Mean variance asset liability management with regime switching

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This paper is concerned with mean variance portfolio selection with liability, regime switching and random coefficients. To tackle the problem, we first study a general non-homogeneous stochastic linear quadratic (LQ) control problem for which two systems of backward stochastic differential equations (BSDEs) with unbounded coefficients are introduced. The existence and uniqueness of the solutions to these two systems of BSDEs are proved by some estimates of BMO martingales and contraction mapping method. Then we obtain the optimal state feedback control and optimal value for the stochastic LQ problem explicitly. Finally, closed form efficient portfolio and efficient frontier for the original mean variance problem are presented.

Key words. Mean variance; liability; non-homogeneous stochastic LQ problem; regime switching; unbounded coefficients

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1 Introduction

Mean variance portfolio selection theory, initiated in the seminal work pioneered by Markowitz [20] in a single period setting, concerns the optimal trade-off between the expected return and the risk of terminal wealth level when one wants to allocate his/her wealth among a basket of securities. It has become a cornerstone of classical financial theory since its inception, and is a paramount model used by hedging fund, investment bank and other financial industry. Li and Ng [15] extended Markowitz’s work to the discrete-time multi-period setting. Continual time mean

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variance problems were extensively studied by linear quadratic optimal control theory; see, e.g. in
[1], [10], [17], [18], [30], [31] and the references therein.

On the other hand, the asset liability management (ALM, for short) is one of the most important
problem in risk management and insurance. Chiu and Li [6] investigated firstly in a continuous
time setting the ALM under the mean variance criteria. Xie, Li and Wang [26] considered this
problem with liability process driven by another correlated Brownian motion. Wei and Wang [22]
found a time-consistent open-loop equilibrium strategy for the mean variance ALM problem. Zeng
and Li [28] studied this problem in a jump diffusion market. Chang [3] concerned this
problem in which interest rate follows the Vasicek model. Chen, Yang and Yin [4] generalized the model
of [6] to a setting in which the coefficients and liability process were modulated by a continuous
time Markov chain and geometric Brownian motion respectively. With liability being described as
Brownian motion with drift, Xie [25] studied mean variance ALM with deterministic and Markov
chain modeled coefficients. In the above Markov chain modulated models, the market parameters,
such as the interest rate, stock appreciation rates and volatilities are assumed to be deterministic
functions of \( t \) for each given regime \( i \).

In practice, however, these market parameters are affected by the uncertainties caused by noises.
Thus, it is too restrictive to set market parameters as constants even if the market status is known.
From practical point of view, it is necessary to allow the market parameters to depend on both the
noises and the Markov chain. In this paper, we study mean variance ALM with regime switching
and random coefficients. The liability is exogenously given by an Itô process and can not be
controlled (see Remark 2.1). By Lagrange duality, the first part of the original mean variance
ALM happens to be a stochastic linear quadratic (LQ) control problem with regime switching and
random coefficients.

Since the pioneering work of Wonham [24], stochastic LQ theory has been extensively studied
by numerous researchers. For instance, Bismut [2] was the first one who studied stochastic LQ
problems with random coefficients. In order to obtain the optimal random feedback control, he
formally derived a stochastic Riccati equation (SRE). But he could not solve the SRE in the general
case. It is Kohlmann and Tang [13], for the first time, that established the existence and uniqueness
of the one-dimensional SRE. Tang [21] made another breakthrough and proved the existence and
uniqueness of the matrix valued SRE with uniformly positive control weighting matrix. Chen,
Li and Zhou [5] studied the indefinite stochastic LQ problem which is different obviously from its
deterministic counterpart. Kohlmann and Zhou [14] established the relationship between stochastic
LQ problems and backward stochastic differential equations (BSDEs). Hu and Zhou [9] solved the
stochastic LQ problem with cone control constraint. Please refer to Chapter 6 in Yong and Zhou
[27] for a systematic accounts on this subject.

Stochastic LQ problems for Markovian regime switching system were studied in Li and Zhou
[16], Wen, Li and Xiong [23] and Zhang, Li and Xiong [29] where sufficient and necessary conditions
of the existence of optimal control, weak closed-loop solvability, open-loop solvability and closed-
loop solvability were established. But the coefficients are assumed to be deterministic functions of
$\tau$ for each given regime $i$ in the above three papers.

This paper further explores general stochastic LQ problem with regime switching and random coefficients. Compared with our previous work [8], non-homogeneous terms emerge in both the state process and cost functional in the present LQ problem. Two related systems of BSDEs are introduced: the first one is the so called system of SRE whose solvability is established by slightly modifying our previous argument in [8]. By contrast, the existing argument cannot deal with the second one because its coefficients, which depend on the solution of the first one, are unbounded. The main idea to overcome this difficulty is first to get some estimates of BMO martingales and then apply contraction mapping method. The solvability of the second one constitutes the major technique contribution of this paper. Eventually we obtain the optimal control and optimal value of the LQ problem through these two systems of BSDEs and some verification arguments. As for the original mean variance problem, we still need to find the Lagrange multiple which is achieved by splitting another related system of linear BSDEs and some delicate analysis.

This paper is organized as follows. Section 2 introduces a continuous time mean variance ALM model with regime switching and random coefficients. In Section 3, we prove the global solvability of two systems of BSDEs and a general stochastic LQ problem. In Section 4, we apply the general results to solve the mean variance ALM problem introduced in Section 2.

## 2 Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space on which are defined a standard $n$-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_n(t))'$ and a continuous-time stationary Markov chain $\alpha_t$ valued in a finite state space $\mathcal{M} = \{1, 2, \ldots, \ell\}$ with $\ell > 1$. We assume $W(t)$ and $\alpha_t$ are independent processes. The Markov chain has a generator $Q = (q_{ij})_{\ell \times \ell}$ with $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{\ell} q_{ij} = 0$ for every $i \in \mathcal{M}$. Define the filtrations $\mathcal{F}_t = \sigma\{W(s), \alpha_s : 0 \leq s \leq t\} \uparrow \mathcal{N}$ and $\mathcal{F}_t^W = \sigma\{W(s) : 0 \leq s \leq t\} \uparrow \mathcal{N}$, where $\mathcal{N}$ is the totality of all the $\mathbb{P}$-null sets of $\mathcal{F}$. 


We will use the following notation throughout the paper:

\[ L_\infty^F(\Omega; \mathbb{R}) = \{ \xi : \Omega \to \mathbb{R} \mid \xi \text{ is } F_T\text{-measurable, and essentially bounded} \}, \]

\[ L_2^2(0, T; \mathbb{R}) = \{ \phi : [0, T] \times \Omega \to \mathbb{R} \mid \phi(\cdot) \text{ is an } \{ F_t \}_{t \geq 0}\text{-adapted process with} \]

\[ \|\phi\| = \left( \mathbb{E} \int_0^T |\phi(t)|^2 dt \right)^{\frac{1}{2}} < \infty \}, \]

\[ L_2^{2, \text{loc}}(0, T; \mathbb{R}) = \{ \phi : [0, T] \times \Omega \to \mathbb{R} \mid \phi(\cdot) \text{ is an } \{ F_t \}_{t \geq 0}\text{-adapted process with} \]

\[ \int_0^T |\phi(t)|^2 dt < \infty \text{ almost surely (a.s.)} \}, \]

\[ L_\infty^F(0, T; \mathbb{R}) = \{ \phi : [0, T] \times \Omega \to \mathbb{R} \mid \phi(\cdot) \text{ is an } \{ F_t \}_{t \geq 0}\text{-adapted essentially} \]

bounded process}.

These definitions are generalized in the obvious way to the cases that \( F \) is replaced by \( F^W \) and \( \mathbb{R} \) by \( \mathbb{R}^n, \mathbb{R}^{n \times m} \text{ or } \mathbb{S}^n \), where \( \mathbb{S}^n \) is the set of symmetric \( n \times n \) real matrices. If \( M \in \mathbb{S}^n \) is positive definite (positive semidefinite), we write \( M > (\geq) 0 \). We write \( A > (\geq) B \) if \( A, B \in \mathbb{S}^n \text{ and} \)

\( A - B > (\geq) 0 \). In our argument, \( t, \omega \), “almost surely” (a.s.) and “almost everywhere” (a.e.) may be suppressed for notation simplicity in some circumstances when no confusion occurs.

Consider a financial market consisting of a risk-free asset (the money market instrument or bond) whose price is \( S_0 \) and \( m \) risky securities (the stocks) whose prices are \( S_1, \ldots, S_m \). Assume \( m \leq n \), i.e., the number of risky securities is no more than the dimension of the Brownian motion. The financial market is incomplete if \( m < n \). These asset prices are driven by stochastic differential equations (SDEs):

\[
\begin{cases}
dS_0(t) = r(t, \alpha_t)S_0(t)dt, \\
S_0(0) = s_0 > 0,
\end{cases}
\]

and

\[
\begin{cases}
dS_k(t) = S_k(t) \left( (\mu_k(t, \alpha_t) + r(t, \alpha_t))dt + \sum_{j=1}^n \sigma_{kj}(t, \alpha_t)dW_j(t) \right), \\
S_k(0) = s_k > 0,
\end{cases}
\]

where \( r(t, i) \) is the interest rate process and \( \mu_k(t, i) \) and \( \sigma_k(t, i) := (\sigma_{k1}(t, i), \ldots, \sigma_{kn}(t, i)) \) are the mean excess return rate process and volatility rate process of the \( k \)th risky security corresponding to a market regime \( \alpha_t = i \), for every \( k = 1, \ldots, m \) and \( i \in \mathcal{M} \).

Define the mean excess return vector

\[ \mu(t, i) = (\mu_1(t, i), \ldots, \mu_m(t, i))', \]
Then the surplus value of the investor
\[ \gamma(t) := (\pi_1(t), \ldots, \pi_m(t))' \]
is called a portfolio of the investor. The admissible portfolio set is defined as
\[ \mathcal{U} = L^2_{\mathcal{F}}(0, T; \mathbb{R}^m). \]

Then the investor’s asset value \( \gamma(t) \) corresponding to a portfolio \( \pi(t) \) is the unique strong solution of the SDE:
\[
\begin{align*}
\frac{d\gamma(t)}{\gamma(t)} &= \mu(t, \alpha_t)dt + \sigma(t, \alpha_t)dW(t), \\
\gamma(0) &= \gamma_0, \quad \alpha_0 = \alpha_0.
\end{align*}
\]

Besides the asset value above, the investor has to pay for some liability \( l(t) \) whose value is modeled as an Itô process
\[
\begin{align*}
dl(t) &= \rho(t, \alpha_t)l(t) - \rho(t, \alpha_t)dt, \\
l(0) &= l_0, \quad \alpha_0 = \alpha_0.
\end{align*}
\]

Then the surplus value of the investor \( X(t) := \gamma(t) - l(t) \) is governed by
\[
\begin{align*}
dX(t) &= \mu(t, \alpha_t)X(t) + \sigma(t, \alpha_t)dW(t), \\
X(0) &= x := \gamma_0 - l_0, \quad \alpha_0 = \alpha_0.
\end{align*}
\]

**Remark 2.1** The liability process is modeled as a geometric Brownian motion in [6], [4], and as a Brownian motion with drift in [3], [25], [26], [28]. As explained in [28], “The liability here is in a generalized sense. We understand it as the subtraction of the real liability and the stochastic income of the investor...A negative liability means that the stochastic income of the investor is bigger than his/her real liability”. We can also interpret the liability as the total value of the investor’s non-tradable assets.

**Assumption 1** For all \( i \in \mathcal{M} \),
\[
\begin{align*}
r(\cdot, \cdot, i), \quad b(\cdot, \cdot, i) &\in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}), \\
\mu(\cdot, \cdot, i) &\in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^m), \\
\rho(\cdot, \cdot, i) &\in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^n), \\
\sigma(\cdot, \cdot, i) &\in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{m \times n}).
\end{align*}
\]
and \( \sigma(t, i)\sigma(t, i)' \geq \delta I_m \) with some constant \( \delta > 0 \), for a.e. \( t \in [0, T] \), where \( I_m \) is the \( m \)-dimensional identity matrix.

For a given expectation level \( z \in \mathbb{R} \), the investor’s mean-variance asset-liability portfolio selection problem is

\[
\text{Minimize} \quad \text{Var}(X(T)) = \mathbb{E}[(X(T) - z)^2],
\]

\[
\text{s.t.} \quad \begin{cases}
\mathbb{E}(X(T)) = z, \\
\pi \in \mathcal{U}.
\end{cases}
\]  

(2.3)

We shall say that the mean-variance problem (2.3) is feasible for a given \( z \) if there is a portfolio \( \pi \in \mathcal{U} \) which satisfies the target constraint \( \mathbb{E}(X(T)) = z \).

The following result gives necessary and sufficient conditions for feasibility of (2.3) for any \( z \in \mathbb{R} \).

**Theorem 2.2** Suppose that Assumption 1 holds. Let \( (\psi(t, i), \xi(t, i)) \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^n), i = 1, \ldots, \ell \) be the unique solutions of systems of linear BSDEs:

\[
\begin{aligned}
d\psi(t, i) &= - \left( r(t, i)\psi(t, i) + \sum_{j=1}^\ell q_{ij}\psi(t, j) \right) dt + \xi(t, i)'dW_t, \\
\psi(T, i) &= 1, \quad \text{for all } i \in \mathcal{M}.
\end{aligned}
\]  

(2.4)

Then the mean-variance problem (2.3) is feasible for any \( z \in \mathbb{R} \) if and only if

\[
\mathbb{E} \int_0^T |\psi(t, \alpha_t)\mu(t, \alpha_t) + \sigma(t, \alpha_t)\xi(t, \alpha_t)|^2 dt > 0.
\]  

(2.5)

**Proof:** For any \( \pi \in \mathcal{U} \) and any real number \( \beta \), set a portfolio \( \pi^\beta(t) := \beta \pi(t) \). Let \( X^\beta(t) \) be the wealth process corresponding to \( \pi^\beta \). Then \( X^\beta(t) = X^0(t) + \beta Y(t) \), where

\[
\begin{aligned}
dX^0(t) &= [r(t, \alpha_t)X^0(t) + b(t, \alpha_t)]dt + \rho(t, \alpha_t)'dW(t), \\
X^0(0) &= x, \quad \alpha_0 = i_0,
\end{aligned}
\]

and

\[
\begin{aligned}
dY(t) &= [r(t, \alpha_t)Y(t) + \pi(t)'\mu(t, \alpha_t)]dt + \pi(t)'\sigma(t, \alpha_t)dW(t), \\
Y(0) &= 0, \quad \alpha_0 = i_0.
\end{aligned}
\]

We first prove the “if” part. Let \( \pi(t) = \psi(t, \alpha_t)\mu(t, \alpha_t) + \sigma(t, \alpha_t)\xi(t, \alpha_t) \), then \( \pi \in \mathcal{U} \). Applying Itô’s lemma to \( Y(t)\psi(t, \alpha_t) \), we have

\[
\begin{aligned}
\mathbb{E}(X(T)) &= \mathbb{E}(X^0(T)) + \beta \mathbb{E}(Y(T)) \\
&= \mathbb{E}(X^0(T)) + \beta \mathbb{E} \int_0^T \pi(t)'(\psi(t, \alpha_t)\mu(t, \alpha_t) + \sigma(t, \alpha_t)\xi(t, \alpha_t))dt \\
&= \mathbb{E}(X^0(T)) + \beta \mathbb{E} \int_0^T |\psi(t, \alpha_t)\mu(t, \alpha_t) + \sigma(t, \alpha_t)\xi(t, \alpha_t)|^2 dt.
\end{aligned}
\]

(2.6)
Notice that $\mathbb{E}(X^0(T))$ is a constant independent of $\pi$, then under (2.5), for any $z \in \mathbb{R}$, there exists $\beta \in \mathbb{R}$ such that $\mathbb{E}(X(T)) = z$.

Conversely, suppose that (2.3) is feasible for any $z \in \mathbb{R}$. Then for any $z \in \mathbb{R}$, there is a $\pi \in \mathcal{U}$, such that $\mathbb{E}(X(T)) = \mathbb{E}(X^0(T)) + \mathbb{E}(Y(T)) = z$. Notice that $\mathbb{E}(X^0(T))$ is independent of $\pi$, thus it is necessary that there is a $\pi \in \mathcal{U}$ such that $\mathbb{E}(Y(T)) \neq 0$. It follows from $\mathbb{E}(Y(T)) = \mathbb{E}\int_0^T \pi(t)'(\psi(t, \alpha_t)\mu(t, \alpha_t) + \sigma(t, \alpha_t)\xi(t, \alpha_t))dt$ that (2.5) is true. □

To avoid trivial cases, we assume (2.5) from now on. This allows us to deal with the constraint $\mathbb{E}(X(T)) = z$ by Lagrangian method.

We introduce a Lagrange multiplier $-2\lambda \in \mathbb{R}$ and consider the following relaxed optimization problem:

$$
\text{Minimize} \quad \mathbb{E}(X(T) - z)^2 - 2\lambda(\mathbb{E}X(T) - z) = \mathbb{E}(X(T) - (\lambda + z))^2 - \lambda^2 =: \hat{J}(\pi, \lambda),
$$

s.t. $\pi \in \mathcal{U}$.

Because Problem (2.3) is a convex optimization problem, Problems (2.3) and (2.6) are linked by the Lagrange duality theorem (see Luenberger [19])

$$
\min_{\pi \in \mathcal{U}, \mathbb{E}(X(T)) = z} \text{Var}(X(T)) = \max_{\lambda \in \mathbb{R}} \min_{\pi \in \mathcal{U}} \hat{J}(\pi, \lambda).
$$

This allows us to solve Problem (2.3) by a two-step procedure: First solve the relaxed problem (2.6), then find a $\lambda^*$ to maximize $\lambda \mapsto \min_{\pi \in \mathcal{U}} \hat{J}(\pi, \lambda)$.

### 3 A general linear quadratic control problem

Problem (2.6) is a stochastic linear quadratic control problem with indefinite running cost. We shall address ourselves a general stochastic LQ problem in this section. Consider the following $\mathbb{R}$-valued linear stochastic differential equation (SDE):

$$
\begin{aligned}
\frac{dX(t)}{dt} &= [A(t, \alpha_t)X(t) + B(t, \alpha_t)'u(t) + b(t, \alpha_t)]dt \\
&\quad + [C(t, \alpha_t)'X(t) + u(t)'D(t, \alpha_t)' + \rho(t, \alpha_t)']dW(t), \quad t \geq 0,
\end{aligned}
$$

\begin{align}
X(0) &= x, \quad \alpha_0 = i_0,
\end{align}

where $A(t, \omega, i)$, $B(t, \omega, i)$, $b(t, \omega, i)$, $C(t, \omega, i)$, $D(t, \omega, i)$, $\rho(t, \omega, i)$ are all $\{\mathcal{F}^W_t\}_{t \geq 0}$-adapted processes of suitable sizes for $i \in \mathcal{M}$, $x \in \mathbb{R}$ and $i_0 \in \mathcal{M}$ are known.

The class of admissible controls is defined as the set

$$
\mathcal{U} := L^2_{\mathbb{F}}(0, T; \mathbb{R}^m).
$$
If \( u(\cdot) \in \mathcal{U} \) and \( X(\cdot) \) is the associated (unique) solution of (3.1), then we refer to \((X(\cdot), u(\cdot))\) as an admissible pair.

The general stochastic linear quadratic optimal control problem (stochastic LQ problem, for short) is stated as follows:

\[
\begin{aligned}
\begin{cases}
\text{Minimize} & J(x, i_0, u(\cdot)) \\
\text{subject to} & (X(\cdot), u(\cdot)) \text{ admissible for (3.1)},
\end{cases}
\end{aligned}
\]  

(3.2)

where the cost functional is given as the following quadratic form

\[
J(x, i_0, u(\cdot)) := E\left[ \int_0^T \left( Q(t, \alpha_t)(X(t) - q(t, \alpha_t))^2 + (u(t) - p(t, \alpha_t))^T R(t, \alpha_t)(u(t) - p(t, \alpha_t)) \right) dt \\
+ G(\alpha_T)(X(T) - g(\alpha_T))^2 \right].
\]  

(3.3)

The associated value function is defined as

\[
V(x, i_0) := \inf_{u \in \mathcal{U}} J(x, i_0, u(\cdot)), \ x \in \mathbb{R}, \ i_0 \in \mathcal{M}.
\]

To make sure the well-posedness of Problem (3.2), we put the following assumption.

**Assumption 2**  For all \( i \in \mathcal{M} \),

\[
\begin{aligned}
A(t, \omega, i), & \ b(t, \omega, i), \ q(t, \omega, i) \in L^\infty_{F^W}(0, T; \mathbb{R}^m), \ B(t, \omega, i), \ p(t, \omega, i) \in L^\infty_{F^W}(0, T; \mathbb{R}^m), \\
C(t, \omega, i), & \ \rho(t, \omega, i) \in L^\infty_{F^W}(0, T; \mathbb{R}^n), \ D(t, \omega, i) \in L^\infty_{F^W}(0, T; \mathbb{R}^{n \times m}), \\
Q(t, \omega, i) \in L^\infty_{F^W}(0, T; \mathbb{R}^n), & \ R(t, \omega, i) \in L^\infty_{F^W}(0, T; \mathbb{R}^{m}), \\
G(\omega, i) \in L^\infty_{F^W}(\Omega; \mathbb{R}), & \ g(\omega, i) \in L^\infty_{F^W}(\Omega; \mathbb{R}).
\end{aligned}
\]

**Assumption 3**  There exists a constant \( \delta > 0 \) such that at least one of the following cases holds.

(i) Standard case: \( Q \geq 0, \ R \geq \delta I_m \) and \( G \geq \delta \).

(ii) Singular case: \( Q \geq 0, \ R \geq 0, \ G \geq \delta \) and \( D'D \geq \delta I_m \).

Under Assumption 3, clearly we have \( J(x, i_0, u(\cdot)) \geq 0 \), for all \((x, i_0, u) \in \mathbb{R} \times \mathcal{M} \times \mathcal{U}\). Problem (3.2) is said to be solvable, if there exists a control \( u^*(\cdot) \in \mathcal{U} \) such that

\[
J(x, i_0, u^*(\cdot)) \leq J(x, i_0, u(\cdot)), \ \forall \ u(\cdot) \in \mathcal{U},
\]

in which case, \( u^*(\cdot) \) is called an optimal control for Problem (3.2), and the optimal value is

\[
V(x, i_0) = J(x, i_0, u^*(\cdot)).
\]

Apparently, Problem (2.6) is a special case of Problem (3.2).
3.1 Linear BSDEs with unbounded coefficients

To tackle Problem (3.2), we first introduce the following systems of ($\ell$-dimensional) BSDE (remind that the arguments $t$ and $\omega$ are suppressed):

$$
\begin{aligned}
    dP(i) &= -\left[(2A(i) + C(i)C(i))P(i) + 2C(i)\Lambda(i) + Q(i)
               + H(P(i), \Lambda(i), i) + \sum_{j\in\mathcal{M}} q_{ij} P(j)\right]dt + \Lambda(i)'dW, \\
    P(T, i) &= G(i), \\
    R(i) + P(i)D(i)'D(i) &> 0, \quad \text{for all } i \in \mathcal{M},
\end{aligned}
$$

(3.4)

where

$$
H(t, \omega, P, \Lambda, i) = -[PB(i) + D(i)'(PC(i) + \Lambda)]'(R(i) + PD(i)'D(i))^{-1}[PB(i) + D(i)'(PC(i) + \Lambda)].
$$

To show the above BSDE has a solution in the sequel, we need the concept of BMO martingales. Here we recall some facts about BMO martingales; see Kazamaki [11]. A process $\int_0^t \Lambda(s)'dW(s)$ is a BMO martingale on $[0, T]$ if and only if its BMO$_2$ normal on $[0, T]$ is finite, namely,

$$
\left\|\int_0^t \Lambda(s)'dW(s)\right\|_{\text{BMO}_2} := \sup_{\tau \leq T} \left(\text{ess sup}_\tau \mathbb{E}\left[\int_\tau^T |\Lambda(s)|^2 ds \right] \right)^{\frac{1}{2}} < \infty,
$$

here and hereafter the sup$_{\tau \leq T}$ is taken over all $\{\mathcal{F}_t^W\}_{t \geq 0}$-stopping times $\tau \leq T$. The Doléans-Dade stochastic exponential $\mathcal{E}(\int_0^t \Lambda(s)'dW(s))$ of a BMO martingale $\int_0^t \Lambda(s)'dW(s)$ is a uniformly integrable martingale. Moreover, if $\int_0^t \Lambda(s)'dW(s)$ and $\int_0^t Z(s)'dW(s)$ are both BMO martingales, then under the probability measure $\mathbb{P}$ defined by $\frac{d\mathbb{P}}{d\mathbb{P}}|_{\mathcal{F}_T^W} = \mathcal{E}(\int_0^T Z(t)'dW(t))$, $\tilde{W}(\cdot) := W(\cdot) - \int_0^t Z(s)ds$ is a standard Brownian motion, and $\int_0^t \Lambda(s)'d\tilde{W}(s)$ is a BMO martingale. The following space plays an important role in our argument

$$
L^{2,\text{BMO}}_{\mathcal{F}_T^W}(0, T; \mathbb{R}^n) := \left\{ \phi \in L^2_{\mathcal{F}_T^W}(0, T; \mathbb{R}^n) \left| \int_0^T \phi(s)'dW(s) \text{ is a BMO martingale} \right. \right\}.
$$

The equation (3.4) is referred to as the stochastic Riccati equation. By a solution to (3.4), we mean a pair of adapted processes $(P(i), \Lambda(i))_{i \in \mathcal{M}}$ satisfying (3.4) and $(P(i), \Lambda(i)) \in L^{2,\text{BMO}}_{\mathcal{F}_T^W}(0, T; \mathbb{R}) \times L^{2,\text{BMO}}_{\mathcal{F}_T^W}(0, T; \mathbb{R}^n)$ for all $i \in \mathcal{M}$. Furthermore, a solution of (3.4) is called nonnegative (resp. uniformly positive) if $P(i) \geq 0$ (resp. $P(i) \geq c$ for some constant $c > 0$) for all $i \in \mathcal{M}$.

We shall use $c$ to represent a generic positive constant which does not depend on $t$ and can be different from line to line.

**Lemma 3.1** Under Assumptions 2 and 3, BSDE (3.4) admits a unique uniformly positive solution $(P(i), \Lambda(i))_{i \in \mathcal{M}}$. 

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Proof: According to Theorems 3.5 (resp. Theorem 3.6) of [8], there exists a unique nonnegative (resp. uniformly positive) solution \((P(i), \Lambda(i))_{i \in \mathcal{M}}\) to BSDE (3.4) under Assumptions 2 and 3 (i) (resp. 3 (ii)). Note that Assumption 3 (i) is stronger than the standard assumption in Theorem 3.5 of [8]. So it remains to show that the solution of (3.4) is actually uniformly positive under Assumptions 2 and 3 (i).

Let \(c_1 > 0\) be any constant such that \(P(i) \leq c_1\), and \(c_2 > 0\) be any constant such that
\[
2A(i) + C(i)'C(i) + q_{ii} - \frac{2c_1}{\delta}|B(i) + D(i)'C(i)|^2 > -c_2, \text{ for all } i \in \mathcal{M}.
\]

Consider the following \(\ell\)-dimensional BSDE:
\[
\begin{cases}
  dP(i) = -\left[(2A(i) + C(i)'C(i) + q_{ii})P(i) + 2C(i)\Lambda(i) + Q(i)\right]dt + \Lambda(i)'dW, \\
P(T, i) = \delta, \\
R(i) + P(i)D(i)'D(i) > 0, \text{ for all } i \in \mathcal{M}.
\end{cases}
\]

(3.5)

This is a decoupled system of BSDEs. From Theorem 4.1 and Theorem 5.2 of [9], the \(i\)th equation in (3.5) admits a unique, hence maximal solution (see page 565 of [12] for its definition) \((P(i), \Lambda(i)) \in L^\infty_F(0, T; \mathbb{R}) \times \mathcal{L}^{2,BMO}_F(0, T; \mathbb{R}^n)\), and \(P(i) \geq 0\) for all \(i \in \mathcal{M}\). From the proof of Theorem 3.5 of [8], the solution \((P(i), \Lambda(i))_{i \in \mathcal{M}}\) of (3.4) could be approximated by solutions of a sequence of BSDEs with Lipschitz generators. Thus we can use comparison theorem for multidimensional BSDEs (see for example Lemma 3.4 of [8]) and then pass to the limit to get
\[
P(i) \geq P(i), \text{ for all } i \in \mathcal{M}.
\]

(3.6)

Let \(g : \mathbb{R}^+ \rightarrow [0, 1]\) be a smooth truncation function satisfying \(g(x) = 1\) for \(x \in [0, c_1]\), and \(g(x) = 0\) for \(x \in [2c_1, +\infty)\). Notice that \(c_1 \geq P(i) \geq P(i)\), so \((P(i), \Lambda(i)) \) is still a solution of the \(i\)th equation in BSDE (3.5) when \(H(P, \Lambda, i)\) is replaced by \(H(P, \Lambda, i)g(P)\) in the generator.

Notice that for \(P = P(i), \Lambda = \Lambda(i)\), we have
\[
H(P, \Lambda, i)g(P)
\geq \frac{1}{\delta} |PB(i) + PD(i)'C(i) + D(i)'\Lambda|^2 g(P)
= -\frac{P^2}{\delta} |B(i) + D(i)'C(i)|^2 g(P) - \frac{P}{\delta} (B(i) + D(i)'C(i))'D(i)'\Lambda g(P) - \frac{1}{\delta} |D(i)'\Lambda|^2 g(P)
\geq -\frac{2c_1P}{\delta} |B(i) + D(i)'C(i)|^2 - \frac{P}{\delta} (B(i) + D(i)'C(i))'D(i)'\Lambda g(P) - \frac{1}{\delta} |D(i)'\Lambda|^2 g(P).
\]

The following BSDE
\[
\begin{cases}
  dP = -\left[ -c_2P + 2C(i)'\Lambda - \frac{P}{\delta} (B(i) + D(i)'C(i))'D(i)'\Lambda g(P) - \frac{1}{\delta} |D(i)'\Lambda|^2 g(P)\right]dt + \Lambda' dW, \\
P(T) = \delta,
\end{cases}
\]

(3.5)
admits a solution \((\delta e^{-c_2(T-t)}, 0)\). Then the maximal solution argument (Theorem 2.3 of [12]) gives
\[
P(t, i) \geq \delta e^{-c_2(T-t)} \geq \delta e^{-c_2T}.
\]
Combine with (3.6), we obtain that the solution \((P(i), \Lambda(i))\) of (3.4) is actually uniformly positive under Assumptions 2 and 3 (i).

In order to solve the non-homogeneous stochastic LQ problem (3.2), we need to consider another system of BSDEs besides BSDEs (3.4).

Let \((P(i), \Lambda(i))\) be the unique uniformly positive solution to (3.4). Set
\[
\Gamma(i) = (R(i) + P(i)D(i)'D(i))^{-1}(P(i)B(i) + D(i)'(P(i)C(i) + \Lambda(i))).
\]

We consider the following system of \((\ell\)-dimensional) linear BSDEs,
\[
\begin{aligned}
dK(i) &= -\left[[P(i)B(i) + D(i)'(P(i)C(i) + \Lambda(i))]'(R(i) + P(i)D(i)'D(i))^{-1}
\times [D(i)'(P(i)\rho(i) - L(i)) - K(i)B(i) - R(i)p(i)] + A(i)K(i) + C(i)'L(i)
\right.
\end{aligned}
\]
\[
\left. -P(i)(C(i)'\rho(i) + b(i)) - \rho(i)'\Lambda(i) + q(i)Q(i) + \sum_{j \in \mathcal{M}} q_{ij}K(j) \right] dt
\]
\[
+ L(i)'dW
\]
\[
K(T, i) = G(i)g(i), \text{ for all } i \in \mathcal{M}.
\]

Even though (3.7) is linear, the coefficients are unbounded as \(\Lambda(i)\) (hence \(\Gamma(i)\)) is unbounded. And the equations in (3.7) are coupled through the term \(\sum_{j \in \mathcal{M}} q_{ij}K(j)\). Up to our knowledge, no existing literature could be directly applied to (3.7). Next we will address ourselves to the solvability of (3.7) which is the main technique contribution of this paper.

The following lemma is called the John-Nirenberg inequality, which can be found in Theorem 2.2 of [11].

**Lemma 3.2 (John-Nirenberg Inequality)** Suppose \(\phi \in L^2_{\mathcal{F}_T}^{2,\text{BMO}}(0, T; \mathbb{R}^n)\) and
\[
\left\| \int_0^T \phi(s)'dW(s) \right\|_{\text{BMO}_2} < 1.
\]

Then for any \(\{\mathcal{F}_t^W\}_{t \geq 0}\)-stopping time \(\tau \leq T\),
\[
\mathbb{E} \left[ e^{\int_\tau^T |\phi(s)|^2 ds} \right] \leq \frac{1}{1 - \left\| \int_0^T \phi(s)'dW(s) \right\|_{\text{BMO}_2}}.
\]

From the last lemma, we immediately have the following estimate.
Lemma 3.3 Suppose \( \phi \in \mathcal{L}^2_{\mathcal{F}^W}(0, T; \mathbb{R}^n) \). Then for each constant \( p \in (0, 2) \), there exists a positive constant \( c_p \) such that
\[
E \left[ e^{\int_\tau^T |\phi(s)|^p ds} \bigg| \mathcal{F}_\tau \right] \leq c_p,
\]
for any \( \{\mathcal{F}_t^W\}_{t \geq 0} \)-stopping time \( \tau \leq T \).

Proof: Denote \( M(\cdot) = \int_0^\cdot \phi(s) dW(s) \). Let \( 0 < \varepsilon < \frac{1}{\|M\|_{\text{BMO}}} \), then \( \|\varepsilon M\|_{\text{BMO}} = \varepsilon \|M\|_{\text{BMO}} < 1 \). For each \( p \in (0, 2) \), we have
\[
0 < c = \sup_{x \geq 0} (x^p - \varepsilon^2 x^2) < \infty,
\]
so \( |x|^p \leq \varepsilon^2 x^2 + c \). Applying the John-Nirenberg inequality to \( \varepsilon M \),
\[
E \left[ e^{\int_\tau^T |\phi(s)|^p ds} \bigg| \mathcal{F}_\tau \right] \leq E \left[ e^{\int_\tau^T (\varepsilon^2 |\phi(s)|^2 + c) ds} \bigg| \mathcal{F}_\tau \right] \leq e^{\varepsilon^2 T} E \left[ e^{\int_\tau^T |\phi(s)|^2 ds} \bigg| \mathcal{F}_\tau \right] \leq \frac{e^{\varepsilon^2 T}}{1 - \|\varepsilon M\|_{\text{BMO}}}.
\]
\( \square \)

Also as a corollary of Lemma 3.2, we have

Lemma 3.4 Suppose \( \phi \in \mathcal{L}^2_{\mathcal{F}^W}(0, T; \mathbb{R}^n) \). For any \( p \geq 1 \), there is a generic constant \( K_p \) such that
\[
E \left[ \left( \int_\tau^T \phi(s)^2 ds \right)^p \bigg| \mathcal{F}_\tau \right] \leq K_p \left\| \int_0^\cdot \phi(s)'dW(s) \right\|_{\text{BMO}}^{2p},
\]
for any \( \{\mathcal{F}_t^W\}_{t \geq 0} \)-stopping time \( \tau \leq T \).

Lemma 3.5 Let \( p \in (0, 2), a, f \) are \( \mathbb{R} \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes, and \( \beta, \phi \) are \( \mathbb{R}^n \)-valued \( \{\mathcal{F}_t^W\}_{t \geq 0} \)-adapted processes such that
\[
|a| \leq |\phi|^p, \ |\beta| \leq |\phi|, \ |f| \leq |\phi|^2,
\]
and \( \phi \in \mathcal{L}^2_{\mathcal{F}^W}(0, T; \mathbb{R}^n) \). Then for any \( \xi \in \mathcal{L}^\infty_{\mathcal{F}^W}(\Omega; \mathbb{R}) \), the following 1-dimensional BSDE
\[
\begin{cases}
-dY = (aY + \beta'Z + f) dt - Z' dW, \\
Y(T) = \xi,
\end{cases}
\]  
(3.8)
admits a unique solution \( (Y, Z) \in \mathcal{L}^\infty_{\mathcal{F}^W}(0, T; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}^W}(0, T; \mathbb{R}^n) \).

Proof: Introduce the processes
\[
J(t) = \exp \left( \int_0^t a(s) ds \right),
\]
and

\[ N(t) = \mathcal{E} \left( \int_0^t \beta(s)\,dW(s) \right). \]

Note that \( N(t) \) is a uniformly integrable martingale, thus \( \tilde{W}(t) := W(t) - \int_0^t \beta(s)\,ds \) is a Brownian motion under the probability \( \tilde{P} \) defined by

\[ \frac{d\tilde{P}}{dP} |_{\mathcal{F}_T} = N(T). \]

Moreover, \( \int_0 a(s)1_0^t\,d\tilde{W}(s) \) and \( \int_0 f(s)1_0^t\,d\tilde{W}(s) \) are two BMO martingales under \( \tilde{P} \).

Set \( Y(t) = J(t)^{-1} \tilde{E}_t \left[ J(T)\xi + \int_t^T J(s)f(s)\,ds \right] \), where \( \tilde{E} \) is the expectation w.r.t. the probability measure \( \tilde{P} \). Then \( Y(T) = \xi \). Using the Cauchy-Schwartz inequality, Lemmas 3.3 and 3.4, we have

\[
|Y(t)| \leq \tilde{E}_t \left[ |\xi|e^{J_t^t a(r)dr} + \int_t^T |f(s)|e^{J_t^t a(r)dr}\,ds \right]
\leq \tilde{E}_t \left[ c e^{J_t^t |\phi(r)|pdr} + e^{J_t^t |K(r)|pdr} \int_t^T |\phi(s)|^2ds \right]
\leq c + \left( \tilde{E}_t \left[ e^{2J_t^t |\phi(r)|dr} \right] \right)^{\frac{1}{2}} \left( \tilde{E}_t \left[ \left( \int_t^T |\phi(s)|^2ds \right)^2 \right] \right)^{\frac{1}{2}}
\leq c,
\]

so \( Y \in L_\mathcal{F}_T^\infty(0,T;\mathbb{R}) \). Moreover,

\[
\tilde{E}\left( J(T)\xi + \int_0^T J(s)f(s)\,ds \right)^2 \leq \tilde{E} \left( J(T)\xi \right)^2 + \int_0^T J(s)|f(s)|^2\,ds^2
\leq 2\tilde{E}e^{J_0^T |a(r)|dr} + 2\tilde{E} \left[ \left( \int_0^T |f(s)|e^{J_0^T |a(r)|dr}\,ds \right)^2 \right]
\leq 2\tilde{E}e^{J_0^T |\phi(r)|pdr} + 2\tilde{E} \left[ \left( \int_0^T |\phi(s)|^2e^{J_0^T |\phi(r)|pdr}\,ds \right)^2 \right]
\leq c + 2\tilde{E} \left[ e^{J_0^T |\phi(r)|pdr} \int_0^T |\phi(s)|^2ds \right]^2
\leq c + \tilde{E} \left[ e^{2J_0^T |\phi(r)|pdr} \left( \int_0^T |\phi(s)|^2ds \right)^2 \right]
\leq c + \tilde{E} \left[ e^{4J_0^T |\phi(r)|pdr} + \left( \int_0^T |\phi(s)|^2ds \right)^2 \right]< \infty,
\]

where we have used the elementary inequality \( 2ab \leq a^2 + b^2 \), Cauchy-Schwartz inequality, Lemmas 3.3 and Lemma 3.4. Thus

\[
J(t)Y(t) + \int_0^t J(s)f(s)\,ds = \tilde{E}_t \left[ J(T)\xi + \int_0^T J(s)f(s)\,ds \right]
\]
is a square integrable martingale under $\tilde{P}$. By the martingale presentation theorem, there exists $\tilde{Z} \in L^2_{\mathcal{F}_{\infty}}(0, T; \mathbb{R}^n)$ such that

$$J(t)Y(t) + \int_0^t J(s)f(s)ds = J(0)Y(0) + \int_0^t \tilde{Z}(s)'d\tilde{W}(s).$$

As a consequence

$$d(J(t)Y(t)) = \tilde{Z}(t)'d\tilde{W}(t) - J(t)f(t)dt.$$

Set $Z(t) = J(t)^{-1}\tilde{Z}(t)$. By Itô’s lemma,

$$dY(t) = d(J(t)^{-1} \cdot J(t)Y(t)) = (-a(t)J(t)^{-1})J(t)Y(t)dt + J(t)^{-1}(\tilde{Z}(t)'d\tilde{W}(t) - J(t)f(t)dt)
= -a(t)Y(t)dt + Z(t)'d\tilde{W}(t) - J(t)f(t)dt
= -a(t)Y(t)dt + Z(t)'dW(t) - Z(t)'\beta(t)dt - f(t)dt.$$

Thus $(Y, Z)$ satisfies $(3.8)$. Because

$$\tilde{\mathbb{E}}_t\left[\int_\tau^T |Z(s)|^2ds\right] = \tilde{\mathbb{E}}_t\left[(\int_\tau^T Z(s)'d\tilde{W}(s))^2\right]
= \tilde{\mathbb{E}}_t\left[(Y(\tau) - \xi - \int_\tau^T (a(s)Y(s) + f(s))ds)^2\right]
\leq c + c\tilde{\mathbb{E}}_t\left[(\int_\tau^T (|\phi(s)|^p + |\phi(s)|^2)ds)^2\right]
\leq c,$$

for any stopping time $\tau \leq T$, it yields that $\int_0^T Z(s)'d\tilde{W}(s)$ is a BMO martingale under $\tilde{P}$. Consequently $\int_0^T Z(s)'dW(s)$ is a BMO martingale under $P$. This shows that $(Y, Z)$ is a solution of $(3.8)$ in $L^\infty_{\mathcal{F}_{\infty}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}_{\infty}}(0, T; \mathbb{R}^n)$.

Let us prove the uniqueness. Suppose $(Y, Z)$ and $(\hat{Y}, \hat{Z})$ are both solutions of $(3.8)$ in $L^\infty_{\mathcal{F}_{\infty}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}_{\infty}}(0, T; \mathbb{R}^n)$. Set

$$\Delta Y = Y - \hat{Y}, \Delta Z = Z - \hat{Z}.$$

Then $(\Delta Y, \Delta Z)$ satisfies the following BSDE:

$$\Delta Y(t) = \int_t^T (a(s)\Delta Y(s) + \beta(s)'\Delta Z(s))dt - \int_t^T \Delta Z(s)'dW(s).$$

By Itô’s lemma, it follows

$$J(t)\Delta Y(t) = -\int_t^T J(s)\Delta Z(s)'d\tilde{W}(s).$$

We get $\Delta Y = 0$ by taking conditional expectation $\tilde{\mathbb{E}}_t$ on both sides. Thus

$$\tilde{\mathbb{E}}_t\left[\int_t^T (J(s)\Delta Z(s))^2ds\right] = \tilde{\mathbb{E}}_t\left[\left(\int_t^T J(s)\Delta Z(s)'d\tilde{W}(s)\right)^2\right] = \tilde{\mathbb{E}}\left[(J(t)\Delta Y(t))^2\right] = 0,$$
so $\Delta Z = 0$ as $J > 0$. This completes the proof of the uniqueness. □

Consider the following system of linear BSDEs:

$$
\begin{cases}
-dK(i) = \left[ a(i)K(i) + \beta(i)'L(i) + f(i) + \sum_{j \neq i} q_{ij}K(j) \right] dt - L(i)'dW, \\
K(T, i) = \xi(i), \text{ for all } i \in \mathcal{M}.
\end{cases}
$$

**Theorem 3.6** Let $p \in (0, 2)$, $a(i), f(i)$ are $\mathbb{R}$-valued $\{\mathcal{F}^W_t\}_{t \geq 0}$-adapted processes, and $\beta(i), \phi(i)$ are $\mathbb{R}^n$-valued $\{\mathcal{F}^W_t\}_{t \geq 0}$-adapted processes such that

$$
|a(i)| \leq |\phi(i)|^p, \ |\beta| \leq |\phi|, \ |f(i)| \leq |\phi(i)|^2,
$$

and $\phi(i) \in L^2_{\mathcal{F}^W}(0, T; \mathbb{R}^n)$, for any $i \in \mathcal{M}$. Then for any $\xi(i) \in L^\infty_{\mathcal{F}^W}(\Omega; \mathbb{R})$, the following system of linear BSDEs:

$$
\begin{cases}
-dK(i) = \left[ a(i)K(i) + \beta(i)'L(i) + f(i) + \sum_{j \neq i} q_{ij}K(j) \right] dt - L(i)'dW, \\
K(T, i) = \xi(i), \text{ for all } i \in \mathcal{M},
\end{cases} \quad (3.9)
$$

admits a unique solution $(K(i), L(i))_{t \in \mathcal{M}}$ such that

$$(K(i), L(i)) \in L^\infty_{\mathcal{F}^W}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}^W}(0, T; \mathbb{R}^n), \text{ for all } i \in \mathcal{M}.$$

**Proof:** We introduce the processes

$$
J(t, i) = \exp \left( \int_0^t a(s, i)ds \right),
$$

and

$$
N(t, i) = \mathcal{E} \left( \int_0^t \beta(s, i)'dW(s) \right).
$$

Note that $N(t, i)$ is a uniformly integrable martingale, thus $\tilde{W}'(t) := W(t) - \int_0^t \beta(s, i)ds$ is a Brownian motion under the probability $\tilde{\mathbb{P}}'$ defined by

$$
\left. \frac{d\tilde{\mathbb{P}}'}{d\mathbb{P}} \right|_{\mathcal{F}^W_T} = N(T, i), \text{ for all } i \in \mathcal{M}.
$$

By Lemma 3.5, for any $U \in L^\infty_{\mathcal{F}^W}(0, T; \mathbb{R}^\ell)$ and $i \in \mathcal{M}$, the following 1-dimensional linear BSDE has a unique solution $(K(i), L(i)) \in L^\infty_{\mathcal{F}^W}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}^W}(0, T; \mathbb{R}^n),$

$$
\begin{cases}
-dK(i) = \left[ a(i)K(i) + \beta(i)'L(i) + f(i) + \sum_{j \neq i} q_{ij}U(j) \right] dt - L(i)'dW, \\
K(i, T) = \xi(i).
\end{cases}
$$

We call the map $U \mapsto (K(1), ..., K(\ell))$ as $\Theta$.  

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By Lemma 3.3, \( \mathbb{E}_t\left[e^{f_T (a(r,i))dr}\right], i \in \mathcal{M} \) are all uniformly bounded, so there exists a constant \( c_3 > 0 \) such that

\[
2\ell \max_{i,j \in \mathcal{M}} q_{ij} \max_{i \in \mathcal{M}} \operatorname{ess sup}_{(t,\omega) \in [0,T] \times \Omega} \mathbb{E}_t\left[e^{f_T (a(r,i))dr}\right] < c_3. \tag{3.10}
\]

For \( U \in L^\infty_{\mathcal{F}_T}(0,T;\mathbb{R}^\ell) \), we introduce a new norm

\[
|U|_\infty := \max_{i \in \mathcal{M}} \operatorname{ess sup}_{(t,\omega) \in [0,T] \times \Omega} e^{c_3 t}|U(t,i)|.
\]

Let \( \mathcal{B} \) be the set of \( U \in L^\infty_{\mathcal{F}_T}(0,T;\mathbb{R}^\ell) \) with \( |U|_\infty < \infty \). For any \( U, \tilde{U} \in \mathcal{B} \), set \( K = \Theta(U) \), \( \tilde{K} = \Theta(\tilde{U}) \), and

\[
\Delta K(t,i) = K(t,i) - \tilde{K}(t,i), \text{ and } \Delta U(t,i) = U(t,i) - \tilde{U}(t,i).
\]

Then by Itô’s lemma,

\[
J(t,i)\Delta K(t,i) = \mathbb{E}_t\left[\int_t^T J(s,i) \sum_{j \neq i} q_{ij} \Delta U(s,j)ds\right].
\]

It follows that

\[
e^{c_3 t}|\Delta K(t,i)| \leq e^{c_3 t} J(t,i)^{-1} \mathbb{E}_t\left[\int_t^T e^{-c_3 s} J(s,i) \sum_{j \neq i} q_{ij} e^{c_3 s} |\Delta U(s,j)|ds\right]
\]

\[
\leq \ell \max_{i,j \in \mathcal{M}} q_{ij} e^{c_3 T} \mathbb{E}_t\left[\int_t^T e^{-c_3 s} e^{f_t (a(r,i))dr} ds\right] |\Delta U|_\infty
\]

\[
\leq \ell \max_{i,j \in \mathcal{M}} q_{ij} \mathbb{E}_t\left[\int_t^T e^{-c_3 (s-t)} ds e^{f_t (a(r,i))dr} \right] |\Delta U|_\infty
\]

\[
\leq \ell \max_{i,j \in \mathcal{M}} q_{ij} \frac{1}{c_3} \mathbb{E}_t\left[e^{f_T (a(r,i))dr}\right] |\Delta U|_\infty.
\]

Thus we have

\[
|\Delta K|_\infty \leq \ell \max_{i,j \in \mathcal{M}} q_{ij} \max_{i \in \mathcal{M}} \operatorname{ess sup}_{(t,\omega) \in [0,T] \times \Omega} \mathbb{E}_t\left[e^{f_T (a(r,i))dr}\right] \frac{1}{c_3} |\Delta U|_\infty \leq \frac{1}{2} |\Delta U|_\infty,
\]

where (3.10) is used in the last inequality. Therefore, \( \Theta \) is a strict contraction mapping on \( \mathcal{B} \) endowed with the norm \( | \cdot |_\infty \). Because \( (\mathcal{B}, | \cdot |_\infty) \) is a complete metric space, the map \( \Theta \) admits a unique fixed point which is the unique solution to the \( \ell \)-dimensional BSDE (3.9). \( \square \)

**Corollary 3.7** Under Assumptions 2 and 3, BSDE (3.7) admits a unique solution \((K(i), L(i))_{i \in \mathcal{M}}\) such that

\[
(K(i), L(i)) \in L^\infty_{\mathcal{F}_T}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n), \text{ for all } i \in \mathcal{M}.
\]
Proof: Set

\[ a(i) = A(i) - B(i)'\Gamma(i) + q_{ii}, \]
\[ \beta(i) = C(i) - D(i)\Gamma(i), \]
\[ f(i) = q(i)Q(i) - P(i)(C(i)'\rho(i) + b(i)) - \rho(i)'\Lambda(i) + (P(i)D(i)'\rho(i) - R(i)p(i))'\Gamma(i), \]

then (3.7) can be rewritten as

\[
\begin{cases}
-dK(i) = \left[ a(i)K(i) + \beta(i)'L(i) + f(i) + \sum_{j \neq i} q_{ij}K(j) \right] dt - L(i)'dW, \\
K(T, i) = G(i)g(i), \text{ for all } i \in \mathcal{M},
\end{cases}
\]

whence admits a unique solution \((K(i), L(i))_{i \in \mathcal{M}}\) such that

\[
(K(i), L(i)) \in L_{F^W}^\infty(0, T; \mathbb{R}) \times L_{F^W}^{2, BMO}(0, T; \mathbb{R}^n), \text{ for all } i \in \mathcal{M},
\]

noting \((P(i), \Lambda(i)) \in L_{F^W}^\infty(0, T; \mathbb{R}) \times L_{F^W}^{2, BMO}(0, T; \mathbb{R}^n)\) from Lemma 3.1, and the coefficients satisfy the assumptions of Theorem 3.6, i.e. \(|a(i)| \leq c|\Lambda(i)|, \ |\beta(i)| \leq c|\Lambda(i)|, \ |f(i)| \leq c|\Lambda(i)|. \] □

3.2 Solution to Problem (3.2)

Theorem 3.8 Suppose that Assumptions 2 and 3 hold. Let \((P(t, i), \Lambda(t, i))_{i \in \mathcal{M}}\) and \((K(t, i), L(t, i))_{i \in \mathcal{M}}\) be the unique solutions of (3.4) and (3.7), respectively. Then Problem (3.2) has an optimal control, as a feedback function of the time \(t\), the state \(X\), and the market regime \(i\),

\[
u^*(t, X, i) = -\left( R(t, i) + P(t, i)D(t, i)'D(t, i) \right)^{-1} \\
\times \left[ \left( P(t, i)D(t, i)'C(t, i) + P(t, i)B(t, i) + D(t, i)'\Lambda(t, i) \right)X \\
+ P(t, i)D(t, i)'\rho(t, i) - K(t, i)B(t, i) - D(t, i)'L(t, i) - R(t, i)p(t, i) \right].  \] (3.11)

Moreover, the corresponding optimal value is

\[
V(x, i_0) = P(0, i_0)x^2 - 2K(0, i_0)x + \mathbb{E}[G(\alpha_T)g(\alpha_T)^2] \\
+ \mathbb{E}\int_0^T \left[ P(t, \alpha_t)\rho(t, \alpha_t)'\rho(t, \alpha_t) - 2K(t, \alpha_t)b(t, \alpha_t) - 2\rho(t, \alpha_t)'L(t, \alpha_t) \\
+ Q(t, \alpha_t)q(t, \alpha_t)^2 + p(t, \alpha_t)'R(t, \alpha_t)p(t, \alpha_t) \\
- \left[ D(t, \alpha_t)'(P(t, \alpha_t)\rho(t, \alpha_t) - L(t, \alpha_t)) - K(t, \alpha_t)B(t, \alpha_t) - R(t, \alpha_t)p(t, \alpha_t) \right]' \\
\times \left( R(t, \alpha_t) + P(t, \alpha_t)D(t, \alpha_t)'D(t, \alpha_t) \right)^{-1} \\
\times \left[ D(t, \alpha_t)'(P(t, \alpha_t)\rho(t, \alpha_t) - L(t, \alpha_t)) - K(t, \alpha_t)B(t, \alpha_t) - R(t, \alpha_t)p(t, \alpha_t) \right] \right] dt.  \] (3.12)

Lemma 3.9 Under the conditions of Theorem 3.8, \(u^*(t, X(t), \alpha_t) \in L_{F^W}^2(0, T; \mathbb{R}^m)\).
Proof: In light of the length of many equations, \("(t, X(t), \alpha_t)\) will be suppressed when no confusion occurs in the sequel. Substituting (3.11) into the state process (3.1) (with \("i\) replaced by \("\alpha_t\)\), we have

\[
\begin{align*}
\d X &= \left[ \left( A - B'(R + PD'D)\right) - 1 (PD'C + PB + D'\Lambda) \right] X \\
& \quad - B'(R + PD'D) - 1 (PD'\rho - KB - DL - Rp) + b \right] \d t \\
& \quad + \left[ \left( C - D(R + PD'D)\right) - 1 (PD'C + PB + D'\Lambda) \right] X \\
& \quad - D(R + PD'D) - 1 (PD'\rho - KB - DL - Rp) + \rho \right] \d W \\
X(0) &= x_0, \quad \alpha_0 = i_0.
\end{align*}
\]

(3.13)

By the basic theorem on PP. 756-757 of Gal'chuk [7], the SDE (3.13) admits a unique strong solution. For \((P(i), \Lambda(i))_{i \in M}\), \((K(i), L(i))_{i \in M}\), the unique solution of (3.4) and (3.7) respectively, and \(X(t)\), the solution of (3.13), applying Itô’s lemma to \(P(t, \alpha_t)X(t)^2 - 2K(t, \alpha_t)X(t)\), we have

\[
\begin{align*}
\int_0^t \left( Q(X - q)^2 + (u^* - p)'R(u^* - p) \right) \d t + P(t, \alpha_t)X(t)^2 - 2K(t, \alpha_t)X(t) \\
&= P(0, i_0)x^2 - 2K(0, i_0)x + \int_0^t \left[ P\rho' - 2 Kb - 2 L\rho + Qq^2 + p' R \rho \\
& \quad - (PD'\rho - KB - DL - Rp)'(R + PD'D)^{-1}(PD'\rho - KB - DL - Rp) \right] \d s \\
& \quad + \int_0^t \left[ 2 (PX - K)(CX + Du^* + \rho) + X^2 \Lambda - 2XL \right] \d W \\
& \quad + \int_0^t \left\{ X^2 \sum_{j,j' \in M} (P(s, j) - P(s, j'))I_{\{\alpha_{s^-} = j\}} - 2X \sum_{j,j' \in M} (K(s, j) - K(s, j'))I_{\{\alpha_{s^-} = j\}} \right\} \d \tilde{N}^{j,j},
\end{align*}
\]

where \((N^{j,j})_{j,j' \in M}\) are independent Poisson processes each with intensity \(q_{j', j}\), and \(\tilde{N}^{j,j} = N^{j,j} - q_{j', j} t, \quad t \geq 0\) are the corresponding compensated Poisson martingales under the filtration \(\mathcal{F}\).

Because \(X(t)\) is continuous, the stochastic integrals in the last equation are local martingales. Thus there exists an increasing sequence of stopping times \(\tau_n\) such that \(\tau_n \uparrow +\infty\) as \(n \to +\infty\) such that

\[
\begin{align*}
\mathbb{E} \left[ \int_0^{t \wedge \tau_n} \left( Q(X - q)^2 + (u^* - p)'R(u^* - p) \right) \d s + P(t \wedge \tau_n)X(t \wedge \tau_n)^2 - 2K(t \wedge \tau_n)X(t \wedge \tau_n) \right] \\
&= P(0, i_0)x^2 - 2K(0, i_0)x + \mathbb{E} \int_0^{t \wedge \tau_n} \left[ P\rho' - 2 Kb - 2 L\rho + Qq^2 + p' R \rho \\
& \quad - (PD'\rho - KB - DL - Rp)'(R + PD'D)^{-1}(PD'\rho - KB - DL - Rp) \right] \d s \\
& \quad + \int_0^t \left\{ X^2 \sum_{j,j' \in M} (P(s, j) - P(s, j'))I_{\{\alpha_{s^-} = j\}} - 2X \sum_{j,j' \in M} (K(s, j) - K(s, j'))I_{\{\alpha_{s^-} = j\}} \right\} \d \tilde{N}^{j,j},
\end{align*}
\]

(3.14)

for any stopping time \(t \leq T\).
Under Assumptions 2 and 3 (i), we have
\[
\mathbb{E} \left[ \int_0^{t \wedge \tau_n} \delta |u^* - p|^2 ds \right]
\]
\[
\leq \mathbb{E} \left[ \int_0^{t \wedge \tau_n} (u^* - p)^i R(u^* - p) ds + P(t \wedge \tau_n) \left( X(t \wedge \tau_n) - \frac{K(t, \alpha t)}{P(t, \alpha t)} \right)^2 \right]
\]
\[
\leq P(0, i_0) x^2 - 2K(0, i_0) x + \mathbb{E} \int_0^{t \wedge \tau_n} \left( P\rho' \rho - 2Kb - 2L'\rho + Qq^2 + p' R \rho \right) ds + \mathbb{E} \left[ \frac{K(t \wedge \tau_n)^2}{P(t \wedge \tau_n)} \right]
\]
\[
\leq P(0, i_0) x^2 - 2K(0, i_0) x + \mathbb{E} \int_0^{T} \left( P\rho' \rho - 2Kb - 2L'\rho + Qq^2 + p' R \rho \right) ds + \mathbb{E} \left[ \frac{K(t \wedge \tau_n)^2}{P(t \wedge \tau_n)} \right]
\]
\[
\leq c,
\]
where c is independent of n. Letting n \to \infty, it follows from the monotone theorem that
\[
u^*(t, X(t), \alpha t) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m).
\]
Under Assumptions 2 and 3 (ii), we have
\[
\mathbb{E} \left[ P(t \wedge \tau_n) \left( X(t \wedge \tau_n) - \frac{K(t \wedge \tau_n)}{P(t \wedge \tau_n)} \right)^2 \right]
\]
\[
\leq P(0, i_0) x^2 - 2K(0, i_0) x + \mathbb{E} \int_0^{t \wedge \tau_n} \left( P\rho' \rho - 2Kb - 2L'\rho + Qq^2 + p' R \rho \right) ds + \mathbb{E} \left[ \frac{K(t \wedge \tau_n)^2}{P(t \wedge \tau_n)} \right]
\]
\[
\leq c.
\]
By Lemma 3.1, there exists constant c_4 > 0 such that P(i) \geq c_4, for all i \in M. Therefore
\[
c_4 \mathbb{E} \left[ X(t \wedge \tau_n)^2 \right] \leq \mathbb{E} \left[ P(t \wedge \tau_n) X(t \wedge \tau_n)^2 \right]
\]
\[
\leq 2 \mathbb{E} \left[ P(t \wedge \tau_n) \left( X(t \wedge \tau_n) - \frac{K(t \wedge \tau_n)}{P(t \wedge \tau_n)} \right)^2 \right] + 2 \mathbb{E} \left[ \frac{K(t \wedge \tau_n)^2}{P(t \wedge \tau_n)} \right] \leq c.
\]
Letting n \to \infty, it follows from Fatou’s lemma that
\[
\mathbb{E} \left[ X(t)^2 \right] \leq c,
\]
for any stopping time t \leq T. This further implies
\[
\mathbb{E} \int_0^{t \wedge T} X(s)^2 ds \leq \int_0^T \mathbb{E} \left[ X(s)^2 \right] ds \leq cT.
\]
(3.15)
By Itô’s Lemma, we have
\[
X(t)^2 = x^2 + \int_0^t \left[ (u^*)' D' Du^* + 2X(D'C + B)' u^* + 2\rho' D u^*
\right.
\]
\[
+ (2A + C'C)X^2 + 2X(b + C'\rho) + \rho' \rho \] ds
\]
\[
+ \int_0^t 2X(CX + Du^* + \rho)' dW.
\]
Because $X(t)$ is continuous, it follows that $2X(CX + Du^* + \rho) \in L^2_F$ \text{loc}(0, T; \mathbb{R}^n)$. Therefore, there exists an increasing localizing sequence $\theta_n \uparrow \infty$ as $n \to \infty$, such that

$$x^2 + \mathbb{E} \int_0^{T \land \theta_n} (u^*)'D' Du^* \, ds$$

$$= \mathbb{E} \left[ X(T \land \theta_n)^2 \right] - \mathbb{E} \int_0^{T \land \theta_n} \left[ 2X(D'C + B)'u^* + 2\rho' Du^* + (2A + C'\rho)X^2 + 2X(b + C'\rho) + \rho' \rho \right] \, ds.$$

Let $\delta > 0$ be given in Assumption 3. By Assumption 2 and (3.15), the above by the elementary inequality $2ab \leq \frac{a^2}{\delta} + \frac{b^2}{2}$ leads to

$$\delta \mathbb{E} \int_0^{T \land \theta_n} |u^*(s, X(s), \alpha_s)|^2 \, ds$$

$$\leq c + c\mathbb{E} \int_0^{T \land \theta_n} \left[ X(s)^2 + 2|X(s)| + 1 \right] |u^*| \, ds$$

$$\leq c + \frac{4\epsilon^2}{\delta} + (c + \frac{4\epsilon^2}{\delta}) \mathbb{E} \int_0^{T \land \theta_n} X(s)^2 \, ds + \frac{\delta}{2} \mathbb{E} \int_0^{T \land \theta_n} |u^*(s, X(s), \alpha_s)|^2 \, ds$$

$$\leq c + \frac{\delta}{2} \mathbb{E} \int_0^{T \land \theta_n} |u^*(s, X(s), \alpha_s)|^2 \, ds.$$

After rearrangement, it follows from the monotone convergence theorem that

$$u^*(t, X(t), \alpha_t) \in L^2_F(0, T; \mathbb{R}^m).$$

\[\square\]

**Proof of Theorem 3.8**: The proof is similar to that of Theorem 4.2 of [8] via applying Itô’s Lemma to $P(t, \alpha_t)X(t)^2 - 2K(t, \alpha_t)X(t)$, so we leave the details to the diligent readers. \[\square\]

**Remark 3.10** Set $(h(t, i), \eta(t, i)) = \left( \frac{K(t,i)}{P(t,i)} - \frac{K(t,i)A(t,i)}{P(t,i)^2} + \frac{L(t,i)}{P(t,i)} \right)$, then $(h(t, i), \eta(t, i))_{i \in M}$ is the solution of the following $\ell$-dimensional linear BSDE

$$dh(i) = \left\{ \left[ A(i) + C(i)'C(i) + C(i)'\frac{\Lambda(i)}{P(t)} + \frac{Q(i)}{P(t)} - C(i)'D(i)\Gamma(i) \right] h(i) - \left[ C(i) + \frac{\Lambda(i)}{P(t)} - D(i)\Gamma(i) \right] \eta(i) - (D(i)'\rho(i) - \frac{R(i)p(i)}{P(t)})\Gamma(i) - \frac{q(i)Q(i)}{P(t)} \right\} dt + \eta(i)'dW,$$

$$h(T, i) = g(i), \text{ for all } i \in M. \quad (3.16)$$

Applying Itô’s Lemma to $K(t, \alpha_t)h(t, \alpha_t)$ on $[0, T]$, the optimal value (3.12) could be represented by
Solution to problem (2.3)

In this section we apply the general results obtained in the previous section to Problem (2.3). In this case, \(A = r, B = \mu, C = 0, D' = \sigma, Q = 0, p = 0, G = 1\). Recall that "\((t, \alpha_t)\)" or "\((t, i)\)" are often suppressed where no confusion occurs for simplicity.

In this case, (3.4) becomes

\[
\begin{align*}
\frac{dP(i)}{dt} &= -\left[2rP(i) - \frac{1}{P(i)}(P(i)\mu + \sigma\Lambda(i))^\prime(\sigma\sigma')^{-1}(P(i)\mu + \sigma\Lambda(i)) + \sum_{j \in M} q_{ij}P(j)\right]dt + \Lambda(i)dW, \\
P(T, i) &= 1, \\
P(i, t) > 0, \text{ for all } i \in M,
\end{align*}
\]

(4.1)

(3.7) becomes

\[
\begin{align*}
\frac{dK(i)}{dt} &= -\left[(r - \frac{1}{P(i)}(P(i)\mu + \sigma\Lambda(i))^\prime(\sigma\sigma')^{-1}\mu)K(i) - \frac{1}{P(i)}(P(i)\mu + \sigma\Lambda(i))^\prime(\sigma\sigma')^{-1}\sigma L(i) \\
&\quad + (P(i)\mu + \sigma\Lambda(i))^\prime(\sigma\sigma')^{-1}\sigma \rho - P(i)b - \rho'\Lambda(i) + \sum_{j \in M} q_{ij}K(j)\right]dt + L(i)dW, \\
K(T, i) &= \lambda + z, \text{ for all } i \in M,
\end{align*}
\]

(4.2)
and (3.16) becomes
\[
\begin{align*}
\frac{dh(i)}{dt} &= \left\{ rh(i) + \mu'(\sigma')^{-1}\sigma\eta(i) + \frac{1}{\rho(i)}\Lambda(i)'(\sigma'(\sigma')^{-1}\sigma - I_n)\eta(i) \\
&\quad - \left[ \frac{1}{P(t)}(P(i)\mu + \sigma\Lambda(i))'\sigma'(\sigma')^{-1}\sigma\rho - b - \frac{1}{P(t)}\rho\Lambda(i) \right] \\
&\quad + \frac{1}{P(t)} \sum_{j \in \mathcal{M}} q_{ij}P(j)(h(i) - h(j)) \right\} dt + \eta(i)'dW, \\
\end{align*}
\]
(4.3)
\]
\[
\begin{align*}
h(T, i) &= \lambda + z, \quad \text{for all } i \in \mathcal{M}.
\end{align*}
\]

**Theorem 4.1** Suppose that Assumption 1 holds. Let \((P(t, i), \Lambda(t, i))_{i \in \mathcal{M}}\) and \((K(t, i), L(t, i))_{i \in \mathcal{M}}\) be the unique solutions of (4.1) and (4.2), respectively. Then Problem (2.6) has an optimal control, as a feedback function of the time \(t\), the state \(X\), and the market regime \(i\),
\[
\pi^*(t, X, i) = -\frac{1}{P(t, i)}(\sigma(t, i)\sigma(t, i)' - 1) \left[ (P(t, i)\mu(t, i) + \sigma(t, i)\Lambda(t, i))X \\
+ P(t, i)\sigma(t, i)\rho(t, i) - (K(t, i)\mu(t, i) + \sigma(t, i)L(t, i)) \right].
\]
Moreover, the corresponding optimal value is
\[
\min_{\pi \in \mathcal{U}} \hat{J}(\pi, \lambda) = P(0, i_0)x^2 - 2K(0, i_0)x + (\lambda + z)^2 - \lambda^2 + \mathbb{E} \int_0^T \left[ P\rho' - 2(Kb + \rho'L) \\
- \frac{1}{P}(P\sigma\rho - (K\mu + \sigma L))'(\sigma\rho - (K\mu + \sigma L)) \right] dt.
\]

**Remark 4.2** The optimal control and optimal value in Theorem 4.1 can be rewritten in terms of the unique solution \((h(t, i), \eta(t, i))_{i \in \mathcal{M}}\) of (4.3):
\[
\pi^*(t, X, i) = -\frac{1}{P(\sigma\rho)}(P\sigma + \sigma\Lambda)(X - h) + P\sigma\rho - P\sigma\eta,
\]
and
\[
\min_{\pi \in \mathcal{U}} \hat{J}(\pi, \lambda) = P(0, i_0)(x - h(0, i_0))^2 - \lambda^2 + \mathbb{E} \int_0^T \sum_{j \in \mathcal{M}} q_{ij}P(j)(h(t, \alpha_i) - h(t, j))^2 dt \\
+ \mathbb{E} \int_0^T P(\rho - \eta)'(I_n - \sigma'(\sigma\rho)^{-1}\sigma)(\rho - \eta) dt.
\]
(4.4)

**Remark 4.3** If \(m = n = 1\) and \(r(\cdot, i), \mu(\cdot, i), \sigma(\cdot, i), b(\cdot, i), \rho(\cdot, i)\) are deterministic functions of \(t\) for all \(i \in \mathcal{M}\), then \(\Lambda(i) = L(i) = \eta(i) \equiv 0\). Accordingly, (4.1) (4.2), (4.3) and (4.4) degenerate to the ODEs (17), (18), (23) and Eq. (26) in Xie [25] respectively with \(\rho(t) \equiv 1\) (Here we take the notation of \(\rho(t)\) used in [25]).

By the Lagrange duality relationship (2.7), we need to find \(\lambda^* \in \mathbb{R}\) which attains the optimal value \(\min_{\pi \in \mathcal{U}} \hat{J}(\pi, \lambda)\). Notice that \((h(t, i), \eta(t, i))_{i \in \mathcal{M}}\) depends on \(\lambda\), we need to separate \(\lambda\) from the equations of \((h(t, i), \eta(t, i))_{i \in \mathcal{M}}\).
Let \((h_1(t,i), \eta_1(t,i))\in M\) and \((h_2(t,i), \eta_2(t,i))\in M\) be, respectively, the unique solutions of the following two systems of linear BSDEs,

\[
\begin{align*}
\begin{cases}
\frac{dh_1(i)}{dt} = & \left\{ rh_1(i) + \mu'(\sigma'^{-1})\sigma\eta_1(i) + \frac{1}{P(i)}\Lambda(i)'(\sigma'^{-1})\sigma - \sigma R\eta_1(i) \\
& - \left[ \frac{1}{P(i)}(P(i)\mu + \sigma\Lambda(i))'(\sigma'^{-1})\sigma - \sigma - \frac{1}{P(i)}\rho'\Lambda(i) \right] \\
& + \frac{1}{P(i)} \sum_{j\in M} q_{ij} P(j)(h_1(i) - h_1(j)) \right\} dt + \eta_1(i)'dW,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
\frac{dh_2(i)}{dt} = & \left\{ rh_2(i) + \mu'(\sigma'^{-1})\sigma\eta_2(i) + \frac{1}{P(i)}\Lambda(i)'(\sigma'^{-1})\sigma - \sigma R\eta_2(i) \\
& + \frac{1}{P(i)} \sum_{j\in M} q_{ij} P(j)(h_2(i) - h_2(j)) \right\} dt + \eta_2(i)'dW,
\end{cases}
\end{align*}
\]

Then by uniqueness of the solution of (4.3), it is not hard to verify \(h = h_1 + (\lambda + z)h_2\) and \(\eta = \eta_1 + (\lambda + z)\eta_2\). For notation simplicity, we denote

\[
P_0 := P(0,i_0), \quad h_{1,0} := h_1(0,i_0), \quad h_{2,0} := h_2(0,i_0).
\]

Then from (4.4), we have

\[
\begin{align*}
\min_{\pi\in\mathcal{D}} \hat{J}(\pi, \lambda) = & P_0 \left( x - h_{1,0} - (\lambda + z)h_{2,0} \right)^2 - \lambda^2 + E \int_0^T \sum_{j\in M} q_{\alpha i j} P(j)(h_1(\alpha_t) - h_1(j))^2 dt \\
& + (\lambda + z)^2 E \int_0^T \sum_{j\in M} q_{\alpha i j} P(j)(h_2(\alpha_t) - h_2(j))^2 dt \\
& + 2(\lambda + z) E \int_0^T \sum_{j\in M} q_{\alpha i j} P(j)(h_1(j) - h_1(\alpha_t))(h_2(\alpha_t) - h_2(j)) dt \\
& + E \int_0^T \left[ P(\rho - \eta_1 - (\lambda + z)\eta_2)'(I_n - \sigma'(\sigma'^{-1})\sigma)(\rho - \eta_1 - (\lambda + z)\eta_2) \right] dt \\
= & -(1 - P_0 h_{2,0}^2 - M_1)\lambda^2 + 2M_2 + (P_0 h_{2,0}^2 + M_1)z - P_0 h_{2,0}(x - h_{1,0})^2,
\end{align*}
\]

where

\[
M_1 := E \int_0^T \sum_{j\in M} q_{\alpha i j} P(j)(h_2(\alpha_t) - h_2(j))^2 dt + E \int_0^T P h_{2}(I_n - \sigma'(\sigma'^{-1})\sigma)\eta_2 dt,
\]

\[
M_2 := E \int_0^T \sum_{j\in M} q_{\alpha i j} P(j)(h_1(\alpha_t) - h_1(j))(h_2(\alpha_t) - h_2(j)) dt - E \int_0^T P(\rho - \eta_1)'(I_n - \sigma'(\sigma'^{-1})\sigma)\eta_2 dt,
\]

\[
M_3 := E \int_0^T \sum_{j\in M} q_{\alpha i j} P(j)(h_1(\alpha_t) - h_1(j))^2 dt + E \int_0^T P(\rho - \eta_1)'(I_n - \sigma'(\sigma'^{-1})\sigma)(\rho - \eta_1) dt.
\]
By Theorem 5.11 of [8], \(0 < P_0h_{2,0}^2 + M_1 < 1\). Thus \(\lambda \mapsto \min_{\pi \in \mathcal{U}} \hat{J}(\pi, \lambda)\) is a strictly concave function, so its stationary point
\[
\lambda^* = \frac{M_2 + (P_0h_{2,0}^2 + M_1)z - P_0h_{2,0}(x - h_{1,0})}{1 - P_0h_{2,0}^2 - M_1}
\]
is the unique maximizer, which leads to
\[
\max_{\lambda \in \mathbb{R}} \min_{\pi \in \mathcal{U}} \hat{J}(\pi, \lambda) = \frac{P_0h_{2,0}^2 + M_1}{1 - P_0h_{2,0}^2 - M_1} z^2 + \frac{2}{1 - P_0h_{2,0}^2 - M_1} \left[ M_2 - P_0h_{2,0}(x - h_{1,0}) \right] z
\]
\[
+ M_3 + P_0(x - h_{1,0})^2 + \frac{[M_2 - P_0h_{2,0}(x - h_{1,0})]^2}{1 - P_0h_{2,0}^2 - M_1}
\]
\[
= \frac{P_0h_{2,0}^2 + M_1}{1 - P_0h_{2,0}^2 - M_1} \left( z - \frac{P_0h_{2,0}(x - h_{1,0}) - M_2}{P_0h_{2,0}^2 + M_1} \right)^2
\]
\[
- \frac{[M_2 - P_0h_{2,0}(x - h_{1,0})]^2}{P_0h_{2,0}^2 + M_1} + M_3 + P_0(x - h_{1,0})^2.
\]
The above analysis boils down to the following theorem.

**Theorem 4.4** The optimal portfolio of Problem (2.3) corresponding to \(\mathbb{E}(X(T)) = z\), as a feedback function of the time \(t\), the wealth level \(X\), and the market regime \(i\), is
\[
\pi^*(t, X, i) = -\frac{1}{P} (\sigma \sigma')^{-1} \left[ (P \mu + \sigma L)(X - h_1 - (\lambda^* + z)h_2) + P \sigma \rho - P \sigma \eta \right],
\]
where
\[
\lambda^* = \frac{M_2 + (P_0h_{2,0}^2 + M_1)z - P_0h_{2,0}(x - h_{1,0})}{1 - P_0h_{2,0}^2 - M_1}.
\]
The mean-variance frontier is
\[
\text{Var}(X(T)) = \frac{P_0h_{2,0}^2 + M_1}{1 - P_0h_{2,0}^2 - M_1} \left( \mathbb{E}(X(T)) - \frac{P_0h_{2,0}(x - h_{1,0}) - M_2}{P_0h_{2,0}^2 + M_1} \right)^2
\]
\[
- \frac{[M_2 - P_0h_{2,0}(x - h_{1,0})]^2}{P_0h_{2,0}^2 + M_1} + M_3 + P_0(x - h_{1,0})^2
\]
with \(0 < P_0h_{2,0}^2 + M_1 < 1\).

**Remark 4.5** If there is no liability, i.e. \(b(t, i) \equiv 0\), \(\rho(t, i) \equiv 0\), then \(h_1(t, i) \equiv 0\), \(\eta_1(t, i) \equiv 0\), \(M_2 = M_3 = 0\) and Theorem 4.4 degenerates to Theorem 5.11 of [8].

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