (24) offers a kind of “line of death” rather than a strict correlation.

We have compiled from the literature measurements of the X-ray and radio spectral indices for 16 PWNe. They are tabulated in Table 1 and Figure 5 displays the radio and X-ray spectral indices. Note that in all but one cases $\alpha_x > 0.5$, which is a bounding value for non-thermal emission from shock accelerated electrons. Remarkably, all the PWNe obey the line of death predicted based on the DSR model of PWN emission, which is clearly supportive of the DSR model, although more data is highly desirable to increase the statistical significance of this finding.

6 DISCUSSION AND CONCLUSIONS

The radiation produced by a charged particle in the presence of stochastic magnetic field differs significantly from the standard synchrotron radiation generated in the presence of large-scale regular magnetic field only. The corresponding radiative process, called diffusive synchrotron radiation (DSR), can produce a variety of spectral shapes depending on the parameter combination.

This paper describes a special regime of DSR, namely that produced in the presence of a strong stochastic magnetic field with structure on a wide variety of scale, but without any regular field. The emission spectra in this regime are obtained from the analysis of the general DSR theory presented in Toptygin & Fleishman (1987a), Toptygin et al. (1987) and Fleishman (2005). We emphasize, however, that even though the general theory includes the special case studied in this paper, the latter has not been clearly formulated and particularly discussed yet.

Stochastically turbulent magnetic fields are common in astrophysical objects. In particular, it is widely believed that relatively strong stochastic component of the magnetic field can be generated by the highly relativistic wind in PWNe (Kazimura et al. 1998). Interestingly, the idea of highly turbulent magnetic field in a PWN came from observations much earlier: Reynolds & Aller (1988) interpreted the presence of sharp filament edges in 3C 58 as a result of highly diffusive motion of the relativistic electrons due to enhanced Alfvén turbulence; specifically, Reynolds & Aller (1988) found that $\delta B / B > 0.1$ on scales of about $10^{11}$ cm is needed to provide the required diffusion coefficient.

Accordingly, we applied this specific DSR regime to interpret the whole broad-band PWN spectra by DSR produced by a single nebular population of the relativistic electrons without additional radio electrons. We reached a remarkably good agreement between the model and observations for two the most studied objects, the Crab Nebula and 3C 58, and also found a statistical evidence in favour of this model based on analysis of the correlations between radio and X-ray spectral indices for a large number of PWNe.

We should mention two shortcomings of simplest version of the DSR model for PWNe presented here. First, in this paper we assumed that the random magnetic field is statistically isotropic, which is typically appropriate for calculating the radiation intensity, but might be highly incorrect for calculating the polarization of radiation. As we mentioned in §4 in the case of anisotropic stochastic magnetic field the degree of DSR polarization can be as high as 50% or more. There are two possible avenues of reconcile the DSR model with the high observed polarization. First, the magnetic turbulence might be substantially anisotropic. This conclusion conforms to modern models of the turbulence generation in astrophysical objects and fast shocks. Second, we considered only the random component of the magnetic field and ignored completely any regular component, which is likely also present in the nebula. Nonetheless, that even this simplest version of the DSR model gives excellent fit to the broadband PWN spectra with only a single power-law population of electrons, which is not possible with the standard synchrotron model. We believe that this success of the DSR model calls for developing the DSR theory further to include the joint effect of strong random and regular fields and to accurately calculate the degree of polarization in various DSR regimes.

We conclude that the interpretation of the broad-band PWN spectra using the DSR model is self-consistent and also offers more ways to observationally study the properties of turbulent magnetic fields such as those produced by relativistic wind in the PWNe.

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Table 1. Radio and X-ray Spectral Indices of PWNe

| Name              | Radio Spectral Index $^{a}$ | Ref.$^{b}$ | X-ray Spectral Index $^{c}$ | Ref.$^{b}$ |
|-------------------|-----------------------------|-----------|----------------------------|-----------|
| G16.7+0.1         | 0.15 ± 0.05                 | 3         | 0.11 ± 0.29                | 4         |
| G21.5−0.9         | 0.0                         | 1         | 0.90 ± 0.2                 | 5         |
| G29.7−0.3 (Kes 75)| 0.30 ± 0.20                 | 3         | 0.92 ± 0.04                | 2         |
| G54.1+0.3         | 0.13 ± 0.05                 | 6         | 0.64 ± 0.18                | 2         |
| G74.9+1.2 (CTB87) | 0.26                        | 7         | 1.48 ± 0.56                | 8         |
| G130.7+3.1 (3C58) | 0.10 ± 0.02                 | 9         | 0.92 ± 0.04                | 2         |
| G184.6−5.8 (Crab) | 0.30 ± 0.04                 | 10        | 1.14 ± 0.01                | 2         |
| G189.1+3.0 (IC443)| 0.02 ± 0.10                 | 11        | 0.7 ± 0.10                 | 11        |
| G263.9−3.3 (Vela X)| 0.10                       | 12        | 0.50 ± 0.04                | 2         |
| G291.0−0.1 (MSH 11−62)| 0.29 ± 0.05   | 13        | 0.9 ± 0.2                  | 13        |
| G380.4−1.2 (MSH 15−52)| 0.05 ± 0.05 | 14        | 0.90 ± 0.20                | 14        |
| G434−2.3          | 0.3                         | 15        | 0.77 ± 0.08                | 15        |
| 0540−69           | 0.25 ± 0.1                  | 16        | 1.09 ± 0.11                | 2         |
| N157B             | 0.19                        | 17        | 1.28 ± 0.12                | 2         |
| B0543-0685        | 0.10 ± 0.05                 | 18        | 0.90 ± 0.4                 | 18        |

$^{a}$ The radio spectral index ($S \propto f^{-\alpha}$) of the remnant or of its pulsar wind nebula segment.

$^{b}$ References: 1. Green (2004). see also Green D.A., 2004, ‘A Catalogue of Galactic Supernova Remnants (2004 January version)’, Mullard Radio Astronomy Observatory, Cavendish Laboratory, Cambridge, United Kingdom (available at http://www.mrao.cam.ac.uk/surveys/surs); 2. Gotthelf (2003); 3. Bock & Gaensler (2005); 4. Helfand et al. (2003); 5. Matheson & Safi-Harb (2005); 6. Velusamy & Becker (1998); 7. Morsi & Reich (1987); 8. Asaoka & Koyama (1990); 9. Green (1986); 10. Bietenholz et al. (1997); 11. Olbert et al. (2001); 12. Hales et al. (2004); 13. Harrus et al. (1998); 14. Gaensler & Wallace (2003); 15. Romani et al. (2003); 16. Manchester et al. (1993); 17. Lazendic et al. (2000); 18. Gaensler et al. (2003)

$^{c}$ The X-ray spectral index ($S \propto f^{-\alpha}$) of the remnant or of its pulsar wind nebula segment.
Figure 1. Single particle DSR spectra for $\nu = 1.5$ and $\gamma = 10^6$, where $\nu$ is the spectral index of the turbulence, $\gamma$ the Lorentz factor of the particle, and $\omega_{pe}$ the plasma frequency. Cases of small-scale ($\omega_0/\omega_{pe} = 10^2$) and large-scale ($\omega_0/\omega_{pe} = 10^{-4} \text{ and } 10^{-7}$) random magnetic field are shown. The shape of the DSR radiation spectrum changes significantly as the largest scale of the field ($L_0 = 2\pi c/\omega_0$) changes as described in the text; in particular, the regime of large-scale magnetic field (red/solid and green/dashed curves) differs substantially from the regime of the small-scale field (blue/dash-dotted curve) which is described in Fleishman (2006b).
Figure 2. Broad-band DSR spectra as produced by three different power-law distributions (equation 21) of ultra-relativistic nebular electrons, which extend from a Lorentz factor $\gamma_1 = 0.3 \times 10^6$ to ones of $\gamma_2 = 3 \times 10^6$, $3 \times 10^7$, and $3 \times 10^8$, respectively. The values of the model parameters were chosen so as to match the spectral energy distribution of the Crab Nebula (see text, §5.1). The values adopted for the spectral index of the turbulence, $\nu$, the energy index of the power-law electron distribution, $\xi$, the plasma frequency, $\omega_{pe}$, and the mean square of the random magnetic field, $\langle B^2 \rangle$, are indicated in the figure.
Figure 3. DSR radiation spectra for a power-law magnetic field spectrum characterized by different spectral indices, $\nu$. The green/dashed curve is $\nu = 1.2$, the red/solid curve is $\nu = 1.5$. In addition, the blue/dash-dotted curve shows the standard synchrotron spectrum for the same power-law electron distribution and the same magnetic field energy density. Arrows indicate the changes of the break frequencies which separate various spectral asymptotes. A region with the standard non-thermal spectrum, $P_\nu \propto \nu^{-\alpha_{ntb}}$, is present in all cases, although at differing frequency ranges and at different levels. Beyond this region, the spectra differ significantly from each other. As in Figure 2, the values for the energy index of the power-law electron distribution, $\xi$, the plasma frequency, $\omega_p$, and the mean square of the random magnetic field $\langle B^2 \rangle$ are indicated in the figure.
Figure 4. The model DSR spectra and observed broad-band spectrum of 3C 58. The radio data are taken from Green (1986), Green & Scheuer (1992), Morsi & Reich (1987), and Salter et al. (1989). The infrared upper limits are from Green & Scheuer (1992) and Green (1994). The X-ray data are from Davelaar et al. (1986), Helfand et al. (1995), Becker et al. (1982), Torii et al. (2000), and Gotthelf et al. (2006), with plotted gamma-ray value being cited in Torii et al. The values adopted for the energy index of the power-law electron distribution, $\xi$, the plasma frequency, $\omega_{pe}$, and the mean square of the random magnetic field $\langle B^2 \rangle$ are indicated in the figure.
Figure 5. A correlation plot of the X-ray ($\alpha_X$) vs. radio ($\alpha_{\text{Radio}}$) spectral indices for the 16 PWNe for which the spectral indices were available in the literature. The dotted line is a linear fit: $\alpha_X = 0.84 \alpha_{\text{Radio}} + 0.74$, with the correlation coefficient $R = 0.33$. The dashed line is the “line of death” predicted within the simplest version of the DSR model based on equation (24) as described in §5.3. The radio and X-ray spectral indices and their sources are listed in Table 1.