Asynchronous Stochastic Block Coordinate Descent with Variance Reduction

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Abstract

Asynchronous parallel implementations for stochastic optimization have received huge successes in theory and practice recently. Asynchronous implementations with lock-free are more efficient than the one with writing or reading lock. In this paper, we focus on a composite objective function consisting of a smooth convex function $f$ and a block separable convex function, which widely exists in machine learning and computer vision. We propose an asynchronous stochastic block coordinate descent algorithm with the accelerated technology of variance reduction (AsySBCDVR), which are with lock-free in the implementation and analysis. AsySBCDVR is particularly important because it can scale well with the sample size and dimension simultaneously. We prove that AsySBCDVR achieves a linear convergence rate when the function $f$ is with the optimal strong convexity property, and a sublinear rate when $f$ is with the general convexity. More importantly, a near-linear speedup on a parallel system with shared memory can be obtained.

Keywords: stochastic optimization, block coordinate descent, parallel computing, lock-free

1. Introduction

Stochastic optimization technologies in theory and practice are emerging recently due to the demand of handling large scale data. Specifically, stochastic gradient descent (SGD) algorithms with various kinds of acceleration technologies (Zhang, 2004; Johnson and Zhang, 2013; Schmidt et al., 2013; Ghadimi and Lan, 2013; Reddi et al., 2013) were proposed to processing large scale smooth convex or nonconvex problems. Also, stochastic coordinate descent (SCD) algorithms (Takáč, 2014; Zhao et al., 2014; Lu and Xiao, 2015) and stochastic dual coordinate ascent algorithms (Shalev-Shwartz and Zhang, 2014; Shalev-Shwartz, 2016) were proposed. For non-smoothing problems, the corresponding proximal algorithms were proposed (Xiao and Zhang, 2014; Lin et al., 2014; Nitanda, 2014). Basically, these algorithms are sequential algorithms which can not be directly used in parallel environment.

To scale up the stochastic optimization algorithms, asynchronous parallel implementations (Richtárik and Takáč, 2016; Chaturapruek et al., 2015; Liu et al., 2015; Recht et al., 2011; Reddi et al., 2015; Lian et al., 2016; Liu and Wright, 2015; Zhao and Li, 2016; Huo and Huang, 2016a; Lian et al., 2016; Avron et al., 2013; Hsieh et al., 2013; Mania et al., 2015; Huo et al., 2016; Huo and Huang, 2016b) have been proposed recently, and received huge successes.
Among these asynchronous parallel implementations, the ones with lock-free are more efficient than the ones with writing or reading lock, because they can achieve near-linear speedup which is the ultimate goal of the parallel computation. In this paper, we focus on the asynchronous parallel implementations with lock-free.

There have been several asynchronous parallel implementations for the stochastic optimization algorithms which are totally free of reading and writing locks. For example, Zhao and Li (2016) proposed an asynchronous parallel algorithm for SVRG and proved the linear convergence. Liu and Wright (2015) proposed an asynchronous parallel algorithm for SCD and proved the linear convergence. Avron et al. (2015) proposed an asynchronous parallel algorithm for SCD to solve a linear system. Mania et al. (2015) proposed a perturbed iterate framework to analyze the asynchronous parallel algorithms of SGD, SCD and sparse SVRG. Huo and Huang (2016a) proposed an asynchronous SGD algorithm with variance reduction on non-convex optimization problems and proved the convergence. Lian et al. (2016) proposed an asynchronous stochastic optimization algorithm with zeroth order and proved the convergence.

In this paper, we focus on a composite objective function as follows.

$$\min_{x \in \mathbb{R}^n} F(x) = f(x) + g(x)$$

where

$$f(x) = \frac{1}{l} \sum_{i=1}^{l} f_i(x), \quad f_i : \mathbb{R}^n \mapsto \mathbb{R}$$

is a block separable, closed, convex, and extended real-valued function. Given a partition \( \{G_1, \cdots, G_k\} \) of \( n \) coordinates of \( x \), we can write \( g(x) \) as \( g(x) = \sum_{j=1}^{k} g_{G_j}(x_{G_j}) \). The formulation \( \Pi \) covers many machine learning and computer vision problems, for example, Lasso (Tibshirani, 1996), group Lasso (Roth and Fischer, 2008), sparse multiclass classification (Blondel et al., 2013), and so on. Note that, Liu and Wright (2015) consider the formulation \( \Pi \) with the constraints that \( |G_j| = 1 \) for all \( j \). Thus, the formulation in (Liu and Wright, 2015) is a special case of \( \Pi \). Each iteration in (Liu and Wright, 2015) only modifies a single component of \( x \) which is an atomic operation in the parallel system with shared memory. However, we need to modify a block coordinate \( G_j \) of \( x \) with lock-free for each iteration, which are more complicated in the asynchronous parallel analysis than the atomic updating in (Liu and Wright, 2015). Due to the complication induced by the block representation of \( g(x) \), there have been no asynchronous stochastic block coordinate descent algorithm with lock-free proposed for handle formulation \( \Pi \), especially on the theoretical analysis.

In this paper, we propose an asynchronous stochastic block coordinate descent algorithm with the accelerated technology of variance reduction (AsySBCDVR), which are with lock-free in the implementation and analysis. We prove that AsySBCDVR achieves a linear convergence rate when the function \( f \) is with the optimal strong convexity property, and a sublinear rate when \( f \) is with the general convexity. More importantly, a near-linear speedup on a parallel system with shared memory can be obtained.

AsySBCDVR is particularly important because it can scale well with the sample size and dimension simultaneously. In the big data era, both of the sample size and dimension could be huge at the same time as modern data collection technologies evolve, which demands that the learning algorithms can process large scale datasets with large sample size and high dimension. AsySBCDVR is based on a doubly stochastic scheme which randomly
choose a set of samples and a block coordinate for each iteration. Thus, AsySBCDVR can process large scale datasets. Especially, the technology of variance reduction is used to accelerate AsySBCDVR such that AsySBCDVR has the linear or sublinear convergence in different conditions. Thus, AsySBCDVR can scale well with the sample size and dimension simultaneously.

We organize the rest of the paper as follows. In Section 2, we give some preliminaries. In section 3, we propose our AsySBCDVR algorithm. In Section 4, we prove the convergence rate for AsySBCDVR. Finally, we give some concluding remarks in Section 5.

2. Preliminaries

In this section, we introduce the condition of optimal strong convexity and three different Lipschitz constants and give the corresponding assumptions, which are critical to the analysis of AsySBCDVR.

Optimal Strong Convexity: Let $F^*$ denote the optimal value of (1), and let $S$ denote the solution set of $F$ such that $F(x) = F^*$, $\forall x \in S$. Firstly, we assume that $S$ is nonempty (i.e., Assumption 1), which is reasonable to (1).

**Assumption 1** The solution set $S$ of (1) is nonempty.

Based on $S$, we define $\mathcal{P}_{S(x)} = \arg\min_{y \in S} \|y - x\|^2$ as the Euclidean-norm projection of a vector $x$ onto $S$. Then, we assume that the convex function $f$ is with the optimal strong convexity (i.e., Assumption 2).

**Assumption 2 (Optimal strong convexity)** The convex function $f$ has the condition of optimal strong convexity with parameter $l > 0$ with respect to the optimal set $S$, which means that, $\exists l$ such that, $\forall x$, we have

$$F(x) - F(\mathcal{P}_{S(x)}) \geq \frac{l}{2} \|x - \mathcal{P}_{S(x)}\|^2 \quad (2)$$

As mentioned in Liu and Wright (2015), the condition of optimal strong convexity is significantly weaker than the normal strong convexity condition. And several examples of optimally strongly convex functions that are not strongly convex are provided in (Liu and Wright, 2015).

Lipschitz Smoothness: Let $\Delta_j$ denote the zero vector in $\mathbb{R}^n$ except that the block coordinates indexed by the set $\mathcal{G}_j$. We define the normal Lipschitz constant ($L_{nor}$), block restricted Lipschitz constant ($L_{res}$) and block coordinate Lipschitz constant ($L_{max}$) as follows.

**Definition 1 (Normal Lipschitz constant)** $L_{nor}$ is the normal Lipschitz constant for $\nabla f_i$ ($\forall i \in \{1, \cdots, l\}$) in (1), such that, $\forall x$ and $\forall y$, we have

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_{nor} \|x - y\| \quad (3)$$

**Definition 2 (Block Restricted Lipschitz constant)** $L_{res}$ is the block restricted Lipschitz constant for $\nabla f_i$ ($\forall i \in \{1, \cdots, l\}$) in (1), such that, $\forall x$, and $\forall j \in \{1, \cdots, k\}$, we have

$$\|\nabla f_i(x + \Delta_j) - \nabla f_i(x)\| \leq L_{res} \|\Delta_j\| \quad (4)$$
In this section, we propose our AsySBCDVR. AsySBCDVR is designed for the parallel environment with distributed memory. It can also work in the parallel environment with shared memory, such as multi-core processors and GPU-accelerators, but it can also work in the parallel environment with distributed memory.

Assumption 3 (Lipschitz smoothness) The function \( f_i (\forall i \in \{1, \ldots, l\} \) is Lipschitz smooth with \( L_{nor} \), \( L_{res} \), and \( L_{max} \) as defined above, we assume that the function \( f_i (\forall i \in \{1, \ldots, l\} \) is Lipschitz smooth with \( L_{nor} \), \( L_{res} \), and \( L_{max} \) (i.e., Assumption 3). In addition, we define \( \Lambda_{res} = \frac{L_{max}}{L_{nor}}, \Lambda_{nor} = \frac{L_{max}}{L_{res}} \).

Assumption 3 (Block Coordinate Lipschitz constant) \( L_{max} \) is the block coordinate Lipschitz constant for \( \nabla f_i \) (\( \forall i \in \{1, \ldots, l\} \)) in (5), such that, \( \forall x, \forall j \in \{1, \ldots, k\} \), we have

\[
\max_{j=1,\ldots,k} \| \nabla f_i(x + \Delta_j) - \nabla f_i(x) \| \leq L_{max} \| (\Delta_j)g_j \| \tag{5}
\]

(5) is equivalent to the formulation (6).

\[
f_i(x + \Delta_j) \leq f_i(x) + \langle \nabla g_j f_i(x), (\Delta_j)g_j \rangle + \frac{L_{max}}{2} \| (\Delta_j)g_j \|^2 \tag{6}
\]

Based on \( L_{nor} \), \( L_{res} \), and \( L_{max} \) as defined above, we assume that the function \( f_i (\forall i \in \{1, \ldots, l\} \) is Lipschitz smooth with \( L_{nor} \), \( L_{res} \), and \( L_{max} \) (i.e., Assumption 3). In addition, we define \( \Lambda_{res} = \frac{L_{max}}{L_{nor}}, \Lambda_{nor} = \frac{L_{max}}{L_{res}} \).

3. Algorithm

In this section, we propose our AsySBCDVR. AsySBCDVR is designed for the parallel environment with shared memory, such as multi-core processors and GPU-accelerators, but it can also work in the parallel environment with distributed memory.

In the parallel environment with shared memory, all cores in CPU or GPU can read and write the vector \( x \) in the shared memory simultaneously without any lock. Besides randomly choosing a sample set and a block coordinate, AsySBCDVR is also accelerated by the variance reduction. Thus, AsySBCDVR has two-layer loops. The outer layer is to parallelly compute the full gradient \( \nabla f(x^s) = \frac{1}{|B|} \sum_{i=1}^{l} \nabla f_i(x^s) \), where the superscript \( s \) denotes the \( s \)-th outer loop. The inner layer is to parallelly and repeatedly update the vector \( x \) in the shared memory. Specifically, all cores repeat the following steps independently and concurrently without any lock:

1. Read: Read the vector \( x \) from the shared memory to the local memory without reading lock. We use \( x_i^{s+1} \) to denote its value, where the subscript \( t \) denotes the \( t \)-th inner loop.

2. Compute: Randomly choose a mini-batch \( B \) and a block coordinate \( j \) from \( \{1, \ldots, k\} \), and locally compute \( \nabla g_j^{s+1} = \frac{1}{|B|} \sum_{i \in B} \nabla g_j f_i(x_i^{s+1}) - \frac{1}{|B|} \sum_{i \in B} \nabla g_j f_i(x^s) \).

3. Update: Update the block \( j \) of the vector \( x \) in the shared memory as \( (x_i^{s+1})_j \leftarrow \mathcal{P}_{g_j} \left( (x_i^{s+1})_j - \frac{\nabla g_j^{s+1}}{\| \nabla g_j^{s+1} \|^2} \right) \) without writing lock.

The detailed description of AsySBCDVR is presented in Algorithm 1. Note that \( \nabla g_j^{s+1} \) computed locally is the approximation of \( \nabla g_j f(x_i^{s+1}) \), and the expectation of \( \nabla g_j^{s+1} \) on \( B \) is equal to \( \nabla f(x_i^{s+1}) \) as follows.

\[
\mathbb{E} \nabla g_j^{s+1} = \mathbb{E} \left( \frac{1}{|B|} \sum_{i \in B} \nabla f_i(x_i^{s+1}) - \frac{1}{|B|} \sum_{i \in B} \nabla f_i(x^s) + \nabla f(x^s) \right) \tag{7}
\]
Thus, $\tilde{x}_t^{s+1}$ is called a stochastic gradient of $f(x)$ at $\hat{x}_t^{s+1}$.

Because AsySBCDVR does not use the reading lock, the vector $\hat{x}_t^{s+1}$ read into the local memory may be inconsistent to the vector $x_t^{s+1}$ in the shared memory, which means that some components of $\hat{x}_t^{s+1}$ are same with the ones in $x_t^{s+1}$, but others are different to the ones in $x_t^{s+1}$. However, we can define a set $K(t)$ of inner iterations, such that,

$$x_t^{s+1} = \hat{x}_t^{s+1} + \sum_{t' \in K(t)} B_{t'}^{s+1} \Delta_{t'}^{s+1}$$

(8)

where $t' \leq t-1$, $(\Delta_{t'}^{s+1})_{g_{j(t')}} = \mathcal{P}_{g_{j(t')}} \frac{\gamma}{\max g_{j(t')}} \left( (x_{t'}^{s+1})_{g_{j(t')}} - \frac{\gamma}{\max v_{t'_{s+1}}} \right)$.

Assumption 4 (Non zero of $B_{t'}^s$) For all inner iterations $t$ in AsySBCDVR, $\forall t' \in K(t)$, we have that $B_{t'}^s \neq 0$.

Assumption 5 (Bound of delay) There exists a upper bound $\tau$ such that $\tau \geq t - \min \{ t' | t' \in K(t) \}$ for all inner iterations $t$ in AsySBCDVR.

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**Algorithm 1** Asynchronous Stochastic Block Coordinate Descent with Variance Reduction (AsySBCDVR)

**Input:** $\gamma$, $S$, and $m$.

**Output:** $x^S$.

1. Initialize $x^0 \in \mathbb{R}^d$, $p$ threads.
2. for $s = 0, 1, 2, S - 1$ do
3. \hspace{1em} $\tilde{x}^s \leftarrow x^s$
4. \hspace{1em} All threads parallelly compute the full gradient $\nabla f(\tilde{x}^s) = \frac{1}{T} \sum_i \nabla f_i(\tilde{x}^s)$
5. \hspace{1em} For each thread, do:
6. \hspace{2em} for $t = 0, 1, 2, m - 1$ do
7. \hspace{3em} Randomly sample a mini-batch $\mathcal{B}$ from \{1, ..., $l$\} with equal probability.
8. \hspace{3em} Randomly choose a block $j(t)$ from \{1, ..., $k$\} with equal probability.
9. \hspace{3em} Compute $\tilde{v}_t^{s+1} = \frac{1}{|\mathcal{B}|} \sum_{j \in \mathcal{B}} \nabla f_i(\tilde{x}_t^{s+1}) - \frac{\gamma}{|\mathcal{B}|} \sum_{j \in \mathcal{B}} \nabla g_{j(t)} f_i(\tilde{x}^s) + \nabla g_{j(t)} f(\tilde{x}^s)$.
10. \hspace{3em} $(x_t^{s+1})_{g_{j(t)}} \leftarrow \mathcal{P}_{g_{j(t)}} \frac{\gamma}{\max g_{j(t)}} \left( (x_t^{s+1})_{g_{j(t)}} - \frac{\gamma}{\max v_{t'_{s+1}}} \right)$.
11. \hspace{3em} $(x_t^{s+1})_{g_{j(t)}} \leftarrow (x_t^{s+1})_{g_{j(t)}}$.
12. \hspace{3em} end for
13. $x_{s+1} \leftarrow x_{s+1}$
14. end for
4. Convergence Analysis

In this section, we follow the analysis of [Liu and Wright, 2013] and prove the convergence rate of AsyDSCDVR (Theorem 8). Specifically, AsySBCDVR achieves a linear convergence rate when the function $f$ is with the optimal strong convexity property, and a sublinear rate when $f$ is with the general convexity.

Before providing the theoretical analysis, we give the definitions of $\hat{x}_{t,t'}^s$, $\overline{x}_{t+1}^s$ and the explanation of $x_t^s$ used in the analysis as follows.

1. $\hat{x}_{t,t'}^s$: Assume the indices in $K(t)$ are sorted in the increasing order, we use $K(t)_{t'}$ to denote the $t'$-th index in $K(t)$.

$$\hat{x}_{t,t'}^s = x_t^s + \sum_{t''=1}^{t'} \left( B_{K(t)_{t''}}^{s+1} \Delta_{K(t)_{t''}}^{s+1} \right) = \hat{x}_t^s + \sum_{t''=1}^{t'} (x_{K(t)_{t''}+1}^s - x_{K(t)_{t''}}^s)$$

Thus, we have that

$$\hat{x}_t^s = \hat{x}_{t,0}^s$$

$$x_t^s = \hat{x}_{t,K(t)}^s$$

$$x_t^s - \hat{x}_t^s = \sum_{t''=0}^{\lfloor K(t) \rfloor - 1} \left( B_{K(t)_{t''}}^{s+1} \Delta_{K(t)_{t''}}^{s+1} \right) = \sum_{t''=0}^{\lfloor K(t) \rfloor - 1} (\hat{x}_{t,t''+1}^s - \hat{x}_{t,t''}^s)$$

$$\nabla f(x_t^s) - \nabla f(\hat{x}_t^s) = \sum_{t''=0}^{\lfloor K(t) \rfloor - 1} (\nabla f(\hat{x}_{t,t''+1}^s) - \nabla f(\hat{x}_{t,t''}^s))$$

2. $\overline{x}_{t+1}^s$: $\overline{x}_{t+1}^s$ is defined as:

$$\overline{x}_{t+1}^s \overset{\text{def}}{=} \mathcal{P}_{\gamma \max g} \left( x_t^s - \frac{\gamma}{L_{\max}} \hat{\nu}_t^s \right)$$

Based on (14), it is easy to verify that $(\overline{x}_{t+1}^s)_{g_j(t)} = (x_{t+1}^{s+1})_{g_j(t)}$. Thus, we have $E_{j(t)}(x_{t+1}^s - x_t^s) = \frac{1}{k} (\overline{x}_{t+1}^s - x_t^s)$. It means that $\overline{x}_{t+1}^s - x_t^s$ captures the expectation of $x_{t+1}^s - x_t^s$.

3. $x_t^s$: As mentioned previously, AsySBCDVR does not use any locks in the reading and writing. Thus, in the line 10 of Algorithm 1 $x_t^s$ (left side of ‘←’) updated in the shared memory may be inconsistent with the idea one (right side of ‘←’) computed by the proximal operator. In the analysis, we use $x_t^s$ to denote the idea one computed by the proximal operator. Same as mentioned in [Mania et al., 2015], there might not be an actual time the idea ones exist in the shared memory, except the first and last iterates for each outer loop. It is noted that, $x_0^s$ and $x_m^s$ are exactly what is stored in shared memory. Thus, we only consider the idea $x_t^s$ in the analysis.

Then, we give two inequalities in Lemma 4 and 5 respectively. Based on Lemma 4 and 5 we prove that $E\|x_{t-1}^s - \overline{x}_t^s\|^2 \leq \rho E\|x_t^s - \overline{x}_{t+1}^s\|^2$ (Lemma 6), where $\rho > 1$ is a user defined parameter. Then, we prove the monotonicity of the expectation of the objectives $EF(x_{t+1}^s) \leq EF(x_t^s)$ (Lemma 7). Note that the analyses only consider the case $|B| = 1$ without loss of generality. The case of $|B| > 1$ can be proved similarly.
Lemma 4 For \( \| \nabla f(x^s_t) - \nabla f(\bar{x}^s_t) \| \) in each iteration of AsySBCDVR, we have its upper bound as
\[
\| \nabla f(x^s_t) - \nabla f(\bar{x}^s_t) \| \leq L_{res} \sum_{t' \in K(t)} \| \Delta^s_{t'} \| 
\]
(15)

Proof Based on (9), we have that
\[
\| \nabla f(x^s_t) - \nabla f(\bar{x}^s_t) \| = \left\| \sum_{t' = 0}^{\mid K(t) \mid - 1} \nabla f(\bar{x}^s_{t,t'+1}) - \nabla f(\bar{x}^s_{t,t'}) \right\| 
\]
(16)

\[
\leq \sum_{t' = 0}^{\mid K(t) \mid - 1} \| \nabla f(\bar{x}^s_{t,t'+1}) - \nabla f(\bar{x}^s_{t,t'}) \| \leq L_{res} \sum_{t' = 0}^{\mid K(t) \mid - 1} \| \bar{x}^s_{t,t'+1} - \bar{x}^s_{t,t'} \| 
\]
\[
= L_{res} \sum_{t' = 0}^{\mid K(t) \mid - 1} \| B^s_{K(t),t'} \Delta^s_{K(t),t'} \| \leq L_{res} \sum_{t' = 0}^{\mid K(t) \mid - 1} \| B^s_{K(t),t'} \| \| \Delta^s_{K(t),t'} \| \leq L_{res} \sum_{t' \in K(t)} \| \Delta^s_{t'} \| 
\]
This completes the proof. ■

Lemma 5 In each iteration of AsySBCDVR, \( \forall x \), we have the following inequality.
\[
\left\langle (\tilde{v}^s_t)_{g_j(t)} + \frac{L_{max}}{\gamma} \Delta^s_t, (x^s_{t+1} - x)_{g_j(t)} \right\rangle + g_{\tilde{g}_j(t)}(x^s_{t+1})_j - g_{\tilde{g}_j(t)}(x^s_t) \leq 0
\]
(17)

Proof The problem solved in lines 8 of Algorithm 3 is as follows
\[
x^s_{t+1} = \arg \min_x \left\langle (\tilde{v}^s_t)_{g_j(t)} + \frac{L_{max}}{2\gamma} \| x - x^s_t \|_{g_j(t)} \right\rangle + g_{\tilde{g}_j(t)}(x^s_t) \leq 0
\]
(18)

If \( x^s_{t+1} \) is the solution of (17), the solution of optimization problem (19) is also \( x^s_{t+1} \), according to the subdifferential version of Karush-Kuhn-Tucker (KKT) conditions (Ruszczynski, 2006).
\[
P(x) = \min_x \left\langle (\tilde{v}^s_t)_{g_j(t)} + \frac{L_{max}}{\gamma} (x^s_{t+1} - x^s_t)_{g_j(t)} \right\rangle + g_{\tilde{g}_j(t)}(x^s_t)
\]
(19)

\[\text{s.t. } x_{\tilde{g}_j(t)} = (x^s_t)_{\tilde{g}_j(t)} \]
Thus, we have that \( P(x) \geq P(x^s_{t+1}) \), \( \forall x \), which leads to (17). This completes the proof. ■

Lemma 6 Let \( \rho \) be a constant that satisfies \( \rho > 1 \), and define the quantities \( \theta_1 = \frac{\rho^2 - \rho^2 + 1}{1 - \rho^2} \)
and \( \theta_2 = \rho^2 - \frac{m}{\rho} \). Suppose that the nonsmooth strong parameter \( \gamma > 0 \) satisfies \( \gamma \leq \min \left\{ \frac{1}{4(L_{res}(1+\theta_1)+\Lambda_{nor}(1+\theta_2))}, \frac{1}{2} k^{1/2} + 2 \Lambda_{nor} \theta_2 + \Lambda_{res} \theta_1 \right\} \), we have
\[
\mathbb{E} \| x^s_{t-1} - x_t^s \|^2 \leq \rho \mathbb{E} \| x^s_t - x_{t+1}^s \|^2
\]
(20)
Proof According to (A.8) in [Liu and Wright (2015)], we have
\[
\|x_t^s - x_{t-1}^s + \bar{\gamma} \| + \frac{\gamma}{L_{\max}} \|x_t^s - \Delta_t^s\| \leq \|x_t^s - x_{t-1}^s\| + \frac{\gamma}{L_{\max}} \|x_t^s - \Delta_t^s\| + \gamma \Lambda_{\text{res}} \left( \sum_{t' \in K(t-1)} \|\Delta_{t'}^s\| + \sum_{t' \in K(t)} \|\Delta_{t'}^s\| \right)
\]
\[
+ \gamma \Lambda_{\text{nor}} (\|x_{t-1}^s - \bar{x}\| + \|x_{t-1}^s - \bar{x}\|)
\]
\[
\leq 2\|x_t^s - x_{t-1}^s\| + 2\gamma \Lambda_{\text{res}} \sum_{t' = t-1}^{t-1} \|\Delta_{t'}^s\| + \Lambda_{\text{nor}} \sum_{t' = 0}^{t-1} \|\Delta_{t'}^s\|
\]
\[
\leq 2\|x_t^s - x_{t-1}^s\| + 2\gamma (\Lambda_{\text{res}} + \Lambda_{\text{nor}}) \|\Delta_0^s\|
\]
where the first inequality uses the nonexpansive property of \(P_{\Delta_t^s}\), the fifth inequality uses A.7 of [Liu and Wright (2015)], the sixth inequality comes from \(\|x_t^s - \bar{x}\| = \|\sum_{t' = 0}^{t-1} \Delta_{t'}^s\| \leq \sum_{t' = 0}^{t-1} \|\Delta_{t'}^s\|\).

If \(t = 1\), we have that \(K(0) = \emptyset\) and \(K(1) \subseteq \{0\}\). Thus, according to (22), we have
\[
\|x_1^s - \bar{x}\| \leq 2\|x_0^s - \bar{x}\| + 2\gamma (\Lambda_{\text{res}} + \Lambda_{\text{nor}}) \|\Delta_0^s\|
\]
(23)

Substituting (23) into (21), and taking expectations, we have
\[
\mathbb{E}[x_0^s - \bar{x}]^2 - \mathbb{E}[x_0^s - \bar{x}]^2 \leq 2\mathbb{E}[\|x_0^s - \bar{x}\| x_0^s - \bar{x}]^2 - 2\gamma \mathbb{E}(\|x_0^s - \bar{x}\| x_0^s - \bar{x})^2 \leq 4\mathbb{E}(\|x_0^s - \bar{x}\| x_0^s - \bar{x})^2 + 4\gamma (\Lambda_{\text{res}} + \Lambda_{\text{nor}}) \mathbb{E}(\|x_0^s - \bar{x}\| \|\Delta_0^s\|)
\]
(24)
where the last inequality uses A.13 in (Liu and Wright, 2015). Further, we have the upper bound of $\mathbb{E}\left(\|x_t^s - \bar{x}_{t+1}^s\|\|\Delta_t^s\|\right)$ as

$$
\mathbb{E}\left(\|x_t^s - \bar{x}_{t+1}^s\|\|\Delta_t^s\|\right) \leq \frac{1}{2}\mathbb{E}\left(k^{-\frac{1}{2}}\|x_t^s - \bar{x}_{t+1}^s\|^2 + k^{\frac{1}{2}}\|\Delta_t^s\|^2\right)
$$

(25)

$$
= \frac{1}{2}\mathbb{E}\left(k^{-\frac{1}{2}}\|x_t^s - \bar{x}_{t+1}^s\|^2 + k^{\frac{1}{2}}\mathbb{E}\left(\|\Delta_t^s\|\right)^2\right) = \frac{1}{2}\mathbb{E}\left(k^{-\frac{1}{2}}\|x_t^s - \bar{x}_{t+1}^s\|^2 + k^{\frac{1}{2}}\mathbb{E}|x_t^s - \bar{x}_{t+1}^s|^2\right)
$$

$$
= k^{-\frac{1}{2}}\mathbb{E}|x_t^s - \bar{x}_{t+1}^s|^2
$$

Substituting (25) into (24), we have

$$
\mathbb{E}|x_0^s - \bar{x}_1^s|^2 - \mathbb{E}|x_t^s - \bar{x}_2^s|^2 \leq k^{-\frac{1}{2}}(4 + 4\gamma (\Lambda_{res} + \Lambda_{nor})) \mathbb{E}\left(\|x_0^s - \bar{x}_1^s\|^2\right)
$$

(26)

which implies that

$$
\mathbb{E}|x_0^s - \bar{x}_1^s|^2 \leq \left(1 - \frac{4 + 4\gamma (\Lambda_{res} + \Lambda_{nor})}{\sqrt{k}}\right)^{-1} \mathbb{E}|x_t^s - \bar{x}_2^s|^2 \leq \rho \mathbb{E}|x_t^s - \bar{x}_2^s|^2
$$

(27)

where the last inequality follows from . Thus, we have (20) for $t = 1$.

$$
\rho^{-1} \leq 1 - \frac{4 + 4\gamma (\Lambda_{res} + \Lambda_{nor})}{\sqrt{k}} \Leftrightarrow \gamma \leq \frac{k^{1/2}(1 - \rho^{-1}) - 4}{4(\Lambda_{res} + \Lambda_{nor})}
$$

(28)

Next, we consider the cases for $t > 1$. For $t - 1 - \tau \leq t' \leq t - 1$ and any $\beta > 0$, we have

$$
\mathbb{E}\left(\|x_t^s - \bar{x}_{t+1}^s\|\|\Delta_t^s\|\right) \leq \frac{1}{2}\mathbb{E}\left(k^{-\frac{1}{2}}\beta^{-1}\|x_t^s - \bar{x}_{t+1}^s\|^2 + k^{\frac{1}{2}}\beta\|\Delta_t^s\|^2\right)
$$

(29)

$$
= \frac{1}{2}\mathbb{E}\left(k^{-\frac{1}{2}}\beta^{-1}\|x_t^s - \bar{x}_{t+1}^s\|^2 + k^{\frac{1}{2}}\beta\mathbb{E}\left(\|\Delta_t^s\|\right)^2\right)
$$

$$
= \frac{1}{2}\mathbb{E}\left(k^{-\frac{1}{2}}\beta^{-1}\|x_t^s - \bar{x}_{t+1}^s\|^2 + k^{\frac{1}{2}}\beta\mathbb{E}|x_t^s - \bar{x}_{t+1}^s|^2\right)
$$

$$
\leq \frac{1}{2}\mathbb{E}\left(k^{-\frac{1}{2}}\beta^{-1}\|x_t^s - \bar{x}_{t+1}^s\|^2 + k^{\frac{1}{2}}\beta \rho^{-t'}\mathbb{E}|x_t^s - \bar{x}_{t+1}^s|^2\right)
$$

$$
\beta = \rho^{t'-1} \leq k^{-\frac{1}{2}}\beta \rho^{t'-1} \mathbb{E}|x_t^s - \bar{x}_{t+1}^s|^2
$$

We assume that (20) holds $\forall t' < t$. By substituting (22) into (21) and taking expectation on both sides of (21), we can have

$$
\mathbb{E}\left(\|x_{t-1}^s - \bar{x}_1^s\|^2 - \|x_t^s - \bar{x}_{t+1}^s\|^2\right) \leq 2\mathbb{E}\left(\|x_{t-1}^s - \bar{x}_1^s\|^2 \|x_t^s - \bar{x}_{t+1}^s - x_{t-1}^s + \bar{x}_1^s\|\right)
$$

$$
\leq 2\mathbb{E}\left(\|x_{t-1}^s - \bar{x}_1^s\| \left(2\|x_t^s - \bar{x}_{t-1}^s\| + 2\gamma \left(\Lambda_{res} \sum_{t'=t-1}^{t-1} \|\Delta_{t'}^s\| + \Lambda_{nor} \sum_{t'=0}^{t-1} \|\Delta_{t'}^s\|\right)\right)\right)
$$

$$
= 4\mathbb{E}\left(\|x_{t-1}^s - \bar{x}_1^s\|^2 \|x_{t-1}^s - \bar{x}_1^s\|\Delta_{t-1}^s\| + \Lambda_{nor} \sum_{t'=0}^{t-1} \|\Delta_{t-1}^s\|\right)
$$

(30)
\[
\begin{align*}
\leq 4k^{-1/2} & E(\|x^*_{t-1} - \bar{x}^*_t\|^2) + \\
4\gamma k^{-1/2} & E(\|x^*_{t} - \bar{x}^*_t\|^2) \left( \Lambda_{res} \sum_{t'=l-1}^{t-1} \rho^{t'-l} + \Lambda_{nor} \sum_{t'=0}^{t-1} \rho^{t'-l} \right) \\
= & (4 + 4\gamma (\Lambda_{res} + \Lambda_{nor}))k^{-1/2} E(\|x^*_{t-1} - \bar{x}^*_t\|^2) + \\
4\gamma k^{-1/2} & E(\|x^*_{t-1} - \bar{x}^*_t\|^2) \left( \Lambda_{res} \sum_{t'=1}^r \rho^{t'} + \Lambda_{nor} \sum_{t'=1}^{t-1} \rho^{t'} \right) \\
= & k^{-1/2} E(\|x^*_{t-1} - \bar{x}^*_t\|^2) \left( 4 + 4\gamma \Lambda_{res} \left( 1 + \frac{\rho^{1/2} - \rho^{1/2} + \gamma}{1 - \rho^{1/2}} \right) + 4\gamma \Lambda_{nor} \left( 1 + \frac{\rho^{1/2} - \rho^{1/2}}{1 - \rho^{1/2}} \right) \right) \\
= & k^{-1/2} E(\|x^*_{t-1} - \bar{x}^*_t\|^2) \cdot (4 + 4\gamma (\Lambda_{res} (1 + \theta_1) + \Lambda_{nor} (1 + \theta_2)))
\end{align*}
\]

where the third inequality uses \eqref{eq:ineq1}. Based on \eqref{eq:ineq2}, we have that

\[
E(\|x^*_{t-1} - \bar{x}^*_t\|^2) \leq \left( 1 - k^{-1/2} (4 + 4\gamma (\Lambda_{res} (1 + \theta_1) + \Lambda_{nor} (1 + \theta_2))) \right)^{-1} E(\|x^*_{t} - \bar{x}^*_t\|^2)
\]

\[
\leq \rho E(\|x^*_{t} - \bar{x}^*_t\|^2)
\]

where the last inequality follows from

\[
\rho^{-1} \leq 1 - k^{-1/2} (4 + 4\gamma (\Lambda_{res} (1 + \theta_1) + \Lambda_{nor} (1 + \theta_2))) \iff \gamma \leq \frac{k^{1/2}(1 - \rho^{-1}) - 4}{4 (\Lambda_{res} (1 + \theta_1) + \Lambda_{nor} (1 + \theta_2))}
\]

This completes the proof. \hfill \blacksquare

**Lemma 7** Let \(\rho\) be a constant that satisfies \(\rho > 1\), and define the quantities \(\theta_1 = \frac{\rho^{1/2} - \rho^{1/2} + \gamma}{1 - \rho^{1/2}}\) and \(\theta_2 = \frac{1}{1 - \rho^{1/2}}\). Suppose the nonnegative steplength parameter \(\gamma > 0\) satisfies \(\gamma \leq \min\left\{ \frac{k^{1/2}(1 - \rho^{-1}) - 4}{4 (\Lambda_{res} (1 + \theta_1) + \Lambda_{nor} (1 + \theta_2))}, \frac{k^{1/2}}{2k^{1/2} + 2\Lambda_{nor} \theta_2 + \Lambda_{res} \theta_1} \right\} \). The expectation of the objective function \(EF(x^*_t)\) is monotonically decreasing, i.e., \(EF(x^*_{t+1}) \leq EF(x^*_t)\).

**Proof** Take expectation \(F(x^*_{t+1})\) on \(j(t)\), we have that

\[
\begin{align*}
\mathbb{E}_{j(t)} F(x^*_{t+1}) &= \mathbb{E}_{j(t)} \left( f(x^*_t + \Delta^*_t) + g(x^*_{t+1}) \right) \\
\leq & \mathbb{E}_{j(t)} \left( f(x^*_t) + \left\langle \nabla g_{j(t)} f(x^*_t), (\Delta^*_t) g_{j(t)} \right\rangle + \frac{L_{\max}}{2} \| (\Delta^*_t) g_{j(t)} \|^2 \right) \\
&+ g g_{j(t)} \left( x^*_{t+1} g_{j(t)} \right) + \sum_{j' \neq j(t)} g g_{j'} \left( x^*_{t+1} g_{j'} \right) \\
= & f(x^*_t) + \frac{k - 1}{k} g(x^*_t) + \mathbb{E}_{j(t)} \left( \left\langle \nabla g_{j(t)} f(x^*_t), (\Delta^*_t) g_{j(t)} \right\rangle + \frac{L_{\max}}{2} \| (\Delta^*_t) g_{j(t)} \|^2 \right)
\end{align*}
\]
+ g_{j(t)} \left( (x^*_{t+1})_j g_{j(t)} \right)
= F(x^*_t) + E_j(t) \left( \langle \hat{\nabla}_j f(x^*_t), (\Delta^*_j) g_{j(t)} \rangle + \frac{L_{\max}}{2} \| (\Delta^*_j) g_{j(t)} \|_2^2 + g_{j(t)} \left( (x^*_{t+1})_j g_{j(t)} \right) \right. \\
- g_{j(t)} \left( (x^*_t)_j g_{j(t)} \right) + \left\langle \nabla g_{j(t)} f(x^*_t) - (\hat{\nabla}_j f(x^*_t))_j \right\rangle \right)
\leq F(x^*_t) + E_j(t) \left( \frac{L_{\max}}{\gamma} \| (\Delta^*_j) g_{j(t)} \|_2^2 + \frac{L_{\max}}{2} \| (\Delta^*_j) g_{j(t)} \|_2^2 + \\
+ \left\langle \nabla g_{j(t)} f(x^*_t) - (\hat{\nabla}_j f(x^*_t))_j \right\rangle \right)
= F(x^*_t) + E_j(t) \left( \left( \frac{L_{\max}}{2} - \frac{L_{\max}}{\gamma} \right) \| (\Delta^*_j) g_{j(t)} \|_2^2 \right) + E_j(t) \left\langle \nabla g_{j(t)} f(x^*_t) - (\hat{\nabla}_j f(x^*_t))_j \right\rangle \right)
= F(x^*_t) + \frac{L_{\max}}{k} \left( \frac{1}{2} - \frac{1}{\gamma} \right) \|\bar{t}^*_{t+1} - x^*_t\|_2^2 + E_j(t) \left\langle \nabla g_{j(t)} f(x^*_t) - (\hat{\nabla}_j f(x^*_t))_j \right\rangle \right)
\end{equation}

where the first inequality uses \( (6) \), and the second inequality uses \( (17) \) in Lemma 5. Consider the expectation of the last term on the right-hand side of \( (34) \), we have

\begin{equation}
\begin{align*}
\mathbb{E} \left\langle \nabla g_{j(t)} f(x^*_t) - (\hat{\nabla}_j f(x^*_t))_j \right\rangle 
&= \mathbb{E} \left\langle \nabla g_{j(t)} f(x^*_t) - (\nabla f_{it}(\bar{x}^*_t) - \nabla f_{it}(\bar{x}^*_t) + \nabla f(\bar{x}^*_t))_j \right\rangle, (\Delta^*_j) g_{j(t)} \right) \\
&= \mathbb{E} \left\langle \nabla g_{j(t)} f(x^*_t) - (\nabla f_{it}(\bar{x}^*_t) - \nabla f_{it}(x^*_t) + \nabla f(\bar{x}^*_t) - \nabla f(\bar{x}^*_t))_j \right\rangle, (\Delta^*_j) g_{j(t)} \right) \\
&= \mathbb{E} \left\langle \nabla g_{j(t)} f(x^*_t) - \nabla g_{j(t)} f(\bar{x}^*_t), (\Delta^*_j) g_{j(t)} \right) \right) \\
&+ \mathbb{E} \left\langle \nabla g_{j(t)} f_{it}(\bar{x}^*_t) - \nabla g_{j(t)} f_{it}(x^*_t), (\Delta^*_j) g_{j(t)} \right) \right) \\
&\leq \mathbb{E} \left( \|\nabla f(x^*_t) - \nabla f(\bar{x}^*_t)\|_2 \|\bar{t}^*_{t+1} - x^*_t\| \right) + \mathbb{E} \left( \|\nabla g_{j(t)} f_{it}(x^*_t) - \nabla g_{j(t)} f_{it}(\bar{x}^*_t)\|_2 \|\bar{t}^*_{t+1} - x^*_t\| \right) \\
&= \frac{1}{k} \mathbb{E} \left( \|\nabla f(x^*_t) - \nabla f(\bar{x}^*_t)\|_2 \|\bar{t}^*_{t+1} - x^*_t\| \right) + \frac{1}{k} \mathbb{E} \left( \|\nabla f_{it}(x^*_t) - \nabla f_{it}(\bar{x}^*_t)\|_2 \|\bar{t}^*_{t+1} - x^*_t\| \right) \\
&\leq 2L_{nor} \|x^*_t - \bar{x}^*_t\|_2 \|\bar{t}^*_{t+1} - x^*_t\| + L_{res} \sum_{t' \in \mathcal{K}(t)} \|\Delta^*_j \|_2 \|\bar{t}^*_{t+1} - x^*_t\| \\
&\leq 2L_{nor} \sum_{t' = 0}^{t-1} \|\Delta^*_j \|_2 \|\bar{t}^*_{t+1} - x^*_t\| + L_{res} \sum_{t' = 0}^{t-1} \|\Delta^*_j \|_2 \|\bar{t}^*_{t+1} - x^*_t\| \\
&\leq 2L_{nor} \sum_{t' = 0}^{t-1} \frac{\rho \frac{t-t'}{k^{3/2}} \mathbb{E} \|\bar{t}^*_{t+1} - x^*_t\|_2^2 + L_{res} \sum_{t' = t-\tau}^{t-1} \frac{\rho \frac{t-t'}{k^{3/2}} \mathbb{E} \|\bar{t}^*_{t+1} - x^*_t\|_2^2}{1 - \rho^2} \\
&= k^{-3/2} \left( 2L_{nor} \frac{\rho^2 - \rho^{2\tau}}{1 - \rho^2} + L_{res} \frac{\rho^2 - \rho^{t-\tau}}{1 - \rho^2} \right) \cdot \mathbb{E} \|\bar{t}^*_{t+1} - x^*_t\|^2 \\
&= k^{-3/2} \left( 2L_{nor} \theta_2 + L_{res} \theta_1 \right) \mathbb{E} \|\bar{t}^*_{t+1} - x^*_t\|^2 
\end{align*}
\end{equation}
where the first inequality uses the Cauchy-Schwarz inequality (Callebaut, 1965), the third inequality uses (3) and (4), the sixth inequality uses (29).

By taking expectations on both sides of (33) and substituting (34), we have

$$
\mathbb{E} F(x_{t+1}^s) \\
\leq \mathbb{E} F(x_t^s) + \frac{L_{\text{max}}}{k} \left( \frac{1}{2} - \frac{1}{\gamma} \right) \mathbb{E} \| x_{t+1}^s - x_t^s \|^2 + \mathbb{E} \left( \nabla g_{j(t)} f(x_t^s) - \left( \hat{v}_t^s \right) g_{j(t)}, \left( \Delta_t^s \right) g_{j(t)} \right) \\
\leq \mathbb{E} F(x_t^s) - \frac{1}{k} \cdot \left( L_{\text{max}} \left( \frac{1}{2} - \frac{1}{\gamma} \right) - \frac{2L_{\text{nor}} \theta_2 + L_{\text{res}} \theta_1}{k^{1/2}} \right) \mathbb{E} \| x_{t+1}^s - x_t^s \|^2
$$

where $L_{\text{max}} \left( \frac{1}{2} - \frac{1}{\gamma} \right) - \frac{2L_{\text{nor}} \theta_2 + L_{\text{res}} \theta_1}{k^{1/2}} \geq 0$ because that $\gamma^{-1} \geq \frac{1}{2} + \frac{2L_{\text{nor}} \theta_2 + L_{\text{res}} \theta_1}{k^{1/2}}$. This completes the proof.

**Theorem 8** Let $\rho$ be a constant that satisfies $\rho > 1$, and define the quantity $\theta' = \frac{\rho^\gamma - \rho^{-1}}{\rho - 1}$.

Suppose the nonnegative step length parameter $\gamma > 0$ satisfies $1 - \Lambda_{\text{nor}} \gamma - \frac{\theta' \theta}{n} - \frac{2\left( \Lambda_{\text{res}} \theta_1 + \Lambda_{\text{nor}} \theta_2 \right) \gamma}{n^{1/2}} \geq 0$. If the optimal strong convexity holds for $f$ with $l > 0$, we have

$$
\mathbb{E} F(x^s) - F^* \leq \frac{L_{\text{max}}}{2 \gamma} \left( 1 + \frac{1}{2k^{(l+1)/2}} \right) \cdot \left( \| x_0^0 - \mathcal{P}_S(x_0^0) \|^2 + \frac{2\gamma}{L_{\text{max}}} \left( \mathbb{E} F(x_0^0) - F^* \right) \right)
$$

If $f$ is a general smooth convex function, we have

$$
\mathbb{E} F(x^s) - F^* \leq \frac{nL_{\text{max}} \| x_0^0 - \mathcal{P}_S(x_0^0) \|^2 + 2\gamma k \left( F(x_0^0) - F^* \right)}{2\gamma k + 2m\gamma s}
$$

**Proof** We have that

$$
\| x_{t+1}^s - \mathcal{P}_S(x_t^s) \|^2 \leq \| x_t^s + \Delta_t^s - \mathcal{P}_S(x_t^s) \|^2
$$

$$
= \| x_t^s - \mathcal{P}_S(x_t^s) \|^2 - \| \Delta_t^s \|^2 - 2 \left( \langle \mathcal{P}_S(x_t^s) - x_t^s - \Delta_t^s \rangle, \langle \Delta_t^s \rangle \right) g_{j(t)}
$$

$$
\leq \| x_t^s - \mathcal{P}_S(x_t^s) \|^2 - \| \Delta_t^s \|^2 + \frac{2\gamma}{L_{\text{max}}} \left( \langle \mathcal{P}_S(x_t^s) - x_{t+1}^s \rangle, \langle \hat{v}_t^s \rangle g_{j(t)} \rangle \right) + \frac{2\gamma}{L_{\text{max}}} \left( \langle g_{j(t)} (\mathcal{P}_S(x_t^s) - x_{t+1}^s) \rangle \right)
$$

$$
= \| x_t^s - \mathcal{P}_S(x_t^s) \|^2 - \| \Delta_t^s \|^2 + \frac{2\gamma}{L_{\text{max}}} \left( \langle \mathcal{P}_S(x_t^s) - x_{t+1}^s \rangle, \langle \hat{v}_t^s \rangle g_{j(t)} \rangle \right) + \frac{2\gamma}{L_{\text{max}}} \left( \langle \mathcal{P}_S(x_t^s) - x_{t+1}^s \rangle, \langle \hat{v}_t^s \rangle g_{j(t)} \rangle \right)
$$

where the first inequality comes from the definition of function $\mathcal{P}_S(x) = \arg \min_{y \in S} \| y - x \|^2$, and the second inequality uses (17) in Lemma 3. For the expectation of $T_1$, we have

$$
\mathbb{E}(T_1) = \mathbb{E} \left( \langle \mathcal{P}_S(x_t^s) - x_t^s \rangle, \langle \hat{v}_t^s \rangle g_{j(t)} \rangle \right)
$$

(39)
\[ \begin{align*}
&= \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - x_t^s, \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}^s) + \nabla f(\tilde{x}^s) \rangle \\
&= \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - x_t^s, \nabla f_{i_t}(\tilde{x}_t^s) \rangle + \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - x_t^s, \nabla f(\tilde{x}^s) \rangle + \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - x_t^s, \nabla f(\tilde{x}^s) \rangle \\
&= \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - \tilde{x}_t^s, \nabla f_{i_t}(\tilde{x}_t^s) \rangle + \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - \tilde{x}_t^s, \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&= \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - \tilde{x}_t^s, \nabla f_{i_t}(\tilde{x}_t^s) \rangle + \frac{1}{k} \mathbb{E} \langle \tilde{x}_t^s - x_t^s, \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&= \frac{1}{k} \mathbb{E} \langle \mathcal{P}_S(x_t^s) - \tilde{x}_t^s, \nabla f_{i_t}(\tilde{x}_t^s) \rangle + \frac{1}{k} \mathbb{E} \langle \tilde{x}_t^s - x_t^s, \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&+ \frac{1}{k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \langle \tilde{x}_{t,t'} - \tilde{x}_{t,t'+1}, \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&\leq \frac{1}{k} \mathbb{E} \langle f_{i_t}(\mathcal{P}_S(x_t^s)) - f_{i_t}(x_t^s) \rangle + \frac{1}{k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \langle f_{i_t}(\tilde{x}_{t,t'}^s) - f_{i_t}(\tilde{x}_{t,t'+1}^s), \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&+ \frac{1}{k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \left( f_{i_t}(\tilde{x}_{t,t'}^s) - f_{i_t}(\tilde{x}_{t,t'+1}^s) + \frac{L_{\max}}{2} \|\tilde{x}_{t,t'} - \tilde{x}_{t,t'+1}\|^2 \right) \\
&= \frac{1}{k} \mathbb{E} \langle f_{i_t}(\mathcal{P}_S(x_t^s)) - f_{i_t}(x_t^s) \rangle + \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \langle \tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s, \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&+ \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \left( \|\tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s\|^2 \right) \\
&= \frac{1}{k} \mathbb{E} \langle f_{i_t}(\mathcal{P}_S(x_t^s)) - f_{i_t}(x_t^s) \rangle + \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \langle \tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s, \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&+ \frac{L_{\max}}{k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \left( \|\tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s\|^2 \right) \\
&= \frac{1}{k} \mathbb{E} \langle f_{i_t}(\mathcal{P}_S(x_t^s)) - f_{i_t}(x_t^s) \rangle + \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \langle \tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s, \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&+ \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \left( \|\tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s\|^2 \right) \\
&= \frac{1}{k} \mathbb{E} \langle f_{i_t}(\mathcal{P}_S(x_t^s)) - f_{i_t}(x_t^s) \rangle + \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \langle \tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s, \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&+ \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \left( \|\tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s\|^2 \right) \\
&= \frac{1}{k} \mathbb{E} \langle f_{i_t}(\mathcal{P}_S(x_t^s)) - f_{i_t}(x_t^s) \rangle + \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \langle \tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s, \nabla f_{i_t}(\tilde{x}_t^s) - \nabla f_{i_t}(\tilde{x}_t^s) \rangle \\
&+ \frac{L_{\max}}{2k} \mathbb{E} \sum_{t'=0}^{|K(t)|-1} \left( \|\tilde{x}_{t,t'}^s - \tilde{x}_{t,t'+1}^s\|^2 \right)
\end{align*}\]
where the fifth equality comes from that $x_t^s$ is independent to $i_t$, the sixth equality uses Lemma [2], the first inequality uses the convexity of $f$ and Lemma [2], the second inequality uses Lemma [2]. For the expectation of $T_2$, we have

$$E(T_2) = E(\langle \Delta_\ell^s, (\hat{\alpha}^\ell \circ \tilde{g}_{\ell^s}) \rangle)$$

$$= E(\langle \Delta_\ell^s, (\nabla f_{i_t}(x_t^s) - \nabla f_{i_t}(\bar{x}^s) + \nabla f(x^s)) \rangle)$$

$$= E(\langle \Delta_\ell^s, (\nabla f_{i_t}(x_t^s) - \nabla f_{i_t}(x_t^s) + \nabla f_{i_t}(\bar{x}^s) - \nabla f_{i_t}(\bar{x}^s) + \nabla f(x^s)) \rangle)$$

$$= E(\langle \Delta_\ell^s, (\nabla f_{i_t}(x_t^s) - \nabla f_{i_t}(x_t^s)) \rangle $$

$$+ E(\langle \Delta_\ell^s, (\nabla f_{i_t}(\bar{x}^s) - \nabla f_{i_t}(\bar{x}^s)) \rangle) + E(\langle \Delta_\ell^s, \nabla g_{\ell^s}(\bar{x}^s) \rangle)$$

$$\leq \frac{1}{k} E(\|\bar{x}_{t+1}^s - x_t^s\| \|\nabla f_{i_t}(x_t^s) - \nabla f_{i_t}(\bar{x}^s)\|) + \frac{1}{k} E(\|\bar{x}_{t+1}^s - x_t^s\| \|\nabla f_{i_t}(\bar{x}^s) - \nabla f_{i_t}(\bar{x}^s)\|) + E(\langle \Delta_\ell^s, \nabla g_{\ell^s}(\bar{x}^s) \rangle)$$

$$\leq \frac{L_{res}}{k} \left( \sum_{t' \in K(t)} \|x_{t+1}^s - x_t^s\| \|\Delta_\ell^s\| \right)$$

$$\leq \frac{L_{res}}{k} E(\|\bar{x}_{t+1}^s - x_t^s\| \|\Delta_\ell^s\| \|\bar{x}^s - x^s\|) + E(\langle \Delta_\ell^s, \nabla g_{\ell^s}(\bar{x}^s) \rangle)$$

$$\leq \frac{L_{res}}{k} \left( \sum_{t' \in K(t)} \|x_{t+1}^s - x_t^s\| \|\Delta_\ell^s\| \right)$$

$$\leq \frac{L_{nor}}{k} E(\|\bar{x}_{t+1}^s - x_t^s\| \|\Delta_\ell^s\| \|\bar{x}^s - x^s\|) + E(\langle \Delta_\ell^s, \nabla g_{\ell^s}(\bar{x}^s) \rangle)$$

$$\leq \frac{L_{res}}{k} \left( \sum_{t' \in K(t)} \|x_{t+1}^s - x_t^s\| \|\Delta_\ell^s\| \right)$$
where the second inequality uses Lemma 4, the fourth inequality uses (29). By substituting the upper bounds from (39) and (40) into (38), we have

\[

\begin{align*}
\mathbb{E}[\|x^s_{t+1} - \mathcal{P}_S(x^s_s)\|^2] &\leq \mathbb{E}[\|x^s_t - \mathcal{P}_S(x^s_s)\|^2] - \frac{1}{k} \mathbb{E}[\|x^s_t - x^s_{t+1}\|^2] + \frac{2\gamma}{L_{\max} k} \mathbb{E}(f(\mathcal{P}_S(x^s_s)) - f(x^s_t)) + \frac{\gamma^2 \theta'}{k^2} \mathbb{E}(\|x^s_t - x^s_{t+1}\|^2) \\
&+ \frac{2\gamma}{L_{\max} k} \left( \frac{L_{res}\theta_1 + L_{nor}\theta_2}{k^{3/2}} \mathbb{E}(\|x^s_t - x^s_{t+1}\|^2) + \mathbb{E}(\langle \Delta_t \rangle_{\mathcal{G}_j(t)}, \nabla \mathcal{G}_j(t) f(\bar{x}^s) \rangle) \\
&+ \frac{1}{k} \mathbb{E}g(\mathcal{P}_S(x^s_t)) - \mathbb{E}g(x^s_t) + \frac{k-1}{k} \mathbb{E}g(x^s_t) \right) \\
&= \mathbb{E}[\|x^s_t - \mathcal{P}_S(x^s_s)\|^2] + \frac{2\gamma}{L_{\max} k} \mathbb{E}(f(\mathcal{P}_S(x^s_s)) - f(x^s_t)) - \frac{1}{k} \left( 1 - \frac{\gamma \theta'}{k} - \frac{2(L_{res}\theta_1 + L_{nor}\theta_2)\gamma}{k^{1/2}L_{\max}} \right) \mathbb{E}(\|x^s_t - x^s_{t+1}\|^2) + \frac{2\gamma}{L_{\max} k} \left( \mathbb{E}(\langle \Delta_t \rangle_{\mathcal{G}_j(t)}, \nabla \mathcal{G}_j(t) f(\bar{x}^s) \rangle) \\
&+ \frac{1}{k} \mathbb{E}g(\mathcal{P}_S(x^s_t)) - \mathbb{E}g(x^s_t) + \frac{k-1}{k} \mathbb{E}g(x^s_t) \right)
\end{align*}
\]

We consider a fixed stage \( s+1 \) such that \( x^{s+1}_0 = x^s_m \). By summing the the inequality (42) over \( t = 0, \ldots, m - 1 \), we obtain

\[

\begin{align*}
\mathbb{E}[\|x^{s+1} - \mathcal{P}_S(x^{s+1})\|^2] &\leq \mathbb{E}[\|x^s - \mathcal{P}_S(x^s)\|^2] + \sum_{t'=0}^{m-1} \frac{2\gamma}{L_{\max} k} \mathbb{E}(f(\mathcal{P}_S(x^{s+1}_{t'})) - f(x^{s+1}_{t'})) - \frac{1}{k} \sum_{t'=0}^{m-1} \left( 1 - \frac{\gamma \theta'}{k} - \frac{2(L_{res}\theta_1 + L_{nor}\theta_2)\gamma}{k^{1/2}L_{\max}} \right) \cdot \mathbb{E}(\|x^{s+1}_{t'} - x^{s+1}_{t'+1}\|^2) \\
&+ \frac{2\gamma}{L_{\max} k} \sum_{t'=0}^{m-1} \mathbb{E}(\langle \Delta_{t'} \rangle_{\mathcal{G}_j(t')}, \nabla \mathcal{G}_j(t') f(\bar{x}^s) \rangle) \\
&+ \frac{2\gamma}{L_{\max} k} \sum_{t'=0}^{m-1} \left( \frac{1}{k} \mathbb{E}g(\mathcal{P}_S(x^{s+1}_{t'})) - \mathbb{E}g(x^{s+1}_{t'+1}) + \frac{n-1}{k} \mathbb{E}g(x^{s+1}_{t'+1}) \right)
\end{align*}
\]
\[
= \mathbb{E}[x^s - \mathcal{P}_S(x^s)]^2 + \sum_{t'=0}^{m-1} \frac{2\gamma}{L_{\text{max}}k} \mathbb{E}\left(f(\mathcal{P}_S(x_{t'}^{s+1})) - f(x_{t'}^{s+1})\right) - \\
\sum_{t'=0}^{m-1} \frac{1}{k} \left(1 - \frac{\gamma \tau \theta'}{k} - \frac{2(L_{\text{res}} \theta_1 + L_{\text{nor}} \theta_2) \gamma}{k^{1/2} L_{\text{max}}}\right) \cdot \mathbb{E}(\|x_{t'}^{s+1} - \overline{x}_{t'}^{s+1}\|^2) \\
+ \frac{2\gamma}{L_{\text{max}}} \mathbb{E}\left(x_{t'}^{s+1} - x^s, \nabla f(\overline{x}^s)\right) \\
+ \frac{2\gamma}{L_{\text{max}}} \sum_{t'=0}^{m-1} \left(\frac{1}{k} \mathbb{E}g(\mathcal{P}_S(x_{t'}^{s+1})) - \mathbb{E}g(x_{t'}^{s+1}) + \frac{k-1}{k} \mathbb{E}g(x_{t'}^{s+1})\right)
\]
\[
\leq \mathbb{E}[x^s - \mathcal{P}_S(x^s)]^2 + \sum_{t'=0}^{m-1} \frac{2\gamma}{L_{\text{max}}k} \mathbb{E}\left(f(\mathcal{P}_S(x_{t'}^{s+1})) - f(x_{t'}^{s+1})\right) - \\
\sum_{t'=0}^{m-1} \frac{1}{k} \left(1 - \frac{\gamma \tau \theta'}{k} - \frac{2(L_{\text{res}} \theta_1 + L_{\text{nor}} \theta_2) \gamma}{k^{1/2} L_{\text{max}}}\right) \cdot \mathbb{E}(\|x_{t'}^{s+1} - \overline{x}_{t'}^{s+1}\|^2) \\
+ \frac{2\gamma}{L_{\text{max}}} \sum_{t'=0}^{m-1} \left(\frac{1}{k} \mathbb{E}g(\mathcal{P}_S(x_{t'}^{s+1})) - \mathbb{E}g(x_{t'}^{s+1}) + \frac{k-1}{k} \mathbb{E}g(x_{t'}^{s+1})\right)
\]
\[
= \mathbb{E}[x^s - \mathcal{P}_S(x^s)]^2 + \sum_{t'=0}^{m-1} \frac{2\gamma}{L_{\text{max}}k} \mathbb{E}\left(f(\mathcal{P}_S(x_{t'}^{s+1})) - f(x_{t'}^{s+1})\right) - \\
\sum_{t'=0}^{m-1} \frac{1}{k} \left(1 - \frac{\gamma \tau \theta'}{k} - \frac{2(L_{\text{res}} \theta_1 + L_{\text{nor}} \theta_2) \gamma}{k^{1/2} L_{\text{max}}}\right) \cdot \mathbb{E}(\|x_{t'}^{s+1} - \overline{x}_{t'}^{s+1}\|^2) \\
+ \frac{2\gamma}{L_{\text{max}}} \sum_{t'=0}^{m-1} \left(\frac{1}{k} \mathbb{E}g(\mathcal{P}_S(x_{t'}^{s+1})) - \mathbb{E}g(x_{t'}^{s+1}) + \frac{k-1}{k} \mathbb{E}g(x_{t'}^{s+1})\right)
\]
\[
\leq \mathbb{E}[x^s - \mathcal{P}_S(x^s)]^2 + \frac{2\gamma}{L_{\text{max}}k} \sum_{t'=0}^{m-1} (F^* - \mathbb{E}F(x_{t'}^{s+1})) + \frac{2\gamma}{L_{\text{max}}} \sum_{t'=0}^{m-1} (\mathbb{E}F(x_{t'}^{s+1}) - \mathbb{E}F(x_{t'}^{s+1})) \\
- \sum_{t'=0}^{m-1} \frac{1}{k} \left(1 - \Lambda_{\text{nor}} \gamma - \frac{\gamma \tau \theta'}{n} - \frac{2(L_{\text{res}} \theta_1 + L_{\text{nor}} \theta_2) \gamma}{k^{1/2} L_{\text{max}}}\right) \cdot \mathbb{E}(\|x_{t'}^{s+1} - \overline{x}_{t'}^{s+1}\|^2) \\
\leq \mathbb{E}[x^s - \mathcal{P}_S(x^s)]^2 + \frac{2\gamma}{L_{\text{max}}k} \sum_{t'=0}^{m-1} (F^* - \mathbb{E}F(x_{t'}^{s+1})) + \frac{2\gamma}{L_{\text{max}}} (\mathbb{E}F(x^s) - \mathbb{E}F(x_{t'}^{s+1}))
\]

where the second inequality uses \([3]\), the final inequality comes from \(1 - \Lambda_{\text{nor}} \gamma - \frac{\gamma \tau \theta'}{n} - \frac{2(L_{\text{res}} \theta_1 + L_{\text{nor}} \theta_2) \gamma}{k^{1/2} L_{\text{max}}} \geq 0\). Define \(S(x^s) = \mathbb{E}[x_t - \mathcal{P}_S(x^s)]^2 + \frac{2\gamma}{L_{\text{max}}} \mathbb{E}(F(x^s) - F^*)\). According to
where the second inequality comes from the monotonicity of $\mathbb{E}F(x^s_i)$. According to (43), we have

$$S(x^s) \leq S(x^0) - \frac{2m\gamma s}{L_{\text{max}}k} \mathbb{E}(F(x^s) - F^*)$$

(44)

Thus, the sublinear convergence rate (37) for general smooth convex function $f$ can be obtained from (44).

If the optimal strong convexity for the smooth convex function $f$ holds with $l > 0$, we have (45) as proved in (A.28) of [Liu and Wright (2015)]:

$$\mathbb{E}(F(x^s) - F^*) \geq \frac{L_{\text{max}}l}{2(l\gamma + L_{\text{max}})} S(x^s)$$

(45)

Thus, substituting (46) into (43), we have

$$S(x^{s+1}) \leq S(x^s) - \frac{2m\gamma l}{2k(l\gamma + L_{\text{max}})} S(x^{s+1})$$

(46)

Based on (46), we have (47) by induction.

$$S(x^s) \leq \left(1 + \frac{1}{2m\gamma l} \right)^s S(x^0)$$

(47)

Thus, the linear convergence rate (36) for the optimal strong convexity on $f$ can be obtained from (47).

5. Conclusion

In this paper, we propose an asynchronous stochastic block coordinate descent algorithm with the accelerated technology of variance reduction (AsySBCDVR), which are with lock-free in the implementation and analysis. AsySBCDVR is particularly important because it can scale well with the sample size and dimension simultaneously. We prove that AsySBCDVR achieves a linear convergence rate when the function $f$ is with the optimal strong convexity property, and a sublinear rate when $f$ is with the general convexity. More importantly, a near-linear speedup on a parallel system with shared memory can be obtained.

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