Stability of Confined Vortex Sheets

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Abstract

We propose a simple model for the evolution of an inviscid vortex sheet in a potential flow in a channel with parallel walls. This model is obtained by augmenting the Birkhoff-Rott equation with a potential field representing the effect of the solid boundaries. Analysis of the stability of equilibria corresponding to flat sheets demonstrates that in this new model the growth rates of the unstable modes remain unchanged as compared to the case with no confinement. Thus, in the presence of solid boundaries the equilibrium solution of the Birkhoff-Rott equation retains its extreme form of instability with the growth rates of the unstable modes increasing in proportion to their wavenumbers.

1 Introduction

Shear layers play an important role in fluid mechanics as they appear in many flows of industrial and geophysical significance when boundary layers separate from solid objects. A key property of shear layers is that under typical conditions they are unstable and undergo the Kelvin-Helmholtz instability as a result of which the vorticity from the shear layer rolls up into big vortices, see, e.g., Figure 1. When occurring recurrently, this phenomenon can in turn give rise to a turbulent cascade.

Inviscid vortex sheets, represented as one-dimensional (1D) curves across which the tangential velocity component exhibits a discontinuity and evolving under their own induction in a potential flow, have been frequently invoked as a mathematical abstraction of actual viscous shear layers [9]. In unbounded domains they admit an elegant description in terms of the Birkhoff-Rott equation. This singular integro-differential equation has a number of interesting properties — in particular, its equilibrium solution representing a flat (undeformed) vortex sheet is highly unstable to small-wavelength perturbations, a fact that underlies the ill-posedness of the Birkhoff-Rott model [7]. As a result, computational studies involving the Birkhoff-Rott equation typically require some regularization in order to track its long-time evolution, usually in the form of the well-known “vortex-blob” approach [4] or using the more recent Euler-alpha strategy [2]. A generic feature characterizing the evolution of (regularized) inviscid vortex sheets is roll-up producing localized vortex spirals [5]. There are many interesting mathematical questions concerning various aspects of the Birkhoff-Rott equation and we
refer the reader to the collection [1] and the monographs [6] for further details on this topic. We add that, as was shown in [8], despite the severe instability of the Birkhoff-Rott equation, the equilibrium corresponding to the flat sheet can be efficiently stabilized using methods of modern control theory.

While the Birkhoff-Rott equation has been originally applied on unbounded or laterally unbounded domains (i.e., domains periodic in the streamwise direction and unbounded in the transverse direction), in this study we consider vortex sheets confined to a bounded domain with parallel walls representing a channel. By analyzing the Birkhoff-Rott equation modified to account for the presence of such solid boundaries, we show that, interestingly, confinement does not change the stability properties of the sheet equilibrium. More specifically, in the presence of solid boundaries the growth rates of the unstable modes remain unchanged as compared to the original case with no confinement. The structure of the paper is as follows: in the next section we first recall the Birkhoff-Rott equation and show how it can be modified to account for the effect of solid boundaries; then, in Section 3 we study the stability of the equilibria in the new model, whereas some final comments are deferred to Section 4.

2 Inviscid Vortex Sheets

In this section we introduce a model for an inviscid vortex sheet confined in a channel with two parallel walls located symmetrically above and below the sheet in its equilibrium configuration, cf. Figure 2. We assume that the flow is periodic, with period $2\pi$, in the streamwise direction.
Figure 2: Schematic representation of a confined vortex sheet. The thick brown line represents the equilibrium configuration $\tilde{z}(\gamma), \gamma \in [0, 2\pi]$.

$x$. As is common in the study of such problems [9], we will extensively use the complex representation of different quantities and will identify a point $(x, y) \in \mathbb{R}^2$ in the 2D space with $z = x + iy \in \mathbb{C}$ in the complex plane, where $i = \sqrt{-1}$ is the imaginary unit. Let then $z(\gamma, t)$ denote the position of a point (in the fixed frame of reference) on the sheet which corresponds to the circulation parameter $\gamma \in [0, 2\pi]$ and some time $t$. The quantity $\gamma$ represents a way of parameterizing the sheet and for sheets of constant intensity is proportional to the arc-length of the curve. Periodicity of the sheet then implies

$$z(\gamma + 2\pi, t) = z(\gamma, t) + 2\pi, \quad \gamma \in [0, 2\pi]. \quad (1)$$

When the sheet is in a laterally unbounded domain, its evolution is governed by the Birkhoff-Rott equation [9]

$$\frac{\partial z^*}{\partial t}(\gamma, t) = V(z(\gamma, t)) := \frac{1}{4\pi i} \text{pv} \int_0^{2\pi} \cot \left( \frac{z(\gamma, t) - z(\gamma', t)}{2} \right) d\gamma', \quad (2)$$

where $z^*$ denotes the complex conjugate of $z$, the integral on the right-hand side (RHS) is understood in Cauchy’s principal-value sense and $V(z) = (u - iv)(z)$ represents the complex velocity at the point $z$ with $u$ and $v$ the horizontal and vertical velocity components (":=" means “equal to by definition”).

We now consider time evolution of a confined vortex sheet on a domain $\Omega := \{(x, y) : x \text{ is } 2\pi\text{-periodic,} \\
- \ b < y < b \}$, with straight boundaries $\partial\Omega$ located at $y = \pm b$ for some $b > 0$, cf. Figure 2.

Distance $b$ will serve as the main parameter in the problem. Since there is no flow through these solid boundaries, the velocity in the flow must satisfy the following conditions on its wall-normal component

$$\Im \left( V(x \pm ib) \right) = 0. \quad (3)$$
Since our model is inviscid, the tangential (slip) velocity component on the walls \( \mathbb{R} \left( V(x \pm ib) \right) \) need not vanish. In order to satisfy condition (3), the velocity field induced by the vortex sheet, given by the RHS of the Birkhoff-Rott equation (2), must be augmented by including a suitable potential velocity field \( W(z) := \frac{\partial \phi}{\partial x}(z) - i \frac{\partial \phi}{\partial y}(z) \) expressed in terms of a potential \( \phi \). This potential is constructed to cancel the velocity induced by the vortex sheet on the boundary \( y = \pm b \), such that the modified Birkhoff-Rott equation takes the following form

\[
\frac{\partial z^*}{\partial t}(\gamma, t) = V(z(\gamma, t)) + W(z(\gamma, t))
\]

\[
= \frac{1}{4\pi i} \text{pv} \int_0^{2\pi} \cot \left( \frac{z(\gamma, t) - z(\gamma', t)}{2} \right) \, d\gamma' + \frac{\partial \phi}{\partial x}(z(\gamma, t)) - i \frac{\partial \phi}{\partial y}(z(\gamma, t)), \quad (4a)
\]

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0 \quad \text{in } \Omega, \quad (4b)
\]

\[
\phi(x, y) = \phi(x + 2\pi, y), \quad x \in (0, 2\pi), \quad -b < y < b, \quad (4c)
\]

\[
\frac{\partial \phi}{\partial y} \bigg|_{y=\pm b} = \Im \left[ \frac{1}{4\pi i} \int_0^{2\pi} \cot \left( \frac{x \pm ib - z(\gamma', t)}{2} \right) \, d\gamma' \right], \quad x \in (0, 2\pi), \quad (4d)
\]

where relations (4c)–(4d) are the boundary conditions for the Laplace equation (4b) defining the potential \( \phi \). We emphasize that the potential depends linearly on the velocity induced by the vortex sheet on the solid boundaries \( y = \pm b \), cf. (4d). Equation (2) and system (4) are complemented with a suitable initial condition \( z(\gamma, 0) = z_0(\gamma) \), \( \gamma \in [0, 2\pi] \). As can be readily verified, they admit an equilibrium solution (a fixed point) \( \tilde{z}(\gamma, t) := \gamma \) for \( \gamma \in [0, 2\pi] \), i.e., \( \frac{\partial}{\partial t} \tilde{z}(\gamma, t) = 0 \), which corresponds to a flat (undeformed) sheet, cf. Figure 2. The stability of this equilibrium solution, and in particular how it depends on the parameter \( b \) defining confinement, is the main question addressed in this paper.

We note that since the sheet intensity, which is assumed constant and equal to unity, represents the difference between the tangential velocity components on both sides of the sheet, the horizontal velocity component in the flow is defined up to an arbitrary constant \( u_0 \) which, however, does not affect the stability properties of the sheet. To see this, we perform a change of coordinates to a moving frame of reference \( Z(t, \gamma) := z(t, \gamma) - u_0 t \), such that equation (4a) becomes \( \frac{\partial Z^*}{\partial t}(\gamma, t) = V(Z(\gamma, t)) + u_0 t + W(Z(\gamma, t) + u_0 t) - u_0 \). It is now clear that the Jacobian of the RHS of this equation, and hence also the stability properties of the sheet, do not depend on the constant term \( u_0 \). Therefore, without loss of generality, below we will assume that \( u_0 = 0 \). In addition, under the assumption of unit sheet strength, at the equilibrium configuration the circulation parameter \( \gamma \) can be identified with the coordinate \( x \), such that we have \( \tilde{z}(x, t) = x \); thus, for simplicity, hereafter we will use \( x \) in lieu of \( \gamma \) to parameterize the equilibrium. The stability of confined equilibria described by system (4) is studied in the next section.

### 3 Stability Analysis

In this section we first review the stability properties of unconfined sheet equilibria governed by the Birkhoff-Rott equation (2), which are classical results [3, 10], and then proceed to analyze the stability of confined sheet equilibria described by system (4). As a starting point,
we perturb the equilibrium state infinitesimally as
\[ z(x, t) = x + \varepsilon \zeta(x, t), \] (5)
for some \( 0 < \varepsilon \ll 1 \). Here, \( \zeta(x, t) \) is a perturbation represented in the periodic setting, cf. (1), as
\[ \zeta(x, t) = \sum_{k=-\infty}^{\infty} \hat{\zeta}_k(t)e^{ikx}, \] (6)
in which \( \hat{\zeta}_k \in \mathbb{C}, \ k \in \mathbb{Z} \), are the Fourier coefficients. Then, we obtain the linearized equation for the evolution of the perturbation \( \zeta(x, t) \) as follows [10]
\[
\frac{\partial \zeta^*}{\partial t}(x, t) = V'[\zeta](x, t) := -\frac{1}{8\pi i} \text{pv} \int_0^{2\pi} \frac{\zeta(x, t) - \zeta(x', t)}{\sin^2 \left( \frac{x-x'}{2} \right)} dx'.
\]
\[
= \sum_{k=-\infty}^{\infty} \hat{\zeta}_k(t)e^{ikx} \left[ -\frac{1}{8\pi i} \text{pv} \int_0^{2\pi} \frac{1 - e^{-ikx'}}{\sin^2 \left( \frac{x'}{2} \right)} dx' \right]
\]
\[
= \sum_{k=-\infty}^{\infty} \hat{\zeta}_k(t)e^{ikx} \left[ -\frac{1}{4i} \text{pv} \int_0^{1} \frac{1 - \cos(2\pi k x') - i \sin(2\pi k x')}{\sin^2(\pi x')} dx' \right]
\]
\[
= \frac{1}{2i} \sum_{k=1}^{\infty} k \hat{\zeta}_k(t)e^{ikx} - \frac{1}{2i} \sum_{k=1}^{\infty} k \hat{\zeta}_{-k}(t)e^{-ikx},
\] (7)
where \( V'[\zeta] \) is a linear operator acting on \( \zeta \) obtained as the linearization of the RHS of equation (2) around the equilibrium configuration. Since
\[
\frac{\partial \zeta^*}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{d\hat{\zeta}_k^*}{dt}e^{-ikx} = \sum_{k=1}^{\infty} \frac{d\hat{\zeta}_k^*}{dt}e^{-ikx} + \sum_{k=1}^{\infty} \frac{d\hat{\zeta}_{-k}^*}{dt}e^{ikx} + \frac{d\hat{\zeta}_0^*}{dt},
\] (8)
equating coefficients of the Fourier components in (7) and (8) corresponding to different wavenumbers \( k \), we obtain an infinite system of linear ordinary differential equations (ODEs) for the evolution of the coefficients \( \hat{\zeta}_k \) in a block-diagonal form
\[
\frac{d\hat{\zeta}_0}{dt} = 0,
\] (9a)
\[
\frac{d}{dt} \begin{bmatrix} \hat{\zeta}_k \\ \hat{\zeta}_{-k}^* \end{bmatrix} = \frac{i}{2} k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\zeta}_k \\ \hat{\zeta}_{-k}^* \end{bmatrix} =: A_k \begin{bmatrix} \hat{\zeta}_k \\ \hat{\zeta}_{-k}^* \end{bmatrix}, \quad k = 1, 2, \ldots .
\] (9b)
corresponding to the original continuous integro-differential problem. Since this latter system has a block-diagonal structure, key insights about the infinite-dimensional problem can be obtained by analyzing just a single block, independently from all other blocks. The eigenvalues corresponding to each diagonal block with matrix \( A_k, \ k \geq 1, \) cf. (9b), are \( \lambda_k = \pm \frac{k}{2} \). Thus, we see that at each wavenumber \( k \) there is an unstable and stable mode and the growth rate of the former is proportional to the wavenumber \( k \), such that small-scale perturbations always become more unstable. This extreme form of instability underlies the ill-posedness of the initial-value problem for the Birkhoff-Rott equation [7].
We now go on to analyze the stability of equilibria of confined vortex sheets governed by system (4) whose linearization takes the form

\[ \frac{\partial \zeta^*}{\partial t} (x,t) = V'[\zeta](x,t) + W'[\zeta](x,t) \]  

(10)

in which \( W'[\zeta](x) = \frac{\partial \phi'}{\partial x}(x,y = 0) - i \frac{\partial \phi'}{\partial y}(x, y = 0) \) represents the linearization of the potential velocity \( W \) in (4a) around the equilibrium. At every instant of time \( t \) the linearized potential \( \phi' \) solves the problem

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi' = 0 \]  

in \( \Omega \),

\[ \phi'(x,y) = \phi'(x + 2\pi, y), \quad x \in (0, 2\pi), \ -b < y < b, \]

(11a)

\[ \frac{\partial \phi'}{\partial y} \bigg|_{y=\pm b} = \Im \left[ V'[\zeta](x \pm ib, t) \right], \quad x \in (0, 2\pi), \]

(11b)

where

\[ V'[\zeta](x \pm ib, t) = \frac{1}{8\pi i} \oint_{|z|=1} \frac{\zeta(x', t) \, dx'}{\sin^2 \left( \frac{x \pm ib - x'}{2} \right)}. \]

(12)

Noting (6) and the fact that \( V'[\zeta] \) is linear in the perturbation \( \zeta \), the RHS in the boundary condition (11c) can be expressed as

\[ \Im \left[ V'[\zeta](x \pm ib, t) \right] = \Im \left[ \frac{1}{8\pi i} \sum_{k=-\infty}^{\infty} \tilde{\zeta}_k(t) I_k(x \pm ib) \right], \]

(13)

where \( I_k(x \pm ib) := \int_0^{2\pi} \frac{e^{ikx'} \, dx'}{\sin^2 \left( \frac{x \pm ib - x'}{2} \right)}, \quad k \in \mathbb{Z}, \)

(14)

represents the linearized complex velocity induced by the Fourier component of the perturbation \( \zeta \) with wavenumber \( k \), cf. (6), on the solid boundaries.

Our goal is to eliminate the linearized potential \( \phi' \) such that the term \( W'[\zeta] \) in (10) can be expressed explicitly as a function of the perturbation \( \zeta \), which in turn will allow us to assess the effect of the confinement on the stability of the equilibrium \( \tilde{\zeta} \). This will be done in three simple steps described in the following three sections.

### 3.1 Evaluation of integrals \( I_k \) in (14)

The integrals \( I_k, k \in \mathbb{Z} \), are defined in the classical (Riemann) sense (i.e., they are nonsingular) and will be evaluated using the calculus of residues. To this end, we map the integral (14) to the positively oriented unit circle \( |z| = 1 \) on the complex plane \( \mathbb{C} \) using the change of variables \( z = e^{ix} \), such that \( dz = iz \, dx \) and (14) becomes

\[ I_k(\xi) = 4i \oint_{|z|=1} \frac{z^k \, dz}{e^{i\xi} - 2z + e^{-i\xi}z^2} = 4i e^{ik} \oint_{|z|=1} \frac{z^k \, dz}{(z - e^{i\xi})^2}, \quad \xi := x \pm iy. \]

(15)

In order to evaluate the integral in (15) using the calculus of residues, we decompose its integrand expression into partial fractions, noting that \( k \) can take both positive and negative
\[
\frac{z^k}{(z - e^{i\xi})^2} = \frac{A_1}{z} + \cdots + \frac{A_{|k|}}{z^k} + \frac{B_1}{z - e^{i\xi}} + \frac{B_2}{(z - e^{i\xi})^2} + C_0 + C_1 z + \cdots + C_{|k|-2} z^{|k|-2}, \quad (16)
\]

where \(A_1, \ldots, A_{|k|}, B_1, B_2, C_0, C_1, \ldots, C_{|k|-2} \in \mathbb{C}\), such that

\[
\oint_{|z|=1} \frac{z^k \, dz}{(z - e^{i\xi})^2} = 2\pi i \left[ A_1 + B_1 \right].
\]

We note that \(A_1 = \cdots = A_{|k|} = 0\) if \(k \geq 0\) and \(C_0 = C_1 = \cdots = C_{|k|-2} = 0\) if \(k \leq 0\). We then have the following four cases depending on the signs of \(k\) and \(y = \Im(\xi) = \pm b\):

- \(k \geq 0\) and \(y = b > 0\) (\(|e^{i\xi}| < 1\)), such that \(A_1 = 0, B_1 = k e^{(k-1)i\xi}\) and

\[
\oint_{|z|=1} \frac{z^k \, dz}{(z - e^{i\xi})^2} = 2\pi i \left[ k e^{(k-1)i\xi} \right],
\]

- \(k \geq 0\) and \(y = -b < 0\) (\(|e^{i\xi}| > 1\)), such that \(A_1 = B_1 = 0\), and

\[
\oint_{|z|=1} \frac{z^k \, dz}{(z - e^{i\xi})^2} = 0,
\]

- \(k < 0\) and \(y = b > 0\) (\(|e^{i\xi}| < 1\)), such that \(A_1 = -B_1 = -k e^{(k-1)i\xi}, A_1 + B_1 = 0\), and

\[
\oint_{|z|=1} \frac{z^k \, dz}{(z - e^{i\xi})^2} = 0,
\]

- \(k < 0\) and \(y = -b < 0\) (\(|e^{i\xi}| > 1\)), such that \(A_1 = -k e^{(k-1)i\xi}, B_1 = 0\) and

\[
\oint_{|z|=1} \frac{z^k \, dz}{(z - e^{i\xi})^2} = -2\pi i \left[ k e^{(k-1)i\xi} \right].
\]

Thus, we finally obtain

\[
I_k(\xi) = \begin{cases} 
-8\pi k e^{ik\xi}, & k \geq 0 \text{ and } \Im(\xi) > 0 \\
+8\pi k e^{-ik\xi}, & k < 0 \text{ and } \Im(\xi) < 0, \\
0, & \text{otherwise}
\end{cases}, \quad (17)
\]

a result which can be verified by approximating integrals (15) numerically.

### 3.2 Solution of System (11) for Linearized Potential

Given the linearity of (11a) and the form of the boundary condition in (11c), which involves a superposition of Fourier components with different wavenumbers \(k\), we represent the perturbation potential in the form of a series

\[
\phi'(x, y) = \sum_{k=0}^{\infty} \phi'_k(x, y), \quad \text{where} \quad \phi'_k(x, y) := \frac{1}{2} k e^{-kb} \left[ P_h(y) e^{ikx} + P_h^*(y) e^{-ikx} \right] \quad (18)
\]
for some functions \( P_k : [-b, b] \to \mathbb{C} \), \( k = 0, 1, \ldots \). Using formulas (17) and representation (18), the boundary condition (11c) becomes equivalent to the following set of relations

\[
\frac{\partial \phi'_k}{\partial y}\big|_{y=b} = \Im \left[ ik \hat{\zeta}_k e^{-kb} e^{i k x} \right] = \frac{k e^{-kb}}{2} \left( \hat{\zeta}_k e^{i k x} + \hat{\zeta}_k^* e^{-i k x} \right), \quad (19a)
\]

\[
\frac{\partial \phi'_k}{\partial y}\big|_{y=-b} = \Im \left[ ik \hat{\zeta}_{-k} e^{-kb} e^{-i k x} \right] = \frac{k e^{-kb}}{2} \left( \hat{\zeta}_{-k} e^{-i k x} + \hat{\zeta}_{-k}^* e^{i k x} \right), \quad k \geq 0, \quad (19b)
\]

such that system (11) reduces to a family of 1D boundary-value problems

\[
\frac{d^2 P_k(y)}{dy^2} - k^2 P_k(y) = 0, \quad \text{in } (-b, b), \quad k = 0, 1, 2, \ldots, \quad (20a)
\]

\[
\frac{d}{dy} P_k(b) = \hat{\zeta}_k, \quad \frac{d}{dy} P_k(-b) = \hat{\zeta}_{-k}^*. \quad (20b)
\]

Their solutions are, noting that \( \hat{\zeta}_0 \equiv 0 \),

\[
P_0(y) = 0, \quad (21a)
\]

\[
P_k(y) = (\hat{\zeta}_k - \hat{\zeta}_{-k}^*) \frac{\cosh(ky)}{2k \sinh(kb)} + (\hat{\zeta}_k + \hat{\zeta}_{-k}^*) \frac{\sinh(ky)}{2k \cosh(kb)}, \quad k \geq 1, \quad (21b)
\]

such that the perturbation potential finally becomes

\[
\phi'(x, y) = \sum_{k=1}^{\infty} \left\{ \left( \hat{\zeta}_k - \hat{\zeta}_{-k}^* \right) \frac{\cosh(ky)}{2k \sinh(kb)} + \left( \hat{\zeta}_k + \hat{\zeta}_{-k}^* \right) \frac{\sinh(ky)}{2k \cosh(kb)} \right\} e^{i k x} \quad (22)
\]

\[
\left[ \left( \hat{\zeta}_k - \hat{\zeta}_{-k}^* \right) \frac{\cosh(ky)}{2k \sinh(kb)} + \left( \hat{\zeta}_k + \hat{\zeta}_{-k}^* \right) \frac{\sinh(ky)}{2k \cosh(kb)} \right] \right\} e^{-i k x}.
\]

### 3.3 Stability of Modified System

In order to evaluate the term \( W' [\hat{\zeta}] (x, t) \) in (10), we need to compute the partial derivatives of the perturbation potential (22) at the location of the unperturbed sheet \( \hat{\zeta} \), i.e., at \( y = 0 \),

\[
\frac{\partial \phi'}{\partial x}\big|_{y=0} = \frac{i}{2} \sum_{k=1}^{\infty} k e^{-kb} \left[ ik P_k(0) e^{i k x} - ik P_k(0) \hat{\zeta}_k^* e^{-i k x} \right], \quad (23a)
\]

\[
\frac{\partial \phi'}{\partial y}\big|_{y=0} = \frac{i}{2} \sum_{k=1}^{\infty} k e^{-kb} \left[ \frac{d}{dy} P_k(0) e^{i k x} - \frac{d}{dy} P_k(0) \hat{\zeta}_k^* e^{-i k x} \right], \quad (23b)
\]

such that, using (21), we obtain

\[
W' [\hat{\zeta}] (x, t) = \frac{i}{2} \sum_{k=1}^{\infty} \frac{k e^{-kb}}{\sinh(2kb)} \left[ \left( e^{-kb} \hat{\zeta}_k - e^{kb} \hat{\zeta}_{-k}^* \right) e^{i k x} - \left( e^{kb} \hat{\zeta}_k^* - e^{-kb} \hat{\zeta}_{-k} \right) e^{-i k x} \right]. \quad (24)
\]

Inserting this expression into (10) and rewriting this equation in terms of Fourier components as was done earlier for the original problem (9), we obtain an infinite system of ODEs for the
Figure 3: Equilibria deformed as $\tilde{z} + \epsilon u_1$ with the unstable eigenvectors $u_1$ of (red dashed line) the unconfined problem (9) and (blue solid lines) the confined problem (25) with $b = 0.01, 0.05, 0.25$ (more deformed sheets correspond to smaller values of $b$). In all cases we have $k = 1$ and $\epsilon = 0.02$, whereas the eigenvectors are normalized to have the same mean-square vertical displacement.

Fourier coefficients preserving the block-diagonal structure of the original problem, with the block corresponding to the wavenumber $k$ given by

$$
\frac{d\hat{\zeta}_0}{dt} = 0,
$$

$$
\frac{d}{dt} \begin{bmatrix} \hat{\zeta}_k \\ \hat{\zeta}^*_{-k} \end{bmatrix} = \frac{i}{2} \begin{bmatrix} f_k(b) & -(1 + g_k(b)) \\ (1 + g_k(b)) & -f_k(b) \end{bmatrix} \begin{bmatrix} \hat{\zeta}_k \\ \hat{\zeta}^*_{-k} \end{bmatrix} = \begin{bmatrix} csch(2kb) & -coth(2kb) \\ coth(2kb) & -csch(2kb) \end{bmatrix} \begin{bmatrix} \hat{\zeta}_k \\ \hat{\zeta}^*_{-k} \end{bmatrix} =: \tilde{\mathbf{A}}_k \begin{bmatrix} \hat{\zeta}_k \\ \hat{\zeta}^*_{-k} \end{bmatrix}, \quad k = 1, 2, \ldots,
$$

where the expressions

$$
f_k(b) := \frac{1}{\sinh(2kb)},
$$

$$
g_k(b) := \frac{e^{-2kb}}{\sinh(2kb)}
$$

represent “corrections” due to confinement effects.

Since when $k > 0$ we have $\lim_{b \to 0} f_k(b), g_k(b) = \infty$ and $\lim_{b \to \infty} f_k(b), g_k(b) = 0$, the terms representing the effect of confinement in matrix $\tilde{\mathbf{A}}_k$ vanish when the distance $b$ increases such that, as expected, the unconfined case is recovered in the limit $b \to \infty$. On the other hand, these terms become dominant as the solid walls approach the vortex sheet, cf. Figure 2. However, the eigenvalues of matrix $\tilde{\mathbf{A}}_k$ do not depend on the distance $b$. They are given by $\tilde{\lambda}_k = \pm \frac{i}{2}$ and are identical to the eigenvalues of matrix $\mathbf{A}_k$, cf. (9b), describing the linearized evolution of unconfined vortex sheets. Thus, interestingly, within the inviscid model considered here, confinement has no effect on the growth rates of the unstable modes.

We now consider how confinement affects the form of unstable eigenmodes and sheet equilibria deformed as $\tilde{z} + \epsilon u_k$ by the unstable modes $u_k$ with wavenumber $k$ are shown in Figure 3.
for \( k = 1 \) and different values of parameter \( b \). Since functions \( f_k(b) \) and \( g_k(b) \), cf. (26a)–(26b), vanish very rapidly as \( b \) increases, differences between the unstable modes in the unconfined and confined cases are significant only for very small values of \( b \), of order \( O(10^{-1}) \) or less, and are essentially imperceptible when \( b \) is \( O(1) \) or larger, which is the more relevant case from the practical point of view. The effect of the confinement is to steepen the profile of the unstable modes.

4 Conclusions

We have proposed a simple model for the evolution of an inviscid vortex sheet in a potential flow in a confined geometry representing a channel with parallel walls. It is obtained by augmenting the standard Birkhoff-Rott equation with a potential field representing the effect of the solid boundaries. Next we considered the stability of equilibria corresponding to flat sheets and demonstrated through analytical computations that the presence of the solid boundaries does not affect the block-diagonal structure of the modified Birkhoff-Rott equation linearized around the equilibrium and, more importantly, the growth rates of the unstable modes remain unchanged with respect to the case with no confinement. Thus, in the presence of solid boundaries the equilibrium solution of the Birkhoff-Rott equation retains its extreme form of instability with the growth rates of the unstable modes increasing in proportion to their wavenumbers.

The validity of the findings reported above is certainly restricted by the assumptions inherent in our highly-idealized model, most importantly, the assumption that the flow is irrotational away from the vortex sheet. Due to boundary-layer effects, this assumption is definitely not going to be satisfied for small values of the distance \( b \). As regards the dynamics of the vortex sheet itself, viscous effects can be accounted for using the “vortex-blob” regularization approach [4] or with the more recently proposed Euler-alpha strategy [2]. As a topic of future investigation, it will be interesting to consider confinement effects on such regularized models.

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