ADDITIONAL REDUCTIONS IN THE K-CONSTRAINED MODIFIED KP HIERARCHY

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ABSTRACT. Additional reductions in the modified k-constrained KP hierarchy are proposed. As a result we obtain generalizations of Kaup-Broer system, Korteweg-de Vries equation and a modification of Korteweg-de Vries equation that belongs to the modified k-constrained KP hierarchy. We also propose solution generating technique based on binary Darboux transformations for the obtained equations.

1. INTRODUCTION

The algebraic constructions of the well-known Kyoto group [1], which are called the Sato theory, play an important role in the contemporary theory of nonlinear integrable systems of mathematical and theoretical physics. The leading place in these investigations is occupied by the theory of equations of Kadomtsev-Petviashvili type (KP hierarchy) and their generalizations and applications [1–3].

One of known generalizations of the KP hierarchy arise as a result of k-symmetry constraints (so-called k-cKP hierarchy) that were investigated in [4–8]. k-cKP hierarchy are closely connected with so-called KP equation with self-consistent sources (KPSCS) [9–12]. Multicomponent k-constraints of the KP hierarchy were introduced in [13] and investigated in [14–18]. This extension of k-cKP hierarchy contains vector (multicomponent) generalizations of such physically relevant systems like the nonlinear Schrödinger equation, the Yajima-Oikawa system, a generalization of the Boussinesq equation, and the Melnikov system.

The modified k-constrained KP (k-cmKP) hierarchy was proposed in [19,20]. It contains, for example, the vector Chen-Lee-Liu, the modified KdV (mKdV) equation and their multi-component extensions. The k-cmKP hierarchy and dressing methods for it via integral transformations were investigated in [21–23].

In [24–26] (2+1)-dimensional extensions of the k-cKP hierarchy ((2+1)-dimensional k-cKP hierarchy) were introduced and dressing methods via differential transformations were investigated. Some systems of this hierarchy were investigated via binary Darboux transformations in [22,23]. This hierarchy was also rediscovered recently in [26,27]. Matrix generalizations of (2+1)-dimensional k-cKP hierarchy were considered in [28,29].

In this paper our aim is to consider additional reductions of the k-cmKP hierarchy that lead to new generalizations of well-known integrable systems. We also investigated dressing methods for the obtained systems via integral transformations that arise from Binary Darboux Transformations (BDT).

Key words and phrases. solitons, binary Darboux transformation, modified constrained Kadomtsev-Petviashvili hierarchy, Grammian solutions.
This work is organized as follows. In Section 2 we present a short survey of results on constraints for the KP hierarchy including the k-cmKP hierarchy. In Section 3 we investigate Lax representations obtained as a result of additional reductions in the k-cmKP hierarchy and corresponding nonlinear systems. Section 4 presents results on dressing methods for Lax pairs obtained in Section 3. In the final section, we discuss the obtained results and mention problems for further investigations.

2. Symmetry constraints of the KP hierarchy

Let us recall some basic objects and notations concerning KP hierarchy, modified KP hierarchy, their multicomponent k-constraints and their (2+1)-extensions. A Lax representation of the KP hierarchy is given by

\[ L_{t_n} = [B_n, L], \quad n \geq 1, \tag{1} \]

where \( L = D + U_1 D^{-1} + U_2 D^{-2} + \ldots \) is a scalar pseudodifferential operator, \( t_1 := x \), \( D := \frac{\partial}{\partial x} \), and \( B_n := (L^n)_+ := (L^n)_{>0} = D^n + \sum_{i=0}^{n-2} u_i D^i \) is the differential operator part of \( L^n \). The consistency condition (zero-curvature equations), arising from the commutativity of flows (1), is

\[ B_{n,t} - B_{k,t} + [B_n, B_k] = 0. \tag{2} \]

Let \( \tilde{B}_n \) denote the formal transpose of \( B_n \), i.e., \( \tilde{B}_n := (-1)^n D^n + \sum_{i=0}^{n-2} (-1)^i D^i u_i^\top \), where \( \top \) denotes the matrix transpose. We will use curly brackets to denote the action of an operator on a function whereas, for example, \( B_n q \) means the composition of the operator \( B_n \) and the operator of multiplication by the function \( q \). The following formula holds for \( B_n q \) and \( B_n \{ q \} \):

\[ B_n q := (B_n q)_{=0} = B_n q - (B_n \{ q \})_{>0}. \]

In the case \( k = 2, n = 3 \) formula (2) presents a Lax pair for the Kadomtsev-Petviashvili equation. Its Lax pair was obtained in [30] (see also [31]).

The multicomponent k-constraints of the KP hierarchy is given by

\[ L_{t_n} = [B_n, L], \tag{3} \]

with the k-symmetry reduction

\[ L_k := L^k = B_k + \sum_{i=1}^m \sum_{j=1}^m q_i m_{ij} D^{-1} r_j = B_k + \mathbf{q} \mathbf{M}_0 D^{-1} \mathbf{r}^\top, \tag{4} \]

where \( \mathbf{q} = (q_1, \ldots, q_m) \) and \( \mathbf{r} = (r_1, \ldots, r_m) \) are vector functions, \( \mathbf{M}_0 = (m_{ij})_{i,j=1}^m \) is a constant \( m \times m \) matrix. In the scalar case \( (m = 1) \) we obtain k-constrained KP hierarchy [4][8]. The hierarchy given by (3)-(4) admits the Lax representation (here \( k \in \mathbb{N} \) is fixed):

\[ [L_k, M_n] = 0, \quad L_k = B_k + \mathbf{q} \mathbf{M}_0 D^{-1} \mathbf{r}^\top, \quad M_n = \partial_{t_n} - B_n. \tag{5} \]

Lax equation (5) is equivalent to the following system:

\[ [L_k, M_n]_{>0} = 0, \quad M_n \{ \mathbf{q} \} = 0, \quad M_n^* \{ \mathbf{r} \} = 0. \tag{6} \]

Below we will also use the formal adjoint \( \tilde{B}_n^* := (\tilde{B}_n)^* = (-1)^n D^n + \sum_{i=0}^{n-2} (-1)^i D^i u_i^\ast \) of \( B_n \), where \( \ast \) denotes the Hermitian conjugation (complex conjugation and transpose).

For \( k = 1 \), the hierarchy given by (6) is a multi-component generalization of the AKNS hierarchy. For \( k = 2 \) and \( k = 3 \), one obtains vector generalizations of the Yajima-Oikawa and Melnikov [9][11] hierarchies, respectively. An essential
extension of the k-cKP hierarchy is its (2+1)-dimensional generalization introduced in [24, 25] and rediscovered in [26, 27].

In [19, 20], a k-constrained modified KP (k-cmKP) hierarchy was introduced and investigated. Dressing methods for k-cmKP hierarchy under additional D-Hermitian reductions were also investigated in [21, 22]. At first we recall the definition of the modified KP hierarchy. A Lax representation of this hierarchy is given by

\[ L_{t_n} = [B_n, L], \quad n \geq 1, \]  

(7)

where \( L = D + U_0 + U_1 D^{-1} + U_2 D^{-2} + \ldots \) and \( B_n := (L^n)_{>0} := D^n + \sum_{i=1}^{n-1} u_i D^i \) is the purely differential operator part of \( L^n \). The consistency condition arising from the commutativity of flows (7), is

\[ B_n, t_e - B_{k,t_n} + [B_n, B_k] = 0. \]  

(8)

The multicomponent k-constraints of the modified KP hierarchy are given by the following system:

\[ L_{t_n} = [B_n, L], \]  

(9)

with the k-symmetry reduction

\[ L_k := L_k = B_k - \sum_{i=1}^{m} \sum_{j=1}^{m} q_{i,j} D^{-1} r_j D = B_k - q \mathcal{M}_0 D^{-1} r^\top D, \]  

(10)

where \( q = (q_1, \ldots, q_m) \) and \( r = (r_1, \ldots, r_m) \) are vector functions, \( \mathcal{M}_0 = (m_{ij})_{i,j=1}^m \) is a constant \( m \times m \) matrix. The hierarchy (9)-(10) admits the Lax representation (here \( k \in \mathbb{N} \) is fixed):

\[ [L_k, M_n] = 0, \quad L_k = B_k - q \mathcal{M}_0 D^{-1} r^\top D, \]  

\[ M_n = \alpha_n \partial_{t_n} - B_n, \quad B_n = D^n + \sum_{i=1}^{n-1} u_i D^i. \]  

(11)

We can rewrite the Lax pair (11) in the following way:

\[ [L_k, M_n] = 0, \quad L_k = B_k - q \mathcal{M}_0 r^\top + q \mathcal{M}_0 D^{-1} r_x^\top, \]  

\[ M_n = \alpha_n \partial_{t_n} - B_n, \quad B_n = D^n + \sum_{i=1}^{n-1} u_i D^i. \]  

(12)

From Lax representation for k-cKP hierarchy (12) and representation (12) we come to conclusion that equation \([L_k, M_n] = 0 \) in (11) is equivalent to the following system:

\[ [L_k, M_n]_{>0} = 0, \quad M_n\{q\} = 0, \quad (M_n^\top\{r_x\}) = 0 \quad (M_n^\top{1} = 0) \]

1. since \([L_k, M_n]\{1\} = 0 \). We can rewrite the last equation in the following form:

\[ (D^{-1} M_n^\top D)\{r\} = 0 \] to keep the order of differentiation equal to \( n \). As a result we obtain:

\[ [L_k, M_n]_{>0} = 0, \quad M_n\{q\} = 0, \quad (D^{-1} M_n^\top D)\{r\} = 0. \]  

(13)

The hierarchy (11) contains vector generalizations of the Chen-Lee-Liu \((k = 1)\), the modified multi-component Yajima-Oikawa \((k = 2)\) and Melnikov \((k = 3)\) hierarchies. Consider some equations that can be obtained from (11) under certain choice of \( k \) and \( n \) (see [23]).

1. \( k = 1, n = 2 \). Then (11) becomes

\[ L_1 = D - q \mathcal{M}_0 D^{-1} r^\top D, \quad M_2 = \alpha_2 \partial_{t_2} - D^2 + 2 q \mathcal{M}_0 r^\top D. \]  

(14)

In this case equation (13) becomes the following system:

\[ \alpha_2 q_{t_2} - q_{xx} + 2 q \mathcal{M}_0 r^\top q_x = 0, \quad \alpha_2 r_{t_2}^\top + r_{xx}^\top + 2 r_x^\top q \mathcal{M}_0 r^\top = 0. \]  

(15)
Under additional Hermitian conjugation reduction: $\alpha_2 = i$, $\mathcal{M}_0 = -\mathcal{M}_0^*$, $r^\top = q^*$ ($L_1^* = -D^{-1}L_1D$, $M_2^* = D^{-1}M_2D$) in (15) we obtain the Chen-Lee-Liu equation:

$$i\mathbf{q}_{t_2} - \mathbf{q}_{xx} + 2q\mathcal{M}_0\mathbf{q}^*\mathbf{q}_x = 0.$$  \hbox{(16)}

(2) $k = 1, n = 3$. In this case (11) takes the form:

$$\begin{align*}
L_1 &= D - q\mathcal{M}_0D^{-1}r^\top D, \\
M_3 &= \alpha_3\partial_3 - D^3 + 3q\mathcal{M}_0r^\top D^2 + 3[q_x\mathcal{M}_0r^\top - (q\mathcal{M}_0r^\top)^2]D, \\
\end{align*}$$

and equations (17)

$$\begin{align*}
\alpha_3q_{t_3} &= q_{xxx} - 3(q\mathcal{M}_0r^\top)q_{xx} - 3(q_x\mathcal{M}_0r^\top - (q\mathcal{M}_0r^\top)^2)q_x, \\
\alpha_3r_{t_3} &= r_{xxx} + 3r^\top(\mathcal{M}_0r^\top) + 3r^\top(r\mathcal{M}_0r^\top + (q\mathcal{M}_0r^\top)^2). \\
\end{align*}$$

After reduction of Hermitian conjugation: $\alpha_3 = 1$, $r^\top = q^*$, $\mathcal{M}_0 = -\mathcal{M}_0^*$ ($L_1^* = -D^{-1}L_1D$, $M_3^* = D^{-1}M_3D$) (18) becomes:

$$\begin{align*}
q_{t_3} &= q_{xxx} + 3(q\mathcal{M}_0q^*)q_{xx} - 3(q_x\mathcal{M}_0q^* - (q\mathcal{M}_0q^*)^2)q_x. \\
\end{align*}$$

(3) $k = 2, n = 3$. After additional reduction in (11): $\alpha_2 = i$, $u_1 := iu$, $u = u(x, t_2) \in \mathbb{R}$, $\mathcal{M}_0 = \mathcal{M}_0^*$ Lax pair in (13) reads:

$$\begin{align*}
[L_2, M_2] &= 0, L_2 = D^2 + iuD - q\mathcal{M}_0D^{-1}q^*D, \\
M_2 &= i\partial_{t_2} - D^2 - iuD, \\
\end{align*}$$

and equation (13) becomes the modified Yajima-Oikawa equation:

$$i\mathbf{q}_{t_2} = \mathbf{q}_{xx} + i\mathbf{q}_x, \quad u_{t_2} = 2(q\mathcal{M}_0q^*)_x.$$ 

In the next section we will introduce additional reductions in Chen-Lee-Liu hierarchy. As a result we will obtain generalizations of the Kaup-Broer system, Korteweg-de Vries equation, modified Korteweg-de Vries equation and their scalar coupled versions.

3. ADDITIONAL REDUCTIONS IN THE MODIFIED k-CONSTRAINED KP HIERARCHY

For further convenience let us make a change in formulae (11):

$$\begin{align*}
\mathbf{q} &\rightarrow \mathbf{q}, \quad \mathbf{r} \rightarrow \mathbf{r}, \quad \mathcal{M}_0 \rightarrow \tilde{\mathcal{M}}_0, \\
\end{align*}$$

\hbox{(20)}

After the change (20) the hierarchy (11) reads:

$$\begin{align*}
[L_k, M_n] &= 0, L_k = B_k - \mathbf{q}\tilde{\mathcal{M}}_0D^{-1}\mathbf{r}^\top D, \\
M_n &= \alpha_n\partial_{t_n} - B_n, \\
B_n &= D^n + \sum_{i=1}^{n-1} u_i D^i. \\
\end{align*}$$

\hbox{(21)}

Let us make the additional reduction in (21):

$$\begin{align*}
\mathbf{q} := (q_1, \ldots, q_m, -v - \beta D^{-1}\{u\}, 1), \\
\tilde{\mathcal{M}}_0 = \begin{pmatrix} \mathcal{M}_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{r} := (r_1, \ldots, r_m, 1, \beta D^{-1}\{u\}) = (r_1, 1, \beta D^{-1}\{u\}),
\end{align*}$$

\hbox{(22)}

where $\mathcal{M}_0$ is $(m \times m)$-constant matrix, $\mathbf{q}$ and $\mathbf{r}$ are $m$-component vectors, $u$ and $v$ are scalar functions, $\beta \in \mathbb{R}$, $D^{-1}\{u\}$ denotes indefinite integral of the function $u$ with respect to $x$. After reduction (22) k-cmKP hierarchy (21) takes the form:

$$\begin{align*}
[L_k, M_n] &= 0, L_k = B_k - \mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}^\top D + v + \beta D^{-1}u, \\
M_n &= \alpha_n\partial_{t_n} - B_n, \\
B_n &= D^n + \sum_{i=1}^{n-1} u_i D^i. \\
\end{align*}$$

\hbox{(23)}

In the following subsections we will investigate hierarchy (23) in case $k = 1$. 

3.1. Reductions of the Chen-Lee-Liu system. Let us put \( k = 1, n = 2 \). Then Lax pair (23) becomes:

\[
[L_1, M_2] = 0, \quad L_1 = D - qM_0D^{-1}r^\top D + \beta D^{-1}u + v, \\
M_2 = \alpha_2\partial_\tau - D^2 + 2(qM_0r^\top - v)D.
\] (24)

A system that corresponds to equation (24) has the form:

\[
\begin{align*}
\alpha_2q_{t_2} &= q_{xx} - 2(qM_0r^\top - v)q_x, \\
\alpha_2\partial_{t_2} - u_{xx} + 2(u(qM_0r^\top - v))_x = 0, \\
-\alpha_2v_{t_2} + 2\beta u_x + v_{xx} - 2(qM_0r^\top - v)v_x = 0.
\end{align*}
\] (25)

Consider additional reductions of Lax pair (24) and system (25).

1. Assume that \( M_0 = -M_0^\# \), \( r^\top = q^\ast \), \( v = -2\text{Im}(\beta D^{-1}\{u\}) \) \( (L_1^* = -DL_1D^{-1}, \ M_2^* = DM_2D^{-1}) \). Then equation (25) takes the form:

\[
\begin{align*}
\alpha_2q_{t_2} &= q_{xx} - 2(2\text{Im}(\beta D^{-1}\{u\}) + qM_0q^\ast)q_x, \\
\alpha_2\partial_{t_2} - u_{xx} + 2(2(2\text{Im}(\beta D^{-1}\{u\}) + qM_0q^\ast))_x = 0.
\end{align*}
\]

2. Let us put \( M_0 = 0 \) in operators \( L_1 \) and \( M_2 \): \( L_1 = D + \beta D^{-1}u + v, \ M_2 = \alpha_2\partial_\tau - D^2 - 2vD \). Then equation (25) becomes the Kaup-Broer system:

\[
\alpha_2u_{t_2} + u_{xx} - 2(uv)_x = 0, \quad -\alpha_2v_{t_2} + 2\beta u_x + v_{xx} + 2vv_x = 0.
\] (26)

In case \( u = 0 \) in (26) we obtain the Burgers equation: \( -\alpha_2v_{t_2} + v_{xx} - vv_x = 0 \).

3. Consider the case \( u = 0 \) in operators \( L_1 \) and \( M_2 \) : \( L_1 = D - qM_0D^{-1}r^\top D + v, \ M_2 = \alpha_2\partial_\tau - D^2 + 2(v + qM_0r^\top)D \). Then (25) reads:

\[
\begin{align*}
\alpha_2q_{t_2} &= q_{xx} - 2(q_0M_0r^\top - v)q_x, \\
\alpha_2\partial_{t_2} - u_{xx} + 2r^\top (qM_0r^\top - v), \\
-\alpha_2v_{t_2} + v_{xx} - (qM_0r^\top - v)v_x = 0.
\end{align*}
\]

3.2. Reductions of the modification of Korteweg-de Vries system (19). Now let us consider the hierarchy (23) in case \( k = 1, n = 3 \). Then its Lax pair \( L_1, M_3 \) in (23) reads:

\[
[L_1, M_3] = 0, \quad L_1 = D - qM_0D^{-1}r^\top D + \beta D^{-1}u + v, \ M_3 = \alpha_3\partial_\tau - D^2 - 3(v - qM_0r^\top)D^2 - 3((qM_0r^\top - v)^2 - qM_0r^\top + \beta u + v_x)D.
\] (27)

Commutator equation in (27) is equivalent to the system:

\[
\begin{align*}
-\alpha_3v_{t_3} + v_{xx} + 3v_{xx} + 9v^2_v + 3v^2_x + 3v^3_x + 6\beta(uv)_x + \\
+3\left\{(qM_0r^\top)^2 - qM_0r^\top\right\}v_x - 3qM_0r^\top v_{xx} - \\
-6qM_0r^\top v_{xx} - 3\beta(qM_0r^\top u)_x - 3\beta qM_0r^\top u_x = 0, \\
\alpha_3\partial_{t_3} - u_{xx} + 2qM_0r^\top q_x + \\
+3\left\{(qM_0r^\top - v)^2 - qM_0r^\top + v_x + \beta u\right\}q_x, \\
\alpha_3r_{t_3} = r_{xx} - 3(qM_0r^\top - v) - 3(qM_0r^\top)_x \\
+3(qM_0r^\top - v)^2 - qM_0r^\top v_x + \beta u, \\
\alpha_3u_{t_3} = u_{xx} + 3(v - qM_0r^\top) - 3(u)(v - qM_0r^\top)_x \\
+3(v - qM_0r^\top - v)^2 - qM_0r^\top v_x + \beta u.
\end{align*}
\] (28)

Consider additional reductions in Lax pair (27) and corresponding system (28).
Assume that $v = -2i \text{Im}(\beta D^{-1}\{u\})$, $q^* = r^\top$, $u \in \mathbb{R}$, $M_0 = -M_0^\top$ ($L_1^* = -DL_1D^{-1}$, $M_3^* = -DM_3D^{-1}$). Then system (23) takes the form:

$$\begin{align*}
\alpha_3 q_{t3} &= q_{xxx} - 3(2i \text{Im}(\beta u) + q M_0 q^*)q_{xx} + \alpha_3 q_{u3} = u_{xxx} + 3 \{u(2i \text{Im}(\beta u) + q M_0 q^*)\}x + 3 \{(q M_0 q^* + 2i \text{Im}(\beta u))^2 - q_x M_0 q^* + \beta u - 2i \text{Im}(\beta u)\}x.
\end{align*}$$

(29)

(a) Let us assume that in addition to reductions described in item 1 functions $q$ and $u$ with matrix $M_0$ are real-valued (i.e., matrix $M_0$ is skew-symmetric: $M_0^\top = -M_0$) and $v = 0$. Then the scalar $q M_0 q^* = 0$ since $q M_0 q^* = -(q M_0 q^*)^\top$ and equation (29) reads:

$$\begin{align*}
\alpha_3 q_{t3} &= q_{xxx} - 3q_x M_0 q^* q_x + 3\beta u q_x,
\alpha_3 q_{u3} &= u_{xxx} - 3(q_x M_0 q^*)x + 6\beta u q_x.
\end{align*}$$

(30)

(2) Let us put $M_0 = 0$ in operators $L_1$, $M_3$ (27):

$L_1 = D + \beta D^{-1}u + v$, $M_3 = \alpha_3 \partial_3 - D^3 - 3vD^2 - 3(v^2 + v_x + \beta u)D$.

Then equation (23) takes the form:

$$\begin{align*}
-\alpha_3 v_{t3} + v_{xxx} + 3v_{xx} + 6v^2 v_x + 3v_x^2 + 6\beta(u) = 0,
\alpha_3 u_{t3} = u_{xxx} - 3(vu) + 3(u v^2 + v_x + \beta u) = 0.
\end{align*}$$

(31)

(a) Under additional restrictions $v = -2i \text{Im}(D^{-1}\{\beta u\})$ ($L_1^* = -DL_1D^{-1}$, $M_3^* = -DM_3D^{-1}$) in item 2 we obtain a complex generalization of the modified Korteweg-de Vries equation:

$$\begin{align*}
\alpha_3 q_{t3} &= u_{xxx} + 6i(u M_0(D^{-1}\{\beta u\}))x + 3 \{u(-4i M_0(D^{-1}\{\beta u\}))^2 - 2i \text{Im}(\alpha u + \beta u)\}x.
\end{align*}$$

(32)

In the real case ($\beta \in \mathbb{R}$, $u$ is a real-valued function, $v = 0$) operators $L_1$ and $M_3$ take the form: $L_1 = D + \beta D^{-1}u$, $M_3 = \beta \partial_3 - D^3 - 3\beta u D$, and we obtain KdV equation in (32):

$$\begin{align*}
\alpha_3 u_{t3} &= u_{xxx} + 6\beta u q_x.
\end{align*}$$

(33)

(3) Let us put $u = 0$ in Lax pair (27): $L_1 = D - q M_0 D^{-1}r^\top D + v$, $M_3 = \alpha_3 \partial_3 - D^3 - 3v - q M_0 r^\top D^2 - 3((q M_0 r^\top - v) - q M_0 r^\top + v_x)D$. Equation (28) becomes:

$$\begin{align*}
-\alpha_3 v_{t3} + v_{xxx} + 3v_{xx} + 6v^2 v_x + 3v_x^2 + 3v^2 &= 0,
\alpha_3 q_{t3} &= q_{xxx} + 3(q - q M_0 r^\top)q_{xx} + 3 \{(q M_0 r^\top - v)^2 - q M_0 r^\top + v_x\}q_x,
\alpha_3 q_{r^\top} &= r^\top_{xxx} - 3(r^\top_{xx} v - r^\top r^\top + v_x) + 3 \{(q M_0 r^\top - v)^2 - q M_0 r^\top + v_x\}r^\top.
\end{align*}$$

(34)

4. Dressing methods for K-cmKP hierarchy

In this section our aim is to elaborate dressing methods for the k-cmKP hierarchy (11). At first we recall a main result from paper (35). Let $1 \times K$-matrix functions $\varphi$ and $\psi$ be solutions of linear problems with $(2+1)$-dimensional generalization of the operator $L_k$ (11) with more general differential part $B_k$:

$$L_k \{\varphi\} = \varphi \Lambda, L_k^\top \{\psi\} = \psi \Lambda, \Lambda, \Lambda \in \text{Mat}_{K \times K}(\mathbb{C}),
L_k = \beta_k \partial_{x_k} + B_k + q M_0 D^{-1}r^\top, B_k = \sum_{j=0}^k u_j D^j.$$
Introduce a binary Darboux transformation (BDT) in the following way:

\[ W = I - \varphi (C + D^{-1} \{ \psi^T \varphi \})^{-1} D^{-1} \psi^T := I - \varphi \Delta^{-1} D^{-1} \psi^T, \quad (36) \]

where \( C \) is a \( K \times K \)-constant nondegenerate matrix. The inverse operator \( W^{-1} \) has the form:

\[ W^{-1} = I + \varphi D^{-1} \left( C + D^{-1} \{ \psi^T \varphi \} \right)^{-1} \psi^T = I + \varphi D^{-1} \Delta^{-1} \psi^T. \quad (37) \]

The following theorem is proven in \[35\].

**Theorem 1.** \[37\] The operator \( \tilde{L}_k := WL_k W^{-1} \) obtained from \( L_k \) in \[36\] via BDT has the form

\[
\tilde{L}_k := WL_k W^{-1} = \beta_k \partial_{\tau_k} + \tilde{B}_k + \tilde{q} M_0 D^{-1} \tilde{r}^T + \Phi M D^{-1} \psi^T, \quad (38)
\]

where

\[
\begin{align*}
M &= CA - \hat{A}^T C, \quad \Phi = \varphi \Delta^{-1}, \quad \Psi = \psi \Delta^{-1}, \quad \Delta = C + D^{-1} \{ \psi^T \varphi \}, \\
\tilde{q} &= W \{ q \}, \quad \tilde{r} = W^{-1} \tau \{ r \}. 
\end{align*}
\]

\( \hat{u}_j \) are scalar coefficients depending on functions \( \varphi, \psi \) and \( u_i, \ i = 0, j \). In particular,

\[
\hat{u}_k = u_k, \quad \hat{u}_{k-1} = u_{k-1}, \ldots.
\]

The exact forms of all the coefficients \( \hat{u}_j \) can be found in \[35\].

Using the previous theorem we obtain the following result for \((2+1)\)-generalization of operator \( L_k \) from the k-cmKP hierarchy \[10\]:

**Theorem 2.** Let \((1 \times K)\)-vector functions \( \varphi \) and \( \psi \) satisfy linear problems:

\[
\begin{align*}
L_k \{ \varphi \} &= \varphi \Lambda, \quad L_k^T \{ \psi \} = \psi \hat{\Lambda}, \quad \Lambda, \hat{\Lambda} \in \text{Mat}_{K \times K}(\mathbb{C}), \\
L_k &= \beta_k \partial_{\tau_k} + B_k - q M_0 D^{-1} r^T D, \quad B_k = \sum_{i=1}^k u_i D_i. \quad (40)
\end{align*}
\]

Then the operator \( \tilde{L}_k \) transformed via operator

\[
W_m := w_0^{-1} W = w_0^{-1} \left( I - \varphi \Delta^{-1} D^{-1} \psi^T \right) = I - \varphi \Delta^{-1} D^{-1} \{ D^{-1} \{ \psi \} \}^T D, \quad (41)
\]

where

\[
\begin{align*}
w_0 &= I - \varphi \Delta^{-1} D^{-1} \{ \psi \}, \quad \Delta = C + D^{-1} \{ \psi^T \varphi \}, \\
\Delta &= C + D^{-1} \{ \psi^T \varphi \}
\end{align*}
\]

has the form:

\[
\tilde{L}_k := W_m L_k W_m^{-1} = \beta_k \partial_{\tau_k} + \tilde{B}_k - \tilde{q} M_0 D^{-1} \tilde{r}^T D + \tilde{\Phi} M D^{-1} \psi^T D, \quad (42)
\]

where

\[
\begin{align*}
M &= CA - \hat{A}^T C, \quad \tilde{\Phi} = -W_m \{ \varphi \} C^{-1} = \varphi \Delta^{-1}, \\
\tilde{\Psi} &= D^{-1} \{ W_m^{-1} \{ \psi \} \} C^{-1} = D^{-1} \{ \psi \} \Delta^{-1}, \quad \tilde{q} = W_m \{ q \}, \\
\tilde{r} &= D^{-1} W_m^{-1} D \{ r \}, \quad \Delta = C + D^{-1} \{ \psi^T \varphi \}. \quad (43)
\end{align*}
\]

**Proof.** Let us check that

\[
w_0^{-1} = I - \varphi \Delta^{-1} D^{-1} \{ \psi^T \}, \quad \Delta = C + D^{-1} \{ \psi^T \varphi \}.
\]

In order to do that we have to verify the equality \( w_0 w_0^{-1} = I \):

\[
w_0 w_0^{-1} = I - \varphi \Delta^{-1} D^{-1} \{ \psi^T \} - \varphi \Delta^{-1} D^{-1} \{ \psi^T \}
\]
Analogously can be verified that \( w_0^{-1}w_0 = I \). By Theorem 1 we obtain:

\[
W_mL_kW_m^{-1} = w_0^{-1}W \left( \beta_k\partial_r + B_k - qM_0\tilde{r}_x + qM_0D^{-1}\tilde{r}_x^T \right) W^{-1}w_0 = \\
\beta_k\partial_r + (W_mL_kW_m^{-1})_{\geq 0} - w_0^{-1}W\{q\}M_0D^{-1}(W^{-1}\tau \{r_x\})\} w_0 + \\
+ w_0^{-1}\Phi MD^{-1}\Psi^T w_0
\]

(44)

We shall point out that: \( \Psi^T w_0 = \Delta^{-1}\psi^T (I - \varphi\Delta^{-1}D^{-1}\{\psi^T\} = (\Delta^{-1}D^{-1}\{\psi^T\})_x = \Psi^T x \). We shall also observe that:

\[
(W_1^{1,\tau} \{r_x\})^T w_0 = (r_x^T - D^{-1}\{r_x^T \varphi\} \Delta^{-1}\psi^T ) (I - \varphi\Delta^{-1}D^{-1}\{\psi^T\}) = \\
(r^T - D^{-1}\{r^T \varphi\} \Delta^{-1}D^{-1}\{\psi^T\})_x = (D^{-1}W_1^{1,\tau}D\{r\})_x = \tilde{r}_x
\]

Thus (44) can be rewritten as:

\[
\tilde{L}_k = W_mL_kW_m^{-1} = w_0^{-1}W \left( \beta_k\partial_r + B_k - qM_0\tilde{r}_x + qM_0D^{-1}\tilde{r}_x^T \right) W^{-1}w_0 = \\
\beta_k\partial_r + (W_mL_kW_m^{-1})_{\geq 0} + qM_0D^{-1}\tilde{r}_x^T - \Phi MD^{-1}\Psi^T = \\
\beta_k\partial_r + (W_mL_kW_m^{-1})_{\geq 0} + qM_0\tilde{r}_x - \Phi MD^{-1}\Psi^T - qM_0D^{-1}\tilde{r}_x D^{-1} - \\
+ \Phi MD^{-1}\Psi D
\]

(45)

Using that \( \tilde{L}_k \{1\} = \tilde{u}_0 = 0 \) we obtain the form of \( \tilde{B}_k \). I.e., \( \tilde{B}_k := (W_mL_kW_m^{-1})_{\geq 0} + \\
qM_0\tilde{r}_x - \Phi MD^{-1}\Psi^T = \sum_{j=1}^k \tilde{u}_j D^j \).

Theorem 2 provides us with a dressing method for k-cmKP hierarchy (11). I.e., the following corollary directly follows from the previous theorem:

**Corollary 1.** Assume that operators \( L_k \) and \( M_n \) in (11) satisfy Lax equation: \([L_k, M_n] = 0\). Let functions \( \varphi \) and \( \psi \) satisfy equations:

\[
L_k \{\varphi\} = \varphi\Lambda, \ L_k \{\psi\} = \psi\tilde{\Lambda}, \ \Lambda, \tilde{\Lambda} \in Mat_{K \times K}(\mathbb{C}), \\
M_n \{\varphi\} = 0, \ M_n \{\psi\} = 0.
\]

Then transformed operators \( \tilde{L}_k = W_mL_kW_m^{-1} \) (see (42) with \( \beta_k = 0 \)) and

\[
\tilde{M}_n = W_mM_nW_m^{-1} = \alpha_n\partial_n - D^n - \sum_{i=1}^{n-1} \tilde{u}_i D^i
\]

via transformation \( W_m \) (47) also satisfy Lax equation: \([\tilde{L}_k, \tilde{M}_n] = 0\)

**Proof.** It can be checked directly that: \([\tilde{L}_k, \tilde{M}_n] = [W_mL_kW_m^{-1}, W_mM_nW_m^{-1}] = W_m[L_k, M_n]W_m^{-1} = 0\). The exact form of operators \( \tilde{L}_k \) and \( \tilde{M}_n \) follows from Theorem 2.

The following corollary follows from Corollary 1 and Theorem 2:

**Corollary 2.** Suppose that functions \( \varphi \) and \( \psi \) satisfy equations (46) with operators \( L_k \) and \( M_n \) defined by (29) then transformed operators have the form:

\[
\tilde{L}_k = B_k - qM_0D^{-1}\tilde{r}_x^T D + \Phi MD^{-1}\Psi^T D + \tilde{v} + \beta D^{-1}\tilde{u}, \\
\tilde{M}_n = \alpha_n\partial_n - \tilde{B}_n, \ \tilde{B}_n = D^n + \sum_{i=1}^{n-1} \tilde{u}_i D^i,
\]

(48)
\[ \mathcal{M} = CA - \tilde{A}^\top C, \tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi\tilde{\Delta}^{-1}, \]
\[ \tilde{\Psi} = D^{-1}\{W_m^{-1}\{\psi\}\}C^{-1}\tilde{\Psi} = D^{-1}\{\psi\}\Delta^{-1}, \tilde{q} = W_m\{q\}, \]
\[ \tilde{r} = W_m^{-1}\{r\}, \Delta = -C + D^{-1}\{\psi^\top\varphi_x\}, \Delta = C + D^{-1}\{\psi^\top\varphi\}, \]
\[ \tilde{u} = W_m^{-1}\{u\}, \tilde{v} = W_m\{v\} + \beta D^{-1}W_m^{-1}\{u\} - \beta W_m\{D^{-1}\{u\}\}. \]

As it was shown in previous Sections the most interesting systems arise from the k-cmKP hierarchy (11) and its reduction (23) after a Hermitian conjugation reduction. Our aim is to show that under additional restrictions binary Darboux Transformation \( W_m \) (11) preserves this reduction.

**Proposition 1.**

1. Let \( \tilde{\psi} = \varphi_x \) and \( C = -C^* \) in the dressing operator \( W_m \). Then the operator \( W_m \) is D-unitary (\( W_m^{-1} = D^{-1}W_m D \)).

2. Let the operator \( L_k \) (11) be D-Hermitian: \( L_k = DL_k D^{-1} \) (D-skew-Hermitian: \( L_k = -DL_k D^{-1} \)) and \( M_n \) (11) be D-Hermitian (D-skew-Hermitian). Then the operator \( L_k = W_m L_k W_m^{-1} \) (see (42) transformed via the D-unitary operator \( W_m \)) is D-Hermitian (D-skew-Hermitian).

3. Assume that the conditions of items 1 and 2 hold. Let \( \tilde{\Lambda} = \Lambda \) in the case of D-Hermitian \( L_k \) (\( \tilde{\Lambda} = -\tilde{\Lambda} \) in D-skew-Hermitian case). We shall also assume that the function \( \varphi \) satisfies the corresponding equations in formulae (46). Then \( \mathcal{M} = \mathcal{M}^* \) (\( \mathcal{M} = -\mathcal{M}^* \)) and \( \tilde{\Psi} = \tilde{\Phi} \) (see formulae 39). 

In subparagraph 4.1 we will show how one can use methods described in Theorem 2 and its corollaries in order to obtain solutions of KdV equation (33) and its generalization (30).

**4.1. Solution generating technique for system (30) and KdV equation (33).** We shall consider equation (30) in case the dimension of vector \( \mathbf{q} \) and matrix \( \mathcal{M}_0 \) is even. I.e., \( m = 2\tilde{m}, \tilde{m} \in \mathbb{N} \) (in this situation skew-symmetric matrix \( \mathcal{M}_0 \) can be non-degenerate). Assume that the skew-symmetric matrix \( \mathcal{M}_0 \) in (30) and vector-function \( \mathbf{q} \) has the form:

\[ \mathcal{M}_0 = \begin{pmatrix} 0_{\tilde{m}} & I_{\tilde{m}} \\ -I_{\tilde{m}} & 0_{\tilde{m}} \end{pmatrix}, \quad \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) = (q_{11}, q_{12}, \ldots, q_{1\tilde{m}}, q_{21}, q_{22}, \ldots, q_{2\tilde{m}}), \]

where \( 0_{\tilde{m}} \) is a \( \tilde{m} \times \tilde{m} \)-dimensional matrix consisting of zeros, \( I_{\tilde{m}} \) is an identity matrix with the dimension \( \tilde{m} \times \tilde{m} \). Equation (30) after notation \( \tilde{u} := u \) can be rewritten in the following form:

\[ \alpha_3 q_{1,t_3} = q_{1,xxx} - 3(q_{1,x} q_{1}^\top - q_{2,x} q_{2}^\top) q_{1,x} + 3\beta \tilde{u} q_{1,x}, \]
\[ \alpha_3 q_{2,t_2} = q_{2,xxx} - 3(q_{1,x} q_{2}^\top - q_{2,x} q_{1}^\top) q_{2,x} + 3\beta \tilde{u} q_{2,x}, \]
\[ \alpha_3 \tilde{u}_{t_3} = \tilde{u}_{xxx} - 3(\tilde{u}(q_{1,x} q_{2}^\top - q_{2,x} q_{1}^\top)) x + 6\beta \tilde{u}_{xx}. \]

In this subsection our aim is to consider the case \( \tilde{m} = 1 \) (although the corresponding solution generating technique can be generalized to the case of an arbitrary natural \( \tilde{m} \)). In this situation \( \mathbf{q}_1 = q_1 \) and \( \mathbf{q}_2 = q_2 \) are scalars. We shall suppose that \( K = 2\tilde{K} \) is an even natural number. Assume that the function \( \varphi \) is \( (1 \times K) \)-vector solution of the system:

\[ L_{10}\{\varphi\} = \varphi_x + \beta D^{-1}\{u\varphi\} = \varphi\Lambda, \quad \Lambda \in \text{Mat}_{K \times K}(\mathbb{C}), \beta \in \mathbb{R}, \]
\[ M_{30}\{\varphi\} = \alpha_3 \varphi_{t_3} - \varphi_{xxx} - 3\beta u \varphi_x = 0, \]

with a number \( u \in \mathbb{R} \).
Using Theorem $2$ and Proposition $1$ we obtain that dressed operators $\tilde{L}_{10}$ and $M_{30}$ via operator $W_m$ with skew-Hermitian matrix $C$ and $\psi = \tilde{\varphi}_x$ has the form:

$$
\begin{align*}
\tilde{L}_{10} &= W_mL_{10}W_m^{-1} = D + \tilde{\Phi}MD^{-1}\tilde{\Phi}^*D + \beta D^{-1}\tilde{u} + \tilde{v} \\
M_{30} &= W_mM_{30}W_m^{-1} = \alpha_3\partial_{t_3} - D^3 - (\tilde{v} + \tilde{\Phi}M\tilde{\Phi}^*)D^2 - 3\left((\tilde{\Phi}M\tilde{\Phi}^* + \tilde{v})^2 + \tilde{\Phi}_x M\tilde{\Phi}^* + \tilde{v}_x + \beta \tilde{u}\right)D,
\end{align*}
$$

(53)

where $M = CL - \Lambda^*C^*$, $\tilde{\Phi} = \varphi \tilde{\Delta}^{-1}$, $\tilde{u} = u - D\{\tilde{\varphi} \tilde{\Delta}^{-1}D^{-1}\{\varphi^T u\}\}$, $\tilde{v} = \beta(\tilde{\Phi}D^{-1}\{\varphi^* u\} - D^{-1}\{u \varphi\}\tilde{\Phi}^*)$, $\tilde{\Delta} = -C + D^{-1}\{\varphi^* \varphi_x\}$. It has to be pointed out that the function $\tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi \Delta^{-1}$ satisfies equation: $M_{30}\{\tilde{\Phi}\} = 0$ because $M_{30}\{\tilde{\Phi}\} = W_mM_{30}W_m^{-1}(W_m\{\varphi\}C^{-1}) = 0$.

Now we assume that function $\varphi$, matrices $C$ and $\Lambda$ are real. In this case $\tilde{v} = \tilde{v}^T = \beta(\tilde{\Phi}D^{-1}\{\varphi^T u\} - D^{-1}\{u \varphi\}\tilde{\Phi}^*)^T = -\tilde{v} = 0$.

Let us put

$$
\Lambda = \text{diag}(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \ldots, \lambda_{K1}, \lambda_{K2}), \lambda_{ij} \in \mathbb{R},
$$

(54)

where elements $C_{ij}$ are $(2 \times 2)$-matrices of the form:

$$
C_{ij} = \begin{pmatrix}
0 & 1 \\
\frac{1}{\lambda_{ij}} & 0 \\
\end{pmatrix},
$$

(55)

Under such a choice of $C$ and $\Lambda$ we obtain that $2\hat{K} \times 2\hat{K}$-dimensional matrix $M = CA - \Lambda^*C^T$ has the block form: $M = (M_{ij})_{\hat{K} \times \hat{K}}$, where $M_{ij} = M_0$ (see formula $(54)$ in case $\tilde{m} = 1$). Let us denote by: $1_{\hat{K}} = (I_2, \ldots, I_2)$ matrix that consists of $K$ $(2 \times 2)$-dimensional identity matrices $I_2$. Then $M = -1_{\hat{K}}^T M_{30} 1_{\hat{K}}$.

Let us put $u = \text{const}$ and choose solution of system $(52)$ in the form: $\varphi = (\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{K1}, \varphi_{K2})$, $\varphi_{ij} = \exp\left\{\left(\frac{1}{4} \lambda_{ij} + \gamma_{ij}\right)x + a_{ij}t\right\}$, where $\gamma_{ij} = \sqrt{\frac{1}{4} \lambda_{ij}^2 - \beta u}$, $a_{ij} = \left\{\left(\frac{1}{4} \lambda_{ij} + \gamma_{ij}\right)^3 + 3 \beta u \left(\frac{1}{4} \lambda_{ij} + \gamma_{ij}\right)\right\}/\alpha_3$. $(2\hat{K} \times 2\hat{K})$-matrix $\tilde{\Delta}$ then takes the block form:

$$
\tilde{\Delta} = -C + D^{-1}\{\varphi^T \varphi_x\} = (\tilde{\Delta}_{ij})_{\hat{K} \times \hat{K}} = 
$$

$$
= \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1\hat{K}} \\
\alpha_{11} + \alpha_{12} & \alpha_{12} + \alpha_{13} & \ldots & \alpha_{1\hat{K}} + \alpha_{1\hat{K}} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{11} + \alpha_{1\hat{K}} & \alpha_{12} + \alpha_{1\hat{K}} & \ldots & \alpha_{1\hat{K}} + \alpha_{1\hat{K}} \\
\end{pmatrix} - \frac{1}{\lambda_{11} + \lambda_{12}} = 
$$

$$
\tilde{\Delta}_{ij} = \begin{pmatrix}
\alpha_{11} & \alpha_{12} \alpha_{13} & \ldots & \alpha_{1\hat{K}} \\
\alpha_{11} + \alpha_{12} & \alpha_{12} + \alpha_{13} & \ldots & \alpha_{1\hat{K}} + \alpha_{1\hat{K}} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{11} + \alpha_{1\hat{K}} & \alpha_{12} + \alpha_{1\hat{K}} & \ldots & \alpha_{1\hat{K}} + \alpha_{1\hat{K}} \\
\end{pmatrix},
$$

(56)

where $\alpha_{ij} = \frac{1}{4} \lambda_{ij} + \gamma_{ij}$. Functions $q = (q_1, q_2) = \varphi \tilde{\Delta}^{-1}1_{\hat{K}}$ and $\tilde{u} = u - D\{\varphi \tilde{\Delta}^{-1}D^{-1}\{\varphi^T u\}\}$ will be solutions of system $(51)$.

We shall point out that in case $\beta = 0$, $\tilde{K} = 1$, $\alpha_3 = 1$ we obtain the following solution of the real version of the mKdV-type equation (equation $(51)$ with $\tilde{u} = 0$):

$$
q = (q_1, q_2) = \begin{pmatrix}
-\frac{2(\lambda_{11} + \lambda_{12})\varphi_{12}}{(\lambda_{11} + \lambda_{12})\varphi_{11}x + \lambda_{11}t - z} \\
-\frac{2(\lambda_{11} + \lambda_{12})\varphi_{11}}{(\lambda_{11} + \lambda_{12})\varphi_{11}x + \lambda_{11}t - z} \\
\end{pmatrix},
$$

$$
\varphi_{1j} = e^{\lambda_{1j}x + \lambda_{1j}t}, \lambda_{1j} > 0, j = 1, 2.
$$
It is also possible to choose other types of matrices $C$ and $\Lambda$ in \((4.1)\) and \((55)\). In particular the following remark holds:

**Remark 1.** In case $\hat{K} = 1$ vector of functions $\varphi = (\varphi_1, \varphi_2)$, $\varphi_1 = \cos(x\lambda_{12} + (3\lambda_{11}^2\lambda_{12} - \lambda_{12}^3)t + \frac{\pi}{4})e^{x\lambda_{11} + (\lambda_{11}^3 - 3\lambda_{11}\lambda_{12})t}$, $\varphi_2 = \sin(x\lambda_{12} + (3\lambda_{11}^2\lambda_{12} - \lambda_{12}^3)t + \frac{\pi}{4})e^{x\lambda_{11} + (\lambda_{11}^3 - 3\lambda_{11}\lambda_{12})t}$. will be a solution of the system \((52)\) with $u = 0$ and $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ -\lambda_{12} & \lambda_{11} \end{pmatrix}$. The corresponding solution generating technique given by \((4.1)-(4.2)\) in case $\hat{K} = 1$, $C_{\hat{K}} = C_1 = \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix}$ gives us a solution of mKdV-type equation \((57)\) with $\hat{u} = 0$ that coincides with a solution obtained in \([36]\).

Now we will consider solution generating technique for KdV \((53)\). For this purpose we assume that function $\varphi$, matrices $\Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_\hat{K})$ and $C = \text{diag}(C_1, \ldots, C_\hat{K})$ are real and have the form:

$$
\Lambda_j = \begin{pmatrix} 0 & \lambda_j \\ \lambda_j & 0 \end{pmatrix}, \ C_j = \begin{pmatrix} 0 & -c_j \\ c_j & 0 \end{pmatrix}.
$$

(57)

In this case we obtain that the matrix $M = CA - \Lambda^T C^T$ consists of zeros in \((58)\). Consider the following solution of system \((52)\):

$$
\varphi = (\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{\hat{K}1}, \varphi_{\hat{K}2}),
\varphi_{j1} = e^{\gamma_j x + a_j t} \cosh\left(\frac{\lambda_j}{2} x + b_j t\right), \ \varphi_{j2} = e^{\gamma_j x + a_j t} \sinh\left(\frac{\lambda_j}{2} x + b_j t\right),
$$

where $\gamma_j = \sqrt{\frac{1}{4} \lambda_j^2 - \beta u}$, $a_j = (\gamma_j^3 + \frac{3}{2} \gamma_j \lambda_j^2 + 3\beta u \gamma_j) / \alpha_3$, $b_j = \left(3\gamma_j^2 \frac{\lambda_j}{2} + \frac{\lambda_j^3}{2} + \frac{3}{2} \beta u \lambda_j\right) / \alpha_3$ and $\lambda_j$, $\alpha_3$, $\beta$, $u \in \mathbb{R}$. Thus, $\hat{v} = 0$ and we obtain Lax pair for KdV equation in \((53)\):

$$
\hat{L}_{10} = D + \beta D^{-1} \hat{u}, \ \hat{M}_{30} = \alpha_3 \partial_{t_3} - D^3 - 3\beta \hat{u} D.
$$

Formula

$$
\hat{u} = u - D \left\{ \varphi \tilde{\Delta}^{-1} D^{-1} \{ \varphi^T u \} \right\} := u + \hat{u},
$$

(58)

gives us a finite density solution of equation \((58)\). In particular, if $\hat{K} = 1$ and $c_1 = \frac{1}{\sqrt{\gamma_1}}$ we obtain the following solution:

$$
\hat{u} = u + \frac{2\gamma_1^2}{\beta \cosh^2 (\gamma_1 x + a_1 t)}.
$$

(59)

Now we shall substitute $\hat{u}$ \((58)\) in KdV equation \((53)\):

$$
\alpha_3 \partial_{t_3} \hat{u} = \hat{u}_{xxx} + 6\beta \hat{u}_{xx} + 6\beta u \hat{u}_x.
$$

(60)

The corresponding pair of operators have the form: $L_1 = D + \beta D^{-1} (u + u)$, $M_3 = \alpha_3 \partial_{t_3} - D^3 - 3\beta \hat{u} D - 3\beta u D$. We have two ways to obtain soliton solutions (that are rapidly decreasing at both infinities in contradistinction to finite density solutions \((60)\) that tend to an arbitrary real number $u$) for KdV from formula \((58)\):

1. By taking the limit $u \to 0$ in \((58)-(60)\).
2. By making a change of the independent variables: $\tilde{x} := x + 6\alpha_3^{-1} \beta u t_3$, $\tilde{t}_3 := t_3$ and $\tilde{v}(\tilde{x}, \tilde{t}_3) := u(x, t_3)$ in equation \((60)\) and solutions \((58)-(59)\). This change corresponds to the change of differential operators in the Lax pair for equation \((60)\) consisting of operators $L_1$ and $M_3$: $\alpha_3 \partial_{\tilde{t}_3} = \alpha_3 \partial_{t_3} - 3\beta u D$. 


5. Conclusions

In this paper we obtain new generalizations (23) of the modified k-cKP (k-cmKP) hierarchy (11). The obtained hierarchy also generalizes the BKP hierarchy [36,38] which is the special case of the k-cmKP hierarchy. Dressing methods elaborated via BDT-type operators (Section 4) give rise to exact solutions of the integrable systems that hierarchy (23) contains. In particular, soliton solutions for general hierarchies (11). The obtained hierarchy also generalizes the BKP hierarchy [36–38] corresponding integrable systems. Consider as an example Lax pair from the (1+1)-BDk-cmKP hierarchy and investigate solution generating technique for the dressing methods. In particular in our forthcoming papers we plan to introduce (2+1)-BDk-cmKP hierarchy that was investigated in [33]:

\[ P_{1,1} = D + c_1 M_2(q) M_0 D^{-1} r^T + c_1 q M_0 D^{-1} (M_2^a(r))^T + c_0 q M_0 D^{-1} r^T = = D + (\alpha_2 q_2 M_0 D^{-1} r^T - \alpha_2 q M_0 D^{-1} r_2^T + q_{xx} M_0 D^{-1} r^T - q M_0 D^{-1} r_{xx} - u q M_0 D^{-1} r^T - q M_0 D^{-1} r^T u) + c_0 q M_0 D^{-1} r^T, \]

(61)

It was shown in [33] that the Lax equation \([P_{1,1}, M_2] = 0\) in (61) is equivalent to the system:

\[ [P_{1,1}, M_2] = 0, \quad c_1 M_2^a(q) + c_0 M_2(q) = 0, \quad c_1 (M_2^a)^2(r) + c_0 M_2^a(r) = 0. \]

(62)

that is equivalent to the generalization of the AKNS system. In case \(c_0 = 1, c_1 = 0\) we obtain AKNS system in (62):

\[ \alpha_2 q_2 - q_{xx} - u q = 0, -\alpha_2 r_t - r_{xx} - u r = 0, \quad u = 2 q M_0 r^T. \]

Assume that the scalar function \(f\) satisfies equations \(P_{1,1}(f) = f \lambda, M_2(f) = 0\). We shall introduce the notations \(\tilde{M}_2 := f^{-1} M_2 f, \tilde{M}_2 := D \tilde{M}_2 D^{-1}, \tilde{P}_{1,1} := f^{-1} P_{1,1} f, \tilde{q} := f^{-1} q, \tilde{r} := D^{-1} (r^T f)\) and consider the following gauge transformations

\[ \tilde{M}_2 = f^{-1} M_2 f = \alpha_2 \partial_t - D^2 - 2 \tilde{u} D, \quad \tilde{M}_2 f = f^{-1} f_x, \]

\[ \tilde{P}_{1,1} := f^{-1} P_{1,1} f = D + f^{-1} f_x + c_1 f^{-1} M_2(q) M_0 D^{-1} r^T f + c_0 f^{-1} q M_0 D^{-1} (M_2^a(r))^T f = = D - c_1 \tilde{M}_2(q) M_0 D^{-1} (\tilde{r}^T D - c_1 \tilde{q} M_0 D^{-1} (\tilde{M}_2^a(\tilde{r}))^T D - c_0 \tilde{q} M_0 D^{-1} \tilde{r}^T D. \]

The equation \([\tilde{M}_2, \tilde{P}_{1,1}] = 0\) is equivalent to the following system:

\[ [\tilde{P}_{1,1}, \tilde{M}_2] = 0, c_1 \tilde{M}_2^a(q) + c_0 \tilde{M}_2(q) = 0, \quad c_1 (\tilde{M}_2^a)^2(\tilde{r}) + c_0 \tilde{M}_2^a(\tilde{r}) = 0. \]

(63)

or in the equivalent form (after notation \(q_0 := \tilde{q}, r_0 := \tilde{r}\)):

\[ [\tilde{P}_{1,1}, \tilde{M}_2] = 0, \quad q_1 = \tilde{M}_2(q_0), \quad r_1 = \tilde{M}_2^a(r_0), \]

\[ c_1 \tilde{M}_2(q_1) + c_0 \tilde{M}_2(q_0) = 0, \quad c_1 \tilde{M}_2^a(r_1) + c_0 \tilde{M}_2^a(r_0) = 0. \]

(64)
System (64) is the generalization of the Chen-Lee-Liu system (case \(c_1 = 0, c_0 = 1\)). In case of additional reduction \(\alpha_2 \in i\mathbb{R}, c_0 = 0, c_1 \in \mathbb{R}, M^*_0 = -M_0, r = \bar{q}\) reads as following:

\[
\begin{align*}
\alpha_2 q_{0,t_2} - q_{0,xx} + 2c_1(q_1 M_0 q_0^* + q_0 M(q_1^*) q_{0,x} - q_0 = 0,
\alpha_2 q_{1,t_2} - q_{1,xx} + 2c_1(q_1 M_0 q_0^* + q_0 M(q_1^*) q_{1,x} = 0.
\end{align*}
\]

(65)

We shall also point out that the extension of the k-cmKP hierarchy (23) can also be generalized to the matrix case. It leads to matrix generalizations of integrable systems that hierarchy (23) contains (including Chen-Lee-Liu (16) and modified-type KdV equation (19)). In particular, the matrix generalization of the modified KdV-type equation (19) differs from the well-known matrix mKdV equation that was investigated by the inverse scattering method in [39].

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