Bulk and Boundary Dynamics in BTZ Black Holes

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**Abstract:**

Recently, the AdS/CFT conjecture of Maldacena has been investigated in Lorentzian signature by Balasubramanian *et al.* We extend this investigation to Lorentzian BTZ black hole spacetimes, and study the bulk and boundary behaviour of massive scalar fields both in the non-extremal and extremal case. Using the bulk-boundary correspondence, we also evaluate the two-point correlator of operators coupling to the scalar field at the boundary of the spacetime, and find that it satisfies thermal periodic boundary conditions relevant to the Hawking temperature of the BTZ black hole.

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1 Introduction

There has been much enthusiasm on the equivalence of string theory on $\text{AdS}_{d+1} \times \mathcal{M}$ spacetimes and conformal field theory on the boundary of the anti-de Sitter space. The equivalence was conjectured by Maldacena [1], and further developed for concrete computational tests by Gubser, Klebanov and Polyakov [3] and by Witten [3]. Two significant aspects of the conjecture were quickly recognized and lot of activity has happened around these themes. On the one hand, one can use the bulk theory (string theory or supergravity) to study various topics in the boundary theory, such as correlation functions [3, 4, 5], potentials between charged objects [5] and so on. On the other hand, the holographic aspect of the conjecture may realize the vision in the original holography conjecture of t’Hooft and Susskind [6, 7, 8]; that the holographic projection to the boundary of the spacetime keeps track of all information which falls into a black hole in the bulk and is thus never lost.

The latter aspect has been studied by Maldacena and Strominger [11, 12], who studied BTZ black holes [13], which have an asymptotic (local) anti-de Sitter geometry. The BTZ black hole is constructed from the AdS space by periodic identifications. It can be viewed as a Lorentzian orbifold, and as a simple example of a more general class of Lorentzian orbifolds constructed from AdS space, recently investigated in [14]. Strominger showed [11] how one can derive the Bekenstein entropy of the BTZ black hole (see also the work of Birmingham et. al. [15]) and, together with Maldacena [12], also investigated other aspects relevant to the information problem. In particular, they argued that the conformal field theory at the boundary has a natural vacuum state analogous to the Minkowski vacuum in Rindler space, which in the BTZ coordinates (analogous to the Rindler coordinates) appears as a thermal multiparticle state. These kind of results are necessary for understanding the black hole evaporation process purely in terms of the boundary theory. We shall perform a related calculation in this paper: we evaluate the two-point function of boundary operators in the BTZ coordinates, and show that it satisfies thermal periodicity conditions relevant to the Hawking temperature of the BTZ black hole.

Despite this progress, a much more detailed understanding of the conjecture and the equivalence between bulk states and boundary states is needed before the information problem can fully be understood. One step which would be helpful for the information problem would be a Hamiltonian version of Maldacena’s conjecture, which would allow one to study dynamical aspects such as the time evolution of states. Originally, the conjecture has been conjectured by Maldacena [1]. There are also other black holes which have an asymptotic local anti-de Sitter geometry, e.g. the Schwarzschild black hole in AdS space [9] which is relevant for Yang-Mills theories at finite temperature [8, 10].
was formulated using a Euclidian signature, however a Lorentz signature version is needed for the Hamiltonian framework. Recently, the conjecture was studied in Lorentz signature by Balasubramanian et. al. [16], who laid out the general framework for the mapping between the bulk field modes and boundary states. In Lorentz signature, the former are divided in two classes of modes. The non-normalizable modes act as fixed backgrounds, and their boundary values act as sources for operators in the boundary theory. Turning on a background non-normalizable mode thus changes the Hamiltonian of the boundary theory. On the other hand, normalizable modes are quantized in the bulk, and correspond to states in the Hilbert space of a given boundary Hamiltonian.

In [16], the bulk-boundary correspondence was studied in AdS using two different coordinate systems. It would be desirable to extend the work for black holes. In this paper, we extend the approach of [16] to 2+1 dimensional BTZ black holes. In particular, we study bulk scalar field modes in BTZ coordinates. The BTZ coordinates are analogous to the Schwarzschild coordinates for “ordinary” black holes. They cover the region outside the (outer) horizon. Schwarzschild modes play a central role in studies of dynamical black holes: the quantum state of the black hole (the Kruskal vacuum) appears to be thermally excited in the Schwarzschild coordinates, with temperature \( T_H \). Thus it is an important question to understand how these modes are understood from the boundary theory point of view. One immediate observation is that BTZ black holes lose one powerful feature of AdS spaces. The periodic identifications break the global \( SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \) isometry group of the \( AdS_3 \) space down to a \( \mathbb{R} \times SO(2) \) subgroup. Thus, we lose the power of the \( SL(2,\mathbb{R}) \) representation theory in identifying bulk modes with boundary states. The reduced global isometry group is something to keep in mind when investigating BTZ black holes: although the spacetime is locally AdS, some aspects of AdS spaces do not apply.

The plan of the paper is as follows. In section 2 we quickly review the key features and coordinate systems of 2+1 dimensional non-extremal and extremal BTZ black holes. In section 3 we compute the two-point function of boundary operators in BTZ coordinates acting on the Poincaré vacuum state. In section 4 we discuss their asymptotic conformal symmetries and global isometries. In section 5 we solve the massive scalar field equations in BTZ coordinates, first for non-extremal and then for extremal black holes, and classify the mode solutions into non-normalizable and normalizable modes. Section 6 is a brief summary.

We should note that solutions to the massive scalar field equation in a non-extremal BTZ black hole spacetime have been studied before e.g. by Ichinose and Satoh [17] (see also [18]) and by Lee et. al. [19, 20]. However, [17] contains only a partial list of solutions, and all of the quoted references look at the solutions with boundary conditions which are not directly applicable for the classification of non-normalizable and normalizable modes.
within the context of the Lorentzian BTZ/CFT conjecture.

2 BTZ Coordinates

AdS$_3$ is defined as the hyperboloid $-X_0^2 - X_1^2 + X_2^2 + X_3^2 = -\Lambda^2$ embedded in a space $\mathbb{R}^{2,2}$ with metric $ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2$. The induced metric in the AdS$_3$ space can be written in global coordinates covering the whole manifold,

$$ds^2 = -(R^2 + \Lambda^2)dt^2 + \frac{\Lambda^2}{R^2 + \Lambda^2}dR^2 + R^2 d\theta^2,$$

where the radial coordinate $R \geq 0$ and the angle coordinate is periodic with $0 \leq \theta \leq 2\pi$. The time coordinate is also periodic with $0 \leq t \leq 2\pi$, but one can go to the covering space CAdS$_3$ with $t \in \mathbb{R}$. Another often used coordinate system is the Poincaré system, which covers only half of the AdS manifold. The metric is

$$ds^2 = \frac{\Lambda^2}{y^2} [dy^2 + dw_+ dw_-]$$

with $y \geq 0$, $w_\pm = x_1 \pm x_0$; $x_1, x_0 \in \mathbb{R}$.

The BTZ black hole is obtained from AdS$_3$ by periodic identifications. For this construction, it is useful to introduce a third coordinate system, dividing the AdS space into three regions and defining coordinates for each patch \[\text{[13]}\]. (For Penrose diagrams of AdS$_3$ in the three different coordinate systems, see e.g. \[\text{[16]}\].) Since we are interested only in the region corresponding to the exterior of the black hole, the relevant coordinate patch is given by

$$X_0 = \hat{r} \cosh \hat{\phi} \quad X_1 = \sqrt{\hat{r}^2 - \Lambda^2} \sinh \hat{t}$$
$$X_2 = \hat{r} \sinh \hat{\phi} \quad X_3 = \sqrt{\hat{r}^2 - \Lambda^2} \cosh \hat{t}$$

with $\hat{r} \geq \Lambda$, $\hat{\phi}, \hat{t} \in \mathbb{R}$. The metric takes the form

$$ds^2 = -(\hat{r}^2 - \Lambda^2)d\hat{t}^2 + \frac{\Lambda^2}{\hat{r}^2 - \Lambda^2}d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2.$$  

This can be rewritten in a form where we can recognize a non-extremal black hole. We make the coordinate transformation

$$\hat{r}^2 = \Lambda^2 \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right) (r^2 \geq r_+^2)$$
$$\begin{pmatrix} \hat{t} \\ \hat{\phi} \end{pmatrix} = \frac{1}{\Lambda} \begin{pmatrix} -r_+ & -r_- \\ r_- & r_+ \end{pmatrix} \begin{pmatrix} t/\Lambda \\ \phi \end{pmatrix}$$

\[\text{To make contact with the notation in \[\text{[13]}\], replace } X_0, X_1, X_2, X_3 \text{ by } U, V, X, Y.\]
and the metric becomes
\[ ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\Lambda^2 r^2} \, dt^2 + \Lambda^2 \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} \, dr^2 + r^2 \left( d\phi - \frac{r^+ r_-}{\Lambda r^2} \, dt \right)^2. \] (7)

At this stage \( \phi \in \mathbb{R} \). The BTZ black hole is obtained through the periodic identification
\[ \phi \sim \phi + 2\pi. \] (8)

The parameters \( r_\pm \) above then correspond to the radii of the outer and inner horizons, and they are related to the mass \( M \) and the angular momentum \( J \) of the black hole by
\[ MA^2 = r_+^2 + r_-^2 \] (9)
\[ JA = 2r_+ r_- . \] (10)

Note that the periodic identification (8) corresponds to the identification
\[ (\hat{t}, \hat{\phi}) \sim (\hat{t} - 2\pi r_- / \Lambda, \hat{\phi} + 2\pi r_+ / \Lambda) \] (11)
in the coordinates (3). So in that coordinate system, for a generic rotating black hole, the identification (11) acts on both coordinates \( \hat{t} \) and \( \hat{\phi} \). If the angular momentum is zero, \( r_- = 0 \), and just the angular coordinate \( \hat{\phi} \) is periodically identified. The identification (11) corresponds to a Lorentz boost in the \( X_0 - X_2 \) and \( X_1 - X_3 \) planes.

We also note that the metric (4) can formally be related to the form (1) in the global coordinate system, by the analytic continuation
\[ \hat{t} = i t ; \ \hat{\phi} = i \theta ; \ \hat{r} = i R. \] (12)

We can also relate it to the Poincaré metric with the coordinate transformation
\[ w_\pm = \frac{\sqrt{r^2 - \Lambda^2}}{\hat{r}} e^{\hat{\phi} \pm \hat{t}} \]
\[ y = \frac{\Lambda}{\hat{r}} e^{\hat{\phi}}. \] (13)

One can easily check that this transforms (2) to the form (4).

The above construction works fine for the non-extremal case \( r_+ > r_- \). In the extremal case the coordinate transformation (3) becomes singular, but the metric (7) is still fine, reducing to the form
\[ ds^2 = -\frac{(r^2 - r_0^2)^2}{\Lambda^2 r^2} \, dt^2 + \Lambda^2 \frac{r^2}{(r^2 - r_0^2)^2} \, dr^2 + r^2 \left( d\phi - \frac{r_0^2}{\Lambda r^2} \, dt \right)^2. \] (14)
In this case we can relate the metric to the Poincaré form (2) using the transformation
\begin{align*}
w_- &= \frac{\Lambda}{2r_0} e^{2r_0(\phi - t)/\Lambda} \\
w_+ &= \phi + t - \frac{\Lambda r_0}{r^2 - r_0^2} \\
y &= \frac{\Lambda}{\sqrt{r^2 - r_0^2}} e^{r_0(\phi - t)/\Lambda}.
\end{align*}

(15)

3 Two-point Functions

There have been many studies of correlation functions in the boundary theory using the bulk-boundary correspondence [4]. To our knowledge, these investigations have mostly been performed in the Poincaré coordinate system. For black hole applications, it would be of interest to examine how the machinery works in the BTZ coordinates (7). As an illustrative example we shall investigate the two-point function of operators coupling to the boundary values of a massive scalar field. Let us quickly review some basic points. A massive scalar field $\Phi$ with a mass $\mu$ in an AdS$_{d+1}$ space with Euclidean signature has a unique solution to its equation of motion. In Poincaré coordinates,
\begin{equation}
\text{ads}^2 = \frac{dy^2 + d\vec{x}_d^2}{y^2},
\end{equation}

(16)

the solution behaves as $\Phi \to y^{2h-} \Phi(\vec{x})$ near the boundary ($y \to 0$). The boundary field $\Phi_0(\vec{x})$ is of mass dimension $2h_-$ and acts as a source to an operator $O(\vec{x})$ of mass dimension $2h_+$ in the boundary theory, with the parameters $h_\pm$ given by
\begin{equation}
h_\pm = \frac{1}{4}(d \pm \sqrt{d^2 + 4\mu^2}).
\end{equation}

(17)

On the other hand, the bulk field $\Phi(y, \vec{x})$ can be obtained from the boundary field $\Phi_0(\vec{x})$ using the bulk-boundary Green’s function. We focus on $2+1$ dimensions and parametrize the boundary coordinates $\vec{x}$ by $w_{\pm}$, so the Poincaré metric takes the form (2). Then the bulk field $\Phi$ is obtained from the boundary values $\Phi_0$ by
\begin{equation}
\Phi(y, w_+, w_-) = \int dw'_+ dw'_- K_P(y, w_+, w_-; w'_+, w'_-) \Phi_0(w'_+, w'_-)
\end{equation}

(18)

where the bulk-boundary Green’s function $K$ is
\begin{equation}
K_P(y, w_+, w_-; w'_+, w'_-) = c \left(\frac{y}{y^2 + \Delta w_+ \Delta w_-}\right)^{2h_+}
\end{equation}

(19)
with $\Delta w_\pm = w_\pm - w'_\pm$. A scaling argument can be used to see that

$$\Phi \rightarrow y^{2h} \Phi_0(w_+, w_-)$$

(20)

near the boundary.

The above relations are given in the Euclidian signature. In the case of a Lorentzian signature, the relation (18) between the bulk field and its boundary value becomes a bit more complicated. The bulk field is no longer uniquely determined by its boundary value, but it is determined up to the addition of a normalizable mode [16]. The normalizable modes have a subleading asymptotic behavior $y^{2h} \phi_0(w_+, w_-)$ near the boundary, compared with the scaling (20) of the non-normalizable modes. In our two-point function calculation, we will be interested in the leading asymptotic behaviour near the boundary. We will ignore the contribution from the non-normalizable modes and work in the Lorentzian signature in the same manner as in the Euclidean signature.

We would like to examine how these relations are modified in the coordinate transformation to the BTZ coordinates (7),

$$w_\pm = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{2\pi T_\pm u_\pm}$$

$$y = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{\pi [T_+ u_+ + T_- u_-]}$$

(21)

where

$$T_\pm \equiv \frac{r_+ \mp r_-}{2\pi \Lambda}$$

$$u_\pm \equiv \phi \pm t/\Lambda.$$  

(22)

On the other hand, for the two-point function calculation it is sufficient to simplify things and only examine the region $r \gg r_\pm$ in the BTZ coordinates, where the coordinate transformation (21) between the Poincaré and the BTZ coordinates reduces to the simplified form

$$w_\pm = e^{2\pi T_\pm u_\pm}$$

$$y = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{\pi [T_+ u_+ + T_- u_-]}.$$  

(23)

The Poincaré boundary coordinates $w_\pm$ are related to the BTZ boundary coordinates $u_\pm$ by an exponential transformation. The boundary coordinates $u_\pm$ are also involved in the transformation between the radial coordinates $y$ and $r$. This fits nicely with what
happens to the scaling relation (20). The boundary field $\Phi_0$ has a conformal dimension $(h, \bar{h}) = (h_-, h_-)$, so it transforms as

$$\Phi_0(w_+, w_-) = \left( \frac{dw_+}{du_+} \right)^{-h_-} \left( \frac{dw_-}{du_-} \right)^{-h_-} \Phi_0(u_+, u_-)$$

(24)

under (23). Then, using (23) and (24), we see that the scaling relation (20) becomes

$$\Phi \to r^{-2h_-} \Phi_0(u_+, u_-) .$$

(25)

Starting from (18), and using the coordinate transformation (23), the scaling relation (24), the scaling of $dw_\pm$, and $h_+ = 1 - h_-$, we find that the relation (18) between the bulk field and the boundary field becomes

$$\Phi(r, u_+, u_-) = \int du'_+ du'_- K_{\text{BTZ}}(r, u_+, u_-; u'_+, u'_-) \Phi_0(u'_+, u'_-)$$

(26)

where the bulk-boundary Green’s function takes the form

$$K_{\text{BTZ}}(r, u_+, u_-; u'_+, u'_-)=c' \frac{\left( \frac{r^2-r'^2}{r^2} \right)^{h_+} e^{-2\pi h_+[T_+\Delta u_+ + T_-\Delta u_-]} \left\{ \frac{r^2-r'^2}{r^2} + (1-e^{-2\pi T_+} \Delta u_+) (1-e^{-2\pi T_-} \Delta u_-) \right\}^{2h_+}}{\left( \frac{r^2-r'^2}{r^2} + (1-e^{-2\pi T_+} \Delta u_+) (1-e^{-2\pi T_-} \Delta u_-) \right)^{2h_+}}$$

(27)

with $\Delta u_\pm = u_\pm - u'_\pm$. Note that this is manifestly invariant under translations of boundary coordinates. To describe a black hole, we also need to remember to take into account the periodic identification (8). This is done using the method of images. We shift the angle coordinate $\phi$ by integer multiples of $2\pi$ and add an infinite sum in front of the Green’s function. In terms of the null coordinates $u_+, u_-$, the final result becomes

$$K_{\text{BTZ}}(r, u_+, u_-; u'_+, u'_-) =c' \sum_{n=-\infty}^{\infty} \frac{\left( \frac{r^2-r'^2}{r^2} \right)^{h_+} e^{-2\pi h_+[T_+\Delta u_+ + T_-\Delta u_-+(T_+ + T_-)2\pi n]} \left\{ \frac{r^2-r'^2}{r^2} + (1-e^{-2\pi T_+} (\Delta u_+ +2\pi n)) (1-e^{-2\pi T_-} (\Delta u_- +2\pi n)) \right\}^{2h_+}}{\left( \frac{r^2-r'^2}{r^2} + (1-e^{-2\pi T_+} (\Delta u_+ +2\pi n)) (1-e^{-2\pi T_-} (\Delta u_- +2\pi n)) \right)^{2h_+}}$$

(28)

Next, we shall use these results to calculate (ignoring overall coefficients) the two point correlator $\langle O(u_+, u_-)O(u'_+, u'_-) \rangle$. We proceed as in [3] and evaluate the surface integral

$$I(\Phi) = \lim_{r_s \to \infty} \int_{T_s} du_+ du_- \sqrt{h} \Phi(\hat{e}_r \cdot \nabla) \Phi,$$

(29)

where $T_s$ is the surface $r = r_s$, $h$ its induced metric (from the BTZ metric (7)) and $\hat{e}_r$ is the unit vector normal to the surface. The induced area element and the radial projection of the gradient are found to be

$$\sqrt{h} \hat{e}_r \cdot \nabla = \frac{r^2}{\Lambda} \frac{r}{\Lambda} \partial_r .$$

(30)

See [17] for a similar discussion for the bulk Green’s function.
To evaluate the radial derivative of $\Phi$, we use the asymptotic form of (26) as $r \to \infty$:

$$
\Phi(r, u, u_0) \sim r^{-2h_+} \int du'_+ du'_- \frac{e^{-2\pi h_+ [T_{+} \Delta u_+ + T_{-} \Delta u_-]}}{(1 - e^{-2\pi T_{+} \Delta u_+})^{2h_+} (1 - e^{-2\pi T_{-} \Delta u_-})^{2h_+}} \Phi_0(u'_+, u'_-) .
$$

(31)

(For notational simplicity, from now on we will suppress the infinite sum over the periodic images. It is included as in (28).) The radial derivative becomes

$$
\sqrt{\hbar} \hat{e}_r \cdot \nabla \Phi \sim (-2h_+) r^{2h_+} \int du'_+ du'_- \cdot \cdot \cdot .
$$

(32)

Finally, substituting (32) into (29) and using (25), we end up with

$$
I(\Phi) \sim \int du_+ du_- du'_+ du'_- \Phi_0(u_+, u_-) \frac{e^{-2\pi h_+ [T_{+} \Delta u_+ + T_{-} \Delta u_-]}}{(1 - e^{-2\pi T_{+} \Delta u_+})^{2h_+} (1 - e^{-2\pi T_{-} \Delta u_-})^{2h_+}} \Phi_0(u'_+, u'_-) .
$$

Thus, we can read off the two point correlator

$$
\langle \mathcal{O}(u_+, u_-)\mathcal{O}(u'_+, u'_-) \rangle \sim \frac{e^{-2\pi h_+ [T_{+} \Delta u_+ + T_{-} \Delta u_-]}}{(1 - e^{-2\pi T_{+} \Delta u_+})^{2h_+} (1 - e^{-2\pi T_{-} \Delta u_-})^{2h_+}} .
$$

(34)

This agrees with the result that would be obtained simply from starting with the two-point correlator

$$
\langle \mathcal{O}(w_+, w_-)\mathcal{O}(w'_+, w'_-) \rangle \sim \frac{1}{(\Delta w_+ \Delta w_-)^{2h_+}}
$$

(35)

and using the scaling relations for dimension $(h_+, h_+)$ operators under the coordinate transformation $w_\pm \mapsto u_\pm$. The physical interpretation of the result is in agreement with the proposal of Maldacena and Strominger in [12]. The correlator (35) is the two-point function for dimension $(h_+, h_+)$ operators in a vacuum state, the Poincaré vacuum. The coordinate transformation (23) induces a mapping of the operators $\mathcal{O}(w_+, w_-)$ to new operators $\mathcal{O}(u_+, u_-)$ by a Bogoliubov transformation. The new operators $\mathcal{O}(u_+, u_-)$ see the Poincaré vacuum state as an excited density matrix. We can see that in fact they see the Poincaré vacuum state as a thermal bath of excitations in the BTZ modes, since the two-point function (34) can be seen to be periodic under the imaginary shifts [21]

$$
(\Delta t, \Delta \phi) \mapsto (\Delta t + i\beta_H, \Delta \phi + i\Omega_H \beta_H)
$$

(36)

where $\beta_H$ is the inverse Hawking temperature $1/T_H$ and $\Omega_H$ is the angular velocity of the outer horizon,

$$
T_H = \frac{r_+^2 - r_2^2}{2\pi \Lambda^2 r_+} ; \quad \Omega_H = \frac{r_-}{\Lambda r_+} .
$$

(37)

For a more detailed discussion on relating the bulk states with boundary states in the framework of quantized operators, see [16].
The angular velocity $\Omega_H$ can be interpreted as a chemical potential. Similar periodicity conditions were also discussed in [17, 22] in the context of Hartle-Hawking Green’s functions in the bulk of BTZ spacetime. Incidentally, note that it is the Hawking temperature $T_H$ which appears as the periodicity. There are two other temperature parameters as well, the “left” and “right” temperatures $T_{\pm}$ which appear in the boundary coordinate transformation (23). In [12], the stress tensor components $T_{\pm\pm}$ for the left movers and the right movers were evaluated, and they turned out to depend on the left and right temperatures. If one views the BTZ black hole as a part of a D1-D5 brane system [1, 12], these are related to the temperatures of the left and right moving excitations on the effective string. It is satisfying that the Hawking temperature $T_H$ appears where it is expected, in the periodicity of the boundary Green’s function. It still remains to be better understood if the Poincaré vacuum really is the appropriate vacuum state for the system. For example, it would be interesting to see if it was equal to the Hartle-Hawking vacuum.

4 Asymptotic Symmetries and Global Isometries

In the remainder of the paper, we shall discuss the symmetries of the BTZ spacetime and investigate the mode solutions to the massive scalar field equation in BTZ backgrounds.

Brown and Henneaux discovered [23] that AdS$_3$ has an asymptotic conformal symmetry generated by two copies of the Virasoro algebra. In the global coordinates, the asymptotic symmetry generators $l_n, \bar{l}_n$ take the form

$$l_n = i e^{i n w} \left[ 2 \partial_w - \frac{\Lambda^2 n^2}{R^2} \partial_{\bar{w}} - i n R \partial_R \right]$$

where $n \in \mathbb{Z}$ and $w = t + \theta, \bar{w} = t - \theta$. The generators $\bar{l}_n$ are obtained by exchanging $w$ with $\bar{w}$. The generators obey the Virasoro algebra

$$[l_m, l_n] = (m - n)l_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$

similarly for $\bar{l}_n$. Furthermore, $[l_m, \bar{l}_n] = 0$. The central charge $c$ was found to be $c = 3\Lambda/2G$.

In addition to the infinite dimensional asymptotic symmetry group, the AdS$_3$ inherits the SO(2,2) $\cong$ SL(2,$\mathbb{R})_L \times$ SL(2,$\mathbb{R})_R$ global isometry group from the space $\mathbb{R}^{2,2}$ through the embedding $-X_0^2 - X_1^2 + X_2^2 + X_3^2 = -\Lambda^2$. The generators of the group SO(2,2) consist of the rotation generators

$$L_{ab} = X_a \partial_b - X_b \partial_a$$

in the $ab = 01, 23$ planes, and the boost generators

$$J_{ab} = X_a \partial_b + X_b \partial_a$$
in the $ab = 02, 03, 12, 13$ planes. The $\text{SL}(2, \mathbb{R})_L$ generators are the linear combinations
\begin{align}
L_1 &= (-L_{01} + L_{23})/2 ; \\
L_2 &= (J_{12} - J_{03})/2 ; \\
L_3 &= (J_{02} + J_{13})/2
\end{align}
(40)
and the $\text{SL}(2, \mathbb{R})_R$ generators are
\begin{align}
\bar{L}_1 &= (-L_{01} - L_{23})/2 ; \\
\bar{L}_2 &= (J_{12} + J_{03})/2 ; \\
\bar{L}_3 &= (J_{02} - J_{13})/2 .
\end{align}
(41)
The $L$ generators commute with the $\bar{L}$ generators, and both sets obey the commutation relations
\begin{align}
[L_1, L_2] &= -L_3 ; \\
[L_1, L_3] &= L_2 ; \\
[L_2, L_3] &= L_1 .
\end{align}
(42)
The global isometry algebra $\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$ is isomorphic to the global conformal algebra generated by the Virasoro generators $l_0, l_\pm, \bar{l}_0, \bar{l}_\pm$. Hence one would like to construct linear combinations $L_0, L_\pm$ out of $L_1, L_2, L_3$ which satisfy the commutation relations
\begin{align}
[L_0, L_\pm] &= \mp L_\pm ; \\
[L_+ , L_-] &= 2L_0 .
\end{align}
(43)
In the global coordinates (1), it was found that the convenient linear combinations are
\begin{align}
L_0 &= iL_1 ; \\
L_\pm &= \pm L_2 + iL_3,
\end{align}
(44)
and the explicit representation of (43) is
\begin{align}
L_0 &= i\partial_w \\
L_\pm &= i\frac{\sqrt{R^2 + \Lambda^2}}{2R} e^{\pm iw} \left\{ \frac{2R^2 + \Lambda^2}{R^2 + \Lambda^2} \partial_w - \frac{\Lambda^2}{R^2 + \Lambda^2} \partial_w \mp iR\partial_R \right\} ,
\end{align}
(45)
where $w = t + \theta, \bar{w} = t - \theta$. The $\bar{L}$ generators are obtained by exchanging $w \leftrightarrow \bar{w}$.

In Poincaré coordinates, the convenient linear combinations turned out to be
\begin{align}
L_0 &= -L_2 ; \\
L_\pm &= i(L_1 \pm L_3)
\end{align}
(46)
with the explicit representations
\begin{align}
L_0 &= -\frac{1}{2} y\partial_y - w_+ \partial_+ \\
L_- &= i\Lambda \partial_+ \\
L_+ &= -\frac{i}{\Lambda} [w_+ y\partial_y + w_+^2 \partial_+ + y^2 \partial_-]
\end{align}
(47)
where $w_\pm = x_1 \pm x_0$. The $\bar{L}$ generators are obtained by exchanging $w_+ \leftrightarrow w_-$.\footnote{Our convention is such that the generators will match with those of [13] after the notational replacement $X_0, X_1, X_2, X_3 \mapsto U, V, X, Y$.}
Note that although both cases represent the same algebra (43), they are different in one aspect. In the global coordinates, (45) gives a hermitian $L_0$, and $L_\pm$ are adjoint operators: $L_+^\dagger = L_-$. (This corresponds to the convention in [27].) Near the boundary ($R \to \infty$), the generators $L_0, L_\pm$ reduce to the asymptotic symmetry generators $l_0, l_\pm$. Thus, $L_0, L_\pm$ represent the bulk extensions of the generators of the global conformal transformations. However, in Poincaré coordinates, $L_0$ is antihermitian, and $L_\pm$ are not adjoint operators. The reason for the difference is that in global coordinates $L_0 \sim L_1$, but in Poincaré coordinates $L_0 \sim L_2$. One can see that only if $L_0 \sim L_1$, it is possible to construct linear combinations of $L_{1,2,3}$ such that $L_0$ is hermitian and $L_\pm$ are adjoint operators. The transformation from global to Poincaré coordinates induces a linear transformation in the $\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$ algebra, the basis transforming according to (46) and (44). As a result, the hermiticity properties of the generators are altered. The lesson is that one should be careful in discussing the unitarity properties of the bulk and boundary states corresponding to the Poincaré modes.

What is the representation of (43) in the BTZ coordinates (3)? The easiest way to find the representation is to start from the global coordinate representation (45), and use the analytic continuation trick (12). The result is

$$L_0 = -\hat{\partial}_w,$$

$$L_\pm = -\sqrt{\hat{r}^2 - \Lambda^2} \frac{e^{\pm \hat{w}}}{2\hat{r}} \left\{ \frac{2\hat{r}^2 - \Lambda^2}{\hat{r}^2 - \Lambda^2} \partial_{\hat{w}} + \frac{\Lambda^2}{\hat{r}^2 - \Lambda^2} \partial_{\hat{\bar{w}}} \mp \hat{r} \partial_{\hat{r}} \right\},$$

(48)

where $\hat{w} = \hat{t} + \hat{\phi}$, $\hat{\bar{w}} = \hat{t} - \hat{\phi}$. Another way to derive these is to use (3) and to find suitable linear combinations of (40). The linear combinations are now

$$L_0 = -L_3 ; \quad L_\pm = L_1 \mp L_2 .$$

(49)

As in the Poincaré representation, $L_0$ is antihermitian, and $L_\pm$ are not adjoint operators. The linear transformation between the global generators (45) and (48) can be found from (49) and (44).

The asymptotic symmetry generators in the BTZ coordinates are equally easy to find. The BTZ metric (4) has the same asymptotic form as the AdS$_3$ metric (1), so the asymptotic generators $l_n, \bar{l}_n$ are obtained from (48) by replacing $R, \theta$ with $\hat{r}, \hat{\phi}$. The generators (48) correspond to the global symmetries of the AdS$_3$ space. An important point however is that after the periodic identifications, (3), (4), the global isometry group is no longer $\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$, but a subgroup. In the BTZ spacetime the global Killing vectors are $\partial/\partial t$, $\partial/\partial \phi$ or $\partial/\partial \hat{t}$, $\partial/\partial \hat{\phi}$. These generate time translations and rotations. So the global isometry group is broken to $\mathbb{R} \times \text{SO}(2)$. The corresponding generators above are $L_0$ and $\bar{L}_0$, while $L_\pm, \bar{L}_\pm$ no longer generate global spacetime symmetry.
transformations. Thus, we can redefine the global isometry generators to be $-2iL_0$ and $-2i\bar{L}_0$, these are hermitian and equal to the asymptotic Virasoro generators $l_0$ and $\bar{l}_0$. Note that the vacuum state $|0\rangle$ of the CFT has to respect the global symmetries of the bulk. In the AdS$_3$ case this means the condition $l_n|0\rangle = 0$ for $n = 0, \pm 1$. However, after the periodic identifications it is sufficient that $l_0|0\rangle = 0$ only. (Similarly for $\bar{l}$'s.)

5 Mode Solutions

In Lorentz signature, the field equations of massive scalar fields in AdS backgrounds have both non-normalizable and normalizable solutions. Explicit forms of normalizable solutions in AdS spacetimes have been worked out in [24, 25, 26, 16]. We will now study the mode solutions for massive scalar fields in non-extremal and extremal BTZ black hole backgrounds. In the non-extremal case, the mode solutions have been studied before in [17, 19, 20] but these studies have used different boundary conditions from the ones that we will adopt. Our goal is to classify the modes into non-normalizable and normalizable modes, the former being fixed classical backgrounds that give source terms in the boundary theory, while the latter are quantized and correspond to states in the boundary theory.

The scalar field equation is

$$\square \Phi = \frac{\mu^2}{\Lambda^2} \Phi, \quad (50)$$

where $\square$ is the d'Alembertian in the appropriate coordinate system, and $\mu^2$ stands for the mass$^2$ and coupling to the Ricci scalar $\mathcal{R}$,

$$\mu^2 \equiv m^2 + \lambda \mathcal{R} = m^2 - \frac{6\lambda}{\Lambda^2}. \quad (51)$$

For example, $\lambda = 0$ (1/8) for minimally (conformally) coupled scalars. The mass term $m^2$ includes contributions from the Kaluza-Klein reduction, if we have started from higher dimensions, e.g. BTZ$\times S^3 \times M_4$ would be relevant for the D1+D5 system.

5.1 Global Isometries

In solving the equation (50), we can make use of the global isometries generated by $\partial/\partial t$, $\partial/\partial \phi$ and make the separation of variables

$$\Phi = e^{-i\omega t + in\phi} f_{n\omega}, \quad (52)$$

where $f_{n\omega}$ only depends on the radial variable. If we make the periodic identification (8), $n$ is quantized and takes integer values. We can now immediately see that in the non-extremal case the solutions $\Phi_{n\omega}$ are eigenmodes of $L_0, \bar{L}_0$. Both eigenvalues are imaginary, which is expected since we obtained the generators by an euclidian continuation.
from the global generators \((45)\) and they turned out to be antihermitian. Before the periodic identifications, the global symmetry is \(\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R\). Its representations in the context of tachyon solutions of \((50)\) and states of \(\text{SL}(2, \mathbb{R})\) WZW models have been discussed in \([18]\). After the identifications, the global isometry group is \(\mathbb{R} \times \text{SO}(2)\) generated by \(l_0 = 2i\partial_w\) and \(\bar{l}_0 = -2i\partial_{\bar{w}}\). The modes belong to a representation labelled by real \((l_0, \bar{l}_0)\) conformal weights \((h, \bar{h})\), with

\[
\begin{align*}
h &= \omega - n/\Lambda \\
\bar{h} &= \omega + n/\Lambda .
\end{align*}
\]  

### 5.2 The non-extremal BTZ black hole

We first solve for mode solutions in the non-extremal black hole coordinate system \((7)\). In this case we can first follow the discussion by Ichinose and Satoh \([17]\). The d’Alembertian is

\[
\Box = -\frac{1}{r^2 N^2} \left[ r^2 \partial_t^2 - \left( \frac{r^2}{\Lambda^2} - M \right) \partial_\phi^2 + J \partial_t \partial_\phi \right] + \frac{1}{r} \partial_r (r N^2 \partial_r) \]

where

\[
N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\Lambda^2 r^2} .
\]

Next, we use the global isometries and separate the variables as in \((52)\). The function \(f_{n\omega}\) depends on the radial coordinate, we make a coordinate transformation and take it to be \(v = r^2/\Lambda^2\). The field equation now reduces to the radial equation

\[
f''_{n\omega} + \frac{\Delta'}{\Delta} f'_{n\omega} + \frac{1}{4\Delta^2} \{ n(Mn - J\omega) - \mu^2 \Delta - (n^2 - \omega^2 \Lambda^2) \} f_{n\omega} = 0 ,
\]

where

\[
\Delta \equiv (v - v_+)(v - v_-) = (v - r_+^2/\Lambda^2)(v - r_-^2/\Lambda^2) .
\]

We then substitute the ansatz

\[
f_{n\omega} = (v - v_+)^\alpha (v - v_-)^\beta g_{n\omega}
\]

where the exponents \(\alpha, \beta\) are given by

\[
\begin{align*}
\alpha^2 &= -\frac{1}{4(v_+ - v_-)^2} (r_+ \omega - r_- n/\Lambda)^2 \\
\beta^2 &= -\frac{1}{4(v_+ - v_-)^2} (r_- \omega - r_+ n/\Lambda)^2
\end{align*}
\]

and \(g_{n\omega}\) is a function of a rescaled radial variable

\[
u = \frac{v - v_-}{v_+ - v_-} .
\]
Note that in general $\alpha, \beta$ are imaginary. Using the Hawking temperature $T_H$ and angular velocity of the outer horizon $\Omega_H$, given by (37), $\alpha$ becomes

$$\alpha = \pm \frac{i}{4\pi T_H} (\omega - \Omega_H n) .$$

(60)

This maps the radial equation to the hypergeometric equation

$$u(1-u)g''_{n\omega} + \{c - (a + b + 1)u\}g'_{n\omega} - abg_{n\omega} = 0 ,$$

(61)

where the parameters are

$$a = \alpha + \beta + h_+$$
$$b = \alpha + \beta + h_-$$
$$c = 2\beta + 1$$

(62)

with

$$h_\pm = \frac{1}{2}(1 \pm \nu) \equiv \frac{1}{2}(1 \pm \sqrt{1 + \mu^2}) .$$

(63)

Note that (63) is the same parameter as (17) (with $d = 2$). Here the coordinate $u$ has the range $1 \leq u \leq \infty$, between the outer horizon and the boundary at infinity.

The solutions of (61) depend on the parameter $\nu$, in particular whether it is integer or not. Let us work out the solutions case by case.

5.2.1 $\nu$ not integer

To find which solutions could be normalizable and which ones cannot, let us first investigate their behaviour near the boundary $u = \infty$. In this region, the two linearly independent solutions for the radial modes are labelled by $h_\pm$, they are

$$f_{n\omega}^{(+)} = (u - 1)^{\alpha}u^{-h_+ - \alpha} F(\alpha + \beta + h_+, \alpha - \beta + h_+; 1 + \nu; u^{-1})$$
$$f_{n\omega}^{(-)} = (u - 1)^{\alpha}u^{-h_- - \alpha} F(\alpha + \beta + h_-, \alpha - \beta + h_-; 1 - \nu; u^{-1})$$

(64)

where we have used the radial variable $u$. $F$ is a hypergeometric function. The asymptotic behaviour near the boundary is

$$f_{n\omega}^{(+)} \sim u^{-h_+}$$
$$f_{n\omega}^{(-)} \sim u^{-h_-}$$

(65)

Now recall $h_\pm = (1 \pm \nu)/2$. Thus, for $\nu > 1$, $\Phi_{n\omega}^{(+)}$ is a candidate normalizable mode, and $\Phi_{n\omega}^{(-)}$ is the non-normalizable mode coupling to operators of dimension $2h_+$ in the boundary theory. However, for $0 < \nu < 1$, both modes are well behaved at the boundary. The situation is therefore similar to the global mode case in [16].
Let us now examine the behaviour of the modes \( \Phi^{(\pm)} \) close to the outer horizon \( u = 1 \).

Using the linear transformation relations of the hypergeometric functions, we find,

\[
f_{n\omega}^{(\pm)} = A_\pm (u - 1)^\alpha u^\beta F(\alpha + \beta + h_\pm, \alpha + \beta + h_\pm; 2\alpha + 1; 1 - u) \\
+ B_\pm (u - 1)^{-\alpha} u^{-\beta} F(-\alpha - \beta + h_\pm, -\alpha - \beta + h_\pm; -2\alpha + 1; 1 - u),
\]

where

\[
A_\pm = \frac{\Gamma(1 \pm \nu)\Gamma(-2\alpha)}{\Gamma(-\alpha - \beta + h_\pm)\Gamma(-\alpha + \beta + h_\pm)} \\
B_\pm = \frac{\Gamma(1 \pm \nu)\Gamma(2\alpha)}{\Gamma(\alpha + \beta + h_\pm)\Gamma(\alpha - \beta + h_\pm)}.
\]

Note that unlike the global modes in [16], in this case the denominators of the two coefficients have no poles, since \( \alpha, \beta \) are imaginary. In fact, we can see that the coefficients, and hence the two terms in (66) are complex conjugates. Hence, near the horizon the modes have the behaviour

\[
f^{(\pm)} \sim e^{i\theta_0}(u - 1)^\alpha + e^{-i\theta_0}(u - 1)^{-\alpha}
\]

where the phase factor \( \theta_0 \) is given by the ratio of the two coefficients (67). Introducing a “tortoise coordinate” \( r_* = \frac{1}{4\pi T_H} \ln(u - 1) \), we can write the normalizable modes \( \Phi^{(+)}_{n\omega} \) in the form

\[
\Phi^{(+)}_{n\omega} \sim e^{-i\omega t + i\nu \phi} \cos[(\omega - \Omega H n)r_* + \theta_0]
\]

where we recognize the solution as a superposition of infalling and outgoing plane waves, with a relative phase shift \( 2\theta_0 \). It is then obvious that the radial component of the scalar current vanishes near the horizon. This is the natural boundary condition to satisfy.

Ichinose and Satoh [17] were interested in the geometric entropy associated with the scalar field and used a different boundary condition. They introduced an ultraviolet cutoff and imposed Dirichlet boundary conditions at a stretched horizon, thereby obtaining the correct logarithmic divergence for the associated geometric entropy. This boundary condition would imply quantized frequencies, but is not the correct one to impose in the present setup. In our case the frequencies (and thus the conformal weights \( h, \bar{h} \)) will remain continuous. We also remark that the modes in (69) are related to the BTZ Kruskal modes in the same manner as for Schwarzschild black holes the asymptotic “out” modes

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\[6\] At this point a reader familiar with the global mode analysis of [16] might wonder why we are not analyzing the two regions in the reverse order. This is because in [16] it was more important that the modes were well behaved in the origin, whereas in our case the behaviour at the boundary is more important. Note also the difference with the absorption coefficient calculations. There the primary boundary condition is that there are only infalling waves at the horizon; the solution near the boundary will then be a superposition of a non-normalizable and a normalizable mode, see [20].
are related to the Kruskal modes. If the scalar field is in a vacuum state with respect to
the Kruskal modes, the Hartle-Hawking vacuum, the modes (33) are thermally excited at
the BTZ Hawking temperature $T_H$. Note however that in section 3 the two-point point
function (34) was evaluated at the Poincaré vacuum. Thus it would be interesting to see
what is the relation of the Hartle-Hawking and Poincaré vacuum.

5.2.2 $\nu$ a nonnegative integer

For completeness, we investigate also the case $\nu$ a nonnegative integer, not studied in
[17]. To investigate the behaviour at the boundary, it is convenient to map the equation
(61) into a new one involving a coordinate $w = 1/(1 - u)$ and a rescaled mode function
$\bar{g}_{n\omega} = w^{-a}g_{n\omega}$ where $a$ is the parameter given by (62). The resulting equation is
\[
w(1 - w)\ddot{g}_{n\omega} + [C - (A + B + 1)w]\dot{g}_{n\omega} - AB\bar{g}_{n\omega} = 0
\] (70)
where
\[
A = \alpha + \beta + h_-
\]
\[
B = -\alpha + \beta + h_-
\]
\[
C = 2h_- = 1 - \nu
\] (71)

The boundary now corresponds to $w = 0$. Consider first the case $\nu = 0$. In the boundary
region we have the two linearly independent solutions
\[
\bar{g}_{n\omega}^{(1)} = F(A, B; 1; w)
\] (72)
\[
\bar{g}_{n\omega}^{(2)} = \ln w F(A, B; 1; w) + \sum_{l=1}^{\infty} k_l w^l,
\] (73)

with the coefficients
\[
k_l \equiv \frac{(A)_l(B)_l}{(l!)^2}[\psi(A + l) - \psi(A) + \psi(B + l) - \psi(B) - 2\psi(l + 1) + 2\psi(l)].
\]

Therefore, the asymptotic behaviour of the corresponding radial modes $f_{n\omega}^{(1,2)}$ is found to be
\[
f_{n\omega}^{(1)} \sim \frac{1}{\sqrt{u}} \rightarrow 0
\]
\[
f_{n\omega}^{(2)} \sim \frac{\ln u}{\sqrt{u}} \rightarrow 0.
\] (74)

Both modes vanish at the boundary, and correspond to normalizable modes. The situation
is then similar to the case $0 < \nu < 1$. If $\nu = 1, 2, \ldots$, the two linearly independent solutions

\footnote{We have chosen $b \geq a$.}
are

$$\tilde{g}^{(1)}_{n\omega} = w' F(A + \nu, B + \nu; 1 + \nu; w)$$

$$\tilde{g}^{(2)}_{n\omega} = \ln w \tilde{g}^{(1)}_{n\omega} + \sum_{n=1}^{\infty} k_{n,\nu} w^n - \sum_{n=1}^{\nu} l_n w^{\nu-n}, \quad (75)$$

with the coefficients

$$k_{n,\nu} \equiv \frac{(A + \nu)_n (B + \nu)_n}{(1 + \nu)_n n!} [\psi(A + \nu + n) - \psi(A + \nu)$$

$$\psi(B + \nu + n) - \psi(B + \nu) - \psi(\nu + 1 + n) + \psi(\nu + 1) - \psi(1)]$$

$$l_n \equiv \frac{(n - 1)!(-\nu)_n}{(1 - A - \nu)_n (1 - B - \nu)_n} \quad (76)$$

where \((-\nu)_n \equiv (-\nu)(\nu-1) \cdots (-\nu+n-1)\). In particular, one can check that the coefficient \(l_n\) multiplying a term \(w^0\) is nonvanishing. The asymptotic behaviour near the boundary can then be found to satisfy

$$f^{(1)} \sim \frac{1}{u^{h_+}}$$

$$f^{(2)} \sim \frac{1}{u^{h_-}}. \quad (77)$$

Thus, \(f^{(1)}\) is the candidate normalizable mode, and \(f^{(2)}\) is the candidate non-normalizable mode coupling to a boundary operator of dimension \(2h_+\).

Using the linear transformation relations, one can find the behaviour of the mode \(f^{(1)}\) near the horizon, \(w \to \infty\). After some algebra, one can see that it will again be a sum of two complex conjugate terms, with the familiar form

$$f^{(1)} \sim e^{i\theta_0} (u - 1)^\alpha + e^{-i\theta_0} (u - 1)^{-\alpha}. \quad (78)$$

Again, it is a superposition of infalling and outgoing waves, and the radial component of the flux vanishes.

### 5.3 The extremal BTZ black hole

We now find the mode solutions in the extremal case. In this case the two regular singular points of the radial differential equation at the two horizons will merge, and the equation can be mapped to a confluent hypergeometric differential equation. Let us start again from the radial equation (55). For the extremal black hole, \(J = M\Lambda\), and

$$\Delta = (v - v_0)^2 = \left(\frac{r^2}{\Lambda^2} - M/2\right). \quad (79)$$

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We first define a variable \( x = 1/(v-v_0) \), for which the radial equation (55) takes the form
\[
f'' + \frac{1}{4}(A^2 - Bx^{-1} - \mu^2 x^{-2})f = 0 ,
\]
with the parameters
\[
A^2 \equiv r_0^2(\omega - \Omega_H n)^2, \\
B \equiv \Lambda^2(\omega^2 - \Omega_H^2 n^2) ,
\]
where \( \Omega_H = 1/\Lambda \) is the angular velocity and \( r_0 = \sqrt{MA^2/2} \) is the radius of the horizon.

Next, we rewrite the equation (80) as a Whittaker equation, using a new variable \( y = iAx \):
\[
f'' + \left( -\frac{1}{4} + \frac{\kappa}{y} + \frac{\frac{1}{4} - \lambda^2}{y^2} \right) f = 0
\]
where the parameters are
\[
\kappa \equiv iB/A = i\Lambda^2 r_0^2 (\omega + \Omega_H n), \\
\frac{1}{4} - \lambda^2 \equiv -\frac{1}{4}\mu^2 .
\]
The last equation gives \( \lambda = \lambda_{\pm} \equiv \pm \sqrt{1 + \mu^2}/2 \). Note that we could have flipped the sign in the definitions of \( y \) and \( \kappa \). Thus, a convenient choice for the two linearly independent solutions of (82) is
\[
f^{(+)} = M_{\kappa,\lambda_+}(y) + M_{-\kappa,\lambda_+}(-y) \\
f^{(-)} = M_{\kappa,\lambda_-}(y) + M_{-\kappa,\lambda_-}(-y) ,
\]
where \( M_{\kappa,\lambda}(y) \) is a Whittaker function [28]. We chose both of the solutions to be real valued. The radial functions have the following asymptotic behaviour at the boundary
\[
f_{m+}^{(+) \sim r^{-2h_+}} \\
f_{m-}^{(-) \sim r^{-2h_-}} .
\]
\( f^{(-)} \) blows up at the boundary and is the candidate non-normalizable mode, whereas \( f^{(+)} \) decays and is the candidate normalizable mode. Near the horizon \( f^{(+)} \) has an oscillatory behavior characterized by
\[
M_{\kappa,\lambda}(y) \sim e^{y/2} y^{-\kappa} .
\]
The radial current vanishes since \( f^{(+)} \) was chosen to be real.
6 Summary

We studied various features of the bulk-boundary correspondence in BTZ black hole spacetimes. We evaluated the bulk-boundary Green’s function for massive scalar fields in the region far from the outer horizon, and used it to calculate the Poincaré vacuum two-point function for operators in the boundary theory which couple to the boundary values of the scalar field in BTZ coordinates. The two-point function satisfies thermal boundary conditions, signaling that the Poincare vacuum looks like a thermal bath at the Hawking temperature $T_H$ of the black hole.

Then, we reviewed symmetry properties of BTZ black holes and examined mode solutions to the massive field equations in non-extremal and extremal backgrounds. The goal was to classify the solutions into non-normalizable and normalizable modes according to the Lorentzian version [10] of Maldacena’s conjecture. This is a technical step, but we expect the knowledge of the modes to be useful in further studies of BTZ black holes, especially because the normalizable modes are thought to correspond to states in the Hilbert space of the boundary theory. They are analogous to the Schwarzschild modes for the familiar three dimensional black holes. Such modes play a central role in studies of black hole evaporation, so the hope is that understanding them as states in the boundary theory will be helpful for the discussion of the evaporation process purely in the boundary theory, according to the holographic principle.

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