Scaling of distributions of sums of positions for chaotic dynamics at band-splitting points

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Abstract – The stationary distributions of sums of positions of trajectories generated by the logistic map have been found to follow a basic renormalization group (RG) structure: a non-trivial fixed-point multi-scale distribution at the period-doubling onset of chaos and a Gaussian trivial fixed-point distribution for all chaotic attractors. Here we describe in detail the crossover distributions that can be generated at chaotic band-splitting points that mediate between the aforementioned fixed-point distributions. Self-affinity in the chaotic region imprints scaling features to the crossover distributions along the sequence of band-splitting points. The trajectories that give rise to these distributions are governed first by the sequential formation of phase-space gaps when, initially uniformly distributed, sets of trajectories evolve towards the chaotic band attractors. Subsequently, the summation of positions of trajectories already within the chaotic bands closes those gaps. The possible shapes of the resultant distributions depend crucially on the disposal of sets of early positions in the sums and the stoppage of the number of terms retained in them.

Introduction. – A few years ago \([1]\) a possible generalization of the central limit theorem (CLT) was put forward, as suitable for strongly correlated variables and that would have as its stationary distribution the so-called q-Gaussian function \([1]\). Subsequently, it was surmised that a fitting model system for the observation of this generalization would be the period-doubling accumulation point of the logistic map \([2]\). This development led to increased interest and discussion \([3–9]\) about whether sums of correlated deterministic variables at vanishing, or near vanishing, Lyapunov exponent \(\lambda\) give rise to a general type of non-Gaussian stationary distribution.

As it turned out \([3,5,6]\), the distributions resembling q-Gaussians at the period-doubling accumulation point require unusual, specific procedures to be obtained. The first one is to work with a small but positive Lyapunov exponent \(\lambda \gtrsim 0\). The second is to discard an initial tract of consecutive positions in the ensemble dynamics, the disposal of a “transient”, before evaluating the sum of the remaining positions. And the third is to stop the summation at a finite number of terms. When the transient set of terms is not discarded the resulting distribution would show an irregular, jagged, serrated shape, whereas if the summation continues towards a larger and larger total number of terms the distribution approaches a Gaussian shape. The q-Gaussian–like distributions were observed along a sequence of values of the map control parameter \(\mu\) that in latter studies \([6]\) were identified as those approximately obeying the Huberman-Rudnick scaling law \([10]\), the power law that relates distance in the control parameter space to Feigenbaum’s universal constant \(\delta\), or, equivalently, the number \(2^n, n = 0, 1, 2, \ldots\), of bands of the chaotic attractors.

Here we provide a thorough rationalization, backed by ample evidence, of the properties of sums of consecutive positions and their distributions for ensembles of trajectories associated with the sequence of chaotic \(2^n\)-band attractors of the logistic map. We add to previous
understanding [7–9] on the distributions of sums of positions at the period-doubling accumulation point for trajectories initiated within the attractor or with an ensemble of them uniformly distributed across the entire phase space (the domain of the map). In the former case [7,8] the support of the stationary distribution is the multifractal set that makes up the Feigenbaum attractor and its amplitude follows its multifractal nature. For the latter case [9] we demonstrated that the stationary distribution reached only when \( \mu = \mu_\infty \) at period doubling or at band-splitting points, see \( \mu = \mu_\infty = 1.401155189092 \ldots \). When \( \mu \) is shifted to values larger than \( \mu_\infty \), \( \Delta \mu \equiv \mu - \mu_\infty > 0 \), the attractors are (mostly) chaotic and consist of \( 2^n \) bands, \( n = 0, 1, 2, \ldots \), where \( 2^n \sim \Delta \mu^{-\kappa} \), \( \kappa = \ln 2/\ln \delta \), and \( \delta = 4.669201609102 \ldots \) is the universal constant that measures both the rate of convergence of the values of \( \mu = \mu_n \) to \( \mu_\infty \) at period doubling or at band-splitting points, see fig. 1(a). The Misiurewicz \( (M_n) \) points, are attractor merging crises, where multiple pieces of an attractor merge together at the position of an unstable periodic orbit [13]. The \( M_n \) points can be determined by evaluation of the trajectories with initial condition \( x_0 = 0 \) for different values of \( \mu \), as these orbits follow the edges of the chaotic bands until, at \( \mu = \mu_n \), the unstable orbit of period \( 2^n \) reaches the merging crises [13].

**Fig. 1:** (Colour on-line) (a) Attractor bands (in black) and gaps between them (white horizontal regions) in logarithmic scales, \(-\log(|\mu - \mu_\infty|)\) and \(-\log(|x|)\) in the horizontal and vertical axes, respectively. The band-splitting points \( M_n \) (circles) follow a straight line indicative of power-law scaling. The vertical white strips are periodic attractor windows. (b) Sequential gap formation for \( M_2 \) by an ensemble of trajectories with initial conditions uniformly distributed along the map phase space. Black dots represent absolute values of trajectory positions \(|x_t|\) at iteration time \( t \). See text.
Distributions of sums of chaotic positions

Trajectories initiated inside a $2^n$-band attractor consist of an interband periodic motion of period $2^n$ and an intraband chaotic motion. Trajectories initiated outside a $2^n$-band attractor exit progressively a family of sets of gaps formed in phase space between the $2^n$ bands. This family of sets of gaps starts with the largest gap formed around the first unstable orbit, or first repeller, of period $2^0$, followed by two gaps containing the second positions of the second repeller of period $2^1$, and so on, see fig. 1(b). The widths of the gaps diminish in a power-law fashion as their numbers $2^k$, $k = 0, 1, 2, \ldots$, for each set increase. We follow the dynamics towards the $M_n$, $n = 0, 1, 2, \ldots$, attractors by setting a uniformly distributed ensemble of initial conditions across phase space, $-1 \leq x_0 \leq 1$, and record the normalized number of bins $W_t$, in a fine partition of this interval, that still contain trajectories at iteration time $t$. The results are shown in fig. 2, where we observe an initial power-law decay in $W_t$ with logarithmic oscillations followed by a transition into a stable regime, a plateau with a fixed value of $W_t$, when (practically) all trajectories become contained and remain in the bands of the attractor.

The properties of $W_t$ show discrete scale invariance associated with powers of 2 characteristic of unimodal maps. The number of logarithmic oscillations in the regime when trajectories flow towards the attractor coincides with the number of consecutive sets of gaps that need to be formed at the $M_n$ points, whereas the final constant level of $W_t$ coincides with the total number of bins that comprise the total width of the $2^n$ bands of the attractors. We notice that these properties when observed along the plateau entry points labeled $t^n*$ shown in fig. 2 obey the Huberman-Rudnick scaling law since the times $t^n*$ are related to the $2^n$ bands of the $M_n$ points and these in turn are given by $\Delta t^n = \delta^{-n}$.

Sums of positions and their distributions at band-splitting points. – We consider now the sum of consecutive positions $x_t$ starting with an iteration time $t = N_s$ up to a final iteration time $t = N_s + N_f$ of a trajectory with initial condition $x_0$ and control parameter value $\mu$ fixed at an $M_n$ point, $n = 0, 1, 2, \ldots$, i.e.

$$X(x_0, N_s, N_f; \mu_n) \equiv \sum_{t=N_s}^{N_s+N_f} x_t. \quad (1)$$

We studied a collection of these sums for trajectories started from a uniform distribution of initial conditions in the entire interval $-1 \leq x_0 \leq 1$ with different values of $n$, $N_s$ and $N_f$, and we also evaluated their corresponding histograms and finally their distributions by centering and normalization of the histograms. Clearly, stationary distributions require $N_f \to \infty$ and, unless there is some unusual circumstance, they are not dependent on the value of $N_s$. We know [7,8] that for all chaotic attractors ($\Delta \mu > 0$) the stationary distribution is Gaussian and that in the
limit $\Delta \mu = 0$ the stationary distribution is of an exceptional kind with intricate multiscale features [4,9]. Here we explore other distributions that can be obtained when $N_s$ and $N_f$ are varied and identify the dynamical properties that give rise to them.

The observation of $q$-Gaussian–like distributions in refs. [3,6] involved a large value of discarded terms $N_s$ before sums similar to that in eq. (1) were evaluated. Also, it was found necessary to limit the number of summands to a finite number $N_f$ to prevent the distribution approach a Gaussian form. For example in ref. [3] a fixed value of $N_s = 2^{12}$ was reported to be used for sums evaluated at attractors with a number of bands $2^n$ with $n$ in the range 4 to 8. These sums were terminated, respectively, with values $N_f = 2^n$ with $n_f$ in the range from 9 to 17. In these studies the values of $\Delta \mu$ were not precisely fixed at band-splitting points as we do here but the dynamical properties we describe are equivalent. We can understand the effect of the values of $n$, $N_s$ and $N_f$ used in terms of the dynamics of trajectories from the knowledge gained in the previous section. In refs. [3,6], the starting times $t = N_s$ in the sums in eq. (1) satisfy the condition $t^*_n \ll N_s$. We can conclude with the assistance of fig. 2, that the terms discarded in those studies comprise the flow of trajectories towards the attractors plus a significant segment of dynamics within the chaotic bands, therefore all of the terms contained in the sums correspond to the dynamics within the chaotic bands.

As a representative example we show in fig. 3 the distributions $P(Y; N_s, N_f; \mu_n)$ for the sums in eq. (1), with $Y = X - \langle X \rangle$, and where $\langle X \rangle$ is the average of $X$ over $x_0$. In this figure $n = 5$ and $N_s = 2^8$, and $N_f$ takes the values $N_f = 2^5$, $2^7$, $2^{13}$ and $2^{17}$, respectively, in panels (a), (b), (c) and (d). In (a) the sum comprises only one visit to each band and the structure of the distribution is the result of one cycle intraband motion of the ensemble of trajectories. In (b) the sum contains already about $2^4 = 16$, band cycles, for which we obtain a distribution with $q$-Gaussian–like shape but sharp drops at the edges. In (c) the $q$-Gaussian–like shape is disappearing after 256 band cycles, while in (d), when there are 4096 band cycles, we observe already the stationary Gaussian form. The same distribution progression pattern shown in fig. 3 is observed at other $M_n$ points. Furthermore, the sums and their distributions for any value of $n$ can be reproduced by rescaling consistent with Huberman-Rudnick law. This is illustrated in fig. 4, where we show in panels (a), (b) and (c) the resemblance of the centered sums $X$ for the band merging points $M_3$, $M_4$ and $M_5$, respectively. In panel (d) we show the distributions $P$ for these sums without rescaling of the horizontal axis $Y$.

**Summary and discussion.** – We have shown that there is an ample variety of distributions $P(Y; N_s, N_f; \mu_n)$ associated with the family of sums of iterated positions, as in eq. (1), obtained from an ensemble of trajectories started from a uniform distribution of initial conditions in the interval $-1 \leq x_0 \leq 1$. The shapes of these distributions vary with $N_s$ and $N_f$ but there is scaling property with respect to $n$. All the types of distributions obtained can be understood from the knowledge of the dynamics that these trajectories follow, both when flowing towards the chaotic-band attractors and when already within these attractors. There exists throughout the family of chaotic band attractors with $\lambda > 0$ an underlying scaling property, displayed, e.g., by the self-affine structure in fig. 1(a). This scaling property is present all over, here highlighted by: i) The sequential formation of gaps shown in fig. 1(b). ii) The number of bins $W_t$ still containing trajectories at iteration time $t$, shown in fig. 2, both for its initial decay with logarithmic oscillations and the final constant regime. And iii) the different classes of sums and their distributions obtained for a given value of $n$ are reproduced for other values of $n$ under appropriate rescaling, as
shown in fig. 4. For adeptness and precision purposes we chose here to study the family of Misjurewicz points $M_n$ but similar, equivalent, results are obtained for chaotic attractors between these points.

The discussion about the types of distributions $P(Y; N_s, N_f; \mu_0)$ is assisted by recalling [7,8] the RG framework associated with summation of positions. Positions $x_t$ for trajectories within chaotic-band attractors can be decomposed as $x_t = \mathbf{x}_t + \delta x_t$, where $\mathbf{x}_t$ is chosen to be (for example) fixed at the center of the band visited at time $t$ and $\delta x_t$ is the distance of $x_t$ from $\mathbf{x}_t$. When the number of bands $2^n$ is large all the values of $\delta x_t$ are small. The sum in eq. (1) can be written as

$$X \equiv \mathbf{X} + \delta X, \quad \mathbf{X} = \sum_{t=N_s}^{N_f} \mathbf{x}_t, \quad \delta X = \sum_{t=N_s}^{N_f} \delta x_t,$$

(2)

where $\mathbf{X}$ captures the interband periodic (and therefore correlated) motion and $\delta X$ consists of the intraband chaotic (and therefore random) motion. As discussed in refs. [7,8] the action of the RG transformation, summation, is driven by $\delta x_t$ and results in gradual widening of all the chaotic bands, such that eventually for a sufficiently large number of summands all of them merge into a single band. When $0 \leq N_s \leq t^*_N$ gap formation competes with band widening, while when $t^*_N \leq N_s$ band widening develops unimpeded. When $0 \leq N_s \leq t^*_N$ the combined processes of the dynamical evolution of the ensemble of trajectories and the repeated RG transformation is dominated initially by gap formation but it is always followed by gap merging. Initially, the distributions for these sums resemble the jagged multiscale shape of the stationary distribution for the nontrivial fixed point at $\Delta \mu = 0$ but they necessarily evolve towards the Gaussian distribution of the trivial fixed point present for $\Delta \mu > 0$ [7,8]. When $t^*_N \ll N_s$, as in refs. [2,3,5,6], the trajectory positions considered in the sums are all contained within the attractor bands and from the first term $t = N_f$ the gaps begin to close due to the action of $\delta x_t$ that is akin to an independent random variable. As we have shown in fig. 3, when the number of summands grows the shape of the distribution evolves by first eliminating the initial serrated patterns, developing a symmetrical shape that shows possible long tails but that end in a sharp drop (the claimed $q$-Gaussian type), and finally the approach to the Gaussian stationary distribution. All of the above can be observed for each $2^n$-band chaotic attractor, basically from $n \geq 1$, and when a self-affine family of these attractors is chosen, like the Misjurewicz points $M_n$ the sums and their distributions can be rescaled such that they just about match for all $n$, as shown in fig. 4, where the sums where started at $N_s \approx t^*_N$.

Concisely, the elimination of a large enough set of early positions in the sums for a given $n$, such that the location of its first term $N_s$ is located inside the plateau of $W_t$ in fig. 2, ensures that the sums capture only the dynamics within the $2^n$-band attractor. Therefore the shape of the distributions are dominated by the uncorrelated chaotic contributions $\delta x_t$, that as $t$ increases evolves towards the final Gaussian shape. A non-Gaussian distribution can only be obtained if there is a finite number of summands $N_f$. Self-affinity in the chaotic-band family of attractors, provides scaling properties to the distributions of sums of positions that are described by an appropriate use of the Huberman-Rudnick power-law expression.

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