Dynamics and symmetries on the noncommutative plane*

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Dedicated to the 70th birthday of Jerzy (Jurek) Lukierski

1 Introduction

Sometime in 1996 Jurek visited us in Bielefeld and asked about a dynamical particle model showing the presence of the second central charge of the planar Galilean group. This was a new and very interesting question. Soon we agreed that such a model must necessarily contain higher order time derivatives of the particle coordinates within a Lagrangian. After Bielefeld Jurek visited Wojtek Zakrzewski in Durham, and we had a long and intense discussion over several weeks by exchanging a lot of e-mails and faxes. In this way our first common paper [1] on the twofold centrally extended Galilean group and noncommutative geometry was born. But besides that it was the beginning of a deep friendship and successful collaboration that has continued up to now.

The aim of the present paper is to highlight the main results of our common work on nonrelativistic particle models on the noncommutative plane and round them off by some new results. Therefore this paper is neither a review of the whole field, nor a critical comparison with the work done by other colleagues (I apologize to all whose papers will not be cited in the following).

The following considerations will concentrate on the noncommutative aspects of classical mechanics. Neither quantum effects nor quantum field theory will be considered.

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* extended version of a lecture presented at the XXII. Max Born Symposium, Wroclaw, 27. – 29. Sept. 2006
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2 Lagrangian for a free exotic particle

A particle characterized by the two central charges \( m \) and \( -\Theta \) of the planar Galilean group will be called exotic. \( m \) and \( \Theta \) appear in the following Lie-brackets (represented by Poisson-brackets (PBs)) between the translation generators \( P_i \) and the boost generators \( K_i \)

\[
\{P_i, K_j\} = m\delta_{ij} , \quad \{K_i, K_j\} = -\Theta \epsilon_{ij} .
\]

(1)

In order to find a Lagrangian whose Noether charges for boosts satisfy (1) we must add the second time derivative of the coordinates \( \ddot{x}_i \) to the usual variables \( x_i \) and \( \dot{x}_i \). As shown in [1] the most general one-particle Lagrangian, which is at most linearly dependent on \( \ddot{x}_i \), leading to the Euler-Lagrange equations of motion which are covariant w.r.t. the planar Galilei group, is given, up to gauge transformations, by

\[
L = \frac{m}{2} \dot{x}_i^2 + \frac{\Theta}{2} \epsilon_{ij} \dot{x}_i \dddot{x}_j .
\]

(2)

Using the 1st-order formalism (2) may be rewritten as

\[
L = P_i \dot{x}_i + \frac{\Theta}{2} \epsilon_{ij} y_i \dot{y}_j - H(y, P)
\]

(3)

with

\[
H(y, P) = y_i P_i - \frac{m}{2} y_i^2 .
\]

(4)

(3) describes a constrained system, because we have

\[
\frac{\partial L}{\partial \dot{y}_i} = - \frac{\Theta}{2} \epsilon_{ij} y_j .
\]

(5)

Therefore the PBs, obtained by means of the Faddeev-Jackiw procedure, take a non-standard form

\[
\{x_i, P_j\} = \delta_{ij} , \quad \{y_i, y_j\} = -\frac{1}{\Theta} \epsilon_{ij} .
\]

(6)

All other PBs vanish.

For the conserved boost generator we obtain

\[
K_i = -mx_i - \Theta \epsilon_{ij} y_j + P_i t
\]

(7)

and therefore, due to (6), the PB resp. commutator of two boosts is nonvanishing

\[
\{K_i, K_j\} = -\Theta \epsilon_{ij} .
\]

(8)
The Lagrangian (3) shows that the phase space is 6-dimensional. In order to split off two internal degrees of freedom, we have to look for a Galilean invariant decomposition of the 6-dim phase space into two dynamically independent parts: a 4-dim external and a 2-dim internal part. This decomposition is achieved by the transformation ([1], [2]) \((x, P, y) \rightarrow (X, P, Q)\) with

\[
y_i = \frac{P_i}{m} + \frac{Q_i}{\Theta}
\]

and

\[
x_i = X_i - \epsilon_{ij} \frac{Q_j}{m}
\]

leading to the following decomposition of the Lagrangian (3)

\[
\mathcal{L} = \mathcal{L}_{\text{ext}} + \mathcal{L}_{\text{int}}
\]

with

\[
\mathcal{L}_{\text{ext}} = P_i \dot{X}_i + \frac{\Theta}{2m^2} \epsilon_{ij} P_i \dot{P}_j - \frac{P_i^2}{2m}
\]

\[
\mathcal{L}_{\text{int}} = \frac{m}{2\Theta^2} Q_i^2 + \frac{1}{2\Theta} \epsilon_{ij} Q_i \dot{Q}_j.
\]

From (9) and the PBs (6) it now follows that the new coordinates \(X_i\) are noncommutative

\[
\{X_i, X_j\} = \frac{\Theta}{m^2} \epsilon_{ij}.
\]

The remaining nonvanishing PBs are

\[
\{X_i, P_j\} = \delta_{ij}, \quad \{Q_i, P_j\} = -\Theta \epsilon_{ij}.
\]

**Conclusion:** The particle Lagrangian (2) containing \(\ddot{x}_i\) leads to a nonvanishing commutator of two boosts. But in order to obtain noncommutative coordinates we are forced to decompose the 6-dim phase space in a Galilean invariant manner into two dynamically independent 4-dim external and 2-dim internal phase spaces.

Generalization of (2): If we add to (2) a term \(f(\dddot{x}_i^2)\), the obtained Lagrangian is the most general one involving, in a Galilean quasi-invariant manner, the variables \(x_i, \dot{x}_i\) and \(\dddot{x}_i\).

Then one can show\(^1\)

i) the PB of the two boosts (8) will not change,

ii) the new 8-dim phase space may be decomposed again in a Galilean invariant manner into two dynamically independent 4-dim parts, an external and an internal one.

\(^1\)Details will be published elsewhere
### 3 Commutative - versus noncommutative plane

The subalgebra of the Galilean algebra containing only translations and boosts is given in the cases of, respectively, their one- or two-fold central extensions by

| one-fold centrally extended                  | two-fold centrally extended                  |
|----------------------------------------------|----------------------------------------------|
| \{P_i, K_j\} = m\delta_{ij}                  | \{P'_i, K'_j\} = m\delta_{ij}               |
| \{P_i, P_j\} = 0                            | \{P'_i, P'_j\} = 0                          |
| \{K_i, K_j\} = 0                            | \{K'_i, K'_j\} = -\Theta\epsilon_{ij}      |

Obviously both are related by the transformations

\[
K'_i = K_i - \frac{\Theta}{2m} \epsilon_{ij} P_j, \quad P'_i = P_i.
\]  

(13)

To this corresponds the following point transformation between noncommutative coordinates \(X_i\) and commutative ones \(q_i\)

\[
X_i = q_i - \frac{\Theta}{2m^2} \epsilon_{ij} P_j
\]

(14)

as can be read off immediately from the form of \(\mathcal{L}_{\text{ext}}\) in (10).

Now the question arises: **What to use in physics, the commutative or the non-commutative plane?**

**Answer:** For free particles both possibilities are equivalent. But in the case of a nontrivial interaction one has to use the commutative (noncommutative) plane, if a local potential or gauge interaction is given in terms of \(q_i\) \((X_i)\).

### 4 General form of noncommutative mechanics

Up to now noncommutativity has been described by a constant \(\Theta\) in the PB (11). But it is possible to get \(\Theta\) as a function of \(X\) and \(P\) if one considers external Lagrangians more general than (10).

To do this consider a very general class of Lagrangians given by

\[
\mathcal{L} = P_i \dot{X}_i + \tilde{A}_i(X, P) \dot{P}_i - H(P, X)
\]

(15)

leading to the PBs

\[
\{X_i, X_j\} \sim \epsilon_{ij} \tilde{B}, \quad \tilde{B} := \epsilon_{k\ell} \partial_{P_k} \tilde{A}_\ell(X, P)
\]

(16)

with

\[
\{P_i, P_j\} = 0.
\]
We dispense with the reproduction of the more complicated form of \( \{ X_i, P_j \} \). Again by the point transformation

\[
X_i \to q_i = X_i - \tilde{A}_i(X, P)
\]

we obtain commuting coordinates \( q_i \) as follows from

\[
P_i \dot{X}_i + \tilde{A}_i \dot{P}_i = P_i \dot{q}_i + \frac{d}{dt}(\tilde{A}_i P_i) .
\]

**Examples:**

1) \( \tilde{A}_i = f(P^2)(X \cdot P) P_i \) (18)
   leading to the PBs of the phase space variables

\[
\{ X_i, X_j \} = \frac{f(P^2)}{1 - P^2 f(P^2)} \epsilon_{ij} L, \quad L := \epsilon_{kl} X_k P_l
\]

\[
\{ X_i, P_j \} = \delta_{ij} + \frac{f(P^2)}{1 - P^2 f(P^2)} P_i P_j .
\]

A particular example is given by \( f(P^2) = \frac{\Theta}{1 + P^2} \) and therefore \( \frac{1}{1 - P^2} = \Theta \).

This gives exactly Snyder’s NC-algebra presented in 1947 [3].

Another case, defined by

\[
f(P^2) = \frac{2}{P^2},
\]

can be related to a deformed Galilei algebra (to be discussed in the next section).

2) \( \tilde{A}_i = \tilde{A}_i(P) \) (21)
   leading to the PBs

\[
\{ X_i, X_j \} = \epsilon_{ij} \tilde{B}(P^2), \quad \{ X_i, P_j \} = \delta_{ij} .
\]

\( \tilde{B} \) is the Berry curvature for the semiclassical dynamics of electrons in condensed matter (cp. [4] and the literature cited therein).

We may generalize (15) to the most general 1st-order Lagrangian

\[
\mathcal{L} = (P_i + A_i(X, P)) \dot{X}_i + \tilde{A}_i(X, P) \dot{P}_i - H(P, X) .
\]

Here \( A_i(X) \) describes standard electromagnetic interaction (cp. section 6). A particular case of a \( P \)-dependent \( A_i \) has been considered in [5]. A detailed discussion of this general Lagrangian, leading to a noncommutative structure, is still under consideration.
5 Lagrangian realization of the $\tilde{k}$-deformed Galilei algebra as a symmetry algebra

In 1991 Jurek and his collaborators Nowicki, Ruegg and Tolstoy invented the $k$-deformed Poincaré algebra [6]. By rescaling the Poincaré generators and $k$, the corresponding nonrelativisitic limit, the $\tilde{k}$-deformed Galilei algebra, has been derived by Giller et al. [7] and, in a different basis, by Azcarraga et al. [8]. In this section we will describe a Lagrangian realization of the latter.

Again we look at the classical Lagrangian (15) specified by (18) and (20) together with the following choice of the Hamiltonian
\[ H = \tilde{k} \ln(P^2/2) . \] (24)

According to (19) we obtain the PBs
\[ \{X_i, X_j\} = -\frac{2}{P^2} \varepsilon_{ij} L \quad \text{and} \quad \{X_i, P_j\} = \delta_{ij} - \frac{2}{P^2} P_i P_j \] (25)

which lead, together with the Hamiltonian (24), to the EOM
\[ \dot{P}_i = 0 \quad \text{and} \quad \dot{X}_i = -\frac{2\tilde{k}}{P^2} P_i . \] (26)

Then we may define “pseudo-boosts” $K_i$
\[ K_i = P_i t + \frac{P^2}{2\tilde{k}} X_i \] (27)

which are conserved. They satisfy, together with $P_i$ and $H$, the PB-algebra
\[ \{K_i, P_j\} = \delta_{ij} P^2 - \frac{P_i P_j}{\tilde{k}} , \]
\[ \{K_i, H\} = -P_i , \quad \{K_i, K_j\} = 0 . \] (28)

Together with the standard algebra of translations (represented by $P_i$ and $H$) and rotations (represented by $L$) the relations (28) build the $\tilde{k}$-deformed Galilei algebra derived in [8].

The limit $\tilde{k} \to \infty$ leads to a divergent Hamiltonian (24). Therefore, the $\tilde{k}$-deformation does not have a standard “no-deformation limit”. Nevertheless the PB-algebra (28) gives the standard expressions in this limit.
6 Electromagnetic interaction and the Hall effect

How to introduce electromagnetic (e.m.) interaction into $L_{\text{ext}}$ (10)?

In the commutative case we have the principle of minimal e.m. coupling

$$P_i \dot{X}_i - \frac{P_i^2}{2m} \rightarrow (P_i + eA_i(X, t)) \dot{X}_i - \frac{P_i^2}{2m} + eA_0(X, t), \tag{29}$$

called the minimal addition rule, which is equivalent, due to the point transformation $P_i \rightarrow P_i - eA_i$, to the minimal substitution rule

$$P_i \dot{X}_i - \frac{P_i^2}{2m} \rightarrow P_i \dot{X}_i - \frac{(P_i - eA_i)^2}{2m} + eA_0(X, t). \tag{30}$$

In the noncommutative case the equivalence of minimal addition and minimal substitution rule is not valid. Therefore we have to consider two different ways of introducing the minimal e.m. coupling:

**Minimal addition** (Duval-Horvathy [9], called DH-model):

$$L_{\text{ext}} \rightarrow L_{\text{e.m.}} = L_{\text{ext}} + e(A_i \dot{X}_i + A_0), \tag{31}$$

which, as usual, is quasiinvariant w.r.t. standard gauge transformations

$$A_\mu(X, t) \rightarrow A_\mu(X, t) + \partial_\mu \Lambda(X, t). \tag{32}$$

**Minimal substitution** (Lukierski-Stichel-Zakrzewski [10], called L.S.Z.-model):

$$H_{\text{ext}} = \frac{P_i^2}{2m} \rightarrow H_{\text{e.m.}} = \frac{(P_i - e\dot{A}_i)^2}{2m} - e\dot{A}_0. \tag{33}$$

The corresponding Lagrangian is quasiinvariant w.r.t. generalized gauge transformations, given in infin. form by

$$\delta \dot{A}_\mu(X, t) := \dot{A}'_\mu(X + \delta X, t) - \dot{A}_\mu(X, t) = \partial_\mu \Lambda(X, t) \tag{34}$$

with

$$\delta X_i = -e\Theta \epsilon_{ij} \partial_j \Lambda \tag{35}$$

supplemented by

$$\delta P_i = e \partial_i \Lambda. \tag{36}$$

Note, that the coordinate transformations (34) are area preserving.

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2 The gauge fields in this model we provide with a hat in order to distinguish them from the corresponding quantities in the DH-model.
It turns out that both models are related to each other by a noncanonical transformation of phase space variables supplemented by a classical Seiberg-Witten transformation of the corresponding gauge potentials:

If we denote the phase space variables and potentials for

– the DH-model by \((\eta, P, A_\mu)\)

– the L.S.Z.-model by \((X, P, \hat{A}_\mu)\)

then we find

\[
\eta_i(X, t) = X_i + e \Theta \epsilon_{ij} \hat{A}_j(X, t) \quad (37)
\]

\[
P_i = P_i - e \hat{A}_i(X, t) \quad (38)
\]

with the resp. field strengths related by

\[
\hat{F}_{\mu\nu}(X, t) = \frac{F_{\mu\nu}(\eta, t)}{1 - e \Theta B(\eta, t)} . \quad (39)
\]

The Seiberg-Witten transformation between the resp. gauge fields is more involved and will not be reproduced here (for details cp. [10]).

These results lead to an interesting by-product:

Consider the PBs of coordinates in both models, given by

\[
\{\eta_i, \eta_j\} = \frac{\Theta m^2 \epsilon_{ij}}{1 - e \Theta B(\eta, t)} \quad \text{and} \quad \{X_i, X_j\} = \frac{\Theta}{m^2} \epsilon_{ij} \quad (40)
\]

then the foregoing results implicitly give the coordinate transformation between a model with a constant noncommutativity parameter \(\Theta\) and one with an arbitrary coordinate-dependent noncommutativity function \(\Theta(X, t)\) (this result has been rediscovered quite recently in [11]).

Now the question arises, which of both models has to be used for physical applications? Let us look at one example, the Quantum Hall effect, in the limit of large e.m. fields. In the case of the DH-model [9] the Hall law

\[
\dot{X}_i = \epsilon_{ij} \frac{E_j}{B} \quad (41)
\]

is valid at the critical magnetic field

\[
B_{\text{crit}} = (e \Theta)^{-1} . \quad (42)
\]

Then it follows from the field transformation law (39) that, for the L.S.Z.-model, the Hall law is valid in the limit of large e.m. fields as required. In order to see this in more detail we have to consider the EOM for the L.S.Z.-model formulated in terms of the gauge-invariant phase space variables \(\eta\) and \(P\). From (29) and the corresponding PBs we obtain \((e = 1, m = 1)\)

\[
\dot{\eta}_i = (1 + \Theta \hat{B}) P_i - \Theta \epsilon_{ij} \dot{E}_j \quad (43)
\]

\[
\dot{P}_i = \dot{B} \epsilon_{ij} P_j + \dot{E}_i . \quad (44)
\]
For the particular case of homogeneous e.m. fields we obtain finally
\[ \dot{\eta}_i = \dot{B} \epsilon_{ij} \eta_j + \dot{E}_i \]  
(45)
leading to the Hall law (41) in the high field limit.
Note that (45) has the same functional form as in the commutative case.

7 Supersymmetry

In the following, we supersymmetrize the e.m. coupling models treated in the last section. To do that we follow the treatment in section 3 of [12]. For that, we consider standard $N = 2$ SUSY characterized by
\[ H = \frac{i}{2} \{ Q, \bar{Q} \} \]  
(46)
and
\[ \{ Q, Q \} = \{ \bar{Q}, \bar{Q} \} = 0 . \]  
(47)
In order to construct the supercharge $Q$, satisfying (46), we start with the common structure of the bosonic Hamiltonian for both models ($e = 1, m = 1$)
\[ H_b = \frac{1}{2} \left( P_i^2 + W_i^2 (X) \right) \]  
(48)
with
\[ P_i = P_i \quad \text{for the DH-model} \]
and
\[ P_i = P_i - A_i \quad \text{for the L.S.Z.-model}. \]
Note that, in accordance with the quantized form of (46), the potential term in (48) is chosen to be positive
\[ A_0 = -\frac{1}{2} W_i^2 . \]  
(49)
In order to add to (48) its fermionic superpartner, we supplement the bosonic phase space variables with fermionic coordinates $\psi_i (\bar{\psi}_i)$ satisfying canonical PBs
\[ \{ \psi_i, \bar{\psi}_j \} = -i \delta_{ij} . \]  
(50)
Now we assume
\[ Q = i (P_i + i W_i) \psi_i \]  
(51)
such that (48) is valid. But now the relations (47) are fulfilled only if the following two conditions are satisfied:

\[ \{ \mathcal{P}_i, \mathcal{P}_j \} = \{ W_i, W_j \} \]  
and  
\[ \{ \mathcal{P}_i, W_j \} = \{ \mathcal{P}_j, W_i \} . \]  

It can be shown that (53) is satisfied automatically in both models, whereas (52) fixes the magnetic field in terms of \( W_i \) (same form for both models):

\[ B = \frac{\Theta}{2} \epsilon_{ij\ell} \partial_k W_i \partial_\ell W_j . \]  

The connection between B-field (54) and electric potential \( A_0 \) (49) takes a simple form in the case of rotational invariance. From

\[ W_i(\mathbf{X}) = \partial_i W(\mathbf{r}) \]  

we obtain

\[ A_0(\mathbf{r}) \leftarrow -\frac{1}{2} (W'(\mathbf{r}))^2 \]  
and

\[ B(\mathbf{r}) \leftarrow -\frac{\Theta}{r} A_0'(\mathbf{r}) . \]

As an example, consider the harmonic oscillator. Then

\[ A_0 = -\frac{\omega^2}{2} r^2 \]  
and we obtain a homogeneous \( B \)-field of strength

\[ B = \Theta \omega^2 . \]

8 Miscellaneous results

The Galilean invariant decomposition of 6-dim phase space into invariant subspaces (cp. section 2) only holds for \( m \neq 0 \). For the case of vanishing mass we have to live with a 6-dim phase space as long as we keep Galilean invariance (for a reduction to 4-dim phase space in other (interacting) cases cp. [13]). But the \( m = 0 \) model shows a higher symmetry: exotic Galilean conformal symmetry supplemented by an additional hidden \( O(2,1) \) symmetry [14].

Other interesting results treat homogeneous e.m. fields \( E_i \) and \( B \) as elements of either an enlarged exotic Galilean algebra [4] or of an enlarged Galilean conformal algebra [14].

It is outside the scope of this paper to discuss these interesting results in more detail.
9 Outlook

Dynamics on the noncommutative plane is a fascinating field. Much has still to be done – hopefully in continuing my very successful collaboration with Jurek as well as with Wojtek.

10 Acknowledgements

I’m grateful to J. Azcarraga, S. Gosh, J. Lukierski and W. Zakrzewski for discussions and helpful remarks.

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