Quark and gluon entanglement in the proton on the light cone at intermediate $x$

Adrian Dumitru$^*$ and Eric Kolbusz$^†$

Department of Natural Sciences, Baruch College, CUNY, 17 Lexington Avenue, New York, NY 10010, USA and The Graduate School and University Center, The City University of New York, 365 Fifth Avenue, New York, NY 10016, USA

In QCD with $N_c$ colors the anti-symmetric valence quark color space singlet state $\sim \epsilon_{i_1 \ldots i_N_c} |i_1, \ldots, i_N_c\rangle$ of the proton corresponds to the reduced density matrix $\rho_{ij} = (1/N_c) \delta_{ij}$ for a single color degree of freedom. Its degenerate spectrum of eigenvalues, $\lambda_i = 1/N_c$, the purity $\text{tr}\rho^2 = 1/N_c$, and the von Neumann entropy $S_{\text{VN}} = \log(N_c)$ all indicate maximal entanglement of color.

On the other hand, for $N_c \to \infty$ the spatial wave function of the proton factorizes into valence quark wave functions determined by a mean field (E. Witten, Nucl. Phys. B 160 (1979) p. 57) where there is no entanglement of spatial degrees of freedom.

A model calculation at $N_c = 3$ using a simple three quark light-front wave function by Brodsky and Schlumpf, predicts percent level entanglement of spatial degrees of freedom.

Using light-cone perturbation theory we also derive the density matrix associated with the four parton $(qqqg)$ Fock state. Tracing out the quarks, we construct the reduced density matrix for the degrees of freedom of the gluon, which encodes its entanglement with the sources. Our expressions provide the dependence of the density matrix on the soft cutoff $x$ for the gluon light-cone momentum, and on the collinear and ultraviolet regulators. Numerical results obtained in a simple approximation indicate stronger entanglement for the gluon (with $x_g < \langle x_q \rangle$) than for quarks in the three quark Fock state.

Contents

I. Introduction 2

II. Density Matrix for the three quark Fock state 3
   A. Numerical estimates for the three quark density matrix 6
   B. Color and spin wave function 7
   C. Violation of Bell-CHSH inequality 8

III. Density Matrix for the three quark and one gluon state at $O(g^2)$ 9
   A. The density matrix for the four-parton $(qqqg)$ state 10
   B. Trace of $\rho_{qqqg}$ 12
   C. $O(g^2)$ virtual correction to the three-quark density matrix 14
   D. Reduced density matrix for the gluon momentum degree of freedom 17
      1. Reduced density matrix for the $x_g$ degree of freedom 17

IV. Summary 18

Acknowledgements 19

References 19

$^*$Electronic address: adrian.dumitru@baruch.cuny.edu
$^†$Electronic address: ekolbusz@gradcenter.cuny.edu
I. INTRODUCTION

The near future may present exciting opportunities to search experimentally for color entanglement in QCD [1]. For example, color entanglement has been argued to break Transverse Momentum Dependent QCD factorization in the production of hadrons with high transverse momentum (and a transverse momentum imbalance) in proton-proton collisions [2–4]. It has also been proposed that colored quarks and gluons in the wave function of the proton are entangled [5–7], these references focusing specifically on gluons with small light-cone momentum fractions $x$, and that the entropy in the final state of high-energy Deeply Inelastic electron-proton Scattering (DIS) experiments, or of hadronic collisions, may reflect their entanglement [6]. This proposal is currently under active investigation [9–14]. A non-zero “entropy of ignorance” could also arise without tracing over entangled degrees of freedom, however, just from the fact that the available measurements provide only limited information on the density matrix, effectively “zeroing out” some of its matrix elements [15].

The density matrix and the entanglement entropy of small-$x$ gluons in a hadron has been computed in the “Color Glass Condensate” (CGC) framework for high-energy QCD in ref. [7]; see, also, ref. [16] for a relation of the von Neumann entropy of small-$x$ gluons to their quantum phase space (Wigner) distribution. The computation of Kovner and Lublinsky [7] has been generalized in ref. [17] to the “high density saturation regime” at low transverse momentum (where quasi-particles emerge).

The evolution of the density matrix for soft small-$x$ gluons with rapidity $Y = \log 1/x$ has been derived recently in ref. [18]. This is obtained by tracing over gluons with rapidity less than $Y$ (or light-cone momentum fractions greater than $x$). The authors find that this evolution equation is of Lindblad form, describing the non-unitary evolution of the density matrix of an open system. The purity of the density matrix decreases with increasing rapidity $Y$.

Ref. [19] describes a correspondence at weak coupling between highly occupied black hole states of soft gravitons and the state of high gluon occupation numbers encountered in the proton at small $x$. Dvali and Vempalopan argue that upon tracing out the sources at higher $x$ the entropy of soft degrees of freedom attains its maximal value permitted by unitarity, and that it is proportional to the area times a Goldstone scale squared.

A substantial amount of work has been done to understand the regime of high gluon occupation number at small light-cone momentum fraction $x$, as we have just outlined. Our approach here is complementary in that we consider the regime of relatively large $x$ where the proton may be composed of only a few particles (“partons”). In sec. III we consider the emission of one single gluon from a three-quark leading Fock state. We will not require the gluon to be soft and so we recover the dependence of the density matrix on its light-cone momentum fraction. We shall also obtain the dependence of the density matrix on the collinear regulator, which in ref. [7] is implicit in their parameter $\mu^2$ (the average color charge density squared per unit transverse area, see refs. [21–23]).

Before considering gluon emission, however, in the next sec. II we first approximate the proton (for $N_c = 3$ colors) by a three-quark state. The color-space wave function $\sim e_{n_1 n_2 n_3}$ corresponds to maximal entanglement of color, in that the reduced density matrix $\rho_{nn'}$ obtained after tracing out all other degrees of freedom has a degenerate spectrum of eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$. We then compute reduced density matrices over various spatial (momentum) degrees of freedom. Using a three-quark light-cone wave function from the literature [24, 25], we find numerically that these density matrices exhibit a high purity $\text{tr} \rho^2 > 0.95$ and low von Neumann entropy $S_{\text{vN}} \lesssim 0.15$ (nats). In this regard, these model wave functions are close to the $N_c \rightarrow \infty$ limit (at fixed $g^2 N_c$) where the spatial wave function of the proton factorizes into valence quark wave functions [26].

Kharzeev has argued [5] that the scattering of a probe located at $x^- = 0$ off the proton would lead to “information scrambling” and suppression of off-diagonal elements of the density matrix of the proton due an average over the phase of its wave function (see, also, sec. III in ref. [8]). The issue of entanglement and entropy production in particle production in high-energy collisions has been addressed also in refs. [3, 7], and has since been revisited in some of the references mentioned above. Here, however, we consider entanglement of various degrees of freedom in a proton per se.

Finally, let us mention that $n$-body quantum correlations manifest also in a non-trivial impact parameter and transverse momentum dependence of color charge correlations in the proton [22, 23, 27, 28], as well as in Bose-Einstein correlations of small-$x$ gluons [29, 30]. The focus of the current paper is on entanglement of degrees of freedom in

---

1 We refer to ref. [20] for a recent review of collider searches for non-linear gluon dynamics.
the proton.

The remainder of the paper is organized as follows. In the following sec. II we consider the density matrix of the three-quark Fock state. In sec. III we compute the leading perturbative correction and the density matrix for the Fock state containing three quarks and a gluon. Sec. IV contains a brief summary.

II. DENSITY MATRIX FOR THE THREE QUARK FOCK STATE

In the absence of gluons protons are made of $N_c = 3$ “valence” quarks and we can write the proton state $| P \rangle = | P^+, \vec{P} = 0 \rangle$ as

$$ | P \rangle = \int [dx_i] \int [d^2k_i] \Psi_{qqq}(k_1; k_2; k_3)| k_1; k_2; k_3 \rangle, \quad (1) $$

where

$$ [dx_i] = \delta \left( 1 - \sum_i x_i \right) \prod_i \frac{dx_i}{2x_i}, \quad (2) $$

$$ [d^2k_i] = 4 \cdot 16\pi^3 \delta \left( \sum_i \vec{k}_i \right) \prod_i \frac{d^2k_i}{16\pi^3}. \quad (3) $$

$k_1, k_2, k_3$ denote the “coordinates” (light-cone momentum fractions and transverse momenta) for the three quarks. Above we only write the (symmetric$^3$) spatial wave function $\Psi_{qqq}(k_1; k_2; k_3)$ which is our focus in this section. The color, flavor, spin wave function is discussed below in sec. II B. Restricting to the three quark state corresponds to a light front constituent quark model.

We normalize the proton state as

$$ \langle K | P \rangle = 16\pi^3 P^+ \delta(P^+ - K^+) \delta(\vec{P} - \vec{K}) . \quad (4) $$

With the standard normalization of quark states,

$$ \langle p | k \rangle = 16\pi^3 k^+ \delta(p^+ - k^+ \delta(\vec{p} - \vec{k}) \quad (5) $$

this leads to the following normalization condition for the three-quark wave function:

$$ \frac{1}{2} \int [dx_i] \int [d^2k_i] | \Psi_{qqq} |^2 = 1 . \quad (6) $$

In what follows it will be useful to factor out the center of momentum (COM) constraint by transforming to the new coordinates $^3$

$$ \xi = \frac{x_1}{x_1 + x_2}, \quad \eta = 1 - x_3 = x_1 + x_2, $$

$$ \vec{Q} = -\vec{k}_3, \quad \vec{q} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2 + \vec{k}_3) + \vec{Q}(1 - \xi), \quad (7) $$

with $0 < \xi, \eta < 1$. Then

$$ 2 [dx_i] = \frac{d\xi}{2\xi(1 - \xi)} \frac{d\eta}{2\eta(1 - \eta)} , \quad \frac{1}{4} [d^2k_i] = \frac{d^2q}{16\pi^3} \frac{d^2Q}{16\pi^3}. \quad (8) $$

---

$^2$ For an introduction into the light-front formalism and its application in QCD we refer to refs. [31–35]. We write three momenta as $k = (k^+, \vec{k}) = (xP^+, \vec{k})$ where $x$ corresponds to the fractional light cone momentum and $\vec{k}$ to the transverse momentum.

$^3$ That is, $\Psi_{qqq}$ is symmetric under exchange of any two quarks: $\Psi_{qqq}(k_1; k_2; k_3) = \Psi_{qqq}(k_2; k_1; k_3)$ etc.
\begin{equation}
\int \frac{d\xi}{2\xi(1-\xi)} \frac{d\eta}{2\eta(1-\eta)} \frac{d^2q}{16\pi^3} \frac{d^2Q}{16\pi^3} |\Psi_{qqq}|^2 = 1 .
\end{equation}

FIG. 1: Diagrammatic illustration of the density matrix \( \rho_{\alpha\alpha'} \) for the three-quark state. The dashed vertical line indicates the insertion of \( |\alpha'|\langle \alpha | \) into \( \langle P|P \rangle \), where \( \alpha, \alpha' \) denote sets of longitudinal and transverse momenta, and colors of the quarks in \( |P\rangle \) and \( \langle P| \), respectively.

The density operator is \( \hat{\rho} = |P\rangle \langle P| \). To obtain its matrix elements we project the proton state \( |P\rangle \) and its dual \( \langle P| \) on three-quark states \( \alpha \equiv \{ x_i, \vec{k}_i \} = \{ \xi, \eta, \vec{q}, \vec{Q} \} \) and \( \alpha' \equiv \{ x'_i, \vec{k}'_i \} = \{ \xi', \eta', \vec{q}', \vec{Q}' \} \), respectively:

\begin{equation}
\rho_{\alpha\alpha'} (2\pi)^3 \delta \left( 1 - \sum_i x_i \right) \delta \left( \sum_i \vec{k}_i \right) (2\pi)^3 \delta \left( 1 - \sum_i x'_i \right) \delta \left( \sum_i \vec{k}'_i \right) = \langle \alpha|P \rangle \langle P|\alpha' \rangle .
\end{equation}

This is shown diagrammatically in fig. 1. With 4

\begin{equation}
\langle \alpha|P \rangle = \Psi_{qqq}^*(\alpha) (2\pi)^3 \delta \left( 1 - \sum_i x_i \right) \delta \left( \sum_i \vec{k}_i \right),
\end{equation}

finally, the density matrix describing the pure (three-quark) state is given by

\begin{equation}
\rho_{\alpha\alpha'} = \Psi_{qqq}^*(\alpha') \Psi_{qqq}^*(\alpha) .
\end{equation}

Its trace over all degrees of freedom is equal to

\begin{equation}
\text{tr}_\alpha \rho_{\alpha\alpha} \equiv \text{tr}_{\xi,\eta,\vec{q},\vec{Q}} \rho_{\alpha\alpha} \equiv \int \frac{d\xi}{2\xi(1-\xi)} \frac{d\eta}{2\eta(1-\eta)} \frac{d^2q}{16\pi^3} \frac{d^2Q}{16\pi^3} |\Psi_{qqq}(\xi, \eta, \vec{q}, \vec{Q})|^2 = 1 ,
\end{equation}

where

\begin{equation}
d\alpha \equiv \frac{d\xi}{2\xi(1-\xi)} \frac{d\eta}{2\eta(1-\eta)} \frac{d^2q}{16\pi^3} \frac{d^2Q}{16\pi^3} = \frac{1}{2} [d\xi_i] [d^2k_i] .
\end{equation}

To arrive at eq. (13) we define the trace of the density operator via

\begin{equation}
\text{tr} \hat{\rho} = \frac{\int d\alpha \int d\alpha' \rho_{\alpha\alpha'} \langle \alpha'|\alpha \rangle}{\frac{1}{4} \langle P|P \rangle}
\end{equation}

4 In eq. (10) we factor out the product of \( \delta \)-functions for the COM constraints so that the density matrix satisfies the familiar normalization condition \( \text{tr}_\alpha \rho = 1 \), eq. (13).
and use
\[ (\alpha'|\alpha) = (16\pi^3)^3 k_1^+ k_2^- k_3^- \delta(k_1^2 - k_1) \delta(k_2^2 - k_2) \delta(k_3^2 - k_3) , \]
\[ \int d\alpha' \langle \alpha'|\alpha \rangle = \frac{1}{2} (2\pi)^3 \delta(1 - \sum x_i) \delta(\sum \vec{k}_i) . \] (17)

Of course, the above pure density matrix is idempotent:
\[ \int d\alpha' \rho_{\alpha \alpha'} \rho_{\alpha'\beta} = \int d\alpha' \Psi^{\alpha'}_{\text{qqq}}(\alpha') \Psi^{\alpha'}_{\text{qqq}}(\alpha) \Psi^{\beta}_{\text{qqq}}(\beta) \Psi^{\beta}_{\text{qqq}}(\alpha) = \rho_{\alpha\beta} . \] (18)

Reduced density matrices can be constructed by tracing over a subset of the degrees of freedom, for example
\[ \rho_{\xi\xi'} = \text{tr}_{\eta,\vec{q},\vec{Q}} \rho_{\alpha\alpha'} = \int \frac{d\eta}{2\pi(1-\eta)} \frac{d^2q}{16\pi^3} \frac{d^2Q}{16\pi^3} \Psi^{\alpha}_{\text{qqq}}(\xi,\eta,\vec{q},\vec{Q}) \Psi^{\beta}_{\text{qqq}}(\xi,\eta,\vec{q},\vec{Q}) , \] (19)
where \( \alpha = \{\xi,\eta,\vec{q},\vec{Q}\} \) and \( \alpha' = \{\xi',\eta',\vec{q}',\vec{Q}'\} \). This density matrix describes a mixed state (except if the "system" degree of freedom \( \xi \) factorizes from the \( \eta,\vec{q},\vec{Q} \) degrees of freedom of the "environment") since some of the degrees of freedom have been "traced out". Strong entanglement of \( \xi \) with the \( \eta,\vec{q},\vec{Q} \) degrees of freedom will suppress off-diagonal elements of the reduced density matrix \( \rho_{\xi\xi'} \) because the integral (19) will be small when either \( \xi \) or \( \xi' \) is different from its value in the entangled state.

We may also normalize the reduced density matrix as follows:\(^5\)
\[ \tilde{\rho}_{\xi\xi'} = \frac{d\xi}{\sqrt{2\xi(1-\xi) 2\xi'(1-\xi')}} \rho_{\xi\xi'} . \] (20)

With this normalization the sum of eigenvalues \( \lambda_i \) is equal to 1 and we may compute the von Neumann entanglement entropy
\[ S_{\text{vn}} = - \sum \lambda_i \log \lambda_i . \] (21)

We use the natural logarithm, so the entropy is measured in nats rather than in bits.

The purity \( 0 \leq p_{\xi} \leq 1 \) of the reduced density matrix is given by the trace of its square (or by \( \sum_i \lambda_i^2 \)):
\[ p_{\xi} = \int \frac{d\xi}{2\pi(1-\xi)} \int \frac{d\xi'}{2\pi(1-\xi')} \rho_{\xi\xi'} \rho_{\xi'\xi} \]
\[ = \int \frac{d\xi}{2\pi(1-\xi)} \int \frac{d\xi'}{2\pi(1-\xi')} \int \frac{d\eta}{2\pi(1-\eta)} \frac{d^2q}{16\pi^3} \frac{d^2Q}{16\pi^3} \int \frac{d\eta'}{2\pi(1-\eta')} \frac{d^2q'}{16\pi^3} \frac{d^2Q'}{16\pi^3} \Psi^{\alpha}_{\text{qqq}}(\xi,\eta,\vec{q},\vec{Q}) \Psi^{\beta}_{\text{qqq}}(\xi,\eta,\vec{q},\vec{Q}) \Psi^{\alpha}_{\text{qqq}}(\xi,\eta,\vec{q},\vec{Q}) \Psi^{\beta}_{\text{qqq}}(\xi,\eta,\vec{q},\vec{Q}) . \] (22)

Note that if the \( \xi \) degree of freedom factorizes, \( \Psi^{\alpha}_{\text{qqq}}(\xi,\eta,\vec{q},\vec{Q}) = \phi(\xi) \rho(\eta,\vec{q},\vec{Q}) \), then the purity is \( p_{\xi} = 1 \). However, in general such factorization does not occur, e.g. due to the "COM constraint" \( x_1 + x_2 + x_3 = 1, \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \), or due to correlations of longitudinal and transverse quark momenta, so that \( p_{\xi} < 1 \). In the absence of many-body

\(^5\) We employ a grid of pivot points in the \( \xi - \xi' \) plane such that \( d\xi = d\xi' \) and where the value of \( \xi \) at the \( i^{th} \) pivot point is \( i/(N_{\text{piv}} + 1) \), \( i = 1, 2, \ldots, N_{\text{piv}} \). Hence, \( d\xi = 1/(N_{\text{piv}} + 1) \). Also, the Jacobian factor must reduce to \( 1/(2\xi(1-\xi)) \) on the \( \xi' = \xi \) diagonal, and it must factorize into a function of \( \xi \) times that same function of \( \xi' \), which fixes it to be \( 1/\sqrt{2\xi(1-\xi) 2\xi'(1-\xi')} \).
correlations encoded in the Hamiltonian we expect that due to the COM constraint alone the impurity of the reduced density matrix is of order

\[ 1 - \text{tr} \rho^2 = O(N_c^{-1}) \]  

in the limit of many colors, \( N_c \to \infty \).

We can also integrate out the longitudinal degrees of freedom to construct reduced density matrices over either \( \vec{q} \) or \( \vec{Q} \). For example,

\[ \rho_{q,q'} = \text{tr}_{q',\xi,q,\vec{q}} \rho_{\alpha \alpha'} = \int \frac{dq_y}{16\pi^3} \frac{d\xi}{2\pi(1-\xi)} \frac{d\eta}{2\eta(1-\eta)} \frac{d^2Q}{16\pi^3} \Psi^*_qqq(\xi,\eta,\vec{q},\vec{Q}) \Psi_qqq(\xi,\eta,\vec{q}',\vec{Q}) . \]  

Here, too, in order to obtain a dimensionless density matrix with properly normalized eigenvalues we should rescale as follows:

\[ \tilde{\rho}_{q,q'} = dq_x \rho_{q,q'} . \]  

A. Numerical estimates for the three quark density matrix

For numerical estimates we employ a simple model for the three quark wave function \( \Psi_qqq \) due to Schlumpf and Brodsky [24, 25]

\[ \Psi_qqq \left( \left\{ x_i, \vec{k}_i \right\} \right) = N_{\text{HO}} \sqrt{x_1x_2x_3} e^{-\mathcal{M}^2/2\beta^2} \]  

where \( \mathcal{M}^2 = \sum \frac{k_i^2 + m_q^2}{x_i} \) is the invariant mass squared of the non-interacting three-quark system [36]. The normalization of this “harmonic oscillator” wave function follows from eq. (6). The non-perturbative parameters \( m_q = 0.26 \text{ GeV} \) and \( \beta = 0.55 \text{ GeV} \) have been tuned in Ref. [25] to the electromagnetic radius, \( R_p = 0.76 \text{ fm} \), the magnetic moments of proton and neutron, \( \mu_{p,n} = 2.81/-1.66 \), and the axial vector coupling \( g_A = 1.25 \).

The quoted references also present a power-law wave function

\[ \Psi_qqq \left( \left\{ x_i, \vec{k}_i \right\} \right) = N_{\text{PWR}} \sqrt{x_1x_2x_3} \left( 1 + \frac{\mathcal{M}^2}{\beta^2} \right)^{-p} . \]  

The corresponding parameters for this wave function (28) are \( p = 3.5, m = 0.263 \text{ GeV}, \beta = 0.607 \text{ GeV} \) [25]. Note that this wave function does not factorize into a product of one quark wave functions, not even in the absence of the COM constraint.

The only dimensional parameters in the above wave functions are \( \beta^2 \) and \( m_q^2 \). The elements of the normalized density matrix are dimensionless, so they will only involve the ratio \( m_q^2/\beta^2 \). This quantity could also be expressed in terms of the product of quark mass and proton radius, or as proton mass times radius \( \beta^2 \) squared.

The Brodsky-Schlumpf light-front model wave function exhibits reasonable behavior which is consistent with the empirical knowledge of the structure of the proton at light-cone momentum fractions \( x \sim 0.1 - 0.5 \). Nevertheless, it would clearly be interesting in the future to compare to density matrices obtained from three-quark wave functions \( \Psi_qqq \) which represent solutions of a LF Hamiltonian with interactions [37, 38]. Also, light-front wave functions at moderate \( x \) may become available from lattice QCD via a large momentum expansion of equal-time Euclidean correlation functions in instant quantization [39–41]. Last but not least, the future electron-ion collider EIC will provide valuable observational constraints on the light-front wave functions [42, 43].

Transforming to unconstrained internal degrees of freedom [36] we have that \( x_1x_2x_3 = \eta^2 (1-\eta) \xi (1-\xi) \) and

\[ \mathcal{M}^2 = \frac{Q^2}{\eta(1-\eta)} + \frac{m_q^2}{1-\eta} + \frac{q^2 + m_q^2}{\eta \xi (1-\xi)} . \]  

In fig. 2 we visualize the LO density matrix \( \rho_{\xi\xi'} \) for the HO wave function. It is clear from the figure that off-diagonal matrix elements are not strongly suppressed and that this density matrix represents a nearly separable state. Indeed,
we obtain that the purity is \( p_\xi \approx 0.98 \), and that the entanglement entropy is low, \( S_{vN} \approx 0.06 \); analogous values for other degrees of freedom and wave functions are listed in table I. This indicates that the current model wave function produces only weak entanglement of spatial degrees of freedom. In the limit \( N_c \to \infty \), with \( g^2 N_c \) fixed, the spatial wave function of the proton factorizes into \( N_c \) valence quark wave functions which are determined by a mean field [26], where spatial degrees of freedom belonging to different quarks would not be entangled.

| d.o.f. | w.f. | purity \( p_\xi \) | \( S_{vN} \) |
|-------|------|-------------------|------------------|
| \( \xi \) | HO | 0.983 | 0.052 |
| \( \xi \) | PWR | 0.992 | 0.029 |
| \( \eta \) | HO | 0.946 | 0.14 |
| \( \eta \) | PWR | 0.962 | 0.10 |
| \( Q_x \) | HO | 0.985 | 0.046 |
| \( Q_x \) | PWR | 0.980 | 0.058 |
| \( q_x \) | HO | 0.985 | 0.046 |
| \( q_x \) | PWR | 0.980 | 0.058 |

TABLE I: Numerical results for the purity and von-Neumann entropy (in nats) of the reduced density matrices over the degree of freedom specified in the first column. The three-quark model wave functions are given in eqs. (27, 28), respectively [24, 25]. The numerical uncertainty is estimated at a few units on the last quoted digit.

**B. Color and spin wave function**

We wrote the symmetric spatial wave function of the proton in eq. (1). That should be multiplied by the color, and (flavor-)spin wave functions. Let us first restore the anti-symmetric color space wave function, so now the proton state \( |P \rangle = |P^+, P = 0 \rangle \) is written as

\[
|P \rangle = \int [dx_i] \int [d^2 k_i] \Psi_{qqq} \left( \{ x_i, k_i \} \right) \frac{1}{\sqrt{6}} \sum_{j_1,j_2,j_3} \epsilon_{j_1,j_2,j_3} \left| \{ x_i, k_i, j_i \} \right>,
\]

where \( j_1, j_2, j_3 = 1 \ldots 3 \) denote the colors of the quarks.
The three-quark state vectors now include labels for the colors of the quarks, \( \alpha \equiv \{ x_i, \bar{c}_i, n_i \} \), and eq. (11) becomes

\[
\langle \alpha | P \rangle = \frac{1}{\sqrt{6}} \epsilon_{n_1 n_2 n_3} \Psi_{qqq} \left( \left\{ x_i, \bar{c}_i \right\} \right) (2\pi)^3 \delta \left( 1 - \sum_i x_i \right) \delta \left( \sum_i \bar{c}_i \right),
\]

Note that \( \Psi_{qqq} \left( \left\{ x_i, \bar{c}_i \right\} \right) \) is invariant under rotations in color space, therefore it does not depend on the quark colors \( n_i \).

The pure state density matrix \( \rho_{\alpha \alpha'} \) from eqs. (10,12) now reads

\[
\rho_{\alpha \alpha'} = \frac{1}{6} \epsilon_{n_1 n_2 n_3} \epsilon_{n'_1 n'_2 n'_3} \Psi_{qqq}^* \left( \left\{ x'_i, \bar{c}'_i \right\} \right) \Psi_{qqq} \left( \left\{ x_i, \bar{c}_i \right\} \right).
\]

Evidently, this factorizes into a density matrix over color space times one over momentum space because so does the pure state (30) from which it has been constructed.

When tracing over the degrees of freedom of the “environment”, we may now also sum over some or all of the colors using

\[
\text{tr}_{n_3} \equiv \frac{1}{6} \sum_{n_3} \epsilon_{n_1 n_2 n_3} \epsilon_{n'_1 n'_2 n'_3} = \frac{1}{6} \left( \delta_{n_1 n'_1} \delta_{n_2 n'_2} - \delta_{n_1 n'_2} \delta_{n_2 n'_1} \right),
\]

\[
\text{tr}_{n_2,n_3} \equiv \frac{1}{6} \sum_{n_2,n_3} \epsilon_{n_1 n_2 n_3} \epsilon_{n'_1 n'_2 n'_3} = \frac{1}{3} \delta_{n_1 n'_1},
\]

\[
\text{tr}_{n_1,n_2,n_3} \equiv \frac{1}{6} \sum_{n_1,n_2,n_3} \epsilon_{n_1 n_2 n_3} \epsilon_{n_1'n_2'n_3} = 1.
\]

Hence, the reduced density matrix shown in previous sections should be understood as the density matrix obtained after tracing out all quark colors.

On the other hand, if we trace out all \textit{but} one color degree of freedom then

\[
\rho_{nn'} = \frac{1}{3} \delta_{nn'} \int \frac{d\xi}{2\xi(1-\xi)} \frac{d\eta}{2\eta(1-\eta)} \frac{d^2 q}{16\pi^3} \frac{d^2 Q}{16\pi^3} \left| \Psi_{qqq}(\xi,\eta,\bar{q},\bar{Q}) \right|^2 = \frac{1}{3} \delta_{nn'}.
\]

Note that all off-diagonal matrix elements are zero and that this density matrix is clearly not a separable state. Its purity is \( \sum_{n,n'} \rho_{nn'} \rho_{n'n} = \frac{1}{3} \), i.e. the inverse of the dimension of the fundamental representation of SU(3), which reflects the entanglement of color. In fact, entanglement is maximal as all eigenvalues of this reduced density matrix are equal (its spectrum is degenerate). For general \( N_c \) we have \( \rho_{nn'} = \frac{1}{N_c} \delta_{nn'} \), \( \text{tr} \rho^2 = \frac{1}{N_c} \), and \( S_{\text{CV}} = \log N_c \).

Lastly, we also multiply by the flavor-spin wave function. The quark-gluon vertex is flavor independent and we will always trace out flavor degrees of freedom. Hence we write

\[
|P\rangle = \int [dx_i] \int [d^2 k_i] \Psi_{qqq} \left( \left\{ x_i, \bar{c}_i \right\} \right) \frac{1}{\sqrt{6}} \sum_{j_1 j_2 j_3} \epsilon_{j_1 j_2 j_3} \left| \left\{ x_i, \bar{c}_i, j_i \right\} \right| |S\rangle,
\]

with

\[
|S\rangle = \frac{1}{\sqrt{12}} \left( 2 |\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle + \langle \uparrow \leftrightarrow \downarrow \rangle \right), \quad \langle S | S \rangle = 1.
\]

Once again, the LO reduced density matrix shown in sec. II should be understood as the density matrix obtained after tracing over all quark helicities. We refer the reader to ref. [44] for an analysis of entanglement of valence and sea spin in the proton, and its relation to chiral symmetry breaking.

C. Violation of Bell-CHSH inequality

In this section we show how color correlations described by the density operator

\[
\hat{\rho} = \frac{1}{6} \epsilon_{i_1 i_2 i_3} \epsilon_{i_1' i_2' i_3'} |i_1, i_2, i_3\rangle \langle i_1', i_2', i_3'|,
\]

(37)
violates a Bell-CHSH [45–47] inequality, indicating that some color correlations are “stronger than classically possible”. We consider here the simplest case, a bipartite subsystem of two quarks and “measurements” within a SU(2) subalgebra of color-SU(3). This maps onto the standard system of two qubits.

Consider the Bell-CHSH operator

\[ C_{\text{CHSH}} = A_1(B_1 + B_2) + A_2(B_1 - B_2) \]  

where the \( A_i \) represent the results of two measurements on one part of the system and \( B_i \) the results of independent measurements on another part. The expectation value of \( C_{\text{CHSH}} \) describes the correlation of these measurements:

\[ \langle C_{\text{CHSH}} \rangle = \text{tr} C_{\text{CHSH}} \hat{\rho} = \frac{1}{6} \sum \epsilon_{i_1 i_2 i_3} \epsilon_{i_1' i_2' i_3'} \langle i_1' i_2' i_3' | C_{\text{CHSH}} | i_1 i_2 i_3 \rangle . \]

The summation in the previous expression is over the quark colors \( i_1, i_2, i_3, i_1', i_2', i_3' \).

The measurement operators \( A_i, B_i \) act in color subspace 1 and 2, respectively, and we construct them from the generators of the first SU(2) subalgebra of color-SU(3), i.e. the first three Gell-Mann matrices \( \lambda_1 = \sigma_1 \oplus 0, \lambda_2 = \sigma_2 \oplus 0, \lambda_3 = \sigma_3 \oplus 0 \). Similarly, we introduce the identity corresponding to that subalgebra, \( I = I_{2 \times 2} \oplus 0 \), as well as \( I_3 = 1 \oplus 1 \oplus 1 \). Hence, our self adjoint measurement operators are \( A_i \otimes I \otimes I_3 \) and \( I \otimes B_i \otimes I_3 \), and

\[ \langle C_{\text{CHSH}} \rangle = \frac{1}{6} \sum \epsilon_{i_1 i_2 i_3} \epsilon_{i_1' i_2' i_3'} [\langle i_1' | A_1 | i_1 \rangle \langle i_2' | B_1 + B_2 | i_2 \rangle + \langle i_1' | A_2 | i_1 \rangle \langle i_2' | B_1 - B_2 | i_2 \rangle] . \]

To obtain a bound on the classical correlation we replace each of the \( A_i \) and \( B_i \) by the identity \( I \) times one of the eigenvalues of \( \sigma_1, \sigma_2, \sigma_3 \) which are +1 or −1. Therefore, these classical measurements commute. For any combination of eigenvalues \( C_{\text{CHSH}} \) either takes the value \( \frac{2}{3} \) or \( -\frac{2}{3} \). Hence, for any classical probability distribution of eigenvalues, i.e. measurement outcomes, we have that

\[ -\frac{2}{3} \leq \langle C_{\text{CHSH}} \rangle_{cl} \leq \frac{2}{3} . \]

The quantum mechanical expectation value of \( C_{\text{CHSH}} \) violates this bound for certain operators \( A_i, B_i \). As an example, for \( A_1 = \lambda_1, A_2 = \lambda_3, B_1 = −(\lambda_1 + \lambda_3)/\sqrt{2}, B_2 = (\lambda_3 - \lambda_1)/\sqrt{2} \) we obtain maximal violation, \( \langle C_{\text{CHSH}} \rangle_q = \frac{2\sqrt{2}}{3} \):

\[ -\frac{2\sqrt{2}}{3} \leq \langle C_{\text{CHSH}} \rangle_q \leq \frac{2\sqrt{2}}{3} . \]

The violation of Bell-CHSH inequalities indicates that beyond a classical approximation, an accurate description of color charge correlations in the proton at large and moderate \( x \) requires accounting for entanglement and quantum correlations, (see, also, refs. [22, 28]). In the future, we hope to apply our approach to improve on classical models of color charge fluctuations in the proton at moderately small \( x \) [48–55], and to study the sensitivity of specific observables to quantum color correlations.

### III. DENSITY MATRIX FOR THE THREE QUARK AND ONE GLUON STATE AT \( \mathcal{O}(g^2) \)

In this section we consider the emission of a gluon from one of the quarks. These corrections give density matrices over the Hilbert space of three quarks and a gluon. They generate a \( |qqqg\rangle \) Fock state component in the proton, and the corresponding density matrix \( \rho_{qqqg} \).

We need to also consider virtual corrections (see fig. 4) due to the exchange of a gluon within \(|P⟩\) or \(⟨P|\). The next-to-leading order corrections do not affect \(⟨P|P⟩\) (see III.D in ref. [21]). Hence, when we trace \( ρ \) over all degrees of freedom, the contributions from real emissions and virtual corrections must cancel to restore \( \text{tr} \ ρ = \text{tr} \rho_{qqq}^{\text{LO}} = 1 \). We will check this explicitly below.

\( ^{6} \sigma_i \) are the Pauli matrices and \( \oplus \) denotes the matrix direct sum \( A \oplus B = \text{diag}(A, B) \).
A. The density matrix for the four-parton (qqgg) state

To compute the corrections to the density matrix at order $g^2$ we begin with the emission of a gluon from the first quark\(^7\), i.e., from the quark with momentum $k_1$, color $i_1$, and helicity $h_1$: \[
|P^+, \vec{P} = 0\rangle_{\mathcal{O}(g)} = \int [dx_i] \int [d^2k_i] \Psi_{qqg}(k_i) \frac{1}{\sqrt{6}} \sum_{j_1,j_2,j_3} \epsilon_{j_1,j_2,j_3} 2g \sum_{\sigma ma} (t^a)_{mj_1} \int_{x_1}^{x_3} \frac{dx_g}{x_g} \frac{d^2k_g}{16\pi^3} \frac{1}{2(x_1 - x_g)} \sum_{h_1,h_2} \frac{1}{P^+} \hat{\psi}_{q \rightarrow qg}^{(\sigma h_1)}(k_1; k_1 - k_g, k_g) |m, k_1 - k_g, h; j_2, k_2, h_2; j_3, k_3, h_3\rangle \langle h_1, h_2, h_3|S \otimes |a, k_g, \sigma\rangle .
\] (43)

Here, a mother quark $\{j_1, k_1, h_1\}$ splits into a daughter quark $\{m, k_1 - k_g, h\}$ and a gluon $\{a, k_g, \sigma\}$, producing a $qqgg$ Fock state in the proton; $x$ is a cutoff on the light-cone momentum fraction of the gluon required by the soft singularity in QCD.

The light-cone gauge Fock space amplitude for the $qg$ state of the quark in $D = 4$ dimensions is \[\hat{\psi}_{q \rightarrow qg}(p; k_q, k_g) = \frac{p^+ \sqrt{1 - z}}{n^2 + \Delta^2} [(2 - z) \hat{n} \cdot \hat{\epsilon}_\sigma + iz h \hat{n} \times \hat{\epsilon}_\sigma] \delta_{hh_1} \, ,\] (44)

where $z = k_g^+/p^+$, $k_g^+/p^+ = 1 - z$, $\hat{n} = \vec{k}_g - z\vec{p}$, and $\Delta^2 = z^2 m_{\text{col}}^2$ is a “quark mass” regulator (in the light-cone energy denominator) for the collinear DGLAP singularity; we will sometimes take $m_{\text{col}}^2 \rightarrow 0$ when possible. The cross product in the second term is taken in two transverse dimensions, $\hat{\alpha} \times \hat{\beta} = \epsilon^{ij} a^i b^j$. Also, since we take the mass of the quarks to zero we assume that their helicity is conserved; therefore, at times we will drop the superscript $h$ (for the helicity of the daughter quark) on $\hat{\psi}$. There are two more analogous contributions on the r.h.s. of eq. (43) corresponding to gluon emission from quark 2 or quark 3, respectively.

Proceeding, we compute the overlap with a prescribed 3-quark, 1-gluon state. In order to completely characterize such a state we need to also keep track of which quark $j$ the gluon was emitted from, we do this explicitly by a superscript: \[\alpha^{(j)}_{qqgg} = \{\{n_1, k_1, h_1\}, a, k_g, \sigma\} \, .\] (45)

By analogy to eq. (43) we write the corresponding state vector as \[|\alpha^{(j)}\rangle = |n_1, k_1 - \delta_{j1} k_g, h_1; n_2, k_2 - \delta_{j2} k_g, h_2; n_3, k_3 - \delta_{j3} k_g, h_3\rangle \otimes |a, k_g, \sigma\rangle \] (46)

We then obtain: \[\langle \alpha^{(j)}_{qqgg} | P \rangle = \frac{\theta}{\sqrt{6}} (2\pi)^3 \delta \left(1 - \sum_i x_i\right) \delta \left(\sum_i \hat{k}_i\right) \langle h_1, h_2, h_3|S \rangle \sum_m \left[ \delta_{j1} \epsilon_{mn_2n_3} (t^a)_{nm} \frac{1}{k_1^+} \hat{\psi}_{q \rightarrow qg}^{(\sigma h_1)}(k_1; k_1 - k_g, k_g) + \delta_{j2} \epsilon_{n_1mn} (t^a)_{nm} \frac{1}{k_2^+} \hat{\psi}_{q \rightarrow qg}^{(\sigma h_2)}(k_2; k_2 - k_g, k_g) + \delta_{j3} \epsilon_{n_1n_2m} (t^a)_{nm} \frac{1}{k_3^+} \hat{\psi}_{q \rightarrow qg}^{(\sigma h_3)}(k_3; k_3 - k_g, k_g) \right] \Psi_{qqg}(k_1; k_2; k_3) \, .\] (47)

It is clear from this expression that $k_1, k_2, k_3$ denote the momenta of the parent quarks so that their longitudinal (transverse) momenta add to $P^+$ (zero). Also, in each term there is a $\Theta$-function which ensures that the LC momentum $(x_3 - x_g)P^+$ of the daughter quark is positive; we do not write it explicitly.

\(^7\) Recall that three momenta are given by $k_1 = (x_1 P^+, \vec{k}_1), k_g = (x_g P^+, \vec{k}_g)$ and so on. From here onward we consider a proton with vanishing transverse momentum to make the following expressions more compact.
Like in eq. (10), the density matrix describing 3q+1g states is given by the direct product of the previous expression, as represented in diagram 3; modulo the respective COM constraints:

\[ \rho^{(jj')}_{\alpha \alpha'} (2\pi^3) \delta \left( 1 - \sum_i x_i \right) \delta \left( \sum_i \vec{k}_i \right) \delta \left( 1 - \sum_i x_i' \right) \delta \left( \sum_i \vec{k}_i' \right) = \langle \alpha^{(j)} | P \rangle \langle P | \alpha'^{(j')} \rangle . \]  

(48)

Hence, the density matrix for the \(|qqqg\rangle\) state is

\[ \rho^{(jj')}_{\alpha \alpha'} = \frac{g^2}{6} \langle h_1, h_2, h_3 | S \rangle \langle S | h_1', h_2', h_3' \rangle \sum_{mm'} \left[ \delta_{j'1} \epsilon_{m'n_2'n_3'}(t^{a'}) \epsilon^{*}_{m'n_1'}(t^{a}) \frac{1}{k_{1}^2} \tilde{\psi}_{q \rightarrow qg}(k_1' - k_g, k_g) \\
\quad + \delta_{j'2} \epsilon_{m'n_2'n_3'}(t^{a'}) \epsilon^{*}_{m'n_1'}(t^{a}) \frac{1}{k_{2}^2} \tilde{\psi}_{q \rightarrow qg}(k_2' - k_g, k_g) \\
\quad + \delta_{j'3} \epsilon_{m'n_2'n_3'}(t^{a'}) \epsilon^{*}_{m'n_1'}(t^{a}) \frac{1}{k_{3}^2} \tilde{\psi}_{q \rightarrow qg}(k_3' - k_g, k_g) \right] \\
\times \left[ \delta_{j1} \epsilon_{mn_2n_3}(t^{a}) \frac{1}{k_{1}^2} \tilde{\psi}_{q \rightarrow qg}(k_1 - k_g, k_g) \\
\quad + \delta_{j2} \epsilon_{n_1mn_3}(t^{a}) \frac{1}{k_{2}^2} \tilde{\psi}_{q \rightarrow qg}(k_2 - k_g, k_g) \\
\quad + \delta_{j3} \epsilon_{n_1mn_3}(t^{a}) \frac{1}{k_{3}^2} \tilde{\psi}_{q \rightarrow qg}(k_3 - k_g, k_g) \right] \Psi^{*}_{qqq}(k_1'; k_2'; k_2') \Psi_{qqq}(k_1; k_2; k_3) . \]  

(49)

Let us consider first the term where the gluon emission occurs from quarks 1 and 1’, respectively, i.e. \( j = j' = 1 \). To make the following expressions more compact we will trace right away over quark helicities and gluon polarization, so we first compute

\[ \sum_{h_1, h_2, h_3, \sigma} \left| \langle h_1, h_2, h_3 | S \rangle \right|^2 \frac{1}{k_{1}^2} \frac{1}{k_{1}^2} \tilde{\psi}_{q \rightarrow qg}^{*}(k_1'; k_g', k_g) \tilde{\psi}_{q \rightarrow qg}(k_1; k_1 - k_g, k_g) \]

\[ = 2\vec{n} \cdot \vec{n}' \sqrt{(1 - z})(1 - z') \left( \frac{1}{n^2 + \Delta^2} \right) \left( 2 - z + z' + zz' \right) , \]  

where \( z = x_g/x_1, \ z' = x_g'/x_1', \ \vec{n} = \vec{k}_g - z\vec{k}_1, \ \vec{n}' = \vec{k}_g' - z'\vec{k}_1, \ \Delta^2 = z^2m^2_{\text{col}}, \ \Delta'^2 = z'^2m^2_{\text{col}}. \) With this, the first contribution to the density matrix for the \(|qqqg\rangle\) state becomes

\[ \rho^{(11')}_{\alpha \alpha'} = g^2 \sum_{mm'} \epsilon_{m'n_2'n_3'}(t^{a'}) \epsilon_{mn_2n_3}(t^{a}) \frac{1}{k_{1}^2} \frac{1}{k_{1}^2} \tilde{\psi}_{q \rightarrow qg}^{*}(k_1'; k_g', k_g) \tilde{\psi}_{q \rightarrow qg}(k_1; k_1 - k_g, k_g) \]

\[ \times \left[ \delta_{j1} \epsilon_{mn_2n_3}(t^{a}) \frac{1}{k_{1}^2} \tilde{\psi}_{q \rightarrow qg}(k_1 - k_g, k_g) \\
\quad + \delta_{j2} \epsilon_{n_1mn_3}(t^{a}) \frac{1}{k_{2}^2} \tilde{\psi}_{q \rightarrow qg}(k_2 - k_g, k_g) \\
\quad + \delta_{j3} \epsilon_{n_1mn_3}(t^{a}) \frac{1}{k_{3}^2} \tilde{\psi}_{q \rightarrow qg}(k_3 - k_g, k_g) \right] \Psi^{*}_{qqq}(k_1'; k_2'; k_2') \Psi_{qqq}(k_1; k_2; k_3) . \]  

(50)
\[ \Theta(x_1 - x_g) \Theta(x_1' - x_g') \vec{n} \cdot \vec{n}' \frac{\sqrt{(1 - z)(1 - z')}}{(n^2 + \Delta^2)(n'^2 + \Delta'^2)} (2 - z - z' + zz') . \] (51)

(Here, the matrix indices \(\alpha, \alpha'\) exclude quark helicities and gluon polarization.) There are two more analogous contributions corresponding to gluon emission from quarks 2,2' or from quarks 3,3', respectively.

Now we derive the density matrix for the case where the gluon in \(|P\rangle\) is emitted by quark 1 while that in \(|P\rangle\) is emitted by quark 2. Using
\[
\sum_{h_1,h_2,h_3} |\langle h_1,h_2,h_3|S\rangle|^2 h_1 h_2 = \frac{1}{3} ,
\] (52)
\[
\sum_{h_1,h_2,h_3} |\langle h_1,h_2,h_3|S\rangle|^2 h_1 h_3 = -\frac{2}{3} ,
\] (53)
\[
\sum_{h_1,h_2,h_3} |\langle h_1,h_2,h_3|S\rangle|^2 h_2 h_3 = -\frac{2}{3} ,
\] (54)
we can use the following expression to trace over quark helicities and gluon polarization:
\[
\sum_{h_1,h_2,h_3,\sigma} |\langle h_1,h_2,h_3|S\rangle|^2 \left( \frac{1}{k_1' k_2'} \tilde{\psi}_{q\rightarrow qg}^{(\sigma)}(k_2';k'_2 - k'_g, k'_g) \tilde{\psi}_{q'\rightarrow qg}^{(\sigma)}(k_1;k_1 - k_g, k_g) \right)^2 \frac{\sqrt{(1 - z)(1 - z')}}{(n^2 + \Delta^2)(n'^2 + \Delta'^2)} (2 - z - z' + \frac{1}{2}zz'(1 + \langle h_1 h_2 \rangle)) ,
\] (55)
where \(z = x_g / x_1\), \(z' = x'_g / x'_1\), \(\vec{n} = \vec{k}_g - z\vec{k}_1\), \(\vec{n}' = \vec{k}_g - z'\vec{k}_2\), \(\Delta^2 = z^2 m_{\text{col}}^2\), \(\Delta'^2 = z'^2 m_{\text{col}}^2\). Then,
\[
\rho_{\alpha\alpha'}^{(12')} = \frac{g^2}{3} \sum_{m,m'} \epsilon_{m_1'm_2'}(\vec{k}_1') \epsilon_{m_2n_3}^{(\sigma)}(\vec{k}_2') \epsilon_{m_3n_2}^{(\sigma)}(\vec{k}_3') \Psi_{q'qg}^{(\sigma)}(k_1';k_2';k_3') \Psi_{q'qg}^{(\sigma)}(k_2;k_2;k_3)
\]
\[
\Theta(x_1 - x_g) \Theta(x_1' - x_g') \vec{n} \cdot \vec{n}' \frac{\sqrt{(1 - z)(1 - z')}}{(n^2 + \Delta^2)(n'^2 + \Delta'^2)} (2 - z - z' + \frac{1}{2}zz'(1 + \langle h_1 h_2 \rangle)) .
\] (56)
There are five more analogous contributions corresponding to emission from quark pairs (13'), (21'), (23'), (31'), (32') respectively.

**B. Trace of \(\rho_{qqgg}\)**

As a first step we need to determine the integration measure over the gluon (spatial) degrees of freedom. With
\[
\langle \alpha'^{(1')}|\alpha^{(1)}\rangle = \langle k_1' - k_g'; k_2'; k_3'|k_1 - k_g; k_2; k_3 \rangle \langle k_g'|k_g \rangle
\]
\[
= (16\pi^3)^4 k_g^+ \delta(k_g' - k_g) (k_1'^+ - k_g') \delta(k_1' - k_g' - k_1 + k_g) k_2'^+ \delta(k_2' - k_2) k_3'^+ \delta(k_3' - k_3)
\] (57)
\[
= (16\pi^3)^4 k_g^+ \delta(k_g' - k_g) (k_1'^+ - k_g') \delta(k_1' - k_g' - k_1 + k_g) k_2'^+ \delta(k_2' - k_2) k_3'^+ \delta(k_3' - k_3)
\] (58)
we define
\[
\frac{d\alpha'^{(1')}}{2} = \frac{dx'_1 dx'_2 dx'_3}{8 x'_1 x'_2 x'_3} \delta(1 - \sum x'_i) \frac{d^2 k'_1 d^2 k'_2 d^2 k'_3}{(2\pi)^6} \frac{\delta(\sum \vec{k}_i')}{16\pi^3} \frac{x'_1}{x'_1 - x_g}
\] (59)
and
\[
\frac{d\alpha^{(1)}}{2} = \frac{dx_1 dx_2 dx_3}{8 x_1 x_2 x_3} \delta(1 - \sum x_i) \frac{d^2 k_1 d^2 k_2 d^2 k_3}{(2\pi)^6} \frac{\delta(\sum \vec{k}_i)}{16\pi^3} \frac{x_1}{x_1 - x_g}
\] (60)
so that
\[
\int d\alpha'^{(1')} \langle \alpha'^{(1')}|\alpha^{(1)}\rangle = \frac{1}{2} (2\pi)^3 \delta(1 - \sum x_i) \delta(\sum \vec{k}_i) ,
\] (61)
\[
\int \Delta \alpha (1) \left( \alpha (1') \big| \alpha (1) \right) = \frac{1}{2} \left( 2\pi \right)^3 \delta (1 - \sum x'_i) \delta (\sum \bar{k}'_i)
\]

which is analogous to eq. (17).

Similarly, with
\[
\left( \alpha (1') | \alpha (1) \right) = \left( k'_1; k'_2 - k'_3 | k_1 - k_2; k_3 \right) \langle k_g | k_g \rangle = \left( 16\pi^3 \right)^4 \frac{\bar{k}'_j}{\bar{k}'_j} \delta (k'_g - k_g) k_1^+ \delta (k'_1 - k_1 + k_g) k_2^+ \delta (k'_2 - k_2 - k_g) k_3^+ \delta (k'_3 - k_3)
\]

and
\[
\frac{d\alpha (1')}{d\alpha (1')} = \frac{1}{2} \frac{d^2 k'_1}{(2\pi)^3} \frac{d^2 k'_2}{(2\pi)^3} \frac{d^2 k'_3}{(2\pi)^3} \delta (1 - \sum x'_i) \delta (\sum \bar{k}'_i),
\]

we obtain
\[
\int \Delta \alpha (1) \left( \alpha (1') | \alpha (1) \right) = \frac{1}{2} \left( 2\pi \right)^3 \delta (1 - \sum x'_i) \delta (\sum \bar{k}'_i),
\]

\[
\int \Delta \alpha (1) \left( \alpha (1') | \alpha (1) \right) = \frac{1}{2} \left( 2\pi \right)^3 \delta (1 - \sum x'_i) \delta (\sum \bar{k}'_i).
\]

The trace over \( \rho^{(ij)} \) is again given by the expression in eq. (15) on the left,
\[
\frac{1}{3 \langle s | P | P \rangle} \int d\alpha \int d\alpha' \left( \alpha (ij) | \alpha (ij) \right) \rho^{(ij)} = \int d\alpha \rho^{(ij)}.
\]

The expression on the left is analogous to the trace of a matrix, \( \sum_{i,j} M_{ij} \langle \epsilon | \epsilon \rangle \) where \( \langle \epsilon | \epsilon \rangle = \delta_{ij} \). On the r.h.s. of the above, in \( \rho^{(ij)} \), the primed quark and gluon momenta have to be expressed in terms of the unprimed momenta as determined by the \( \delta \)-functions in eqs. (58, 63), respectively.

We first trace eq. (51) over the quark degrees of freedom by summing over their colors, and integrating over their longitudinal and transverse momenta. This leads to the reduced gluon density matrix
\[
\rho^{(j'=j)} = 2g^2 \text{tr} t^a t^{a'} \frac{1}{2} \int [d^2 k] \Theta (x_1 - x_g) \Theta (x_1 - x'_g) \Psi^*_{qq} (k_1 - k_g) \Psi_{qq} (k_1; k_2; k_3)
\]
\[
\frac{\bar{n} \cdot \bar{n}' \sqrt{(1 - z)(1 - z')}}{(n^2 + \Delta^2)(n'^2 + \Delta'^2)} (2 - z - z' + zz'),
\]

where now \( z = x_g / x_1 \), \( z' = x'_g / (x_1 - x_g + x'_g) \), \( \bar{n} = \bar{k}_g - z \bar{k}_1 \), \( \bar{n}' = \bar{k}_g - z' \bar{k}_1 \). \( \Delta^2 = z^2 m_{q}^2 \), \( \Delta'^2 = z'^2 m_{q}^2 \).

The indices \( \alpha \) now refer exclusively to the longitudinal and transverse momentum of the gluon. This expression includes a factor of 3 to account for the diagrams where the gluon emission occurs from quarks 2,2' or 3,3', respectively (this uses the symmetry of the three quark wave function under exchange of any two quarks).

As a final step, we trace out the gluon by summing over \( a' = a \) and integrating over
\[
\int \frac{dx_g}{16\pi^3} \int \frac{d^2 k_g}{(1 - x_g)}.
\]

Here we encounter a UV divergent integral over the transverse momentum \( \bar{k}_g \) of the gluon. To regularize this quantity we subtract a UV contribution which we shall add back to the \( \mathcal{O} (g^2) \) virtual correction to \( \rho^{(ij)} \) in the following sec. III C. It is given by the \( \Lambda \)-dependent part of the integrand of the \( \mathcal{O} (g^2) \) contribution to the wave function renormalization factor
\[
\sqrt{Z_q (x_1) Z_q (x'_1) Z_q (x_2) Z_q (x'_2) Z_q (x_3) Z_q (x'_3)} - 1 =
\]
\[
- \frac{1}{2} C_q (x_1; \Lambda) - \frac{1}{2} C_q (x'_1; \Lambda) - \frac{1}{2} C_q (x_2; \Lambda) - \frac{1}{2} C_q (x'_2; \Lambda) - \frac{1}{2} C_q (x_3; \Lambda) - \frac{1}{2} C_q (x'_3; \Lambda) - \frac{1}{2} C_q (x_3; \Lambda).
\]

(62)
which is written explicitly in eqs. (80, 82). This gives the regularized trace

\[
\text{tr } \rho^{(j')} = 8g^2 \frac{1}{2} \int [dx_g] \int [d^2 k_g] \frac{dx_g}{x_g} \left| \Psi_{qqq}(k_1; k_2; k_3) \right|^2 \left[ 1 + \left( 1 - \frac{x_g}{x_1} \right)^2 \right] \int \frac{d^2 k_g}{16\pi^3} \left[ \frac{1}{k_g^2 + \Delta^2} - \frac{1}{k_g^2 + \Lambda^2} \right]
\]

where \( \Lambda = (x_g/x_1) M_{\text{UV}} \) while \( \Delta = (x_g/x_1) m_{\text{col}} \). If the integral over \( x_g \) is dominated by \( x_g \), much less than typical quark light-cone momentum fractions then we can replace the upper limit by \( \langle x_q \rangle \) and use the normalization condition (6) for the three-quark wave function to simplify further:

\[
\simeq \frac{g^2}{\pi^2} \log \left( \frac{x_q}{x} \right) \log \frac{M_{\text{UV}}^2}{m_{\text{col}}^2}.
\]

The previous expressions exhibit a dependence on the IR cutoffs, \( x \) for the soft singularity and \( m_{\text{col}} \) for the collinear singularity, and on the UV regulator \( M_{\text{UV}} \) in \( D = 4 \) dimensions. A dependence of the entanglement entropy on the logarithm of the UV cutoff has also been found in ref. [7].

Similarly, to trace eq. (56) over quark degrees of freedom we set \( k_1 - k_g = k_1', k_2 = k_2' - k_g', k_3 = k_3, n_i' = n_i \):

\[
\rho^{(12')}_{\alpha\alpha'} = -\frac{g^2}{3} \int [dx_g] \int [d^2 k_i] \left( \frac{n_i \cdot n_i'}{n_i^2 + \Delta^2} \right) \left( \frac{1 - z - z' + \frac{1}{2} zz'}{n_i^2 + \Delta^2} \right)
\]

\[
\times \Theta(x_1 - x_g) \Theta(1 - (x_2 + x_g')) \Psi_{qqq}(k_1 - k_g; k_2 + k_3; k_3) \Psi_{qqq}(k_1; k_2; k_3),
\]

with \( z = x_g/x_1, z' = x_g'(x_2 + x_g'), n_i = \vec{k}_g - z \vec{k}_1, n_i' = \vec{k}_g' - z'(\vec{k}_2 + \vec{k}_3), \Delta = zm_{\text{col}}, \Delta' = z'm_{\text{col}} \). The trace over the remaining gluon degrees of freedom is obtained by summing over \( a' = a \), setting \( k_g = k_g' \), and integrating with the measure

\[
\int \frac{dx_g}{x_g} \int \frac{d^2 k_g}{16\pi^3} \frac{x_1 - x_g}{x_1 - x_g}.
\]

Hence,

\[
\text{tr } \rho^{(12')} = -\frac{2g^2}{3} \int \frac{dx_g}{x_g} \int \frac{d^2 k_g}{16\pi^3} \int [dx_i] \left[ \Theta(x_1 - x_g) \Theta(1 - x_2 - x_g) \right]
\]

\[
\times \Psi_{qqq}(k_1 - k_g; k_2 + k_g; k_3) \Psi_{qqq}(k_1; k_2; k_3)
\]

\[
\cdot \frac{x_1 x_2}{(x_2 + x_g)(x_1 - x_g)} \frac{1}{(n_i^2 + \Delta^2)(n_i'^2 + \Delta'^2)} \left( 2 - z - z' + \frac{1}{2} zz'(1 + \langle h_1 h_2 \rangle) \right).
\]

Note that the integral over \( \vec{k}_g \) converges in the UV because some of the transverse momentum arguments of \( \Psi_{qqq} \) are shifted by \( \pm \vec{k}_g \). This expression cancels against eq. (90).

C. \( O(g^2) \) virtual correction to the three-quark density matrix

In this section we derive the corrections to the LO three-quark density matrix from sec. II. These are due to i) the emission and reabsorption of a gluon by a quark, and ii) the exchange of a gluon by two distinct quarks, either in \( |P> \) or in \( <P| \); see fig. 4.

We begin with the former correction which amounts to multiplying each quark state vector in eq. (1) or (30) by a wave function renormalization factor \( Z_q^{1/2}(x_i) \). The factor \( Z_q(x_i) \) equals 1 minus the \( O(g^2) \) correction [21]

\[
C_q(x_i) = \frac{1}{2x_i} \int \frac{dx_g}{x_g} \frac{d^2 k_g}{2\pi^3} \frac{1}{2x_g} \sum_{n,a,\sigma} \left| \psi_{q\to aq}(p; k_q, k_g)/P^+ \right|^2.
\]

The three-momenta satisfy \( p = k_q + k_g; a, n \) are the colors of the daughter gluon and quark, respectively; and \( \sigma \) denotes the polarization of the gluon.
The quark wave function renormalization factor is UV divergent:

$$C_q(x_1) = \frac{g^2 C_F}{8\pi^2} \int_{x/x_1}^1 \frac{dz}{z} \left[ 1 + (1 - z)^2 \right] A_0(z m_{\text{col}})$$  \hspace{1cm} (77)

$$A_0(\Delta) = 4\pi \int \frac{d^3n}{(2\pi)^2} \frac{1}{n^2 + \Delta^2} .$$  \hspace{1cm} (78)

$m_{\text{col}}$ denotes a “quark mass like” regulator of the collinear singularity. One may use dimensional regularization [21, 56, 57] to regularize $C_q(x_1)$. Here, instead we employ a scheme where we subtract the contribution from a mass scale in the ultraviolet, $\Lambda^2 = z^2 m_{\text{UV}}^2$. We define the regularized function

$$A_0^{\text{reg}}(\Lambda/\Delta) = A_0(\Delta) - A_0(\Lambda) = 4\pi \int \frac{d^3n}{(2\pi)^2} \left[ \frac{1}{n^2 + \Delta^2} - \frac{1}{n^2 + \Lambda^2} \right] = \log \frac{\Lambda^2}{\Delta^2} .$$  \hspace{1cm} (79)

This function is now used in eq. (77) to obtain

$$C_q(x_1; x, \frac{m_{\text{UV}}}{m_{\text{col}}}) = 2g^2 C_F \int_x^{x_1} \frac{dx_g}{x_g} \int \frac{d^3n}{16\pi^3} \frac{x_1}{x_1 - x_g} \frac{x_1 - x_g}{x_1} \left[ 1 + (1 - \frac{x_g}{x_1})^2 \right] \left[ \frac{1}{n^2 + \Delta^2} - \frac{1}{n^2 + \Lambda^2} \right] ,$$  \hspace{1cm} (80)

where $\Delta^2 = x_g^2 m_{\text{col}}^2/x_1^2$, $\Lambda^2 = x_g^2 m_{\text{UV}}^2/x_1^2$. In effect we have added to $-C_q(x_1)$ the infinite term

$$\frac{g^2 C_F}{2\pi} \int_{x/x_1}^1 \frac{dz}{z} \left[ 1 + (1 - z)^2 \right] \int \frac{d^3n}{(2\pi)^2} \frac{1}{n^2 + \Delta^2} .$$  \hspace{1cm} (81)

This is the same UV divergent contribution we previously subtracted from tr $\rho^{(11')}$. 

In all, the first $\mathcal{O}(g^2)$ correction to the three-quark density matrix amounts to multiplying eqs. (10, 12) by

$$\sqrt{Z_q(x_1) Z_q(x_2) Z_q(x_3) Z_q(x'_1) Z_q(x'_2) Z_q(x'_3)}$$

$$= 1 - \frac{C_q(x_1; x, \frac{m_{\text{UV}}}{m_{\text{col}}})}{2} - \frac{C_q(x_2; x, \frac{m_{\text{UV}}}{m_{\text{col}}})}{2} - \frac{C_q(x_3; x, \frac{m_{\text{UV}}}{m_{\text{col}}})}{2} - \frac{C_q(x'_1; x, \frac{m_{\text{UV}}}{m_{\text{col}}})}{2} - \frac{C_q(x'_2; x, \frac{m_{\text{UV}}}{m_{\text{col}}})}{2} - \frac{C_q(x'_3; x, \frac{m_{\text{UV}}}{m_{\text{col}}})}{2} .$$  \hspace{1cm} (82)

Here, we discard contributions beyond $\mathcal{O}(g^2)$. This generates a correction factor for the trace of the three-quark density matrix:

$$1 - 3 \frac{1}{2} \int [dx_1] \int [d^2k_1] |\Psi_{qqq}(k_1; k_2; k_3)|^2 C_q(x_1; x, \frac{m_{\text{UV}}}{m_{\text{col}}}) .$$  \hspace{1cm} (83)

The $\mathcal{O}(g^2)$ correction cancels against the contribution from the trace of the density matrix for the $|qqqq\rangle$ state, eq. (72).
We now move on to the second kind of $O(g^2)$ correction due to the exchange of a gluon by two quarks. Let quark 1 emit and quark 2 absorb the gluon in $|P\rangle$:

$$|P^+, \vec P = 0\rangle_{O(g^2)} = \int [dk_1] [d^2k_1] \Psi_{qq\bar q} (k_1; k_2; k_3) \frac{1}{\sqrt{6}} \sum_{j_1j_2j_3} \sum_{h_i,h_i'} \langle h_1, h_2, h_3 | S \rangle$$

$$4g^2 \sum_{\sigma,a,n,m} \langle t^\alpha \rangle_{m_j}(t^\alpha)_{n_j} \int \frac{d^d k}{d^4 k} \frac{\Theta(\min(x_1, 1 - x_2) - x_3)}{2(1 - x_3)} \frac{1}{P^+} \hat \psi_{qg-qg}(k_1; k_1 - k_g, k_g) \frac{1}{2(x_2 + x_3)} \frac{1}{P^+} \hat \psi_{qg-qg}(k_2, k_2 + k_g, k_g) \frac{1}{P^+} \hat \psi_{qg-qg}(k_2, k_2 + k_g, k_g)$$

$$\Theta(1 - x_1 + x_2 - x_3) \delta(h_1) \delta(h_2) \delta(h_3).$$

(There are analogous contributions corresponding to gluon exchanges between quarks 1 and 3, and 2 and 3.) We now multiply by

$$\langle P | \alpha' \rangle = \frac{1}{\sqrt{6}} \langle \alpha \rangle_{n_1n_2n_3} \Psi_{qq\bar q} (k_1; k_2; k_3) \langle h_1, h_2, h_3 | S \rangle \sum_{j_1j_2j_3} \sum_{h_i,h_i'} \langle h_1, h_2, h_3 | S \rangle \sum_{j_1j_2j_3} \sum_{h_i,h_i'} \langle h_1, h_2, h_3 | S \rangle$$

We can trace out the quark helicities and sum over gluon polarizations with the help of

$$\sum_{\sigma} \sum_{h_1,h_2,h_3} \langle h_1, h_2, h_3 | S \rangle \| \hat \psi_{qg-qg}(k_1; k_1 + k_g, k_g) \frac{1}{P^+} \hat \psi_{qg-qg}(k_2 - k_g, k_g; k_g) \frac{1}{P^+} \hat \psi_{qg-qg}(k_2 - k_g, k_g; k_g)$$

$$\Theta(1 - x_1 + x_2 - x_3) \delta(h_1) \delta(h_2) \delta(h_3).$$

(84)

(85)

(86)

(87)

(88)

(89)

Tracing out the quarks, we get

$$\text{tr} \rho^{(12)} \frac{1}{2} \int [dk_1] \int [d^2 k_1] \int_0^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \Theta(1 - x_1 + x_2) \Theta(1 - x_1 + x_2)$$

$$\langle h_1, h_2 | S \rangle \sum_{j_1j_2j_3} \sum_{h_i,h_i'} \langle h_1, h_2, h_3 | S \rangle \sum_{j_1j_2j_3} \sum_{h_i,h_i'} \langle h_1, h_2, h_3 | S \rangle$$

This cancels against eq. (75) which is easily checked by renaming $k_1 \leftrightarrow k_2$. The complete UV finite virtual correction due to a gluon exchange in $|P\rangle$ or $\langle P |$ includes $\text{tr} \rho^{(13)}$, $\text{tr} \rho^{(23)}$, $\text{tr} \rho^{(1'2')}$, $\text{tr} \rho^{(1'3')}$, $\text{tr} \rho^{(2'3')}$; this amounts to replacing $\langle h_1 h_2 | \rightarrow \langle h_1 h_2 + h_1 h_3 + h_2 h_3 / 3 = -\frac{1}{3}$ and multiplying eq. (90) by a factor of 6.
D. Reduced density matrix for the gluon momentum degree of freedom

In this subsection we collect the expressions for the reduced density matrix \(\rho_{k_g k_g'}\) which describes the entanglement of the momentum of the gluon with other degrees of freedom which we have traced over. This density matrix is of block-diagonal form with the first 1 \times 1 block given by the number

\[
1 - 3 \frac{1}{2} \int [dx_1] \int [d^2 k_1] |\Psi_{qqq}(k_1; k_2; k_3)|^2 C_q(x_1; x, M_{UV}/m_{col})
+ 4g^2 \frac{1}{2} \int [dx_1] \int [d^2 k_1] \int dx_g \frac{d^2 g}{x_g} \frac{1}{16\pi^3} \sqrt{\frac{x_1 x_2}{x_g}} \frac{x_1 x_2}{x_1 + x_2} \Theta(1 - (x_1 + x_2)) \Theta(x_2 - x_g)
\left(\frac{1}{n^2 + \Delta^2} \right)^{z + \frac{1}{3} z' z'}. \tag{91}
\]

In the last term, \(z = x_g/(x_1 + x_g), z' = x_g/x_2, \bar{n} = \vec{k}_g - z(\vec{k}_1 + \vec{k}_g), \bar{n}' = \vec{k}_g - z'\bar{k}_2, \Delta^2 = z^2 m_{col}^2, \Delta'^2 = z'^2 m_{col}^2\). The expression for \(C_q(x_1; x, M_{UV}/m_{col})\) is given in eq. (80).

The next block is given by the sum of two contributions. The first is the matrix

\[
\rho_{k_g k_g'}^A = 8g^2 \frac{1}{2} \int [dx_1] \int [d^2 k_1] \Theta(x_1 - x_g) \Theta(1 - x_1 + x_g + x'_g) \Psi_{qqq}^*(k_1 - k_g + k'_g; k_2; k_3) \Psi_{qqq}(k_1; k_2; k_3)
\left(\frac{1}{n^2 + \Delta^2} \right)^{z + \frac{1}{3} z' z'}. \tag{92}
\]

where \(z = x_g/x_1, z' = x'_g/(x_1 - x_g + x'_g), \bar{n} = \vec{k}_g - z\vec{k}_1, \bar{n}' = \vec{k}_g - z'(\vec{k}_1 - \vec{k}_g + \vec{k}_g'), \Delta^2 = z^2 m_{col}^2, \Delta'^2 = z'^2 m_{col}^2\). Along the diagonal of this block one adds

\[
- 8g^2 \frac{1}{2} \int [dx_1] \int [d^2 k_1] |\Psi_{qqq}(k_1; k_2; k_3)|^2 \left[1 + (1 - z)^2\right] \frac{1 - z}{k_g^2 + \Lambda^2}, \tag{93}
\]

with \(\Lambda = zM_{UV}\). To perform the trace over this contribution one sets \(k'_g = k_g\), which implies \(z' = z\) and \(\bar{n}' = \bar{n}\), and integrates with the measure (69) which includes a Jacobian \(x_1/(x_1 - x_g) = 1/(1 - z)\). This cancels the second term in eq. (91).

The second contribution is

\[
\rho_{k_g k_g'}^B = -8g^2 \frac{1}{2} \int [dx_1] \int [d^2 k_1] \bar{n} \cdot \bar{n}' \frac{\sqrt{(1 - z)(1 - z')}}{(n^2 + \Delta^2)(n^2 + \Delta'^2)} \Theta(x_1 - x_g) \Theta(1 - (x_2 + x'_g)) \Psi_{qqq}^*(k_1 - k_g; k_2 + k'_g; k_3) \Psi_{qqq}(k_1; k_2; k_3), \tag{94}
\]

where now \(z = x_g/x_1, z' = x'_g/(x_2 + x'_g), \bar{n} = \vec{k}_g - z\vec{k}_1, \bar{n}' = \vec{k}_g - z'(\vec{k}_2 + \vec{k}_g'), \Delta^2 = z^2 m_{col}^2, \Delta'^2 = z'^2 m_{col}^2\). To trace this matrix one again sets \(k'_g = k_g\) and integrates with the measure (69). This cancels the last term in eq. (91).

1. Reduced density matrix for the \(x_g\) degree of freedom

We can trace the expressions from the previous section over the gluon transverse momentum to obtain the density matrix \(\rho_{x_g x_g'}\) for the last remaining degree of freedom corresponding to the light-cone momentum fraction of the gluon. To render the result in as simple a form as possible we will restrict to \(x_g, x'_g\) much less than the typical quark momentum fraction \((x_q)\). Accordingly, when integrating over \(x_g\) we assume that the cutoff \(x\) for the soft singularity is much less than \((x_g)\).\(^8\)

\(^8\) However, we also assume that \(\alpha_s \log \frac{(x_g)}{x} \ll 1\) so that a resummation of the density matrix to all orders in this parameter (see ref. [18]) is not required.
The first $1 \times 1$ block of $\rho_{x_g x_g'}$ is then given by

$$1 - \frac{g^2}{\pi^2} \log \frac{\langle x_g \rangle}{x} \log \frac{M_{UV}^2}{m_{col}^2} + \int \frac{dx_g}{x_g} F(x_g^2 m_{col}^2),$$  \hspace{1cm} (95)$$

with

$$F(x_g^2 m_{col}^2) = 16 g^2 \frac{1}{2} \int [dx_i] \int [d^2 k_i] \frac{d^2 k_g}{16 \pi^3 k_g^2 + \frac{1}{x_g^2} m_{col}^2} \Psi_{qqq}(k_1; k_2; k_3) \Psi_{qqq}(x_1, k_1 + k_g; x_2, k_2 - k_g; x_3, k_3).$$  \hspace{1cm} (96)$$

The second block is given by

$$\rho_{x_g x_g'} = \frac{g^2}{\pi^2} \log \frac{M_{UV}^2}{m_{col}^2} - F(\max(x_g^2, x_g'^2) \cdot m_{col}^2).$$  \hspace{1cm} (97)$$

Note that taking the trace involves an integration over $dx_g/x_g$. Hence, for proper normalization of the eigenvalues the r.h.s. of eq. (97) should be multiplied by $dx_g/\sqrt{x_g^2 x_g^{'2}}$ in order to transform the trace operation to a sum over $x_g$-bins; compare to eq. (20).

For illustration we proceed to determine the spectrum of the above density matrix numerically. We again use the “harmonic oscillator” three-quark input wave function from ref. [24, 25] and set the remaining parameters as follows: a small coupling constant $\alpha_s = g^2/4\pi = 0.1$ and a fairly large collinear regulator $m_{col} = 1$ GeV so that the perturbative calculation should apply, $\langle x_g \rangle = 0.3$, log $M_{UV}^2/m_{col}^2 = 4$, and the soft cutoff $x = 0.1$. Even for such fairly large cutoff on the gluon light-cone momentum we obtain a low purity of $\text{tr} \rho = 0.52$: two eigenvalues of the density matrix are close to 0.5 while the others are close to 0. This purity is substantially lower than the purity of the reduced density matrices for the three quark Fock state (c.f. table I).

We emphasize again that the density matrix written in eqs. (95 – 97) is approximate. As such, even though it is symmetric and its trace is equal to 1 it may violate the positivity requirement on the eigenvalues. For the set of parameters mentioned above we find numerically that the absolute value of the most negative eigenvalue is 50 times smaller than the smallest positive eigenvalue. For greater coupling $\alpha_s$ or a substantially smaller cutoff $x$, however, the magnitude of the most negative eigenvalue increases and so the above reduced density matrix becomes unphysical.

**IV. SUMMARY**

In this paper we have analyzed entanglement of degrees of freedom in the light-cone wave function of the proton at intermediate parton momentum fractions. In sec. II we focused on the three quark Fock state which should dominate for large $x$. When one traces the pure density matrix for the anti-symmetric $\langle i_1 \cdots i_N, \bar{i}_1, \cdots, \bar{i}_N \rangle$ color state over all but one color degree of freedom then the spectrum of eigenvalues of the resulting reduced density matrix $\rho_{ij} = \frac{1}{N_c} \delta_{ij}$ is degenerate, and the von Neumann entropy is $S_{NN} = \log N_c$, indicating maximal entanglement of color.

On the other hand, in the limit of many colors, the spatial proton wave function should factorize into a product of $N_c$ one-body quark wave functions [26] where spatial degrees of freedom belonging to different quarks would not be entangled.

For $N_c = 3$, we used a model three-quark wave function from the literature [24, 25] to find weak entanglement of spatial degrees of freedom (longitudinal or transverse quark momenta); the reduced density matrices exhibit purities of 95% or greater. These model wave function involve as the only dimensionless physical parameter that the density matrix may depend on, the product of constituent quark mass and proton radius, or alternatively the mass of the proton times its radius.

However, to check whether, indeed, the known large-$x$ structure of the proton requires weak entanglement of spatial degrees of freedom, it would be interesting to repeat the analysis with three-quark wave functions which actually solve a light-front Hamiltonian with interactions [37]. Also, one could check entanglement in light-front wave functions obtained via “Large Momentum Effective Theory” from lattice QCD [39–41].

In sec. III we included the $|qqgg\rangle$ Fock state via light-cone perturbation theory. Tracing over quark degrees of freedom and gluon helicity and color we obtained the reduced density matrix for the gluon momentum degree of freedom in sec. III D. In $D = 4$ space-time dimensions the trace of that density matrix receives UV divergent contributions due
to the integration over the gluon transverse momentum. Upon regularization, the contributions from “real emissions” and “virtual corrections” cancel. However, even though the sum of eigenvalues does not depend on the UV regulator, nor on the collinear regulator or the soft cutoff, their spectrum does (and therefore so does the purity and the von Neumann entropy). In sec. III D 1 we further trace over the gluon transverse momentum to write the reduced density matrix for the remaining gluon light-cone momentum fraction degree of freedom in a particularly simple form by employing a small-$x_g$ ($\ll \langle x_g \rangle$) approximation. We obtain numerically that even for rather weak coupling, $x_g$ appears to be more strongly entangled with the traced-out “environment” than quark momentum fractions in the three-quark Fock state. This is, at least qualitatively consistent with the suggestion that entanglement grows stronger with decreasing $x$ [5–7, 18, 19].

Acknowledgements

We thank Alex Kovner, Vladimir Skokov, and Raju Venugopalan for useful discussions. We also acknowledge support by the DOE Office of Nuclear Physics through Grant DE-SC0002307, and The City University of New York for PSC-CUNY Research grant 64025-00 52. The figures have been prepared with Jaxodraw [58].

[1] C. Aidala, Spin-momentum correlations, aharonov-bohm, and color entanglement in quantum chromodynamics, https://indico.fnal.gov/event/19854/ (2019).
[2] P. J. Mulders and T. C. Rogers (2011), 1102.4569.
[3] T. C. Rogers and P. J. Mulders, Phys. Rev. D 81, 094006 (2010), 1001.2977.
[4] C. A. Aidala and T. C. Rogers (2021), 2108.12319.
[5] D. E. Kharzeev (2021), 2108.08792.
[6] D. E. Kharzeev and E. M. Levin, Phys. Rev. D 95, 114008 (2017), 1702.03489.
[7] A. Kovner and M. Lublinsky, Phys. Rev. D 92, 034016 (2015), 1506.05394.
[8] A. Kovner, M. Lublinsky, and M. Serino, Phys. Lett. B 792, 4 (2019), 1806.01089.
[9] Z. Tu, D. E. Kharzeev, and T. Ulrich, Phys. Rev. Lett. 124, 062001 (2020), 1904.11974.
[10] D. E. Kharzeev and E. Levin, Phys. Rev. D 104, L031503 (2021), 2102.09773.
[11] N. Armesto, F. Dominguez, A. Kovner, M. Lublinsky, and V. Skokov, JHEP 05, 025 (2019), 1901.08080.
[12] G. Dvali and R. Venugopalan (2021), 2106.11989.
[13] K. Zhang, K. Hao, D. Kharzeev, and V. Korepin (2021), 2110.04881.
[14] V. Andreiev et al. (H1), Eur. Phys. J. C 81, 212 (2021), 2011.01812.
[15] H. Duan, C. Akkaya, A. Kovner, and V. V. Skokov, Phys. Rev. D 101, 036017 (2020), 2001.01726.
[16] Y. Hagiwara, Y. Hatta, B.-W. Xiao, and F. Yuan, Phys. Rev. D 97, 094029 (2018), 1801.00087.
[17] H. Duan, A. Kovner, and V. V. Skokov (2021), 2111.06475.
[18] N. Armesto, F. Dominguez, A. Kovner, M. Lublinsky, and V. Skokov, JHEP 05, 025 (2019), 1901.08080.
[19] G. Dvali and R. Venugopalan (2021), 2106.11989.
[20] A. Morreale and F. Salazar, Universe 7, 312 (2021), 2108.08254.
[21] A. Dumitru and R. Paatelainen, Phys. Rev. D 103, 034026 (2021), 2010.11245.
[22] A. Dumitru, H. Mäntysaari, and R. Paatelainen, Phys. Lett. B 820, 136560 (2021), 2103.11682.
[23] A. Dumitru and R. Paatelainen (2021), 2106.12623.
[24] F. Schlumpf, Phys. Rev. D 47, 4114 (1993), [Erratum: Phys.Rev.D 49, 6246 (1994)], hep-ph/9212250.
[25] S. J. Brodsky and F. Schlumpf, Phys. Lett. B 329, 111 (1994), hep-ph/9402214.
[26] E. Witten, Nucl. Phys. B 160, 57 (1979).
[27] A. Dumitru, G. A. Miller, and R. Venugopalan, Phys. Rev. D 98, 094004 (2018), 1808.02501.
[28] A. Dumitru, V. Skokov, and T. Stebel, Phys. Rev. D 101, 054004 (2020), 2001.04516.
[29] T. Altinoluk, N. Armesto, G. Beuf, A. Kovner, and M. Lublinsky, Phys. Lett. B 751, 448 (2015), 1503.07126.
[30] A. Kovner, M. Li, and V. V. Skokov (2021), 2105.14971.
[31] G. Lepage and S. J. Brodsky, Phys. Rev. D 22, 2157 (1980).
[32] A. Haridranath, in International School on Light-Front Quantization and Non-Perturbative QCD (1996), hep-ph/9612244.
[33] S. J. Brodsky, H.-C. Pauli, and S. S. Pinsky, Phys. Rept. 301, 299 (1998), hep-ph/9705477.
[34] S. J. Brodsky, D. S. Hwang, B.-Q. Ma, and I. Schmidt, Nucl. Phys. B 593, 311 (2001), hep-th/0003082.
[35] M. Burkardt, Adv. Nucl. Phys. 23, 1 (1996), hep-ph/9505259.
[36] B. L. G. Bakker, L. A. Kondratyuk, and M. V. Terentev, Nucl. Phys. B 158, 497 (1979).
[37] S. Xu, C. Mondal, J. Lan, X. Zhao, Y. Li, and J. P. Vary (BLFQ), Phys. Rev. D 104, 094036 (2021), 2108.03909.
[38] E. Shuryak and I. Zahed (2022), 2202.00167.
[39] X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang, and Y. Zhao, Rev. Mod. Phys. 93, 035005 (2021), 2004.03543.
[40] X. Ji and Y. Liu (2021), 2106.05310.
[41] Y. Liu, Y. Zhao, and A. Schäfer, *Light-front wavefunction from lattice QCD through large-momentum effective theory*. https://www.snowmass21.org/docs/files/summaries/TF/SNOWMASS21-TF2_TF5-CompF2_CompF0-044.pdf (2021), [Online; accessed 19-December-2021].

[42] *Proceedings, Probing Nucleons and Nuclei in High Energy Collisions: Dedicated to the Physics of the Electron Ion Collider: Seattle (WA), United States, October 1 - November 16, 2018* (WSP, 2020), 2002.12333.

[43] R. Abdul Khalek et al. (2021), 2103.05419.

[44] S. R. Beane and P. Ehlers, Mod. Phys. Lett. A 35, 2050048 (2019), 1905.03295.

[45] J. S. Bell, Physics Physique Fizika 1, 195 (1964).

[46] J. S. Bell, Rev. Mod. Phys. 38, 447 (1966).

[47] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).

[48] S. Schlichting and B. Schenke, Phys. Lett. B 739, 313 (2014), 1407.8458.

[49] H. Mäntysaari and B. Schenke, Phys. Rev. Lett. 117, 052301 (2016), 1603.04349.

[50] H. Mäntysaari and B. Schenke, Phys. Rev. D 94, 034042 (2016), 1607.01711.

[51] H. Mäntysaari and B. Schenke, Phys. Lett. B 772, 832 (2017), 1703.09256.

[52] H. Mäntysaari and B. Schenke, Phys. Rev. D 98, 034013 (2018), 1806.06783.

[53] H. Mäntysaari, N. Mueller, and B. Schenke, Phys. Rev. D 99, 074004 (2019), 1902.05087.

[54] H. Mäntysaari, K. Roy, F. Salazar, and B. Schenke, Phys. Rev. D 103, 094026 (2021), 2101.02464.

[55] S. Demirci, T. Lappi, and S. Schlichting, Phys. Rev. D 103, 094025 (2021), 2101.03791.

[56] G. Beuf, Phys. Rev. D 94, 054016 (2016), 1606.00777.

[57] H. Hämminen, T. Lappi, and R. Paatelainen, Annals Phys. 393, 358 (2018), 1711.08207.

[58] D. Binosi, J. Collins, C. Kaufhold, and L. Theussl, Comput. Phys. Commun. 180, 1709 (2009), 0811.4113.