Generating infinite-dimensional algebras from loop algebras by expanding Maurer Cartan forms.

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Abstract

It is shown that the expansion methods developed in refs. [1] can be generalized so that they permit to study the expansion of algebras of loops, both when the compact finite-dimensional algebra and the algebra of loops have a decomposition into two subspaces.
I. INTRODUCTION

Let \( G(M) = G(S^1) = \text{Map}(S^1; G) \) be, the group of smooth mappings (loops) \( z \rightarrow g(z) \) of the circle \( S^1 = \{ z \in \mathbb{C} / |z| = 1 \} \) into a simple, compact and connected finite-dimensional Lie group \( G \). The group structure is defined by the pointwise multiplication of functions \((\hat{g} \hat{g})(z) = \hat{g}(z)g(z)\). \( \text{Map}(S^1; G) \) is an infinite-dimensional group, the loop group \( LG \), the elements of which can be represented by

\[
g(z) = e^{\alpha^a(z)T_a}, \quad a = 1, \cdots, r = \dim G
\]  

where \( T_a = -T^a \) are the generators of the finite-dimensional Lie algebra \( \mathfrak{g} \), \([T_a, T_b] = f_{ab}^c T_c\). For elements near the identity,

\[
g(z) \approx 1 + \alpha^a(z)T_a. \quad (2)
\]

Making a Laurent expansion of \( \alpha^a(z) \) on the circle

\[
\alpha^a(z) = \sum_{n=-\infty}^{\infty} \alpha^a_{-n} z^n \quad (3)
\]

expression (2) reads

\[
g(z) \approx 1 + \sum_{n=-\infty}^{\infty} \alpha^a_{-n} T_a z^n = 1 + \sum_{n=-\infty}^{\infty} \alpha^a_{-n} T^m_a, \quad T^m_a \equiv T_a z^n \quad (4)
\]

where \( T^m_a \) are the generators of the algebra \( \hat{\mathfrak{g}} \equiv \mathfrak{g}(S^1) \). We may now write the commutation relations of the Lie algebra in terms of the generators \( T^m_a \). The commutators of the finite-dimensional \( \mathfrak{g} \) then imply

\[
[T^m_a, T^n_b] = f_{ab}^c T^m_{c+n}. \quad (5)
\]

Eqs. (5) are the defining relations of the loop algebra associated with \( \mathfrak{g} \), that is the algebra \( \hat{\mathfrak{g}} = LG = \text{Map}(S^1, \mathfrak{g}) \) of the loop group \( LG \). The original finite-dimensional Lie algebra \( \mathfrak{g} \) is reproduced by the generators \( T^0_a \); they correspond to the generators of the group of the constant maps \( S^1 \rightarrow G \) which is isomorphic to \( G \). With the previous conventions, \( T^m_a = -T^m_{-a} \) since, \( z \) being of unit modulus, \( z^* = z^{-1} \).

On the other hand, if \( \{ \omega^a(g) \}, a = 1, \cdots, r = \dim G, \) is the basis determined by the (dual, left-invariant) Maurer–Cartan one-forms on \( G \); then, the Maurer-Cartan equations that characterize \( \mathfrak{g} \), in a way dual to its Lie bracket description, are given by

\[
d\omega^c = -\frac{1}{2} C_{ab}^c \omega^a \wedge \omega^b, \quad a, b, c = 1, \cdots, r = \dim G.
\]

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In direct analogy we can say that if \( \{ \omega^{a,n} \}, \; i = 1, ..., r = \text{dim} G, \; n \in \mathbb{Z} \) is the basis determined by the (dual, left-invariant) Maurer–Cartan one-forms on \( LG \); then, the corresponding Maurer-Cartan equations that characterize the algebra \( \hat{G} \), are given by

\[
d\omega^{c,l} = -\frac{1}{2} f_{(a,m)(b,n)}^{(c,l)} \omega^{a,n} \wedge \omega^{b,m}, \quad a, b, c = 1, ..., r = \text{dim} G; \quad l, m, n \in \mathbb{Z}. \tag{6}
\]

\[
d\omega^{c,l} = -\frac{1}{2} \delta_{m+n} f_{ab}^{c} \omega^{a,n} \wedge \omega^{b,m}, \quad a, b, c = 1, ..., r = \text{dim} G; \quad l, m, n \in \mathbb{Z}. \tag{7}
\]

The purpose of this paper is to generalize the expansion procedures developed in ref. [1] so that it permits to study the expansion of the algebras of loops when both the compact finite-dimensional algebra \( G \) and the loop algebra (which is an infinite-dimensional algebra \( \hat{G} \)) have a decomposition into two subspaces \( V_0 \oplus V_1 \).

This article is organized as follow: In section II we consider the rescaling of the group parameters. In section III we study (i) the expansion of the loop algebras when the compact finite-dimensional algebra \( G \) has a decomposition into two subspaces \( G = V_0 \oplus V_1 \) (ii) the conditions under which the expanded algebra closes (iii) the closure of the expanded algebra when \( V_0 \) is a subalgebra. In section IV we study the expansion of the loop algebra (which is an infinite-dimensional algebra \( \hat{G} \)), where this algebra \( \hat{G} \) admits a decomposition \( \hat{G} = V_0 \oplus V_1 \). The expansion of \( \hat{G} = V_0 \oplus V_1 \) when \( \{V_0, V_1\} \) satisfy the condition of symmetric coset is considered in section V. Section VI concludes the work with a brief comment.

II. RESCALING OF THE GROUP PARAMETERS AND THE EXPANSION PROCEDURE

Let \( LG \) be the loop group, of local coordinates \( g^a(z), \; a = 1, ..., r = \text{dim} G \). Let \( \hat{G} \) be its algebra of basis \( \{T^a_n\} \), which may be realized by left-invariant generators \( T^a_n(g) \) on the group manifold. Let \( \hat{G}^* \) be the coalgebra, and let \( \{\omega^{a,n}(g)\}, \; i = 1, ..., r = \text{dim} G, \; n \in \mathbb{Z}, \) be the basis (dual, i.e., \( \omega^{a,n}(T^b_m) = \delta^{a}_m \delta^{n}_b \)) determined by the Maurer-Cartan one-form on \( LG \). Then, when \( [T^m_a, T^n_b] = f_{ab}^{c} T^m_c T^{n+n} \), the Maurer-Cartan equations read

\[
d\omega^{c,l} = -\frac{1}{2} f_{(a,m)(b,n)}^{(c,l)} \omega^{a,n} \wedge \omega^{b,m}, \quad a, b, c = 1, ..., r = \text{dim} G; \quad l, m, n \in \mathbb{Z}. \tag{7}
\]

Let \( \theta \) be the left-invariant canonical form on \( LG \),

\[
\theta(g) = g^{-1} dg = e^{-ig_a^n T^a_n} d e^{ig_a^n T^a_n} \equiv \omega^{a,n} T^a_n, \quad a = 1, ..., r = \text{dim} G; \quad n \in \mathbb{Z} \tag{8}
\]
Since
\[ e^{-A}de^A = dA + \frac{1}{2} [dA, A] + \frac{1}{3!} [[dA, A], A] + \frac{1}{4!} [[[dA, A], A], A] + \cdots \] (9)
one obtains, for \( A = g_{a,n} T^{a,n} \), the expansion of \( \theta (g) \) as polynomials in the group coordinates \( g^{a,n} \):
\[
\theta (g) = e^{-ig_{a,n} T^{a,n}} de^{ig_{a,n} T^{a,n}} \\
= idg_{a_1,n_1} T^{a_1,n_1} + \frac{i^2}{2!} [dg_{a_2,n_2} T^{a_2,n_2}, g_{a_3,n_3} T^{a_3,n_3}] \\
+ \frac{i^3}{3!} [[dg_{a_2,n_2} T^{a_2,n_2}, g_{a_3,n_3} T^{a_3,n_3}], g_{a_4,n_4} T^{a_4,n_4}] \\
+ \frac{i^4}{4!} [[[dg_{a_2,n_2} T^{a_2,n_2}, g_{a_3,n_3} T^{a_3,n_3}], g_{a_4,n_4} T^{a_4,n_4}], g_{a_5,n_5} T^{a_5,n_5}] \\
+ \cdots \cdots 
\] (10)
where the indices \( a_1, a_2, a_3 \cdots = 1, 2, \ldots, \dim \mathcal{G} \), and \( n_i \in \mathbb{Z} \). Factoring the coordinates and their derivatives in the Lie brackets
\[
\theta (g) = idg_{i_1,n_1} T^{i_1,n_1} + \frac{i^2}{2!} dg_{i_2,n_2} g_{i_3,n_3} [ T^{i_2,n_2}, T^{i_3,n_3}] \\
+ \frac{i^3}{3!} dg_{i_2,n_2} g_{i_3,n_3} g_{i_4,n_4} [[ T^{i_2,n_2}, T^{i_3,n_3}], T^{i_4,n_4}] \\
+ \frac{i^4}{4!} dg_{i_2,n_2} g_{i_3,n_3} g_{i_4,n_4} g_{i_5,n_5} [[[ T^{i_2,n_2}, T^{i_3,n_3}], T^{i_4,n_4}], T^{i_5,n_5}] \\
+ \cdots \cdots 
\] (11)
Using the commutation relation (5) we have
\[
[T^{a_2,n_2}, T^{a_3,n_3}] = i f_{h_1}^{a_2,a_3} T^{h_1,n_2+n_3} 
\] (12)
\[
[[T^{a_2,n_2}, T^{a_3,n_3}], T^{a_4,n_4}] = i^2 f_{h_2}^{a_2,a_3} f_{h_1}^{a_4,a_5} T^{h_3,n_2+n_3+n_4} 
\] (13)
\[
[[[T^{i_2,n_2}, T^{i_3,n_3}], T^{i_4,n_4}], T^{i_5,n_5}] = i^3 f_{h_3}^{a_2,a_3} f_{h_2}^{a_4,a_5} f_{h_1}^{a_6,a_7} T^{h_4,n_2+n_3+n_4+n_5} 
\] (14)
so that (11) takes the form
\[
\theta (g) = idg_{a,n} T^{a,n} + \frac{i^3}{2!} dg_{a_2,n_2} g_{a_3,n_3} f_{a}^{a_2,a_3} T^{a_2,n_2+n_3} \\
+ \frac{i^5}{3!} dg_{a_2,n_2} g_{a_3,n_3} g_{a_4,n_4} f_{h_1}^{a_2,a_3} f_{a}^{h_1,a_4} T^{a_2,n_2+n_3+n_4} \\
+ \frac{i^7}{4!} dg_{a_2,n_2} g_{a_3,n_3} g_{a_4,n_4} g_{a_5,n_5} f_{h_1}^{a_2,a_3} f_{h_2}^{h_1,a_4} f_{a}^{h_2,a_5} T^{a_2,n_2+n_3+n_4+n_5} \\
+ \cdots \cdots 
\] (15)
expression that can be rewritten as
\[ \theta (g) = [i d g_{a,n} + \frac{i^3}{2!} \delta^{(n_2+n_3)} dg_{a_2,n_2} g_{a_3,n_3} f^{a_2,a_3}_a + \frac{i^5}{3!} \delta^{(n_2+n_3+n_4)} dg_{a_2,n_2} g_{a_4,n_4} g_{a_5,n_5} f^{a_2,a_3}_a f^{h_1,i_4}_a + \frac{i^7}{4!} \delta^{(n_2+n_3+n_4+n_5)} dg_{a_2,n_2} g_{a_3,n_3} g_{a_4,n_4} g_{a_5,n_5} f^{a_2,a_3}_a f^{h_1,i_4}_a f^{h_2,a_5}_a + \cdots] T^{a,n} \omega_{a,n} \] (16)

Therefore, the Maurer-Cartan 1-forms, \( \omega_{a,n}(g) \), as a polynomial in the coordinates of the group \( g_{a,n} \) is given by
\[ \omega_{a,n} = i d g_{a,n} + \frac{i^3}{2!} \delta^{(n_1+n_2)} dg_{a_1,n_1} g_{a_2,n_2} f^{a_1,a_2}_a + \sum_{2}^{\infty} \frac{i^{2\beta+1}}{(\beta+1)!} \delta^{(n_2+n_3+\cdots+n_{\beta+1})} dg_{a_1,n_1} g_{a_2,n_2} g_{a_{\beta+1},n_{\beta+1}} f^{a_1,a_2}_a f^{h_{1,a_2}_h}_f h_{h_{\beta+1},a_{\beta+1}} f^{h_{\beta-1,a_{\beta-1}}}_a \] (18)

From (18) we can see that the rescaling of some coordinates \( g_{i,a} \)
\[ g_{a,n} \rightarrow \lambda g_{a,n} \] (19)
will generate an expansion of Maurer-Cartan 1-forms \( \omega_{i,n}(g, \lambda) \) as a sum of 1-forms \( \omega_{i,n}(g) \) on \( LG \) multiplied by the corresponding powers of \( \lambda^\alpha \) of \( \lambda \). This means that the expansion (18) exists and can be expressed as
\[ \omega_{i,n} = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega_{i,n;\alpha} \] (20)

It should be noted that in the case \( n = 0 \) and \( n_1 = n_2 = \cdots = n_{\beta+1} = 0 \) the equation (18) takes the form
\[ \omega_{a,0} = [i \delta^{a_1}_a + \frac{i^3}{2!} g_{a_2,0} f^{a_1,a_2}_a + \sum_{2}^{\infty} \frac{i^{2\beta+1}}{(\beta+1)!} g_{a_2,0} g_{a_{\beta+1},0} f^{a_1,a_2}_a f^{h_{1,a_2}_h}_f h_{h_{\beta+1},0} f^{h_{\beta-1,a_{\beta-1}}}_a f^{h_{\beta-2,a_{\beta-2}}}_a f^{h_{\beta-1,a_{\beta-1}}}_a \] (21)
That is, the equation (18) reduces to the equation (2.5) of ref. [1].
III. EXPANSION OF LOOP ALGEBRAS $\hat{G}$ WHEN $G = V_0 \oplus V_1$

In this section we consider the expansion of the loop algebras $\hat{G}$ when the compact finite-dimensional algebra $G$ has a decomposition into two subspaces $G = V_0 \oplus V_1$ (ii) and we study the conditions under which the expanded algebra closes. The case when $V_0$ is a subalgebra is also analyzed.

We consider the splitting of $\hat{G}^*$ into the sum of two vector subspaces

$$\hat{G}^* = V_0^* \oplus V_1^*, \quad (22)$$

$V_0^*$, $V_1^*$ being generated by the Maurer-Cartan forms $\omega^{a_0,n}(g)$, $\omega^{a_1,n}(g)$ of $\hat{G}^*$ with indices corresponding, respectively, to the unmodified and modified parameters,

$$g^{a_0,n} \rightarrow g^{a_0,n}, \quad g^{a_1,n} \rightarrow \lambda g^{a_1,n}, \quad a_0(a_1) = 1, \ldots, \dim V_0 (\dim V_1), \quad n \in \mathbb{Z}. \quad (23)$$

In general, the series of $\omega^{a_0,n}(g, \lambda) \in V_0^*$, $\omega^{a_1,n}(g, \lambda) \in V_1^*$ will involve all powers of $\lambda$

$$\omega^{a_p,n}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{a_p,n;\alpha}(g)$$

$$= \omega^{a_p,n;0}(g) + \lambda \omega^{a_p,n;1}(g) + \lambda^2 \omega^{a_p,n;2}(g) + \ldots, \quad p = 0, 1 \quad (24)$$

where $\omega^{a_p,n}(g, 1) = \omega^{a_p,n}(g)$.

With the above notation, the Maurer-Cartan equations (6) for $\hat{G}$ can be rewritten as

$$d\omega^{c_s,l} = -\frac{1}{2} f^{c_s,l}_{ap,q} \omega^{a_p,n} \omega^{b_q,m} \quad (p,q,s = 0,1) \quad (25)$$

where $a_p, b_q = 1, \ldots, \dim V_0 (\dim V_1); \quad l, n, m \in \mathbb{Z}$ and where

$$\omega^{c_s,l} = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{c_s,l;\alpha} \quad (26)$$

$$\omega^{a_p,n} = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{a_p,n;\alpha} \quad (27)$$

$$\omega^{b_q,m} = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{b_q,m;\alpha}. \quad (28)$$

Introducing into the Maurer-Cartan (25) we have

$$\sum_{\alpha=0}^{\infty} \lambda^\alpha d\omega^{c_s,l;\alpha} = -\frac{1}{2} f^{c_s,l}_{ap,q} \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{a_p,n;\alpha} \sum_{\beta=0}^{\infty} \lambda^\beta \omega^{b_q,m;\beta} \quad (29)$$
and, using the eq. (A.1) from Ref. [1], the Maurer-Cartan equations are expanded in powers of $\lambda$:

\[
\sum_{\alpha=0}^{\infty} a^{\alpha} d\omega^{e_s;l;\alpha} = -\frac{1}{2} c_{a_p,n}^{e_s;l} \sum_{\alpha=0}^{\infty} \lambda^\alpha \sum_{\beta=0}^{\infty} \omega^{a_p,n;\beta} \omega^{a_q,m;\alpha-\beta} \quad (30)
\]

\[
= \sum_{\alpha=0}^{\infty} \lambda^\alpha \left( -\frac{1}{2} c_{a_p,n}^{e_s;l} \sum_{\alpha=0}^{\infty} \omega^{a_p,n;\beta} \omega^{a_q,m;\alpha-\beta} \right).
\]

The equality of the two $\lambda$-polynomials in (30) requires the equality of the coefficients of equal power $\lambda^\alpha$. This implies that the coefficients one-forms $\omega^{a_p,m;\alpha}$ satisfy the identities

\[
d\omega^{e_s;l;\alpha} = -\frac{1}{2} c_{a_p,n}^{e_s;l} b_{q,m} \sum_{\beta=0}^{\infty} \omega^{a_p,n;\beta} \omega^{b_q,m;\alpha-\beta},
\]

where $p, q, s = 0, 1; a_p, b_q = 1, \ldots, \dim V_0 (\dim V_1); l, n, m \in \mathbb{Z}$.

We can rewrite (31) in the form

\[
d\omega^{e_s;l;\alpha} = -\frac{1}{2} C_{(a_p,n;\beta)}^{(e_s,l;\alpha)} (b_{q,m;\gamma}) \omega^{a_p,n;\beta} \omega^{b_q,m;\gamma}
\]

\[
C_{(a_p,n;\beta)}^{(e_s,l;\alpha)} (b_{q,m;\gamma}) = \delta^\alpha_{\beta+\gamma} c_{a_p,n}^{e_s;l} b_{q,m} \quad (33)
\]

that is,

\[
C_{(a_p,n;\beta)}^{(e_s,l;\alpha)} (b_{q,m;\gamma}) = \begin{cases} 0 & \text{if } \beta + \gamma \neq \alpha \\ c_{a_p,n}^{e_s;l} b_{q,m} & \text{if } \beta + \gamma = \alpha \end{cases} \quad (34)
\]

where $a_p, b_q, c_s: 1, 2, \ldots, \dim G, l, n, m \in \mathbb{Z}$ and $\alpha, \beta: 0, 1, 2, \ldots$.

Now we ask, under which conditions the 1-forms $\omega^{c_0;0;\alpha_0}, \omega^{c_1;l;\alpha_1}$ generate new infinite dimensional algebras. The answer is given by the following analysis: consider the one-forms

\[
\{ \omega^{a_0,l;\alpha_0}, \omega^{a_1,l;\alpha_1} \} = \{ \omega^{a_0,l;0}, \omega^{a_0,l;1}, \ldots, \omega^{a_0,l;N_0}, \omega^{a_1,l;0}, \omega^{a_1,l;1}, \ldots, \omega^{a_1,l;N_1} \}
\]

with $\alpha_0 = 0, \ldots, N_0, \alpha_1 = 0, \ldots, N_1, l \in \mathbb{Z}$. The conditions under which these forms generate new algebras are found by demanding that the algebra generated by eq. (35) is closed under the exterior derivative $d$ and that the Jacobi identities for the new algebra are satisfied.

In fact, to find the conditions under which the algebra is closed, we write:

\[
d\omega^{c_0,l;\alpha_0} = -\frac{1}{2} c_{a_0,n}^{c_0,l} b_{q,m} \sum_{\beta=0}^{\infty} \omega^{a_0,n;\beta} \omega^{b_0,m;\alpha-\beta}
\]

\[
= -\frac{1}{2} c_{a_0,n}^{c_0,l} b_{q,m} \sum_{\beta=0}^{\infty} \omega^{a_0,n;\beta} \omega^{b_0,m;\alpha-\beta} - \frac{1}{2} c_{a_0,n}^{c_0,l} b_{l,m} \sum_{\beta=0}^{\infty} \omega^{a_0,n;\beta} \omega^{b_1,m;\alpha-\beta}
\]
\[- \frac{1}{2} c_{a_1,n}^{c_{t,l}} b_{0,m} \sum_{\beta=0}^{\alpha} \omega^{a_1,n;\beta} b_{0,m;\alpha-\beta} - \frac{1}{2} c_{a_1,n}^{c_{t,l}} b_{1,m} \sum_{\beta=0}^{\alpha} \omega^{a_1,n;\beta} b_{1,m;\alpha-\beta} \]  

(36)

which implies that

\[
d \omega^{c_0,l;N_0} = - \frac{1}{2} c_{a_0,n}^{c_{t,l}} b_{0,m} \left[ \omega^{a_0,n;0} b_{0,m;N_0} + \ldots + \omega^{a_0,n;N_0} b_{0,m;0} \right] 
- \frac{1}{2} c_{a_0,n}^{c_{t,l}} b_{1,m} \left[ \omega^{a_0,n;0} b_{1,m;N_0} + \ldots + \omega^{a_0,n;N_0} b_{1,m;0} \right] 
- \frac{1}{2} c_{a_1,n}^{c_{t,l}} b_{0,m} \left[ \omega^{a_1,n;0} b_{0,m;N_0} + \ldots + \omega^{a_1,n;N_0} b_{0,m;0} \right] 
- \frac{1}{2} c_{a_1,n}^{c_{t,l}} b_{1,m} \left[ \omega^{a_1,n;0} b_{1,m;N_0} + \ldots + \omega^{a_1,n;N_0} b_{1,m;0} \right].
\]  

(37)

\[
d \omega^{c_1,l;N_1} = - \frac{1}{2} c_{a_0,n}^{c_{t,l}} b_{0,m} \left[ \omega^{a_0,n;0} b_{0,m;N_1} + \ldots + \omega^{a_0,n;N_1} b_{0,m;0} \right] 
- \frac{1}{2} c_{a_0,n}^{c_{t,l}} b_{1,m} \left[ \omega^{a_0,n;0} b_{1,m;N_1} + \ldots + \omega^{a_0,n;N_1} b_{1,m;0} \right] 
- \frac{1}{2} c_{a_1,n}^{c_{t,l}} b_{0,m} \left[ \omega^{a_1,n;0} b_{0,m;N_1} + \ldots + \omega^{a_1,n;N_1} b_{0,m;0} \right] 
- \frac{1}{2} c_{a_1,n}^{c_{t,l}} b_{1,m} \left[ \omega^{a_1,n;0} b_{1,m;N_1} + \ldots + \omega^{a_1,n;N_1} b_{1,m;0} \right].
\]  

(38)

Wherefrom we can see that the 1-forms \( b_{1;m;N_0} \) and \( b_{1;m;N_0} \), corresponding to the terms identified by the symbols \((i), (ii), (iii)\) and \((iv)\) in the equation \(37\), belong to the base \((35)\) if and only if

\[ N_0 \leq N_1. \]  

(39)

On the other hand, the 1-forms \( b_{0;m;N_1} \) and \( b_{0;m;N_1} \), corresponding to the terms identified by the symbols \((v), (vi), (vii)\) and \((viii)\) in the equation \((38)\), belong to the base \((35)\) if and only if

\[ N_1 \leq N_0. \]  

(40)

From \((39-40)\) it follows trivially that the conditions under which the expanded algebra closes is

\[ N_0 = N_1. \]  

(41)
Let $\mathcal{G} = V_0 \oplus V_1$, where now $V_0$ is a subalgebra $L_0$ of $\mathcal{G}$. From the commutation relation

$$[T_{a,n}, T_{b,m}] = f_{ab}^{c} T_{c,n+m} = f_{a,n}^{c,l} b_{m} X_{c,l}$$

(42)

$a_p, b_q = 1, \ldots, \dim V_0 (\dim V_1)$; $l, n, m \in \mathbb{Z}$. From (42) we can see that $L_0 = \{T_{a,0}\}$ generates a subalgebra given by

$$[T_{a,0}, T_{b,0}] = f_{ab}^{c} X_{c,0} = f_{a,0}^{c,l} b_{0} T_{c,0}.$$  (43)

From (43) we see that

$$f_{a,0}^{c,l} b_{0} = c_{ab}^{e} \delta_{0}^{n} = 0, \text{ para } n \neq 0, \ n \in \mathbb{Z}.\quad (44)$$

Using (44) in the expansion

$$\omega^{a,n}(g) = [\delta^{a,n}_{(b,m)} + \frac{1}{2!} f_{b,n}^{a,0} c_{m} g^{c,m} + o(g^2)] dg^{a,n}$$

(45)

we find that under the rescaling

$$g^{a,0} \rightarrow g^{a,n}, \ g^{a,n} \rightarrow \lambda g^{a,n} (n \neq 0),$$

$$a, (0) = 1, \ldots, \dim V_0$$

$$a, n = 1, \ldots, \dim V_1.$$

$$V_1 = \{T_{a,n}\} \text{ with } n \neq 0\quad (46)$$

the expansion of $\omega^{a,0}(g, \lambda)$ ($\omega^{a,n}(g, \lambda)$ with $n \neq 0$) starts with the power $\lambda^0$ ($\lambda^1$). In fact, for $\omega^{a,0}(g)$ we have

$$\omega^{a,0}(g) = \left[\delta^{a,0}_{(b,n)} + \frac{1}{2!} f_{b,n}^{a,0} c_{m} g^{c,m} + o(g^2)\right] dg^{a,n}$$

$$= dg^{a,0} + \frac{1}{2!} f_{b,n}^{a,0} c_{m} g^{c,m} dg^{a,n} + o(g^3)$$

$$= dg^{a,0} + \frac{1}{2!} \left( f_{b,0}^{a,0} c_{0} g^{c,0} dg^{b,0} + f_{b,0}^{a,0} c_{n} g^{c,n} dg^{b,0} \right)$$

$$+ \frac{1}{2!} \left( f_{b,n}^{a,0} c_{0} g^{c,0} dg^{b,n} + f_{b,n}^{a,0} c_{m} g^{c,m} dg^{b,n} \right) + o(g^3)\quad (47)$$
which implies that under the rescaling $g^{a,0} \rightarrow g^{a,0}$, $g^{a,n} \rightarrow \lambda g^{a,n}$ ($n \neq 0$),

$$\omega^{a,0}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{a,0;\alpha}(g) \quad (48)$$

while for $\omega^{a,l}(g)$, with $l \neq 0$, we have

$$\omega^{a,l}(g) = \left[ \delta^{(a,l)}_{(b,n)} + \frac{1}{2!} f^{a,l}_{b,n} c_m g^{c,m} + o(g^2) \right] dg^{b,n} \quad (49)$$

Therefore the expansion of $\omega^{a,l}(g, \lambda)$ starts with the power $\lambda^1$

$$\omega^{a,n}(g, \lambda) = \sum_{\alpha=1}^{\infty} \lambda^{\alpha} \omega^{a,n;\alpha}(g) \quad (50)$$

However, for computation purposes it is better to spread the sum from zero and assume that $\omega^{a,n;0} = 0$ for $n \neq 0$. Thus we have that Eqs. (48-50) can be summarized as:

$$\omega^{a,n}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{a,n;\alpha}(g) \quad (51)$$

$$\omega^{a,n;0} = 0 \text{ for } n \neq 0.$$  

Inserting (51) into the Maurer-Cartan equations $d\omega^{c,l} = -\frac{1}{2} f^{c,l}_{a,n} b_m \omega^{a,n} \omega^{b,m}$, we have

$$\sum_{\alpha=0}^{\infty} \lambda^{\alpha} d\omega^{c,l;\alpha} = -\frac{1}{2} f^{c,l}_{a,n} b_m \left( \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{a,n;\alpha} \right) \left( \sum_{\beta=0}^{\infty} \lambda^{\beta} \omega^{b,m;\beta} \right) \quad (52)$$

$$= -\frac{1}{2} f^{c,l}_{a,n} b_m \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \sum_{\beta=0}^{\alpha} \omega^{a,n;\beta} \omega^{b,m;\alpha-\beta}$$

$$= \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \left( -\frac{1}{2} f^{c,l}_{a,n} b_m \sum_{\beta=0}^{\alpha} \omega^{a,n;\beta} \omega^{b,m;\alpha-\beta} \right).$$

The equality of the coefficients of equal power $\lambda^{\alpha}$ leads to the equation

$$d\omega^{c,l;\alpha} = -\frac{1}{2} f^{c,l}_{a,n} b_m \sum_{\beta=0}^{\alpha} \omega^{a,n;\beta} \omega^{b,m;\alpha-\beta} \quad (53)$$

$$= -\frac{1}{2} \delta^{l}_{n+m} f^{c}_{a,b} \sum_{\beta=0}^{\alpha} \omega^{a,n;\beta} \omega^{a,m;\alpha-\beta}$$
which can be rewritten as

\[ d\omega^{c,l;\alpha} = -\frac{1}{2} f^{(c,l;\alpha)}_{(a,n;\beta)} (b,m;\gamma) \omega^{a,n;\beta} \omega^{b,m;\gamma} \]  

(54)

where

\[ f^{(c,l;\alpha)}_{(a,n;\beta)} (b,m;\gamma) = \delta^{\alpha}_{\beta+\gamma} f^{c,l}_{a,n} b,m \delta^{\alpha}_{\beta} \delta^{l}_{n+m} f^{c}_{ab} \]  

(55)

\[ \omega^{a,n;0} = 0 \text{ for } n \neq 0. \]

A. Analysis of \( \hat{G}(N) \) for the cases \( N = 0,1, \ldots \)

Consider the form of equations (53).

For \( \alpha = 0 \) we find:

\[ d\omega^{c,l;0} = -\frac{1}{2} \delta^{l}_{n+m} f^{c}_{ab} \omega^{a,n;0} \omega^{b,m;0} \]  

(56)

but \( \omega^{a,n;0} = 0 \) for \( n \neq 0 \), we have

\[ d\omega^{c,0;0} = -\frac{1}{2} f^{c}_{ab} \omega^{a,0;0} \omega^{b,0;0}. \]  

(57)

For \( \alpha = 1 \) we find:

\[ d\omega^{c,l;1} = -\frac{1}{2} \delta^{l}_{n+m} f^{c}_{ab} \sum_{\beta=0}^{1} \omega^{a,n;\beta} \omega^{b,m;1-\beta} \]  

\[ = -\frac{1}{2} \delta^{l}_{n+m} f^{c}_{ab} \omega^{a,n;0} \omega^{b,m;1} - \frac{1}{2} \delta^{l}_{n+m} f^{c}_{ab} \omega^{a,n;1} \omega^{b,m;0} \]

\[ = -\frac{1}{2} \delta^{l}_{n+m} f^{c}_{ab} \omega^{a,0;0} \omega^{b,l;1} - \frac{1}{2} \delta^{l}_{n+m} f^{c}_{ab} \omega^{a,1;0} \omega^{b,l;0} \]

\[ = -\frac{1}{2} f^{c}_{ab} \omega^{a,0;0} \omega^{b,l;1} - \frac{1}{2} f^{c}_{ab} \omega^{a,1;0} \omega^{b,l;0} \]

\[ = -\frac{1}{2} f^{c}_{ab} \omega^{a,0;0} \omega^{b,l;1} + \frac{1}{2} f^{c}_{ab} \omega^{a,1;0} \omega^{b,l;0} \]

\[ = -\frac{1}{2} f^{c}_{ab} \omega^{a,0;0} \omega^{b,l;1} - \frac{1}{2} f^{c}_{ab} \omega^{a,0;0} \omega^{b,l;1} \]

\[ = -f^{c}_{ab} \omega^{a,0;0} \omega^{b,l;1} \]

\[ d\omega^{c,l;1} = -f^{c}_{ab} \omega^{a,0;0} \omega^{b,l;1}. \]  

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In summary

\[ \alpha = 0 : \quad d\omega^{c,0;0} = -\frac{1}{2} f_{ab} \omega^{a,0;0} \omega^{b,0;0}; \quad (59) \]
\[ \alpha = 1 : \quad d\omega^{c,n;1} = -f_{ab} \omega^{a,0;0} \omega^{b,n;1}; \]
\[ \alpha \geq 2 : \quad d\omega^{c,l;\alpha} = -\frac{1}{2} \delta_{l}^{n+m} f_{ab} \sum_{\beta=0}^{\alpha} \omega^{a,n;\beta} \omega^{b,m;\alpha-\beta}. \]

so that \( \hat{\mathcal{G}}(0) \) is given by

\[ d\omega^{c,0;0} = -\frac{1}{2} f_{ab} \omega^{a,0;0} \omega^{b,0;0}; \quad (60) \]

and \( \hat{\mathcal{G}}(1) \) is given by

\[ d\omega^{c,0;0} = -\frac{1}{2} f_{ab} \omega^{a,0;0} \omega^{b,0;0}; \quad (61) \]
\[ d\omega^{c,n;1} = -f_{ab} \omega^{a,0;0} \omega^{b,n;1}. \]

From the first equation we can see a non-trivial result: while for a finite-dimensional Lie algebra \( \mathcal{G}(0) = \mathcal{G} \), for the loop algebra \( \hat{\mathcal{G}}(0) \neq \hat{\mathcal{G}} \) but \( \hat{\mathcal{G}}(0) = \hat{\mathcal{G}} \) where \( \mathcal{G} \) is the compact Lie algebra.

V. THE CASE \( \hat{\mathcal{G}} = V_0 \oplus V_1 \) IN WHICH \( V_1 \) IS A SYMMETRIC COSET

It is possible to consider the infinite-dimensional algebra as \( \hat{\mathcal{G}} = V_0 \oplus V_1 \) where \( V_0 \) is generated by the infinite set of generators given by

\[ \{ ..., T_{a,-4}, T_{a,-2}, T_{a,0}, T_{a,2}, T_{a,4} ... \} \quad (62) \]

and where \( V_1 \) is generated by

\[ \{ ..., T_{a,-3}, T_{a,-1}, T_{a,1}, T_{a,3} ... \}. \quad (63) \]

From the commutation relation

\[ [T_{a,n}, T_{b,m}] = f_{ab}^{c} T_{c,n+m} \quad (64) \]

we clearly see that the condition for a symmetric coset is to satisfy:

\[ [V_0, V_0] \subset V_0 \quad (65) \]
\[ [V_0, V_1] \subset V_1 \]
\[ [V_1, V_1] \subset V_0. \]
It is therefore interesting to study the expansion of the infinite-dimensional algebra expanded with this choice of $V_0$ and $V_1$. For convenience we distinguish the generators $T_{a,n}$ where the index $n$ is even from the case when the index is odd. The most natural choice is to use a subscript zero (one), $n_0$ ($n_1$), for even values (odd). Thus (62-64) take the form:

$$\{T_{a,n_0}\} = \{\ldots, T_{a,-4}, T_{a,-2}, T_{a,0}, T_{a,2}, T_{a,4} \ldots\} \quad (66)$$

$$\{T_{a,n_1}\} = \{\ldots, T_{a,-3}, T_{a,-1}, T_{a,1}, T_{a,3} \ldots\} \quad (67)$$

$$[T_{a,n_0}, T_{b,m_0}] = f^{c}_{ab} T_{c,n_0+m_0} = f^{c}_{a,n_0} b,m_0 T_{c,l_0} \quad (68)$$

$$[T_{a,n_0}, T_{b,m_1}] = f^{c}_{ab} T_{c,n_0+m_1} = f^{c}_{a,n_0} b,m_1 T_{c,l_1}$$

$$[T_{a,n_1}, T_{b,m_1}] = f^{c}_{ab} T_{c,n_1+m_1} = f^{c}_{a,n_1} b,m_1 T_{c,l_0}$$

From where we see that the conditions of symmetric cosets for the structure constants are given by

$$f^{c}_{a,n_0} b,m_0 = f^{c}_{a,n_0} b,m_1 = f^{c}_{a,n_1} b,m_1 = 0 \quad (69)$$

The idea is: (a) to find the expansions of $\omega^{i,n_0}(g, \lambda)$ and $\omega^{i,n_1}(g, \lambda)$; (b) to replace the expansions in the Maurer-Cartan equations and (c) to find the conditions under which are generated new algebras.

To find the expansions of $\omega^{a,n_0}(g, \lambda)$ and $\omega^{a,n_1}(g, \lambda)$ we must study the general expansion of $\omega^{a,n_0}(g)$ and $\omega^{a,n_1}(g)$ in terms of the coordinates and then analyze the behavior under the following rescaling:

$$g^{a,n_0} \rightarrow g^{a,n_0}, \quad g^{a,n_1} \rightarrow \lambda g^{a,n_1} \quad (70)$$

$$n_0 = \ldots, -4, -2, 0, 2, 4, \ldots$$

$$n_1 = \ldots, -3, -1, 1, 3, \ldots$$

For $\omega^{a,n_0}(g)$ we find

$$\omega^{a,n_0}(g) = \left[ \delta^{(a,n_0)}_{(b,m)} + \frac{1}{2!} f^{a,n_0}_{b,m} g^{c,l} + o(g^3) \right] d g^{b,m} \quad (71)$$

$$= \delta^{(a,n_0)}_{(b,m)} d g^{b,m} + \frac{1}{2!} f^{a,n_0}_{b,m} c,l g^{c,l} d g^{b,m} + o(g^3)$$

$$= d g^{b,n_0} + \frac{1}{2!} f^{a,n_0}_{b,m} c,l g^{c,l} d g^{b,m} + o(g^3)$$

$$= d g^{b,n_0} + \frac{1}{2!} f^{a,n_0}_{b,m} c,l_0 g^{c,l_0} d g^{b,m_0} + \frac{1}{2!} f^{a,n_0}_{b,m_1} c,l_1 g^{c,l_1} d g^{b,m_1} + o(g^3).$$
Analyzing higher order terms we find that if you rescale the parameters as in (70), then \( \omega^{a,n_0}(g, \lambda) \) contains only even powers of \( \lambda \). The proof is a direct generalization of the procedure used in ref. [1]. For this it is useful to write the condition (69) as

\[
 f_{a,n}^{c,l} b,m_q = 0, \text{ for } s \neq (p + q) \mod 2. \tag{72}
\]

Performing the same procedure for \( \omega^{a,n_1}(g, \lambda) \) we find that appear in the expansion only odd powers of \( \lambda \). Thus we have

\[
 \omega^{a,n_0}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{2\alpha} \omega^{a,n_0;2\alpha}(g) \tag{73}
\]

\[
 \omega^{a,n_1}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{2\alpha+1} \omega^{a,n_1;2\alpha+1}(g) \tag{74}
\]

which can be written as

\[
 \omega^{a,n_p}(g, \lambda) = \omega^{a,n_\bar{\alpha}}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\bar{\alpha}} \omega^{a,n_\bar{\alpha};\alpha}(g); \tag{77}
\]

\[
 \bar{\alpha} = \alpha \mod 2, \quad p = 0, 1.
\]

Replacing (74) in the Maurer-Cartan equations, we obtain the following set of equations:

\[
 d\omega^{c,l;\alpha} = -\frac{1}{2} f^{c,l;\alpha} (a,n;\beta) (b,m;\gamma) \omega^{a,n;\beta} b,m;\gamma \tag{75}
\]

where

\[
 f^{c,l;\alpha} (a,n;\beta) (b,m;\gamma) = f_{a,n}^{c,l} b,m; \delta^{\alpha}_{\beta+\gamma} \tag{76}
\]

\[
 \bar{\alpha} = \alpha \mod 2, \quad \bar{\beta} = \beta \mod 2, \quad \bar{\gamma} = \gamma \mod 2.
\]

Performing the same procedure developed in ref. [1], we find that the expanded algebra (74) closes when the coefficients of the expansion are truncated at orders that satisfy the conditions

\[
 N_1 = N_0 - 1, \quad \text{or} \tag{77}
\]

\[
 N_1 = N_0 + 1.
\]

Now we consider some examples:
1. *The case in which \( N_1 = 0, \hat{\mathcal{G}} (0, 0) :*

If \( N_1 = 0 \) we have the trivial case \( \hat{\mathcal{G}} (0, 0) = \hat{\mathcal{G}} (0) : \)

\[
d\omega^{c, l_0; 0} = -\frac{1}{2} f^{(c, l_0; 0)}(a, m_0; 0) \omega^{a, n_0; 0} \omega^{b, m_0; 0}
\]

which can be written as

\[
d\omega^{c, l_0; 0} = -\frac{1}{2} f^{c, l_0}_{a, n_0} b, m_0 \omega^{a, n_0; 0} \omega^{b, m_0; 0}.
\] (78)

This means that, \( \hat{\mathcal{G}} (0, 0) \) is the subalgebra \( \mathcal{L}_0 = \{ T_{a, n_0} \} \) of the original infinite-dimensional algebra \( \hat{\mathcal{G}}. \)

2. *The case in which \( \hat{\mathcal{G}} (0, 1) \) is obtained as an Inönü-Wigner contraction of \( \hat{\mathcal{G}} :*

Consider now the case \( \hat{\mathcal{G}} (0, 1) \)

\[
d\omega^{c, l_0; 0} = -\frac{1}{2} f^{c, l_0}_{a, n_0} b, m_0 \omega^{a, n_0; 0} \omega^{b, m_0; 0}
\] (79)

\[
d\omega^{c, l_1; 1} = -\frac{1}{2} f^{(c, l_1; 1)}(a, n_1; \gamma) d\omega^{a, n_1; \gamma} b, m_1; \gamma
\]

\[
= -\frac{1}{2} f^{(c, l_1; 1)}(a, n_0; 0)(b, m_1; 1) \omega^{a, n_0; 0} \omega^{b, m_1; 1} + f^{(c, l_1; 1)}(a, n_1; 1)(b, m_0; 0) \omega^{a, n_1; 1} \omega^{b, m_0; 0}
\]

\[
= -f^{c, l_1}_{a, n_0} b, m_1 \omega^{a, n_0; 0} \omega^{b, m_1; 1}.
\] (80)

This means that \( \hat{\mathcal{G}} (0, 1) \) is given by

\[
d\omega^{c, l_0; 0} = -\frac{1}{2} f^{c, l_0}_{a, n_0} j, m_0 \omega^{a, n_0; 0} \omega^{b, m_0; 0}
\] (81)

\[
d\omega^{c, l_1; 1} = -f^{c, l_1}_{a, n_0} b, m_1 \omega^{a, n_0; 0} \omega^{b, m_1; 1}
\] (82)

i.e. \( \hat{\mathcal{G}} (0, 1) \) corresponds to the Inönü-Wigner contraction of \( \hat{\mathcal{G}} \) with respect to \( \mathcal{L}_0 = \{ T_{a, n_0} \} \):

In fact, consider the Inönü-Wigner contraction of

\[
[T_{a, n_0}, T_{b, m_0}] = f^{c, l_0}_{a, n_0} T_{c, n_0 + m_0} = f^{c, l_0}_{a, n_0} b, m_0 T_{c, l_0}
\] (83)

\[
[T_{a, n_0}, T_{b, m_1}] = f^{c, l_1}_{a, n_0} T_{c, n_0 + m_1} = f^{c, l_1}_{a, n_0} b, m_1 T_{c, l_1}
\] (84)

\[
[T_{a, n_1}, T_{b, m_1}] = f^{c, l_1}_{a, n_1} T_{c, n_1 + m_1} = f^{c, l_1}_{a, n_1} b, m_1 T_{c, l_1}.
\] (85)
Rescaling the generators of the coset space \( \hat{G}/\mathcal{L}_0 \): \( T_{a,n_0} = Y_{a,n_0} \), \( T_{a,n_1} = \lambda Y_{a,n_1} \), we have

\[
[Y_{a,n_0}, Y_{b,m_0}] = f^{c}_{ab} Y_{c,n_0+m_0} = f_{a,n_0}^{c,l_0} Y_{c,l_0}
\]

(86)

\[
[Y_{a,n_0}, Y_{b,m_1}] = f^{c}_{ab} Y_{c,n_0+m_1} = f_{a,n_0}^{c,l_1} Y_{c,l_1}
\]

(87)

\[
[Y_{a,n_1}, Y_{b,m_1}] = \lambda^{-2} f^{c}_{ab} Y_{c,n_1+m_1} = \lambda^{-2} f_{a,n_1}^{c,l_0} Y_{c,l_0}.
\]

(88)

Taking the limit \( \lambda \to \infty \) one finds

\[
[Y_{a,n_0}, Y_{b,m_0}] = f^{c}_{ab} Y_{c,n_0+m_0} = f_{a,n_0}^{c,l_0} Y_{c,l_0}
\]

(89)

\[
[Y_{a,n_0}, Y_{b,m_1}] = f^{c}_{ab} Y_{c,n_0+m_1} = f_{a,n_0}^{c,l_1} Y_{c,l_1}
\]

(90)

\[
[Y_{a,n_1}, Y_{b,m_1}] = 0.
\]

(91)

That is, the unique structure constants that are nonzero are \( f_{a,n_0}^{c,l_0} \) \( b,m_0 \) and \( f_{a,n_0}^{c,l_1} \) \( b,m_1 \).

This means that the equations

\[
d\omega^{c,l_0;0} = -\frac{1}{2} f_{a,n_0}^{c,l_0} b,m_0 \omega^{a,n_0;0} \omega^{b,m_0;0}
\]

(92)

\[
d\omega^{c,l_1;1} = -f_{a,n_0}^{c,l_1} b,m_1 \omega^{a,n_0;0} \omega^{b,m_1;1}
\]

(93)

correspond to the Inönü-Wigner contraction of \( \hat{G} \) with respect to \( \mathcal{L}_0 = \{T_{a,n_0}\} \). Notice that the odd sector of the \( \hat{G} \) algebra becomes abelian after contraction.

3. The Case \( \hat{G}(2,1) \)

In this case we have,

\[
d\omega^{a,l_0;0} = -\frac{1}{2} f_{a,n_0}^{a,l_0} b,m_0 \omega^{a,n_0;0} \omega^{b,m_0;0}
\]

(94)

\[
d\omega^{c,l_1;1} = -f_{a,n_0}^{c,l_1} b,m_1 \omega^{a,n_0;0} \omega^{b,m_1;1}
\]

(95)

\[
d\omega^{c,l_0;2} = -\frac{1}{2} f^{(c,l_0;2)}_{(a,n_0;0)(b,m_0;0)} \omega^{a,n_0;0} \omega^{b,m_0;2} + f^{(c,l_0;2)}_{(a,n_0;0)(b,m_0;0)} \omega^{a,n_0;2} \omega^{b,m_0;0} + f^{(c,l_0;2)}_{(a,n_1;0)(b,m_1;1)} \omega^{a,n_1;1} \omega^{b,m_1;1}
\]

(96)
Thus $\hat{G}(2,1)$ is given by

\begin{align}
\delta\omega^{c,0;0}_{a,n_0} &= -\frac{1}{2} f_{a,n_0}^{c,l_0} b_{m_0} \omega^{a,n_0;0}_b \omega^{b,m_0;0} \\
\delta\omega^{c,1;1}_{a,n_1} &= -f_{a,n_0}^{c,l_1} b_{m_1} \omega^{a,n_0;0}_b \omega^{b,m_1;1} \\
\delta\omega^{c,2;0}_{a,n_0} &= -f_{a,n_0}^{c,l_0} b_{m_0} \omega^{a,n_0;0}_b \omega^{b,m_0;0} - \frac{1}{2} f_{a,n_1}^{c,l_0} b_{m_1} \omega^{a,n_1;1}_b \omega^{b,m_1;1}.
\end{align}

and is generated by

\begin{align}
\{ \omega^{a,n_0;0}_a, \omega^{a,n_1;1}_a, \omega^{a,n_0;2}_a \}
\end{align}

\begin{align}
n_0 &= \ldots, -4, -2, 0, 2, 4, \\
n_1 &= \ldots, -3, -1, 1, 3, 
\end{align}

VI. COMMENT

We have shown that the expansion methods developed in refs. [1] (see also [5], [6]) can be generalized so that they permit to study the expansion of the algebras of loops both when the compact finite-dimensional algebra $G$ and the loop algebra (which is an infinite-dimensional algebra $\hat{G}$) have a decomposition into two subspaces $V_0 \oplus V_1$.

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