Counting ramified coverings and intersection theory on Hurwitz spaces II
(Local structure of Hurwitz spaces and combinatorial results)

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Abstract

The Hurwitz space is a compactification of the space of rational functions of a given degree. We study the intersection of various strata of this space with its boundary. A study of the cohomology ring of the Hurwitz space then allows us to obtain recurrence relations for certain numbers of ramified coverings of a sphere by a sphere with prescribed ramifications. Generating functions for these numbers belong to a very particular subalgebra of the algebra of power series.

1 Introduction

1.1 Preliminaries

This paper is a continuation of the paper [9] by S. Lando and the author. Here we use the same general framework and the same notations.

In [9] we introduced Hurwitz spaces $\mathcal{H}_n$ and their projectivizations $\mathbb{P}\mathcal{H}_n$ (along the lines of [11]) and the Lyashko-Looijenga map $LL$. We also described various strata in Hurwitz spaces and proved several relations satisfied by the homology classes represented by these strata. The degree of

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the Lyashko-Looijenga map on each stratum and the corresponding Hurwitz number were related to the coupling of the stratum with a power of a particular 2-cohomology class $\Psi_n \in H^2(\mathbb{P}\mathcal{H}_n)$.

A stratum $\Sigma_{\kappa_1,\ldots,\kappa_c} \in \mathcal{H}_n$ is described by a set of $c$ partitions of positive integers $d_1,\ldots,d_c$. A generic function lying in this stratum has $c$ critical values of multiplicities $d_1,\ldots,d_c$. The elements of the partitions correspond to the multiplicities of the critical points. In the sequel we often omit in the above notation the trivial partitions $\kappa = 1$ corresponding to simple critical points. We denote by $\mu_{\kappa_1,\ldots,\kappa_c}$ the degree of the LL map on the stratum $\Sigma_{\kappa_1,\ldots,\kappa_c}$.

A Hurwitz number $h_{\kappa_1,\ldots,\kappa_c}$ is the number of ramified coverings with $c$ prescribed ramification points and ramification types $\kappa_1,\ldots,\kappa_c$, each covering being counted with weight $1/|\text{Aut}|$ (where $|\text{Aut}|$ is the number of automorphisms of the covering).

The precise relation formula obtained in [9] is the following:

$$h_{\kappa_1,\ldots,\kappa_c} = \frac{\mu_{\kappa_1,\ldots,\kappa_c}}{|\text{Aut}\{\kappa_1,\ldots,\kappa_c\}|} = \frac{1}{n!} \frac{\langle \Psi^d_n, \mathbb{P}\Sigma_{\kappa_1,\ldots,\kappa_c} \rangle}{|\text{Aut}\{d_1,\ldots,d_c\}|},$$

where $d$ is the dimension of the projectivized stratum $\mathbb{P}\Sigma_{\kappa_1,\ldots,\kappa_c}$.

Codimension 1 strata have special names: the caustic $C_n = \Sigma_2$ and the Maxwell stratum $M_n = \Sigma_{1,2}$. We also use boundary strata $\Delta_n$ and $\Delta_{\rho,q}$ (see [9]).

### 1.2 The results

In the present paper we use the above results to derive recurrence relations (expressed as partial differential equations on generating functions) on some Hurwitz numbers corresponding to certain strata in the Hurwitz space. These relations have a meaning similar to the WDVV equation in quantum cohomology: intersecting a stratum with the boundary of the Hurwitz space, we split the initial Riemann sphere into several spheres with smaller numbers of marked points. These results are formulated in Section 2.

Let $\Sigma$ be a stratum (i.e., the closure of the set of rational functions with prescribed multiplicities of critical points and values) in the Hurwitz space $\mathcal{H}_n$. We first study the intersection of $\Sigma$ with the boundary of the Hurwitz space (i.e., with the set of stable rational functions defined on singular nodal curves). Our main geometrical result is that the neighborhood of this intersection locally looks like a smaller Hurwitz space (Theorem 5). In particular,
this allows us to find the multiplicity of the intersection. This investigation of the geometry of the Hurwitz space is carried out in Section 3.

In Section 4 we use the cohomological relations of [9] and the results of Section 3 to prove the recurrence relations of Section 2.

Finally, in Section 5 we give some explicit formulas for Hurwitz numbers, resulting from our recurrence relations. Here are some examples of these enumerative results. Let \( \kappa_1, \ldots, \kappa_c \) be partitions such that the sum of all their elements equals \( 2n - 2 \). As above, \( h_{\kappa_1, \ldots, \kappa_c} \) is the corresponding Hurwitz number. Let \( \alpha \) be the partition 2 and \( \beta \) the partition 3. They correspond to critical values whose preimage contains, respectively, a unique double critical point or a unique triple critical point. In the notation below we omit the partition 1 that corresponds to a simple ramification point.

**Theorem 1**

\[
\begin{align*}
    h_{\alpha, \alpha} &= \frac{3}{4} (27n^2 - 137n + 180) \frac{n^{n-6} (2n - 6)!}{(n - 3)!}, \\
    h_{\alpha, \beta} &= 8 (6n^2 - 37n + 60) \frac{n^{n-7} (2n - 7)!}{(n - 4)!}, \\
    h_{\beta, \beta} &= \frac{4}{9} (256n^3 - 2787n^2 + 10448n - 13440) \frac{n^{n-8} (2n - 8)!}{(n - 4)!}, \\
    h_{\alpha, \alpha, \alpha} &= \frac{1}{8} (729n^3 - 6723n^2 + 21026n - 22680) \frac{n^{n-7} (2n - 8)!}{(n - 4)!}.
\end{align*}
\]

We have an algorithm that allows one to find similar formulas for other cases and a conjecture about the general form of the answer.

We also prove that some generating series for the numbers of ramified coverings belong to a very special subalgebra of the algebra of formal power series and discuss the properties of this subalgebra. Namely, fix \( c \) points on the sphere and \( c \) partitions \( \kappa_1, \ldots, \kappa_c \) of positive integers \( d_1, \ldots, d_c \). Denote by \( h_{\kappa_1, \ldots, \kappa_c}(n) \) the number of the \( n \)-sheeted coverings of the sphere by a sphere, ramified over the \( c \) points with ramification types \( \kappa_i \) and, in addition, having \( d(n) = 2n - 2 - \sum d_i \) fixed simple ramification points. Denote by \( f_{\kappa_1, \ldots, \kappa_c} \) the series

\[
f_{\kappa_1, \ldots, \kappa_c}(t) = \sum \frac{h_{\kappa_1, \ldots, \kappa_c}(n)}{d(n)!} t^n.
\]
Theorem 2  The power series \( f_{\kappa_1, \ldots, \kappa_c} \) belongs to the algebra generated by the power series

\[
\sum_{n \geq 1} \frac{n^{n-1}}{n!} t^n \quad \text{and} \quad \sum_{n \geq 1} \frac{n^n}{n!} t^n.
\]

We believe that this subalgebra may play an important role in the combinatorial theory of ramified coverings of the sphere.

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2 Recurrence relations for Hurwitz numbers

Here we write out recurrence relations for some Hurwitz numbers. We introduce generating functions whose coefficients are these Hurwitz numbers. The recurrence relations can then be expressed as partial differential equations with an infinite number of terms for each generating function. These relations are proved in the two following sections (Sections 3 and 4).

Let \( \Sigma = \Sigma_{\kappa_1, \ldots, \kappa_c} \) be a stratum in \( \mathcal{H}_n \). The main idea of the proof is to combine the expression

\[
h_{\kappa_1, \ldots, \kappa_c} = \frac{1}{n!} |\text{Aut}\{\kappa_1, \ldots, \kappa_c\}| \langle \mathbb{P}\Sigma, \Psi_n^{\dim\mathbb{P}\Sigma} \rangle
\]

for the Hurwitz numbers (Theorems 3 and 4 in [9]) and the expression

\[
\Psi_n = \frac{1}{2n(n-1)} \sum_{p+q=n} pq \Delta_{p,q}
\]

for the cohomology class \( \Psi_n \) (Proposition 4.9 in [9]). From these two identities we deduce:

Corollary 2.1

\[
2n(n-1) \frac{h_{\kappa_1, \ldots, \kappa_c}}{|\text{Aut}\{\kappa_1, \ldots, \kappa_c\}|} = \frac{1}{n!} \sum_{p+q=n} pq \langle (\mathbb{P}\Sigma \cap \Delta_{p,q}) , \Psi_n^{\dim\mathbb{P}\Sigma - 1} \rangle.
\]
The right-hand side of the last equality can be interpreted as a sum of degrees of the $LL$ map on the subvarieties of the from $\Sigma \cap \Delta_{p,q}$. A point of this intersection is a meromorphic function defined on a curve with at least 2 irreducible components. We will see that it can happen that the curve actually consists of more than 2 components, even for a generic point of the intersection. Consider a particular irreducible component of the curve, and suppose that the restrictions of the stable meromorphic functions to this component are of degree $m$. Then these restrictions form a stratum in the space $H_m$. Knowing the degrees of the $LL$ map on such strata for each component of the curve (or, equivalently, the corresponding Hurwitz numbers), one can find the degree of $LL$ on the intersection $\Sigma \cap \Delta_{p,q}$.

In order to be able to use this method, we will restrict our attention to particular classes of strata $\Sigma$, whose intersection with $\Delta_{p,q}$ is not too difficult to understand. Moreover, these classes must satisfy the following property.

**Definition 2.2** A class of strata $\Sigma$ is called stable under intersection with $\Delta$ if, for any $p, q$, the restrictions of the functions $f \in \Sigma \cap \Delta_{p,q}$ to any given component of the nodal curve, form a stratum $\Sigma'$ that belongs to the same class.

Here is a list of several classes of strata stable under intersection with $\Delta$. The classes are described by properties that should be satisfied by a generic function of a stratum $\Sigma$ in order for $\Sigma$ to belong to the class.

1. All strata.
2. All critical values (and therefore all critical points) are simple.
3. All critical values are simple except maybe one.
4. There is a critical point with the maximal possible multiplicity (i.e., multiplicity $n - 1$ for functions of degree $n$).
5. The values at distinct critical points are distinct (but the critical points can be multiple).
6. The values at distinct critical points are distinct and the multiplicities of the critical points are equal to 1 or 2.
2.1 Strata with distinct critical values at distinct critical points

In this subsection we consider ramified coverings with particular ramification types. Namely, we consider rational functions $f$ such that the preimage under $f$ of every critical value of $f$ contains only one critical point. We introduce a generating function for the corresponding Hurwitz numbers and write down a partial differential equation on this generating function. A closed formula for such Hurwitz numbers is still unknown, but we have found many new formulas for particular cases.

**Definition 2.3** A stratum $\Sigma$ is called *simple* if for a generic function $f \in \Sigma$, every critical value of $f$ is attained at a unique (possibly multiple) critical point. In other words, the set of partitions $\kappa_1, \ldots, \kappa_c$ that describes the stratum $\Sigma$ (see Introduction) contains only partitions consisting of a single element.

**Example 2.4** The caustic is a simple stratum, while the Maxwell stratum is not.

Before proceeding we briefly describe the intersection of a simple stratum $\Sigma$ with $\Delta_{p,q}$. Consider a function $f$ in the open part of $\Sigma$ that tends to a stable meromorphic function in the intersection $\Sigma \cap \Delta_{p,q}$. Then, generically, two critical values of $f$ get glued together.

The monodromies corresponding to these critical values are cyclic permutations (i.e., have only one cycle of length greater than 1), because each critical value has a unique critical point in its preimage. When the critical values are glued together the monodromies are multiplied. The two cycles in the monodromies have $k \geq 2$ elements in common and their product is a permutation with $k$ cycles. One can see that this implies that the rational curve splits into $k$ components on which the meromorphic function is not constant. If $k \geq 3$, there is one additional component, on which the function is a constant. The other $k$ components are glued to this additional one. The cases $k = 0$ and $k = 1$ do not interest us, because one can see that in these cases the rational curve does not split.

We introduce the following notation for the simple strata. Let

\[ Y = y_1^{m_1} y_2^{m_2} \cdots y_{n-1}^{m_{n-1}} \]
be a monomial of degree $2n - 2$ in variables $y_1, \ldots, y_{n-1}$, the variable $y_i$ having degree $i$, i.e.,

$$\sum i m_i = 2n - 2.$$ 

Denote by $\Sigma(Y)$ the simple stratum such that a generic function $f \in \Sigma(Y)$ has $m_1$ simple critical points, $m_2$ double critical points, $\ldots$, $m_{n-1}$ critical points of order $n - 1$. Denote by $h(Y)$ the Hurwitz number corresponding to the stratum $\Sigma$. Finally, denote by $|\text{Aut}(Y)|$ the number $m_1! \ldots m_{n-1}!$.

Consider the following generating function:

$$G(t, y_1, y_2, \ldots) = \sum_{n=1}^{\infty} \sum_{Y} \frac{h(Y)}{|\text{Aut}(Y)|} t^n Y.$$ 

The second sum is taken over the monomials $Y$ of degree $2n - 2$.

The first terms of $G$ are

$$G = t + \left(\frac{1}{4} y_1^2\right) t^2 + \left(\frac{1}{6} y_1^4 + \frac{1}{2} y_1^2 y_2 + \frac{1}{6} y_2^2\right) t^3 +$$

$$+ \left(\frac{1}{6} y_1^6 + \frac{9}{8} y_1^4 y_2 + \frac{2}{3} y_1^2 y_3 + \frac{3}{2} y_1^2 y_2^2 + y_1 y_2 y_3 + \frac{1}{6} y_2^3 + \frac{1}{8} y_3^2\right) t^4 + \ldots.$$ 

The first term $t$ is added by convention; it does not really correspond to a stratum.

We are going to prove that the function $G$ satisfies a partial differential equation with an infinite number of terms. In order to describe the terms we introduce the notion of a decomposition list. To each decomposition list corresponds a term of the partial differential equation.

**Definition 2.5** A list of non-negative numbers $(a_1, \ldots, a_k; i, j)$ is called a *decomposition list* if it is possible for two cyclic permutations with cycles of lengths $i+1$ and $j+1$ to have a product with cycle lengths $a_1 + 1$, $\ldots$, $a_k + 1$.

More precisely, consider two permutations $\sigma_1$ and $\sigma_2$ that are cycles of lengths $i+1$ and $j+1$. We suppose that $\sigma_1$ and $\sigma_2$ act on $i+j+2-k$ elements and their cycles have $k \geq 2$ elements in common. Thus each of the $i+j+2-k$ elements participates in at least one of the cycles. Suppose that the product
σ₁σ₂ has k cycles of lengths \(a_1 + 1, \ldots, a_k + 1\). Then \((a_1, \ldots, a_k; i, j)\) is a decomposition list. Conversely, for any decomposition list one can find two permutations satisfying the above description.

This definition is equivalent to the following conditions:

\[2 \leq k \leq \min(i + 1, j + 1)\]

and

\[a_1 + \ldots + a_k = i + j - 2k + 2.\]

Now to each decomposition list we assign three positive integers (related to each other).

**Definition 2.6** Let \(L = (a_1, \ldots, a_k; i, j)\) be a decomposition list. Denote by \(q(L)\) the number of ways to choose \(k\) integers \(b_i, 0 \leq b_i \leq a_i\) whose sum is equal to \(i - k + 1\). Denote by \(p(L)\) the number

\[p(L) = (a_1 + 1) \ldots (a_k + 1) q(L).\]

Finally, denote by \(m(L)\) the number \(m(L) = p(L)/(a_k + 1)\).

The numbers \(q(L)\) and \(p(L)\) are symmetric with respect to permutations of the \(a_r\) and with respect to the transposition of \(i\) and \(j\). The number \(p(L)\) is the number of ways to decompose a given permutation with cycles of lengths \(a_1 + 1, \ldots, a_k + 1\) into a product of two cyclic permutations with cycles of lengths \(i + 1\) and \(j + 1\). The numbers \(m(L)\) for various decomposition lists \(L\) will play the role of coefficients in the equation on \(G\).

Denote by \(D_0\) the differential operator \(D_0 = t \partial/\partial t\) and by \(D_i, i \geq 1,\) the differential operator \(D_i = \partial/\partial y_i\). Instead of writing \(D_i f\) we will usually right \(f_i\). Thus \((G_1 G_2)_0\) means \(t \partial/\partial t \left(\partial G/\partial y_1 \cdot \partial G/\partial y_2\right)\).

**Theorem 3** The generating function \(G\) satisfies the following partial differential equation.

\[2t^2 \frac{\partial^2 G}{\partial t^2} = \sum m(L) y_i y_j (G_{a_1} \ldots G_{a_{k-1}})_0 (G_{a_k})_0,\]

where the sum is taken over all decomposition lists \(L = (a_1, \ldots, a_k; i, j)\).
The first terms of the partial differential equation are:

\[
2t^2 \frac{\partial^2 G}{\partial t^2} =
\]

\[
= \ y_1^2 \left( (G_0)_0 \right)^2 + 6 y_1 y_2 (G_0)_0 (G_1)_0 +
\]

\[
+ 4 y_2^2 (G_0)_0 (G_2)_0 + 4 y_2^2 \left( (G_1)_0 \right)^2 + y_2^2 ((G_0)^2)_0 (G_0)_0 +
\]

\[
+ 8 y_1 y_3 (G_0)_0 (G_2)_0 + 4 y_1 y_3 \left( (G_1)_0 \right)^2 +
\]

\[
+ 2 y_2 y_3 ((G_0)_0)_0 (G_1)_0 + 8 y_2 y_3 (G_0 G_1)_0 (G_0)_0 +
\]

\[
+ 10 y_2 y_3 (G_0)_0 (G_3)_0 + 20 y_2 y_3 (G_1)_0 (G_2)_0 + \ldots
\]

Here we have added up the coefficients of similar terms arising from decomposition lists that differ only in the order of the numbers \(a_1, \ldots, a_{k-1}\) or in the order of \(i\) and \(j\).

In Section 5 we give some explicit formulas for the coefficients of \(G\) resulting from Theorem 3. Some of them were known before, but many are new.

### 2.2 Strata with at most one multiple critical value

In this subsection we consider the strata \(\Sigma\) such that a generic function in \(\Sigma\) has only one multiple critical value (and any number of simple critical values). The degrees of \(LL\) on such strata, as well as the corresponding Hurwitz numbers, were first given by Hurwitz without a complete proof. Today several proofs are known: a reconstruction of Hurwitz’s proof by V. Strehl [12], a combinatorial proof by I. Goulden and D. Jackson [6], and an algebro-geometric proof by T. Ekedahl, S. Lando, M. Shapiro and A. Vainstein [4]. Here we give a new algebro-geometric proof.

More precisely, we will introduce a generating function \(F\) whose coefficients are the above Hurwitz numbers and prove that it satisfies a partial differential equation with an infinite number of terms. This equation allows one to find all the coefficients of \(F\).
Definition 2.7  A stratum $\Sigma$ is called \textit{primitive} if a generic function $f \in \Sigma$ has a unique multiple critical value (that can be attained at several critical points of different multiplicities). In other words, the set of partitions $\kappa_1, \ldots, \kappa_c$ that describes the stratum contains at most one partition different from 1.

As above, before proceeding we will briefly describe the intersection of a primitive stratum $\Sigma$ with $\Delta_{p,q}$. Consider a function $f$ in the open part of $\Sigma$, that tends to a stable meromorphic function in $\Sigma \cap \Delta_{p,q}$. Generically, two critical values of $f$ get glued together. There are two possibilities.

First, the two critical values can be simple. In that case the rational curve splits into two components. The preimages of the multiple critical value are distributed (in some way) between these two components. This situation corresponds to the first term on the right-hand side of the equality in Theorem 4.

Second, the multiple critical value can get glued with a simple critical value. In that case the rational curve also splits into two components. But now one of the preimages of the multiple critical value tends to the intersection point of the two components. This intersection point is, in general, a critical point for the functions on both components. Such a situation corresponds to the second term (the infinite sum) on the right-hand side of the equality in Theorem 4.

Primitive strata can be described by monomials in an infinite number of variables $x_1, x_2, \ldots$. Let $X = x_1^{m_1} \cdots x_n^{m_n}$ be a monomial of degree $n$, the variable $x_i$ having degree $i$, i.e., $\sum i m_i = n$. We denote by $\Sigma(X)$ the primitive stratum in $H_n$ such that the unique multiple critical value has $m_1$ simple preimages, $m_2$ double preimages, $\ldots$, $m_n$ preimages of multiplicity $n$.

To the monomial $X$ we assign the following numbers. The number $|\text{Aut}(X)|$ is the number $m_1! \cdots m_n!$. The number $R(X)$ is the number of simple critical values of a generic function in the stratum $\Sigma(X)$; it is equal to $n + p - 2$, where $p = \sum m_i$ is the number of factors in the monomial $X$. Finally $h(X)$ is the number of ramified coverings of the sphere with $R(X)$ simple ramification points and one ramification point with ramification type given by the partition $\kappa = 1^{m_2} \ldots (n-1)^{m_n}$. (In other words, $h(X) = h_\kappa$.)

Note that, unfortunately, a preimage of multiplicity $k \geq 2$ is a critical point of multiplicity $k - 1$. 
We introduce the following generating function

\[ F(x_1, x_2, \ldots) = \sum_{n=1}^{\infty} \sum_{X} \frac{n \cdot h(X)}{R(X)! |\text{Aut}(X)|} X t^n, \]

where the second sum is taken over the monomials \(X\) of degree \(n\).

Hurwitz’s formula for the the Hurwitz numbers \(h(X)\) gives the following expression for \(F\):

\[ F = \sum_{n=1}^{\infty} \frac{n^{p-2}}{p!} \prod_{i=1}^{p} \frac{k_i}{k_i!} x_{k_i} t^n. \]

The first terms of \(F\) are

\[ F = t x_1 + t^2 \left( \frac{1}{2} x_1^2 + x_2 \right) + t^3 \left( \frac{1}{2} x_1^3 + 2 x_1 x_2 + \frac{3}{2} x_3 \right) + \]

\[ + t^4 \left( \frac{2}{3} x_1^4 + 4 x_1^2 x_2 + \frac{9}{2} x_1 x_3 + 2 x_2^2 + \frac{8}{3} x_4 \right) + \ldots \]

The very first term \(tx_1\) is added by convention. It does not really correspond to a stratum.

**Theorem 4** The function \(F\) satisfies the following partial differential equation

\[ 2 t^2 \frac{\partial}{\partial t} \left( \frac{F}{t} \right) = \left( t \frac{\partial F}{\partial t} \right)^2 + \sum_{i,j \geq 1} (i+j) x_{i+j} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} \]

This equation and the initial condition \(F = tx_1 + \ldots\) determine all the coefficients of \(F\). The proof of the theorem is completely independent of the Hurwitz formula.

**Remark 2.8** Goulden and Jackson \[6\] obtained by purely combinatorial methods a similar partial differential equation for a generating function that enumerates the same numbers. The idea of their method is as follows. The Hurwitz number \(h(X)\) with \(X = x_{k_1} \ldots x_{k_p}\) is equal (up to a simple combinatorial factor) to the number of shortest possible transitive factorizations into transpositions of a given permutation \(\sigma \in S_n\) whose cycles have lengths \(k_1, \ldots, k_p\). Multiplying the last transposition of the factorization by \(\sigma\), we obtain a new permutation \(\sigma'\), which is again factorized into transpositions,
but not necessarily transitively (i.e., the group generated by the transpositions may not be transitive). Considering various possible cases one can express the number of shortest transitive factorizations for the initial stratum in terms of similar numbers for several other strata, which is then encoded in a partial differential equation on the generating function. From the algebro-geometric point of view, the case when the factorization of $\sigma'$ is not transitive corresponds to the situation when the curve of definition of the meromorphic function $f$ splits into 2 components. The case when the factorization of $\sigma'$ remains transitive corresponds to other possible degenerations. Thus our equation is different from that of Goulden and Jackson in that we have managed to get rid of all the degenerations except the splitting of the rational curve. The price to pay is that our equation contains nontrivial coefficients coming from the algebraic geometry (they are multiplicities of intersections of some strata). Therefore we think that our equation cannot be easily obtained by purely combinatorial methods.

3 Studying the intersection of a stratum $\Sigma$ with $\Delta_{p,q}$

The proofs of Theorems 3 and 4 are obtained by studying the intersection of primitive and simple strata with $\Delta_{p,q}$. (Recall that $\Delta_{p,q}$ is the closure of the set of stable functions defined on curves with two irreducible components, the functions being of degree $p$ on one component and $q$ on the other one.) This intersection usually consists of a very large number of irreducible components. The multiplicities of the intersection can be different for different irreducible components. Below the word component means a union of irreducible components.

For a precise definition of the strata $\Delta_{p,q}$ and of the cohomology class $\Psi_n$ see in [9], Definitions 4.8 and 2.11.

3.1 The $k$-eared strata

Here we introduce the notion of $k$-eared stratum and prove that every component of the intersection of a simple or primitive stratum $\Sigma$ with $\Delta_{p,q}$ is contained in a $k$-eared stratum. Let $\Sigma$ be any stratum in $\mathcal{H}_n$. Denote by $d$ the dimension of $\mathbb{P}\Sigma$; thus a generic function $f$ in $\Sigma$ has $d + 1$ critical values.
Lemma 3.1 Let $\Sigma'$ be a component of the intersection $\Sigma \cap \Delta_{p,q}$. Suppose that a generic function in $\Sigma'$ has less than $d$ critical values. Then $\langle \mathbb{P}\Sigma', \Psi_{d-1} \rangle = 0$.

Remark 3.2 Since our goal is to find the sum of $\langle \mathbb{P}\Sigma', \Psi_{d-1} \rangle$ over all the components of the intersection $\Sigma \cap \Delta_{p,q}$, what the lemma actually says is that we can discard the components where more than two critical values get glued together.

Actually, there are no components satisfying the conditions of the lemma, but this fact is much more difficult to prove than the lemma itself and the lemma is sufficient for our purposes.

Proof of the lemma. The image of $\Sigma'$ under $LL$ has dimension at most $d - 1$ (i.e., less than the dimension of $\Sigma'$ itself). On the other hand, the 2-cohomology class $\Psi_n$ is the pull-back of a 2-cohomology class on the image of $LL$ (see the proof of Theorem 4 in [9]). Therefore the number $\langle \mathbb{P}\Sigma', \Psi_{d-1} \rangle$ can be computed in the image of $LL$, where it is equal to 0 for dimension reasons. \hfill \diamond

Now we will consider a particular kind of subvarieties of $\mathcal{H}_n$, called the $k$-eared strata, and study their neighborhoods in $\mathcal{H}_n$. As we will soon see, all the intersections of $\Delta_{p,q}$ with simple and primitive strata lie in such subvarieties. Therefore, studying their neighborhoods will help us to find the multiplicities of the intersections of $\Delta_{p,q}$ with simple and primitive strata.

A 2-eared stratum is consists of stable meromorphic functions defined on a 2-component curve, such that the restrictions of the functions to each component belong to some given strata, while the common point of the two components is a critical point of given multiplicity on each component of the curve.

A $k$-eared stratum for $k \geq 3$ is consists of stable meromorphic functions defined on a $(k + 1)$-component curve. One of the components is central. The meromorphic function is constant on this component. All the other $k$ components are called peripheral. They do not intersect each other, but each of them intersects the central component at one point. The restriction of the meromorphic function to a peripheral component belongs to a given stratum. The point of intersection of a peripheral component with the central component is a critical point of given multiplicity.

In order to give a precise definition of a $k$-eared stratum, let us first define a stratum with a distinguished point. Recall that a stratum is described
by an unordered set of partitions (see Definition 3.3 in [9]), each partition corresponding to a critical value of the stable function, and an element \( i \) in a partition corresponding to a critical point of multiplicity \( i \). We can similarly describe a stratum of stable functions with a distinguished point on the curve \( S \). In order to encode the necessary information about the distinguished point we must make the following choices. If we want the value of the function at the distinguished point to be a critical value, we chose one of the partitions \( \kappa \) in the set that describes the stratum. Otherwise, if we want the value at the distinguished point to be an ordinary value, we do not choose anything. Suppose we have chosen a partition \( \kappa \). Now we must decide whether the distinguished point will be a critical point or not. In the first case we choose one of the elements \( i \) of the partition; in the second case we choose nothing. Thus our choice encodes one of the following possibilities: either the distinguished point is a critical point of multiplicity \( i \) with critical value of type \( \kappa \); or the distinguished point is a noncritical point, but the value of the function at the distinguished point is a critical value of type \( \kappa \); or the value of the function at the distinguished point is noncritical.

Now a stratum of functions with a distinguished point is (the closure of) the set of all stable functions defined on nodal curves \( S \) with one distinguished point, the distinguished point satisfying the above conditions.

Let \( k \geq 2 \) be a natural number. Divide \( n \) into a sum of \( k \) natural numbers \( n = n_1 + \ldots + n_k, n_i \geq 1 \). Divide the set of poles \( \{1, \ldots, n\} \) into \( k \) parts with \( n_1, \ldots, n_k \) elements. Choose \( k \) strata \( \Sigma_1, \ldots, \Sigma_k \) of stable functions with a distinguished point in the spaces \( H_{n_1}, \ldots, H_{n_k} \).

**Definition 3.3** Let \( k = 2 \). Consider the set of all stable meromorphic functions defined on 2-component curves, such that the restrictions of the function to the components belong to open parts of the strata \( \Sigma_1, \Sigma_2 \) and the intersection point of the components is the distinguished point for both components. This set is called an *open 2-eared stratum*. Its closure in \( \mathcal{H}_n \) is called a *2-eared stratum*.

Let \( k \geq 3 \). Consider the set of all stable meromorphic functions defined on \((k+1)\)-component curves with one central and \( k \) peripheral components, such that the restriction of the function to the central component is a constant, while its restrictions to the peripheral components belong to open parts of the strata \( \Sigma_1, \ldots, \Sigma_k \), while the intersection point of each peripheral component with the central one is the distinguished point on the peripheral component. This set is called an *open \( k \)-eared stratum*. Its closure in \( \mathcal{H}_n \) is called a *\( k \)-eared stratum*. 
Proposition 3.4 Let $\Sigma$ be a simple or a primitive stratum. Consider an irreducible component $\Sigma'$ of the intersection $\Sigma \cap \Delta_{p,q}$ such that
\[
\left\langle \mathbb{P}\Sigma', \Psi_n^{\dim \mathbb{P}\Sigma'} \right\rangle \neq 0.
\]
Then $\Sigma'$ is contained in a $k$-eared stratum for some $k$. Moreover, we can, in a natural way, assign a decomposition list $L$ (Definition 2.5) to the component $\Sigma'$.

Proof. According to Lemma 3.1 a generic stable function $(S', f') \in \Sigma'$ has one critical value less than a generic function $(S, f) \in \Sigma$. (The function $f'$ is not the derivative of $f$, just another stable function.) Therefore as $(S, f)$ approaches $(S', f')$, two of its critical values are glued together, and their monodromies are multiplied.

First suppose $\Sigma$ is a primitive stratum. Then $f$ has only one critical value that is not simple. Therefore, among the two critical values that get glued together, at least one is simple, i.e., the corresponding monodromy is a transposition. When we multiply a transposition by another permutation $\sigma$, two cases are possible. Either the two permuted elements of the transposition belong to two different cycles of $\sigma$, in which case these cycles are merged into one cycle in the product. Or the two permuted elements of the transposition belong to the same cycle of $\sigma$, in which case this cycle splits into two cycles in the product.

Since $(S', f')$ lies in $\Delta_{p,q}$, we know that the monodromies of $f'$ do not act transitively on the set of poles, but have at least two transitivity components corresponding to the irreducible components of $S'$. It follows that only the second of the above two cases is possible. It is easy to see that the set of monodromies of $f'$ has exactly two transitivity components. Therefore the stable function $(S', f')$ belongs to a 2-eared stratum.

We assign to $\Sigma'$ the decomposition list $L = (a_1, a_2; i, 1)$, where $i + 1$ is the length of the cycle of $\sigma$ that splits into two, and $a_1 + 1$, $a_2 + 1$ are the lengths of the two cycles obtained by the splitting. (The last number 1 means that the transposition from the above discussion is a cycle of length $1 + 1 = 2$.)

Now suppose that $\Sigma$ is a simple stratum. Then the monodromies of the critical points of $f$ that get glued together are cycles. When two cyclic permutations are multiplied, several cases are possible. For simplicity, let
us momentarily forget about the elements of the permutation that do not belong to either of the two cycles. (i) If the cycles do not have common elements, their product is just a permutation with two cycles of the same lengths as the cycles that we multiply. (ii) If the cycles have one common element, then their product is just one cycle. (iii) If the cycles have \( k \geq 2 \) elements in common, then their product is a permutation with \( k \) cycles.

Again, we know that the set of monodromies of \( f' \) has at least two transitivity components. It follows that only case (iii) is possible, and that the actual number of transitivity components is \( k \). Thus the curve \( S' \) has \( k \) components on which \( f' \) is not constant. It remains to check that for \( k \geq 3 \) it has only one component on which \( f' \) is constant. This follows from the fact that the elements of two cycles form a unique transitivity component under the action of these two cycles, provided the cycles have at least one common element. Thus we have proved that \( (S', f') \) belongs to a \( k \)-eared stratum.

We assign to the component \( \Sigma' \) the decomposition list \( L = (a_1, \ldots, a_k; i, j) \), where \( i, j \) are the multiplicities of the two critical values that are glued together, and \( a_1 + 1, \ldots, a_k + 1 \) are the lengths of cycles in the product of their monodromies.

\[ \diamond \]

### 3.2 Computing the multiplicity of intersection

Here we study the local geometry of \( k \)-eared strata, which allows us to compute the multiplicity of the intersection of a simple or a primitive stratum \( \Sigma \) with \( \Delta_{p,q} \).

**Proposition 3.5** Let \( k \geq 3 \). Consider a \( k \)-eared stratum \( \Sigma \) and a generic point in its image under the LL map. The preimage of this point in \( \Sigma \) is isomorphic to a finite number of copies of the compactified moduli space \( \overline{M}_k \).

**Proof.** The restriction of the stable meromorphic function to any peripheral component is determined, up to a finite number of possibilities, by its image under \( LL \). Its restriction to the central component is a constant, and the constant is also determined by the image under \( LL \). The only thing that can vary is the disposition on the central component of the \( k \) points at which it intersects the peripheral components. These dispositions form the space \( \overline{M}_k \). Thus each point of a \( k \)-eared stratum is naturally contained in a subvariety isomorphic to \( \overline{M}_k \). Note that the same is true for a projectivized \( k \)-eared stratum \( \mathbb{P}\Sigma \). \( \diamond \)
Notation 3.6 We introduce the following notation that the reader must bear in mind in the sequel:

\[ \overline{\mathcal{M}}_k \equiv A_k \subset B_k \supset \Sigma' \subset \Sigma \cap \Delta_{p,q} \subset \mathcal{H}_n. \]

Here \( \Sigma \) is a primitive or a simple stratum, \( \Sigma' \) a component of its intersection with \( \Delta_{p,q} \). Further, \( B_k \) is the \( k \)-eared stratum that contains \( \Sigma' \) (according to Proposition 3.4), and \( A_k \) is a subvariety of \( B_k \) (as in Proposition 3.5) containing a point of \( \Sigma' \) and isomorphic to the moduli space \( \overline{\mathcal{M}}_k \). The image of \( A_k \) under the LL map is a point. In general we will assume that the subvariety \( A_k \) is chosen generically inside \( B_k \).

Now we study the neighborhood of \( A_k \) in \( \mathcal{H}_n \). To do that, we must introduce, following [4] a new kind of Hurwitz spaces, more general than the one we have used up to now. The neighborhood of a \( k \)-eared stratum will be very similar to such Hurwitz spaces.

Consider a list of nonnegative integers \( a_1, \ldots, a_k \). Let \( \sigma \) be a permutation of \( k + \sum a_r \) elements with cycle lengths \( a_1 + 1, \ldots, a_k + 1 \).

Definition 3.7 We call an indexed Hurwitz space \( \mathcal{H}(a_1, \ldots, a_k) \) the space of all stable meromorphic functions \( (S, f) \), considered up to an additive constant, with \( k \) poles of orders \( a_1 + 1, \ldots, a_k + 1 \).

We call a decorated Hurwitz space \( \mathcal{H}_\sigma \) the space of the same stable functions as above endowed with the following additional information. The preimages under \( f \) of the half-line \( [A, +\infty] \) are numbered. (Here \( A \) is a sufficiently large real number such that there are no critical values of \( f \) on the semi-closed interval \( [A, +\infty) \).) The monodromy around \( \infty \) (i.e., the permutation of the numbered preimages obtained by going around \( \infty \) in the counterclockwise direction) is equal to \( \sigma^{-1} \).

It is easy to see that an indexed Hurwitz space \( \mathcal{H}(a_1, \ldots, a_k) \) is the quotient of a decorated Hurwitz space \( \mathcal{H}_\sigma \) by the group \( \mathbb{Z}/(a_1 + 1)\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/(a_k + 1)\mathbb{Z} \) acting by cyclic renumberings of the preimages of \( [A, +\infty] \) around each pole.

Indexed Hurwitz spaces were studied in detail in [4] (for curves of all genera). It was shown there that for \( k \geq 3 \) such Hurwitz spaces are cones over \( \overline{\mathcal{M}}_k \); in particular, they are not smooth. Decorated Hurwitz spaces are
smooth and can be seen to be fiber bundles over $\overline{M}_k$, as we now briefly describe.

Indeed, suppose we have fixed the positions of $k$ points on $\mathbb{CP}^1$. Then the principle part of the stable meromorphic function at the neighborhood of a pole of order $a + 1$ can be written as

$$\left(\frac{u}{z}\right)^{a+1} + v_1 \left(\frac{u}{z}\right)^a + \ldots + v_a \frac{u}{z},$$

where $z$ is the variable and $(u, v_1, \ldots, v_a)$ is a set of parameters. Multiplying $u$ by an $(a+1)$-th root of unity $\varepsilon$ and each $v_i$ by $\varepsilon^i$ amounts to a renumbering of the preimages of the interval $[A, +\infty]$. Putting together the above sets of parameters for all the poles, we obtain a description of a fiber of the projection from $H_\sigma$ to $\overline{M}_k$. When $u = 0$, the point $z = 0$ is no longer a pole of the function. In this case we must glue a new component to the curve $S$ at the point $z = 0$. The coordinate on this new component will be $w = u/z$. Thus on this new component a pole of order $a + 1$ will be preserved.

For indexed Hurwitz spaces, the parameters $u$ and $v_i$ are defined only up to multiplication by some powers of an $(a+1)$-th root of unity. More precisely, assign to $u$ the weight 1 and to $v_i$ the weight $i$. Consider the algebra of polynomials in the variables $u$ and $v_i$, such that the weight of each of their monomials is a multiple of $a + 1$. Then the algebra of functions on a fiber of $H(a_1, \ldots, a_k)$ over $\overline{M}_k$ is the tensor product of such algebras for $a_1, \ldots, a_k$.

Consider a $k$-eared stratum $B_k$ in $H_n$. Let $(S, f)$ be a generic point of this stratum. Let $w$ be the value of $f$ on the central component of $S$ and $w'$ a point close to $w$. Let us number those preimages of $w'$ that lie close to the central component of $S$. Denote by $\sigma$ the permutation that these preimages undergo as $w'$ goes around $w$. Finally, consider the subvariety $A_k$ obtained from the initial generic point of the $k$-eared stratum by changing the positions of the intersection points on the central component without touching the peripheral components.

**Theorem 5** For $k \geq 3$, the neighborhood of $A_k$ in $H_n$ is isomorphic to the neighborhood of $\overline{M}_k \times \{0\}$ in $H_\sigma \times \mathbb{C}^d$ for some positive integer $d$.

**Proof.** As in the proof of Proposition denote by $(S', f')$ a stable function in $A_k$ and by $(S, f)$ a stable function in $H_n$ close to it.
The function \((S', f') \in A_k\) is almost a ramified covering of \(\mathbb{CP}^1\) by the curve \(S'\). The only difference with a ramified covering is that the central component of \(S'\) is sent to a point by \(f'\). The idea of the proof is that any deformation of \((S', f')\) can be regarded as a set of independent deformations of the ramification structure at the neighborhoods of the critical values of \(f'\). Deforming the special critical value (the value of \(f'\) on the central component) looks like deforming a point of \(\overline{\mathcal{M}}_k\) inside \(\mathcal{H}_\sigma\). Deforming any other critical value looks simply like moving off the origin in a complex vector space of some dimension.

Consider a point \((S, f)\) in \(\mathcal{H}_n\) close to \((S', f')\). Then, if \(W\) is a critical value of the function \(f'\) whose monodromy is a permutation \(\rho\), the function \(f\) will have several critical values close to \(W\), whose product of monodromies is equal to \(\rho\).

First suppose \(W\) is an ordinary critical value (not the value of \(f'\) on the central component of \(S'\)).

The simplest case is when there is only one critical point in the preimage of \(W\), in other words, the monodromy \(\rho\) has only one cycle of length \(p > 1\). Then at the neighborhood of the critical point in question one can choose a local coordinate \(z\) such that the function \(f'\) locally equals \(z^p + W\). Now a deformation of \(f'\) “at the neighborhood of the critical value \(W\)” can be obtained in the following way. Consider a small disc \(D_W\) surrounding \(W\). Cut out from the curve \(S'\) the connected component of \((f')^{-1}(D_W)\) containing the critical point \(z = 0\). Instead, glue into the hole thus obtained the preimage of \(D_W\) under the function \(z^p + W + (\lambda_1 z^{p-1} + \ldots + \lambda_p)\) with sufficiently small complex numbers \(\lambda_i\). Thus we obtain a new curve \(S\) with a new stable meromorphic function \(f\) on it. The set of such deformations is parameterized by the \(\lambda_i\)s and the initial function \((S', f')\) corresponds to \(\lambda_1 = \ldots = \lambda_p = 0\).

The complex structure along the gluing contour on the new surface \(S\) is uniquely reconstructed in such a way that the function \(f\) becomes meromorphic. Alternatively, we can cut out from \(S'\) the preimage of a disc slightly smaller than \(D_W\) and glue back the preimage of a disc slightly bigger than \(D_W\). The gluing then identifies two preimages of an open annulus and the complex structure extends automatically.

Now suppose that there are several critical points whose image is equal to \(W\). This case is actually just as simple, because we can perform the above operation of cutting off a small disc and replacing it by another one independently for all the critical points.

It remains to consider the case when \(W = w\) is the value of \(f'\) on the
central component of $S'$. First of all, the preimage of $w$ can contain critical points that do not lie on the central component of $S'$. These can be deformed by the same procedure as above.

Now let us see how the neighborhood of the central component of $S'$ can be deformed. First, as above, cut off from the curve $S'$ the connected component of $(f')^{-1}(D_w)$ containing the central component. Now consider a stable meromorphic function $g$ in $\mathcal{H}_\sigma$ close to the base $\mathcal{M}_k$; this means that the critical values of $g$ are close to each other. More precisely, we suppose that by adding an appropriate constant to $g$ we can make all its critical values fit inside the disc $D_w$. Then $g$ determines a ramified covering of the disc $D_w$ with monodromy $\sigma$. Therefore we can glue the preimage $g^{-1}(D_w)$ instead of $(f')^{-1}(D_w)$ into the curve $S'$. We will obtain a new curve $S$ with a stable meromorphic function $f$ on it, close to $(S', f')$.

This finishes the proof.

Having described the neighborhood of $A_k$, we describe the intersections of this neighborhood with the stratum $\Sigma$ and with $\Delta_{p,q}$. These intersections look like the varieties $s_L$ and $\delta_r$ below.

Consider an indexed Hurwitz space $\mathcal{H}(a_1, \ldots, a_k)$ and a decorated Hurwitz space $\mathcal{H}_\sigma$, where $\sigma$ is a permutation with cycle lengths $a_1 + 1, \ldots, a_k + 1$. Let $L = (a_1, \ldots, a_k; i, j)$ be a decomposition list.

**Definition 3.8** Denote by $s_L \subset \mathcal{H}_\sigma$ the stratum consisting of stable meromorphic functions with exactly two critical points of multiplicities $i$ and $j$. (The corresponding critical values are automatically distinct.) Denote by $\overline{s}_L \subset \mathcal{H}(a_1, \ldots, a_k)$ the similarly defined stratum in the indexed Hurwitz space. (In other words, $\overline{s}_L$ is the quotient of $s_L$ by the action of the group $\bigoplus \mathbb{Z}/(a_r + 1)\mathbb{Z}$.)

Denote by $\delta_r \subset \mathcal{H}_\sigma$ the stratum consisting of the stable meromorphic functions defined on 2-component curves, the $r$th pole being on the first component, and all the other poles on the second component. Denote by $\overline{\delta}_r \in \mathcal{H}(a_1, \ldots, a_k)$ the similarly defined stratum in the indexed Hurwitz space. (Again $\overline{\delta}_r$ is the quotient of $\delta_r$ by the action of the group $\bigoplus \mathbb{Z}/(a_r + 1)\mathbb{Z}$.)

Let $(S, f)$ be a generic point of the intersection $\Sigma \cap \Delta_{p,q}$, where $\Sigma$ is a simple or a primitive stratum. The point $(S, f)$ is contained in the variety $A_k$ (see Notation 3.6). Denote by $L$ the decomposition list corresponding to the point $(S, f)$ (see Proposition 3.4).
Proposition 3.9  Let $k \geq 3$.

(i) The intersection of the neighborhood of $A_k$ with the stratum $\Sigma$ is isomorphic to $s(L) \times \mathbb{C}^{d_s}$ for some integer $d_s$.

(ii) The intersection of the neighborhood of $A_k$ with $\Delta_{p,q}$ is isomorphic to $\bigcup \delta_r \times \mathbb{C}^{d_r}$ for some integers $d_r$, where the union is taken over the numbers $r \in \{1, \ldots, k\}$ such that the restriction of $f$ on the $r$th peripheral component of $S$ is of degree $p$ or of degree $q$.

Proof.  (i) This is a consequence of the proof of Theorem 5. Indeed, to move off $A_k$ into $\Sigma$ we must deform the critical values of $f$ in the following way. For an ordinary critical value (i.e., not the value of $f$ on the central component of $S$): just move the critical value itself, preserving the multiplicity of the critical point in its preimage. This corresponds to moving off the origin in some vector space $\mathbb{C}^{d_s}$. For the value of $f$ on the central component of $S$: this value should be decomposed into two critical values, $k$ cycles of lengths $a_1 + 1, \ldots, a_k + 1$ being represented as a product of two cycles of lengths $i + 1$ and $j + 1$. This is precisely the description of the stratum $s_L$.

(ii) This, again, follows from the proof of Theorem 5. To move off $A_k$ into $\Sigma$ we must deform the critical values of $f$ in the following way. For an ordinary critical value (i.e., not the value of $f$ on the central component of $S$): deform it in any possible way. For the value of $f$ on the central component of $S$: it should be resolved in such a way that only the $r$th peripheral component (for one of the numbers $r$ described in the Proposition) of the curve $S$ remains a separate component, while all the other components merge together. This is precisely the description of $\delta_r$.  

Now we should compute the multiplicity of the intersection of $s(L)$ and $\delta_r$. Actually, this is easier to do for $\overline{s(L)}$ and $\overline{\delta_r}$. We start with a simple geometrical description of the stratum $\overline{s(L)}$.

Recall that $p(L)$ is the number of ways to decompose $\sigma$ into a product of two cycles with lengths $i + 1$ and $j + 1$, while

$$q(L) = p(L)/(a_1 + 1) \ldots (a_k + 1)$$

(see Definition 2.6).

Proposition 3.10 For $k \geq 3$ the stratum $\overline{s_L} \subset \overline{\mathcal{H}(a_1, \ldots, a_k)}$ is a union of $q(L)$ straight lines, each line being contained in a fiber of the bundle $\overline{\mathcal{H}(a_1, \ldots, a_k)} \rightarrow \overline{\mathcal{M}_k}$.
Proof. This is almost obvious. Up to transformations \( f \mapsto af + b \), there are exactly \( p(L) \) stable meromorphic functions with poles of multiplicity \( a_1 + 1, \ldots, a_k + 1 \), with two critical points of multiplicities \( i \) and \( j \), with numbered preimages of \([A, +\infty]\) and with a given monodromy \( \sigma \). Forgetting the numbering, gives us \( q(L) \) functions, still up to transformations \( f \mapsto af + b \). Since a point of \( \mathcal{H}_\sigma \) is a meromorphic function up to an additive constant, the assertion follows. \( \diamond \)

Proposition 3.11 For \( k \geq 3 \), the multiplicity of intersection of the stratum \( s(L) \) and \( \delta_r \) in \( \mathcal{H}(a_1, \ldots, a_k) \) equals

\[
\frac{1}{a_r + 1} q(L).
\]

The multiplicity of intersection of the stratum \( s(L) \) and \( \delta_k \) (here we have chosen \( r = k \)) in \( \mathcal{H}_\sigma \) equals \( m(L) \) (see Definition 2.10).

Remark 3.12 The first of the two multiplicities is not necessarily an integer. This is due to the fact that \( \mathcal{H}(a_1, \ldots, a_k) \) is not a smooth variety and \( s(L) \) and \( \delta_r \) are Cartier divisors, that is, algebraic subvarieties that are not necessarily locally equal to zero sets of functions. Here is the simplest example of such situation.

Consider the cone \( z^2 = x^2 + y^2 \) in \( \mathbb{C}^3 \). Let \( l_1 \) and \( l_2 \) be two straight lines belonging to this cone. We claim that the multiplicity of their intersection is equal to \( 1/2 \). Indeed, consider the plane tangent to the cone along \( l_1 \). Then its intersection with the cone is \( 2l_1 \). Similarly, the intersection of the cone with the plane tangent to it along \( l_2 \) equals \( 2l_2 \). Now we can move slightly both planes, and the intersection of the three subvarieties: the two shifted planes and the cone will consist of two points. Thus \( 2l_1 \cap 2l_2 = 2 \), so \( l_1 \cap l_2 = 1/2 \).

Proof of Proposition 3.11 First let \( k \geq 3 \).

The stratum \( \delta_r \) intersects each fiber of the bundle \( \mathcal{H}(a_1, \ldots, a_k) \) transversally, while the stratum \( s(L) \) is a union of straight lines, each of which is contained in one of these fibers. Therefore we can restrict our attention to one fiber and find the multiplicity of intersection inside it.

The principle part of the \( r \)th pole of \( f \) can be written as

\[
\left( \frac{u}{z} \right)^{a_r + 1} + v_1 \left( \frac{u}{z} \right)^{a_r} + \ldots + v_{a_r} \frac{u}{z}.\]
First consider the space $\mathbb{C}^{a_r+1}$ with coordinates $u, v_1, \ldots, v_{a_r}$ endowed with the action of $\mathbb{C}^* \lambda : u \mapsto \lambda u, v_i \mapsto \lambda^i v_i$. In this space we consider the hyperplane $u = 0$ and an orbit of the $\mathbb{C}^*$ action that is not contained in this hyperplane. The multiplicity of the intersection between the two (at the origin) is equal to 1.

Now we factor this space by the action of $\mathbb{Z}/(a_r+1)\mathbb{Z}$ (seen as the subgroup of roots of unity of order $a_r+1$ in $\mathbb{C}^*)$. The image of the hyperplane is the subvariety $u^{a_r+1} = 0$. The image of the $\mathbb{C}^*$ orbit is a straight line. The multiplicity of the intersection between the two equals $1/(a_r+1)$.

It follows that the intersection of a straight line in a fiber of the bundle $\mathcal{H}(a_1, \ldots, a_k) \to \overline{\mathcal{M}}_k$ with the stratum $\overline{\delta}_r$ has multiplicity $1/(a_r + 1)$. Since the stratum $\overline{s}(L)$ consists of $q(L)$ such lines, the first assertion of the proposition follows. The multiplicity of intersection of $s(L)$ and $\delta$ is equal to $(a_1 + 1) \cdot \ldots \cdot (a_k + 1)$ times the multiplicity of intersection of $\overline{s}(L)$ and $\overline{\delta}_k$, whence the second assertion.

In Figure 1 we have symbolically shown the space $\mathcal{H}(a_1, \ldots, a_k)$ as a cone over $\overline{\mathcal{M}}_k$ and the strata $s(L)$ and $\delta_r$ inside it.

Figure 1: The horizontal line is $\overline{\mathcal{M}}_k$. The space $\mathcal{H}(a_1, \ldots, a_k)$ is a cone over $\overline{\mathcal{M}}_k$; we have shown several fibers of this cone. The plane is the stratum $\overline{\delta}_r$. The bold oblique line is one of the lines composing $\overline{s}(L)$.

Now we consider the case $k = 2$. For $k = 2$ the stratum $s(L)$ in $\mathcal{H}_\sigma$ is defined exactly as before, while $\delta$ is now just the set of stable functions defined on 2-component curves. One can see that if $L = (a_1, a_2; i, j)$ is a decomposition list, then $q(L) = \min(i, j, a_1 + 1, a_2 + 1)$.

**Proposition 3.13** For $k = 2$, the multiplicity of intersection of the stratum $\overline{s}(L)$ and $\overline{\delta}$ in $\mathcal{H}(a_1, \ldots, a_k)$ equals

$$\frac{a_1 + a_2 + 2}{(a_1 + 1)(a_2 + 2)} q(L).$$
The multiplicity of intersection of the stratum \( s(L) \) and \( \delta \) in \( \mathcal{H}_\sigma \) equals \((a_1 + a_2 + 2)q(L)\).

**Remark 3.14** The reader may have noticed that the case \( k = 2 \) has been excluded from all the previous propositions. Indeed, since the moduli space \( \overline{\mathcal{M}}_2 \) does not exist, the geometry of the intersection of \( \Sigma \) with \( \Delta_{p,q} \) in the neighborhood of a \( k \)-eared stratum is different for \( k = 2 \) and \( k \geq 3 \). However, not surprisingly, the formula for the multiplicity of the intersection happens to be the same. Indeed, in the case \( k = 2 \), the same stratum \( \delta \) corresponds to two different decomposition lists: \((a_1, a_2; i, j)\) and \((a_2, a_1; i, j)\). (Recall that for \( k \geq 3 \), the points \((S, f)\) of \( \delta_r \) are stable functions, such that the \( r \)th pole of \( f \) lies on a separate component of \( S \). But for \( k = 2 \), saying that the second pole lies on a separate component is the same as saying that the first pole lies on a separate component.) The corresponding numbers \( m(L) \) (Definition 2.6) are equal to \((a_1 + 1)q(L)\) and \((a_2 + 1)q(L)\). Adding them, we obtain \((a_1 + a_2 + 2)q(L) = (i + j)q(L)\), as in the assertion of the proposition.

**Proof of Proposition 3.13** Let us consider the space \( \hat{\mathcal{H}}_\sigma \) of stable meromorphic functions \((S, f)\) with two poles of multiplicities \( a_1 + 1 \) and \( a_2 + 1 \) and with *one distinguished point on the curve \( S \).* This space can be treated as in the previous proposition (for \( k \geq 3 \)), because it is a fiber bundle over the moduli space \( \overline{\mathcal{M}}_3 \), i.e., over a point. There is a forgetful map from \( \hat{\mathcal{H}}_\sigma \) to \( \mathcal{H}_\sigma \), that forgets the distinguished point and, if there appeared a component of \( S \) on which \( f \) is constant, and that contains only two special points, contracts it.

The stratum \( \delta \) has two preimages under the forgetful map. The first one, \( \hat{\delta}_1 \), consists of the functions defined on 2-component curves, with the second pole and the distinguished point on one component and the first pole on the other. The second one, \( \hat{\delta}_2 \) also consists of the functions defined on 2-component curves, but with the second pole on one component and the first pole and the distinguished point on the other. We also denote by \( \hat{s}(L) \) the preimage of \( s(L) \).

The multiplicities of intersections of \( \hat{\delta}_1 \) and \( \hat{\delta}_2 \) with \( \hat{s}(L) \) are computed as in the previous proposition. They are equal to, respectively, \( q(L)(a_1 + 1) \) and \( q(L)(a_2 + 1) \).

The multiplicity of intersection of \( \delta \) with \( s(L) \) is equal to their sum, that is, \((a_1 + a_2 + 2)q(L)\). \( \Box \)
4 Proofs of the relations on generating functions

Once we have studied the intersection between a primitive or a simple stratum and \( \Delta_{p,q} \) we can prove Theorems 3 and 4.

**Proof of Theorem 3.** Consider a monomial \( Y \) in variables \( t, y_1, y_2, \ldots \). Suppose its degree in \( t \) equals \( n \); thus it corresponds to a stratum in \( \mathbb{P}H_n \). The coefficient of this monomial in \( G \) is equal, up to a combinatorial coefficient, to the degree of the \( LL \) map of the corresponding stratum \( \Sigma(Y) \).

Let us look at the ways in which the monomial \( Y \) can arise on the right-hand side of the equality of Theorem 3. Consider a decomposition list \( L = (a_1, \ldots, a_k; i, j) \) and \( k \) monomials in the generating function \( G \). This data determines exactly one term on the right-hand side of the equality of Theorem 3. We suppose that this term contributes to the coefficient of \( X \). We denote by \( p \) the degree in \( t \) of the \( k \)th monomial. It is easy to see that the same data also determines a \( k \)-eared stratum that contains a component of intersection of \( \Sigma(Y) \) and \( \Delta_{p,q} \).

Recall that the combinatorial coefficient, by which the degree of \( LL \) is multiplied in \( G \), equals

\[
\frac{1}{n!|\text{Aut}(Y)|}
\]

As we multiply the monomials, the factors \( n! \) are combined into a multinomial coefficient

\[
\binom{n}{n_1, \ldots, n_k}
\]

that enumerates the ways to distribute the \( n \) numbered poles among the \( k \) components. Similarly, the coefficients \( |\text{Aut}(Y)| \) enumerate the number of ways to distribute the critical points of the same multiplicity among the \( k \) components.

As we apply the operator \( D_{a_i} \) to the \( i \)th monomial, it is multiplied by its degree in the variable \( y_{a_i} \). This corresponds to the number of choices of a critical point of multiplicity \( a_i \) that will become the intersection point of the \( i \)th peripheral component with the central component in the \( k \)-eared stratum.

The coefficient \( m(L) \) is the multiplicity of the intersection of \( \Sigma(Y) \) with \( \Delta_{p,q} \) (see Propositions 3.11 and 3.13 and Remark 3.21).
Finally, according to the formula of Corollary 2.1, this should be multiplied by \(pq\) and divided by \(2^n(n-1)\). The multiplication by \(pq\) is achieved by applying \(D_0\) to the \(k\)th monomial and to the product of the first \(k-1\) monomials. Instead of dividing the resulting monomial by \(2^n(n-1)\), we multiply by \(2^n(n-1)\) the corresponding monomial on the left-hand side of the equality, by writing

\[
2t^2 \frac{\partial^2 G}{\partial t^2}
\]

instead of just \(G\).

Thus each term on the right-hand side of the equality corresponds to a \(k\)-eared stratum that contains a component of \(\Sigma(Y) \cap \Delta_{p,q}\), and correctly computes the contribution of this component. All the components of the intersection are taken into account. This finishes the proof. ∎

**Proof of Theorem 4.** Consider a monomial \(X\) in variables \(t, x_1, x_2, \ldots\). Suppose its degree in \(t\) equals \(n\); thus it corresponds to a stratum in \(\mathbb{P}H_n\). The coefficient of this monomial in \(F\) is equal, up to a combinatorial coefficient, to the degree of the \(LL\) map of the corresponding stratum \(\Sigma(X)\).

Let us look at the ways, in which the monomial \(X\) can arise on the right-hand side of the equality of Theorem 4. The right-hand side of the equality of Theorem 4 contains the term

\[
(t \frac{\partial F}{\partial t})^2
\]

and the terms

\[
(i + j)x_{i+j} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j}
\]

for all \(i, j\).

Let us choose one of the above terms and two monomials of the function \(F\). We denote by \(p\) and \(q\) their degrees in the variable \(t\). We suppose that as we apply the chosen term to the two monomials we obtain a contribution to the coefficient of the monomial \(X\).

Then the choice of the term in the equation and of two monomials determines a 2-eared stratum that constitutes a component of the intersection \(\Sigma(X) \cap \Delta_{p,q}\). The term \((t \frac{\partial F}{\partial t})^2\) corresponds to the case when two simple critical values get glued together. The terms of the second type correspond to the case when the multiple critical value is glued together with a simple
critical value; a cycle of length $i + j$ is then multiplied by a transposition and is decomposed into a cycle of length $i$ and a cycle of length $j$.

Recall that the combinatorial coefficient by which the degree of $LL$ is multiplied to obtain the coefficient of $F$ equals

\[
\frac{1}{(n-1)! |\text{Aut}(X)|} = \frac{n}{n! |\text{Aut}(X)|}.
\]

As we multiply two monomials, the factors $n!$ combine to

\[
\binom{n!}{p! q!},
\]

which enumerates the ways to distribute the $n$ numbered poles among the two components of the 2-eared stratum. Similarly, the factors $|\text{Aut}(X)|$ account for the number of ways to distribute the preimages of the only multiple critical value among the two components.

As we apply the operator $t \partial / \partial t$ to a monomial, we multiply it by the degree of the stable functions in the corresponding stratum. This gives the number of ways to choose a preimage of a regular (noncritical) value of our function. The chosen point will be the intersection point between the two components of the nodal curve in our 2-eared stratum. Similarly, when we differentiate a monomial with respect to the variable $x_i$, this counts the number of ways to choose a preimage of multiplicity $i$ of the only multiple critical value. The chosen preimage will again be the intersection point of the two components of the nodal curve.

The coefficient $i + j$ is the multiplicity of the intersection of $\Sigma(X)$ and $\Delta_{p, q}$ along the 2-eared stratum (Proposition 3.13). According to the formula in Corollary 2.11, it remains to multiply the coefficient of our monomial by $pq$, and to divide it by $2n(n-1)$. The factor $pq/n$ appears automatically because of the factor $n$ in the numerator of the combinatorial coefficient

\[
\frac{n}{n! |\text{Aut}(X)|}
\]

by which the degree of $LL$ is multiplied. Instead of dividing the result by $2(n-1)$, we multiply by $2(n-1)$ the corresponding monomial on the left-hand side of the equality, by writing

\[
2t^2 \frac{\partial}{\partial t} \left( \frac{F}{t} \right)
\]
instead of just $F$.

Thus the terms on the right-hand side enumerate all the components of the intersection $\Sigma(X) \cap \Delta_{p,q}$ and each term computes correctly the contribution of the corresponding component.

This finishes the proof. ⋄

5 Combinatorial results

5.1 Some explicit formulas for Hurwitz numbers

Here we give some explicit formulas for the coefficients of the function $G$. Some of them were known before, but others are new.

We start by listing the already known expressions for the coefficients of $G$ and the corresponding Hurwitz numbers.

In Theorem 6 below $y_*$ means “any $y_i$”; two asterisks do not necessarily denote the same subscript. As usual, $|\text{Aut}|$ is the number of ways to permute the variables $y_i$ preserving their subscripts $i$.

**Theorem 6**

1. The coefficient of \( t^n y_{n-1} y_\ast \ldots y_\ast \) in $G$ equals $n^{k-3}/|\text{Aut}|$. The corresponding Hurwitz number equals $n^{k-3}$.

2. The coefficient of \( t^n y_{n-2} y_\ast \ldots y_\ast \) in $G$ equals \[
\frac{(k - 2)(n - 1)^{k-3}}{|\text{Aut}|}.
\]
The corresponding Hurwitz number equals $(k - 2)(n - 1)^{k-3}$.

3. The coefficient of \( t^n y_i y_1^{2n-2-i} \) in $G$ equals, for $i \neq 1$,
\[
\frac{n^{n-i-3}(i+1)^{i+1}}{(n-i-1)!(i+1)!}.
\]
The corresponding Hurwitz number equals
\[
\frac{n^{n-i-3}(i+1)^{i+1}(2n-2-i)!}{(n-i-1)!(i+1)!}.
\]

4. The coefficient of \( t^n y_1^{2n-2} \) in $G$ equals $n^{n-3}/n!$. The corresponding Hurwitz number equals $n^{n-3}(2n-2)!/n!$.
Proof.
1. The strata in questions are polynomial strata: if we make a change of 
variables so that the critical point corresponding to the factor $y_{n-1}$ is sent to 
$\infty$, all the functions of these strata will become polynomials. The degrees of 
the Lyashko-Looijenga map in the case of polynomials were found in [8].
2. The strata in question are some particular strata of the versal defor-
mation of the singularity $D_n$. The degree of the Lyashko-Looijenga map on 
these strata was found in [10].
3 and 4. These are primitive strata and the corresponding degree is given 
by the Hurwitz formula (see Section 2.2).

Now we give some new formulas for the coefficients of $G$ and the corre-
sponding Hurwitz numbers. We start with the following example.

Example 5.1 Consider the stratum $\Sigma$ in $\mathbb{P}H_n$ such that a generic function $f \in \Sigma$ has two double critical points while the remaining $2n-6$ critical points 
are simple. The Hurwitz number corresponding to the stratum $\Sigma$ equals

$$\frac{3}{4} \frac{(2n-6)!}{n!} \cdot n^{n-5} \cdot (n-1)(n-2)(27n^2-137n+180).$$

This example has the following generalization.

Let $M$ be a monomial in variables $y_2, y_3, \ldots$ (without $y_1$). Denote by $k$ 
the sum of the indices of its factors and by $r$ the number of the factors (the 
total degree of the monomial). Further denote by $s$ the maximum among 
the indices of the factors of $M$ and the number $\lceil (k + 1)/2 \rceil$ (where $\lceil \cdot \rceil$ is the 
integer part).

Conjecture 5.2 The coefficient of $t^n y_1^{2n-2-k} M$ in $G$ equals

$$\frac{n^{n-3-k+r}}{n!} P_M(n),$$

where $P_M$ is a polynomial of degree $k$ with rational coefficients divisible by 
$(n-1) \ldots (n-s)$. The degree of the $LL$ map on the corresponding stratum 
equals

$$|\text{Aut}(M)| \frac{(2n-2-k)!}{n!} n^{n-3-k+r} P_M(n).$$

Here, as usual, $|\text{Aut}(M)|$ is the number of ways to permute the factors of $M$ 
preserving their indices.
Although we do not know how to prove Conjecture 5.2, there is an algorithm that checks that the conjecture is true for any given monomial $M$.

Here are some simplest polynomials $P_M$:

$$P_1(n) = 1,$$

$$P_{y_2}(n) = \frac{9}{2} \cdot (n - 1)(n - 2),$$

$$P_{y_3}(n) = \frac{9}{2} \cdot (n - 1)(n - 2)(n - 3),$$

$$P_{y_4}(n) = \frac{32}{3} \cdot (n - 1)(n - 2)(n - 3)(n - 4),$$

$$P_{y_2y_3}(n) = \frac{3}{8} \cdot (n - 1)(n - 2)(27n^2 - 137n + 180),$$

$$P_{y_2y_4}(n) = 8 \cdot (n - 1)(n - 2)(n - 3)(6n^2 - 37n + 60),$$

$$P_{y_2y_3y_4}(n) = \frac{2}{9} \cdot (n - 1)(n - 2)(n - 3)(256n^3 - 2787n^2 + 10448n - 13440),$$

$$P_{y_2y_3y_4y_4}(n) = \frac{1}{48} \cdot (n - 1)(n - 2)(n - 3)(729n^3 - 6723n^2 + 21026n - 22680).$$

We have not been able to find a general pattern for these polynomials.

### 5.2 The algebra of generating functions

Let us fix $c$ points on $\mathbb{CP}^1$ and assign a positive integer $d_i$ and a partition $\kappa_i$ of $d_i$, $1 \leq i \leq c$, to each marked point. Consider $n$-sheeted coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ with simple preimages of $\infty$, ramified over the $c$ distinguished points with ramification types $\kappa_i$ and, in addition, having $d(n) = 2n - 2 - \sum d_i$ other fixed simple ramification points.

Denote by $h_{\kappa_1, \ldots, \kappa_c}(n)$ the corresponding Hurwitz number (the number of the above coverings counted with coefficient $1/|\text{Aut}|$).

We regroup these Hurwitz numbers into a generating function

$$f_{m_1, \ldots, m_c}(t) = \sum_n \frac{h_{\kappa_1, \ldots, \kappa_c}(n)}{d(n)!} t^n.$$

We will prove that all such generating functions belong to the following algebra $\mathcal{A}$. 

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**Definition 5.3** Denote by \( \mathcal{A} \) the subalgebra of the algebra of power series in one variable \( t \), generated by the series

\[
Y = \sum_{n \geq 1} \frac{n^{n-1}}{n!} t^n \quad \text{and} \quad Z = \sum_{n \geq 1} \frac{n^n}{n!} t^n
\]

We start by a precise description of the algebra \( \mathcal{A} \).

**Proposition 5.4** The algebra \( \mathcal{A} \) is isomorphic to \( \mathbb{C}[X, X^{-1}] \), where \( X = 1 - Y \), \( X^{-1} = 1 + Z \).

**Proof.** All we have to prove is the equality \((1 - Y)(1 + Z) = 1\). This equality follows from the identity

\[
\sum_{p+q=n} \frac{n!}{p!q!} p^p q^{q-1} = (n - 1)n^{n-1},
\]

which follows from Abel’s identity

\[
\sum_{p+q=n} \frac{n!}{p!q!} (a + p)^p (b + q)^q = \frac{(a + b + n)^n}{b},
\]

by letting \( a \) tend to 0, then subtracting \( n^n/b \) from both sides of the identity, and finally letting \( b \) tend to 0. For a proof of Abel’s identity see [1], [3]; see also [5] for a computer-generated proof. \( \triangle \)

Denote by \( A_n \) the following number

\[
A_n = \sum_{p+q=n} \frac{n!}{p!q!} p^p q^q = n! \sum_{k=0}^{n-2} \frac{n^k}{k!}.
\]

For some combinatorial properties of this sequence see [4]. (Exercise: using Abel’s identity prove that the two expressions defining \( A_n \) are equal.)

**Proposition 5.5** The algebra \( \mathcal{A} \) is formed by the formal power series of the form

\[
a + \sum_{n \geq 1} \frac{L(n)n^n + P(n)A_n}{n!} t^n,
\]

where \( L \) is a Laurent polynomial and \( P \) just a polynomial in \( n \).
Proof. We will actually prove two more precise statements.

(i) The subalgebra of $\mathcal{A}$ generated by $X$ (or by $Y$) is the algebra of power series as above such that the Laurent polynomial $L$ contains only negative powers of $n$ and the polynomial $P$ is equal to 0.

(ii) The subalgebra of $\mathcal{A}$ generated by $X^{-1}$ (or by $Z$) is the algebra of power series as above such that the Laurent polynomial $L$ is actually a polynomial.

Assertion (i) follows from the equality
\[
Y^k = k \sum_{n \geq 1} \frac{(n-k+1) \ldots (n-1)}{n!} t^n,
\]
which is proved by induction on $k$. Denote by $D$ the differential operator $D = t \partial / \partial t$. It acts on power series by multiplying the coefficient of $t^n$ by $n$. The step of induction is carried out using the equality
\[
DY = Z = \frac{Y}{1-Y}.
\]

From this equality we deduce
\[
D \left( \frac{Y^{k+1}}{k+1} - \frac{Y^k}{k} \right) = \left( Y^k - Y^{k-1} \right) \frac{Y}{1-Y} = -Y^k.
\]

Knowing that the free term of $Y^{k+1}$ equals 0, we can deduce the expression of $Y^{k+1}$ from that of $Y^k$.

The proof of assertion (ii) is similar, although, as far as we know, there is no simple formula for $Z^k$.

By definition of the sequence $A_n$, we have
\[
\sum_{n \geq 1} \frac{A_n}{n!} t^n = Z^2.
\]

Therefore what we must prove is that the series $D^k Z$ and $D^k (Z^2)$ for $k \geq 0$ can be expressed as polynomials in $Z$ and vice versa.

From $(1 - Y)(1 + Z) = 1$ and $Z = DY$ we find
\[
DZ = Z(1 + Z)^2.
\]

It follows by induction that $D^k Z$ can be expressed as a polynomial in $Z$ of degree $2k + 1$ with positive integer coefficients. Hence $D^k (Z^2)$ can be expressed
as a polynomial in $Z$ of degree $2k + 2$ with positive integer coefficients. Since we have obtained exactly once every possible degree of polynomials in $Z$, it follows that any polynomial in $Z$ is a linear combination of the series $D^k Z$ and $D^k(Z^2)$.

We do not know whether the algebra $\mathcal{A}$ has already appeared in the literature. It certainly plays an important role in the understanding of ramified coverings of the sphere, as shows Theorem 2 from Introduction. We restate it here.

**Theorem 2** For any partitions $\kappa_1, \ldots, \kappa_c$, the series $f_{\kappa_1, \ldots, \kappa_c}(t)$ belongs to the algebra $\mathcal{A}$.

**Conjecture 5.6** If we write the series $f_{\kappa_1, \ldots, \kappa_c}(t)$ in the form of Proposition 5.5 then the corresponding polynomial $P(n)$ is equal to 0.

Why the polynomial $P$ always equals 0 remains mysterious. In the proof of Theorem 2, the function $f_{\kappa_1, \ldots, \kappa_c}$ is obtained as a sum of products of similar functions. Another way to obtain $f_{\kappa_1, \ldots, \kappa_c}$ as a different sum of products of similar functions can be obtained from a generalization of Theorem 3. (The relation of Theorem 3 itself gives a method for calculating the elements of the algebra $\mathcal{A}$ that correspond to various generating functions $f_{\kappa_1, \ldots, \kappa_c}(t)$ corresponding to simple strata.) However, in both methods, the polynomials $P$ that correspond to various terms of the sum are not equal to 0. They cancel out only after the addition.

**Remark 5.7** Theorem 2 resembles very much another theorem on the ramified coverings of a torus. Fix a finite number of ramification points on a torus and consider $n$-sheeted coverings of the torus having ramifications of prescribed types over the marked points. The numbers of such ramifications can be regrouped in a generating function. It is known (see [2]) that this generating function will always belong to a very particular subalgebra of the algebra of power series, namely, to the algebra of quasimodular forms (that is, the algebra generated by the three Eisenstein series $E_2$, $E_4$, $E_6$). This important result was predicted by mirror symmetry for elliptic curves.

**Proof of Theorem 2** We will prove the theorem by induction on $c$. First of all note that the base of induction (the cases $c = 0, 1$) is an immediate
consequence of the Hurwitz formula (see [7]):

\[ h_\kappa = \frac{(2n - 2 - d(\kappa))!}{(n - d(\kappa) - m)! |\text{Aut}(\kappa)|} \prod_{i=1}^{m} \frac{(k_i + 1)^{k_i+1}}{(k_i + 1)!} (n^{n-d(\kappa)-3}) \]

where \( \kappa = (k_1, \ldots, k_m) \), \( d(\kappa) = k_1 + \ldots + k_m \), and \(|\text{Aut}(\kappa)|\) is the number of ways to permute the numbers \( k_i \) preserving their values.

Now we will show that each generating function \( f_{\kappa_1, \ldots, \kappa_c}(t) \) can be obtained from similar generating functions involving \( c - 1 \) instead of \( c \) partitions by applying sums, products, and the operator \( D = t \partial / \partial t \). Since all these operations preserve the algebra \( A \), if all the generating functions for \( c - 1 \) partitions belong to \( A \), so does the final generating function. The proof goes in the spirit of [6].

The monodromy of a ramification point with ramification type \( \kappa = (k_1, \ldots, k_m) \) is a permutation whose cycle lengths are \( k_1 + 1, \ldots, k_m + 1 \) and, in addition, some number of cycles of length 1. For simplicity we will call \( \kappa \) the cycle type of the permutation. Consider a factorization of the identity permutation on \( n \) elements into a product of \( c \) permutations \( \sigma_1, \ldots, \sigma_c \) with cycle types \( \kappa_1, \ldots, \kappa_c \) and \( d(n) \) transpositions \( \tau_1, \ldots, \tau_{d(n)} \) (in that order) such that the group generated by all these permutations acts transitively on the \( n \) elements. Such a factorization is called a transitive factorization. It is easy to see that the number \( d(n) \) is the smallest possible number of transpositions that should be added to the other permutations so that a transitive factorization would exist. It follows that every transitive factorization is the set of monodromies of a ramified covering of a sphere by a sphere. Moreover, the number of all transitive factorizations equals \( n! h_{\kappa_1, \ldots, \kappa_c}(n) \).

Before proceeding with the proof, let us give an example of what will happen. Consider the generating function \( f_{1,1} \) corresponding to the case when \( c = 2 \) and the two special critical values are actually just simple critical values; in other words, the corresponding monodromies are transpositions. A product of two transpositions can be either a cycle of length 3 (corresponding to a double critical point), or a permutation with two cycles of lengths 2 (a critical value attained at two simple critical points), or the identity permutation with two distinguished elements (the ramified covering splits into two irreducible components that intersect at one point). Therefore we will need three simpler generating functions: \( f_2 \), \( f_{1^2} \), and \( f_0 \) (the latter corresponds to the case \( c = 0 \)). We have the following equality:

\[ f_{1,1} = 3f_2 + 2f_{1^2} + \frac{1}{2} (Df_0)^2. \]
The coefficient 3 is due to the fact that there are three ways do decompose a cycle of length 3 into a product of 2 transpositions. Similarly, the coefficient 2 is due to the fact that a permutation with 2 cycles of length 2 can be decomposed into a product of 2 transpositions in two ways. The operator $D$ is applied to the function $f_\emptyset$ because if there are $p$ and $q$ sheets in the two transitivity components, then there are, respectively, $p$ and $q$ ways to choose their intersection point on each of the components (and the operator $D$ multiplies by $p$ the coefficient of $t^p$). Finally, the coefficient $1/2$ is due to the fact that the two transitivity components are indistinguishable from each other. The reader can check that the equality is true using the expressions

\[
\begin{align*}
    f_\emptyset &= \sum_{n \geq 1} \frac{n^{n-3}}{n!} t^n = -\frac{1}{12}(X - 1)(2X^2 + 5X + 5), \\
    Df_\emptyset &= \sum_{n \geq 1} \frac{n^{n-2}}{n!} t^n = -\frac{1}{2}(X - 1)(X + 1), \\
    f_{12} &= 2 \sum_{n \geq 1} (n - 3)(n - 2)(n - 1) \frac{n^{n-4}}{n!} t^n = \frac{1}{2}(X - 1)^4, \\
    f_2 &= \frac{9}{2} \sum_{n \geq 1} (n - 2)(n - 1) \frac{n^{n-4}}{n!} t^n = -\frac{1}{8}(X - 1)^3(3X + 1), \\
    f_{1,1} &= \sum_{n \geq 1} (2n - 2)(2n - 3) \frac{n^{n-3}}{n!} t^n = -\frac{1}{2}(X - 1)^2(2X - 3).
\end{align*}
\]

We will show that similar relations hold in the general case. We do not need to specify the coefficients in these relations; it suffices to know that they are some fixed rational numbers. Now we proceed with the proof.

Let us see what happens to a transitive factorization as we multiply the permutations $\sigma_{c-1}$ and $\sigma_c$. We obtain a new list of permutations, whose product is equal to the identity permutation. However the group generated by these permutations is not necessarily transitive. Denote by $k$ the number of its transitivity components. Each transitivity component is a subset of the set \{1, \ldots, n\}. For each component we actually obtain a transitive factorization of the identity permutation acting on this subset. The elements of each of the partitions $\kappa_1, \ldots, \kappa_{c-2}$ are distributed among the transitivity components. The $d(n)$ transpositions are also distributed among the transitivity components. As for the product $\sigma_{c-1}\sigma_c$, we will separate its cycles
into two groups. The first group consists of those cycles that have at least one element in common with a cycle of length greater than 1 either in \( \sigma_{c-1} \) or in \( \sigma_c \). The cycles in this first group can have length 1 (singletons) or greater than 1. The cycles of length 1 will be called \textit{special} singletons. The second group consists of all the other cycles. All these cycles are automatically singletons. If the number \( n \) is big enough, the second group will be much bigger than the first one. The cycles of length greater than 1 and the special singletons of the product \( \sigma_{c-1} \sigma_c \) are again distributed in some way among the \( k \) transitivity components. Thus to every transitive factorization with cycle types \( \kappa_1, \ldots, \kappa_c \) we can assign unambiguously the following data: \( k \) transitive factorizations acting on some subsets of the set \( \{1, \ldots, n\} \); each of these transitive factorizations is composed of \( c-1 \) permutations with some given cycle types, plus the necessary number of transpositions; some number of special singletons is specified among the singletons of the \( (c-1) \)th permutation.

Working backwards, one can see that the above data allows us to reconstruct several possible transitive factorizations with cycle types \( \kappa_1, \ldots, \kappa_c \). The number of possibilities is equal to the product of two factors.

The first factor is a multinomial coefficient that counts the number of ways to order the \( d(n) \) transpositions. These transpositions are already ordered inside each of the \( k \) transitive factorizations, and we have to order them all respecting these partial orders. As we multiply two generating functions, this first factor appears automatically, because of the coefficient \( d(n)! \) in the denominator of the coefficient of \( t^n \) in \( f_{\kappa_1, \ldots, \kappa_c} \).

The second factor is the number of ways to reconstruct the permutations \( \sigma_{c-1} \) and \( \sigma_c \) knowing their product. More precisely, the \( (c-1) \)th permutations in the \( k \) transitive factorizations can be regrouped into a unique permutation acting on the set \( \{1, \ldots, n\} \). This permutation has some cycles of lengths greater than 1 and, in addition, a given number of special singletons. We must decompose this permutation into a product of two permutations \( \sigma_{c-1} \) and \( \sigma_c \) (with given cycle types) in such a way that all nonspecial singletons remain singletons in \( \sigma_{c-1} \) and \( \sigma_c \). Moreover, the \( k \) subsets of the subdivision must merge into a unique transitivity component after we have replaced the product \( \sigma_{c-1} \sigma_c \) by two permutations \( \sigma_{c-1} \) and \( \sigma_c \). The number of such factorizations is the second factor. This factor, divided by some product of factorials due to indistinguishable transitivity components, will play the role of the coefficient in the expression for \( f_{\kappa_1, \ldots, \kappa_c} \).

Now we can precisely describe how the generating function \( f_{\kappa_1, \ldots, \kappa_c} \) is
expressed from similar generating functions assigned to lists of $c-1$ partitions. First we make a list of all possible ways to divide the elements of each of the partitions $\kappa_1, \ldots \kappa_{c-2}$ into $k$ parts and to distribute among the same $k$ parts the cycle lengths and the special singletons of each possible product of the permutations $\sigma_{c-1}$ and $\sigma_c$. This gives us a finite list of possibilities; each possibility can be described by a table:

\[
\begin{align*}
&\kappa_1^{(1)}, \ldots, \kappa_{c-1}^{(1)}, m^{(1)}; \\
&\kappa_1^{(2)}, \ldots, \kappa_{c-1}^{(2)}, m^{(2)}; \\
&\cdots \\
&\kappa_1^{(k)}, \ldots, \kappa_{c-1}^{(k)}, m^{(k)}.
\end{align*}
\]

Here $\kappa_i^{(j)}$ for $i \leq c-2$ is the partition formed by those elements of the partition $\kappa_i$ that were assigned to the $j$th part (empty partitions are allowed); $\kappa_{c-1}^{(j)}$ is the similar partition assigned to the restriction to the $j$th part of the product $\sigma_{c-1} \sigma_c$; finally, $m^{(j)}$ is the number of special singletons in the $j$th part.

To each row of this table we assign the generating function corresponding to the partitions of this row. The coefficients of this generating function count the number of transitive factorizations in the corresponding transitivity component. Further, denote by $d^{(j)}$ the sum of the elements of $\kappa_{c-1}^{(j)}$. Then we apply to the corresponding generating function the operator

\[
(D - d^{(j)}) \cdots (D - d^{(j)} - m^{(j)} + 1).
\]

This corresponds to choosing $m^{(j)}$ special singletons among all the available singletons. Now we multiply the obtained expressions for each row. The product thus obtained is further multiplied by the number of ways to reconstruct the permutations $\sigma_{c-1}$ and $\sigma_c$ knowing the permutation $\sigma_{c-1} \sigma_c$, and then divided by the number of automorphisms of the set of rows of the table. The sum of such terms over all possible tables gives us an expression for the generating function $f_{\kappa_1, \ldots, \kappa_c}$.

This finishes the proof.

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