$USp(2k)$ Matrix Model: Nonperturbative Approach to Orientifolds

H. Itoyama

and

A. Tokura

Department of Physics,
Graduate School of Science, Osaka University,
Toyonaka, Osaka, 560 Japan

Abstract

We discuss theoretical implications of the large $k$ $USp(2k)$ matrix model in zero dimension. The model appears as the matrix model of type $IIB$ superstrings on a large $T^6/Z^2$ orientifold via the matrix twist operation. In the small volume limit, the model behaves four dimensional and its $T$ dual is six-dimensional worldvolume theory of type $I$ superstrings in ten spacetime dimensions. Several theoretical considerations including the analysis on planar diagrams, the commutativity of the projectors with supersymmetries and the cancellation of gauge anomalies are given, providing us with the rationales for the choice of the Lie algebra and the field content. A few classical solutions are constructed which correspond to Dirichlet $p$-branes and some fluctuations are evaluated. The particular scaling limit with matrix $T$ duality transformation is discussed which derives the $F$ theory compactification on an elliptic fibered $K3$.

---

1This work is supported in part by the Grant-in-Aid for Scientific Research Fund (2126,97319) from the Ministry of Education, Science and Culture, Japan.
I. Introduction

This paper discusses in depth theoretical implications of the $\text{USp}(2k)$ matrix model in zero dimension introduced in ref. [1]. A particular emphasis will be given to the aspects of the model as a nonperturbative framework to deal with orientifold compactification.

Gauge fields and strings have governed our thoughts on unified theory of all forces including gravity and constituents for more than two decades. One of our current theoretical endeavours is, it seems, to take gauge fields as dynamical variables of noncommuting matrix coordinates [2] to construct string theory from matrices. This approach strives for overcoming some of the difficulties of the first quantized superstring theory, which have led to an inevitable impasse: one may list, among other things, the existence of infinitely degenerate perturbative vacua, the problem of supermoduli etc. One dimensional matrix model [3] of $M$ theory [4] has obtained successes on the agreement of the spectrum and other properties with the low energy eleven-dimensional supergravity theory while the zero-dimensional model [5] of type $\text{IIB}$ superstrings lays its basis on the correspondence [6, 7, 8, 5] with the first quantized action of the Schild type gauge [9] and appears to be numerically accessible. We will often refer to the latter case as reduced model. See refs. [10] for some of the references on the subsequent developments.

We would like to show that the reduced model presented in this paper descends from the first quantized nonorientable type $\text{I}$ superstring theory [11], which is believed to be related to heterotic string theory [12] by S duality [4, 13]. In this sense, it is expected that the model is exposed to phenomenological questions of particle physics by the presence of gauge bosons, matter fermions and other properties. As is pointed out in ref. [1], the model, at the same time, captures one of the exact results in string theory, namely the $F$ theory compactification on an elliptic fibered $K3$, which is originally deduced geometrically [14] from the $SL(2,\mathbb{Z})$ duality [15]; it is nonetheless exact quantum mechanically.

In the next section, the definition of the $\text{USp}(2k)$ matrix model introduced in ref. [1] is recalled. The relationship of the parts not involving the fields in the fundamental representation with the type $\text{IIB}$ matrix model is given precisely, by introducing projectors onto the $\text{USp}$ adjoint as well as the antisymmetric representation. This is found to be useful in developing the analysis in remaining sections. The definition of the model appears to be rather ad hoc at first. In the subsequent three sections, we will show that our model passes in fact several stringent criteria which the large $k$ reduced model of orientifold must satisfy. We will be able to provide the rationales for our choice of the Lie algebra $\text{usp}$, for the choice of the number of the noncommuting coordinates belonging to the adjoint representation and that to the antisymmetric representation, and finally for the number of multiplets needed,
denoted by \( n_f \), belonging to the fundamental representation.

The most basic notion of the large \( k \) reduced models is that the dense set of Feynman diagrams in the large \( k \) limit forms the string worldsheet \([10]\). This is not limited to a combinatorial equivalence. The reduced \( U(2k) \) Yang-Mills model goes to the string action in the Schild gauge. The Lie algebra \( u(2k) \) becomes isomorphic to the area preserving diffeomorphisms on a sphere. In section 3, we begin with showing how this fact is extended to nonorientable strings. We examine the role played by the matrix \( F \) in large \( k \) \( USp \) Feynman diagrams, ignoring the diagrams coming from the fields belonging to the fundamental representation. This is combined with the analysis relating \( F \) to the worldsheet involution in the large \( k \) limit, telling us that the surfaces created by the dense set of Feynman diagrams are nonorientable. The correspondence with the first quantized operator approach confirms that \( F \) is a matrix analog of the twist operation. This is strengthened by showing that equation (3.9) changes sign under the matrix T duality transformation \([17]\). In section 4, we examine the commutativity of the projectors with dynamical as well as kinematical supersymmetry. The cases which pass this criterion with eight dynamical and eight kinematical supercharges are found to be very scarce. The field content of our model stands as the most natural choice.

In section 5, we discuss the role played by the fields in the fundamental representation and the cancellation of gauge anomalies. Obviously, these fields create boundaries of the surfaces.

Combining these analyses in sections 3, 4 and 5, we conclude that the model in its original form is the large \( k \) reduced model of type \( IIB \) superstrings on a large \( T^6/\mathbb{Z}_2 \) orientifold. In the other limit, namely the small volume limit in which the model behaves as in four dimensional flat spacetime, the \( T \) duality transformation takes this model into the six-dimensional worldvolume theory representing type \( I \) superstrings in ten spacetime dimensions. The anomaly cancellation of this worldvolume gauge theory in section 5 selects \( n_f = 16 \), telling us that this is the matrix counterpart of the original Green-Schwarz cancellation leading to \( SO(32) \) type \( I \) nonorientable superstrings.

In section 6, we turn to constructing classical solutions which correspond to a D-string and two (anti-)parallel D-strings. A formula for the one-loop effective action on a general background is obtained. This is used to evaluate the potential between two antiparallel D-strings. Evidently, two additional dimensions are not generated in this naive large \( k \) limit. These solutions are straightforwardly generalized to solutions representing a \( Dp \)-brane and parallel \( Dp \)-branes, which we illustrate in the case of \( p = 3 \) in section 7. In section 8, applying some of the results obtained in sections 3, 4 and 5, we supplement the discussion of ref. \([1]\) on the connection with the \( F \) theory compactification on an elliptic fibered \( K3 \).
II. Definition of the $USp(2k)$ matrix model

We adopt a notation that the inner product of the two $2k$ dimensional vectors $u_i$ and $v_i$ invariant under $USp(2k)$ are

$$\langle u, v \rangle = u_i F^{ij} v_j , \quad (2.1)$$

$$F^{ij} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} \quad (2.2)$$

$I_k$ is the unit matrix. The raising and lowering of the indices are done by $F = F^{ij}$ and $F^{-1} = F_{ij}$. The element $X$ of the $usp(2k)$ Lie algebra satisfying $X^t F + FX = 0$ and $X^\dagger = X$ can be represented as

$$X = \begin{pmatrix} M & N \\ N^* & -M^t \end{pmatrix} \quad (2.3)$$

with $M^\dagger = M$ and $N^t = N$. It is sometimes convenient to adopt the tensor product notation:

$$X = \left( \frac{1 + \sigma^3}{2} \right) \otimes M + \left( \frac{1 - \sigma^3}{2} \right) \otimes (-M^t) + \sigma^+ \otimes N + \sigma^- \otimes N^* \quad (2.4)$$

where $\sigma^1$, $\sigma^2$ and $\sigma^3$ are Pauli matrices, and $\sigma^\pm \equiv (\sigma^1 \pm i\sigma^2)/2$. On the other hand, the element $Y$ of the antisymmetric representation of the $USp(2k)$ is

$$Y = \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \quad (2.5)$$

with $B^t = -B$ and $C^t = -C$. The hermiticity condition can be imposed. In the tensor product notation, eq. (2.5) becomes then

$$\left( \frac{1 + \sigma^3}{2} \right) \otimes A + \left( \frac{1 - \sigma^3}{2} \right) \otimes A^t + \sigma^+ \otimes B + \sigma^- \otimes (-B^*) \quad (2.6)$$

with $A^\dagger = A$ and $B^t = -B$.

Let us recall the definition of the $USp(2k)$ matrix model in zero dimension introduced in ref. [1].

Our zero-dimensional model can be written, by borrowing $N = 1$, $d = 4$ superfield notation in the Wess-Zumino gauge. One simply drops all spacetime dependence of the fields but keeps all grassmann coordinates as they are:

$$S \equiv S_{vec} + S_{asym} + S_{fund} \quad (2.7)$$
The chiral superfields introduced above are

\[ S_{\text{vec}} = \frac{1}{4g^2} Tr \left( \int d^2\theta W^\alpha W_\alpha + \text{h.c.} + 4 \int d^2\theta d^2\bar{\theta} \Phi \Phi + e^{2V} \Phi e^{-2V} \right) \]

\[ S_{\text{asym}} = \frac{1}{g^2} \int d^2\theta d^2\bar{\theta} \left( T^{ij} (e^{2V})^i_j + T^{ij} \bar{T}^{ij} (e^{-2V})^i_j \right) + \frac{1}{g^2} \left\{ \sqrt{2} \int d^2\theta \bar{T}^{ij} (\Phi_{\text{asym}})^i_j T_{k\ell} + \text{h.c.} \right\} \]

\[ S_{\text{fund}} = \frac{1}{g^2} \sum_{f=1}^{n_f} \left[ \int d^2\theta d^2\bar{\theta} \left( Q^{(f)}_i + Q^{(f)}_j + \bar{Q}^{(f)}_i + \bar{Q}^{(f)}_j \right) + \left\{ \int d^2\theta \left( m^{(f)}_i \bar{Q}^{(f)}_i + \sqrt{2} \bar{Q}^{(f)}_i \Phi \right) + \text{h.c.} \right\} \right] . \]

The chiral superfields introduced above are

\[ W_\alpha = -\frac{1}{8} D\bar{D} e^{-2V} D\bar{D} e^{2V} , \Phi = \Phi + \sqrt{2} \theta \psi_\Phi + \theta F_\Phi , \quad (2.8) \]

\[ Q_i = Q_i + \sqrt{2} \theta \psi Q_i + \theta F_Q , \quad (2.9) \]

\[ D_\alpha = \frac{\partial}{\partial \theta^\alpha} , \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^\alpha} \quad (2.10) \]

\[ V = -\theta \sigma^m \bar{\theta} v_m + i \theta \theta \bar{\psi} \lambda - i \bar{\theta} \theta \bar{\psi} \lambda + \frac{1}{2} \theta \theta \bar{\psi} \lambda D \quad (2.11) \]

We represent the antisymmetric tensor superfield \( T_{ij} \) as

\[ Y \equiv (TF)^i_j = \begin{pmatrix} A & B \\ C & A^\dagger \end{pmatrix} \quad (2.12) \]

with \( B^i = -B, C^i = -C \). We define \( \bar{Y} \) similarly.

In terms of components, the action reads, with indices suppressed,

\[ S_{\text{vec}} = \frac{1}{g^2} Tr \left( \frac{1}{4} \psi_m \psi_m - [D_m, \Phi]^\dagger [D_m, \Phi] - i \lambda \sigma^m [D_m, \lambda] - i \bar{\psi} \sigma^m [D_m, \bar{\psi}] \right) \]

\[ -i \sqrt{2} [\lambda, \psi] \Phi^i - i \sqrt{2} [\bar{\lambda}, \bar{\psi}] \bar{\Phi}^i \]

\[ + \frac{1}{g^2} Tr \left( \frac{1}{2} D\bar{D} - D\Phi^\dagger + F_\Phi^\dagger F_\Phi \right) \quad (2.13) \]

\[ S_{\text{asym}} = \frac{1}{g^2} \left\{ \left( (D_m T)^* (D_m T) - i \bar{\psi} T^\dagger \sigma^m D_m \psi T - i \sqrt{2} T^* \lambda (\text{asym}) \psi T + i \sqrt{2} \bar{\psi} T^\dagger \lambda (\text{asym}) T \right) \right. \]

\[ \left. - (D_m \bar{T})^* (D_m \bar{T}) - i \bar{\psi} \bar{T}^\dagger \sigma^m D_m \bar{\psi} \bar{T} - i \sqrt{2} \bar{T}^* \lambda (\text{asym}) \bar{\psi} \bar{T} + i \sqrt{2} \bar{\psi} \bar{T}^\dagger \lambda (\text{asym}) \bar{T} \right\} \]

\[ - 2 (\Phi_{\text{asym}} T^*) (\Phi_{\text{asym}} T) - 2 (\bar{T} \Phi_{\text{asym}}) (\bar{T}^* \Phi_{\text{asym}}) \]
\[-\sqrt{2}(\bar{\psi}_T\psi^{(asym)}_T) + \bar{T}\psi^{(asym)}_T + \psi_T\Psi^{(asym)}_T \]
\[-\sqrt{2}(\bar{\psi}_T\psi^{(asym)}_T)^* + T^*\psi^{(asym)}_T + \psi_T\Psi^{(asym)}_T \]
\[+ \sqrt{2}\bar{T}F^{(asym)}_\Phi T + \sqrt{2}T^*F^{(asym)}_\Phi T^* + \bar{T}D^{(asym)}T + \bar{T}^*D^{(asym)}T^* \]  \(2.14\)

\[S_{fund} = + \frac{1}{g^2} \sum_{f=1}^{n_f} \left[ (\mathcal{D}_m Q_{(f)})^*(\mathcal{D}^m Q_{(f)}) - i\bar{\psi}_{Q(f)} \sigma^m \mathcal{D}_m \psi_{Q(f)} + i\sqrt{2}Q_{(f)} \lambda \psi_{Q(f)} - i\sqrt{2} \bar{\psi}_{Q(f)} \bar{\lambda} \right] \]
\[+ \frac{1}{g^2} \sum_{f=1}^{n_f} \left[ (\mathcal{D}_m \bar{Q}_{(f)})^*(\mathcal{D}^m \bar{Q}_{(f)}) - i\bar{\psi}_{\bar{Q}(f)} \sigma^m \mathcal{D}_m \psi_{\bar{Q}(f)} - i\sqrt{2} \bar{\psi}_{\bar{Q}(f)} \bar{\lambda} \right] \]
\[+ \frac{1}{g^2} \sum_{f=1}^{n_f} \left[ Q_{(f)}^* \Phi^i Q_{(f)} + \bar{Q}_{(f)} \Phi^i \bar{Q}^*_{(f)} + Q_{(f)} \Phi \bar{Q}^*_{(f)} \right] \]
\[-2Q_{(f)}^* \Phi^i \Phi Q_{(f)} - 2\bar{Q}_{(f)} \Phi^i \Phi \bar{Q}^*_{(f)} \]
\[-\sqrt{2}(\bar{\psi}_{\bar{Q}(f)} \psi_{Q(f)} + \bar{\psi}_{Q(f)} \psi_{\bar{Q}(f)}) \]
\[-\sqrt{2} \bar{\psi}_{\bar{Q}(f)} \psi_{\bar{Q}(f)} + Q_{(f)}^* \bar{\psi} \bar{\psi}_{\bar{Q}(f)} + \bar{Q}_{(f)} \psi \psi_{Q(f)} \]
\[+ \sqrt{2}\bar{Q}_{(f)} F_{\Phi^i} Q_{(f)} + \sqrt{2} Q_{(f)}^* F_{\Phi}^i Q_{(f)} \]  \(2.15\)

where

\[D_{i}^{\ j} = [\Phi^i, \Phi]_j + \sum_{f=1}^{n_f} (Q_{(f)}^j Q_{(f)}_i + \bar{Q}_{(f)}^j \bar{Q}_{(f)}^*_{i}) + 2T^{*}jkT^{*}_{ki} + 2\bar{T}^{*}jk\bar{T}^{*}_{ki} \]  \(2.16\)

\[F_{\Phi^i}^{\ j} = - \sum_{f=1}^{n_f} (\sqrt{2}Q_{(f)}^j \bar{Q}_{(f)}^*_{i}) - \sqrt{2}T^{*}jkT^{*}_{ki} \]  \(2.17\)

Here \(\mathcal{D}_m = iv_m\) with \(v_m\) in appropriate representations. \(\Phi_{(asym)}\), \(\psi^{(asym)}\) and \(F_{\Phi}^{(asym)}\) are the fields in anti-symmetric representation.

Let us now find a relationship of \(S_{vec} + S_{asym}\) in eq. \(2.7\) with the reduced action of the four dimensional \(N = 4\) supersymmetric Yang-Mills written again in terms of superfields. This latter action in turn is related in the component form to the reduced action of the ten-dimensional \(N = 1\) Yang-Mills, which is nothing but the type \(IIB\) matrix model \[3\].

First note that \(S_{vec} + S_{asym}\) in eq. \(2.7\) is written as

\[S_{vec} + S_{asym} \equiv S_{adj+asym} = \frac{1}{4g^2} Tr \left( \int d^2\theta W^\alpha W_\alpha + h.c. + 4 \int d^2\theta d^2\bar{\theta} \Phi^{ij} e^{2V} \Phi_i e^{-2V} \right) \]
\[ + \frac{1}{\sqrt{2}g^2} Tr \left( \int d^2 \theta d^2 \bar{\theta} \epsilon^{ijk} [\Phi_i, [\Phi_j, \Phi_k]] + h.c. \right) , \]  

(2.18)

where we have introduced the notation

\[ \Phi_1 \equiv \Phi , \ \Phi_2 \equiv Y , \ \Phi_3 \equiv \tilde{Y} . \]  

(2.19)

The form of eq. (2.18) is nothing but the reduced action of \( d = 4, \mathcal{N} = 4 \) super Yang-Mills, which we denote by \( S_{\mathcal{N}=4}^{d=4} \):

\[ S_{\text{adj+asym}} = S_{\mathcal{N}=4}^{d=4} . \]  

(2.20)

It is expedient to introduce the projector acting on \( U(2k) \) matrices:

\[ \hat{\rho}_+ \bullet \equiv \frac{1}{2} \left( \bullet \mp F^{-1} \bullet F \right) . \]  

(2.21)

The action of \( \hat{\rho}_- \) and that of \( \hat{\rho}_+ \) take any \( U(2k) \) matrix into the matrix lying in the adjoint representation of \( USp(2k) \) and that in the antisymmetric representation respectively. We can therefore write

\[ V = \hat{\rho}_- V , \ \Phi_1 = \hat{\rho}_- \Phi_1 , \ \Phi_i = \hat{\rho}_+ \Phi_i , \ i = 2, 3 , \]  

(2.22)

where the symbols with underlines lie in the adjoint representation of \( U(2k) \).

We now invoke the well-known fact that the action of \( d = 4, \mathcal{N} = 4 \) super Yang-Mills can be obtained from the dimensional reduction of \( d = 10, \mathcal{N} = 1 \) super Yang-Mills down to four dimensions \([RS] \). This is stated as

\[ S_{\mathcal{N}=4}^{d=4} (u_m, \Phi_i, \lambda, \psi_i, \bar{\Phi}_i, \bar{\lambda}, \bar{\psi}_i) = S_{\mathcal{N}=1}^{d=10} (u_M, \Psi) , \]

\[ S_{\mathcal{N}=1}^{d=10} (u_M, \Psi) = \frac{1}{g^2} Tr \left( \frac{1}{4} [u_M, u_N] [u^M, u^N] - \frac{1}{2} \bar{\Psi} \Gamma^M [u_M, \Psi] \right) . \]  

(2.23)

Here

\[ \Phi_i = \frac{1}{\sqrt{2}} (v_{3+i} + iv_{6+i}) , \]

and

\[ \Psi = (\lambda, 0, \psi_1, 0, \psi_2, 0, \psi_3, 0, 0, \bar{\lambda}, 0, \bar{\psi}_1, 0, \bar{\psi}_2, 0, \bar{\psi}_3)^t , \]  

(2.24)

which is a thirty-two component Majorana-Weyl spinor satisfying

\[ C \bar{\Psi}^t = \Psi , \ \Gamma_{11} \bar{\Psi} = \Psi \ . \]  

(2.25)

With regard to eqs. (2.24) and (2.25), the same is true for objects with underlines. The ten dimensional gamma matrices have been denoted by \( \Gamma^M \). We will not spell out their explicit form which is determined from eqs. (2.23),(2.24).
What we have established through the argument above are summarized as the following formulas useful in later sections.

\[ S_{\text{adj+asym}} = S_{N=1}^{d=10}(\hat{\rho}_{b\mp} \chi_{M}, \hat{\rho}_{f\mp} \chi) , \]

where \( \hat{\rho}_{b\mp} \) is a matrix with Lorentz indices and \( \hat{\rho}_{f\mp} \) is a matrix with spinor indices.

\[ \hat{\rho}_{b\mp} = \text{diag}(\hat{\rho}_{-}, \hat{\rho}_{-}, \hat{\rho}_{-}, \hat{\rho}_{+}, \hat{\rho}_{+}, \hat{\rho}_{+}, \hat{\rho}_{+}, \hat{\rho}_{+}) \]

\[ \hat{\rho}_{f\mp} = \hat{\rho}_{-1(4)} \otimes \begin{pmatrix} 1_{(2)} & 0 \\ 1_{(2)} & 0 \end{pmatrix} + \hat{\rho}_{+1(4)} \otimes \begin{pmatrix} 0 & 1_{(2)} \\ 0 & 1_{(2)} \end{pmatrix}. \]

The notable properties of the model discussed in [1] are, among other things,
1) it possesses eight dynamical and eight kinematical supersymmetries.
2) translations in six out of ten directions are broken.
We will discuss implication of these in subsequent sections.

III. \( USp(2k) \) planar diagrams, matrix twist and matrix T dual

We now discuss \( USp(2k) \) planar diagrams to see how they create nonorientable surfaces approximated by the dense set of Feynman diagrams. We set aside the fields lying in the fundamental representation in this section. We ignore fermions as well. It is well-known that the large \( k \) expansion of ordinary \( U(2k) \) pure Yang-Mills theory in arbitrary dimensions is a topological (genus) expansion of the two-dimensional (discretized) surfaces created by the Feynman diagrams [10]. It is simple to see how this is modified by \( USp(2k) \) Feynman diagrams where some of them are in the adjoint while the others are in the nonadjoint (antisymmetric).

Recall that the propagator in the \( U(2k) \) gauge theory is

\[ \langle \chi m s \chi n i \rangle = \delta_i^s \delta_j^m \delta_{mn} D = \text{figure 1} \]

(3.1)

From now on we ignore the \( D \) function as its dependence on the arguments is irrelevant to the present discussion. The three and four point vertices are depicted in figure 2.

Let \( \mathcal{G} \) be a \( U(2k) \) Feynman diagram. Its dependence on \( g^2 \) and on \( k \) denoted by \( r(\mathcal{G}) \) is known to be

\[ r(\mathcal{G}) = (g^2k)^{\varepsilon-\nu} k^\chi, \]

\[ \chi = \mathcal{F} - \mathcal{E} + \mathcal{V} = 2 - 2\mathcal{H}, \]

(3.2)
where $E$ is the number of external lines in $G$, which is also the number of edges of the surface, while $V$ is the number of three and four point vertices in $G$ and is on the surface. The number of faces or index loops and the number of holes of the surface are denoted by $F$ and by $H$ respectively.

In $USp$ Feynman diagrams, eq. (3.1) is modified to

$$\langle v_M v_N \rangle = 2k_i^2 \sum_{a=1}^{2k^2 \pm k} (t^a)^s (t^a)^j \delta_{MN}$$

$$= (\hat{\rho}_r^s r_i^j) \delta_{MN}$$

$$= \text{figure 3} \quad (3.3)$$

Here we have treated the adjoint and nonadjoint cases collectively. Similarly, let $G$ be a $USp$ Feynman diagram. As the propagator contains the second term which reduces the number of index loops by one, $r(G)$ depends upon how many times double lines representing propagators cross. Clearly

$$r(G) = r(G; c) = (g^2 k)^{E-V} k^{E-c}, \quad (3.4)$$

where $c$ denotes the number of crossings.
We still need to show that $c$ denotes the number of cross caps and not the number of boundaries. Let us recall that, according to the present point of view, the two-dimensional surface swept by a string is formed by the dense set of Feynman diagrams. To render this more tangible and more than a combinatorial argument, we note that, via the Schild gauge correspondence, the algebra acting on the functions on the string world sheet must be isomorphic to the large $k$ limit of the appropriate Lie algebra acting on matrices. For this, it is enough to adopt the argument of Pope and Romans[19] on area-preserving diffeomorphisms on $RP^2$ and the large $k$ limit of the $usp(2k)$ Lie algebra in the present context. Consider first the sphere parametrized by three coordinates $x^i, i = 1, 2, 3$ such that $x^i x^i = 1$. The complete set of functions on the sphere is the spherical harmonics represented by

$$Y^{(p)}(x^i) = a_{i_1 \cdots i_p} x^{i_1} \cdots x^{i_p},$$

where $a_{i_1 \cdots i_p}$ are totally symmetric and traceless constants. The algebra of area preserving diffeomorphisms is defined by a bracket of two functions $A(x^i)$ and $B(x^i)$:

$$\{A, B\} \equiv \epsilon_{ijk} x^j \partial_k A \partial_k B. \tag{3.6}$$

When $A = Y^{(m)}$, $B = Y^{(n)}$, a finite sum of irreducible polynomials $Y^{(p)}$, $|m - n| \leq p \leq m + n - 1$ is generated. This algebraic structure is obtained by the large $k$ limit of the $su(2k)$ Lie algebra in the form of maximal $su(2)$ embeddings:

$$\Lambda^{(p)} = a_{i_1 \cdots i_p} \Sigma^{i_1} \cdots \Sigma^{i_p}, p = 1 \sim k - 1. \tag{3.7}$$

Here, $\Sigma^i$ are the $su(2)$ generators in the $2k$-dimensional representation. On the other hand, $RP^2$ geometry is obtained from the sphere by the antipodal identification $x^i \rightarrow -x^i$, under which the harmonics splits into even and odd ones. Only the odd ones are responsible for forming the algebra of area-preserving diffeomorphism on $RP^2$: this is clear from eq. (3.6). We see that the diffeomorphisms of $RP^2$ are generated by the large $k$ limit of the generators

$$\Lambda^{(2p-1)} = a_{i_1 \cdots i_{2p-1}} \Sigma^{i_1} \cdots \Sigma^{i_{2p-1}}, p = 1, 2, \cdots k. \tag{3.8}$$
As shown by Pope and Romans, the algebra formed by eq. (3.8) is the Lie algebra \( u_{sp} \). This concludes that the diagrams generated from the propagator (eq. (3.3)) and the vertices contain \( RP^2 \). The theory we are constructing via matrices is the reduce model of nonorientable strings.

To extend the above argument to higher genera with crosscaps, let us note that the role of the matrix \( F \) can be seen by the correspondence with the twist operation in the operator formalism of the first quantized string. Ten of the noncommuting coordinates \( v_M \), which are dynamical variables, satisfy

\[
v_i^t = -Fv_iF^{-1}, \quad i \in \{0, 1, 2, 3, 4, 7\} \equiv M_-
\]

\[
v_I^t = Fv_IF^{-1}, \quad I \in \{5, 6, 8, 9\} \equiv M_+ .
\]  

(3.9)

The \( v_M \) are noncommuting counterparts of the ten string coordinates \( X_M \). That this is more than just an analogy is clear as the limit exists from our action to the string action of the Schild type gauge. Taking the transpose is interpreted as flipping the direction of an arrow drawn on a string. The operation \( F \) is the matrix analog of the twist operation \( \Omega \). The classical counterpart of eq. (3.9) is therefore

\[
X_i(\bar{z}, z) = -\Omega X_i(z, \bar{z})\Omega^{-1}, \quad i \in M_-
\]

\[
X_I(\bar{z}, z) = \Omega X_I(z, \bar{z})\Omega^{-1}, \quad I \in M_+ .
\]  

(3.10)

The presence of four dimensional fixed surfaces (orientifold surfaces, O3s) becomes clear from this equation (3.10). We conclude that our model is a matrix model on a large volume \( T^6/Z^2 \) orientifold. This is consistent with that the translations in six out of ten directions are broken.

The \( T \) duality transformation plays an interesting role in matrix models as it relates worldvolume theories of various dimensions via Fourier transforms. We will now find how the matrix \( T \) dual behaves under \( F \). First, let us impose periodicities with period \( 2\pi R \) for \( L \) out of the ten coordinates. Recall that

\[
Y_\ell \equiv \hat{T}[X_\ell] \equiv X_{\ell R} - X_{\ell L} ,
\]  

(3.11)

\[
\hat{T}[X_\ell](\bar{z}, z) = \begin{cases} 
  +\Omega \hat{T}[X_\ell](z, \bar{z})\Omega^{-1} & \text{if } \ell \in M_- \\
  -\Omega \hat{T}[X_\ell](z, \bar{z})\Omega^{-1} & \text{if } \ell \in M_+ 
\end{cases} .
\]  

(3.12)

To impose periodicities on infinite size matrices \( v_\ell \), we decompose \( v_\ell \) into blocks of \( n \times n \) matrices. Specify each individual block by an \( L \)-dimensional row vector \( \vec{a} \) and an \( L$-$

\[2\]In the context of ref. [3], see ref. [20].
dimensional column vector $\vec{b}$: $(v_\ell)_{\vec{a},\vec{b}} \equiv \sqrt{\alpha'} \langle \vec{a} | \hat{v}_\ell | \vec{b} \rangle$. Let the shift vector be

$$(U(i))_{\vec{a},\vec{b}} = \left( \prod_{j(\neq i)} \delta_{a_j,b_j} \right) \delta_{a_i,b_{i+1}} . \quad (3.13)$$

The condition to be imposed is

$$U(i)v_\ell U(i)^{-1} = v_\ell - \delta_{\ell,i} R / \sqrt{\alpha'} . \quad (3.14)$$

The solution in the Fourier transformed space is

$$\langle \vec{x} | \hat{\hat{v}}_\ell | \vec{x}' \rangle = -i \left( \frac{\partial}{\partial x'_\ell} + i \hat{\nu}_\ell(\vec{x}) \right) \delta^{(L)}(\vec{x} - \vec{x}') , \quad (3.15)$$

$$\hat{\nu}_\ell(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}^L} \hat{\nu}_\ell(\vec{k}) \exp \left( \frac{-i \vec{k} \cdot \vec{x}}{R} \right) ,$$

$$\hat{R} \equiv \alpha' / R . \quad (3.16)$$

The matrix $T$ dual is nothing but the Fourier transform: it interchanges the radius parameter $R$ setting the period of the original matrix index with the dual radius $\hat{R}$ which is the period of the space Fourier conjugate to the matrix index. Let us write

$$\hat{T} (v_\ell)_{\vec{a},\vec{b}} \equiv \langle \vec{x} | \hat{\hat{v}}_\ell | \vec{x}' \rangle . \quad (3.17)$$

Multiply eq. (3.13) written in the bracket notation

$$\langle \vec{d} | \hat{\nu}_\ell | \vec{a} \rangle = \mp \sum_{\vec{b},\vec{c}} \langle \vec{a} | \hat{\hat{F}} | \vec{b} \rangle \langle \vec{b} | \hat{\nu}_\ell | \vec{c} \rangle \langle \vec{c} | \hat{\hat{F}}^{-1} | \vec{d} \rangle \quad (3.18)$$

by $\langle \vec{x} | \vec{a} \rangle \langle \vec{d} | \vec{x}' \rangle = \langle \vec{a} | \vec{x} \rangle^* \langle \vec{x}' | \vec{d} \rangle^*$. Sum over $\vec{a}$ and $\vec{d}$. From the left hand side, we obtain

$$- \left[ -i \frac{\partial}{\partial x'_\ell} - \hat{\nu}_\ell(-\vec{x}') \right] \delta^{(L)}(\vec{x}' - \vec{x}) . \quad (3.19)$$

We find

$$\hat{T} [v_\ell]^t = \begin{cases} +\hat{T} [F] \hat{T} [v_\ell] \hat{T} [F^{-1}] & \text{if } \ell \in \mathcal{M}_- \\ -\hat{T} [F] \hat{T} [v_\ell] \hat{T} [F^{-1}] & \text{if } \ell \in \mathcal{M}_+ \end{cases} , \quad (3.20)$$

provided

$$\hat{\nu}_\ell(-\vec{x}') = -\hat{\nu}_\ell(\vec{x}') . \quad (3.21)$$

It is satisfying to see that the sign change of eq. (3.20) from eq. (3.12) under the matrix $T$ dual is in accordance with the sign change of eq. (3.10) from eq. (3.10).

One can now imagine imposing periodicities with periods depending on the directions and letting some of the radii zero. The $T$ duality provides worldvolume gauge theories in various dimensions. We will discuss a few cases later.
IV. \(USp\) projector and supersymmetry

We will now derive a set of conditions under which the projectors \(\hat{\rho}_{b\mp}\), \(\hat{\rho}_{f\mp}\), which act respectively on \(\varphi_M\) and \(\Psi\), and dynamical \(\delta^{(1)}\) as well as kinematical \(\delta^{(2)}\) supersymmetry commute. Our choice for \(\hat{\rho}_{b\mp}\) and that for \(\hat{\rho}_{f\mp}\) emerge as the case which passes the tight constraint of having eight dynamical and eight kinematical supersymmetries. Let us start with

\[
\delta^{(1)} \varphi_M = i\bar{\epsilon}\Gamma_M \Psi \quad (4.1)
\]

\[
\delta^{(1)} \Psi = \frac{i}{2} [\varphi_M, \varphi_N] \Gamma^{MN} \epsilon \quad (4.2)
\]

\[
\delta^{(2)} \varphi_M = 0 \quad (4.3)
\]

\[
\delta^{(2)} \Psi = \xi \quad (4.4)
\]

Let us write generically

\[
\varphi_M \equiv \delta_M^N \hat{\rho}_{b\mp} \varphi_N \quad (4.5)
\]

\[
\Psi_A \equiv \delta_{AB} \hat{\rho}_{f\mp} \Psi_B \quad (4.5)
\]

The condition \([\hat{\rho}_{b\mp}, \delta^{(1)}] \varphi_M = 0\) together with eq. (4.1) gives

\[
\sum_{A=1}^{32} \left( \epsilon \Gamma_M \right)_A \left( \hat{\rho}^{(A)}_{f\mp} - \hat{\rho}^{(M)}_{b\mp} \right) \Psi_A = 0 \quad (4.6)
\]

with index \(M\) not summed. The condition

\[
\left[ \hat{\rho}_{f\mp}, \delta^{(1)} \right] \Psi \bigg|_{\varphi_M \rightarrow \hat{\rho}_{b\mp} \varphi_M} = 0 \quad (4.7)
\]

together with eq. (4.2) provides

\[
\left( 1 - \hat{\rho}^{(A)}_{f\mp} \right) \left[ \hat{\rho}^{(M)}_{b\mp}, \hat{\rho}^{(N)}_{b\mp} \varphi_N \right] \left( \Gamma^{MN} \epsilon \right)_A = 0 \quad (4.8)
\]

The restriction at eq. (4.7) comes from the fact that eq. (4.2) is true only on shell. Eq. (4.3) does not give us anything new while \([\hat{\rho}_{f\mp}, \delta^{(2)}] \Psi = 0\) with eq. (4.4) gives

\[
\xi_A 1 = \xi_A \hat{\rho}^{(A)}_{f\mp} 1 \quad (4.9)
\]

with index \(A\) not summed.

In order to proceed further, we rewrite eq. (4.5) explicitly as

\[
\hat{\rho}^{(M)}_{b\mp} \equiv \Theta(M \in \mathcal{M}_-) \hat{\rho}_- + \Theta(M \in \mathcal{M}_+) \hat{\rho}_+ \quad (4.10)
\]

\[
\hat{\rho}^{(A)}_{f\mp} \equiv \Theta(A \in \mathcal{A}_-) \hat{\rho}_- + \Theta(A \in \mathcal{A}_+) \hat{\rho}_+ \quad (4.10)
\]
where
\[ \mathcal{M}_- \cup \mathcal{M}_+ = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \quad \mathcal{M}_- \cap \mathcal{M}_+ = \emptyset, \quad (4.11) \]
\[ \mathcal{A}_- \cup \mathcal{A}_+ = \{1, 2, 5, 6, 9, 10, 13, 14, 19, 20, 23, 24, 27, 28, 31, 32\}, \quad \mathcal{A}_- \cap \mathcal{A}_+ = \emptyset. \quad (4.12) \]

We find that eq. (4.6) gives
\[ (\bar{\epsilon} \Gamma_M)_{A+} = (\bar{\epsilon} \Gamma_M)_{A-} = 0, \quad (4.13) \]
while eq. (4.8) gives
\[ (\Gamma_{M-N} \epsilon)_{A-} = 0 \]
\[ (\Gamma_{M-N} \epsilon)_{A+} = (\bar{\epsilon} \Gamma_{M-N} \epsilon)_{A+} = 0. \quad (4.14) \]

Equation (4.9) gives
\[ \xi_{A-} = 0. \quad (4.15) \]

As we consider the case of eight kinematical supersymmetries, the number of elements of the sets denoted by \( \mathcal{A}_\pm \) must be
\[ \sharp(\mathcal{A}_-) = 8 \quad \text{and} \quad \sharp(\mathcal{A}_+) = 8. \quad (4.16) \]

Eqs. (4.13) and (4.14) are regarded as the ones which determine the anticommuting parameter \( \epsilon \), and the sets \( \mathcal{A}_+, \mathcal{A}_-, \mathcal{M}_+ \) and \( \mathcal{M}_- \). In addition they must satisfy the conditions (4.11), (4.12) and (4.16).

We search for solutions by first trying out as an input an appropriate thirty-two component anticommuting parameter \( \epsilon \) satisfying Majorana-Weyl condition.

Given \( \epsilon \), we see if we can determine \( \mathcal{A}_+, \mathcal{A}_-, \mathcal{M}_+ \) and \( \mathcal{M}_- \) successfully. Our strategy is:

(i) calculate \( (\bar{\epsilon} \Gamma_M)_{A} \) and \( (\Gamma_{MN} \epsilon)_{A} \) for all \( M, N \) and \( A \).

(ii) calculate \( \sum_{A} (\bar{\epsilon} \Gamma_{M_1} A (\bar{\epsilon} \Gamma_{M_2}) A \). If this value is nonzero, the both indices \( M_1 \) and \( M_2 \) belong to either \( \mathcal{M}_- \) or \( \mathcal{M}_+ \). We can, therefore, divide \( \mathcal{M}_- \cup \mathcal{M}_+ \) into two sets.

(iii) from eq. (4.14) we see that if \( (\Gamma_{M-N} \epsilon)_{A} \neq 0 \), then \( A \in \mathcal{A}_+ \). If \( (\Gamma_{M-N} \epsilon)_{A} \neq 0 \) or \( (\Gamma_{M+N} \epsilon)_{A} \neq 0 \), then \( A \in \mathcal{A}_- \). Use the results of (i) and (ii) to determine \( \mathcal{A}_- \) and \( \mathcal{A}_+ \). We must then check if \( \sharp(\mathcal{A}_-) = 8, \sharp(\mathcal{A}_+) = 8 \) and \( \mathcal{A}_- \cap \mathcal{A}_+ = \emptyset \). If these are not satisfied,
our original input $\epsilon$ is not a solution.

**(iv)** From eq. (4.13) we see that if $\epsilon_{\Gamma_M}^A \neq 0$ then $A \in A_-$, and if $\epsilon_{\Gamma_M^+}^A \neq 0$ then $A \in A_+$. Determine $A_-$ and $A_+$. If $A_-$ and $A_+$ determined this way are consistent with the result from (iii), we obtain a solution to eqs. (4.13) and (4.14). This also determines $M_-$ and $M_+$ as we have two ways of choosing them from (ii).

We have tried out many cases, some of which we will describe. The case leading to our model is

$$\epsilon = (\epsilon_0, 0, \epsilon_1, 0, \epsilon_0, 0, \epsilon_0, 0, \epsilon_1, 0, 0, 0, 0)^t.$$  \hspace{1cm} (4.17)

Note that $\epsilon_0$, $\epsilon_1$, $\bar{\epsilon}_0$ and $\bar{\epsilon}_1$ are two-component anticommuting parameters. From step (ii), we see $M_- \cup M_+$ are divided into two sets:

$$\{\{ 0, 1, 2, 3, 4, 7 \}\} \text{ and } \{\{ 5, 6, 8, 9 \}\}. \hspace{1cm} (4.18)$$

From step (iii), we find

$$A_- = \{\{ 1, 2, 5, 6, 19, 20, 23, 24 \}\},$$
$$A_+ = \{\{ 9, 10, 13, 14, 27, 28, 31, 32 \}\}. \hspace{1cm} (4.19)$$

From step (iv), we obtain

$$M_- = \{\{ 0, 1, 2, 3, 4, 7 \}\},$$
$$M_+ = \{\{ 5, 6, 8, 9 \}\}. \hspace{1cm} (4.20)$$

We conclude that

$$\hat{\rho}_{b\mp} = \text{diag}(\hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+),$$
$$\hat{\rho}_{f\mp} = \hat{\rho}_- 1_{(4)} \otimes \begin{pmatrix} 1_{(2)} & 0 \\ 0 & 1_{(2)} \end{pmatrix} + \hat{\rho}_+ 1_{(4)} \otimes \begin{pmatrix} 0 & 1_{(2)} \\ 1_{(2)} & 0 \end{pmatrix}, \hspace{1cm} (4.21)$$

which are the projectors of our model.

Among other cases, we have tried the following one:

$$\epsilon = (\epsilon_0, 0, \epsilon_1, 0, \epsilon_2, 0, \epsilon_3, 0, 0, \epsilon_0, 0, \epsilon_1, 0, \epsilon_2, 0, \epsilon_3)^t.$$  \hspace{1cm} (4.22)

From step (ii), we obtain

$$\{\{ 0, 1, 2, 3, 4, 7 \}\} \text{ and } \{\{ 5, 6, 8, 9 \}\}. \hspace{1cm} (4.23)$$
We find that \( A_- \) and \( A_+ \) determined from step (iii) do not satisfy \( A_- \cap A_+ = \emptyset \).

We have examined the following cases (and their permutations) as well with no success:

\[
\epsilon = (\epsilon_0, 0, \epsilon_1, 0, \epsilon_2, 0, \epsilon_3, 0, 0, \epsilon_0, 0, -\epsilon_1, 0, -\epsilon_2, 0, -\epsilon_3)^t
\]
\[
\epsilon = (\epsilon_0, 0, \epsilon_1, 0, 0, \epsilon_3, 0, 0, \epsilon_0, 0, \epsilon_1, 0, 0, 0, 0, \epsilon_3)^t
\]
\[
\epsilon = (\epsilon_0, 0, \epsilon_1, 0, 0, \epsilon_3, 0, 0, \epsilon_0, 0, \epsilon_1, 0, 0, -\epsilon_3)^t.
\]
\[
\epsilon = (\epsilon_0, 0, \epsilon_1, 0, 0, 0, \epsilon_3, 0, \epsilon_0, 0, -\epsilon_1, 0, 0, 0, -\epsilon_3)^t.
\]
\[
\epsilon = (\epsilon_0, 0, \epsilon_1, 0, 0, 0, \epsilon_1, 0, 0, \epsilon_0, 0, \epsilon_1, 0, 0, \epsilon_3)^t.
\] (4.24)

There is, however, another solution which we have found. Let

\[
\epsilon = (\epsilon_0, 0, \epsilon_1, 0, 0, 0, 0, 0, \epsilon_0, 0, \epsilon_2, 0, \epsilon_3)^t.
\] (4.25)

The consistent sets

\[
A_- = \{1, 2, 5, 6, 27, 28, 31, 32\},
\]
\[
A_+ = \{9, 10, 13, 14, 19, 20, 23, 24\},
\] (4.26)
\[
M_- = \{4, 7\},
\]
\[
M_+ = \{0, 1, 2, 3, 5, 6, 8, 9\}.
\] (4.27)

are obtained from steps (i), (ii), (iii) and (iv). The projectors (4.10) are

\[
\hat{\phi}_{b\Xi} = \text{diag}(\hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+3, \hat{\rho}_+3, \hat{\rho}_+3, \hat{\rho}_+3)
\]
\[
\hat{\rho}_f \Xi = \hat{\rho}_-1(4) \otimes \begin{pmatrix} 1_{(2)} & 0 \\ 0 & 0 \end{pmatrix} + \hat{\rho}_+1(4) \otimes \begin{pmatrix} 0 & 1_{(2)} \\ 1_{(2)} & 0 \end{pmatrix}.
\] (4.28)

This is the case considered in ref. [20, 21] in the context of \( M \) theory compactification to the lightcone heterotic strings (with \( \epsilon_0, \epsilon_1, \bar{\epsilon}_2 \) and \( \bar{\epsilon}_3 \) in eq. (4.25) all real).

V. The role of the fundamental representation and anomaly cancellation of worldvolume theory

So far, we have ignored the fields in the fundamental representation. These fields do not contribute to the diagrams in spherical topology. They are irrelevant to the questions
concerning with the spacetime coordinates. They create, however, disk diagrams and higher genera with boundaries and are responsible for creating an open string sector. This is in fact required as nonorientable closed strings by themselves are not consistent. It is well-known that the simplest way to establish the consistency is through the (global) cancellation of dilaton tadpoles between disk and $RP^2$ diagrams Refs. [22], [23], leading to the $SO(32)$ Chan-Paton factor. This survives toroidal compactifications with/without discrete projection Ref. [24]. It should be that the sum of an infinite set of diagrams of the matrix model contributing to the disk/$RP^2$ geometry yields the string partition function of the disk/$RP^2$ diagram. The Chan-Paton trace at the boundary corresponds to the trace with respect to the flavor index. The $n_f$ should therefore be fixed by the tadpole cancellation. The flavor symmetry of the model is the local gauge symmetry of strings.

The lack of the combinatorial argument and the absence of the vertex operator construction at this moment, however, prevent us from proceeding to such calculation via matrices. Instead, we will examine gauge anomalies of worldvolume theories by taking the $T$ dual and subsequently the zero volume limit of $T^6/Z^2$. In particular, let us do this for all six adjoint directions. The resulting theory is the six dimensional worldvolume gauge theory obeying eq. (3.21) with matter in the antisymmetric and fundamental representation. This is the type I superstrings in ten spacetime dimensions. This case is also the first nontrivial case of getting a potentially anomalous theory. In fact, by acting

$$\Gamma_{(6)} \equiv \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^7$$

on $\Psi$, we see that the adjoint fermions $\lambda$ and $\psi_1$ have chirality plus while $\psi_{2,3}$ have chirality minus. The fermions in the fundamental representation have chirality minus. The standard technology to compute nonabelian anomalies is provided by the family’s index theorem and the descent equations Refs. [11], [25]. We find that the condition for the anomaly cancellation:

$$tr_{adj}F^4 - tr_{asym}F^4 - n_ftrF^4$$

$$= (2k + 8)trF^4 + 3\left(trF^2\right)^2 - \left((2k - 8)trF^4 + 3\left(trF^2\right)^2\right) - n_ftrF^4$$

$$= (16 - n_f)trF^4 = 0 \quad ,$$

where we have indicated the traces in the respective representations. The case $n_f = 16$ is selected by the consistency of the theory. In the case discussed in eq. (4.28), we conclude from similar calculation that the anomaly cancellation of the worldvolume two-dimensional gauge theory selects sixteen complex fermions.

VI. One-loop effective action and D-string solutions
A. one-loop effective action

In this subsection, we will establish a formula for the one-loop effective action of the $USp$ matrix model on a generic bosonic background\[^{[1]}\]. Let us first find one-loop fluctuations on a generic classical solution of the $USp(2k)$ matrix model. We write

\[
\begin{align*}
    v_m &= p_m + ga_m, \quad (m = 0 \sim 3), \quad \lambda = \chi_0 + g\phi_0 \\
    v_I &= p_I + ga_I, \quad (I = 4 \sim 9), \quad \psi_i = \chi_i + g\phi_i, \quad (i = 1 \sim 3)
\end{align*}
\]

with $(p_m, p_I, \chi_0, \chi_i)$ a configuration satisfying equations of motion. In order to fix the gauge invariance we add the ghost and the gauge fixing term

\[
S_{gfgh} = \frac{1}{2} Tr \left( [p_M, a^M]^2 - [p^K, b][p_K, c] \right),
\]

where $c$ and $b$ are, respectively, the ghosts and the antighosts lying in the adjoint representation of $USp(2k)$. Denote by $S^{(2)}$ the part in $S_{adj+asym}$ which is quadratic in $a$ and $\phi$. The one-loop effective action $W_{\text{one-loop}}$ is

\[
W_{\text{one-loop}} = -i \log \int [da_m][da_I][d\phi_0][d\tilde{\phi}_0][d\phi_i][d\tilde{\phi}_i][dc][db] \exp \left[ iS^{(2)} + iS_{gfgh} \right].
\]

Instead of resorting to the direct gaussian integrations of the expression above, let us use eqs. (2.26) and (2.27).

In the same way as eq. (6.1), we decompose $\underline{\underline{M}}$ and $\Psi$ into the backgrounds and the quantum fluctuations. Let us denote the fluctuations by $\underline{M}^{(fl)}$ and $\Psi^{(fl)}$. Then from eq. (2.26) we have

\[
S^{(2)} = S_{N=1}^{d=10}^{(2)} (\hat{\rho}_{b+\underline{M}}^{(fl)}, \hat{\rho}_{f+\Psi}^{(fl)}) ,
\]

where $S_{N=1}^{d=10}^{(2)} (\hat{\rho}_{b+\underline{M}}^{(fl)}, \hat{\rho}_{f+\Psi}^{(fl)})$ is the part in the action of $d = 10, N = 1$ super Yang-Mills which is quadratic in the fluctuations. As the variables are explicitly projected either onto $USp(2k)$ adjoint or onto antisymmetric matrices, we can safely replace the integration measure by that of the $u(2k)$ Lie algebra valued matrices. We obtain

\[
W_{\text{one-loop}} = -i \log \int [da_m^{(fl)}][da_i^{(fl)}][dc][db] \exp \left[ iS_{N=1}^{d=10}^{(2)} (\hat{\rho}_{b+\underline{M}}^{(fl)}, \hat{\rho}_{f+\Psi}^{(fl)}) + iS_{gfgh}(\hat{\rho}_{-b}, \hat{\rho}_{-c}) \right]
\]

\[
= \frac{1}{2} \log \det \left( O_b \hat{\rho}_{b+} \right) - \frac{1}{2} \log \det \left( O_f \hat{\rho}_{f+} \left( \frac{1 + \Gamma_{11}}{2} \right) \right) - \log \det \left( \hat{\rho}_K \hat{F}_K \hat{\rho}_{-} \right)
\]

where

\[
\begin{align*}
    O_b^M &= -\delta_L^M \hat{P}_K \hat{P}_K + 2i \hat{F}_{L}^M, \quad O_f = -\Gamma_M \hat{P}_M, \\
    \hat{P}_K &\cdot = [p_K, \cdot], \quad \hat{F}_{KL} \cdot = i[[p_K, p_L], \cdot].
\end{align*}
\]

\[^{3}\] The solutions we will construct in the next subsection and in section 7 are relevant only in the large $k$ limit. We will, therefore, ignore the fields lying in the fundamental representation.
In obtaining eq. (6.6), we have set all fermionic backgrounds $\chi_0$ and $\chi_i$ to zero. As a consequence, the one-loop effective action on a generic bosonic background is given by

$$W_{\text{one-loop}} = \left(\frac{6}{2} - \frac{4}{2} - 1\right) Tr \log \left(\hat{P}_K \hat{P}_K \hat{\rho}_-\right) + \left(\frac{4}{2} - \frac{4}{2} - 1\right) Tr \log \left(\hat{P}_K \hat{P}_K \hat{\rho}_+\right) + W_b + W_f,$$

(6.8)

$$W_b = \frac{1}{4} Tr \log \left[ \left(\delta^M_L + \frac{4}{(P_K \hat{P}_K)^2} \hat{F}^N_L \hat{F}^M_N\right) \hat{\rho}_{b\mp}\right],$$

(6.9)

$$W_f = -\frac{1}{4} Tr \log \left[ \left(1 + \frac{i}{2 \hat{P}_K \hat{P}_K} \Gamma^{MN} \hat{F}_{MN}\right) \hat{\rho}_{f\mp} \left(\frac{1 + \Gamma_{11}}{2}\right)\right].$$

(6.10)

We put the matrix $\hat{F}_{MN}$ into the following form with respect to the Lorentz indices.

$$\hat{F}_{MN} = \left(\begin{array}{cccccccc}
0 & -\hat{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hat{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\hat{B}_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \hat{B}_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\hat{B}_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_3 & 0 \\
0 & 0 & 0 & 0 & \hat{B}_4 & 0 & 0 & 0 \\
0 & 0 & 0 & \hat{B}_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \hat{B}_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_5 & 0
\end{array}\right),$$

(6.11)

When the classical configuration is BPS saturated, $\hat{F}_{MN} = 0$ and $W_{\text{one-loop}}$ vanishes.

**B. D-string solution**

Let us construct a few particular classical bosonic solutions of the model. We set the fields lying in fundamental representation of $USp(2k)$ to zero. Equation of motion is

$$[p_N, [p^M, p^N]] = 0.$$

(6.12)

There are three cases of solutions representing a D-string configuration, depending upon which two directions the worldsheet extends to infinity. When both of the directions are the adjoint directions, say $v_0$ and $v_1$, the nonvanishing components are

$$p_0 = \left(\frac{1 + \sigma^3}{2}\right) \otimes \mathbf{x} + \left(\frac{1 - \sigma^3}{2}\right) \otimes (-\mathbf{x}^t), \quad p_1 = \left(\frac{1 + \sigma^3}{2}\right) \otimes \mathbf{\pi} + \left(\frac{1 - \sigma^3}{2}\right) \otimes (-\mathbf{\pi}^t).$$

(6.13)

\(^4\) The calculation in what follows parallels those of refs. [5, 26].
When both are in the antisymmetric directions, say \( v_5 \) and \( v_8 \), the nonvanishing components are
\[
p_5 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \mathbf{x} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \mathbf{x}^t, \quad p_8 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \mathbf{\pi} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \mathbf{\pi}^t.
\] (6.14)

When one is in the adjoint direction, say \( v_0 \), and the other is in the antisymmetric direction, say \( v_8 \),
\[
p_0 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \mathbf{x} + \left( \frac{1 - \sigma^3}{2} \right) \otimes (-\mathbf{x}^t), \quad p_8 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \mathbf{\pi} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \mathbf{\pi}^t.
\] (6.15)

In above expressions, \( \mathbf{x} \) and \( \mathbf{\pi} \) are infinite size matrices with the commutator \([\mathbf{\pi}, \mathbf{x}] = -i\).

Let us now turn to the solutions representing two parallel D-strings and two anti-parallel D-strings. We will illustrate this by the most interesting case that the two D strings are extended in the two directions (\( v_5 \) and \( v_8 \)) of antisymmetric representations separated by \( d \) in the \( v_4 \) direction which is the adjoint direction. The nonvanishing components are
\[
p_5 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{x}^t & 0 \\ 0 & \mathbf{x}^t \end{pmatrix},
\]
\[
p_8 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{\pi} & 0 \\ 0 & \mathbf{\pi} \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{\pi}^t & 0 \\ 0 & \mathbf{\pi}^t \end{pmatrix},
\]
\[
p_4 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} -d/2 & 0 \\ 0 & d/2 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} d/2 & 0 \\ 0 & -d/2 \end{pmatrix},
\] (6.16)

for two parallel D-strings, and
\[
p_5 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{x}^t & 0 \\ 0 & \mathbf{x}^t \end{pmatrix},
\]
\[
p_8 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{\pi} & 0 \\ 0 & -\mathbf{\pi} \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} \mathbf{\pi}^t & 0 \\ 0 & -\mathbf{\pi}^t \end{pmatrix},
\]
\[
p_4 = \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} -d/2 & 0 \\ 0 & d/2 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} d/2 & 0 \\ 0 & -d/2 \end{pmatrix},
\] (6.17)

for two anti-parallel D-strings.

### C. force between antiparallel D-string

We would like to determine the scale of our spacetime given by the model. This can be done by computing the force mediating two classical objects which are by themselves a non-BPS configuration. We will evaluate the \( W_b \) and the \( W_f \) in the case of the two antiparallel
D-strings separated by distance \( d \), which have been constructed in the last subsection. We compute the force exerting with each other. From eq. (6.17) we have \( \hat{P}^0 = \hat{P}^1 = \hat{P}^2 = \hat{P}^3 = \hat{P}^6 = \hat{P}^7 = \hat{P}^9 = 0 \), \( \hat{B}_1 = \hat{B}_2 = \hat{B}_3 = \hat{B}_5 = 0 \), \( \hat{P}_K \hat{P}^K = (\hat{P}^4)^2 + (\hat{P}^5)^2 + (\hat{P}^8)^2 \), \( \hat{P}^4 = \frac{d}{2} \hat{B}_4 \) and, after some algebra, we obtain

\[
[\hat{P}^5, \hat{P}^8] = -i \hat{B}_4 , \quad [\hat{P}^4, \hat{P}^5] = 0 , \quad [\hat{P}^4, \hat{P}^8] = 0 .
\]

(6.18)

When we take trace with Lorentz indices in (6.9) and with spinor indices in (6.10), we arrive at the following expressions:

\[
W_b = \frac{1}{2} Tr \left[ \log \left( 1 - \frac{4 \hat{B}_4 \hat{B}_4}{(\hat{P}_K \hat{P}^K)^2} \right) \hat{\rho}_+ \right]
\]

(6.19)

\[
W_f = -Tr \left[ \log \left( 1 - \frac{1}{(\hat{P}_K \hat{P}^K)^2} \hat{\rho}_+ \right) + \log \left( 1 - \frac{1}{(\hat{P}_K \hat{P}^K)^2} \hat{B}_4 \hat{B}_4 \right) \hat{\rho}_+ \right]
\]

(6.20)

In appendix A, the eigenvalues of \( \hat{B}_4 \hat{B}_4 \), their degeneracies and the eigenmatrices are determined. We compile the results at table 1 for the antisymmetric eigenmatrices and at table 2 for the adjoint eigenmatrices.

| the eigenvalue of \( \hat{B}_4 \hat{B}_4 \) | the degeneracy |
|------------------------------------------|---------------|
| 4                                       | \( k^2 - k \) |
| 0                                       | \( k^2 \)     |

table 1

| the eigenvalue of \( \hat{P}_K \hat{P}^K \) | the degeneracy |
|-----------------------------------------------|----------------|
| \( d^2 + 4n + 2 \)                           | \( k \)         |

| the eigenvalue of \( \hat{P}_K \hat{P}^K \) | the degeneracy |
|-----------------------------------------------|----------------|
| \( d^2 + 4n + 2 \)                           | \( k \)         |

| the eigenvalue of \( \hat{B}_4 \hat{B}_4 \) | the degeneracy |
|-----------------------------------------------|---------------|
| 4                                             | \( k^2 + k \) |
| 0                                             | \( k^2 \)     |

| the eigenvalue of \( \hat{P}_K \hat{P}^K \) | the degeneracy |
|-----------------------------------------------|----------------|
| \( d^2 + 4n + 2 \)                           | \( k \)         |

Using these tables, we obtain

\[
W_b = \frac{k}{2} \sum_{n=0}^{\infty} \log \left( 1 - \frac{16}{(d^2 + 4n + 2)^2} \right) ,
\]

(6.21)

\[
W_f = -2k \sum_{n=0}^{\infty} \log \left( 1 - \frac{4}{(d^2 + 4n + 2)^2} \right) .
\]

(6.22)
Putting all these together, we find
\[ W_{\text{one-loop}} = -\frac{k}{2} \log \left[ \left( \frac{d^2}{4} \right)^{-4} \frac{d^2/4 + 1/2}{d^2/4 - 1/2} \left( \frac{\Gamma \left( \frac{d^2}{4} + \frac{1}{2} \right)}{\Gamma \left( \frac{d^2}{4} \right)} \right)^8 \right] = -\frac{k}{2} \left\{ \frac{8}{d^6} + \mathcal{O}(d^{-8}) \right\} \, . \] (6.23)

This potential provides the asymptotic behavior of the force mediating two antiparallel D-strings. From this we conclude that the dimension of spacetime is ten at least in this naive large \( k \) limit.

VII. Construction of D3-brane solutions

It is not difficult to extend the construction of the D-string solutions in the previous section to general D\( p \)-brane solutions. We will illustrate this by a D3-brane, two parallel D3-branes and multiple D3-branes which are parallel to one another.

Let us first consider a D3-brane solution. When the worldvolume extends in \( v_5, v_8, v_6 \) and \( v_9 \) directions, the nonvanishing components are given by
\[
\begin{align*}
p_5 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes x_1 + \left( \frac{1 - \sigma^3}{2} \right) \otimes x^t_1, \\
p_8 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \pi_1 + \left( \frac{1 - \sigma^3}{2} \right) \otimes \pi^t_1, \\
p_6 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes x_2 + \left( \frac{1 - \sigma^3}{2} \right) \otimes x^t_2, \\
p_9 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \pi_2 + \left( \frac{1 - \sigma^3}{2} \right) \otimes \pi^t_2.
\end{align*}
\] (7.1)

It is straightforward to check that this configuration satisfies equation of motion. In the above expression, \( x_1, x_2, \pi_1 \) and \( \pi_2 \) are operators (infinite matrices) with the commutators
\[
\left[ \pi_1, x_1 \right] = -i \sqrt{\frac{V_4}{k}}, \quad \left[ \pi_2, x_2 \right] = -i \sqrt{\frac{V_4}{k}}. \quad (7.2)
\]

Here we must take the limit of \( k \rightarrow \infty \) with \( V_4/k \) fixed to \( (\alpha')^2 \).

Now let us calculate the value of the action. We have
\[
\begin{align*}
\left[ p^5, p^8 \right] &= \sigma^3 \otimes i\alpha'1_k, \\
\left[ p^6, p^9 \right] &= \sigma^3 \otimes i\alpha'1_k.
\end{align*} \quad (7.3)
\]
When we substitute these into the action,

\[
S = \frac{1}{g^2(\alpha')^2} Tr \left( \frac{1}{2} [p^5, p^8][p_5, p_8] + \frac{1}{2} [p^6, p^9][p_6, p_9] \right)
\sim \frac{1}{g^2(\alpha')^2} V_4 = T_{3\text{-brane}} V_4. \tag{7.4}
\]

Here \( g^2 \) is regarded as string coupling \( g_{st} \). This is consistent with the D-brane action which is given by the tension times the volume of the D-brane. Therefore it is appropriate to think of the above solution as a D3-brane solution.

Next, take two parallel D3-branes which are separated by distance \( d \) in the \( v_4 \) direction. The nonvanishing components are

\[
\begin{align*}
p_5 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} x_1^t & 0 \\ 0 & x_1^t \end{pmatrix} \\
p_8 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_1 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} \pi_1^t & 0 \\ 0 & \pi_1^t \end{pmatrix} \\
p_6 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} x_2 & 0 \\ 0 & x_2 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} x_2^t & 0 \\ 0 & x_2^t \end{pmatrix} \\
p_9 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} \pi_2 & 0 \\ 0 & \pi_2 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} \pi_2^t & 0 \\ 0 & \pi_2^t \end{pmatrix} \\
p_4 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} -d/2 & 0 \\ 0 & d/2 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} d/2 & 0 \\ 0 & -d/2 \end{pmatrix} \tag{7.5}
\end{align*}
\]

Finally let us consider \( N \) parallel D3-branes which are separated in the \( v_4 \) and \( v_7 \) directions. We denote the position of the \( i \)-th D3-brane by \( v_4 = d_4^{(i)} \) and \( v_7 = d_7^{(i)} \). The worldvolume extends in the \( v_5, v_8, v_6 \) and \( v_9 \) directions. The nonvanishing components are

\[
\begin{align*}
p_5 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} x_1 \\ \cdot \cdot \cdot \\ x_1 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} x_1^t \\ \cdot \cdot \cdot \\ x_1^t \end{pmatrix} \\
p_8 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} \pi_1 \\ \cdot \cdot \cdot \\ \pi_1 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} \pi_1^t \\ \cdot \cdot \cdot \\ \pi_1^t \end{pmatrix} \\
p_6 &= \left( \frac{1 + \sigma^3}{2} \right) \otimes \begin{pmatrix} x_2 \\ \cdot \cdot \cdot \\ x_2 \end{pmatrix} + \left( \frac{1 - \sigma^3}{2} \right) \otimes \begin{pmatrix} x_2^t \\ \cdot \cdot \cdot \\ x_2^t \end{pmatrix}
\end{align*}
\]
\[
p_9 = \left(\frac{1 + \sigma^3}{2}\right) \otimes \begin{pmatrix} \pi_2 \\ \cdots \\ \pi_2 \end{pmatrix} + \left(\frac{1 - \sigma^3}{2}\right) \otimes \begin{pmatrix} \pi_2^t \\ \cdots \\ \pi_2^t \end{pmatrix}
\]

\[
p_4 = \left(\frac{1 + \sigma^3}{2}\right) \otimes \begin{pmatrix} d_4^{(1)} \\ \cdots \\ d_4^{(N)} \end{pmatrix} + \left(\frac{1 - \sigma^3}{2}\right) \otimes \begin{pmatrix} -d_4^{(1)} \\ \cdots \\ -d_4^{(N)} \end{pmatrix}
\]

\[
p_7 = \left(\frac{1 + \sigma^3}{2}\right) \otimes \begin{pmatrix} d_7^{(1)} \\ \cdots \\ d_7^{(N)} \end{pmatrix} + \left(\frac{1 - \sigma^3}{2}\right) \otimes \begin{pmatrix} -d_7^{(1)} \\ \cdots \\ -d_7^{(N)} \end{pmatrix}
\] (7.6)

VIII. F theory on an elliptic fibered K3

We will now show that the model is able to describe the F theory compactification on an elliptic fibered K3 [14, 27]. Our objective here is to demonstrate that the matrix model in fact derives one of the very few exact results in critical string theory. While the original construction of Vafa is purely geometrical in nature, our model provides an action principle and path integrals to the F theory compactification.

In sections 4, 5 and 6, we have seen that our model is the matrix model of type IIB superstrings on a large \(T^6/Z^2\) orientifold. The coupling constant has no spacetime dependence and is a bona fide parameter. One can make the coupling space-dependent by taking the matrix T dual in various ways to go to higher dimensional worldvolume gauge theories as we have already discussed in the previous sections. The coupling constant then starts running with the coordinates labelling the quantum moduli space, \(\langle v \rangle\), which is denoted by \(\vec{u}\). This is in accordance with the marginal scalar deformation of the original action to a type of nonlinear \(\sigma\) model. The background field appearing through this procedure is a massless axion-dilaton field. The running coupling constant is, therefore, identified as the space-dependent axion-dilaton background field \(\lambda(\vec{u})\).

Let \(\vec{u}\) be the complex coordinates on a complex \(n\)-dimensional base space \(B_n\). F theory compactification of an elliptically fibered \(C-Y\) \((n + 1)\) fold \(M_{n+1}\) on the base \(B_n\) is defined by saying that the \(u\)-dependent axion-dilaton background field of type IIB superstrings on \(B_n \times R^{9-2n}\) is the modular parameter of the fiber \(T^2\) as a function of \(\vec{u}\). We would like to show that this is in fact the case in our matrix model. To provide F theory set-up as a reduced model for the case \(n = 1\), we are going to send the period \(R\) of the four out of the six adjoint directions \(v_0, v_1, v_2, v_3\) to zero and to take the matrix T dual. The resulting model in the limit of vanishing mass parameters is type IIB on a large \(T^2/Z^2\) orientifold,
namely on $CP^1$, equipped with sixteen $D7$-branes. Coupling starts running as we turn on the mass parameters. Following Sen\cite{27}, we would now like to take the scaling limit

$$\tilde{R} \rightarrow \infty ,$$

$$m_i \tilde{R} \rightarrow \text{finite} \quad i = 1, \sim 4$$

$$m_i \tilde{R} \rightarrow \infty \quad i = 5, \sim 16 \quad \text{(8.1)}$$

simultaneously taking the matrix $T$ dual. The second and the third lines of this equation come from the consistency with the $RR$ charge counting. The resulting worldvolume theory around one of the four $O7$s is the $d = 4, \mathcal{N} = 2$ supersymmetric $USp(2k)$ gauge theory with one massless antisymmetric hypermultiplet and four fundamental hypermultiplets with masses $m_i$. The special properties of this theory valid for all $k$ are that it is UV finite and that at least low energy physics is the same for all $k$\cite{28}. One can, therefore, deduce the $u$ dependence of the coupling of the model in the large $k$ limit by simply looking at the $k = 2$ case, namely, the $SU(2)$ susy gauge theory with four flavours. The $u$ dependence of the coupling $\lambda$ is supplied by the work of Seiberg-Witten\cite{29}. The work of Sen\cite{27} shows that the way the modular parameter of the bare torus in the massless limit is dressed by the four mass parameters in the SW curve of the massive four flavour case is mathematically identical to the description of $F$ theory in the neighborhood of the constant coupling. One can therefore safely conclude that the coupling $\lambda(u)$ of the model is in fact the modular parameter of the spectral torus. This is what we wanted to show.

\textbf{IX. Acknowledgements}

We thank Shinji Hirano, Toshihiro Matsuo and Asato Tsuchiya for helpful discussion on this subject.
Appendix  A

In appendix A we will determine the eigenvalues of the operators \( \hat{B}_4 \hat{B}_4 \) and \( \hat{P}_K \hat{P}_K \). We consider the both cases that the eigenmatrices are in the adjoint and the antisymmetric representations in \( USp(2k) \). These eigenvalues and their degeneracy are needed in order to calculate the one-loop effective action.

Suppose that an operator \( \hat{O} \) has an adjoint action on a \( 2k \times 2k \) matrix \( a \):

\[
\hat{O} \ a = \ [o, \ a] \ ,
\]

(A.1)

Here \( o \) is the \( 2k \times 2k \) matrix. Let us first consider the case that the matrix \( a \) is given by eq. (2.6). Note that the operator \( \hat{B}_4 = i[\hat{P}_5, \hat{P}_8] \) is represented by the matrix \( b_4 = -\sigma^3 \otimes \sigma^3 \otimes 1_{(k/2)} \).

It is not difficult to see that the eigenvalues of \( \hat{B}_4 \hat{B}_4 \) are either 0 or 4. For the 0 eigenvalue we simply solve \( \hat{B}_4 a_{(0 sym)}^{(0)} = 0 \) and the eigenmatrices are

\[
\begin{align*}
&\left( \frac{1+\sigma^3}{2} \right) \otimes 1_{(2)} \otimes H_0 + \left( \frac{1-\sigma^3}{2} \right) \otimes (1_{(2)} \otimes H_0)^t \ , \\
&\left( \frac{1+\sigma^3}{2} \right) \otimes \sigma^3 \otimes H_3 + \left( \frac{1-\sigma^3}{2} \right) \otimes (\sigma^3 \otimes H_3)^t \ ,
\end{align*}
\]

\( \sigma^+ \otimes \sigma^1 \otimes A_1 + \sigma^- \otimes \{-(\sigma^1 \otimes A_1)^*\} \ , \)

\( \sigma^+ \otimes \sigma^2 \otimes A_2 + \sigma^- \otimes \{-(\sigma^2 \otimes A_2)^*\} \ , \)

(A.2)

Since the \( (k/2) \times (k/2) \) matrices satisfy \( H_{0,3}^1 = H_{0,3}^t \), \( A_1^t = -A_1 \) and \( A_2^t = A_2 \), the degeneracy is \( k^2 \). As for the eigenvalue 4, the solution is

\[
\begin{align*}
&\left( \frac{1+\sigma^3}{2} \right) \otimes (\sigma^1 \otimes H_1 + \sigma^2 \otimes H_1) + \left( \frac{1-\sigma^3}{2} \right) \otimes (\sigma^1 \otimes H_1 + \sigma^2 \otimes H_1)^t \ , \\
&\left( \frac{1+\sigma^3}{2} \right) \otimes (\sigma^1 \otimes H_2 - \sigma^2 \otimes H_2) + \left( \frac{1-\sigma^3}{2} \right) \otimes (\sigma^1 \otimes H_2 - \sigma^2 \otimes H_2)^t \ ,
\end{align*}
\]

\( \sigma^+ \otimes (1_{(2)} \otimes A_0 + \sigma^3 \otimes A_0) + \sigma^- \otimes (1_{(2)} \sigma^3 \otimes A_0)^* \)

\( \sigma^+ \otimes (1_{(2)} \otimes A_3 - \sigma^3 \otimes A_3) + \sigma^- \otimes (1_{(2)} \sigma^3 \otimes A_3)^* \)

(A.3)

and the degeneracy is \( k^2 - k \) because of \( H_{1,2}^1 = H_{1,2}^t \) and \( A_{0,3}^t = -A_{0,3} \).

Let us now calculate the eigenvalues of the operator \( \hat{P}_K \hat{P}_K = \frac{i}{4} \hat{B}_4 \hat{B}_4 + \hat{P}_5 \hat{P}_5 + \hat{P}_8 \hat{P}_8 \). Clearly \( \hat{B}_4 \hat{B}_4 \) and \( \hat{P}_5 \hat{P}_5 + \hat{P}_8 \hat{P}_8 \) are simultaneously diagonalized. When \( \hat{P}_5 \hat{P}_5 + \hat{P}_8 \hat{P}_8 \)
acts on the eigenstates with eigenvalue 4 of $\hat{B}_4 \hat{B}_4$, we replace $\hat{B}_4 \hat{B}_4$ by its eigenvalue. Let $\hat{P} \equiv \hat{P}_5 \hat{B}_4 / 2\sqrt{2}$ and $\hat{Q} \equiv \hat{P}_8 / \sqrt{2}$. We obtain

$$[\hat{P}, \hat{Q}] = -i.$$ 

The eigenvalues of $\hat{P}_5 \hat{P}_5 + \hat{P}_8 \hat{P}_8 = 2(\hat{P} \hat{P} + \hat{Q} \hat{Q})$ are those of the harmonic oscillator and are given by $4n + 2$ with integer $n$. The degeneracy is $k$ for large $k$. We summarize the results in table 1.

| the eigenvalue of $\hat{B}_4 \hat{B}_4$ | the degeneracy | the eigenvalue of $\hat{P}_K \hat{P}_K$ | the degeneracy |
|-------------------------------------|----------------|--------------------------------------|----------------|
| 4                                  | $k^2 - k$      | $d^2 + 4n + 2$                       | $k$            |
| 0                                  | $k^2$          |                                      |                |

Our calculation of the effective action does not require the case in which the eigenvalue of $\hat{B}_4 \hat{B}_4$ is zero.

Similarly, the eigenmatrices lying in the adjoint representation (eq. (2.4)) can be determined. The difference is the off-diagonal degrees of freedom, which change the degeneracy of $\hat{B}_4 \hat{B}_4$ eigenvalues. The degeneracy of the $\hat{P}_K \hat{P}_K$ eigenvalues is the same in the previous case. Summing up the adjoint case, we obtain table 2.

| the eigenvalue of $\hat{B}_4 \hat{B}_4$ | the degeneracy | the eigenvalue of $\hat{P}_K \hat{P}_K$ | the degeneracy |
|-------------------------------------|----------------|--------------------------------------|----------------|
| 4                                  | $k^2 + k$      | $d^2 + 4n + 2$                       | $k$            |
| 0                                  | $k^2$          |                                      |                |
References

[1] H. Itoyama and A. Tokura, preprint hep-th/9708123, Prog. Theor. Phys. 99 (1998) to appear.

[2] E. Witten, Nucl. Phys. B460 (1996)335.

[3] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, Phys. Rev. D55 (1997)5112.

[4] E. Witten, Nucl. Phys. B443 (1995)85.

[5] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B498 (1997)467; M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, preprint hep-th/9705128.

[6] Y. Nambu, p. 1 in “Quark Confinement and Field Theory” (John Wiley & Sons, New York, 1977).

[7] I. Bars, Phys. Lett. 245B (1990) 35.

[8] D.B. Fairlie, P. Fletcher and C.Z. Zachos, J. Math Phys. 31 (1990) 1088.

[9] A. Schild, Phys. Rev. D16 (1977)1722.

[10] V. Periwal, Phys. Rev. D55 (1997) 1711; L. Motl, preprint hep-th/9612198; N. Kim and S.J. Rey, Nucl.Phys. B504 (1997) 189; R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl.Phys. B500 (1997) 43; A. Fayyazuddin and D.J. Smith, preprint hep-th/9703208; T. Banks and L. Motl, preprint hep-th/9703218; N. Kim and S.J. Rey, preprint hep-th/9705132; S. Sethi and L. Susskind, preprint hep-th/9702101; T. Banks and N. Seiberg, preprint hep-th/9702187; D. A. Lowe, Phys. Lett. 403B (1997) 243 ; S.J. Rey, Nucl.Phys. B502 (1997) 170 ; P. Horava, Nucl.Phys. B505 (1997) 84; M. Li, Nucl. Phys. B499 (1997) 149; M.B. Green and M. Gutperle, Phys. Lett. B398 (1997) 69; I. Chepelev, Y. Makeenko and K. Zarembo, Phys. Lett. 400B (1997)43; A. Fayyazuddin and D.J. Smith, Int. J. Mod. Phys. Lett. A12 (1997) 1447; A. Fayyazuddin, Y. Makeenko, P. Olsen, D. J. Smith and K. Zarembo, preprint hep-th/9703038; T. Yoneya, Prog. Theor. Phys. 97 (1997) 949; B. Sathiapalan, Int. J. Mod. Phys. Lett. A12 (1997) 1301; C.F. Kristjansen and P. Olesen, Phys. Lett. B405 (1997) 45; L. Chekhov and K. Zarembo, Int. J. Mod. Phys. Lett. A12 (1997) 2331; I. Chepelev and A. A. Tseytlin, preprint hep-th/9705126; K.-J. Hamada, Phys.Rev. D56 (1997) 7503; O. A. Solovev, preprint hep-th/9707043; N. Kitsunezaki and J. Nishimura, preprint hep-th/9707162; H. Sugawara, preprint hep-th/9708029; S. Hirano and M. Kato, preprint hep-th/9708039.
B. P. Mandal and S. Mukhopadhyay, preprint hep-th/9709098; I. Oda, preprint hep-th/9710030; A.K. Biswas, A.K. Kumar and G. Sengupta, preprint hep-th/9711040; T. Suyama and A. Tsuchiya, preprint hep-th/9711073.

[11] M. B. Green and J. H. Schwarz, Phys. Lett. 149B (1984) 117.

[12] D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985)502, Nucl. Phys. B256 (1985)253, Nucl. Phys. B267 (1986)75.

[13] J. Polchinski and E. Witten, Nucl. Phys. B460 (1996)525.

[14] C. Vafa, Nucl.Phys.B469(1996)403.

[15] C. M. Hull and P. K. Townsend, Nucl. Phys. B438 (1995)109

[16] G. ’t Hooft, Nucl.Phys. B72 (1974) 461.

[17] W. Taylor IV , Phys. Lett. B394 (1997)283; O.J. Ganor, S. Ramgoolam and W. Taylor IV, Nucl.Phys.B492(1997)191.

[18] L. Brink, J. Scherk and J. H. Schwarz, Nucl.Phys. B121 (1977) 77.

[19] C. N. Pope and L. Romans, Class. Quantum Grav. 7 (1990) 97.

[20] U.H. Danielsson and G. Ferretti, Int. J. Mod. Phys. A12 (1997)4581; L. Motl, preprint hep-th/9612198; N. Kim and S.J. Rey, Nucl.Phys. B504 (1997)189.

[21] S. Kachru and E. Silverstein, Phys. Lett. 396B (1997)70; D. A. Lowe, Nucl.Phys. B501 (1977) 134, Phys. Lett. 403B (1997)243; T. Banks, N. Seiberg and E. Silverstein, Phys. Lett. 401B (1997)30.

[22] M. B. Green and J. H. Schwarz, Phys. Lett. 151B (1985)21

[23] H. Itoyama and P. Moxhay, Nucl. Phys. B293 (1987)685

[24] Y. Arakane, H. Itoyama, H. Kunitomo and A. Tokura, Nucl. Phys. B486 (1997)149.

[25] B. Zumino, W. Yong-Shi and A. Zee, Nucl.Phys. B239 (1984) 477 .

[26] I. Chepelev, Y. Makeenko and K. Zarembo, Phys. Lett. 400B (1997)43 ; I. Chepelev and A. A. Tseytlin, preprint hep-th/9705120

[27] A. Sen, Nucl.Phys.B475(1996)562.

[28] M.R. Douglas, D.A. Lowe, J.H. Schwarz, Phys.Lett.B394(1997)297; O. Aharony, J. Sonnenschein, S. Yankielowicz and S. Theisen, Nucl.Phys. B493 (1997) 177 .
[29] N. Seiberg and E. Witten, \textit{Nucl. Phys.} \textbf{B426} (1994)19, \textit{Nucl. Phys.} \textbf{B431} (1994)484.