Counting curves which move with threefolds

Herbert Clemens and Holger P. Kley

July 1998

Abstract

Let $X$ be a (possibly nodal) $K$-trivial threefold moving in a fixed ambient space $P$. Suppose $X$ contains a continuous family of curves, all of whose members satisfy certain unobstructedness conditions in $P$. A formula is given for computing the corresponding virtual number of curves, that is, the number of curves on a generic deformation of $X$ “contributed by” the continuous family on $X$.

0 Introduction

0.1 Suppose $X_0$ is a projective threefold with at worst ordinary node singularities which is embedded in a smooth projective manifold $P$, and that

$$Z_0 \subseteq X_0$$

is a connected curve. Assume that

$$H^1(Z, N_{Z/P}) = 0$$

for all $\{Z\} \in J'$, an open set in the Hilbert scheme of $P$, with $J'$ containing the connected component $I'$ of $\{Z_0\}$ in the Hilbert scheme of $X_0$. Suppose in addition that

$$\omega_{X_0} \otimes \mathcal{O}_Z \cong \mathcal{O}_Z$$

for all $\{Z\} \in I'$. Then the expected dimension of the set of curves $\{Z\} \in I'$ which deform to a generic deformation of $X_0$ is zero. The purpose of this paper is to compute the (virtual) number $\gamma(I')$ of such curves under certain additional assumptions.

We work in the setting in which $X_0$ is the zero-scheme of a regular section of a vector bundle on $P$ such that, via pull-back and push-forward, $I'$ is given as the zero-scheme of the associated section $\sigma_0$ of the associated bundle $V$ on $J'$. Then

$$\gamma(I') := \deg(c_{\text{top}}(V)),$$
which can be computed as the geometric intersection number of \( \sigma_0(J') \) with the zero-section \( z_V(J') \) of \( V \). Using Fulton-MacPherson intersection theory \(^3\), one rescales \( \sigma_0(J') \) by multiplication by larger and larger constants to create a “homotopy” between \( \sigma_0(J') \) and the normal cone \( C_{I' \setminus J'} \subset V|_{I'} \) of \( I' \) in \( J' \), so that

\[
\gamma(I') = z_V(J') \cdot C_{I' \setminus J'}
\]

can be calculated as an intersection product in \( V|_{U} \).

Under the above assumptions, we reinterpret the sheaf of obstructions to deformation as the sheaf of Kähler differentials on \( I' \) with logarithmic poles along the locus of curves passing through the nodes of \( X_0 \), thereby allowing the computation of \( \gamma(I') \) in terms of the geometry of \( I' \).

The results of the present paper do not guarantee that \( I' \) contributes \( \gamma(I') \) rigid curves to a general deformation \( X_t \) of \( X_0 \); a priori, the count \( \gamma(I') \) is purely virtual. In some cases, however, the precise structure of the obstruction sheaf does enable rigidity results; see \(^10\) and the remarks following example 4.3 of the present work.

Furthermore, our computations are local in that we work with a single connected component of the Hilbert scheme of \( X_0 \). The global (virtual) number of curves of given degree and genus is computed—at least in case \( g = 0, 1 \)—via Gromov-Witten invariants in the fundamental papers of Kontsevich \(^{13}\), Givental \(^{8, 7}\) and Lian, Liu, Yau \(^{15}\). Also, a symplectic treatment of Gromov-Witten invariants for nodal \( X_0 \) appears in \(^{14}\).

The paper is organized as follows: In \(^1\) we establish some general properties of Hilbert schemes, allowing \( X_0 \) to have arbitrary isolated singularities and \( Z \) and \( X_0 \) to be of arbitrary positive dimension. Let \( d = \dim Z \); then by Serre duality, \( H^d(N^\vee_{Z \setminus X_0} \otimes \omega_Z) \) is isomorphic to \( H^0(N_{Z \setminus X_0})^\vee \) which is the cotangent space to \( I' \) at \( \{ Z \} \). The key result—Lemma \(^{13}\)—establishes a relative form of this fact: \( \Omega^1_I \) is the dth higher direct image of the relative conormal tensor the relative dualizing sheaf. The assumptions and notation established in \(^1\) will be used throughout the work.

Then in \(^2\) we explore the case in which \( X_0 \) is a smooth threefold, \( Z \subset X_0 \) a curve, and \( \omega_{X_0} \otimes \omega_Z \cong \omega_X \). These assumptions give an adjunction isomorphism between \( H^1(N_{Z \setminus X_0}) \) and \( H^1(N_{Z \setminus X_0} \otimes \omega_Z) \), which in light of the results of \(^1\), gives an isomorphism between \( \Omega^1_I \) and the first higher direct image of the relative normal bundle; this is Proposition \(^2, 4\). Given the role of \( H^1(N_{Z \setminus X_0}) \) in the obstruction theory of the Hilbert scheme, the resulting formula for \( \gamma(I') \) in Corollary \(^4, 4\) should not be too surprising. The main technical difficulty arises from the failure of \( I' \) to be smooth in general.

In \(^3\), we extend the computation of \(^2\) to the case in which \( X_0 \) has ordinary nodes, which—at least for enumerative purposes—introduces logarithmic poles along the locus of curves passing through the nodes of \( X_0 \). We will assume that the generic curve parameterized by \( I' \) does not pass through the nodes of \( X_0 \) which enables us to use the technical results of the previous section.

Finally, in \(^4\), we illustrate our formulas with three examples.
0.2 Conventions and Notation All schemes are separated and of finite type over the field $\mathbb{C}$ of complex numbers. If $Z$ is a closed subscheme of $X$, we denote by $\mathcal{I}_{Z/X}$ the ideal sheaf, by $\text{Spec}(\bigoplus \mathcal{I}_{n}^{-1}\mathcal{O}_X) \to Z$ the normal cone of $Z$ in $X$, and by $\mathcal{N}_{Z/P} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_{Z/P}, \mathcal{O}_P) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}_{Z/X}/\mathcal{I}_{Z/X}^2, \mathcal{O}_Z)$ the normal sheaf of $Z$ in $X$. If $X$ is smooth, $\mathcal{T}_X := (\mathcal{O}_X^1)\wedge$ is the tangent sheaf of $X$. When appropriate, the notations $\omega_X$ or $\omega_{X/X'}$ refer to dualizing or relative dualizing sheaves, not canonical sheaves.

We generally pass between the notions of locally-free sheaf and vector bundle without comment, but we use roman type $E \to X$ to denote the geometric vector bundle associated to a locally-free $\mathcal{O}_X$-module $E$. In this case, $z_E : X \to E$ is the zero-section of $E$.

For flat families of schemes we use the notation $F \to F'$; that is, a prime (') indicates the base of a flat family.

Finally if $Z \hookrightarrow X$ is a closed embedding, we denote by $\{Z\}$ the corresponding point in the Hilbert scheme of $X$.

0.3 Acknowledgments We wish to express our gratitude to Y. Ruan for explaining to us his symplectic methods of computing Gromov-Witten invariants of Calabi-Yau threefolds with nodes, and to L. Ein for suggesting the connection with the log-complex. Those discussions and the the previous results of the second author for elliptic curves on Calabi-Yau complete intersections were the genesis for the results in this paper. We also thank the referee for several corrections and suggestions for improving the exposition.
of $s_0$ has isolated singularities. We require that $\mathcal{E}$ be sufficiently ample, in the sense that for all $\{Z\} \in J'$,

\begin{equation}
H^1(Z, \mathcal{E} \otimes \mathcal{O}_Z) = 0.
\end{equation}

Let $E \to P$ denote the geometric vector bundle associated to $\mathcal{E}$. There is a natural surjection

$$\mathcal{I}_{zE}(P) \to \mathcal{I}_{X_0}(P) \cong \mathcal{I}_{X_0}(P)/\mathcal{I}_2,$$

which restricts to give a sequence of isomorphisms:

\begin{equation}
\mathcal{E}^\vee \otimes \mathcal{O}_{X_0} \cong \left( \mathcal{I}_{z(P)}\mathcal{E}/\mathcal{I}_2\mathcal{E} \right) \otimes \mathcal{O}_{X_0} \\
\cong \mathcal{I}_{X_0}(P)/\mathcal{I}_2 \mathcal{E} \\
\cong \mathcal{I}_{X_0}(P)/\mathcal{I}_2 X_0 \\
\cong \mathcal{I}_{X_0}(P)/\mathcal{I}_2 X_0 \\
\cong \mathcal{I}_{X_0}(P)/\mathcal{I}_2 X_0 \\
\cong \mathcal{I}_{X_0}(P)/\mathcal{I}_2 X_0.
\end{equation}

1.3 Next consider the incidence scheme or universal family $J \subset J' \times P$ with projections

\begin{equation}
J \quad \xrightarrow{q} \quad P \\
\downarrow \quad p \quad \downarrow \quad p \\
J'
\end{equation}

Let

$$\mathcal{V} := p_* q^* \mathcal{E} \quad \text{and} \quad \sigma_0 := p_* q^* s_0.$$

Then because of (1.2.2) (see [11, Thm. 1.5]), $\mathcal{V}$ is locally free and there is a scheme-theoretic equality

\begin{equation}
I' := \text{Hilb}^{X_0}(J') = (\sigma_0 = 0).
\end{equation}

Notice that that by shrinking $J'$, we may assume $I'$ is connected.

1.4 For the remainder of this section, consider an arbitrary cartesian square

\begin{equation}
\begin{array}{ccc}
B & \xrightarrow{b} & J \\
\downarrow^{p^B} & & \downarrow^p \\
B' & \xrightarrow{b'} & J'
\end{array}
\end{equation}

where we allow $B'$ to be an object in the analytic category. Setting

$$A' := B' \times_{J'} I',$$
pulls back to a cartesian square
\[
\begin{array}{ccc}
A & \to & I \\
\downarrow^{p^A} & & \downarrow^{p^0} \\
A' & \to & I'
\end{array}
\]
Let
\[q_0 := q|_I : I \to X_0\]
be the second projection, and set
\[q_B := q \circ b \quad \text{and} \quad q_A := q_0 \circ a.
\]
In these situations, we often will need the ideal sheaves
\[J_B := I_B / B \times P \quad \text{and} \quad I_A := I_A / A' \times X_0.
\]
In the cases \(B' = J'\) (and \(A' = I'\)), we simplify to
\[J := J_J / J \times P \quad \text{and} \quad I := I_I / I \times X_0.
\]

\[\text{1.5 Lemma} \quad \text{There is an isomorphism of exact sequences of } \mathcal{O}_A\text{-modules:}
\]
\[
\begin{array}{ccccccccc}
0 & \to & q_B^* E^* \otimes \mathcal{O}_A & \to & b^* (I / I^2) \otimes \mathcal{O}_A & \to & a^* (J / J^2) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & q_B^* E^* \otimes \mathcal{O}_A & \to & (J_B / J_B^2) \otimes \mathcal{O}_A & \to & (J_A / J_A^2) & \to & 0
\end{array}
\]

\[\text{1.6 Proof: Let } Z \text{ be a fiber of } A \to A', \text{ which we identify with its image } Z \subset X_0. \text{ In light of (1.2.3), either left-hand arrow restricts on } Z \text{ to the standard morphism of conormal sheaves.}
\]
\[J_{X_0 \setminus P} / J_{X_0 \setminus P}^2 \otimes \mathcal{O}_Z \to J_{Z \setminus P} / J_{Z \setminus P}^2.
\]
We assumed that \(Z\) is a local complete intersection in the smooth variety \(P\), so away from the singularities of \(X_0\), this is injective. So, since \(\dim Z \geq 1\) and \(X_0\) has isolated singularities, it is generically injective. Suppose the kernel is supported at some \(z \in Z\).

Since \(J_{X_0 \setminus P} / J_{X_0 \setminus P}^2 \otimes \mathcal{O}_Z\) is locally free, we may restrict to an affine neighborhood \(\text{Spec } R\) of \(z\) over which the kernel corresponds to a non-zero submodule \(M \subset R^{\oplus k}\) of a free module and assume that \(M\) is annihilated by the maximal ideal \(m\) of \(z\). Then any projection of \(M\) to the various summands is annihilated by \(m\), and one of these must be non-zero, so is some non-zero ideal \(\pi(M) \subset R\) with proper support \(z\). Therefore \(m\) is an (embedded) associated prime of \(R\).
But $Z$, being a local complete intersection, is Cohen-Macaulay, and hence has no embedded components. (See, e.g., [16, Thm. 17.3].) We conclude that the morphism is everywhere injective.

Next observe that the natural map

$$I_{Z(P) \setminus E} \longrightarrow I_{X_0 \setminus s_0(P)} \cong I_{X_0 \setminus P}$$

pulls back to a surjection

$$q_0^* I_{Z(P) \setminus E} \longrightarrow q_0^* I_{X_0 \setminus P}$$

on $I' \times P$, whence an exact sequence

$$q_0^* I_{Z(P) \setminus E} \longrightarrow \mathcal{J} \longrightarrow I \longrightarrow 0$$

which remain exact after tensoring with $\mathcal{O}_Z$. Thus, both sequences restrict to exact sequences on each fiber of $p_A$, so by Nakayama’s lemma, they are exact.

Finally, the natural surjection $b'^* \mathcal{J} \rightarrow \mathcal{J}_B$ induces a surjection of the middle terms. Since both are locally free of the same rank, it must be an isomorphism. It follows that the natural map between the right-hand terms must also be an isomorphism.

1.7 Let $\omega := \omega_{A/A'}$ be the relative dualizing sheaf. The first two non-trivial sheaves in Lemma 1.5 are locally free $\mathcal{O}_A$-modules. Therefore, by Verdier duality,

$$(b')^* \mathcal{V} \cong \mathcal{O}_{A'} = \mathcal{H}om_{A'}(p_A^* q_A^* E, \mathcal{O}_{A'})$$

(1.7.1)

$$\cong R^d (p_A^* \circ \mathcal{H}om_A(\underline{\omega}, \mathcal{E})) (q_A^* E)$$

$$= R^d p_A^* (q_A^* E \otimes \omega)$$

and

(1.7.2)

$$(b')^* \Omega^1_{J'} \otimes \mathcal{O}_{A'} = \mathcal{H}om_{A'}((b')^* \mathcal{J}_P \otimes \mathcal{O}_{A'}, \mathcal{O}_{A'})$$

$$\cong R^d (p_A^* \circ \mathcal{H}om_A(\underline{\omega}, \mathcal{J})) (\mathcal{H}om_A (b^* (\mathcal{J}/\mathcal{J}^2) \otimes \mathcal{O}_A, \mathcal{O}_A))$$

$$= R^d p_A^* (b^* (\mathcal{J}/\mathcal{J}^2) \otimes \omega),$$

where we have used the infinitesimal properties of the Hilbert scheme to make the identification $p_A (b^* (\mathcal{J}/\mathcal{J}^2)) \cong \mathcal{J}_P$. We combine these calculations:

1.8 Lemma There is a commutative diagram

$$\begin{array}{ccc}
R^d p_A^* (q_B^* E \otimes \omega) & \xrightarrow{q_{P0}} & R^d p_A^* (b^* (\mathcal{J}/\mathcal{J}^2) \otimes \omega) \rightarrow R^d p_A^* (a^* (\mathcal{J}/\mathcal{J}^2) \otimes \omega) \rightarrow 0 \\
\downarrow l & & \downarrow l \downarrow l \\
(b')^* \mathcal{V} \otimes \mathcal{O}_A & \xrightarrow{(b')^* \sigma_0} & (b')^* \Omega^1_{J'} \otimes \mathcal{O}_A \rightarrow (a')^* \Omega^1_{J'} \rightarrow 0
\end{array}$$

with exact rows and all vertical maps isomorphisms.
1.9 Proof: Exactness of the top row results from Lemma 1.5 after applying $\mathbb{R}^d p_A^*$ (which preserves right exactness because the fiber-dimension of $p_A^*$ is $d$).

Next note that since $I'$ is the zero-scheme of $\sigma_0$, there is a surjection
$$V^\lor \otimes \mathcal{O}_{I'} \rightarrow I' - \rightarrow \mathcal{I}_I' \rightarrow \Omega^1_{\mathcal{I}_I'} - \rightarrow 0$$
and hence, using the standard exact sequence of differentials, a commutative diagram with exact rows and columns:

$$
\begin{array}{c c c c}
V^\lor \otimes \mathcal{O}_{I'} & \rightarrow & \Omega^1_{\mathcal{I}_I'} \otimes \mathcal{O}_{I'} & \rightarrow & \Omega^1_{I'} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{I}_I' \otimes \Omega^1_{\mathcal{I}_I'} & \rightarrow & \Omega^1_{\mathcal{I}_I'} & \rightarrow & \Omega^1_{I'} & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & 0
\end{array}
$$

(1.9.1)

Now the exactness of the bottom row of the Lemma follows by pull-back to $A'$.

Next the left two vertical isomorphism are (1.7.1) and (1.7.2), and commutativity of the left-hand square follows via duality, Nakayama’s lemma, and pull-back to $A'$ from [11, Prop. 1.6]. Finally, a diagram chase establishes the existence of the third vertical isomorphism and the commutativity of the right-hand square.

1.10 For the derived functors associated to the functor

$$(1.10.1) \ T_A := p_A^* \circ \mathcal{H}om_A(\_ , \mathcal{O}_A)$$

we have by (1.1.2) that
$$\mathbb{R}^1 T_A (b^* (\mathcal{I} / \mathcal{I}^2) \otimes \mathcal{O}_A) = 0.$$ 

So from Lemma 1.5 we obtain the exact sequence

$$(1.10.2) \ 0 \rightarrow T_A (a^* (\mathcal{I} / \mathcal{I}^2)) \rightarrow (b')^* \mathcal{I}_{I'} \otimes \mathcal{O}_{A'} \rightarrow (b')^* V \otimes \mathcal{O}_{A'} \rightarrow \mathbb{R}^1 T_A (a^* (\mathcal{I} / \mathcal{I}^2)) \rightarrow 0.$$ 

We will see that, when $d = 1$, the sheaf $\mathbb{R}^1 T_A (\mathcal{I} / \mathcal{I}^2)$ measures the obstruction to moving the curves $\{Z\} \in I'$ when the regular section $s_0$ of $\mathcal{E}$ is deformed to a generic section $s_t$.

1.11 Let

$$V := \text{Spec} (\text{Sym}^* (V^\lor))$$

be the geometric vector bundle associated to $V$. Define the normal cone to $I'$ in $J'$ to be

$$C_{I' \setminus J'} := \text{Spec} \left( \bigoplus_{r=0}^{\infty} \mathcal{O}_{J'_{I'} / J'_{I'}} \right)$$
There is a canonical surjection

\[(b')^* (\mathcal{J}_{I',J'}) \to \mathcal{J}_{A',B'},\]

so from the left-hand square of (1.9.1), we obtain a commutative diagram of \(\mathcal{O}_{A'}\)-algebras

\[
\begin{array}{ccc}
\text{Sym}^* ((b')^* \mathcal{V} \otimes \mathcal{O}_{A'}) & \longrightarrow & \text{Sym}^* ((b')^* \Omega^1_{J'} \otimes \mathcal{O}_{A'}) \\
\bigoplus_{r=0}^{\infty} \left( J^r_{I' \setminus J'}/J^{r+1}_{I' \setminus J'} \right) & \text{−−−−→} & \text{Sym}^* ((b')^* \Omega^1_{J'} \otimes \mathcal{O}_{A'}) \\
\bigoplus_{r=0}^{\infty} \left( J^r_{A' \setminus B'}/J^{r+1}_{A' \setminus B'} \right) & \text{−−−−→} & \text{Sym}^* ((b')^* \Omega^1_{J'} \otimes \mathcal{O}_{A'})
\end{array}
\]

whence a commutative diagram of morphisms of cones over \(A'\):

\[
\begin{array}{ccc}
C_{A' \setminus B'} & \longrightarrow & A' \times \mathcal{V} C_{J' \setminus J'} \\
\bigoplus_{r=0}^{\infty} \left( J^r_{A' \setminus B'}/J^{r+1}_{A' \setminus B'} \right) & \text{−−−−→} & \bigoplus_{r=0}^{\infty} \left( J^r_{A' \setminus B'}/J^{r+1}_{A' \setminus B'} \right)
\end{array}
\]

(See [3, Ch. 4] and [2, §1].) Note that the left-hand vertical arrows in (1.11.1) are surjective, so that the right-hand vertical arrows in (1.11.2) are closed embeddings.

1.12 Now (1.3.1) is precisely the statement that the square

\[
\begin{array}{ccc}
I' & \longrightarrow & J' \\
\downarrow i & & \downarrow \sigma_0 \\
J' & \longrightarrow & V
\end{array}
\]

is Cartesian. Then we define the class

\[
\gamma(I') := Z(\sigma_0) := z^r_J[J'] \in \text{CH}_*(I')
\]

to be the localized top Chern class of [8, §14.1], where it is shown that

\[
i_* \gamma(I') = c_{top}(V) \cap [J'] \in \text{CH}_*(J'),
\]

i.e., that this class represents the top Chern class of \(V\). In the language of [2], this class is the virtual fundamental class

\[
\gamma(I') = [I', F^*],
\]
where $F^*$ is the complex $[V^\gamma|_I \to \Omega^1_{V^\gamma}]$ (in degrees $-1$ and $0$); see ‘the basic example’ in [loc. cit., §6]. See also [1, §1]. We remark that although the definition of $\gamma(I')$ in no way depends on the completeness of $I'$, its enumerative significance does.

2 Smooth threefolds

2.1 It is at this point that we make our final assumptions, namely assume:

\[(2.1.1)\quad d = \dim Z = 1\]

so that in particular, since $E \otimes \mathcal{O}_X \cong N_{X_0/P}$, (1.1.2) implies (1.2.2). Furthermore, we assume that

\[(2.1.2)\quad \dim X_0 = 3\]

and that for all $\{Z\} \in I'$,

\[(2.1.3)\quad \omega_{X_0} \otimes \mathcal{O}_Z \cong \mathcal{O}_Z \quad \text{and} \quad h^0(\mathcal{O}_Z) = 1.\]

Finally, we assume that

\[(2.1.4)\quad X_0 \text{ is smooth,}\]

although this will be weakened in §3.

2.2 From (2.1.3) we immediately have

\[(2.2.1)\quad p_0^*q_0^*(\omega_{X_0}) =: \mathcal{K}\]

for some line bundle $\mathcal{K}$ on $I'$, and we set

$$\mathcal{K}_A' := (a')^*\mathcal{K} \sim p_A^*q_A^*(\omega_{X_0}).$$

Using (2.1.1), (2.1.4), (1.1.2) and Riemann-Roch, we compute:

\[\dim I' = \chi(N_{Z/P}) = \chi(N_{Z/X_0}) + \chi(E \otimes \mathcal{O}_Z) = \deg \omega_Z + 2(1 - g_a(Z)) + \text{rk} V = \text{rk} V.\]

Consequently,

$$\gamma(I') \in \text{CH}_0(I').$$
2.3 Now \((2.1.1), (2.1.2),\) and \((2.1.4)\) imply that \((\mathcal{J}/\mathcal{J}^2)\) is locally free of rank two, so the relative adjunction isomorphism

\[\omega_{A/A'} \otimes a^* \Lambda^2 (\mathcal{J}/\mathcal{J}^2) \cong q^*_A(\omega_{X_0})\]

induces an isomorphism:

\[
\omega_{A/A'} \otimes (\mathcal{J}_A/\mathcal{J}_A^2) \cong \text{Hom}_A((\mathcal{J}_A/\mathcal{J}_A^2), \omega_{A/A'} \otimes \Lambda^2 (\mathcal{J}_A/\mathcal{J}_A^2))
\]

\[\cong \text{Hom}_A((\mathcal{J}_A/\mathcal{J}_A^2), q^*_A(\omega_{X_0}))
\]

\[\cong \text{Hom}_A((\mathcal{J}_A/\mathcal{J}_A^2), (p^A)^*(\mathcal{K}_{A'})).\]

Thus, by the projection formula and Lemma 1.8, Proposition

2.4 Proposition There is an isomorphism

\[R^1 p_A^*(\text{Hom}_A((\mathcal{J}_A/\mathcal{J}_A^2), \mathcal{O}_A)) \otimes \mathcal{K}_A \cong (a')^* \Omega^1_{J'}.\]

2.5 A further consequence of the local freeness of \((\mathcal{J}_A/\mathcal{J}_A^2)\) is that we have an isomorphism of derived functors

\[R^p A((\mathcal{J}_A/\mathcal{J}_A^2), \mathcal{O}_A)) \overset{\cong}{\rightarrow} R^p \mathcal{J}_A/\mathcal{J}_A^2),\]

which, when combined with \((1.10.2)\) and Proposition 2.4, yields the exact sequence

\[(2.5.2)\quad (b')^* \mathcal{T}_{J'} \otimes \mathcal{O}_{A'} \rightarrow (b')^* \mathcal{V} \otimes \mathcal{O}_{A'} \overset{\delta}{\rightarrow} \text{Hom}_{A'}(\mathcal{K}_{A'}, (a')^* \Omega^1_{J'}) \rightarrow 0.\]

2.6 Let \(\mathfrak{C}\) be the set of components of \(C_{I' \setminus J'}\). For \(C \in \mathfrak{C}\), let \(m(C)\) be the geometric multiplicity of \(C\) in \(C_{I' \setminus J'}\), \(S' = S'(C) \subset I'\) its support, \(p = p(C) \in S' \subset I'\) the generic point of \(S'\), \(k = k(S')\) the function field of \(S'\), and \(C_p := C \times_{I'} p\) the fiber of \(C\) over \(p\). Following our conventions, \(p^S: S \rightarrow S'\) is the pullback to \(S'\) of \(p^0: I \rightarrow I'\).

Now the surjective sheaf morphisms

\[
\mathcal{V} \otimes \mathcal{O}_{I'} \rightarrow \text{Hom}_{I'}(\mathcal{K}, \Omega^1_{I'}) \rightarrow \text{Hom}_{I'}(\mathcal{K}, \Omega^1_{S'}).\]

give rise a surjective composition

\[
\mathcal{V} \otimes k \rightarrow \text{Hom}_k(\mathcal{K} \otimes k, \Omega^1_{I'}) \rightarrow \text{Hom}_k(\mathcal{K} \otimes k, \Omega^1_{S'} \otimes k)
\]

of maps of \(k\)-vector spaces, which we can view as a morphism

\[(2.6.1)\quad V|_p \rightarrow (T^\vee_{S'} \otimes K^{-1}_{S'})|_p\]

of varieties over \(k\).

2.7 Lemma Assume that \(p^S\) is generically smooth. Then given \(\kappa \in H^0(\omega_{X_0}),\) the composition

\[
C_p \xrightarrow{(1.11.2)} V|_p \xrightarrow{(2.6.1)} (T^\vee_{S'} \otimes K^{-1}_{S'})|_p \xrightarrow{\text{gp} \cdot q^*_p(\kappa)} T^\vee_{S'}|_p
\]

is the constant map to zero. Furthermore, if \(p_0^0 q^*_p(\kappa)\) does not vanish at \(p\), then \(C_p\) is the geometric kernel of the vector-space surjection \(V|_p \rightarrow (T^\vee_{S'} \otimes K^{-1}_{S'})|_p\).
2.8 Proof: The second assertion follows from the first since (2.6.1) is surjective and
\[ \dim_k C_p + \dim S' = \dim J' = \text{rk} V. \]

To prove the first assertion, let \( \pi': \tilde{J}' \to J' \) be the blow-up of \( J' \) along \( I' \). Then the epimorphism
\[ \bigoplus_{j=0}^{\infty} J^j_{I' \setminus J'} \longrightarrow \bigoplus_{j=1}^{\infty} \left( J^j_{I' \setminus J'}/J^{j+1}_{I' \setminus J'} \right) \]
of graded \( O_{J'} \)-algebras induces—via \( \text{Proj} \)— a closed embedding
\[ P(C_{I' \setminus J'}) \hookrightarrow \tilde{J}' \]
over \( I' \hookrightarrow J' \), with (scheme-theoretic) image the exceptional divisor.

Now let \( U \subset S' \) be a small analytic neighborhood of a general point in \( S' \) and let \( R \) be any line bundle over \( U \) which is a sub-cone of \( C|_U \); in other words, \( R \) is a ray in \( C|_U \). Now such an \( R \) determines (and is determined by) a section
\[ \phi: U \to P(C)|_U \]
of the projection \( P(C)|_U \to U \). Let \( 0 \in \Delta \) be a one-dimensional disk with parameter \( t \). Set
\[ Y' := \Delta \times U. \]
Shrinking \( U \) as necessary, we can construct an embedding
\[ g' = g'_R: Y' \to \tilde{J}' \]
such that
\[ g'(0, u) = \phi(u) \quad \text{for all } u \in U, \]
and
\[ g'(\Delta \times U) \subset J' \setminus I' \subset \tilde{J}'. \]

Let \( f := \pi' \circ g': Y' \to J' \), and consider the scheme
\[ W' := (b \circ f)^{-1}(I'). \]
Then \( W' \cong \text{Spec} \left( \mathbb{C}[t]/(t^m) \right) \times U \), where \( m = m(C) \). Observe that under the canonical identification \( U \cong \{0\} \times U = W'_\text{red} \), we have isomorphisms
\[ R \cong (C_{W' \setminus Y'})_{\text{red}} \cong C_{W' \setminus Y'} \times_{W'} W'_{\text{red}}. \]
Keeping the notation of 1.4, we have a fiber square

\[
\begin{array}{ccc}
W := W' \times_{I'} I & \xrightarrow{w} & I \\
| & | & | \\
W' & \xrightarrow{w'} & I'
\end{array}
\]

and we set \( J_W := J_{W' \times I' \times X_a} \). Then applying the functoriality in Lemma 1.5—once with \( B' = Y' \), once with \( B' = W'_\text{red} \)—we see that the exact sequence

\[
(f')^* \mathcal{T}_{J'} \otimes \mathcal{O}_{W'} \xrightarrow{\delta} \mathcal{H}om_{W'}(\mathcal{K}_{W'}, (w')^* \Omega_{I'}) \rightarrow 0.
\]

of (2.5.2) restricts on \( W'_\text{red} \cong U \) to

\[
\begin{array}{ccc}
\mathcal{T}_{J'} \otimes \mathcal{O}_U & \xrightarrow{\delta} & \mathcal{H}om_U(\mathcal{K}_U, \Omega_{I'} \otimes \mathcal{O}_U) \\
| & | & | \\
\mathcal{T}_{J'} \otimes \mathcal{O}_U & \xrightarrow{\delta} & \mathcal{H}om_U(\mathcal{K}_U, \Omega_{I'} \otimes \mathcal{O}_U) \rightarrow 0.
\end{array}
\]

Let \( \mathcal{R} \) be the sheaf of sections of \( R \). Then starting with (2.8.1), we have a sequence of morphisms

\[
\mathcal{R} \cong C_{W' \times I' \times W'} U \xrightarrow{[1.1.2]} \mathcal{V} \otimes \mathcal{O}_U \xrightarrow{\delta} R^1_{p*}(N_{U \times I' \times X_a}) \xrightarrow{\text{rest.}} \mathcal{H}om_U(\mathcal{K}_U, \Omega_{I'_U}) \xrightarrow{p^*q^* \kappa} \Omega_{I'_U}.
\]

Then the image of \( \Gamma(\mathcal{R}) \) in \( \Gamma \left( R^1_{p*}(N_{U \times I' \times X_a}) \right) \) consists of the obstructions to extending the family \( W \rightarrow W' \) to a family of subschemes of \( X_0 \) over the base \( \text{Spec} \mathbb{C}[t]/(t^{m+1}) \times U \), so that in the language of [4, Ch. 2], this image contains only sections associated to the normal cone. Essentially, these are the obstructions to curvilinear deformation. Shrinking \( U \) as needed, Cor. 2.6 of [loc.cit.] with \( r = d = 1 \) states that the composition

\[
\mathcal{R} \rightarrow \mathcal{V} \otimes \mathcal{O}_U \rightarrow \Omega_{I'_U}
\]

is zero. (It is here that the assumption on the generic smoothness of \( p^S \) becomes necessary.) But by construction, the map \( R \rightarrow V|_U \) factors through \( C \), and since \( U \subset S' \) is general and \( R \) is an arbitrary ray in \( C \), the Lemma follows.

\[
\begin{array}{c}
\kappa_S \in H^0(\omega_{X_a})
\end{array}
\]

such that \( (p_0)_* q_0^*(\kappa_S) \) does not vanish identically on \( S' \). Then

\[
\gamma(I') = \sum_{C \in \mathcal{E}} m(C) c_{\text{top}} \left( \mathcal{K}_{S'}^{-1} \otimes \Omega_S^{1}(C) \right) \cap [S'(C)].
\]
2.10 Proof: By the excess intersection formula and the linearity of the intersection product,

\[ \gamma(I') = \sum_{C \in \mathcal{C}} m(C) \deg(z_V^I[C]). \]

Now since each \( S' \) is smooth, Lemma 2.7 and the surjectivity of (2.6.1) imply, by dimension, that there is an exact sequence of vector bundles over \( S' \)

\[ 0 \longrightarrow C \longrightarrow V|_{S'} \longrightarrow T_{S'}^\vee \otimes K_{S'}^{-1} \longrightarrow 0 \]

so that

\[ z_V^I[C] = c_{\text{top}}(\Omega^1_{S'} \otimes K_{S'}^{-1}) \cap [S']. \]

The desired formula follows immediately. ♣

3 Threefolds with nodes

3.1 In this section, we make the same assumptions as in 1.1, 1.2 and 2.1, except that we weaken (2.1.4) and assume instead that

(3.1.1) the singularities of \( X_0 \) are a set \( \Xi \) of ordinary double points.

Moreover, we require that

\[ \text{Obs}(Z, X_0) \subseteq H^1 \left( Z, \left( \mathcal{J}_{Z \setminus X_0}/\mathcal{J}_{Z \setminus X_0}^2 \right)^\vee \right) \]

for every curve \( Z = q_0(I_{y'}) \) with \( y' \in I' \), where

\[ \text{Obs}(Z, X_0) \subseteq \text{Ext}^1_Z \left( \mathcal{J}_{Z \setminus X_0}/\mathcal{J}_{Z \setminus X_0}^2, \mathcal{O}_Z \right) \]

is the space of obstructions generated by (obstructed) curvilinear deformations of \( Z \) in \( X_0 \). (See [12, p. 29ff.]) Note that away from points \( y' \in I' \) which represent curves passing through nodes of \( X_0 \), the techniques of §2 apply. In particular, away from such points, the reduction to the use of the curvilinear obstruction space in the following proceeds just as in the proof of Lemma 2.7.

3.2 We continue to use the notation established in 1.4 and 2.6. For each \( C \in \mathcal{C} \), assume

(3.2.1) The support \( S' = S'(C) \) is smooth and the morphism

\[ p^S: S := S' \times_I I \rightarrow S' \]

is generically smooth.
(3.2.2) There is a section
\[ \kappa_S \in H^0(X_0, \omega_{X_0}) \]
such that \( p_S^* q_S^*(\kappa_S) \) does not vanish identically on \( S' \).

(3.2.3) If
\[ q_S(S) \cap \Xi \neq \emptyset, \]
then \( X_0 \) contains a surface \( Y_S \), smooth at
\[ Y_S \cap \Xi =: \{ x_1^S, \ldots, x_r^S(S) \} \]
such that
\[ q_S(S) \subseteq Y_S. \]

(3.2.4) The scheme
\[ D_{S'}^i = p^S(q_S^{-1}(x_i^S)) \]
is smooth divisor in \( S' \) for all \( i \).

(3.2.5) Either
1. \( D_{S'} := \sum_{i=1}^r D_{S'}^i \) is a normal-crossing divisor on \( S' \), or
2. \( I' \) is smooth connected, so that \( C = \{ C \} \) with \( S'(C) = I' \).

3.3 Theorem Under assumptions \([3.2.1]–[3.2.5]\)
\[ \gamma(I') = \sum_{C \in \mathcal{E}} m(C) \left( c_{\text{top}} \left( \Omega_{S'(C)} \right) \cap [S'(C)] \right), \]
where each \( \Omega_{S'} \) is a locally free sheaf which is an extension
\[ 0 \rightarrow X_{S'}^{-1} \otimes \Omega_{S'}^1 \rightarrow \Omega_{S'} \rightarrow \bigoplus_{i=1}^{r(S)} \mathcal{O}_{D_{S'}^i} \rightarrow 0, \]

3.4 Proof: The calculation of the contributions of components \( C \) such that
\[ q_S(S) \cap \Xi = \emptyset, \]
is just as in Corollary \([2.3]\). So by \([3.2.3]\), we need only treat the components of \( C \) for which all curves of \( S = S(C) \) lie in a surface \( Y_S \subseteq X_0 \). For such a \( C \), and \( x_S^z \in Y_S \cap \Xi \), let \( Z \rightarrow X_0 \) be a fiber of \( S \) over a point of \( D_{S'}^1 \); that is, \( Z \) is a curve parameterized by \( S' \) which passes through the node \( x_S^z \) of \( X_0 \). Since the
embedding dimension of our threefold singularity is four, we can choose local generators
\[ \{a_1, a_2\} \cup \{b_1, \ldots, b_{\dim P - 4}\} \]
for the ideal \( J_{Y_S \setminus P} \) near \( x^*_S \) in such a way that the \( b_k \) locally generate the ideal of a smooth fourfold containing \( X_0 \).

By Lemma 1.5, there is an exact sequence
\[
0 \longrightarrow q^* \mathcal{E} \otimes \mathcal{O}_S \xrightarrow{q^*s_0} J/J^2 \otimes \mathcal{O}_S \longrightarrow J/S \otimes \mathcal{O}_S \longrightarrow 0
\]
whence the exact sequence
\[
0 \longrightarrow \mathcal{H}om_S (J/S, \mathcal{O}_S) \longrightarrow \mathcal{H}om_S (J/S, \mathcal{O}_S) \longrightarrow q^*_S \mathcal{E} \longrightarrow \mathcal{E}xt^1_S (J/S, \mathcal{O}_S) \longrightarrow 0.
\]

Now since \( X_0 \) is the zero-scheme of \( s_0 \), there is some trivialization of \( \mathcal{E} \) near \( x^*_S \) with respect to which the local expression for \( s_0 \) is
\[ s_0 = (a_1 f_1 + a_2 f_2, b_1, \ldots, b_{\dim P - 4}), \]
with \( \{f_1, f_2\} \) cutting out the point \( x^*_S \) in \( Y_S \). So under the map
\[ \mathcal{H}om_Z \left( J/Z, J/Z \otimes \mathcal{O}_Z, \mathcal{O}_Z \right) \longrightarrow \mathcal{E} \otimes \mathcal{O}_Z, \]
the images of the \( (\dim P - 4) \) homomorphisms
\[ b_j \mapsto \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \]
generate a codimension one subspace of the vector space \( E|_{x^*_S} \). We denote the one-dimensional quotient as \( C_i \).

Now the mapping
\[ \mathcal{H}om_S (J/S, \mathcal{O}_S) \longrightarrow q^*_S \mathcal{E} \]
is surjective away from \( q^{-1}S(\Xi) \), and since the generators \( a_1 \) and \( a_2 \) must give sections of \( \mathcal{E} \otimes \mathcal{O}_Z \) which vanish at \( x^*_S \), we conclude from (3.4.2) that there is a natural isomorphism
\[
\bigoplus_i q^*_S C_i \sim \mathcal{E}xt^1_S (J/S, \mathcal{O}_S).
\]

Recall that we have a functor
\[ T^S := p_S^* \circ \mathcal{H}om_S (\_, \mathcal{O}_S). \]
Applying $RT^S$ to the sequence \((3.4.1)\), we obtain an exact sequence
\[
0 \rightarrow T^S (I_S/I^2_S) \rightarrow T_S' \otimes O_S' \rightarrow V \otimes O_S' \rightarrow R^1T^S (I_S/I^2_S) \rightarrow 0.
\]

Because the fiber dimension of $p^S$ is one, the Grothendieck spectral sequence for $T^S$ degenerates at $E_2$. We therefore obtain an exact sequence:
\[
(3.4.5) \quad 0 \rightarrow R^1p^S_* (\mathcal{H}om_S (I_S/I^2_S, O_S)) \rightarrow R^1T^S (I_S/I^2_S) \rightarrow p^S_* \mathcal{E}xt^1_S (I_S/I^2_S, O_S) \rightarrow 0.
\]

Thus, we have the commutative diagram of $O_{S'}$-modules

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{W}_{S'} & \longrightarrow & R^1p^S_* (\mathcal{H}om_S (I_S/I^2_S, O_S)) \longrightarrow 0 \\
\downarrow & & \downarrow \\
\mathcal{V} \otimes O_{S'} & \longrightarrow & R^1T^S (I_S/I^2_S) \longrightarrow 0 \\
\downarrow & & \downarrow \\
\bigoplus_i O_{D_{i,S'}} & \longrightarrow & p^S_* \mathcal{E}xt^1_S (I_S/I^2_S, O_S) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

with exact rows and columns (where $\mathcal{W}_{S'} := \ker(\varepsilon)$).

Let $y' \in S'$; we have a map
\[
\varepsilon_{y'} : V_{y'} := (V/m_{y',S'} V) \rightarrow \bigoplus_i \left( O_{D_{i,S'}} / m_{y',S'} O_{D_{i,S'}} \right)
\]
induced from $\varepsilon$ in \((3.4.6)\). Referring to \((1.11.2)\), the condition \((3.1.2)\) says that, under the geometric inclusion $C \subset V|_{S'}$, we have:
\[
(3.4.7) \quad \varepsilon_{y'}(C \cap V_{y'}) = 0.
\]

(It is here that we use the fact that the rays of the normal cone correspond to curvilinear obstructions just as in the proof of Lemma 2.7.)

We next claim that there is a natural surjection
\[
(3.4.8) \quad R^1p^S_* (\mathcal{H}om_S (I_S/I^2_S, O_S)) \rightarrow \Omega^1_{S'} \otimes \mathcal{K}^{-1}_{S'}.
\]

To see this, let
\[
(3.4.9) \quad U := (S \setminus q^{-1}_S(\Xi)) \rightarrow S
\]
be the natural inclusion. Then referring to (2.3), we have morphisms

\[
\begin{align*}
\omega_{S/S'} \otimes (I_S^2/2S) &\longrightarrow j_*j^* (\omega_{S/S'} \otimes (I_S^2/2S)) \\
&\longrightarrow j_*j^* (\mathcal{H}om_S (I_S/2S, \omega_{S/S'} \otimes \Lambda^2 (I_S/2S))) \\
\longrightarrow \mathcal{H}om_S (I_S/2S, j_*j^* \omega_{X_0}) \\
&\leftarrow \mathcal{H}om_S (I_S/2S, q_*^\omega_{X_0}) \\
&\leftarrow \mathcal{H}om_S (I_S/2S, o_S) \otimes (p^S)^* k_{S'}.
\end{align*}
\]

(3.4.10)

all of which restrict to isomorphisms over $U$. Moreover, the fiber dimension of $p^S$ is one, so we have surjections

\[
R^1 p^S_* (\omega_{S/S'} \otimes (I_S/2S)) \longrightarrow R^1 p^S_* (\mathcal{H}om_S (I_S/2S, j_*j^* \omega_{X_0})) \leftarrow R^1 p^S_* (\mathcal{H}om_S (I_S/2S, o_S)) \otimes k_{S'}
\]

which are isomorphisms modulo torsion supported on $D_{S'}$. But by Lemma 1.8, there is a surjection

\[
R^1 p^S_* (\omega_{S/S'} \otimes (I_S/2S)) \twoheadrightarrow \Omega_{1}^{S'} \otimes o_{S'} \xrightarrow{\text{rest.}} \Omega_{1}^{S'}
\]

and thus, since $\Omega_{1}^{S'}$ is locally free, we can deduce the existence of the surjection (3.4.8), and therefore, from (3.4.6), as surjection

\[
W_{S'} \rightarrow \Omega_{1}^{S'} \otimes k_{S'}^{-1}.
\]

Set

\[
\mathcal{Q}_{S'} :=
\]

Referring to (3.4.6), let

\[
\mathcal{Q}_{S'} := \frac{V \otimes o_{S'}}{\ker (W_{S'} \rightarrow \Omega_{1}^{S'} \otimes k_{S'}^{-1})}.
\]
Now a diagram chase completes (3.4.6) to a commutative diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
W_{S'} & \to & R^1p^S_{S'}(\mathcal{H}om_{S}(\mathcal{I}_{S}/\mathcal{I}_{S}^2, \mathcal{O}_S)) \\
\downarrow & \downarrow & \downarrow \\
\Omega_{S'}^1 \otimes K_{S'}^{-1} & \to & \mathcal{Q}_{S'} \\
\end{array}
\] (3.4.8)

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
V \otimes \mathcal{O}_{S'} & \to & R^1T^S(\mathcal{I}_{S}/\mathcal{I}_{S}^2) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{Q}_{S'} & \to & \mathcal{Q}_{S'} \\
\end{array}
\] (3.4.11)

with exact columns and horizontal epimorphisms.

Finally consider (3.2.5). In case 1, \(W_{S'}\) is locally free because the normal-crossing divisor \(D_{S'}\) imposes independent conditions: if \(y' \in S'\) lies in the divisors \(D^{S'}_{j_1}, \ldots, D^{S'}_{j_r}\), then near \(y'\), the left-hand column of (3.4.11) looks like

\[
0 \to \bigoplus_{j=1}^r \mathcal{O}_{S'}(-D^{S'}_{j_1}) \oplus \mathcal{O}_{S'}^{r'} \to \mathcal{O}_{S'}^{r'+r'} \to \bigoplus_{j=1}^r \mathcal{O}_{D^{S'}_{j_1}} \to 0.
\]

Then away from \(D_{S'}\), the cone component \(C\) must be, as in the smooth case, exactly the geometric kernel of the epimorphism of vector bundles

\[W_{S'} \to T^S_{S'} \otimes K_{S'}^{-1}.\]

By (3.4.7) therefore, \(C\) must be a geometric sub-bundle of \(V|_{S'}\), so that \(\mathcal{Q}_{S'}\) is locally free. Then, proceeding just as in the proof of Corollary 2.9, we have that

\[
\gamma(I') = \sum_{C \in \mathcal{C}} m(C)c_{\text{top}}(\mathcal{Q}_{S'}) \cap [S'(C)].
\]

In case 2 of (3.2.5), i.e., if \(S' = I'\) is smooth, \(R^1p^S_{S'}(\mathcal{H}om_{S}(\mathcal{I}_{S}/\mathcal{I}_{S}^2, \mathcal{O}_S))\) is locally free by semicontinuity and the local theory of the Hilbert scheme. Consequently, since the surjection of (3.4.8) is generically injective (Proposition 2.4), it is an isomorphism. Thus,

\[R^1T^S(\mathcal{I}_{S}/\mathcal{I}_{S}^2) \xrightarrow{\sim} \mathcal{Q}_{S'}.\]

Now \(C\) is just the geometric normal bundle of \(S' = I'\) in \(J'\) and keeping in mind the infinitesimal properties of the Hilbert scheme, (1.10.2) (with \(A' = S' = I'\)) realizes \(R^1T^S(\mathcal{I}_{S}/\mathcal{I}_{S}^2)\) as the excess normal bundle of the diagram

\[
\begin{array}{ccc}
J' & \to & V \\
\downarrow_{\sigma_0} & & \\
J' & \xrightarrow{z'} & V
\end{array}
\]
The desired formula is now an immediate consequence of the excess intersection formula.

Finally, we have the following analogue of Proposition 2.4:

3.5 Theorem If, in Theorem 3.3, $I'$ itself is smooth and, for the generic point $p'$ of each component $D'$ of $D = D'$, the curves $I_{p'}$ are locally smooth at the node $x'$, then

$$R^1T(\mathcal{J}/\mathcal{J}^2) \cong \mathcal{K}_{I'}^{-1} \otimes \Omega_{I'}^1[\log D] = \Omega_{I'}.$$  

3.6 Proof: We refer to the proof of Theorem 3.3 in the case $S' = I'$ (in which case $I_S = I$ and $p_S = p^0$). We have seen that the kernel of the surjection

$$R^1\mathcal{p}_0^*(\mathcal{Hom}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_I)) \to \Omega_{I'}^1 \otimes \mathcal{K}_{I'}^{-1}$$  

given in (3.4.8) is supported along $D$. On the other hand, by (3.4.5), the domain of (3.6.1) is a subsheaf of the locally free sheaf $\mathcal{Q}_{I'}$ and hence, so is its kernel. Since $I'$ is smooth, it follows that (3.6.1) must in fact be an isomorphism. So, by (3.4.11),

$$R^1T(\mathcal{J}/\mathcal{J}^2) = \mathcal{Q}_{I'}$$  

and is therefore locally free. Now apply the functor $R\mathcal{Hom}(\_, \mathcal{O}_{I'})$ to the exact sequence (3.4.5) to obtain the exact sequence

$$0 \to \mathcal{Q}_{I'} \to \mathcal{J}_{I'} \otimes \mathcal{K}_{I'} \to \bigoplus_i \mathcal{O}_{D^i}(D^i) \to 0$$  

since, by the standard divisorial exact sequence,

$$\mathcal{E}xt^1_{\mathcal{J}_{I'}}(\mathcal{O}_{D^i}, \mathcal{O}_{I'}) \cong \mathcal{O}_{D^i}(D^i).$$

Thus it suffices to show that, at a general point $y'$ of any fixed $D^i$, the sections of $\ker \tau$ lie in $\mathcal{K}_{I'} \otimes \mathcal{J}_{I'}[\log D^i]$. Choose coordinates $y'_0, \ldots$ on $I'$ near $y' = 0$ such that

$$D^i = \{y'_0 = 0\}.$$

We refer to the (3.4.9) and (3.4.10) in the proof of Theorem 3.3. We analyze the local behavior of the mapping

$$(\mathcal{J}/\mathcal{J}^2) \otimes \omega_{I'/I'} \to \mathcal{Hom}(\mathcal{J}/\mathcal{J}^2, j_* j^* \mathcal{q}_0^* \omega_{\mathcal{X}_0})$$

in terms of analytic local coordinates

$$\{c_1, c_2\} \cup \{a_1, a_2\} \cup \{b_1, \ldots, b_{\text{dim } p - 1}\}$$
defined on a small analytic neighborhood of the node \( x^i \in P \). (Compare \((2.3.1)\)).

Since \( j \) is the inclusion of an open subset with complement of codimension 2, then locally

\[
j_* j^* q^* \omega_{X_0} \cong q^* \omega_{X_0}.
\]

We may assume that upon restricting to the smooth analytic fourfold \( F \) given by

\[
\{ b_1 = \cdots = b_{\dim P - 4} = 0 \},
\]

\( X_0 \) has local equation

\[
a_1 c_1 + a_2 c_2 = 0,
\]

the curve \( I_0 \) is locally given by

\[
c_1 = a_1 = a_2 = 0,
\]

and the incidence scheme \( I \) is given locally in \( I' \times F \) by

\[
a_1 = a_2 = 0
\]

\[
c_1 + c_1 f_0(c_1, c_2) + c_2 g_0(c_1, c_2) = ky_0',
\]

with \( k \neq 0 \) and \( f_0, g_0 \) vanishing at \((0,0)\). Also, \( \omega_{X_0} \) has local generator

\[
(3.6.4) \quad \text{Res}_{X_0} \frac{da_1 \wedge da_2 \wedge dc_1 \wedge dc_2}{a_1 c_1 + a_2 c_2} = \pm \frac{da_1 \wedge dc_1 \wedge dc_2}{a_1} = \pm \frac{da_2 \wedge dc_1 \wedge dc_2}{c_1},
\]

whereas \( \omega_{I/I'} \) has local generator \( dc_2 \). Thus, keeping \((2.1.3)\) in mind, \((3.6.3)\) specializes at the curve \( I_0 \) to the map

\[
(3.6.5) \quad \left( \mathcal{J}_{I_0/X_0} / \mathcal{J}_{I_0/X_0}^2 \right) \otimes \omega_{X_0} \rightarrow \mathcal{H}om_{\mathcal{O}_{I_0}} \left( \mathcal{J}_{I_0/X_0} / \mathcal{J}_{I_0/X_0}^2, \mathcal{O}_{I_0} \right)
\]

given by

\[
a_1 \mapsto \begin{pmatrix} a_j \\ c_1 \end{pmatrix} \mapsto \pm da_i \wedge da_1 \wedge dc_2, \\
(3.6.6) \quad c_1 \mapsto \begin{pmatrix} a_i \\ c_1 \end{pmatrix} \mapsto \pm da_i \wedge dc_1 \wedge dc_2 \wedge 0.
\]

But, by \((3.6.4)\), the image of \((3.6.6)\) consist entirely of homomorphisms which vanish at the point \( x^i \). Thus, near \( y' = 0 \), the mapping

\[
R^1 p'_0(\omega_{I/I'} \otimes \mathcal{J}/\mathcal{J}^2) \rightarrow R^1 p'_0(\mathcal{H}om_{\mathcal{O}}(\mathcal{J}/\mathcal{J}^2, q_0^* \omega_{X_0}))
\]
is a surjection with torsion kernel, so that, from (3.4.3) and the projection formula we have the exact sequence

\[ 0 \rightarrow R^1 p_0^\ast (\omega_{I/I'} \otimes (J/J^2)) \otimes K_{I'}^{-1} \rightarrow R^1 T(J/J^2) \otimes p_0^\ast \mathcal{E}xt_f^1 (J/J^2, \mathcal{O}_I) \rightarrow 0. \]

So, as in (3.6.2), we have an exact sequence

\[ 0 \rightarrow (R^1 T(J/J^2))^\vee \rightarrow \left( \frac{R^1 p_0^\ast (\omega_{I/I'} \otimes (J/J^2))}{\text{some torsion sheaf}} \otimes K_{I'}^{-1} \right)^\vee \rightarrow \bigoplus_i \mathcal{O}_{D_i}(D_i) \rightarrow 0, \]

where the middle term is isomorphic to $T_I \otimes K_{I'}$. But by Verdier duality:

\[ R^1 T(J/J^2) = R^1 (p_0^\ast \mathcal{H}om_I(\bigwedge \omega_{I/I'}))(J/J^2 \otimes \omega_{I/I'}) = \mathcal{H}om_I(p_0^\ast ((J/J^2) \otimes \omega_{I/I'}), \mathcal{O}_I). \]

So, by local freeness and by (3.6.2), we obtain an exact sequence

\[ p_0^\ast ((J/J^2) \otimes \omega_{I/I'}) \xrightarrow{\rho} T_I \otimes K_{I'} \xrightarrow{\tau} \bigoplus D_i \mathcal{O}_{D_i}(D_i) \rightarrow 0, \]

where the image of $\rho$ consists entirely of homomorphisms which vanish at $x^i$ and so lie in $K_{I'} \otimes T_{I'}[\log D_i].$

\[ \square \]

\section{Applications}

We conclude this paper with three applications of the formulas of Corollary 2.9 and Theorem 3.3.

\subsection{Example:} This application was suggested by A. Bertram and M. Thaddeus. Let $C$ be a hyperelliptic curve of genus 4 and let

\[ X_0 = C^{(3)} \]

be the third symmetric power of $C$. Embed $X_0$ in

\[ P = C^{(7)} \]

which is a $\mathbb{P}^3$ bundle over the Jacobian $J(C)$. Thus, rational curves in $P$ are unobstructed and $X_0$ is the zero-scheme of a section of

\[ \mathcal{E} = \mathcal{L}^{\oplus 4}, \]

where $\mathcal{L}$ is the line bundle given by the divisor

\[(\text{basept.} + C^{(6)}).\]
Let
\[ I' = W_3^1 \cong C \]
be the Hilbert scheme of \( g^1_3 \)'s on \( C \) so that
\[ I \subseteq I' \times C^{(3)} \]
becomes the tautological \( \mathbb{P}^1 \)-fibration over \( I' \). Under the Abel-Jacobi map
\[ X_0 \to \text{Pic}^3(C), \]
the fibers of \( I/I' \) are contracted to double points of the theta divisor \( \Theta \), which is itself the image of \( X_0 \). These are canonical singularities, so \( \omega_{X_0} \) is the pullback of \( \mathcal{O}(\Theta) \). Thus
\[ (p^0)^* (\mathcal{O}(\Theta) \otimes \mathcal{O}_C) = q^*_0 \omega_{X_0} \]
so that Corollary 2.9 yields the well-known fact that the number of \( g^1_3 \)'s on a generic curve of genus four is
\[ \gamma(I') = c_1(\omega_C) - (\Theta \cdot C) = 6 - 4 = 2. \]

4.2 Example: Our second application is again not new, being the subject of [1]. Let \( X_0 \) be the Fermat quintic hypersurface in \( P = \mathbb{P}^4 \). Let \( I' \) be the Hilbert scheme of lines in \( X_0 \). Following [1], \( I'_{\text{red}} \) is the union of 50 Fermat quintic plane curves meeting transversely in pairs at 375 points, these points being exactly the flex point of the Fermat plane curves. Using Plücker coordinates in the Grassmannian of \( \mathbb{P}^4 \), one computes that the local analytic structure of \( I' \) away from the crossings is given by
\[ \mathbb{C}[x, y]/(y^2) \]
while at each of the 375 crossings it is given by
\[ \mathbb{C}[x, y]/(x^3 y^2, x^2 y^3). \]
The components of the normal cone are computed from this local analytic structure using the local primary decomposition
\[ (x^3 y^2, x^2 y^3) = (y^2) \cap (x^2) \cap (x, y)^5. \]
One computes that the normal cone has one component of multiplicity 2 over each Fermat quintic curve \( F \) and one component of multiplicity 5 over each crossing point. Hence, by Corollary 2.9, the number of lines on the general quintic threefold is
\[ \gamma(I') = 50 \cdot 2 \cdot c_1(\omega_F) + 375 \cdot 5 \cdot c_0(\omega_{\text{pt}}) = 2875. \]
4.3 Example The setting for the final application was first considered \[3\], and later in \[11\], \[5\] and \[10\]. To the authors’ best knowledge, however, the calculations below are new.

Consider either of the following cases: Choose

\begin{align}
(4.3.1) \quad & g_1, \alpha_2 \in \Gamma(\mathbb{P}^4, \mathcal{O}(4)) \quad \text{and} \quad g_2, \alpha_1 \in \Gamma(\mathbb{P}^4, \mathcal{O}(1)) \\
\text{or} \quad & g_1, \alpha_2 \in \Gamma(\mathbb{P}^4, \mathcal{O}(3)) \quad \text{and} \quad g_2, \alpha_1 \in \Gamma(\mathbb{P}^4, \mathcal{O}(2))
\end{align}

sufficiently general so that both the K3 surface
\[ Y := \{ g_1 = g_2 = 0 \} \]
and the del Pezzo surface
\[ S := \{ g_2 = \alpha_1 = 0 \} \]
are smooth, and such that the quintic threefold
\[ X_0 := \{ \alpha_1 g_1 + \alpha_2 g_2 = 0 \} \]
has only ordinary nodes, all of which—16 in case (4.3.1) and 36 in case (4.3.2)—lie on \( S \cap Y \). In \[11\], it is shown that, despite the existence of the nodes,
\[ N_{Y \setminus X_0} = \omega_Y \quad \text{and} \quad N_{S \setminus X_0} = \omega_S. \]

For any curve \( C \) in \( S \), we have the exact sequence
\[ 0 \rightarrow H^0(C, N_{C \setminus S}) \rightarrow H^0(C, N_{C \setminus X_0}) \rightarrow H^0(C, N_{S \setminus X_0} \otimes \mathcal{O}_C), \]
and, since \( \omega_S \) is ample and \( h^1(\mathcal{O}_S) = 0 \),
\[ h^0(N_{C \setminus X_0}) = h^0(\mathcal{N}_{C \setminus S}) = h^0(\mathcal{O}_S(C)) - 1. \]

Thus the linear system
\[ I' := |\mathcal{O}_S(C)| \]
is a connected component of the Hilbert scheme of \( X_0 \). If the \( g_i \) and the \( \alpha_i \) are sufficiently general, the divisor of curves passing through at least one node is a simple normal-crossing divisor consisting of hyperplanes. So we may apply Theorem 3.5 as long as
\[ H^1(C, \mathcal{N}_{C \setminus \mathbb{P}^4}) = 0 \]
for all \( C \) in \( I' \).

For example, in case (4.3.1), the lines in the plane \( S \) contribute
\[ \gamma(I') = c_2(\Omega^1_{\mathbb{P}^2}[\log 16\mathbb{P}^1]) \cap [\mathbb{P}^2] = 91 \]
lines to the general quintic threefold $X$, the conics in $S$ contribute

$$c_5(\Omega^1_{\mathbb{P}^5}[\log \mathbb{P}^4]) \cap [\mathbb{P}^5] = 2002$$

conics to $X$, and the cubic curves in $S$ contribute

$$c_9(\Omega^1_{\mathbb{P}^9}[\log \mathbb{P}^9]) \cap [\mathbb{P}^9] = 2002$$

cubic elliptic curves to $X$.

In case (4.3.2),

$$S \cong \text{Bl}_5 \text{ pts.} \mathbb{P}^2$$

and the lines in $\mathbb{P}^2$ contribute

$$c_2(\Omega^1_{\mathbb{P}^2}[\log \mathbb{P}^1]) \cap [\mathbb{P}^2] = 595.$$ twisted cubics to $X$, and the hyperplane sections of $S$ contribute

$$c_4(\Omega^1_{\mathbb{P}^4}[\log \mathbb{P}^3]) \cap [\mathbb{P}^4] = 46.376$$
degree 4 elliptic curves to $X$.

Note that one can perform analogous constructions and computation on the other types of complete-intersection Calabi-Yau threefolds. Moreover, in any of these cases, the contributions of curves lying on the K3 surface $Y$ can be calculated; see [1] (curves of genus 1) and [10] (where the results of the present paper are applied to curves of higher genus). In fact, it is shown there that if $Y$ has Picard number 2, then the curves coming from the primitive linear system on $Y$ not generated by the hyperplane sections contribute only geometrically rigid curves to a general deformation of $X_0$.

References

[1] A. Albano and S. Katz, *Lines on the Fermat quintic threefold*, Trans. Amer. Math. Soc. 324 (1991), no. 1, 353–368.

[2] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), no. 1, 45–88.

[3] H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, Inst. Hautes Études Sci. Publ. Math. 58 (1983), 19–38.

[4] *Cohomology and obstructions*, Preprint, math.AG/9809127, 1998.

[5] T. Ekedahl, T. Johnsen, and D.E. Sommervoll, *Isolated rational curves on K3-fibered Calabi-Yau threefolds*, Preprint, alg-geom9710010.

[6] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, vol. 2, Springer-Verlag, 1984.
[7] A.B. Givental, *Elliptic Gromov-Witten invariants and the generalized mirror conjecture*, Preprint, mathAG/9803053.

[8] ______, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices 1996 (1996), no. 13, 613–663.

[9] T. Graber and R. Pandharipande, *Localization of virtual classes*, Preprint, alg-geom/9708001.

[10] H.P. Kley, *On the existence of curves in k-trivial threefolds*, Preprint available at www.math.utah.edu/~kley, 1998.

[11] ______, *Rigid curves in complete intersection Calabi-Yau threefolds*, to appear in Compositio Math., preprint available at www.math.utah.edu/~kley, 1998.

[12] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, vol. 32, Springer, 1996.

[13] M.L. Kontsevich, *Enumeration of rational curves via torus actions*, The Moduli Space of Curves (Texel Island, 1994), Progress in Mathematics, vol. 129, Birkhäuser, 1995, pp. 335–368.

[14] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov Witten invariants of Calabi-Yau threefolds I*, Preprint, mathAG/9803036.

[15] B. Lian, K. Liu, and S.T. Yau, *Mirror principle I*, Preprint, alg-geom/9712011.

[16] H. Matsumura, *Commutative ring theory*, Cambridge studies in advanced mathematics, vol. 8, Cambridge University Press, 1986.