Schur–Weyl duality for tensor powers of the Burau representation

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Abstract

Artin’s braid group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$.

For complex parameters $q_1, q_2$ such that $q_1 q_2 \neq 0$, the group $B_n$ acts on the vector space $E = \sum_i C e_i$ with basis $e_1, \ldots, e_n$ by

$\sigma_i \cdot e_i = (q_1 + q_2) e_i + q_1 e_{i+1}, \quad \sigma_i \cdot e_{i+1} = -q_2 e_i,$

$\sigma_i \cdot e_j = q_1 e_j$ if $j \neq i, i + 1$.

This representation is (a slight generalization of) the Burau representation. If $q = -q_2 / q_1$ is not a root of unity, we show that the algebra of all endomorphisms of $E \otimes^r$ commuting with the $B_n$-action is generated by the place-permutation action of the symmetric group $S_r$ and the operator $p_1$, given by

$p_1(e_1 \otimes e_2 \otimes \cdots \otimes e_r) = q^{r-1} \sum_{i=1}^n e_i \otimes e_2 \otimes \cdots \otimes e_r.$

Equivalently, as a $(C B_n, P_r(\lfloor n \rfloor q))$-bimodule, $E \otimes^r$ satisfies Schur–Weyl duality, where $P_r(\lfloor n \rfloor q)$ is a certain subalgebra of the partition algebra $P_r(\lfloor n \rfloor q)$ on $2r$ nodes with parameter $\lfloor n \rfloor q = 1 + q + \cdots + q^{n-1}$, isomorphic to the semigroup algebra of the "rook monoid" studied by W. D. Munn, L. Solomon, and others.

1 Introduction

Although most results hold over any field of characteristic zero, we will work over the complex field $\mathbb{C}$ throughout this paper, so we abbreviate $\otimes \mathbb{C}$ to $\otimes$, $\dim \mathbb{C}$ to $\dim$, etc. The partition algebra $P_r(n)$ (see Sect. 6) was introduced independently by Martin [31,32] and Jones [23] in connection with the Potts model in mathematical physics, as the generic centralizer algebra $\text{End}_{W_n}(E^{\otimes r})$ for the natural permutation representation $E$ of the Weyl group $W_n$ of $\text{GL}(E) \cong \text{GL}_n(\mathbb{C})$, acting diagonally on $E^{\otimes r}$. By [23], there is a Schur–Weyl duality for $E^{\otimes r}$ as a $(C W_n, P_r(n))$-bimodule; this was recently applied [4] to explain stability properties of Kronecker coefficients and Deligne [8] gave a categorical framework for these algebras.
In this paper, we replace $E$ by the (unreduced) Burau representation $E$ of Artin’s braid group $B_n$ and ask for a combinatorial description of the centralizer algebra $\text{End}_{B_n}(E^{\otimes r})$, with $B_n$ acting diagonally on the tensor power. The paper [22] focused attention on the class of representations of $B_n$ in which the image $T_i$ of the braid group generators $\sigma_i$ satisfy a quadratic relation $(T_i - q_1)(T_i - q_2) = 0$. The Burau representation $E$ is the simplest non-trivial example of such; we choose to work with its general two-parameter version defined in Sect. 2. Most of our results depend not on $q_1, q_2$ but only on their negative ratio $q = -q_2/q_1$.

If $q$ is not a root of unity, our main result (Theorem 9.4) is that $E^{\otimes r}$ satisfies Schur–Weyl duality as a $(\mathbb{C}B_n, \mathcal{P}_r([n]_q))$-bimodule, where $\mathcal{P}_r([n]_q)$ is the subalgebra of the partition algebra $\mathcal{P}_r([n]_q)$ spanned by partial permutation diagrams, at parameter the quantum integer $[n]_q = 1 + q + \cdots + q^{n-1}$. At $q = 1$, the Burau representation is isomorphic to the natural permutation representation of the Weyl group $W_n \cong S_n$ of $\text{GL}(E)$, and generically $\text{End}_{w_n}(E^{\otimes r}) \cong \mathcal{P}_r(n)$, so we think of $\mathcal{P}_r([n]_q)$ as a $q$-analogue of the partition algebra $\mathcal{P}_r(n)$; however, it is not a flat deformation since the two algebras have different dimensions. A different $q$-analogue was studied in [19]; it does have the same dimension as the partition algebra.

Corollary 9.6 gives a second new instance of Schur–Weyl duality involving the braid group, in which the algebra $\mathcal{P}_r([n]_q)$ is replaced by its specialization $\mathcal{P}_r(n)$ at $q = 1$.

The key technical result (Theorem 5.2) is topological, which states that as long as $q = -q_2/q_1$ is not a root of unity, then the Zariski closure of the image of the reduced Burau representation $B_n \to \text{GL}(F)$ contains $\text{SL}(F)$. This leads to our third new instance of Schur–Weyl duality (Theorem 5.3) for the space $F^{\otimes k}$ regarded as a representation of the group $B_n \times S_k$, with $B_n$ acting diagonally and $S_k$ acting by place-permutation.

In Sect. 8, we study the algebra $\mathcal{P}_r(z)$ for $z \neq 0$ an arbitrary complex parameter, from the viewpoint of iterated inflations in the theory of cellular algebras, which enables a quick proof of its semisimplicity and an easy derivation of its irreducible representations. Theorem 8.6, the main result of Sect. 8, gives a new presentation of $\mathcal{P}_r(z)$ by generators and relations. As a consequence (Corollary 8.7) we deduce that $\mathcal{P}_r(z) \cong \mathcal{P}_r(1)$ for any $z \neq 0$. In particular, $\mathcal{P}_r([n]_q)$ is isomorphic to the semigroup algebra of the rook monoid (symmetric inverse semigroup) studied by Munn [36,37], Solomon [41], and others.

Proposition 3.6 implies that $E$ is a semisimple $\mathbb{C}B_n$-module if and only if $[n]_q \neq 0$. So the assumption $[n]_q \neq 0$ implies that $E^{\otimes r}$ is also semisimple as a $\mathbb{C}B_n$-module, by [7, p. 88]; thus by the Jacobson density theorem [21, § 4.3] or [29, Chap. XVII, Theorem 3.2] it follows that $E^{\otimes r}$ satisfies the double-centerizer property (the natural map from $\mathbb{C}B_n$ to $\text{End}_{Z}(E^{\otimes r})$ is surjective, where $Z = \text{End}_{B_n}(E^{\otimes r})$). Thus, the primary task is to identify the centerizer $Z = \text{End}_{B_n}(E^{\otimes r})$ combinatorially, which we do when $q$ is not a root of unity. If $q$ is an $l$th root of unity for $l \neq n$ but $[n]_q \neq 0$ then a description of $Z$ is an interesting open problem.

2 Artin’s braid group

Artin’s braid group $B_n$ is the group defined by generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the braid relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i - j| > 1.$$ (1)
The classical \(n\)-dimensional (unreduced) Burau representation \(B_n \to \text{GL}(E)\), introduced in [6], has a number of definitions [3]. An algebraic definition (see [22]) may be given in terms of a basis \(e_1, \ldots, e_n\) by letting \(\sigma_i\) act by

\[
\sigma_i \cdot e_j = \begin{cases} 
  e_j & \text{if } j \neq i, i+1, \\
  te_{j-1} & \text{if } j = i+1, \\
  (1-t)e_j + e_{j+1} & \text{if } j = i,
\end{cases}
\]  

where \(0 \neq t \in \mathbb{C}\). In other words, if \(Q' = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}\) then \(\sigma_i\) acts via the \(n \times n\) block matrix

\[
\sigma_i \mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & Q' & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix},
\]

where as usual \(I_k\) denotes the \(k \times k\) identity matrix.

We now (slightly) generalize the Burau representation to depend on two parameters \(q_1, q_2\) by setting \(t = -q_2/q_1\) in Eq. (2) and scaling by \(q_1\) to obtain

\[
\sigma_i \cdot e_j = \begin{cases} 
  q_1e_j & \text{if } j \neq i, i+1, \\
  -q_2e_{j-1} & \text{if } j = i+1, \\
  (q_1 + q_2)e_j + q_1e_{j+1} & \text{if } j = i.
\end{cases}
\]  

In other words, if we define \(Q = \begin{bmatrix} q_1 + q_2 & -q_2 \\ q_1 & 0 \end{bmatrix}\) then \(\sigma_i\) acts on \(E\) via the \(n \times n\) block diagonal matrix

\[
\beta_i = \begin{bmatrix} q_1I_{i-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & q_1I_{n-i-1} \end{bmatrix}.
\]

The map defined by \(\sigma_i \mapsto \beta_i\) is a representation \(B_n \to \text{GL}(E)\) of the braid group \(B_n\). Its linear extension \(\mathbb{C}B_n \to \text{End}(E)\) to the group algebra \(\mathbb{C}B_n\) factors through (via \(\sigma_i \mapsto T_i \mapsto \beta_i\)) the two-parameter Iwahori–Hecke algebra \(H_n(q_1, q_2)\), defined (as in [2]) by generators \(T_1, \ldots, T_{n-1}\) subject to the relations

\[
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i \quad \text{if } |i-j| > 1
\]

\[
(T_i - q_1)(T_i - q_2) = 0.
\]

We always assume that \(q_1, q_2 \neq 0\), so that the generators \(T_i\) are invertible elements in \(H_n(q_1, q_2)\), with

\[
T_i^{-1} = (T_i - q_1 - q_2)/(q_1q_2).
\]

Although it is easy to eliminate one of the parameters, we prefer to carry both along, as explained in the remark below. Doing so causes no essential difficulty.

**Remark 2.1** Set \(q = -q_2/q_1\). There are well-known algebra isomorphisms

\[
H_n(q_1, q_2) \cong H_n(-1, q), \quad H_n(q_1, q_2) \cong H_n(1, -q)
\]
defined by sending \( T_i \mapsto -q_i T_i \), \( T_i \mapsto q_i T_i \), respectively. Moreover, the map \( T_i \mapsto q^{-1/2}T_i \) defines an algebra isomorphism

\[
H_n(-q^{-1/2}, q^{1/2}) \cong H_n(-1, q),
\]

so \( H_n(-q^{-1/2}, q^{1/2}) \cong H_n(q_1, q_2) \). The algebra \( H_n(-q^{-1/2}, q^{1/2}) \), the “balanced” form of the Iwahori–Hecke algebra, is often preferred in the theory of quantum groups. The generalized Burau representation makes equal sense in all three of these popular one-parameter versions of \( H_n(q_1, q_2) \).

3 Decomposing the Burau representation

In this section we introduce a bilinear form in order to decompose \( E \) as a \( \mathbb{C}B_n \)-module under appropriate conditions. The algebra \( H_n(q_1, q_2) \) is used as a technical aid in obtaining the decomposition; it will not be used anywhere else in this paper.

By direct computation, we notice the following explicit eigenvectors for the \( \beta_i \) operators defined in the previous section.

**Lemma 3.1** Assume that \( q_1 q_2 \neq 0 \). For any \( i = 1, \ldots, n - 1 \) the operator \( \beta_i \) has eigenvectors:

(a) \( e_{i-1}, e_i + e_{i+1}, e_{i+2}, \ldots, e_n \) with eigenvalue \( q_1 \).

(b) \( f_i := q_2 e_i + q_1 e_{i+1} \) with eigenvalue \( q_2 \).

In particular, \( \beta_i \) is diagonalizable if and only if \( q_1 \neq q_2 \) and \( f_0 := e_1 + \cdots + e_n \) is a simultaneous eigenvector for all the \( \beta_i \).

**Proof** Parts (a), (b) are easily checked. Observe that \( f_i = q_2 e_i + q_1 e_{i+1} \) and \( e_i + e_{i+1} \) are linearly dependent if and only if \( q_1 = q_2 \), which proves the diagonalizability claim. \( \square \)

**Remark 3.2** If \( q_1 = q_2 \), then the \( \beta_i \) have only one eigenvalue and the corresponding eigenspace has dimension \( n - 1 \).

By Lemma 3.1, the simultaneous eigenvector \( f_0 \) spans a line

\[
L = \mathbb{C}f_0 = \mathbb{C}(e_1 + \cdots + e_n) \subset E.
\]

The line \( L \) is an \( H_n(q_1, q_2) \)-submodule, hence also a \( \mathbb{C}B_n \)-submodule, of \( E \). Let \( F \) be the subspace

\[
F = \sum_{i=1}^{n-1} \mathbb{C}f_i = \sum_{i=1}^{n-1} \mathbb{C}(q_2 e_i + q_1 e_{i+1})
\]

spanned by the eigenvectors in Lemma 3.1(b). Since by assumption \( q_1, q_2 \) are nonzero, the spanning vectors are linearly independent, so \( F \) is a subspace of \( E \) of dimension \( n - 1 \).

We leave the elementary proof of the following to the reader.

**Lemma 3.3** For any \( i, j = 1, \ldots, n - 1 \) the action of \( \sigma_i \in B_n \) on \( f_j \in F \) is given by the rules

(a) \( \sigma_i \cdot f_j = q_{1j} f_j \) if \( j \neq i - 1, i, i + 1 \).

(b) \( \sigma_i \cdot f_{i-1} = q_{1i-1} f_{i-1} + q_1 f_i \).

(c) \( \sigma_i \cdot f_i = q_2 f_i \).

(d) \( \sigma_i \cdot f_{i+1} = -q_2 f_i + q_1 f_{i+1} \).
Lemma 3.5 Suppose that $q$ is asymmetric bilinear form $\langle - \rangle$. Hence, $F$ is also an $H_n(q_1, q_2)$-submodule of $E$, with the $T_i$ acting on $F$ by the same formulas.

We call $F$ the (generalized) reduced Burau representation. For concreteness, the matrices of the action of the $\sigma_i$ with respect to the $(f_j)$-basis of $F$ are listed below:

$$
\sigma_1 \mapsto \begin{bmatrix}
q_2 & -q_2 & 0 & \cdots & 0 \\
0 & q_1 & 0 & \cdots & 0 \\
0 & 0 & q_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & q_1 \\
\end{bmatrix}, \quad 
\sigma_{n-1} \mapsto \begin{bmatrix}
q_1 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & q_1 & 0 & 0 \\
0 & \cdots & 0 & q_1 & 0 \\
0 & \cdots & 0 & q_1 & q_2 \\
\end{bmatrix},
$$

$$
\sigma_i \mapsto \begin{bmatrix}
q_1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & q_1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & q_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & q_1 \\
\end{bmatrix}, \quad (1 < i < n - 1),
$$

where the unique diagonal $q_2$-entry is in the $i$th row and column of the matrix of $\sigma_i$.

Remark 3.4 Note that the determinant of the $\sigma_i$ on $F$ is $q_1^{n-2}q_2$.

Next, we aim to show that $E = L \oplus F$ under suitable hypotheses. To that end, we define a symmetric bilinear form $\langle - , - \rangle$ on $E$ by

$$
\langle e_i, e_j \rangle = \delta_{ij}q^{i-1} \tag{6}
$$

extended bilinearly, where $q = -q_2/q_1$. Let $J = \text{diag}(1, q, \ldots, q^{n-1})$ be the matrix of the form. Since $q_1q_2 \neq 0$, it follows that $\det J \neq 0$, so the form is nondegenerate.

Lemma 3.5 Suppose that $q_1q_2 \neq 0$. Then, $F = L^\perp$, the orthogonal complement with respect to the form.

Proof We have $\langle f_0, f_i \rangle = \langle q_2 e_i + q_1 e_{i+1}, f_0 \rangle = q_2 q^{i-1} + q_1 q^i = 0$ for all $i = 1, \ldots, n - 1$. Hence, $F \subset L^\perp$. Since $\dim L^\perp = n - 1$ by the standard theory of bilinear forms, it follows by dimension comparison that the inclusion is equality.

Proposition 3.6 Suppose that $q_1q_2 \neq 0$. Set $q = -q_2/q_1$ and $[n]_q = 1 + q + \cdots + q^{n-1}$.

(a) $E = L \oplus F$ if and only if $[n]_q \neq 0$. When this holds, it is an orthogonal decomposition as $H_n(q_1, q_2)$-modules, hence also as $\mathbb{C}B_n$-modules.

(b) If $n > 2$, then $F$ is irreducible as a $\mathbb{C}B_n$-module if and only if $[n]_q \neq 0$. (If $n = 2$, then $F$ is irreducible for any $q$.)

Proof (a) We have $E = L \oplus F$ if and only if $L \cap F = 0$. As $\dim L = 1$ and $F = L^\perp$, this fails if and only if $L \subset F$, i.e., $L \subset L^\perp$, which is true if and only if the restriction of the form to $L$ is degenerate (i.e., $L$ is a degenerate subspace). Since

$$
\langle f_0, f_0 \rangle = \langle e_1 + \cdots + e_n, e_1 + \cdots + e_n \rangle = 1 + q + \cdots + q^{n-1} = [n]_q
$$
it follows that the subspace $L$ is degenerate if and only if $[n]_q = 0$.

(b) First, observe that it is immediate from parts (b), (d) of Lemma 3.3 that any $CB_n^*$-submodule of $F$ containing any one of the basis elements $f_i$ must be equal to $F$ itself.

Now let $S$ be a proper $CB_n^*$-submodule of $F$. It is not the zero module, so it contains at least one nonzero vector

$$x = x_1 f_1 + \cdots + x_{n-1} f_{n-1},$$

where the $x_i \in \mathbb{C}$ are not all zero. Applying the formulas in Lemma 3.3, we see for any $i = 1, \ldots, n-1$ that

$$-q_1 x + \sigma_i \cdot x = \left((q_2 - q_1) x_i + q_1 x_{i-1} - q_2 x_{i+1}\right)f_i = q_1 \left((-q - 1)x_i + x_{i-1} + qx_{i+1}\right)f_i,$$

where we stipulate that $x_0 = x_n = 0$ to make the edge cases $i = 1, n - 1$ sensible. If the factor $(-q - 1)x_i + x_{i-1} - qx_{i+1} \neq 0$ for any $i$, then $f_i \in S$ and thus $S$ must contain $F$ (by the previous paragraph), so $S = F$. In other words, in order for a proper submodule $S$ to exist, it is necessary that

$$(-q - 1)x_i + x_{i-1} + qx_{i+1} = 0 \quad \text{for all } i = 1, \ldots, n - 1$$

so the $x_i$ satisfy a homogeneous linear system, and hence (as the $x_i$ are not all zero) the determinant of its coefficient matrix must be zero. The coefficient matrix in question is a banded tridiagonal $(n - 1) \times (n - 1)$ matrix of the form

$$A_n = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & c & a & b & \\ & & c & a \\ & & & & \end{bmatrix},$$

where $a = -(q + 1), b = q, c = 1$. Determinants of tridiagonal matrices are known as continuants and satisfy the simple recursion [35, Chapter XIII]

$$\det A_n = a \det A_{n-1} - bc \det A_{n-2}. \quad (7)$$

(This can be checked directly by basic linear algebra.) In the present case, this amounts to the formula $D_n = -(q + 1)D_{n-1} - qD_{n-2}$ where we have set $D_n = \det A_n$. Since $D_3 = [3]_q$, $D_4 = -[4]_q$, it follows by induction that $D_n = (-1)^{n+1}[n]_q$ for all $n \geq 3$. We conclude that in order to have a proper submodule of $F$ it is necessary that $[n]_q = 0$. That is, $[n]_q \neq 0$ implies that $F$ is irreducible.

Conversely, if $[n]_q = 0$ then by the proof of part (a) we have $L \subset F$, and $L$ is a proper submodule since $\dim F > 1$. 

\[\square\]

\textbf{Remark 3.7} (i) Notice that the decomposition $E = L \oplus F$ holds at $q = 1$.

(ii) The proof shows that $L \subset F$ whenever $[n]_q = 0$. 

\[\square\]
4 The centralizer $\text{End}_{B_n}(E)$

Henceforth, we set $[n]_q = 1 + q + \cdots + q^{n-1}$ and assume that $[n]_q \neq 0$, where $q = -q_2/q_1$, so that $E = L \oplus F$. Note that $[n]_q = 0$ implies that $q^n = 1$ and thus that $q$ is a root of unity. So if $q$ is not a root of unity, then $[n]_q \neq 0$. We wish to compute the algebra $\text{End}_{B_n}(E)$ of operators commuting with the image of the generalized Burau representation $B_n \rightarrow \text{End}(E)$. This is the algebra of all $X \in \text{End}(E)$ satisfying

$$\beta_i X = X \beta_i \iff \beta_i X \beta_i^{-1} = X$$

for all $i = 1, \ldots, n - 1$.

Let $P_0$ in $\text{End}(E)$ be the orthogonal projection onto $L$, defined by sending $v \mapsto v_0$, where $v = v_0 + v_0'$ (uniquely) for $v_0 \in L, v_0' \in F$.

**Theorem 4.1** Assume that $q_1 q_2 \neq 0$ and $[n]_q \neq 0$, where $q = -q_2/q_1$. The algebra $\text{End}_{B_n}(E)$ of endomorphisms commuting with the generalized Burau action of $B_n$ on $E$ is spanned by the identity operator $1 = \text{id}_E$ and the operator defined by the $n \times n$ matrix

$$P = \begin{bmatrix}
1 & q & q^2 & \cdots & q^{n-1} \\
1 & q & q^2 & \cdots & q^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & q & q^2 & \cdots & q^{n-1}
\end{bmatrix}. $$

**Proof** Since $E = L \oplus F$ as $\mathbb{C}B_n$-modules, it follows from Schur’s Lemma that

$$\text{End}_{B_n}(E) \cong \text{End}_{B_n}(L) \oplus \text{End}_{B_n}(F)$$

and thus $\dim \text{End}_{B_n}(E) = 2$. Since the identity operator $1 = \text{id}_E$ evidently commutes with the action of $B_n$, we only need a second, linearly independent, commuting operator.

We claim that $P_0$ commutes with the image of $B_n$; that is, $P_0$ commutes with $\beta_i$ for all $i = 1, \ldots, n - 1$. To see this, write $v = v_0 + v_0'$ for $v_0 \in L$ and $v_0' \in F = L^\perp$. Then, $\beta_i$ acts as $q_1$ on $v_0$ by Lemma 3.1 and $\beta_i(v_0') \in F$, so

$$\beta_i P_0 (v) = \beta_i (v_0) = q_1 v_0,$$

$$P_0 \beta_i (v) = P_0 (\beta_i (v_0) + \beta_i (v_0')) = P_0 (q_1 v_0) = q_1 v_0.$$

This proves the claim. As $P_0$ is clearly not a scalar multiple of the identity, we see that $\text{End}_{B_n}(E) \cong \mathbb{C} \text{id}_E \oplus \mathbb{C} P_0$.

To compute the matrix of $P_0$ with respect to the standard basis $e_1, \ldots, e_n$ we calculate

$$\frac{(e_1, e_1 + \cdots + e_n)}{(e_1 + \cdots + e_n, e_1 + \cdots + e_n)} = \frac{q^{i-1}}{1 + q + \cdots + q^{n-1}}.$$

This shows that the matrix of $P_0$ is $(1 + q + \cdots + q^{n-1})^{-1}$ times the matrix $P = (q^{i-1})_{1 \leq i, j \leq n}$, which completes the proof.

**Remark 4.2** (i) Under the hypothesis of the theorem, the operator $1 - P_0$ (where $1 = \text{id}_E$) is projection onto $F$. It also commutes with the action of $B_n$. Hence, the set $(1 - P_0, P_0)$ is another basis of $\text{End}_{B_n}(E)$.

(ii) The matrix $P$ factors as $P = U J$ where $U$ is the matrix of all ones and $J$ the matrix of the form $(-, -)$. Also, $[n]_q = \text{trace}(P) = \text{trace}(J)$.
5 Schur–Weyl duality for $F^\otimes k$

As always, we assume that $q_1 q_2 \neq 0$. Assume that $q = -q_2/q_1$ is not a root of unity. In this section, we establish a new instance of Schur–Weyl duality for the tensor power $F^\otimes k$, regarded as a representation of the group $B_n \times S_k$, where $S_k$ is the symmetric group on $k$ letters acting by place-permutation and $B_n$ acts diagonally. In particular, we show that $End_{B_n}(F^\otimes k)$ is spanned by the image of the action of $S_k$.

Lemma 5.1 Put $G = \rho(B_n)$, where $\rho : B_n \to GL(F)$ is the reduced Burau representation. Let $\overline{G}$ be the Zariski closure of $G$ in $GL(F)$. If $\overline{G}$ contains $SL(F)$, then

$$End_{B_n}(F^\otimes k) = End_{GL(F)}(F^\otimes k).$$

Proof First, we observe that $End_{G}(F) = End_{G}(F)$. Indeed, if $A \in End_{G}(F)$, then its commutant algebra

$$Comm(A) = \{X \in End(F) : AX =XA\}$$

is the solution set of the linear system $AX -XA = 0$, so $Comm(A)$ is a Zariski-closed subset of $End(F)$ and thus $Comm(A) \cap GL(F)$ is Zariski-closed in $GL(F)$. Now $G \subset Comm(A) \cap GL(F)$, so $\overline{G} \subset Comm(A) \cap GL(F)$, and hence $A \in End_{\overline{G}}(F)$. This proves that $End_{G}(F) \subset End_{\overline{G}}(F)$, and thus (the opposite inclusion being obvious) we have the desired equality

$$End_{\overline{G}}(F) = End_{G}(F).$$

Now by hypothesis $SL(F) \subset \overline{G} \subset GL(F)$. Thus, we have

$$End_{SL(F)}(F^\otimes k) \supset End_{\overline{G}}(F^\otimes k) \supset End_{GL(F)}(F^\otimes k).$$

Now $End_{SL(F)}(F^\otimes k) = End_{GL(F)}(F^\otimes k)$ since $GL(F)$ and $SL(F)$ differ only by scalars, so it follows that all of the inclusions displayed above are equalities. Since $End_{B_n}(F^\otimes k) = End_{G}(F^\otimes k)$ holds by definition, we are done. \hfill \Box

The following is the key observation.

Theorem 5.2 Assume that $q_1 q_2 \neq 0$ and that $q = -q_2/q_1$ is not a root of unity. Then, the Zariski closure (in $GL(F)$) of the image of the reduced Burau representation $\rho : B_n \to GL(F)$ contains $SL(F)$. Hence,

$$End_{B_n}(F^\otimes k) = End_{GL(F)}(F^\otimes k).$$

Although elementary, the proof of Theorem 5.2 is rather technical, so we defer it to Appendix A. Here, we explore consequences of the theorem.

We write $\lambda \vdash k$ to indicate that $\lambda$ is a partition of $k$, meaning that $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ is a tuple of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$ and $\lambda_1 + \cdots + \lambda_{\ell(\lambda)} = k$. The number $\ell(\lambda)$ of parts of $\lambda$ is its length.

In his tome on the classical groups, Weyl defined the *enveloping algebra* of a group representation $G \to GL(U)$ to be the subalgebra of $End(U)$ generated by the image of the representation. This coincides with the image of the group algebra $\mathbb{C}G$ under the natural linear extension $\mathbb{C}G \to End(U)$ of the representation.
Theorem 5.3  Assume that \( q_1 q_2 \neq 0 \) and \( q = -q_2 / q_1 \) is not a root of unity. Then, the space \( F^{\otimes k} \), regarded as a group representation of the direct product \( B_n \times S_k \), satisfies Schur–Weyl duality, in the sense that the enveloping algebra of each group equals the centralizer of the other. In particular,

\[
F^{\otimes k} \cong \bigoplus_{\lambda \vdash k, \ell(\lambda) \leq n-1} \Delta(\lambda) \otimes \text{Sp}^\lambda,
\]

where \( \Delta(\lambda) \) is the Schur module for \( \text{GL}(F) \) of highest weight \( \lambda \), regarded as \( \mathbb{C} B_n \)-module via the representation \( B_n \rightarrow \text{GL}(F) \), and \( \text{Sp}^\lambda \) is the Specht module for the symmetric group \( S_k \) indexed by \( \lambda \).

Proof  As we have \( \text{End}_{B_n}(F^{\otimes k}) = \text{End}_{\text{GL}(F)}(F^{\otimes k}) \) by Theorem 5.2, the first claim follows from classical Schur–Weyl duality (see, e.g., [15, 26, 39, 43]), which goes back to Schur [40]. The decomposition in the last part then follows by well-known standard arguments in the theory of semisimple algebras. □

Remark 5.4  Let \( T \) be a tableau of shape \( \lambda \), where \( \lambda \vdash k \). Let \( R(T), C(T) \) be the subgroups of \( S_k \) preserving, respectively, the numbers in the rows, columns of \( T \). In the group algebra \( \mathbb{C} S_k \), define

\[
a_T = \sum_{\sigma \in R(T)} \sigma, \quad b_T = \sum_{\sigma \in C(T)} \text{sgn}(\sigma) \sigma.
\]

These elements of \( \mathbb{C} S_k \), and the products \( a_T b_T, b_T a_T \) in either order, are Young symmetrizers. Then (see, e.g., [1, 9]), we have isomorphisms:

(i)  \( \Delta(\lambda) \cong b_T a_T (F^{\otimes k}) \).
(ii)  \( \text{Sp}^\lambda \cong \mathbb{C} S_k b_T a_T \).

It makes no difference here if \( b_T a_T \) is replaced by \( a_T b_T \). The isomorphism in (i) is as \( \mathbb{C} \text{ GL}(F) \)-modules, which induces an isomorphism as \( \mathbb{C} B_n \)-modules. Furthermore, \( \Delta(\lambda) \) (resp., \( \text{Sp}^\lambda \)) has a basis indexed by the set of all semistandard (resp., standard) tableaux of shape \( \lambda \).

6 The partition algebra

Fix a complex number \( z \neq 0 \). The partition algebra \( \mathcal{P}_r(z) \) was introduced independently by Martin and Jones [23, 31, 32] in connection with the Potts model in mathematical physics, as a generalization of the Temperley–Lieb [42] and Brauer [5] algebras. It is a “diagram algebra” with a graphical basis consisting of graphs on \( 2r \) nodes, depending on a parameter \( z \). We refer the reader to [18] for a convenient summary of many basic properties of \( \mathcal{P}_r(z) \).

Recall that \( \mathcal{P}_r(z) \) has a basis \( \mathcal{P}_r \) consisting of all set partitions of the set \( \{1, \ldots, r, 1', \ldots, r'\} \). Equivalently, \( \mathcal{P}_r \) may be identified with the set of equivalence relations on \( \{1, \ldots, r, 1', \ldots, r'\} \). The various (disjoint) subsets of a set partition \( d \) in \( \mathcal{P}_r \) are called the blocks of \( d \). Conventionally, \( d \) is depicted as a graph with \( 2r \) nodes, arranged in two parallel horizontal rows, numbered \( 1, \ldots, r \) on the top and \( 1', \ldots, r' \) on the bottom, with two nodes connected by a path if and only if they lie in the same block. Thus, the connected components of the graph determine the blocks of the set partition (and the graphical depiction is not necessarily unique).
Composition of graphs $d_1, d_2$ in $\mathcal{P}_r$ is defined by stacking $d_1$ above $d_2$, identifying the top row of $d_2$ with the bottom row of $d_1$, and omitting any connected components contained entirely in the middle two (identified) rows. The result is always another graph (set partition in $\mathcal{P}_r$) that we denote by $d_1 \circ d_2$. Multiplication $d_1 d_2$ in the partition algebra $\mathcal{P}_r(z)$ is then defined by setting

$$d_1 d_2 = z^N (d_1 \circ d_2),$$

where $N = \text{the number of omitted connected components}$. The linear extension of this rule defines an associative multiplication on $\mathcal{P}_r(z)$.

The composition rule $(d_1, d_2) \mapsto d_1 \circ d_2$ makes $\mathcal{P}_r$ into a monoid, called the partition monoid. The specialization $\mathcal{P}_r(1)$ is isomorphic to the semigroup algebra $\mathbb{C}\mathcal{P}_r$, consisting of all formal $\mathbb{C}$-linear combinations of the elements of $\mathcal{P}_r$, with multiplication in $\mathbb{C}\mathcal{P}_r$ the linear extension of the monoid operation on $\mathcal{P}_r$.

The only fact about $\mathcal{P}_r(z)$ that we need is the following.

**Lemma 6.1** [18] The partition algebra $\mathcal{P}_r(z)$ is generated by the diagrams of the form

$$s_i = \begin{bmatrix} \cdots & \mathbf{x} & \cdots \end{bmatrix}, \quad p_j = \begin{bmatrix} \cdots & \mathbf{e} & \cdots \end{bmatrix},$$

and

$$p_{i+\frac{1}{2}} = \begin{bmatrix} \cdots & \mathbf{y} & \cdots \end{bmatrix}$$

for $i = 1, \ldots, r - 1$ and $j = 1, \ldots, r$. In $p_j$ column $j$ is the unusual one, while in $s_i$ and $p_{i+\frac{1}{2}}$ the same is true of columns $i, i+1$.

**Remark 6.2**

(i) See [18, Thm. 1.11] for a set of defining relations on the above generators that determine the partition algebra.

(ii) It is known [32, 33] that the set $\Lambda_{cr} = \{\lambda \vdash k : k = 0, 1, \ldots, r\}$ indexes the isomorphism classes of irreducible $\mathcal{P}_r(z)$-modules.

**7 Example**

Assume that $q_1 q_2 \neq 0$ and that $q = -q_2/q_1$ is not a root of unity. Our next overarching goal is to compute the centralizer algebra $\operatorname{End}_{B_n}(E \otimes^{cr})$ of the tensor powers of the unreduced Burau representation $E$. In this section, we work through a motivating example (the case $r = 2$) that suggests the general result to follow.

Thanks to our assumptions, we know that $E = L \oplus F$ as $\mathbb{C}B_n$-modules. Thus, we have

$$E \otimes E = (L \oplus F) \otimes (L \oplus F)$$

$$\cong (L \otimes L) \oplus (L \otimes F \oplus F \otimes L) \oplus (F \otimes F)$$

as $\mathbb{C}B_n$-modules. From the standard isomorphism $X \otimes Y \cong Y \otimes X$ for group representations $X, Y$ we deduce that $L \otimes F \cong F \otimes L$. From Theorem 5.3, we have the semisimple decomposition as $\mathbb{C}B_n$-modules

$$F \otimes F \cong \begin{cases} \text{Sym}^2 F \oplus \wedge^2 F & \text{if } n > 2, \\ \text{an irreducible module} & \text{if } n = 2 \end{cases}$$

(10)
since $F \otimes F$ is one dimensional, hence irreducible, when $n = 2$. We note that $\text{Sym}^2 F \cong \Delta(2)$ and $\wedge^2 F \cong \Delta(1^2)$ as $\mathbb{C} \text{GL}(F)$-modules, and hence also as $\mathbb{C} B_n$-modules.

It is easy to check that $L \otimes L$, $L \otimes F$, and the irreducible components on the right hand side of (10) are pairwise non-isomorphic modules. Thus, by Schur’s Lemma it follows that for $n > 2$ we have

$$\text{End}_{B_n}(E \otimes E) \cong \text{End}_{B_n}(L \otimes L) \oplus \text{End}_{B_n}(L \otimes F \oplus F \otimes L)$$

$$\oplus \text{End}_{B_n}(\text{Sym}^2 F) \oplus \text{End}_{B_n}(\wedge^2 F)$$

while for $n = 2$ we have

$$\text{End}_{B_n}(E \otimes E) \cong \text{End}_{B_n}(L \otimes L) \oplus \text{End}_{B_n}(L \otimes F \oplus F \otimes L)$$

$$\oplus \text{End}_{B_n}(F \otimes F).$$

In light of the isomorphism $L \otimes F \cong F \otimes L$ already mentioned, it follows by Schur’s Lemma that

$$\dim \text{End}_{B_n}(E \otimes E) = \begin{cases} 1^2 + 2^2 + 1^2 + 1^2 = 7 & \text{if } n > 2, \\ 1^2 + 2^2 + 1^2 = 6 & \text{if } n = 2. \end{cases}$$

Now we consider the above computation from a different perspective, which sheds much light on the general situation. Let $p$ be the pseudo-projection $E \twoheadrightarrow L$ defined by the matrix $P = (q^{i,j} - 1)_{1 \leq i, j \leq n}$ of Theorem 4.1, with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $E$. Then, the linear operators

$$p_1 = p \otimes \text{id}_E, \quad p_2 = \text{id}_E \otimes p$$

are both elements of $\text{End}_{B_n}(E \otimes E)$. Since the diagonal action of $B_n$ evidently commutes with place-permutations, the swap operator $s : E \otimes E \rightarrow E \otimes E$ defined by $s(v_1 \otimes v_2) = v_2 \otimes v_1$ also belongs to $\text{End}_{B_n}(E \otimes E)$. Let $A$ be the subalgebra of $\text{End}(E \otimes E)$ generated by $p_1, p_2, s$. Since the generators commute with the action of $B_n$, we have an inclusion $A \subseteq \text{End}_{B_n}(E \otimes E)$.

Now the relations

$$s^2 = 1, \quad p_j^2 = [n]_q p_j \quad (j = 1, 2), \quad p_1 p_2 = p_2 p_1,$$

$$p_2 = sp_1 s, \quad sp_1 p_2 = p_1 p_2 s = p_1 p_2$$

are easily checked to hold in $A$, where $1 = \text{id}_{E \otimes E}$. These relations imply that $A$ is spanned over $\mathbb{C}$ by the seven endomorphisms

$$\{1, s, p_1, p_2, sp_1, p_1 s, p_1 p_2\}$$

since any other word in the generators reduces to one of these.

**Lemma 7.1** Assume that $q = -q_2/q_1$ is not a root of unity.

(a) If $n = 2$, then the elements in (16) satisfy the linear dependence relation

$$p_1 - sp_1 - p_1 s + p_2 - (1 + q)(1 - s) = 0$$

and $\dim A = 6$. 

(b) If \( n > 2 \), then the spanning elements in (16) are linearly independent, hence a basis of \( \mathcal{A} \), and thus \( \text{dim } \mathcal{A} = 7 \). Thus, \( \mathcal{A} \) is the algebra defined by generators \( s, p_1, p_2 \) subject to the defining relations (15).

In both cases, \( \mathcal{A} = \text{End}_H(E \otimes E) \).

Proof First, we note that the operators in (16) take \( e_i \otimes e_j \) to

\[
e_i \otimes e_j, e_j \otimes e_i, q^{-1}f_0 \otimes e_i, q^{-1}e_j \otimes f_0, q^{-1}e_j \otimes f_0, q^{-1}(i-1)(j-1)f_0 \otimes f_0,
\]

respectively. Assume that scalars \( x_j \in \mathbb{C} \) exist such that

\[
x_1 1 + x_2 s + x_3 p_1 + x_4 p_2 + x_5 sp_1 + x_6 p_1 s + x_7 p_1 p_2 = 0.
\]

Then, the mapping on the left hand side takes \( e_i \otimes e_j \) to zero, for all \( i, j = 1, \ldots, n \). In particular, by taking \( i = j \) we obtain

\[
(x_1 + x_2)e_i \otimes e_i + (x_3 + x_6)q^{-1}f_0 \otimes e_i + (x_4 + x_5)q^{-1}e_i \otimes f_0 + x_7 q^{-1}(i-1)^2 f_0 \otimes f_0 = 0,
\]

where \( f_0 = e_1 + \cdots + e_n \). This holds for all \( i \), so we conclude that

\[
x_2 = -x_1, \quad x_6 = -x_3, \quad x_5 = -x_4, \quad x_7 = 0.
\]

So we are reduced to finding \( x_1, x_3, x_4 \). Putting this information back into the first equation and setting \( i = 1, j = 2 \) gives

\[
x_1e_1 \otimes e_2 - x_1e_2 \otimes e_1 + x_3f_0 \otimes e_2 + x_4q e_1 \otimes f_0 - x_4e_2 \otimes f_0 - x_3qf_0 \otimes e_1 = 0.
\]

Looking at the coefficients of \( e_1 \otimes e_2, e_2 \otimes e_1 \), respectively, yields the equations

\[
x_1 + x_3 + x_4q = 0, \quad -x_1 - x_4 - x_3q = 0.
\]

If \( n = 2 \), then these equations have the non-trivial solution \( x_3 = x_4 = 1, x_1 = -(1 + q) = -[2]_q \). This yields the dependence relation that proves (a). If \( n > 2 \), then there are additional equations coming from the coefficients of \( e_1 \otimes e_3, e_3 \otimes e_1 \) which force \( x_3 = x_4 = x_1 = 0 \), proving the linear independence claim in (b). Once we know that the seven spanning elements in (16) are linearly independent, it follows that \( \text{dim } \mathcal{A} = 7 \) and thus the inclusion \( \mathcal{A} \subseteq \text{End}_H(E \otimes E) \) must be an equality, by dimension comparison. This gives the second claim in (b), since the algebra defined by the generators and relations is at most seven dimensional.

The last claim is now clear as well, since in the \( n = 2 \) case it is easy to check that the six spanning elements in (16) are linearly independent, once a chosen dependent operator has been eliminated. \( \square \)

Recall that the Weyl group \( W_n \subset \text{GL}(E) \) acts diagonally on tensor space \( E^\otimes r \), by restriction of the diagonal action of \( \text{GL}(E) \). In this situation, there is a natural commuting action of the partition algebra \( \mathcal{P}_r(n) \) on \( r \) nodes with parameter \( n \), and Schur–Weyl duality holds for \( E^\otimes r \) regarded as a \( (\mathbb{C}W_n, \mathcal{P}_r(n)) \)-bimodule. In particular, the centralizer algebra \( \text{End}_{\mathbb{C}W_n}(E^\otimes r) \) is the image of the representation \( \mathcal{P}_r(n) \to \text{End}(E^\otimes r) \).

Our final result of this section makes a connection with the partition algebra. To make that connection, we replace the parameter \( n \) by its \( q \)-analogue \([n]_q\).

**Proposition 7.2** The generic seven-dimensional algebra defined by generators \( p_1, p_2, s \) subject to the relations (15) of this section may be identified with the subalgebra \( \mathcal{P}_2([n]_q) \) of the partition algebra \( \mathcal{P}_2([n]_q) \) with parameter \([n]_q\) spanned by all partition diagrams on two nodes with no horizontal edges.
Proof This follows from the presentation [18, Thm. 1.11] of the partition algebra along with results of this section. The generators of \( P_r([nq]) \) are denoted by \( p_i, s_j, p_{i+1/2} \) in [18], for \( i = 1, \ldots, r \) and \( j = 1, \ldots, r - 1 \). Let \( S \) be the subalgebra generated by the \( p_i, s_j \).

In the case \( r = 2 \) one can check that the defining relations on our \( p_1, p_2 \) are satisfied in \( P_2([nq]) \), where we identify our \( p_i \) with those elements of [18] and identify \( s = s_1 \). Thus, our algebra is isomorphic to a quotient of the corresponding subalgebra \( P'_2([nq]) \).

By dimension comparison, the two algebras are equal. \( \square \)

To complete the connection to the partition algebra, we note that the basis elements in the algebra \( P'_2([nq]) \) can be written in terms of generators as

\[ 1, s, p_1, p_2, p_1s, sp_1, p_1p_2 \]

and these elements correspond, respectively, with the seven partition diagrams listed below:

These are precisely the partition diagrams in \( P_2([nq]) \) which depict partial permutations.

8 The algebra \( P'_r(z) \) and its representations

Now we study the subalgebra \( P'_r(z) \) of \( P_r(z) \) spanned by all graphs satisfying the conditions of Definition 8.1(a). The main results of this section are:

(i) A description of \( P'_r(z) \) by generators and relations.
(ii) A proof that \( P'_r(z) \) is semisimple if \( z \neq 0 \).
(iii) A construction of the irreducible representations \( \{ C(\lambda) : \lambda \in \Lambda \} \) of \( P'_r(z) \), where \( \Lambda \) is the union of the set of partitions of \( k \), as \( k \) runs from 0 to \( r \).

We will also see that \( P'_r(z) \) has a basis in natural bijection with the set of partial permutations (bijections between subsets) on \( \{1, \ldots, r\} \).

Definition 8.1 We need the following notation.

(a) Let \( P'_r \) be the subset of \( P_r \) consisting of all \( d \) in \( P_r \) satisfying the properties:

\[ \bullet \text{ every block of } d \text{ has cardinality } 1 \text{ or } 2. \]
\[ \bullet \text{ every block of cardinality } 2 \text{ contains exactly one element from } \{1, \ldots, r\} \text{ and one from } \{1', \ldots, r'\}. \]

(b) Every \( d \in P'_r \) determines a unique bijection \( bi(d) \) between two subsets of \( \{1, \ldots, r\} \), sending \( i \) to \( j \) if and only if \( \{i, j'\} \) is a block of \( d \) of cardinality 2. Such bijections are partial permutations on \( \{1, \ldots, r\} \).

(c) Let \( P'_r(z) \) be the subalgebra of \( P_r(z) \) with basis \( P'_r \).

Composition \( (d_1, d_2) \mapsto d_1 \circ d_2 \) makes \( P'_r \) into a submonoid of the partition monoid \( P_r \). Setting \( f_i = bi(d_i) \) for \( i = 1, 2 \) we have the composite map \( f_1f_2 \) defined by

\[ x(f_1f_2) = (x f_1)f_2 \]

where we write maps on the right of their arguments (for compatibility with diagrammatic composition). Then, \( bi(d_1 \circ d_2) = f_1f_2 \), so \( bi \) is a monoid isomorphism of \( P'_r \) onto the monoid of all partial permutations of \( \{1, \ldots, r\} \) (bijections between subsets of \( \{1, \ldots, r\} \)).
under functional composition. In other words, $P'_r$ is isomorphic to the “symmetric inverse semigroup” studied by Munn [36, 37], Solomon [41], and others.

Solomon observed that $P'_r$ is isomorphic to the monoid of $r \times r$ matrices (under matrix multiplication) with at most one nonzero entry, equal to 1, in each row and column. Such matrices enumerate the ways to place non-attacking rooks on an $r \times r$ chessboard, so Solomon introduced the term “rook monoid” for $P'_r$. Observe that the algebra $P'_r(1)$ is the semigroup algebra $\mathbb{C} P'_r$ of the rook monoid.

**Definition 8.2** If $d \in P'_r$ we define its rank, written rank$(d)$, to be the number of edges in $d$. (This coincides with the rank of its rook matrix realization, in the usual sense of rank of a matrix.)

There is a unique element of $P'_r$ of rank zero. Elements of $P'_r$ of maximum rank $r$ correspond to permutations of $\{1, \ldots, r\}$ under the map $d \mapsto \text{bi}(d)$, so the symmetric group $S_r$ is isomorphic to a subgroup of $P'_r$. Furthermore,

$$\dim P'_r(z) = |P'_r| = \sum_{k=0}^{r} \binom{r}{k}^2 k!$$

as there are $\binom{r}{k}^2 k!$ partial permutations of rank $k$, for each $k = 0, 1, \ldots, r$.

Henceforth, we will identify $d \in P'_r$ with its underlying bijection $\text{bi}(d)$, thus identifying $P'_r$ with the rook monoid of partial permutations.

**Lemma 8.3** The rook monoid $P'_r$ is isomorphic to the submonoid of $P_r$ generated by the diagrams

$$s_i = \begin{bmatrix} \cdots & \times & \cdots \end{bmatrix} \quad \text{and} \quad p_j = \begin{bmatrix} \cdots & \cdot & \cdots \end{bmatrix}$$

for $i = 1, \ldots, r-1$, $j = 1, \ldots, r$, where the crossing in $s_i$ is in columns $i$, $i+1$ and column $j$ is the only column in $p_j$ without an edge. Hence, $P'_r(z)$ is the subalgebra of $P_r(z)$ generated by the same diagrams.

**Proof** The $s_i$ generate a copy of the symmetric group $S_r$. This is the set of diagrams in $P'_r(z)$ of full rank $r$. The diagram $d_k = p_1 \cdots p_{k-1}p_k$ is a diagram of rank $r-k$. Left or right multiplication by a permutation diagram in $S_r$ acts on $d_k$ to permute the nodes in the top or bottom row, respectively, thus generating all diagrams of rank $r-k$, for each $k = 0, 1, \ldots, r$.

Let $J$ be the set of diagrams of the form $\prod_{j \in J} p_j$, such that $J$ is a subset of $\{1, \ldots, r\}$. If $i \neq j$, then $p_i, p_j$ commute, so the order of the factors in the product is irrelevant.

**Lemma 8.4** Any diagram $d$ in $P'_r$ may be written in the form

$$d = pw = wp'$$

for some $w \in S_r$, some $p, p' \in J$.

**Proof** Given $d$, we may extend $d$ to a permutation $w$ in $S_r$ by adding edges (usually in more than one way). Pre-multiplying $w$ by an appropriate diagram $p \in J$ erases those added edges; similarly for post-multiplication. □
The extension of $d$ to a permutation $w$ in Lemma 8.4 is in general far from unique. Our next task is to find a way to describe all such extensions. Suppose that $d \in P_r'$. Then, $d$ is a bijection mapping a subset $J \subset \{1, \ldots, r\}$ onto another subset, and there exists a unique map $d^+ : \{1, \ldots, r\} \to \{0, 1, \ldots, r\}$ such that

$$x(d^+) = \begin{cases} xd & \text{if } x \in J, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, if $r = 8$ the partial permutation $d = \begin{array}{cccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$ defines the map $d^+$ given by

$$d^+ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 0 & 5 & 4 & 0 & 6 & 7 \end{pmatrix}$$

in the obvious extension of the usual two-line notation for permutations.

A cycle $(i_1i_2 \ldots i_m)$ is the map $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_m \mapsto i_1$ as usual in the theory of symmetric groups. A link $[j_1j_2 \ldots j_m]$ is the map $j_1 \mapsto j_2 \mapsto \cdots \mapsto j_m \mapsto 0$. In both cases, all other indices are fixed. Disjoint cycles or links commute.

Munn [36] (cf. also [14]) observed that every partial permutation can be written as a product of disjoint cycles and links, and this expression is unique apart from the order of the (commuting) factors. For example, the partial permutation $d$ above can be expressed in the cycle–link notation as

$$d = [1, 2, 3](4, 5)[8, 7, 6].$$

By replacing all links in this expression by the corresponding cycle, we obtain a permutation $w(d) \in S_r$ that we call the canonical extension of $d$. So in the above example, $w(d) = (1, 2, 3)(4, 5)(8, 7, 6)$. The cycles in $w(d)$ corresponding to links in $d$ will be called link–cycles. For instance, the link–cycles for $d = [1, 2, 3](4, 5)[8, 7, 6]$ are $(1, 2, 3)$, $(8, 7, 6)$.

**Lemma 8.5** Let $d \in P_r'$ be a partial permutation, regarded as a bijection $d : X \to Y$ where $X, Y$ are subsets of $\{1, \ldots, r\}$. Let $w(d)$ in $S_r$ be the canonical extension of $d$ to a permutation.

(a) Let $X' = \{1, \ldots, r\} - X$ and $Y' = \{1, \ldots, r\} - Y$. Then,

$$d = pw(d) = w(d)p',$$

where $p = \prod_{j \in X} p_j, p' = \prod_{j \in Y'} p_j$.

(b) If $w \in S_r$ is any extension of $d$, then $w$ is a product of disjoint cycles obtained from the cycle–link decomposition of $d$ by joining one or more of its link–cycles in $w(d)$.

**Proof** (a) is diagrammatically obvious. To prove (b), notice that in order to find an extension $w$ of $d$, one has to choose an isolated node in the bottom row of $d$ for each isolated node in the top row of $d$. In other words, pick a bijection from $X'$ onto $Y'$. The canonical choice preserves the cycles in $w(d)$; every other choice joins one or more link–cycles together. For example, if $d = [1, 2, 3](4, 5)[8, 7, 6]$ then the two possible extensions of $d$ are $w_1 = w(d) = (1, 2, 3)(4, 5)(8, 7, 6)$ and $w_2 = (1, 2, 3, 8, 7, 6)(4, 5)$.  

**Theorem 8.6** The algebra $P_r'(z)$ of partial permutations is isomorphic to the algebra generated by $s_i, p_j$ for $i = 1, \ldots, r - 1$ and $j = 1, \ldots, r$ subject to the relations...
(a) $p_i^2 = zp_jp_i = p_ip_i (i \neq j)$.
(b) $s_i^2 = 1, s_is_{i+1}s_i = s_{i+1}s_is_i, s_is_j = s_js_i$ if $|i - j| > 1$.
(c) $sp_ip_{i+1} = p_ip_{i+1}s_i, sp_is_{i+1}s_i = s_{i+1}s_is_i, sp_{i+1}si = sp_{i+1}si$ if $j \neq i, i + 1$.

**Proof** That the generators satisfy these relations is easily checked by direct computation with diagrams (see also the proof of Theorem 1.11 in [18]). It follows that $P'_r(z)$ is a homomorphic image of the algebra $A$ defined by the given generators and relations, so $\dim A \geq \dim P'_r(z)$. To finish, it suffices to show that $\dim A \leq \dim P'_r(z)$.
The defining relations for $A$ imply the following extension of the first relation in (c):

\[(i, j) p_ip_j = p_ip_j (i, j) = p_ip_j (i \neq j),\]  

(18)

where $(i, j)$ is the transposition that swaps $i$ with $j$. This follows by an easy argument that we leave to the reader.

Let $x$ be any word in $A$ in the generators. The relation $s_ip_is_i = p_i + 1$ in (c) is equivalent to $s_ip_i = p_i + 1$.

Thus, we may use the last two relations in (c) to commute all the $s_i$ to the right of all the $p_j$. Since the $p_j$ all commute with each other, their order is immaterial. If any $p_i$ is repeated, we can use the first relation in (a) to eliminate that repetition, at the expense of introducing a scalar multiple. This shows that $A$ is spanned by the set of words in the generators of the form

\[pw \text{ where } p = \prod_{j \in X} p_j, \quad w \in S_r,\]

and $X$ is a subset of $\{1, \ldots, r\}$. Of course, $w$ is expressed by some product of the $s_i$ (in more than one way) as usual in the Coxeter presentation of symmetric groups.

To finish, it suffices to prove that the spanning set \{pw\} in the previous paragraph can be reduced to one in bijection with the set of partial permutations. Let $d$ be a partial permutation diagram. The canonical word $pw(d)$ is one expression in $A$ for $d$. If $pw$ is any other such word, then it is obtained from $pw(d)$ by inserting one or more transpositions $(i, j)$ between $p, w$ according to rule (18), hence $pw = pw(d)$ in $A$. Such insertions correspond precisely to joining one or more link–cycles in $w(d)$, according to Lemma 8.5(b). This completes the proof.

Theorem 8.6 immediately implies that $P'_r(z)$ is isomorphic to the semigroup algebra $C P'_r$ of the rook monoid, for any $z \neq 0$, a (perhaps surprising) fact which does not seem to have been noticed before.

**Corollary 8.7** For any $z \neq 0$, there is an algebra isomorphism $P'_r(z) \cong P'_r(1) = C P'_r$.

**Proof** The isomorphism is defined on generators by sending $s_i \mapsto s_i$ and $p_j \mapsto z^{-1}p_j$ for all $i = 1, \ldots, r - 1, j = 1, \ldots, r$.

Next, we wish to prove the semisimplicity of $P'_r(z)$ and describe its irreducible representations. Although the semisimplicity follows from Corollaries 8.7 and 2.19 of [41], which goes all the way back to results of Munn in the 1950s, we prefer to give a more efficient modern approach, exploiting the idea of “iterated inflation” introduced by König.
and Xi [27,28] into the theory of cellular algebras [11]. We will follow the recent paper [13] which in particular gives a useful criterion for verifying that a given algebra is an iterated inflation of cellular algebras.

**Definition 8.8** If \( d \) in \( \mathcal{P}'_r \) is a partial permutation, regarded as a bijection \( d : X \to Y \), where \( X, Y \) are subsets of \( \{1, \ldots, r\} \), we write

\[
\text{dom}(d) = X = \{x_1, \ldots, x_k\}, \quad \text{im}(d) = Y = \{y_1, \ldots, y_k\}
\]

for the domain and image of \( d \), where it is assumed that \( x_1 < x_2 < \cdots < x_k, y_1 < y_2 < \cdots < y_k \). The **underlying permutation** of \( d \) is the element \( \pi(d) \) in \( S_k \) defined by the rule

\[
\pi(d) = j \iff x_i d = y_j.
\]

Diagrammatically, this amounts to renumbering the nodes in the domain and image of the corresponding graph.

For example, the partial permutation \( d = \begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot
\end{array} \) has the associated triple

\[
(\text{dom}(d), \pi(d), \text{im}(d)) = (\{1, 2, 4, 5, 7, 8\}, \pi, \{2, 3, 4, 5, 6, 7\})
\]

where \( \pi(d) = \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 6 \end{pmatrix} \in S_6 \) in the usual two-line notation.

Given any \((X, \pi, Y)\) such that \( X, Y \) are subsets of \( \{1, \ldots, r\} \) of the same cardinality \( k \), and \( \pi \in S_k \), there is a corresponding partial permutation \( d \) in \( \mathcal{P}'_r \) such that \((\text{dom}(d), \pi(d), \text{im}(d)) = (X, \pi, Y)\). In other words, \( d \) is determined uniquely by its triple \((\text{dom}(d), \pi(d), \text{im}(d))\).

**Lemma 8.9** Suppose that \( d_1, d_2 \in \mathcal{P}'_r \), regarded as partial permutations. Put \( Z = \text{im}(d_1) \cap \text{dom}(d_2) \). Then:

(a) \( \text{rank}(d_1 \circ d_2) = |Z| \).
(b) \( \text{dom}(d_1 \circ d_2) = (Z)d_1^{-1}, \) the preimage of \( Z \) under \( d_1 \).
(c) \( \text{im}(d_1 \circ d_2) = (Z)d_2, \) the image of \( Z \) under \( d_2 \).
(d) If \( d_1 d_2 = z^N(d_1 \circ d_2) \) is the product in \( \mathcal{P}'_r(z) \), then

\[
N = r - |\text{im}(d_1) \cup \text{dom}(d_2)|.
\]

**Proof** Parts (a)–(c) follow by observing that the paths from top to bottom in the composite diagram formed by stacking \( d_1 \) above \( d_2 \) all pass through the points in \( Z = \text{im}(d_1) \cap \text{dom}(d_2) \). Part (d) follows by noticing that the connected components in the middle of the composite diagram is the cardinality of the intersection of the complements of \( \text{im}(d_1) \) and \( \text{dom}(d_2) \) in \( \{1, \ldots, r\} \); then apply De Morgan’s law.

For the remainder of this section, we mostly follow the notational conventions of [13]; but we will use * to denote anti-involutions. We need to work with based vector spaces. For our purposes, a based vector space is a pair \((U, U)\) consisting of a given vector space \( U \) along with a chosen basis \( U \) (we use script letters for the basis). Fix the parameters \( r \) and \( z \neq 0 \) and consider the following data:

\[
\begin{align*}
A &= \mathcal{P}'_r(z), & A &= \mathcal{P}'_r, \\
B(k) &= \mathcal{S}_k, & B(k) &= S_k, \\
U(k) &= \mathcal{U}(k), & U(k) &= \{M \subseteq \{1, \ldots, r\} : |M| = k\}.
\end{align*}
\]
Then, \((A, A), (B(k), B(k)), (U(k), U(k))\) are all based vector spaces. The algebra \(A\) has a \(\mathbb{C}\)-linear anti-involution \(d \mapsto d^*\) defined on the diagram basis by letting \(d^*\) (for \(d \in A\)) be the diagram obtained from \(d\) by flipping it upside down (reflecting across its horizontal axis of symmetry). There is a similar anti-involution (also denoted by \(*\)) on the group algebra \(B(k) = C_{S_k}\) defined by \(w \mapsto w^{-1}\) for all \(w \in B(k) = S_k\). It is clear that

\[
A \cong \bigoplus_{k=0}^{r} U(k) \otimes B(k) \otimes U(k) \tag{19}
\]

as vector spaces. Indeed, the isomorphism is defined by sending any \(a \in A\) to \(\text{dom}(a) \otimes \pi(a) \otimes \text{im}(a)\) and extending linearly. Thus, we may identify \(A\) with the right hand side of (19). It is also clear that (under this identification) the anti-involutions satisfy the compatibility condition

\[
(u \otimes b \otimes v)^* = v \otimes b^* \otimes u \tag{20}
\]

for any \(u, v \in U(k), b \in B(k)\), and any \(k = 0, 1, \ldots, r\). Now define maps

\[
\phi_k : A \times U(k) \to U(k), \quad \theta_k : A \times U(k) \to B(k)
\]

by the rules

\[
\phi_k(a, u) = \begin{cases} 
  x^{r-\text{rank}(a)}(u)a^{-1} & \text{if } u \subseteq \text{im}(a), \\
  0 & \text{otherwise},
\end{cases}
\]

\[
\theta_k(a, u) = \begin{cases} 
  \pi(a') & \text{if } u \subseteq \text{im}(a), \\
  0 & \text{otherwise},
\end{cases}
\]

where \(a'\) is the restriction of \(a\) to the preimage \((u)a^{-1}\), regarded as a partial permutation \((u)a^{-1} \to u\). We claim that the multiplication rule (8) in the algebra \(A = \mathcal{T}_r^*(z)\) satisfies the relation

\[
(a(u \otimes b \otimes v)) \equiv \phi_k(a, u) \otimes \theta_k(a, u)b \otimes v \mod J(< k) \tag{21}
\]

for any \(u, v \in U(k), b \in B(k), a \in A\), where

\[
J(< k) = \bigoplus_{j<k} U(j) \otimes B(j) \otimes U(j).
\]

Recall that \(\text{rank}(a)\) is the number of edges in \(a\), where \(a \in A\). To see the claim, observe that by Lemma 8.9 the rank of \(x = a(u \otimes b \otimes v)\) is \(|\text{im}(a) \cap u|\), which is always \(\leq k\) since \(u \in U(k)\). Furthermore, \(\text{rank}(x) = k\) if and only if \(u \subseteq \text{im}(a)\), in which case \(\pi(x) = \pi(a')b\) and \(x\) has a scalar factor of \(z^{r-\text{rank}(a)}\).

Properties (20), (21) verify the hypotheses of [13, Theorem 1], so by that result \(A\) is a cellular algebra with respect to the anti-involution \(*\) and the cell datum \((\Lambda, M, C)\), where

\[
\Lambda = \{\lambda : \lambda \vdash k \text{ and } 0 \leq k \leq r\}
\]

and (for \(\lambda \vdash k\))

\[
M(\lambda) = U(k) \times \text{Tab}(\lambda), \quad C^k_{(x, y); (y, y)} = x \otimes C^k_{X, Y} \otimes y.
\]

Here, \(\text{Tab}(\lambda)\) is the set of standard tableaux of shape \(\lambda\) and \(\{C^k_{X, Y}\}\) is any cellular basis of \(B(k) = C_{S_k}\). (Cellular bases of symmetric group algebras exist; see, e.g., [11, 25, 38].)
Remark 8.10  Strictly speaking, we should write $\Lambda$ as the set of all pairs $(k, \lambda)$ such that $k = 0, 1, \ldots, r$ and $\lambda \vdash k$. But one can recover the value of $k$ from $\lambda$ itself, so we have simplified the notation accordingly.

The importance of realizing $A = P'_r(z)$ as an iterated inflation is that we now have easy access to its representations. This enables an easy proof of the second part of the following.

Theorem 8.11  The algebra $A = P'_r(z)$ is an iterated inflation of group algebras of symmetric groups. Furthermore, if $z \neq 0$ we have that:

(a) The cell modules of $A$ are indexed by the set $\Lambda$. If $\lambda \vdash k$, the cell module $C(\lambda)$ corresponding to a given $\lambda \in \Lambda$ has the form
\[
C(\lambda) = U(k) \otimes Sp^k
\]
with action for any $a \in A$ given by
\[
a(x \otimes z) = \phi_k(a, x) \otimes \theta_k(a, x)z
\]
for all $x \in U(k)$, $z \in Sp^k$. Here, $Sp^k$ is the cell module for $B(k) = \mathbb{C}S_k$ indexed by $\lambda$, which may be identified with the (irreducible) Specht module indexed by $\lambda$.

(b) Define a $B(k)$-valued $C$-linear bilinear form $\psi_k$ on $U(k)$ by the linear extension of the rule
\[
\psi_k(y, u) = \begin{cases} 
z^r - k \cdot 1_B & \text{if } y = u, \\
0 & \text{otherwise} \end{cases}
\]
for any $y, u \in U(k)$. Then, $\psi_k(y, u)^* = \psi_k(u, y)$ and
\[
(x \otimes c \otimes y)(u \otimes b \otimes v) \equiv x \otimes c\psi_k(y, u)b \otimes v \mod J(<k).
\]

(c) $A$ is semisimple, and the set of cell modules $\{C(\lambda) : \lambda \in \Lambda\}$ forms a complete set of irreducible $A$-modules, up to isomorphism.

Proof  Part (a) is the first half of [13, Proposition 3]. Part (b) is essentially the same as [13, Proposition 2].

For part (c) we use the standard bilinear form $\langle \cdot, \cdot \rangle$ on $C(\lambda)$, which by the second part of [13, Proposition 3], is related to the standard bilinear form $\langle \cdot, \cdot \rangle_\lambda$ on $Sp^k$ by
\[
\langle x \otimes z, y \otimes w \rangle = \langle z, \psi_k(x, y)w \rangle_\lambda = \langle \psi_k(y, x)z, w \rangle_\lambda
\]
for any $x, y \in U(k)$, any $z, w \in Sp^k$. Graham and Lehrer [11] showed that $A$ is semisimple (and the cell modules are its irreps) if and only if the bilinear form is nondegenerate on each cell module. That this is so in our situation follows from the nondegeneracy of $\langle \cdot, \cdot \rangle_\lambda$ and $\psi_k$. This completes the proof. □

Remark 8.12  (i) There are other approaches to proving the semisimplicity of $P'_r(z)$. For instance, one could adapt the arguments in [41] to find explicit formulas for primitive central idempotents in $A = P'_r(z)$. Another approach is based on Corollary 8.7, which immediately implies the semisimplicity of $P'_r(z)$ for $z \neq 1$, since Munn proved that $\mathbb{C}P'_r = P'_r(1)$ is semisimple. We have included the above arguments to demonstrate the relevance of the language of cellular algebras in providing perspective on existing results in the literature.
(ii) The vector space $\mathcal{U}(k)$ is itself a $\mathcal{P}'_r(z)$-module, with action defined on generators by $a \cdot u = \phi_k(a, u)$ for $a \in \mathcal{A}$, $u \in \mathcal{U}(k)$.

(iii) Grood [14] gives a purely combinatorial construction of the irreducible representations of $\mathbb{C}\mathcal{P}'_r = \mathcal{P}'_r(1)$, in terms of a natural generalization of standard tableaux. This description can easily be adapted to $\mathcal{P}'_r(z)$.

(iv) Observe that the set $\Lambda$ in Theorem 8.11(a) is equal to the set $\Lambda_{\leq r}$ in Remark 6.2(ii).

In other words, the (isomorphism classes of) irreducible representations of $\mathcal{P}_r(z)$ and $\mathcal{P}'_r(z)$ are indexed by the same set.

To conclude this section, we describe the module structure of $\mathcal{U}(k)$.

**Proposition 8.13** $\mathcal{U}(k)$ is the vector space spanned by $\mathcal{U}(k)$, the set of subsets of $\{1, \ldots, r\}$ of cardinality $k$. The action of $\mathcal{P}'_r(z)$ on $\mathcal{U}(k)$ is determined by

$$w \cdot u = u', \quad p_j \cdot u = \begin{cases} zu & \text{if } j \notin u, \\ 0 & \text{otherwise} \end{cases}$$

for all $w \in S_r$, $j = 1, \ldots, r$, $u \in \mathcal{U}(k)$, where $u' = (u)w^{-1}$ is the preimage of $u$ under $w$.

**Proof** If $w \in S_r$, then $\text{im}(w) = \{1, \ldots, r\}$. Hence, $\phi_k(w, u) = u'$. Furthermore, if $j = 1, \ldots, r$ then $\text{im}(p_j) = \{1, \ldots, r\} - \{j\}$. If $u \in \mathcal{U}_k$, then $u \subseteq \text{im}(p_j)$ if and only if $j \notin u$, in which case the preimage of $u$ under $p_j$ is $u$ itself, and $\phi_k(p_j, u) = zu$. Otherwise, $\phi_k(p_j, u) = 0$. \qed

### 9 The main result

In this section, we prove our main result, that Schur–Weyl duality holds for the tensor power $E^{\otimes r}$ of the unreduced Burau representation, assuming that $q_1q_2 \neq 0$ and $q = -q_2/q_1$ is not a root of unity.

The space $E^{\otimes r}$ is a representation of the group $B_n$ with $B_n$ acting diagonally, via:

$$b \cdot (v_1 \otimes \cdots \otimes v_r) = (b \cdot v_1) \otimes \cdots \otimes (b \cdot v_r)$$

(22)

for any $b \in B_n$, $v_j \in E$ for $j = 1, \ldots, r$. It is also a representation of $\mathcal{P}'_r([n]_q)$, with (left) action defined on generators by

$$s_i \cdot (v_1 \otimes \cdots \otimes v_r) = v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_r$$

$$p_j \cdot (v_1 \otimes \cdots \otimes v_r) = v_1 \otimes \cdots v_{j-1} \otimes P v_j \otimes v_{j+1} \otimes \cdots \otimes v_r$$

(23)

for any $i = 1, \ldots, r - 1$, $j = 1, \ldots, r$. In the above, $P$ is the pseudo-projection defined in Theorem 4.1, acting solely in the $j$th tensor place, and the action of the $s_i$ induces the usual place-permutation action of the symmetric group $S_r$. That (23) gives a well-defined action of $\mathcal{P}'_r([n]_q)$ requires proof, as follows.

**Lemma 9.1** Equations (23) make $E^{\otimes r}$ into a left $\mathcal{P}'_r([n]_q)$-module.

**Proof** We only need to check that the operators corresponding to the action of the generators $s_j$, $p_j$ satisfy the defining relation of $\mathcal{P}'_r([n]_q)$ in Theorem 8.6. The fact that $P^2 = [n]_q P$ immediately implies that $P^2 = [n]_q p_j$. It is clear from the definition of the action that $p_i p_j = p_j p_i$ for any $i \neq j$. So the relations in Theorem 8.6(a) are satisfied. The
relations in Theorem 8.6(b) are standard for the place-permutation action of symmetric groups. Finally, the relations in Theorem 8.6(c) are easily verified from the definitions of the action.

\[ \square \]

Remark 9.2 One can also define the action of \( P'((n)_q) \) directly, without invoking Theorem 8.6. Given a partial permutation \( d \in P' \), it is clear that

\[ d \cdot (v_1 \otimes \cdots \otimes v_r) = u_1 \otimes \cdots \otimes u_r \]

where \( u_i = v_j \) if \( d \) takes \( i \) to \( j \) and \( u_i = P v_i \) otherwise. See [41, Lemma 5.4] for further details.

By Theorem 4.1, the actions of \( P \) and \( B_n \) on \( E \) commute. It follows that the actions defined in (22), (23) commute, so they make \( E \otimes r \) into a \((C B_n, P'((n)_q))\)-bimodule.

Proposition 9.3 Set \( n = \dim E \). Assume that \( q_1 q_2 \neq 0 \) and \( q = -q_2/q_1 \) is not a root of unity. As a \((C B_n, P'((n)_q))\)-bimodule, we have a multiplicity-free semisimple decomposition

\[ E \otimes r \cong \bigoplus_{k=0}^{r} \bigoplus_{X \subset \{1, \ldots, r\}} \Delta(\lambda) \otimes C(\lambda). \]

Proof We begin by decomposing \( E \otimes r \) as a \( C B_n \)-module. To that end, we use the projection \( P_0 \) onto \( L \) defined in the proof of Theorem 4.1. We have \( 1 = P_0 + (1 - P_0) \) where \( 1 = \text{id}_E \) is the identity map on \( E \). This orthogonal idempotent decomposition induces the vector space decomposition

\[ E = P_0 E \oplus (1 - P_0) E = L \oplus F; \]

see Remark 4.2. Now we expand the \( r \)th tensor power of the identity map \( 1 = \text{id}_E \) to obtain

\[ 1 \otimes r = (P_0 + (1 - P_0))^\otimes = \sum_{X \subset \{1, \ldots, r\}} \delta_1(X) \otimes \cdots \otimes \delta_r(X) \]

where we define the symbol \( \delta_j(X) \) for \( X \subset \{1, \ldots, r\} \) by

\[ \delta_j(X) = \begin{cases} 1 - P_0 & \text{if } j \in X \\ P_0 & \text{otherwise.} \end{cases} \]

Applying the maps in the above decomposition to tensor space \( E \otimes r \) gives the decomposition

\[ E \otimes r = \bigoplus_{X \subset \{1, \ldots, r\}} \delta_1(X) E \otimes \cdots \otimes \delta_r(X) E. \]

This is a decomposition as \( C B_n \)-modules. Its isotypic components are of the form

\[ I(k) := \bigoplus_{X \subset \{1, \ldots, r\} : |X| = k} \delta_1(X) E \otimes \cdots \otimes \delta_r(X) E \cong \binom{r}{k} E \otimes r \otimes l_{r-k}, \]

where \( k = 0, 1, \ldots, r \) and where the isomorphism follows from the standard commutativity (up to isomorphism) of tensor products.

Each summand of \( I(k) \) is a tensor product of \( k \) copies of \( F \) and \( r - k \) copies of \( L \) in some order, determined by the indexing subset \( X \) of cardinality \( k \). For instance, the summand
indexed by \( X = \{1, \ldots, k\} \) is \( F^{\otimes k} \otimes L^{\otimes r-k} \). Plugging in the result of Theorem 5.3 for \( F^{\otimes k} \) we obtain
\[
F^{\otimes k} \otimes L^{\otimes r-k} \cong \bigoplus_{\lambda: k; \ell(\lambda) \leq n-1} \Delta(\lambda) \otimes Sp^k \otimes L^{\otimes r-k}.
\]
Every other summand of \( I(k) \) is isomorphic to the above, as \( \mathbb{C} B_n \)-modules, by an isomorphism that permutes the tensor places according to the given indexing subset \( X \). Thus, we have
\[
I(k) = \bigoplus_{X \subset \{1, \ldots, r\}; |X| = k} \left( \bigoplus_{\lambda: k; \ell(\lambda) \leq n-1} \Delta(\lambda) \otimes Sp^k \otimes \left( \bigoplus_{X \subset \{1, \ldots, r\}; |X| = k} L^{\otimes r-k} \right) \right),
\]
where the copy of \( \Delta(\lambda) \otimes Sp(\lambda) \) in each term on the right hand side is spread across the tensor places in \( X \). Switching the order of summation using commutativity of tensor products, we obtain that up to isomorphism
\[
I(k) \cong \bigoplus_{\lambda: k; \ell(\lambda) \leq n-1} \Delta(\lambda) \otimes Sp^k \otimes \left( \bigoplus_{X \subset \{1, \ldots, r\}; |X| = k} L^{\otimes r-k} \right).
\]
In this decomposition, the copies of \( \Delta(\lambda) \otimes Sp^k \) in the right hand side are spread across tensor positions indexed by \( X \), while the \( r-k \) factors of \( L \) are in tensor positions indexed by the complement \( \{1, \ldots, r\} - X \). Hence, the fact that \( PP_0 = [n]_q P_0 \) implies that \( p_j \) acts on \( \bigoplus_{X \subset \{1, \ldots, r\}; |X| = k} L^{\otimes r-k} \) as the scalar \([n]_q \) if \( j \notin X \) and as zero otherwise. By Proposition 8.13, it follows that if we set \( z = [n]_q \) and regard \( U(k) \) as a \( \mathcal{P}'_r([n]_q) \)-module, then
\[
U(k) \cong \bigoplus_{X \subset \{1, \ldots, r\}; |X| = k} L^{\otimes r-k}.
\]
The result thus follows from Theorem 8.11. The proof is complete. \( \square \)

Now we finally obtain our main result.

**Theorem 9.4** Suppose that \( q_1 q_2 \neq 0 \) and \( q = -q_2/q_1 \) is not a root of unity. As a \((\mathbb{C} B_n, \mathcal{P}'_r([n]_q))\)-bimodule, \( E^{\otimes r} \) satisfies Schur–Weyl duality, in the sense that the enveloping algebra of each action is equal to the centralizer algebra for the other:
\[
\text{im}(\mathbb{C} B_n) = \text{End}_{\mathcal{P}'_r([n]_q)}(E^{\otimes r}), \quad \text{im}(\mathcal{P}'_r([n]_q)) = \text{End}_{\mathbb{C} B_n}(E^{\otimes r}).
\]
The action of \( \mathcal{P}'_r([n]_q) \) is faithful if and only if \( n > r \), in which case we have an isomorphism \( \mathcal{P}'_r([n]_q) \cong \text{End}_{\mathbb{C} B_n}(E^{\otimes r}) \).

**Proof** Since the actions of \( \mathbb{C} B_n, \mathcal{P}'_r([n]_q) \) commute, we have inclusions
\[
\text{im}(\mathbb{C} B_n) \subset \text{End}_{\mathcal{P}'_r([n]_q)}(E^{\otimes r}), \quad \text{im}(\mathcal{P}'_r([n]_q)) \subset \text{End}_{\mathbb{C} B_n}(E^{\otimes r}).
\]
By the standard theory of (split) semisimple algebras, it suffices to show that either of these inclusions is equality. By Proposition 9.3 and Schur’s Lemma, we have
\[
\dim \text{End}_{\mathbb{C} B_n}(E^{\otimes r}) = \sum_{k=0}^r \sum_{\lambda: k; \ell(\lambda) \leq n-1} (\dim C(\lambda))^2.
\]
If \( n > r \), then the length restriction in the second sum above is vacuous, and the right hand side is equal to the dimension of \( \mathcal{P}'_r([n]_q) \), so the equality
\[
\text{im}(\mathcal{P}'_r([n]_q)) = \text{End}_{\mathbb{C} B_n}(E^{\otimes r})
\]
follows by dimension comparison, and furthermore the action of $P'_r([n]_q)$ is faithful. This completes the proof if $n > r$.

Otherwise the action of $P'_r([n]_q)$ is not faithful, and by Proposition 9.3 the $C(\lambda)$ for $\ell(\lambda) > n - 1$ do not appear as constituents of the representation $E^{\otimes r}$. Hence, their corresponding matrix algebras appear in the kernel of the representation $P'_r([n]_q) \to \text{End}(E^{\otimes r})$. Thus, the right hand side of Eq. (24) is still equal to the dimension of the image of the representation, so again the needed equality follows by dimension comparison. $\square$

In particular, the algebra $\text{End}_{B_n}(E^{\otimes r})$ is spanned by endomorphisms arising from the (appropriately scaled) action of rook monoid elements, and is generated by the endomorphisms coming from the place-permutation action of $S_r$ and by $p_1$, since the other $p_j$ are all conjugate to $p_1$ by elements of $S_r$.

**Remark 9.5**  
(i) Bowing to tradition in the literature on diagram algebras, we formulated our main results in terms of the pseudo-idempotent operator $P$ of Theorem 4.1. From some viewpoints (clearing denominators, compatibility with the standard definitions of the partition, and Brauer algebras) it makes sense to use $P$ instead of the actual projection operator $P_0$. On the other hand, there is nothing to prevent from using $P_0$ in place of $P$. Doing so leads to a statement of Schur–Weyl duality in which the semigroup algebra $P'_r(1)$ plays the role of $P'_r([n]_q)$. We leave the details to the reader.

(ii) Solomon [41] proved an instance of Schur–Weyl duality for the rook monoid. In terms of our notation, he considers $E^{\otimes r}$ as a representation of $\text{GL}(F)$ with $\text{GL}(F)$ acting trivially on $L$, and computes the centralizer $\text{End}_{\text{GL}(F)}(E^{\otimes r})$, proving that it is a quotient of the semigroup algebra of the rook monoid. Thus, his Schur–Weyl duality is closely related to ours, although our approach is self-contained and independent of [41].

We conclude this section by considering the effects of setting $q = 1$. In that case, the representation $B_n \to \text{End}(E^{\otimes r})$ factors through the Weyl group $W_n \subset \text{GL}(E)$, where $W_n$ is isomorphic to the symmetric group on $n$ letters, and the Burau representation $E$ becomes isomorphic to the standard $n$-dimensional representation of $W_n$. Here, we know by the work of Martin and Jones (see [18]) that $\text{End}_{B_n}(E^{\otimes r}) = \text{End}_{W_n}(E^{\otimes r})$ is the image of the full partition algebra $P_r(n)$ at parameter $n$.

On the other hand, at $q = 1$ we have $[n]_q = n$, and one can ask for a determination of the centralizer algebra $\text{End}_{P'_r(n)}(E^{\otimes r})$ for $P'_r(n)$. At first glance this appears to be a separate problem, but in fact it leads to yet another new instance of Schur–Weyl duality, as follows.

**Corollary 9.6**  For any $n > 2$, there exists a complex number $q$ such that $[n]_q = n$ and $q$ is not a root of unity. Fix such a value of $q = -q_2/q_1$ in the Burau representation. Regarded as a $(\mathbb{C}B_n, P'_r(n))$-bimodule, $E^{\otimes r}$ satisfies Schur–Weyl duality, in the sense that the enveloping algebra of each action is equal to the centralizer algebra for the other:

$$\text{im}(\mathbb{C}B_n) = \text{End}_{P'_r(n)}(E^{\otimes r}), \quad \text{im}(P'_r(n)) = \text{End}_{B_n}(E^{\otimes r}).$$

The action of $P'_r(n)$ is faithful if and only if $n > r$, in which case we have an isomorphism $P'_r(n) \cong \text{End}_{B_n}(E^{\otimes r})$. 
Proof} Values of $q$ such that $[n]_q = n$ are obtained by solving the polynomial equation

$$1 + x + x^2 + \cdots + x^{n-1} = n \iff (1 - n) + x + x^2 + \cdots + x^{n-1} = 0.$$

If $n > 2$, then non-root of unity solutions exist, as the product of the roots of the polynomial is, up to sign, equal to $|1 - n| = n - 1$ and $n - 1 > 1$. Let $q$ be such a solution. Then, $[n]_q = n$ and hence $P'_r([n]_q) = P'_r(n)$. The result then follows from Theorem 9.4. \qed

Remark 9.7 Suppose that $n > 2$, and fix a non-root of unity value of $q$, as in Theorem 9.6. Then, the algebra $\text{End}_{P'_r(n)}(E^\otimes r)$ does not depend on $q$, but is spanned by linear endomorphisms of the form $g \otimes \cdots \otimes g$, where $g$ is the image of a braid group element under the Burau representation $B_n \rightarrow \text{GL}(E)$, which depends on $q$.

10 Dimensions of $P'_r(z)$-irreps

Assume that $q$ is not a root of unity and $q_1 q_2 \neq 0$. For any $z \neq 0$, Theorem 8.11(a) gives one way of computing dimensions of the irreducible $P'_r(z)$-modules, in terms of dimensions of Specht modules for symmetric groups. We now apply the Schur–Weyl duality statement in Theorem 9.4 to obtain a second, more combinatorial, method of computing those dimensions. By Schur–Weyl duality, we know that:

The multiplicities of the irreducible representations of $B_n$ in $E^\otimes r$ are equal to the dimensions of the irreducible representations of the corresponding centralizer algebra.

The proof of Proposition 9.3 shows that this set of modules is precisely the set of irreducible polynomial $GL(F) \cong GL_{n-1}(\mathbb{C})$-modules appearing in $\bigoplus_{k=0}^{r} F^\otimes k$. Note that the action of $B_n$ on this module is given by composition with the representation $B_n \rightarrow \text{GL}(F)$. Thus, the set

$$\Lambda(n, r) = \{ \lambda \vdash k : 0 \leq k \leq r \text{ and } \ell(\lambda) \leq n - 1 \}$$

indexes the set of modules in question. By semisimplicity, we may write

$$E^\otimes r \cong \bigoplus_{\lambda \in \Lambda(n, r)} c^n_{\lambda} \Delta(\lambda),$$

where $c^n_{\lambda} = \dim \text{Hom}_{B_n}(\Delta(\lambda), E^\otimes r) = \dim \text{Hom}_{GL(F)}(\Delta(\lambda), E^\otimes r)$ is the desired multiplicity.

We may assume that $n > r$, in which case the value of $c^n_{\lambda}$ does not depend on $n$, so we write $c^r_{\lambda}$ for this limiting value. These numbers may be computed by induction on $r$ (holding $n > r$ fixed) using the well-known Pieri rule [10] for tensor products of polynomial $GL_{n-1}(\mathbb{C})$ representations. Notice that for $n > r$ the indexing set $\Lambda(n, r)$ is equal to

$$\Lambda(r) = \{ \lambda \vdash k : 0 \leq k \leq r \},$$

which also does not depend on $n$. This is the indexing set of the irreducible $P'_r(z)$-modules (up to isomorphism). Set $\delta_{r, \lambda} = 1$ if $\lambda \in \Lambda(r)$ and 0 otherwise. Then, for any $\lambda \in \Lambda(r)$, we claim that

$$c^r_{\lambda} = \delta_{r-1, \lambda} c^{r-1}_{\lambda} + \sum_{\mu} c^{r-1}_{\mu},$$

where the summation is over all $\mu \in \Lambda(r - 1)$ obtained from $\lambda$ by removing one box from its Young diagram. This recursion computes $\dim C(\lambda) = c^r_{\lambda}$, where $C(\lambda)$ is the irreducible
\( P'(\{n\}_q) \)-module indexed by \( \lambda \). Since for \( z \neq 0 \), we have \( P'(\{n\}_q) \cong P'(z) \) by Corollary 8.7, this also computes the dimension of the corresponding irreducible \( P'(z) \)-module.

To prove the claim, we may as well specialize \((q_1, q_2)\) to \((1, -q)\). Then, \( L \cong \mathbb{C} \) is isomorphic to the trivial module. Assume by induction that \( E \otimes r - 1 \cong \bigoplus_{\mu \in \Lambda(r-1)} c_{\mu}^{r-1} \Delta(\mu) \).

Tensoring both sides by \( E = L \oplus F \cong \mathbb{C} \oplus F \) gives

\[
E^{\otimes r} \cong \bigoplus_{\mu \in \Lambda(r-1)} c_{\mu}^{r-1} \Delta(\mu) \oplus \bigoplus_{\mu \in \Lambda(r-1)} c_{\mu}^{r-1} \Delta(\mu) \otimes F.
\]

The Pieri rule (in a special case) says that \( \Delta(\mu) \otimes F \) is isomorphic to a direct sum of \( \Delta(\lambda) \), each with multiplicity one, for every partition \( \lambda \) obtained from \( \mu \) by adding one box to its Young diagram. Combining this with the above decomposition justifies Eq. (25).

Tabulating the values of \( c_{\lambda}^{r} \) for \( r = 0, 1, 2, 3, 4 \) gives the following table, in which the number in the final column is the sum of squares of the \( c_{\lambda}^{r} \) in its row.

| \( r \) | \( \emptyset \) | (1) | (2) | (2, 1) | (1, 2) | (1, 1, 1) | (2, 2) | (1, 2, 1) | (2, 3) | (1, 2, 1) | (2, 2, 1) | \( \dim P'(z) \) |
|------|----------|-----|-----|--------|--------|----------|--------|----------|--------|----------|--------|----------|
| 0    | 1        |     |     |        |        |          |        |          |        | 1        |        |          |
| 1    | 1        | 1   |     |        |        |          |        |          |        | 2        |        |          |
| 2    | 1        | 2   | 1   | 1      |        |          |        |          |        | 7        |        |          |
| 3    | 1        | 3   | 3   | 3      | 1      | 2        | 1      |          |        | 34       |        |          |
| 4    | 1        | 4   | 6   | 4      | 8      | 4        | 1      | 3        | 2      | 3        | 1      | 209      |

We can also compute these numbers using a Bratteli diagram. For our purposes, a Bratteli diagram is a rooted tree with vertices given by Young diagrams, constructed by the following recursive algorithm, assuming that the first \( r - 1 \) rows are already constructed:

- Copy the vertices in row \( r - 1 \) to row \( r \), and add an edge from each copied vertex \( \mu \) to its corresponding vertex \( \mu \) in the previous row.
- Add the partitions of \( r \) to the end of the \( r \)th row, and draw an edge from vertex \( \lambda \) in the \( r \)th row to vertex \( \mu \) in the preceding row if and only if \( \lambda \) is obtainable from \( \mu \) by adding one box.

To get started, the initial row \( r = 0 \) contains a single vertex labeled by the empty partition \( \emptyset \). We display the first 5 rows of this graph in Fig. 1, noting that the dimension of the irreducible \( P'(z) \)-module \( C(\lambda) \) is equal to the number of paths of \( r \) edges from the root to \( \lambda \) in the graph. Figure 1 coincides with [17, Fig. 1].

Remark 10.1 The above method also computes the \( c_{\lambda}^{n,r} \) if \( n \leq r \). Simply observe that \( c_{\lambda}^{n,r} = 0 \) if \( \ell(\lambda) \geq n \) and is equal to \( c_{\lambda}^{r} \) otherwise, so to compute dimensions of the irreducible representations of the centralizer algebra \( \text{End}_{B_n}(E^{\otimes r}) \), we merely omit all nodes from the Bratteli diagram with \( n \) or more rows of boxes.

11 Analogues of Schur algebras

We continue to assume that \( q_1 q_2 \neq 0 \) and \( q = -q_2/q_1 \) is not a root of unity. To avoid complexity, it is often useful to replace the infinite-dimensional group algebra \( \mathbb{C}B_n \) in Theorem 9.4 by its finite-dimensional image \( S'_q(n, r) \). So we let \( S'_q(n, r) \) be the enveloping algebra of the \( B_n \)-action; that is, the image of the representation \( \mathbb{C}B_n \to \text{End}(E^{\otimes r}) \). Then, Schur–Weyl duality implies that

\[
S'_q(n, r) = \text{End}_{P'_q(\{n\}_q)}(E^{\otimes r})
\]
as algebras. Regarding $E^\otimes r$ as an $(S'_q(n, r), P'_r([n]_q))$-bimodule, notice that Schur–Weyl duality still holds, so we lose no information by this replacement. Since $P'_r([n]_q)$ contains a copy of the symmetric group $S_r$, it is clear that $S'_q(n, r)$ is a subalgebra of the well-known Schur algebra (see [12,34])

$$S(n, r) = \text{End}_{S_r}(E^\otimes r)$$

appearing in classical Schur–Weyl duality. Since all the $p_j$ defined in (23) are conjugate in $P'_r(z)$ for any $z$, it follows that

$$S'_q(n, r) = S(n, r) \cap \text{Comm}(p_1) = \{ X \in S(n, r) : p_1X = xp_1 \}$$

(26)
is the subalgebra of $S(n, r)$ consisting of elements that commute with the operator $p_1$.

**Example 11.1** The Schur algebra $S(2, 2)$ is ten dimensional, with [34, § 2.2] a faithful matrix realization in $\text{End}(E \otimes E)$ as the set of matrices of the form

$$\begin{bmatrix}
 x_1 & x_2 & x_2 & x_3 \\
 x_4 & x_5 & x_6 & x_7 \\
 x_4 & x_6 & x_5 & x_7 \\
 x_8 & x_9 & x_9 & x_{10}
\end{bmatrix},$$

where $x_1, \ldots, x_{10}$ are arbitrary. Taking the intersection with $\text{Comm}(p_1)$ shows that $S'_q(2, 2)$ is the set of matrices of the form

$$\begin{bmatrix}
 x_1 & qx_4 & qx_4 & q^2x_8 \\
 x_4 & x_1 + (q - 1)x_4 & qx_8 & qx_4 + (q - 1)qx_8 \\
 x_4 & qx_8 & x_1 + (q - 1)x_4 & qx_4 + (q - 1)qx_8 \\
 x_4 & x_1 + (q - 1)x_4 & x_4 + (q - 1)x_8 & x_1 + 2(q - 1)x_4 + (q - 1)^2x_8
\end{bmatrix},$$

Fig. 1  Bratteli diagram for $r = 4$
where \( x_1, x_4, x_8 \) are arbitrary. So \( \dim S'_q(2, 2) = 3 \).

The irreducible \( S'_q(n, r) \)-modules are precisely the restrictions to \( \mathbb{C}B_n \) of the homogeneous polynomial representations of \( \text{GL}(F) \cong \text{GL}_{n-1}(\mathbb{C}) \) of degrees 0, 1, \ldots, \( r \) and are indexed by the set \( \Lambda(n, r) \).

Suppose that \( n > 2 \) and assume that \( q \) is chosen (as in Corollary 9.6) so that \( [n]_q = n \) while maintaining the condition that \( q \) is not a root of unity. In this situation, part of the Schur–Weyl duality statement in Corollary 9.6 is that the enveloping algebra \( S'(n, r) \) of the \( B_n \)-action is the centralizer algebra

\[
S'(n, r) = \text{End}_{P'_q(n)}(E^{Sr}).
\]

This algebra is the specialization at \( q = 1 \) of the algebra \( S'_q(n, r) \). Notice that it is independent of \( q \) because the action of \( P'_q(n) \) is independent of \( q \). This subalgebra of the classical Schur algebra makes sense over any commutative ring. It may be interesting to characterize its irreducible representations over a field of positive characteristic.

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**A Appendix: Proof of Theorem 5.2**

Assume that \( q_1 q_2 \neq 0 \) and \( q = -q_2/q_1 \) is not a root of unity. In this appendix (which is independent of Sects. 6–10) we give an elementary proof that the Zariski closure \( C \) contains \( \text{SL}(F) \), where \( G = \rho(B_n) \) is the image of the reduced Burau representation \( \rho : B_n \rightarrow \text{GL}(F) \). This result is needed to obtain Schur–Weyl duality for \( F^{\otimes k} \) (see Theorem 5.3), which is needed for the proof of the main results in Sect. 9. In case \( q \) is transcendental, the result is a special case of [30, Theorem B], obtained by different methods.

We get the following result from Lemma 3.3 by an elementary induction on \( k \), which is left to the reader.

**Lemma A.1** Let \( k \) be a positive integer. For any \( i, j = 1, \ldots, n-1 \) the action of \( \sigma_i^k \in B_n \) on \( f_j \in F \) is given by the rules

(a) \( \rho(\sigma_i^k)f_j = q_1^k f_j \) if \( j \neq i-1, i, i+1 \).

(b) \( \rho(\sigma_i^k)f_{i-1} = q_1^k f_{i-1} + q_1 \Phi_k(q_1, q_2) f_i \).

(c) \( \rho(\sigma_i^k)f_i = q_2^k f_i \).

(d) \( \rho(\sigma_i^k)f_{i+1} = -q_2^k \Phi_k(q_1, q_2) f_i + q_1^k f_{i+1} \)

where \( \Phi_k(q_1, q_2) = \sum_{j=0}^{k-1} q_1^j q_2^{k-1-j} \). \( q_1 \neq q_2 \).

Set \( \Phi_k = \Phi_k(q_1, q_2) \) for short. For convenience of reference, the matrices of the operators \( \rho(\sigma_i^k) \) with respect to the \( \{ f_i \} \) basis are listed below:

\[
\rho(\sigma_1^k) = \begin{bmatrix}
q_2^k & 0 & \cdots & 0 \\
0 & q_1^k & 0 & \cdots \\
0 & 0 & q_1^k & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & q_1^k
\end{bmatrix}, \quad \rho(\sigma_{n-1}^k) = \begin{bmatrix}
q_1^k & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & q_1^k & 0 & 0 \\
0 & \cdots & 0 & q_1^k & 0 \\
0 & \cdots & 0 & q_1^k & q_2^k
\end{bmatrix}
\]
Henceforth, we identify GL(F) ∼= GL_{n-1}(C) by means of the basis \{f_i\}.

Recall (Remark 3.4) that det(σ_j) = q_1^{-a} q_2 for all j. The proof of Theorem 5.2 falls naturally into two cases, depending whether q_1^{-a} q_2 is or is not a root of unity. The non-root of unity case requires the following lemma.

**Lemma A.2** Suppose that q_1^{-a} q_2 is not a root of unity. If g ∈ G, 0 ≠ z ∈ C, then zg = zI_{n-1}g ∈ \overline{G}.

**Proof** It is well known [24] that the "full twist" θ_n = Δ_n^2 ∈ B_n, where

\[ Δ_n = (σ_1 \cdots σ_{n-1})(σ_1 \cdots σ_{n-2}) \cdots (σ_1σ_2)σ_1, \]

generates the center of the braid group B_n. (An easy exercise [24, Exercise 1.3.2] gives the alternate formula θ_n = (σ_1 \cdots σ_{n-1})^n.) Thus, by Schur's Lemma it follows that ρ(θ_n) = λI_{n-1} for some λ ∈ C. Now an easy calculation shows that λ = (q_1^{-a} q_2)^n. As λ is not a root of unity, the Zariski closure of the subgroup generated by ρ(θ_n) is \( H = \{ zI_{n-1} : z \neq 0 \} \) and so zI_{n-1} ∈ \overline{G} for every 0 ≠ z ∈ C. Since \overline{G} is a group, for any g ∈ G it follows that zg = zgI_{n-1} ∈ \overline{G}. \□

Let \( \{e_{ij}\}_{i,j=1}^{n-1} \) be the standard basis of matrix units for matrix space, defined in terms of the Kronecker delta symbols by e_{ij} = (δ_{ik}δ_{jl})_{k,l=1}^{n-1}. Set

\[ E(i) = I_{n-1} - e_{ii} = \sum_{j \neq i} e_{ij}. \]

We also need the constants

\[ a = \frac{q}{1+q}, \quad b = \frac{1}{1+q}. \]

Notice that a = \(- q_2/(q_1 - q_2), b = q_1/(q_1 - q_2).\)

**Proposition A.3** Assume that q_1q_2 ≠ 0 and q = \(- q_2/q_1\) is not a root of unity. Let \overline{G} be the Zariski closure of G = ρ(B_n).

(a) If q_1^{-a} q_2 is not a root of unity, then \overline{G} contains the one-parameter subgroups H_1, \ldots, H_{n-1} where

\[ H_i = (1 - δ_{i1})b(1 - z_i)e_{ii-1} + z_ie_{ii} \]

\[ + (1 - δ_{i,n-1})a(1 - z_i)e_{ii+1} + E(i) \]

for nonzero complex scalars \( z_1, \ldots, z_{n-1}. \)

(b) If q_1^{-a} q_2 is a root of unity, then \overline{G} contains the one-parameter subgroups K_1, \ldots, K_{n-1} where

\[ K_i = (1 - δ_{i1})b(w_i - w_i^{2^{-a}})e_{ii-1} + w_i^{2^{-a}}e_{ii} \]
for nonzero complex scalars $w_1, \ldots, w_{n-1}$.

**Proof** (a) It follows from Lemma A.2 that $q_1^{-1} \rho(\sigma_i) \in \overline{G}$ for all $i = 1, \ldots, n-1$. Hence, $(q_1^{-1} \rho(\sigma_i))^k = q_1^{-k} \rho(\sigma_i^k)$ belongs to $\overline{G}$ for all $i$ and all $k \geq 0$. By Lemma A.1, it follows by an easy calculation that

$$(q_1^{-1} \rho(\sigma_i))^k = (1 - \delta_{i,1})b(1 - (-q)^k)e_{i,i-1} + (-q)^k e_{ii} + (1 - \delta_{i,n-1})a(1 - (-q)^k)e_{i,i+1} + E(i).$$

Notice that this matrix depends only on $q = -q_2/q_1$ and lies in $H_i$ for any $k \geq 0$. Since $-q$ is not a root of unity, it follows that the intersection $H_i \cap \overline{G}$ is infinite. Thus, $H_i \subset \overline{G}$, since $H_i$ is a closed one-parameter group.

(b) Since $q_1^{n-2}q_2$ is a root of unity, there is a positive integer $d$ such that $(q_1^{n-2}q_2)^d = 1$, hence $q_2^d = q_1^{d(2-n)}$. Since $q = -q_2/q_1$, it follows that

$$q^d = (-1)^d q_1^{d(1-n)}.$$ 

Now $q$ is not a root of unity, so $q_1$ cannot be a root of unity. By Lemma A.1, if $k$ is a non-negative integer we have

$$\rho(\sigma_i^{kd}) = (1 - \delta_{i,1})b(q_1^{kd} - q_1^{kd(2-n)})e_{i,i-1} + q_1^{kd(2-n)} e_{ii} + (1 - \delta_{i,n-1})a(q_1^{kd} - q_1^{kd(2-n)})e_{i,i+1} + q_1^{kd} E(i).$$

Since $q_1$ is not a root of unity, the powers $q_1^{kd}$ are distinct for all $k$ and thus the matrices $\rho(\sigma_i^{kd})$ are also distinct for all $k$. Notice that $\rho(\sigma_i^{kd})$ belongs to $K_i$ for all $k$. Hence, the cardinality of $K_i \cap \overline{G}$ is infinite. This forces $K_i \subset \overline{G}$, as $K_i$ is a closed one-parameter group.

**Corollary A.4** Assume that $q_1q_2 \neq 0$ and $q = -q_2/q_1$ is not a root of unity. Let $\overline{G}$ be the Zariski closure of $G = \rho(B_n)$.

(a) If $q_1^{n-2}q_2$ is not a root of unity, then the Lie algebra $\text{Lie}(\overline{G})$ contains the elements

$$u_i = (1 - \delta_{i,1})b e_{i,i-1} + e_{ii} + (1 - \delta_{i,n-1})a e_{i,i+1}$$

for $i = 1, \ldots, n-1$.

(b) If $q_1^{n-2}q_2$ is a root of unity, then $\text{Lie}(\overline{G})$ contains the elements

$$v_i = (1 - \delta_{i,1})b (n-1) e_{i,i-1} + (2-n) e_{ii} + (1 - \delta_{i,n-1})a (n-1) e_{i,i+1} + E(i)$$

for $i = 1, \ldots, n-1$.

**Proof** For (a), take the derivative $d/dz_i$ at $z_i = 1$ of the one-parameter subgroup $H_i$. Similarly, for (b) take the derivative $d/dw_i$ at $w_i = 1$ of the one-parameter subgroup $K_i$. □
Proposition A.5  Assume that \( q_1 q_2 \neq 0 \) and \( q \) is not a root of unity. Suppose that \( n \geq 3 \).

(a) The Lie algebra generated by \( u_1, \ldots, u_{n-1} \) equals \( \mathfrak{gl}_{n-1} \).

(b) The Lie algebra generated by \( v_1, \ldots, v_{n-1} \) equals \( \mathfrak{sl}_{n-1} \).

Proof  (a) Let \( \mathfrak{g} \) be the Lie algebra generated by \( u_1, \ldots, u_{n-1} \). Since \( \mathfrak{g} \subseteq \mathfrak{gl}_{n-1} \), it suffices to show the reverse containment. We will argue that \( e_{ij} \in \mathfrak{g} \) for all \( i, j = 1, \ldots, n-1 \), making use of the standard commutator formula \([e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}\).

Assume that \( n \geq 4 \). Direct computation shows that

\[
[u_1, [u_1, u_2]] + [u_1, u_2] = 2a(ab - 1)e_{12} - 2a^2 e_{13}.
\]

As \( a \neq 0 \), it follows that

\[
A_2' = \frac{1}{2a}( [u_1, [u_1, u_2]] + [u_1, u_2]) = (ab - 1)e_{12} - ae_{13}
\]

is in \( \mathfrak{g} \). Then, \( A_1 = u_1 \) and \( A_2 = bu_1 - A_2' = be_{12} + e_{13} + ae_{14} \) are both in \( \mathfrak{g} \). Clearly \( A_1, A_2 \) are linearly independent.

Next, we recursively compute elements \( A_k \) for \( k = 2, \ldots, n-1 \) by defining

\[
A_k = \frac{1}{a} [A_{k-1}, u_k] \quad \text{for all } k = 3, \ldots, n-1.
\]

Then, by direct computation we have

\[
A_k = \begin{cases} 
be_{1,k-1} + e_{1k} + ae_{1,k+1} & \text{if } k = 3, \ldots, n-2, \\
be_{1,n-2} + e_{1,n-1} & \text{if } k = n-1.
\end{cases}
\]

We claim that the elements \( A_1, A_2, \ldots, A_{n-1} \) are linearly independent.

To see this, identify each \( A_k \) with its coordinate vector with respect to the basis \( \{e_{11}, \ldots, e_{1,n-1}\} \) of the first row of matrix space. Let \( M \) be the matrix whose rows are those coordinate vectors in order. Then, \( M \) is the \((n-1) \times (n-1)\) tridiagonal banded matrix

\[
M = \begin{bmatrix}
1 & a & 0 & \cdots & 0 \\
b & 1 & a & \ddots & \vdots \\
0 & b & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & a \\
0 & \cdots & 0 & b & 1
\end{bmatrix}
\]

with \( a \)'s on the super-diagonal and \( b \)'s on the sub-diagonal. Setting \( D_n = \det M \) we see by Eq. (7) that \( D_n \) satisfies

\[
D_n = D_{n-1} - abD_{n-2} \quad (n \geq 5),
\]

where \( D_3 = 1 - ab \) and \( D_4 = 1 - 2ab \). Thus, we obtain the following. \( \Box \)

Lemma A.6  \( D_n = \frac{[n]_q}{(1+q)^{n-1}} \) for all \( n \geq 3 \).
Proof First check directly that $D_3$ and $D_4$ satisfy the given formula. Assume by induction that $D_{n-2}$ and $D_{n-1}$ satisfy the formula. Applying the recursion (27) gives

$$D_n = \frac{[n-1]_q}{(1+q)^{n-2}} - \frac{q}{(1+q)^2} \frac{[n-2]_q}{(1+q)^{n-3}} = \frac{(1+q)[n-1]_q - q[n-2]_q}{(1+q)^{n-1}}$$

and the result follows from the identity $(1+q)[n-1]_q - q[n-2]_q = [n]_q$. \hfill \blacksquare

Lemma A.6 proves the claim, as $q$ is not a root of unity. The claim implies that $\mathbb{C}A_1 + \mathbb{C}A_2 + \cdots + \mathbb{C}A_{n-1} = \sum_{j=1}^{n-1} \mathbb{C}e_{ij}$. Hence, $e_{ij} \in \mathfrak{g}$ for all $j = 1, \ldots, n-1$. Now we finish the proof quite easily, by observing that

$$[e_{ij}, u_{i+1}] = \begin{cases} -be_{i+1,j} & \text{if } i + 1 \neq j, \\ -be_{jj} + be_{j-1,j} + e_{j-1,j} + ae_{j-1,j+1} & \text{if } i + 1 = j \neq n - 1, \\ -be_{n-1,n-1} + be_{n-2,n-2} + e_{n-2,n-1} & \text{if } i + 1 = j = n - 1. \end{cases}$$

This implies that $[e_{ij}, u_{i+1}] = -be_{i+1,j}$ modulo a linear combination of elements of the form $e_{k\ell}$ where $k < i + 1$. Assuming by induction that the $e_{k\ell} \in \mathfrak{g}$ for all $k < i + 1$, it follows from the fact that $b \neq 0$ that $e_{i+1,j} \in \mathfrak{g}$ for all $j$. This completes the proof of (a) in case $n \geq 4$.

The case $n = 3$ must be handled separately, and is simpler. Notice that in this case

$$A'_2 = \frac{1}{2a}([u_1, [u_1, u_2]] + [u_1, u_2]) = (ab - 1)e_{12} \in \mathfrak{g}.$$ 

Since $ab - 1 = 0$ if and only if $[3]_q = 1 + q + q^2 = 0$, we see that $ab - 1 \neq 0$ as $q$ is not a root of unity. Hence, $e_{12} \in \mathfrak{g}$ and $e_{11} = u_1 - ae_{12} \in \mathfrak{g}$. Thus, $[e_{11}, u_2] = -be_{21} \in \mathfrak{g}$, which implies that $e_{21} \in \mathfrak{g}$ and $e_{22} = u_2 - be_{21} \in \mathfrak{g}$, completing the proof of (a) in the case $n = 3$.

(b) Now let $\mathfrak{g}_1 = \langle u_1, \ldots, u_{n-1} \rangle$ be the Lie algebra in part (a), and set $\mathfrak{g}_2 = \langle v_1, \ldots, v_{n-1} \rangle$. Notice that

$$u_k = \frac{1}{n-1} (I_{n-1} - v_k)$$

for all $k$. Thus, $[u_k, u_j] = c[v_i, v_j]$ with $c = 1/(n - 1)^2$. Hence, $\mathfrak{g}'_1 = \mathfrak{g}'_2$, that is, the derived algebras of $\mathfrak{g}_1, \mathfrak{g}_2$ coincide. By part (a), $\mathfrak{g}_1 = \mathfrak{s}l_{n-1}$, so $\mathfrak{g}'_1 = \mathfrak{s}l_{n-1} = \mathfrak{s}l_{n-1} \subset \mathfrak{g}_2$. As it is obvious that $\mathfrak{g}_2 \subset \mathfrak{s}l_{n-1}$, it follows that $\mathfrak{g}_2 = \mathfrak{s}l_{n-1}$. This completes the proof of Proposition A.5. \hfill \blacksquare

We now have all the tools needed to prove Theorem 5.2.

Proof of Theorem 5.2 It is well known (see, e.g., [16, 20]) that closed subgroups of $\text{GL}(\mathbb{F})$ contain the one-parameter subgroups generated by all elements of their Lie algebra; indeed, that can be taken as an equivalent definition of the Lie algebra for such groups. By Proposition A.5, it easily follows that $\mathcal{G}$ contains $\text{SL}_{n-1} \cong \text{SL}(\mathbb{F})$, so we are done. (If $n = 2$ there is nothing to prove, as $\text{dim} \mathbb{F} = 1$ in that case.) \hfill \blacksquare

Remark A.7 (i) It is easy to see that $\mathcal{G} = \text{GL}(\mathbb{F})$ if and only if $q_n^{n-2}q_2$ is not a root of unity. The sufficiency of this condition follows from Proposition A.5(a). For its necessity, observe that if $\xi = q_n^{n-2}q_2$ is a root of unity (of order $d$, say) then elements of $\mathcal{G}$ satisfy the polynomial equation

$$\prod_{p=0}^{d-1} (\text{det}(X_{ij}) - \xi^p) = 0$$

and not all elements of $\text{GL}(\mathbb{F})$ satisfy it.
(ii) When \( q_1^{n-2} q_2 \) is a primitive \( d \)th root of unity, for \( d > 1 \), we have a strict containment \( G \nsubsetneq \text{SL}(F) \), because the generators of \( G \) do not have determinant one.
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