Solvability of the boundary-value problem for a mixed equation involving hyper-Bessel fractional differential operator and bi-ordinal Hilfer fractional derivative

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In a rectangular domain, a boundary-value problem is considered for a mixed equation with a regularized Caputo-like counterpart of hyper-Bessel differential operator and the bi-ordinal Hilfer's fractional derivative. By using the method of separation of variables a unique solvability of the considered problem has been established. Moreover, we have found the explicit solution of initial-boundary problems for the heat equation with the regularized Caputo-like counterpart of the hyper-Bessel differential operator with the non-zero starting point.

KEYWORDS
bi-ordinal Hilfer's derivative, boundary-value problems, fractional wave equation, hyper-Bessel fractional differential operator, sub-diffusion equation

MSC CLASSIFICATION
35M12, 35R11

1 INTRODUCTION

The study of fractional order differential equations has been attracting many scientists because of its adequate and interesting applications in modeling of real-life problems related to several fields of science. Initial-value problems (IVPs) and boundary-value problems (BVPs) involving the Riemann–Liouville and Caputo derivatives attract most interest (see, for instance, previous works). Especially, studying IVPs and BVPs for the sub-diffusion, fractional wave equations are well-studied (see previous works). BVPs for mixed-type equations are also an interesting target for many authors (see previous works).

Introducing a generalized Riemann–Liouville fractional derivatives (later called Hilfer’s derivative) has opened a new gate in the research of fractional calculus. Therefore, one can find several works devoted to studying this operator in various problems. We also note that in 1968, M. M. Dzhrbashyan and A. B. Nersesyan introduced the following integral-differential operator.
\[ D^\sigma_{0+}g(x) = I^1_{0+} D^n_{0+} \cdots D^1_{0+} D^\sigma_{0+} g(x), \quad n \in \mathbb{N}, \quad x > 0, \] (1)

which is more general than Hilfer’s operator. Here \( I^\alpha_{0+} \) and \( D^\alpha_{0+} \) are the Riemann–Liouville fractional integral and the Riemann–Liouville fractional derivative of order \( \alpha \) respectively (see Definition 2.1), and \( \sigma_n \in (0, n] \) is defined by

\[ \sigma_n = \sum_{j=0}^{n} \gamma_j - 1 > 0, \quad \gamma_j \in (0, 1]. \]

There are some works,\(^22-25\) related with this operator. New wave of investigations involving this operator might appear due to the translation of original work\(^22\) in FCAA\(^25\) (see also Ahmad et al.\(^26\)).

In addition, from announcing the concept of hyper-Bessel fractional differential derivative by I. Dimovski,\(^27\) several articles have been published dedicated to studying problems containing this type of operators (see previous works\(^28-31\)). For instance, the combination of fractional diffusion equation and wave equation were investigated in different domains in previous works\(^32,33\).

We also refer that for solving partial differential equations (PDEs) powerful techniques have been using by several mathematicians so far. For example, in previous works\(^34,35\) authors presented a new development of methodology based on Adomian decomposition method for solving PDEs and system of PDEs. Also, while in Kumar and Zeidan,\(^36\) numerical solution of a non-linear fractional diffusion equation with advection and reaction terms is constructed, Lie group method is applied to investigate the symmetry group of transformations under which the governing time-fractional PDE remains invariant\(^37\) (see also previous works\(^38,39\)).

In this work, we investigate a boundary value problem for a mixed equation involving the sub-diffusion equation with Caputo-like counterpart of a hyper-Bessel fractional differential operator and the fractional wave equation with Hilfer’s bi-ordinal derivative in a rectangular domain. The theorem on the uniqueness and the existence of the solution has been proved. The main method is based on separation variables which is applicable and convenient to write solution explicitly rather than other methods.

The rest of the paper is organized as follows: In the preliminaries section we provide necessary information on Mittag-Leffler functions (Section 2.1), hyper-Bessel functions (Section 2.2.), bi-ordinal Hilfer’s fractional derivatives (Section 2.3), and on differential equation involving bi-ordinal Hilfer’s fractional derivatives (Section 2.4). Auxiliary result is formulated in Theorem 1. In Section 3, we formulate the main problem and state our main result in Theorem 2. In Appendix one can find detailed arguments of the proof of Theorem 1.

2 | PRELIMINARIES

In this section we present some definitions and auxiliary results related to the generalized Hilfer derivative and fractional hyper-Bessel differential operator which will be used in the sequel. We start recalling the definition of the Mittag-Leffler function.

2.1 | Important properties of the Mittag-Leffler function

The two parameter Mittag-Leffler (M-L) function is an entire function given by

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R}. \] (2)

**Lemma 1** (see Podlubny\(^3\)). Let \( \alpha < 2, \beta \in \mathbb{R} \) and \( \frac{\alpha}{2} \leq \mu < \min\{\pi, \pi \alpha\} \). Then the following estimate holds for some \( M > 0 \):

\[ |E_{\alpha,\beta}(z)| \leq \frac{M}{1 + |z|}, \quad \mu \leq |\arg z| \leq \pi, \quad |z| \geq 0. \]

The Laplace transform of the Mittag-Leffler function is given in the following lemma.
Lemma 2. \((7)\). For any \(\alpha > 0\), \(\beta > 0\) and \(\lambda \in \mathbb{C}\), we have

\[
\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}, \quad (\text{Re}(s) > |\lambda|^{1/\alpha}),
\]

where the Laplace transform of a function \(f(t)\) is defined by

\[
\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st}f(t)dt.
\]

Lemma 3. If \(\alpha > 0\) and \(\beta \in \mathbb{C}\), then the following recurrence formula holds:

\[
E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z).
\]

Later, we use the properties of a Wright-type function studied by A. Pskhu, defined as

\[
e_{\mu,\delta}^{\alpha}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \mu)\Gamma(\delta - \beta n)}, \quad \alpha > 0, \alpha > \beta.
\]

M-L function can be determined by Wright-type function as a special case \(E_{\alpha,\beta}(z) = e_{\beta,1}^{0}(z)\). So, we can record some properties of M-L function which can be reduced from the Wright-type function’s properties.

Lemma 4 \((40)\). If \(\pi \geq |\arg z| > \frac{\pi}{2} + \varepsilon, \varepsilon > 0\), then the following relations are valid for \(z \to \infty\):

\[
\lim_{|z| \to \infty} E_{\alpha,\beta}(z) = 0,
\]

\[
\lim_{|z| \to \infty} zE_{\alpha,\beta}(z) = -\frac{1}{\Gamma(\beta - \alpha)}.
\]

2.2 | Regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator

Definition 1. The Riemann–Liouville fractional integral \(I_{\alpha+}^a f(t)\) and derivative \(D_{\alpha+}^n f(t)\) of order \(\alpha\) are defined by

\[
I_{\alpha+}^a f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)d\tau / (t-\tau)^{1-\alpha},
\]

\[
D_{\alpha+}^n f(t) = \left(\frac{d}{dt}\right)^n I_{\alpha+}^{n-\alpha} f(t), \quad n - 1 < \alpha < n,
\]

where \(\Gamma(\alpha)\) is Euler’s gamma-function.

Definition 2. The Erdelyi–Kober (E-K) fractional integral of a function \(f(t) \in C_\mu\) with arbitrary parameters \(\delta > 0, \gamma \in \mathbb{R}\) and \(\beta > 0\) is defined as

\[
I_{\beta,\alpha+}^{\gamma,\delta} f(t) = \frac{t^{\alpha(n+\delta)}}{\Gamma(\delta)} \int_a^t (t^{\beta} - \tau^{\beta})^{\delta-1} t^{\beta\gamma} f(\tau) d\tau.
\]
which can be reduced up to a weight to $\int_\tau^a f(t)$ (Riemann–Liouville fractional integral) at $\gamma = 0$ and $\beta = 1$, and Erdelyi–Kober fractional derivative of $f(t) \in C^{(n)}_\mu$ for $n - 1 < \delta \leq n, n \in \mathbb{N}$, is defined by

$$D^{\gamma, \delta}_{\beta, \alpha+} f(t) = \prod_{j=1}^n \left( \gamma + j \frac{d}{dt} \right) \left( H^{\gamma+\delta, \alpha-\delta}_{\beta, \alpha+} f(t) \right),$$

where $C^{(n)}_\mu$ is the weighted space of continuous functions defined as

$$C^{(n)}_\mu = \{ f(t) = t^p f(t); p > \mu, f \in C^{(n)}[0, \infty) \} , \ C^{(0)}_\mu = C^{(0)}_\mu \text{ with } \mu \in \mathbb{R}.$$

**Definition 3.** Regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator for $\theta < 1$, $0 < \alpha \leq 1$ and $t > a \geq 0$ is defined in terms of the E-K fractional order operator

$$c \left( \int_0^a \frac{d}{dt} \right)^\alpha f(t) = (1 - \theta)^\alpha t^{\alpha(1-\theta)} D^{-\alpha, \alpha}_{1-\theta, \alpha+} (f(t) - f(a))$$

or in terms of the hyper-Bessel differential (R-L type) operator

$$c \left( \int_0^a \frac{d}{dt} \right)^\alpha f(t) = \left( \int_0^a \frac{d}{dt} \right)^\alpha f(t) - \frac{f(a)(t^{(1-\theta)} - a^{(1-\theta)})^-\alpha}{(1-\theta)^\alpha \Gamma(1-\alpha)},$$

where

$$\left( \int_0^a \frac{d}{dt} \right)^\alpha f(t) = \begin{cases} (1 - \theta)^\alpha t^{\alpha(1-\theta)} D^{0, a}_{1-\theta, \alpha+} f(t) & \text{if } \theta < 1, \\ (\theta - 1)^\alpha t^{\alpha(1-\theta)} D^{-1, \alpha}_{1-\theta, \alpha+} f(t) & \text{if } \theta > 1, \end{cases}$$

is a hyper-Bessel fractional differential operator.$^{28}$

From (4) for $a = 0$ we obtain the definition presented in Garra et al.$^{28}$ and also Caputo FDO is the particular case of Caputo-like counterpart hyper-Bessel operator at $\theta = 0$.

**Theorem 1.** Assume that the following conditions hold:

- $\tau \in C[0, 1]$ such that $\tau(0) = \tau(1) = 0$ and $\tau' \in L^2(0, 1)$,
- $f(\cdot, t) \in C^3[0, 1]$ and $f(x, \cdot) \in C_\mu[a, T]$ such that $f(0, t) = f(\pi, t) = f_{\text{ext}}(0, t) = f_{\text{ext}}(1, t) = 0$, and $\frac{\partial^3}{\partial x^3} f(\cdot, t) \in L^1(0, 1)$.

Then, in $\Omega = \{ 0 < x < 1, a < t < T \}$, the problem of finding the solution of the equation

$$c \left( \int_0^a \frac{d}{dt} \right)^\alpha u(x, t) - u_{xx}(x, t) = f(x, t),$$

satisfying the conditions

$$u(0, t) = 0, u(1, t) = 0, a \leq t \leq T,$$

$$u(x, a+) = \tau(x), \ 0 \leq x \leq 1,$$

has a unique solution given by

$$u(x, t) = \sum_{k=1}^\infty \left[ r_k E_{\alpha, 1} \left( \frac{(k\pi)^2}{p^a}(t^p - a^p)^a \right) + G_k(t) \right] \sin(k\pi x),$$

(5)
where $p = 1 - \theta$ and

$$
G_k(t) = \frac{1}{p^\mu \Gamma(\alpha)} \int_a^t (t^p - \tau^p)^{\alpha-1} f_k(\tau) d(\tau^p) - \frac{(kr)^2}{p^{2\alpha}} \int_a^t (t^p - \tau^p)^{2\alpha-1} E_{\alpha,2\alpha} \left[ -\frac{(kr)^2}{p^\alpha} (\tau^p)^\alpha \right] f_k(\tau) d(\tau^p),
$$

$$
\tau_k = 2 \int_0^1 \tau(x) \sin(k\pi x) dx. \quad f_k(t) = 2 \int_0^t f(x, t) \sin(k\pi x) dx, \quad k = 1, 2, 3, \ldots
$$

In fact, for $\alpha = 0$, Theorem 1 implies the result of Al-Musalhi et al. (2020) (see Theorem 3.1). For the detailed proof of Theorem 1 see the appendix.

### 2.3 The bi-ordinal Hilfer fractional differential operator

**Definition 4.** Hilfer's derivative $D_{1+}^{\alpha,\mu}$ of order $\alpha$ ($n - 1 < \alpha \leq n$, $n \in \mathbb{N}$) of type $\mu$ ($0 \leq \mu \leq 1$) is defined by

$$
D_{1+}^{\alpha,\mu} f(t) = I_{0+}^{\mu(n-a)} \left( \frac{d}{dt} \right)^n I_{0+}^{(1-\mu)(n-a)} f(t). \quad (6)
$$

Then in Bulavatsky, V. M. Bulavatsky considered generalized Hilfer's derivative in the form

$$
D_{1+}^{(\alpha,\beta)\mu} f(t) = I_{0+}^{\mu(1-\alpha)} \frac{d}{dt} I_{0+}^{(1-\mu)(1-\beta)} f(t),
$$

where $0 < \alpha, \beta \leq 1$, $0 \leq \mu \leq 1$.

In the same way, one can present bi-ordinal Hilfer fractional derivative of orders $\alpha$ ($n - 1 < \alpha \leq n$), $\beta$ ($n - 1 < \beta \leq n$) and of type $\mu \in [0, 1]$ by the following relation

$$
D_{1+}^{(\alpha,\beta)\mu} f(t) = I_{0+}^{\mu(n-a)} \left( \frac{d}{dt} \right)^n I_{0+}^{(1-\mu)(n-\beta)} f(t). \quad (7)
$$

In general, (7) is also preserved as (6) in terms of its interpolation concept. Specifically, when $\mu = 0$, (7) gives the Riemann–Liouville fractional derivative of $\beta$ order and for $\mu = 1$, the bi-ordinal Hilfer derivative expresses the Caputo fractional derivative of order $\alpha$.

Similarly, the bi-ordinal Hilfer’s fractional derivative of orders $\gamma \in (1, 2]$, and $\beta \in (1, 2]$ and type $\mu \in [0, 1]$ can be written as a special case of (7) for $n = 2$:

$$
D_{1+}^{(\gamma,\beta)\mu} f(t) = I_{0+}^{\mu(2-\gamma)} \left( \frac{d}{dt} \right)^2 I_{0+}^{(1-\mu)(2-\beta)} f(t). \quad (8)
$$

Here we present the formula for the Laplace transform of (8) which will be used later:

$$
\mathcal{L}\{D_{1+}^{(\alpha,\beta)\mu} f(t)\} = s^{\beta+\mu(\alpha-\beta)} \mathcal{L}\{f(t)\} - s^{1-\mu(2-a)} \left[ I_{0+}^{(1-\mu)(2-\beta)} f(t) \right]_{t=0+} - s^{-\mu(2-a)} \left[ \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} f(t) \right]_{t=0+}. \quad (9)
$$

**Remark 1**. The bi-ordinal Hilfer’s derivative $D_{0+}^{(\alpha,\beta)\mu} g(t)$ can be written as

$$
D_{0+}^{(\alpha,\beta)\mu} g(t) = I_{0+}^{\mu(n-a)} \left( \frac{d}{dt} \right)^n I_{0+}^{(1-\mu)(n-\beta)} g(t) = I_{0+}^{\mu(n-a)} \left( \frac{d}{dt} \right)^n I_{0+}^{\gamma(1-\mu)\mu} g(t) = I_{0+}^{\mu(n-a)} D_{0+}^\gamma g(t) = I_{0+}^{\gamma} D_{0+}^\gamma g(t), \quad (10)
$$
for \( t \in [0, T] \), where \( \gamma = \beta + \mu(n - \beta) \) and \( \delta = \beta + \mu(a - \beta) \).

Considering the Remark 1, it is possible to show that Dzhurbashyan-Nersesyan fractional differential operator (1) can be reduced up to the bi-ordinal Hilfer’s fractional differential operator for \( n = 1 \), that is,

\[
D_0^n g(t) = I_{0+}^{1-\gamma} D_0^\gamma g(t).
\]

### 2.4 Differential equation involving bi-ordinal Hilfer derivative

Let us consider the following problem:

Find a solution of the equation

\[
D_t^{\gamma, \beta, \mu} u(t) + \lambda u(t) = f(t), \quad (1 < \gamma, \beta \leq 2, \ 0 \leq \mu \leq 1).
\]  

(11)

satisfying the initial conditions

\[
\lim_{t \to t_0^+} I_{t_0}^{1-\mu} u(t) = \xi_0,
\]  

(12)

\[
\lim_{t \to t_0^+} \frac{d}{dt} I_{t_0}^{1-\mu} u(t) = \xi_1,
\]  

(13)

where \( f(t) \) is a given function and \( \lambda, \xi_0, \xi_1 = \text{const} \).

If \( f \in C^1_{-1}(0, +\infty) \), then the problem (11)-(13) has an unique solution represented by

\[
u(t) = \xi_0 t^{\beta-2(1-\mu)} E_{\beta, \mu(2-\gamma)-1}(-\lambda t^\beta) + \xi_1 t^{\mu(\beta-1)(1-\mu)} E_{\beta, \mu(2-\gamma)}(-\lambda t^\beta) + \int_0^t (t-\tau)^{\beta-1} E_{\beta, \delta}(-\lambda(t-\tau)^\delta) f(\tau)d\tau,
\]

(14)

where \( \delta = \beta + \mu(\gamma - \beta), q = \beta + \mu(2 - \beta) \).

In fact, applying the Laplace transform (11) by means of (9) and considering initial conditions (12), (13) yield

\[
\mathcal{L}\{u\} = \frac{\xi_0 s^{1-\mu(2-\gamma)} + \xi_1 s^{-\mu(2-\gamma)} + \mathcal{L}\{f\}}{s^{\beta+\mu(q-\beta)} + \lambda},
\]

(15)

where \( \mathcal{L}\{u\} \) and \( \mathcal{L}\{f\} \) are the Laplace transform of functions \( u \) and \( f \), respectively.

According to Lemma 2, the Laplace transform of the Mittag-Leffler function\(^5\)\(^7\) is as follows

\[
\mathcal{L}^{-1}\left\{ \frac{s^{1-\mu(2-\gamma)}}{s^{\beta+\mu(q-\beta)} + \lambda} \right\} = t^{\beta-2(1-\mu)} E_{\beta, \mu(2-\gamma)}(-\lambda t^\beta),
\]

\[
\mathcal{L}^{-1}\left\{ \frac{s^{-\mu(2-\gamma)}}{s^{\beta+\mu(q-\beta)} + \lambda} \right\} = (t-\tau)^{\beta-1} \mathcal{L}\{f\}_E_{\beta, \mu(2-\gamma)}(-\lambda(t-\tau)^\delta) f(\tau)d\tau,
\]

where \( \mathcal{L}^{-1} \) is an inverse Laplace transform operator.

Considering above evaluations and after applying the inverse Laplace transform to (15), we can write the solution of (11)-(13) in the form (14).

We refer readers to the work\(^25\)\(^26\) and Ahmad et al.\(^26\) where the Dzhurbashyan–Nersesyan operator was the main target, which has more general character and the problem (11)-(13) can be obtained as a particular case.
3 | THE STATEMENT OF THE MAIN PROBLEM AND ITS INVESTIGATIONS

Let us consider the following equation

\[
 f(x, t) = \begin{cases} 
 C\left(t^{\theta} \frac{d}{dt}\right)^{\alpha} u(x, t) - u_{xx}(x, t), & (x, t) \in \Omega_1, \\
 D_{t}^{(\gamma, \beta, \mu)} u(x, t) - u_{xx}(x, t), & (x, t) \in \Omega_2, 
\end{cases}
\]  
\[ (16) \]

in a domain \( \Omega = \Omega_1 \cup \Omega_2 \cup Q \). Here \( \Omega_1 = \{(x, t) : 0 < x < 1, \ a < t < b \} \), \( \Omega_2 = \{(x, t) : 1 < x < 1, \ 0 < t < a \} \), \( Q = \{(x, t) : 0 < x < 1, \ t = a \} \), \( a, b \in \mathbb{R}^+ \) such that \( a < b, \ 0 < a \leq 1, \ \theta < 1 \), \( 1 < \gamma, \beta < 2, \ 0 \leq \mu \leq 1 \), \( f(x, t) \) is a given function, \( C\left(t^{\theta} \frac{d}{dt}\right)^{\alpha} \) is the regularized Caputo-like counterpart of the hyper-Bessel operator defined as in (3), \( D_{t}^{(\gamma, \beta, \mu)} \) is the bi-ordinal Hilfer’s derivative defined as in (8).

**Problem.** Find a solution of (16) in \( \Omega \), satisfying regularity conditions

\[
 u(\cdot, t) \in C[0, 1] \cap C^2[0, 1], \ C\left(t^{\theta} \frac{d}{dt}\right)^{\alpha} u(x, \cdot) \in C[a, b], \\
 t^{2-q}u, \ t^{2-q}u_{x} \in C(\Omega_2), \ t^{2-q}D_{t}^{(\gamma, \beta, \mu)} u(x, t) \in C(\Omega_2), u_{xx} \in C(\Omega) 
\]

and the boundary-initial conditions

\[
 u(0, t) = 0, \ 0 \leq t \leq b, \\ u(1, t) = 0, \ 0 \leq t \leq b, \\ \lim_{t \to 0^+} t^{(1-\mu)(2-\beta)} u(x, t) = \phi(x), \ 0 \leq x \leq 1, 
\]

as well as the gluing conditions

\[
 \lim_{t \to a^-} t^{(1-\mu)(2-\beta)} u(x, t) = \lim_{t \to a^+} t \phi(x), \ 0 \leq x \leq 1, \\
 \lim_{t \to a^-} \frac{d}{dt} t^{(1-\mu)(2-\beta)} u(x, t) = \lim_{t \to a^+} (t - a)^{(1-\beta)} u_{t}(x, t), \ 0 < x < 1, 
\]

here \( \phi(x) \) is a given function, \( q = \beta + \mu(2 - \beta) \).

This is very important as anyone in the world can reproduce the provided results easily at any time. The key motivation to formulate this problem is a possible application in diffusion-wave processes, which will be described by the mixed equation as Equation (16).\textsuperscript{41} For example, a gas movement in a channel surrounded by porous medium will be governed by the mixed parabolic-hyperbolic type equation, because inside of the channel movement will be described by the wave equation, in porous media by diffusion equation.\textsuperscript{42} Practical importance of the parabolic part of the considered mixed equation can be seen in Garra et al.\textsuperscript{28} Regarding the remained part we note that use bi-ordinal Hilfer derivative generalizes classical wave equation and both fractional generalizations: the Riemann–Liouville and the Caputo cases.

Our choice of the method of separation of variables is motivated by the considered domain which allows us to use this powerful method. We, as well, note that there is a method of Green’s function, which is successfully applied for fractional diffusion-wave equations by A.Pskhu.\textsuperscript{40,43} However, in our case, there are certain difficulties linked to the unknown properties of the hyper-Bessel operator which did not allow us to use this tool.

First we introduce the following new notations:

\[
 \lim_{t \to 0^+} \frac{d}{dt} t^{(1-\mu)(2-\beta)} u(x, t) = \psi(x), \ 0 < x < 1, \\
 \lim_{t \to a^+} u(x, t) = \tau(x), \ 0 \leq x \leq 1, 
\]

here \( \tau(x) \) and \( \psi(x) \) are unknown functions to be found later.
Using the method of separation of variables for solving the homogeneous equation corresponding to (16), that is searching for a solution as \( u(x, t) = T(t)X(x) \) and considering (17) and (18) in homogeneous case, yield the following problem:

\[
X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(1) = 0.
\]  

(24)

It is obvious that (24) is a Sturm–Liouville problem on finding eigenvalues and eigenfunctions, and it has the following solution:

\[
\lambda_k = (k\pi)^2, \quad X_k(x) = \sin(k\pi x), \quad k = 1, 2, 3, \ldots.
\]

(25)

Considering the fact that the system of eigenfunctions \( \{X_k(x)\} \) in (25) forms the orthogonal basis in \( L^2(0, 1) \), we look for the solution \( u(x, t) \) and given function \( f(x, t) \) in the form of series expansions as follows:

\[
u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x),
\]

(26)

\[
f(x, t) = \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x),
\]

(27)

where \( u_k(t) \) is unknown function to be found, \( f_k(t) \) is known and given by

\[
f_k(t) = 2 \int_0^1 f(x, t) \sin(k\pi x)dx.
\]

Substituting (26) and (27) into Equation (16) in \( \Omega_1 \) and considering initial condition (22) gives the following fractional differential equation

\[
C \left(t^\theta \frac{d}{dt}\right)^\alpha u_k(t) + (k\pi)^2 u_k(t) = f_k(t)
\]

with initial condition

\[
u_k(a^+) = \tau_k,
\]

where \( \tau_k \) is the coefficient of series expansion of \( \tau(x) \) in terms of orthogonal basis (24), that is,

\[
\tau_k = 2 \int_0^1 \tau(x) \sin(k\pi x)dx.
\]

(28)

After finding the solution of this problem, then considering (25) we can write the solution of (16) in \( \Omega_1 \) satisfying the conditions (17), (18), and (23) stated in (10).

Now by using the solution (10), we evaluate \( (t-a)^{1-(1-\theta)\alpha} u_t(x, t) \):

\[
(t-a)^{1-(1-\theta)\alpha} u_t(x, t) = \sum_{k=1}^{\infty} \left[\frac{(k\pi)^2}{p^{\alpha-1}} \tau_k E_{\alpha, \alpha} \left(-\frac{(k\pi)^2}{p^\alpha} (t^\theta - a^\theta)^\alpha\right) + (t-a)^{1-\alpha} G_k(t)\right] \sin(k\pi x),
\]

where \( p = 1 - \theta \).

Considering above given evaluations, we obtain the following relation on \( Q \) deduced from \( \Omega_1 \) as \( t \to a^+ \):

\[
\lim_{t \to a^+} (t-a)^{1-(1-\theta)\alpha} u_t(x, t) = \sum_{k=1}^{\infty} \left[\frac{(k\pi)^2}{\Gamma(\alpha)p^{\alpha-1}} \tau_k\right] \sin(k\pi x).
\]

(28)
Now we establish another relation on $Q$ which will be reduced from $\Omega_2$.

According to the method of separation of variables, considering (26), (27) and initial conditions (19), (22), we obtain the following problem finding a solution of equation

$$D_t^\{\beta, \mu\}W(t) + \lambda_k W(t) = f_k(t),$$

satisfying the initial conditions

$$\lim_{t \to 0^+} I_t^{(1-\mu)(2-\beta)} W(t) = \varphi_k,$$

$$\lim_{t \to 0^+} \frac{d}{dt} I_t^{(1-\mu)(2-\beta)} W(t) = \psi_k.$$

It is obvious that (14) is the solution for above-given problem. Hence, using the solution (14) and taking (26) into account we write the solution of (16) in $\Omega_2$ satisfying (19) and (22) as

$$u(x, t) = \sum_{k=1}^{\infty} W_k(t) \sin(k\pi x),$$

(29)

where

$$W_k(t) = \varphi_k t^{\beta-2(1-\mu)E_{\delta, \delta+\mu(2-\gamma)-1}}(-\lambda_k t^\delta) + \psi_k t^{\mu+\beta-1(1-\mu)E_{\delta, \delta+\mu(2-\gamma)}(-\lambda_k t^\delta)} + \int_0^t (t-\tau)^{\delta-1} E_{\delta, \delta} \left[-\lambda_k(t-\tau)\right] f_k(\tau) d\tau,$$

here $\delta = \beta + \mu(\gamma - \beta)$ and

$$\varphi_k = 2 \int_0^1 \varphi(x) \sin(k\pi x) dx$$

and $\psi_k$ is not yet known.

Now by using (29) we simplify $\lim_{t \to 0^-} I_t^{(1-\mu)(2-\beta)} W_k(t)$ and $\lim_{t \to 0^-} \frac{d}{dt} I_t^{(1-\mu)(2-\beta)} W_k(t)$ as follows

$$\lim_{t \to 0^-} I_t^{(1-\mu)(2-\beta)} W_k(t) = \varphi_k E_{\delta, 1}(-\lambda_k a^\delta) + \psi_k a E_{\delta, 2}(-\lambda_k a^\delta) + \int_0^a (s-a)^{\delta+q-1} E_{\delta+\delta+q} \left[-\lambda_k(s-a)\right] f_k(s) ds,$$

(30)

$$\lim_{t \to 0^-} \frac{d}{dt} I_t^{(1-\mu)(2-\beta)} W_k(t) = -\varphi_k \lambda_k a^{\delta-1} E_{\delta, \delta}(-\lambda_k a^\delta) + \psi_k E_{\delta, 1}(-\lambda_k a^\delta) + \int_0^a (s-a)^{\delta+q-2} E_{\delta+\delta+q-1} \left[-\lambda_k(s-a)\right] f_k(s) ds,$$

(31)

After substituting (30) and (23) into gluing condition (20) and substituting (31), (28) into the gluing condition (21), we obtain the following the system of linear algebraic equations with respect to $\tau_k$ and $\psi_k$:

$$\begin{align*}
\varphi_k E_{\delta, 1}(-\lambda_k a^\delta) + \psi_k a E_{\delta, 2}(-\lambda_k a^\delta) + \int_0^a (s-a)^{\delta+q-1} E_{\delta+\delta+q} \left[-\lambda_k(s-a)\right] f_k(s) ds &= \tau_k, \\
\varphi_k \lambda_k a^{\delta-1} E_{\delta, \delta}(-\lambda_k a^\delta) - \psi_k E_{\delta, 1}(-\lambda_k a^\delta) - \int_0^a (s-a)^{\delta+q-2} E_{\delta+\delta+q-1} \left[-\lambda_k(s-a)\right] f_k(s) ds &= \frac{\lambda_k}{\Gamma(a^\mu+1)} \tau_k.
\end{align*}$$

(32)
From (32), we find $\psi_k$ and $r_k$:

$$
\psi_k = \frac{B_k}{\Delta_k} \varphi_k + \frac{C_k}{\Delta_k},
$$

\begin{equation}
(33)
\end{equation}

$$
\tau_k = \left( E_{k,1}(\lambda_k a^\delta) + \frac{B_k}{\Delta_k} E_{k,2}(\lambda_k a^\delta) \right) \varphi_k + \frac{C_k}{\Delta_k} E_{k,2}(\lambda_k a^\delta)
+ \int_0^a (a-s)^{\delta+q-1} E_{k,\delta+q} \left[ -\lambda_k (a-s)^\delta \right] f_k(s) ds,
\end{equation}

(34)

where

$$\Delta_k = E_{k,1}(\lambda_k a^\delta) + \frac{\lambda_k a}{\Gamma(\alpha) p^{\alpha-1}} E_{k,2}(\lambda_k a^\delta),$$

$$B_k = -\frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} E_{k,1}(\lambda_k a^\delta) + \lambda_k a^{\delta-1} E_{k,\delta}(\lambda_k a^\delta),$$

$$C_k = -\frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} \int_0^a (a-s)^{\delta+q-1} E_{k,\delta+q} \left[ -\lambda_k (a-s)^\delta \right] f_k(s) ds - \int_0^a (a-s)^{\delta+q-2} E_{k,\delta+q-1} \left[ -\lambda_k (a-s)^\delta \right] f_k(s) ds,$$

here $\lambda_k = (k\pi)^2$, $q = (1-\mu)(2-\beta)$.

First of all, we will find an estimate for $B_k$ by using Lemma 1:

$$|B_k| \leq \frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} |E_{k,1}(\lambda_k a^\delta)| + \lambda_k a^{\delta-1} |E_{k,\delta}(\lambda_k a^\delta)|$$

$$\leq \frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} \frac{M}{1+\lambda_k a^\delta} + \lambda_k a^{\delta-1} \frac{M}{1+\lambda_k a^\delta}$$

$$\leq \frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} \frac{M}{\lambda_k a^\delta} + \lambda_k a^{\delta-1} \frac{M}{\lambda_k a^\delta} = \frac{M p^{1-\alpha}}{a^{\delta-1}} + \frac{M}{a} = M_1 < \infty, \quad (M_1 = \text{const}).$$

Now let us find the upper bound of $C_k$ after integrating by parts the integrals in it:

$$|C_k| \leq \frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} \int_0^a |a-s|^{\delta+q-1} |E_{k,\delta+q}(\lambda_k (a-s)^\delta)| |f_k(s)| ds$$

$$+ |f_k(0)| a^{\delta+q} |E_{k,\delta+q}(-\lambda_k a^\delta)| + \int_0^a |a-s|^{\delta+q-1} |E_{k,\delta+q}(\lambda_k (a-s)^\delta)| |f_k'(s)| ds$$

$$\leq \frac{p^{1-\alpha}}{\lambda_k \Gamma(\alpha)} \int_0^a |a-s|^{\delta+q-1} \left[ \frac{M}{1+\lambda_k a^\delta} \right] |f_{2k}(s)| ds + |f_k(0)| a^{\delta+q} M$$

$$+ \int_0^a |a-s|^{\delta+q-1} \left[ \frac{M}{1+\lambda_k a^\delta} \right] |f_k'(s)| ds$$

$$\leq \frac{p^{1-\alpha}}{\lambda_k \Gamma(\alpha)} \int_0^a |a-s|^{q-1} M |f_{2k}(s)| ds + \frac{|f_k(0)| M a^q}{\lambda_k} + \int_0^a |a-s|^{q-1} \frac{M |f_k'(s)| ds}{\lambda_k}$$

$$= \frac{1}{\lambda_k} \left[ \frac{p^{1-\alpha} M C_1 a^q}{\Gamma(\alpha) q} + |f_k(0)| M a^q + \frac{M C_2 a^q q}{q} \right] = M_2, \quad M_2 > 0,$$

where $|f_{2k}(t)| \leq C_1$, $|f_k'(t)| \leq C_2$, $f_{2k}(t) = -\lambda_k f_k(t)$, $f_k'(t) = 2 \int_0^1 f(t, x) \sin(k\pi x) dx$. 

From (32), we find $\psi_k$ and $r_k$:
Note that on above inequalities we imply that \( f(\cdot, t) \in C^1(0, 1), f_{xx}(\cdot, t) \in L^1(0, 1) \) and \( f \in C^1_{-1}(L^1(0, 1); 0, a) \) for convergence of the last integrals.

By using above evaluations, we find the estimate for \(|\psi_k|\):

\[
|\psi_k| \leq \frac{1}{|\Delta_k|} [ |B_k||\varphi_k| + |C_k|] \leq \frac{1}{|\Delta_k|} \left[ \frac{M_1|\varphi_{1k}|}{\lambda_k} + \frac{M_2}{\lambda_k} \right] \leq \frac{1}{|\Delta_k|} \left( \frac{2M_1^2}{\lambda_k} + 2|\varphi_{1k}|^2 + \frac{M_2}{\lambda_k} \right) = \frac{M_3}{\lambda_k} \tag{35}
\]

where \( M_3 > 0, \varphi_{1k} = -\sqrt{\lambda_k} \varphi_k \) and we assume \( \varphi' \in L^2(0, 1) \), provided that \( \Delta_k \neq 0 \) for any \( k \).

From (34) and in the same way one can show that

\[
\begin{align*}
|\tau_k| & \leq \left| E_{\delta,1}(-\lambda_k \alpha^\delta) \right| + \left| \frac{B_k}{\Delta_k} \right| \left| E_{\delta,2}(-\lambda_k \alpha^\delta) \right| \left| \varphi_k \right| + \left| \frac{C_k}{\Delta_k} \right| \left| E_{\delta,2}(\lambda \alpha^\delta) \right| \left| \varphi_k \right| + \left| \varphi_k \right| \\
& \quad + \int_0^a |a - s|^q |f_k(s)| \, ds \\
& \leq \left| \frac{M_1}{1+\lambda_k \alpha^\delta} + \frac{B_k}{\Delta_k} \right| \left| \frac{M}{1+\lambda_k \alpha^\delta} \right| \left| \varphi_k \right| + \left| \frac{C_k}{\Delta_k} \right| \left| \frac{M}{1+\lambda_k \alpha^\delta} \right| \left| \varphi_k \right| \\
& \quad + \int_0^a |a - s|^q |f_k(s)| \, ds \\
& \leq \frac{M_1|\varphi_k|}{\lambda_k \alpha^\delta} \left( 1 + \frac{B_k}{\Delta_k} \right) + \frac{C_k}{\Delta_k} \left( \frac{M}{1+\lambda_k \alpha^\delta} + \frac{M}{\lambda_k} \int_0^a |a - s|^q |f_k(s)| \, ds \right) \\
& \leq \frac{1}{\lambda_k} \left[ \frac{M}{\alpha^\delta} \left( |\varphi_k| + \frac{B_k}{\Delta_k} |\varphi_k| + \left| \frac{C_k}{\Delta_k} \right| \right) + \frac{MC_0 \alpha^\delta}{q} \right] = \frac{M_4}{\lambda_k},
\end{align*}
\]

where \( M_4 > 0, \varphi \in L^1(0, 1), |f_k(t)| \leq C_0. \)

The system of linear equations (32) is equivalent to the considered problem in terms of existing the solution. For that reason, if \( \Delta_k \neq 0 \) for any \( k \), (32) has only one solution or the considered problem's solution is unique, if it exists. Therefore, we show that \( \Delta_k \) is not equal to zero for any sufficiently large \( k \).

By using Lemma 4, the behavior of \( \Delta_k \) at \( k \to \infty \) can be written as follows:

\[
\lim_{k \to \infty} \Delta_k = \lim_{|z| \to -\infty} \left[ E_{\delta,1}(z) + \frac{p^{1-\delta}}{\Gamma(a) \alpha^{\delta-1}} z E_{\delta,2}(z) \right] = \frac{p^{1-\delta}}{\Gamma(a) \Gamma(2-\delta) \alpha^{\delta-1}}
\]

This proves that \( \Delta_k \neq 0 \) for sufficiently large \( k \).

For proving the existence of the solution, we need to show uniform convergence of series representations of \( u(x, t), u_{xx}(x, t), C \left( \frac{\partial}{\partial t} \right)^a u(x, t) \) and \( D_t^{(\rho, \beta)} u(x, t) \) by using the solution (5) and (29) in \( \Omega_1 \) and \( \Omega_2 \) respectively.

In Al-Musalhi et al., \(^{29}\) the uniform convergence of series of \( u(x, t) \) and \( u_{xx}(x, t) \) was shown for \( t > 0 \). Similarly, for \( t > a \), we obtain the following estimate:

\[
|u(x, t)| \leq \sum_{k=1}^{\infty} \left( \frac{|\tau_k|}{p^a + (k \pi)^2 |t^p - a^p|^a} + \frac{1}{(k \pi)^2} \int_a^t |t^p - \tau^p|^{a-1} f_{2k}(\tau) \, d(\tau^p) \right) \\
+ \int_a^t \frac{|t^p - \tau^p|^{2a-1}}{p^a + (k \pi)^2 |t^p - a^p|^a} |f_{2k}(\tau)| \, d(\tau^p),
\]

where \( |f_{2k}(t)| \leq C_1, f_{2k}(t) = 2 \int_0^1 f_{xx}(x, t) \sin(k \pi x) \, dx. \)
Since (36) and \( \frac{\partial}{\partial x} f(\cdot, t) \in L^1[0, 1] \), then the above series converges and hence, by the Weierstrass M-test the series of \( u(x, t) \) is uniformly convergent in \( \Omega_1 \).

The series of \( u_{\text{est}}(x, t) \) is written in the form below

\[
 u_{\text{est}}(x, t) = -\sum_{k=1}^{\infty} (k\pi)^2 \left( \tau_k E_{a,1} \left[ \frac{(k\pi)^2}{p^a} (t - a)^{p^a} \right] + G_k(t) \right) \sin(k\pi x).
\]

We obtain the following estimate by considering the inequality (36):

\[
 |u_{\text{est}}(x, t)| \leq M \sum_{k=1}^{\infty} \left( \frac{M_4}{p^a + (k\pi)^2 |t^p - \tau^p|^a} \right)^{\frac{1}{a}} + \frac{1}{(k\pi)^2} \int_{a}^{t} |t^p - \tau^p|^{\alpha-1} |f_{\text{est}}(\tau)| d(\tau^p)
 + \int_{a}^{t} \left| \frac{t^p - \tau^p}{p^a + (k\pi)^2 |t^p - \tau^p|^a} \right| |f_{\text{est}}(\tau)| d(\tau^p),
\]

where \( f_{\text{est}}(t) = 2 \int_{0}^{1} \frac{\partial}{\partial x} f(x, t) \sin(k\pi x) dx \) and \( f(0, t) = f(1, t) = f_{\text{est}}(0, t) = f_{\text{est}}(1, t) = 0 \).

Since \( \frac{\partial}{\partial x} f(\cdot, t) \in L^1[0, 1] \), one can make sure that this above series is convergent.

Thus, the series in the expression of \( u_{\text{est}}(x, t) \) is bounded by a convergent series which is uniformly convergent according to the Weierstrass M-test. Then, the series of \( C \left( \frac{\partial}{\partial t} \right)^{\alpha} u(x, t) \) which can be written by

\[
 C \left( \frac{\partial}{\partial t} \right)^{\alpha} u(x, t) = -\sum_{k=1}^{\infty} (k\pi)^2 \left( \tau_k E_{a,1} \left[ -\frac{(k\pi)^2}{p^a} (t - a)^{p^a} \right] + G_k(t) \right) \sin(k\pi x) + f(x, t),
\]

has uniform convergence which can be showed in the same way to the uniform convergence of the series of \( u_{\text{est}}(x, t) \) (see Al-Musalhi et al.\(^{29}\)).

Now we need to show that the series of \( t^{2-q} u(x, t) \) and its derivatives should converge uniformly in \( \Omega_2 \) by using (29). We estimate

\[
 |t^{2-q} u(x, t)| \leq \sum_{k=1}^{\infty} \left( |\varphi_k| |E_{\delta,\delta+2(\tau-\gamma)}(-\lambda_k t^\delta)| + |\psi_k| |t E_{\delta,\delta+2(\tau-\gamma)}(-\lambda_k t^\delta)| \right)
 + |f_k(0)| |t^{\delta+2-q} |E_{\delta,\delta+1}(-\lambda_k t^\delta)| + t^{2-q} \int_{0}^{t} |t - \tau|^{\delta} |E_{\delta,\delta+1}(-\lambda_k(t - \tau)^\delta)| |f_k'(\tau)| d\tau.
\]

Consider estimates of the Mittag-Leffler function (see Lemma 1)

\[
 |t^{2-q} u(x, t)| \leq \sum_{k=1}^{\infty} \left( |\varphi_k| \frac{M}{1 + \lambda_k t^\delta} + |\psi_k| \frac{M}{1 + \lambda_k t^\delta} + |f_k(0)| |t^{\delta+2-q} |M| 1 + \lambda_k t^\delta + t^{2-q} \int_{0}^{t} |t - \tau|^{\delta} \frac{M}{1 + \lambda_k(t - \tau)^\delta} |f_k'(\tau)| d\tau \right),
\]

where \( f_k'(t) = \int_{0}^{1} f'_k(x, t) \sin(k\pi x) dx \), \( f'_k(\cdot, t) \in L^1(0, 1) \) and \( \varphi'(x) \in L^2(0, 1) \). Then the series is convergent or in other words the functional series represents \( |u(x, t)| \) is bounded by the convergent series. According to the Weierstrass M-test this functional series converges uniformly in \( \Omega_2 \).
In the similar way one can show that

\[
|u_{xx}(x, t)| \leq \sum_{k=1}^{\infty} (k\pi)^2 \left( |\varphi_k||t^{\beta-2}(1-\mu)E_{\delta,\delta+\mu(2-\gamma)}(-\lambda_k t^\delta)| + |f_k(0)||t^\delta||E_{\delta,\delta+1}(-\lambda_k t^\delta)|
+ \int_0^t |t - \tau|^\delta E_{\delta,\delta+1}(-\lambda_k(t - \tau)^\delta) |f_k'(\tau)| d\tau \right).
\]

Considering (35), and the properties of Mittag-Leffler function, we write the following estimate for \(u_{xx}(x, t)\) in \(\Omega_2\)

\[
|u_{xx}(x, t)| \leq \sum_{k=1}^{\infty} \left[ |\varphi_k| \frac{M t^{\beta-2}(1-\mu)}{1 + \lambda_k t^\delta} + \frac{M_3 t^{\mu+(\beta-1)(1-\mu)}}{1 + \lambda_k t^\delta} + \frac{M |f_k(0)||t^\delta|}{1 + \lambda_k t^\delta}
+ \int_0^t |t - \tau|^\delta \frac{M}{1 + \lambda_k(t - \tau)^\delta} |f_k'(\tau)| d\tau \right].
\]

where \(f_k'(t) = 2 \int_0^1 f_{xx}(x, t) \sin(k\pi x) dx\), and also it implies that \(\varphi' \in L^1(0, 1)\), \(f(x, \cdot) \in C^1_{-1}(0, a)\), \(f_{xx}(\cdot, t) \in L^1(0, 1)\).

From the last inequality, we can see that the functional series \(u_{xx}(x, t)\) is bounded by convergent series and it means that this functional series converges uniformly in \(\Omega_2\), according to the Weierstrass M-test.

Using the equation in \(\Omega_2\), we write \(D_t^{(\gamma, \beta)\mu}u(x, t)\) in the form

\[D_t^{(\gamma, \beta)\mu}u(x, t) = u_{xx}(x, t) + f(x, t)\]

and its uniform convergence can be shown in a similar way.

Finally, considering the Weierstrass M-test, the above arguments prove that Fourier series in (5) and (29) converge uniformly in the domains \(\Omega_1\) and \(\Omega_2\). This is the proof that the considered problem’s solution exists in \(\Omega\).

The intention of this paper was to prove the uniqueness and existence of the solution to the problem (16)–(21), as we summarize in the following theorem.

**Theorem 2.** If the following conditions

1. \(\Delta_k \neq 0\), for all \(k = 1, 2, 3, \ldots\)
2. \(\varphi \in C[0, 1]\) and \(\varphi' \in L^2(0, 1)\),
3. \(f(\cdot, t) \in C^2[0, 1]\) and \(f \in C^1_{-1}(L^1(0, 1); 0, a)\), \(f(x, \cdot) \in C_{\mu}(a, b)\), such that \(f(0, t) = f(1, t) = 0\), \(f_{xx}(0, t) = f_{xx}(1, t) = 0\) and \(\frac{\partial f}{\partial x}f(\cdot, t) \in L^1(0, 1)\) hold, then there exists a unique solution of the considered problem (16)-(21).

## 4 Conclusion

As an interesting target for studying the boundary-value problems for mixed hyperbolic-parabolic type equations, it has been arising the necessity to generalize these kinds of problems. At the same time, while generalizing these problems by using fractional derivatives it is also important to use a more general definition of fractional differentiation. In this paper, we investigated the boundary-value problem considering both concerns aforementioned above. We also generalized the results given in Al-Musalhi et al.\(^{29}\) by presenting a more general definition of regularized Caputo-like counterpart hyper-Bessel fractional differential operator at arbitrary starting point. Furthermore, the connection was established between the given data and the uniqueness and existence of the solution.
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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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**APPENDIX A**

Here we explain the derivation of the series $\mathcal{C}\left(\frac{\partial}{\partial t}\right)^{\alpha} u(x, t)$ in (5). Using relation (4) we get

$$
\mathcal{C}\left(\frac{\partial}{\partial t}\right)^{\alpha} u(x, t) = \sum_{k=0}^{\infty} \left(\frac{\partial}{\partial t}\right)^{\alpha} \left(\tau_k E_{\alpha,1} \left[-\frac{(k\pi)^2}{p^\alpha}(t^p - a^p)^\alpha\right] + G_k(t)\right) - \frac{\tau_k(t^p - a^p)^\alpha}{p^{-\alpha}\Gamma(1-\alpha)} \sin(k\pi x).
$$
The hyper-Bessel derivative of the Mittag-Leffler function is
\[
\left(t^\alpha \frac{\partial}{\partial t}\right)^a \tau_k E_{a,1} \left[-\frac{(k\pi)^2}{p^a} (t^\alpha - a^\alpha)^a\right] = \tau_k p^\alpha (t^\alpha - a^\alpha)^{-\alpha} E_{a,1-a} \left[-\lambda (t^\alpha - a^\alpha)^a\right].
\]

Using Lemma 2.3, we can write the last expression as
\[
\left(t^\alpha \frac{\partial}{\partial t}\right)^a \tau_k E_{a,1} \left[-\frac{(k\pi)^2}{p^a} (t^\alpha - a^\alpha)^a\right] = \frac{\tau_k p^\alpha (t^\alpha - a^\alpha)^{-\alpha}}{\Gamma(1-a)} + \tau_k (k\pi)^2 E_{a,1} \left[-\frac{(k\pi)^2}{p^a} (t^\alpha - a^\alpha)^a\right].
\]

Then evaluating \(\left(t^\alpha \frac{\partial}{\partial t}\right)^a G(t)\) gives that
\[
\left(t^\alpha \frac{\partial}{\partial t}\right)^a G(t) = \left(t^\alpha \frac{\partial}{\partial t}\right)^a \left(f_k^*(t) + \lambda^* \int_0^t (t^\alpha - a^\alpha)^{\alpha-1} E_{a,a} \left[\lambda^*(t^\alpha - a^\alpha)^a\right] f_k^*(\tau)d(\tau^p)\right).
\]

where \(\lambda^* = -\frac{k\pi}{p^a}\) and \(f_k^*(t) = \frac{1}{p\Gamma(a)} \int_0^t (t^\alpha - s^\alpha)^{\alpha-1} f_k(\tau)d(\tau^p)\).

The second term in the last expression can be simplified using the Erd'elyi-Kober fractional derivative for \(n = 1\),
\[
-\lambda_k t^{-p\alpha} \left(1 - \alpha + \frac{t}{p} \frac{d}{dt}\right) \int_a^{\Gamma(1-a)} \left(1 - \alpha + \frac{t}{p} \frac{d}{dt}\right) E_{a,1} \left[\lambda^*(t^\alpha - s^\alpha)^a\right] f_k^*(s)d(s^p)
\]

\[
= -\lambda_k t^{-p\alpha} \left(1 - \alpha + \frac{t}{p} \frac{d}{dt}\right) \int_a^{\Gamma(1-a)} E_{a,1} \left[\lambda^*(t^\alpha - s^\alpha)^a\right] f_k^*(s)d(s^p) -
\]

\[
- \lambda_k \frac{t^{-p\alpha+1}}{p} \frac{d}{dt} \left(t^{-p(1-a)} \int_a^{\Gamma(1-a)} E_{a,1} \left[\lambda^*(t^\alpha - s^\alpha)^a\right] f_k^*(s)d(s^p)\right)
\]

\[
= -\lambda_k (1 - \alpha) t^{-p\alpha} \int_a^{\Gamma(1-a)} E_{a,1} \left[\lambda^*(t^\alpha - s^\alpha)^a\right] f_k^*(s)d(s^p) +
\]

\[
+ \lambda_k (1 - \alpha) t^{-p\alpha} \int_a^{\Gamma(1-a)} E_{a,1} \left[\lambda^*(t^\alpha - s^\alpha)^a\right] f_k^*(s)d(s^p)\]

\[
- \lambda_k t^{-p\alpha} \int_a^{\Gamma(1-a)} \lambda^*(t^\alpha - s^\alpha)^{\alpha-1} E_{a,a} \left[\lambda^*(t^\alpha - s^\alpha)^a\right] f_k^*(s)d(s^p)\]

\[
= -\lambda_k \left(f_k^*(t) + \lambda^* \int_0^t (t^\alpha - a^\alpha)^{\alpha-1} E_{a,a} \left[\lambda^*(t^\alpha - a^\alpha)^a\right] f_k^*(\tau)d(\tau^p)\right) = -\lambda_k G_k(t).
\]

Hence, we get
\[
C \left(t^\alpha \frac{\partial}{\partial t}\right)^a u(x, t) = -\sum_{k=0}^{\infty} (k\pi)^2 \left[\tau_k E_{a,1} \left(\frac{(k\pi)^2}{p^a} (t^\alpha - a^\alpha)^a\right) + G_k(t)\right] \sin(k\pi x) + f(x, t).
\]
This proves that solution (5) satisfies the equation

$$C\left(\int_t^\infty \frac{\partial}{\partial t} \right) u(x, t) - u_{xx}(x, t) = f(x, t).$$

We would like to note that using the result of this work, one can consider fractional PDE with the Bessel operator considering local\(^4\) and non-local boundary value problems.\(^1\) In that case the Fourier–Bessel series will play an important role. The other possible applications are related with the consideration of more general operators in space variables. For instance, in Ruzhansky et al,\(^4\) general positive operators have been considered, and the results of this paper can be extended to that setting as well.