Surreal Analysis:
An Analogue of Real Analysis for Surreal Numbers

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Abstract
Surreal numbers, which were discovered as part of analyzing combinatorial games, possess a rich numerical structure of their own and share many arithmetic and algebraic properties with real numbers. In order to develop the theory of surreal numbers beyond simple arithmetic and algebra, mathematicians have initiated the formulation of surreal analysis, the study of surreal functions and calculus operations. In this paper, we extend their work with a rigorous treatment of transcendental functions, limits, derivatives, power series, and integrals. In particular, we propose surreal definitions of three new analytic functions using truncations of Maclaurin series. Using a new representation of surreals, we present formulae for limits of sequences and functions (hence derivatives). Although the class of surreals is not Cauchy complete, we can still characterize the kinds of surreal sequences that do converge, prove the Intermediate Value Theorem, and establish the validity of limit laws for surreals. Finally, we show that some elementary power series and infinite Riemann sums can be evaluated using extrapolation, and we prove the Fundamental Theorem of Calculus for surreals so that surreal functions can be integrated using antidifferentiation. Extending our study to defining other analytic functions, evaluating power series in generality, finding a consistent method of Riemann integration, proving Stokes’ Theorem to further generalize surreal integration, and solving differential equations remains open.

1. Introduction
Since their invention by John Conway in 1972, surreal numbers have intrigued mathematicians who wanted to investigate the behavior of a new number system. Even though surreal numbers were constructed out of attempts to describe the endgames of two-player combinatorial games like Go and Chess, they are a number system in their own right and have many properties in common with real numbers. Conway demonstrated in [Con01] that out of a small collection of definitions, numerous arithmetic and algebraic similarities could be found between reals and surreals. Using a creation process, starting with the oldest number (called “0”) and progressing toward more non-trivial numbers, Conway proved that the surreals contain both the reals and the ordinals. After defining basic arithmetic operations (comparison, negation, addition, and multiplication) for surreals, he showed that the surreals contain never-before-seen numbers, such as $\omega^5 - (\omega + 3\pi)^2 \times \omega^{-\omega}$, that arise out of combining reals and ordinals ($\omega$ is the first transfinite ordinal). By determining the properties of surreal arithmetic operations, Conway studied the algebraic structure of surreals, concluding that the surreals form an object (called “No”) that shares all properties with a totally ordered field, except that its elements form a proper class. With surreal arithmetic and algebra in place, developing surreal analysis, the study of functions and calculus, is the next step in building

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the theory of surreal numbers. Below, we discuss earlier work on each of these aspects of analysis and introduce our own results.

The study of surreal functions began with polynomials, which were constructed using the basic arithmetic operations that Conway introduced in [Con01]. Subsequently, Gonshor found a definition of \( \exp(x) \) that satisfies such fundamental properties as \( \exp(x + y) = \exp(x) \cdot \exp(y) \) for all \( x, y \in \mathbb{N} \) [Gon86]. Moreover, Kruskal defined \( 1/x \), and Bach defined \( \sqrt{x} \) [Con01]. In this paper, we show that Gonshor’s method for defining \( \exp(x) \) can be utilized to define \( \arctan(x) \), \( -\log(1 - x) \), and \( \frac{1}{1-x} \), as is independently observed by Costin in [Cos12].

Regarding surreal calculus, surreal sequences are restricted to a limit-ordinal number of terms in the literature, but such sequences are only “convergent” (to numbers) when their actual limits are approximated [All85]. Moreover, earlier work has not defined the limit, and hence the derivative, of a surreal function. Using a new representation (called “universal”) of surreals, we show in this paper how to define convergence for surreal sequences. Although we show that \( \mathbb{N} \) is not Cauchy complete, we characterize the kinds of surreal sequences that do converge, and we establish the Intermediate Value Theorem for surreals without using completeness. We also present a method of finding the limit of surreal functions and demonstrate that this method allows derivatives to be computed correctly. Finally, we discuss consequences of our definition of limits, such as the validity of limit laws.

Conway and Norton initiated the study of surreal calculus by introducing a preliminary analogue of Riemann integration on surreals [Con01]. The “Conway-Norton” integral failed to have properties standard in real calculus, however, such as the translation invariance property: \( \int_a^b f(x)dx = \int_{a-t}^{b-t} f(x+t)dx \), for any surreal function \( f(x) \) and \( a, b, t \in \mathbb{N} \). While Fornasiero fixed this issue in [For04], the new integral, like its predecessor, yields \( \exp(\omega) \) instead of the desired \( \exp(\omega) - 1 \) for \( \int_0^\omega \exp(x)dx \). The difficulty of creating a definition of integration for surreals is attributed in [Cos10] to the fact that the topological space of surreals is totally disconnected when the standard notion of openness is used; i.e. given any open interval \( (a, b) \in \mathbb{N} \), \( (a, b) = (a, g) \cup (g, b) \), where \( g \in (a, b) \) is a gap and the intervals \( (a, g) \) and \( (g, b) \) are open under the standard notion of openness [Mun00]. In this paper, we present a method of evaluating certain infinitely long “Riemann” sums by extrapolation from the naturals to the ordinals, and we show that this method can correctly integrate polynomials and exponential functions.\(^1\) We also prove that the Fundamental Theorem of Calculus holds for surreals as long as we have a definition of integration that satisfies certain necessary properties, so once an appropriate definition of integration is found, surreal functions can be integrated using the method of antidifferentiation. A method of Riemann integration that works for all functions over all intervals is, however, yet to be discovered.

Other earlier work on surreal calculus has concentrated on power series. Since Conway found a way of expressing each surreal number as a Conway normal form, Alling and Costin have both pursued methods of using normal forms to evaluate power series for known surreal functions [All87, Cos12]. Progress using this method is currently hindered by the inability to find a correct definition of integration on surreals. While the applicability of our method of integration to evaluating power series remains to be studied, we present in this paper an extrapolative method of evaluating certain power series as a possible alternative approach.

There are numerous motivations for studying surreal analysis. Because it has many interesting similarities to real analysis, surreal analysis may also have applications in science, engineering, and other areas of mathematics. Real analysis is a powerful branch of mathematics that has influenced numerous fields, and if fully developed, a theory of surreal analysis might lead to even

\(^1\)Because the surreals contain the ordinals, “Riemann” sums of infinite length are considered, unlike in real analysis, where the term “Riemann” requires sums of finite length.
more mathematical advancements. Particularly, Costin has hypothesized that using normal forms
to define surreal power series may allow currently unsolvable ordinary and partial differential equa-
tions (DEs) to be solved [Cos12]. Instead of approximating the solutions to such DEs, scientists,
engineers, and mathematicians would then be able to determine these solutions exactly, a result
that would impact all of their respective disciplines.

To summarize our results, we enlarge the list of known surreal functions by adding three new
definitions, and we demonstrate that limits of surreal sequences can be evaluated in a manner
similar to what is done in real analysis. Furthermore, we establish methods of evaluating limits,
derivatives, and integrals of surreal functions that produce results analogous to those obtained in
real analysis. By studying the consequences of our definitions, we show that the surreals possess
analytic connections to real numbers along with arithmetic and algebraic similarities. Because we
have consistent surreal definitions of basic calculus operations, we believe that our methods can be
applied to yield solutions of currently unsolvable DEs.

The rest of this paper is organized as follows. Section 2 discusses definitions and basic proper-
ties of surreals, and Section 3 introduces our definitions for three new surreal functions. Section 4
motivates and explains our approach to developing surreal calculus using universal representations,
and Section 5 explains our method of evaluating limits of sequences and discusses the characteris-
tics of convergent sequences. Section 6 discusses limits and derivatives of functions, includes the
Intermediate Value Theorem, and also considers limit laws, and Section 7 concerns our methods of
evaluating power series and integrals, both with Riemann sums and with the Fundamental Theo-
rem of Calculus. Finally, Section 8 concludes the paper with a summary of our results as well as a
discussion of open problems.

2. Definitions and Basic Properties

In this section, we review all basic definitions and properties of surreals and introduce our own
definitions and conventions. Throughout the rest of the paper, unqualified terms such as “number,”
“sequence,” and “function” refer to surreal objects only. Any reference to real objects will include
the descriptor “real” to avoid ambiguity. For an easy introduction to surreal numbers, see [Knu74].

2.1. Numbers

Conway constructed numbers recursively, as described in the following definition:

Definition 1 ([Con01]). (1) Let L and R be two sets of numbers. If there do not exist a ∈ L and
b ∈ R such that a ≥ b, there is a number denoted as {L | R} with some name x. (2) For every
number named x, there is at least one pair of sets of numbers (Lx, Rx) such that x = {Lx | Rx}.

As is suggested by their names, Lx is the left set of x, and Rx is the right set of x. Conway also
represents the number x = {Lx | Rx} as x = {xL | xR}, where the left options xL and right options
xR run through all members of Lx and Rx, respectively. (We explain the meaning of assigning a
name x to a form {L | R} later.) Having introduced the construction of numbers, we now consider
properties of surreals. Let No be the class of surreal numbers, and for all a ∈ No, let No< a be the
class of numbers < a and No> a be the class of numbers > a. The following are basic arithmetic
properties of numbers:

Definition 2 ([Con01]). Let x1, x2 ∈ No. Then,

\[ x_1 + x_2 = \begin{cases} x_1 + x_2 & \text{if } x_1, x_2 \geq 0 \\ x_1 + x_2 & \text{if } x_1, x_2 < 0 \\ \text{undefined} & \text{otherwise} \end{cases} \]

Moreover, solving surreal DEs is another method of finding definitions of new surreal functions.
1. **Comparison:** $x_1 \leq x_2$ iff (no $x_1^L \geq x_2$ and no $x_2^R \leq x_1$); $x_1 \geq x_2$ iff $x_2 \leq x_1$; $x_1 = x_2$ iff ($x_1 \geq x_2$ and $x_1 \leq x_2$); $x_1 < x_2$ iff ($x_1 \leq x_2$ and $x_1 \neq x_2$); $x_1 > x_2$ iff $x_2 < x_1$.

2. **Negation:** $-x_1 = \{-x_1^R \mid -x_1^L\}$.

3. **Addition:** $x_1 + x_2 = \{x_1^L + x_2, x_1 + x_2^L \mid x_1^R + x_2, x_1 + x_2^R\}$.

4. **Multiplication:** $x_1 \times x_2 = \{x_1^L x_2 + x_1 x_2^L - x_1^L x_2^L, x_1 x_2^R + x_1 x_2^R - x_1^R x_2^R \mid x_1^L x_2^R + x_1 x_2^R - x_1^L x_2^L, x_1^R x_2 + x_1 x_2^L - x_1^R x_2^L\}$.

The first part of Definition 2 yields the following theorem relating $x, x^L,$ and $x^R$:

**Theorem 3 ([Con01]).** For all $x \in \mathbb{N}$, $x^L < x < x^R$.

Let $\text{On}$ be the class of ordinals, and for all $\alpha \in \text{On}$, let $\text{On}_{<\alpha}$ be the set of ordinals $\prec \alpha$ and $\text{On}_{>\alpha}$ be the class of ordinals $\succ \alpha$. Of relevance is the fact that $\text{On} \subset \mathbb{N}$; in particular, if $\alpha \in \text{On}$, $\alpha$ has the representation $\alpha = \{\text{On}_{<\alpha} \mid \}$. Ordinals can be combined to yield “infinite numbers,” and the multiplicative inverses of such numbers are “infinitesimal numbers.” Representations of the form $\{L \mid R\}$ are known as genetic formulae. The name “genetic formula” highlights the fact that numbers can be visualized as having birthdays; i.e. for every $x \in \mathbb{N}$, there exists $\alpha \in \text{On}$ such that $x$ has birthday $\alpha$ (Theorem 16 of [Con01]). We write $b(x) = \alpha$ if the birthday of $x$ is $\alpha$. Because of the birthday system, the form $\{L \mid R\}$ represents a unique number, as stated below:

**Theorem 4 ([Con01]).** Let $x \in \mathbb{N}$. Then, $x = \{L_x \mid R_x\}$ iff $x$ is the oldest number greater than the elements of $L_x$ and less than the elements of $R_x$.

Because numbers are defined recursively, mathematical induction can be performed on the birthdays of numbers; i.e. we can hypothesize that a statement holds for older numbers and use that hypothesis to show that the statement holds for younger numbers.

We now demonstrate how a form $\{L \mid R\}$ can have a name $x$. To do this, we need two tools: (1) Basic arithmetic properties; and (2) $\mathbb{R} \subset \mathbb{N}$. The first tool is established in Definition 2. The second tool results from the construction of numbers. We now illustrate the construction process:

1. **Day 0:** $0 = \{\}\$ is taken as a “base case” for the construction of other numbers.

2. **Day 1:** 0 can belong in either the left or right set of a new number, so we get two numbers, named so: $1 = \{0\}$ and $-1 = \{\emptyset\}$.

3. **Day 2:** We can now use the numbers 0, ±1 in the left and right sets of newer numbers still, which we name so: $2 = \{1\}$, $1/2 = \{0 \mid 1\}$, $-2 = \{-1\}$, and $-1/2 = \{-1 \mid 0\}$.

4. All dyadic rationals are created on finite days.

5. **Day $\omega$:** All other reals are created, so $\mathbb{R} \subset \mathbb{N}$.

It is easy to show that for any $\{L \mid R\}, \{L' \mid R'\} \in \mathbb{N}$ such that their names, say $a = \{L \mid R\}, b = \{L' \mid R'\}$, satisfy $a, b \in \mathbb{R}$, an arithmetic property governing the reals $a, b$ also holds for the surreals $\{L \mid R\}, \{L' \mid R'\}$. For example, the sum of two surreals equals the sum of their names; i.e. $\{0\} = 1$ and $\{1\} = 2$, so $\{0\} + \{0\} = 1 + 1 = 2 = \{1\}$, where we use part 3 of Definition 2 to do the surreal addition. Therefore, assigning names like 1 to $\{0 \mid \}$ and 2 to $\{1 \mid \}$ does make sense.

In [Con01], Theorem 21 states that every number can be uniquely represented as a formal sum over ordinals $\sum_{i \in \text{On}_{<\alpha}} r_i \cdot \omega^i$, where the coefficients $r_i$ satisfy $r_i \in \mathbb{R}$. This representation is called the **normal form** of a number.

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3The method by which these transfinite sums are evaluated is not relevant to the rest of the paper but is discussed thoroughly in [Con01].
2.2. Gaps

Unlike its real analogue, the surreal number line is riddled with gaps, which are defined as follows:

**Definition 5 ([Con01])**. Let L and R be two classes of numbers such that $L \cup R = \mathbb{N}$. If there do not exist $a \in L$ and $b \in R$ such that $a \geq b$, the form $\{L \mid R\}$ represents a gap.

Gaps are Dedekind sections of $\mathbb{N}$. The Dedekind completion of $\mathbb{N}$, which contains all numbers and gaps, is denoted $\mathbb{N}^\text{D}$. Basic arithmetic operations (except for negation) on $\mathbb{N}^\text{D}$ are different from those on $\mathbb{N}$ [For04]. Three gaps worth identifying are (1) $\text{On} = \{\mathbb{N}\}$, the gap larger than all surreals (surreal version of infinity); (2) $\text{Off} = -\text{On}$, the gap smaller than all surreals (surreal version of $-\infty$); and (3) $\infty = \{+\text{finite and - numbers} \mid +\text{infinite numbers}\}$, the object called infinity and denoted $\infty$ in real analysis. The gap $\text{On}$ is important for the purpose of evaluating limits of sequences and functions. Throughout the rest of the paper, we say that a sequence is of length $\text{On}$ if its elements are indexed over all elements of the proper class of ordinals $\text{On}$.

The real number line does not have gaps because it is Dedekind complete. The definition of gaps is not Dedekind complete. The definitions of functions are also inductive; the left and right options of $f$ for all such $x$, we have $f^L(x, x^L, x^R, f(x^L), f(x^R)) < f^R(x, x^L, x^R, f(x^L), f(x^R))$.

For example, $f(x) = x^2$ is represented as $f(x) = \{2xx^L - x^L^2, 2xx^R - x^R^2 \mid xx^L + xx^R - x^L x^R\}$, where the left options $f^L$ can be either $2xx^L - x^L^2$ or $2xx^R - x^R^2$ and the right options $f^R$ can be $xx^L + xx^R - x^L x^R$.

The definitions of functions are also inductive; the left and right options of $f$ are “simpler” than $f$ itself. In particular, $f^L$ and $f^R$ can be “simpler” than $f$ in three ways: (1) Some option of $f(x)$ is $f(x^L)$ or $f(x^R)$ ($f$ evaluated at an older number); (2) the functions $f^L$, $f^R$ are themselves “simpler” than $f$ (e.g. $f$ is transcendental, while $f^L$, $f^R$ are rational); and (3) $f$ satisfies a combination of both (1) and (2). In Section 3, we use the fact that functions are defined inductively to illustrate the validity of our new function definitions.

It is important for the purpose of studying surreal analysis to define what it means for a surreal function to be continuous. To this end, we present our definition of continuity as follows, after first introducing our definition of open classes in $\mathbb{N}$. Notice that open classes are the surreal analogues of the open sets discussed in real analysis.

**Definition 7.** An interval is open if it (1) has endpoints in $\mathbb{N} \cup \{\text{On, Off}\}$; and (2) does not contain its endpoints.\(^4\) A class is open if it has the form $\bigcup_{i \in I} A_i$, where $I$ is a proper set and the $A_i$ are open intervals.

\(^4\)We use the notation $(a, b)$ to denote an open interval and the notation $[a, b]$ to denote a closed interval.
Remark. In Definition 7 above, it is important to note that the open intervals whose union is an open class must be indexed over a proper set. This stipulation is crucial to the compactness arguments we present in Section 7. Also, as an example, notice that the interval \((\text{Off}, \infty)\) is open, even though it has \(\infty\) as an endpoint, because it can be expressed as a union over the (proper) set of integers: \((\text{Off}, \infty) = \bigcup_{i \in \mathbb{N}} (\text{Off}, i)\). On the other hand, notice that the interval \((\infty, \text{On})\) is not open because it cannot be expressed as a union over a proper set. Finally, observe that our notion of openness is not equivalent to locality; i.e. it is not equivalent to the following statement: “A space \(S\) is open if every point in \(S\) has a neighborhood contained in \(S\).” For example, the interval \((\infty, \text{On})\) does satisfy the requirement that every point in the interval has a neighborhood in it, but according to our definition, this interval is not open. Nonetheless, our notion of openness does imply locality.

Definition 8. Let \(A\) be a subinterval of \(\text{No}\), and let \(f : A \to \text{No}\) be a function. Then \(f\) is continuous on \(A\) if for any class \(B\) open in \(\text{No}\), \(f^{-1}(B)\) is open in \(A\).

3. Three New Surreal Functions

In this section, we propose definitions of three new surreal functions, namely \(\arctan(x)\), \(n\log(x) = -\log(1 - x)\), and \(\frac{1}{1-x}\), using integer truncations of their respective Maclaurin series, and we show that our definitions allow the new surreal functions to match their real analogues on their domains.

Maclaurin expansions are useful for defining surreal functions [Gon86]. Given a real analytic function \(f(z)\), \(n \in \mathbb{N}\), and \(z \in \text{No}\), let \([z]_n\) denote the \(n\)-truncation of the Maclaurin expansion of \(f(z)\); i.e. \([z]_n = \sum_{i=0}^{n} \frac{f^{(i)}(0)z^i}{i!}\). Also, let \(s(f(x))\) denote the surreal analogue of the real function \(f(x)\) just in this section.

Definition 9. On the domain \((-\frac{1}{\infty}, 1 + \frac{1}{\infty})\) we define the following functions:

\[
\begin{align*}
s(\arctan(x)) &= \left\{-\frac{\pi}{4}, s(\arctan(x^L)) + \left[\frac{x - x^L}{1 + xx^L}\right]_{4n+3}, s(\arctan(x^R)) + \left[\frac{x - x^R}{1 + xx^R}\right]_{4n+1}\right\}; \\
s(\arctan(L)) &= \left[\frac{x - x^R}{1 + xx^R}\right]_{4n+3}, s(\arctan(L)) - \left[\frac{x - x^L}{1 + xx^L}\right]_{4n+1}; \\
s(n\log(x)) &= \left\{s(n\log(x^L)) + \left[\frac{x - x^L}{1 - x^L}\right]_n, s(n\log(x^R)) + \left[\frac{x - x^R}{1 - x^R}\right]_{2n+1}\right\}; \\
s(n\log(R)) - \left[\frac{x - x^L}{1 - x^L}\right]_n, s(n\log(L)) - \left[\frac{x - x^R}{1 - x^R}\right]_{2n+1}\right\}; \\
s(\frac{1}{1-x}) &= \left\{1/2, s\left(\frac{1}{1-x^L}\right), (x-x^L)s\left(\frac{1}{1-x^L}\right), s\left(\frac{1}{1-x^R}\right), (x-x^R)s\left(\frac{1}{1-x^R}\right)\right\}; \\
&= \left[\frac{1}{1-x^R}\right]_{2n+1} - \left[(x-x^R)(x)^m\right]_{2n+1}k, s\left(\frac{1}{1-x^L}\right) - \left[(x-x^L)(x)^m\right]_{2n+1}k, s\left(\frac{1}{1-x^R}\right) - \left[(x-x^R)(x)^m\right]_{2n+1}k\right\},
\end{align*}
\]

under the following condition: if any left or right option of \(x\) does not lie in \((-1 + \frac{1}{\infty}, 1 - \frac{1}{\infty})\), then the elements involving that option are ignored.
We restrict the functions in Definition 9 to the interval \((-1 + \frac{1}{\infty}, 1 - \frac{1}{\infty})\) because the Maclaurin series expansions of all three functions converge on this interval only. This also explains why we must ensure that none of the \(x^L\) and \(x^R\) are outside \((-1 + \frac{1}{\infty}, 1 - \frac{1}{\infty})\). To check that the three definitions are reasonable, we must verify that the real functions \(\arctan(x)\), \(n\log(x)\), and \(\frac{1}{1-x}\) agree with their surreal analogues when their arguments are real; i.e. \(f(x) = s(f(x))\) for all real \(x\) in the domain of \(s(f(x))\). To this end, we state and prove the following theorem:

**Theorem 10.** For all real \(x \in (-1 + \frac{1}{\infty}, 1 - \frac{1}{\infty})\), \(\arctan(x) = s(\arctan(x))\), \(n\log(x) = s(n\log(x))\), and \(\frac{1}{1-x} = s\left(\frac{1}{1-x}\right)\).

**Proof.** Because \(s(\arctan(0)) = \arctan(0) = 0\), \(s(n\log(0)) = n\log(0) = 0\), and \(s\left(\frac{1}{1-x}\right) = \frac{1}{0} = 1\), we have base cases for all three functions. We can now use induction upon the complexity of the functions, as described in Subsection 2.3. Thus \(s(f(x^L)) = f(x^L)\) and \(s(f(x^R)) = f(x^R)\), where \(f(x)\) is any one of \(\arctan(x)\), \(n\log(x)\), and \(\frac{1}{1-x}\). In the rest of the proof, we only consider \(\arctan(x)\) because the proofs for the other two functions are similar.

To show that \(s(\arctan(x))^L < s(\arctan(x))^R\), we need the following four inequalities: \(\left[\frac{x-x^L}{1+xx^L}\right]_{4n+3} < \arctan\left(\frac{x-x^L}{1+xx^L}\right) = \arctan(x) - \arctan(x^L)\), \(\left[\frac{x-x^R}{1+xx^R}\right]_{4n+1} < \arctan\left(\frac{x-x^R}{1+xx^R}\right) = \arctan(x) - \arctan(x^R)\), \(\left[\frac{x^R-x}{1+xx^R}\right]_{4n+3} < \arctan\left(\frac{x^R-x}{1+xx^R}\right) = \arctan(x^R) - \arctan(x)\), and \(\left[\frac{x^L-x}{1+xx^L}\right]_{4n+1} < \arctan\left(\frac{x^L-x}{1+xx^L}\right) = \arctan(x^L) - \arctan(x)\). But all these requirements hold because it is clear from the Maclaurin series expansion of \(\arctan(x)\) that \([z]_{4n+3} < \arctan(z)\) when \(z > 0\) and \([z]_{4n+1} < \arctan(z)\) when \(z < 0\). We next show that \(s(\arctan(x))^L\) and \(s(\arctan(x))^R\) both “approach” \(\arctan(x)\). Suppose \(x > 0\). Then, 0 is a possible \(x^L\), so we take \(x^L = 0\). Observe \(\lim_{n \to \infty} [z]_{4n+3} = \arctan(z) - \frac{1}{\infty}\) and \(\lim_{n \to \infty} [-z]_{4n+1} = \arctan(z) + \frac{1}{\infty}\), so \([z]_{4n+3} | -[z]_{4n+1} = \arctan(z)\). Now suppose \(x < 0\). Then, 0 is a possible \(x^R\), so we take \(x^R = 0\). Observe \(\lim_{n \to \infty} [z]_{4n+1} = \arctan(z) - \frac{1}{\infty}\) and \(\lim_{n \to \infty} [-z]_{4n+3} = \arctan(z) + \frac{1}{\infty}\), so \([z]_{4n+1} | -[z]_{4n+3} = \arctan(z)\). It follows that \(s(\arctan(x)) = \arctan(x)\).

We found genetic definitions of the functions \(\arctan(x)\), \(n\log(x)\), and \(\frac{1}{1-x}\) using their Maclaurin series, but these definitions only work on the restricted domain \((-1 + \frac{1}{\infty}, 1 - \frac{1}{\infty})\). To avoid such domain restrictions, it would certainly be interesting to establish a more general method of defining surreal functions without utilizing their Maclaurin series. As an example, an alternate genetic formula for \(\frac{1}{1-x}\) can be obtained by substituting \(1 - x\) for \(x\) in Kruskal’s definition of the function \(1/x\); this alternate formula works as long as \(x \neq 1\) [Con01].

## 4. Surreal Calculus

In this section, we present the difficulties in doing calculus with the methods established in earlier work, motivate our own approach to surreal calculus, and discuss universal representations, a concept key to our approach.

### 4.1. Motivations

Infinitely long sequences of real numbers are standard in real analysis. However, in earlier work on surreals, particularly [Gon86] and [For04], sequences (and series, which are sequences of partial sums) are restricted to have limit-ordinal length. The “need” for such a restriction can be explained

\[\text{By “approach,” we mean “become sufficiently close,” so that } \{s(\arctan(x))^L | s(\arctan(x))^R\} = \arctan(x).\]
informally as follows. Suppose we have a sequence \( \mathfrak{A} = a_1, a_2, \ldots \) of length \( \text{On} \). It is possible that for every \( m \in \text{On} \), \( b(a_i) > m \) for all \( i \in \text{On}_{>n} \) and some \( n \in \text{On} \). In an attempt to create a genetic formula for the limit of \( \mathfrak{A} \), we can write \( \lim_{i \to \text{On}} a_i = \{L | R\} \). But, because \( b(a_i) \) can be made arbitrarily large by taking \( i \) large enough, the elements of sets \( L \) and \( R \) would depend on the options of all terms in some subsequence (with length \( \text{On} \)) of \( \mathfrak{A} \). So, the cardinalities of \( L \) and \( R \) would have initial ordinals that are not less than \( \text{On} \), which is impossible because for all \( \alpha \in \text{On} \), \( \alpha < \text{On} \). Thus \( L \) and \( R \) would be too large to be sets, and the genetic formula of \( \lim_{i \to \text{On}} a_i \) would fail to satisfy Definition 1.\(^6\) However, if \( c_1, c_2, \ldots \) is a sequence of length \( \alpha \) where \( \alpha \) is a limit-ordinal, then there exists \( m \in \text{On} \) such that for all \( i \in \text{On}_{<\alpha} \), \( m > b(c_i) \), so \( b(c_i) \) is bounded. Thus, in any reasonable genetic formula \( \{L | R\} \) for \( \lim_{i \to \alpha} c_i \), \( L \) and \( R \) would be small enough to be sets, and \( \{L | R\} \) would satisfy Definition 1. It is for this reason that related work has found the need to restrict the length of sequences.

While it does preserve Conway’s construction of numbers (Definition 1), restricting sequences to have limit-ordinal length has an undesirable outcome, as shown in the following theorem:

**Theorem 11.** Let \( b \in \text{No} \). Then, there does not exist an eventually nonconstant sequence \( \mathfrak{A} = t_1, t_2, \ldots \) of length \( \alpha \), where \( \alpha \) is a limit-ordinal, such that for every (surreal) \( \varepsilon > 0 \), there is an \( N \in \text{On}_{<\alpha} \) satisfying \( |t_n - b| < \varepsilon \) whenever \( n \in \text{On}_{>N} \cap \text{On}_{<\alpha} \).

**Proof.** Suppose such a sequence \( \mathfrak{A} \) exists, and assume without loss of generality that none of the \( t_i \) are equal to \( b \). (If any \( t_i \) are equal to \( b \), discard them; the remaining subsequence has the same limit as the original sequence.) Now let \( z \) be the smallest ordinal such that \( z > \sup\{|b|, b(t_1), b(t_2), \ldots |\} \), and let \( \varepsilon = 1/\omega^2 \). Then there exists some \( N \in \text{On}_{<\alpha} \) such that for all \( n \in \text{On}_{>N} \cap \text{On}_{<\alpha} \), (1) \( t_n \neq b \); and (2) \( -\varepsilon < t_n - b < 1/\omega^2 = \varepsilon \). (\( t_n \neq b \) holds for all \( n \in \text{On}_{<\alpha} \).) When (1) and (2) are combined, either \( b - 1/\omega^2 < t_n < b \) or \( b < t_n < b + 1/\omega^2 \) holds. Thus, in any genetic formula for \( t_n \), there is at least one left or right option whose birthday is \( \geq z \). So for all \( n \in \text{On}_{>N} \cap \text{On}_{<\alpha} \), \( b(t_n) \geq z \), which is a contradiction because we chose \( z \) so that \( b(t_n) < z \). \( \square \)

From Theorem 11, we conclude that nonconstant sequences of limit-ordinal length are unhelpful for doing calculus because they do not have surreal limits (by “limit” we mean the \( \varepsilon-\delta \) notion). We must consider \( \text{On} \)-length sequences in \( \text{No} \), not only for the above reasons, but also in light of work by Sikorski, who showed that in a field of character \( \omega_\mu \) (which is an initial regular ordinal number), we need to consider sequences of length \( \omega_\mu \) to obtain convergence for nontrivial sequences [Sik48]. Thus, in \( \text{No} \), which has character \( \text{On} \), we need to consider sequences of length \( \text{On} \).

Now, let us discuss a similar problem faced with finding limits of functions. Suppose \( f(x) \) is a function whose domain is \( \text{No} \). Then, by Definition 27, there exists \( \delta > 0 \) such that for some \( \beta \in \text{On} \) and for all \( \varepsilon \in (0, 1/\omega^3) \), \( \lim_{x \to a} f(x) - \varepsilon < f(x) < \lim_{x \to a} f(x) + \varepsilon \) whenever \( |x - a| < \delta \). Thus, we can make \( b(\delta) \) arbitrarily large by taking \( \beta \) sufficiently large, so any reasonable genetic formula \( \{L | R\} \) for \( \lim_{x \to a} f(x) \) would depend on values of \( x \) of arbitrarily large birthday. Thus, \( L \) and \( R \) would again be too large to be sets. Because differentiation is taking the limit of the function \( \frac{f(x+h) - f(x)}{h} \) as \( h \to 0 \), we also cannot differentiate functions while still satisfying Definition 1. We conclude that a different representation (which we call universal in Subsection 4.2) that allows \( L \) and \( R \) to be proper classes is necessary for limits and derivatives to work for surreals.

As discussed in Section 1, current methods of integration have all been problematic. While the integrals defined in [Con01] and [For04] involve the partitioning of the interval of integration, thus resembling Riemann integrals, only a finite number of partitions is used. In Section 7, we show that partitioning the interval of integration into arbitrarily many subintervals allows the surreal integrals
of polynomials and exponentials to match those of real calculus. We also prove the Fundamental Theorem of Calculus, which allows us to integrate functions using antiderivatives, as long as a consistent definition of integration can be found.

4.2. Universal Representations

In this subsection, we define universal representations and show that they preserve all basic properties of numbers. Next, we discuss universal representations of functions. Finally, we discuss two possible representations of gaps: normal forms and universal representations.

By Theorem 4, \( x = \{ L \mid R \} \) iff \( x \) is the oldest number greater than the elements of \( L \) and less than the elements of \( R \). Suppose \( A \) and \( B \) are nonempty sets such that for all \( a \in A \), \( a < x \) and for all \( b \in B \), \( b > x \). Then, if \( \mathcal{L} = L \cup A \) and \( \mathcal{R} = R \cup B \), \( x = \{ \mathcal{L} \mid \mathcal{R} \} \) as well; i.e. the identity of \( x \) remains unchanged when \( L \) is replaced with \( \mathcal{L} \) and \( R \) with \( \mathcal{R} \). Now suppose \( A \) and \( B \) are, more generally, nonempty classes satisfying \( a < x \) and \( b > x \) for all \( a \in A, b \in B \). Although the object \( \{ \mathcal{L} \mid \mathcal{R} \} \) now does not obey Conway’s original construction of numbers, we observe that \( \{ \mathcal{L} \mid \mathcal{R} \} \) still represents \( x \), since every \( \ell \in \mathcal{L} \) has \( \ell < x \), every \( r \in \mathcal{R} \) has \( r > x \), and \( x \) still satisfies Theorem 4. We find it useful for the purpose of developing surreal analysis to make the following definition, which introduces universal representations and is indeed a special case of the observation made earlier in this paragraph:

**Definition 12.** For all \( x \in \text{No} \), the universal representation of \( x \) is \( x = \{ \text{No}_{<x} \mid \text{No}_{>x} \} \).

Even when considering universal representations, we use notational conventions similar to the ones described in Section 2; i.e. \( x = \{ x^\mathcal{L} \mid x^\mathcal{R} \} = \{ L_x \mid R_x \} \), where we write “\( \mathcal{L} \), “\( \mathcal{R} \)” instead of “\( L \), “\( R \)” to distinguish universal representations from genetic formulae. The following theorem allows us to use universal representations of numbers in all basic arithmetic operations and still obtain the same answers given by Definition 2:

**Theorem 13.** Every property in Definition 2 holds when the numbers \( x_1 = \{ L_{x_1} \mid R_{x_1} \} \) and \( x_2 = \{ L_{x_2} \mid R_{x_2} \} \) are written in their respective universal representations.

**Proof.** We only prove the addition property because the proofs for the others are similar. Let \( y = x_1 + x_2 \). Then, by part three of Definition 2, \( y = \{ x^L_1 + x_2, x_1 + x^L_2 \mid x^R_1 + x_2, x_1 + x^R_2 \} \).

Now allow \( x_1 \) and \( x_2 \) to be written as universal representations; i.e. \( x_1 = \{ \text{No}_{<x_1} \mid \text{No}_{>x_1} \} \) and \( x_2 = \{ \text{No}_{<x_2} \mid \text{No}_{>x_2} \} \). We claim that the new object \( z = \{ x^L_1 + x_2, x_1 + x^L_2 \mid x^R_1 + x_2, x_1 + x^R_2 \} \) is the universal representation of \( y \). If \( a < y \) and \( b > y \), then \( \text{No}_{<a} \subseteq L_z \) and \( \text{No}_{>b} \subseteq R_z \), since there exist \( x^L_1 \in \text{No}_{<x_1} \) and \( x^R_1 \in \text{No}_{>x_1} \) such that \( x^L_1 = a - x_2 \) and \( x^R_1 = b - x_2 \). Therefore, \( \text{No}_{<y} \subseteq L_z \) and \( \text{No}_{>y} \subseteq R_z \), which, when combined, gives us \( \text{No} \backslash \{ y \} \subseteq L_z \cup R_z \). But, \( L_z \cup R_z \subseteq \text{No} \) because \( z^L < y < z^R \), so \( y \) is the only number between the elements of \( L_z \) and those of \( R_z \). Thus, the object \( z \) is the universal representation of \( y \).

Gaps can be represented in two ways: (1) As normal forms (formal sums over ordinals); and (2) as universal representations. We begin by presenting the normal forms of gaps, which are described in great detail in [Con01]. All gaps can be classified into two types, Type I and Type II. Type I and Type II gaps have the following normal forms:

- **Type I:** \( \sum_{i \in \text{On}} r_i \cdot \omega^y_i \), or
- **Type II:** \( \sum_{i \in \text{On}_{<a}} r_i \cdot \omega^y_i \oplus (\pm \omega^y) \),

9
where in both sums the \( r_i \) are nonzero real numbers and \((y_i)\) is a descending sequence. In the Type II sum, \( \alpha \in \Omega \), \( \Theta \) is a gap whose right class contains all of the \( y_i \), and the operation \( \oplus \) denotes the sum of a number \( n \) and gap \( g \), defined by: \( n \oplus g = \{ n + g^L | n + g^R \} \). Also, \( \omega^\Theta = \{ 0, a \cdot \omega^L | b \cdot \omega^R \} \), where \( a, b \in \mathbb{R}_{>0} \) and \( l \in \mathcal{L}_\Theta, r \in \mathcal{R}_\Theta \).

It is easy to see that gaps can be represented in their own universal representations, for if we have a gap \( g \), we can write it as \( g = \{ \text{No}_{<g} | \text{No}_{>g} \} \). In particular, we consider \( \Omega = \{ \text{No} \} \) and \( \text{Off} = \{ | \text{No} \} \) to be universal representations of the gaps \( \Omega \) and \( \text{Off} \) respectively.

5. Sequences of Numbers and their Limits

In this section, we present a universal representation for the limit of an \( \Omega \)-length sequence and provide examples demonstrating how our universal representation can be used to evaluate the limits of sequences. We show that \( \text{No} \) is not Cauchy complete, and we characterize the kinds of sequences that do converge.

5.1. Evaluation of Limits of Sequences

The approach we take to defining the limit of an \( \Omega \)-length sequence is analogous to the method Conway uses in introducing the arithmetic properties of numbers in Chapter 0 of [Con01]. Specifically, we first define the limit of an \( \Omega \)-length sequence to be a certain universal representation and then prove that this definition is a reasonable one; i.e. show that it is equivalent to the usual \( \varepsilon\)-\( \delta \) definition. We now define limits of \( \Omega \)-length sequences as follows, after first introducing our notation for the limit of a sequence:

**Definition 14.** Let \( \mathfrak{A} = a_1, a_2, \ldots \) be an \( \Omega \)-length sequence. Then, define

\[
\ell(\mathfrak{A}) := \left\{ a : a < \sup \left( \bigcup_{i \geq 1} \mathcal{L}_{a_i} \right) \right\}.
\]

**Definition 15.** Let \( \mathfrak{A} = a_1, a_2, \ldots \) be an \( \Omega \)-length sequence. We say that the limit of \( \mathfrak{A} \) is \( \ell \) and write \( \lim_{i \to \Omega} a_i = \ell \) if the expression on the right-hand-side of (1) in Definition 14 is a universal representation and \( \ell = \ell(\mathfrak{A}) \).

**Remark.** In the above definition, we make no distinction as to whether \( \ell(\mathfrak{A}) \) is a number or a gap. Definition 15 holds in both cases, although we do not say that surreal sequences approaching gaps are convergent, just like we do not say real sequences approaching \( \pm \infty \) are convergent.

Before we state and prove Theorem 17, which proves the equivalence of Definition 15 with the standard \( \varepsilon\)-\( \delta \) definition, we need the following lemma:

**Lemma 16.** Let \( \mathfrak{A} = a_1, a_2, \ldots \) be an \( \Omega \)-length sequence, and let \( \mathfrak{B} = a_k, a_{k+1}, \ldots \) be an \( \Omega \)-length sequence. Then \( \ell(\mathfrak{B}) = \ell(\mathfrak{A}) \).

**Proof.** We need to show that the following two statements hold: (1) \( \sup \left( \bigcup_{i \geq k} \mathcal{L}_{a_i} \right) = \sup \left( \bigcup_{i \geq 1} \mathcal{L}_{a_i} \right) \) and (2) \( \inf \left( \bigcup_{i \geq k} \mathcal{R}_{a_i} \right) = \inf \left( \bigcup_{i \geq 1} \mathcal{R}_{a_i} \right) \). We first prove (1). Let \( M = \left( \bigcup_{i \geq k} \mathcal{L}_{a_i} \right) \) and \( N = \left( \bigcup_{i \geq 1} \mathcal{L}_{a_i} \right) \). Note that \( \bigcup_{j \geq i} \mathcal{L}_{a_i} \subseteq \bigcup_{j \geq i+1} \mathcal{L}_{a_j} \). Therefore, \( P = \left( \bigcup_{1 \leq i < k} \mathcal{L}_{a_i} \right) \subseteq M \). But \( P \cup M = N \), implying \( M = N \). So, \( \sup(M) = \sup(N) \). We now...
prove (2). Let $S = \bigcup_{i \geq k} \bigcap_{j \geq i} R_{a_j}$ and $T = \bigcup_{i \geq 1} \bigcap_{j \geq i} R_{a_j}$. Note that $\bigcap_{j \geq i} R_{a_j} \subseteq \bigcap_{j \geq i+1} R_{a_j}$. Therefore, $U = \bigcup_{1 \leq i < k} \bigcap_{j \geq i} R_{a_j} \subseteq S$. But $U \cup S = T$, implying $S = T$. So $\inf(S) = \inf(T)$. Statements (1) and (2) suffice to show that $\ell(\mathbb{B}) = \ell(\mathbb{A})$. □

Remark. Lemma 16 also holds for $\mathbb{R}$ and supports our intuition that the first terms of a sequence have no bearing on the limit of that sequence.

**Theorem 17.** Let $\mathfrak{A} = a_1, a_2, \ldots$ be an On-length sequence. If $\lim_{i \to \text{On}} a_i = \ell(\mathfrak{A}) \in \text{No}$, then for every (surreal) $\varepsilon > 0$, there is an $N \in \text{On}$ satisfying $|a_n - \ell(\mathfrak{A})| < \varepsilon$ whenever $n \in \text{On}_{>N}$. Conversely, if $\ell$ is a number such that for every (surreal) $\varepsilon > 0$, there is an $N \in \text{On}$ satisfying $|a_n - \ell| < \varepsilon$ whenever $n \in \text{On}_{>N}$, then $\lim_{i \to \text{On}} a_i = \ell$.

**Proof.** For the forward direction, we must prove that for every $\varepsilon > 0$, there exists $N \in \text{On}$ such that whenever $n \in \text{On}_{<N}$, $|a_n - \ell(\mathfrak{A})| < \varepsilon$. Split $\mathfrak{A}$ into two subsequences, $\mathfrak{A}_+ = b_1, b_2, \ldots$ being the subsequence of all terms $\geq \ell(\mathfrak{A})$ and $\mathfrak{A}_- = c_1, c_2, \ldots$ being the subsequence of all terms $\leq \ell(\mathfrak{A})$.

Note that the limit of an On-length subsequence equals the limit of its parent sequence. If either $\mathfrak{A}_+$ or $\mathfrak{A}_-$ has ordinal length (they cannot both be of ordinal length because $\mathfrak{A}$ has length On), by Lemma 16, we can redefine $\mathfrak{A} := a_\beta, a_{\beta+1}, \ldots$ for some $\beta \in \text{On}$ such that the tail of the new sequence $\mathfrak{A}$ lies entirely in either $\mathfrak{A}_+$ or $\mathfrak{A}_-$, depending on which subsequence has length On. Let us assume that both $\mathfrak{A}_+, \mathfrak{A}_-$ have length On.

Observe $|b_n - \ell(\mathfrak{A})| = b_n - \ell(\mathfrak{A})$. Suppose there does not exist $N_1 \in \text{On}$ such that whenever $n \in \text{On}_{>N_1}$, $b_n - \ell(\mathfrak{A}) < \varepsilon$ for some $\varepsilon > 0$. Then for arbitrarily many $n > N_1$, $b_n \geq \ell(\mathfrak{A}) + \varepsilon$. Thus, $y = \inf \left( \bigcup_{i \geq 1} \bigcap_{j \geq i} R_{b_n} \right) \geq \ell(\mathfrak{A}) + \varepsilon$, a contradiction because $y = \ell(\mathfrak{A})$ if the expression on the right-hand-side of (1) is the universal representation of $\ell(\mathfrak{A})$. Therefore, there exists $N_1 \in \text{On}$ such that whenever $n > N_1$, $b_n - \ell(\mathfrak{A}) < \varepsilon$. A similar argument shows that there exists $N_2 \in \text{On}$ such that whenever $n > N_2$, $c_n - \ell(\mathfrak{A}) < \varepsilon$. Then $N = \max\{N_1, N_2\}$ satisfies Definition 15.

If $\mathfrak{A}_+$ is of ordinal length, then instead of $N = \max\{N_1, N_2\}$ we have $N = N_2$. Similarly, if $\mathfrak{A}_-$ is of ordinal length, then instead of $N = \max\{N_1, N_2\}$ we have $N = N_1$.

For the other direction, if $\lim_{i \to \text{On}} a_i \neq \ell$, then there are two cases to consider. The first case is that the expression on the right-hand-side of (1) in Definition 14 is not a universal representation. This would imply that

$$\inf \left( \bigcup_{i \geq 1} \bigcap_{j \geq i} R_{a_j} \right) - \sup \left( \bigcup_{i \geq 1} \bigcap_{j \geq i} L_{a_j} \right) > \varepsilon,$$

for some $\varepsilon > 0$, because otherwise we would have elements of the right class of a number smaller than elements of the left class. But since the $a_i$ can be made arbitrarily close to $\ell$ by taking $i$ sufficiently large, we can pick $x \in \bigcup_{i \geq 1} \bigcap_{j \geq i} R_{a_j}$ and $y \in \bigcup_{i \geq 1} \bigcap_{j \geq i} L_{a_j}$ such that $|x - \ell| < \varepsilon/2$ and $|y - \ell| < \varepsilon/2$. By the Triangle Inequality, $|x - y| \leq |x - \ell| + |y - \ell| < \varepsilon/2 + \varepsilon/2 < \varepsilon$, which contradicts the claim in (2). Thus, it follows that the expression on the right-hand-side of (1) in Definition 14 is a universal representation.

The second case is that $\ell(\mathfrak{A})$ is a gap, not a number. Since the expression on the right-hand-side of (1) in Definition 14 is a universal representation, we have that

$$\inf \left( \bigcup_{i \geq 1} \bigcap_{j \geq i} R_{a_j} \right) = \sup \left( \bigcup_{i \geq 1} \bigcap_{j \geq i} L_{a_j} \right) = \ell(\mathfrak{A}).$$

Now suppose that $\ell(\mathfrak{A}) < \ell$. Using notation from the proof of the first case above, we know that for every $\varepsilon > 0$, we have $|y - \ell| < \varepsilon$. If we pick $\varepsilon$ such that $\ell - \varepsilon > \ell(\mathfrak{A})$, then we have $y > \ell(\mathfrak{A})$. 

11
which contradicts (3). Thus, \( \ell(\mathfrak{A}) \neq \ell \). By analogous reasoning in which we replace “left” with “right,” we find that \( \ell(\mathfrak{A}) \neq \ell \). Finally, we have \( \ell(\mathfrak{A}) = \ell \), so \( \ell(\mathfrak{A}) \) cannot be a gap. This completes the proof of the theorem.

It is only natural to wonder why in our statement of Theorem 17 we restrict our consideration to sequences approaching numbers. The issue with extending the theorem to describe gaps is nicely demonstrated in the example sequence \( \mathfrak{A} = a_1, a_2, \ldots \) defined by \( a_i = \omega^{1/i} \). Substituting \( \mathfrak{A} \) into the expression on the right-hand-side of (1) in Definition 14 yields \( \ell(\mathfrak{A}) = \infty \). However, it is clearly not true that we can make \( a_i \) arbitrarily close to \( \infty \) by picking \( i \) sufficiently large, because for every surreal \( \varepsilon \in (0, 1) \) and every \( i \in \text{Ord} \), we have that \( a_i - \varepsilon > \infty \). In all, the equivalence of the universal representation and \( \varepsilon-\delta \) notions of limits holds for numbers but does not generalize to gaps. To ensure that the definition of a limit of an \( \text{On} \)-length sequence holds for all such sequences, we must rely on the universal representation notion of a limit in our definition.

We next consider how our method of evaluating limits of sequences using universal representations can be employed to completely characterize the types of Cauchy sequences that do converge as well as those that do not.

5.2. Cauchy Sequences

We can now distinguish between sequences that converge (to numbers), sequences that approach gaps, and sequences that neither converge nor approach gaps. For the real numbers, all Cauchy sequences converge; i.e. \( \mathbb{R} \) is Cauchy complete. However, \( \mathbb{N} \) is not Cauchy complete; there are Cauchy sequences of numbers that approach gaps. But because it is only natural to wonder what characterizes convergent sequences, we devote the rest of this subsection to determining what types of Cauchy sequences converge (to numbers) and what types do not. Let us begin our formal discussion of Cauchy sequences by defining them as follows:

**Definition 18.** Let \( \mathfrak{A} = a_1, a_2, \ldots \) be a sequence of length \( \text{On} \). Then \( \mathfrak{A} \) is a Cauchy sequence if for every (surreal) \( \varepsilon > 0 \) there exists \( N \in \text{On} \) such that whenever \( m, n \in \text{On} \), \( |a_m - a_n| < \varepsilon \).

As follows, we show that \( \mathbb{N} \) is not Cauchy complete by providing an example of a sequence that satisfies Definition 18 but approaches a gap.

**Example 19.** Let \( \mathfrak{A} = 1, 1 + 1/\omega, 1 + 1/\omega + 1/\omega^2, 1 + 1/\omega + 1/\omega^2 + 1/\omega^3, \ldots \). Note that \( \mathfrak{A} \) is a Cauchy sequence because it satisfies Definition 18; i.e. for every \( \varepsilon > 0 \), there exists \( N \in \text{On} \) such that whenever \( m, n \in \text{On} \), \( |\sum_{i \in \text{On} \leq m} 1/\omega^i - \sum_{i \in \text{On} \leq n} 1/\omega^i| < \varepsilon \). It is easy to check that the universal representation of \( \ell(\mathfrak{A}) \) in this example, it is easy to check that the universal representation obtained for \( \ell(\mathfrak{A}) \) is that of the object \( \sum_{i \in \text{On}} 1/\omega^i \). However, as explained in Subsection 4.2, \( \sum_{i \in \text{On}} 1/\omega^i \) is the normal form of a gap, so \( \mathfrak{A} \) is a Cauchy sequence that approaches a gap, a result that confirms the fact that \( \mathbb{N} \) is not Cauchy complete.

Four key steps comprise our strategy for classifying Cauchy sequences: (1) First prove that sequences approaching Type II gaps are not Cauchy; (2) second conclude that Cauchy sequences either converge, approach Type I gaps, or diverge; (3) third prove that only a certain kind of Type I gap can be approached by Cauchy sequences; and (4) fourth prove that Cauchy sequences that do not approach such Type I gaps are convergent. We execute this strategy as follows.

To prove that sequences approaching Type II gaps are not Cauchy, we need a restriction on the definition of gaps. We restrict Conway’s original definition of gaps as follows:
Definition 20. A surreal gap is any Dedekind section of \( \mathbb{N}_o \) that cannot be represented as either \( \{ \mathbb{N}_{<x} \mid \mathbb{N}_{\geq x} \} \) or \( \{ \mathbb{N}_{\leq x} \mid \mathbb{N}_{>x} \} \) for some \( x \in \mathbb{N}_o \). Further, objects of the form \( \{ \mathbb{N}_{<x} \mid \mathbb{N}_{\geq x} \} \) or \( \{ \mathbb{N}_{\leq x} \mid \mathbb{N}_{>x} \} \) are defined to be equal to \( x \).

Remark. From now on, the unqualified word “gap” refers only to gaps of the type described in Definition 20.

Lemma 21. Let \( \mathfrak{A} = a_1, a_2, \ldots \) If \( \lim_{i \to \mathbb{O}_n} a_i = g \) for some gap \( g \) of Type II, then the sequence \( \mathfrak{A} \) is not Cauchy.

Proof. Suppose \( \mathfrak{A} \) is Cauchy. It follows that \( |a_i^{\mathfrak{A}} - a_j^{\mathfrak{A}}| \) can be made arbitrarily close to 0 if \( i, j \in \mathbb{O}_n \) are taken sufficiently large. Then, if \( \ell(\mathfrak{A}) \) denotes the universal representation of \( g \), \( |\ell(\mathfrak{A}) - \ell(\mathfrak{A})^{\mathfrak{A}}| \) can be made arbitrarily close to 0. It follows that \( |\ell(\mathfrak{A}) - g| \) can be made arbitrarily close to 0 if \( i, j \in \mathbb{O}_n \) are taken sufficiently large. As discussed in the previous section, we know that \( g = \sum_{i \in \mathbb{O}_n} r_i \omega^{y_i} + (\pm \omega^\Theta) \) for some gap \( \Theta \). Also, as described in Subsection 2.1, \( h = \sum_{i \in \mathbb{O}_n} r_i \omega^{y_i} \) is a number, so \( |g - \ell(\mathfrak{A})^{\mathfrak{A}}| = |h' + (\pm \omega^\Theta)| \), where \( h' \in \mathbb{N}_o \). Now, \( h' + (\pm \omega^\Theta) \) is a gap, and \( \Theta > \text{Off} \) because \( \omega^{\text{Off}} = 1/\mathbb{O}_n \) is not a gap by Definition 20. Because we can make \( h' + (\pm \omega^\Theta) \) smaller than any (surreal) \( \varepsilon > 0 \), pick \( \varepsilon = \omega^r \) for some number \( r < \Theta \) (which is possible to do because \( \Theta > \text{Off} \)). Then we can either have (1) \( h' + \omega^\Theta > h' + \omega^r \); or (2) \( h' + \omega^\Theta < h' + \omega^r \). In case (1) let \( z \) denote the largest power of \( \omega \) in the normal form of \( h' \). Clearly, \( z > \Theta \) and \( z > r \), so the largest power of \( \omega \) in the normal form of \( h' + \omega^r \) is \( z \). But it follows that \( h' + \omega^r < \omega^\Theta \), a contradiction. Similarly, in case (2) let \( z \) denote the largest power of \( \omega \) in the normal form of \( h' \). Clearly, \( z < \Theta \), so the largest power of \( \omega \) in the normal form of \( h' + \omega^r \) is \( z \). But it follows that \( h' + \omega^r < \omega^\Theta \), a contradiction. Thus we have the lemma. \( \square \)

It might seem like the gap restriction of Definition 20 was imposed as a convenient means of allowing Lemma 21 to hold. Nevertheless, there is sound intuitive reasoning for why we must restrict gaps in this way. In Definition 5, gaps are defined to be Dedekind sections of \( \mathbb{N}_o \). This means that sections like \( 1/\mathbb{O}_n = \{ \mathbb{N}_{<0} \mid \mathbb{N}_{>0} \} \) are gaps. But if we allow objects such as \( 1/\mathbb{O}_n \) to be gaps, we can create similar “gaps” in the real line by claiming that there exist: (1) for each \( a \in \mathbb{R} \), an object \( a > a \) and less than all reals \( a > a \); and (2) another object \( a < a \) and greater than all reals \( a < a \). However, such objects are not considered to be “gaps” in the real line. Additionally, without Definition 20, \( \mathbb{O}_n \)-length sequences like \( 1, 1/2, 1/4, \ldots \) would be said to approach the gap \( 1/\mathbb{O}_n \) rather than the desired limit 0, and in fact no \( \mathbb{O}_n \)-length sequences would converge at all.

For these reasons, we must restrict the definition of gaps.

Now, Cauchy sequences must therefore either converge, approach Type I gaps, or diverge. Let us next consider the case of Cauchy sequences approaching Type I gaps. Designate a Type I gap \( g = \sum_{i \in \mathbb{O}_n} r_i \cdot \omega^{y_i} \) to be a Type Ia gap iff \( \lim_{i \to \mathbb{O}_n} y_i = \text{Off} \) and to be a Type Ib gap otherwise. We now state and prove the following lemma about Type I gaps:

Lemma 22. Let \( \mathfrak{A} = a_1, a_2, \ldots \) be a Cauchy sequence. If \( \lim_{i \to \mathbb{O}_n} a_i = g \) for some gap \( g \) of Type I, then \( g \) is a gap of Type Ia.

Proof. Suppose \( g = \sum_{i \in \mathbb{O}_n} r_i \cdot \omega^{y_i} \) is of Type Ib. Because the \( y_i \) are a decreasing sequence that does not approach \( \text{Off} \), \( y_i \) are bounded below by some number, say \( b \). Now since \( \mathfrak{A} \) is Cauchy, it follows that \( |a_i^{\mathfrak{A}} - a_j^{\mathfrak{A}}| \) can be made arbitrarily close to 0 if \( i, j \in \mathbb{O}_n \) are taken sufficiently large. Then, if \( \ell(\mathfrak{A}) \) denotes the universal representation of \( g \), \( |\ell(\mathfrak{A}) - \ell(\mathfrak{A})^{\mathfrak{A}}| \) can be made arbitrarily close to 0. It follows that \( |\ell(\mathfrak{A}) - g| \) can be made arbitrarily close to 0 if \( i, j \in \mathbb{O}_n \) are taken sufficiently large. In particular, we can choose \( \ell(\mathfrak{A}) \) so that \( |\ell(\mathfrak{A}) - g| < \omega^b \). Then, either (1)
\[ \ell(\mathfrak{A}) > g > \ell(\mathfrak{A}) - \omega^b; \text{ or } (2) \ell(\mathfrak{A}) < g < \ell(\mathfrak{A}) + \omega^b. \] In case (1), the largest exponent \( z \) of \( \omega \) in the normal form of the (positive) object \( \ell(\mathfrak{A}) - g \) satisfies \( z \geq y_\alpha \) for some \( \alpha \in \text{On} \). Therefore, \( z > b \), so clearly \( \ell(\mathfrak{A}) - g > \omega^b \), a contradiction. In case (2), the largest exponent \( z \) of \( \omega \) in the normal form of the (positive) object \( g - \ell(\mathfrak{A}) \) satisfies \( z = y_\alpha \) for some \( \alpha \in \text{On} \). Therefore, \( z > b \), so clearly \( g - \ell(\mathfrak{A}) > \omega^b \), a contradiction. Thus, we have the lemma. \( \square \)

**Remark.** Consider the case of a Cauchy sequence \( \mathfrak{A} = a_1, a_2, \ldots \) that approaches a Type Ia gap \( g \). Note that \( \mathfrak{A} \) approaches \( g \) iff it is equivalent to the sequence \( \mathfrak{A} \) defined by successive partial sums of the normal form of \( g \). Here, two sequences \((a_n)\) and \((b_n)\) are considered equivalent iff for every (surreal) \( \varepsilon > 0 \) there exists \( N \in \text{On} \) such that whenever \( n > N \), \(|a_n - b_n| < \varepsilon\).

From Lemmas 21 and 22, we know that there is a Cauchy sequence that approaches a gap iff the gap is of Type Ia (the reverse direction follows easily from the properties of the normal form of a Type Ia gap). We now prove that Cauchy sequences that do not approach gaps of Type Ia must converge (to numbers).

**Theorem 23.** Let \( \mathfrak{A} = a_1, a_2, \ldots \) be a Cauchy sequence that does not approach a gap of Type Ia. Then \( \lim_{i \to \text{On}} a_i \in \text{No} \).

**Proof.** We first prove that \( \mathfrak{A} \) is bounded. Let \( \alpha \in \text{On} \) such that for all \( \beta, \gamma \in \text{On} \geq \alpha \), we have that \(|a_\beta - a_\gamma| < 1\). Then for all \( \beta \geq \alpha \), by the Triangle Inequality we have that \(|a_\beta| < |a_\alpha| + 1\). So, for all \( \beta \in \text{On} \), we have that \(|a_\beta| \leq \max\{|a_1|, |a_2|, \ldots, |a_\alpha| + 1\}|. Thus \( \mathfrak{A} \) is bounded.

Now consider the class \( C = \{ x \in \text{No} : x < a_\alpha \text{ for all } \alpha \text{ except ordinal-many} \} \). We next prove that \( \sup(C) \in \text{No} \). If not, then \( \sup(C) \) is a gap, say \( g \), and \( g \neq \text{On} \) since \( \mathfrak{A} \) is bounded. Because \( \mathfrak{A} \) is Cauchy, we claim that \( D = \{ x \in \text{No} : x > a_n \text{ for all } \alpha \text{ except ordinal-many} \} \) satisfies \( \inf(D) = g \). If this claim were untrue, then we can find two numbers \( p, q \in (\sup(C), \inf(D)) \) (there are at least two numbers in this interval because of the gap restriction of Definition 20) so that \(|a_\beta - a_\gamma| > |p - q|\) for \( \text{On-many } \beta, \gamma \in \text{On} \), which contradicts the fact that \( \mathfrak{A} \) is Cauchy. Thus the claim holds. Now \( \mathfrak{A} \) satisfies Definition 15, so \( \lim_{i \to \text{On}} a_i = g \), where by assumption \( g \) must be a gap of Type II or a gap of Type Ib. But, by Lemmas 21 and 22, \( \mathfrak{A} \) cannot be Cauchy, a contradiction. So \( \sup(C) = \inf(D) \in \text{No} \).

We finally prove that for every \( \varepsilon > 0 \), we can find \( \alpha \in \text{On} \) so that for every \( \beta \in \text{On} \geq \alpha \), we have \(|a_\beta - \sup(C)| < \varepsilon\). Suppose the contrary, so that for some \( \varepsilon > 0 \) we have (1) \( a_\beta \leq \sup(C) - \varepsilon \) for \( \text{On-many } \beta \); or (2) \( a_\beta \geq \sup(C) + \varepsilon \) for \( \text{On-many } \beta \). In the first case, we find that there is an upper bound of \( C \) that is less than \( \sup(C) \), which is a contradiction, and in the second case, we find that there is a lower bound of \( D \) that his greater than \( \inf(D) \), which is again a contradiction. Thus we have the theorem. \( \square \)

### 5.3. Examples of Limit Evaluation

We now present three representative examples of surreal sequences, and we evaluate their limits using Definition 15.

**Example 24.** Let \( \mathfrak{A} = 1, 2, 3, \ldots \), the sequence of positive ordinals. We have:

\[
\mathcal{L}_{\ell(\mathfrak{A})} = \left\{ a : a < \sup \left( \bigcup_{i \geq 1} \bigcap_{j \geq i} \mathcal{L}_{a_j} \right) \right\} = \left\{ a : a < \sup \left( \bigcup_{i \geq 1} \{ c : c < i \} \right) \right\},
\] (4)
which follows from the fact that \( \mathcal{L}_{a_j} \) contains all numbers less than \( j \) in this case. But the union in (4) is clearly equal to \( \mathbb{N}_0 \), so \( \mathcal{L}_{\ell(\mathfrak{A})} = \mathbb{N}_0 \). We also have:

\[
\mathcal{R}_{\ell(\mathfrak{A})} = \left\{ b : b > \inf \left( \bigcup_{i \geq 1} \mathcal{R}_{a_i} \right) \right\} = \left\{ b : b > \inf \left( \bigcup_{i \geq 1} \emptyset \right) \right\},
\]

which holds because \( \mathcal{R}_{a_j} \) contains all numbers greater than \( j \) in this case. Again, we can simplify the union in (5) by realizing that there are no elements in this union. Thus, \( \mathcal{R}_{\ell(\mathfrak{A})} = \emptyset \). But since \( \ell(\mathfrak{A}) = \{ \mathcal{L}_{\ell(\mathfrak{A})} \mid \mathcal{R}_{\ell(\mathfrak{A})} \} = \{ \mathbb{N}_0 \} = \mathbb{O}_n \) is a universal representation of a gap, we have \( \lim_{i \to \mathbb{O}_n} i = \mathbb{O}_n \). This confirms our intuition that a steadily increasing sequence approaches the surreal notion of “infinity.”

**Example 25.** Let \( \mathfrak{A} = 1/2, 1/4, 1/8, \ldots \), the sequence of positive ordinal powers of 1/2. We have

\[
\mathcal{L}_{\ell(\mathfrak{A})} = \left\{ a : a < \sup \left( \bigcup_{i \geq 1} \mathcal{L}_{a_i} \right) \right\} = \left\{ a : a < \sup \left( \bigcup_{i \geq 1} \{ c : c < (1/2)^i \} \right) \right\}.
\]

Clearly, 0 is the maximum element in the union of (6). So, \( \mathcal{L}_{\ell(\mathfrak{A})} = \mathbb{N}_0 \). Similarly, we have

\[
\mathcal{R}_{\ell(\mathfrak{A})} = \left\{ b : b > \inf \left( \bigcup_{i \geq 1} \mathcal{R}_{a_i} \right) \right\} = \left\{ b : b > \inf \left( \bigcup_{i \geq 1} \{ c : c > (1/2)^i \} \right) \right\}.
\]

By analogous reasoning, we find from (7) that \( \mathcal{R}_{\ell(\mathfrak{A})} = \mathbb{N}_0 \). Since \( \ell(\mathfrak{A}) = \{ \mathcal{L}_{\ell(\mathfrak{A})} \mid \mathcal{R}_{\ell(\mathfrak{A})} \} = \{ \mathbb{N}_0 \mid \mathbb{N}_0 \} = 0 \) is a universal representation of a number, Definition 15 yields \( \lim_{i \to \mathbb{O}_n} 1/2^i = 0 \), a result we might have expected from studying the real analogue of \( \mathfrak{A} \).

**Example 26.** Let \( \mathfrak{A} = -1/2, 1/4, -1/8 \ldots \). We have

\[
\mathcal{L}_{\ell(\mathfrak{A})} = \left\{ a : a < \sup \left( \bigcup_{i \geq 1} \mathcal{L}_j \right) \right\} = \left\{ a : a < \sup \left( \bigcup_{i \geq 1} \{ c : c < -(1/2)^m \} \right) \right\},
\]

where \( m = i \) when the \( i \)th term is negative and \( m = i + 1 \) when the \( i \)th term is positive. Similarly, we find that

\[
\mathcal{R}_{\ell(\mathfrak{A})} = \left\{ b : b > \inf \left( \bigcup_{i \geq 1} \mathcal{R}_j \right) \right\} = \left\{ b : b > \inf \left( \bigcup_{i \geq 1} \{ c : c > (1/2)^m \} \right) \right\},
\]

where \( m = i \) when the \( i \)th term is positive and \( m = i + 1 \) when the \( i \)th term is negative. From (8) and (9), we see that \( \mathcal{L}_{\ell(\mathfrak{A})} = \mathbb{N}_0 \) and \( \mathcal{R}_{\ell(\mathfrak{A})} = \mathbb{N}_0 \). Because \( \ell(\mathfrak{A}) = \{ \mathcal{L}_{\ell(\mathfrak{A})} \mid \mathcal{R}_{\ell(\mathfrak{A})} \} = \{ \mathbb{N}_0 \mid \mathbb{N}_0 \} = 0 \) is a universal representation, Definition 15 yields \( \lim_{i \to \mathbb{O}_n} (-1/2)^i = 0 \), a result we again might have expected from studying the real analogue of \( \mathfrak{A} \).

### 6. Limits and Derivatives of Functions

In this section, we present a universal representation for the limit of a function, prove the Intermediate Value Theorem for surreals without completeness, and provide an example of how our universal representation can be used to find a derivative. We also consider consequences of our universal representations for limits and derivatives. In particular, we prove that two key limit laws, addition and multiplication, hold for surreal functions. We now show how the limit of a function can be evaluated using a universal representation.
6.1. Evaluation of Limits of Functions

The following is our definition of the limit of a surreal function. Notice that the definition is analogous to that of the limit of an On-length surreal sequence, Definition 15.

Definition 27. Let f be a function defined on an open interval surrounding a, except possibly at a. We say that f(x) converges to a limit l as x → a and write that \( \lim_{x \to a} f(x) = l \) if the expression in (10) is the universal representation of l.

\[
\left\{ p : p < \sup \left( \bigcup_{b<x<a} \bigcap_{x<y<a} \mathcal{L}_f(y) \right) \right\} < \inf \left( \bigcup_{a<x<c} \bigcap_{a<y<x} \mathcal{R}_f(y) \right) = \ell
\]

We now prove that the universal representation notion of the limit of a surreal function is equivalent to the standard \( \varepsilon \)-\( \delta \) definition.

Theorem 28. Let \( f(y) \) be defined on \( (b,c) \subseteq \text{No} \) containing a, except possibly at a. If \( \lim_{y \to a} f(y) = \ell \in \text{No} \), then for every (surreal) \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( 0 < |y-a| < \delta, \) \( |f(y) - \ell| < \varepsilon. \) Conversely, if \( \ell \) is a number such that for every (surreal) \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( 0 < |y-a| < \delta, \) \( |f(y) - \ell| < \varepsilon, \) then \( \lim_{y \to a} f(y) = \ell. \)

Proof. For the forward direction, we must prove that for every \( \varepsilon > 0, \) there exists \( \delta > 0 \) such that whenever \( 0 < |y-a| < \delta, \) \( |f(y) - \ell| < \varepsilon. \) We must consider \( x \to a \) (1) from below, and (2) from above. We first consider case (1). Suppose that for some \( \varepsilon > 0, \) we cannot find \( \delta > 0 \) such that whenever \( 0 < x-a < \delta, \) \( |f(x) - \ell| < \varepsilon. \) Then, for every \( x \in (b,a), \) there is always a value of \( y \in [x,a) \) such that \( |f(y) - \ell| \geq \varepsilon. \) This contradicts the fact that sup \( \left( \bigcup_{b<x<a} \bigcap_{x<y<a} \mathcal{L}_f(y) \right) = \ell, \) because \( |f(y) - \ell| \geq \varepsilon \) implies that sup \( \left( \bigcup_{b<x<a} \bigcap_{x<y<a} \mathcal{L}_f(y) \right) \) should be either > or < \( \ell. \) We now consider case (2). Suppose that for some \( \varepsilon > 0, \) we cannot find \( \delta > 0 \) such that whenever \( 0 < a-x < \delta, \) \( |f(x) - \ell| < \varepsilon. \) Then, for every \( x \in (a,c), \) there is always a value of \( y \in (a,x] \) such that \( |f(y) - \ell| \geq \varepsilon. \) This contradicts the fact that inf \( \left( \bigcup_{a<x<c} \bigcap_{a<y\leq x} \mathcal{R}_f(y) \right) = \ell, \) because \( |f(y) - \ell| \geq \varepsilon \) implies that inf \( \left( \bigcup_{a<x<c} \bigcap_{a<y\leq x} \mathcal{R}_f(y) \right) \) should be either > or < \( \ell. \) Cases (1) and (2) suffice to show that \( \lim_{y \to a} f(y) = \ell. \)

For the other direction, if \( \lim_{y \to a} f(y) \neq \ell, \) there are two cases to consider. The first case is that the expression in (10) in Definition 27 is not a universal representation. This would imply that

\[
\inf \left( \bigcup_{a<x<c} \bigcap_{a<y\leq x} \mathcal{R}_f(y) \right) - \sup \left( \bigcup_{b<x<a} \bigcap_{x<y<a} \mathcal{L}_f(y) \right) > \varepsilon,
\]

for some \( \varepsilon > 0, \) because otherwise we would have elements of the right class of a number smaller than elements of the left class. But since \( f(y) \) can be made arbitrarily close to \( \ell \) by taking \( y \) sufficiently close to \( a, \) we can pick \( p \in \bigcup_{a<x<c} \bigcap_{a<y\leq x} \mathcal{R}_f(y) \) and \( q \in \bigcup_{b<x<a} \bigcap_{x<y<a} \mathcal{L}_f(y) \) such that \( |p-\ell| < \varepsilon/2 \) and \( |q-\ell| < \varepsilon/2. \) By the Triangle Inequality, \( |p-q| \leq |p-\ell| + |q-\ell| < \varepsilon/2 + \varepsilon/2 < \varepsilon, \) which contradicts the claim in (11). Thus, it follows that the expression in (10) in Definition 27 is a universal representation.

The second case is that the expression in (10) in Definition 27, which for brevity we denote as \( \ell', \) is a gap, not a number. Since the expression in (10) in Definition 27 is a universal representation,
Now suppose that $\ell' < \ell$. Using notation from the proof of the first case above, we know that for every $\varepsilon > 0$, we have $|q - \ell| < \varepsilon$. If we pick $\varepsilon$ such that $\ell - \varepsilon > \ell'$, then we have $q > \ell'$, which contradicts (12). Thus, $\ell' \neq \ell$. By analogous reasoning in which we replace “left” with “right,” we find that $\ell' \neq \ell$. Finally, we have $\ell' = \ell$, so $\ell'$ cannot be a gap. Thus, we have the theorem. \qed

Remark. As with limits of On-length sequences, we restrict Theorem 28 to functions that converge to numbers because the standard $\varepsilon$-$\delta$ definition does not generalize to gaps. Also, notice that the expression on the right-hand-side of (1) in Definition 14 as well as Definition 15 can be easily modified to provide a definition of a limit of a function $f(x)$ as $x \to \text{On}$ or $x \to \text{Off}$. Finally, the notions of $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ are also preserved in Theorem 28. Specifically, the left class of $\lim_{x \to a} f(x)$ describes the behavior of $f(x)$ as $x \to a^-$ and the right class of $\lim_{x \to a} f(x)$ describes the behavior of $f(x)$ as $x \to a^+$.

6.2. Intermediate Value Theorem

Even though the standard proof of the Intermediate Value Theorem (IVT) on $\mathbb{R}$ requires completeness, we show in this subsection that we can prove the IVT for surreals without using completeness by using Definitions 7 and 8 along with the following results regarding connectedness in No. The proofs below are surreal versions of the corresponding proofs from [Mun00].

Definition 29. A class $T \subset \text{No}$ is connected if there does not exist a separation of $T$; i.e. there does not exist a pair of disjoint nonempty classes $U, V$ that are open in $T$ such that $T = U \cup V$.

Lemma 30. Every convex class $T \subset \text{No}$ is connected.

Proof. Suppose the pair of classes $U, V$ forms a separation of $T$. Then, take $u \in U$ and $v \in V$, and assume without loss of generality that $u < v$ ($u \neq v$ because $U \cap V = \emptyset$). Because $T$ is convex, we have that $[u, v] \subset T$, so consider the pair of classes $U' = U \cap [u, v], V' = V \cap [u, v]$. Notice that (1) because $U \cap V = \emptyset$, we have that $U' \cap V' = \emptyset$; (2) $u \in U'$ and $v \in V'$, so neither $U'$ nor $V'$ is empty; (3) $U'$ and $V'$ are open in $[u, v]$; and (4) clearly $U' \cup V' = [u, v]$. So, the pair $U', V'$ forms a separation of $[u, v]$.

Now consider $w = \text{sup}(U')$. If $w \in \text{No}$, then we have two cases: (1) $w \in V'$ and (2) $w \in U'$. In case (1), because $V'$ is open, there is an interval contained in $V'$ of the form $[x, w]$, for some number or gap $x$. Then $x$ is an upper bound of $U'$ that is less than $w$, because all numbers between $w$ and $v$ inclusive are not in $U'$, which contradicts the definition of $w$. In case (2), because $U'$ is open, there is an interval contained in $U'$ of the form $[w, y]$, for some number or gap $y$. But then any $z \in [w, y]$ satisfies both $z \in U'$ and $z > w$, which again contradicts the definition of $w$.

If $w$ is a gap, then notice that there is no $x \in V'$ such that both $x < w$ and $(x, w) \subset V'$ hold, because then $w \neq \text{sup}(U')$. Thus, the class $V = \{x \in V' : x > w\}$ is open, for no intervals contained in $V'$ lie across $w$. Now, $w = \{U' \mid V\}$, but because $U'$ and $V$ are open, they are unions of intervals indexed over sets. Consequently, $w = \{L \mid R\}$ for some pair of sets $L, R$, so $w \in \text{No}$, which contradicts our assumption that $w$ is a gap. \qed

Lemma 31. If $f$ is continuous on $[a, b]$, then the image $f([a, b])$ is connected.

Proof. Suppose $f([a, b])$ is not connected. Then there exists a separation $U, V$ of $f([a, b])$; i.e. there exists a pair of disjoint nonempty open classes $U, V$ such that $f([a, b]) = U \cup V$. Then, we have the following: (1) $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint because $U, V$ are disjoint; (2) $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty because the map from $[a, b]$ to the image $f([a, b])$ under $f$ is clearly surjective; (3) $f^{-1}(U)$ and $f^{-1}(V)$ are open in $[a, b]$ because $f$ is continuous, so the preimages of open classes
are open in \([a, b]\); and (4) \([a, b] = f^{-1}(U) \cup f^{-1}(V)\) because any point whose image is in \(U\) or \(V\) must be in the corresponding preimage \(f^{-1}(U)\) or \(f^{-1}(V)\). Thus, the pair \(f^{-1}(U), f^{-1}(V)\) forms a separation of \([a, b]\). But this contradicts Lemma 30, because intervals are convex, so \(f([a, b])\) is connected.

\[\square\]

**Theorem 32 (IVT).** If \(f\) is continuous on \([a, b] \subset \mathbb{No}\), then for every \(u \in \mathbb{No}\) that lies between \(f(a)\) and \(f(b)\), there exists a number \(p \in [a, b]\) such that \(f(p) = u\).

**Proof.** Assume that neither \(f(a) = u\) nor \(f(b) = u\) (if either of these were true, we would have the theorem). Consider the classes \(U = f([a, b]) \cap (\mathbb{Off}, u)\) and \(V = f([a, b]) \cap (u, \mathbb{On})\). Notice that (1) \(U \cap V = \emptyset\) because \((\mathbb{Off}, u) \cap (u, \mathbb{On}) = \emptyset\); (2) neither \(U\) nor \(V\) is empty because either \(f(a) < f(b)\) so \(f(a) \in U\) and \(f(b) \in V\), or \(f(a) > f(b)\) so \(f(a) \in V\) and \(f(b) \in U\); and (3) both \(U\) and \(V\) are open in \(f([a, b])\) (but not necessarily in \(\mathbb{No}\)) because each is the intersection of \(f([a, b])\) with an open ray. Now assume there is no \(p \in [a, b]\) such that \(f(p) = u\). Because \(f([a, b]) = U \cup V\), we have that the pair \(U, V\) is a separation for \(f([a, b])\), so \(f([a, b])\) is not connected. But this violates Lemma 31, so there is a \(p \in [a, b]\) such that \(f(p) = u\).

The IVT holds for surreals does not of course prevent functions from reaching numbers at gaps, but it does prevent continuous functions from having isolated zeroes at gaps.\(^7\) More precisely, a continuous function \(f\) can reach 0 at a gap \(g\) iff for every (surreal) \(\varepsilon > 0\) there exists a zero of \(f\) in some open interval of width \(\varepsilon\) containing \(g\). The only continuous functions we know of that reach a number at gap are constant on an open interval containing the gap, but we do not yet know whether it is impossible for a continuous function to reach a number at a gap without being locally constant.

### 6.3. Derivatives and Limit Laws

It is important to note that derivatives are limits of functions; i.e. \(\frac{d}{dt} f(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}\). Therefore, derivatives can be evaluated using Definition 27. The following example demonstrates this method of differentiation:

**Example 33.** Let us evaluate \(\frac{d}{dt} t^2\) using Definition 27. We must find \(\lim_{h \to 0} \frac{(t+h)^2-t^2}{h}\) or \(\lim_{h \to 0} 2t + h\), which is defined at least for \(h \in (-1, 1)\). Observe that \(\mathcal{L}_{2t+h} = \{t + t^{\mathbb{L}} + h, 2t + h^{\mathbb{L}}\}\) and \(\mathcal{R}_{2t+h} = \{t + t^{\mathbb{R}} + h, 2t + h^{\mathbb{R}}\}\). Thus, we get:

\[
\frac{d}{dt} t^2 = \left\{ p : p < \sup \left( \bigcup_{-1 < x < 0} \bigcap_{-h < x < 0} \{t + t^{\mathbb{L}} + h, 2t + h^{\mathbb{L}}\} \right) \right\}.
\]

Now pick \(z \in \mathbb{No}_{<2t}\). Then there exists \(h \in [x, 0)\) such that \(h > z - 2t\). So for some choice of \(h \in [x, 0)\), there exists \(h^{\mathbb{L}}\) with \(z - 2t = h^{\mathbb{L}}\), or equivalently \(z = 2t + h^{\mathbb{L}}\). Therefore, we know that \(z \in \left( \bigcup_{-1 < x < 0} \bigcap_{-h < x < 0} \{t + t^{\mathbb{L}} + h, 2t + h^{\mathbb{L}}\} \right)\) for all \(z \in \mathbb{No}_{<2t}\). Also, \(t + t^{\mathbb{L}} + h, 2t + h^{\mathbb{L}} < 2t\) is always true for \(h \in [x, 0)\), so \(\sup \left( \bigcup_{-1 < x < 0} \bigcap_{-h < x < 0} \{t + t^{\mathbb{L}} + h, 2t + h^{\mathbb{L}}\} \right) = 2t\). Thus, the left class is \(\mathbb{No}_{<2t}\). Next pick \(z \in \mathbb{No}_{>2t}\). Then there exists \(h \in (0, x]\) such that \(h < z - 2t\). So for

\(^7\)The zero-function reaches 0 at every gap.
some choice of \( h \in (0, x] \), there exists \( h^* \) with \( z - 2t = h^* \), or equivalently \( z = 2t + h^* \). Therefore, \( z \in \left( \bigcup_{0 < x < 1} \bigcap_{0 < h < x} \{ t + t^* + h, 2t + h^* \} \right) \) for all \( z \in \text{No}_{>2t} \). Also, \( t + t^* + h, 2t + h^* > 2t \) is always true for \( h \in (0, x] \), so \( \inf \left( \bigcup_{0 < x < 1} \bigcap_{0 < h < x} \{ t + t^* + h, 2t + h^* \} \right) = 2t \). Thus, the right class is \( \text{No}_{>2t} \). Finally, we have \( \frac{d}{dt}t^2 = 2t \), as usual.

Evaluating limits and derivatives of functions can be made easier through the use of limit laws. We introduce two key limit laws in the following theorem:

**Theorem 34.** Let \( a \in \text{No} \) and \( f, g \) be functions, and suppose that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist. Then, the following hold: (1) Addition: \( \lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \); and (2) Multiplication: \( \lim_{x \to a} (f \cdot g)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \).

**Proof.** We only consider addition, because the arguments for multiplication are similar to those for addition. If we let \( \lim_{x \to a} f(x) = \ell(f) \) and \( \lim_{x \to a} g(x) = \ell(g) \), then we want to show that \( \ell(f) + \ell(g) = \sup \left( \bigcup_{b < x < c} \bigcap_{x \leq y < a} \mathcal{L}(f + g)(y) \right) = \inf \left( \bigcup_{a < c < d} \bigcap_{a < y \leq x} \mathcal{R}(f + g)(y) \right) \). We first show that \( \ell(f) + \ell(g) \not< \inf \left( \bigcup_{b < x < a} \bigcap_{x \leq y < a} \mathcal{L}(f + g)(y) \right) \). Suppose for now that we have \( \ell(f) + \ell(g) < \inf \left( \bigcup_{b < x < a} \bigcap_{x \leq y < a} \mathcal{L}(f + g)(y) \right) \). Since \( \mathcal{L}(f + g)(y) = \{ f(y) + g(y), f(y) + g(y) \} \), for some \( \varepsilon, \delta > 0 \), either \( f(y) \geq \ell(f) + \varepsilon \) or \( g(y) \geq \ell(g) + \delta \) or both are true for some \( y \) in an arbitrarily small interval \( (x, a) \). This contradicts \( \lim_{x \to a} f(x) = \ell(f) \) or \( \lim_{x \to a} g(x) = \ell(g) \) or both, and so our supposition was false. A similar proof can be used to show that \( \ell(f) + \ell(g) \not> \sup \left( \bigcup_{b < x < a} \bigcap_{x \leq y < a} \mathcal{L}(f + g)(y) \right) \).

We next show that \( \ell(f) + \ell(g) \not< \inf \left( \bigcup_{b < x < a} \bigcap_{x \leq y < a} \mathcal{R}(f + g)(y) \right) \). Suppose for now that we have \( \ell(f) + \ell(g) < \inf \left( \bigcup_{b < x < a} \bigcap_{x \leq y < a} \mathcal{R}(f + g)(y) \right) \). Since \( \mathcal{R}(f + g)(y) = \{ f(y)^* + g(y), f(y) + g(y)^* \} \), for some \( \varepsilon, \delta > 0 \), either \( f(y) \geq \ell(f) + \varepsilon \) or \( g(y) \geq \ell(g) + \delta \) or both are true for some \( y \) in an arbitrarily small interval \( (x, a) \). This contradicts \( \lim_{x \to a} f(x) = \ell(f) \) or \( \lim_{x \to a} g(x) = \ell(g) \) or both, and so our supposition was false. A similar proof can be used to show that \( \ell(f) + \ell(g) \not> \inf \left( \bigcup_{b < x < a} \bigcap_{x \leq y < a} \mathcal{R}(f + g)(y) \right) \). \( \square \)

**Remark.** Note that in the proof of Theorem 34 above, we made use of the properties of the universal representation for the limit of a function. However, limit laws can also be proven using the \( \varepsilon, \delta \) definition of the limit alone. That both of these proofs work demonstrates the equivalence, established in Theorem 28, between the notion of the limit as a universal representation and the usual \( \varepsilon, \delta \) notion of the limit.

# 7. Power Series and Integrals

In this section, we present our methods of evaluating power series and infinite “Riemann” sums. We also prove the Fundamental Theorem of Calculus as a method of evaluating integrals easily, as long as we have a definition of integration. A consistent genetic definition or universal representation of Riemann integration that works for all functions nevertheless remains to be discovered.

## 7.1. Power Series

The evaluation of power series in real analysis usually entails finding a limit of a sequence of partial sums. Because Definition 15 allows us to find limits of sequences, it might seem as though evaluating power series as limits of partial sum sequences is possible. However, it is often the case that we do
not know a closed-form expression for the $\alpha^{\text{th}}$ partial sum of a series, where $\alpha \in \text{On}$. Without such an expression, we cannot determine the left and right options of the $\alpha^{\text{th}}$ partial sum and therefore cannot use Definition 15. Also, suppose $\zeta$ is a limit-ordinal. Then by Theorem 11, we cannot claim that the $\zeta^{\text{th}}$ partial sum is the limit of previous partial sums, for this limit might be a gap. The next example illustrates how the partial sums of an $\text{On}$-length series can become “stuck” at a gap:

**Example 35.** Consider the series $s = \sum_{i \in \text{On}} 1/2^i$. The sum of the first $\omega$ terms, $\sum_{i \in \text{On} \cap \omega} 1/2^i$, is the gap $g$ between real numbers with real part less than 2 and numbers with real part at least 2. By the definition of gap addition described in [Con01], it is clear that $g + \varepsilon = g$ for all infinitesimals $\varepsilon$. But note that all remaining terms in the series, namely $1/2^2, 1/2^{2+1}, \ldots$, are infinitesimals, so the sequence of partial sums for the entire series is: $1, 3/2, 7/4, \ldots, g, g, \ldots, g, g, \ldots$. So, if we define the sum of a series to be the limit of its partial sums, we find that $s = g$, a result that is against our intuition that $s = 2$ (which is true for the real series $\sum_{i=0}^{\infty} 1/2^i$).

Because of the problem described in Example 35, we cannot use the standard notion of “sum” in order to create surreal power series that behave like real power series. Our solution to this problem is to extrapolate natural partial sums (which can be evaluated using part 3 of Definition 2) to ordinal partial sums. We define this method of extrapolation as follows:

**Definition 36.** Let $\sum_{i=1}^{\text{On}} a_i$ be a series. Suppose that for all $n \in \mathbb{N}$, $\sum_{i=1}^{\text{On}} a_i = f(n)$, where $f(n)$ can be expressed as a fixed linear combination of finitely many rational and exponential functions. Then for all $\alpha \in \text{On}$, define $\sum_{i=1}^{\alpha} a_i := f(\alpha)$.

**Remark.** We call a fixed linear combination of finitely many rational and exponential functions a “closed-form expression.” Notice that closed-form expressions are well-defined.

Definition 36 is intended to be used with Definition 15. Specifically, $\sum_{i=1}^{\text{On}} a_i = \lim_{\alpha \to \text{On}} f(\alpha)$, which is easy to evaluate using Definition 15. The rule in Definition 36 only works when the $n^{\text{th}}$ partial sum of a series can be expressed as a fixed linear combination of finitely many known surreal functions. In other cases, our method of extrapolation ceases to work. If we return to the case in Example 35, we see that Definition 36 does indeed yield $\sum_{i \in \text{On}} 1/2^i = 2$, as desired. Moreover, we find that $1/e^x = 1 + x + x^2 + \ldots$ holds on the interval $-1 < x < 1$, as usual. However, the power series of other known functions including $e^x$, $\arctan(x)$, and $\text{nl}(x)$ cannot be evaluated using Definition 36. Nevertheless, extrapolation seems to be useful for evaluating power series, and further investigation might lead to a more general method of evaluating such series.

### 7.2. Integrals and the Fundamental Theorem of Calculus

Real integrals are usually defined as limits of Riemann sums. Because Definition 15 gives us the limits of $\text{On}$-length sequences and since we know how to evaluate certain kinds of sums using Definition 36, we now discuss how we can evaluate certain Riemann sums. We define distance and area, which are necessary for integration. We now define distance in $\text{No}^2$ to be analogous to the distance metric in $\mathbb{R}^2$:

**Definition 37.** Let $A = (a_1, a_2), B = (b_1, b_2) \in \text{No}^2$. Then the distance $AB$ from $A$ to $B$ is defined to be $AB = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$.

It is shown in [All87] that distance as defined in Definition 37 satisfies the standard properties of distance that holds in $\mathbb{R}$. Also, in $\mathbb{R}^2$, a notion of the area of a rectangle exists. We define the area of a rectangle in $\text{No}^2$ to be analogous to the area of a rectangle in $\mathbb{R}^2$:

**Definition 38.** If $ABCD$ is a rectangle in $\text{No}^2$, its area is $[ABCD] = AB \cdot BC$. 

20
Because we have defined the area of a rectangle in \( \mathbb{No}^2 \), we can consider Riemann sums. It is easy to visualize a Riemann sum in which the interval of integration is divided into finitely many subintervals. However, when the number of subintervals is allowed to be any ordinal, it is not clear what adding up the areas of an infinite number of rectangles means. For this reason, earlier work has restricted Riemann sums to have only finitely many terms. Just as we did with power series earlier in this section, we make an “extrapolative” definition for what we want the \( \alpha \)th Riemann sum of a function to be, thereby defining what we mean by an infinite Riemann sum:

**Definition 39.** Let \( f(x) \) be a function continuous except at finitely many points on \( [a, b] \). Suppose that for all \( n \in \mathbb{N} \) and \( c, d \in [a, b] \) such that \( c \leq d \), \( g(n, c, d) = \sum_{i=0}^{n} \frac{d-c}{n} f \left( c + i \left( \frac{d-c}{n} \right) \right) \), where \( g(n, c, d) \) can be expressed as a fixed linear combination of finitely many rational and exponential functions. Then, for all \( \alpha \in \Omega \), define the \( \alpha \)th Riemann sum of \( f \) on \( [a, b] \) to be \( g(\alpha, a, b) \).

For functions like polynomials and exponentials, Definition 39 in combination with Definition 15 evaluates integrals correctly. In particular, if \( g(\alpha, a, b) \) is known for some function \( f(x) \) on the interval \( [a, b] \), then \( \int_{a}^{b} f(x) dx = \lim_{\alpha \to \Omega} g(\alpha, a, b) \). We demonstrate this method as follows:

**Example 40.** Let us evaluate \( \int_{a}^{b} \exp(x) dx \) by using our “extrapolative notion” of Riemann sums. In this case, we have the following, which results when we use Definition 39:

\[
g(\alpha, a, b) = \frac{b-a}{\alpha} \sum_{i=0}^{\alpha} \exp \left( a + i \left( \frac{b-a}{\alpha} \right) \right)
= \frac{(a-b) \exp(a+a/\alpha)}{\alpha(\exp(a/\alpha) - \exp(b/\alpha))} \left( -1 + \exp \left( \frac{(b-a)(\alpha+1)}{\alpha} \right) \right)
\]

It is now easy to see that \( \lim_{\alpha \to \Omega} g(\alpha, a, b) = \exp(b) - \exp(a) \), as desired. In the case where \( a = 0 \) and \( b = \omega \), we have \( \int_{0}^{\omega} \exp(x) dx = \exp(\omega) - 1 \), which resolves the issue with the integration methods used in [Con01] and [For04].

In real calculus, limits of Riemann sums are difficult to evaluate directly for most functions. In order to integrate such functions, the notion of antiderivatives is used. However, we require the Fundamental Theorem of Calculus (FTC) in order to say that finding an antiderivative is the same as evaluating an indefinite integral. Because we can differentiate surreal functions using Definition 27, we can also find antiderivatives of surreal functions. In order to compute indefinite surreal integrals using antiderivatives, we show that the FTC holds on \( \mathbb{No} \).

We now state and prove the surreal analogue of the Extreme Value Theorem (EVT), which is required to prove the FTC. To prove the EVT, we need four lemmas regarding compactness.

**Lemma 41 ([For04]).** Let \( A \) be a proper set of subintervals open in \( \mathbb{No} \), where every subinterval has endpoints in \( \mathbb{No} \cup \{ \text{On}, \text{Off} \} \). If \( A \) is a covering of \( \mathbb{No} \), then \( A \) has a finite subcovering.\(^8\)

The proofs of the following lemmas and the EVT use the standard arguments (see [Mun00] for these proofs) but with modifications made to work with the new notion of compactness.

**Lemma 42.** Let \( A \) be a subinterval of \( \mathbb{No} \). Then \( A \) is compact if every covering of \( A \) by subintervals open in \( \mathbb{No} \) has a finite subcovering.

\(^8\)The result in Lemma 41 is a special form of compactness, where sets are replaced by subintervals. We use the unqualified word “compactness” to refer to this special form only.
Proof. Let $\mathcal{B} = \{\mathcal{B}_\alpha\}$ be a covering of $A$ by subintervals open in $A$. For each $\mathcal{B}_\alpha$, find a subinterval $\mathcal{C}_\alpha$ open in $\mathbb{N}$ such that $\mathcal{B}_\alpha = \mathcal{C}_\alpha \cap A$. Then $\mathcal{C} = \{\mathcal{C}_\alpha\}$ is a covering of $A$ by subintervals open in $\mathbb{N}$, so $\mathcal{C}$ has a finite subcovering, say $\{\mathcal{C}_n\}$, for some $n \in \mathbb{N}$. Thus, $\{\mathcal{B}_n\}$, a finite subcovering of $\mathcal{B}$, covers $A$, from which it follows that $A$ is compact. \hfill \square

Lemma 43. Let $[a, b] \subset \mathbb{N}$. Suppose $A$ is a set of subintervals open in $[a, b]$. If $A$ is a covering of $[a, b]$, then $A$ has a finite subcovering.

Proof. Let $\mathcal{B} = \{\mathcal{B}_\alpha\}$ be a covering of $[a, b]$ by subintervals open in $\mathbb{N}$. Now $\bigcup_\alpha \mathcal{B}_\alpha = (c, d)$ for some $c, d \in \mathbb{N}$. Then if $c' \in (c, a), d' \in (b, d)$, $\mathcal{C} = \mathcal{B} \cup \{\text{On}, c'\} \cup \{d', \text{On}\}$ is an open covering of $\mathbb{N}$ that contains subintervals whose endpoints are in $\mathbb{N} \cup \{\text{On}, \text{Off}\}$. By Lemma 41, $\mathcal{C}$ has a finite subcovering $\mathcal{C}'$. If $\{\text{Off}, c'\} \in \mathcal{C}'$, discard $\{\text{Off}, c'\}$; if $\{d', \text{On}\} \in \mathcal{C}'$, discard $\{d', \text{On}\}$. The resulting subcovering of $\mathcal{C}'$ is a finite subcovering of $\mathcal{B}$ that covers $[a, b]$. So, by Lemma 42, the closed interval $[a, b]$ is compact. \hfill \square

Lemma 44. Let $A$ be a closed subinterval of $\mathbb{N}$, and let $f : A \to \mathbb{N}$ be continuous. Then $f(A)$ is compact.

Proof. Let $\mathcal{B}$ be a covering of $f(A)$ by subintervals open in $\mathbb{N}$. Note that for all $B \in \mathcal{B}$, $f^{-1}(B)$ is a union of subintervals open in $\mathbb{N}$ by Definition 8 because $f$ is continuous. Then $\mathcal{C} = \{f^{-1}(B) : B \in \mathcal{B}\}$ is a covering of $A$ with subintervals open in $\mathbb{N}$. By Lemma 43, $A$ is compact because $A$ is closed. Therefore, $\mathcal{C}$ has a finite subcovering, say $\{f^{-1}(B_i) : B_i \in \mathcal{B}, i \in \mathbb{N}_{\leq n}\}$ for some $n \in \mathbb{N}$. Then $\{B_i : B_i \in \mathcal{B}, i \in \mathbb{N}_{\leq n}\}$ is a finite subcovering of $\mathcal{B}$ that covers $f(A)$. So, by Lemma 42, $f(A)$ is compact. \hfill \square

Theorem 45 (EVT). Let $A$ be a closed subinterval of $\mathbb{N}$, and let $f : A \to \mathbb{N}$ be continuous. Then there exists $c, d \in A$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in A$.

Proof. Because $A$ is a closed subinterval of $\mathbb{N}$, $A$ is compact by Lemma 43, and because $f$ is continuous, $f(A)$ is compact by Lemma 44. Suppose that $\sup(f(A))$ is a gap. Then $f(A)$ cannot be covered by a finite number of open subintervals with endpoints in $\mathbb{N} \cup \{\text{On}, \text{Off}\}$, so $f(A)$ is not compact, which is a contradiction. Thus, $\sup(f(A)) = d \in \mathbb{N}$, and by a similar argument, $\inf(f(A)) = c \in \mathbb{N}$. Now suppose that $f$ does not attain an absolute maximum value on $A$. Then, $f(A)$ is either $[c, d)$ or $(c, d]$ and is therefore not compact, which is again a contradiction. So, $f$ attains an absolute maximum value $d$ on $A$, and by a similar argument, $f$ attains an absolute minimum value $c$ on $A$. \hfill \square

To prove the FTC on $\mathbb{N}$, we also need (at least) a characterization of the definite integral of a function $f$ on the interval $[a, b]$ that works for all continuous functions $f$. (Our extrapolative method of evaluating Riemann sums only works for functions that satisfy the conditions of Definition 39.) We present our characterization of integration as follows:

Definition 46. The definite integral of a continuous function $f$ on an interval $[a, b]$ is a function $T(a, b)$ that satisfies the following three properties: (1) If for all $x \in [a, b]$ we have that $f(x) = c$ for some $c \in \mathbb{N}$, $T(a, b) = c(b - a)$. (2) If $m$ is the absolute minimum value of $f$ on $[a, b]$ and $M$ is the absolute maximum value of $f$ on $[a, b]$, then $m(b - a) \leq T(a, b) \leq M(b - a)$; and (3) for any number $c \in [a, b]$, $T(a, c) + T(c, b) = T(a, b)$.

Remark. Note that in Definition 46, we do not characterize $T(a, b)$ completely; we merely specify the properties that a definite integral must have in order for the FTC to be true. In fact, $T(a, b)$ can be any function having these properties, and the FTC will still hold. Observe that if we
only consider functions that satisfy the requirements of Definition 39, our extrapolative notion of Riemann sums does satisfy Definition 46.

We are now ready to state and prove the FTC on No.

**Theorem 47 (FTC).** If \( f \) is continuous on \([a, b] \subset \text{No}\), then the function \( g \) defined for all \( x \in [a, b] \) by \( g(x) = \int_a^x f(t)dt \) is continuous on \([a, b]\) and satisfies \( g'(x) = f(x) \) for all \( x \in (a, b) \).

**Proof.** The standard proof of the FTC from real analysis (see [Spi08] for this proof) works for surreals because: (1) We can find derivatives of functions using Definition 27; (2) the Extreme Value Theorem holds on \text{No}; and (3) we have defined integration in Definition 46. We outline the proof as follows.

Pick \( x, x+h \in (a, b) \). Then by Definition 46, \( g(x+h) - g(x) = \int_x^{x+h} f(t)dt \), so as long as \( h \neq 0 \),

\[
\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt. \tag{13}
\]

Suppose \( h > 0 \). We know that \( f \) is continuous on \([x, x+h]\), so by the EVT, there exist \( c, d \in [x, x+h] \) such that \( f(c) \) is the absolute minimum value of \( f \) on \([x, x+h]\) and \( f(d) \) is the absolute maximum value of \( f \) on \([x, x+h]\). By Definition 46, we know that \( h \cdot f(c) \leq \int_x^{x+h} f(t)dt \leq h \cdot f(d) \), so substituting the result of (13) we have:

\[
f(c) \leq \frac{g(x+h) - g(x)}{h} \leq f(d), \tag{14}
\]

which holds when \( h < 0 \) too, but the argument is similar so we omit it. If we let \( h \to 0 \), it is clear by the Squeeze Theorem (which follows from Definition 27 and Theorem 28) that \( g'(x) = f(x) \).

One issue with relying solely upon antidifferentiation to evaluate integrals is that surreal functions do not have unique primitives. For example, in the case of the function \( f(x) = 1 \), there are many possible continuous primitives in addition to \( F(x) = x \), including

\[
F(x) = \begin{cases} x & : x < \infty \\ x - 1 & : x > \infty. \end{cases} \tag{15}
\]

The piecewise function \( F(x) \) above is continuous because both \( x \) and \( x - 1 \) approach \( \infty \) as \( x \) approaches \( \infty \), which would not be the case if \( x - 1 \) were replaced by, say, \( x - \omega \) in the definition of \( F \). We believe that the issue of selecting the primitive of a function can be resolved by utilizing the notion of function complexity described in Subsection 2.3. In particular, we present the following rule as a means of determining what the primitive of a function should be:

**Definition 48.** The primitive of a continuous function \( f \) is the simplest continuous function \( F \) such that \( F' = f \).

Definition 48 eliminate functions like \( F(x) \) in (15) as possible primitives of \( f(x) = 1 \), for the expression \( x - 1 \) is more complicated than the expression \( x \) and functions that do require piecewise representations are more complicated than functions that do not.

Using Theorem 47 in combination with Definition 48, it is easy to integrate functions using the method of antidifferentiation, as long as we have a definition of integration. Note that this method of integration works for all known surreal functions whose antiderivatives are also known, not just functions for which we can evaluate limits of Riemann sums using the method of extrapolation in combination with Definition 15.
8. Conclusions and Open Questions

In this research, we extended the theory of surreal analysis by identifying issues with methods employed in earlier work and by creating definitions that are more consistent with their real analogues. We found new formulae for surreal versions of the functions arctan(x), − log(1−x), and \( \frac{1}{1-x} \). Moreover, we introduced methods of determining the limit of a sequence and the limit (and hence the derivative) of a function. Finally, we proposed a method of evaluating elementary power series and Riemann sums, and we proved the Fundamental Theorem of Calculus for surreals. Our new definitions as well as their consequences show that there are numerous analytic connections between the surreals and reals, along with the arithmetic and algebraic relationships found in earlier work.

Several open questions remain. In order to complete the analogy between real and surreal functions, consistent genetic formulae of other transcendental functions, such as \( \sin(x) \), along with their necessary properties remain to be found. Furthermore, methods of extending such genetic formulae to work on domains larger than the region of convergence of a power series need to be studied. The most significant open problem that remains in surreal analysis is finding a genetic formula or a universal representation for the definite integral of a function. Such a definition of integration must also satisfy the requirements of Definition 46. Two other aspects of real calculus that remain incomplete for surreals are power series and DEs. A method of evaluating power series in greater generality remains to be developed, one that does not depend on the form of the \( \alpha \)th partial sum. In addition, using such a method to evaluate power series should allow basic properties, such as \( f(x) = (\text{power series of } f(x)) \) on its region of convergence, to hold. To extend surreal analysis even further, it is necessary to investigate functions of multiple variables as well as more general versions of the results presented in this paper. For example, a future study could consider proving a surreal version of Stokes’ Theorem (see [Spi71]) as a generalization of the FTC once a consistent notion of surreal differential forms has been developed.

A comprehensive study of DEs remains to be performed, and as part of such a study, many questions should be answered. The following are two possible questions about the behavior of an analytic function \( f(x) \) under basic calculus operations that such a study should answer: (1) What is \( \frac{d^\alpha}{dx^\alpha} f(x) \) for any \( \alpha \in \text{On} \)? and (2) Does the function that results from integrating \( f(x) \) some ordinal \( \alpha \) number of times and then differentiating \( \alpha \) times equal \( f(x) \) for all \( \alpha \in \text{On} \)? Finding answers to such questions would settle the structure of surreal DEs and would determine whether surreal DEs are more general than their real analogues. Once a consistent method of solving DEs has been created, this method should be applied to DEs whose solutions are not known. Solving surreal DEs would not only allow power series to be evaluated [Cos12], but also impact fields as diverse as mathematics, science, and engineering by permitting solutions of currently unsolvable DEs to be found.

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