Decoupled oscillations and resonances of three neutrinos in matter

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Abstract

In a previous paper, we gave a new theoretical framework in which three neutrino mixing in matter are discussed. Rigorous analytical solutions are obtained. In present paper an approximate method is developed for studying three flavor neutrino oscillations in matter in the new framework. Using condition of $\Delta m_1^2 \ll \Delta m_2^2$, decoupled resonances, which is discovered by Kuo et al. previously, are obtained without small angle approximation. Calculation and approximation is consistent and rigorous. All the formulae appear in simple symmetric form and some physical characteristic are more explicit. Around the two resonant points, the mass eigenvalue whose eigenfunction does not take part in the oscillation process, can be easily calculated to higher precision. We find, the oscillation amplitude is dominated by vacuum mixing and is smaller than 1 for smaller A resonance, but it is not in the larger A. This is characteristic for the three flavor oscillations, but do not exist in two flavor.

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I. INTRODUCTION

Recently new experimental results for neutrino appear rapidly [1]. More and more theoretical articles are published attempting at fixing their mass and mixing angles [2]. Solar neutrino and some other experiments relate to the mixing and oscillations of the neutrino in matter [1–3]. For studying these data, theoretical methods for solving the propagation and oscillation equations of neutrino in matter are required. Many important results have been obtained for two flavors. An important fact, discovered more than ten years ago, is that there is resonance of the mixing angle in matter due to the interaction of charged current [4]. Because of the increase of mixing parameters the situation is more complex in three flavors than two. An important advance was gained in the research of MSW oscillation in three flavors in 1986. It was found that the three flavor oscillations are approximately decoupled into two independent oscillations and resonances between two flavors [5]. Many important results from the researches of two flavors oscillation can be used in the three flavor case [6]. Many works have been done and papers are published. They have been used in analyzing the experimental results, analytically and numerically [6,7]. However a new theoretical framework and analytical method may be helpful to this task. We have made a series of efforts in this respect. In a previous paper, the author gave a new theoretical framework to discuss three neutrino mixing in matter [8]. Results are given analytically in explicit, symmetric, and very simple form. Now we apply them to the approximate solutions and discuss three flavor oscillations in matter.

In section II, the results obtained in the first paper [8] and will be used in the paper, are quoted.

In section III, using and only using the condition of $\Delta m_1^2 \ll \Delta m_2^2$, a strict approximate mathematical method is developed to deal with the eigenvalue equation of mass in matter for three neutrinos. By the method, a cubic algebra equation is reduced into two of quadratic, in two regions around $A \sim \Delta m_1^2$ and $A \sim \Delta m_2^2$ respectively. Thus, we get four solutions at each region, but only three of them satisfies our condition imposed on eigenvalues used for reducing the equation. In this way, decoupled oscillations and resonances can be read out directly. In addition, the solution which does not take part in the resonance can be calculated to higher precision which will be very useful in derivating propagating equation in a followed paper.

In section IV, a consistent rigorous mathematical derivation is presented using our the-
oretical framework [8] to expose the behaviors of the mixing parameters in matter around $A \sim \Delta m_1^2$ and $A \sim \Delta m_2^2$ when $\Delta m_1^2 \ll \Delta m_2^2$. Approximately decoupled oscillation and resonance behavior are obtained between $U_{e1}^m$ and $U_{e2}^m$ around the $A \sim \Delta m_1^2$, and between $U_{e2}^m$ and $U_{e3}^m$ around $A \sim \Delta m_2^2$ respectively. This phenomena is the partner of the oscillation and resonance between $\cos \theta^m$ and $\sin \theta^m$ in two flavors [4]. In three flavor, this had been discovered as the decoupled resonances of $\theta^m_1$ around $\Delta m_1^2$ and $\theta^m_3$ around $\Delta m_2^2$ by T.K. Kuo and his collaborator previously [5]. However, there are some phenomena which are not discussed or proved explicitly before. In the lower resonance, the amplitude, $\sqrt{U_{e1}^{m2} + U_{e2}^{m2}} \approx \sqrt{\eta_{11}^2 + \eta_{22}^2} < 1$. It is determined by the vacuum parameters. However, in the higher resonance, $\sqrt{U_{e2}^{m2} + U_{e3}^{m2}} \approx 1$ which is independent on the vacuum mixing parameters. There is notable difference between two flavor and three flavor oscillation and resonance in matter, even if in the case of decoupled resonances in three flavors. When the vacuum mixing angles are not small, the new observation is meaningful.

In the end, in section V, the results are discussed and the conclusion is given.

II. THEORETICAL FRAMEWORK

We begin from the general equation of the eigenvalue problem and some of the rigorous analytical solutions obtained in a previous paper [8]. All symbols and definitions used in that paper will be used in here without explanation. The equation dominated the eigenvalues of three neutrinos in matter is.

$$\lambda^2 + a\lambda^2 + b\lambda + c = 0$$ (1)

where

$$a = -(A + \Delta m_2^2)$$ (2)

$$b = -(\Delta m_1^2)^2 + \left[\Delta m_1^2 (\eta_{11} - \eta_{22}) + \Delta m_2^2 (\eta_{11} + \eta_{22})\right] A$$ (3)

and

$$c = (\Delta m_1^2)^2 \Delta m_2^2 - \Delta m_1^2 \left[\Delta m_2^2 (\eta_{11} - \eta_{22}) - \Delta m_1^2 \eta_{33}\right] A$$ (4)

The solutions for $\nu_e$ mixing matrix elements in matter are
\[ U^{e_i}_{m} = N^u \sum_{i=1}^{3} \frac{\eta_i^2}{\lambda^u - \Delta_i} \]  
\[ u = 1, 2, 3 \]  
(5)

where \( N^u \) is the normalizing constant.

\[ N^u = \left[ \sum_{i=1}^{3} \left( \frac{\eta_i}{\lambda^u - \Delta_i} \right)^2 \right]^{-\frac{1}{2}} \]  
\[ u = 1, 2, 3 \]  
(6)

and \( \Delta_i \)'s are

\[ \Delta_1 = -\frac{1}{2} (m_2^2 - m_1^2) = -\Delta m_1^2 \]
\[ \Delta_2 = \frac{1}{2} (m_2^2 - m_1^2) = \Delta m_1^2 \]
\[ \Delta_3 = m_3^2 - \frac{1}{2} (m_2^2 + m_1^2) = \Delta m_2^2 \]  
(7)

The mass eigenvalues in matter expressed by \( \lambda_u \) are

\[ M^2_u = \lambda_u + \frac{1}{2} (m_1^2 + m_2^2) \]  
(8)

Remembering that we have proved

\[ \eta_1^2 + \eta_2^2 + \eta_3^2 = 1 \]  
(9)

it is easy to see

\[ U^m_{e1} + U^m_{e2} + U^m_{e3} = 1 \]  
(10)

This relation is helpful to our task below. Now we make use of these formulae and the condition \( \Delta m_1 \ll \Delta m_2 \) to derive approximate solutions in which decoupled resonances emerged explicitly.

III. APPROXIMATE SOLUTIONS OF EIGENVALUES

A. Dimensionless Equation

Before we begin work, it is convenient to make the equation dimensionless in favor of estimating the quantitative order for each term in the Eq.(1)-Eq.(4). Taking \( \lambda, \Delta m_1^2 \) and \( \Delta m_2^2 \) in the unit of \( A \), we introduce the following variable transformation.
\[
\lambda = \frac{\Lambda}{A}, \quad \Delta m^2_1 = \frac{\Delta m^2_1}{A} \text{ and } \Delta m^2_2 = \frac{\Delta m^2_2}{A}
\] (11)

We get

\[
\overline{\lambda}^3 + a\overline{\lambda}^2 + b\overline{\lambda} + c = 0 \tag{12}
\]

where

\[
a = -(1 + \Delta m^2_2) \tag{13}
\]

\[
b = -(\overline{\Delta m}^2_1)^2 + [\Delta m^2_1 (\eta_{11} - \eta_{22}) + \Delta m^2_2 (\eta_{11} + \eta_{22})] \tag{14}
\]

and

\[
c = (\overline{\Delta m}^2_1)^2 \Delta m^2_2 - \Delta m^2_1 [\Delta m^2_2 (\eta_{11} - \eta_{22}) - \Delta m^2_1 \eta_{33}] \tag{15}
\]

Three flavor neutrino oscillations can be decoupled approximately into two independent oscillations between two flavors and resonances, if the neutrino masses have the hierarchical characteristic, or more exactly if \(\Delta m^2_1 \ll \Delta m^2_2\), where \(\Delta m^2_1 = \frac{(m^2_3 - m^2_1)}{2}\) and \(\Delta m^2_2 = \frac{m^2_3 - (m^2_2 + m^2_1)}{2}\). This fact have been observed by T.K.Kuo and his collaborator previously. Now, we derive it in this new theoretical framework.

**B. Solutions around \(A \sim \Delta m^2_1\)**

First, we search for the approximate solutions around the region \(A \sim \Delta m^2_1\). In present case,

\[
\overline{\Delta m}^2_1 \sim 1 \quad \text{and} \quad \overline{\Delta m}^2_2 \gg 1 \tag{16}
\]

1. **Solution to the small eigenvalues \(\lambda \sim \Delta m^2_1\).**

We try to search for the solutions which satisfy \(\lambda \sim \Delta m^2_1\). Then \(\overline{\lambda} \sim \overline{\Delta m}^2_1 \sim 1\). Neglecting all the terms which is of order 1 in Eq.(12)-Eq.(15), we obtain the following approximate equation,
\[
\lambda^2 - (\eta_{11} + \eta_{22}) \lambda - \left[ (\Delta m_1^2)^2 - \Delta m_1^2 (\eta_{11} - \eta_{22}) \right] = 0
\]  

(17)

There are two solutions for it

\[
\lambda_\pm = \frac{(\eta_{11} + \eta_{22}) \mp \sqrt{(\eta_{11} + \eta_{22})^2 + 4 \left[ (\Delta m_1^2)^2 - \Delta m_1^2 (\eta_{11} - \eta_{22}) \right]}}{2}
\]

(18)

Both of them satisfy the condition \( \lambda \sim \Delta m_1^2 \sim 1 \). Thus we obtain two solutions

\[
\lambda_{1,2} = A (\eta_{11} + \eta_{22}) \mp \sqrt{A^2 (\eta_{11} + \eta_{22})^2 + 4 \left[ (\Delta m_1^2)^2 - A \Delta m_1^2 (\eta_{11} - \eta_{22}) \right]}
\]

(19)

2. Solution to the large eigenvalue \( \lambda \sim \Delta m_2^2 \).

We try to search for the third solution as \( \lambda \sim \Delta m_2^2 \). That is the solution satisfying \( \lambda \sim \Delta m_2^2 \gg 1 \). We can neglect the terms whose orders are equal or smaller than \( \Delta m_2^2 \) in Eq.(12)-Eq.(15). We get an approximate equation as

\[
\overline{\lambda}^2 - \left( 1 + \Delta m_2^2 \right) \overline{\lambda} + \Delta m_2^2 (\eta_{11} + \eta_{22}) = 0
\]

(20)

There are two solutions

\[
\overline{\lambda}_\pm = \frac{1}{2} \left[ 1 + \Delta m_2^2 \pm \sqrt{(1 + \Delta m_2^2)^2 - 4 \Delta m_2^2 (\eta_{11} + \eta_{22})} \right]
\]

(21)

but, only the solution \( \overline{\lambda}_+ \) satisfies \( \overline{\lambda} \sim \Delta m_2^2 \) which is used for obtaining the reduced equation. Therefore we get a maximum solution

\[
\lambda_3 = \frac{1}{2} \left[ A + \Delta m_2^2 + \left( A + \Delta m_2^2 \right) \sqrt{1 - \frac{4 A \Delta m_2^2 (\eta_{11} + \eta_{22})}{(A + \Delta m_2^2)^2}} \right]
\]

(22)

Remaining to the order \( A \), we get

\[
\lambda_3 \approx \Delta m_2^2 + A \eta_{33}
\]

(23)
C. Solutions around $A \sim \Delta m_2^2$.

When $A \sim \Delta m_2^2$

$$\Delta m_2^2 \sim 1 \quad \text{and} \quad \Delta m_1^2 \ll 1 \quad (24)$$

1. Larger solution $\lambda \sim \Delta m_2^2$.

We try to search for the solutions it satisfy $\lambda \sim \Delta m_2^2$. Neglecting all the terms having the order smaller than $\left(\Delta m_2^2\right)^2$, from Eq.(12)-Eq.(15), we obtain an approximate equation

$$\lambda^2 - (1 + \Delta m_2^2)\lambda + \Delta m_2^2 (\eta_{11} + \eta_{22}) \lambda = 0 \quad (25)$$

There are two solutions for it

$$\lambda_{\pm} = \frac{1}{2} \left(1 + \Delta m_2^2\right) \pm \frac{1}{2} \sqrt{(1 + \Delta m_2^2)^2 - 4 \Delta m_2^2 (\eta_{11} + \eta_{22})} \quad (26)$$

Both of them satisfy the condition $\lambda \sim \Delta m_2^2$. Thus we obtain

$$\lambda_{2,3} = \frac{1}{2} (A + \Delta m_2^2) \mp \frac{1}{2} \sqrt{(A + \Delta m_2^2)^2 - 4A \Delta m_2^2 (\eta_{11} + \eta_{22})} \quad (27)$$

2. Smaller solution $\lambda \sim \Delta m_1^2$

We try to search for the third solution in $\lambda \sim \Delta m_1^2 \ll 1$. In Eq.(12)-Eq.(15), neglecting the terms whose order are equal or smaller $\left(\Delta m_1^2\right)^2$, we obtain an approximate equation

$$(\eta_{11} + \eta_{22}) \lambda - \Delta m_1^2 (\eta_{11} - \eta_{22}) = 0 \quad (28)$$

Its solution is

$$\lambda = \frac{\Delta m_1^2}{\eta_{11} + \eta_{22}} (\eta_{11} - \eta_{22}) \quad (29)$$
That is

\[ \lambda_1 = \Delta m_1^2 \frac{\eta_{11} - \eta_{22}}{\eta_{11} + \eta_{22}} \]  

(30)

In fact, in our present method, higher precision can be reached for this solution by neglecting only the terms in order \((\Delta m_1^2)^3\). Now we have a twice algebra equation

\[
(1 + \Delta m_2^2)\lambda^2 - \left[ \Delta m_1^2 (\eta_{11} - \eta_{22}) + \Delta m_2^2 (\eta_{11} + \eta_{22}) \right] \lambda - \left[ (\Delta m_1^2)^2 - \Delta m_1^2 (\eta_{11} - \eta_{22}) \right] \Delta m_2^2 + (\Delta m_1^2)^2 \eta_{33} = 0 \]

(31)

There are two solutions for this equation, but only one of them satisfies \(\bar{\lambda} \sim \Delta m_1^2\) which is used to reduce the equation. It is the smaller one. The solution is a minimum solution for \(A \sim \Delta m_2^2\). When we discuss propagating equation in a followed paper, this higher precision solution is very useful, but we are not necessary write it here explicitly.

**IV. APPROXIMATELY DECOUPLED RESONANCE BEHAVIORS OF** \(U_{e1}^M, U_{e2}^M, U_{e3}^M\)

In section II, we have introduced a set of solutions, Eq.(5) for the \(U_{e,u}^m\) from a previous paper [8], because we are interested only in the mixing associated with \(\nu_e\) in matter. They are expressed as functions of the vacuum parameters and potential \(A\). Now we use the equation to discuss the approximately decoupled resonant behaviors of \(U_{e1}^m, U_{e2}^m,\) and \(U_{e3}^m\).

**A. Resonant form for** \(\lambda_1\) **and** \(\lambda_2\) **around** \(A \sim \Delta m_1^2\).

When \(A \sim \Delta m_1^2\), we have get three approximate solutions. They are expressed in Eq.(19) and Eq.(22). There is a possible resonance between eigenstates of \(\lambda_1\) and \(\lambda_2\) when the quantity in the square root in Eq.(19) takes the minimum value. It can be reached when the following condition of \(A = A_l\) is satisfied:

\[
A_l = 2 \Delta m_1^2 \frac{\eta_{11} - \eta_{22}}{(\eta_{11} + \eta_{22})^2} \]

(32)

Introducing
\[ \rho_l = A (\eta_{11} + \eta_{22}) - 2\Delta m_1^2 \frac{\eta_{11} - \eta_{22}}{\eta_{11} + \eta_{22}} \text{ and } \delta_l = \Delta m_1^2 \frac{2\eta_1 \eta_2}{\eta_{11} + \eta_{22}} \]  

(33)

we can rewrite the solutions Eq.(19) as

\[ \lambda_{1,2} = \frac{1}{2} \left[ A (\eta_{11} + \eta_{22}) \mp \sqrt{\rho_l^2 + 4\delta_l^2} \right] \quad A \sim \Delta m_1^2 \]  

(34)

It is clear that \( 4\delta_l^2 > 0 \) when \( \eta_1 \eta_2 \neq 0 \). There is a possible resonance between eigenstates of \( \lambda_1 \) and \( \lambda_2 \) when the condition \( A = A_l \) is satisfied. Because \( \lambda_3 \sim \Delta m_2^2 \gg \lambda_{1,2} \sim \Delta m_1^2 \), this resonance is decoupled approximately from \( \lambda_3 \).

Taking the traditional representation and let \( U = U_2 U_3 U_1 \), we have

\[ \frac{\eta_{11} - \eta_{22}}{(\eta_{11} + \eta_{22})^2} = \frac{c_1^2 c_2^2 - s_1^2 c_3^2}{(c_1^2 c_2^2 + s_1^2 c_3^2)^2} = \frac{\cos 2\theta_1}{c_3^2} \]  

(35)

Thus, the lower resonance condition, Eq.(32), becomes

\[ A = A_l = 2\Delta m_1^2 \frac{\cos 2\theta_1}{c_3^2} \]  

(36)

It is the same as the result obtained by Kuo et al.

**B. The behaviors of \( U_{e1}^m \), \( U_{e2}^m \) and \( U_{e3}^m \) in the neighborhood of \( A \sim \Delta m_1^2 \).**

We can use the equations of \( U_{e1}^m \) and \( U_{e2}^m \) and take the approximation as follows

\[ U_{e1,2}^m \approx N^{1,2} \left( \frac{\eta_1^2}{\lambda_{1,2} + \Delta m_1^2} + \frac{\eta_2^2}{\lambda_{1,2} - \Delta m_1^2} \right) \]  

(37)

Correspondingly, we have

\[ N^{1,2} \approx \left[ \left( \frac{\eta_1}{\lambda_{1,2} + \Delta m_1^2} \right)^2 + \left( \frac{\eta_2}{\lambda_{1,2} - \Delta m_1^2} \right)^2 \right]^{-\frac{1}{2}} \]  

(38)

Substituting the Eq.(38) into the Eq.(37), we obtain

\[ U_{e1,2}^m \approx \frac{\lambda_{1,2} - \Delta m_1^2}{|\lambda_{1,2} - \Delta m_1^2|} \frac{\alpha_{1,2} \sqrt{\eta_1^2 + \eta_2^2}}{\alpha_{1,2}^2 + (\eta_1^2 + \eta_2^2) \delta_l^2} \]  

(39)

Where

\[ \alpha_{1,2} = \sqrt{\eta_1^2 + \eta_2^2} \left( \frac{\lambda_{1,2} - \Delta m_1^2}{\eta_1^2 + \eta_2^2} \right) \]  

(40)
Using Eq.(34) of $\lambda_{1,2}$, we get
\[
\alpha_{1,2} = \sqrt{\eta_1^2 + \eta_2^2} \left( \frac{1}{2} \rho_l \pm \frac{1}{2} \sqrt{\rho_l^2 + 4\delta_l^2} \right) \tag{41}
\]
Then
\[
U_{e1,2}^m \approx \frac{\lambda_{1,2} - \Delta m_1^2}{|\lambda_{1,2} - \Delta m_1^2| \alpha_{1,2}} \sqrt{\frac{\eta_1^2 + \eta_2^2}{2}} \left( 1 \mp \frac{\rho_l}{\sqrt{\rho_l^2 + 4\delta_l^2}} \right) \tag{42}
\]
The $\eta_3^m$ is not independent. It is determined by $U_{e1}^m + U_{e2}^m + U_{e3}^m = 1$. from it, we obtain
\[
U_{e3}^m \approx \eta_3 \tag{43}
\]
When the resonance condition $\rho_l = 0$ is satisfied, we have
\[
U_{e1}^m \approx \frac{\lambda_{1,2} - \Delta m_1^2}{|\lambda_{1,2} - \Delta m_1^2| \alpha_{1,2}} \sqrt{\frac{\eta_1^2 + \eta_2^2}{2}} \tag{44}
\]

C. Resonant form for $\lambda_2$ and $\lambda_3$ around $A \sim \Delta m_2^2$.

We have obtained three solutions when $A \sim \Delta m_2^2$. They are expressed in Eq.(27) and Eq.(30). There is a possible resonance between eigenstates of $\lambda_2$ and $\lambda_3$ when the quantity in the square root takes a minimum value. It is easy to show that the minimum can be obtained when the following condition of $A = A_h$ is satisfied:
\[
A_h = \Delta m_2^2 (\eta_{11} + \eta_{22} - \eta_{33}) \tag{45}
\]
Let
\[
\rho_h = A - \Delta m_2^2 (\eta_{11} + \eta_{22} - \eta_{33}) \quad \text{and} \quad \delta_h = \Delta m_2 \sqrt{\eta_{11} + \eta_{22} \eta_3} \tag{46}
\]
the $\lambda_{2,3}$ can be rewrite as
\[
\lambda_{2,3} = \frac{1}{2} \left( A + \Delta m_2^2 \mp \sqrt{\rho_h^2 + 4\delta_h^2} \right) \tag{47}
\]
It is clear that $4\delta_h^2 > 0$ when $(\eta_{11} + \eta_{22}) \eta_{33} \neq 0$. We obtain a possible resonance. Because $\lambda_1 \ll \Delta m_2^2 \sim \lambda_{2,3}$, this resonance between eigenstates of $\lambda_2$ and $\lambda_3$ is decoupled approximately from $\lambda_1$. 

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When we take the traditional representation and let \( U = U_2 U_3 U_1 \), we have
\[
\eta_1 = c_1 c_3, \quad \eta_2 = s_1 c_3 \quad \text{and} \quad \eta_3 = s_3 \quad (48)
\]

Then,
\[
\eta_{11} + \eta_{22} - \eta_{33} = c_1^2 c_3^2 + s_1^2 c_3^2 - s_3^2 = \cos 2\theta_3 \quad (49)
\]

The higher resonance condition, Eq.(45), becomes
\[
A_h = \Delta m_2^2 \cos 2\theta_3 + O \left( \Delta m_1^2 \right) \quad (50)
\]

It is the same as the result obtained by Kuo et al. at large \( A \) resonance.

**D. The behaviors of \( U_{e1}^m, U_{e2}^m \) and \( U_{e3}^m \) in the neighborhood of \( A \sim \Delta m_2^2 \).**

When \( A \sim \Delta m_2^2 \), the higher resonance, \( \lambda_1 \ll \lambda_{2,3} \). We can use the equations of \( U_{e2}^m \) and \( U_{e3}^m \) and make the approximation as follows

\[
U_{e2,3}^m \approx N^{2,3} \left[ \frac{\eta_1^2}{\lambda_{2,3}} + \frac{\eta_2^2}{\lambda_{2,3}} + \frac{\eta_3^2}{\lambda_{2,3} - \Delta m_2^2} \right] \quad (51)
\]

Correspondingly, we have

\[
N^{2,3} \approx \left[ \left( \frac{\eta_1}{\lambda_{2,3}} \right)^2 + \left( \frac{\eta_2}{\lambda_{2,3}} \right)^2 + \left( \frac{\eta_3}{\lambda_{2,3} - \Delta m_2^2} \right)^2 \right]^{-\frac{1}{2}} \quad (52)
\]

Substituting the Eq.(52) into the Eq.(51), we obtain

\[
U_{e2,3}^m \approx \frac{\lambda_{2,3} \left( \lambda_{2,3} - \Delta m_2^2 \right)}{\left| \lambda_{2,3} \left( \lambda_{2,3} - \Delta m_2^2 \right) \right|} \frac{\alpha_{2,3}}{\sqrt{\alpha_{2,3}^2 + \delta_h^2}} \quad (53)
\]

Where

\[
\alpha_{2,3} = \lambda_{2,3} - \Delta m_2^2 \left( \eta_1^2 + \eta_2^2 \right) \quad (54)
\]

Using Eq.(47) of \( \lambda_{2,3} \), we get

\[
\alpha_{2,3} = \frac{1}{2} \rho_h + \frac{1}{2} \sqrt{\rho_h^2 + 4\delta_h^2} \quad (55)
\]

Then
The $U_{e1}^m$ is not independent. It is determined by $U_{e1}^m + U_{e2,3}^m + U_{e3}^m = 1$. From it, we obtain

$$U_{e1}^m \approx 0$$

When the resonance condition $\rho_h = 0$ is satisfied, we have

$$U_{e2,3}^m \approx \frac{\lambda_{2,3} (\lambda_{2,3} - \Delta m_2^2)}{||\lambda_{2,3} (\lambda_{2,3} - \Delta m_2^2)||} \frac{\alpha_{2,3}}{|\alpha_{2,3}|} \sqrt{\frac{1}{2} \left(1 \pm \frac{\rho_h}{\sqrt{\rho_h^2 + 4\delta_h}}\right)}$$

\[ (58) \]

V. DISCUSSION AND CONCLUSION

In situation of three neutrinos in matter, the mixing is generally different to two neutrinos. They do not occur in any plane spanned by two of the three eigenvectors $|\nu_{m,u}^m\rangle$ ($u = 1, 2, 3$). We can not use a simple mixing angle to describe their mixing as done in two neutrino situation. However in case of $\Delta m_1^2 \ll \Delta m_2^2$, we have two approximately decoupled resonances. In each case the oscillation do occurs between two mass eigenvectors, but there is important difference from two neutrinos situation.

Around $A \sim \Delta m_1^2$, matter effect make $U_{e1}^m$ and $U_{e2}^m$ variation when $A$ change. The variation has oscillation characteristic and resonance. Oscillation occurs between $|\nu_{m,1}^m\rangle$ and $|\nu_{m,2}^m\rangle$ but it has an amplitude $\sqrt{U_{e1}^m + U_{e2}^m} = \sqrt{\eta_1^2 + \eta_2^2} < 1$. This amplitude is determined by the vacuum mixing parameters. In plane spanned by $|\nu_{m,1}^m\rangle$ and $|\nu_{m,2}^m\rangle$, there exist a mixing angle $\beta_t$ which varies with $A$. The $U_{e1}^m = a \cos \beta_t$ and $U_{e2}^m = a \sin \beta_t$ with $a = \sqrt{\eta_1^2 + \eta_2^2} < 1$. In this meaning, we have oscillation and resonance in a plane spanned by $|\nu_{m,1}^m\rangle$ and $|\nu_{m,2}^m\rangle$. It constitutes only a part of $|\nu_e\rangle$. The $|\nu_{m,3}^m\rangle$ contribution is nonzero and has a approximately constant mixing coefficient $U_{e3}^m |\nu_{m,3}^m\rangle \simeq \eta_3 |\nu_{m,3}^m\rangle$. Noticeably $|\nu_{m,3}^m\rangle$ enter only due to vacuum mixing but not matter effect. It is not oscillation with the potential $A$ change. When $U_{e3} = \eta_3$ is not small, the nature is important. It is different with two neutrino oscillation and resonance in matter.

Around $A \sim \Delta m_2^2$, the case is same. However, because of the small mass, the mixing contribution of $|\nu_{m,1}^m\rangle$ is negligible to $U_{e2}^m$ and $U_{e3}^m$ oscillation. There exist a angle $\beta_h$ which varies with $A$ in a plane spanned by the eigenvector $|\nu_{m,2}^m\rangle$ and $|\nu_{m,3}^m\rangle$. The $U_{e2} = \cos \beta_h$
and $U_{\alpha_3}^m = \sin \beta$ with an amplitude 1. The oscillation and resonance occur when potential $A$ change. In this case, $U_{\alpha_1}^m \approx 0$ whatever the values of $\eta_i$ is taken. Then the oscillation and resonance is more as one of two neutrinos.

We have used a theoretical framework developed in a previous paper which is convenient for dealing with the three flavor neutrino oscillations in matter. In condition of $\Delta m_1^2 \ll \Delta m_2^2$, approximately decoupled resonance behaviors is studied and discussed in details. New interesting physical results are obtained. There are important differences in resonant phenomena between two flavors and three flavors even in the case of decoupled resonances, in particular when $\eta_3$ is large. In traditional mixing angle description, there is mixing order problem. In some case, it may mask the physical characteristic and lead to misunderstanding.

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