Revisiting Parametricity: Inductives and Uniformity of Propositions

ABHISHEK ANAND∗, Cornell University
GREG MORRISETT, Cornell University

Reynold’s parametricity theory captures the property that parametrically polymorphic functions behave uniformly: they produce related results on related instantiations. In dependently typed programming languages, such relations and uniformity proofs can be expressed internally, and generated as a program translation.

We present a new parametricity translation for a significant fragment of Coq. Previous translations of parametrically polymorphic propositions allowed non-uniformity. For example, on related instantiations, a function may return propositions that are logically inequivalent (e.g. True and False). We show that uniformity of polymorphic propositions is not achievable in general. Nevertheless, our translation produces proofs that the two propositions are logically equivalent and also that any two proofs of those propositions are related. This is achieved at the cost of potentially requiring more assumptions on the instantiations, requiring them to be isomorphic in the worst case.

Our translation augments the previous one for Coq by carrying and compositionally building extra proofs about parametricity relations. It is made easier by a new method for translating inductive types and pattern matching. The new method builds upon and generalizes previous such translations for dependently typed programming languages.

Using reification and reflection, we have implemented our translation as Coq programs1. We obtain several stronger free theorems applicable to an ongoing compiler-correctness project. Previously, proofs of some of these theorems took several hours to finish.

1 INTRODUCTION

Krishnaswami and Dreyer [2013] summarize Reynold’s work on parametricity in the following perfect way:

Reynolds [1983] famously introduced the concept of relational parametricity with a fable about data abstraction. Professors Bessel and Descartes, each teaching a class on complex numbers, defined them differently in the first lecture, the former using polar coordinates and the latter using (of course) cartesian coordinates. But despite accidentally trading sections after the first lecture, they never taught their students anything false, since after the first class, both professors proved all their theorems in terms of the defined operations on complex numbers, and never in terms of their underlying coordinate representation.

Reynolds formalized this idea by giving a semantics for System F in which each type denoted not just a set of well-formed terms, but a logical relation between them, defined recursively on the type structure of the language. Then, the fact that well-typed client programs were insensitive to a specific choice of implementation could be formalized in terms of their taking logically related inputs to logically related results. Since the two constructions of the complex numbers share the same interface, and it is easy to show they are logically related at that interface, any client of the interface must return equivalent results regardless of which implementation of the interface is used.

In Reynold’s work and subsequent work for other modern languages (e.g. OCaml [Crary 2017]), the logical relations for types are meta-theoretic (not defined in the programming language being studied). In contrast, in dependently typed programming languages such as Coq, one can express

∗abhishek.anand.iitg@gmail.com
1https://github.com/aa755/paramcoq-iff
within the language such logical relations and the proofs that programs are related. Thus, recent works [Bernardy et al. 2010, 2012; Keller and Lasson 2012] have defined program translations that translate types to their logical relations. Because terms can appear in types in dependently typed languages, these translations translate both terms and types. An amazing aspect of the translation of terms is that it produces proofs of the corresponding abstraction theorems: Let \([T]\) denote the parametricity translation of the type \(T\). For closed terms \(t\) and \(T\), if \(t:T\) (\(t\) has type \(T\)) in System F, Reynold’s abstraction theorem says that \((t,t)\) is in the relation \([T]\). The proof of this theorem is in the meta-theory. In contrast, in Coq, amazingly, the proof is precisely \([t]\), the translation of \(t\).

In Coq, parametricity is a powerful tool to obtain not only statements of free theorems [Wadler 1989], but also free Coq proofs of those theorems. In our recent compiler correctness project, we have used the implementation\(^2\) by Keller and Lasson [2012] to automatically obtain for free several Coq proofs that otherwise took many hours to manually write. For example, by polymorphically defining the big-step operational semantics of some intermediate languages, we were able to obtain for free (Section 6) that the semantics are preserved when we change the representation from de Bruijn (the representation used in the compiler’s source language) to named-variable bindings (the representation used in the backend). However, as we explain below, Keller and Lasson’s translation produces useless abstraction theorems for polymorphic propositions or relations. In contrast, our translation lets us obtain the uniformity of even the polymorphically defined, undecidable relations (e.g. observational equivalence).

Undecidable relations are particularly problematic because, as we explain next, they cannot be equivalently redefined in a way that allows reaping the benefits of the existing translation [Keller and Lasson 2012]. In proof assistants such as Coq, some amount of logic can be done using the boolean datatype. A predicate over a type \(X\) can be represented as a function of type \(X \rightarrow \text{bool}\). Given a polymorphic function, say \(f\), returning a \(\text{bool}\), Coq’s parametricity translation produces a proof that on different, parametrically related instantiations, \(f\) will produce the same boolean value. However, undecidable predicates (or \(n\)-ary relations in general) cannot be defined this way, because Coq functions are \(\text{computable}\): a term of type \(\text{bool}\) must eventually \(\text{compute}\) to one of the two boolean values: \(\text{true}\), \(\text{false}\). One can cheat and use a strong version of the axiom of excluded middle to make such definitions. However, the axiom is provably \(\text{non-parametric}\) [Keller and Lasson 2012, Sec. 5.4.2]. Hence parametricity translations cannot generate abstraction theorems for definitions using the axiom.

Proof Assistants based on dependent types (e.g. Agda [Norell 2009], Coq, F* [Swamy et al. 2016], Idris [Brady 2013], LEAN [de Moura et al. 2015], Nuprl [Constable et al. 1986]) have another, perhaps more idiomatic mechanism for defining propositions/relations. For example, dependent function types can be used to express universal quantification. Using such quantification, one can easily define undecidable relations. An \(n\)-ary relation is just a function that takes \(n\) arguments and returns a proposition. In Coq, \(\text{Prop}\) is a special universe whose inhabitants are intended to be types denoting logical propositions. In the “propositions as types, proofs as programs” tradition, by “\(P\) is a proposition”, we mean \(P:\text{Prop}\), and by “\(p\) is a proof of \(P\)”, we mean \(p:P\).

Propositions enjoy a special status in Coq. For example, by restricting pattern matching on proofs (Section 2.2), Coq ensures that one can consistently assume the proof irrelevance axiom which says that any two proofs of a proposition are equal. Also, as a result, Coq’s compiler can erase all proofs [Letouzey 2004] to a dummy term.

The existing parametricity translation [Keller and Lasson 2012] translates propositions and proofs as well. However, propositions are treated just like other types, and proofs are treated just like

\(^2\)https://github.com/mlasson/paramcoq
members of other types. As a result, \texttt{Prop}, which is a universe and whose inhabitants are propositions (types), is treated differently than \texttt{bool} which is not a universe, and whose members are not types: they are mere data constructors: \texttt{true} and \texttt{false}. \[\texttt{[bool]}\], the parametricity relation for the type \texttt{bool} relates \texttt{true} with \texttt{true} and \texttt{false} with \texttt{false}, and relates nothing else. In contrast, propositions (types) \(P_1\) and \(P_2\) are related by \[\texttt{[Prop]}\] if there is any relation, say \(R\), between the proofs of \(P_1\) and \(P_2\). Note that there exist relations even between logically inequivalent types. For example, \(\lambda (t : \text{True}) (f : \text{False})\), True is a relation between the propositions \texttt{True} and \texttt{False}. This means that polymorphically defined propositions may have logically inequivalent meanings in related instantiations. Thus, abstraction theorems for polymorphic propositions, as generated by the existing parametricity translation [Keller and Lasson 2012], are useless.

In the context of the previous paragraph, the main advantage of our translation is that it additionally ensures/requires:

(1) logical equivalence of the related propositions: \(P_1 \leftrightarrow P_2\)

(2) triviality of the relation: \(\forall (p_1 : P_1) (p_2 : P_2), R p_1 p_2\)

Here, \(R\) is the relation between the proofs of the propositions \(P_1\) and \(P_2\). The usefulness of the first property was already explained above. The second is useful when instantiating an interface that includes proofs. For example, an interface describing a semigroup (in abstract algebra) in Coq may also contain fields representing the proofs of associativity equations. To use parametricity to obtain free proofs that polymorphic functions over semigroups behave uniformly, one needs to provide two instantiations of the semigroup interface, and prove that all the fields, including the proof fields, are related. The triviality property makes it trivial to prove that the proof fields are related. Previously, it took one of us several hours to do one of these proofs. The Appendix (Section A.5.1) provides a Coq statement of the proof, in case the reader wants to independently assess the difficulty.

There is a cost to achieving the above two properties for polymorphic propositions: our abstraction theorem may make stronger assumptions in some cases. For example, consider Coq’s polymorphic equality proposition, which is defined using indexed induction:

\[
\text{Inductive eq } (T : \text{Type}) \ (x : T) : T \rightarrow \text{Prop} := \text{eq_refl} : eq \ T \ x \ x
\]

This syntax says that \texttt{eq} is a family of propositions (types) and for any type \(T\) and \(x\) of type \(T\), \texttt{eq_refl} is a proof that \(x\) is equal to itself. Because Coq’s typehood judgements are preserved under computation, for closed \(x\) and \(y\), the proposition \(eq \ T \ x \ y\) asserts that the normal forms of \(x\) and \(y\) are the same. Thus Coq lets us define propositions that make logical observations that no computation can make: by parametricity, all functions of the type \(\forall \ T : \text{Type}, T \rightarrow T \rightarrow \text{bool}\) are constant functions. In Section 3, we see that for indexed-inductive propositions to behave uniformly, the parametricity relation between the two instantiations of the index type may need to be one-to-one. Also, for universal quantification (\(\forall\)), the relation for the quantified type may need to be total.

After analysing the uniformity requirements for Coq’s mechanisms for defining new propositions (Section 3, 4), we explain our new parametricity translation that ensures these requirements (Section 5). We call our new translation the IsoRel translation because in the worst case, the two instantiations of type variables need to be isomorphic. In contrast, we call the old translation [Keller and Lasson 2012] the AnyRel translation, because one can pick any relation between \(\texttt{true}\) and \(\texttt{false}\). For convenience, mentions of Coq constants are usually hyperlinked to their definition, if defined in this paper or in Coq’s standard library. Also, to take advantage of syntax highlighting, we recommend reading this paper in color.
the two instantiations, as long as each item in the interface respects the relation. In this sense, Reynold’s original parametricity translation of types can be considered an AnyRel translation.

Our IsoRel translation excludes propositions that mention types of higher universes ($\text{Type}_i$ for $i > 0$) at certain places (Section 5.3). For example, in universal quantification, the quantified type must be in $\text{Prop}$ or $\text{Type}_0$, which is also denoted by $\text{Set}$ in Coq. Also, in inductively defined propositions, the types of indices and the types of arguments of constructors (except the parameters of the type) must be in $\text{Set}$ or $\text{Prop}$. $\text{Set}$ and $\text{Prop}$ suffice for many concrete applications, such as correctness of computer systems (e.g. compilers, operating systems) and cyber-physical systems. For example, in $\text{Set}$, one can define natural, rational, and real numbers, functions and infinitely branching trees of real numbers, and abstract syntax trees used by compilers. Our restrictions may be problematic for some applications such as proving the consistency of powerful logics [Anand and Rahli 2014].

The AnyRel translation serves as a core of our IsoRel translation. The IsoRel translation adds extra proofs about the AnyRel translations of types and propositions. The main challenge is to compositionally build the extra proofs of new type and proposition constructions from the corresponding proofs of their subcomponents. Because understanding the AnyRel translation is crucial for understanding our IsoRel translation, we first present our version of the AnyRel translation in Section 2. Our AnyRel translation is similar to the one by Keller and Lasson [2012], except for the translation of inductive types and pattern matching. Our AnyRel translation of inductive types (Section 2.3) and pattern matching (Section 2.4) simplifies our IsoRel translation because it allows us to use the $\text{Prop}$ universe for defining the parametricity relations of those types. As explained above, the $\text{Prop}$ universe is well-suited for defining logical relations. Our AnyRel translation of inductive types and pattern matching is inspired by a translation by Bernardy et al. [2012, Sec 5.4]. However, we uncover and fix a subtle flaw in how they translate indexed-inductive types and pattern matching on inhabitants of those types.

Summary of Contributions:
- For a significant fragment of Coq, a new parametricity translation (IsoRel) that augments our version of the AnyRel translation to enforce the uniformity of polymorphically defined propositions (Section 3-5). The IsoRel translation uses the proof irrelevance and function extensionality axioms.
- For indexed-inductive types and pattern matching, a new AnyRel translation (Section 2.3, 2.4) which has proof-irrelevance properties that simplify the IsoRel translation and are also independently useful. The AnyRel translation does not use any axiom.
- An application of parametricity translations (AnyRel, IsoRel) to obtain for free many tedious Coq proofs about compiler correctness (Section 6). We show a theorem (observational equivalence respects $\alpha$ equality) that the IsoRel translation can prove but the AnyRel cannot.

2 ANYREL TRANSLATION

In this section, we present the AnyRel translation that forms the core of the IsoRel translation described in the next sections. As mentioned above, unlike the IsoRel translation, the AnyRel translation does not ensure the uniformity of propositions, and treats propositions (types) just like other types, and treats proofs just like members of other types. First, we describe the translation of a core calculus of Coq that excludes inductive constructions. This core is exactly the Calculus of Constructions (CoC) [Coquand and Huet 1988]. Although our presentation is very similar to the one by Keller and Lasson [2012], it highlights why we will later need a new translation for inductive types. Then we add inductive types to the calculus and compare, in the setting of Coq, the
existing AnyRel translations of inductive constructions and associated constructs such as pattern-matching (Section 2.2). Finally, we describe our new translation (Section 2.3, 2.4), which is inspired by the compared translations. Our translation has proof irrelevance properties that simplify the IsoRel translation. Also, we uncover and fix a subtle flaw in one of the compared translations.

2.1 Core Calculus
The following grammar describes the language of CoC (both terms and types):

\[
\begin{align*}
  s & ::= \text{Prop} \mid \text{Type}_i \\
  A, B & ::= x \mid s \mid \forall x : A, B \mid \lambda x : A, B \mid (AB)
\end{align*}
\]

where \( x \) ranges over variables and \( i \) ranges over natural numbers. \( s \) denotes universes (also known as sorts in the literature). The translation often needs four extra variables for each variable in the input. Just to avoid capture, without loss of generality, we assume that there are five disjoint classes of variables and the input only has variables from the first class, and has no repeated bound variables. We assume that \( z, r, 4, \) and \( 5 \) are injective functions that respectively map variables of the first class to variables of the next four classes. Semantic concepts such as \( \alpha \)-equality, reduction, typehood are totally agnostic to this distinction between classes of variables. Finally, for any term \( A, A_2 \) denotes the term obtained by replacing every variable \( v \) by \( v_2 \).

For now, we define \( \hat{s} := s \). Let \( c \) be some variable of the first class.

\[\llbracket \hat{s} \rrbracket := \lambda (c : s)(c_2 : s), c \to c_2 \to \hat{s}\]

\[\llbracket x \rrbracket := x_r\]

\[\llbracket \forall x : A, B \rrbracket := \lambda (x_4 : \forall x : A, B)(x_3 : \forall x_2 : A_2, B_2), \forall (x : A)(x_2 : A_2)(x_r : [A]x x_2), [B](x_4 x)(x_3 x_2)\]

\[\llbracket \lambda x : A, B \rrbracket := \lambda (x : A)(x_2 : A_2)(x_r : [A]x x_2), [B]\]

\[\llbracket (A B) \rrbracket := ([A]B B)\]

The translation of contexts is obvious from the translation of the \( \hat{\lambda} \) case:

\[\llbracket \emptyset \rrbracket := \emptyset\]

\[\llbracket \Gamma, x : A \rrbracket := \llbracket \Gamma \rrbracket, x : A, x_2 : A_2, x_r : [A]x x_2\]

As examples, \( \llbracket \forall A : \text{Type} \, \text{A} \to \text{A} \rrbracket \beta \) reduces to the relation \( \lambda (A_4 : \forall A : \text{Type}, A \to A)(A_3 : \forall A_2 : \text{Type}, A_2 \to A_2), \forall (A_2 : \text{Type})(A_r : A \to A_2 \to \text{Type})(a : A)(a_2 : A_2), A_r a_2 \to A_r (x_3 A a)(x_2 A_2 a)\) and \( \llbracket \lambda (A : \text{Type})\, a : A \rrbracket \) is \( \lambda (A_2 : \text{Type})(A_r : A \to A_2 \to \text{Type})(a : A)(a_2 : A_2)(a_r : A_r a_2), a_r\).

Problem with the above definition of \( \hat{s} \) is that for a closed \( T : \text{Type}_i \), \( \llbracket T \rrbracket \) is a relation of type \( T \to T_2 \to \text{Prop} \). In Coq, logical relations typically return propositions. Thus one may instead desire the following type: \( T \to T_2 \to \text{Prop} \), which is what we get by defining \( \hat{s} := \text{Prop} \). As explained in the previous section, inhabitants of the \text{Prop} universe enjoy a special status in Coq’s logic and
compiler. Unfortunately, Keller and Lasson [2012, Sec. 4.2] observed that having $\hat{s} := \text{Prop}$ breaks the abstraction theorem above for the typehood judgement $\text{Type}_i : \text{Type}_{i+1}$.

Keller and Lasson [2012, Sec. 4.2] consider a different calculus (CIC\(_r\)), which has two chains of universes $\text{Type}_i$ and $\text{Set}_i$. The latter chain does not have the rule $\text{Set}_i : \text{Set}_{i+1}$ and thus they are able to have $\hat{\text{Set}}_i := \text{Prop}$. However, without that rule, the higher universes in the latter chain may have limited utility. Also, although they defined an embedding from CIC\(_r\) to Coq, they didn’t define any embedding of any fragment of Coq into CIC\(_r\). Thus, it is not clear how their theory applies to Coq. Indeed, their implementation for Coq always picks $\hat{s} := s$.

Instead of switching to a different calculus, we consider Coq. Note that the relations for the lowermost universe can live in $\text{Prop}$, i.e., we can define $\text{Type}_0 := \text{Prop}$ and $\hat{s} := s$ otherwise. For $i > 0$, the abstraction theorem for $\text{Type}_0 : \text{Type}_i$ $\beta$-reduces to the following, which typechecks in Coq: $(\lambda (A \: A_2 : \text{Type}_0), A \rightarrow A_2 \rightarrow \text{Prop}) : (\text{Type}_0 \rightarrow \text{Type}_0 \rightarrow \text{Type}_i)$.

To follow Coq’s convention, we will henceforth write $\text{Set}$ instead of $\text{Type}_0$.

In the next subsection, we will see that the desire to have $\hat{\text{Set}} := \text{Prop}$ has major implications on how the inductive types are translated.

### 2.2 Previous Translations of Inductive Types and Propositions: Comparison

In the above core calculus, the only way to form new types was to form dependent function types. One can also inductively define new types and propositions in Coq. For example, below we have a Peano-style inductive definition of natural numbers:

```coq
Inductive nat : Set :=
| O : nat
| S : nat -> nat.
```

One can write functions by pattern matching on inductive data/proofs. For example, below are the definitions of the predecessor function (left) and a logical predicate (right) asserting that the input is zero.

```coq
Definition pred (n: nat) : nat :=
match n with
| O => O
| S n => n
end.

Definition isZero (n: nat) : Prop :=
match n with
| O => True
| S _ => False
end.
```

Bernardy et al. [2012] presented two ways to translate inductive types and pattern matching: the inductive style translation and the deductive style translation. The two methods are, according to the authors, isomorphic in their Agda-like setting where there is no universe analogous to $\text{Prop}$. However, in Coq, as we explain next, the deductive style is more suitable for translating inductive types, and the inductive style is the only choice (among the two) for inductive propositions. Also, in the next subsection, we will uncover and fix a subtle flaw in the deductive-style translation. Below, for $\text{nat}$, we have the inductive-style translation (left) and the deductive-style translation (right).

```coq
Inductive nat_r : nat -> nat -> Set :=
| O_r : nat_r O O
| S_r : ∀ n n_2 : nat, nat_r n n_2 -> nat_r (S n) (S n_2).

Fixpoint nat_r (n n_2 : nat) : Set :=
match n, n_2 with
| O, O => True
| S m, S m_2 => nat_r m m_2
| _, _ => False
end.

Definition O_r : nat_r O O := I.
Definition S_r (n n_2 : nat) (n_r : nat_r n n_2) : nat_r (S n) (S n_2) := n_r.
```

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The inductive-style translation is straightforward. Roughly speaking, given an inductive \(I : T\), it introduces a new inductive \(I := [T]\ I\ I\). For each constructor \(c : C\), \(I\) has the constructor \(c_r : [C] c\ c\). In both the styles, \([\ ]\) is extended to define \([I] := I_r\) and \([c] := c_r\).

The deductive style translation defines the same relation by structural recursion. The constructors are translated separately. \(I : \text{True}\) is the constructor of the inductively defined proposition \(\text{True}\).

The translation of Coq’s \texttt{match} construct depends on how the type of the discriminatee, which must be inductive (or coinductive), is translated. Below, we have the inductive-style (left) and the deductive-style (right) translation of the above-defined predecessor function. Again, the inductive-style translation is straightforward. We just translate each subterm of the \texttt{match} construct. The deductive style translation of an inductive type is not an inductively defined type. Thus, in the deductive style, we cannot do a pattern match on the translation of the discriminatee. Instead, we pattern match on the original discriminatee \(n\) and \(n_2\). In the cases when the constructors are different, the type of the argument \(n_r\) computes to \(\text{False}\) (see the last branch in the definition of \(\text{nat}_r\)). For any type \(T\), and \(p : \text{False}\), \(\text{False} \_\text{rect} T p\) has type \(T\). (Readers who find it odd that we apply \(n_r\) to the \texttt{match} term, and lambda bind it with refined types in each branch may wish to read “The One Rule of Dependent Pattern Matching in Coq” [Chlipala 2011, Sec 8.2].)

\begin{verbatim}
Definition pred_r (n n_2 : nat) :=
  match n with
  | O => O
  | S m m_2 m_3 => m
end.

Note that in the inductive style translation, we pattern match on the translated discriminatee, whose type is the translated inductive (fully applied). To ensure the consistency of the proof irrelevance axiom, Coq has a proof-elimination restriction that ensures that one can pattern match on proofs to only create proofs. (There is an exception called singleton elimination, which we describe in the next subsection.) Recall that a term \(p\) is a proof iff its type’s type is \(\text{Prop}\) (\(p\’s\) type is a proposition). If we define \(\text{Set} := \text{Prop}\), the above inductive-style translation of \(\text{pred}\) is well-typed in Coq, because it matches on proofs to create proofs. However, the inductive-style translation of the above-defined \texttt{isZero} predicate violates the proof-elimination restriction:

\textbf{Lem 2.1. When \texttt{Set} := \texttt{Prop}, the inductive style translation of large elimination can be ill typed.}

\textbf{Proof.} Below is the inductive-style translation of the above-defined \texttt{isZero} predicate:

\begin{verbatim}
Definition isZero_r (n n_2 : nat) :=
  match n with
  | O => True_r
  | S m m_2 m_3 => False_r
end.

It pattern-matches on a proof \((n_r\) has type \(\text{nat}_r\ n\ n\), which has type \(\text{Prop}\)) to produce a relation, and not a proof. Note that \((\text{isZero} n) \rightarrow (\text{isZero} n_2) \rightarrow \text{Prop}\) does not have type \(\text{Prop}\).

Indeed, if Coq allowed one to match on proofs and produce the \texttt{True} proposition on one proof and the \texttt{False} proposition on another (e.g., consider the definition \texttt{isZero} when \texttt{nat} is declared in the \texttt{Prop} universe), one can easily refute proof irrelevance, which says that any two proofs of a proposition are equal (logically indistinguishable).

In contrast, the deductive-style translation doesn’t suffer from this problem, because the resultant pattern matches are on discriminatees of the original inductive type, and not the translated one.
(nat has type Set, not Prop). Thus, regarding the proof-elimination restriction, the translatability of pattern matches in the deductive style is independent of how we define Set. Indeed, the deductive-style translation of isZero happily typechecks when we define Set := Prop or Set := Set. Thus, the deductive-style translation of inductive types and corresponding pattern matching allows more flexibility in the choice of Set. Although propositions (members of the universe Prop) are also called types in the literature, in this paper, the word “type” usually refers to terms that are members of Set or Type, but not Prop.

The deductive style has other advantages over the inductive style: In the inductive style, proofs that are by induction on variables of the translated inductive type are often difficult. We explain this at the beginning of Section 2.3. Also, the deductive-style translation enables proofs by computation, e.g., natr computes when the two numbers are in normal form.

One can also inductively define logical propositions in Coq. The story for translating inductive propositions is the opposite: the deductive-style violates the proof-elimination restriction.

Lemma 2.2. The deductive-style translation of inductive propositions can be ill-typed.

Proof. Replace Set by Prop in the above definition of nat. Then, nat can be seen as the “True” proposition and its members, e.g. O, can be considered proofs of the proposition. The deductive-style translation of nat, as shown above (natr), would then be ill-typed because it would then match on proofs (of nat) to produce propositions, not proofs. □

In summary, the deductive-style translation is more suitable for translating inductive types, and the inductive-style translation is the only choice (among the 2) for inductive propositions. Thus, unfortunately, one has to implement both styles to translate Coq in a way that allows Set := Prop. This is what we do in our AnyRel translation, because having Set := Prop greatly simplifies our IsoRel translation.

Keller and Lasson [2012, Sec. 4.4] show a third approach, which is a hybrid approach, but only for a simple example which has no indices (indices are explained in the next subsection): they don’t provide a general translation for inductive types. Thus we exclude (just) that part of their paper from further consideration. Also, their implementation always uses the inductive-style translation and always chooses Set := Set.

The inductive-style translation is quite simple and well explained and implemented by Keller and Lasson [2012, Sec. 4.3]. In the next subsection, we turn our attention to the deductive-style translation, which is more complex than the inductive-style translation.

For the rest of this paper, we define Set := Prop 4.

2.3 Deductive-style Translation of Indexed-Inductive Types

The above subsection established that to ensure that Set := Prop, inductive types (but not inductive propositions) should be translated in the deductive style. This section takes a closer look at our deductive-style translation, especially of indexed-inductive definitions. Our translation is inspired by the translation by Bernardy et al. [2012]. However, while implementing it for Coq, we found that it is even more complex than the way it was presented in the literature [Bernardy 2011; Bernardy et al. 2012]. The additional complexity is fundamental in nature and has nothing to do with Coq.

4Ensuring Set=Prop, which simplifies the IsoRel translation, is problematic in the presence of universe-polymorphic inductive types, regardless of whether we choose the deductive-style translation or the inductive-style translation. The problem arises from limitations in the expressivity of Coq’s universe polymorphism. We discuss the problem in the Appendix (Section A.1). As mentioned before, our IsoRel translation does not work for higher universes anyway, for much more fundamental reasons (Section 5.3).
One can mutually inductively define an infinite family of types/propositions using Coq’s indexed-inductive definitions. Below is a typical indexed-inductive definition. The type \( \text{Vec} \ T \ m \) is just like the type \( \text{list} \ T \) except that its inhabitants must have length exactly \( m \).

```coq
Inductive Vec (T : Set) : forall (m : nat), Set := nilV: Vec T O | consV: forall (n : nat), T -> Vec T n -> Vec T (S n).
```

Note that the constructor \( \text{consV} \) takes a \( \text{Vec} \ T \ n \) and constructs a \( \text{Vec} \ T \ (S \ n) \): the input and output

The arguments of the type that vary in the definition are called \textit{indices}. The other arguments are called \textit{parameters}. In the above type, \( T \) is a parameter and \( m \) is an index. Coq requires that the parameters be listed before ‘:’ and the indices be listed after ‘:’. In general, the definition of a member of the family can depend on the definition of other members of the family. Thus, even if we have a variable, say \( v \) whose type is a specific member of an inductive family (as determined by the indices), to do a proof by induction on \( v \), one has to consider all the members of the family. In particular, the property that is being proved by induction must be \textit{well-defined} for all members of the family. This often makes such proofs difficult, because one needs to generalize over indices (see Section 4.2 for an example). We will see below that the inductive-style translation of an inductive type with \( n \) indices produces an inductive with \( 3n + 2 \) indices!

The deductive-style translation of the above type, as presented in previous literature is flawed in a subtle way. Below, we have first the (correct) inductive-style translation [Bernardy et al. 2012, page 24, middle] and deductive-style translation from the literature ([Bernardy et al. 2012, page 21, top], Bernardy [2011, page 31]). We have adapted these from Agda-like syntax to Coq. The authors claimed that the two styles are isomorphic.

```coq
Inductive Vecr (T T2 : Set) : forall (m : nat), Prop := nilVr: Vecr T T2 O O O => (nilV T) (nilV T2). | consVr: forall (n : nat), (m : nat_r m m2) (vn : Vec T m) (vn2 : Vec T2 m2), Prop := Vecr T T2 T2 T2 (S n) (S n2 n2 r) (consV T n t vn) (consV T2 t2 t2 vn2).
```

The argument \( m_r \) is \textit{unused} and \textit{irrelevant} in the deductive-style translation (\( \text{Vec}_r \)), which is a recursive function (and not an inductive). Thus, one can prove by induction on \( v \) that for all \( m m2, m_r m_{R1} m_{R2}, \forall v, v_2 \), the proposition \( \text{Vec}_r \ m m2 m_{R1} \ v v_2 \) is equal to \( \text{Vec}_r \ m m2 m_{R2} \ v v_2 \). This is not the case in the inductive-style translation. For example, the constructor \( \text{nilV}_r \) \textit{requires} \( m_r \) to be (definitionally) equal to \( O_r \). Thus, to prove that the two styles are isomorphic in this example, one needs to at least prove that \( \forall \ (m_r : \text{nat}_r O O) \), \( m_r = O_r \). It just so happens that this is provable for this example of \( \text{Vec} \). However, in general, the index type may not be concrete: it may be a type variable. Also, it may be in a higher universe, in which case, its relation need not be in \textit{Prop}. In that case, we get to pick \textit{any} relation for the type, and we can easily pick a relation \( R \) such that for some \( x \) and \( y \), there are multiple distinct inhabitants in the type \( R \times y \). For example, we can pick \( R := \lambda x y. \) \text{bool} \). Also, \textit{we will see in Section 2.4 that the translation of \texttt{match} terms requires proofs
like the above, that \( \forall (m_r : \text{nat}_r, O O), m_r = O_r \). Thus, even when provable, \([\square]\) will need to cook up these proofs: it is not clear how to do that automatically.

Thus we strengthen the propositions returned in the deductive-style translation to add the above-mentioned equality constraints. Here is the corrected version:

```
Fixpoint Vec, (T T2 : Set) (T_r : T \to T2 \to \text{Prop})
  (m m2 : \text{nat}) (m_r : \text{nat}_r, m m2) (v : \text{Vec} T m) (v_2 : \text{Vec} T2 m2) : \text{Prop} :=
(m, v, v_2)
| \text{nilV}, \text{nilV} \Rightarrow \lambda m_r, m_r = O_r ,
| \text{consV} n t v n_2, \text{consV} n_2 t_2 v n_2 \Rightarrow \lambda m_r ,
  \{ n_r : \text{nat}_r, n n_2 \& T_r t t_2 \& \text{Vec}_r T T2 T_r, n n_2 n_r vn v n_2 \& m_r = (S_r n n_2 n_r) \}
| \_ \_ \Rightarrow \lambda \_ \_ , \text{False end} m_r .
```

After adding the equality constraints, the deductive-style translation is isomorphic to the inductive-style translation. If an inductive constructor has recursive arguments that are functions, our proof of the isomorphism needs the function extensionality axiom.

The AnyRel translation does not use any axiom. Preservation of reduction is typically a step in proving the abstraction theorem [Keller and Lasson 2012, Lemma 2]. Thus, we need to be careful in using axioms or opaque definitions because at certain places, they may block reduction. (The IsoRel translation described after this section uses axioms, but nevertheless achieves preservation of reduction.)

The only reason we add the equality constraints is that, as mentioned above, the proofs of those constraints are needed in the translation of pattern matches. These constraints added significant complexity to our implementation of \([\square]\), even after we were able to simplify the constraints a bit, as explained in the rest of this subsection.

In general, an indexed-inductive type may have several indices. Also, the types of the later indices may be dependent on the previous indices or parameters. Below is an example:

```
Inductive isNil : \forall (n: \text{nat}) (v: \text{Vec nat n}), \text{Set} :=
isNil : \forall (v v : \text{Vec nat O}), \text{isNil O v} v .
```

It is tricky to even state the equality constraints of the dependent indices (e.g. \( v \) in the example above) because the types of the two sides of the equality will not be definitionally equal. We will illustrate this soon. While implementing \([\square]\), the main source of complexity came even later, when implementing the translation of pattern matches. There, we had to not only “rewrite” with the proofs of these equality constraints one by one, but also show that the proofs are each equal to the canonical equality proof (eq_refl). Fortunately, we found a much simpler way: we define a generalized equality type that can, in one step, assert the equality of all the corresponding indices. Here is such an equality type for translating isNil:

```
Inductive isNil_indicesEq(n n2 : nat) (n_r : nat_r, n n2) (v : \text{Vec nat n}) (v2 : \text{Vec nat n2})
  (v_r : \text{Vec}_r \text{nat nat nat}_r, n n2 n_r v v_2) : \forall (in_r : \text{nat}_r, n n2) (i v_r : \text{Vec}_r \text{nat nat nat}_r, n n2 i n_r v v_2), \text{Prop} :=
isNil_refl : isNil_indicesEq n n2 n_r v v_2 v_r n_r v_r .
```

This generalized equality type asserts that the indices \( n_r \) and \( v_r \) are equal to the indices \( i n_r \) and \( i v_r \). The types of \( v_r \) and \( i v_r \) are different (not convertible, for the purpose of typechecking). Thus, it is ill-typed to just write \( v_r = i v_r \). Unlike JMeq [McBride 2002], our generalized equality type simultaneously asserts the equality of sequences of dependent indices. This greatly simplifies our translation of pattern matching, where now just generating one match on the proof of this generalized equality type changes all the indices, and changes the only one proof to the canonical form, which is isNil_refl.
2.4 Pattern Matching (deductive-style)

We already saw some examples of deductive-style translations of pattern-matching (e.g. pred) on non-indexed inductive types. Implementing the translation of pattern-matches on indexed-inductives is more complex, as we will illustrate with an example. The main goal of this subsection is to discharge the claim made in the previous subsection that the equality constraints described in the previous subsection are crucial for translating pattern matches.

We consider the following pattern-matching function over the indexed-inductive isNil defined in the previous section:

Definition isNilRec (P : ∀ (n : nat) (v : Vec nat n), isNil n v → Set)
(f : ∀ vv : Vec nat O, P O vv (isNil vv)) (n : nat) (v : Vec nat n) (d : isNil n v) : P n v d :=
match d with
| isnil x ⇒ f x end.

It can be considered an induction/recursion principle for isNil. It takes an f that works for the canonical forms (there is only 1), and returns a function that works for an arbitrary member of the inductive family. The deductive-style translation of the above function is shown in Figure 1. Unfortunately, it is the most complex example presented in this paper. In general, for every argument, the translation has three arguments: see the clause for λ in the definition of [ ] (Section 2.1). As we enter each pattern match, the return type gets refined: the discriminee is replaced by the constructor applied to its arguments and the indices are replaced with the indices returned by the constructor. Inside the first two pattern matches, n and n2 each become O, d becomes isnil x, . . . . However, nnr, which is the proof that n and n2 are related, doesn’t change to O.

The translation of the original body of the match, f x, is f x n x2 x2. The two outermost pattern matches bring x and x2 in scope, but not xr. In general, they bring the original constructor arguments and the 2 versions in scope. However, we also have to bring to scope the proofs that the corresponding constructor arguments are related, e.g. that x and x2 are related. In the deductive
Definition `isNilRec` $(P : \forall (n : \text{nat}) (v : \text{Vec nat } n), \text{isNil } n v \rightarrow \text{Set})$

$(P_2 : \forall (n_2 : \text{nat}) (v_2 : \text{Vec nat } n_2), \text{isNil } n_2 v_2 \rightarrow \text{Set})$

$(P_r : \forall (n n_2 : \text{nat}) (n_r : \text{nat}, n n_2) (v : \text{Vec nat } n) (v_2 : \text{Vec nat } n_2)$

$(v_r : \text{Vec n nat nat n} n_2 n_r v v_2) (d : \text{isNil } n v)$

$(d_2 : \text{isNil } n_2 v_2) (d_r : \text{isNil n} n_2 n_r v v_2 v_r d d_2), P n v d \rightarrow P_2 n_2 v_2 d_2 \rightarrow \text{Prop}$

$(f : \forall vv : \text{Vec nat O, O O vv (isnil vv)}) (f_2 : \forall vv_2 : \text{Vec nat O, O O vv_2 (isnil vv_2)})$

$(f_r : \forall (vv vv_2 : \text{Vec nat O}) (vv_r : \text{Vec n nat nat n_r O O O r vv vv_2}),$

$P_r O O O_r v v_2 v_r (\text{isnil } vv) (\text{isnil } vv_2) (\text{isnil_r v v_2 v_r}) (f vv) (f_2 vv_2))$

$(n n_2 : \text{nat}) (n_r : \text{nat n n_2}) (v : \text{Vec nat n}) (v_2 : \text{Vec nat n_2})$

$(v_r : \text{Vec n nat nat n_r n n_2 n_r v v_2}) (d : \text{isNil } n v)$

$(d_2 : \text{isNil } n_2 v_2) (d_r : \text{isNil n} n_2 n_r v v_2 v_r d d_2):$

$P_r \ldots n_r \ldots v_r \ldots d_r (\text{isNilRec f \ldots d}) (\text{isNilRec f_2 \ldots d_2}) :=$

$\text{match } d \text{ as } \ldots \text{ in } \ldots \text{return } \ldots \text{ with}$

$| \text{isnil } x \Rightarrow \text{match } d_2 \text{ as } \ldots \text{ in } \ldots \text{return } \ldots \text{ with}$

$| \text{isnil } x_2 \Rightarrow \lambda (n n_r : \text{nat n n_2}) (v : \text{Vec n nat nat n_r O O n n_r x x_2})$

$(dd_r : \text{isNil n_r O O n n_r x x_2 v v_r (isnil x) (isnil x_2)}),$

$\text{match } dd_r \text{ with}$

$| \text{existT x_r pdeq} \Rightarrow$

$\text{(match pdeq as } \ldots \text{ in } \ldots \text{return } \ldots \text{ with}$

$| \text{isNil_refl f x x_r} :$

$(P_r O O O_r x x_2 x_r) (\text{isnil x}) (\text{isnil x_2}) (\text{isnil_r x x_2 x_r}) (f x) (f_2 x_2))$

$\text{end): (P_r O O n n_r x x_2 vv_r (isnil x) (isnil x_2) (\text{existT x_r pdeq}) (f x) (f_2 x_2))}$

$\text{end}$

$\text{end n_r v_r d_r}.$

Fig. 1. Translation of pattern matching requires the equality constraints

style translation, these proofs are packed as dependent pairs in the proof that the two discriminants $(d$ and $d_2$) are related. In this case, that proof is $dd_r$. See the definition of `isNil`, to understand how the type of $dd_r$ computes to a dependent pair. In general, if the constructor has $n$ arguments, this type would compute to the type of nested dependent pairs containing a total of $n+1$ items. The first $n$ pattern matches on $dd_r$ will ensure that all the free variables of the translation of the body are in scope. For example, in Figure 1, the 3rd innermost `match` brings $x_r$ in scope. However, the type of the translation of the body needs rewriting. In Figure 1, we have shown the type (as checked by Coq), of the innermost pattern match. This is the expected return type, which as described above, has $nn_r$ instead of $O_r$, etc. The type of the translation of the body, which is the innermost body in the translation, is also shown and aligned to the expected return type. Note that $nn_r$ in the outer type needs to change to $O_r$, $vv_r$ to $x_r$, and the dependent pair `(existT x_r pdeq)` needs to change to `(isnil_r x x_2 x_r)`. The latter computes to the dependent pair `(existT x_r (isNil_refl \ldots))` Thus, the last change is essentially to change `pdeq` to the canonical proof `(isNil_refl \ldots)`, as hinted in the previous subsection. All these changes are achieved by just one pattern match on `pdeq`, the proof of the generalized equality type described in the previous subsection.

The general scheme for translating pattern matches can be found in the Appendix (Section A.3).
2.5 Fixpoints (recursive functions)

Our translation of \texttt{fix} (or \texttt{Fixpoint}) terms is largely as described by Keller and Lasson [2012]. A minor change was required because we translate inductive types and corresponding pattern matches in the deductive style. It is explained in Appendix A.4.

2.6 Summary

In this section, we presented the AnyRel translation that will serve as the core of the IsoRel translation described in the rest of the paper. The main advantage of the translation in this section over the AnyRel translation implemented by Keller and Lasson [2012] is that we have $\texttt{Set} := \texttt{Prop}$, which means that relations for types in the $\texttt{Set}$ universe enjoy the proof irrelevance property, which is useful not only in the IsoRel translation, but in other applications as well. Ensuring $\texttt{Set} := \texttt{Prop}$ required a deductive-style translation of inductive types. We found that the deductive-style translation of pattern matches on inhabitants of indexed-inductive types requires strengthening the deductive-style translation of those types with equality constraints that were erroneously missing in the literature [Bernardy 2011; Bernardy et al. 2012]. Stating and using those equality constraints becomes challenging for inductive types with multiple, dependent indices. We showed how to simplify the construction. We also showed that the deductive-style translation does not work for inductively defined propositions: those need to be translated in the inductive style, as does pattern matching on proofs of those propositions.

We have implemented our AnyRel translation as functions in Coq itself. Using reification and reflection [Malecha and Sozeau 2014], we have used those Coq functions to translate several examples. The translated program is delivered to the reflection mechanism which ensures that the result is well-typed before adding it to Coq’s environment of definitions and declarations. Also, our translation produces all the implicit arguments, and is thus immune to the incompleteness of Coq’s type inference mechanism.

3 UNIFORMITY OF PROPOSITIONS

We begin this section by describing why $\llbracket \texttt{Prop} \rrbracket$ is too weak to ensure the uniformity of propositions. Then and in the next section, we develop the main technical lemmas needed to ensure the uniformity. In Section 5, we use these lemmas in the IsoRel translation $\llbracket \texttt{iso} \rrbracket$, which ensures the uniformity of propositions. We believe the lemmas in Section 3 and 4 are independently interesting and useful.

Recall (Section 2.1) that $\llbracket \texttt{Prop} \rrbracket := \lambda (P P_2 : \texttt{Prop}) , P \rightarrow P_2 \rightarrow \texttt{Prop}$. If we have $\theta : \texttt{Prop}$, Theorem 1 says $\llbracket \theta \rrbracket : \theta \rightarrow \theta_2 \rightarrow \texttt{Prop}$. In applications of parametricity, $\theta$ would typically denote a proposition in one instantiation and $\theta_2$ would denote the corresponding proposition in the other instantiation. In the example at the beginning of Section 1, one instantiation is the cartesian representation of complex numbers, and the other instantiation is the polar representation of complex numbers. $\theta$ and $\theta_2$, in the respective instantiations, could be the proposition that addition is commutative.

We want the two propositions to mean the same in both the instantiations. However, the statement (type) of $\llbracket \theta \rrbracket$, which is the proof that $\theta$ and $\theta_2$ are parametrically related, is too weak. $\llbracket \theta \rrbracket$ is merely a relation between $\theta$ and $\theta_2$. As explained in Section 1, there is a relation even between logically inequivalent propositions, such as \texttt{True} and \texttt{False}. In contrast, if we instead had $\theta : \texttt{bool}$, Theorem 1 says: $\llbracket \theta \rrbracket : \texttt{bool} \rightarrow \theta \rightarrow \theta_2$, where \texttt{bool} is the deductive-style translation of the inductive type \texttt{bool}, which has only two constructors: \texttt{true} and \texttt{false}. We hope that from the previous section, it is clear that \texttt{bool} implies that either \texttt{true} or \texttt{false} reduces to \texttt{true}, or \texttt{false} reduce to \texttt{false}. 13
The main goal of this paper is to strengthen the translation of the universe \texttt{Prop} to get uniformity properties similar to the type \texttt{bool}. In Section 1, we identified and motivated two properties that we wish to have for the relations between (proofs of) propositions:

\textbf{Definition IffProps} \{ \texttt{A B : Prop} \} \{ \texttt{R : A \rightarrow B} \} : \texttt{Prop} := \texttt{A \leftrightarrow B}.

\textbf{Definition CompleteRel} \{ \texttt{A B : Prop} \} \{ \texttt{R : A \rightarrow B} \} : \texttt{Prop} := \forall (a : A) (b : B), R a b.

To ensure these properties, in the IsoRel translation, we define the translation of \texttt{Prop} in a way that is equivalent to the following:

\[
\llbracket \text{Prop} \rrbracket_{\text{iso}} := \lambda (A A_2 : \text{Prop}), \{ R : A \rightarrow A_2 \rightarrow \text{Prop} & \text{IffProps} R \land \text{CompleteRel} R \}.
\]

Instead of returning an arbitrary relation, the IsoRel translation requires the relation to come bundled (as a dependent pair) with proofs of the above two properties (of the relation) of interest. This paper is mainly about tackling the far-reaching consequences of the above change. The contributions of the previous section, although independently interesting, were made to ensure that we can have \texttt{Set} := \texttt{Prop} (instead of \texttt{Set} := \texttt{Set}), which makes it easy to tackle some of the consequences. In \llbracket \text{iso} \rrbracket, other parts of \llbracket \text{iso} \rrbracket also need to be updated to cope with the change in the translation of \texttt{Prop}, so that we get essentially the same abstraction theorem as before (Theorem 1).

For example, as we will see in this section, we also need to bundle relations for types with some properties. We will see in the next subsections that the relations produced by translating the types mentioned in propositions may need to have one or both of the following properties:

\textbf{Definition OneToOne} \{ \texttt{A B : Set} \} \{ \texttt{R : A \rightarrow B} \} : \texttt{Prop} :=

\[
(\forall (a : A) (b_1 b_2 : B), R a b_1 \rightarrow R a b_2 \rightarrow b_1 = b_2) \land (\forall (b : B) (a_1 a_2 : A), R a_1 b \rightarrow R a_2 b \rightarrow a_1 = a_2).
\]

\textbf{Definition Total} \{ \texttt{A B : Set} \} \{ \texttt{R : A \rightarrow B} \} : \texttt{Type} :=

\[
(\forall (a : A), \{ b : B & (R a b) \}) \times (\forall (b : B), \{ a : A & (R a b) \}).
\]

The Total property says that for all \(a : A\) there exists a related \(b : B\) and vice versa. A relation satisfying both of the above properties can be considered an isomorphism. Thus, in the worst case, the IsoRel translation produces free Coq proofs justifying the commonly held belief that isomorphic instantiations of interfaces have the same logical properties. However, as we will see in this section, \textit{many propositions need} neither of the above properties to behave uniformly. Here is an example where any relation works for the first argument \(T\):

\textbf{Definition PNone} := \lambda (T : \texttt{Set}) (f : T \rightarrow \texttt{nat}) (a b : T), \ (f a = f b).

The AnyRel translation of the argument \(f\) already implies that on related inputs, \(f\) produces equal numbers. Some need only one: the next two polymorphic propositions respectively only need the Total and OneToOne properties for the first argument \(T\).

\textbf{Definition PTot} := \lambda (T : \texttt{Set}) (f : T \rightarrow \texttt{nat}), \forall (t : T), f t = \texttt{O}.

\textbf{Definition POne} := \lambda (T : \texttt{Set}) (f : \texttt{nat} \rightarrow T), \forall (n : \texttt{nat}), f n = f (S (S n)).

We will see that we need the Total property for universally quantified types and types of arguments of inductive constructors. Also, we need the OneToOne property for index types of inductively defined propositions, such as the equality proposition. To allow such fine-grained analysis, for now, unlike for propositions, we don’t globally assume the Total and OneToOne properties for relations produced by translating types. We could have done that by defining:

\[
\llbracket \text{Set} \rrbracket_{\text{iso}} := \lambda (A A_2 : \texttt{Set}), \{ R : A \rightarrow A_2 \rightarrow \texttt{Prop} \land \text{Total} R \times \text{OneToOne} R \}.
\]

In the AnyRel translation described in the previous section, had we not ensured that \(\texttt{Set} := \texttt{Prop}\), and instead chosen \(\texttt{Set} := \texttt{Set}\), we would also need (Section 3.2) to consider compositionally build proofs of a third property about parametricity relations of types, which seems hard but doable:

\textbf{Definition irrelevant} \{ \texttt{A A_2 : Set} \} \{ R : A \rightarrow A_2 \rightarrow \texttt{Set} \} := \forall (a : A) (a_2 : A_2) (p1 p2 : R a a_2), p1 = p2.
Because we have \( \text{Set} := \text{Prop} \), the type of \( R \) in the above definition becomes \( A \rightarrow A_2 \rightarrow \text{Prop} \) and thus the above property becomes a trivial consequence of the proof irrelevance axiom. Also, in Section 4.2, we will see that our IsoRel translation needs that (or a similar) axiom anyway.

The abstraction theorem (Theorem 1) for the AnyRel translation says that for closed terms \( t \) and \( T \), if \( t : T \), then \( \llbracket t \rrbracket : (\llbracket T \rrbracket \rightarrow \text{Prop}) \). The main change now is that in some cases, \( \llbracket T \rrbracket_{iso} \) may be a dependent pair: of a relation and some proofs about the relation. Thus, we may need to project out the relation from \( \llbracket T \rrbracket_{iso} \).

The above change in the translation of \( \text{Prop} \) means that for relations of propositional variables, we get to assume the two extra properties. However, for composite propositions, we must build the proofs of those two properties while assuming the property for the subcomponents, if any. Fortunately, starting from the universes, there are only two ways to construct new types or propositions in Coq: dependent function types and inductive types. Although one can also construct propositions or types by pattern matching and returning different types in each branch, recursively, those types always originate from the two primitive mechanisms mentioned above. When viewed through the lens of logic, dependent function types correspond to universal quantification, and one can construct inductive types that correspond to familiar logical constructs such as existential quantification.

In the next two subsections, we see how to compositionally build the proofs for the two ways to build new propositions, and the additional assumptions (Total or OneToOne property) needed about relations of types mentioned in the propositions. Then, in the next section, we will see how to compositionally build the proofs of Total and OneToOne properties for AnyRel translations of types. Finally, in Section 5, we use these constructions of proofs in \( \llbracket \text{iso} \rrbracket \). Propositions where types of higher universes (Type\( _i \) for \( i > 0 \)) occur at certain places are excluded for fundamental reasons (Section 5.3).

All the proofs in this and the next section (except Section 3.3) were originally done in Coq. The appendix (Section A.5) has pointers to Coq proofs submitted as anonymous supplementary material. Appendix A.8 summarizes the main lemmas of this and the next section as tables.

### 3.1 Universal Quantification

Consider \( A : \text{Set}, B : A \rightarrow \text{Prop} \rightarrow (\forall (a:A), B a) : \text{Prop} \). By Theorem 1, we have \( \llbracket A : \text{Set}, B : A \rightarrow \text{Prop} \rrbracket \rightarrow \llbracket (\forall (a:A), B a) : \text{Prop} \rrbracket \). By unfolding definitions and \( \beta \) reduction, we get: \( A : \text{Set}, B : \text{Set}, A r : A \rightarrow A_2 \rightarrow \text{Set}, B A \rightarrow \text{Prop}, B_2 : A_2 \rightarrow \text{Prop}, B_i : (\forall (a:A) (a_2 : A_2), A_r a a_2 \rightarrow (B a) \rightarrow (B_2 a_2) \rightarrow \text{Prop} \rightarrow \llbracket (\forall (a:A), B a) \rrbracket : (\forall (a:A), B a) \rightarrow (\forall (a_2:A_2), B_2 a_2) \rightarrow \text{Prop} \)

We wish to prove that the relation \( \llbracket (\forall (a:A), B a) \rrbracket \) has the IffProps and CompleteRel properties. Because \( \llbracket \text{iso} \rrbracket \) will be structurally recursive (Section 5), we have 2 hypotheses asserting that \( B_r \) already has the 2 properties, i.e., \( \text{iHrec} : (\forall (a:A) (a_2 : A_2), (a_r : A_r a_2), \text{IffProps} (B_r a a_2 a_r) \) and \( \text{chRec} : (\forall (a:A) (a_2 : A_2), (a_r : A_r a_2), \text{CompleteRel} (B_r a a_2 a_r) \). Using \( \text{chHrec} \), it is trivial to prove the following:

**Lemma 3.1.** \( \text{CompleteRel} (\llbracket (\forall (a:A), B a) \rrbracket) \)

In contrast, IffProps \( (\llbracket (\forall (a:A), B a) \rrbracket \) which \( \beta \)-reduces to \( (\forall (a:A), B a) \leftrightarrow (\forall (a_2 : A_2), B_2 a_2) \), is impossible to prove without additional assumption(s). As a counterexample, take \( A \) to be a non-empty type, \( A_2 \) to be an empty type (e.g. \( \text{False} \)), and \( B \) and \( B_2 \) to be \( \lambda \_ \_ \text{False} \). A simple and sufficient assumption is Total \( A_r \):

**Lemma 3.2.** \( \text{Total} A_r \rightarrow \text{iffProps} (\llbracket (\forall (a:A), B a) \rrbracket) \)

Using \( \text{iHrec} \), the proof is straightforward. We defer the discussion of the necessity of the Total assumption to Section 3.3. In summary, universal quantifications behave uniformly if the relation corresponding to the quantified type is Total.
3.2 Inductively defined propositions

We already saw an indexed-inductive proposition (the polymorphic equality proposition) in Section 1. In Coq, relations and predicates are often defined using indexed-induction. Here is the definition of $\le$ on natural numbers:

\[
\text{Inductive le (n : nat) : nat } \to \text{ Prop} := \\
\text{le_n : le n n} \\
\text{le_S : } \forall \ m \ \text{nat}, \ \text{le n m } \to \ \text{le n (S m)}. \\
\]

Unlike universal quantification, inductively defined propositions come in infinitely many shapes. For example, there can be an arbitrary number of parameters, indices, constructors and arguments of constructors. To explain the key ideas, we consider just one type, which is an indexed version of the W type [Martin-Löf 1984] and can be understood as trees with possibly infinite branching. W types can be used to encode a large class of inductively defined types [Abbott et al. 2004; Dybjer 1997].

\[
\text{Inductive IWP (I : Set) (A : Set) (AI : A } \to \text{ I) (BI : } \forall \ a : A, \ B a } \to \text{ I) } : \ \forall \ (i : I), \ \text{Prop} := \\
\text{pnode : } \forall \ (a : A) \ (\text{branches : } \forall \ b : B a, \ \text{IWP I A B AI BI (BI a b)}) \ , \ \text{IWP I A B AI BI (AI a)}. \\
\]

I is the type of indices. There is only one index type. This may be a loss of convenience, but is not a loss of generality, because one can use (dependent) pairs to encode multiple, dependent indices. The type A encodes the non-recursive arguments. Given any $a : A$, $B a$ denotes the branching factor of a node of the tree: see the branches argument of the constructor pnode. For example, we can choose $B := \lambda (a : A), \ \text{bool}$ for binary (proof) trees. The function AI determines the index of the return type of the constructor. Similarly, the function BI determines the indices of the subtrees in the branches argument of the constructor pnode. Appendix A.5 shows how to encode the above-defined relation le as an instance of IWP.

Using IWP, we proved in Coq the uniformity properties for the large class of inductive propositions encodable using IWP. Otherwise, this proof may have needed reasoning about a deep embedding of Coq’s inductives. Our implementation, which although is inspired by the uniformity proofs for IWP, directly translates each inductive, without using the encoding. This has several advantages. Users don’t have to use unnatural encodings of their inductive propositions. Even if the encoding could be automated, users may prefer to directly understand how the translation works for their definitions, instead of understanding how it is obtained via an encoding. Below, although we mainly focus on the uniformity proof for IWP, we include hints for generalizing the construction to other inductive propositions.

As in the previous subsection, in the translated context, we need to prove the IffProps and the CompleteRel properties for [IWP I A B AI BI i]. Because IWP returns a Prop, it is translated in the inductive style (Section 2.2). Let IWP, denote the inductive-style translation of IWP. We explain the proof of the CompleteRel property and one direction of the IffProps property. We conveniently prove both the properties simultaneously. We will use the following abbreviations in a translated context: $W := \text{IWP I A B AI BI}$, $W_2 := \text{IWP I2 A2 B2 AI2 BI2}$, $W_r := \text{IWP r I2 I r A2 A2 r B B2 B2 AI AI2 AI2 BI BI2 BI2}$, $\text{npodew} := \text{pnode I A B AI BI}$, $\text{npodew2} := \text{pnode I2 A2 B2 AI2 BI2}$

**Lemma 3.3.** Total $A_r \to (\forall (a : A) (a_2 : A_2) (a_r : A_r a a_2), \ \text{Total (B_r a a_2 a_r))) \to \text{OneToOne I}_r$

$\to (p : W i), (W_2 i_2 \land \forall y : W_2 i_2, \ W_r i i_2 i_r p y)$.

Note that $i_r$ has type $I_r , i_2$. We proceed by induction on $p$. The corresponding proof term is a structurally recursive function which pattern matches on $p$. (Our translation directly produces fully elaborated Gallina proof terms, and not LTac proof scripts which have less well-defined semantics.) In the inductive step, we have, for some $a$ and branches, $p := (\text{npodew a branches}) : W (AI a)$. Note that the pattern matching (induction) refines the index of the discriminate $p$ from $i$ to $(AI a)$. Note that the pattern matching (induction) refines the index of the discriminate $p$ from $i$ to $(AI a)$.
Also, \( i_r \) now has type \( I_r \) \((\text{Ai} \ a) \ i_2 \). It is straightforward to use the induction hypothesis and the Total property for \( \mathcal{A}_r \) and \( \mathcal{B}_r \) to obtain \( a_2 \) and \( \text{branches}_2 \) such that \((\text{pnodew}_2 \ a_2 \ \text{branches}_2)\): \( \mathcal{W}_2 \ (\text{AiL}_2 \ a_2) \). Total \( \mathcal{A}_r \) also provides an \( a_r : (\mathcal{A}_r \ a \ a_2) \). We are not done yet even for the left conjunct because it needs something of type \( \mathcal{W}_2 \ i_2 \). Thus we need a proof of \( \text{AiL}_2 a_2 = i_2 \). This is where the OneToOne property of \( I_r \) comes to the rescue. Recall that we have \( i_r : (I_r \ (\text{Al} \ a) \ i_2) \). Also \( \mathcal{[AI]} = \mathcal{AI} \ a \ a_2 \ a_r \), which has type \( I_r \ (\text{Ai} \ a) \ (\text{AiL}_2 \ a_2) \). Thus, we get the needed equality by invoking the hypothesis OneToOne \( I_r \). Now we can substitute \( i_2 \) with \((\text{AiL}_2 \ a_2) \) everywhere (all hypotheses and the conclusion). In general, this rewriting step has to be done for each index of an inductive proposition and rewriting everywhere becomes important, especially while implementing the translation, when the later indices are dependent. Now \( \text{pnodew}_2 \ a_2 \ \text{branches}_2 \) is a proof of the left conjunct.

The right conjunct now has type: \( \forall \ y : \mathcal{W}_2 \ (\text{AiL}_2 \ a_2) \), \( \mathcal{W}_r \ (\text{Ai} \ a) \ (\text{AiL}_2 \ a_2) \ i_r \ (\text{pnodew} \ a \ \text{branches}) \ y \).

Now we pick an arbitrary \( y \) and use proof irrelevance for the proposition \( \mathcal{W}_2 \ (\text{AiL}_2 \ a_2) \) to produce a proof that \( y = \text{pnodew}_2 \ a_2 \ \text{branches}_2 \) and then substitute the former with the latter. This step is crucial: we don’t have the CompleteRel property for \( \mathcal{A}_r \): \( \mathcal{A} \) is not a proposition. We are only assuming the Total property for \( \mathcal{A}_r \). Thus, if we had analyzed the original \( y \) by pattern matching on it, we would have obtained an \( a_2 \) that may be different from \( a_2 \) and unrelated to \( a \).

Next, we use proof irrelevance for the proposition \( I_r \ (\text{Ai} \ a) \ (\text{AiL}_2 \ a_2) \) to replace \( i_r \) with \((\text{AI}_r \ a \ a_2 \ a_r) \). Had we not ensured that \( \mathcal{Set} := \mathcal{Prop} \), and instead chosen \( \mathcal{Set} := \mathcal{Set} \), we would be unable to invoke proof irrelevance here and may need to explicitly assume irrelevant \( I_r \). In general, this rewriting has to be done for each index, from the leftmost index to the rightmost index, because that is the order of dependencies. Then we can use the constructor \( \text{pnode}_r \), and the induction hypothesis to finish the proof.

**Corollary 3.4.** Total \( \mathcal{A}_r \rightarrow (\forall \ (a: \mathcal{A}) \ (a_2: \mathcal{A}_2) \ (a_r: \mathcal{A}_r \ a \ a_2), \ \text{Total} \ (\mathcal{B}_r \ a \ a_2 \ a_r)) \rightarrow \text{OneToOne} \ I_r \rightarrow (\text{IffProps} (\mathcal{W}_r \ i \ i_2 \ i_r) \ \land \ \text{CompleteRel} (\mathcal{W}_r \ i \ i_2 \ i_r)) \)

### 3.3 Necessity of our assumptions

In the previous two subsections, to prove the uniformity of the two canonical constructions of propositions, we sometimes needed to assume the Total and/or OneToOne property for the translations of types mentioned in those propositions. Now we consider the necessity of the two assumptions.

**Lemma 3.5.** Suppose \( \mathcal{U} : \mathcal{Set}, \ \mathcal{V} : \mathcal{Set} \) are closed, and that there is a tool \( T \) than can, for any closed \( P : \mathcal{Set} \rightarrow \mathcal{Prop} \) whose body does not mention types of higher universes, produce a proof of \( P \ U \leftrightarrow P \ V \). Then there exists a Total and OneToOne relation between \( U \) and \( V \).

**Proof.** Define \( \text{isoTypes} := \lambda \ A \ A_2 : \mathcal{Set}, \exists \ (f : A \rightarrow A_2) \ (g : A_2 \rightarrow A), \ \forall \ (s : A, \ g (f \ s) = s) \land (\forall \ (s : A_2, \ f (g \ s) = s)). \) Now, invoke the tool \( T \) on \( P := (\text{isoTypes} \ U) : (\mathcal{Set} \rightarrow \mathcal{Prop}) \) to get a proof of \( (\text{isoTypes} \ U \ V) \leftrightarrow (\text{isoTypes} \ U \ V) \), which implies \( \text{isoTypes} \ U \ V \), which implies that there exists a Total and OneToOne relation between \( U \) and \( V \). \( \square \)

In contrast, there are examples where our translations will make unnecessary assumptions. Suppose \( f : \text{nat} \rightarrow \text{bool} \) is a closed function that always returns \( \text{false} \). Now consider \( (\lambda (T: \mathcal{Set}), \ \forall (n: \text{nat}) \ \text{if} \ f \ n \ \text{then} \ (\forall (t: T), \ t = t) \ \text{else} \ \text{True}) : (\mathcal{Set} \rightarrow \mathcal{Prop}). \) In this case, because the returned proposition has a quantification on \( T \), our translation would require Total \( T_r \). However, a smarter translation could figure out in some cases that \( f \) always returns \( \text{false} \) and thus make the quantification disappear. However, it is impossible to determine whether an arbitrary closed function of type \( \text{nat} \rightarrow \text{bool} \) always returns \( \text{false} \). Thus, there will be examples where every such tool makes unnecessary assumptions.
4 TOTAL AND ONE-TO-ONE PROPERTIES OF RELATIONS OF TYPES

In the above section, we saw that to ensure the uniformity of propositions, the AnyRel translation of types appearing in propositions may need to have the Total or OneToOne properties. In this section, we consider all the ways to construct new canonical types in the universe Set and show how to build the compositional proofs of the Total and OneToOne properties. As mentioned before, we only consider the lowermost universe (Set) in the IsoRel translation.

4.1 Dependent Function Types

We have \( A : \text{Set}, B : A \rightarrow \text{Set}, \Gamma (a : A), B a: \text{Set} \). In the translated context, we need to prove Total \( \forall (a : A), B a \) and OneToOne \( \forall (a : A), B a \). The assumptions Total \( A_r \) and \( \forall (a : A) (a_2 : A_2) (a_r : A_r a_2), \) Total \( (B_r a a_2 a_r) \) are not sufficient to prove Total \( \forall (a : A), B a \). As a counterexample, consider \( A, A_2 := \text{bool}, B, B_2 := \lambda _, \text{bool}; A_r := \lambda (a a_2 : \text{bool}), \text{True}; \) and \( B_r := \lambda \,_.(b b_2 : \text{bool}), b = b_2. \) \( \forall (a : A), B a \) := \( \lambda (x : A)(x_2 : A_2)(x_2 a_2 x_2), B_r x x_2 x_r (x_2 x) (x_2 x_2) \) relates nothing to \( \lambda (x : \text{bool}), x \). Intuitively, because \( A_r \) is a complete relation, \( \forall (a : A), B a \) only relates constant functions. The above counterexample was mainly enabled by the coarseness of \( A_r \). Let \( A_r \) be Total but not OneToOne. Indeed, the proof is easy after adding the assumption OneToOne \( A_r \):

\[
\text{Lemma 4.1. } \quad \text{Total } A_r \rightarrow (\forall (a : A) (a_2 : A_2) (a_r : A_r a_2), \text{Total } (B_r \ a a_2 a_r) \rightarrow \text{OneToOne } A_r
\]

Consider the proof of one side. Given an arbitrary \( f : (\forall (a : A), B a) \), using the totality of \( A_r \) and \( B_r \), it is easy to cook up an \( f_2 : (\forall (a_2 : A_2), B_2 a_2) \). Then we need to prove \( \forall (a : A), B a \) \( f \) \( f_2 \). For this part, we needed the hypothesis OneToOne \( A_r \) and proof irrelevance of the relation \( A_r \).

\[
\text{Lemma 4.2. } \quad \text{Total } A_r \rightarrow (\forall (a : A) (a_2 : A_2) (a_r : A_r a_2), \text{OneToOne } (B_r \ a a_2 a_r))
\]

The proof is straightforward. To prove equality of functions, it uses the dependent function extensionality axiom, which is believed to be consistent with the proof irrelevance axiom in Coq: \( \forall \{A : \text{Type}\} \{B : A \rightarrow \text{Type}\}, \forall (f : B A \rightarrow x : A, B x), (\forall x, f x = g x) \rightarrow f = g. \)

4.2 Inductive Types

The Total and the OneToOne properties of the AnyRel translations of inductive types boil down to the same properties for the types of arguments of their constructors. Let \( c \) be a constructor of an inductive type (family) \( I \). It is useful to classify the arguments of \( c \) into two categories: those that are recursive (whose types mention \( I \)) and those that are not. For example, in the constructor \textit{pnode} in Section 3.2, \( a \) is a non-recursive argument and \textit{branches} is a recursive argument. The non-recursive arguments are easy to tackle. Because \( \parallel \parallel_{i=0} \) will be (Section 5) structurally recursive, we can assume that we already have the Total and the OneToOne properties for the types of those arguments. The recursive arguments are hard to tackle. Their types mention members of the type family \( I \), and we don’t yet have their proofs of the Total and OneToOne properties yet: we are in the process of building that. Thus we need to carefully analyse the types of the recursive arguments and build the recursive proofs of the Total and the OneToOne properties in a way that satisfies Coq’s termination (well-definedness) checker for recursive functions.

Fortunately, Coq has a strict-positivity restriction on the shape of the types of recursive arguments of constructors. These types must be of the form \( \forall (t_1 : T_1) \rightarrow \ldots \rightarrow (t_m : T_m), (I \ldots, \ldots, \ldots) \). Coq’s strict-positivity restriction is a bit more permissive. For example, the type \( \text{nat} \rightarrow \text{list}(f \ldots) \) is acceptable as a type of a constructor argument. Inductives with such constructors are called nested inductives. Our theory and implementation
where \( I \ldots \) represents \( I \) applied to enough arguments so that it becomes a type. Also, the types \( T_i \) must not mention \( I \). (Thus, we can assume Total \( \{ T_i \} \) and OneToOne \( \{ T_i \} \).) So, the types of recursive arguments are (dependent) function types returning the inductive to which the constructor belongs. \( m \) can be 0, as in the definition of natural numbers or lists.

Fortunately, in the previous subsection, we already saw how to compositionally construct the Total and OneToOne properties for (dependent) function types. Those proofs were non-trivial. Thus, we encapsulate those constructions as reusable lemmas and use them in the IsoRel translation of inductives. For example, the lemma totalPiHalf below is the combinator for one direction of the Total property.

**Definition** IsoRel := \( \lambda (A A_2 : \text{Set}), \{ A_r : A \rightarrow A_2 \rightarrow \text{Prop} & (\text{Total} \ A_r) \times (\text{OneToOne} \ A_r) \} \).

**Definition** TotalHalf \( \{ A : \text{Set}, (A_r : A 

: \rightarrow A_2 \rightarrow \text{Prop}) : \text{Type} := \forall (a : A), \{ a_2 : A_2 \& (A_r \ a_a_2) \} \).**

**Definition** anyRelPi \( \{ A : \text{Set}\} (A_r : A \rightarrow A_2 \rightarrow \text{Prop}) \{ B : A \rightarrow \text{Set} \} \{ B_2 : A_2 \rightarrow \text{Set} \}

\( (B_r : \forall a a_2, A_r \ a_a_2 \rightarrow (B \ a \rightarrow (B_2 a_2) \rightarrow \text{Prop}) (f : \forall a, B a) \ (f_2 : \forall a_2, B_2 a_2) : \text{Prop} := \forall a \ a_2 (A_r \ a_a_2), B_2 \ a_a_2 (f_a \ f_2 a_2) \).

**Lemma** totalPiHalf := \( \forall \ (A : \text{Set}) (A_r : \text{IsoRel} \ A \ A_2) \{ B : A \rightarrow \text{Set} \} \{ B_2 : A_2 \rightarrow \text{Set} \}

\( (B_r : \forall a a_2, (\pi_1 \ A_r) \ a_a_2 \rightarrow (B \ a \rightarrow (B_2 a_2) \rightarrow \text{Prop}) (B^\text{Tot} : \forall a \ a_2 (A_r \ a_a_2), \text{TotalHalf} (B_2 \ a_a_2)) \text{TotalHalf} (\text{anyRelPi} (\pi_1, A_r) B_r) \).

We have a similar combinator for the other direction, and similar combinators, one for each direction of the OneToOne property. If the type of the recursive constructor argument has nested function types, we nest the appropriate combinator to get the proof of one direction of the Total or OneToOne property. For example, in the type \( \forall (t_1 : T_1) \ (t_2 : T_2) \ldots \ (t_m : T_m) \), \((\ldots)\) mentioned above, there will be an \( m \)-level nesting. In the base case, when the type is just \((\ldots)\), recursively call the proof (of one half of the Total or OneToOne property) currently being recursively defined.

In the above discussion, we saw how to construct the proofs of one direction of the Total and OneToOne properties of types of all arguments (both recursive and non-recursive) of all constructors. Now we explain how we use these proofs to build the proofs of the same properties of the AnyRel translations of inductive types. As in Section 3.2, we use a \( W \) type to illustrate the construction. However, our implementation directly translates inductive types. The type below is the same as the proposition \( \pi^W \) in Section 3.2, except that we change its universe \( \text{Prop} \) to \( \text{Set} \) and change names to avoid clashes.

**Inductive** IWT \( \{ A : \text{Set}, \{ A_r : A \rightarrow A \rightarrow \text{Prop} \} \}

\( \text{tnode} : \forall (a : A) (\text{branches} : \forall b : B a, \text{IWT} \ I \ A \ B \ A \ I \ (B \ a \ b)), \text{IWT} \ I \ A \ B \ A \ I \ (A \ a) \).

Again, we use the following abbreviations in a translated context: \( \text{WT} := \text{IWT} \ I \ A \ B \ A \ I \ B \ L_i \), \( \text{WT}_2 := \text{IWT} \ I_2 \ A_2 \ B_2 \ A_2 \ B_2 \ L_2 \), \( \text{WT}_r := \text{IWT} \ I_2 \ I_r \ A \ A_2 \ A_r \ B \ B_2 \ B_r \ A \ A_2 \ A_2 \ A_r \ B \ B_2 \ B_r \ L_i \ B_2 \ B_r \), \( \text{tnodew} := \text{tnode} \ I \ A \ B \ A \ I \ B_2 \ A_2 \ B_2 \ B_2 \.

**Lemma 4.3.** Total \( A_r \rightarrow (\forall (a : A) (a_2 : A_2) (a_r : A_r \ a_a_2), \text{Total} (B_r \ a_a_2)) \rightarrow (\forall (a : A) (a_2 : A_2) (a_r : A_r \ a_a_2), \text{OneToOne} (B_r \ a_a_2) \rightarrow \text{OneToOne} I_r \rightarrow \text{Total} (\text{WT}_r \ i \ i_2 i_r) \).

For one direction of totality, given a \( f : (\text{WT} \ I) \), we need to produce a \( t_2 : (\text{WT}_2 \ i_2) \), and prove \( \text{WT}_r \ i \ i_2 i_r \). This proof is by induction on \( f \). Note that \( B \) serves as a domain type in the type of \( \text{branches} \) in \( \text{IWT} \) and that in the combinator totalPiHalf shown above, both the Total and OneToOne properties are needed for the relation for the domain type. This is because we needed both properties for the domain type in Lemma 4.1. Therefore, here we needed both properties for \( B_r \) to produce the argument \( \text{branches}_{2} \) in \( t_2 \). We also needed \( \text{OneToOne} \ I_r \) for the same reason we needed it in...
Lemma 3.3: to do rewriting in indices. The construction generalizes to other inductives, subject to the limitations discussed in Section 5.3.

The proof of the OneToOne property is straightforward, except at one place:

\[ \text{OneToOne } A_r \rightarrow (\forall (a:A) (a_2:A_2) (a_1:A_1 a a_2), \text{Total } (B_r a a_2 a_1)) \rightarrow \text{OneToOne } (W_i r i_2 i_1) \]

The difficulty unsurprisingly involves indices. First, in the above lemma, note that we don’t need any property about \( I_r \). Also, recall that in Lemma 4.2, we only needed the Total property for the domain type. Therefore, here, we need only the Total property for \( B_r \).

Given \( t : (W_i t) \), \( t_2 : (W_i t_2) \), \( t_2 : (W_i t_2) \), \( t_r : W_i t_2 i_2 t t_2 \), and \( t_2 : W_i t_2 i_2 i_1 t t_2 \), we need to produce a proof of \( t_2 = t_2 \). The proof begins by pattern matching (induction) on \( t \) and then another (nested) pattern match on \( t_2 \). In general, inductives may have several constructors. In cases where the constructors from the two pattern matches are different, we’re done because \( t_r \) computes to False (see Section 2.3). We are now left only with cases that have the same constructor. Back to the concrete example, we now have for some \( a \), branches, \( a_2 \), and branches, \( t := \text{tnodew } a \text{ branches} \) and \( t_2 := \text{tnodew } a \text{ branches } \). \( t_2 \) and \( t_2 \) now have type \( W_i t_2 (A_l a_2) \), and we need to prove \( t_2 = t_2 \). The obvious step now is to do a (nested) pattern match on \( t_2 \). However, this is illegal. As explained in Section 2.3, for indexed inductive types, the definition of the type for one index may depend on the definition for other indices. Therefore, to do induction on an indexed inductive type, the property being proved by induction must be well-typed for all indices. Also, an equality is only well-typed if both sides have the same type. Thus, when we do a pattern match on \( t_2 \), the index \( (A_l a_2) \) of its type gets generalized to a fresh variable, say \( i_2 \). Then the type of \( t_2 \) becomes \( W_i t_2 \), and thus the types of \( t_2 \) and \( t_2 \) become non-convertible.

A common solution to such problems is to state the equality in a more general type. We can generalize the statement \( t_2 = t_2 \) to the statement that the dependent pair of \( (A_l a_2) \) and \( t_2 \) and the dependent pair of \( (A_l a_2) \) and \( t_2 \) are equal in the sigma type \( \{ i_2 : i_2 \text{ & } W_t t_2 i_2 \} \). Now when we pattern match on \( t_2 \), the type of the RHS of the equality remains unchanged. The rest of the proof is straightforward.

Finally, we have to undo the generalization of the equality statement. For that, we use the following lemma from Coq’s standard library, which although unprovable [Hofmann and Streicher 1998], is a consequence of proof irrelevance (or the UIP (Unicity of Identity Proofs) axiom).

\[ \text{Lemma inj_pair2: } \forall (U : \text{Type}) (P : U \rightarrow \text{Type}) (p : U) (x y : P p), \text{existT } p x = \text{existT } p y \rightarrow x = y. \]

In general, an inductive type may have several (say \( n \)) indices. Our translation then uses \( n \) nested dependent pairs. Also, the above lemma is then invoked \( n \) times.

5 ISOREL TRANSLATION

Now we use the lemmas developed in the previous two sections to define the IsoRel translation. Those lemmas are summarized in tables in Appendix A.8.

In Section 3, we saw how to systematically produce proofs of the two desirable properties (IffProps and CompleteRel) for AnyRel translations of propositions. In the IsoRel translation, we augment the AnyRel translation to ensure that parametricity relations of propositions always come bundled with those two properties. We wish to define:

\[ \text{[Prop]}_{\text{iso}} := \lambda (A A_2; \text{Prop}), \{ R : A \rightarrow A_2 \rightarrow \text{Prop} & \text{IffProps } R & \text{CompleteRel } R \}. \]

\[ \text{Lemma 5.1. For any } A: \text{Prop}, B: \text{Prop}, \text{ and } R: (A \rightarrow B \rightarrow \text{Prop}), (\text{IffProps } R & \text{CompleteRel } R) \leftrightarrow (\text{Total } R \times \text{OneToOne } R). \]

\[ \text{Proof. OneToOne } R \text{ is a trivial consequence of proof irrelevance. Also, using proof irrelevance, it is straightforward to prove Total } R \leftrightarrow (\text{IffProps } R & \text{CompleteRel } R). \]
Thus we instead choose the following equivalent definition:

\[
\text{Definition } \text{IsoRel} := \lambda (A_2 : \text{Set}), \{ A_r : A \rightarrow A_2 \rightarrow \text{Prop} \ & \ (\text{Total } A_r) \times (\text{OneToOne } A_r) \}. \\
\text{Prop}_{\text{iso}} := \lambda (A_2 : \text{Prop}), \text{IsoRel } A A_2.
\]

Also, when propositions mention types, we may need the AnyRel parametricity relations of those types to have the Total or OneToOne property. In Section 4, we saw how to systematically build these properties for types in Set. Thus, we can choose to define:

\[
\text{Set}_{\text{iso}} := \lambda (A_2 : \text{Set}), \text{IsoRel } A A_2.
\]

This choice is not ideal because the proofs of the desirable properties of many propositions don’t need one or both of the two bundled properties of the types mentioned in the propositions. We saw three examples (PNone, PTot, POne) in Section 3. We use a 2-stage process in our IsoRel translation. In the first stage, which we call the weak IsoRel translation and denote by \(\llbracket\text{iso}\rrbracket\), we always bundle the relations for types with both the two properties. \(\llbracket\text{iso}\rrbracket\) is structurally recursive and implemented in Coq (Gallina). In the 2\(^{nd}\) stage (Section 5.2), we attempt to remove unused assumptions from the generated abstraction theorems. For efficiency, this 2\(^{nd}\) stage is implemented as an OCaml plugin for Coq. We denote the composition of the two stages by \(\llbracket\text{isio}\rrbracket\), and call it the (strong) IsoRel translation.

It is natural to consider a 1-phase approach where the main translation itself determines the minimally needed assumptions on type variables. We considered and rejected that approach because it seemed very complex to implement. A discussion can nevertheless be found in Appendix A.6.

5.1 \(\llbracket\text{iso}\rrbracket\) (weak IsoRel translation)

First we define the following functions to construct and destruct IsoRel.

\[
\text{Definition } \text{mkIsoRel}(A A_2 : \text{Set}) (A_r : A \rightarrow A_2 \rightarrow \text{Prop}) (A_r,\text{tot} : \text{Total } A_r) \\
(\lambda c c_2 : \text{OneToOne } A_r) : \text{IsoRel } A A_2 := \text{existT } A_r (A_r,\text{tot}, A_r,\text{one}).
\]

\[
\text{Definition } \text{projRel}(A A_2 : \text{Set}) (A_r,\text{iso} : \text{IsoRel } A A_2) : A \rightarrow A_2 \rightarrow \text{Prop} := \pi_1 A_r,\text{iso}.
\]

W.r.t. \(\llbracket\rrbracket\), the main change in \(\llbracket\text{iso}\rrbracket\) is that the parametricity relations of types and propositions come bundled with proofs. As a result, we often have to project out relations from bundles before applying them. Let \(\pi \Delta t\) denote \(\text{projRel } A A_2 t\) if \(A\) has type \(\text{Prop}\) or \(\text{Set}\), and just \(t\) otherwise. In our implementation, wherever needed, our reifier invokes Coq’s typechecker and includes this information (a flag indicating that a term has type \(\text{Prop}\) or \(\text{Set}\)) in the reified terms. \(\llbracket\text{iso}\rrbracket\) needs this information for the domain and codomain types of \(\Pi\) types, the argument types of \(\lambda\) terms, and the return types of \text{match} and \text{fix} terms. The desired correctness property of \(\llbracket\text{iso}\rrbracket\) is: for closed \(t\) and \(T\), if \(t : T\), then we must have \(\llbracket t \rrbracket_{\text{iso}} : ((\pi T) \llbracket T \rrbracket_{\text{iso}}) t t\).

\[
\llbracket\text{Prop}\rrbracket_{\text{iso}} := \lambda (c c_2 : \text{Prop}), \text{IsoRel } c c_2.
\]

\[
\llbracket\text{Set}\rrbracket_{\text{iso}} := \lambda (c c_2 : \text{Set}), \text{IsoRel } c c_2.
\]

For \(i > 0\), we have:

\[
\llbracket\text{Type}_i\rrbracket_{\text{iso}} := \lambda (c c_2 : \text{Type}_i), c \rightarrow c_2 \rightarrow \text{Type}_i
\]

\[
\llbracket x \rrbracket_{\text{iso}} := x_r
\]

\[
\llbracket \lambda x : A B \rrbracket_{\text{iso}} := \lambda (x_2 : A) (x_2_2 : A_2) (x_r : (\pi A (A) (\text{iso}) x x_2)), \llbracket B \rrbracket_{\text{iso}}
\]

\[
\llbracket (A B) \rrbracket_{\text{iso}} := \llbracket A \rrbracket_{\text{iso}} B B_2 \llbracket B \rrbracket_{\text{iso}}
\]

The translation of dependent function types/propostions has two cases. First, we define the following relation, which is the same as the AnyRel translation, except that if necessary, it projects out the relations of the domain and the codomain type.

\[
\llbracket \forall x : A B \rrbracket_{\Pi} := \lambda (x_4 : \forall x : A B)(x_5 : \forall x_2 : A_2 B_2), \forall (x_2 : A)(x_r : (\pi A (A) (\text{iso}) x x_2)), (\pi B (B) (\text{iso}) x_4 x)(x_5 x_2)
\]

If \(\forall x : A B\) has type \text{Type}_i where \(i > 0\), then we have

\[
\llbracket \forall x : A B \rrbracket_{\text{iso}} := \llbracket \forall x : A B \rrbracket_{\Pi}
\]
If $\forall x : A. B$ has type Set or Prop (depending on the type of $B$) then we have

$$\forall x : A. B \mid_{iso} := \text{mkIsoRel} (\forall x : A. B) \ (\forall x : A. B) \ (\forall x : A. B )$$

Here $ptot$ and $pone$ respectively are the proofs of the Total and OneToOne properties, whose construction was explained in Section 3.1 (if $B$ : Prop) or Section 4.1 (otherwise): It is important to prefer the construction in Section 3.1 (also see Lemma 5.1) because that uses fewer assumptions and thus increases the potency of the 2nd phase described in the next subsection. More details can be found in Appendix B.5.2.

Just like the case for $\Pi$ type, if an inductive type is in the Set or Prop universe, we bundle its relation with the two proof terms produced as explained in Section 3.2 (for inductive propositions) or Section 4.2 (otherwise).

The translation of the match and the fix constructs are nearly the same as in the AnyRel translation. There was a small change needed in the return types. Coq’s kernel requires every pattern match to include a return type (which is a function of the discriminee and its indices). The AnyRel translation of a match term (say $t$) of type $T$, is a match term whose return type is $\llbracket T \rrbracket t t_2$. In the IsoRel translation, the return type is $(\pi_T \llbracket T \rrbracket_{iso}) t t_2$. A similar change was needed in the translation of fixpoints.

5.1.1 Correctness. As explained before, w.r.t. $\llbracket \rrbracket$, the only changes in $\llbracket \rrbracket_{iso}$ are: 1) The relations produced by $\llbracket \rrbracket$ of types/propositions in Set or Prop are now paired with proofs of Total and OneToOne properties. 2) As a result, at some places, we project the relations out of the pairs. In Sections 3 and 4, we explained in detail how to construct the proofs of Total and OneToOne properties. Those constructions were originally done and proved correct in Coq. Except for the construction of those proofs, the correctness argument for $\llbracket \rrbracket_{iso}$ is almost identical to the correctness argument for $\llbracket \rrbracket$: one proves that the translation preserves substitution, then reduction, and finally typehood [Keller and Lasson 2012, Lemma 2, Theorem 1].

In Appendix B, we discuss a formal (but not machine checked) proof of correctness of $\llbracket \rrbracket_{iso}$ for a CoC-like core calculus. The formalized calculus excludes inductive types and associated constructs such as pattern matching and fixpoints. However, we illustrate that $\llbracket \rrbracket_{iso}$ correctly translates the W type (which can encode inductive types) and its recursion principle (which can encode pattern matching and fixpoints). We also show that $\llbracket \rrbracket_{iso}$ preserves $\iota$ reduction of that recursion principle: intuitively, axioms don’t block preservation of reduction because we use axioms only in proofs of the Total and OneToOne properties. We have also tested $\llbracket \rrbracket_{iso}$ on a large variety of inductives (e.g. multiple and dependent indices, multiple constructors, various shapes of arguments of constructors). Recall that Coq typechecks the result of $\llbracket \rrbracket_{iso}$ (after reflection): so soundness is not a concern.

5.2 Eliminating Unused Hypotheses

As mentioned before, $\llbracket \rrbracket_{iso}$ has a post-processing stage where the user can ask the system to strengthen an abstraction theorem generated by $\llbracket \rrbracket_{iso}$. In Section 3, we saw that our proofs of the desirable properties (IffProps, CompleteRel) of propositions may not need one or both of the two properties (Total, OneToOne) about the relations of types mentioned in the propositions. Similarly, the proof of the Total or OneToOne property for relations of composite types may not need one or both of the two properties (Total, OneToOne) of subcomponents (Section 4). Thus, we expect the proofs produced by $\llbracket \rrbracket_{iso}$ to not mention some of the hypotheses. We want to strengthen the statements of the theorems produced by $\llbracket \rrbracket_{iso}$ by pruning the unused hypotheses.

There are many ways to define what it means for a variable $x$ (e.g. a hypothesis) to be unused in a term (e.g. a proof) $p$. We say that a variable $x$ is definitionally unused in $p$ if $\exists$ a term $p'$ such that $p'$ is definitionally equal to $p$ and the free variables of $p'$ does not include $x$. It is easy to
exactly determine whether \( x \) is definitionally unused in \( p \): just strongly normalize \( p \) and check if \( x \) occurs in the free variables of the normal form. However, for some realistic applications, strong normalization often ran for hours and then ran out of memory on our machines with 32GB RAM. So, we use a publically available Coq plugin that avoids normalizing many subterms (e.g. whose free variables do not include \( x \)), and is yet guaranteed to return the exact answer. This plugin runs within a few seconds in all our applications so far. If it succeeds in eliminating \( x \), it also returns the term \( p' \) where \( x \) does not occur free. \( p' \) can be considered a proof of a stronger theorem which does not have the hypothesis \( x \).

As an example, consider a polymorphic proposition of the form \( \lambda (T:\text{Set}), \theta \), where \( \theta \) is some term. \( \llbracket \lambda (T:\text{Set}), \theta \rrbracket_{\text{iso}} := \lambda (T:\text{Set}) (T_2:\text{Set}) (T_r: \text{IsoRel } T T_2), \llbracket \theta \rrbracket_{\text{iso}}. \) We \( \eta \)-expand \( T_r \), say as variables \( R, RTot, ROne \), and then use the above-mentioned plugin, hoping one or both of \( RTot, ROne \) disappear in \( \llbracket \theta \rrbracket_{\text{iso}} \).

A more effective approach would be to aim for the following definition: a variable \( x \) is logically unused in \( p \) if \( \exists \) a term \( p' \) such that \( p' \) is propositionally equal to \( p \) and the free variables of \( p' \) does not include \( x \). It is impossible to solve this variant exactly, but we believe there are heuristics that would yield better results (stronger theorems) than the above approach in some applications.

### 5.3 Limitations of the IsoRel translation

\( \llbracket \rrbracket_{\text{iso}} \) fails for propositions where types of higher universes occur at certain places. In universal quantification, the quantified type must be in \( \text{Set} \) or \( \text{Prop} \). In inductively defined propositions, the types of indices and the types of arguments of constructors (except the parameters of the type) must be in \( \text{Set} \) or \( \text{Prop} \). In Section 3, we saw that the relations of the types at those positions may need to have the \( \text{Total} \) and/or the \( \text{OneToOne} \) properties. Unfortunately, it is not possible to systematically produce the proofs of those properties for types in higher universes:

Suppose we redefined, for \( i > 0 \), \( \llbracket \text{Type}_i \rrbracket_{\text{iso}} \) to be just like \( \llbracket \text{Set} \rrbracket_{\text{iso}} \). Then, the abstraction theorem for \( \text{Set:Type}_1 \) fails. Now, \( \llbracket \text{Set} \rrbracket_{\text{iso}} \) needs to be augmented to also produce the proofs of the \( \text{Total} \) and the \( \text{OneToOne} \) property for the relation \( \lambda (A A_2: \text{Set}), \text{IsoRel } A A_2 \). The latter property is not provable:

**Lemma 5.2.** There is no axiom-free proof of \( \text{OneToOne } (\lambda (A A_2: \text{Set}), \text{IsoRel } A A_2) \)

**Proof.** It is easy to produce a \( \text{Total} \) and \( \text{OneToOne} \) relation between the types \( \text{nat} \) and \( \text{nat} \), and between the types \( \text{nat} \) and \( \text{list True} \). Then, it is easy to see that \( \text{OneToOne } (\lambda (A A_2: \text{Set}), \text{IsoRel } A A_2) \) implies \( \text{nat} = \text{list True} \), which is unprovable in Coq: looking at the definition of \( = \) (Section 1), it is obvious that if \( u = v \) and \( u \) and \( v \) are closed, they must be definitionally equal. \( \square \)

\( \text{nat = list True} \) may be provable using the univalence axiom [The Univalent Foundations Program 2013, Sec. 2.10]. However, that axiom refutes UIP (Unicity of Identity Proofs) which is useful in many Coq developments. For example, the proof of the inj_pair2 lemma used in Section 4.2 uses the UIP axiom (proof irrelevance implies UIP). Also, UIP is needed for the justification of erasing (equality) proofs during the compilation of Coq programs [Letouzey 2004].

As an example of the above limitation, the IsoRel translation fails on the following because the index type is in a higher universe.

**Inductive** isNat : \( \forall (A: \text{Set}), \text{Prop} := \text{isnat : isNat } \text{nat} \).

**Set / Set.** Indeed, the IsoRel abstraction theorem for isNat is easily refutable.

Nevertheless, as explained in Section 1, \( \text{Set} \) and \( \text{Prop} \) suffice for many practical application domains, especially verification of computer and physical systems.

As explained in Section 5.1, \( \llbracket \text{Vx:A.B} \rrbracket_{\text{iso}} \) works differently in the cases when \( B: \text{Prop} \) and \( B: \text{Set} \). Thus, \( \llbracket \rrbracket_{\text{iso}} \) may produce ill-typed results for terms whose typing derivations use the rule Prop.
The interface has two type variables: \( Tm \) for the type of terms and \( BTm \) for the type of bound terms [Howe 1989, Sec. 2]. In the \( \lambda \) term \( \lambda x.t \), \((x, t)\) can be considered a bound term. Bound terms only support the \text{applyBtm} operation. 

\text{applyBtm}(x, t) u \text{ represents } t[u/x]. \text{ To define big step evaluation, given a term } (Tm), \text{ we need to figure out what kind of a term it is: a } \lambda, \text{ an application, a number, or a variable. The } \text{tmKind} \text{ operation does just that. It also allows limited access to subterms of a term.} 

Note that the interface never allows direct access to variables and can be instantiated even with de Bruijn terms and de Bruijn substitution (for \text{applyBtm}). Now, as shown in Figure 2, we can polymorphically define not only the big-step evaluation semantics (evaln), but also a notion of observational equivalence (obseq).

\( \llbracket \text{evaln} \rrbracket \) is a proof that on related instantiations of the above interface, on related inputs evaln produces related outputs. Given two concrete implementations of lambda terms and bound terms, say \( LTm \) and \( LBTm \), we instantiate \( Tm, Tm_2 := LTm; BTm, BTm_2 := LBTm; Tm_r, BTm_r := \alpha \) equality. This instantiation of \( \llbracket \text{evaln} \rrbracket \) is a proof that on \( \alpha \) equal inputs, evaln produces \( \alpha \) equal outputs.

\footnote{Our interface abstracts over both named and de Bruijn style variable bindings, and thus we were able to use parametricity (AnyRel translation) to also obtain the proof that the big-step operational semantics is preserved when changing the representation from de Bruijn indices to named-variable representation.}
The idea of globally enforcing that parametricity relations satisfy some desirable properties was inspired by Krishnaswami and Dreyer [2013]: they globally enforce a zigzag-completeness property. However, that property is unrelated to our work which enforces Total and OneToOne properties.

For applications described in Section 6, Coq developments typically employ rewriting [Sozeau 2010] and other proof-search mechanisms. For example, Cohen et al. [2013] use a library of proof search hints to semi-automatically refine algorithms (e.g. Strassen’s matrix product) from simple
data structures to complex but efficient ones. Proof search mechanisms have less well-defined semantics and reliability properties. Our translation directly produces fully elaborated proof terms. It is more automatic and preserves the meaning of polymorphic propositions.

Zimmermann and Herbelin [2015] built a Coq plugin to transfer theorems across isomorphisms. Instead of using proof-search mechanisms, they structurally recurse over the statement of the to-be-transferred theorem. However, they consider a smaller class of propositions. Inductively defined propositions were not considered. Also, propositions produced by pattern-matching (e.g. \texttt{obsEq} in Section 6) were not considered.

Transfer tools also exist for other proof assistants such as Isabelle/HOL [Huffman and Kunčar 2013]. However, our problem is more general because HOL doesn’t have dependent types.

Several works [Atkey et al. 2014; Bernardy et al. 2015; Krishnaswami and Dreyer 2013] have constructed \textit{meta-theoretic} parametric models of variants of dependent type theory. Such models may be useful in proving the consistency of such type theories and justify various useful extensions. Our focus is not on the consistency of Coq or justifying extensions to Coq. Like Keller and Lasson [2012], our translation produces proofs \textit{expressed in} Coq (Gallina) that are useful (Section 6) without needing any extension to Coq.

There is one approach that is even more general than our \textit{weak} IsoRel translation (⟦⟧iso): Homotopy Type Theory (HoTT) [The Univalent Foundations Program 2013] is an area of active research. It aims to serve as a foundation for full-fledged proof assistants like Coq. The main advantage of HoTT is that it validates the \textit{univalence principle} which says that, isomorphic types (more generally, equivalent types), even those in higher universes, are \textit{equal}. Also, as usual, every function, including the ones that return propositions, produces equal outputs on equal inputs. Equal propositions are, of course, logically equivalent! Thus, HoTT may be able to provide some of the benefits that our weak IsoRel translation provides in Coq. However, those benefits come at a cost. Section 5.3 explained that univalence refutes UIP and why that can be problematic in Coq.

As mentioned before, our \textit{strong} IsoRel translation (⟦⟧sIso) does not always require the two instantiations to be isomorphic. In Section 3, we saw examples where one or both of the \texttt{Total} and \texttt{OneToOne} assumptions are not needed. In contrast, to use univalence to conclude that two types are equal, one needs to always provide an isomorphism (more generally, an equivalence [The Univalent Foundations Program 2013, Sec. 4]). Thus, even in HoTT, a version of our \textit{strong} IsoRel translation may be useful. Also, there it may be able to work for \textit{all} universes (Section 5.3).

\textbf{Conclusion.} We presented a new parametricity translation for a significant fragment of Coq. Unlike the existing translations, it ensures that parametrically related propositions are logically equivalent. This allows us to obtain free proofs that polymorphic propositions behave uniformly.

Our goal was to develop a principled way to get free Coq proofs for our compiler-verification project. We believe that our translation would be useful in many other application domains as well. Our implementation and test-suite are publically available on Github: https://github.com/aa755/paramcoq-iff.

\textbf{ACKNOWLEDGMENTS}

We thank Marc Lasson for help with understanding his and Chantal Keller’s implementation of the paramcoq plugin, in particular the proof obligations that it generates. We also thank the anonymous ICFP 2017 reviewers for their detailed and constructive feedback.

This material is based upon work supported by the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.
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A  APPENDIX

A.1 Universe-polymorphic inductive types are problematic for $\texttt{Set} := \texttt{Prop}$

Consider the following universe polymorphic inductive type:

\[
\text{Inductive \texttt{list}@\{i\} (A : \texttt{Type}@\{i\}) : \texttt{Type}@\{i\} :=}
\]
\[
| \texttt{nil} : \texttt{list} A \\
| \texttt{cons} : A \rightarrow \texttt{list} A \rightarrow \texttt{list} A. \\
\]

The definition can be considered quantified over the universe $i$. The AnyRel parametricity translation of $\texttt{list}$, whether in the inductive style or in the deductive style, would also have to be polymorphic. Recall from Section 2 that we have $\texttt{Type}_0 := \texttt{Prop}$ and $\texttt{s}$ otherwise. To the best of our knowledge, Coq’s syntax for universe polymorphism is too restrictive to allow a definition like the following (see the type of $A_r$):

\[
\text{Fixpoint \texttt{list}_R}@\{i\} (A : \texttt{Type}@\{i\}) (A_2 : \texttt{Type}@\{i\}) (A_r : A \rightarrow A_2 \rightarrow \texttt{if} i \text{ is } 0 \text{ then } \texttt{Prop} \text{ else } \texttt{Type}@\{i\}) ... \\
\]

An inductive-style translation would also suffer from the same problem. The problem doesn’t arise if we choose $\texttt{s} := s$ for every universe. Then, the type of $A_r$ would simply be $A \rightarrow A_2 \rightarrow \texttt{Type}@\{i\}$.

A.2 Deductive-style AnyRel translation of inductive types: the general case

Consider a general inductive type $T$ of the form:

\[
\text{Inductive } T \ (p_1 : P_1) \ldots (p_n : P_n) : \forall (i_1 : I_1) \ldots (i_k : I_k), s := \\
| c_1 : C_1 \\
| \ldots \\
| cm : C_m. \\
\]

Recall from Section 2.1 that $s$ denotes a universe ($\texttt{Prop}$ or $\texttt{Type}_i$). Now we describe the deductive-style translation of $T$. First, we define the corresponding generalized equality proposition, as explained in Section 2.3:

\[
\text{Inductive } T_{\text{indicesEq}} \\
(p_1 : P_1) (p_{1_2} : P_{1_2}) (p_{1_r} : \prod P_{1_2} p_{1_2} \ldots (p_n : P_n) (p_{n_2} : P_{n_2}) (p_{n_r} : \prod P_{n_2} p_{n_2}) \\
(i_1 : I_1) (i_{1_2} : I_{1_2}) (i_{1_r} : \prod I_{1_2} i_{1_2} \ldots (i_k : I_k) (i_{k_2} : I_{k_2}) (i_{k_r} : \prod I_{k_2} i_{k_2}) : \\
\forall (i_{1_r} : \prod I_{1_2} i_{1_2}) \ldots (i_{k_r} : \prod I_{k_2} i_{k_2}) : \texttt{Prop} := \\
| T_{\text{refl}} : T_{\text{indicesEq}} p_{1_1} p_{1_r} \ldots p_n p_{n_2} p_{n_r} i_1 i_{1_2} \ldots i_k i_{k_2} i_{k_r} : \\
\]

In the future, instead of generating one such inductive proposition for each inductive type, we plan to have only one for each class of inductives that have the same number of indices (e.g. $T$ has $k$ indices).

Let $t$ be a variable of the first class (Section 2.1) such that $t$ is distinct from any variable in the above definitions. Now, we can define the AnyRel relation for the above inductive type ($T$):
Fixpoint $T_r (p_1 : P_1) (p_1 : P_1) (p_1 : [P_1] p_1 p_1) \ldots (p_n : P_n) (p_n : P_n) (p_n : [P_n] p_n p_n)$

$(i_1 : I_1) (i_1 : I_1) (i_1 : [I_1] i_1 i_1) \ldots (i_k : I_k) (i_k : I_k) (i_k : [I_k] i_k i_k)$

$(t : T p_1 \ldots p_n i_1 \ldots i_k) (t_2 : T p_2 \ldots p_n i_2 \ldots i_k) \{ \text{struct } t \} : \hat{s} :=$

match $t$ in $T$ \ldots $i_1 \ldots i_k$ return $\text{Ret}_o$ with

\[\vdots\]

| $cu \ a_1 \ldots a_l \Rightarrow$
  | \hspace{1cm} match $t_2$ in $T$ \ldots $i_2 \ldots i_k$ return $\text{Ret}_i$ with
  | \hspace{1cm} \vdots
  | $al : [A] a_1 a_2 \& \ldots \& [A] a_1 a_2 \&$
  | $\ldots \& T \text{indicesEq } p_1 p_1 p_1 p_1 p_1 \ldots p_n p_n p_n p_n p_n$

$CI_1 CI_1 \ldots CI_k CI_k CI_k CI_k i_1 \ldots i_k$

\[\vdots\]

| $cu \ldots \Rightarrow \lambda i_1 \ldots i_k r, \text{False}$
| \hspace{1cm} \vdots
| \hspace{1cm} end
| \hspace{1cm} $i_1 \ldots i_k$.

In the above, $1 \leq u \leq m$, $1 \leq v \leq m$, and $u \neq v$. Also, $Cu$, the type declaration for $cu$ in the definition of $T$ is:

$\forall (a_1 : A_1) \ldots (a_l : A_l), T p_1 \ldots p_n CI_1 \ldots CI_k.$

$\text{Ret}_o$ is $\forall (i_1 \ldots : [I_1] i_1 i_1) \ldots (i_k \ldots : [I_k] i_k i_k), \hat{s}.$

$\text{Ret}_i$ is a refined version of $\text{Ret}_o$, where the variables $i_1, \ldots, i_k$ are respectively substituted with $CI_1, \ldots, CI_k$.

The translation of constructors (e.g. $cu$) of $T$ is straightforward. Note that the abstraction theorem (Theorem 1) already determines the type of the result of the translation. The result of translating a constructor is a function that packages some of its arguments into dependent pairs whose types were shown in the above definition. The innermost member of such dependent pairs is always the canonical proof of the corresponding generalized equality proposition. For example, for the constructors of $T$, the innermost member is always of the form $T\text{refl} \ldots$.

In the case of mutual inductive definitions, we produce mutually recursive functions.

A.3 Deductive-style AnyRel translation of pattern matching on inductive types: the general case

Now we will see how to translate a pattern match on a discriminee of the inductive type $T$ defined above (Section A.2). Consider a term

$m :=$

match $(d : D)$ as $t$ in $T$ \ldots $i_1 \ldots i_k$ return $R$ with

\[\vdots\]

| $cu \ a_1 \ldots a_l \Rightarrow bu$
| \hspace{1cm} \vdots
| \hspace{1cm} end.
Note that the discriminee $d$ has type $D$. In Coq, $d$ need not be a variable. In the representation of terms in Coq’s kernel, the type of discriminee is not stored. Our reifier computes that type and includes it in the reified terms. Below, we will see that $D$ is needed in the translation. Intuitively, the translation uses $\llbracket d \rrbracket$, the translation of the discriminee. Note that $\llbracket d \rrbracket \llbracket D \rrbracket d_2$. $D$ must be of the form $T dP_1 \ldots dP_n dI_1 \ldots dI_k$.

Recall [Chlipala 2011, Sec 8.2] that in $m$, the return type $R$ can mention the variables $i_1 \ldots i_k$ and $t$. (Also, those variables are bound only in $R$.) In other words, the return type of a match is a function of the discriminee and the indices of the (co-)inductive type of the discriminee. While checking each branch, Coq substitutes those variables in $R$ to values corresponding to the constructor of the branch. For example, $bu$ must be of type:

$$R \ [ CI_1 / i_1, \ldots, CI_k / i_k; (cu \ dP_1 \ldots dP_n \ a_1 \ldots a_l ) / t ]$$

The translation of the match term $m$ shown above is:

$$\begin{align*}
\text{match } d \text{ as } t \text{ in } T \ldots i_1 \ldots i_k \text{ return } Ret_{out} \text{ with} \\
&: \\
&| cu \ a_1 \ldots a_l \Rightarrow \\
&\quad \text{match } d_2 \text{ as } t_2 \text{ in } T \ldots i_2 \ldots i_{k_2} \text{ return } Ret_{in} \text{ with} \\
&\quad : \\
&\quad | cu \ a_1 \ldots a_l \Rightarrow \lambda \ i_1 \ldots i_k \ t_r, \\
&\quad \quad \text{match } t_r \text{ in } \ldots \text{ return } \ldots \text{ with} \\
&\quad \quad | \text{existT } a_1 \ t_r \Rightarrow \\
&\quad \quad \quad \ldots \\
&\quad \quad \text{match } t_r \text{ in } \ldots \text{ return } \ldots \text{ with} \\
&\quad \quad | \text{existT } a_r \ pdeq \Rightarrow \\
&\quad \quad \quad \text{match } pdeq \text{ as } \ldots \text{ return } \ldots \text{ with} \\
&\quad \quad \quad | T_{refl} \Rightarrow [bu] \\
&\quad \quad \quad \text{end} \\
&\quad \quad \text{end} \\
&\quad \quad \ldots \\
&\quad \quad \ldots \\
&\quad \quad | cu \ldots \Rightarrow \lambda \ i_1 \ldots i_k \ t_r, \text{False_rect } t_r \\
&\quad \quad : \\
&\quad \quad \text{end} \\
&\quad \quad \ldots \\
&\quad \quad \ldots \\
&\quad \llbracket d_{I_1} \rrbracket \ldots \llbracket d_{I_k} \rrbracket \llbracket d \rrbracket.
\end{align*}$$

Next, we describe the terms $Ret_{out}$ and $Ret_{in}$ mentioned in the above definition. Given these, it should be easy to figure out the return types (of the inner match terms) that have been denoted by $\ldots$ for brevity. Also, our implementation (as a Coq function) is publically available in a Github repository (https://github.com/aa755/paramcoq-iff).

First we define the term $ma$ which is obtained by replacing the discriminee $d$ in $m$ by the variable $t$:

$$\begin{align*}
\text{match } d_2 \text{ as } t_2 \text{ in } T \ldots i_2 \ldots i_{k_2} \text{ return } Ret_{in} \text{ with} \\
&: \\
&| cu \ a_1 \ldots a_l \Rightarrow \\
&\quad \text{match } t_r \text{ in } \ldots \text{ return } \ldots \text{ with} \\
&\quad | \text{existT } a_r \ pdeq \Rightarrow \\
&\quad \quad \text{match } pdeq \text{ as } \ldots \text{ return } \ldots \text{ with} \\
&\quad \quad | T_{refl} \Rightarrow [bu] \\
&\quad \quad \text{end} \\
&\quad \quad \ldots \\
&\quad \quad \ldots \\
&\quad \quad | cu \ldots \Rightarrow \lambda \ i_1 \ldots i_k \ t_r, \text{False_rect } t_r \\
&\quad \quad : \\
&\quad \quad \text{end} \\
&\quad \quad \ldots \\
&\quad \quad \ldots \\
&\quad \llbracket d_{I_1} \rrbracket \ldots \llbracket d_{I_k} \rrbracket \llbracket d \rrbracket.
\end{align*}$$
\[ ma := \]
match \( t \) as \( t \) in \( T \) \( \ldots \) \( i_1 \ldots i_k \) return \( R \) with
\[ \begin{align*}
| cu \ a_1 \ldots a_l \Rightarrow bu \\
\end{align*} \]
end.

Note that in the above definition of the term \( ma \), the occurrence of \( t \) after \( as \) is a bound variable and not substitutable. The occurrence at the position of discriminee is substitutable. \( Ret_{out} \) and \( Ret_{in} \) are obtained by performing substitutions in the following term:
\[ Ret := \forall (i_1r :: \llbracket I \rrbracket \ i_1 i_2) \ldots (i_kr :: \llbracket I_k \rrbracket \ i_k i_2k) (t_r :: \llbracket D \rrbracket \ t t_2), \llbracket R \rrbracket \ ma ma_2 . \]

Now, we can define \( Ret_{out} \) and \( Ret_{in} \) as follows:
\[ Ret_{out} := Ret \left[ dI_1 / i_1, \ldots, dIk / i_k, (cu \ dP_1 \ldots dP_n a_1 \ldots a_l) / t \right] \]
\[ Ret_{in} := Ret \left[ CI_1 / i_1, \ldots, CI_k / i_k, (cu \ dP_1 \ldots dP_n a_1 \ldots a_l) / t \right] \]

### A.4 AnyRel translation of fixpoints

Our translation of \( \text{fix} \) (or \( \text{Fixpoint} \)) terms is largely as described by Keller and Lasson [2012]. Roughly speaking, \( \llbracket \text{fix} F \rrbracket \) is just \( \text{fix} \llbracket F \rrbracket \). The translation of \( \text{fix} \) terms depends a tiny bit on how the inductives are translated. Unlike in Agda, each \( \text{fix} \) term in Coq has a designated \text{struct} argument of an inductive type. Coq requires that any recursive call should be made on a structural subterm of the \text{struct} argument. Coq can often infer the \text{struct} argument and in this paper, we have usually omitted the annotations stating the \text{struct} argument. In \( \text{Vec}r \), (Section 2.3), which is the deductive-style translation of the type \( \text{Vec}r \), \( v \) is the \text{struct} argument. Suppose we are translating \( \text{fix} F \), where \( F \) is of the form \( \lambda \ldots (v : I) \ldots \). Suppose the \text{struct} argument is \( v \). If \( I \) was translated in inductive style (e.g. when \( I \) is a proposition), we must pick \( v_r \) as the \text{struct} argument in the translation of \( \text{fix} F \). Coq guarantees that \( F \) only makes recursive calls on subterms of \( v \), which are obtained by pattern matching on \( v \). In the inductive-style translation of \( F \), those matches will be translated to pattern matching on \( v_r \). In contrast, if \( I \) was translated in deductive style (e.g. when \( I \) is a type), those matches will be translated into matches on \( v \) and \( v_2 \). Thus we can choose either \( v \) or \( v_2 \) as the \text{struct} argument. We choose \( v \).

A problem not mentioned in the literature, but partially addressed in the implementation by Keller and Lasson [2012], is that the translation of \( \text{fix} F \) needs to generate unfolding equations of the form \( \text{fix} F = F (\text{fix} F) \). For some pathological programs, these equations are \text{unprovable}.

Marc Lasson gave us the following example where the unfolding equation is unprovable:
\[ \text{Fixpoint} \ zero (A : \text{Type}) (x : A) (p : x = x) \{ \text{struct} \ p \} := 0. \]
To ensure strong normalization, a \( \text{fix} \) term only reduces (unfolds) when the \text{struct} argument is in head normal form. In the definition above, it is impossible to prove that \( p \) is equal to something in the head normal form [Hofmann and Streicher 1998].

### A.5 Locating proofs in the supplementary material

For proofs in Section 3 and Section 4, see the files \( \text{Pi.v} \) and \( \text{IWTP.v} \) in the supplementary material.

#### A.5.1 The importance of the triviality property.
See the admitted lemma in the file \( \text{triviality.v} \) in the supplementary material.

### A.6 1-phase strong IsoRel translation

As mentioned in Section 5, it is natural to consider another design, where the main translation itself determines the minimally needed assumptions on type variables (or variables denoting type
families) by, e.g., analysing the bodies of lambda terms, and directly uses the appropriately minimal type for type variables.

Such a translation seems hard to implement for several reasons. It would be non-compositional, while translating an application of some function $F$ to some type $T$, we may need to prune the translation of $T$ depending on the translation of $F$.

Also, we are only interested in removing the top level arguments of an abstraction theorem. It is not clear whether there is an advantage (disadvantage?) to removing the arguments of $\lambda$ subterms that appear elsewhere.

### A.7 Abstraction theorems for $\text{obseq}$

#### A.7.1 $\llbracket \text{obseq} \rrbracket$ (AnyRel translation).

\[
\forall (Tm Tm_2 : \text{Set}) (Tm_r : Tm \to Tm_2 \to \text{Prop}) (BTm BTm_2 : \text{Set}) (BTm_r : BTm \to BTm_2 \to \text{Prop})
\]
\[
(\text{applyBtm} : BTm \to Tm \to Tm) (\text{applyBtm}_2 : BTm_2 \to Tm_2 \to Tm)
\]
\[
(\text{applyBtm}_r : \forall (b : BTm) (b_2 : BTm_2) (b_r : BTm_r b b_2) (a : Tm) (a_2 : Tm_2) (a_r : Tm_r a a_2),
\]
\[
Tm_r (\text{applyBtm} b a) (\text{applyBtm}_2 b_2 a_2)
\]
\[
(\text{tmKind} : Tm \to \text{TMKind} Tm BTm) (\text{tmKind}_2 : Tm_2 \to \text{TMKind} Tm_2 BTm_2)
\]
\[
(\text{tmKind}_r : \forall (a : Tm) (a_2 : Tm_2) (a_r : Tm_r a a_2),
\]
\[
\text{TmKind}_r Tm Tm_2 Tm_r BTm BTm_2 BTm_r (\text{tmKind} a) (\text{tmKind}_2 a_2)
\]
\[
(tl : Tm) (tl_2 : Tm_2) (tl_r : Tm r tl tl_2) (tr : Tm) (tr_2 : Tm_2) (tr_r : Tm_r tr tr_2),
\]
\[
(\text{obseq} \text{TM} BTm \text{applyBtm} \text{tmKind} tl tr)
\]
\[
(\text{obseq} Tm_2 BTm_2 \text{applyBtm} \text{tmKind}_2 tl_2 tr_2)
\]
\[
\to \text{Prop}.
\]

#### A.7.2 $\llbracket \text{obseq} \rrbracket_{\text{iso}}$ (weak IsoRel translation). Recall that for any relation $R$ between any two propositions $A$ and $B$, Total $R$ is logically equivalent to $(\text{IffProps} R \land \text{CompleteRel} R)$.

\[
\forall (Tm Tm_2 : \text{Set}) (Tm_r : \text{IsoRel} Tm Tm_2) (BTm BTm_2 : \text{Set}) (BTm_r : \text{IsoRel} BTm BTm_2)
\]
\[
(\text{applyBtm} : BTm \to Tm \to Tm) (\text{applyBtm}_2 : BTm_2 \to Tm_2 \to Tm)
\]
\[
(\text{applyBtm}_r : \forall (b : BTm) (b_2 : BTm_2) (b_r : BTm_r b b_2) (a : Tm) (a_2 : Tm_2) (a_r : \pi_1 Tm_r a a_2),
\]
\[
\pi_1 Tm_r (\text{applyBtm} b a) (\text{applyBtm}_2 b_2 a_2)
\]
\[
(\text{tmKind} : Tm \to \text{TMKind} Tm BTm) (\text{tmKind}_2 : Tm_2 \to \text{TMKind} Tm_2 BTm_2)
\]
\[
(\text{tmKind}_r : \forall (a : Tm) (a_2 : Tm_2) (a_r : \pi_1 Tm_r a a_2),
\]
\[
\text{TmKind}_r Tm Tm_2 (\pi_1 Tm_r) BTm BTm_2 (\pi_1 BTm_r) (\text{tmKind} a) (\text{tmKind}_2 a_2)
\]
\[
(tl : Tm) (tl_2 : Tm_2) (tl_r : \pi_1 Tm_r tl tl_2) (tr : Tm) (tr_2 : Tm_2) (tr_r : \pi_1 Tm_r tr tr_2),
\]
\[
\text{IsoRel} (\text{obseq} \text{TM} BTm \text{applyBtm} \text{tmKind} tl tr) (\text{obseq} Tm_2 BTm_2 \text{applyBtm} \text{tmKind}_2 tl_2 tr_2).
\]

#### A.7.3 $\llbracket \text{obseq} \rrbracket_{\text{sIso}}$ (strong IsoRel translation). The conclusion is the same as before ($\llbracket \text{obseq} \rrbracket_{\text{iso}}$) but 3 assumptions (OneToOne $Tm_r$, Total $BTm_r$, OneToOne $BTm_r$) were removed by the second phase (Section 5.2).
∀ (Tm Tm₂ : Set) (Tmₚ : Tm → Tm₂ → Prop) (Tmₚ tot : Total Tmₚ)
(BTm BTm₂ : Set) (BTmₚ : BTm → BTm₂ → Prop)
(applyBTm : BTm → Tm → Tm) (applyBTm₂ : BTm₂ → Tm₂ → Tm₂)
(applyBTmₚ : ∀ (b : BTm) (b₂ : BTm₂) (bₚ : BTmₚ b b₂) (a : Tm) (a₂ : Tm₂) (aₚ : Tmₚ a a₂),
Tmₚ (applyBTm b a) (applyBTm₂ b₂ a₂))
(tmKind : Tm → TmKind Tm BTm) (tmKind₂ : Tm₂ → TmKind Tm₂ BTm₂)
(tmKindₚ : ∀ (a : Tm) (a₂ : Tm₂) (aₚ : Tmₚ a a₂),
Tmₚ (tmKind a a₂))
(tl : Tm) (tl₂ : Tm₂) (tlₚ : Tmₚ tl tl₂) (tr : Tm) (tr₂ : Tm₂) (trₚ : Tmₚ tr tr₂),
IsoRel (obseq Tm BTm applyBTm tmKind tl tr) (obseq Tm₂ BTm₂ applyBTm₂ tmKind₂ tl₂ tr₂).

A.8 Tabulation of assumptions in lemmas in Section 3 and 4

A.8.1 Canonical Propositions. Recall that for any relation \( R \) between any two propositions \( A \) and \( B \), Total \( R \) is logically equivalent to (IffProps \( R \) ∧ CompleteRel \( R \)). Also, OneToOne \( R \) is a trivial consequence of proof irrelevance.

Universal Quantification. (∀x:A,B):Prop

| proof of | assumptions on | axioms | lemma |
|----------|----------------|--------|-------|
| IffProps \[∀x:A,B\] | Total | 3.2 |
| CompleteRel \[∀x:A,B\] | | 3.1 |

Inductive propositions. (IWP I A B AI BI i):Prop

| proof of | assumptions on | axioms | lemma |
|----------|----------------|--------|-------|
| IffProps \[I\] \[A\] \[B\] | OneToOne Total Total | 3.4 |
| CompleteRel \[I\] \[A\] \[B\] | OneToOne Total Total | proof irrelevance 3.4 |

For general inductive propositions, index types behave like (regarding the use of assumptions) \( I \), types of non-recursive arguments (Section 4.2) to constructors (except parameters) behave like \( A \), and the domain types in recursive arguments behave like \( B \).

A.8.2 Canonical Types.

dependent function types. (∀x:A,B):Set

| proof of | assumptions on | axioms | lemma |
|----------|----------------|--------|-------|
| \[A\] \[B\] | Total, OneToOne Total | proof irrelevance 4.1 |
| OneToOne \[A\] \[B\] | OneToOne Total | function extensionality 4.2 |

Inductive types. (IWT I A B AI BI i):Set

| proof of | assumptions on | axioms | lemma |
|----------|----------------|--------|-------|
| \[I\] \[A\] \[B\] | Total, OneToOne Total, OneToOne | proof irrelevance 4.3 |
| OneToOne \[A\] \[B\] | OneToOne Total | proof irrelevance, function extensionality 4.4 |
B CORRECTNESS OF THE WEAK ISOREL TRANSLATION

In this section, we discuss a formal proof of correctness of \( \llbracket \cdot \rrbracket_{iso} \) for CoC\(^{-}\), a CoC-like core calculus.

Figure 3 (adapted from [Keller and Lasson 2012]) shows the subtyping rules of CoC\(^{-}\). W.r.t. CoC, the omissions are \( \text{Type}_0 : \text{Type}_1 \) and \( \text{Prop} : \text{Set} \). Recall that \( \text{Type}_0 \) is written as \( \text{Set} \) in Coq. We can add back the former rule if Coq gave us terms that make explicit all uses of that subtyping rule (Section 5.3). For example, it would be sensible for a future version of Coq’s typechecker to furnish a typing derivation for terms that it deems well-typed. The problem with the latter rule is explained below in Appendix B.2.1.

Figure 4 (adapted from [Keller and Lasson 2012]) shows the typing rules of CoC\(^{-}\). \( \equiv \Gamma \) is essentially Coq’s \( \beta \)-equivalence, except that it maintains an invariant (Appendix B.1) that prevents capture during the translation. The only omission (highlighted) is that when constructing a proposition using universal quantification, one can only quantify over types in \( \text{Set} \) or \( \text{Prop} \). Our proof of the uniformity of universal quantification (Lemma 3.2) needs the \( \text{Total} \) property for the relation of the quantified type. We were unable to systematically build that property for types in higher universes (Section 5.3, Section 4).

Recall that \( \llbracket \cdot \rrbracket_{iso} \) is implemented as a \emph{structurally recursive} function in Coq (Gallina). Its input is obtained by a reifier that translates the OCaml representation of Coq terms to a Coq datatype. We use the inverse operation (reflection) to declare the output \( \llbracket \cdot \rrbracket_{iso} \) in Coq’s environment, but only after Coq typechecks the output of reflection. (We use a monad to automate these steps.)

The grammar of CoC\(^{-}\) is essentially the grammar of CoC presented in Section 2.1, except that we make explicit some implementation details: Recall (Section 5.1) that \( \llbracket \cdot \rrbracket_{iso} \) needs to make different choices depending on whether a type is in the universe \( \text{Set} \), \( \text{Prop} \) or \( \text{Type}_i \) \((i > 0)\). For example, it needs to pair the relations of types/propositions in \( \text{Set} \) or \( \text{Prop} \) with proofs of \( \text{Total} \) and \( \text{OneToOne} \) properties. As a result, at some places (e.g. \( \llbracket \lambda \ldots \rrbracket_{iso} \)), it needs to project the relations out of such pairs. To ensure the simplicity of \( \llbracket \cdot \rrbracket_{iso} \), we push the task of determining the universe of types to the reifier, which has access to Coq’s typechecker. The terms produced by the reifier has flags indicating the universe information wherever needed (Section 5.1).

We make these flags explicit in the grammar of CoC\(^{-}\): In \( \llbracket \lambda (x:A), B \rrbracket \), we use a 2-letter subscript respectively denoting the universes of \( A \) and \( B \). The letters are: \( S \) for \( \text{Set} \), \( P \) for \( \text{Prop} \), and \( T \) for \( \text{Type}_i \) \((i > 0)\). For example, the syntax \( \llbracket \forall_{SP} (x:A), B \rrbracket \) implies \( A : \text{Set} \) and \( B : \text{Prop} \). Similarly, \( \llbracket \lambda_{SP} (x:A), B \rrbracket \) implies \( A : \text{Set} \). We will omit the subscripts in contexts where they do not matter.

Unlike \( \llbracket \cdot \rrbracket \), even for terms in CoC\(^{-}\), \( \llbracket \cdot \rrbracket_{iso} \) produces terms that are \emph{not} in CoC\(^{-}\) (not even in CoC). For example, \( \llbracket \text{Set} \rrbracket_{iso} = \lambda (A A_2 : \text{Set}) , \text{IsoRel} A A_2 \). \text{IsoRel} is defined using \( \Sigma \) types, which are missing in CoC\(^{-}\). Instead of defining an extended core calculus for interpreting the output of \( \llbracket \cdot \rrbracket_{iso} \), we take the luxury of interpreting it in Coq (CiC). Also, our translation invokes (transparent) lemmas proved in Coq. In the proofs in this section, we assume that the proof terms corresponding to those lemmas indeed have the types proven in Coq.
B.1 Avoiding variable capture in parametricity translations

As mentioned before (Section 5.1.1), except for the construction of proofs of the Total and One-ToOne properties, the correctness argument for \( \llbracket \text{iso} \rrbracket \) is almost identical to the correctness argument for \( \llbracket \rrbracket \): one proves that the translation preserves substitution, then reduction, and finally typehood [Keller and Lasson 2012, Lemma 2, Theorem 1]. However, we needed to make some assumptions of those theorems explicit. (We have done some parts of the proof in Coq, just to increase confidence in our paper proof.) For example, Theorem 1 of Keller and Lasson [2012] doesn’t hold for input terms that have shadowed bound variables: \( \llbracket \lambda (x: \text{nat}) (x: \text{Vec nat } x), x \rrbracket := \lambda (y: \text{nat}) (y_2: \text{nat}) (y_3: \text{nat}, x x_2) (x: \text{Vec nat } x) (x_2: \text{Vec nat } x_2) (x_3: \text{Vec}_r \text{nat } x, x x_2, x_r) \), which is ill-typed: the highlighted arguments to \( \text{Vec}_r \) have incorrect type. The problem is easily rectified by \( \alpha \) renaming the input: \( \llbracket \lambda (y: \text{nat}) (x: \text{Vec nat } y), x \rrbracket := \lambda (y: \text{nat}) (y_2: \text{nat}) (y_3: \text{nat}_r y y_2) (x: \text{Vec nat } y) (x_2: \text{Vec nat } y_2) (x_3: \text{Vec}_r \text{nat } y, y r x x_2), x_r \).

Because \( \llbracket \text{iso} \rrbracket \) uses \( \llbracket \rrbracket \) at its core, it suffers from the same problem. In general, a natural way to fix the problem is to \( \alpha \)-rename the input to ensure that there are no repeated bound variables. However, a weaker condition suffices: the input must be in Barendregt’s convention. Formally, a closed term should have no shadowed bound variables (nested bound variables with same name). Open terms in a typing context, say \( \Gamma \), must satisfy an additional property: their bound variables should be distinct from variables in \( \Gamma \). We believe that using the weaker condition simplified some of our proofs in the next two subsections.

Recall that we have 5 disjoint classes of variables (Section 2.1). The input must only have variables of the first class: to avoid capture the other classes are reserved for use by the translation. Also, the variable \( c \) must not occur in the input because it is reserved for translating universes. We believe that having separate classes of variables resulted in simpler proofs. Similar techniques have been used before in mechanized proofs about CPS translation [Dargaye and Leroy 2007].
safe₁⁻¹ denotes a conjunction of such capture-safety conditions on the input \( t \) in the context that binds variables \( l \): bound variables of \( t \) are disjoint from the variables \( l \), there is no shadowing of bound variables in \( t \), \( \text{freeVars} \) \( t \) \( \subseteq \) \( l \), all variables in \( t \) are of the first class, and the variable \( c \) does not occur in \( t \). In \( \lambda x : A . B \) and \( \forall x : A . B \), the “no shadowing” condition ensures that the variable \( x \) does not occur in the bound variables of \( B \). \( \text{safe}_1 \) additionally requires that \( x \) does not occur in the bound variables of \( A \). We believe this additional condition is not necessary, but our current proof of Lemma B.1 uses it.

\( \text{safe}_1 \) is sufficient for \( \llbracket t \rrbracket_{\text{iso}} \) to be well-defined up to \( \alpha \), thus eliminating the possibility of capture:

**Lemma B.1.** \( \text{safe}_1 t_1 \rightarrow \text{safe}_1 t_2 \rightarrow t_1 =_\alpha t_2 \rightarrow \llbracket t_1 \rrbracket_{\text{iso}} =_\alpha \llbracket t_2 \rrbracket_{\text{iso}} \)

**B.2 Preservation of substitution**

Because the typing rules of CoC mention \( \beta \) equivalence (Figure 4), in this and the next subsection, we prove that \( \llbracket \rrbracket_{\text{iso}} \) preserves \( \beta \) equivalence.

In a context that binds variables \( l \), consider the term \((\lambda (x : A), b) t \). In CoC, this term will \( \beta \) reduce to \( b [ t / x] \). Thus, we need to characterize \( \llbracket b [ t / x] \rrbracket_{\text{iso}} \). As explained in the previous subsection, we require that the input to \( \llbracket \rrbracket_{\text{iso}} \) satisfies the \( \text{safe}_1 \) property. Thus, in CoC, we use a substitution operation that preserves it. Let \( b[t/x]_l \) denote the substitution of \( t \) for \( x \) in \( b \), performed in the following way: First \( b \) is \( \alpha \) renamed to \( b' \), such that its bound variables are disjoint from all the variables of \( t \), and \( \text{safe}_{x; b'} \). Finally, we perform a naive structurally recursive substitution, say \( \text{unsafeSubst} \), of \( t \) for \( x \) in \( b' \), without doing any further \( \alpha \) renaming. It is easy to prove that \( \text{safe}_1 ((\lambda (x : A), b) t) \) implies \( \text{safe}_1 (b[t/x]_l) \)

To understand \( \llbracket b[t/x]_l \rrbracket_{\text{iso}} \), it is helpful to understand the free variables of \( \llbracket b \rrbracket_{\text{iso}} \). Let \( \text{lv}_2 \) denote a function from lists of variables to lists of variables, such that \( \text{lv}_2 \), \( l \) \( l \) \( ++ \) \( (\lambda x, x_2) l \) \( ++ \) \( (\lambda x, x_2) l \). Intuitively, for every variable \( x \) in \( l \), the list \( (\text{lv}_2, l) \) contains not only \( x \) but also \( x_2 \) and \( x_3 \).

**Lemma B.2.** \( \text{safe}_1 t \rightarrow \text{freeVars} \llbracket t \rrbracket_{\text{iso}} \subseteq \text{lv}_2 \) (\( \text{freeVars} \) \( t \))

The proof is by structural induction on \( t \).

Thus, in \( b[t/x]_l \) \( \text{iso} \), if we perform the substitution after the translation of \( b \), we will need to substitute for not only \( x \) but also \( x_2 \) and \( x_3 \):

**Lemma B.3.** \( \text{safe}_{c; b} \) \( b \rightarrow \) \( \text{safe}_1 t \rightarrow \llbracket b[t/x]_l \rrbracket_{\text{iso}} =_\alpha \llbracket b \rrbracket_{\text{iso}} [t/x] [t_2/x_2] [\llbracket t \rrbracket_{\text{iso}}/x_3] \)

Note that the RHS of the equation uses the regular capture-avoiding substitution. We only need the input of \( \llbracket \rrbracket_{\text{iso}} \) to be safe. The proof is tedious but straightforward. We begin by rewriting with \( \alpha \) equality to replace the substitution operations on both sides with \( \text{unsafeSubst} \), which is structurally recursive because it does not have to do \( \alpha \) renaming before recursing under binders. Then the proof proceeds by structural recursion on \( b' \). For rewriting, we use Lemma B.1 and the following lemma about bound variables of translations:

**Lemma B.4.** \( \text{safe}_1 t \rightarrow \text{boundVars} \llbracket t \rrbracket_{\text{iso}} \subseteq c :: \text{lv}_2 \) (\( \text{boundVars} \) \( t \))

**B.2.1 Prop \( \not\supseteq \) Set.** Our proof of Lemma B.3 crucially depends on the fact that substitution does not change the universe flags in \( \forall \). Thus, \( \llbracket \rrbracket_{\text{iso}} \) makes the same decision before and after the substitution; Appendix B.5.2 presents \( \forall \ldots \rrbracket_{\text{iso}} \) in much more detail than Section 5.1.

For the correctness of our implementation, it is also important to ensure that on well-typed inputs, the reifier produce the same flags before and after the substitution. This is why allowing the rule Prop \( \Rightarrow \) Set in the input may be problematic. If we had Prop \( \Rightarrow \) Set, it would be legal to
substitute a proposition, say False, for a variable X:Set. For example, the term ((\lambda (X:Set), \forall (x:nat), X) False) would be well typed. Our reifier reifies ((\lambda (X:Set), \forall (x:nat), X) False) as ((\lambda (X:Set), \forall_S (x:nat), X) False), but reifies the \beta redex (\forall (x:nat), False) as (\forall_S \pi (x:nat), False). \square_{iso} will thus make different decisions (different combinators for the Total proof) because of the difference in flags. Thus the end-to-end translation (\square_{iso} composed with the reifier and reflector) would not preserve this \beta reduction.

Preservation of definitional equality is necessary, at least in the presence of inductive types. If closed terms \(u\) and \(v\) are definitionally equal, then eq\_refl\(u = v\). The corresponding abstraction theorem holds iff the end-to-end translations of \(u\) and \(v\) are definitionally equal.

Using Prop \rightarrow Set is not always a problem: many other parts of \square_{iso} do not differentiate between the two. For example, the reduction of ((\lambda (X:Set), \forall (x:X), nat) False) is preserved.

### B.3 Preservation of \(\beta\) equivalence

\(\beta\) equivalence (\(\equiv\)), which is used in the typing rules in Figure 4, is the conditionally reflexive, symmetric, transitive closure of the \(\beta\)-reduction explained in the previous subsection. Reflexivity only holds for safe terms: \(t \equiv\Gamma t\) iff safevars\(t\). vars \(\Gamma\) denotes the variables of the typing context \(\Gamma\). For example, vars \([x::nat, y::bool]) = [x, y]\). Overloading notation, below, safe\(\Gamma t\) will denote safevars\(t\).

In \(\equiv\), the \(\beta\) reductions steps may occur even in subterms, even under binders: when recursing under a binder, we add the variable to the context.

In a context that binds the variables \(l\), the term ((\lambda (x:A), b) t) \(\beta\) reduces in CoC\(^{-}\) to \(b[t/x]_l\).

\(\square_{iso} ((\lambda (x:A), b) t) := ((\lambda (x:A) (x_2:A_2) (x_r:\ldots)), \square_{iso} b) t \square_{iso} t \equiv\Gamma t\), which is definitionally equivalent in Coq to \(\square_{iso} [t/x] [t_2/x_2] [\square_{iso} t/x_r]\), which is exactly the RHS of Lemma B.3.

Using Lemma B.3, it is easy to prove the following:

**Lemma B.5.** \(u \equiv\Gamma v \rightarrow \square_{iso} u \equiv \square_{iso} v\)

On the RHS, we have Coq's definitional equivalence (\(\equiv\)), which is unconditionally reflexive. As mentioned before, only the input to \square_{iso} needs to be in Barandregt's convention.

### B.4 Preservation of subtyping

The typing rules of CoC\(^{-}\) (Figure 4) mention the subtyping relation (Figure 3). Thus, we prove that \square_{iso} preserves the subtyping relation. The predicate safeC lifts the safe property to contexts, ensuring that all types in the context are safe inputs to \square_{iso}. :>_1 and :>_2 are respectively the subtyping relations of Coq (CIC), not CoC\(^{-}\).

**Lemma B.6.** safe\(\Gamma U \rightarrow safeV\rightarrow safeC \Gamma \rightarrow \Gamma \vdash U :>_1 V \rightarrow \square_{iso} :>_1 u :U \rightarrow \square_{iso} :>_1 u' :U_2 \rightarrow \square_{iso} :>_1 v :V \rightarrow \square_{iso} :>_1 v' :V_2 \rightarrow \square_{iso} :>_1 (\pi_U \ [U]_iso u u' :>_1 (\pi_V \ [V]_iso v v'))

The proof is straightforward, by induction on the derivation of \(\Gamma \vdash U :>_1 V\).

### B.5 Preservation of typehood

**Theorem 2** (Abstraction Theorem). safe\(\Gamma a \rightarrow safeB \rightarrow safeC \Gamma \rightarrow \Gamma \vdash a : B \rightarrow \square_{iso} :>_1 a : B \land \square_{iso} :>_1 a_2 : B_2 \land \square_{iso} :>_1 a : \pi_B \ [B]_iso a a_2\)

The proof is by induction on the derivation of \(\Gamma \vdash a : B\). In the next three subsubsections, we will look at the three cases that are most different between \square_{iso} and \square.

**B.5.1 universes.** The interesting cases are \(\Gamma \vdash Set : Type_i\) and \(\Gamma \vdash Prop : Type_i\). For \(i \geq 0\), \[Type_i]_{iso} = \[Type_i]\), so the proofs for the cases \(\Gamma \vdash Type_i : Type_{i+1}\) for \square_{iso} are the same as the proofs for \square.
Because \( \text{Prop} \) and \( \text{Set} \) are closed terms, it suffices to consider the empty context. Below, we consider \( \text{Prop} : \text{Type}_1 \). The other case is similar.

We need to prove \( [\text{Prop}]_{\text{iso}} : \pi_{\text{Type}_1} [\text{Type}_1]_{\text{iso}} \) \( \text{Prop} \) Prop, which is (on unfolding definitions) \( \lambda (c : \text{Prop}) \). \( \text{IsoRel} c \) \( c : \text{Prop} \rightarrow \text{Prop} \rightarrow \text{Type}_1 \), which boils down to \( c : \text{Prop}, c : \text{Prop} \vdash [\text{Prop}] : [\text{Type}_1]_{\text{iso}} \) \( \text{Prop} \). \( \vdash c \rightarrow c : \text{Prop} \rightarrow \text{Prop} \rightarrow \text{Type}_1 \), which (using Coq's typing rules for Inductives) mainly boils down to \( c : \text{Prop}, c : \text{Prop} \vdash \lambda (c : \text{Prop}) \). \( \vdash \lambda (c : \text{Prop}) \). \( \vdash [\text{Prop}]_{\text{iso}} : \text{iso} \). \( \vdash [\text{Prop}]_{\text{iso}} : \text{iso} \). \( \vdash [\text{Prop}]_{\text{iso}} : \text{iso} \). Note that we interpret the result of \( [\text{Prop}]_{\text{iso}} \) in Coq, not CoC. Thus, we were able to use the subtyping rules omitted in CoC.°

B.5.2 \( \forall \) (rules \( \forall_1 \) and \( \forall_2 \) in Figure 4). W.r.t. \( [\ ] \), in \( [\text{Prop}]_{\text{iso}} \), the interesting cases are when \( \Gamma \vdash (\forall x : A. B) : \text{Set} \) and \( \Gamma \vdash (\forall x : A. B) : \text{Prop} \). We first explain how \( [\text{Prop}]_{\text{iso}} \) works in these cases in more detail (this presentation is slightly different, but equivalent to the one in Section 5.1). Correctness would then be obvious. In these cases, our implementation merely invokes one of the following definitions (combinators) that have been already accepted (deemed well-typed) by Coq:

Definition \( \lambda (f_1 : \forall a : A_1, B_1 a) (f_2 : \forall a : A_2, B_2 a) : \text{IsoRel} (\forall a : A_1, B_1 a) (\forall a : A_2, B_2 a) := \exists T (\lambda (f_1 : \forall a : A_1, B_1 a) (f_2 : \forall a : A_2, B_2 a)) \Rightarrow \forall (a_1 : A_1) (a_2 : A_2) \Rightarrow (p : \pi_1 A_r a_1 a_2), \pi_1 (f_1 a_1) (f_2 a_2)

(\ldots).

Definition \( \lambda (f_1 : \forall a : A_1, B_1 a) (f_2 : \forall a : A_2, B_2 a) : \text{IsoRel} (\forall a : A_1, B_1 a) (\forall a : A_2, B_2 a) := \exists T (\lambda (f_1 : \forall a : A_1, B_1 a) (f_2 : \forall a : A_2, B_2 a)) \Rightarrow \forall (a_1 : A_1) (a_2 : A_2) \Rightarrow (p : \pi_1 A_r a_1 a_2), \pi_1 (f_1 a_1) (f_2 a_2)

(\ldots).

The bodies of these definitions are dependent pairs whose first components are essentially the AnyRel translations of \( \Pi \) types. The second components are huge and thus shown as \( \ldots \); they are proofs of the Total and OneToOne properties, which were already explained respectively in Section 3.1 (also Lemma 5.1) and Section 4.1. \( [\text{Prop}]_{\text{iso}} \) merely refers to one of these two constants by name (the string “\( \text{piSet} \)” or “\( \text{piProp} \)” and then applies the six arguments. The reflector turns those strings to references to the above definitions.

Using these definitions (instead of constructing their bodies by hand in \( [\text{Prop}]_{\text{iso}} \)) greatly simplified our implementation and proofs. Many proofs, e.g. Lemma B.2, did not have to reason about the horrendously complex bodies of those definitions. In Lemma B.2, we only had to perform the substitution on the arguments to the constant, which are relatively very simple, as we will show soon. Even in this subsection, we don’t need to reason about the correctness of the proof parts shown above as \( \ldots \), because Coq has already checked them for us! Below, we will merely argue that the arguments to the above definitions are of correct types.

When \( \Gamma \vdash (\forall x : A. B) : \text{Prop} \), as indicated by the flags \( \forall_{SP} \) or \( \forall_{PP} \), \( [\text{Prop}]_{\text{iso}} \) invokes the lemma \( \text{piProp} \). When \( \Gamma \vdash (\forall x : A. B) : \text{Set} \), as indicated by the flags \( \forall_{SP} \) or \( \forall_{PS} \), \( [\text{Prop}]_{\text{iso}} \) invokes the lemma \( \text{piSet} \). The lemma \( \text{piProp} \) uses fewer assumptions about the arguments \( A_r \) and \( B_r \), because it exploits proof irrelevance. Thus, it is important to prefer the lemma \( \text{piProp} \). Note that \( \text{Prop} \vdash \Gamma : \text{Set} \), even though \( \text{Prop} \not\vdash \Gamma : \text{Set} \). In both cases, the 6 arguments are the same: \( A_1 : A, A_2 : A, A_r : [A]_{\text{iso}}, B_1 : \lambda (x : A), B_2 : \lambda (x : A), B_2, \) and \( B_r : \lambda (x : A) (x : A_{\text{iso}}) (x : \pi_1 [A]_{\text{iso}} x x_2), [B]_{\text{iso}} \). These arguments are in the context \( [\Gamma]_{\text{iso}} \).
We will consider the case $\Gamma \vdash (\forall x : A. B) : \text{Prop}$ (rule $\forall_2$ in Figure 4, with $A : \text{Set}$). The other cases are similar. It is easy to check that for the above instantiation, the return type is correct (exactly what the abstraction theorem needs). The correctness of the types of the arguments follows from the induction hypotheses (abstraction theorems for the two premises of the rule in Figure 4). The two induction hypotheses (after unfolding definitions) are: $\Gamma \vdash A : \text{Set} \land \Gamma \vdash A_2 : \text{Set} \land \Gamma \vdash \exists x : A. x_1. x_2 : B_2. x_1. x_2$, with $\Gamma \vdash \forall x : A_1. x_3 : A_1. x_4 : A_2. x_1. x_2. x_r : \pi_1 A_1 x_3 x_4. x_1. x_2. x_r : \text{Prop} \land 
abla I \vdash A : \text{isoRel} A A_2$ and
$\Gamma \vdash B_1. B_2 : \text{Prop} \land 
abla I \vdash A : \text{isoRel} B_1 B_2$.

Lemma B.7. $\lambda (\text{freeVars} t_2 = \text{map} (\lambda x, x_2) (\text{freeVars} t)$

B.5.3 $\lambda$ (rule $\text{ABS in Figure 4}$). Now we consider the case $\Gamma \vdash (\lambda x : A. B) : (\forall x : A. C)$.

W.r.t. $\Gamma$, in $\Gamma$, the interesting case is when the type $\forall x : A. C$ is in the universe $\text{Set}$ or $\text{Prop}$. Consider the case when $(\forall x : A. C) : \text{Prop}$. We need to prove that in the typing context $\Gamma : \text{iso}$, $\Gamma : \text{iso}$ has type $\Gamma : \text{iso}$. We need to prove that in the typing context $\Gamma : \text{iso}$, $(\lambda x : A. C) : \text{iso}$, which (as explained in the previous subsection) is $(\pi_1 (\pi_\text{Prop} A_2 A_1 \text{iso} \ldots)) (\lambda x : A. B) (\lambda x : A_2. B_2)$. Coq’s definitional equality includes $\delta$ and $\eta$ reductions (definition unfolding and pattern matching). After unfolding the definition of $\pi_\text{Prop}$, we get a dependent pair. Then acts on the pair ($\eta$ reduction) to produce the first component, which is essentially the AnyRel translation of $\forall x : A. C$. The second component (proofs of Total and OneToOne properties) get thrown away by $\pi_1$. The rest of this proof is essentially the same as that for the AnyRel translation ($\square$).

B.6 Translation of the W type and its induction principle

$IWT_{\text{ind}}$ is a general induction (recursion) principle for the type $IWT$ in Section 4.2.

Definition $IWT_{\text{ind}} :=$

$$\lambda (i : A : \text{Set}) (b : A \rightarrow \text{Set}) (a i : A \rightarrow a) (b i : \forall a : A, B a \rightarrow a) (\forall x : A. B) (\lambda x : A. x) (\lambda x : A_2. B_2) (\lambda x : A_2. B_2) (\lambda x : A_2. B_2) (\lambda x : A_2. B_2)$

Just as inductive types can be encoded as instantiations of $IWT$, Coq’s pattern matching and fixpoints (recursive functions) can be encoded as instantiations of $IWT_{\text{ind}}$. Thus, we checked that $\Gamma : \text{iso}$ and $\Gamma : \text{iso}$ succeed and are of correct type. We also checked that in the most general context, $\Gamma : \text{iso}$ preserves the reduction (unfolding $f x$ and $i$ reduction of pattern matching) of $IWT_{\text{ind}}$. (As explained in the above subsections, preservation of reduction is a step in proving that $\Gamma : \text{iso}$ preserves typing.)

In Coq, reductions can happen even under binders. Thus, below we pick terms $LHS$ and $RHS$ which observe the reduction of $IWT_{\text{ind}}$ in the most general context. $LHS$ and $RHS$ are the same except the highlighted part. $LHS$ reduces to $RHS$. 
We observed that $\llbracket \text{LHS} \rrbracket_{iso}$ and $\llbracket \text{RHS} \rrbracket_{iso}$ succeed and that $\llbracket \text{LHS} \rrbracket_{iso}$ is definitionally equal to $\llbracket \text{RHS} \rrbracket_{iso}$. One way to check that terms $u$ and $v$ are definitionally equal is to ask Coq to check $(\text{eq_refl} : (u = v))$. We used this method.

Although we did only one reduction experiment for $\text{IWTind}$, because Coq’s reductions are preserved under substitutions (and how $\llbracket \rrbracket_{iso}$ translates $\lambda$ and application terms), we have hereby proved that reductions of $\text{IWTind}$ in all well-typed instantiations are preserved.

In our implementation repository (https://github.com/aa755/paramcoq-iff), the experiments in this subsection can be found in the file test-suite/iso/IWTS.v