Bonnet and Isotropically Isothermic Surfaces in 4-Dimensional Space Forms

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Abstract
We study the Bonnet problem for surfaces in 4-dimensional space forms, namely, to what extent a surface is determined by the metric and the mean curvature. Two isometric surfaces have the same mean curvature if there exists a parallel vector bundle isometry between their normal bundles that preserves the mean curvature vector fields. We deal with the structure of the moduli space of congruence classes of isometric surfaces with the same mean curvature and with properties inherited on a surface by this structure. The study of this problem led us to a new conformally invariant property, called isotropic isothermicity, that coincides with the usual concept of isothermicity for surfaces lying in totally umbilical hypersurfaces, and is related to lines of curvature and infinitesimal isometric deformations that preserve the mean curvature vector field. The class of isotropically isothermic surfaces includes the one of surfaces with a vertically harmonic Gauss lift and particularly the minimal surfaces, and overlaps with that of isothermic surfaces without containing the entire class. We show that if a simply connected surface is not proper Bonnet, which means that the moduli space is a finite set, then it admits either at most one, or exactly three Bonnet mates. For simply connected proper Bonnet surfaces, the moduli space is either 1-dimensional with at most two connected components diffeomorphic to the circle, or the 2-dimensional torus. We prove that simply connected Bonnet surfaces lying in totally geodesic hypersurfaces of the ambient space as surfaces of non-constant mean curvature always admit Bonnet mates that do not lie in any totally umbilical hypersurface. Such surfaces either admit exactly three Bonnet mates, or they are proper Bonnet with moduli space the torus. We show that isotropic isothermicity characterizes the proper Bonnet surfaces, and we provide relevant conditions for non-existence of Bonnet mates for compact surfaces. Moreover, we study compact surfaces that are locally proper Bonnet, and we prove

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that the existence of a uniform substructure on the local moduli spaces characterizes surfaces with a vertically harmonic Gauss lift that are neither minimal, nor superconformal. In particular, we show that the only compact, locally proper Bonnet surfaces with moduli space the torus, are those with nonvanishing parallel mean curvature vector field and positive genus.

**Keywords** Mean curvature · Bonnet problem · Isothermic surfaces · Conformal invariants · Innitesimal deformations · Superconformal surfaces

**Mathematics Subject Classification** 53C42 · 53A10

1 Introduction

The theory of isometric or conformal immersions deals with the study of isometric or conformal invariants of immersions, aiming at the possible classification of the immersions with respect to these invariants. In the classical theory of surfaces in a complete, simply connected 3-dimensional space form \(Q^3_c\) of curvature \(c\), a basic problem is to investigate to what extent several geometric data determine a surface up to congruence, and furthermore, to study and classify the exceptional surfaces that are not uniquely determined by certain data.

In 1867, Bonnet [4] raised the problem to what extent a surface in \(Q^3_c\) is determined by the metric and the mean curvature. This naturally leads to the following question: given an isometric immersion \(f: M \to Q^3_c\) of a 2-dimensional oriented Riemannian manifold, how many noncongruent to \(f\) isometric immersions of \(M\) into \(Q^3_c\) can exist with the same mean curvature with \(f\)? Any noncongruent to \(f\) such surface is called a Bonnet mate of \(f\). A generic surface in \(Q^3_c\) is uniquely determined by the metric and the mean curvature. The exceptions are called Bonnet surfaces. Several aspects of the Bonnet problem have been studied by Bonnet [4], Cartan [6], Tribuzy [55], Chern [11], Roussos and Hernandez [53], Kenmotsu [40], and Smyth and Tinaglia [54] among many others. It turns out that a simply connected surface \(f: M \to Q^3_c\) either admits at most one Bonnet mate, or the moduli space of all isometric immersions of \(M\) into \(Q^3_c\) that have the same mean curvature with \(f\), is the circle \(S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}\). In the latter case, the surface is called proper Bonnet. It has been shown by Bonnet [4] and Lawson [42] that simply connected surfaces with constant mean curvature are proper Bonnet, unless they are totally umbilical. For compact surfaces, Lawson and Tribuzy [43] proved that a surface of non-constant mean curvature in \(Q^3_c\) admits at most one Bonnet mate. It still remains an open problem if there exist compact surfaces of non-constant mean curvature in \(\mathbb{R}^3\) that do admit a Bonnet mate.

The Bonnet problem for surfaces in \(Q^3_c\) is closely related to the extensively studied class of isothermic surfaces in \(Q^3_c\). It was shown by Raffy [52] that a proper Bonnet surface is isothermic away from its isolated umbilic points. Afterwards, Graustein [27] proved that an isothermic Bonnet surface is proper Bonnet. His characterization of proper Bonnet surfaces involving the isothermicity has been used by Bobenko and Eitner [2] for the classification of simply connected, umbilic-free proper Bonnet surfaces of non-constant mean curvature in \(\mathbb{R}^3\). On the other hand, Kamberov et
al. [39] described all simply connected, umbilic-free Bonnet pairs in $\mathbb{R}^3$ in terms of isothermic surfaces. Recently, Jensen et al. [36] provided sufficient conditions in terms of isothermicity, for non-existence of Bonnet mates for compact surfaces.

An umbilic-free surface $f : M \to Q_c^3$ is called isothermic if it admits a conformal curvature line parametrization around every point. This is equivalent to the co-closeness of the principal connection form of the surface, which is a globally defined 1-form on $M$. Isothermicity is a conformally invariant property that appears in several problems where a surface is not uniquely determined by certain geometric invariants. As a matter of fact, isothermic surfaces admit an amount of transformations that preserve geometric data, and they are characterized by the existence of these transformations (cf. [31]). Classical examples of isothermic surfaces in $Q_c^3$ are the umbilic-free surfaces with constant mean curvature, and particularly, the minimal surfaces, as well as their Möbius transformations. The notion of isothermicity has been extended for surfaces in the Euclidean space with arbitrary codimension by Palmer [49]. Isothermicity in arbitrary codimension is again a conformally invariant property, and such isothermic surfaces inherit the most of the properties of those in 3-dimensional space forms. For instance, they are characterized by the existence of analogous transformations. However, in codimension greater than one, the isothermicity implies flatness of the normal bundle of the surface, and this restricts the class of isothermic surfaces from including the minimal surfaces and their Möbius transformations.

There is a characterization of isothermic surfaces in $\mathbb{R}^3$ which has no higher codimensional analogue, namely, an umbilic-free surface in $\mathbb{R}^3$ is isothermic if and only if it locally admits a nontrivial infinitesimal isometric deformation that preserves the mean curvature. Probably the only recent proof of this result can be found in [41], where this characterization of isothermicity has been extended to discrete surfaces in $\mathbb{R}^3$. As mentioned in [12], this result dates back to the 19th century, and it seems to have been almost forgotten until this reference, since surfaces that admit such deformations had in the meantime been studied, without establishing a correlation with isothermicity (see Section 6.1 of the survey [32]). The theory of infinitesimal isometric deformations of surfaces and submanifolds in the Euclidean space has a long and rich history, as can be seen in the surveys [32,33] and is still developing (cf. [15,18,37]). In particular, the relation of isothermicity with the Bonnet problem for surfaces in $\mathbb{R}^3$ verifies very elegantly the quote of Efimov stated in [33], that “the theory of infinitesimal isometric deformations is the differential of the theory of isometric deformations”.

Besides space forms, the Bonnet problem has been studied for surfaces in homogeneous 3-manifolds [25], and it was recently raised for surfaces in static 3-manifolds [44].

The Bonnet problem for surfaces in 4-dimensional space forms $Q^4_c$ has been studied in [51]. Two isometric surfaces in $Q^4_c$ are said to have the same mean curvature if there exists a parallel vector bundle isometry between their normal bundles that preserves the mean curvature vector fields. Most of the results in [51] concern compact surfaces and are global in nature.

In this paper, we focus mainly on local aspects of the Bonnet problem for surfaces in $Q^4_c$. The local study of the problem led us to a new conformally invariant property, which has a similar effect on the Bonnet problem for surfaces in $Q^4_c$ with that
of isothermicity on the classical Bonnet problem. This property is called isotropic isothermicity and we discuss it first.

We introduce the notion of isotropically isothermic surfaces in $\mathbb{Q}^4_c$, generalizing the one of isothermic surfaces in $\mathbb{Q}^3_c$, as follows: using the two isotropic parts of the Hopf differential of an oriented surface $f : M \to \mathbb{Q}^4_c$, we introduce two differential 1-forms $\Omega^+ = \frac{\partial}{\partial \gamma^+}$ and $\Omega^- = \frac{\partial}{\partial \gamma^-}$, called the mixed connection forms of $f$. The form $\Omega^\pm$ is defined away from pseudo-umbilic points of $f$, i.e., the points where the curvature ellipse of $f$ is a circle, at which the normal curvature satisfies $\pm K_N \geq 0$. For an umbilic-free surface lying in some totally umbilical hypersurface of $\mathbb{Q}^4_c$, both mixed connection forms coincide with the principal connection form of the surface. Extending naturally the definition of isothermic surfaces in $\mathbb{Q}^3_c$, we call a surface $f : M \to \mathbb{Q}^4_c$ isotropically isothermic if at least one of the mixed connection forms is defined and co-closed on the whole $M$. If this occurs for both mixed connection forms, then $f$ is called strongly isotropically isothermic.

It turns out that isotropic isothermicity is a property invariant under conformal changes of the metric of the ambient space. Examples of isotropically isothermic surfaces in $\mathbb{Q}^4_c$ are the non-superconformal surfaces with a vertically harmonic Gauss lift, the minimal superconformal surfaces, and their Möbius transformations, away from isolated points. In particular, non-superconformal minimal surfaces are strongly isotropically isothermic away from pseudo-umbilic points. We note that, as follows from [30,51], surfaces with a vertically harmonic Gauss lift are the analogues in $\mathbb{Q}^3_c$ of constant mean curvature surfaces in $\mathbb{Q}^3_c$, and particularly, superconformal surfaces with a vertically harmonic Gauss lift generalize the totally umbilical surfaces. The class of strongly isotropically isothermic surfaces includes the one of isothermic surfaces lying in totally umbilical hypersurfaces of the ambient space; however, we show that there exist isothermic surfaces in $\mathbb{R}^4$ which are not isotropically isothermic.

For surfaces in $\mathbb{R}^4$, we prove that isotropic isothermicity is related to infinitesimal isometric deformations that preserve the mean curvature, and that strong isotropic isothermicity involves the principal curvature lines, studied in [26,29], along which the second fundamental form of the surface points in the direction of a principal axis of the curvature ellipse. For an infinitesimal isometric deformation of a surface $f : M \to \mathbb{R}^4$, we define the parallel preservation in the normal bundle under the deformation, of quantities related to the second fundamental form of $f$, in such a way that the deformation is trivial if and only if it preserves parallelly in the normal bundle the mean curvature vector field and the Hopf differential, i.e. the second fundamental form. We note that parallel preservation of the mean curvature vector field in the normal bundle implies preservation of its length and of the normal curvature. Our first result is the following.

**Theorem 1** Let $f : M \to \mathbb{R}^4$ be an oriented surface, free of pseudo-umbilic points. The surface $f$ is isotropically isothermic if and only if it locally admits a nontrivial infinitesimal isometric deformation that preserves parallelly in the normal bundle, the mean curvature vector field and an isotropic part of the Hopf differential. Moreover, $f$ is strongly isotropically isothermic if and only if it is isotropically isothermic and admits a conformal principal curvature line parametrization around every point.
For umbilic-free superconformal surfaces with nowhere-vanishing mean curvature vector field, we show that isotropic isothermicity is related to the mean-directional curvature lines, studied in [47], along which the second fundamental form of the surface points in the direction of the mean curvature vector. It is known that such surfaces have a holomorphic Gauss lift (cf. [19]). All these superconformal surfaces in \( \mathbb{R}^4 \) have been locally parametrized in terms of minimal surfaces by Dajczer and Tojeiro [16] and Moriya [48].

**Theorem 2** Let \( f : M \to \mathbb{R}^4 \) be an oriented, umbilic-free superconformal surface with nowhere-vanishing mean curvature vector field. The following are equivalent:

(i) The surface \( f \) is isotropically isothermic.

(ii) There exists a conformal mean-directional curvature line parametrization around every point of \( M \).

(iii) Locally, the surface \( f \) admits a nontrivial infinitesimal isometric deformation that preserves, parallelly in the normal bundle the mean curvature vector field, and the holomorphicity of a Gauss lift of \( f \).

Isotropically isothermic superconformal surfaces in \( \mathbb{R}^4 \) satisfying the conditions of the above theorem can be obtained as compositions, either of superminimal surfaces in the 4-sphere with a stereographic projection, or of holomorphic curves in \( \mathbb{R}^4 \) with inversions.

To the best of our knowledge, the notion of isotropic isothermicity is the only generalization of isothermicity for surfaces in \( \mathbb{Q}^3_c \) that allows surfaces with nonflat normal bundle. Moreover, apart from the parallel preservation in the normal bundle, there is no other known concept of preservation of exterior geometric data under infinitesimal deformations of submanifolds in codimension greater than one. As far as we know, this is also the first time that the aforementioned curvature lines appear in a problem that is not related exclusively to their own interest.

Transformations of isotropically isothermic surfaces will be the subject of a forthcoming paper.

The rest of our results concern the Bonnet problem. For an isometric immersion \( f : M \to \mathbb{Q}^4_c \), we denote by \( \mathcal{M}(f) \) the moduli space of congruence classes of all isometric immersions of \( M \) into \( \mathbb{Q}^4_c \) that have the same mean curvature with \( f \). Every nontrivial class in \( \mathcal{M}(f) \) is called a *Bonnet mate* of \( f \), and the surface \( f \) is called *proper Bonnet* if it admits infinitely many Bonnet mates. The structure of the moduli space for compact surfaces has been studied in [51]. The following result determines the possible structure of \( \mathcal{M}(f) \) for simply connected surfaces.

**Theorem 3** Let \( f : M \to \mathbb{Q}^4_c \) be a simply connected oriented surface.

(i) If \( f \) is not proper Bonnet, then it admits either at most one Bonnet mate, or exactly three.

(ii) If \( f \) is proper Bonnet, then the moduli space \( \mathcal{M}(f) \) is a space diffeomorphic to a manifold. Moreover, \( f \) is characterized according to the structure of \( \mathcal{M}(f) \) as follows:

**Tight:** The moduli space is one dimensional with at most two connected components, each one diffeomorphic to \( S^1 \simeq \mathbb{R}/2\pi\mathbb{Z} \).
Flexible: The moduli space is diffeomorphic to the torus $S^1 \times S^1$.

In particular, $f$ admits at most one Bonnet mate if $M$ is homeomorphic to $S^2$.

It has been proved in [51] that simply connected surfaces in $Q^4_c$ with a vertically harmonic Gauss lift, which are neither minimal, nor superconformal, are proper Bonnet. In particular, it was shown that non-minimal surfaces with parallel mean curvature vector field which are not totally umbilical, are flexible. Surfaces with nonvanishing parallel mean curvature vector field lie as constant mean curvature surfaces in some totally umbilical hypersurface of $Q^4_c$ (cf. [8,58]).

The following theorem implies that there exist flexible proper Bonnet surfaces in $Q^4_c$ that do not lie in any totally umbilical hypersurface. Such surfaces in our result arise as Bonnet mates of surfaces in $Q^4_c$, which are given by the composition of a proper Bonnet surface with non-constant mean curvature in $Q^3_c$ with a totally geodesic inclusion. The following theorem also shows that the simply connected Bonnet pairs in $Q^3_c$, give rise to Bonnet quadruples in $Q^4_c$.

**Theorem 4** Let $f : M \to Q^4_c$ be a simply connected oriented surface, which is the composition of a non-minimal Bonnet surface $F : M \to Q^3_c$ with a totally geodesic inclusion. Every Bonnet mate of $F$ in $Q^3_c$ determines two Bonnet mates $f^-$ and $f^+$ of $f$ in $Q^4_c$ that do not lie in any totally geodesic hypersurface. The surface $f^\pm$ lies in some totally umbilical hypersurface of $Q^4_c$ if and only if $F$ has constant mean curvature. Moreover, either $f$ admits exactly three Bonnet mates, or it is a flexible proper Bonnet surface.

We show that proper Bonnet surfaces are isotropically isothermic away from isolated points, and that strong isotropic isothermicity characterizes the flexible surfaces away from their isolated pseudo-umbilic points. In particular, the umbilic-free flexible surfaces obtained by the above theorem are furthermore isothermic. We also prove a result analogous to that of Graustein [27], which implies that a simply connected, Bonnet and strongly isotropically isothermic surface is proper Bonnet. This result indicates that the most natural class to look for simply connected Bonnet surfaces which are not proper Bonnet, is that of half or strongly totally non-isotropically isothermic surfaces that are surfaces in which either at least one, or both of mixed connection forms, respectively, are everywhere defined and nowhere co-closed.

In the sequel, we deal with compact surfaces. It has been proved in [51] that compact surfaces in $Q^4_c$ whose both Gauss lifts are not vertically harmonic, admit at most three Bonnet mates. The following theorem shows that for such surfaces, and in contrast to the simply connected case, additional assumptions involving isotropic isothermicity are restrictive for the existence of Bonnet mates. It is inspired by a recent result of Jensen–Musso–Nicolodi [36] for surfaces in $\mathbb{R}^3$.

**Theorem 5** Let $f : M \to Q^4_c$ be a compact-oriented surface whose both Gauss lifts are not vertically harmonic. If $f$ is either isotropically isothermic, or half totally non-isotropically isothermic, on an open dense and connected subset of $M$, then it admits at most one Bonnet mate. In particular, $f$ does not admit any Bonnet mate, if it is either strongly isotropically isothermic, or strongly totally non-isotropically isothermic, on such a subset of $M$. 
Thereafter, we study locally proper Bonnet surfaces. A surface $f : M \to \mathbb{Q}_4^4$ is
called locally proper Bonnet if every point of $M$ has a neighbourhood, restricted to
which $f$ is proper Bonnet. If such a surface is non-minimal, then for any sufficiently
small neighbourhood $U$ of every $p \in M$, there exists a submanifold $L^n(p), 1 \leq n \leq 2,$
of the torus $\mathbb{S}^1 \times \mathbb{S}^1$, that is also a submanifold of the moduli space $\mathcal{M}(f|_U)$. The
surface $f$ is called uniformly locally proper Bonnet if there exists a submanifold $L^n,$
$1 \leq n \leq 2,$ of the torus, having the above property for every $p \in M$. In particular, if
this submanifold is the torus itself, then $f$ is called locally flexible.

The following results concern compact surfaces that are locally proper Bonnet. A
basic ingredient of their proofs is an index theorem that we obtain using the mixed
connection forms, which extends the Poincaré–Hopf index theorem for surfaces in $\mathbb{Q}_c^3$
with isolated umbilics. The following theorem characterizes compact surfaces with
a vertically harmonic Gauss lift that are neither minimal, nor superconformal, as the
only compact, uniformly locally proper Bonnet surfaces in $\mathbb{Q}_c^4$.

**Theorem 6** Let $f : M \to \mathbb{Q}_4^4$ be a non-minimal, compact-oriented surface. The surface $f$
is uniformly locally proper Bonnet if and only if it has a vertically harmonic
and non-holomorphic Gauss lift.

Our next result concerns superconformal surfaces. We mention that Fujioka [24]
found a class of simply connected surfaces with nonflat normal bundle in the hyperbolic
4-space that can be deformed by preserving the length of the mean curvature vector
field. A careful look on the conditions that he imposed in order to obtain this class,
shows that these surfaces are superconformal and proper Bonnet in our sense. For
compact surfaces, we prove the following.

**Theorem 7** There do not exist compact-oriented superconformal surfaces in $\mathbb{Q}_c^4$ that
are locally proper Bonnet.

The following theorem shows that the compact, locally flexible proper Bonnet
surfaces in $\mathbb{Q}_c^4$ have parallel mean curvature vector field. From [8,58], it follows that
such a surface lies as a constant mean curvature surface in some totally umbilical
hypersurface of $\mathbb{Q}_c^4$. Jointly with Theorem 4, this gives a strong generalization of a
result due to Umehara [56].

**Theorem 8** A compact-oriented surface $f : M \to \mathbb{Q}_c^4$ is locally flexible proper Bonnet
if and only if it has nonvanishing parallel mean curvature vector field and the genus
of $M$ is positive.

The paper is organized as follows: in Sect. 2, we fix the notation and we give
some preliminaries. In Sect. 3, we introduce the mixed connection forms of surfaces
in $\mathbb{Q}_c^4$, and we prove an index theorem that will be used for the proofs of Theorems
6-8. We also provide some applications, among them, a short proof of a result due
to Asperti [1]. In Sect. 4, we introduce the concept of isotropic isothermicity, we
prove that it is a conformally invariant property, and we give some examples. We also
investigate its relation with isothermicity and with lines of curvature. The last part of the
section concerns infinitesimal isometric deformations, and there we prove Theorems
1 and 2. In Sect. 5, we set up the framework for the study of the Bonnet problem.

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Sect. 6 is devoted to simply connected surfaces. We prove a theorem that provides detailed information about the structure of the moduli space, and we give the proofs of Theorems 3 and 4. In the last part of the section, we study proper Bonnet surfaces and we prove that they are isotropically isothermic. We also show that such surfaces admit conformal metrics of constant curvature $-1$, away from points at which some Gauss lift is vertically harmonic. Section 7 deals with compact surfaces. We investigate the effect of isotropic isothermicity on the structure of the moduli space, and we give the proof of Theorem 5. Finally, we study locally proper Bonnet surfaces, and we prove Theorems 6, 7 and 8.

2 Preliminaries

Throughout the paper, $M$ is a connected, oriented 2-dimensional Riemannian manifold. A surface $f : M \to \mathbb{Q}_c^n$, $n = 3, 4$, is an isometric immersion into the complete, simply connected $n$-dimensional space form of curvature $c$.

Let $f : M \to \mathbb{Q}_c^4$ be a surface. We denote by $N_f M$ the normal bundle of $f$ and by $\nabla^\perp, R^\perp$ the normal connection and its curvature tensor, respectively. The orientations of $M$ and $\mathbb{Q}_c^4$ induce an orientation on the normal bundle of $f$. The normal curvature $K_N$ of $f$ is given by $K_N = \langle R^\perp(e_1, e_2)e_4, e_3 \rangle$, where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are positively oriented orthonormal frame fields of $TM$ and $N_f M$, respectively, and $\langle \cdot, \cdot \rangle$ stands for the Riemannian metric of $\mathbb{Q}_c^4$. Notice that if $\tau$ is an orientation-reversing isometry of $\mathbb{Q}_c^4$, then $f$ and $\tau \circ f$ have opposite normal curvatures. The Gaussian curvature $K$ of $M$ and the normal curvature satisfy the equations:

$$d\omega_{12} = -K\omega_1 \wedge \omega_2, \quad d\omega_{34} = -K_N\omega_1 \wedge \omega_2,$$

where $\{\omega_k\}$ is the dual frame field of $\{e_k\}$, $1 \leq k \leq 4$, and its corresponding connection forms $\omega_{kl} = -\omega_{lk}$, $1 \leq k, l \leq 4$, are given by

$$d\omega_k = \sum_{m=1}^{4} \omega_{km} \wedge \omega_m, \quad 1 \leq k \leq 4. \tag{2}$$

If $M$ is compact, the Euler–Poincaré characteristics $\chi, \chi_N$ of $TM$ and $N_f M$, are respectively given by

$$\chi = \frac{1}{2\pi} \int_M K, \quad \chi_N = \frac{1}{2\pi} \int_M K_N.$$

Let $\alpha : TM \times TM \to N_f M$ be the second fundamental form of $f$. The shape operator $A_\xi$ of $f$ with respect to $\xi \in N_f M$ is the symmetric endomorphism of $TM$ defined by $\langle A_\xi X, Y \rangle = \langle \alpha(X, Y), \xi \rangle$. The surface $f$ is said to have flat normal bundle if $K_N \equiv 0$ on $M$. This is equivalent to the existence for every $p \in M$, of an orthonormal basis of $T_p M$ that diagonalizes all shape operators of $f$ at $p$. 

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The curvature ellipse of \( f \) at each \( p \in M \) is defined by

\[
E_f(p) = \{ \alpha(X, X) : X \in T_p M, \|X\| = 1 \}.
\]

It is indeed an ellipse on \( N_f M(p) \) centred at the mean curvature vector \( H(p) = \text{trace}\alpha(p)/2 \), which may degenerate into a line segment or a point. It is parametrized by

\[
\alpha(X_\theta, X_\theta) = H(p) + \cos 2\theta \left( \frac{\alpha_{11} - \alpha_{22}}{2} \right) + \sin 2\theta \alpha_{12},
\]

where \( X_\theta = \cos \theta e_1 + \sin \theta e_2, \alpha_{kl} = \alpha(e_k, e_l), k, l = 1, 2, \) and \( \{e_1, e_2\} \) is an orthonormal basis of \( T_p M \). The ellipse degenerates into a line segment or a point if and only if the vectors \( \frac{\alpha_{11} - \alpha_{22}}{2} \) and \( \alpha_{12} \) are linearly dependent, or equivalently, if \( R^\perp = 0 \) at \( p \) (cf. [28]). Moreover, at a point where the curvature ellipse is nondegenerate, \( K_N \) is positive if and only if the orientation induced on the ellipse as \( X_\theta \) traverses positively the unit tangent circle, coincides with the orientation of the normal plane. The lengths \( \lambda_1, \lambda_2 \) of the semi-axes of \( E_f \), satisfy at any point the relations (cf. [45])

\[
\lambda_1^2 + \lambda_2^2 = \|H\|^2 - (K - c), \quad \lambda_1 \lambda_2 = \frac{1}{\pi} A(E_f) = \frac{1}{2} |K_N|, \tag{3}
\]

where \( A(E_f) \) is the area of the curvature ellipse. Therefore, at every point of \( M \), we have that

\[
\|H\|^2 - (K - c) \geq |K_N|.
\]

A point \( p \in M \) is called pseudo-umbilic if the curvature ellipse is a circle at \( p \), and the set \( M_0(f) \) of pseudo-umbilic points of \( f \) is characterized as follows:

\[
M_0(f) = \left\{ p \in M : \|H\|^2 - (K - c) = |K_N| \right\}.
\]

A surface for which any point is pseudo-umbilic is called superconformal. A pseudo-umbilic point is called umbilic if the circle degenerates into a point. By setting

\[
M_0^{\pm}(f) = \{ p \in M_0(f) : \pm K_N \geq 0 \},
\]

it follows that \( M_0(f) = M_0^+(f) \cup M_0^-(f) \), and that the set \( M_1(f) \) of umbilic points is

\[
M_1(f) = M_0^+(f) \cap M_0^-(f) = \{ p \in M : \|H\|^2 = K - c \}.
\]
2.1 Complexification and Associated Differentials

The complexified tangent bundle $TM \otimes \mathbb{C}$ of a $2$-dimensional oriented Riemannian manifold $M$, decomposes into the eigenspaces of the complex structure $J$, denoted by $T^{(1,0)}M$ and $T^{(0,1)}M$, corresponding to the eigenvalues $i$ and $-i$, respectively.

The second fundamental form of a surface $f: M \to \mathbb{Q}_c^2$ can be $\mathbb{C}$-bilinearly extended to $TM \otimes \mathbb{C}$ with values in the complexified normal bundle $N_{fM} \otimes \mathbb{C}$ and then decomposed into its $(k, l)$-components $\alpha^{(k,l)}$, $k + l = 2$, which are tensors of $k$ many 1-forms vanishing on $T^{(0,1)}M$ and $l$ many 1-forms vanishing on $T^{(1,0)}M$. For a positively oriented local orthonormal frame field $\{e_1, e_2\}$ of $TM$, the Hopf invariant $\mathcal{H}(e_1, e_2)$ of $f$ with respect to $\{e_1, e_2\}$ is the local section of $N_{fM} \otimes \mathbb{C}$ defined by

$$\mathcal{H}(e_1, e_2) = 2\alpha^{(2,0)}(e_1, e_1) = \frac{\alpha_{11} - \alpha_{22}}{2} - i\alpha_{12}, \quad \alpha_{kl} = \alpha(e_k, e_l), \quad k, l = 1, 2. \quad (4)$$

Let $J^\perp$ be the complex structure of $N_{fM}$ defined by the metric and the orientation. The complexified normal bundle decomposes as follows:

$$N_{fM} \otimes \mathbb{C} = N_{f}^{-} M \oplus N_{f}^{+} M$$

into the eigenspaces $N_{f}^{-} M$ and $N_{f}^{+} M$ of $J^\perp$, corresponding to the eigenvalues $i$ and $-i$, respectively. Any section $\xi \in N_{fM} \otimes \mathbb{C}$ is decomposed as $\xi = \xi^- + \xi^+$, where

$$\xi^\pm = \pi^\pm(\xi),$$

and the projection $\pi^\pm: N_{fM} \otimes \mathbb{C} \to N_{f}^\pm M$ is given by

$$\pi^\pm(\xi) = \frac{1}{2}(\xi \pm iJ^\perp \xi), \quad \xi \in N_{fM} \otimes \mathbb{C}.$$ 

A section $\xi$ of $N_{fM} \otimes \mathbb{C}$ is called isotropic if at any point of $M$, either $\xi = \xi^-$, or $\xi = \xi^+$. This is equivalent to $\langle \xi, \bar{\xi} \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the $\mathbb{C}$-bilinear extension of the metric. Notice that $\langle \xi, \eta \rangle = 0$ for $\xi \in N_{f}^{-} M$ and $\eta \in N_{f}^{+} M$ implies that either $\xi = 0$, or $\eta = 0$. According to the above decomposition, the Hopf invariant of $f$ with respect to $\{e_1, e_2\}$ splits into isotropic parts as $\mathcal{H}(e_1, e_2) = \mathcal{H}^-(e_1, e_2) + \mathcal{H}^+(e_1, e_2)$, where

$$\mathcal{H}^\pm(e_1, e_2) = \frac{1}{2} \left( \frac{\alpha_{11} - \alpha_{22}}{2} \pm J^\perp \alpha_{12} \pm iJ^\perp \left( \frac{\alpha_{11} - \alpha_{22}}{2} \pm J^\perp \alpha_{12} \right) \right). \quad (5)$$

The length of $\mathcal{H}^\pm(e_1, e_2)$ is independent of the frame field $\{e_1, e_2\}$, and the function $\|\mathcal{H}^\pm\|$ given by

$$\|\mathcal{H}^\pm\| = \sqrt{2} \|\mathcal{H}^\pm(e_1, e_2)\| = \sqrt{\|H\|^2 - (K - c)^2KN} \quad (6)$$

vansishes precisely on $M_0^\pm(f)$.  

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Let $E$ be a complex vector bundle over $M$ equipped with a connection $\nabla^E$. An $E$-valued differential $\Psi$ of $r$-order is an $E$-valued $r$-covariant tensor field on $M$ of holomorphic type $(r,0)$. The $r$-differential $\Psi$ is called holomorphic (cf. [3]) if its covariant derivative $\nabla^E \Psi$ has holomorphic type $(r+1,0)$. Let $(U,z=x+iy)$ be a local complex coordinate on $M$. The Wirtinger operators are defined on $U$ by

$$\partial = \partial_x = (\partial_x - i \partial_y)/2, \quad \bar{\partial} = \bar{\partial}_y = (\partial_x + i \partial_y)/2,$$

where $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$. On $U$, the differential $\Psi$ has the form $\Psi = \psi dz^r$, where $\psi: U \to E$ is given by $\psi = \Psi(\partial, \ldots, \partial)$. Then, $\Psi$ is holomorphic if and only if

$$\nabla^E_{\bar{\partial}} \psi = 0,$$

i.e. $\psi$ is a holomorphic local section. For later use, we need the following result (cf. [3,10]).

**Lemma 9** Assume that the $E$-valued differential $\Psi$ is holomorphic and let $p \in M$ be such that $\Psi(p) = 0$. Let $(U,z)$ be a local complex coordinate with $z(p) = 0$. Then either $\Psi \equiv 0$ on $U$, or $\Psi = z^m \Psi^*$, where $m$ is a positive integer and $\Psi^*(p) \neq 0$.

Let $f: M \to \mathbb{Q}^4_c$ be an oriented surface. In terms of a local complex coordinate $(U,z = x+iy)$, the metric $d\mathbf{s}^2$ of $M$ is written as $d\mathbf{s}^2 = \lambda^2 |dz|^2$, where $\lambda > 0$ is the conformal factor. Setting $e_1 = \partial_x/\lambda$ and $e_2 = \partial_y/\lambda$, the components of $\alpha$ are given by

$$\alpha^{(2,0)} = \alpha(\partial, \partial) dz^2, \quad \alpha^{(0,2)} = \overline{\alpha^{(2,0)}}, \quad \alpha^{(1,1)} = \alpha(\partial, \bar{\partial})(dz \otimes d\bar{z} + d\bar{z} \otimes dz),$$

where

$$\alpha(\partial, \partial) = \frac{\lambda^2}{2} \mathcal{H}(e_1, e_2) \quad \text{and} \quad \alpha(\partial, \bar{\partial}) = \frac{\lambda^2}{2} \mathcal{H}. \quad (7)$$

The Hopf differential of $f$ is the quadratic $N_f M \otimes \mathbb{C}$-valued differential $\Phi = \alpha^{(2,0)}$ with local expression $\Phi = \alpha(\partial, \partial) dz^2$. According to the decomposition of $N_f M \otimes \mathbb{C}$, the Hopf differential splits into isotropic parts as

$$\Phi = \Phi^- + \Phi^+, \quad \text{where} \quad \Phi^\pm = \pi^\pm \circ \Phi.$$

The following has been proved in [51, Lemma 8].

**Lemma 10** (i) The zero sets of $\Phi^\pm$ and $\Phi$ are $M_0^\pm(f)$ and $M_1(f)$, respectively.

(ii) The surface $f$ is superconformal with normal curvature $\pm K_N \geq 0$ if and only if $\Phi^\pm \equiv 0$. In particular, if $f$ is superconformal, then $K_N$ vanishes precisely on $M_1(f)$.

On $(U,z)$, the differential $\Phi^\pm$ has the expression

$$\Phi^\pm = \phi^\pm dz^2, \quad (8)$$
and the compatibility equations for $f$ can be written as follows:

\[(\text{Gauss})\quad (\log \lambda^2)_{\bar{z}} - \frac{2}{\lambda^2} \left( \langle \phi^-, \overline{\phi^-} \rangle + \langle \phi^+, \overline{\phi^+} \rangle \right) + \frac{\lambda^2}{2} (\|H\|^2 + c) = 0, \quad (9)\]

\[(\text{Codazzi})\quad \nabla_{\bar{\overline{\partial}}}^1 \phi^- = \frac{\lambda^2}{2} \nabla_{\bar{\overline{\partial}}}^1 H^-, \quad \nabla_{\bar{\overline{\partial}}}^1 \phi^+ = \frac{\lambda^2}{2} \nabla_{\bar{\overline{\partial}}}^1 H^+, \quad (10)\]

\[(\text{Ricci})\quad R^\perp (\partial, \bar{\partial}) = \frac{2}{\lambda^2} (\phi^- \wedge \overline{\phi^-} + \phi^+ \wedge \overline{\phi^+}), \quad (11)\]

where $R^\perp$ is the $\mathbb{C}$-trilinear extension of the normal curvature tensor, and the wedge product satisfies $(\xi \wedge \zeta) \eta = (\zeta, \eta)\xi - (\xi, \eta)\zeta$, for $\xi, \zeta, \eta \in N_{fM} \otimes \mathbb{C}$. It follows from (8) and (10) that $\Phi$ is holomorphic if and only if the mean curvature vector field $H$ is parallel in the normal connection.

### 2.2 Twistor Spaces and Gauss Lifts

Let $f : M \to \mathbb{R}^4$ be an oriented surface. We recall that (see for instance [51, Sect. 4.2]) the Grassmannian $Gr(2, 4)$ of oriented 2-planes in $\mathbb{R}^4$, is isometric to the product $S^2_+ \times S^2_-$ of two spheres of radius $1/\sqrt{2}$. Accordingly, the Gauss map $g : M \to Gr(2, 4)$ of $f$, decomposes into a pair of maps as $g = (g_+, g_-) : M \to S^2_+ \times S^2_-$. For surfaces in not necessarily flat space forms $Q^4_c$, the geometric information encoded in the components $g_+$ and $g_-$ of the Gauss map of a surface in $\mathbb{R}^4$ is encoded in the Gauss lifts of the surface to the twistor bundle of $Q^4_c$.

We briefly recall some facts about the twistor theory of 4-dimensional space forms (cf. [19,33,43]). The twistor bundle $\mathcal{Z}$ of $Q^4_c$ is the set of all pairs $(p, \tilde{J})$, where $p \in Q^4_c$ and $\tilde{J}$ is an orthogonal complex structure on $T_pQ^4_c$, endowed with the twistor projection $\varrho : \mathcal{Z} \to Q^4_c$, defined by $\varrho(p, \tilde{J}) = p$. The twistor bundle is a $O(4)/U(2)$-bundle over $Q^4_c$ associated to $O(Q^4_c)$, the principal $O(4)$-bundle of orthonormal frames in $Q^4_c$, which has two connected components. More precisely, at a point $p \in Q^4_c$, any orthonormal frame $e = (e_1, e_2, e_3, e_4)$ of $T_pQ^4_c$ determines an orthogonal complex structure $\tilde{J}_e$, given by

$$\tilde{J}_e e_1 = e_2, \quad \tilde{J}_e e_3 = e_4, \quad \tilde{J}_e^2 = -I.$$

Every orthogonal complex structure on $T_pQ^4_c$ can be written in the above form for some orthonormal frame of $T_pQ^4_c$. In particular, $\tilde{J}_e = \tilde{J}_{\tilde{e}}$ if and only if $\tilde{e} = eA$ for some $A \in U(2)$. Therefore, the set of all orthogonal complex structures on $T_pQ^4_c$ is $O(4)/U(2)$ and has two connected components diffeomorphic to $SO(4)/U(2) = \{ \tilde{J}_e : e \text{ is a } \pm \text{ oriented frame of } T_pQ^4_c \}$. Hence, the twistor bundle is

$$\mathcal{Z} = O(Q^4_c) \times_{O(4)} O(4)/U(2) = O(Q^4_c)/U(2)$$

and its two connected components are denoted by $\mathcal{Z}_+$ and $\mathcal{Z}_-$. Each projection $\varrho_\pm : \mathcal{Z}_\pm \to Q^4_c$ is a $S^2$-fibre bundle over $Q^4_c$, where $\varrho_\pm$ is the restriction of $\varrho$ on $\mathcal{Z}_\pm$. 

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There is a one-parameter family of Riemannian metrics $g_t$, $t > 0$, defined on $\mathcal{Z}$ that make $\mathcal{Q}_+$ and $\mathcal{Q}_-$ Riemannian submersions. With respect to the decomposition of the tangent bundle of $\mathcal{Z}_\pm$ into horizontal and vertical subbundles as $T\mathcal{Z}_\pm = T^h\mathcal{Z}_\pm \oplus T^v\mathcal{Z}_\pm$, the metric $g_t$ is given by the pull-back of the metric of $\mathcal{Q}_c^4$ to the horizontal subspaces and by adding the $t^2$-fold of the canonical metric of the fibres.

Let $G_{r2}(T\mathcal{Q}_c^4)$ be the Grassmann bundle of oriented 2-planes tangent to $\mathcal{Q}_c^4$. There are projections $\Pi_+: G_{r2}(T\mathcal{Q}_c^4) \to \mathcal{Z}_+$ and $\Pi_-: G_{r2}(T\mathcal{Q}_c^4) \to \mathcal{Z}_-$ defined as follows: if $\zeta \in T_p\mathcal{Q}_c^4$ is an oriented 2-plane, then $\Pi_+(p, \zeta)$ is the complex structure on $T_p\mathcal{Q}_c^4$ corresponding to the rotation by $+\pi/2$ on $\zeta$ and the rotation by $\pm\pi/2$ on $\zeta^\perp$. The Gauss lift $G_f: M \to G_{r2}(T\mathcal{Q}_c^4)$ of an oriented surface $f: M \to \mathcal{Q}_c^4$ is defined by $G_f(p) = (f(p), f_*T_p M)$. The Gauss lifts of $f$ to the twistor bundle are the maps

$$G_+: M \to \mathcal{Z}_+ \quad \text{and} \quad G_-: M \to \mathcal{Z}_-, \quad \text{where} \quad G_\pm = \Pi_\pm \circ G_f.$$

At any point $p \in M$, the Gauss lift $G_\pm$ is given by $G_\pm(p) = (f(p), \tilde{J}_\pm(f(p)))$, where

$$\tilde{J}_\pm(f(p)) = \begin{cases} f_* \circ J(p), & \text{on } f_* T_p M, \\ \pm J(p), & \text{on } N_f T_p M. \end{cases}$$

Let $\{e_k\}_{1 \leq k \leq 4}$ be a positively oriented, local adapted orthonormal frame field of $\mathcal{Q}_c^4$, where $\{e_1, e_2\}$ is in the orientation of $TM$. Denote by $\{\omega_k\}_{1 \leq k \leq 4}$ the corresponding coframe and by $\omega_{kl}$, $1 \leq k, l \leq 4$, the connection forms given by (2). Locally, the pull-back of $g_t$ on $M$ under $G_\pm$, is related to the metric $ds^2$ of $M$ (cf. [23,34]) as follows

$$G_\pm^*(g_t) = ds^2 + \frac{t^2}{4} \left( (\omega_{13} \mp \omega_{24})^2 + (\omega_{23} \pm \omega_{14})^2 \right). \quad (12)$$

The Gauss lift $G_\pm: M \to (\mathcal{Z}_\pm, g_t)$ is called conformal if its induced metric $G_\pm^*(g_t)$ is conformal to $ds^2$, and is called isometric if $G_\pm^*(g_t) = ds^2$. The following has been proved in [34, Prop. 8.2].

**Proposition 11** Let $f: M \to \mathcal{Q}_c^4$ be an oriented surface. The Gauss lift $G_\pm: M \to (\mathcal{Z}_\pm, g_t)$ of $f$ is either conformal, or isometric, if and only if either (i), or (ii), respectively, holds:

(i) The surface $f$ is either minimal, or superconformal with normal curvature $\pm K_N \geq 0$.

(ii) The surface $f$ is minimal and superconformal with normal curvature $\pm K_N \geq 0$.

Adopting the notation of [34], there exists an almost complex structure $\mathcal{J}_+$ on $\mathcal{Z}$ that makes $(\mathcal{Z}_\pm, g_t)$ a Hermitian manifold. The Gauss lift $G_\pm: M \to (\mathcal{Z}_\pm, g_t)$ is called holomorphic if it is holomorphic with respect to $\mathcal{J}_+$. The following has been proved in [34, Prop. 8.1].

**Proposition 12** Let $f: M \to \mathcal{Q}_c^4$ be an oriented surface. The Gauss lift $G_\pm: M \to (\mathcal{Z}_\pm, g_t)$ of $f$ is holomorphic if and only if $f$ is superconformal with normal curvature $\pm K_N \geq 0$.

Immediate consequence of Propositions 11 and 12 is the following.
Proposition 13 Let $f : M \to \mathbb{Q}_c^4$ be an oriented surface with nowhere-vanishing mean curvature vector field. The Gauss lift $G_{\pm} : M \to (\mathbb{Z}_{\pm}, g_t)$ of $f$ is holomorphic if and only if it is conformal.

The Gauss lift $G_{\pm} : M \to (\mathbb{Z}_{\pm}, g_t)$ is called vertically harmonic if its tension field has vanishing vertical component with respect to the decomposition $T\mathbb{Z}_{\pm} = T^h\mathbb{Z}_{\pm} \oplus T^v\mathbb{Z}_{\pm}$. The following has been proved in [51, Prop. 9].

Proposition 14 Let $f : M \to \mathbb{Q}_c^4$ be an oriented surface with mean curvature vector field $H$. The following are equivalent:

(i) The Gauss lift $G_{\pm} : M \to (\mathbb{Z}_{\pm}, g_t)$ of $f$ is vertically harmonic.

(ii) The differential $\Phi_{1\pm}$ is holomorphic.

(iii) The section $H_{\pm}$ is anti-holomorphic.

(iv) $\nabla_{\mathcal{J}X} H = \pm \mathcal{J} \nabla_{\mathcal{J}X} H$, for any $X \in TM$.

For later use, we need the following consequence of Theorem 8.1. in [34]. Notice that for a local orthonormal frame field $\{e_3, e_4\}$ of $N_f M$, the covariant differential of the mean curvature vector field $H = H^3 e_3 + H^4 e_4$ is given by

$$\nabla_{\mathcal{J}X} H = \sum_{a=3}^4 (dH^a + \sum_{b=3}^4 H^{b \omega_{ba}}) \otimes e_a = \sum_{a=3}^4 \sum_{j=1}^2 H^a_j \omega_j \otimes e_a. \quad (13)$$

Proposition 15 Let $f : M \to \mathbb{Q}_c^4$ be an oriented surface. The squared length of the vertical component $\tau^v(G_{\pm})$ of the tension field of the Gauss lift $G_{\pm} : M \to (\mathbb{Z}_{\pm}, g_t)$ of $f$, is given by

$$\|\tau^v(G_{\pm})\|^2 = 4 \left( (H^3_1 \mp H^4_2)^2 + (H^3_2 \pm H^4_1)^2 \right),$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are positively oriented local orthonormal frame fields of $TM$ and $N_f M$, respectively, and $H^a_j, j = 1, 2, a = 3, 4$, is given by (13).

Proof It follows immediately from the proof of [34, Thm. 8.1], where the components of the tension field of $G_{\pm}$ have been computed (see also the proof of [51, Prop. 9]).

Remark 16 (i) Proposition 14 and Lemma 10(ii) imply that any superconformal surface $f : M \to \mathbb{Q}_c^4$ with $\pm K_N \geq 0$ has vertically harmonic Gauss lift $G_{\pm}$.

(ii) From Proposition 14, it follows that both Gauss lifts are vertically harmonic if and only if the mean curvature vector field of the surface is parallel in the normal connection.

(iii) In the case of $\mathbb{R}^4$, $(\mathbb{Z}_{\pm}, g_t)$ is isometric to the product $\mathbb{R}^4 \times S^2(t)$. The Grassmann bundle is trivial $Gr_2(\mathbb{R}^4) \cong \mathbb{R}^4 \times Gr(2, 4)$ and the Gauss lift of $f$ to the Grassmann bundle is given by $G_f = (f, g)$, where $g = (g_+, g_-) : M \to S^2_+ \times S^2_-$ is the Gauss map of $f$. The Gauss lift $G_{\pm}$ of $f$ to the twistor bundle is then given by $G_{\pm} = (f, \sqrt{2t} g_{\pm})$, and it is vertically harmonic if and only if $g_{\pm}$ is harmonic.

(iv) Lagrangian surfaces in $\mathbb{R}^4$ with conformal or harmonic Maslov form constitute examples of surfaces with the component $g_+$ or $g_-$, respectively, harmonic (cf. [7]).
3 The Mixed Connection Forms of Surfaces in $\mathbb{Q}_c^4$

Let $f : M \rightarrow \mathbb{Q}_c^4$ be an oriented surface with $M_0^\pm (f)$ isolated and consider a positively oriented local orthonormal frame field $\{e_1, e_2\}$ of $TM$ defined on an open $U \subset M \setminus M_0^\pm (f)$. By virtue of (5) and (6), the frame field $\{e_1, e_2\}$ determines a unique orthonormal frame field $\{e_3^\pm, e_4^\pm\}$ of $N_f U$ such that

$$H^\pm (e_1, e_2) = \frac{1}{2} \|H^\pm\| (e_3^\pm \pm ie_4^\pm), \tag{14}$$

where

$$e_3^\pm = \|H^\pm\|^{-1} \left( \frac{\alpha_{11} - \alpha_{22}}{2} \pm J^\perp \alpha_{12} \right), \quad e_4^\pm = J^\perp e_3^\pm, \tag{15}$$

and $\alpha_{kl} = \alpha(e_k, e_l), k, l = 1, 2$. Define the 1-form $\Omega^\pm (e_1, e_2)$ on $U$ by

$$\Omega^\pm (e_1, e_2) = 2\omega_{12} \pm \omega_{34}^\pm, \tag{16}$$

where the connection forms $\omega_{12}$ and $\omega_{34}^\pm$ correspond to the dual frame field of $\{e_1, e_2, e_3^\pm, e_4^\pm\}$ and are given by (2). The following proposition shows that $\Omega^\pm (e_1, e_2)$ is independent of the frame field $\{e_1, e_2\}$ and thus well defined on $M \setminus M_0^\pm (f)$.

**Proposition 17** Let $f : M \rightarrow \mathbb{Q}_c^4$ be an oriented surface. If $M_0^\pm (f)$ is isolated, then

(i) There exists a 1-form $\Omega^\pm$ on $M \setminus M_0^\pm (f)$ such that

$$\Omega^\pm \big|_U = \Omega^\pm (e_1, e_2) \tag{17}$$

for every positively oriented local orthonormal frame field $\{e_1, e_2\}$ of $TM$, defined on an open $U \subset M \setminus M_0^\pm (f)$.

(ii) The exterior derivative of $\Omega^\pm$ is globally defined on $M$ and satisfies

$$d\Omega^\pm = -(2K \pm K_N) dM, \tag{18}$$

where $dM$ is the volume element of $M$.

(iii) For every $p \in M_0^\pm (f)$, the limit

$$I^\pm (p) = \lim_{r \to 0} \frac{1}{2\pi} \int_{S_r(p)} \Omega^\pm \tag{19}$$

exists, where $S_r(p)$ is a positively oriented geodesic circle of radius $r$ centred at $p$.

**Proof** (i) Let $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ be positively oriented orthonormal frame fields on an open, simply connected $U \subset M \setminus M_0^\pm (f)$. Consider the frame fields $\{e_3^\pm, e_4^\pm\}$ and $\{\tilde{e}_3^\pm, \tilde{e}_4^\pm\}$ of $N_f U$ determined by $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$, respectively, from (14). Since $U$ is simply-connected, it follows that there exists a smooth function $\tau$ on $U$ such that
\[ \vec{e}_1 - i \vec{e}_2 = \exp(i \tau)(e_1 - i e_2). \]
Moreover, from (4) and (14), we obtain for the normal frame fields that
\[ \vec{e}_3^\pm \pm i \vec{e}_4^\pm = \exp(2i \tau)(e_3^\pm \pm i e_4^\pm). \]
These relations imply that
\[ \vec{\omega}_{12} = \omega_{12} + d \tau \quad \text{and} \quad \vec{\omega}_{34}^\pm = \omega_{34}^\pm 2d \tau. \]
Therefore, from (16), it follows that
\[ \Omega^\pm(\vec{e}_1, \vec{e}_2) = \Omega^\pm(e_1, e_2). \]

By virtue of the above, we define \( \Omega^\pm \) by (17), for an arbitrary positively oriented orthonormal frame field \( \{e_1, e_2\} \), on every simply connected \( U \subset M \smallsetminus M_0^\pm(f) \). It is clear that \( \Omega^\pm \) is globally defined on \( M \smallsetminus M_0^\pm(f) \), and that (17) also holds for frame fields defined on non-simply connected subsets \( U \subset M \smallsetminus M_0^\pm(f) \).

(ii) Using part (i) and (1), exterior differentiation of (16) yields that (18) holds on \( M \smallsetminus M_0^\pm(f) \). Since the right-hand side of (18) is defined globally on \( M \), the proof follows.

(iii) Let \( p \in M_0^\pm(f) \). Consider positively oriented geodesic circles \( S_{r_1}(p) \) and \( S_{r_2}(p) \) centred at \( p \), with \( r_2 < r_1 \), and denote by \( D \) the annular region bounded by these circles. Stokes’ theorem yields that
\[ \int_{S_{r_1}(p)} \Omega^\pm - \int_{S_{r_2}(p)} \Omega^\pm = \int_D d \Omega^\pm. \]
Part (ii) implies that the right-hand side of the above tends to zero as \( r_1, r_2 \to 0 \). Therefore, any sequence \( \int_{S_{r_n}(p)} \Omega^\pm \) with \( r_n \to 0 \) is a Cauchy sequence, and thus, it converges. This completes the proof. \( \square \)

**Remark 18** Let \( F: M \to \Omega^3_c \) be an umbilic-free oriented surface with shape operator \( A \) and corresponding principal curvatures \( k_1, k_2 \), with \( k_1 > k_2 \). Every point of \( M \) has a neighbourhood \( U \) on which there exists a principal frame field \( \{e_1, e_2\} \) of \( F \), i.e. a positively oriented orthonormal frame field of \( T U \) such that \( Ae_l = k_l e_l, l = 1, 2 \). Since a principal frame field is unique up to sign on its domain, there exists a 1-form \( \Omega \) on \( M \) such that \( \Omega|_U = \omega_{12} \), where \( \omega_{12} \) is the connection form corresponding to the dual coframe of a principal frame field \( \{e_1, e_2\} \) on \( U \subset M \). We call \( \Omega \) the principal connection form of \( F \).

The following proposition shows that the mixed connection forms \( \Omega^- \) and \( \Omega^+ \) are the natural generalizations to surfaces in 4-dimensional space forms, of the principal connection form \( \Omega \) of surfaces in 3-dimensional space forms.

**Proposition 19** Assume that \( f: M \to \Omega^+_c \) is the composition of an umbilic-free oriented surface \( F: M \to \Omega^3_c \), \( \tilde{c} \geq c \), with a totally umbilical inclusion \( j: \Omega^3_c \to \Omega^+_c \). Then, \( \Omega^- = \Omega^+ = 2\Omega \), where \( \Omega \) is the principal connection form of \( F \).

**Proof** Let \( \xi \) be the unit normal vector field of \( F \) in \( \Omega^3_c \), and \( A \) the shape operator of \( F \) with respect to \( \xi \). As in Remark 18, let \( k_1, k_2 \), with \( k_1 > k_2 \) be the corresponding principal curvatures of \( F \) and consider a principal frame field \( \{e_1, e_2\} \) of \( F \) on \( U \subset M \).

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Proposition 17(i) and (16) imply that $\Omega^\pm|_U = \Omega^\pm(e_1, e_2) = 2\omega_{12} \pm \omega_{34}^\pm$. Moreover, for the second fundamental form $\alpha$ of $f$ we have that $\alpha_{11} - \alpha_{22} = (k_1 - k_2)j_\star \xi$ and $\alpha_{12} = 0$, where $\alpha_{1,l} = \alpha(e_k, e_l), k, l = 1, 2$. Then, from (15) it follows that $e_3 = e_3^\pm = j_\star \xi$. Since $j_\star \xi$ is parallel in the normal connection of $f$, we obtain that $\omega_{34}^- = \omega_{34}^+ = 0$. Then, Proposition 17(i) and Remark 18 imply that $\Omega^-|_U = \Omega^+|_U = 2\Omega|_U$, and this completes the proof. \hfill $\Box$

Assume that $f : M \to \mathbb{C}$ is a surface with $M_0^\pm(f)$ isolated. Proposition 17(i) allows us to express locally the mixed connection form $\Omega^\pm$, by using (16) for the normalized basic vectors fields corresponding to a complex coordinate. Consider a local complex coordinate $(U, z = x + iy)$ on $M$ and set $e_1 = \partial_z / \lambda, e_2 = \partial_x / \lambda$, where $\lambda > 0$ is the conformal factor. From (7) and (8), it follows that $\phi^\pm = (\lambda^2/2)\mathcal{H}^\pm(e_1, e_2)$. By virtue of (14), this implies that

$$\phi^\pm = \frac{\lambda^2}{4}\|\mathcal{H}^\pm\|(e_3^\pm \mp i e_4^\pm) \quad \text{on} \quad U \setminus M_0^\pm(f).$$

The connection form $\omega_{12}$ of the dual frame field of $\{e_1, e_2\}$ is given by $\omega_{12} = \star d \log \lambda$, where $\star$ is the Hodge star operator. In particular, exterior differentiation gives $d\omega_{12} = \Delta \log \lambda \omega_1 \wedge \omega_2$, where $\Delta = 4\lambda^{-2}\partial \bar{\partial}$ is the Laplacian on $M$, and (1) implies that the Gaussian curvature of $M$ is given by $K = -\Delta \log \lambda$. If $\omega_{34}^\pm$ is the connection form of the dual frame field of $\{e_3^\pm, e_4^\pm\}$, then according to Proposition 17(i), the expression of $\Omega^\pm$ in terms of the complex coordinate $z$ is

$$\Omega^\pm = \star d \log \lambda^2 \pm \omega_{34}^\pm \quad \text{on} \quad U \setminus M_0^\pm(f).$$

Proposition 20 Let $f : M \to \mathbb{C}$ be an oriented surface with $M_0^\pm(f)$ isolated. Consider a simply connected complex chart $(U, z)$ on $M$, with $U \cap M_0^\pm(f) = \{p\}$ and $z(p) = 0$. If there exists a positive integer $m$ such that the differential $\Phi^\pm$ satisfies

$$\Phi^\pm = z^m \hat{\Phi}^\pm \quad \text{on} \quad U, \quad \hat{\Phi}^\pm(p) \neq 0,$$

then $I^\pm(p) = -m$.

**Proof** Let $\Phi^\pm = \phi^\pm dz^2$ on $U$, where $\phi^\pm$ is given by (20) on $U \setminus \{p\}$. For $r > 0$, consider a positively oriented geodesic circle $S_r(p) = \partial B_r(p) \subset U$. Stokes’ theorem implies that $\int_{S_r(p)} \star d \log \lambda = -\int_{B_r(p)} K \omega_1 \wedge \omega_2$, and since the Gaussian curvature is bounded on $B_r(p)$, from Proposition 17(iii) and (21) we obtain that

$$I^\pm(p) = \pm \lim_{r \to 0} \frac{1}{2\pi} \int_{S_r(p)} \omega_{34}^\pm.$$  

Assume that $\hat{\Phi}^\pm$ is given by $\hat{\Phi}^\pm = \hat{\phi}^\pm dz^2$ on $U$. Since $\hat{\phi}^\pm \in N_f^\pm U$ and $\hat{\phi}^\pm \neq 0$ everywhere on $U$, there exist $R \in C^\infty(U; (0, +\infty))$ and an orthonormal frame field $\{e_3, e_4\}$ of $N_f U$, such that $\hat{\phi}^\pm = R(e_3 \pm ie_4)$. Then, from (20) and (22), it follows that

$$\frac{\lambda^2}{2}\|\mathcal{H}^\pm\|(e_3^\pm \mp i e_4^\pm) = z^m R(e_3 \pm ie_4) \quad \text{on} \quad U \setminus \{p\}.$$  

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Let \( c(s), s \in [0, 2\pi], \) be a parametrization of \( S_r(p) \) as a simple closed curve. There exists a smooth function \( \tau(s), s \in [0, 2\pi], \) such that along \( c, \) the frame fields \( \{e_3^\pm, e_4^\pm\} \) and \( \{e_3, e_4\} \) are related by

\[
e_3^\pm(s) \pm ie_4^\pm(s) = e^{\mp i\tau(s)}(e_3(s) \pm ie_4(s)). \tag{25}
\]

Therefore,

\[
\frac{1}{2\pi} \int_{S_r(p)} \omega_3^\pm - \frac{1}{2\pi} \int_{S_r(p)} \omega_34 = \frac{1}{2\pi} \int_{S_r(p)} d\tau. \tag{26}
\]

We argue that the right-hand side of (26) is equal to \( \mp m. \) From (24) and (25) it follows that along \( c \) we have

\[
\frac{\lambda(s)^2 \|\mathcal{H}^\pm\|(s)}{2R(s)} = (z(s))^m e^{\pm i\tau(s)}. \]

Let \( k(s) \) be the function at the left-hand side of the above. Since \( k(s) > 0, s \in [0, 2\pi], \) it follows that

\[
\log k(s) = \log((z(s))^m e^{\pm i\tau(s)}). \]

Differentiating the above with respect to \( s, \) then integrating from 0 to \( 2\pi, \) and taking into account that \( k(0) = k(2\pi), \) we obtain that

\[
0 = \log k(2\pi) - \log k(0) = m \int_0^{2\pi} \frac{z'(s)}{z(s)} \, ds \pm i \int_0^{2\pi} \tau'(s) \, ds,
\]

or, equivalently

\[
\frac{1}{2\pi} \int_{S_r(p)} d\tau = \mp \frac{m}{2\pi i} \int_{z(S_r(p))} \frac{dw}{w} = \mp m. \tag{27}
\]

Since \( \omega_34 \) is defined everywhere on \( U \) and \( K_N \) is bounded on \( B_r(p), \) by using (1), we obtain that

\[
\lim_{r \to 0} \int_{S_r(p)} \omega_34 = \lim_{r \to 0} \int_{B_r(p)} d\omega_34 = - \lim_{r \to 0} \int_{B_r(p)} K_N \omega_1 \wedge \omega_2 = 0.
\]

Therefore, by taking limits in (26) and using (23) and (27), the proof follows. \( \square \)

**Theorem 21** Let \( f : M \to \mathbb{Q}_c^4 \) be a compact-oriented surface. If \( M_0^\pm(f) \) is isolated, then

\[
2\chi \pm \chi_N = \sum_{p \in M_0^\pm(f)} I^\pm(p).
\]

**Proof** Assume that \( M_0^\pm(f) \neq \emptyset \) and let \( M_0^\pm(f) = \{p_1, \ldots, p_k\}, \) where \( k \) is a positive integer. For a sufficiently small \( r > 0, \) let \( M_r = M \setminus (B_r(p_1) \cup \cdots \cup B_r(p_k)), \) where
$B_r(p_j)$ is the geodesic ball of radius $r$, centred at $p_j$, $j = 1, \ldots, k$. Stokes’ theorem implies that

$$\int_{M_r} d\Omega^\pm = - \sum_{j=1}^{k} \int_{S_r(p_j)} \Omega^\pm,$$

where $\Omega^\pm$ is the form of Proposition 17(i), and $S_r(p_j) = \partial B_r(p_j)$ is positively oriented with respect to its interior. From the above and (18), we obtain that

$$2\chi \pm \chi_N = -\frac{1}{2\pi} \lim_{r \to 0} \int_{M_r} d\Omega \pm = \sum_{j=1}^{k} \frac{1}{2\pi} \lim_{r \to 0} \int_{S_r(p_j)} \Omega \pm,$$

and the proof follows from (19). If $M_{0}^\pm(f) = \emptyset$, the proof follows by integrating (18) on $M$.

In the sequel, we provide some applications of Theorem 21. The first one is a short proof of the following result due to Asperti [1].

**Theorem 22** If a compact 2-dimensional Riemannian manifold immerses isometrically into $Q^4_c$ with everywhere nonvanishing normal curvature, then it is homeomorphic either to the sphere $S^2$, or to the real projective space $\mathbb{R}P^2$.

**Proof** Let $\tilde{M}$ be a compact 2-dimensional Riemannian manifold, and $f : \tilde{M} \to Q^4_c$ an isometric immersion with $K_N \neq 0$ everywhere. Assume that $\tilde{M}$ is oriented and that $\pm K_N > 0$. Then, $M_{0}^\pm(f) = \emptyset$ and Theorem 21 implies that $2\chi = \pm \chi_N$. Since $\pm \chi_N > 0$, it follows that $\chi > 0$ and thus, $\tilde{M}$ is homeomorphic to $S^2$. If $\tilde{M}$ is non-orientable, then we apply the previous procedure to the lift of $f$ to the orientable double covering of $\tilde{M}$, and we conclude that $\tilde{M}$ is homeomorphic to $\mathbb{R}P^2$.

We mention here that a long-standing open problem posed by S.S. Chern [9, p. 45] is to investigate the existence of compact surfaces of negative Gaussian curvature in $\mathbb{R}^4$. In this direction, we obtain the following result.

**Theorem 23** Let $f : M \to Q^4_c$ be an isometric immersion of a compact-oriented 2-dimensional Riemannian manifold $M$. If $c \geq 0$ and the normal curvature of $f$ does not change sign, then the Gaussian curvature $K$ of $M$ satisfies $\max K \geq 0$.

**Proof** Arguing indirectly, suppose that $\max K < 0$. Since $c \geq 0$, this implies that $M_{1}(f) = \emptyset$. Since $K_N$ does not change sign, we may assume that $\pm K_N \geq 0$. Therefore, $M_{0}^\pm(f) = \emptyset$, and as in the proof of Theorem 22, we obtain that $M$ is homeomorphic to $S^2$. Then, the Gauss-Bonnet theorem implies that there exist points of $M$ with positive Gaussian curvature, and this is a contradiction.

Immediate consequences of the above theorem are the following corollaries; the first one has been proved by Peng and Tang [50] for surfaces in $\mathbb{R}^4$.  

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Corollary 24 Let \( f : M \rightarrow Q^4_c \), \( c \geq 0 \), be an isometric immersion of a compact-oriented 2-dimensional Riemannian manifold \( M \). If the normal curvature of \( f \) is constant, then there exists a point of \( M \) with nonnegative Gaussian curvature.

Corollary 25 Let \( M \) be a compact-oriented 2-dimensional Riemannian manifold with Gaussian curvature \( K < 0 \). If there exists an isometric immersion \( f : M \rightarrow Q^4_c \), \( c \geq 0 \), then its normal curvature satisfies \( \min K_N < 0 < \max K_N \).

4 Isotropically Isothermic Surfaces

We introduce here the notion of isotropically isothermic surfaces in 4-dimensional space forms, as a generalization of the notion of isothermic surfaces in 3-dimensional space forms. We recall that an umbilic-free surface \( F : M \rightarrow Q^3_c \) is called isothermic if it admits a conformal curvature line parametrization around every point. This is equivalent (see for instance [35]) with the co-closeness of the principal connection form \( \Omega \) of \( F \). Inspired by Proposition 19, we give the following definitions.

Let \( f : M \rightarrow Q^4_c \) be an oriented surface with \( M_0^\pm(f) = \emptyset \). A point \( p \in M \) is called a \( \pm \) isotropically isothermic point for \( f \) if \( d \times \Omega^\pm(p) = 0 \). The surface \( f \) is called \( \pm \) (totally non) isothermic if every point is \( \pm \) (non-)isotropically isothermic. Moreover, \( f \) is called strongly (totally non) isothermic if it is both \( + \) and \( - \) (totally non) isotropically isothermic. In the sequel, a \( \pm \) (totally non) isothermic surface is simply called (half totally non) isothermic, whenever we do not need to distinguish between the signs.

The following lemma provides a characterization of isotropically isothermic points in terms of a complex coordinate. Notice that if \( f : M \rightarrow Q^4_c \) is a surface with \( M_0^\pm(f) = \emptyset \), then for every complex chart \((U, z)\) on \( M \), there exists a smooth complex function \( h^\pm \) on \( U \), such that the section \( \phi^\pm \) of \( N_f^\pm U \) given by (8) satisfies on \( U \) the relation

\[
\nabla^\pm_\bar{\partial} \phi^\pm = h^\pm \phi^\pm.
\] (28)

Lemma 26 Let \( f : M \rightarrow Q^4_c \) be an oriented surface with \( M_0^\pm(f) = \emptyset \). A point \( p \in M \) is \( \pm \) isotropically isothermic for \( f \) if and only if for every complex chart \((U, z)\) around \( p \), the function \( h^\pm \) satisfies

\[
\text{Im } h^\pm_z(p) = 0.
\]

Proof Let \((U, z = x + iy)\) be a complex chart around \( p \) and set \( e_1 = \partial_x / \lambda, e_2 = \partial_y / \lambda \), where \( \lambda > 0 \) is the conformal factor. Consider the frame field \( \{e_3^\pm, e_4^\pm\} \) of \( N_f^\pm U \) determined by \( \{e_1, e_2\} \) from (14). Then (20) and (21) hold on \( U \). From (28) and (20) it follows that

\[
\nabla^\pm_\bar{\partial} \phi^\pm = \frac{\lambda^2}{4} \|H^\pm\| h^\pm \left( e_3^\pm \pm i e_4^\pm \right) \text{ on } U.
\] (29)

Differentiating (20) with respect to \( \bar{\partial} \) in the normal connection, we obtain

\[
\nabla^\pm_\bar{\partial} \phi^\pm = \frac{1}{4} \left( \bar{\partial} (\lambda^2 \|H^\pm\|) \mp i \lambda^2 \|H^\pm\| \omega_{34}^\pm(\bar{\partial}) \right) \left( e_3^\pm \pm i e_4^\pm \right).
\]
The above and (29) yield that
\[
h^\pm = \tilde{\partial} \log(\lambda^2 \|H^\pm\|) \mp i\omega^\pm_{34}(\tilde{\partial}). \tag{30}
\]
Differentiating (30) with respect to \(z\), and taking the imaginary part yields
\[
\frac{4}{\lambda^2} \Im h^\pm_z = \mp \left( e_1(\log \lambda)\omega^\pm_{34}(e_1) + e_2(\log \lambda)\omega^\pm_{34}(e_2) + e_1(\omega^\pm_{34}(e_1)) + e_2(\omega^\pm_{34}(e_2)) \right).
\]
From (21) and the above, we obtain that \(d\star\Omega^\pm = -(4/\lambda^2)\Im h^\pm_z \omega_1 \wedge \omega_2\), and this completes the proof.

\[\square\]

**Proposition 27** Let \(f : M \to \mathbb{Q}^4\) be an oriented surface with \(M^\pm_0(f) = \emptyset\). The surface \(f\) is \(\pm\) isotropically isothermic if and only if for every simply connected complex chart \((U, z)\), the section \(\phi^\pm\) given by (8) has the form
\[
\phi^\pm = D^\pm \xi^\pm, \tag{31}
\]
where \(D^\pm \in C^\infty(U; (0, +\infty))\), and \(\xi^\pm\) is a nowhere-vanishing holomorphic section of \(N^\pm f U\).

**Proof** Let \((U, z)\) be a simply connected complex chart. Since \(M^\pm_0(f) = \emptyset\), the section \(\phi^\pm\) is given on \(U\) by (20). Appealing to Proposition 17(i), we express \(\Omega^\pm\) on \(U\) in terms of \(z\), by (21).

Assume that \(f\) is \(\pm\) isotropically isothermic. From (21) it follows that \(d\star\omega^\pm_{34} = 0\) and thus, there exists a smooth positive function \(r^\pm\) on \(U\) such that
\[
\omega^\pm_{34} = \mp d\log r^\pm. \tag{32}
\]
Taking into account (20), we define \(D^\pm\) and \(\xi^\pm\), respectively, by
\[
D^\pm = \frac{\lambda^2 \|H^\pm\|}{4r^\pm} \quad \text{and} \quad \xi^\pm = r^\pm(e^\pm_3 \pm ie^\pm_4). \tag{33}
\]
Differentiating \(\xi^\pm\) with respect to \(\tilde{\partial}\) in the normal connection yields
\[
\nabla_{\tilde{\partial}} \xi^\pm = \frac{1}{r^\pm} \left( (\log r^\pm)_z \mp i\omega^\pm_{34}(\tilde{\partial}) \right) (e^\pm_3 \pm ie^\pm_4). \tag{34}
\]
From the above and (32), it follows that \(\xi^\pm\) is holomorphic.

Conversely, assume that (31) holds on \(U\). By setting \(r^\pm = \|\xi^\pm\|/\sqrt{2}\), from (31) and (20), we obtain (33). Therefore, (34) is valid. Since \(\xi^\pm\) is holomorphic, from (34) we obtain (32). Equations (21) and (32) imply that \(d\star\Omega^\pm = 0\) on \(U\). Since \(U\) is arbitrary, it follows that \(f\) is \(\pm\) isotropically isothermic.

\[\square\]
The characterization of isotropic isothermicity provided by Proposition 27, also
makes sense for oriented surfaces immersed in orientable 4-dimensional Riemannian
manifolds of not necessarily constant sectional curvature and can be used as the defi-
nition of isotropic isothermicity for such surfaces.

**Proposition 28** Let $N$ be a Riemann surface and $F : N \to \mathbb{Q}_c^4$ a conformal immersion. The property of $F$ equipped with its induced metric being isotropically isothermic is invariant under conformal changes of the metric of $\mathbb{Q}_c^4$.

**Proof** Let $f : M \to \mathbb{Q}_c^4$ be the isometric immersion induced by $F$, where $M = (N, ds^2)$ and $ds^2 = F^*\langle \cdot, \cdot \rangle$. Consider the Riemannian manifold $\tilde{\mathbb{Q}}_c^4$, obtained from $\mathbb{Q}_c^4$ by the conformal change $\langle \cdot, \cdot \rangle_\mu = \mu^2 \langle \cdot, \cdot \rangle$ of its metric, where $\mu \in C^\infty(\mathbb{Q}_c^4; (0, +\infty))$, equipped with the same orientation with $\mathbb{Q}_c^4$. The conformal immersion $F$ induces the isometric immersion $\tilde{f} : \tilde{M} \to \tilde{\mathbb{Q}}_c^4$, where $\tilde{M} = (N, d\tilde{z}^2)$ and $d\tilde{z}^2 = \mu^2 ds^2$.

Assume that $f$ is $\pm$ isotropically isothermic. We argue that $\tilde{f}$ is also $\pm$ isotropically isothermic. The normal bundles of $f$ and $\tilde{f}$ coincide as vector bundles over $N$, and since their bundle metrics are conformal, they have the same complex structure $J$. It follows easily (see for instance [17]) that the second fundamental forms $\alpha, \tilde{\alpha}$, and the normal connections $\nabla^\perp, \tilde{\nabla}^\perp$, of $f$ and $\tilde{f}$, respectively, are related by

$$\tilde{\alpha}(X, Y) = \alpha(X, Y) - \frac{1}{\mu} \langle X, Y \rangle (\text{grad}\, \mu)^\perp \quad \text{and} \quad \tilde{\nabla}^\perp_X \eta = \nabla^\perp_X \eta + \frac{1}{\mu} (\text{grad}\, \mu, X) \eta,$$

for all $X, Y \in TN$ and $\eta \in N_f M = N_f \tilde{M}$, where grad denotes the gradient with respect to $\langle \cdot, \cdot \rangle$. Let $(U, z)$ be a complex chart on $\tilde{M}$ with conformal factor $\tilde{\lambda}$. Then, $(U, z)$ is also a complex chart on $M$ with conformal factor $\lambda = \tilde{\lambda}/\mu$. From the first equation in (35), it follows that the Hopf differentials $\Phi, \tilde{\Phi}$ of $f, \tilde{f}$, respectively, coincide. In particular, if $\Phi^\pm$ is given by (8) and $\Phi^\pm = \tilde{\Phi}^\pm d\tilde{z}^2$ on $U$, then $\phi^\pm = \tilde{\phi}^\pm$. Proposition 27 implies that $\phi^\pm = D^\pm \xi^\pm$, where $D^\pm$ is a smooth positive function on $U$ and $\xi^\pm$ a nowhere-vanishing $\nabla^\perp$-holomorphic local section. Then, we have that

$$\tilde{\phi}^\pm = \tilde{\phi}^\pm = \tilde{D}^\pm \tilde{\xi}^\pm,$$

where $\tilde{D}^\pm = \mu D^\pm$ and $\tilde{\xi}^\pm = \frac{1}{\mu} \xi^\pm$.

Since $\tilde{\xi}^\pm$ is $\tilde{\nabla}^\perp$-holomorphic, from the second equation in (35), we obtain that $\tilde{\xi}^\pm$ is $\tilde{\nabla}^\perp$-holomorphic. From Proposition 27, it follows that $\tilde{f}$ is $\pm$ isotropically isothermic. \hfill \Box

**Remark 29** Adopting the notation of the proof of Proposition 28, by using (15), (16), and Proposition 17(i), it is easy to see that if the metric $\langle \cdot, \cdot \rangle_\mu$ has constant curvature, then the corresponding mixed connection forms $\Omega^\pm, \tilde{\Omega}^\pm$ of $f$ and $\tilde{f}$, are related by $\tilde{\Omega}^\pm = \Omega^\pm + 2* d \log \mu$.

**Example 30** We provide classes of isotropically isothermic surfaces $f : M \to \mathbb{Q}_c^4$. The surfaces in the classes (iii) and (iv) below are always strongly isotropically isothermic.

(i) **Surfaces with a vertically harmonic Gauss lift (neither minimal, nor superconfor-
mal):**
Assume that the Gauss lift $G_\pm$ of $f$ is vertically harmonic and that $M_0^{\pm}(f) = \emptyset$. Proposition 14 implies that $\Phi^\pm$ is holomorphic, and from Proposition 27, it follows that $f$ is $\pm$ isotropically isothermic. According to Proposition 28, by appropriate conformal changes of the metric of (possibly part of) $Q_\varepsilon^4$, we obtain from $f$ other $\pm$ isotropically isothermic surfaces in $Q_\varepsilon^4$ whose corresponding Gauss lift $\tilde{G}_\pm$ is not vertically harmonic.

(ii) Minimal superconformal surfaces:
Assume that $f$ is minimal and superconformal, with $M_0^0(f) = \emptyset$. For the Hopf differential $\Phi^1$, Lemma 10(i) implies that $\Phi^\pm \equiv 0$, and thus, $\Phi \equiv \Phi^\pm$. The Codazzi equation yields that $\Phi^1$ is holomorphic, and from Proposition 27, it follows that $f$ is $\pm$ isotropically isothermic. Since the superconformal property is conformally invariant, by virtue of Proposition 28, we obtain from $f$, non-minimal superconformal surfaces in $Q_\varepsilon^4$ that are isotropically isothermic.

(iii) Non-superconformal minimal surfaces:
Assume that $f$ is minimal with $M_0(f) = \emptyset$. The Codazzi equation implies that the Hopf differential of $f$ is holomorphic, and Proposition 27 yields that $f$ is strongly isotropically isothermic. Proposition 28 implies that under appropriate conformal changes of the metric of $Q_\varepsilon^4$, the surface $f$ gives rise to non-minimal, strongly isotropically isothermic surfaces in $Q_\varepsilon^4$. In particular, since the flatness of the normal bundle of a surface in $Q_\varepsilon^4$ is a conformally invariant property, it follows that such a surface has nonflat normal bundle, if the normal bundle of $f$ is nonflat.

(iv) Isothermic surfaces in totally umbilical hypersurfaces:
Assume that $f$ is the composition of an umbilic-free surface $F: M \to Q_\varepsilon^3$, $\varepsilon \geq c$, with a totally umbilical inclusion. Proposition 19 implies that $f$ is strongly isotropically isothermic if and only if $F$ is isothermic.

4.1 Lines of Curvature

We recall that (cf. [26,29]) a principal direction of an oriented surface $f: M \to Q_\varepsilon^4$ at $p \in M$, is a line in $T_pM$ generated by a unit vector which makes extremal the length of $\alpha(X, X)$, where $\alpha$ is the second fundamental form of $f$, and $X$ varies on the unit circle of $T_pM$. If $p \in M \smallsetminus M_0(f)$, then there exist four principal directions of $f$ at $p$. The principal curvature lines of $f$ are those curves on $M \smallsetminus M_0(f)$ which are tangent to principal directions.

An oriented surface $f: M \to Q_\varepsilon^4$ is called isothermic (cf. [49]) if around every point of $M$, there exists a complex chart with the property that its corresponding basic vector fields diagonalize at every point of its domain, all shape operators of $f$. It is straightforward to show that a surface is isothermic if and only if around every point of $M$, there exists a complex chart $(U, z = x + iy)$ such that $\alpha(e_1, e_2) = 0$ at every point of $U$, where $e_1 = \partial_x/\lambda$, $e_2 = \partial_y/\lambda$, and $\lambda > 0$ is the conformal factor.

Proposition 31 Let $f: M \to Q_\varepsilon^4$ be an oriented surface with $M_0(f) = \emptyset$.

(i) Assume that $f$ is strongly isotropically isothermic. Then it admits a conformal principal curvature line parametrization around every point. In particular, $f$ is isothermic if it has flat normal bundle.
(ii) If \( f \) is isotropically isothermic and admits a conformal principal curvature line parametrization around every point, then it is strongly isotropically isothermic. In particular, if \( f \) is isothermic and isotropically isothermic, then it is strongly isotropically isothermic.

**Proof** Let \( p \in M \). Since \( \mathcal{E}(p) \) is not a circle, from [51, Lemma 6], it follows that there exist positively oriented local orthonormal frame fields \( \{ e_1, e_2 \} \) of \( TM, \{ e_3, e_4 \} \) of \( N_f M \), on a neighbourhood \( U \) of \( p \), and \( \kappa, \mu \in C^\infty(U) \) with \( \kappa > |\mu| \), such that \( \alpha_{11} - \alpha_{22} = 2\kappa e_3 \) and \( \alpha_{12} = \mu e_4 \), where \( \alpha_{kl} = \alpha(e_k, e_l), k, l = 1, 2 \). In particular, from the proof of [51, Lemma 6], it follows that \( e_3 \) is in the direction of the major axis of \( \mathcal{E}_f \), and \( \kappa, |\mu| \) are the lengths its semi-axes at every point of \( U \). Then, (15) implies that \( e_3^+ = e_3^- \), and thus, \( \omega_{34}^+ = \omega_{34}^- \). From Proposition 17(i) it follows that

\[
\Omega^+ + \Omega^- = 4\omega_{12} \quad \text{on} \quad U,
\]  

where \( \omega_{12} \) is the connection form corresponding to the dual frame field of \( \{ e_1, e_2 \} \).

(i) Since \( \Omega^+ \) and \( \Omega^- \) are both co-closed, from (36) it follows that \( d\star\omega_{12} = 0 \). Therefore, there exists a positive function \( \lambda \) on \( U \) such that \( \star\omega_{12} = -d\log \lambda \). This implies that the forms \( \lambda^{-1}\omega_1, \lambda^{-1}\omega_2 \) are closed, and thus, there exist smooth functions \( x, y \) on \( U \) such that \( dx = \lambda^{-1}\omega_1, dy = \lambda^{-1}\omega_2 \). Then, \( z = x + iy \) is a complex coordinate on \( U \) with conformal factor \( \lambda \), such that \( e_1 = \partial_x/\lambda, e_2 = \partial_y/\lambda \). In particular, if \( f \) has flat normal bundle, then (3) implies that \( \mu = 0 \). Therefore, \( a_{12} = 0 \), and thus, \( f \) is isothermic.

(ii) Suppose that \( f \) is \( \pm \) isotropically isothermic and consider a conformal principal curvature line parametrization \( (U, z = x + iy) \) around \( p \in M \), with conformal factor \( \lambda > 0 \). Then, the connection form of the dual frame field of \( \{ \tilde{e}_1 = \partial_x/\lambda, \tilde{e}_2 = \partial_y/\lambda \} \) is given by \( \tilde{\omega}_{12} = \star d\log \lambda \).

We claim that there exists a conformal principal curvature line parametrization on \( U \), with normalized basic vector fields \( e_1, e_2 = Je_1 \), such that \( \alpha_{11} \) is a vertex of \( \mathcal{E}_f \) determined by the major axis at any point of \( U \). Indeed, in the case where \( \alpha(\tilde{e}_1, \tilde{e}_1) \) is a vertex of \( \mathcal{E}_f \) determined by the minor axis, we consider the frame field \( \{ e_1, e_2 \} \) given by \( e_1 - ie_2 = \exp(i\pi/4)(\tilde{e}_1 - i\tilde{e}_2) \). Then, the connection form of its dual frame field is given by \( \omega_{12} = \tilde{\omega}_{12} \), and the vector field \( \alpha_{11} \) is a vertex of \( \mathcal{E}_f \) determined by the major axis. Since \( \omega_{12} \) is co-closed, as in the proof of part (i), it follows that there exists a complex coordinate with normalized basic vector fields \( e_1 \) and \( e_2 \).

For the frame field \( \{ e_1, e_2 \} \), equation (36) is valid. Since \( d\star\Omega^\pm = 0 \), from (36) it follows that \( d\star\Omega^\mp = 0 \) and thus, \( f \) is strongly isotropically isothermic. The rest of the proof is obvious. \( \square \)

The following example shows that the converse of Proposition 31(i) is not true in general. Bearing in mind Example 30(iv), it also shows that the classes of isothermic and isotropically isothermic surfaces overlap, but no one of these classes is contained in the other.

**Example 32** Isothermic surfaces in \( \mathbb{R}^4 \) that are strongly totally non-isotropically isothermic:
Let \( \gamma_j : I_j \to \mathbb{R}^2 \) be a smooth curve parametrized by its arc length \( s_j \), where \( I_j \) is an open interval, \( j = 1, 2 \). Let \( n_j \) be the normal vector field of \( \gamma_j \) such that \( \{ t_j = \dot{\gamma}_j, n_j \} \) is positively oriented, where the dot denotes the derivative with respect to \( s_j \), \( j = 1, 2 \). By setting \( M = I_1 \times I_2 \) and \( z = s_1 + is_2 \), it is clear that \( z \) is a global complex coordinate on \( M \) with basic vector fields \( e_1, e_2 \), where \( e_j = \partial/\partial s_j \), \( j = 1, 2 \). Moreover, the connection form of the corresponding coframe of \( \{e_1, e_2\} \) satisfies \( \omega_{12} = 0 \). We consider the product surface \( f : M \to \mathbb{R}^4, f = \gamma_1 \times \gamma_2 \). Then, the adapted to \( f \) frame field

\[
\{ f_*e_1 = (t_1, 0), N_1 = (n_1, 0), f_*e_2 = (0, t_2), N_2 = (0, n_2) \}
\]

is positively oriented in \( \mathbb{R}^4 \). Therefore, \( J^\perp N_1 = -N_2 \). Let \( k_j \) be the curvature of \( \gamma_j \), \( j = 1, 2 \). For the second fundamental form \( \alpha \) of \( f \), we have \( \alpha_{11} = k_1 N_1, \alpha_{22} = k_2 N_2 \), and \( \alpha_{12} = 0 \), where \( \alpha_{kl} = \alpha(e_k, e_l) \), \( k, l = 1, 2 \). Since \( \alpha_{12} = 0 \), it follows that \( f \) is isothermic.

Assume furthermore that \( f \) is umbilic-free, or equivalently that there do not exist points \( (s_1, s_2) \) on \( M \) such that \( k_1(s_1) = k_2(s_2) = 0 \), and set

\[
e_3 = \frac{\alpha_{11} - \alpha_{22}}{\|\alpha_{11} - \alpha_{22}\|} = \frac{1}{\sqrt{k_1^2 + k_2^2}}(k_1 N_1 - k_2 N_2), \quad e_4 = J^\perp e_3.
\]

Then, (15) implies that \( e_3 = e_3^- = e_3^+ \). Since \( \omega_{12} = 0 \), from Proposition 17(i) and (16), it follows that \( f \) is strongly isotropically isothermic if and only if \( \omega_{34} \) is co-closed. An easy computation shows that at every point of \( M \), the equation \( d \ast \omega_{34} = 0 \) is equivalent to the differential equation

\[
k_1 \ddot{k}_2 - \ddot{k}_1 k_2 + 2k_1 k_2 \frac{(\dot{k}_1)^2 - (\dot{k}_2)^2}{k_1^2 + k_2^2} = 0
\]

(37)

for the curvatures of \( \gamma_1 \) and \( \gamma_2 \), where each dot denotes a derivative of \( k_j \) with respect to \( s_j \), \( j = 1, 2 \). Clearly, if \( k_j(s_j) = c_j s_j, 0 \neq c_j \in \mathbb{R}, j = 1, 2 \), and \( c_1 \neq c_2 \), then for \( s_1 s_2 > 0 \), it follows from (37) that \( f \) is strongly totally non-isotropically isothermic.

We recall (cf. [47]) that a mean-directional curvature line of an oriented surface \( f : M \to \mathbb{Q}^4_c \), is a curve on \( M \) which is tangent at every point to a unit vector field, in which image under the second fundamental form of \( f \) is parallel to the mean curvature vector field. There exist two families of mean-directional curvature lines, in which common singularities are the minimal points of \( f \) and the points where the ellipse of curvature \( E_f \) degenerates into a line segment, parallel to the mean curvature vector.

**Proposition 33** Let \( f : M \to \mathbb{Q}^4_c \) be an umbilic-free superconformal surface with nowhere-vanishing mean curvature vector field. The surface \( f \) is isotropically isothermic if and only if it admits a conformal mean-directional curvature line parametrization around every point.
Proof Since \( M_1(f) = \emptyset \), by virtue of Lemma 10(ii), we may assume that \( \pm K_N < 0 \). Then, Lemma 10(ii) implies that \( \Phi \equiv 0 \), and from Proposition 14, it follows that the Gauss lift \( G_\pm \) of \( f \) is vertically harmonic. Consider the orthonormal frame field \( \{ e_3 = H/\| H \|, e_4 = J^\perp e_3 \} \) of the normal bundle. Using Proposition 14(iv) and (2), we obtain that the connection form of its dual frame field is given by

\[
\omega_{34} = \mp \ast d \log \| H \|. \tag{38}
\]

Let \( r > 0 \) be the radius of \( \mathcal{E}_f \) at every point of \( M \) and consider a positively oriented local orthonormal frame field \( \{ e_1, e_2 \} \) of \( TM \), such that

\[
\alpha_{11} = (\| H \| + r)e_3, \quad \alpha_{22} = (\| H \| - r)e_3.
\]

Since \( \mathcal{H}_\mp(e_1, e_2) = 0 \), from (5) and the above it follows that

\[
\Omega^\pm = 2\omega_{12} - \ast d \log \| H \|, \tag{39}
\]

where \( \omega_{12} \) is the connection form corresponding to the dual frame field of \( \{ e_1, e_2 \} \).

A conformal parametrization in which coordinate curves are mean-directional curvature lines exists around every point of \( M \), if and only if \( \omega_{12} \) is co-closed. The proof follows immediately from (39).

\( \square \)

4.2 Infinitesimal Deformations

Let \( f : M \to \mathbb{R}^4 \) be an oriented surface and denote by \( N \) the underlying Riemann surface of \( M \), such that \( M = (N, ds^2) \). A deformation of the immersion \( f \) is a smooth map \( F : I \times N \to \mathbb{R}^4 \) with \( F(0, \cdot) = f \), where \( I \subset \mathbb{R} \) is an open interval containing 0. For every \( t \in I \), we denote by \( f_t \) the isometric immersion \( F(t, \cdot) : M_t \to \mathbb{R}^4 \), where \( M_t = (N, ds^2_t) \). At any point of \( N \), the Taylor expansion of \( f_t \) around \( t = 0 \) is

\[
f_t = f + tT + o(t),
\]

where \( T = F_\ast \partial / \partial t |_{t=0} = \delta f_t \), and \( \delta = (d/dt)|_{t=0} \) is the variational operator. The deformation \( F \) is called isometric if \( ds_t^2 = ds^2 \) for every \( t \in I \) and is called infinitesimal isometric if \( \delta ds_t^2 = 0 \). If \( F \) is infinitesimal isometric, then the section \( T \in \Gamma(f_\ast(T\mathbb{R}^4)) \) defined above is called the bending field of \( F \), and by using the Taylor expansion of \( f_t \), it follows that it satisfies

\[
\langle \tilde{\nabla}_X T, f_\ast Y \rangle + \langle f_\ast X, \tilde{\nabla}_Y T \rangle = 0, \quad X, Y \in TM, \tag{40}
\]

where \( \tilde{\nabla} \) is the connection of \( f_\ast(T\mathbb{R}^4) \).

Every section \( T \) of \( f_\ast(T\mathbb{R}^4) \) satisfying (40) is called a bending field, and such sections always exist; the variational vector field \( T \) of an isometric deformation of \( f \) produced by a smooth one-parameter family of isometries of \( \mathbb{R}^4 \) satisfies (40) and is
called a trivial bending field. A bending field $T$ is trivial (cf. [17]) if and only if there exist constant vectors $C \in \Lambda^2\mathbb{R}^4$ and $v \in \mathbb{R}^4$, such that

$$T = C \cdot f + v,$$

where the dot multiplication of a simple 2-vector $X \wedge Y \in \Lambda^2\mathbb{R}^4$ with $Z \in \mathbb{R}^4$ is defined by $X \wedge Y \cdot Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$, extends linearly to every element of $\Lambda^2\mathbb{R}^4$ and is skew-symmetric with respect to the inner product of $\mathbb{R}^4$. An infinitesimal isometric deformation is called either trivial, or nontrivial, if its bending field is either trivial on $M$, or nontrivial on an open and dense subset of $M$, respectively. Two bending fields $T_1, T_2 \in \Gamma(f^*(T\mathbb{R}^4))$ are equivalent, and we identify them, if there exist $0 \neq c \in \mathbb{R}$ and a trivial bending field $T_0$, such that $T_2 = cT_1 + T_0$.

Every bending field $T \in \Gamma(f^*(T\mathbb{R}^4))$ determines a unique infinitesimal isometric deformation of the form:

$$f_t = f + tT,$$  \hspace{1cm} (41)

for $t$ in some fixed interval $I$, which is always assumed to be sufficiently small for our purposes. In the sequel, we deal only with deformations of the above form, and we write $F : I \times M \to \mathbb{R}^4$ to denote such a deformation of the surface $f : M \to \mathbb{R}^4$.

For the proofs of Theorems 1 and 2, we need a version of the fundamental theorem of infinitesimal isometric deformations, recently proved in [15] in invariant form, in terms of moving frames. A statement of the fundamental theorem in this context, and also some auxiliary results can be found in the survey paper [33]. Because it turned out to be impossible for the author to find detailed proofs, or even proofs of some of these results (some references in [33] are in Russian, and others are really hard to find), and arguments involving moving frames jointly with Taylor expansions are quite delicate, we also provide neat proofs of everything that we use to obtain our results.

**Lemma 34** Let $F : I \times M \to \mathbb{R}^4$ be an infinitesimal isometric deformation.

(i) If $M$ is simply connected, then every orthonormal frame field $\{e_1, e_2\}$ on $M$ extends to a smooth with respect to $t$, orthonormal frame field $\{e_1(t), e_2(t)\}$ on $M_t$, with dual frame field $\{\omega_1(t), \omega_2(t)\}$ and corresponding connection form $\omega_{12}(t)$, such that

$$\delta e_j(t) = \delta \omega_j(t) = 0, \quad j = 1, 2, \quad \text{and} \quad \delta\omega_{12}(t) = 0.$$  \hspace{1cm} (42)

(ii) Every orthonormal frame field $\{e_3, e_4\}$ of $N_f M$ locally extends to a smooth with respect to $t$, local orthonormal frame field $\{e_3(t), e_4(t)\}$ of $N_f M_t$.

**Proof** (i) Let $\{e_1, e_2\}$ be an orthonormal frame field on $M$. Applying the Gram–Schmidt process with respect to the metric $ds_t^2$, to the frame field $\{e_1, e_2\}$, we obtain a smooth with respect to $t$, orthonormal frame field $\{\tilde{e}_1(t), \tilde{e}_2(t)\}$ on $M_t$, with $\tilde{e}_j(0) = e_j$, $j = 1, 2$. Since $\delta ds_t^2 = 0$, from the Taylor expansions $\tilde{e}_j(t) = e_j + t\delta \tilde{e}_j(t) + o(t)$, $j = 1, 2$, it follows that $\delta \tilde{e}_1(t) = u e_2$ and $\delta \tilde{e}_2(t) = -u e_1$, for some $u \in C^\infty(M)$. Then, the orthonormal frame field on $M_t$ defined by $e_1(t) + i e_2(t) = \exp(itu)(\tilde{e}_1(t) + i\tilde{e}_2(t))$, depends smoothly on $t$ and satisfies $e_j(0) = e_j$ and $\delta e_j(t) = 0$, $j = 1, 2$. Let $\{\omega_1(t), \omega_2(t)\}$ be the dual frame field of $\{e_1(t), e_2(t)\}$ and $\omega_{12}(t)$ its connection form. Since the coefficients of the corresponding powers of $t$ in the Taylor expansions of $e_j(t)$
and $\omega_j(t)$ are dual, from $\delta e_j(t) = 0$, it follows that $\delta \omega_j(t) = 0$, $j = 1, 2$. Moreover, using the Taylor expansions of all the involved forms in $\omega_j(t) = \omega_j(t) \wedge \omega_r(t)$, $j, r = 1, 2$, and comparing the coefficients of $t$, we obtain that $\delta \omega_{12}(t) = 0$.

(ii) We claim that every point of $M$ has a neighbourhood on which, there exists a smooth with respect to $t$, local orthonormal frame field $\{\tilde{e}_3(t), \tilde{e}_4(t)\}$ of $N_f, M_t$. Indeed, for $p \in M$, there exists a neighbourhood $U$ of $p$, such that $x$ is a chart in $U$. A local orthonormal frame field $\{\tilde{e}_1, \tilde{e}_2\}$ of $N_f, M_t$ can be chosen so that $\tilde{e}_a(0) = \tilde{e}_a, a = 3, 4$, there exists $\tau \in C^\infty(U)$ such that $\tilde{e}_3 = \tau e_3 = \exp(\tau)(\tilde{e}_3 - i\tilde{e}_4)$. Then, the orthonormal frame field $\{\tilde{e}_3(t), \tilde{e}_4(t)\}$ of $N_f, U_t$, where $\tilde{e}_4(t) = \tau (\tilde{e}_3(t) - i\tilde{e}_4(t))$ depends smoothly on $t$ and the claim follows.

Let $\{e_3, e_4\}$ be a $\pm$ oriented orthonormal frame field of $N_f M$. For $p \in M$, consider a frame field $\{\tilde{e}_3(t), \tilde{e}_4(t)\}$ of $N_f, U_t$ as in the claim proved above. By setting $\tilde{e}_a(0) = \tilde{e}_a, a = 3, 4$, there exists $\tau \in C^\infty(U)$ such that $e_3 = \tau e_3 = \exp(\tau)(e_3 - i e_4)$. Then, the orthonormal frame field $\{e_3(t), e_4(t)\}$ of $N_f, U_t$ given by $e_3(t) = \tau (e_3(t) - i e_4(t))$ depends smoothly on $t$, and $e_a(0) = e_a, a = 3, 4$. □

Let $F : I \times M \to \mathbb{R}^4$ be an infinitesimal isometric deformation. An adapted to $F$ orthonormal frame field is a smooth with respect to $t$ frame field $\{e_k(t)\}_{1 \leq k \leq 4}$, such that $\{e_1(t), e_2(t)\}$ and $\{e_3(t), e_4(t)\}$ are positively oriented orthonormal frame fields of $T M_t$ and $N_f, M_t$, respectively, and the former satisfies (42). For such a frame field, we denote by $\omega_{kl}(t), 1 \leq k, l \leq 4$, the connection forms of its corresponding coframe, and by $\{e_k(t)\}_{1 \leq k \leq 4}$ the adapted to $f_i$ frame field given by

$$e_j(t) = f_i e_j(t), \quad j = 1, 2, \quad \text{and} \quad e_a(t) = e_a(t), \quad a = 3, 4.$$  

Then, the Gauss and Weingarten formulae for $f_i$ imply that

$$\tilde{\nabla}^t e_k(t) = \sum_{l=1}^{4} \omega_{kl}(t) e_l(t), \quad 1 \leq k \leq 4, \quad (43)$$

where $\tilde{\nabla}^t$ stands for the connection of $f_i^*(T \mathbb{R}^4)$, and $\tilde{\nabla}^0 = \tilde{\nabla}$. In order to simplify the notation, we also set $e_k(0) = e_k, \omega_k(0) = \omega_{kl}$, for $1 \leq k, l \leq 4$.

**Lemma 35** Let $F : I \times M \to \mathbb{R}^4$ be an infinitesimal isometric deformation with bending field $T$. If $\{e_k(t)\}_{1 \leq k \leq 4}$ is an adapted to $F$ orthonormal frame field, then there exists a unique section $W$ of $f^*(\Lambda^2 T \mathbb{R}^4)$ such that the variations of $e_k(t)$ are
given by
\[ \delta \varepsilon_k(t) = W \cdot \varepsilon_k, \quad 1 \leq k \leq 4, \quad \text{with} \quad \delta \varepsilon_j(t) = \tilde{\nabla}_{e_j} T, \quad j = 1, 2, \quad (44) \]
and the variations \( \varphi_{kl} = \delta \omega_{kl}(t) \) of the connection forms, by
\[ \varphi_{kl} = \langle \tilde{\nabla} W \cdot \varepsilon_k, \varepsilon_l \rangle, \quad 1 \leq k, l \leq 4, \quad (45) \]
where \( \tilde{\nabla} \) is the connection of \( f^*(\Lambda^2 T R^4) \).

**Proof** Differentiating the relations \( \langle \varepsilon_k(t), \varepsilon_l(t) \rangle = \delta_{kl} \), where \( \delta_{kl} \) is the Kronecker’s delta, we obtain that \( \langle \delta \varepsilon_k(t), \varepsilon_l(t) \rangle = -\langle \varepsilon_k, \delta \varepsilon_l(t) \rangle, \quad 1 \leq k, l \leq 4 \). Therefore, by setting \( w_{kl} = \langle \delta \varepsilon_k(t), \varepsilon_l(t) \rangle, \quad 1 \leq k, l \leq 4 \), it follows that the section \( W \) of \( f^*(\Lambda^2 T R^4) \) given by

\[ W = - \sum_{1 \leq k < l \leq 4} w_{kl} \varepsilon_k \wedge \varepsilon_l \]
satisfies \( \delta \varepsilon_k(t) = W \cdot \varepsilon_k, \quad 1 \leq k \leq 4 \) and is clearly unique. Furthermore, Lemma 34(i) yields that \( \varepsilon_j(t) = e_j + o(t) \), and thus, \( f_{t_s} e_j(t) = f_{t_s} e_j + o(t), \quad j = 1, 2 \). Using (41) in the right-hand side of the last relation, we obtain that

\[ \varepsilon_j(t) = f_{t_s} e_j + t \tilde{\nabla}_{e_j} T + o(t), \quad j = 1, 2. \]
The above implies that \( \delta \varepsilon_j(t) = \tilde{\nabla}_{e_j} T, \quad j = 1, 2 \), and (44) follows.

Moreover, from (43), we have that
\[ \omega_{kl}(t) = \langle \tilde{\nabla} t \varepsilon_k(t), \varepsilon_l(t) \rangle, \quad 1 \leq k, l \leq 4, \]
and from the Taylor expansions \( \varepsilon_k(t) = \varepsilon_k + t \delta \varepsilon_k(t) + o(t), \quad 1 \leq k \leq 4 \), we obtain
\[ \tilde{\nabla} t \varepsilon_k(t) = \tilde{\nabla} \varepsilon_k + t \tilde{\nabla} \delta \varepsilon_k(t) + o(t), \quad 1 \leq k \leq 4. \]
Using again the Taylor expansions of \( \varepsilon_l(t), \quad 1 \leq l \leq 4 \), the above two relations give
\[ \omega_{kl}(t) = \omega_{kl} + t \left( \langle \tilde{\nabla} \delta \varepsilon_k(t), \varepsilon_l \rangle + \langle \tilde{\nabla} \varepsilon_k, \delta \varepsilon_l(t) \rangle \right) + o(t), \quad 1 \leq k, l \leq 4. \]
The above and (44) imply that
\[ \varphi_{kl} = \langle \tilde{\nabla} (W \cdot \varepsilon_k), \varepsilon_l \rangle + \langle \tilde{\nabla} \varepsilon_k, W \cdot \varepsilon_l \rangle, \quad 1 \leq k, l \leq 4. \]
Equation (45) follows immediately from the above, by using that the formulae
\[ \tilde{\nabla}_X (V \cdot f_s Y) = (\tilde{\nabla}_X V) \cdot f_s Y + V \cdot \tilde{\nabla}_X f_s Y \quad \text{and} \quad \langle V \cdot f_s X, f_s Y \rangle = -(f_s X, V \cdot f_s Y) \quad (46) \]
hold for any \( V \in \Gamma(f^*(\Lambda^2 T R^4)) \) and \( X, Y \in TM \). \( \square \)
The following is the fundamental theorem of infinitesimal isometric deformations in terms of moving frames. The main idea of the proof is contained in [46], where the theorem has been proved in terms of local coordinates.

**Theorem 36** Assume that \( f : M \to \mathbb{R}^4 \) is a simply connected oriented surface.

(i) Let \( F : I \times M \to \mathbb{R}^4 \) be an infinitesimal isometric deformation. If \( \{e_k(t)\}_{1 \leq k \leq 4} \) is an adapted to \( F \) orthonormal frame field, then the variations \( \{\varphi_{kl}\}_{1 \leq k, l \leq 4} \) of the connection forms of its dual frame field satisfy the fundamental system

\[
\varphi_{12} = 0 \quad \text{and} \quad \varphi_{kl} = -\varphi_{lk}, \quad 1 \leq k, l \leq 4, \tag{47}
\]

\[
\sum_{j=1}^{2} \omega_j \wedge \varphi_{ja} = 0, \quad a = 3, 4, \tag{48}
\]

\[
d\varphi_{ja} = \sum_{r=1}^{2} \omega_{jr} \wedge \varphi_{ra} + \sum_{b=3}^{4} (\varphi_{jb} \wedge \omega_{ba} + \omega_{jb} \wedge \varphi_{ba}), \quad j = 1, 2, \quad a = 3, 4, \tag{49}
\]

\[
\sum_{a=3}^{4} (\varphi_{1a} \wedge \omega_{a2} + \omega_{1a} \wedge \varphi_{a2}) = 0, \quad d\varphi_{34} = \sum_{j=1}^{2} (\varphi_{3j} \wedge \omega_{j4} + \omega_{3j} \wedge \varphi_{j4}). \tag{50}
\]

(ii) Let \( \{e_1, e_2, e_3, e_4\} \) be positively oriented orthonormal frame fields of \( TM \) and \( N_f M \), respectively, and \( \{\omega_{kl}\}_{1 \leq k, l \leq 4} \) the connection forms of the dual frame field of \( \{e_k\}_{1 \leq k \leq 4} \). To every solution, \( \{\varphi_{kl}\}_{1 \leq k, l \leq 4} \) of the fundamental system corresponds a unique bending field \( T \). Moreover, for the infinitesimal isometric deformation \( F \) determined by \( T \), the frame field \( \{e_k\}_{1 \leq k \leq 4} \) locally extends to an adapted to \( F \) local orthonormal frame field, such that the variations of the connection forms of its corresponding coframe are the \( \{\varphi_{kl}\}_{1 \leq k, l \leq 4} \).

**Proof** (i) The Taylor expansions of the connection forms are

\[
\omega_{kl}(t) = \omega_{kl} + t \varphi_{kl} + o(t), \quad 1 \leq k, l \leq 4, \tag{51}
\]

which imply that \( \varphi_{kl} = -\varphi_{lk}, \quad 1 \leq k, l \leq 4 \). In particular, Lemma 34(i) yields that \( \varphi_{12} = 0 \) and (47) follows. Taking into account that \( e_j(t) = e_j + o(t), \; j = 1, 2 \), and using (51) to compare the coefficients of \( t \) in the relations \( \omega_{ja}(t)(e_r(t)) = \omega_{ra}(t)(e_j(t)) \), \( j, r = 1, 2 \), for \( a = 3, 4 \), we obtain (48). The remaining equations of the fundamental system follow using (51) and comparing the \( t \)-terms in the relations \( d\omega_{kl}(t) = \sum_{m=1}^{4} \omega_{km}(t) \wedge \omega_{ml}(t) \), for \( 1 \leq k, l \leq 4 \).

(ii) For a solution \( \{\varphi_{kl}\}_{1 \leq k, l \leq 4} \) of the fundamental system, consider the sections

\[
V_j = - \sum_{1 \leq k < l \leq 4} \varphi_{kl}(e_j)\varepsilon_k \wedge \varepsilon_l, \quad j = 1, 2.
\]
of $f^*(\Lambda^2 T\mathbb{R}^4)$, where $\varepsilon_j = f_* e_j$, $j = 1, 2$, and $\varepsilon_a = e_a$, $a = 3, 4$. Since the bundle $f^*(\Lambda^2 T\mathbb{R}^4)$ is flat, there exists a parallel vector bundle isometry $P : f^*(\Lambda^2 T\mathbb{R}^4) \to M \times \mathbb{R}^6$, where $M \times \mathbb{R}^6$ is the trivial bundle over $M$, equipped with its canonical connection $\hat{\nabla}$. Consider the 1-form $\omega \in \Gamma(T^* M \otimes \mathbb{R}^6)$ given by $\omega = \hat{V}_1 \omega_1 + \hat{V}_2 \omega_2$, where $\hat{V}_j = PV_j$, $j = 1, 2$. Its exterior derivative satisfies

$$d\omega(e_1, e_2) = \hat{\nabla}_{e_1} \omega(e_2) - \hat{\nabla}_{e_2} \omega(e_1) - \omega([e_1, e_2])$$

$$= \hat{\nabla}_{e_1} \hat{V}_2 - \hat{\nabla}_{e_2} \hat{V}_1 + \omega_{12}(e_1) \hat{V}_1 + \omega_{12}(e_2) \hat{V}_2$$

$$= P \left( \hat{\nabla}_{e_1} V_2 - \hat{\nabla}_{e_2} V_1 + \omega_{12}(e_1) V_1 + \omega_{12}(e_2) V_2 \right).$$

Using (43) and all the equations of the fundamental system apart from (48), it follows that the quantity in the last parenthesis is equal to zero, and therefore, $\omega$ is closed. Since $M$ is simply connected, there exists a unique, up to a constant vector in $\mathbb{R}^6$, section $\hat{\nabla} : M \to \mathbb{R}^6$ such that $d\hat{\nabla} = \omega$. This implies that $V = P^{-1} \hat{\nabla}$ is the unique, up to a constant vector in $\Lambda^2 \mathbb{R}^4$, section of $f^*(\Lambda^2 T\mathbb{R}^4)$ satisfying

$$\hat{\nabla}_{e_j} V = V_j, \quad j = 1, 2.$$ (52)

Consider furthermore the sections $T_j = V \cdot \varepsilon_j$, $j = 1, 2$, of $f^*(T\mathbb{R}^4)$. Using (48), it follows that

$$\hat{\nabla}_{e_1} T_2 - \hat{\nabla}_{e_2} T_1 + \omega_{12}(e_1) T_1 + \omega_{12}(e_2) T_2 = 0,$$

and since the bundle $f^*(T\mathbb{R}^4)$ is flat, arguing as above, we conclude that there exists a unique, up to a constant vector in $\mathbb{R}^4$, section $T$ of $f^*(T\mathbb{R}^4)$ such that

$$\hat{\nabla}_{e_j} T = T_j = V \cdot \varepsilon_j, \quad j = 1, 2.$$ (53)

Using the second equation in (46), the above implies that $T$ is a bending field. In particular, $T$ is uniquely determined up to a trivial bending field.

Let $F : I \times M \to \mathbb{R}^4$ be the infinitesimal isometric deformation determined by $T$. Lemma 34 implies that $\{e_k\}_{1 \leq k \leq 4}$ locally extends to an adapted to $F$ local orthonormal frame field $\{\tilde{e}_k(t)\}_{1 \leq k \leq 4}$. For simplicity, we may assume that this occurs globally. Let $W$ be the section of Lemma 35 corresponding to $\{\tilde{e}_k(t)\}_{1 \leq k \leq 4}$. From (44) and (53), it follows that $(W - V) \cdot \varepsilon_j = 0, j = 1, 2$. Therefore, $W = V - u \varepsilon_3 \wedge \varepsilon_4$ for some $u \in C^\infty(M)$. From (45), by differentiating the last relation and using (52) and (43), we obtain that the variations $\{\tilde{\varphi}_{kl}\}_{1 \leq k, l \leq 4}$ of the connection forms $\{\tilde{\omega}_{kl}(t)\}_{1 \leq k, l \leq 4}$ of the dual frame field of $\{\tilde{e}_k(t)\}_{1 \leq k \leq 4}$ are given by

$$\tilde{\varphi}_{j3} = \varphi_{j3} + u \omega_{j4}, \quad \tilde{\varphi}_{j4} = \varphi_{j4} - u \omega_{j3}, \quad j = 1, 2, \quad \text{and} \quad \tilde{\varphi}_{34} = \varphi_{34} + du.$$ (54)

Consider the adapted to $F$ orthonormal frame field $\{e_k(t)\}_{1 \leq k \leq 4}$, given by $e_j(t) = \tilde{e}_j(t)$, $j = 1, 2$, and $e_3(t) + i e_4(t) = \exp(itu)(\tilde{e}_3(t) + i \tilde{e}_4(t))$. For the connection
forms \{\omega_{kl}(t)\}_{1 \leq k, l \leq 4} of its corresponding coframe, we have

\[ \omega_{j3}(t) + i \omega_{j4}(t) = e^{it\omega}(\tilde{\omega}_{j3}(t) + i \tilde{\omega}_{j4}(t)), \quad j = 1, 2, \]

and

\[ \omega_{34}(t) = \tilde{\omega}_{34}(t) - t du. \]

Differentiating the above relations with respect to \( t \) and using (54), it follows that \( \delta \omega_{kl}(t) = \varphi_{kl}, 1 \leq k, l \leq 4, \) and this completes the proof. \( \square \)

**Corollary 37** Let \( F: \mathbb{I} \times M \to \mathbb{R}^4 \) be an infinitesimal isometric deformation of a simply connected oriented surface. The deformation is trivial if and only if for every adapted to \( F \) orthonormal frame field \( \{e_k(t)\}_{1 \leq k \leq 4} \), the variations of the connection forms of its corresponding coframe vanish. In particular, \( \varphi_{34} = 0 \) implies that \( \varphi_{kl} = 0, 1 \leq k, l \leq 4. \)

**Proof** Let \( \{e_k(t)\}_{1 \leq k \leq 4} \) be an adapted to \( F \) orthonormal frame field and consider the corresponding section \( W \) of Lemma 35. The bending field \( T \) of \( F \) is trivial if and only if \( \nabla_j T = C \cdot \epsilon_j, j = 1, 2, \) where \( C \) is a constant vector in \( \Lambda^2 \mathbb{R}^4 \). Using (44), this is equivalent to \( (W - C) \cdot \epsilon_j = 0, j = 1, 2. \) The last relation holds if and only if \( W = C - u \epsilon_3 \land \epsilon_4, \) for some \( u \in C^\infty(M). \) The rest of the proof follows by repeating the part of the proof of Theorem 36(ii), concerning the infinitesimal deformation. \( \square \)

Let \( F: \mathbb{I} \times M \to \mathbb{R}^4 \) be an infinitesimal isometric deformation. Consider a smooth with respect to \( t, \) section \( \xi(t) \in N_f M_t. \) We say that \( F \) preserves \( \xi(t) \) parallelly in the normal bundle, if around every point of \( M, \) there exists a local orthonormal frame field \( \{e_3(t), e_4(t)\} \) of \( N_f M_t \) that depends smoothly on \( t, \) such that

\[ \delta \omega_{34}(t) = 0 \quad \text{and} \quad \delta(\xi(t), e_a(t)) = 0, \quad a = 3, 4, \]

where \( \omega_{34}(t) \) is the connection form of the dual frame field of \( \{e_3(t), e_4(t)\}. \) In the case where \( \xi(t) = H_{f_t}, \) we say that \( F \) preserves the mean curvature vector field parallelly in the normal bundle.

Let \( \Psi(t) \) be a \( N_f M_t \otimes \mathbb{C} \)-valued quadratic differential that depends smoothly on \( t. \) If \( \{e_1(t), e_2(t)\} \) is a smooth with respect to \( t, \) positively oriented local orthonormal frame field on \( M_t \) with dual frame field \( \{\omega_1(t), \omega_2(t)\}, \) then \( \Psi(t) \) has the local expression

\[ \Psi(t) = \psi(t)(\omega_1(t) + i \omega_2(t))^2, \quad \psi(t) \in N_f M_t \otimes \mathbb{C}. \]

We say that \( F \) preserves \( \Psi(t) \) parallelly in the normal bundle, if for every local orthonormal frame field, \( \{e_1(t), e_2(t)\} \) on \( M_t \) satisfying (42), \( F \) preserves parallelly.
in the normal bundle the real and the imaginary parts of $\psi(t)$. In particular, if $\Psi(t) = \Phi^{\pm}(t)$, where $\Phi(t)$ is the Hopf differential of $f_1$, we say that $F$ preserves parallelly in the normal bundle, the differential $\Phi^{\pm}$.

**Proposition 38** Let $F: I \times M \rightarrow \mathbb{R}^4$ be an infinitesimal isometric deformation of a simply connected oriented surface. Suppose that $\{e_k(t)\}_{1 \leq k \leq 4}$ is an adapted to $F$ orthonormal frame field with $\varphi_{34} = 0$. Then

(i) The deformation $F$ preserves parallelly in the normal bundle, the mean curvature vector field and the differential $\Phi^{\pm}$ if and only if

$$\varphi_{13} = *\varphi_{23} \quad \text{and} \quad \varphi_{14} = *\varphi_{24} = \mp\varphi_{23}. \quad (55)$$

(ii) Assume that $M_{0}^{\mp}(f) = \emptyset$. If (55) holds and $\varphi_{23}$ is nowhere vanishing, then

$$\varphi_{23} = L \cos \phi \omega_1 \mp \sin \phi \omega_2, \quad (56)$$

for functions $L > 0$ and $\phi$ on $M$, satisfying

$$d \log L = *\Omega^{\mp} \quad \text{and} \quad e_3^{\mp} = \cos \phi e_3 + \sin \phi e_4, \quad (57)$$

where $\{\omega_1, \omega_2\}$ is the dual frame field of $\{e_1, e_2\}$, and $e_3^{\mp}$ is given by (14).

**Proof** (i) From (5), it follows that the differential $\Phi^{\pm}(t)$ is given by

$$\Phi^{\pm}(t) = \frac{1}{2} \left( \psi^{\pm}(t) \pm i J^{\perp} \psi^{\pm}(t) \right) \left( \omega_1(t) + i \omega_2(t) \right)^2,$$

where $J^{\perp}$ is the complex structure of $N_{f_1} M_t$,

$$\psi^{\pm}(t) = \sum_{a=3}^{4} \psi^{\pm}_a(t) e_a(t) = \frac{\alpha_{11}(t) - \alpha_{22}(t)}{2} \pm J^{\perp}_t \alpha_{12}(t),$$

and $\alpha_{jr}(t) = \alpha_{f_1}(e_j(t), e_r(t)), \ j, r = 1, 2$, where $\alpha_{f_1}$ is the second fundamental form of $f_1$. Equation (43) yields that

$$\alpha_{jr}(t) = \sum_{a=3}^{4} \omega_{ja}(t)(e_r(t)) e_a(t), \quad j, r = 1, 2. \quad (58)$$

Using the above, it follows that

$$\psi^{\pm}_a(t) = \frac{\omega_{1a}(t)(e_1(t)) - \omega_{2a}(t)(e_2(t))}{2} \pm (-1)^b \omega_{1b}(e_2(t)), \quad a, b = 3, 4, \quad b \neq a,$$
and that the components $H_a(t), a = 3, 4,$ of the mean curvature vector field $H_{f_t}$ of $f_t$, with respect to the frame field $\{e_3(t), e_4(t)\}$, are given by

$$H_a(t) = \langle H_{f_t}, e_a(t) \rangle = \frac{\omega_{1a}(t)(e_1(t)) + \omega_{2a}(t)(e_2(t))}{2}, \quad a = 3, 4.$$ 

Therefore, we have that

$$\delta \psi_a^\pm(t) = \frac{\varphi_{1a}(e_1) - \varphi_{2a}(e_2)}{2} = \pm (-1)^a \varphi_{1b}(e_2), \quad a, b = 3, 4, \quad b \neq a,$$

$$\delta H_a(t) = \frac{\varphi_{1a}(e_1) + \varphi_{2a}(e_2)}{2}, \quad a = 3, 4.$$ 

Taking into account (48), it follows that $\delta H_a(t) = 0$ is equivalent to $\varphi_{1a} = \ast \varphi_{2a}, a = 3, 4$. Moreover, it is clear that $F$ preserves $\Phi^\pm$ if and only if it preserves the section $\psi^\pm(t)$, parallelly in the normal bundle. Provided $\delta H_a(t) = 0$, it follows that the equations $\delta \psi_a^\pm(t) = 0, a = 3, 4,$ are equivalent to $\varphi_{14} = \mp \varphi_{23}$, and this completes the proof.

(ii) Consider $\phi \in \mathcal{C}^\infty(M)$ that satisfy the second equation in (57). By substituting $\alpha_{jr}, j, r = 1, 2,$ from (58) for $t = 0$, into (15), we obtain that

$$2\|H^\pm\| \cos \phi = \omega_{13} \pm \omega_{24}(e_1) - (\omega_{23} \mp \omega_{14})(e_2),$$

$$2\|H^\pm\| \sin \phi = \omega_{14} \mp \omega_{23}(e_1) - (\omega_{24} \pm \omega_{13})(e_2).$$

Using (55) to express all the variations $\varphi_{ja}, j = 1, 2, a = 3, 4,$ in terms of $\varphi_{23}$, and taking into account that $\varphi_{34} = 0$, it follows that the equations in (50) are equivalent. By virtue of the above relations, it follows that (50) is equivalent to

$$\varphi_{23}(e_1) \sin \phi \pm \varphi_{23}(e_2) \cos \phi = 0.$$

Since $\varphi_{23}$ is nowhere vanishing, the above implies that there exists a positive function $L \in \mathcal{C}^\infty(M)$ such that (56) is valid. It remains to prove that $L$ satisfies the first equation in (57).

Consider the coframe $\{\theta_1, \theta_2\}$ on $M$, given by

$$\theta_1 = \varphi_{23}, \quad \theta_2 = \ast \theta_1,$$

with corresponding connection form $\theta_{12}$ determined by the relations $d\theta_1 = \theta_{12} \wedge \theta_2$ and $d\theta_2 = -\theta_{12} \wedge \theta_1$. Using (56), it can be easily deduced that

$$\theta_{12} = \omega_{12} \mp d\phi + \ast d \log L.$$

On the other hand, by using the first equation in (55), from (49), we obtain that

$$d\theta_j = d\varphi_{r3} = (-1)^r(-\omega_{12} \pm \omega_{34}) \wedge \theta_r, \quad j, r = 1, 2, \quad j \neq r.$$
and thus, $\theta_{12} = -\omega_{12} \pm \omega_{34}$. The last relation and the above expression of $\theta_{12}$ imply that

$$2\omega_{12} \mp (\omega_{34} + d\phi) = -d \log L.$$  

From the second equation in (57), we obtain that $\omega_{34} + d\phi = \omega_{34}^\mp$, where $\omega_{34}^\mp$ is the connection form of the dual frame field of $\{e_3^\mp, e_4^\mp\}$. By virtue of Proposition 17(i), the first equation in (57) follows from the above relation.  

The following is the infinitesimal analogue of the uniqueness part of the fundamental theorem of surfaces in $\mathbb{R}^4$.

**Theorem 39** An infinitesimal isometric deformation $F: I \times M \to \mathbb{R}^4$ of an oriented surface is trivial if and only if it preserves parallelly in the normal bundle, the mean curvature vector field and the Hopf differential.

**Proof** If $F$ is trivial, then Corollary 37 implies that around every point of $M$, there exists an adapted to $F$ local orthonormal frame field, such that the variations of all of the connection forms of its corresponding coframe vanish. Proposition 38(i) yields that $F$ preserves parallelly in the normal bundle, the mean curvature vector field and both isotropic parts of the Hopf differential.

Conversely, assume that $F$ preserves parallelly in the normal bundle, the mean curvature vector field and the Hopf differential. Then, around every point of $M$, there exists an adapted to $F$ local orthonormal frame field with $\varphi_{34} = 0$. Since $F$ preserves both isotropic parts of the Hopf differential, from (55), it follows that $\varphi_{kl} = 0$ for $1 \leq k, l \leq 4$. Then, Corollary 37 implies that $F$ is trivial.  

**Proof of Theorem 1:** Without loss of generality, suppose that $M$ is simply connected; otherwise, we argue on a simply connected neighbourhood around every point of $M$.

Assume that $F: I \times M \to \mathbb{R}^4$ is a nontrivial infinitesimal isometric deformation that preserves parallelly in the normal bundle, the mean curvature vector field and the isotropic part $\Phi^\mp$ of the Hopf differential. Then, every point of $M$ has a neighbourhood $U$ on which, there exists an adapted to $F$ local orthonormal frame field with $\varphi_{34} = 0$. Corollary 37 and (55) imply that $\varphi_{23} \neq 0$ on the open and dense subset of $U$, on which the bending field of $F$ is nontrivial. Then, from the first equation in (57), it follows that $d\star \Omega^\mp = 0$ on this subset and thus, on $U$. This shows that $f$ is $\mp$ isotropically isothermic.

Conversely, assume that $f$ is $\mp$ isotropically isothermic. Then, there exists a smooth positive function $L$ on $M$, satisfying the first equation in (57). Let $\{e_1, e_2\}$ be a positively oriented orthonormal frame field of $TM$. Since $M_0(f) = 0$, the frame field $\{e_1, e_2\}$ determines the orthonormal frame fields $\{e_3^\mp, e_4^\mp\}$ and $\{e_3^\pm, e_4^\pm\}$ of $N_fM$, given by (14). By setting $e_a = e_a^\pm$, $a = 3, 4$, we consider $\phi \in C^\infty(M)$ satisfying the second equation in (57). Then, we define $\varphi_{23}$ by (56), $\varphi_{34} = 0$, and the remaining $\varphi_{kl}$, $1 \leq k, l \leq 4$, from equations (55) and (47). It is straightforward to check that $\{\varphi_{kl}\}_{1 \leq k, l \leq 4}$ satisfy the fundamental system with respect to the connection forms $\{\omega_{kl}\}_{1 \leq k, l \leq 4}$ of the dual frame field of $\{e_k\}_{1 \leq k \leq 4}$. From Theorem 36(ii), it follows that the solution $\{\varphi_{kl}\}_{1 \leq k, l \leq 4}$ determines a unique bending field $T$. In particular, since
$\varphi_{34} = 0 \neq \varphi_{23}$ everywhere on $M$, Corollary 37 implies that $T$ is nontrivial. Moreover, for the infinitesimal isometric deformation determined by $T$, Theorem 36(ii) implies that $\{e_k\}_{1 \leq k \leq 4}$ locally extend to an adapted to $F$ local orthonormal frame field, such that the variations of the connection forms of its corresponding coframe are the $\{\varphi_{kl}\}_{1 \leq k, l \leq 4}$. From Proposition 38(i), it follows that $F$ preserves parallelly in the normal bundle, the mean curvature vector field and the differential $\Phi^\pm$. The rest of the proof follows immediately from Proposition 31.

For the proof of Theorem 2, we need the following lemma. We recall from Proposition 12 that the Gauss lift $G_\pm$ of a superconformal surface $f: M \to \mathbb{R}^4$ with $\pm K_N \geq 0$, is holomorphic.

**Lemma 40** Let $f: M \to \mathbb{R}^4$ be an oriented superconformal surface with $\pm K_N \geq 0$ and nowhere-vanishing mean curvature vector field. Assume that $F: I \times M \to \mathbb{R}^4$ is an infinitesimal isometric deformation that preserves parallelly in the normal bundle the mean curvature vector field. Then, $F$ preserves parallelly in the normal bundle the differential $\Phi^\pm$ if and only if it preserves the holomorphicity of the Gauss lift $G_\pm: M \to (\mathbb{Z}, g_1)$ of $f$.

**Proof** Since $F$ preserves parallelly in the normal bundle the mean curvature vector field, around every point of $M$, there exists an adapted to $F$ local orthonormal frame field $\{e_k(t)\}_{1 \leq k \leq 4}$ with $\varphi_{34} = 0$. In particular, from the proof of Proposition 38(i), it follows that $\varphi_{1a} = *\varphi_{2a}, a = 3, 4$.

Since $\pm K_N \geq 0$, Lemma 10(ii) implies that $\Phi^\pm \equiv 0$. Therefore, using (58) for $t = 0$, from (5) and (6), we obtain that $(\omega_{23} \pm \omega_{14}) = *((\omega_{13} \mp \omega_{24})$. Moreover, since $H \neq 0$ everywhere on $M$, a simple computation shows that $\omega_{13} \mp \omega_{24}$ is nowhere vanishing.

Let $G_\pm(t)$ be the Gauss lift of $f_t$ into $(\mathbb{Z}, g_1)$. From (12), we have that

$$G^*_\pm(t)(g_1) = ds^2_t + \frac{1}{4} \left((\omega_{13}(t) \mp \omega_{24}(t))^2 + (\omega_{23}(t) \pm \omega_{14}(t))^2\right).$$

Proposition 12 implies that $F$ preserves the holomorphicity of $G_\pm$ if and only if $\delta G^*_\pm(t)(g_1) = 0$. Differentiating the above with respect to $t$, and using that $\varphi_{1a} = *\varphi_{2a}, a = 3, 4$, and that $(\omega_{23} \pm \omega_{14}) = *((\omega_{13} \mp \omega_{24}) \neq 0$ everywhere on $M$, we obtain that $\delta G^*_\pm(t)(g_1) = 0$ is equivalent to (55). The proof follows from Proposition 38(i). \hfill $\square$

**Proof of Theorem 2:** The equivalence of (i) and (ii) has been proved in Proposition 33.

We argue that (i) is equivalent to (iii). Since $M_1(f) = \emptyset$, Lemma 10(ii) yields that $K_N \neq 0$ everywhere on $M$ and therefore, it also implies that either $\Phi^+ \equiv 0$, or $\Phi^- \equiv 0$ on $M$. Assume that $\Phi^\pm \equiv 0$ on $M$. Since $M_1(f) = \emptyset$, from Lemma 10(i), it follows that $M^+_\pm(f) = \emptyset$. Hence, every positively oriented local orthonormal frame field $\{e_1, e_2\}$ of $TM$ determines the local orthonormal frame field $\{e_3, e_4\}$ of $N_f M$, given by (14). By virtue of Lemma 40, the equivalence of (i) and (iii) follows by repeating the proof of Theorem 1, using the frame field $\{e_3 = H/\|H\|, e_4 = J^\perp e_3\}$ instead of $\{e_3^\pm, e_4^\pm\}$, to show the converse implication. \hfill $\square$
5 The Moduli Space of Isometric Surfaces with the Same Mean Curvature

We recall briefly some facts from [51], about isometric surfaces in $\mathbb{Q}_c^4$ with the same mean curvature. Let $M$ be a 2-dimensional oriented Riemannian manifold, and $f, \tilde{f}: M \rightarrow \mathbb{Q}_c^4$ isometric immersions with mean curvature vector fields $H$ and $\tilde{H}$, respectively. The surfaces $f, \tilde{f}$ are said to have the same mean curvature, if there exists a parallel vector bundle isometry $T: N_f M \rightarrow N_{\tilde{f}} M$ such that $TH = \tilde{H}$. If $f$ and $\tilde{f}$ have the same mean curvature and they are noncongruent, then the pair $(f, \tilde{f})$ is called a pair of Bonnet mates.

Assume that $f, \tilde{f}: M \rightarrow \mathbb{Q}_c^4$ have the same mean curvature and let $T: N_f M \rightarrow N_{\tilde{f}} M$ be a parallel vector bundle isometry satisfying $TH = \tilde{H}$. After an eventual composition of $\tilde{f}$ with an orientation-reversing isometry of $\mathbb{Q}_c^4$, we may hereafter suppose that $T$ is orientation-preserving. Let $\alpha, \tilde{\alpha}$ be the second fundamental forms of $f$ and $\tilde{f}$, respectively. The section of $\text{Hom}(TM \times TM, N_f M)$ given by $DT_f, \tilde{f} = \alpha - T^{-1} \circ \tilde{\alpha}$ is traceless and measures how far the surfaces deviate from being congruent. Its $\mathbb{C}$-bilinear extension decomposes into its $(k, l)$-components, $k+l = 2$, and the $(2, 0)$-part is given by

$$Q^{T}_{f, \tilde{f}} = (DT_f, \tilde{f})^{(2, 0)} = \Phi - T^{-1} \circ \tilde{\Phi},$$

where $\Phi$ and $\tilde{\Phi}$ are the Hopf differentials of $f$ and $\tilde{f}$, respectively. The following has been proved in [51, Lemma 12].

**Lemma 41** Let $f, \tilde{f}: M \rightarrow \mathbb{Q}_c^4$ be non-minimal surfaces and $T: N_f M \rightarrow N_{\tilde{f}} M$ an orientation-preserving parallel vector bundle isometry satisfying $TH = \tilde{H}$. Then

(i) The quadratic differential $Q^{T}_{f, \tilde{f}}$ is holomorphic and independent of $T$.

(ii) The normal curvatures of the surfaces are equal and the curvature ellipses $\mathcal{E}_f, \mathcal{E}_{\tilde{f}}$ are congruent at any point of $M$. In particular, $M^\pm_0(f) = M^\pm_0(\tilde{f})$.

By virtue of Lemma 41(i), we assign to each pair of non-minimal surfaces $(f, \tilde{f})$ with the same mean curvature, a holomorphic quadratic differential denoted by $Q_{f, \tilde{f}}$, which is called the distortion differential of the pair and is given by

$$Q_{f, \tilde{f}} = \Phi - T^{-1} \circ \tilde{\Phi}.$$  

The distortion differential of such a pair is simply denoted by $Q$, whenever, there is no danger of confusion.

Let $f, \tilde{f}: M \rightarrow \mathbb{Q}_c^4$ be non-minimal surfaces with the same mean curvature. It is clear that $Q \equiv 0$ if and only if $f$ and $\tilde{f}$ are congruent. If $(f, \tilde{f})$ is a pair of Bonnet mates, then according to Lemmas 9 and 41(i), the zero set $Z$ of $Q$ consists of isolated
points only. With respect to the decomposition $N_f M \otimes \mathbb{C} = N_f^+ M \oplus N_f^- M$, the distortion differential splits as

$$Q = Q^- + Q^+, \quad \text{where} \quad Q^\pm = \pi^\pm \circ Q.$$  

From Lemma 41(i), it follows that both differentials $Q^-$ and $Q^+$ are holomorphic, and $Q^\pm$ is given by

$$Q^\pm = \Phi^\pm - T^{-1} \circ \Phi^\pm. \quad (59)$$  

Lemma 9 implies that either $Q^\pm \equiv 0$, or the zero set $Z^\pm$ of $Q^\pm$ consists of isolated points only.

For an oriented surface $f : M \to \mathbb{Q}_e^4$, we denote by $M(f)$ the moduli space of congruence classes of all isometric immersions of $M$ into $\mathbb{Q}_e^4$, that have the same mean curvature with $f$.

Assume that $f : M \to \mathbb{Q}_e^4$ is a non-minimal oriented surface. Since the distortion differential of a pair of Bonnet mates does not vanish identically, the moduli space can be written as follows:

$$M(f) = N^- (f) \cup N^+ (f) \cup \{f\},$$

where

$$N^\pm (f) = \{ \tilde{f} : Q^\pm_{f, \tilde{f}} \neq 0 \}/\text{Isom}^+(\mathbb{Q}_e^4),$$

$\{f\}$ is the trivial congruence class, and $\text{Isom}^+(\mathbb{Q}_e^4)$ is the group of orientation-preserving isometries of $\mathbb{Q}_e^4$. Moreover, the moduli space decomposes into disjoint components as

$$M(f) = M^* (f) \cup M^- (f) \cup M^+ (f) \cup \{f\},$$

where

$$M^\pm (f) = N^\pm (f) \setminus N^\mp (f) = \{ \tilde{f} : Q^\pm_{f, \tilde{f}} \equiv Q^\pm_{\tilde{f}, \tilde{f}} \}/\text{Isom}^+(\mathbb{Q}_e^4),$$

and

$$M^* (f) = N^- (f) \cap N^+ (f) = \{ \tilde{f} : Q^-_{f, \tilde{f}} \neq 0 \text{ and } Q^+_{f, \tilde{f}} \neq 0 \}/\text{Isom}^+(\mathbb{Q}_e^4).$$

In order to simplify the notation in the sequel, we set $\bar{M}^\pm (f) = M^\pm (f) \cup \{f\}$.

Hereafter, whenever we refer to a surface in the moduli space, we mean its congruence class. A surface $f : M \to \mathbb{Q}_e^4$ is called a Bonnet surface if $M(f) \setminus \{f\} \neq \emptyset$. Any $\tilde{f} \in M(f) \setminus \{f\}$ is called a Bonnet mate of $f$. A Bonnet surface $f$ is called proper Bonnet if it admits infinitely many Bonnet mates.
5.1 Bonnet Mates

In view of Lemma 41(ii), we denote by $M_0 = M_0^- \cup M_0^+$ and $M_1$, the set of pseudo-umbilic and umbilic points of a pair of non-minimal Bonnet mates, respectively.

**Proposition 42** If $\tilde{f} \in \mathcal{N}^\pm(f)$, then there exists $\theta^\pm \in C^\infty(M \setminus M_0^\pm; (0, 2\pi))$, such that the distortion differential of the pair $(f, \tilde{f})$ satisfies on $M \setminus M_0^\pm$ the relation

$$Q^\pm = (1 - e^{\mp i\theta^\pm})\Phi^\pm.$$  \hspace{1cm} (60)

Moreover, $Q^\pm$ vanishes precisely on $M_0^\pm$, which consists of isolated points only.

**Proof** From [51, Lemma 14, Prop. 15], it follows that $M_0^\pm \subset Z^\pm$ is isolated, and there exists $\theta^\pm \in C^\infty(M \setminus Z^\pm; (0, 2\pi))$ such that (60) is valid on $M \setminus Z^\pm$. It remains to prove that $M_0^\pm = Z^\pm$.

Arguing indirectly, assume that there exists $p \in Z^\pm \setminus M_0^\pm$. Then, Lemma 10(i) implies that $\Phi^\pm(p) \neq 0$. Since $Q^\pm$ and $\Phi^\pm$ are smooth and $\Phi^\pm(p) \neq 0$, from (60), it follows that the function $k = \exp(\mp i\theta^\pm)$ extends smoothly at $p$, with $k(p) = 1$.

We claim that $\theta^\pm$ extends smoothly at $p$. We first show that the limit of $\theta^\pm$ at $p$ exists; assume to the contrary that there exist sequences $p_n, q_n \in M \setminus Z^\pm, n \in \mathbb{N}$, converging at $p$, such that $\theta^\pm(p_n) \to 0$ and $\theta^\pm(q_n) \to 2\pi$. Since $\theta^\pm$ is continuous on $M \setminus Z^\pm$, for every $r > 0$, there exists $s_r \in B_r(p) \setminus \{p\}$ such that $\theta^\pm(s_r) = \pi$, or equivalently, $k(s_r) = -1$. On the other hand, since $k$ is continuous at $p$, there exists $\tilde{r} > 0$ such that $|k - 1| < 1/2$ on $B_{\tilde{r}}(p)$. This is a contradiction and thus, the limit of $\theta^\pm$ at $p$ exists. Since $k$ is smooth and $\theta^\pm$ extends continuously at $p$, the claim follows.

Let $(U, z)$ be a complex chart with $U \cap Z^\pm = \{p\}$. From Lemmas 41(i) and 9, it follows that there exists a positive integer $m$ such that $Q^\pm = z^m\Psi^\pm$ on $U$, and $\Psi^\pm(p) \neq 0$. Using (60), this is equivalent to

$$(1 - e^{\mp i\theta^\pm})\phi^\pm = z^m\psi^\pm, \quad \psi^\pm(p) \neq 0, \hspace{1cm} (61)$$

where $\phi^\pm$ is given by (8), and $\Psi^\pm = \psi^\pm dz^2$ on $U$. Differentiating (60) with respect to $\partial$ in the normal connection and using the holomorphicity of $Q^\pm$, we obtain

$$\left(h^\pm(1 - e^{\mp i\theta^\pm}) \pm ie^{\mp i\theta^\pm}\theta^\pm_z\right)\phi^\pm = 0,$$

where $h^\pm$ is given by (28). Since $\phi^\pm \neq 0$ everywhere on $U$, the above implies that

$$\theta^\pm_z = \mp ih^\pm(1 - e^{\mp i\theta^\pm}), \quad \theta^\pm_z = \pm ih^\pm(1 - e^{\mp i\theta^\pm}).$$

Using that $\theta^\pm(p) = 0$ or $2\pi$, from the above relation, we obtain that all derivatives of $\theta^\pm$ vanish at $p$. Therefore, differentiation of (61) $m$-times with respect to $\partial$ in the normal connection yields that $m!\psi^\pm(p) = 0$. This is a contradiction, and the proof follows. \hfill $\Box$

The following lemma is essential for our results.
Lemma 43 Let $M$ be a simply connected, oriented 2-dimensional Riemannian manifold with a global complex coordinate $z$, and $f: M \to \mathbb{C}$ a surface with $M_0^\pm(f)$ isolated. Consider the differential equation

$$
\theta_{\bar{z}}^\pm = \mp i h^\pm(1 - e^{\pm i \theta^\pm}), \quad \theta_z^\pm = \pm i h^\mp(1 - e^{\mp i \theta^\pm}), \tag{62}
$$

where $h^\pm$ is given by (28) on $M \setminus M_0^\pm(f)$, and $\theta^\pm \in C^\infty(M \setminus M_0^\pm(f); \mathbb{R})$. Then, the graph of any solution of (62) is an integral surface of the distribution $D^\pm$ on $\mathbb{R} \times (M \setminus M_0^\pm(f))$, defined by the 1-form

$$
\rho^\pm = d\theta^\pm \mp i h^\mp(1 - e^{\mp i \theta^\pm}) dz \pm i h^\pm(1 - e^{\pm i \theta^\pm}) d\bar{z}. \tag{63}
$$

We have that

(i) Any solution $\theta^\pm \in C^\infty(M \setminus M_0^\pm(f); \mathbb{R})$ of (62) satisfies the equations

$$
A^\pm e^{\pm 2i \theta^\pm} - 2i (\text{Im} A^\pm) e^{\pm i \theta^\pm} - A^\mp = 0, \tag{64}
$$

$$
\theta_{\bar{z}z}^\pm = \mp A^\pm(1 - e^{\pm i \theta^\pm}), \tag{65}
$$

where

$$
A^\pm = i \left( h_z^\pm - |h^\pm|^2 \right) = -\text{Im} h_z^\pm + i (\text{Re} h_z^\pm - |h^\pm|^2). \tag{66}
$$

(ii) Assume that $h^\pm$ extends smoothly on $M$. Then, $D^\pm$ is involutive on $\mathbb{R} \times M$ if and only if $A^\pm \equiv 0$ on $M$. If $D^\pm$ is involutive, then its maximal integral surfaces are graphs of solutions of (62) on $M$. In particular, any solution of (62) on $M$ is equivalent modulo $2\pi$, either to a harmonic function $\theta^\pm \in C^\infty(M; (0, 2\pi))$, or to the constant function $\theta^\pm \equiv 0$, and the space of the distinct modulo $2\pi$ solutions can be smoothly parametrized by $S^1 \cong \mathbb{R}/2\pi \mathbb{Z}$.

(iii) If (62) has a harmonic solution $\theta^\pm \in C^\infty(M \setminus M_0^\pm(f); (0, 2\pi))$, then $h^\pm$ extends smoothly on $M$, and $A^\pm \equiv 0$.

Proof It is clear that the graph of any solution of (62) is an integral surface of $D^\pm$.

(ii) Assume that $\theta^\pm \in C^\infty(M \setminus M_0^\pm(f); \mathbb{R})$ satisfies (62). From (62) it follows that

$$
\theta_{\bar{z}z}^\pm = \mp A^\pm(1 - e^{\pm i \theta^\pm}) \quad \text{and} \quad \theta_{zz}^\pm = \mp A^\pm(1 - e^{\mp i \theta^\pm}),
$$

where $A^\pm$ is given by (66). Since $\theta_{\bar{z}z}^\pm = \theta_{zz}^\pm$, the above implies (64) and (65).

(iii) From (63) and (66), it follows that $\rho^\pm$ and $A^\pm$ can be smoothly extended on $\mathbb{R} \times M$ and $M$, respectively. The Frobenius Theorem yields that $D^\pm$ is involutive if and only if $\rho^\pm \wedge d\rho^\pm \equiv 0$ on $\mathbb{R} \times M$, or equivalently, $A^\pm \equiv 0$ on $M$.

Assume that $D^\pm$ is involutive on $\mathbb{R} \times M$, and let $\Sigma$ be a maximal integral surface. Then $\rho^\pm \equiv 0$ on $\Sigma$. Since $M$ is simply connected and $\rho^\pm$ is defined globally on $\mathbb{R} \times M$, from (63) it follows that $\Sigma$ is the graph of a solution of (62) on $M$.

Let $\theta^\pm \in C^\infty(M; \mathbb{R})$ be a solution of (62) on $M$. Since $A^\pm \equiv 0$ on $M$, from (65) it follows that $\theta^\pm$ is harmonic. Clearly, $\theta^\pm + 2k\pi$ also satisfies (62) for every $k \in \mathbb{Z}$. 

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The surface f is of the following holds, then it extends to a harmonic function is equivalent modulo 2π. From the above two relations, it follows that the function θ ± is harmonic on M \ M_0 ±. Therefore, any solution of (62) on M is smoothly parametrized by S^1 ≃ \mathbb{R}/2\pi\mathbb{Z}.

(iii) Let θ ± ∈ C^∞(M \ M_0 ± (f); (0, 2\pi)) be a harmonic function satisfying (62). Since θ ± is bounded with isolated singularities, it extends to a harmonic function θ ± ∈ C^∞(M; [0, 2\pi]). We claim that θ ± does not attain the values 0 and 2\pi on M. Arguing indirectly, assume that there exists a point at which θ ± attains the value 0 or 2\pi. Then θ ± has an interior minimum or maximum, respectively, and the maximum principle implies that θ ± ≡ 0 or 2\pi, respectively, on M. This is a contradiction, since θ ±(p) ∈ (0, 2\pi) for every p ∈ M \ M_0 ±(f). Therefore, θ ± ∈ C^∞(M; (0, 2\pi)). From (62), it follows that h ± extends smoothly at every point of M_0 ±(f). Since θ ± is harmonic, (65) implies that A ± ≡ 0 on M.

Proposition 44 If f̂ ∈ N ±(f), then the function θ ± of Proposition 42 satisfies (62) on U \ M_0 ± for every simply connected complex chart (U, z) on M. Moreover, if one of the following holds, then it extends to a harmonic function θ ± ∈ C^∞(M; (0, 2\pi)).

(i) There exists f̂ ∈ N ±(f) ∩ N ±(f).
(ii) The surface f is ± isotropically isothermic on M \ M_0 ±.

Proof Let (U, z) be a simply connected complex chart on M. In the proof of Proposition 42, it has been shown that θ ± satisfies (62) on U \ M_0 ±. We claim that if (i) or (ii) holds, then θ ± is harmonic on U \ M_0 ±.

(i) To unify the notation, set f_1 = f̂, θ_1 ± = θ ± and f_2 = f̂. Proposition 42 implies that there exists θ_j ± ∈ C^∞(M \ M_0 ±; (0, 2\pi)) such that the distortion differential Q_j of the pair (f, f_j) satisfies

Q_j ± = (1 - e^{\pm iθ_j ±})\Phi ± \quad on \ M \ M_0 ±

for j = 1, 2, where \Phi is the Hopf differential of f. Moreover (cf. [51, Lemma 17] and its proof), the distortion differential Q of the pair (f_1, f_2) satisfies

Q ± = T \circ (Q_1 ± - Q_2 ±),

where T : N_f M → N_{f_1} M is an orientation and mean curvature vector field-preserving, parallel vector bundle isometry. From the above two relations, it follows that

Q ± = (e^{\pm iθ_2 ±} - e^{\pm iθ_1 ±})T \circ \Phi ± \quad on \ M \ M_0 ±.
Since $f_2 \in N^\pm(f_1)$, it is clear that $f_1 \in N^\pm(f_2)$. Proposition 42 implies that $Q^\pm$ vanishes precisely on $M_0^\pm$, and from the above, it follows that $\theta_j^\pm \neq \theta_j^\pm$ everywhere on $M \setminus M_0^\pm$. Since $\theta_j^\pm$, $j = 1, 2$, satisfies (62) on $U \setminus M_0^\pm$, from Lemma 43(i), it follows that it also satisfies (64). At every point of $U \setminus M_0^\pm$, equation (64) viewed as a polynomial equation, has the distinct roots $1, e^{\mp i\theta_1^\pm}, e^{\mp i\theta_2^\pm}$. Hence, $A^\pm \equiv 0$ on $U \setminus M_0^\pm$ and the claim follows by virtue of (65).

(ii) Arguing indirectly, assume that $\theta^\pm$ is not harmonic on $U \setminus M_0^\pm$. Appealing to Lemma 43(i), equation (65) implies that there exists $p \in U \setminus M_0^\pm$ such that $A^\pm(p) \neq 0$. On the other hand, Lemma 26 and (66) yield that $\text{Re} \ A^\pm \equiv 0$ on $U \setminus M_0^\pm$. Since $\text{Re} \ A^\pm(p) = 0 \neq \text{Im} \ A^\pm(p)$, equation (64) implies that $\exp(\pm i\theta^\pm(p)) = 1$. This is a contradiction since $\theta^\pm$ takes values in $(0, 2\pi)$, and the claim follows.

Since $\theta^\pm$ is a harmonic function satisfying (62) on $U \setminus M_0^\pm$, Lemma 43(iii) implies that $h^\pm$ extends smoothly on $U$ and $A^\pm \equiv 0$ on $U$. From Lemma 43(ii), it follows that $\theta^\pm$ extends to a harmonic function on $U$ with values in $(0, 2\pi)$, satisfying (62) on $U$. Since $U$ is arbitrary, this completes the proof.

\[\square\]

### 6 Simply Connected Surfaces

#### 6.1 The Structure of the Moduli Space

We study here the moduli space $\mathcal{M}(f)$ for simply connected surfaces $f : M \to \mathbb{Q}_c^4$. The following proposition determines the structure of $\mathcal{M}(f)$ for such compact surfaces.

**Proposition 45** Let $f : M \to \mathbb{Q}_c^4$ be an oriented surface. If $M$ is homeomorphic to $S^2$, then $f$ admits at most one Bonnet mate.

**Proof** If both Gauss lifts of $f$ are not vertically harmonic, then [51, Thm. 2] implies that $f$ admits at most one Bonnet mate. Assume that $f$ has a vertically harmonic Gauss lift. We claim that $f$ is superconformal. Indeed, if $f$ is non-minimal, then [51, Thm. 3] yields that it is superconformal. If $f$ is minimal, the claim follows by a well-known result of Calabi [5]. Then, [51, Thm. 5(i)] implies that $f$ admits at most one Bonnet mate.

By virtue of the above proposition, in the sequel, we focus on non-compact surfaces. The following theorem provides information about the structure of the moduli space of non-minimal such surfaces.

**Theorem 46** Let $M$ be a non-compact, simply connected, oriented 2-dimensional Riemannian manifold, and $f : M \to \mathbb{Q}_c^4$ a non-minimal surface. Then

(i) Either there exists at most one Bonnet mate of $f$ in $\mathcal{M}^\pm(f)$, or the component $\mathcal{M}^\pm(f)$ is diffeomorphic to $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$.

(ii) We have that $\mathcal{M}^\times(f) \neq \emptyset$ if and only if $\mathcal{M}^\sim(f) \neq \emptyset \neq \mathcal{M}^\dagger(f)$. If $\mathcal{M}^\times(f) \neq \emptyset$, then there is a one-to-one correspondence between Bonnet mates $f^\sim \in \mathcal{M}^\times(f)$ and pairs $f^\dagger$, $f^\ddagger$ with $f^\pm \in \mathcal{M}^\pm(f)$, such that the distortion differential of the
pair \((f, \tilde{f})\) is given by
\[
Q = Q_{f,f^-} + Q_{f,f^+},
\]
where \(Q_{f,f^\pm}\) is the distortion differential of the pair \((f, f^\pm)\).

(iii) The surface \(f\) is proper Bonnet if and only if either \(\tilde{M}^-(f) = S^1\), or \(\tilde{M}^+(f) = S^1\).

(iv) The moduli space \(\mathcal{M}(f)\) can be parametrized by the product \(\tilde{M}^-(f) \times \tilde{M}^+(f)\). In particular, if \(f\) is proper Bonnet, then \(\mathcal{M}(f)\) is a smooth manifold.

For the proof of the above theorem, we need the following.

**Proposition 47** Let \(M\) be a simply connected oriented 2-dimensional Riemannian manifold with a global complex coordinate \(z\), and \(f : M \to \mathbb{Q}_e^4\) a non-minimal surface with \(M^\pm_0(f)\) isolated.

(i) If \(\tilde{f} \in \mathcal{M}^\pm(f)\) and \(\mathcal{M}^\pm(f) \setminus \{\tilde{f}\} \neq \emptyset\), then there exists a harmonic function \(\theta \pm \in C^\infty(M; (0, 2\pi))\) satisfying (62) on \(M\), such that the distortion differential of the pair \((f, \tilde{f})\) is given by (60) on \(M\).

(ii) If \(h^\pm\) can be smoothly extended on \(M\), then the distinct modulo \(2\pi\) solutions of (62) on \(M\) determine noncongruent surfaces in \(\mathcal{M}^\pm(f)\). In particular, any solution \(\theta^\pm \in C^\infty(M; (0, 2\pi))\) determines a unique \(\tilde{f} \in \mathcal{M}^\pm(f)\) such that the distortion differential of the pair \((f, \tilde{f})\) is given by (60) on \(M\).

**Proof** (i) Propositions 42 and 44 imply that there exists \(\theta^\pm \in C^\infty(M \setminus M^\pm_0; (0, 2\pi))\) satisfying (62) on \(M \setminus M^\pm_0\), such that the distortion differential \(Q\) of the pair \((f, \tilde{f})\) is given by (60) on \(M \setminus M^\pm_0\). Let \(\hat{f} \in \mathcal{M}^\pm(f) \setminus \{\tilde{f}\}\). Then, \([51, \text{Lemma } 17(ii)]\) implies that \(\hat{f} \in \mathcal{M}^\pm(f) \cap \mathcal{M}^\pm(\tilde{f})\). From Proposition 44(i), it follows that \(\theta^\pm\) extends to a harmonic function \(\theta^\pm \in C^\infty(M; (0, 2\pi))\). In particular, from the proof of Proposition 44 it follows that \(\theta^\pm\) satisfies (62) on \(M\). From Lemma 10(i) and Proposition 42, it follows that \(Q\) and \(\Phi^\pm\) vanish precisely on \(M^\pm_0\). Since \(\theta^\pm\) is defined on the whole \(M\), it is clear that \(Q\) is given by (60) on \(M\).

(ii) Assume that \(h^\pm\) extends smoothly on \(M\). For a solution \(\theta^\pm\) of (62) on \(M\), consider the quadratic differential
\[
\Psi = \Phi^\pm + e^{\mp i\theta^\pm} \Phi^\pm.
\]
By using (8), it is straightforward to check that \(\Psi\) satisfies equations (9) and (11) with respect to \(\nabla^\perp, R^\perp, H\). Since \(\theta^\pm\) satisfies (62), by using (8), it follows that \(\Phi - \Psi\) is holomorphic. Therefore, \(\Psi\) satisfies the Codazzi equation. By the fundamental theorem of submanifolds, there exists a unique (up to congruence) isometric immersion \(\tilde{f} : M \to \mathbb{Q}_e^4\) and an orientation-preserving parallel vector bundle isometry \(T : N_f M \to N_{\tilde{f}} M\), such that the Hopf differential \(\tilde{\Phi}\) and the mean curvature vector field \(\tilde{H}\) of \(\tilde{f}\) are given by \(\tilde{\Phi} = T \circ \Psi\) and \(\tilde{H} = TH\), respectively. Clearly, \(\tilde{f}\) is congruent to \(f\) if and only if \(\theta^\pm \equiv 0 \mod 2\pi\). If \(\tilde{f}\) is noncongruent to \(f\), then the distortion differential of the pair \((f, \tilde{f})\) satisfies \(Q^\pm \equiv 0\) and thus, \(\tilde{f} \in \mathcal{M}^\pm(f)\). In
Proof of Theorem 46: Since $M$ is non-compact, the Uniformization Theorem implies that it is conformally equivalent either to the complex plane, or to the unit disc. Therefore, $M$ admits a global complex coordinate $z$.

(i) Assume that there exist at least two Bonnet mates of $f$ in $\mathcal{M}(f)$, and let $\tilde{f} \in \mathcal{M}(f)$. Proposition 42 implies that $M_0^{\pm}$ is isolated. Since $\mathcal{M}(f) \setminus \{\tilde{f}\} \neq \emptyset$, from Proposition 47(i), it follows that the differential equation (62) has a harmonic solution $\theta \in C^\infty(M \setminus M_0^{\pm}; (0, 2\pi))$. Then, Lemma 43(ii-ii) implies that the space of the distinct modulo $2\pi$ solutions of (62) can be smoothly parametrized by $\mathbb{S}^1$. The proof follows by virtue of Proposition 47(ii).

(ii) Assume that there exists $\tilde{f} \in \mathcal{M}(f)$ and consider the quadratic differentials

$$\Psi_{f^-} = \Phi - Q^- \quad \text{and} \quad \Psi_{f^+} = \Phi - Q^+,$$

where $\Phi$ is the Hopf differential of $f$, and $Q$ is the distortion differential of the pair $(f, \tilde{f})$. We argue that $\Psi_{f^-}$ and $\Psi_{f^+}$ satisfy the compatibility equations with respect to $\nabla^\perp, R^\perp, H$. From Lemma 41(i), it follows that $Q^{\pm}$ is holomorphic, and thus, the differential $\Psi_{f^\pm}$ satisfies the Codazzi equation. Lemma 10(i) and Proposition 42 yield that $\Phi^{\pm}$ and $Q^{\pm}$ vanish precisely on $M_0^{\pm}$. Therefore, $\Psi_{f^\pm}(p) = \Phi(p)$ at any $p \in M_0^{\pm}$, and thus, $\Psi_{f^\pm}$ satisfies the algebraic equations (9) and (11) on $M_0^{\pm}$. Moreover, since $\tilde{f} \in \mathcal{M}(f)$, Proposition 42 implies that there exist $\theta^\pm, \theta^\pm$ with $\theta^\pm \in C^\infty(M \setminus M_0^{\pm}; (0, 2\pi))$ such that $Q^{\pm}$ is given by (60) on $M \setminus M_0^{\pm}$. Using (60) and (8), it follows that $\Psi_{f^\pm}$ satisfies the equations (9) and (11) on $M \setminus M_0^{\pm}$. The fundamental theorem of submanifolds implies that there exist unique Bonnet mates $f^-, f^+: M \to \mathbb{Q}^4_{\pm}$ of $f$, such that the Hopf differential $\Phi_{f^\pm}$ of $f^\pm$ is given by $\Phi_{f^\pm} = T_{\pm} \circ \Psi_{f^\pm}$, where $T_{\pm}: N_fM \to N_{f^\pm}M$ is an orientation and mean curvature vector field-preserving, parallel vector bundle isometry. From Lemma 41(i), it follows that the distortion differential of the pair $(f, f^\pm)$ is $Q^{\pm}$, and thus, $f^\pm \in \mathcal{M}(f)$.

Conversely, assume that there exist $f^-, f^+$ with $f^\pm \in \mathcal{M}(f)$ and consider the quadratic differential $\Psi = \Psi^- + \Psi^+$ with

$$\Psi^- = \Phi^- - Q_{f,f^-} \quad \text{and} \quad \Psi^+ = \Phi^+ - Q_{f,f^+},$$

where $Q_{f,f^\pm}$ is the distortion differential of the pair $(f, f^\pm)$. Lemma 41(i) implies that $Q_{f,f^-}$ and $Q_{f,f^+}$ are both holomorphic and, thus, $\Psi$ satisfies the Codazzi equation. From Lemma 10(i) and Proposition 42, it follows that $\Psi^{\pm}$ vanishes precisely on $M_0^{\pm}$. Furthermore, Proposition 42 implies that there exist $\theta^-, \theta^+$ with $\theta^\pm \in C^\infty(M \setminus M_0^{\pm}; (0, 2\pi))$ such that

$$Q_{f,f^\pm} = (1 - e^{\pi i \theta^\pm})\Phi^\pm \quad \text{on} \quad M \setminus M_0^{\pm}.$$ 

Using the above and (8), it follows that $\Psi$ satisfies (9) and (11) on $M \setminus M_0$. Taking into account that $\Psi^{\pm}(p) = 0$ at any $p \in M_0^{\pm}$, from the above and (8), we obtain
that $\Psi$ also satisfies (9) and (11) at any point of $M_0$. The fundamental theorem of submanifolds and Lemma 41(i) imply that there exists a unique Bonnet mate $\tilde{f}$ of $f$, such that the distortion differential of the pair $(f, \tilde{f})$ is $Q = Q_f.f^- + Q_f.f^+$. Clearly, $\tilde{f} \in \mathcal{M}^s(f)$. The rest of the proof is now obvious.

(iii) Assume that $f$ is proper Bonnet. Then at least one of the disjoint components of $\mathcal{M}(f)$ is infinite. From part (ii), it follows that at least one of $\mathcal{M}^-(f)$ and $\mathcal{M}^+(f)$ is infinite. If $\mathcal{M}^+(f)$ is infinite, then part (i) implies that $\mathcal{M}^+(f) = S^1$. The converse is obvious.

(iv) From Proposition 42 and the proof of part (i), it follows that if $\mathcal{M}^\pm(f) \neq \emptyset$, then there exists a one-to-one correspondence between Bonnet mates of $f$ in $\mathcal{M}^\pm(f)$, and solutions $\theta^\pm \in \mathcal{C}^\infty(M \setminus M_0^\pm, (0, 2\pi))$ of (62). Using part (ii), we deduce that the moduli space is parametrized by the pairs $(\theta^-, \theta^+)$, for those solutions $\theta^\pm \in \mathcal{C}^\infty(M \setminus M_0^\pm, (0, 2\pi))$ of (62) that correspond to surfaces in $\mathcal{M}^\pm(f)$. Obviously, according to this parametrization, $\theta^\pm \equiv 0$ correspond to $\mathcal{M}^\pm(f)$. It is now clear that $\mathcal{M}(f)$ can be parametrized by $\tilde{\mathcal{M}}^-(f) \times \tilde{\mathcal{M}}^+(f)$. In particular, if $f$ is proper Bonnet, then parts (iii) and (i) imply that the moduli space is a smooth manifold. □

Remark 48 From the proof of Theorem 46(i), it follows that if $\tilde{\mathcal{M}}^\pm(f)$ is diffeomorphic to $S^1$, then its parametrization is induced by the parametrization of the space of the distinct modulo $2\pi$ solutions of (62). In the proof of Lemma 43(ii), the parametrization $\theta^\pm_t, t \in S^1$ of these solutions is such that

$$\theta^\pm_t(p) = t, \quad t \in S^1,$$

at some $p \in M$. Obviously, this parametrization depends on $p$ and is not unique, unless the solutions of (62) are constant. In this case, from (62), it follows that $h^\pm \equiv 0$ on $M$. Then, (28) and Proposition 14 imply that the Gauss lift $G_{\pm}$ of $f$ is vertically harmonic.

Proof of Theorem 3: If $M$ is compact, the proof follows from Proposition 45. Moreover, if $f$ is minimal, then it is known (cf. [14,22,57]) that either $\mathcal{M}(f) = \{f\}$, or $\mathcal{M}(f) = S^1$. Assume that $M$ is non-compact and $f$ is non-minimal.

(i) If $f$ is not proper Bonnet, then Theorem 46(iii) and (i) imply that $f$ admits at most one Bonnet mate in each one of $\mathcal{M}^-(f)$ and $\mathcal{M}^+(f)$. If $\mathcal{M}^-(f) \neq \emptyset \neq \mathcal{M}^+(f)$, then Theorem 46(ii) yields that $f$ admits exactly three Bonnet mates.

(ii) If $f$ is proper Bonnet, then Theorem 46(iii) implies that either $\mathcal{M}^-(f) = S^1$, or $\mathcal{M}^+(f) = S^1$. Assume that $\tilde{\mathcal{M}}^\pm(f) = S^1$. From Theorem 46(i) and (iv), it follows that $f$ is either tight, or flexible, if there exist either at most one, or infinitely many Bonnet mates of $f$ in $\mathcal{M}^\pm(f)$, respectively. □

6.2 Bonnet Surfaces in $Q^3_\mathbb{C} \subset Q^4_\mathbb{C}$

We study here Bonnet surfaces lying in totally geodesic hypersurfaces of the ambient space.
Lemma 49 Let \( f : M \to \mathbb{Q}_c^4 \) be an oriented surface, which is the composition of a non-minimal Bonnet surface \( F : M \to \mathbb{Q}_c^3 \) with a totally geodesic inclusion \( j : \mathbb{Q}_c^3 \to \mathbb{Q}_c^4 \). For every Bonnet mate \( \tilde{F} \) of \( F \) in \( \mathbb{Q}_c^3 \), we have that \( \tilde{f} = j \circ \tilde{F} \in \mathcal{M}^*(f) \).

**Proof** Let \( \tilde{F} : M \to \mathbb{Q}_c^3 \) be a Bonnet mate of \( F \). Denote by \( \xi, \tilde{\xi} \) the unit normal vector fields of \( F \) and \( \tilde{F} \) in \( \mathbb{Q}_c^3 \), respectively, and by \( h \) their common mean curvature function. Then, the mean curvature vector fields of \( f \) and \( \tilde{f} \) are given by \( H = hj_*\xi \) and \( \tilde{H} = hj_*\tilde{\xi} \), respectively. The parallel vector bundle isometry \( T : N_f M \to N_{\tilde{f}} M \) given by \( Tj_*\xi = j_*\tilde{\xi}, T(J^\perp j_*\xi) = \tilde{J}^\perp j_*\tilde{\xi} \) preserves the mean curvature vector fields, where \( J^\perp \) and \( \tilde{J}^\perp \) are the complex structures of \( N_f M \) and \( N_{\tilde{f}} M \), respectively. Therefore, \( \tilde{f} \in \mathcal{M}(f) \). Since the image of the second fundamental form of \( f, \tilde{f} \) is contained in the line bundle spanned by \( j_*\xi, j_*\tilde{\xi} \), respectively, from Lemma 41(i) and the definition of \( T \), it follows that the zeros of the distortion differential of the pair \((f, \tilde{f})\) satisfy \( Z^- = Z^+ = Z \). Hence, \( \tilde{f} \in \mathcal{M}^*(f) \). \( \square \)

**Proof of Theorem 4:** Let \( f = j \circ F \), where \( j : \mathbb{Q}_c^3 \to \mathbb{Q}_c^4 \) is a totally geodesic inclusion, and denote by \( \xi \) the unit normal of \( F \) in \( \mathbb{Q}_c^3 \). Since \( M \) is simply connected and \( F \) is a Bonnet surface, the result of Lawson-Tribuzy [43] implies that \( M \) is non-compact. Let \( \tilde{F} : M \to \mathbb{Q}_c^3 \) be a Bonnet mate of \( F \). From Lemma 49, it follows that \( j \circ \tilde{F} \in \mathcal{M}^*(f) \) and Theorem 46(ii) implies that there exist Bonnet mates \( f^- \) and \( f^+ \) of \( f \), with \( f^\pm \in \mathcal{M}^\pm(f) \). In particular, since any Bonnet mate of \( f \) lying in some totally geodesic \( \mathbb{Q}_c^3 \subset \mathbb{Q}_c^4 \) belongs to \( \mathcal{M}^*(f) \), the surface \( f^\pm \) does not lie in any totally geodesic hypersurface of \( \mathbb{Q}_c^4 \).

Assume that \( f^\pm \) lies in some totally umbilical \( \mathbb{Q}_c^3 \subset \mathbb{Q}_c^4, \tilde{c} > c \). Proposition 42 implies that \( M_1 \) is isolated. Let \((U, z)\) be a complex chart with \( U \cap M_1 = \emptyset \). Then, there exist \( \varphi, \varphi^\pm \in C^\infty(U) \) such that the Hopf differentials \( \Phi, \Phi_{f^\pm} \) of \( f \) and \( f^\pm \), respectively, are given by

\[
\Phi = \frac{\lambda^2}{2} e^{i\varphi} \sqrt{\|H\|^2 - Ke_3dz^2} \quad \text{and} \quad \Phi_{f^\pm} = \frac{\lambda^2}{2} e^{i\varphi^\pm} \sqrt{\|H\|^2 - Ke_3^\pm dz^2}, \quad (68)
\]

where \( \lambda > 0 \) is the conformal factor, \( e_3 = j_*\xi \), and \( e_3^\pm \in N_{f^\pm} M \) is a smooth unit vector field, parallel to the line segment that the ellipse of curvature of \( f^\pm \) degenerates. Consider an orientation and mean curvature vector field-preserving, parallel vector bundle isometry \( T_{f^\pm} : N_{f^\pm} M \to N_{f^\pm} M \). Appealing to Lemma 41(i) and using (68), it follows that the distortion differential \( Q_{f, f^\pm} \) of the pair \((f, f^\pm)\) is given on \( U \) by

\[
Q_{f, f^\pm} \equiv Q_{f^\pm, f^\pm} = \frac{\lambda^2}{4} \sqrt{\|H\|^2 - K} \left(e^{i\varphi}(e_3 \pm ie_4) - e^{i\varphi^\pm}(e_3^\pm \pm i e_4^\pm)\right) dz^2, \quad (69)
\]

where \( e_4 = J^\perp e_3, e_3^\pm = T_{f^\pm} e_3^\pm \) and \( e_4^\pm = J^\perp e_3^\pm \). On the other hand, according to Proposition 42 there exists \( \theta^\pm \in C^\infty((U; (0, 2\pi)) \) such that \( Q_{f, f^\pm} \) is given by (60) on \( U \). Substituting \( \Phi_{f^\pm} \) from (68) into (60), and using (69), we obtain that

\[ e_3^\pm \pm i e_4^\pm = e^{i(\varphi - \varphi^\pm \pm \theta^\pm)}(e_3 \pm ie_4) \quad \text{on} \ U. \]
Moreover, since $Q_{f,f^\pm} = 0$, from Lemma 41(i) and (68), it follows that

$$\varepsilon_3^\pm i\varepsilon_4^\pm = e^{i(\varphi - \varphi^\pm)}(e_3^\mp i e_4) \quad \text{on} \ U.$$

From the last two equations, we obtain that $\theta^\pm = \pm 2(\varphi - \varphi^\pm) \mod 2\pi$. Then, the above implies that

$$\omega_{34}^\pm = \frac{1}{2} d\theta^\pm + \omega_{34},$$

where $\omega_{34}$ and $\omega_{34}^\pm$ are the connection forms corresponding to the dual frame fields of $\{e_3, e_4\}$ and $\{\varepsilon_3^\pm, \varepsilon_4^\pm\}$, respectively. Since $f$ and $f^\pm$ lie in totally umbilical hypersurfaces and $T_{\pm}$ is parallel, it follows that the vector fields $e_3$ and $\varepsilon_3^\pm$ are parallel in the normal connection of $f$. Therefore, the last relation yields that $\theta^\pm$ is constant on $U$.

Proposition 44 implies that $\theta^\pm$ satisfies (62) on $U$. From (62), it follows that $h^\pm \equiv 0$ on $U$. Then, (28) and Proposition 14 yield that the section $H^\pm$ is anti-holomorphic on $U$. Since $H = he_3$, where $h$ is the mean curvature function of $F$, this implies that $h$ is constant on $U$. Since $U$ is arbitrary and $M_1$ is isolated, it follows that $h$ is constant on $M$.

Conversely, if $F$ has constant mean curvature function, then $f$ and its Bonnet mates have nonvanishing parallel mean curvature vector field. From [8,58], it follows that $f^\pm$ lies in some totally umbilical hypersurface of $Q^4_c$.

Moreover, from Theorem 46(i) and (iv), it is clear that either $f$ admits exactly three Bonnet mates, or it is flexible proper Bonnet.

6.3 Proper Bonnet Surfaces

We study here non-minimal proper Bonnet surfaces $f : M \to Q^4_c$. From Proposition 45, it follows that if $f : M \to Q^4_c$ is a simply connected proper Bonnet surface, then $M$ is non-compact, and therefore, it admits a global complex coordinate $z$. By virtue of Theorem 46(iii-iv), we focus on surfaces with $\bar{M}^\pm(f) = S^1$. For such a surface, Proposition 42 implies that $M_0^\pm(f)$ consists of isolated points only.

We need some facts about absolute value type functions (cf. [20] or [21]). Let $M$ be a 2-dimensional oriented Riemannian manifold. A function $u \in C^\infty(M; [0, +\infty))$ is called of absolute value type if for all $p \in M$ and any complex coordinate $z$ around $p$, there exists a nonnegative integer $m$ and a smooth positive function $u_0$ on a neighbourhood $U$ of $p$, such that

$$u = |z - z(p)|^m u_0 \quad \text{on} \ U.$$

If $m > 0$, then $p$ is called a zero of $u$ of multiplicity $m$. It is clear that if an absolute value type function $u$ does not vanish identically, then its zeros are isolated and they have well-defined multiplicities. Furthermore, the Laplacian $\Delta \log u$ is still defined and smooth at the zeros of $u$.
Proposition 50 Let \( f : M \to \mathbb{Q}_c^4 \) be a simply connected oriented surface with \( \mathcal{M}_\pm(f) = S^1 \). Consider a complex chart \((U, z)\) on \( M \), with \( U \cap M_0^\pm(f) = \{p\} \) and \( z(p) = 0 \). Then

(i) There exists a positive integer \( m \), such that differential \( \Phi^\pm \) satisfies

\[
\Phi^\pm = z^m \hat{\Phi}^\pm \quad \text{on} \quad U, \quad \hat{\Phi}^\pm(p) \neq 0.
\]

(ii) The function \( \| \mathcal{H}^\pm \| \) is of absolute value type on \( M \). The multiplicity of its zero \( p \in M_0^\pm(f) \) is the integer \( m \).

Proof (i) Let \( \tilde{f} \in \mathcal{M}_\pm(f) \). From Proposition 47(i), it follows that there exists \( \theta^\pm \in C^\infty(U; (0, 2\pi)) \) such that the distortion differential of the pair \((f, \tilde{f})\) is given by

\[
Q = \left(1 - e^{\mp i\theta^\pm}\right)\Phi^\pm \quad \text{on} \quad U.
\]

Proposition 42 implies that \( p \) is the only zero of \( Q \) in \( U \). Lemmas 41(i) and 9 yield that there exists a positive integer \( m \) such that

\[
Q = z^m \hat{Q}^\pm \quad \text{on} \quad U, \quad \hat{Q}^\pm(p) \neq 0.
\]

By setting \( \hat{Q}^\pm = \left(1 - e^{\mp i\theta^\pm}\right)^{-1} \hat{\Phi}^\pm \), the proof follows from the above expressions of \( Q \).

(ii) Let \( z = x + iy \) and set \( e_1 = \partial_x / \lambda \), \( e_2 = \partial_y / \lambda \), where \( \lambda > 0 \) is the conformal factor. If \( \hat{\Phi}^\pm = \hat{\phi}^\pm z^2 \) on \( U \), then part (i) implies that \( \phi^\pm = z^m \hat{\phi}^\pm \), where \( \phi^\pm \) is given by (8) on \( U \). Consequently, from (20) it follows that

\[
\| \mathcal{H}^\pm \| = |z|^m u, \quad \text{where} \quad u = \sqrt{2\lambda} \| \hat{\phi}^\pm \| \quad \text{is smooth and positive}.
\]

Clearly, the multiplicity of \( p \) is \( m \).

Lemma 51 Let \( M \) be an oriented 2-dimensional Riemannian manifold with a global complex coordinate \( z \), and \( f : M \to \mathbb{Q}_c^4 \) a surface with \( M_0^\pm(f) = \emptyset \). The 1-forms \( a_1^\pm, a_2^\pm \) on \( M \) given by

\[
a_1^\pm = d \log \| \mathcal{H}^\pm \| - *\Omega^\pm, \quad a_2^\pm = *a_1^\pm,
\]

vanish precisely at the points where the Gauss lift \( G_\pm \) of \( f \) is vertically harmonic. Moreover,

(i)

\[
da_2^\pm = \left(\Delta \log \| \mathcal{H}^\pm \| - 2K_N^{-1}\|N^\pm\| \right) dM = \frac{4}{\lambda^2} \Re h^\pm_\tau dM,
\]

(ii)

\[
a_1^\pm \wedge a_2^\pm = \frac{\| \tau^\pm(G^\pm) \|^2}{4 \| \mathcal{H}^\pm \|^2} dM = \frac{4}{\lambda^2} |h^\pm|^2 dM,
\]

\( \square \)
where $\lambda > 0$ is the conformal factor, and $h^\pm$ is given by (28) on $M$.

**Proof** Let $z = x + iy$ and set $e_1 = \partial_x / \lambda, e_2 = \partial_y / \lambda$. Consider the frame field $\{e_3^\pm, e_4^\pm\}$ of $N_f M$ determined by $\{e_1, e_2\}$ from (14). Then, (20) and (21) hold on $M$, and as in the proof of Lemma 26, we obtain (29) and (30). Using (21), from (30), it follows that

$$a_1^\pm = \frac{2}{\lambda} (\Re h^\pm \omega_1 + \Im h^\pm \omega_2),$$

(70)

where $\{\omega_1, \omega_2\}$ is the dual frame field of $\{e_1, e_2\}$. Proposition 14 and (28) imply that $h^\pm(p) = 0$ if and only if the Gauss lift $G_\pm$ of $f$ is vertically harmonic at $p$. Therefore, from (70), it follows that $a_1^\pm$ vanishes precisely at the points where $G_\pm$ is vertically harmonic.

(i) Differentiating $\omega_{34}^\pm = \omega_{34}(e_1)\omega_1 + \omega_{34}(e_2)\omega_2$ and using (1) and the fact that $\omega_{12} = \ast d \log \lambda$, we obtain

$$K_N = \mp (e_1(\log \lambda)\omega_{34}(e_2) - e_2(\log \lambda)\omega_{34}(e_2) + e_1(\omega_{34}(e_2)) - e_2(\omega_{34}(e_1))).$$

Differentiating (30) with respect to $z$, taking the real part, and using the above and that $\Delta \log \lambda = -K$, yields $(4/\lambda^2) \Re h_z^\pm = \Delta \log \|H^\pm\| - 2K \mp K_N$. On the other hand, taking into account (18), by exterior differentiation of $a_2^\pm$ we obtain that $da_2^\pm = (\Delta \log \|H^\pm\| - 2K \mp K_N) dM$, and this completes the proof.

(ii) Let $H = H^\pm e_3^\pm + H^4\pm e_4^\pm$ be the the mean curvature vector field of $f$. Then,

$$H^\pm = \frac{1}{2} (H \pm i J^\perp H) = \frac{1}{2} (H^\pm \mp i H^4\pm)(e_3^\pm \pm ie_4^\pm).$$

Differentiating the above with respect to $\partial$ in the normal connection, we obtain from (10) that

$$\nabla_\partial^\perp \phi^\pm = \frac{\lambda^2}{4} \left( \partial (H^\pm \mp i H^4\pm) \mp i \omega_{34}(\partial)(H^\pm \mp i H^4\pm) \right) (e_3^\pm \pm ie_4^\pm).$$

From (29) and the above, it follows that

$$h^\pm = \frac{\lambda}{2} \left( \frac{H_1^\pm \mp H_2^4\pm}{\|H^\perp\|} \mp i \frac{H_2^3\pm \pm H_1^4\pm}{\|H^\perp\|} \right),$$

where $H_j^\pm, j = 1, 2, a = 3, 4$, is given by (13). Then, (70) implies that

$$a_1^\pm = u^\pm \omega_1 - v^\pm \omega_2, \quad \text{where} \quad u^\pm = \frac{H_1^3\pm \mp H_2^4\pm}{\|H^\perp\|}, \quad v^\pm = \frac{H_2^3\pm \pm H_1^4\pm}{\|H^\perp\|}. \quad (71)$$

From the above two relations, it follows that

$$a_1^\pm \wedge a_2^\pm = \left( (u^\pm)^2 + (v^\pm)^2 \right) dM = \frac{4}{\lambda^2} |h^\pm|^2 dM,$$
where \( dM = \omega_1 \wedge \omega_2 \). On the other hand, from Proposition 15 and (71), we obtain that

\[
\| \tau^v(G_{\pm}) \|^2 = 4\| \mathcal{H}_{\pm} \|^2 \left( (u^\pm)^2 + (v^\pm)^2 \right),
\]

and the proof follows from the last two equations.

**Theorem 52** Let \( f : M \to \mathbb{Q}_c^4 \) be a simply connected oriented surface. If \( \tilde{M}_{\pm}(f) = S^1 \), then

(i) The Gauss lift \( G_{\pm} \) of \( f \) is vertically harmonic at any point of \( M_{\pm}^0(f) \).

(ii) The surface \( f \) is \( \pm \) isotropically isothermic on \( M \setminus M_{\pm}^0(f) \), and the following differential equation is valid on the whole \( M \)

\[
\Delta \log \| \mathcal{H}_{\pm} \| - 2K \mp K_N = \frac{\| \tau^v(G_{\pm}) \|^2}{4\| \mathcal{H}_{\pm} \|^2}. \tag{72}
\]

(iii) The forms \( a_1^\pm, a_2^\pm \) of Lemma 51 satisfy on \( M \setminus M_{\pm}^0(f) \) the relations

\[
da_1^\pm = 0 \quad \text{and} \quad da_2^\pm = a_1^\pm \wedge a_2^\pm. \tag{73}
\]

Conversely, if \( f \) is non-minimal with \( M_{\pm}^0(f) = \emptyset \), and (ii) or (iii) holds, then \( \tilde{M}_{\pm}(f) = S^1 \).

**Proof** Let \( \tilde{f} \in \mathcal{M}_{\pm}(f) \). Proposition 42 yields that \( M_{\pm}^0(f) \) is isolated. From Proposition 47(i), it follows that there exists a harmonic function \( \theta^\pm \in C^\infty(M; (0, 2\pi)) \) satisfying (62) on \( M \). Lemma 43(iii) implies that \( h^\pm \) extends smoothly on \( M \) and \( A^\pm \equiv 0 \). Then, from (66), it follows that

\[
\text{Im} h^\pm \equiv 0 \quad \text{and} \quad |h^\pm|^2 \equiv \text{Re} h^\pm \quad \text{on} \quad M. \tag{74}
\]

(i) Since \( h^\pm \) extends smoothly on \( M \), equation (28) holds on \( M \). From Lemma 10(i) and (28), we obtain that

\[
\nabla^\perp g \phi^\pm(p) = 0 \quad \text{for any} \quad p \in M_{\pm}^0(f).
\]

Appealing to Proposition 14, this is equivalent with the vertical harmonicity of \( G_{\pm} \) at \( p \).

(ii) By virtue of Lemma 26, the first equation in (74) implies that \( f \) is \( \pm \) isotropically isothermic on \( M \setminus M_{\pm}^0(f) \). Using Lemma 51, the second equation in (74) yields that (72) holds on \( M \setminus M_{\pm}^0(f) \). From Proposition 50(ii) it follows that the left-hand side of (72) can be smoothly extended on \( M \). Therefore, (72) is valid on the whole \( M \).

(iii) From the definition of \( a_1^\pm \), it follows that the first equation in (73) is equivalent with the fact that \( f \) is \( \pm \) isotropically isothermic on \( M \setminus M_{\pm}^0(f) \). Moreover, Lemma 51 implies that the second equation in (73) is equivalent with the second equation in (74) on \( M \setminus M_{\pm}^0(f) \).
Conversely, assume that $f$ is non-minimal and $M_0^\pm(f) = \emptyset$. As above, we obtain that each one of (ii) and (iii) is equivalent to (74). From (74) and (66), it follows that $A^\pm \equiv 0$ on $M$, and Lemma 43(ii) implies that the space of the distinct modulo $2\pi$ solutions of (62) on $M$ is parametrized by $S^1$. From Proposition 47(ii), it follows that $N^\pm(f) = S^1$. \hfill $\square$

**Corollary 53** Let $f : M \to Q_c^4$ be a simply connected surface. If $N^\pm(f) = S^1$ and $\tau^v(G^\pm) \neq 0$ everywhere, then the conformal metric
\[
\hat{ds}^2 = \frac{||\tau^v(G^\pm)||^2}{4||H^\pm||^2}ds^2
\]
has Gaussian curvature $\hat{K} = -1$.

**Proof** By virtue of Theorem 52(i), it follows that $M_0^\pm(f) = \emptyset$. Consider the forms $a_1^\pm, a_2^\pm$ of Lemma 51. Proposition 15 and (71) yield that
\[
ds^2 = a_1^\pm \otimes a_1^\pm + a_2^\pm \otimes a_2^\pm \text{ on } M.
\]
Let $a_{12}^\pm$ be the connection form corresponding to the coframe $\{a_1^\pm, a_2^\pm\}$. Then,
\[
da_{12}^\pm = a_1^\pm \wedge a_{12}^\pm \text{ and } da_{12}^\pm = -\hat{K}a_1^\pm \wedge a_{12}^\pm.
\]
Since $N^\pm(f) = S^1$, the first equation of the above and the second relation in (73) yield that $a_{12}^\pm = a_2^\pm$. Using the second equation of the above, this implies that $da_{12}^\pm = -\hat{K}a_1^\pm \wedge a_{12}^\pm$, and the proof follows by virtue of the second relation in (73). \hfill $\square$

**Remark 54** (i) Theorems 46(i) and 52(i) imply that a surface $f$ admits at most one Bonnet mate in $M^\pm_0(f)$, if there exists a point $p \in M_0^\pm(f)$ at which the Gauss lift $G^\pm$ of $f$ is not vertically harmonic. From Theorem 46(iv), it follows that $f$ is not proper Bonnet if there exists an umbilic point at which $H$ is non-parallel. This extends a result of Roussos and Hernandez [53, Thm. 1B].

(ii) Equation (72) extends the Ricci-like condition satisfied by the non-superconformal surfaces in which Gauss lift $G^\pm$ is vertically harmonic (cf. [51, Prop. 23(iii)]).

(iii) For umbilic-free surfaces in $\mathbb{R}^3$, the analogues of (72) and (73), are due to Colares-Kenmotsu [13] and Chern [11], respectively.

From Theorems 46(iv) and 52(ii), it follows that a flexible proper Bonnet surface is strongly isotropically isothermic away from its isolated pseudo-umbilic points. The following proposition shows that a Bonnet, strongly isotropically isothermic surface is proper Bonnet. The analogous result for isothermic Bonnet surfaces in $Q_c^3$ is due to Graustein [27].

**Proposition 55** Let $f : M \to Q_c^4$ be a non-minimal, simply connected oriented surface. If $f$ is $\pm$ isotropically isothermic, then either $N^\pm(f) = \{f\}$, or $N^\pm(f) = S^1$. In particular, if $f$ is Bonnet and strongly isotropically isothermic, then either $M(f) = S^1$, or $M(f) = S^1 \times S^1$. 

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Proof Assume that there exists a Bonnet mate \( \tilde{f} \in M^\pm(f) \). From Proposition 42, it follows that there exists \( \theta^\pm \in C^\infty(M; (0, 2\pi)) \), such that the distortion differential of the pair \((f, \tilde{f})\) is given by (60) on \( M \). Proposition 44(ii) yields that \( \theta^\pm \) is harmonic.

If \( M \) is compact, then the maximum principle implies that \( \theta^\pm \) is constant. From Lemma 41(i) and (60), it follows that \( \Phi^\pm \) is holomorphic, and Proposition 14 implies that the Gauss lift \( G^\pm \) of \( f \) is vertically harmonic. From [51, Thm. 3], it follows that \( f \) is superconformal. Then, [51, Prop. 10] yields that \( \overline{\mathcal{M}}^\pm(f) = S^1 \). Lemma 10(i) implies that \( M^\pm_0(f) = M \) and this contradicts the fact that \( f \) is \( \pm \) isotropically isothermic.

Therefore, if \( M \) is homeomorphic to \( S^2 \), then \( \overline{\mathcal{M}}^\pm(f) = \{f\} \).

Assume that \( M \) is non-compact and let \( \varepsilon \) be a global complex coordinate on \( M \). Proposition 44 yields that \( \theta^\pm \) satisfies (62) on \( M \). Then, Lemma 43(iii-ii) implies that the space of the distinct modulo \( 2\pi \) solutions of (62) is parametrized by \( S^1 \). From Proposition 47(ii), we obtain that \( \overline{\mathcal{M}}^\pm(f) = S^1 \). In particular, if \( f \) is Bonnet and strongly isotropically isothermic, the proof follows by using Theorem 46(iv).

Example 56 Tight proper Bonnet surfaces in \( \mathbb{R}^4 \) that are strongly isotropically isothermic and isothermic, and they have a vertically harmonic Gauss lift:

We consider the product in \( \mathbb{R}^4 \) of two plane curves \( \gamma_1, \gamma_2 \), as in Example 32, and we adopt the notation used there. Assume that the curvature of \( \gamma_j \) is \( k_j(s_j) = cs_j \), \( j = 1, 2 \), with \( 0 \neq c \in \mathbb{R} \), and we restrict the product surface \( f \) such that \( f : M \rightarrow \mathbb{R}^4 \) is simply connected and umbilic free. Clearly, \( f \) has flat normal bundle and does not lie in any totally umbilical hypersurface of \( \mathbb{R}^4 \). Moreover, from (37), it follows that \( f \) is strongly isotropically isothermic.

It has been proved by Hasegawa [30, Example 1] that the Gauss lift \( G^- \) of \( f \) is vertically harmonic. Since \( f \) is neither minimal, nor superconformal, from [51, Thm. 4, Prop. 25], it follows that \( \overline{\mathcal{M}}^-(f) = S^1 \).

Since \( f \) is \( \pm \) isotropically isothermic, Proposition 55 implies that either \( \overline{\mathcal{M}}^+(f) = \{f\} \), or \( \overline{\mathcal{M}}^+(f) = S^1 \). We claim that \( \overline{\mathcal{M}}^+(f) = \{f\} \). Arguing indirectly, assume that \( \overline{\mathcal{M}}^+(f) = S^1 \). Then, from Theorem 52(ii), it follows that

\[
\Delta \log \| \mathcal{H}^+ \| - 2K = \frac{\| \tau^v(G^+) \|^2}{4\| \mathcal{H}^+ \|^2}.
\]

On the other hand, since \( \overline{\mathcal{M}}^-(f) = S^1 \), Theorem 52(ii) yields that

\[
\Delta \log \| \mathcal{H}^- \| - 2K = 0.
\]

Since \( K_N = 0 \) everywhere on \( M \), it follows that \( \| \mathcal{H}^- \| = \| \mathcal{H}^+ \| \) and the above two relations imply that the Gauss lift \( G^+ \) of \( f \) is vertically harmonic. Therefore, the mean curvature vector field of \( f \) is parallel in the normal connection, and thus (cf. [8,58]), \( f \) lies in some totally umbilical hypersurface of \( \mathbb{R}^4 \). This is a contradiction and the claim follows. From Theorem 46(iv), we deduce that \( \mathcal{M}(f) = S^1 \).
7 Compact Surfaces

7.1 The Effect of Isotropic Isothermicity

We study here the effect of isotropic isothermicity on the structure of the moduli space $\mathcal{M}(f)$ for compact surfaces.

**Theorem 57** Let $f : M \to \mathbb{Q}_4^*$ be a compact-oriented surface, and $V$ an open and dense subset of $M$. If one of the following holds, then $\mathcal{N}(f) = \emptyset$.

(i) The Gauss lift $G_\pm$ of $f$ is not vertically harmonic and $f$ is $\pm$ isotropically isothermic on $V$.

(ii) The set $V$ is connected and $f$ is totally non- $\pm$ isotropically isothermic on $V$.

**Proof** If (i) or (ii) holds, then from Proposition 14 or Examples 30(ii-iii), respectively, it follows that $f$ is non-minimal. Arguing indirectly, assume that there exists a Bonnet mate $\tilde{f} \in \mathcal{N}(f)$. Then, Proposition 42 implies that $M_0^\pm$ is isolated and that there exists $\theta^\pm \in C^\infty(M \setminus M_0^\pm ; (0, 2\pi))$, such that the distortion differential $Q$ of the pair $(f, \tilde{f})$ satisfies (60) on $M \setminus M_0^\pm$.

(i) Since $V$ is dense, it follows that $f$ is $\pm$ isotropically isothermic on $M \setminus M_0^\pm$. Then, Proposition 44(ii) implies that $\theta^\pm$ extends to a bounded harmonic function on $M$, which has to be constant by the maximum principle. By virtue of Lemma 41(i), from (60), it follows that $\Phi^\pm$ is holomorphic. Proposition 14 yields that the Gauss lift $G_\pm$ of $f$ is vertically harmonic, and this is a contradiction.

(ii) From the definition of non- $\pm$ isotropically isothermic points, it follows that $M_0^\pm \subset M \setminus V$. Therefore, $\theta^\pm$ is defined everywhere on $V$. Let $(U, z)$ be a complex chart with $U \subset V$. Proposition 44 implies that $\theta^\pm$ satisfies (62) on $U$. From Lemma 26, it follows that $\text{Im } h^\pm_\zeta \neq 0$ everywhere on $U$. Appealing to Lemma 43(i), (66) and (65) yield that $\Delta \theta^\pm$ is nowhere vanishing on $U$. Since $U$ is an arbitrary subset of the connected $V$, we deduce that either $\Delta \theta^\pm > 0$, or $\Delta \theta^\pm < 0$, on $V$. Since $V$ is dense in $M \setminus M_0^\pm$, it follows by continuity that either $\Delta \theta^\pm \geq 0$, or $\Delta \theta^\pm \leq 0$, on $M \setminus M_0^\pm$. As in the proof of [36, Thm. 2], it can be shown that either $\theta^\pm$, or $-\theta^\pm$ can be extended to a subharmonic function on $M$ which attains a maximum, and thus, it has to be constant by the maximum principle for subharmonic functions. As in the proof of part (i), it follows that the Gauss lift $G_\pm$ of $f$ is vertically harmonic. Then, example 30(i) implies that $f$ is $\pm$ isotropically isothermic on $V$, which is a contradiction. $\square$

**Proof of Theorem 5**: Since $G_\pm$ is not vertically harmonic and $f$ is either $\pm$ isotropically isothermic, or totally non- $\pm$ isotropically isothermic, on $V$, Theorem 57 implies that $\mathcal{N}(f) = \emptyset$. On the other hand, since $G_\mp$ is not vertically harmonic, from [51, Thm. 13(i)], it follows that there exists at most one Bonnet mate of $f$ in $\mathcal{M}(f)$. Therefore, $f$ admits at most one Bonnet mate. In particular, if $f$ is either strongly isotropically isothermic, or strongly totally non-isotropically isothermic, on $V$, then Theorem 57 implies that $\mathcal{N}(f) = \mathcal{N}(f) = \emptyset$, and thus, $f$ does not admit any Bonnet mate. $\square$

The following consequence of Theorem 5 shows that the result of [36] can be strengthened.
Corollary 58 Let $F: M \to \mathbb{Q}_c^3$ be a compact-oriented surface and $j: \mathbb{Q}_c^3 \to \mathbb{Q}_c^4$ a totally geodesic inclusion. If the mean curvature of $F$ is not constant and $F$ is either isothermic, or totally non-isothermic, on an open dense and connected subset $V$ of $M$, then $f = j \circ F$ does not admit any Bonnet mate in $\mathbb{Q}_c^4$.\\

**Proof** From Proposition 19, it follows that $f$ is strongly (totally non) isothermic on $V$ if and only if $F$ is (totally non) isothermic on $V$. The proof follows immediately from Theorem 5. \qed

### 7.2 Locally Proper Bonnet Surfaces

An oriented surface $f: M \to \mathbb{Q}_c^4$ is called *locally proper Bonnet*, if every point of $M$ has a simply connected neighbourhood $U$ such that $f|_U$ is proper Bonnet. Notice that if $f|_U$ is non-minimal, then Theorem 46(iv) implies that $\mathcal{M}(f|_U)$ is a smooth manifold.

**Proposition 59** Let $f: M \to \mathbb{Q}_c^4$ be a locally proper Bonnet surface. Then

(i) Either $f$ is minimal, or $\text{int}\{p \in M : H(p) = 0\} = \emptyset$.

(ii) If $f$ is non-minimal, then for every $p \in M$, there exists a submanifold $L^n(p)$, $1 \leq n \leq 2$, of the torus $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \simeq \mathbb{R}/2\pi \mathbb{Z}$, with the property that $L^n(p)$ is also a submanifold of $\mathcal{M}(f|_U)$ for every sufficiently small simply connected neighbourhood $U$ of $p$. In particular, for every point of $M$, a submanifold of the torus with this property is either $\mathbb{S}^1_- = \mathbb{S}^1 \times \{0\}$, or $\mathbb{S}^1_+ = \{0\} \times \mathbb{S}^1$.

**Proof** (i) Arguing indirectly, assume that $f$ is non-minimal and $\text{int}\{p \in M : H(p) = 0\} \neq \emptyset$. Then, for a boundary point $\bar{p}$ of $\{p \in M : H(p) = 0\}$, there exists a simply connected complex chart $(U, z)$ around $\bar{p}$ such that $f|_U$ is proper Bonnet and non-minimal. By virtue of Theorem 46(iii), we may assume that $\mathcal{M}^\pm(f|_U) = \mathbb{S}^1$. Let $\tilde{f} \in \mathcal{M}^\pm(f|_U)$. From Proposition 42, it follows that $M^\pm_0(f|_U)$ is isolated. Since $M^\pm_0(f|_U) = M^\pm_0(f) \cap U$, we may assume that $\tilde{p}$ and $U$ are such that $M^\pm_0(f|_U) = \emptyset$. Then, the Codazzi equation and (28) imply that

$$h^\pm \equiv 0 \quad \text{on} \quad U \cap \text{int}\{p \in M : H(p) = 0\}.$$\\

According to Proposition 47(i), there exists a harmonic function $\theta^\pm \in C^\infty(U; (0, 2\pi))$ satisfying (62) on $U$, such that the distortion differential of the pair $(f|_U, \tilde{f})$ is given by (60) on $U$. From (62) and the above, it follows that the harmonic function $\theta^\pm$ is constant on $U \cap \text{int}\{p \in M : H(p) = 0\}$ and thus, constant on $U$. Then, (62) yields that $h^\pm \equiv 0$ on $U$. Proposition 14 and (28) imply that the Gauss lift $G_\pm$ of $f$ is vertically harmonic on $U$. From [51, Prop. 23(ii)] we know that $\|H\|^2$ is an absolute value type function on $U$. Since $\|H\|^2$ vanishes on an open subset of $U$, it follows that $H \equiv 0$ on $U$. This is a contradiction, since $f|_U$ is non-minimal.

(ii) Assume that $f$ is non-minimal and let $p \in M$. There exists a simply connected complex chart $(V, z)$ around $p$ such that $f|_V$ is proper Bonnet. From part (i), it follows that $f|_V$ is non-minimal and Theorem 46(iii) implies that either $\mathcal{M}^-_0(f|_V) = \mathbb{S}^1$, or $\mathcal{M}^+_0(f|_V) = \mathbb{S}^1$. Assume that $\mathcal{M}^\pm_0(f|_V) = \mathbb{S}^1$. By virtue of Remark 48, we
parametrize $\hat{M}_p^\pm(f|V)$ such that (67) is valid at $p$, and we write $\hat{M}_p^\pm(f|V) = S^1$. For every sufficiently small simply connected neighbourhood $U$ of $p$, we have that $U \subset V$ and therefore, $\hat{M}_p^\pm(f|U) = S^1$. Appealing to Theorem 46(iv), it is clear that $S_1^\pm$ is a submanifold of $\mathcal{M}(f|U)$.

Let $f: M \to \mathbb{Q}_c^4$ be a non-minimal locally proper Bonnet surface. By virtue of Proposition 59(ii), we give the following definition; the surface $f$ is called uniformly locally proper Bonnet if there exists a submanifold $L^n$, $1 \leq n \leq 2$, of the torus $S^1 \times S^1$, $S^1 \simeq \mathbb{R}/2\pi \mathbb{Z}$, with the property that for every $p \in M$, $L^n$ is also a submanifold of $\mathcal{M}(f|U)$ for every sufficiently small simply connected neighbourhood $U$ of $p$. In this case, $L^n$ is called a deformation manifold for $f$. Moreover, $f$ is called locally flexible proper Bonnet if the torus $S^1 \times S^1$ is a deformation manifold for $f$.

**Lemma 60** A surface $f: M \to \mathbb{Q}_c^4$ is uniformly locally proper Bonnet with deformation manifold $S_1^\pm$ if and only if every point of $M$ has a simply connected neighbourhood $U$ such that $\hat{M}_U^\pm(f|U) = S^1$. If $S_1^\pm$ is a deformation manifold for $f$, then the set $M_0^\pm(f|U)$ is isolated.

**Proof** Assume that $S_1^\pm$ is a deformation manifold for $f$. Then, every point of $M$ has a simply connected neighbourhood $U$ such that $S_1^\pm$ is a submanifold of $\mathcal{M}(f|U)$. From Theorem 46(iv), it follows that $\hat{M}_U^\pm(f|U) = S^1$. The converse follows in a similar manner with the proof of Proposition 59(ii).

Suppose now that $S_1^\pm$ is a deformation manifold for $f$ and arguing indirectly, assume that $M_0^\pm(f) = S_1^\pm$ has an accumulation point $p$. Then, there exists a neighbourhood $U$ of $p$ such that $\hat{M}_U^\pm(f|U) = S^1$. Proposition 42 implies that $M_0^\pm(f|U)$ is isolated. This is a contradiction, since $M_0^\pm(f|U) = M_0^\pm(f) \cap U$.

For the proof of the following theorem, we recall (cf. [20]) that if $M$ is compact and $u \neq 0$ is an absolute value function on $M$, then

$$\int_M \Delta \log u = -2\pi N(u),$$

where $N(u)$ is the number of zeros of $u$, counted with multiplicities.

**Theorem 61** Let $f: M \to \mathbb{Q}_c^4$ be a non-minimal, compact-oriented surface. The surface $f$ is uniformly locally proper Bonnet with deformation manifold $S_1^\pm$ if and only if the Gauss lift $G_\pm(f)$ is vertically harmonic and non-holomorphic.

**Proof** Assume that $S_1^\pm$ is a deformation manifold for $f$. Lemma 60 yields that $M_0^\pm(f)$ is isolated. Then, Lemma 10(ii) and Proposition 12 imply that the Gauss lift $G_\pm(f)$ is non-holomorphic. From Lemma 60 and Theorem 52(ii), it follows that (72) is valid on the whole $M$. Integrating (72) on $M$ yields

$$\int_M \Delta \log \|H^\pm\| - \int_M (2K \pm K_N) = \int_M \frac{\|\tau^v(G_\pm)\|^2}{4\|H^\pm\|^2}.$$  

(75)
From Lemma 60 and Proposition 50(ii), it follows that \( \| \mathcal{H}^\pm \| \) is an absolute value function on \( M \) with isolated zeros. Therefore, we have

\[
\int_M \Delta \log \| \mathcal{H}^\pm \| = -2\pi N(\| \mathcal{H}^\pm \|).
\]

On the other hand, Theorem 21 and Propositions 20 and 50(i) imply that

\[
\int_M (2K \pm K_N) = -2\pi N(\| \mathcal{H}^\pm \|).
\]

From the above two relations, it follows that the left-hand side of (75) vanishes identically. Therefore, (75) implies that \( \| \tau_v(G^\pm) \| \equiv 0 \) on \( M \), and this shows that the Gauss lift \( G^\pm \) of \( f \) is vertically harmonic.

Conversely, assume that the Gauss lift \( G^\pm \) of \( f \) is vertically harmonic and non-holomorphic. By virtue of Lemma 10(ii), Proposition 12 implies that \( M \neq M^\pm_0(f) \).

From [51, Prop. 23(ii)], it follows that \( M^\pm_0(f) \) is isolated, and that the mean curvature vector field of \( f \) does not vanish on any open subset of \( M \). Then, [51, Thm. 4, Prop. 25] imply that every point of \( M \) has a simply connected neighbourhood \( U \) such that \( \mathcal{M}^\pm(f|_U) = S^1 \). The proof now follows from Lemma 60.

**Proof of Theorem 6:** By virtue of Lemma 59(ii), either \( S^1_- \), or \( S^1_+ \), is a deformation manifold for \( f \). The proof follows immediately from Theorem 61.

**Proof of Theorem 7:** For minimal surfaces, the result is known (cf. [38,57]). Let \( f : M \to Q^4_c \) be a non-minimal, compact superconformal surface and arguing indirectly, assume that \( f \) is locally proper Bonnet.

We claim that the normal curvature of \( f \) does not change sign. By virtue of Lemma 59(ii) and Theorem 46(iii), every point of \( M \) has a neighbourhood \( U \) such that either \( \mathcal{M}^-_0(f|_U) = S_1^- \), or \( \mathcal{M}^+_0(f|_U) = S_1^+ \). Then, Proposition 42 implies that either \( M^-_0(f|_U) \), or \( M^+_0(f|_U) \) is isolated. Since \( M^\pm_0(f|_U) = M^\pm_0(f) \cap U \) and \( M_1(f) = M^-_0(f) \cap M^+_0(f) \), we deduce that \( M_1(f) \) is isolated. From Lemma 10(ii), it follows that the normal curvature of \( f \) vanishes at isolated points only, and this proves the claim.

Assume that \( \pm K_N \geq 0 \). Lemma 10(ii) implies that \( \Phi^\pm \equiv 0 \). Therefore, \( \mathcal{M}^\pm(f|_U) = \emptyset \) for every \( U \subset M \). Since \( f \) is locally proper Bonnet, from Theorem 46(iii) and Lemma 60, it follows that \( f \) is uniformly locally proper Bonnet with deformation manifold \( S^1_\pm \). Then, Theorem 61 implies that the Gauss lift \( G^\pm \) is vertically harmonic and non-holomorphic. On the other hand, since \( \Phi^\pm \equiv 0 \), from Proposition 14 it follows that \( G^\pm \) is vertically harmonic. Since both Gauss lifts of \( f \) are vertically harmonic, the mean curvature vector field of \( f \) is parallel in the normal connection. Therefore, \( K_N \equiv 0 \) on \( M \). Proposition 12 then implies that \( G^\pm \) is holomorphic, which is a contradiction.

**Corollary 62** There do not exist uniformly locally proper Bonnet surfaces in \( Q^4_c \) of genus zero.
Proof Arguing indirectly, assume that $M$ is homeomorphic to $\mathbb{S}^2$ and let $f: M \to \mathbb{Q}_c^4$ be a uniformly locally proper Bonnet surface. By virtue of Lemma 59(ii), assume that $\mathbb{S}_1^{\pm}$ is a deformation manifold for $f$. Theorem 61 implies that the Gauss lift $G_{\pm}$ of $f$ is vertically harmonic. Then, from [51, Thm. 3], it follows that $f$ is superconformal. This contradicts Theorem 7.

Proof of Theorem 8: Assume that $f$ is locally flexible proper Bonnet. From [22], it follows that $f$ is non-minimal. Since both $\mathbb{S}_1^{\pm}$ and $\mathbb{S}_1^1$ are deformation manifolds for $f$, Theorem 61 implies that both Gauss lifts of $f$ are vertically harmonic. Therefore, $f$ has nonvanishing parallel mean curvature vector field. Moreover, Corollary 62 yields that $\text{genus}(M) > 0$.

Conversely, assume that $f$ has nonvanishing parallel mean curvature vector field and that $\text{genus}(M) > 0$. Since $M$ is not homeomorphic to $\mathbb{S}^2$, it follows that $f$ is not totally umbilical, and thus, Lemma 10(i) yields that the Hopf differential $\Phi$ of $f$ does not vanish identically on $M$. On the other hand, the Codazzi equation implies that $\Phi$ is holomorphic. Therefore, from Lemmas 9 and 10(i), it follows that the umbilic points of $f$ are isolated. Then, [51, Prop. 26(iii)] implies that every point of $M$ has a simply connected neighbourhood $U$ such that $\mathcal{M}(f|_U) = \mathbb{S}_1^1 \times \mathbb{S}_1^1$. This completes the proof.

An immediate consequence of Theorems 4 and 8 is the following result due to Umehara [56].

Theorem 63 Let $F: M \to \mathbb{Q}_c^3$ be a non-minimal, compact-oriented surface with $\text{genus}(M) > 0$. The surface $F$ is locally proper Bonnet if and only if it has constant mean curvature.

Proof Let $j: \mathbb{Q}_c^3 \to \mathbb{Q}_c^4$ be a totally geodesic inclusion and set $f = j \circ F$. From Theorem 4, it follows that $F$ is locally proper Bonnet if and only if $f$ is locally flexible. Theorem 8 implies that $f$ is locally flexible if and only if it has parallel mean curvature vector field, or equivalently, if the mean curvature of $F$ is constant.

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