Generalized characteristic polynomials of graph bundles

Dongseok Kim a, Hye Kyung Kim b, Jaeun Lee c,∗

a Department of Mathematics, Kyonggi University, Suwon 443-760, Republic of Korea
b Mathematics, Catholic University of Taegu, Kyongsan 712-702, Republic of Korea
c Department of Mathematics, Yeungnam University, Kyongsan 712-749, Republic of Korea

Received 29 May 2007; accepted 25 March 2008
Available online 20 May 2008
Submitted by R.A. Brualdi

Abstract

In this paper, we find computational formulae for generalized characteristic polynomials of graph bundles. We show that the number of spanning trees in a graph is the partial derivative (at (0, 1)) of the generalized characteristic polynomial of the graph. Since the reciprocal of the Bartholdi zeta function of a graph can be derived from the generalized characteristic polynomial of a graph, consequently, the Bartholdi zeta function of a graph bundle can be computed by using our computational formulae.

© 2008 Elsevier Inc. All rights reserved.

AMS classification: 05C50; 05C25; 15A15; 15A18

Keywords: Generalized characteristic polynomials; Graph bundles; The Bartholdi zeta functions

1. Introduction

One of classical invariants in graph theory is the characteristic polynomial which comes from the adjacency matrices. It displays not only graph theoretical properties but also algebraic prospec-
tives, such as spectra of graphs. There have been many meaningful generalizations of characteristic polynomials [4,10]. In particular, we are interested in one found by Cvetkovic et al. as a polynomial on two variables [4],

This research was supported by the Yeungnam University research grants in 2007.
∗ Corresponding author.
E-mail addresses: dongseok@kgu.ac.kr (D. Kim), hkkim@cu.ac.kr (H.K. Kim), julee@yu.ac.kr (J. Lee).
$$FG(\lambda, \mu) = \det(\lambda I - (A(G) - \mu D(G))).$$

The zeta functions of finite graphs \cite{1,2,6} feature of Riemann’s zeta functions and can be considered as an analogue of the Dedekind zeta functions of a number field. It can be expressed as the determinant of a perturbation of the Laplacian and a counterpart of the Riemann hypothesis \cite{15}. Bartholdi introduced the Bartholdi zeta function $Z_G(u, t)$ of a graph $G$ together with a comprehensive overview and problems on the Bartholdi zeta functions \cite{1}. He also showed that the reciprocal of the Bartholdi zeta function of $G$

$$Z_G(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{\nu - \varepsilon} \det[I - A(G)t + (1 - u)(\varpi(G) - (1 - u)I)t^2].$$

Kwak et al. studied the Bartholdi zeta functions of some graph bundles having regular fibers \cite{9}. Mizuno and Sato also studied the zeta function and the Bartholdi zeta function of graph coverings \cite{11,12}. Recently, it was shown that the Bartholdi zeta function $Z_G(u, t)$ can be found as the reciprocal of the generalized characteristic polynomials $FG(\lambda, \mu)$ with a suitable substitution \cite{7}.

The aim of the present article is to find computational formulae for generalized characteristic polynomials $FG(\lambda, \mu)$ of graph bundles and its applications. For computational formulae, we show that if the fiber of the graph bundle is a Schreier graph, the conjugate class of the adjacency matrix has a representative whose characteristic polynomial can be computed efficiently using the representation theory of the symmetric group. We also provide computational formulae for generalized characteristic polynomials of graph bundles $G \times F$ where the images of $\phi$ lie in an abelian subgroup $\Gamma$ of $\text{Aut}(F)$. To demonstrate the efficiency of our computation formulae, we calculate the generalized characteristic polynomials of some $K_n$-bundles $K_{1,m} \times K_n$ and find the number of spanning trees of $K_{1,m} \times K_n$ which is a standard model of network with hubs. Its adjacency matrix, known as a “kite”, is one of important examples in matrix analysis.

The outline of this paper is as follows. First, we review the terminology of the generalized characteristic polynomials and show that the number of spanning trees in a graph is the partial derivative (at $(0, 1)$) of the generalized characteristic polynomial of the given graph in Section 2. Next, we study a similarity of the adjacency matrices of graph bundles and find computational formulae for generalized characteristic polynomial $FG(\lambda, \mu)$ of graph bundles in Section 3. In Section 4, we find the generalized characteristic polynomial of $K_{1,m} \times K_n$ and find the number of spanning trees of $K_{1,m} \times K_n$.

2. Generalized characteristic polynomials and complexity

In this section, we review the definitions and useful properties of the generalized characteristic polynomials and find the number of spanning trees in a graph using the generalized characteristic polynomials.

Let $G$ be an undirected finite simple graph with vertex set $V(G)$ and edge set $E(G).$ Let $\nu_G$ and $\varepsilon_G$ denote the number of vertices and edges of $G$, respectively. An adjacency matrix $A(G) = (a_{ij})$ is the $\nu_G \times \nu_G$ matrix with $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ otherwise. The degree matrix $\varpi(G)$ of $G$ is the diagonal matrix whose $(i, i)$th entry is the degree $d_G(v_i)$ of $v_i$ in $G$ for each $1 \leq i \leq \nu_G$. The complexity $\kappa(G)$ of $G$ is the number of spanning trees in $G$. An automorphism of $G$ is a permutation of the vertex set $V(G)$ that preserves the adjacency. By $|X|$, we denote the cardinality of a finite set $X$. The set of automorphisms forms a permutation group, called the automorphism group $\text{Aut}(G)$ of $G$. The characteristic polynomial of $G$, denoted by $\Phi(G; \lambda)$, is the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G).$ Cvetkovic et al.
introduced a polynomial on two variables of $G$, $F_G(\lambda, \mu) = \det(\lambda I - (A(G) - \mu \mathcal{D}(G)))$ as a generalization of characteristic polynomials of $G$ [4], for example, the characteristic polynomial of $G$ is $F_G(\lambda, 0)$ and the characteristic polynomial of the Laplacian matrix $\mathcal{D}(G) - A(G)$ of $G$ is $(-1)^{\nu_G} F_G(-\lambda, 1)$.

In [7], it was shown that the Bartholdi zeta function $Z_G(u, t)$ of a graph can be obtained from the polynomial $F_G(\lambda, \mu)$ with a suitable substitution as follows:

$$Z_G(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{\nu_G} t^{\nu_G} F_G\left(\frac{1}{t} - (1 - u)^2 t, (1 - u)t\right)$$

and

$$F_G(\lambda, \mu) = \frac{\lambda^{\nu_G}}{(1 - \mu^2)^{\nu_G}} Z_G\left(1 - \frac{\lambda \mu}{1 - \mu^2}, \frac{1 - \mu^2}{\lambda}\right)^{-1}.$$

The complexities for various graphs have been studied [11,13]. In particular, Northshield showed that the complexity of a graph $G$ can be given by the derivative

$$f'_G(1) = 2(\varepsilon_G - \nu_G) \kappa(G)$$

of the function $f_G(u) = \det(I - uA(G) + u^2(\mathcal{D}(G) - I))$, for a connected graph $G$ [13]. By considering the idea of taking the derivative, we find that the complexity of a finite graph can be expressed as the partial derivative of the generalized characteristic polynomial $F_G(\lambda, \mu)$ evaluated at $(0, 1)$.

**Theorem 2.1.** Let $F_G(\lambda, \mu) = \det(\lambda I - (A(G) - \mu \mathcal{D}(G)))$ be the generalized characteristic polynomial of a graph $G$. Then the number of spanning trees in $G$, $\kappa(G)$, is

$$\frac{1}{2\varepsilon_G} \frac{\partial F_G}{\partial \mu} \bigg|_{(0, 1)} = \prod_k \det(B_{0, 1} + M^k).$$

where $\varepsilon_G$ is the number of edges of $G$.

**Proof.** Let $B_{\lambda, \mu} = \lambda I - (A(G) - \mu \mathcal{D}(G)) = ((b_{\lambda, \mu})_{ij})$ and let $B_{\lambda, \mu}^k = ((B_{\lambda, \mu}^k)_{ij})$ denote the matrix $B_{\lambda, \mu}$ with each entry of $k$th row replace the corresponding partial derivative with respect to $\mu$. Then

$$\frac{\partial}{\partial \mu} \det((b_{\lambda, \mu})) = \sum_{\sigma} \text{sgn}(\sigma) \frac{\partial}{\partial \mu} \left(\prod_i (b_{\lambda, \mu})_{i\sigma(i)}\right) = \sum_k \sum_{\sigma} \text{sgn}(\sigma) \prod_i (b_{\lambda, \mu})^k_{i\sigma(i)} = \sum_k (\det B_{\lambda, \mu}^k).$$

Since $(b_{0, 1})^k_{ij} = d^G_i \delta_{ij} + a_{ij}(\delta_{kj} - 1) = d_i \delta_{ij} - a_{ij} + a_{ij} \delta_{ki}$,

$$\frac{\partial F_G}{\partial \mu} \bigg|_{(0, 1)} = \prod_k \det(B_{0, 1} + a_{ij}(\delta_{ki})).$$

Let $(M^k)_{ij} = a_{ij} \delta_{ki}$ and let $(C_{0, 1})_{ij}$ be the cofactor of $b_{ij}$ in $B_{0, 1}$. Then

$$\det(B_{0, 1} + M^k) = \sum_j (B_{0, 1} + M^k)_{kj} (-1)^{k+j} (C_{0, 1})_{kj} = d_k (C_{0, 1})_{kk}.$$
Since $\det B_{0,1} = 0$, $(C_{0,1})_{ij} = \kappa(G)$ for all $i$ and $j$ [3]. Hence

$$\frac{\partial F_G}{\partial \mu} \bigg|_{(0,1)} = \sum_k d_k(C_{0,1})_{kk} = \kappa(G) \sum_k d_k = 2\varepsilon_G \kappa(G).$$

Therefore

$$\kappa(G) = \frac{1}{2\varepsilon_G} \frac{\partial F_G}{\partial \mu} \bigg|_{(0,1)}.$$

3. Generalized characteristic polynomials of graph bundles

Let $G$ be a connected graph and let $\tilde{G}$ be the digraph obtained from $G$ by replacing each edge of $G$ with a pair of oppositely directed edges. The set of directed edges of $G$ is denoted by $E(\tilde{G})$. By $e^{-1}$, we mean the reverse edge to an edge $e \in E(\tilde{G})$. We denote the directed edge $e$ of $\tilde{G}$ by $uv$ if the initial and the terminal vertices of $e$ are $u$ and $v$, respectively. For a finite group $G$, the action of $\text{Aut}\ G$ on $G$ is a function $\phi : E(\tilde{G}) \to G$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\tilde{G})$. We denote the set of all $G$-voltage assignments of $G$ by $C^1(G; G)$.

Let $F$ be another graph and let $\phi \in C^1(G; \text{Aut}(F))$. Now, we construct a graph $G \times^\phi F$ with the vertex set $V(G \times^\phi F) = V(G) \times V(F)$, and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \times^\phi F$ if either $u_1 u_2 \in E(\tilde{G})$ and $v_2 = \phi(u_1 u_2) v_1$ or $u_1 = u_2$ and $v_1 v_2 \in E(F)$. We call $G \times^\phi F$ the $F$-bundle over $G$ associated with $\phi$ (or, simply a graph bundle) and the first coordinate projection induces the bundle projection $p^\phi : G \times^\phi F \to G$. The graphs $G$ and $F$ are called the base and the fibre of the graph bundle $G \times^\phi F$, respectively. Note that the map $p^\phi$ maps vertices to vertices, but the image of an edge can be either an edge or a vertex. If $F = K_n$, the complement of the complete graph $K_n$ of $n$ vertices, then an $F$-bundle over $G$ is just an $n$-fold graph covering over $G$. If $\phi(e)$ is the identity of $\text{Aut}(F)$ for all $e \in E(\tilde{G})$, then $G \times^\phi F$ is just the Cartesian product of $G$ and $F$, details can be found in [8].

Let $\phi$ be an $\text{Aut}(F)$-voltage assignment of $G$. For each $\gamma \in \text{Aut}(F)$, let $\tilde{G}_{(\phi, \gamma)}$ denote the spanning subgraph of the digraph $\tilde{G}$ whose directed edge set is $\phi^{-1}(\gamma)$. Thus the digraph $\tilde{G}$ is the edge-disjoint union of spanning subgraphs $\tilde{G}_{(\phi, \gamma)}$, $\gamma \in \text{Aut}(F)$. Let $V(G) = \{u_1, u_2, \ldots, u_{v_G}\}$ and $V(F) = \{v_1, v_2, \ldots, v_{v_F}\}$. We define an order relation $\leq$ on $V(G \times^\phi F)$ as follows: for $(u_i, v_k), (u_j, v_l) \in V(G \times^\phi F)$, $(u_i, v_k) \leq (u_j, v_l)$ if and only if either $k < \ell$ or $k = \ell$ and $i \leq j$. Let $P(\gamma)$ denote the $v_F \times v_F$ permutation matrix associated with $\gamma \in \text{Aut}(F)$ corresponding to the action of $\text{Aut}(F)$ on $V(F)$, i.e., its $(i, j)$-entry $P(\gamma)_{ij} = 1$ if $\gamma(v_i) = v_j$ and $P(\gamma)_{ij} = 0$ otherwise. Then for any $\gamma, \delta \in \text{Aut}(F)$, $P(\delta \gamma) = P(\delta) P(\gamma)$. Kwak and Lee expressed the adjacency matrix $A(G \times^\phi F)$ of a graph bundle $G \times^\phi F$ as follows.

**Theorem 3.1** [8]. Let $G$ and $F$ be graphs and let $\phi$ be an $\text{Aut}(F)$-voltage assignment of $G$. Then the adjacency matrix of the $F$-bundle $G \times^\phi F$ is

$$A(G \times^\phi F) = \left( \sum_{\gamma \in \text{Aut}(F)} P(\gamma) \otimes A(\tilde{G}_{(\phi, \gamma)}) \right) + A(F) \otimes I_{v_G},$$

where $P(\gamma)$ is the $v_F \times v_F$ permutation matrix associated with $\gamma \in \text{Aut}(F)$ corresponding to the action of $\text{Aut}(F)$ on $V(F)$, and $I_{v_G}$ is the identity matrix of order $v_G$. 
A graph $F$ is called a Schreier graph if there exists a subset $S$ of $S_{vF}$ such that $S^{-1} = S$ and the adjacency matrix $A(F)$ of $F$ is $\sum_{s \in S} P(s)$. Such a subset $S$ is called the connecting set of the Schreier graph $F$. Notice that a Schreier graph with connecting set $S$ is a regular graph of degree $|S|$ and most regular graphs are Schreier graphs [5, Section 2.3]. The definition of the Schreier graph here is different that of original one. But, they are basically identical [5, Section 2.4]. Clearly, every Cayley graph is a Schreier graph.

Now, if the fiber $F$ of $G \times \phi F$ is a Schreier graph with connecting set $S$, then one can have that

$$A(G \times \phi F) = \left( \sum_{\gamma \in \text{Aut}(F)} P(\gamma) \otimes A(\hat{G}_{(\phi, \gamma)}) \right) + \sum_{s \in S} P(s) \otimes I_{vG},$$

where $P(\gamma)$ is the $vF \times vF$ permutation matrix associated with $\gamma \in \text{Aut}(F)$ corresponding to the action of $\text{Aut}(F)$ on $V(F)$, and $I_{vG}$ is the identity matrix of order $vG$. Let $\Gamma$ be the subgroup of $S_{vF}$ generated by $\{\phi(e), s : e \in E(\hat{G}), s \in S\}$. A representation $\rho$ of a group $\Gamma$ over the complex numbers is a group homomorphism from $\Gamma$ to the general linear group $GL(r, \mathbb{C})$ of invertible $r \times r$ matrices over $\mathbb{C}$. The number $r$ is called the degree of the representation $\rho$ [14]. Suppose that $\Gamma \leq S_\ell$ is a permutation group on $\Omega$. It is clear that $P : \Gamma \rightarrow GL(r, \mathbb{C})$ is defined by $\gamma \rightarrow P(\gamma)$, where $P(\gamma)$ is the permutation matrix associated with $\gamma \in \Gamma$ corresponding to the action of $\Gamma$ on $\Omega$, is a representation of $\Gamma$. It is called the permutation representation. Let $\rho_1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\Gamma$ and let $f_i$ be the degree of $\rho_i$ for each $1 \leq i \leq \ell$, where $f_1 = 1$ and $\sum_{i=1}^\ell f_i^2 = |\Gamma|$. It is well-known that the permutation representation $P$ can be decomposed as the direct sum of irreducible representations: $\rho = \bigoplus_{i=1}^\ell m_i \rho_i$ [14]. In other words, there exists a unitary matrix $M$ of order $|\Gamma|$ such that

$$M^{-1} P(\gamma) M = \bigoplus_{i=1}^\ell (I_{m_i} \otimes \rho_i(\gamma))$$

for any $\gamma \in \Gamma$, where $m_i \geq 0$ is the multiplicity of the irreducible representation $\rho_i$ in the permutation representation $P$ and $\sum_{i=1}^\ell m_i f_i = vF$. Notice that $m_1 \geq 1$ because it represents the number of orbits under the action of the group $\Gamma$. Since $F$ is a regular graph of degree $|S|$, one can see that

$$\mathcal{D}(G \times \phi F) = I_{vF} \otimes (\mathcal{D}(G) + |S|I_{vG}).$$

Now, one can have that the matrix $A(G \times \phi F) - \mu \mathcal{D}(G \times \phi F)$ is similar to

$$\bigoplus_{i=1}^\ell I_{m_i} \otimes \left( \sum_{\gamma \in \text{Aut}(F)} \rho_i(\gamma) \otimes A(\hat{G}_{(\phi, \gamma)}) - I_{f_i} \otimes \mu(\mathcal{D}(G) + |S|I_{vG}) \right) + \left( \sum_{s \in S} \rho_i(s) \otimes I_{vG} \right).$$

By summarizing these, we obtain the following theorem.

**Theorem 3.2.** Let $G$ be a connected graph and let $F$ be a Schreier graph with connecting set $S$. Let $\phi : E(\hat{G}) \rightarrow \text{Aut}(F)$ be a permutation voltage assignment. Let $\Gamma$ be the subgroup of the symmetric group $S_{vF}$ generated by $\{\phi(e), s : e \in E(\hat{G}), s \in S\}$. Furthermore, let $\rho_1, \rho_2, \ldots, \rho_\ell$
be the irreducible representations of $\Gamma$ having degree $f_1, f_2, \ldots, f_\ell$, respectively. Then the matrix $A(G \times_\phi F) - \mu \mathcal{D}(G \times_\phi F)$ is similar to
\[
\bigoplus_{i=1}^\ell I_{m_i} \otimes \left( \sum_{\gamma \in \text{Aut}(F)} \rho_i(\gamma) \otimes A(\tilde{G}_{(\phi,\gamma)}) - I_{f_i} \otimes \mu(\mathcal{D}(G)) + |S|I_{\nu_G} \right)
\]
\[
+ \left( \sum_{s \in S} \rho_i(s) \right) \otimes I_{\nu_G},
\]
where $m_i \geq 0$ is the multiplicity of the irreducible representation $\rho_i$ in the permutation representation $P$ and $\sum_{i=1}^\ell m_i f_i = v_F$.

It is easy to see that the following theorem follows immediately from Theorem 3.2.

**Theorem 3.3.** Let $G$ be a connected graph and let $F$ be a Schreier graph with connecting set $S$. Let $\phi : E(\tilde{G}) \to \text{Aut}(F)$ be a permutation voltage assignment. Let $\Gamma$ be the subgroup of the symmetric group $S_{\nu_F}$ generated by $\{\phi(e), s : e \in E(\tilde{G}), s \in S\}$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\Gamma$ having degree $f_1, f_2, \ldots, f_\ell$, respectively. Then the characteristic polynomial $F_{G \times_\phi F}(\lambda, \mu)$ of a graph bundle $G \times_\phi F$ is
\[
\prod_{i=1}^\ell \det \left[ I_{f_i} \otimes \left( (\lambda + \mu|S|)I_{\nu_G} + \mu \mathcal{D}(G) \right) - \sum_{\gamma \in \text{Aut}(F)} \rho_i(\gamma) \otimes A(\tilde{G}_{(\phi,\gamma)}) - \left( \sum_{s \in S} \rho_i(s) \right) \otimes I_{\nu_G} \right]^{m_i},
\]
where $m_i \geq 0$ is the multiplicity of the irreducible representation $\rho_i$ in the permutation representation $P$ and $\sum_{i=1}^\ell m_i f_i = v_F$.

Let $F = \overline{K}_n$ be the trivial graph on $n$ vertices. Then any $\text{Aut}(\overline{K}_n)$-voltage assignment is just a permutation voltage assignment defined in [5], and $G \times_\phi \overline{K}_n = G^\phi$ is just an $n$-fold covering graph of $G$. In this case, it may not be a regular covering. Now, the following comes from Theorem 3.3.

**Corollary 3.4.** Let $G$ be a connected graph and let $F = \overline{K}_n$. The characteristic polynomial $F_{G^\phi}(\lambda, \mu)$ of the connected covering $G^\phi$ of a graph $G$ derived from a permutation voltage assignment $\phi : E(\tilde{G}) \to S_n$ is
\[
F_G(\lambda, \mu) \times \prod_{i=2}^\ell \det \left[ I_{f_i} \otimes \left[ \lambda I_{\nu_G} + \mu \mathcal{D}(G) \right] - \sum_{\gamma \in \Gamma} \rho_i(\gamma) \otimes A(\tilde{G}_{(\phi,\gamma)}) \right]^{m_i},
\]
where $m_i \geq 0$ is the multiplicity of the irreducible representation $\rho_i$ in the permutation representation $P$ and $\sum_{i=1}^\ell m_i f_i = n$.

Next, we consider the characteristic polynomial depending on two variables of graph bundles $G \times_\phi F$ where the images of $\phi$ lie in an abelian subgroup $\Gamma$ of $\text{Aut}(F)$ and the fiber $F$ is $r$-regular.
In this case, for any $\gamma_1, \gamma_2 \in \Gamma$, the permutation matrices $P(\gamma_1)$ and $P(\gamma_2)$ are commutative and $R_F = rI_{v_F}$.

It is well known (see [3]) that every permutation matrix $P(\gamma)$ commutes with the adjacency matrix $A(F)$ of $F$ for all $\gamma \in \text{Aut}(F)$. Since the matrices $P(\gamma)$, $\gamma \in \Gamma$, and $A(F)$ are all diagonalizable and commute with each other, they can be diagonalized simultaneously. i.e., there exists an invertible matrix $M$ such that $M^{-1}P(\gamma)M$ and $M^{-1}A(F)M$ are diagonal matrices for all $\gamma \in \Gamma$. Let $\lambda_{(\gamma,1)}, \ldots, \lambda_{(\gamma,v_F)}$ be the eigenvalues of the permutation matrix $P(\gamma)$ and let $\lambda_{(F,1)}, \ldots, \lambda_{(F,v_F)}$ be the eigenvalues of the adjacency matrix $A(F)$. Then

$$
M^{-1}_F P(\gamma)_F M_F = \begin{bmatrix}
\lambda_{(\gamma,1)} & 0 \\
0 & \ddots \\
0 & \lambda_{(\gamma,v_F)}
\end{bmatrix}
$$
and

$$
M^{-1}_F A(F)_F M_F = \begin{bmatrix}
\lambda_{(F,1)} & 0 \\
0 & \ddots \\
0 & \lambda_{(F,v_F)}
\end{bmatrix}.
$$

Using these similarities, we find that

$$
(M_F \otimes I_{v_G})^{-1}\left(\sum_{\gamma \in \Gamma} P(\gamma) \otimes A(\vec{G}(\phi,\gamma)) + A(F) \otimes I_{v_G}\right)(M_F \otimes I_{v_G})
= \bigoplus_{i=1}^{v_F} \left(\sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\vec{G}(\phi,\gamma)) + \lambda_{(F,i)} I_{v_G}\right).
$$

It is not hard to show that the degree matrix $R(G \times^\phi F)$ is equal to $I_{v_F} \otimes R(G) + R(F) \otimes I_{v_G}$ and

$$
(M \otimes I_{v_G})^{-1}R(G \times^\phi F)(M \otimes I_{v_G}) = I_{v_F} \otimes R(G) + rI_{v_F} \otimes I_{v_G}
= \bigoplus_{i=1}^{v_F} (R(G) + rI_{v_G}).
$$

By summarizing these facts, we find the following theorem.

**Theorem 3.5.** Let $G$ be a connected graph and let $F$ be a connected regular graph of degree $r$. If the images of $\phi \in C^1(G; \text{Aut}(F))$ lie in an abelian subgroup of $\text{Aut}(F)$, then the matrix $A(G \times^\phi F) - \mu R(G \times^\phi F)$ is similar to

$$
\bigoplus_{i=1}^{v_F} \left(\lambda_{(F,i)} - r\mu\right)I_{v_G} + \left(\sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\vec{G}(\phi,\gamma)) - \mu R(G)\right).
$$

Notice that the Cartesian product $G \times F$ of two graphs $G$ and $F$ is a $F$-bundle over $G$ associated with the trivial voltage assignment $\phi$, i.e., $\phi(e) = 1$ for all $e \in E(\vec{G})$ and $A(G) = A(\vec{G})$. The following corollary comes from this observation.
Corollary 3.6. For any connected graph $G$ and a connected $r$-regular graph $F$, the matrix $A(G \times F) - \mu \mathcal{D}(G \times F)$ of the Cartesian product $G \times F$ is similar to
\[
\bigoplus_{i=1}^{v_F} ((\lambda_{(F,i)} - r \mu)I_{v_G} + (A(G) - \mu \mathcal{D}(G))).
\]
In particular, if $G$ is a regular graph of degree $d$, then the matrix $A(G \times F) - \mu \mathcal{D}(G \times F)$ of the cartesian product $G \times F$ is
\[
\bigoplus_{i=1}^{v_F} \bigoplus_{j=1}^{v_G} (\lambda_{(F,i)} + \lambda_{(G,j)} - (r + d)\mu),
\]
where $\lambda_{(G,j)}$ ($1 \leq j \leq v_G$) and $\lambda_{(F,i)}$ ($1 \leq i \leq v_F$) are the eigenvalues of $G$ and $F$, respectively.

Now, we consider the case the images of $\phi \in C^1(G; \text{Aut}(F))$ lie in an abelian subgroup of $\text{Aut}(F)$. A vertex-and-edge weighted digraph is a pair $D_\omega = (D, \omega)$, where $D$ is a digraph and $\omega : V(D) \cup E(D) \to \mathbb{C}$ is a function. We call $\omega$ the vertex-and-edge weight function on $D$. Moreover, if $\omega(e^{-1}) = \overline{\omega(e)}$, the complex conjugate of $\omega(e)$, for each edge $e \in E(D)$, we say that $\omega$ is unitary. Given any vertex-and-edge weighted digraph $D_\omega$, the adjacency matrix $A(D_\omega) = (a_{ij})$ of $D_\omega$ is a square matrix of order $|V(D)|$ defined by
\[
a_{ij} = \sum_{e \in E(v_i, v_j)} \omega(e)
\]
and where $E(v_i, v_j)$ is the set of all directed edge from $v_i$ to $v_j$ in $D$ and the degree matrix $\mathcal{D}_{D_\omega}$ is the diagonal matrix whose $(i, i)$th entry is $\omega(v_i)$. We define
\[
F_{D_\omega}(\lambda, \mu) = \det(\lambda I - (A(D_\omega) - \mu \mathcal{D}_{D_\omega})).
\]
For any $\Gamma$-voltage assignment $\phi$ of $G$, let $\omega_1(\phi) : V(\vec{G}) \cup E(\vec{G}) \to \mathbb{C}$ be the function defined by
\[
\omega_1(\phi)(v) = \deg_G(v), \quad \omega_1(\phi)(e) = \lambda_{(\phi(e), i)}
\]
for $e \in E(\vec{G})$ and $v \in V(\vec{G})$ where $i = 1, 2, \ldots, v_F$. Using Theorem 3.5, we have the following theorem.

Theorem 3.7. Let $G$ be a connected graph and let $F$ be a connected regular graph of degree $r$. If the images of $\phi \in C^1(G; \text{Aut}(F))$ lie in an abelian subgroup of $\text{Aut}(F)$, then the characteristic polynomial $F_{G \times \phi F}(\lambda, \mu)$ of a graph bundle $G \times^\phi F$ is
\[
\prod_{i=1}^{v_F} F_{\vec{G}_{\omega_1(\phi)}}(\lambda + r \mu - \lambda_{(F,i)}, \mu).
\]
Now, the following corollary follows immediately from Corollary 3.6.

Corollary 3.8. For any connected graph $G$ and a connected $r$-regular graph $F$, the characteristic polynomial $F_{G \times F}(\lambda, \mu)$ of the cartesian product $G \times F$ is
\[
\prod_{i=1}^{v_F} F_G(\lambda + r \mu - \lambda_{(F,i)}, \mu).
\]
In particular, if $G$ is a regular graph of degree $d$, then the characteristic polynomial $F_{G \times F}(\lambda, \mu)$ of the cartesian product $G \times F$ is
where \( \lambda_{(G,j)} (1 \leq j \leq v_G) \) and \( \lambda_{(F,i)} (1 \leq i \leq v_F) \) are the eigenvalues of \( G \) and \( F \), respectively.

4. Generalized characteristic polynomial of \( K_{1,m} \times K_n \)

In this section, we find the generalized characteristic polynomial of \( K_{1,m} \times K_n \) and find the number of spanning trees of \( K_{1,m} \times K_n \). As we mentioned in introduction, \( K_{1,m} \times K_n \) is a typical model for networks with hubs thus, we will count its spanning trees. Since \( K_n \) features many nice structures, we discuss the generalized characteristic polynomial of graph bundles with a fiber, Cayley graph.

Let \( \mathcal{A} \) be a finite group with identity \( \text{id}_\mathcal{A} \) and let \( S \) be a set of generators for \( \mathcal{A} \) with the properties that \( S = S^{-1} \) and \( \text{id}_\mathcal{A} \notin S \), where \( S^{-1} = \{ x^{-1} \mid x \in \Omega \} \). The Cayley graph \( \text{Cay}(\mathcal{A}, S) \) is a simple graph whose vertex-set and edge-set are defined as follows:

\[
V(\text{Cay}(\mathcal{A}, S)) = \mathcal{A} \quad \text{and} \quad E(\text{Cay}(\mathcal{A}, S)) = \{|(g, h)| g^{-1}h \in S\}.
\]

From now on, we assume that \( \mathcal{A} \) is an abelian group of order \( n \). Let \( G \) be a graph and let \( \phi : E(\tilde{G}) \to \mathcal{A} \) be an \( \mathcal{A} \)-voltage assignment. Notice that the left action \( \mathcal{A} \) on the vertex set \( \mathcal{A} \) of \( \text{Cay}(\mathcal{A}, S) \) gives a group homomorphism from \( \mathcal{A} \) to \( \text{Aut}(\text{Cay}(\mathcal{A}, S)) \). Let \( P \) be the permutation representation of \( \mathcal{A} \) corresponding to the action. Then the map \( \tilde{\phi} : E(\tilde{G}) \to \text{Aut}(\text{Cay}(\mathcal{A}, S)) \) defined by \( \tilde{\phi}(e) = P(\phi(e)) \) for any \( e \in E(\tilde{G}) \) is an \( \text{Aut}(\text{Cay}(\mathcal{A}, S)) \)-voltage assignment. We also denote it by \( \phi \). Notice that every irreducible representation of an abelian group is linear.

For convenience, let \( \chi_1 \) be the principal character of \( \mathcal{A} \) and \( \chi_2, \ldots, \chi_n \) be the other \( n - 1 \) irreducible characters of \( \mathcal{A} \). Now, by Theorem 3.2, we have that the matrix \( A(\mathcal{A} \times \text{Cay}(\mathcal{A}, S)) = \mu \mathcal{D}(\mathcal{A} \times \text{Cay}(\mathcal{A}, S)) \) is similar to

\[
(|S|(1 - \mu))I_{v_G} + (A(\tilde{G}) - \mu \mathcal{D}(\tilde{G}))
\]

\[
\oplus \bigoplus_{i=2}^{n} \left( \chi_i(S) - |S|\mu \right) I_{v_G} + \left( \sum_{\gamma \in \mathcal{A}} \chi_i(\gamma) A(\tilde{G}(\phi, \gamma)) - \mu \mathcal{D}(\tilde{G}) \right),
\]

where \( \chi_i(S) = \sum_{s \in S} \chi_i(s) \) for each \( i = 2, 3, \ldots, n \). By Theorem 3.7, we have that the characteristic polynomial \( F_{G \times \text{Cay}(\mathcal{A}, S)}(\lambda, \mu) \) of a graph bundle \( G \times \phi \text{Cay}(\mathcal{A}, S) \) is

\[
F_G(\lambda + |S|(\mu - 1), \mu) \times \prod_{i=2}^{n} F_{\tilde{G}_{\omega_i(\phi)}}(\lambda + |S|\mu - \chi_i(S), \mu),
\]

where \( \omega_i(\phi) : V(\tilde{G}) \cup E(\tilde{G}) \to \mathcal{C} \) be the function defined by

\[
\omega_i(\phi)(v) = \text{deg}_G(v), \quad \omega_i(\phi)(e) = \chi_i(\phi(e))
\]

for \( e \in E(\tilde{G}) \) and \( v \in V(\tilde{G}) \) where \( i = 2, 3, \ldots, n \). Let \( K_n \) be the complete graph on \( n \) vertices. Then \( K_n \) is isomorphic to \( \text{Cay}(\mathcal{A}, \mathcal{A} - \{\text{id}_\mathcal{A}\}) \) for any group \( \mathcal{A} \) of order \( n \). Since \( \chi_i(\mathcal{A} - \{\text{id}_\mathcal{A}\}) = n - 1 \) and \( \chi_i(\mathcal{A} - \{\text{id}_\mathcal{A}\}) = -1 \) for each \( i = 2, 3, \ldots, n \), we have

\[
F_{G \times K_n}(\lambda, \mu) = F_G(\lambda + (n - 1)(\mu - 1), \mu) \times \prod_{i=2}^{n} F_{\tilde{G}_{\omega_i(\phi)}}(\lambda + (n - 1)\mu + 1, \mu).
\]
Moreover, if \( \phi \) is the trivial voltage assignment, then

\[
F_{G \times K_n}(\lambda, \mu) = F_G(\lambda + (n - 1)(\mu - 1), \mu) \times F_G(\lambda + (n - 1)\mu + 1, \mu)^{n-1}.
\]

Let \( G \) be the complete bipartite graph \( K_{1,m} \) which is also called a star graph. Notice that \( K_{1,m} \) is a tree and hence every graph bundle \( K_{1,m} \times \phi F \) is isomorphic to the Cartesian product \( K_{1,m} \times F \) of \( K_{1,m} \) and \( F \). It is known [7] that for any natural numbers \( s \) and \( t \)

\[
F_{K_{1,m}}(\lambda, \mu) = (\lambda + t\mu)^s(\lambda + s\mu)^t - st
\]

and hence \( F_{K_{1,m}}(\lambda, \mu) = (\lambda + \mu)^{m-1}(\lambda + \mu)(\lambda + m\mu) - m \). Now, we can see that

\[
F_{K_{1,m} \times \phi K_n}(\lambda, \mu) = F_{K_{1,m}}(\lambda + (n - 1)(\mu - 1), \mu) \times F_{K_{1,m}}(\lambda + (n - 1)\mu + 1, \mu)^{n-1}
\]

\[
= [\lambda + n\mu - (n - 1)]^{m-1}[\lambda + (m + n - 1)\mu - (n - 1)] - m
\]

\[
\times [\lambda + n\mu + 1]^{(m-1)(n-1)}[\lambda + (m + n - 1)\mu + 1] - m]^{n-1}.
\]

Now, by applying Theorem 2.1, we have that the number of spanning trees of \( K_{1,m} \times K_n \) is

\[
\kappa(K_{1,m} \times K_n) = n^{n-2}(m + n + 1)^{n+1}(n + 1)^{(m-1)(n-1)}.
\]

References

[1] L. Bartholdi, Counting pathes in graphs, Enseign. Math. 45 (1999) 83–131.
[2] H. Bass, The Ihara Selberg zeta function of a tree lattice, Int. J. Math. 3 (1992) 717–797.
[3] N. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, London, 1993.
[4] D.M. Cvetkovic, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1979.
[5] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley, New York, 1987.
[6] Y. Ihara, On discrete subgroups of the two by two projective linear group over \( p \)-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.
[7] H.K. Kim, J. Lee, A generalized characteristic polynomial of a graph having a semifree action, Discrete Math. 308 (2008) 555–556.
[8] J.H. Kwak, J. Lee, Characteristic polynomials of some graph bundles II, Linear and Multilinear Algebra 32 (1992) 61–73.
[9] J.H. Kwak, J. Lee, M.Y. Sohn, Bartholdi zeta functions of graph bundles having regular fibers, Eur. J. Combin. 26 (2005) 593–605.
[10] R. Lipton, N. Vishnoi, Z. Zalcstein, A Generalization of the Characteristic Polynomial of a Graph, CC Technical Report, GIT-CC-03-51. <http://citeseer.ist.psu.edu/642697.html>.
[11] H. Mizuno, I. Sato, On the weighted complexity of a regular covering of a graph, J. Combin. Theory Ser. B 89 (2003) 17–26.
[12] H. Mizuno, I. Sato, Bartholdi zeta functions of graph coverings, J. Combin. Theory Ser. B 89 (2003) 27–41.
[13] S. Northshield, A note on the zeta function of a graph, J. Combin. Theory Ser. B 74 (1998) 408–410.
[14] J.-P. Serre, Linear Representations of Finite Groups, Springer-Verlag, New York, 1977.
[15] H. Stark, A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996) 126–165.