Abstract. We propose constructions of \(k\)-graphs from combinatorial input and initiate a study of their spectral theory. Guided by geometric insight, we obtain several new series of \(k\)-graphs using cube complexes covered by Cartesian products of trees, for \(k \geq 3\). The constructions yield rank \(k\) Cuntz-Krieger algebras for all \(k \geq 2\). We introduce Ramanujan \(k\)-graphs satisfying optimal spectral gap property, and show explicitly how to construct such \(k\)-graphs.

1. Introduction

The main thrust of this paper is to construct new series of examples of \(k\)-graphs for all \(k \geq 2\) using cube complexes. Towards this end we formalise a combinatorial definition of a finite \(k\)-graph \(\Lambda\) which is decoupled from geometrical realisations. The definition is formulated using a directed coloured graph with edges of \(k\) colours, together with rules to uniquely form paths of length two and length three in different colours.

The study of \(k\)-graphs and their \(C^*\)-algebras originates in work of Robertson and Steger in [26], where they found a generalisation of the Cuntz-Krieger algebras from topological Markov shifts introduced in [3]. The construction of [26] was motivated by an observation of Spielberg [30] clarifying that a free group \(\Gamma\) on finitely many generators, viewed as the fundamental group of a finite connected graph, admits an action on the boundary \(\Omega\) of its Cayley graph such that in the associated crossed product \(C(\Omega) \rtimes \Gamma\) one finds generating partial isometries \(s_w\) for an ordinary Cuntz-Krieger algebra \(A\) associated to a matrix \(M\) that records incidence in the universal covering graph. This provided the basis for defining a higher rank Cuntz-Krieger algebra \(A\) in [26]: the input consists of a finite alphabet and a family of commuting \((0,1)\)-matrices \(M_1, M_2, \ldots, M_k\), with \(k \geq 1\), having entries in the alphabet, satisfying a number of conditions, and controlling the formation of words \(w\) of \(\mathbb{N}^k\)-valued shape.

Shortly after [26], Kumjian and Pask [12] defined a higher rank, or \(k\)-graph, as an abstract concept consisting of a category \(\Lambda\) with a \(\mathbb{N}^k\)-valued degree of morphisms modelling, roughly speaking, the formation of words from [26]. The definition of \(\Lambda\) is concise but allows nevertheless for associating a \(C^*\)-algebra \(C^*(\Lambda)\) that is generated by partial isometries and satisfies Cuntz-Krieger type relations, recovering the definition from [26]. The study of \(k\)-graph \(C^*\)-algebras expanded fast into a very active direction of research in operator algebras touching many other fields of mathematics. Many results are now known that describe properties on a \(k\)-graph that transfer to \(C^*(\Lambda)\) such as simplicity or ideal structure.

The explicit examples of higher rank Cuntz-Krieger algebras \(A\) from [26] feature \(k = 2\). There the foundation for the construction of the \(C^*\)-algebra is a group action on the boundary of an affine \(\tilde{A}_2\) building, from which one extracts a suitable alphabet and defines two commuting matrices with the
properties (H0)-(H3) specified in [26], see also [11]. Now, even though $k$-graphs were defined some time ago, not many explicit examples are known for $k \geq 3$, see though [17].

Our combinatorial definition is easy to employ in concrete situations, as we shall illustrate. The definition also has the advantage that it bypasses the need to prescribe matrices as in [26], possibly of large size, and conditions on them encompassing formation of words before a $C^*$-algebra can be defined. In our Example 5.10 the 3-graph has 25 vertices and so the associated incidence matrices will be $25 \times 25$. We compute them with the help of MAGMA, after defining the 3-graph using our combinatorial definition, with the purpose of estimating the joint spectral gap of the 3-graph.

One of our main results establishes the existence of an infinite family of combinatorial $k$-graphs constructed from $k$-cube complexes. We record it below:

**Theorem 1.1.** (Cf. Theorem [3.4]) Given $P$ a one-vertex $k$-cube complex with $k \geq 2$, there is a one-vertex $k$-graph $\Lambda(P)$ obtained by setting the edges in the 1-skeleton of $\Lambda(P)$ to be the directed edges of $P$. Specifically, each geometric edge in $P$ gives rise to two distinct edges in the 1-skeleton of $\Lambda(P)$, of same colour and opposite orientation.

The series of examples of $k$-graphs we obtain from Theorem 1.1 differ from, for example, the $k$-graphs constructed in [11], [25], [26], [27], since these papers only considered the case $k = 2$. Moreover, our examples are new for $k = 2$, see Remark 3.8. Our construction also differs from the more recent [16] when $k = 2$ and [17] for $k \geq 3$, both of which emulate [26] and [11]. The core idea in these references is as follows: Given a cell complex $\mathcal{X}$, each $k$-dimensional cell in $\mathcal{X}$ becomes a vertex in a $k$-graph $\Lambda(\mathcal{X})$, and for two such cells there is an edge $\Lambda(\mathcal{X})$ if the given cells are adjacent via a $(k-1)$-cell, for $k \geq 2$. In this construction one may take pointed cells as vertices, or unpointed. Another way to distinguish our higher rank graphs from the ones in [26] is to compute the products of coordinate matrices. The products of coordinate matrices in [26] have to be (0,1)-matrices, but this is not necessarily the case in our construction (see Example 5.10).

To place our definition in the context of similar developments, we recall that a recipe for constructing 2-graphs was proposed in [12, Section 6], starting from two distinct directed graphs on the same set of vertices with commuting vertex matrices. A step forward was made in [6], see their Remark 2.3, where a certain associativity type condition was identified as sufficient. In [9], the authors distilled these earlier attempts at constructing higher rank graphs and landed on a prescription requiring a skeleton, or a $k$-coloured graph (where the colouring refers to edges and employs $k$ distinct colours) and a collection of building blocks termed "squares" that satisfy compatibility requirements, see [9, Theorem 4.4]. The "squares" here are certain coloured-graph morphisms.

A further simplification of the prescription of a $k$-graph from its skeleton, seen as a $k$-coloured graph, has been employed in concrete examples such as [13, Example 7.7] and [15, Section 8.2]. In fact, this last example articulates the requirements on the coloured graph that inspired our conditions (F1) and (F2) in Definition 3.1. It is interesting to note that the validity of the associativity condition (F2), for $k \geq 3$, is a priori highly nontrivial. There are connections to finding solutions of the Yang-Baxter equation, see for example [34] and [32].

There is a strong connection between geometry of CW-complexes, groups and semigroup actions, higher rank graphs and the theory of $C^*$-algebras. The difficulty is that there are many ways to associate $C^*$-algebras to groups, semigroups and CW-complexes, and this can lead to both isomorphic and non-isomorphic $C^*$-algebras. For the higher rank graphs, there is a canonical way to associate a $C^*$-algebra, cf [12], but it happens that non-isomorphic $k$-rank graphs lead to the same $C^*$-algebra. This conclusion is often achieved through computation of K-theory and applications of the powerful Kirchberg-Phillips classification machinery for purely infinite simple unital nuclear $C^*$-algebras.
One interesting question is what is a "genuine" higher rank? Meaning that our $k$-rank graph cannot be obtained by some standard procedure from graphs of smaller ranks. We suggest to address this question by introducing the spectral theory of combinatorial higher rank graphs. So far the spectrum of strongly connected higher rank graphs was considered in [10] and [13], through Perron-Frobenius theory, which lead to new explicit constructions of von Neumann factors. We generalise the results of [13] by constructing infinite series of III$_\lambda$ factors for any $k$, and infinitely many values of $\lambda$.

We want to stress the following simple but important point about our construction of $k$-graphs: Recall that for an undirected graph with (vertex) adjacency matrix $A$, we have $A(v, w) = 1 = A(w, v)$ if vertices $v, w$ are connected. Thus the adjacency matrix is symmetric and the eigenvalues are real. Now, in a $k$-graph, we have directed edges in the various colours in its 1-skeleton. There is no reason why the adjacency matrix should be symmetric. What can be said in general about a $k$-graph is that, if it is strongly connected, then its associated coordinate matrices jointly admit a unimodular Perron-Frobenius eigenvector, [10]. Now, in our construction of the $k$-graph $\Lambda(P)$ from a $k$-cube complex $P$, the procedure is such that it assigns to each undirected edge in the 1-skeleton of $P$ a pair of edges with opposite orientation in $\Lambda(P)$. As a consequence, the adjacency matrix for the complex in direction $i$ is the same as the coordinate matrix $M_i$ of $\Lambda(P)$ in colour $i$, for all $i = 1, \ldots, k$. Thus all our constructions of $k$-graphs have symmetric matrices.

With this in mind, we suggest a new class of higher rank graphs, which we call Ramanujan $k$-rank graphs, see Section 5.2. Their coordinate matrices are symmetric, so all eigenvalues are real and it makes sense to consider the spectral gap. We will show that our graphs satisfy the optimal spectral gap, and this distinguishes them from the examples of higher rank graphs that have appeared in the literature so far.

The structure of the paper is the following: In a preliminary section 2 we collect conventions and results about categories, groups acting on products of trees, $k$-cube groups and associated cube complexes, $k$-graphs and their $C^*$-algebras. In Section 3 we formalise our definition of combinatorial $k$-graphs, show how this relates to [12], and construct explicit examples of one-vertex $k$-graphs from $k$-cube groups and their complexes, see Theorem 3.4. We then show that the resulting $k$-graphs are strongly connected in the sense of [10] and rigid in the sense of [14], in particular they are aperiodic and yield classifiable $C^*$-algebras in the sense of the Kirchberg-Phillips classification. In Section 4 we show how to obtain $k$-graphs on several vertices from covers of one-vertex complexes, and discuss some of their properties, in particular we show that once again the resulting $C^*$-algebras are covered by the Kirchberg-Phillips classification theory. In Corollary 5.3 we expand the scope of the constructions of 2-graphs in [13, Example 7.7] leading to factors of type $\text{III}_{1/2}$ and give an explicit infinite family giving type $\text{III}_{1/(2L)^2}$ factors, with $L$ an arbitrary integer. In the final section 5.2 we introduce the notion of Ramanujan $k$-graphs and show that there are infinite families of such $k$-graphs, see Theorem 5.9. Example 5.10 details an explicit Ramanujan 3-graph on 25 vertices with optimal spectral gap. The 3-graph moreover features the interesting property that while it arises from an infinite 3-cube group $\Gamma_1$ which is also an irreducible lattice, in the cover with 25 sheets each of the three distinct alphabets in $\Gamma_1$ generates a finite group of order 25.

Acknowledgment. We thank Mark Lawson and Aidan Sims for useful comments. This research was initiated during a visit by A.V. to Oslo in connection to the Master Class on "Equilibrium states in semigroup theory, K-theory and number theory", 4-6 November, 2019, supported by the Trond Mohn Foundation through the project "Pure Mathematics in Norway". It was completed while N.L. was a Research Fellow in the Cluster of Excellence Mathematics Münster at WWU, Germany. She thanks for warm hospitality and excellent working conditions in the Cluster.
2. Preliminaries

2.1. Categories. We follow the principles laid out in [4] Chapter II, §1.1 and §1.2]. A category $\mathcal{C}$ consists of objects $\text{Obj}(\mathcal{C})$ and morphisms $\text{Hom}(\mathcal{C})$. We blur the distinction between $\mathcal{C}$ and $\text{Hom}(\mathcal{C})$, and refer to the latter as the elements of $\mathcal{C}$. To each element $f \in \mathcal{C}$ we associate two objects, its origin $o_{\mathcal{C}}(f)$ and terminus $t_{\mathcal{C}}(f)$, and the category itself is seen as a collection of elements endowed with a partial product governed by compatibility of objects. More precisely, two elements $f, g \in \mathcal{C}$ have a product $fg$ provided that the terminus of $f$ is the origin of $g$, thus the product in the category is a reverse composition. The following rules for the product need to hold in $\mathcal{C}$: if $fg$ and $gh$ are defined, then $f(gh)$ and $(fg)h$ are defined, too, and we have $f(gh) = (fg)h$. These rules allow us to form paths in $\mathcal{C}$ similarly to the formation of paths on a directed graph. At this moment we point out that this convention for product is the opposite of the one utilized in $k$-graph literature.

To each object $v$ corresponds a unique element with origin and terminus equal to $v$. Such elements are referred to as identity elements, since for each $f \in \mathcal{C}$, the product on the left with the element corresponding to the origin of $f$ is defined and equal to $f$, as is the product on the right with the element corresponding to its terminus.

2.2. Directed graphs. By a directed graph $D$ we mean a set $D^0$ of vertices and a set $D^1 \subset D^0 \times D^0$ of edges with an orientation, allowing loops. We shall assume that $D^0$ is finite. We form a path $e\,f$ when $t_D(e) = o_D(f)$ for $e, f \in D^1$, and extend this to finite directed paths $f_1f_2\ldots f_m$ on $D$, and likewise in the case of a finite number of graphs on the same vertex set, see Section 3. By a $k$-coloured graph for $k \geq 1$ we mean a graph with a surjective map $c$ from $D^1$ onto a finite set of size $k$ whose elements are colours.

2.3. Complexes covered by products of trees. We start by introducing our definition of $k$-cube complex. Then we expand on the case of one-vertex $k$-cube complexes, for which we follow the notation and approach of [32] and [17]. We refer to [28] for details on CAT(0) complexes and to [8] for the basic theory of CW complexes. We use the letter $T$ for an arbitrary regular tree, and $T_l$ for the regular $l$-valent tree, where $l \geq 1$.

**Definition 2.1.** Let $k \geq 1$ be a positive integer. A CW complex $\mathcal{X}$ is a $k$-dimensional cube complex, or a $k$-cube complex, if its universal cover is a Cartesian product of $k$ trees $T_1 \times T_2 \times \cdots \times T_k$, each of which has finite constant valency.

In the case of a one-vertex $k$-cube complex $P$, an equivalent definition is as the quotient space $P = Z\backslash G$ of a group $G$ with a free and transitive action on a product $Z = T_1 \times T_2 \times \cdots \times T_k$ of $k$ trees. Such $G$ are called $k$-cube groups, see Definition 2.3. For general $k$-cube complexes with more than one vertex, the similar definition as a quotient space $\mathcal{X} = Z\backslash G$ can be enforced upon replacing transitive action with cocompact action.

We leave the case of trees with possibly non-constant and/or infinite valency for future discussion.

To describe a $k$-cube complex for $k \geq 2$ it is useful to recall the formalism of 2-complexes (or square complexes) covered by products of two trees, see e.g. [32].

A square complex $S$ is a 2-dimensional combinatorial cell complex with 1-skeleton consisting of a graph $G(S) = (V(S), E(S))$ with set of vertices $V(S)$, and set of oriented edges $E(S)$, and with 2-cells arising from a set of squares that are combinatorially glued to the graph $G(S)$. More precisely, let $e \mapsto e^{-1}$ denote orientation reversal of an edge $e \in E(S)$, and suppose that $(e_1, e_2, e_3, e_4)$ is a 4-tuple of oriented edges in $E(S)$ with the origin of $e_{i+1}$ equal to the terminus of $e_i$ (for $i$ modulo 4). A square $□ = (e_1, e_2, e_3, e_4)$ is the orbit of $(e_1, e_2, e_3, e_4)$ under the dihedral action generated by
cyclically permuting the edges \(e_i\) and by the reverse orientation map

\[
(e_1, e_2, e_3, e_4) \mapsto (e_4^{-1}, e_3^{-1}, e_2^{-1}, e_1^{-1}).
\] (2.1)

As customary, we think of a square-shaped 2-cell glued to the (topological realization of the) respective edges of the graph \(G(S)\).

A vertical/horizontal structure (in short, a VH-structure) on a square complex is given by a bipartite structure of the set of unoriented edges \(E(S) = E_V \sqcup E_H\) such that for every vertex \(v \in V(S)\) the link at \(v\) is the complete bipartite graph on the resulting partition \(E(v) = E(v)_V \sqcup E(v)_H\), with \(E(v)\) denoting the set of oriented edges with origin \(v\). For more details on square complexes, we refer the reader to, for example, [2]. Torsion free cocompact lattices \(\Gamma\) in the link at \(v\) is the complete bipartite graph on the resulting partition \(E(v) = E(v)_V \sqcup E(v)_H\), with \(E(v)\) denoting the set of oriented edges with origin \(v\). For more details on square complexes, we refer the reader to, for example, [2]. Torsion free cocompact lattices \(\Gamma\) in \(\text{Aut}(T_m) \times \text{Aut}(T_l)\) with \(m, l \geq 1\), not interchanging the factors and considered up to conjugation, correspond uniquely to finite square complexes \(S\) with a VH-structure of partition size \((m, l)\) up to isomorphism. Further, simply transitive torsion free lattices not interchanging the factors correspond to finite square complexes with only one vertex and a VH-structure, necessarily of constant partition size.

2.4. **One-vertex \(k\)-cube complexes.** We first look at the case when \(k = 2\). Let \(S\) be a square complex with one vertex \(v \in S\) and a VH-structure \(E(S) = E_V \sqcup E_H\). Passing from the origin to the terminus of an oriented edge induces a fixed point free involution on \(E(v)_V\) and on \(E(v)_H\). Thus the partition size is necessarily a tuple \((2m, 2l)\) of even integers, \(m, l \geq 1\). The lattice identified with \(\pi_1(S, v)\) admits a description in terms of two generating subsets \(A, B\), see [32] Definition 5).

**Definition 2.2.** A vertical/horizontal structure, or VH-structure, in a group \(G\) is an ordered pair \((A, B)\) of finite subsets \(A, B \subseteq G\) such that the following hold.

1. Taking inverses induces fixed point free involutions on \(A\) and \(B\).
2. The union \(A \cup B\) generates \(G\).
3. The product sets \(AB\) and \(BA\) have size \(#A \cdot #B\) and satisfy \(AB = BA\).
4. The sets \(AB\) and \(BA\) do not contain 2-torsion.

The tuple \((#A, #B)\) is called the valency vector of the VH-structure in \(G\).

If a group \(G\) admits a VH-structure \((A, B)\) of valency vector \((#A, #B)\), then by [2] Section 6.1, there is a square complex \(S_{A,B}\) with one vertex and a VH-structure in the sense of subsection 2.3. The set of oriented edges of \(S_{A,B}\) is the disjoint union \(E(S_{A,B}) = A \sqcup B\), with the orientation reversion map given by \(e \mapsto e^{-1}\), and with \(A, B\) labelling the edges in vertical and horizontal direction, respectively. The link of \(S_{A,B}\) in \(v\) is the complete bipartite graph with vertices labelled by \(A\) and \(B\), see [32] Lemma 1], and [1] Theorem C] implies that the universal cover of \(S_{A,B}\) is a product of trees. Conversely, given a square complex \(S\) with a VH-structure \((A, B)\) and a single vertex, its fundamental group (i.e. the fundamental group of its topological realisation) admits a VH-structure of valency \((#A, #B)\), see [24] Proposition 5.7]. We refer to it as a \((#A, #B)\)-group. Example 3.7 shows a \((4, 4)\)-complex with associated \((4, 4)\)-group.

To describe the geometric squares of \(S_{A,B}\), note that a relation \(ab = b'a'\) in \(G\) with \(a, a' \in A\) and \(b, b' \in B\) (not necessarily pairwise distinct), as prescribed by Definition 2.2 leads to four algebraic relations obtained upon cyclic permutation and inversion, namely

\[
ab = b'a', \quad a^{-1}b' = ba^{-1}, \quad a'^{-1}b'^{-1} = b^{-1}a^{-1} \quad \text{and} \quad a'b'^{-1} = b'^{-1}a.
\] (2.2)

This leads to the definition of a geometric square as a tuple of four Euclidean squares. All four vertices in each square coalesce into the single vertex \(v\) of \(S_{A,B}\) when we glue the edges according to labels and orientation. Before we introduce our convention, we recall briefly two other conventions for describing geometric squares.
2.5. One convention - see e.g. Rattaggi. The formalism of a geometric square seen as a 4-tuple of squares in Euclidean space is well-known. In \[22\] Figure 4.1, page 182, for example, the group relation \(ab = b'a'\) is reflected by the 4-tuple of squares having edges labelled according to a one-way cyclic permutation in counterclockwise direction, see below:

\[(2.3)\]

\[\begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}\]

The notation means that if \(S_O\) is regarded as a reference square, then \(S_H\) is obtained by reflection in the horizontal direction (about \(b\)), \(S_V\) by reflection in the vertical direction (about \(a\)) and, finally, \(S_R\) arises from rotation counterclockwise by \(\pi\). Our use of \(S_O, S_R, S_H, S_V\) as notation for the squares is inspired by \[17\], Section 2).

The geometric square associated to \(ab = b'a'\) in \[22\] and visualised in \(2.3\) is given by

\[\{(a, b, a', b'), (a', b', a, b), (a^{-1}, b^{-1}, a', b^{-1}), (a^{-1}, b^{-1}, a', b^{-1})\}.

2.6. A second convention - see Kimberley-Robertson. In \[11\], Kimberley-Robertson adopted a two-direction labelling of their squares which to a group relation \(ab = b'a'\) assigns a 4-tuple of squares according to the convention below:

\[(2.4)\]

\[\begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}\]

The geometric square associated to \(ab = b'a'\) and visualised in \(2.4\) is given by

\[\{(a, b, b', a'), (a^{-1}, b^{-1}, b^{-1}, a^{-1}), (a^{-1}, b^{-1}, b^{-1}, a), (a', b^{-1}, b^{-1}, a')\}.

2.7. Our convention. We depart from both these conventions of labelling edges in squares determining a geometric square (and we swap the letters for vertical and horizontal directions): we keep the cyclic permutation in counterclockwise direction from \[22\] but choose labelling of edges as "starting" at one vertex by going out in both horizontal and vertical direction, similar to \[11\]. This convention will facilitate our constructions of \(k\)-graphs later on.

\[(2.5)\]

\[\begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}\]

Explicitly, we define a geometric square as visualised in \(2.5\) to be a tuple

\[(2.6)\]

\[\{(a, b, a^{-1}, b^{-1}), (a^{-1}, b^{-1}, a, b), (a', b^{-1}, a^{-1}, b'), (a^{-1}, b', a', b^{-1})\},

where any two squares are seen as equivalent.

Since for our purposes it will be important to keep track of how such squares arise, we introduce the following more precise notation: for \(a \in A\) and \(b \in B\), we let

\[(2.7)\]

\[S_O^{a,b} := (a, b, a^{-1}, b^{-1}),\]
where $a', b'$ are the unique elements in $A$ and $B$, respectively, such that $ab = b'a'$. We refer to $ab$ as the vertical-horizontal pair of edges in $S_O^{ab}$ and to $b'a'$ as the horizontal-vertical pair of edges in $S_O^{ab}$.

In [32], the last named author generalised VH-structure to the $k$-dimensional case, as follows.

**Definition 2.3.** (See [32] Definition 7) A $k$-cube structure in a group $G$ is an ordered $k$-tuple $(A_1, \ldots, A_k)$ of finite subsets $A_i \subseteq G$ such that the following hold for all $i, j = 1, \ldots, k, i \neq j$:

1. Taking inverses induces fixed point free involutions on $A_i$.
2. The union $\cup A_i$ generates $G$.
3. The product sets $A_iA_j$ and $A_jA_i$ have size $\#A_i \cdot \#A_j$ and $A_iA_2 = A_2A_i$.
4. The sets $A_iA_j$ and $A_2A_i$ do not contain 2-torsion.
5. The group $G$ acts simply transitively on a Cartesian product of $k$ trees.

The tuple $(\#A_1, \ldots, \#A_k)$ is the valency vector of the $k$-cube structure in $G$, and $A_1, \ldots, A_k$ are generating sets of $G$.

We note that each pair $(A_i, A_j) \subseteq G$ for $i, j = 1, \ldots, k$ with $i \neq j$ forms a subgroup $G_{i,j}$ of $G$ equipped with a VH-structure. This observation can be used to show that to a given $k$-cube group $G$ with generating family $(A_1, \ldots, A_k)$ there is an associated $k$-cube complex $P_{(A_1, \ldots, A_k)}$. Its 2-dimensional cells are prescribed by the square complexes $S_{A_i, A_j}$ obtained from each $(\#A_i, \#A_j)$ group $G_{i,j}$ for $i \neq j$.

**Remark 2.4.** In studying $k$-cube groups and one-vertex $k$-cube complexes, we will move freely between two equivalent interpretations. Starting from a $k$-cube group $G$ defined algebraically through properties (1)-(5) in Definition 2.3, the associated quotient space $Z \backslash G$ with $Z$ a product of $k$ trees is a $k$-cube complex with one vertex. Its construction as a CW complex from $j$-cells for $0 \leq j \leq k$ is detailed in [17, Definition 2.3]. Conversely, one may define a $k$-cube group $G$ from combinatorial data by starting with $k$ finite sets of even cardinalities (encoding edges), and building up by induction (on dimension of cells) a $k$-dimensional complex with the necessary compatibility to yield generating sets $A_1, A_2, \ldots, A_k$ for $G$, see [17, Definition 2.4].

We now recall briefly the second procedure to obtain a $k$-cube complex $P$ for $k \geq 2$, see [17, Section 2] for details. The 0-dimensional cells of $P$ form a set $V$ of vertices, and the 1-dimensional cells form a set $E$ of edges of $P$. The set of edges partitions into $k$ subsets as $E = E_1 \sqcup \cdots \sqcup E_k$, with the convention that if an edge $e$ is in $E_j$ then $e^{-1} \in E_j$ for $j = 1, \ldots, k$. We refer to $E_i$ as the subset of edges of colour $i$, for $i = 1, \ldots, k$, where we assume that there are $k$ distinct colours. The 2-cells are obtained from geometric squares of the form $S_O = (a, b, a'^{-1}, b'^{-1})$ as the equivalence class $\{S_O, S_R, S_H, S_V\}$ described in [26], where $a, a' \in E_i$ and $b, b' \in E_j$ for $i \neq j$. By a geometric square we mean any square in $\{S_O, S_R, S_H, S_V\}$. Similar to [17], for distinct colours $i \neq j$ we let

$$F(i, j) = \{S = (a, b, a'^{-1}, b'^{-1}) \mid S \text{ is a geometric square with } a, a' \in E_i, b, b' \in E_j\},$$

and we denote by $S^{ij}$ a generic square in $F(i, j)$ for all $i \neq j$ in $\{1, \ldots, k\}$. The 3-cells are determined by geometric cubes, all whose 6 faces are geometric squares, see figure 7 for a (generic) such cube.
More precisely, the 6 faces of the cube are geometric squares \((S_{1}^{ij}, S_{2}^{ij}, S_{3}^{ij}, S_{4}^{ij}, S_{5}^{ij}, S_{6}^{ij})\), with
\[
S_{1}^{ij} = (a_{1}, b_{1}, a_{2}^{-1}, b_{2}^{-1}), \text{ front face}
\]
\[
S_{2}^{ij} = (a_{2}, c_{3}, a_{3}^{-1}, c_{2}^{-1}), \text{ right face}
\]
\[
S_{3}^{ij} = (b_{2}, c_{2}, b_{3}^{-1}, c_{3}^{-1}), \text{ bottom face}
\]
\[
S_{4}^{ij} = (b_{1}, c_{3}, b_{4}^{-1}, c_{4}^{-1}), \text{ top face}
\]
\[
S_{5}^{ij} = (a_{1}, c_{4}, a_{4}^{-1}, c_{3}^{-1}), \text{ left face}
\]
\[
S_{6}^{ij} = (a_{4}, b_{1}, a_{3}^{-1}, b_{3}^{-1}), \text{ back face}
\]

In particular, any one of the 6 geometric squares in this list is given subject to the equivalence relation (2.6), and the geometric cube can be equivalently presented with any one of the 8 vertices in the bottom-left position. We stress that the labels \(a_{1}, \ldots, a_{4}, b_{1}, \ldots, b_{4}, c_{1}, \ldots, c_{4}\) here are formal symbols that keep track of how the cubes are glued in the complex. As already mentioned, if \(e\) is a label for an edge, then \(e^{-1}\) is the label recording orientation reversal.

For \(4 \leq l \leq k\), the \(l\)-dimensional cells are \(l\)-cubes, see [17].

A given \(k\)-cell in a \(k\)-cube complex \(P\) has a topological realisation as the product of intervals \([0,1]^{k}\). By [17], any \(k\)-cell in \(P\) has a topological realisation as the product of intervals \([0,1]^{k}\). Denoting by \(\varepsilon_{i}\) the standard basis elements in \(\mathbb{R}^{k}\), for \(i = 1, \ldots, k\), we view a geometric edge in \(P\) as having degree \(\varepsilon_{i}\) if it lies in the span of the generator \(\varepsilon_{i}\) in its topological realisation. This agrees with the degree of paths in higher rank graphs in section 3.

2.8. Examples of cube groups. The cube groups in this section were introduced in [24], see also [32]. They are the first explicit examples of groups acting freely and transitively on products of \(k\) trees of constant valencies, for \(k \geq 3\). We refer to them as RSV-groups.

There are two distinct cases: The first is that the trees have same valency. The construction of the groups in this case is of arithmetic nature. In order to recall the simplest example in the case \(k = 3\) we recall quickly the main ingredients of the construction.

For \(q\) an odd prime, let \(\delta \in \mathbb{F}_{q}^{\times}\) be a generator of the multiplicative group of the field with \(q^{2}\) elements. If \(i,j \in \mathbb{Z}/(q^{2} - 1)\mathbb{Z}\) satisfy \(i \neq j \pmod{q - 1}\), then \(1 + \delta^{j-i} \neq 0\), and it follows that there is a unique \(x_{i,j} \in \mathbb{Z}/(q^{2} - 1)\mathbb{Z}\) with \(\delta^{x_{i,j}} = 1 + \delta^{j-i}\).

Set \(y_{i,j} := x_{i,j} + i - j\), and note that \(\delta^{y_{i,j}} = \delta^{x_{i,j} + i - j} = (1 + \delta^{j-i}) \cdot \delta^{i-j} = 1 + \delta^{i-j}\). Define
\[
l(i,j) := i - x_{i,j}(q-1),
\]
\[
k(i,j) := j - y_{i,j}(q-1),
\]
and further let $M \subseteq \mathbb{Z}/(q^2 - 1)\mathbb{Z}$ be a union of cosets under $(q - 1)\mathbb{Z}/(q^2 - 1)\mathbb{Z}$ with $\#M = k$.

If $q$ is odd, it was shown in [24] that the following groups act freely and transitively on a product of $k$ trees:

$$\Gamma_{M,\delta} = \left\{ a_i, i \in M \mid a_i a_j = a_{k(i,j)}a_{l(i,j)} \text{ for all } i, j \in M \text{ with } i \neq j \pmod{q - 1} \right\}$$

**Example 2.5.** The smallest example in dimension $k = 3$ arises with $q = 5$ and $M$ equal to the collection of cosets $i \in \mathbb{Z}/24\mathbb{Z}$ with $i$ not dividing 4. This group, denoted $\Gamma_1$, acts vertex transitively on the product of three regular trees $T_6 \times T_6 \times T_6$ and has the presentation

$$\Gamma_1 = \left\langle a_1, a_5, a_9, a_{13}, a_{17}, a_{21}, a_{23}, b_2, b_6, b_{10}, b_{14}, b_{18}, b_{22}, c_3, c_7, c_{11}, c_{15}, c_{19}, c_{23} \right\rangle$$

$$\begin{array}{l}
a_i a_{i+12} = b_i b_{i+12} = c_i c_{i+12} = 1 \text{ for all } i,
\quad a_1 b_{17} a_{22}, a_1 b_{10} b_6, a_1 b_{14} a_{21} b_{14}, a_1 b_{18} a_5 b_{18},
a_1 b_{22} a_{17} b_2, a_5 b_{22} a_{21} b_6, a_5 b_{22} a_{21} b_2, a_5 b_{22} a_9 b_{22},
a_1 c_3 a_{17} c_3, a_1 c_7 a_{13} c_9, a_1 c_{11} a_9 c_{11}, a_1 c_{15} a_{11} c_{23}, a_1 c_{19} a_5 c_{19},
a_1 c_7 a_{21} c_7, a_5 c_3 a_{17} c_3, a_5 c_7 a_{13} c_5, a_5 c_7 a_{13} c_9,
b_2 c_3 b_{18} c_3, b_2 c_7 b_{10} c_{11}, b_2 c_{11} b_3 c_7, b_2 c_{15} b_{22} c_{15}, b_2 c_{19} b_6 c_{19},
b_2 c_3 b_{18} c_3, b_6 c_3 b_{22} c_7, b_6 c_7 b_{22} c_3, b_6 c_3 b_{10} c_{23}.
\end{array}$$

Thus $\Gamma_1$ is a $3$-cube group with $A_1 = \{a_1, a_5, a_9, a_{13}, a_{17}, a_{21}\}$ and similar descriptions for $A_2$ and $A_3$. It is an arithmetic group, so it is residually finite. Of interest to us is the fact that it admits quotients of order $5^l, l \in \mathbb{N}$, see Example 5.10.

In [24], the second-named author with Rungtanapirom and Stix also constructed $k$-cube groups acting on a product of trees of distinct constant valencies. Explicitly, for any set of size $k$ of distinct odd primes $p_1, \ldots, p_k$, there is a group acting simply transitively on a product of $k$ trees of valencies $p_1 + 1, \ldots, p_k + 1$, obtained using Hamiltonian quaternion algebras.

**Example 2.6.** For $p_1 = 3, p_2 = 5, p_3 = 7$, there is an explicit presentation of a group acting simply transitively on a product of three trees $T_4 \times T_6 \times T_8$, see [24]. Indeed, with $i, j, k$ denoting the quaternions, let

$$\begin{array}{l}
a_1 = 1 + j + k, \quad a_2 = 1 + j - k, \quad a_3 = 1 - j - k, \quad a_4 = 1 - j + k, \\
b_1 = 1 + 2i, \quad b_2 = 1 + 2j, \quad b_3 = 1 + 2k, \quad b_4 = 1 - 2i, \quad b_5 = 1 - 2j, \quad b_6 = 1 - 2k, \\
c_1 = 1 + 2i + j + k, \quad c_2 = 1 - 2i + j + k, \quad c_3 = 1 + 2i - j + k, \quad c_4 = 1 + 2i + j - k, \\
c_5 = 1 - 2i - j - k, \quad c_6 = 1 + 2i - j - k, \quad c_7 = 1 - 2i + j + k, \quad c_8 = 1 - 2i - j + k.
\end{array}$$

Then $a_{i-1} = a_{i+2}, b_{i-1} = b_{i+3},$ and $c_{i-1} = c_{i+4}$. The required group is given by

$$\Gamma_2 = \left\langle a_1, b_1, c_1, \ldots, a_4, b_6, c_8 \mid a_1 b_1 a_4 b_2, a_1 b_2 a_4 b_1, a_1 b_3 a_2 b_1, a_1 b_4 a_2 b_3, a_1 b_5 a_1 b_6, a_2 b_2 a_2 b_6, a_1 c_1 a_2 b_8, a_1 c_2 a_4 c_8, a_1 c_3 a_2 c_2, a_1 c_4 a_3 c_3, a_1 c_5 a_1 c_6, a_1 c_7 a_4 c_1, a_2 c_1 a_4 c_6, a_2 c_4 a_2 c_7, b_1 c_1 b_5 c_4, b_1 c_2 b_1 c_5, b_1 c_3 b_6 c_1, b_1 c_4 b_5 c_3, b_1 c_5 b_2 c_3, b_1 c_7 b_1 c_8, b_2 c_1 b_3 c_2, b_2 c_2 b_5 c_5, b_2 c_4 b_5 c_3, b_2 c_7 b_6 c_4, b_3 c_1 b_6 c_6, b_3 c_4 b_6 c_3 \right\rangle.$$

This is also denoted $\Gamma_{3,5,7}$, see [24].

**2.9. Higher rank graphs.** We recall the definition of a $k$-graph due to Kumjian and Pask [12]. For an integer $k \geq 1$, view $\mathbb{N}^k$ as a monoid under pointwise addition. A $k$-graph is a countable small category $\Lambda$ together with an assignment of a degree $d(\mu) \in \mathbb{N}^k$ to every morphism $\mu \in \Lambda$ such that for all $\mu, \nu, \pi \in \Lambda$ the following hold

1. $d(\mu \nu) = d(\mu) + d(\nu)$; and
(2) Whenever \(d(\pi) = m + n\), there is a unique factorisation \(\pi = \mu \nu\) such that \(d(\mu) = m\) and \(d(\nu) = n\).

Condition (2) is known as the factorisation property in the \(k\)-graph. The composition in \(\mu \nu\) is understood in the sense of morphisms, thus the source \(s(\mu)\) of \(\mu\) equals the range \(r(\nu)\) of \(\nu\). Note that the morphisms of degree 0 (in \(\mathbb{N}^k\)) are the identity morphisms in the category. Denote this set by \(\Lambda^0\), and refer to its elements as vertices of \(\Lambda\). With \(e_1, \ldots, e_k\) denoting the generators of \(\mathbb{N}^k\), the set \(\Lambda^{e_i} = \{\lambda \in \Lambda \mid d(\lambda) = e_i\}\) consists of edges (or morphisms) of degree \(e_i\), for \(i = 1, \ldots, k\). We write \(v\Lambda^n\) for the set of morphisms of degree \(n \in \mathbb{N}^k\) with range \(v\).

Throughout this paper we are concerned with \(k\)-graphs where \(\Lambda^0\) and all \(\Lambda^{e_i}, i = 1, \ldots, k\), are finite. A \(k\)-graph \(\Lambda\) so that \(0 < \# v\Lambda^n < \infty\) for all \(v \in \Lambda^0\) and all \(n \in \mathbb{N}^k\) is source free and row-finite. Following [10], a finite \(k\)-graph \(\Lambda\) is strongly connected if \(v\Lambda w \neq \emptyset\) for all vertices \(v, w \in \Lambda^0\).

The coordinate matrices \(M_1, \ldots, M_k \in \text{Mat}_{\Lambda^0}(\mathbb{N})\) of \(\Lambda\) are \(\Lambda^0 \times \Lambda^0\) matrices with

\[
M_i(v, w) = |v\Lambda^e w|.
\]

By the factorisation property, the matrices \(M_i\) pairwise commute for \(i = 1, \ldots, k\). For \(n = (n_i)_{i=1}^k \in \mathbb{N}^k\), we define

\[
M^n := \prod_{i=1}^k M_i^{n_i}.
\]

We denote the spectral radius of a square matrix \(B\) by \(\rho(B)\), and we let

\[
\rho(\Lambda) := (\rho(M_1), \rho(M_2), \ldots, \rho(M_k)) \in [0, \infty)^k.
\]

For \(m \in \mathbb{Z}^k\) we write \(\rho(\Lambda)^m\) for the product \(\prod_{i=1}^k \rho(M_i)^{m_i}\).

Given a row-finite, source free \(k\)-graph \(\Lambda\), its associated \(C^*\)-algebra \(C^*(\Lambda)\) is the universal \(C^*\)-algebra generated by a family \(\{s_\mu \mid \mu \in \Lambda\}\) of partial isometries satisfying

(CK1) \(\{s_v \mid v \in \Lambda^0\}\) is a family of mutually orthogonal projections;

(CK2) \(s_\mu s_v = s_\mu\nu\) whenever \(s(\mu) = r(\nu)\);

(CK3) \(s_\mu s_\mu = s_{s(\mu)}\) for all \(\mu\);

(CK4) \(s_v = \sum_{\mu \in v\Lambda^n} s_\mu s_\mu^*\) for all \(v \in \Lambda^0, n \in \mathbb{N}^k\).

3. Construction of \(k\)-graphs from \(k\)-cube groups: the one-vertex case

We start with the announced definition of a combinatorial higher rank graph. Later we illustrate it with a large supply of \(k\)-graphs with one vertex, for \(k \geq 2\). In Section 4 we use the results of this section combined with concrete constructions of covering maps to produce higher rank graphs with more than one vertex. We stress that the definition below is not applicable to all \(k\)-graphs, in particular all our examples will be row-finite and source-free.

**Definition 3.1.** Let \(k \geq 2\) be a positive integer. A combinatorial \(k\)-graph \(\Lambda\) is a directed graph with set of vertices \(V\), a finite set, and set of edges \(E \subset V \times V\) that is a disjoint union \(E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k\) with \(E_i\) for \(i = 1, \ldots, k\) regarded as edges of colour \(i\), together with a bijection \(\phi\) on the set \(Y\) of all directed paths of length two of edges of colours given by ordered pairs \((i, j)\) with \(i \neq j\) in \(\{1, 2, \ldots, k\}\), satisfying the following properties:

(F1) The bijection \(\phi: Y \to Y\) is such that for \(xy\) a path of length two with \(x\) of colour \(i\) and \(y\) of colour \(j\) we have \(\phi(xy) = y'x'\), where \(y'\) has colour \(j\), \(x'\) has colour \(i\) and the origin and terminus vertices of the paths \(xy\) and \(y'x'\) coincide. We write this as \(xy \sim y'x'\).
(F2) For all \( x \in E_i, \ y \in E_j \) and \( z \in E_l \) so that \( xyz \) is a path on \( E \), where \( i, j, l \) are distinct

colours, if \( x_1, x_2, x^2 \in E_i, \ y_1, y_2, y^2 \in E_j \) and \( z_1, z_2, z^2 \in E_l \) satisfy

\[
xy \sim y^1x^1, x^1z \sim z^1x^2, y^1z^1 \sim z^2y^2
\]

and

\[
yz \sim z_1y_1, xz_1 \sim z_2x_1, x_1y_1y_2 \sim y_2x_2,
\]

it follows that \( x_2 = x^2, y_2 = y^2 \) and \( z_2 = z^2 \).

Remark 3.2. It is possible to express the bijection \( \phi \) using notation from [12]. If \( C \) and \( D \) are
directed 1-graphs with common set of vertices \( V = C^0 = D^0 \), distinct sets of edges \( C^1, D^1 \), and
commuting vertex matrices, let

\[
C^1 \ast D^1 = \{(x, y) \in C^1 \times D^1 \mid t(x) = o(y)\}.
\]

Then the bijection \( \phi \) in (F1) is given by its restrictions \( \phi_{i,j} : E_i^1 \ast E_j^1 \rightarrow E_j^1 \ast E_i^1 \), for all \( i \neq j \) in
\( \{1, 2, \ldots, k\} \).

Note that condition (F2) in Definition 3.1 is vacuous when \( k = 2 \). We refer to \( E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k \)
as the 1-skeleton of \( \Lambda \).

Proposition 3.3. Every combinatorial \( k \)-graph \( \Lambda \) given by \( V, E \) and the bijection \( \phi \) together with
(F1) and (F2) as in Definition 3.1 is a \( k \)-graph, where \( s(f) = t_E(f) \) and \( r(f) = o_E(f) \) for \( f \in E \).

Proof. The given data of \( V, E \) and \( \phi \) can be viewed as the category with objects given by \( V \) and
elements given by finite partial products governed by compatibility of objects as follows: each edge in
\( E \) is an element in the category with its colour \( i \) determining uniquely a degree in \( \mathbb{N}^k \). Next,
a product \( f_1f_2 \) with \( f_1, f_2 \in E \) is defined when the terminus of \( f_1 \) is the origin of \( f_2 \). Property
(F1) ensures that \( f_1f_2 \) has a well-defined size, or degree, in \( \mathbb{N}^k \) determined by the colours \( i_1 \) of \( f_1 \)
and \( i_2 \) of \( f_2 \). Using (F2) we define uniquely elements \( f_1f_2f_3 \) if the natural compatibility conditions
\( t_E(f_1) = o_E(f_2), t_E(f_2) = o_E(f_3) \) hold, and get a well-defined degree for \( f_1f_2f_3 \) in \( \mathbb{N}^k \) determined
by the colours \( i_j \) of \( f_j \) for \( j = 1, 2, 3 \). Using induction we define elements \( f_1f_2 \cdots f_m \) on edges of
colour \( i_j \) for \( 1 \leq j \leq m \) and obtain a unique degree by (F2). This procedure yields a category, which
we continue to denote \( \Lambda \), together with a functor \( d : \Lambda \rightarrow \mathbb{N}^k \) with a built-in unique factorisation
property. By swapping origin and terminus of edges to range and source when we view the edge in
the category, we view any \( f_1f_2 \cdots f_m \) as a composition of morphisms in the sense of [12]. \( \square \)

Theorem 3.4. Given \( P \) a one-vertex \( k \)-cube complex with \( k \geq 2 \), there is a one-vertex \( k \)-graph \( \Lambda(P) \)
defined by setting the edges in the 1-skeleton of \( \Lambda(P) \) to be the directed edges of \( P \). Specifically, each
geometric edge in \( P \) gives rise to two distinct edges in the 1-skeleton of \( \Lambda(P) \), of same colour and
opposite orientation.

We prove this theorem in stages. First we prove the case \( k = 2 \), where the statement is a consequence of the description of a one-vertex 2-complex as \( S_{A,B} \) for a VH-structure \( A, B \). Then we
prove the case \( k = 3 \) by employing geometric cubes. The general case \( k \geq 3 \) follows by induction.

Lemma 3.5. Assume that \( S_{A,B} \) is a one-vertex square complex with associated group \( G \) given by
a VH-structure \((A, B)\) with \#A and \#B both even positive integers. Suppose that

\[
A = \{a_1, \ldots, a_L, a_{L+1}, \ldots, a_{2L}\} \text{ and } B = \{b_1, \ldots, b_K, b_{K+1}, \ldots, b_{2K}\},
\]

with \( a_r a_{L+r} = 1 \) in \( G \) for all \( r = 1, \ldots, L \) and \( b_s b_{K+s} = 1 \) in \( G \) for all \( s = 1, \ldots, K \), with \( K, L \geq 1 \).
In particular, for each \( r = 1, \ldots, L \), we have that \( a_r \) and \( a_{L+r} \) label the same geometric edge in \( S_{A,B} \),
but with opposite origin and terminus. Similarly for \( b_s \) and \( b_{K+s} \) for \( s = 1, \ldots, K \).
Then there is a combinatorial 2-graph with \( E(S) = E_1 \cup E_2 \) in the sense of Definition 3.1 obtained by associating to each \( a_r \) for \( r = 1, \ldots, 2L \) a directed edge \( a_r \) in \( E_1 \), and to each \( b_s \) for \( s = 1, \ldots, 2K \) a directed edge \( b_s \) in \( E_2 \).

**Proof.** We have that for each \( a_r \in A \) and \( b_s \in B \), with \( r = 1, \ldots, 2L \), \( s = 1, \ldots, 2K \), there are unique \( a_{l(r,s)} \in A \) and \( b_{m(r,s)} \in B \), with \( l(r,s) \in \{1, \ldots, 2L\} \) and \( m(r,s) \in \{1, \ldots, 2K\} \), such that \( a_r b_s = b_{m(r,s)} a_{l(r,s)} \).

In particular, \( a_r b_s \) is contained as a vertical-horizontal pair of edges in a unique geometric square in the family

\[
S_s^{a_r b_s}, r = 1, \ldots, 2L, s = 1, \ldots, 2K, s = O, H, V, R,
\]

with \( b_{m(r,s)} a_{l(r,s)} \) forming the horizontal-vertical pair of edges in \( S_s^{a_r b_s} \) starting and ending at the same vertices as \( a_r b_s \) (which we recall coalesce to the single vertex \( v \) of \( S_{A,B} \)).

Let \( xy \) be a path of length two with \( x \in E_1 \) and \( y \in E_2 \). Then \( xy \) is uniquely determined by \( x = a_r \) for some \( r = 1, \ldots, 2L \) and \( y = b_s \) for some \( s = 1, \ldots, 2K \). Let now \( a_r b_s, a_{l(r,s)} \) and \( b_{m(r,s)} \) be as above. Then \( y' := b_{m(r,s)} \in E_2 \) and \( x' := a_{l(r,s)} \in E_1 \) determine a unique path of length two \( y'x' \) so that \( xy \sim y'x' \).

This defines the required bijection \( \phi : Y \to Y \) with \( \phi(xy) = y'x' \) on the set \( Y \) of all paths of length two of distinct colours. \( \square \)

**Example 3.6.** A simple construction of a 2-graph based on this procedure recovers a known example, see [20, 34] and [14] Example 11.1(1)], where \( \theta(i,j) = (i,j) \) is the identity permutation of the set \( \{1,2\} \times \{1,2\} \). More precisely, starting with the \( (2,2) \)-group \( G = \mathbb{Z} \times \mathbb{Z} \) with \( A = \{a, a^{-1}\} \) and \( B = \{b, b^{-1}\} \), we have the commutation relation \( ab = ba \) as the basis for a geometric square \( S_{O}^{a,b} \), and an application of Lemma 3.5 gives a graph with edge set a disjoint union of \( E_1 = \{a_1, a_2\} \) and \( E_2 = \{b_1, b_2\} \) with the bijection \( \phi \) on the set of paths of length two of distinct colours read off from \( S_{O}^{a,b}, S_{H}^{a,b}, S_{V}^{a,b} \) and \( S_{R}^{a,b} \):

\[
a_1 b_1 \sim a_1 b_2 a_1, a_2 b_1 \sim b_2 a_1, a_2 b_1 \sim b_1 a_2 \text{ and } a_2 b_2 \sim b_2 a_2.
\]

**Example 3.7.** We now present a 2-graph from this recipe where the group \( G \) is not of product type. As we will explain, figure 2 shows an example of a \((4,4)\)-group \( G \), cf. [32].

![Figure 2](image-url)

The four squares are geometric squares representing the 2-cells of an associated complex \( S_{A,B} \), where \( A = \{a_1, a_2, a_3, a_4\} \) for \( a_3 = a_1^{-1} \) and \( a_4 = a_2^{-1} \), and \( B = \{b_1, b_2, b_3, b_4\} \) for \( b_3 = b_1^{-1} \) and \( b_4 = b_2^{-1} \). Here \( L = K = 2 \), cf. Lemma 3.5. With our convention in (2.7) we have, from left to right, \( S_{O}^{a_1,b_1}, S_{O}^{a_1,b_3}, S_{O}^{a_1,b_4} \) and \( S_{O}^{a_2,b_1} \).
The associated 2-graph \( \Lambda(S_{A,B}) \) from Lemma \[3.5\] has 1-skeleton whose edges are given by the disjoint union of \( E_1 = \{a_1, a_2, a_3, a_4\} \) and \( E_2 = \{b_1, b_2, b_3, b_4\} \). Let us now describe explicitly the bijection \( \phi : Y \to Y \).

We have 16 paths of length two of the form \( xy \), where \( x \in E_1 \) and \( y \in E_2 \), given by all the possible \( a_ib_j \) with \( i, j = 1, \ldots, 4 \). Correspondingly, we have all possible vertical-horizontal pairs of edges \( a_ib_j \) in the collection of geometric squares

\[
S_{a_i, b_j}, i, j = 1, \ldots, 4, * = O, V, R, H.
\]

Pick for each \( a_ib_j \) the unique \( a_{l(i,j)} \in A \) and \( b_{m(i,j)} \in B \) such that \( b_{m(i,j)}a_{l(i,j)} \) is the corresponding horizontal-vertical pair of edges in the same square, and let \( y' = b_{m(i,j)} \), \( x' = a_{l(i,j)} \) as prescribed by the proof of Lemma \[3.5\].

Explicitly, corresponding to the horizontal-vertical pairs of edges in the geometric square

\[
\{s_{O1}, s_{V1}, s_{R1}, s_{H1}\},
\]

it is seen that \( \phi(a_1b_1) = b_4a_3 \), \( \phi(a_2b_1) = b_1a_1 \), \( \phi(a_1b_2) = b_3a_3 \) and \( \phi(a_2b_2) = b_2a_1 \).

Similarly, from the geometric square

\[
\{s_{O2}, s_{V2}, s_{R2}, s_{H2}\},
\]

we get \( \phi(a_1b_3) = b_1a_4 \), \( \phi(a_2b_3) = b_3a_2 \), \( \phi(a_2b_4) = b_1a_3 \) and \( \phi(a_4b_1) = b_3a_1 \); from

\[
\{s_{O3}, s_{V3}, s_{R3}, s_{H3}\},
\]

we get \( \phi(a_1b_4) = b_2a_2 \), \( \phi(a_3b_2) = b_4a_4 \), \( \phi(a_4b_4) = b_2a_3 \) and \( \phi(a_2b_2) = b_4a_1 \); finally, from the geometric square

\[
\{s_{O4}, s_{V4}, s_{R4}, s_{H4}\},
\]

we get \( \phi(a_2b_1) = b_2a_4 \), \( \phi(a_4b_2) = b_1a_2 \), \( \phi(a_2b_4) = b_3a_4 \) and \( \phi(a_4b_3) = b_4a_2 \); this describes the bijection \( \phi \) completely.

The link of \( S_{A,B} \) at its vertex \( v \) is the complete bipartite graph of type \((4,4)\), see Figure \[3\].

![Figure 3. The link of the complex](image)

It follows that \( G := \pi_1(S_{A,B}, v) \) is a \((4,4)\)-group. In fact, \( G \) is the fundamental group of a CAT(0) complex with Gromov link condition, see \[7\]. We remind that every edge of the complex belongs to four squares, see figure \[4\] for a fragment of the universal cover of the complex showing the edge \( a_1 \) belonging to four squares (in the universal cover).
Remark 3.8. The group with the same VH-structure as in Figure 2 appears also in [11], Section 7, as the group $2 \times 2.37$ in their list. However, their $2$-graph is different from the one in Example 7.7 since, if we translate the notions of [11] into higher rank graphs, the $2$-graph corresponding to the group $2 \times 2.37$ would have sixteen vertices. In general, the $2$-graphs of [11] corresponding to $(2m, 2n)$ groups give $2$-graphs with $4mn$ vertices, $4(m - 1)mn$ edges of one colour and $4mn(n - 1)$ edges of another colour. The $2$-graphs of [25], [27] have $3(q^2 + q + 1)$ vertices and $3(q^2 + q + 1)q$ edges of each colour for $q$ being a prime power different from $3$.

Proof. (Proof of Theorem 3.4, the general case.) Let $P_{A_1, \ldots, A_k}$ be a $k$-cube complex associated with a $k$-cube group $G$ with underlying structure determined by the ordered tuple $(A_1, \ldots, A_k)$. For each $i = 1, \ldots, k$ we write

$$A_i = \{a_i^1, \ldots, a_{L_i}^1, a_{L_i+1}^1, \ldots, a_{2L_i}^1\},$$

with the convention that $a_{r}^{i}a_{L_i+r}^{i}$ in $G$ for $1 \leq r \leq L_i$. Define a graph $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k$ by the assignment that to each $a_i^j$ corresponds a directed edge $a_i^j$ in $E_i$, with $i = 1, \ldots, k$ and $r = 1, \ldots, 2L_i$.

Suppose that $xy$ is a path of length two in $E$ with $x \in E_i$ and $y \in E_j$ for distinct $i, j$ in $\{1, \ldots, k\}$. Then there are $a_i^r \in A_i$ and $b_s^j \in A_j$ for unique $r = 1, \ldots, 2L_i$ and $s = 1, \ldots, 2L_j$, such that $x = a_i^r$ and $y = b_s^j$. Since we have a cube complex, there is an associated square complex $P_{A_i, A_j}$ with corresponding group $G_{A_i, A_j}$, where we choose the convention that $A_i$ is vertical and $A_j$ horizontal direction. Lemma 3.5 implies that there is a unique path of length two $y'x'$ with $y' \in A_j$ and $x' \in A_i$, corresponding to a unique square with vertical-horizontal and horizontal-vertical pairs given by

$$a_i^r a_j^s = a_m^{l(r,s)} a_l^{l(r,s)},$$

such that $xy \sim y'x'$. Here $a_m^{l(r,s)} \in A_j$ and $a_l^{l(r,s)} \in A_i$ are uniquely determined. This provides the desired bijection $\phi$ and settles requirement (F1) of Definition 3.1.

Next suppose that we are given a path $xyz$ with $x \in E_i$, $y \in E_j$, $z \in E_l$ for distinct colours $i, j, l$ in $\{1, \ldots, k\}$.

To begin with, there is a unique square $S_{ij}^{l}$ which contains a vertical-horizontal pair $a_i^r a_j^s$ with $a_i^r \in A_i$, $a_j^s \in A_j$ so that $x = a_i^r$ and $y = a_j^s$. Upon completing the square $S_{ij}^{l}$ to $a_i^r a_j^s a_i^{p_1} a_j^{p_1}$, as in the beginning of the proof, for unique $r^1 \in \{1, \ldots, 2L_i\}$ and $s^1 \in \{1, \ldots, 2L_j\}$, we have

$$xy \sim y^1x^1$$

for $x^1 = a_i^{p_1}$ and $y^1 = a_j^{p_1}$.
Next we use \(x^1\) and \(z\) to extract a geometric square \(S^{ij}_2\), determined by \(a^i_t, a^t_i = a^i_t, a^t_i\), for unique \(t^1 \in \{1, \ldots, 2L_i\}\) and \(r^2 \in \{1, \ldots, 2L_j\}\), so that
\[
x^1z \sim z^1x^2 \text{ for } z = a^i_t, z^1 = a^t_i, \text{ and } x^2 = a^t_i.
\]

Finally, by using \(y^1, z^1\) we extract a geometric square \(S^{ij}_3\), determined by \(a^j_{s^1}, a^t_{s^1} = a^j_{s^1}, a^t_{s^1}\), for unique \(s^2 \in \{1, \ldots, 2L_j\}\) and \(t^2 \in \{1, \ldots, 2L_j\}\), so that
\[
y^1z^1 \sim z^2x^2 \text{ for } y^2 = a^j_{s^2}, \text{ and } z^2 = a^t_{s^2}.
\]

The squares \(S^{ij}_1, S^{ij}_2\) and \(S^{ij}_3\) fit into a unique geometric cube such that \(a^i_t, a^t_i, a^t_i\) is a path joining two vertices in the cube at the longest possible distance. Now, in this geometric cube, we have also obtained the path \(a^i_t, a^t_i, a^t_i, a^t_i\) that joins the same vertices in the cube and is opposite to \(a^i_t, a^t_i, a^t_i, a^t_i\).

If we perform the same argument starting with \(y, z\) to obtain \(yz \sim z_1y_1\), followed by \(x, z_1\) to obtain \(xz_1 \sim z_2x_1\) and finally \(x_1, y_1\) to get \(x_1y_1 \sim y_2z_2\), we find unique \(r_2 \in \{1, \ldots, 2L_i\}, s_2 \in \{1, \ldots, 2L_j\}\) and \(t_2 \in \{1, \ldots, 2L_j\}\) such that
\[
x_2 = a^i_{r_2}, y_2 = a^j_{s_2}, z_2 = a^t_{r_2}.
\]

Moreover, we have that \(a^i_t, a^t_i, a^t_i, a^t_i\) is another path in the geometric cube from above, joining the same vertices at longest possible distance, and being opposite to \(a^i_t, a^t_i, a^t_i, a^t_i\). Since there can only be one such path of longest distance opposite to \(a^i_t, a^t_i, a^t_i, a^t_i\) in a geometric cube, we must have
\[
a^j_{s_2} = a^j_{s_2}, a^t_{r_2} = a^t_{r_2}, \text{ and } a^t_{r_2} = a^t_{r_2}.
\]

Then \(x_2 = x^2, y_2 = y^2\) and \(z_2 = z^2\), as required to fulfill condition (F2).

To visualise the argument in the proof of Theorem 3.4, we refer to the generic geometric cube in figure 5. Let \(x = a_1\) (or, for consistency, \(x\) is a directed edge in \(E_i\) labelled with \(a_1 \in A_i\)), \(y = b_1\), and \(z = c_3\). The argument produces the path \(a^i_t, a^j_{s_2}, a^t_{r_2}\) given by \(c_1b_3a_3\) following the squares \(S^{ij}_1 = (a_1, b_1, a_2^{-1}, b_2^{-1}), S^{ij}_2 = (a_2, c_3, a_3^{-1}, c_2^{-1})\) and \(S^{ij}_3 = (b_2, c_2, b_3^{-1}, c_1^{-1})\). Alternatively, it produces the path \(a^j_{s_2}, a^t_{r_2}, a^t_{r_2}\) following the squares \((b_1, c_3, b_4^{-1}, c_4^{-1}), (a_1, c_4, a_4^{-1}, c_1^{-1})\) and \((a_4, b_4, a_3^{-1}, b_3^{-1})\).

**Example 3.9.** In figure 5, we present a geometric cube which is part of the data of the 3-cube group \(\Gamma_2\) from Section 2.8.

![Figure 5. A geometric cube for the \(\Gamma_2\) group](image)

The generating sets of \(\Gamma_2\) are \(A_1 = \{a_1, a_2, a_1^{-1}, a_2^{-1}\}, A_2 = \{b_1, b_2, b_3, b_1^{-1}, b_2^{-1}, b_3^{-1}\}\) and \(A_3 = \{c_1, c_2, c_3, c_4, c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}\}\). There are \(((\#A_1) \cdot (\#A_2) \cdot (\#A_3)/2)^3 = 24\) cubes in total, where
the factor $2^3$ in the denominator corresponds to the fact that there are 8 vertices in the cube, and we can complete the cube starting with three edges of distinct colours from any one of them.

The cube in figure 5 is obtained from the triple $(a_1, b_1, c_2)$ of edges in the three alphabets by completing its faces with geometric squares. With the notation of figure 4 the faces $S_{12}^1$, $S_{23}^2$ and $S_{31}^3$ arise, respectively, from the group relations $a_1 b_1 a_2 b_2$, $a_2 c_2 a_1 c_3$ and $b_2 c_2 b_1 c_3$ (identified with $b_2^{-1} c_3^{-1} b_2 c_4^{-1}$). The remaining three faces correspond to the geometric squares $b_1 c_2 b_1 c_5$, $a_1 c_1 a_2 c_8$ (identified with $a_1 c_1 a_2 c_4^{-1}$) and $a_2 b_4 a_2 b_2$ (identified with $a_2^{-1} b_1^{-1} a_1 b_2$).

3.1. Aperiodicity. Aperiodicity of a higher rank graph is an important property, because together with cofinality it implies simplicity of the associated $C^*$-algebra, and further implies pure infiniteness if every vertex can be reached from a loop with an entrance. We next investigate aperiodicity of $\Lambda(P)$ from Theorem 3.4.

We recall the necessary facts and notation from [12]. Let $\Lambda$ be a $k$-graph. If $m = (m_i), q = (q_i) \in \mathbb{N}^k$, we write $m \preceq q$ if $m_i \leq q_i$ for all $i = 1, \ldots, k$. By $\Omega_k$ we denote the $k$-graph with vertex set $\Omega^0 = \mathbb{N}^k$ and set of elements (morphisms) consisting of pairs $(m,n) \in \mathbb{N}^k \times \mathbb{N}^k$ with $m \leq n$ and $d(m,n) = n - m$. The set $\Lambda^\infty$ of infinite paths consists of degree preserving functors $\omega : \Omega_k \rightarrow \Lambda$. An infinite path $\omega$ is aperiodic provided that for every $q \in \mathbb{N}^k$ and all $p \in \mathbb{Z}^k \setminus \{0\}$, there is $(m, n) \in \Omega_k$ such that $m + p \geq 0$ and $\omega(m + p + q, n + p + q) \neq \omega(m + p, n + p)$. The $k$-graph $\Lambda$ satisfies the aperiodicity condition (A) provided that for every $v \in \Lambda^0$ there is an aperiodic path $\omega$ with $r(\omega) = v$.

In the present case, the existence of an aperiodic infinite path will be provided by the theory of rigid $k$-monoids from [14]. Given a $k$-cube complex $P$, the one-vertex $k$-graph $\Lambda(P)$ determines a $k$-monoid in the sense of [14] by reversing range and source of elements, with alphabets $E_i = \{a^i_1, \ldots, a^i_{2L_i}\}$ for $i = 1, \ldots, k$ where each $L_i \geq 1$.

Lemma 3.10. Suppose that $P$ is a one-vertex $k$-cube complex with underlying structure $(A_1, \ldots, A_k)$ for $i = 1, \ldots, k$, where each $A_i$ is of the form $\{a^i_1, \ldots, a^i_{2L_i}\}$ and $a^i_r a^i_{L_i+r} = 1$ in the associated group for all $1 \leq r \leq L_i$. Let $\Lambda(P)$ be the associated $k$-graph with 1-skeleton $E$ as in Theorem 3.4. Then

(1) $\Lambda(P)$ is left rigid: for every $x' \in E_i, y' \in E_j$ with $i \neq j$ there are unique $x \in E_i, y \in E_j$ such that $xy' \sim yx'$.

(2) $\Lambda(P)$ is right rigid: for every $x \in E_i, y \in E_j$, $i \neq j$, there are unique $x' \in E_i, y' \in E_j$ such that $xy' \sim yx'$.

The two properties of being rigid above arise from the fact that the link of the vertex $v$ in $P$ has no multiple edges. Therefore, every top-left corner and every bottom-right corner appear exactly once in a geometric square. The formal proof is below.

Proof. Suppose that $x' \in E_i$ and $y' \in E_j$ for $i \neq j$ in $\{1, \ldots, k\}$. Since we have a cube complex, there is an associated square complex $P_{A_i,A_j}$. By our construction of $\Lambda(P)$, there are $a^i_r \in A_i$ and $b^j_s \in A_j$ for unique $r = 1, \ldots, 2L_i$ and $s = 1, \ldots, 2L_j$ such that $x' = a^i_r$ and $y' = b^j_s$. In the square complex $P_{A_i,A_j}$ there is a unique square of the form

$$S_O = (a^i_r, (b^j_s)^{-1}, (a^i_r)^{-1}, (b^j_s)^{-1})$$

for $g \in \{1, \ldots, 2L_i\}$ and $h \in \{1, \ldots, 2L_j\}$. The associated $S_H$ is $(a^i_r, b^j_s, (a^i_r)^{-1}, (b^j_s)^{-1})$, and thus $a^i_r b^j_s = b^j_s a^i_r$ in $G_{A_i,A_j}$. Letting $x = a^i_r$ and $y = b^j_s$ gives $xy' \sim yx'$ in $\Lambda(P)$, as claimed in (1).

Part (2) is similar. Starting this time with $x \in E_i$ and $y \in E_j$ for distinct $i, j$ in $\{1, \ldots, k\}$, we find $a^i_r \in A_i$ and $b^j_s \in A_j$ for unique $r \in \{1, \ldots, 2L_i\}$ and $s \in \{1, \ldots, 2L_j\}$ so that $x = a^i_r$ and $y = b^j_s$. 
Consider the unique square
\[ S_O = ((a_i^j)^{-1}, b_i^j, a_{m_i}^j, (b_n^j)^{-1}) \]
in \( P_{A_i, A_j} \), and form its associated \( S_V \), which is \((a_i^j, b_n^j, (a_m^j)^{-1}, (b_i^j)^{-1})\). Then \( a_i^j b_n^j = b_n^j a_m^j \) in \( G_{A_i, A_j} \), so letting \( x' = a_m^j \) and \( y' = b_n^j \) leads to \( xy' \sim yx' \) in \( \Lambda(P) \), as claimed in (2).

\[ \square \]

**Corollary 3.11.** The graph \( \Lambda(P) \) satisfies the aperiodicity condition for every one-vertex \( k \)-cube complex \( P \). In particular, \( C^*(\Lambda(P)) \) is simple.

**Proof.** Let \( P \) be a one-vertex \( k \)-complex, which we may assume as in the hypothesis of Lemma 3.10. Since \( \Lambda(P) \) satisfies left and right rigidity by Lemma 3.10, it follows from [14, Corollary 11.10 and Lemma 4.15] that \( \Lambda(P) \) is effective and hence admits an aperiodic infinite path. As there is only one vertex, the aperiodicity condition is satisfied. Since \( \Lambda(P) \) is also cofinal, \( C^*(\Lambda) \) is simple by [12, Proposition 4.8].

The constructions of [26] produce a purely infinite simple rank two Cuntz-Krieger algebra \( \mathcal{A} \). This uses in a crucial way the fact that every word \( w \) of a given shape \( m = (m_1, m_2) \in \mathbb{N}^2 \) admits at least two distinct extensions \( w', w'' \), in the sense that the origin of \( w', w'' \) (with suitable interpretation) equals the terminus of \( w \), and both have same shape \( e_j \) for \( j = 1, 2 \).

For a row-finite and source free \( k \)-graph \( \Lambda \), [12, Proposition 4.9] provided conditions that imply \( C^*(\Lambda) \) is purely infinite simple. A slight adjustment was identified in [29, Proposition 8.8], which we present here (writing cycle instead of loop): given a finitely aligned \( k \)-graph \( \Lambda \), a morphism \( \mu \in \Lambda \setminus \Lambda^0 \) is a cycle with an entrance if \( s(\mu) = r(\mu) \) and there exists \( s(\mu) \Lambda \) having \( d(\alpha) \leq d(\mu) \) and being distinct from the initial segment of \( \mu \) of degree \( d(\alpha) \). Thus for some factorisation \( \mu = \mu_1 \mu_2 \) where \( d(\mu_1) = n \leq d(\mu) \), there exists \( \alpha \neq \mu_1 \) with \( d(\alpha) = n \) and \( r(\alpha) = r(\mu_1) \). Therefore, upon swapping source and range and replacing them with terminus and origin, and by interpreting the constructions of [26] in terms of higher rank graphs, the existence of a cycle with an entry requires that for a given \( \mu_2 \) there are two distinct extensions, with the additional property that the origin of \( \mu_2 \) is the terminus of one of the extensions. As we will show below, our \( k \)-graphs satisfy the stronger aperiodicity condition used in [26].

**Proposition 3.12.** Let \( P \) be a one-vertex \( k \)-complex as in the hypothesis of Lemma 3.10 for \( k \geq 2 \) and let \( \Lambda(P) \) be the associated one-vertex \( k \)-graph. If \( L_i \geq 2 \) for \( i = 1, \ldots, k \), then the vertex in \( \Lambda(P) \) supports at least two distinct cycles of length two in colour \( i \), hence \( C^*(\Lambda(P)) \) is purely infinite. Furthermore, \( C^*(\Lambda(P)) \) falls under the Kirchberg-Phillips classification theory and is thus determined by its \( K \)-theory.

**Proof.** Fix \( i \in \{1, \ldots, k\} \) with \( L_i \geq 2 \). Then we can form the length-two cycles \( \mu = a_1^i a_{L_i+1}^i \) and \( \nu = a_2^i a_{L_i+2}^i \) based at \( v \) with \( d(\mu) = 2 = d(\nu) \). Now \( a_2^i \) provides an edge \( \alpha \) with nontrivial degree \( d(\alpha) \leq d(\mu) \) which is an entry to \( \mu \) not already contained in \( \mu \). We conclude that \( C^*(\Lambda(P)) \) is purely infinite. The last claim follows by [29, Corollary 8.15], see also [8, Remark 5.2], by appealing to the classification result in [19].

\[ \square \]

4. **Construction of \( k \)-graphs with several vertices**

In this section we present our construction of \( k \)-graphs with several vertices, for \( k \geq 2 \), and provide examples and applications.

To construct \( k \)-graphs with more than one vertex we need a procedure to obtain \( k \)-cube complexes with several vertices. It is known that in a complex with several vertices one cannot consistently...
identify the label of an edge with the label as generator in the fundamental group. We come around this challenge by introducing additional layers of labels, corresponding to covers with \( N \) sheets, similar to what is done for \( N = 2 \) in [15, Section 8.1]. In general, this is a hard problem, since there are examples of such complexes without non-trivial finite covers, cf. [2]. Even when the complexes under consideration are known to admit \( N \)-covers, corresponding to subgroups of index \( N \) of the fundamental group, see e.g. [8, Theorem 1.38], it is difficult to construct covers explicitly. One challenge is that the subgroups can be defined in many different ways. For us a cover will be defined "by picture", meaning that it is explicitly defined by the images of vertices, edges, faces and so on. In all our pictures, the covering map amounts to forgetting the upper indexes, and we can explicitly see that we have a local homeomorphism at each point.

Recall that in a \( k \)-cube complex with generating structure \( A_1, \ldots, A_k \) we view edges as being coloured, with each \( A_i \) for \( i = 1, \ldots, k \) endowed with a distinct colour. For a complex \( \mathcal{X} = \tilde{P} \) obtained as an \( N \)-cover of a one-vertex \( k \)-cube complex \( P_{A_1, \ldots, A_k} \), the coloured edges are given by \( p^{-1}(A_i) \) for \( i = 1, \ldots, k \), where \( p : \mathcal{X} = \tilde{P} \rightarrow P_{A_1, \ldots, A_k} \) is the covering map. A \( k \)-complex, when viewed as undirected, is always connected.

**Proposition 4.1.** Suppose that \( G \) is a \( k \)-cube group with associated \( k \)-cube complex \( P_{A_1, \ldots, A_k} \). Then \( P_{A_1, \ldots, A_k} \) admits a double cover \( p : \tilde{P} \rightarrow P_{A_1, \ldots, A_k} \) with \( \tilde{P} \) a complex with 2 vertices.

We stress that the content of this proposition is that there is an explicit double cover, which is prescribed "by picture" on 2-cells and 3-cells, see Lemmas 4.3 and 4.4. Before presenting the proof we point out a consequence.

**Corollary 4.2.** Each \( k \)-cube group \( G \) admits a subgroup of index 2.

**Proof.** By e.g. [8, Theorem 1.38], for a given path-connected, locally path-connected, and semilocally simply connected space \( X \) there is a bijection between the set of basepoint preserving isomorphism classes of path-connected covering spaces \( p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0) \) and the set of subgroups of \( \pi_1(X, x_0) \). The correspondence associates the subgroup \( p_* (\pi_1(\tilde{X}, \tilde{x}_0)) \) to the covering \( (\tilde{X}, \tilde{x}_0) \), and the number of sheets of the covering equals the index of \( p_* (\pi_1(\tilde{X}, \tilde{x}_0)) \) in \( \pi_1(X, x_0) \), see [8, Proposition 1.32]. Applying this to the 2-cover from Proposition 4.1 yields existence of a subgroup of \( G \) of index 2. □

In order to motivate our constructions of coverings, we review a construction of a 2-cover of the complex associated to the torus \( \mathbb{T}^2 \).

![Figure 6. A double cover of the torus](image)

Recall from, for example [8, Page 14], that \( \mathbb{T}^2 \) is obtained from a 2-cell given by a square with pairs of opposite edges having same orientation and label \( a \) (vertically) or \( b \) (horizontally), by gluing it onto the wedge of two circles. A double cover with two vertices arises from two squares with oriented edges having distinct labels \( a^1, a^2, b^1 \) and \( b^2 \), with the upper index 1 or 2 indicating the
source vertex, as shown in figure \[6\]. The complex is obtained by attaching the two squares (the 2-cells) to the graph with vertices 1 and 2 in Figure \[6\]. If \( P \) is the complex associated to the \((2,2)\)-group in Example \[3,6\], then the cover \( \tilde{P} \) just described will lead to a 2-graph with two vertices, see Proposition \[4,7\]; see also Figure 2 in \[15, Section 8.1\].

The proof of Proposition \[4,1\] relies on a couple of lemmas, with the first one detailing an explicit double cover for the 2-cells of the given \( k \)-cube complex.

**Lemma 4.3.** Suppose that we have a one-vertex square complex \( S = S_{A,B} \) with \( \text{VH-structure} \ (A,B) \) and vertex \( v \). Then there is a 2-cover \( p : \tilde{S} \to S \) given by a square complex \( \tilde{S} \) with two vertices \( v_1 \) and \( v_2 \) whose squares are given by the prescription: The inverse image of a geometric square \( S^O_b = (a,b,c^{-1},d^{-1}) \) in \( S_{A,B} \) consists of two geometric squares, \( S_1 = (a^1,b^2,(c^2)^{-1},(d^1)^{-1}) \) and \( S_2 = (a^2,b^1,(c^1)^{-1},(d^2)^{-1}) \) in \( \tilde{S} \), and the covering map is determined by

\[
p(*^1) = p(*^2) = * \text{ for } * = a, b, c, d.
\]

Equivalently, if we denote \( v_1 = 1 \) and \( v_2 = 2 \), the covering \( p \) is depicted on the squares by

\[
\begin{array}{ccc}
2 & \begin{array}{c} b^2 \\ 1 \\ a^1 \\
\end{array} & 1 \\
& S_1 & c^2 \\
1 & d^1 & 2 \\
\end{array}
\begin{array}{ccc}
1 & \begin{array}{c} b^1 \\ 2 \\ a^2 \\
\end{array} & 1 \\
& S_2 & c^3 \\
2 & d^2 & 1 \\
\end{array}
\begin{array}{c}
1 \\
& \begin{array}{c} b \\ 1 \\
\end{array} \\
\end{array}
\begin{array}{c}
S^O_{b^*} \\
\end{array}
\begin{array}{c}
c \\
\end{array}
\end{array}
\]

Note that in the 2-cover there are two geometric edges for each geometric edge in \( S_{A,B} \), so for example to \( a \in S_{A,B} \) having origin and terminus the vertex 1 (identified with \( v \)) there will correspond \( a^1 \) and \( a^2 \) in \( \tilde{S} \), with \( a^1 \) having origin 1 and terminus 2, and \( a^2 \) with origin 2 and terminus 1.

**Proof.** We need only observe that \( p(S^O_b) = S^O_b = p(S^O_{b^*}) \) for \( * = O, H, V, R \). Thus the square complex \( \tilde{S} \) is well-defined. The map \( p \) is a local homeomorphism because it is defined on the cells, and sends edges to edges and vertices to vertices. \( \square \)

**Lemma 4.4.** Let \( P_{A_1,\ldots,A_k} \) be a \( k \)-cube complex associated with a \( k \)-cube group \( G \) with underlying structure determined by the ordered tuple \( (A_1,\ldots,A_k) \), with \( \#A_i = 2L_i \) for every \( i = 1,\ldots,k \). There is a 2-cover \( \tilde{P} \) of \( P_{A_1,\ldots,A_k} \) determined as follows: On each 2-dimensional cell, the covering \( p \) is defined in Lemma \[4.3\]. On a 3-dimensional geometric cube, such as described in figure \[1\] where we assume \( a_i \in A_i, b_j \in A_j \) and \( c_l \in A_l \) for \( r, s, t = 1,\ldots,4 \) and \( i, j, l \in \{1,\ldots,k\} \), the cover \( \tilde{C} \) of \( C \) consists of two geometric cubes, see figure \[7\] with labelling of edges \( \{a^r_\epsilon\}, \{b^r_\epsilon\} \) and \( \{c^r_\epsilon\} \) for \( \epsilon = 1, 2 \), and with the covering map given by

\[
p : \tilde{C} \to C, p(a^r_1) = p(a^r_2) = a_r, r = 1,\ldots,4,
\]

and similarly for \( p(b^l_\epsilon) \) and \( p(c^l_\epsilon) \). For \( 4 \leq l \leq k \), the map \( p \) is defined on an arbitrary \( l \)-cube by its prescription on the underlying 3-cubes.

**Proof.** At \( k = 3 \), it suffices to verify that \( p \) on \( \tilde{C} \) is well-defined. But this is clear from the construction of the map \( p \) in \[4.2\]. The map \( p \) is constructed recursively on higher dimensional cubes: if \( C \) is an \( l \)-cube for \( 4 \leq l \leq k \), then \( p \) is prescribed consistently on all \((l-1)\)-dimensional faces of \( C \), similarly to how \[4.2\] is obtained on 3-cubes from its prescription in \[4.1\] on squares. \( \square \)

**Proof of Proposition 4.7.** This follows by applying Lemmas 4.3 and 4.4. \( \square \)
To obtain $k$-complexes with $N$ vertices for $N > 2$, there are several ways to use $k$-cube groups. Many of the $k$-cube groups are residually finite, so, because of the 1-to-1 correspondence between the subgroups of index $N$ and $N$-covers of the corresponding $k$-cube complex, we can get infinitely many $k$-cube complexes with $N$ vertices. In principle, different subgroups of the same index $N$ can lead to different coverings. If $N > 3$, then the labelling of vertices is hard to sort out and we do not know of an explicit prescription similar to (4.2).

**Proposition 4.5.** Suppose that $G$ is a residually finite $k$-cube group with $k \geq 2$. To each normal subgroup $H$ of $G$ of finite index $N$ there is a $k$-complex $\mathcal{X}$ with $N$ vertices obtained by the following prescription: let $Q : G \to S_N$ the homomorphism obtained by composing the embedding of $G/H$ into the symmetric group on $N$ letters $S_N$ given by Cayley’s theorem with the quotient map $q : G \to G/H$.

For $a \in G$, the permutation $Q(a)$ in $S_N$ encodes the edges in the complex, with $a^n$ labelling an edge from the vertex $n$ to the vertex $n' = Q(a)(n)$ for $n = 1, \ldots, N$.

**Proof.** Suppose that $H$ is a subgroup of $G$ so that $(G : H) = N$. The complex $\mathcal{X}$ is described by associating to each square $S_a = (a, b, (a')^{-1}, (b')^{-1})$ in the one-vertex complex of $G$ a total of $N$ squares

$$(Q(a), Q(b), Q((a')^{-1}), Q((b')^{-1}))$$

in the new, $N$-vertex complex. Indeed, applying $Q$ to the relation $ab = b'a'$ in $G$ gives the identity $Q(a)Q(b) = Q(b')Q(a')$ in $S_N$, which in turn yields $N$ squares in the complex $\mathcal{X}$ determined by

$$Q(a)Q(b)(n) = Q(b')Q(a')(n), \text{ for all } n = 1, \ldots, N.$$ 

More precisely, for each $n$, let $m = Q(a)Q(b)(n) = Q(b')Q(a')(n)$ and consider the square with labelling $(a')^n(b')^s$ in horizontal-vertical direction from vertex $n$ to vertex $m$ via vertex $s$, and with labelling $b'^s a'^r$ in horizontal-vertical direction via vertex $r$, where $m = Q(a)(r), r = Q(b)(n), Q(a')(n) = s$ and $Q(b')(s) = m$, see the figure, where 1 in the right square denotes the vertex in the complex of $G$:

(4.3)
We extend the notions of left and right rigid to combinatorial \( k \)-graphs with more than one vertex in the natural way. The idea is that being rigid means that if two edges can form a corner (either bottom-left or top-right), then they do form a unique corner.

**Definition 4.6.** Suppose that \( \Lambda \) is a \( k \)-graph as in Definition 3.1. Then \( \Lambda \) is right rigid if for \( x \in E_i \) and \( y \in E_j \) edges of distinct colours \( i \neq j \) so that \( o(x) = o(y) \), there are unique \( x' \in E_i \) and \( y' \in E_j \) with \( xy' \sim yx' \). Left rigid is defined in a similar way.

**Proposition 4.7.** Suppose that \( P \) is a \( k \)-cube complex with an \( N \)-cover \( \tilde{P} \rightarrow P \) where \( k \geq 2 \) and \( N \geq 1 \), and let \( \mathcal{X} = \tilde{P} \) be the associated \( k \)-cube complex with \( N \) vertices from Proposition 4.5.

Then there is a combinatorial \( k \)-graph \( \Lambda(\mathcal{X}) \) defined by sending a vertex of \( \mathcal{X} \) to a vertex in \( \Lambda(\mathcal{X}) \), and by sending an edge of \( \mathcal{X} \) to two edges in \( \Lambda(\mathcal{X}) \), of same colour, but opposite orientation. Furthermore, \( \Lambda(\mathcal{X}) \) is strongly connected and left and right rigid.

**Proof.** The proof is the same as in Theorem 3.4. For every path of length two \( xy \) with \( x \in E_i \) and \( y \in E_j \) for \( i \neq j \) so that \( o(y) = t(x) \), there is a unique geometric square \( S_l \) in \( \mathcal{X} \) in which \( xy \) corresponds to a vertical-horizontal pair of edges. Then the corresponding horizontal-vertical pair of edges gives rise to \( x' \in E_i \) and \( y' \in E_j \) such that \( xy \sim y'x' \), and this defines uniquely the bijection \( \phi \) on the set \( Y \) of paths of length two of distinct colours required in (F1). The condition (F2) holds similarly to the proof of Theorem 3.4 because any \( ijl \)-coloured path will be contained in a unique \( 3 \)-cube in \( \mathcal{X} \).

To show that \( \Lambda(\mathcal{X}) \) is strongly connected let \( v, w \) be distinct vertices. Then there is an undirected path \( y_1 y_2 \ldots y_m \) in the 1-skeleton of the one-vertex \( k \)-complex \( P \) with \( v = o(y_1) \) and \( w = t(y_m) \). In particular, \( v_i = t(y_i), v_{i+1} = o(y_i) \) are adjacent vertices in the complex for \( 1 \leq i \leq m-1 \) (identifying \( v = v_1 \)). Our construction of the \( k \)-graph gives two directed edges, of opposite orientation, with source \( v_i \) and terminus \( v_{i+1} \), respectively the opposite, for each \( i = 1, \ldots, m-1 \). This allows to form directed paths in \( \Lambda(P) \) from \( v \) to \( w \) and from \( w \) to \( v \), as needed.

To see that \( \Lambda(\mathcal{X}) \) is rigid, it suffices to note that every vertex in the cover \( \tilde{P} \) has the same link as the one vertex of \( P \), and in particular its link contains no multiple edges. Therefore the proof of Lemma 3.10 carries through.

**Corollary 4.8.** The \( k \)-graph \( \Lambda(\mathcal{X}) \) from Proposition 4.7 satisfies the aperiodicity condition.

**Proof.** Since \( \Lambda(\mathcal{X}) \) is rigid, the existence of an aperiodic path in \( \Lambda(\mathcal{X}) \) based at a given vertex is guaranteed as in the one-vertex case, see [14, Lemma 4.15 and Corollary 11.10].

**Proposition 4.9.** Assume the hypotheses of Proposition 4.7. Suppose moreover that \( P = P_{A_1, \ldots, A_k} \) with \( A_i \), given as in Lemma 3.10 for \( i = 1, \ldots, k \). If \( \#A_i \geq 4 \) for all \( i = 1, \ldots, k \), then every vertex \( \Lambda(\mathcal{X}) \) supports at least two cycles of each colour. In particular, \( C^*(\Lambda(\mathcal{X})) \) is purely infinite and therefore classifiable by the Kirchberg-Phillips classification theory.

**Proof.** Let \( \Lambda(\mathcal{X})^0 \) denote the vertices, or identities, in our \( k \)-graph. Since every geometric edge in \( P \) gives rise to two, twin edges in \( \Lambda(\mathcal{X}) \), we have that for each colour \( i \in \{1, \ldots, k\} \), every vertex \( v \in \Lambda(\mathcal{X})^0 \) admits at least \( 2\#A_i \) incident edges, namely edges with origin or terminus \( v \). Furthermore, by our construction of \( \Lambda(\mathcal{X}) \) we also know that each edge is contained in a length-two cycle. Thus, for a given \( v \in \Lambda(\mathcal{X})^0 \), there are at least two cycles \( \mu = x_1 x_2 \) and \( \nu = x_3 x_4 \) based at \( v \) and consisting of edges of colour \( i \), with the terminus of \( x_1 \) possibly distinct from the terminus of \( x_3 \). Then \( x_2 \) is an entry to \( \mu \) of smaller degree and not already contained in \( \mu \). In this consideration the vertex \( v \) already supports a cycle with an entrance, but since our \( k \)-graph is strongly connected we could have chosen a cycle \( \mu \) based at a different vertex \( w \) and apply the same consideration. Now [29, Proposition 8.8 and Corollary 8.15] apply to give the claimed conclusion.
Corollary 4.10. For any \( k \geq 2 \) there is a strongly connected two-vertex \( k \)-graph \( \Lambda \) such that \( \#(v_1 \Lambda^i v_2) \) and \( \#(v_2 \Lambda^i v_1) \) are even positive integers for all \( i = 1, \ldots, k \), where \( \Lambda^0 = \{v_1, v_2\} \).

Proof. This follows by Propositions 4.7 and 4.1. \( \square \)

We next present an explicit construction of an infinite family of \( k \)-graphs with two vertices, for \( k \geq 2 \), obtained from a double cover of a one-vertex \( k \)-cube complex. The construction was partly outlined in [15, Section 8.1], as corresponding to the uniform labelling \( l_u \), and explicit factorisation rules of the 2-vertex graph were given in the case of the mixed labelling \( l_m \). Here we describe completely the case \( l_u \) as an application of Proposition 4.7.

Proposition 4.11. For \( k \geq 2 \) and any \( k \)-tuple \( (L_1, \ldots, L_k) \) of positive integers there exists an aperiodic strongly connected 2-vertex \( k \)-rank graph \( \Lambda \) with \( |v \Lambda^i w| = 2L_i \), where \( v \) and \( w \) are the vertices in \( \Lambda \) and \( i = 1, 2 \).

Proof. Fix \( k \geq 2 \) and for each \( i = 1, \ldots, k \) let \( L_i \) be an alphabet with \( L_i \) letters. Let \( F_i = F_{L_i} \) be the free group generated by \( L_i \) for each \( i = 1, \ldots, k \). The product group \( F_1 \times \cdots \times F_k \) acts simply and transitively on the product of trees \( T_2L_1 \times \cdots \times T_2L_k \), and produces in the quotient a complex \( P \) with one vertex and skeleton a wedge of \( \sum_{i=1}^k L_i \) circles. The 2-cells in \( P \) arise from pairs \( a \in L_i, b \in L_j \) for \( i \neq j \) with the commutation relation \( ab = ba \) as in Example 3.6; for each such pair, there is a torus glued to the wedge of circles. By induction, glue in 3-cubes in the complex, then \( l \)-cubes for all \( 4 \leq l \leq k \). By considering the double cover of each square as prescribed in figure 6, we obtain (by induction on dimension of \( l \)-cells) a complex \( \tilde{P} \) with 2-vertices.

Applying Proposition 4.7 produces a \( k \)-graph \( \Lambda := \Lambda(P) \) with the desired property: for each geometric edge in \( P \), say having label \( a \in L_i \), there are two geometric edges labelled \( a^1 \) and \( a^2 \) in \( \tilde{P} \), and each of these gives exactly one edge in the associated \( k \)-graph \( \Lambda \) between the two vertices. \( \square \)

Example 4.12. To illustrate Proposition 4.11 suppose that \( k = 2 \) and \( L_1 = L_2 = 1 \). Note that in particular this example will not satisfy the hypothesis of Proposition 4.9. The associated \( \Lambda \) has 1-skeleton the graph with two vertices in figure 6, where we identify \( v \) as vertex 1 and \( w \) as vertex 2. If we view the coloured edges in direction \( e_1 \in \mathbb{N}_2 \) as labelled by \( L_1 \) and in direction \( e_2 \in \mathbb{N}_2 \) to be labelled by \( L_2 \), then by Proposition 4.11 we have

\[
\begin{align*}
   w\Lambda^{e_1}v &= \{a^1, a^2\}, \quad w\Lambda^{e_2}v = \{b^1, b^2\}, \quad v\Lambda^{e_1}w = \{a^2, a^1\} \quad \text{and} \quad v\Lambda^{e_2}w = \{b^2, b^1\}.
\end{align*}
\]

The 8 factorisation rules are:

\[
\begin{align*}
   a^1b^2 &= b^1a^2, a^2b^2 = b^1a^1, a^1b^1 = b^2a^2, a^2b^1 = b^1a^1, \\
   a^2b^1 &= b^2a^1, a^1b^1 = b^2a^2, a^2b^2 = b^1a^1, a^1b^1 = b^2a^2.
\end{align*}
\]

If \( k = 2 \) and \( L_1 = L_2 = 2 \), then the corresponding 2-graph on two vertices has the same 1-skeleton, and for example \( \#v\Lambda^{e_2}w = \#w\Lambda^{e_1}v = 4 \), and similarly in colour \( e_2 \). This graph satisfies the hypothesis of Proposition 4.9.

5. Applications

5.1. Von Neumann algebras from strongly connected \( k \)-graphs. We give a large supply of von Neumann type III factors from \( k \)-graphs as in [13], for infinitely many values of \( \lambda \) in \( (0, 1] \). We start with some preparation.

The notion of adjacency operator in \( i \)-direction for a \( k \)-cube complex, where \( i \in \{1, \ldots, k\} \) was introduced in [24, Section 6]. The basic ingredients are as follows: Let \( X \) be a \( k \)-cube complex
with vertex set (of its 1-skeleton) denoted $X_0$ and with universal cover a product $T_1 \times T_2 \times \cdots \times T_k$ of regular trees. For each $i = 1, \ldots, k$ and $V, W \in X_0$, we write $V \sim_i W$ if the two vertices in the complex are adjacent in the $i$-direction of $X$. The adjacency operator $A_i$ in $i$-direction is defined on $L^2(X_0)$ by

$$A_i(f)(V) = \sum_{W \sim_i V} f(W).$$

Since all complexes considered here are locally finite in a strong sense, meaning that at every vertex there are finitely many edges in each direction $i$, or of each colour $i$, for $i \in \{1, \ldots, k\}$, the operators $A_i$ become $|X_0| \times |X_0|$ matrices. It was further observed in [24] Remark 6.4 that whenever each pair of edges starting at a vertex of $X$ in direction $i, j$, with $i \neq j$, belong to a unique square in $X$, then $A_i$ and $A_j$ commute.

**Proposition 5.1.** Let $X$ be a $k$-cube complex with $N$ vertices covered by a cartesian product of $k$ trees with valencies $n_1, n_2, \ldots, n_k \in \mathbb{Z}^+$, respectively, where $k \geq 2$ and $N \geq 1$. Let $\Lambda(X)$ be the associated $k$-graph as in Proposition 4.7. Then

$$\rho(\Lambda(X)) = (n_1, n_2, \ldots, n_k).$$

**Proof.** The assumption on $X$ says that there are $n_i$ edges (disregarding orientation) of colour $i$ for each $i = 1, \ldots, k$. The graph $\Lambda(X)$ is constructed by assigning two edges in its skeleton, of opposite orientation, for each geometric edge in $X$. Therefore, if $M_i$ denotes the coordinate matrix of $\Lambda(X)$ in colour $i$, we have that $M_i$ is the same as the adjacency operator in $i$-direction $A_i$. Thus it is a symmetric matrix with largest positive eigenvalue equal to the valency of the tree in colour $i$. Hence, $\rho(M_i) = n_i$ for each $i = 1, \ldots, k$, as claimed.

**Remark 5.2.** The graph of Proposition 4.11 satisfies $\rho(\Lambda) = (2L_1, \ldots, 2L_k)$.

Given a strongly connected $k$-graph $\Lambda$, it was shown in [10] Corollary 4.6 that $C^*(\Lambda)$ admits KMS states at inverse temperature $\beta = 1$ for the one-parameter action $\alpha: \mathbb{R} \to \text{Aut} C^*(\Lambda)$, the so-called preferred dynamics, characterised by $\alpha_t(s_\mu) = e^{it \log \rho(\Lambda) - d(\mu)} s_\mu$, $t \in \mathbb{R}$, $\mu \in \Lambda$. Following [13], define $S := \{\rho(\Lambda) d(\mu) - d(\nu) \mid \mu, \nu \in \Lambda \text{ are cycles}\}$ and let $\lambda := \sup \{s \in S \mid s < 1\}$. By the main result of [13], Theorem 3.1, we have $\lambda \in (0, 1]$ and the von Neumann algebra generated by the image of $C^*(\Lambda)$ in the GNS representation $\pi_\varphi$ corresponding to an extremal KMS state $\varphi$ is the injective type III$_\lambda$ factor.

Our application here is motivated by [13] Example 7.7, see also [18] and [35]. It consists of producing an infinite family of von Neumann factors $(\pi_\varphi(C^*(\Lambda)))''$ of type III$_\lambda$ associated to $k$-graphs in this fashion. Recall from [13] Section 6 that the group of periods of a strongly connected graph $\Lambda$ is defined as $\mathcal{P}_\Lambda = \mathcal{P}_\varphi^+ - \mathcal{P}_\varphi^+$, where for an arbitrary vertex $v \in \Lambda$, $\mathcal{P}_\varphi^+$ is the subsemigroup of $d(v\Lambda v)$ of $\mathbb{N}^k$. Equivalently, $\mathcal{P}_\Lambda$ is the subgroup of $\mathbb{Z}^k$ determined as $\{d(\mu) - d(\nu) \mid \mu, \nu \text{ are cycles in } \Lambda\}$. As shown in [13] Theorem 7.3], the set $S$ above is the closure inside the positive real half-line of the set $\{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}$.

**Corollary 5.3.** For $k \geq 2$ and any $k$-tuple $(L_1, \ldots, L_k)$ of positive integers, there is a III$_\lambda$ von Neumann factor $(\pi_\varphi(C^*(\Lambda)))''$, where $\pi_\varphi$ is the GNS representation of $C^*(\Lambda)$ corresponding to an extremal KMS$_1$ state $\varphi$, and the type is determined as

$$\lambda = \sup\{(2L_1)^{m_1} (2L_2)^{m_2} \cdots (2L_k)^{m_k} \mid (m_1, m_2, \ldots, m_k) \in \mathcal{P}_\Lambda\} \cap (0, 1].$$

In particular, if $L_1 = \cdots = L_k = L$, then $\lambda = (2L)^{-2}$. 

Proof. Fix $k \geq 2$ and positive integers $L_1, \ldots, L_k$. Let $\Lambda$ be the associated $k$-graph with two vertices and spectral radii $\rho(\Lambda) = (2L_1, \ldots, 2L_k)$ given by Proposition 4.11. Then $\Lambda$ is strongly connected, and we may apply Theorem 3.1 of [13] to obtain the claimed von Neumann factors.

The remaining task is to compute $P_\Lambda$. By our construction of $\Lambda$ it is not hard to see that $P_\Lambda$ is generated by $m \in \mathbb{Z}^k$ where either $m_i = 2$ for a unique $i \in \{1, \ldots, k\}$ while $m_l = 0$ at $l \neq i$, or $m_i = m_j = 1$ for some $i \neq j$ in $\{1, \ldots, k\}$ and $m_l = 0$ for $l \notin \{i, j\}$. If $L_1 = L_2 = \cdots = L_k = L$, then $\rho(\Lambda)^m = (2L)^{\sum_{i=1}^k m_i}$ with $m \in P_\Lambda$, and because $\sum_{i=1}^k m_i \in 2\mathbb{Z}$, the required type is attained as $\lambda = (2L)^{-2}$.

It was pointed out in [13, Remark 7.6] that the von Neumann type of the factors arising from extremal KMS$_1$ states depends only on the skeleton of the $k$-graph under consideration, and not on its factorisation rules. In the case of our examples in Corollary 5.3, this means that the type of the von Neumann factor depends only on the complex $\tilde{P}$ built up as a 2-cover of the one-vertex complex $(T_{2L_1} \times \cdots \times T_{2L_k})/(F_1 \times \cdots \times F_k)$.

5.2. Spectral theory of $k$-graphs. Alon and Boppana prove that asymptotically in families of finite $(q+1)$-regular graphs $X_n$ with diameter tending to $\infty$ the largest absolute value of a non-trivial eigenvalue $\lambda(X_n)$ of the adjacency operator $A_{X_n}$ has limes inferior $\lim_{n \to \infty} \lambda(X_n) \geq 2\sqrt{q}$.

Now, instead of graphs we may consider cube complexes covered by products of trees $T_1 \times \cdots \times T_k$, such that $T_i$ has valency $q_i$ and look at adjacency operators $A_i$ in direction $i$ corresponding to an individual tree $T_i$.

**Definition 5.4.** Let $X$ be a finite $k$-cube complex that has constant valency $q_i+1$ in all directions $i = 1, \ldots, k$. Then $X$ is a **cubical Ramanujan complex**, if for each $i \in \{1, \ldots, k\}$, the eigenvalues $\lambda$ of $A_i$ either satisfy the equality $\lambda = \pm (q_i+1)$ or the bound

$$\lambda \leq 2\sqrt{q_i}.$$  

Each such complex yields a $k$-graph $\Delta$ such that $\rho(\Delta) = (q_1+1, q_2+1, \ldots, q_k+1)$.

There are explicit constructions of Ramanujan cube complexes for several infinite families in [24]. We consider next the complexes from [24] corresponding to congruence subgroups of arithmetic lattices. We reformulate some results of [24] in the light of the present paper.

**Theorem 5.5.** For $p$ a prime, $l$ a positive integer, and $N \geq 2$, there are infinitely many $k$-cube complexes with $N$ vertices covered by products of $k$ trees, where $k \leq p-1$ and each tree is of valency $p^l+1$, satisfying optimal spectral properties, namely with a spectral gap the interval $[2\sqrt{q}, q+1]$, for $q = p^l$.

**Proof.** Such $k$-cube complexes were constructed in [24, Section 6]. They correspond to congruence quotients of arithmetic groups. The number of vertices of such complexes is given by the order of the group $\text{PGL}(2, p^l)$.

**Remark 5.6.** There are also non-residually finite complexes which have interesting $k$-graphs, although they do not necessarily exhibit the optimal spectral gap. Such complexes with one vertex were constructed in [24, Section 5]. Applying Lemma 4.4, we get such complexes with 2 vertices for all values $k \geq 1$.

Now we extend the notion of the Ramanujan cube complexes to higher-rank graphs.

**Definition 5.7.** We say that a coordinate matrix of a $k$-graph is $L$-regular for $L \in \mathbb{N}$, if the sum of all row entries is equal to $L$. 

\textbf{Definition 5.8.} Let $\Lambda$ be a $k$-graph with $L_i$-regular coordinate matrices $M_1, \ldots, M_k$ having positive second eigenvalue $\lambda_i$. We say that the $k$-graph $\Lambda$ is Ramanujan if

$$\lambda_i \leq 2\sqrt{L_i - 1} \text{ for all } i = 1, \ldots, k.$$ 

\textbf{Theorem 5.9.} For each $k \geq 2$, there is an infinite family of Ramanujan $k$-graphs with $N \geq 2$ vertices. More precisely, $N$ is determined as the index of congruence subgroups of the RSV-groups $\Gamma_{M, \delta}$ from subsection 2.8.

\textit{Proof.} For a complex $\mathcal{X}$ with $N$ vertices, there is a $k$-graph $\Lambda(\mathcal{X})$ with $N$ vertices by an application of Proposition 4.7. Applying this to an arbitrary complex $\mathcal{X}$ as in Theorem 5.5 gives that the $k$-graph satisfies the optimal spectral gap requirement of Definition 5.8. \hfill $\square$

We note that both one-vertex cube complexes covered by product of $k$ trees and one-vertex higher rank graphs are trivially Ramanujan, so we will require in addition the number of vertices to be greater than, for example, the maximum of $(L_i - 1)^2$, $i = 1, \ldots, k$.

\textbf{Example 5.10.} We now describe an explicit Ramanujan 3-graph with 25 vertices in the above infinite family. Let $p = 5$ and consider the group $\Gamma_1$ from section 2.8 acting on a product of three trees with valencies $(6, 6, 6)$. Let $P$ denote the one-vertex 3-complex associated to $G$. The existence of the claimed 3-graph is assured by Proposition 4.5 and Proposition 4.7 because $\Gamma_1$ has a quotient $L$ of order 25 (indeed, it has quotients of order 5 for all $l \geq 1$). Let $\mathcal{X}$ denote the resulting complex with 25 vertices, and let $\Lambda$ be its associated 3-graph given by Proposition 4.7.

It turns out that in the cover, certain subsets of generators of $\Gamma_1$ already generate a group of order 25, as may be verified using MAGMA. For example, the image $Q(a_1)$ in $S_{25}$ is the product of disjoint cycles

$$Q(a_1) = (1, 15, 24, 8, 17)(2, 11, 25, 9, 18)(3, 12, 21, 10, 19)(4, 13, 22, 6, 20)(5, 14, 23, 7, 16).$$

With the notation of Proposition 4.5 we have isomorphisms of groups

$$L \cong (Q(a_1), Q(a_5), Q(a_9)),$$

$$L \cong (Q(b_2), Q(b_6), Q(b_{10})),$$

$$L \cong (Q(c_3), Q(c_7), Q(c_{11})).$$

thus all three groups in the right hand side are abstractly isomorphic to a finite group of order 25. Let $K_1$, $K_2$ and $K_3$, respectively, denote the Cayley graphs of the finite group of order 25 coming from the three preceding isomorphisms.

This means that while the presentation of the infinite group $\Gamma_1$ requires generators of all three colours $a_i, b_j, c_k$ (as $\Gamma_1$ is irreducible), in the presentation of the finite group of order 25, generators of only one colour suffice. The finite cover is the complex $\mathcal{X}$, and fixing each colour yields the Cayley graph of a finite group of order 25. In other words, each of the generating sets $A_1$, $A_2$ and $A_3$ give Cayley graphs in three different sets of generators (of the three different colours) of the same finite group.

The adjacency matrices $M_i$ of the Cayley graphs $K_i$, $1 = 1, 2, 3$, may be computed using MAGMA, using that $Q(G)$ acts as permutations in $S_{25}$. As noted in the proof of Proposition 5.1, the adjacency matrices $M_1, M_2, M_3$ of the complex are also the adjacency matrices of the 3-graph $\Lambda$. Each $M_i$ is 6-regular in the sense of Definition 5.7 for $1 = 1, 2, 3$, as may be seen from the concrete description of the matrices obtained with MAGMA. An application of Theorem 5.9 gives that $\Lambda$ is a Ramanujan 3-graph, so the second largest eigenvalue $\lambda_i$ of $M_i$ is dominated by $2\sqrt{5}$, for $i = 1, 2, 3$. It turns out in this case that all three matrices $M_1, M_2, M_3$ are equal.
In this example, the spectral gap is strictly in the optimal bound, namely, the second eigenvalue of $M_1$ is dominated by $3^{24}$, according to MAGMA computations. This bound is better than the theoretically predicted $2\sqrt{5}$.

Also using MAGMA shows that the product $M_1M_2M_3$ is not a $(0, 1)$-matrix, which distinguishes this example from [26] and all papers inspired by it. For example, the diagonal entries in $M_1M_2M_3$ are all equal to 12. The remaining entries are 6, 7 or 15.

6. Matrices of the Cayley graph of an order 25-group

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