REGULARIZATION OF SIDEWAYS PROBLEM FOR A TIME FRACTIONAL DIFFUSION EQUATION WITH NONLINEAR SOURCE

TRAN BAO NGOC, NGUYEN HUY TUAN, AND MOKHTAR KIRANE

Abstract. In this paper, we consider an inverse problem for a time-fractional diffusion equation with a nonlinear source. We prove that the considered problem is ill-posed, i.e. the solution does not depend continuously on the data. The problem is ill-posed in the sense of Hadamard. Under some weak a priori assumptions on the sought solution, we propose a new regularization method for stabilizing the ill-posed problem. We also provide a numerical example to illustrate our results.

Keywords: Ill-posed, Regularization method, Caputo’s fractional derivatives, Fourier transform.

1. Introduction

In this article, we consider the following concentration identification problem (CIP) for the time fractional nonlinear diffusion equation:

\[ D_\alpha^t u(x,t) - u_{xx}(x,t) = f(x,t,u(x,t)), \quad x > 0, \quad t > 0, \quad 0 < \alpha < 1, \] (1.1)

with the Cauchy condition and initial condition

\[ u(x,0) = 0, \quad x \geq 0, \] (1.2)
\[ u(0,t) = g(t), \quad t \geq 0, \] (1.3)
\[ u_x(0,t) = h(t), \quad t \geq 0, \] (1.4)

where \( u \) is the solute concentration (see [1]), \( f \) is the source term defined later. The function \( g(t) \) and \( h(t) \) denote the solute concentration and the measurement datum of dispersion flux, respectively, on the left boundary. We will recover the solute concentration \( u(x,t) \) in the region \( \{ (x,t), 0 \leq x < 1, t > 0 \} \) from the measurement data of source terms \( f(x,t,u(x,t)) \) and boundary concentrations \( g(t), h(t) \). The fractional derivative \( \partial^\alpha u / \partial t^\alpha \) is the Caputo fractional derivative of order \( \alpha \) defined by [2]

\[ \frac{\partial^\alpha u}{\partial t^\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha} \quad \text{for} \quad 0 < \alpha < 1, \] (1.5)
\[ \frac{\partial^\alpha u}{\partial t^\alpha}(t) = \frac{\partial u}{\partial t}, \quad \text{for} \quad \alpha = 1, \] (1.6)

where \( \Gamma(.) \) is the Gamma function. Problem (1.1)-(1.4) is an ill-posed inverse problem (see Lemma 2.1); the solution does not depend continuously on the given data, i.e., any small perturbation in the given data may cause a large change to the solution. The sideways problem with classical derivative has been considered by many authors. In 1995, Teresa Regińska [3] solved a sideways heat problem which consists in applying the wavelet basis decomposition of measured data in the quarter plane...
(x ≥ 0, t ≥ 0). In 1999, F. Berntsson [4] investigated the following sideways heat equation

\[ ku_{xx} = u_t, \quad 0 < x < 1, \quad t ≥ 0, \]
\[ u(1, t) = g(t), \quad t ≥ 0, \]
\[ u_x(1, t) = h(t), \quad t ≥ 0, \]
\[ u(x, 0) = 0, \quad 0 < x < 1, \]

where \( g, h \) were the functions to be defined later. He tried to use the spectral method to determine the temperature \( u(x, t) \) for \( 0 ≤ x < 1 \) from temperature measurements \( g = u(1, \cdot) \) and heat-flux measurements \( h = u_x(1, \cdot) \). Later on, in 2010, T. Wei [5] proposed a spectral regularization method for the following time fractional advection-dispersion equation

\[ 0D_t^α u + bu_x = au_{xx}, \quad 0 < x < L, \quad t > 0, \]
\[ u(0, t) = f(t), \quad t ≥ 0, \]
\[ u_x(0, t) = g(t), \quad t ≥ 0, \]
\[ u(x, 0) = 0, \quad 0 < x < L. \]

where \( f, g \) were the functions to be defined later.

Recently, the homogeneous problem, i.e, \( f ≡ 0 \) in Eq. (1.1) has been considered by some authors, for example, see [5, 9, 7, 8]. Although there are many papers on the homogeneous case of the identification problem, the inhomogeneous case has not been intensively investigated. The inhomogeneous case is first studied by G.H. Zheng [1]. Very recently, Tuan and his group [9] have studied a more general case of inhomogeneous source term in the form \( f(x, t) \). Until now, to our knowledge, the problem (1.1)-(1.4) with a generalized source term \( f(x, t, u) \) has not been studied. Obviously, the nonlinear problem is much more challenging. In case of the homogeneous problem, we can transform the solution \( u \) into the linear equation

\[ u(1, t) = \mathcal{T}(g, h), \quad (1.7) \]

where \( \mathcal{T} \) is a linear unbounded operator. Then, there are many choices of stability term for regularization that have been proposed. The main idea is to replace the operator \( \mathcal{T} \) with a class of linear bounded operator. However, when the right hand side of (1.1) depend on \( u \), it is impossible to represent \( u \) as (1.7). Thus, the techniques and methods in previous papers on the homogeneous case cannot be applied directly to solve the nonlinear inhomogeneous problem. Now, we describe our new ideas for the nonlinear inhomogeneous problem. The sought solution of Problem (1.1)-(1.4) can be represented by a nonlinear integral equation containing some instability terms, see (2.14). Our main objectives are to find a suitable integral equation for approximating the exact solution by replacing the instability terms with regularization terms and then show that the solution of regularized problem converges to the exact solution. Notice that the following sideways problem for time fractional diffusion equation

\[ -u_x(x, t) = D_t^α u(x, t) + f(x, t, u(x, t)), \quad x > 0, \quad t > 0, \]
\[ u(1, t) = g(t), \quad t ≥ 0, \]
\[ \lim_{x \to +\infty} u(x, t) = u(x, 0) = 0, \quad t ≥ 0, \quad (1.8) \]

has been studied by M. Kirane et al [10]. Our problem (1.1)-(1.4) is more complicated than (1.8) since there are two boundary functions in (1.3)-(1.4) to be investigated. Moreover, in this paper, we first give the convergence rate in \( H^p \) norm which is not considered in [10] and some previous papers [9, 5, 6, 7, 8].
The outline of this work is as follows. In Section 2, we present the ill-posedness of the problem. In Section 3, we propose our regularization method and convergence estimates for the regularized solution and the sought solution are given in both $L^2$- and $H^p$-norm based on the a priori assumptions. Finally, in Section 4 we implement a numerical example to illustrate the theoretical results.

2. ILL-POSEDNESS OF THE NONLINEAR PROBLEM

2.1. The mild solution of Problem (1.1)-(1.4). To apply the Fourier transform, thanks to [11], we extend all functions in this paper to the whole line $-\infty < t < +\infty$ by defining them to be zero for $t < 0$. The Fourier transform of $L^2(\mathbb{R})$ function $v(t)$ ($-\infty < t < \infty$) is defined by

$$F(v)(\omega) := \hat{v}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(t)e^{-i\omega t} dt, \quad \omega \in \mathbb{R}.$$  

We denote by $\|v\|_{L^2(\mathbb{R})}$ the $L^2(\mathbb{R})$ norm, i.e.,

$$\|v\|_{L^2(\mathbb{R})} = \left( \int_{\mathbb{R}} |v(\omega)|^2 d\omega \right)^{\frac{1}{2}},$$

and by $\|v\|_{H^p(\mathbb{R})}$ the $H^p(\mathbb{R})$ norm, i.e.,

$$\|v\|_{H^p(\mathbb{R})} = \left( \int_{\mathbb{R}} (1 + \omega^2)^p |\hat{v}(\omega)|^2 d\omega \right)^{\frac{1}{2}}.$$  

Note that when $p = 0$, $H^p(\mathbb{R}) = H^0(\mathbb{R}) = L^2(\mathbb{R})$.

Applying Fourier transform with respect to variable $t$ to the problem (1.1) - (1.4), we obtain the following second order differential equation

$$\begin{cases}
(i\omega)^{\alpha} \hat{u}(x,\omega) - \hat{u}_{xx}(x,\omega) = \hat{f}(x,\omega, u(x,\omega)), & x > 0, \omega \in \mathbb{R}, \\
\hat{u}(0,\omega) = \hat{g}(\omega), & \omega \in \mathbb{R}, \\
\hat{u}_x(0,\omega) = \hat{h}(\omega), & \omega \in \mathbb{R}.
\end{cases}$$  

(2.9)

Put

$$k(\omega) := (i\omega)^{\frac{\alpha}{2}} = \Re(k(\omega)) + i\Im(k(\omega)),$$

(2.10)

where the real and image parts of $k(\omega)$ are respectively

$$\Re(k(\omega)) := |\omega|^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4}, \quad \Im(k(\omega)) = |\omega|^{\frac{\alpha}{2}} \text{sign}(\omega) \sin \frac{\alpha \pi}{4}.$$  

(2.11)

Multiplying the first equation of (2.9) by $\frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)}$ and integrating two sides on $[0; x]$, we derive

$$\int_0^x \frac{(i\omega)^{\alpha} \hat{u}(z,\omega) - \hat{u}_{zz}(z,\omega)}{k(\omega)} \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} dz = \int_0^x \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} \hat{f}(z,\omega, u(z,\omega)) dz.$$  

(2.12)
By applying integration by parts to the left side of (2.12), and combining the second and third equation of (2.9), we obtain
\[
\tilde{u}(x, \omega) = \cosh \left( k(\omega)x \right) \tilde{g}(\omega) + \frac{\sinh \left( k(\omega)x \right)}{k(\omega)} \tilde{h}(\omega) - \int_0^x \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} \tilde{f}(z, \omega, u(z, \omega)) dz,
\]
for \( x \geq 0, \omega \in \mathbb{R} \). Applying the inverse Fourier transform to (2.13), we have
\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega)x \right) \tilde{g}(\omega)e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sinh \left( k(\omega)x \right)}{k(\omega)} \tilde{h}(\omega)e^{i\omega t} d\omega
\]
\[-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^x \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} \tilde{f}(z, \omega, u(z, \omega)) e^{i\omega t} dz d\omega,
\]
for \( x \geq 0, t \in \mathbb{R} \).

Note that the real part of \( k(\omega) \) is an increasing positive function of \( \omega \). Therefore the terms \( \cosh \left( k(\omega)x \right), \frac{\sinh \left( k(\omega)x \right)}{|k(\omega)|} \) increase rather quickly when \( |\omega| \to \infty \): small errors in high-frequency components can blow up and completely destroy the solution for \( 0 < x < 1 \). Therefore, the problem is severely ill-posed and regularization methods are required for finding the approximate solution of our problem.

2.2. **Some notations.** We prove the following lemma which will be important to obtain the main results.

**Lemma 2.1.** Let \( \Re(z) \) be the real part of any complex number \( z \). If \( \Re(z) > 0 \) then we have

a) \( |\cosh(z)| \leq e^{\Re(z)} \);

b) \( \left| \frac{\sinh(\lambda z)}{z} \right| \leq \lambda e^{\lambda \Re(z)} \).

**Proof.** a) We have
\[
|\cosh(z)| = \frac{|e^z + e^{-z}|}{2} = \frac{|e^{\Re(z) + i\Im(z)} + e^{-\Re(z) - i\Im(z)}|}{2}
\]
\[
= \frac{|e^{\Re(z)}(\cos \Im(z) + i \sin \Im(z)) + e^{-\Re(z)}(\cos \Im(z) - i \sin \Im(z))|}{2}
\]
\[
= \frac{\left| (e^{\Re(z)} + e^{-\Re(z)}) \cos \Im(z) + i \sin \Im(z) (e^{\Re(z)} - e^{-\Re(z)}) \right|}{2}
\]
\[
= \frac{\sqrt{(e^{\Re(z)} + e^{-\Re(z)})^2 \cos^2 \Im(z) + (e^{\Re(z)} - e^{-\Re(z)})^2 \sin^2 \Im(z)}}{2}
\]
\[
= \frac{\sqrt{e^{2\Re(z)} + 2 \cos 2\Im(z) + e^{-2\Re(z)}}}{2}
\]
\[
\leq \frac{e^{\Re(z)} + e^{-\Re(z)}}{2} = \cosh \Re(z) \leq e^{\Re(z)} \text{ for } \Re(z) > 0.
\]
b) First, we have
\[ \left| \frac{\sinh(z\lambda)}{z} \right| = \left| \int_0^\lambda \cosh(sz)\,ds \right| \leq \int_0^\lambda |\cosh(sz)|\,ds \leq \int_0^\lambda e^{\Re(z)s}\,ds = \frac{e^{\lambda \Re(z)} - 1}{\Re(z)}. \] (2.15)

Second, we have
\[ e^{\lambda \Re(z)} - 1 = \sum_{k=0}^{+\infty} \frac{\left(\lambda \Re(z)\right)^k}{k!} - 1 = \sum_{k=1}^{+\infty} \frac{\left(\lambda \Re(z)\right)^k}{k!} = \lambda \sum_{k=1}^{+\infty} \frac{\left(\lambda \Re(z)\right)^{k-1}}{k!} = \lambda \sum_{l=0}^{+\infty} \frac{\left(\lambda \Re(z)\right)^{l}}{(l + 1)!} \\leq \lambda \sum_{l=0}^{+\infty} \frac{\left(\lambda \Re(z)\right)^{l}}{l!} \leq \lambda e^{\lambda \Re(z)}. \] (2.16)

The inequality in part b) follows from (2.15) and (2.16). □

**Lemma 2.2.** Let \( \epsilon, \xi, p, \gamma \) be positive real numbers. If \( \epsilon \) satisfies the following condition
\[ \epsilon < \left[ \frac{\xi \gamma \cos \frac{\alpha \pi}{4}}{p} \right]^{\frac{1}{\xi}}, \] (2.17)
then we have
\[ (1 + \omega^2)^p \exp \left( 2(x - 1 - \gamma)\omega^\xi \cos \frac{\alpha \pi}{4} \right) \leq (1 + \epsilon^{-2})^p \exp \left( 2(x - 1 - \gamma)\epsilon^{-\xi} \cos \frac{\alpha \pi}{4} \right) \] (2.18)
for all \( 0 \leq x < 1 \) and \( \omega \geq \frac{1}{\epsilon} \).

**Proof.** For \( 0 \leq x < 1 \), we define
\[ \Lambda(\omega) := (1 + \omega^2)^p \exp \left( 2(x - 1 - \gamma)\omega^\xi \cos \frac{\alpha \pi}{4} \right), \]
for \( 0 \leq x \leq 1 \). Let us denote
\[ \Gamma(\omega) = 2wp(1 + \omega^2)^{p-1} \exp \left( 2(x - 1 - \gamma)\omega^\xi \cos \frac{\alpha \pi}{4} \right) \]
then
\[ \frac{d\Lambda}{d\omega} = \Gamma(\omega) \left[ 1 - \frac{(1 + \omega^2)(1 - x + \gamma) \cos \frac{\alpha \pi}{4} \cdot \xi \omega^{\xi-2}}{p} \right] \]
\[ \leq \Gamma(\omega) \left[ 1 - \frac{\omega^{2\gamma \cos \frac{\alpha \pi}{4} \cdot \xi \omega^{\xi-2}}}{p} \right] \]
\[ \leq \Gamma(\omega) \left[ 1 - \frac{\xi \gamma \cos \frac{\alpha \pi}{4} \cdot \xi \omega^{\xi}}{p} \right] \leq \Gamma(\omega) \left[ 1 - \frac{\xi \gamma \cos \frac{\alpha \pi}{4} \cdot \epsilon^{-\xi}}{p} \right] \] (2.19)
since \( \omega \geq \frac{1}{\epsilon} \). It follows from (2.17) that \( 1 - \frac{\xi \gamma \cos \frac{\alpha \pi}{4} \cdot \epsilon^{-\xi}}{p} < 0 \). Therefore, the inequality (2.19) implies \( \frac{d\Lambda}{d\omega} \leq 0 \) for \( \omega \geq \frac{1}{\epsilon} \). The proof is completed. □
2.3. **An example of ill-posedness for problem (1.1)-(1.4)**. In this subsection, we give an example of ill-posedness by choosing the function \( f \) as follows

\[
f(z, t, u(z, t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{2} e^{-k(\omega)} \hat{u}(z, \omega) e^{i\omega t} d\omega \tag{2.20}
\]

or

\[
\tilde{f}(z, \omega, u(z, \omega)) = \frac{1}{2} e^{-k(\omega)} \hat{u}(z, \omega), \tag{2.21}
\]

or for all \((z, \omega) \in [0; 1] \times [0; +\infty)\) and it is extended to zero for all \((z, \omega) \in [0; 1] \times (-\infty; 0)\). The ill-posedness of the problem (1.1)-(1.4) corresponding to the above function \( f \) can be proved by using the following lemmas.

**Lemma 2.3.** Let \( f \) be defined as (2.20). Then for any \((g, h) \in \left(L^2(\mathbb{R})\right)^2\), Problem (2.14) has unique solution \( u^*(g, h) \in C([0, 1]; L^2(\mathbb{R}))\).

**Proof.** Let us set

\[
\Phi(v)(x, t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega) x \right) \hat{g}(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( \frac{k(\omega) x}{k(\omega)} \right) \hat{h}(\omega) e^{i\omega t} d\omega
\]

or

\[
\Phi(v)(x, t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega) x \right) \hat{g}(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( \frac{k(\omega) x}{k(\omega)} \right) \hat{h}(\omega) e^{i\omega t} d\omega
\]

\[
- \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} e^{-k(\omega) \hat{v}(z, \omega) e^{i\omega t} dzd\omega},
\]

for all \( v \in C([0, 1]; L^2(\mathbb{R}))\). For any \((g, h) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})\), we claim that the following equation

\[
v(x, t) = \Phi(v)(x, t) \tag{2.22}
\]

has a unique solution in \( C([0, 1]; L^2(\mathbb{R}))\). For \( v_1, v_2 \in C([0, 1]; L^2(\mathbb{R})) \) and \( 0 \leq x \leq 1 \), we shall prove that

\[
\|\Phi(v_1)(x, \cdot) - \Phi(v_2)(x, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{2} \|v_1 - v_2\|_{C([0, 1]; L^2(\mathbb{R}))}^2. \tag{2.23}
\]

Indeed, we have

\[
\Phi(v_1)(x, t) - \Phi(v_2)(x, t) = - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} e^{-k(\omega) \hat{v}_1(z, \omega) - \hat{v}_2(z, \omega) dz e^{i\omega t} d\omega
\]

\[
= \mathcal{F}^{-1} \left( - \frac{1}{2} \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} e^{-k(\omega) \hat{v}_1(z, \omega) - \hat{v}_2(z, \omega) dz \right)(t).
\]
By using the Parseval’s identity, we obtain
\[
\|\Phi(v_1)(x, \cdot) - \Phi(v_2)(x, \cdot)\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{+\infty} |\Phi(v_1)(x, t) - \Phi(v_2)(x, t)|^2 dt
\]
\[
= \frac{1}{2} \int_{-\infty}^{+\infty} \int_0^x \sinh \left( \frac{k(\omega)(x - z)}{k(\omega)} \right) e^{-k(\omega)} |\hat{v}_1(z, \omega) - \hat{v}_2(z, \omega)|^2 dz \, d\omega. \tag{2.24}
\]

Applying the Hölder inequality to (2.24), we derive
\[
\left| \int_0^x \sinh \left( \frac{k(\omega)(x - z)}{k(\omega)} \right) e^{-k(\omega)} |\hat{v}_1(z, \omega) - \hat{v}_2(z, \omega)| \, dz \right|^2 \\
\leq x \int_0^x \left| \sinh \left( \frac{k(\omega)(x - z)}{k(\omega)} \right) \right|^2 \left| e^{-k(\omega)} |\hat{v}_1(z, \omega) - \hat{v}_2(z, \omega)| \right|^2 \, dz \\
\leq \int_0^x (x - z)^2 \exp \left( 2\Re(k(\omega))(x - z) \right) \exp \left( -2\Re(k(\omega)) \right) |\hat{v}_1(z, \omega) - \hat{v}_2(z, \omega)|^2 \, dz \\
\leq \int_0^x |\hat{v}_1(z, \omega) - \hat{v}_2(z, \omega)|^2 \, dz, \tag{2.25}
\]
where we have used the Lemma 2.1 as follows
\[
\left| \sinh \left( \frac{k(\omega)(x - z)}{k(\omega)} \right) \right|^2 \leq (x - z)^2 \exp \left( 2\Re(k(\omega))(x - z) \right). \tag{2.26}
\]

It follows from (2.24), and (2.25) that
\[
\|\Phi(v_1)(x, \cdot) - \Phi(v_2)(x, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{2} \int_{-\infty}^{+\infty} \int_0^x |\hat{v}_1(z, \omega) - \hat{v}_2(z, \omega)|^2 \, dz \, d\omega \\
\leq \frac{1}{2} \int_0^x \|\hat{v}_1(z, \cdot) - \hat{v}_2(z, \cdot)\|_{L^2(\mathbb{R})}^2 dz \\
\leq \frac{1}{2} \int_0^x \|v_1(z, \cdot) - v_2(z, \cdot)\|_{L^2(\mathbb{R})}^2 dz \\
\leq \frac{1}{2} \|v_1 - v_2\|^2_{C([0,1];L^2(\mathbb{R}))}. \tag{2.27}
\]

Consequently, we get
\[
\|\Phi(v_1) - \Phi(v_2)\|^2_{C([0,1];L^2(\mathbb{R}))} \leq \frac{1}{2} \|v_1 - v_2\|^2_{C([0,1];L^2(\mathbb{R}))}, \tag{2.29}
\]
so \(\Phi\) is a contract mapping on \(C([0,1]; L^2(\mathbb{R}))\). Thus, there is a unique fixed point of \(\Phi\) in \(C([0,1]; L^2(\mathbb{R}))\) which is denoted by \(u^*(g,h)\), i.e. \(\Phi(u^*(g,h)) = u^*(g,h)\). We obtain the existence and uniqueness of the solution of (2.22). Hence, the problem (2.14) has a unique solution \(u^*(g,h)\) \(\in C([0,1]; L^2(\mathbb{R}))\).

\[\square\]

**Lemma 2.4.** Let \(f\) be defined as (2.20). Then for any \((g,h) \in \left(L^2(\mathbb{R})\right)^2\), Problem (2.14) is unstable.
Proof. To show the instability of \( u \), we construct the functions \( g_0 = h_0 = 0 \) and \( (g_n, h_n) \) defined by the Fourier transform, as follows:

\[
\tilde{g}_n(\omega) = \frac{1}{k(\omega)} \chi[n; n + \frac{1}{n}] (\omega), \quad \tilde{h}_n(\omega) = \chi[n; n + \frac{1}{n}] (\omega),
\]

for all \( \omega \in \mathbb{R} \). It is easy to check that

\[
\|g_n - g_0\|_{L^2(\mathbb{R})} + \|h_n - h_0\|_{L^2(\mathbb{R})} \rightarrow 0,
\]

when \( n \rightarrow 0 \). Indeed, we have

\[
\|g_n - g_0\|^2_{L^2(\mathbb{R})} = \int_n^{n + \frac{1}{n}} \frac{1}{k(\omega)}^2 d\omega.
\]

We note that \( \omega^\alpha > 1 \) for all \( \omega \in \left[ n; n + \frac{1}{n} \right] \). So

\[
\int_n^{n + \frac{1}{n}} \frac{1}{\omega^\alpha} d\omega < \int_n^{n + \frac{1}{n}} d\omega < \frac{1}{n} \rightarrow 0 \quad \text{when} \ n \rightarrow +\infty.
\]

This implies \( \|g_n - g_0\|^2_{L^2(\mathbb{R})} \rightarrow 0 \) when \( n \rightarrow +\infty \). Moreover,

\[
\|h_n - h_0\|^2_{L^2(\mathbb{R})} = \|h_n\|^2_{L^2(\mathbb{R})} = \|\tilde{h}_n\|^2_{L^2(\mathbb{R})} = \int_n^{n + \frac{1}{n}} \frac{1}{\omega} d\omega = \frac{1}{n} \rightarrow 0 \quad \text{when} \ n \rightarrow +\infty. \tag{2.30}
\]

Let \( u_n(g_n, h_n) \) and \( u(g_0, h_0) \) be two solutions of Problem \((2.14)\) corresponding to \((g_n, h_n)\) and \((g_0, h_0)\) respectively. The existence of \( u_n(g_n, h_n) \) and \( u(g_0, h_0) \) has been proved in Lemma 2.3. We will show that \( \|u_n(g_n, h_n) - u(g_0, h_0)\|_{L^2(\mathbb{R})} \) does not converge to zero when \( n \rightarrow +\infty \). Since

\[
\tilde{u}_n(g_n, h_n)(x, \omega) = \cosh \left( k(\omega)x \right) \frac{1}{k(\omega)} \chi[n; n + \frac{1}{n}] + \frac{\sinh (k(\omega)x)}{k(\omega)} \chi[n; n + \frac{1}{n}] (\omega)
\]

\[
- \frac{1}{2} \int_0^x \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} e^{-k(\omega)} \tilde{u}_n(g_n, h_n)(z, \omega) dz
\]

\[
= e^{k(\omega)x} \frac{k(\omega)}{k(\omega)} \chi[n; n + \frac{1}{n}] (\omega) - \frac{1}{2} \int_0^x \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} e^{-k(\omega)} \tilde{u}_n(g_n, h_n)(z, \omega) dz. \tag{2.31}
\]

We get

\[
\tilde{u}_n(g_n, h_n)(x, \omega) - \tilde{u}(g_0, h_0)(x, \omega) = A(x, \omega) - B(x, \omega), \tag{2.32}
\]

where

\[
A(x, \omega) := e^{k(\omega)x} \frac{k(\omega)}{k(\omega)} \chi[n; n + \frac{1}{n}] (\omega)
\]

and

\[
B(x, \omega) := \frac{1}{2} \int_0^x \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} e^{-k(\omega)} \left[ \tilde{u}_n(g_n, h_n)(z, \omega) - \tilde{u}(g_0, h_0)(z, \omega) \right] dz.
\]

If \( n \) is large enough, then

\[
\left| A(x, \omega) \right| = \frac{e^{R(k(\omega))x}}{\omega^{\frac{3}{2}}} = \frac{\exp \left( x \cos \frac{\alpha \pi}{4}, \omega \frac{\alpha}{2} \right)}{\omega^{\frac{3}{2}}} \geq \sqrt{6} \omega, \tag{2.33}
\]

where

\[
R(k(\omega)) = \frac{\alpha}{2} \int_0^\infty \frac{1}{\omega} d\omega.
\]
for \( \omega \in \left[ n; n + \frac{1}{n} \right] \). On the other hand, we have

\[
\left| B(x, \omega) \right|^2 \leq \frac{1}{2} \int_0^x \left| \tilde{a}_n(g_n, h_n)(z, \omega) - \tilde{a}(g_0, h_0)(z, \omega) \right|^2 dz
\]

by using the same way in (2.25). Hence,

\[
\int_{[n; n + \frac{1}{n}]} \left| B(x, \omega) \right|^2 d\omega \leq \int_{-\infty}^{+\infty} \left| B(x, \omega) \right|^2 d\omega
\]

\[
\leq \frac{1}{2} \int_0^x \int_{-\infty}^{+\infty} \left| \tilde{a}_n(g_n, h_n)(z, \omega) - \tilde{a}(g_0, h_0)(z, \omega) \right|^2 d\omega dz
\]

\[
\leq \frac{1}{2} \int_0^x \| \tilde{a}_n(g_n, h_n)(z, \cdot) - \tilde{a}(g_0, h_0)(z, \cdot) \|^2_{L^2(\mathbb{R})} dz
\]

\[
\leq \frac{1}{2} \int_0^x \| u_n(g_n, h_n)(z, \cdot) - u(g_0, h_0)(z, \cdot) \|^2_{L^2(\mathbb{R})} dz. \tag{2.34}
\]

It follows from (2.32), (2.33), and (2.34) that

\[
\| u_n(g_n, h_n)(x, \cdot) - u(g_0, h_0)(x, \cdot) \|^2_{L^2(\mathbb{R})}
\]

\[
= \int_{-\infty}^{+\infty} |A(x, \omega) - B(x, \omega)|^2 d\omega
\]

\[
\geq \int_{[n; n + \frac{1}{n}]} \left( \frac{|A(x, \omega)|^2}{2} - |B(x, \omega)|^2 \right) d\omega
\]

\[
\geq \int_{[n; n + \frac{1}{n}]} 3\omega^2 d\omega - \int_{[n; n + \frac{1}{n}]} |B(x, \omega)|^2 d\omega
\]

\[
\geq \int_{[n; n + \frac{1}{n}]} 3\omega^2 d\omega - \frac{1}{2} \int_0^x \| u_n(g_n, h_n)(z, \cdot) - u(g_0, h_0)(z, \cdot) \|^2_{L^2(\mathbb{R})} dz
\]

\[
\geq \int_{[n; n + \frac{1}{n}]} 3\omega^2 d\omega - \frac{1}{2} \sup_{0 \leq z \leq 1} \| u_n(g_n, h_n)(z, \cdot) - u(g_0, h_0)(z, \cdot) \|^2_{L^2(\mathbb{R})}, \tag{2.35}
\]

where we have used the following inequalities

\[
|z_1 - z_2|^2 \geq \left| \frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} \right|,
\]

for all complex numbers \( z_1 \) and \( z_2 \). From (2.35) we have

\[
\| u_n(g_n, h_n)(x, \cdot) - u(g_0, h_0)(x, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{1}{2} \sup_{0 \leq z \leq 1} \| u_n(g_n, h_n)(z, \cdot) - u(g_0, h_0)(z, \cdot) \|^2_{L^2(\mathbb{R})} \geq n
\]

\[
\tag{2.36}
\]

since

\[
\int_{[n; n + \frac{1}{n}]} 3\omega^2 d\omega = \left[ \left( n + \frac{1}{n} \right)^3 - n^3 \right] \geq n.
\]

The left side of (2.36) is less than \( \frac{3}{2} \sup_{0 \leq z \leq 1} \| u_n(g_n, h_n)(x, \cdot) - u(g_0, h_0)(x, \cdot) \|^2_{L^2(\mathbb{R})} \). Hence, we obtain

\[
\sup_{0 \leq z \leq 1} \| u_n(g_n, h_n)(x, \cdot) - u(g_0, h_0)(x, \cdot) \|^2_{L^2(\mathbb{R})} \geq \frac{2}{3} n. \tag{2.37}
\]

The above inequality implies that Problem (2.14) is ill-posed in the Hadamard sense in \( L^2 \)-norm.
Since \(0\), the problem (3.40) can be rewritten as apply the truncation method. Let 3.1. implies that the fractional case of parabolic problem is “less ill-posed” than the classical one. In order to obtain a stable approximate solution of the problem, we can see that the degree of ill-posedness for the classical parabolic problem is \(\left|\frac{\alpha}{2}\right|\). Since \(0 < \alpha < 1\), we have that \(\exp\left(\frac{1}{\delta}\right)\) grows at a slower rate than \(\exp\left(\frac{1}{\delta}\right)\) does. This implies that the fractional case of parabolic problem is “less ill-posed” than the classical one.

3. Regularization and error estimate for problem 2.14

3.1. Regularized solution. In order to obtain a stable approximate solution of the problem, we apply the truncation method. Let \((g^\delta, h^\delta) \in (L^2(\mathbb{R}))^2\) be the measured data which satisfy
\[
\|g^\delta - g\|_{L^2(\mathbb{R})} + \|h^\delta - h\|_{L^2(\mathbb{R})} \leq \delta,
\]
where the constant \(\delta > 0\) is called the error level. We present the following regularization problem
\[
\tilde{w}(x, \omega) = \cosh (k(\omega)x) \hat{g}^\delta(\omega) + \frac{\sinh (k(\omega)x)}{k(\omega)} \hat{h}^\delta(\omega) - \int_0^x \frac{\sinh (k(\omega)(x - z))}{k(\omega)} \hat{f}(z, \omega, w(z, \omega)) dz,
\]
for \(x \geq 0, \omega \in \mathbb{R}\), where \(\epsilon := \epsilon(\delta) > 0\) is a regularization parameter and
\[
\begin{bmatrix}
\hat{g}^\delta(\omega) & \hat{h}^\delta(\omega) \\
\hat{f}(z, \omega, w(z, \omega))
\end{bmatrix}
:= \begin{bmatrix}
g^\delta(\omega) & h^\delta(\omega) \\
\hat{f}(z, \omega, w(z, \omega))
\end{bmatrix} \chi_{[-\frac{1}{4}, \frac{1}{4}]}(\omega).
\]
The problem (3.40) can be rewritten as
\[
w(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh (k(\omega)x) \hat{g}^\delta(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sinh (k(\omega)x)}{k(\omega)} \hat{h}^\delta(\omega) e^{i\omega t} d\omega
\]
\[- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^x \sinh (k(\omega)(x - z)) \hat{f}(z, \omega, w(z, \omega)) e^{i\omega t} dz d\omega,
\]
for \(x \geq 0, t \in \mathbb{R}\).

The following lemma will show that the regularized problem (3.41) is well-posed.

Lemma 3.1. Let \(\epsilon > 0, \delta > 0\) and \(g^\delta, h^\delta \in L^2(\mathbb{R})\). Assume that \(f \in L^\infty([0,1] \times \mathbb{R} \times \mathbb{R})\) satisfies the following condition
\[
|f(x, t, v_1) - f(x, t, v_2)| \leq K|v_1 - v_2|, \quad 0 \leq x \leq 1, \quad t \in \mathbb{R}, \quad v_1, v_2 \in \mathbb{R}
\]
for a constant \(K > 0\) independent of \(x, t, v_1, v_2\). Then Problem (3.41) has a unique solution denoted by \(u^\delta \in C([0, 1]; L^2(\mathbb{R}))\).
Proof. Define

\[ \Theta_{\epsilon, \delta}(w)(x, t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega) x \right) \hat{g}_{\epsilon}^\delta(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( k(\omega) x \right) \hat{h}_{\epsilon}^\delta(\omega) e^{i\omega t} d\omega \]

\[ - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} \hat{f}_{\epsilon}(z, \omega, w(z, \omega)) e^{i\omega t} dzd\omega, \]

for all \( w \in C([0, 1]; L^2(\mathbb{R})) \).

We show that the problem (3.41) has a unique solution by proving that \( \Theta_{\epsilon, \delta} \) has a unique fixed point in \( C([0, 1]; L^2(\mathbb{R})) \). For \( w_1, w_2 \in C([0, 1]; L^2(\mathbb{R})) \) and \( 0 \leq x \leq 1 \), we will show the following estimate

\[ \| \Theta_{\epsilon, \delta}^m(w_1)(x, \cdot) - \Theta_{\epsilon, \delta}^m(w_2)(x, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \left( K \exp \left( e^{-\frac{\alpha}{4}} \cos \frac{\alpha \pi}{4} \right) \right)^2 m \| w_1 - w_2 \|_{C([0, 1]; L^2(\mathbb{R}))}^2 \]

(3.43)

for all integer numbers \( m \geq 1 \). For \( m = 1 \), we have

\[ \Theta_{\epsilon, \delta}(w_1)(x, t) - \Theta_{\epsilon, \delta}(w_2)(x, t) \]

\[ = - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} \left[ \hat{f}_{\epsilon}(z, \omega, w_1(z, \omega)) - \hat{f}_{\epsilon}(z, \omega, w_2(z, \omega)) \right] dz e^{i\omega t} d\omega \]

\[ = \mathcal{F}^{-1} \left( - \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} \left[ \hat{f}_{\epsilon}(z, \omega, w_1(z, \omega)) - \hat{f}_{\epsilon}(z, \omega, w_2(z, \omega)) \right] dz \right)(t). \]

(3.44)

By applying the Parseval’s Theorem to (3.44), we get

\[ \| \Theta_{\epsilon, \delta}(w_1)(x, \cdot) - \Theta_{\epsilon, \delta}(w_2)(x, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[ = \int_{-\infty}^{+\infty} \left| \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} \left[ \hat{f}_{\epsilon}(z, \omega, w_1(z, \omega)) - \hat{f}_{\epsilon}(z, \omega, w_2(z, \omega)) \right] dz \right|^2 d\omega. \]

(3.45)

Applying the Hölder inequality to (3.45), we obtain

\[ \left| \int_{0}^{x} \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} \left[ \hat{f}_{\epsilon}(z, \omega, w_1(z, \omega)) - \hat{f}_{\epsilon}(z, \omega, w_2(z, \omega)) \right] dz \right|^2 \]

\[ \leq x \int_{0}^{x} \left| \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} \right|^2 \left| \hat{f}_{\epsilon}(z, \omega, w_1(z, \omega)) - \hat{f}_{\epsilon}(z, \omega, w_2(z, \omega)) \right|^2 dz \]

\[ \leq x \int_{0}^{x} (x - z)^2 \exp \left( 2\Re(k(\omega))(x - z) \right) \left| \hat{f}_{\epsilon}(z, \omega, w_1(z, \omega)) - \hat{f}_{\epsilon}(z, \omega, w_2(z, \omega)) \right|^2 dz, \]

where

\[ \left| \frac{\sinh \left( k(\omega) (x - z) \right)}{k(\omega)} \right|^2 \leq (x - z)^2 \exp \left( 2\Re(k(\omega))(x - z) \right) \]

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by using the Lemma 2.1. Therefore,
\[
\|\Theta_{e,\delta}(w_1(x,\cdot) - \Theta_{e,\delta}(w_2(x,\cdot))\|_{L^2(\mathbb{R})}^2 \\
\leq x \int_0^x \int_{-\infty}^{+\infty} (x-z)^2 \exp \left(2R(k(\omega))(x-z)\right) \left| \tilde{f}_e(z,\omega, w_1(z,\omega)) - \tilde{f}_e(z,\omega, w_2(z,\omega)) \right|^2 d\omega dz \\
\leq x \int_0^x \int_{-\frac{1}{2}}^{\frac{1}{2}} (x-z)^2 \exp \left(2R(k(\omega))(x-z)\right) \left| \tilde{f}(z,\omega, w_1(z,\omega)) - \tilde{f}(z,\omega, w_2(z,\omega)) \right|^2 d\omega dz \\
\leq x^3 \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \int_0^x \| \tilde{f}(z,\cdot, w_1(z,\cdot)) - \tilde{f}(z,\cdot, w_2(z,\cdot)) \|_{L^2(\mathbb{R})}^2 dz \\
\leq \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \int_0^x \| f(z,\cdot, w_1(z,\cdot)) - f(z,\cdot, w_2(z,\cdot)) \|_{L^2(\mathbb{R})}^2 dz \\
\leq K^2 \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \int_0^x \| w_1(z,\cdot) - w_2(z,\cdot) \|_{L^2(\mathbb{R})}^2 dz,
\]
(3.46)
where the Lipschitz property (3.42) has been used. This immediately implies that
\[
\|\Theta_{e,\delta}(w_1(x,\cdot) - \Theta_{e,\delta}(w_2(x,\cdot))\|_{L^2(\mathbb{R})}^2 \leq K^2 \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \| w_1 - w_2 \|_{C([0,1];L^2(\mathbb{R}))}^2,
\]
so (3.43) holds for \( m = 1 \). Assume that (3.43) holds for \( m = j, j \geq 1 \), i.e.,
\[
\|\Theta_{e,\delta}^j (w_1(x,\cdot) - \Theta_{e,\delta}^j (w_2(x,\cdot))\|_{L^2(\mathbb{R})}^2 \leq \left( K \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \right)^{2j} \frac{z_j}{j!} \| w_1 - w_2 \|_{C([0,1];L^2(\mathbb{R}))}^2.
\]
We are going to prove that (3.43) holds for \( m = j + 1 \). Indeed, it follows from (3.46) that
\[
\|\Theta_{e,\delta}^{j+1} (w_1(x,\cdot) - \Theta_{e,\delta}^{j+1} (w_2(x,\cdot))\|_{L^2(\mathbb{R})}^2 \\
\leq K^2 \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \int_0^x \|\Theta_{e,\delta}^j (w_1(z,\cdot) - \Theta_{e,\delta}^j (w_2(z,\cdot))\|_{L^2(\mathbb{R})}^2 dz \\
\leq K^2 \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \int_0^x \left( K \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \right)^{2j} \frac{z_j}{j!} \| w_1 - w_2 \|_{C([0,1];L^2(\mathbb{R}))}^2 dz \\
\leq \left( K \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \right)^{2(j+1)} \frac{z_j}{j!} \| w_1 - w_2 \|_{C([0,1];L^2(\mathbb{R}))}^2.
\]
(3.47)
By induction principle, we conclude that (3.43) holds for all integer numbers \( m \geq 1 \). Now, by taking the supremum of (3.47) in the variable \( x \), we derive
\[
\|\Theta_{e,\delta}^m (w_1) - \Theta_{e,\delta}^m (w_2)\|_{C([0,1];L^2(\mathbb{R}))}^2 \leq \left( K \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \right)^{2m} \frac{1}{m!} \| w_1 - w_2 \|_{C([0,1];L^2(\mathbb{R}))}^2.
\]
(3.48)
On the other hand, because
\[
\lim_{m \to +\infty} \left( K \exp \left(2x \epsilon^{-\frac{3}{2}} \cos \frac{\alpha \pi}{4}\right) \right)^{2m} \frac{1}{m!} = 0.
\]
It follows from (3.48) that there exists an integer number \( m_* \geq 1 \) such that \( \Theta_{e,\delta}^{m_*} \) is a contraction mapping. Thus, there exists a unique fixed point of \( \Theta_{e,\delta}^{m_*} \in C([0,1];L^2(\mathbb{R})) \) which denoted by \( w_*^\delta \), i.e.,
\[
\Theta_{e,\delta}^{m_*} (w_*^\delta) = w_*^\delta.
\]
Hence \( \Theta_{e,\delta}^{m_*} (\Theta_{e,\delta} (w_*^\delta)) = \Theta_{e,\delta} (w_*^\delta) \), i.e., \( \Theta_{e,\delta} (w_*^\delta) \) is also a fixed point of \( \Theta_{e,\delta}^{m_*} \) in \( C([0,1];L^2(\mathbb{R})) \). This implies that \( \Theta_{e,\delta} (w_*^\delta) = w_*^\delta \), due to the uniqueness of the fixed point.
3.2. $L^2$ estimate. From now on, let $\delta > 0$ be the error level and $g^\delta, h^\delta \in L^2(\mathbb{R})$ be the measured data satisfy (3.39). Let $\epsilon := \epsilon(\delta) > 0$ be the regularization parameter and let $u^\delta_\epsilon$ be the regularized solution of (3.41) respectively.

**Theorem 3.1.** Let $f$ be defined by Lemma 3.1 and assume that the problem (2.14) has a unique (exact) solution $u$ such that

$$\int_{-\infty}^{+\infty} \exp(2(1-x)\Re(k(\omega))) |\hat{u}(x,\omega)|^2 \, d\omega < M_1^2, \quad 0 \leq x < 1,$$

for $M_1 > 0$, then

$$\|u^\delta_\epsilon(x,.) - u(x,.)\|_{L^2(\mathbb{R})} \leq C_1 \exp\left(xe^{-\frac{\alpha}{4}} \cos \frac{\alpha \pi}{4}\right) \delta + C_1 \exp\left((x-1)e^{-\frac{\alpha}{4}} \cos \frac{\alpha \pi}{4}\right)$$

where $C_1 = 4e^{K^2} + 2M_1 e^{K^2}$. As a consequence, if we choose $\epsilon = \left(\frac{\cos \frac{\alpha \pi}{4}}{\ln \frac{1}{\delta}}\right)^\frac{1}{2}$ then

$$\|u^\delta_\epsilon(x,.) - u(x,.)\|_{L^2(\mathbb{R})} \leq C_1 \delta^{1-x}, \quad 0 \leq x < 1.$$

**Remark 3.1.** If $f = 0$ and $h = 0$ then since (2.13), we have

$$\hat{u}(x,\omega) = \cosh(\frac{k(\omega)x}{2}) \hat{g}(\omega).$$

Then the left-hand side of (3.49) is

$$\int_{-\infty}^{+\infty} \exp(2(1-x)\Re(k(\omega))) |\hat{u}(x,\omega)|^2 \, d\omega = \int_{-\infty}^{+\infty} \exp(2(1-x)\Re(k(\omega))) \left| \cosh(\frac{k(\omega)x}{2}) \right|^2 \hat{g}(\omega)^2 \, d\omega$$

where we have used $|\cosh(z)| \leq e^{\Re(z)}$ since Lemma 2.1. Moreover, we get

$$k(\omega) \int_{0}^{1} \hat{u}(z,\omega) dz = \frac{e^{k(\omega)} - e^{-k(\omega)}}{2} \hat{g}(\omega).$$

This implies that

$$\hat{u}(1,\omega) + k(\omega) \int_{0}^{1} \hat{u}(z,\omega) dz = e^{k(\omega)} \hat{g}(\omega).$$

Hence, we get

$$\int_{-\infty}^{+\infty} \exp(2\Re(k(\omega))) \left| \hat{g}(\omega) \right|^2 \, d\omega \leq 2 \int_{-\infty}^{+\infty} |\hat{u}(1,\omega)|^2 \, d\omega + 2 \int_{-\infty}^{+\infty} |\omega|^\alpha \int_{0}^{1} |\hat{u}(z,\omega)|^2 \, dz \, d\omega$$

$$= 2 \int_{-\infty}^{+\infty} |\hat{u}(1,\omega)|^2 \, d\omega + 2 \int_{0}^{1} \int_{-\infty}^{+\infty} |\omega|^\alpha |\hat{u}(z,\omega)|^2 \, d\omega \, dz$$

$$= 2 \|u(1,.)\|_{L^2(\mathbb{R})}^2 + 2 \|u\|_{L^2(0,1; H^{\frac{\alpha}{2}}(\mathbb{R}))}^2.$$  (3.55)

Thus, if $u \in L^2(0,1; H^{\frac{\alpha}{2}}(\mathbb{R}))$ then (3.55) holds. Therefore, we can say that the condition (3.49) is nature and makes sense.
Proof. We only consider $0 \leq x < 1$ throughout this proof. We recall that

$$u_\varepsilon(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega)x \right) \hat{g}_\varepsilon(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( k(\omega)x \right) \hat{h}_\varepsilon(\omega) e^{i\omega t} d\omega$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega)(x-z) \right)}{k(\omega)} \hat{f}_\varepsilon(z, \omega, u_\varepsilon(z, \omega)) e^{i\omega t} dz d\omega$$

and

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega)x \right) \hat{g}(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( k(\omega)x \right) \hat{h}(\omega) e^{i\omega t} d\omega$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega)(x-z) \right)}{k(\omega)} \hat{f}(z, \omega, u(z, \omega)) e^{i\omega t} dz d\omega.$$  (3.56)

Applying Lemma 3.1, there exists a unique function $u_\varepsilon \in C([0, 1] \cap L^2(\mathbb{R}))$ satisfying

$$u_\varepsilon(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega)x \right) \hat{g}_\varepsilon(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( k(\omega)x \right) \hat{h}_\varepsilon(\omega) e^{i\omega t} d\omega$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega)(x-z) \right)}{k(\omega)} \hat{f}_\varepsilon(z, \omega, u_\varepsilon(z, \omega)) e^{i\omega t} dz d\omega.$$  (3.58)

where $\left[ \hat{g}_\varepsilon(\omega) \quad \hat{h}_\varepsilon(\omega) \quad \hat{f}_\varepsilon(z, \omega, w(z, \omega)) \right] := \left[ \hat{g}(\omega) \quad \hat{h}(\omega) \quad \hat{f}(z, \omega, w(z, \omega)) \right] \chi_{\left[ \frac{1}{2}, \frac{1}{2} \right]}(\omega)$. In order to establish an estimate for $\| u_\varepsilon(x, .) - u(x, .) \|_{L^2(\mathbb{R})}$, we introduce the quantity $P_\varepsilon(u)$ defined as follows

$$P_\varepsilon(u)(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega)x \right) \hat{g}_\varepsilon(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( k(\omega)x \right) \hat{h}_\varepsilon(\omega) e^{i\omega t} d\omega$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega)(x-z) \right)}{k(\omega)} \hat{f}_\varepsilon(z, \omega, u_\varepsilon(z, \omega)) e^{i\omega t} dz d\omega,$$  (3.59)

where $u$ is the exact solution. The triangle inequality shows that

$$\| u_\varepsilon(x, .) - u(x, .) \|_{L^2(\mathbb{R})}^2 \leq 2 \| u_\varepsilon(x, .) - u_\varepsilon(x, .) \|_{L^2(\mathbb{R})}^2 + 2 \| u_\varepsilon(x, .) - u(x, .) \|_{L^2(\mathbb{R})}^2.$$  (3.60)

Next, we divide the proof into three steps.

**Step 1:** Estimate for $\| u_\varepsilon(x, .) - u_\varepsilon(x, .) \|_{L^2(\mathbb{R})}^2$. It follows from (3.56) and (3.58) that

$$u_\varepsilon(x, t) - u_\varepsilon(x, t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega)x \right) \left[ \hat{g}_\varepsilon(\omega) - \hat{g}(\omega) \right] e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( k(\omega)x \right) \left[ \hat{h}_\varepsilon(\omega) - \hat{h}(\omega) \right] e^{i\omega t} d\omega$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{x} \frac{\sinh \left( k(\omega)(x-z) \right)}{k(\omega)} \left[ \hat{f}_\varepsilon(z, \omega, u_\varepsilon(z, \omega)) - \hat{f}(z, \omega, u(z, \omega)) \right] e^{i\omega t} dz d\omega.$$
which can be represented by the inverse Fourier transform as

\[ u^\delta_t(x, t) - u_e(x, t) \]

\[ \begin{aligned} &\mathcal{F}^{-1} \left( \cosh \left( k(\omega)x \right) \left[ \hat{g}_e^\delta(\omega) - \hat{g}_e(\omega) \right] + \frac{\sinh \left( k(\omega)x \right)}{k(\omega)} \left[ \hat{h}_e^\delta(\omega) - \hat{h}_e(\omega) \right] \right)(t) \\
&+ \mathcal{F}^{-1} \left( -\int_0^x \frac{\sinh \left( k(\omega)(x-z) \right)}{k(\omega)} \left[ \hat{f}_e(z, \omega, u^\delta_e(z, \omega)) - \hat{f}_e(z, \omega, u_e(z, \omega)) \right] dz \right)(t). \end{aligned} \]  

(3.61)

By applying the Parseval's Theorem to (3.61), we derive

\[ \begin{aligned} &\| u^\delta_t(x, \cdot) - u_e(x, \cdot) \|^2_{L^2(\mathbb{R})} \\
&\leq 2 \int_{-\infty}^{+\infty} \left\{ \cosh \left( k(\omega)x \right) \left| \hat{g}_e^\delta(\omega) - \hat{g}_e(\omega) \right|^2 + \frac{\sinh \left( k(\omega)x \right)}{k(\omega)} \left| \hat{h}_e^\delta(\omega) - \hat{h}_e(\omega) \right|^2 \right\} d\omega \\
&+ 2 \int_{-\infty}^{+\infty} \left| \int_0^x \frac{\sinh \left( k(\omega)(x-z) \right)}{k(\omega)} \left[ \hat{f}_e(z, \omega, u^\delta_e(z, \omega)) - \hat{f}_e(z, \omega, u_e(z, \omega)) \right] dz \right|^2 d\omega. \end{aligned} \]  

(3.62)

We continue to estimate \( I_1 \) as follows:

\[ \begin{aligned} I_1 &\leq 2 \int_{-\infty}^{+\infty} \left| \cosh \left( k(\omega)x \right) \left| \hat{g}_e^\delta(\omega) - \hat{g}_e(\omega) \right|^2 + 2 \int_{-\infty}^{+\infty} \left| \frac{\sinh \left( k(\omega)x \right)}{k(\omega)} \right|^2 \left| \hat{h}_e^\delta(\omega) - \hat{h}_e(\omega) \right|^2 d\omega \\
&\leq 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \cosh \left( k(\omega)x \right) \right|^2 \left| \hat{g}_e^\delta(\omega) - \hat{g}_e(\omega) \right|^2 d\omega + 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\sinh \left( k(\omega)x \right)}{k(\omega)} \right|^2 \left| \hat{h}_e^\delta(\omega) - \hat{h}(\omega) \right|^2 d\omega \\
&\leq 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp \left( 2x |\Re(k(\omega))| \right) \left| \hat{g}_e^\delta(\omega) - \hat{g}(\omega) \right|^2 d\omega + 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \exp \left( 2x |\Re(k(\omega))| \right) \left| \hat{h}_e^\delta(\omega) - \hat{h}(\omega) \right|^2 d\omega, \end{aligned} \]

where

\[ \begin{aligned} &\left| \cosh \left( k(\omega)x \right) \right|^2 \leq \exp \left( 2x |\Re(k(\omega))| \right), \\
&\left| \frac{\sinh \left( k(\omega)x \right)}{k(\omega)} \right|^2 \leq x^2 \exp \left( 2x |\Re(k(\omega))| \right) \end{aligned} \]

by applying Lemma 2.1. On the other hand, for \(|\omega| \leq \frac{1}{e}\) we have

\[ \exp \left( 2x |\Re(k(\omega))| \right) \leq \exp \left( 2x e^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right). \]

Thus we get

\[ I_1 \leq 2 \exp \left( 2x e^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \| \hat{g}_e^\delta - \hat{g} \|^2_{L^2(\mathbb{R})} + 2x^2 \exp \left( 2x e^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \| \hat{h}_e^\delta - \hat{h} \|^2_{L^2(\mathbb{R})}. \]
Consequently, we can obtain

\[ I_1 \leq 4 \exp \left( 2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \delta^2, \quad (3.63) \]

where the fact that \( x \leq 1 \) and the condition (3.39) have been used. Applying the Hölder’s inequality and Lemma 2.1, it yields we obtain

\[ I_2 \leq \int_{-\alpha}^{\alpha} \int_{-\infty}^{\infty} \left| \sinh \left( \frac{k(\omega)(x-z)}{k(\omega)} \right) \right| \left| \hat{f}_\epsilon(z, \omega, u^\delta(z, \omega)) - \hat{f}_\epsilon(z, \omega, u(\epsilon, z, \omega)) \right|^2 dz d\omega \]

\[ \leq x \int_0^x \int_{-\frac{1}{2}}^{\frac{1}{2}} (x-z)^2 \exp \left( 2(x-z) \Re(k(\omega)) \right) \left| \hat{f}(z, \omega, u^\delta(z, \omega)) - \hat{f}(z, \omega, u(\epsilon, z, \omega)) \right|^2 d\omega dz. \]

It follows from \( x(x-z)^2 \leq 1 \) and \( \exp \left( 2(x-z) \Re(k(\omega)) \right) \leq \exp \left( 2(x-z)\epsilon^{-\frac{a}{2}} \cos \frac{\alpha \pi}{4} \right) \) for \( 0 \leq z \leq x \leq 1, |\omega| \leq \frac{1}{\epsilon} \) that

\[ I_2 \leq \exp \left( 2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \int_0^x \exp \left( -2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \times \]

\[ \times \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \hat{f}(z, \omega, u^\delta(z, \omega)) - \hat{f}(z, \omega, u(\epsilon, z, \omega)) \right|^2 d\omega dz. \]

Using Paserval’s identity and the Lipschitz condition (3.42), we obtain

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \hat{f}(z, \omega, u^\delta(z, \omega)) - \hat{f}(z, \omega, u(\epsilon, z, \omega)) \right|^2 d\omega dz \]

\[ \leq \| \hat{f}(z, , u^\delta(z, .)) - \hat{f}(z, , u(\epsilon, z, .)) \|^2_{L^2(\mathbb{R})} \]

\[ = \| f(z, , u^\delta(z, .)) - f(z, , u(\epsilon, z, .)) \|^2_{L^2(\mathbb{R})} \]

\[ \leq K^2 \| u^\delta(z, .) - u(\epsilon, z, .) \|^2_{L^2(\mathbb{R})}. \]

Therefore, we derive

\[ I_2 \leq K^2 \exp \left( 2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \int_0^x \exp \left( -2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) K^2 \| u^\delta(z, .) - u(\epsilon, z, .) \|^2_{L^2(\mathbb{R})} dz. \]

\[ \leq K^2 \exp \left( 2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \int_0^x \mathcal{Y}_1(z) dz, \quad (3.64) \]

where we put

\[ \mathcal{Y}_1(z) := \exp \left( -2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \| u^\delta(z, .) - u(\epsilon, z, .) \|^2_{L^2(\mathbb{R})} \geq 0 \]

for \( 0 \leq z \leq x \). It follows from (3.62), (3.63), and (3.64) that

\[ \| u^\delta(x, .) - u(\epsilon, x, .) \|^2_{L^2(\mathbb{R})} \leq 2I_1 + 2I_2 \]

\[ \leq 8 \exp \left( 2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \delta^2 + 2K^2 \exp \left( 2x_0^{-a} \cos \frac{\alpha \pi}{4} \right) \int_0^x \mathcal{Y}_1(z) dz, \quad (3.65) \]

which is equivalent to

\[ \mathcal{Y}_1(x) \leq 8\delta^2 + 2K^2 \int_0^x \mathcal{Y}_1(z) dz. \quad (3.66) \]
Applying the Gronwall’s inequality to (3.67), we obtain
\[ \mathcal{V}_1(x) \leq 8\delta^2 e^{2K^2 x}. \]

Hence,
\[ \|u_\epsilon(x,\cdot) - u(x,\cdot)\|^2_{L^2(\mathbb{R})} \leq 8e^{2K^2} \exp \left( 2x\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \delta^2. \] (3.67)

**Step 2:** Estimate for \( \|u_\epsilon(x,\cdot) - u(x,\cdot)\|^2_{L^2(\mathbb{R})} \). We have
\[ \|u_\epsilon(x,\cdot) - u(x,\cdot)\|^2_{L^2(\mathbb{R})} \leq 2\|u_\epsilon(x,\cdot) - \mathcal{P}_\epsilon(u)(x,\cdot)\|^2_{L^2(\mathbb{R})} + 2\|\mathcal{P}_\epsilon(u)(x,\cdot) - u(x,\cdot)\|^2_{L^2(\mathbb{R})}. \] (3.68)

We split this step into three sub-steps. The first one is estimating \( \|u_\epsilon(x,\cdot) - \mathcal{P}_\epsilon(u)(x,\cdot)\|^2_{L^2(\mathbb{R})} \) and the second one is estimating \( \|\mathcal{P}_\epsilon(u)(x,\cdot) - u(x,\cdot)\|^2_{L^2(\mathbb{R})} \). The last one will combine two first ones to obtain an estimate for \( \|u_\epsilon(x,\cdot) - u(x,\cdot)\|^2_{L^2(\mathbb{R})} \).

**Sub-step 2a:** Estimate for \( \|u_\epsilon(x,\cdot) - \mathcal{P}_\epsilon(u)(x,\cdot)\|^2_{L^2(\mathbb{R})} \). By subtracting the equations (3.58) and (3.59), we derive
\[ u_\epsilon(x,t) - \mathcal{P}_\epsilon(u)(x,t) = \mathcal{F}^{-1} \left( - \int_0^x \sinh \left( \frac{k(\omega)(x-z)}{k(\omega)} \right) \left[ \tilde{f}_\epsilon(z,\omega,u_\epsilon(z,\omega)) - \tilde{f}_\epsilon(z,\omega,u(z,\omega)) \right] dz \right)(t). \]

By using the same proof as for (3.64), we obtain
\[ \|u_\epsilon(x,\cdot) - \mathcal{P}_\epsilon(u)(x,\cdot)\|^2_{L^2(\mathbb{R})} \leq \mathcal{K}^2 \exp \left( 2x\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \int_0^x \mathcal{Z}_1(z)dz, \] (3.69)
where \( \mathcal{Z}_1(z) := \exp \left( -2\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \|u_\epsilon(z,\cdot) - u(z,\cdot)\|^2_{L^2(\mathbb{R})} \geq 0 \) for all \( 0 \leq z \leq x \).

**Sub-step 2b:** Estimate for \( \|\mathcal{P}_\epsilon(u)(x,\cdot) - u(x,\cdot)\|^2_{L^2(\mathbb{R})} \). By subtracting the equations (3.58) and (3.59), we obtain
\[ \mathcal{P}_\epsilon(u)(x,t) - u(x,t) \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cosh \left( k(\omega)x \right) \left[ \dot{g}_\epsilon(\omega) - \dot{g}(\omega) \right] e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sinh \left( k(\omega)x \right) \left[ \dot{h}_\epsilon(\omega) - \dot{h}(\omega) \right] e^{i\omega t} d\omega \]
\[ - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^x \sinh \left( \frac{k(\omega)(x-z)}{k(\omega)} \right) \left[ \tilde{f}_\epsilon(z,\omega,u(z,\omega)) - \tilde{f}(z,\omega,u(z,\omega)) \right] e^{i\omega t} dz d\omega, \]
which can be represented by the inverse Fourier transform as follows
\[ \mathcal{P}_\epsilon(u)(x,t) - u(x,t) \]
\[ = \mathcal{F}^{-1} \left( \cosh \left( k(\omega)x \right) \left[ \dot{g}_\epsilon(\omega) - \dot{g}(\omega) \right] + \sinh \left( k(\omega)x \right) \left[ \dot{h}_\epsilon(\omega) - \dot{h}(\omega) \right] \right)(t) \]
\[ + \mathcal{F}^{-1} \left( - \int_0^x \sinh \left( \frac{k(\omega)(x-z)}{k(\omega)} \right) \left[ \tilde{f}_\epsilon(z,\omega,u(z,\omega)) - \tilde{f}(z,\omega,u(z,\omega)) \right] \right)(t) \]
\[ = \mathcal{F}^{-1} \left( -\hat{u}(x,\omega)\chi_R([-\frac{1}{2},\frac{1}{2}])\right)(\omega) \] (3.70)
Taking $L^2(\mathbb{R})$-norm of $P_\epsilon(u)(x, t) - u(x, t)$ and applying the Parseval’s identity, we get
\[ \|P_\epsilon(u)(x, .) - u(x, .)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}\setminus[-\frac{1}{8}, \frac{1}{8}]} |\hat{u}\epsilon(x, \omega)|^2 d\omega. \]
Therefore,
\[ \|P_\epsilon(u)(x, .) - u(x, .)\|_{L^2(\mathbb{R})}^2 \]
\[ = \int_{\mathbb{R}\setminus[-\frac{1}{8}, \frac{1}{8}]} \exp\left(2(x-1)\Re (k(\omega))\right) |\hat{u}\epsilon(x, \omega)|^2 \exp\left(2(1-x)\Re (k(\omega))\right) d\omega \]
\[ \leq \exp\left(2(x-1)\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right) \int_{\mathbb{R}\setminus[-\frac{1}{8}, \frac{1}{8}]} \exp\left(2(1-x)\Re (k(\omega))\right) |\hat{u}\epsilon(x, \omega)|^2 d\omega, \]
where we have used that
\[ \exp\left(2(x-1)\Re (k(\omega))\right) = \exp\left(2(x-1)|\omega|^2 \cos \frac{\alpha\pi}{4}\right) \leq \exp\left(2(x-1)\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right), \]
for all $|\omega| \leq \frac{1}{\epsilon}$. On the other hand, it follows from the assumption (3.77) that
\[ \int_{\mathbb{R}\setminus[-\frac{1}{8}, \frac{1}{8}]} \exp\left(2(1-x)\Re (k(\omega))\right) |\hat{u}\epsilon(x, \omega)|^2 d\omega \leq M_1^2. \]
Hence,
\[ \|P_\epsilon(u)(x, .) - u(x, .)\|_{L^2(\mathbb{R})}^2 \leq M_1^2 \exp\left(2(x-1)\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right). \] (3.71)

**Sub-step 2c:** Estimate for $\|u_\epsilon(x, .) - u(x, .)\|_{L^2(\mathbb{R})}$. Combining (3.68), (3.69) and (3.71), we derive
\[ \|u_\epsilon(x, .) - u(x, .)\|_{L^2(\mathbb{R})} \leq 2M_1^2 \exp\left(2(x-1)\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right) \]
\[ + 2K^2 \exp\left(2\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right) \int_0^x \mathcal{Z}_1(z)dz. \] (3.72)
This implies that
\[ \mathcal{Z}_1(x) \leq 2M_1^2 \exp\left(-2\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right) + 2K^2 \int_0^x \mathcal{Z}_1(z)dz. \] (3.73)
Therefore,
\[ \mathcal{Q}_1(x) \leq 2M_1^2 \exp\left(-2\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right) e^{2K^2x} \]
by applying Gronwall’s inequality to (3.73). Thus,
\[ \|u_\epsilon(x, .) - u(x, .)\|_{L^2(\mathbb{R})} \leq 2M_1^2 e^{2K^2x} \exp\left(2(x-1)\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right). \] (3.74)

**Step 3:** Estimate for $\|u_\epsilon^\delta(x, .) - u(x, .)\|_{L^2(\mathbb{R})}$. Since (3.60), (3.67) and (3.74), we obtain
\[ \|u_\epsilon^\delta(x, .) - u(x, .)\|_{L^2(\mathbb{R})} \leq 16\epsilon^{2K^2} \exp\left(2\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right) \delta^2 + 4M_1^2 e^{2K^2} \exp\left(2(x-1)\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right). \]
So
\[ \|u_\epsilon^\delta(x, .) - u(x, .)\|_{L^2(\mathbb{R})} \leq 4\epsilon^{2K^2} \exp\left(\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right) \delta + 2M_1 e^{K^2} \exp\left((x-1)\epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha\pi}{4}\right), \] (3.75)
i.e., (3.50) is proved. In addition, substituting $\epsilon = \left(\frac{\cos \frac{\alpha\pi}{4}}{\ln \frac{1}{\delta}}\right)^{\frac{2}{\alpha}}$ into (3.75) we get
\[ \|u_\epsilon^\delta(x, .) - u(x, .)\|_{L^2(\mathbb{R})} \leq C_1 \delta^{1-x}. \] (3.76)
3.3. $H^p$ estimate. In Theorem 3.1, an estimate of $\|u^\delta(x, .) - u(x, .)\|_{L^2(\mathbb{R})}$ was given according to the a priori condition (3.49). To obtain a result in $H^p(\mathbb{R})$, we assume that the exact solution $u$ satisfies the stronger condition (3.77). Recall that $H^p(\mathbb{R})$ is defined in Section 2.

**Theorem 3.2.** Let $f$ be defined by Lemma 3.1. Suppose that there exist constants $M_2 > 0$, $\gamma > 0$ and $\mu > \max\{4 - \alpha; 4p - \alpha\}$ such that

$$
\int_{-\infty}^{+\infty} \exp \left(2(1 + \gamma - x)|\omega|^{\frac{q}{2}} \cos \frac{\alpha \pi}{4}\right) |\hat{u}(x, \omega)|^2 d\omega < M_2^2, \quad 0 \leq x < 1. \tag{3.77}
$$

Let us choose $\delta$ such that

$$
\left(\frac{A + B}{\ln \frac{1}{\delta}}\right)^{\frac{2}{\alpha + \mu}} < \min \left\{1; \left(\frac{B}{A}\right)^{\frac{2}{\alpha + \mu}} \cdot \left[\frac{\mu + \mu \gamma \cos \frac{\alpha \pi}{4}}{p}\right]^{\frac{2}{\alpha + \mu}}\right\} \tag{3.78}
$$

and choose $\epsilon = \left(\frac{A + B}{\ln \frac{1}{\delta}}\right)^{\frac{2}{\alpha + \mu}}$ then

$$
\|u^\delta(x, .) - u(x, .)\|_{H^p(\mathbb{R})} \leq C_2 \delta^{-\frac{B}{\alpha + \mu}}, \tag{3.79}
$$

for $0 \leq x < 1$ where $A = \frac{p}{2} + 1 + K^2 2^p, \quad B = \frac{1}{2} \gamma \cos \frac{\alpha \pi}{4}, q = \max\{2; \frac{\alpha}{2}; 2p\}, \quad C_2 = 4 + 2M_2$.

**Proof.** First, we have

$$
\|u^\delta(x, .) - u(x, .)\|_{H^p(\mathbb{R})}^2 \leq 2\|u^\delta(x, .) - u_\epsilon(x, .)\|_{H^p(\mathbb{R})}^2 + 2\|u_\epsilon(x, .) - u(x, .)\|_{H^p(\mathbb{R})}^2. \tag{3.80}
$$

We split the proof into three steps as follows

**Step 1:** Estimate $\|u^\delta(x, .) - u_\epsilon(x, .)\|_{H^p(\mathbb{R})}^2$. From (3.61), we get

$$
\|u^\delta(x, .) - u_\epsilon(x, .)\|_{H^p(\mathbb{R})}^2 \leq 2 \int_{-\infty}^{+\infty} (1 + \omega^2)^p \left| \cosh \left( k(\omega)x \right) \left[ \hat{g}^\delta(\omega) - \hat{g}_\epsilon(\omega) \right] + \sinh \left( k(\omega)x \right) \left[ \hat{h}^\delta(\omega) - \hat{h}_\epsilon(\omega) \right] \right|^2 d\omega
$$

$$
= J_1 + 2 \int_{-\infty}^{+\infty} (1 + \omega^2)^p \int_0^x \frac{\sinh \left( k(\omega)(x - z) \right)}{k(\omega)} \left[ \hat{f}_\epsilon(z, \omega, u^\delta(z, \omega)) - \hat{f}_\epsilon(z, \omega, u_\epsilon(z, \omega)) \right] dz d\omega. \tag{3.81}
$$

The quantities $J_1$ and $J_2$ are estimated by the same way as Theorem 3.1. Firstly, we note that

$$
(1 + \omega^2)^p \leq (1 + \epsilon^{-2})^p \quad \text{and} \quad \exp \left(2x \Re(k(\omega))\right) \leq \exp \left(2xe^{-\frac{\alpha \pi}{4}} \cos \frac{\alpha \pi}{4}\right) \quad \text{for all} \quad |\omega| \leq \frac{1}{\epsilon},
$$

so

$$
J_1 \leq 2 (1 + \epsilon^{-2})^p \exp \left(2xe^{-\frac{\alpha \pi}{4}} \cos \frac{\alpha \pi}{4}\right) \|\hat{g}^\delta - \hat{g}\|_{L^2(\mathbb{R})}^2
$$

$$
+ 2 (1 + \epsilon^{-2})^p \exp \left(2xe^{-\frac{\alpha \pi}{4}} \cos \frac{\alpha \pi}{4}\right) \|\hat{h}^\delta - \hat{h}\|_{L^2(\mathbb{R})}^2.
$$
This implies
\[ \|u^\delta(x, \cdot) - u(x, \cdot)\|_{H^p(\mathbb{R})}^2 \leq 4 (1 + \epsilon^{-2})^p \exp \left( 2x\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \delta^2, \]  
where the condition (3.39) has been used. Secondly, it follows from
\[ \exp \left( 2(x-z)H(k(\omega)) \right) \leq \exp \left( 2(x-z)e^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right), \]
for all \(|\omega| \leq \frac{1}{\epsilon}\) and \(z \leq x\) that
\[ J_2 \leq (1 + \epsilon^{-2})^p K^2 \exp \left( 2x\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \int_0^x \exp \left( -2z\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \|u^\delta_\epsilon(z, \cdot) - u_\epsilon(z, \cdot)\|_{L^2(\mathbb{R})}^2 dz, \]
where the Lipschitz property (3.42) has been used. Moreover, because of the fact that
\[ \|u^\delta_\epsilon(z, \cdot) - u_\epsilon(z, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \|u^\delta_\epsilon(z, \cdot) - u_\epsilon(z, \cdot)\|_{H^p(\mathbb{R})}^2, \]
we obtain
\[ J_2 \leq K^2 (1 + \epsilon^{-2})^p \exp \left( 2x\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \int_0^x \mathcal{Y}_2(z) dz, \]  
where
\[ \mathcal{Y}_2(z) = \exp \left( -2z\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \|u^\delta_\epsilon(z, \cdot) - u_\epsilon(z, \cdot)\|_{H^p(\mathbb{R})}^2 \geq 0, \]
for all \(0 \leq z \leq x\). Now, by associating (3.81) with (3.82), we derive
\[ \|u^\delta_\epsilon(x, \cdot) - u_\epsilon(x, \cdot)\|_{H^p(\mathbb{R})}^2 \leq 8 (1 + \epsilon^{-2})^p \exp \left( 2x\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \delta^2 + 2K^2 (1 + \epsilon^{-2})^p \exp \left( 2x\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \int_0^x \mathcal{Y}_2(z) dz. \]
This implies that
\[ \mathcal{Y}_2(x) \leq 8 (1 + \epsilon^{-2})^p \delta^2 + 2K^2 (1 + \epsilon^{-2})^p \int_0^x \mathcal{Y}_2(z) dz. \]
Applying the Gronwall’s inequality, we get
\[ \mathcal{Y}_2(x) \leq 8 (1 + \epsilon^{-2})^p \delta^2 \exp \left( 2K^2 (1 + \epsilon^{-2})^p x \right). \]
Therefore,
\[ \|u^\delta_\epsilon(x, \cdot) - u_\epsilon(x, \cdot)\|_{H^p(\mathbb{R})}^2 \leq 8 (1 + \epsilon^{-2})^p \exp \left( 2x\epsilon^{-\frac{2}{p}} \cos \frac{\alpha \pi}{4} \right) \exp \left( 2K^2 (1 + \epsilon^{-2})^p x \right) \delta^2. \]  

\underline{Step 2:} Estimate \(\|u_\epsilon(x, \cdot) - u(x, \cdot)\|_{H^p(\mathbb{R})}^2\). We divide this step into three sub-steps since
\[ \|u_\epsilon(x, \cdot) - u(x, \cdot)\|_{H^p(\mathbb{R})}^2 \leq 2\|u_\epsilon(x, \cdot) - \mathcal{P}_\epsilon(u)(x, \cdot)\|_{H^p(\mathbb{R})}^2 + 2\|\mathcal{P}_\epsilon(u)(x, \cdot) - u(x, \cdot)\|_{H^p(\mathbb{R})}^2. \]  
The first and the second steps are estimating \(\|u_\epsilon(x, \cdot) - \mathcal{P}_\epsilon(u)(x, \cdot)\|_{H^p(\mathbb{R})}^2\) and \(\|\mathcal{P}_\epsilon(u)(x, \cdot) - u(x, \cdot)\|_{H^p(\mathbb{R})}^2\) respectively. The last sub-step will combine two first sub-steps to derive an estimate for \(\|u_\epsilon(x, \cdot) - u(x, \cdot)\|_{H^p(\mathbb{R})}^2\).

\underline{Sub-step 2a:} Estimating \(\|u_\epsilon(x, \cdot) - \mathcal{P}_\epsilon(u)(x, \cdot)\|_{H^p(\mathbb{R})}^2\). It follows from (3.58) and (3.59) that
\[ u_\epsilon(x, t) - \mathcal{P}_\epsilon(u)(x, t) = \mathcal{F}^{-1} \left( - \int_0^x \sinh \left( k(\omega)(x-z) \right) \frac{k(\omega)}{k(\omega)} \left[ \tilde{f}_\epsilon(z, \omega, u_\epsilon(z, \omega)) - \tilde{f}_\epsilon(z, \omega, u(z, \omega)) \right] dz \right) (t). \]
Hence,
\[
\|u_\epsilon(x, \cdot) - \mathcal{P}_\epsilon(u)(x, \cdot)\|^2_{\mathcal{H}^p(\mathbb{R})} = \int_{-\infty}^{+\infty} (1 + \omega^2)^p \left| \int_0^x \sinh \left( \frac{k(\omega)(x-z)}{k(\omega)} \right) \left[ \hat{f}_\epsilon(z, \omega, u(z, \omega)) - \hat{f}_\epsilon(z, \omega, u_\epsilon(z, \omega)) \right] dz \right|^2 d\omega.
\]

We note that
\[
\left| \sinh \left( \frac{k(\omega)(x-z)}{k(\omega)} \right) \right|^2 \leq (x-z)^2 \exp \left( 2(x-z)\Re(k(\omega)) \right),
\]
and
\[
\exp \left( 2(x-z)\Re(k(\omega)) \right) \leq \exp \left( 2(x-z)\epsilon^{-\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right) \leq \exp \left( 2(x-z)\epsilon^{-\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right), \quad |\omega| \leq \frac{1}{\epsilon},
\]
since \( \epsilon^{-\frac{\alpha+\mu}{2}} < \epsilon^{-\frac{\alpha+\mu}{2}} \) for \( \frac{1}{\epsilon} > 1 \). By a similar argument as in \((3.82)\), we obtain
\[
\|u_\epsilon(x, \cdot) - \mathcal{P}_\epsilon(u)(x, \cdot)\|^2_{\mathcal{H}^p(\mathbb{R})} \leq K^2 (1 + \epsilon^{-2})^p \exp \left( 2x\epsilon^{-\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right) \int_0^x Z_2(z) dz, \quad (3.85)
\]
where
\[
Z_2(z) := \exp \left( -2z\epsilon^{-\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right) \|u(z, \cdot) - u_\epsilon(z, \cdot)\|^2_{\mathcal{H}^p(\mathbb{R})} \geq 0,
\]
for all \( 0 \leq z \leq x \).

**Sub-step 2b:** Estimate \( \|\mathcal{P}_\epsilon(u)(x, \cdot) - u(x, \cdot)\|^2_{\mathcal{H}^p(\mathbb{R})} \). The equations \((3.59)\) and \((3.57)\) show that
\[
\mathcal{P}_\epsilon(u)(x, t) - u(x, t) = \mathcal{F}^{-1} \left( -\hat{u}(x, \omega)\chi_{\mathbb{R}\setminus\left[-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right]}(\omega) \right)
\]
where \( \hat{u}(x, \omega) \) is given in \((2.13)\). So
\[
\|\mathcal{P}_\epsilon(u)(x, \cdot) - u(x, \cdot)\|^2_{\mathcal{H}^p(\mathbb{R})} = \int_{\mathbb{R}\setminus\left[-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right]} \exp \left( 2(1 + \gamma - x)|\omega|^{\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right) |\hat{u}(x, \omega)|^2 (1 + \omega^2)^p \exp \left( 2(x-\gamma-1)|\omega|^{\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right) d\omega.
\]
Since \( \delta \) satisfies \((3.78)\), we imply that \( \epsilon \) satisfies the condition \((2.17)\). Applying Lemma \(2.2\) with \( \xi = \frac{\alpha + \mu}{2} > 0 \), we get
\[
(1 + \omega^2)^p \exp \left( 2(x-\gamma-1)|\omega|^{\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right) \leq (1 + \epsilon^{-2})^p \exp \left( 2(x-\gamma-1)\epsilon^{-\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right),
\]
for all \( |\omega| \leq \frac{1}{\epsilon} \). Using the assumption \((3.77)\) we have
\[
\int_{\mathbb{R}\setminus\left[-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right]} \exp \left( 2(1 + \gamma - x)|\omega|^{\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right) |\hat{u}(x, \omega)|^2 d\omega \leq M_2^2.
\]
Therefore,
\[
\|\mathcal{P}_\epsilon(u)(x, \cdot) - u(x, \cdot)\|^2_{\mathcal{H}^p(\mathbb{R})} \leq M_2^2 (1 + \epsilon^{-2})^p \exp \left( 2(x-\gamma-1)\epsilon^{-\frac{\alpha+\mu}{2}} \cos \frac{\alpha \pi}{4} \right).
\]
Sub-step 2c: Estimate for the term $\|u_t(x,.) - u(x,.)\|_{H^p(\mathbb{R})}^2$. Combining (3.84) and (3.85), (3.86), we get

$$\|u_t(x,.) - u(x,.)\|_{H^p(\mathbb{R})}^2 \leq 2M_2^2 (1 + \epsilon^{-2})^p \exp \left( 2(x - \gamma - 1)e^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) + 2K^2 (1 + \epsilon^{-2})^p \int_0^x Z_2(z)dz,$$

which implies that

$$Z_2(x) \leq 2M_2^2 (1 + \epsilon^{-2})^p \exp \left( -2(\gamma + 1)e^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) + 2K^2 (1 + \epsilon^{-2})^p \int_0^x Z_2(z)dz.$$

Applying the Gronwall’s inequality, we obtain

$$Z_2(x) \leq 2M_2^2 (1 + \epsilon^{-2})^p \exp \left( -2(\gamma + 1)e^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) \exp \left( 2K^2 (1 + \epsilon^{-2})^p x \right). \tag{3.87}$$

This leads to

$$\|u(x,.) - u_t(x,.)\|_{H^p(\mathbb{R})}^2 \leq 2M_2^2 (1 + \epsilon^{-2})^p \times \exp \left( 2(x - \gamma - 1)e^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) \exp \left( 2K^2 (1 + \epsilon^{-2})^p \right). \tag{3.88}$$

Step 3: Estimate for $\|u_t(x,.) - u(x,.)\|_{H^p(\mathbb{R})}$. It follows from (3.80), (3.83), and (3.88) that

$$\|u_t(x,.) - u(x,.)\|_{H^p(\mathbb{R})} \leq 16 (1 + \epsilon^{-2})^p \exp \left( 2xe^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) \exp \left( 2K^2 (1 + \epsilon^{-2})^p \right) \times \exp \left( 2K^2 (1 + \epsilon^{-2})^p \right).$$

So

$$\|u_t(x,.) - u(x,.)\|_{H^p(\mathbb{R})} \leq 4 (1 + \epsilon^{-2})^{p/2} \exp \left( xe^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) \exp \left( K^2 (1 + \epsilon^{-2})^p \right) \delta + 2M_2 (1 + \epsilon^{-2})^{p/2} \exp \left( (x - \gamma - 1)e^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) \exp \left( K^2 (1 + \epsilon^{-2})^p \right).$$

Now we consider the estimate (3.79). Since $\delta$ satisfies the condition (3.78), we know that $\frac{1}{\epsilon} > 1$. This follows from $0 \leq x \cos \frac{\alpha \pi}{4} \leq 1$ that

$$(1 + \epsilon^{-2})^{p/2} \leq \left( e^{\epsilon^{-2}} \right)^{p/2},$$

$$\exp \left( xe^{-\frac{\alpha + \mu}{2} \cos \frac{\alpha \pi}{4}} \right) \leq e^{-\frac{x}{2}},$$

$$\exp \left( K^2 (1 + \epsilon^{-2})^p \right) \leq e^{K^2(2\epsilon^{-2})^p},$$

and

$$\left( e^{\epsilon^{-2}} \right)^{p/2} e^{-\frac{x}{2}} e^{K^2(2\epsilon^{-2})^p} = e^{\frac{p}{2} e^{-2} + e^{-\frac{x}{2}} + K^2(2\epsilon^{-2})^p} \leq e^{Ae^{-q}}$$

where $q = \max \left\{ 2; \frac{\alpha}{2}; 2p \right\}, \quad A = \frac{p}{2} + 1 + K^2(2p)$. It follows from $\mu > \max \{ 4 - \alpha; 4p - \alpha \}$ that

$$\frac{\alpha + \mu}{2} \geq \max \left\{ 2; \frac{\alpha}{2}; 2p \right\} = q.$$ Hence, $e^{-q} \leq e^{-\frac{\alpha + \mu}{2}}$ and

$$\left( e^{\epsilon^{-2}} \right)^{p/2} e^{-\frac{x}{2}} e^{K^2(2\epsilon^{-2})^p} \leq e^{Ae^{-\frac{\alpha + \mu}{2}}}.$$
Combining the above arguments, we derive
\[
4 \left( 1 + e^{-2} \right)^{p/2} \exp \left( x e^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \exp \left( K^2 \left( 1 + e^{-2} \right)^p \right) \delta \leq 4e^{Ae^{-\frac{a+b}{2}}} \delta. \tag{3.89}
\]

On the other hand, we also have
\[
\left( 1 + e^{-2} \right)^{p/2} \leq \left( e^{e^{-2}} \right)^{p/2},
\exp \left( (x - \gamma - 1) e^{-\frac{a+b}{2}} \cos \frac{\alpha \pi}{4} \right) \leq \exp \left( -\gamma \cos \frac{\alpha \pi}{4} e^{-\frac{a+b}{2}} \right),
\exp \left( K^2 \left( 1 + e^{-2} \right)^p \right) \leq e^{K^2(2e^{-2})^p},
\]
and
\[
\left( e^{e^{-2}} \right)^{p/2} \exp \left( -\gamma \cos \frac{\alpha \pi}{4} e^{-\frac{a+b}{2}} \right) e^{K^2(2e^{-2})^p} = e^p e^{e^{-2} + K^22e^{-2p}} = e^p e^{e^{-2} + K^22e^{-2p}} \exp \left( -\gamma \cos \frac{\alpha \pi}{4} e^{-\frac{a+b}{2}} \right)
\leq e^{Ae^{-q-2Be^{-\frac{a+b}{2}}}},
\]
where \( B = \frac{1}{2} \gamma \cos \frac{\alpha \pi}{4} \). From \( 3.78 \), we have \( \epsilon < \left( \frac{B}{A} \right)^{\frac{1}{2}-q-q} \). So \( Ae^{-q} \leq Be^{-\frac{a+b}{2}} \). We conclude that the following inequality holds
\[
\left( e^{e^{-2}} \right)^{p/2} \exp \left( -\gamma \cos \frac{\alpha \pi}{4} e^{-\frac{a+b}{2}} \right) e^{K^2(2e^{-2})^p} \leq e^{-Be^{-\frac{a+b}{2}}}. \tag{3.90}
\]

The above arguments imply that
\[
2M_2 \left( 1 + e^{-2} \right)^{p/2} \exp \left( (x - \gamma - 1) e^{-\frac{a+b}{2}} \cos \frac{\alpha \pi}{4} \right) \exp \left( K^2 \left( 1 + e^{-2} \right)^p \right) \leq 2M_2 e^{-Be^{-\frac{a+b}{2}}}. \tag{3.90}
\]

From \( 3.88 \), \( 3.89 \), and \( 3.90 \), we derive
\[
\| u^\delta (\cdot, t) - u(\cdot, t) \|_{H^p(\mathbb{R})} \leq 4e^{Ae^{-\frac{a+b}{2}}} \delta + 2M_2 e^{-Be^{-\frac{a+b}{2}}}. \tag{3.91}
\]

By some simple computations, we get
\[
\| u^\delta (\cdot, t) - u(\cdot, t) \|_{H^p(\mathbb{R})} \leq C_2 \delta \pi^{\frac{B}{\alpha}}.
\]
where \( C_2 = 4 + 2M_2 \), i.e., the inequality \( 3.79 \) is proved. \( \square \)

**Remark 3.2.** To obtain the convergence, we need the strong assumption \( 3.77 \) on the exact solution \( u(x, t) \). The techniques are not new and come from applying Gronwall’s inequality. In the next theorem, we will present a new way to deal with a weaker assumption \( 3.92 \) on the exact solution \( u(x, t) \). Indeed, the assumption \( 3.92 \) is much better than the assumption \( 3.77 \) of the previous Theorem.

**Theorem 3.3.** Assume that the problem \( 2.14 \) has a unique (exact) solution \( u \) such that
\[
\int_{-\infty}^{+\infty} \exp \left( 2(1 + \gamma - x)R(k(\omega)) \right) |\hat{u}(x, \omega)|^2 d\omega < M_3^2, \quad 0 \leq x < 1, \tag{3.92}
\]
for \( M_3 > 0 \). Let us choose the regularization parameter \( \epsilon \) such that
\[
\epsilon < \left[ \frac{\alpha \gamma \cos \frac{\alpha \pi}{4}}{2p} \right]^\frac{2}{\alpha}. \tag{3.93}
\]
then
\[
\| u_\epsilon^\delta(x, .) - u(x, .) \|_{H^p(\mathbb{R})} \leq C_3 \left( 1 + \epsilon^{-2} \right)^p \exp \left( x e^{-\frac{x}{p}} \cos \frac{\alpha \pi}{4} \right) \delta
\]
\[
+ C_3 \left( 1 + \epsilon^{-2} \right)^p \exp \left( (x - \gamma - 1)e^{-\frac{x}{2}} \cos \frac{\alpha \pi}{4} \right),
\]
(3.94)

for all \(0 \leq x < 1\) where
\[
C_3 = \max \left\{ 5e^{K^2} ; 3M_3 \left( \epsilon K^2 + 1 \right) \right\}.
\]

**Proof.** Using the triangle inequality, we have
\[
\| u_\epsilon^\delta(x, .) - u(x, .) \|^2_{H^p(\mathbb{R})} \leq 3\| u_\epsilon^\delta(x, .) - u_\epsilon(x, .) \|^2_{H^p(\mathbb{R})} + 3\| u_\epsilon(x, .) - \mathcal{P}_\epsilon(u)(x, .) \|^2_{H^p(\mathbb{R})}
\]
\[
+ 3\| \mathcal{P}_\epsilon(u)(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})} \leq 3 \left( 1 + \epsilon^{-2} \right)^p \| u_\epsilon^\delta(x, .) - u_\epsilon(x, .) \|^2_{L^2(\mathbb{R})}
\]
\[
+ 3 \left( 1 + \epsilon^{-2} \right)^p \| u_\epsilon(x, .) - \mathcal{P}_\epsilon(u)(x, .) \|^2_{L^2(\mathbb{R})} + 3\| \mathcal{P}_\epsilon(u)(x, .) - u(x, .) \|^2_{H^p(\mathbb{R})}.
\]
(3.95)

Moreover, we have
\[
\| u_\epsilon(x, .) - \mathcal{P}_\epsilon(u)(x, .) \|^2_{L^2(\mathbb{R})} \leq \| u_\epsilon(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})} + \| \mathcal{P}_\epsilon(u)(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})}
\]
\[
\leq \| u_\epsilon(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})} + \| \mathcal{P}_\epsilon(u)(x, .) - u(x, .) \|^2_{H^p(\mathbb{R})}.
\]
(3.96)

Therefore, by combining (3.95) and (3.96), we obtain the following inequality
\[
\| u_\epsilon^\delta(x, .) - u(x, .) \|^2_{H^p(\mathbb{R})} \leq 3 \left( 1 + \epsilon^{-2} \right)^p \| u_\epsilon^\delta(x, .) - u_\epsilon(x, .) \|^2_{L^2(\mathbb{R})}
\]
\[
+ 3 \left( 1 + \epsilon^{-2} \right)^p \| u_\epsilon(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})}
\]
\[
+ 3 \left[ (1 + \epsilon^{-2})^p + 1 \right] \| \mathcal{P}_\epsilon(u)(x, .) - u(x, .) \|^2_{H^p(\mathbb{R})}.
\]
(3.97)

The term \(\| u_\epsilon^\delta(x, .) - u_\epsilon(x, .) \|^2_{L^2(\mathbb{R})}\) can be similarly estimated as (3.67)
\[
\| u_\epsilon^\delta(x, .) - u_\epsilon(x, .) \|^2_{L^2(\mathbb{R})} \leq 8e^{2K^2} \exp \left( 2xe^{-\frac{x}{2}} \cos \frac{\alpha \pi}{4} \right) \delta^2.
\]
(3.98)

Next, we divide this proof into three steps. The first step is estimating \(\| u_\epsilon(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})}\), the second step is estimating \(\| \mathcal{P}_\epsilon(u)(x, .) - u(x, .) \|^2_{H^p(\mathbb{R})}\), and the last step is obtaining an estimate for \(\| u_\epsilon^\delta(x, .) - u(x, .) \|^2_{H^p(\mathbb{R})}\).

**Step 1:** Estimate \(\| u_\epsilon(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})}\). We have
\[
\| u_\epsilon(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})} \leq 2\| u_\epsilon(x, .) - \mathcal{P}_\epsilon(u)(x, .) \|^2_{L^2(\mathbb{R})} + 2\| \mathcal{P}_\epsilon(u)(x, .) - u(x, .) \|^2_{L^2(\mathbb{R})}.
\]
(3.99)

In addition, the inequality (3.69) also holds under the assumption (3.92)
\[
\| u_\epsilon(x, .) - \mathcal{P}_\epsilon(u)(x, .) \|^2_{L^2(\mathbb{R})} \leq K^2 \exp \left( 2xe^{-\frac{x}{2}} \cos \frac{\alpha \pi}{4} \right) \int_0^x Z_1(z)dz,
\]
(3.100)

where \(Z_1(z) := \exp \left( -2ze^{-\frac{z}{2}} \cos \frac{\alpha \pi}{4} \right) \| u_\epsilon(z, .) - u(z, .) \|^2_{L^2(\mathbb{R})} \geq 0\) for all \(0 \leq z \leq x\). It follows from (3.70) that
\[
\mathcal{P}_\epsilon(u)(x, .) - u(x, .) = \mathcal{F}^{-1} \left( -\hat{u}(x, \omega) \chi_{\mathbb{R}\setminus[-\frac{1}{2}; \frac{1}{2}]}(\omega) \right).
\]
(3.101)
Using the assumption (3.92), we get

\[ \| \mathcal{P}_\epsilon (u)(x, \cdot) - u(x, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[ = \int_{\mathbb{R} \setminus [-\frac{1}{2} : \frac{1}{2}]} \exp \left( 2(1 + \gamma - x) \Re (k(\omega)) \right) |\hat{u}(\omega, \omega)|^2 \exp \left( 2(x - \gamma - 1) \Re (k(\omega)) \right) d\omega \]

\[ \leq \exp \left( 2(x - \gamma - 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \int_{\mathbb{R} \setminus [-\frac{1}{2} : \frac{1}{2}]} \exp \left( 2(1 + \gamma - x) \Re (k(\omega)) \right) |\hat{u}(\omega, \omega)|^2 d\omega \]

\[ \leq M_3^2 \exp \left( 2(x - \gamma - 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right). \tag{3.102} \]

It follows from (3.99), (3.100), and (3.102) that

\[ \| u_\epsilon(x, \cdot) - u(x, \cdot) \|_{L^2(\mathbb{R})}^2 \leq 2K^2 \exp \left( 2x \epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \int_0^x Z_1(z) dz \]

\[ + 2M_3^2 \exp \left( 2(x - \gamma - 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right). \]

Multiplying two sides of the above inequality by \( \exp \left( 2x \epsilon^{-\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \), we obtain

\[ Z_1(x) \leq 2M_3^2 \exp \left( - 2(\gamma + 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) + 2K^2 \int_0^x Z_1(z) dz. \]

By applying the Gronwall inequality, we derive

\[ Z_1(x) \leq 2M_3^2 \exp \left( - 2(\gamma + 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) e^{2K^2 x}, \]

which leads to

\[ \| u_\epsilon(x, \cdot) - u(x, \cdot) \|_{L^2(\mathbb{R})}^2 \leq 2M_3^2 \exp \left( 2(x - \gamma - 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) e^{2K^2 x}. \tag{3.103} \]

**Step 2:** Estimate \( \| \mathcal{P}_\epsilon (u)(x, \cdot) - u(x, \cdot) \|_{H^p(\mathbb{R})}^2 \). It follows from (3.101) that

\[ \| \mathcal{P}_\epsilon (u)(x, \cdot) - u(x, \cdot) \|_{H^p(\mathbb{R})}^2 \]

\[ = \int_{\mathbb{R} \setminus [-\frac{1}{2} : \frac{1}{2}]} \exp \left( 2(1 + \gamma - x) \Re (k(\omega)) \right) |\hat{u}(\omega, \omega)|^2 (1 + \omega^2)^p \exp \left( 2(x - \gamma - 1) \Re (k(\omega)) \right) d\omega. \]

Since \( \epsilon \) satisfies (3.93) and (2.17), we apply the Lemma 2.2 for \( \xi = \frac{\alpha}{2} > 0 \) in order to obtain

\[ (1 + \omega^2)^p \exp \left( 2(x - \gamma - 1) \Re (k(\omega)) \right) = (1 + \omega^2)^p \exp \left( 2(x - \gamma - 1) |\omega|^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right) \]

\[ \leq (1 + \epsilon^{-2})^p \exp \left( 2(x - \gamma - 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right), \]

for all \( |\omega| \leq \frac{1}{\epsilon} \). On the other hand, since (3.92), we have

\[ \int_{\mathbb{R} \setminus [-\frac{1}{2} : \frac{1}{2}]} \exp \left( 2(1 + \gamma - x) \Re (k(\omega)) \right) |\hat{u}(\omega, \omega)|^2 d\omega \leq M_3^2. \]

Thus,

\[ \| \mathcal{P}_\epsilon (u)(x, \cdot) - u(x, \cdot) \|_{H^p(\mathbb{R})}^2 \leq M_3^2 \left( 1 + \epsilon^{-2} \right)^p \exp \left( 2(x - \gamma - 1) \epsilon^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{4} \right). \tag{3.104} \]
Step 3: Estimate \( \| u^\delta_i(x,.) - u(x,.) \|_{H^p(\mathbb{R})}^2 \). Combining (3.97), (3.98), (3.103), and (3.104), we get

\[
\| u^\delta_i(x,.) - u(x,.) \|_{H^p(\mathbb{R})}^2 \leq 24e^{2K^2} (1 + \varepsilon^{-2})^p \exp \left( 2x - \frac{\alpha \pi}{4} \right) \delta^2 \\
+ 6M_3e^{2K^2} (1 + \varepsilon^{-2})^p \exp \left( 2(x - \gamma - 1) - \frac{\alpha \pi}{4} \right) \\
+ 3M_3 \left[ (1 + \varepsilon^{-2})^p + 1 \right] (1 + \varepsilon^{-2})^p \exp \left( 2(x - \gamma - 1) - \frac{\alpha \pi}{4} \right).
\]

The inequality \( (1 + \varepsilon^{-2})^p + 1 \leq 2 (1 + \varepsilon^{-2})^p \) implies that

\[
\| u^\delta_i(x,.) - u(x,.) \|_{H^p(\mathbb{R})} \leq 5e^{K^2} (1 + \varepsilon^{-2})^p/2 \exp \left( x - \frac{\alpha \pi}{4} \right) \delta \\
+ 3M_3e^{K^2} (1 + \varepsilon^{-2})^p/2 \exp \left( (x - \gamma - 1) - \frac{\alpha \pi}{4} \right) \\
+ 3M_3 (1 + \varepsilon^{-2})^p \exp \left( (x - \gamma - 1) - \frac{\alpha \pi}{4} \right).
\]

(3.105)

The inequality (3.105) leads to

\[
\| u^\delta_i(x,.) - u(x,.) \|_{H^p(\mathbb{R})} \leq C_3 (1 + \varepsilon^{-2})^p \exp \left( x - \frac{\alpha \pi}{4} \right) \delta \\
+ C_3 (1 + \varepsilon^{-2})^p \exp \left( (x - \gamma - 1) - \frac{\alpha \pi}{4} \right),
\]

where \( C_3 = \max \left\{ 5e^{K^2}; 3M_3 \left( e^{K^2} + 1 \right) \right\} \). \( \square \)

**Remark 3.3.** In the case of finite time, the problem can be solved by using the Fourier truncation method. In the future, we will investigate this method to solve the problem.

**Remark 3.4.** The boundary conditions (1.3)-(1.4) are given on the left boundary. In the case that the boundary conditions are given on the right boundary, i.e.,

\[
\begin{align*}
\hat{u}(1, \omega) = g_1(1), \\
\hat{u}_x(1, \omega) = h_1(1).
\end{align*}
\]

Multiplying the first equation of (2.9) by \( \frac{\sinh(k(\omega)(z-x))}{k(\omega)} \) and integrating two sides on \([x; 1]\), we derive

\[
\hat{u}(x, \omega) = \cosh(k(\omega)(1-x)) \hat{g}_1(\omega) + \frac{\sinh(k(\omega)(1-x))}{k(\omega)} \hat{h}_1(\omega) \\
- \int_x^1 \frac{\sinh(k(\omega)(z-x))}{k(\omega)} \hat{f}(z, \omega, u(z, \omega)) dz,
\]

(3.106)

for \( x \geq 0, \omega \in \mathbb{R} \). Therefore, the problem corresponding to the right-boundary conditions can be treated in the same way as the problem corresponding to the left-boundary conditions.

### 4. Numerical Example

In this section, we present a simple numerical example to show the efficiency of the method. The numerical example is implemented for \( t \in [0, 2\pi] \). Consider the inverse problem (1.1)-(1.4) according to \( g(t) = t^2, h(t) = -2t^2 \) and

\[
f(x, t, u(x, t)) = \frac{u(x, t)}{1 + u^2(x, t)} + \tilde{f}(x, t)
\]

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Table 1. The relative error between \( u \) and \( u^\epsilon \) for \( \alpha = 0.4 \) with \( \omega_{\text{max}}^1 = 16.9339 \), \( \omega_{\text{max}}^2 = 20.9183 \), \( \omega_{\text{max}}^3 = 24.9027 \) and \( \omega_{\text{max}}^4 = 31.8755 \).

| \( x \)  | \( \omega_{\text{max}}^1 \) | \( \omega_{\text{max}}^2 \) | \( \omega_{\text{max}}^3 \) | \( \omega_{\text{max}}^4 \) |
|--------|--------|--------|--------|--------|
| \( x = 0.15 \) | 0.1378 | 0.0941 | 0.0756 | 0.0580 |
| \( x = 0.25 \) | 0.1581 | 0.1015 | 0.0821 | 0.0639 |
| \( x = 0.35 \) | 0.1472 | 0.1157 | 0.0942 | 0.0758 |
| \( x = 0.45 \) | 0.1869 | 0.1386 | 0.1138 | 0.0966 |
| \( x = 0.55 \) | 0.2160 | 0.1688 | 0.1411 | 0.1270 |
| \( x = 0.65 \) | 0.2396 | 0.2089 | 0.1729 | 0.1631 |
| \( x = 0.75 \) | 0.2729 | 0.2478 | 0.2034 | 0.1971 |
| \( x = 0.85 \) | 0.2872 | 0.2777 | 0.2268 | 0.2240 |
| \( x = 0.95 \) | 0.3292 | 0.3022 | 0.2447 | 0.2430 |

where

\[
\tilde{f}(x,t) = e^{-2x} \left( \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^2}{1 + e^{-4x}t^4 - 4t^2} \right).
\]

Note that \( f \) satisfies the Lipschitz condition

\[
|f(x,t,v_1) - f(x,t,v_2)| \leq |v_1 - v_2|, \quad 0 \leq x \leq 1, \quad t \in \mathbb{R}, \quad v_1, v_2 \in \mathbb{R}.
\]

Moreover, we can easily verify that

\[ u(x,t) = e^{-2x}t^2 \]

is the exact solution of the problem (1.1)-(1.4). The problem is numerically solved for \( \alpha = 0.4 \) and \( \alpha = 0.7 \). The noisy data \( g^\delta \), \( h^\delta \) are given by

\[
g^\delta(t) = g(t) \left( 1 + \frac{1}{\sqrt{\pi}} \delta \text{ rand(size(.))} \right),
\]

\[
h^\delta(t) = h(t) \left( 1 + \frac{1}{\sqrt{\pi}} \delta \text{ rand(size(.))} \right),
\]

where \( \delta \) is the noisy level and \( \text{rand(size(.))} \) is a random matrix with elements in \([-1; 1]\). By using the formula (3.41), we can compute the regularized solution \( u^\delta \) with respect to the noisy data \( g^\delta \), \( h^\delta \) and the regularized parameter \( \epsilon (\omega_{\text{max}}) \). The relative error between the exact solution \( u \) and the regularized solution \( u^\delta \) is approximated as

\[
\text{Error} = \left( \frac{\sum_{l=0}^{N} |u(x, t_l) - u^\delta(x, t_l)|^2}{\sum_{l=0}^{N} |u(x, t_l)|^2} \right)^{1/2},
\]

for fixed space point \( x \in (0,1) \). Here

\[ t_l = l \Delta t, \quad \Delta t = \frac{2\pi}{N}, \quad l = 0, N, \]

and we choose \( N = 512 \). Table 1 and 2 show the errors between the exact solution and the regularized solution for \( \alpha = 0.4 \) and \( \alpha = 0.7 \). We can see that the errors are increasing when \( \omega_{\text{max}} \) and \( x \) are increasing. So the regularized parameter should be chosen larger to get more exact results. The figures show the regularized solutions for some values of the noisy level \( \delta \). These images show that the errors are smaller when the noisy level \( \delta \) is smaller.
Table 2. The relative error between $u$ and $u^c_\omega$ for $\alpha = 0.7$ with $\omega_{\text{max}1} = 16.9339$, $\omega_{\text{max}2} = 20.9183$, $\omega_{\text{max}3} = 24.9027$ and $\omega_{\text{max}4} = 31.8755$.

| $x$  | $\omega_{\text{max}1}$ | $\omega_{\text{max}2}$ | $\omega_{\text{max}3}$ | $\omega_{\text{max}4}$ |
|------|----------------|----------------|----------------|----------------|
| $x = 0.15$ | 0.1358 | 0.0996 | 0.0805 | 0.0622 |
| $x = 0.25$ | 0.1562 | 0.1244 | 0.1024 | 0.0817 |
| $x = 0.35$ | 0.2175 | 0.1779 | 0.1502 | 0.1255 |
| $x = 0.45$ | 0.2891 | 0.2620 | 0.2249 | 0.1946 |
| $x = 0.55$ | 0.3785 | 0.3583 | 0.3032 | 0.2690 |
| $x = 0.65$ | 0.4551 | 0.4230 | 0.3585 | 0.3245 |
| $x = 0.75$ | 0.5118 | 0.4682 | 0.3916 | 0.3604 |
| $x = 0.85$ | 0.5652 | 0.4989 | 0.4153 | 0.3874 |
| $x = 0.95$ | 0.5902 | 0.5256 | 0.4372 | 0.4127 |

Figure 1. A comparison between the exact solution and its computed regularization solution corresponding to $\delta$.

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(T.B. Ngoc) Department of Mathematical Economics, Banking University of Ho Chi Minh City, Vietnam

Department of Mathematics and Computer Science, VNUHCM-University of Science, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam
E-mail address: tranbaongoc@hcmuaf.edu.vn

(N.H. Tuan) Applied Analysis Research Group Faculty of Mathematics and Statistics Ton Duc Thang University Ho Chi Minh City, Vietnam
E-mail address: nguyenhuytuan@tdt.edu.vn

(M. Kirane) Lasie, Faculté des Sciences et Technologies, Université de La Rochelle, Avenue M. Crépeau, La Rochelle, Cedex 17042, France