Nonlinear Deterministic Filter for Inertial Navigation and Bias Estimation with Guaranteed Performance

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Abstract—Unmanned vehicle navigation concerns estimating attitude, position, and linear velocity of the vehicle the six degrees of freedom (6 DoF). It has been known that the true navigation dynamics are highly nonlinear modeled on attitude, position, and linear velocity of the vehicle the six proposed filter. The filter ensures systematic convergence of the error components starting from almost any initial condition. Also, the errors converge asymptotically to the origin. Experimental results validates the robustness of the initial condition. Also, the errors converge asymptotically to convergence of the error components starting from almost any initial condition. Experimental results validates the robustness of the proposed filter.

I. INTRODUCTION

Navigation solutions are key element in autonomous vehicles [1,2]. Navigation estimation algorithms considers the vehicle orientation (attitude), position, and linear velocity to be completely unknown and require estimating the aforementioned three elements [3,4]. Navigation solutions become indispensable if global positioning systems (GPS) are unreliable. A set of measurements is required for the estimation process. The vehicle’s orientation can be estimated using for instance, inertial measurement unit (IMU) [5–9], while vehicle’s pose (orientation and position) can be estimated using IMU and vision unit [10]. Recently, other potential solutions emerged to estimate the vehicle’s pose as well as map the unknown environment such as nonlinear deterministic filter for simultaneous localization and mapping (SLAM) [11,12] and nonlinear stochastic filter for SLAM [1]. The family of pose and SLAM filters requires linear velocity to be known. In practice, linear velocity is hard to obtain in GPS-denied regions and it is challenging to reconstruct.

Traditionally, inertial navigation used to addressed using Gaussian filters, such as Kalman filter [13], extended Kalman filter [14], unscented Kalman filter [15], and particle filter which is non-Gaussian filter [16] among others. Gaussian filters are based on linear approximation while particle filters do not have clear measure for optimal performance. However, navigation dynamics of a vehicle navigating in three dimensional (3D) space are highly nonlinear. Also, the dynamics cannot be classified as right or left invariant. Thereby, a recent development of filters for inertial navigation on the Lie Group have been developed, for instance invariant extended Kalman filter (IEKF) on the Lie Group of $\mathbb{SE}_2(3)$ [17], a Riccati filter design [18], and a nonlinear stochastic filter on the Lie group of $\mathbb{SE}_2(3) = \mathbb{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \subset \mathbb{R}^{5 \times 5}$ [3,4]. The filters in [3,4,17,18] are developed directly on $\mathbb{SE}_2(3)$, however measures of transient and steady-state can be defined.

To sum up, the navigation dynamics are highly nonlinear posed on the Lie group of $\mathbb{SE}_2(3)$. Also, the transient and steady-state error has to be taken into account. It is worth noting that navigation filters relies on gyroscope and accelerometer measurements, and therefore, uncertainties in gyroscope and accelerometer measurements should be addressed. Hence, this paper consider the previously mentioned challenges through the following set of contributions: (1) A nonlinear filter on the Lie Group of $\mathbb{SE}_2(3)$ for inertial navigation with predefined measures of transient and steady-state performance is proposed; (2) the closed loop errors are shown to be almost globally asymptotically stable, and when provided with data from low-cost IMU and feature sensors; and (3) the presence of unknown bias in IMU measurements is successfully tackled.

The rest of the paper is organized as follows: Section II presents important notation and identities, and introduces the true navigation problem. Section III present the concept of guaranteed measures of transient and steady-state performance and proposes a novel nonlinear filter. Section IV shows experimental results. Finally, Section V concludes the work.

II. NAVIGATION FRAMEWORK

A. Preliminaries

Let $\mathbb{R}$ set of real numbers, and $\mathbb{R}^{n \times m}$ denote denote an $n$-by-$m$ real real numbers. $\mathbb{I}_n$ is an $n$-by-$n$ identity matrix and $0_{n \times m}$ is an $n$-by-$m$ dimensional matrix of zeros. $\{I\}$ stands for the fixed inertial-frame and $\{B\}$ corresponds to the fixed body-frame attached to a vehicle moving in 3D space. The vehicle’s orientation in 3D space, known as attitude, is described by $R \in \mathbb{SO}(3)$ where $\mathbb{SO}(3)$ is a short-hand notation for Special Orthogonal Group given by

$$\mathbb{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} | RR^T = R^T R = \mathbb{I}_3, \text{det (} R \text{) = +1}\}$$

where $\text{det (} \cdot \text{)}$ stands for a determinant. $\mathfrak{s}o(3)$ is the Lie algebra of $\mathbb{SO}(3)$ denoted by

$$\mathfrak{s}o(3) = \{x \in \mathbb{R}^{3 \times 3} | x^T = -x, x \in \mathbb{R}^3\}$$

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\begin{equation}
\begin{bmatrix} x \\ \omega_x \end{bmatrix}_x = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{s} \mathfrak{o} (3), \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{equation}

The following mappings are defined
\begin{equation}
\text{vex}(\begin{bmatrix} x \\ \omega_x \end{bmatrix}_x) = x, \quad \forall x \in \mathbb{R}^3
\end{equation}
\begin{equation}
P_a (M) = \frac{1}{2} (M - M^\top) \in \mathfrak{s} \mathfrak{o} (3), \quad \forall M \in \mathbb{R}^{3 \times 3}
\end{equation}
\begin{equation}
\mathcal{Y} (M) = \text{vex}(P_a (M)) \in \mathbb{R}^3, \quad \forall M \in \mathbb{R}^{3 \times 3}
\end{equation}
Define $||R||_1$ as the Euclidean distance of $R \in \mathfrak{s} \mathfrak{o} (3)$ such that
\begin{equation}
||R||_1 = \frac{1}{4} \text{Tr} (I_3 - R) \in [0, 1]
\end{equation}
For more information about attitude and pose notation visit [5,10]. The extended form of the Special Euclidean Group denoted by $\mathbb{S}\mathbb{E}_2 (3) = \mathfrak{s} \mathfrak{o} (3) \times \mathbb{R}^3 \times \mathbb{R}^3 \subset \mathbb{R}^{5 \times 5}$ is defined by
\begin{equation}
\mathbb{S}\mathbb{E}_2 (3) = \{ X \in \mathbb{R}^{5 \times 5} \mid R \in \mathfrak{s} \mathfrak{o} (3), P, V \in \mathbb{R}^3 \}
\end{equation}
\begin{equation}
X = \begin{bmatrix} R & P & V \\ 0_{1 \times 3} & 1 & 0 \\ 0_{1 \times 3} & 0 & 1 \end{bmatrix} \in \mathbb{S}\mathbb{E}_2 (3)
\end{equation}
where $X$, $R$, $P$, and $V$ denote a homogeneous navigation matrix, rigid-body’s orientation, position, and linear velocity, respectively. Let $\mathcal{U}_M = \mathfrak{s} \mathfrak{o} (3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ be a submanifold of $\mathbb{R}^{5 \times 5}$ where
\begin{equation}
\mathcal{U}_M = \{ u (\begin{bmatrix} \Omega \\ \omega_x \end{bmatrix}_x, V, a, \kappa) \mid \Omega \in \mathfrak{s} \mathfrak{o} (3), V, a \in \mathbb{R}^3, \kappa \in \mathbb{R} \}
\end{equation}
\begin{equation}
u_1 (\begin{bmatrix} \Omega \\ \omega_x \end{bmatrix}_x, V, a, \kappa) = \begin{bmatrix} \Omega \\ \omega_x \end{bmatrix}_x \in \mathcal{U}_M
\end{equation}
with $\Omega \in \mathbb{R}^3$ and $a \in \mathbb{R}^3$ being the vehicle’s true angular velocity and the apparent acceleration composed of all non-gravitational forces affecting the vehicle. It should be remarked that $R, \Omega, a \in \{B\}$ and $P, V \in \{I\}$.

B. Navigation Dynamics and Measurements

The true 3D navigation dynamics are highly nonlinear described by [3,4]
\begin{equation}
\begin{cases}
\dot{\hat{X}} = XU - GX \\
\dot{\hat{X}} = XU - GX
\end{cases}
\end{equation}

where $b_1$ and $b_2$ denote unknown bias. Consider $n$ features in the environment where $p_i \in \mathbb{R}^3$ denotes the $i$th feature position $p_i \in \{I\}$ for all $i = 1, 2, \ldots, n$. Let the $i$th feature measurement be
\begin{equation}
y_i = R^\top (p_i - P) + b_i + n_i \in \mathbb{R}^3
\end{equation}
where $b_i$ is an unknown bias and $n_i$ is noise.
Assumption 1: At least three non-collinear features available for measurement at each time instant.

C. Navigation Matrix: Estimate, Error, and Measurements Setup

Let the estimate of the true homogeneous navigation matrix $\hat{X} \in \mathbb{S}\mathbb{E}_2 (3)$ defined in (6) be
\begin{equation}
\hat{X} = \begin{bmatrix} \hat{R} & \hat{P} & \hat{V} \\ 0_{1 \times 3} & 1 & 0 \\ 0_{1 \times 3} & 0 & 1 \end{bmatrix} \in \mathbb{S}\mathbb{E}_2 (3)
\end{equation}
where $\hat{R} \in \mathfrak{s} \mathfrak{o} (3)$, $\hat{P} \in \mathbb{R}^3$, and $\hat{V} \in \mathbb{R}^3$ denote estimates of the true orientation, position, and velocity, respectively. Define the error between $X$ and $\hat{X}$ as
\begin{equation}
\hat{X} = X - \hat{X}^{-1} = \begin{bmatrix} \hat{R} & \hat{P} & \hat{V} \\ 0_{1 \times 3} & 1 & 0 \\ 0_{1 \times 3} & 0 & 1 \end{bmatrix}
\end{equation}
with $\hat{R} = RR^\top$, $\hat{P} = P - \hat{R} \hat{P}$, and $\hat{V} = V - \hat{R} \hat{V}$. A set of measurements have to be defined to be subsequently used as part of the filter design, part of the following definitions can be found at [3,4,19]. Let us begin by defining the error:
\begin{equation}
\hat{y}_i = y_i - \hat{X}^{-1} \hat{y}_i = y_i - \hat{X} \hat{y}_i
\end{equation}
\begin{equation}
\hat{y}_i = ([y_i, 0, 0])^\top \in \mathcal{M}, \quad p_i = \hat{R} \hat{y}_i = \hat{R} \hat{y}_i, \quad \hat{p}_i = \hat{R} \hat{y}_i = \hat{R} \hat{p}_i - \hat{R} \hat{P}, \quad \hat{p}_i = \hat{p}_i - \hat{R} \hat{p}_i, \quad \hat{P} = P - \hat{R} \hat{P}.
\end{equation}
Define the following components: $s_T = \sum_{i=1}^{n} s_i$, $p_c = \frac{1}{s_T} \sum_{i=1}^{n} s_i p_i$, and $\mathcal{M} = \sum_{i=1}^{n} s_i (p_i - p_c)^T (p_i - p)^T = 2 \sum_{i=1}^{n} s_i p_i p_i^T - s_T p_c p_c^T$ such that
\begin{equation}
\mathcal{M} = \sum_{i=1}^{n} s_i (p_i - p_c)^T - s_T p_c p_c^T
\end{equation}
where $s_i > 0$ denotes the sensor confidence level of the $i$th landmark, and $n \geq 3$ as defined in Assumption 1. One finds $\sum_{i=1}^{n} s_i (p_i - p_c) y_i^T \hat{R} = \sum_{i=1}^{n} s_i (p_i - p_c) (p_i - P)^T \hat{R}$ which means that
\begin{equation}
\sum_{i=1}^{n} s_i (p_i - p_c) y_i^T \hat{R} = \sum_{i=1}^{n} s_i (p_i - p_c) y_i^T \hat{R} = \sum_{i=1}^{n} s_i (p_i - p_c) (p_i - P)^T \hat{R}
\end{equation}
Additionally, the following result can be obtained $\sum_{i=1}^{n} s_i \hat{y}_i \hat{R}_c = \sum_{i=1}^{n} s_i (p_i - \hat{R}_c \hat{P} - \hat{R}^\top) = \sum_{i=1}^{n} s_i (p_i - \hat{R}^\top) P$ such that $\sum_{i=1}^{n} s_i \hat{y}_i = s_T \hat{R}_c \hat{P} - P = (I_3 - \hat{R}_c \hat{P}) P$. Note that $\hat{R} \rightarrow I_3$ indicates that $\hat{P}_c \rightarrow \hat{P}$ and $\hat{R} \rightarrow I_3$ implying that $\hat{R}^\top \hat{P}_c = \hat{P}_c = \hat{P}$. Summing up the above derivations, the following set
expressed in terms of vector measurements will be used in the filter design [3,4]:

\[
\begin{cases}
    p_c = \frac{1}{T} \sum_{i=1}^{n} s_i p_i, & s_T = \sum_{i=1}^{n} s_i \\
    M = \sum_{i=1}^{n} s_i p_i p_i^T - s_T p_c p_c^T \\
    \tilde{y}_i = p_i - \tilde{R} y_i - \tilde{P} \\
    M\tilde{R} = \sum_{i=1}^{n} s_i (p_i - p_c) y_i^T \tilde{R}^T \\
    \tilde{R}^T \tilde{P} = \sum_{i=1}^{n} s_i y_i \tilde{y}_i
\end{cases}
\] (16)

D. Nonlinear Filter Framework and Error Dynamics

Lemma 1: Let \( \tilde{R} \in \mathbb{S}^3 \), \( M = M^T \in \mathbb{R}^{3 \times 3} \). Define \( \bar{M} = \text{Tr}\{M\}I_3 - M \) such that \( \lambda_{\text{min}} = \lambda(\bar{M}) \) and \( \lambda_{\text{max}} = \dot{\lambda}(\bar{M}) \) are the minimum and the maximum eigenvalues of \( \bar{M} \), respectively. Then, one has

\[
\frac{\lambda_{\text{max}}}{2} (1 + \text{Tr}\{\tilde{R}\}) ||M\tilde{R}||_1 \leq ||\mathbf{Y}(M\tilde{R})||^2 \leq 2\lambda_{\text{max}}||M\tilde{R}||_1
\] (17)

Proof: See [5], Lemma 1.

From Lemma 1, define \( \bar{M} = \text{Tr}\{M\}I_3 - M \) given that \( \lambda(M) = \{\lambda_1, \lambda_2, \lambda_3\} \) with \( \lambda_3 \geq \lambda_2 \geq \lambda_1 \). In view of Assumption (1), at least two of the eigenvalues in the set \( \lambda(M) \) are greater than zero and therefore \( \lambda(\bar{M}) = \{\lambda_3 + \lambda_2, \lambda_3 + \lambda_1, \lambda_2 + \lambda_1\} \) see Section 4.2 in [5].

III. NONLINEAR NAVIGATION FILTER DESIGN

A. Systematic Convergence

Consider the following error in view of the measurements in (16):

\[
e = [e_1, e_2, e_3, e_4]^T = \left[ ||M\tilde{R}||_1, \tilde{P}^T \tilde{R} \right]^T \in \mathbb{R}^4
\] (18)

Define \( \xi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) as a positive smooth and time-decreasing function [20]:

\[
\xi_i(t) = (\xi_i^0 - \xi_i^\infty) \exp(-\xi_i t) + \xi_i^\infty, \quad \forall i = 1, 2, \ldots, 4
\] (19)

with \( \xi_i^0 > 0 \) being upper bound of the known large set, \( \xi_i^\infty > 0 \) being upper bound of the small set, and \( \xi_i > 0 \) being convergence rate of \( \xi_i(t) \) from \( \xi_i^0 \) to \( \xi_i^\infty \). Define the error \( e_i \) as [20]:

\[
e_i = \xi_i N(E_i)
\] (20)

where \( E_i \in \mathbb{R} \) denotes the transformed error which is unconstrained, and \( N(E_i) \) denotes a smooth function which is differentiable, strictly increasing, and bounded, for more information see [5,11]. The inverse transformation of (21) is given by

\[
E_i = N^{-1}(e_i/\xi_i)
\] (21)

The inverse transformation in (21) is defined by [5,11,20]

\[
E_i = \frac{1}{2\xi_i} \left[ \ln \frac{\xi_i + e_i/\xi_i}{\xi_i - e_i/\xi_i} \right], \quad \xi_i > 0 \quad \text{if} \quad e_i(0) \leq 0
\]

\[
E_i = \frac{1}{2\xi_i} \left[ \ln \frac{\xi_i + e_i/\xi_i}{\xi_i - e_i/\xi_i} \right], \quad \xi_i > 0 \quad \text{if} \quad e_i(0) < 0
\] (22)

Let us define

\[
\Delta_i = \frac{1}{2\xi_i} \frac{\partial N^{-1}(e_i/\xi_i)}{\partial (e_i/\xi_i)} = \frac{1}{2\xi_i} \left( \frac{1}{\xi_i + e_i/\xi_i} + \frac{1}{\xi_i - e_i/\xi_i} \right)
\] (23)

Accordingly, the dynamics of \( \dot{E}_i \) are

\[
\dot{E}_i = \Delta_i \left( \frac{d}{dt} e_i - \dot{\xi}_i e_i \right)
\] (24)

Or to put simply

\[
\dot{E} = \begin{bmatrix} \Delta_R & 0_{1 \times 3} \\ 0_{3 \times 1} & \Delta_P \end{bmatrix} \begin{bmatrix} \frac{d}{dt} e - \frac{\mu_R}{\mu_P} & 0_{1 \times 3} \end{bmatrix} e
\] (25)

such that \( \mu_R = \dot{\xi}_1/\xi_1, \mu_P = \text{diag}(\dot{\xi}_2/\xi_2, \dot{\xi}_3/\xi_3, \dot{\xi}_4/\xi_4) \). \( \Delta_R = \Delta_1 \), and \( \Delta_P = \text{diag}(\Delta_2, \Delta_3, \Delta_4) \) where \( \mu_R, \Delta_R \in \mathbb{R} \) and \( \mu_P, \Delta_P \in \mathbb{R}^{3 \times 3} \). For more information of orientation, pose, and SLAM filters with prescribed performance visit [5,11].

B. Nonlinear Navigation Filter with Bias Compensation

In this Subsection the unknown bias inevitably present in measurements of angular velocity and acceleration is accounted for. From (9), let \( \tilde{b}_1 \) and \( \tilde{b}_a \) denote the estimates of \( b_1 \) and \( b_a \), respectively. Define the bias error as

\[
\begin{cases}
    \tilde{b}_1 = b_1 - \tilde{b}_1 \\
    \tilde{b}_a = b_a - \tilde{b}_a
\end{cases}
\] (26)

Consider the following nonlinear filter design on the Lie Group of \( \mathbb{S}E_2(3) \):

\[
\begin{cases}
    \dot{\tilde{R}} = \tilde{R}[\Omega_m - b_1 I_3] - [w_\Omega]_\times \tilde{R} \\
    \dot{\tilde{P}} = \tilde{V} - [w_\Omega]_\times \tilde{P} - w_P \tilde{P} - k_a \Delta P E_P \\
    \dot{\tilde{V}} = \tilde{R}(a_m - b_1) - [w_\Omega]_\times \tilde{V} - w_a
\end{cases}
\]

(such that)

\[
\begin{cases}
    w_\Omega = -k_w (E_R \Delta R + 1) \mathbf{Y}(M\tilde{R}) \\
    w_P = [p_c - \tilde{R}^T \tilde{P}]_\times w_\Omega - \ell_P \tilde{R}^T \tilde{P} - k_a \Delta P E_P \\
    w_a = -\tilde{g} + k_a \delta \left[ [w_\Omega]_\times - \Delta P \right] E_P \\
    \tilde{b}_1 = -\gamma b_1 \Delta R E_R + 1 \tilde{R}^T \mathbf{Y}(M\tilde{R}) \\
    \tilde{b}_a = -\gamma_a \Delta P E_P
\end{cases}
\] (28)

with \( k_w, k_v, \ell_P, \ell_P, \gamma_b, \gamma_a \), and \( \gamma_a \) being positive constants, \( w_\Omega, w_P, \) and \( w_a \) denoting correction factors \( \forall w_\Omega, w_P, w_a \in \mathbb{R}^3 \), \( \mathbf{Y}(M\tilde{R}) = \text{veex}(\mathbf{P}_a(M\tilde{R})) \), \( E_R = E_1 \), and \( E_P = [E_2, E_3, E_4]^T \). Also, \( U_m = u([\Omega_m - \tilde{b}_1]_\times, 0_{3 \times 1}, a_m - b_a, 1) \in \mathcal{U}_M \) and \( W = u([w_\Omega]_\times, w_P, w_a, 1) \in \mathcal{U}_M \).

Theorem 1: Consider the navigation dynamics in (28) coupled with feature measurements (output \( \tilde{y}_i = X^{-1} \tilde{p}_i \)) for all \( i = 1, 2, \ldots, n \), angular velocity measurements \( \Omega_m = \Omega + b_1 \), and acceleration measurements \( a_m = a + b_a \). Let Assumption 1 hold. Combine the nonlinear filter design in (27) and (28) with the measurements of \( \tilde{y}_i, \Omega_m, \) and \( a_m \). Let the design parameters \( k_w, k_v, \ell_P, \ell_P, \gamma_b, \gamma_a, \delta \). \( \delta_i > \delta \geq |e_i(0)|, \xi_i^0 > \xi_i^\infty \) be selected as positive constants. Define the set

\[
S = \{ (E_R, E_P, \tilde{V}, \tilde{b}_1, \tilde{b}_a) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \mid E_R = 0, E_P = \tilde{V} = \tilde{b}_1 = \tilde{b}_a = 0_{3 \times 1} \}
\] (29)
given that $E_R, E_P \in L_\infty$ and $\tilde{R}(0)$ does not belong to the unattractive set defined in [5]. Then, 1) the errors $E_R, E_P, \tilde{V}, \tilde{b}_\Omega$, and $b_v$ converge asymptotically to $S$ in (29), 2) the trajectory of $\tilde{R}$ converges asymptotically to $I$, and 3) the trajectories of $\tilde{P}$ converge asymptotically to the origin.

**Proof:** From (8) and (27), the orientation error dynamics are
\[
\dot{\tilde{R}} = -\tilde{R}[\tilde{R}(\Omega_m - \Omega)\times + \tilde{R}[w_\Omega]_\times
\]
(30)
where $[\tilde{R}\Omega]_\times = \tilde{R}[\Omega]_\times \tilde{R}^T$. From (30), (8), and (27), the position error dynamics are
\[
\dot{\tilde{P}} = \tilde{V} - \tilde{R}[\dot{\tilde{P}}]_\times \tilde{R}(\Omega_m - \Omega) + \tilde{R}w_V
\]
(31)
From (30), (8), and (27), the velocity error dynamics are
\[
\dot{\tilde{V}} = -\tilde{R}(a_m - a) - \tilde{R}[\dot{\tilde{V}}]_\times \tilde{R}(\Omega_m - \Omega) + \tilde{g} + \tilde{R}w_a
\]
(32)
Given that $\Omega_m = \Omega + b_\Omega$ and $a_m = a + b_a$, one has
\[
\begin{align*}
\frac{d}{dt}||\tilde{M}\tilde{R}||_1 & = -\frac{1}{2} \mathcal{Y}(\tilde{M}\tilde{R})^T(\tilde{R}b_\Omega - w_\Omega) \\
\frac{d}{dt}\tilde{P} & = \tilde{R}^T\tilde{V} - \left[\tilde{P} - p_e + \tilde{R}^T\dot{\tilde{P}}_\Omega\right]_\times \tilde{R}b_\Omega \\
\frac{d}{dt}\tilde{V} & = -\left[\tilde{R}^T\tilde{V}\right]_\times \tilde{R}b_\Omega - [w_\Omega]_\times \tilde{R}^T\tilde{V} + \tilde{R}b_a + \tilde{R}^T\tilde{g} + \tilde{R}w_a
\end{align*}
\]
(33)
From (33) and (25), one obtains
\[
\begin{align*}
\dot{E}_R & = \Delta R \left( \frac{d}{dt}||\tilde{M}\tilde{R}||_1 - \mu R||\tilde{M}\tilde{R}||_1 \right) \\
\dot{E}_P & = \Delta P \left( \frac{d}{dt}\tilde{R}^T\tilde{P}_\Omega - \mu P\tilde{R}^T\dot{\tilde{P}}_\Omega \right)
\end{align*}
\]
(34)
Consider selecting the following Lyapunov function candidate $L_T = L_T(\tilde{M}\tilde{R})_1, E_R, E_P, \tilde{R}^T\tilde{V}, \tilde{R}b_\Omega, \tilde{R}b_a$
\[
L_T = L_R + L_{PV}
\]
(35)

Let $\lambda_{\mathcal{M}} = \lambda(\mathcal{M})$. The function $L_R$ is selected as
\[
L_R = E_R^2 + L_1
\]
(36)
where $L_{Rb_\Omega} = 2||\tilde{M}\tilde{R}||_1 + \frac{1}{2\gamma_b}||\tilde{b}_\Omega||^2 + \frac{1}{2\gamma_b\lambda_{\mathcal{M}}} \mathcal{Y}(\tilde{M}\tilde{R})^T \tilde{R}b_\Omega$. $L_{Rb_a}$ in (36) follows
\[
\varepsilon_1 \left[ \frac{\frac{1}{2\gamma_b\lambda_{\mathcal{M}}} - \frac{1}{4\gamma_b\lambda_{\mathcal{M}}}}{\frac{2}{\gamma_b\lambda_{\mathcal{M}}}} \right] \leq L_1 \leq \varepsilon_1 \left[ \frac{\frac{1}{4\gamma_b\lambda_{\mathcal{M}}}}{\frac{2}{\gamma_b\lambda_{\mathcal{M}}}} \right]
\]
(37)
with $\varepsilon_1 = \left[ \sqrt{||\tilde{M}\tilde{R}||_1, ||\tilde{R}b_\Omega||^T} \right]$ and $\sqrt{2\lambda_{\mathcal{M}}||\tilde{R}M||_1} \geq ||\mathcal{Y}(\tilde{M}\tilde{R})||$ as defined in (17), Lemma 1. Both $A_4$ and $A_\lambda$ become positive given that
\[
\frac{1}{\gamma_b} \geq \frac{1}{16\gamma_b^2\lambda_{\mathcal{M}}}
\]
(37)
Let $\gamma_b$ be selected as in (37). Recall (34) and (33). The derivative of (36) is
\[
\dot{L}_R \leq -c_R(\Delta R^2 + 1)||\tilde{M}\tilde{R}||_1 - c_b||\tilde{R}b_\Omega||^2 + \frac{c_b}{\gamma_b\sqrt{2\lambda_{\mathcal{M}}}} ||\tilde{M}\tilde{R}||_1 ||\tilde{R}b_\Omega||
\]
which can be expressed as
\[
\dot{L}_R \leq -\varepsilon_1^T \left[ \frac{c_R}{\gamma_b\sqrt{2\lambda_{\mathcal{M}}}} \frac{c_b}{\gamma_b\sqrt{2\lambda_{\mathcal{M}}}} \right] \varepsilon_1
\]
(38)
where $\varepsilon_1 = \left[ \sqrt{||\tilde{M}\tilde{R}||_1, ||\tilde{R}b_\Omega||} \right]^T$. It can be deduced that $A_2$ become positive by selecting $c_Rc_b \geq \frac{c_b^2}{2\gamma_b\lambda_{\mathcal{M}}}$ such that
\[
\gamma_b \geq \sqrt{\frac{8c_R^2(1 + Tr(\tilde{R}))) + K_M}{2\sqrt{3}(w_k - 1)\Delta M}}
\]
(39)
where $K_M = (\sqrt{3}\gamma_b + (2(w_k - 1)\lambda_{\mathcal{M}} - \gamma_0)k_e^2)\Delta M$. Consider selecting $\gamma_b$ as in (39) and defining $A_{-\Delta_2} = \Delta(A_2)$. Thus, $L_R$ in (38) follows the inequality below
\[
\dot{L}_R \leq -\Delta_2 ||R||^2 - c_R E_R^2 \Delta_2 ||\tilde{M}\tilde{R}||_1
\]
(40)
Let us turn our attention to the second portion of the $L_T$ definition in (35). Define the following Lyapunov function candidate:
\[
L_{PV} = \frac{1}{2}||E_P||^2 + \frac{1}{2\gamma_a}||\tilde{R}^T\tilde{V}||^2 + \frac{1}{2\gamma_a}||\tilde{R}b_a||^2 - \delta E_P^T\tilde{R}^T\tilde{V} + \delta_3 b_a^T\tilde{R}^T\tilde{V}
\]
(41)
It is straightforward to show that $L_{PV}$ in (41) follows
\[
\varepsilon_2 \left[ \frac{\frac{1}{\gamma_a^2}}{\frac{2}{\gamma_a}} \frac{0}{\frac{1}{\gamma_a}} \frac{0}{\frac{1}{2\gamma_a}} \right] \leq L_{PV} \leq \varepsilon_2 \left[ \frac{\frac{1}{\gamma_a^2}}{\frac{2}{\gamma_a}} \frac{0}{\frac{1}{\gamma_a}} \frac{0}{\frac{1}{2\gamma_a}} \right] \varepsilon_2
\]
(42)
with $\varepsilon_2 = \left[ ||E_P||, ||\tilde{R}V||, ||\tilde{R}b_a|| \right]^T$. It becomes apparent that $A_3, A_\varepsilon$, and in turn $L_{PV}$ can be made positive, if the following holds
\[
\frac{1}{2\gamma_a} - \frac{3}{2\gamma_a} > 0 \text{ and can be enabled by selecting } \delta, \delta_a, k_a, \gamma_a > 0
\]
(43)
Let $k_a$ be selected as in (42). Recall (34) and (33). The time derivative of $L_{PV}$ in (41) is as follows:
\[
\dot{L}_{PV} \leq -\varepsilon_{PV}^T \left[ \frac{\frac{k_a}{\gamma_a}}{\frac{1}{\gamma_a}} \frac{\frac{k_a}{\gamma_a}}{\frac{1}{\gamma_a}} \frac{\frac{\delta c_3}{2}}{\frac{1}{\gamma_a}} \frac{\delta c_4}{2} \right] \varepsilon_{PV}
\]
(44)
where \( \varepsilon_{PV} = \left[ |E_P|, |\tilde{V}^T \tilde{R}| \right]^T \). It becomes clear that \( A_4 \) in (43) can be made positive by selecting
\[
\frac{1}{\delta} > \frac{k_3^2 c_3^2 + 4 k_c c_c c_2}{4 c_3 c_1 k_v}
\] (44)
Let \( \delta \) be selected as in (44) and define \( \lambda_{A_4} = \lambda(A_4) \). One may rewrite the inequality in (43) as follows:
\[
\hat{L}_{PV} \leq -\left[ \begin{array}{c}
|E_P| \\
|\tilde{R}b_a|
\end{array} \right]^T \left[ \begin{array}{c}
\frac{\lambda_{A_4}}{\delta} \\
\frac{\delta_a}{\delta}
\end{array} \right] \left[ \begin{array}{c}
|E_P| \\
|\tilde{R}b_a|
\end{array} \right] + c_b(|\tilde{V}| + |E_P| + |\tilde{R}b_a|) |I_3 - \tilde{R}|_F + c_b|\tilde{R}b_a||E_P| + c_b|\tilde{R}b_a||\tilde{R}^T \tilde{V}|
\] (45)
It can be shown that \( A_5 \) is positive if the following holds:
\[
\delta_a < \frac{4 \lambda_{A_4}}{c_b^2}
\] (46)
Consider selecting \( \delta_a \) as in (46) and defining \( \lambda_{A_5} = \lambda(A_5) \).
Define \( \varepsilon_T = \left[ ||E_P||, \sqrt{|M \tilde{R}|}_1, |\tilde{R}b_a| \right] \) and let \( c_n = \max\left\{ 4 X_M c_y, c_b \right\} \). Hence, one obtains
\[
\hat{L}_T \leq -\varepsilon_T^T \left[ \begin{array}{c}
\frac{\lambda_{A_5}}{\delta} + \frac{c_n 1_{2 \times 1}}{\lambda_{A_5}}
\end{array} \right] \varepsilon_T - \frac{\lambda M_k W_y^2 \Delta^2_R}{4(1 + \text{Tr}\{R\})} |M \tilde{R}|_1
\] (47)
Select the parameters of \( A_2 \) and \( A_3 \) such that \( 4 \lambda_{A_4} \lambda_{A_5} > c_b^2 \) guaranteeing the positive definiteness of \( A_T \). Accordingly, the following result is obtained:
\[
\hat{L}_T \leq -\lambda(A_T)||\varepsilon_T||^2 - \frac{\lambda M_k W_y^2 \Delta^2_R}{4(1 + \text{Tr}\{R\})} |M \tilde{R}|_1
\] (48)
Recall that \( \varepsilon_T = \left[ ||E_P||, \sqrt{|M \tilde{R}|}_1, |\tilde{R}b_a| \right] \) where \( \varepsilon_P = \left[ \begin{array}{c}
|E_P| \\
|\tilde{R}b_a|
\end{array} \right]^T \), and \( \varepsilon_{PV} = \left[ \begin{array}{c}
|E_P| \\
|\tilde{V}^T \tilde{R}|
\end{array} \right]^T \). In accordance with the definition of \( L_T \) in (35), the inequality in (48) indicates that \( \hat{L}_T \) is negative for all \( L_T > 0 \), and \( \hat{L}_T = 0 \) only when \( L_T = 0 \). This ensures asymptotic convergence of \( L_T \) enabling the guaranteed performance of transient and steady-state error in (18). This completes the proof.

For implementation purposes, the nonlinear navigation filter with guaranteed performance for unknown bias outlined in (27) and detailed in (28) is presented in discrete form. Let \( \Delta T \) denote a small sample time step. The complete discrete implementation steps are detailed in Algorithm 1.

**Algorithm 1:** Discrete nonlinear filter.

**Initialization:**
1. Set \( \tilde{R}[0] \in \mathbb{S}^3 \), \( \hat{P}[0] \in \mathbb{R}^3 \), \( \tilde{V}[0] \in \mathbb{R}^3 \), and \( \hat{b}[0] = \hat{b}_a[0] = 0_{3 \times 1} \).
2. Select \( k_u, k_v, k_a, \xi_P, \gamma_v, \gamma_a, \delta, \delta_a = \delta > |e_1(0)|, \xi_0 > |e_1(0)| \) and \( \xi_\infty \) as positive constants, and the sample \( k = 0 \).

**while** (1) **do**

**/* Prediction step */**
3. \( \tilde{X}_{k+1|k} = \left[ \begin{array}{c}
\tilde{R}_{k|k} \\
\hat{P}_{k|k} \\
\hat{V}_{k|k}
\end{array} \right] = \left[ \begin{array}{c}
\Omega_m[k] - b_0[k] \\
0_{1 \times 3} \\
0_{1 \times 3}
\end{array} \right] \) and \( \hat{U}_k = \left[ \begin{array}{c}
\Omega_m[k] - b_0[k] \\
0_{1 \times 3} \\
0_{1 \times 3}
\end{array} \right] \)

**/* Update step */**
4. \( \lambda_{k+1} = \tilde{X}_{k+1|k} \text{exp}(\Delta T \tilde{U}_k) \)
5. \( p_c = \frac{1}{\Delta T} \sum_{i=1}^{n} s_i p_i[k] \), \( s_T = \sum_{i=1}^{n} s_i \)
6. \( M \tilde{R}_k = \sum_{i=1}^{n} s_i (p_i[k] - p_T) \hat{y}_1[k] \hat{R}_{k+1|k} \)
7. \( \tilde{P} = \sum_{i=1}^{n} s_i (p_i[k] - p_T) \hat{y}_1[k] \hat{R}_{k+1|k} \)
8. \( \xi_i[k] = (\xi_0 - \xi_\infty) \exp(-\Delta T \xi) + \xi_\infty \)
9. if \( e_1[k] > \xi_i[k] \) then
10. \( \xi_i[k] = e_1[k] + \epsilon \), \( \epsilon \) is a small constant

**end if**
11. \( E_i = \frac{1}{\Delta T} \log \left( \frac{\xi_i[k]}{|\Sigma + e_1[k]| |\Sigma - e_1[k]|} \right) \)
12. \( \Delta_1 = \frac{1}{2} \text{sgn}(|\Sigma + e_1[k]| |\Sigma - e_1[k]|) \)

**end for**
13. \( \Delta P[k] = \left[ \begin{array}{c}
\Delta \tilde{R} \\
\Delta P_T \\
\Delta P_e
\end{array} \right] = \text{diag}(\Delta_2, \Delta_3, \Delta_4) \)
14. \( \hat{b}_{1}[k+1] = \hat{b}_{1}[k] - \Delta T \gamma_0 \hat{R}_{k+1|k} \hat{X}(M \hat{R}_{k}) \)
15. \( \hat{b}_{1}[k+1] = \hat{b}_{1}[k] - \Delta T \gamma_0 \hat{R}_{k+1|k} \hat{X}(M \hat{R}_{k}) \)
16. \( W_k = \left[ \begin{array}{c}
\hat{w}_v[k] \\
\hat{w}_a[k]
\end{array} \right] \)
17. \( \tilde{X}_{k+1|k+1} = \text{exp}(-W_k \Delta T) \tilde{X}_{k+1|k} \)
18. \( k = k + 1 \)

**end while**
a sampling rate of 20Hz and the IMU measurements were obtained at a sampling rate of 200 Hz using ADIS16448. The dataset does not include values of real world features, as a result, a set of virtual landmarks were assigned randomly in accordance with Assumption 1. The camera parameters were calibrated through a Stereo Camera Calibrator Application and the images are not distorted, see Fig. 1. More details can be found about EuRoC dataset in [21]. Consider the initial value of orientation, position and linear velocity to be:

$$\hat{\mathbf{R}}_0 = \hat{\mathbf{R}}(0) = \mathbf{I}_3, \quad \hat{\mathbf{P}}(0) = \hat{\mathbf{V}}(0) = [0, 0, 0]^T$$

Let the initial bias estimate \( \hat{b}_\Omega(0) = \hat{b}_a(0) = [0, 0, 0]^T \) and the design parameters be defined as \( k_w = 3, k_v = 4, k_a = \ell_P = 4, \gamma_b = 2, \gamma_a = 3, \delta = 0.15, \ell = 1.2[1, 1, 1]^T \), \( \xi^\infty = [0.03, 0.08, 0.08, 0.08]^T \), and

$$\xi^0 = \hat{\delta} = \left[ \frac{1.3||\hat{\mathbf{R}}(0)||}{2\hat{\mathbf{R}}(0)^T P_\xi(0)} \right] + 2[0.25, 1, 1, 1]^T$$

Fig. 1. This image presents an example of right and left feature detection and tracking performed by a stereo camera via the Computer Vision System Toolbox with MATLAB R2020b. The annotated photograph is adopted from the EuRoC dataset [21].

Fig. 2 shows the experimental performance of the proposed filter in discrete form. The dataset that was used in this Section is the Vicon Room 2 01 dataset [21]. Fig. 2 presents robust convergence starting from large error in initialization to the neighborhood of the origin. This can be confirmed from the left portion of Fig. 2 which shows accurate tracking performance. Hence, the proposed solution is promising to be implemented on a cheap electronic kit.

V. CONCLUSION

In this paper, the navigation problem of a vehicle traveling in 3D space has been addressed in deterministic sense. The vehicle’s orientation, position, and linear velocity have been estimated successfully through a proposed nonlinear filter on \( \mathbb{SE}_2(3) \). The proposed solution successfully accounts and compensates for the unknown bias inevitably present in angular velocity and acceleration measurements. According to the experimental results, the proposed filter showed strong tracking capabilities of the unknown pose and estimation of the unknown linear velocity of the vehicle.

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Appendix

Quaternion Representation of the Proposed Filter

The unit-quaternion is a four-element vector \( Q = [q_0, q_1, q_2, q_3]^T \in \mathbb{S}^3 \) where \( q_0 \in \mathbb{R} \) and \( q = [q_1, q_2, q_3]^T \in \mathbb{R}^3 \) such that

$$\mathbb{S}^3 = \{ Q \in \mathbb{R}^4 \mid ||Q|| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1 \}$$

The inverse of the rotation is represented by the inverse of a unit-quaternion defined by \( Q^{-1} = [ -q_0 \ -q_1 \ -q_2 \ -q_3 ]^T \in \mathbb{S}^3 \). Let \( \odot \) stand for the quaternion product between two unit-quaternions. The quaternion multiplication between \( Q_1 = [ q_{01} \ q_{11} \ q_{21} \ q_{31} ]^T \in \mathbb{S}^3 \) and \( Q_2 = [ q_{02} \ q_{12} \ q_{22} \ q_{32} ]^T \in \mathbb{S}^3 \) is defined by

$$Q_1 \odot Q_2 = \left[ \begin{array}{c} q_{01}q_{02} - q_1q_2 \\ q_{01}q_{02} + q_1q_2 + q_0q_1 + q_0q_2 \\ q_{01}q_{02} + q_1q_2 - q_0q_1 - q_0q_2 \end{array} \right]$$

For \( \Omega \in \mathbb{R}^3 \), define \( \psi(\Omega) = [0, \Omega]^T \in \mathbb{R}^4 \) and \( \mathcal{T}(\psi(\Omega)) = \Omega \in \mathbb{R}^3 \). For more details of Quaternion to/from \( \mathbb{SO}(3) \) representation, visit [22]. Recall the measurements in (16) and define

$$\begin{align*}
\hat{y}_i &= p_i - \mathcal{T}\left( \hat{Q} \odot \psi(y_i) \odot \hat{Q}^{-1} \right) - \hat{P} \\
\Phi_q &= M\hat{R} = \sum_{i=1}^{n} s_i [p_i - p_c]^T \eta_i \mathcal{R}_Q \\
v_q &= \hat{R}^T P_\xi = \frac{1}{\tau^2} \sum_{i=1}^{n} s_i y_i \\
||M\hat{R}||_1 &= \frac{1}{4} \text{Tr}(M - \Phi_q)
\end{align*}$$

where \( \mathcal{R}_Q = (\hat{q}_0^2 - ||\hat{q}||^2)\mathbf{I}_3 + 2\hat{q}\hat{q}^T + 2\hat{q}_0 [\hat{q}]_x \). Thus, the error vector in (18) is \( e = [||M\hat{R}||_1, v_q]^T \in \mathbb{R}^4 \). The equivalent quaternion representation of the filter can be obtained as follows:

$$\begin{align*}
\Gamma(\Omega) &= \left[ \begin{array}{cc} 0 & -\Omega^T \\ \Omega & -[\Omega]_x \end{array} \right], \quad \Psi(\Omega) = \left[ \begin{array}{cc} 0 & -\Omega^T \\ \Omega & [\Omega]_x \end{array} \right] \\
\hat{Q} &= \frac{1}{2} \Gamma \left( \hat{\Omega}_m - \hat{b}_\Omega \right) \hat{Q} - \frac{1}{2} \Psi(\hat{q}_1) \hat{Q} \\
\hat{P} &= \hat{V} - [w_\Omega]_x \hat{P} - w_V \\
\hat{V} &= \mathcal{T} \left( \hat{Q} \odot \psi(a_m - b_m) \odot \hat{Q}^{-1} \right) - [w_\Omega]_x \hat{V} - w_a \\
w_\Omega &= -k_w(E_R \Delta_R + 1) \Psi(\Phi_q) \\
w_V &= [p_c - v_q]_x w_\Omega - \ell_P v_q - k_v \Delta_P E_P \\
w_a &= -\tilde{g} + k_a [\delta [w_\Omega]_x - \Delta_P] E_P \\
\dot{\hat{b}}_\Omega &= -\gamma_b (\Delta_R E_R + 1) \mathcal{T} \left( \hat{Q}^{-1} \odot \psi(\Phi_q) \odot \hat{Q} \right) \\
\dot{\hat{b}}_a &= -\gamma_a \delta \mathcal{T} \left( \hat{Q}^{-1} \odot \psi(E_P) \odot \hat{Q} \right)
\end{align*}$$

(49)
Fig. 2. Dataset Vicon Room 201: True trajectory versus estimated trajectory plotted using the proposed nonlinear discrete navigation filter. In the right portion, the true trajectory is plotted in green solid-line while the estimated trajectory is plotted in blue dash-line. The true 3-axes orientation is plotted in red dash-line. The final destination is marked with a star. The features are marked by black circles. In the right portion, a blue solid-line is used to demonstrate error components of: orientation $||\hat{R}R^\top||$, position $||P - \hat{P}||$, velocity $||V - \hat{V}||$, bias in angular velocity $||b_\Omega - \hat{b}_\Omega||$, and bias in acceleration $||b_a - \hat{b}_a||$.

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