EQUIVARIANT HOLOMORPHIC ANOMALY EQUATION

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Abstract. In [16] the fundamental relationship between stable quotient invariants and the B-model for local \( \mathbb{P}^2 \) in all genera was studied under some specialization of equivariant variables. We generalize the argument of [16] to full equivariant settings without the specialization. Our main results are the proof of holomorphic anomaly equations for the equivariant Gromov-Witten theories of local \( \mathbb{P}^2 \) and local \( \mathbb{P}^3 \). We also state the generalization to full equivariant formal quintic theory of the result in [17].

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0. Introduction

0.1. Equivariant local \( \mathbb{P}^n \) theories. Equivariant local \( \mathbb{P}^n \) theories can be constructed as follows. Let the algebraic torus

\[ T_{n+1} = (\mathbb{C}^*)^{n+1} \]

act with the standard linearization on \( \mathbb{P}^n \) with weights \( \lambda_0, \ldots, \lambda_n \) on the vector space \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \). Let \( \overline{M}_g(\mathbb{P}^n, d) \) be the moduli space of stable maps to \( \mathbb{P}^n \) equipped with the canonical \( T_{n+1} \)-action, and let

\[ C \rightarrow \overline{M}_g(\mathbb{P}^n, d), \ f : C \rightarrow \mathbb{P}^n, \ S = f^*\mathcal{O}_{\mathbb{P}^n}(1) \rightarrow C \]

Date: July 2018.
be the standard universal structures.

The equivariant Gromov-Witten invariants of the local $\mathbb{P}^n$ are defined via the equivariant integrals

$$N^\text{GW}_{g,d} = \int_{[\mathcal{M}_g(\mathbb{P}^n,d)]^{\text{vir}}} e\left(-R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(-n-1)\right).$$

The integral (1) defines a rational function in $\lambda_i$

$$N^\text{GW}_{g,d} \in \mathbb{Q}(\lambda_0, \ldots, \lambda_n).$$

Over the moduli space of stable quotients, there is a universal curve

$$\pi : \mathcal{C} \to \overline{\mathcal{Q}}_g(\mathbb{P}^n, d)$$

with a universal quotient

$$0 \longrightarrow S \longrightarrow \mathbb{C}^N \otimes \mathcal{O}_C \overset{\text{qu}}{\longrightarrow} Q \longrightarrow 0.$$ 

The equivariant stable quotient invariants of the local $\mathbb{P}^n$ are defined via the equivariant integrals

$$N^{SQ}_{g,d} = \int_{[\mathcal{Q}_g(\mathbb{P}^n,d)]^{\text{vir}}} e\left(-R\pi_*S\right).$$

The integral (3) also defines a rational function in $\lambda_i$

$$N^{SQ}_{g,d} \in \mathbb{Q}(\lambda_0, \ldots, \lambda_n).$$

We refer the reader to [16, Section 1] for a more leisurely treatment of stable quotients.

In [16] it was observed that the analysis of $I$-function in [21] plays important role in the study of local $\mathbb{P}^n$ theories. But the result in [21] holds only after the specialization to $(n+1)$-th root of unity $\zeta_{n+1}$,

$$\lambda_i = \zeta_{n+1}^i.$$ 

In order to generalize the results in [16] to full equivariant theories, one needs the analogous generalization of the results in [21] to full equivariant settings. This will be studied in Appendix.

0.2. Holomorphic anomaly for $K\mathbb{P}^2$. We state the precise form of the holomorphic anomaly equations for local $\mathbb{P}^2$. Denote by $K\mathbb{P}^2$ the total space of the canonical bundle over $\mathbb{P}^2$. Let $H \in H^2(K\mathbb{P}^2, \mathbb{Q})$ be the hyperplane class obtained from $\mathbb{P}^2$, and let
be the Gromov-Witten and stable quotient series respectively (involving the evaluation morphisms at the markings). The relationship between the Gromov-Witten and stable quotient invariants of \( K_{\mathbb{P}^2} \) is proven in [7] in case \( 2g - 2 + n > 0 \):

\[
\mathcal{F}_{g,m}^{GW}(Q(q)) = \mathcal{F}_{g,m}^{SQ}(Q(q)),
\]

where \( Q(q) \) is the mirror map,

\[
I_{K_{\mathbb{P}^2}}^1(q) = \log(q) + 3 \sum_{d=1}^{\infty} (-q)^d \frac{(3d - 1)!}{(d!)^3},
\]

\[
Q(q) = \exp \left( I_{K_{\mathbb{P}^2}}^1(q) \right) = q \cdot \exp \left( 3 \sum_{d=1}^{\infty} (-q)^d \frac{(3d - 1)!}{(d!)^3} \right).
\]

To state the holomorphic anomaly equations, we need the following additional series in \( q \).

\[
L(q) = (1 + 27q)^{-\frac{1}{3}} = 1 - 9q + 162q^2 + \ldots,
\]

\[
C_1(q) = q \frac{d}{dq} I_{K_{\mathbb{P}^2}}^1,
\]

\[
A_2(q) = \frac{1}{L^3} \left( 3 \frac{d^2}{dq^2} C_1 - \frac{C_1}{C_1} + 1 - \frac{L^3}{2} \right).
\]

We also need new series \( L_i(q) \) defined by roots of following degree 3 polynomial in \( L \) for \( i = 0, 1, 2 \):

\[
(1 + 27q)L^3 - (\lambda_0 + \lambda_1 + \lambda_2)L^2 + (\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)L - \lambda_0 \lambda_2 \lambda_3,
\]

with initial conditions, \( L_i(0) = \lambda_i \).

Let \( f_2 \) be the polynomial of degree 2 in variable \( x \) over \( \mathbb{C}(\lambda_0, \lambda_1, \lambda_2) \) defined by

\[
f_2(x) := (\lambda_0 + \lambda_1 + \lambda_2)x^2 - 2(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)x + 3 \lambda_0 \lambda_1 \lambda_2.
\]
The ring
\[ G_2 := \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[L_0^{\pm 1}, L_1^{\pm 1}, L_2^{\pm 1}, f(L_0)^{-\frac{1}{2}}, f(L_1)^{-\frac{1}{2}}, f(L_2)^{-\frac{1}{2}}] \]
will play a basic role in our paper. Consider the free polynomial rings in the variables \( A_2 \) and \( C_1^{-1} \) over \( G_2 \),
(5) \[ G_2[A_2], \ G_2[A_2, C_1^{-1}] . \]
We have canonical maps
(6) \[ G_2[A_2] \to \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]], \ G_2[A_2, C_1^{-1}] \to \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]] \]
given by assigning the above defined series \( A_2(q) \) and \( C_1^{-1} \) to the variables \( A_2 \) and \( C_1^{-1} \) respectively. Therefore we can consider elements of the rings (5) either as free polynomials in the variables \( A_2 \) and \( C_1^{-1} \) or as series in \( q \).

Let \( F(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]] \) be a series in \( q \). When we write
\[ F(q) \in G_2[A_2] , \]
we mean there is a canonical lift \( F \in G_2[A_2] \) for which
\[ F \to F(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]] \]
under the map [6]. The symbol \( F \) without the argument \( q \) is the canonical lift. The notation
\[ F(q) \in G_2[A_2, C_1^{-1}] \]
is parallel.

Let \( T \) be the standard coordinate mirror to \( t = \log(q) \),
\[ T = I_{K^2}^{K^2}(q) . \]
Then \( Q(q) = \exp(T) \) is the mirror map.

**Conjecture 1.** For the stable quotient invariants of \( K\mathbb{P}^2 \),
(i) \( F^{SQ}_g(q) \in G_2[A_2] \) for \( g \geq 2 \),
(ii) \( F^{SQ}_g \) is of degree at most \( 3g - 3 \) with respect to \( A_2 \),
(iii) \( \frac{\partial^k F^{SQ}_g}{\partial T^k} \) \( q \in G_2[A_2, C_1^{-1}] \) for \( g \geq 1 \) and \( k \geq 1 \),
(iv) \( \frac{\partial^k F^{SQ}_g}{\partial T^k} \) is homogeneous of degree \( k \) with respect to \( C_1^{-1} \).

Here, \( F^{SQ}_g = F^{SQ}_{g,0} \).

**Conjecture 2.** The holomorphic anomaly equations for the stable quotient invariants of \( K\mathbb{P}^2 \) hold for \( g \geq 2 \):
\[
\frac{1}{C^2_1} \frac{\partial F^{SQ}_g}{\partial A_2} = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial F^{SQ}_{g-1}}{\partial T} \frac{\partial F^{SQ}_i}{\partial T} + \frac{1}{2} \frac{\partial^2 F^{SQ}_{g-1}}{\partial T^2}.
\]
The derivative of $F_{SQ}^g$ with respect to $A_2$ in the above equation is well-defined since

$$F_{SQ}^g \in G_2[A_2]$$

by part (i) of Conjecture 2. By parts (ii) and (iii),

$$\frac{\partial F_{SQ}^g}{\partial T}, \frac{\partial^2 F_{SQ}^g}{\partial T^2} \in G_2[A_2, C_1^{-1}]$$

are both of degree 2 in $C_1^{-1}$. Hence, the holomorphic anomaly equation of Conjecture 2 may be viewed as holding in $G[A_2]$ since the factors of $C_1^{-1}$ on both sides cancel. If we use the specializations by primitive third root of unity $\zeta$

$$\lambda_i = \zeta^i,$$

the holomorphic anomaly equations here for $K\mathbb{P}^2$ recover the precise form presented in [1, (4.27)] via B-model physics.

Conjecture 2 determine $F_{SQ}^g \in G_2[A_2]$ uniquely as a polynomial in $A_2$ up to a constant term in $G_2$. The degree of the constant term can be bounded. Therefore Conjecture 2 determine $F_{SQ}^g$ from the lower genus theory together with a finite amount of date.

We will prove the following special cases of the conjectures in Section 6.

**Theorem 3.** Conjecture 1 holds for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$\lambda_i \neq \lambda_j \text{ for } i \neq j,$$

$$(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0.$$

**Theorem 4.** Conjecture 2 holds for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$\lambda_i \neq \lambda_j \text{ for } i \neq j,$$

$$(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0.$$

### 0.3. Holomorphic anomaly equations for $K\mathbb{P}^3$

We state the precise form of the holomorphic anomaly equations for local $\mathbb{P}^3$. Since the study of local $\mathbb{P}^3$ will be parallel to the study of local $\mathbb{P}^2$, we will sometime use the same notations for local $\mathbb{P}^2$ and local $\mathbb{P}^3$. Since the study of two theories are logically independent in our paper, the indication of each notation will be clear from the context. Denote by $K\mathbb{P}^3$ the total space of the canonical bundle over $\mathbb{P}^3$. Let $H \in H^2(K\mathbb{P}^3, \mathbb{Q})$ be the hyperplane class obtained from $\mathbb{P}^3$, and let
be the Gromov-Witten and stable quotient series respectively with \(a + b = m\) (involving the evaluation morphisms at the markings). The relationship between the Gromov-Witten and stable quotient invariants of \(\mathbb{P}^3\) is proven in [7] in case \(2g - 2 + n > 0\):

(7) \[ F_{g,m}[a,b](Q(q)) = F_{g,m}[a,b](q) , \]

where \(Q(q)\) is the mirror map,

\[
I_{1}^{\mathbb{P}^3}(q) = \log(q) + 4 \sum_{d=1}^{\infty} q^d \frac{(4d - 1)!}{(d!)^4},
\]

\[
Q(q) = \exp \left( I_{1}^{\mathbb{P}^3}(q) \right) = q \cdot \exp \left( 4 \sum_{d=1}^{\infty} q^d \frac{(4d - 1)!}{(d!)^4} \right).
\]

To state the holomorphic anomaly equations, we need the following additional series in \(q\).

\[
L(q) = (1 - 4^4 q)^{-\frac{1}{4}} = 1 + 64q + 10240q^2 + \ldots,
\]

\[
C_1(q) = q \frac{d}{dq} I_{1}^{\mathbb{P}^3},
\]

\[
A_2(q) = q \frac{d}{dq} C_1 \frac{C_1}{C_1}.
\]

We will define extra series \(B_2(q), B_4(q) \in \mathbb{C}[[q]]\) in [56]. We also need new series \(L_i(q)\) defined by roots of following degree 4 polynomial in \(L\) for \(i = 0, 1, 2, 3\):

\[
(1 - 4^4 q)L^4 - s_1 L^3 + s_2 L^2 - s_3 L + s_4 ,
\]

with initial conditions,

\[
L_i(0) = \lambda_i.
\]

Here, \(s_k\) is \(k\)-th elementary symmetric function in \(\lambda_0, \ldots, \lambda_3\). Let \(f_3\) be the polynomial of degree 3 in variable \(x\) over \(\mathbb{C}(\lambda_0, \ldots, \lambda_3)\) defined by

\[
f_3(x) := s_1 x^3 - 2s_2 x^2 + 3s_3 x - 4s_4.
\]
The ring
\[ G_3 := \mathbb{C}(\lambda_0, \ldots, \lambda_3)[L_0^{\pm 1}, \ldots, L_3^{\pm 1}, f(L_0)^{-\frac{1}{2}}, \ldots, f(L_3)^{-\frac{1}{2}}] \]
will play a basic role in our paper. Consider the free polynomial rings in the variables \( A_2, B_2, B_4 \) and \( C_1^{-1} \) over \( G_3 \),
\( G_3[ A_2, B_2, B_4, C_1^{\pm 1}] \).

We have canonical map
\[ G_3[A_2, B_2, B_4, C_1^{\pm 1}] \to \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]] \]
given by assigning the above defined series \( A_2(q), B_2(q), B_4(q) \) and \( C_1(q) \) to the variables \( A_2, B_2, B_4 \) and \( C_1 \) respectively. Therefore we can consider elements of the rings (8) either as free polynomials in the variables \( A_2, B_2, B_4 \) and \( C_1 \) or as series in \( q \).

Let \( F(q) \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]] \) be a series in \( q \). When we write \( F(q) \in G_3[A_2, B_2, B_4, C_1^{\pm 1}] \),
we mean there is a canonical lift \( F \in G_3[A_2, B_2, B_4, C_1^{\pm 1}] \) for which
\[ F \to F(q) \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]] \]
under the map (9). The symbol \( F \) without the argument \( q \) is the canonical lift. The notation \( F(q) \in G_3[A_2, B_2, B_4, C_1^{\pm 1}] \)
is parallel.

Let \( T \) be the standard coordinate mirror to \( t = \log(q) \),
\[ T = I_1^{KP^3}(q) \]
Then \( Q(q) = \exp(T) \) is the mirror map.

**Conjecture 5.** For the stable quotient invariants of \( K\mathbb{P}^3 \),
(i) \( F_{g,a+b[a,b]}^{SQ}(q) \in G_3[A_2, B_2, B_4, C_1^{\pm 1}] \) for \( g \geq 2 \),
(ii) \( F_{g,a+b}^{SQ} \) is of degree at most \( 2(3g - 3) \) with respect to \( A_2 \),
(iii) \( \frac{\partial^{k} F_{g,a+b}^{SQ}(q)}{\partial t^{k}} \in G_3[A_2, B_2, B_4, C_1^{\pm 1}] \) for \( g \geq 1 \) and \( k \geq 1 \).

Here, \( F_{g,a+b}^{SQ} = F_{g,0}^{SQ}[0, 0] \).

**Conjecture 6.** The holomorphic anomaly equations for the stable quotient invariants of \( K\mathbb{P}^3 \) hold for \( g \geq 2 \):
\[ \frac{L^2}{4C_1} \frac{\partial F_{g}^{SQ}}{\partial A_2} + \frac{-2s_1L^4 - C_1(3B_2L^2 - s_1L^6)}{4C_1^2} \frac{\partial F_{g}^{SQ}}{\partial B_4} = \sum_{i=1}^{g-1} F_{g-i,1}^{SQ}[0, 1] F_{i,1}^{SQ}[1, 0] + F_{g-i,1}^{SQ}[1, 1], \]
The derivative of $F^{SQ}_{g}$ with respect to $A_2$, $B_2$ and $B_4$ in the above equations is well-defined since

$$F^{SQ}_{g} \in G_3[A_2, B_2, B_4, C_i^{\pm 1}]$$

by part (i) of Conjecture 6.

We will prove the following special cases of the conjectures in Section 6.

**Theorem 7.** Conjecture 5 holds for the choices of $\lambda_0, \ldots, \lambda_3$ such that

\[
\begin{align*}
\lambda_i & \neq \lambda_j \text{ for } i \neq j, \\
4s_2^2 - s_1 s_3 & = 0, \\
2s_2^3 - 27s_1^2 s_4 & = 0.
\end{align*}
\]

**Theorem 8.** Conjecture 6 holds for the choices of $\lambda_0, \ldots, \lambda_3$ such that

\[
\begin{align*}
\lambda_i & \neq \lambda_j \text{ for } i \neq j, \\
4s_2^2 - s_1 s_3 & = 0, \\
2s_2^3 - 27s_1^2 s_4 & = 0.
\end{align*}
\]

For Calabi-Yau 3-folds, holomorphic anomaly equations were first discovered in physics ([2]). Also there were many studies to understand holomorphic anomaly equations mathematically ([12, 16, 17, 19]). But less is known for higher dimensional Calabi-Yau manifolds in physics. It might be interesting question to find physical arguments for the holomorphic anomaly equation for $K\mathbb{P}^3$ proposed in our paper.

0.4. Holomorphic anomaly for equivariant formal quintic invariants. A particular twisted theory on $\mathbb{P}^4$ related to the quintic 3-fold was introduced in [17]. Let the algebraic torus

$$T = (\mathbb{C}^*)^5$$

act with the standard linearization on $\mathbb{P}^4$ with weights $\lambda_0, \ldots, \lambda_4$ on the vector space $H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(1))$.

Let

$$C \to \overline{M}_g(\mathbb{P}^4, d), \quad f : C \to \mathbb{P}^4, \quad S = f^*O_{\mathbb{P}^4}(-1) \to C$$

be the universal curve, the universal map, and the universal bundle over the moduli space of stable maps — all equipped with canonical
T-actions. We define the formal quintic invariants by

$$\tilde{N}_{g,d}^{\text{GW}} = \int_{[\overline{M}_g(P^4,d)]^{vir}} e(R\pi_*(S^{-5})),$$

(11)

where $e(R\pi_*(S^{-5}))$ is the equivariant Euler class defined after localization. More precisely, on each $T$-fixed locus of $\overline{M}_g(P^4,d)$, both $R^0\pi_*(S^{-5})$ and $R^1\pi_*(S^{-5})$ are vector bundles with moving weights, so

$$e(R\pi_*(S^{-5})) = \frac{c_{\text{top}}(R^0\pi_*(S^{-5}))}{c_{\text{top}}(R^1\pi_*(S^{-5}))}$$

is well-defined. The integral (11) is homogeneous of degree 0 in localized equivariant cohomology and defines a rational function in $\lambda_i$, $\tilde{N}_{g,d}^{\text{GW}} \in \mathbb{C}(\lambda_0, \ldots, \lambda_4)$.

Let $g \geq 2$. The associated Gromov-Witten generating series is

$$\tilde{F}_g^{\text{GW}}(Q) = \sum_{d=0}^{\infty} \tilde{N}_{g,d}^{\text{GW}} Q^d \in \mathbb{C}[[Q]].$$

Let

$$I^Q_0(q) = \sum_{d=0}^{\infty} q^d \frac{(5d)!}{(d!)^5}, \quad I^R_0(q) = \log(q) I^Q_0(q) + 5 \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left( \sum_{r=d+1}^{5d} \frac{1}{r} \right).$$

We define the generating series of stable quotient invariants for formal quintic theory by the wall-crossing formula for the true quintic theory which has been recently proven by Ciocan-Fontanine and Kim in [6],

$$\tilde{F}_g^{\text{GW}}(Q(q)) = I^Q_0(q)^{2g-2} \cdot \tilde{F}_g^{\text{SQ}}(q)$$

(12)

with respect to the true quintic mirror map

$$Q(q) = \exp \left( \frac{I^Q_0(q)}{I^Q_0(q)} \right) = q \cdot \exp \left( 5 \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left( \sum_{r=d+1}^{5d} \frac{1}{r} \right) \right).$$

Denote the B-model side of (12) by

$$\tilde{F}_g^{\text{B}}(q) = I^Q_0(q)^{2g-2} \tilde{F}_g^{\text{SQ}}(q).$$

In order to state the holomorphic anomaly equations, we require several series in $q$. First, let

$$L(q) = (1 - 5^5 q)^{-\frac{1}{5}} = 1 + 625q + 117185q^2 + \ldots.$$
Let $D = q \frac{d}{dq}$, and let

$$C_0(q) = I_0^Q, \quad C_1(q) = D \left( \frac{I_1^Q}{I_0^Q} \right).$$

We define

$$K_2(q) = - \frac{1}{L^5} \frac{DC_0}{C_0},$$

$$A_2(q) = \frac{1}{L^5} \left( - \frac{1}{5} \frac{DC_1}{C_1} - \frac{2}{5} \frac{DC_0}{C_0} - \frac{3}{25} \right),$$

$$A_4(q) = \frac{1}{L^4} \left( - \frac{1}{25} \left( \frac{DC_0}{C_0} \right)^2 - \frac{1}{25} \left( \frac{DC_0}{C_0} \right) \left( \frac{DC_1}{C_1} \right) 
\quad + \frac{1}{25} D \left( \frac{DC_0}{C_0} \right) + \frac{2}{25^2} \right),$$

$$A_6(q) = \frac{1}{31250L^{15}} \left( 4 + 125D \left( \frac{DC_0}{C_0} \right) + 50 \left( \frac{DC_0}{C_0} \right) \left( 1 + 10D \left( \frac{DC_0}{C_0} \right) \right) 
\quad - 5L^5 \left( 1 + 10 \left( \frac{DC_0}{C_0} \right) + 25 \left( \frac{DC_0}{C_0} \right)^2 + 25D \left( \frac{q \frac{d}{dq} C_0}{C_0} \right) \right) 
\quad + 125D^2 \left( \frac{DC_0}{C_0} \right) - 125 \left( \frac{DC_0}{C_0} \right)^2 \left( \left( \frac{DC_1}{C_1} \right) - 1 \right) \right).$$

For the full equivariant formal quintic theory, we need extra series $B_1, B_2, B_3, B_4 \in \mathbb{C}[[q]]$ obtained from $I$-function of quintic. We will give the exact definitions of these extra series in forthcoming paper [15]. These series are closely related to the extra generators in [12], where the formal quintic theory was studied in genus 2 case with connections to real quintic theory.

We also need new series $L_i(q)$ defined by roots of following degree 5 polynomial in $L$ for $i = 0, \ldots, 4$:

$$(1 - 5^5 q) L^5 - s_1 L^4 + s_2 L^3 - s_3 L^2 + s_4 L - s_5,$$

with initial conditions,

$$L_i(0) = \lambda_i.$$

Here $s_k$ is $k$-th elementary symmetric function in $\lambda_0, \ldots, \lambda_4$. Let $f_4$ be the polynomial of degree 4 in variable $x$ over $\mathbb{C}(\lambda_0, \ldots, \lambda_4)$ defined by

$$f_4(x) := s_1 x^4 - 2 s_2 x^3 + 3 s_3 x^2 - 4 s_3^2 - 5 s_4.$$
The ring
\[ G_Q := \mathbb{C}(\lambda_0, \ldots, \lambda_4)[L_0^{\pm 1}, \ldots, L_4^{\pm 1}, f_4(L_0)^{-\frac{1}{2}}, \ldots, f_4(L_4)^{-\frac{1}{2}}] \]
will play a basic role in formal quintic theory.

Let \( T \) be the standard coordinate mirror to \( t = \log(q) \),
\[ T = \frac{I_1^Q(q)}{I_0^Q(q)}. \]

Then \( Q(q) = \exp(T) \) is the mirror map. Let
\[ G_Q[A_2, A_4, A_6, B_1, B_2, B_3, B_4, C_0^{\pm 1}, C_1^{-1}, K_2] \]
be the free polynomial ring over \( G_Q \).

**Conjecture 9.** For the series \( \tilde{F}_g^B \) associated to the formal quintic,

1. \( \tilde{F}_g^B(q) \in G_Q[A_2, A_4, A_6, B_1, \ldots, B_4, C_0^{\pm 1}, C_1^{-1}, K_2] \) for \( g \geq 2 \),
2. \( \frac{\partial^k \tilde{F}_g^B}{\partial T^k}(q) \in G_Q[A_2, A_4, A_6, B_1, \ldots, B_4, C_0^{\pm 1}, C_1^{-1}, K_2] \) for \( g \geq 1, k \geq 1 \),
3. \( \frac{\partial^k \tilde{F}_g^B}{\partial T^k} \) is homogeneous with respect to \( C_1^{-1} \) of degree \( k \).

**Conjecture 10.** The series \( \tilde{F}_g^B \) associated to the formal quintic satisfy some holomorphic anomaly equations which specialize to
\[
\frac{1}{C_0^2 C_1^2} \frac{\partial \tilde{F}_g^B}{\partial A_2} - \frac{1}{5C_0^2 C_1^2} \frac{\partial \tilde{F}_g^B}{\partial A_4} K_2 + \frac{1}{50C_0^2 C_1^2} \frac{\partial \tilde{F}_g^B}{\partial A_6} K_2^2 = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \tilde{F}_g^{B-i}}{\partial T} \frac{\partial \tilde{F}_g^B}{\partial T} + \frac{1}{2} \frac{\partial^2 \tilde{F}_g^{B-1}}{\partial T^2},
\]
\[ \frac{\partial \tilde{F}_g^B}{\partial K_2} = 0, \]
with the choice of \( \lambda_i = \zeta_5^i \).

We expect holomorphic anomaly equations in Conjecture 10 to hold in the ring

\[ G_Q[A_2, A_4, A_6, B_1, B_2, B_3, B_4, C_0^{\pm 1}, C_1^{-1}, K_2]. \]

**Remark 11.** If we specialize \( \lambda_i \) to (the power of) primitive fifth root of unity \( \zeta_5 \),
\[ \lambda_i = \zeta_5^i, \]

(13)
the expected equations in Conjecture 10 exactly matches the conjectural holomorphic anomaly equation \[ (2.52) \] for the true quintic theory and this was the main result in [17]. Also the ring (13) can be reduced to the Yamaguchi-Yau ring introduced in [20] for the true quintic theory only with the choice of specialization (14). This explains why the specialization (14) used in [17] is the natural choice.

**Theorem 12.** Conjecture 9 holds for the choices of \( \lambda_0, \ldots, \lambda_4 \) such that

\[
\lambda_i \neq \lambda_j \quad \text{for} \quad i \neq j,
\]

\[
2s_1s_3 = s_2^2,
\]

\[
8s_1^2s_4 = s_2^3,
\]

\[
80s_1^2s_5 = s_2^4.
\]

where \( s_k \) is \( k \)-th elementary symmetric function in \( \lambda_0, \ldots, \lambda_4 \).

Theorems 12 will be proven and the precise form of holomorphic anomaly equations in Conjecture 10 will appear in [15].

0.5. **Acknowledgments.** I thank S. Guo, H. Iritani, F. Janda, B. Kim, A. Klemm, M. C.-C. Liu, G. Oberdieck, Y. Ruan and E. Scheidegger for discussions over the years about the invariants of Calabi-Yau geometries and holomorphic anomaly equation. I am especially grateful to R. Pandharipande for the suggestion of this project and useful discussions. I thank F. Janda for correcting the statements of Conjecture 9 and Theorem 12. I was supported by the grant ERC-2012-AdG-320368-MCSK.

1. Localization Graph

1.1. **Torus action.** Let \( T = (\mathbb{C}^*)^{n+1} \) act diagonally on the vector space \( \mathbb{C}^{n+1} \) with weights

\[
-\lambda_0, \ldots, -\lambda_n.
\]

Denote the \( T \)-fixed points of the induced \( T \)-action on \( \mathbb{P}^n \) by

\[
p_0, \ldots, p_n.
\]

The weights of \( T \) on the tangent space \( T_{p_j}(\mathbb{P}^n) \) are

\[
\lambda_j - \lambda_0, \ldots, \lambda_j - \lambda_j, \ldots, \lambda_j - \lambda_n.
\]

\[\footnote{Our functions \( K_2 \) and \( A_{2k} \) are normalized differently with respect to \( C_0 \) and \( C_1 \). The dictionary to exactly match the notation of [11] (2.52) is to multiply our \( K_2 \) by \( (C_0C_1)^2 \) and our \( A_{2k} \) by \( (C_0C_1)^{2k} \).}\]
There is an induced $\mathbb{T}$-action on the moduli space $\overline{Q}_{g,k}(\mathbb{P}^n, d)$. The localization formula of $[11]$ applied to the virtual fundamental class $[\overline{Q}_{g,k}(\mathbb{P}^n, d)]^{\text{vir}}$ will play a fundamental role in our paper. The $\mathbb{T}$-fixed loci are represented in terms of dual graphs, and the contributions of the $\mathbb{T}$-fixed loci are given by tautological classes. The formulas here are standard, see $[13, 18]$.

1.2. Graphs. Let the genus $g$ and the number of markings $k$ for the moduli space be in the stable range
\begin{equation}
2g - 2 + k > 0.
\end{equation}
We can organize the $\mathbb{T}$-fixed loci of $\overline{Q}_{g,k}(\mathbb{P}^n, d)$ according to decorated graphs. A decorated graph $\Gamma \in \mathcal{G}_{g,k}(\mathbb{P}^n)$ consists of the data $(\mathcal{V}, \mathcal{E}, \mathcal{N}, \mathcal{g}, \mathcal{p})$ where
(i) $\mathcal{V}$ is the vertex set,
(ii) $\mathcal{E}$ is the edge set (including possible self-edges),
(iii) $\mathcal{N} : \{1, 2, \ldots, k\} \to \mathcal{V}$ is the marking assignment,
(iv) $\mathcal{g} : \mathcal{V} \to \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying
\[ g = \sum_{v \in \mathcal{V}} \mathcal{g}(v) + h^1(\Gamma) \]
and for which $(\mathcal{V}, \mathcal{E}, \mathcal{N}, \mathcal{g})$ is stable graph$^3$.
(v) $\mathcal{p} : \mathcal{V} \to (\mathbb{P}^n)^{\mathbb{T}}$ is an assignment of a $\mathbb{T}$-fixed point $\mathcal{p}(v)$ to each vertex $v \in \mathcal{V}$.

The markings $\mathcal{L} = \{1, \ldots, k\}$ are often called legs.

To each decorated graph $\Gamma \in \mathcal{G}_{g,k}(\mathbb{P}^n)$, we associate the set of fixed loci of
\[ \sum_{d \geq 0} [\overline{Q}_{g,k}(\mathbb{P}^n, d)]^{\text{vir}} q^d \]
with elements described as follows:
(a) If $\{v_i \mid p(v) = p_i\}$, then $f^{-1}(p_i) \mathcal{L}$ is a disjoint union of connected stable curves of genera $\mathcal{g}(v_{i_1}), \ldots, \mathcal{g}(v_{i_j})$ and finitely many points.
(b) There is a bijective correspondence between the connected components of $\mathcal{C} \setminus \mathcal{D}$ and the set of edges and legs of $\Gamma$ respecting vertex incidence where $\mathcal{C}$ is domain curve and $\mathcal{D}$ is union of all subcurves of $\mathcal{C}$ which appear in (a).

We write the localization formula as
\[ \sum_{d \geq 0} [\overline{Q}_{g,k}(\mathbb{P}^n, d)]^{\text{vir}} q^d = \sum_{\Gamma \in \mathcal{G}_{g,k}(\mathbb{P}^n)} \text{Cont}_\Gamma. \]

$^3$Corresponding to a stratum of the moduli space of stable curves $\overline{M}_{g,n}$. 

While $G_{g,k}(\mathbb{P}^n)$ is a finite set, each contribution $\text{Cont}_\Gamma$ is a series in $q$ obtained from an infinite sum over all edge possibilities (b).

1.3. **Unstable graphs.** The moduli spaces of stable quotients

$$\overline{Q}_{0,2}(\mathbb{P}^n, d) \quad \text{and} \quad \overline{Q}_{1,0}(\mathbb{P}^n, d)$$

for $d > 0$ are the only cases where the pair $(g, k)$ does not satisfy the Deligne-Mumford stability condition (15).

An appropriate set of decorated graphs $G_{0,2}(\mathbb{P}^n)$ is easily defined: The graphs $\Gamma \in G_{0,2}(\mathbb{P}^n)$ all have 2 vertices connected by a single edge. Each vertex carries a marking. All of the conditions (i)-(v) of Section 1.2 are satisfied except for the stability of $(V, E, N, \gamma)$. The localization formula holds,

$$\sum_{d \geq 1} [\overline{Q}_{0,2}(\mathbb{P}^n, d)]^\text{vir} q^d = \sum_{\Gamma \in G_{0,2}(\mathbb{P}^n)} \text{Cont}_\Gamma,$$

For $\overline{Q}_{1,0}(\mathbb{P}^n, d)$, the matter is more problematic — usually a marking is introduced to break the symmetry.

2. **Basic correlators**

2.1. **Overview.** We review here basic generating series in $q$ which arise in the genus 0 theory of quasimap invariants. The series will play a fundamental role in the calculations of Sections 3 - 6 related to the holomorphic anomaly equation for $K\mathbb{P}^2$.

We fix a torus action $T = (\mathbb{C}^*)^3$ on $\mathbb{P}^2$ with weights $^4$

$$-\lambda_0, -\lambda_1, -\lambda_2$$

on the vector space $\mathbb{C}^3$. The $T$-weight on the fiber over $p_i$ of the canonical bundle

$$\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2$$

is $-3\lambda_i$. The toric Calabi-Yau $K\mathbb{P}^2$ is the total space of $\mathcal{O}_{\mathbb{P}^2}(1)$.

$^4$The moduli spaces $\overline{Q}_{0,0}(\mathbb{P}^n, d)$ and $\overline{Q}_{0,1}(\mathbb{P}^n, d)$ are empty by the definition of a stable quotient.

$^5$The associated weights on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ are $\lambda_0, \lambda_1, \lambda_2$ and so match the conventions of Section 0.1.
2.2. First correlators. We require several correlators defined via the Euler class of the obstruction bundle,
\[ e(\text{Obs}) = e(R^1\pi_*\mathcal{S}^3) \]
associated to the $K\mathbb{P}^2$ geometry on the moduli space $\mathcal{Q}_{g,n}(\mathbb{P}^2,d)$. The first two are obtained from standard stable quotient invariants. For $\gamma_i \in H^*_T(\mathbb{P}^2)$, let
\[ \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle_{g,n,d}^{\text{SQ}} = \int_{[\mathcal{Q}_{g,n}(\mathbb{P}^2,d)]^{\text{vir}}} e(\text{Obs}) \cdot \prod_{i=1}^n \text{ev}^*_i(\gamma_i) \psi^{a_i}, \]
\[ \langle \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle \rangle_{0,n}^{\text{SQ}} = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \langle \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}, t, \ldots, t \rangle \rangle_{0,n+k,d}^{\text{SQ}}, \]
where, in the second series, $t \in H^*_T(\mathbb{P}^2)$. We will systematically use the quasimap notation $0^+$ for stable quotients,
\[ \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle_{0,n}^{0+,0^+} = \langle \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle \rangle_{0,n}^{\text{SQ}}. \]

2.3. Light markings. Moduli of quasimaps can be considered with $n$ ordinary (weight 1) markings and $k$ light (weight $\epsilon$) markings\footnote{See Sections 2 and 5 of \cite{5}.}
\[ \mathcal{Q}_{g,n|k}^{0+,0^+}(\mathbb{P}^2,d). \]
Let $\gamma_i \in H^*_T(\mathbb{P}^2)$ be equivariant cohomology classes, and let $\delta_j \in H^*_T([\mathbb{C}^3/\mathbb{C}^*])$ be classes on the stack quotient. Following the notation of \cite{13}, we define series for the $K\mathbb{P}^2$ geometry,
\[ \langle \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; \delta_1, \ldots, \delta_k \rangle \rangle_{0,n|k,d}^{0+,0^+} = \int_{[\mathcal{Q}_{g,n|k}^{0+,0^+}(\mathbb{P}^2,d)]^{\text{vir}}} e(\text{Obs}) \cdot \prod_{i=1}^n \text{ev}^*_i(\gamma_i) \psi^{a_i} \cdot \prod_{j=1}^k \hat{\text{ev}}^*_j(\delta_j), \]
\[ \langle \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle \rangle_{0,n}^{0+,0^+} = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \langle \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; t, \ldots, t \rangle \rangle_{0,n+k,d}^{0+,0^+}. \]
where, in the second series, \( t \in H_T^1(\mathbb{C}^3/\mathbb{C}^*) \).

For each \( T \)-fixed point \( p_i \in \mathbb{P}^2 \), let
\[
e_i = e(T_{p_i}(\mathbb{P}^2)) \cdot (-3\lambda_i)
\]
be the equivariant Euler class of the tangent space of \( K\mathbb{P}^2 \) at \( p_i \). Let
\[
\phi_i = -3\lambda_i \prod_{j \neq i} (H - \lambda_j) / e_i, \quad \phi^i = e_i \phi_i \in H_T^*(\mathbb{P}^2)
\]
be cycle classes. Crucial for us are the series
\[
S_i(\gamma) = e_i \left( \left\langle \frac{\phi_i}{z - \psi}, \gamma \right\rangle \right)_{0+0+},
\]
\[
V_{ij} = \left\langle \left\langle \frac{\phi_i}{x - \psi}, \frac{\phi_j}{y - \psi} \right\rangle \right\rangle_{0+0+}.
\]

Unstable degree 0 terms are included by hand in the above formulas. For \( S_i(\gamma) \), the unstable degree 0 term is \( \gamma|_{p_i} \). For \( V_{ij} \), the unstable degree 0 term is \( \delta_{ij} e_i (x + y) \).

We also write
\[
S(\gamma) = \sum_{i=0}^{2} \phi_i S_i(\gamma).
\]

The series \( S_i \) and \( V_{ij} \) satisfy the basic relation
\[
e_i V_{ij}(x, y) e_j = \sum_{k=0}^{2} S_i(\phi_k)|_{z=x} S_j(\phi^k)|_{z=y} \]
proven\(^7\) in \([7]\).

Associated to each \( T \)-fixed point \( p_i \in \mathbb{P}^2 \), there is a special \( T \)-fixed point locus,
\[
\overline{Q}_{0,k|m}^{0+,0+}(\mathbb{P}^2, d)^{T,p_i} \subset \overline{Q}_{0,k|m}^{0+,0+}(\mathbb{P}^2, d),
\]
where all markings lie on a single connected genus 0 domain component contracted to \( p_i \). Let \( \text{Nor} \) denote the equivariant normal bundle of \( Q_{0,n|k}^{0+,0+}(\mathbb{P}^2, d)^{T,p_i} \) with respect to the embedding (19). Define
\[
\left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; \delta_1, \ldots, \delta_k \right\rangle_{0^n,0+k,0,0+p_i}^{0+,0+} =
\]
\[
\int_{\overline{Q}_{0,n|k}^{0+,0+}(\mathbb{P}^2, d)^{T,p_i}} \frac{e(\text{Obs})}{e(\text{Nor})} \cdot \prod_{i=1}^{n} eV_i^*(\gamma_i) \psi^{a_i} \cdot \prod_{j=1}^{k} \hat{e}V_j^*(\delta_j),
\]

\(^7\)In Gromov-Witten theory, a parallel relation is obtained immediately from the WDDV equation and the string equation. Since the map forgetting a point is not always well-defined for quasimaps, a different argument is needed here \([7]\).
\[ \langle \gamma_1 \psi_1^{\alpha_1}, \ldots, \gamma_n \psi_n^{\alpha_n} \rangle_{0,n}^{0+,0+,p_i} = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \langle \gamma_1 \psi_1^{\alpha_1}, \ldots, \gamma_n \psi_n^{\alpha_n}; t, \ldots, t \rangle_{0,n|k,\beta}^{0+,0+,p_i}. \]

2.4. Graph spaces and I-functions.

2.4.1. Graph spaces. The big I-function is defined in \([5]\) via the geometry of weighted quasimap graph spaces. We briefly summarize the constructions of \([5]\) in the special case of \((0+,0+)-\)stability. The more general weightings discussed in \([5]\) will not be needed here.

As in Section 2.3, we consider the quotient \(\mathbb{C}^3/\mathbb{C}^*\) associated to \(\mathbb{P}^2\). Following \([5]\), there is a \((0+,0+)-\)stable quasimap graph space

\[(20) \quad \mathcal{Q}G_{g,n|k,d}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*]) . \]

A \(\mathbb{C}\)-point of the graph space is described by data

\[ ((C, x, y), (f, \varphi) : C \rightarrow [\mathbb{C}^3/\mathbb{C}^*] \times [\mathbb{C}^2/\mathbb{C}^*]) . \]

By the definition of stability, \(\varphi\) is a regular map to

\[ \mathbb{P}^1 = \mathbb{C}^2//\mathbb{C}^* \]

of class 1. Hence, the domain curve \(C\) has a distinguished irreducible component \(C_0\) canonically isomorphic to \(\mathbb{P}^1\) via \(\varphi\). The standard \(\mathbb{C}^*\)-action,

\[(21) \quad t \cdot [\xi_0, \xi_1] = [t\xi_0, \xi_1], \quad \text{for } t \in \mathbb{C}^*, [\xi_0, \xi_1] \in \mathbb{P}^1, \]

induces a \(\mathbb{C}^*\)-action on the graph space.

The \(\mathbb{C}^*\)-equivariant cohomology of a point is a free algebra with generator \(z\),

\[ H_{\mathbb{C}^*}^*(\text{Spec} (\mathbb{C})) = \mathbb{Q}[z] . \]

Our convention is to define \(z\) as the \(\mathbb{C}^*\)-equivariant first Chern class of the tangent line \(T_0\mathbb{P}^1\) at \(0 \in \mathbb{P}^1\) with respect to the action \([21]\),

\[ z = c_1(T_0\mathbb{P}^1) . \]

The \(T\)-action on \(\mathbb{C}^3\) lifts to a \(T\)-action on the graph space \([20]\) which commutes with the \(\mathbb{C}^*\)-action obtained from the distinguished domain.
component. As a result, we have a $T \times \mathbb{C}^*$-action on the graph space and $T \times \mathbb{C}^*$-equivariant evaluation morphisms

$$\text{ev}_i : \mathcal{QG}^{0+,0+}_{g,n|k,\beta}([\mathbb{C}^3/\mathbb{C}^*]) \to \mathbb{P}^2, \quad i = 1, \ldots, n,$$

$$\hat{\text{ev}}_j : \mathcal{QG}^{0+,0+}_{g,n|k,\beta}([\mathbb{C}^3/\mathbb{C}^*]) \to [\mathbb{C}^3/\mathbb{C}^*], \quad j = 1, \ldots, k.$$

Since a morphism

$$f : C \to [\mathbb{C}^3/\mathbb{C}^*]$$

is equivalent to the data of a principal $G$-bundle $P$ on $C$ and a section $u$ of $P \times_{\mathbb{C}^*} \mathbb{C}^3$, there is a natural morphism

$$C \to EC^* \times_{\mathbb{C}^*} \mathbb{C}^3$$

and hence a pull-back map

$$f^* : H^*_{\mathbb{C}^*}([\mathbb{C}^3/\mathbb{C}^*]) \to H^*(C).$$

The above construction applied to the universal curve over the moduli space and the universal morphism to $[\mathbb{C}^3/\mathbb{C}^*]$ is $T$-equivariant. Hence, we obtain a pull-back map

$$\hat{\text{ev}}^*_j : H^*_T(\mathbb{C}^3, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}[z] \to H^*_T(\mathcal{QG}^{0+,0+}_{g,n|k,\beta}([\mathbb{C}^3/\mathbb{C}^*]), \mathbb{Q})$$

associated to the evaluation map $\hat{\text{ev}}_j$.

2.4.2. $I$-functions. The description of the fixed loci for the $\mathbb{C}^*$-action on

$$\mathcal{QG}^{0+,0+}_{g,0|k,d}([\mathbb{C}^3/\mathbb{C}^*])$$

is parallel to the description in [4] §4.1 for the unweighted case. In particular, there is a distinguished subset $M_{k,d}$ of the $\mathbb{C}^*$-fixed locus for which all the markings and the entire curve class $d$ lie over $0 \in \mathbb{P}^1$. The locus $M_{k,d}$ comes with a natural proper evaluation map $\text{ev}_\bullet$ obtained from the generic point of $\mathbb{P}^1$:

$$\text{ev}_\bullet : M_{k,d} \to \mathbb{C}^3/\mathbb{C}^* = \mathbb{P}^2.$$

We can explicitly write

$$M_{k,d} \cong M_d \times 0^k \subset M_d \times (\mathbb{P}^1)^k,$$

where $M_d$ is the $\mathbb{C}^*$-fixed locus in $\mathcal{QG}^{0+,0+}_{0,0,d}([\mathbb{C}^3/\mathbb{C}^*])$ for which the class $d$ is concentrated over $0 \in \mathbb{P}^1$. The locus $M_d$ parameterizes quasimaps of class $d$,

$$f : \mathbb{P}^1 \to [\mathbb{C}^3/\mathbb{C}^*],$$

with a base-point of length $d$ at $0 \in \mathbb{P}^1$. The restriction of $f$ to $\mathbb{P}^1 \setminus \{0\}$ is a constant map to $\mathbb{P}^2$ defining the evaluation map $\text{ev}_\bullet$. 
As in [3, 4, 8], we define the big $I$-function as the generating function for the push-forward via $ev_\bullet$ of localization residue contributions of $M_{k,d}$. For $t \in H^*_T([\mathbb{C}^3/\mathbb{C}^*], \mathbb{Q}) \otimes \mathbb{Q} [z]$, let

$$Res_{M_{k,d}}(t^k) = \prod_{j=1}^{k} \hat{ev}_j^*(t) \cap Res_{M_{k,d}}[QG_{0,0}]_{k,d}([\mathbb{C}^3/\mathbb{C}^*])^{\text{vir}}$$

$$= \prod_{j=1}^{k} \hat{ev}_j^*(t) \cap [M_{k,d}]^{\text{vir}} e(\text{Nor}_{M_{k,d}}^{\text{vir}}),$$

where $\text{Nor}_{M_{k,d}}^{\text{vir}}$ is the virtual normal bundle.

**Definition 13.** The big $I$-function for the $(0+, 0+)$-stability condition, as a formal function in $t$, is

$$I(q, t, z) = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} ev_\bullet^*(Res_{M_{k,d}}(t^k)).$$

2.4.3. Evaluations. Let $\tilde{H} \in H^*_T([\mathbb{C}^3/\mathbb{C}^*])$ and $H \in H^*_T(\mathbb{P}^2)$ denote the respective hyperplane classes. The $I$-function of Definition 13 is evaluated in [5].

**Proposition 14.** For $t = t\tilde{H} \in H^*_T([\mathbb{C}^3/\mathbb{C}^*], \mathbb{Q})$,

$$I(t) = \sum_{d=0}^{\infty} q^d e^{(H + dz)/z} \prod_{k=0}^{2d-1} (-3H - kz) \frac{\prod_{i=0}^{2} \prod_{k=1}^{d} (H - \lambda_i + kz)}{\prod_{i=0}^{2} \prod_{k=1}^{d} (H - \lambda_i + kz)}.$$

 Observe that the $I$-function has following expansion after restriction $t = 0$,

$$I|_{t=0} = 1 + \frac{I_{1,0} H}{z} + \frac{I_{2,0} H^2 + I_{2,1}(\lambda_0 + \lambda_1 + \lambda_2) H}{z^2} + O\left(\frac{1}{z^3}\right),$$

where

$$I_{1,0}(q) = \sum_{d=1}^{\infty} \frac{(3d - 1)!}{(d!)^3} (-q)^d,$$

$$I_{2,0}(q) = \sum_{d=1}^{\infty} \frac{(3d - 1)!}{(d!)^3} \left(3\text{Har}[3d - 1] - 3\text{Har}[d]\right) (-q)^d,$$

$$I_{2,1}(q) = \sum_{d=1}^{\infty} \frac{(3d - 1)!}{(d!)^3} \text{Har}[d] (-q)^d.$$

Here $\text{Har}[d] := \sum_{k=1}^{d} \frac{1}{k}$. 
We return now to the functions $S_i(\gamma)$ defined in Section 2.3. Using Birkhoff factorization, an evaluation of the series $S(H^j)$ can be obtained from the $I$-function, see [13]:

\[
S(1) = I,
\]

\[
S(H) = \frac{z \frac{d}{dt} S(1)}{z \frac{d}{dt} S(1)|_{t=0, H=1, z=\infty}},
\]

\[
S(H^2) = \frac{z \frac{d}{dt} S(H) - (\lambda_0 + \lambda_1 + \lambda_2) N_2 S(H)}{\left(z \frac{d}{dt} S(H) - (\lambda_0 + \lambda_1 + \lambda_2) N_2 S(H)\right)|_{t=0, H=1, z=\infty}}.
\]

For a series $F \in \mathbb{C}[[\frac{1}{z}]]$, the specialization $F|_{z=\infty}$ denotes constant term of $F$ with respect to $\frac{1}{z}$. Here, $N_2$ is series in $q$ defined by

\[
N_2(q) := \frac{d}{dq} \left( \frac{q \frac{d}{dq} I_{2,1}}{1 + q \frac{d}{dq} I_{1,0}} \right).
\]

2.4.4. Further calculations. Define small $I$-function

\[
\overline{I}(q) \in H^*_T(\mathbb{P}^2, \mathbb{Q})[[q]]
\]

by the restriction

\[
\overline{I}(q) = I(q, t)|_{t=0}.
\]

Define differential operators

\[
D = q \frac{d}{dq}, \quad M = H + z D.
\]

Applying $z \frac{d}{dt}$ to $\overline{I}$ and then restricting to $t = 0$ has same effect as applying $M$ to $\overline{I}$

\[
\left[ \left(z \frac{d}{dt}\right)^k \overline{I}\right]|_{t=0} = M^k \overline{I}.
\]

The function $\overline{I}$ satisfies following Picard-Fuchs equation

\[
(M - \lambda_0)(M - \lambda_1)(M - \lambda_2) + 3q M(3M + z)(3M + 2z) \overline{I} = 0
\]

implied by the Picard-Fuchs equation for $I$,

\[
\left( \prod_{j=0}^2 \left(z \frac{d}{dt} - \lambda_j\right) + 3q \left(z \frac{d}{dt}\right) \left(3 \left(z \frac{d}{dt}\right) + z\right) \left(3 \left(z \frac{d}{dt}\right) + 2z\right) \right) \overline{I} = 0.
\]

The restriction $\overline{I}|_{H=\lambda_i}$ admits following asymptotic form

\[
\overline{I}|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i} z + R_{2,i} z^2 + \ldots\right)
\]
with series \( \mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]] \).

A derivation of (25) is obtained in [21] via the Picard-Fuchs equation (24) for \( \mathcal{I}_{|H=\lambda_i} \). The series \( \mu_i \) and \( R_{k,i} \) are found by solving differential equations obtained from the coefficient of \( z^k \). For example,

\[
\lambda_i + D\mu_i = L_i, \\
R_{0,i} = \left( \frac{\lambda_i \prod_{j \neq i}(\lambda_i - \lambda_j)}{f(L_i)} \right)^{\frac{1}{2}}.
\]

Define the series \( C_1 \) and \( C_2 \) by the equations

\[
C_1 = z \frac{d}{dt} S(1)|_{z=\infty, t=0, H=1}, \\
C_2 = \left( z \frac{d}{dt} S(H) - (\lambda_0 + \lambda_1 + \lambda_2)N_2 S(H) \right)|_{z=\infty, t=0, H=1}.
\]

The following relation was proven in [21],

\[
C_1^2 C_2 = (1 + 27q)^{-1}.
\]

From the equations (23) and (25), we can show the series \( S_i(1) = S(1)|_{H=\lambda_i}, \ S_i(H) = S(H)|_{H=\lambda_i}, \ S_i(H^2) = S(H^2)|_{H=\lambda_i} \) have the following asymptotic expansions:

\[
S_i(1) = e^\mu_i \left( R_{00,i} + R_{01,i}z + R_{02,i}z^2 + \ldots \right), \\
S_i(H) = e^\mu_i \frac{L_i}{C_1} \left( R_{10,i} + R_{11,i}z + R_{12,i}z^2 + \ldots \right), \\
S_i(H^2) = e^\mu_i \frac{L_i^2}{C_1 C_2} \left( R_{20,i} + R_{21,i}z + R_{22,i}z^2 + \ldots \right).
\]

We follow here the normalization of [21]. Note

\[
R_{0k,i} = R_{k,i}.
\]

As in [21] Theorem 4], we expect the following constraints.

**Conjecture 15.** For all \( k \geq 0 \), we have

\[
R_{k,i} \in G_2.
\]

Conjecture [15] is the main obstruction for the proof of Conjecture [1] and [2]. By the same argument of Section 6, we obtain the following result.

**Theorem 16.** Conjecture [15] implies Conjecture [1] and [2].

By applying asymptotic expansions (28) to (23), we obtain the following results.
Lemma 17. We have
\[
R_{1p+1,i} = R_{0p+1,i} + \frac{DR_{0p,i}}{L_i};
\]
\[
R_{2p+1,i} = R_{1p+1,i} + \frac{DR_{1p,i}}{L_i} + \left(\frac{DL_i}{L_i^2} - \frac{X}{L_i}\right) - (\lambda_0 + \lambda_1 + \lambda_2)N_2 \frac{R_1k,i}{L_i},
\]
with \( X = \frac{DC_1}{C_1} \).

From Lemma 17, we obtain results for \( \mathcal{S}(H)|_{H=\lambda_i} \) and \( \mathcal{S}(H^2)|_{H=\lambda_i} \).

Lemma 18. Suppose Conjecture 15 is true. Then for all \( k \geq 0 \), we have for all \( k \geq 0 \),
\[
R_{1k,i} \in \mathcal{G}_2,
\]
\[
R_{2k,i} = Q_{2k,i} - \frac{R_{1k-1,i}}{L} X - (\lambda_0 + \lambda_1 + \lambda_2)N_2 \frac{R_{1k,i}}{L_i},
\]
with \( Q_{2k,i} \in \mathcal{G}_2 \).

2.5. Determining \( DX \) and \( N_2 \). The following relation was proven in [16].

\[
X^2 - (L^3 - 1) X + DX - \frac{2}{9}(L^3 - 1) = 0.
\]

By the above result, the differential ring
\[
\mathcal{G}_2[X, DX, DDX, \ldots]
\]
is just the polynomial ring \( \mathcal{G}_2[X] \). Denote by \( \text{Coeff}(x^i y^j) \) the coefficient of \( x^i y^j \) in
\[
\sum_{k=0}^{2} e^{-\frac{\mu_{i+1}}{x}} \text{Coeff}_{z=x} \phi_k \text{Coeff}_{z=y} \phi^k.
\]

From (18) and (28), we obtain the following equation.
\[
\text{Coeff}(x^2) + \text{Coeff}(y^2) - \text{Coeff}(xy) = 0.
\]

Above equation immediately yields the following relation.
\[
N_2 = -\frac{1}{2}C_2 + \frac{1}{2}L^3.
\]

3. Higher genus series on \( \overline{M}_{g,n} \)

3.1. Intersection theory on \( \overline{M}_{g,n} \). We review here the now standard method used by Givental [9, 10, 14] to express genus \( g \) descendent correlators in terms of genus 0 data.

Let \( t_0, t_1, t_2, \ldots \) be formal variables. The series
\[
T(c) = t_0 + t_1 c + t_2 c^2 + \ldots
\]
in the additional variable \( c \) plays a basic role. The variable \( c \) will later be replaced by the first Chern class \( \psi_i \) of a cotangent line over \( \overline{M}_{g,n} \),

\[
T(\psi_i) = t_0 + t_1 \psi_i + t_2 \psi_i^2 + \ldots ,
\]

with the index \( i \) depending on the position of the series \( T \) in the correlator.

Let \( 2g - 2 + n > 0 \). For \( a_i \in \mathbb{Z}_{\geq 0} \) and \( \gamma \in H^*(\overline{M}_{g,n}) \), define the correlator

\[
\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n} = \sum_{k \geq 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{i=1}^k T(\psi_{n+i}) .
\]

In the above summation, the \( k = 0 \) term is

\[
\int_{\overline{M}_{g,n}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} .
\]

We also need the following correlator defined for the unstable case,

\[
\langle \langle 1, 1 \rangle \rangle_{0,2} = \sum_{k > 0} \frac{1}{k!} \int_{\overline{M}_{0,2+k}} \prod_{i=1}^k T(\psi_{2+i}) .
\]

For formal variables \( x_1, \ldots, x_n \), we also define the correlator

(32)

\[
\langle \langle \frac{1}{x_1 - \psi}, \ldots, \frac{1}{x_n - \psi} \bigg| \gamma \rangle \rangle_{g,n}
\]

in the standard way by expanding \( \frac{1}{x_i - \psi} \) as a geometric series.

Denote by \( L \) the differential operator

\[
L = \frac{\partial}{\partial t_0} - \sum_{i=1}^\infty t_i \frac{\partial}{\partial t_{i-1}} = \frac{\partial}{\partial t_0} - t_1 \frac{\partial}{\partial t_0} - t_2 \frac{\partial}{\partial t_1} - \ldots .
\]

The string equation yields the following result.

**Lemma 19.** For \( 2g - 2 + n > 0 \), we have \( L \langle \langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n} = 0 \) and

\[
L \langle \langle \frac{1}{x_1 - \psi}, \ldots, \frac{1}{x_n - \psi} \bigg| \gamma \rangle \rangle_{g,n} = \left( \frac{1}{x_1} + \ldots + \frac{1}{x_n} \right) \langle \langle \frac{1}{x_1 - \psi}, \ldots, \frac{1}{x_n - \psi} \bigg| \gamma \rangle \rangle_{g,n} .
\]

After the restriction \( t_0 = 0 \) and application of the dilaton equation, the correlators are expressed in terms of finitely many integrals (by the
dimension constraint). For example,

\[
\begin{align*}
\langle\langle 1, 1, 1 \rangle \rangle_{0,3} &= \frac{1}{1 - t_1}, \\
\langle\langle 1, 1, 1, 1 \rangle \rangle_{0,4} &= \frac{t_2}{(1 - t_1)^3}, \\
\langle\langle 1, 1, 1, 1, 1 \rangle \rangle_{0,5} &= \frac{t_3}{(1 - t_1)^4} + \frac{3t_2^2}{(1 - t_1)^5}, \\
\langle\langle 1, 1, 1, 1, 1, 1 \rangle \rangle_{0,6} &= \frac{t_4}{(1 - t_1)^5} + \frac{10t_2t_3}{(1 - t_1)^6} + \frac{15t_2^3}{(1 - t_1)^7}.
\end{align*}
\]

We consider \(\mathbb{C}(t_1)[t_2, t_3, \ldots]\) as \(\mathbb{Z}\)-graded ring over \(\mathbb{C}(t_1)\) with

\[\deg(t_i) = i - 1 \quad \text{for} \quad i \geq 2\]

Define a subspace of homogeneous elements by

\[\mathbb{C}\left[\frac{1}{1 - t_1}\right][t_2, t_3, \ldots]_{\text{Hom}} \subset \mathbb{C}(t_1)[t_2, t_3, \ldots].\]

We easily see

\[\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} |_{t_0=0} \in \mathbb{C}\left[\frac{1}{1 - t_1}\right][t_2, t_3, \ldots]_{\text{Hom}}.\]

Using the leading terms (of lowest degree in \(\frac{1}{1 - t_1}\)), we obtain the following result.

**Lemma 20.** The set of genus 0 correlators

\[\left\{ \langle\langle \ldots, 1 \rangle \rangle_{0,n} |_{t_0=0} \right\}_{n \geq 4}\]

freely generate the ring \(\mathbb{C}(t_1)[t_2, t_3, \ldots]\) over \(\mathbb{C}(t_1)\).

By Lemma 20, we can find a unique representation of \(\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \rangle \rangle_{g,n} |_{t_0=0}\) in the variables

\[\left\{ \langle\langle \ldots, 1 \rangle \rangle_{0,n} |_{t_0=0} \right\}_{n \geq 3}.
\]

The \(n = 3\) correlator is included in the set (33) to capture the variable \(t_1\). For example, in \(g = 1\),

\[
\begin{align*}
\langle\langle 1, 1 \rangle \rangle_{1,2} |_{t_0=0} &= \frac{1}{24} \left( \frac{\langle\langle 1, 1, 1, 1 \rangle \rangle_{0,5} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle \rangle_{0,3} |_{t_0=0}} - \frac{\langle\langle 1, 1, 1 \rangle \rangle_{0,4}^2 |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle \rangle_{0,3}^2 |_{t_0=0}} \right), \\
\langle\langle 1 \rangle \rangle_{1,1} |_{t_0=0} &= \frac{1}{24} \frac{\langle\langle 1, 1, 1 \rangle \rangle_{0,4} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle \rangle_{0,3} |_{t_0=0}}.
\end{align*}
\]
A more complicated example in $g = 2$ is

$$\langle\langle, 2, 0|_{t_0 = 0} = \frac{1}{1152} \langle\langle 1, 1, 1, 1, 1 \rangle\rangle_{0, 6}|_{t_0 = 0} - \frac{7}{1920} \langle\langle 1, 1, 1, 1 \rangle\rangle_{0, 5}|_{t_0 = 0} \langle\langle 1, 1, 1, 1 \rangle\rangle_{0, 4}|_{t_0 = 0} + \frac{1}{360} \langle\langle 1, 1, 1 \rangle\rangle_{0, 3}|^3_{t_0 = 0}.$$

**Definition 21.** For $\gamma \in H^*({\overline{M}}_{g,k})$, let

$$P^{a_1, \ldots, a_n, \gamma}_{g,n}(s_0, s_1, s_2, \ldots) \in \mathcal{Q}(s_0, s_1, \ldots)$$

be the unique rational function satisfying the condition

$$\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle\rangle_{g,n}|_{t_0 = 0} = P^{a_1, a_2, \ldots, a_n, \gamma}_{g,n}|_{s_i = \langle\langle 1, \ldots, 1 \rangle\rangle_{0, i+3}} = 0.$$

**Proposition 22.** For $2g - 2 + n > 0$, we have

$$\langle\langle 1, \ldots, 1 | \gamma \rangle\rangle_{g,n} = P^{0, \ldots, 0, \gamma}_{g,n}|_{s_i = \langle\langle 1, \ldots, 1 \rangle\rangle_{0, i+3}}.$$

**Proof.** Both sides of the equation satisfy the differential equation

$$L = 0.$$

By definition, both sides have the same initial conditions at $t_0 = 0$. □

**Proposition 23.** For $2g - 2 + n > 0$,

$$\langle\langle \frac{1}{x_1 - \psi_1}, \ldots, \frac{1}{x_n - \psi_n} | \gamma \rangle\rangle_{g,n} = e^{\langle\langle 1, 1 \rangle\rangle_{0, 2}(\sum \frac{1}{x_i})} \sum_{a_1, \ldots, a_n} P^{a_1, \ldots, a_n, \gamma}_{g,n}|_{s_i = \langle\langle 1, \ldots, 1 \rangle\rangle_{0, i+3}} \frac{x_1^{a_1 + 1} \cdots x_n^{a_n + 1}}{a_1 \cdots a_n}.$$

**Proof.** Both sides of the equation satisfy differential equation

$$L - \sum \frac{1}{x_i} = 0.$$

Both sides have the same initial conditions at $t_0 = 0$. We use here

$$L\langle\langle 1, 1 \rangle\rangle_{0, 2} = 1, \quad \langle\langle 1, 1 \rangle\rangle_{0, 2}|_{t_0 = 0} = 0.$$

There is no conflict here with Lemma 19 since $(g, n) = (0, 2)$ is not in the stable range. □
3.2. The unstable case \((0, 2)\). The definition given in (32) of the correlator is valid in the stable range
\[ 2g - 2 + n > 0. \]
The unstable case \((g, n) = (0, 2)\) plays a special role. We define
\[ \left\langle \left\langle \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right\rangle \right\rangle_{0, 2} \]
by adding the degenerate term
\[ \frac{1}{x_1 + x_2} \]
to the terms obtained by the expansion of \( \frac{1}{x_i - \psi_i} \) as a geometric series. The degenerate term is associated to the (unstable) moduli space of genus 0 with 2 markings.

**Proposition 24.** We have
\[ \left\langle \left\langle \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right\rangle \right\rangle_{0, 2} = e^{(1,1)_{0,2}} \left( \frac{1}{x_1 + x_2} \right)^2. \]

**Proof.** Both sides of the equation satisfy differential equation
\[ L - \sum_{i=1}^{2} \frac{1}{x_i} = 0. \]
Both sides have the same initial conditions at \(t_0 = 0\). \(\square\)

3.3. Local invariants and wall crossing. The torus \(T\) acts on the moduli spaces \(\overline{M}_{g,n}(\mathbb{P}^2, d)\) and \(\overline{Q}_{g,n}(\mathbb{P}^2, d)\). We consider here special localization contributions associated to the fixed points \(p_i \in \mathbb{P}^2\).

Consider first the moduli of stable maps. Let
\[ \overline{M}_{g,n}(\mathbb{P}^2, d)^{T:p_i} \subset \overline{M}_{g,n}(\mathbb{P}^2, d) \]
be the union of \(T\)-fixed loci which parameterize stable maps obtained by attaching \(T\)-fixed rational tails to a genus \(g\), \(n\)-pointed Deligne-Mumford stable curve contracted to the point \(p_i \in \mathbb{P}^2\). Similarly, let
\[ \overline{Q}_{g,n}(\mathbb{P}^2, d)^{T:p_i} \subset \overline{Q}_{g,n}(\mathbb{P}^2, d) \]
be the parallel \(T\)-fixed locus parameterizing stable quotients obtained by attaching base points to a genus \(g\), \(n\)-pointed Deligne-Mumford stable curve contracted to the point \(p_i \in \mathbb{P}^2\).

Let \(\Lambda_i\) denote the localization of the ring
\[ \mathbb{C}[\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}] \]
at the three tangent weights at $p_i \in \mathbb{P}^2$. Using the virtual localization formula \cite{[11]}, there exist unique series

$$S_{p_i} \in \Lambda_i[[\psi]][[Q]]$$

for which the localization contribution of the $T$-fixed locus $\overline{M}_{g,n}(\mathbb{P}^2, d)^{T,p_i}$ to the equivariant Gromov-Witten invariants of $K\mathbb{P}^2$ can be written as

$$\sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,n}(K\mathbb{P}^2,d)^{T,p_i}]_{vir}} \psi_1^{a_1} \cdots \psi_n^{a_n} =$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int_{M_{g,n+k}} H_{g}^{p_i} \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{j=1}^{k} S_{p_i}(\psi_{n+j}) .$$

Here, $H_{g}^{p_i}$ is the standard vertex class,

$$(34) \quad \frac{e(E_g^* \otimes T_{p_i}(\mathbb{P}^2))}{e(T_{p_i}(\mathbb{P}^2))} \cdot \frac{e(E_g^* \otimes (-3\lambda_i))}{(-3\lambda_i)} ,$$

obtained the Hodge bundle $E_g \to \overline{M}_{g,n+k}$.

Similarly, the application of the virtual localization formula to the moduli of stable quotients yields classes

$$F_{p_i,k} \in H^*(\overline{M}_{g,n+k}) \otimes \Lambda_i$$

for which the contribution of $Q_{g,n}(\mathbb{P}^2, d)^{T,p_i}$ is given by

$$\sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_{g,n}(K\mathbb{P}^2,d)^{T,p_i}]_{vir}} \psi_1^{a_1} \cdots \psi_n^{a_n} =$$

$$\sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{M_{g,n+k}} H_{g}^{p_i} \psi_1^{a_1} \cdots \psi_n^{a_n} F_{p_i,k} .$$

Here $\overline{M}_{g,n+k}$ is the moduli space of genus $g$ curves with markings

$$\{p_1, \cdots, p_n\} \cup \{\hat{p}_1 \cdots \hat{p}_k\} \in C^{ns} \subset C$$

satisfying the conditions

(i) the points $p_i$ are distinct,

(ii) the points $\hat{p}_j$ are distinct from the points $p_i$, with stability given by the ampleness of

$$\omega_C(\sum_{i=1}^{m} p_i + \epsilon \sum_{j=1}^{k} \hat{p}_j)$$

for every strictly positive $\epsilon \in \mathbb{Q}$.
The Hodge class $H_{g}^{p}$ is given again by formula (34) using the Hodge bundle

$$E_g \rightarrow \overline{M}_{g,n|k}.$$  

**Definition 25.** For $\gamma \in H^*(\overline{M}_{g,n})$, let

$$\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i,\infty} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{j=1}^{k} S_{p_i}(\psi_{n+j}),$$

$$\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i,0+} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n|k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} F_{p_i,k}.$$  

**Proposition 26** (Ciocan-Fontanine, Kim [7]). For $2g - 2 + n > 0$, we have the wall crossing relation

$$\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i,\infty}(Q(q)) = \langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i,0+}(q)$$

where $Q(q)$ is the mirror map

$$Q(q) = \exp(I_{K\mathbb{P}^2}(q)).$$

Proposition 26 is a consequence of [7, Lemma 5.5.1]. The mirror map here is the mirror map for $K\mathbb{P}^2$ discussed in Section 0.2. Propositions 22 and 26 together yield

$$\langle \langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n}^{p_i,\infty} = P_{0,0,0,\gamma}^{0,0,0,0,0}(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,\infty}, \langle \langle 1, 1, 1 \rangle \rangle_{0,4}^{p_i,\infty}, \ldots),$$

$$\langle \langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n}^{p_i,0+} = P_{0,0,0,\gamma}^{0,0,0,0,0}(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,0+}, \langle \langle 1, 1, 1 \rangle \rangle_{0,4}^{p_i,0+}, \ldots).$$

Similarly, using Propositions 23 and 26, we obtain

$$e^{\langle (1,1) \rangle_{0,2}^{p_i,\infty} \left( \sum_{a_1,\ldots,a_n} \frac{1}{x_1^{a_1+1} \cdots x_n^{a_n+1}} \prod_{a_1,\ldots,a_n} P_{a_1,\ldots,a_n,\gamma}^{a_1,\ldots,a_n,\gamma}(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,\infty}, \langle \langle 1, 1, 1 \rangle \rangle_{0,4}^{p_i,\infty}, \ldots) \right)},$$

and

$$e^{\langle (1,1) \rangle_{0,2}^{p_i,0+} \left( \sum_{a_1,\ldots,a_n} \frac{1}{x_1^{a_1+1} \cdots x_n^{a_n+1}} \prod_{a_1,\ldots,a_n} P_{a_1,\ldots,a_n,\gamma}^{a_1,\ldots,a_n,\gamma}(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,0+}, \langle \langle 1, 1, 1 \rangle \rangle_{0,4}^{p_i,0+}, \ldots) \right)}.$$
4. Higher genus series on $K\mathbb{P}^2$

4.1. **Overview.** We apply the localization strategy introduced first by Givental \[9, 10, 14\] for Gromov-Witten theory to the stable quotient invariants of local $\mathbb{P}^2$. The contribution $\text{Cont}_{\Gamma}(q)$ discussed in Section 1 of a graph $\Gamma \in G_g(\mathbb{P}^2)$ can be separated into vertex and edge contributions. We express the vertex and edge contributions in terms of the series $S_i$ and $V_{ij}$ of Section 2.3.

4.2. **Edge terms.** Recall the definition of $V_{ij}$ given in Section 2.3,

\[(36) \quad V_{ij} = \langle \langle \frac{\phi_i}{x-\psi}, \frac{\phi_j}{y-\psi} \rangle \rangle_{0,2}^{0+,0+}.
\]

Let $\nabla_{ij}$ denote the restriction of $V_{ij}$ to $t = 0$. Via formula (16), $\nabla_{ij}$ is a summation of contributions of fixed loci indexed by a graph $\Gamma$ consisting of two vertices connected by a unique edge. Let $w_1$ and $w_2$ be $T$-weights. Denote by $\nabla_{ij}^{w_1,w_2}$

the summation of contributions of $T$-fixed loci with tangent weights precisely $w_1$ and $w_2$ on the first rational components which exit the vertex components over $p_i$ and $p_j$.

The series $\nabla_{ij}^{w_1,w_2}$ includes both vertex and edge contributions. By definition (36) and the virtual localization formula, we find the following relationship between $\nabla_{ij}^{w_1,w_2}$ and the corresponding pure edge contribution $E_{ij}^{w_1,w_2}$,

\[
E_{ij}^{w_1,w_2} e_i V_{ij}^{w_1,w_2} e_j = \sum_{a_1,a_2} e_{a_1} \frac{(1)_{\psi(a_1+1)}^{\frac{p_j}{w_2}}}{x_1(a_1)^{a_1}} \frac{(1)_{\psi(a_2+1)}^{\frac{p_i}{w_1}}}{x_2(a_2)^{a_2}} (-1)^{a_1+a_2} \nabla_{ij}^{w_1,w_2} w_1^{a_1} w_2^{a_2} x_1^{a_1} x_2^{a_2}.
\]

After summing over all possible weights, we obtain

\[
e_i \left( \nabla_{ij} - \frac{\delta_{ij}}{e_i(x+y)} \right) e_j = \sum_{w_1, w_2} e_i \nabla_{ij}^{w_1,w_2} e_j.
\]

The above calculations immediately yield the following result.

---

\[8\]We use the variables $x_1$ and $x_2$ here instead of $x$ and $y$.\]
Lemma 27. We have

\[
\left[ e^{-\langle 1,1 \rangle_{p_1,0^+}} e^{-\langle 1,1 \rangle_{p_2,0^+}} e^{\left( \nabla_{ij} - \frac{\delta_{ij}}{e_{ij}(x+y)} \right)} e_{x_1^{a_1-1} x_2^{a_2-1}} \right] =
\sum_{w_1, w_2} \frac{\langle 1,1 \rangle_{p_1,0^+}}{w_1} e^{\langle 1,1 \rangle_{p_2,0^+}} (-1)^{a_1+a_2} \frac{E_{w_1, w_2}}{w_1^{a_1} w_2^{a_2}}.
\]

The notation \([\ldots] x_1^{a_1-1} x_2^{a_2-1}\) in Lemma 27 denotes the coefficient of \(x_1^{a_1-1} x_2^{a_2-1}\) in the series expansion of the argument.

4.3. A simple graph. Before treating the general case, we present the localization formula for a simple graph\(^9\). Let \(\Gamma \in G_g(\mathbb{P}^2)\) consist of two vertices and one edge, \(v_1, v_2 \in \Gamma(V)\), \(e \in \Gamma(E)\) with genus and \(T\)-fixed point assignments \(g(v_i) = g_i\), \(p(v_i) = p_i\).

Let \(w_1\) and \(w_2\) be tangent weights at the vertices \(p_1\) and \(p_2\) respectively. Denote by \(\text{Cont}_{\Gamma, w_1, w_2}\) the summation of contributions to \((37)\)

\[
\sum_{d > 0} q^d \left[ Q_g(K\mathbb{P}^2, d) \right]^{\text{vir}}
\]

of \(T\)-fixed loci with tangent weights precisely \(w_1\) and \(w_2\) on the first rational components which exit the vertex components over \(p_1\) and \(p_2\). We can express the localization formula for \((37)\) as

\[
\left\langle \frac{1}{w_1 - \psi} \right| H^{p_1}_{g_1} \right\|_{w_1, w_2} e^{w_1, w_2} \left\langle \frac{1}{w_2 - \psi} \right| H^{p_2}_{g_2} \right\|_{g_2, 1}
\]

which equals

\[
\sum_{a_1, a_2} \frac{\langle 1,1 \rangle_{p_1,0^+}}{w_1^{a_1}} e^{\langle 1,1 \rangle_{p_2,0^+}} \frac{P_{\langle 1 \rangle_{g_1,1}^{a_1-1}} H^{p_2}_{g_2}}{w_1^{a_1} w_2^{a_2}} e^{\langle 1,1 \rangle_{g_2,1}^{a_2-1}} P_{\langle 1 \rangle_{g_2,1}^{a_2}}
\]

where \(H^{p_i}_{g_i}\) is the Hodge class \((34)\). We have used here the notation

\[
P_{\langle 1 \rangle_{g_1,1}^{k_1}, \ldots, \langle 1 \rangle_{g_n,1}^{k_n}} H^{p_i}_{h, n} =
\]

\[
P_{h_1, \ldots, h_n} H^{p_i}_{h_1} \left( \langle 1 \rangle_{g_1,1}^{k_1}, \ldots, \langle 1 \rangle_{g_n,1}^{k_n} \right)
\]

and applied \((35)\).  
\(^9\)We follow here the notation of Section 1.
After summing over all possible weights $w_1, w_2$ and applying Lemma 27, we obtain the following result for the full contribution

$$\text{Cont}_\Gamma = \sum_{w_1, w_2} \text{Cont}_{\Gamma, w_1, w_2}$$

of $\Gamma$ to $\sum_{d \geq 0} q^d [\overline{Q}_g(K\mathbb{P}^2, d)]^\text{vir}$.

**Proposition 28.** We have

$$\text{Cont}_\Gamma = \sum_{a_1, a_2 > 0} P \left( \psi^{a_1-1} | H^{p_{g_1}}_{g_1, 1} \right) P \left( \psi^{a_2-1} | H^{p_{g_2}}_{g_2, 1} \right) \cdot (-1)^{a_1+a_2} \left[ e^{-\frac{(1, 1)p_{g_1}}{x_1}} e^{-\frac{(1, 1)p_{g_2}}{x_2}} \frac{\delta_{ij}}{x_1 + x_2} \right] x_1^{a_1-1} x_2^{a_2-1}.$$ 

4.4. **A general graph.** We apply the argument of Section 4.3 to obtain a contribution formula for a general graph $\Gamma$.

Let $\Gamma \in G_{g, 0}(\mathbb{P}^2)$ be a decorated graph as defined in Section 1. The flags of $\Gamma$ are the half-edges\footnote{Flags are either half-edges or markings.}. Let $F$ be the set of flags. Let $w : F \to \text{Hom}(T, \mathbb{C}^*) \otimes \mathbb{Q}$ be a fixed assignment of $T$-weights to each flag.

We first consider the contribution $\text{Cont}_{\Gamma, w}$ to

$$\sum_{d \geq 0} q^d [\overline{Q}_g(K\mathbb{P}^2, d)]^\text{vir}$$

of the $T$-fixed loci associated $\Gamma$ satisfying the following property: the tangent weight on the first rational component corresponding to each $f \in F$ is exactly given by $w(f)$. We have

$$\text{Cont}_{\Gamma, w} = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}^F_{>0}} \prod_{v \in V} \text{Cont}^A_{\Gamma, w}(v) \prod_{e \in E} \text{Cont}_{\Gamma, w}(e).$$

The terms on the right side of (38) require definition:

- The sum on the right is over the set $\mathbb{Z}^F_{>0}$ of all maps $A : F \to \mathbb{Z}_{>0}$ corresponding to the sum over $a_1, a_2$ in Proposition 28.
• For $v \in V$ with $n$ incident flags with $w$-values $(w_1, \ldots, w_n)$ and $A$-values $(a_1, a_2, \ldots, a_n)$,

$$\text{Cont}_{\Gamma,w}(v) = \frac{\mathbb{P} \left[ \psi_1, \ldots, \psi_n \Bigg| \prod_{j=1}^n \frac{\psi_j \cdot \mathcal{H}_{\mathcal{G}(v)}^{p(v),0+}}{\mathcal{G}(v)} \right] p(v,0+) \prod_{j=1}^n w_j^{a_j}}{w_1^{a_1} \cdots w_n^{a_n}}.$$

• For $e \in E$ with assignments $(p(v_1), p(v_2))$ for the two associated vertices and $w$-values $(w_1, w_2)$ for the two associated flags,

$$\text{Cont}_{\Gamma,w}(e) = e_{\langle\langle 1,1 \rangle\rangle} \prod_{j=1}^n \frac{\mathbb{P} \left[ \psi_j \cdot \mathcal{H}_{\mathcal{G}(v)}^{p(v),0+} \right] p(v,0+) \prod_{j=1}^n w_j^{a_j}}{w_1^{a_1} \cdots w_n^{a_n}}.$$

The localization formula then yields (38) just as in the simple case of Section 4.3.

By summing the contribution (38) of $\Gamma$ over all the weight functions $w$ and applying Lemma 27, we obtain the following result which generalizes Proposition 28.

**Proposition 29.** We have

$$\text{Cont}_{\Gamma} = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_+^2 \times \mathbb{Z}_+^2} \prod_{v \in V} \text{Cont}_{\Gamma,v}^{A} \prod_{e \in E} \text{Cont}_{\Gamma,e}^{A},$$

where the vertex and edge contributions with incident flag $A$-values $(a_1, \ldots, a_n)$ and $(b_1, b_2)$ respectively are

$$\text{Cont}_{\Gamma,v}^{A} = \mathbb{P} \left[ \psi_1, \ldots, \psi_n \Bigg| \prod_{j=1}^n \frac{\psi_j \cdot \mathcal{H}_{\mathcal{G}(v)}^{p(v),0+}}{\mathcal{G}(v)} \right] p(v,0+) \prod_{j=1}^n w_j^{a_j},$$

$$\text{Cont}_{\Gamma,e}^{A} = (-1)^{b_1+b_2} \left[ e^{-\langle\langle 1,1 \rangle\rangle} \prod_{j=1}^n \frac{\mathbb{P} \left[ \psi_j \cdot \mathcal{H}_{\mathcal{G}(v)}^{p(v),0+} \right] p(v,0+) \prod_{j=1}^n w_j^{a_j}}{w_1^{a_1} \cdots w_n^{a_n}} \right] \left( \mathcal{V}_{ij} - \frac{1}{e_i(x+y)} \right) e_j \left. \right|_{x_{1}^{b_1-1}, \ldots, x_{n}^{b_n-1}} \left. \right|_{x_{1}^{b_1-1}, \ldots, x_{n}^{b_n-1}},$$

where $p(v_1) = p_i$ and $p(v_2) = p_j$ in the second equation.

4.5. **Legs.** Let $\Gamma \in \mathbb{G}_{g,n}(\mathbb{P}^2)$ be a decorated graph with markings. While no markings are needed to define the stable quotient invariants of $K\mathbb{P}^2$, the contributions of decorated graphs with markings will appear in the proof of the holomorphic anomaly equation. The formula for the contribution $\text{Cont}_{\Gamma}(H, \ldots, H)$ of $\Gamma$ to

$$\sum_{d \geq 0} q^d \prod_{j=0}^n \text{ev}^j(H) \cap \left[ Q_{g,n}(K\mathbb{P}^2, d) \right]^{\text{vir}}$$

is given by the following result.

\[^{11}\text{In case } e \text{ is self-edge, } v_1 = v_2.\]
Proposition 30. We have
\[
\text{Cont}_T(H, \ldots, H) = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{>0}} \prod_{v \in V} \text{Cont}_A^T(v) \prod_{e \in E} \text{Cont}_A^T(e) \prod_{l \in L} \text{Cont}_A^T(l),
\]
where the leg contribution is
\[
\text{Cont}_A^T(l) = (-1)^{A(l)-1} \left[ e^{-\frac{(\langle 1, 1 \rangle_{0,2}^0)^+}{z} \bar{S}_p(l)(H)} \right]^{A(l)-1}.
\]
The vertex and edge contributions are same as before.

The proof of Proposition 30 follows the vertex and edge analysis. We leave the details as an exercise for the reader. The parallel statement for Gromov-Witten theory can be found in [9, 10, 14].

5. Vertices, edges, and legs

5.1. Overview. Using the results of Givental [9, 10, 14] combined with wall-crossing [7], we calculate here the vertex and edge contributions in terms of the function \( R_k \) of Section 2.4.4.

5.2. Calculations in genus 0. We follow the notation introduced in Section 3.1. Recall the series
\[
T(c) = t_0 + t_1 c + t_2 c^2 + \ldots.
\]

Proposition 31. (Givental [9, 10, 14]) For \( n \geq 3 \), we have
\[
\langle \langle 1, \ldots, 1 \rangle \rangle_{0,n}^{p_i,\infty} = (\sqrt{\Delta_i})^{2g-2+n} \left( \sum_{k \geq 0} \frac{1}{k!} \int_{M_{0,n+k}} T(\psi_n+1) \cdots T(\psi_n+k) \right) |_{t_0=0, t_1=0, t_j \geq 2 = (-1)^j Q_{j+1}, i}
\]
where the functions \( \sqrt{\Delta_i}, Q_{l,i} \) are defined by
\[
\bar{S}_i^\infty(1) = e_i \langle \langle \psi_i, 1 \rangle \rangle_{0,2}^{p_i,\infty} = e^{\frac{(\langle 1, 1 \rangle_{0,2}^0)^+}{z} \bar{S}_p(1)} \left( 1 + \sum_{l=1}^{\infty} Q_{l,i}^l z^l \right).
\]

The existence of the above asymptotic expansion of \( \bar{S}_i^\infty(1) \) can also be proven by the argument of [14, Theorem 5.4.1]. Similarly, we have an asymptotic expansion of \( \bar{S}_i(1) \),
\[
\bar{S}_i(1) = e^{\frac{(\langle 1, 1 \rangle_{0,2}^0)^+}{z} \sum_{l=0}^{\infty} R_{l,i}^l z^l}.
\]
By \((28)\), we have
\[
\langle\langle 1, 1 \rangle\rangle_{0,2}^{p_{i,0}^+} = \mu_i.
\]
After applying the wall-crossing result of Proposition \(26\) we obtain
\[
\langle\langle 1, \ldots, 1 \rangle\rangle_{0,\infty}^{p_{i,0}^+}(Q(q)) = \langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}^{p_{i,0}^+}(q),
\]
\[
\mathcal{S}_i^\infty(1)(Q(q)) = \mathcal{S}_i(1)(q),
\]
where \(Q(q)\) is mirror map for \(K\mathbb{P}^2\) as before. By comparing asymptotic expansions of \(\mathcal{S}_i^\infty(1)\) and \(\mathcal{S}_i(1)\), we get a wall-crossing relation between \(Q_{l,i}\) and \(R_{l,i}\),
\[
\sqrt{\Delta_i}(Q(q)) = \frac{1}{R_{0,i}(q)},
\]
\[
Q_{l,i}(Q(q)) = \frac{R_{l,i}(q)}{R_{0,i}(q)} \text{ for } l \geq 1.
\]
We have proven the following result.

**Proposition 32.** For \(n \geq 3\), we have
\[
\langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}^{p_{i,0}^+} = \frac{1}{R_{0,i}(q)}.
\]

Another simple consequence of Proposition 32 is the following basic property.

**Corollary 33.** For \(n \geq 3\), we have
\[
\langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}^{p_{i,0}^+} \in \mathbb{C}[R_{0,i}^{\pm 1}, R_{1,i}, R_{2,i}, \ldots].
\]

5.3. **Vertex and edge analysis.** By Proposition 29, we have decomposition of the contribution to \(\Gamma \in G_g(\mathbb{P}^2)\) to the stable quotient theory of \(K\mathbb{P}^2\) into vertex terms and edge terms
\[
\text{Cont}_\Gamma = \frac{1}{|Aut(\Gamma)|} \sum_{A \in \mathbb{Z}_{>0}^E} \prod_{v \in V} \text{Cont}_A^A(v) \prod_{e \in E} \text{Cont}_A^A(e).
\]

**Lemma 34.** Suppose Conjecture 15 is true. Then we have
\[
\text{Cont}_A^A(v) \in G_2.
\]
Proof. By Proposition 29,  
\[ \text{Cont}^A_\Gamma(v) = P \left[ \psi_1^{q_1-1}, \ldots, \psi_n^{q_n-1} \right] H^{p(v)_{0,+}}_{g(v)_{0,n}}. \]

The right side of the above formula is a polynomial in the variables  
\[ \frac{1}{\langle(1, 1, 1)\rangle_{p(v),0,+}} \quad \text{and} \quad \left\{ \langle(1, \ldots, 1)\rangle_{p(v),0,+} | t_i = 0 \right\}_{n \geq 4} \]

with coefficients in \( \mathbb{C}(\lambda_0, \lambda_1, \lambda_2) \). The Lemma then follows from the evaluation (39), Corollary 33, and Conjecture 15. \( \square \)

Let \( e \in E \) be an edge connecting the \( T \)-fixed points \( p_i, p_j \in \mathbb{P}^2 \). Let the \( A \)-values of the respective half-edges be \( (k, l) \).

**Lemma 35.** Suppose Conjecture 15 is true. Then we have \( \text{Cont}^A_\Gamma(e) \in G[X] \) and

- the degree of \( \text{Cont}^A_\Gamma(e) \) with respect to \( X \) is 1,
- the coefficient of \( X \) in \( \text{Cont}^A_\Gamma(e) \) is

\[ (-1)^{k+l} \frac{3L_i L_j R_{1-k-l}}{L^3} \]

Proof. By Proposition 29,  
\[ \text{Cont}^A_\Gamma(e) = (-1)^{k+l} \left[ e^{-\frac{\mu_k}{x} - \frac{\mu_l}{y}} e_i \left( \nabla_{ij} - \frac{\delta_{ij}}{e_i(x+y)} \right) e_j \right]_{x^{k-1}y^{l-1}}. \]

Using also the equation
\[ e_i \nabla_{ij}(x, y)e_j = \sum_{r=0}^{2} S_i(\phi_r)|_{z=x} S_j(\phi^r)|_{z=y} \]

we write \( \text{Cont}^A_\Gamma(e) \) as

\[ \left[ (-1)^{k+l} e^{-\frac{\mu_k}{x} - \frac{\mu_l}{y}} \sum_{r=0}^{2} S_i(\phi_r)|_{z=x} S_j(\phi^r)|_{z=y} \right]_{x^{k-1}y^{l-1} - x^{k+1}y^{l-2} + \ldots + (-1)^{k-1} x^{k+l-1}} \]

where the subscript signifies a (signed) sum of the respective coefficients. If we substitute the asymptotic expansions (28) for
\[ \bar{S}_i(1), \bar{S}_i(H), \bar{S}_i(H^2) \]
in the above expression, the Lemma follows from Conjecture 15, Lemma 18 and (31). \( \square \)
5.4. **Legs.** Using the contribution formula of Proposition [30],

\[ \text{Cont}_{\Gamma}^{A}(l) = (-1)^{A(l)-1} \left[ e^{-\frac{((1,1))^{p(l),0+}}{2} S_{p(l)}(H)} \right]_{z^{A(l)-1}}, \]

we easily conclude under the assumption of Conjecture [15]

\[ C_1 \cdot \text{Cont}_{\Gamma}^{A}(l) \in G_2. \]

6. **Holomorphic Anomaly for \( K\mathbb{P}^2 \)**

6.1. **Proof of Theorem [3]** By definition, we have

\[ A_2(q) = \frac{1}{L^3} \left( 3X + 1 - \frac{L^3}{2} \right). \]

Conjecture [15] was proven in Appendix for the choices of \( \lambda_0, \lambda_1, \lambda_2 \) such that

\[(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0.\]

Hence, statement (i),

\[ \mathcal{F}_g^{SQ}(q) \in G_2[A_2], \]

follows from Proposition [29] and Lemmas [34] - [35]. Statement (ii), \( \mathcal{F}_g^{SQ} \) has at most degree 3g - 3 with respect to \( A_2 \), holds since a stable graph of genus \( g \) has at most 3g - 3 edges. Since

\[ \frac{\partial}{\partial T} = q \frac{\partial}{C_1 \partial q}, \]

statement (iii),

\[ \frac{\partial^k \mathcal{F}_g^{SQ}}{\partial T^k}(q) \in G_2[A_2][C_1^{-1}], \]

follows since the ring

\[ G_2[A_2] = G_2[X] \]

is closed under the action of the differential operator

\[ D = q \frac{\partial}{\partial q} \]

by [29]. The degree of \( C_1^{-1} \) in (41) is 1 which yields statement (iv). \( \square \)
6.2. **Proof of Theorem 2.** Let $\Gamma \in G_g(\mathbb{P}^2)$ be a decorated graph. Let us fix an edge $f \in E(\Gamma)$:

- if $\Gamma$ is connected after deleting $f$, denote the resulting graph by $\Gamma^0_f \in G_{g-1,2}(\mathbb{P}^2)$,

- if $\Gamma$ is disconnected after deleting $f$, denote the resulting two graphs by $\Gamma^1_f \in G_{g_1,1}(\mathbb{P}^2)$ and $\Gamma^2_f \in G_{g_2,1}(\mathbb{P}^2)$

where $g = g_1 + g_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph $\Gamma^0_f$ in case • should be viewed as sum of 2 graphs $\Gamma^0_f, (1, 2) + \Gamma^0_f, (2, 1)$.

Similarly, in case ••, we will sum over the ordering of $g_1$ and $g_2$. As usual, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.

By Proposition 29, we have the following formula for the contribution of the graph $\Gamma$ to the stable quotient theory of $K\mathbb{P}^2$,

$$
\text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}^E} \prod_{v \in V} \text{Cont}^A_{\Gamma}(v) \prod_{e \in E} \text{Cont}^A_{\Gamma}(e).
$$

Let $f$ connect the $T$-fixed points $p_i, p_j \in \mathbb{P}^2$. Let the $A$-values of the respective half-edges be $(k, l)$. By Lemma 35, we have

$$
\frac{\partial \text{Cont}^A_{\Gamma}(f)}{\partial X} = (-1)^{k+l} \frac{3L_i L_j R_{1-k-1,i} R_{1-l-1,j}}{L^3}.
$$

• If $\Gamma$ is connected after deleting $f$, we have

$$
\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}^E} \left( \frac{L^3}{3C^2} \right) \frac{\partial \text{Cont}^A_{\Gamma}(f)}{\partial X} \prod_{v \in V} \text{Cont}^A_{\Gamma}(v) \prod_{e \in E, e \neq f} \text{Cont}^A_{\Gamma}(e) = \frac{1}{2} \text{Cont}_{\Gamma^0_f}(H, H).
$$

The derivation is simply by using (42) on the left and Proposition 30 on the right.
If $\Gamma$ is disconnected after deleting $f$, we obtain
\[
\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathcal{A}_{\geq 0}} \left( \frac{L^3}{3C_1^2} \right) \frac{\partial \text{Cont}^A_\Gamma(f)}{\partial X} \prod_{v \in V} \text{Cont}^A_\Gamma(v) \prod_{e \in E, e \neq f} \text{Cont}^A_\Gamma(e) = \frac{1}{2} \text{Cont}^1_\Gamma(H) \text{Cont}^2_\Gamma(H)
\]
by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in G_\gamma(\mathbb{P}^2)$ and using the vanishing
\[
\partial \text{Cont}^A_\Gamma(v) = 0
\]
of Lemma 34, we obtain
\[
(43) \quad \left( \frac{L^3}{3C_1^2} \right) \frac{\partial}{\partial X} \langle \rangle_{g,0}^{\text{SQ}} = \frac{1}{2} \sum_{i=1}^{g-1} \langle H \rangle_{g-i,1}^{\text{SQ}} \langle H \rangle_{i,1}^{\text{SQ}} + \frac{1}{2} \langle H, H \rangle_{g-1,2}^{\text{SQ}}.
\]
We have followed here the notation of Section 0.2. The equality (43) holds in the ring $G_2[A_2, C_1^{-1}]$.

Since $A_2 = \frac{1}{L^2}(3X + 1 - \frac{L^3}{2})$ and $\langle \rangle_{g,0}^{\text{SQ}} = \mathcal{F}_{g}^{\text{SQ}}$, the left side of (43) is, by the chain rule,
\[
\frac{1}{C_1^2} \frac{\partial \mathcal{F}_{g}^{\text{SQ}}}{\partial A_2} \in G_2[A_2, C_1^{-1}].
\]
On the right side of (43), we have
\[
\langle H \rangle_{g-i,1}^{\text{SQ}} = \mathcal{F}_{g-i,1}^{\text{SQ}}(q) = \mathcal{F}_{g-i,1}^{\text{GW}}(Q(q)),
\]
where the first equality is by definition and the second is by wall-crossing [4]. Then,
\[
\mathcal{F}_{g-i,1}^{\text{GW}}(Q(q)) = \frac{\partial \mathcal{F}_{g-i,1}^{\text{GW}}}{\partial T}(Q(q)) = \frac{\partial \mathcal{F}_{g-i}^{\text{SQ}}}{\partial T}(q)
\]
where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing [4], so we conclude
\[
\langle H \rangle_{g-i,1}^{\text{SQ}} = \frac{\partial \mathcal{F}_{g-i}^{\text{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]].
\]
Similarly, we obtain
\[
\langle H \rangle_{i,1}^{\text{SQ}} = \frac{\partial \mathcal{F}_{i}^{\text{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]],
\]
\[
\langle H, H \rangle_{g-1,2}^{\text{SQ}} = \frac{\partial^2 \mathcal{F}_{g-1}^{\text{SQ}}}{\partial T^2}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]].
\]
Together, the above equations transform (43) into exactly the holomorphic anomaly equation of Theorem 4.

\[
\frac{1}{C_1^2} \frac{\partial F^\text{SQ}}{\partial A_2} = 2 \sum_{i=1}^{g-1} \frac{\partial F^\text{SQ}_{g-1}}{\partial T} \frac{\partial F^\text{SQ}_i}{\partial T} - \frac{1}{2} \frac{\partial^2 F^\text{SQ}_{g-1}}{\partial T^2} + \frac{1}{2} \frac{\partial F^\text{SQ}_{g-1}}{\partial T} \frac{\partial T}{\partial T} (q)
\]

as an equality in \(\mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]\).

The series \(L\) and \(A_2\) are expected to be algebraically independent. Since we do not have a proof of the independence, to lift holomorphic anomaly equation to the equality

\[
\frac{1}{C_1^2} \frac{\partial F^\text{SQ}}{\partial A_2} = 2 \sum_{i=1}^{g-1} \frac{\partial F^\text{SQ}_{g-1}}{\partial T} \frac{\partial F^\text{SQ}_i}{\partial T} + \frac{1}{2} \frac{\partial^2 F^\text{SQ}_{g-1}}{\partial T^2}
\]

in the ring \(G_2[A_2, C_1^{-1}]\), we must prove the equalities

\[
\langle H \rangle^\text{SQ}_{g-1,i,1} = \frac{\partial F^\text{SQ}_{g-1}}{\partial T}, \quad \langle H \rangle^\text{SQ}_{i,1} = \frac{\partial F^\text{SQ}_i}{\partial T},
\]

\[
\langle H, H \rangle^\text{SQ}_{g-1,2} = \frac{\partial^2 F^\text{SQ}_{g-1}}{\partial T^2}
\]

in the ring \(G_2[A_2, C_1^{-1}]\). The lifting follow from the argument in Section 7.3 in [16].

We do not study the genus 1 unpointed series \(F^\text{SQ}_1(q)\) in the paper, so we take

\[
\langle H \rangle^\text{SQ}_{1,1} = \frac{\partial F^\text{SQ}_1}{\partial T}, \quad \langle H, H \rangle^\text{SQ}_{1,2} = \frac{\partial^2 F^\text{SQ}_1}{\partial T^2}
\]

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using Proposition 30.

7. Holomorphic anomaly for \(K\mathbb{P}^3\)

7.1. Overview. We fix a torus action \(T = (\mathbb{C}^*)^4\) on \(\mathbb{P}^3\) with weights\(^{12}\)

\[-\lambda_0, \ldots, -\lambda_3\]

on the vector space \(\mathbb{C}^4\). The \(T\)-weight on the fiber over \(p_i\) of the canonical bundle

\[
\mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \mathbb{P}^3
\]

\(^{12}\)The associated weights on \(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\) are \(\lambda_0, \ldots, \lambda_3\) and so match the conventions of Section 0.1.
is $-4\lambda_i$. The toric Calabi-Yau $K\mathbb{P}^3$ is the total space of (45). The basic generating series and other essential objects defined in Section 2 – Section 5 can be defined similarly for $K\mathbb{P}^3$. We will not repeat the definitions of these objects unless necessary.

7.2. I-functions.

7.2.1. Evaluations. Let $\tilde{H} \in H^+_4([\mathbb{C}^4/\mathbb{C}^*])$ and $H \in H^+_4(\mathbb{P}^3)$ denote the respective hyperplane classes. The I-function of Definition 13 for $K\mathbb{P}^3$ is evaluated in [5].

**Proposition 36.** For $t = t\tilde{H} \in H^+_*([\mathbb{C}^4/\mathbb{C}^*], \mathbb{Q})$,

\[
I(t) = \sum_{d=0}^{\infty} q^d e^{t(H+dz)/z} \frac{\prod_{k=0}^{4d-1} (-4H - kz)}{\prod_{i=0}^{3}\prod_{k=1}^{d}(H - \lambda_i + kz)}. \tag{46}
\]

We define the series $I_{i,j}$ by following expansion of the I-function after restriction $t = 0$,

\[
I|_{t=0} = 1 + \begin{array}{l}
I_{10}H/zh + I_{20}H^2/zh^2 + I_{30}H^3/zh^3 + I_{31}H^2/zh^2 + O(1/z^4).
\end{array}
\]

For example,

\[
I_{10}(q) = \sum_{d=1}^{\infty} 4^{(4d-1)!/(d!)^4} q^d \in \mathbb{C}[[q]],
\]

\[
I_{20}(q) = \sum_{d=1}^{\infty} 4^{(4d-1)!/(d!)^4} \left(4\text{Har}[4d-1] - 4\text{Har}[d]\right) q^d \in \mathbb{C}[[q]],
\]

\[
I_{21}(q) = \sum_{d=1}^{\infty} 4 s_1^{(4d-1)!/(d!)^4} \text{Har}[d] q^d \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]].
\]

Here $\text{Har}[d] := \sum_{k=1}^{d} \frac{1}{k}$.

\^\text{13} In fact the contents of Section 2 – 5 can be stated universally for all $K\mathbb{P}^n$. 
We return now to the functions $S_i(\gamma)$ defined in Section 2.3. We define the following additional series in $q$:

\[
C_1 = 1 + D I_{10}, \quad J_{10} = \frac{I_{10} + DI_{20}}{C_1}, \quad J_{11} = \frac{DI_{21}}{C_1},
\]

\[
J_{20} = \frac{I_{20} + DI_{30}}{C_1}, \quad J_{21} = \frac{I_{21} + DI_{31}}{C_1}, \quad J_{22} = \frac{I_{22} + DI_{32}}{C_1},
\]

\[
C_2 = 1 + D J_{10}, \quad K_{10} = \frac{J_{10} + DJ_{20}}{C_2},
\]

\[
K_{11} = \frac{J_{11} + DJ_{21} - (DJ_{11})J_{10}}{C_2}, \quad K_{12} = \frac{DJ_{22} - (DJ_{11}J_{11})}{C_2},
\]

\[
C_3 = 1 + D K_{10}.
\]

Here, $D = q \frac{d}{dq}$. The following relations were proven in \[21\],

\[
C_2 = C_3,
\]

\[
C_2^2 C_2^2 = (1 - 4^4 q)^{-1}.
\]

Using Birkhoff factorization, an evaluation of the series $S(H^j)$ can be obtained from the $I$-function, see \[13\]:

\[
S(1) = I,
\]

\[
S(H) = \frac{z \frac{d}{dt} S(1)}{C_1},
\]

\[
S(H^2) = \frac{z \frac{d}{dt} S(H) - (DJ_{11})S(H)}{C_2},
\]

\[
S(H^3) = \frac{z \frac{d}{dt} S(H^2) - (DK_{11})S(H^2) - (DK_{12})S(H)}{C_3}.
\]

### 7.2.2. Further calculations

Define small $I$-function

\[
\tilde{I}(q) \in H^*_T(\mathbb{P}^3, \mathbb{Q})[[q]]
\]

by the restriction

\[
\tilde{I}(q) = I(q, t)|_{t=0}.
\]

Define differential operators

\[
D = q \frac{d}{dq}, \quad M = H + zD.
\]
Applying $z \frac{d}{dt}$ to $I$ and then restricting to $t = 0$ has same effect as applying $M$ to $I$

$$\left[ \left( z \frac{d}{dt} \right)^k I \right]_{t=0} = M^k I.$$ 

The function $I$ satisfies following Picard-Fuchs equation

$$\left( \prod_{j=0}^{3} (M - \lambda_j) - 4qM(4M + z)(4M + 2z)(4M + 3z) \right)I = 0$$

implied by the Picard-Fuchs equation for $I$,

$$\left( \prod_{j=0}^{3} \left( z \frac{d}{dt} - \lambda_j \right) - q \prod_{k=0}^{3} \left( 4z \frac{d}{dt} + kz \right) \right)I = 0.$$

The restriction $I|_{H=\lambda_i}$ admits following asymptotic form

$$I|_{H=\lambda_i} = e^{\mu_i/z} \left( R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \ldots, \lambda_d)[[q]]$.

A derivation of (50) is obtained in [21] via the Picard-Fuchs equation (49) for $I|_{H=\lambda_i}$. The series $\mu_i$ and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of $z^k$. For example,

$$\lambda_i + D\mu_i = L_i,$$

$$R_{0,i} = \left( \frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)} \right)^{1/2}.$$ 

From the equations (48) and (50), we can show the series

$$\mathcal{S}_i(1) = \mathcal{S}(1)|_{H=\lambda_i} , \quad \mathcal{S}_i(H) = \mathcal{S}(H)|_{H=\lambda_i} , \quad \mathcal{S}_i(H^2) = \mathcal{S}(H^2)|_{H=\lambda_i} , \quad \mathcal{S}_i(H^3) = \mathcal{S}(H^3)|_{H=\lambda_i}$$

have the following asymptotic expansions:

$$\mathcal{S}_i(1) = e^{\mu_i/2} \left( R_{00,i} + R_{01,i}z + R_{02,i}z^2 + \ldots \right),$$

$$\mathcal{S}_i(H) = e^{\mu_i} \frac{L_i}{C_1} \left( R_{10,i} + R_{11,i}z + R_{12,i}z^2 + \ldots \right),$$

$$\mathcal{S}_i(H^2) = e^{\mu_i} \frac{L_i^2}{C_1 C_2} \left( R_{20,i} + R_{21,i}z + R_{22,i}z^2 + \ldots \right),$$

$$\mathcal{S}_i(H^3) = e^{\mu_i} \frac{L_i^3}{C_1 C_2 C_3} \left( R_{30,i} + R_{31,i}z + R_{32,i}z^2 + \ldots \right).$$

We follow here the normalization of [21]. Note

$$R_{0k,i} = R_{k,i}.$$ 

As in [21] Theorem 4, we expect the following constraints.
Conjecture 37. For all $k \geq 0$, we have

$$R_{k,i} \in G_3.$$  

Conjecture 37 is the main obstruction for the proof of Conjecture 5 and 6. By the same argument of Section 7, we obtain the following result.

Theorem 38. Conjecture 37 implies Conjecture 5 and 6.

By applying asymptotic expansions (51) to (48), we obtain the following results.

Lemma 39. We have

$$R_{1,p+1,i} = R_{0,p+1,i} + \frac{D R_{0,p,i}}{L_i},$$

$$R_{2,p+1,i} = R_{1,p+1,i} - E_{11,i} R_{1,k,i} + \frac{D R_{1,p,i}}{L_i} + \left(\frac{D L_i}{L_i^2} - \frac{A_2}{L_i}\right) R_{1,p,i},$$

$$R_{3,p+1,i} = R_{2,p+1,i} - E_{21,i} R_{2,k,i} - E_{22,i} R_{1,k,i} + \frac{D R_{2,p,i}}{L_i} + \left(2 \frac{D L_i}{L_i^2} - \frac{A_2}{L_i} - \frac{D C_2}{C_2} \right) R_{1,p,i}$$

with

$$E_{11,i} = \frac{D J_{11}}{L_i}, \quad E_{21,i} = \frac{D K_{11}}{L_i}, \quad E_{22,i} = \frac{C_2}{L_i^2} D K_{12}.$$  

7.3. Determining $DA_2$ and new series. The following relation was proven in [16].

$$A_2^2 + (L^4 - 1)A_2 + 2DA_2 - \frac{3}{16}(L^4 - 1) = 0.$$  

By the above result, the differential ring

$$G_3[A_2, DA_2, DDA_2, \ldots]$$

is just the polynomial ring $G[A_2]$. The second equation in (47) yields the following relation.

$$2A_2 + 2 \frac{DC_2}{C_2} = L^4 - 1.$$  

Denote by $\text{Coeff}(x^i y^j)$ the coefficient of $x^i y^j$ in

$$\sum_{k=0}^{3} e^{-\frac{\phi}{2} - \frac{\phi^2}{2}} S_i(\phi_k)|_{z=x} S_i(\phi_k)|_{z=y}.$$
From (18) and (51), we obtain the following equations.

\[ \text{Coeff}(x^2) - \frac{1}{2}\text{Coeff}(xy) = 0, \]
\[ \text{Coeff}(x^4) - \text{Coeff}(x^3y) + \frac{1}{2}\text{Coeff}(x^2y^2) = 0. \]

Above equations immediately yields the following relations.

\[ E_{11,i} = \frac{E_{21,i}}{2} - \frac{s_1L^2}{2C_1L_i} + \frac{s_1L^4}{2L_i}, \]
\[ E_{22,i} = \frac{L^4(s_1^2(-3 + 2C_1L^2 + C_2^2L^4) - 4s_2(-1 + C_1^2))}{8C_2^2L_i^2} - \frac{s_1(-3L^2 + C_1L^4)}{4C_1L_i}E_{21,i} - \frac{3}{8}E_{21}. \]

We define the series \( B_2 \) and \( B_4 \) which appeared in the introduction by

\[ B_2 = L_iE_{21,i}, \quad B_4 = DB_2. \]

Note that \( B_2(q), B_4(q) \in \mathbb{C}[[q]]. \)

From Lemma 39 with the relations (52), (54) and (55), we obtain results for \( \mathbb{S}(H)_{|H=\lambda_i}, \mathbb{S}(H^2)_{|H=\lambda_i}, \) and \( \mathbb{S}(H^3)_{|H=\lambda_i}. \)

**Lemma 40.** Suppose Conjecture 37 is true. Then for all \( k \geq 0, \)
we have for all \( k \geq 0, \)

\[ R_{1,k,i}, R_{2,k,i}, R_{3,k,i} \in G_3[A_2, B_2, B_4, C_1^{\pm 1}]. \]

### 7.4. Vertex, edge, and leg analysis.

By parallel argument as in Section 4, we have decomposition of the contribution to \( \Gamma \in G_{g,k}(\mathbb{P}^3) \) to the stable quotient theory of \( K\mathbb{P}^3 \) into vertex terms, edge terms and leg terms

\[ \text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{>0}} \prod_{v \in V} \text{Cont}_\Gamma^A(v) \prod_{e \in E} \text{Cont}_\Gamma^A(e) \prod_{l \in L} \text{Cont}_\Gamma^A(l). \]

The following lemmas follow from the argument in Section 5.

**Lemma 41.** Suppose Conjecture 37 is true. Then we have

\[ \text{Cont}_\Gamma^A(v) \in G_3. \]

Let \( e \in E \) be an edge connecting the \( T \)-fixed points \( p_i, p_j \in \mathbb{P}^3. \) Let the \( A \)-values of the respective half-edges be \( (k, l). \)

**Lemma 42.** Suppose Conjecture 37 is true. Then we have

\[ \text{Cont}_\Gamma^A(e) \in G_3[A_2, B_2, B_4, C_1^{\pm 1}]. \]
Lemma 43. Suppose Conjecture 37 is true. Then we have
\[
\text{Cont}_A^\Gamma(l) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}].
\]

7.5. Proof of Theorem 7. Conjecture 37 can be proven for the choices of \(\lambda_0, \ldots, \lambda_3\) such that
\[
\lambda_i \neq \lambda_j \text{ for } i \neq j,
\]
\[
4s_2^2 - s_1s_3 = 0,
\]
\[
2s_2^3 - 27s_1^2s_4 = 0.
\]
by the argument in Appendix. Hence, statement (i),
\[
\mathcal{F}_{g,a+b}[a, b](q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}],
\]
follows from the arguments in Proposition 29 and Lemmas 41 - 43.
Statement (ii), \(\mathcal{F}_{g}^{\text{SQ}}\) has at most degree \(2(3g - 3)\) with respect to \(A_2\), holds since a stable graph of genus \(g\) has at most \(3g - 3\) edges. Since
\[
\frac{\partial}{\partial T} = \frac{q}{C_1} \frac{\partial}{\partial q},
\]
statement (iii),
\[
(57) \quad \frac{\partial^k \mathcal{F}_{g}^{\text{SQ}}}{\partial T^k}(q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}],
\]
follows from divisor equation in stable quotient theory and statement (i). \(\square\)

7.6. Proof of Theorem 6: first equation. Let \(\Gamma \in \mathbb{G}_g(\mathbb{P}^3)\) be a decorated graph. Let us fix an edge \(f \in E(\Gamma)\):

- if \(\Gamma\) is connected after deleting \(f\), denote the resulting graph by
  \[
  \Gamma^0_f \in \mathbb{G}_{g-1,2}(\mathbb{P}^3),
  \]
- if \(\Gamma\) is disconnected after deleting \(f\), denote the resulting two graphs by
  \[
  \Gamma^1_f \in \mathbb{G}_{g_1,1}(\mathbb{P}^3) \quad \text{and} \quad \Gamma^2_f \in \mathbb{G}_{g_2,1}(\mathbb{P}^3)
  \]
where \(g = g_1 + g_2\).

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph \(\Gamma^0_f\) in case • should be viewed as sum of 2 graphs
\[
\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)}.
\]
Similarly, in case ••, we will sum over the ordering of \(g_1\) and \(g_2\). As usual, the summation will be later compensated by a factor of \(\frac{1}{2}\) in the formulas.
By the argument in Section 5.3, we have the following formula for the contribution of the graph $\Gamma$ to the stable quotient theory of $K\mathbb{P}^3$,

$$\text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \prod_{v \in V} \text{Cont}^A_\Gamma(v) \prod_{e \in E} \text{Cont}^A_\Gamma(e).$$

Let $f$ connect the $T$-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the $A$-values of the respective half-edges be $(k, l)$. Denote by $D_1$ the differential operator

$$\frac{L^2}{4C_1} \frac{\partial}{\partial A_2} + \frac{-2s_1L^4 - C_1(3B_2L^2 - s_1L^6)}{4C_2} \frac{\partial}{\partial B_4}.$$

By Lemma 39 and the explicit formula for $\text{Cont}^A_\Gamma(f)$ in Lemma 35, we have

$$(58) \quad D_1 \text{Cont}^A_\Gamma(f) = (-1)^{k+l} \left( \frac{L^2 L_i R_{2k-1,i} R_{1l-1,j}}{C_1^2 C_2} + \frac{L_i L_j^2 R_{1k-1,i} R_{2l-1,j}}{C_1^2 C_2} \right).$$

• If $\Gamma$ is connected after deleting $f$, we have

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} D_1 \text{Cont}^A_\Gamma(f) \prod_{v \in V} \text{Cont}^A_\Gamma(v) \prod_{e \in E, e \neq f} \text{Cont}^A_\Gamma(e)
= \text{Cont}^{\cdot}_{\Gamma(f)}(H, H^2) + \text{Cont}^{\cdot}_{\Gamma(f)}(H^2, H).$$

The derivation is simply by using (58) on the left and the argument in Proposition 30 on the right.

• If $\Gamma$ is disconnected after deleting $f$, we obtain

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} D_1 \text{Cont}^A_\Gamma(f) \prod_{v \in V} \text{Cont}^A_\Gamma(v) \prod_{e \in E, e \neq f} \text{Cont}^A_\Gamma(e)
= \text{Cont}^{\cdot}_{\Gamma(f)}(H) \text{Cont}^{\cdot}_{\Gamma(f)}(H^2) + \text{Cont}^{\cdot}_{\Gamma(f)}(H^2) \text{Cont}^{\cdot}_{\Gamma(f)}(H)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in G_9(\mathbb{P}^3)$ and using the vanishing

$$\frac{\partial \text{Cont}^A_\Gamma(v)}{\partial A_2} = 0, \quad \frac{\partial \text{Cont}^A_\Gamma(v)}{\partial B_4} = 0$$

of Lemma 41, we obtain

$$(59) \quad D_1 \langle \rangle^{\text{SQ}}_{g,0} = \sum_{i=1}^{g-1} \langle H \rangle_{g-i,1}^{\text{SQ}} \langle H^2 \rangle_{i,1}^{\text{SQ}} + \langle H, H^2 \rangle_{g-1,2}^{\text{SQ}}.$$

$^{14}$Lemma 35 is stated for $K\mathbb{P}^2$, but parallel statement holds for $K\mathbb{P}^3$. 
We have followed here the notation of Section 0.3. The equality (59) holds in the ring $G_3[A_2, B_2, B_4, B_1^\pm 1]$.

7.7. Proof of Theorem [6]: second equation. By the argument in Section 5.3, we have the following formula for the contribution of the graph $\Gamma$ to the stable quotient theory of $K\mathbb{P}^3$,

$$\text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \prod_{v \in V} \text{Cont}_\Gamma^A(v) \prod_{e \in E \setminus \{f\}} \text{Cont}_\Gamma^A(e).$$

Let $f$ connect the $T$-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the $A$-values of the respective half-edges be $(k, l)$. Denote by $D_2$ the differential operator

$$\frac{2L^2}{C_1(L^4 - 1 - 2A_2)} \frac{\partial}{\partial B_2}.$$ 

By Lemma [39] and the explicit formula for $\text{Cont}_\Gamma^A(f)$ in Lemma [35], we have

$$D_2 \text{Cont}_\Gamma^A(f) = (-1)^{k+l} \frac{2L_i L_j R_{1,k-1,i} R_{1,l-1,j}}{C_1^2}.$$ 

• If $\Gamma$ is connected after deleting $f$, we have

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} D_2 \text{Cont}_\Gamma^A(f) \prod_{v \in V} \text{Cont}_\Gamma^A(v) \prod_{e \in E \setminus \{f\}} \text{Cont}_\Gamma^A(e) = \text{Cont}_{\Gamma_f^0}(H, H).$$

The derivation is simply by using (60) on the left and the arguments in Proposition [30] on the right.

• If $\Gamma$ is disconnected after deleting $f$, we obtain

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} D_2 \text{Cont}_\Gamma^A(f) \prod_{v \in V} \text{Cont}_\Gamma^A(v) \prod_{e \in E \setminus \{f\}} \text{Cont}_\Gamma^A(e) = \text{Cont}_{\Gamma_f^1}(H) \text{Cont}_{\Gamma_f^2}(H)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in G_9(\mathbb{P}^3)$ and using the vanishing

$$\frac{\partial \text{Cont}_\Gamma^A(v)}{\partial B_2} = 0$$

\footnote{Lemma [35] is stated for $K\mathbb{P}^2$, but parallel statement holds for $K\mathbb{P}^3$.}
of Lemma 41 we obtain

\begin{equation}
D_2 \langle g,0 \rangle = \sum_{i=1}^{g-1} \langle H \rangle_{g-1,1} \langle H \rangle_{i,1} + \langle H, H \rangle_{g-1,2}.
\end{equation}

We have followed here the notation of Section 0.3. The equality (61) holds in the ring \( G_3[A_2, B_2, B_4, C_1^{\pm 1}] \).

On the right side of (61), we have

\begin{equation}
\langle H \rangle_{g-1,1} = \mathcal{F}_{g-1,1}(1,0)(q) = \mathcal{F}_{g-1,1}(1,0)(Q(q)),
\end{equation}

where the first equality is by definition and the second is by wall-crossing (7). Then,

\begin{equation}
\langle H \rangle_{g-1,1} = \frac{\partial \mathcal{F}_{g-1}(q)}{\partial T}(Q(q)) = \frac{\partial \mathcal{F}_{g-1}(q)}{\partial T},
\end{equation}

where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing (7), so we conclude

\begin{equation}
\langle H \rangle_{g-1,1} = \frac{\partial \mathcal{F}_{g-1}(q)}{\partial T} \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]].
\end{equation}

Similarly, we obtain

\begin{equation}
\langle H \rangle_{i,1} = \frac{\partial \mathcal{F}_{i}(q)}{\partial T} \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]],
\end{equation}

\begin{equation}
\langle H, H \rangle_{g-1,2} = \frac{\partial^2 \mathcal{F}_{g-1}(q)}{\partial T^2} \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]].
\end{equation}

Together, the above equations transform (61) into exactly the second holomorphic anomaly equation of Theorem 8

\begin{equation}
\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_{g-1}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-1}}{\partial T} \frac{\partial \mathcal{F}_{i}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}}{\partial T^2}.
\end{equation}

as an equality in \( \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]] \). To lift holomorphic anomaly equation to the equality

\begin{equation}
\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_{g-1}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-1}}{\partial T} \frac{\partial \mathcal{F}_{i}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}}{\partial T^2}
\end{equation}

in the ring \( G_3[A_2, B_2, B_4, C_1^{\pm 1}] \), we must prove the equalities

\begin{equation}
\langle H \rangle_{g-1,1} = \frac{\partial \mathcal{F}_{g-1}}{\partial T}, \quad \langle H \rangle_{i,1} = \frac{\partial \mathcal{F}_{i}}{\partial T},
\end{equation}

\begin{equation}
\langle H, H \rangle_{g-1,2} = \frac{\partial^2 \mathcal{F}_{g-1}}{\partial T^2}.
\end{equation}
in the ring $G_3[A_2, B_2, B_4, C_1^{+1}]$. The lifting follow from the argument in Section 7.3 in [16].

We do not study the genus 1 unpointed series $F_{1}^{SQ}(q)$ in the paper, so we take

$$
\langle H \rangle_{1,1}^{SQ} = \frac{\partial F_{1}^{SQ}}{\partial T},
\langle H, H \rangle_{1,2}^{SQ} = \frac{\partial^2 F_{1}^{SQ}}{\partial T^2}.
$$

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using the argument in Proposition 30.

8. Appendix

8.1. Overviews. In section 0.1 the equivariant Gromov-Witten invariants of the local $\mathbb{P}^n$ were defined,

$$
N_{g,d}^{GW} = \int_{[\mathcal{M}_{g}^{vir}(\mathbb{P}^n,d)]} e \left( -R\pi_{*}f^{*}O_{\mathbb{P}^n}(-n-1) \right).
$$

We associate Gromov-Witten generating series by

$$
\mathcal{F}_{g}^{GW,n}(Q) = \sum_{d=0}^{\infty} N_{g,d}^{GW} Q^d \in \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[Q]].
$$

Motivated by mirror symmetry (11, 12, 19), we can make the following predictions about the genus $g$ generating series $\mathcal{F}_{g}^{GW,n}$.

(A) There exist a finitely generated subring $G \subset \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[Q]]$ which contains $\mathcal{F}_{g}^{GW,n}$ for all $g$.

(B) The series $\mathcal{F}_{g}^{GW,n}$ satisfy holomorphic anomaly equations, i.e. recursive formulas for the derivative of $\mathcal{F}_{g}^{GW,n}$ with respect to some generators in $G$.

8.1.1. $I$-function. $I$-function defined by

$$
I_n = \sum_{d=0}^{\infty} \prod_{k=1}^{(n+1)d-1} \frac{(-n+1)H-kz}{\prod_{i=0}^{n} \prod_{k=1}^{d} (H+kz-\lambda_i)} q^d \in H^*_T(\mathbb{P}^n, \mathbb{C})[[q]],
$$

is the central object in the study of Gromov-Witten invariants of local $\mathbb{P}^n$ geometry. See [16], [17] for the arguments. Several important properties of the function $I_n$ was studied in [21] after the specialization

$$
\lambda_i = \zeta_{n+1}^i.
$$
where $\zeta_{n+1}$ is primitive $(n+1)$-th root of unity. For the study of full equivariant Gromov-Witten theories, we extend the result of [21] without the specialization (63).

8.1.2. Picard-Fuchs equation and Birkhoff factorization. Define differential operators

$$D = q \frac{d}{dq}, \quad M = H + zD.$$  

The function $I_n$ satisfies following Picard-Fuchs equation

$$\left( \prod_{i=0}^{n} (M - \lambda_i) - q \prod_{k=0}^{n} (- (n+1)M - kz) \right) I_n = 0.$$  

The restriction $I_n|_{H=\lambda_i}$ admits following asymptotic form

$$(64) \quad I_n|_{H=\lambda_i} = e^{\mu_i} \left( R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i$, $R_{k,i} \in \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[q]]$.

A derivation of (64) is obtained from [4, Theorem 5.4.1] and the uniqueness lemma in [4, Section 7.7]. The series $\mu_i$ and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of $z^k$. For example,

$$\lambda_i + D\mu_i = L_i,$$

where $L_i(q)$ is the series in $q$ defined by the root of following degree $(n+1)$ polynomial in $\mathcal{L}$

$$\prod_{i=0}^{n}(\mathcal{L} - \lambda_i) - (-1)^{n+1}q\mathcal{L}^{n+1}.$$  

with initial conditions,

$$\mathcal{L}_i(0) = \lambda_i.$$  

Let $f_n$ be the polynomial of degree $n$ in variable $x$ over $\mathbb{C}(\lambda_0, \ldots, \lambda_n)$ defined by

$$f_n(x) := \sum_{k=0}^{n} (-1)^k k s_{k+1} x^{n-k},$$

where $s_k$ is $k$-th elementary symmetric function in $\lambda_0, \ldots, \lambda_n$. The ring

$$\mathcal{G}_n := \mathbb{C}(\lambda_0, \ldots, \lambda_n)[L_0^{\pm 1}, \ldots, L_n^{\pm 1}, f_n(L_0)^{-\frac{1}{2}}, \ldots, f_n(L_n)^{-\frac{1}{2}}]$$

will play a basic role.

The following Conjecture was proven under the specialization (63) in [21, Theorem 4].
Conjecture 44. For all $k \geq 0$, we have

$$R_{k,i} \in G_n.$$ 

Conjecture 44 for the case $n = 1$ will be proven in Section 8.3. Conjecture 44 for the case $n = 2$ will be proven in Section 8.4 under the specialization (72). In fact, the argument in Section 8.4 proves Conjecture 44 for all $n$ under the specialization which makes $f_n(x)$ into power of a linear polynomial.

8.2. Admissibility of differential equations. Let $R$ be a commutative ring. Fix a polynomial $f(x) \in R[x]$. We consider a differential operator of level $n$ with following forms.

$$P(A_{lp}, f)[X_0, \ldots, X_{n+1}] = DX_{n+1} - \sum_{n \geq l \geq 0, p \geq 0} A_{lp}D^pX_{n-l},$$

where $D := \frac{d}{dx}$ and $A_{lp} \in R[x]_f := R[x][f^{-1}]$. We assume that only finitely many $A_{lp}$ are not zero.

Definition 45. Let $R_k$ be the solutions of the equations for $k \geq 0$,

$$P(A_{lp}, f)[X_k, \ldots, X_{k+n}] = 0,$$

with $R_0 = 1$. We use the conventions $X_i = 0$ for $i < 0$. We say differential equations (66) is admissible if the solutions $R_k$ satisfies for $k \geq 0$,

$$R_k \in R[x]_f.$$ 

Remark 46. Note that the admissibility of $P(A_{lp}, f)$ in Definition 45 do not depend on the choice of the solutions $R_k$.

Lemma 47. Let $f$ be a degree one polynomial in $x$. Each $A \in R[x]_f$ can be written uniquely as

$$A = \sum_{i \in \mathbb{Z}} a_if^i$$

with finitely many non-zero $a_i \in R$. We define the order $\text{Ord}(A)$ of $A$ with respect to $f$ by smallest $i$ such that $a_i$ is not zero. Then

$$P(A_{lp}, f)[X_0, \ldots, X_{n+1}] := DX_{n+1} - \sum_{n \geq l \geq 0, p \geq 0} A_{lp}D^pX_{n-l} = 0,$$
is admissible if following condition holds:

\[
\begin{align*}
\text{Ord}(A_{l0}) & \leq -2, \\
\text{Ord}(A_{l1}) & \leq 0, \\
\text{Ord}(A_{lp}) & \leq p + 1 \quad \text{for } p \geq 2.
\end{align*}
\]

(67)

Proof. The proof follows from simple induction argument. □

Lemma 48. Let \( f \) be a degree two polynomial in \( x \). Denote by

\[ R_f \]

the subspace of \( \mathbb{R}[x]_f \) generated by \( f^i \) for \( i \in \mathbb{Z} \). Each \( A \in R_f \) can be written uniquely as

\[ A = \sum_{i \in \mathbb{Z}} a_i f^i \]

with finitely many non-zero \( a_i \in \mathbb{R} \). We define the order \( \text{Ord}(A) \) of \( A \in R_f \) with respect to \( f \) by smallest \( i \) such that \( a_i \) is not zero. Then

\[
\mathcal{P}(A_{lp}, f)[X_0, \ldots, X_{n+1}] := DX_{n+1} - \sum_{n \geq l \geq 0, p \geq 0} A_{lp} D^p X_{n-l} = 0,
\]

is admissible if following condition holds:

\[
\begin{align*}
A_{lp} &= B_{lp} \quad \text{if } p \text{ is odd}, \\
A_{lp} &= \frac{df}{dx} \cdot B_{lp} \quad \text{if } p \text{ is even},
\end{align*}
\]

where \( B_{lp} \) are elements of \( R_f \) with

\[
\begin{align*}
\text{Ord}(B_{l0}) & \leq -2, \\
\text{Ord}(B_{lp}) & \leq \left\lfloor \frac{p-1}{2} \right\rfloor \quad \text{for } p \geq 1.
\end{align*}
\]

Proof. Since \( f \) is degree two polynomial in \( x \), we have

\[
\frac{d^2 f}{dx^2}, \left(\frac{df}{dx}\right)^2 \in R_f.
\]

Then the proof of Lemma follows from simple induction argument. □

8.3. Local \( \mathbb{P}^1 \).
8.3.1. Overview. In this section, we prove Conjecture 44 for the case $n = 1$. Recall the $I$-function for $KP^1$,

$$I_1(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{2d-1} (-2H - kz)}{\prod_{i=0}^{d} \prod_{k=1}^{d}(H - \lambda_i + kz)} q^d.$$  \hspace{1cm} (68)

The function $I_1$ satisfies following Picard-Fuchs equation

$$\left( (M - \lambda_0)(M - \lambda_1) - 2qM(2M + z) \right) I_1 = 0 .$$  \hspace{1cm} (69)

Recall the notation used in above equation,

$$D = q \frac{d}{dq}, \ M = H + zD.$$

The restriction $I_1|_{H=\lambda_i}$ admits following asymptotic form

$$I_1|_{H=\lambda_i} = e^{\mu_i/z} \left( R_{0,i} + R_{1,i} z + R_{2,i} z^2 + \ldots \right)$$  \hspace{1cm} (70)

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1)[[q]]$. The series $\mu_i$ and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of $z^k$ in (69). For example, we have for $i = 0, 1$,

$$\lambda_i + D\mu_i = L_i ,$$

$$R_{0,i} = \left( \frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f_1(L_i)} \right)^{\frac{1}{2}} ,$$

$$R_{1,i} = \left( \frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)} \right)^{\frac{1}{2}} .$$

$$\left( \frac{-16s_1^2s_2^2 + 88s_2^3 + (27s_1^3s_2 - 132s_1s_2^2) L_i + (-12s_1^4 + 54s_1^3s_2)L_i^2}{24s_1(L_i s_1 - 2s_2)^3} + \frac{12\lambda_1^2 - 9\lambda_1 \lambda_{i+1} + \lambda_{i+1}^2}{24(\lambda_i^2 - \lambda_i \lambda_{i+1})} \right).$$

Here $s_1 = \lambda_0 + \lambda_1$ and $s_2 = \lambda_0 \lambda_1$. In the above expression of $R_{1,i}$, we used the convention $\lambda_2 = \lambda_0$.

8.3.2. Proof of Conjecture 44. We introduce new differential operator $D_i$ defined by for $i = 0, 1$,

$$D_i = (DL_i)^{-1} D.$$

By definition, $D_i$ acts on rational functions in $L_i$ as the ordinary derivation with respect to $L_i$. If we use following normalizations,

$$R_{k,i} = f_1(L_i)^{-\frac{1}{2}} \Phi_{k,i}$$
the Picard-Fuchs equation (73) yields the following differential equations,

\[ \mathcal{D}_i \Phi_{p,i} - A_{00,i} \Phi_{p-1,i} - A_{01,i} \mathcal{D}_i \Phi_{p-1,i} - A_{02,i} \mathcal{D}^2 \Phi_{p-1,i} = 0, \]

(71)

with

\[
A_{00,i} = \frac{-s_i^3 s_j^2 + (s_i^3 s_j + 8s_i s_j^2)L_i + (2s_i^4 - 9s_i^2 s_j)L_i^2}{4(L_i s_j - 2s_j)^4},
\]

\[
A_{01,i} = \frac{2s_i s_j^2 + (-s_i^2 s_j - 8s_j^2)L_i + (-s_i^3 + 10s_i s_j)L_i^2 - s_i^2 L_i^3}{2(L_i s_j - 2s_j)^3},
\]

\[
A_{02,i} = \frac{s_j^2 - 2(s_i s_j)L_i + (s_i^2 + s_j)L_i^2 - s_i L_i^3}{(L_i s_j - 2s_j)^2}.
\]

Here \( s_k \) is the \( k \)-th elementary symmetric functions in \( \lambda_0, \lambda_1 \). Since the differential equations (71) satisfy the condition (67), we conclude the differential equations (71) is admissible.

8.3.3. Gomov-Witten series. By the result of previous subsection, we obtain the following result which verifies the prediction (A) in Section 8.1.

**Theorem 49.** For the Gromov-Witten series of \( K\mathbb{P}^1 \), we have

\[ \mathcal{F}_{g}^{GW,1}(Q(q)) \in G_1, \]

where \( Q(q) \) is the mirror map of \( K\mathbb{P}^1 \) defined by

\[ Q(q) := q \cdot \exp\left(2 \sum_{d=1}^{\infty} \frac{(2d-1)!}{(d!)^2} q^d\right). \]

Theorem 49 follows from the argument in [16]. The prediction (B) in Section 8.1 is trivial statement for \( K\mathbb{P}^1 \).

8.4. Local \( \mathbb{P}^2 \).

8.4.1. Overview. In this section, we prove Conjecture 44 for the case \( n = 2 \) with following specializations,

\[ \lambda_i \neq \lambda_j \text{ for } i \neq j, \]

(72)

\[ (\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0. \]

For the rest of the section, the specialization (72) will be imposed. Recall the \( I \)-function for \( K\mathbb{P}^2 \).

\[ I_2(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{d-1} (-3H - kz)}{\prod_{i=0}^{d} \prod_{k=1}^{d}(H - \lambda_i + kz)} q^d. \]
The function $I_2$ satisfies following Picard-Fuchs equation

\begin{equation}
(M - \lambda_0)(M - \lambda_1)(M - \lambda_2) + 3qM(3M + z)(3M + 2z)I_2 = 0
\end{equation}

Recall the notation used in above equation,

$$D = q \frac{d}{dq}, \quad M = H + zD.$$ 

The restriction $I_2|_{H=\lambda_i}$ admits following asymptotic form

$$I_2|_{H=\lambda_i} = e^{\mu_i/z} \left( R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$. The series $\mu_i$ and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of $z^k$ in (73). For example,

$$\lambda_i + D\mu_i = L_i,$$

$$R_{0,i} = \left( \frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f_2(L_i)} \right)^{\frac{1}{2}}.$$ 

8.4.2. Proof of Conjecture 44. We introduce new differential operator $D_i$ defined by

$$D_i = (DL_i)^{-1}D.$$

If we use following normalizations,

$$R_{k,i} = f_2(L_i)^{-\frac{1}{2}}\Phi_{k,i}$$

the Picard-Fuchs equation (73) yields the following differential equations,

\begin{equation}
D_i\Phi_{p,i} - A_{00,i}\Phi_{p-1,i} - A_{01,i}D_i\Phi_{p-1,i} - A_{02,i}D_i^2\Phi_{p-1,i} - A_{10,i}\Phi_{p-2,i} - A_{11,i}D_i\Phi_{p-2,i} - A_{12,i}D_i^2\Phi_{p-2,i} - A_{13,i}D_i^3\Phi_{p-2,i} = 0
\end{equation}

with $A_{jl,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[L_i, f_2(L_i)^{-1}]$. We give the exact values of $A_{jl,i}$ for reader’s convinence.

$$A_{00,i} = \frac{s_1}{9(s_1L_i - s_2)^5} \left( s_1s_2^3 + (-4s_1^2s_2^2 + 3s_3^3)L_i 
+ (-s_1^3s_2 + 12s_1s_2^2)L_i^2 + (11s_1^4 - 36s_1^2s_2)L_i^3 \right),$$
\[
A_{01,i} = \frac{-s_1}{3(s_1 L_i - s_2)^4} \left( s_2^3 - 4(s_1 s_2^2)L_i + (3s_1^2 s_2 + 9s_2^2)L_i^2 
\right.
\left. + (3s_1^3 - 21s_1 s_2)L_i^3 + 3s_1^2 L_i^4 \right),
\]
\[
A_{02,i} = \frac{-1}{3(s_1 L_i - s_2)^3} \left( s_2^3 - 5(s_1 s_2^2)L_i + 9s_1^2 s_2 L_i^2 + (-6s_1^3 - 3s_1 s_2)L_i^3 
\right. 
\left. + 6s_1^2 L_i^4 \right),
\]
\[
A_{10,i} = \frac{s_1^2 L_i}{27(s_1 L_i - s_2)^8} \left( (8s_1^2 s_2^5 - 21s_2^6) + (-48s_1^3 s_2^4 + 126s_1 s_2^5)L_i + (120s_1^4 s_2^3 
\right. 
\left. - 315s_1^2 s_2^4)L_i^2 + (-124s_1^5 s_2^2 + 264s_1^3 s_2^3 + 144s_1 s_2^4)L_i^3 
\right. 
\left. + (12s_1^6 s_2 + 153s_1^4 s_2^2 - 432s_1^2 s_2^3)L_i^4 + (60s_1^7 - 342s_1^5 s_2 
\right. 
\left. + 432s_1^3 s_2^3)L_i^5 + (-33s_1^6 + 108s_1^4 s_2)L_i^6 \right),
\]
\[
A_{11,i} = \frac{-s_1 L_i}{27(s_1 L_i - s_2)^8} \left( (8s_1^2 s_2^5 - 21s_2^6) + (-48s_1^3 s_2^4 + 126s_1 s_2^5)L_i 
\right. 
\left. + (120s_1^4 s_2^3 - 315s_1^2 s_2^4)L_i^2 + (-124s_1^5 s_2^2 + 264s_1^3 s_2^3 + 144s_1 s_2^4)L_i^3 
\right. 
\left. + (12s_1^6 s_2 + 153s_1^4 s_2^2 - 432s_1^2 s_2^3)L_i^4 + (60s_1^7 - 342s_1^5 s_2 
\right. 
\left. + 432s_1^3 s_2^3)L_i^5 + (-33s_1^6 + 108s_1^4 s_2)L_i^6 \right),
\]
\[
A_{12,i} = \frac{s_1}{9(s_1 L_i - s_2)^7} \left( - s_2^6 + 9s_1 s_2^5 L_i + (-32s_1^2 s_2^4 - 9s_2^5)L_i^2 
\right. 
\left. + (57s_1^3 s_2^3 + 60s_1 s_2^4)L_i^3 + (-48s_1^4 s_2^2 - 171s_1^2 s_2^3)L_i^4 
\right. 
\left. + (9s_1^5 s_2 + 237s_1^3 s_2^2 + 27s_1 s_2^3)L_i^5 + (9s_1^6 - 144s_1^4 s_2 - 90s_1^2 s_2^2)L_i^6 
\right. 
\left. + (9s_1^5 + 108s_1^3 s_2)L_i^7 - 18s_1^4 L_i^8 \right),
\]
\[
A_{13,i} = \frac{-3L_i^2 s_2^2 - 3L_i s_1 s_2 + s_2^2)}{27(s_1 L_i - s_2)^6} \left( -3L_i^3 s_1 + 3L_i^2 s_2^2 - 3L_i s_1 s_2 + s_2^2 \right) \right).
\]

Here $s_k$ is the $k$-th elementary symmetric functions in $\lambda_0, \lambda_1, \lambda_2$. Since the differential equations \((74)\) satisfy the condition \((67)\), we conclude that the differential equations \((74)\) is admissible.
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