Abstract. We aim to maximize the number of first-quadrant lattice points in a convex domain with respect to reciprocal stretching in the coordinate directions. The optimal domain is shown to be asymptotically balanced, meaning that the stretch factor approaches 1 as the “radius” approaches infinity. In particular, the result implies that among all \( p \)-ellipses (or Lamé curves), the \( p \)-circle encloses the most first-quadrant lattice points as the radius approaches infinity.

The case \( p = 2 \) corresponds to minimization of high eigenvalues of the Dirichlet Laplacian on rectangles, and so our work generalizes a result of Antunes and Freitas. Similarly, we generalize a Neumann eigenvalue maximization result of van den Berg, Bucur and Gittins.

The case \( p = 1 \) remains open: which right triangles in the first quadrant with two sides along the axes will enclose the most lattice points, as the area tends to infinity?

1. Introduction

Among ellipses of given area centered at the origin and symmetric about both axes, which one encloses the most integer lattice points in the open first quadrant? One might guess the optimal ellipse would be circular, but a non-circular ellipse can enclose more lattice points, as shown in Figure 1. Nonetheless, optimal ellipses must become more and more circular as the area increases to infinity, by a striking result of Antunes and Freitas [2].

To formulate the problem more precisely, consider the number of positive-integer lattice points lying in the elliptical region

\[
\left( \frac{x}{s^{-1}} \right)^2 + \left( \frac{y}{s} \right)^2 \leq r^2,
\]

where the ellipse has “radius” \( r > 0 \) and semiaxes proportional to \( s^{-1} \) and \( s \). Notice that the area \( \pi r^2 \) of the ellipse is independent of the “stretch factor” \( s \). Denote by \( s = s(r) \) a value (not necessarily unique) of the stretch factor that maximizes the lattice point count. Then the theorem of Antunes and Freitas says \( s(r) \to 1 \) as \( r \to \infty \). In other words, optimal ellipses become circular in the infinite limit. (Their theorem was stated differently, in terms of minimizing the \( n \)-th eigenvalue of the Dirichlet Laplacian on rectangles. Section 10 explains the connection.)
Figure 1. Circle $s = 1$ and ellipse $s = 1.15$, for radius $r = 4.96$. The ellipse encloses three more points than the circle, as shown in bold.

An analogous result for optimal ellipsoids becoming spherical has been proved recently in three dimensions by van den Berg and Gittins [4], again in the eigenvalue formulation. The higher dimensional case remains open.

This paper extends the result of Antunes and Freitas from circles to essentially arbitrary concave curves in the first quadrant that decrease between the intercept points $(0, 1)$ and $(1, 0)$. The “ellipses” in this situation are the images of the concave curve under rescaling by $s^{-1}$ and $s$ in the horizontal and vertical directions, respectively. Our main result, Theorem 2, shows under a mild monotonicity hypothesis on the second derivative of the curve that $s(r) \to 1$ as $r \to \infty$. Thus the most “balanced” curve in the family will enclose the most lattice points in the limit.

Theorem 3 allows the curvature to blow up or degenerate at the intercept points, which permits us to treat the family of $p$-ellipses for $1 < p < \infty$. In each case the $p$-circle is asymptotically optimal for the lattice counting problem in the first quadrant. The case $p = 1$ in Section 9 appears to be an open problem. Our numerical evidence suggests that the right triangle enclosing the most lattice points in the open first quadrant (and with right angle at the origin) does not approach a 45–45–90 degree triangle as $r \to \infty$. Instead one seems to get an infinite limit set of optimal triangles.

Finally, if one counts lattice points in the closed first quadrant (that is, counting points on the axes as well), then the results reverse direction from maximization to minimization of the lattice count. Theorem 5 shows that the value $s = s(r)$ minimizing the number of enclosed lattice points will tend to 1 as $r \to \infty$. In the case of circles and ellipses, this result was obtained recently by van den Berg, Bucur and Gittins [3]; as explained in Section 10, they showed that the maximizing rectangle for the $n$-th eigenvalue of the Neumann Laplacian must approach a square as $n \to \infty$.

The method for general concave curves in this paper builds on the framework of Antunes and Freitas for ellipses. Specifically, first we develop a non-sharp bound on the counting function (Proposition 6) in order to control the stretch factor $s(r)$. Then we prove more precise lattice counting estimates (Proposition 7) of Krätzel type, relying on a lemma of van der Corput discussed in Appendix A. These estimates enable us to show $s(r) \to 1$ as $r \to \infty$, in the proof of Theorem 2.
2. Assumptions and definitions

The first quadrant is the open set \( \{(x, y) : x, y > 0\} \).
Assume throughout the paper that \( \Gamma \) is a concave, strictly decreasing curve in the first quadrant that intercepts the \( x \)- and \( y \)-axes at \( x = L \) and \( y = M \) respectively. Write \( \text{Area}(\Gamma) \) for the area enclosed by the curve \( \Gamma \) and the \( x \)- and \( y \)-axes.

We represent the curve \( \Gamma \) by \( y = f(x) \) for \( x \in [0, L] \), so that \( f \) is a concave strictly decreasing function, and of course \( f \) is continuous. In particular
\[
M = f(0) > f(x) > f(L) = 0
\]
whenever \( x \in (0, L) \). Denote the inverse function of \( f(x) \) by \( g(y) \) for \( y \in [0, M] \), so that \( g \) also is concave and strictly decreasing.

We define a rescaling of the curve by parameter \( r > 0 \):
\[
r\Gamma = \text{image of } \Gamma \text{ under the radial scaling } \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},
\]
and define an area-preserving stretch of the curve by:
\[
\Gamma(s) = \text{image of } \Gamma \text{ under the diagonal scaling } \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix},
\]
where \( s > 0 \). In other words, \( \Gamma(s) \) is obtained from \( \Gamma \) after compressing the \( x \)-direction by \( s \) and stretching the \( y \)-direction by \( s \). Define the counting function for \( r\Gamma(s) \) by
\[
N(r, s) = \text{number of positive-integer lattice points lying inside or on } r\Gamma(s)
\]
\[
= \# \{(j, k) \in \mathbb{N} \times \mathbb{N} : k \leq rsf(js/r) \}.
\]
For each \( r > 0 \), we consider the set
\[
S(r) = \arg\max_{s>0} N(r, s)
\]
consisting of the \( s \)-values that maximize the number of first-quadrant lattice points enclosed by the curve \( r\Gamma(s) \). The set \( S(r) \) is well-defined because the maximum is indeed attained, as the following argument shows. The curve \( r\Gamma(s) \) has \( x \)-intercept at
$rs^{-1}L$, which is less than 1 if $s > rL$ and so in that case the curve encloses no positive-integer lattice points. Similarly if $s < (rM)^{-1}$, then $r\Gamma(s)$ has height less than 1 and contains no lattice points in the first quadrant. Thus for each fixed $r > 0$, if $s$ is sufficiently small or sufficiently large then the counting function $N(r, s)$ equals zero, while obviously for intermediate values of $s$ the integer-valued function $s \mapsto N(r, s)$ is bounded. Hence $N(r, s)$ attains its maximum at some $s > 0$.

For later reference, we write down the above bound on optimal $s$-values.

**Lemma 1** ($r$-dependent bound on optimal stretch factors).

$S(r) \subset [(rM)^{-1}, rL]$ whenever $r \geq 2(LM)^{-1/2}$.

**Proof.** The curve $r\Gamma(s)$ with the particular choice $s = \sqrt{L/M}$ has horizontal and vertical intercepts equal to $r\sqrt{LM}$. That intercept value is $\geq 2$ since $r \geq 2(LM)^{-1/2}$. Hence by concavity, $r\Gamma(s)$ encloses the point $(1, 1)$, and so the maximum of the counting function $s \mapsto N(r, s)$ is greater than zero. Now argue as above to conclude $S(r) \subset [(rM)^{-1}, rL]$.  

3. Main results

Recall in the next theorem that $g$ is the inverse function of $f$, as illustrated in Figure 2.

**Theorem 2** (Optimal concave curve is asymptotically balanced). Assume $(\alpha, \beta) \in \Gamma$ is a point in the first quadrant such that $f \in C^2[0, \alpha]$ with $f' < 0$ on $(0, \alpha]$ and $f'' < 0$ on $[0, \alpha]$, and similarly that $g \in C^2[0, \beta]$ with $g' < 0$ on $(0, \beta]$ and $g'' < 0$ on $[0, \beta]$. Further suppose $f''$ is monotonic on $[0, \alpha]$ and $g''$ is monotonic on $[0, \beta]$.

If the intercepts of $\Gamma$ are equal ($L = M$), then the optimal stretch factor for maximizing $N(r, s)$ approaches 1 as $r$ tends to infinity, with

$$S(r) \subset [1 - O(r^{-1/6}), 1 + O(r^{-1/6})],$$

and the maximal lattice count has asymptotic formula

$$\max_{s > 0} N(r, s) = r^2 \text{Area}(\Gamma) - rL + O(r^{2/3}).$$

The theorem is proved in Section 6. Slight improvements to the decay rate $O(r^{-1/6})$ and the error term $O(r^{2/3})$ are possible, as explained after Proposition 7.

**Example** (Connection to Dirichlet eigenvalues of the Laplacian on rectangles). When $\Gamma$ is a quarter circle, the counting function $N(r, s)$ counts the Dirichlet eigenvalues of the Laplacian on a rectangle. This case of Theorem 2 corresponds to an eigenvalue minimization result of Antunes and Freitas [2], as explained in Section 10.

For minimizing eigenvalues on general domains, and not just on rectangles, the optimal shapes up to the 15th eigenvalue have been determined numerically with impressive accuracy [1, 11]. This numerical work does not, regrettably, provide a clear indication of a limiting optimal shape. Nonetheless, we agree with Antunes and Freitas’s remark that the “most natural guess” is that the shape minimizing the
nth eigenvalue will approach a disk as \( n \to \infty \). This is known to occur if the area normalization is strengthened to a perimeter normalization [5].

**More general curves for lattice counting.** We want to weaken the smoothness and monotonicity assumptions in Theorem 2. We start with a definition of piecewise smoothness.

**Definition** \((PC^2)\).

(i) We say a function \( f \) is piecewise \( C^2 \)-smooth on a half-open interval \((0, \alpha]\) if \( f \) is continuous and a partition \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_l = \alpha \) exists such that \( f \in C^2(0, \alpha_1] \) and \( f \in C^2[\alpha_{i-1}, \alpha_i] \) for \( i = 2, \ldots, l \). Write \( PC^2(0, \alpha] \) for the class of such functions.

(ii) Write \( f' < 0 \) to mean that \( f' \) is negative on the subintervals \((0, \alpha_1]\) and \([\alpha_{i-1}, \alpha_i]\) for \( i = 2, \ldots, l \), with the derivative being taken in the one-sided senses at the partition points \( \alpha_1, \ldots, \alpha_l \). The meaning of \( f'' < 0 \) is analogous.

(iii) We will label partition points using the same letter as for the right endpoint. In particular, the partition for \( g \in PC^2(0, \beta] \) is \( 0 = \beta_0 < \cdots < \beta_\ell = \beta \).

For the next theorem, take a point \((\alpha, \beta) \in \Gamma \) lying in the first quadrant and suppose we are given continuous functions

\[
\delta : (0, \infty) \to (0, \alpha), \quad \epsilon : (0, \infty) \to (0, \beta),
\]

and numbers \( a_1, a_2, b_1, b_2 > 0 \) that satisfy the following conditions as \( r \to \infty \):

\[
\begin{align*}
\delta(r) &= O(r^{-2a_1}), \quad f''(\delta(r))^{-1} = O(r^{1-4a_2}), \\
\epsilon(r) &= O(r^{-2b_1}), \quad g''(\epsilon(r))^{-1} = O(r^{1-4b_2}).
\end{align*}
\]  

(The second condition in (1) says that \( f''(x) \) cannot be too small as \( x \to 0 \).) Let

\[
e = \min \{ \frac{1}{6}, a_1, a_2, b_1, b_2 \}.
\]

Now we extend Theorem 2 to a larger class of concave curves. As always, the curve \( \Gamma \) is assumed to be concave and strictly decreasing in the first quadrant.

**Theorem 3** (Optimal concave curve is asymptotically balanced).

Assume \( f \in PC^2(0, \alpha] \) with \( f' < 0 \) and \( f'' < 0 \), and \( f'' \) is monotonic on \((\alpha_{i-1}, \alpha_i] \) for \( i = 1, \ldots, l \). Similarly assume \( g \in PC^2(0, \beta] \) with \( g' < 0 \) and \( g'' < 0 \), and \( g'' \) is monotonic on \((\beta_{j-1}, \beta_j] \) for \( j = 1, \ldots, \ell \). Suppose the continuous functions \( \delta(r) \) and \( \epsilon(r) \) satisfy conditions (1) and (2).

If the intercepts of \( \Gamma \) are equal \( (L = M) \), then the optimal stretch factor for maximizing \( N(r, s) \) approaches 1 as \( r \) tends to infinity, with

\[
S(r) \subset [1 - O(r^{-e}), 1 + O(r^{-e})],
\]

and the maximal lattice count has asymptotic formula

\[
\max_{s > 0} N(r, s) = r^2 \text{Area}(\Gamma) - rL + O(r^{1-2e}).
\]

The proof is presented in Section 7.
Example 4 (Optimal $p$-ellipses for lattice point counting). Fix $1 < p < \infty$, and consider the $p$-circle
\[ \Gamma : |x|^p + |y|^p = 1, \]
which has equal intercepts $L = M = 1$. That is, the $p$-circle is the unit circle for the $\ell^p$-norm on the plane. Then the $p$-ellipse
\[ r\Gamma(s) : |sx|^p + |s^{-1}y|^p \leq r^p \]
has first-quadrant counting function
\[ N(r, s) = \# \{(j, k) \in \mathbb{N} \times \mathbb{N} : (js)^p + (ks^{-1})^p \leq r^p\}. \]

We will show that the $p$-ellipse containing the maximum number of positive-integer lattice points must approach a $p$-circle in the limit as $r \to \infty$, with
\[ S(r) \subset [1 - O(r^{-e}), 1 + O(r^{-e})] \]
where $e = \min \{1/6, 1/2p \}$.

Theorem 2 fails to apply to $p$-ellipses when $p \neq 2$, because the second derivative of the curve (see $f''(x)$ below) blows up at $x = 0$ when $1 < p < 2$, and vanishes there when $2 < p < \infty$. Instead we will apply Theorem 3.

To verify that the $p$-circle satisfies the hypotheses of Theorem 3, we let $\alpha = \beta = 2^{-1/p}$ and choose
\[ \delta(r) = r^{-1/p}, \quad \epsilon(r) = r^{-1/p}, \]
for all large $r$. Then $\delta(r) = r^{-2a_1}$ with $a_1 = 1/2p$. Next,
\[
\begin{align*}
    f(x) &= (1 - x^p)^{1/p}, \\
    f'(x) &= -x^{p-1}(1 - x^p)^{-1+1/p}, \\
    f''(x) &= -(p - 1)x^{p-2}(1 - x^p)^{-2+1/p},
\end{align*}
\]
so that
\[ |f''(\delta(r))|^{-1} \leq (\text{const.}) r^{-2/p}, \]
and hence $a_2 = 1/2p$ in (1). Thus $f$ satisfies hypothesis (1).

Further, the interval $(0, \alpha)$ can be partitioned into subintervals on which $f''$ is monotonic, because the third derivative
\[ f'''(x) = -(p - 1)x^{p-3}(1 - x^p)^{-3+1/p}((1 + p)x^p + p - 2) \]
vanishes at most once in the unit interval.

The calculation is the same for $g$, and so the desired conclusion for $p$-ellipses follows from Theorem 3, when $1 < p < \infty$.

For $p = \infty$, the $\infty$-circle is a Euclidean square and the $\infty$-ellipse is a rectangle. Many different rectangles of given area can contain the same number of lattice points, for example, a $4 \times 1$ rectangle and a $2 \times 2$ square, and so the maximizing $s$-values need not approach 1 as $r \to \infty$. This and other issues one might want to address for the counting function when $p = \infty$ can be handled using the explicit formula
\[ N(r, s) = \lfloor rs^{-1} \rfloor \lfloor rs \rfloor. \]

The case $p = 1$ seems to be an open problem, as discussed in Section 9.
Incidentally, an explicit estimate on the number of lattice points in the full $p$-ellipse (all four quadrants) was obtained by Krätzel [10, Theorem 2] for $p \geq 2$.

**Lattice points in the closed first quadrant, and Neumann eigenvalues.** Our results have analogues for lattice point counting in the closed (rather than open) first quadrant, as we now explain. When counting nonnegative integer lattice points, which means we include lattice points on the axes, the counting function for $r\Gamma(s)$ is

$$N(r, s) = \# \{(j, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : k \leq rs f(js/r)\},$$

where $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$. Define

$$S(r) = \operatorname{argmin}_{s>0} N(r, s).$$

In other words, the set $S(r)$ consists of the $s$-values that minimize the number of lattice points inside the curve $r\Gamma(s)$ in the closed first quadrant.

**Theorem 5** (Optimal concave curve is asymptotically balanced). *If the intercepts of $\Gamma$ are equal ($L = M$), then under the assumptions of Theorem 2 the optimal stretch factor for minimizing $N(r, s)$ approaches 1 as $r$ tends to infinity:*

$$S(r) \subset [1 - O(r^{-1/6}), 1 + O(r^{-1/6})],$$

$$\min_{s>0} N(r, s) = r^2 \text{Area}(\Gamma) - rL + O(r^{2/3}),$$

*and under the assumptions of Theorem 3 we have similarly that:*

$$S(r) \subset [1 - O(r^{-e}), 1 + O(r^{-e})],$$

$$\min_{s>0} N(r, s) = r^2 \text{Area}(\Gamma) - rL + O(r^{1-2e}).$$

The proof is in Section 8.

Consequently, we reprove a recent theorem of van den Berg, Bucur and Gittins [3] saying that the optimal rectangle of area 1 for maximizing the $n$-th Neumann eigenvalue of the Laplacian approaches a square as $n \to \infty$. See Section 10 for a detailed discussion, and explanation of why the approach in this paper is simpler.

4. **Two-term upper bound on counting function**

In order to control the stretch factor when proving our main results later in the paper, we now develop a two-term upper bound on the lattice point counting function. The leading order term of the bound is simply the area inside the curve, and thus is best possible, while the second term scales like the length of the curve and so has the correct order of magnitude.

Recall $\Gamma$ is the graph of $y = f(x)$, where $f$ is concave and strictly decreasing on $[0, L]$, with $f(0) = M$, $f(L) = 0$. We do not assume $f$ is differentiable, in the next proposition.

**Proposition 6** (Two-term upper bound on counting function). *Let $C = M - f(L/2)$. 
Figure 3. Positive integer lattice count satisfies $N \leq \text{Area}(\Gamma) - \text{Area(triangles)}$, in proof of Proposition 6(a).

(a) The number $N$ of positive-integer lattice points lying inside $\Gamma$ in the first quadrant satisfies

$$N \leq \text{Area}(\Gamma) - \frac{1}{2} C$$

provided $L \geq 1$.

(b) The number $N(r, s)$ of positive-integer lattice points lying inside $r \Gamma(s)$ in the first quadrant satisfies

$$N(r, s) \leq r^2 \text{Area}(\Gamma) - \frac{1}{2} Crs$$

whenever $r \geq s/L$.

Proof. Part (a).

Clearly $N$ equals the total area of the squares of sidelength 1 having upper right vertices at positive-integer lattice points inside the curve $\Gamma$. Consider also the right triangles of width 1 formed by secant lines on $\Gamma$ (see Figure 3), that is, the triangles with vertices $(i - 1, f(i - 1)), (i, f(i)), (i - 1, f(i))$, where $i = 1, \ldots, \lfloor L \rfloor$. These triangles lie above the squares by construction, and lie below $\Gamma$ by concavity. Hence

$$N + \text{Area(triangles)} \leq \text{Area}(\Gamma).$$

Since $f$ is decreasing, we find

$$\text{Area(triangles)} = \sum_{i=1}^{\lfloor L \rfloor} \frac{1}{2} (f(i - 1) - f(i))$$

$$= \frac{1}{2} (f(0) - f(\lfloor L \rfloor))$$

$$\geq \frac{1}{2} (M - f(L/2)) = \frac{1}{2} C$$

because $\lfloor L \rfloor \geq L/2$ when $L \geq 1$. Combining (4) and (5) proves Part (a).

Part (b). Simply replace $\Gamma$ in Part (a) with the curve $r \Gamma(s)$, meaning we replace $L, M, f(x)$ with $rs^{-1}L, rsM, rsf(sx/r)$ respectively. □
5. Two-term counting estimates with explicit remainder

We start with a result for $C^2$-smooth curves. What matters in the following proposition is that the right side of estimate (6) has the form $O(r^q)$ for some $q < 1$, and that the $s$-dependence in the estimate can be seen explicitly. The detailed dependence on the functions $f$ and $g$ will not be important for our purposes.

**Proposition 7** (Two-term counting estimate). Take a point $(\alpha, \beta) \in \Gamma$ lying in the first quadrant, and assume that $f \in C^2[0, \alpha]$ with $f' < 0$ on $(0, \alpha]$ and $f'' < 0$ on $[0, \alpha]$, and similarly that $g \in C^2[0, \beta]$ with $g' < 0$ on $(0, \beta]$ and $g'' < 0$ on $[0, \beta]$, and further suppose $f''$ is monotonic on $[0, \alpha]$ and $g''$ is monotonic on $[0, \beta]$.

(a) The number $N$ of positive-integer lattice points inside $\Gamma$ in the first quadrant satisfies:

$$|N - \text{Area}(\Gamma) + (L + M)/2| \leq 6 \left( \int_{0}^{\alpha} |f''(x)|^{1/3} \, dx + \int_{0}^{\beta} |g''(y)|^{1/3} \, dy \right) + 175 \left( \max_{[0, \alpha]} \frac{1}{|f''|^{1/2}} + \max_{[0, \beta]} \frac{1}{|g''|^{1/2}} \right) + \frac{1}{4}(|f'(\alpha)| + |g'(\beta)|) + 7.$$

(b) The number $N(r, s)$ of positive-integer lattice points lying inside $r\Gamma(s)$ in the first quadrant satisfies (for $r, s > 0$):

$$|N(r, s) - r^2 \text{Area}(\Gamma) + r(s^{-1}L + sM)/2| \leq 6r^{2/3} \left( \int_{0}^{\alpha} |f''(x)|^{1/3} \, dx + \int_{0}^{\beta} |g''(y)|^{1/3} \, dy \right) + 175r^{1/2} \left( \max_{[0, \alpha]} \frac{s^{-3/2}}{|f''|^{1/2}} + \max_{[0, \beta]} \frac{s^{3/2}}{|g''|^{1/2}} \right) + \frac{1}{4}(s^2|f'(\alpha)| + s^{-2}|g'(\beta)|) + 7. \quad (6)$$

Proposition 7 and its proof are closely related to work of Krätzel [10, Theorem 1]. We give a direct proof below for two reasons: we want the estimate (6) that depends explicitly on the stretching parameter $s$, and we want a proof that can be modified to use a weaker monotonicity hypothesis, in Proposition 8.

A better bound on the right side of (6), giving order $O(r^{\theta+\epsilon})$ with $\theta = 131/208 \approx 0.63 < 2/3$, can be found in work of Huxley [8], with precursors in [7, Theorems 18.3.2 and 18.3.3]. That bound is difficult to prove, though, and the improvement is not important for our purposes since it leads to only a slight improvement in the rate of convergence for $S(r)$, namely from $O(r^{-1/6})$ to $O(r^{(\theta-1)/2})$ in Theorem 2.

**Proof.** Part (a). We divide the region under $\Gamma$ into three parts. Let $N_1$ count the lattice points lying to the left of the line $x = \alpha$ and above $y = \beta$, and $N_2$ count the lattice points to the right of $x = \alpha$ and below $y = \beta$, and $N_3$ count the lattice points...
in the remaining rectangle \((0, \alpha] \times (0, \beta]\). That is,

\[
N_1 = \sum_{0 < m \leq \alpha} \sum_{\beta < n \leq f(m)} 1 = \sum_{0 < m \leq \alpha} (\lfloor f(m) \rfloor - \lfloor \beta \rfloor),
\]

\[
N_2 = \sum_{0 < n \leq \beta} \sum_{\alpha < m \leq g(n)} 1 = \sum_{0 < n \leq \beta} (\lfloor g(n) \rfloor - \lfloor \alpha \rfloor),
\]

\[
N_3 = \lfloor \alpha \rfloor \lfloor \beta \rfloor.
\]

In terms of the \textit{sawtooth} function \(\psi\), defined by

\[
\psi(x) = x - \lfloor x \rfloor - \frac{1}{2},
\]

one can evaluate

\[
N_1 = \sum_{0 < m \leq \alpha} (f(m) - \psi(f(m)) - 1/2 - \lfloor \beta \rfloor).
\]

Then we apply the Euler–Maclaurin summation formula

\[
\sum_{0 < m \leq \alpha} f(m) = \int_0^\alpha f(x) \, dx - \psi(\alpha)f(\alpha) + \psi(0)f(0) + \int_0^\alpha f'(x)\psi(x) \, dx
\]

(which we observe for later reference holds whenever \(f\) is piecewise \(C^1\)-smooth) to deduce that

\[
N_1 = \int_0^\alpha f(x) \, dx - \psi(\alpha)f(\alpha) + \psi(0)f(0) + \int_0^\alpha f'(x)\psi(x) \, dx
\]

\[
- \sum_{0 < m \leq \alpha} \psi(f(m)) - \lfloor \alpha \rfloor (1/2 + \lfloor \beta \rfloor)
\]

\[
= \int_0^\alpha f(x) \, dx - \psi(\alpha)\beta - M/2 + \int_0^\alpha f'(x)\psi(x) \, dx
\]

\[
- \sum_{0 < m \leq \alpha} \psi(f(m)) - \lfloor \alpha \rfloor (1/2 + \lfloor \beta \rfloor).
\]

Similarly

\[
N_2 = \int_0^\beta g(y) \, dy - \psi(\beta)\alpha - L/2 + \int_0^\beta g'(y)\psi(y) \, dy
\]

\[
- \sum_{0 < n \leq \beta} \psi(g(n)) - \lfloor \beta \rfloor (1/2 + [\alpha]),
\]
and so
\[ N = N_1 + N_2 + N_3 \]
\[ = \int_0^\alpha f(x) \, dx + \int_0^\beta g(y) \, dy - [\alpha] [\beta] - (L + M)/2 \]
\[ - \psi(\alpha) \beta - [\alpha]/2 - \psi(\beta) \alpha - [\beta]/2 \]
\[ + \int_0^\alpha f'(x) \psi(x) \, dx + \int_0^\beta g'(y) \psi(y) \, dy \]
\[ - \sum_{0 < m \leq \alpha} \psi(f(m)) - \sum_{0 < n \leq \beta} \psi(g(n)) \]
\[ = \text{Area}(\Gamma) - (L + M)/2 + \int_0^\alpha f'(x) \psi(x) \, dx + \int_0^\beta g'(y) \psi(y) \, dy \]
\[ - \sum_{0 < m \leq \alpha} \psi(f(m)) - \sum_{0 < n \leq \beta} \psi(g(n)) + \text{remainder} \quad (7) \]
where
\[ \text{remainder} = -(\alpha - [\alpha]) (\beta - [\beta]) + (\alpha - [\alpha] + \beta - [\beta])/2. \quad (8) \]

This remainder lies between 0 and 1, since 0 ≤ −xy + (x + y)/2 ≤ 1 when x, y ∈ [0, 1].

We estimate the sum of sawtooth functions in (7) by using Lemma 12 (which is due to van der Corput): since \( f'' \) is monotonic and nonzero on \([0, \alpha] \), the lemma implies
\[ \left| \sum_{0 < m \leq \alpha} \psi(f(m)) \right| \leq 6 \int_0^\alpha |f''(x)|^{1/3} \, dx + 175 \max_{[0,\alpha]} \frac{1}{|f''|^{1/2}} + 3 \quad (9) \]
and similarly
\[ \left| \sum_{0 < n \leq \beta} \psi(g(n)) \right| \leq 6 \int_0^\beta |g''(y)|^{1/3} \, dy + 175 \max_{[0,\beta]} \frac{1}{|g''|^{1/2}} + 3. \quad (10) \]

To estimate the integrals of \( f'\psi \) and \( g'\psi \) in (7), we introduce the antiderivative of the sawtooth function, \( \Psi(t) = \int_0^t \psi(z) \, dz \), and observe that \(-1/8 \leq \Psi(t) \leq 0 \) for all \( t \in \mathbb{R} \). By integration by parts and the fact that \( f'' < 0 \), we have
\[ \left| \int_0^\alpha f'(x) \psi(x) \, dx \right| = \left| \left[ f'(x) \Psi(x) \right]_{x=0}^{x=\alpha} - \int_0^\alpha f''(x) \Psi(x) \, dx \right| \]
\[ \leq \frac{1}{8} |f'(\alpha)| + \frac{1}{8} \left| \int_0^\alpha f''(x) \, dx \right| \]
\[ = \frac{1}{8} |f'(\alpha)| + \frac{1}{8} (f'(0) - f'(\alpha)) \]
\[ \leq \frac{1}{4} |f'(\alpha)| \quad (11) \]

since \( f'(\alpha) \leq f'(0) \leq 0 \). The same argument gives
\[ \left| \int_0^\beta g'(y) \psi(y) \, dy \right| \leq \frac{1}{4} |g'(\beta)|. \quad (12) \]
Combining (7)–(12) completes the proof of Part (a).

Part (b). Simply apply Part (a) to the curve $r\Gamma(s)$ by replacing $L, M, f(x), g(y), \alpha, \beta$ with $rs^{-1}L, rsM, rsf(sx/r), rs^{-1}g(s^{-1}y/r), rs^{-1}\alpha, rs\beta$ respectively. \(\square\)

**Advanced counting estimate.** The hypotheses in the last result are somewhat restrictive. In particular, we would like to handle infinite curvature or zero curvature at the $x$- and $y$-intercepts of the curve $\Gamma$, which means $f''$ must be allowed to blow up or vanish at $x = 0$. Further, we would like to relax the monotonicity assumption on $f''$. The next result achieves these goals.

Two numbers $\delta$ and $\epsilon$ appear in the next Proposition. Their role in the proof is that on the intervals $0 < x \leq \delta$ and $0 < y \leq \epsilon$ we bound the sawtooth function trivially with $|\psi| \leq 1/2$. On the remaining intervals we seek cancellations.

**Proposition 8** (Two-term counting estimate for more general curve). Take a point $(\alpha, \beta) \in \Gamma$ lying in the first quadrant, and assume $f \in PC^2(0, \alpha]$ with $f' < 0$ and $f'' < 0$, and that $f''$ is monotonic on $[\alpha_i-1, \alpha_i]$ for $i = 1, \ldots, l$. Similarly assume $g \in PC^2(0, \beta]$ with $g' < 0$ and $g'' < 0$, and that $g''$ is monotonic on $[\beta_{j-1}, \beta_j]$ for $j = 1, \ldots, \ell$.

(a) If $\delta \in (0, \alpha)$ and $\epsilon \in (0, \beta)$ then the number $N$ of positive-integer lattice points inside $\Gamma$ in the first quadrant satisfies:

$$|N - \text{Area}(\Gamma) + (L + M)/2| \leq 6\left( \int_0^\alpha |f''(x)|^{1/3} \, dx + \int_0^\beta |g''(y)|^{1/3} \, dy \right) + 175\left( \frac{1}{|f''(\delta)|^{1/2}} + \frac{1}{|g''(\epsilon)|^{1/2}} \right) + 350\left( \sum_{i=1}^l \frac{1}{|f''(\alpha_i)|^{1/2}} + \sum_{j=1}^{\ell} \frac{1}{|g''(\beta_j)|^{1/2}} \right) + \frac{1}{4}\left( \sum_{i=1}^l |f'(\alpha_i)| + \sum_{j=1}^{\ell} |g'(\beta_j)| \right) + \frac{1}{2}(\delta + \epsilon) + 3(l + \ell) + 1.$$

(b) If functions $\delta : (0, \infty) \to (0, \alpha)$, $\epsilon : (0, \infty) \to (0, \beta)$, are given, then the number $N(r, s)$ of positive-integer lattice points inside $r\Gamma(s)$ in the first quadrant satisfies (for $r, s > 0$):

$$|N(r, s) - r^2 \text{Area}(\Gamma) + rs^{-1}L + sM)/2| \leq 6r^{2/3}\left( \int_0^\alpha |f''(x)|^{1/3} \, dx + \int_0^\beta |g''(y)|^{1/3} \, dy \right) + 175r^{1/2}\left( \frac{s^{-3/2}}{|f''(\delta(r))|^{1/2}} + \frac{s^{3/2}}{|g''(\epsilon(r))|^{1/2}} \right) + 350r^{1/2}\left( \sum_{i=1}^l s^{-3/2}|f''(\alpha_i)|^{1/2} + \sum_{j=1}^{\ell} s^{3/2}|g''(\beta_j)|^{1/2} \right) + \frac{1}{4}\left( \sum_{i=1}^l s^2|f'(\alpha_i)| + \sum_{j=1}^{\ell} s^{-2}|g'(\beta_j)| \right) + \frac{r}{2}(s^{-1}\delta(r) + s\epsilon(r)) + 3(l + \ell) + 1. \quad (13)$$
The integral of $|f''|^{1/3}$ appearing in the conclusion of Proposition 8 is finite, because by Hölder’s inequality and the fact that $f'' < 0$ and $f$ is decreasing, we have

$$\int_0^{\alpha_1} |f''(x)|^{1/3} \, dx \leq \alpha_1^{2/3} \left| \int_0^{\alpha_1} f''(x) \, dx \right|^{1/3} = \alpha_1^{2/3} |f'(0^+) - f'(\alpha_1)|^{1/3} < \infty.$$  

The integral of $|g''|^{1/3}$ is finite for the same reasons.

**Proof.** Part (a). The lattice point counting equation (7) holds just as in the proof of Proposition 7, and so the task is to estimate each of the terms on the right side of that equation.

Estimate (9) on the sum of the sawtooth function is no longer valid, because $f''$ is no longer assumed to be monotonic on the whole interval $[0, \alpha]$. To control this sawtooth sum, we first observe

$$\left| \sum_{0 < m \leq \delta} \psi(f(m)) \right| \leq \frac{1}{2} \delta$$

since $|\psi| \leq 1/2$ everywhere. Next, we have $\delta \in (\alpha_{j-1}, \alpha_j]$ for some $j \in \{1, \ldots, l\}$, and

$$\left| \sum_{\delta < m \leq \alpha_j} \psi(f(m)) \right| \leq 6 \int_{\delta}^{\alpha_j} |f''(x)|^{1/3} \, dx + 175 \max \left\{ \frac{1}{|f''(\delta)|^{1/2}}, \frac{1}{|f''(\alpha_j)|^{1/2}} \right\} + 3$$

by Lemma 12 applied on the interval $(\delta, \alpha_j]$. Applying that lemma again on each interval $(\alpha_{i-1}, \alpha_i]$ with $i = j + 1, \ldots, l$ gives that

$$\left| \sum_{\alpha_{i-1} < m \leq \alpha_i} \psi(f(m)) \right| \leq 6 \int_{\alpha_{i-1}}^{\alpha_i} |f''(x)|^{1/3} \, dx + 175 \max \left\{ \frac{1}{|f''(\alpha_{i-1})|^{1/2}}, \frac{1}{|f''(\alpha_i)|^{1/2}} \right\} + 3.$$

By summing the last three displayed inequalities, we deduce a sawtooth bound

$$\left| \sum_{0 < m \leq \alpha} \psi(f(m)) \right|$$

$$\leq \frac{1}{2} \delta + 6 \int_{\delta}^{\alpha} |f''(x)|^{1/3} \, dx + \frac{175}{|f''(\delta)|^{1/2}} + \sum_{i=j}^{l-1} \frac{350}{|f''(\alpha_i)|^{1/2}} + \frac{175}{|f''(\alpha)|^{1/2}} + 3(l - j + 1)$$

$$\leq \frac{1}{2} \delta + 6 \int_0^{\alpha} |f''(x)|^{1/3} \, dx + \frac{175}{|f''(\alpha)|^{1/2}} + \sum_{i=1}^{l} \frac{350}{|f''(\alpha_i)|^{1/2}} + 3l.$$  \hspace{1cm} (14)

Next, we adapt estimate (11) on the integral of $f'\psi$ by simply applying the same argument on each interval $[\alpha_{i-1}, \alpha_i]$, hence finding

$$\left| \int_0^{\alpha} f'(x)\psi(x) \, dx \right| \leq \sum_{i=1}^{l} \left[ \frac{1}{8} |f'(\alpha_i)| + \frac{1}{8} (f'(\alpha_{i-1}) - f'(\alpha_i)) \right]$$

$$\leq \frac{1}{4} \sum_{i=1}^{l} |f'(\alpha_i)|.$$  \hspace{1cm} (15)
By combining (7), (8) with (14), (15) and the analogous estimates on \( g \), we complete the proof of Part (a).

Part (b). Apply Part (a) to the curve \( r \Gamma(s) \) by replacing \( L, M, f(x), g(y), \alpha, \beta, \delta, \epsilon \) with \( rs^{-1}L, rsM, rsf(sx/r), rs^{-1}g(s^{-1}y/r), rs^{-1}\alpha, rs\beta, rs^{-1}\delta(r), rs\epsilon(r) \) respectively. \( \square \)

6. Proof of Theorem 2

Step 1 — Estimate on remainder terms. From Proposition 7(b) we have that

\[
|N(r,s) - r^2 \text{Area}(\Gamma) + r(s^{-1}L + sM)/2| 
\leq O(r^{2/3}) + (s^{-3/2} + s^{3/2})O(r^{1/2}) + (s^2 + s^{-2})O(1) + O(1)
\]

for \( r > 0 \), where the constants implicit in the \( O(\cdot) \)-terms depend only on the curve \( \Gamma \) and are independent of \( s \).

Step 2 — \( S(r) \) is bounded above and away from 0. Applying (16) with \( s = 1 \) gives that

\[
r^2 \text{Area}(\Gamma) - cr/2 \leq N(r,1)
\]

for all large \( r \), where the constant \( c > 0 \) depends only on the curve \( \Gamma \). Suppose \( r \) is large enough that this estimate holds, and that \( r \geq 2(LM)^{-1/2} \). Let \( s \in S(r) \). Then \( r \geq s/L \) by Lemma 1, and so Proposition 6(b) (which uses concavity of the curve \( \Gamma \)) applies to give

\[
N(r,s) \leq r^2 \text{Area}(\Gamma) - Crs/2.
\]

The choice “\( s = 1 \)” must give a smaller lattice point count than obtained from the maximizing value \( s \), which means \( N(r,1) \leq N(r,s) \). Hence we may combine the preceding inequalities to deduce that

\[
s \leq c/C.
\]

Interchanging the roles of the horizontal and vertical axes, we similarly find

\[
s^{-1} \leq \tilde{c}/\tilde{C}
\]

for some constants \( \tilde{c}, \tilde{C} > 0 \). Thus the set \( S(r) \) is bounded above and bounded away from 0, for all large \( r \).

Step 3 — \( S(r) \) approaches \( \{1\} \) as \( r \to \infty \). Let \( s \in S(r) \), so that \( s \) and \( s^{-1} \) are bounded above independently of \( r \) by Step 2, for all large \( r \). Then the right side of (16) has the form \( O(r^{2/3}) \) for all large \( r \), with the constant implicit in this “\( O \)” term being independent of \( s \). Thus

\[
N(r,s) \leq r^2 \text{Area}(\Gamma) - r(s^{-1}L + sM)/2 + O(r^{2/3}),
\]

\[
N(r,1) \geq r^2 \text{Area}(\Gamma) - r(L + M)/2 - O(r^{2/3}),
\]

as \( r \to \infty \). Since \( N(r,1) \leq N(r,s) \) as above, we conclude

\[
s^{-1}L + sM \leq L + M + O(r^{-1/3}).
\]
Taking \( L = M \) implies \( s = 1 + O(r^{-1/6}) \), by Lemma 13, which proves the first claim in the theorem. For the second claim, when \( s \in S(r) \) we have

\[
N(r, s) = r^2 \text{Area}(\Gamma) - rL + O(r^{2/3})
\]

by (16), using also that \( 1 \leq (s + s^{-1})/2 \leq 1 + O(r^{-1/3}) \) by (18) with \( L = M \).

7. Proof of Theorem 3

Step 1 — Estimate on remainder terms. We will show that the right side of estimate (13) in Proposition 8(b) is bounded by

\[
O(r^{2/3}) + s^{-3/2}O(r^{1-2a_2}) + s^{3/2}O(r^{1-2b_2}) + (s^{-3/2} + s^{3/2})O(r^{1/2})
\]

\[
+ (s^2 + s^{-2})O(1) + s^{-1}O(r^{1-2a_1}) + sO(r^{1-2b_1}) + O(1)
\]

for \( r \geq 1 \), where the constants implicit in the \( O(\cdot) \)-terms depend only on the curve \( \Gamma \) and are independent of \( s \).

The first two terms on the right of (13) are obviously \( O(r^{2/3}) \). For the next term, observe by assumption in (1) that

\[
\frac{r^{1/2}s^{-3/2}}{|f''(\delta(r))|^{1/2}} = s^{-3/2}O(r^{1-2a_2}),
\]

and similarly for the analogous term involving \( g'' \). Since \( f''(\alpha_i) \) and \( g''(\beta_j) \) are constant, the corresponding terms in (13) can be estimated by \((s^{-3/2} + s^{3/2})O(r^{1/2})\). Similarly, the terms involving \( f'(\alpha_i) \) and \( g'(\beta_j) \) can be estimated by \((s^2 + s^{-2})O(1)\).

Next, \( s^{-1}r\delta(r) = s^{-1}O(r^{1-2a_1}) \) by the assumption in (1), and similarly for \( \epsilon(r) \). And, of course, \( 3(l + \ell) + 1 \) is constant. That concludes the estimation of (13).

Step 2 — \( S(r) \) is bounded above and away from 0. By Step 1, the right side of estimate (13) is bounded by \( O(r) \) when \( s = 1 \), and so Proposition 8(b) with \( s = 1 \) gives that

\[
r^2 \text{Area}(\Gamma) - cr/2 \leq N(r, 1)
\]

for all large \( r \). Here the constant \( c > 0 \) depends only on the curve \( \Gamma \). Now the boundedness of the set \( S(r) \) can be shown exactly as in the proof of Theorem 2.

Step 3 — \( S(r) \) approaches \( \{1\} \) as \( r \to \infty \). Let \( s \in S(r) \), so that Step 2 implies that \( s \) and \( s^{-1} \) are bounded above independently of \( r \), for all large \( r \). Then by Step 1, the right side of estimate (13) is bounded by

\[
O(r^{2/3}) + O(r^{1-2a_2}) + O(r^{1-2b_2}) + O(r^{1-2a_1}) + O(r^{1-2b_1}) \leq O(r^{1-2\epsilon})
\]

(19)

for all large \( r \). Here the constants implicit in the \( O(\cdot) \)-terms depend only on the curve \( \Gamma \) and are independent of \( s \). Thus by (13),

\[
N(r, s) \leq r^2 \text{Area}(\Gamma) - r(s^{-1}L + sM)/2 + O(r^{1-2\epsilon}),
\]

\[
N(r, 1) \geq r^2 \text{Area}(\Gamma) - r(L + M)/2 - O(r^{1-2\epsilon}),
\]
as \( r \to \infty \). Since choosing “\( s = 1 \)” must give a smaller lattice point count than we get from the maximizing value \( s \), we see that \( N(r, 1) \leq N(r, s) \). Combining the preceding displayed inequalities therefore gives

\[
s^{-1}L + sM \leq L + M + O(r^{-2\epsilon}). \tag{20}
\]

Hence if \( L = M \) then \( s = 1 + O(r^{-\epsilon}) \) as \( r \to \infty \), by Lemma 13.

Further, when \( s \in S(r) \) the lattice count has asymptotic formula

\[
N(r, s) = r^2 \text{Area}(\Gamma) - rL + O(r^{1\epsilon})
\]

by (13) and (19), using also that \( 1 \leq (s + s^{-1})/2 \leq 1 + O(r^{-2\epsilon}) \) by (20) with \( L = M \).

8. Proof of Theorem 5

First we need a two-term bound on the counting function in the closed first quadrant. Assume \( f \) is concave and strictly decreasing on \([0, L]\), with \( f(0) = M, f(L) = 0 \). Then we have the following analogue of Proposition 6, although the constant \( C \) is slightly different than in that result.

**Proposition 9** (Two-term lower bound on counting function). Let \( C = M - f(L/4) \).

(a) The number \( N \) of nonnegative-integer lattice points lying inside \( \Gamma \) in the closed first quadrant satisfies:

\[
N \geq \text{Area}(\Gamma) + \frac{1}{2}C. \tag{21}
\]

(b) The number of nonnegative-integer lattice points lying inside \( r\Gamma(s) \) in the closed first quadrant satisfies (for \( r, s > 0 \)):

\[
N(r, s) \geq r^2 \text{Area}(\Gamma) + \frac{1}{2}Cr \epsilon.
\]

**Proof.** Part (a). Clearly \( N \) equals the total area of the squares of sidelength 1 having lower left vertices at nonnegative integer lattice points inside the curve \( \Gamma \). The union of these squares contains \( \Gamma \), since the curve is decreasing.

We separate the proof into cases according to the value of \( L \).

Case (i): Suppose \( L \leq 2 \), so that \( L/4 \leq 1/2 \). Consider a rectangle whose lower left vertex sits on the curve at \( x = L/4 \), and has vertices

\[(L/4, f(L/4)), \quad (1, f(L/4)), \quad (1, M), \quad (L/4, M).\]

By construction, this rectangle lies inside the union of squares of sidelength 1, and it lies above \( \Gamma \) because the curve is decreasing. Hence

\[
N \geq \text{Area}(\Gamma) + \text{Area(rectangle)}
\]

\[= \text{Area}(\Gamma) + (1 - L/4)(M - f(L/4))\]

\[\geq \text{Area}(\Gamma) + \frac{1}{2}(M - f(L/4))\]

as desired.

Case (ii): Suppose \( L \geq 2 \). Consider the right triangles of width 1 formed by tangent lines on \( \Gamma \), that is, the triangles with vertices \((i, f(i)), (i+1, f(i)), (i+1, f(i) + f'(i+))\),
where \( i = 0, 1, \ldots, \lfloor L \rfloor - 1 \). These triangles all lie above the horizontal axis, since by concavity \( f(i) + f'(i+) \geq f(i+1) \geq 0 \); the last inequality explains why the biggest \( i \)-value we consider is \( \lfloor L \rfloor - 1 \).

Thus these triangles lie inside the union of squares of sidelength 1, and lie above \( \Gamma \) by concavity. Hence

\[
\mathcal{N} \geq \text{Area}(\Gamma) + \text{Area}(\text{triangles}).
\]

To complete the proof of Case (ii), we estimate

\[
\text{Area}(\text{triangles}) \geq \frac{1}{2} \sum_{i=1}^{\lfloor L \rfloor - 1} |f'(i+)| \\
\geq \frac{1}{2} \sum_{i=1}^{\lfloor L \rfloor - 1} (f(i-1) - f(i)) \quad \text{by concavity} \\
= \frac{1}{2} (f(0) - f(\lfloor L \rfloor - 1)) \\
\geq \frac{1}{2} (M - f(L/4))
\]

because \( \lfloor L \rfloor - 1 \geq L/2 \geq L/4 \) when \( L \geq 2 \).

Part (b). Replace \( \Gamma \) in Part (a) with the curve \( r\Gamma(s) \), meaning we replace \( L, M, f(x) \) with \( rs^{-1}L, rsM, rsf(sx/r) \) respectively. \( \square \)

**Proof of Theorem 5.** The number of lattice points on the axes inside \( r\Gamma(s) \) is

\[
\lfloor Lr/s \rfloor + \lfloor M rs \rfloor + 1 = Lr/s + M rs + \rho(r,s)
\]

where \( |\rho(r,s)| \leq 1 \). Thus our two counting functions \( \mathcal{N}(r,s) \) and \( N(r,s) \) (which include and exclude the points on the axes, respectively) are related by

\[
\mathcal{N}(r,s) = N(r,s) + r(s^{-1}L + sM) + \rho(r,s).
\]  

(22)

Suppose first that the assumptions of Theorem 2 hold.
Step 1 — Estimate on remainder terms. By Step 1 in Section 6 (the proof of Theorem 2) and using (22), we find for $r > 0$ that

$$|\mathcal{N}(r, s) - r^2 \text{Area}(\Gamma) - r(s^{-1}L + sM)/2| \leq O(r^{2/3}) + (s^{-3/2} + s^{3/2})O(r^{1/2}) + (s^2 + s^{-2})O(1) + O(1).$$

(23)

Step 2 — $\mathcal{S}(r)$ is bounded above and away from 0. Applying (23) with $s = 1$ gives that

$$r^2 \text{Area}(\Gamma) + cr/2 \geq \mathcal{N}(r, 1)$$

(24)

for all large $r$, where the constant $c > 0$ depends only on the curve $\Gamma$. Suppose $r$ is large enough that this estimate holds. Let $s \in \mathcal{S}(r)$. Then Proposition 9(b) applies to give

$$\mathcal{N}(r, s) \geq r^2 \text{Area}(\Gamma) + Crs/2.$$

Since $\mathcal{N}(r, 1) \geq \mathcal{N}(r, s)$ by choice of $s$ as a minimizer for $\mathcal{N}(r, \cdot)$, we deduce from the preceding inequalities that

$$s \leq c/C.$$

Interchanging the roles of the horizontal and vertical axes, we similarly find

$$s^{-1} \leq \tilde{c}/\tilde{C}$$

for some constants $\tilde{c}$ and $\tilde{C}$. Thus the set $\mathcal{S}(r)$ is bounded above and bounded away from 0 for all large $r$.

Step 3 — $\mathcal{S}(r)$ approaches $\{1\}$ as $r \to \infty$. Let $s \in \mathcal{S}(r)$, so that $s$ and $s^{-1}$ are bounded above independently of $r$ by Step 2, for all large $r$. Then the right side of estimate (23) has the form $O(r^{2/3})$ for all large $r$, with the constant implicit in this “O” term being independent of $s$. From two applications of estimate (23) we deduce

$$\mathcal{N}(r, s) \geq r^2 \text{Area}(\Gamma) + r(s^{-1}L + sM)/2 - O(r^{2/3}),$$

$$\mathcal{N}(r, 1) \leq r^2 \text{Area}(\Gamma) + r(L + M)/2 + O(r^{2/3}),$$

as $r \to \infty$. Since $\mathcal{N}(r, 1) \geq \mathcal{N}(r, s)$ as above, we deduce that

$$s^{-1}L + sM \leq L + M + O(r^{-1/3}).$$

Suppose $L = M$. Then $s = 1 + O(r^{-1/6})$ as $r \to \infty$ by Lemma 13. Also, estimate (23) implies for $s \in \mathcal{S}(r)$ that

$$\mathcal{N}(r, s) = r^2 \text{Area}(\Gamma) + rL + O(r^{2/3}),$$

where we used that $(s^{-1} + s)/2 = 1 + O(r^{-1/3})$ by above. This finishes the proof of Theorem 5 when the assumptions of Theorem 2 hold.

Now suppose the assumptions of Theorem 3 hold. For Step 1, we use Step 1 in Section 7 (the proof of Theorem 3) together with (22) to show:

$$|\mathcal{N}(r, s) - r^2 \text{Area}(\Gamma) - r(s^{-1}L + sM)/2|$$

$$\leq O(r^{2/3}) + s^{-3/2}O(r^{1-2\alpha_2}) + s^{3/2}O(r^{1-2\beta_2}) + (s^{-3/2} + s^{3/2})O(r^{1/2})$$

$$+ (s^2 + s^{-2})O(1) + s^{-1}O(r^{1-2\alpha_1}) + sO(r^{1-2\beta_1}) + O(1)$$

(25)
for $r \geq 1$, where the constants implicit in the $O(\cdot)$-terms depend only on the curve $\Gamma$ and are independent of $s$.

Then Step 2 proceeds exactly as above, except using (25) instead of (23). For Step 3, proceed again as above, noting that the right side of the estimate (25) has the form $O(r^{1-2\epsilon})$ for all large $r$ (see Step 3 in Section 7), with the constant implicit in this “$O$” term being independent of $s$. Continuing as in Step 3 above shows that $s = 1 + O(r^{-\epsilon})$ and $N(r, s) = r^2 \text{Area}(\Gamma) + rL + O(r^{1-2\epsilon})$. This completes the proof of Theorem 5 when the assumptions of Theorem 3 hold.

9. Open problem for 1-ellipses — lattice points in right triangles

Lattice point maximization for right triangles appears to be an open problem, in the following sense. The $p$-circle when $p = 1$ is a diamond with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. It intersects the first quadrant in the line segment $\Gamma$ joining the points $(0, 1)$ and $(1, 0)$. Here $L = M = 1$. Stretching the 1-circle in the $x$- and $y$-directions gives a 1-ellipse

$$|sx| + |s^{-1}y| = 1,$$

which together with the coordinate axes forms a right triangle of area $1/2$ in the first quadrant, with one vertex at the origin and hypotenuse $\Gamma(s)$ joining the vertices at $(s^{-1}, 0)$ and $(0, s)$.

As previously, we write $S(r)$ for the set of $s$-values that maximize the number of positive-integer (first quadrant) lattice points below or on $r\Gamma(s)$, when $r > 0$. See Figure 5 for an example showing that $s = 1$ need not be a maximizer: the 45–45–90 degree triangle does not always enclose the most lattice points.

The goal is to understand the limiting behavior of the maximizing $s$-values. In particular, does $S(r)$ converge to $\{1\}$ as $r \to \infty$? We proved the answer is “Yes” for $p$-ellipses when $1 < p < \infty$ (see Example 4), but for $p = 1$ it seems to be an open
problem. In fact, we expect the answer is “No” when \( p = 1 \), and we support this belief with numerical and theoretical evidence in the remainder of this section.

**Why is the 1-ellipse not covered by our theorems?** For the \( p \)-ellipse with \( p = 1 \), Theorem 3 does not apply because \( f \) is linear and so \( f'' \equiv 0 \). Specifically, in the proof we see inequalities (9) and (10) are no longer useful, since their right sides are infinite. The situation cannot easily be rescued, because the left side of (6) need not even be \( o(r) \). For example, when \( s = 1 \) and \( r \) is an integer, by evaluating the number \( N(r, 1) \) of lattice points under the curve \( y = r - x \) we find

\[
N(r, 1) - r^2 \text{Area}(\Gamma) + r(L + M)/2 = \frac{1}{2} r(r - 1) - \frac{1}{2} r^2 + r = \frac{1}{2} r,
\]

which is of order \( r \) and hence has the same order as the “boundary term” \( r(L + M)/2 \) on the left side. Thus the method breaks down completely for \( p = 1 \). We seek instead to illuminate the situation through numerical investigations.

**How can one efficiently maximize the lattice counting function for the 1-ellipse?** A brute force method of counting how many lattice points lie under the line \( r \Gamma(s) \), and then varying \( s \) to maximize that number of lattice points, is simply unworkable in practice. The counting function \( N(r, s) \) jumps up and down in value as \( s \) varies, sometimes jumping quite rapidly, and a brute force method of sampling at a finite collection of \( s \)-values can never be expected to capture all such jump points or their precise locations.

Instead, for a given \( r \) we should pre-identify the possible jump values of \( s \), and use that information to count the lattice points. We start with the simple observation that a lattice point \((j, k)\) lies under the line \( r \Gamma(s) \) if and only if

\[
sj + s^{-1}k \leq r,
\]

which is equivalent to

\[
js^2 - rs + k \leq 0.
\]  

(26)

For this quadratic inequality to have a solution, the discriminant must be nonnegative, \( r^2 - 4jk \geq 0 \), and thus we need only consider lattice points beneath the hyperbola \( r^2 = 4xy \). For each such lattice point, equality holds in (26) for two positive \( s \)-values, namely

\[
s_{\min}(j, k; r) = \frac{r - \sqrt{r^2 - 4jk}}{2j}, \quad s_{\max}(j, k; r) = \frac{r + \sqrt{r^2 - 4jk}}{2j}.
\]

The geometrical meaning of these values can be understood, as follows: as \( s \) increases from 0 to \( \infty \), one endpoint of the line segment \( r \Gamma(s) \) slides up on the \( y \)-axis while the other endpoint moves left on the \( x \)-axis. The line segment passes through the point \((j, k)\) twice: first when \( s = s_{\min}(j, k; r) \) and again when \( s = s_{\max}(j, k; r) \). The point \((j, k)\) lies below the line when \( s \) belongs to the closed interval between these two values.
Thus the counting function is

\[ N(r, s) = \# \{(j, k) : s_{\min}(j, k; r) \leq s \leq s_{\max}(j, k; r) \} \]

\[ = \sum_{j, k > 0} 1_{s_{\min}(j, k; r) \leq s} - \sum_{j, k > 0} 1_{s_{\max}(j, k; r) < s} \]

where we sum only over positive-integer lattice points with \(4jk \leq r^2\).

The last formula says that the counting function \(N(r, s)\) equals the number of values \(s_{\min}(j, k; r)\) that are less than or equal to \(s\) minus the number of values \(s_{\max}(j, k; r)\) that are less than \(s\). To facilitate the evaluation in practice, one should sort the list of values of \(s_{\min}(j, k; r)\) into increasing order, and similarly sort the list of values of \(s_{\max}(j, k; r)\). The numbers in these two lists are the only numbers where \(N(r, s)\) can change value, as \(s\) increases. In particular, when \(s\) increases to \(s_{\min}(j, k; r)\), the point \((j, k)\) is picked up by the line segment for the first time and so \(N(r, s)\) increases by 1. When \(s\) increases strictly beyond \(s_{\max}(j, k; r)\), the point \((j, k)\) is dropped by the line segment and so \(N(r, s)\) decreases by 1. Note the counting function might increase or decrease by more than 1 at some \(s\)-values, if the sorted lists of \(s_{\min}\) and \(s_{\max}\) values have repeated entries (arising from lattice points that are picked up by, or else dropped by, the line segment at the same \(s\)-value).

After sorting the \(s_{\min}\) and \(s_{\max}\) lists, we evaluate the maximum of \(N(r, s)\) by scanning through the two lists, increasing a counter by 1 at each number in the sorted \(s_{\min}\) list, and decreasing the counter just after each number in the sorted \(s_{\max}\) list. The largest value achieved by the counter is the maximum of \(N(r, s)\), and \(S(r)\) consists of the closed interval or intervals of \(s\)-values on which this maximum count is attained.

By this method, we can maximize the lattice counting function for the 1-ellipse in a computationally efficient manner, for any given \(r > 0\).

When presenting the results of this method graphically, in Figure 6, we plot only the largest \(s\) value in \(S(r)\), because the family of 1-ellipses is invariant under the map \(s \mapsto 1/s\) and so the smallest value in \(S(r)\) will be just the reciprocal of the largest value.

Numerical evidence in Figure 6 suggests that the set \(S(r)\) does not converge to \(\{1\}\) as \(r \to \infty\). Indeed, the plotted values appear to cluster at a large number of heights, which presumably have some number theoretic significance.

**A possible counterexample?** Inspired by the numerical calculations in Figure 6, one might seek a counterexample at \(s = \sqrt{2}\). We will show below that this value of \(s\) gives a substantially higher count of lattice points than \(s = 1\), for a certain sequence of \(r\)-values tending to infinity. This observation suggests (but does not prove) that \(\sqrt{2}\) or some number close to it should belong to \(S(r)\), for those \(r\)-values. To be clear: we have not found a proof of this claim, but doing so would provide a counterexample to the idea that the set \(S(r)\) converges to \(\{1\}\) as \(r \to \infty\).
Figure 6. Optimal s-values for maximizing the number of lattice points in the 1-ellipse (triangle). The graph plots sup S(r) versus r. The plotted r-values are multiples of $\sqrt{3}/10$, an irrational number chosen in the hope of exhibiting generic behavior.

To compare the counting functions for $s = 1$ and $s = \sqrt{2}$, we first notice that for $s = 1$ the counting function for the 1-circle is given by

$$N(r, 1) = \lfloor r \rfloor \lfloor r - 1 \rfloor / 2, \quad r > 0.$$  

At $s = \sqrt{2}$ the slope of the 1-ellipse is $-2$, and for the special choice $r = \sqrt{2}(m + 1/2)$ with $m \geq 1$ the counting function can be evaluated explicitly as

$$N(r, \sqrt{2}) = m^2.$$  

We further choose $m$ such that $r \in (n - 1/4, n)$ for some integer $n$, noting that an increasing sequence of such $m$-values can be found due to the density in the unit interval of multiples of $\sqrt{2}$ modulo 1. Then, writing $r = n - \epsilon$ where $\epsilon < 1/4$, we have

$$N(r, \sqrt{2}) - N(r, 1) = m^2 - (n - 1)(n - 2)/2$$

$$= \frac{1}{2}(r^2 - \sqrt{2}r + 1/2) - \frac{1}{2}(r + \epsilon - 1)(r + \epsilon - 2)$$

$$\geq \frac{1}{2}r - \text{(constant)}.$$  

Hence $\lim \sup_{r \to \infty} (N(r, \sqrt{2}) - N(r, 1))/r \geq 1/2$, and so $s = \sqrt{2}$ can give (for certain choices of $r$) a substantially higher count of lattice points than $s = 1$, as we wanted to show.

10. **Connection between counting function maximization and eigenvalue minimization**

Maximizing a counting function is morally equivalent to minimizing the size of the things being counted. Let us apply this general principle to the case of the circle

$$\Gamma : x^2 + y^2 = 1 \quad \text{in the first quadrant},$$  

and its associated ellipses $r\Gamma(s)$. In this section, $L = M = 1$ and Area($\Gamma$) = $\pi/4$. 


Minimizing eigenvalues of the Dirichlet Laplacian on rectangles. Write
\[ \{\lambda_n(s) : n = 1, 2, 3, \ldots\} = \{(js)^2 + (ks)^2 : j, k = 1, 2, 3, \ldots\} \]  
so that \( \lambda_n(s) \) is the \( n \)th eigenvalue of the Dirichlet Laplacian on a rectangle of side lengths \( s^{-1}\pi \) and \( s\pi \). (The eigenfunctions have the form \( \sin(jsx)\sin(ksy) \).) Then the lattice point counting function is the eigenvalue counting function, because
\[
N(r, s) = \#\{(j, k) : (js)^2 + (ks)^2 \leq r^2\} = \#\{n : \lambda_n(s) \leq r^2\}.
\]
Define
\[ S_*(n) = \arg\min_{s > 0} \lambda_n(s), \]
so that \( S_*(n) \) is the set of \( s \)-values that minimize the \( n \)th eigenvalue.

The next result says that the rectangle minimizing the \( n \)th eigenvalue will converge to a square as \( n \to \infty \), and so Theorem 2 and its proof generalize the Dirichlet eigenvalue minimization result and proof by Antunes and Freitas.

**Corollary 10** (Antunes and Freitas [2, Theorem 2.1]; optimal Dirichlet rectangle is asymptotically balanced). The optimal stretch factor for minimizing \( \lambda_n(s) \) approaches 1 as \( n \to \infty \), with
\[ S_*(n) \subset [1 - O(n^{-1/12}), 1 + O(n^{-1/12})], \]
and the minimal Dirichlet eigenvalue satisfies the asymptotic formula
\[ \min_{s > 0} \lambda_n(s) = \frac{4}{\pi}n + \left(\frac{4}{\pi}\right)^{3/2} n^{1/2} + O(n^{1/3}). \]

**Proof.** Step 1 — The plan is to adapt the 3-step proof of Theorem 2 to deal with eigenvalues rather than the counting function.

Step 2 — \( S_*(n) \) is bounded above. Let \( s \in S_*(n) \), and put \( r = \sqrt{\lambda_n(s)} \). Notice \( r > s \) by the formula (27) for the eigenvalues, and so Proposition 6(b) gives a lower bound on the \( n \)th eigenvalue:
\[
n \leq \lambda_n(s)\pi/4 - C\sqrt{\lambda_n(s)}s/2.
\]
Discarding the second term on the right implies that \( \lambda_n(s) \geq 4n/\pi \), and so
\[ n \leq \lambda_n(s)\pi/4 - C'\sqrt{ns}. \]  
(28)

Next, the eigenvalues \( \lambda_n(1) \) of the square having side length \( \pi \) satisfy
\[ n \geq \lambda_n(1)\pi/4 - c\sqrt{\lambda_n(1)/2} \]
for all large \( n \), by substituting \( r = \sqrt{\lambda_n(1)} - \delta \) into (17) and letting \( \delta \to 0 \). Hence \( \lambda_n(1) \leq (\text{const.})n \), and so
\[ n \geq \lambda_n(1)\pi/4 - c'\sqrt{n}. \]  
(29)

Combining inequalities (28) and (29) and noting that \( \lambda_n(s) \leq \lambda_n(1) \) (because \( s \in S_*(n) \) minimizes the \( n \)th eigenvalue), we see that
\[ s \leq \frac{c'}{C'^2}. \]
for all large \( n \). Hence \( s \) is bounded above by an absolute constant, for all large \( n \). The boundedness of \( s \in S_s(n) \) for small \( n \) is obvious from the formula for the eigenvalues \( \lambda_n(s) \). So the set \( S_s(n) \) is bounded above, independent of \( n \).

Similarly \( S_s(n) \) is bounded below away from 0, as we see by replacing \( s \) with \( s^{-1} \), in other words, interchanging the roles of the \( x \)- and \( y \)-axes.

Step 3 — \( S_s(n) \) approaches \( \{1\} \) as \( n \to \infty \). In this step we follow closely the approach of Antunes and Freitas [2, Section 3(d)], simply expressing their argument using results from our paper.

Let \( s \in S_s(n) \), so that \( s \) and \( s^{-1} \) are bounded above independently of \( n \), by Step 2. Then the right side of estimate (16) has the form \( O(r^{2/3}) \) for all large \( r \), with the constant implicit in this “\( O \)” term being independent of \( s \). By first substituting \( r = \sqrt{\lambda_n(s)} \) into (16), and then substituting \( r = \sqrt{\lambda_n(s)} - \delta \) into (16) and letting \( \delta \to 0 \), we find

\[
n = \lambda_n(s)\pi/4 - \sqrt{\lambda_n(s)(s^{-1} + s)/2} + O(\lambda_n(s)^{1/3})
\]  

for all \( n \), where we also substituted \( L = M = 1 \). Note the function \( \lambda \mapsto \lambda\pi/4 - \sqrt{\lambda(s^{-1} + s)/2} \) is increasing for all large \( \lambda > 0 \), since \( s \) and \( s^{-1} \) are bounded. Since also \( \lambda_n(s) \leq \lambda_n(1) \) (which holds by the minimizing property of \( s \in S_s(n) \)), we deduce that

\[
n \leq \lambda_n(1)\pi/4 - \sqrt{\lambda_n(1)(s^{-1} + s)/2} + O(\lambda_n(1)^{1/3})
\]  

for all large \( n \). Further, by re-deriving (30) for \( s = 1 \) we obtain

\[
n = \lambda_n(1)\pi/4 - \sqrt{\lambda_n(1)} + O(\lambda_n(1)^{1/3}).
\]

Combining the last two formulas shows that \( s^{-1} + s \leq 2 + O(\lambda_n(1)^{-1/6}) \) as \( n \to \infty \), and so \( s = 1 + O(\lambda_n(1)^{-1/12}) \) by Lemma 13. The remainder term can be replaced by \( O(n^{-1/12}) \), because \( \lambda_n(1) \) is comparable to \( n \) by the above asymptotic formula.

Lastly, substituting \( (s^{-1} + s)/2 = 1 + O(n^{-1/6}) \) into the asymptotic (30) and solving for \( \lambda_n(s) \) shows that

\[
\lambda_n(s) = \frac{4}{\pi} n + \left(\frac{4}{\pi}\right)^{3/2} n^{1/2} + O(n^{1/3})
\]

as desired. \( \square \)

**Remark.** Step 2 in the proof relied on Proposition 6 to bound the stretch factor \( s \) of the optimal rectangle. Note that Proposition 6 is simpler in both statement and proof than the corresponding Theorem 3.1 of Antunes and Freitas [2], which contains an additional lower order term with an unhelpful sign.

**Remark.** One would like to prove using only the definition of the counting function that

\[
S_s(n) \to 1 \quad \text{if and only if} \quad S(r) \to 1,
\]

or in other words that the rectangle minimizing the \( n \)th eigenvalue will converge to a square if and only if the ellipse maximizing the number of lattice points converges to a circle. Then Corollary 10 would follow qualitatively from Theorem 2. Our attempts
to find such an abstract equivalence have failed due to possible multiplicities in the eigenvalues. Perhaps an insightful reader will see how to succeed where we have failed.

Maximizing eigenvalues of the Neumann Laplacian on rectangles. If one considers lattice points in the closed first quadrant, that is, allowing also the lattice points on the axes, then one obtains the Neumann eigenvalues of the rectangle having side lengths \( s^{-1} \pi \) and \( s \pi \):

\[
\{ \mu_n(s) : n = 1, 2, 3, \ldots \} = \{ (js)^2 + (ks^{-1})^2 : j, k = 0, 1, 2, \ldots \}.
\]

Notice the first eigenvalue is always zero: \( \mu_1(s) = 0 \) for all \( s \). The lattice point counting function is once again an eigenvalue counting function, because

\[
\mathcal{N}(r, s) = \# \{(j, k) : (js)^2 + (ks^{-1})^2 \leq r^2 \} = \# \{ n : \mu_n(s) \leq r^2 \}.
\]

The appropriate problem is to maximize the \( n \)th eigenvalue (rather than minimizing as in the Dirichlet case), and so we let

\[
\mathcal{S}_s(n) = \text{argmax}_{s > 0} \mu_n(s).
\]

The corollary below says that the rectangle maximizing the \( n \)th Neumann eigenvalue will converge to a square as \( n \to \infty \).

**Corollary 11** (van den Berg, Bucur and Gittins [3]; optimal Neumann rectangle is asymptotically balanced). The optimal stretch factor for maximizing \( \mu_n(s) \) approaches 1 as \( n \to \infty \), with

\[
\mathcal{S}_s(n) \subset [1 - O(n^{-1/12}), 1 + O(n^{-1/12})],
\]

and the maximal Neumann eigenvalue satisfies the asymptotic formula

\[
\max_{s > 0} \mu_n(s) = \frac{4}{\pi} n - \left( \frac{4}{\pi} \right)^{3/2} n^{1/2} + O(n^{1/3}).
\]

**Proof.** Step 1 — We will adapt the 3-step proof of Theorem 5.

Step 2 — \( \mathcal{S}_s(n) \) is bounded above. Let \( n \geq 2 \) and continue to take \( \Gamma \) to be the quarter-circle of radius 1. Let \( s \in \mathcal{S}_s(n) \), put \( r = \sqrt{\mu_n(s)} - \delta \) into Proposition 9(b), and then let \( \delta \to 0 \) to obtain

\[
n \geq \mu_n(s) \pi/4 + C \sqrt{\mu_n(s)} s/2
\]

for all \( n \). Because \( s \in \mathcal{S}_s(n) \) maximizes the \( n \)th eigenvalue, we have \( \mu_n(s) \geq \mu_n(1) \), and so

\[
n \geq \mu_n(1) \pi/4 + C \sqrt{\mu_n(1)} s/2.
\]

Also, the eigenvalues \( \mu_n(1) \) of the square having side length \( \pi \) are known to satisfy

\[
n \leq \mu_n(1) \pi/4 + C \sqrt{\mu_n(1)} s/2
\]

for all large \( n \), by substituting \( r = \sqrt{\mu_n(1)} \) into (24). For later reference, observe the preceding inequality implies \( \mu_n(1) \geq (\text{const.}) n \) for large \( n \).

Combining the last two displayed inequalities shows that \( s \leq c/C \) for all large \( n \). For small \( n \), the formula for the eigenvalues \( \mu_n(s) \) implies that the maximizing values \( s \in \mathcal{S}_s(n) \) are bounded. Hence the set \( \mathcal{S}_s(n) \) is bounded above. Similarly it is bounded
below away from 0, by interchanging the roles of the $x$- and $y$-axes (replacing $s$ with $s^{-1}$ in the argument above).

Step 3 — $\mathcal{S}_s(n)$ approaches \{1\} as $n \to \infty$. Let $s \in \mathcal{S}_s(n)$, so that $s$ and $s^{-1}$ are bounded above independently of $n$, by Step 2. Then the right side of estimate (23) has the form $O(r^{2/3})$ for $r \geq 1$, with the constant implicit in this “$O$” term being independent of $s$. By substituting $r = \sqrt{\mu_n(s)} - \delta$ into (23) and letting $\delta \to 0$, and then substituting $r = \sqrt{\mu_n(1)}$ into (23) and remembering that $L = M = 1$ for the circle, we find that

$$n \geq \mu_n(s)\pi/4 + \sqrt{\mu_n(s)(s^{-1} + s)/2} - O(\mu_n(s)^{1/3}),$$

$$n \leq \mu_n(1)\pi/4 + \sqrt{\mu_n(1)} + O(\mu_n(1)^{1/3}),$$

as $n \to \infty$. Since $\mu_n(1) \leq \mu_n(s)$ by definition of the maximizing set $\mathcal{S}_s(n)$, we deduce by combining the preceding inequalities that $s^{-1} + s \leq 2 + O(\mu_n(s)^{-1/6})$. Hence $s = 1 + O(\mu_n(s)^{-1/12})$ as $n \to \infty$, by Lemma 13. The remainder term is bounded by $O(n^{-1/12})$, since $\mu_n(s) \geq \mu_n(1) \geq (\text{const.})n$.

For the final claim in the Corollary, first substitute $r = \sqrt{\mu_n(s)}$ into (23), and then substitute $r = \sqrt{\mu_n(s)} - \delta$ into (23) and let $\delta \to 0$, to obtain that

$$n = \mu_n(s)\pi/4 + \sqrt{\mu_n(s)(s^{-1} + s)/2} + O(\mu_n(s)^{1/3})$$

for all $n \geq 2$. Hence $\mu_n(s)$ is comparable to $n$, and so the remainder term can be replaced by $O(n^{1/3})$. We substitute $(s^{-1} + s)/2 = 1 + O(n^{-1/6})$ into the asymptotic formula (31) and solve for $\mu_n(s)$ to find

$$\mu_n(s) = \frac{4}{\pi}n - \left(\frac{4}{\pi}\right)^{3/2}n^{1/2} + O(n^{1/3}).$$

\hfill \Box

Remark. The lower bound on the counting function in Proposition 9, which we used in Step 2 to control the stretch factor $s$ of the optimal rectangle, is simpler in both statement and proof than the corresponding Lemma 2.2 by van den Berg et al. [3]. Further, our Proposition 9 holds for all $r > 0$, whereas [3, Lemma 2.2] holds only for $r \geq 2s$. Consequently we need not establish an \textit{a priori} bound on $s$ as was done in [3, Lemma 2.3].

Those authors did obtain a slightly better rate of convergence than we do, by calling on sophisticated lattice counting estimates of Huxley; see the comments after Proposition 7.

\textbf{Appendix A. The van der Corput sum}

The next lemma is due to van der Corput [6, Satz 5], and is central to the proofs of Proposition 7 and Proposition 8. We formulate the lemma as in Krätzel [9, Korollar zu Satz 1.5, p. 24]. The constants in the lemma are interesting only for being modest in size.

Recall the sawtooth function $\psi(x) = x - \lfloor x \rfloor - 1/2$. 


Lemma 12. Let $a < b$, and suppose $h \in C^2[a, b]$ with $h''$ monotonic and nonzero on $[a, b]$. Then

$$\left| \sum_{a < n \leq b} \psi(h(n)) \right| \leq 6 \int_a^b |h''(t)|^{1/3} \, dt + 175 \max_{[a, b]} \frac{1}{|h''|^{1/2}} + 3.$$ 

Proof. Krätzel’s lemma has “$+2$” as the final term. We have added 1 to correct a gap in the proof, as follows.

Following Krätzel’s notation, suppose $|h''| \geq \lambda > 0$. His proof of Lemma 12 relies on two earlier results, [9, Korollar zu Satz 1.3, p. 16] and [9, Korollar zu Satz 1.4, p. 23], both of which are plainly false when $\lambda > 121$ if $a$ and $b$ are sufficiently close together and the interval $[a, b]$ contains an integer.

This difficulty is fixed if one assumes $|h''| \leq 121$, because then $\lambda \leq 121$ and the two earlier results are valid as written. Hence we obtain Lemma 12 with “$+2$” as the final term, under the assumption $|h''| \leq 121$. (Note the $\lambda$-values used in Krätzel’s proof of Lemma 12 are values of $|h''|$ at certain points in the interval, and so we are not free to simply choose smaller $\lambda$-values when calling on the earlier two results.)

Now suppose $|h''| > 121$ at some point in the interval. In view of the monotonicity, we may suppose $|h''|$ is increasing with $|h''(b)| > 121$ at the right endpoint; the argument is similar if $|h''|$ is decreasing with $|h''(a)| > 121$ at the left endpoint. Let $c \in [a, b)$ be the rightmost point at which $|h''(c)| \leq 121$, or else if no such point exists then let $c = a$. If $c > a$ then $|h''| \leq 121$ on $[a, c]$ and so by the case already proved,

$$\left| \sum_{a < n \leq c} \psi(h(n)) \right| \leq 6 \int_a^c |h''(t)|^{1/3} \, dt + 175 \max_{[a, c]} \frac{1}{|h''|^{1/2}} + 2;$$

if $c = a$ then obviously the same inequality holds. On $(c, b]$ we observe since $|\psi| \leq \frac{1}{2} \leq 6(121)^{1/3} \leq 6|h''|^{1/3}$ that

$$\left| \sum_{c < n \leq b} \psi(h(n)) \right| \leq \frac{1}{2} (b - c + 1) \leq 6 \int_c^b |h''(t)|^{1/3} \, dt + 1.$$ 

Combining the two inequalities finishes the proof of the lemma. \qed

Incidentally, Lemma 12 is used in the proof of a lattice counting estimate due to Krätzel [10, Theorem 1] (see [10, Lemma 2]). That theorem is stated for general $C^2$-smooth convex domains, but since the lemma requires $|f''|$ to be monotonic, Krätzel’s theorem holds in fact only for that subclass of $C^2$-convex domains.

**Appendix B. An elementary lemma**

The following result was used repeatedly, for $s > 0$ and $0 < t < 1$.

**Lemma 13.**

$$s + s^{-1} \leq 2 + t \implies |s - 1| \leq 3\sqrt{t}$$
Proof. Taking the square root on both sides of the inequality
\[(s^{1/2} - s^{-1/2})^2 = s + s^{-1} - 2 \leq t\]
and then using that the number 1 lies between \(s^{1/2}\) and \(s^{-1/2}\), we find
\[|s^{1/2} - 1| \leq t^{1/2}.
Hence \(1 - t^{1/2} \leq s^{1/2} \leq 1 + t^{1/2}\), and now squaring both sides and using that \(t < t^{1/2}\) (when \(t < 1\)) proves the lemma.

\[\square\]

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