Transposition mirror symmetry construction
and period integrals

Susumu TANABÉ

Abstract. In this note we study several conditions to be imposed on a mirror symmetry candidate to the generic multi-quasihomogeneous Calabi-Yau variety defined in the product of the quasihomogeneous projective spaces. We propose several properties for a Calabi-Yau complete intersection variety so that its period integrals can be expressed by means of quasihomogeneous weights of its mirror symmetry candidate as it has been suggested by Berglund, Candelas et alii (Theorem 3.1). As a corollary, we see certain duality between the monodromy data and the Poincaré polynomials of the Euler characteristic for the pairs of our varieties (Theorem 4.1).

1 Introduction and Notations.

The transposition mirror construction has been proposed by P. Berglund and T. Hübisch [3] as a trial to generalize so called Greene-Plesser mirror construction that comprises mirror pairs of Fermat type hypersurfaces. Later, in the article [2], in relying on the transposition method, the authors have proposed a natural hypothesis on the period integrals associated to the complete intersection (CI) Calabi-Yau variety \( X \) that is supposed to be a mirror symmetry to the generic multi-quasihomogeneous Calabi-Yau variety \( Y \) of codimension \( \ell \) defined in the product of the quasihomogeneous projective spaces \( \mathbb{P}(\tau_1)^{(g_1 (1), \ldots, g_{\tau_1 + 1})} \times \cdots \times \mathbb{P}(\tau_k)^{(g_1 (k), \ldots, g_{\tau_k + 1})} \). They mean under the notion of the mirror symmetry between \( X \) and \( Y \) an interchange between geometric symmetry \( (G_X, G_Y) \) and quantum symmetry \( (Q_Y, Q_X) \) groups of each varieties [4] (See Definition 1, Theorem 4.2 below).

That is to say, for the mirror symmetry pair \( X \) and \( Y \), the following isomorphisms holds,

\[ Q_X \cong G_Y, \]

\[ Q_Y \cong G_X. \]

In this article we propose certain sufficient conditions on \( X \) and \( Y \) so that their hypothesis ([2] §3.3) on the period integrals holds (Theorem 3.1). Namely the period integrals defined on \( X \) can be expressed by means of quasihomogeneous weights of its mirror symmetry \( Y \) and vice versa. It will be shown that our sufficient conditions entail the mirror symmetry between \( X \) and \( Y \) in the above sense of [4] (see Theorem 4.2).

AMS Subject Classification: 14J32 primary), 32S25, 32C65 (secondary).
Key words and phrases: mirror symmetry, hypergeometric functions, monodromy.
partially supported by Hokkaido University, ICTP (Trieste).
During the entire article we shall restrict ourselves to the case $\ell = k$.

In accordance with the suggestion on the mirror symmetry to the generic multi-quasihomogeneous Calabi-Yau made in [2], we shall consider the following system of equations on $X := (\mathbb{C}^\times)^n$ as defining equations of $X = X_1$,

$$X_s = \{ x \in \mathbb{T}^n; f_1(x) + s_1(f_2(x) - 1) = \cdots = f_{2k-1}(x) + s_k(f_{2k}(x) - 1) = 0 \},$$

with system (1.1) below. Here we use the notation $s = (s_1, \cdots, s_k) \in \mathbb{T}^k$, $I = (1, 1, \cdots, 1) \in \mathbb{R}^k$.

$$f_1(x) = x^{\tau_{1}^{(1)}} + \cdots + x^{\tau_{1}^{(1)}},$$

$$f_2(x) = \prod_{j \in I^{(1)}} x_j + 1,$$

$$\vdots$$

$$f_{2i-1}(x) = x^{\tau_{1}^{(i)}} + \cdots + x^{\tau_{1}^{(i)}},$$

$$f_{2i}(x) = \prod_{j \in I^{(i)}} x_j + 1,$$

$$\vdots$$

$$f_{2k-1}(x) = x^{\tau_{1}^{(k)}} + \cdots + x^{\tau_{1}^{(k)}},$$

$$f_{2k}(x) = \prod_{j \in I^{(k)}} x_j + 1.$$

Here $I^{(j)}, 1 \leq j \leq k$ are sets of indices which are complementary one another in such a way that $\cup_{q \in [1,k]} I^{(q)} = \{1, \cdots, n\}$ and $I^{(q)} \cap I^{(q')} = \emptyset$ if $q \neq q'$. From now on we shall make use of the notations $\tau_\nu := |I^{(\nu)}|$ and $b^q := \sum_{\nu=1}^q \tau_\nu$. Additionally we suppose that

$$\sum_{\nu=1}^k \tau_\nu = b^k = n.$$

The equation $f_{2j-1}(x)$ (resp. $f_{2j}(x)$) is defined by the monomials with powers $\tau^{(j)}_{1}, \cdots, \tau^{(j)}_{r_j} \in \mathbb{Z}^n$ (resp. $\tau^{(j)}_{r_j+1} \in \mathbb{Z}^n$) such that for the weight vector $g^{(q)} = (0, \cdots, 0, g_1^{(q)}, \cdots, g_{r_q}^{(q)}, 0, \cdots, 0) \in \mathbb{Z}_{\geq 0}^n$, $1 \leq q \leq k$ the following quasihomogeneity condition holds,

$$Q_j^{(q)} := < \tau^{(j)}_{1}, g^{(q)} > = \cdots = < \tau^{(j)}_{r_j}, g^{(q)} > = < \tau^{(j)}_{r_j+1}, g^{(q)} >, 1 \leq j \leq k.$$

This means that the point $\tau^{(j)}_{r_j+1}$ belongs to the $(\tau_j - 1)$-dimensional hyperplane generated by $\tau^{(j)}_{1}, \tau^{(j)}_{1}, \cdots, \tau^{(j)}_{r_j}$. The following condition shall be imposed if we suppose that $X_s$ is a Calabi-Yau variety:

$$\sum_{j=1}^k Q_j^{(q)} = \sum_{i=1}^{r_q} g_i^{(q)} = < g^{(q)}, (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0) >, 1 \leq q \leq k.$$
We will impose this condition on (1.1) in the further arguments.

In addition to that we assume that for each element \( \lambda \in \text{Aut}(X_j) \), of the group automorphism of the hypersurface \( X_j = \{ x \in \mathbb{T}^n; f_{2j-1}(x) = 0 \} \) the relation \((\lambda_* f_{2j-1})(x) = \lambda_*(x^{\tau_{2j-1}})\) holds. More precisely, every \( \lambda \in \text{Aut}(X_s) \) admits the following decomposition

\[
\lambda = q \cdot g \cdot h,
\]

where \( q \in Q_X \), \( g \in G_X \) and \( h \in H \) each of which is a non cyclic element of \( \text{Aut}(X_s) \). The subgroup \( Q_X \cong \prod_{q=1}^{k} \mathbb{Z}Q^{(q)} \), with \( Q^{(q)} = L.C.M.(Q_1^{(q)}, \ldots, Q_k^{(q)}) \) is a cyclic group generated by \( k \) different cyclic actions \( q^{(\nu)}, 1 \leq \nu \leq k \), corresponding to the quasihomogeneity,

\[
q^{(\nu)}: (x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_{2^{n\nu}} - 1, e^{\frac{2n\nu \pi i}{2}} x_{2^{n\nu}-1+1}, \ldots, e^{\frac{2n\nu \pi i}{2}} x_{2^{n\nu}}, x_{2^{n\nu}+1}, \ldots, x_n).
\]

The group (called geometric symmetry) \( G_X \) consists of elements \( g \notin Q_X \) of the following form

\[
g_s: (x_1, \ldots, x_n) \rightarrow (e^{\frac{2n \alpha_1 \pi i}{2}} x_1, e^{\frac{2n \alpha_2 \pi i}{2}} x_2, \ldots, e^{\frac{2n \alpha_n \pi i}{2}} x_n),
\]

for some \( d > 0 \) and \((\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \). Here we remark that some of \( \alpha_i \) can be zero. The group \( H \) is the non cyclic part of the group \( \text{Aut}(X_s) \).

**Definition 1** In the following decomposition,

\[
\text{Aut}(X_s) \cong Q_X \times G_X \times H,
\]

we call \( Q_X \) (resp. \( G_X \)) the quantum symmetry (resp. geometric symmetry) of \( X_s \).

Further we apply so called Cayley trick to (1.1) to get a polynomial

\[
F(x, s, y) = \sum_{j=1}^{k} y_{2j-1}(f_{2j-1}(x) + s_j) + \sum_{j=1}^{k} y_{2j} f_{2j}(x),
\]

with \( L = n + 3k \) terms. The procedures (2.3), (2.4) below explain why we consider this polynomial \( F(x, s, y) \) to calculate the period integrals associated to \( X_s \). One may consult [13] and [14] for more details on the utility of the Cayley trick in the calculus of period integrals.

To manipulate the polynomial \( F(x, s, y) \) we introduce the notation \( a^\nu := \tau_1 + \cdots + \tau_\nu + 3\nu = b^\nu + 3\nu \). In particular, the \( a^{i-1} + 1 \)-th term of \( F(x, s, y) \) corresponds to \( y_{2i-1} x^{a_{i-1}^{(i)}} \) and the \( a^i - 3 = (a^{i-1} + \tau_i) \)-th term - \( y_{2i-1} x^{a_{i}^{(i)}} \). The \( (a^{i} - 1) \)-th term - \( y_{2i} \prod_{j \in I^{(i)}} x_j \). The \( (a^i - 2) \)-th term - \( y_{2i} \).

From the polynomial \( F(x, s, y) \) we construct a matrix \( L \) that consists of the row vectors \( r \)-th term of which corresponds to the power of the \( r \)-th monomial term present in \( F(x, s, y) \). For example, the row vector \( \tilde{v}_q^{(\nu)} \) corresponds to the row number \( q^{\nu-1} + \nu \) of the matrix \( L \),

\[
\tilde{v}_q^{(\nu)} = (\tilde{v}_q^{(\nu)}, 0, \cdots, 0, 1, 0, \cdots, 0),
\]

\[1 \leq \nu \leq k, 1 \leq q \leq \tau_\nu\].

Next we look at the system of linear equations \( \Xi = (\xi_1, \ldots, \xi_L) \),

\[
i^L \cdot \Xi = (1, \cdots, 1, z_1, \cdots, z_k),
\]

(1.5)
that is equivalent to the relation,
\[ \Xi \equiv \tau L^{-1}, \quad (1, \cdots, 1, z_1, \cdots, z_k). \]

Thus we get linear functions \((\xi_1(z), \cdots, \xi_L(z))\) that will be later denoted by \((\mathcal{L}_1(0, z, 0), \cdots, \mathcal{L}_k(0, z, 0))\) in the notation (2.5) below. These linear functions will play essential role in the calculus of the period integrals.

We define a \((n \times k)\)-matrix \(V^\Lambda\) as follows:
\[ V^\Lambda := (v^{(1)}_{\tau_1+1}, \cdots, v^{(k)}_{\tau_k+1}), \]
where \(v^{(q)}_{\tau_q+1}\) is a \(n\)-row vector that corresponds to \(\text{supp}(f_{2q}) \setminus \{0\}\). By virtue of the quasihomogeneity (1.2) we can define a \(k \times k\) matrix as follows:
\[ \hat{Q} := \begin{pmatrix} Q^{(1)}_1 & \cdots & Q^{(k)}_1 \\ \vdots & \ddots & \vdots \\ Q^{(1)}_k & \cdots & Q^{(k)}_k \end{pmatrix} = \tau L \cdot \begin{pmatrix} \hat{g}^{(1)}_1, \cdots, \hat{g}^{(k)}_1 \end{pmatrix}. \]

For the simplicity of the formulation, we will use a diagonal matrix
\[ G = \text{diag}(g^{(1)}_1, \cdots, g^{(k)}_1, g^{(2)}_1, \cdots, g^{(k)}_2, \cdots, g^{(k)}_k). \]

We introduce a \(n \times n\) matrix:
\[ L^\Lambda := \begin{pmatrix} v^{(1)}_{\tau_1} \\ v^{(1)}_{\tau_2} \\ \vdots \\ v^{(1)}_{\tau_k} \\ v^{(2)}_{\tau_1} \\ \vdots \\ v^{(k)}_{\tau_1} \\ v^{(k)}_{\tau_k} \end{pmatrix}. \]

We construct a matrix \(T^L\) constructed from the transposed matrix \(L^\Lambda\) after some proper permutations of the rows and columns such that each row of \(T^L\) corresponds to a vertex of a polynomial
\[ T F(x, s, y) = \sum_{j=1}^k y_{2j-1}(T f_{2j-1}(x) + s_j) + \sum_{j=1}^k y_{2j} T f_{2j}(x), \]

for the polynomials,
\[ T f_{2q-1}(x) = x^{T v^{(q)}_1} + \cdots + x^{T v^{(q)}_n}, \]
\[ T f_{2q}(x) = \prod_{\ell \in T I^{(q)}} x^\ell + 1, \quad 1 \leq q \leq k. \]

Here we impose the condition
\[ \{\tau_1, \cdots, \tau_k\} = \{|T I^{(1)}|, \cdots, |T I^{(k)}|\} = \{\tilde{\tau}_1, \cdots, \tilde{\tau}_k\} = \{|I^{(1)}|, \cdots, |I^{(k)}|\}. \]
Thus we can define an one to one mapping
\[ \nu : [1, k] \rightarrow [1, k], \]
such that \( |TI(\nu(j))| = \tau(\nu(j)) = \tilde{\tau}_j = |I(j)| \). Further we impose a condition \( \nu^2 = id \) so that \( T(F(x, s, y)) = F(x, s, y) \) holds. In a way parallel to (1.7) – (1.10), we can define the weight system
\[ (\vec{T} \vec{g}^{(1)}, \ldots, \vec{T} \vec{g}^{(k)}), \]
\[ (1.2)^T \quad TQ_j^{(q)} = \langle \vec{T} \vec{v}_r^{(j)}, \vec{T} \vec{g}^{(q)} \rangle, \quad 1 \leq r \leq \tilde{\tau}_q, 1 \leq j, q \leq k. \]

We impose a condition necessary for Calabi-Yau property of \( Y \)
\[ (1.3)^T \quad \sum_{j=1}^k TQ_j^{(q)} = \sum_{i=1}^{\tilde{\tau}_q} Tg_i^{(q)}, \quad 1 \leq q \leq k. \]

It is easy to see that the equations of \( T(1.1) \) define a CI in \( F(\tilde{\tau}_1), \ldots, \tau(\nu(T)), \ldots, g^{(k)} \) with \( Tg_{\tilde{\tau}_q+1} = TQ_j^{(q)} \).

In analogy with the Definition 1 we define the following decomposition
\[ \text{Aut}(Y_s) \cong \mathcal{Q}_Y \times \mathcal{G}_Y \times T\mathcal{H}, \]
for
\[ Y_s = \{ x \in T^n; Tf_1(x) + s_1(Tf_2(x) - 1) = \cdots = Tf_{2k-1}(x) + s_k(Tf_2k(x) - 1) = 0 \}. \]
The quantum symmetry \( \mathcal{Q}_Y \cong \prod_{k=1}^k \mathbb{Z}_{\tau(\nu_1)}, \) with \( \mathcal{T}(\mathcal{Q}) = L.C.M.(TQ_1^{(q)}, \ldots, TQ_{\tilde{\tau}_q}^{(q)}) \) is a cyclic group corresponding to the quasihomogeneity and \( \mathcal{G}_Y \) the remaining cyclic part called geometric symmetry. The group \( T\mathcal{H} \) is the remaining non cyclic part of \( \text{Aut}(Y_s) \).

We introduce matrices analogous to the case of (1.1),
\[ (1.7)^T \quad TV^\Lambda := (t(T \vec{v}^{(\nu(1))}), t(T \vec{v}^{(\nu(2))}), \ldots, t(T \vec{v}^{(\nu(k))})), \]
where \( T\vec{v}^{(\nu(q))} \) is a \( n \)-row vector which corresponds to \( \text{supp}(Tf_{2\nu(q)}) \setminus \{0\} \). More precisely, the \( j \)-th column of the matrix \( TV^\Lambda \) equals to
\[ t(T \vec{v}_{\tau(r)+1}^{(\nu(j))}) = t(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0), \quad \tau(\nu(j)) = \tilde{\tau}_j, \]
here \( b^{(\nu(j))} := \sum_{\tau(r)=1}^{\nu(j)} \tau(r). \)
\[ (1.8)^T \quad T\hat{Q} := \begin{pmatrix} TQ_1^{(1)} & \cdots & TQ_k^{(1)} \\ \vdots & \ddots & \vdots \\ TQ_1^{(k)} & \cdots & TQ_k^{(k)} \end{pmatrix} = t(TV^\Lambda) \cdot \begin{pmatrix} t(T \vec{g}^{(1)}), \ldots, t(T \vec{g}^{(k)}) \end{pmatrix}, \]
\[ (1.9)^T \quad TG = \text{diag}(Tg_1^{(1)}, \ldots, Tg_1^{(2)}, \ldots, Tg_1^{(k)}), \ldots, Tg_1^{(k)}, Tg_1^{(1)}). \]
the Mellin transform
the fibre integral
(2)
the classical HGF of Gauss can be expressed by means of the integral,
the notion of the generalized HGF in the sense of Mellin-Barnes-Pinchere [9]. After this formulation,
without thimble. It is useful to understand the calculus of the Mellin transform in connection with

Finally we remark that due to the property \( \nu^2 = id \) and the definition (1.7)\( ^T \), there exists a
permutation matrix \( \lambda \in SL(n, \mathbb{Z}) \) such that

\[
\lambda \cdot V^\Lambda = T^V^\Lambda, \lambda \cdot T^L_\Lambda = T^L_\Lambda.
\]

2 Mellin transform of period integrals

In this section we review the results on the period integrals to be used for the verification of the
hypothesis in [2] in the subsequent section. See for the detail of proofs [13], [14].

Let us consider the Leray’s coboundary (see [15]) to define the period integral that is equivalent
to the period integral of the variety \( X_\gamma \subset H_n(\mathbb{T}^n \setminus \cup_{i=1}^k \{ x \in \mathbb{T}^n : f_{2i-1}(X) + s_i = 0 \} \cup_{t=1}^l \{ x \in \mathbb{T}^n : f_{2t}(X) = 0 \} ) \) such that \( \Re(f_{2i-1}(X) + s_i) |_{\gamma} < 0, \Re(f_{2t}(X)) |_{\gamma} < 0 \). Further on central object of
our study is the following fibre integral,

\[
I_{x^1, \gamma}(s) = \int_\gamma (f_1(x) + s_1)^{-\zeta_1} \cdots (f_{2k-1}(x) + s_{k-1})^{\zeta_{2k-1}} (f_{2k}(x))^{\zeta_{2k}} \frac{dx}{x^s},
\]

and its Mellin transform,

\[
M_{x^1, \gamma}^\xi(z) := \int_\Pi s^z f_{x^1, \gamma}(s) \frac{ds}{s^1},
\]

for certain cycle \( \Pi \) homologous to \( \mathbb{R}^k \) which avoids the singular loci of \( I_{x^1, \gamma}(s) \) (cf. [11]). Thus
the fibre integral \( I_{x^1, \gamma}^\xi(s) \) is a ramified function on the torus \( \mathbb{T}^k \). We introduce the notation \( \gamma^\Pi := \cup_{(s) \in \Pi} ((\gamma, s)) \). One shall not confuse it with the thimble of Lefschetz, because \( \gamma^\Pi \) is rather a tube
without thimble. It is useful to understand the calculus of the Mellin transform in connection with
the notion of the generalized HGF in the sense of Mellin-Barnes-Pinchere [9]. After this formulation,
the classical HGF of Gauss can be expressed by means of the integral,

\[
_2F_1(\alpha, \beta, \gamma|s) = \frac{1}{2\pi i} \int_{z_0}^{z_0 + i\infty} (-s)^z \frac{\Gamma(z + \alpha)\Gamma(z + \beta)\Gamma(-z)}{\Gamma(z + \gamma)} dz, \quad -\Re \alpha, -\Re \beta < z_0.
\]

We can introduce \( (w_{2k}') \cdots w_{2k''} \) natural quasihomogeneous weight of \( (y_1, \cdots, y_k) \) so that \( F(x, 0, y) \)
of (1.4) gets the quasihomogeneous zero weight with respect to the variables \( (x, y) \). Next we modify
the Mellin transform

\[
M_{x^1, \gamma}^\xi(z) = e(\zeta) \int_{S^{k-1}((w') \times \gamma^\Pi)} \frac{x^i \omega s^{-1} \omega_{\Omega}(\omega \wedge \wedge s)}{(\omega_1(f_1(x) + s_1) + \cdots + \omega_{2k}(f_{2k}(x)))^{i_1 + \cdots + i_{2k} + 2k}} dx
\]
\[ c(\zeta) \int_{\mathbb{R}^+} \sigma^{m_1 + \cdots + m_2 + 2k} \frac{d\sigma}{\sigma} \int_{S^2} \omega^\zeta \Omega_0(\omega) \int_{\mathbb{R}} x^\zeta dx \int_{\Pi} s^\zeta e^{\sigma(f_1(x)+s_1)+\cdots+\omega_k(f_2k(x))} ds \]

with \( c(\zeta) = \frac{\Gamma\left(m_1 + \cdots + m_2 + 2k\right)}{\Gamma\left(m_1\right)} \). Here we made use of notations \( S_{2k-1}^2(\omega'') \) for the set of columns and rows of the matrix \( L \in \mathbb{L} \).

In the above transformation we used a classical interpretation of Dirac’s delta function as a residue:

\[ \int_{\mathbb{R}^+} \gamma \int_{\mathbb{R}^+} e^{\gamma f_j(x) + \sigma} \gamma dy \wedge dx = \frac{\Gamma(\gamma + 1)}{\Gamma(\zeta_j + 1)} \int_{\gamma} (f_j(x) + s_j)^{-\zeta_j - 1} dx. \]

We will rewrite, up to constant multiplication, the expression obtained as a modification of \( M_{\gamma,\zeta}^h(x) \) into the following form,

\[ \int_{(\mathbb{R}^+)^{2k} \times \gamma^n} e^{\Psi(T) x^1 + 1} y^\zeta + 1 s x \frac{dx}{x^1} \wedge \frac{dy}{y^1} \wedge \frac{ds}{s^1} \]

where

\[ \Psi(T) = T_1(X, s, y) + \cdots + T_L(x, s, y) = F(x, s, y), \]

in which each term \( T_i(x, s, y) \) stands for a monomial in variables \( (x, s, y) \) of the phase function (1.4).

We transform the above integral into the following form,

\[ \int_{(\mathbb{R}^+)^{2k} \times \gamma^n} e^{\Psi(T(x,s,y)) x^1 + 1} s^2 y^\zeta + 1 s x \frac{dx}{x^1} \wedge \frac{dy}{y^1} \wedge \frac{ds}{s^1} \]

\[ = (\det L)^{-1} \int_{(\mathbb{R}^+)^{2k} \times \gamma^n} e^{\sum_{a \in \ell} T_a} \prod_{a \in \ell} T_a^{L_a(i,x,z)} \wedge \frac{dT_a}{T_a} \]

\[ = (-1)^{m_1 + \cdots + m_2 + 2k} (\det L)^{-1} \int_{(\mathbb{R}^+)^{2k} \times \gamma^n} e^{\sum_{a \in \ell} T_a} \prod_{a \in \ell} T_a^{L_a(i,x,z)} \wedge \frac{dT_a}{T_a} \]

Here \( L_a(\mathbb{R}^+)^{2k} \times \gamma^n) \) means the image of the chain in \( \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{R}} \times \mathbb{C}^{\mathbb{R}} \) into that in \( \mathbb{C}^{\mathbb{R}} \) induced by the transformation (2.3). We define \( -L_a(\mathbb{R}^+)^{2k} \times \gamma^n) = \{ (-T_1, \cdots, -T_L) \in \mathbb{C}^{\mathbb{R}}; \{T_1, \cdots, T_L\} \in L_q(\mathbb{R}^+)^{2k} \times \gamma^n) \}, \forall T_a > 0, a \in \{1, L\} \). The second equality of (2.4) follows from Proposition 2.1, 3) below that can be proven in a way independent of the argument to derive (2.4). We will denote the set of columns and rows of the matrix \( L \) by \( I \),

\[ I := \{1, \cdots, L\}. \]

Here we remember the condition \( L = n + 3k \) imposed on (1.4).

The following notion helps us to formulate the result in a compact manner.

**Definition 2** A meromorphic function \( g(z) \) is called \( \Delta \)-periodic for \( \Delta \in \mathbb{Z}_{>0} \), if

\[ g(z) = h(e^{2\pi i \zeta_1}, \cdots, e^{2\pi i \zeta_k}), \]

for some rational function \( h(\zeta_1, \cdots, \zeta_k) \).
For the CI (1.1) (i.e. we can construct \( F(x, s, y) \) for which the matrix \( L \) is non-degenerate), we have the following statement.

**Proposition 2.1** 1) For any cycle \( \Pi \in H_0(T^k \setminus S.S.I^\xi_{x,\gamma}(s)) \) the Mellin transform (2.1) can be represented as a product of \( \Gamma \)-function factors up to a \( \Delta \)-periodic function factor \( g(z) \),

\[
M^\xi_{i,\gamma}(z) = g(z) \prod_{\alpha \in I} \Gamma(L_a(i, z, \zeta)),
\]

with

\[
(2.5) \quad L_a(i, z, \zeta) = \left( \sum_{j=1}^n A^a_j(i_j + 1) + \sum_{\ell=1}^k B^a_\ell z_\ell + \sum_{\ell=1}^{2k} D^a_\ell(\zeta_\ell + 1) \right) \Delta, \quad a \in I.
\]

Here the following matrix \( \Delta^{-1}T = (L)^{-1} \) has integer elements,

\[
(2.6) \quad T = (A^a_1, \ldots, B^a_1, \ldots, B^a_k, D^a_1, \ldots, D^a_{2k})_{1 \leq a \leq L},
\]

with \( G.C.D.(A^a_1, \ldots, A^a_n, B^a_1, \ldots, B^a_k, D^a_1, \ldots, D^a_{2k}) = 1 \), for all \( a \in I \). In this way \( \Delta > 0 \) is uniquely determined.

The coefficients of (2.5) satisfy the following properties for each index \( a \in I \):

a) Either \( L_a(i, z, \zeta) = \frac{\Delta}{z_\ell} z_\ell \), i.e. \( A^a_1 = \cdots = A^a_n = 0, B^a_1 = \cdots = B^a_k = 0, B^a_\ell = 1. \)

b) Or \( L_a(i, z, \zeta) = \frac{\Delta}{z_{2\ell-1}} (z_{2\ell-1} + \zeta_{2\ell-1} - z_\ell) \),

c) Or

\[
L_a(i, z, \zeta) = \left( \sum_{j=1}^n A^a_j(i_j + 1) + \sum_{\ell=1}^k B^a_\ell(z_\ell - \zeta_{2\ell-1} - 1) \right) \Delta.
\]

2) For each fixed index \( 1 \leq \ell \leq n, 1 \leq q \leq k \), the following equalities take place:

\[
(2.7) \quad \sum_{a \in I} A^a_\ell = 0, \quad \sum_{a \in I} B^a_q = 0.
\]

3) The following relation holds among the linear functions \( L_a, a \in I \):

\[
\sum_{a \in I} L_a(i, z, \zeta) = \zeta_1 + \cdots + \zeta_{2k} + 2k.
\]

In the sequel, especially from (3.2) of the next section, we will make use of the notation as follows,

\[
(2.8) \quad (w_1^{(a)}, w_2^{(a)}, \ldots, w_n^{(a)}; p_1^{(a)}, \ldots, p_k^{(a)}; q_1^{(a)}, \ldots, q_{2k}^{(a)}) = \left( \frac{A^a_1}{\Delta}, \ldots, \frac{A^a_n}{\Delta}, \frac{B^a_1}{\Delta}, \ldots, \frac{B^a_k}{\Delta}, \ldots, \frac{D^a_1}{\Delta}, \ldots, \frac{D^a_{2k}}{\Delta} \right)_{1 \leq a \leq L}.
\]

In view of the Proposition 2.1, we introduce the subsets of indices \( a \in I = \{1, 2, \ldots, L\} \) as follows.

**Definition 3** The subset \( I^+_q \subset I \) (resp. \( I^-_q, I^0_q \)) consists of the indices \( a \) such that the coefficient \( B^a_q \) of \( L_a(i, z, \zeta) \) (2.5) is positive (resp. negative, zero).

From this proposition we get the following.
Corollary 2.2 The integral \( I_{\xi_{\varphi_1}}^\gamma(s) \) satisfies the hypergeometric system of Horn type as follows:

\[
L_q(\varphi_s,s)I_{\xi_{\varphi_1}}^\gamma(s) := \left[ P_{q,1}(\varphi_s,\zeta) - s^q Q_{q,1}(\varphi_s,\zeta) \right] I_{\xi_{\varphi_1}}^\gamma(s) = 0, \quad 1 \leq q \leq k
\]

with

\[
P_{q,1}(\varphi_s,\zeta) = \prod_{a \in I_q^+} \prod_{j=0}^{B_q^n-1} (\mathcal{L}_a(i, -\varphi_s, \zeta) + j),
\]

\[
Q_{q,1}(\varphi_s,\zeta) = \prod_{\bar{a} \in I_q^-} \prod_{j=0}^{-B_q^n-1} (\mathcal{L}_{\bar{a}}(i, -\varphi_s, \zeta) + j),
\]

where \( I_q^+, I_q^- \), \( 1 \leq q \leq k \) are the sets of indices defined in Definition 3.

The degree of two operators \( P_{q,1}(\varphi_s,\zeta) \), \( Q_{q,1}(\varphi_s,\zeta) \) are equal. Namely,

\[
deg P_{q,1}(\varphi_s,\zeta) = \sum_{a \in I_q^+} B_q^n = - \sum_{\bar{a} \in I_q^-} B_q^n = deg Q_{q,1}(\varphi_s,\zeta).
\]

## 3 Hypothesis by Berglund, Candelas et alii

In this section, we apply the results from §2 on the period integrals to a class of Calabi-Yau varieties (1.1) studied in the framework of Landau-Ginzburg vacua theory.

Our aim is to find out sufficient conditions so that the (Mellin transform of) the period integrals on \( X_s \) can be expressed by means of quasihomogeneous weight data of \( Y \). The main theorem of this section is Theorem 3.1.

Before proceeding to the proof of the Theorem 3.1, we write down concretely the matrix \( L \) in taking (1.1), (1.4) into account, and we have the following row vectors of \( L \):

\[
\bar{v}^{(\nu)}_q = (v^{(\nu)}_{q,1}, v^{(\nu)}_{q,2}, \ldots, v^{(\nu)}_{q,n}, 0, 0, 0, 0, 0, \ldots, 0),
\]

\( 1 \leq \nu \leq k, 1 \leq q \leq \tau_{\nu} \). In using the notations (2.8), §2 for the column vectors of the matrix \( L^{-1} \), one can deduce the following system for each \( \nu \) and \( q \):

\[
v^{(\nu)}_{q,1} w_1^{(a_{\nu})} + v^{(\nu)}_{q,2} w_2^{(a_{\nu})} + \cdots + v^{(\nu)}_{q,n} w_n^{(a_{\nu})} = -1 \text{ for } a_{\nu}^{\nu} + 1 < q < a_{\nu}^{\nu} + \tau_{\nu},
\]

\[= 0 \text{ otherwise.}
\]

\[
v^{(1)}_{1,j} P_q^{(1)} + v^{(2)}_{2,j} P_q^{(2)} + \cdots + v^{(1)}_{r_1,j} P_q^{(r_1)} + v^{(2)}_{r_2,j} P_q^{(r_2+4)} + \cdots + v^{(2)}_{r_2,j} P_q^{(r_2+r_2+3)} + \cdots + v^{(k)}_{r_k,j} P_q^{(\delta_{\nu}^{\nu}-3)}
\]

\[= -1 \text{ for } j \in I^{(q)},
\]

\[= 0 \text{ for } j \in I^{(q)}, r \neq q.
\]

Here we shall remark that the system (3.2) for a fixed \( \nu \) (resp. (3.3) for a fixed \( q \)) consists of \( n \)-linear independent equations with respect to unknowns \( \{ w_1^{(a_{\nu})}, \ldots, w_n^{(a_{\nu})} \} \), (resp. \( \{ P_q^{(j)} : j \in I_q \} \)).
Here we made use of the notation $I_A$ the set of indices $\{1, \ldots, L\} \setminus \cup_{\nu=1}^k \{a^{\nu-2}, a^{\nu-1}, a^{\nu}\}$. One sees other necessary conditions on $w_j^{(a^{\nu-1})}$,

$$\sum_{j \in I^{(\nu)}} w_j^{(a^{\nu-1})} = 1,$$

$$\sum_{j \in I^{(\nu)}} w_j^{(a^{\nu})} = -1.$$  

This can be seen from the fact that the product of the $(a^{\nu-1})$–th column of the matrix $L^{-1}$ with the $(a^{\nu-1})$–th row of $L$ is equal to 1. On the other hand $n+2\nu$–th column of $L$ with the $j$–th row $(1 \leq j \leq n+2)$ of $L^{-1}$ is equal to 0 which entails $w_j^{(a^{\nu-1})} + w_j^{(a^{\nu})} = 0$. In addition to that, we see

$$\sum_{j \in I^{(\nu)}} w_j^{(a^{\nu-1})} = \sum_{j \in I^{(\nu)}} w_j^{(a^{\nu})} = 0,$$

for $\nu' \neq \nu$.

In this situation we deduce the following system from (3.3),

$$(3.6) \quad \begin{pmatrix} \vec{P}_1 \\ \vec{P}_2 \\ \vdots \\ \vec{P}_k \end{pmatrix} \cdot L_A = - t V^A,$$

with $\vec{P}_q = (p_q^{(1)}, p_q^{(2)}, \ldots, p_q^{(1)}, \ldots, p_q^{(n+3k-3)})$. If we denote by $\vec{L}_j$ the $j$–th column vector of the matrix $L_A$, we have the following equation derived directly from (3.6):

$$\begin{pmatrix} z_1 \vec{P}_1 + z_2 \vec{P}_1 + \cdots + z_k \vec{P}_k, L_j \end{pmatrix} + z_q = 0 \text{ if } j \in I^{(q)}.$$

In view of (1.12), we get

$$(3.8) \quad (T L_A)^{-1} T V^A = (L_A)^{-1} V^A,$$

that yields

$$(3.9) \quad \begin{pmatrix} T \vec{g}^{(1)} \\ \vdots \\ T \vec{g}^{(k)} \end{pmatrix} \cdot T G^{-1} \cdot (T L_A)^{-1} T V^A$$

where we used the notation $\vec{a}_q := \sum_{\nu=1}^q (\vec{r}_\nu + 3)$. Let us introduce a permutation matrix $\nu^* \in SL(k, \mathbb{Z})$ whose $j$–th column equals to

$$(\nu(j))^{-1} \cdot \nu(j)^{n-(\nu(j))} t(0, \ldots, 0, 1, 0, \ldots, 0).$$
The last expression (3.9) turns out to be $-\nu^*$. This fact can be seen by the relation $L \cdot L^{-1} = \text{id}_L$ and the Proposition 2.1, a, b, c.

We can define a set of indices $T I_A$ for (1.4)$^T$ analogous to $I_A$. We have the indices of $I_A$, $1 = j_1 < \cdots < j_n = L - 4$ and those of $T I_A$, $1 = i_1 < \cdots < i_n = L - 4$. In this situation we consider two conditions on the CI (1.1),

$$
(3.10)
\begin{pmatrix}
T \xi_{i_1}(z) \\
\vdots \\
T \xi_{i_n}(z)
\end{pmatrix}
= G \cdot V^A \cdot 
\begin{pmatrix}
T \xi^{(1)}(z) \\
\vdots \\
T \xi^{(k)}(z)
\end{pmatrix},
$$

for linear functions $(T \xi^{(1)}(z), \cdots, T \xi^{(k)}(z))$ and

$$(3.10)^T
\begin{pmatrix}
\xi_{j_1}(z) \\
\vdots \\
\xi_{j_n}(z)
\end{pmatrix}
= T \rho \cdot T V^A \cdot 
\begin{pmatrix}
\xi^{(1)}(z) \\
\vdots \\
\xi^{(k)}(z)
\end{pmatrix},
$$

for possibly another $k$–tuple of linear functions $(\xi^{(1)}(z), \cdots, \xi^{(k)}(z))$.

Due to the condition (1.11) on the set of indices $T I_A$, we have for some permutation matrices $\rho, T \rho \in SL(n, Z)$,

$$
\rho \cdot V^A = G^{-1} \cdot \left( t(g^{(1)}), \cdots, t(g^{(k)}) \right),
$$

$$
T \rho \cdot T V^A = T G^{-1} \cdot \left(t(T g^{(1)}), \cdots, t(T g^{(k)})\right).
$$

On these $\rho$ and $T \rho$, we impose the following conditions,

$$(3.11)
G \cdot \rho = ^t (G \cdot \rho),
$$

$$(3.11)^T
T G \cdot T \rho = ^t (T G \cdot T \rho).
$$

Under these conditions we formulate the following Theorem that verifies an hypothesis proposed by [2], §3.3 under certain conditions.

**Theorem 3.1** The Mellin transform $M_{0, \gamma}^0(z)$ of the period integral $I_{0, \gamma}^0(s)$ for the Calabi-Yau CI (1.1) has the following form up to a $\Delta$–periodic function in the sense of Definition 2, if it satisfies the conditions (3.10), (3.10)$^T$, (3.11), (3.11)$^T$,

$$(3.12)
M_{0, \gamma}^0(z(\xi)) = \prod_{\nu=1}^k \prod_{j=1}^{r_\nu} \frac{\Gamma(T g^{(\nu)}_j \xi^{(\nu)})}{\Gamma(\sum_{\nu=1}^k Q^{(\nu)}_q \xi^{(\nu)})}.
$$

Here $z(\xi) = (z_1(\xi), \cdots, z_k(\xi))$ is a $k$–tuple of linear functions in variables $\xi = (\xi^{(1)}, \cdots, \xi^{(k)})$ defined by the relation (3.10)$^T$.

In a symmetric way the Mellin transform $M_{0, \tau \gamma}^0(z(T \xi))$ for the Calabi-Yau CI, (1.1)$^T$ admits an expression as follows up to a $\Delta$–periodic function,

$$(3.12)^T
M_{0, \tau \gamma}^0(z(T \xi)) = \prod_{\nu=1}^k \prod_{j=1}^{r_\nu} \frac{\Gamma(g^{(\nu)}_j T \xi^{(\nu)})}{\Gamma(\sum_{\nu=1}^k Q^{(\nu)}_q T \xi^{(\nu)})}.
$$

The functions $T \xi^{(\nu)}$ are defined by the relation (3.10).

To prove the Theorem we prepare the following lemma.
Lemma 3.2 Under the conditions imposed on (1.1) in the theorem 3.1, the Mellin transform \( M_{0,\gamma}^0(z) \) of the period integral \( I_{\nu,\gamma}^0(s) \) for the CI (1.1) admits up to a \( \Delta \)-periodic function (in the sense of Definition 2) an expression as follows,

\[
M_{0,\gamma}^0(z) = \prod_{i=1}^{k} \Gamma(z_i) \prod_{j \in I_\lambda} \Gamma(\xi_j(z)).
\]

Proof It is necessary to show that special solutions of the system (1.5) satisfy,

\[
\xi_{a \nu-2}(z) = \xi_{a \nu-1}(z) = z_\nu, \quad \xi_{a \nu}(z) = 1 - z_\nu.
\]

To see this, we remark that the system below is a direct consequence of the relation \( LL^{-1} = id_L \).

\[
\sum_{i=1}^{n} v_{(a \nu)}^{(\nu,i)} w_i^{(a \nu)} + q_{2 \mu - 1}^{(a \nu)} = 0, \quad a \nu - 1 \leq q \leq a \nu - 1 + \tau_\mu, \quad \mu \in [1, k].
\]

The last equality can be deduced from (3.4) and (3.5). Let \( \mu(j) \in [1, k] \) be an index such that \( j \) belongs to \( I^{(\mu(j))} \). Then we have

\[
\sum_{i \in I_\lambda(j)} v_{(a \nu)}^{(\mu(j),i)} p^{(a \nu)}_{i} + p^{(a \nu)}_{q} = 0,
\]

for \( q \in [1, k] \).

We get the following relation,

\[
q_{2 \nu - 1}^{(a \nu)} = q_{2 \nu}^{(a \nu)} = 1,
\]

\[
q_{2 \nu - 1}^{(a \nu')} = q_{2 \nu}^{(a \nu')} = 0, \quad \text{for } \nu' \neq \nu.
\]

The equality \( q_{2 \nu}^{(a \nu')} = 0 \) can be derived from the fact that the product of the \( a \nu - 1 \)-th column of \( L^{-1} \) with the \( a \nu' - 1 \)-th row of \( L \) is equal to 0. On the other hand, the product of the \( (n + 2 \nu) \)-th column of \( L \) with the \( a \nu' - 1 \)-th row of \( L^{-1} \) is equal to \( q_{2 \nu - 1}^{(a \nu')} + q_{2 \nu}^{(a \nu')} = 0 \). This proves (3.17). The product of the \( a \nu - 1 \)-th column of \( L^{-1} \) with the \( a \nu' - 1 \)-th row of \( L \) is equal to \( q_{2 \nu - 1}^{(a \nu')} + p_{\nu}^{(a \nu')} = 0 \). One deduces the equality \( p_{\nu}^{(a \nu')} = 0 \) for \( \nu \neq \nu' \). The product of \( (a \nu - 1) \)-th column of \( L^{-1} \) with the \( (a \nu' - 2) \)-th row of \( L \) is equal to \( q_{2 \nu - 1}^{(a \nu')} + p_{\nu}^{(a \nu' - 1)} = 0 \). As a consequence, we have

\[
p_{\nu}^{(a \nu' - 1)} = 0 \quad \text{for } \nu \neq \nu',
\]

\[
p_{\nu}^{(a \nu' - 1)} = -q_{2 \nu - 1}^{(a \nu')} = 1, \quad p_{\nu}^{(a \nu')} = -1,
\]

because \( p_{\nu}^{(a \nu')} + p_{\nu}^{(a \nu' - 1)} = 0 \). Additionally we see that

\[
q_{\nu}^{(a \nu' - 1)} = 0, \quad r \neq 2 \nu - 1.
\]
In summary, by virtue of (3.14), (3.16), (3.18),
\[ \xi_{a^\nu}(z) = \sum_{i=1}^{n} w_i^{(a^\nu)} + \sum_{r=1}^{2k} q_r^{(a^\nu)} + \sum_{q=1}^{k} p_q^{(a^\nu)} z_q = -1 + 1 + 1 - z_\nu = 1 - z_\nu. \]

On the other hand (3.14), (3.19), (3.18), yield
\[ \xi_{a^\nu-1}(z) = 1 + 0 - 1 + z_\nu = z_\nu. \]

As for the function \( \xi_{a^\nu-2}(z) \) it is easy to see that all elements of the \((a^\nu - 2)\)th column of \( L^{-1} \) consist of zeros except the \((n + 2k + \nu)\)th element which is equal to 1.

We thus have the equality,
\[ M_{0,\gamma}(z) = \prod_{i=1}^{k} \Gamma(z_i) \Gamma(1 - z_i) \prod_{j \in \Lambda} \Gamma(\xi_j(z)) = \prod_{i=1}^{k} \frac{\pi}{\sin \pi z_i} \Gamma(z_i) \prod_{j \in \Lambda} \Gamma(\xi_j(z)). \]

Q.E.D.

Proof of the Theorem 3.1 Our main task is to show the following relation,

\[ (3.20) \quad \nu^* \cdot \begin{pmatrix} 1 - z_1 \\ \vdots \\ 1 - z_k \end{pmatrix} = \hat{T} \begin{pmatrix} \xi^{(1)}(z) \\ \vdots \\ \xi^{(k)}(z) \end{pmatrix}, \]

for the permutation matrix \( \nu^* \in SL(k, \mathbb{Z}) \) introduced just after the formula (3.9). To do this, first of all we modify the relation,

\[ (3.21) \quad \hat{T} \hat{Q} = \begin{pmatrix} T \hat{g}^{(1)} \\ \vdots \\ T \hat{g}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} T \rho^{-1} \cdot T G^{-1} \cdot (T \hat{g}^{(1)}), \ldots, (T \hat{g}^{(k)}) \end{pmatrix}, \]

that can be derived from (1.8)\(^T\). Here we remark that after the definition of \( T \rho \in SL(n, \mathbb{Z}) \) introduced just before (3.11) the following relation holds,

\[ (3.22) \quad T V^\Lambda = (T \rho)^{-1} . T G^{-1} \cdot (T \hat{g}^{(1)}), \ldots, (T \hat{g}^{(k)}). \]

From this relation and (3.11)\(^T\) we see that

\[ (3.23) \quad \begin{pmatrix} T \hat{g}^{(1)} \\ \vdots \\ T \hat{g}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} T \rho^{-1} \cdot T G^{-1} \cdot (T \hat{g}^{(1)}), \ldots, (T \hat{g}^{(k)}) \end{pmatrix} = \begin{pmatrix} T \hat{g}^{(1)} \\ \vdots \\ T \hat{g}^{(k)} \end{pmatrix} . T V^\Lambda. \]

By virtue of the condition (3.10)\(^T\), the following equality holds

\[ (3.24) \quad \begin{pmatrix} T \hat{g}^{(1)} \\ \vdots \\ T \hat{g}^{(k)} \end{pmatrix} . T V^\Lambda \cdot \begin{pmatrix} \xi^{(1)}(z) \\ \vdots \\ \xi^{(k)}(z) \end{pmatrix} = \begin{pmatrix} T \hat{g}^{(1)} \\ \vdots \\ T \hat{g}^{(k)} \end{pmatrix} . T G^{-1} \cdot \begin{pmatrix} \xi_{j_1}(z) \\ \vdots \\ \xi_{j_n}(z) \end{pmatrix}. \]

The combination of (3.6) and (3.8) entails that the expression (3.24) is equal to

\[ - \begin{pmatrix} T \hat{g}^{(1)} \\ \vdots \\ T \hat{g}^{(k)} \end{pmatrix} . T G^{-1} \cdot (L_\Lambda)^{-1} V^\Lambda \cdot \begin{pmatrix} 1 - z_1 \\ \vdots \\ 1 - z_k \end{pmatrix} = - \begin{pmatrix} T \hat{g}^{(1)} \\ \vdots \\ T \hat{g}^{(k)} \end{pmatrix} . T G^{-1} \cdot (T L_\Lambda)^{-1} T V^\Lambda \cdot \begin{pmatrix} 1 - z_1 \\ \vdots \\ 1 - z_k \end{pmatrix}. \]
From (3.9) it follows that the last expression equals to
\[ \nu^* \cdot \left( \begin{array}{c} 1 - z_1 \\ \vdots \\ 1 - z_k \end{array} \right). \]
This means (3.20) and consequently,
\[ \sum_{\nu=1}^{k} TQ_{q}^{(\nu)} \xi^{(\nu)}(z) = 1 - z_q, \quad 1 \leq q \leq k. \]
From the last equality we can directly derive the relation to be proved
\[ \prod_{j=1}^{\nu} \Gamma(\xi_j(z)) = \prod_{\nu=1}^{k} \tau_{\nu} \prod_{j=1}^{\nu} \Gamma( Tg_{j}^{(\nu)} \xi^{(\nu)}), \]
which proves (3.12). The formula (3.12)\(^T\) can be proven in a parallel way. Q.E.D.

4 Duality between monodromy data and Poincaré polynomials

In connection with the mirror symmetry, we consider the structural algebra of the CI (1.1) of dimension \( n - k \) denoted by \( X = X_1 \),
\[ A_X := \frac{C[x]}{(f_1 + f_2 - 1, \cdots f_{2k-1} + f_{2k} - 1)C[x]}, \]
and a natural filtration on it,
\[ A_X^j := \bigoplus C\{ x^\alpha \in A_X; (\alpha, \bar{g}^{(1)}) = j_1, \cdots, (\alpha, \bar{g}^{(k)}) = j_k \}, \]
with the Poincaré polynomial,
\[ P_A_X(\lambda) = \sum_{j \in \mathbb{Z}_{>0}} \dim(A_X^j) \lambda_1^{j_1} \cdots \lambda_k^{j_k}. \]
In an analogous way, we define corresponding notions of the CI \( Y \) defined by (1.1)\(^T\),
\[ A_Y := \frac{C[x]}{(f_1 + f_2 - 1, \cdots f_{2k-1} + f_{2k} - 1)C[x]}, \]
\[ A_Y^j := \bigoplus \{ x^\alpha \in A_Y; (\alpha, T\bar{g}^{(1)}) = j_1, \cdots, (\alpha, T\bar{g}^{(k)}) = j_k \}, \]
Transposition

\[ P_{A_Y}(\lambda) = \sum_{j \in \mathbb{Z}_{\geq 0}} \dim(A_Y^j)\lambda^j. \]

In this situation the classical result due to [6] gives us,

\[ PA_X(\lambda) = \frac{\prod_{q=1}^k \prod_{i=1}^k (1 - \lambda^{Q^{(r)}} q)}{\prod_{q=1}^k \prod_{i=1}^k (1 - \lambda^{g^{(r)}} q)}, \quad PA_Y(\lambda) = \frac{\prod_{q=1}^k \prod_{i=1}^k (1 - \lambda^{T Q^{(r)}} q)}{\prod_{q=1}^k \prod_{i=1}^k (1 - \lambda^{T g^{(r)}} q)}. \]

Further we introduce the variables \((t_1, \ldots, t_k) \in \mathbb{T}^k\) such that

\[ \prod_{i=1}^k s^{T Q^{(q)}}_i = t_q, \quad 1 \leq q \leq k. \]

If we assume that \( \text{rank} \ T \tilde{Q} = k \), this equation is always solvable with respect to the variables \( s = s(t) \). We consider the Mellin inverse transform of \( M^0_{\tilde{Q}}(z(\xi)) \) associated to the CI (1.1),

\[ U_\alpha(s) = \int \tilde{\Pi}_\alpha \prod_{i=1}^k \prod_{j=1}^k \bigg( \frac{\Gamma(T g_j^{(r)}(\xi)^{(r)}(z))}{\Gamma(T Q^{(r)} q(\xi)^{(r)}(z))} \bigg)^{s-\xi} d\xi, \]

where \( \tilde{\Pi}_\alpha \subset \mathbb{R}^k \) is a cycle avoiding the singular loci of the integrand. It is easy to check

\[ U_\alpha(s(t)) = \text{det}(T \tilde{Q})^{-1} \int_{\tilde{T} \tilde{Q}^{-1}(\tilde{\Pi}_\alpha)} \prod_{i=1}^k \prod_{j=1}^k \bigg( \frac{\Gamma(T g_j^{(r)}(\xi)^{(r)}(z))}{\Gamma(T Q^{(r)} q(\xi)^{(r)}(z))} \bigg)^{s-\xi} d\xi^1 \cdot \cdots \cdot d\xi^k. \]

The following system of differential equations annihilates the inverse Mellin transform \( U_\alpha(s(t)) \),

\[ L_\nu(t_\nu, \vartheta_\nu) U_\alpha(s(t)) = 0, \quad 1 \leq \nu \leq k, \]

where

\[ L_\nu(t_\nu, \vartheta_\nu) = \left( \prod_{j=1}^r g_j^{(r)}, 1, \ldots, 1 \right) \prod_{q=1}^k \prod_{r=0}^{T Q^{(r)} - 1} \left( - T g_j^{(r)}(\vartheta_\nu) + r \right) - \nu \prod_{q=1}^k \prod_{r=0}^{T Q^{(r)} - 1} \left( \sum_{\mu=1}^k T Q^{(r)} q(\vartheta_\nu) + r \right), \quad 1 \leq \nu \leq k. \]

We denote by \( \chi_\nu \) the degree of the operator \( L_\nu(t, \vartheta) : \chi_\nu = \sum_{q=1}^k T Q^{(r)} q + \sum_{r=1}^k T g_j^{(r)} = T g^{(r)} q_k + 1 \) that has already been introduced in (1.3). \( \check{L}_\nu(t_\nu, \vartheta_\nu) \) as follows,

\[ \check{L}_\nu(t_\nu, \vartheta_\nu) := \prod_{j=1}^r g_j^{(r)}, 1, \ldots, 1 \prod_{q=1}^k \prod_{r=0}^{T Q^{(r)} - 1} \left( - T g_j^{(r)}(\vartheta_\nu) + r \right) - \nu \prod_{q=1}^k \prod_{r=0}^{T Q^{(r)} - 1} \left( T Q^{(r)} q(\vartheta_\nu) + r \right). \]

On the \( \chi_\nu \) dimensional solution space of the operator \( \check{L}_\nu(t_\nu, \vartheta_\nu) \), we consider the monodromy \( M^{(0)}_\nu \in GL(\chi_\nu, \mathbb{C}) \) (resp. \( M^{(\infty)}_\nu \in GL(\chi_\nu, \mathbb{C}) \)) around the point \( t_\nu = 0 \) (resp. \( t_\nu = \infty \)). Then we have the following characteristic polynomials of the monodromy that can be easily calculated from the expression \( \check{L}_\nu(t_\nu, \vartheta_\nu) \),

\[ \text{det}(M^{(\infty)}_\nu - \lambda_\nu \cdot \text{id}_{\chi_\nu}) = \prod_{\nu=1}^k (1 - \lambda^{T Q^{(\nu)} q}), \quad \text{det}(M^{(0)}_\nu - \lambda_\nu \cdot \text{id}_{\chi_\nu}) = \prod_{\nu=1}^k (1 - \lambda^{T g^{(\nu)} q}). \]
As a consequence the rational function defined by
\[ M_X(\lambda_1, \ldots, \lambda_k) := \prod_{q=1}^{k} \frac{\det(M_q^{(\infty)}) - \lambda_q \cdot id_{X_q}}{\det(M_q^{(0)}) - \lambda_q \cdot id_{X_q}} \]
has a form

\[ M_X(\lambda_1, \ldots, \lambda_k) = \frac{\prod_{q=1}^{k} \prod_{\nu=1}^{k} (1 - \lambda_q^\nu \cdot Q^{(q)}_\nu)}{\prod_{q=1}^{k} \prod_{\tau=1}^{k} (1 - \lambda_q^\tau \cdot g^{(q)}_\tau)} \tag{4.3} \]

For the rational function \( M_Y(\lambda_1, \ldots, \lambda_k) \) defined in a parallel way to the function \( M_X(\lambda_1, \ldots, \lambda_k) \), we have
\[ M_Y(\lambda_1, \ldots, \lambda_k) = \frac{\prod_{\nu=1}^{k} \prod_{\tau=1}^{k} (1 - \lambda_q^\nu \cdot Q^{(q)}_\nu)}{\prod_{\tau=1}^{k} \prod_{\nu=1}^{k} (1 - \lambda_q^\nu \cdot g^{(q)}_\tau)} \]

Let \( \bar{Y} \) be the compactification of \( CI \subset \mathbb{T}^n \) in the product of quasihomogeneous projective spaces \( \mathbb{P} := \mathbb{P}^{(r_1)}(T_{g_1}, \ldots, T_{g_{r_1+1}}) \times \cdots \mathbb{P}^{(r_k)}(T_{g_1}, \ldots, T_{g_{r_k+1}}) \). The coherent sheaf on \( \mathbb{P} \) \( \mathcal{O}_\mathbb{P}(\zeta) \), \( \zeta = (\zeta_1, \ldots, \zeta_k) \) is defined by sections on the open set \( U_I = \{x \in \mathbb{C}^n; x_i \neq 0, i \in I\} \) that are given by
\[ \Gamma(U_I, \mathcal{O}_\mathbb{P}(\zeta)) := \oplus \mathbb{C}\{x^\alpha; \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n, \alpha_i \geq 0 \text{ for } i \notin I, (T_{g_i}^{(q)}, \alpha) = \zeta_q, 1 \leq q \leq k\}. \]

We define the coherent sheaf \( \mathcal{O}_Y(\zeta) \) by sections on the open set \( U_I \cap Y \),
\[ \Gamma(U_I, \mathcal{O}_Y(\zeta)) := \oplus \mathbb{C}\{x^\alpha \in A_Y; \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n, \alpha_i \geq 0 \text{ for } i \notin I, (T_{g_i}^{(q)}, \alpha) = \zeta_q, 1 \leq q \leq k\}. \]

We introduce the Euler characteristic for this sheaf,
\[ \chi(\mathcal{O}_Y(\zeta)) := \sum_{i=0}^{n-k} (-1)^i \dim H^i(\mathcal{O}_Y(\zeta)). \]

After [6, 7] the Poincaré polynomial of the Euler characteristic,
\[ P\mathcal{O}_Y(t_1, \ldots, t_k) := \sum_{\zeta \in (\mathbb{Z}_{\geq 0})^k} \chi(\mathcal{O}_Y(\zeta)) t_1^{\zeta_1} \cdots t_k^{\zeta_k} \]
admits an expression as follows,

\[ P\mathcal{O}_Y(t_1, \ldots, t_k) = \frac{\prod_{q=1}^{k} \prod_{\nu=1}^{k} (1 - t_q^{\nu} \cdot Q^{(q)}_\nu)}{\prod_{\nu=1}^{k} \prod_{\tau=1}^{k} (1 - t_q^{\tau} \cdot g^{(q)}_\tau)} \tag{4.4} \]

For the sheaf \( \mathcal{O}_X(\zeta) \) defined analogously to \( \mathcal{O}_Y(\zeta) \), we consider the Poincaré polynomial of the Euler characteristics
\[ P\mathcal{O}_X(t_1, \ldots, t_k) := \sum_{\zeta \in (\mathbb{Z}_{\geq 0})^k} \chi(\mathcal{O}_X(\zeta)) t_1^{\zeta_1} \cdots t_k^{\zeta_k}. \]

We have the following expression after [6] and [7],
\[ P\mathcal{O}_X(t_1, \ldots, t_k) = \frac{\prod_{q=1}^{k} \prod_{\nu=1}^{k} (1 - t_q^{\nu} \cdot Q^{(q)}_\nu)}{\prod_{\nu=1}^{k} \prod_{\tau=1}^{k} (1 - t_q^{\tau} \cdot g^{(q)}_\tau)}. \]
If we compare (4.1), (4.3) and (4.4) we get the following statement.

**Theorem 4.1** For the Calabi-Yau CI’s $X$ and $Y$ defined by (1.1) and $(1.1)^T$ we have the following relations,

$$M_X(\lambda_1, \cdots, \lambda_k) = P\mathcal{O}_Y(\lambda_1, \cdots, \lambda_k),$$

$$M_Y(\lambda_1, \cdots, \lambda_k) = P\mathcal{O}_X(\lambda_1, \cdots, \lambda_k),$$

if they satisfy sufficient conditions of Theorem 3.1.

In this situation one can easily derive the existence of the following symmetry from the Theorem 3.1.

**Theorem 4.2** Assume that $X$, (1.1) and $Y$, $(1.1)^T$ satisfy all conditions imposed on them in Theorem 3.1, then there is a symmetry between geometric symmetry and quantum symmetry as follows.

$$\mathcal{Q}_X \cong \prod_{q=1}^{k} Z_{\bar{Q}^{(q)}}, \cong \mathcal{G}_Y,$$

$$\mathcal{Q}_Y \cong \prod_{q=1}^{k} Z_{\bar{Q}^{(q)}}, \cong \mathcal{G}_X.$$

**Proof** The isomorphism $\mathcal{Q}_X \cong \prod_{q=1}^{k} Z_{\bar{Q}^{(q)}}$ and its analogy on $Y_s$ is clear from the quasihomogeneity of the systems (1.1) and $(1.1)^T$. The existence of the cyclic action $\prod_{q=1}^{k} Z_{\bar{Q}^{(q)}}$ on $X_s$ can be read off from the monodromy actions of $M_q^{(\infty)}$, $1 \leq q \leq k$ on the solutions to the system (4.2) that are defined along vanishing cycles on $X_s$. Here we remark that the monodromy group is a subgroup of $\text{Aut}(X_s)$ in view of the integer power transform from $(s_1, \cdots, s_k)$ to $(t_1, \cdots, t_k)$ defined just after (4.1). We remark here that the monodromy actions of $M_q^{(\infty)}$, $1 \leq q \leq k$ consist part of the cyclic action $\prod_{q=1}^{k} Z_{\bar{Q}^{(q)}}$ because every $T_{g_{j}^{(q)}}$, $j = 1, \cdots, \hat{r}_q$ divides $\bar{T_{\bar{Q}^{(q)}}}$. As for the monodromy of the solution to the system (4.2)’ an analogous argument holds. Q.E.D.

**Remark 1** As we have not calculated the global monodromy of the integrals $U_n(s(t))$, (4.3), it is not proper to talk about the existence of a mirror symmetry between $X$ and $Y$. We gathered, however, the monodromy data which correspond to certain limit of the values $t_\ell \to 0, \ell \neq \nu$. Roughly speaking, this procedure can be interpreted as the selection of a special face of the Newton polyhedron $\Delta(F(x, s, y))$ (1.4) for the calculus of the monodromy data.

In the article [5], authors studied the period integral of $X$ at certain limit of the parameter values and they deduced informations on the analogies Gromov-Witten invariants of $Y$. They call this duality “local mirror symmetry”. It is probable that our Theorem 4.1 is one of numerous aspects of the local mirror symmetry.

## 5 Dual nef-partition interpretation

In this section we show that under certain condition the transposition mirror construction corresponds to the notion of the dual nef-partion due to L.Borisov [1].

First of all we consider the following set of vectors defined after (1.1), (1.2).

$$\bar{v}_1^{(q)} - \bar{v}_{\hat{r}_q+1}^{(q)}$$

$$\vdots$$

$$\bar{v}_{\hat{r}_q}^{(q)} - \bar{v}_{\hat{r}_q+1}^{(q)}$$

(5.1) $1 \leq q \leq k$
This gives rise to a partition of \( I_\Lambda \) with \( \sharp I_\Lambda = n \). We set

\[
\Delta_q := \text{convex hull of } (\{0\} \cup \bigcup_{j=1}^{\tau_q} \{\vec{v}_j^{(q)} - \vec{v}_{\tau_q+1}^{(q)}\}).
\]

Further on we impose the following condition on the Minkowski sum of \( \Delta_q \),

\[
dim(\Delta_1 + \cdots + \Delta_k) = n - k.
\]

We introduce the \((n - k)\) dimensional integral lattice

\[
V_Z = \{x \in \mathbb{Z}^n; \langle x, \vec{g}^{(q)} \rangle = 0, 1 \leq q \leq k\}.
\]

After this notation each \( \Delta_q \) is located on \( V_R = V_Z \times \mathbb{R} \cong \mathbb{R}^{n-k} \). Consequently \( \Delta_1 + \cdots + \Delta_k \subset V_R \).

There exists a piecewise linear function \( \phi_q \) such that \( \phi_q(y_1 + y_2) \leq \phi_q(y_1) + \phi_q(y_2) \). Namely we shall define it as follows,

\[
\phi_q(y) = -\min_{x \in \Delta_q} \langle x, y \rangle.
\]

We construct a set of polyhedra dual to the set \( \{\Delta_1, \cdots, \Delta_k\} \),

\[
\Delta_q^* = \{\vec{m} \in \mathbb{R}^n; (5.5)_\ell, (5.5)'_\ell, 1 \leq \ell \leq k\}.
\]

The conditions \((5.5)_\ell, (5.5)'_\ell\) look like the following,

\[
\langle \vec{m}, \vec{v}_j^{(q)} - \vec{v}_{\tau_q+1}^{(q)} \rangle \geq -1, 1 \leq j \leq \tau_q.
\]

\[
\langle \vec{m}, \vec{v}_i^{(q)} - \vec{v}_{\tau_q+1}^{(q)} \rangle \geq 0, 1 \leq i \leq \tau_q, q \neq \ell.
\]

We can construct the set \( \{\Delta_1^*, \cdots, \Delta_k^*\} \) by means of the vertices vectors \( \vec{m}_1^{(q)}, \cdots, \vec{m}_\tau^{(q)} \), \( 1 \leq \ell \leq k \) which are defined by the following set of equalities and inequalities.

\[
\langle \vec{m}_r^{(q)}, \vec{v}_j^{(q)} - \vec{v}_{\tau_q+1}^{(q)} \rangle = -1, j \neq r, 1 \leq j \leq \tau_q.
\]

\[
\langle \vec{m}_r^{(q)}, \vec{v}_r^{(q)} - \vec{v}_{\tau_q+1}^{(q)} \rangle = -1.
\]

\[
\langle \vec{m}_r^{(q)}, \vec{v}_j^{(q)} - \vec{v}_{\tau_q+1}^{(q)} \rangle = 0.
\]

for \( q \in [1, k] \) and \( j \in [1, \tau_q] \setminus j_q \) for some index \( j_q \) associated to \( \vec{m}_r^{(q)} \).

By virtue of the condition \((5.3)\) we have a set of \((n - k)\) independent equations corresponding to the equalities \((5.6)_1, (5.6)_3\) above.

We see that we may choose as \( \vec{m}_r^{(q)} \in M_R = M_Z \otimes \mathbb{R} \) induced from the mapping

\[
pr : \mathbb{Z}^n \rightarrow \frac{\mathbb{Z}^n}{\sum_{q=1}^k \mathbb{Z}^{\vec{g}^{(q)}}} := M_Z \cong \mathbb{Z}^{n-k},
\]
in view of the relation (1.2).

Therefore we may uniquely determine the set of vectors \( \{ \tilde{m}_r^{(\ell)} \} \in M_R \) as solutions to a system of \((n-k)\) equations (5.6)\(_1\), (5.6)\(_3\) under certain compatibility condition.

To formulate this compatibility condition, let us denote by
\[
P = (\hat{(m)}^{(1)}_1, \cdots, \hat{(m)}^{(1)}_{r_1}, \hat{(m)}^{(2)}_1, \cdots, \hat{(m)}^{(k)}_k),
\]
a \(n \times n\) matrix whose \(b^{\ell-1} + r\) th column corresponds to \(m_r^{(\ell)}\). The system (5.6)\(_*\) can be realized by the following matrix equation if it is solvable,
\[
(5.7)

\begin{align*}
(L_A - \rho \cdot V^A \cdot t V^A) \cdot P &= T L_A - T \rho \cdot T V^A \cdot t (T V^A).
\end{align*}
\]
The solvability of this equation can be understood as the compatibility mentioned above. Let us formulate a sufficient condition for the solvability of (5.7).

**Lemma 5.1** Let us assume that all conditions imposed on (1.1) and \(T(1.1)\) in Theorem 3.1 are satisfied. Furthermore we assume that it is possible to make \(\lambda = id_n\) in (1.12) by means of the rearrangements of rows and columns in \(L_A\). Assume that \(G = id_n\) (resp. \(T G = id_n\)) in (1.9) (resp. \(T(1.9)\)). Then the equation (5.7) is solvable with respect to \(P\).

**proof** The existence of \(k\) linearly independent eigenvectors \(\vec{v}_1^{(0)}, \cdots, \vec{v}_k^{(0)} \in R^n\) such that
\[
^t \vec{v}_\ell^{(0)} \cdot (L_A - \rho \cdot V^A \cdot t V^A) = ^t \vec{v}_\ell^{(0)} \cdot (T L_A - T \rho \cdot T V^A \cdot t (T V^A)) = \vec{0} \in R^k,
\]
is a necessary condition for the solvability of the equation (5.7). It is also a sufficient condition for the solvability as the following relations show,
\[
^t (L_A - \rho \cdot V^A \cdot t V^A) \in GL(W),
\]
\[
^t P \cdot t (L_A - \rho \cdot V^A \cdot t V^A) = t (T L_A - T \rho \cdot T V^A \cdot t (T V^A)) \in GL(W),
\]
for the vector space \(W := \frac{R^n}{\sum_{\ell=1}^k R^0_{\vec{v}_\ell^{(0)}}} \cong R^{n-k}\).

We will see further that
\[
^t (L_A - \rho \cdot V^A \cdot t V^A) \cdot V^A = t (T L_A - T \rho \cdot T V^A \cdot t (T V^A)) \cdot V^A = 0 \in End(R^n, R^k),
\]
under the imposed conditions.

Firt we remark that \(^t (\rho \cdot V^A) \cdot V^A = t \hat{Q}\) in view of (1.8) due to the condition \(G = id_n\). Thus it is enough to show the equality
\[
(5.8)

(\hat{L}_A)^{-1} V^A = V^A.
\]
The left hand side of (5.8) is, in its turn, equal to,
\[
-(^t \hat{P}_1, \cdots, ^t \hat{P}_k) \cdot t \hat{Q},
\]
by virtue of (3.6). Let us introduce a matrix,
\[
\hat{P} = \begin{pmatrix}
\hat{P}_1^{(1)} & \cdots & \hat{P}_1^{(k)} \\
\vdots & \ddots & \vdots \\
\hat{P}_k^{(1)} & \cdots & \hat{P}_k^{(k)}
\end{pmatrix},
\]
\[
(5.9)

for \( T\xi^{(q)}(z) = \sum_{r=1}^{k} p_r^{(q)} (1 - z_r) \). With this notation the matrix (5.9) equals to
\[
(5.10) \quad -V^A T\tilde{P} T\tilde{Q},
\]
if we assume (3.10) for \( G = id_n \). Under the conditions imposed on (1.1) and \( T(1.1) \) in Theorem 3.1 we have
\[
\begin{pmatrix}
1 - z_1 \\
\vdots \\
1 - z_k
\end{pmatrix} = \tilde{Q} \begin{pmatrix}
T\xi^{(1)}(z) \\
\vdots \\
T\xi^{(k)}(z)
\end{pmatrix},
\]
which is a mere analogy to (3.10). In making use of this equality we see that (5.10) is equal to \( V^A \).

To see the equality
\[
T (T\Lambda - T\rho T V^A : T (TV^A)) \cdot V^A = 0 \in \text{End}(\mathbb{R}^n, \mathbb{R}^k),
\]
first we prove the equality
\[
T (T\Lambda - T\rho T V^A : T (TV^A)) \cdot V^A = 0,
\]
in a way parallel to the proof of (5.8). Further we see that \( TV^A = V^A \) under the condition \( \lambda = id_n \).

Q.E.D.

As the matrices \( L_\Lambda - \rho \cdot V^A : T V^A \) and \( T\Lambda - T\rho T V^A : T (TV^A) \) have rank \( (n - k) \) because of the condition (5.3), the columns of the matrix \( P \) are determined as elements in \( M_\mathbb{R} \). We recall that the matrices \( \rho \cdot V^A \) and \( T\rho T V^A \) have been introduced to formulate the conditions (3.11) and (3.11). It is clear that the columns of the solution \( P \) to this equation satisfy the equations (5.6), and, in particular, determine \( j_\ell \) of (5.6).

The vectors (5.1) admit another interpretation. We introduce unit vectors
\[
\epsilon_q = (0, \cdots, 1_q, 0, \cdots, 0) \in \mathbb{Z}^k, 1 \leq q \leq k.
\]
Let us consider a \((n + k)\) dimensional cone \( \sigma \) with \((n + k)\) generators
\[
\tilde{v}_0^{(q)} = (0, \cdots, 0, \epsilon_q) \in \mathbb{Z}^{n+k},
\]
\[
\tilde{v}_j^{(q)} = (\tilde{v}_j^{(q)} - \tilde{v}_{q+1}^{(q)}, \epsilon_q) \in \mathbb{Z}^{n+k}, 1 \leq j \leq q, 1 \leq q \leq k.
\]
The dual cone \( \hat{\sigma} \) to the cone \( \sigma = \mathbb{R}_{\geq 0} (v_0^{(1)}, v_1^{(1)}, \cdots, v_{\tau_1}^{(1)}, \cdots, v_{\tau_k}^{(k)}) \) is defined as
\[
\hat{\sigma} = \{ y \in \mathbb{R}^{n+k}; \langle x, y \rangle \geq 0 \text{ for all } x \in \sigma \}.
\]
The generators of the dual cone \( \hat{\sigma} \) are given by the vectors,
\[
\hat{m}_0^{(\ell)} := (0, \cdots, 0, \epsilon_\ell),
\]
\[
\hat{m}_r^{(\ell)} := (\hat{m}_r^{(\ell)}, \epsilon_\ell), 1 \leq r \leq \tau_\ell, 1 \leq \ell \leq k
\]
with \( \hat{m}_r^{(\ell)} \) satisfying the equations (5.6). It is evident that the following inequalities hold for the above vectors,
\[
\langle \tilde{v}_j^{(q)}, \hat{m}_r^{(\ell)} \rangle \geq 0.
\]
Conversely all vectors \( \hat{m} \) satisfying the conditions
\[
\langle \tilde{v}_j^{(q)}, \hat{m} \rangle \geq 0
\]
for every $\tilde{v}^{(q)}$ must be a linear combination of $\tilde{m}^{(\ell)}$s with positive coefficients as they form a basis of $M_R \times R^k$ the dual space to $V_R \times R^k$. Thus we get the generators of the dual cone $\Delta$ by means of the vertices of $\{\Delta_1^*, \ldots, \Delta_k^*\}$. This is an example of realization of [1], Theorem 4.6.

Let us formulate a statement on the dual partition $\{\Delta_1^*, \ldots, \Delta_k^*\}$ that can be characterized as a dual nef-partition to $\{\Delta_1, \ldots, \Delta_k\}$. We spare the space to formulate the definition of the notion of dual nef-partition (see [1] Definition 4.2) by making it clear in the proof of the following statement.

**Proposition 5.2** We assume the following two conditions on $(1.1)$ and $T(1.1)$. a. $\Delta_q \cap \Delta_\ell = \{0\}$ for $q \neq \ell$. b. It is possible to choose an integer entry matrix $P$ in the matrix equation $(5.7)$. Then the following statements hold.

1. $\{\Delta_1, \ldots, \Delta_k\}$ is a nef-partition of the Minkowski sum $\Delta_1 + \cdots + \Delta_k$.
2. $\{\Delta_1^*, \ldots, \Delta_k^*\}$ is a nef-partition of the Minkowski sum $\Delta_1^* + \cdots + \Delta_k^*$ dual to $\{\Delta_1, \ldots, \Delta_k\}$.
3. The transposed polynomials $Tf_{q\ell-1}/(Tf_{q\ell} - 1)$ are obtained from $\Delta_q^*$ by means of the torus closed embedding,

$$\prod_{i=1}^{\ell} \Pi_{\tau_i y_{\ell_q}}^{-1}(x_{\ell_q}) \rightarrow \Pi_{y_{q\ell}}^{n-1}(x_1, \ldots, x_n) \mapsto \Pi_1 \ldots \Pi_{\ell_q} \prod_{\tau_i y_{\ell_q} \leq \tau_i} x_{\ell_q}^{(\nu_{\ell_q} - \nu_{\ell_q+1} e_{\ell_q})},$$

where $e_q = (0, \ldots, 1^q, 0, \ldots, 0) \in Z^n$.

**Proof**

1. The condition $P \in End(Z)$ entails the integral linear property of the function $\phi_\ell(y)$ defined in $(5.4)$. The convexity of $\phi_\ell : M_R \rightarrow R$ is guaranteed by the following facts. First the number of vertices in $\Delta_\ell$ is less than $(n - k + 1) = dimM_R + 1$. Secondly the condition a.

   It is easy to see from the equation $(5.7)$ that

$$\phi_q(m_q^{\ell}) = \delta_q, 1 \leq r \leq \tau_\ell.$$  

The existence of an integral upper convex piecewise linear function $\phi_q$, $1 \leq q \leq k$ satisfying $(5.11)$ is equivalent to the definition of nef-partition $\{\Delta_1, \ldots, \Delta_k\}$ of the Minkowski sum $\Delta_1 + \cdots + \Delta_k$ whose dimension is equal to $dimV_R$ after $(5.3)$.

2. After the condition b,

$$\psi_\ell(x) = -\min_{y \in \Delta} \langle x, y \rangle,$$

is an integral piecewise linear function on $V_R$. The relation $\Delta_q^* \cap \Delta_\ell^* = \{0\}$ for $q \neq \ell$ is clear from the existence of the function $(5.11)$. The number of vertices in $\Delta_\ell^*$ is less than $(n - k + 1) = dimV_R + 1$.

Thus we see that $\psi_\ell(x)$ is an upper convex function. This shows that $\{\Delta_1^*, \ldots, \Delta_k^*\}$ is a nef-partition of the Minkowski sum $\Delta_1^* + \cdots + \Delta_k^*$. The existence of an integral upper convex piecewise linear function $(5.11)$ shown before means that $\{\Delta_1^*, \ldots, \Delta_k^*\}$ is a dual nef-partition to $\{\Delta_1, \ldots, \Delta_k\}$.

3. First of all we remark that the transposed construction entails,

$$\sum_{\ell=1}^k Tg_j^{(q)}(v_j^{(\nu(q))} - \tilde{v}_j^{(\nu(q))}) = 0.$$

By virtue of the convexity of the function $\sum_{\ell=1}^k \phi_\ell$ (see [10] §2.3) or equivalently

$$\Delta_1 + \cdots + \Delta_k = \{x \in V_R : \langle x, y \rangle \geq -\sum_{\ell=1}^k \phi_\ell(y), y \in M_R\},$$
the mapping (5.7) is a closed torus embedding. Thus a polynomial whose Newton diagram equals to $\Delta^{\ast}_\ell$ in $(y_1, \ldots, y_n)$ variables coincides with a polynomial obtained as a sum of monomials with exponents from the rows of the RHS of (5.7) i.e. $\prod_{\ell=1}^k \prod_{1 \leq j \leq \bar{\tau}_\ell} T^{y_{\nu(\ell)}-T^{y_{\nu(\ell)}}}_{\bar{\tau}_\ell+1}$. Further argument is parallel to that in [4], §3. Q.E.D.

6 Examples

Example 6.1, Schimmrigk variety As a simple, but non-trivial example we recall the following example whose period integral has been studied in [2],

$$f_1(x) = \sum_{i=0}^3 x_i^3, f_2(x) = x_1x_2x_3 + 1,$$

$$f_3(x) = \sum_{i=1}^3 x_ix_i^2 + 3, f_4(x) = x_0x_4x_5x_6 + 1.$$
whose period integral can be expressed through its Mellin transform,

\[ \sigma_n \]

up to periodic functions.

For each fixed \( T \), we can calculate the Mellin transform of the period integral,

\[ M_{0, \gamma}^0(z) = \Gamma(-\frac{1}{3}(z_1 - 1) + \frac{1}{9}(z_2 - 1), \xi^{(2)} = -\frac{1}{3}(z_2 - 1). \]

For \( f(x) \) as well as for \( T f(x) \) we can calculate the Mellin transform of the period integral,

\[ M_{0, \gamma}^0(z) = \Gamma(-\frac{1}{3}(z_1 - 1) + \frac{1}{9}(z_2 - 1))^{\frac{1}{3}}(z_1)\Gamma(z_2) = \frac{\Gamma(\xi^{(1)})\Gamma(\xi^{(2)})}{\Gamma(3\xi^{(1)} + \xi^{(2)})\Gamma(3\xi^{(2)})}, \]

up to \( 2^T - \)-periodic functions.

Analogously we can look at the CI defined on \( T^{2n+1} \)

\[ f_1(x) = \sum_{i=0}^{n} x_i^n, f_2(x) = x_1x_2 \cdots x_n + 1, \]

\[ f_3(x) = \sum_{i=1}^{n} x_i x_{i+n}, f_4(x) = x_0x_{n+1}x_{n+2} \cdots x_{2n} + 1, \]

whose period integral can be expressed through its Mellin transform,

\[ M_{0, \gamma}^0(z) = \frac{\Gamma(\xi^{(1)})^n\Gamma(\xi^{(2)})^{n+1}}{\Gamma(n\xi^{(1)} + \xi^{(2)})\Gamma(n\xi^{(2)})}, \]

up to \( n^n - \)periodic functions.

It is worthy to notice that our matrix \( L \) satisfies an interesting condition below.

**The magic square condition** For each fixed \( q \in [1, k] \), we can find a single valued mapping \( \sigma : b \in I_\Lambda \rightarrow \{1, \cdots, n\} \) such that

\[ p^b_q = w_{\sigma(b)}, \text{ for all } b \in I_\Lambda. \]
This condition plays central rôle in the interpretation of the strange duality found by Arnol’d on the interchange between Gabrielov number and Dolgachev number from the point of view of the mirror symmetry [8].

**Example 6.2** We consider an example of an hypersurface studied in [2]. We have the following data after the notations above,

\[ f_1(x) = x_1^7 + x_2^2 x_4 + x_3^2 x_5 + x_4^3 + x_5^3, \quad f_2 = x_1 x_2 x_3 x_4 x_5. \]

\[
L = \begin{bmatrix}
7 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 7 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 7 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix},
\]

\[
L^{-1} = \begin{bmatrix}
\frac{6}{39} & -1/49 & -1/49 & -\frac{2}{39} & -\frac{2}{39} & 0 & 1/7 & -1/7 \\
-\frac{2}{147} & \frac{19}{147} & -\frac{2}{147} & -\frac{11}{147} & -\frac{4}{147} & 0 & 2/21 & -2/21 \\
-\frac{2}{147} & -\frac{19}{147} & \frac{19}{147} & -\frac{4}{147} & -\frac{147}{147} & 0 & 2/21 & -2/21 \\
-1/21 & -1/21 & -1/21 & \frac{7}{21} & -2/21 & 0 & 1/3 & -1/3 \\
-1/21 & -1/21 & -1/21 & -2/21 & \frac{7}{21} & 0 & 1/3 & -1/3 \\
1/7 & 1/7 & 1/7 & 2/7 & 2/7 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1/7 & -1/7 & -1/7 & -2/7 & -2/7 & 1 & 1 & -1 \\
\end{bmatrix}
\]

We have therefore the Mellin transform of the period integral associated to the CI \( x \in (\mathbb{C}^x)^5 \); \( f_1(x) + s_1 = 0, f_2(x) + 1 = 0 \) up to 7–periodic functions,

\[
M_{0,\gamma}^0(z) = \Gamma(-\frac{1}{7}(z_1 - 1))^3 \Gamma(-\frac{2}{7}(z_1 - 1))^2 \Gamma(z_1) = \frac{\Gamma(\xi(1))^3 \Gamma(2\xi(1))^2}{\Gamma(7\xi(1))}.
\]

After the construction (1.4)' we see that

\[
T f_1(x) = x_1^7 + x_2^2 + x_3^2 + x_4 x_5 + x_3 x_5, \quad T f_2 = x_1 x_2 x_3 x_4 x_5.
\]

We then have the Mellin transform of the period integral associated to the CI \( x \in (\mathbb{C}^x)^5 \); \( T f_1(x) + s_1 = 0, T f_2(x) + 1 = 0 \) up to 147–periodic functions,

\[
M_{0,\gamma}^0(z) = \Gamma(-\frac{1}{7}(z_1 - 1)) \Gamma(-\frac{2}{21}(z_1 - 1))^2 \Gamma(-\frac{1}{3}(z_1 - 1))^2 \Gamma(z_1) = \frac{\Gamma(3\xi(1)) \Gamma(2\xi(1))^2 \Gamma(7\xi(1))^2}{\Gamma(21\xi(1))}.
\]

**References**

[1] V.V. Batyrev and L.A. Borisov *Dual cones and Mirror symmetry for generalized Calabi-Yau manifolds*, Comm.Math.Phys. **168**, (1995), 493-533.
[2] P. Berglund, Ph. Candelas, X. de la Ossa, A. Font, T. Hubsch, D. Jancic et F. Quevedo, *Periods for Calabi-Yau variety and Landau-Ginzburg vacua*, Nuclear Physics B **419**, (1994), pp.352-403.

[3] P. Berglund, T. Hubsch, *A generalized construction of mirror manifolds*, Nuclear Physics B **393** (1993), no. 1-2, pp.377–391.

[4] P. Berglund, S. Katz, *Mirror symmetry constructions: A Review*, Mirror symmetry, II, 87–113, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997.

[5] T. M. Chiang, A. Klemm, S.-T. Yau et E. Zaslow, *Local mirror symmetry: Calculations and interpretations*, Adv.Theor.Math.Phys. 3, no.1, (1999), pp.495-565.

[6] I. Dolgachev, *Weighted projective varieties*, Lecture Notes in Math. **956**, pp. 34-71. Springer Verlag, 1982.

[7] V. V. Golyshev, *Riemann-Roch Variations*, Izvestia Math. **65** (2001), no. 5, pp.853-887.

[8] M. Kobayashi, *Duality of weights, Mirror symmetry and Arnold’s strange duality*, alg-geom/9502004.

[9] I. Norlund, *Hypergeometric functions*, Acta Math. **94**, (1955/56), pp.289-349.

[10] T. Oda, *Convex Bodies and Algebraic Geometry*, An introduction to the Theory of Toric Varieties, Ergebnisse der Math. (3) 15, Springer Verlag, 1988.

[11] M. Passare, T. Sadykov, A. G. Tsikh, *Nonconfluent hypergeometric functions in several variables and their singularities*, to appear in Compositio Mathematica.

[12] S. Tanabe, *Hodge structure of fibre integrals associated to the affine hypersurface in a torus*, preprint, math.AG/0206126.

[13] S. Tanabe, *Transformée de Mellin des intégrales- fibres associées à l’intersection complète non-dégénérée*, preprint math.AG/0405399

[14] S. Tanabe, *On Horn-Kapranov uniformisation of the discriminantal loci*, preprint math.AG/0503262

[15] V. A. Vassiliev, *Ramified integrals, singularities and Lacunas*, Kluwer Academic Publishers, Dordrecht, 1995.

Independent University of Moscow
Bol'shoj Vlasievskij perenlok 11,
Moscow, 121002,
Russia
E-mails: tanabe@mccme.ru,
tanabe@mpim-bonn.mpg.de