The complete evaluation of Rogers Ramanujan and other continued fractions with elliptic functions

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Abstract

In this article we present evaluations of continued fractions studied by Ramanujan. More precisely we give the complete polynomial equations of Rogers-Ramanujan and other continued fractions, using tools from the elementary theory of the Elliptic functions. We see that all these fractions are roots of polynomials with coefficients depending only on the inverse elliptic nome-q and in some cases the Elliptic Integral-K. In most of simplifications of formulas we use Mathematica.
1 Introductory definitions and formulas

For $|q| < 1$, the Rogers Ramanujan continued fraction (RRCF) is defined as

$$R(q) := \frac{q^{1/5} q^1 q^2 q^3}{1+1+1+\cdots} \tag{1}$$

We also define

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \tag{2}$$

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n) = (q; q)_{\infty} \tag{3}$$

$$\Phi(-q) := \prod_{n=1}^{\infty} (1 + q^n) = (-q; q)_{\infty} \tag{4}$$

and also hold the following relations of Ramanujan

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)} \tag{5}$$

$$\frac{1}{R^6(q)} - 11 - R(q) = \frac{f^6(-q)}{q f^6(-q^5)} \tag{6}$$

From the Theory of Elliptic Functions we have:

Let

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin(t)^2}} dt \tag{7}$$

It is known that the inverse elliptic nome $k = k_r$, $k^2_r = 1 - k^2_r$ is the solution of

$$\frac{K(k'_r)}{K(k)} = \sqrt{r} \tag{8}$$

In what it follows we assume that $r \in \mathbb{R}_+$. When $r$ is rational then $k_r$ is algebraic.

$$k_r = \frac{8q^{1/2}\Phi(-q)^{12}}{1 + \sqrt{1 + 64q\Phi(-q)^{24}}} \tag{9}$$

We can write the functions $f$ and $\Phi$ using elliptic functions. It holds

$$\Phi(-q) = 2^{2^{-1/6} q^{-1/24} (k_r)^{1/12}} (k'_r)^{1/6} \tag{10}$$

$$f(-q)^8 = \frac{2^{5/3}}{\pi^4} q^{-1/3}(k_r)^{2/3}(k'_r)^{8/3} K(k_r)^4 \tag{11}$$
also holds
\[ f(-q^2)^6 = \frac{2k_r k'_r K(k_r)^3}{\pi^3 q^{1/2}} \] (12)

From [B,G] it is known that
\[ R'(q) = \frac{1}{5q} - \frac{5}{6} f(-q)^4 R(q) \sqrt{R(q)^{-5} - 11 - R(q)^5} \] (13)

2 Evaluations of Rogers Ramanujan Continued Fraction

**Theorem 2.1**
If \( q = e^{-\pi \sqrt{r}} \) and
\[ a = a_r = \left( \frac{k'_r}{k'_{25r}} \right)^2 \sqrt{\frac{k_r}{k_{25r}}} M_5(r)^{-3} \] (14)

Then
\[ R(q) = \left( -\frac{11}{2} - \frac{a_r}{2} + \frac{1}{2} \sqrt{125 + 22a_r + a_r^2} \right)^{1/5} \] (15)

Where \( M_5(r) \) is root of: \((5x - 1)^5(1 - x) = 256(k_r)^2(k'_r)^2x\).

**Proof.**
Suppose that \( N = n^2 \mu \), where \( n \) is positive integer and \( \mu \) is positive real then it holds that
\[ K[n^2 \mu] = M_n(\mu)K[\mu] \] (16)

Where \( K[\mu] = K(k_\mu) \)

The following formula for \( M_5(r) \) is known
\[ (5M_5(r) - 1)^5(1 - M_5(r)) = 256(k_r)^2(k'_r)^2 M_5(r) \] (17)

Thus if we use (5) and (10) and the above consequence of the Theory of Elliptic Functions, we get:
\[ R^{-5}(q) - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)} = a = a_r \]

Solving with respect to \( R \) we get the result.

The relation between \( k_{25r} \) and \( k_r \) is
\[ k_r k_{25r} + k'_r k'_{25r} + 2 \cdot 4^{1/3}(k_r k_{25r} k'_r k'_{25r})^{1/3} = 1 \] (18)

We will try to evaluate \( k_{25r} \). For this we set
\[ k_{25r} k_r = w^2 \] (19)
then setting directly to (17) the following parametrization of \( w \) (see also [B3] pg.280):

\[
    w = \sqrt{\frac{L(18 + L)}{6(64 + 3L)}}
\]

we get

\[
    \left(\frac{k_{25r}}{w^{1/2}}\right)^{1/2} = \frac{w^{1/2}}{(k_r)^{1/2}} = \frac{1}{2} \sqrt{4 + \frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)^2 + \frac{1}{2} \sqrt{\frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)}}
\]

where

\[
    M = 18 + L / 64 + 3L
\]

From the above relations we get also

\[
    -\frac{k_r - w}{\sqrt{k_r} w} = \frac{k_{25r} - w}{\sqrt{k_{25r}} w} = \sqrt{\frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)}
\]

We can consider the above equations as follows: Taking an arbitrary number \( L \) we construct an \( w \). Now for this \( w \) we calculate the two numbers \( k_r \) and \( k_{25r} \). Thus when we know the \( w \), the \( k_r \) and \( k_{25r} \) are given by (20) and (21).

The result is: We can set a number \( L \) and from this calculate the two inverse elliptic nome’s, or equivalently, find easy solutions of (17). But we don’t know the \( r \). One can see (from the definition of \( k_r \)) that the \( r \) can be evaluated from equation

\[
    r = r_{k_r} = r[L] = \frac{K^2(k_r')}{K^2(k_r)}
\]

or

\[
    r = r_{k_{25r}} = r[25L] = \frac{1}{5} \frac{K^2(k_{25r}')}{K^2(k_{25r})}
\]

However there is no way to evaluate the \( r \) in a closed form, such as roots of polynomials, or else. Some numerical evaluations as we will see, indicate as that even \( k_r \) are algebraic numbers, the \( r \) are not rational.

**Theorem 2.2**

Set

\[
    A_L = a \left( \frac{K^2(k_L')}{K^2(k_L)} \right) = \frac{(k_L)^3(1 - (k_L)^2)M_5(L) - 3}{(k_L)^2w - w^5}
\]

then

\[
    R \left( e^{-\pi \sqrt{r[L]}} \right) = \left( -\frac{11}{2} \frac{A_L}{2} + \frac{1}{2} \sqrt{125 + 22A_L + A_L^2} \right)^{1/5}
\]

where the \( k_L \) and \( w \) are given by (19) and (20).

**Example.**

Set \( L = 1/3 \) then

\[
    w = \frac{1}{3} \sqrt{\frac{11}{78}}
\]
and

\[ k_{25r} = \frac{1}{3} \sqrt{\frac{11}{78}} \left( \frac{-4(\frac{11}{13})^{1/6} + (\frac{13}{11})^{1/6}}{\sqrt{6}} \right) + \frac{1}{2} \left\{ \frac{2}{3} \left( -4 \left( \frac{11}{13} \right)^{1/6} + \left( \frac{13}{11} \right)^{1/6} \right)^2 \right\} \]

and

\[ k_r = \frac{\frac{1}{3} \sqrt{\frac{11}{78}}}{\left( \frac{-4(\frac{11}{13})^{1/6} + (\frac{13}{11})^{1/6}}{\sqrt{6}} \right) + \frac{1}{2} \left\{ \frac{2}{3} \left( -4 \left( \frac{11}{13} \right)^{1/6} + \left( \frac{13}{11} \right)^{1/6} \right)^2 \right\}^2} \]

where the \( r \) is given by

\[ r = \frac{K^2 \left( \sqrt{1 - k_r^2} \right)}{K^2 (k_r)} \]

Now we can see (The results are known in the Theory of Elliptic Functions) how we can found evaluations of \( R(q) \) when \( r \) is given and \( k_r \) is known:

From (19) it is

\[ L = -9 + 9w^2 + \sqrt{3} \sqrt{27 + 74w^2 + 27w^4} \]  \hspace{1cm} (27)

from the relation between \( M \) and \( L \) we get

\[ M = \frac{1}{64} \left( 9 - 9w^2 + \sqrt{81 + 222w^2 + 81w^4} \right) \]  \hspace{1cm} (28)

Hence from (20)

\[ t = \frac{w - k_r}{\sqrt{k_r} w} \]  \hspace{1cm} (29)

also

\[ t = \sqrt{\frac{2}{3}} \left( \frac{1}{y^{1/6}} - 4y^{1/6} \right) \]  \hspace{1cm} (30)

where \( y = M/L \). Hence \((k = k_r)\):

\[ \frac{M}{L} = \left( \frac{\sqrt{3}(k - w) + \sqrt{3k^2 + 26kw + 3w^2}}{8\sqrt{2kw}} \right)^6 \]  \hspace{1cm} (31)

or

\[ \frac{1}{64} \left( \frac{9 - 9w^2 + \sqrt{81 + 222w^2 + 81w^4}}{-9 + 9w^2 + \sqrt{81 + 222w^2 + 81w^4}} \right) = \left( \frac{\sqrt{3}(k - w) + \sqrt{3k^2 + 26kw + 3w^2}}{8\sqrt{2kw}} \right)^6 \]
or
\[
-9 + 9w^2 + \sqrt{81 + 222w^2 + 81w^4} = \left( \frac{\sqrt{3(k - w)} + \sqrt{3k^2 + 26kw + 3w^2}}{8\sqrt{2w}} \right)^3
\]
(32)

setting now
\[
k_r^* = \left( \frac{-1 + 4p^2 + \sqrt{1 - 2p^2 + 16p^4}}{\sqrt{6p}} \right)^2
\]
(33)

and
\[
w = \frac{6^{1/4}p^{3/2}}{\sqrt{-1 + 64p^6 + \sqrt{1 + 88p^6 + 4096p^{12}}}}
\]
(34)

\[
W = -1 + 4p^2 + \sqrt{1 - 2p^2 + 16p^4}
\]

and
\[
T = -1 + 64p^6 + \sqrt{1 + 88p^6 + 4096p^{12}}
\]

we have
\[
k_r^* w = k_r
\]

Also
\[
p = \left( \frac{T(2 + T)}{216 + 128T} \right)^{1/6} = \left( \frac{W(2 + W)}{6 + 8W} \right)^{1/2}
\]
(35)

But
\[
w = 6k_r \left( \frac{W + 2}{(6 + 8W)W} \right)
\]
(34a)

\[
T = \sqrt{6W^2 k_r} \left( \frac{W(2 + W)}{6 + 8W} \right)
\]

where the equation for finding \(W\) from \(k_r\) is
\[
-108k_r^2 \left( \frac{W(W + 2)}{8W + 6} \right)^{5/2} + \sqrt{6k_r} W^2 \left( 1 - 64 \left( \frac{W(W + 2)}{8W + 6} \right)^{3/2} \right) + 3W^4 \left( \frac{W(W + 2)}{8W + 6} \right)^{1/2} = 0
\]
(36)

We give the complete polynomial equation of \(p\) arising from (35):
\[
k_r^2 + 2\sqrt{6k_r} k_r^2 p - 24k_r^2 p^2 - 10\sqrt{6k_r} k_r^2 p^3 + 240k_r^2 p^4 + 32\sqrt{6k_r} k_r^2 p^5 + 
+(54 - 1388k_r^2 + 54k_r^4)p^6 - 128\sqrt{6k_r} k_r^2 p^7 + 3840k_r^2 p^8 + 640\sqrt{6k_r} k_r^2 p^9 - 
-6144k_r^2 p^{10} - 2048\sqrt{6k_r} k_r^2 p^{11} + 4096k_r^2 p^{12} = 0
\]
(37)

It is evident that the Rogers Ramanujan Continued Fraction is a polynomial equation with coefficients depending by \(k_r\).
From (32) we have
\[ \sqrt{6k_r^*} = \frac{-1 + 4p^2 + \sqrt{1 - 2p^2 + 16p^4}}{p} \]
where \( p \) is root of (36).

Using Mathematica we get the following simplification formula for
\[ x = \sqrt{k_r^*} = \frac{1}{\sqrt{k_{25r}}} \]

\[ k_r^2 + 4k_r^2k_r x - 6k_r^2x^2 + 20k_r^2k_r x^3 + 15x^4 - 16k_r^2k_r x^5 + (16 - 52k_r^2 + 16k_r^4)x^6 + 
+ 16k_r^2k_r x^7 + 15k_r^2x^8 - 20k_r^2k_r x^9 - 6k_r^2x^{10} - 4k_r^2k_r x^{11} + k_r^2x^{12} = 0 \quad (38) \]

set now
\[ c_r = \frac{k_r^2(k_r^*)^5}{(k_r^*)^4 - k_r^2} \]

and
\[ G(q) = (R^{-5}(q) - 11 - R^5(q))^{1/3} \]

then

**Theorem 2.3**

i) \[ 3125c_r^2 - 6250c_r^{5/3}G(q) + 4375c_r^{4/3}G^2(q) - 1500c_rG^3(q) + 275c_r^{2/3}G^4(q) + 
+ 2c_r^{1/3}(-13 + 128k_r^2k_r^*)G^5(q) + G^6(q) = 0 \quad (39a) \]

Also

ii) \[ k_r^6 + k_r^3(-16 + 10k_r^3)w + 15k_r^2w^2 - 20k_r^2w^3 + 15k_r^2w^4 + k_r(10 - 16k_r^2)w^5 + w^6 = 0 \quad (39b) \]

Once we know \( k_r \) we can calculate \( w \) from the above equation and hence the \( k_{25r} \). Hence the problem reduces to solve 6-th degree equations. The first is (16) and the second is (39b).

**Proof.**

i) \[ R(q)^{-5} - 11 - R^5(q) = a_r = \frac{k_r^2(1 - k_r^2)}{w(k_r^2 - w^4)}M_5(r)^{-3} \]

\[ = \frac{k_r^2(k_r^*)^5}{(k_r^*)^4 - k_r^2}M_5(r)^{-3} \]

and \( M_5(r) \) satisfies \((5x - 1)^5(1 - x) = 256(k_r)^2(k_r^*)^2x\).

After elementary algebraic calculations we get the result.

ii) From (35) we get:
\[ -\sqrt{6}t - 3U + 108t^2U^5 + 64\sqrt{6}tU^6 = 0 \]
and 
\[ t = \frac{k_r}{W^2} = \frac{w}{6U^2} \]

and
\[-32\sqrt{6}wU^6 + (9 - 54w^2)U^3 + 3\sqrt{6}w = 0 \quad (a)\]

\[ U = \left( \frac{W(W + 2)}{8W + 6} \right)^{1/2} \quad (b) \]

and also
\[ \left( \frac{W(W + 2)}{8W + 6} \right)^{1/2} = \frac{w}{6k_r}W \quad (c) \]

Hence solving the system we obtain the 6-th degree equation.

**Corollary**

The solution of (39b) with respect to \( k_r \) when we know \( w \) is

\[ \frac{w^{1/2}}{(k_r)^{1/2}} = \frac{1}{2} \sqrt{4 + \frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)^2 + \frac{1}{2} \sqrt{\frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)^2}} \]

(40)

Where

\[ w = \sqrt{\frac{L(18 + L)}{6(64 + 3L)}} \]

\[ M = \frac{18 + L}{64 + 3L} \]

**Theorem 2.3**

\[ R'(q) = \frac{2^{4/3}(k_r)^{5/12}(k'_r)^{5/3}}{5k_{25r}^{1/12}(k_{25r}')^{1/3} \sqrt{M_5(r)}} \times \]

\[ \left( -\frac{11}{2} - \frac{a_r}{2} + \frac{1}{2} \sqrt{125 + 22a_r + a_r^2} \right)^{1/5} K^2(k_r) \frac{\pi^2 q}{\pi^2 q} \]

(41)

**Proof.**

Combining (11) and (10) and Theorem 2.1 we get the proof.

**Evaluations.**

\[ R(e^{-2\pi}) = \frac{-1}{2} - \frac{\sqrt{5}}{2} + \sqrt{\frac{5 + \sqrt{5}}{2}} \]

\[ R'(e^{-2\pi}) = 8 \sqrt{\frac{2}{5} \left( 9 + 5\sqrt{5} - 2\sqrt{50 + 22\sqrt{5}} \right) \frac{e^{2\pi}}{\pi^3} \Gamma \left( \frac{5}{4} \right)^4} \]

Sumarizing our results we can say that:

1) Theorem 2.1 is quite usefull for evaluating \( R(q) \) when we know \( k_r \) and \( k_{25r} \). But this it was known already to Ramanujan by using the function \( X(-q) = (-q; q^2)_{\infty} \), (see [8]).

2) Theorem 2.2 is more kind of a Lemma rather a Theorem and it might help
for further research.

3) Theorem 2.3 is a proof that the Rogers Ramanujan continued fraction is a root of a polynomial equation with coefficients the $k_r$ where $r$ positive real.

4) Theorem 2.4 is a consequence of a Ramanujan integral first proved by Andrews (see [5]) and it is usefull for evaluations of $R'(e^{-\pi\sqrt{r}})$, $r \in \mathbb{Q}$.

The above theorems can used to derive also modular equations of $R(q)$, from the modular equations of $k_r$. More precise we can guess an equality with the help of a methematical pacage (for example in Mathematica there exist the command 'recognize'), and then proceed to proof, using the Theorems which we present in this article. We follow this prosedure with other fractions (the Rogers Ramanujan is little dificult) such as the qubic or Ramanujan-Gollnitz-Gordon.

These two last continued fractions are more easy to handle. The elliptic function theory and the sigular moduli $k_r$ will excract and give us several proofs of modular indenties.

### 3 The H-Continued Fraction

Heng Huat Chan and Sen-Shan Huang [11] studied the Ramanujan Gollnitz-Gordon continued fraction

$$H(q) := \frac{q^{1/2}}{1 + q + 1 + q^3 + 1 + q^5 + 1 + q^7 + \cdots}$$

where $|q| < 1$.

In a paper of C. Adiga and T. Kim [2] one can find the next identity for this fraction

$$H(q)^{-1} - H(q) = \frac{M^2(q^2)}{M^2(q^4)}$$

where

$$M(q) = \frac{q^{1/8} - q^1 - q^2 - q^3 - \cdots}{1 + q^1 + q^2 + 1 + q^3 + \cdots} = q^{1/8}(q^2; q^2)_{\infty}$$

Next we will use some properties of the inverse Elliptic Nome and show how this can help us to evaluate the H-fraction. For to complete our pur pose we need the relation between $k_r$ and $k_{4r}$. There holds the following

**Lemma 3.1**

$$k_{4r} = \frac{1 - k'_r}{1 + k'_r}$$

and

$$K[4r] = \frac{1 + k'_r}{2}K[r]$$

**Proof.**

For (43) see ([B3], pg. 102, 215). The identity for $K[4r]$ is known from the theory of elliptic functions

**Theorem 3.1**

$$H(q) = -P + \sqrt{P^2 + 1}$$
where

\[ P = \frac{k_r}{(1 - k'_r)} \]

or

\[ k_r = \frac{4(H - H^3)}{(1 + H^2)^2} \quad (48) \]

**Proof.**

It is known that, under some conditions in the sequence \( b_n \) (see [16]) it holds

\[
\frac{1}{1 + b_1} \frac{-b_2}{1 + b_1 + b_2} \cdots = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} b_k
\]

(49)

Hence if we set \( b_n = q^n, \, |q| < 1 \), then

\[
M(q) = \theta_2(q^{1/2}) = q^{-1/8} \sqrt{\frac{k_{r/4}K(k_{r/4})}{2\pi}}
\]

(50)

where

\[
\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}
\]

\[
\sum_{n=0}^{\infty} q^{n(n+1)/2} = 1/2q^{-1/8}\theta_2(q^{1/2})
\]

Using Lemma 3.1 and identity (41) we get the proof.

**Theorem 3.2**

If \( ab = \pi^2 \), then

\[
\left( H(e^{-a}) + 2 - \frac{1}{H(e^{-a})} \right) \left( H(e^{-b}) + 2 - \frac{1}{H(e^{-b})} \right) = 8
\]

(51)

**Proof.**

Set

\[
\psi(q) = \sum_{n=0}^{\infty} q^{(n+1)n/2}
\]

(52)

and

\[
\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}
\]

(53)

Identity (49) becomes.

If \( ab = 4\pi^2 \)

\[
\left( 2 - \frac{\psi(e^{-a})^2}{e^{-a/4}\psi(e^{-2a})^2} \right) \left( 2 - \frac{\psi(e^{-b})^2}{e^{-b/4}\psi(e^{-2b})^2} \right) = 8
\]

(54)

From [B3] pg.43 we have if \( ab = 2\pi \), then

\[
\psi(e^{-a^2}) = \frac{\sqrt{b}}{2\sqrt{a}} e^{a^2/8} \phi(-e^{-b^2/2})
\]

(55)
\[
\psi(e^{-2a^2}) = \frac{\sqrt{b/2}}{2\sqrt{a}} e^{a^2/4} \phi(-e^{-b^2/4})
\]  (56)

Hence if \(ab = \pi^2/4\), ([B3] pg.98)

\[
\left(1 - \frac{\phi(e^{-a})}{\phi(-e^{-a})}\right) \left(1 - \frac{\phi(e^{-b})}{\phi(-e^{-b})}\right) = 2
\]  (57)

But this is equivalent to

\[
k_{1/(4r)}' = \frac{1 - k_r'}{1 + k_r'}
\]  (58)

(For details [B3] pg.98, 102 and 215). Which is equivalent to

\[
k_r = k_{1/r}'.
\]  (59)

But this is true from the definition of the modulus-\(k\) (see relation (7)). This completes the proof.

**Corollary.**

If \(ab = \pi^2\)

\[
(1 + \sqrt{2} + H(e^{-a}))(1 + \sqrt{2} + H(e^{-b})) = 2(2 + \sqrt{2})
\]  (60)

**Proof.**
This follows from Theorem 3.2 and as in [B3] pg. 84

**Evaluations.**

\[
H\left(e^{-\pi/2}\right) = \sqrt{1 + 2\sqrt{2} - 2\sqrt{2 + \sqrt{2}}}
\]

\[
H\left(e^{-\pi\sqrt{2}}\right) = \sqrt{3 + 2\sqrt{2} - 2\sqrt{4 + 3\sqrt{2}}}
\]

Now it is easy to see how we can construct modular equations of these continued fractions from the modular equations of the inverse elliptic nome. For example for the \(H\) continued fraction we give the second degree modular equation:

**Theorem 3.3**

\[H^2(q) = \frac{H(q^2) - H^2(q^2)}{1 + H(q^4)}\]  (61)

**Proof.**
If \(ab = 4\) and \(k_r = k(e^{-\pi\sqrt{q}}), q_r = e^{-\pi\sqrt{r}}\) then

\[(1 + k_a)(1 + k_b) = 2\]

which can be written as

\[(1 + k_{4a})(1 + k_{4b}) = 2\]

But from Theorem 3.1

\[k_a = \frac{4H^2(q^{1/2})}{(H^2(q^{1/2}) - 1)^2}\]
and the result follows after elementary algebraic computations.

Also from (47) and (60) one can get

\[ \sqrt{k_r'} = \frac{H(q^2) + 2H(q^2) - 1}{H(q^2) - 2H(q^2) - 1} \quad (62) \]

For to proceed we must mention that the relation between \(k_{9r}\) and \(k_r\) is given by

\[ \sqrt{k_r k_{9r}} + \sqrt{k_r' k_{9r}} = 1 \quad (63) \]

4 The Ramanujan’s Cubic Continued Fraction

Let

\[ V(q) := \frac{q^{1/3} + q^2 q^2 + q^4 q^3 + q^6}{1 + \frac{1}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \ldots}}} \quad (64) \]

is the Ramanujan’s cubic continued fraction, then

**Lemma 4.1**

\[ V(q) = 2^{-1/3} (k_{9r})^{1/4} (k_{9r}')^{1/6} \]

\[ \frac{1}{(k_r)^{1/12} (k_{9r})^{1/2}} \]

where the \(k_{9r}\) are given by (61)

**Proof.**

It is known (see and [14] pg. 596) that

\[ V(q) = q^{1/3} \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty} \]

But

\[ \Phi(-q) = (-q, q)_\infty = \frac{1}{(q, q^2)_\infty} \]

thus

\[ V(q) = q^{1/3} \frac{\Phi(-q^3)_{\infty}}{\Phi(-q)} \]

and equation (63) follows from (8).

**Lemma 4.2**

If

\[ G(x) = \frac{x}{\sqrt{2\sqrt{x} - 3x + 2x^{3/2} - 2\sqrt{x^3} \sqrt{1 - 3\sqrt{x} + 4x - 3x^{1/2} + x^2}}} \]

and

\[ k_{9r} = \frac{w}{k_r} \]

and

\[ k_{9r}' = \frac{(1 - \sqrt{w})^2}{k_r'} \]
then  
\[ k_r = G(w) \]  \hspace{1cm} (66)

**Proof.**

Set the values of \( k_r \) and \( k_9r \) in (61).

Also holds

\[
\frac{1}{(V(q)V(q^3))^{12}} = 256w \left( \frac{1 - w^2}{G(w)^2} \right)^2 \left( 1 - \frac{G(w)^2 G^{-1}(w/G(w))^2}{w^2} \right)^3
\]

If we set  
\[ W = 2 - 3\sqrt{w} + 2w - 2(1 - \sqrt{w})\sqrt{1 - \sqrt{w} + w} \]  \hspace{1cm} (67)

then

\[
V(q) = \frac{(k'_r)^{2/3}u^{1/4}}{21^{1/3}(k_r)^{1/3}(1 - \sqrt{w})} = \frac{(W - w^{3/2})^{1/3}W^{-1/6}}{21^{1/3}(1 - \sqrt{w})} \]  \hspace{1cm} (68)

after solving (65) with respect to \( w \) and making the simplifications we arrive at

\[
2V^3(q) = \frac{\sqrt{W}}{(1 + \sqrt{W})^2} \]  \hspace{1cm} (69)

and

\[
(k_r)^2 = \sqrt{W} \left( \frac{2 + \sqrt{W}}{1 + 2\sqrt{W}} \right)^3 \]  \hspace{1cm} (70)

Hence we get the following equation

\[
(k_r)^{2/3} = Z^2 \frac{\sqrt{2}V(q)^{3/2} + Z^3}{-\sqrt{2}V(q)^{3/2} + 2Z^3} \]  \hspace{1cm} (71)

Where \( Z = \sqrt[12]{W(q)} \). Reducing the above equation in polynomial form we have

\[
sk_r^{2/3} + sZ^2 - 2k_r^{2/3}Z^3 + Z^5 = 0 \]  \hspace{1cm} (72)

and

\[
s^2 = 2V^3(q) = \frac{Z^6}{(1 + Z^6)^2} \]  \hspace{1cm} (73)

From these two equations we arrive to

**Theorem 4.1**

Set \( T = \sqrt{1 - 8V(q)^3} \) then holds the next equation

\[
(k_r)^2 = \frac{(1 - T)(3 + T)^3}{(1 + T)(3 - T)^3} \]  \hspace{1cm} (74)

**Corollary 4.1**

If \( X = \sqrt{W(q)} = \frac{1 + T}{1 - T} \) and \( Y = \sqrt{W(q^3)} \), then

\[
X^{1/2} \left( \frac{2 + X}{1 + 2X} \right)^{3/2} = 2 \left( \frac{1 + 2Y}{2 + Y} \right)^{3/4} Y^{1/4} \]  \hspace{1cm} (75)

\[
+ Y^{1/2} \left( \frac{2 + Y}{1 + 2Y} \right)^{3/4} \]
The duplication formula is

**Proposition 4.1**

Set \( u = T(q^2) \), \( v = T(q) \), then

\[
\frac{\sqrt{(1-u)(3+u)^{3/2}}}{\sqrt{(1+u)(3-u)^{3/2}}} = \frac{(3-v)^{3/2}\sqrt{1+v-4v^{3/2}}}{(3-v)^{3/2}\sqrt{1+v+4v^{3/2}}} \quad (76)
\]

We can simplify the problem of finding modular equations of degree 3 using the Cubic continued fraction. As someone can see with direct algebraic calculations and with definitions of \( W, V(q) \) and Lemma 4.2 there holds:

**Proposition 4.2**

If \( k_{91r} = \frac{w}{k_r} \), then

\[
w = \left( \frac{1 - 4V(q)^3 - 8V(q)^6 - \sqrt{1 - 8V(q)^3}}{4V(q)^3 (1 - 2V(q)^3 - \sqrt{1 - 8V(q)^3})} \right)^2 \quad (77)
\]

The Ramanujan’s modular equation which relates \( V(q) \) and \( V(q^3) \) is

\[
V(q^3) = V(q)^3 \frac{1 - V(q^3) + V(q^3)^2}{1 + 2V(q^3) + 4V(q^3)^2} \quad (78)
\]

(see [10]) one can get from the above formula and Proposition 4.2 the following:

**Proposition 4.3**

\[
k_{81r} = \left( \frac{1 + 2V(q^3)^2 - \sqrt{1 - 8V(q^3)^3}}{1 + 2V(q^3)^2 + \sqrt{1 - 8V(q^3)^3}} \right)^2 k_r \quad (79)
\]

**Corollary 4.2**

Set \( u = H(q) \), \( v = H(q^6) \) and

\[
t = \frac{4T(q)}{(1 + T(q))(3 - T(q))}
\]

, then

\[
\frac{\sqrt{u^4 - 6u^2 + 1(u^2 + 2v - 1)}}{(u^2 + 1)(v^2 + 2v - 1)} = t \quad (80)
\]

**Proof.**

Set \( q \to q^3 \) in (60) and then use Proposition 4.2.

**Evaluations**

a) \( V(e^{-\pi}) = \frac{1}{2^{2/3}} \left( -67 - 39\sqrt{3} + (9 + 6\sqrt{3})\sqrt{2(12 + 7\sqrt{3})} \right) \)

b) \( \frac{(3 - T(e^{-\pi\sqrt{3}}))^3(3 + T(e^{-\pi\sqrt{2}}))^3}{(1 - T(e^{-\pi\sqrt{2}}))(1 + T(e^{-\pi\sqrt{2}}))} = 5832 \)
Using the tables of \( k_r \) we can find a wide number of evaluations for the cubic continued fraction.

c) From [13] we have

\[
b_{M,N} = \frac{Ne^{-\left(\frac{N-4}{4}\right)\sqrt{M}}\psi/2(-e^{-\pi\sqrt{MN}})\phi^{2}(-e^{-2\pi\sqrt{MN}})}{\psi^{2}(-e^{-\pi\sqrt{MN}})\phi^{2}(-e^{-2\pi\sqrt{MN}})}
\]

Then from Theorem 4.1

\[
b_{M,3}^{2} = \frac{9(1 - T^{2})}{T^{2}(9 - T^{2})}
\]

where

\[
\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^{2}}
\]

\[
\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}
\]

\(|q| < 1.\)

5 Other Continued Fractions

Section 1.

Another continued fraction is

\[
S(q) = q^{1/8} \frac{q}{1} + \frac{q^{2}}{1+1} + \frac{q^{3}}{1+1} + \frac{q^{4}}{1+1} + \frac{q^{5}}{1+1} + \ldots
\]

for which it is known that

\[
S(q) = q^{1/8} \frac{(-q^{2}; q^{2})_{\infty}}{(-q; q^{2})_{\infty}}
\]

after using Euler’s Theorem: \((-q, q)_{\infty} = 1/(q; q^{2})_{\infty}\) and making some simplifications and rearrangements in the products we find

\[
S(q) = q^{1/8} \frac{\Phi(-q^{2})^{2}}{\Phi(-q)}
\]

Now making use of (8) we get

\[
S(q) = 2^{-1/6}(k_{4r})^{1/6}(k'_{r})^{1/6}
\]

Using the relation between \( k_{4r} \) and \( k_{r} \) form Lemma 3.1, we get

**Theorem 5.1**

\[
S(q) = \frac{(k_{r})^{1/4}}{\sqrt{2}}
\]
Hence the fraction $S$ is the inverse elliptic nome and as someone can see there holds a very large number of modular equations, but since it is trivial we not mention here.

**Section 2.**

The continued fraction

$$Q(q) = \frac{q^{1/2}}{1 - q + (1 - q)(q^2 + 1) + (1 - q)(q^4 + 1) + (1 - q)(q^6 + 1) + \ldots} \quad (85)$$

which is known that

$$Q(q) = q^{1/3} \frac{(q^4; q^4)_\infty^2}{(q^2; q^4)_\infty^2} = M(q^2)^2 \quad (86)$$

it becomes

$$Q(q) = \ldots = q^{1/2} f(-q^4)^2 \Phi(-q^2)^2 = M(q^2)^2 \quad (87)$$

or

**Theorem 5.2**

$$Q(q) = \frac{1}{\pi} K(k_{4r}) \sqrt{k_{4r}^4} = \frac{1}{2\pi} K(k_r) k_r \quad (88)$$

**Proof.**

It follows from the relation between $Q$ and $M$.

**Evaluation**

$$Q(e^{-\pi \sqrt{2}}) = \left(\frac{\sqrt{2} - 1}{\sqrt{2\pi}}\right) \frac{\Gamma(9/8)}{\Gamma(5/8)}$$

**Theorem 5.3**

If

$$u = \frac{Q(q)}{Q(q^2)}, \quad v = \frac{Q(q^3)}{Q(q^5)}$$

then

$$v^4 + u^4 - v^3u^3 + 6v^2u^2 - 16vu = 0 \quad (89)$$

**Proof.**

From relation (59), we get

$$4\sqrt{uv} + \sqrt{(u^2 - 4)(u^2 - 4)} = \sqrt{(v^2 + 4)(u^2 + 4)}$$

after some simplification we get the result
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