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Optimum Polar Codes Encoder over Binary Discrete Memory-less Channels

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Abstract. In modern communication systems, there is an increasing need for more capacity-achieving coding schemes to fulfill high data rates demands. Polar codes are very promising candidates to achieve near Shannon’s capacity and improved systems’ reliability due to their polarized channel construction idea which depends on utilizing the good channel in data transmission and consumes the noisy ones in frozen bits transmission as pilots. This paper presented the performance of polar codes for binary-erasure-channel, binary-symmetric channel, and additive-white-Gaussian-noise-channel at various rates and different design parameter to find the optimum ones for these channels. The numerical analysis showed that when the rate increased the system bit error rate performance degrades, while it is improved when the design parameters are optimized for all the studied channels.

1. Introduction

A new type of channel coding is Polar codes which provide more reliability to modern communication systems over any binary input symmetric discrete memory-less channel such as binary- erasure-channel (BEC), binary -symmetric -channel (BSC), and additive- white- Gaussian -noise (AWGN) channel. Erdal Arikan invented Polar codes in 2009 [1], that proved the first outcome for the channel polarization to achieve the symmetric capacity for BEC channel only which is the highest rate when N goes to ∞ where N is the code length. On the other hand, he didn’t study the optimum performance of polar codes over BSC and AWGN channels. Also, the symmetric capacity referred to mutual-information linking the channel input and output, given regularly distributed input, and its use to measure the rate of information transmission. Noted that transmitted bit through BEC may be received accurately with probability = (1−ε) or to be lost with erasure probability ε. Also for BSC, the transmitted bit is received perfectly with probability = (1−p), or turn over with crossover transition probability = p. Finally, for the AWGN channel, the transmitted bits are received correctly depending on the systems’ signal to noise ratio (SNR) value.

Polar codes depends on a new idea for encoding called channel polarization Which produced N bit channel indices, in the encoding operation the N copies bit channels are divided into good channels
indices (noiseless) which used to transmit K information bits and bad channels indices (noisy) which allocated to transmit N-K excess bits which are called frozen bit.

The primary design methodology includes channel polarization, encoding scheme and the formation methods of polar codes over BSC are offered in [2] and the realistic manufacture way for polar codes over AWGN-channels in [3]. It explored for the practical implementation of polar codes which are complexity efficient and perform well for BEC and BSC. Using the method of recursive estimation of Bhattacharyya parameters of bit-channels is discussed in [4] which provided a low-complexity method for polar codes encoding operation ensuring the same bit error rate than which introducing in [1]. It showed that polar codes are appropriate for channel coding and able to accomplish the best possible performance for numerous other vital troubles in information theory such as loss in resource compression in [5]. All at once polar codes can be efficiently constructed for general input alphabets this is presenting in [6]. Also, the bit error rate (BER) performance for different data lengths and codeword rates using polar codes are straddling many scenarios for reliability and high throughput is investigated in [7].

In this paper, presenting the optimum polar code's performance over BEC, BSC, and AWGN channel which providing the lowest bit error rate over these channels. Our work relied on constructing the polar codes by polarized the channel into N copies by determined the initial Bhattacharyya parameter according to the design values $\varepsilon$, $p$ and SNR [8] for BEC, BSC, and AWGN, respectively. We find in our extensive simulations The optimum design values of $\varepsilon$, $p$ and SNR for polar code construction among a range of available design values. Which resulted in a polar codes constructions produced enhanced performance when the value of $\varepsilon$, $p$ and SNR are optimum.

This paper is planned as follows: offering brief polar codes fundamentals such as its basic idea channel polarization, Bhattacharyya parameter estimation, and the decoding operation in section 2. Stating our system model by constructing polar codes by Arikan’s Bhattacharyya bounds is in section 3, the numerical analysis of the performance of the polar code is discussed in section 4, and finally, the overall research is concluded in section 5.

2. Polar codes fundamentals

Polar codes have newly concerned a large interest because they are the initial codes realize the capacity of a binary –discrete- memory-less channel (B-DMC), by a large codeword. The polar codes construction relies on the channel polarization phenomenon which is the major idea to establish its’ encoding scheme. With a relatively low complexity for encoder and decoder. A pair of essential parameters should be calculated to attain the capacity of B-DMC, the tow parameters are used to measure both the rate and reliability of the communication system. the first one is the symmetric capacity or mutual information and can be calculated by [1]

$$I(W) \triangleq \sum_{y \in Y} \sum_{x \in X} W(y|x) \log \frac{W(y|x)}{\frac{1}{2} W(y|0) + \frac{1}{2} W(y|1)}$$

(1)

where x, y is the input and output of the polarized bit channels, respectively, and $w(y|x)$ is the probability that y is the begotten output given that the input is x. While the second one is Bhattacharyya parameter and can be deliberated by [1]

$$Z(W) \triangleq \sum_{y \in Y} \sqrt{W(y|0) W(y|1)}$$

(2)

Original Polar code can define by three parameters (N, K, and $A^c$) where N is the polarized bit channels or the length of a codeword, K is the integer of data bits which we need to transmit it per codeword, and $A^c$ is the indices of N-K integer corresponding to the locations of fixed bits from the set $\{0,1,\ldots,N-1\}$ Figure1. Show an example of polar codes encoding operation
for \( \{N, K, A^c\} = \{4, 2, \{0, 2\}\} \) with complexity \( O(N \log_2 N) \). For polar code the encoding process for K information bits [9]. The codeword rate is \( R = K/N \). And \( n = \log_2 N \) \( F = F \otimes F \ldots \otimes F \) is Kronecker product of Erkan’s polarizing kernel matrix (n copies).

\[
F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}
\]

(3)

And the generator matrix \( G_N = B_N F^{\otimes n} \). Noted that \( G_N = F^{\otimes n} \) in the case of symmetric polar codes and \( G_N = B_N F^{\otimes n} \) in case of non-symmetric polar codes where \( B_N \) is known as permutation or bit reversal matrix.

\[
X^N = G_N \cdot U^N
\]

(4)

According to the companionship of K-information bits and [2], we write equation (4) in the form

\[
X^N = U^N_\alpha G_N \left( A \right) \oplus U^N_{\alpha^c} G_N \left( A^c \right)
\]

(5)

**Figure 1.** Polar codes encoder for \( \{N, K, A^c\} = \{4, 2, \{0, 2\}\} \)

where \( A = \{0, 1, \ldots, N-1\} \) corresponding to the set of information bits \( U^N_\alpha \). And \( \left( A^c \right) = \{0, 1, \ldots, N - 1\} \) corresponding to the N-K fixed bits \( U^N_{\alpha^c} \).

### 2.1. Channel Polarization

Channel polarization [10], is a conversion of a given B-DMC \( W \) to produce a vector channel \( W^i_N : 1 \leq i \leq N \) such that, as \( N \) becomes larger, \( I(W^i_N) \) goes towards 0 or 1. The channel polarization process contains two phases one is the channel combining phase and the other is the channel splitting phase.

### 2.2. Bhattacharyya parameter based channel construction

The Channel Polarization process begins with calculates the value of Bhattacharyya parameter \( Z(W) \) for the type of channel over BEC, BSC, and AWGN. According to the first polar code construction, the value of Bhattacharyya parameter is the basic parameter to Estimate the noiseless bit channels and the noisy bit channels. After polarization, polar codes encoder arrange this bit channels indices to send data bits in the good Bhattacharyya parameter channels which tend to 0 and transmit frozen bits in the bad Bhattacharyya parameter channel which tend to 1.

### 2.3. The Decoding

Successive cancellation (SC) decoder [11] is famous for polar codes. Mainly activate in the opposite direction of the encoder structure Figure 1 from right to left. Using a decoding process that looks like a trail of the classic belief propagation algorithm [12]. The likelihoods change in the invalidate way,
using a double of likelihood conversion equations to choose on and guess $u_0^N$ the transmitted bit. Knowing that a block error has occurred when $\hat{u}_0^N \neq u_0^N$ where $\hat{u}_0^N$ is the received bit.

3. System model
The initial construction of polar code is by Bhattacharyya parameter bounds of the bit channels, this scheme using Bhattacharyya parameters to determine the good bit channels to send information on it and select the frozen bits to the bad bit channel. The Bhattacharyya parameters of bit channels changing easily as $2Z(W) - Z^2(W)$ and $Z^2(W)$ at every polarized transformation. Noted that there is a significant adaptation to be used, due to the non-universality of polar codes as shown in [13]. The initial value for Bhattacharyya parameters for bit-channels is differing due to the type of the channel and is calculated as in Table 1.

| Type of channel | Affected parameter | initial Z(W) | Good bit channel | Bad bit channel |
|-----------------|--------------------|--------------|------------------|-----------------|
| BEC             | Erasure probability | ε            | $2Z(W) - Z^2(W)$ | $Z^2(W)$        |
|                 | symmetric probability | $p$        | $2\sqrt{p(1-p)}$ | $Z^2(W)$        |
| BSC             | symmetric probability | $p$        | $2Z(W) - Z^2(W)$ | $Z^2(W)$        |
| AWGN            | SNR                | $e^{(-K/N) = (E_b/N_0)}$ | $2Z(W) - Z^2(W)$ | $Z^2(W)$        |

Presenting the system model is in Figure 2 which the original recursive algorithm for the encoding operation needs a primary value and this was proposed by Arikan corresponding to the most unpleasant BER. This started the Bhattacharyya parameter value of BEC, BSC and AWGN channel. Therefore it has many initial values according to the channel type as we mention Table 1.

![Figure 2. The studied system model](image)

4. Numerical analysis
The polar codes construction over BEC, BSC and AWGN channel is discussed using the initial value of Arikan’s Bhattacharyya for every channel. In Figure 3 observe that the performance of polar codes over of BEC through different rates and same initial Bhattacharyya parameter value which is the design erasure probability equal to result in best BER in case of codeword rate equal to 0.25, given the least value of BER over rate=0.75. And produce moderate BER value in case of rate = 0.5. For example the BER equal to $\approx 0.5 \times 10^{-2}$ in case of a rate equal to 0.25 and erasure probability equal to 0.4. So, the incrementing in codeword rate from 0.25 to 0.5 to 0.75 caused degradation in the communication system performance over the same design erasure probability. So our research aims to find the optimum value for the initial value of Bhattacharyya parameter to provide less BER with acceptable codeword rate.
Also in Figure 4, noting that in the case of BSC the incrementing in codeword rate from 0.25 to 0.5 to 0.75 caused degradation in the communication system performance over the same design value of transition probability. For example when the transition probability equal to 0.1 the BER $\approx 0.75 \times 10^{-2}$ in case of codeword rate 0.25, produced the highest value for BER over rate = 0.75 and moderate value BER at rate = 0.5.

And in Figure 5 the BER over AWGN channel with the same design SNR = 0 is changing ascending in relation to the incrementing of codeword rate. So for our work we have to use the exhaustive search technique for the common issue to find the optimized values for initial Bhattacharyya parameters which are the erasure probability, transition probability, and SNR to optimize the communication system performance with acceptable codeword rates over the various channel types.

In Figure 6 the effect of various design values of $\varepsilon$ for BEC is discussed showing that the optimum design value $\varepsilon$ for BEC is equal to 0.75, therefore, this value of design produce the best polar code performance for the communication system BER among the other design values. Also, in Figure 7 there is an optimum design value for $p$ that equal to 0.31 for best polar code performance over BSC. Also, in Figure 8 the various values for the design SNR causing a mild changing in the BER performance. Finally, the optimization analysis for the used design value determined that the optimum values are $\varepsilon = 0.75$ for BEC and $p = 0.31$ for BSC using exhaustive search technique.
Figure 5. The effect of changing the rate at $N = 128$ in case of AWGN

Figure 6. The effect of various design values on system BER performance for BEC

Figure 7. The effect of various design values on system BER performance for BSC
5. Conclusion
This paper studied the performance of polar codes for three different channels, namely, BEC, BSC, and AWGN channel at various rates. The numerical analysis proved that when the rate increased BER system performance degrades for all the studied channels. Then, a straightforward search to find the best design parameters for different channel models is performed. The analysis showed that the performance is improved when the design parameters are optimized for all the considered symmetric channels.

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