THE GENERALIZED AUSLANDER-REITEN DUALITY ON AN
EXACT CATEGORY

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Abstract. We introduce a notion of generalized Auslander-Reiten duality on
a Hom-finite Krull-Schmidt exact category $C$. This duality induces the gen-
eralized Auslander-Reiten translation functors $\tau$ and $\tau^-$. They are mutually
quasi-inverse equivalences between the stable categories of two full subcate-
gories $C_r$ and $C_l$ of $C$. A non-projective indecomposable object lies in the
domain of $\tau$ if and only if it appears as the third term of an almost split con-
flation; dually, a non-injective indecomposable object lies in the domain of $\tau^-$
if and only if it appears as the first term of an almost split conflation.

1. Introduction

Throughout $k$ denotes a commutative artinian ring. The categories we consider
are $k$-linear, Hom-finite, Krull-Schmidt and skeletally small.

Recall that an abelian category $A$ is said to have enough almost split sequences
provided that each non-projective indecomposable object appears as the third term
of an almost split sequence, and that each non-injective indecomposable object
appears as the first term of an almost split sequence. Lenzing and Zuzua proved
that an Ext-finite abelian category $A$ has enough almost split sequences if and only
if it has Auslander-Reiten duality; see [8, Theorem 1.1]. We observe that the notion
of Auslander-Reiten duality applies to exact categories.

Inspired by [5], we introduce the notion of generalized Auslander-Reiten duality
for an exact category $C$. This duality induces the generalized Auslander-Reiten
translation functors $\tau$ and $\tau^-$, which are defined on the stable categories of two full
subcategories $C_r$ and $C_l$ of $C$. Then $C$ has Auslander-Reiten duality in the sense of
[8] if and only if $C_l = C = C_r$. In general, $C_r$ and $C_l$ are not equal to $C$.

We prove that a non-projective indecomposable object lies in $C_r$ if and only if it
appears as the third term of an almost split conflation, and that a non-injective inde-
composable object lies in $C_l$ if and only if it appears as the first term of an almost
split conflation; see Proposition 2.4. We prove that the generalized Auslander-
Reiten translation functors $\tau$ and $\tau^-$ are mutually quasi-inverse equivalences be-
 tween the projectively stable category of $C_r$ and the injectively stable category of
$C_l$; see Proposition 3.4.

In Section 4, we describe the full subcategories $C_r$ and $C_l$, and the generalized
Auslander-Reiten translation functors $\tau$ and $\tau^-$ in the case that $C$ is the category
of finitely presented representations of a locally finite interval-finite quiver. This
description depends on the results of [8].

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2. Two full subcategories

Let $k$ be a commutative artinian ring and $\hat{k}$ be the minimal injective cogenerator. Denote by $k\text{-mod}$ the category of finitely generated $k$-modules and by $D = \text{Hom}_k(-, \hat{k})$ the Matlis duality.

Let $\mathcal{C}$ be a $k$-linear exact category. Recall that an exact category is an additive category $\mathcal{C}$ together with a collection $\mathcal{E}$ of kernel-cokernel pairs $(i,d)$ which satisfies the axioms in \cite{2} Appendix A; compare \cite{10} Section 2. Here, a kernel-cokernel pair $(i,d)$ means a sequence of morphisms $X \rightarrow Y \rightarrow Z$ satisfying that $i$ is the kernel of $d$ and $d$ is the cokernel of $i$. A kernel-cokernel pair $(i,d)$ in $\mathcal{E}$ is called a conflation, while $i$ is called an inflation and $d$ is called a deflation. For two objects $X$ and $Y$ in $\mathcal{C}$, we denote by $\text{Ext}_k^d(X,Y)$ the set of equivalent classes of conflations $Y \rightarrow E \rightarrow X$. Given a conflation $\eta: Y \rightarrow E \rightarrow X$, for each $f: Z \rightarrow X$, we denote by $\eta.f = \text{Ext}_k^d(f,Y)(\eta)$ the conflation obtained by the pullback of $\eta$ along $f$. Dually, for each $g: Y \rightarrow Z$, we denote by $g.\eta = \text{Ext}_k^d(X,g)(\eta)$ the conflation obtained by the pushout of $\eta$ along $g$.

Recall from \cite{8} Section 2] that a morphism $f: X \rightarrow Y$ is projectively trivial provided that for each object $Z$, the induced map $\text{Ext}_k^d(f,Z): \text{Ext}_k^d(Y,Z) \rightarrow \text{Ext}_k^d(X,Z)$ is zero. We observe that $f$ is projectively trivial if and only if $f$ factors through each deflation $E \rightarrow Y$ ending at $Y$. Dually, a morphism $f$ is injectively trivial provided that for each object $Z$, the induced map $\text{Ext}_k^d(Z,f): \text{Ext}_k^d(Z,X) \rightarrow \text{Ext}_k^d(Z,Y)$ is zero.

Given two objects $X$ and $Y$, we denote by $\mathcal{P}(X,Y)$ the set of projectively trivial morphisms from $X$ to $Y$. Then $\mathcal{P}$ forms an ideal of $\mathcal{C}$. The projectively stable category $\mathcal{C}$ of $\mathcal{C}$ is the factor category $\mathcal{C}/\mathcal{P}$. Given a morphism $f: X \rightarrow Y$, we denote by $\overline{f}$ its image in $\mathcal{C}$. We denote by $\text{Hom}_\mathcal{C}(X,Y) = \text{Hom}_\mathcal{C}(X,Y)/\mathcal{P}(X,Y)$ the set of morphisms in $\mathcal{C}$. Given an object $Z$, we mention that $\text{Ext}_k^d(Z,-)$ is a contravariant functor from $\mathcal{C}$ to the category of $k$-modules. We observe that an object $P$ becomes zero in $\mathcal{C}$ if and only if $P$ is projective in $\mathcal{C}$.

Dually, we denote by $\mathcal{I}(X,Y)$ the set of injectively trivial morphisms from $X$ to $Y$. The injectively stable category $\mathcal{C}$ of $\mathcal{C}$ is the factor category $\mathcal{C}/\mathcal{I}$. Given a morphism $f: X \rightarrow Y$, we denote by $\overline{f}$ its image in $\mathcal{C}$. We denote by $\text{Hom}_\mathcal{C}(X,Y) = \text{Hom}_\mathcal{C}(X,Y)/\mathcal{I}(X,Y)$ the set of morphisms in $\mathcal{C}$. We mention that $\text{Ext}_k^d(Z,-)$ is a functor from $\mathcal{C}$ to the category of $k$-modules. We observe that an object $I$ becomes zero in $\mathcal{C}$ if and only if $I$ is injective in $\mathcal{C}$.

Recall that a morphism $v: E \rightarrow Y$ is right almost split if it is not a retraction and each $f: Z \rightarrow Y$ which is not a retraction factors through $v$. Dually, a morphism $u: X \rightarrow E$ is left almost split if it is not a section and each $f: X \rightarrow Z$ which is not a section factors through $u$. A conflation $\eta: X \rightarrow E \rightarrow Y$ is an almost split conflaction if $u$ is left almost split and $v$ is right almost split. Recall that an object whose endomorphism algebra is local is called strongly indecomposable. We mention that given an almost split conflation $X \rightarrow E \rightarrow Y$, the objects $X$ and $Y$ are strongly indecomposable; see \cite{11} Proposition II.4.4. We refer to \cite{2} Chapter V] for the general properties of almost split conflations.

We observe the fact that for each non-split conflation $\delta: X \rightarrow Y \rightarrow Z$, there exists some $\gamma \in D \text{Ext}_k^d(Z,X)$ such that $\gamma(\delta) \neq 0$. The following lemma is essentially due to \cite{8} Proposition 3.1]; compare \cite{6} Theorem 9.3 and Corollary 9.4].
Lemma 2.1. Let $\eta: X \to E \to Y$ be an almost split conflation and let $\gamma \in D\text{Ext}^1_C(Y,X)$ \textup{with } $\gamma(\eta) \neq 0$. Then we have the following statements.

(1) For each $M$, we have a non-degenerated $k$-bilinear map

\[ \langle -,- \rangle_M : \text{Hom}_C(M,X) \times \text{Ext}^1_C(Y,M) \to \hat{k}, \quad \langle f,\mu \rangle \mapsto \gamma(f,\mu). \]

If moreover $\text{Hom}_C(M,X) \in k\text{-mod}$ for each $M$, then the induced map

\[ \phi_{Y,M} : \text{Hom}_C(M,X) \to D\text{Ext}^1_C(Y,M), \quad f \mapsto \langle f,\mu \rangle_M, \]

is an isomorphism and natural in $M$ with $\gamma = \phi_{Y,X}(\text{Id}_X)$. 

(2) For each $M$, we have a non-degenerated $k$-bilinear map

\[ \langle -,- \rangle_M : \text{Ext}^1_C(M,X) \times \text{Hom}_C(Y,M) \to \hat{k}, \quad (g,\mu) \mapsto \gamma(\mu,g). \]

If moreover $\text{Hom}_C(Y,M) \in k\text{-mod}$ for each $M$, then the induced map

\[ \psi_{X,M} : \text{Hom}_C(Y,M) \to D\text{Ext}^1_C(M,X), \quad g \mapsto \langle -,- \rangle_M, \]

is an isomorphism and natural in $M$ with $\gamma = \psi_{X,Y}(\text{Id}_Y)$.

Proof. (1) It is sufficient to show that the $k$-bilinear map $\langle -,- \rangle_M$ is non-degenerated for each $M$. The naturalness of $\phi_Y : \text{Hom}_C(-,X) \to D\text{Ext}^1_C(Y,-)$ follows from a direct verification.

On the one hand, assume that $\mu : M \to E' \to Y$ is a non-split conflation. Since $\eta$ is almost split, we obtain the following commutative diagram

\[
\begin{array}{ccc}
M & \longrightarrow & E' \\
| \downarrow f | \downarrow \eta | \\
X & \longrightarrow & E \\
\end{array}
\]

Here, the left square is a pushout diagram. Hence $f,\mu = \eta$ and $\langle f,\mu \rangle_M = \gamma(\eta) \neq 0$.

On the other hand, let $\overline{f} : M \to X$ be a nonzero morphism in $\overline{C}$. We have that $f$ is not injectively trivial in $C$. Hence there exists some object $N$ and some $\mu \in \text{Ext}^1_C(N,M)$ such that $f,\mu = \text{Ext}^1_C(N,f)(\mu)$ is non-split. Since $\eta$ is almost split, it can be obtained by a pullback of $f,\mu$ along some $h : Y \to N$.

\[
\begin{array}{ccc}
M & \longrightarrow & E_1 \\
| \downarrow f | \downarrow \eta | \\
X & \longrightarrow & E \\
\end{array}
\]

Hence we have $\eta = (f,\mu).h = f.(\mu,h)$ and then $\langle f,\mu,h \rangle_M = \gamma(\eta) \neq 0$.

(2) The proof is similar. \qed

The following lemma is a slight modification of [3, Proposition 4.1].

Lemma 2.2. Let $Y$ be a strongly indecomposable object in $C$.

(1) Assume $D\text{Ext}^1_C(Y,Z) \in k\text{-mod}$ for each $Z$. If the functor $D\text{Ext}^1_C(Y,-)$ is isomorphic to $\text{Hom}_C(-,Y')$ for some $Y'$, which has a non-injective strongly indecomposable direct summand, then there exists an almost split conflation ending at $Y$. 


Proof. (1) Let $\phi: \text{Hom}_C(-, Y') \to D\text{Ext}_k^1(Y, -)$ be an isomorphism of functors. Set $\gamma = \phi_Y(\text{id}_Y)$. By the naturalness of $\phi$, for each object $M$ and each morphism $f: M \to Y'$, we have
$$\phi_M(\tilde{f}) = D\text{Ext}_k^1(Y, f)(\gamma) = \gamma \circ \text{Ext}_k^1(Y, f).$$
It follows that $\phi_M(\tilde{f})(\mu) = \gamma(f, \mu)$ for each $\mu \in \text{Ext}_k^1(Y, M)$.

Let $X$ be a non-injective strongly indecomposable direct summand of $Y'$. Then the isomorphism $\phi_X$ induces a non-degenerated $k$-bilinear map
$$\langle -, - \rangle_X: \text{Hom}_C(X, Y') \times \text{Ext}_k^1(Y, X) \to \tilde{k}, \quad (\tilde{f}, \mu) \mapsto \gamma(f, \mu).$$
Let $H \subseteq \text{Hom}_C(X, Y')$ be the subset formed by non-sections. Observe that $H$ is a $k$-submodule since $X$ is strongly indecomposable. We have that $Z(X, Y') \subseteq H$ since $X$ is non-injective. Hence $H/Z(X, Y')$ is a proper $k$-submodule of $\text{Hom}_C(X, Y')$. Then there exists some nonzero $\delta \in D\text{Hom}_C(X, Y')$ such that $\delta(\tilde{f}) = 0$ for each $f \in H$. By the non-degenerated bilinear form $\langle -, - \rangle_X$, there exists some non-split conflation $\eta: X \xrightarrow{\delta} E \to Y$ such that $\delta = \langle -, \eta \rangle_X$. Then we have $\langle \tilde{f}, \eta \rangle_X = 0$ for each $f \in H$.

We claim that $u$ is left almost split. Indeed, we observe that $u$ is not a section. Assume that $h: X \to M$ is not a section. Then for each $g: M \to Y'$, the morphism $g \circ h$ is not a section and thus lies in $H$. Hence we have $\langle g \circ h, \eta \rangle_X = 0$. Consider the non-degenerated $k$-bilinear map
$$\langle -, - \rangle_M: \text{Hom}_C(M, Y') \times \text{Ext}_k^1(Y, M) \to \tilde{k}, \quad (\tilde{f}, \mu) \mapsto \gamma(f, \mu),$$
induced by $\phi_M$. For each $g: M \to Y'$, we have
$$\langle \tilde{f}, \eta \rangle_M = \gamma(g, (h, \eta)) = \langle g \circ h, \eta \rangle_X = 0.$$
(1) Assume that $X \in \mathcal{C}_r$ and $X \simeq Y$ in $\mathcal{C}$. Then the object $Y$ lies in $\mathcal{C}_r$. 
(2) Assume that $X \in \mathcal{C}_l$ and $X \simeq Y$ in $\mathcal{C}$. Then the object $Y$ lies in $\mathcal{C}_l$.

Proof. (1) We observe that $\overline{\text{Ext}}^1_\mathcal{E}(X, -) \simeq \overline{\text{Ext}}^1_\mathcal{E}(Y, -)$ as functors. Then the result follows. The proof of (2) is similar. □

As a consequence of Lemmas 2.1 and 2.2, we have the following description of indecomposable objects in $\mathcal{C}_r$ and $\mathcal{C}_l$.

**Proposition 2.4.** Let $Y$ be an indecomposable object in $\mathcal{C}$.

(1) Assume that $Y$ is non-projective. Then $Y \in \mathcal{C}_r$ if and only if there exists an almost split conflation ending at $Y$.

(2) Assume that $Y$ is non-injective. Then $Y \in \mathcal{C}_l$ if and only if there exists an almost split conflation starting at $Y$. □

### 3. An equivalence between stable subcategories

Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt exact category. For each $Y$ in $\mathcal{C}_r$, we fix some isomorphism of functors

$$
\phi_Y : \overline{\text{Hom}}(-, \tau Y) \longrightarrow \overline{\text{Ext}}^1_\mathcal{E}(Y, -).
$$

Then $\tau$ gives a map from the objects of $\mathcal{C}_r$ to $\mathcal{C}$. Dually, for each $X$ in $\mathcal{C}_l$, we fix some isomorphism of functors

$$
\psi_X : \overline{\text{Hom}}(-, \tau^- X) \longrightarrow \overline{\text{Ext}}^1_\mathcal{E}(-, X).
$$

Then $\tau^-$ gives a map from the objects of $\mathcal{C}_l$ to $\mathcal{C}$.

**Lemma 3.1.** Let $X$ and $Y$ be two objects in $\mathcal{C}$.

(1) If $X, Y \in \mathcal{C}_r$ and $X \simeq Y$ in $\mathcal{C}$, then we have $\tau X \simeq \tau Y$ in $\mathcal{C}$.

(2) If $X, Y \in \mathcal{C}_l$ and $X \simeq Y$ in $\mathcal{C}$, then we have $\tau^- X \simeq \tau^- Y$ in $\mathcal{C}$.

Proof. (1) We observe that $\overline{\text{Hom}}(-, \tau X) \simeq \overline{\text{Hom}}(-, \tau Y)$, since they are both isomorphic to $\overline{\text{Ext}}^1_\mathcal{E}(X, -) \simeq \overline{\text{Ext}}^1_\mathcal{E}(Y, -)$. Then the result follows from Yoneda’s lemma. The proof of (2) is similar. □

**Lemma 3.2.** Let $Y$ be an object in $\mathcal{C}$.

(1) If $Y \in \mathcal{C}_r$, then $\tau Y \in \mathcal{C}_l$ and $Y \simeq \tau^- Y$ in $\mathcal{C}$.

(2) If $Y \in \mathcal{C}_l$, then $\tau^- Y \in \mathcal{C}_r$ and $Y \simeq \tau Y$ in $\mathcal{C}$.

Proof. We only prove (1). We may assume that $Y$ is indecomposable and non-projective. By Lemma 2.2(1), there exists some almost split conflation $X \to E \to Y$. Then we have $\overline{\text{Hom}}(-, X) \simeq \overline{\text{Ext}}^1_\mathcal{E}(Y, -)$ and $\overline{\text{Hom}}(-, Y) \simeq \overline{\text{Ext}}^1_\mathcal{E}(-, X)$ by Lemma 2.1. We then obtain $X \in \mathcal{C}_l$. It follows from Yoneda’s lemma that $\tau Y \simeq X$ in $\mathcal{C}$, and $\tau^- X \simeq Y$ in $\mathcal{C}$. Hence we have $\tau Y \in \mathcal{C}_l$ by Lemma 2.3(2). Then we have $\tau^- Y \simeq \tau^- X \simeq Y$ in $\mathcal{C}$. Here, the first isomorphism follows from Lemma 3.1(2). □

We denote by $\mathcal{C}_r$ the image of $\mathcal{C}_r$ under the canonical functor $\mathcal{C} \to \mathcal{C}$, and by $\mathcal{C}_l$ the image of $\mathcal{C}_l$ under the canonical functor $\mathcal{C} \to \mathcal{C}$. We will make $\tau$ into a functor from $\mathcal{C}_r$ to $\mathcal{C}_l$, and $\tau^-$ into a functor from $\mathcal{C}_l$ to $\mathcal{C}_r$. 
For each morphism $f: Y \to Y'$ in $C_r$, define the morphism $\tau(f): \tau Y \to \tau Y'$ in $\mathcal{C}_l$ such that the following diagram commutes

\[
\begin{array}{c}
\text{Hom}_C(-, \tau Y) \xrightarrow{\tau Y} D \text{Ext}_E^1(Y, -) \\
\text{Hom}_C(-, \tau(f)) \downarrow \quad \downarrow D \text{Ext}_E^1(f, -) \\
\text{Hom}_C(-, \tau Y') \xrightarrow{\tau Y'} D \text{Ext}_E^1(Y', -).
\end{array}
\]

Here, the existence and uniqueness of $\tau(f)$ are guaranteed by Yoneda’s lemma. Then it follows that $\tau$ is a functor from $C_r$ to $\mathcal{C}_l$. Moreover, if $f$ is projectively trivial, then $D \text{Ext}_E^1(f, -) = 0$ and thus $\tau(f) = 0$ in $\mathcal{C}_l$. Hence $\tau$ induces a functor from $C_r$ to $\mathcal{C}_l$ which we still denote by $\tau$.

Similarly, we have a functor $\tau^-: \mathcal{C}_l \to C_r$. For each $\overline{g}: X \to X'$ in $\mathcal{C}_l$, the morphism $\tau^-(\overline{g}): \tau^-X \to \tau^-X'$ is given by the following commutative diagram

\[
\begin{array}{c}
\text{Hom}_C(\tau^- X', -) \xrightarrow{\psi_{X'}} D \text{Ext}_E^1(-, X') \\
\text{Hom}_C(\tau^- (\overline{g}), -) \downarrow \quad \downarrow D \text{Ext}_E^1(-, \overline{g}) \\
\text{Hom}_C(\tau^- X, -) \xrightarrow{\psi_X} D \text{Ext}_E^1(-, X).
\end{array}
\]

We want to show that the functors $\tau$ and $\tau^-$ are mutually quasi-inverse equivalences between $C_r$ and $\mathcal{C}_l$.

For each $Y \in \mathcal{C}_r$, we set

$$\theta_Y = \psi_{\tau Y, Y}^{-1}(\phi_{\tau Y, Y}(\text{Id}_{\tau Y})): \tau^- \tau Y \to Y$$

in $C_r$. Dually, for each $X \in \mathcal{C}_l$, we set

$$\xi_X = \phi_{\tau X, X}^{-1}(\psi_{\tau X, X}(\text{Id}_{\tau X})): X \to \tau \tau^- X$$

in $\mathcal{C}_l$.

**Lemma 3.3.** Let $\theta_Y$ and $\xi_X$ be as above.

1. The morphism $\theta_Y$ is natural in $Y$, and for each morphism $f: Y \to Y'$ in $C_r$ we have

   $$\tau(f) = \phi_{\tau Y, \tau Y'}^{-1}(\psi_{\tau Y, \tau Y'}(f \circ \theta_Y)).$$

2. The morphism $\xi_X$ is natural in $X$, and for each morphism $\overline{g}: X \to X'$ in $\mathcal{C}_l$ we have

   $$\tau^-(\overline{g}) = \psi_{\tau X', \tau X}^{-1}(\phi_{\tau X', \tau X}(\xi_X \circ \overline{g})).$$

**Proof.** (1) For each $f: Y \to Y'$ in $C_r$, we have the following commutative diagram

\[
\begin{array}{c}
\text{Hom}_C(\tau Y, \tau Y) \xrightarrow{\phi_{\tau Y, \tau Y}} D \text{Ext}_E^1(Y, \tau Y) \xleftarrow{\psi_{\tau Y, \tau Y}} \text{Hom}_C(\tau^- \tau Y, Y) \\
\text{Hom}_C(\tau Y, \tau(f)) \downarrow \quad \downarrow D \text{Ext}_E^1(f, \tau Y) \\
\text{Hom}_C(\tau Y, \tau Y') \xrightarrow{\phi_{\tau Y, \tau Y'}} D \text{Ext}_E^1(Y, \tau Y') \xleftarrow{\psi_{\tau Y, \tau Y'}} \text{Hom}_C(\tau^- \tau Y', Y').
\end{array}
\]
The left square commutes by the definition of \( \tau(f) \), and the right square commutes since the isomorphism \( \psi_{Y,Y} \) is natural. By a diagram chasing, we have 

\[
\tau(f) = \phi_{Y,Y}^{-1}(\psi_{Y,Y}(f \circ \theta_{Y})).
\]

We have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{C}(\tau Y', \tau Y') & \xrightarrow{\phi_{Y',Y'}} & D \text{Ext}_{C}^{1}(Y', \tau Y') \\
\downarrow \text{Hom}_{C}(\tau Y, \tau Y') & & \downarrow \text{Hom}_{C}(\tau Y, \tau Y') \\
\text{Hom}_{C}(\tau Y, \tau Y') & \xrightarrow{\psi_{Y,Y}^{'}} & \text{Hom}_{C}(\tau \tau Y', \tau Y').
\end{array}
\]

The right square commutes by the definition of \( \tau \tau f \). By a diagram chasing, we have 

\[
\tau f = \phi_{Y,Y}^{-1}(\psi_{Y,Y}(\theta_{Y} \circ \tau f)).
\]

We then obtain \( f \circ \theta_{Y} = \theta_{Y} \circ \tau f \). It follows that \( \theta_{Y} \) is a natural transformation.

(2) The proof is similar. \( \square \)

The following result strengthens [8] Proposition 3.3.

**Proposition 3.4.** The natural transformations \( \theta \) and \( \xi \) are both isomorphisms. Hence, the functors \( \tau \) and \( \tau \tau \) are quasi-inverse to each other.

**Proof.** We only prove that \( \theta_{Y} \) is an isomorphism for each \( Y \in C \). We may assume that \( Y \) is indecomposable and non-projective in \( C \). Let \( \alpha = \psi_{Y,\tau Y}(\text{Id}_{\tau Y}) \) in \( D \text{Ext}_{C}^{1}(\tau Y, \tau Y) \) and let \( \beta = \phi_{Y,Y}(\text{Id}_{Y}) \) in \( D \text{Ext}_{C}^{1}(Y, \tau Y) \). By the definition of \( \theta_{Y} \), we have \( \beta = \psi_{Y,Y}(\theta_{Y}) \). Consider the following commutative diagram 

\[
\begin{array}{ccc}
\text{Hom}_{C}(\tau Y, \tau Y) & \xrightarrow{\psi_{Y,Y}} & \text{Hom}_{C}(\tau \tau Y, \tau Y) \\
\downarrow \text{Hom}_{C}(\tau Y, \tau Y) & & \downarrow \text{Hom}_{C}(\tau Y, \tau Y) \\
\text{Hom}_{C}(\tau Y, \tau Y) & \xrightarrow{\psi_{Y,Y}} & \text{Hom}_{C}(\tau Y, \tau Y).
\end{array}
\]

Since \( \psi_{Y,Y}(\text{Id}_{\tau Y}) = \alpha \) and \( \text{Hom}_{C}(\tau Y, \theta_{Y})(\text{Id}_{\tau Y}) = \theta_{Y} \), we obtain 

\[
\beta = D \text{Ext}_{C}^{1}(\theta_{Y}, \tau Y)(\alpha) = \alpha \circ D \text{Ext}_{C}^{1}(\theta_{Y}, \tau Y).
\]

By Lemma 2.2(1), there exists an almost split conflation \( \eta: X \to Y \to Y \). By Lemma 2.1(1), we have a natural isomorphism \( \phi': \text{Hom}_{C}(\tau Y, \tau Y) \to D \text{Ext}_{C}^{1}(Y, \tau Y) \) such that \( \phi'_{X}(\text{Id}_{X})(\eta) \neq 0 \). Setting \( \beta' = \phi'_{X}(\text{Id}_{X}) \), we have \( \beta'(\eta) \neq 0 \). By Yoneda’s lemma, there exists some \( s: X \to \tau Y \) such that \( \text{Hom}_{C}(\tau, s) = \phi_{Y,Y}^{-1} \circ \phi' \). We obtain 

\[
\beta' = \phi'_{X}(\text{Id}_{X}) = (\phi_{Y,Y} \circ \text{Hom}_{C}(X, s))(\text{Id}_{X}) = \phi_{Y,Y}(\tau).
\]

Consider the following commutative diagram 

\[
\begin{array}{ccc}
\text{Hom}_{C}(\tau Y, \tau Y) & \xrightarrow{\psi_{Y,Y}} & \text{Hom}_{C}(\tau Y, \tau Y) \\
\downarrow \text{Hom}_{C}(\tau Y, \tau Y) & & \downarrow \text{Hom}_{C}(\tau Y, \tau Y) \\
\text{Hom}_{C}(X, \tau Y) & \xrightarrow{\phi_{Y,Y}} & \text{Hom}_{C}(X, \tau Y).
\end{array}
\]
Since $\phi_{Y,Y}(\text{id}_Y) = \beta$ and $\text{Hom}_C(s, \tau Y)(\text{id}_Y) = \xi$, we obtain

$$\beta' = D\text{Ext}^1_Y(Y, s)(\beta) = \beta \circ \text{Ext}^1_Y(Y, s) = \alpha \circ \text{Ext}^1_Y(\theta_Y, \tau Y) \circ \text{Ext}^1_Y(Y, s).$$

Therefore we have

$$0 \neq \beta'(\eta) = \alpha(\langle s, \eta \rangle \theta_Y) = \alpha(\langle s, \eta \theta_Y \rangle),$$

which implies that the conflation $\eta \theta_Y$ is non-split. It follows that $\theta_Y : \tau^{-1} Y \to Y$ is a retraction in $C$, since $\eta$ is almost split. Hence $\theta_Y$ is an isomorphism in $C$ since we have already known $\tau^{-1} Y \simeq Y$ in $C$. \hfill \Box

The following lemma shows that $(\tau^{-1}, \tau)$ forms an adjoint pair, with unit $\xi$ and counit $\theta$; see [9, Section IV.1].

**Lemma 3.5.** We have $\tau(\theta_Y) \circ \xi_Y = \text{id}_Y$ for each $Y \in C$, and $\tau^{-1} \circ \tau^{-1}(\xi_Y) = \text{id}_{\tau^{-1} Y}$ for each $X \in C$.

**Proof.** We only prove the first equality. We have the following commutative diagram:

$$
\begin{array}{c}
\text{Hom}_C(\tau Y, \tau^{-1} Y) & \phi_{\tau^{-1} Y, \tau Y} & D\text{Ext}^1_Y(\tau^{-1} Y, \tau Y) & \psi_{\tau Y, \tau^{-1} Y} & \text{Hom}_C(\tau^{-1} Y, \tau^{-1} Y) \\
\downarrow \text{Hom}_C(\tau Y, \tau^{-1} Y) & \downarrow \phi_Y & \downarrow D\text{Ext}^1_Y(Y, \tau Y) & \downarrow \psi_Y & \downarrow \text{Hom}_C(Y, \tau^{-1} Y).
\end{array}
$$

The left square commutes by the definition of $\tau(\theta_Y)$, and the right square commutes since $\psi_Y$ is natural. By the definitions of $\phi_Y$ and $\xi_Y$, we have

$$\phi_{\tau^{-1} Y, \tau Y}(\psi_{\tau Y, \tau^{-1} Y}(\theta_Y)) = \text{id}_Y,$$

and

$$\phi_{\tau^{-1} Y, \tau Y}(\phi_{\tau^{-1} Y, \tau Y}(\theta_Y)) = \xi_Y.$$

By the above commutative diagram, we have

$$\text{id}_Y = \text{Hom}_C(\tau Y, \tau(\theta_Y)) \circ \xi_Y = \tau(\theta_Y) \circ \xi_Y. \hfill \Box$$

**Definition 3.6.** Let $C$ be a Hom-finite Krull-Schmidt exact category. We call the sextuple obtained above

$$\{C, C_i, \phi, \psi, \tau, \tau^{-1}\}$$

the generalized Auslander-Reiten duality on $C$ and call the functors $\tau$ and $\tau^{-1}$ the generalized Auslander-Reiten translation functors.

We observe that the functor $\tau$ depends on $\phi$ and the functor $\tau^{-1}$ depends on $\psi$. Then $C$ has Auslander-Reiten duality in the sense of [8] if and only if $C_i = C = C_r$.

**Remark 3.7.** Let $(C, E)$ be a Frobenius category. Then the projectively stable category and the injectively stable category of $C$ are the same and have a natural triangulated structure. The generalized Auslander-Reiten duality on $C$ gives the generalized Serre duality on $C$ in the sense of [8]. More precisely, we have $(C)_r = C_r$ and $(C)_l = C_l$. Let $\Sigma$ be the translation functor of $C$. Then the functor $\tau\Sigma : C_r \to C_l$ gives the generalized Serre functor of $C$. 
4. The category of finite presented representations

From now on, we let \( k \) be a field and \( Q = (Q_0, Q_1) \) be a quiver. Here, \( Q_0 \) is the set of vertices and \( Q_1 \) is the set of arrows. Given an arrow \( \alpha: a \to b \), denote by \( s(\alpha) = a \) its source and by \( t(\alpha) = b \) its target. A path \( p \) of length \( l \geq 1 \) is a sequence of arrows \( \alpha_1 \cdots \alpha_l \) such that \( s(\alpha_i) = t(\alpha_i) \) for each \( i = 1, 2, \ldots, l - 1 \). We let \( s(p) = s(\alpha_1) \) and \( t(p) = t(\alpha_l) \). A left infinite path is an infinite sequence of arrows \( \alpha_1 \alpha_2 \cdots \alpha_n \cdots \) such that \( s(\alpha_i) = t(\alpha_{i+1}) \) for each \( i \geq 1 \). Dually, a right infinite path is an infinite sequence of arrows \( \cdots \alpha_n \cdots \alpha_1 \) such that \( s(\alpha_{i+1}) = t(\alpha_i) \) for each \( i \geq 1 \).

Recall that a quiver \( Q \) is locally finite if for each \( a \in Q_0 \), the set of arrows starting at \( a \) or ending at \( a \) is finite. A quiver \( Q \) is interval-finite if for any \( a, b \in Q_0 \), the set of paths \( p \) with \( s(p) = a \) and \( t(p) = b \) is finite. We will assume that \( Q \) is locally finite and interval-finite.

A representation \( M \) of \( Q \) is locally finite dimensional if \( M(a) \) is finite dimensional for each \( a \in Q_0 \), and is finite dimensional if moreover \( \bigoplus_{a \in Q_0} M(a) \) is finite dimensional. Denote by \( \text{rep}(Q) \) the category of locally finite dimensional representations, and by \( \text{rep}^b(Q) \) the full subcategory formed by finite dimensional representations. Then the Matlis duality induces a duality \( D: \text{rep}(Q) \to \text{rep}(Q^{op}) \), which sends \( \text{rep}^b(Q) \) into \( \text{rep}^b(Q^{op}) \). Here, \( Q^{op} \) is the opposite quiver of \( Q \).

Recall from [3] Section 2 that the path-category \( Q \) of \( Q \) has \( Q_0 \) as the set of objects; if \( x, y \in Q_0 \), the morphisms from \( x \) to \( y \) are the linear combinations of paths from \( x \) to \( y \). It is well known that \( \text{rep}(Q) \) is equivalent to the category of contravariant functors from \( Q \) to \( k \)-mod. A contravariant functor \( F: Q \to k\text{-mod} \) is finitely presented, if there exists an exact sequence of functors \( \bigoplus_i \text{Hom}_{k}(\cdot, x_i) \to \bigoplus_j \text{Hom}_{k}(\cdot, y_j) \to F \to 0 \), for finitely many \( x_i, y_j \in Q \). The exact sequence is called a presentation of \( F \). A representation of \( Q \) is finitely presented if the corresponding contravariant functor is finitely presented. Denote by \( \text{rep}^+(Q) \) the full subcategory of \( \text{rep}(Q) \) formed by the finitely presented representations. We note that \( \text{rep}^+(Q) \) is a Hom-finite hereditary abelian subcategory, which is closed under extensions in \( \text{rep}(Q) \); moreover, we have \( \text{rep}^b(Q) \subseteq \text{rep}^+(Q) \); see [3] Proposition 1.15.

Denote by \( A = kQ \) the path algebra of \( Q \). Let \( P_1 \xrightarrow{f} P_0 \to M \to 0 \) be a minimal presentation of \( M \). Then the cokernel of \( \text{Hom}_A(f, A) \) is called the transpose \( \text{Tr} M \) of \( M \). Here, \( \text{Tr} M \) has no nonzero projective direct summands. We observe that a morphism \( f: X \to Y \) in \( \text{rep}^+(Q) \) is projectively trivial if and only if it factors through a projective object, since \( \text{rep}^+(Q) \) has enough projectives. Then we obtain a duality \( \text{Tr}: \text{rep}^+(Q) \to \text{rep}^+(Q^{op}) \). Here, \( \text{rep}^+(Q) \) can be embedded in \( \text{rep}^+(Q) \) as a full subcategory, since \( \text{rep}^+(Q) \) is hereditary. Then we have a contravariant functor

\[
\text{Tr}: \text{rep}^+(Q) \to \text{rep}^+(Q^{op}).
\]

The following lemma is contained in the proof of [3] Theorem 2.8.

**Lemma 4.1.** Let \( L, M \in \text{rep}(Q) \).

1. If \( M \) lies in \( \text{rep}^+(Q) \), then there exists an isomorphism \( \text{Hom}(L, D \text{Tr} M) \cong D \text{Ext}^1(L, M) \), which is natural in \( L \) and \( M \).

2. If \( M \) lies in \( \text{rep}^b(Q) \), then there exists an isomorphism \( \text{Hom}(\text{Tr} DM, L) \cong D \text{Ext}^1(L, M) \), which is natural in \( L \) and \( M \). \( \square \)
The following lemma is due to [3] Theorem 2.8, Corollary 2.9 and Propositions 3.6.

**Lemma 4.2.** Let $M \in \text{rep}^+(Q)$ be an indecomposable representation.

1. If $M$ is non-projective, then there exists an almost split sequence $0 \to D\text{Tr}M \to E \to M \to 0$ in $\text{rep}(Q)$.
2. If $M$ is non-injective and lies in $\text{rep}^b(Q)$, then there exists an almost split sequence $0 \to M \to E \to \text{Tr}DM \to 0$ in $\text{rep}^+(Q)$.
3. Assume that $0 \to M \to E \to N \to 0$ is an exact sequence in $\text{rep}(Q)$. Then it is an almost split sequence in $\text{rep}^+(Q)$ if and only if it is an almost split sequence in $\text{rep}(Q)$ with $M \in \text{rep}^b(Q)$. □

**Lemma 4.3.** Let $f: X \to Y$ be an injectively trivial morphism in $\text{rep}^+(Q)$. If $Y$ has no nonzero injective direct summands and lies in $\text{rep}^b(Q)$, then we have $f = 0$.

**Proof.** We may assume that $Y$ is indecomposable. By Lemma 4.2(2), we have an almost split sequence $0 \to Y \to E \to \text{Tr}DY \to 0$ in $\text{rep}^+(Q)$. By Lemma 4.1(1), we have $\text{Hom}(X, D\text{Tr}DY) \simeq D\text{Ext}^1(\text{Tr}DY, X)$. By Lemmas 2.1(1), we have $\text{Hom}(X, Y) \simeq D\text{Ext}^1(\text{Tr}DY, X)$. Observe that $D\text{Tr}DY \simeq Y$. Then we have $\text{Hom}(X, Y) \simeq \text{Hom}(X, Y)$. Then the result follows. □

Given a collection $S$ of objects, denote by $\text{add} S$ the category of direct summands of finite direct sums of objects in $S$.

**Proposition 4.4.** Let $Q$ be a locally finite interval-finite quiver. Set $\mathcal{C} = \text{rep}^+(Q)$. Then we have

$$\mathcal{C}_r = \{X \in \mathcal{C}|\text{Tr}X \in \text{rep}^b(Q^{\text{op}})\}$$

and

$$\mathcal{C}_l = \text{add}\{X|X \in \text{rep}^b(Q) \text{ or } X \text{ is an injective object in } \mathcal{C}\}.$$ 

Moreover, the functors $D\text{Tr}$ and $\text{Tr}D$ induce the generalized Auslander-Reiten translation functors.

Denote by $(\mathcal{C}_r)_P$ the full subcategory of $\mathcal{C}_r$ formed by objects by objects without nonzero projective direct summands, and by $(\mathcal{C}_l)_I$ the full subcategory of $\mathcal{C}_l$ formed by objects without nonzero injective direct summands. We obtain the induced functors $D\text{Tr}: (\mathcal{C}_r)_P \to (\mathcal{C}_l)_I$ and $\text{Tr}D: (\mathcal{C}_l)_I \to (\mathcal{C}_r)_P$. Since $\mathcal{C}$ is hereditary and has enough projectives, the canonical functor $(\mathcal{C}_r)_P \to C_r$ is an equivalence. By Lemma 4.3 the canonical functor $(\mathcal{C}_l)_I \to \mathcal{C}_l$ is an equivalence. Then we have the induced functors $D\text{Tr}: \mathcal{C}_r \to \mathcal{C}_l$ and $\text{Tr}D: \mathcal{C}_l \to \mathcal{C}_r$.

**Proof.** We observe that $\text{Tr}P = 0$ for each projective object $P$. Then the first equality follows from Proposition 2.3(1) and Lemma 4.2(1) and (3). The second equality follows from Proposition 2.4(2) and Lemma 4.2(2) and (3).

Let $X \in \mathcal{C}$. By Lemmas 4.1(1) and 4.3, for each $Y \in (\mathcal{C}_r)_P$, we have an isomorphism $\text{Hom}(X, D\text{Tr}Y) \simeq D\text{Ext}^1(Y, X)$, which is natural in $X$ and $Y$ since $D\text{Tr}Y$ has no nonzero injective direct summands. Similarly, for each $Y \in (\mathcal{C}_l)_I$, we have $\text{Hom}(\text{Tr}DY, X) \simeq D\text{Ext}^1(X, Y)$, which is natural in $X$ and $Y$. Then the result follows from the construction of generalized Auslander-Reiten translation functors. □

Combining Proposition 4.4 and [3] Theorem 3.7, we have the following direct consequence.
Corollary 4.5. The category $\text{rep}^+(Q)$ has Auslander-Reiten duality if and only if $Q$ has neither left infinite path nor right infinite path, or else $Q$ is of the form
\[
\cdots \circ \longrightarrow \diamond \longrightarrow \circ \longrightarrow \cdots
\]
\hfill $\square$

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