THE FORMATION OF ECCENTRIC COMPACT BINARY INSPIRALS AND THE ROLE OF GRAVITATIONAL WAVE EMISSION IN BINARY–SINGLE STELLAR ENCOUNTERS

JOHAN SAMSING1, MORGAN MACLEOD2, AND ENRICO RAMIREZ-RIUZ2

1 Dark Cosmology Centre, Niels Bohr Institute, University of Copenhagen, Juliane Maries Vej 30, DK-2100 Copenhagen, Denmark
2 Department of Astronomy and Astrophysics, University of California, Santa Cruz, CA 95064, USA

ABSTRACT

The inspiral and merger of eccentric binaries leads to gravitational waveforms distinct from those generated by circularly merging binaries. Dynamical environments can assemble binaries with high eccentricity and peak frequencies within the LIGO band. In this paper, we study binary–single stellar scatterings occurring in dense stellar systems as a source of eccentrically inspiraling binaries. Many interactions between compact binaries and single objects are characterized by chaotic resonances in which the binary–single system undergoes many exchanges before reaching a final state. During these chaotic resonances, a pair of objects has a non-negligible probability of experiencing a very close passage. Significant orbital energy and angular momentum are carried away from the system by gravitational wave (GW) radiation in these close passages, and in some cases this implies an inspiral time shorter than the orbital period of the bound third body. We derive the cross section for such dynamical inspiral outcomes through analytical arguments and through numerical scattering experiments including GW losses. We show that the cross section for dynamical inspirals grows with increasing target binary semi-major axis $a$ and that for equal-mass binaries it scales as $a^{9/7}$. Thus, we expect wide target binaries to predominantly contribute to the production of these relativistic outcomes. We estimate that eccentric inspirals account for approximately 1% of dynamically assembled non-eccentric merging binaries. While these events are rare, we show that binary–single scatterings are a more effective formation channel than single–single captures for the production of eccentrically inspiraling binaries, even given modest binary fractions.

Key words: binaries: close – galaxies: star clusters: general – gravitational waves – stars: black holes – stars: kinematics and dynamics – stars: neutron

Online-only material: color figures

1. INTRODUCTION

The density of stars in galactic nuclei and in the centers of some globular clusters (GCs) can be more than a million times higher than that in our solar neighborhood (Lightman & Shapiro 1978). In such cases, a primordial binary will undergo a close encounter with at least one other star with high probability within its lifetime (e.g., Rasio et al. 2007). It is in these environments, called dense stellar systems, that binary populations will no longer be truly primordial as their stellar composition, eccentricity, and period distributions will be largely determined by past interactions with other stars (e.g., McMillan 1991; Hut et al. 1992; Ivanova et al. 2003, 2005b, 2006, 2008, 2010; Hopman et al. 2006; Fregeau 2008). This transformation of binary systems was envisioned by Hills (1976), who suggested that exchanging neutron stars into preexisting binaries might be a natural way to form X-ray binaries as byproducts.

Dynamical friction causes the heaviest stars and primordial binaries to concentrate toward the cluster’s core (Meylan & Heggie 1997; Fregeau et al. 2002, 2009). Since the heaviest stars tend to be left in the binary following such three-body encounters (this can be understood as a consequence of the tendency toward energy equipartition, in which the lighter star would have the highest velocity in the final state), binaries are quite effective at soaking up heavy stars such as neutron stars and heavy white dwarfs (Hills & Fullerton 1980; Sigurdsson & Phinney 1993, 1995; Heggie et al. 1996), even if none of them originally had a companion.

After such an exchange, the binary will not only be slightly wider but also heavier, which will result in gravitational focusing being more effective. The binary’s cross section for encounters will thus be larger than before the exchange. For this reason, a binary likely to undergo one exchange over some time period is likely to have several more encounters coming rapidly after the first exchange (Sigurdsson & Phinney 1993). The tendency to exchange the heaviest compact stars also has the consequence that the rates of ejection of binaries involved in three-body exchanges are less than those predicted by models in which all stars have equal masses. The recoil speeds of the light, single stars are consequently larger.

A large fraction of the encounters where the field star approaches within approximately a binary semi-major axis (SMA), $a_0$, of the binary center of mass (COM) result in resonant interactions (RIs), in which the three stars wander for a long time on chaotic orbits and approach each other repeatedly (Heggie 1975; Hut 1993). During these chaotic encounters, the stars have many opportunities for close encounters. If the stars are compact, angular momentum loss due to gravitational radiation may become a noticeable effect during close passages (Peters 1964) and could cause the two stars to be driven together. It is the interplay between binaries and compact objects in such dense environments and their ability to manufacture eccentric merging binaries in three-body exchanges that forms the main topic of this work.

Our main goal in this paper is to study how the inclusion of gravitational wave (GW) losses modifies the compact binary outcomes that originate from three-body scatterings, in particular during RIs. The inclusion of GW losses into the binary–single dynamical system, we argue, introduces a new potential outcome in which a pair of objects may dynamically inspiral and merge while the three-body system is still in resonance. These
outcomes are rare, and they are typically only realized during RIs. Chaotic, resonant orbits augment the probability of very close passages when compared with direct interactions (DIs), and they can produce systems with correspondingly short GW inspiral times. Gültekin et al. (2006) first explored the cross section for these inspiral outcomes in the context of IMBH formation and growth. A surprising result of Gültekin et al.’s simulations is that the cross section for inspiral outcomes increases with increasing binary SMA. This is perhaps counterintuitive because one might expect that the cross section for relativistic outcomes would be largest in very tight binaries. However, we will show that this is a natural consequence of resonant binary–single interactions and that the scaling with binary SMA can be analytically derived. As a consequence of this scaling, the rate of inspirals assembled in single–binary interactions dominates over those assembled in single–single encounters by a factor of about 5 in a typical GC.

In this paper, we explore the cross section for dynamical inspiral outcomes during binary–single interactions through numerical experiments and analytic calculations. In Section 2, we review some of the dynamical properties and outcomes of binary–single interactions. In order to gain insight into how the inclusion of GW losses modifies binary–single interaction dynamics, in Section 3 we summarize the results of binary–single scatterings with point masses in Newtonian gravity. Readers familiar with previous work in binary–single dynamics may wish to skip to Section 4, in which we describe the inclusion of post-Newtonian (PN) corrections to the binary–single system equation of motion. Section 5 describes the formation of dynamical inspirals from RIs between hard binaries (HBs) and single objects. We explain the origin of these inspirals through numerical scattering experiments, and we use our results to motivate an analytic derivation of the scaling of the inspiral cross section with binary SMA. In Section 6, we show that dynamical inspirals give rise to high eccentricity inspirals that emit GWs in the frequency range 10–10^4 Hz. This is the frequency window that will be observed with future instruments such as advanced LIGO⁵ (Harry & the LIGO Scientific Collaboration 2010; LIGO Scientific Collaboration et al. 2013) and advanced VIRGO⁶ (Degallaix et al. 2013). We compare this process with eccentric inspirals arising from single–single interactions and show that the cross section is greatly enhanced in binary–single interactions. In Section 7, we extend our calculations to consider binaries containing white dwarfs, we discuss binary lifetimes and the role of GW emission, and we estimate whether the products of binary–single interactions are ejected or retained in their host stellar system. Finally, we estimate the rates of eccentric inspirals given typical GC core properties.

2. BINARY–SINGLE ENCOUNTERS

Binary–single stellar encounters in dense stellar systems may be broadly divided into a few well-defined categories. In the majority of encounters, the incoming object passes the binary on a hyperbolic trajectory with a pericenter distance that is large in comparison with the binary separation (Heggie 1975). The passage time is greater than the binary’s orbital period, and the binary is subjected to a weak perturbation (WP). A strong perturbation (SP) is possible (Heggie 1975) when the incoming object approaches the binary on a hyperbolic trajectory that happens to pass at a distance comparable to the binary SMA.

In this case, the interaction time is less than or similar to the binary’s orbital period.

The accumulation of WPs and SPs across the lifetime of a binary in a dense stellar system modifies the expected eccentricity and SMA distributions as compared with more isolated binaries. To quantify this effect, one must rely on integrations of the coevolution of binaries and their parent clusters over the cluster’s relaxation time (e.g., Aarseth & Lecar 1975; Hills 1975a, 1975b; Heggie 1975; Lightman & Shapiro 1978; McMillan 1986; Baumgardt et al. 2002; Fregeau et al. 2003, 2009; Ivanova et al. 2005a; Fregeau & Rasio 2007).

2.1. Close Interactions and Their Cross Section

A close interaction (CI), by contrast, occurs when the incoming object passes within a sphere of influence marked by the binary’s separation. In these cases, the gravitational interaction between all three bodies may be of similar strength, and the outcomes are chaotic. In this work, we will focus on CIs and the dramatic role they play in reshaping binaries. Figure 1 shows a schematic overview of the different interactions and their expected outcomes.

We define a CI as having occurred when the third body passes within a distance \( r_{CI} \) from the binary COM. We choose \( r_{CI} \) as the distance from the COM to the lighter object in the binary,\(^①\)

\[
r_{CI} = \frac{m_2}{m_1 + m_2}a_0, \tag{1}
\]

where \(1, 2\) are the binary members in order of ascending mass \((m_2 > m_1)\), \(3\) is the incoming object, and \(m_1 + m_2\) is the mass of the target binary. This value is always between \(a_0/2\) (if \(m_1 = m_2\)) and \(a_0\) (if \(m_2 \gg m_1\)).

Whether a CI will occur is analytically predictable given the impact parameter, \(b\), and velocity, \(v_\infty\), of the third body relative to the target binary. At large separations between the binary and the incoming object, the fact that the binary is composed of two objects is unimportant, and thus the encounter can realistically be treated as the interaction between two point masses: the binary with total mass \(m_{bin} = m_1 + m_2\) and the incoming object with mass \(m_3\). In this case, a given distance of closest approach between the incoming single and the COM of the binary, \(r_{min}\), corresponds directly to an impact parameter, \(b\), defined at infinity (Sigurdsson & Phinney 1993),

\[
b = r_{min} \sqrt{1 + \frac{2GM_{tot}}{r_{min}v_\infty^2}}, \tag{2}
\]

where \(v_\infty\) is the initial relative velocity at infinity of the binary COM and the single object, and \(M_{tot} = m_{bin} + m_3\). The second term in this expression corresponds to the gravitational focusing of trajectories from an initially large impact parameter to a closer pericenter distance. Because the argument of the square root is always larger than unity, \(b\) is always greater than \(r_{min}\).

If we now consider the interactions with a closest approach less than the sphere of the binary, \(r_{CI}\), then we see that all encounters with impact parameter less than the corresponding \(b_{CI} = b(r_{CI})\) will have \(r_{min} < r_{CI}\). Therefore, all encounters coming from within the area \(\sigma_{CI} = \pi b_{CI}^2\) will lead to an interaction with \(r_{min} \leq r_{CI}\). This area \(\sigma_{CI}\) is defined as the cross section for a CI. Given the definition of \(b\) above, this may be written

\[
\sigma_{CI} = \pi b_{CI}^2 = \pi\left(\frac{2GM_{tot}}{r_{CI}v_\infty^2}\right)^2. \tag{3}
\]

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① http://www.ligo.caltech.edu/
② https://www.cscma.infn.it/advirgo/
Figure 1. Schematic illustration of binary–single interactions and their final states. The top panel shows three different types of interactions. The top left panel shows a weak perturbation (WP) where the single encounter is only weakly perturbing the binary, but over several orbital periods. The top right panel shows a short but strongly perturbing encounter (SP). A close interaction (CI) is shown in the middle panel. The evolution of the system from this CI channel can further be divided into the two interaction channels: DI and RI. These are illustrated in the middle panel. The RI channel can be decomposed into intermediate binary–single states (IMSs), where an intermediate binary is formed with a bound companion. Several IMSs are created and destroyed in the chaotic RI before a final state is reached. The RI erases any information of initial conditions. On the other hand, the DI channel is very fast, and as a result, the endstate depends sensitively on the initial state. Which channel dominates depends particularly on the mass ratio between the objects and the velocity of the incoming object. The set of endstates from both the RI and the DI interactions are listed in the middle panel, where the individual interaction diagrams are defined in the bottom panel. There is generally a similar final state scheme for each permutation of the objects.

Whether the first (geometric) or second (gravitational focus) term in parentheses dominates depends on the relative binding energy of the binary and the kinetic energy of the incoming object.

Given a distribution of single stars, the CI cross section, $\sigma_{CI}$, gives an estimate of how often such interactions can occur. As $\sigma_{CI}$ increases, the more encounters will be focused into the binary system. In a stellar system with an isotropic stellar density, $n$, and typical relative velocity, $v_\infty$, this rate of CIs per binary may be approximated as

$$\Gamma_{CI} \simeq n \sigma_{CI} v_\infty.$$  

Thus, given a stellar distribution, the cross section is the only factor that determines the relative rates of different processes. For this reason, a significant effort will be invested in deriving the cross sections of the various outcomes of CIs as fractions of the total CI cross section. In the following section, we explore the role of the relative energy of the binary and the single object in shaping binary–single interactions.

2.2. Hard and Soft Target Binaries

The relative velocity of the binary and the single object, $v_\infty$, as compared with the characteristic velocity of a binary, $v_c$, determines the outcomes that are possible in a binary–single interaction. A binary's characteristic velocity is defined as (Hut & Bahcall 1983)

$$v_c^2 = G \frac{m_1 m_2 (m_1 + m_2 + m_3)}{m_3 (m_1 + m_2)} \frac{1}{a_0^2}.$$  

This velocity is written such that if the relative velocity at infinity is larger than $v_c$ ($v_\infty > v_c$), then the total energy of the three-body system is positive (Heggie 1975).

A binary with $v_\infty > v_c$ is described as a soft binary relative to its environment. The cross section for CI, Equation (3), is well approximated by the binary's geometrical cross section, $\pi r_{CI}^2$. Because the velocity at infinity is greater than the binary's orbital velocity, the binary appears nearly static during the interaction. The resultant encounters can thus be viewed mainly as two-body interactions that are well described by impulsive approximations (Heggie 1975; Hut 1983). Additionally, with $v_\infty > v_c$, the incoming body carries a large amount of energy in comparison with the binary's binding energy. That excess of energy can effectively be utilized to split the binary (Heggie 1975).

HBs are characterized by $v_\infty < v_c$. In this case, the cross section for CI is dominated by the gravitational focus term, and

$$\sigma_{CI} \simeq \frac{2 \pi G m_{tot} r_{CI}^2}{v_\infty^2}.$$  

Thus, in this limit, $\sigma_{CI} \propto a_0/v_\infty^2$. Further, the energy carried from the encounter into the system is relatively small, and a temporary bound triple state can be formed (Hut 1983).

In dense stellar systems, the HB limit is typically the relevant limit for the steady-state binary population. Equation (5) can be rewritten for equal mass encounters as

$$v_\infty \approx 36.5 (m/M_\odot)^{1/2} (a_0/\text{AU})^{-1/2} \text{ km s}^{-1}.$$  \hfill (7)

Values for $v_\infty$ are in the $10^{\sim}50$ km s$^{-1}$ range for galactic GCs (Lightman & Shapiro 1978). Thus, any binaries with SMA smaller than $\approx 1$ AU will be in the HB limit. In clusters, HBs tend to be the ones that survive as encounters tend to split soft binaries (Heggie 1975). Further, on the basis of a statistical trend toward energy equipartition (Heggie 1975; Hills 1975b), HBs tend to become harder (as energy is transferred from the binary to the single on an unbound orbit), whereas soft binaries get softened or disrupted (as the incoming single star pumps energy into the system before leaving). This natural selection makes an HB population even harder and causes a soft binary population to evaporate.

Binary–single CIs involving HBs may be decomposed into DIs and RIs. DIs are brief, two-body interactions that occur and the binary–single system is transformed. The IMSs themselves are unstable because they are disrupted every time the current bound single object makes a close passage. Over the course of several such IMS changes (through three-body interaction knots), the triple system evolves chaotically, losing memory of the initial conditions with which the single object first entered the binary (Heggie 1975). Rare outcomes may be achieved with higher likelihood in RIs for the simple reason that the single object makes many randomized close passages through the binary system. This is particularly significant when GW radiation is included into the three-body equation of motion because there is a non-negligible probability that a very close (and thus highly dissipative) passage will take place.

2.3. Outcomes of Close Interactions

In the previous section we have described how CIs arise in binary–single star encounters and how their likelihood can be quantified by their cross section, $\sigma_{CI}$. During a CI, the system is in a three-body state, but no three-body state is stable (Hut 1993) and the system will thus invariably evolve (through the DI or the RI channel) into one out of the several possible final states or outcomes as illustrated in Figure 1. In general, there is a given cross section for each of these possible outcomes to occur. In Section 2.4, we describe how we compute these outcome cross sections statistically on the basis of the fraction of binary–single scatterings that can generate a given outcome. In the two sections below we describe in detail the particular final outcomes expected from CI interactions.

2.3.1. Outcomes from Newtonian Gravity

In Newtonian gravity, a binary–single interaction can result in a binary with an unbound companion, a collision, or three
unbound objects. The cases in which a binary is left behind may be further subdivided on the basis of the properties of the surviving binary (Heggie 1975; Hut & Bahcall 1983). If the binary is composed of the original two objects (1, 2), then we refer to the encounter as a fly-by even though the endstate binary may be the result of a more complex interaction than the fly-by label suggests. If instead the binary is composed of one of the original binary members and the third body, we denote the encounter as an exchange. In this case, the binary may either be (1, 3) or (2, 3). An outcome in which all three members are mutually unbound is possible when the total system energy is positive, $v_\infty > v_c$. This outcome is denoted as an ionization. Collisions are possible at all values of $v_\infty$, but they are most likely to occur at negative total binding energies where the gravitational focus cross sections of the individual objects are larger.

### 2.3.2. Inspiraling Binaries due to GW Emission

With GW emission included in the three-body equation of motion, a new outcome is possible: dynamical inspirals. Inspirals are characterized by the gravitational radiation driven inspiral of an IMS binary while the third object is bound to the binary. Inspirals are particularly likely to occur during RIs. The magnitude of GW emission depends strongly on the distance of closest approach between two objects (e.g., Peters 1964). In relatively widely separated binaries, inspirals do not result from tightly bound circular orbits, but rather they are the product of orbits of very high eccentricity in which the objects experience close pericenter passages that generate significant GW emission and thus substantially reduce their orbital energy and angular momentum. High eccentricity orbits are most readily achieved in the chaotic environment of RIs, where despite the $e = 0$ initial conditions we impose on the binaries, the angular momenta of the three bodies rerandomized and approach an isotropic distribution with increasing number of passages.

Figure 3 shows an inspiral from one of our simulations. The binary–single interaction happens at the left of the plot and then propagates toward the right, terminating with the inspiral. One important feature of this interaction is that the bulk of the energy losses occur in three-body knots, where the relative orbital angular momenta of the bodies are randomized and the objects undergo very close pericenter passages, which in turn give rise to the spikes seen in the energy loss rate. Inspirals are of particular interest, as we will show in this paper, because they occur more frequently in widely separated target binaries and give rise to eccentric compact object mergers.

### 2.4. Numerical Approach

Here we study the outcomes of binary–single interactions and their associated cross sections by performing large sets of numerical scattering experiments. To this end, we have developed a new $N$-body code to integrate the equation of motion of the three bodies using a fourth order Hermite integration scheme. The equation of motion including the effect from GW emission is discussed in Section 4.1. For a full description of the code and the exact state classification criteria employed, the reader is referred to Appendices A and B. For each scattering experiment, the target binary was randomly orientated in phase and orbital plane orientation.

We estimate the cross section numerically for a given outcome type $O_t$ by performing $N_{\text{tot}}$ binary–single interactions with isotropic sampling across a disk at infinity with radius $b$. If the total number of outcomes of type $O_t$ from that scattering set is denoted by $N_t$, then the corresponding cross section for outcome $O_t$ can be estimated by

$$\sigma_t = \frac{N_t}{N_{\text{tot}}} \pi b^2$$ (8)
with a corresponding error given by

$$\Delta \sigma_i = \frac{\sqrt{N_i}}{N_{\text{tot}}} \pi b^2. \quad (9)$$

This, in turn, implies a rate of a given outcome $O_i$,

$$\Gamma_i \simeq n \sigma_i v_\infty \quad (10)$$

expected from a distribution of single objects with number density $n$ and typical relative velocity $v_\infty$. Thus, the rate of outcomes of type $O_i$ compared with the rate of CIs is defined by the ratio of their cross sections, $\Gamma_i/\Gamma_{\text{CI}} = \sigma_i/\sigma_{\text{CI}}$.

3. NEWTONIAN POINT-PARTICLE LIMIT

To gain insight and to provide a direct link to previous studies in Newtonian gravity, we will first describe the most salient features of binary—single encounters of point masses in Newtonian gravity. These interactions and their final states, or outcomes, are well-studied numerically and theoretically, especially in the pioneering series of work by Hut & Bahcall (1983), Hut (1983, 1993), Heggie & Hut (1993), Goodman & Hut (1993), McMillan & Hut (1996), Heggie et al. (1996). More recent work by Fregeau et al. (2004) and Fregeau & Rasio (2007) have extended such studies to calculate the probability of collisions and the coevolution of binaries and their host clusters.

When the three objects are equal point masses, the outcome of an interaction will always be either a fly-by, an exchange, or an ionization. These outcomes were described in Section 2.3.1. In this section, we calculate their associated cross section over a broad range of encounter velocities $v_\infty/v_c$ by using a series of numerical scattering experiments. In our equal mass case,

$$\frac{v_\infty}{v_c} = v_\infty \sqrt{\frac{2a_0}{3m}}, \quad (11)$$

thus any defining characteristics of the system can be rescaled using this ratio. We perform a total of $8 \times 10^5$ binary—single scatterings divided into 40 sets each with $2 \times 10^3$ interactions. For each scattering experiment, the target binary is randomly orientated in phase and orbital plane. The velocities of the encounters for the 40 sets are equally spaced in $\log v$ from $v_\infty/v_c = 0.01$ to 8. The maximum impact parameter, $b_{\text{max}}$, is kept fixed for all scatterings at 5$a_0$. In this setup, outcomes from all the three interaction channels—WP, SP, and CI—will occur depending on $v_\infty/v_c$. Our numerical approach is closely related to the one used in Hut & Bahcall (1983). We also refer the reader to Appendices A and B for further details on our numerical approach.

Figure 4 shows the results from our scattering experiments. Both panels show the cross sections for exchange, fly-by, and ionization as a function of $v_\infty/v_c$. The upper panel includes outcomes from all interactions including DIs and RIs, while the lower panel shows the outcomes coming from the RIs only. In what follows, we detail the outcomes and their dependence with $v_\infty/v_c$.

3.1. Low Velocity ($v_\infty/v_c \ll 1$)

At low velocities, gravitational focus leads to all interactions happening via the CI channel. Therefore, all final state outcomes will be a result from either the DI or the RI channel. Since the total energy of the three-body system is initially negative and no bound triple state can form a stable final state (Hut 1993), the only possible outcome is a binary (carrying the negative energy part in form of binding energy) and a single unbound object. Depending on which two objects form the binary, the outcome will either be labeled as an exchange or a fly-by.

Within the CI channel, the probability of a given outcome depends on whether the binary has experienced an RI or a DI. If the outcome is a result of the RI channel, then any permutation of the three objects in the final state is equally likely since the RI erases any memory of the binary’s initial configuration. As a result, the exchange and fly-by outcomes have the same cross

![Figure 4](image-url)

**Figure 4.** Integrated cross sections for the classical outcomes: exchange (brown triangles), ionization (green squares), and fly-by (orange stars) as a function of $v_\infty/v_c$, where $v_\infty$ is the relative velocity of the incoming object at infinity and $v_c$ the characteristic velocity given by Equation (5). The dashed lines show analytical approximations to the exchange (Equation (13)) and ionization (Equation (14)) cross sections. The vertical dotted lines indicate two characteristic velocities, the gravitational focusing velocity $v_{\text{foc}}/v_c$ ($b_{\text{max}} = 5a_0$) ≈ 0.28 and the velocity that divides the system into having total positive or negative energy, $v_\infty = v_c$. Top: Cross sections calculated from all interactions including RIs and DIs. Bottom: Cross sections only including endstates coming from RI encounters. The channel erases any information about initial conditions, and all the three objects have thus an equal probability to be kicked out. As a result, the fly-by and exchange cross sections are identical. Because a fly-by cannot result from a DI, the exchange and fly-by cross sections are separated in the top panel. As can be clearly seen, the cross section for an RI is independent of $v_\infty$ as long as $v_\infty < v_{\text{foc}}$. Each plot is based on a total of $8 \times 10^5$ scatterings. (A color version of this figure is available in the online journal.)
section when the system has evolved through an RI. This can be seen in the lower panel of Figure 4.

For interactions passing through the DI channel, fly-bys have a negligible probability to occur. The reason is that a DI is characterized by having only a single interaction that in the majority of cases leads to an exchange between the incoming object and one of the binary members. A typical fly-by involves at least two closest IMS pairs, leading to these interactions being classified as arising from the RI channel. This leads to the cross section difference between the exchange and fly-by when all interactions are included as seen in the upper panel of Figure 4.

The critical velocity that defines the transition to all interactions happening through the CI channel, $\nu_{\text{loc}}$, is found from Equation (3),

$$\frac{\nu_{\text{loc}}}{\nu_c} = \sqrt{2} \left( \frac{a_0}{b_{\text{max}}} \right),$$

which in our numerical setup with $b_{\text{max}} = 5a_0$ gives $\nu_{\text{loc}}/\nu_c = 0.28$. This critical velocity transition is illustrated with a vertical dotted line in Figure 4. It is clear in the lower panel in Figure 4 that this line accurately separates the plot into two regimes. The cross sections are approximately flat to the left of this line, when $\nu_\infty < \nu_c$. This tells us that the relative numbers of RIs and DIs are nearly constant and, as a result, are independent of the exact impact parameter and encounter velocity as long as the interaction is a CI.

3.2. Intermediate Velocity ($\nu_\infty/\nu_c \approx 1$)

At intermediate velocities, the resultant encounters are a mixture of CIs, SPs, and WPs, and the velocity dependence shapes the resultant cross sections. CIs can still occur at intermediate velocities, but their probability decreases as $\sigma_{\text{CI}} \propto (\nu_\infty/\nu_c)^{-2}$, as given by Equation (3). This scaling solely determines the shape of the exchange cross section in this regime, since exchanges only can happen via a CI. This is seen in Figure 4, where the exchange cross section is observed to clearly transition from being flat at low velocities to decreasing as $\nu_\infty/\nu_c$ at intermediate velocities.

WPs and SPs happen with increasing frequency as the velocity is increased since more encounters pass through the binary instead of making a CI. These perturbative encounters necessarily result in a fly-by classification since the encounter never comes close enough to make an exchange, thus leading to a velocity dependent increase in the associated cross section.

3.3. High Velocity ($\nu_\infty/\nu_c > 1$)

In high velocity interactions ($\nu_\infty > \nu_c$), the total energy of the three-body system is positive and ionization becomes a possible outcome. Ionization occurs when all three objects are unbound with respect to each other. This outcome dominates over the exchange outcome in this high velocity regime as seen in the upper panel in Figure 4.

Because of the high velocity, CIs are rare. The CI cross section is determined by the geometrical term in Equation (3). Since the geometrical term only depends on the size of the target binary, the occurrence of a CI is independent of velocity. In contrast to the intermediate velocity range, the observed steep decrease in both the exchange and ionization cross sections as the velocity increases is a result of properties of the interactions themselves, rather than a varying number of CIs.

As observed in the lower panel of Figure 4, RIs do not occur at high velocity. All outcomes from the CI channel are, therefore, only arising from the DI channel. The main reason for this is that the incoming object enters the binary with such a high velocity that the pair appears to be approximately stationary. The majority of interactions between the single and the binary will therefore be a DI between the incoming single and its nearest binary object. The problem therefore reduces to a two-body interaction between the encounter and one of the binary members. This setup has an analytical solution, and cross sections for exchange and ionization can be analytically estimated in this so-called impulsive regime. This was first done by Hut (1983), who calculated in this high velocity regime the exchange cross section

$$\sigma_{\text{ex}} = \frac{320 \pi a_0^2}{81 \nu_\infty^6},$$

and the ionization cross section

$$\sigma_{\text{ion}} = \frac{40 \pi a_0^2}{9 \nu_\infty^2}.$$  (14)

These scalings are also shown in Figure 4. The similarity of this three-body scattering problem to atomic physics can be seen by comparing the exchange scenario, in the limit where one of the binary members are very light, with electron capture (or charge transfer) in heavy nucleus interactions (Shakeshaft & Spruch 1979).

4. GRAVITATIONAL WAVE LOSSES AND THREE-BODY DYNAMICS

In this section, we describe how general relativity (GR) corrections are included into the equation of motion in our three-body integration code, and we highlight the dynamical consequences of these loss terms.

4.1. Adding General Relativistic Corrections

In this work, we include the energy and angular momentum losses by GW radiation using the PN formalism (Blanchet 2006). In this formalism, the acceleration experienced by an object of mass $m_1$ due to the gravitational force from a second object of mass $m_2$ is expanded in series as

$$a = a_0 + c^{-2} a_2 + c^{-4} a_4 + c^{-6} a_6 + O(c^{-6}).$$

The standard Newtonian force per unit mass, $a_0$ is

$$a_0 = -\frac{Gm_2}{r_{12}^2} \hat{r}_{12},$$

where the separation vector is $r_{12} = r_1 - r_2$, its magnitude is $r_{12} = |r_{12}|$, and its direction is $\hat{r}_{12} = r_{12}/r_{12}$. The terms $a_2$ and $a_4$ account for the periastron shift. The leading order term that represents the radiation of energy and momentum from the system, $a_5$, is also known as the 2.5PN term. This term takes the following form:

$$a_5 = \frac{4}{5} \frac{G^2 m_1 m_2}{r_{12}^4} \left[ \left( \frac{2Gm_1}{r_{12}} - \frac{8Gm_2}{r_{12}} - \nu_{12}^2 \right) \nu_{12} + (\hat{r}_{12} \cdot \nu_{12}) \left( \frac{52Gm_2}{3r_{12}} - \frac{6Gm_1}{r_{12}} + 3 \nu_{12}^2 \right) \hat{r}_{12} \right].$$

where the relative velocity scalar, $\nu_{12}$, and the vector, $\nu_{12}$, are defined following the same conventions as in Blanchet.
We use the modified acceleration $a = a_0 + c^{-5}a_5$ in our numerical treatment instead of the Newtonian $a_0$. A fundamental difference between the purely Newtonian acceleration and the 2.5PN acceleration is that $a_5$ depends not only on the separation between the objects but also on their relative velocity.

Including the conservative terms $a_2$ and $a_3$ is key when describing hierarchical systems whose evolution is secular (Naoz et al. 2013; Antonini et al. 2014). However, in strong binary–single scatterings hierarchical configurations of this form never arise. This is because the single bound object has a pericenter distance with respect to the binary of the order of the SMA. As a result, the single object cannot undergo many orbits without experiencing an SP. Binary–single interactions are therefore dominated by highly chaotic motions. Orbital changes due to the $a_2$ and $a_3$ terms will only modify the outcome of individual scatterings, but when averaged over a statistical ensemble of encounters such effects are not expected to alter the final outcomes. The conservative terms are not expected to influence our results that are solely based on dissipative effects.

The energy and angular momentum losses through the 2.5PN term should coincide with those calculated using the quadrupolar formalism for two bodies. To this end, the orbit-averaged equations for the time dependent evolution of SMA, $a$, and eccentricity, $e$, of a two-body system emitting GWS derived by Peters (1964) have provided a useful test framework to many authors:

$$\frac{da}{dt} = \frac{64}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{e^2 a^3 (1 - e^2)^{7/2}} \left( 1 + \frac{74}{24} e^2 + \frac{37}{96} e^4 \right),$$  \hspace{1cm} (18)

and

$$\frac{da}{de} = \frac{12 a}{19 e} \left[ \frac{1}{(1 - e^2)}\left( 1 + (121/304)e^2 \right) \right].$$  \hspace{1cm} (19)

By including the comparable 2.5PN terms directly in our three-body integration of the equation of motion, we can capture losses in three-body interaction knots as well as reproduce Equations (18) and (19) in the case where the system develops strong hierarchy and two bodies evolve following the secular evolution described by Peters (1964). In Appendices A and B, we show comparisons between the orbit-averaged Equations (18) and (19) and a direct numerical integration in our code.

With the inclusion of losses to GW radiation, binaries have a finite lifetime. If, for example, we consider a binary with objects of equal mass, $m$, and a circular orbit with initial SMA $a_0$, Equation (18) reduces to the form $da/dt \propto (m/a)^3$ with the solution

$$t_{\text{life}}(a_0) = 1.6 \times 10^{17} \left( \frac{a_0}{a_{\text{au}}} \right)^4 \left( \frac{m}{M_\odot} \right)^{-3} \text{yr}. \hspace{1cm} (20)$$

Here $t_{\text{life}}$ is the GW inspiral time, or the time it takes for the initial binary to evolve from $a = a_0$ to $a = 0$. The dependence on the SMA to the fourth power makes the lifetime very sensitive to small changes in $a_0$. In the other limit, where the initial eccentricity $e_0$ is not far from unity, the inspiral time is

$$t_{\text{life}}(a_0, e_0) \approx t_{\text{life}}(a_0) \frac{768}{425} \left( 1 - e_0^2 \right)^{7/2}. \hspace{1cm} (21)$$

The lifetime of a very eccentric binary is shorter than that of a binary in a circular orbit with similar SMA because as the eccentricity increases, the pericenter distance, which is given by $r_{\text{min}} = (1 - e)a$, decreases. This results in a higher GW flux every pericenter passage, which in turn decreases the lifetime and gradually circularizes the orbit of the binary.

An analytical solution for the coupled evolution in $a$ and $e$ also exists (Peters 1964):

$$a(e) = \frac{c_0 a_{12/19}^{12/19}}{1 - e^2} \left( 1 + \frac{121}{304} e^2 \right)^{870/2299} ,$$  \hspace{1cm} (22)

where $c_0$ is a constant with dimensions of length, set according to the initial conditions $(a, e)$ of the binary system. From this expression we see that in the high eccentricity limit, where $e \approx 1$, the SMA scales as $a(e) \propto (1 - e)^{-1}$. As a result, the orbital SMA (and thus also the orbital energy) must change by many orders of magnitude before the eccentricity becomes significantly less than unity. Inspiraling binaries thus only become approximately circular during the last phases of their inspiral.

### 4.2. Significance of PN Corrections

The binary’s compactness determines many of the important dynamical properties of the system, especially the importance of PN corrections and collisions. A dimensionless compactness can be defined as (Blanchet 2006)

$$\gamma = \frac{Gm}{r c^2}.$$  \hspace{1cm} (23)

Using $\gamma$, we can write the acceleration, $a = a_0 + c^{-5}a_5$, in terms of the dimensionless radius and mass, $\tilde{r} = r/r_u$ and $\tilde{m} = m/m_u$, where $r_u$ and $m_u$ are arbitrary length and mass scales, respectively. In these units, the resultant acceleration is $\tilde{a} = a/(Gm_u/r_u^2)$, and we have

$$\tilde{a} = \tilde{a}_0(\tilde{m}, \tilde{r}) + \gamma^{5/2} \tilde{a}_5(\tilde{m}, \tilde{r}, \tilde{v}). \hspace{1cm} (24)$$

For systems that are strongly relativistic, the SMA $a_0 \approx Gm/c^2$, and as a result, PN corrections become very important. For weakly PN systems, $a_0 \gg Gm/c^2$ and the compactness of the orbit provides an estimate for the importance of the PN corrections to the equation of motion of a circular, $e \approx 0$, orbit. However, a key point that we emphasize in this work is that measuring the strength of the PN corrections only in terms of the compactness of the initial binary orbit can be misleading. In chaotic three-body interactions, the eccentric orbits and close passages that arise make it possible for strong PN corrections to be realized even in systems with initially wide SMA. As we will discuss later, the initial compactness of the binary system still determines the probability that a very strong encounter will occur.

Close approaches in eccentric orbits lead to strong PN corrections to the equation of motion. They also may lead to direct collisions. The maximal strength of PN corrections to the acceleration is therefore set by the physical size and mass of the objects, rather than by the initial SMA of their orbits. This can be quantified by calculating the compactness $\gamma$ for the interacting objects themselves using their mass and radius. For example, if the objects are black holes, their compactness $\gamma \approx 1$, and PN corrections can therefore reach their maximal strength. If the constituent objects are not black holes, then $\gamma < 1$, and the magnitude of the maximal PN corrections for that three-body system is reduced. Neutron stars have typical dimensionless compactness of $\gamma \approx 0.2$, while a 0.6$M_\odot$ white dwarf is characterized by $\gamma \approx 10^{-4}$. Interacting WDs will therefore generally collide before PN corrections become strong.
If a system of N interacting objects is only composed of BHs, then the dynamics of the system becomes scale free (e.g., Shapiro & Teukolsky 1983; Gültük et al. 2006). The reason is that the equation of motion scales with the masses of the BHs, as do the BH gravitational radii. For example, for a binary–single interaction involving three equal mass BHs, the expected dynamics for a system with $a_0 = 10^{-3}$ AU and $m_{BH} = 1 M_\odot$ will be equivalent to that of a system with $a_0 = 10^{-1}$ AU and $m_{BH} = 10^2 M_\odot$. This allows us to identify dynamically similar systems that occur in different astrophysical contexts. If the N interacting objects are not BHs, then the system loses its scale-free behavior as the object radius no longer scales with mass. Neutron stars, for example, exhibit a relatively constant radius across their observed mass range (Steiner et al. 2010), while white dwarfs have an inverse mass radius relationship $R_{WD} \propto m_{WD}^{-1/3}$.

4.3. Energy Losses

The effects of GW energy loss can be most easily seen by examining Equation (17) in the context of a circular binary of equal mass objects. In that case, $\mathbf{\dot{r}} \cdot \mathbf{v}_{12} = 0$, leaving only the first term in Equation (17). For equal mass objects, the term in parentheses in Equation (17) evaluates to a negative number, and the direction of $\mathbf{a}$ is determined by $-\mathbf{v}_{12}$, directly against the motion of the two bodies. As a result, the orbiting objects essentially experience a drag force,

$$F_{2.5PN} = \frac{32\sqrt{2}}{5} \frac{G^{7/2}}{c^5} \left(\frac{m}{r}\right)^{9/2}. \quad (25)$$

This follows directly from Equation (17) by substituting $v = \sqrt{2Gm/r}$. The energy leaving the system per unit time can be easily calculated by using $\Delta E_{\text{orb}} = \text{force} \times \text{distance} = F_{2.5PN}2\pi r$, from which it follows that

$$\frac{dE}{dt} \simeq \frac{\Delta E_{\text{orb}}}{T_{\text{orb}}} = -\frac{64}{5} \frac{G^4}{c^7} \left(\frac{m}{r}\right)^5, \quad (26)$$

where $T_{\text{orb}} = 2\pi(2Gm/r^3)^{-1/2}$ is the orbital period. One should notice that the distance $r$ is changing as a function of time with a rate that can be calculated by using the Newtonian relation $dE/dr = -Gm^2/2r^2$.

The above formalism can be extended to a binary–single interaction. The distribution of GW energy radiated during a resonant encounter is shown in Figure 5. The upper panel shows how energy from the system is depleted as new intermediate binary–single states are created. The fractional energy loss is relatively small, especially for binaries with large SMA, but at each encounter the binaries are effectively hardened and the relative likelihood for the system to undergo a collision or a merger is increased. The lower panel shows the cumulative distribution of the fractional energy loss between the initial state and the final state for the same set of interactions. Figure 6 shows the corresponding cumulative distributions of the number of IMSs (top panel) and the number of close pairs (bottom panel) in a binary–single interaction. The number of close pairs is greater than the number of IMSs since it also includes all close passings that can occur within a single state (see Figure 1). For the set of scatterings ending with an unbound companion (exchange or fly-by), the number of three-body interactions are reduced when GR is included. For example, it can be seen in Figure 6 that 20% of all scatterings will have more than about 50 CIs when GR is not included. On the other hand, when GR is included only about 5% of all scatterings will experience a similar number of CIs. The reason is simply that the possibility of the system inspiraling when GR is included truncates the chain of resonance interactions.

5. THE FORMATION OF DYNAMICAL INSPIRALS

With the inclusion of energy and angular momentum losses from GW emission, a new class of dynamical outcomes appears, which we denote here as inspirals. These are interactions in which two of the objects inspiral and merge while all three objects are still in a bound three-body state, i.e., before one of the classical outcomes of exchange, fly-by, or ionization is achieved. An example of an inspiral endstate is shown in Figure 3.

In order to understand how the inclusion of GR corrections changes the binary–single outcome landscape, we recompute the Newtonian scattering experiments shown in Figure 4 with the addition of the 2.5PN term in the equation of motion. Our results are illustrated in Figure 7. The revised cross sections include inspirals and collisions between solar mass black holes with an initial binary SMA of $10^{-4}$ AU. The top panel shows the resultant cross sections from all interaction channels including DIs and RIs, while the bottom panel includes only endstates
Figure 6. Number of three-body interactions between equal mass BHs arising from binary–single scatterings. Both panels include only states from the RI channel. The target binary is circular and is chosen to be initially very hard with \(a_0 = 10^{-5}\) AU and \(m_{BH} = 1\) M\(_\odot\). The black lines indicate scatterings where GR corrections have been added, while blue lines show experiments with no GR corrections included. The two plots differ in the way the number of interactions are counted. Top: number of times an intermediate binary–single state (IMS) is observed to occur during a resonant interaction. Bottom: number of times a new closest pair has been identified during the resonant interaction. A high number of close pairs indicates highly chaotic motion during the encounter (see Figure 1) that occurs between each IMS.

(A color version of this figure is available in the online journal.)

arising from the RI channel. By comparing the two panels, one can conclude that inspirals are about three times more likely to arise from an RI than a DI. This holds true across SMAs, an observation that will become useful when we derive the analytical treatment for inspiral occurrence in Section 5.2.

Another important point is that the cross section for inspirals is approximately flat when \(v_\infty < v_{foc}\). This implies that the probability that an inspiral will occur is not sensitive to the exact value of the impact parameter, \(b\), or velocity, \(v_\infty\), as long as the single object experiences a CI with the binary. The lack of a dependence on the initial conditions arises because nearly all inspirals are generated from RIs (for which memory of the initial conditions is rapidly lost through ensuing resonances) and because the fraction of RIs and DIs is approximately constant for \(v_\infty \ll v_c\) (see Section 3.1). This observation makes it possible to write the probability of an outcome being an inspiral, given the interaction is a CI, as

\[
P_{\text{insp}} \equiv N_{\text{insp}}/N_{\text{CI}},
\]

and the corresponding inspiral cross section as

\[
\sigma_{\text{insp}} = P_{\text{insp}}\sigma_{\text{CI}},
\]

\[
\simeq P_{\text{insp}} \frac{3\pi G m a_0}{v_\infty^2},
\]

where the last equality holds for the equal mass case. This factorization is useful in the sense that it separates the contribution coming from the chaotic RIs from the standard focusing cross section that simply acts as a weight factor. It is important to notice that \(P_{\text{insp}}\) depends on the compactness of the initial binary, i.e., its SMA \(a_0\) and mass \(m_{\text{bin}}\), as we will show in Section 5.2.

### 5.1. Phase Space Distribution of Inspirals

Figure 8 shows distributions of the orbital parameters \((a, e)\) for all exchange and fly-by binaries (orange) and intermediate state binaries (blue) from \(2 \times 10^4\) HB binary–single interactions. The division at \(a/a_0\) indicates energy conservation between the newly formed binary with SMA \(a\) and the initial binary with SMA \(a_0\). The target binary must shrink if the single object becomes unbound, i.e., exchange and fly-by binaries have \(a < a_0\), while IMS binaries have \(a > a_0\).
Inspirals appear in gray in the right-hand panel in Figure 8, where the 2.5PN term is included in the equation of motion. These inspirals form from the subset of IMS binaries that merge while the three-body system is still bound and are therefore (mainly) initially created with \( a > a_0 \). Since GWs in general carry energy out of the system before an endstate is reached, then IMSs can flow across the initial \( a/a_0 = 1 \) border line. This means that all outcome distributions are slightly changed when GR is included. Inspiral states are, however, those that experience the highest energy losses.

Immediately after an inspiraling binary has formed, it evolves according to Equation (22). Several of these evolutionary trajectories are shown with thin black lines in Figure 8. GW emission circularizes the binary as its SMA is decreased. This migrates binaries from their initial formation region in the right-hand side of the \((a, e)\) phase space to the lower left. Therefore, the exact location of the inspiral event in Figure 8 depends on when the system was identified in the code (see Appendix A for a discussion of the selection criteria for states). It is therefore not necessarily representative of the binary’s initially assembled position in the formation locus for inspirals.

The phase space accessible for inspirals depends on \( \gamma \) (Equation (23)). At particular \((a, e)\) combinations with close pericenter approaches, direct collisions can also occur. A direct tradeoff can then be found between the number of collisions and the number of inspirals. The rates for these particular endstates cannot be independent because they originate from a similar phase space region. Not surprisingly, extended objects produce relatively fewer inspirals and more collisions than compact ones. The importance of the object’s size is illustrated in Figure 8, in which we plot the boundaries defined by the BH and NS diameters, respectively.

### 5.2. Analytic Derivation of Inspiral Cross Sections

In this section, we develop an analytical understanding of what determines the occurrence rate of inspirals and collisions, including how the outcomes depend on the initial SMA and on the mass of the target binary. Each IMS is characterized by three parameters: the SMA \((a)\) and eccentricity \((e)\) of the IMS binary and the orbital period of the bound companion, which we denote here as the isolation time \((t_{iso})\). Since the single object is bound to the binary during an IMS, \(t_{iso}\) is finite. It then follows that if an IMS binary is formed with \(t_{fbd} < t_{iso}\), then the binary will inspiral before the return of the bound companion. The lifetime, \(t_{iso}\), is determined by Equations (18) and (19) but can be estimated by Equations (20) and (21) in the circular and eccentric limits, respectively. In all of the following calculations, we assume the HB limit \((v_{\infty} \ll v_c)\).

The probability of a particular outcome being an inspiral can be estimated by considering the fraction of states during an RI that satisfies \(t_{fbd}(a, e) < t_{iso}(a)\). The isolation time \(t_{iso}\) is described by Kepler’s law

\[
\frac{t_{iso}}{2\pi} = \sqrt{\frac{a_{bs}^3}{Gm_{tot}}}.
\]  

where \(a_{bs}\) is the SMA of the hierarchical triple. This SMA, \(a_{bs}\), can be expressed in terms of the initial binary SMA, \(a_0\), and the SMA of the IMS binary, \(a\), by making use of energy...
conservation

\[ E_{\text{tot}} \approx - \frac{G m_1 m_2}{2a_0} = E_{\text{bin}} + E_{\text{bs}} = - \frac{G m_1}{2a} - \frac{G m_{\text{bin}} m_{\text{sin}}}{2a_{\text{bs}}}, \]

(30)

where “bin” and “sin” respectively refer to the binary and the single bound object in the hierarchical triple. In the equal mass case, Equation (30) reduces to

\[ a_{\text{bs}} = \frac{2a_0}{1 - 1/a_{\text{t}}}, \]

(31)

such that

\[ t_{\text{insp}} = \left( \frac{2}{1 - 1/a_{\text{t}}} \right)^{3/2} 2\pi \sqrt{\frac{a_0}{G m_{\text{tot}}}}, \]

(32)

where \( a_{\text{t}} = a/a_0 \) and the last term in Equation (32) is the orbital time of the initial binary system, \( T_{\text{orb,0}} \). Equation (32) relates the normalized SMA, \( a_{\text{t}} \), of a given IMS binary to the time it remains isolated from its bound companion. Since \( a_{\text{t}} > 1 \) during a resonance, it follows that \( t_{\text{insp}} > T_{\text{orb,0}} \).

We can now compare \( t_{\text{insp}} \) with \( t_{\text{life,0}} \), which in the high eccentricity limit is given by Equation (21). The ratio \( F_{\text{insp}} = t_{\text{life,0}}/t_{\text{insp}} \) describes the lifetime relative to the binary isolation time and can be written as

\[ F_{\text{insp}} = \frac{C F c^5}{G} \left( \frac{a_0}{m} \right)^{5/2} (1 - e^2)^{7/2} a_{\text{t}}^{5/2} (a_{\text{t}} - 1)^{-3/2}, \]

(33)

where \( C_F = (3\sqrt{3})/(680\pi\sqrt{2}) \approx 1.7 \times 10^{-3} \). If \( F_{\text{insp}} < 1 \), the binary will inspiral before the third body returns. If, on the other hand, \( F_{\text{insp}} > 1 \), another three-body encounter will take place. The boundary defined by \( F_{\text{insp}} = 1 \) produces a clear division in the \((a_{\text{t}}, e)\) phase space plane, clearly separating IMSs that will inspiral from those that can be followed by further three-body interactions (Figure 9).

Defining the allowed phase space region for inspirals as \( \Delta_{\text{insp}} = 1 - e \) and setting \( F_{\text{insp}} = 1 \) in Equation (33), we get

\[ \Delta_{\text{insp}} \approx \frac{1}{2} \frac{G^{5/7}}{C_F^{2/7} e^{10/7}} \left( \frac{m}{a_0} \right)^{5/7} a_{\text{t}}^{-5/7} (a_{\text{t}} - 1)^{-3/7}, \]

(34)

which implies \( \Delta_{\text{insp}} \propto (m/a_0)^{5/7} \). Assuming that the \((a, e)\) sampling of IMSs is relatively uniform where \( e \sim 1 \), as observed in Figure 9, we conclude that the number of IMSs within the inspiral region is \( \propto (m/a_0)^{5/7} \). This means that the probability that an outcome will be an inspiral, given that the interaction is a CI (see 27), scales as

\[ P_{\text{insp}} \propto \left( \frac{m}{a_0} \right)^{5/7} \propto \gamma^{5/7}, \]

(35)

such that

\[ \sigma_{\text{insp}} = P_{\text{insp}} \sigma_\text{CI} \propto a_0^{2/7} \frac{m^{12/7}}{e^2_{\text{in}}}. \]

(36)

This illustrates that the cross section for inspirals is expected to increase with the SMA of the target binary. The dominant inspiral-producing targets in a cluster are thus not extremely compact binaries but are instead wide ones.

Collisions occupy a similar phase space region to that populated by inspirals, with the size of the interacting objects and the initial SMA of the target binary determining their relative cross sections. If an IMS binary is formed with a periapsis
which leads to the result that the probability of a collision is
\( P_{\text{coll}} \propto a_0^{-1} \). The associated cross section, \( \sigma_{\text{coll}} \), can be estimated using Equation (28), and it is thus independent of \( a_0 \).

If we compare Equations (34) and (37), we can see that the probability of a collision (\( \propto a_0^{-1} \)) decreases faster than the inspiral probability (\( \propto a_0^{-5/7} \)) as \( a_0 \) increases. This means that collisions will occupy a progressively smaller fraction of the available inspiral phase space as the SMA of the target binary increases. Inspirals arising from widely separated binaries are therefore less likely to be depleted by collisions, which in turn makes widely separated binaries even better targets for inspiral production.

5.3. Numerical Determination of the Cross Section

Figure 10 shows the formation probability and corresponding cross sections of inspirals and collisions as a function of initial SMA derived using numerical scattering experiments. The symbols show results from our numerical simulations, while the dashed lines show the results from our analytical estimates giving by Equation (36). As discussed in Section 5.2, the inspiral cross section increases with SMA. This is because the gravitational focusing cross section for a CI increases faster with SMA (\( \propto a_0 \)) than the probability of an inspiral decreases (\( \propto a_0^{-5/7} \)).

As can be seen in Figure 10, the numerical and analytical scalings are in agreement in the asymptotic limit but show small differences in slope at low SMA. These differences are caused by having neglected a series of physical effects in the analytical scaling, such as collisions and GW energy losses before the interaction has reached its final end state. However, these corrections are only important for target binaries in the high compactness limit. From an astrophysical perspective, these binaries are believed to be a negligible target population as these they are expected to merge before a CI can take place. The reader is referred to Section 7 for further discussion.

Since we have now shown that inspirals are a likely outcome even from widely separated binaries, it is important to compare them with mergers arising from the widely discussed single–single GW capture scenario (Hansen 1972; Stephens et al. 2011; Kocsis & Levin 2012; East & Pretorius 2012; East et al. 2013).

5.4. Comparison with Single–Single Capture

Inspirals resulting from binary–single interactions and mergers resulting from single–single GW capture can create binaries with extremely short merger times and, in some cases, with very high eccentricity. Comparing the formation probabilities for eccentric mergers arising from both mechanisms is thus of great interest.

A single–single capture occurs when two objects pass close enough to each other that the resulting GW energy losses are larger than their initial positive energy. To first order, the energy radiated away during the first passage can be obtained by integrating the GW energy losses along the initial, unperturbed unbound orbit (Hansen 1972):

\[
\Delta E = -\frac{2}{15} \frac{G^7}{c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)^{1/2}}{r_{\text{min}}(a, e)^{7/2}} h(e),
\]

where \( r_{\text{min}} = a(1 - e) \) is the minimum distance between the two objects in the unbound orbit, and \( h(e) \) is a dimensionless constant for which \( h(e = 1) = 425\pi/(8\sqrt{2}) \). For capture to occur, we require \( \Delta E > (1/2)\mu v_{\infty}^2 \), where \( \mu \) is the reduced mass. Combining this with Equation (38), we find the maximum allowed \( r_{\text{min}} \) for a capture, which we denote as \( r_{\text{cap}} \) (Lee 1993),

\[
r_{\text{cap}} = \left( \frac{85\pi}{6\sqrt{2}} \right)^{2/7} \frac{G m_1^{2/7} m_2^{2/7} (m_1 + m_2)^{3/7}}{c^{10/7} v_{\infty}^{7/2}}.
\]
All single–single encounters with pericenter distance smaller than \( r_{\text{cap}} \) become bound. In comparison with the CI interaction cross section derived in Section 2.1, the cross section for a single–single interaction with pericenter distance less than \( r_{\text{p,max}} \) can be written as

\[
\sigma_{\text{SS}}(r_{\text{min}} < r_{\text{p,max}}) \simeq \frac{2\pi Gm_{\text{tot}} r_{\text{p,max}}}{v_{\infty}^2}. \tag{40}
\]

The capture cross section can be estimated by inserting \( r_{\text{p,max}} = r_{\text{cap}} \) in Equation (40),

\[
\sigma_{\text{cap}} = 2\pi \left( \frac{85\pi}{6\sqrt{2}} \right)^{2/7} \frac{G^2 m_1^{2/7} m_2^{2/7} (m_1 + m_2)^{10/7}}{\epsilon^{10/7} v_{\infty}^{18/7}}. \tag{41}
\]

This cross section can then be compared directly with the cross section for inspirals arising from binary–single encounters. The ratio between the two cross sections can be approximated using Equation (36),

\[
\frac{\sigma_{\text{insp}}}{\sigma_{\text{cap}}} \propto \left( \frac{a_0 v_{\infty}^2}{m} \right)^{2/7}. \tag{42}
\]

The number of inspirals relative to single–single captures is then expected to increase with \( a_0 \) and \( v_{\infty} \) but decrease as the mass increases.

Figure 11 shows the numerically derived ratio of binary–single inspirals to single–single captures based on \( 8 \times 10^5 \) binary–single scatterings. The two mechanisms have similar cross sections for tight binaries and typical cluster velocity dispersions. For binary SMAs larger than \( 10^{-3} \) AU, binary–single inspiral interactions clearly dominate. This implies that inspirals resulting from binary–single interactions may contribute substantially to the inspiraling and eccentric merging binary population in GCs. In the next section, we will explore the particularly interesting case of binaries that pass through the LIGO detector frequency band with high eccentricity.

6. ECCENTRIC INSPIRALS IN THE LIGO BAND

Compact merging binaries will be observed by advanced LIGO in the near future (Harry & the LIGO Scientific Collaboration 2010; Mandel & O’Shaughnessy 2010; Abadie et al. 2010; LIGO Scientific Collaboration et al. 2013). To detect these inspirals, templates must be convolved with the timeseries data from the interferometer (Abadie et al. 2010; The NRAR Collaboration et al. 2013; Nitz et al. 2013; Brown et al. 2013). The waveforms of relatively high eccentricity differ from those of circular binaries. For example, Huerta & Brown (2013) find that for eccenticities greater than about \( e \approx 0.2 \), the match to circular templates is degraded by more than 50%. An understanding of the quantity and origin of eccentric binaries that pass through the LIGO band is therefore extremely important for future GW searches.

In the GW inspirals and mergers, one might expect that the majority of binaries will be nearly circular when entering the LIGO band, since GWs carry away both energy and angular momentum at a rate such that the circularization time is similar to the merging time (Peters 1964; Gültekin et al. 2004, 2006). However, as we show in this paper, the dynamical inspiral states formed in binary–single encounters are formed with very high initial eccentricity and rapid merger times. As a result, most of these dynamical formed inspirals will be directly observable in the LIGO band at the time of formation, i.e., when they are still highly eccentric. In what follows, we explore in detail the fraction of highly eccentric LIGO sources one expects to come from binary–single interactions as well as making a direct comparison with highly eccentric inspirals formed via single–single interactions.

To quantify the number of eccentric binary mergers in our scattering experiments, we use an approximate form for the gravitational peak frequency (Wen 2003),

\[
f_{\text{GW}} = \frac{1}{\pi} \sqrt{\frac{Gm_{\text{tot}}}{a^3}} (1 + e)^{1.954} (1 - e^2)^{1.5}, \tag{43}
\]

where \( \sqrt{a^3/Gm_{\text{tot}}} \) is the orbital time, \( T_{\text{orb}} \).

6.1. Eccentric Binaries from Binary–Single Interactions

The eccentricity distribution of binaries resulting from binary–single interactions includes binaries that evolve into the LIGO band and binaries that are born in the LIGO band. Figure 12 shows the results from binary–single interactions between NSs with 1.4 \( M_\odot \) masses and 12 km radius for different
to Equation (22). The dotted black lines show a few of these evolutionary tracks. The two dashed black lines show constant gravitational peak frequencies $f_{\text{GW}} = 10^1, 10^2$ Hz, which have been chosen to illustrate the sensitivity window range for advanced LIGO (Harry & the LIGO Scientific Collaboration 2010; LIGO Scientific Collaboration et al. 2013) as well as future advanced VIRGO (Degallaix et al. 2013).

By comparing the orbit evolution trajectories in Figure 12 with the lines of constant $f_{\text{GW}}$, we can see that they are parallel for $\log(1 - e^2) \ll 0$. This is because the evolution of $a$ for both scales as $(1 - e^2)^{-1}$. This implies that high eccentricity mergers that are not born in the LIGO band cannot evolve into it with high eccentricity. The binaries that are identified inside the LIGO band are thus the only ones that are able to be detected with high eccentricity. This set of binaries is the dynamically formed inspirals. From the $(a,e)$ distributions shown in the top panel in Figure 12 one can calculate the corresponding $f_{\text{GW}}$ distributions by making use of Equation (43) (bottom panel in Figure 12). The values of $f_{\text{GW}}$ are observed to change only slightly during inspiral, since the binaries spiral in with almost constant peak frequency. As observed in Figure 12, target binaries with $a \sim 10^{-2} - 10^{-3}$ AU produce inspirals with $f_{\text{GW}}$ distributions that peak around the most sensitive LIGO frequency $\approx 200$ Hz. The relative normalizations of the distributions shown in the bottom panel of Figure 12 can be derived from Figure 10.

6.2. Eccentric Binaries from Single–Single Capture

Once a binary is formed via single–single GW capture, its subsequent evolution can be followed in the $(a,e)$ plane according to Equation (22). By comparing with arguments presented above for the binary–single capture case, we can conclude that if binaries formed through single–single capture are not formed with $f_{\text{GW}}$ that places them in the LIGO band, they will circularize before LIGO can observe them as eccentric binaries.

To estimate the cross section for highly eccentric LIGO sources resulting from single–single captures, we first rewrite Equation (43) in the equal mass case and in the high eccentricity limit ($\sim 1$),

$$r_0 \approx \frac{2^{2/3} \pi Gm}{3f_0^{4/3}}$$

where $m$ is the mass of each of the objects, and $r_0$ is the required pericenter distance for an eccentric binary to have a peak $f_0$. It then follows that all encounters with pericenter distance $r_{\text{min}} < r_0$ will have $f_{\text{GW}} > f_0$. Therefore, the cross section for a single–single encounter having $f_{\text{GW}} > f_0$ can be simply calculated by setting $r_{\text{p,max}} = r_0$ in Equation (40),

$$\sigma_{\text{SS}}(f_{\text{GW}} > f_0) = \frac{4\pi Gm}{v_\infty^2} \left(\frac{r_0 - 2r_{\text{obj}}}{v_\infty}\right).$$

To account for the object’s finite size ($r_{\text{obj}}$), we have subtracted the cross section for direct collisions in Equation (45). The velocity dependence ($v^{-2}_\infty$) in Equation (45) implies that the cross section for high eccentricity single–single captures scales as the gravitational focusing cross section. The single–single capture cross section scales as $v_\infty^{-4/7}$ such that $\sigma_{\text{SS}}(f_{\text{GW}} > f_0)/\sigma_{\text{cap}} \propto v_\infty^{-8/7}$. As the velocity increases, the single–single high eccentricity cross section relative to the capture cross section will also increase.

(A color version of this figure is available in the online journal.)

Figure 12. Distribution of orbital parameters in the $(a, 1 - e^2)$ plane and the corresponding gravitational peak frequency $f_{\text{GW}}(a,e)$ for all endstate binaries resulting from binary–single interactions between NSs with $1.4 M_\odot$ masses and 12 km radii. The relative velocity between the encounter and the target binary is $v_\infty = 10$ km s$^{-1}$. The plot includes the classical outcomes exchange and fly-by (plus symbols) and the GR outcome inspirals (squares). Different colors denote different initial SMAs of the target binary. Top: orbital parameters plus symbols and fly-by ($\times$). The dashed black line shows a few examples of the inspiral orbital evolution due to GW that are approximately representative of the advanced LIGO window. The dotted black lines show a few examples of the inspiral orbital evolution due to GW radiation given by Equation (22). When $(1 - e^2) \ll 1$, these evolutionary tracks are parallel to the gravitational peak frequency lines. This implies that if a binary with high eccentricity is not formed in the LIGO band, then it will never evolve into it with high eccentricity. Inspirals are therefore the only states arising from a binary–single interaction that will have the potential of being observable as high eccentric mergers. Bottom: distributions of gravitational peak frequencies from all identified inspirals. These distributions stay almost unchanged during the inspiral since the binaries evolve with approximately constant GW frequency. The sensitivity of LIGO peaks around $\sim 200$ Hz. We note here that future spaceborne interferometers such as LISA will be sensitive to lower frequencies, with a peak around 10$^{-2}$ Hz. The dashed black line shows the eccentricity distribution expected from merging binaries resulting from single–single captures. For illustration purposes, all histograms have been normalized to their peak values.

Initial SMA of the target binary. The top panel shows the distribution of all binaries in the $\log(a, 1 - e^2)$ plane immediately after final-state identification. Inspirals are shown with large square symbols. The distribution of inspirals is not static. Instead, each binary evolves because of GW radiation according
experiments of single–single objects. In Figure 13, we show the different cross sections and corresponding scalings for the various outcomes expected from single–single and binary–single encounters.

6.3. Comparison between Binary–Single and Single–Single

In previous sections, we have computed the scalings for the cross sections of binary–single interactions and single–single captures; a summary of our results is given in Figure 13. We now turn our attention to the relative normalization of eccentric inspirals arising from binary–single and single–single capture as a function of binary SMA and GW frequency threshold. Figure 14 shows the normalization of the numerically computed inspiral cross sections for high eccentricity binaries as a function of binary–single and single–single capture cross sections. Solid lines show inspirals formed by binary–single interactions, and dashed lines show inspirals formed by single–single captures. The resultant high eccentricity binaries formed via binary–single and single–single encounters have different gravitational peak frequencies at formation as shown in Figure 12. Each line defined by \( f_{\text{GW}} \) denotes a cross section that only includes inspirals that are born with a gravitational frequency above the given threshold. Bottom: ratio between the single–single and binary–single cross sections shown in the top panel. As described in the text, both high eccentricity single–single and binary–single inspirals scale as \( \propto v^{-2} \). This makes the ratio independent of velocity.

\[
\frac{\sigma_{\text{insp}}(f_{\text{GW}} > f_0)}{\sigma_{\text{SS}}(f_{\text{GW}} > f_0)} \approx \frac{3}{4} \frac{P_{\text{insp}} a_0^2}{f_0} \propto a_0^{2/7} f_0^{2/3}.
\]  

The estimation of \( P_{\text{insp}} \) in this limit is given by Equation (35). Our numerical and analytical results strongly suggest that the cross section for the formation of eccentric compact binary inspirals is significantly larger in the binary–single case than in the single–single case. As shown in Section 7, this also leads to
Here we turn our attention to the implications of our results and illustrate how they change with the inclusion of a more extended binary companion by calculating scatterings for WD–NS binaries in Section 7.1. We discuss the merger lifetime and resulting COM kicks in Sections 7.2 and 7.3, respectively. We provide a simple estimate of typical event rates in dense stellar systems in Section 7.4. Finally, we present our conclusions in Section 7.5.

7.1. Target Binaries Containing White Dwarfs

We have seen that wider binary SMAs lead to an enhancement in the cross section for inspiral outcomes in the case of binaries composed of NSs and BHs. In widely separated binaries, the binary members need not be compact objects. In this section, we consider the case where the target binary contains a white dwarf (WD) companion (Thompson et al. 2009).

WDs have a well-defined mass-radius relationship, which takes the following form for lower-mass WDs,

$$r_{\text{WD}} \simeq \frac{1}{m_{\text{WD}}^{1/3}} \left( \frac{18\pi^2}{10} \frac{h^2 (m_p/0.5)^{-5/3}}{Gm_e} \right),$$

$$\approx 2.9 \times 10^9 (m_{\text{WD}}/M_\odot)^{-1/3} \text{cm},$$

where $m_e$ is the electron mass and $m_p$ the proton mass (Carroll & Ostlie 1996).

Another characteristic scale imposed by the size of the WD is the separation at which the WD fills its Roche lobe,

$$a_{\text{MT}} \simeq r_{\text{WD}} \frac{0.6 q^{2/3} + \ln (1 + q^{1/3})}{0.49 q^{2/3}},$$

where $q = m_{\text{WD}}/m_{\text{NS}}$ (Eggleton 1983). In WD–NS binaries containing moderately massive WDs, the resulting mass transfer is stable, and the binary overcomes the destabilizing effects produced by GW radiation due to the ongoing mass transfer (e.g., Marsh et al. 2004; Paschalidis et al. 2009).

The phase space of NS–NS binary outcomes that result from NS scatterings including a companion WD are shown in the upper panel of Figure 16, which can be directly compared with the upper panel of Figure 12. These experiments involve a 1.4 $M_\odot$ NS encountering a WD–NS binary containing a 0.5 $M_\odot$ WD and a 1.4 $M_\odot$ NS. A comparison with Figure 15 shows the increased importance of collisions in the WD–NS target case when compared with NS–NS targets. However, we see that inspiral outcomes between two NSs are still possible, despite the presence of the WD. By contrast, inspirals between the WD and the NS typically do not occur due because of the extended radius of the WD (see, e.g., Willems et al. 2007 for double WDs seen by LISA6). However, the cross section for inspirals is reduced somewhat as compared with NS–NS target binaries. This is partially due to the fact that there is one (rather than three) possible pairwise combination that can result in double NS binaries. Additionally, in tight binaries with $a \approx a_{\text{MT}}$, collisions with the WD play an important role in depleting inspiral outcomes (Lee et al. 2010). The hierarchy of masses in the system also likely plays a role by somewhat reducing the typical number of resonances (Sigurdsson & Phinney 1993). Despite these effects, which tend to deplete the number of inspiral outcomes, we find that NS–NS inspirals have a larger

\[\text{http://lisa.nasa.gov/}\]

the interesting conclusion that the rate of production of inspirals in GCs is dominated by binary–single encounters, even when the fraction of compact objects in binaries is relatively modest. As argued in Section 1, we expect the fraction of binaries containing compact objects to be enhanced in the cores of GCs. This enhancement will further aid the assembly of inspirals by binary–single encounters.

7. DISCUSSION

We have discussed the formation of eccentric inspirals in the context of binary–single interactions and compared them with the more widely discussed single–single capture scenario. The expected outcomes for binary–single and single–single interactions of equal mass NSs are shown in Figure 15. The solid black line shows the binary–single CI cross section. Other outcomes shown are subcategories of the CI cross section. The solid red line shows exchange, the solid gray line shows inspirals, and the solid purple line shows collisions. The green line shows binaries with merger lifetimes less than a Hubble time, which will be discussed in Section 7.2. Similarly, the dashed black line shows the total cross section for single–single capture, while the dashed red line shows only eccentric captures for which $f_{GW} > 10$ Hz, and the dashed purple line shows the collision cross section. As we emphasized in the previous section, most inspirals occur with $f_{GW} \gtrsim 10$ Hz, so the inspiral cross section may be directly compared with the eccentric component of the single–single cross section. The upper $x$-axis label shows the GW inspiral lifetime for binaries separated by a given initial SMA (bottom $x$-axis labels).

Figure 15. Summary of relevant outcome cross sections arising from binary–single and single–single encounters between equal mass NSs. Each NS has a mass of 1.4 $M_\odot$ and radius of 12 km. The dashed lines show results from single–single encounters, while the solid lines show results from binary–single interactions. The black solid line shows the CI cross section, the dark gray line shows the inspiral cross section, and the purple and brown lines show the cross sections for collisions and exchanges, respectively. The green line shows the cross section for binaries that merge in less than a Hubble time. The black dashed line shows the single–single capture cross section, and the red dashed line shows the cross section for single–single high eccentric (e $\sim$ 1) binary with gravitational peak frequency $f_{GW} > 10$ Hz. The vertical, black dashed line shows the single–single pericenter distance for a capture $r_{\text{cap}}$. We note that the scaling between lines depends on velocity, which here is assumed to be 10 km s$^{-1}$.

(A color version of this figure is available in the online journal.)
cross section than single–single captures with $f_{\text{low}} > 10$ Hz as long as the binary SMA $a_0 \gtrsim 10^{-3}$ AU. Thus, we still expect wide binaries containing WDs to contribute meaningfully to the eccentric inspiral channel, in particular if they dominate the NS-hosting binary population as in Grindlay et al. (2006). A concern for systems containing extended objects is that tidal dissipation may play an important role in modifying the dynamics (e.g., McMillan 1986), an effect we ignore here and hope to implement in future work.

### 7.2. Binary Lifetimes

Even if the initial binary lifetime is greater than a Hubble time, $t_{\text{Hubble}}$, a fraction of binaries that undergo a scattering will be either deposited or exchanged into orbits with very short lifetimes (Clausen et al. 2012). Thus, a fraction of even very widely separated binaries can produce mergers with $t_{\text{life}} < t_{\text{Hubble}}$. Figure 17 shows the distribution of final binary lifetimes realized following binary–single scatterings with varying binary SMA. In the classical point mass limit, we see that an approximate power law distribution is produced. The inclusion of GW radiation and finite radii introduces two physical scales that break the self-similarity of the problem. The hard cutoff corresponds to the scales of the objects themselves and depletion by collisions. The inspiral population manifests itself as a knee at scales corresponding to the typical pericenter distances of the rapid inspiral outcomes.

The cross section for creation of binary products whose lifetime are less than a Hubble time is plotted in Figure 15 for encounters involving NS. The key feature of this cross section is that it does not vanish when $a_0 \gtrsim 10^{-1.7}$ AU, where $t_0 > t_{\text{Hubble}}$. Instead, this cross section remains approximately flat. The reason for this is that resultant binaries generally have a much smaller pericenter distance than the target binary and therefore also a shorter lifetime as seen in Figure 17.

#### 7.3. Retention or Ejection of Binary–Single Outcomes

A remaining question is whether final binaries resulting from binary–single interactions are kicked out or whether they merge in situ. Kicks relative to the initial COM occur when a fraction of the initial binary’s binding energy is transferred to the relative motion of the binary and the single (Phinney & Sigurdsson 1991). We denote the resulting binary kick velocity as $v_{\text{kick}}$. The associated hardening of these binaries leads to a shorter binary lifetime (since $t_{\text{life}} \propto a^4$), and one therefore expects that a high kick velocity is associated with a short lifetime. A binary that receives a high-velocity kick will therefore not necessarily merge outside of its environment.

This tradeoff between lifetime and kick velocity is evident in Figure 18. The figure shows a scatter plot of kick velocity $v_{\text{kick}}$ and survival distance, defined as $v_{\text{kick}} \times t_{\text{life}}$, for all endstate NS binaries with respect to the initial COM. We use the survival distance to estimate where the binary will merge. Radius and escape velocity for a typical GC are shown with dashed lines. In this simple calculation, only final binaries in the upper right quadrant merge outside the cluster. If we now assume that binary SMAs are lognormally distributed and we only consider binaries that merge in less than a Hubble time, $t_{\text{life}} < t_{\text{Hubble}}$.
time (below the dash-dotted line in Figure 18), we calculate that \( \sim 30\% \) (10\%) of all merging binaries arising from NS–NS (NS–WD) targets are kicked out with a median distance of \( \sim 80 \) (50) kpc. While there is little direct evidence that close double neutron star binaries can form and merge in GCs, the double neutron star system PSR B2127+11C in the Galactic GC M15 (Anderson et al. 1990) is an example of such a system and has \( t_{\text{life}} \approx 2 \times 10^8 \) yr.

The retention or ejection of binaries has implications for cluster dynamics and merger-induced transients such as, for example, short gamma-ray bursts (Belczynski et al. 2006; Lee \& Ramirez-Ruiz 2007). If binaries are retained, they participate in the continued cluster evolution acting as a heat source or sink, depending on their SMA. In some cases the binary distribution may reach a steady state (e.g., Ivanova et al. 2005a). Merging binaries are expected to show environmental dependence in their electromagnetic signatures (Panaitescu et al. 2001; Rosswog & Ramirez-Ruiz 2002; Metzger \& Berger 2012; Kelley et al. 2013; Rosswog et al. 2013).

If a relativistic (short gamma-ray burst) or a mildly relativistic mass ejection resulted from the merger of two compact objects, the resulting afterglow could then, at least in part, be due to the interaction of the ejecta with the stellar winds of the red giant cluster members (De Colle et al. 2012). Because of the large stellar density in the cluster core, the external shock would then take place within a more dense medium than the intergalactic medium (Lee et al. 2010). In addition, the merger sites of compact binaries will determine whether we expect the electromagnetic signatures of binary mergers to statistically trace the GC distribution around galaxies (Grindlay et al. 2006; Lee et al. 2010; Church et al. 2011) or the galactic potential (Bloom et al. 1999; Rosswog et al. 2003; Belczynski et al. 2006; Zheng \& Ramirez-Ruiz 2007; Zemp et al. 2009; Fong et al. 2010; Kelley et al. 2010; Fong \& Berger 2013).

7.4. Rates

Given distributions of target binaries and single encounters, we can convert the calculated cross sections into event rates. In this section we present some simple order-of-magnitude estimates of the rates of dynamical NS–NS inspirals achieved in GC environments. We denote the total number of NSs by \( N_{\text{NS}} \) and assume that some fraction \( f_b = N_{\text{bin}} / N_{\text{NS}} \) is in binary systems (target binaries). The remaining fraction remains single (encounter population), \( f_s = 1 - f_b \). The target binaries are distributed according to their SMA \( dN_{\text{bin}} / da \), which we assume is uniform in log \( a \), \( dN_{\text{bin}} / da \propto a^{-1} \). The differential rate of inspirals per SMA can then be written as

\[
\frac{d\Gamma_{\text{imp}}}{da} = \frac{dN_{\text{bin}}}{da} n_s \sigma_{\text{imp}} v_{\infty},
\]

where \( n_s \) is the number density of single NSs, \( n_s = f_s N_{\text{NS}} / V_{\text{core}} \), and \( V_{\text{core}} \) is the volume of the cluster core over which both single and binary objects are distributed. To obtain the total rate of inspirals, we integrate over the binary distribution,

\[
\Gamma_{\text{imp}} = \int \frac{d\Gamma_{\text{imp}}}{da} da.
\]

We note here that while we need to evaluate this integral for a given binary distribution and inspiral cross section as a function of SMA, it will generally scale as \( \Gamma_{\text{imp}} \propto N_{\text{NS}}^2 f_b (1 - f_b) v_{\infty}^{-1} \). Below we provide some rate estimates based on simple examples that describe the distribution of NSs in GCs.

In a typical GC, there may be as many as \( N_{\text{NS}} \approx 10^3 \), for example, as modeled in the case of M15 by Murphy et al. (2011), whose best fit model has 1500 NSs with a half-mass radius of 0.17 pc. In what follows, we take \( V_{\text{core}} = (0.17 \text{ pc})^3 \), a typical relative velocity \( v_{\infty} = 10 \text{ km s}^{-1} \), and \( N_{\text{NS}} = 10^{3} \). If \( 30\% \) of these NSs are in NS–NS binaries distributed between \( 10^{-3} \) and 1 AU in SMA (\( f_b = 0.3 \)), the rate of NS inspirals will be

\[
\Gamma_{\text{imp}}^{(\text{NS–NS})} \approx 0.7 \text{ yr}^{-1} \text{Gpc}^{-3}.
\]
merger rate assembled in clusters. This estimate should be treated as an upper limit because, for example, if the NS is in a binary with a main sequence star, the effects of collisions will be more significant than those with a WD companion.

These same assumptions imply a rate of single–single NS captures in GCs,

$$\Gamma_{\text{cap}} = f_s N_{\text{NS}} \nu_c \sigma_{\text{cap}} v_\infty \approx 0.5 \text{ yr}^{-1} \text{ Gpc}^{-3},$$  

(54)

where we note that the velocity dependence in this case is $v_\infty^{-1/7}$. By the same token, we can calculate the rate of eccentric binaries in the LIGO band arising from single–single encounters

$$\Gamma_{\text{GW}}(f_{\text{GW}} > 10 \text{ Hz}) \approx 0.15 \text{ yr}^{-1} \text{ Gpc}^{-3},$$  

(55)

which has a velocity dependence of $v_\infty^{-1}$. Thus, if the binary fraction $f_b > 0.18$ (for WD–NS binaries) or $f_b > 0.08$ (for NS–NS binaries), the binary–single channel will dominate the formation of eccentric NS inspirals over the widely discussed single–single channel.

We can also compare with the number of non-eccentric mergers that occur from dynamical interactions. These are defined in our scattering experiments as those binaries arising from either an exchange or fly-by interaction whose lifetime is less than a Hubble time, $t_{\text{life}} < t_{\text{Hubble}}$. If we take our NS–NS target binary simulations as representative, non-eccentric merger outcomes have a rate of approximately

$$\Gamma_{\text{merge}}^{(\text{NS–NS})} \approx 120 \text{ yr}^{-1} \text{ Gpc}^{-3}.$$  

(56)

Binaries with $t_{\text{life}} < t_{\text{Hubble}}$ are thus more common by a factor of approximately 160 than inspirals. Grindlay et al. (2006), whose rate estimate is in rough agreement with Equation (56), concludes that $\sim 10\%$ of all mergers may be dynamically assembled in GCs. The remainder of mergers are expected to arise from binaries assembled in the field (e.g., Dominik et al. 2012, 2013). However, the exact fraction of mergers in clusters depends sensitively on the distribution of wide binaries containing compact objects, which is difficult to constrain observationally. If this estimate is correct, then the inspiral rate represents a $\sim 1\%$ fraction of the anticipated total compact object merger rate assembled in cluster.\(^7\)

Normalized to the rate of eccentric NS mergers from single–single capture for which $f_{\text{GW}} > 10$ Hz, we can write a hierarchy of rates as

$$\frac{\Gamma_{\text{GW}}(f_{\text{GW}} > 10 \text{ Hz})}{\Gamma_{\text{cap}}^{(\text{NS–NS})}} \approx 1 : 2 : 5 : 800.$$  

(57)

The expected number, and correspondingly the number density, of BHs in GCs remains uncertain. Mass segregation, for example, has been argued to give rise to a BH-dominated subsystem that collapses and dynamically decouples from the remainder of the stellar system (Spitzer 1969; Kulkarni et al. 1993; Zwart et al. 2007, Mapelli et al. 2011, 2013; and Morscher et al. 2013) has suggested that the Spitzer instability and mass segregation may not be as effective, leading to the retention of a non-negligible fraction of black holes. The extremes of either scenario suggest that NS–NS inspirals rather than BH–BH inspirals should dominate the inspiral rate because of either BH ejection or ineffective segregation of BHs into a dense subcluster. However, the interaction rates of black holes in clusters will only be definitively revealed by ongoing N-body and Monte Carlo cluster modeling efforts.

7.5. Significance of Eccentric Inspirals

We have demonstrated that binary–single scatterings are likely to dominate the production of eccentric binaries. In such GW-driven inspirals, the energy change is much more rapid than the angular momentum change, such that the circularization time and inspiral time are similar, $t_{\text{insp}} \approx t_{\text{circ}}$ (Peters 1964). One consequence of this is that binaries whose peak frequency, Equation (43), is at lower frequency than the LIGO band will enter the LIGO band with relatively low eccentricity since these objects tend to circularize as they inspiral. This can be seen most clearly in the trajectories drawn in Figures 12 and 16. For a binary to be seen as eccentric in a given waveband, it must have been formed with high eccentricity in that band. Eccentric inspirals produce gravitational waveforms that are distinct from those of circularly inspiraling binaries (Königsdörffer & Gopakumar 2006; Stephens et al. 2011; East et al. 2012; Gold & Bruegmann 2013; East & Pretorius 2012; Gold et al. 2012; Huerta & Brown 2013). These may be so distinct that non-circular binaries will go undetected without uniquely created waveform templates (East et al. 2013; Huerta & Brown 2013), and the timing between premerger GW bursts will contain valuable information about the equation of state. Close encounters in these systems can also lead to tidal deformations strong enough to crack the crust of the NS and tap into the $\sim 10^{46}$ erg stored in elastic energy, potentially generating flaring activity prior to the merger (Tsang et al. 2012; Tsang 2013). In contrast to quasi-circular NS–NS mergers, eccentric binary mergers can also result in massive disks even for equal mass binaries (East & Pretorius 2012).

Neutron stars that merge with high eccentricity have potentially unique gravitational and post-merger electromagnetic signatures (e.g., Lee et al. 2010; East & Pretorius 2012). The merger of these binaries may eject copious neutron rich material in tidal tails that will synthesize significantly larger masses of r-process rich material (Lee et al. 2010; Rosswog et al. 2013) than the widely discussed, non-eccentric binary mergers (Lattimer & Schramm 1974; Rosswog et al. 1999; Rosswog & Liebendörfer 2003; Rosswog 2005; Lee & Ramirez-Ruiz 2007; Metzger et al. 2010; Roberts et al. 2011; Bauswein et al. 2013; Kasen et al. 2013; Barnes & Kasen 2013; Tanaka & Hotokezaka 2013; Grossman et al. 2014).

Multi-messenger astronomy offers tantalizing prospects for probing the nature of compact objects, their binary assembly, evolution, and eventual merger (Rosswog 2007a, 2007b; Bloom O’Leary et al. 2006). In this case, dynamical ejections through binary–binary and binary–single interactions might deplete the cluster black hole population after the formation of some initial binaries (Kulkarni et al. 1993; Sigurdsson & Hernquist 1993). However, these interactions would also produce inspirals and mergers (O’Leary et al. 2006; O’Leary et al. 2007), perhaps even leading to the runaway formation of a massive black hole (Portegies Zwart et al. 2004). More recent work by Portegies Zwart et al. (2007), Mapelli et al. (2011, 2013), and Morscher et al. (2013) has suggested that the Spitzer instability and mass segregation may not be as effective, leading to the retention of a non-negligible fraction of black holes. The extremes of either scenario suggest that NS–NS inspirals rather than BH–BH inspirals should dominate the inspiral rate because of either BH ejection or ineffective segregation of BHs into a dense subcluster. However, the interaction rates of black holes in clusters will only be definitively revealed by ongoing N-body and Monte Carlo cluster modeling efforts.
et al. 2009; Lee et al. 2010; Faber & Rasio 2012; Metzger & Berger 2012; Lehner et al. 2012; Kelley et al. 2013; Nissanka et al. 2013; Palenzuela et al. 2013a, 2013b; Berger et al. 2013; Tanvir et al. 2013; Bartos et al. 2013), in addition to possible insights into the origin of r-process nucleosynthetic elements and short gamma-ray bursts (Lattimer & Schramm 1974; Lee & Ramirez-Ruiz 2002; Rosswog & Ramirez-Ruiz 2003; Rosswog 2004; Miller 2005; Roberts et al. 2011; Bauswein et al. 2013).

An eccentric GW signal detection might be one of the most exciting prospects, as it would provide a clear signature of the dynamical binary assembly process. In the explicit absence of such detection, the use of eccentric waveform template searches could help exclude a significant dynamically assembled population of merging compact binaries in dense stellar systems.

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APPENDIX A

N-BODY INTEGRATOR WITH GW ENERGY LOSS CORRECTION

We use a Fourth-Order Hermite Integrator with a variable time step to evolve the N-body system. The dynamical effect from GW radiation is included using the PN formalism (Blanchet 2006) by modifying the Newtonian acceleration term from \( a_0 \) to \( a_0 + c^{-3}a_1 \) as described in Section 4.1. This modified PN expansion of the acceleration is strictly valid only for two isolated objects. However, one can still make use of this approach without introducing significant errors for \( N > 2 \) objects since the 2.5PN term has a much steeper dependence on the distance \( r \) than the Newtonian acceleration (\( r^{-5/2} \) versus \( r^{-2} \) for a circular binary). The contribution from the closest pair will therefore always dominate. Further justification for this formalism can be found in Gültekin et al. (2006). The 2.5PN term is the first term in the expansion that acts like an energy sink, i.e., carries energy out of the system. The energy loss from this term is, when orbit averaged, equivalent to the loss calculated from the quadrupole formalism described in Peters (1964). A comparison between the two approaches is shown in Figure 19, which plots the orbital evolution in the \((a, e)\) plane for a binary that inspirals (top panel) because of GW radiation and for a single object that captures another single one by emitting GW (bottom panel). The black solid lines are from our N-body code, where the red dots show the results from solving for \((a, e)\), using the quadrupole formalism: Equations (18) and (19). These tests were found to be in very good agreement, as can be seen in Figure 19.

To speed up the binary–single scattering experiments, we have propagated the encounter from infinity to a distance \( r_{\text{proj}} \) from the COM of the target binary by modeling the binary–single system as a two-body system. The distance \( r_{\text{proj}} \) was chosen to be a fraction of the maximum value of either \( r_{bs} \) or \( a_0 \), where \( r_{bs} \) is the minimum distance between the COM of the binary and the interloper in the two-body frame and \( a_0 \) is the SMA of the binary. This approach ignores the effect from the binary’s dipole gravitational field on the encounter for \( r > r_{\text{proj}} \), but the error is insignificant. Further details on the errors related to this strategy can be found in Hut & Bahcall (1983).

APPENDIX B

IDENTIFYING STATES

B.1. Binary–Single State

Following Fregeau et al. (2004), we state that the three interacting objects are in a binary–single state if the binary objects are bound to each other and the tidal force from the single at the binary’s apocenter \( (F_{\text{tid}}) \) is smaller than the relative force at apocenter \( (F_{rel}) \) by some fraction \( \delta_{\text{tid}}, \) i.e., if \( F_{\text{tid}}/F_{rel} < \delta_{\text{tid}}. \) The two force terms are simply given by

\[
F_{\text{tid}} = \frac{m_{\text{bin},1}m_{\text{bin},2}}{[a(1+e)]^2} \quad (B1)
\]

and

\[
F_{\text{rel}} \approx \frac{2(m_{\text{bin},1} + m_{\text{bin},2})m}{r^3} a(1+e), \quad (B2)
\]

where \( m_{\text{bin},i} \) is the mass of binary object \( i, m_s \) is the mass of the single object, \( r \) is the distance between the single object and the COM of the binary, and \( a, e \) are the SMA and eccentricity of the binary, respectively.

If a three-body state is identified as a binary–single state and the single object is unbound from the binary, the state is labeled either as an exchange or a fly-by, depending on which objects the binary is composed of. If the single object is indeed bound to the binary, the state is denoted as an IMS. There is no dependence on \( \delta_{\text{tid}} \) if the single object is unbound, but the chosen value for \( \delta_{\text{tid}} \) will truncate the identified \((a, e)\) distribution of IMSs simply because not all configurations satisfy the threshold requirement. For this work we use \( \delta_{\text{tid}} = 0.5 \) for identifying an IMS and \( \delta_{\text{tid}} = 0.1 \) for identifying an exchange or a fly-by.

B.2. Inspirals

A binary with a bound single companion that inspirals because of GW radiation is denoted as an inspiral. Since the binaries that inspirals have a bound companion, the inspiral state is a subclass of the IMS discussed above. In these cases, the \((a, e)\) values for the orbital parameters of the inspiraling binary are set at initial identification, when the three-body state is identified as an IMS. The value for this first set of \((a, e)\) depends strongly on the threshold \( \delta_{\text{tid}} \) since a smaller \( \delta_{\text{tid}} \) allows more time for the binary to spiral in. However, the total number of inspirals is not affected, and therefore the resulting cross sections are also not sensitive to the choice of \( \delta_{\text{tid}} \).

B.3. Collisions

We assume in all scattering experiments that the objects are rigid spheres with radius \( r_i \). We say that object \( i \) and \( j \) have collided if these spheres ever overlap, \( r_{ij} < r_i + r_j \). To distinguish collisions from inspirals, we say that collisions are colliding objects that are not in an IMS binary. This definition is practical, but there is a gray zone between collisions and inspirals. One can, for example, have an IMS binary with initial pericenter distance \( r_{\text{min}} < r_i + r_j \) or a configuration where enough GW energy is radiated away such that two objects collide before an IMS is identified by the code. In general, this overlap is only important at the very smallest binary SMAs, in which the SMA

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begins to become comparable to the size of the objects of order $10^{-5}$ AU for solar mass compact objects. At larger separations, any sensitivity is lost because the number of inspirals greatly dominates over the number of direct impacts.

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![Figure 19](image-url)
