The finite Fourier Transform and projective 2-designs

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Abstract
There are several approaches to define an eigenvector decomposition of the finite Fourier Transform (Fourier matrix), which is in some sense unique, and at best resembles the eigenstates of the quantum harmonic oscillator.

A solution given by Balian and Itzykson [9] in 1986 for prime dimensions \( d = 3 \mod 4 \) is revisited. It is shown, that by applying the Weyl-Heisenberg matrices to this eigenvector basis, a projective 2-design is generated.

1 Motivation: The Quantum Harmonic Oscillator

The position operator \( \mathbf{X} \) and the impulse operator \( \mathbf{P} \) are each defined on a dense subset of \( \mathcal{L}^2(\mathbb{R}) \) via the equations \((\mathbf{X} f)(x) = xf(x)\), and \((\mathbf{P} f)(x) = -i \frac{d}{dx} f(x)\).

Here we have set \( \hbar = 1 \). They fulfill the Canonical commutation relation \([\mathbf{X}, \mathbf{P}] = i \mathbf{I}\). When we set all physical parameters to 1, the Hamiltonian for the quantum harmonic oscillator is \( \mathbf{H} = \frac{1}{2}(\mathbf{X}^2 + \mathbf{P}^2) \). Its eigenvalues resp. energy levels are \( n + \frac{1}{2} \) with \( n = 0, 1, 2, \ldots \). The corresponding eigenstates are \( \psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2} \) with the Hermite polynomials \( H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \). All \( \psi_n \) constitute also a particular choice of eigenstates for the Fourier Transform

\[
(F f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i2\pi xy} f(x) dx.
\]

\( F \) has the 4 eigenvalues \( \pm 1, \pm i \): \((F \psi_n)(x) = (-i)^n \psi_n(x)\).

To \( \mathbf{X} \) and \( \mathbf{P} \) a two-parameter strongly continuous group can be assigned, the so called Weyl-Heisenberg group of unitary operators

\[
\mathbf{W}(r, s) := e^{i(r \mathbf{P} + s \mathbf{X})} = e^{-\frac{ir^2}{2}} e^{i \mathbf{P}} e^{i \mathbf{X}}.
\]

With \( r, s \in \mathbb{R} \to \mathbf{W}(r, s) \) they make up the unique unitary, irreducible, projective representation of the additive group \( \mathbb{R} \times \mathbb{R}, [31] \).

Now we turn to the finite dimensional counterparts.
2 Weyl-Heisenberg Matrices and Fourier Matrix

Let $d$ be the dimension of a finite-dimensional complex vector space.

We use throughout the paper the notation of bra’s $\langle,|$ (row-vectors) and ket’s $|,\rangle$ (column-vectors). Let $|e_r\rangle$, with $r \in \mathbb{Z}_d$ be the standard basis. Addition of indices is always modulo $d$, so e.g. for bra’s $|e_r + s\rangle := |e_{(r+s)(\text{mod} \, d)}\rangle$.

Let $U|e_r\rangle = e^{i\frac{2\pi r}{d}} |e_r\rangle$ and $V|e_r\rangle = |e_{r+1}\rangle$ or in matrix form:

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e^{\frac{2\pi i}{d}} & 0 & \cdots & 0 \\ 0 & 0 & e^{\frac{4\pi i}{d}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\frac{2i(d-1)\pi}{d}} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let $\tau = e^{i\frac{\pi (d+1)}{d}} = -e^{\frac{4\pi i}{d}}$, and for $r, s \in \mathbb{Z}_d$ for odd $d$, resp. $r, s \in \mathbb{Z}_{2d}$ for even $d$

$$W_{(r,s)} := \tau^{rs}V^rU^s.$$

The case of even $d$ needs $2d \times 2d$ matrices due to the phase added. We will look only at the odd case in the following.

These matrices generate the so-called finite Weyl-Heisenberg group, and make up the unique, irreducible, projective representation of the additive group $\mathbb{Z}_d \times \mathbb{Z}_d$.

The factor $\tau^{rs}$ is the analogue of the factor $e^{-\frac{irs}{d}}$ in the infinite dimensional case and simplifies calculations significantly (APPLEBY [2]).

In some papers $\hat{\tau} = e^{i\frac{\pi (d^2+1)}{d}} = (-1)^d e^{\frac{4\pi i}{d}}$ is used instead of $\tau$. Sometimes inverse matrices are used in the definition. Furthermore for odd $d$ the matrix $U' = \text{diag}(e^{\frac{2\pi i}{d}}, \ldots, e^{\frac{2\pi i}{d}}, 1, e^{\frac{4\pi i}{d}}, \ldots, e^{\frac{2i(d-1)\pi}{d}})$ is often used, as it reflects the axial symmetry in the infinite case (e.g. SINGH and CARROLL [30]).

In contrast to the infinite case, we don’t have uniquely determined infinitesimal generators of the Weyl-Heisenberg group in finite dimensions (as $X, P$). Therefore we have also no uniquely defined finite counterpart of the quantum harmonic oscillator and its eigenstates.

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1 **Remark on the notation:** Due to the fact that $U$ and $V$ can also be considered as generalizations of the Pauli Matrices $\mathbf{Z}, \mathbf{X}$, defined in dimension $d = 2$ (as rotations of the Bloch Sphere around the Z- and X-axis), in several papers these letters are used instead. We use the letters $U$ and $V$ as e.g. SCHWINGER [27] and in many subsequent physics papers, to emphasize the fact that we work on qudits ($d \geq 2$) and not just qubits.

The letter $W$ stands for Weyl, who brought the matrices above onstage in physics first [33]. Very often $D$ for Displacement is used instead of $W$. 

2
But we have at least a finite counterpart to the Fourier-Transform: The \textit{Fourier-Matrix} (also called \textit{Finite} or \textit{Discrete Fourier-Transformation} or \textit{Schur-Matrix}) is the \( d \times d \) matrix

\[
\mathbf{F} = \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} e^{\frac{2\pi i rs}{d}} |e_r\rangle \langle e_s|.
\]

In matrix form

\[
\mathbf{F} = \frac{1}{\sqrt{d}} \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & e^{\frac{2\pi i}{d}} & e^{\frac{4\pi i}{d}} & \ldots & e^{\frac{2(d-1)\pi i}{d}} \\
1 & e^{\frac{4\pi i}{d}} & e^{\frac{8\pi i}{d}} & \ldots & e^{\frac{4(d-1)\pi i}{d}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e^{\frac{2(d-1)\pi i}{d}} & e^{\frac{4(d-1)\pi i}{d}} & \ldots & e^{\frac{(d-1)^2\pi i}{d}}
\end{pmatrix}.
\]

As \( \mathbf{F}^4 = \mathbf{I} \), the possible eigenvalues of \( \mathbf{F} \) are \( \pm 1, \pm i \), as in the infinite case. The multiplicity of the eigenvalues as function of the dimension \( d \) is given by the following table (see \[8\]).

| \( d \) \( = 4k \) | \( k + 1 \) | \( k \) | \( k \) | \( k - 1 \) |
|-----------------|-----|---|---|-----|
| \( d \) \( = 4k + 1 \) | \( k + 1 \) | \( k \) | \( k \) | \( k \) |
| \( d \) \( = 4k + 2 \) | \( k + 1 \) | \( k + 1 \) | \( k \) | \( k \) |
| \( d \) \( = 4k + 3 \) | \( k + 1 \) | \( k + 1 \) | \( k + 1 \) | \( k \) |

There is long and ongoing history to define an, in some sense unique, eigenvector decomposition of the Fourier Matrix.

Proposals come from Mathematicians, Physicists, and Electrical and Electronics Engineers (see examples in \[1,4,9,13,14,17,18,20–22,34,37\]).

The approaches are either starting with the continuous harmonic oscillator eigenstates (e.g. sampling at equidistant points), or focusing on algebraic methods.

Here we propose a characterization in terms of Quantum Designs, specifically projective 2-designs.

### 3 Projective 2-designs

Let \( \{|\psi_i\rangle : 1 \leq i \leq N\} \) be \( N \) normed vectors and \( \{\mathbf{P}_i := |\psi_i\rangle \langle \psi_i| : 1 \leq i \leq N\} \) be the corresponding projection matrices. There are many equivalent definitions, when these sets form a projective 2-design, see e.g. \[10,12,19,25,26,32,33,36\], (historically) starting with the condition, that

\[
\frac{1}{N} \sum_{i=1}^{N} f(\mathbf{P}_i) = \int_{\mathbf{CP}^{n-1}} f(\mathbf{P})d\mathbf{P}.
\]
for any homogeneous polynomial \( f \) of degree 2, and the normed unitary invariant (Haar-)integral on the right side.

This is equivalent to

\[
\frac{1}{N} \sum_{i=1}^{N} P_i \otimes P_i = \frac{2}{d(d+1)} \Pi_{\text{sym}} \tag{1}
\]

where \( \Pi_{\text{sym}} \) is the orthogonal projection on the symmetric subspace of \( \mathbb{C}^d \otimes \mathbb{C}^d \).

Another equivalent condition is, that the inequality

\[
\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\text{tr}(P_i P_j))^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\langle \psi_i | \psi_j \rangle|^4 \geq \frac{2}{d(d+1)}
\]

becomes an equality. Well know examples are \([10,11]\):

- SIC - (conjectured to exist for all \( d \in \mathbb{N} \)): \( N = d^2 \),
- Complete sets of MUBs - (exist for all \( d = \text{prime power} \)): \( N = d(d+1) \)
- Clifford group applied to any vector (for \( d = \text{prime} \)): \( N = d(d^2-1) \)

In the last example the Clifford group for prime \( d \) is also an example of so-called unitary 2-design. We only refer to projective 2-designs in this paper and skip the attribute projective frequently.

4 An example for \( d = 3 \) and some numerical search

The Fourier matrix for \( d=3 \)

\[
F = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha
\end{pmatrix}, \quad \alpha = e^{2\pi i/3}.
\]

has the 3 eigenvalues \( \pm 1 \) and \( i \), and the (up to phases) unique normed eigenvectors are

\[
|\psi_1\rangle = \frac{1}{\sqrt{6+2\sqrt{3}}} \begin{pmatrix} 1 + \sqrt{3} \\ 1 \\ 1 \end{pmatrix},
|\psi_{-1}\rangle = \frac{1}{\sqrt{6-2\sqrt{3}}} \begin{pmatrix} 1 - \sqrt{3} \\ 1 \\ 1 \end{pmatrix},
|\psi_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}
\]

We apply the \( d^2 \) Weyl-Heisenberg matrices to each of the \( d = 3 \) vectors and get a set of \( d^3 = 27 \) vectors \( \{W_{(r,s)}|\psi_x\rangle : x = \pm 1, i \text{ and } 0 \leq r, s, \leq 2\} \).

These vectors form a 2-design! Actually \( |\psi_i\rangle \) is also a fiducial vector for a SIC-POVM, so due to the additivity of the 2-design property the first 2 eigenvectors generate also a 2-design of 18 vectors.

Numerical search found no complete eigenvector basis of the Fourier matrix for each \( d = 4, 5, 6 \), that generates a 2-design like above. But for \( d = 7, 11 \) there were found in each case (seemingly unique) solutions.

For \( d = 5 \) there is a (unique) basis of 2 eigenvectors of eigenvalue 1, which together with the unique eigenvector for eigenvalue \(-1\) generates a set of \( 3d^2 = 75 \) vectors, that form a 2-design.

In all these case the eigenvectors can be taken to be real-valued only.
To construct the appropriate eigenvector basis for primes \( d = 4k + 3 = 3, 7, 11, \ldots \) we need some preparation.

5 Clifford Group

In the following section \( d \) is taken to be odd\(^2\).

The Clifford group is defined as normalizer of the Weyl-Heisenberg group \( \tau^q \mathbf{W}(r,s), 0 \leq q, r, s \leq d-1 \), in the group of unitary matrices \( U \in U(d) \). It includes the Weyl-Heisenberg group itself as normal subgroup. We are interested here in the subgroup of \( U \in U(d) \), for which for all \( r, s \in \mathbb{Z}_d \)

\[
U \mathbf{W}(r,s) U^{-1} = \mathbf{W}(r',s')
\]

for some \( r', s' \in \mathbb{Z}_d \) (with no additional phase involved!), and call it restricted Clifford group \( \mathcal{C}(d) \). It is well known \([2,10,11]\), that the restricted Clifford group is a projective representation of \( SL(2,\mathbb{Z}_d) \) with the group homomorphism \( h \) defined via

\[
G = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightarrow \begin{bmatrix} r' \\ s' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}
\]

where \( U_G \) is a representative, which is fixed up to an overall phase factor.

The Fourier matrix is an element of the Clifford group:

\[
\mathbf{F} \mathbf{W}(r,s) \mathbf{F}^{-1} = \mathbf{W}(-s,r) \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow \mathbf{F}
\]

5.1 The Subgroup of elements commuting with \( \mathbf{F} \)

Let \( \mathcal{F}(d) \subset \mathcal{C}(d) \) be the subgroup of elements of the restricted Clifford group, that commute with the Fourier matrix. In terms of \( SL(2,\mathbb{Z}_d) \) this means

\[
\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \Rightarrow \alpha = \delta, \gamma = -\beta.
\]

Therefore \( \mathcal{F}(d) = \{ e^{i\xi} U_G \} \) with \( G \in FSL(2,\mathbb{Z}_d) \)

\[
FSL(2,\mathbb{Z}_d) = \left\{ G = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} : \det(G) = \alpha^2 + \beta^2 = 1 \right\}
\]

\( \mathcal{F}(d) \) and \( FSL(2,\mathbb{Z}_d) \) are abelian (commutative) groups.

If \( d \) is an odd prime then \( FSL(2,\mathbb{Z}_d) \) is a cyclic group (see \([6,7,9,14]\)) with order

\[
|FSL(2,\mathbb{Z}_d)| = \begin{cases} d-1 & \text{if } d = 4k + 1 \\ d+1 & \text{if } d = 4k + 3. \end{cases}
\]

\(^2\)For the definition of the restricted Clifford group given here the choice of the phases \( \tau^{rs} \) for \( \mathbf{W}(r,s) \) is essential. Therefore again in case of even \( d \), one would have to choose \( \mathbb{Z}_{2d} \) instead of \( \mathbb{Z}_d \).
5.2 Representations of $C(d)$ and especially $FC(d)$ in $C^d$

We assume $d$ to be an odd prime. An explicit representation of $C(d)$ is given by Appleby [3] (see also [23]). We restrict to the subgroup $FSL(2, \mathbb{Z}_d)$, and get, except when $\beta = 0$ (which means in $FSL(2, \mathbb{Z}_d)$: $\alpha = \pm 1 \to G = \pm I$).

$$G = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \to U_G = \frac{e^{i\theta}}{\sqrt{d}} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \tau(\frac{1}{d}(ar^2 - 2rs + \alpha s^2)) |e_r\rangle \langle e_s|$$

The Weyl-Heisenberg matrices are an orthogonal basis of all matrices. Therefore we can also expand the elements above in this basis. An according representation (see [9], and Athanasiu, Floratos, Nicolis [7]) is (except the case $\alpha = 1, \beta = 0$).

$$G = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \to U_G = \frac{e^{i\eta}}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \tau(\frac{1}{2(1-\alpha)}(r^2 + s^2)) W_{(r,s)}$$

The overall phases factors $e^{i\theta}$ resp. $e^{i\eta}$ can be chosen such, that the presentation becomes de-projectivized (ordinary resp. faithfull). We don’t need this here, and set the factors to 1.

6 2-designs from the Fourier matrix

From here on we assume $d$ to be prime with $d = 4k + 3$.

In this case equation (2), with $e^{i\eta} = 1$, simplifies to the $d$ matrices

$$R_m := \frac{1}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \tau^m(r^2 + s^2) W_{(r,s)}, \ 0 \leq m \leq d - 1$$

We get this form, when we set (mod $d$)

$$m = \frac{\beta}{2(1-\alpha)} \iff \alpha = \frac{4m^2 - 1}{4m^2 + 1}, \ \beta = \frac{4m}{4m^2 + 1}$$

These maps are well defined, as $4m^2 + 1 = 0$ (mod $d$) has no solution (resp. $-1$ is no quadratic residue) for $d = 4k + 3$.

It was observed by Balian and Itzykson ([9], 1986) that for primes $d = 4k + 3$ these matrices provide a unique common orthogonal basis of eigenvectors of $F$. Up to a phase $F$ corresponds to $m = \frac{d-1}{2}$. See also [6,7,11].

They play also a role in the concept of MUB-balanced states [1,4,34].

Now we can state the central result of this paper
Theorem 1. Let \( d = 4k + 3 \) be a prime. Let \( \{ |\psi_i \rangle : 1 \leq i \leq d \} \) be the unique common orthogonal basis of eigenvectors of the \( d \) matrices \( R_m = \frac{1}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \tau^{m(r^2 + s^2)} W_{(r,s)} \), \( 0 \leq m \leq d - 1 \).

The \( d^3 \) vectors \( \{ W_{(k,l)} |\psi_i \rangle : 1 \leq i \leq d, 0 \leq k, l \leq d - 1 \} \) form a \( 2 \)-design.

Actually, in the following, we are going to work with the \( d \) corresponding projection matrices \( \{ P_i := |\psi_i \rangle \langle \psi_i | : 1 \leq i \leq d \} \) on the eigenvectors. Let
\[
P_i^{(k,l)} = W_{(k,l)} P_i W_{(k,l)}^{-1} \quad \text{with } 0 \leq i, k, l \leq d - 1.
\]

We are going to prove
\[
\frac{1}{d^3} \sum_{i=0}^{d-1} \sum_{k=0}^{d-1} \sum_{l=1}^{d-1} P_i^{(k,l)} \otimes P_i^{(k,l)} = \frac{2}{d(d + 1)} \Pi_{\text{sym}} \quad (5)
\]

For this we need some preparation

- In the next subsection we sum up some features of the Weyl-Heisenberg matrices and of their tensor products, which we use subsequently.

- In the subsection afterwards we expand the projection matrices \( P_i \) on the eigenvectors in terms of the Weyl-Heisenberg matrices. We don’t calculate them explicitly, but just show some of their properties, based on relations of the Weyl-Heisenberg matrices and the \( R_m \), which allow us

- in the final subsection to prove the theorem.

6.1 Some properties of Weyl-Heisenberg matrices

First we summarize some well-known relations on the Weyl-Heisenberg matrices \( [2,3,10,11, \text{etc.} \ (d \text{ odd})] \).
\[
W_{(k,l)} W_{(r,s)} = \tau^{2(rl - sk)} W_{(r,s)} W_{(k,l)} = \tau^{(rl - sk)} W_{(k+r,l+s)} \quad (6)
\]
\[
\text{tr} \left( W_{(k,l)} W_{(k',l')}^* \right) = \begin{cases} 
  d & \text{if } k = k', l = l' \text{ (mod } d) \\
  0 & \text{else}.
\end{cases} \quad (7)
\]

The last equation describes the already mentioned orthogonality of the Weyl-Heisenberg matrices. Immediate consequence is (see also \[35\])
\[
\sum_{k=0}^{d-1} \sum_{l=0}^{d-1} W_{(k,l)} W_{(r,s)} W_{(k,l)}^{-1} \otimes W_{(k,l)} W_{(r',s')} W_{(k,l)}^{-1} = \begin{cases} 
  d^2 (W_{(r,s)} \otimes W_{(-r,-s)}) & \text{if } r' = -r, s' = -s \text{ (mod } d) \\
  0 & \text{else.}
\end{cases} \quad (8)
\]
We furthermore notice the following formula

\[ \text{SWAP} = \sum_{q_1=0}^{d-1} \sum_{q_2=0}^{d-1} (|e_{q_2}\rangle \langle e_{q_1}| \otimes |e_{q_1}\rangle \langle e_{q_2}|) = \frac{1}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} W_{(r,s)} \otimes W_{(-r,-s)} \]

This is a special case of \( \text{SWAP} = \frac{1}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} g(r,s) \otimes g^*(r,s) \) for any orthogonal matrix base \( \{g(r,s)\} \), \( 0 \leq r, s \leq d-1 \), where the standard norm \( \|g(r,s)\|^2_2 = d \). See the paper by Siewert [29], where this relation is extensively exploited. An immediate consequence is (see also [24]:

\[ \Pi_{\text{sym}} = \frac{1}{2} (I \otimes I + \text{SWAP}) = \frac{1}{2} \left( I \otimes I + \frac{1}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} W_{(r,s)} \otimes W_{(-r,-s)} \right) \]

### 6.2 Some properties of \( R_m \) and the projection matrices \( P_i \)

**Lemma 1.** Let \( d = 4k + 3 \) be prime, and \( R_m \) as in definition [3]. For \( 0 \leq m, m' \leq d-1 \)

\[ \text{tr} \left( R_m \right) = 1, \quad \text{tr} \left( R_m R_m^* \right) = \begin{cases} 1 & \text{if } m \neq m' \\ \frac{1}{d} & \text{if } m = m'. \end{cases} \]  

**Proof.** \( \text{tr} \left( R_m \right) = 1 \) follows directly, and further using [7], after short calculation

\[ \text{tr} \left( R_m R_m^* \right) = \frac{1}{d} \left( \sum_{r=0}^{d-1} \tau(m-m')r^2 \right)^2 = \begin{cases} (\pm i)^2 = -1 & \text{if } m \neq m' \\ \frac{1}{d} & \text{if } m = m'. \end{cases} \]

Here we used, that for prime numbers \( d = 4k + 3 \) the quadratic Gauss sum are \( \sum_{r=0}^{d-1} \tau ar^2 = (\frac{a}{d}) i\sqrt{d} \) with the Legendre symbol \( (\frac{a}{d}) = \pm 1 \), for \( a \neq 0 \).

Next we define auxiliary matrices \( X_m \) as orthonormalization of the \( R_m \).

**Lemma 2.** Let \( d = 4k + 3 \) be prime, and

\[ X_m := \frac{1}{\sqrt{d}+1} R_m + \kappa I, \quad \kappa = \frac{\sqrt{d}+1-1}{d\sqrt{d}+1}, \quad 0 \leq m \leq d-1 \]  

then for \( 0 \leq m, m' \leq d-1 \)

\[ \text{tr} \left( X_m \right) = 1, \quad \text{tr} \left( X_m X_m^* \right) = \begin{cases} 1 & \text{if } m \neq m' \\ 0 & \text{if } m = m'. \end{cases} \]  

**Proof.** The proof is straight forward using the definition and equations [10].

Next we derive an Ansatz for the projection matrices on the eigenvectors of \( R_m \) via the auxiliary matrices \( X_m \). This helps us to prove some properties.
Lemma 3. Let $d = 4k + 3$ be prime, and let

$$P_i := \sum_{m=0}^{d-1} \lambda_{im} X_m \quad 0 \leq i, m \leq d - 1$$

be the projection matrices on the common eigenvectors of all $R_m$ in arbitrary order. Then for all $0 \leq i \leq d - 1$

$$P_i = \frac{1}{d^{\sqrt{d + 1}} + 1} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} p_i^{(r,s)} W_{(r,s)} + \kappa I$$

with

$$\kappa = \frac{\sqrt{d + 1} - 1}{d^{\sqrt{d + 1}}}, \quad p_i^{(r,s)} = \sum_{m=0}^{d-1} \lambda_{im} \tau^m(r^2 + s^2) \quad 0 \leq r, s, i \leq d - 1$$

and we have

$$p_i^{(0,0)} = 1 \quad 0 \leq i \leq d - 1$$

$$(p_i^{(r,s)})^* = p_i^{(-r,-s)} \quad 0 \leq r, s, i \leq d - 1$$

$$\| (p_i^{(r,s)})_{0 \leq i \leq d-1} \|^2 = d \quad 0 \leq r, s \leq d - 1$$

Proof. The matrix $A = (\lambda_{im})_{0 \leq i \leq d-1; 0 \leq m \leq d-1}$ transfers between the $d$ orthonormal matrices $X_m$ and the also orthonormal $d$ matrices $P_i$. Therefore it must be unitary, which we use below.

We insert $X_m$ resp. $R_m$ in definition (13) and get

$$P_i = \frac{1}{d^{\sqrt{d + 1}} + 1} \left( \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \sum_{m=0}^{d-1} \lambda_{im} \tau^m(r^2 + s^2) W_{(r,s)} \right) + \sum_{m=0}^{d-1} \lambda_{im} \kappa I$$

By applying the trace on (13), and using $\text{tr}(P_i) = 1$ and equation (12) we get $\sum_{m=0}^{d-1} \lambda_{im} = 1$, for all $0 \leq i \leq d - 1$. This proves (14) with the definition of $p_i^{(r,s)}$ as in (15). Now for the properties of these coefficients.

- $p_i^{(0,0)} = 1$ follows again from $\sum_{m=0}^{d-1} \lambda_{im} = 1$, for all $0 \leq i \leq d - 1$
- $(p_i^{(r,s)})^* = p_i^{(-r,-s)}$ follows as $P_i^* = P_i$ and $W_{(r,s)}^* = W_{(r,-s)}$.
- $\| (p_i^{(r,s)})_{0 \leq i \leq d-1} \|^2 = d$ is valid, as the vector $(p_i^{(r,s)})_{0 \leq i \leq d-1}$ is for any $0 \leq r, s \leq d - 1$ the unitary transform of the vector $(\tau^m(r^2 + s^2))_{0 \leq m \leq d-1}$ by $A$, and this vector has obviously squared norm $\| (\tau^m(r^2 + s^2))_{0 \leq m \leq d-1} \|^2 = d$.

Remark: When $P_x$ is an fiducial projector for a Weyl-Heisenberg covariant SIC, then $|p_i^{(r,s)}|^2 = 1$. See also [5][24][35] for the SIC-related background.
6.3 Proof of Theorem 1

We want to prove equation \((5)\). Let

\[
x = \frac{1}{d^3(d + 1)} \sum_{i=0}^{d-1} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} P_i^{(k,l)} \otimes P_i^{(k,l)} = \frac{1}{d^3(d + 1)} \sum_{i=0}^{d-1} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} W_{(k,l)} P_i W_{(k,l)}^{-1} \otimes W_{(k,l)} P_i W_{(k,l)}^{-1}
\]

When we insert \(P_i\), as described in Lemma \(3\) the sums reduce significantly, due to the equation \((5)\) about sums of tensor-products of Weyl-Heisenberg matrices. We get

\[
x = \frac{1}{d^3(d + 1)} \sum_{i=0}^{d-1} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} p_i^{(r,s)} (-r,-s) W_{(r,s)} \otimes W_{(-r,-s)}
\]

\[
+ \sum_{i=0}^{d-1} p_i^{(0,0)} 2(\sqrt{d + 1} - 1) I \otimes I + \sum_{i=0}^{d-1} (\sqrt{d + 1} - 1)^2 I \otimes I
\]

We use \((16a)\) \(p_i^{(0,0)} = 1\) and \((16b)\) \(p_i^{(r,s)} \cdot p_i^{(-r,-s)} = |p_i^{(r,s)}|^2\) and get

\[
x = \frac{1}{d^3(d + 1)} \sum_{i=0}^{d-1} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} |p_i^{(r,s)}|^2 W_{(r,s)} \otimes W_{(-r,-s)} + \frac{1}{d(d + 1)} I \otimes I
\]

And finally we use the key equation \((16c)\) \(\| (p_i^{(r,s)})_{0 \leq i \leq d-1} \|_2^2 = |p_0^{(r,s)}|^2 + \ldots + |p_{d-1}^{(r,s)}|^2 = d\) to get

\[
x = \frac{1}{d(d + 1)} \left( \frac{1}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} W_{(r,s)} \otimes W_{(-r,-s)} + I \otimes I \right) = \frac{2}{d(d + 1)} \Pi_{\text{sym}}
\]

according the formula for \(\Pi_{\text{sym}}\) in \((9)\). \(\square\)

7 State space

SŁOMCZYŃSKI and SZYMUSIAR [28] noticed, that \(\{P_i := |\psi_i\rangle \langle \psi_i| : 1 \leq i \leq N\}\) is a 2-design, iff for any matrix \(\rho\) with \(\text{tr}(\rho) = 1\)

\[
\rho = (d + 1) \sum_{i=1}^{N} \frac{d}{N} \text{tr}(\rho P_i) P_i - I
\]

This equation can e.g. be proven by using equation \((\Pi)\) for 2-designs, with \(\Pi_{\text{sym}} = \frac{1}{2} (I \otimes I + \text{SWAP})\)

\[
\sum_{i=1}^{N} P_i \otimes P_i = \frac{N}{d(d + 1)} (I \otimes I + \text{SWAP}) .
\]

Multiplying it with \(I \otimes \rho\), applying the partial trace, and using (SIEWERT [29])

\[
\text{tr}[2] ((I \otimes \rho) \cdot \text{SWAP}) = \rho
\]

we get \(\sum_{i=1}^{N} \text{tr}(\rho P_i) P_i = \frac{N}{d(d + 1)} (\text{tr}(\rho)I + \rho)\).
As a consequence we state

**Corollary 1.** Let \( d = 4k + 3 \) be a prime. Let \( \{ P_i := |\psi_i\rangle\langle\psi_i| : 1 \leq i \leq d \} \) be the projection matrices on the unique common orthogonal basis of eigenvectors of the \( d \) matrices \( R_m \) as in Theorem \( 1 \) and \( P_i^{(k,l)} = W_{(k,l)} P_i W_{(k,l)}^{-1} \) with \( 0 \leq i, k, l \leq d - 1 \). Then for any matrix \( \rho \) with \( \text{tr}(\rho) = 1 \) (e.g. density matrices)

\[
\rho = (d + 1) \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \rho_i^{(k,l)} P_i^{(k,l)} - I \quad \text{with} \quad \rho_i^{(k,l)} = \frac{1}{d^2} \text{tr}(\rho P_i^{(k,l)})
\]

This result can be seen as counterpart of the equation for the \( d^2 \) projection matrices of a SIC 2-designs \( \rho = (d + 1) \sum_{i=1}^{d^2} \rho_i P_i - I \), with \( \rho_i = \frac{1}{d} \text{tr}(\rho P_i) \) which (as "measurement in the sky") attracted a great deal of attention in the context of the QBism approach to quantum theory by Fuchs et al. [15,16].

### 8 Concluding remarks

One can try to generalize the construction given here to further dimensions \( d \). The rank 1 projection matrices of SICs, as well as complete sets of MUBs are projective 2-designs, which, like the construction here, are covariant under the Weyl-Heisenberg groups. Are there more similar 2-designs?

Finally it would be interesting, if and how the 2-design property could possibly be related to the quantum harmonic oscillator.

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References

[1] Amburg, I., Sharma, R., Sussman, D.M., and Wootters, W.K.: States that "look the same" with respect to every basis in a mutually unbiased set, arXiv:quant-ph/1407.4074

[2] Appleby, D.M.: SIC-POVMs and the extended Clifford group, J. Math. Phys., 46, 052107 (2005), arXiv:quant-ph/0412001

[3] Appleby, D.M.: Properties of the extended Clifford group with applications to SIC-POVMs and MUBs, arXiv:quant-ph/0909.5233

[4] Appleby, D. M., Bengtsson, I., and Dang, H. B.: Galois Unitaries, Mutually Unbiased Bases, and MUB-balanced states, arXiv:quant-ph/1409.7987

[5] Appleby, D. M., Bengtsson, I., Flammia, S., and Goyeneche, D.: Tight Frames, Hadamard Matrices and Zauner’s Conjectur, J. Phys. A 52, 295301 (2019), arXiv:quant-ph/1903.06721

[6] Athanasiu, G.G., Floratos, E.G., and Nicolis, S. Holomorphic Quantization on the Torus and Finite Quantum Mechanics, J.Phys. A29 :6737,(1996), arXiv:hep-th/9509098

[7] Athanasiu, G.G., Floratos, E.G., and Nicolis, S. Fast Quantum Maps J.Phys. A31 (1998) L655, arXiv:math-ph/9805012

[8] Auslander, L. and Tolimieri, R.: Is computing with the finite Fourier Transform pure or applied mathematics?, Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 6, 847–897.

[9] Balian, R., and Itzykson, C.: Observations sur la mécanique quantique finie. C. R. Acad. Sci. Paris Sér. I Math., 303 no. 16 (1986), 773–778.

[10] Bengtsson, I., and Życzkowski, K.: On discrete structures in finite Hilbert spaces, arXiv:quant-ph/1701.07902

[11] Bengtsson, I., and Życzkowski, K.: Geometry of Quantum States: An Introduction to Quantum Entanglement, Cambridge University Press; 2nd edition (2017)

[12] Delsarte, P. Goethals, J.M. and Seidel, J.J.: Spherical Codes and Designs, Geom. Dedicata 6 (1977), 363–388.

[13] Dickinson, B.W., and Steiglitz, K.: Eigenvectors and functions of the discrete Fourier transform, IEEE Transactions on Acoustics, Speech, and Signal Processing, 30 (1982), no. 1, 25–3.

[14] Floratos E.G., and Leontaris, G.K.: Uncertainty relation and non-dispersive states in Finite Quantum Mechanics, Phys.Lett. B412, (1997) 35-41 arXiv:hep-th/970615
[15] Fuchs, C.A, and and Schack, R.: A Quantum-Bayesian Route to Quantum-State Space, arXiv:quant-ph/0912.4252

[16] Fuchs, C.A, and and Stacey, B.C.: QBism: Quantum Theory as a Hero’s Handbook, Axioms 6(3) (2017), arXiv:quant-ph/1703.07901

[17] Grünbaum, F.A.: The eigenvectors of the discrete Fourier transform: A version of the Hermite functions, Journal of Mathematical Analysis and Applications, 88 (1982), 355–363.

[18] Gurevich, S., and Hadani, R.: On the diagonalization of the discrete Fourier transform, arXiv:cs.IT/0808.3281

[19] Hoggar, S.G.: $t$-Designs in Projective Spaces, European J. Combin. 3 (1982), 233–254.

[20] Kuznetsov A., and Kwasnicki M.: Minimum Hermite-Type eigenbasis of the discrete Fourier transform, arXiv:math.CA/1706.08740

[21] McClellan, J.H. and Parks, T.W.: Eigenvalue and Eigenvector Decomposition of the Discrete Fourier Transform, IEEE Trans. Audio Electroacoustics AU-20 (1972), no. 1, 66–74.

[22] Morton, P.: On the Eigenvectors of Schur’s Matrix, J. Number Theory 12 (1980), 122–127.

[23] Neuhauser, M.: An explicit construction of the Metaplectic Representation over a Finite Field, Journal of Lie Theory 12 (2002) 15–30

[24] Ostrovskyi, V., and Yakymenko, D.: Geometric properties of SIC-POVM tensor square, Lett. Math. Phys. 112, 7 (2022), arXiv:quant-ph/1911.05437

[25] Renes, J.M, Blume-Kohout, R., Scott, A.J., and Caves, C.M.: Symmetric Informationally Complete Quantum Measurement, J. Math. Phys. 45, 2171 (2004), arXiv:quant-ph/0310075

[26] Roy, A. and Scott, A.J.: Weighted complex projective 2-designs from bases: optimal state determination by orthogonal measurements, J. Math. Phys. 48, 072110 (2007), arXiv:quant-ph/0703025

[27] Schwinger, J.: Unitary Operator Bases, Proc. Nat. Acad. Sci. U.S.A. 46 (1960), 570–579.

[28] Słomczyński, W. and Szymusiak, A.: Morphorphic POVMs, generalised qplexes, and 2-designs, Quantum 4, 338 (2020), arXiv:quant-ph/1911.12456

[29] Siewert, J.: On orthogonal bases in the Hilbert-Schmidt space of matrices, J. Phys. Commun. 6, 055014 (2022) arXiv:quant-ph/2205.06035

[30] Singh, A., and Carroll, M.: Modeling Position and Momentum in Finite-Dimensional Hilbert Spaces via Generalized Clifford Algebra, arXiv:quant-ph/1806.10134
[31] Thirring, W.: *Lehrbuch der Mathematischen Physik, Bd.3*, Springer, Wien, New York, (1979).

[32] Waldron, S.: *An Introduction to Finite Tight Frames*, Springer, New York, (2018).

[33] Weyl, H.: *Gruppentheorie und Quantenmechanik*, 2. Auflage 1931, Reprint by Wissenschaftliche Buchgesellschaft, Darmstadt, 1981.

[34] Wootters, W.K. and Sussman, D. M.: Discrete Phase Space and Minimum-Uncertainty States, arXiv:quant-ph/0704.1277

[35] Yakymenko, D.: On the continuous Zauner conjecture, arXiv:quant-ph/2112.05875

[36] Zauner, G.: *Quantendesigns*, Ph.D. Thesis, Vienna, 1999 (in german language). - English translation in *International Journal of Quantum Information (IJQI)*, Volume 9, Issue 1, (2011), 445–507 (online available at http://www.gerhardzauner.at/documents/gz-quantumdesigns.pdf).

[37] Zhang, S and Vourdas, A.: Analytic Representation of Finite Quantum Systems, *J. Phys. A: Math.* 37 (2004) 8349-8363, arXiv:quant-ph/0504014