POSTULATION OF GENERIC LINES AND ONE DOUBLE LINE IN $\mathbb{P}^n$ IN VIEW OF GENERIC LINES AND ONE MULTIPLE LINEAR SPACE

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Abstract. A well-known theorem by Hartshorne–Hirschowitz ([HH81]) states that a generic union $X \subset \mathbb{P}^n$, $n \geq 3$, of lines has good postulation with respect to the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$. So a question that arises naturally in studying the postulation of non-reduced positive dimensional schemes supported on linear spaces is the question whether adding a $m$-multiple linear space $m\mathbb{P}^r$ to $X$ can still preserve its good postulation, which means in classical language that, whether $m\mathbb{P}^r$ imposes independent conditions on the linear system $|\mathcal{I}_X(d)|$. Recently, the case of $r = 0$, i.e., the case of lines and one $m$-multiple point, has been completely solved by several authors ([CCG16], [AB14], [Bal15]) starting with Carlini–Catalisano–Geramita, while the case of $r > 0$ was remained unsolved, and this is what we wish to investigate in this paper. Precisely, we study the postulation of a generic union of $s$ lines and one $m$-multiple linear space $m\mathbb{P}^r$ in $\mathbb{P}^n$, $n \geq r + 2$. Our main purpose is to provide a complete answer to the question in the case of lines and one double line, which says that the double line imposes independent conditions on $|\mathcal{I}_X(d)|$ except for the only case $\{n = 4, s = 2, d = 2\}$. Moreover, we discuss an approach to the general case of lines and one $m$-multiple linear space, $(m \geq 2, r \geq 1)$, particularly, we find several exceptional such schemes, and we conjecture that these are the only exceptional ones in this family. Finally, we give some partial results in support of our conjecture.

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1. Introduction

To understand the geometry of a closed subscheme $X$ as an embedded scheme in $\mathbb{P}^n$, one of the first points of interest is considering the postulation problem, i.e. determining the number of conditions imposed by asking hypersurfaces of any degree to contain $X$. In terms of sheaf cohomology, we would like to know the rank of the restriction maps

$$\rho(d) : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(X, \mathcal{O}_X(d)).$$

We say that $X$ has maximal rank or good postulation or expected postulation if $\rho(d)$ has maximal rank, i.e. it is either injective or surjective, for each $d \geq 0$. This amounts to saying that one or other of the integers $h^0(\mathcal{I}_X(d)), h^1(\mathcal{I}_X(d))$ is zero, and this shows that the property that $X$ imposes independent conditions to degree $d$ hypersurfaces can be interpreted cohomologically.

On the other hand, this classical problem is equivalent to computing the Hilbert function of $X$. Let $HF(X, d)$ be the Hilbert function of $X$ in degree $d$, namely, $HF(X, d) = h^0(\mathcal{O}_{\mathbb{P}^n}(d)) - h^0(\mathcal{I}_X(d))$, i.e. the rank of $\rho(d)$. In order to determine the Hilbert function of $X$ in some degree $d$, there is an expected value for it given by a naive count of conditions. This value is determined by assuming that $X$ imposes independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$ of degree $d$ hypersurfaces in $\mathbb{P}^n$, i.e.,

$$h^0(\mathcal{I}_X(d)) = \max \left\{ h^0(\mathcal{O}_{\mathbb{P}^n}(d)) - h^0(\mathcal{O}_X(d)), 0 \right\},$$

or equivalently

$$HF(X, d) = \min \{ HP(X, d), HP(\mathbb{P}^n, d) \},$$

where $HP(X, d)$ is the Hilbert polynomial of $X$. In [CCG10], the authors called a scheme $X$ with such Hilbert function for all $d \geq 0$, has bipolynomial
Hilbert function. It always holds $HF(X, d) \leq \min\{HP(X, d), HP(\mathbb{P}^n, d)\}$, so a natural question to ask is: when does this inequality turn into equality?

An important observation is that the postulation problem depends not only on the numerical data involved in it, but also on the position of the components of $X$. If $X$ is in sufficiently general position one expects that $X$ has bipolynomial Hilbert function, and therefore has good postulation, but this naive guess is in general false.

When we restrict our attention to the special class of schemes $X \subset \mathbb{P}^n$ which supported on unions of generic linear spaces, there is much interest in the postulation problem. In this situation it is noteworthy that the notions of good postulation and bipolynomial Hilbert function coincide. Original investigation mostly concentrated on the reduced cases (see e.g. [Bal10], [Bal11], [CCG10], [CCG12], [Der07], [DS02], [HH81], [GO81]): if $\dim X = 0$, i.e. $X$ is a generic collection of points in $\mathbb{P}^n$, it is well known that $X$ has good postulation (see [GMR83]); if $\dim X = 1$, we have a brilliant result due to Hartshorne and Hirschowitz, going back to 1982 [HH81], which states that a generic collection of lines in $\mathbb{P}^n$, $n \geq 3$, has good postulation; as soon as we go up to $\dim X > 1$, the postulation problem becomes more and more complicated. The extent of our ignorance in this situation is illustrated by the fact that the complete answer to the postulation problem even in two-dimensional case is not yet known (see [Bal11], [CCG10] for a generic union of lines and a few planes, and [Bal10] for a generic union of lines and a linear space).

On the other hand there is also a lot of interest on the postulation of non-reduced schemes supported on linear spaces: concerning the zero-dimensional case, i.e. fat points schemes, the postulation problem is a field of active research in algebraic geometry which has occupied researchers’s minds for over a century, but, despite all the progress made on this problem, it is still very live and widely open in its generality (see e.g. [AH95], [CH14], [Har04], and also [Cil00] for a survey of results, related conjectures and open questions); concerning the positive dimensional case, the postulation problem turns out to be far too complicated and giving a complete answer appears to be ambitious and quite difficult, this is why that it has never been systematically studied and in fact it had remained unsolved for a long time. Apparently the work of Carlîni–Catalisano–Geramita [CCG16] is a turning point in this story, which together with the recent papers [AB14] and [Bal15] shows that a generic union $X$ of $s$ lines and one $m$-fat point in $\mathbb{P}^n$, $n \geq 3$, always has good postulation in degree $d$ except for the cases $\{n = 3, m = d, 2 \leq s \leq d\}$, (see for the proof: [CCG16] in the case of $n \geq 4$, [AB14] in the case of $\{n = 3, m = 3\}$, and [Bal15] in the case of $\{n = 3, m \geq 4\}$). As far as we know, this is the only complete knowledge of the postulation in the case of non-reduced positive dimensional schemes supported on linear spaces (for other related results see
This paper was motivated by an attempt to go further in this direction, namely one may ask about the generic union of lines and one $m$-multiple line and one may hope that behaves well with respect to the postulation problem modulo a certain list of exceptions. The main result of this paper is solving the case of $m = 2$, i.e. the case of a generic union of lines and one double line. More precisely, we prove the following theorem:

**Theorem 1.1.** Let $n, d \in \mathbb{N}$, and $n \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $s \geq 1$ lines and one double line. Then $X$ has good postulation, i.e.,

$$h^0(\mathcal{I}_X(d)) = \max \left\{ \binom{d + n}{n} - (nd + 1) - s(d + 1), 0 \right\},$$

except for the only case $\{n = 4, s = 2, d = 2\}$.

Geometrically, the theorem says that one generic double line in $\mathbb{P}^n$ imposes independent conditions to hypersurfaces of given degree $d$ containing $s$ generic lines, with the exception of only the case $\{n = 4, s = 2, d = 2\}$.

A first generalization is asking not only for $m = 2$, but also for $m > 2$ arbitrary. Inspired by the question involving an arbitrary multiple line instead of a double line, and based on several examples, we conjecture that a similar result should hold, in analogy with the statement of Theorem 1.1. Namely we formulate the following conjecture:

**Conjecture 1.2.** Let $n, d, r \in \mathbb{N}$, and $n \geq r + 2 \geq 3$. The scheme $X \subset \mathbb{P}^n$ consisting of $s \geq 1$ generic lines and one generic $m$-multiple line, $(m \geq 2)$, always has good postulation, except for the cases $\{n = 4, m = d, 2 \leq s \leq d\}$.

From this conjecture we can deduce the important relation of the failure of $X$ to have good postulation in degree $d$, with the multiplicity of a multiple line, the dimension of the ambient space, and the number of apparent simple lines. This seems to be fairly general behavior, which leads us to advance in more general situation. Indeed, one can push the problem we are facing even further, in the sense of substituting a multiple line with a multiple linear space of any dimension, and try to make a conjecture which parallels the above one.

Based on a similar analogy and some further evidence, we propose a conjecture as follows, which is significantly stronger than the former.

**Conjecture 1.2.** Let $n, d, r \in \mathbb{N}$, and $n \geq r + 2 \geq 3$. The scheme $X \subset \mathbb{P}^n$ consisting of $s \geq 1$ generic lines and one $m$-multiple linear space of dimension $r$, $(m \geq 2)$, always has good postulation, except for the cases $\{n = r + 3, m = d, 2 \leq s \leq d\}$.

Of course, this conjecture coincides with the previous one for $r = 1$. As we shall see in §7, there are results that make the conjecture rather plausible. It is worth mentioning that the case $m = 1$ is considered in [Bal10], where Ballico
proved that a generic disjoint union of lines and one linear space always has good postulation [Bal10, Proposition 1].

We want to finish by pointing out that a non-reduced scheme $X$ supported on generic union of linear spaces always has exceptions, a phenomenon that does not happen when $X$ is reduced (according to a conjecture in [CCG10]). In fact, the “bad behavior” of $X$ is always related to the multiple component of it.

The structure of the paper. Section 2 contains background material. To be more explicit, after recalling basic definitions and notations on the schemes of multiple linear spaces in §2.1, we then give, in §2.2, some lemmas and some elementary observations which are extremely useful in dealing with the postulation problem. Next, in §2.3 we collect the known results concerning the postulation of lines as well as of lines and one multiple point, that are necessary for our proofs in Sections 4–6, in addition, we look at the Hilbert polynomial of multiple linear spaces. We will study the postulation of our schemes by a degeneration approach, namely, degeneration of two skew lines in such a way that the resulting degenerated scheme would be a sundial, in the sense of [CCG11], thanks to a theorem of Carlini–Catalisano–Geramita on the postulation of sundials in any projective space; all this is represented in §2.4.

Section 3 making up the core of the paper devoted to the outline of the proof of our main theorem, Theorem 1.1. We begin this section with the exceptional case $\{n = 4, s = 2, d = 2\}$ of the theorem. We make explicit, in §3.1, the geometric reason that prevents a scheme consisting of one generic double line and two generic simple lines in $\mathbb{P}^4$ from imposing independent conditions to quadric hypersurfaces. Moreover, we solve completely the case of $d = 2$ of the theorem. In §3.2 we explain a rephrasing of Theorem 1.1, that is Theorem 3.2. So our goal will be to prove Theorem 1.1 in the reformulation of Theorem 3.2. §3.3 describes in detail our strategy for proving Theorem 3.2, which is based on geometric constructions of specialized and degenerated schemes, the well-known Horace lemma, and the intersection theory on a hyperplane or on a smooth quadric surface. We would like to point out that our method of degenerations owed to the works [CCG11, CCG16].

In order to apply the strategy we will use an induction procedure which has difficult but delicate beginning steps for $n = 3$ and $n = 4$. In Sections 4 and 5 we prove Theorem 3.2 for, respectively, $n = 3$ and $n = 4$, setting the stage for our induction approach. While, the proof for the general case $n \geq 5$ will be carried out in Section 6.

Conjecture 1.2, which geometrically amounts to saying that one generic $m$-multiple linear space $m\mathbb{P}^r$ in $\mathbb{P}^n$, $(n \geq r + 2 \geq 3)$, fails to impose independent conditions to degree $d$ hypersurfaces through $s$ generic lines if and only if
\{ n = r + 3, m = d, 2 \leq s \leq d \}, is stated and discussed in Section 7, where we prove it for the exceptional cases, and we describe completely what happen for \( d = m \).

Finally, in Appendix, \( \S 8 \), we collect several numerical lemmas needed for our proofs in Sections 5 and 6.

2. Background

We work throughout over an algebraically closed field \( k \) with characteristic zero.

2.1. Notations. Given a closed subscheme \( X \) of \( \mathbb{P}^n \), \( \mathcal{I}_X \) will denote the ideal sheaf of \( X \). If \( X, Y \) are closed subschemes of \( \mathbb{P}^n \) and \( X \subset Y \), then we denote by \( \mathcal{I}_{X,Y} \) the ideal sheaf of \( X \) in \( \mathcal{O}_Y \).

If \( F \) is a coherent sheaf on the scheme \( X \), for any integer \( i \geq 0 \) we use \( h^i(X, F) \) to denote the \( k \)-vector space dimension of the cohomology group \( H^i(X, F) \). In particular, when \( X = \mathbb{P}^n \), we will often omit \( X \) and we will simply write \( h^i(F) \).

A (fat) point of multiplicity \( m \), or an \( m \)-multiple point, with support \( P \in \mathbb{P}^n \), denoted \( mP \), is the zero-dimensional subscheme of \( \mathbb{P}^n \) defined by the ideal sheaf \((\mathcal{I}_P)^m\), i.e. the \( (m - 1)^{th} \) infinitesimal neighbourhood of \( P \). In case \( P \in X \) for any smooth variety \( X \subset \mathbb{P}^n \), we will write \( mP \mid_X \) for the \( (m - 1)^{th} \) infinitesimal neighborhood of \( P \) in \( X \), that is the schematic intersection of the \( m \)-multiple point \( mP \) of \( \mathbb{P}^n \) and \( X \) with \((\mathcal{I}_{P,X})^m\) as its ideal sheaf.

Similarly, if \( L \subset \mathbb{P}^n \) is a line (resp. linear space), the closed subscheme of \( \mathbb{P}^n \) supported on \( L \) and defined by the ideal sheaf \((\mathcal{I}_L)^m\) is called a (fat) line of multiplicity \( m \) (resp. linear space), or an \( m \)-multiple line (resp. linear space), and is denoted by \( mL \).

Let \( m_1, \ldots, m_s \) be positive integers and let \( X_1, \ldots, X_s \) be \( s \) closed subschemes of \( \mathbb{P}^n \). We denote by

\[ m_1X_1 + \cdots + m_sX_s \]

the schematic union of \( m_1X_1, \ldots, m_sX_s \), i.e. the subscheme of \( \mathbb{P}^n \) defined by the ideal sheaf \((\mathcal{I}_{X_1})^{m_1} \cap \ldots \cap (\mathcal{I}_{X_s})^{m_s}\).

2.2. Preliminary lemmas. The basic tool for the study of the postulation problem is the so called Castelnuovo’s inequality, that is an immediate consequence of the well-known residual exact sequence (for more details see e.g. [AH95, Section 2]).

We first recall the notion of residual scheme ([Ful84, \( \S 9.2.8 \)]).

Definition 2.1. Let \( X, Y \) be closed subschemes of \( \mathbb{P}^n \).

(i) The closed subscheme of \( \mathbb{P}^n \) defined by the ideal sheaf \((\mathcal{I}_X : \mathcal{I}_Y)\) is called the residual of \( X \) with respect to \( Y \) and denoted by \( \text{Res}_Y(X) \).
(ii) The schematic intersection $X \cap Y$ defined by the ideal sheaf $(I_X + I_Y)/I_Y$ of $O_Y$ is called the trace of $X$ on $Y$ and denoted by $Tr_Y(X)$.

We note that the generally valid identity for ideal sheaves $(I_X_1 \cap I_X_2 : I_Y) = (I_X_1 : I_Y) \cap (I_X_2 : I_Y)$ implies that the residual of the schematic union $X_1 + X_2$ is the schematic union of the residuals.

**Lemma 2.2** (Castelnuovo’s Inequality). Let $d, e \in \mathbb{N}$, and $d \geq e$. Let $H \subseteq P^n$ be a hypersurface of degree $e$, and let $X \subseteq P^n$ be a closed subscheme. Then

$$h^0(P^n, I_X(d)) \leq h^0(P^n, I_{Res_H(X)}(d-e)) + h^0(H, I_{Tr_H(X)}(d)).$$

This lemma, especially after the outstanding work of Hirschowitz [Hir81], is the basis for a standard method of working inductively with degree to solve the postulation problem and particularly is central to our proofs in the present paper (Sections 4–6).

The following remark is quite immediate.

**Remark 2.3.** Let $n, d, s, s' \in \mathbb{N}$, $s' < s$. Let $W_s = X_1 + \cdots + X_s \subset P^n$ be the schematic union of non-intersecting closed subschemes $X_i$.

- (i) If $h^1(I_{W_s}(d)) = 0$, then $h^1(I_{W_{s'}}(d)) = 0$.
- (ii) If $h^0(I_{W_{s'}}(d)) = 0$, then $h^0(I_{W_s}(d)) = 0$.

The following lemma shows that how to add a collection of points lying on a linear space $\Pi \subset P^n$ to a scheme $X \subset P^n$, in such a way that imposes independent conditions on the linear system of degree $d$ hypersurfaces passing through $X$ for a given degree $d$ [CCG10, Lemma 2.2].

**Lemma 2.4.** Let $d \in \mathbb{N}$. Let $X \subseteq P^n$ be a closed subscheme, and let $P_1, \ldots, P_s$ be generic points on a linear space $\Pi \subset P^n$.

If $h^0(I_X(d)) = s$ and $h^0(I_{X+\Pi}(d)) = 0$, then $h^0(I_{X+P_1+\cdots+P_s}(d)) = 0$.

### 2.3. What results were previously known.

As a key question in the direction of studying the postulation problem of a scheme $X \subset P^n$ supported on unions of generic linear spaces, one can ask: *What is the Hilbert polynomial of $X$?* When $X$ is reduced, Derksen answered this question by giving a formula for computing the Hilbert polynomial of $X$ (see [Der07] for details). Moreover, the Hilbert polynomial of a multiple linear space is well-known, and it is not difficult to verify it by a count of parameters, that can be found in e.g. [Bal16, §2] and [DHST14, Lemma 2.1].

**Lemma 2.5.** Let $n, d, r \in \mathbb{N}$, $r < n$ and $1 \leq m \leq d$. Let $\Pi \subset P^n$ be a linear space of dimension $r$, then

$$HP(m\Pi, d) = \sum_{i=0}^{m-1} \binom{r + d - i}{r} \binom{n + i - r - 1}{i}.$$
Indeed, the requirement for a degree $d$ hypersurface in $\mathbb{P}^n$ to contain $m\Pi$, i.e. to have multiplicity $m$ along the linear space $\Pi$, imposes the number of conditions on it, which is at most the right hand side of (1).

In our case, i.e. the case of double line, one knows that for a hypersurface to contain a double line $2L$ is equivalent to saying that it is singular along the line $L$, and Lemma above asserts that $2L$ in $\mathbb{P}^n$ imposes $(nd + 1)$ independent conditions to degree $d$ hypersurfaces.

Now we recall a few results on the postulation of schemes supported on generic linear spaces which we will use to prove our Theorem 1.1 in §§4–6. We start with a spectacular result due to Hartshorne and Hirschowitz, about the generic lines.

**Theorem 2.6** (Hartshorne–Hirschowitz). [HH81, Theorem 0.1] Let $n, d \in \mathbb{N}$, and $n \geq 3$. Let $X \subset \mathbb{P}^n$ be a generic union of $s$ lines. Then $X$ has good postulation, i.e.,

$$h^0(\mathcal{I}_X(d)) = \max \left\{ \binom{d + n}{n} - s(d + 1), 0 \right\}.$$ 

As a first step for positive dimensional non-reduced cases, in [CCG16], [AB14], and [Bal15] the authors examined the postulation problem for a generic collection of skew lines and one fat point in $\mathbb{P}^n$, and they found out that when $n \geq 4$ these schemes have good postulation, but when $n = 3$ there are several defective such schemes. Now, one can present these results simultaneously in a theorem as follows.

**Theorem 2.7.** Let $n, m, d \in \mathbb{N}$, and $n \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $s \geq 1$ lines and one fat point of multiplicity $m \geq 2$. Then $X$ has good postulation, i.e.,

$$h^0(\mathcal{I}_X(d)) = \max \left\{ \binom{d + n}{n} - \binom{m + n - 1}{n} - s(d + 1), 0 \right\},$$

except for the cases $\{n = 3, m = d, 2 \leq s \leq d\}$.

Since we will apply the theorem for the case $m = 2$ and $d \geq 3$ several times in the next sections, it is convenient to restate it as follows.

**Corollary 2.8.** Let $n, d \in \mathbb{N}$, and $n, d \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $s \geq 1$ lines and one double point. Then $X$ has good postulation in degree $d$, i.e.

$$h^0(\mathcal{I}_X(d)) = \max \left\{ \binom{d + n}{n} - (n + 1) - s(d + 1), 0 \right\}.$$
2.4. A degeneration approach. A natural approach to the postulation problem is to argue by degeneration. In view of the fact that we have the semicontinuity theorem for cohomology groups in a flat family, one may use the degenerations and the semicontinuity theorem in order to be able to better handle the postulation of generic configuration of linear spaces. Specifically, if one can prove that the property of having good postulation is satisfied in the special fiber, i.e. the degenerate scheme, then one may hope to obtain the same property in the general fiber, i.e. the original scheme.

In the celebrated paper [HH81] Hartshorne and Hirschowitz investigated a new degeneration technique to attack the postulation problem for a generic union of lines. In fact, they degenerate two skew lines in $\mathbb{P}^3$ in such a way that the resulting scheme becomes a “degenerate conic with an embedded point” (which also was used in [Hir81]). Even more generally, one can push this trick of “adding nilpotents” further, to give a degeneration of two skew lines in higher dimensional projective space $\mathbb{P}^n$, $n \geq 3$, this is what the authors introduced in [CCG10, Definition 2.7] and called a (3-dimensional) sundial.

According to the terminology of [HH81], we say that $C$ is a degenerate conic if $C$ is the union of two intersecting lines $L$ and $M$, so $C = L + M$.

**Definition 2.9.** Let $L$ and $M$ be two intersecting lines in $\mathbb{P}^n$, $n \geq 3$, and let $T \cong \mathbb{P}^3$ be a generic linear space containing the degenerate conic $L + M$. Let $P$ be the singular point of $L + M$, i.e. $P = L \cap M$. We call the scheme $L + M + 2P|_T$ a degenerate conic with an embedded point or a (3-dimensional) sundial.

One can show a sundial is a flat limit inside $\mathbb{P}^n$ of a flat family whose general fiber is the disjoint union of two lines, i.e. a sundial is a degeneration of two generic lines in $\mathbb{P}^n$, $n \geq 3$. This is the content of the following lemma (see [HH81, Example 2.1.1] for the case $n = 3$, and [CCG10, Lemma 2.5] for the general case).

**Lemma 2.10.** Let $X_1 \subset \mathbb{P}^n$, $n \geq 3$, be the disjoint union of two lines $L_1$ and $M$. Then there exists a flat family of subschemes $X_i \subset (X_1) \cong \mathbb{P}^3$, $(i \in k)$, whose general fiber is the union of two skew lines and whose special fiber is the sundial $X_0 = M + L + 2P|(X_1)$, where $L$ is a line and $M \cap L = P$.

Note that we can also easily degenerate a simple generic point and a degenerate conic to a sundial. Therefore, a sundial is either a degeneration of two generic lines, or a degeneration of a scheme which is the union of a degenerate conic and a simple generic point.

We recall here the main result in [CCG11] which guarantees that a generic collection of sundials will behave well with respect to the postulation problem.

**Theorem 2.11.** [CCG11, Theorem 4.4] Let $n, d \in \mathbb{N}$, and $n \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $x$ sundials and $y$ lines. Then $X$ has...
good postulation, i.e.

\[ h^0(\mathcal{I}_X(d)) = \max \left\{ \binom{d+n}{n} - (2x+y)(d+1), 0 \right\}. \]

### 3. Outline of the proof of Theorem 1.1

Now we have all the necessary tools to tackle our main theorem, Theorem 1.1.

#### 3.1. The Exceptional Case

We look for the case where \( X \) fails to have good postulation. Actually, there is only one exception in this infinite family, namely the case \( \{n = 4, s = 2\} \) which, \( H^0(\mathcal{I}_X(2)) \) has dimension one instead of zero. As we will see below, this exceptional case arises from geometric reason, although the proof follows from numerical reason.

Now we prove the following proposition, which completely describes the case \( d = 2 \) of Theorem 1.1:

**Proposition 3.1.** The scheme \( X \subset \mathbb{P}^n, n \geq 3, \) consisting of \( s \geq 1 \) generic lines and one double line \( 2L \) has good postulation in degree \( d = 2 \), i.e.,

\[ h^0(\mathcal{I}_X(2)) = \max \left\{ \binom{n+2}{n} - (2n+1) - 3s, 0 \right\} \]

except for the only case \( \{n = 4, s = 2\} \).

**Proof.** The sections of \( \mathcal{I}_X(2) \) correspond to quadric hypersurfaces in \( \mathbb{P}^n \) which, in order to contain \( 2L \), have to be cones whose vertex contains the line \( L \).

If \( n = 3 \), we obviously have \( h^0(\mathcal{I}_X(2)) = 0 \), as expected.

If \( n \geq 4 \), we consider the projection \( X' \) of \( X \) from \( L \) onto a generic linear subspace \( \mathbb{P}^{n-2} \subset \mathbb{P}^n \), hence \( X' \) is a scheme consisting of \( s \) generic lines in \( \mathbb{P}^{n-2} \).

It follows that

\[ h^0(\mathbb{P}^n, \mathcal{I}_X(2)) = h^0(\mathbb{P}^{n-2}, \mathcal{I}_{X'}(2)). \]

In case \( n > 4 \), by Hartshorne–Hirschowitz theorem (Theorem 2.6) we get

\[ h^0(\mathbb{P}^{n-2}, \mathcal{I}_{X'}(2)) = \max \left\{ \binom{2+n-2}{2} - 3s, 0 \right\} = \max \left\{ \binom{n}{2} - 3s, 0 \right\}, \]

and we get the conclusion.

In case \( n = 4 \), \( X' \) is a generic union of \( s \) lines in \( \mathbb{P}^2 \). Hence, for \( s > 2 \) it is immediate to see that \( h^0(\mathcal{I}_{X'}(2)) = 0 \). For \( s \leq 2 \) we have \( h^0(\mathcal{I}_{X'}(2)) = \binom{2-s+2}{2} = \binom{4-s}{2} \), on the other hand the expected value for \( h^0(\mathcal{I}_X(2)) \) is \( \max \{6-3s, 0\} \).

Thus for \( s = 1 \), \( h^0(\mathcal{I}_X(2)) = 3 \), as expected; but for \( s = 2 \), \( h^0(\mathcal{I}_X(2)) = 1 \) while the expected one is 0, which is what we wanted to show. \( \square \)
3.2. **Rephrasing Theorem 1.1.** From what we have observed in the previous subsection, it remains to verify Theorem 1.1 for $d \geq 3$, asserts that schemes $X \subset \mathbb{P}^n$ consisting of $s$ generic lines and one generic double line have good postulation for all $d \geq 3$, i.e., $h^0(\mathcal{I}_X(d)) = 0$ or $h^1(\mathcal{I}_X(d)) = 0$. (Note that the case $d = 1$ of Theorem 1.1 being trivial, so we omit it.)

First note that, as $X$ varies in a flat family, by the semicontinuity of cohomology, the condition of good postulation, is clearly an open condition on the family of $X$. Hence to prove Theorem 1.1, it is enough to find any scheme of $s$ lines and one double line, or even any scheme which is a specialization of a flat family of $s$ lines and one double line, which has good postulation.

Given $n$ and $d$, suppose one can choose $s$ so that:

$$(d + n) \choose n = (nd + 1) + s(d + 1)$$

and suppose one can find $X$ so that $h^0(\mathcal{I}_X(d)) = h^1(\mathcal{I}_X(d)) = 0$. Then if one removes some lines from $X$, one gets a scheme $X' \subset X$ such that $h^1(\mathcal{I}_{X'}(d))$ will still be zero; and if one adds some lines to $X$, one gets a scheme $X'' \supset X$ such that $h^0(\mathcal{I}_{X''}(d))$ will still be zero (by Remark 2.3). In other words, the good postulation for that given $n, d$ and the unique integer $s$, that gives the equality above, implies the good postulation for the same $n, d$ and any $s$ whatsoever.

Unfortunately for given $n, d$ one can not always find such an $s$. Therefore we will make adjustments by adding some collinear points to $X$, to get a similar equality. In particular, we prove the following theorem which, by Remark 2.3, implies our main Theorem 1.1:

**Theorem 3.2.** Let $n, d \in \mathbb{N}$, and $n, d \geq 3$. Let

$$r = \left\lfloor \frac{(d+n)}{n} - (nd + 1) \right\rfloor; \quad q = \left( \frac{d+n}{n} - (nd + 1) - r(d + 1) \right).$$

Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $r$ lines $L_1, \ldots, L_r$, one double line $2L$ and $q$ points $P_1, \ldots, P_q$ lying on a generic line $M$. Then $X$ has good postulation, i.e.,

$$h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) = \left( \frac{d+n}{n} - (nd + 1) - r(d + 1) - q \right) = 0.$$

From our discussion above, Theorem 1.1 follows immediately from this theorem. Indeed, to prove Theorem 1.1 for that $n, d$ and any $r' \leq r$, simply remove the $q$ points and $r - r'$ lines, then the corresponding $h^1(\mathcal{I}(d))$ will be zero; to prove it for $r'' > r$, first add the line $M$ passing through the $q$ collinear points, then add $r'' - r - 1$ disjoint lines, then the corresponding $h^0(\mathcal{I}(d))$ will be zero.

**Notation 3.3.** We denote by $S(n, d)$ and $S^*(n, d)$ for $d \geq 3$, the statement of Theorem 1.1 and the statement of Theorem 3.2, respectively.
Remark 3.4. As we have seen, the statement $S^*(n, d)$ implies $S(n, d)$. On the other hand, the converse also follows directly from Lemma 2.4.

3.3. Strategy of the Proof. We illustrate our general strategy explicitly to prove Theorem 3.2 for a generic scheme $X \subset \mathbb{P}^n$ consisting of one double line, $r$ simple lines and $q$ collinear points, as follows: The difficulty with proving a property like “good postulation” is that, it is very hard to lay hands on a generic scheme $X$. Our approach to overcome to this difficulty is to start with a special scheme, which is obtained by several different kind of specializations and degenerations, and then use semicontinuity theorem for cohomology groups to discover the same property for generic scheme $X$. The next step is to reduce the postulation problem of our scheme, via Castelnuovo’s inequality, to the study of the postulation of a residual scheme and a trace scheme, that is *La méthode d’Horace*, elaborated by A. Hirschowitz [Hir81].

To be more precise, for $n \geq 4$ we specialize $x$ simple lines into a fixed hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ and degenerate $q'$ other pairs of simple lines to sundials, further, we specialize these sundials into $\mathbb{P}^{n-1}$ unless their singular points, which requires a capability of guessing the right specialization. Thus if one can choose these numbers correctly, that is in such a way that the numbers $x$ and $q'$ are sufficiently many to comply with the induction hypothesis on degree (see Appendix, Lemma 8.1), then the residual has good postulation, while the trace is a scheme in $\mathbb{P}^{n-1}$, which is more complicated to verify because of the appearance of $q'$ degenerate conics and one double point. Now to handle the problem involving the postulation of the trace scheme we specialize $\bar{r}$ lines, $\bar{q}$ simple points, and the double point into a fixed hyperplane $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$, then we take again residual and trace. Of course, the numbers $\bar{r}$ and $\bar{q}$ should not be too numerous, and we have to find these numbers satisfying all the necessary inequalities (see Appendix, Lemma 8.2). This time the trace consists of $\bar{r}$ lines, some simple points, and one double point, which by Corollary 2.8 has good postulation, while the residual consists of $q'$ degenerate conics, $(x - \bar{r})$ lines and some simple points, which we will degenerate it to a scheme consisting of $q'$ sundials, $(x - \bar{r})$ lines and some points, that by Theorem 2.11 has good postulation (these arrangements contain a lot of technical details which can be found in Appendix, Lemma 8.2).

This argument for the trace of $X$ can be applied in cases $n \geq 5$, but unfortunately does not cover the case $n = 4$, where forced intersection of lines appear in $\mathbb{P}^{n-2} = \mathbb{P}^2$. In fact the case $n = 4$ will be taken care of by a smooth quadric surface $Q$ and a way of specialization which is considerably different from that mentioned above. Explicitly, we specialize one line of each of the degenerate conics, together with $\hat{r}$ simple lines, into the same ruling on $Q$, moreover, we specialize $\hat{q}$ simple points onto $Q$, then we take again residual and trace. Surely, the numbers $\hat{r}$ and $\hat{q}$ should not be so much, and we have to find these
numbers satisfying all the necessary inequalities (see Appendix, Lemma 8.3). Now the current residual consisting of one double point, \((x - \hat{r} + q')\) lines and some simple points will be verified by Corollary 2.8, while the current trace, which is a scheme in \(Q\), will be verified by applying some results from internal geometry of \(Q\).

What about in \(\mathbb{P}^3\)? Actually, the most difficult part of the proof is the case of \(\mathbb{P}^3\). Our approach to this case uses extremely an ad hoc method which is done via specializing as many lines as is needed into a smooth quadric surface instead of a plane, and then, if necessary, degenerating some other pairs of simple lines to sundials (and even more specializing sundials and points), that requires a case by case discussion. Here the role of the smooth quadric is explained by the property of having two rulings of skew lines and by the known results from intersection theory on it. We notice that also in the case of \(\mathbb{P}^3\) our method can then be applied under certain numerical conditions, and this is why the proof splits into three specific cases \(d \equiv 0 \pmod{3}\), \(d \equiv 1 \pmod{3}\) and \(d \equiv 2 \pmod{3}\), which described exactly in Section 4. In fact, our method can be safely applied for \(d \equiv 0 \pmod{3}\), as well as for \(d \equiv 1 \pmod{3}\), but a slight complication arises in the case of \(d \equiv 2 \pmod{3}\), where we have to consider a different specialization, which is done by placing the support of the double line into the smooth quadric.

Summing up, the method for proving our Theorem 3.2, based on the induction on degree \(d\), breaks down into three parts: \(n = 3\), \(n = 4\), and \(n \geq 5\), which we have to investigate each of them separately in §§4–6.

We would like to point out that to make the strategy applicable, many verifications are needed because of the messy arithmetic involved (see §8).

Since to prove the property of good postulation, according to our strategy, we will use in the sequel Castelnuovo’s inequality and the semicontinuity of cohomology several times, it will be useful to consider the following remark.

**Remark 3.5.** With the hypotheses of Theorem 3.2, let \(\tilde{X}\) be the scheme obtained from \(X\) by combining specializations and degenerations via a fixed hypersurface \(H \subset \mathbb{P}^n\) of degree \(e\).

If \(h^0(I_{\text{Res}_H(\tilde{X})(d-e)}) = 0\) and \(h^0(H, I_{\text{Tr}_H(\tilde{X})}(d)) = 0\), then by Castelnuovo’s inequality (Lemma 2.2) we have \(h^0(I_{\tilde{X}}(d)) = 0\), and this implies, by the semicontinuity of cohomology, \(h^0(I_X(d)) = 0\).

4. **Proof in \(\mathbb{P}^3\)**

In this section we prove Theorem 3.2 in \(\mathbb{P}^3\).

We start with a useful observation concerning the behaviour of certain one-dimensional subschemes of a smooth quadric surface \(Q \cong \mathbb{P}^1 \times \mathbb{P}^1\) with respect
to the linear system of curves of type \((a, b)\), which we will often use in the sequel (for a proof see [HH81, Lemma 2.3]).

**Lemma 4.1.** Let \(\alpha, \beta, \gamma, \delta, d \in \mathbb{N}\), and let \(Q \subset \mathbb{P}^3\) be a smooth quadric. Let the scheme \(W \subset Q\) be a generic union of \(\alpha\) lines belonging to the same ruling of \(Q\), \(\beta\) simple points, \(\gamma\) simple points lying on a line belonging to the same ruling of the \(\alpha\) lines, and \(\delta\) double points. If the following conditions are satisfied:

1. \(\alpha(d + 1) + \beta + \gamma + 3\delta = (d + 1)^2\);
2. \(\delta \leq d + 1\);
3. \(\gamma \leq d + 1\);
4. if \(d > \alpha\) then \(\delta \leq \frac{d+1-\gamma}{2} + (d - \alpha - 1) \left\lfloor \frac{d+1}{2} \right\rfloor\), otherwise \(\delta = 0\);

then \(h^1(Q, \mathcal{I}_W(d)) = h^0(Q, \mathcal{I}_W(d)) = 0\).

Before we begin our investigations in the case of \(\mathbb{P}^3\), we recall some elementary facts about the geometry on the smooth quadric surface \(Q\): the divisor class group of \(Q\) is \(\mathbb{Z} \oplus \mathbb{Z}\), generated by a line in each of the two rulings; by the type we mean the class in \(\mathbb{Z} \oplus \mathbb{Z}\); the curves on \(Q\) are those of type \((a, b)\) with \(a, b \geq 0\); by convention \(\mathcal{O}_Q(d) = \mathcal{O}_Q(d, d)\); finally \(h^0(Q, \mathcal{O}_Q(a, b)) = (a+1)(b+1)\).

Now we state and prove Theorem 3.2 in \(\mathbb{P}^3\), that is:

**\(S^*(3, d):\)** Let \(d \geq 3\) and

\[
q = \left( \frac{d+3}{3} \right) - (3d+1) - r(d+1).
\]

Let the scheme \(X \subset \mathbb{P}^3\) be a generic union of \(r\) lines \(L_1, \ldots, L_r\), one double line \(2L\) and \(q\) points \(P_1, \ldots, P_q\) lying on a generic line \(M\). Then \(X\) has good postulation, i.e.,

\[
h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) = \left( \frac{d+3}{3} \right) - (3d+1) - r(d+1) - q = 0.
\]

**Proof.** In order to start the induction argument we need to establish the base cases \(d = 3, 4\).

First consider the case \(d = 3\). In this case we have \(r = 2\) and \(q = 2\), therefore \(X = 2L + L_1 + L_2 + P_1 + P_2 \subset \mathbb{P}^3\).

Fix a generic plane \(H \subset \mathbb{P}^3\), and consider the scheme \(\tilde{X}\) obtained from \(X\) by specializing the line \(L_1\) and the points \(P_1, P_2\) into \(H\). By abuse of notation, we will again denote these specialized line and points by \(L_1\) and \(P_1, P_2\). (Keeping in mind that in the sequel, by abuse of notation, we will always denote the specialized components by the same letters as the original ones.)

We have \(\text{Res}_H(\tilde{X}) = 2L + L_2\), then it is obvious that

\[
h^0(\mathcal{I}_{\text{Res}_H(\tilde{X})}(2)) = 0.
\]
Also, $\text{Tr}_H(\tilde{X}) = 2R|_H + L_1 + S + P_1 + P_2 \subseteq H$, where $L \cap H = R$ and $L_2 \cap H = S$. Since $L_1$ is a fixed component for the sections of $\mathcal{I}_{\text{Tr}_H(\tilde{X})}(3)$, we get that
\[ h^0(H, \mathcal{I}_{\text{Tr}_H(\tilde{X})}(3)) = h^0(H, \mathcal{I}_{\text{Tr}_H(\tilde{X})-L_1}(2)). \]
Since the double point $2R|_H$ imposes 3 independent conditions on $|\mathcal{O}_H(2)|$, and the points $P_1, P_2, S$ are generic points in $H$, we easily get that
\[ h^0(H, \mathcal{I}_{\text{Tr}_H(\tilde{X})-L_1}(2)) = \binom{2+2}{2} - 3 - 3 = 0. \]
Thus by Remark 3.5 we have $h^0(\mathcal{I}_X(3)) = 0$, that is, $X$ has good postulation in degree 3.

Now consider the case $d = 4$. We have $r = 4$ and $q = 2$, then $X$ is the schematic union: $X = 2L + L_1 + L_2 + L_3 + L_4 + P_1 + P_2 \subseteq \mathbb{P}^3$.

Let $Q$ be a smooth quadric surface, and let $\tilde{X}$ be the scheme obtained from $X$ by specializing three of the lines $L_i$ in such a way that $L_1, L_2, L_3$ become lines of the same ruling on $Q$, and by specializing the points $P_1, P_2$ onto $Q$.

Then we get $\text{Res}_Q(\tilde{X}) = 2L + L_4 \subseteq \mathbb{P}^3$, and it clearly follows that
\[ h^0(\mathcal{I}_{\text{Res}_Q(\tilde{X})}(2)) = 0. \]

Consider the trace of $\tilde{X}$ on $Q$, that is
\[ \text{Tr}_Q(\tilde{X}) = 2R|_Q + 2R|_Q + L_1 + L_2 + L_3 + S_1 + S_2 + P_1 + P_2 \subseteq Q, \]
where $L \cap Q = R_1 + R_2$ and $L_4 \cap Q = S_1 + S_2$. Note that the scheme $\text{Tr}_Q(\tilde{X})$ is generic union in $Q$ of three lines belonging to the same ruling of $Q$, four simple points and two double points, hence we can apply Lemma 4.1, with $(\alpha = 3, \beta = 4, \gamma = 0, \delta = 2, d = 4)$, and we obtain
\[ h^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{X})}(4)) = 0. \]
So by Remark 3.5 it follows that $h^0(\mathcal{I}_X(4)) = 0$. Hence the case $d = 4$ is done.

Now assume $d \geq 5$. We consider three cases, and we proceed by induction on $d$. Let $Q$ be a smooth quadric surface in $\mathbb{P}^3$.

**Case $d \equiv 0 \pmod{3}$.** Write $d = 3t, t \geq 2$. Then
\[ r = \frac{(t+1)(3t+2)}{2} - 3, \quad q = 2. \]
We have $X = 2L + L_1 + \cdots + L_r + P_1 + P_2 \subseteq \mathbb{P}^3$. Since $2t + 1 \leq r$, we specialize $2t + 1$ of the lines $L_i$ in such a way that $L_1, \ldots, L_{2t+1}$ become lines of the same ruling on $Q$, and we denote by $\tilde{X}$ the specialized scheme. We have $\text{Res}_Q(\tilde{X}) = 2L + L_{2t+2} + \cdots + L_r + P_1 + P_2 \subseteq \mathbb{P}^3$, which is the generic union of
one double line, $\frac{(3t+1)}{2} - 3$ lines and two points, so by the induction hypothesis it follows that

$$h^0(\mathcal{I}_{\text{Res}(\tilde{X})}(d-2)) = 0.$$ 

Now we treat the trace scheme

$$Tr_Q(\tilde{X}) = 2R_1|Q| + 2R_2|Q| + L_1 + \cdots + L_{2t+1}$$

$$+ S_{1,2t+2} + S_{2,2t+2} + \cdots + S_{1,r} + S_{2,r} \subseteq Q,$$

where $L \cap Q = R_1 + R_2$ and $L_i \cap Q = S_{1,i} + S_{2,i}$, $(2t + 2 \leq i \leq r)$. Note that the points $R_1, R_2, S_{1,i}, S_{2,i}, (2t + 2 \leq i \leq r)$, are generic points on $Q$. That is $Tr_Q(\tilde{X})$ consists of $2t + 1$ lines of the same ruling on $Q$, two generic double points and $t(3t + 1) - 6$ generic simple points, then we can easily check that $Tr_Q(\tilde{X})$ satisfies the conditions of Lemma 4.1, with $(\alpha = 2t + 1, \beta = t(3t + 1) - 6, \gamma = 0, \delta = 2)$, and this implies

$$h^0(Q, \mathcal{I}_{Tr_Q(\tilde{X})}(d)) = 0.$$

Hence by Remark 3.5 we get $h^0(\mathcal{I}_X(d)) = 0$.

**Case $d \equiv 1 \pmod{3}$.** Write $d = 3t + 1, t \geq 2$. Then

$$r = \frac{(t+1)(3t+4)}{2} - 3, \quad q = 2.$$ 

In this case we have $X = 2L + L_1 + \cdots + L_r + P_1 + P_2 \subseteq \mathbb{P}^3$.

We wish to construct a specialization of $X$ so that the expected vanishing $h^0(\mathcal{I}_X(d)) = 0$ is obtained. In order to do this, we introduce the specialization $\tilde{X}$ of $X$ in the following way:

- specialize the points $P_1, P_2$ onto $Q$;
- specialize the first $2t + 1$ lines $L_i$ in such a way that they become lines of the same ruling on $Q$, and call the resulting set of lines $X_1$;
- degenerate the next $2t - 2$ pairs of lines $L_i$, so that they become $2t - 2$ sundials $\hat{C}_i = C_i + 2N_i, (1 \leq i \leq 2t - 2)$, where $C_i$ is a degenerate conic and $2N_i$ is a double point with support at the singular point of $C_i$, furthermore, specialize the points $N_1, \ldots, N_{2t-2}$ onto $Q$, and call the resulting scheme of sundials $X_2$, that is

$$X_2 = \hat{C}_1 + \cdots + \hat{C}_{2t-2},$$

with the property that the singular points of $\hat{C}_i$ lie on $Q$;
- leave the remaining simple lines $L_i$, which are $r - (2t + 1) - 2(2t - 2) = \frac{(3t-5)}{2} + 2$ lines, generic outside $Q$, and call this collection of lines $X_3$; notice that we can do the above specialization because of the inequality $r \geq 2t + 1 + 2(2t - 2)$. Then by letting

$$\tilde{X} = 2L + X_1 + X_2 + X_3 + P_1 + P_2 \subseteq \mathbb{P}^3,$$
we get the desired specialization of $X$.

Now we perform the process of verifying the residual and the trace on this specialized scheme $\tilde{X}$. We obtain

$$Tr_{Q}(\tilde{X}) = 2R_1|_Q + 2R_2|_Q + X_1 + Tr_{Q}(X_2) + Tr_{Q}(X_3) + P_1 + P_2 \subset Q,$$

where $L \cap Q = R_1 + R_2; \ Tr_{Q}(\tilde{C}_i) = 2N_i|_Q + C_i \cap Q$, and $C_i \cap Q$ is a union of two simple points, $(1 \leq i \leq 2t - 2)$, therefore $Tr_{Q}(X_2)$ consists of $2t - 2$ double points and $4t - 4$ simple points; moreover, $Tr_{Q}(X_3)$ consists of $t(3t - 5) + 4$ simple points. Hence $Tr_{Q}(\tilde{X})$ is generic union in $Q$ of $2t + 1$ lines belonging to the same ruling of $Q$, $2t$ double points and $t(3t - 1) + 2$ simple points. An easy computation, yields that the scheme $Tr_{Q}(\tilde{X})$ verifies the conditions of Lemma 4.1, with $(\alpha = 2t + 1, \beta = t(3t - 1) + 2, \gamma = 0, \delta = 2t)$, then we have

$$h^0(Q, \mathcal{I}_{Tr_Q(\tilde{X})}(d)) = 0.$$

So we are done with $Tr_{Q}(\tilde{X})$. If we can prove $h^0(\mathcal{I}_{Res_Q(\tilde{X})}(d - 2)) = 0$ then, by Castelnuovo’s inequality, we get $h^0(\mathcal{I}_{\tilde{X}}(d)) = 0$.

Here we consider the residual scheme

$$Res_{Q}(\tilde{X}) = 2L + C_1 + \cdots + C_{2t-2} + X_3 \subset \mathbb{P}^3.$$

In order to compute $h^0(\mathcal{I}_{Res_Q(\tilde{X})}(d - 2))$, we need to construct a specialization of $Res_{Q}(\tilde{X})$, and take again the residual and the trace with respect to $Q$.

First, let $M_{1,i}, M_{2,i}$ be the two lines which form the degenerate conic $C_i$, $(1 \leq i \leq 2t - 2)$, this means $C_i = M_{1,i} + M_{2,i}$. Pick a line $L' \subset X_3$. Now let $\tilde{R}$ be the scheme obtained from $Res_{Q}(\tilde{X})$ by specializing the degenerate conics $C_i$ and the lines $L, L'$ in such a way that the lines $M_{1,1}, \ldots, M_{1,2t-2}$ and $L, L'$ become $2t$ lines of the same ruling on $Q$ (the lines $M_{2,1}, \ldots, M_{2,2t-2}$ and the other $\frac{t(3t-5)}{2} + 1$ lines of $X_3$ remain generic lines, not lying on $Q$).

From this specialization we have

$$Res_{Q}(\tilde{R}) = L + M_{2,1} + \cdots + M_{2,2t-2} + (X_3 - L'),$$

that is generic union of $\frac{t(3t-1)}{2}$ lines in $\mathbb{P}^3$, hence by Hartshorne–Hirschowitz theorem (Theorem 2.6) we immediately get, (note that $d = 3t + 1$),

$$h^0(\mathcal{I}_{Res_Q(\tilde{R})}(d - 4)) = \left(\frac{d - 4 + 3}{3}\right) - \frac{t(3t - 1)}{2}(d - 4 + 1) = 0.$$
On the other hand, $M_{2,i}$ meets $Q$ in the two points which are $M_{1,i} \cap M_{2,i}$, that is contained in $M_{1,i}$, and another point, which we denote by $S_i$. Thus

$$Tr_Q(\tilde{R}) = 2L |_Q + M_{1,1} + \cdots + M_{1,2t-2} + L' + S_1 + \cdots + S_{2t-2} + Tr_Q(X_3 - L') \subset Q,$$

where $Tr_Q(X_3 - L')$ is made by $t(3t - 5) + 2$ generic points. Therefore the scheme $Tr_Q(\tilde{R})$ is generic union in $Q$ of one double line, $2t - 1$ lines, such that all of these $2t$ lines are placed in the same ruling of $Q$, and $3t(t - 1)$ points. Considering $Q$ as $\mathbb{P}^1 \times \mathbb{P}^1$ and assuming these $2t$ lines belong to the first ruling of $Q$, we see that each of these lines is a curve of type $(1, 0)$ on $Q$.

Note that the double line $2L |_Q$ and the lines $M_{1,i}, L'$, $(1 \leq i \leq 2t - 2)$, are fixed components for the curves of $H^0(Q, \mathcal{I}_{Tr_Q(\tilde{R})}(d - 2, d - 2))$, since $d - 2 \geq 2t + 1$. Now set $\Lambda = 2L |_Q + M_{1,1} + \cdots + M_{1,2t-2} + L' \subset Q$, which is of type $(2t + 1, 0)$. Hence by removing the fixed component $\Lambda$, and by using the fact that the scheme $Tr_Q(\tilde{R}) - \Lambda$ is generic union of $3t(t - 1)$ simple points, moreover by recalling the equality $d = 3t + 1$, we deduce

$$h^0(Q, \mathcal{I}_{Tr_Q(\tilde{R})}(d - 2, d - 2)) = h^0(Q, \mathcal{I}_{Tr_Q(\tilde{R})}(d - 2, d - 2) - \Lambda) = h^0(Q, \mathcal{I}_{Tr_Q(\tilde{R})}(t - 2, 3t - 1)) = h^0(Q, \mathcal{O}_Q(t - 2, 3t - 1)) = (t - 1)3t - 3t(t - 1) = 0.$$

This together with $h^0(\mathcal{I}_{Res_Q(\tilde{R})}(d - 4)) = 0$, implies that

$$h^0(\mathcal{I}_{\tilde{R}}(d - 2)) = 0,$$

consequently, by semicontinuity, $h^0(\mathcal{I}_{Res_Q(\tilde{X})}(d - 2)) = 0$. So we conclude that $h^0(\mathcal{I}_{\tilde{X}}(d)) = 0$, and from here, by Remark 3.5, we get $h^0(\mathcal{I}_X(d)) = 0$.

**Case $d \equiv 2 \pmod{3}$**. Write $d = 3t + 2, t \geq 1$. Then

$$r = \frac{3t(t + 3)}{2}, \quad q = t + 3.$$

We have $X = 2L + L_1 + \cdots + L_r + P_1 + \cdots + P_{t+3} \subset \mathbb{P}^3$, where $P_1, \ldots, P_{t+3}$ are points lying on a generic line $M$.

Realize $Q$ as $\mathbb{P}^1 \times \mathbb{P}^1$. We specialize $2t$ of the lines $L_i$ and the lines $L, M$ in such a way that $L_1, \ldots, L_{2t}$ and $L, M$ become $2t + 2$ lines of the first ruling on $Q$, i.e. each has type $(1, 0)$, and we denote by $\tilde{X}$ the specialized scheme (note that this is possible since $r \geq 2t + 2$). It is clear from this specialization that the points $P_1, \ldots, P_{t+3}$ become points on the line $M$ belonging to the first ruling of $Q$. 
First we consider the residual scheme

\[ \text{Res}_Q(\tilde{X}) = L + L_{2t+1} + \cdots + L_r \subset \mathbb{P}^3, \]

that is generic union of \( r - 2t + 1 = \frac{(t+1)(3t+2)}{2} \) lines, so according to Hartshorne–Hirschowitz theorem we get

\[ h^0(\mathcal{I}_{\text{Res}_Q(\tilde{X})}(d - 2)) = \left( \frac{d - 2 + 3}{3} \right) - \frac{(t + 1)(3t + 2)}{2}(d - 2 + 1) = 0. \]

Then we are left with the trace scheme, which is

\[ Tr_Q(\tilde{X}) = 2L|_Q + L_1 + \cdots + L_{2t} + X_1 + P_1 + \cdots + P_{t+3} \subset Q, \]

where \( X_1 = Tr_Q(L_{2t+1} + \cdots + L_r) \). Using the fact that each \( L_i \) meets \( Q \) at two points, \((2t + 1 \leq i \leq r)\), it follows that \( X_1 \) is made by \( 2(r - 2t) = t(3t + 5) \) simple points.

Observe that the double line \( 2L|_Q \) and the lines \( L_1, \ldots, L_{2t} \) are fixed components for the curves of \( H^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{X})}(d, d)) \), (note that \( d \geq 2t + 2 \)). Set \( \Lambda = 2L|_Q + L_1 + \cdots + L_{2t} \subset Q \), which has type \((2t + 2, 0)\). Removing the fixed component \( \Lambda \) implies that

\[ h^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{X})}(d, d)) = h^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{X}) - \Lambda}(d - (2t + 2), d)) = h^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{X}) - \Lambda}(t, 3t + 2)). \]

Hence we need to show that \( h^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{X}) - \Lambda}(t, 3t + 2)) = 0 \), where \( \text{Tr}_Q(\tilde{X}) - \Lambda = X_1 + P_1 + \cdots + P_{t+3} \subset Q \). To see this, we wish to construct a specialization of \( \text{Tr}_Q(\tilde{X}) - \Lambda \) with the desired vanishing, we then must verify the residual and the trace in this new situation.

We start by choosing \( t \) lines \( M_1, \ldots, M_t \) of the first ruling on \( Q \), \( M_i \neq M \). Next, let \( Y \) be the scheme obtained from \( \text{Tr}_Q(\tilde{X}) - \Lambda \) by specializing the \( t(3t + 3) \) points of \( X_1 \) onto the lines \( M_i \) in such a way that each of these lines contains exactly \( 3t + 3 \) of these points, and by specializing the remaining \( 2t \) points of \( X_1 \) onto the line \( M \) (this is possible because \( t(3t + 5) = 2t + t(3t + 3) \)).

Now suppose that \( C \) is a curve of \( H^0(Q, \mathcal{I}_Y(t, 3t + 2)) \), i.e. a curve on \( Q \) of type \((t, 3t + 2)\) containing \( Y \). As we have just seen, the line \( M \) and also each line \( M_i \), \((1 \leq i \leq t)\), contains \( 3t + 3 \) points of \( Y \). The fact that \( C \) contains these points forces \( C \) to have the lines \( M, M_i \) as fixed components (since otherwise \( C \) must intersect \( M \) (resp. \( M_i \)) at \( 3t + 2 \) points, while \( C \) already pass through the \( 3t + 3 \) points of \( M \) (resp. \( M_i \)), which is impossible); but the number of these lines is \( t + 1 \) and they are placed in the first ruling, which is a contradiction with the type \((t, 3t + 2)\) of \( C \). So such a \( C \) cannot exist, i.e., we have proved that \( h^0(Q, \mathcal{I}_Y(t, 3t + 2)) = 0 \). Then by semicontinuity one can deduce that
\[ h^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{\mathcal{X}})}(t, 3t + 2)) = 0, \] which is equivalent to
\[ h^0(Q, \mathcal{I}_{\text{Tr}_Q(\tilde{\mathcal{X}})}(d, d)) = 0. \]

Finally, from Remark 3.5 we get the conclusion. \(\square\)

5. Proof in \(\mathbb{P}^4\)

In this section we will prove Theorem 3.2 for the case \(n = 4\), which for convenience we state again.

\(\mathcal{S}^*(4, d)\): Let \(d \geq 3\) and
\[ r = \left\lfloor \frac{(d+4)}{4} - \frac{(4d+1)}{d+1} \right\rfloor; \quad q = \left( \frac{d+4}{4} \right) - (4d+1) - r(d+1). \]

Let the scheme \(X \subset \mathbb{P}^4\) be a generic union of \(r\) lines \(L_1, \ldots, L_r\), one double line \(2L\) and \(q\) points \(P_1, \ldots, P_q\) lying on a generic line \(M\). Then \(X\) has good postulation, i.e.,
\[ h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) = \left( \frac{d+4}{4} \right) - (4d+1) - r(d+1) - q = 0. \]

**Proof.** Let us begin with the case \(d = 3\). In this case we have \(r = 5\), and \(q = 2\), so \(X = 2L + L_1 + \cdots + L_5 + P_1 + P_2 \subset \mathbb{P}^4\).

Pick a generic hyperplane \(H \subset \mathbb{P}^4\). Now specialize the lines \(L, L_1\) and also the points \(P_1, P_2\) into \(H\), and denote by \(\tilde{X}\) the specialized scheme.

On the one hand we obtain \(\text{Res}_H(\tilde{X}) = L + L_2 + \cdots + L_5 \subset \mathbb{P}^3\), that is, \(\text{Res}_H(\tilde{X})\) is union of 5 generic lines. Thus by Hartshorne–Hirschowitz theorem, Theorem 2.6, we immediately get
\[ h^0(\mathcal{I}_{\text{Res}_H(\tilde{X})}(2)) = \left( \frac{2+4}{4} \right) - 15 = 0. \]

On the other hand we have
\[ \text{Tr}_H(\tilde{X}) = 2L|_H + L_1 + S_2 + \cdots + S_5 + P_1 + P_2 \subset H, \]
where \(L_i \cap H = S_i, (2 \leq i \leq 5)\). This means that \(\text{Tr}_H(\tilde{X})\) is generic union of one double line, one simple line, and 6 simple points in \(H \cong \mathbb{P}^3\). As we observed in Section 4, \(\mathcal{S}^*(3, 3)\) holds, which implies that \(\mathcal{S}(3, 3)\) holds. Now from \(\mathcal{S}(3, 3)\), with \(s = 1\), we get that the scheme \(2L|_H + L_1 \subset H \cong \mathbb{P}^3\) has good postulation in degree 3, i.e.,
\[ h^0(\mathcal{I}_{2L|_H + L_1}(3)) = \left( \frac{3+3}{3} \right) - 10 - 4 = 6. \]
Since $P_1, P_2$ and $S_i, (2 \leq i \leq 5)$, are 6 generic points in $H$, we get

$$h^0(H, \mathcal{I}_{\text{Tr}_H(\tilde{X})}(3)) = 0.$$  

Now by Remark 3.5 it follows that $h^0(\mathcal{I}_X(3)) = 0$.

Let us consider the case $d = 4$. Then $r = 10$ and $q = 3$. We observe that

$$X = 2L + L_1 + \cdots + L_{10} + P_1 + P_2 + P_3 \subset \mathbb{P}^4,$$

where $P_1, P_2, P_3$ are generic points lying on the line $M$.

Fix a generic hyperplane $H \subset \mathbb{P}^4$. Let $\tilde{X}$ be the scheme obtained from $X$ by specializing the lines $L$ and $L_1, L_2, L_3$ into $H$.

We have

$$\text{Res}_H(\tilde{X}) = X_1 + P_1 + P_2 + P_3 \subset \mathbb{P}^4,$$

where $X_1 = L + L_4 + \cdots + L_{10}$.

Applying Hartshorne–Hirschowitz theorem to $X_1$, which is union of 8 generic lines in $\mathbb{P}^4$, yields

$$h^0(\mathcal{I}_{X_1}(3)) = \binom{3+4}{4} - 32 = 3;$$

and also to $X_1 + M$, which is union of 9 generic lines in $\mathbb{P}^4$, yields

$$h^0(\mathcal{I}_{X_1 + M}(3)) = \max \left\{ \binom{3+4}{4} - 36, 0 \right\} = 0.$$

Hence by Lemma 2.4 we get

$$h^0(\mathcal{I}_{\text{Res}_H(\tilde{X})}(3)) = 0.$$

Moreover, we have

$$\text{Tr}_H(\tilde{X}) = 2L|_H + L_1 + L_2 + L_3 + S_4 + \cdots + S_{10} \subset H,$$

where $L_i \cap H = S_i, (4 \leq i \leq 10)$.

By setting $X_2 = 2L|_H + L_1 + L_2 + L_3$, we see that $X_2$ is generic union in $H \cong \mathbb{P}^3$ of one double line and 3 simple lines, so by $S(3, 4)$, with $s = 3$, we obtain

$$h^0(\mathcal{I}_{X_2}(4)) = \binom{4+3}{3} - 13 - 15 = 7.$$

Notice that the points $S_4, \ldots, S_{10}$ are 7 generic points in $H$, therefor

$$h^0(H, \mathcal{I}_{\text{Tr}_H(\tilde{X})}(4)) = 0.$$

This together with $h^0(\mathcal{I}_{\text{Res}_H(\tilde{X})}(3)) = 0$ implies that $h^0(\mathcal{I}_{\tilde{X}}(4)) = 0$, and from here, by semicontinuity, it follows the conclusion, which finishes the proof in this case.

Now assume $d \geq 5$. The rest of the proof will be by induction on $d$. 
We start by letting
\[ r' = \left\lfloor \frac{(d+3)}{4} - \frac{4(d - 1) + 1}{d} - q \right\rfloor; \]
\[ q' = \left( \frac{d+3}{4} \right) - \frac{4(d - 1) + 1}{d} - r'd - q; \]
\[ x = r - r' - 2q'; \]
further, noting that \( r', q', x \geq 0 \) (see Appendix, Lemma 8.1).

Recall that the scheme
\[ X = 2L + L_1 + \cdots + L_r + P_1 + \cdots + P_q \subset \mathbb{P}^4, \]
is generic union of the double line \( 2L \), the \( r \) simple lines \( L_i \), and the \( q \) points \( P_i \) belonging to the generic line \( M \).

Fix a generic hyperplane \( H \subset \mathbb{P}^4 \). In order to prove that \( X \) has good postulation in degree \( d \), we construct a scheme \( \tilde{X} \) obtained from \( X \) by combining specializations and degenerations as follows:

- specialize the first \( x \) lines \( L_i \) into \( H \), and call the resulting set of lines \( X_1 \);
- degenerate the next \( q' \) pairs of lines \( L_i \), so that they become \( q' \) sundials
  \[ \tilde{C}_i = C_i + 2N_i|_H; \quad (1 \leq i \leq q'), \]
  where \( C_i \) is a degenerate conic, \( H_i \cong \mathbb{P}^3 \) is a generic linear space containing \( C_i \) and \( 2N_i|_H \) is a double point in \( H_i \) with support at the singular point of \( C_i \), furthermore, specialize \( \tilde{C}_i \) in such a way that \( C_i \subset H \), but \( 2N_i|_H \not\subset H \), and call the resulting scheme of sundials \( X_2 \), that is
  \[ X_2 = \tilde{C}_1 + \cdots + \tilde{C}_{q'}, \]
  with the property that the degenerate conics \( C_i \) lie in \( H \), but \( 2N_i|_H \not\subset H \);
- leave the remaining simple lines \( L_i \), which are \( r' = r - x - 2q' \) lines, generic not lying in \( H \), and call this collection of lines \( X_3 \);

then let
\[ \tilde{\tilde{X}} = 2L + X_1 + X_2 + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^4. \]

We need to show that \( h^0(\mathcal{I}_{\tilde{\tilde{X}}}(d)) = 0 \), which clearly implies that \( h^0(\mathcal{I}_X(d)) = 0 \). To do that, by Castelnuovo’s inequality, it would be enough to show that \( h^0(\mathcal{I}_{\text{Res}_H(\tilde{\tilde{X}})}(d - 1)) = 0 \), and \( h^0(\mathcal{I}_{\text{Tr}_H(\tilde{\tilde{X}})}(d)) = 0 \).

First we verify the residual, which is
\[ \text{Res}_H(\tilde{\tilde{X}}) = 2L + \text{Res}_H(X_2) + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^4, \]
where $\text{Res}_H(X_2) = N_1 + \cdots + N_q'$. Recall that the points $P_i$ are $q$ generic lying on the line $M$. In order to apply Lemma 2.4 to get $h^0(\mathcal{I}_{\text{Res}_H}(\tilde{X})(d-1)) = 0$, it suffices to prove the two following equalities

$$h^0(\mathcal{I}_{2L + \text{Res}_H(X_2)+X_3}(d-1)) = q;$$

$$h^0(\mathcal{I}_{2L + \text{Res}_H(X_2)+X_3+M}(d-1)) = 0.$$

By the induction hypothesis we have that $\mathcal{S}^c(4, d-1)$ holds, then $\mathcal{S}(4, d-1)$ holds. Now by applying $\mathcal{S}(4, d-1)$ to the scheme $2L + X_3$, which consists of one double line and $r'$ generic lines, we get

$$h^0(\mathcal{I}_{2L+X_3}(d-1)) = \left(\frac{d+3}{4}\right) - (4(d-1) + 1) - r'd = q + q'.$$

Since $\text{Res}_H(X_2)$ consists of $q'$ generic points, it immediately follows

$$(2) \quad h^0(\mathcal{I}_{2L+X_3+\text{Res}_H(X_2)}(d-1)) = q.$$

In the same way, by applying $\mathcal{S}(4, d-1)$ to the scheme $2L + X_3 + M$, which consists of one double line and $r' + 1$ generic lines, we get

$$h^0(\mathcal{I}_{2L+X_3+M}(d-1)) = \max\left\{\left(\frac{d+3}{4}\right) - (4(d-1) + 1) - (r'+1)d, 0\right\}$$

$$= \max\{q + q' - d, 0\},$$

and therefore

$$(3) \quad h^0(\mathcal{I}_{2L+X_3+M+\text{Res}_H(X_2)}(d-1)) = \max\{q - d, 0\} = 0.$$

Hence by (2) and (3) we get

$$h^0(\mathcal{I}_{\text{Res}_H}(\tilde{X})(d-1)) = 0,$$

so we are done with the residual scheme.

Now we treat the trace scheme $Tr_H(\tilde{X})$, which we denote by $T$ for short, that is

$$T = Tr_H(\tilde{X}) = 2R|_H + X_1 + C_1 + \cdots + C_{q'} + X'_3 \subset H \cong \mathbb{P}^3,$$

where $L \cap H = R$ thus $2L \cap H = 2R|_H$ is a double point in $H$, and $X'_3 = Tr_H(X_3)$ is a generic collection of $r'$ simple points; moreover, recall that $X_1$ is made by $x$ generic lines, where $x = r - r' - 2q'$ as defined before.

We must prove that $h^0(H, \mathcal{I}_T(d)) = 0$. In order to do this, we wish to construct a specialization of $T$, with the desired vanishing, but this time our specialization will be via a smooth quadric surface. Since our investigations of $T$ will be done in $H$, as the ambient space, so for simplicity of notation we will from now on write $\mathbb{P}^3$ instead of $H$, as well as, $2R$ instead of $2R|_H$. 

Let \( Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \) be a smooth quadric in \( \mathbb{P}^3 \). Notations and terminology concerning \( Q \) are those of the Section 4. Let

\[
\hat{r} = \left\lfloor \frac{(d + 1)^2 - (d + 2)q' - 2x}{d - 1} \right\rfloor,
\]

\[
\hat{q} = (d + 1)^2 - (d + 2)q' - (d - 1)\hat{r} - 2x.
\]

Note that \( \hat{r} \geq 0 \), and so \( \hat{q} \geq 0 \) (see Appendix, Lemma 8.3 (i)).

To begin, let \( M_{1,i}, M_{2,i} \) be the two lines which form the degenerate conic \( C_i \), \( (1 \leq i \leq q') \), this means \( C_i = M_{1,i} + M_{2,i} \), and let \( S_1, \ldots, S_r \) be the points of \( X_3' \). Because of the inequalities \( \hat{r} \leq x \) and \( \hat{q} \leq r' \), (both are proved in Appendix, Lemma 8.3 (ii), (iii)), we can specialize \( T \) in the following way:

Let \( \tilde{T} \) be the scheme obtained from \( T \) by specializing the degenerate conics \( C_i \) and \( \hat{r} \) lines \( L_1, \ldots, L_{\hat{r}} \) of \( X_1 \) in such a way that the lines \( M_{1,1}, \ldots, M_{1,q'} \) and \( L_1, \ldots, L_{\hat{r}} \) become lines belonging to the first ruling of \( Q \), and by specializing \( \hat{q} \) points \( S_1, \ldots, S_{\hat{q}} \) of \( X_3' \) onto \( Q \) (the lines \( M_{2,1}, \ldots, M_{2,q'} \) and the other lines \( L_{\hat{r}+1}, \ldots, L_x \) of \( X_1 \), also the remaining points \( S_{\hat{q}+1}, \ldots, S_r' \) of \( X_3' \) and the point \( R \), remain generic not lying on \( Q \)).

Next, we perform the process of treating the residual and the trace of the specialized scheme \( \tilde{T} \), with respect to \( Q \), to get \( h^0(\mathcal{I}_{\tilde{T}}(d)) = 0 \).

We have

\[
\text{Res}_Q(\tilde{T}) = 2R + M_{2,1} + \cdots + M_{2,q'} + L_{\hat{r}+1} + \cdots + L_x + S_{\hat{q}+1} + \cdots + S_r' \subset \mathbb{P}^3.
\]

Observe that the scheme \( \text{Res}_Q(\tilde{T}) - (S_{\hat{q}+1} + \cdots + S_r') \) is generic union of one double point and \( q' + x - \hat{r} \) lines in \( \mathbb{P}^3 \), and that \( d - 2 \geq 3 \), thus by Corollary 2.8 we get

\[
h^0(\mathcal{I}_{\text{Res}_Q(\tilde{T}) - (S_{\hat{q}+1} + \cdots + S_r')}(d - 2)) = \left( \begin{array}{c} d - 2 + 3 \\ 3 \end{array} \right) - 4 - (q' + x - \hat{r})(d - 1) = r' - \hat{q},
\]

(the last equality is proved in Appendix, Lemma 8.3 (v)). Moreover, the points \( S_{\hat{q}+1}, \ldots, S_r' \) are \( r' - \hat{q} \) generic points, so we immediately get

\[
h^0(\mathcal{I}_{\text{Res}_Q(\tilde{T})}(d - 2)) = 0.
\]

Now it remains to consider the trace scheme. We first notice that \( M_{2,i} \) meets \( Q \) in the two points which are \( (M_{1,i} \cap M_{2,i}) \) and another point, which we denote by \( S_i' \), also recall that \( M_{1,i} \subset Q \), so we have that \( C_i \cap Q = M_{1,i} + S_i' \), \( (1 \leq i \leq q') \). Similarly, \( L_j \), \( (\hat{r} + 1 \leq j \leq x) \), meets \( Q \) in two points, then \( \text{Tr}_Q(L_{\hat{r}+1} + \cdots + L_x) \) is a collection of \( 2(x - \hat{r}) \) points, which we denote by \( T_1 \).
Thus we obtain
\[ T r_Q(\tilde{T}) = L_1 + \cdots + L_{\hat{r}} + T_1 + M_{1,1} + \cdots + M_{1,q'} \]
\[ + S'_1 + \cdots + S'_{q'} + S_1 + \cdots + S_{\hat{q}} \subset Q. \]

Since the lines \( M_{1,i} \) and \( L_j \), \((1 \leq i \leq q'; 1 \leq j \leq \hat{r})\), are contained in the first ruling of \( Q \), furthermore \( d \geq q' + \hat{r} \) (see Appendix, Lemma 8.3 (iv)), then all of these lines are fixed components for the curves of \( H^0(Q, I_{T r_Q(\tilde{T})}(d, d)) \). Set
\[ \Lambda = L_1 + \cdots + L_{\hat{r}} + M_{1,1} + \cdots + M_{1,q'} \subset Q. \]

Now by removing the fixed component \( \Lambda \), and by using the fact that the points \( S'_1, S_k, \ (1 \leq i \leq q'; 1 \leq k \leq \hat{q}) \), are generic on \( Q \), as well as the points of \( T_1 \), we conclude that
\[ h^0(Q, I_{T r_Q(\tilde{T})}(d, d)) = h^0(Q, I_{T r_Q(\tilde{T}) - \Lambda}(d - q' - \hat{r}, d)) \]
\[ = (d - q' - \hat{r} + 1)(d + 1) - (q' + \hat{q} + 2x - 2\hat{r}) \]
\[ = (d + 1)^2 - q'(d + 2) - \hat{r}(d - 1) - 2x - \hat{q} \]
\[ = 0. \]

Putting together \( h^0(I_{Res_Q(\tilde{T})}(d - 2)) = 0 \) and \( h^0(Q, I_{T r_Q(\tilde{T})}(d, d)) = 0 \) we have \( h^0(I_{\tilde{T}}(d)) = 0 \), therefore, by semicontinuity, we have \( h^0(I_{T}(d)) = 0 \). This completes the proof.  \( \square \)

6. Proof in \( \mathbb{P}^n \) for \( n \geq 5 \)

We come to the general case \( n \geq 5 \). Now we have the bases for our inductive approach, we are ready to prove Theorem 3.2 in the general setting.

\( S^*(n, d) \): Let \( n, d \in \mathbb{N} \), and \( n \geq 5, d \geq 3 \). Let
\[ r = \left\lfloor \frac{(d+n)}{d} - (nd+1) \right\rfloor; \quad q = \left( \frac{d+n}{n} \right) - (nd+1) - r(d+1). \]

Let the scheme \( X \subset \mathbb{P}^n \) be a generic union of \( r \) lines \( L_1, \ldots, L_r \), one double line \( 2L \) and \( q \) points \( P_1, \ldots, P_q \) lying on a generic line \( M \). Then \( X \) has good postulation, i.e.,
\[ h^1(I_X(d)) = h^0(I_X(d)) = \left( \frac{d+n}{n} \right) - (nd+1) - r(d+1) - q = 0. \]

Proof. We will prove the theorem by induction on \( d \). We proceed to the general case of \( n \geq 5 \), noting that \( S^*(3, d) \) and \( S^*(4, d) \) have been proved.

To begin, let
\[ r' = \left\lfloor \frac{(d-1+n)}{d} - (n(d-1) + 1) - q \right\rfloor; \]
\[ q' = \left( \frac{d-1+n}{n} \right) - (n(d-1) + 1) - r'd - q; \]
\( x = r - r' - 2q' \);

we can check that \( r', q', x \geq 0 \) (see Appendix, Lemma 8.1).

Let \( H \subset \mathbb{P}^n \) be a generic hyperplane. For the purpose of getting \( h^0(\mathcal{I}_X(d)) = 0 \), we wish to find a scheme \( \tilde{X} \) obtained from \( X \) by combining specializations and degenerations so that the desired vanishing can be achieved. Now we construct the required \( \tilde{X} \) in the following way, which is analogous to the one used in \( \mathbb{P}^4 \) in the previous section:

- specialize the first \( x \) lines \( L_i \) into \( H \), and call the resulting set of lines \( X_1 \);
- degenerate the next \( q' \) pairs of lines \( L_i \), so that they become \( q' \) sundials

\[
\tilde{C}_i = C_i + 2N_i|_{H_i}; \quad (1 \leq i \leq q'),
\]

where \( C_i \) is a degenerate conic, \( H_i \cong \mathbb{P}^3 \) is a generic linear space containing \( C_i \) and \( 2N_i|_{H_i} \) is a double point in \( H_i \) with support at the singular point of \( C_i \), furthermore, specialize \( \tilde{C}_i \) in such a way that \( C_i \subset H_i \), but \( 2N_i|_{H_i} \not\subset H_i \), and call the resulting scheme of sundials \( X_2 \), that is

\[
X_2 = \tilde{C}_1 + \cdots + \tilde{C}_{q'};
\]

- leave the remaining simple lines \( L_i \), which are \( r' = r - x - 2q' \) lines, generic not lying in \( H \), and call this collection of lines \( X_3 \);

then let

\[
\tilde{X} = 2L + X_1 + X_2 + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^n.
\]

To show that \( h^0(\mathcal{I}_{\tilde{X}}(d)) = 0 \), by Castelnuovo’s inequality, our goal will be to show that the following vanishings

\[
h^0(\mathcal{I}_{\text{Res}_H(\tilde{X})}(d-1)) = 0; \quad h^0(H, \mathcal{I}_{\text{Tr}_{\mathbb{P}^n}(\tilde{X})}(d)) = 0.
\]

With regard to residual, we have

\[
\text{Res}_H(\tilde{X}) = 2L + \text{Res}_H(X_2) + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^n,
\]

where \( \text{Res}_H(X_2) = N_1 + \cdots + N_{q'} \).

By the induction hypothesis we know that \( S^*(n, d - 1) \) holds, which implies that \( S(n, d - 1) \) holds (note that \( n \geq 5 \), then if \( d - 1 = 2 \), by Proposition 3.1 we also have that \( S(n, 2) \) holds). So we can apply \( S(n, d - 1) \) to the scheme \( 2L + X_3 \), as well as, to the scheme \( 2L + X_3 + M \), therefore

\[
h^0(\mathcal{I}_{2L+X_3}(d-1)) = \binom{d - 1 + n}{n} - (n(d - 1) + 1) - r'd = q + q'.
\]
\[
h^0(I_{2L+X_3+M}(d-1)) = \max \left\{ \left( \frac{d-1+n}{n} \right) - (n(d-1) + 1) - (r' + 1)d, 0 \right\}
= \max\{q + q' - d, 0\}.
\]

Observe that \(Res_H(X_2)\) is made by \(q'\) generic points, so we get
(4) \[
h^0(I_{2L+X_3+Res_H(X_2)}(d-1)) = q;
\]
(5) \[
h^0(I_{2L+X_3+M+Res_H(X_2)}(d-1)) = \max\{q - d, 0\} = 0.
\]

Having (4) and (5), moreover, recalling that the points \(P_i\) are \(q\) generic points lying on the line \(M\), we can now apply Lemma 2.4, hence
\[
h^0(I_{Res_H}(\tilde{x})(d-1)) = 0,
\]
as we wanted.

Now, we consider trace scheme \(Tr_H(\tilde{X})\), which we denote by \(T\) for short,
\[
T = Tr_H(\tilde{X}) = 2R|_H + X_1 + C_1 + \cdots + C_q' + X_3' \subset H \cong \mathbb{P}^{n-1},
\]
where \(L \cap H = R\) thus \(2L \cap H = 2R|_H\) is a double point in \(H\), and \(X_3' = Tr_H(X_3)\) is a generic collection of \(r'\) simple points, which we denote by \(S_1, \ldots, S_{r'}\). In addition, recall that \(X_1\) is made by \(x\) generic lines, where \(x = r - r' - 2q'\)
as defined before.

For simplicity in the notation, we will henceforward write \(\mathbb{P}^{n-1}\) instead of \(H\), as well as, \(2R\) instead of \(2R|_H\).

In order to verify the scheme \(T\), we make a specialization \(\tilde{T}\) of \(T\) via a fixed hyperplane as follows: we start by setting
\[
\tilde{r} = \left\lfloor \frac{d+n-2}{n-2} \right\rfloor - (n-1) - r + r';
\]
\[
\tilde{q} = \left( \frac{d+n-2}{n-2} \right) - (n-1) - \tilde{r}d - r + r',
\]
also noting that \(\tilde{r}, \tilde{q} \geq 0\) (Appendix, Lemma 8.2 (i)). Pick a generic hyperplane \(H'\) in \(\mathbb{P}^{n-1}\). Now using the inequalities \(\tilde{r} \leq x\) and \(\tilde{q} \leq r'\), (both are proved in Appendix, Lemma 8.2 (ii), (iii)), we specialize the lines \(L_1, \ldots, L_x\) of \(X_1\), also the points \(S_1, \ldots, S_q\) of \(X_3'\) and the point \(R\) into \(H'\), and we denote by \(\tilde{T}\) the specialized scheme (note that the other lines \(L_{r+1}, \ldots, L_x\) of \(X_1\), the degenerate conics \(C_i\), and the other points of \(X_3'\) remain generic outside \(H'\)).

Now in order to prove that \(h^0(\mathbb{P}^{n-1}, I_T(d)) = 0\), by semicontinuity, our next goal will be to prove that \(h^0(\mathbb{P}^{n-1}, I_{\tilde{T}}(d)) = 0\).

\(L_i\) meets \(H'\) at one point, \((\tilde{r} + 1 \leq i \leq x)\), so \(Tr_{H'}(L_{r+1} + \cdots + L_x)\) is a union of \(x - \tilde{r}\) points, which we denote by \(T_1\). Moreover, \(C_j\) meets \(H'\) in two
points, \((1 \leq j \leq q')\), then \(\text{Tr}_{H'}(C_1 + \cdots + C_{q'})\) is a collection of \(2q'\) points, which we denote by \(T_2\). Accordingly with these notations, we have

\[
\text{Tr}_{H'}(\tilde{T}) = 2R|_{H'} + L_1 + \cdots + L_q + T_1 + T_2 + S_1 + \cdots + S_q \subset H' \cong \mathbb{P}^{n-2}.
\]

First we apply Corollary 2.8 to the scheme \(2R|_{H'} + L_1 + \cdots + L_q\), which implies that

\[
h^0(H', \mathcal{I}_{2R|_{H'} + L_1 + \cdots + L_q}(d)) = \binom{d + n - 2}{n - 2} - (n - 1) - r(d + 1) = \bar{q} - \bar{r} + r' = \bar{q} - \bar{r} + 2q' + x,
\]

next, by the fact that the schematic union \((T_1 + T_2 + S_1 + \cdots + S_q)\) is a generic union of \(x - \bar{r} + 2q' + \bar{q}\) simple points, we immediately get

\[
h^0(H', \mathcal{I}_{\text{Tr}_{H'}(\tilde{T})}(d)) = 0,
\]

so we are finished with the trace scheme.

Then we are left with the residual of \(\tilde{T}\) with respect to \(H' \cong \mathbb{P}^{n-2}\), which is

\[
\text{Res}_{H'}(\tilde{T}) = R + C_1 + \cdots + C_{q'} + L_{q'+1} + \cdots + L_x + S_{q+1} + \cdots + S_{q'} \subset \mathbb{P}^{n-1}.
\]

It is the existence of the degenerate conics \(C_i\) that impedes us to directly investigate the residual scheme. Our method to afford this difficulty is to take a degeneration of \(\text{Res}_{H'}(\tilde{T})\), but using a different way to do so. Indeed, according to the observation of §2.4 saying that a sundial can be considered as a degeneration of a degenerate conic together with a simple point, we then degenerate \(q'\) points \(S_{q+1}, \ldots, S_{q+q'}\) together with \(q'\) conics \(C_i\) so that they become \(q'\) sundials \(\bar{C}_i\) having singularity at these points, (it is possible because \(q' \leq r' - \bar{q}\), Appendix, Lemma 8.2 (iii)). We set \(\Gamma = R + S_{q+q'+1} + \cdots + S_{q'}\).

Let \(Y\) be the scheme obtained from \(\text{Res}_{H'}(\tilde{T})\) by this degeneration, more precisely,

\[
Y = \bar{C}_1 + \cdots + \bar{C}_{q'} + L_{\bar{q}+1} + \cdots + L_x + \Gamma \subset \mathbb{P}^{n-1}.
\]

The scheme \(Y - \Gamma\) is generic union of \(q'\) sundials and \(x - \bar{r}\) lines in \(\mathbb{P}^{n-1}\), so by Theorem 2.11 it has good postulation, in other words

\[
h^0(\mathbb{P}^{n-1}, \mathcal{I}_{Y - \Gamma}(d - 1)) = \binom{d - 1 + n - 1}{n - 1} - (2q' + x - \bar{r})d = r' - q' - \bar{q} + 1,
\]

the computations to get the last equality can be found in Appendix, Lemma 8.2 (iv). Since \(\Gamma\) is generic union of \(r' - \bar{q} + 1 - q'\) points, it then immediately
follows that
\[ h^0(\mathbb{P}^{n-1}, \mathcal{I}_Y(d - 1)) = 0, \]
and from here, again by semicontinuity, we obtain
\[ h^0(\mathbb{P}^{n-1}, \text{Res}_{H}(\tilde{T})(d - 1)) = 0. \]
This together with (6), by Castelnuovo’s inequality, yields that
\[ h^0(\mathbb{P}^{n-1}, \mathcal{I}_\tilde{T}(d)) = 0, \]
and this is in fact what we wanted to show, hence the proof is complete. □

7. On Conjecture 1.2

Now coming back to our Conjecture 1.2, we will prove it only in a special case.

7.1. Some evidence for Conjecture 1.2. The main result of this paper, Theorem 1.1, attracts our attention to a natural class of objects that is schemes \( X \) of lines and one fat linear space in projective space. In fact, the geometry of the exception that we determined in Theorem 1.1 leads us to conjecture that it can be generalized somehow to the families of lines and one fat linear space. The basic motivation lies in the fact that, no defective cases with respect to the linear system \( |\mathcal{I}_X(d)| \) have been discovered, unless \( d = m \), where \( m \) is the multiplicity of that linear space. So we hope the following conjecture, which exactly describes the failure of \( X \) to have good postulation.

**Conjecture 7.1** (Conjecture 1.2 of the Introduction). Let \( n, d, r \in \mathbb{N} \), and \( n \geq r + 2 \geq 3 \). The scheme \( X \subset \mathbb{P}^n \) consisting of \( s \geq 1 \) generic lines and one \( m \)-multiple linear space \( m\Pi \), \( (m \geq 2) \), with \( \Pi \cong \mathbb{P}^r \subset \mathbb{P}^n \), always has good postulation, except for the cases
\[ \{ n = r + 3, m = d, 2 \leq s \leq d \} . \]

This conjecture would be in perfect analogy with Theorem 1.1. Note that it is a hard problem to prove it in general case, and doing so requires the most sophisticated investigations with a lot of technical details, in the setting of specialization and degeneration.

Now we show that the conjecture is true for the special case of \( d = m \), which is in the center of our attention. Before proceeding to state and prove it, let us introduce the following integer \( \alpha(n,d,r,m) \), for all integers \( n, r, d, m \) with \( n > r \) and \( d \geq m - 1 \), which we will use throughout this section:
\[ \alpha(n,d,r,m) = \sum_{i=0}^{m-1} \binom{r + d - i}{r} \binom{n + i - r - 1}{i} . \]
Observe that $\alpha_{n,d,r,m}$ is exactly the Hilbert polynomial of $m\Pi$ in degree $d$, Lemma 2.5. Moreover, when $d = m$ with a straightforward computation, one easily sees that:

$$\alpha_{n,m,r,m} = \binom{n+m}{n} - \binom{n+m-r-1}{n-r-1}.$$ 

**Proposition 7.2.** The scheme $X \subset \mathbb{P}^n$, $n \geq r + 2 \geq 3$, consisting of $s \geq 1$ generic lines and one $m$-multiple linear space $m\Pi$, $(m \geq 2)$, with $\Pi \cong \mathbb{P}^r \subset \mathbb{P}^n$, has good postulation in degree $m$, i.e.,

$$h^0(\mathcal{I}_X(m)) = \max \left\{ \binom{n+m}{n} - \alpha_{n,m,r,m} - s(d+1), 0 \right\}$$

$$= \max \left\{ \binom{n+m-r-1}{n-r-1} - s(d+1), 0 \right\},$$

except for $\{n = r + 3, 2 \leq s \leq d\}$, in which case the defect is $\binom{s}{2}$.

**Proof.** First notice that, the sections of $\mathcal{I}_X(m)$ correspond to degree $m$ hypersurfaces in $\mathbb{P}^n$ which, in order to contain $m\Pi$, have to be cones whose vertex contains the linear space $\Pi$.

For $n = r + 2$, it is easy to see that the linear system $|\mathcal{I}_X(m)|$ is empty, i.e. $h^0(\mathcal{I}_X(m)) = 0$, that is what was expected.

For $n \geq r + 3$, let us consider the projection $X'$ of $X$ from $\Pi$ into a generic linear subspace $\mathbb{P}^{n-r-1} \subset \mathbb{P}^n$. Then we have that the scheme $X'$ consists of $s$ generic lines in $\mathbb{P}^{n-r-1}$, also that the following equality

$$h^0(\mathbb{P}^n, \mathcal{I}_X(m)) = h^0(\mathbb{P}^{n-r-1}, \mathcal{I}_{X'}(m)).$$

In case $n > r + 3$, we have $n - r - 1 \geq 3$, therfor from Hartshorne–Hirschowitz theorem 2.6 it follows that

$$h^0(\mathbb{P}^{n-r-1}, \mathcal{I}_{X'}(m)) = \max \left\{ \binom{m+n-r-1}{n-r-1} - s(d+1), 0 \right\},$$

which is the expected value for $h^0(\mathbb{P}^n, \mathcal{I}_X(m))$, so we are done in this case.

In case $n = r + 3$, $X'$ is a generic union of $s$ lines in $\mathbb{P}^2$. Hence, if $s > m$ we obviously have $h^0(\mathcal{I}_{X'}(m)) = 0$, as expected. If $s \leq m$ we have $h^0(\mathcal{I}_{X'}(m)) = \binom{m+2s+2}{2}$, on the other hand the expected value for $h^0(\mathcal{I}_X(m))$ is

$$\max \left\{ \binom{m+2}{2} - s(m+1), 0 \right\},$$

which we denote by $\exp h^0(\mathcal{I}_X(m))$. Thus for $s = 1$, we get that $h^0(\mathcal{I}_X(m)) = \binom{m+1}{2}$, as expected; but for $2 \leq s \leq m$, we get that $h^0(\mathcal{I}_X(m)) \neq \exp h^0(\mathcal{I}_X(m))$ and the defect is

$$h^0(\mathcal{I}_X(m)) - \exp h^0(\mathcal{I}_X(m)) = \binom{s}{2},$$
which finishes the proof. □

7.2. Final remark. A complete proof for Conjecture 1.2 will be a substantial effort, however, we believe that a method analogous to that presented in §3.3, can be successfully applied for studying postulation problem for a generic scheme of lines and one fat linear space in \( \mathbb{P}^n \), and we plan to study this problem in the future. Indeed, if one can provide a proof for Conjecture 1.2 for a generic union of lines and one fat line in \( \mathbb{P}^n \), then even interestingly enough, one may hope to generalize this approach to the cases of lines and one fat linear space, that seems to be quite difficult. Actually, compared with the proof we gave in this paper, in the case of lines and one fat linear space we are forced to divide the proof in much more steps. While an argument analogous to Theorem 1.1 works in a more complicated way for the higher dimensional ambient projective spaces, investigations in two initial ambient spaces cause troubles, and this is why we leave it for the future.

8. Appendix: Calculations

**Lemma 8.1.** Let \( n \geq 5, d \geq 3 \) or \( n = 4, d \geq 5 \). Let

\[
 r = \left\lfloor \frac{(d+n)-(nd+1)}{d+1} \right\rfloor, \quad q = \left(\frac{d+n}{n}\right) - (nd+1) - r(d+1);
\]

\[
 r' = \left\lfloor \frac{(d-1+n)-(n(d-1)+1)-q}{d} \right\rfloor, \quad q' = \left(\frac{d-1+n}{n}\right) - (n(d-1)+1) - r'd - q.
\]

Then

(i) \( r' \geq 0 \);

(ii) \( r - r' - 2q' \geq 0 \).

**Proof.** (i) Since \( q \leq d \), we have

\[
 \left(\frac{d-1+n}{n}\right) - (n(d-1)+1) - q \geq \left(\frac{d-1+n}{n}\right) - (n(d-1)+1) - d,
\]

so in order to show that \( r' \geq 0 \) it is enough to show that

\[
 (7) \left(\frac{d-1+n}{n}\right) - (n(d-1)+1) \geq d.
\]

First consider the case \( n = 4 \) and \( d \geq 5 \), then we obviously have

\[
 \left(\frac{d-1+n}{n}\right) - (n(d-1)+1) - d = \left(\frac{d+3}{4}\right) - 5d + 3 \geq 0.
\]
Now consider the case \( n \geq 5 \) and \( d \geq 3 \). Notice that the function \( \binom{d-1+n}{n} - (n(d - 1) + 1) \) is an increasing function in \( n \), hence to get the conclusion it suffices to prove the inequality (7) only for \( n = 5 \). Now by letting \( n = 5 \), we easily see that

\[
\binom{d-1+5}{5} - (5(d - 1) + 1) - d
= \binom{d+4}{5} - 6d + 4 \geq 0,
\]

the last inequality is surely holds for \( d \geq 3 \).

(ii) We have to prove that

\[
\left\lfloor \frac{\binom{d+n}{n} - (nd + 1)}{d+1} \right\rfloor \geq r' + 2q'.
\]

Since \( r' \) and \( q' \) are integers, the inequality above is equivalent to the following

\[
\frac{\binom{d+n}{n} - (nd + 1)}{d+1} \geq r' + 2q',
\]

hence, it is enough to prove that

\[
\binom{d+n}{n} - (nd + 1) - (d+1)r' - 2(d+1)q' \geq 0.
\]
We have
\[
\binom{d+n}{n} - (nd + 1) - (d+1)r' - 2(d+1)q' \\
= \binom{d+n}{n} - 2(d+1)\binom{d-1+n}{n} + 2(d+1)(n(d-1) + 1) - (nd + 1) \\
+ 2(d+1)q + (d+1)(2d-1)r' \\
\geq \binom{d+n}{n} - 2(d+1)\binom{d-1+n}{n} + 2(d+1)(n(d-1) + 1) - (nd + 1) \\
+ 2(d+1)q + (d+1)(2d-1) \left\{ \frac{d-1+n}{n} - \frac{(d+1)(n(d-1) + 1) - q}{d} \right\} \\
= \frac{1}{d} \left\{ d\binom{d+n}{n} - (d+1)\binom{d-1+n}{n} - (n-1) + q(d+1) - d(d+1)(2d-1) \right\} \\
= \frac{1}{d} \left\{ (n-1)\binom{d-1+n}{n} - (n-1) + q(d+1) - d(d+1)(2d-1) \right\} \\
\geq \frac{1}{d} \left\{ (n-1)\binom{d-1+n}{n} - \binom{d-1+n}{n} - d(d+1)(2d-1) \right\} \\
= \frac{1}{d} \left\{ (n-2)\binom{d-1+n}{n} - d(d+1)(2d-1) \right\}.
\]

For \( n \geq 5 \), we get
\[
(n-2)\binom{d-1+n}{n} - d(d+1)(2d-1) \\
\geq 3\binom{d+4}{5} - d(d+1)(2d-1) \\
= \frac{1}{40}d(d+1)\{(d+2)(d+3)(d+4) - 80d + 40\} \geq 0,
\]
it is quite immediate to check that the last inequality holds for all \( d \geq 3 \), hence we are done in the case \( n \geq 5 \).

For \( n = 4 \), we get
\[
(n-2)\binom{d-1+n}{n} - d(d+1)(2d-1) \\
= \frac{1}{12}d(d+1)\{(d+2)(d+3) - 24d + 12\} \\
= \frac{1}{12}d(d+1)(d-1)(d-18),
\]
so this is positive for all \( d \geq 18 \). This is what we wanted to show for \( n = 4 \) and \( d \geq 18 \), then we are left with \( 5 \leq d \leq 17 \). Now by a direct computation we get the desired inequality \( r - r' - 2q' \geq 0 \) in the case \( n = 4 \) with \( 5 \leq d \leq 17 \) as follows:

\[
\begin{array}{c|cccccccccc}
\hline
\text{d} & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
\hline
r - r' - 2q' & 7 & 9 & 2 & 10 & 9 & 10 & 17 & 9 & 30 & 34 & 17 & 35 & 32 \\
\hline
\end{array}
\]

\[\square\]

**Lemma 8.2.** Let \( n \geq 5 \) and \( d \geq 3 \). With the notations as in Lemma 8.1, let

\[
\bar{r} = \left\lfloor \frac{(d+n-2)}{n-2} - \frac{(n-1) - r + r'}{d} \right\rfloor,
\]

\[
\bar{q} = \left(\frac{d + n - 2}{n - 2}\right) - (n - 1) - \bar{r}d - r + r'.
\]

Then

(i) \( \bar{r} \geq 0 \);
(ii) \( \bar{r} \leq r - r' - 2q' \);
(iii) \( r' \geq q' + \bar{q} \);
(iv) \( (d+n-2) - (r - r' - \bar{r})d = r' - q' - \bar{q} + 1 \).

**Proof.** (i) We will verify that

\[
\left(\frac{d + n - 2}{n - 2}\right) - (n - 1) - r + r' \geq 0.
\]

Since

\[
r \leq \frac{(d+n)}{d+1}; \quad r' \geq \frac{(d-1+n)}{d} - \frac{(n(d-1)+1)-q}{d} - 1,
\]

we have

\[
r - r' \leq \frac{1}{d+1} \left\{ \left(\frac{d+n}{d+1}\right) - (nd+1) \right\}
- \frac{1}{d} \left\{ \left(\frac{d-1+n}{n}\right) - (n(d-1)+1) - q \right\} + 1
= \frac{1}{d(d+1)} \left\{ d\left(\frac{d+n}{n}\right) - (d+1)\left(\frac{d-1+n}{n}\right) - (n-1) + q(d+1) \right\} + 1
= \frac{1}{d(d+1)} \left\{ (n-1)\left(\frac{d-1+n}{n}\right) - (n-1) + q(d+1) + d(d+1) \right\}.
\]
Then we get
\[
\left( \frac{d+n-2}{n-2} \right) - (n-1) - r + r' \\
\geq \left( \frac{d+n-2}{n-2} \right) - (n-1) \\
- \frac{1}{d(d+1)} \left\{ (n-1) \left( \frac{d-1+n}{n} \right) - (n-1) + 2d(d+1) \right\} \\
= \frac{A}{d(d+1)},
\]
where
\[
A = d(d+1) \left( \frac{d+n-2}{n-2} \right) - (n-1) \left( \frac{d-1+n}{n} \right) - (n-1)(d^2+d-1) - 2d(d+1).
\]

A straightforward computation, yields
\[
A = \frac{1}{n} \left( \frac{d+n-2}{n-2} \right) (nd^2 - d^2 + d) - (n-1)(d^2+d-1) - 2d(d+1).
\]

Since \( n \geq 5, d \geq 3 \) we have \( n(n-1) \leq \binom{d+n-2}{n-2} \), and from here it follows
\[
A \geq (n-1)(nd^2 - d^2 + d - (d^2+d-1)) - 2d(d+1) \\
= d^2(n^2 - 3n) - 2d + n - 1 \geq 0,
\]
which completes the proof.

(ii) In order to prove that \( \bar{r} \leq r - r' - 2q' \), it suffices to prove that
\[
\frac{\binom{d+n-2}{n-2} - (n-1) - r + r'}{d} \leq r - r' - 2q' + 1,
\]
which is equivalent to the following
\[
r(d+1) - r'(d+1) - \binom{d+n-2}{n-2} + (n-1) - 2q'd + d \geq 0.
\]

From the definitions of \( r \) and \( r' \), moreover the inequality \( q' \leq d - 1 \), we get
\[
r(d+1) - r'(d+1) - \binom{d+n-2}{n-2} + (n-1) - 2q'd + d \\
\geq \binom{d+n}{n} - (nd+1) - (d+1) - \frac{d+1}{d} \left\{ \binom{d-1+n}{n} - (n(d-1) + 1) - q \right\} \\
- \binom{d+n-2}{n-2} + (n-1) - 2d^2 + 3d,
\]
which, by an easy computation, is equal to
\[
\frac{1}{d} \left\{ (n-1) \binom{n+d-2}{n} + n(d-1) - 2d^3 + 2d^2 - 2d + 1 + q(d+1) \right\}.
\]
Now we observe that
\[
(n-1) \binom{n+d-2}{n} + n(d-1) - 2d^3 + 2d^2 - 2d + 1 + q(d+1)
\geq (n-1) \binom{n+d-2}{n} + n(d-1) - 2d^3 + 2d^2 - 2d + 1,
\]
hence, we will be done if we prove that
\[
(n-1) \binom{n+d-2}{n} + n(d-1) - 2d^3 + 2d^2 - 2d + 1 \geq 0.
\]
For \( n \geq 6 \), we have
\[
(n-1) \binom{n+d-2}{n} + n(d-1) - 2d^3 + 2d^2 - 2d + 1 
\geq 5 \binom{d+4}{6} - (2d^3 - 2d^2 - 4d + 5),
\]
which, for \( d \geq 3 \), is positive, as we wanted.
For \( n = 5 \), the inequality (8) becomes:
\[
4 \binom{d+3}{5} - (2d^3 - 2d^2 - 3d + 4) \geq 0,
\]
which is true for \( d \geq 5 \), so we are left with \( d = 3, 4 \) in the case of \( n = 5 \). But direct computations show that also these cases satisfy the required inequality \( \bar{r} \leq r' - r' - 2q' \). More precisely, if \( d = 3 \), we have \( \bar{r} = 3 \) and \( r' - r' - 2q' = 5 \); if \( d = 4 \), we have \( \bar{r} = 5 \) and \( r' - r' - 2q' = 11 \).

(iii) We want to prove that \( q' + \bar{q} \leq r' \). By the inequalities \( q', \bar{q} \leq d - 1 \), which implies \( q' + \bar{q} \leq 2d - 2 \), and also by the following one
\[
r' \geq \frac{(d-1+n)}{d} - \frac{(n(d-1)+1) - q}{d} - 1,
\]
it is enough to prove that
\[
\frac{(d-1+n)}{d} - \frac{(n(d-1)+1) - q}{d} - 1 \geq 2d - 2,
\]
i.e.
\[
\left( \frac{d-1+n}{n} \right) - (n(d-1)+1) - q - 2d^2 + d \geq 0.
\]
Using \( q \leq d \), we have to show that
\[
\binom{d - 1 + n}{n} - (n(d - 1) + 1) - 2d^2 \geq 0,
\]
or, equivalently,
\begin{equation}
\binom{d - 1 + n}{n} - n(d - 1) \geq 2d^2 + 1. \tag{9}
\end{equation}

Notice that the function \( \binom{d-1+n}{n} - n(d-1) \) is an increasing function in \( n \). For \( n = 5 \), the inequality \( (9) \) becomes
\[
\binom{d + 4}{5} \geq 2d^2 + 5d - 4,
\]
which holds for \( d \geq 4 \). So it remains to check \( q' + \bar{q} \leq r' \) in the case of \( d = 3 \) with \( n = 5 \). In this case we can directly compute that \( r' = 3, q' = 1, \bar{q} = 0 \). Hence the case \( n = 5 \) is done.

For \( n = 6 \), the inequality \( (9) \) becomes
\[
\binom{d + 5}{6} \geq 2d^2 + 6d - 5,
\]
which holds for \( d \geq 4 \). So we are left with \( d = 3 \). A direct computation in the case of \( d = 3 \) with \( n = 6 \) yields that \( r' = 4, q' = 2, \bar{q} = 0 \). So we are done for \( n = 6 \).

Finally, for \( n = 7 \), the inequality \( (9) \) becomes
\[
\binom{d + 6}{7} \geq 2d^2 + 7d - 6,
\]
which is true for any \( d \geq 3 \). Now, since \( \binom{d-1+n}{n} - n(d-1) \) is an increasing function in \( n \), we have proved \( (9) \) for all \( n \geq 7 \) and \( d \geq 3 \). That finishes the proof of part (iii).

(iv) We must check that
\[
\binom{d + n - 2}{n - 1} - (r - r' - \bar{r})d = r' - q' - \bar{q} + 1,
\]
that is
\begin{equation}
\binom{d + n - 2}{n - 1} + (r'd + q') + (\bar{r}d + \bar{q} - r') = rd + 1. \tag{10}
\end{equation}
From the definitions of \( q', \bar{q} \) we have
\[
r'd + q' = \binom{d - 1 + n}{n} - n(d - 1) - 1 - q;
\]
\[ \tilde{r}d + \tilde{q} - r' = \left(\frac{d + n - 2}{n - 2}\right) - (n - 1) - r. \]

Now using these equalities and an easy computation yields
\[
\left(\frac{d + n - 2}{n - 1}\right) + (r'd + q') + (\tilde{r}d + \tilde{q} - r') = \left(\frac{d + n}{n}\right) - nd - r - q,
\]
which by \( q = \left(\frac{d+n}{n}\right) - (nd + 1) - r(d + 1) \) is equal to \( r(d + 1) \), that is what we wanted (10). \( \Box \)

**Lemma 8.3.** Let \( d \geq 5 \). Let \( r, r', q, q' \) be as in Lemma 8.1 in the case \( n = 4 \).

Let \( \hat{r} = \left\lfloor \left(\frac{(d + 1)^2 - (d + 2)q' - 2(r - r' - 2q')}{d - 1}\right) \right\rfloor \), \( \hat{q} = (d + 1)^2 - (d + 2)q' - (d - 1)\hat{r} - 2(r - r' - 2q'). \)

Then
\[
\begin{align*}
\text{(i) } & \hat{r} \geq 0; \\
\text{(ii) } & \hat{r} \leq r - r' - 2q'; \\
\text{(iii) } & \hat{q} \leq r'; \\
\text{(iv) } & q' + \hat{r} \leq d; \\
\text{(v) } & r' - \hat{q} = \left(\frac{d+1}{3}\right) - 4 - (d - 1)(r - r' - \hat{r} - q').
\end{align*}
\]

**Proof.** (i) We need to show that
\[
(d + 1)^2 - (d + 2)q' - 2(r - r' - 2q') \geq 0,
\]
that is
\[
(d + 1)^2 - (d - 2)q' - 2(r - r') \geq 0.
\]
Recall:
\[
r = \left\lfloor \frac{\left(\frac{d+4}{4}\right) - 4d - 1}{d + 1}\right\rfloor, \quad q = \left(\frac{d + 4}{4}\right) - 4d - 1 - r(d + 1);
\]
\[
r' = \left\lfloor \frac{\left(\frac{d+3}{4}\right) - 4d + 3 - q}{d}\right\rfloor, \quad q' = \left(\frac{d + 3}{4}\right) - 4d + 3 - q - r'd.
\]

Let us start by computing \( (d - 2)q' + 2(r - r') \):
\[
\begin{align*}
(d - 2)q' + 2r - 2r' & = (d - 2)\left(\frac{d + 3}{4}\right) - (d - 2)(4d - 3) \\
& - (d - 2)q - (d^2 - 2d + 2)r' + 2r \\
& \leq \frac{A}{d(d + 1)},
\end{align*}
\]
where
\[
A = (d^2 + d)(d - 2)\left(\frac{d + 3}{4}\right) - (d^2 + d)(d - 2)(4d - 3)
\]
\[
- (d^2 + d)(d - 2)q - (d + 1)(d^2 - 2d + 2)\left\{\left(\frac{d + 3}{4}\right) - 4d + 3 - q\right\}
\]
\[
+ 2d\left\{\left(\frac{d + 4}{4}\right) - 4d - 1 - (d + 1)\right\}
\]
\[
= 2d\left(\frac{d + 4}{4}\right) - 2(d + 1)\left(\frac{d + 3}{4}\right) - 2(d^2 + d + 3) + 2q(d + 1)
\]
\[
= 6\left(\frac{d + 3}{4}\right) - 2(d^2 + d + 3) + 2q(d + 1).
\]

Therefore we get
\[
(d - 2)q' + 2(r - r') \leq \frac{A}{d(d + 1)}
\]
\[
= \frac{(d + 2)(d + 3)}{4} - \frac{2(d^2 + d + 3)}{d^2 + d} + \frac{2q}{d},
\]
by noting that \(\frac{2(d^2 + d + 3)}{d^2 + d} \geq 2\) and that \(q \leq d\), it immediately follows
\[
(d - 2)q' + 2(r - r') \leq \frac{(d + 2)(d + 3)}{4}.
\]

Now from here we have
\[
(d + 1)^2 - (d - 2)q' - 2(r - r') \geq (d + 1)^2 - \frac{(d + 2)(d + 3)}{4}
\]
\[
= \frac{3d^2 + 3d - 2}{4} \geq 0,
\]
and this finishes the proof.

(ii) In order to check that \(\hat{r} \leq r - r' - 2q'\), it suffices to check that
\[
\frac{(d + 1)^2 - (d + 2)q' - 2(r - r' - 2q')}{d - 1} \leq r - r' - 2q' + 1,
\]
that is
\[
(d + 1)^2 - (d + 2)q' - 2(r - r' - 2q') \leq (d - 1)(r - r' - 2q') + (d + 1),
\]
or, equivalently
\[
(d + 1)(r - r') - dq' - (d + 1)^2 + (d - 1) \geq 0.
\]
Again, using the definitions of \( r \) and \( r' \), moreover the inequality \( q' \leq d - 1 \), one gets

\[
\begin{align*}
(d + 1)(r - r') - dq' - (d + 1)^2 + (d - 1) \\
\geq (d + 1)r - (d + 1)r' - 2(d^2 + 1) \\
\geq \left( \frac{d + 4}{4} \right) - (4d + 1) - (d + 1) \\
- \frac{d + 1}{d} \left\{ \left( \frac{d + 3}{4} \right) - 4d + 3 - q \right\} - 2(d^2 + 1),
\end{align*}
\]

which, by a short computation similar to that in part (i), is equal to

\[
\frac{1}{d} \left\{ 3 \left( \frac{d + 3}{4} \right) - (2d^3 + d^2 + 3d + 3) + q(d + 1) \right\}.
\]

Now we have

\[
\begin{align*}
3 \left( \frac{d + 3}{4} \right) - (2d^3 + d^2 + 3d + 3) + q(d + 1) \\
\geq 3 \left( \frac{d + 3}{4} \right) - (2d^3 + d^2 + 3d + 3) \\
= \frac{1}{8}(d^4 - 10d^3 + 3d^2 - 18d - 24),
\end{align*}
\]

which, in fact for \( d \geq 10 \) is positive, as required. Then it remains to check that the cases \( 5 \leq d \leq 9 \) satisfy \( \hat{r} \leq r - r' - 2q' \). Computing each of these cases, we get the conclusion:

| \( d \) | \( \hat{r} \) | \( r - r' - 2q' \) |
|------|---|------------------|
| 5    | 5 | 7                |
| 6    | 6 | 9                |
| 7    | 2 | 2                |
| 8    | 5 | 10               |
| 9    | 4 | 9                |

(iii) To prove \( \hat{q} \leq r' \), by noting that \( \hat{q} \leq d - 2 \), it is enough to prove

\[
\frac{(d + 3)}{4} - 4d + 3 - q \geq d - 1,
\]

i.e.

\[
(d + 3) - 4d + 3 - q - d(d - 1) \geq 0.
\]

(11)
Observe that
\[
\begin{align*}
\binom{d+3}{4} - 4d + 3 - q - d(d-1) \\
\geq \binom{d+3}{4} - 4d + 3 - d - d(d-1) \\
= \binom{d+3}{4} - (d^2 + 4d - 3).
\end{align*}
\]

For \( d \geq 5 \), it is immediate to see that
\[
\binom{d+3}{4} - (d^2 + 4d - 3) \geq 0,
\]
which gives (11).

(iv) We will show that \( d - q' - \hat{r} \geq 0 \). We have
\[
\begin{align*}
d - q' - \hat{r} &\geq d - q' - \frac{1}{d-1}((d+1)^2 - (d-2)q' - 2r + 2r') \\
&= \frac{1}{d-1}(2r - 2r' - q' - 3d - 1),
\end{align*}
\]
moreover,
\[
\begin{align*}
2r - 2r' - q' - 3d - 1 \\
&= 2r + (d-2)r' - \binom{d+3}{4} + d - 4 + q \\
&\geq \frac{A}{d(d+1)},
\end{align*}
\]
where,
\[
A = 2d\left(\binom{d+4}{4}\right) - 2d(4d+1) - 2d(d+1) + (d-2)(d+1)\left(\binom{d+3}{4}\right) \\
-(d-2)(d+1)(4d-3) - (d-2)(d+1)q - d(d+1)(d-2) \\
-(d^2 + d)\left(\binom{d+3}{4}\right) + (d^2 + d)(d - 4) + (d^2 + d)q.
\]
After simple computations, one can easily find that

\[
A = 2d \left( \frac{d+4}{4} \right) - 2(d+1) \left( \frac{d+3}{4} \right) - (4d^3 + 5d^2 + d + 6) + 2(d+1)q
\]

\[
= 6 \left( \frac{d+3}{4} \right) - (4d^3 + 5d^2 + d + 6) + 2(d+1)q
\]

\[
\geq 6 \left( \frac{d+3}{4} \right) - (4d^3 + 5d^2 + d + 6),
\]

which is positive for \(d \geq 11\), hence we are left with \(5 \leq d \leq 10\). Now by direct calculations we get \(q' + \hat{r} \leq d\) in these cases as follows:

| \(d\) | \(q'\) | \(\hat{r}\) | \(q' + \hat{r}\) |
|------|------|------|------|
| 5    | 0    | 5    | 5    |
| 6    | 0    | 6    | 6    |
| 7    | 5    | 2    | 7    |
| 8    | 2    | 5    | 7    |
| 9    | 4    | 4    | 8    |
| 10   | 5    | 4    | 9    |

(v) We have to verify that

\[
r' - \hat{q} = \left( \frac{d+1}{3} \right) - 4 - (d-1)(r - r' - \hat{r} - q'),
\]

that is

\[(12) \quad (d-1)\hat{r} + \hat{q} + (d-2)r' + (d-1)q' = (d-1)r - \left( \frac{d+1}{3} \right) + 4.
\]

Rewrite the left hand side as

\[
((d-1)\hat{r} + \hat{q} + (d-2)q' - 2r') + (dr' + q').
\]

Recalling that

\[
(d-1)\hat{r} + \hat{q} + (d-2)q' - 2r' = (d+1)^2 - 2r;
\]

\[
dr' + q' = \left( \frac{d+3}{4} \right) - 4d + 3 - q,
\]

the left hand side of (12) becomes:

\[
(d+1)^2 - 2r + \left( \frac{d+3}{4} \right) - 4d + 3 - \left( \frac{d+4}{4} \right) + 4d + 1 + (d+1)r
\]
\[(d + 1)^2 - \binom{d + 3}{3} + 4 + (d - 1)r = -\binom{d + 1}{3} + 4 + (d - 1)r,\]

and we are done. \(\square\)

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