GENERIC NORMS AND METRICS ON COUNTABLE ABELIAN GROUPS

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Abstract. For a countable abelian group $G$ we investigate the generic properties of the space of all invariant metrics on $G$. We prove that for every such an unbounded group $G$, i.e. group which has elements of arbitrarily high order, there is a dense set of invariant metrics on $G$ with which it is isometric to the rational Urysohn space, and a comeager set of invariant metrics such that the completion is isometric to the Urysohn space. This generalizes results of Cameron and Vershik, Niemiec, and the author.

Then we prove that for every $G$ such that $G \cong \bigoplus_N \mathbb{Z}$ there is a comeager set of invariant metrics on $G$ such that all of them give rise to the same metric group after completion. If moreover $G$ is unbounded, then using a result of Melleray and Tsankov we get that the completion is extremely amenable. Finally, we observe that all the known universal abelian Polish groups, e.g. the Shkarin’s group, are extremely amenable.

Introduction

For an unbounded countable abelian group $G$, J. Melleray and T. Tsankov prove in [9] that the set of all invariant metrics on $G$ with which it is extremely amenable is comeager in the Polish space of all invariant metrics on $G$. That result motivated the work presented in this paper. We focus on two main themes:

Groups isometric to the Urysohn space. In [2], Cameron and Vershik prove that there is an invariant metric on $\mathbb{Z}$ with which it is isometric to the rational Urysohn space. In particular, the completion is isometric to the Urysohn space and thus the Urysohn space has a structure of a monothetic abelian group. Niemiec in [10] prove that the Shkarin’s universal abelian Polish group is isometric to the Urysohn space and its canonical countable dense subgroup (which is $\bigoplus_N \mathbb{Q}/\mathbb{Z}$)

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is isometric to the rational Urysohn space. Here we generalize these results by proving that actually every unbounded countable abelian group, i.e. group having elements of arbitrarily high order, admits an invariant metric with which it is isometric to the rational Urysohn space. In particular, this covers the cases of \( \mathbb{Z} \) and \( \bigoplus \mathbb{N} \mathbb{Q}/\mathbb{Z} \). We prove something more general stated in the theorem below.

**Theorem 0.1.** For any unbounded countable abelian group \( G \), the set of all invariant metrics with which it is isometric to the rational Urysohn space is dense in the space of all invariant metrics on \( G \). Moreover, the set of all invariant metrics on \( G \) with which the completion is isometric to the Urysohn space is dense \( G_\delta \).

**Generic metrics.** We motivate the next topic by the following facts. We recall that in the Polish space of all countable graphs, those that are isomorphic to the random graph form a comeager subset. One cannot expect literally the same property for a metric structure, however it is true that in the Polish space of all metrics on a countable set, those whose completion is isometric to the Urysohn universal space form a comeager set. Analogously, one can show that in the Polish space of all norms on the countable infinite-dimensional vector space over \( \mathbb{Q} \), those whose completion is isometric to the Gurarij space ([5]) form a comeager set.

Our aim is to find generic metrics on countable abelian groups. There are of course many countable abelian groups to consider. However, we can prove the following result.

**Theorem 0.2.** Let \( G \) be a countable abelian group such that \( G \cong \bigoplus \mathbb{N} G \). Then there is a comeager set of invariant metrics on \( G \) such that all of them give rise to the same metric group after the completion.

Theorem 0.2 raises the question on the number of generic abelian Polish metric groups. That is, one may ask whether there is a single abelian Polish metric group \( \mathcal{G} \) such that for every countable abelian group \( G \) with the property that \( G \cong \bigoplus \mathbb{N} G \), the completion of \( G \) with respect to the generic metric provided by Theorem 0.2 is isometrically isomorphic to \( \mathcal{G} \). We shall later see that it is not the case.

When the group \( G \) is moreover unbounded we get the following.

**Corollary 0.3.** Let \( G \) be an unbounded countable abelian group such that \( G \cong \bigoplus \mathbb{N} G \). Then there is a comeager set of invariant metrics on \( G \) such that all of them give rise to the same metric group after the completion which is isometric to the Urysohn space and extremely amenable.
We also derive the extreme amenability of the universal Polish groups.

**Corollary 0.4.** The universal abelian Polish groups of Shkarin [13] and the author [3] are extremely amenable.

1. Definitions and preliminaries

A metric $d$ on an abelian group $G$ is invariant if for every $a, b, c \in G$ we have $d(a, b) = d(a + c, b + c)$. A norm on $G$ is a function $\lambda : G \to \mathbb{R}_0^+$ from $G$ to non-negative reals which attains zero only at $0_G \in G$ and satisfies for every $a, b \in G$, $\lambda(a) = \lambda(-a)$ and $\lambda(a + b) \leq \lambda(a) + \lambda(b)$. There is a one-to-one correspondence between invariant metrics and norms on abelian groups: for a norm $\lambda$, $d_\lambda(a, b) := \lambda(a - b)$ defines an invariant metric, and for an invariant metric $d$, $\lambda(a) := d(a, 0)$ defines a norm.

Consider the set of all norms, respectively invariant metrics, on some countable abelian group $G$. It can be viewed as a subset of $\mathbb{R}^G$, resp. $\mathbb{R}^{G \times G}$. In both cases, one can easily check it is a $G_\delta$ set, thus a Polish space (we refer the reader to [7] for facts needed about Polish spaces). It turns out it is more convenient for us to work with norms, rather than invariant metrics, so we shall do so mostly in the sequel.

Let us denote the Polish space of norms on $G$ by $N_G$. Later, when the group is known from the context, respectively it is fixed, we shall just write $N$. Clearly, the space $N_G$ is homeomorphic with the Polish space of all invariant metrics on $G$ via the formula above.

The content of this paper is to investigate generic properties of the space of norms on countable abelian groups. We recall one important result of Melleray and Tsankov in that direction that we shall apply in our paper.

The “official” definition of an unbounded abelian group is postponed till later when it is used. Let us say it means the group contains elements of arbitrarily high order.

**Theorem 1.1** (Melleray, Tsankov, Theorem 6.4 in [9]). Let $G$ be a countable unbounded abelian group. Then the set $E = \{\lambda \in N_G : (G, \lambda) \text{ is extremely amenable}\}$ is dense $G_\delta$ in $N_G$.

We note that Melleray and Tsankov formulated the theorem with invariant metrics rather than norms which is, as noted above, however equivalent.

**Definition 1.2.** Let $G$ be an abelian group and $A \subseteq G$ a symmetric subset containing zero. A partial norm on $A$ is a function $\lambda : A \to \mathbb{R}_0^+$ satisfying the following requirements:
• $\lambda(x) = 0$ iff $x = 0$, for $x \in A$,
• $\lambda(x) = \lambda(-x)$ for $x \in A$,
• $\lambda(x) \leq \sum_{i=1}^{n} \lambda(x_i)$, where $n$ is arbitrary, $x, x_1, \ldots, x_n \in A$ and $x = \sum_{i=1}^{n} x_i$.

If $\lambda$ satisfies all the conditions except the first one, then it is called a partial seminorm. If it satisfies the first and the second condition we call it a partial pre-norm.

**Lemma 1.3.** Let $G$ be an abelian group, $A \subseteq G$ some symmetric subset containing zero and $\lambda_A : A \rightarrow \mathbb{R}_0^+$ a partial norm on $A$. Then for any subset $A \subseteq B \subseteq \langle A \rangle \leq G$, where $\langle A \rangle$ is the subgroup of $G$ generated by $A$, there exists a partial seminorm $\lambda_B : B \rightarrow \mathbb{R}_0^+$ which extends $\lambda_A$ on $A$.

In particular, if $A$ is finite and $B = \langle A \rangle = G$, then $\lambda_B$ is a norm on $G$ extending $\lambda_A$.

**Proof.** Take $b \in B$ and set

$$\lambda_B(b) = \inf \{ \sum_{i=1}^{n} \lambda_A(a_i) : (a_i)_{i=1}^{n} \subseteq A, b = \sum_{i=1}^{n} a_i \}.$$ 

Since $B \subseteq \langle A \rangle$, there exist $(a_i)_{i=1}^{n} \subseteq A$ such that $b = \sum_{i=1}^{n} a_i$. It directly follows from the definition that $\lambda_B$ satisfies all the conditions of a partial seminorm. Moreover, if $A$ is finite, the the infimum from the definition of $\lambda_B$ might be replaced by the minimum and it follows that $\lambda_B$ in that case is a partial norm. \hfill \Box

**Remark 1.4.** Notice that the partial seminorm $\lambda_B$ is actually the greatest extension of $\lambda_A$; i.e. if $\lambda : B \rightarrow \mathbb{R}_0^+$ is any partial seminorm on $B$ that extends $\lambda_A$, then $\lambda \leq \lambda_B$; and if $A$ is finite, then $\lambda_B$ is the greatest partial norm extending $\lambda_A$.

When we are given a symmetric subset $A \subset G$ containing zero of some abelian group and also some pre-norm $\rho' : A \rightarrow \mathbb{R}_0^+$ then we can get, using the same formula as in the proof above, a greatest partial seminorm determined by $\rho'$; or, in the case $A$ is finite, a greatest partial norm determined by $\rho'$. We state that explicitly in the next fact and omit the easy proof.

**Fact 1.5.** Let $A \subseteq G$ be a symmetric subset containing zero of some abelian group. Let $\rho' : A \rightarrow \mathbb{R}_0^+$ be a pre-norm. Then the formula, applied for every $x \in A$,

$$\rho(x) = \inf \{ \sum_{i=1}^{n} \rho'(x_i) : x = \sum_{i=1}^{n} x_i, (x_i)_{i=1}^{n} \subseteq A \}$$
gives a partial seminorm on $A$, resp. partial norm on $A$ if $A$ is finite.

In the text below we shall be interested in the possibilities how to extend partial norms. We recall the notion of a Katětov function on a metric space which corresponds, in the terminology of continuous model theory, to the quantifier-free type over a metric space. Let $X$ be a metric space and $f : X \to \mathbb{R}^+_0$ a function. It is called Katětov if it satisfies for every $x,y \in X$

$$|f(x) - f(y)| \leq d_X(x,y) \leq f(x) + f(y).$$

The function $f$ can be then viewed as a prescription of distances of a new point to the points of $X$ in the sense that we can define a one-point extension $X \cup \{x_f\}$ and define the distance of $x_f$ to a point $y \in X$ as $f(y)$.

We start with some algebraic definition of elements in abelian groups that have the potential to realize Katětov functions.

**Definition 1.6.** Let $G$ be an abelian group and $A \subseteq G$ a symmetric subset (containing zero). Let $g \in G \setminus A$. We define the distance of $g$ from $A$ in $G$, denoted by $\text{dist}_{G}(g,A)$, as the length of the shortest oriented path between $g$ and $A$ in the oriented Cayley graph of the group with generators $\{g\} \cup A$, i.e. there is an edge going from $a$ to $g + a$, however there might not be an edge in the other direction.

Now let $G$ be an abelian group, $A \subseteq G$ a finite symmetric subset containing zero and $\lambda_A$ a partial norm on $A$. Set $\bar{A} = \{a-b : a,b \in A\}$. Note that $A \subseteq \bar{A}$ and that $\bar{A}$ is again a finite symmetric subset containing zero. Let $\bar{\lambda}_A$ be the greatest extension of $\lambda_A$ onto $\bar{A}$ guaranteed by Lemma 1.3 and note that it induces a metric $d_A$ on $A$ defined as $d_A(a,b) = \bar{\lambda}_A(a-b)$ for $a,b \in A$. Let $f : A \to \mathbb{R}^+$ be a Katětov function defined on $A$ with respect to the metric $d_A$. We are now interested whether it is possible to find an element $g \in G \setminus A$ and a partial norm $\lambda$ on $\{g-a, a - g : a \in A\} \cup \bar{A}$ which extends $\bar{\lambda}_A$ and such that for every $a \in A$ we have $f(a) = \lambda(g-a)$.

Set $m = \min\{\min_{a \neq 0 \in A} \bar{\lambda}_A(a), \min_{a \in A} f(a)\}$ and $M = \max\{\max_{a \in \bar{A}} \bar{\lambda}_A(a), \max_{a \in A} f(a)\}$. Then we have the following proposition.

**Proposition 1.7.** Suppose that there exists $g \in G \setminus \bar{A}$ such that $\text{dist}(g, \bar{A}) > 2\frac{M}{m}$. Then there exists a partial norm $\lambda$ on $\{g-a, a - g : a \in A\} \cup \bar{A}$ which extends $\bar{\lambda}_A$ and such that for every $a \in A$ we have $f(a) = \lambda(g-a)$.

**Proof.** Set $B = \bar{A} \cup \{g-a, a - g : a \in A\}$. Note that $B$ is also a symmetric set containing zero. Define $\lambda$ on $B$ as follows: for any $b \in B$...
such that
\[ \lambda(b) = \begin{cases} \bar{\lambda}_A(b) & b \in \bar{A}, \\ f(x) & \text{if } b = g - x, \text{ or } b = x - g. \end{cases} \]

It suffices to check that \( \lambda \) is a partial norm on \( B \). The first two conditions of the definition of a partial norm are easily checked. We claim that also the last condition is satisfied. Suppose otherwise. Then there are \( b \in B \) and \( (b_i)_{i=1}^n \subseteq B \) such that \( b = \sum_{i=1}^n b_i \) and \( \lambda(b) > \sum_{i=1}^n \lambda(b_i) \). We have that \( n \leq \frac{M}{m} \) as otherwise
\[ M \geq \lambda(b) > n \cdot m > \frac{M}{m} \cdot m = M, \]
a contradiction.

Moreover, without loss of generality we may also suppose that for no \( i \neq j \leq n \) we have \( b_i = g - a_i \) and \( b_j = a_j - g \), for some \( a_i, a_j \in A \). Indeed, suppose otherwise. Then since \( g - a_i + a_j - g = a_j - a_i \in \bar{A} \) and the function \( f \) is Katětov we have that \( \lambda(g - a_i) + \lambda(a_j - g) = f(a_i) + f(a_j) \geq d_A(a_j, a_i) = \bar{\lambda}_A(a_j - a_i) \). Thus we can replace the pairs \( b_i, b_j \) by \( a_j - a_i \).

**Case 1:** \( b \in \bar{A} \). We claim that there must be some \( i \leq n \) such that \( b_i \) is equal to \( g - a \) or \( a - g \) for some \( a \in A \). Indeed, otherwise we get into a contradiction since \( \bar{\lambda}_A \) is a partial norm. By the argument in the paragraph above there is a single sign \( \varepsilon \in \{1, -1\} \) such that for every \( b_i \notin \bar{A} \) we have \( b_i = \varepsilon \cdot g - \varepsilon \cdot a_i \) for some \( a_i \in A \). Suppose that \( \varepsilon = 1 \), the other case is analogous. It follows that there is some \( m \leq n \) such that \( b = m \cdot g + \sum_{i=1}^n c_i \), where \( (c_i)_{i=1}^n \subseteq \bar{A} \). Thus we get that \( \text{dist}(g, \bar{A}) \leq 2n < 2\frac{M}{m} \), a contradiction.

**Case 2:** \( b = g - a \), or \( b = a - g \), for some \( a \in A \). Let us say \( b = g - a \), the other case is analogous. Suppose at first that for all \( i \leq n \) we have \( b_i \notin \bar{A} \). Then we again get \( \text{dist}(g, \bar{A}) \leq 2n < 2\frac{M}{m} \), a contradiction.

Thus we suppose that for some \( i \leq n \) we have \( b_i \notin \bar{A} \). As in Case 1 we may suppose that there is a single sign \( \varepsilon \in \{1, -1\} \) such that for every \( b_i \notin \bar{A} \) we have \( b_i = \varepsilon \cdot g - \varepsilon \cdot a_i \) for some \( a_i \in A \).

Suppose at first that \( \varepsilon = -1 \). Then there is some \( m \leq n + 1 \) such that \( m \cdot g = \sum_{i=1}^n c_i \), where \( (c_i)_{i=1}^n \subseteq \bar{A} \). Thus we again get that \( \text{dist}(g, \bar{A}) \leq n \leq 2\frac{M}{m} \), a contradiction..

Now suppose that \( \varepsilon = 1 \). Suppose without loss of generality that \( b_1 = g - a_1 \), for some \( a_1 \in A \). Then, since \( n > 1 \), \( \sum_{i=2}^n b_i = a_1 - a \in \bar{A} \).

If for any \( 2 \leq i \leq n \) we have \( b_i = g - a_i \), for some \( a_i \in A \), then we get that \( \text{dist}(g, \bar{A}) < 2n < 2\frac{M}{m} \), a contradiction. Thus suppose that for
every \(2 \leq i \leq n\) we have \(b_i \in \bar{A}\). Then we claim that we may suppose that \(n = 2\). Indeed, we have that \(\sum_{i=2}^{n} b_i = a_1 - a \in \bar{A}\) and thus \(\bar{\lambda}_A(a_1 - a) \leq \sum_{i=2}^{n} \bar{\lambda}_A(b_i)\). So we are left with the case that

\[
\lambda(g - a) = f(a) > \lambda(g - a_1) + \lambda(a_1 - a) = f(a_1) + \bar{\lambda}_A(a_1 - a) = f(a_1) + d_A(a_1, a),
\]

which is again a contradiction with the fact that \(f\) is Katětov. \(\square\)

**Definition 1.8.** We shall call an abelian group \(G\) unbounded if it either contains an element of infinite order or it contains elements of arbitrarily high finite orders.

**Lemma 1.9.** Let \(G\) be an unbounded abelian group, \(A \subseteq G\) a finite symmetric subset containing zero and \(R > 0\) a real number. Then there exists \(g \in G\) such that \(\text{dist}(g, A) > R\).

**Proof.** Set \(B = \{n_1 \cdot a_1 + \ldots + n_i \cdot a_i : a_1, \ldots, a_i \in A, n_1, \ldots, n_i \geq 0, n_1 + \ldots + n_i \leq R\}\). Note that \(B\) is again finite symmetric and containing zero. It suffices to show that there is \(g \in G\) such that \(n \cdot g \notin B\), for every \(0 < n \leq R\).

If there is \(b \in B\) such that \(b\) has infinite order, then it clearly suffices to take \(N \cdot b\), for \(N\) sufficiently large, as \(g\). So suppose that every \(b \in B\) has finite order. Let \(N\) be the maximum of orders of elements from \(B\). Suppose there is no such \(g \in G\), thus for every \(g \in G\) there are \(n \leq R\) and \(b \in B\) such that \(n \cdot g = b\). However, then follows that the order of every \(g \in G\) is bounded by \(R \cdot N\). That is a contradiction with unboundedness of \(G\). \(\square\)

2. Groups isometric to the rational Urysohn space

We recall here that the Urysohn universal metric space \(\mathbb{U}\) is the unique Polish metric space (i.e. complete and separable) satisfying the property that a partial isometry between two finite subsets extends to an autoisometry of the whole space. It was constructed by Urysohn in [14]. We refer to Chapter 5 in [12] for more information about this space.

The Urysohn space has one distinguished countable dense set which is called the rational Urysohn space and denoted by \(\mathbb{QU}\). It is the unique countable metric space with rational distances that again has the property that any partial isometry between two finite subsets extends to an autoisometry of the whole space. The following well-known fact gives another characterization of \(\mathbb{QU}\) which we will use.
Fact 2.1. Let $X$ be a countable metric space with rational distances. Then $X$ is isometric to $\mathbb{Q}$ if and only if for every finite subset $F \subseteq X$ and every rational Katětov function $f : F \to \mathbb{Q}$ there exists $x \in X$ realizing $f$, i.e. $\forall y \in F \left( d(x, y) = f(y) \right)$.

Lemma 2.2. Let $G$ be an abelian group, $F \subseteq G$ a finite symmetric subset containing zero and let $\rho$ be a partial norm on $F$. Then for any $\varepsilon > 0$ there exists a rational partial norm $\rho_R$ on $F$ such that for any $f \in F$ we have $|\rho(f) - \rho_R(f)| < \varepsilon$.

Proof. Enumerate $F$ as $(f_i)_{i \leq n}$ such that for $i < j \leq n$ we have $\rho(f_i) \geq \rho(f_j)$, and $f_n = 0$. Let $\delta = \min\{\varepsilon, \min\{\rho(f) - \rho(g) : f, g \in F, \rho(f) \neq \rho(g)\}\}$. Moreover, choose positive rational numbers $(r_i)_{i \leq n}$ such that

- for any $i, j < n$ if $\rho(f_i) = \rho(f_j)$, then $r_i = r_j$;
- for any $i < n$ we have $i\varepsilon/n > r_i - \rho(f_i) > (i - 1)\varepsilon/n$.

Now for $i < n$ we define $\rho_R(f_i) = r_i$, and for $f_n = 0$ we set $\rho_R(f_n) = 0$. We claim $\rho_R$ is as desired. Clearly, it is rational and for any $i \leq n$ we have $\varepsilon > \rho_R(f_i) - \rho(f_i) \geq 0$. If we check that $\rho_R$ is a partial norm then we will be done.

First, for any $i \leq n$ we have $\rho_R(f_i) = 0$ iff $f_i = f_n = 0$, and also for any $i < j < n$, if $f_i = -f_j$ then $\rho_R(f_i) = \rho_R(f_j)$. So it remains to check that for any $i_1, \ldots, i_k, i < n$ such that $f_i = f_{i_1} + \ldots + f_{i_k}$ we have $\rho_R(f_i) \leq \rho_R(f_{i_1}) + \ldots + \rho_R(f_{i_k})$. If for some $j \leq k$ we have that $i_j \leq i$, then $\rho(f_{i_j}) \geq \rho(f_i)$, thus also by the choice of $r_i$'s, $\rho_R(f_{i_j}) \geq \rho_R(f_i)$, and there is nothing to prove. So suppose that for all $j \leq k$ we have $i < i_j$. Then since $\rho_R(f_i) - \rho(f_i) > \rho_R(f_{i_j}) - \rho(f_{i_j}) > 0$, for all $j \leq k$, and since $\rho(f_i) \leq \rho(f_{i_1}) + \ldots + \rho(f_{i_k})$, we are done. \hfill $\Box$

Theorem 2.3. Let $G$ be a countable unbounded abelian group. Then there exists a norm $\lambda$ on $G$ such that $(G, \lambda)$ is isometric to the rational Urysohn space.

Moreover, the set of all norms on $G$ with which it is isometric to the rational Urysohn space is dense.

Proof. The proof uses Lemmas 1.9 and 1.3 and Proposition 1.7 and follows the standard construction of the rational Urysohn space. Enumerate $G$ as $\{g_n : n \in \mathbb{N}\}$ and let $\{(A_i, f_i) : i \in \mathbb{N}\}$ be an enumeration with infinite repetition of all pairs $(A, f)$, where $A$ is a finite rational metric space and $f : A \to \mathbb{Q}$ a rational Katětov function over $A$.

By induction, we shall produce finite symmetric sets $F_i, i \in \mathbb{N}_0$, containing zero, with partial rational norm $\lambda_i$ on $F_i$ such that for every $n$, $F_n \subseteq F_{n+1}$ and $\lambda_n \subseteq \lambda_{n+1}$, $G = \bigcup_n F_n$, and such that $G$ with the metric induced by the norm $\lambda = \bigcup_n \lambda_n$ is isometric to the rational
Urysohn space.

Set $F_0 = \{0\}$ and $\lambda_0$ be the trivial norm on $F_0$. Now suppose that
for some $n$ even, $F_n$ and $\lambda_n$ on $F_n$ have been defined. We define
$F_{n+1}$ and $\lambda_{n+1}$. Take the element $g = g_{n/2+1}$. If $g \in F_n$ then we do
nothing, i.e. set $F_{n+1} = F_n$ and $\lambda_{n+1} = \lambda_n$. So suppose that $g \notin F_n$.
We set $F_{n+1} = F_n \cup \{g, -g\}$. We need to extend $\lambda_n$. We distinguish cases:

- $g \in \langle F_n \rangle$: then we use Lemma 1.3 to extend the partial norm
  $\lambda_n$ on $F_n$ to a partial norm $\lambda_{n+1}$ on $F_{n+1}$,
- $g \notin \langle F_n \rangle$: then we can set
  1. $\lambda_{n+1}(a) = \lambda_n(a)$ if $a \in F_n$,
  2. set $m$ to be the minimal positive integer such that for some
     $f \in F_n$, $mg = f$; if no such $m$ exists (in particular, $g$ has an
     infinite order), or $f = 0$, then we set $\lambda_{n+1}(a) = \lambda_{n+1}(-a) = 1$;
     otherwise, we set $\lambda_{n+1}(a) = \lambda_{n+1}(-a) = \lambda_n(f)/m$.

In any case, it is easy to check that $\lambda_{n+1}$ is a partial norm on
$F_{n+1}$.

Now suppose that for some $n$ odd, $F_n$ and $\lambda_n$ on $F_n$ have been defined.
Set $G_n = F_n = \{a-b: a, b \in F_n\}$ and extend $\lambda_n$ to $\rho_n$ on $G_n$ by Lemma
1.3. Then $\rho_n$ induces a metric $d_n$ on $F_n$, as usual, by $d_n(a, b) = \rho_n(a-b)$,
for $a, b \in F_n$.

Set $(A, f) = (A_n, f_n)$. If there is no subset of $(F_n, d_n)$ isometric to
$A$ then do nothing and set $F_{n+1} = F_n$ and $\lambda_{n+1} = \lambda_n$. Otherwise, take
some subset $B \subseteq F_n$ isometric to $A$ and consider $f$ to be defined on
$B \cong_{iso} A$. We can clearly extend $f$ to the whole $F_n$, still denoted by
$f$, so that it is still Katětov. Just set for instance $f(x) = \min\{f(a) +
\lambda_n(a, x) : a \in B\}$, for $x \in F_n$.

Set $m = \min\{\min_{x \neq 0 \in G_n} \rho_n(x), \max_{x \in F_n} f(x)\}$ and
$M = \max\{\max_{x \in G_n} \rho_n(x), \max_{x \in F_n} f(x)\}$. By Lemma 1.9 we can find
some element $g \in G$ such that $\text{dist}(g, G_n) > 2M/m$. Then by Proposition
1.7 we can extend $G_n$ to $F_{n+1} = G_n \cup \{g - a, a - g : a \in F_n\}$ and $\rho_n$
to a partial norm $\lambda'_{n+1}$ on $F'_{n+1}$ such that $f(x) = \lambda'_{n+1}(g - x)$ for every
$x \in F_n$. If $B$ was the only subset of $(F_n, d_n)$ isometric to $A$ then we
are done and set $F_{n+1} = F'_{n+1}$ and $\lambda_{n+1} = \lambda'_{n+1}$. Otherwise, we repeat
the same procedure. Note that there could be at most finitely many
subsets of $(F_n, d_n)$ that are isometric to $A$, thus we are finished after
finitely many repetitions of the procedure.

When the induction is finished we have that $G = \bigcup_n F_n$ since at the
$n$-th step, for $n$ even, we have guaranteed that $g_{n/2+1}$ is contained in
Moreover, $G$ with the metric induced by the norm $\lambda = \bigcup_n \lambda_n$ is isometric to the rational Urysohn space. By Fact 2.1 it suffices to check it satisfies the rational one-point extension property. However if we take some finite $B \subseteq G$ and one-point extension determined by a rational Katětov function $f$ on $B$, then we can find $n$ such that $B \subseteq F_n$, $B$ is isometric to $A_n$ and $f$ on $B$ corresponds to $f_n$ on $A_n$. Then we have guaranteed that the Katětov function is realized in $F_{n+1}$.

Finally, we show how to get the “moreover part” from the statement of the theorem, i.e. that the set of norms with which $G$ is isometric to the rational Urysohn space is actually dense.

Take any finite symmetric subset $F \subseteq G$ containing zero, let $\rho$ be an arbitrary partial norm on $F$ and $\varepsilon > 0$ arbitrary. Then using Lemma 2.2 we get a partial rational norm $\rho_R$ on $F$ such that for every $f \in F$ we have $|\rho(f) - \rho_R(f)| < \varepsilon$. We just set $F_0 = F$ and $\lambda_0 = \rho_R$. Then we continue the induction as above and obtain at the end a norm $\lambda$ with which $G$ is isometric to the rational Urysohn space and is $\varepsilon$-close on a finite subset $F$ to the partial norm $\rho$.

**Corollary 2.4.** If $G$ is unbounded, then the set of norms \{\(\lambda : (G, \lambda) \text{ is isometric to } U\)\} is comeager.

**Proof.** Consider the set of norms $\lambda$ on $G$ satisfying the following condition:

(2.1) \(\forall \varepsilon > 0 \forall F \subseteq G\) finite symmetric and containing zero, \(\forall f : F \to \mathbb{Q}\) Katětov with respect to $F$ \(\exists g \in G \forall a \in F(|f(a) - \lambda(g - a)| < \varepsilon)\).

It is well-known and straightforward to prove using standard arguments that for any $\lambda$ satisfying (2.1) we have that the completion $(G, \lambda)$ is isometric to the Urysohn space.

Moreover, an immediate computation gives that (2.1) is a $G_\delta$ condition. Any norm $\lambda$ with which $G$ is isometric to the rational Urysohn space certainly satisfies (2.1), thus it follow from Theorem 2.3 that the condition defines a dense $G_\delta$ set. □

**Remark 2.5.** We note that there is one example in the literature of a countable abelian group which admits a norm with which it is isometric to the rational Urysohn space, yet it is not unbounded. It is the countable Boolean group (see [11]). The case of exponent 2 is obviously special and it is open whether other bounded countable abelian groups admit such a norm (see the open problems in [10], where it is proved
that groups of exponent 3 do not admit such a norm). We conjecture that they do not.

3. Generic norms

For a norm $\lambda$ on $G$ denote by $(G,\lambda)$ the completion. We shall call a norm $\lambda$ on a countable abelian group $G$ generic if the set $\{\rho \in \mathcal{N}_G : (G,\lambda) \cong (G,\rho)\}$ is comeager. In other words, a countable abelian group $G$ admits a generic norm if all the norms on $G$ except those coming from a meager set give rise to the same normed group after the completion. It follows from Theorem 1.1 that if $\lambda$ is a generic norm on a countable unbounded abelian group $G$, then $(G,\lambda)$ is extremely amenable.

Definition 3.1. Let $G$ be a countable abelian group. We call $G$ infinitely-summed if $G \cong \bigoplus_{n \in \mathbb{N}} G_n$. In particular, notice that if $G$ is infinitely-summed then it is not finitely generated.

Let $G$ be an infinitely-summed countable abelian group. Write $G$ as $\bigoplus_{n \in \mathbb{N}} G_n$, where for each $n \in \mathbb{N}$, $G_n \cong G$. For each $i \in \mathbb{N}$, let $\phi_i : G_1 \to G_i$ be an isomorphism. Fix some infinite generating sequence $(d'_n)_n$ of $G_1$ such that for every $n$, $d'_{n+1} \notin \langle d'_i : i \leq n \rangle$ and $G_1 = \langle d'_n : n \in \mathbb{N} \rangle$. Let $D = \bigcup_i \bigcup_n \{\phi_i(d'_n)\}$, i.e. $D$ is the set of generators of $G$. Fix also an enumeration $\{d_n : n \in \mathbb{N}\}$ of $D$. For every set of integers (in all cases in the sequel finite) $A \subseteq \mathbb{N}$, let $F_A \leq G$ be the subgroup $\langle d_i : i \in A \rangle$. Note that for any finite $A \subseteq \mathbb{N}$ there exists disjoint finite $A' \subseteq \mathbb{N}$ and a bijection $\phi : A \to A'$ which uniquely determines an isomorphism between $\tilde{\phi} : F_A \to F_{A'}$ determined by sending $d_i$ to $d_{\phi(i)}$ for $i \in A$. Let $\Phi$ be the set of all isomorphisms between finitely generated subgroups of the form $F_A$ and $F_{A'}$ which are determined by some bijection between $A$ and $A'$. Obviously, for most choices of generating sets $D$, not all bijections between two finite subsets $A, A' \subseteq \mathbb{N}$ of the same size give rise to isomorphisms between $F_A$ and $F_{A'}$, which do not have to be isomorphic at all. We write $A \equiv A'$ if there does exist a bijection between $A$ and $A'$ which gives rise to an isomorphism between $F_A$ and $F_{A'}$. If we want to specify the isomorphism we write $A \equiv_{\phi} A'$, where $\phi$ is the bijection between $A$ and $A'$ and $\tilde{\phi}$ the corresponding isomorphism between $F_A$ and $F_{A'}$.

Also, for each finite $A \subseteq \mathbb{N}$, let $| \cdot |_A : F_A \to \mathbb{N}$ be the length function (i.e. norm) associated to the generating set $\{d_i : i \in A\}$, i.e. the distance from 0 in the graph metric of the Cayley graph of $F_A$ with $\{d_i : i \in A\}$ as a generating set. If there is no danger of confusion then
we write just $| \cdot |$ instead of $| \cdot |_A$.

Suppose now that $A, A' \subseteq \mathbb{N}$ are two finite subsets such that $A \equiv_\phi A'$ witnessed by some $\phi \in \Phi$. Suppose also that $F_A$ is equipped with a norm $\rho$ and $F_{A'}$ with a norm $\rho'$. Then we write $(A, \rho) \equiv_{\phi, \varepsilon} (A', \rho')$ if for every $f \in F_A$ we have $|\rho(f) - \rho'(\tilde{\phi}(f))| < \varepsilon \cdot |f|$.

Finally, suppose that $G$ is equipped with a norm $\lambda$. Let again $A, A'$ be two finite subsets of integers such that for some $\phi \in \Phi$ we have $A \equiv_\phi A'$ and let $\varepsilon > 0$. Then we write $\delta^\lambda_\phi(A, A') < \varepsilon$ if for every $a \in A$ we have $\lambda(d_a - \tilde{\phi}(d_a)) = \lambda(d_a - d_{\phi(a)}) < \varepsilon$. Notice that this is equivalent to saying that for every $f \in F_A$ we have $\lambda(f - \tilde{\phi}(f)) < \varepsilon \cdot |f|$. Again, we shall suppress the upper index $\lambda$ from $\delta^\lambda_\phi$ when it is clear from the context.

Next lemma shows that the conditions $(A, \rho) \equiv_{\phi, \varepsilon} (A', \rho')$ is determined on finite sets.

**Lemma 3.2.** Let $A, A' \subseteq \mathbb{N}$ be two finite subsets such that $A \equiv_\phi A'$. Then there exists a finite subset $C \subseteq F_A$ such that for any norm $\rho$ on $F_A$ and norm $\rho'$ on $F_{A'}$, if for every $a \in C$ we have $|\rho(a) - \rho'(\tilde{\phi}(a))| < \varepsilon$, then $|\rho(f) - \rho'(\tilde{\phi}(f))| < \varepsilon \cdot |f|$ for every $f \in F_A$; i.e. $(A, \rho) \equiv_{\phi, \varepsilon} (A', \rho')$.

**Proof.** We proved the lemma in case of free abelian groups in [3] (Proposition 3.2). Precisely, we prove the lemma for the case when $G$ is the free abelian group with countably many generators and $(d_n)_n$ is an enumeration of these free generators. That says, that for two finitely generated free abelian groups $F, F'$ with free generators $f_1, \ldots, f_n, f'_1, \ldots, f'_n$, and an isomorphism $\phi_F : F \to F'$ determined by sending $f_i$ to $f'_i$; for $i \leq n$, there is some finite subset $C_F \subseteq F$ such that for any two norms $\chi$, resp. $\chi'$, on $F$, resp. on $F'$, if $|\chi(a) - \chi'(\phi_F(a))| < \varepsilon$, then $|\chi(x) - \chi'(\phi_F(x))| < \varepsilon \cdot |x|_F$, where $|x|_F$ is now the length of $x$ in free generators of $F$.

We show how the general case follows from the particular one above. Take finite $A, A' \subseteq \mathbb{N}$ as in the statement of the lemma. $F_A$ is finitely generated abelian group, thus it is isomorphic to a sum of a finitely generated free abelian group and a finite abelian group. We may ignore the finite part and assume that $F_A$ is isomorphic to a free abelian group. Indeed, if we find the desired finite subset $C \subseteq F_A$ in such a case, then we might take a sum of such $C$ with the finite group. Thus suppose that $F_A$ is isomorphic to a finitely generated free abelian group, and choose free generators $f_1, \ldots, f_n$ of $F_A$. It is a well-known and easy to observe that the length function $| \cdot |_A$ and the length function $| \cdot |_F$ obtained with respect to the free generators $f_1, \ldots, f_n$ are bi-Lipschitz equivalent. In particular, there is some $K$ such that $| \cdot | \leq K \cdot | \cdot |_F$. Thus
we may apply the result for free abelian groups with free generators with \( \varepsilon/K \) to obtain the result for \( F_A \) with generators \( \{ d_a : a \in A \} \). \( \square \)

The following is the main theorem of this section and the whole paper.

**Theorem 3.3.** Let \( G \) be an infinitely-summed group. Then \( G \) admits a generic norm \( \lambda \).

From now on, fix an infinitely-summed group \( G \), the enumerated set of generators \( D = \{ d_n : n \in \mathbb{N} \} \) as above, and the set of bijections \( \Phi \) as above.

Let \( \mathcal{G} \subseteq \mathcal{N} \) denote the set of all norms \( \lambda \) on \( G \) satisfying the following condition:

\[
\forall \varepsilon > \varepsilon' > 0, \forall A_0 \subseteq A \subseteq G \text{ finite such that } F_A \text{ is equipped with a norm } \rho_A, \text{ and } (A_0, \rho_A) \equiv_{\text{id}, \varepsilon} (A_0, \lambda) \exists A' \subseteq \mathbb{N}, \exists \phi \in \Phi \text{ such that } (A, \rho_A) \equiv_{\phi, \varepsilon'} (A', \lambda) \text{ and } \delta_\phi(A_0, \phi[A_0]) < \varepsilon.
\]

In order to prove Theorem 3.3, we shall prove that \( \mathcal{G} \) is dense \( G_\delta \) and that for any \( \lambda, \rho \in \mathcal{G} \) we have \( (G, \lambda) = (G, \rho) \). That will give that \( G \) admits a generic norm. Moreover in case \( G \) is unbounded, we show that there is \( \lambda \in \mathcal{G} \) such that \( (G, \lambda) \) is isometric to the rational Urysohn space. That will complete the proof of Theorem 3.3.

The first step showed in the next lemma is simple.

**Lemma 3.4.** \( \mathcal{G} \) is \( G_\delta \).

**Proof.** This follows immediately from the following simple observations. For any \( \phi \) and \( \varepsilon > 0 \) and finite \( A, A' \) and \( A_0 \subseteq A \) and a norm \( \rho \) on \( F_A \), both

\[
(A, \rho) \equiv_{\phi, \varepsilon} (A', \lambda)
\]

and

\[
\delta_\phi(A_0, \phi[A_0]) < \varepsilon
\]

are open conditions, resp. define open neighborhoods of \( \lambda \). Indeed, that follows from Lemma 3.2 which says that the first condition is determined on a finite subset of \( F_A \).

Then one can check that all the universal quantifiers in (3.1) can be taken over countable sets and one obtains that (3.1) defines a \( G_\delta \) subset.

Next we want to show that all the norms from \( \mathcal{G} \) give rise to the same normed group after the completion. Note that the condition (3.1) is similar to the condition on vector space norms which give rise to the
Gurarij space, the homogeneous separable Banach space constructed by Gurarij in [5]. The following proposition is thus similar to the main result of [8] where they prove the uniqueness of the Gurarij space.

**Proposition 3.5.** For any two \( \lambda, \lambda' \in \mathcal{G} \) we have that \((G, \lambda)\) and \((G, \lambda')\) are isometrically isomorphic.

**Proof.** Consider two norms \( \lambda, \lambda' \in \mathcal{N} \). Let \((i_j)_j\) be an enumeration of \( \mathbb{N} \) with an infinite repetition.

By induction, for every \( j \in \mathbb{N} \) we shall construct two finite sequences \((a^j_i)_{i=0}^{2^j-1} \subseteq \mathbb{N}\) and \((b^j_i)_{i=0}^{2^j} \subseteq \mathbb{N}\) such that

1. for every \( j \in \mathbb{N}, i_j \in \{a^j_i : i \leq 2j - 1\} \cap \{b^j_i : i \leq 2j\} \), i.e. there are \( i, i' \) such that \( i_j = a^j_i = b^j_{i'} \);

2. for every \( j \in \mathbb{N} \) there is some \( \phi_j \in \Phi \) such that \( \phi_j(a^j_i) = b^j_i \), for every \( i \leq 2j - 1 \), and

   \[
   (a^j_i)_{i=0}^{2^j-1}, \lambda) \equiv_{\phi_j, 1/2^{j-1}} ((b^j_i)_{i=0}^{2^j-1}, \lambda')
   \]

   and \( \psi_j \in \Phi \) such that \( \psi_j(b^j_i) = a^{j+1}_i \), for every \( i \leq 2j \), and

   \[
   ((b^j_i)_{i=0}^{2^j}, \lambda') \equiv_{\psi_j, 1/2^j} ((a^{j+1}_i)_{i=0}^{2^j}, \lambda);
   \]

3. for every \( j \in \mathbb{N} \) we have

   \[
   \delta^\lambda_{\psi_j \circ \phi_j} ((a^j_i)_{i=0}^{2^j-1}, (a^{j+1}_i)_{i=0}^{2^j-1}) < 1/2^{2j-1}
   \]

   and

   \[
   \delta^\lambda'_{\phi_j \circ \psi_j} ((b^j_i)_{i=0}^{2^j}, (b^{j+1}_i)_{i=0}^{2^j+1}) < 1/2^j.
   \]

Note that in particular for every \( j \in \mathbb{N} \) we have

\[
(3.2) \quad a^{j+1}_i = \psi_j \circ \phi_j(a^j_i), \forall i \leq 2j - 1,
\]

\[
b^{j+1}_i = \phi_{j+1} \circ \psi_j(b^j_i), \forall i \leq 2j.
\]

Suppose at first that such sequences have been constructed. Denote by \( \mathbb{G} \) the completion of \((G, \lambda)\) and by \( \mathbb{G}' \) the completion of \((G, \lambda')\). By [3], for each \( i \in \mathbb{N} \) we have that the sequence \((g^j_i)_j\), where \( g^j_i = d_{a^j_i} \) for all \( i, j \), is Cauchy in \((G, \lambda)\), thus it has the limit, denoted by \( g_i \), in \( \mathbb{G} \). Analogously by [3], for each \( i \in \mathbb{N} \) the sequence \((h^j_i)_j\), where \( h^j_i = d_{b^j_i} \) for all \( i, j \), is Cauchy in \((G, \lambda')\) and we denote by \( h_i \) the limit in \( \mathbb{G}' \). We claim that \( \langle (g_i)_i \rangle \) is a dense subgroup in \( \mathbb{G} \) and \( \langle (h_i)_i \rangle \) is a dense subgroup in \( \mathbb{G}' \). We prove the former, the latter is analogous. Since \( G \) is dense in \( \mathbb{G} \) it suffices to show that for any \( g \in G \) and any \( \varepsilon > 0 \) there
exists \( g' \in \langle (g_i) \rangle \) such that \( \lambda(g' - g) < \varepsilon \). Take some finite \( C \subseteq \mathbb{N} \) such that \( g \in F_C \) and let \( k = |g|_C \). There exists \( N \) such that for every \( j \geq N \) and \( i \leq 2j - 1 \) we have
\[
\lambda(g^j_i - g_i) < \varepsilon/k.
\]
Also, for each \( c \in C \) we can find \( i_c \) and \( j_c \geq N \) such that \( g^j_c - g_i < \varepsilon/k \), for every \( c \in C \), it follows there is an element \( g' \in \langle g_i \rangle \) such that \( \lambda(g' - g) < \varepsilon \), and the claim is proved.

Next we claim that the map sending \( g_i \) to \( h_i \), for each \( i \in \mathbb{N} \), can be extended to an isometric isomorphism \( \Psi : \langle g_i : i \in \mathbb{N} \rangle \to \langle h_i : i \in \mathbb{N} \rangle \).

Indeed, let \( F_g \) be the free abelian group generated by \( (g_i)_i \) and \( F_h \) be the free abelian group generated by \( (h_i)_i \). We equip \( F_g \), resp. \( F_h \), with \( \lambda \), resp. \( \lambda' \), which may now be considered as seminorms; i.e. the kernel of the canonical projection from \( F_g \) onto \( \langle g_i : i \in \mathbb{N} \rangle_C \) is equal to the set of zero elements of \( \lambda \) considered as a seminorm on \( F_g \). Analogously for \( F_h \).

By abusing the notation, denote also by \( \Psi \) the canonical isomorphism between \( F_g \) and \( F_h \). It suffices to check that for any \( g \in F_g \) we have
\[
\lambda(g) = \lambda'(\Psi(g)).
\]

Denote also for every \( j \in \mathbb{N} \), by \( \Psi^j \) the homomorphism from \( F_g \) onto \( \langle g^j_i : i \leq 2j - 1 \rangle \) which sends \( g_i \) to \( g^j_i \), for \( i \leq 2j - 1 \), and \( g_i \) to 0 for \( i > 2j - 1 \). Analogously, denote by \( \Psi^j \) the homomorphism from \( F_h \) onto \( \langle h^j_i : i \leq 2j \rangle \) which sends \( h_i \) to \( h^j_i \), for \( i \leq 2j \), and \( h_i \) to 0 for \( i > 2j \).

Then for some \((K_n)_n\) such that \( K_n \to 0 \)
\[
\lambda(g) = \lim_j \lambda(\Psi^j(g)) = \lim_j \lambda'(\Psi^j \circ \Psi(g)) + K_n = \lambda'(\Psi(g)).
\]
The first equality follows from the definition, the second from (2), and the last one again from the definition and the fact that \( K_n \to 0 \).

It follows that we may uniquely extend \( \Psi \) to \( G \), which we shall still denote by \( \Psi \) and which is an isometric isomorphism between \( G \) and \( G' \).

It remains to find the sequences.

We will proceed by induction. We show the first odd and even step of the induction and then the general odd and even step of the induction.

Set \( a^1_1 = i_1 \). By (3.1), there exists \( b^1_1 \) such that \( \langle \{a^1_1\}, \lambda \rangle \equiv_{\phi_{1/2}} \langle \{b^1_1\}, \lambda' \rangle \) for some \( \phi_1 \in \Phi \).

Now if \( b^1_1 = i_1 \) then take as \( b^2_1 \) an arbitrary natural number; otherwise, take \( b^2_1 = i_1 \). Again by (3.1), there exist \( a^2_1, a^2_2 \in \mathbb{N} \) such that
\begin{itemize}
  \item \( \langle \{a^2_1, a^2_2\}, \lambda \rangle \equiv_{\phi_{1/4}} \langle \{b^1_1, b^2_1\}, \lambda' \rangle \), for some \( \phi_2 \in \Phi \),
\end{itemize}
• $\delta_{\phi^{-1}_2 \phi_1} (\{a_i^1\}, \{a_i^2\}) < 1/2$; in other words, $\lambda(d_{a_i^1} - d_{a_i^2}) < 1/2$.

Now suppose that we have found sequences $(a_i^{n-1})_{i=0}^{2n-3}$ and $(b_i^{n-1})_{i=0}^{2n-2}$. We shall find $(a_i^n)_{i=0}^{2n-1}$ and $(b_i^n)_{i=0}^{2n-2}$. Since by assumption we have that $((a_i^{n-1})_{i=0}^{2n-3}, \lambda) \equiv \phi_{n-1, 1/2^{2n-3}} (b_i^{n-1}, \lambda)$, using (3.1) we can find $(a_i^n)_{i=0}^{2n-2}$ and $\psi_{n-1} \in \Phi$ such that $((b_i^{n-1})_{i=0}^{2n-2}, \lambda') \equiv \psi_{n-1, 1/2^{2n-2}} ((a_i^n)_{i=0}^{2n-2}, \lambda)$ and moreover

$$\delta_{\psi_{n-1} \circ \phi_{n-1}} (\{a_i^{n-1}\}_{i=0}^{2n-3}, (a_i^n)_{i=0}^{2n-3}) < 1/2^{2n-3}.$$ 

If $i_n \in (a_i^n)_{i=0}^{2n-2}$ then we set $a_{2n-1}^n$ to be any natural number. Otherwise, we set $a_{2n-1}^n = i_n$.

Then analogously, using (3.1), we find $(b_i^n)_{i=0}^{2n-1}$ and $\phi_n \in \Phi$ such that $((a_i^n)_{i=0}^{2n-1}, \lambda) \equiv \phi_{n, 1/2^{2n-1}} ((b_i^n)_{i=0}^{2n-1}, \lambda')$ and moreover

$$\delta_{\phi_n \circ \psi_{n-1}} (\{b_i^{n-1}\}_{i=0}^{2n-2}, (b_i^n)_{i=0}^{2n-2}) < 1/2^{2n-2}.$$ 

Again, if $i_n \in (b_i^n)_{i=0}^{2n-1}$ then we set $b_{2n}^n$ to be any natural number. Otherwise, we set $b_{2n}^n = i_n$. This finishes the induction and the proof. 

To finish the proof of Theorem 3.3 we need to prove that $\mathcal{G}$ is dense. Notice that so far we have not yet even proved that $\mathcal{G}$ is non-empty. The next proposition will do.

**Proposition 3.6.** $\mathcal{G}$ is dense.

Moreover, if $G$ is unbounded, then the subset $\{\lambda \in \mathcal{G} : (G, \lambda) \equiv_{iso} \mathbb{Q}\} \subseteq \mathcal{G}$ is dense.

**Proof.** Let $F$ be a finite symmetric subset of $G$ containing zero and $\rho$ a partial norm on $F$. By extending $\rho$ if necessary, we may suppose without loss of generality that for some finite $C \subseteq \mathbb{N}$, we have $\langle F \rangle = F_C$. Then by $\bar{\rho}$ we shall denote the extension of $\rho$ to $\langle F \rangle = F_C$ obtained using Lemma 1.3. We begin with a claim.

**Claim 3.7.** Let $C \subseteq \mathbb{N}$ be finite and let $\rho$ be norm on $F_C$. Then for every $\varepsilon > 0$ there exists a rational norm $\rho_R$ on $F_C$ such that $(C, \rho_R) \equiv_{id, \varepsilon} (C, \rho_R)$.

To prove the claim, we use Lemma 3.2 to find a finite set $F \subseteq F_C$ such that for any two norms $\rho_1, \rho_2$ on $F_C$, if $|\rho_1(f) - \rho_2(f)| < \varepsilon$, for every $f \in F$, then $(C, \rho_1) \equiv_{id, \varepsilon} (C, \rho_2)$. We may suppose that $F$ is finite symmetric containing zero. Then we use Lemma 2.2 to find a partial rational norm $\rho_R$ on $F$ such that $|\rho(f) - \rho_R(f)| < \varepsilon$, for every
$f \in F$. Then the extension of $\rho_R$, still denoted by $\rho_R$, to the whole $F_C$, obtained by Lemma 1.3, is the desired rational norm.

Let us denote a norm on some $F_C$, for $C$ finite, \textit{finitely generated} if it is obtained as an extension using Lemma 1.3 of some partial norm defined on a finite symmetric subset of $F_C$ containing zero. Let us now enumerate all triples $T_n = (B_n, A_n, \rho_n)$, where $B_n \subseteq A_n \subseteq \mathbb{N}$ are finite and $\rho_n$ is a finitely generated rational norm on $F_{A_n}$. Moreover, suppose there is an infinite repetition of each such a triple in the enumeration.

By induction, we shall construct an increasing sequence of finite sets $(C_n)_n$, i.e. $C_n \subseteq C_m$, for $n < m$, and an increasing sequence of rational norms $(\lambda_n)_n$, i.e. $\lambda_n \subseteq \lambda_m$, for $n < m$, such that

(1) $C_1 = C$ and $\lambda_1 = \rho_R$;
(2) $\bigcup_n C_n = \mathbb{N}$;
(3) for each $n$, $\lambda_n$ is a norm on $F_{C_n}$;
(4) for every $n$, if there is $B'_n \subseteq C_n$ such that

$$
(B'_n, \lambda_n) \equiv_{\phi', 1/2^n} (B_n, \rho_n),
$$

then there is $A'_n \subseteq C_n$ such that

$$
(A_n, \rho_n) \equiv_{\phi, 0} (A'_n, \lambda_n)
$$

and

$$
\delta_{\phi \circ \phi'}(B'_n, \phi \circ \phi'[B'_n]) < 1/2^n.
$$

We shall now proceed to the induction. The first step has been already done, i.e. we set $C_1 = C$ and $\lambda_1 = \rho_R$ as obtained from the claim above.

Let us now describe the general step. Suppose we have produced finite set $C_n \subseteq$ and rational norm $\lambda_n$ on $F_{C_n}$. Consider now the triple $T_n = (B_n, A_n, \rho_n)$. Suppose that there are some $B'_n \subseteq C_n$ and $\phi' \in \Phi$ such that $\phi[B'_n] = B_n$ and $(B'_n, \lambda_n) \equiv_{\phi', 1/2^n} (B_n, \rho_n)$. There can be at most finitely many such $B'_n$'s. To simplify the notation and proof, we shall suppose there is just one such $B'_n \subseteq C_n$, and actually $B'_n = B_n$, and thus $\phi' = \text{id}$. If there are more such finite subsets of $C_n$, the procedure that follows is repeated (finitely many times). If there is no such a finite subset $B'_n$, then we set $C'_{n+1} = C_n$ and $\lambda'_{n+1} = \lambda_n$ and use the procedure below to extend $C'_{n+1}$ to $C_{n+1}$ and $\lambda'_{n+1}$ to $\lambda_{n+1}$.

Thus we suppose that $B_n \subseteq C_n$ and $(B_n, \lambda_n) \equiv_{\text{id}, 1/2^n} (B_n, \rho_n)$. Since $G$ is infinitely-summed we can find $A'_n$ disjoint with $C_n$ such that there is some $\phi \in \Phi$ which is a bijection between $A_n$ and $A'_n$. Set $C'_{n+1} = \ldots$
We shall now define a rational norm \( \lambda_{n+1} \) on \( F_{C_{n+1}} \) which extends \( \lambda_n \).

For each \( c \in B_n \), set \( h_c \) to be \( \min \{ 1/2^n, \lambda_n(d_c) + \rho_n(d_c) \} \). For \( x \in F_{C_n} \cup F_{A_n} \cup \{ d_c - \phi(d_c), \phi(d_c) - d_c : c \in B_n \} \) we set \( \chi(x) \) as follows:

\[
\chi(x) = \begin{cases} 
\lambda_n(x) & \text{if } x \in F_{C_n}, \\
\rho_n(y) & \text{if } x = \phi(y), y \in F_{A_n}, \\
h_c & \text{if } x = \epsilon(d_c - \phi(d_c)), \text{ where } c \in B_n, \epsilon \in \{1, -1\}.
\end{cases}
\]

We use Fact 1.5 to get a partial norm \( \lambda'_{n+1} \) on \( F_{C_n} \cup F_{A_n} \cup \{ d_c - \phi(d_c), \phi(d_c) - d_c : c \in B_n \} \). Note that for each \( c \in B_n \) we have \( \lambda'_{n+1}(d_c - \phi(d_c)) \leq 1/2^n \). We need to check that for each \( x \in F_{C_n} \cup F_{A_n} \) we have \( \lambda'_{n+1}(x) = \chi(x) \). Then we could use Lemma 1.3 again to extend \( \lambda'_{n+1} \) to \( F_{C_{n+1}} \) still denoted by \( \lambda'_{n+1} \). It will follow that \( (A_n, \rho_n) \equiv \phi_0 (A'_n, \lambda'_{n+1}) \) and that \( \delta_\phi(B_n, \phi[B_n]) \leq 1/2^n \).

We need to check that for any \( x_1, \ldots, x_k \in F'_{n+1} \) such that \( x = \sum_{i=1}^k x_i \in F'_{n+1} \) we have \( \chi(x) \leq \sum_{i=1}^k \chi(x_i) \).

We have two cases: \( x \in F_{C_n} \) and \( x \in F_{A_n} \). We shall treat only the first one, the second one is analogous.

**Case 1:** Suppose that \( x \in F_{C_n} \). Since \( G \) is abelian we may suppose that there are \( k_1, k_2 \leq k \) such that for every \( 1 \leq i \leq k_1 \) we have \( x_i \in F_{C_n} \), for every \( k_1 < i \leq k_2 \) we have \( x_i \in F_{A_n} \) and for every \( k_2 < i \leq k \) we have \( x_i \in \{ d_c - \phi(d_c), \phi(d_c) - d_c : c \in B_n \} \). Moreover, for every \( k_2 < i \leq k \) we may suppose that \( \chi(x_i) = 1/2^n \). Otherwise, \( \chi(x_i) = \lambda_n(d_c) + \rho_n(d_c) \), for some \( c \in B_n \), i.e. \( x_i \) is equal to \( d_c - \phi(d_c) \) or \( \phi(d_c) - d_c \). In that case we would replace \( x_i \) by a pair \( d_c, -\phi(d_c) \), resp. \( \phi(d_c), -d_c \) without increasing the sum \( \sum_{i=1}^k \chi(x_i) \). Set \( z_1 = \sum_{i=1}^{k_1} x_i \), \( z_2 = \sum_{i=k_1+1}^{k_2} x_i \) and \( z_3 = \sum_{i=k_2+1}^k x_i \). Since \( x = z_1 + z_2 + z_3 \), it follows that \( z_3 = -z_2 + \phi^{-1}(z_2) \). Since \( (B_n, \lambda_n) \equiv \phi_0 (B_n, \rho_n) \) we get that \( z_2 \in F_{B_n} \) and \( |\lambda_n(z_2) - \rho_n(z_2)| < |z_2|/2^n \). Since

\[
\sum_{i=k_2+1}^k \chi(x_i) = |z_2|/2^n,
\]

we get that

\[
\lambda_n(z_1) + \rho_n(z_2) + |z_2|/2^n \geq \lambda_n(x)
\]

and we are done.

Finally, set \( C_{n+1} = C_{n+1} \cup \{ n \} \) and extend \( \lambda'_{n+1} \) to \( \lambda_{n+1} \) on \( F_{C_{n+1}} \) arbitrarily.
When the induction is finished we get $\lambda = \bigcup_n \lambda_n$ is a norm on $G$. We check that $\lambda \in G$. Take any $\varepsilon > \varepsilon' > 0$, finite subsets $A_0 \subseteq A \subseteq \mathbb{N}$. Let $\rho$ be some norm on $F_A$ such that $(A_0, \rho) \equiv_{id, \varepsilon} (A_0, \lambda)$. Using Claim 3.7 we can find a rational norm $\rho_R$ on $F_A$ such that $(A_0, \rho_R) \equiv_{id, \varepsilon} (A_0, \lambda)$. Moreover, thanks to Lemma 3.2 we may suppose that $\rho_R$ is finitely generated. Then by the construction, there is some $\phi$ such that $A \equiv_{\phi} \phi[A]$, $\phi[A] \subseteq C_{n+1}$ and

$$(A, \rho) \equiv_{\phi, \varepsilon'} (\phi[D], \lambda)$$

and

$$\delta_{\phi}(A, \phi[A]) < 1/2^n < \varepsilon,$$

and we are done.

Also, since at the beginning the finite symmetric subset $F$ and a partial norm $\rho$ on $F$ were arbitrary, it shows that $G$ is dense.

Finally, we show how to get the “moreover” part from the statement of the proposition. If $G$ is unbounded then we combine the two induction procedures from this proof and the proof of Theorem 2.3 into one. Besides the enumeration of triples $(T_n)_n$ as above, consider also the enumeration $\{(A_i, f_i) : i \in \mathbb{N}\}$ (again with infinite repetition) of all pairs $(A, f)$, where $A$ is a finite rational metric space and $f : A \rightarrow \mathbb{Q}$ a rational Katětov function over $A$. Then we divide the induction procedure into odd and even steps. During odd steps, we take care of triples $(T_n)_n$ as above. During even steps, we take care of pairs $(A_n, f_n)_n$ as in the proof of Theorem 2.3. It follows that after the induction we get a norm $\lambda \in G$ such that $(G, \lambda)$ is isometric to the rational Urysohn space. □

**Corollary 3.8.** Let $G$ be an unbounded infinitely-summed countable Abelian group. There exists an Abelian Polish metric group $\mathbb{G}$ which is extremely amenable and isometric to the Urysohn space such that for comeager-many norms $\lambda$ on $G$ we have

$$\mathbb{G} = (G, \lambda).$$

In particular, for every such $G$ there is a norm $\lambda$ such that $(G, \lambda)$ is extremely amenable and isometric to the rational Urysohn space.

**Remark 3.9.** Although the results above show that for every infinitely-summed countable Abelian group $G$ there is the corresponding Abelian Polish metric group $\mathbb{G}$, one might ask whether there is actually a single generic Abelian Polish metric group $\mathbb{H}$. That is, whether for every
infinitely-summed countable Abelian group $G$ and a generic metric $\rho$ on $G$, we have $(G, \rho) = \mathbb{H}$.

The answer to that question is negative. We shall see in the next section that the Shkarin universal group and the group of the author (from [3]) are two different generic Abelian Polish groups (their direct sum is another different example).

### 3.1. Extremely amenable universal abelian groups.

In this section, we observe that the known universal abelian Polish groups are extremely amenable, and provide another proof that they are, with their norm, isometric to the Urysohn space. More precisely, we formulate a certain extension property of these groups (analogous to the extension property of the Gurarij Banach space), show that this extension property is a dense $G_δ$ property, and then use our results and results of Melleray and Tsankov to show that the extension property defines these groups uniquely up to isometric isomorphism and that they are extremely amenable and isometric to the Urysohn space.

Since these groups are constructed using Fraïssé theoretic methods, we shall assume here that the reader has a basic knowledge of this area. We refer the reader to Chapter 7 in [6] as a reference to the Fraïssé theory.

The first example of a universal abelian Polish group, in the sense that every abelian Polish group, or every second-countable abelian Hausdorff group, embeds via topological isomorphism as a subgroup, was provided by Shkarin in [13]. This group was further investigated by Niemiec in [10]. The group was constructed as a normed group, however it is universal only in the topological sense. It does not contain isometrically as a subgroup every separable abelian normed group. A normed group universal in this stronger sense was constructed by the author in [3].

Let us describe how these universal groups were constructed. The Shkarin’s group, further denoted by $\mathbb{G}_S$, is constructed as follows: First one considers the class of all finite groups equipped with rational-valued norms. Then one can check that this class is a Fraïssé class, thus it has a Fraïssé limit. It is straightforward to check that the limit is a countable abelian group, denoted by $G_S$, which is algebraically isomorphic to $\bigoplus_\mathbb{N} \mathbb{Q}/\mathbb{Z}$, equipped with a rational norm $\lambda_S$. In particular, $G_S$ is an infinitely-summed unbounded group. Then one takes the completion $(G_S, \lambda_S)$ to obtain the group $\mathbb{G}_S$. 
In the sequel, we shall use the following notation. For two finite groups $G, H$ which are isomorphic via some $\phi$, and which are equipped with norms $\lambda_G$ and $\lambda_H$ respectively, and for some $\varepsilon > 0$ we shall write
$$(G, \lambda_G) \equiv_{\phi, \varepsilon} (H, \lambda_H)$$
to express that they are $\varepsilon$-isomorphic via $\phi$, i.e. for every $g \in G$ we have $|\lambda_G(g) - \lambda_H(\phi(g))| < \varepsilon$.

Consider now the following set $S$ of norms $\lambda$ on $G_S = \bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$, i.e. a subset of $\mathcal{N}_{G_S}$:
$$(3.3) \quad \forall \varepsilon > 0 \forall G_0 \leq G_1 \leq G_S \text{ finite}, \forall \rho \text{ a rational norm on } G_1$$
$$(G_0, \rho) \equiv_{\text{id}, \varepsilon} (G_0, \lambda) \text{ then } \exists G_0 \leq G_1' \leq G_S \text{ isomorphic to } G_1$$
via some $\phi$ such that $(G_1', \lambda) \equiv_{\phi, \varepsilon} (G_1, \rho)$.

First, it is clear, by the definition of a Fraïssé limit, that $\lambda_S \in S$; moreover, that the set $S$ is dense. Secondly, one can easily check that $(3.3)$ define a $G_\delta$ subset of $\mathcal{N}_{G_S}$. Thus $S$ is a dense $G_\delta$ set.

It also follows from the proofs of Shkarin and Niemiec that for any $\lambda \in S$, the group $(G_S, \lambda)$ is also universal replicating the approximation arguments for $\lambda_S$. It is analogous to the case of metrics on a countable set satisfying $(2.1)$ from the proof of Corollary 2.4. Then applying Theorem 3.3, Corollary 2.1 and Theorem 1.1 of Melleray and Tsankov we get the following corollary.

**Corollary 3.10.** There exists a generic norm $\lambda$ on $G_S = \bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$ such that $(G_S, \lambda)$ is the Shkarin’s group, which is thus extremely amenable and isometric to the Urysohn space.

Analogous arguments give the following corollary.

**Corollary 3.11.** There exists a generic norm $\lambda$ on the infinitely generated free abelian group $G$ such that $(G, \lambda)$ is the metrically universal abelian group from [3], which is thus also extremely amenable and isometric to the Urysohn space.

These two universal groups thus also provide examples of two generic, yet not isometrically isomorphic, abelian Polish metric groups.

4. Problems

In our opinion, the most interesting and challenging problem is to investigate similar properties of the spaces of metrics on non-abelian countable groups. There one can distinguish two cases, which coincide
in the case of abelian groups: the space of all continuous left-invariant metrics (we comment on the ‘continuity’ below) and the space of bi-invariant metrics. In the case of bi-invariant metrics, one can again easily check that for any countable group $G$, the set of all bi-invariant metrics, is a closed subset $\mathbb{R}^{G \times G}$, thus a Polish space.

We proved the following result in [4].

**Theorem 4.1.** There exists a generic (in the space of metrics bounded by 1) bi-invariant metric, bounded by 1, $d$ on $F_\infty$, the free group of countably many generators, such that $(F_\infty, d)$ is isometric to the Urysohn sphere and metrically universal for the class of separable groups with bi-invariant metrics bounded by 1.

We do not know whether for some countable non-abelian groups the subset of bi-invariant metrics with which these groups are extremely amenable is comeager; i.e. we do not know whether it is possible to generalize the Melleray and Tsankov’s result to the non-abelian situation. We also do not know about generic bi-invariant metrics on non-abelian groups different from free groups.

Regarding the general left-invariant metrics, first thing to observe is that while bi-invariant metrics make the group operations continuous, in fact Lipschitz, this is no longer true for general left-invariant metrics. The corresponding general norms on groups (that do not necessarily make the group topological) were considered in the literature, see e.g. [1]. However, in most cases it is reasonable to consider only such metrics that do make the group operations continuous. For a group $G$ and a left-invariant metric $d$ on $G$, the group operations are continuous if and only if for every $g \in G$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $h \in G$, if $\lambda_d(h) < \delta$, then $\lambda_d(g^{-1} \cdot g \cdot h) < \varepsilon$, where $\lambda_d$ is the corresponding norm. Let us call such norms and metrics continuous.

The main problem is that we do not know how to code the continuous norms and metrics on a non-abelian group as a Polish space. Indeed, the straightforward computation gives that they form an $F_{\sigma\delta}$ subset of $\mathbb{R}^G (\mathbb{R}^{G^2})$, thus not necessarily a space with Polish topology. Even some special subsets of continuous norms such as uniformly discrete norms seem not to be Polish, but rather $F_\sigma$ subsets of $\mathbb{R}^G$. A special subclass of uniformly discrete norms, often considered in geometric group theory, that is the class of all proper norms again seems to be $F_{\sigma\delta}$.

However, we do not exclude the possibility that a better computation reveals that these are Polish spaces - in a natural way.
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