A Parameter-free Algorithm for Convex-Concave Min-max Problems

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Abstract

Parameter-free optimization algorithms refer to algorithms whose convergence rate is optimal with respect to the initial point without any learning rate to tune. They are proposed and well-studied in the online convex optimization literature. However, all the existing parameter-free algorithms can only be used for convex minimization problems. It remains unclear how to design a parameter-free algorithm for convex-concave min-max problems. In fact, the best known convergence rates of the algorithms for solving these problems depend on the size of the domain, rather than on the distance between initial point and the optimal solution. In this paper, we provide the first parameter-free algorithm for several classes of convex-concave problems and establish corresponding state-of-the-art convergence rates, including strictly-convex-strictly-concave min-max problems and min-max problems with non-Euclidean geometry. As a by-product, we utilize the parameter-free algorithm as a subroutine to design a new algorithm, which obtains fast rates for min-max problems with a growth condition. Extensive experiments are conducted to verify our theoretical findings and demonstrate the effectiveness of the proposed algorithm.

1. Introduction

In this paper, we are interested in the following problem

\[ \min_{x} \max_{y} F(x, y) \tag{1} \]

where \( X \subset \mathbb{R}^n \), \( Y \subset \mathbb{R}^m \) are convex and compact sets. This problem has broad applications in machine learning, e.g., stochastic AUC maximization (Ying et al., 2016), generative adversarial nets (Goodfellow et al., 2014), robust optimization (Ben-Tal et al., 2009), and adversarial training (Madry et al., 2017). In this paper, we consider the case that \( F(x, y) \) is convex in \( x \) and concave in \( y \).

The canonical method for solving the convex-concave game is the primal-dual gradient method (Nemirovski et al., 2009). For example, in the Euclidean setup, primal-dual gradient method is performing gradient descent on the primal variable \( x \) and gradient ascent on the dual variable \( y \) simultaneously. Nemirovski et al. (2009) proved that the averaged solution of primal-dual gradient method has good convergence guarantees in terms of the duality gap, which is

\[ \max_{y \in Y} F(\bar{x}_T, y) - \min_{x \in X} F(x, \bar{y}_T) \leq \frac{D^2}{2\eta T} + \frac{1}{2} \eta G^2, \tag{2} \]

where \((\bar{x}_T, \bar{y}_T) = (\frac{1}{T} \sum_{t=1}^{T} x_t, \frac{1}{T} \sum_{t=1}^{T} y_t)\) is the averaged solution, \( D \) is the diameter of the domain, \( G \) is an upper bound on the norm of the gradient. To get the tightest bound for the RHS of (2), we can choose \( \eta = \frac{D}{G \sqrt{T}} \) and end up with \( \frac{DG}{\sqrt{T}} \) bound for the duality gap.

Strangely enough, the optimal learning rate scheme in the primal-dual gradient method completely ignores the effect of the initialization. Regardless of how close the initialization is to the optimal solution, this algorithm ends up with the same complexity guarantees. This is counter-intuitive: we would expect to be able to obtain a faster convergence if the initialization is closer to the optimal solution. Yet, the hardness results in the optimization literature show that \( \Omega\left(\frac{DG}{\sqrt{T}}\right) \) complexity lower bound is unimprovable (Nemirovskij & Yudin, 1983; Juditsky et al., 2011). This naturally motivates the following question:

Under mild conditions, can we design a first-order algorithm for solving convex-concave min-max problems, whose computational complexity provably adapts to the distance between initialization and the optimal solution?

In other words, the goal is to obtain

\[ \tilde{O}\left(\text{dist}(x_0, x_\ast) + \text{dist}(y_0, y_\ast)\right) \]

convergence rate, where \( \text{dist} \) is some metric depending on the geometry of the problem.

We give an affirmative answer to this question by designing a new algorithm called Coin-Betting-Min-Max (CB-Min-Max). We design our algorithm based on the decoupling of

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1The \( \tilde{O} \) notation hides poly-logarithmic terms.
the primal-dual structure and on the coin-betting technique in parameter-free online learning (Orabona & Pál, 2016; Cutkosky & Orabona, 2018a). Specifically, we run a constrained version of coin-betting algorithm on the primal and dual variables simultaneously.

Despite the algorithmic simplicity of our solution, there are two main difficulties in the analysis. First, the definition of the performance metric (e.g., duality gap defined in the LHS of (2)) makes the optimal point achieved in the metric explicitly dependent on the averaged solution of the algorithm, and this particular optimal point may be very different from the optimal solution of the original convex-concave min-max problem. Second, due to the discrepancy between the optimal point achieved in the duality gap and the optimal solution of the original problem, the traditional analysis of parameter-free algorithms does not give initialization-dependent bound with respect to the optimal solution, and hence it can only achieve the worst-case bound which depends on the size of the domain.

To address these issues, we show that the additional assumption of strict convexity/strict concavity of the objective function suffices to prove that our proposed algorithm achieves a convergence rate which explicitly depends on the distance between the initial point and the optimal solution. Several extensions under different settings are also analyzed.

Our contributions are summarized as follows.

- We design a parameter-free algorithm for solving a broad class of convex-concave problems. To the best of our knowledge, this is the first time that a parameter-free algorithm is proposed for min-max problems.
- We prove state-of-the-art convergence rates for the proposed algorithm under various settings, including strictly-convex-strictly-concave min-max problems, and min-max problems with non-euclidean geometry. We also design an algorithm based on the proposed parameter-free algorithm and obtain fast rates for convex-concave min-max problems with a growth condition. To the best of our knowledge, leveraging function growth condition to obtain these improved rates for min-max problems is the first shown in this work.
- We verify our theoretical results by conducting both synthetic experiments and distributionally robust optimization on benchmark datasets. We empirically show that our algorithm exhibits good performance in practice.

2. Related Work

Convex-Concave Min-max Optimization Convex-concave Min-max Optimization is widely studied in optimization literature, and it is closely related to the variational inequality. The work of (Korpelevich, 1976) proposed the extragradient method for solving variational inequalities, and this was later extended into non-euclidean space (e.g., mirror-prox (Nemirovski, 2004), dual extrapolation (Nesterov, 2007)). The stochastic version of mirror-prox was proposed by (Juditsky et al., 2011). Hsieh et al. (2020) proposed variable stepsize scaling for extragradient method to improve the algorithm’s performance. In nonsmooth case, Nemirovski et al. (2009) analyzed the primal-dual gradient method in non-euclidean space. (Nedić & Ozdaglar, 2009) considered subgradient methods for solving min-max problems and provided per-iteration convergence rate estimates on the solutions. Monteiro & Svaiter (2010) designed hybrid proximal extragradient methods with a different performance measure. Bach & Levy (2019) provided a universal algorithm for solving variational inequalities, which adapts to noise and smoothness.

There are several papers considering specific cases in convex-concave min-max optimization, including functions with a bilinear term (Nesterov, 2005; Chambolle & Pock, 2011; Chen et al., 2014; 2017; He & Monteiro, 2016; Liu et al., 2018; Daskalakis et al., 2018; Liang & Stokes, 2019; Gidel et al., 2019; Mokhtari et al., 2020; Azizian et al., 2020; Bailey et al., 2020), smooth or strongly-convex-(strongly)-concave (Nesterov & Scrimali, 2006; Zhao, 2019; Lin et al., 2020; Yan et al., 2020), the last-iterate convergence (Abernethy et al., 2019; Daskalakis & Panageas, 2018; Golowich et al., 2020). There are also some papers about establishing lower bounds in various cases (Ouyang et al., 2013; Zhang et al., 2019; Ibrahim et al., 2020).

However, none of these works provide an upper bound for duality gap which explicitly depends on the distance between the initialization and the optimal solution.

Parameter-Free Online Convex Optimization In Online Convex Optimization (OCO) (Gordon, 1999; Zinkevich, 2003), the aim of the learner is to minimize the regret w.r.t. any fixed predictor. Most of the OCO algorithm require some knowledge of the competitor, for example, its norm, in order to achieve the smallest regret (see, e.g., Orabona, 2019). Hence, it becomes impossible to compete uniformly with all competitors, unless the algorithm has some knowledge of the future. Morally speaking, the OCO setting is a strict generalization of the setting of stochastic optimization of convex functions and competing with any fixed predictor corresponds exactly to design convex optimization algorithms that have optimal dependency on the distance between the initial point and the optimal solution. Again, without some knowledge of the norm of the optimal solution, classic optimization algorithms fails to get the right dependency.
Parameter-free algorithms avoid setting step sizes completely and get the optimal dependency on the competitor up to poly-log terms (Streeter & McMahan, 2012). The core idea of these algorithms is to use Follow-the-Regularized-Leader\(^2\) (FTRL) (Shalev-Shwartz, 2007; Abernethy et al., 2008) with a time-varying linearithmic (non-strongly convex) regularizer (e.g., Streeter & McMahan, 2012; Orabona, 2014; Cutkosky & Boahen, 2016; 2017; Kotlowski, 2019; Kempka et al., 2019). These algorithms can also be viewed as betting schemes through the duality between regret and reward (McMahan & Orabona, 2014; Orabona & Pál, 2016; Cutkosky & Orabona, 2018a; Cutkosky & Sarlos, 2019).

As far as we know, this is the first application of parameter-free algorithms to convex-concave min-max problems.

3. Setting and Assumptions

Notation Denote \(\| \cdot \|\) by Euclidean norm. Define \((\cdot, \cdot)\) as the inner product in Euclidean space. Define II as the orthogonal projection operator. We denote vectors by bold letters, e.g., \(x, g\).

Setting and Assumptions As we said in the introduction, we are interested in the following optimization problem

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x, y) \tag{3}
\]

where \(\mathcal{X} \subseteq \mathbb{R}^m, \mathcal{Y} \subseteq \mathbb{R}^n\) are convex and compact sets.

We will assume that \(F\) is convex in the first argument and concave in the second one. Moreover, we assume to have access to a first-order black-box optimization oracle \(\tilde{g} = (\tilde{g}^x, \tilde{g}^y)\) at any point \((x, y)\). where \(\tilde{g}^x \in \partial_x F(x, y)\), the subgradient of \(F\) w.r.t. its first argument, and \(\tilde{g}^y \in \partial_y (-F(x, y))\), the subgradient of \(-F\) w.r.t. its second argument.

We will also assume that the subgradients \(\tilde{g}^x\) and \(\tilde{g}^y\) have bounded support. In particular, we assume w.l.o.g. that \(\|\tilde{g}^x\| \leq 1\) and \(\|\tilde{g}^y\| \leq 1\) for all \(x \in \mathcal{X}\) and \(y \in \mathcal{Y}\). Note that this assumption is necessary to use parameter-free algorithms (Orabona & Pál, 2016) and it is reasonable because we assume bounded domains.

4. Algorithm and Theoretical Analysis

4.1. Main Idea

The core idea of our approach is to decouple the primal problem and dual problem in (1), and by then utilize no-regret algorithms to perform the optimization. This is not a new idea by any means, see, for example, (Abernethy et al., 2018). However, we need to be particularly careful

\footnote{Dual Averaging (Nesterov, 2009) is a specialization of FTIR to linear functions.}

Algorithm 1 Constrained Coin-betting OCO

\begin{algorithm}
\caption{Constrained Coin-betting OCO}
\begin{algorithmic}[1]
\Input{Convex and compact feasible set \(\mathcal{X}\), initial point \(x_0 \in \mathcal{X}, \epsilon' > 0\)}
\State \(x_0 = x_0\)
\For {$t = 0, \ldots, T - 1$}
\State \(x_t = \Pi_x (\bar{x}_t), y_t = \Pi_y (\bar{y}_t)\)
\State Receive subgradients \(\tilde{g}^x_t\)
\State \(g^x_t = \frac{1}{t} (\tilde{g}^x_t + \|\tilde{g}^x_t\| \cdot \frac{\bar{x}_t - x_t}{\|\bar{x}_t - x_t\|})\) (Define \(0/0 = 0\))
\State \(\bar{x}_{t+1} = \frac{1}{t} \sum_{j=1}^{t} (\mathcal{g}^x_j, \bar{x}_j)\)
\EndFor
\State \Return \(\bar{x}_T = \frac{1}{T} \sum_{t=1}^{T} x_t\).
\end{algorithmic}
\end{algorithm}

of the dependency on the initial point. For this reason, we first introduce a Lemma, which is a simple application of Abernethy et al. (2018, Theorem 9).

Suppose to use an OCO algorithm fed with losses \(\ell_t(x) = F(x, y_t)\) that produces the iterates \(x_t\) and another OCO algorithm fed with losses \(h_t(y) = -F(x_t, y)\) that produces the iterates \(y_t\). Then, we can state the following Lemma (with detailed proof in Appendix A).

\begin{lemma}
Let \(\bar{x}_T = \frac{1}{T} \sum_{t=1}^{T} x_t, \bar{y}_T = \frac{1}{T} \sum_{t=1}^{T} y_t\). Then, we have
\[
\max_{y \in \mathcal{Y}} F(\bar{x}_T, y) - \min_{x \in \mathcal{X}} F(x, \bar{y}_T) \leq \frac{R_T(x'_T) + R_T(y'_T)}{T},
\]
\end{lemma}

where \(R_T(y) = \sum_{t=1}^{T} h_t(y_t) - \sum_{t=1}^{T} h_t(y), R_T(x) = \sum_{t=1}^{T} \ell_t(x_t) - \sum_{t=1}^{T} \ell_t(x), x'_T = \arg\min_{x \in \mathcal{X}} F(x, \bar{y}_T), \) and \(y'_T = \arg\max_{y \in \mathcal{Y}} F(\bar{x}_T, y)\).

In words, the above Lemma says that we can use two OCO algorithms to minimize the problem in (2). In particular, the convergence rate depends on the sum of the average regret of the two OCO algorithms. Note that different from similar results, here we care about considering the regret of the OCO algorithms w.r.t. specific points. This will be critical in our improved rates.

4.2. Algorithm Design and Analysis

Following the main idea in Section 4.1, our proposed algorithm is presented in Algorithm 2, which can be seen as simultaneous usage of Algorithm 1 on primal variable \(x\) and dual variable \(y\). Algorithm 1 is a parameter-free algorithm which works for constrained problems. The design of Algorithm 1 comes from the unconstrained coin-betting algorithm in (Orabona & Pál, 2016) and a black-box reduction from a constrained coin-betting algorithm to an unconstrained one (Cutkosky & Orabona, 2018b).

The idea is to first establish the regret bound of a unconstrained surrogate function \(\tilde{\ell}_t\) over the sequence \(\{\bar{x}_t\}\) as in Lemma 2, and then use the black-box reduction to get the regret bound
Algorithm 2 CB-Min-Max($\epsilon', x_0, y_0, T$)

**Input:** $\epsilon' > 0, x_0 \in \mathcal{X}, y_0 \in \mathcal{Y}$

1: $x_0 = x_0, y_0 = y_0$
2: for $t = 0, \ldots, T - 1$ do
3: $x_t = \Pi_{\mathcal{X}}(\bar{x}_t), y_t = \Pi_{\mathcal{Y}}(\bar{y}_t)$
4: Receive subgradients $\bar{g}_t = (\bar{g}_t^x, \bar{g}_t^y)$
5: $g_t^x = \frac{1}{2} \left( \bar{g}_t^x + \|\bar{g}_t^y\| \cdot \frac{\bar{x}_t - x_t}{\|\bar{x}_t - x_t\|} \right)$ (Define $0/0 = 0$)
6: $g_t^y = \frac{1}{2} \left( \bar{g}_t^y + \|\bar{g}_t^x\| \cdot \frac{\bar{y}_t - y_t}{\|\bar{y}_t - y_t\|} \right)$ (Define $0/0 = 0$)
7: $\bar{x}_{t+1} = -\sum_{j=1}^t g_j^x (\epsilon' - \sum_{j=1}^t \langle g_j^x, x_j \rangle)$
8: $\bar{y}_{t+1} = \sum_{j=1}^t g_j^y (\epsilon' + \sum_{j=1}^t \langle g_j^y, y_j \rangle)$
9: end for
10: **Return** $\bar{x}_T = \frac{1}{T} \sum_{t=1}^T x_t, \bar{y}_T = \frac{1}{T} \sum_{t=1}^T y_t$. 

**Corollary 1.** Algorithm 2 guarantees that

$$
\max_{y \in \mathcal{Y}} F(\bar{x}_T, y) - \min_{x \in \mathcal{X}} F(x, \bar{y}_T) \leq O \left( \frac{1}{T} \left( \frac{\|x_0 - x'_T\| \sqrt{\ln (1 + \|x_0 - x'_T\| T)}}{\sqrt{T}} + \frac{\|y_0 - y'_T\| \sqrt{\ln (1 + \|y_0 - y'_T\| T)}}{\sqrt{T}} \right) \right),
$$

where $x'_T \in \arg \min_{x \in \mathcal{X}} F(x, \bar{y}_T)$, and $y'_T \in \arg \max_{y \in \mathcal{Y}} F(\bar{x}_T, y)$. A parallel version of Lemma 3 in terms of $y$ is

$$
\frac{1}{T} \sum_{t=1}^T h_t(y_t) - \frac{1}{T} \sum_{t=1}^T h_t(y) \leq O \left( \frac{1}{T} \left( \frac{\|y_0 - y\| \sqrt{\ln (1 + \|y_0 - y\| T)}}{\sqrt{T}} \right) \right).
$$

**Proof.** Note that $h_t(y) = -F(x_t, y)$. Taking $x = x'_T \in \arg \min_{x \in \mathcal{X}} F(x, \bar{y}_T)$ in Lemma 3, $y = y'_T \in \arg \max_{y \in \mathcal{Y}} F(\bar{x}_T, y)$ in (7) and combining Lemma 2, we can prove the Corollary.

**Remark:** By Corollary 1, we are able to show that Algorithm 2 enjoys an $O \left( \frac{D_X \ln(D_X T) + D_Y \ln(D_Y T)}{\sqrt{T}} \right)$ convergence rate in terms of duality gap, where $D_X$ and $D_Y$ are the diameters of the domain $\mathcal{X}$ and $\mathcal{Y}$ respectively. The reason is that we do not know whether $x'_T, (y'_T)$ is close to $x_*, (y_*)$ or not, so we cannot get the convergence rate dependent on the distance between the initial point and the optimal solution. In this case, only the worst case bound (dependent on the diameter of the domain) is applicable.

To address this issue, we introduce one mild assumption which helps us obtain the desired bound with explicit dependence on the distance between the initialization and the optimal solution.

**In particular, we will use the following assumption:**

**Assumption 1.** $F(x, y)$ is strictly convex in $x$ and strictly concave in $y$ and the unique optimal point $(x_*, y_*)$ lies in the interior of $\mathcal{X} \times \mathcal{Y}$.

**Remark:** This assumption is crucial to get the desired bound due to the following reasons. First, it can guarantee that both $x'_T$ and $y'_T$ are unique, which avoid the complicated set-valued analysis. Second, we can utilize this assumption to show that $x'_T, (y'_T)$ gets close to $x_*, (y_*)$ when the algorithm runs a sufficiently large number of iterations.

We formalize these insights into the following key Lemma.
Lemma 4. Suppose Assumption 1 holds. Define \( x'_T = \arg\min_{x \in X} F(x, y_T) \). Then, for Algorithm 2 we have
\[
\frac{1}{T} \sum_{t=1}^{T} \ell_t(x_t) - \frac{1}{T} \sum_{t=1}^{T} \ell_t(x'_T) \leq O\left( \frac{1}{T} + \|x_0 - x_*\| \sqrt{\ln(1 + \|x_0 - x_*\|T)} \right).
\]

Proof. By Assumption 1, we can uniquely define both \( x_* \) and \( x'_T \) as the following: \( x_* = \arg\min_{x \in X} F(x, y_* \) and \( x'_T = \arg\min_{x \in X} F(x, y_T) \).

For brevity of notation, denote by \( C_T(x) = \frac{1}{T} + \frac{x \sqrt{\ln(1 + xT)}}{\sqrt{T}} \). Note that \( C_T(x) \) is non-decreasing in \( x \) and \( \lim_{T \to \infty} C_T(x) = 0 \) for any \( x \in \mathbb{R}_+ \).

Note that
\[
\frac{1}{T} \sum_{t=1}^{T} \ell_t(x_t) - \frac{1}{T} \sum_{t=1}^{T} \ell_t(x'_T) \leq O\left( C_T(||x_0 - x_*||) \right)
\]

By the convergence rate of the Algorithm 2 established in Corollary 1, we know that
\[
F(x_*, y_*) - F(x_*, y_T) \leq F(x_*, y_*) - \min_{x \in X} F(x, y_T)
\]
\[
\leq \max_{y \in Y} F(x_*, y_T) - \min_{x \in X} F(x, y_T)
\]
\[
\leq O\left( C_T(D_X) + C_T(D_Y) \right),
\]
where \( D_X \) is the diameter of \( X \) and \( D_Y \) is the diameter of \( Y \).

We now claim that \( y_T \to y_* \) when \( T \to \infty \). Let’s see why.

Note that \( F(x, y) \) is strictly concave in terms of \( y \). Define \( \bar{F}(y) = F(x_*, y) \). Taking the limit for \( T \to \infty \) in (9), for the sequence \( \{y_t\}_{i=1}^{\infty} \), we have \( \bar{F}(\bar{y}_t) \to \bar{F}(y_*) \). Given that the domain is bounded, we can extract a convergent subsequence \( \{\bar{y}_t\} \subset \{y_t\} \) and we assume that \( \bar{y}_t \to \bar{y} \).

By the continuity of \( \bar{F}(y) \) in terms of \( y \), we know that \( \bar{F}(\bar{y}_t) \to \bar{F}(\bar{y}) \). Now, \( \bar{F}(\bar{y}_t) \) is also a subsequence of the convergent sequence \( \bar{F}(\bar{y}_t) \), then \( \bar{F}(\bar{y}_t) \to \bar{F}(\bar{y}) \). Since \( y_* \) is uniquely defined, this implies that \( \bar{y} = y_* \). This means that any convergent subsequence of \( \{\bar{y}_t\}_{i=1}^{\infty} \) converges to \( y_* \), so \( y_T \to y_* \).

Our next claim is that the mapping \( \arg\min_{x \in X} F(x, y) \) is a continuous function in terms of \( y \).

First, define \( H(y) = \arg\min_{x \in X} F(x, y) \). By the compactness of \( Y \), we have a sequence \( y_k \to y_* \). Define \( x_k = H(y_k) \) and \( x_* = H(y_*) \). By the compactness of \( X \), there exists a convergent subsequence \( x_{k_i} \) of \( x_k \), and we denote its limit by \( \bar{x} \). From the above, we have \( F(x_{k_i}, y_k) \leq F(x, y_k) \) for all \( x \in X \), that implies \( F(\bar{x}, y_* \leq F(x, y_*) \) for any \( x \in X \). In particular, this implies \( F(\bar{x}, y_*) \leq F(x_*, y_*) \). By the uniqueness of the minimizer, we must have \( \bar{x} = x_* \). Given that any convergent subsequence converges to \( x_* \), this means that \( x_k \to x_* \) and hence \( \arg\min_{x \in X} F(x, y) \) is a continuous function in terms of \( y \).

Combining the above two claims, by the definition of \( x_* \) and \( x'_T \), we obtain that \( x'_T \to x_* \) when \( T \to \infty \). Hence \( \|x_* - x'_T\| = o(1) \). Plugging it in (8), we have
\[
\frac{1}{T} \sum_{t=1}^{T} \ell_t(x_t) - \frac{1}{T} \sum_{t=1}^{T} \ell_t(x'_T) \leq O\left( C_T(||x_0 - x_*||) \right).
\]

With Lemma 4 and a parallel argument in terms of \( y \), we are able to prove the following Theorem 1.

Theorem 1. Suppose Assumption 1 holds. Then, Algorithm 2 guarantees that
\[
\max_{y \in Y} F(x_T, y) - \min_{x \in X} F(x, y_T) \leq O\left( \frac{1}{T} + \frac{||x_0 - x_*|| \sqrt{\ln(1 + ||x_0 - x_*||T)}}{\sqrt{T}} \right)
\]
\[
+ \frac{||y_0 - y_*|| \sqrt{\ln(1 + ||y_0 - y_*||T)}}{\sqrt{T}}.
\]

Remark: Theorem 1 shows that Algorithm 2 enjoys a \( O\left( \frac{\text{dist}(x_0, x_*) + \text{dist}(y_0, y_*)}{\sqrt{T}} \right) \) convergence rate, where \( \text{dist}(a, b) = ||a - b|| \).

5. Extensions

In this section, we discuss two possible extensions of our Algorithm 2. In Section 5.1, we consider extending our algorithm to the non-Euclidean space. One particular interest is simplex setup, due to emerging applications in distributionally robust optimization (Namkoong & Duchi, 2016). In Section 5.2, we consider an extension of our algorithm when the function satisfies some growth conditions, in which we establish improved rates.

5.1. Non-Euclidean Space: Simplex Setup

Min-max optimization in Non-Euclidean spaces is an important topic, which has broad applications in machine learning (e.g., distributionally robust optimization). In this subsection, we focus on the simplex setup by considering the following problem \( \min_{x \in X} \max_{p \in \Delta_n} F(x, p), \) where
\[
\Delta_n = \left\{ (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \mid 0 \leq p_i \leq 1, \sum_{i=1}^{n} p_i = 1 \right\}.
\]
We first present two key Lemmas (Lemma 5 and Lemma 6) which is a synthesis of Algorithm 1 and Algorithm 3. It is Algorithm 3.

Remark 5.2. Fast Rates: Functions with Growth Condition

In this subsection, we consider the following assumptions.

Assumption 2. \( \|x - x^*\| \le c_1 (F(x, y^*) - F(x, y_\ast))^{\theta} \) and \( \|y - y^*\| \le c_2 (F(x^*, y^*) - F(x_\ast, y))^{\theta} \), where \( c_1 > 0, \ c_2 > 0, \ \theta < 1 \), \( (x_\ast, y_\ast) \) is the optimal solution of the original problem (1).

Remark: The Assumption 2 is a generalization of growth condition in minimization problems (Li, 2013; Yang & Lin, 2018).

Assumption 3. \( H_1(y) = \arg \min_{x \in \mathcal{X}} F(x, y) \) is a \( L_y \)-Lipschitz mapping, \( H_2(x) = \arg \max_{y \in \mathcal{Y}} F(x, y) \) is a \( L_x \)-Lipschitz mapping, i.e., \( \|H_1(y_1) - H_1(y_2)\| \le L_y \|y_1 - y_2\| \) for any \( y_1, y_2 \in \mathcal{Y} \), and \( \|H_2(x_1) - H_2(x_2)\| \le L_x \|x_1 - x_2\| \) for any \( x_1, x_2 \in \mathcal{X} \).
Algorithm 5 Restart-CB-Min-Max

Input: $x_0, y_0, \epsilon$
1: for $s = 1, \ldots, S$ do
2: \quad $(\tilde{x}_s, \tilde{y}_s) = \text{CB-Min-Max}(\epsilon', \tilde{x}_{s-1}, \tilde{y}_{s-1}, T_s)$
3: end for
4: Return $\tilde{x}_S, \tilde{y}_S$

Remark: Assumption 3 is closely related to the Aubin Property (Aubin, 1984), which is usually employed to characterize the Lipschitz behavior of solution set for convex optimization problems. Examples satisfying this assumption can be found in (Dontchev & Rockafellar, 2009) (e.g., Example 3B.6, Exercise 3C.5, etc.).

Under Assumption 2 and 3, we can design a restart version of the algorithm, which is presented in Algorithm 5. We will prove that Algorithm 5 enjoys faster convergence rate.

We first provide convergence guarantee for one-stage of Algorithm 5, which is presented in Theorem 3.

**Theorem 3.** Suppose Assumptions 2 and 3 hold. Running 2 for $T$ iterations yields

$$
\max_{y \in Y} F(\tilde{x}_T, y) - \min_{x \in X} F(x, \tilde{y}_T) 
\leq O \left( \frac{1}{T} + \frac{\text{ObjGap}^\theta(x_0, y_0) \ln T}{\sqrt{T}} \right),
$$

where $\text{ObjGap}(x_0, y_0) = \|(F(x_0, y_0) - F(x_*, y_*) + (F(x_0, y_0) - F(x_*, y_*)\right)$.

**Proof.** Following the proof of Lemma 4, from (8), we know that

$$
\frac{1}{T} \sum_{t=1}^{T} \ell_t(x_t) - \frac{1}{T} \sum_{t=1}^{T} \ell_t(x'_T) 
\leq O(C_T(\|x_0 - x_*\| + \|x_* - x'_T\|)),
$$

where $\ell_t(x) = F(x, y_t), C_T(x) = \frac{1}{T} + \frac{x}{\sqrt{T \ln(1+\theta^2T)}}$, $x'_T = \arg \min_x F(x, \tilde{y}_T)$. By the growth condition in Assumption 2, we know that

$$
\|x_0 - x_*\| \leq c_1 (F(x_0, y_*) - F(x_*, y_*))^\theta
$$

By the Lipschitz property of the argmin map, we have $\|x_0 - x'_T\| = \|\arg\min_x F(x, y_*) - \arg\min_x F(x, \tilde{y}_T)\| \leq L_Y \|y_T - y_*\|$. Note that we have $\|y_T - y_*\| \leq c_2 (F(x_0, y_*) - F(x_*, y_*))^\theta$, so we have

$$
\|x_0 - x'_T\| \leq c_2 L_Y (F(x_0, y_*) - F(x_*, y_*))^\theta
\leq c_2 L_Y \left( F(x_0, y_*) - \min_{x_0} F(x, \tilde{y}_T) \right)^\theta
\leq O \left( \frac{c_2 L_Y D^\theta}{\theta^2/2} \right),
$$

where the last inequality holds due to the convergence guarantee established in Corollary 1.

Combining (12), (13) and (14), we have

$$
\frac{1}{T} \sum_{t=1}^{T} \ell_t(x_t) - \frac{1}{T} \sum_{t=1}^{T} \ell_t(x'_T) 
\leq O \left( \frac{1}{T} + \frac{1}{T^{1+\theta}} + \frac{(F(x_0, y_*) - F(x_*, y_*))^\theta \ln(T)}{\sqrt{T}} \right),
$$

By a parallel argument in terms of $y$, we have

$$
\frac{1}{T} \sum_{t=1}^{T} h_t(y_t) - \frac{1}{T} \sum_{t=1}^{T} h_t(y'_T) 
\leq O \left( \frac{1}{T} + \frac{1}{T^{1+\theta}} + \frac{(F(x_*, y_0) - F(x_*, y_*))^\theta \ln(T)}{\sqrt{T}} \right),
$$

where $h_t(x) = -F(x, y_t), y'_T = \arg \max_y F(x_T, y)$. Combining (15) and (16), we know that

$$
\max_{y \in Y} F(\tilde{x}_T, y) - \min_{x \in X} F(x, \tilde{y}_T) 
\leq O \left( \frac{1}{T} + \frac{1}{T^{1+\theta}} + \frac{\text{ObjGap}^\theta(x_0, y_0) \ln T}{\sqrt{T}} \right),
$$

where $\text{ObjGap}(x_0, y_0) = \|(F(x_0, y_0) - F(x_*, y_*)) + (F(x_0, y_0) - F(x_*, y_*))\|$. $\Box$

We then introduce the Theorem 4, which illustrates the improved rate achieved by Algorithm 5.

**Theorem 4.** Suppose Assumptions 2 and 3 hold. Assume $\text{ObjGap}(\tilde{x}_0, \tilde{y}_0) = \|(F(\tilde{x}_0, \tilde{y}_0) - F(x_*, y_*) + (F(x_0, y_0) - F(x_*, y_*)) \leq \epsilon_0$. Define $\epsilon_s = \epsilon_0/2^s$. Run Algorithm 5 for $S = \lceil \log(\epsilon_0/\epsilon) \rceil$ stages with $T_s = \tilde{O}(\frac{1}{\epsilon^2})$, and we can guarantee that $\max_{x \in X} F(\tilde{x}_S, y) - \min_{x \in X} F(x, \tilde{y}_S) \leq \epsilon$. The total iteration complexity is $\tilde{O}(1/\epsilon^{2(1-\theta)})$.

**Proof.** Define $\text{DualityGap}(\tilde{x}_0, \tilde{y}_0) = \max_{x \in X} F(\tilde{x}_0, y) - \min_{x \in X} F(x, \tilde{y}_0)$. We know that $\text{ObjGap}(\tilde{x}_0, \tilde{y}_0) \leq \text{DualityGap}(\tilde{x}_0, \tilde{y}_0)$. By invoking the subroutine Algorithm 2 to run $T_0 = \tilde{O}(\frac{1}{\epsilon^2})$ iterations, we know that the duality gap at the new point will be decreased to $\epsilon_1 = \epsilon_0/2$. Then the Algorithm 5 restarts by setting the new point as the initial point, and then invokes the subroutine Algorithm 2 to run $T_1 = \tilde{O}(\frac{1}{\epsilon_1^2})$ number of iterations, and then it restarts again. Algorithm 5 repeats this process under it reaches $\epsilon$-duality gap. We know that we have $S = \lceil \log(\epsilon_0/\epsilon) \rceil$ stages and hence the total complexity is $\sum_{s=0}^{S} \tilde{O}(\frac{1}{\epsilon^2}) = \tilde{O}(1/\epsilon^{2(1-\theta)})$. $\Box$
6. Experiments

In this section, we conduct experiments to justify the effectiveness of our proposed algorithm. We consider two problems: one is a synthetic problem, and the another is distributionally robust optimization.

**Synthetic Problem** In the first experiment, we consider the following synthetic min-max problem:

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x, y) := \frac{\rho}{4} x^4 + xy - \frac{\rho}{4} y^4, \tag{18}
\]

where \( \mathcal{X} = \{ x \mid |x| \leq R_x \} \) and \( \mathcal{Y} = \{ y \mid |y| \leq R_y \} \), \( \rho, R_x, R_y \) are all positive constants. We can see that the function \( F \) in (18) is an strictly-convex-strictly-concave min-max problem, and the optimal solution is \((0, 0)\). In our experiment, we set \( R_x = R_y = 5 \), \( \rho = 0.5 \). We choose the same initial point \((x_0, y_0)\) for both primal-dual gradient method with and CB-Min-Max, and report the distance to the optimal solution versus the number of iterations. The learning rate of the primal dual gradient method is set to be \( \frac{D}{\sqrt{T}} \), where \( D \) is the diameter of the domain, \( G \) is the gradient’s upper bound and \( T \) is the number of iterations. For CB-Min-Max, we set all gradients in Algorithm 2 to be scaled by its upper bound \( G \) to make sure that the scaled gradient has norm smaller than 1. The experimental results are presented in Figure 1. It is observed that for various initialization points, Algorithm 2 consistently converges much faster than the primal-dual gradient method.

**Distributionally Robust Optimization** In the second experiment, we consider the following distributionally robust optimization problem:

\[
\min_{\|\mathbf{w}\| \leq R} \max_{n \in \Delta_n} \frac{1}{n} \sum_{i=1}^{n} p_i \ell_i(w) + \frac{\lambda}{2} \frac{\|\mathbf{p} - \frac{1}{n} \mathbf{1_n}\|^2}{n} + \frac{\rho}{2} \|\mathbf{w}\|^2, \tag{19}
\]

where \( \ell_i(w) = \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i) \) with \((\mathbf{x}_i, y_i) \in \mathbb{R}^d \times \{-1, +1\}\) being the feature-label pair and \( \mathbf{w} \in \mathbb{R}^m \) being the model parameter, \( \mathbf{p} \) is a probability vector, \( n \) is the number of training examples, \( \mathbf{1}_n = [1; 1; \ldots; 1] \) with length \( n \). In our experiment, we set \( R = 10^5 \), \( \lambda = \rho = 10^{-4} \). Because of the added regularizer, the function becomes strictly-convex-strictly-concave and hence satisfies our Assumption 1. We compare two algorithms: the primal-dual gradient descent (Nemirovski et al., 2009) and our Algorithm 4 (CB-Min-Max-Simplex). In this setup, primal-dual gradient descent method updates \( w \) by gradient descent and updates \( p \) by exponential gradient ascent simultaneously. The learning rates are set to be \( \frac{2 \sqrt{T}}{G_w \sqrt{T}} \) and \( \frac{\log(n)}{G_p} \) respectively for primal variable and dual variable, where \( G_w \) is the 2-norm of gradient in terms of \( w \), and \( G_p \) is infinity-norm of the gradient in terms of \( p \). \( T \) is the number of iterations. Both algorithms start from \( w_0 = 0 \), \( p_0 = \left[ \frac{1}{n}; \frac{1}{n}; \ldots; \frac{1}{n} \right] \) and run \( T = 1000 \) iterations (each iteration amounts to one pass of the training set). We test our algorithms on four benchmark datasets from libsvm website \(^3\) (SensIT Vehicle (combined), dna, gisette, protein). The detailed experimental settings can be found in the Appendix C. We report the training loss and test loss, as shown in the Figure 2. It can be observed that our proposed parameter-free algorithm (i.e., Algorithm 4) significantly outperforms the standard primal-dual gradient method.

7. Conclusion

In this paper, we design and analyze the first parameter-free algorithm for a class of convex-concave min-max problems. Various extensions are proposed, including min-max problems with non-Euclidean geometry and fast rate results for min-max problems with function growth condition. Experimental results show the superior performance of the proposed algorithm. In the future, we plan to study the lower bound for parameter-free algorithms in min-max problems.

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\(^3\)https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
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A. Proofs in Section 4

A.1. Proof of Lemma 1

Proof. Note that $\ell_t(x) = F(x, y_t)$, and $h_t(y) = -F(x_t, y)$. By Jensen’s inequality, we have

$$F(x_T, y) - F(x, y_T) \leq \frac{1}{T} \sum_{t=1}^{T} F(x_t, y) - \frac{1}{T} \sum_{t=1}^{T} F(x, y_t)$$

$$= \frac{1}{T} \sum_{t=1}^{T} F(x_t, y) - \frac{1}{T} \sum_{t=1}^{T} F(x_t, y_t)$$

$$+ \frac{1}{T} \sum_{t=1}^{T} F(x_t, y_t) - \frac{1}{T} \sum_{t=1}^{T} F(x, y_t)$$

$$= \frac{1}{T} \sum_{t=1}^{T} (h_t(y_t) - h_t(y)) + \frac{1}{T} \sum_{t=1}^{T} (\ell_t(x_t) - \ell_t(x)) .$$

Taking $x = x'_T \in \arg \min_{x \in X} F(x, y_T)$ and $y = y'_T \in \arg \max_{y \in Y} F(x_T, y)$, we get the stated result.

B. Proofs in Section 5.1

B.1. Proof of Lemma 5

Proof.

$$KL(p, \pi) = \sum_{i=1}^{d} p_i \ln \frac{p_i}{\pi_i} = \sum_{i=1}^{d} p_i \ln \frac{p_i}{q_i} + \sum_{i=1}^{d} p_i \ln \frac{q_i}{\pi_i}$$

$$= KL(q, \pi) + \sum_{i=1}^{d} q_i \ln \frac{q_i}{\pi_i} + \sum_{i=1}^{d} (p_i - q_i) \ln \frac{q_i}{\pi_i}$$

$$\leq KL(q, \pi) + \sum_{i=1}^{d} q_i \ln \frac{q_i}{\pi_i} + \max_{i} \ln (q_i/\pi_i, 0) \sum_{i=1}^{d} |p_i - q_i| .$$

B.2. Proof of Lemma 6

Proof. The proof follows Corollary 6 in (Orabona & Pál, 2016) and Jensen’s inequality in terms of $p$.

B.3. Proof of Theorem 2

Proof. By combining Lemma 6, Lemma 2 and Lemma 1, we can get the following bound

$$\max_{p \in \Delta_n} F(x_T, p) - \min_{x \in X} F(x, p_T) \leq O \left( \frac{1}{T} + \frac{\|x_0 - x_T\|}{\ln (1 + \|x_0 - x_*\|T)} \sqrt{T} \right)$$

$$+ \sqrt{\ln T + KL(p'_T, p_0)} .$$

(20)

where $x'_T = \arg \min_{x \in X} F(x, p_T)$ and $p'_T = \arg \max_{p \in \Delta_n} F(x_T, p)$. Note that (20) is the counterpart of Corollary 1 in the probability simplex setup. By Lemma 6 (taking $\pi = p_0$, $p = p'_T$ and $q = p_*$), we know that

$$KL(p'_T, p_0) \leq KL(p'_T, p_*)$$

$$+ \max_{i} \ln \frac{p_{*, i}}{p_{0, i}} \|p'_T - p_*\|_1 + KL(p_*, p_0).$$

(21)
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Figure 3. Comparison of different algorithms for the distributionally robust optimization problem (19) on letter, mnist, madelon, pendigits datasets. PDG stands for primal-dual gradient method, CB-Min-Max-Simplex stands for Algorithm 4.

Then, we are able to follow the proof of Lemma 4 to show that $KL(p_T', p_\star) \to 0$ (this implies that $\max_i \max (\ln p_{i,0} / p_{i,1}, 0) \|p_T' - p_\star\|_1 \to 0$ by Pinsker’s inequality). In addition, it is proved in Lemma 4 that $x_T' \to x_\star$. Combining these facts with (21), the theorem is proved.

C. Detailed Experimental Settings

In this section, we provide dataset preparation details for Section 6.

- SensIT Vehicle (combined): all the classes with label 2 and 3 are regarded as class $-1$, and the class 1 is regarded as class 1.
- dna: all the classes with label 2 and 3 are regarded as class $-1$, and the class 1 is regarded as class 1.
- gisette: we use the original dataset without any preprocessing.
- protein: all the classes with label 0 and 2 are regarded as class $-1$, and the class 1 is regarded as class 1.

D. More Experimental Results

In this section, we present more experimental results to verify that our proposed Algorithm 4 is better than primal-dual gradient method. We aim to solve the distributionally robust optimization problem (19) and the hyperparameter settings of each algorithm are the same as described in Section 6. We conduct our experiments on another 4 datasets from libsvm website. The dataset preparation is listed as the following:

- letter: all the classes with label 1 to label 25 are regarded as class $-1$, and the class 26 is regarded as class 1.
- mnist: all the classes with label 0 to label 8 are regarded as class $-1$, and the class 9 is regarded as class 1.
- madelon: we use the original dataset without any preprocessing.
- pendigits: all the classes with label 0 to label 8 are regarded as class $-1$, and the class 9 is regarded as class 1.

The experimental results are presented in Figure 3. It is observed that our proposed Algorithm 4 converges much faster then primal-dual gradient method.