ABSTRACT. We consider a free interpolation problem in Nevanlinna and Smirnov classes and find a characterization of the corresponding interpolating sequences in terms of the existence of harmonic majorants of certain functions. We also consider the related problem of characterizing positive functions in the disc having a harmonic majorant. An answer is given in terms of a dual relation which involves positive measures in the disc with bounded Poisson balayage. We deduce necessary and sufficient geometric conditions, both expressed in terms of certain maximal functions.

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1. Introduction and Statement of Results

1.1. Interpolating sequences for the Nevanlinna Class. Let $\Lambda$ be a discrete sequence of points in the unit disk $\mathbb{D}$. For a space of holomorphic functions $X$, the interpolation problem consists in describing the trace of $X$ on $\Lambda$, i.e. the set of restrictions $X|\Lambda$, regarded as a sequence space. One approach is to fix a target space $l$ and look for conditions so that $X|\Lambda = l$. An alternative approach, known as free interpolation, is to require that $X|\Lambda$ be ideal, i.e. stable under multiplication by $\ell^\infty$. See [Nik02, Section C.3.1 (Volume 2)], in particular, Theorem C.3.1.4, for functional analytic motivations. This approach is natural for those spaces that are stable under multiplication by $H^\infty$, the space of bounded holomorphic functions on $\mathbb{D}$. For Hardy and Bergman spaces both definitions turn out to be equivalent, with the usual choice of $l$ as an $\ell^p$ space with an appropriate weight (see [ShHSh], [Sc93]).

The situation changes for the non-Banach classes we have in mind, namely the Nevanlinna class

$$N = \left\{ f \in \text{Hol}(\mathbb{D}) : \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta < \infty \right\}$$

and the related Smirnov class

$$N^+ = \left\{ f \in N : \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(e^{i\theta})| \, d\theta \right\}.$$

We briefly discuss the known results. Naftalevič [Na56] described the sequences $\Lambda$ for which the trace $N|\Lambda$ coincides with the sequence space $l_{Na}^\Lambda = \{(a_\lambda) : \sup_{\lambda} (1 - |\lambda|) \log^+ |a_\lambda| < \infty\}$ (we state the precise result after Proposition 1.11). The choice of $l_{Na}$ is motivated by the fact that $\sup_{\lambda} (1 - |\lambda|) \log^+ |f(\lambda)| < \infty$ for $f \in N$, and this growth is attained. Unfortunately, the growth condition imposed in $l_{Na}$ forces the sequences to be confined in a finite union of Stolz angles. Consequently a big class of Carleson sequences (i.e. sequences such that $H^\infty|\Lambda = \ell^\infty$), namely those containing a subsequence tending tangentially to the boundary, cannot be interpolating in the sense of Naftalevič. This does not seem natural, for $H^\infty$ is in the multiplier space of $N$. In a sense, the target space $l_{Na}$ is “too big”. Further comments on Naftalevič’s result can be found in [HaMa01] and below, after Proposition 1.11.

For the Smirnov class, Yanagihara [Ya74] proved that in order that $N^+|\Lambda$ contains the space $l_{Ya}^\Lambda = \{(a_\lambda) : \sum_{\lambda} (1 - |\lambda|) \log^+ |a_\lambda| < \infty\}$, it is sufficient that $\Lambda$ is a Carleson sequence. However there are Carleson sequences such that $N^+|\Lambda$ does not embed into $l_{Ya}^\Lambda$ [Ya74 Theorem 3] : the target space $l_{Ya}^\Lambda$ is “too small”.

We now turn to the definition of free interpolation.

Definition. A sequence space $l$ is called ideal if $\ell^\infty \subset l$, i.e. whenever $(a_n)_n \in l$ and $(\omega_n)_n \in \ell^\infty$, then also $(\omega_n a_n)_n \in l$.

Definition. Let $X$ be a space of holomorphic functions in $\mathbb{D}$. A sequence $\Lambda \subset \mathbb{D}$ is called free interpolating for $X$ if $X|\Lambda$ is ideal. We denote $\Lambda \in \text{Int} X$.

Remark 1.1. For any function algebra $X$ containing the constants, $X|\Lambda$ is ideal if and only if $\ell^\infty \subset X|\Lambda$. 
The inclusion is obviously necessary. In order to see that it is sufficient notice that, by assumption, for any $(\omega_\lambda)_{\lambda} \in \ell^\infty$ there exists $g \in X$ such that $g(\lambda) = \omega_\lambda$. Thus, if $(f(\lambda))_{\lambda} \in X|\Lambda$, the sequence of values $(\omega_\lambda f(\lambda))_{\lambda}$ can be interpolated by $fg \in X$.

It is then clear that $\text{Int} \ N^+ \subset \text{Int} \ N$.

Since the Nevanlinna and Smirnov classes contain the constants, free interpolation for these classes entails the existence of a nonzero function $f \in N$ vanishing on $\Lambda$, hence the Blaschke condition $\sum_{\lambda \in \Lambda} (1 - |\lambda|) < \infty$ is necessary and will be assumed throughout this paper.

Given the Blaschke product $B_\Lambda = \prod_{\lambda \in \Lambda} b_\lambda$ with zero-sequence $\Lambda$, denote $B_\lambda = B_{\Lambda \setminus \{\lambda\}} = B_\Lambda / b_\lambda$. Here $b_\lambda = (|\lambda|/\lambda)(\lambda - z)(1 - \lambda z)^{-1}$. Define then

$$\varphi_\Lambda(z) := \begin{cases} \log |B_\lambda(\lambda)|^{-1} & \text{if } z = \lambda \in \Lambda \\ 0 & \text{if } z \notin \Lambda \end{cases}$$

**Definition.** We say that a Borel measurable function $\varphi$ defined on the unit disc admits a **positive harmonic majorant** if and only if there exists a positive harmonic function $h$ on the unit disc such that $h(z) \geq \varphi(z)$ for any $z \in \mathbb{D}$.

Let $\text{Har}(\mathbb{D})$ denote the space of harmonic functions in $\mathbb{D}$ and $\text{Har}^+(\mathbb{D})$ the subspace of its positive functions. Consider also the Poisson kernel in $\mathbb{D}$:

$$P(z, \zeta) = P_z(\zeta) = \text{Re} \left( \frac{\zeta + z}{\zeta - z} \right) = \frac{1 - |z|^2}{|\zeta - z|^2}. $$

Our characterization of interpolating sequences for the Nevanlinna class is as follows. Note that the existence of a harmonic majorant occurs at two junctures: first, to decide which sequences of points are free interpolating, second, to identify the trace space that arises for those sequences which are indeed free interpolating.

**Theorem 1.2.** Let $\Lambda$ be a sequence in $\mathbb{D}$. The following statements are equivalent:

(a) $\Lambda$ is a free interpolating sequence for the Nevanlinna class $N$.

(b) The trace space is given by:

$$N|\Lambda = l_N := \{(a_\lambda)_{\lambda} : \exists h \in \text{Har}^+(\mathbb{D}) \text{ such that } h(\lambda) \geq \log^+ |a_\lambda|, \lambda \in \Lambda\}.$$  

(c) $\varphi_\Lambda$ admits a harmonic majorant.

(d) There exists $C > 0$ such that for any sequence of nonnegative numbers $\{c_\lambda\},$

$$\sum_{\lambda \in \Lambda} c_\lambda \varphi_\Lambda(\lambda) = \sum_{\lambda \in \Lambda} c_\lambda \log |B_\lambda(\lambda)|^{-1} \leq C \sup_{\zeta \in \partial \mathbb{D}} \sum_{\lambda \in \Lambda} c_\lambda P_\lambda(\zeta).$$

We recall that any positive harmonic function on the unit disk is the Poisson integral of a positive measure on the unit circle,

$$h(z) = P[\mu](z) = \int_{\partial \mathbb{D}} P_z(\zeta) d\mu(\zeta).$$

We will say that a harmonic function is **quasi-bounded** if and only if it admits an absolutely continuous boundary measure (for the reasons for this terminology, see [He69, pp. 6–7]). The analogous result for the Smirnov class will, as can be expected, involve quasi-bounded harmonic functions.
Theorem 1.3. Let $\Lambda$ be a sequence in $\mathbb{D}$. The following statements are equivalent:

(a) $\Lambda$ is a free interpolating sequence for the Smirnov class $N^+$.
(b) The trace space is given by

\[ N^+|\Lambda = l_{N^+} := \{(a_\lambda) : \exists h \in \text{Har}_+(\mathbb{D}) \text{ quasi-bounded} : h(\lambda) \geq \log^+ |a_\lambda|, \, \lambda \in \Lambda \}. \]

(c) $\varphi_{\Lambda}$ admits a quasi-bounded harmonic majorant.

The classical Carleson condition characterizing interpolating sequences for bounded analytic functions in the unit disc is $\sup_{\mathbb{D}} \varphi_{\Lambda} < \infty$, hence statements (c) in both results above can be viewed as Carleson-type conditions.

In view of Theorems 1.2 and 1.3, it seems natural to ask whether the measure $\mu$ such that $\varphi_{\Lambda} \leq P[\mu]$ can be obtained from $\Lambda$ in a canonical way. We do not have an answer to this question, but with Propositions 1.11 and 1.12 it is easy to construct examples that discard natural candidates, such as the (weighted) sum of Dirac masses

\[ \mu = \sum_{\lambda} (1 - |\lambda|) \delta_{\lambda}, \]  

its Poisson balayage (see definition below) $d\mu = \sum_{\lambda} (1 - |\lambda|) P_\lambda(\zeta)d\sigma$.

1.2. Positive harmonic majorants. The condition in Theorem 1.2 (d) arises in the solution of a problem of independent interest:

**Problem.** Which functions $\varphi : \mathbb{D} \to \mathbb{R}^+$ admit a (positive) harmonic majorant?

Answers to this problem lead to rather precise theorems about the permissible decrease of the modulus of bounded holomorphic functions, e.g. Corollary 1.5 below. See [Hay], [LySe97]; [EiES] also provides a survey of such results. The existence of harmonic majorants is relevant as well to the study of zero-sequences for Bergman and related spaces of holomorphic functions [Lu96].

An answer to the problem of positive harmonic majorants can be given in dual terms (see [NiTh] for another, more theoretical, characterization). The Poisson balayage (or swept-out function) of a finite positive measure $\mu$ in the closed unit disc is defined as

\[ B(\mu)(\zeta) = \int_{\mathbb{D}} P_\zeta(\zeta)d\mu(z) \]

for $\zeta \in \partial\mathbb{D}$. We will be interested in the class of measures having bounded balayage. Recall that Carleson measures are those finite positive measures whose balayage has bounded mean oscillation [Gar81, Theorem VI.1.6, p. 229]; this is also an easy consequence of the $H^1$-BMO duality [Gar81, Theorem VI.4.4, p. 245]. Hence positive measures with bounded balayage form a subclass of the usual Carleson measures. It is easy to see (cf. Section 6) that positive measures with bounded balayage are precisely those which operate against positive harmonic functions, that is, those measures $\mu$ for which there exist a constant $C = C(\mu)$ such that

\[ \int_{\mathbb{D}} h(z)d\mu(z) \leq Ch(0) \]

for any positive harmonic function in the unit disc $\mathbb{D}$.

Theorem 1.4. Let $\varphi$ be a nonnegative Borel function on the unit disk $\mathbb{D}$. Then there exists a (positive) harmonic function $h$ such that $h(z) \geq \varphi(z)$ for any $z \in \mathbb{D}$ if and only if there exists a
constant \( C = C(\varphi) \) such that for every positive measure \( \mu \),
\[
\int_{\mathbb{D}} \varphi(z) \, d\mu(z) \leq C \sup_{\zeta \in \partial \mathbb{D}} \int_{\mathbb{D}} P_{\zeta}(\zeta) \, d\mu(z).
\]

The necessity of the condition is obvious, while the sufficiency follows from a convenient version of a classical result in Convex Analysis, known as Minkowski-Farkas Lemma. The characterization of interpolating sequences in the Nevanlinna class in dual terms given by condition (d) in Theorem 1.2 follows from this result.

This can be applied to study the decrease of a non-zero bounded analytic function in the disc along a given non-Blaschke sequence.

**Corollary 1.5.** Let \( \Lambda \) be a separated non-Blaschke sequence and \((\varepsilon_{\lambda})_{\lambda \in \Lambda}\) a sequence of positive values. Then there exists a non-zero function \( f \in H^\infty(\mathbb{D}) \) with \( |f(\lambda)| < \varepsilon_{\lambda}, \lambda \in \Lambda \), if and only if \( \Lambda \) is the union of a Blaschke sequence and a sequence \( \Gamma \) for which there exists a universal constant \( C = C(\Gamma) \) such that
\[
\sum_{\gamma \in \Gamma} c_\gamma \log \varepsilon_\gamma^{-1} \leq C \sup_{\zeta \in \partial \mathbb{D}} \sum_{\gamma \in \Gamma} c_\gamma P_\gamma(\zeta)
\]
for any sequence of nonnegative numbers \((c_\gamma)_{\gamma \in \Gamma}\).

We do not know what is the corresponding analogue of Theorem 1.4 for quasi-bounded harmonic functions (i.e. for the Smirnov class). In proving the sufficiency of such an analogue, one should deduce that the measure \( \mu \) giving \( h = P[\mu] \) is absolutely continuous with respect to the Lebesgue measure. This extra condition on \( \mu \) does not seem to follow in an obvious way from our proof, based on Farkas’ Lemma.

For the problem of harmonic majorants it is desirable to obtain criteria which, although only necessary or sufficient, are more geometric and easier to check than the duality condition of Theorem 1.4.

Recall that the Stolz angle with vertex \( \zeta \in \partial \mathbb{D} \) and aperture \( \alpha \) is defined by
\[
\Gamma_\alpha(\zeta) := \{ z \in \mathbb{D} : |z - \zeta| \leq \alpha(1 - |z|^2) \}.
\]
In our considerations the angle \( \alpha \) is of no importance, so we will write \( \Gamma(\zeta) \) for the generic Stolz angle with aperture \( \alpha \). Given a function \( f \) from \( \mathbb{D} \) to \( \mathbb{R}^+ \), the non-tangential maximal function is defined as
\[
Mf(\zeta) := \sup_{\Gamma(\zeta)} f.
\]
Let \( \sigma \) be the normalized Lebesgue measure on \( \partial \mathbb{D} \). Consider the weak-\( L^1 \) space
\[
L^1_{w}(\partial \mathbb{D}) = \{ f \text{ measurable} : \sup_{t > 0} t \sigma\{ \zeta : |f(\zeta)| > t \} < \infty \}
\]
and let
\[
L^1_{w,0}(\partial \mathbb{D}) = \{ f \text{ measurable} : \lim_{t \to \infty} t \sigma\{ \zeta : |f(\zeta)| > t \} = 0 \}.
\]

It is well-known that the non-tangential maximal function of the Poisson transform of a positive finite measure belongs to \( L^1_w \) [Gar81, Theorem 5.1, p. 28]. A more careful analysis shows that if \( \mu \) is absolutely continuous, then its Poisson transform is in \( L^1_{w,0} \). This and some easy estimates imply the following result.
Proposition 1.6. (a) If \( \varphi \) admits a harmonic majorant, then \( M \varphi \in L^1_w(\partial \mathbb{D}) \).
(b) If \( \varphi \) admits a positive quasi-bounded harmonic majorant, then \( M \varphi \in L^1_{w,0}(\partial \mathbb{D}) \).
(c) If \( M \varphi \in L^1(\partial \mathbb{D}) \), then the function \( \varphi \) admits \( P[M \varphi] := P[M \varphi d\sigma] \) as a quasi-bounded harmonic majorant.

As far as necessary conditions are concerned, there is a way to improve the previous result by using another maximal function.

Definition. Given a nonnegative function \( f \) defined on the unit circle, its Hardy-Littlewood maximal function is given by

\[
f^*(x) := \sup_{I \ni x} \frac{1}{\sigma(I)} \int_I f.
\]

For \( \varphi \) a nonnegative function on the disc, define

\[
\varphi^H(\zeta) := \sup_{z \in \mathbb{D}} \varphi(z) \chi^*_I(z),
\]

where \( I_z \) is the “Privalov shadow” interval

(1.1)

\[
I_z := \{ \zeta \in \partial \mathbb{D} : z \in \Gamma(\zeta) \}.
\]

Proposition 1.7. (a) If \( \varphi \) admits a harmonic majorant, then \( \varphi^H \in L^1_w(\partial \mathbb{D}) \).
(b) If \( \varphi \) admits a quasi-bounded harmonic majorant, then \( \varphi^H \in L^1_{w,0}(\partial \mathbb{D}) \).

We will give some examples in Proposition 7.4 that show that this is indeed stronger than the necessary condition given in the first part of Proposition 1.6 but still falls short of giving a sufficient condition for the existence of a harmonic majorant.

1.3. Geometric criteria for interpolation. We would like to obtain some geometric implications of the analytic conditions given in Theorems 1.2 and 1.3. To begin with, we would like to state the maybe surprising result that separated sequences (with respect to the hyperbolic distance) are interpolating for the Smirnov class (and hence the Nevanlinna class). Recall that a sequence \( \Lambda \) is called separated if \( \delta(\Lambda) := \inf_{\lambda \neq \lambda'} \rho(\lambda, \lambda') > 0 \), where

\[
\rho(z, w) := |b_z(w)| = \left| \frac{z - w}{1 - z \overline{w}} \right|,
\]

is the pseudo-hyperbolic distance.

For such sequences, the values \( \log |B_\lambda(\lambda)|^{-1} \) can always be majorized by the values at \( \lambda \in \Lambda \) of the Poisson integral of an integrable function (see Proposition 4.1), thus the following corollary is immediate from Theorem 1.3.

Corollary 1.8. Let \( \Lambda \) be a separated Blaschke sequence. Then \( \Lambda \in \text{Int } N^+ \) (hence \( \Lambda \in \text{Int } N \)).

More precise conditions can be deduced from Propositions 1.6, 1.7 and (c) in Theorems 1.2 and 1.3.

Corollary 1.9. Let \( \Lambda \) be a sequence in \( \mathbb{D} \).

(a) If \( \Lambda \in \text{Int } N \) then \( \varphi^H_\Lambda \in L^1_w(\partial \mathbb{D}) \). If \( \Lambda \in \text{Int } N^+ \) then \( \varphi^H_\Lambda \in L^1_{w,0}(\partial \mathbb{D}) \).
(b) If \( M \varphi_\Lambda \in L^1(\partial \mathbb{D}) \) then \( \Lambda \in \text{Int } N^+ \) (and hence \( \Lambda \in \text{Int } N \)).
Notice that the necessary conditions obtained by replacing \( \varphi^H_\Lambda \) by \( M\varphi_\Lambda \) in (a) also hold. This in an immediate consequence of the estimate \( \varphi^H_\Lambda \geq M\varphi_\Lambda \).

This result implies the following Carleson-type conditions.

**Corollary 1.10.** (a) If \( \Lambda \in \text{Int } N^+ \), then

\[
(1.2) \quad \lim_{|\lambda| \to 1} (1 - |\lambda|) \log |B_\lambda(\lambda)|^{-1} = 0.
\]

(b) If \( \Lambda \in \text{Int } N \), then

\[
(1.3) \quad \sup_{\lambda \in \Lambda} (1 - |\lambda|) \log |B_\lambda(\lambda)|^{-1} < \infty.
\]

(c) If \( \Lambda \) is Blaschke and

\[
(1.4) \quad \sum_{\lambda \in \Lambda} (1 - |\lambda|) \log |B_\lambda(\lambda)|^{-1} < \infty,
\]

then \( \Lambda \in \text{Int } N^+ \) (and so also \( \Lambda \in \text{Int } N \)).

Condition (1.2) already appeared in [Ya74, Theorem 1] as a necessary condition for the sequence space \( l_{Ya} \) (as defined in the beginning of Section 1.1) to be included in the trace of \( N^+ \).

In some situations the conditions above are indeed a characterization of interpolating sequences. For instance, the weak \( L^1 \)-condition characterizes interpolating sequences lying on a radius, while for sequences approaching the unit circle very tangentially the characterization is given by the strong \( L^1 \)-condition. This is collected in the next results.

**Proposition 1.11.** Assume that \( \Lambda \subset \mathbb{D} \) lies in a finite union of Stolz angles.

(a) \( \Lambda \in \text{Int } N^+ \) if and only if (1.2) holds.

(b) \( \Lambda \in \text{Int } N \) if and only if (1.3) holds.

It should be mentioned that (b) can also be derived from Naftalevič’s result [Na56, Theorem 3]. On the other hand, his full characterization of the sequences such that \( N|\Lambda = l_{Na} \) can also be deduced from Theorem 1.2.

**Corollary** (Naftalevič, 1956). \( N|\Lambda = l_{Na} \) if and only if \( \Lambda \) is contained in a finite union of Stolz angles and (1.3) holds.

Let us consider the other geometric extreme, sequences which in particular only approach the circle in a tangential fashion. Write

\[
(1.5) \quad \mu_\Lambda := \sum_{\lambda} (1 - |\lambda|)\delta_\lambda,
\]

where \( \delta_\lambda \) stands for the Dirac measure at \( \lambda \).

**Proposition 1.12.** If \( \mu_\Lambda \) has bounded balayage, then \( \Lambda \in \text{Int } N \) if and only if \( \Lambda \in \text{Int } N^+ \), and if and only if (1.4) holds.

Note that the condition that \( \mu_\Lambda \) has bounded balayage implies in particular that \( \Lambda \) approaches the circle tangentially. In Section 8 we will see more concrete conditions of geometric separation which are sufficient to imply that \( \mu_\Lambda \) has bounded balayage (Proposition 8.2).
When \( \mu_\Lambda \) has bounded balayage, the trace space will embed into Yanagihara’s target space. More precisely, the following result holds.

**Proposition 1.13.** The following are equivalent:

(a) \( N|\Lambda \subset l_{Ya} \),

(b) \( N^+|\Lambda \subset l_{Ya} \),

(c) \( \mu_\Lambda \) has bounded balayage, i.e. \( \sup_{\zeta \in \partial D} \sum_\lambda (1 - |\lambda|) P_\lambda(\zeta) < \infty \).

Yanagihara considered the sequences such that \( N^+|\Lambda \supset l_{Ya} \). These are automatically in \( \text{Int} N^+ \), since for any Blachke sequence \( l_{Ya} \supset \ell_\infty \). Conversely, Lemma 8.1 (see Section 8) implies that \( l_{Ya} \subset N^+ \), thus if \( \lambda \in \text{Int} N^+ \), then by Theorem 1.3(b) \( N^+|\Lambda \supset l_{Ya} \). Therefore Theorem 1.3 characterizes in particular the sequences studied by Yanagihara.

Altogether, free interpolation for the Nevanlinna and Smirnov classes can be described in terms of the intermediate target spaces \( l_N \) and \( l_{N^+} \). Notice first that always \( N^+|\Lambda \subset l_{N^+} \) and \( N|\Lambda \subset l_N \) (this is proved at the beginning of Section 5). So, \( \lambda \in \text{Int} N^+ \) if and only if \( N^+|\Lambda \supset l_{N^+} \), and \( \lambda \in \text{Int} N \) if and only if \( N|\Lambda \supset l_N \). Observe also that \( l_{Ya} \subset l_{N^+} \subset l_N \subset l_{Na} \).

The paper is organized as follows. The next section is devoted to collecting some basic results on functions in the Nevanlinna class. In Section 3 we prove the sufficiency for interpolation of the conditions (c) of Theorems 1.3 and 1.2. We essentially use a result by Garnett allowing interpolation by \( H_\infty \) functions on sequences which are denser than Carleson sequences, under some decrease assumptions on the interpolated values. In Section 4 we study the necessity of these conditions. We first observe that in the product \( B_\lambda(\lambda) \) appearing in Theorem 1.2 only the factors \( b_\lambda(\lambda') \) with \( \lambda' \) close to \( \lambda \) are relevant. Then we split the sequence into four pieces, thereby reducing the interpolation problem, in a way, to that on separated sequences. The trace space characterization will be discussed in Section 5. In Section 6 we consider measures with bounded balayage, show that they operate against positive harmonic functions and prove Theorem 1.4. In Section 7 we prove Proposition 1.7 and provide examples to show that the sufficient condition is not necessary, and the necessary condition not sufficient. Section 8 is devoted to the proofs of Corollary 1.10, Propositions 1.11, 1.13 and 1.12 as well as the deduction of Naftalevič’s result from Theorem 1.2. Also, we give examples of measures with bounded balayage. In the final section, we exploit the reasoning of Section 8 to construct non-Carleson interpolating sequences for “big” Hardy-Orlicz classes.

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2. Preliminaries

We next recall some standard facts about the structure of the Nevanlinna and Smirnov classes (general references are e.g. [Gar81], [Nik02] or [RosRov]).

A function \( f \) is called outer if it can be written in the form

\[
f(z) = C \exp \left\{ \int_{\partial D} \frac{\zeta + z}{\zeta - z} \log v(\zeta) d\sigma(\zeta) \right\},
\]

where \( |C| = 1, v > 0 \) a.e. on \( \partial D \) and \( \log v \in L^1(\partial D) \). Such a function is the quotient \( f = f_1/f_2 \) of two bounded outer functions \( f_1, f_2 \in H_\infty \) with \( \|f_i\|_\infty \leq 1, i = 1, 2 \). In particular, the weight
\( v \) is given by the boundary values of \( |f_1/f_2| \). Setting \( w = \log v \), we have

\[
\log |f(z)| = P[w](z) = \int_{\partial D} P(z)w(\zeta)d\sigma(\zeta).
\]

This formula allows us to freely switch between assertions about outer functions \( f \) and the associated measures \( w d\sigma \).

Another important family in this context are inner functions: \( I \in H^\infty \) such that \( |I| = 1 \) almost everywhere on \( \partial D \). Any inner function \( I \) can be factorized into a Blaschke product \( B_\Lambda \) carrying the zeros \( \Lambda = \{\lambda_n\}_n \) of \( I \), and a singular inner function \( S \) defined by

\[
S(z) = \exp \left\{ -\int_{\partial D} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right\},
\]

for some positive Borel measure \( \mu \) singular with respect to Lebesgue measure.

According to the Riesz-Smirnov factorization, any function \( f \in N^+ \) is represented as

\[
f = \alpha BSf_1/f_2,
\]

where \( f_1, f_2 \) are outer with \( \|f_1\|_\infty, \|f_2\|_\infty \leq 1 \), \( S \) is singular inner, \( B \) is a Blaschke product and \( |\alpha| = 1 \). Similarly, functions \( f \in N \) are represented as

\[
f = \alpha BS_1 f_1/S_2 f_2,
\]

with \( f_i \) outer, \( \|f_i\|_\infty \leq 1 \), \( S_i \) singular inner, \( B \) is a Blaschke product and \( |\alpha| = 1 \).

In view of the Riesz-Smirnov factorization described above, the essential difference between Nevanlinna and Smirnov functions is the extra singular factor appearing in the denominator in the Nevanlinna case. This is reflected in the corresponding result for free interpolation in \( N \) by the fact that \( \varphi_\Lambda \) is bounded by a harmonic function, not necessarily quasi-bounded.

### 3. From Harmonic Majorants to Interpolation

For a given Blaschke sequence \( \Lambda \subset \mathbb{D} \) set \( \delta_\Lambda = |B_\Lambda(\lambda)| \). The key result in the proof of the sufficient condition is the following theorem by Garnett \cite{Gar77}, that we cite for our purpose in a slightly weaker form (see also \cite{Nik02} as a general source, in particular C.3.3.3(g) (Volume 2) for more results of this kind).

**Theorem.** Let \( \varphi : [0, \infty) \to [0, \infty) \) be a decreasing function such that \( \int_0^\infty \varphi(t) \, dt < \infty \). If a sequence \( a = (a_\lambda)_\lambda \) satisfies

\[
|a_\lambda| \leq \delta_\lambda \varphi(\log \frac{e}{\delta_\lambda}), \quad \lambda \in \Lambda,
\]

then there exists a function \( f \in H^\infty \) such that \( f|\Lambda = a \).

Observe that according to our former notation we have \( \log(e/\delta_\lambda) = 1 + \varphi_\Lambda(\lambda) \).

As we have already noted in Remark \[1.1\], in order to have free interpolation in the Nevanlinna and Smirnov classes, it is sufficient that \( \ell^\infty \subset N|\Lambda \) and \( \ell^\infty \subset N^+|\Lambda \) respectively. Our aim will be to accommodate the decrease given in Garnett’s result by an appropriate function in \( N \) or \( N^+ \). This is the crucial step in the proof of the sufficient conditions of Theorems \[1.3\] and \[1.2\] and it occupies its main part.
Proof of sufficiency of $[1.3](c)$ and $[1.2](c)$. The proof will be presented for the more difficult case of the Nevanlinna class. So, assume that $h \in \text{Har}_+(\mathbb{D})$ majorizes $\varphi_\Lambda$. Then $h$ is the Poisson integral of a positive measure $\mu$ on the circle and the function

$$(3.1)$$

$$g(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)$$

has positive real part in the disc. By Smirnov’s theorem, $g$ is an outer function in some $H^p$, $p < 1$, and therefore in $N^+$ (see [Nik02], in particular A.4.2.3 (Volume 1)). Also $\exp(g)$ is in the Nevanlinna class. By assumption we have

$$\log(1/\delta_\lambda) \leq \text{Re} g(\lambda), \lambda \in \Lambda.$$

Take now $\varphi(t) = (1 + t)^{-2}$, which obviously satisfies the hypothesis of Garnett’s theorem, and set $H = (2 + g)^2$, which is still outer in $N^+$. We have the estimate

$$|H(\lambda)| = |2 + g(\lambda)|^2 \geq (2 + \text{Re} g(\lambda))^2 \geq (1 + \log e/\delta_\lambda)^2 \geq \frac{1}{\varphi(\log(e/\delta_\lambda))},$$

hence the sequence $(\gamma_\lambda)_\lambda$ defined by

$$\gamma_\lambda = \frac{1}{H(\lambda)\varphi(\log(e/\delta_\lambda))}, \lambda \in \Lambda,$$

is bounded by 1.

In order to interpolate $\omega = (\omega_\lambda)_\lambda \in \ell^\infty$ by a function in $N$, split

$$\omega_\lambda = \left(\frac{\omega_\lambda\gamma_\lambda \exp(-g(\lambda))}{\delta_\lambda}\delta_\lambda \varphi(\log e/\delta_\lambda)\right) \cdot \frac{H(\lambda)}{\exp(-g(\lambda))}. $$

Since by hypothesis $(\omega_\lambda\gamma_\lambda \exp(-g(\lambda))/\delta_\lambda)_\lambda$ is bounded, we can apply Garnett’s result to interpolate the sequence

$$a_\lambda = \frac{\omega_\lambda\gamma_\lambda \exp(-g(\lambda))}{\delta_\lambda}\delta_\lambda \varphi(\log e/\delta_\lambda), \lambda \in \Lambda,$$

by a function $f \in H^\infty$. Now $F = f H \exp(g)$ is a function in $N$ with $F|\Lambda = \omega$.

The proof for the Smirnov case is obtained by observing that if the measure $\mu$ is absolutely continuous, then the function $\exp(g)$ is in the Smirnov class and so is the interpolating function $F$. □

4. FROM INTERPOLATION TO HARMONIC MAJORANTS

We first show that in order to construct the appropriate function estimating $\log |B_\lambda(\lambda)|^{-1}$ we only need to consider the factors of $B_\lambda$ given by points $\lambda' \in \Lambda$ which are close to $\lambda$. This is in accordance with the results for some related spaces of functions [HaMa01, Theorem 1], and it obviously implies Corollary [L8]

**Proposition 4.1.** Let $\Lambda$ be a Blaschke sequence. For any $\delta \in (0, 1)$, there exists a quasi-bounded positive harmonic function $h = P[w], w \in L^1(\partial\mathbb{D})$, such that

$$-\log \prod_{\lambda: \rho(\lambda, z) \geq \delta} |b_\lambda(z)| \leq h(z), \quad z \in \mathbb{D},$$
and therefore an outer function $G \in N^+$, where $G = \exp(-g)$ and $g$ is given by $(3.1)$ with $d\mu = w d\sigma$, such that
\[
\prod_{\lambda, \rho(\lambda, z) \geq \delta} |b_\lambda(z)| \geq |G(z)|, \quad z \in \mathbb{D}.
\]

**Proof.** We shall use the intervals $I_z$ introduced in (1.1). In [NPT, p. 124, lines 3 to 17], it is proved that the function $w$ given by
\[
w(\zeta) = c_0 \sum_{\lambda \in \Lambda} \chi_{I_\lambda}(\zeta),
\]
where $c_0$ is an appropriate positive constant, is suitable. At this juncture, the separation hypothesis made in [NPT, Lemma 4] is no longer used. ■

**Proof of the necessity of (1.3) (c) and (1.2) (c).** We will use a dyadic partition of the disk: for any $n \in \mathbb{N}$, let
\[
(4.1) \quad I_{n, k} := \{ e^{i\theta} : \theta \in [2\pi k 2^{-n}, 2\pi (k + 1) 2^{-n}) \}, \quad 0 \leq k < 2^n.
\]
and the associated Whitney partition in “dyadic squares”:
\[
(4.2) \quad Q_{n, k} := \{ re^{i\theta} : e^{i\theta} \in I_{n, k}, 1 - 2^{-n} \leq r < 1 - 2^{-n-1} \}.
\]
Observe that the hyperbolic diameter of each Whitney square $Q_{n, k}$ is bounded between two absolute constants.

We split the sequence into four pieces: $\Lambda = \bigcup_{i=1}^4 \Lambda_i$ such that each piece $\Lambda_i$ lies in a union of dyadic squares that are uniformly separated from each other. More precisely, set
\[
\Lambda_1 = \Lambda \cap Q^{(1)},
\]
where the family $Q^{(1)}$ is given by $\{Q_{2n, 2k}\}_{n, k}$ (for the remaining three sequences we respectively choose $\{Q_{2n, 2k+1}\}_{n, k}$, $\{Q_{2n+1, 2k}\}_{n, k}$, and $\{Q_{2n+1, 2k+1}\}_{n, k}$). In order to avoid technical difficulties we count only those $Q$ containing points of $\Lambda$ (in case $\Lambda \cap Q$ is empty there is nothing to prove). In what follows we will argue on one sequence, say $\Lambda_1$. The arguments are the same for the other sequences.

Our first observation is that, by construction, for $Q, L \in Q^{(1)}, Q \neq L$,
\[
\rho(Q, L) := \inf_{z \in Q, w \in L} \rho(z, w) \geq \delta > 0,
\]
for some fixed $\delta$. In what follows, the letters $j, k, \ldots$ will stand for indices in $\mathbb{N}$ of the form $(n, l), 0 \leq l < 2^n$. The closed rectangles $\overline{Q}_{j}$ are compact in $\mathbb{D}$ so that $\Lambda_1 \cap Q_{j}$ can only contain a finite number of points (they contain at least one point, by assumption). Therefore
\[
0 < m_j := \min_{\lambda \in \Lambda_1 \cap Q_{j}} |B_\lambda(\lambda)|
\]
(note that we consider the entire Blaschke product $B_\lambda$ associated with $\Lambda \setminus \{\lambda\}$). Take $\lambda_j \in Q_{j}$ such that $m_j = |B_{\lambda_j}(\lambda_j)|$.

Assume now that $\Lambda \in \text{Int} \; N$. Since $\ell^\infty \subset N|\Lambda$, there exists a function $f_1 \in N$ such that
\[
f_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\lambda_j\}_j \\ 0 & \text{if } \lambda \notin \{\lambda_j\}_j. \end{cases}
\]
By the Riesz-Smirnov factorization we have

\[(4.3) \quad f_i = B_{\Lambda \setminus \{\lambda_j^1\}} \frac{h_1}{h_2 T_2},\]

where \( T_2 \) is singular inner, \( h_1 \) is some function in \( H^\infty \) and \( h_2 \) is outer in \( H^\infty \). Again, we can assume \( \|h_1\|_\infty \leq 1, i = 1, 2 \). Hence

\[1 = |f_1(\lambda_k^1)| \leq |B_{\Lambda \setminus \{\lambda_j^1\}}(\lambda_k^1)| \cdot \frac{1}{|h_2(\lambda_k^1)T_2(\lambda_k^1)|},\]

and

\[|B_{\Lambda \setminus \{\lambda_j^1\}}(\lambda_k^1)| \geq |h_2(\lambda_k^1)T_2(\lambda_k^1)|, \quad k \in \mathbb{N}.\]

Since \( h_2 T_2 \) does not vanish and is bounded above by 1, the function \( \log |h_2 T_2| \) is a negative harmonic function. By Harnack’s inequality, there exists an absolute constant \( c \geq 1 \) such that

\[\frac{1}{c} |\log |h_2(\lambda_k^1)T_2(\lambda_k^1)|| \leq |\log |h_2(z)T_2(z)|| \leq c |\log |h_2(\lambda_k^1)T_2(\lambda_k^1)||, \quad z \in Q_k,\]

hence

\[|h_2(\lambda_k^1)T_2(\lambda_k^1)|^c \leq |h_2(z)T_2(z)| \leq |h_2(\lambda_k^1)T_2(\lambda_k^1)|^{1/c}, \quad z \in Q_k.\]

This yields

\[(4.4) \quad |(h_2 T_2)^c(\lambda')| \leq |(h_2 T_2)(\lambda_k^1)| \leq |B_{\Lambda \setminus \{\lambda_j^1\}}(\lambda_k^1)|\]

for every \( \lambda' \in \Lambda_1 \cap Q_k \).

Let us now exploit Proposition \[4.1\] By construction, the sequence \( \{\lambda_j^1\}_j \subset \Lambda_1 \) is separated. Therefore, there exists an outer function \( G_1 \) in the Smirnov class such that

\[|B_{\{\lambda_j^1\}}(\lambda_k^1)| \geq |G_1(\lambda_k^1)|, \quad k \in \mathbb{N}.\]

Again, \( G_1 \) is a quotient of two bounded outer functions and we can suppose that \( G_1 \) is outer in \( H^\infty \) with \( \|G_1\|_\infty \leq 1 \). Also, we can use Harnack’s inequality as above to get

\[|G_1(\lambda_k^1)| \geq |G_1(\lambda)|\]

for every \( \lambda' \in \Lambda_1 \cup Q_k \). This together with \[4.4\] and our definition of \( \lambda_k^1 \) give

\[|B_{\Lambda \setminus \{\lambda'\}}(\lambda')| \geq |B_{\Lambda \setminus \{\lambda_k^1\}}(\lambda_k^1)| = |B_{\Lambda \setminus \{\lambda_j^1\}}(\lambda_k^1)| \cdot |B_{\{\lambda_j^1\}}(\lambda_k^1)| \geq |(h_2 T_2)^c(\lambda')| \cdot |G_1^c(\lambda')|\]

for every \( \lambda' \in Q_k \) and \( Q_k \in Q^{(1)} \). Set \( g_1 = (h_2 G_1)^c \) and \( S_1 = T_2^c \); by construction, \( g_1 \) is outer with \( \|g_1\|_\infty \leq 1 \) and \( S_1 \) is singular inner.

Construct in a similar way functions \( g_i, S_i \) for the sequences \( \Lambda_i, i = 2, 3, 4 \) and define the products \( g = \prod_{i=1}^4 g_i \) and \( S = \prod_{i=1}^4 S_i \). Of course \( g \) is outer in \( H^\infty \), and \( S \) is singular inner. So, whenever \( \lambda' \in \Lambda \), there exists \( k \in \{1, 2, 3, 4\} \) such that \( \lambda' \in \Lambda_k \), and hence

\[(4.5) \quad |B_{\Lambda}(\lambda)| \geq |g_k(\lambda)S_k(\lambda)| \geq |g(\lambda)S(\lambda)|.\]

Therefore, the positive harmonic function \( h = - \log |gS| \) satisfies \( h(\lambda) \geq - \log |B_{\Lambda}(\lambda)| \). The proof for \( N^+ \) goes along the same lines, except that singular inner factors do not occur in \[4.5\], and so will not appear in \[4.5\] neither. \( \blacksquare \)
5. The trace spaces

In this short section we prove the trace space characterization of free interpolation given in Theorems 1.2 and 1.3.

In order to see that (b) in each theorem implies free interpolation it suffices to observe that \( \ell^\infty \subset l_{N^+} \subset l_N \) and use Remark 1.1.

For the proof of the converse, we will only consider the situation in the Nevanlinna class, since the case of the Smirnov class is again obtained by removing the singular part of the measure and the singular inner factors.

Assume that \((a_\lambda)_{\lambda} \in N|\Lambda\) and that \(f \in N\) is such that \(f(\lambda) = a_\lambda, \lambda \in \Lambda\). Since \(f\) can be written as \(f = f_1/(S_2 f_2)\), where \(f_1 \in H^\infty, \|f_1\|_\infty \leq 1, S_2\) is singular inner with associated singular measure \(\mu_S\), and \(f_2 \in H^\infty\) is an outer function with \(\|f_2\|_\infty \leq 1\), we can define the positive finite measure \(\mu = \log(1/f_2)\) \(d\sigma + d\mu_S\) which obviously satisfies \(P[\mu](\lambda) \geq \log^+ |a_\lambda|, \lambda \in \Lambda\).

Conversely, suppose that \((a_\lambda)_{\lambda}\) is such that there is a positive finite measure \(\mu\) with \(P[\mu](\lambda) \geq \log^+ |a_\lambda|\). The Radon-Nikodym decomposition of \(\mu\) is given by \(d\mu = w \, d\sigma + d\mu_S\), where \(w \in L^1(\partial \mathbb{D})\) is positive and \(\mu_S\) is a positive finite singular measure. Let \(S\) be the singular inner function associated with \(\mu_S\), and let \(f\) be the function defined by

\[
f(z) = \exp \left( \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} w(\zeta) \, d\sigma(\zeta) \right), \quad z \in \mathbb{D}.
\]

By definition, \(f\) is outer in \(N^+\) and \(F = f/S \in N\). Clearly, \(\log^+ |a_\lambda| \leq \log |F(\lambda)|\), thus \(|a_\lambda| \leq |F(\lambda)|\). Since \(N|\Lambda\) is ideal by assumption, there exists \(f_0 \in N\) interpolating \((a_\lambda)_{\lambda}\).

6. Harmonic majorants and measures with bounded balayage

Let us start by proving that positive measures with bounded balayage are precisely those which operate against positive harmonic functions. Define

\[
B := \{\mu \text{ positive Borel measures on } \mathbb{D} \text{ such that } \sup_{\zeta \in \partial \mathbb{D}} B(\mu)(\zeta) \leq 1\}.
\]

**Proposition 6.1.** Let \(\mu\) be a positive Borel measure on the disk. Then \(\int_{\mathbb{D}} h d\mu\) is finite for any positive harmonic function \(h\) on the disk if and only if there exists some \(c > 0\) such that \(\mu\) has balayage uniformly bounded by \(c\). Furthermore, the relevant constants are related:

\[
\sup_{\zeta \in \partial \mathbb{D}} B(\mu)(\zeta) = \sup_{\zeta \in \partial \mathbb{D}} \int_{\mathbb{D}} P_z(\zeta) \, d\mu(z) = \sup \left\{ \int_{\mathbb{D}} h d\mu : h \in \text{Har}_+(\mathbb{D}), h(0) = 1 \right\},
\]

and for any positive harmonic function \(h\),

\[
h(0) = \max_{\mu \in B} \int_{\mathbb{D}} h d\mu.
\]

**Proof.** Let \(h = P[\nu]\), where \(\nu \geq 0\) is a measure on \(\partial \mathbb{D}\). If \(\mu\) has balayage bounded by \(c\),

\[
\int_{\mathbb{D}} h(z) d\mu(z) = \int_{\partial \mathbb{D}} \int_{\mathbb{D}} P_z(\zeta) d\mu(z) d\nu(\zeta) \leq c \nu(\partial \mathbb{D}) = ch(0).
\]
Conversely, since \( z \mapsto P_z(\zeta) \) is a harmonic function for any fixed \( \zeta \), \( \int_D P_z(\zeta) \, d\mu(z) \) is pointwise defined. Pick a sequence \( \zeta_n \) such that
\[
\lim_{n \to \infty} \int_D P_z(\zeta_n) \, d\mu(z) = \sup_{\zeta \in \partial D} \int_D P_z(\zeta) \, d\mu(z),
\]
where the supremum on the right hand side might a priori be infinite. Since the set \( E := \{ h \in \text{Har}_+(D) : h(0) = 1 \} \) is uniformly bounded on compact sets in \( D \), a normal family argument shows that \( \sup \{ \int_D h d\mu : h \in E \} < \infty \). Observe that the mapping \( z \mapsto P_z(\zeta_n) \) is in \( E \) for every \( \zeta_n, n \in \mathbb{N} \). Hence \( \sup_n \int_D P_z(\zeta_n) d\mu(z) < \infty \).

This proves that \( \mu \) has bounded balayage, and the equalities between constants that we had announced. \( \blacksquare \)

The next result is a refined version of Theorem 1.4 stated in the introduction.

**Theorem 6.2.** Let \( \varphi \) be a nonnegative Borel function on the unit disk. Then there exists a (positive) harmonic function \( h \) such that \( h(z) \geq \varphi(z) \) for any \( z \in \mathbb{D} \) if and only if
\[
M_\varphi := \sup_{\mu \in \mathcal{B}} \int_D \varphi \, d\mu < \infty.
\]
Furthermore,
\[
M_\varphi = \inf \{ h(0) : h \in \text{Har}(\mathbb{D}), h \geq \varphi \}.
\]
That \( 6.1 \) is necessary is clear from the above considerations. In order to prove that it is sufficient, we will reduce ourselves to a discrete version of it. We will use the dyadic squares introduced in (4.2). As in the previous section, choose a point \( z_{n,k} \) in each cube, say
\[
z(Q_{n,k}) = z_{n,k} := (1 - 2^{-n}) \exp(2\pi i k 2^{-n}).
\]
Observe that by Harnack’s inequality, there exists a universal constant \( K \) such that if \( z, z' \) lie in the same Whitney square \( Q_{n,k} \) (as defined in (4.2)), then \( K^{-1} P_z(\zeta) \leq P_{z'}(\zeta) \leq KP_{z'}(\zeta) \), for any \( \zeta \in \partial \mathbb{D} \).

**Lemma 6.3.** The function \( \varphi \) satisfies condition \( 6.1 \) if and only if there exists a constant \( M'_\varphi \) such that for any sequence of nonnegative coefficients \( \{ c_{n,k} \} \) such that
\[
\sup_{\zeta \in \partial D} \sum_{n,k} c_{n,k} P_{z_{n,k}}(\zeta) \leq 1,
\]
then
\[
\sum_{n,k} c_{n,k} \sup_{Q_{n,k}} \varphi \leq M'_\varphi.
\]
Furthermore, \( C^{-1} M_\varphi \leq M'_\varphi \leq CM_\varphi \), where \( C > 1 \) is an absolute constant.

**Proof of Lemma 6.3** Pick \( z_{n,k} \in Q_{n,k} \) such that \( \varphi(z_{n,k}) \geq (\sup_{Q_{n,k}} \varphi)/2 \) and define the measure \( \mu := \sum_{n,k} c_{n,k} \delta_{z_{n,k}} \). Then, if \( \{ c_{n,k} \} \) satisfies (6.2),
\[
\int_D P_z(\zeta) \, d\mu(z) = \sum_{n,k} c_{n,k} P_{z_{n,k}}(\zeta) \leq K \sum_{n,k} c_{n,k} P_{z_{n,k}}(\zeta) \leq K.
\]
So if \( \varphi \) satisfies (6.1),
\[
\sum_{n,k} C_{n,k} \sup_{Q_{n,k}} \varphi \leq 2 \sum_{n,k} C_{n,k} \varphi(z_{n,k}^*) = 2 \int_{\mathbb{D}} \varphi \, d\mu \leq 2KM\varphi.
\]
The converse direction is easier, and left to the reader (it also follows from the proof of the theorem, below).

We now need a classical result in convex analysis. Recall that the convex hull of a subset \( A \subset \mathbb{R}^d \) is defined as
\[
\text{Conv}(A) := \left\{ \sum_{i=1}^{N} \alpha_i a_i : a_i \in A, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}.
\]
If we write \( \mathbb{R}_+ A := \{\lambda x : \lambda \geq 0, x \in A\} \), then the conical convex hull of \( A \) is defined as
\[
\text{Cone}(A) := \text{Conv}(\mathbb{R}_+ A) = \left\{ \sum_{i=1}^{N} \alpha_i a_i : a_i \in A, \alpha_i \geq 0 \right\}.
\]
When \( A \) is a finite set, the conical convex hull is equal to its closure: \( \overline{\text{Cone}(A)} = \text{Cone}(A) \) (for this and other facts, see [Hull]). The key fact for us will be the generalized form of the Minkowski-Farkas Lemma [Hull, Chapter III, Theorem 4.3.4] that we cite here only for finite \( A \). Let \( \langle \cdot, \cdot \rangle \) stand for the standard Euclidean scalar product in \( \mathbb{R}^d \).

**Theorem 6.4.** Let \( (a_j, b_j) \in \mathbb{R}^d \times \mathbb{R}, 1 \leq j \leq N, \) be such that \( X := \{x \in \mathbb{R}^d : \langle a_j, x \rangle \leq b_j\} \neq \emptyset \). Denote \( A := \{(a_j, b_j), 1 \leq j \leq N\} \subset \mathbb{R}^d \times \mathbb{R} \). Then the following properties are equivalent for \( (v, r) \in \mathbb{R}^d \times \mathbb{R} \):

(a) For any \( x \in X \), \( \langle v, x \rangle \leq r \).

(b) \( (v, r) \in \text{Cone}(A) \).

We will use the following special case. For a vector \( v \in \mathbb{R}^d \), the coordinates are denoted by \( v^i, 1 \leq i \leq d \). Also, \( \mathbb{R}^d_+ \) denotes the set of points of \( \mathbb{R}^d \) with nonnegative coordinates.

**Corollary 6.5.** Given \( a_j \in \mathbb{R}^d, 1 \leq j \leq N, \) let \( X_+ := \{x \in \mathbb{R}^d_+ : \langle a_j, x \rangle \leq 1\} \), and suppose that \( X_+ \neq \emptyset \). Then the following properties are equivalent for \( v \in \mathbb{R}^d_+ \):

(a) For any \( x \in X_+, \langle v, x \rangle \leq 1 \).

(b) There exist \( \alpha_j \geq 0, 1 \leq j \leq N \) such that \( \sum_{j=1}^{N} \alpha_j = 1 \) and for any \( i = 1, \ldots, d \),
\[
v^i \leq \sum_{j=1}^{N} \alpha_j a^i_j.
\]

**Proof.** Let \{\( e_i \}_{1 \leq i \leq d} \} be the canonical basis of \( \mathbb{R}^d \) and consider
\[
A := \{(a_j, 1), 1 \leq j \leq N\} \cup \{(-e_i, 0), 1 \leq i \leq d\}.
\]
Then \( X_+ \) corresponds to the \( X \) in Theorem 6.4, from what we see that (a) implies that there exist \( \alpha_j \geq 0, \beta_i \geq 0, 1 \leq j \leq N, 1 \leq i \leq d \), such that
\[
(v, 1) = \sum_{j=1}^{N} \alpha_j (a_j, 1) - \sum_{i=1}^{d} \beta_i (e_i, 0).
\]
When applied to each coordinate, this yields \( 1 = \sum_{j=1}^{N} \alpha_j \) and
\[
\nu^i = \sum_{j=1}^{N} \alpha_j a^i_j - \beta_i \leq \sum_{j=1}^{N} \alpha_j a^i_j.
\]
The converse implication is immediate.

**Proof of Theorem 6.2.** Suppose that \( \varphi \) satisfies (6.1). For each nonnegative integer \( m \), we define
\[
a_j := \left( P_{z_{n,k}}(\exp(ij \cdot 2^{-m}2\pi)) \right)_{0 \leq n \leq m, 0 \leq k \leq 2^n - 1}
\]
d \( = \sum_{n=0}^{m} 2^n \) and
\[
X_+ := \left\{ \{c_{n,k}\}_{0 \leq n \leq m, 0 \leq k \leq 2^n - 1} \in \mathbb{R}_+^d : \sum_{0 \leq n \leq m, 0 \leq k \leq 2^n - 1} c_{n,k} P_{z_{n,k}}(\exp(ij \cdot 2^{-m}2\pi)) \leq 1, \text{ for } 1 \leq j \leq 2^m - 1 \right\}.
\]
Obviously, \( X_+ \) is not empty: for instance \( c_{0,0} = 1 \) and \( c_{n,k} = 0 \) for \( n \geq 1 \) gives a point in \( X_+ \). We claim that any \( \{c_{n,k}\} \in X_+ \) will satisfy (6.2) up to a constant. Indeed, for any \( \theta \in [0, 2\pi) \), there is an index \( j < 2^m \) so that \( j \cdot 2^{-m}2\pi \leq \theta < (j + 1) \cdot 2^{-m}2\pi \), therefore by Harnack’s inequality, for any \( z \) such that \( |z| \leq 1 - 2^{-m} \),
\[
P_z(e^{i\theta}) = P_{z \exp((j \cdot 2^{-m}2\pi - \theta))}(\exp(ij \cdot 2^{-m}2\pi)) \leq K P_z(\exp(ij \cdot 2^{-m}2\pi)).
\]
Therefore \( \{K^{-1}c_{n,k}\} \) satisfies (6.2), and by Lemma 6.3 and the hypothesis, \( \varphi \) satisfies (6.3) with constant \( KM'_{\varphi} \). Corollary 6.5 then implies the existence of positive coefficients \( (\alpha_j^m)_{j=0}^{2^m-1} \) with sum equal to \( KM'_{\varphi} \), such that
\[
\sup_{Q_{n,k}} \varphi \leq \sum_{j=0}^{2^m-1} \alpha_j^m P_{z_{n,k}}(\exp(ij \cdot 2^{-m}2\pi)) = \int_{\partial D} P_{z_{n,k}} d\nu^m,
\]
where \( \nu^m \) is the discrete measure on the circle given by the following combination of Dirac masses:
\[
\nu^m = \sum_{j=0}^{2^m-1} \alpha_j^m \delta_{\exp(ij \cdot 2^{-m}2\pi)}.
\]
Since the mass of \( \nu^m \) is uniformly bounded by \( KM'_{\varphi} \), we can take a weak* limit \( \nu \) of this sequence of measures, so that for any \( (n,k) \),
\[
\sup_{Q_{n,k}} \varphi \leq \int_{\partial D} P_{z_{n,k}} d\nu = h(z_{n,k}),
\]
where \( h := P[\nu] \). Harnack’s inequality now implies that there is an absolute constant \( C_1 \) such that \( C_1 h(z) \geq \varphi(z) \) for any \( z \in D \). This proves the theorem, with the inequality
\[
\inf \{h(0) : h \in \text{Har}(D), h \geq \varphi\} \leq C_1 KM'_{\varphi} \leq CC_1 KM_{\varphi}.
\]
The constants \( C_1, K \) and \( C_1 \) only depend on the discretization we have chosen. Picking a discretization with smaller “squares”, we may make all three constants as close to 1 as we wish. ■

We finish this section with the proof of Corollary 6.5.
Proof of Corollary 1.5. Given a non-Blaschke sequence \( \Lambda \), arguing as in [NPT] one can show that there exists a function \( f \in H^\infty(D) \) in the unit disc with \( |f(\lambda)| < \varepsilon \lambda \) for any \( \lambda \in \Lambda \) if and only if \( \Lambda \) is the union of a Blaschke sequence and a sequence \( \Gamma \) for which there exists a positive harmonic function \( h \) in the unit disc with \( h(\gamma) > -\log \varepsilon \gamma \) for all \( \gamma \in \Gamma \). Then the result follows from Theorem 1.4. ■

7. WEAKER CONDITIONS FOR THE EXISTENCE OF HARMONIC MAJORANTS

In this section we state first a sufficient condition implied by a result of Borichev on a similar problem. On the other hand, we also prove the necessary condition of Proposition 1.7 and show that it is not sufficient.

The following was communicated to us by our colleague Alexander Borichev (see [NTTh] for a proof).

**Theorem 7.1.** Given a collection of nonnegative data \( \{ \varphi_{n,k} \} \subset \mathbb{R}^+ \), there exists a finite positive measure \( \nu \) on \( \partial D \) such that

\[
\frac{\nu(I_{n,k})}{\sigma(I_{n,k})} \geq \varphi_{n,k}
\]

if and only if

\[
S := \sup \left\{ \sum_{(n,k) \in A} \varphi_{n,k} \sigma(I_{n,k}) : \{I_{n,k}\}_{(n,k) \in A} \text{ is a disjoint family} \right\} < \infty.
\]

This is an analogue of the discretized version of Theorem 1.2(d), (as in Lemma 6.3) obtained by considering only measures of type \( \mu_A := \sum_{(n,k) \in A} \sigma(I_{n,k}) \delta_{\varphi_{n,k}} \), and by replacing the Poisson kernel \( P_z \) by the “square” kernels

\[
K_z(e^{i\theta}) := K_{I_z}(e^{i\theta}) := \frac{1}{\sigma(I_z)} \chi_{I_z}(e^{i\theta}).
\]

Here \( I_z \) denote the intervals defined in (1.1) and \( \chi_E \) stands for the characteristic function of \( E \).

The similarity of Theorem 1.2 with this result leads us to an:

**Open Question.** Is condition (d) in Theorem 1.2 still sufficient if we restrict it to \( \{c_\lambda\} \) such that for any \( \lambda \in \Lambda, c_\lambda = 0 \) or \( 1 - |\lambda| = 0 \)?

Theorem 7.1 together with the estimate \( K_z \lesssim P_z \) provide a sufficient (but not necessary) condition for domination by true harmonic functions, which is clearly less restrictive than requiring that \( M\varphi \in L^1(\partial D) \), but easier to check in concrete examples than the characterizing condition of Theorem 1.4.

**Corollary 7.2.** Any positive function \( \varphi \) such that \( \varphi_{n,k} := \sup_{Q_{n,k}} \varphi \) satisfies (7.1) admits a harmonic majorant. On the other hand, the positive harmonic function \( z \mapsto P_z(1) \) does not satisfy (7.1) for certain choices of \( \Lambda \).

**Proof.** It is well known and easy to see that there exists a constant \( c \) such that \( P_z \geq cK_{I_{n,k}} \) for any \( z \in Q_{n,k} \) (the constant \( c \) depends on the aperture \( \alpha \) of the Stolz angle). Therefore, for any \( z \in Q_{n,k} \)

\[
P[\nu](z) \geq c \int_{\partial D} K_{I_{n,k}}(\zeta) d\nu(\zeta) \geq c \frac{\nu(I_{n,k})}{\sigma(I_{n,k})} \geq c \varphi_{n,k} = c \sup_{Q_{n,k}} \varphi,
\]
which proves that $P[(1/c)\nu]$ is the harmonic majorant we are looking for.

To see that the condition is not necessary, consider any $A \subset \{(n, 1) : n \in \mathbb{N}\}$. Then the intervals $I_n,1$ are all disjoint; however $P_{z_n,1}(1) \approx 2^n \approx \sigma(I_n,1)^{-1}$, so that condition (7.1) will fail (the sum is comparable to $\#A$).

In the same way as in Corollary 1.10 Corollary 7.2 and Proposition 1.7 imply the following result. For $Q = Q_{n,k}$, write $I(Q) = I_{n,k}$ (the radial projection of the square to an arc of the circle).

**Corollary 7.3.** Assume that $\Lambda$ is contained in a union $A$ of Whitney squares $Q$ of center $z(Q)$ and that

$$\sup\left\{ \sum_{Q \in A'} (1 - |z(Q)|) \sup_{x \in \Lambda \cap Q} \log |B_{\Lambda}(x)|^{-1} \right\} < \infty,$$

where the supremum is taken over all $A' \subset A$ such that $\{I(Q) : Q \in A'\}$ is a disjoint family, then $\Lambda$ is interpolating for the Nevanlinna class.

We move next to the proof of the necessary condition in terms of the Hardy-Littlewood maximal function.

**Proof of Proposition 1.7.** (a) The problem can be localized, so we may work on the upper half plane, $\mathbb{C}_+: = \{x + iy : y > 0\}$, with $I_{x+iy} = (x - y, x + y)$, restricting ourselves to positive harmonic functions which are Poisson integrals of positive measures with finite mass. Here the Poisson kernel is given by

$$P_{x+iy}(s) = \frac{1}{\pi} \frac{y}{(x - s)^2 + y^2}.$$

For a measurable set $E \subset \mathbb{R}$ write $|E| = \sigma(E)$. Also, we only need to look at boundary points in a fixed bounded interval, say $-1 \leq x \leq 1$.

For any $t > 0$, let $E_t := \{s \in [-1, 1] : \varphi^H(s) > t\}$. For any $s \in E_t$, there exists $z = z(s)$ and $J = J(s) \supset I_z$ such that

$$(7.2) \quad \varphi(z) \int_J \chi_{I_z} > t |J|, \quad \text{i.e.} \quad \varphi(z) |I_z| > t |J|.$$  

By Vitali’s covering lemma, there exist an absolute constant $c_1 \in (0, 1)$ and a disjoint family of intervals $J_j := J(s_j), 1 \leq j \leq N$, such that $\sum_j |J_j| \geq c_1 |E_t|$.

Write $z_j := z(s_j) =: x_j + iy_j$. Note that since the point $z_j$ is contained in the “tent” over $I_{z_j}$ (therefore in the tent over $J_j$) the points $z_j$ are separated in the hyperbolic metric.

Now define new points $z_j'$ in the following way: let $y'_j := |J_j|/2 = y_j |J_j|/|I_{z_j}| \geq y_j$ and $z_j' := x_j + iy'_j$. Note that $|J_j \cap I_{z_j}'| \geq |J_j|/2$.

We claim that $h(z_j') \geq t$, where $h$ is a harmonic majorant of $\varphi$. Indeed, writing $h = P[\mu],

$$h(z_j') = \frac{1}{\pi y'_j} \int_{\mathbb{R}} \frac{1}{1 + \left( \frac{t - x_j}{y'_j} \right)^2} d\mu(t) \geq \frac{1}{\pi y'_j} \int_{\mathbb{R}} \frac{1}{1 + \left( \frac{t - x_j}{y'_j} \right)^2} d\mu(t) = \frac{y'_j}{y_j} h(z_j),$$

and, by (7.2), $h(z_j) \geq \varphi(z_j) > t |J_j|/|I_{z_j}| = t y'_j / y_j$. 

Proposition 7.4. There are functions \( \varphi \) such that \( \varphi^H \in L^1_w(\partial\mathbb{D}) \), but that do not admit a harmonic majorant.

(b) Similarly.

We now give two examples showing that the necessary condition of Proposition 1.7 is strictly stronger than that of Proposition 1.6 but still not sufficient.

Proof. The proof rests on the following family of examples. Note that it is easy to turn those examples into examples of sequences which are (or are not) interpolating for the Nevanlinna class.

Again we will work on \( \mathbb{C}_+ \). Our functions \( \varphi \) will vanish everywhere on the upper half plane, except on the sequence \( \lambda_k := x_k + iy_k \), where \( x_k = k^{-\alpha} \) and \( y_k = k^{-\beta} \). To ensure that \( y_k \leq (x_{k+1} - x_k)^2 \) we take \( \beta \geq 2(\alpha+1) \). With this choice, it can be deduced from Proposition 8.2 (or the remark before Lemma 8.4), that a necessary and sufficient condition for the existence of a harmonic majorant is that \( M\varphi \in L^1 \), that is,

\[
\sum_k \varphi(\lambda_k) y_k < \infty.
\]

We note that

\[
\chi^*_{I_{\lambda_k}}(x) = \frac{2}{1 + \max(1, \frac{|x-x_k|}{y_k})}.
\]

Henceforth we only study data \( \{\varphi_k\} := \{\varphi(\lambda_k)\} \) which are increasing sequences of positive numbers tending to infinity. We also assume that \( \{(\varphi_k y_k + \varphi_{k+1} y_{k+1})/(x_k - x_{k+1})\}\) forms an increasing sequence. Let \( k_0(t) := \min\{k : t < \varphi_k\} \). The necessary condition arising from the fact that \( M\varphi \in L^1_w(\mathbb{R}) \) reads

\[
\sum_{k \geq k_0(t)} y_k \simeq k_0^{1-\beta}(t) \leq \frac{C}{t}, \quad \forall t > 0.
\]

This condition will be assumed for both examples.

Now, for \( k \geq k_0(t) \), define \( J_k := \{x : \varphi_k \chi_k^*(x) > t\} \simeq (x_k - y_k \varphi_k/t, x_k + y_k \varphi_k/t) \), and let \( k_1(t) := \min\{k : J_k \cap J_{k+1} \neq \emptyset\} \). Then,

\[
\bigcup_{k \geq k_1(t)} J_k = (0, x_{k_1(t)} + y_{k_1(t)} \frac{\varphi_{k_1(t)}}{t}) = (0, k_1^{-\alpha}(t) + k_1^{-\beta}(t) \frac{\varphi_{k_1(t)}}{t})
\]

and

\[
\left| \{x : \varphi^H(x) > t\} \right| \simeq k_1^{-\alpha}(t) + k_1^{-\beta}(t) \frac{\varphi_{k_1(t)}}{t} + \frac{2}{t} \sum_{k = k_0(t)}^{k_1(t)-1} \frac{\varphi_k}{k} \simeq k_1^{-\alpha}(t) + \frac{2}{t} \sum_{k = k_0(t)}^{k_1(t)} \frac{\varphi_k}{k}.
\]
In order to prove (a), choose \( \varphi_k := \varepsilon_k k^{-1} \). Since \( t \simeq \varepsilon_k (t) k_0(t) \beta^{-1} \), condition (7.4) becomes that \((\varepsilon_k)_k\) remains bounded above, while the necessary and sufficient condition (see (7.3)) is

\[
\sum_k \varepsilon_k \frac{k}{k} < \infty.
\]

With \( \varepsilon_k := (\log k)^{-1} \), this condition fails, so that \( \varphi \) admits no harmonic majorant.

However, \( k_0(t) \simeq (t \log t)^{1/(\beta - 1)} \). Since \( x_k - x_{k+1} \simeq k^{-\alpha - 1} \), then \( 1/k_1(t)^{\alpha + 1} \simeq \varepsilon_k(t)/(tk) \), thus \( k_1(t) \simeq (t/\varepsilon_k(t))^{1/\alpha} \), and \( k_1(t) \simeq (t \log t)^{1/\alpha} \).

Therefore equation (7.5) becomes

\[
\left| \{ x : \varphi^H(x) > t \} \right| \simeq \frac{1}{t \log t} + 2 \frac{k_1(t)}{t} \sum_{k=k_0(t)} k \log k \simeq \frac{1}{t \log t} + 2 \frac{\log}{t \log k_0(t)} \log \left( \frac{k_1(t)}{k_0(t)} \right)
\]

\[
\leq \frac{1}{t \log t} + C \frac{\varepsilon_k}{t} \leq C',
\]

and this choice of \( \varphi \) does satisfy the necessary condition given in Proposition 1.7.

To prove the second statement in the Lemma, choose \( \varepsilon_k := 1 \). With similar but easier calculations one sees that \( k_0(t) \simeq t^{1/(\beta - 1)} \) and \( k_1(t) \simeq t^{1/\alpha} \). Therefore (7.5) becomes

\[
\left| \{ x : \varphi^H(x) > t \} \right| \simeq \frac{1}{t} + \sum_{k=k_0(t)}^{k_1(t)-1} \frac{1}{k} \simeq \frac{1}{t} + 2 \frac{\log}{t} \left( \frac{k_1(t)}{k_0(t)} \right) \simeq \frac{\log t}{t},
\]

so the \( L^1 \) condition fails for \( \varphi^H \), even though \( \varphi \) satisfies the necessary condition in Proposition 1.6. 

8. Proofs of the geometric conditions

Proof of Corollary 1.10 Since

\[
I_{\lambda} \subset \{ \zeta \in \partial \mathbb{D} : M \varphi_{\lambda}(\zeta) \geq \log |B_{\lambda}(\zeta)|^{-1} \}, \quad \lambda \in \Lambda,
\]

to prove (a) and (b) it suffices to apply condition (a) of Corollary 1.9. Statement (c) follows from the next Lemma applied to \( \varphi_{\lambda} \).

Lemma 8.1. Let \( \varphi : \mathbb{D} \rightarrow \mathbb{R}^+ \) satisfy \( \sum_{\lambda \in \mathbb{D}} (1 - |\lambda|) \varphi(\lambda) < \infty \). Then \( \varphi \) admits a harmonic majorant.

**Proof.** Let \( u = \sum_{\lambda} \varphi(\lambda) \chi_{\lambda} \). By assumption \( u \in L^1(\partial \mathbb{D}) \) and obviously \( M \varphi \leq u \), hence the result follows from Corollary 1.9(b). 

Parts (b) and (c) also follow directly from Theorem 1.2(d), by a simple argument based on the \( \ell^1, \ell^\infty \) duality. 

Proof of Proposition 1.11 It is enough to consider the case where \( \Lambda \) is contained in only one Stolz angle. Indeed, if \( \Lambda = \bigcup_{i=1}^n \Lambda_i \) with \( \Lambda_l \subset \Gamma_{\zeta_l}, l = 1, \ldots, n \), and \( \zeta_i \neq \zeta_j, \) then

\[
\lim_{z \rightarrow \zeta_i, z \in \Gamma_{\zeta_i}} |B_{\Lambda_j}(z)| = 1, \quad j \neq i,
\]
so that \( \log |B_\lambda(\lambda)|^{-1} \) behaves asymptotically like \( \log |B_{\lambda_n}(\lambda)|^{-1} \) in \( \Gamma_\varepsilon \) (here \( \lambda \in \Lambda \)). Also, we can assume that the sequence is radial (this means that we replace the initial sequence by one which is in a uniform pseudo-hyperbolic neighborhood of the initial one; by Harnack’s inequality such a perturbation does not change substantially the behavior of positive harmonic functions).

According to Corollary [1.10] it is enough to prove the sufficiency of the conditions. Let us first show that condition (1.2) implies interpolation in \( N^+ \). In order to construct a function \( w \in L^1(\partial \mathbb{D}) \) meeting the requirement of Theorem [1.3c] assume that \( \Lambda = \{ \lambda_n \}_{n \in [0, 1)} \) is arranged in increasing order and set \( \varepsilon_n = (1 - |\lambda_n|) \log |B_{\lambda_n}(\lambda_n)|^{-1} \). Clearly there exists a decreasing sequence \( (\varepsilon_n)_n \) with \( \varepsilon_n \leq \varepsilon_n, n \in \mathbb{N}, \) and \( \lim n \varepsilon_n = 0 \). Now, if \( I_n = I_{\lambda_n} \), set \( J_n = I_n \setminus I_{n+1}, \) \( \beta_n = \varepsilon_n - \varepsilon_{n+1}, \) and set

\[
\lambda \in w \in \mathbb{D}.
\]

Then \( w \in L^1(\partial \mathbb{D}) \), and

\[
P[w](\lambda_n) \geq \int_{I_n} P(\lambda_n, \xi) \sum_k \frac{\beta_k}{\sigma(I_n)} \chi_{I_n}(\xi) d\sigma(\xi) \gtrsim \sum_k \frac{\beta_k}{\sigma(I_n)} \frac{1}{1 - |\lambda_n|} \int_{I_k} d\sigma(\xi)
\]

This and Theorem [1.3] prove the assertion.

The proof for the Nevanlinna class is even simpler. Set \( d\mu_s = \delta_1 \), the Dirac mass on \( 1 \in \partial \mathbb{D} \). From [1.3] we get

\[
\log |B_{\lambda_n}(\lambda_n)|^{-1} \lesssim \frac{1}{1 - |\lambda_n|} \lesssim P[\mu_s](\lambda_n),
\]

and we finish by applying Theorem [1.2].

**Proof of Proposition [1.12]** By Corollary [1.10c], we already know that [1.4] is a sufficient condition for \( \Lambda \) to be interpolating for \( N^+ \). Conversely, suppose that \( \Lambda \) is interpolating for \( N \), that is, \( \varphi_{\Lambda} \) admits a harmonic majorant. Since \( \mu_\Lambda \) has bounded balayage, then \( \int_\mathbb{D} \varphi_{\Lambda} d\mu_\Lambda < \infty \), which is exactly [1.4].

**Proof of Proposition [1.13]** It is obvious that (a) implies (b). If we assume (c), \( \mu_\Lambda \) will act against any positive harmonic function. Suppose \( F \in N \). As seen in Section [5] there exists a positive harmonic function \( h \) so that \( \log^+ |F| \leq h \). Thus, taking \( \mu_\Lambda \) as in [1.5],

\[
\sum_{\lambda \in \Lambda} (1 - |\lambda|) \log^+ |F(\lambda)| = \int_\mathbb{D} \log^+ |F(\lambda)| d\mu_\Lambda(\lambda) \leq \int_\mathbb{D} h(\lambda) d\mu_\Lambda(\lambda) < \infty.
\]

Finally, to prove that (b) implies (c), suppose that (c) doesn’t hold, i.e. \( g_\Lambda := \sum_{\lambda}(1 - |\lambda|)P_\lambda \) is unbounded. Since \( g_\Lambda \) is lower semi-continuous, this implies that \( g_\Lambda \notin L^\infty(\partial \mathbb{D}) \). Since \( L^\infty \) is the dual of \( L^1 \), there exists \( f \in L^1(\partial \mathbb{D}) \) such that \( \int_\partial \mathbb{D} fg_\Lambda = \infty \). Taking an outer function \( F \in N^+ \) with \( \log |F| = P[f] \) we see that

\[
\sum_{\lambda \in \Lambda} (1 - |\lambda|) \log |F(\lambda)| = \sum_{\lambda \in \Lambda} (1 - |\lambda|) \int_{\partial \mathbb{D}} P_\lambda f = \int_{\partial \mathbb{D}} fg_\Lambda = \infty,
\]

so (b) doesn’t hold.■
Proof of Nafašalević’s theorem. Assume that $\Lambda$ is contained in a finite union of Stolz angles and $(1.3)$ holds. By Proposition $1.11$, $\Lambda \in \text{Int} N$, hence the trace $N|\Lambda$ is given by the majorization condition of Theorem $1.2$(b). Taking as majorizing function the Poisson integral of the sum of the Dirac masses at the vertices, we see that $N|\Lambda \supset l_{Na}$.

Conversely, if $N|\Lambda = l_{Na}$ then the trace is ideal, so $\Lambda$ is free interpolating and by Corollary $1.10$(b) $(1.3)$ holds. According to Theorem $1.2$(b) and the definition of $l_{Na}$, the function

$$\varphi(z) = \begin{cases} 
(1 - |\lambda|)^{-1} & \text{if } z = \lambda \in \Lambda \\
0 & \text{if } z \notin \Lambda 
\end{cases}$$

admits a harmonic majorant $h$. Let $h(z) = P[\mu](z)$ and consider the intervals

$$I_z^\alpha = \{ \zeta \in \partial D : z \in \Gamma_{\alpha}(\zeta) \}.$$

There exist constants $\alpha$ and $C_\alpha$ such that $\mu(I_z^\alpha) > C_\alpha$ for any $z$ such that $h(z) \geq (1 - |z|)^{-1}$.

If $\Lambda$ is not contained in a finite union of Stolz angles, then there is an accumulation point $\zeta \in \partial D$ of $\Lambda'$ such that $\Lambda' \not\subset \Gamma_{\beta}(\zeta)$ for any $\beta$. Pick $\beta > \alpha$; then for $\lambda' \in \Lambda'$, $I_{\lambda'}^\alpha \not\subset \zeta$ and we can construct an infinite subsequence $\Lambda'' \subset \Lambda'$ such that the Privalov shadows $\{I_{\lambda'}^\alpha\}_{\lambda' \in \Lambda''}$ are disjoint. This prevents $h$ from being the Poisson integral of a finite positive measure. ■

We now give an example of a concrete separation condition implying that $\mu_{\Lambda}$ has bounded balayage.

**Proposition 8.2.** Assume that $\Lambda \subset \mathbb{D}$ is contained in the union of a family $A$ of Whitney squares such that

$$|\text{Arg}(z(Q)) - \text{Arg}(z(L))| > g^{-1}(1 - |z(Q)|)$$

for any $Q, L \in A, Q \neq L$, where $z(Q)$ is the center of $Q$ and $g$ is a positive function, with $g(x)/x$ decreasing and

$$\int_0^x g(x) x^2 \, dx < \infty.$$ 

Then $\Lambda \in \text{Int} N$ if and only if $\Lambda \in \text{Int} N^+$, and if and only if

$$\sum_{Q \in A} (1 - |z(Q)|) \sup_{\lambda \in \Lambda \cap Q} \log |B_{\lambda}(\lambda)|^{-1} < \infty.$$ 

Note that this covers some cases where $\mu_{\Lambda}$ has not bounded balayage, even though another measure associated with the sequence will be (see the proof).

In order to prove Proposition $8.2$, consider the “Carleson window” $Q(e^{i\theta}, r)$ centered at $e^{i\theta}$, of side $r$:

$$Q(e^{i\theta}, r) := \{ z \in \mathbb{D} : 1 - |z| \leq r, |\text{Arg}(z) - \theta| \leq r \}.$$

The next result is a Carleson-type condition which implies boundedness of the balayage.

**Lemma 8.3.** Suppose that $\mu(Q(e^{i\theta}, r)) \leq g(r)$, where $g$ is a nondecreasing function on $[0, 2)$ with

$$\int_0^2 g(x) x^2 \, dx < \infty.$$ 

Then $\mu$ is a measure with bounded balayage.
A discrete version of this condition is
\[
\sum_n 2^n \sup_{\theta \in \mathbb{R}} \mu(Q(e^{i\theta}, 2^{-n})) < \infty,
\]
as can be checked by writing a Riemann sum.

**Proof.** For any \( t > 0 \), let \( \Omega_t(\theta) := \{ z \in \mathbb{D} : P_z(e^{i\theta}) \geq t \} \). This is a disk, tangent to the unit circle at the point \( e^{i\theta} \), of radius \( 1/(t+1) \). Therefore \( \Omega_t(\theta) \subset Q(e^{i\theta}, C/t) \) for \( t \geq 1 \), say, with \( C > 0 \) an absolute constant.

Using the distribution function \( \mu(\Omega_t(\theta)) \) and the fact that the measure \( \mu \) is bounded, we get the following estimate for the balayage of \( \mu \):
\[
\int_{\mathbb{D}} P_z(e^{i\theta})d\mu(z) = \int_0^\infty \mu(\Omega_t(\theta))dt \leq C_1 + \int_1^\infty \mu(Q(e^{i\theta}, C/t))dt \\
\leq C_1 + \int_1^\infty g(C/t)dt \leq C_1 + C\int_1^1 \frac{g(x)}{x^2}dx < \infty.
\]

\[ \blacksquare \]

We will now compare measures satisfying the condition in Lemma 8.3 measures with bounded balayage and Carleson measures. Each set is included in the next, and the examples will show that both inclusions are strict.

**Example 1.** Let \( \alpha = \{\alpha_n\} \) be a sequence of nonnegative reals. Let \( \mu_{\alpha} \) be the measure concentrated on the circles centered at the origin of radius \( 1 - 2^{-n} \) given in dual terms by
\[
\int_{\mathbb{D}} f(z)d\mu_{\alpha}(z) := \sum_{n \geq 1} \alpha_n \frac{1}{2\pi} \int_0^{2\pi} f((1 - 2^{-n})e^{i\theta})d\theta.
\]

One can check that \( \mu_{\alpha} \) is a Carleson measure if and only if it has bounded balayage and this happens if and only if \( \sum_n \alpha_n < \infty \). Also \( \mu_{\alpha} \) satisfies the condition in Lemma 8.3 if and only if \( \sum_n \sum_{k \geq n} \alpha_k < \infty \).

**Example 2.** Let \( m \) be a nonnegative-valued function on the interval \([0, 1)\). Let \( \mu_m \) be the measure concentrated on the ray from the origin to 1 given by
\[
\int_{\mathbb{D}} f(z)d\mu_m(z) := \int_0^1 f(x)m(x)dx.
\]

One can check that \( \mu_m \) is a Carleson measure if and only if there exists a constant \( K \) such that
\[
\int_{1-\delta}^1 m(x)dx \leq K\delta, \quad \forall \delta > 0
\]
and \( \mu_m \) is a measure with bounded balayage if and only if it satisfies the condition in Lemma 8.3 which happens if and only if
\[
\int_1^1 \frac{m(x)}{1-x}dx < \infty.
\]

In particular, if we take \( \alpha_k = k^{-\gamma} \), with \( 1 < \gamma \leq 2 \), \( \mu_m \) is a measure with bounded balayage but it does not satisfy the condition in Lemma 8.3 if we take \( m(x) = 1 \), \( \mu_m \) is a Carleson measure, but it does not have bounded balayage.
In view of Proposition 1.12, among other things, it is interesting to understand for which separated sequences \( \Lambda \) the corresponding measure \( \mu_\Lambda \) has bounded balayage. It is easy to see that this is the case when \( |\lambda|/|\lambda| - \lambda'/|\lambda'| | \geq (1 - |\lambda|)^{1/2}, \lambda' \neq \lambda \), but more is true.

**Lemma 8.4.** Suppose that \( g \) is a positive valued function such that \( g(x)/x \) is increasing and

\[
\int_0^\infty \frac{g(x)}{x^2} \, dx < \infty.
\]

Let \( g^{-1} \) stand for the inverse function of \( g \). Then, if we have a sequence \( \Lambda \subset \mathbb{D} \) such that

\[
\left| \frac{\lambda}{|\lambda|} - \frac{\lambda'}{|\lambda'|} \right| \geq g^{-1}(1 - |\lambda|), \quad \forall \lambda' \neq \lambda,
\]

the measure \( \mu_\Lambda \) has bounded balayage.

Examples of such functions \( g \) are given by \( x(\log x)^{-1-\varepsilon} \), with \( \varepsilon > 0 \). In that case, \( g^{-1}(t) \simeq t(\log \frac{1}{x})^{1+\varepsilon} \).

On the other hand, we can see that for the above lemma to hold, we must have \( g^{-1}(t) \gg t \). More precisely, take the sequence in the upper half-plane given by

\[
\lambda_k := e^{-k} + ik^{-1/2}e^{-k}.
\]

Then, \( \text{Re } \lambda_k - \text{Re } \lambda_{k+1} \simeq e^{-k} \), so the sequence \( \{\lambda_k\}_k \) verifies the separation condition in Lemma 8.4 with \( g^{-1}(x) \simeq x(\log \frac{1}{x})^{1/2} \), but

\[
\sum_k (\text{Im } \lambda_k) P_{\lambda_k}(0) \simeq \sum_k \left( \frac{\text{Re } \lambda_k}{\text{Im } \lambda_k} \right)^2 = \sum_k \frac{1}{k} = \infty.
\]

**Proof of Lemma 8.4.** Let \( \theta \in [0, 2\pi) \). By hypothesis, there is at most one \( \lambda \in \Lambda \) such that

\[
\theta \in J_\lambda := \left( \text{Arg}(\lambda) - \frac{1}{2}g^{-1}(1 - |\lambda|), \text{Arg}(\lambda) + \frac{1}{2}g^{-1}(1 - |\lambda|) \right).
\]

Let \( \mu' := \sum_{\lambda' \neq \lambda} (1 - |\lambda'|)\delta_{\lambda'} \). Then

\[
\int_{\mathbb{D}} P_z(e^{i\theta}) \, d\mu_{\Lambda}(z) = (1 - |\lambda|)P_{\lambda}(e^{i\theta}) + \int_{\mathbb{D}} P_z(e^{i\theta}) \, d\mu'(z) \leq C + \int_{\mathbb{D}} P_z(e^{i\theta}) \, d\mu'(z).
\]

By the proof of Lemma 8.3 for this specific value of \( \theta \), we see that it will be enough to check that for some absolute constants \( C_1, C_2 \), one has

\[
\mu'(Q(e^{i\theta}, r)) \leq C_1 g(C_2 r), \quad \text{for } 0 < r < 2.
\]

Consider \( \Sigma_r := \{\lambda' \neq \lambda : \lambda' \in Q(e^{i\theta}, r)\} \). For any \( \lambda' \in \Sigma_r \) we have

\[
\sigma(J_{\lambda'}) = g^{-1}(1 - |\lambda'|) \leq 2|\theta - \text{Arg}(\lambda')| \leq 2r,
\]

so the intervals \( J_{\lambda'} \) are all contained in \([\theta - 3r, \theta + 3r]\). Since they are disjoint, \( \sum_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'}) \leq 6r \).

Using that \( g(x)/x \) is increasing we have

\[
\sup_{\lambda' \in \Sigma_r} \frac{1 - |\lambda'|}{\sigma(J_{\lambda'})} \leq \frac{g\left( \sup_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'}) \right)}{\sup_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'})} \leq \frac{g(2r)}{2r}.
\]
Finally,
\[\mu'(Q(e^{i\theta}, r)) = \sum_{\lambda' \in \Sigma_r} 1 - |\lambda'| \leq \sup_{\lambda' \in \Sigma_r} \frac{1 - |\lambda'|}{\sigma(J_{\lambda'})} \sum_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'}) \leq \frac{g(2r)}{2r} - 6r = 4g(2r).\]

**Proof of Proposition 8.2** For each Whitney square \(Q\) in \(A\), let \(\lambda(Q)\) be the point in \(\Lambda \cap Q\) such that
\[\log |B_{\lambda(Q)}(\lambda(Q))|^{-1} = \max\{\log |B_{\lambda}(\lambda)|^{-1} : \lambda \in \Lambda \cap Q\}.
\]
Let \(\Sigma\) be the sequence formed by \(\{\lambda(Q) : Q \in A\}\). By Lemma 8.4, the corresponding measure \(\mu_{\Sigma}\) has bounded balayage. Therefore, there exists a positive harmonic function \(h\) with \(h(\lambda(Q)) \geq \log |B_{\lambda(Q)}(\lambda(Q))|^{-1}\) if and only if
\[\sum (1 - |\lambda(Q)|) \log |B_{\lambda(Q)}(\lambda(Q))|^{-1} < \infty.\]
According to condition (c) in Theorem 1.2, one deduces that \(\Lambda \in \text{Int } N\) if and only if the last sum converges. Furthermore, when this is the case, the function \(h\) can always be taken quasi-bounded (see Lemma 8.1), so that interpolation can actually be performed in the Smirnov class.

### 9. Hardy-Orlicz Classes

Let \(\phi : \mathbb{R} \rightarrow [0, \infty)\) be a convex, nondecreasing function satisfying

(i) \(\lim_{t \rightarrow \infty} \phi(t)/t = \infty\)

(ii) \(\Delta_2\)-condition: \(\phi(t + 2) \leq M\phi(t) + K, t \geq t_0\) for some constants \(M, K \geq 0\) and \(t_0 \in \mathbb{R}\).

Such a function is called strongly convex (see [RosRov]), and one can associate with it the corresponding Hardy-Orlicz class
\[\mathcal{H}_\phi = \{f \in N^+ : \int_{\partial D} \phi(\log |f(\zeta)|) d\sigma(\zeta) < \infty\},\]
where \(f(\zeta)\) is the non-tangential boundary value of \(f\) at \(\zeta \in \partial D\), which exists almost everywhere. In [Har99], the following result was proved.

**Theorem.** Let \(\phi\) be a strongly convex function satisfying (i), (ii) and the \(V_2\)-condition:
\[2\phi(t) \leq \phi(t + \alpha), \quad t \geq t_1\]
where \(\alpha > 0\) is a suitable constant and \(t_1 \in \mathbb{R}\). Then \(\Lambda \subset \mathbb{D}\) is free interpolating for \(\mathcal{H}_\phi\) if and only if \(\Lambda\) is a Carleson sequence, and in this case
\[\mathcal{H}_\phi|\Lambda = \{a = (a_\lambda)_{\lambda} : |a|_\varphi = \sum_{\lambda \in \Lambda} (1 - |\lambda|) \varphi(\log |a_\lambda|) < \infty\}.\]

The conditions on \(\phi\) imply that for all \(\mathcal{H}_\phi\) there exist \(p, q \in (0, \infty)\) such that \(H^p \subset \mathcal{H}_\phi \subset H^q\). In particular, the \(V_2\)-condition implies the inclusion \(H^p \subset \mathcal{H}_\phi\) for some \(p > 0\). This \(V_2\)-condition has a strong topological impact on the spaces. In fact, it guarantees that metric bounded sets are also bounded in the topology of the space (and so the usual functional analysis tools still apply in this situation; see [Har99] for more on this and for further references). It was not clear whether this was only a technical problem or if there existed a critical growth for \(\phi\) (below exponential growth \(\phi(t) = e^{pt}\) corresponding to \(H^p\) spaces) giving a breakpoint in the behavior of interpolating sequences for \(\mathcal{H}_\phi\).
We can now affirm that this behavior in fact changes between exponential and polynomial growth. Let \( \phi \) be a strongly convex function with associated Hardy-Orlicz space \( \mathcal{H}_\phi \). Assume moreover that \( \phi \) satisfies

\[
\phi(a + b) \leq c(\phi(a) + \phi(b)),
\]

for some fixed constant \( c \geq 1 \) and for all \( a, b \geq t_0 \). The standard example in this setting is \( \phi_p(t) = t^p \) for \( p > 1 \). We have the following result.

**Theorem 9.1.** Let \( \phi : \mathbb{R} \to [0, \infty) \) be a strongly convex function such that (9.1) holds. If there exists a positive weight \( w \in L^1(\partial \mathbb{D}) \) such that \( \phi \circ w \in L^1(\partial \mathbb{D}) \) and \( \varphi_\Lambda \leq P[w] \), then \( \Lambda \in \text{Int} \mathcal{H}_\phi \).

**Proof.** Note first that (9.1) implies that \( \mathcal{H}_\phi \) is an algebra contained in \( N^+ \), hence it is sufficient to interpolate bounded sequences (see Remark 1.1). As in Section 3 we set

\[
g(z) = \int_{\mathbb{D}} \frac{\zeta + z}{\zeta - z} w(\zeta) d\sigma(\zeta).
\]

The reasoning carried out in Section 3 leads to an interpolating function of the form \( fH \exp(g) \), with \( f \in H^\infty \), and \( H = (2 + g)^2 \) out in \( H^p \) for all \( p < 1 \) (note that the measure \( \mu \) defining \( g \) here is absolutely continuous, in fact \( \mu = w \, d\sigma \)). Also, \( H^p \subset \mathcal{H}_\phi \) for any \( p > 0 \) by our conditions on \( \phi \). By construction, \( \int \phi(\log |\exp g|) = \int \phi \circ w < \infty \) so that \( \exp(g) \in \mathcal{H}_\phi \). Since \( \mathcal{H}_\phi \) is an algebra, we deduce that \( fH \exp(g) \in \mathcal{H}_\phi \).

**Example 9.2.** We give an example of an interpolating sequence for \( \mathcal{H}_\phi \) which is not Carleson, thus justifying our claim that there is a breakpoint between Hardy-Orlicz spaces verifying the \( V_2 \)-condition and those that do not.

Consider the functions \( \phi_p \) and let \( \Lambda_0 = \{\lambda_n\}_n \subset \mathbb{D} \) be a Carleson sequence verifying \( I_n \cap I_k = \emptyset \), \( n \neq k \), where \( I_n \) are the arcs defined in (1.1). Since \( \sum_n(1 - |\lambda_n|) < \infty \), there exists a strictly increasing sequence of positive numbers \( (\gamma_n)_n \) such that \( \sum_n(1 - |\lambda_n|)\gamma_n < \infty \) and \( \lim_{n \to \infty} \gamma_n = \infty \). Setting

\[
w = \sum_n \gamma_n^{1/p} \chi_{I_n},
\]

we obtain \( \int \phi_p \circ w = \sum_n(1 - |\lambda_n|)\gamma_n < \infty \) and \( w \in L^1(\partial \mathbb{D}) \) since \( p > 1 \). Associate with \( \Lambda_0 \) a second Carleson sequence \( \Lambda_1 = \{\lambda'_n\}_n \) such that the pseudo-hyperbolic distance between corresponding points satisfies \( |b_{\lambda'_n}(\lambda_n)| = e^{-\gamma_n/p} \). Since \( \gamma_n \to \infty \) the elements of the sequence \( \Lambda = \Lambda_0 \cup \Lambda_1 \) are arbitrarily close and \( \Lambda \) cannot be a Carleson sequence. By construction, \( \log |B_\lambda(\lambda)|^{-1} \leq P[w](\lambda) \) (as before, we may possibly have to multiply \( u \) with some constant \( c \) to have that condition also in the points \( \lambda'_n \), but this operation conserves the integrability condition), and therefore \( \Lambda \in \text{Int} \mathcal{H}_\phi \).

REFERENCES

[EiEs] Eiderman, V. Ya. & Essén, M. Uniqueness theorems for analytic and subharmonic functions (Russian), Algebra i Analiz 14 (2002), no. 6, 1–88; English translation in St. Petersburg Math. J. 14 (2002), no. 6.

[Gar77] J. B. Garnett, Two remarks on interpolation by bounded analytic functions, Banach spaces of analytic functions (Proc. Pelewzynski Conf., Kent State Univ., Kent, Ohio, 1976), pp. 32–40. Lecture Notes in Math., Vol. 604, Springer, Berlin, 1977.

[Gar81] J.B. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[Har99] A. Hartmann, Free interpolation in Hardy-Orlicz spaces, Studia Math. 135 (1999), no. 2, 179–190.
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[HaMa01] A. Hartmann & X. Massaneda, Interpolating sequences for holomorphic functions of restricted growth, Ill. J. Math. 46 (2002), no.3, 929–945.

[Hay] W.K. Hayman, Identity theorems for functions of bounded characteristic, J. London Math. Soc. 58 (1998), no. 2, 127–140.

[He69] Heins, M., Hardy Classes on Riemann Surfaces, Springer Verlag, Berlin, Heidelberg, New York, Lectures Notes in Mathematics no. 98, 1969.

[HULL] Hiriart-Urruty, J.-B., Lemaréchal, C. Convex Analysis and Minimization Algorithms I, Grundlehren der mathematischen Wissenschaften 305, Springer-Verlag, Berlin Heidelberg New York, 1993.

[Lu96] D. Luecking, Zero sequences for Bergman spaces, Complex Variables Theory Appl. 30 (1996), no. 4, 345–362.

[LySe97] Lyubarski, Yu. I. & Seip, K. A uniqueness theorem for bounded analytic functions, Bull. London Math. Soc. 29 (1997), 49–52.

[McC92] J. McCarthy, Topologies on the Smirnov class, J. Funct. Anal. 104 (1992), no. 1, 229–241.

[Na56] A.G. Naftalevič, On interpolation by functions of bounded characteristic (Russian), Vilniaus Valst. Univ. Mokslu Darbai. Mat. Fiz. Chem. Mokslu Ser. 5 (1956), 5–27.

[NPT] Nicolau, A., Pau, J., & Thomas, P. J. Smallness sets for bounded holomorphic functions, Journal d’Analyse Mathématique 82 (2000), 119–148.

[NiTh] Nicolau, A., & Thomas, P. J. Superharmonic envelopes for positive data on the disk, preprint, math/0304482

[Nik86] N.K. Nikolski [Nikol’ski], Treatise on the shift operator, Springer-Verlag, Berlin etc., 1986.

[Nik02] Operators, functions, and systems: an easy reading. Vol. 1, Hardy, Hankel, and Toeplitz; Vol.2, Model Operators and Systems, Mathematical Surveys and Monographs, 92 and 93. American Mathematical Society, Providence, RI, 2002.

[PaTh] Pau, J. & Thomas, P. J. Decrease of bounded holomorphic functions along discrete sets, electronically available from math arXiv : http://arXiv.org/abs/math.CV/0106048 math/0106048

[RosRov] M. Rosenblum & J. Rovnyak, Hardy classes and operator theory. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1985.

[Se93] K. Seip, Beurling type density theorems in the unit disk, Invent. Math., 113 (1993), 21–39.

[ShSh] H.S. Shapiro & A.L. Shields, On some interpolation problems for analytic functions, Amer. J. Math., 83 (1961), 513–532.

[ShSh] J. Shapiro & A. Shields, Unusual topological properties of the Nevanlinna class, Amer. J. Math. 97 (1975), 915–936.

[Ya74] N. Yanagihara, Interpolation theorems for the class $N^+$, Illinois J. Math., 18 (1974), 427–435.

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