The mechanical response of a creased sheet

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We investigate the mechanics of thin sheets decorated by non-interacting creases. The system considered here consists in parallel folds connected by elastic panels. We show that the mechanical response of the creased structure is twofold, depending both on the bending deformation of the panels and the hinge-like intrinsic response of the crease. We show that a characteristic length scale, defined by the ratio of bending to hinge energies, governs whether the structure’s response consists in angle opening or panel bending when a small load is applied. The existence of this length scale is a building block for future works on origami mechanics.

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long dimension of the strip and perpendicular to the reference plane (Fig. 1(b)). The dynamometer is mounted on a translation stage, allowing to pull on the strip along its length. For each position of the stage, a picture of the illuminated strip is taken with a Nikon® digital camera, at a 60° angle with respect to the direction of the light source. The strip’s profiles \( y(x) \) are extracted from these images (Fig. 1(c)) and correlated to the applied force of the dynamometer.

In order to extract the full mechanical behavior of the folded thin sheet, we study a limiting case for which Euler’s elastica yields analytical solutions for the deformation of the panels: a strip creased in the middle of its long dimension, parallel to the short dimension, and pulled from both ends parallel to the strip’s flat state as shown in Fig. 1(a). The two-dimensional elastica problem to solve is the following:

\[
BW\theta_{ss} - F \sin \theta = 0
\]

where \( s \) is the curvilinear coordinate along the profile starting from the crease, \( \theta \) its local angle with respect to the horizontal axis, subscript \( s \) is the curvilinear derivative, \( B = E h^3/12(1-\nu^2) \) is the bending rigidity of the sheet (\( E \) is the Young’s modulus and \( \nu \) the Poisson ratio), \( h \) its thickness and \( W \) its width. The boundary conditions read

\[
\theta(\infty) = 0 \, , \, \theta_s(\infty) = 0 \, , \, \theta(0) = \frac{\pi - \phi}{2}
\]

where \( \phi \) is the current opening angle of the fold. Here the term \( \infty \) signifies that the length of the elastica is large compared to any other length scale in the problem. Solving this equation leads to

\[
\tan \frac{\theta}{4} = \tan \frac{\theta(0)}{4} \exp - \frac{s}{c}
\]

where \( c \) is the elastic length scale of the problem:

\[
c = \sqrt{\frac{BW}{F}}
\]

Now, we use this exact result to extract the opening angle of the fold \( \theta(0) \) and the characteristic length scale \( c \) in a realistic experimental situation, in order to obtain the response of the crease to the loading moment. For this, one needs to transform the profiles \( y(x) \) of Fig. 1(c) into a parametrization \( \theta(s) \).

![Fig. 1. a) Schematic of the experimental setup. b) Picture of the laser-illuminated sheet showing the resulting deformation of the fold. c) Profiles \( y(x) \) extracted from the pictures for various elongations applied to a sheet of thickness \( h = 350\mu m \).](image1)

![Fig. 2. Lin-log plot of \( \tan \theta/4 \) as function of the curvilinear coordinate \( s \) for a representative set of elongations corresponding to the profiles shown in Fig. 1(c), and their respective exponential fits (red lines). The darker the dots, the smaller is the force. The oscillations at the tails of each curve are due to the unavoidable errors in the computation of the slopes \( y'(x) \).](image2)
A valuable byproduct of the previous fit is the current opening angle of the crease \( \phi = 2(\frac{p}{2} - \theta(0)) \) as a function of the applied force. If the crease can be considered as a torsional hinge, one expects a well defined constitutive behavior relating the applied moment to the angle difference of the loaded and rest states of the crease. Fig. 3(c) shows the opening moment \( M \) function of the opening angle \( \phi \) for right portions (blue disks) and left portions (red diamonds) of the strip of the same sheet as in (a). Solid line is a global linear fit. d) The rigidity \( \kappa \) of the crease for different thicknesses. The error bars are 95% confidence intervals.

The experimental results show that the crease stiffness is characterized by a torsional rigidity parameter \( \kappa \) whose dimension is \( N.m \), while the dimension of bending rigidity of the panels \( B \) is \( N.m \). Therefore, the ratio of these two parameters defines a length scale \( L^* \equiv B/\kappa \), which in our case is a few centimeters (see Fig. 4). In order to understand how these two modes of deformations compete, one can compute the variation of energy associated with the deformation of an origami structure around an equilibrium configuration. The simplest origami is the “accordion-like” folded strips, because it is controlled by two degrees of freedom, the rest angle \( \phi_0 \) of the crease protocol leads to reproducible results and, in turn, governs the mechanical properties of the hinge. This feature can be understood qualitatively by noticing that the creation of the crease involves localized plastic deformations, through localized storage of bending elastic energy \( \mathcal{E}_b \). Assuming an ideally plastic behavior of the material, and considering the thickness \( h \) as the crease characteristic radius of curvature, the plastic strain scales as \( \epsilon_p \sim h\phi_p/h = \phi_p \), where \( \phi_p \) is the characteristic angular region in which the irreversible deformations are localized. Using the constitutive relation that relates the plastic strain to the yield stress \( \sigma_Y \), one deduces that \( \phi_p \sim \epsilon_p \sim \sigma_Y/E \) independently of the sample thickness. If one assumes that the opening angle \( \phi_0 \) is a univocal function of \( \phi_p \), this simple estimate shows that \( \phi_0 \) depends on the mechanical properties of the material but not on its thickness. In order to test this result, we prepared additional samples with various thicknesses by using the same protocol, and found that the rest angles always lie between 30° and 40° with no significant dependence on \( h \) (see Fig. 4). Notice that these rest angles are smaller than those obtained from the mechanical tests carried out above. This might be explained by noticing that the crease is subjected to long relaxation processes \( \mathcal{E}_b \) and the mechanical testing might accelerate the convergence to the final rest angle. This is supported by the quick convergence of our loading cycles to a well defined reversible response.
and the length $l$ of panel (see Fig. 6). Under a small applied tension, each element of the structure elongates by a total amount $\delta y$. The response of the hinge results in a contribution $\delta y_c$ to the total elongation given by $\delta y_c \approx (l/2) \cos (\phi_0/2) \delta \phi$, where $\phi = \phi_0 + \delta \phi$ is the opening angle of the crease. As a consequence of the constitutive behavior of the crease, the associated energy cost reads $\delta E_c = \kappa W (\delta \phi)^2 = 4\kappa W (\delta y_c)^2 / (l^2 \cos^2 (\phi_0/2))$.

On the other hand, the bending deformation of the panel contributes to the total elongation $\delta y_c$. Using geometric arguments one finds that $\delta y_c = \delta y_b + \delta y_c$, and to first order in $\delta \phi$, the minimization of $E_{tot} = E_b + E_c$ yields

$$\frac{\delta y_c}{\delta y} = \frac{1}{1 + \frac{\phi_0^2}{3l^2}} = \frac{1}{1 + \frac{l}{3L}}.$$  

Eq. (5) states that for a given strip, when the linear panel size is small compared to the characteristic length scale $L^*$, the longitudinal extension of the structure is mostly due to the opening of the fold. In this limit, the panels can be considered as rigid. Thus, the unfolding mechanism corresponds to what would be expected for usual origami whose deformation along the creases can be deduced from kinematics [10]. In the opposite limit, when the linear size of the elementary panels is large compared to $L^*$, the extension of the structure is governed by the bending of these panels while the creases do not open up a lot. Fig. 6 shows the two opposite responses of a simple accordion like origami to an applied loading when the size of the panels is modified. Using simple arguments [8], it is shown that the energy stored in the crease, and consequently the mechanical parameter $\kappa$, scales as $B/h$, which implies that the characteristic length scale $L^*$ is a linear function of the thickness. Fig. 5 shows that the data are compatible with such a scaling, $L^* \approx 200h$. This large scale separation between $h$ and $L^*$ can be rationalized by noting that the bending modulus $B$ involves the Young modulus $E$, while the crease rigidity $\kappa$ involves plastic deformations, and therefore the yield stress $\sigma_Y$. The large dimensionless ratio of these two material parameters seems to govern this scale separation and thus the origami length scale $L^*$.

In summary, a mechanical characterization of creased sheets is presented, relying on a protocol that yields well defined reversible mechanical behavior and rest fold angle. Our technique provides an accurate measurement of the crease rigidity and the sheet’s bending modulus $B$. We have shown that the rest angle and the origami length scale $L^*$ are governed by the ratio $\sigma_Y/E$ of the material, which appears as the relevant magnifying factor of the thickness in the origami response. This characteristic length scale $L^*$, which we expect to be a generic feature of more elaborate origami-like structures, represents a very simple design tool to predict the overall behavior of folded systems, and in particular whether they fold or bend. Since in general, it is of primary interest that the structure actually unfolds/actuates instead of bending/failing, this characterization also brings insights into the typical constraints on hinge mechanisms that should be associated with a given set of flexible panels to reach the desired function.

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