A numerical method for calculating round slabs based on generalized equations of the finite difference method

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Abstract. Round slabs are widely used in construction equipment. Currently, many different special structures are being built that have a circular shape in the plan. The article provides an analysis of thin isotropic circular slab in pure bending, the action of concentrated forces and slab loaded by a distributed load, supported at the centre. Different boundary conditions are considered. The initial differential equations of slab bending in polar coordinates are approximated by generalized equations of the finite difference method (MCR). The algorithm allows you to take into account the final discontinuities of the desired function, the right part of the original differential equations, as well as discontinuities of derivatives of these functions without thickening the grid and using contour points. It should be noted that for the Central point of the plate, differential bending equations and their numerical approximation in Cartesian coordinates are used, which makes it possible to determine the forces and movements at this point at any load.

The calculation algorithm is reduced to solving a system of differential-algebraic equations composed for field and contour points. The results of the calculation of slabs for these effects are presented. The reliability of the solution is confirmed by the convergence of results on some grids when the step is reduced, by comparing some solutions with existing ones, and by performing static calculation checks. The advantage of this method is the ability to implement solutions with a small number of partitions.

1. Introduction
The purpose of this paper is to apply the generalized equations of the finite difference method (MCR) to the calculation of round isotropic slabs for pure bending, concentrated force, and a plate with a support centre, thus, a numerical solution is used for the calculation. The algorithm allows taking into account the final discontinuities of the desired function, the right part of the original differential equations, and the discontinuities of the derivatives of these functions without thickening the grid and using contour points. The solution of the problems was performed on some grids, the convergence and comparison with existing solutions were numerically investigated, and static calculation checks were performed.

The paper [1] is based on the calculation of round slabs of constant stiffness using generalized equations of the finite difference method. This article is a continuation of publications [2], [3]. [1] contains an extensive bibliography of works that use generalized MCR equations to solve problems of statics, dynamics, and stability of slabs and shells. Analytical calculation of round slabs and solution in rows is given in [4]. In [5], Legendre functions, orthogonal Chebyshev polynomials, and
hypergeometric series are involved in the problems of bending round slabs of constant and variable stiffness under the action of various types of discontinuous loads. The finite element method, which is currently used as the main one for structural analysis, is implemented in computational complexes, for example, SCAD and LIRA.

2. Methods

The resolving differential equation for the transverse bending of thin isotropic slabs of the fourth-order [4] is presented in [1] concerning the dimensionless unknowns \( m \) and was a system of two equations (3.4.1), (3.4.2).

\[
\frac{\partial^2 m}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial m}{\partial \rho} + \frac{\partial^2 m}{\partial \eta^2} = -p ; \quad (1) \quad \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial w}{\partial \rho} + \frac{\partial^2 w}{\partial \eta^2} = -m , \tag{2}
\]

where \( M = \frac{M^{(r)}+M^{(s)}}{1+\mu} \); \( w = \frac{W}{q_0 \rho^2} \); \( p = \frac{q}{q_0} \); \( f = F/\rho; \rho = \frac{r}{a} \).

The numerical solution of the problem is built on a grid with steps \( h \) and \( \tau \) respectively, in the radial and ring directions. In Fig. 1 shows a fragment of a grid; the numbers I, II, III, IV indicate the numbers of elements having a common point \( i, j \).
For the central point where \( \rho_1 = 0 \), the numerical equations are written in the Cartesian coordinate system (3.1.24), (3.1.25) [2]. We give these equations, taking into account the fact that:

\[
\bar{m} = \frac{m}{h^2}, \bar{w} = \frac{w}{h^4}:
\]

\[
m_{i-1,j} + m_{i,j-1} - 4m_{i,j} + m_{i,j+1} + m_{i+1,j} + \frac{h}{2}\left(t_{-uv}\Delta m^\xi_{v} + t_{-uv}\Delta m^\eta_{v} + t_{-uv}\Delta m^\bar{\eta}_{v} + t_{-uv}\Delta m^\bar{\eta}_{v}\right) = 0
\]  

\[
= -\frac{h^2}{4}(t_{p_\eta} + t_{p_\eta} + t_{p_\eta} + t_{p_\eta})
\]

\[
w_{i-1,j} + w_{i,j-1} - 4w_{i,j} + w_{i,j+1} + w_{i+1,j} = -m_{i,j}h^2
\]  

(8)

(9)

Here \( t_{-uv}\Delta m^\xi_{v} \), \( t_{-uv}\Delta m^\eta_{v} \) - linear loads.

Equations (6) - (9), compiled for the calculated points of the field, should be solved taking into account the boundary conditions.

Let us proceed to the difference approximation of the boundary conditions for hinge anchoring on the contour, assuming that at the point, \( j \) the moment is set \( m_{i,j}(\rho) = 1 \). From the conditions on the circuit \( \frac{\partial^2 w}{\partial \eta^2} = 0 \) and equations (2), (4) at the point \( i,j \) we have \( m_{i,j} + \frac{1-\mu}{\rho_i} \frac{\partial w}{\partial \rho} = 1 \). Now, taking into account the expression for \( \frac{\partial w}{\partial \rho_{i,j}} \) (3.4.10) [1] we get: \( m_{ij}(1-\alpha h) - \frac{2\alpha}{h} w_{i-1,j} = 1 \), where

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**Figure 1**. Fragment of a grid with calculated points
\( \alpha = \frac{1 - \mu}{\rho} \)  

(10) With a hard pinch on the circuit \( w_{ij} = \frac{\partial w}{\partial \rho_{ij}} = 0 \) and from the expression for \( \frac{\partial w}{\partial \rho_{ij}} \) we get: \( \frac{2}{h} w_{i+1,j} = -hm_{ij} \) (11).

2.1 Example No. 1.

Calculate a pivotally supported round slab loaded with moments along the contour \([01][02][03]\) (Fig.2), that is, being in a state of pure bending. In this case, the problem becomes one-dimensional and the equations simplify. Write down the equations for the points of the field (6), (7) and the contour (10). The equations for the Central point (8), (9) are written in the Cartesian coordinate system, which are two mutually perpendicular diameters. As an illustration, we present the equations for the field point (4,1), center (1,1), and contour (5,1) at \( h=1/4, [01] 0.3, [02]=0.75 \).

From the solution, we get: at all calculated points \( m_{i,j}=1.5382 \). Since, at the central point (1,1), the moments in the annular and radial directions are equal from (3) \( m_{i,1}^{(\rho)} = m_{i,1}^{(n)} = 1 \) follows, on the circuit with known values of \( m_{5,1} \) and \( m_{5,1}^{(\rho)} \) we also calculated from (3). \( m_{5,1}^{(n)} = 1 \). The same result is obtained for all points of the field, it coincides with smaller steps, \( h = 1/2 \) and \( h= 1/3 \). The value of the
deflection in the centre $w_{1,1} = 0.3845$ coincides with the solution [4]. $w = \frac{5+\mu}{8+64(1+\mu)}$. The exact solution is obtained by the numerical method.

2.2 Example No. 2.

Calculation of a circular slab pinched along the contour for the action of the dimensionless concentrated force $f = 1$ applied in the centre. We approximate the force by a load of the “cross” type, which is distributed within a step according to a linear law (Fig. 3). Determining the ordinate of the linear load from the relation \(I/2 = \frac{i-l}{l} \Delta m_{1,1}^\xi * h * 4 = 1\), we obtain $\Delta m_{1,1}^\xi = 1/2h$, and the term taking into account the linear loads in (8) $h/2 (\frac{i-l}{l} \Delta m_{1,1}^\xi * 4) = 1$. The result obtained, where the total jump is equal to the effective force, applies only to the linear approximation of the load. Further, for the field points, equations (6), (7) are compiled, for the centre point (8), (9) and the contour (11).

![Figure 3.](image)

a) a slab clamped along the contour under the action of a concentrated force;  
b) approximation of concentrated force.

Table 1 shows the values of the generalized moments in the centre, on the contour and deflection at different steps, as well as the solution obtained by the finite element method in the SCAD calculation complex.

|          | $h \frac{1}{2}$ | $h = \frac{1}{3}$ | $h \frac{1}{4}$ | MKE (SCAD) |
|----------|----------------|------------------|----------------|------------|
| $m_{1,1}$ | 0.267          | 0.319            | 0.3556         | 0.384      |
| $m_{51,1}$| -0.067         | -0.064           | -0.0635        | -0.074     |
| $w_{1,1}$ | 0.025          | 0.0205           | 0.01865        | 0.0194     |

Table 1 illustrates the convergence of the solution with decreasing step.
2.3 Example No. 3
Let's consider the calculation of a circular slab with point support in the centre, loaded with a uniformly distributed load p=1 (Fig. 4). We give two solutions. The first method is a solution using the canonical equation of the force method. We remove the support and apply the desired reaction R, which is approximated by linear loads, as in the previous example. Having determined the deflection in the centre of the slab from a uniformly distributed load \( w_{1,1}(p) = 0.01754 \) (the solution was obtained using the generalized MKR equations at \( h = 1/4 \)) and from \( R = 1 \) \( w_{1,1}(R) = 0.01865 \), we obtain the value of the desired reaction R from the condition that the deflection on the central support vanishes to \( 0.01865 \times R + 0.01754 = 0; R = 0.9404 \). The values of moments and deflections are obtained by the method of superposition.

The second method is reduced to calculating a slab with a support in the center on the combined action of a uniformly distributed load and the desired reaction of the support R, which is approximated according to (Fig. 3.) The reaction is included in equation (8). The deflection in the center of the slab is zero \( w_{1,1} = 0 \). The results of the two solutions are the same, they are: \( m_{1,1} = -0.2073; m_{2,1} = 0.01212; m_{3,1} = 0.04359; m_{4,1} = 0.01247; m_{5,1} = -0.06332; R = 0.9402; w_{1,1} = 0; w_{2,1} = 0.00324; w_{3,1} = 0.00381; w_{4,1} = 0.00192 \).

Here are the final values of the moments: \( m_{1,1}^{(\rho)} = m_{1,1}^{(\eta)} = -0.1347 \);
\( m_{5,1}^{(\rho)} = -0.06332; m_{5,1}^{(\eta)} = -0.0189; m_{3,1}^{(\rho)} = 0.0499; m_{3,1}^{(\eta)} = 0.01497 \).

The diagram of moments in the radial direction is shown in Figure 4.

![Figure 4](image-url)

a) a slab with support in the centre
b) diagram \( m^{(\rho)} \) in the radial direction
Perform a static check and project all forces on an axis perpendicular to the slab plane. We define the dimensionless transverse force on the circuit \( m_{l,j}^\rho \) according to the formula (3.4.9) [1], which for the axisymmetric problem has the form:

\[
\frac{2}{h} m_{l-1,j}^\rho - \frac{2}{h} m_{l,j}^\rho + 2 \left( 1 + \frac{h}{2\pi l} \right) m_{l,j}^\rho = -\frac{h}{2} (p_l^l + p_l^{ll}) \]   (12).

Knowing \( m_{5,1}^\rho \); \( m_{4,1}^\rho \) we get \( m_{5,1}^\rho = -0.3805 \). The sum of the transverse forces on the circuit and the reaction of the support \( m_{5,1}^\rho 2\pi \rho_5 R = 3.32 \), the resultant of the distributed load \( \rho_5^2 = 3.14 \). The equilibrium equations are satisfied with an accuracy of 5%, which indicates sufficient accuracy for a numerical solution.

3. Conclusion
An algorithm for calculating round plates is developed based on the generalized equations of the finite difference method. The algorithm is illustrated by some examples that show not only good convergence but also sufficient accuracy of the solution with a small number of partitions. The method can be recommended for use in engineering calculations, when checking calculations and as an additional calculation option, along with other methods.

References

[1] Gabbasov R.F., Gabbasov A.R., Filatov V.V. Numerical construction of discontinuous solutions to problems of structural mechanics.- M. : ASV, 2008. - 280p.-

[2] Gabbasov R.F., Hoang T.A., Uvarova N.B., Lipatova O.N. Calculation of round slabs of constant stiffness at local loads // Industrial and civil construction. 2015 No. 3 - pp. 20-23.

[3] Uvarova N.B., Nikitenko M.A. Application of generalized equations of the finite difference method to the calculation of round slabs for linear loads//Scientific Review. 2017 No. 6 -pp. 39-43.

[4] Timoshenko S.P., Voikovsky-Krieger S. Slabs and shells/trans. from English - M.: Nauka, 1966.- 635p.

[5] Koreneva E.B. Analytical methods for calculating slabs of variable thickness and their practical applications.- M. : ASV, 2009.- 238p.

[6] Varvak P.M. The grid method in the problems of calculating building structures.-M.: Stroyizdat, 1977. p. 154.

[7] Andreev V.I., Yazeyev B.M., Chepurnenko A.S. Axisymmetric bending of a round flexible slab during creep// Bulletin of MGSU. 2014. No 5. pp. 16-24.

[8] Chizhevsky K. G. Calculation of round and ring plates. - Leningrad, 1977. p. 210.

[9] Zolotov A.B., Akimov P.A., Sidorov V.N. Mozgaleva M.L.: Numerical and analytical methods for calculating building structures. M. : ASV. 2009. p. 336.

[10] Calculation of the foundation slabs of the MKE="/"Young Scientist". -.2016. No 1. pp. 163-174.

[11] Strength, stability, fluctuations, t.1. Handbook edited by N. A. Birger, Ya. G. Panovko.-M: Engineering, 1968. p. 831.

[12] Smirnov V.A. Calculation of slabs of complex shape-M.: Stroyizdat, 1978. p. 154.

[13] Andreev VI, Yazeyev BM, Chepurnenko AS On the Bending of a Thin Slab at Nonlinear Creep//Advanced Materials Research. 2014. Vol. 900. pp. 707-710

[14] Sulain Abo Diab, Ibraheem Hassam "Geometrical nonlinear analysis of thin slabs using modified finite element method", Tishreen University Journal for Research and Scientific Studies-Engineering Sciences Vol. (93) No. (4), pp. 175-195, 2017.