This letter derives the probability distribution function (PDF) of received symbols for orthogonal frequency-division multiplexing (OFDM) based massive multiple-input, multiple-output (MMIMO) systems, which uses maximal-ratio combining (MRC) detection. The effects of noise and interferences are evaluated through random variables, and the PDF is then derived from their joint probability and characteristic functions. The bit error rate (BER) using Binary and Quadrature Phase Shift Keying (BPSK, QPSK), and M-ary Amplitude Modulation (AM) waveforms is then analyzed by using this PDF. Simulation and analytical results confirm that the derived equations provide accurate PDF and BER, and therefore, can be used to evaluate the performance of OFDM-MMIMO systems.

Introduction: MMIMO systems have been widely utilized in wireless networks to enhance their spectral and energy efficiencies [1]. Since the number of transmit and receive antennas in MMIMO systems is very large, linear detection has been the focus in a plethora of research works due to its reduced computational complexity compared to non-linear detection. The PDF of received symbols for a linear detector is usually assumed to be a Gaussian distribution. However, a number of research works focused on deriving more accurate PDF approximations for linear detection. The PDF of the received symbols for zero forcing (ZF) detection, which uses Neumann series expansion (NSE), was introduced in 2017 [2], where the authors used a newly derived PDF to evaluate BER for uncoded and coded systems, and enhance the performance of soft-output ZF detection. The ZF operation in NSE was limited to the first order term and Gram matrix was assumed to be a diagonal matrix. As a result, a better approximation can be achieved by considering the ignored co-channel interferences at the receiver. Recently, Hama, in [3], derived PDF of the output of MRC detection with M-PSK and M-QAM symbols. However, the output of MRC detection was normalized by a constant instead of the diagonal components in the Gram matrix.

This letter introduces a different approach to derive the PDF of received symbols for a multi-user OFDM-MMIMO system utilizing MRC detection and Gray-coded M-QAM modulation. The effect of interference and noise is considered as random variables, and their PDFs are then analyzed using their joint probabilities and characteristic functions. Moreover, the derived PDF is used to evaluate BER for this system.

System model: An uplink multi-user OFDM-MMIMO system is considered in this letter. Inter-symbol interference is assumed to be completely eliminated by OFDM by selecting an appropriate cyclic prefix (CP) that exceeds the delay spread of the channel. After removing the CP and applying the fast Fourier transform (FFT) operation, the received signal vector, $Y_n \in C^{N_r \times 1}$, in frequency-domain at the $n^{th}$ subcarrier can be expressed as

$$Y_n = H_n X_n + W_n,$$

where $N_t$ and $N_r$ represent the number of transmit and receive antennas, respectively. $H_n \in C^{N_r \times N_t}$ is channel frequency response matrix with components, $h_{n,j}$, that are complex-valued, zero-mean, Gaussian random variables, that is, $CN(0, \sigma_n^2)$. Their variance per dimension $\sigma_n^2$ is 0.5. $W_n \in C^{N_r \times 1}$ represents complex-valued, zero-mean, additive white Gaussian noise (AWGN), that is, $CN(0, \sigma_n^2)$, where the variance $\sigma_n^2$ is defined as

$$\sigma_n^2 = \frac{N_t E_b}{2 \log_2(M) P_t},$$

where $E_b$ is the average symbol energy and $M$ is the number of signal constellation points. $\gamma_n = E_b/N_0$ represents the signal-to-noise ratio per bit. $X_n \in C^{N_t \times 1}$ represents the transmitted symbols that are uniformly distributed and modulated. Let us define the index $\lambda = \ell, Q$ and let $X_n^\ell$ and $X_n^Q$ be the in-phase and quadrature components of $X_n$. The PDF of these variables for M-QAM waveforms can be expressed as

$$p_{\lambda}(X_n^\lambda) = \frac{1}{\Delta} \sum_{\alpha=\lambda}^{\Delta} \delta(X_n^\lambda - \tilde{X}_n^\lambda),$$

where $\Delta$ is the number of symbols per dimension in the signal constellation, and $\tilde{X}_n^\lambda$ is the constellation point per dimension, that is, $\{-1, 1\}$ for BPSK and QPSK symbols.

Classical MRC detection: In order to detect the transmitted symbols $X_n$, the MRC detector multiplies the received symbols with a weight matrix $H_n^\dagger$, to obtain an estimate $\hat{X}_n$ as

$$\hat{X}_n = H_n^\dagger Y_n.$$

Substituting $Y_n$ from (1), this equation can be rewritten as

$$\hat{X}_n = G_n X_n + H_n^\dagger W_n,$$

where $G_n = H_n^\dagger H_n$ is the Gram matrix with components denoted as $G_{n,j}$. According to (5), the $i^{th}$ component in $X_n$ contains information for the $i^{th}$ user and can be expressed as

$$\hat{X}_i = \frac{\sum_{j=1}^{N_r} |H_{i,j}|^2 X_j + \sum_{j \neq i} G_{i,j} X_j + \sum_{j=1}^{N_r} |H_{i,j}|^2 W_j}{\sum_{j=1}^{N_r} |H_{i,j}|^2}.$$

Before detection, we need to normalize the estimated symbols due to the multiplication with the Gram matrix, thus, the symbols for the $i^{th}$ user is then evaluated from $\hat{X}_i$ as

$$\hat{\beta}_i = \frac{\sum_{j=1}^{N_r} |H_{i,j}|^2 \beta_j + \sum_{j \neq i} G_{i,j} \beta_j + \sum_{j=1}^{N_r} |H_{i,j}|^2 \gamma_j}{\sum_{j=1}^{N_r} |H_{i,j}|^2}.$$

Finally, by substituting $\hat{\beta}_i$ in (6), the received symbols for MRC detection can be expressed as

$$\hat{X}_i = X_i + \frac{N_r}{|H_{i,i}|^2} H_{i,i} X_i + \frac{N_r}{|H_{i,i}|^2} H_{i,i} W_i.$$

Definition of random variables: Undoubtedly, the second and third term in (8) represent the effect of co-channel interferences and enhanced noise in the received symbols for MRC detection. We proceed now with the derivation of the PDF of the received symbol for MRC detection, where utilize $M$-QAM modulation. A number of random variables are defined to simplify the explanation. According to the MMIMO channel model in (1), the $i^{th}$ component of $Y_n$ can be expressed as

$$Y_i = H_{i,i} X_i + \sum_{j=1,j \neq i}^{N_r} H_{i,j} X_j + W_i,$$

where $X_i$ is information which is transmitted by $i^{th}$ user. Let $\alpha_j$ be $\sum_{j=1,j \neq i}^{N_r} H_{i,j} \gamma_j$, (9) is rewritten as

$$Y_i = H_{i,i} X_i + \alpha_i + W_i.$$

Let $\beta_i$ be $\alpha_i + W_i$, this equation is become

$$Y_i = H_{i,i} X_i + \beta_i.$$

According to the definition of $Y_i$ in (11), $\tilde{X}_i$ in (8) can be rewritten as

$$\tilde{X}_i = X_i + \frac{\sum_{j=1}^{N_r} H_{i,j} \beta_j + \sum_{j=1}^{N_r} |H_{i,j}|^2 \gamma_j}{\sum_{j=1}^{N_r} |H_{i,j}|^2}.$$

Let $\eta_j = \sum_{j=1}^{N_r} H_{i,j} \beta_j$, $\xi_i = \sum_{j=1}^{N_r} |H_{i,j}|^2 \gamma_j$, and $Z_i = \eta_j/\xi_i$, (12) can be expressed as

$$\tilde{X}_i = X_i + Z_i.$$
Z, in (13) represents the effect of co-channel interference and noise in the received symbols. Since \(\alpha_i, \beta_i, \eta_i, \text{and } \xi_i\) are from functions of independent random variables, the PDF of \(Z\) can be then determined from the joint probabilities and characteristic functions of these variables.

The PDF of interferences \(\alpha_i\): For M-QAM symbols, the in-phase and quadrature components of \(\alpha_i\) can be expressed as

\[
\alpha_i^r = \sum_{j=1\atop j \neq i}^{N_t} (H_i^r X_j^r - H_i^q X_j^q), \quad (14a)
\]

\[
\alpha_i^q = \sum_{j=1\atop j \neq i}^{N_t} (H_i^r X_j^q + H_i^q X_j^r), \quad (14b)
\]

Since \(H_i^r\) is a Gaussian random variable, \(\alpha_i^r\) is a summation of \((2N_t - 2)\) products of \(H_i^r\) and \(X_j^r\). Let \(U_i^r\) be \(H_i^r X_j^r\), the PDF of this random variable is evaluated from (4) as

\[
p_a(U_i^r) = \frac{1}{\Delta} \frac{1}{2\pi \sigma_u^2} \sum_{v=1}^{\Delta} \exp \left( -\frac{(U_i^r - \overline{U}_i^r)^2}{2\sigma_u^2} \right).
\]

If the PDFs of \(H_i^r\) and \(X_j^r\) are substituted in (15), the result from this integral operation is

\[
p_a(U_i^r) = \frac{1}{\Delta} \frac{1}{2\pi \sigma_u^2} \sum_{v=1}^{\Delta} \exp \left( \frac{-\left(U_i^r - \overline{U}_i^r \right)^2}{2\sigma_u^2} \right).
\]

The characteristic function of \(U_i^r\) is then evaluated from \(p_a(U_i^r)\), and the result is given as

\[
\Phi_a(\omega) = \frac{1}{\Delta} \sum_{v=1}^{\Delta} \exp \left( -\frac{\left(U_i^r - \overline{U}_i^r \right)^2}{2\sigma_u^2} \right).
\]

Since \(\alpha_i^r\) is a summation of \((2N_t - 2)\) terms of \(U_i^r\), the characteristic function of \(\alpha_i^r\) can be evaluated from \(\Phi_a(\omega)\) as

\[
\Phi_a(\omega) = \Phi_a(\omega) \Phi_a(\omega).
\]

Therefore, the PDF of \(\beta_i^r\) can be determined by

\[
p_{\beta_i^r} = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} e^{-j\omega\beta_i^r} \Phi_a(\omega) d\omega.
\]

The PDF of noise and interference \(\beta_i^r; \beta_i^q\) is sum of \(\alpha_i^r\) and \(w_i\), and the characteristic function of this random variable is determined by

\[
\Phi_{\beta_i}(\omega) = \Phi_a(\omega) \Phi_{\eta_i}(\omega).
\]

Evidently, (23) is a sum of zero-mean Gaussian functions, which are multiplied by constant coefficient. Their variance \(\sigma^2_{\beta_i}\) depends on \(k_1, k_2, \ldots, k_\Delta\), and is determined by

\[
\sigma^2_{\beta_i}(k_1, k_2, \ldots, k_\Delta) = \frac{1}{\Delta} \sum_{v=1}^{\Delta} \exp \left( \frac{-2\sigma^2_{\beta_i}}{\Delta \sigma_a^2} \right)
\]

Deriving PDF of \(\eta_i\): \(\eta_i\) is a summation of \((2N_t)\) products of \(H_i^r\) and \(\beta_i^r\). Generally, the PDF of summation of random variables is evaluated through convolution of their PDFs, and the PDF of product of random variables is determined through integral operation. However, \(p_a(H_i^r)\) in (23) is a sum of scaled Gaussian functions and, \(p_a(H_i^r)\) is a Gaussian distributed too. Therefore, we use Equation (6.9) in [5] to simplify the PDF of the sum of product of two zero-means Gaussian random variables. Coefficients in (23) are evaluated separately. If the PDF of \(H_i^r\) and \(\beta_i^r\) are substituted in (6.9) in [5], \(p_a(\eta_i)\) becomes

\[
p_a(\eta_i) = \frac{2(N_t - 2)!}{\Gamma(N_t) \Delta^{2N_t - 2}} \sum_{k_1, k_2, \ldots, k_{N_t} = -2}^{\Delta} \frac{1}{\pi k_1} \times
eq \sum_{k_{N_t}, v=1}^{\Delta} \exp \left( \frac{-\sigma^2_{\beta_i} \Delta}{\sigma_a^2} \right). \]

The PDF of received symbols \(Z_i\): The PDF of \(\xi_i\) was previously derived as [2]

\[
p_a(\xi_i) = \frac{x_i h_i \sigma_a^2}{\sigma^2_{\beta_i} \Gamma(N_t)} \xi_i, i > 0.
\]

Since \(Z_i\) is the ratio of \(\eta_i\) and \(\xi_i\), its PDF is evaluated from their joint probability as [4]

\[
p_a(Z_i) = \int_0^\infty \xi_i p_a(\eta_i, \xi_i) d\xi_i.
\]

If the PDFs of \(\eta_i\) and \(\xi_i\) in (25) and (26) are substituted in (27), the result from this integration is

\[
p_a(Z_i) = \frac{2(N_t - 2)!}{\Gamma(N_t) \Delta^{2N_t - 2}} \sum_{k_1, k_2, \ldots, k_{N_t} = -2}^{\Delta} \frac{1}{\pi k_1} \times
eq \sum_{k_{N_t} = 1}^{\Delta} \exp \left( \frac{-\sigma^2_{\beta_i} \Delta}{\sigma_a^2} \right). \]

Equation (28) represents the PDF of \(Z_i\) for M-QAM waveform. It is worth to point out that the PDF of \(Z_i\) for BPSK and QPSK waveform can be derived from the PDF of \(\alpha_i^r, \beta_i^r\), and \(\eta_i\) using the same approach as for M-QAM. Since \(\Delta = 2\), the PDF of \(Z_i\) for BPSK and QPSK waveform is given as

\[
p_a(Z_i) = \frac{\sigma^2_{\beta_i}}{\Gamma(N_t)} \times
eq \sum_{k_{N_t} = 1}^{\Delta} \exp \left( \frac{-\sigma^2_{\beta_i} \Delta}{\sigma_a^2} \right). \]

The parameter \(\sigma^2_{\beta_i}\) for BPSK and QPSK waveform is defined as

\[
\sigma^2_{\beta_i, \text{BPSK}} = (N_t - 1) \sigma_a^2 + \sigma_{\eta_i}^2,
\]

\[
\sigma^2_{\beta_i, \text{QPSK}} = 2(N_t - 1) \sigma_a^2 + \sigma_{\eta_i}^2.
\]

Figure 1 compares simulation and analytical results for the PDF of \(Z_i\) for OFDM-MMIMO systems utilizing 16-QAM. \(E_b/No\) was \(-10\) dB. The OFDM frame size was chosen to be 1024 symbols, while \(N_t = 5, N_r = 64, 128, 256\). Evidently, the derived equation provides an accurate PDF compared to that of simulation result. These results were also
compared using a 2-sample Kolmogorov–Smirnov test, which has significance level of 5%, thus, confirming that there is no significant difference between analytical and empirical results.

**BER analysis:** BER of OFDM-MMIMO systems can be directly analyzed through $p_i(Z^*_i)$. For BPSK and QPSK and 16-QAM, there are only two constellation points per dimension, and two error events in each dimension. Since the occurrence of these constellation points is uniformly distributed, their symbol error rate $P_e$ can be evaluated by considering only one error event as

$$P_e = \int_0^\infty p_i(Z^*_i + 1) dZ^*_i.$$  

(31)

However, solving this equation with $p_i(Z^*_i)$ directly is complex. In order to solve this issue, the definition of $p_i(Z^*_i)$ in (27) is substituted into (31) as [2]

$$P_e = \int_0^\infty \int_0^\infty \xi p_i((Z^*_i + 1) \xi) p_i(\xi) dZ^*_i d\xi.$$  

(32)

If $p_i(\eta^*_i)$ for QPSK and $p_i(\xi)$ in (26) are substituted in (32), we obtain

$$P_e = \frac{\sigma^2}{\Gamma(N_r, F)} \sum_{i=0}^{N_r-1} \frac{\sigma_i^2}{\Gamma(i)} \sum_{j=0}^{N_r-1} 2^{N_r-2-j} \beta(j, 2) \eta^*_i \eta j.$$  

(33)

If the system uses Gray-coded mapping to produce BPSK or QPSK symbols, their BER $P_e$ is as same as $P_i$ in (33). Similarly, 16-QAM has four constellation points per dimension, that is, $-3, -1, 1,$ and $3$, and six error events. The probability of each error event is also determined by substituting $p_i(\eta^*_i)$ and $p_i(\xi)$ in (25) and (26) in (32) to obtain

$$P_{e_{-16-QAM}} = \frac{(2N_r - 2)!}{\Delta(N_r - 1)!} \sum_{i=0}^{N_r-1} \sum_{j=0}^{N_r-1} 2^{N_r-1-j} \beta(j, 2) \eta^*_i \eta j.$$  

(34)

The BER for 16-QAM waveform is then evaluated as

$$P_{e_{-16-QAM}} = 6P_{e_{-16-QAM}} / 4 \log_2(\Delta)$$  

(35)

Substituting $P_{e_{-16-QAM}}$ from (34) and $\Delta = 4$ in (35), $P_{e_{-16-QAM}}$ becomes

$$P_{e_{-16-QAM}} = \frac{3(2N_r - 2)!}{\Delta(N_r - 1)!} \sum_{i=0}^{N_r-1} \sum_{j=0}^{N_r-1} 2^{N_r-1-j} \beta(j, 2) \eta^*_i \eta j.$$  

(36)

Figure 2 compares simulation and analytical results of BER for the MMIMO systems using QPSK and 16-QAM, where $N_t = 10$, while $N_r = \{64, 128, 256\}$. The block size utilized was 1024 bits. Evidently, both results are closely matched. The small deviation is due to the fact that for the systems with small $N_r$, the PDF of $Z^*_i$ is derived from the joint probability of $\eta_i$ and $\xi_i$, by assuming the variables to be independent random variables, which is only fulfilled if $N_r$ is large enough.

In Figure 3, the analytical results from (36) are now compared to that of the BER analysis in [3]. The block size utilized and $N_t$ were as described in the previous results with $N_r = 5$. The results confirm that the BER from the proposed analysis are closer to the simulation results than that of [3]. It is worth noting that for low $E_b/N_0$, the $P_e$ in (32)

**Conflict of interest:** The authors declare no conflicts of interest.

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**References**

1. Araújo, D.C., et al.: Massive MIMO: Survey and future research topics. *IET Commun.* 10(15), 1938–1946 (2016)

2. Al-Askery, A.J., et al.: Performance analysis of coded massive mimo-ofdm systems using effective matrix inversion. *IEEE Trans. Commun.* 65(12), 5244–5256 (2017)

3. Hama, Y., Ochiai, H.: Performance analysis of matched-filter detector for mimo spatial multiplexing over rayleigh fading channels with imperfect channel estimation. *IEEE Trans. Commun.* 67(5), 3220–3233 (2019)

4. Papoulis, A.: *Probability, Random Variables, and Stochastic Processes*, 4th ed. McGraw-Hill, Boston (2002)

5. Simon, M.K.: *Probability Distributions Involving Gaussian Random Variables a Handbook for Engineers and Scientists.* Springer, New York (2006)