ON THE DIOPHANTINE EQUATION \( f(x)f(y) = f(z)^n \)
CONCERNING LAURENT POLYNOMIALS

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ABSTRACT. By the theory of elliptic curves, we investigate the nontrivial rational parametric solutions of the Diophantine equation \( f(x)f(y) = f(z)^n \), where \( n = 1, 2 \) and \( f(X) \) are some simple Laurent polynomials.

1. Introduction

Let \( f(X) \in \mathbb{Q}[X] \) be a polynomial without multiple roots and \( \text{deg}(f) \geq 2 \). There are many authors considered the integer solutions of the Diophantine equation
\[
(1.1) \quad f(x)f(y) = f(z)
\]
for different polynomials \( f(X) \), such as L. Euler [5] (\( f(X) = X(X + 1)/2 \)), C. Ko [10] (\( f(X) = X^X \), P. Erdős proposed), S. Katayama [8] and A. Baragar [1] (\( f(X) = X(X + 1) \)), K. Kashiwara [7] (\( f(X) = X^2 - 1 \)), Y. Bugeaud [3] and M.A. Bennett [2] (\( f(X) = X^k - 1, k > 3 \)), and Y. Zhang and T. Cai [20, 21] (\( z \) is a square, \( \text{deg}(f) = 2, 3 \)). In 1992 and 1994, A. Schinzel and U. Zannier [13, 14] investigated the number of integer solutions of Eq. (1.1) for monic quadratic polynomials with integer coefficients and gave many important results about it. In 2015, Y. Zhang and T. Cai [22] showed that Eq. (1.1) has infinitely many nontrivial positive integer solutions for \( f(X) = X(X + d) \) with \( d \geq 3 \), infinitely many nontrivial positive integer solutions for \( f(X) = (X - 1)X(X + 1) \), and a rational parametric solution for \( f(X) = X(X - 1)(X^3 - 3) \).

Another interesting Diophantine equation is
\[
(1.2) \quad f(x)f(y) = f(z)^2.
\]

For the related information, we can refer to [12, 17, 9, 2, 19, 6]. In 2007, M. Ulas [18] proved that if \( f(X) = X^2 + k \), where \( k \in \mathbb{Z} \), then Eq. (1.2) has infinitely many rational parametric solutions; if \( f(X) = X(X^2 + X + t) \), where \( t \in \mathbb{Q} \), then Eq. (1.2) has infinitely many rational solutions for all but finitely many \( t \). In 2015, we [23] gave the conditions for \( f(X) = A X^2 + B X + C \) such that Eq. (1.2) has infinitely many nontrivial integer solutions and proved that it has a rational parametric solution for infinitely many irreducible cubic polynomials which solved a problem of [18].

In this paper we consider the nontrivial rational parametric solutions of Eqs. (1.1) and (1.2) for some simple Laurent polynomials. The nontrivial solutions

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Recall that a Laurent polynomial with coefficients in a field $\mathbb{F}$ is an expression of the form

$$f(X) = \sum_{k} a_{k} X^{k}, a_{k} \in \mathbb{F},$$

where $X$ is a formal variable, the summation index $k$ is an integer (not necessarily positive) and only finitely many coefficients $a_{k}$ are non-zero. Here we mainly care about the simple Laurent polynomials

$$f(X) = AX + B + \frac{C}{X},$$

where $A, B, C \in \mathbb{Z}$ and $C \neq 0$.

For Eq. (1.1), by the theory of elliptic curves, we have

**Theorem 1.1.** For $f(X) = AX + B + \frac{C}{X}$ with $ABC \neq 0$, Eq. (1.1) has infinitely many nontrivial rational parametric solutions $(x, y, z)$.

**Theorem 1.2.** For $f(X) = AX + \frac{C}{X}$ with $AC \neq 0$, if there exists a nonzero rational number $T$ such that the elliptic curve

$$E_{2} : Y^{2} = (X - 2ACT^{2})(X + 2ACT^{2})(X - 2AC(2A^{2}T^{4} + (4AC - 1)T^{2} + 2C^{2}))$$

has positive rank, then Eq. (1.1) has infinitely many nontrivial rational solutions $(x, y, z)$.

**Remark 1.1.** When $A = 0$, $f(X) = B + \frac{C}{X}$, Eq. (1.1) becomes

$$\frac{(B^{2} - B)xyz - Cxy + BCxz + BCyz + C^{2}z}{xyz} = 0.$$

If $B = 1$, we can find infinitely many integer solutions of $xy - xz - yz - Cz = 0$ for any nonzero integers $C$, such as

$$(x, y, z) = (k(k + 1 + C), k + 1, k), (k(kl - C - l), kl - C, kl - C - l),$$

where $k, l$ are integer parameters. When $f(X) = 1 + \frac{1}{X}$, it is interesting to note that A. Padoa [5, p.688] studied the Diophantine equation

$$1 + \frac{1}{x} (1 + \frac{1}{y}) = 1 + \frac{1}{z},$$

which is equivalent with

$$(x - z)(y - z) = z(z + 1).$$

If $z$ is given, then we can obtain all couples $x, y$ by finding all pairs of positive integers whose products is $z(z + 1)$, and adding $z$ to each other. J.E.A. Steggall [5, p.688] found the positive integer solutions of Eq. (1.3), by noting that $xy$ must be divisible by $x + y + 1 = a$, and hence $a|x(x+1)$. Hence for any integer $x$, determine a factor $a > x + 1$ of $x(x+1)$; then $y = a - x - 1$, while $z = a - b$, where $b = x(x+1)/a$.

If $B \neq 0, 1$, it is easy to get the rational parametric solutions of $(B^{2} - B)xyz - Cxy + BCxz + BCyz + C^{2}z = 0$, such as

$$(x, y, z) = \left( u, v, \frac{Cuv}{(B^{2} - B)uv + BCu + BCv + C^{2}} \right),$$

where $u, v$ are rational parameters. But it seems difficult to get its infinitely many integer solutions.
Then Eq. (1.1) becomes trivial rational solutions 

\( C \) has infinitely many rational points 

\( z \) nonzero rational number 

Consider the above equation as a quadratic equation of 

Theorem 1.3. For \( f(X) = AX + B + \frac{C}{X} \) with \( ABC \neq 0 \), Eq. (1.2) has infinitely many nontrivial rational solutions \((x, y, z)\). 

Theorem 1.4. For \( f(X) = AX + \frac{C}{X} \), if there exist nonzero integers \( A, C \) and a nonzero rational number \( z \) such that the quartic elliptic curve 

\[ \text{C}_4: v^2 = -4A^3Cz^4y^4 + (A^4z^8 + 4A^3Cz^6 - 2A^2C^2z^4 + 4AC^3z^2 + C^4)y^2 - 4AC^3z^4 \]

has infinitely many rational points \((y, v)\), then Eq. (1.2) has infinitely many nontrivial rational solutions \((x, y, z)\).

Remark 1.2. When \( A = 0 \), \( f(X) = B + \frac{C}{X} \), Eq. (1.2) reduces to 

\[ C((2Byz - Bz^2 + Cy)x - Byz^2 - Cz^2) = 0. \]

If \( BC \neq 0 \), a rational parametric solution of \((2Byz - Bz^2 + Cy)x - Byz^2 - Cz^2 = 0\) is

\[ (x, y, z) = \left( \frac{v^2(Bu + C)}{2Bu - Bv^2 + Cu}, u, v \right), \]

where \( u, v \) are rational parameters. To get its infinitely many integer solutions is a difficult problem.

2. The proofs of Theorems

Proof of Theorem 1.1. For \( f(X) = AX + B + \frac{C}{X} \), let 

\[ x = T, y = u. \]

Then Eq. (1.1) becomes

\[ f(u) = Az(At^2 + Bt + C)u^2 + (-Atz^2 + (At^2 + (B - 1)T + C)Bz - CT)u + Cz(At^2 + BT + C) = 0. \]

Consider the above equation as a quadratic equation of \( u \), if it has rational solutions, then the discriminant

\[ \Delta(z) = a_1z^4 + a_2z^3 + a_3z^2 + a_4z + a_5 \]

should be a square, where

\[ a_1 = A^3T^2, \]

\[ a_2 = -2ABT(At^2 + (B - 1)T + C), \]

\[ a_3 = -A^2(4AC - B^2)T^4 - 2AB(4AC - B^2 + B)T^3 + (B^4 - 8A^2C^2 - 2AB^2C - 2B^3 + 2AC + B^2)T^2 - 2BC(4AC - B^2 + B)T - C^2(4AC - B^2), \]

\[ a_4 = -2BCT(At^2 + (B - 1)T + C), \]

\[ a_5 = C^2T^2. \]

Let us consider the curve \( \text{C}_1: v^2 = \Delta(z) \).

1) If \( 4AC - B^2 \neq 0 \), the discriminant of \( \Delta(z) \) is non-zero as an element of \( \mathbb{Q}(T) \), then \( \text{C}_1 \) is smooth. By the method described in [11, p.77] (or see [3, p.476, Proposition 7.2.1]), \( \text{C}_1 \) is birationally equivalent with the elliptic curve

\[ E_1: Y^2 = (X + 2ACT^2)(X^2 + a_3X + a_6), \]
where $a_3$ is the coefficient of $z^2$ in $\Delta(z)$ and
\[
a_0 = 2ACT^2(A^2(4AC + B^2)T^4 + 2AB(4AC + B^2 - B)T^3 + (8A^2C^2 + 6AB^2C + B^4 - 2B^3C + 2AB^2 + B^3)T^2 + 2BC(4AC + B^2 - B)T + C^2(4AC + B^2)).
\]
Because the map $\varphi_1 : C_1 \to E_1$ is quite complicated, we do not present the explicit equations for the coordinates of it.

It is easy to see that $E_1$ contains the point
\[P = (2ACT^2, 4ABC(1T + C)T^2).\]
By the group law, we have
\[\begin{aligned}
[2] P &= 2AC((2AC - 1)B^2 + 4AC(A + C + 1)B + 2AC(A + C + 1)^2))
\end{aligned}\]
A quick computation reveals that the remainder of the division of the numerator by the denominator with respect to $B$ is equal to
\[4A^2C^2(A + C + 1)(2B + A + C + 1)
\]
and thus is non-zero provided $AC \neq 0$. By the Nagell-Lutz theorem (see [15, p.78]), $P_1$ is of infinite order on $E_1$, and thus $P$ is of infinite order on $E_1$. Hence, the group $E_1(\mathbb{Q}(T))$ is infinite.

Compute the points $[m]P$ on $E_1$ for $m = 2, 3, \ldots$, next calculate the corresponding point $\varphi_1^{-1}([m]P) = (v_m, z_m)$ on $C_1$ and solve the equation $f(u) = 0$ for $u$. Put the calculated roots into the expression for $u$, get various $\mathbb{Q}(T)$-rational solutions $(x, y, z)$ of Eq. (1.1) for $f(X) = AX + B + \frac{C}{T}$.

2) For $4AC - B^2 = 0$, then $f(X) = \frac{2AX+B}{4AX}$, and
\[\Delta(z) = 16A^2(2Az + B)^2\]
\[\times (4A^2T^2z^2 + (-8A^2BT^3 - (8AB^2 - 4AB)T^2 - 2B^3z)z + B^2T^2).
\]
If $4A^2T^2z^2 + (-8A^2BT^3 - (8AB^2 - 4AB)T^2 - 2B^3z)z + B^2T^2 = w^2$, then $\Delta(z)$ is a square. This quadratic equation about $z, w$ can be parameterized by
\[z = \frac{2BT(4A^2T^2 + 2(2B - 1)AT + B^2 + t)}{4A^2T^2 - t^2},
\]
\[w = \frac{BT(4(2t + 1)A^2T^2 + 4(2B - 1)AT + 2B^2t + t^2)}{4A^2T^2 - t^2}.
\]
where \( t \) is a rational parameter. Then
\[
\begin{align*}
    u & = \frac{B(2AT - t)(4A^2T^2 + (2B - 1)2AT + B^2 + t)}{2A(2AT + B)^2(2AT + t)}, \\
    \text{or} & \quad \frac{2A(2AT - t)(4A^2T^2 + 2(2B - 1)AT + B^2 + t)}{2A(2AT + B)^2(2AT + t)}.
\end{align*}
\]

So the rational parametric solutions of Eq. (1.1) are
\[
(x, y, z) = \left( T, \frac{B(2AT - t)(4A^2T^2 + 2(2B - 1)AT + B^2 + t)}{2A(2AT + B)^2(2AT + t)}, \frac{2BT(4A^2T^2 + 2(2B - 1)AT + B^2 + t)}{4A^2T^2 - t^2} \right),
\]
\[
\text{or} \quad \left( T, \frac{B(2AT + B)^2(2AT + t)}{2A(2AT - t)(4A^2T^2 + 2(2B - 1)AT + B^2 + t)}, \frac{2BT(4A^2T^2 + 2(2B - 1)AT + B^2 + t)}{4A^2T^2 - t^2} \right).
\]

Combining 1) and 2), we complete the proof of Theorem 1.1.

\[\square\]

**Example 1.** The point \([2]P\) on \(E_1\) leads to the solutions of Eq. (1.1):
\[
(x, y, z) = \left( T, -\frac{(AT^2 + BT + C)B}{(AT^2 + (B + 1)T + C)A}, -\frac{(AT^2 + (B + 1)T + C)C}{BT} \right),
\]
\[
\text{or} \quad \left( T, -\frac{(AT^2 + (B + 1)T + C)C}{(AT^2 + BT + C)B}, -\frac{(AT^2 + (B + 1)T + C)C}{BT} \right).
\]

**Proof of Theorem 1.2.** For \(f(X) = AX^2 + \frac{C}{X}\), let
\[
x = T, \quad y = u.
\]

Then Eq. (1.1) reduces to
\[
(A^2T^2z + ACz)u^2 + (-ATz^2 - CT)u + ACT^2z + C^2z = 0.
\]

If the above equation has rational solutions \(u\), then the discriminant
\[
\Delta'(z) = A^2T^2z^4 + (-4A^3CT^4 - 8A^2C^2T^2 - 4AC^3 + 2ACT^2)z^2 + C^2T^2
\]
should be a square.

Let us consider the curve \(C_2 : v^2 = \Delta'(z)\). If \(AC \neq 0\), the discriminant of \(\Delta'(z)\) is non-zero as an element of \(\mathbb{Q}(T)\), then \(C_2\) is smooth. By the method described in [11] p.77 (or see [1] p.476, Proposition 7.2.1), \(C_2\) is birationally equivalent to the elliptic curve
\[
E_2 : Y^2 = (X - 2ACT^2)(X + 2ACT^2)(X - 2AC(2A^2T^4 + (4AC - 1)T^2 + 2C^2)).
\]

Hence, if there exists a nonzero rational number \(T\) such that the elliptic curve \(E_2\) has positive rank, then Eq. (1.1) has infinitely many nontrivial rational solutions \((x, y, z)\).

\[\square\]

**Example 2.** When \(A = C = 1, T = 4/3\), the elliptic curve \(E_2\) leads to
\[
E'_2 : U^2 = V^3 - 2212V^2 - 82944V + 183472128,
\]
where \( U = 729Y, V = 81X \). Then

\[
z = \frac{U}{24(V - 2212)}, y = \frac{50(V - 288)}{U}.
\]

Note that the rank of \( E'_2 \) is one, so Eq. (1.1) has infinitely many rational solutions \((4/3, y, z)\) for \( f(X) = X + \frac{1}{X} \), such as

\[
(4/3, 351, 92), \quad (4/3, 231, 288), \quad (4/3, 17849194593, 307689313546).
\]

To get infinitely many positive rational solutions of Eq. (1.1), we need a theorem of Poincaré and Hurwitz (see [16, p.78]) about the density of rational points: If an elliptic curve \( E \) defined over \( \mathbb{Q} \) has positive rank and at most one torsion point of order two, then the set \( E(\mathbb{Q}) \) is dense in \( E(\mathbb{R}) \). The same result holds if \( E \) has three torsion points of order two under the assumption that we have a rational point of infinite order on the bounded branch of the set \( E(\mathbb{R}) \).

If there is a point on \( E'_2 \) satisfying the condition

\[
z = \frac{U}{24(V - 2212)} > 0, \quad y = \frac{50(V - 288)}{U} > 0,
\]

then there are infinitely many rational points on \( E'_2 \) satisfying it. Because the point \((V, U) = (8712, 702000)\) on \( E'_2 \) satisfies the above condition, so Eq. (1.1) has infinitely many positive rational solutions \((4/3, y, z)\) for \( f(X) = X + \frac{1}{X} \).

**Proof of Theorem 1.3.** For \( f(X) = AX + B + \frac{C}{X} \), let

\[y = xT^2, \quad z = xT.\]

Then Eq. (1.2) becomes

\[g(x) = (T - 1)^2(ABT^2x^2 + (ACT^2 + 2ACT + AC)x + BC) = 0.\]

Consider the above equation as a quadratic equation of \( x \), if it has rational solutions, the discriminant

\[
\Delta(T) = A^2C^2T^4 + 4A^2C^2T^3 + (6A^2C^2 - 4AB^2C)T^2 + 4A^2C^2T + A^2C^2
\]

should be a square. Let us consider the curve \( C_3 : v^2 = \Delta(T) \).

1) If \( 4AC - B^2 \neq 0 \), the discriminant of \( \Delta(T) \) is non-zero, then \( C_3 \) is smooth. By the method described in [11] p.77 [or see [4] p.476, Proposition 7.2.1], \( C_3 \) is birationally equivalent with the elliptic curve

\[E_3 : Y^2 = (X + 2A^2C^2)(X^2 + 4AC(AC - B^2)X + 4A^3C^3(AC + 2B^2)).\]

Because the map \( \varphi_3 : C_3 \to E_3 \) is complicated, we omit it.

Note that \( E_3 \) contains the point

\[Q = (2A^2C^2, 8A^3C^3).\]

By the group law, we have

\[Q = (2A^2C^2 - 2AB^2C + B^4, (AC - B^2)(A^2C^2 + 4AB^2C - B^4)).\]
\[ \frac{AC}{Q} = \left( -7A^6C^8 + 88A^7B^2C^7 - 420A^6B^4C^6 + 24A^5B^6C^5 + 50A^4B^8C^4 \\
+ 8A^3B^{10}C^3 - 20A^2B^{12}C^2 + 8AB^{14}C - B^{16} \right) / (4(AC - B^2)^2(A^2C^2 + 4AB^2C - B^4)^2), \]

\( (A^4C^4 - 20A^3B^2C^3 + 6A^2B^4C^2 - 4AB^6C + B^8)(A^8C^8 + 80A^7B^2C^7 - 180A^6B^4C^6 + 656A^5B^6C^5 - 282A^4B^8C^4 - 80A^3B^{10}C^3 + 76A^2B^{12}C^2 - 16AB^{14}C + B^{16}) / (8(AC - B^2)^3(A^2C^2 + 4AB^2C - B^4)^3) \).

A quick computation reveals that the remainder of the division of the numerator by the denominator of the \( X \)-th coordinate of \([4]Q\) with respect to \( B \) is equal to

\[ 16A^3C^3(3B^{10} - 17ACB^8 + 14A^2C^2B^6 + 26A^3C^3B^4 - 9A^4C^4B^2 + A^5C^5). \]

and thus is non-zero provided \( AC \neq 0 \). By the Nagell-Lutz theorem (see [15, p.78]),\([4]Q\) is of infinite order on \( E_3 \), then there are infinitely many rational points on \( E_3 \) for \( 4AC - B^2 \neq 0 \).

For \( m = 2, 3, \ldots \), compute the points \([m]Q\) on \( E_3 \), next calculate the corresponding point \( \varphi^{-1}_3([m]Q) = (v_m, T_m) \) on \( C_3 \) and solve the equation \( g(x) = 0 \) for \( x \). Put the calculated roots into the expression for \( x \), we get various rational solutions \((x, y, z)\) of Eq. (1.2) for \( f(x) = AX + B + C \).

2) For \( 4AC - B^2 = 0 \), then \( f(x) = (2AX + B)^2 \), and

\[ \Delta(T) = A^4B^2(T^2 + 6T + 1)(T - 1)^2. \]

If \( T^2 + 6T + 1 = w^2 \), then \( \Delta(T) \) is a square. This quadratic equation about \( T, w \) can be parameterized by

\[ T = -\frac{2(t - 3)}{t^2 - 1}, w = \frac{t^2 - 6t + 1}{t^2 - 1}, \]

where \( t \) is a rational parameter. Then

\[ x = \frac{(t + 1)^2B}{16A}, \text{ or } -\frac{(t - 1)^2B}{(t - 3)^2A}. \]

So the rational parametric solutions of Eq. (1.2) are

\[
(x, y, z) = \left( -\frac{(t + 1)^2B}{16A}, \frac{(t - 3)^2B}{4(t - 1)^2A}, \frac{(t - 3)(t + 1)B}{8(t - 1)A} \right),
\]

or

\[
-\frac{(t - 1)^2B}{(t - 3)^2A}, -\frac{4B}{(t + 1)^2A}, \frac{2(t - 1)B}{t + 1)(t - 3)A}.
\]

Combining 1) and 2), then the proof of Theorem 1.3 is completed. \( \square \)

**Example 3.** The point \(-Q = (2A^2C^2, -8A^3C^3)\) on \( E_3 \) leads to the solutions of Eq. (1.2):

\[ (x, y, z) = \left( -\frac{(AC - B^2)^2}{4A^2BC}, -\frac{AC - B^2}{B^2}, -\frac{2AB}{AC} \right), \]

or

\[ -\frac{B}{A} - \frac{4AB^2C^2}{(AC - B^2)^2}, \frac{2BC}{AC - B^2}. \]
the point $-2Q = (-A^2C^2 - 2AB^2C + B^4, -(AC - B^2)(A^2C^2 + 4AB^2C - B^4))$
on E_3 leads to the solutions of Eq. (1.2):

$$(x, y, z) = \left( \frac{-4ABC^2}{(AC - B^2)^2}, \frac{B(3AC - B^2)(AC + B^2)^2}{A(A^2C^2 - 6AB^2C + B^4)^2}, \frac{2BC(3AC - B^2)(AC + B^2)}{(AC - B^2)(A^2C^2 - 6AB^2C + B^4)} \right),$$

or

$$(x, y, z) = \left( \frac{C(A^2C^2 - 6AB^2C + B^4)}{B(3AC - B^2)^2(AC + B^2)^2}, \frac{(AC - B^2)^2}{4A^2BC}, \frac{(AC - B^2)(A^2C^2 - 6AB^2C + B^4)}{2AB(AC + B^2)(3AC - B^2)} \right).$$

**Proof of Theorem 1.4.** For $f(X) = AX + \frac{C}{X}$, Eq. (1.2) reduces to

$$(A^2y^2z^2 + ACz^2)x^2 + (-A^2yz^4 - 2ACyz^2 + C^2y)x + ACyz^2 + C^2z^2 = 0.$$ If the above equation has rational solutions $x$, then the discriminant

$$\Delta(y) = -4A^4Cz^4y^4 + (A^4z^8 + 4A^3Cz^6 - 2A^2C^2z^4 + 4AC^3z^2 + C^4)y^2 - 4AC^3z^4$$ should be a square. Let us consider the curve $C_4: v^2 = -4A^3Cz^4y^4 + (A^4z^8 + 4A^3Cz^6 - 2A^2C^2z^4 + 4AC^3z^2 + C^4)y^2 - 4AC^3z^4$.

Therefore, if there exist nonzero integers $A, C$ and a nonzero rational number $z$ such that the elliptic curve $C_4$ has infinitely many rational points $(y, v)$, then Eq. (1.2) has infinitely many nontrivial rational solutions $(x, y, z)$. \[\square\]

**Example 4.** When $A = 1, C = -1, z = 4$, $C_4$ is birationally equivalent to

$$E_4: Y^2 = X^3 + 48577X^2 - 4194304X - 203746705408.$$ Then

$$y = \frac{Y}{64(48577 + X)}, x = \frac{(48577 + X)(X^2 + 97154X - 450Y + 4194304)}{(64X + 3108928 + Y)(64X + 3108928 - Y)}.$$ Note that the rank of $E_4$ is one, so Eq. (1.2) has infinitely many rational solutions $(x, y, 4)$ for $f(X) = X - \frac{1}{X}$, such as

$$(x, y, z) = \left( \frac{18}{13}, \frac{1}{14}, \frac{4}{4} \right), \left( \frac{18971}{1024}, \frac{5891}{4061}, \frac{4}{4} \right), \left( \frac{825}{2204}, \frac{1631}{10244}, \frac{4}{4} \right), \left( \frac{58502431053824}{507945397025551}, \frac{23133182812831}{48912132263775}, \frac{4}{4} \right).$$

In virtue of the theorem of Poincaré and Hurwitz (see [10, p.78]), $E_4$ has infinitely many rational points in every neighborhood of any one of them. If there is a point on $E_4$ satisfying the condition

$$y = \frac{Y}{64(48577 + X)} > 0, \quad x = \frac{(48577 + X)(X^2 + 97154X - 450Y + 4194304)}{(64X + 3108928 + Y)(64X + 3108928 - Y)} > 0,$$

then there are infinitely many rational points on $E_4$ satisfying it. Because the point

$$(X, Y) = \left( \frac{112352}{49}, \frac{79764000}{343} \right)$$
on $E_4$ satisfies the above condition, then Eq. (1.2) has infinitely many positive rational solutions $(x, y, 4)$ for $f(X) = X - \frac{1}{X}$. 

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3. Some related questions

We have studied the rational parametric solutions of Eqs. (1.1) and (1.2) for \( f(X) = AX + B + \frac{C}{X} \), but we don’t get the same results for other simple Laurent polynomials.

**Question 3.1.** For \( f(X) = AX^2 + BX + C + \frac{D}{X} \) or \( AX + B + \frac{C}{X} + \frac{D}{X^2} \), whether Eqs. (1.1) and (1.2) have rational solutions? If they have, are there infinitely many?

To find the integer solutions of Eqs. (1.1) and (1.2) is also an interesting question. Here we list some nontrivial integer solutions of Eq. (1.1) for \( f(X) = X + 1 + \frac{C}{X} \) with \( 1 \leq C \leq 10, -100 \leq x < y \leq 100 \) and \(-100 \leq z \leq 100 \) in Table 1.

| \( C \) | \( (x, y, z) \) |
|-------|------------------|
| 1     | \((-3, -2, 2)\) |
| 2     | \((-2, -1, 1), (-2, -1, 2)\) |
| 3     | \((-4, -2, 8), (-4, 21, -84), (-3, -2, 6), (-2, 2, -12)\) |
| 4     | \((-9, -3, 27), (-4, -3, 12), (-3, -1, 12)\) |
| 5     | \((-16, -4, 64), (-5, -4, 20), (-4, -1, 20), (-2, 2, -20), (2, 3, 30)\) |
| 6     | \((-7, 2, -42), (-7, 3, -42), (-6, -5, 30), (-5, -1, 30)\) |
| 7     | \((-7, -6, 42), (-6, -1, 42), (-5, 3, -35), (3, 13, 91)\) |
| 8     | \((-8, -7, 56), (-7, -1, 56), (-5, 2, -40), (-5, 4, -40), (2, 11, 88), (4, 11, 88)\) |
| 9     | \((-9, -8, 72), (-8, -1, 72)\) |
| 10    | \((-10, -9, 90), (-9, -1, 90)\) |

Table 1. Some integer solutions of Eq. (1.1) for \( f(X) = X + 1 + \frac{C}{X} \)

By some calculations, we can find a lot of Laurent polynomials such that Eqs. (1.1) and (1.2) have integer solutions, but it seems difficult to prove they have infinitely many integer solutions. Hence, we raise

**Question 3.2.** Does there exist a Laurent polynomial \( f(X) = AX + B + \frac{C}{X} \) with \( ABC \neq 0 \) such that Eqs. (1.1) and (1.2) have infinitely many integer solutions?

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