Abstract

Stability of the local gamma factor in the unitary case
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In [RS05], Rallis and Soudry prove the stability under twists by highly ramified characters of the local gamma factor arising from the doubling method, in the case of a symplectic group or orthogonal group $G$ over a local non-archimedean field $F$ of characteristic zero, and a representation $\pi$ of $G$, which is not necessarily generic. This paper extends their arguments to show the stability in the case when $G$ is a unitary group over a quadratic extension $E$ of $F$, thereby completing the proof of the stability for classical groups. This stability property is important in Cogdell, Piatetski-Shapiro, and Shahidi’s use of the converse theorem to prove the existence of a weak lift from automorphic, cuspidal, generic representations of $G(\mathbb{A})$ to automorphic representations of $GL_n(\mathbb{A})$ for appropriate $n$, to which references are given in [RS05].
Stability of the local gamma factor in the unitary case.

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1 Introduction

Let $G$ be either a symplectic or orthogonal group over a local non-archimedean field $F$ of characteristic zero, or a unitary group over a quadratic extension $E$ of $F$. We consider the local gamma factor, associated to an irreducible admissible representation $\pi$ of $G$, by the doubling method of Piatetski-Shapiro and Rallis ([GPSR87], [LR05]). Denote the local gamma factor by $\gamma(\pi, \chi, s, \psi)$, where $\chi$ is a character of $F^*$, and $\psi$ is a fixed non-trivial character $F$. In this paper we treat the last open case of the following result (cf. Theorem 1 in [RS05]).

**Theorem 1.1.** The local gamma factor $\gamma(\pi, \chi, s, \psi)$ is stable, for $\chi$ sufficiently ramified. This means that for two given irreducible admissible representations $\pi_1, \pi_2$ of $G$, there exists an integer $N > 0$, such that

$$\gamma(\pi_1, \chi, s, \psi) = \gamma(\pi_2, \chi, s, \psi),$$

for all characters $\chi$, with conductor having an exponent larger than $N$.

Rallis and Soudry, in [RS05], prove Theorem 1.1 in the symplectic and orthogonal cases, and this paper completes the proof of Theorem 1.1 by extending their arguments to the unitary case.

The stability property of the local gamma factor, under highly ramified twists, is well known for $GL_n \times GL_m$. It was proved by Jacquet and Shalika. For generic representation of split classical groups, it is known thanks to the works of Cogdell, Piatetski-Shapiro and Shahidi. The stability property is a key ingredient in the proof, by the converse theorem, of the existence of a weak lift from automorphic, cuspidal, generic representations of $G(\mathbb{A})$ ($G$ a split classical group) to automorphic representations of $GL_n(\mathbb{A})$ (appropriate $n$), where $\mathbb{A}$ is the adele ring of a given number field. See [RS05] for precise references to the literature. In Theorem 1.1 $\pi$ is any irreducible representation of $G$; even when $G$ is quasi-split, $\pi$ is not necessarily generic. The proof of the stability in this paper follows the argument of [RS05] closely. Therefore, experts in the subject will want to turn to the parts that are new and specific to the unitary case; these are Lemmas 4.9 and 4.10 and some of the details of the calculations in 4.2, the standard material contained in §5.1 through 5.4 and, finally, Proposition 5.8. The reader who is familiar with [RS05] is advised to turn to Proposition 5.8 first, since this elementary, but apparently new, observation is the heart of the matter concerning the extension of the arguments of [RS05] to the unitary case.

Recall that in the local theory of the doubling method, we consider the integrals

$$Z(v_1, \hat{v}_2, f_{\chi, s}) = \int_G \langle \pi(g)v_1, \hat{v}_2 \rangle f_{\chi, s}(i(g, 1)) \, dg. \tag{1.1}$$

Here $v_1$ lies in $V_\pi$—a space for $\pi$, and $\hat{v}_2$ lies in the smooth dual of $V_\pi$, $\hat{V}_\pi$ (affording the contragredient representation $\hat{\pi}$). Thus, $g \mapsto \langle \pi(g)v_1, \hat{v}_2 \rangle$ is a matrix coefficient of $\pi$: $f_{\chi, s}$ is a holomorphic section in an induced representation of the split “doubled” group $H$—induced from the Siegel parabolic subgroup $P$ of $H$, and a character, which is of the form $\chi(\det \cdot)|\det \cdot|^{s-1/2}$. Finally, there is an embedding $i : G \times G \to H$, such that $P \cdot i(G \times G) = P \cdot i(G \times 1)$ is an open and dense subset in $H$. The integrals (1.1) converge absolutely in a right-half-plane and continue meromorphically to the whole plane, being rational functions in $q^{-s}$, where $q$ is the number of elements in the residue field of $F$. The functions $Z(v_1, \hat{v}_2, f_{\chi, s})$ satisfy a functional equation

$$\Gamma(\pi, \chi, s)Z(v_1, \hat{v}_2, f_{\chi, s}) = Z(v_1, \hat{v}_2, M(\chi, s)f_{\chi, s}),$$

where $M(\chi, s)$ is the intertwining operator associated to the element $w = i(1, -1)$. The proportionality factor $\Gamma(\pi, \chi, s)$ is a rational function of $q^{-s}$ which depends only on $\pi$ and $\chi$. Note that $\Gamma(\pi, \chi, s)$ is independent of $\psi$. 


The local gamma factor \( \gamma(\pi, \chi, s, \psi) \) is obtained from \( \omega_{\pi}(-1)\Gamma(\pi, \chi, s) \) (where \( \omega_{\pi} \) is the central character of \( \pi \)) by multiplication by a factor which depends on \( \chi, \psi \) (and \( G \)) and not on \( \pi \). See [RS05], pp. 292–3 for the details. Therefore, Theorem 1.1 will follow from

**Theorem 1.2.** Let \( \pi \) be an irreducible admissible representation of \( G \). Then \( \omega_{\pi}(-1)\Gamma(\pi, \chi, s) \) is stable, for sufficiently ramified \( \chi \). More precisely, there is a positive integer \( N \), such that for all ramified characters \( \chi \) of \( F^* \), with conductor having exponent larger than \( N \), we have

\[
\omega_{\pi}(-1)\Gamma(\pi, \chi, s) = M(\chi, s) f_{\chi, s}(i(-1, 1))
\]

for certain choice of \( f_{\chi, s} \).

A more precise form of this stability is given in Theorem 3.1.

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## 2 Notation and Preliminaries

As far as possible, we keep the notation consistent with [RS05]. Let \( E \) be any local non-archimedean field, of characteristic zero. We denote by \( \mathcal{O}_E \) its ring of integers, and by \( \mathfrak{p}_E \) the prime ideal of \( \mathcal{O}_E \). We assume that the residue field \( \mathcal{O}_E/\mathfrak{p}_E \) has \( q_E \) elements. We denote by \( | \cdot |_E \) the absolute value on \( E \), such that \( |\varpi_E|_E = q_E^{-1} \), for any generator \( \varpi_E \) of \( \mathfrak{p}_E \).

Now let \( E \) be a local non-archimedean field, of characteristic zero with an involution \( \theta \). In certain situations it will be more convenient to denote \( \theta \) by conjugation, so, as a matter of notation, we set

\[
\tau = \theta(e), \text{ for all } e \in E.
\]

Let \( F \) be the fixed field of \( \theta \). Since \( F \) is again a local non-archimedean field of characteristic zero, all of the above notation again applies to \( F \). Further, we may write

\[
E = F \oplus F\omega, \text{ as a vector space, where } \omega \in E - F, \omega^2 = a \in F - \{0\}.
\]

See §5.1 for a proof. Note that we have

\[
\theta(\omega) := \overline{\omega} = -\omega,
\]

and the relation completely determines the involution \( \theta \) of \( E \).

Let \( V \) be a pair \((V, b)\) consisting of an \( m \)-dimensional vector space \( V \) over \( E \) and a sesqui-linear form \( b \) on \( V \) such that

\[
\theta(b(v, u)) = b(u, v) \text{ for all } u, v \in V.
\]

We will also assume that \( b \) is non-degenerate.

Unless otherwise mentioned we will always denote by \( G \) the group of isometries \( \text{Isom}(V) \) of \( V \), considered as an algebraic group over \( F \). A concrete way of doing this is via the “restriction of
scalars” construction. That is we consider $V$ to be a $2m$-dimensional vector space over $F$, and then considering $G$ to be the $F$-linear transformations of $V$ satisfying an additional set of conditions corresponding to $E$-linearity and unitarity with respect to $\theta$. This point of view will be developed in greater detail when we need it, in the proof of Lemma 4.2 below.

It will be convenient to fix a basis $B$ of $V$ as follows. We fix an orthogonal $E$-basis of $V$, $B = \{v_1, \ldots, v_m\}$, such that

\begin{align}
|b(v_1, v_1)| = \cdots = |b(v_k, v_k)| &= q, \\
|b(v_{k+1}, v_{k+1})| = \cdots = |b(v_m, v_m)| &= 1.
\end{align}

The choice of an orthogonal $E$-basis for $V$ satisfying (2.3) is possible by Théorème IX.6.1.1 of Bourbaki Algèbre, [Bou59].

For $x \in \text{Mat}_n(E)$, let $x^* = t^\theta(x)$, where the superscripted $t$ on the left indicates the usual transpose of the matrix, and $\theta(x)$ denotes the “conjugation” operation $\theta$ applied entry-wise to $x$. We set

\begin{align}
T &= \text{diag}(b(v_1, v_1), \ldots, b(v_m, v_m)).
\end{align}

Note that

\begin{align}
T &= T^* \quad \text{and} \quad T^{-1} = (T^{-1})^*,
\end{align}

since $T$ is diagonal with entries in $F$.

Using the basis $B$, we write $G = U_m(E)$ as a matrix group. We have an isomorphism of $G$ with the group

\begin{align}
U_m(T) &= \{g \in \text{GL}_m(E) \mid g^*Tg = T\} \equiv G
\end{align}

We will write the Lie algebra $\mathfrak{g}$ of $G$ in the matrix form

\begin{align}
\mathfrak{g} &\cong \mathfrak{u}_m(F) := \{x \in M_m(E) \mid x^*T + Tx = 0\}.
\end{align}

All representations $\pi$ of $G$, considered here, are assumed to be admissible. We denote by $V_\pi$ a vector space realization of $\pi$, and, if it has a central character, we denote it by $\omega_\pi$. Note that the center $Z(G)$ of $G$ is isomorphic to $U_1(E)$, the elements of norm 1 in $E$, and $Z(G)$’s isomorphic image in $U_m(T)$ is $U_1(E)I_m$.

## 3 The doubling method

Consider $V \times V$, consisting of the doubled space $V \times V$ equipped with the bilinear form $b^* = b \oplus (-b)$. Denote by $H$ the isometry group of $(V \times V, b^*)$. Since the subspace
\[ V^\Delta = \{(v,v) \mid v \in V\} \]

is an \( m \)-dimensional isotropic subspace of \( V \times V \), hence a maximal isotropic subspace \( V \times V \), the group \( H \) is quasi-split. The elements \( (g_1, g_2) \) of \( G \times G \) act on \( V \times V \) by

\[ (g_1, g_2)(v_1, v_2) = (g_1(v_1), g_2(v_2)), \]

and they clearly preserve \( b^* \). Thus we get a natural embedding \( i : G \times G \hookrightarrow H \). Consider the maximal parabolic subgroup \( P_{V^\Delta} \) of \( H \) which preserves \( V^\Delta \). This is a Siegel type parabolic subgroup of \( H \). Its Levi part is isomorphic to \( \text{GL}(V^\Delta) \cong \text{GL}(V) \). Denote the unipotent radical of \( P_{V^\Delta} \) by \( U_{V^\Delta} \). We have the “transversality”.

\[ \text{(3.1)} \quad i(G \times G) \cap P_{V^\Delta} = i(G^\Delta), \]

where

\[ G^\Delta = \{(g, g) \mid g \in G\}. \]

Recall that \( P_{V^\Delta} \setminus H/i(G \times G) \) is finite and contains only one open orbit, which is \( P_{V^\Delta} \cdot i(G \times G) = P_{V^\Delta} \cdot (G \times 1) \). This equality follows from [GPSR87], as in [GPSR87], p. 8. Denote by \( \text{det}(\cdot) \) the algebraic character of \( P_{V^\Delta} \) given by \( P \mapsto \text{det}(P|_{V^\Delta}) \). Let \( \chi \) be a (unitary) character of \( E^* \). Consider, for \( s \in \mathbb{C} \),

\[ \rho_{\chi,s} = \text{Ind}^H_{P_{V^\Delta}} (\chi \circ \text{det}\cdot)|_{\text{det}\cdot} |^{s-\frac{1}{2}}. \]

The induction is normalized as in §3 of [RS05].

Let \( \pi \) be an irreducible representation of \( G \), acting in a space \( V_\pi \). Consider the contragredient representation \( \hat{\pi} \) acting in \( \hat{V}_\pi \), the smooth dual of \( V_\pi \). Denote by \( \langle \cdot, \cdot \rangle \) the canonical \( G \)-invariant bilinear form on \( V_\pi \times \hat{V}_\pi \). Let \( v_1 \in V_\pi \) and \( \hat{v}_2 \in \hat{V}_\pi \), and let \( f_{\chi,s} \in V_{\rho_{\chi,s}} \) be a holomorphic section. The local zeta integrals attached to \( \pi \) by the doubling method are

\[ \text{(3.2)} \quad Z(v_1, \hat{v}_2, f_{\chi,s}) = \int_G \langle \pi(g)v_1, \hat{v}_2 \rangle f_{\chi,s}(i(g,1)) \, dg, \]

By Theorem 3 in [LR03], the integral in (3.2) converges absolutely in a right-half plane and continues to a meromorphic function in the whole plane. This function is rational in \( q^{-s} \). We keep denoting the analytic continuation by \( Z(v_1, \hat{v}_2, f_{\chi,s}) \). Consider the intertwining operator

\[ M(\chi, s) = M(s) : \rho_{\chi,s} \rightarrow \rho_{\theta(\chi)^{-1}, 1-s}, \]

defined, first for \( \text{Re}(s) \gg 0 \), as an absolutely convergent integral

\[ \text{(3.3)} \quad M(\chi, s)f_{\chi,s}(h) = \int_{U_{V^\Delta}} f_{\chi,s}(wu) \, du, \]
and then, by meromorphic continuation, to the whole plane. Here, we take, as in \cite{LR05}, \( w = i(1, -1) \in i(G \times G) \). Note that

\[
w(V^\Delta) = V^{-\Delta} = \{(v, -v) \mid v \in V\}.
\]

The subspace \( V^{-\Delta} \) is a maximal isotropic subspace of \( V \times V \) which is transversal to \( V^\Delta \), i.e., \( V^\Delta \cap V^{-\Delta} = \{0\} \). By Theorem 3 in \cite{LR05}, we have a functional equation (as an identity of meromorphic functions in the whole plane)

\[
(3.4) \quad \Gamma(\pi, \chi, s) Z(v_1, \hat{v}_2, f_{\chi,s}) = Z(v_1, \hat{v}_2, M(\chi, s)f_{\chi,s})
\]

for all \( v_1 \in V_\pi, \hat{v}_2 \in \hat{V}_\pi, f_{\chi,s} \in V_{\rho,s} \) (holomorphic section). The function \( \Gamma(\pi, \chi, s) \) depends on the choice of measure \( d\mu \) made in the definition of \( M(\chi, s) \).

Section 9 of \cite{LR05} explains how to obtain the local gamma factor \( \gamma(\pi, \chi, s, \psi) \) from \( \Gamma(\pi, \chi, s) \). In the Hermitian case, the relation between the two is given in (25), \cite{LR05} as

\[
(3.5) \quad \gamma(\pi, \chi, s, \psi) = \frac{\xi_G(\chi, A)}{C_H(\chi, s, A, \psi)\omega(\pi)(-1)\Gamma(\pi, \chi, s)},
\]

where \( C_H(\chi, s, A, \psi) \) is a certain rational function of \( q^{-s} \), which depends only on \( \chi, \psi \), a certain matrix \( A \), and \( H \)—see §5 of \cite{LR05} for the exact definition—and where \( \xi_G(\chi, A) = \chi^{-1}(\det A)|\det A|^{-s/2} \). It is shown in \cite{LR05}, §§8–9, that \( \gamma(\pi, \chi, s, \psi) \) is independent of the choice of \( A \).

Since by (3.5), \( \gamma(\pi, \chi, s, \psi) \) is obtained from \( \omega(\pi)(-1)\Gamma(\pi, \chi, s) \) by multiplication by a factor which depends only on \( \chi, \psi \) (and \( G \)), and not on \( \pi \), Theorem 1.1 will follow from the explicit formula for \( \omega(\pi)(-1)\Gamma(\pi, \chi, s) \) in Theorem 3.1, valid for \( \chi \) sufficiently ramified, which evidently does not depend on \( \pi \).

**Theorem 3.1.** Let \( \pi \) be an irreducible representation of \( G \). Then there exists a positive integer \( N \), such that for any ramified character \( \chi \) of \( E^\times \), having conductor \( 1 + \mathfrak{P}_E^{N_\chi} \) with \( N_\chi > N \), we have

\[
\omega(\pi)(-1)\Gamma(\pi, \chi, s) = |2|E^2_\chi^{-m}(\pi)(v) \times \int_{\theta(\mathfrak{P}_E^{n_\chi})} \chi^{-1}(\det(I_m - v))d\mu(v),
\]

where

\[
n_\chi := \left[ \frac{N_\chi + 1}{2} \right].
\]

The measure \( d\mu(v) \) is to be specified below, in \cite{LR05}.

We end this section with an explicit description of \( i(g, 1) \), \( g \in G \) as a matrix, following the decomposition

\[
(3.6) \quad V \times V = V^\Delta \oplus V^{-\Delta}
\]
and a choice of a standard basis of $V \times V$ whose Gram matrix, with respect to $b^*$ is $w_{2m}$. Here, we are using the notation

$$w_n = \begin{pmatrix} 1 \\ \cdot \\ 1 \end{pmatrix},$$

for $n \in \mathbb{N}$, as in [RS05]. Let $T$ be the diagonal matrix representing $b$, as defined in (2.4). In order to obtain a standard basis of $V \times V$, in the sense of a basis consistent with the decomposition (3.6) whose Gram matrix (with respect to $b^*$) is $w_{2m}$, we proceed as follows. We let $u_i = \frac{1}{b(v_i,v_i)} v_i$ for $i = 1, \ldots, m$. Then

$$(3.7) \quad \tilde{B} = \left\{ (v_1, v_1), \ldots, (v_m, v_m), \frac{1}{2}(u_m, -u_m), \ldots, \frac{1}{2}(u_1, -u_1) \right\}$$

is a standard basis of $V \times V$. Writing the elements of $G$ as matrices, with respect to $B$, and the elements of $H$ as matrices, with respect to $\tilde{B}$, it is now easy to verify Lemma 3.2.

**Lemma 3.2.** We have, for all $g \in G$,

$$(3.8) \quad i(g, 1) = \begin{pmatrix} \frac{1}{2}(g + I_m) \\ w_m T(g - I_m) \\ \frac{1}{2} w_m T(g + I_m) T^{-1} w_m \end{pmatrix}$$

and

$$(3.9) \quad w = i(1, -1) = \begin{pmatrix} 1 \\ 2w_m T \end{pmatrix} T^{-1} w_m$$

4 Proof of Theorem 3.1

Choose $f_{\chi,s}$ so that it is supported in the open orbit $P_{V \triangle} \cdot i(G \times G) = P_{V \triangle} \cdot i(G \times 1)$, and so that the restriction $f_{\chi,s}|_{i(G \times 1)}$, thought of as a function of $G$, is the characteristic function $\phi_U$ of a small compact open subgroup $U$ of $G$. We assume that $U$ is small enough, so that $V^U \neq 0$. Let $0 \neq v_1 \in V^U$, and choose $\hat{v}_2 \in \hat{V}^\pi$, such that $\langle v_1, \hat{v}_2 \rangle = 1$. Then the integral (3.2) converges for all $s$, and is easily seen to be

$$Z(v_1, \hat{v}_2, f_{\chi,s}) = \int_U \langle \pi(u) v_1, \hat{v}_2 \rangle \, du = m(U) \langle v_1, \hat{v}_2 \rangle = m(U),$$

where $m(U)$ is the measure of $U$, and so, from the functional equation (3.4),

$$\Gamma(\pi, \chi, s) = \frac{1}{m(U)} \int_G \langle \pi(g) v_1, \hat{v}_2 \rangle M(\chi, s) f_{\chi,s}(i(g, 1)) \, dg,$$

for $\text{Re}(s) \ll 0$. Our next task is to compute $M(\chi, s) f_{\chi,s}(i(g, 1))$ for our choice of $f_{\chi,s}$.

From now till (4.62), we assume that $\text{Re}(s) \gg 0$, so that the expression for $M(\chi, s) f_{\chi,s}$ given in (3.2) is valid.
Lemma 4.1. For the above choice of $f_{\chi,s}$, we have, for $\Re(s) \gg 0$,

\begin{align}
(4.3) \quad M(\chi, s)f_{\chi, s}(i(g, 1)) &= |2|^{m(1-2s)} \chi^{-m}(-2) \times \int_G \chi(\det(I_m + h)) |\det(I_m + h)|_E^{s + \frac{m}{2} - \frac{1}{2}} \phi_U(-hg) \, dh.
\end{align}

Proof. Let $\Re(s) \gg 0$, so that the integral in (3.3) converges absolutely. Using (3.8), we have

\begin{align}
(4.4) \quad M(\chi, s)f_{\chi, s}(i(g, 1)) &= \int_{(w_m x)^* = -(w_m x)} f_{\chi, s} \left( w \begin{pmatrix} I_m & x \\ I_m & I_m \end{pmatrix} i(g, 1) \right) \, dx \\
&= \int_{(w_m x)^* = -(w_m x)} f_{\chi, s} \left( \begin{pmatrix} 0 & \frac{1}{2}T^{-1}w_m \\ 2w_m T & 2w_m Tx \end{pmatrix} i(g, 1) \right) \, dx
\end{align}

We choose $dx$ to be the standard measure of matrices which are “skew-hermitian with respect to the second diagonal”, i.e.,

\begin{align}
(4.5) \quad dx = \prod_{i+j \leq m+1} dx_{ij} \prod_{i=1}^m dx_{ij}^{(\omega)}
\end{align}

where $dx_{ij}$ is the Haar measure of $E$ which assigns the measure 1 to $O_E$, and $dx_{ij}^{(\omega)}$ is the $F$-invariant measure of $F\omega$ which assigns the measure 1 to $O_{F\omega}$.

By the choice of $f_{\chi,s}$, we must have, in order for $x$ to make a nonzero contribution to the integral,

\begin{align}
\left( \begin{array}{cc} I_m & x \\ I_m & I_m \end{array} \right) i(g, 1) \in P_{V^\triangle} \cdot i(G \times 1),
\end{align}

i.e.,

\begin{align}
(4.6) \quad \left( \begin{array}{cc} I_m & x \\ I_m & I_m \end{array} \right) \in P_{V^\triangle} \cdot i(G \times 1).
\end{align}

Explicating (4.6), we must solve, for a given $x \in \text{Mat}_m(E)$, such that $(w_m x)^* = -w_m x$,

\begin{align}
(4.7) \quad \left( \begin{array}{cc} 0 & \frac{1}{2}T^{-1}w_m \\ 2w_m T & 2w_m Tx \end{array} \right) &= \left( \begin{array}{cc} E & Y \\ 0 & E^\dagger \end{array} \right) \left( \begin{array}{cc} \frac{1}{2}(h + I_m) & \frac{1}{2}(h - I_m)T^{-1}w_m \\ w_m T(h - I_m) & w_m T(h + I_m)T^{-1}w_m \end{array} \right).
\end{align}

Here $E^\dagger = w_m (E^*)^{-1}w_m$ and $Y$ is such that

\begin{align}
\left( \begin{array}{cc} E & Y \\ 0 & E^\dagger \end{array} \right) \in P_{V^\triangle}.
\end{align}
The condition (4.7) results from (4.6) by substituting the expressions of (4.4) and (3.8).

The system (4.7) has a solution, if and only if

\[(4.8) \ det(I_m - 2xw_mT) \neq 0,\]

in which case, we get

\[(4.9) \ h = \frac{2xw_mT + I_m}{2xw_mT - I_m} = \frac{2xw_mT + I_m}{-c(T^{-1})(I_m + (2xw_mT))^*},\]

\[E = -(2xw_mT + I_m)^{-1}.\]

Note that

\[(4.10) \ det(2xw_mT + I_m) = (-1)^m(\det(-c(T^{-1})(I_m + (2xw_mT))^*)).\]

By (4.9) and (4.10), we have

\[\det h = (-1)^m \rho, \text{ where } \rho = \det(-(I_m + (2xw_mT))^*).\]

Thus, \(\det h(\det h) = 1\), which is to say that \(h \in \det^{-1}(U_1(E))\), and in particular \(h\) is in \(\text{GL}_m(E)\).

Consider the Cayley transform

\[c : \mathfrak{gl}_m(E)' \rightarrow \text{GL}_m'(E),\]

given by

\[c(y) = \frac{I_m + y}{I_m - y},\]

where

\[(4.11) \ \mathfrak{gl}_m(E)' = \{y \in \mathfrak{g} | \ det(I_m + y)(I_m - y) \neq 0\}, \quad \text{GL}_m(E)' = \{t \in G | \ det(t + I_m) \neq 0\}.

Now observe that \(c\) is a bijection from \(\mathfrak{gl}_m(E)\)' to its image \(\text{GL}_m'(E)\). This is most easily proved by the following method. We write down a “formal” inverse for \(c\),

\[c^{-1}(t) = \frac{t - I_m}{t + I_m},\]

meaning that the left- and right-compositions of \(c^{-1}\) with \(c\) are the identity mappings (formally). Then it follows that \(c^{-1}\) is an actual inverse to \(c\) on the largest “natural domain” for \(c\) (i.e. set excluding points where \(c\) fails to be defined for obvious reasons) intersected with the inverse image under \(c\) of the largest natural domain of \(c^{-1}\). It is easy to calculate that these natural domains and their inverse images are as described in (4.11).
The restriction of $c$ from $\mathfrak{gl}(E)$ to $\mathfrak{g}' := \mathfrak{u}(E) \cap \mathfrak{gl}(E)$ is a bijection onto the image $U'_m(E) := U_m(E) \cap \text{GL}_m(E)'$. Note that in the case of the restriction to the Lie algebra of the unitary group, we can drop explicit mention of the requirement that $\det(I_m + y) \neq 0$, since $\det(I_m + y)$ is merely the conjugate $\det(I_m - y)$, and the requirement that the latter is nonzero implies that the former is as well.

Now, (4.9) means that

$$h = -c(2xw_mT).$$

It is easy to show that

$$\left(w_mx\right)^* = -(w_mx) \text{ if and only if } 2xw_mT \in \mathfrak{g}.$$ 

Here the Lie algebra $\mathfrak{g} = \mathfrak{u}(F)$ is written in matrix form as in (2.7). From (4.13), (4.10), and (4.12), we get, for $\text{Re}(s) \gg 0$,

$$M(\chi, s)f(\chi, s)(i(g, 1))$$

$$= |2|^{-m^2} F(-1)^m \times$$

$$\int \chi^{-1}(|\det(I_m - xw_mT)|^s |\det(I_m - xw_mT)|^{-s-m/2+1/2} \phi_U(-c(xw_mT)g) d\mu(y).$$

Letting $y = xw_mT$, we see by (4.13) that the domain of integration in (4.14) in the variable $y$ is $\mathfrak{g}'$. Denote by

$$d\mu(y)$$

the measure $d\mu(y) = dx$, where $dx$ is as in (4.5).

Then

$$M(\chi, s)f(\chi, s)(i(g, 1))$$

$$= |2|^{-m^2} F(-1)^m \times$$

$$\int \chi^{-1}(|\det(I_m - y)|^s |\det(I_m - y)|^{-s-m/2+1/2} \phi_U(-c(y)g) d\mu(y).$$

Now, in (4.16), we want to make the change of variable

$$c(y) = h.$$ 

Lemma 4.2. Let $f$ be a function in $C^\infty_c(G)$ such that the function

$$y \mapsto |f(c(y))||\det(I_m - y)|^{-m},$$

defined on $\mathfrak{g}' = \mathfrak{u}'$, and extended by 0 to $\mathfrak{g}$, is integrable. Then there is a choice of Haar measure on $G$ such that
\[
\int_{g'} f(\epsilon(y)) |\text{det}(I_m - y)|_{E^{-m}} \, d\mu(y) = \int_G f(h) |\text{det}(h)|_{E}^{m/2} \, dh = \int_G f(h) \, dh.
\]

**Proof.** Since the Jacobian of (4.17) is given by a rational function, defined over \( F \), it is enough to compute it over an algebraic closure \( \overline{F} \) of \( F \) containing \( E \). Thus, we may assume, for this proof, that \( F \) is algebraically closed. Over the algebraically closed field \( F \), \( G \) becomes the split group \( \text{GL}_m(F) \). See §5.2 for the elementary verification of this last statement. Throughout this proof we will use \( |\cdot| \) to denote \( |\cdot|_F \). We may replace the first equality of (4.18), to be proved, with the new formula

\[
\int_{\text{gl}_m(F)'} f(\epsilon(y)) |\text{det}(I_m - y)|^{-2m} \, d\mu(y) = \int_{\text{GL}_m(F)'} f(h) |\text{det}(h)|^{-m} \, dh.
\]

We compare the two sides of (4.19) by computing both using the appropriate forms of Weyl’s Integration Formula. Because, over the field \( F \), which by the above argument may be assumed to be algebraically closed, there is only one conjugacy class of Cartan subgroups of \( \text{GL}_m(F) \), the “algebra” form of Weyl’s Integration Formula says that the left-hand side of (4.19) is

\[
\int_{\mathcal{L}(F)} |D(x)| \int_{\text{GL}_m(F)/L(F)} f(\epsilon(\epsilon(g)x)) |\text{det}(I_m - \epsilon(g)x)|^{-2m} \, dg \, dx =
\]

\[
= \int_{\mathcal{L}(F)} |D(x)| \int_{\text{GL}_m(F)/L(F)} f(\epsilon(g)(\epsilon(x))) |\text{det}(I_m - x)|^{-2m} \, dg \, dx =
\]

\[
= \int_{\mathcal{L}(F)} |D(x)| |\text{det}(I_m - x)|^{-2m} \int_{\text{GL}_m(F)/L(F)} f(\epsilon(g)(\epsilon(x))) \, dg \, dx.
\]

Here, \( \epsilon \) is the conjugation map, \( \mathcal{L}(F) \) is the diagonal subalgebra of \( \mathfrak{gl}_m(F) \), \( L \) is the diagonal subgroup of \( \text{GL}_m(F) \); \( D(x) \) is the coefficient of \( t^m \) in the characteristic polynomial \( \text{det}(t I_{\mathfrak{gl}_m} - \text{ad}(x)) \) of the linear endomorphism \( \text{ad}(x) \). Write

\[ x = \text{diag}(x_1, \ldots, x_m), \text{ with } x_i \in F. \]

And denote by

\[ dx = dx_1 \cdots dx_m \]

the product measure. Since \( D(x) \) is the product of the roots of \( \text{GL}_m(F) \), evaluated at \( x \), we have

\[
|D(x)| = \prod_{1 \leq i < j \leq m} |x_i - x_j|^2
\]
Change variables in (4.20), \( \frac{x_i}{1-x_i} = t_i, \ i = 1, \ldots, n \) (reflecting the explicit formula for \( c \)). Then
\[
x_i = \frac{t_i}{(t_i + 1)^n}, \ \ \ \text{reflecting the explicit formula for} \ c^{-1},
\]
and \( dx_i = |2t_i/(t_i + 1)^2| \ d^* t_i \). Using this and (4.21), a simple calculation shows that (4.20) equals, up to a positive constant (a power of \( |2| \)),

\[
(4.22) \quad \int_{L(F)} \prod_{1 \leq i < j \leq m} |t_i - t_j|^2 \prod_{i=1}^m |t_i| \left( \int_{\text{GL}_m(F)/L(F)} f(c(g)t) \, dg \right) \, d^* t,
\]

Here, we use the notation
\[
t = \text{diag}(t_1, \ldots, t_m), \ \text{for} \ t_i \in F - \{0\}, \ i = 1, \ldots, m,
\]
\[
d^* t = d^* t_1 \cdots d^* t_m.
\]

The “group” form of the Weyl integration formula on \( \text{GL}_m(F) \) says that the right-hand side of (4.19) is, up to a positive constant,

\[
\int_{L(F)} |d(t)| \left( \int_{\text{GL}_m(F)/L(F)} f(c(g)t) \det(c(g)t)^m \, dg \right) \, d^* t,
\]

where \( d(t) \) is the coefficient of \( z^m \) in the polynomial \( \det(zI_{gl_m} - (\text{Ad}(t) - gl_m)) \). By clearly \( \det(c(g)t) = \det(t) \), and the resulting factor of \( |\det(t)|^m \) in the inner integrand can be taken out of the inner integral, and we obtain, for the right-hand side of (4.19),

\[
(4.23) \quad \int_{L(F)} |d(t)||\det(t)|^m \left( \int_{\text{GL}_m(F)/L(F)} f(c(g)t) \, dg \right) \, d^* t,
\]

Since \( d(t) \) is the product over all the roots of \( \text{GL}_m(F) \) of the difference of the root from 1, evaluated at \( t \), we have

\[
(4.24) \quad |d(t)| = \prod_{1 \leq i < j \leq m} |t_i - t_j|^2 \prod_{i=1}^m |t_i|^{-m+1}.
\]

Since
\[
|\det(t)| = \prod_{i=1}^m |t_i|,
\]
the equality (4.24) implies that (4.23) equals

\[
(4.25) \quad \int_{L(F)} \prod_{1 \leq i < j \leq m} |t_i - t_j|^2 \prod_{i=1}^m |t_i| \left( \int_{\text{GL}_m(F)/L(F)} f(c(g)t) \, dg \right) \, d^* t,
\]
Since (4.22) and (4.25) are equal, the first equality of Lemma 4.1 is proved.

For the justification that \(|\det(h)| = 1\) for \(h \in G\), see §5.4.

We continue with the proof of Lemma 4.1. We make the change of variable (4.17) in (4.16) (Re(s) is still large enough), and by (4.18), we get that there is a choice of Haar measure \(dh\) on \(G\) such that

\[
M(\chi, s) f_{s,v}(i(g, 1)) = |2|^{s(1-2s)} \chi^{-s} \left( \int_G \chi(\det(I_m + h)) |\det(I_m + h)|^{s + \frac{m}{2}} d\mu(-hg) \right) dh.
\]

This proves Lemma 4.1.

The following facts about the Cayley transform are easy to verify.

**Lemma 4.3.** Let \(v \in G'\) and \(g \in G'\). Assume that the elements \(I_m \pm v \in \text{Mat}_m(E)\) are invertible. Then

- **CT 1** \(I_m - c(v)g = (I_m - v)^{-1}(-c^{-1}(g) - v)(I_m + g)\).
- **CT 2** \(I_m + c(v)g = (I_m - v)^{-1}(I_m + vc^{-1}(g))(I_m + g)\).
- **CT 3** We have \(I_m + c(v)g\) invertible if and only if \(I_m + vc^{-1}(g)\) is invertible, and in this case, examining the ratio of **CT 1** to **CT 2**,

\[
c^{-1}(c(v)g) = (I_m - v)^{-1}(c^{-1}(g) + v)(I_m + vc^{-1}(g))^{-1}(I_m - v).
\]

- **CT 1'** \(I_m - c(v)^{-1}g = (I_m + v)^{-1}(-c^{-1}(g) + v)(I_m + g)\).
- **CT 2'** \(I_m + c(v)^{-1}g = (I_m + v)^{-1}(I_m - vc^{-1}(g))(I_m + g)\).
- **CT 3'** We have \(I_m + c(v)^{-1}g\) invertible if and only if \(I_m - vc^{-1}(g)\) is invertible, and in that case, examining the ratio of **CT 1'** to **CT 2'**,

\[
c^{-1}(c(v)^{-1}g) = (I_m + v)^{-1}(-c^{-1}(g) + v)(I_m - vc^{-1}(g))^{-1}(I_m + v).
\]

Denote, for a matrix \(x \in \text{Mat}_m(E)\),

\[
(4.26) \quad ||x|| = \max_{1 \leq i, j \leq m} |x_{ij}|_E.
\]

This is a norm on \(\text{Mat}_m(E)\). It is not difficult to verify that the norm satisfies these five properties.

**Norm 1** \(||x + y|| \leq \max\{||x||, ||y||\}\), for all \(x, y \in \text{Mat}_m(E)\).

**Norm 2** \(||x + y|| = ||y||\), if \(||x|| < ||y||\).

**Norm 3** \(||k_1 x k_2|| = ||x||\), for all \(k_1, k_2 \in \text{GL}_m(O_E)\).

**Norm 4** If \(||v|| < 1\), then we have

\(I_m - v \in \text{GL}_m(O_E)\),

meaning that \(I_m - v\) is both invertible and integer (possessed of integral entries).
Norm 5 \(|x^*| = |x|.|\)

For the justification of Norm 5, see Lemma 5.7(c). Let \(N\) be a positive even integer such that

\[
q^N > |8|_E^{-1}q^4,
\]

and such that

\[
\epsilon(g(P^{N-2}_E)) \subset U.
\]

From now on, we assume that the conductor \(1 + P_N^{N_x}\) of \(\chi\) is such that \(N_x > N\). We now return to the integral in (4.3), evaluated at \(-g^{-1}\) in place of \(g\). In order for a point \(h\) in the domain of integration to make a non-zero contribution to the integral (i.e., in order for the integrand to be nonzero at \(h\)), we must have \(hg^{-1} = u\), where \(u \in U\). That is, we must have \(h = ug\) for some \(u \in U\). Then, according to (4.3), we have

\[
M(\chi,s)f_{\chi,s}(i(-g^{-1},1)) = 2|2|_E^m(1-2s)\chi_m(-2) \times \sum_{L=-\infty}^{\infty} \int_{||r^{-1}(ug)||=s^L} \chi(\det(I_m + ug))|\det(I_m + ug)|_E^{s+\frac{2}{E} - \frac{1}{2}} du,
\]

for \(\text{Re}(s) \gg 0\). Denote, for \(g \in G\), \(\text{Re}(s) \gg 0\), and \(L \in \mathbb{Z}\),

\[
I_L(\chi,s;g) := \int_{||r^{-1}(ug)||=s^L} \chi(\det(I_m + ug))|\det(I_m + ug)|_E^{s+\frac{2}{E} - \frac{1}{2}} du.
\]

Set

\[
n_\chi = \left[\frac{N_x + 1}{2}\right].
\]

Our main aim from this point is to prove an analogue of Lemmas 4.4–6 from [RS05], namely that

\textbf{Lemma 4.4.} We have

\[
I_L(\chi,s;g) = 0,
\]

for all \(L \geq -n_\chi\).

The first main step towards proving Lemma 4.4 will be making a change of variable in the integral \(I_L(\chi,s;g)\) that will allow us to replace the single integral of \(I_L(\chi,s;g)\) with a double integral. In order to state Lemma 4.4, it is convenient to introduce the following piece of notation.
Definition 4.5. Let \( + (\cdot) \) be the “non-negativity function” from the reals to the non-negative reals. That is, let \( + (\cdot) \) be defined piecewise by

\[
+ r = \begin{cases} 
  r & \text{if } r \geq 0 \\
  0 & \text{if } r < 0
\end{cases}
\]

Let \( - (\cdot) \) be the “non-positivity function” from the reals to the non-negative reals defined analogously so that

\[
(4.32) \quad \text{for all } r \in \mathbb{R}, \quad r = + r - - r, \quad \text{and } |r| = + r + - r.
\]

Lemma 4.6. Suppose that \( L \) satisfies

\[
(4.33) \quad L > -n, \quad \text{equivalently } - L < n.
\]

Then

\[
(4.34) \quad I_L(\chi, s; g) = \frac{1}{\mu(g, L + n)} \int_{g, L + n} \int_{||c^{-1}(ug)|| = q^L} \chi(\det(I_m + c(v)ug))|\det(I_m + c(v)ug)|^{s + \frac{m}{2} - \frac{1}{2}} du d\mu(v).
\]

We start with a lemma of a preliminary nature, useful in making changes of variable.

Lemma 4.7. Let \( g, || \cdot || \), be as above, \( L \in \mathbb{Z} \), \( U \) any subgroup of \( G \) and \( du \) a left Haar measure on \( U \). For \( A \subset G \), let \( 1_A \) denote the characteristic function of \( A \). Let \( a, g \in G \) be fixed, and assume that these elements of \( G \) satisfy the following two conditions,

\[
(4.35) \quad a \in U,
\]

and

\[
(4.36) \quad \text{for } u \in U, \quad ||c^{-1}(ug)|| = L \text{ if and only if } ||c^{-1}(ug)|| = L.
\]

Let \( f \) be a function on \( G \). Assuming that either integral converges, we have

\[
(4.37) \quad \int_U f(u) 1_{||c^{-1}(g)|| - 1}(L)(u) du = \int_U f(au) 1_{||c^{-1}(ug)|| - 1(L)}(u) du.
\]

In other words

\[
(4.38) \quad \int_{||c^{-1}(ug)|| = q^L} f(u) du = \int_{||c^{-1}(ug)|| = q^L} f(au) du.
\]
Proof. Each side of (4.37) is equal to
\[
\int_{U} f(au)1_{(||c^{-1}(g)||^{-1}(L))}(au) \, du,
\]
the left-hand side by the condition of (4.35) that \(a \in U\) and because \(du\) is a Haar measure on \(U\). The right-hand side is equal to the above expression because (4.36) means that
\[
U \cap (||c^{-1}(a \cdot g)||)^{-1}(L) = U \cap (||c^{-1}(a \cdot g)||)^{-1}(L).
\]
Since we are integrating over \(U\), we can drop the intersections with \(U\) for the purposes of the current argument and substitute
\[
(||c^{-1}(a \cdot g)||)^{-1}(L) = (||c^{-1}(a \cdot g)||)^{-1}(L),
\]
on the right-hand side of (4.37). Then, we use the obvious equality
\[
1_{(||c^{-1}(a \cdot g)||)^{-1}(L)}(u) = 1_{(||c^{-1}(a \cdot g)||)^{-1}(L)}(au)
\]
to complete the proof of (4.37).

We derive (4.38) from (4.37) by reinterpreting the multiplication of the integral by the characteristic function as a restriction of the domain of integration.

Proof of Lemma 4.6. Let
\[
(4.39) \quad v \in \mathfrak{g}^{+L+n_{\chi}} := \mathfrak{g}(\mathbb{P}^{L+n_{\chi}})_{E}, \text{ but otherwise arbitrary.}
\]
We can apply Lemma 4.7 with \(a = c(v)\) to rewrite \(I_{L}\), provided that we verify (4.35) and (4.36) in the present instance, which is the aim of the following argument.

As for (4.36), it is easy to see that
\[
(4.40) \quad c(v) \in U,
\]
because, \(^{+L} \geq 0\) (by definition), which together with (4.31), and the assumption on the conductor \(N_{\chi}\) just following (4.28), implies
\[
^{+L} + n_{\chi} \geq n_{\chi} = \left[\frac{N_{\chi} + 1}{2}\right] > \left[\frac{N + 1}{2}\right] > \frac{N - 2}{2},
\]
and then we can apply (4.28) to conclude (4.40).

The verification of (4.36) consists in verifying two implications. In both directions, we will use the fact that (4.39), combined with Norm 4, implies that
\[
(4.41) \quad I_{m} - v \in \text{GL}_{m}(O_{E}).
\]
Note that we will also use (4.33) in the proof of both implications.

**First implication of (4.36).** Suppose $||c^{-1}(ug)|| = q^L$.

We first claim that $I_m + c(v)ug$ is invertible. Lemma (4.3) can be applied because of (4.41). Applying property CT 3 from Lemma 4.3 with "g" = ug, we have

$$I_m + c(v)ug \text{ is invertible if and only if } I_m + vc^{-1}(ug) \text{ is invertible.}$$

However, it can be shown directly that $I_m + vc^{-1}(ug)$ is invertible by using the definitions of $v$ in (4.39) and $I_L(\chi, s; g)$ in (4.30) to observe that

$$||vc^{-1}(ug)|| \leq q^{-(L+n\chi)}||c^{-1}(ug)|| = q^{-n\chi+(L-L)} = q^{-n\chi-L} < 1,$$

where we have used (4.32) in the last equality. Therefore, Property Norm 4 implies that

$$I_m + vc^{-1}(ug) \in GL_m(\mathcal{O}_E) \text{ and in particular, invertible.}$$

We conclude from (4.41) and (4.42) that $I_m + c(v)ug$ is invertible.

Since $I_m + c(v)ug$ is invertible, property CT 3 from Lemma 4.3 gives an expression for $c^{-1}(c(v)ug)$. Using this expression and some previously established facts, we deduce that

$$||c^{-1}(c(v)ug)|| = ||(I_m - v)^{-1}(c^{-1}(ug) + v)(I_m + vc^{-1}(ug))(I_m - v)||$$

$$= ||c^{-1}(ug)|| + ||v|| \text{ (by (4.41), (4.44), and Norm 3)}$$

$$= ||c^{-1}(ug)|| \text{, (by (4.39) and Norm 2)}$$

$$= q^L \text{ (by assumption).}$$

**Second implication of (4.36).** Suppose $||c^{-1}(c(v)ug)|| = q^L$.

This is similar to the first direction, but with CT 1' through CT 3' playing the role of CT 1 through CT 3. The main steps in the proof are as follows. First, we have

$$I_m + ug \text{ is invertible if and only if } I_m - vc^{-1}(c(v)ug) \text{ is invertible.}$$

By similar arguments to those used in the proof of the first implication, we show that

$$I_m - vc^{-1}(c(v)ug) \in GL_m(\mathcal{O}_E).$$

Property CT 3' now applies to give

$$||c^{-1}(ug)|| = ||c^{-1}c(v)ug||$$

$$= ||(I_m + v)^{-1}(-Gc^{-1}(c(v)ug) + v)(I_m - vc^{-1}(c(v)ug))^{-1}(I_m + v)||$$

$$= || - c^{-1}(c(v)ug) + v||$$

$$= ||c^{-1}(c(v)ug)||$$

$$= q^L.$$
The penultimate step is justified by the hypothesis that \( \| \epsilon^{-1}(\epsilon(v)ug) \| = q^L = q^{v-L} > q^{-n_{\chi} + q^L} \), (by (4.39) again) which is clearly at least as large as \( \| v \| = q^{-n_{\chi} - L} \), since \( v \geq 0 \). We then apply property \textbf{Norm 2} to obtain the penultimate equality of (4.40). 

Combining (4.40), (4.45), and (4.46), we see that the element \( a = \epsilon(v) \) of \( G \) satisfies the two hypotheses of Lemma 4.7. Applying (4.38), with \( f(u) = \chi(\det(I_m + ug)) \), we deduce that for \( v \) as in (4.39),

\[
I_L(\chi, s; g, v) := \int_{\| \epsilon^{-1}(ug) \| = q^L} \chi(\det(I_m + \epsilon(v)ug)) \det(I_m + \epsilon(v)ug)|_{E}^{s + \frac{q}{2} - \frac{1}{2}} \, du = I_L(\chi, s; g).
\]

In other words, the value of \( I_L(\chi, s; g, v) \), defined in the first line of the above equality, is independent of \( v \) satisfying (4.39) and equal to \( I_L(\chi, s; g) \). Obviously, averaging the constant \( I_L(\chi, s; g, v) \) over \( \mathfrak{g}^{+L+n_{\chi}} \) and dividing by the total measure \( \mu(\mathfrak{g}^{+L+n_{\chi}}) \) does nothing, so we have

\[
I_L(\chi, s; g) = \frac{1}{\mu(\mathfrak{g}^{+L+n_{\chi}})} \int_{\mathfrak{g}^{+L+n_{\chi}}} I_L(\chi, s; g, v) \, d\mu(v),
\]

where \( d\mu(v) \) is any invariant measure on \( \mathfrak{g} \). In particular, with \( d\mu(v) \) the measure on \( \mathfrak{g} \) in (4.15), we complete the derivation of (4.34) by using the definition of \( I_L(\chi, s; g, v) \) in the previous expression for \( I_L(\chi, s; g) \). \( \Box \)

**Lemma 4.8.** Assume that (4.33) is satisfied. Continuing with the calculation of \( I_L \) from Lemma 4.6 there exist \( a, b \in E^\times \), satisfying the properties

\[
|a|_E = |b|_E = q^{N_{\chi}},
\]

and

\[
|a + b|_E \leq q^{N_{\chi}},
\]

such that

\[
I_L = \frac{1}{\mu(\mathfrak{g}^{+L+n_{\chi}})} \int_{\| \epsilon^{-1}(ug) \| = q^L} \chi(\det(I_m + ug)) \det(I_m + ug)|_{E}^{s + \frac{q}{2} - \frac{1}{2}} \times
\]

\[
\times \int_{v \in \mathfrak{g}^{+L+n_{\chi}}} \psi_0(btrv + atr(\epsilon^{-1}(ug))) \, d\mu(v) \, du,
\]

where \( \psi_0 \) is a fixed character of \( E \) whose conductor is \( \mathcal{O}_E \).
Consequently, we have

\[
|\det(I_m - v)| \text{ and } |\det(I_m + v\chi^{-1}(ug))| = 1.
\]

Using Property CT 2, we may rewrite the integrand of (4.49) as

\[
\chi^{-1}(\det((I_m - v))\chi(\det(I_m + v\chi^{-1}(ug))))\chi(\det(I_m + ug)) \times
\]

\[
|\det(I_m - v)|_{E}^{s + \frac{\eta}{2} + \frac{1}{2}}|\det(I_m + v\chi^{-1}(ug))|_{E}^{s - \frac{\eta}{2} - \frac{1}{2}}|\det(I_m + ug)|_{E}^{s - \frac{m}{2} - \frac{1}{2}} =
\]

\[
= \chi^{-1}(\det((I_m - v))\chi(\det(I_m + ug)))\chi(\det(I_m + v\chi^{-1}(ug)))|\det(I_m + ug)|_{E}^{s - \frac{m}{2} - \frac{1}{2}}.
\]

By using (4.49), we see that the fourth and fifth factors in the product don’t contribute to the first expression, to obtain the latter expression of (4.50).

We now apply Corollary 5.6 to rewrite each of the factors \(\chi^{-1}(\det((I_m - v))\) and \(\chi(\det(I_m + v\chi^{-1}(ug)))\). In the first case \(n = 2L + n\), while in the second \(n = n\), and in both cases \(N = 2n\). Since \(N \leq 2n\), the hypothesis (5.2) is verified. The Corollary now gives \(a, b \in E^x\) satisfying (4.57), with

\[
\chi^{-1}(\det((I_m - v))\chi(\det(I_m + v\chi^{-1}(ug))) = \psi(btrv)\psi( atr(v\chi^{-1}(ug)))
\]

(4.51)

for all \(v\) such that \(v, v\chi^{-1}(ug) \in \mathcal{P}_E^{n}\). By (4.49) (4.51) does indeed apply to the \(v\) considered in the integrand of (4.44). Together with the calculation of (4.50), this proves the Lemma with the exception of the claim (4.48).

In order to verify (4.48), note that, by construction, \(a, b\) are elements of \(E^x\) such that

\[
\chi(1 + x) = \psi_0(ax), \text{ and } \chi^{-1}(1 + x) = \psi_0(bx), \text{ for all } x \in \mathcal{P}_E^{n}.
\]

Consequently, we have

\[
q^0 = 1 = \psi_0(ax + bx) = \psi((a + b)x) \text{ for all } x \in \mathcal{P}_E^{n},
\]

implying that

\[
(a + b)\mathcal{P}_E^{n} \subseteq \mathcal{O}_E, \text{ so that } |a + b|_{E}q^{-n} \leq 1.
\]

This immediately yields (4.48). \(\square\)

Rewriting the single integral \(I_L(\chi, s; g)\) as the iterated integral of (4.34) was the first main step in showing that \(I_L(\chi, s; g) = 0\). From this point, the strategy of the proof consists in changing the order of integration and rewriting the inner integrand in such a way that this inner integrand can be seen to be zero, for all \(u\) in the range of integration of the outer integrand.

**Lemma 4.9.** Let \(L \in \mathbb{Z}\) and assume that \(L \neq 0\). Let \(\psi_0\) be a fixed character of \(E\) with conductor \(\mathcal{O}_E\), \(a, b \in E^x\) such that
Then for \( X \in \mathfrak{gl}_m(E) \) such that

\[
\|X\| = q^L,
\]

we have

\[
\int_{v \in L + n} \psi_0(\text{tr}(vX) + \text{attr}(vX)) \, d\mu(v) = 0.
\]

**Proof.** Note that the expression \( \text{tr}(vX) + \text{attr}(vX) \) is equal to \( \text{tr}(v(bI_m + aX)) \)

As in the proof of Lemma 4.4 of [RS05], at (4.42) we see that in order for the integral (4.54) to be nonzero, we must have

\[
\|bI_m + aX\| \leq |2|^{-1} q^{+L+n}.
\]

Since \( L \neq 0 \), by assumption, \( q^L \neq 1 \), so that (4.53) and the equality of (4.52) together imply that

\[
\|bI_m\| \neq \|aX\|.
\]

Therefore, Property **Norm 2** implies that

\[
\|bI_m + aX\| = \max(\|bI_m\|, \|aX\|).
\]

By the strict inequality of (4.52), (4.53), and the definition of \( ^+L \) before (4.32)

\[
\max(\|bI_m\|, \|aX\|) > |2|^{-1} q^{+L+n}.
\]

so that by (4.56)

\[
\|bI_m + aX\| > |2|^{-1} q^{+L+n}.
\]

With (4.55) this gives a contradiction. Therefore, by the above comments, the integral of (4.54) equals zero.  

**Lemma 4.10.** Let \( \psi_0 \) be a fixed character of \( E \) with conductor \( \mathbb{O}_E \), \( a, b \in E^\times \) satisfying (4.47) and (4.48). Then for
(4.57) \( X \in \mathfrak{g}, \) such that \( \|X\| = 1, \)

we have

\[
\int_{\mathfrak{g}} \psi_\Omega(b \text{tr} v + a \text{tr}(vX)) \, d\mu(v) = 0.
\]

**Proof.** In order for the integral of (4.58) not to vanish, we must have

\[
\|bI_m + aX\| \leq |2|^{-1} q^{n_x+1},
\]

paralleling (4.59). The hypothesis (4.58) is equivalent to assuming that \( a + b \) is of the form \( \varpi^{-n_x} \sigma, \) for \( \varpi \) the generator of \( \mathfrak{P}_E \) and some \( \sigma \in \mathcal{O}_E. \) So in order for the integral of (4.58) to be nonzero, we must have

\[
\|\varpi^{-n_x} \sigma + a(X - I_m)\| \leq |2|^{-1} q^{n_x+1},
\]

By Norm 2, because \( \|\varpi^{-n_x} \sigma\| \leq q^{n_x}, \) this implies that

\[
\|a(X - I_m)\| \leq |2|^{-1} q^{n_x+1},
\]

so by (4.47),

\[
\|X - I_m\| \leq |2|^{-1} q^{n_x-N_x+1},
\]

meaning that

\[
X - I_m \in \mathfrak{gl}_m(\mathfrak{P}_E^{N_x-n_x-1-\nu_E(2)}).
\]

Thus \( X \in I_m + \mathfrak{gl}_m(\mathfrak{P}_E^{N_x-n_x-1-\nu_E(2)}). \) Because of (4.27), this, combined with Proposition 5.8, implies that \( X \notin \mathfrak{g}. \) Thus, we obtain a contradiction with (4.57).

**Completion of Proof of Lemma 4.4.** By applying Lemma 4.9 or 4.10 to Lemma 4.8 depending on whether \( L = 0 \) or \( L \neq 0, \) we deduce the vanishing of \( I_L \) for all \( L \) in the required range of \( L \geq -n_x. \) The reason the hypotheses of Lemma 4.9 are satisfied is that according to Lemma 4.8 we have (4.47) and then (4.27) implies (4.52). The reason the hypotheses of Lemma 4.10 are satisfied in the case \( L = 0 \) is that \( X = c^{-1}(ug) \) satisfies (4.57) for all \( u \) in the domain of integration of the outer integral for \( I_L. \)

**Completion of Proof of Theorem 3.1.** Combining (4.30) and (4.31) and Lemma 4.4, we have, for \( \text{Re}(s) \gg 0, \)

\[
M(\chi, s) f_{\chi, s}(i(-g^{-1}, 1)) = |2|^m(1-2s) \chi^{-m(-2)} \times \int \chi(\det(I_m + ug)) |\det(I_m + ug)|^{s+\varpi-\frac{1}{2}} |\det(I_m + ug)|^{\nu_E(2)} du.
\]
The domain of integration in (4.60) is over \( u \in U \), such that

\[
ug \in \mathcal{C}(\mathcal{P}_E^{m}) \subseteq \mathcal{C}(\mathcal{P}_E^{N-2}) \subseteq U,
\]

where the latter two containments follow from (4.28) and (4.31). This implies that \( M(\chi, s) f_{X,s}(i(g^{-1}, 1)) \) is supported in \( U \). Since we may now assume that \( g \in U \) and \( du \) is the Haar measure on \( U \) with total mass 1, the following general observation applies. For any function \( f \) on \( U \) we have

\[
F \text{ defined by, for } g \in U, F(g) := \int_U f(ug) \, du, \text{ is a constant function equal to } F(1).
\]

We apply this general observation with \( f \) the product of the integrand in (4.60) and the characteristic function of the set

\[
\mathcal{C}^{-1}\left((-\infty, q^{-m})\right).
\]

The result of doing so is to deduce that \( M(\chi, s) f_{X,s}(i(g^{-1}, 1)) \) is constant on \( U \) and equal to \( M(\chi, s) f_{X,s}(i(-1, 1)) \). Note further that

\[
|\det(I_m + u)| = |2^m \det(I_m - (I_m - u)/2)| = |2|^m \left| \det \left( I_m - \frac{I_m - u}{2} \right) \right|
\]

and the latter factor is 1 for \( u \) close to \( I_m \), by Lemma 5.3 part (b). Therefore, the entire factor \( |\det(I_m + u)|^{s + \frac{d-\frac{d}{2}}{2}} \) in the integrand of (4.60) comes out of the integral as \( |2|^m \frac{m(s + \frac{d-\frac{d}{2}}{2})}{2} = |2|^m(2s+m-1) \). These arguments imply that, that the expression for \( M(\chi, s) f_{X,s}(i(-g^{-1}, 1)) \) in (4.60) can be replaced by

\[
M(\chi, s) f_{X,s}(i(-1, 1)) = |2|^m F \chi(-2) \times \int_{\mathcal{C}^{-1}(u)}(\det(I_m + u)) \, du.
\]

We may change variable in (4.61), \( u = \mathcal{C}(v), v \in \mathcal{C}(\mathcal{P}_E^{m}) \), and choose the measure \( \mu(v) \) on \( \mathcal{C}(\mathcal{P}_E^{m}) \) as in (4.15), and get (using \( I_m + \mathcal{C}(v) = 2(I_m - v)^{-1} \)),

\[
M(\chi, s) f_{X,s}(i(-1, 1)) = |2|^m F \chi(-1) \times \int_{\mathcal{C}^{-1}(v)}(\det(I_m - v)) \, d\mu(v).
\]

We have proved (4.62) for \( \Re(s) \gg 0 \). But \( s \) does not appear on the right side of (4.62). Therefore, (4.62) is valid for all \( s \) by analytic continuation. In summary, we have shown that, for all \( s \in \mathbb{C} \),

\[
M(\chi, s) f_{X,s}(i(-g^{-1}, 1)) \text{ is given by the integral of the right side of (4.62) for } g \in U, \text{ independent of } g, \text{ and by } 0 \text{ for } g \notin U.
\]

Now let \( \Re(s) \) be small enough that the expression of (4.62) is valid. By the last sentence of the previous paragraph, we can replace the integral over \( G \) by an integral over \( U \), then the factor
$M(\chi, s)f_{\chi,s}(i(-u^{-1}, 1))$ in the integrand by $M(\chi, s)f_{\chi,s}(i(-1, 1))$. Further, since $U$ is chosen so that $v_1$ is $U$-invariant, and further, since $\langle v_1, \hat{v}_2 \rangle = 1$, the first factor $\langle \pi(-1)v_1, \hat{v}_2 \rangle$ reduces to $\omega_{\pi}(-1)$. So on the right side we are left integrating the constant $M(\chi, s)f_{\chi,s}(i(-1, 1))$ over $U$ the measure on $U$ was chosen to have total mass one. It follows then that (4.63) reduces to

$$\omega_{\pi}(-1)\Gamma(\pi, \chi, s) = M(\chi, s)f_{\chi,s}(i(-1, 1)).$$

The right-hand side of (4.63) is independent of $\chi, s$, and depends only on $\pi$. Therefore, by analytic continuation, (4.63) is valid for all $s$. Substituting the expression of (4.62) for $M(\chi, s)f_{\chi,s}(i(-1, 1))$ into the right-hand side of (4.63), we obtain Theorem 3.1.

5 Appendix: elementary verifications

5.1 Generalities concerning quadratic extensions of local fields.

We begin by proving the claim in the second paragraph of §2. Let $\omega' \in E - F$. Then, since $E$ is a quadratic extension of $F$, there exist $a_0, b \in F$, such that

$$\omega'^2 + a_0\omega' + b = 0.$$ 

Completing the square, we obtain the equivalent equations

$$\omega^2 + a_0\omega + \frac{a_0^2}{4} = \frac{a_0^2}{4} - b,$n

$$\left(\omega' + \frac{a_0}{2}\right)^2 = \frac{a_0^2}{4} - b.$$ 

Now set $\omega = \omega' + \frac{a_0^2}{2}$ and $a = \frac{a_0}{2} - b$, to obtain an $\omega$ and $a$ as claimed. Note that we have only used the field property and $\text{char} E \neq 2$ in this argument (in practice $\text{char} E$ is zero). In order to verify (2.2), note that, since $a \in F = \text{Fix}(\theta)$,

$$\overline{\sigma}^2 = a = \omega^2,$$ 

so that we have

$$\left(\frac{\omega}{\overline{\sigma}}\right)^2 = 1,$$ 

from which we deduce

$$\overline{\sigma} = -\omega,$$ 

since $\omega \in E - F$, by assumption.
5.2 Passage from $G = U_m(E)$ to $GL_m(E) = GL_m(F)$.

We explicate and prove the sentence

Over $\overline{F}$, defined by be an algebraic closure of $F$ containing $E$, $G$ becomes the split group $GL_m(\overline{F})$.

from the proof of Lemma 4.2. In order to make the statement more comprehensible, we break down the passage from $F$ to $\overline{F}$ into two steps, as follows.

- **Step 1.** Replace $F$ with the quadratic extension $E$, over which $G$ becomes $GL_m(E)$.
- **Step 2.** Replace $E$ with $\overline{F} = E$, over which $GL_m(E)$ becomes $GL_m(F) = GL_m(\overline{F})$, hence split.

It is well-known that a split group over an algebraically closed field has a conjugacy class of Cartan subgroups, so this argument also validates the application of the “simple” form of the Weyl integration formulas, (4.20) and (4.23), above.

Of the two steps above, Step 2 amounts to a straightforward tensoring operation on the Lie algebra level. Alternatively, considering the abstract Chevalley group $GL_m$ as a functor from fields (Field) to matrix groups, we may consider this step as a simple substitution of objects belonging to the domain category Field. Therefore, only Step 1 needs any further explanation, for which we introduce the following concepts.

**Definition 5.1.** Let $F \subset E$ be a quadratic extensions of fields of characteristic different from 2. By §5.1, this implies that as a vector space, $E$ is of the form $F \oplus \omega F$, with $\omega^2 = a \in F$. For $V$ an $m$-dimensional vector space over $E$, let $V(F)$ be the $2m$-dimensional $F$-vector space which equals $V$ as a set. If

$$\beta = \{v_1, \ldots, v_m\}$$

is a basis for $V$, then call

$$\beta^F := \{v_1, \omega v_1, \ldots, v_m, \omega v_m\},$$

the corresponding basis for $V(F)$. Fix a $B$ as above so as to consider $\mathfrak{gl}(V(F))$ as the matrix-algebra $\mathfrak{gl}_{2m}(F)$. For an element $X \in \mathfrak{gl}(V(F))$, denote by $X^{(ij)}$ the $ij$th two-by-two block counting from the upper-left of the matrix $X$, for $1 \leq i, j \leq m$. Thus $X^{(ij)} \in \mathfrak{gl}_2(F)$, and its entries will be denoted by $X^{(ij)}_{kl}$ for $1 \leq k, l \leq 2$. Define the operation $\text{transp}$ on $\mathfrak{gl}(V(F))$ by the relation

$$(\text{transp}(X))^{(ij)} = X^{(ji)}$$

for $1 \leq i, j \leq m$, in other words, the transpose operation operating a matrix of 2-by-2 blocks instead of individual entries. Define the operation $\text{conj}$ on $\mathfrak{gl}(V(F))$ by the relations

$$(\text{conj}(X))^{(ij)}_{kl} = (-1)^{k-1}X^{(ij)}_{kl}, \text{ for } 1 \leq i, j \leq m, 1 \leq k, l \leq 2$$

i.e. $\text{conj}$ negates “off-diagonal” entries of each two-by-two block and leaves the diagonal entries alone. Finally, let $\mathfrak{g}$ be the subalgebra of $\mathfrak{gl}(V(F))$ satisfying the conditions
**E-linearity.** Each $X^{(ij)}$ is of the form

$$
\begin{pmatrix}
    f_1 & af_2 \\
    f_2 & f_1
\end{pmatrix}.
$$

**E-unitarity.** We have $\text{conj} \circ \text{transp}(X) = -X$.

**Proposition 5.2.** With $\mathfrak{g}$ defined as above, we have $\mathfrak{g} \cong u_m(E)$ as a Lie algebra over $F$.

**Proof.** Consider the $F$-isomorphism of $V$ with $V(F)$ induced by the identification of the underlying sets. This isomorphism induces natural embedding of $\mathfrak{gl}_m(E)$ into $\mathfrak{gl}_{2m}(F)$. The first condition, **E-linearity**, is equivalent to $X$ belonging to the image of this embedding, that is to $X$’s actually from a “restriction of scalars” from an element of $\mathfrak{gl}_m(E)$. Given that $X$ satisfies the condition of **E-linearity**, it is clear that the inverse image of $X$ belongs to $u_m(F)$ if and only if $X$ satisfies the second condition, of **E-unitarity**. 

Therefore, in order to justify Step 1, we can replace the original task of showing that $u_m(F) \otimes E \cong \mathfrak{gl}_m(E)$, with showing that $\mathfrak{g} \otimes E \cong \mathfrak{gl}_m(E)$. The isomorphism is simply the isomorphism induced on the level of vector space endomorphisms by the vector-space isomorphism $V(F) \otimes E \cong V$. This can be made obvious by showing how a basis of the former maps to a basis of the latter. In order to define the basis, we adopt the notation

$$
E^{(ij)} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Mat}_{2m}(F) \text{ with } \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \text{ as the } ij\text{-th } 2\text{-by-}2 \text{ block and zeros elsewhere.}
$$

Also, we use $e_{ij}$ to denote the (usual) elementary matrix with 1 in the $ij$th position and zeros elsewhere.

Then it is easily calculated that the $E$-basis of $\mathfrak{g} \otimes E$

$$
\left\{ E^{(ii)} \left( \begin{array}{cc} 0 & a \\ 1 & 0 \end{array} \right) \right\}_{1 \leq i \leq m} \bigcup \left\{ \left( E^{(ij)} + E^{(ji)} \right) \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \right\}_{1 \leq i < j \leq m} \bigcup \left\{ \left( E^{(ij)} - E^{ji} \right) \left( I_2 \right) \right\}_{1 \leq i < j \leq m}
$$

maps to the basis of $\mathfrak{gl}_m(E)$

$$
\left\{ e_{ii} \right\}_{1 \leq i \leq m} \bigcup \left\{ \omega(e_{ij} + e_{ji}) \right\}_{1 \leq i < j \leq m} \bigcup \left\{ e_{ij} - e_{ji} \right\}_{1 \leq i < j \leq m}.
$$

**Example.** As an example of the matrix form $\mathfrak{g}$ for $u_m$ just given, we offer the simplest case in which all main features of the situation are visible, namely the case $m = 2$. Then we readily calculate that

$$
\mathfrak{g} = \left\{ \begin{pmatrix}
    a f_2^{(11)} & f_1^{(12)} & a f_2^{(12)} \\
    f_2^{(11)} & f_2^{(12)} & f_1^{(12)} \\
    -f_2^{(12)} & a f_2^{(12)} & -f_2^{(12)} \\
    f_2^{(12)} & -f_1^{(12)} & f_2^{(22)}
\end{pmatrix} \mid f_k^{(ij)} \in F \text{ for } i, j, k \in \{1, 2\} \right\}.
$$

The reader can now verify by inspection the above claims concerning the bases $\left\{ E^{(ii)} \left( \begin{array}{cc} 0 & a \\ 1 & 0 \end{array} \right) \right\}_{1 \leq i \leq m} \bigcup \cdots$ and $\left\{ e_{ii} \right\}_{1 \leq i \leq m} \bigcup \cdots$ in this case of $m = 2$ and see how to extend the arguments to general $m$. 

5.3 General observations concerning Mat\(_n(F)\), for \(F\) a local field, and the absolute value.

For this subsection, let \(F\) be a non-Archimedean local field.

**Lemma 5.3.** Let \(v \in \text{Mat}_n(F)\). Let \(C \in \mathbb{Z}\) and assume that

\[
v \in \text{Mat}_n(F)_C := \text{Mat}_n(\mathcal{P}^C_F).
\]

Then we have the following,

(a) \(\det(I_m - v) \in 1 + \mathcal{P}^C\) and \(\det(I_m - v) \equiv 1 + \text{tr}(v) \mod \mathcal{P}^{2C}\).

(b) If in addition \(C > 0\), then \(|\det(I_m - v)| = 1\), and \(I_m - v\) is invertible.

(c) The conclusions of (b) hold if we make the (equivalent) assumption that \(||v|| < 1\), where ||·|| is the matrix norm defined in \((4.26)\).

**Proof.** For (a), use induction on \(m\). The assertion concerning the determinant is clear for \(m = 1\). For \(m > 1\), expand the determinant in minors along any row or column. All the minors will lie in the set \(1 + \mathcal{P}^C\). All the co-factors will lie in the ideal \(\mathcal{P}^C\) except for the one coming from the diagonal, which is clearly in \(1 + \mathcal{P}^C\). Now it easy to arrive at the conclusion. The assertion concerning the trace is proved by an analogous induction.

For (b), use “ultra-metric” property of the absolute value.

Part (c) follows immediately from part (b) and the definitions of the absolute value and the matrix norm ||·||.

**Lemma 5.4.** For \(N, n\) positive integers such that

\[(5.1)\]

\[n \leq N \leq 2n,\]

the map

\[x \mapsto 1 + x\]

defines a group isomorphism

\[
\mathcal{P}_F^n/\mathcal{P}_F^N \xrightarrow{\cong} 1 + \mathcal{P}_F^n/1 + \mathcal{P}_F^N.
\]

**Proof.** Obviously, the map defined in the lemma has a well-defined inverse. Both the original map and its inverse are defined on the whole quotient group \(\mathcal{P}_F^n/\mathcal{P}_F^N\), and \(1 + \mathcal{P}_F^n/1 + \mathcal{P}_F^N\), respectively. Therefore, the map in the lemma is a set bijection. The condition \(N \leq 2n\) is precisely what’s needed to guarantee that each map respects the group law.

It is well-known that the map \text{Char} taking a group to its character group is a (contravariant) functor on the category \textbf{Group}. Therefore, by Lemma 5.4 the map

\[x \mapsto 1 + x\]
and its inverse induce an isomorphism of the character group of $\mathcal{O}_F/\mathcal{P}_F^N$ with that of $1 + \mathcal{P}_F/1 + \mathcal{P}_F^N$.

Finally, there is the well-known description of the character group of $F$ as the mappings of the form

$$x \mapsto \psi_0(ax),$$

where $\psi_0$ is a fixed character of $F$ whose conductor is $\mathcal{O}_F$, and $a \in F^*$ such that the negative of the valuation $-v_E(a)$ equals the conductor of $\chi$. Summarizing the above discussion, we have proved the following.

**Corollary 5.5.** Let $N, n$ satisfy (5.1). Let $\psi_0$ be a fixed character of $F$ whose conductor is $\mathcal{O}_F$. The characters of $1 + \mathcal{P}_F^n$ with conductor not exceeding $N$ are given precisely by

$$\chi(1 + x) = \psi_0(ax), \text{ for all } x \in \mathcal{P}_F^n,$$

where $a$ is an element of $F^*$ satisfying $-v_F(a)$ equal to the conductor of $\chi$.

Further, putting together Corollary 5.5 and Lemma 5.3, we deduce the following

**Corollary 5.6.** Let $n, N$ be as in (5.1), and $\psi_0$ a fixed character of $F$ whose conductor is $\mathcal{O}_F$. Let $\chi$ be a character of $F^\times$ such that

(5.2) \hspace{1cm} The conductor of $\chi$ is no greater than $N$.

Then there exists a fixed $a \in F^*$ with

(5.3) \hspace{1cm} $-\nu(a)$ equal to the conductor of $\chi$.

such that

(5.4) \hspace{1cm} $\chi(\det(1 - v)) = \psi_0(\text{tr}(v)), \text{ for all } v \in \text{Mat}_m(F) \text{ with } ||v|| \leq q^{-n}.$

### 5.4 The Field Norm and Finite Extensions

Directly from the definition of $G$ in (2.6), we have

$$\det(g)\overline{\det(g)} = 1,$$

so that

$$|\det(g)|_{E}|\overline{\det(g)}|_{E} = 1.$$

Since the field extension $E/F$ is quadratic, the two elements of $\text{Gal}(E/F)$ are $\text{Id}$ and $\theta = \tau$. Thus, the above equality is equivalent to

$$N_{E/F}(\det(g)) = 1.$$

The claim at the end of the proof of Lemma 4.2, namely, that $|\det(g)| = 1$, then follows from part (b) of the following basic lemma:
Lemma 5.7. Let $\mathbb{Q}_p \subset F \subset E$ be a tower of finite field extensions, with $[E : F] = m$ and $[F : \mathbb{Q}_p] = n$. Then we have

(a) For any $x \in E$,
\[ |x|_E = m^n \sqrt{|N_{F/\mathbb{Q}_p}N_{E/F}(x)|_p}. \]

(b) The kernel of $N_{E/F}$ is a subset of the kernel of $|\cdot|_E$, in other words, for any $x \in E$,
\[ N_{E/F}(x) = 1 \text{ implies } |x|_E = 1. \]

(c) $|x|_E = |x|_E$.

Proof. The proof of (a) consists of assembling several standard facts in the theory of finite field extensions of complete normed fields. We use [Gou97], as a reference. By Corollary 5.3.2, there is at most one absolute value on $E$ extending the $p$-adic absolute value on $\mathbb{Q}_p$. By Theorem 5.3.5, this absolute value is given by the formula
\[ |x|_E = m^n \sqrt{|N_{E/\mathbb{Q}_p}(x)|_p}. \]

Now we rewrite the norm inside the radical using the well-known formula
\[ N_{E/\mathbb{Q}_p} = N_{F/\mathbb{Q}_p} \circ N_{E/F}, \]

which appears as Problem 192 on p. 132. This completes the proof of (a). Part (b) is a simple consequence of (a).

Part (c) follows directly from (a) and the calculation
\[ N_{E/F}(x) = x\tau = \tau x = N_{E/F}(\tau). \]

Of course, one can make much more general statements along the lines of Lemma 5.7, for example, replacing $\mathbb{Q}_p$ with a more general non-Archimedean local field, but such generalizations do not concern us here.

5.5 Quantitative separation of $g$ from $I_m$

In the proof of Lemma 4.10 we needed to use the fact that the elements of $g$ cannot get “too close” to the identity $I_m$, in a precise quantitative sense. The purpose of the following proposition is simply to prove this claim.

Proposition 5.8. For $T$ a form matrix as in (2.3)–(2.4), let $g$ be the realization of $u_m$ given by
\[ g = \{xw_mT \mid x \in \mathfrak{gl}_m(E) \text{ and } (xw_m)^* = -xw_m \}. \]
Let 
\[ n > v_E(2). \]

Then there is no \( X \in \mathfrak{g} \) such that \( X - I_m \in \mathfrak{g}_m(\mathcal{P}_E^n) \).

Proof. Suppose otherwise, so that we have

\[ (5.5) \quad xw_m T - I_m \in \mathfrak{gl}_m(\mathcal{P}_E^n). \]

By (2.3) and (2.4), because \( T^{-1} \) is a diagonal matrix with entries of norm 1 or \( q^{-1} \), hence in \( \mathfrak{o} \). Therefore, multiplying an element of \( \mathfrak{gl}_m(\mathcal{P}_E^n) \) on the right by \( T^{-1} \) multiplies each column by an integer. Multiply (5.5) on the right by \( T^{-1} \) to obtain

\[ (5.6) \quad xw_m - T^{-1} \in \mathfrak{gl}_m(\mathcal{P}_E^n). \]

Then apply (\( \cdot \))^* to (5.6). Using part (c) of Lemma 5.7, we have

\[ (xw_m)^* - (T^{-1})^* \in \mathfrak{gl}_m(\mathcal{P}_E^n), \]

so that by the description of \( \mathfrak{g} \) in the hypotheses and (2.5)

\[ (5.7) \quad -xw_m - (T^{-1}) \in \mathfrak{gl}_m(\mathcal{P}_E^n). \]

Add (5.6) and (5.7) to get

\[ 2T^{-1} \in \mathfrak{gl}_m(\mathcal{P}_E^n). \]

By (4.8) and (4.9) this means that

\[ 2b(v_i, v_i) \in \mathcal{P}_E^n, \]

implying

\[ |2|_E q \quad \text{or} \quad |2|_E \times 1 \leq q^{-n}, \]

i.e.,

\[ |2|_E \leq q^{-n-1} \quad \text{or} \quad |2|_E \leq q^{-n}. \]

Since \( n > v_E(2) \), either of these condition will produce a contradiction. \( \Box \)
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