ON THE CLASSIFICATION OF DEGREE 1 ELLIPTIC
THREEFOLDS WITH CONSTANT j-IN Variant

REMK E KLOOSTERMAN

Abstract. We describe the possible M ordell-W eil groups for degree 1 elliptic
thre efold with rational base and constant j-invariant. Moreover, we classify all
such elliptic threefolds if the j-invariant is nonzero. We can use this classi-
cation to describe a class of singular hypersurfaces in P (2; 3; 1; 1; 1) that admit
no variation of Hodge structure. (Remark 9.3.)

1. Introduction

In this paper we work over the field C of complex numbers. Let X ! B be
an elliptic threefold with a (xed) section 0 : B ! X , such that B is a rational
surface. Assume that X is not birationally equivalent to a product E B , with E
an elliptic curve.

Fix now a W eierstr as equation for the generic ber of . As explained in
Section 2 this establishes a degree 6k hypersurface Y P (2k; 3k; 1; 1; 1) that is
birational to X and such that the bration is birationally equivalent to projection
from (1 : 1 : 0 : 0 : 0) onto a plane.

This integer k is not unique. We call the minimal possible k for which such an
Y exists the degree of X ! B . One can easily show that if X is a rational
threefold then the degree equals 1 or 2, and that if X is Calabi-Y au then the degree
equals 3.

For a general point p 2 B we can calculate the j-invariant of the elliptic curve
1 (p). This yields a rational function j( ) : B 9KP 1.

In this paper we study elliptic threefolds of degree 1 with rational base and
constant j-invariant. We would like to classify all such possible threefolds. The
two invariants that interest us are the configuration of singular bers of and the
structure of the M ordell-W eil group M W ( ), the group of rational sections of .
The actual classification we are aiming at in this paper is a classi cation of possi-
bile singular bci of irreducible and reduced degree 6 threefolds Y in P (2; 3; 1; 1; 1)
together with the possibilities for M W ( ). In [8] it is explained how to obtain an
e llip tic threefold X from Y .

One way of constructing elliptic threefolds is taking a cone Y over an elliptic
surface S P (2; 3; 1; 1; 1) P (2; 3; 1; 1; 1). The M ordell-W eil group and the con-
guration of singular bers can be obtained from S . All possibilities for such S have
already been classi ed by Oguiso and Shioda [9]. We refer to such Y as ’obtained
by the cone construction’. We exclude such Y from our classi cation. One can

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show that \( Y \) is a cone over an elliptic surface if and only if the discriminant curve is a union of lines through one point.

We split our considerations in three cases, namely the general one \( j() \neq 0;1728 \), and two special cases \( j() = 1728 \) and \( j() = 0 \).

The case \( j() \neq 0;1728 \) is the easiest one. In this case it is well-known that \( Y \) is given by

\[
y^2 = x^3 + AP^2x + BP^3
\]

with \( A;B \in \mathbb{C} \) and \( P \in \mathbb{C} \setminus \{z_0;z_1;z_2\} \), i.e., \( P \) is homogeneous of degree 2. Our assumptions imply that \( \mathcal{P} = 0 \) is a smooth conic. It turns out that in this case \( \mathcal{M}_W() = (\mathbb{Z}/22^2) \).

The exceptional cases \( j() = 0;1728 \) are more interesting. In these cases one has an equation of the form

\[
y^2 = x^3 + R; \text{ resp. } y^2 = x^3 + Qx
\]

with \( Q \in \mathbb{C} \setminus \{z_0;z_1;z_2\} \) and \( R \in \mathbb{C} \setminus \{z_0;z_1;z_2\} \).

To calculate the group \( \mathcal{M}_W() \) we use the results of [5]. It turns out that \( \mathcal{M}_W() \) is determined by the type of singularities and the configuration of singular points of \( Q = 0 \), resp. \( R = 0 \).

More precisely, the main result of [5] states that \( \mathcal{M}_W() \) is isomorphic to the group of \( \mathcal{W} \) divisors on \( Y \) modulo the Cartier divisors on \( Y \). In our case this can be reformulated as

\[
\text{rank} \mathcal{M}_W() = h^4(Y)_{\text{prim}} = \dim \text{coker } F^2 H^4(\mathbb{P},nY;C) \bigoplus_{p=2}^4 H^4(\mathbb{P},nY;C)
\]

where \( \mathbb{P} \) consists of the points \( f = y = Q = 0 \) and, respectively, \( f = y = R = 0 \).

The Poincare residue map yields a natural surjection from \( \mathbb{C} \setminus \{z_0;z_1;z_2\} \times \mathbb{C} \setminus \{z_0;z_1;z_2\} \) onto \( F^2 H^4(\mathbb{P},nY;C) \). To determine \( H^4_p(Y;C) \) we use three methods. Let \( p \geq 2 \).

1. If \( (Y;p) \) is an isolated singularity and is semi-weighted homogeneous, then there is a method in [ca] to compute an explicit basis for \( H^4_p(Y;C) \), together with the Hodge filtration.

2. If \( (Y;p) \) is not weighted homogeneous, but \( (Y;p) \) is isolated, then there is a classical method of Brion [6] to calculate \( H^4_p(Y) \). This method does not produce the Hodge filtration, and in the weight homogeneous case it is more complicated than D in ca's method.

   This method is implemented in the computer algebra package Singular. Since this case is rather exceptional, we preferred to calculate \( H^4_p(Y;C) \) by using Singular. Hence several of the results in the sequel are only valid up to the correct in implementation of Brion's method in Singular.

3. If \( (Y;p) \) is a non-isolated singularity, but is weighted homogeneous, then the transversal type is an \( AD \) \(-\) surface singularity. To calculate \( H^4_p(Y;C) \) we apply a generalization of D in ca's method, due to Hulek and the author [E].

We list now the possible groups:

Theorem 1.1. Suppose \( Y \) \( \equiv (2;3;1;1;1) \) is a degree 6 hypersurface, corresponding to an elliptic threefold \( X \), not obtained by the cone construction and not birational to a product \( E \times B \). Then \( \mathcal{M}_W() \) is one of the following:

\( (\mathbb{Z}/22^2) \) if \( j() \neq 0;1728 \).
Theorem 1.2. Suppose $Y$ satisfies the conditions of the previous theorem, and suppose that $j() = 1728$.

We have that $MW() = (Z=Z)^2$ if and only if $Q = 0$ defines a double conic and $MW() = Z=Z^2$ if and only if $Q = 0$ is the unique quartic with two $A_3$ singularities.

For $j() = 0$ the number of cases to consider is quite large. One should apply the following program:

1. Determine all possible types of singularities of sextic curves. This is done in [3].
2. For each type of singularity, determine $H^4_p(Y)$.
3. Determine which combinations of singularities are possible on a sextic curve. Here one might restrict oneself to combinations of singularities that yield non-trivial $H^4_p(Y)$.
4. For each configuration, study the relation between $h^4(Y)$ and the position of the singularities.

The second point is completely done in this paper, except for six types of singularities that are both not weighted homogeneous and not isolated. The number of cases to consider at the third and fourth point is quite big. We restrict ourselves to the cases where the sextic is non-reduced, and the case where the sextic has ordinary cusps. (It turns out that if the sextic has a node at $p$ then $H^4_p(Y)$ vanishes, for this reason we study sextic with cusps.)

The curves with only cusps as singularities yield examples for the groups $0;Z^2;Z^4$ and $Z^6$. One can show that $MW() = (Z=Z)^2$ if and only if $R$ defines a triple conic, and $MW() = Z=Z^2$ if and only if $R$ defines a double cubic. This success to provide examples for each of the groups mentioned in Theorem 1.3.

In Section 2 we recall several standard facts concerning elliptic threefolds. In particular we construct our model $Y$. In Section 3 we limit the possibilities for the group $MW()$. This is done by studying the behaviour of $MW()$ under specialization and considering the class function of rational elliptic surfaces [3]. In Section 4 we discuss the possible singularities for quartic and sextic plane curves. This yields a class function of possible singularities on $Y$. In Section 5 we calculate the local cohomology $H^4_p(Y)$ for each possible type of singularity on $Y$. In Section 6 we give some details on how to calculate $\text{rank} MW()$. In the following three sections we give a class function for the cases $j() = 1728$, $j() = 0$ and $R = 0$ is non-reduced and $j() = 0$ and $R = 0$ is a cuspidal sextic. In Section 10 we prove Theorem 1.1.

Notation 1.3. Let $x; y; z_2; z_2$ be coordinates on $P(2;3;1;1;1)$. Throughout this paper $Y = P(2;3;1;1;1)$ is a reduced and irreducible degree 6 hypersurface, containing the point $(1:1:0:0:0)$, and such that $Y$ corresponds to an elliptic fibration with constant $j$-invariant, i.e., $Y$ has a defining equation of the form

$$y^2 = x^3 + AP^2 x + BP^3; \quad y^2 = x^3 + Q x; \text{ or } y^2 = x^3 + R;$$

Here $P; Q; R$ are homogeneous polynomials in $z_2; z_2; z_2$ of degree 2; 4 and 6 respectively and $AB \neq 0$. The curve $C$ is the curve defined by $P = 0, Q = 0$ or $R = 0$ (depending on the case). The set $Q$ consists of the singular points of $C_{\text{red}}$. 
2. Preliminaries

Definition 2.1. An elliptic n-fold is a quadruple \((X;B;\iota;\varrho)\), with \(X\) a smooth projective n-fold, \(B\) a smooth projective n-1-fold, \(\iota:X\to B\) a morphism, such that the generic fiber is a general curve and \(\varrho\) is a section of \(\iota\).

The \(M\) ordell-Wald group of \(\iota\), denoted by \(MW()\), is the group of rational sections \(:B\to X\) with identity element \(0\).

We will focus on the cases \(n=2;3\). Note that in the case \(n=2\) any rational section can be extended to a regular section.

Clearly \(MW()\) is a birational invariant, in the sense that if \(\iota_1:X_1\to B_1,\iota_2:X_2\to B_2\) are elliptic n-folds such that there exists a birational isomorphism \(\iota:X_1\to X_2\) mapping the general fiber of \(\iota_1\) to the general fiber of \(\iota_2\), then \(\iota:MW()\). \(\Box\)

We shall now describe in some detail how to associate to an elliptic n-fold \(X\) a degree 6k hypersurface \(Y\) in the weighted projective space \(P=P(2k;3k;1^{n-1})\) which is birational to \(X\). Here we restrict ourselves to the case where \(B\) is a rational n-1-fold. In this case, the morphism \(\iota\) establishes \(C(X)\) as the function field of an elliptic curve \(E\) over \(C(z_1,\ldots,z_{n-1})\), i.e. \(C(X)=C(x;y;z_1,\ldots,z_{n-1})\) where

\[
y^2 = x^3 + f_1(z_1,\ldots,z_{n-1})x + f_2(z_1,\ldots,z_{n-1})
\]

with \(f_1;f_2\in C(z_1,\ldots,z_{n-1})\). One has a natural isomorphism

\(MW() = \text{E}(C(B))\)

where \(\text{E}(C(B))\) is the group of \(C(B)\)-rational points of \(E\).

Without loss of generality we may assume that \(E\) is a global minimal Weierstrass equation, i.e., \(f_1;f_2\) are polynomials and there is no polynomial \(g\in 2C\) \([z_1,\ldots,z_{n-1}]\) in \(C\) such that \(g^6\) divides \(f_1\) and \(g^6\) divides \(f_2\).

To obtain a hypersurface in \(P\) we need to nd a weighted homogeneous polynomial \(f\) in \(C(x;y;z_1,\ldots,z_{n-1})\) such that \(f^6\) divides \(f_1\) and \(f^6\) divides \(f_2\).

To obtain a hypersurface \(Y\) of degree 6k in \(P\). Let \(k=\text{deg}(f_1)=\text{deg}(f_2)=6\text{deg}(g)\) and define \(P\) and \(Q\) as the polynomials

\[
P = z_0^{4k}f_1(z_1=z_0,\ldots,z_{n-1}=z_0); \quad Q = z_0^{6k}f_2(z_1=z_0,\ldots,z_{n-1}=z_0);
\]

Then

\[
y^2 = x^3 + P(z_0;z_1,\ldots,z_{n-1})x + Q(z_0;z_1,\ldots,z_{n-1})
\]

defines a hypersurface \(Y\) of degree 6k in \(P\). Let \(L\) be the locus where all the partial derivatives of the defining equation vanish. Consider the projection \(\pi:99K\to P\) with center \(L=fz_0=\emptyset\) and its restriction to \(Y\). Then there exists a diagram:

\[
\begin{array}{c}
X \to Y \\
\downarrow \quad \quad \quad \quad \downarrow \\
S \to P^2
\end{array}
\]

Note that \(Y\setminus L=f(1:1:0:0)g\). If \(k=1\) then \(L\) consists of two points, none of which lie on \(Y\). If \(k>1\) then an easy calculation in local coordinates shows that \(P_{\text{sing}}\) is precisely \(L\), that \(L\) and \(L\) are disjoint and that \(Y\) has an isolated singularity at \((1:1:0:0)\). For any \(k\) we have that \(L\) is not dense at \((1:1:0:0)\), and be the blow-up of \(P\) along \(L\). Let \(X_0\) be the strict transform of \(Y\) in \(P\). An easy calculation in local coordinates shows that
X₀ ! Y resolves the singularity of Y at (1 : 1 : 0 : : 0) and that the induced map ₀ :X₀ ! P² is a morphism. Moreover, all bers of ₀ are irreducible curves.

Definition 2.2. The degree of an elliptic n-fold :X ! B, with rational base, is the smallest K such that there is a degree 6k hypersurface Y in P (2k;3k;1⁰⁻¹) birational to .

As remarked above, we can consider the generic ber of as an elliptic curve E over C (z₀;:::;zₙ). In the sequel we consider only elliptic curves such that j() = j(E) is constant, i.e., j(E) 2 C. Most of the sequel will be concentrated on j() 2 f0;1728g. If this is the case then E has complex multiplication.

Lemma 2.3. Let K be an elliptic curve, such that E has complex multiplication over K. Suppose rank E(K) is finite. Then rank E(K) is even.

Proof. Since E(K) is an End(E)-module it follows that

E(K) = E(K)tor \ End(E)

Since End(E) is a free Z-module of rank 2, it follows that E(K) = E(K)tor Z² (as Z-modules), hence E(K) has even Z-rank.

The following minor results will be used several times:

Lemma 2.4. Let V = C be a variety. Let E = C (V) be an elliptic curve such that j(E) 2 C. Suppose j(E) 6 0;1728, then (Z=2Z) is a subgroup of E(C(V)).

Proof. Let E = C be an elliptic curve with j(E) = j(E). Then we can nd a Weierstrass equation y² = x³ + ax + b for E, with a,b 2 C. Let 1, 2, 3 be roots of x³ + ax + b.

Since j(E) 6 0;1728 we have that E is given by

y² = x³ + aP²x + bP³

for some P 2 C(V). For i = 1;2,3 we have that x = iP is a root of x³ + aP²x + bP³, hence x = iP; y = 0 is a point of order 2 on E(C(V)). From this it follows that (Z=2Z) E(C(V)).

Lemma 2.5. Let K be any field not of characteristic 2;3. Let E = K be an elliptic curve with j(E) = 1728 then E(K) contains a point of order 2.

Proof. Since K is not of characteristic 2;3 we have that E has a Weierstrass equation y² = x³ + ax with a 2 K. The point (0;0) is a point of order 2.

3. Possible Mordell-Weil groups & Specialization

We describe now all possible Mordell-Weil groups for elliptic surfaces of degree 1 with constant j-invariant. Using a specialization result this limit the possibilities for Mordell-Weil groups for elliptic threefolds of degree 1. Note that an elliptic surface is rational if and only if its degree is 1. We start by recalling some results from Oguiso and Shioda [6]:

Proposition 3.1. Let : S ! P³ be a rational elliptic surface with j() = 0. Then S is birationally equivalent to a surface in P (2;3;1) given by an equation of the form y² = x² + f(z₀;z₁), with f homogeneous of degree 6. In the following
table we list all possible factorizations of $f$, the contribution of the singular fibres to the Neron-Severi lattice, $\text{rank} M W (\cdot)$ and $\# M W (\cdot)_{\text{tor}}$.

| Factorization | Cont. to NS($S$) | $\text{rank} M W (\cdot)$ | $\# M W (\cdot)_{\text{tor}}$ |
|---------------|------------------|-----------------|-----------------|
| $[1;1;1;1;1]$ | $A_2$ | 6 | 1 |
| $[2;1;1;1;1]$ | $2A_2$ | 4 | 1 |
| $[2;2;1;1]$ | $3A_2$ | 2 | 3 |
| $[3;1;1;1;1]$ | $D_4$ | 4 | 1 |
| $[3;2;1;1]$ | $D_4 + A_2$ | 2 | 1 |
| $[3;3]$ | $2D_4$ | 0 | 4 |
| $[4;1;1;1]$ | $E_6$ | 2 | 1 |
| $[4;2]$ | $E_6 + A_2$ | 0 | 3 |
| $[5;1]$ | $E_8$ | 0 | 1 |

**Proposition 3.2.** Let $S: P^1$ be a rational elliptic surface with $j(\cdot) = 1728$. Then $S$ is birationally equivalent to a surface in $P(2;3;1;1)$ given by an equation of the form $y^2 = x^3 + f(z_0; z_1)x$, with $f$ homogeneous of degree 4. In the following table we list all possible factorizations of $f$, the contribution of the singular fibres to the Neron-Severi lattice, $\text{rank} M W (\cdot)$ and $\# M W (\cdot)_{\text{tor}}$.

| Factorization | Cont. to NS($S$) | $\text{rank} M W (\cdot)$ | $\# M W (\cdot)_{\text{tor}}$ |
|---------------|------------------|-----------------|-----------------|
| $[1;1;1;1;1]$ | $4A_1$ | 4 | 2 |
| $[2;1;1]$ | $D_4 + 2A_1$ | 2 | 2 |
| $[2;2]$ | $2D_4$ | 0 | 4 |
| $[3;1;1;1]$ | $E_6 + A_1$ | 0 | 2 |

**Proposition 3.3.** Let $S: P^1$ be a rational elliptic surface with $j(\cdot)$ constant and $j(\cdot) \not\equiv 0$ (1728). Then $S$ is birationally equivalent to a surface in $P(2;3;1;1)$ given by an equation of the form $y^2 = x^3 + af(z_0; z_1)x^2 + bf(z_0; z_1)^3$, with $\deg(f) = 2$ and $a; b \geq 2$. Then has two fibres with corresponding lattice $D_4$ and $M W (\cdot) = (Z=22)^2$.

Therefore, for a rational elliptic surface with constant $j$-invariant the possible $M$-ordel-$W$ ellgroups are $Z_r^r$, $r \geq 2$, $f_0; g_4; 8; (Z=22)$, $Z_r^r$, $r_2 \leq 2$, $f_0; g_4; 8; (Z=32)$, $Z_r^r$, $r_2 \leq 2$, $f_0; g_2$ and $Z=42$.

We will now prove a specialization result, probably well-known to the experts, that implies that in the threefold case, the $M$-ordel-$W$ ellgroup is a subgroup of one of the above groups.

Let $Y: P(2;3;1;1)$ be an elliptic threefold. Let $Y = f_0 z_0 + a_1 z_1 + a_2 z_2 = 0$ be a line. Let $H = f_0 z_0 + a_1 z_1 + a_2 z_2 = 0$ be the corresponding hyperplane. Then $Y = Y \setminus H$. $P(2;3;1;1)$ is a rational elliptic surface, provided $Y$ is not a component of the discriminant curve of $Y$.

**Lemma 3.4.** The restriction of rational sections to $Y$ defines a group homomorphism $M W (\cdot)! M W (\cdot)$.

**Proof.** A rational section $P^2 \times 99K Y$ can be represented as $[z_0^2; z_1^2; z_2^2]$. Let $[f; g; h; z_0^2; z_1^2; z_2^2; 0]$ where $f, g, h$ are homogeneous polynomials in $z_0, z_1$, and $z_2$ such that $\deg(f) = 2 \deg(h) + 2$ and $\deg(g) = 3 \deg(h) + 3$. 


The indeterminacy locus of such a rational map is contained in $V(f;g;h)$, the locus where $f;g;h$ vanish. We prove now that we can find representatives $f;g;h$ such that $\gcd(f;g;h) = 1$, i.e., the indeterminacy locus of $Y$ is finite. This implies that $j$ is a well-defined rational section.

Assume we have an irreducible non-constant polynomial $p$ dividing $f;g;h$. Then $p^3$ divides

$$f^3 + pfh^4 + qh^6 = g^2;$$

hence $p^4$ divides $g^2$. Write $g = p^2q$, $h = ph_1$ and $f = pf_1$. Now,

$$p^4q^2 = g^2 = f^3 + pfh^4 + qh^6 = p^3(f^3_1 + Afh_1^4p^2 + Bfh_1^6p^3);$$

From this it follows that $p$ divides $f_1$ and $p^2$ divides $f$. This implies that $p^6$ divides $f^3 + pfh^4 + qh^6 = g^2$, hence $p^3$ divides $g$ and

$$[z_0;z_1;z_2] \in \{ f=p^2; g=p^3; z_0h=p; z_1h=p; z_2h=p \}$$

is a well-defined rational map, represented by polynomials and equivalent to the rational map $[z_0;z_1;z_2] \mapsto \{ f; g; z_0h; z_1h; z_2h \}$.

Iterating this process shows that we may assume that $V(f;g;h)$ is a finite set. Since the indeterminacy locus of $Y$ is finite, the restriction $j: \emptyset$! $Y$ is a rational map, which can be extended to a morphism, since $j$ is a curve. If $h$ does not vanish at all points of $j$, then it is a section. If $h$ does vanish along $j$, then the image of $j$ lies in $Y \setminus f_0 = z_1 = z_2 = 0g = f(1:1:0:0:0)$, i.e., the image of $j^*$ coincides with the image of the zero-section. This yields the existence of the map $MW(\emptyset)! MW(\emptyset)$.

Finally, to see that this map is a group homomorphism, note that on both $Y$ and $Y^*$ the addition of sections is defined otherwise.

The following result is probably known to the experts, but we did not find a reference for this in the literature.

Proposition 3.5. Let $\text{red } P^2$ be the reduced curve defined by the vanishing of $4A^2 + 27B^2$. Let $\emptyset$ $P^2$ be a very general line. Then the map $MW(\emptyset)! MW(\emptyset)$ is injective.

Moreover, suppose that $\text{red }$ is neither a union of lines nor an irreducible conic. Then there exists a line $\emptyset$ such that

1. $\emptyset$ is tangent to $\text{red }$ at some point.
2. $\emptyset$ intersects $\text{red }$ in at least two distinct points.
3. The natural map

$$MW(\emptyset)! MW(\emptyset)$$

is injective.

Proof. It suffices to prove that there are at most countably many lines such that

$$MW(\emptyset)! MW(\emptyset)$$

is not injective, since if $\text{red }$ is not the union of lines nor a conic then there are uncountably many lines that satisfy the first and second property, the results follow.

Let $r = \text{rank} MW(\emptyset). W$ like $MW(\emptyset)$ such that $\text{red }$ generates $MW(\emptyset)$.

Consider a section $\emptyset = \emptyset + \sum_{i=1}^{r-1} n_i i.2MW(\emptyset)$, where as

$$[z_0;z_1;z_2] \mapsto \{ f; g; z_0h; z_1h; z_2h \}$$
which some

Corollary 3.7. Follows that other an j is seven. From Propositions 3.2 and 3.5 it follows that M W ( ) is not injective.

Corollary 3.6. Suppose : X ! B is an elliptic threefold of degree 1, j( ) is constant and j( ) 6 0;1728. Then M W ( ) = (Z=2Z)².

Proof. From Lemma 2.3 it follows that (Z=2Z)² M W ( ). From Propositions 3.2 and 3.5 it follows that M W ( ) is a subgroup of either (Z=2Z)², which yields the corollary.

Corollary 3.7. Suppose : X ! B is an elliptic threefold of degree 1, j( ) is constant and equals 1728. Then M W ( ) is one of the following:

\[ (Z=2Z) \; Z^4; (Z=2Z)^2 \]

with r 2 f0;2;4g.

Proof. From Lemma 2.3 it follows that (Z=2Z)² M W ( ), From Lemma 2.3 it follows that the rank is even. From Propositions 3.2 and 3.5 it follows that M W ( ) is a subgroup of either (Z=2Z)² or (Z=2Z)², which yields the corollary.

Corollary 3.8. Suppose : X ! B is an elliptic threefold of degree 1 and j( ) is constant and equals 0. Then M W ( ) is one of the following:

\[ Z^2; (Z=3Z) \; Z^2; (Z=2Z)^2 \]

with r 2 f0;2;4;8g, r 2 f0;2g.

Proof. From Propositions 3.2 and 3.5 it follows that M W ( ) is a subgroup of either Z³, (Z=2Z)² or Z³ Z². From Lemma 2.3 it follows that the rank is even.

To prove the corollary we have to exclude the group Z=2Z. Suppose has a section of order two, i.e., Y is given by an equation of the form y² = x³ + 3 and \( \{ x:0;z_1;z_2 \} \) is a section of order 2. Then for i = 1;2 the morphisms \( \{ x:0;z_1;z_2 \} \) define sections of order 2, where \( \frac{1}{2} = \frac{1}{2} \), hence we can compute two-torsion. In particular, Z=2Z does not occur as possible Mordell-Weil group.

Later we will show that the cases r = 4, r 1 = 8 and r 2 = 2 can only occur in the cone construction case. At this point we will show that in these cases the curve C is a union of lines, but not necessarily through one point.

A sum e that the j-invariant is constant. Then it is well-known that rank M W ( ) equals 2a 4, where a is the number of singular fibers of , i.e., a = \# C \{ counted without multiplicities).\n
Lemma 3.9. Suppose C is not the union of lines. Then y² = x³ + P x has Mordell-Weil rank at most 2.

Proof. If P is of the form P = 0 for some irreducible polynomial x=xP, of degree 2, then for a very general line ' the elliptic surface is an elliptic surface with 2l 0, hence there is a rank 0, since for a very general line the map M W ( ) ! M W ( ) is injective (Proposition 3.3), we have that rank M W ( ) = 0.
Otherwise we can apply the second part of Proposition 3.5. Let \( L \) be a line satisfying the properties mentioned in this proposition. Then \( \cdot \) has \( j = 1728 \) and one of either type \( I_0 \) or \( III \) and hence by Proposition 3.7 has rank at most 2. Since \( \text{MW}(\cdot) \) is injective, it follows that \( \text{MW}(\cdot) \) has rank at most 2.

Lemma 3.10. Suppose \( C \) is not the union of lines. Then \( y^2 = x^3 + \mathcal{Q} \) has \( \text{Morita-Weil} \) rank at most 6.

Proof. The proof is similar to the previous lemma. If \( \mathcal{Q} \) is a quadratic polynomial to the power three then for a general line \( \cdot \) is an elliptic surface with 2 \( I_0 \) bers has therefore rank 0 and hence rank \( \text{MW}(\cdot) = 0 \).

Otherwise we can apply Proposition 3.5. Using this we deduce that \( \cdot \) has a rank of type \( IV;I_0;IV \) or \( II \), and therefore rank \( \text{MW}(\cdot) \leq 6 \), and such that \( \text{MW}(\cdot) \) is injective.

Lemma 3.11. Suppose \( j = 0 \) then 3 \# \( \text{MW}(\cdot) \) if and only if \( Y \) is given by an equation of the form \( y^2 = x^3 + f^2 \), where \( f \) is a cubic polynomial.

If \( (E = 32) \ E^2 \) is a subgroup of \( \text{MW}(\cdot) \) then \( f = 0 \) is an union of lines.

Proof. Suppose \( 3 \# \text{MW}(\cdot) \). Since for a very general line \( \cdot \) the map \( \text{MW}(\cdot) \) is injective Proposition 3.5 it follows from Propositions 3.11 that for such a line \( \cdot \) the intersection \( C \cap \cdot \) consists of three points with multiplicity 2 or one point with multiplicity 2 and one point with multiplicity 4. Hence \( C \) is a double cubic.

Conversely, if \( Y \) is given by \( y^2 = x^3 + f^2 \) then \( x = 0; y = f \) defines a section of order 3.

Suppose \( f = 0 \) is not the union of lines. Then \( C \) is a reduced cubic. Then by Proposition 3.11 there exists a line \( \cdot \), such that \( \text{MW}(\cdot) \) is injective and such that \( \# \cdot \cap C = 2 \), hence \( \text{MW}(\cdot) \) is nine.

4. Singularities of quartic and sextic plane curves

4.1. Quartic curves. The classification of singular quartic curves is well-known. We give a sketch. First assume that \( C \) is reduced. Then either \( C \) is the union of four lines through one point, or \( C \) has at most \( ADE \) singularities. The first case corresponds to the cone construction case, so suppose we are in the latter case. Let \( p_1;\ldots;p_k \) be the singularities of \( C \), let \( M_1;\ldots;M_k \) be the corresponding Mburn lattice. Then \( M_1 \) can be embedded in the lattice corresponding to the affine Dynkin diagram \( E_7 \). This limits the possibilities to \( A_1;\ldots;A_7;D_4;\ldots;D_7 \) and \( E_6;E_7 \).

Assume that \( C \) is non-reduced and that \( \text{Cred} \) is not the union of lines through one point. Then \( C \) has either a double line \( \cdot \) together with a (possibly reducible) conic \( T \), or a double conic. If \( C \) is a double conic, then it has to be irreducible, hence \( \text{Cred} \) is smooth and \( P = \cdot \).

Let \( q \) be a point of the singular locus of \( \text{Cred} \). The above discussion shows that \( (C;q) \) is one of the following singularities

- \( f^2, \text{i.e. } - \) and \( T \) intersect transversely;
- \( f^2, \text{i.e. } - \) is tangent to \( T \);
- \( f^2, \text{i.e. } - \) is a \( (A_1 \text{ singularity}) \), i.e., \( T \) is the union of two lines.

Note that in the second case we have that \( P \) consists of one point.
4.2. Sextic curves. Sextic curves have more possible singularities.

Theorem 4.1. A reduced sextic can have the following singularities [6]:

\[ A_k : x^2 + y^{k+1}, \quad k = 19. \]
\[ B_k : y (x^2 + y^{k-1}), \quad k = 19. \]
\[ C_k : x^3 + y^4. \]
\[ D_k : x^3 + y^5. \]
\[ E_k : x^3 + y^6. \]
\[ B_{k1} : x^5 + y^1, \quad k = 1. \] If \( k = 3 \) then 6 1 12, if \( k > 3 \) then 1 6.
\[ xB_{k1} : x (x^k + y^1), \quad (k; l) 2 f (2; 5); (2; 7); (3; 4); (3; 5); (4; 5) g. \]
\[ yB_{k1} : y (x^k + y^1), \quad (k; l) 2 f (3; 4); (3; 5); (5; 6); (5; 4) g. \]
\[ xyB_{k1} : xy (x^k + y^1), \quad (k; l) 2 f (2; 3); (3; 4) g. \]
\[ C_{k1} : x^5 + y^1 + x^2y^2, \quad k = 2k + 2 = 11 and k \quad n (l), \quad \text{with} \quad n (l) = 15; 14; 12; 11; 9 \text{ for } l = 3; 4; \cdots; 9. \]
\[ yC_{k1} : y (x^k + y^1 + x^2y^2), \quad k = 2k + 1 = 1, \quad k = 2 f 3; 5; 6 g. \] If \( k = 3 \) then 7 1 12. If \( k = 5 \) then 12 f 5; 6 g.
\[ D_{k1} : x^5 + y^1 + x^2y^2, \quad 2p + 3 = q \quad 1. \] If \( k = 3 \) then 9 10 13. Otherwise \( (k; l) 2 f (4; 7); (5; 6); (5; 7); (6; 5); (6; 6); (6; 7) g. \]
\[ E_{k1} : x^5 + y^1 + x^2y^3 + x^2y^2, \quad k = 7. \]
\[ S_{k1} : (x^2 + y^1)^2 + (a_0 + a_1 y) x^{3+k}. \quad a_0 \quad 0; a_1 = 2; C; k = 1; 2; 3. \]
\[ S_{k2} : (x^2 + y^1)^2 + (a_0 + a_1 y) x^{3+k}. \quad a_0 \quad 0; a_1 = 2; C; k = 1; 2; 3. \]

All these singularities are also in A moKd’s list, so one might also use the names given by A moKd. A translation between our names and that of A moKd can be found in [5, Remark 1].

Several of these singularities have distinct Milnor and Tjurina numbers, and are therefore not semisweighted homogeneous.

We do not present a classification of non-reduced sextics here. Essentially, one has either

- a double line with a quartic,
- a double conic with another conic,
- a double cubic,
- a triple line with a cubic,
- a triple line with a double line and a line,
- a triple conic or
- a quadruple line with conic.

The possibilities for the singularities are a combination of the possibilities of singularities for conics, cubics and quartic, and the possible intersection numbers between the components.

5. Calculating \( H^s_p (Y; C) \)

In this section we discuss three approaches to calculate \( H^s_p (Y; Q) \). For each singularity that we encounter, one of these methods applies, except for six types of singularities. We list \( h^s_p (Y) \) for each of the singularities.

5.1. D in ca’s method. Let \((Y; p)\) be a semisweighted isolated hypersurface singularity. We have a local equation of the form \( f_p + q_p = 0 \), such that \((Y; p)\) is a \( -\text{constant deformation of } f_p = 0 \) and \( f_p = 0 \) defines a weighted homogeneous isolated hypersurface singularity.
Let $w_1, \ldots, w_4$ are the weights of the variables, $w_p = w_1 + w_2 + w_3 + w_4$ and let $d_p$ be the (weighted) degree of $f$. Then $D$ in case 3 shows
\[ H^4_p(Y) = \sum_{i=1}^{3} R(f)_{d_{i}p} w_p. \]
Moreover, this direct sum decomposition is just $R(f)^\mathbb{Z}_{d_p} w_p = 0$ and $R(f)^{3d_p} w_p = 0$. This follows from the fact that in all cases $d_p < w_p$ and the existence of a non-degenerated pairing $R(f)_{d_p} w_p$. Hence $R(f)_{4d_p} 2w_p = C$. This implies that $H^4_p(Y; C) = C (2)^k$ with $k = \h^4_p(Y)$.

If $j(\cdot) = 1728$ then all singularities of $Y$ are non-isolated, so for the rest of this subsection assume that $j(\cdot) = 0$.

We have a sem i-weighted hypersurface singularity if and only if the sextic $C$ is reduced at $q = (p)$ and has a weighted homogeneous singularity. This limits us to cases that $C$ has either an ADE singularity, or a $B_{k, l}i; xB_{k, l}i; yB_{k, l}i; xyB_{k, l}$ singularity. We list now the singularities with non-trivial $H^4_p(Y)$.

Proposition 5.1: Suppose $(C; q)$ is a weighted homogeneous singularity of a sextic curve, not a point of order six, and such that $h^4_p(Y; C) \neq 0$. Then $(C; q)$ is one of
\[ A_6; A_5; A_8; A_{11}; A_{14}; A_{17}, \]
\[ E_6, \]
\[ B_{32}; B_{33}; B_{34}, B_{32}; B_{33}; B_{34}. \]

The following lemma will yield a proof of this proposition and the proofs provide basis for $H^4_p(Y; C)$ for each non-trivial case.

Lemma 5.2. Suppose $C$ has an $A_k$ singularity at $q$. If $k \equiv 2 \mod 3$ then $H^4_p(Y; C)$ is isomorphic to $C (2)^2$. Otherwise, $H^4_p(Y; C)$ vanishes.

Proof. We have a local equation for $Y$ of the form
\[ y^2 = x^3 + t^2 + s^{k+1}; \]
Setting weights $6; 3k + 3; 2k + 2; 3k + 3$ for $s; t; x; y$, we obtain $d_p = 6k + 6, w_p = 8k + 14$. Hence $2d_p = w_p = 4k$. The Jacobian ideal is generated by $y; t; x^2; s^k$. Hence $R(f)_{4k} 2$ is spanned by
\[ x^{(k-2)_3} y^{(k+1)_3} \]
This means that $H^4_p(Y) = 0$ if $k \equiv 2 \mod 3$ and $H^4_p(Y) = C (2)^2$ if $k \equiv 2 \mod 3$.

Lemma 5.3. Suppose $C$ has an $D_k$ singularity at $q$ then $H^4_p(Y; C) = 0$.

Proof. We have a local equation for $Y$ of the form
\[ y^2 = x^3 + s^2 + s^k; \]
Setting weights $6; 3k + 6; 2k + 2; 3k + 3$, we obtain $d_p = 6k + 6, w_p = 8k$. Hence $2d_p = 4k$. The Jacobian ideal is generated by $y; t; x^2; (k+1)sk^2 + t^2$. Since $t$ and $x$ have even weight, it follows that each generator of $R(f)_{4k}$ is divisible by $t$. Since $s$ is in the Jacobian ideal, the only possibility is $s = x, t$. Since

\[ \text{From the discussion of singular sextics it follows that only the } S_k \text{ singularities have moduli. Since the } S_k \text{ singularities are not semi-weighted homogenous it turns out that all semi-weighted homogenous singularities are rigid and therefore weighted homogenous.} \]
$(2k \ 2) + (3k \ 6) = 4k$ 7 has no integral solution with $k > 3; > 0$; 0 we have $R(f)_{4k} = 0$.

Lemma 5.4. Suppose $C$ has an $E_k$ singularity at $q$, $k \geq 2$ $f(6;7;8g$ then $H^4_p(Y;C) = C \ (2)^2$ if $k = 6$ and $H^4_p(Y;C) = 0$ otherwise.

Proof. $E_6$ : We have a local equation for $Y$ of the form

$$y^2 = x^3 + t^3 + s^4;$$

Setting weights 3;4;4;6, we obtain $d_p = 12; w_p = 17; 2d_p \ w_p = 7$. The only monomials of weights 7 are $x;ts$ and their classes provide a basis for $R(f_p)$.

$E_7$ : We have a local equation for $Y$ of the form

$$y^2 = x^3 + t^3 + s^3;$$

Setting weights 4;6;6;9, we obtain $2d_p \ w_p = 11$. Since there are no monomials of degree 11, we obtain $R(f_p)_{11} = 0$.

$E_8$ : We have a local equation for $Y$ of the form

$$y^2 = x^3 + t^3 + s^3;$$

Setting weights 6;10;10;15, we obtain $2d_p \ w_p = 19$. Since there are no monomials of degree 19, we obtain $R(f_p)_{19} = 0$.

Remark 5.5. Suppose $(f;p)$ is a weighted homogeneous hypersurface singularity. Let $(ff_p = 0g;0j)$ be a local equation of $(Y;p)$, where $f_p$ is weighted homogeneous. Then $S_p = ff_p = 0g$ describes a surface in $F(w_0;w_1;w_2;w_3; w_4)$. Dimca’s method (as well as the method of Hulek-Kloosterman) relies on the isomorphism $H^4_p(Y;C) = H^2(S_p;C)_{prim}(L)$.

Often one can simplify this calculation. If exactly three of the four weights have a non-trivial common divisor one can apply the following procedure: Suppose $S_p = F(w_0;w_1;w_2;w_3) / F(w_0;w_1;w_2;w_3)$ and $q = w_0$. Then there is an isomorphism $' : F(w_0;w_1;w_2;w_3) / F(w_0;w_1;w_2;w_3)$ by sending $(x_0 : x_1 : x_2 : x_3)$ to $(x_0^3 : x_1 : x_2 : x_3)$.

Let $q_p$ be the equation of $'(S_p)$. Suppose that $q_p$ has an isolated singularity, i.e., $'(S_p)$ is quasi-smooth. Since $'$ is an isomorphism we have then $H^{1;1}(S_p)_{prim} = R(q_p)_{4d w_0; w_1; w_2; w_3}$. We often refer to this as the three weights trick.

The reason to apply the three weights trick is the following: in several cases it turns out that 1 lies in the Jacobian ideal of $q_p$. This in turn implies that $R(q_p) = 0$, and $h^{1;1}(S_p)_{prim} = 0$.

Lemma 5.6. Suppose $C$ has a $B_{k;1}$ singularity and $6 - k$ then $H^4_p(Y;C) = 0$.

Proof. We have a local equation for $(C; q)$ of the form

$$y^2 = x^3 + t^3 + s^4;$$

hence we can set the weights for $s;tx; y$ to be $6; 6; 2k; 3k$. If $2 - k$ we may apply the three weight trick (Remark 5.5). Therefore it is easy to study $y = x^3 + t^3 + s^1$. If $3 - k$ we may also use the three weight trick, in this case we obtain the singularity $y^2 = x + t^2 + s^1$. In both cases $1$ is in the Jacobian ideal, hence the Jacobian ring (and therefore the local cohomology) vanishes.

Recall that in Theorem 4.1 we give a list of possible values $(k;l)$ such that $B_{k;1}$ occurs as a singularity on a sextic.
Lemma 5.7. Suppose C has a B_{k;1} singularity and 6 j k l then
\[ H^i_p(\mathcal{Y};C) = C(2)^2 \] if \( k = 3 \) and \( l \) even, 2; \( 4 \) mod 6.
\[ H^i_p(\mathcal{Y};C) = C(2)^2 \] if \( k = 3 \) and \( l \) odd, 0 mod 6.
\[ H^i_p(\mathcal{Y};C) = C(2)^2 \] if \( (k;l) = (4;6) \).
\[ H^i_p(\mathcal{Y};C) = 0 \] if \( (k;l) = (5;6) \).

Proof. An easy computation shows that if \( (k;l) = (5;6) \) then \( R_{2d_p}w_p = 0 \) and there is no local cohomology.

If \( k = 3 \) then \( R_{2d_p}w_p \) is generated by
\[ \text{xts}^{(l;6)} = \text{xts}^{(l;2)^2} \].

If \( (k;l) = (4;6) \) then \( R_{2d_p}w_p \) is generated by \( \text{xts}t^2 \).

Lemma 5.8. Suppose C has an xB_{k;1}, an yB_{k;1} or an xyB_{k;1} singularity then \( H^i_p(\mathcal{Y};C) \) vanishes.

Proof. We used the computer algebra package Singular to check for every admissible value of \( (k;l) \) (see Theorem 5.3) that \( R_{2d_p}w_p = 0 \).

5.2. Method of Brieskorn. A second class of singularities are non-weighted homogeneous isolated hypersurface singularities.

Let \( f_p \) be a local equation for \( (\mathcal{Y};p) \). Then \( f_p = y^2 + x^3 + q_p(s;t) \). We explain now the method to calculate \( H^i_p(\mathcal{Y}) \). First observe that this group equals \( H^4(\mathcal{F}) \) the part of the cohomology of the Milnor algebra that is invariant under the monodromy.

Now \( H^4(\mathcal{F}) \) is naturally isomorphic to the Milnor algebra of \( (f_p;0) \). The Milnor algebra can be easily calculated. Brieskorn [2] developed a method to calculate the action of the monodromy on \( H^4(\mathcal{F}) \). We will not explain this method, but use the computer algebra package Singular, which contains an implementation of this method.

For computational reasons it is better to let Singular calculate the monodromy action on \( H^4(\mathcal{F}_1) \), where \( F_1 \) is the Milnor algebra of \( q_p(s;t) = 0 \). From this one can deduce the monodromy on \( H^4(\mathcal{F}) \) as follows:

For an arbitrary singularity \( f(k_1,\ldots;k_n) = 0 \) one can identify \( H^0(\mathcal{F}) \) with the Milnor algebra of \( M(f) = \langle k_1,\ldots,k_n \rangle \).

Now, the Milnor algebra of \( x_{n+1}^d + f \) is the direct sum \( \bigoplus_{i=1}^d x_{n+1}^M(f) \). One easily shows that for \( h \in M(f) \) we have \( T_{x_{n+1}^M(f)} = \exp(2jT_{h}) \). Hence the eigenvalues of the monodromy are \( 2j \), \( 2j \), and \( 2j \). One needs to multiply all the eigenvalues of \( T_h \) by all the \( d \)-th roots of unity except for the root of unity 1. In the case \( y^2 + x^3 + q_p(s;t) \) we apply this procedure twice.

The computer algebra package Singular produced the following results:

Proposition 5.9. Suppose \((C;q)\) is a \( C_{k;1} \) or \( D_{k;1} \) or \( F_{k,1} \) or \( S_k \) singularity on a sextic curve. Then
\[ I_q^2(\mathcal{Y};C) = 4 \text{ if } (C;q) = C_{3k;1} \text{ singularity.} \]
\[ I_q^2(\mathcal{Y};C) = 2 \text{ if } (C;q) = C_{k;1} \text{ singularity,} \]
where exactly one of \( k;1 \) is divisible by 3, or \((C;q)\) is either a \( S_1 \) or a \( S_k \) singularity.
5.3. Method of Hulek-Kloosterman. The main method we use works for non-isolated singularities. Let \((Y;p)\) be such a singularity. Since we have a minimal elliptic threefold, such a singularity is one-dimensional, and the transversal types are ADE surface singularities.

There are three classes to distinguish:

- \(j() = 1728\) and \(C\) has an isolated singularity at \(q\).
- \(j() = 1728\) and \(C\) has no isolated singularity at \(q\).
- \(j() = 0\) and \(C\) has a non-isolated singularity at \(q\).

If \(C\) is a quartic then \((C;q)\) is weighted homogeneous. If \(C\) is a sextic, then except for six types of singularities, \((C;q)\) is weighted homogeneous. For the rest of this subsection assume that \((C;q)\) is weighted homogeneous. Then \((Y;q)\) is weighted homogeneous. This implies that we may apply [5, Proposition 7.7], which is a generalization of Dimca's method. We start by giving a short outline of this method:

Let \(f_p\) be a local equation for \((Y;q)\), let \(w_p\) and \(d_p\) be as in D in ca's method. Let \(R(f_p)\) be the Jacobian ring of \(f_p\). Hulek and the author proved that \(H^4\) has pure Hodge structure of weight 4 with \(h^{0,4} = h^{4,0} = 0, h^{2,2} = h^{1,3} = \dim R(f_p)\). To determine \(h^{1,2}\), we need to introduce some notation. The equation \(f_p = 0\) defines a surface \(S_p\) in weighted projective 3-space \(P^3\). Now, Hulek and the author show that \(h^{1,2}(Y) = h^{1,1}(S_p)\).

The Hodge number \(h^{1,1}(S_p)\) can be determined as follows: Let \(q_1, \ldots, q_6\) be the points where all the partials of \(f_p\) vanish. Then \((S_p; q_i)\) is an ADE singularity. If \(q \in P^3\), then let \(M_q\) be the Milnor algebra of \((S_p; q)\).

If \(q \in P^3\), we do the following: we have a degree 6 quotient map \(P^3 \to P^3\). Let \(G\) be the Galois group of this cover. Let \(q \in P^3\). Let \(G_q\) be the stabilizer of \(q\). Let \(g\) be a local equation of \((S_p; q)\) in \(P^3\). Then \(G_q\) acts on the Milnor algebra of \(q\). Let \(M_q\) be the invariant part of \(M\) under \(G_q\). One can show that this definition is independent of the choices made. Let

\[
R(f_p)_{d_p} \cong \ker R(f_p)_{d_p} \cong M_q :.
\]

Then it follows from the work of Steenbrink [11] that

\[
h^{1,1}(S_p) = \dim R(f_p)_{d_p}.
\]

This succeeds in calculating all the Hodge numbers.

Remark 5.10. In addition to the three weights trick [Remark 5.5], there is another trick we can apply. Namely, let \((f_p)\) be the locus, where all the partials of \(f_p\) vanish. Assume that \((f_p) \setminus P^3_{\text{sing}} = \emptyset\). Then

\[
h^{1,1}(S_p) = \bigotimes_{q \in (f_p)} h^{1,1}(S_q).
\]

where \(q\) is the Milnor number of the singularity at \(q\). (This formula holds, since \(S_p\) has only ADE surface singularities. For a proof of this, see e.g. [5, Lemma 6.1].)

As written above we distinguish between three classes. First assume \(j() = 1728\).}

Proposition 5.11. Suppose \((C;q)\) is a singularity of a quartic curve, not a point of order four, such that \(h^1(Y) \neq 0\). Then \((C;q)\) is isolated and one of \(A_3; A_7\).
A sum exists that (C;q) is isolated. From the classification of singular quartics it follows that (C;q) is an ADE singularity.

In all cases it turns out that $d_p = w_p < 0$, hence $H^4_p(Y)$ is of pure $(2;2)$-type. Since $w_p$ and $d_p$ are listed in every proof, we do not mention that $d_p < w_p$. We prove now:

Lemma 5.12. Suppose $C$ has an $A_k$ singularity at $q$ then $H^4_p(Y;C) = C(2)^2$ if $k \equiv 3 \mod 4$ and $H^4_p(Y;C) = 0$ otherwise.

Proof. If $C$ has an $A_k$ singularity at $q$ then $Y$ is locally of the form $f_p = 0$ with

$$f_p = y^2 + x^3 + (t^{k+1} + s^2)x.$$ 

Set the weights of $s;t;x;y$ to be $2k + 2;4;2k + 2;3k + 3$. The sum $w_p$ of the weights equals $7k + 11$. The degree $d_p$ equals $6k + 6$.

To determine $h^{2/2}$ we start by determining $R_{2d_p}w_p = R_{5k+1}$. Since $y;x^2 + t^{k+1} + s^2;t^x$ and $sx$ generate the Jacobian ideal, it follows that

$$R_{5k+1} = \text{span}_{\mathbb{R}}\{t^k; (s^2)^m; (t^{4m} + s^2)^m; 1\}.$$ 

Hence $R_{5k+1} = 0$ if $k \equiv 3 \mod 4$. From this it follows that $h^4_p(Y) = 0$ if $k \equiv 3 \mod 4$.

Suppose that $k \equiv 3 \mod 4$, i.e., $k = 3m + 1$. Our defining equation is of the form

$$y^2 + x^3 + (t^{4m} + s^2)x;$$

We set the weights of $s;t;x;y$ to be $2m + 1;2m + 3;3m$. Now, $d_p = 6m; w_p = 7m + 1$. From this it follows that $R_{2d_p}w_p$ is generated by

$$t^{5m}; t^{3m}; s^m; t^m; s^{2m}; t^{3m}; 1.$$ 

Since $S$ has $A_1$ singularities at $(1:1:0:0)$ and $(1:1:0:0)$. The Milnor algebra is generated by $1$, i.e., $R_{2d_p}w_p$ is spanned by elements of $R_{2d_p}w_p$ that vanish at $(1:1:0:0)$ and $(1:1:0:0)$, hence it is spanned by

$$xt^{5m}; 1$$

and $h^4_p(Y) = 2$.

Lemma 5.13. Suppose $C$ has a $D_k$ singularity at $q$. Then $H^4_p(Y;C) = C(2)^2$ if $k \equiv 3 \mod 4$ and $H^4_p(Y;C) = 0$ otherwise.

Proof. In this case we have a local equation of the form

$$y^2 + x^3 + t(t^{k+2} + s^2)x;$$

The weights here are $2k + 4;4;2k + 2;3k + 3$. Hence $d_p = 6k + 2; w_p = 7k + 5; 2d_p = 5k + 7$. Consider

$$R_{2d_p}w_p = \mathbb{R}[y;x;st] = \mathbb{R}(y^2 + t^{k+1} + s^2; stx; (k + 1)t^{k+2} + s^2)x).$$

It is easy to see that $t^{5+4k}; t^{3+4k}; s^m; t^m; s^{2m}; t^{3+4k}; 5m$ span this vector space. Hence, a necessary condition to have local cohomology is $k \equiv 1 \mod 4$.

Consider the case $k \equiv 3 \mod 4$, i.e., $k = 4m + 3$, then we have a local equation of the form

$$y^2 + x^3 + t(t^{4m+1} + s^2)x.$$
We can normalize the weights such that they become 4m + 1; 2; 4m + 2; 6m + 3.
The degree is 12m + 6, the sum of the weights equals 14m + 8. The vector space \( R_{10m + 4} \) is spanned by
\[ t^{5m + 2}; s^2 t^{m + 1}; x t^{3m + 1}; \]
The partials of \( f_p \) vanish if \( t = u = v = w = 0 \) or if \( t^{4m + 1} + s^2 = x = y = 0 \). These equations yield points \( q_1; q_2 \) where \( S_p \) has an \( A_1 \) singularity. At such a point the Milnor algebra is generated by 1, hence the kernel \( R(f_p) \) of \( M_q \). \( M_q \) consists of functions vanishing at \( q_1 \) and \( q_2 \). So \( R \) is generated by
\[ (t^{4m + 1} + s^2) t^{m + 1}; x t^{3m + 1}; \]
Thus \( H^4_p(Y; C) = C \langle 2 \rangle^2 \).

The case \( k \equiv 1 \mod 4 \) is different. Set \( k = 4m + 1 \). Then we have a local equation of the form
\[ y^2 + u^3 + t(t^{4m + 1} + s^2) x; \]
The weights are 4m + 1; 2; 4m + 2; 6m. This surface is isomorphic to the surface \( S \) given by
\[ y^2 + u^3 + t(t^{4m + 1} + s) x \]
in \( P(4m + 1; 2; 2m; 3m) \). The surface \( S \) is of degree 6m and the sum of the weights is 9m. The only monomials of degree 3m are \( y; x t^m; t^{3m} \). Since \( y \) and \( x t \) are in the Jacobian ideal \( \mathfrak{J} \) turns out that \( R(f_p)_{2d_p} \) is generated by \( t^{3m} \).

The surface \( S \) has an \( A_1 \) singularity at \( q = (1: 1: 0: 0) \). At this point we have a trivial stabilizer. The Milnor algebra \( M_q \) is generated by 1 in local coordinates. Hence all elements of \( R(f_p)_{2d_p} \) have to vanish at \( q \). So \( t^{3m} \in R(f_p)_{2d_p} \) have to vanish at \( q \). Hence \( h^4_p(Y) = 0 \).

Lemma 5.14. Suppose \( C \) has an \( E_k \) singularity at \( q \) then \( H^4_p(Y; C) \) vanishes.

Proof. Case \( E_6 \):
\[ y^2 + u^3 + (s^3 + t^4) x; \]
the weights are 4; 3; 6; 9. This surface is isomorphic to
\[ y^2 + u^3 + (s + t^3) x \]
in \( P(4; 1; 2; 3) \). The degree is 6, the sum of the weights equals 10, whence \( 2d_p = 2 \). The only monomials of degree 2 are \( x \) and \( t^2 \). Since \( x \) is in the Jacobian ideal \( \mathfrak{J} \) follows that \( R_{2d_p} \) is generated by \( t^2 \). As \( S \) has an \( A_1 \) singularity at \( (1: 1: 0: 0) \), all elements of \( R_{2d_p} \) have to vanish at \( (1: 1: 0: 0) \). Since \( t^2 \) does not vanish, we obtain that \( h^4_p(Y) = 0 \).

Case \( E_7 \):
\[ y^2 + u^3 + (s^3 + st^3) x; \]
the weights are 12; 8; 18; 27. This surface is isomorphic to
\[ y + u^3 + (s^3 + st^3) x \]
in \( P(6; 4; 9; 27) \). Since \( 1 \) is in the Jacobian ideal we obtain \( R \) is the zero ring, hence \( H^4_p(Y; C) = 0 \).

We come now to the case where \( (C; q) \) is not an isolated singularity. From Section 4.3 it follows that we only have to consider the following two singularities:

Lemma 5.15. Suppose \( (C; q) \) has local equation \( s^2 t = 0 \) or \( s^2 (s - t^2) = 0 \). Then \( H^4_p(Y) = 0 \).
Proof. In the first case we have a local equation \( y^2 = x^3 + s^2 t s \) for \((Y;p)\). This defines a degree 6 surface \( S \) in \( P(1;2;2;3) \). Hence \( 2d_p \) \( w_p = 4 \). The monomials \( x t; s^5; t^2 s^2 t^2 \) span \( R((t_p)_p) \). The surface \( S \) has two singularities, namely at \( q_1 = (1 : 0 : 0 : 0) \) and \( q_2 = (0 : 1 : 0 : 0) \).

The Milnor algebra \( M_{q_1} \) is generated by 1 (which translates to \( s^3 \) in global coordinates). For \( q_2 \), note that the Milnor algebra is generated by \( 1; s^2 \). The group \( G_{q_2} \) is generated by \( s T \), hence \( M_{q_2} \) is spanned by \( t^2 x t; s^2 t^2 \) and \( R_{2d_p} w_p = 0 \).

Consider now \( y^2 = x^3 + s^2 (s^3 t^2) x \). This defines a surface in \( P(4;2;6;9) \) and is isomorphic to \( y = x^3 + s^2 (s^3 t^2) x \) in \( P(2;1;3;9) \). This surface has \( h^2_{prim} \neq 0 \), hence \( H^4_p (Y;C) = 0 \).

We turn now to the final case, namely \( j() = 0 \) and \( C \) is non-reduced.

Lemma 5.16. Suppose \((Y;p)\) is one of the following singularities

(1) \( y^2 = x^3 + t^2 s \)
(2) \( y^2 = x^3 + t^2 (t \cdot s^3) \)
(3) \( y^2 = x^3 + t^2 s \)
(4) \( y^2 = x^3 + t^2 s^3 \)
(5) \( y^2 = x^3 + t^2 s^2 \)
(6) \( y^2 = x^3 + t^2 s^2 \)

Then \( H^4_p (Y) = 0 \)

Proof. For each case we list a choice for the weights. We then either state that we may apply the three weights trick (Remark 5.5) or we give an outline on how to compute \( R_{2d_p} w_p \):

(1) \( 2;2;2;3 \): (three weights).
(2) \( 2;6;6;9 \): (three weights).
(3) \( 3;1;2;3 \): In this case we have \( 2d_p \) \( w_p = 3 \). A basis for \( R((t_p)_p) \) is \( s; x t \). At \((1:0:0:0)\) we have the following stabilizer: \( x T T^2 x T^3 x T^4 \). The Milnor algebra has basis \( 1; x t; x t^2 \). After taking invariants under the stabilizer we find that \( 1; x t \) span \( M_{G_{p1}} \). Hence \( R_{2d_p} w_p = 0 \).
(4) \( 3;6;8;12 \): (three weights).
(5) \( 2;1;2;3 \): In this case we have \( 2d \) \( w = 4 \). A basis for \( R((t_p)_p) \) is \( x s, x t^2, t^2 s, s^3 \). At \((1:0:0:0)\) we have \( t^2 \) as stabilizer. The Milnor algebra is spanned by \( 1; x t; x t^2; x t^3 \); hence the invariants under the stabilizer are (in global coordinates) \( s^2 x s t^2 s^2 x t^3 \). Hence \( R_{2d_p} w_p = 0 \).
(6) \( 2;1;2;3 \): A basis for \( R((t_p)_p) \) is \( x t^2; x s; t^2 s \). At \((1:0:0:0)\) the stabilizer is generated by \( t \), \( t \) the Milnor algebra is spanned by \( 1; x t \), hence is invariant under the stabilizer, so we can exclude \( s^2 x s \). At \((0:1:0:0)\) we have no stabilizer, the Milnor algebra is spanned by \( 1; x \) in local coordinates, hence \( t^2; x t^3 \) can be excluded. From this it follows that \( R_{2d_p} w_p (t_p) = 0 \).

Lemma 5.17. Suppose \( T \) is a reduced quartic with a double and a triple point. Then either

\( T \) has exactly two singularities, the triple point is a \( D_6 \) singularity and the double point is an \( A_1 \) singularity,

\( T \) has exactly two singularities, the triple point is a \( D_5 \) singularity and the double point is an \( A_1 \) singularity.

ELLIPSE THREEFOLDS WITH CONSTANT j-INVARIANT
T has a $D_4$ singularity an up to $3A_1$ singularities.

Proof. Suppose $q_1$ is a double point and $q_2$ a the triple point. Let $'$ be the line through $q_1$ and $q_2$. Since $(T'; q_1 \ 2$ and $(T'; q_2 \ 3$ it follows that $'$ is a component of $T$. Let $K$ be the residual cubic. Then $q_1$ is a smooth point of $K$ and $q_2$ a double point of $K$. Since $T$ is reduced, we have that $'$ is not a component of $K$. From this it follows that $(K'; q_i)^3 = i$ for $i = 1; 2$. In particular, at $q_2$ we have an $A_1$ singularity. Hence all double points of $T$ are $A_1$ singularities.

Note that if $K$ has an $A_k$ singularity at $q_1$ then $T$ has $D_{3+k}$ singularity. Since $K$ is a cubic we have that $k = 3$.

If $K$ has an $A_3$ singularity at $q_1$ then $K$ is a conic $Q$ together with a line tangent to $Q$ at $q_1$. Hence $T$ has a $D_6$ and an $A_1$ singularity and no other singularities.

If $K$ has an $A_2$ singularity at $q_1$ then $K$ is an irreducible cubic and smooth outside $q_1$. Hence $T$ has a $D_5$ and an $A_1$ singularities.

If $K$ has an $A_1$ singularity at $q_1$ then $K$ has at most 2 other $A_i$ singularities, hence $T$ has a $D_4$ singularity and at most three $A_1$ singularities.

Lemma 5.18. Suppose $C$ is a double line $'$ together with a reduced quartic $T$. Suppose that $Y$ has at least two singularities such that $H^4_p(Y) \neq 0$ then one of the following occurs

- $C$ has at least two cusps, none of them along $'$.
- $'$ is a bitangent of $T$, and $C$ might be smooth or has double points along $C \ ' $.
- $C$ has an $E_5$ singularity, but not along $C \ ' $, and there is a point $p \ C \ ' $ such that $p$ is a $2 f_2; 4g$.
- $C$ has an $A_2$ or $A_3$ singularity, $C$ is smooth along $C \ ' $ and there is a point $q \ C \ ' $ such that $q$ is a $2 f_2; 4g$.
- $C$ has an $A_2$ or $A_5$ singularity not along $C \ ' $ and $C$ has a double point along $C \ ' $.

Proof. Suppose first that $T$ is smooth outside $T \ ' $. Since we have at least two singularities that are not rational smooth, and the singularity $y^2 = x^3 + t^3$ is rational smooth (Lemma 5.18), it follows that $(T'; q_1)$ is smooth. Hence $'$ is a bitangent.

Suppose that there is a singularity $(T'; q_2)$ such that $H^4_p(Y) \neq 0$. Then $(T'; q_2)$ is a $A_2; A_5$ or $E_6$ singularity. Let $q_2 \ T \ ' $. If $T$ is smooth at $q_2$ it follows from Lemma 5.18 that $(T \ q_2) \ f_2; 4g$.

If $(T'; q')$ is an $E_6$ singularity then it follows from Lemma 5.18 that $T$ has no double points. Since a reduced quartic has at most one triple point, this implies that $T$ is smooth outside $q'$. Hence the second singularity such that $H^4_p(Y) \neq 0$, comes from a point in $q_2 \ T \ ' $. From Lemma 5.18 it follows that $(T \ q) \ f_2; 4g$.

If $(T'; q')$ is an $A_2$ or $A_5$ singularity then $(T'; q')$ might be smooth and the intersection number $(T' \ q) \ f_2; 4g$.

Suppose none of the intersections points of $T$ and $'$ yields a non-trivial $H^4_p(Y)$. Then $T$ has at least two singularities with types $A_2; A_5; E_6$. Since the combinations $2E_6, 2A_5, E_6 + A_2$ and $A_5 + A_2$ are not possible, it follows that $T$ has at least two $A_2$ singularities.

Lemma 5.19. Suppose $Q$ is a quartic with an $A_k$ singularity at $q$ and $'$ is a line through $p$, not contained in $Q$. Then $(k; q \ p)$ is one of the following
\( k;2), 1 \ k \ 7. \\
\( k;3), 1 \ k \ 2. \\
\( k;4), 1 \ k \ 7, k \neq 2. \\

Proof. Since \( Q \) is a quartic, we have \( k = 7. \) For a general line \( \ell \) we have \( (Q \ell) = 2. \) This yields the case \( k=2. \)

Suppose now \( k = 2 \) and \( \ell \) is given by \( t=0. \) The quartics locally given by \( st+ s^3 \) or \( st+ s^4 \) yield the cases \((1;3) \) and \((1;4). \)

Suppose now that we have \( k > 1 \) then we have a local equation of the form

\[
t^2 + a_{30}s^3 + a_{12}s^2t + a_{23}st^2 + a_{21}s^3 + a_{10}s^4 + a_{31}s^2t^2 + a_{03}st^3 + a_{04}t^4.
\]

Since \( \ell \) is a component of \( Q, \) we have that either \( a_{30} \) or \( a_{40} \) is nonzero.

If \( a_{30} \neq 0 \) then we have an \( A_2 \) singularity and \( \ell = 3. \)

If \( a_{30} = 0 \) then \( k \neq 3 \) and \( C \ell = 4. \)

Remark 5.20. A straightforward calculation shows that the contact equivalence class of \( \ell \) is a singularity on a quartic curve, is determined by the type of singularity of \( f \) and the intersection number of \( f=0 \) with \( \ell \).

Lemma 5.21. Suppose \( C \) is a triple line \( \ell \) together with a reduced cubic \( K. \) Suppose that \( Y \) has at least two singularities such that \( H^4_p(Y) \neq 0 \) then \( K \) is a cuspidal cubic, and \( \ell \) is an \( A_2 \) singularity.

Proof. Suppose \( K \) has two points \( q_1, q_2, \) not on \( \ell \) yielding non-zero \( H^4_p(Y) \). Then from Lemma 5.16 it follows that \( K \) is singular at \( q_1 \) or \( q_2 \) is an \( A_2 \) singularity of \( K \) yielding non-trivial local cohomology.

Since a cubic has only \( A_1, A_2 \) or \( D_4 \) singularities, and \( A_1, D_4 \) singularities yield rationally smooth points on \( Y, \) it follows that \( (K, \ell) \) is an \( A_2 \) singularity.

Since cuspidal cubics have exactly one singularity, it follows that \( (K, \ell) \) is smooth, hence \( \ell \) is an \( A_2 \) singularity of \( Y, \)

Lemma 5.22. Suppose \( C \) is a quadruple line \( \ell \) together with a reduced conic \( T. \) Then \( Y \) has at most one singular point \( p \) with \( H^4_p(Y) \neq 0. \)

Proof. Let \( q_1 \) and \( q_2 \) be points yielding non-trivial local cohomology. Since \( T \) is a conic \( \ell \) is either smooth or has an \( A_2 \) singularity. Since an isolated \( A_2 \) singularity yields a rational smooth point on \( Y, \) we have that \( q_1, q_2 \) are not tangent to \( T. \) From Lemma 5.16 it follows that \( H^4_p(Y) = 0 \) in this case.

Lemma 5.23. Suppose \( C \) consists of two double lines \( \ell_1, \ell_2 \) together with a reduced conic \( T. \) Suppose that \( Y \) has at least two singularities such that \( H^4_p(Y) \neq 0. \) Then \( \ell_1 \) and \( \ell_2 \) are tangent to \( T. \)

Proof. A point on \( T \) but not in \( T \) \( \ell_1, \ell_2 \) is either an isolated \( A_2 \) singularity of \( C \) or smooth, hence has no non-trivial local cohomology.

From Lemma 5.16 it follows that transversal intersection of \( \ell_1 \) with \( T \) has trivial local cohomology. Hence \( \ell_1 \) and \( T \) are tangent to \( C. \)
Lemma 5.24. Suppose $C$ consists of a smooth double conic $K$ together with a reduced conic $T$. Suppose that $Y$ has at least two singularities such that $H^2_p(Y) \neq 0$ then $C$ and $K$ have common tangents at two intersection points.

Proof. A point on $T$ but not in $T \setminus K$ is either an isolated $A_1$ singularity of $C$ or smooth, hence has trivial local cohomology.

Transversal intersections of $K$ with $T$ have trivial local cohomology. Hence we need at least two points such that $K \setminus T = 2$. Since $K$ and $T$ are conics, this implies that $K$ and $T$ have two intersection points with intersection multiplicity 2.

Lemma 5.25. Suppose $C$ consists of three double lines, not passing through one point or $C$ consists of the union of a triple line with a double and single line, not all three passing through one point. Then there is no point with non-trivial local cohomology.

Proof. Note that all intersections are transversal. Hence the result follows directly from Lemma 5.16.

We still need to determine $H^2_p(Y)$ for singularities of type $(A_2;m)$. Note that for $(A_2;4;k)$ we have local equations
\[(t + s)^2(t^2 + s^{k+1})\]
which are not weighted homogeneous. For singularities of type $(A_2;k)$, for $k = 2, 3, 4g$ we have local equations
\[t^2(ts + (t - s)^k)\]
which are not weighted homogeneous. In total, we have six types of singularities for which we do not have a method to calculate $H^2_p(Y)$.

It remains to consider the cases $(A_1;2)$, for $1 < k < 7$, $(A_1;3)$, $(A_1;4)$ and the case that $Q$ is smooth at the intersection points with $', \ast$ is a bitangent or a quadruple tangent to $Q$.

Lemma 5.26. Suppose we have a singularity of the form
\[y^2 = x^3 + t^2(t + s^{2k})\]
with $k = 2, 3, 4g$. Then $H^2_p(Y)$ is two-dimensional.

Proof. Setting weights $1; 2k; 2k; 3k$, yields $2d = 5k - 1$. Clearly, the degree $2d$, $w_p = 5k - 1$. The degree $2d$, $w_p$ part of $R(f)_{2d} w_p$ is spanned by $s^{5k-1}; t^{2k-1}; x^{3k-1}; t^2; t^{2k}; x^{2k-1}$. At $(0:0:0)$ we have an $A_1$ singularity. The images of $s^{5k-1}$ and $x^{3k-1}$ generate the local Minor algebra, hence $H^2_p(Y)$ is 2-dimensional.

Lemma 5.27. Suppose we have an $(A_1;2)$ singularity then $H^2_p(Y)$ is non-zero if and only if $k = 2, 3, 4g$. If $k = 2, 3, 4g$ then $H^2_p(Y) = C(2)$.

Proof. A local equation is of the form
\[y^2 = x^3 + s^2(t^2 + s^{k+1})\]
Setting weights $6; 3k + 3; 2k + 6; 3k + 9$ shows that we can apply the three weight trick if $3 - 2k + 6$. Hence if $k \not= 6$ mod 3 then $H^2_p(Y) = 0$.

If $k = 3$, we have that $R_{2d} w_p$ is spanned by the images of $x^2; t^2; s^4; t^2$. The local Minor algebra at $y = x = s = 0$ is generated by $1; x$, hence $R_{2d} w_p$ is spanned by $x^2; s^4$ and $H^2_p(Y) = 2$. 
If \( k = 6 \), we have that \( \mathbb{R}_{2d} \) is spanned by the images of \( xs^6; s^6 \). The local M infrared algebra at \( y = x = s = 0 \) is generated by \( 1; x \), hence \( \mathbb{R}_{2d} \) is spanned by \( xs^6; s^6 \) and \( h_1^4(\gamma) = 2 \).

**Lemma 5.28.** Suppose we have an \( \mathbb{A}_2;3 \) or an \( \mathbb{A}_3;4 \) singularity then \( H_1^4(\gamma) = 0 \).

**Proof.** In the first case we have local equation \( y^2 = x^3 + t^2(t^2 + s^3) \). Setting weights \( 3; 2; 4; 6 \), show that we can apply the three weights trick to reduce to the singularity \( y^2 = x^3 + s(s + t^3) \) with weights \( 3; 1; 2; 3 \). From **Lemma 5.16** it follows that this singularity has no local cohomology.

In the second case we have local equation \( y^2 = x^3 + t^2(t^2 + s^4) \). Setting weights \( 6; 3; 8; 12 \), show that we can apply the three weights trick. Hence there is no local cohomology.

Finally, in the case of a triple line and a cubic curve we have the following singularity.

**Lemma 5.29.** Suppose \( (\gamma; p) \) is a singularity of type

\[
y^2 = x^3 + t^3(t + s^3);
\]

Then \( H_1^4(\gamma) \) is two-dimensional.

**Proof.** If we set weights of \( s; t; x; y \) to be \( 1; 3; 4; 6 \), we obtain \( 2d \) \( w = 10 \). The vector space \( \mathbb{R}(p)^{10} \) is spanned by \( xs^6; xt^3; xt^2; t^2s^4; s^6 \). We have a singularity at \( (1; 0; 0; 0) \). The stabilizer at this point is trivial and the M infrared algebra is generated (in global coordinates) by \( s^6; xs^6; ts^3; xts^3 \), hence \( \mathbb{R} \) is generated by \( t^2s^4; xt^2 \).

**6. Determining the Mordell-Weil rank**

To determine the Mordell-Weil rank of an elliptic threefold we use the main result of [5]: Let \( Y \) \( P(2; 3; 1; 1) \) be a hypersurface given by

\[
y^2 = x^3 + P(x + Q);
\]

where \( P \) and \( Q \) are polynomials in \( z_2; z_1; z_0 \) of degree 4 and 6 respectively.

Let \( X \) be the projection from \( (1: 1: 0: 0: 0) \) onto the plane \( fx = y = 0 \). The map is not defined at \( (1: 1: 0: 0: 0) \). Let \( X_0 \) be the blow-up of \( Y \) at \( (1: 1: 0: 0: 0) \). This blow-up resolves the singularity of \( X \) and endows \( X_0 \) with the structure of a Weierstrass elliptic function in the sense of Miranda. Miranda gave a description of which birational transformations one needs to apply in order to obtain an elliptic threefold \( : X \rightarrow S \).

The torsion part of \( \text{MW}(\gamma) \) can be determined by specialization and we will come back to this later. In [5] we gave together with Klaus Hulek a procedure that for general \( Y \) calculates \( \text{rank MW}(\gamma) \). To determine the rank of \( \text{MW}(\gamma) \) one can use that if \( H^4(\gamma; C) \) has a pure weight 4 Hodge structure then

\[
\text{rank } \text{MW}(\gamma) = \text{rank } H^{2,2}(H^4(\gamma; C))/H^4(\gamma; Z) = 1.
\]

In general intersections of the type \( H^{2,2}(H^4(\gamma; C))/H^4(\gamma; Z) \) are hard to calculate. An exception is the case \( H^4(\gamma; C) = H^{2,2}(\gamma; C) \). This is actually always the case in all our examples.
Lemma 6.1. Suppose every non-isolated singularity of $Y$ is weighted homogeneous. Then $H^4(Y;\mathbb{C})$ is pure of type $(2;2)$.

Proof. Consider the exact sequence

$$0 \to H^4_p(Y) \to H^4(Y) \to H^4(Y) / H^4_p(Y) \to 0.$$  

We start by proving that the mixed Hodge structure on $H^4(Y)$ is pure of weight 4. Since $Y$ is smooth, it follows that $H^4(Y)$ has only Hodge weights 4, whereas $H^4(Y)$ has only Hodge weights 4, since $Y$ is proper (both statements can be found in [12, Section 5.3]). Hence, to prove the above claim, it suffices to prove that the Hodge structure on $H^4_p(Y)$ is of pure weight 4.

Suppose $p \geq 2$ and suppose we have a weighted homogeneous singularity at $p$. Then by the results of [23] and of [23], it follows that $H^4_p(Y)$ has pure weight 4. If $(Y;p)$ is not weighted homogeneous then this singularity is isolated. The procedures in the Singular library `cmsg.lib` allow us to calculate the weight filtration on $H^4_p(Y)$, it turns out that for all singularities mentioned in Theorem 4.3, the Hodge structure on $H^4_p(Y)$ is of pure weight 4. From this it follows that $H^4(Y)$ is of pure weight 4.

In order to prove that $H^4(Y)$ is of pure type $(2;2)$, consider $f: Y \to Y$, a resolution of singularities of $Y$. Let $Y' = f^{-1}(Y)$ be a general line. Then $Y' = f^{-1}(Y)$ is irreducible and is a rational elliptic surface. Moreover, since $Y'$ is an ample, we have by Lefschetz' hyperplane theorem that $H^2(Y') \to H^2(Y)$ is injective. From the rationality of $Y$, it follows that $h^{2,0}(Y) = 0$ and therefore $h^{2,0}(Y') = 0$. Using Poincare duality one obtains $h^{3,1}(Y) = h^{1,3}(Y) = 0$. In particular $H^{2,2}(Y') = H^4(Y)$.

We have an exact sequence $H^3(Y) \to H^4(Y) \to H^4(Y')$. Since $E$ is proper it turns out that there the graded piece of weight 4 in $H^3(Y)$ is trivial. Since $H^4(Y)$ is of pure weight 4 this exact sequence in plies that $H^4(Y)$ injects in $H^4(Y')$. The latter Hodge structure is of pure type $(2;2)$, so the same holds for $H^4(Y)$.

Proposition 6.2. Suppose $(Y;p)$ is a semi-weighted homogeneous hypersurface singularity. Then $H^4_p(Y;\mathbb{C}) / H^4_p(Y)$ is surjective.

Proof. Suppose $\mathfrak{m} = 0$. Then there exist positive integers $d$ (divisible by 6), $v_1$, $v_2$, $\ldots$, such that $v_1 + v_2 = v_1 + v_2 = d$, and the gcd of $d$ is 6, $v_1$ and $v_2$ equals 1 and $(y;p)$ is locally given by $y^2 + x^3 + s = 0$. The singularities are the s sixt degree since $C$ is a sextic and we may assume that $+ \to +$ at $m$ and 6.

If both $v_1$ and $v_2$ are divisible by 2 then three of the weights are divisible by 2 and we can apply the 3 weights trick and obtain that $H^4_p(Y) = 0$. The same conclusion holds if both $v_1$ and $v_2$ are divisible by 3.

For all other choices of $(v_1,v_2)$ we used the computer program Singular to calculate $2d$ and $d$. The map $H^4(U) \to H^4_p(Y)$ is surjective. The only triples $(d,v_1,v_2)$ not satisfying this criterion are $d = 12; v_1 = 1; v_2 = 3$ and $d = 12; v_1 = 1; v_2 = 4$. Since the singularity lies on a sextic it turns out that this corresponds to the singularities $y^2 = x^3 + t^3(1 + s^3)$ and $y^2 = x^3 + t^3(t + s^3)$.

For both singularities we know $H^4_p(Y) = 0$.

The case $j = 1728$ can be treated similarly, but turns out to be easier. This finishes the proof.
Summarizing, we have that \( \text{rank } M W(0) = h^4(Y) \leq 1 \), that \( h^4(Y) = 1 \) equals the dimension of the cokernel of

\[
H^4(P_n Y; C) ! \rightarrow p_2 \rightarrow H^4_p(Y)
\]

and that if \( Y \) consists of one point at which \( Y \) has a weighted homogeneous singularity then this cokernel is trivial.

To calculate in practice the cokernel we might use that this cokernel is of pure \((2;2)\)-type, hence it suffices to calculate

\[
\text{coker } G^2_p H^4(U; C) ! \rightarrow G^2_p H^4_p(Y) :
\]

In the sequel, we will only calculate the rank in the case that \((Y; p)\) is weighted homogeneous, hence for the rest of this section assume that \( Y \) has only weighted homogeneous singularities. In the previous section we showed for each weighted homogeneous singularity that \( H^4_p(Y) \) is pure of type \((2;2)\). Hence it suffices to calculate

\[
\text{coker } G^2_p H^4(U; C) ! \rightarrow H^4_p(Y) :
\]

There is a natural map \( C [x; y; z_0; z_1; z_2] ! \mapsto G^2_p H^4(U; C) \) given by

\[
g \frac{g}{f^2} :
\]

Here \( f \) is a defining equation for \( Y \) and \( g \) is the \( \text{standard} \) \( 4\)-form on \( P \) (cf. [5, Section 5]). The Jacobian ideal lies in the kernel of this map (see e.g., [2]). Since \( y \) is in the Jacobian ideal, we get a surjection \( C [x; y; z_0; z_1; z_2] ! \rightarrow H^4(U; C) \).

Then \( H^4(U; C) ! \rightarrow H^4(Y; C) \) can be calculated as follows. In the previous section we provided generators \( g_i; \ldots; g_k \) for \( H^4(Y; C) \). Now the map \( C [x; y; z_0; z_1; z_2] \times C [x; y; z_0; z_1; z_2] \rightarrow H^4(Y; C) \) is given by

\[
G ! \frac{\partial g_i}{\partial g_j} (p); \ldots; \frac{\partial g_i}{\partial g_k} (p) :
\]

We can simplify the calculation of the Modell-Weil rank further: the only interesting cases are \( j() = 0; 1728 \). In that case the biregular with section has an extra automorphism, namely

\[
! : (x; y; z_0; z_1; z_2) ! (x; y; z_0; z_1; z_2)
\]

with \( i^2 = 1 \) if \( j() = 0 \) or

\[
i : (x; y; z_0; z_1; z_2) ! (x; y; z_0; z_1; z_2)
\]

if \( j() = 1728 \).

Let \( i \) be either \( 0 \) or \( 1 \). The action of \( i \) gives \( M W(0) \) the structure of a \( \mathbb{Z} \)-module. In particular the \( \mathbb{Z} \)-rank of \( M W(0) \) is twice the \( \mathbb{Z} \)-rank of \( M W(0) \). If we take a basis \( P_1; \ldots; P_t \) for \( M W(0) \) tor as \( \mathbb{Z} \)-module, then \( P_1; P_1; \ldots; P_t; P_t \) is a basis for \( M W(0) = \mathbb{Z} \)-module.

Then \( i \) acts on \( P_1; P_1 \) as

\[
\begin{array}{lll}
0 & 1 & \text{resp.} & 0 & 1 \\
1 & 1 & \text{resp.} & 1 & 0
\end{array}
\]

This implies that on \( M W(0) \) the only eigenvalues of \( i \) are \( i \) resp. \( i^2 \), and the corresponding two eigenspaces have the same dimension.
The automorphism induces actions on $H^4(\mathcal{Y};\mathcal{C})_{\text{prim}}, H^4_p(\mathcal{Y};\mathcal{C})$ and the graded place $G_{\mathcal{C}}^2 H^4(J;\mathcal{C})$. Recall that we are interested in the calculation of the cokernel of

$$F^3 H^4(J;\mathcal{C})! \rightarrow p^2 H^4_p(\mathcal{Y});$$

The cokernel is a direct sum of the two eigenspaces and both eigenspaces have the same dimension. Hence it suffices to calculate the dimension of the $!^2$ (resp. $!$) eigenspace of the cokernel.

Since $(!) = !$ if $j() = 0$ (resp. $!$ if $j() = 1728$) and $F^3 H^4(J;\mathcal{C})$ is a quotient of

$$(xC \times z_0; z_1, z_2) \subset C \times z_0; z_1, z_2) \frac{1}{\mathcal{E}};$$

it follows that the $!^2$-eigenspace, respectively, the $!$ eigenspace is the cokernel of

$$(xC \times z_0; z_1, z_2) \subset C \times z_0; z_1, z_2) \frac{1}{\mathcal{E}}! \rightarrow H^4_p(\mathcal{Y};\mathcal{C})!^2!;$$

At the level of the local cohomology the same phenomena happens i.e. $F$ acts on monomials of the form $xh(t^6)$ as multiplication by $!^2$ resp. $!$, and on monomials of the form $h(t^6)$ it acts as $!,!^2$ respectively.

Remark 6.3. It should be remarked that on $H^4_p(\mathcal{Y})$ the two eigenspaces have the same dimension. However, on $F^3 H^4(J;\mathcal{C})$ the two eigenspaces have different dimensions, namely 6 and 15. For computational reasons we choose to work with the 6-dimensional space.

7. Classification I: $j() = 1728$

This case is rather easy.

Lemma 7.1. Suppose $j() = 1728$. Then $M \text{W}(!)_{\text{tor}} = Z = 22$ if only if $C$ is a double conic. If $C$ is a double conic then $M \text{W}(!)_{\text{tor}} = (Z = 22)^2$.

Proof. From Lemma 2.2 it follows that $Z = 22$ is a subgroup of $M \text{W}(!)$. Suppose that $M \text{W}(!)_{\text{tor}} > 2$, then for a general line $\tau$ the intersection $C \backslash \tau$ consists of two points with multiplicity 2 by Proposition 5.2. Hence $C$ is a double conic. Conversely, if $C$ is a double conic, then $Y$ is given by $y^2 = x^3 + f^2x$. This threefold has $x = f; y = 0$ and $x = 0; y = 0$ as sections of order 2. Hence $M \text{W}(!)_{\text{tor}} = (Z = 22)^2$.

Theorem 7.2. Suppose $j() = 1728$ and that $C_{\text{red}}$ is not the union of lines through one point. Then $M \text{W}(!)$ is in nine if and only if $C$ is a quartic with two $A_3$ singularities.

Moreover, we have

$M \text{W}(!) = Z = 42$ if and only if $C$ is a double conic,

$M \text{W}(!) = Z = 22$ if and only if $C$ is a quartic with two $A_3$ singularities.

$M \text{W}(!) = Z = 22$ otherwise.

Proof. Suppose $C$ is a quartic with two $A_3$ singularities. A smooth degree 4 curve has Euler characteristic 4. Since the Milnor number of an $A_3$ singularity is 3, we obtain that $C$ has Euler characteristic $4 + 6 = 2$, hence $h^1(C) = 0$, because $h^0(C) = h^2(C) = 1$. This implies that $C$ is a rational curve. Hence without loss of generality we may assume that $C$ is given by $z_0^4 z_1^2 z_2^2$. It remains to show that

$$y^2 = x^3 (z_0^4 z_1^2 z_2^2)$$
has \( M_{\text{ord} \mathcal{W}} \) equal rank 2. Since \( h^2_4(Y) = 2 \) and consists of two points, we have \( \text{rank} \mathcal{W}(2) = 4 \). From the surjectivity of \( H^4(U) \) \( \text{Proposition 5.2} \), it follows that the cokernel \( H^4(U) \) ! \( H^4(Y) \) has dimension at most 3, and since this dimension is even, it follows that rank \( \mathcal{W}(2) \geq 2 \) 8. Note that \( x = z_0^2; y = z_0z_1z_2 \) is a point of infinite order. Hence rank \( \mathcal{W}(2) = 2 \).

Conversely, we have that \( H^4(Y; C) \) is non-zero if and only if \( \text{rank}(C; g) \) is an isolated singularity of type \( A_3, A_7 \) or \( D_7 \). Since \( H^4(U; C) \)! \( H^4(Y; C) \) is surjective for each such singularity, we need to have at least two singularities for positive rank. This means that \( C \) is a quartic with 2 \( A_3 \) singularities.

To finish the proof, note that from Corollary 5.3, it implies that if \( \mathcal{W}(2) \) is non-zero then it is isomorphic to \( \mathbb{Z} \) or \( \mathbb{Z} \times \mathbb{Z} \). From the previous lemma it follows that the latter only occurs if \( C \) is a double conic.

8. Partial classification: case \( j() = 0 \) and \( C \) is non-reduced

Suppose \( C \) is a non-reduced sextic. Consider first the case that \( C \) is a reduced quartic with a double line. In this case we cannot calculate \( H^4_p(Y) \) for the six types of the singularities that occur in this case. For this reason we give a few examples with positive rank.

**Example 8.1.** Suppose \( C \) is the union of a double line \( ' \) and a quartic \( Q \). Then \( \mathcal{W}(2) \) has rank 2 if one of the following occurs

- \( C \) has an \( E_6 \) singularity, and \( ' \) intersects \( Q \) with multiplicity 4 in a smooth point or
- \( C \) has two \( A_3 \) singularities along \( ' \).

**Proof.** In the first case we may assume that, after a change of coordinates if necessary, \( Y \) is given by \( y^2 = x^3 + z_0^2(z_1^3 + z_0z_2^2) \). Since \( H^4(Y) \) is four-dimensional, \( H^4(U) \) ! \( H^4(Y) \) is not the zero map, and the cokernel has even dimension, we have that rank \( \mathcal{W}(2) = 2 \) \( \text{f0}; 2g \). Now \( x = z_0z_2 \) and \( y = z_0z_1^2 \) is a point of infinite order, showing that rank \( \mathcal{W}(2) = 2 \).

In the case we may assume that, after a change of coordinates if necessary, \( Y \) is given by \( y^2 = x^3 + z_0^2(z_1^3 + z_0z_2^2) \). Since \( C \) has two \( A_3 \) singularities it follows that \( H^4(Y) \) is four-dimensional. By the same reasoning as above we have that rank \( \mathcal{W}(2) \geq 2 \) \( \text{f0}; 2g \). The point \( x = z_0z_2; y = z_0z_1^2 \) has clearly infinite order, hence the rank equals 2.

From the results in Section 5 it follows that there are non-reduced sextics, not being a double line with a quartic, that might yield elliptic threefolds with positive rank. In all these cases it turns out that the rank equals 2.

**Theorem 8.2.** Suppose \( C \) is one of the following

- \( C \) is a triple line \( ' \) together with cuspidal cubic \( K \), and \( ' \) is a smooth point of \( K \),
- \( C \) is a conic together with two double lines \( ' \), such that the \( '_{\text{red}} \) are tangent to \( C \) or
- \( C \) is a conic together \( C_0 \) with a double conic \( C_2 \), and \( C_1 \) and \( C_2_{\text{red}} \) intersect in precisely two points with multiplicity 2.

Then \( \mathcal{W}(2) = \mathbb{Z}^2 \).
Proof. Using a specialization argument it follows that in all these cases the torsion part is trivial. In all cases consists of two points, and both points have $h^1_p(Y) = 2$. The map $H^1_p(Y) \to H^1_p(Y)$ is not the zero map by Proposition 6.2, hence the cokernel has dimension at most 3 and therefore $\text{rank} M_W(\cdot) \geq 3$. Since the rank is even, one has $\text{rank} M_W(\cdot) \geq 2 \cdot 0, 2g_2$. In order to prove the results it suffices to give a non-trivial section.

In the first case, without loss of generality we may assume that $Y$ is given by

$$y^2 = x^3 + z_0^3 (z_0^2 z_1 - z_2^2);$$

Then the section $x = z_0 z_2, y = z_0^2 z_1$ is non-torsion.

In the second case, without loss of generality we may assume that $Y$ is given by

$$y^2 = x^3 + (z_0^2 + z_1 z_2) (z_0^2 + z_1 z_2)^2;$$

Then the section $x = z_1 z_2, y = z_0 z_2 z_2$ is non-torsion.

In the third case, without loss of generality we may assume that without loss of generality $Y$ is given by

$$y^2 = x^3 + (z_0^2 + z_1 z_2) (z_0^2 + z_1 z_2)^2;$$

with $C$. The section $x = (z_0^2 + z_1 z_2), y = (P_{1} \cdots P_{m}) z_0 (z_0^2 + z_1 z_2)$ is non-torsion.

9. Case $j(\cdot) = 0$ and $C$ is a cuspidal curve.

Suppose $C$ is a sextic with only cusps. It is well-known that $C$ can have at most 9 cusps. Moreover, at most 3 of such cusps can lie on a line and at most 6 of them on a conic.

We need the following lemma:

Lemma 9.1. Let $p_1, \ldots, p_n, q, m$ be a set of distinct points in $\mathbb{P}^2$, with no four points collinear and no seven points lying on the same conic. Let $K$ be the cokernel of the evaluation map at $p_1, \ldots, p_m$:

$$\vdash: \{z_0, z_1, z_2\} \to \mathbb{C}^m:$$

Then $\dim K = m$ for $m \geq 7$, and $\dim K = 0$ for $m < 5$. For $m = 6$ we have $\dim K = 1$ if all the points lie on a conic, $\dim K = 0$ otherwise.

Proof. If $m \geq 7$ then the $m$ points do not lie on a conic, hence the kernel of $\vdash$ is trivial and the cokernel has dimension $6 - m$.

If $m = 6$ and the points do not lie on a conic then the kernel of $\vdash$ is again trivial and $\dim K = 0$.

If $m = 6$ and the points do lie on a conic then the kernel of $\vdash$ is one-dimensional and so is the cokernel.

If $m < 6$ then $K$ is non-trivial only if the elements in the kernel have a common component. Such a component is necessarily a line and $m \geq 3$. A straightforward calculation shows that if $m \geq 5$ and precisely three of the $m$ points are collinear then the kernel of $\vdash$ has dimension $6 - m$, so $\dim K = 0$.

Let $Y$ be an elliptic threefold of the form $y^2 = x^3 + f(z_0, z_1, z_2)$ where $f = 0$ is a reduced sextic with only cusps as singularities. For each cusp $p_i$ of $f = 0$ a direction $\gamma_i$ such that $C$ intersects $\gamma_i$ with multiplicity 3 at $p_i$.

In Lemma 5.2 we studied the singularity $y^2 = x^3 + t^3 + s^2$. It turns out that $H^1_p(Y)$ is generated by the class of $x$ and $t$. 
Theorem 9.2. The following \(f\) has the following Model-Weyl group:

\[ xC \mathbb{[}z_0; z_1; z_2 \mathbb{]} C [z_0; z_1; z_2] = \mathbb{C} \]

where \((xf_2 + f_4)\) is mapped to \((f_2(p_i)) \mathbb{C} f_4\). To simplify matters, we can decompose the cokernel into eigenspaces for the complex multiplication. One eigenspace is the cokernel of

\[ C [z_0; z_1; z_2] \mathbb{C} f_4 = \mathbb{C} \]

where the other is the cokernel of

\[ xC \mathbb{[}z_0; z_1; z_2 \mathbb{]} C \mathbb{C} f_2 \mathbb{[}f_2(p_i)\mathbb{]} \]

By the above lemma, this map has one-dimensional cokernel if \(m = 6\) and the cusps lie on a conic or \(m = 7\), a two-dimensional cokernel if \(m = 8\) and a three-dimensional cokernel if \(m = 9\). The latter case is well known, it means that the curve \(C\) is the dual of a smooth cubic.

Since both eigenspaces have equal dimension, we obtain the following result.

Theorem 9.2. Let \(f = 0\) be a reduced sextic, with only cusps as singularities. Suppose the cusps are at \(p_1; \ldots; p_m\). Then the elliptic threefold

\[ y^2 = x^3 + f \]

has the following Mordell-Weyl group:

- If \(m = 5\) or \(m = 6\) and the \(p_i\) do not lie on a conic then \(MW(\mathbb{C}) = 0\).
- If \(m = 6\) and the \(p_i\) lie on a conic then \(MW(\mathbb{C}) = \mathbb{Z}^2\).
- If \(m = 7\) then \(MW(\mathbb{C}) = \mathbb{Z}^2 \mathbb{C} Z^6\).

In particular, this shows the existence of the Mordell-Weyl groups \(Z^{2r}\) for \(r = 0; 1; 2; 3; 4\).

Remark 9.3. Suppose \(C\) is a sextic with 9 cusps. Then \(C\) is the dual curve of a smooth cubic. Hence there is a one-dimensional family of sextics with 9 cusps, and hence a one-dimensional family of elliptic threefolds with Mordell-Weyl rank 6. Since the Mordell-Weyl rank is six and \(H^4\) is pure of type \(2; 2\) it follows that \(H^4(Y; \mathbb{Q}) = \mathbb{Q} \mathbb{C} Z^2\). All other cohomology groups, except for \(H^3\), can be calculated using the Lefschetz hyperplane theorem, i.e., \(H^3(Y; \mathbb{Q}) = \mathbb{Q} \mathbb{C} Z^2\) for \(i \neq 0; 1; 3\) and \(H^3(Y; \mathbb{Q}) = 0\) for \(i \neq 0; 2; 3; 4; 6; 8\).

As explained in Section 3 it follows that \(H^3(Y; \mathbb{Q}) = \mathbb{Q} \mathbb{C} Z^6\). All cohomology groups have Hodge structures of Tate type, and there is no variation of Hodge structures possible. In particular, a Torelli type result as obtained by Grothendieck [4, Section 6] in a similar setting is not possible in our case.

10. Possible Mordell-Weyl groups

In the previous section we have seen the existence of the groups \(Z^{2r}\) for \(r = 0; 1; 2; 3; 4\). In order to prove Theorem 14 we have to show the existence of the groups \(Z = 32; (2Z = 2Z)^2\).

Remark 10.1. We have that \(Z = 32\) if and only if \(Y\) has an equation of the form

\[ y^2 = x^3 + f^2 \]

where \(f = 0\) is a cubic. We showed Lemma 3.3 that then \(MW(\mathbb{C}) = Z = 32\) unless \(f = 0\) is the union of three lines, and since we have excluded the cone construction...
case, $f = 0$ is the union of three lines $l_1, l_2, l_3$ without a common intersection point. That means that consists of three points $p_1, p_2, p_3$ and at each point we have a local equation $y^2 = x^3 + (ts)^2$. As explained in Lemma 5.13, we have that $H^4_{p_i}(Y) = 0$, whence $MW(\cdot) = Z = 3Z$ in this case, and $Z = 3Z$ $Z^2$ is not possible.

Remark 10.2. Suppose we have that $MW(\cdot) = Z^8$. We showed before that than $C$ is a reduced sextic, and is a union of six lines, not through one point. That means that for each $p_2$ we have a local equation of the form $y^2 = x^3 + t^m + s^6$ with $Z = 5$. For each such singularity we have $H^4_{p_i}(Y) = 0$, so if $C$ is the union of lines then $MW(\cdot)$ is finite. This shows that $Z^8$ is not possible.

Summarizing we get:

Theorem 10.3. Let $y^2 = x^3 + f$ be an elliptic threefold, $f = 0$ is not the union of lines through one point.

- $MW(\cdot) = (Z = 2Z)^2$ if and only if $f = 0$ is triple conic.
- $MW(\cdot) = (Z = 3Z)$ if and only if $f = 0$ is double cubic.
- Otherwise $MW(\cdot)$ is one of $0; 2Z; 2Z; Z^6$, and all these cases occur.

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