Abstract: A highly strong upper estimate in the modified asymptotic formula for sums of the primes’ reciprocals is proved to be necessary (as well as sufficient) in order the Ramanujan inequality holds true. Some other criteria in similar terms are also obtained.

Keywords: Mertens formula, Gronwall numbers, Ramanujan inequality, Riemann Hypothesis

1. Notations, brief history and main results

As usually, let \( \mathbb{N} \) be a set of all positive integers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( p \) run the set \( \mathbb{P} := \{p_1, p_2, \ldots\} \), \( p_j < p_{j+1} \), of all primes, \( \varepsilon \) is an arbitrary positive number, \( C_y \) stand for positive constants which may depend only on a parameter \( y \); symbols \( \triangleright \) and \( \square \) denote the proof’s beginning and end; \( \log x \) and \( \gamma \) stand (resp.) for the natural logarithm of a positive \( x \) and the Euler-Mascheroni constant:

\[
\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577 215 664 \ldots
\]  

In 1874 F. Mertens [1] proved his famous asymptotic formula

\[
S(x) := \sum_{p \leq x} \log \frac{p}{p - 1} = \log \log x + \gamma + R(x) \text{ with } R(x) = O \left( \frac{1}{\log x} \right). 
\]  

The best known unconditional, (i. e. without assumption of the Riemann Hypothesis (RH)), estimate for this remainder at the moment (2021) seems to be \( R(x) = O(\exp(-c(\log x)^{3/5}(\log \log x)^{-1/5})) \).

In 1984 assuming RH G. Robin [2, Th. 3] has come to the fundamentally stronger estimate: \( |R(x)| < \log x/(8\pi \sqrt{x}) \), \( x > X_0 \).
We will present an integer $N > 1$ as its canonical factorization in primes

$$N := p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}; \quad \alpha_j \in \mathbb{N}_0, \; \alpha_k > 0,$$

where the number $k := k(N)$ and exponents $\alpha_j := \alpha_j(N), \; 1 \leq j \leq k$, are uniquely defined by $N$.

The greatest prime factor $p_k := p_k(N)$ of $N$ will be denoted by $\text{gpf}(N)$.

Let $\sigma(N)$ stand for the arithmetic multiplicative function sum of all divisors of $N \in \mathbb{N}$. The properties of this function are well described in a very informative paper [3], which contains a lot of valuable historical remarks, as well as many definitions, notations and facts widely used in this paper. In particular, in Sect. 5 the classical formula for $\sigma(N)$ is adduced, $N$ being defined in (1.3), namely:

$$\sigma(N) := \prod_{j=1}^{k} (1 + p_j + \ldots + p_j^{\alpha_j}) = \prod_{j=1}^{k} \frac{p_j^{\alpha_j+1} - 1}{p_j - 1} \quad (1.4)$$

T. Gronwall in 1913, basing on (1.2) established the sharp upper order of $\sigma(n)$, namely he proved [4] that:

$$\limsup_{N \to \infty} G(N) = e^\gamma = 1.781 072 \ldots; \; \text{where } G(N) := \frac{\sigma(N)}{N \log \log N}, \quad (1.5)$$

which we will call Gronwall numbers.

S. Ramanujan has noticed (in 1915, the first publication in 1997 [5]) that:

\textit{if RH holds true, then in addition to (1.6) for all $N$ sufficiently large the following strict (Ramanujan) inequality (RI) takes place:}

$$G(N) < e^\gamma, \quad \forall N > n_0. \quad (1.6)$$

Almost 70 years later G. Robin [2, Th 1] proved a paramount assertion, which in a sense complements the Ramanujan’s result, namely:

\textit{if (1.6) holds true for all integers $N > 5040$, then RH is valid.}

We will call (1.6) with $n_0 = 5040$ the \textit{Ramanujan-Robin} inequality (RRI), in which the statement of RH is exhaustively encoded in terms of $\sigma(N)$.

Robin has also shown that \textit{in fact RI (with uncertain $n_0$) and RRI are equivalent}, because if RH holds false, then there are infinitely many $N$’s such that $G(N) > \gamma$.

In this paper we \textit{do not strive} to prove any of these conjectures but rather to reveal the direct interrelation between them and the remainder in the \textit{modified} Mertens formula:

$$S(x) = \log \log \theta(x) + \gamma + Q(x), \quad (1.7)$$

which differs from (1.2) by replacing $x$ in $\log \log$ by the first Chebyshev function $\theta(x) := \sum \{\log p : p \leq x\}$ (cf [6, 3.1]).
Theorem. RRI is equivalent to each of the following three conditions:

\[ \forall \varepsilon > 0 \forall x > 1 : Q(x) < C_\varepsilon x^{-0.5 + \varepsilon} \quad (1.8) \]

\[ \forall \varepsilon > 0 \forall x > 1 : Q(x) > -C_\varepsilon x^{-0.5 + \varepsilon}, \quad (1.9) \]

\[ A_0 := \limsup_{x \to +\infty} Q(x) \sqrt{x \log x} < +\infty. \quad (1.10) \]

In addition, (1.10) necessarily implies that \( A_0 \leq 2\sqrt{2} \); for this reason the situation \( 2\sqrt{2} < A_0 < +\infty \) is logically impossible.

The proof of the Theorem is set forth in Sect. 3; in Section 2 all needed auxillaries and the main Lemma are adduced; in Sect. 4 some corollaries and directions of further research are given.

2. Some known facts and main lemma

Further the well-known assertions are brought together concerning the asymptotic behavior of primes [6, Ch. 5] in their weak form sufficient for our purposes:

Proposition 1. (i) For all \( x > 1 \) one has \( |\theta(x) - x| < C_0 x / \log x \);

(ii) \( p_{k+1} - p_k < C_1 p_k / \log p_k \);

(iii) RH is equivalent to each of the two relationships:

\[ \forall \varepsilon > 0 \exists C_\varepsilon \forall x > 1 : |\theta(x) - x| < C_\varepsilon x^{0.5 + \varepsilon}; \quad |\theta(x) - x| < \frac{\sqrt{x} \log^2 x}{8\pi}, \quad x > X_0. \quad (2.1) \]

For the sequel we will need the (perhaps also well-known) assertion, which follows from Proposition 1(i):

Proposition 2. Let \( \lambda > 1 \); then for all \( x > X_\lambda := \exp(\max(1, 2/(\lambda - 1))) \) one has:

\[ Y = Y(x, \lambda) := \sum_{p > x} \frac{1}{p^\lambda} = \frac{1 + \delta(x, \lambda)}{(\lambda - 1)x^{\lambda - 1} \log x}; \quad |\delta(x, \lambda)| < \frac{C_1}{(\lambda - 1) \log x}. \quad (2.2) \]

\[ \text{In fact, using the integration by parts one obtains} \]

\[ Y = \int_{x^+}^{\infty} \frac{d\theta(t)}{t^\lambda \log t} = \frac{\theta(t)}{t^\lambda \log t} \bigg|_{x^+}^{\infty} - \int_{x^+}^{\infty} \theta(t) \left( \frac{1}{t^\lambda \log t} \right)' \, dt \]

\[ = -\frac{\theta(x^+)}{x^\lambda \log x} + \int_{x}^{\infty} \frac{\theta(t)(\lambda \log t + 1)}{t^{\lambda + 1} \log^2 t} \, dt. \quad (2.3) \]
Analogously, replacing here \( \theta(t) \) by \( t \) one obtains the identity:

\[
J = J(x, \lambda) := \int_{x}^{\infty} \frac{dt}{t^\lambda \log t} = -\frac{x}{x^\lambda \log x} + \int_{x}^{\infty} \frac{t (\lambda \log t + 1)}{t^{\lambda+1} \log^2 t} \, dt
\]

\[
= -\frac{1}{x^{\lambda-1} \log x} + \lambda J + \frac{\beta J}{\log x}; \quad \text{where } 0 < \beta = \beta(x, \lambda) < 1 \text{ for } x > X_\lambda, \quad (2.4)
\]

whence it follows that for all \( x > X_\lambda \) (explanations below):

\[
J(x, \lambda) = \frac{1}{(\lambda - 1) x^{\lambda-1} \log x \left(1 - \frac{\beta}{(\lambda-1) \log x}\right)}
\]

\[
\Rightarrow \quad 0 < J(x, \lambda) - \frac{1}{(\lambda - 1) x^{\lambda-1} \log x} < \frac{2}{(\lambda - 1)^2 x^{\lambda-1} \log^2 x}. \quad (2.5)
\]

Here we have taken into account that since \( x > X_\lambda \) then by virtue of the \( X_\lambda \)-definition the number \( t := 1/(\lambda - 1) \log x \) belongs to the interval \((0, 1/2)\) and hence the inequality \( 1/(1 - t) < 1 + 2t \) holds.

On the other hand, substituting (2.4) from (2.3) and using Proposition 1(i) leads to:

\[
|Y - J| \leq \frac{|\theta(x^+) - x|}{x^\lambda \log x} + \int_{x}^{\infty} \frac{|\theta'(t) - t| (\lambda \log t + 1)}{t^{\lambda+1} \log^2 t} \, dt
\]

\[
\leq \frac{C_0}{\log x} \left(\frac{2}{x^{\lambda-1} \log x} + J\right) < \frac{4C_0}{(\lambda - 1)^2 x^{\lambda-1} \log^2 x}, \quad \forall x > X_\lambda. \quad (2.6)
\]

Joining (2.6) with (2.5) one comes to (2.2) \( \Box \).

The main role in the proof of the Theorem plays the following unconditional assertion, binding Mertens function \( S(x) \) and Gronwall numbers \( G(N) \), which is the most important and complicated part of the paper.

**Lemma.** For any \( k \in \mathbb{N} \) there are a real number \( \delta_k \), \( \{\delta_k\} \to 0 \text{ as } k \to \infty \), and an integer \( N^*_k \) such that \( \text{gpf}(N^*_k) = p_k \) and

\[
\log G(N^*_k) > S(p_k) - \log \log \theta(p_k) - \frac{2\sqrt{2} + \delta_k}{\sqrt{p_k} \log p_k}. \quad (2.7)
\]

▷ 1) First we describe the special construction of \( \{N^*_k\}_{k=1}^{\infty} \) providing (2.7). We’ll suppose that \( k \) is large enough; put \( r = r_k := [\sqrt{\log 2 p_k}], \) and define:

\[
q_1 := p_k, \quad q_m := \max\{p_j : p_j^m \leq 2p_k\}, \quad 2 \leq m \leq r. \quad (2.8)
\]
In other words, \( q_m = q_{m,k} \) is the greatest prime \( \leq (2p_k)^{1/m} \); hence \( q_{m-1} < q_m \) for all \( m, 1 < m \leq r \). From Proposition 3 one may easily deduce that the quantity \( q_{m,k} = (2p_k)^{1/m}(1 - \delta_{k,m}) \); \( 0 \leq \delta_k := \max_{1 < m \leq r} \delta_{k,m} \to 0, \; k \to \infty \).

Let \( \nu = \nu_r := \max\{ j : p_j \leq q_r \} \), \( H := q_r^{\nu+1} \); define the exponents \( \{ \alpha_j \}_{j=1}^k \)

\[
\alpha_j := \left[ \frac{\log H}{\log p_j} \right] - 1, \; \text{if } j < \nu; \quad \alpha_j := \max\{ m \leq r : q_m \geq p_j \}, \; \text{if } j \geq \nu; \quad (2.9)
\]

It is clear that: 1) \( \alpha_{\nu} = \alpha_{\nu+1} = r \), \( p_{\nu} = q_r \), 2) \( \alpha_j \geq \alpha_{j+1}, \; 1 \leq j < k \), 3) the equality \( \alpha_j = m < r \) is equivalent to \( q_{m+1} < p_j \leq q_m \).

Let \( T(x) := \exp(\theta(x)) \) stand for a product of all primes \( p \leq x \).

Now we are able to determine the numbers \( N_k^* \), for which the relationship (2.7) is guaranteed:

\[
N_k^* := \prod_{m=1}^r T(q_m) \cdot \prod_{j=1}^{\nu-1} p_j^{\alpha_j-r} = \prod_{j=1}^k p_j^{\alpha_j}. \quad (2.10)
\]

2) Let’s study the quantity \( \eta = \eta_k := \log N_k^* = E_k + F_k; E_k := \sum_{m=1}^r \theta(q_m), \quad F_k := \sum_{j=1}^{\nu-1} (\alpha_j - r) \log p_j \). Having taken into account the definition (2.8) of \( q_m \) and the relationships: \( \max\{|1 - \theta(q_{m,k})(2p_k)^{-1/m}| : 1 < m \leq r \} \to 0, \; k \to \infty \), \( \nu < p_{\nu} = q_r < (2p_k)^{1/r}, \log H = (r + 1) \log q_r \), one has:

\[
E_k := \theta(p_k) + C_k \sqrt{p_k} + O(p_k^{1/3} \sqrt{\log 2p_k}), \quad C_k \to \sqrt{2}, \; k \to \infty;
\]

\[
0 < F_k < \nu \log H < q_r(r + 1) \log q_r = O(p_k^\varepsilon) \Rightarrow \eta_k - \theta(p_k) \approx \sqrt{2p_k}. \quad (2.11)
\]

3) From (1.4) and (1.2) it follows that

\[
\log G(N_k) = \log \frac{\sigma(N_k)}{N_k} - \log \log \log N_k = \sum_{j=1}^k \log \frac{p_j^{\alpha_j+1} - 1}{p_j^{\alpha_j}(p_j - 1)} - \log \log \eta_k = \sum_{j=1}^k \log \frac{p_j}{p_j - 1} - \sum_{j=1}^k \log \frac{p_j^{\alpha_j+1}}{p_j^{\alpha_j+1} - 1} - \log \log \eta_k = S_k - U_k - V_k. \quad (2.12)
\]

Now with certain \( t_k \) in between of \( \theta(p_k) \) and \( \eta_k \), one has:

\[
V_k - \log \log \theta(p_k) = \frac{\eta_k - \theta(p_k)}{t_k \log t_k} \approx \frac{\sqrt{2}}{\sqrt{p_k \log p_k}}, \; k \to \infty. \quad (2.13)
\]

4) To make sure that the quantity \( U_k \) is also \( \approx \sqrt{2}/\sqrt{p_k \log p_k} \) as \( k \to \infty \), we present it as a sum:
\[ U_k = \sum_{m=1}^{r} U_{k,m}; \quad U_{k,m} := \sum_{q_{m+1} < p_j \leq q_m} \log \frac{p_j^{m+1}}{p_j^{m+1} - 1}, \quad m < r; \]

\[ U_{k,r} := \sum_{j=1}^{\nu-1} \log \frac{p_j^{\alpha_j+1}}{p_j^{\alpha_j+1} - 1}. \quad (2.14) \]

All summands \( U_{k,m} \) here are positive. Using the elementary inequality: \(-t^2 < \log(1 - t) + t < 0, \quad 0 < t < 1/4\), easily deduced from the Taylor formula, one may assert that for \( m < r, \ k > k_0 \) and some \( \delta_{j,m} \in (0, 1)\):

\[ U_{k,m} = \sum_{q_{m+1} < p_j \leq q_m} -\log \left( 1 - \frac{1}{p_j^{m+1}} \right) = \sum_{q_{m+1} < p_j \leq q_m} \left( \frac{1}{p_j^{m+1}} + \frac{\delta_{j,m}}{p_j^{2m+2}} \right), \quad (2.15) \]

5) Recollecting now the definition (2.2) of the quantity \( Y(x, \lambda) \) in Proposition 2, we may rewrite the latter equality as follows:

\[ U_{k,m} = Y(q_{m+1}, m+1) - Y(q_m, m+1) + W_{k,m}; \]

\[ 0 < W_{k,m} < Y(q_{m+1}, 2m+2), \quad 1 \leq m < r. \quad (2.16) \]

Applying the relationship (2.2) with \( \lambda = m + 1, \ 2m + 2, \ x = q_{m+1}, q_m \), and having taken into account that \( q_{m+1} \approx (2p_k)^{1/(m+1)} \), by virtue of defining formula (2.8), one comes to the estimates

\[ U_{k,m} < \frac{1}{mq_{m+1}^{m}} \log q_{m+1} \left( 1 + \frac{C_1}{\log q_{m+1}} \right) < C_2 \ p_k^{-m/(m+1)}; \quad m < r. \quad (2.17) \]

Further, for \( m = r \) due to the fact that \( \alpha_j \log p_j > (r + 1) \log q_r - \log p_\nu \) for \( j < \nu \) (cf. the left part of definition (2.9)) and \( \nu < p_\nu = q_r \), one obtains:

\[ U_{k,r} < 2 \sum_{j=1}^{\nu-1} \frac{1}{p_j^{\alpha_j+1}} < \frac{2\nu p_\nu}{q_r^{\nu+1}} < \frac{2}{q_r^{\nu-1}} < \frac{3}{(2p_k)^{1-1/r}} = O(p_k^{-1+\varepsilon}). \quad (2.18) \]

From these two estimates it follows that for \( k \) large enough:

\[ \sum_{m=2}^{r} U_{k,m} < C_2 p_k^{-2/3}(\log p_k)^{1/2}. \quad (2.19) \]

6) At last, if \( m = 1, \) then again by virtue of Proposition 2, one has

\[ U_{k,1} = Y(q_2, 2) + O(p_k^{-1}) = \frac{1 + O(1/\log x)}{\sqrt{2p_k \log \sqrt{2p_k}}} = \frac{\sqrt{2} + O(1/\log p_k)}{\sqrt{p_k \log p_k}}. \quad (2.20) \]
whence in junction with (2.13) and (2.19) it follows that $U_k \approx \frac{\sqrt{2}}{\sqrt{p_k} \log p_k}$, and joining this with (2.11), (2.12), one comes to the limit relationship:

$$\log G(N_k^*) - (S(p_k) - \log \log \theta(p_k)) \approx \frac{2\sqrt{2}}{\sqrt{p_k} \log p_k}, \quad k \to \infty,$$

which in turn implies (2.7) $\square$.

Now we have got all the tools needed to move forward.

3. Proof of the Theorem

**Sufficiency.** Due to Nicolas result (cf \cite{7}, \cite{2}, Sect. 4) the negation of RH implies $Q(x) = \Omega_{\pm}(x^{-b})$ for some $b \in (0, 0.5)$, i.e. according to the meaning of the symbol $\Omega_{\pm}$, for some $\delta > 0$ and any $X > 0$ there are $y > z > X$ such that $Q(y) > \delta y^{-b}$, $Q(z) < -\delta z^{-b}$, but each of these two inequalities contradicts (resp.) to (1.8), (1.9). Besides, obviously (1.10) $\Rightarrow$ (1.8).

Thus it is proved that each of (1.8), (1.9) and (1.10) implies RH $\square$.

**Necessity.** Let us suppose that (1.10) is false, or more precisely, that $B_0 > 2\sqrt{2}$; then taking into account the relationships (1.2), (1.7) and (1.10), one may conclude that for any fixed $\varepsilon_1 \in (0, B_0 - 2\sqrt{2})$ the set

$$K_{\varepsilon_1} := \left\{ k : Q(p_k) > \frac{2\sqrt{2} + \varepsilon_1}{\sqrt{p_k} \log p_k} \right\}$$

is infinite. (3.1)

But then by virtue of Lemma and the equality $Q(x) = S(x) - \log \log \theta(x) - \gamma$, (cf (1.7), (2.1)) one obtains for all sufficiently large $k \in K_{\varepsilon_1}$

$$\log G(N_k^*) > \gamma + Q(p_k) - \frac{2\sqrt{2} + \delta_k}{\sqrt{p_k} \log p_k} > \gamma + \frac{\varepsilon_1 - \delta_k}{\sqrt{p_k} \log p_k} > \gamma,$$

because $\delta_k \to 0$, $k \to \infty$, whereas $\varepsilon_1 > 0$, and consequently RH holds false.

Further, assuming RH one deduces from (1.2), (1.7) and Proposition 1(iii), that there is $t, (t - x)(t - \theta(x)) < 0$ for which

$$|R(x) - Q(x)| = \left| \frac{\theta(x) - x}{t \log t} \right| < \frac{1 + o(1)}{8\pi \sqrt{x \log x}},$$

and joining this with Robin’s estimate $|R(x)| < \log x/(8\pi \sqrt{x})$, mentioned in Sect. 1, one obtains $|Q(x)| < (1 + o(1)) \log x/(8\pi \sqrt{x})$, which in turn implies (1.9) $\square$.

This completes the Theorem’s proof.
4. Conclusive Remarks.

1) The assertions in Theorem may be presented in a *discrete* form, when \( x \) in (1.2) runs only the sequence of primes \( \{p_j\}_{j=1}^{\infty} \).

2) One may also replace in (1.2) \( \log(p/p - 1) \) by primes reciprocals \( 1/p \).

**Corollary.** The Ramanujan inequality (1.6) (and thus RH) is equivalent to each of the following three unilateral estimates for all sufficiently large \( k \):

\[
\sum_{j=1}^{k} \frac{1}{p_j} < \log \log \theta(p_k) + B_1 + p_k^{-0.5 + \varepsilon}, \quad \forall \varepsilon > 0; \quad (4.2)
\]

\[
\sum_{j=1}^{k} \frac{1}{p_j} > \log \log \theta(p_k) + B_1 - p_k^{-0.5 + \varepsilon}, \quad \forall \varepsilon > 0; \quad (4.3)
\]

\[
\sum_{j=1}^{k} \frac{1}{p_j} < \log \log \theta(p_k) + B_1 + \frac{A_0}{\sqrt{p_k \log p_k}}, \quad A_0 < \infty, \quad (4.4)
\]

where \( B_1 \) stands for the Meissel-Mertens constant:

\[
B_1 := \lim_{k \to \infty} \left( \sum_{j=1}^{k} \frac{1}{p_j} - \log \log \theta(p_k) \right) = \gamma - \sum_p \left( \log \frac{p}{p - 1} - \frac{1}{p} \right) = \gamma - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k} = 0.261497 \ldots \quad (4.5)
\]

For the proof one should use the Theorem and notice that (cf. (1.2))

\[
\sum_{j=1}^{k} \left( \frac{1}{p_j} + \log \left( 1 - \frac{1}{p_j} \right) \right) = B_1 - \gamma + O \left( \frac{1}{p_k} \right). \quad (4.6)
\]

3) Quite recently the author established (combining the method by Ingham [8, Sect. V. 10] with the properties of the so-called locally G-maximal numbers, studied in [9, Sect 2]), that *if in (1.10) \( A_0 < +\infty, \) and thus RH is true, then necessarily* \( 1.5 - \varepsilon < Q(x) \sqrt{x \log x} < 2.5 + \varepsilon, \quad \forall x > X_\varepsilon \) \quad (4.6)

whence one may deduce the relationship:

\[
\max \{ \log G(N) : \text{gpf}(N) = p_k \} = \gamma - \frac{a_k}{\sqrt{p_k \log p_k}};
\]

\[
2\sqrt{2} - 2.5 - \varepsilon < a_k < 2\sqrt{2} - 1.5 + \varepsilon, \quad \forall k > K_\varepsilon, \quad (4.7)
\]

which quantitatively refines the initial Ramanujan inequality (1.6).

These results will be presented in the next author’s papers.
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