On invariant sets of diffeomorphisms

Mehrzad Monzavi and Reza Mirzaei

Abstract

We give a simple upper bound for the upper box dimension of a backward invariant set of a $C^1$-diffeomorphism of a Riemannian manifold. We also estimate an upper bound for the box dimension of a forward invariant set of a $C^1$-mapping with finite Brouwer degree in a Riemannian manifold.

AMS Subject Classification: 58A05, 28A80.

Key words: Riemannian manifold, Invariant set, Box dimension.

1. Introduction

The direct computation of the Hausdorff dimension of invariant compact sets is a problem of high complexity. Therefore, it is interesting to obtain analytic estimates of this dimension. Recently, many research studies have been developed on the investigation of the Hausdorff dimension of invariant sets of discrete dynamical systems. First results in this direction are given in [3] for compact subsets of $\mathbb{R}^n$ that are backward invariant under $C^1$-maps. Wolf in [24] gave Hausdorff dimension estimates (related to the values and behavior of $\det D_x f$ and $\|D_x f\|$) for compact forward invariant sets of $C^1$-diffeomorphisms in $\mathbb{R}^n$. In [13, 14] the conditions of these estimates are weakened using a Lyapunov type function in $\mathbb{R}^n$. In [21], Temam gave upper bounds for the fractal dimension of flow-invariant sets in a Hilbert space, which is proved in [10] for vector fields on Riemannian manifolds. In [7], Franz considered compact invariant sets of $C^1$-diffeomorphisms for which there exists an equivalent splitting of the tangent bundle. Qu and Zhou in [2] generalized the results of [24] to the map on smooth Riemannian manifolds with non-negative Ricci curvature. In [18], Wolf's theorem is generalized to complete Riemannian manifolds without conditions on curvature. For further studies of estimation of upper bounds for Hausdorff dimension of invariant compact sets, one may consult [4, 12, 15, 20, 23]. In the present paper, we estimate an upper bound for Hausdorff and box dimension of compact backward invariant sets of $C^1$-diffeomorphisms on Riemannian manifolds with conditions on $\min S_n(D_x f)$ and $\max |\det D_x f|$. We
also estimate an upper bound for the box dimension of a compact forward invariant set of a $C^1$-mapping with finite Brouwer’s degree on Riemannian manifolds.

2. Preliminaries

We will use the following definitions and facts.

(1) Consider a linear operator $L : E \to E'$ between two Euclidean spaces of dimension $n$ with scalar products $<.,.>_E$ and $<.,.>_{E'}$ respectively. The adjoint operator of $L$ is the unique linear operator $L^* : E' \to E$, determined by $<Lx,y>_E = <x,L^*y>_{E'}$ for all $x \in E$ and $y \in E'$. The eigenvalues of the positive semi-definite operator $\sqrt{L^*L}$ are the singular values of $L$. The singular values are all non-negative, usually listed in order to their size and multiplicity $S_1(L) \geq S_2(L) \geq ... \geq S_n(L)$. The absolute value of the determinant of $T$ is stated as the square root of the determinant of $T^*T$.

(2) Let $M$, $N$ be Riemannian manifolds of dimension $n$, $U$ be an open subset of $M$.
(a) If $f : U \to M$ is a $C^1$-diffeomorphism. We denote the tangent map of $f$ at the point $x \in M$ by $D_xf : T_xM \to T_{f(x)}M$ and the norm of $f$ at that point, is defined by
$$\|D_xf\| = \sup\{|D_xf(v)| : v \in T_xM; |v| = 1\}$$
(b) For $r > 0$ the $r$-neighborhood of a set $F \subset M$ is defined by
$$B_r(F) = \{x \in M : d(x,a) < r \text{ for some } a \in A\}$$
(c) Let $F$ be a non-empty bounded subset of $M$ and $N_\delta(F)$ be the smallest number of balls of radius at most $\delta$ which can cover $F$. The upper box dimension of $F$ is defined (see [5]) by
$$\overline{\dim}_BF = \limsup_{\delta \to 0} \frac{\log(N_\delta F)}{-\log \delta}$$
(d) If $M$ and $N$ are compact orientable manifolds and $f : M \to N$ is a differentiable map and $y \in N$ is a regular value of $f$, then the Brouwer degree of $f$ at $y$ is defined (see [16]) by
$$\text{deg}(f) = \sum_{x \in f^{-1}(y)} \text{sgn}(D_xf)$$
Where, $\text{sgn}(D_x(f))$ equals $+1$ or $-1$ according to $D_x(f)$, which it preserves or reverses orientation.
(e) If \( f : U \to M \) is a \( C^1 \)-diffeomorphism onto its image. A compact subset \( K \) of \( U \) is called forward \( f \)-invariant if \( f(K) \subset K \). If \( K \subset f(K) \) then \( K \) is called backward \( f \)-invariant.

Authors of [1] gave a fractal dimension estimate for the invariant set of a function \( f : U \subset M \to M \) under the conditions

\[
0 < \min S_n(D_xf) < \sqrt{n}^{-1}
\]

and

\[
(\max |det D_xf|)(\min S_n(D_xf))^{d-n} \leq 8^{-n} n^{-\frac{d}{2}}.
\]

Qu and Zhou in [2] weakened the conditions and upgraded the results of [1] under the condition that Ricci curvature is non-negative. In this paper we generalize the results of [1] to complete Riemannian manifolds (without considerations on Ricci curvature). We will prove the following theorems.

**Theorem 1.1.** Let \( U \subset M \) be an open subset of a \( n \)-dimensional Riemannian manifold \( M \), \( f : U \to M \) be a \( C^1 \)-diffeomorphism onto its image and \( K \) be a backward \( f \)-invariant set. If \( 0 < \min S_n(D_xf) < 1 \) and there exists a number \( d \in (0,d] \) such that

\[
(\max |det D_xf|)(\min S_n(D_xf))^{d-n} \leq 1,
\]

then

\[
\text{dim}_B K \leq d.
\]

The following theorem is proved in [24] under the condition \( M = \mathbb{R}^n \). We prove the same result in more general case, when \( M \) is a complete Riemannian manifold.

**Theorem 1.2.** Let \( U \subset M \) be an open subset of a complete Riemannian manifold \( M \) and \( f : U \to M \) be a \( C^1 \)-mapping with the Brouwer degree \( d \). Let \( K \subset U \) be a compact \( f \)-invariant set and suppose that \( f \) has a non-zero Jacobian determinant. Put

\[
b = \lim_{m \to \infty} \frac{1}{m} \log(\min \{|det D_xf^m|, x \in K\})
\]

\[
s = \lim_{m \to \infty} \frac{1}{m} \log(\max \{|D_xf^m|, x \in K\})
\]

If \( b > 0 \), then \( s > 0 \) and

\[
\text{dim}_B K \leq n - \frac{b - \log d}{s} < n
\]
Remark 1.3. Because the Hausdorff dimension of a set is smaller or equal to its upper box dimension, theorems 1.1 and 1.2 also give upper bounds for the Hausdorff dimension of $K$.

Remark 1.4. In Theorem 1.1, the assumption $0 < \min \text{dim}_n(D_xf) < 1$ would be unnecessary provided that the inequality $(\max|\det D_xf|)(\min \text{dim}_n(D_xf))^{d-n} \leq 1$ was strict.

2. Proofs of the theorems

Lemma 2.1 (see [18]). If $K$ is a compact subset of a Riemannian manifold $M$ and $\text{dim}M = n$, then

$$\overline{\text{dim}}_B K \leq n + \limsup_{r \to 0} \frac{\log(\text{vol}(B_r K))}{-\log(r)}$$

Fact 2.2 (see [8, Theorem 2.92]). If $M$ is a Riemannian manifold and $x_0 \in M$, then there is an open ball around $x_0$ such that for any $x, y \in U$ there is a unique geodesic $\gamma$ joining $x$ to $y$ with the length equal to $d(x, y)$.

Remark 2.3 (see [2]). Let $B \subset U$ be an open subset of a Riemannian manifold $M$ and $f : U \to M$ a $C^1$-map. If $B$ is bounded then

$$\text{vol}(f(B)) \geq \inf_{x \in B} |\det D_x f| \text{vol}(B).$$

Remark 2.4. If $U$ is an open subset of a Riemannian manifold $M$ and $f : U \to M$ a $C^1$-diffeomorphism on its image, It is proved in [18] that if $K \subset U$ is a compact forward $f$-invariant set and

$$b = \lim_{m \to \infty} \frac{1}{m} \log(\min\{|\det D_x f^m|, x \in K\})$$

$$s = \lim_{m \to \infty} \frac{1}{m} \log(\max\{|D_x f^m|, x \in K\})$$

as well as $b > 0$, then $s > 0$ and

$$\overline{\text{dim}}_B K \leq n - \frac{b}{s} < n$$

In a similar way we can prove the following theorem.

Theorem 2.5. Let $U$ be an open subset of a Riemannian manifold $M$ and
Let $f : U \to M$ a $C^1$-diffeomorphism on its image. Let $K \subset U$ be a compact backward $f$-invariant set. Define

$$b = \lim_{m \to \infty} \frac{1}{m} \log(\min\{|\det D_x f^{-m}|, x \in K\})$$

$$s = \lim_{m \to \infty} \frac{1}{m} \log(\max\{|D_x f^{-m}|, x \in K\})$$

If $b > 0$, then $s > 0$ and

$$\dim B K \leq n - \frac{b}{s} < n$$

Proof of Theorem 1.1.

Since all norms in $\mathbb{R}^n$ are equivalent, the values of $b$ and $s$ are independent of the norm. Therefore the norm of $D_x f : T_x M \to T_{f(x)} M$ is equal to

$$\|D_x f\| = \sqrt{\alpha_n}$$

Where $\alpha_1$ is the maximum eigenvalue of $|D_x f|^t D_x f$. Thus we have

(1) \quad $S_n((D_x \varphi)^{-1}) = \|D_x \varphi\|^{-1}$

By (1) and the assumption that there exists a number $d \in (0,d]$ such that $(\max|\det D_x f|)(\min S_n(D_x f))^{d-n} \leq 1$, we have

(2) \quad $\max|\det(D_x f)| \leq (\min S_n(D_x f))^{n-d} = ((\max\|\det(D_x f)^{-1}\|)^{-1})^{n-d}$

Using $f^{-1}(K) \subset K$ and (2) we have

$$(\max\|\det(D_x f)^{-1}\|)^{n-d} \leq \min|\det(D_x f^{-1})|$$

Furthermore,

$\min|\det(D_x f^{-m})| = \min|\det(D_x f^{-1}) \cdots \det(D_{f^{-m+1}(x)} f^{-1})| \geq (\min|\det(D_x f^{-1})|)^m$

And

$\max\|D_x f^{-m}\| \leq \max(\|D_x f^{-1}\| \cdots \|D_{f^{-m+1}(x)} f^{-1}\|) \leq (\max|\det(D_x \varphi^{-1})|)^m$

By the assumptions of Theorem 1.1 we have

$$\max|\det(D_x f)| \leq (\min S_n(D_x f))^{n-d} \leq \min S_n(D_x f)$$
Which results
\[ \min \det(D_x f^{-1}) > 1 \]
Therefore,
\[
\frac{b}{s} = \lim_{m \to \infty} \frac{\log(\min |\det D_x \varphi^{-m}| : x \in k)}{\log(\max \|D_x \varphi^{-m}\| : x \in k)} \geq \\
\geq \lim_{m \to \infty} \frac{\log(\min |\det D_x \varphi^{-1}|^m)}{\log(\max \|D_x \varphi^{-1}\|^m)} \geq n - d
\]
Now by Theorem 2.5 we have \( \overline{\dim}_B K < d \).

□

Proof of Theorem 1.2.

Since \( f \) is a \( C^1 \)-mapping, it follows from the definition of \( b \) and \( s \) and the continuity argument that for each \( \delta > 0 \), there exists \( K_\delta \in \mathbb{N} \) and \( \epsilon > 0 \) such that
\[
1 < \exp(k_\delta(b - \delta)) < |\det D_x f^{k_\delta}|
\]
and
\[
||D_x f^{k_\delta}|| < \exp(k_\delta(s + \delta))
\]
for all \( x \in B_\epsilon(K) \). Since the Jacobian determinant is non-zero on \( K \), there exists a neighborhood of \( K \) on which \( f \) can be considered locally as a \( C^1 \)-diffeomorphism onto its image. Let us assume \( B_\epsilon(K) \) is such a neighborhood. It is possible to choose \( \epsilon \) sufficiently small that for each \( x \in K \) and each positive number \( c \leq \epsilon \), \( B_c(x) \) admits the results of Fact 2.2. From now on consider the mapping \( g = f^{k_\delta} \). Notice that \( K \) is also backward \( g \)-invariant. Put
\[
r_m = \epsilon (\exp(k_\delta(s + \delta)))^{-m} < \epsilon
\]
and
\[
B_m = B_{r_m}(K)
\]
for all \( m \in \mathbb{N} \).

(1) Let \( x \in B_1 \), then there exists \( y \in K \) such that \( d(x, y) < r_1 \). By Fact 2.2, there is a minimal geodesic \( \gamma : [0, 1] \to B_1(x) \) from \( x \) to \( y \). So
\[
d(g(x), g(y)) \leq \int_0^1 \left| \frac{d}{dt} g(\gamma(t)) \right| dt \leq \int_0^1 ||D_{\gamma(t)} g|| \left| \frac{d}{dt} \gamma(t) \right| dt \leq
\]

6
\[
\leq (\exp(k_3(s + \delta))) \int_0^1 \left| \frac{d}{dt} \gamma(t) \right| dt = (\exp(k_3(s + \delta)))d(x, y) < (\exp(k_3(s + \delta)))r_1 = \epsilon
\]

(2) Now let \(x \in B_2\), then there exists \(y \in K\) such that \(d(x, y) < r_2\). Similarly, we have
\[
d(g^2(x), g^2(y)) \leq \int_0^1 \left| \frac{d}{dt} g^2(\gamma(t)) \right| dt \leq \\
\leq \int_0^1 \left| D_{\gamma(t)}g^2 \right| \left| \frac{d}{dt} \gamma(t) \right| dt = \int_0^1 \left| D_{\gamma(t)}g \right| \left| D_{g(\gamma(t))}g \right| \left| \frac{d}{dt} \gamma(t) \right| dt \leq \\
\leq (\exp(k_3(s + \delta)))^2 \int_0^1 \left| \frac{d}{dt} \gamma(t) \right| dt = (\exp(k_3(s + \delta)))^2d(x, y) < (\exp(k_3(s + \delta)))^2r_2 = \epsilon
\]

By induction, for any \(m \in \mathbb{N}\) and any \(x \in B_m\), there exists \(y \in K\) such that \(d(g^m(x), g^m(y)) \leq \epsilon\). Since \(g^m(K) \subset K\) and then \(g^m(x) \in B_K\). Thus \(g^m(B_m(K)) \subset B_\epsilon K\) for all \(m \in \mathbb{N}\). Which results,
\[
\text{(5)} \quad \text{vol}(g^m(B_m(K))) \leq \text{vol}(B_\epsilon(K))
\]

\(B_m(K)\) is bounded, so there exist balls \(U_{k_1}, \ldots, U_{k_n}\) such that \(g^m|_{U_{k_i}}\) is a \(C^1\)-diffeomorphism onto its image and \(B_\epsilon(K) \subset \bigcup_{i=1}^n U_{k_i}\). Put
\[
V_{k_i} = U_{k_i} \cap B_\epsilon(K)
\]

\[
V_{k_i} = (U_{k_i} \cap B_\epsilon(K)) \setminus \bigcup_{s=1}^{i-1} V_{k_s}
\]

By Remark 2.3 we have,
\[
\text{(6)} \quad \text{vol}(g^m(V_{k_i})) \geq \inf_{x \in V_{k_i}} |\det D_x g^m| \text{vol}(V_{k_i})
\]

By (3), we have \(\exp(K_\delta(b - \delta)) < |\det D_x g|\), thus \(\exp(K_\delta(b - \delta))^m < |\det D_x g^m|\). Therefore by (6), we have
\[
\text{(7)} \quad \text{vol}(V_{k_i}) \leq \exp(K_\delta(b - \delta))^{-m} \text{vol}(g^m(V_{k_i}))
\]

The sets \(V_{k_i}\) are pairwise disjoint and \(\bigcup_{i=1}^n V_{k_i} = B_\epsilon(K)\). Using (5) and (7) we get
\[
\text{vol}(B_\epsilon(K)) = \sum_{i=1}^n \text{vol}(V_{k_i}) \leq
\]

7
\[ \leq \sum_{i=1}^{n} \exp(K_{\delta}(b - \delta))^{-m} \text{vol}(g^{m}(V_{k_{i}})) \leq d^{km} \exp(K_{\delta}(b - \delta))^{-m} \text{vol}(B_{c}(K)). \]

Therefore,

\[
\limsup_{r \to 0} \frac{\log(\text{vol}(B_{r}(K)))}{-\log(r)} = \limsup_{r \to 0} \frac{\log(\text{vol}B_{m}(K))}{-\log(r_m)} \leq \lim_{m \to \infty} \frac{\log(d^{km} \exp(K_{\delta}(b - \delta))^{-m} \text{vol}(B_{\epsilon}(K)))}{-\log(\exp(k_{\delta}(s + \delta))^{m})} = -\frac{b - \delta - \log d}{s + \delta}
\]

Since \( \delta \) is arbitrary small, then

\[
(8) \quad \limsup_{r \to 0} \frac{\log(\text{vol}(B_{r}(K)))}{-\log(r)} \leq -\frac{b - \log d}{s}
\]

Now by (8) and Lemma 2.1, we get

\[
\dim B_{K} \leq n - \frac{b - \log d}{s}
\]

\[\square\]

References

[1] V. Boichenko, A. Franz, G. Leonov and V. Reitmann, Hausdorff and fractal dimension estimates for invariant sets of non-injective maps. Z. Anal. Anw. 17, 207-23, 1998.

[2] QU Chengqin and Zhou Zuoling, Fractal dimension estimate for invariant set in complete Riemannian manifolds. Chaos, Soliton and Fractals 31, 1165-72, 2007.

[3] A. Douady et J. Oesterle, Dimension de Hausdorff des attracteurs. C. R. Acad. Sci. Paris ser. A 290, 1135-38, 1980.

[4] A. Eden, C. Foias, R. Temam, Local and global Lyapunov exponents. J. Dynam. Differ. Equations. 3, 13377, 1991.

[5] K. Falconer, Fractal geometry. John Wiley and Sons, New York, 1990.

[6] A. Fathi, Expansiveness, Hyperbolicity and Hausdorff dimension. Commun. Math. Phys. 126, 249-62, 1990.

[7] A. Franz, Hausdorff dimension estimates for invariant sets with an equivariant tangent bundle splitting. Nonlinearity 11, 1063-74, 1998.

[8] S. Gallot, D. Hulin and J. Lafontaine, Riemannian geometry. Springer Verlag, Berlin, 1980.

[9] X. Gu, An upper bound for the Hausdorff dimension of a hyperbolic set. Nonlinearity 4, 927-34, 1991.

[10] W. Heineken, fractal dimension estimates for invariant sets of vector fields. Diploma thesis, University of Technology Dresden, 1997.
[11] F. Ledrappier, *Some relations between relation and Lyapunov exponents*. Commun. Math. Phys. 81, 229-38, 1981.

[12] F. Ledrappier, L. S. Young, *Dimension formula for random transformations*. Commun. Math. Phys. 81, 229-38, 1988.

[13] G. A. Leonov, V. A. Bochenko, *Lyapunov’s direct method in the estimation of the Hausdorff dimension of attractors*. Acta Appl. Math. 26, 1-60, 1992.

[14] G. A. Leonov, I. M. Burkin, I. Shepelyawyi, *Frequency methods in oscillation theory*. Kluwer Academic Publishers, Dordrecht, 1996.

[15] G. A. Leonov, D. V. Ponomarenko, V. B. Smirnova, *Frequency methods for nonlinear analysis (Theory and Applications)*. World Scientific, Singapore, 1996.

[16] J. W. Milnor, *Topology from differentiable viewpoint*. The University Press of Virginia, Charlottesville, 1965.

[17] A. Mirle, *Hausdorff dimension estimates for invariant sets of K-1-maps*. Preprint 25/95, DFG-schwerpunktprogramm "Dynamik: Analysis, effiziente Simulation und Ergodentheorie", 1995.

[18] M. Mirzaie, *On fractal dimension of invariant sets*. Mathematical reports 13, 377-84, 2011.

[19] A. Noack, V. Reitmann, *Hausdorff dimension of invariant sets of time-dependent vector fields*. ZAA 15(2), 457-73, 1996.

[20] R. A. Smith, *Some applications of Hausdorff dimension inequalities for ordinary differential equations*. Proc. Roy. Soc. Edinburgh. 104A, 23559, 1986.

[21] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*. Springer, New York, 1988.

[22] P. Thieullen, *Entropy and Hausdorff dimension for infinite-dimensional dynamical systems*. Journal of Dynamics and Differential Equations, 4(1), 127-59, 1992.

[23] A. Yu. Pogromsky, H. Nijmeijer, *On estimates of the Hausdorff dimension of invariant compact sets*. Nonlinearity 13, 927-45. The UK, 2000.

[24] C. Wolf, *On the box dimension of an invariant set*. Nonlinearity 14, 73-9, 2001.

Department of mathematics
Faculty of sciences
I. KH. International university (IKIU)
Qazvin, Iran
mehrzad_monzavi@yahoo.com
r_mirzaie@yahoo.com