ON LOGARITHMIC COEFFICIENTS OF CERTAIN STARLIKE FUNCTIONS RELATED TO VERTICAL STRIP

R. KARGAR

Abstract. In the present paper two certain subclasses of starlike functions associated with vertical strip are considered which are denoted by $S(\alpha, \beta)$ and $M(\delta)$ where $\alpha < 1$, $\beta > 1$ and $\pi/2 \leq \delta < \pi$. The main aim of this paper is to investigate some basic properties of the classes $S(\alpha, \beta)$ and $M(\delta)$ such as, subordination relations, sharp inequalities for sums involving logarithmic coefficients and estimate of logarithmic coefficients.

1. Introduction

Throughout this article $\Delta$ is the open unit disk on the complex plane $\mathbb{C}$. Let $H$ be the class of all analytic functions in $\Delta$ and $A$ be a subclass of $H$ with the normalization $f(0) = f'(0) - 1 = 0$. The subclass of $A$ consisting of all univalent functions $f$ in $\Delta$ is denoted by $U$. We denote by $S^*$ the class of starlike functions and by $K$ the class of convex functions. Also, a function $f \in A$ is said to be close-to-convex, if there is a convex function $g$ such that

$$\text{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \ (z \in \Delta).$$

We denote by $C$ the set of all close-to-convex functions.

It is well-known that the logarithmic coefficients have had great impact in the development of the theory of univalent functions. For example, de Branges by using of this concept, was able to prove the famous Bieberbach’s conjecture [1]. The logarithmic coefficients $\gamma_n := \gamma_n(f)$ of $f \in A$ are defined by

$$(1.1) \quad \log \left\{\frac{f(z)}{z}\right\} = \sum_{n=1}^{\infty} 2\gamma_n z^n \ (z \in \Delta).$$

As an example we consider the rotation of Koebe function

$$k_\epsilon(z) = \frac{z}{(1-\epsilon z)^2} \quad (|\epsilon| = 1).$$

Then the function $k_\epsilon$ has logarithmic coefficients $\gamma_n(k_\epsilon) = \epsilon^n/n, \ n \geq 1$. The inequality $|\gamma_n(f)| \leq 1/n$ holds for each starlike function $f \in U$ and the equality is attained for the rotation of Koebe function, but it is false for the full class $U$, even in order of magnitude. Indeed, if $f \in U$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then by (1.1) it follows that

$$\gamma_1 = \frac{a_2}{2} \quad \text{and} \quad \gamma_2 = \frac{1}{2} \left(a_3 - \frac{a_2^2}{2}\right).$$

By using the Bieberbach conjecture (de Branges Theorem) and applying the Fekete-Szegö inequality (see [7]), the following sharp estimates hold

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) \approx 0.635 \ldots.$$
However, the sharp estimate of $|\gamma_n|$ when $n \geq 3$ and $f \in \mathcal{U}$ it is still open. For more explanation of this issue, it is necessary to point out that there is a bounded and univalent function with logarithmic coefficients $\gamma_n$ such that $\gamma_n \neq O(n^{-0.83})$ [5, p. 242]. Also, there exists a close-to-convex function $f$ such that $|\gamma_n(f)| > 1/n$, [8]. In completing this entry Z. Ye showed that the logarithmic coefficients $\gamma_n$ of each close-to-convex function $f$ in $\mathcal{U}$ satisfy in $|\gamma_n(f)| \leq (A \log n)/n$, where $A$ is an absolute constant, see [32]. Anyway, the problem of the best upper bounds for the logarithmic coefficients of univalent functions is still open, when $n \geq 3$.

Sharp inequalities are known for sums involving logarithmic coefficients. For instance, the logarithmic coefficients $\gamma_n$ of every function $f \in \mathcal{U}$ satisfy the sharp inequality
\[
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6},
\]
and the equality is attained for the Koebe function (see [6, Theorem 4]). Also, for each $f \in \mathcal{U}$ the sharp inequality
\[
\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^2 |\gamma_n|^2 \leq 4 \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^2 \frac{1}{n^2} = \frac{2\pi^2 - 12}{3},
\]
holds (see [25]). On the other hand, in order to prove the Bieberbach conjecture, de Branges first proved the following inequality
\[
\sum_{k=1}^{n} \sum_{m=1}^{\infty} \left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0 \quad (n \geq 1)
\]
or equivalently
\[
\sum_{k=1}^{n} (n-k+1) \left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0 \quad (n \geq 1),
\]
which it is attributed to Milin and then he established the Bieberbach conjecture. We remark that the Milin conjecture (the inequality (1.4)) implies Robertson’s conjecture and, hence, Bieberbach’s conjecture.

In the sequel, we recall some another results that give more information about sums involving logarithmic coefficients. The first is Milin’s Lemma. The Milin’s Lemma ([18]) states for each $f \in \mathcal{U}$ that the inequality
\[
\sum_{n=1}^{N} n|\gamma_n|^2 \leq \sum_{n=1}^{N} \frac{1}{n} + \delta \quad (N = 1, 2, \ldots),
\]
holds where $\delta < 0.312$. The second assertion is Bazilevich’s Theorem (see [5, Section 5.6]) which states for each $f \in \mathcal{U}$ that
\[
\sum_{n=1}^{\infty} n \left|\gamma_n - \frac{1}{n} e^{-i\theta_0}\right| \leq \frac{1}{2} \log \frac{1}{\kappa},
\]
where $e^{-i\theta_0}$ is the direction of maximal growth and the number
\[
\kappa = \lim_{r \to 1^-} (1-r)^2 |f(r)| \quad (0 < r < 1),
\]
is called the Hayman index of $f$, assumed to be positive (see [9]). Indeed, $0 \leq \kappa \leq 1$ and $\kappa = 1$ if and only if $f$ is a rotation of the Koebe function.

Recently, Obradović et al. [21] proved that the logarithmic coefficients $\gamma_n$ of any $f \in \mathcal{U}(\lambda)$ where
\[
\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left|\frac{z}{f(z)}\right|^2 f'(z) - 1 < \lambda, \quad z \in \Delta, \ 0 < \lambda \leq 1 \right\},
\]
satisfy the sharp inequality
\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \left( \frac{\pi^2}{6} + 2Li_2(\lambda) + Li_2(\lambda^2) \right), \]
where \( Li_2 \) denotes the dilogarithm function. Also, the logarithmic coefficients \( \gamma_n \) of \( f \in U(1) \) satisfy the inequality (1.3). For more details and interesting properties of the family \( U(\lambda) \), the reader may refer to [20]. Again, in [21, Theorem 2], the authors also proved that the logarithmic coefficients \( \gamma_n \) of \( f \in G(a) \) where \( G(a) \) denotes the class of locally univalent normalized analytic functions \( f \) in the unit disk \( \Delta \) satisfying the condition
\[ \text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < 1 + \frac{a}{2} (a > 0, z \in \Delta), \]
satisfy the inequalities
\[ \sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \leq \frac{a^2}{4(a + 2)}, \]
\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{a^2}{4} Li_2((1 + a)^{-2}) \]
and
\[ |\gamma_n| \leq \frac{a}{2(a + 1)n} \quad (n = 1, 2, \ldots). \]
The class \( G(a) \) has been studied extensively by Kargar et al. [11], Maharana et al. [17], Obradović et al. [19], and Ponnusamy and Sahoo [22].

In the sequel, we recall some special functions. First, we recall from [23, p. 42] that if \( f(0) = 0 \) and \(-\alpha < \text{Re} f < \beta \) for \( z \in \Delta \), then we have
\[ f(z) \prec \frac{1}{\pi i} \log \left( \frac{1 - e^{-\frac{2\pi i}{1 - \alpha}}}{1 - z} \right) =: \mathcal{L}_{\alpha,\beta}(z), \]
where "\( \prec \)" denotes the subordination relation. Like to the function \( \mathcal{L}_{\alpha,\beta}(z) \), Kuroki and Owa [15] introduced the function \( P_{\alpha,\beta}(z) \) as follows
\[ P_{\alpha,\beta}(z) := 1 + \frac{\beta - \alpha}{\pi i} \log \left( \frac{1 - e^{2\pi i / (1 + \alpha)}}{1 - z} \right), \]
where \( \beta > 1 \) and \( \alpha < 1 \). The function \( P_{\alpha,\beta}(z) \) is a convex univalent function in \( \Delta \) and has the form
\[ P_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \]
where
\[ B_n = \frac{\beta - \alpha}{n \pi} i \left( 1 - e^{2\pi i / (1 + \alpha)} \right) \quad (n = 1, 2, \ldots). \]
Also, \( P_{\alpha,\beta}(z) \) maps \( \Delta \) onto a convex domain
\[ \Omega_{\alpha,\beta} := \{ w \in \mathbb{C} : \alpha < \text{Re} w < \beta \} \]
conformally. Recently, the function \( P_{\alpha,\beta}(z) \) has been studied by many works, for example [10, 15, 30, 31].

Also, we recall an another analytic function by
\[ B_{\delta}(z) := \frac{1}{2i \sin \delta} \log \left( \frac{1 + ze^{i\delta}}{1 + ze^{-i\delta}} \right) \quad (z \in \Delta), \]
due to Dorff [2] and studied in [3], [4], [12], [13], [14]. The function $B_\delta(z)$ maps $\Delta$ onto the convex hull of three points (one of which may be that point at infinity) on the boundary of $\Omega_\delta$

$$\Omega_\delta := \left\{ w : \frac{\delta - \pi}{2 \sin \delta} < \text{Re} w < \frac{\delta}{2 \sin \delta} \right\}.$$  

In other words, the image of $\Delta$ may be a vertical strip when $\pi/2 \leq \delta < \pi$, while in other cases, a half strip, a trapezium, or a triangle. Also, the function $B_\delta(z)$ is convex univalent in $\Delta$ and has the form

$$B_\delta(z) = \sum_{n=1}^{\infty} A_n z^n \quad (z \in \Delta),$$

where

$$A_n = \frac{(-1)^{n-1} \sin n\delta}{n \sin \delta} \quad (n = 1, 2, \ldots).$$

Here, we recall two certain subclasses of starlike functions which are connected with the functions $P_{\alpha,\beta}(z)$ and $B_\delta(z)$. First, let $S(\alpha,\beta)$ denote the class of all functions $f \in A$ which satisfy in the following two–sided inequality

$$\alpha < \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \beta \quad (\alpha < 1, \beta > 1).$$

By definition of subordination, $f \in S(\alpha,\beta)$ if, and only if

$$\frac{zf'(z)}{f(z)} < P_{\alpha,\beta}(z) \quad (z \in \Delta),$$

where $P_{\alpha,\beta}(z)$ given by (1.6). The class $S(\alpha,\beta)$ was introduced in [15] and studied in [16] and [31].

Also, for $\pi/2 \leq \delta < \pi$, the function $f \in A$ belongs to the class $M(\delta)$, if $f$ satisfies

$$1 + \frac{\delta - \pi}{2 \sin \delta} < \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\delta}{2 \sin \delta} \quad (z \in \Delta).$$

Moreover, by definition of subordination, $f \in M(\delta)$ if, and only if

$$\left( \frac{zf'(z)}{f(z)} - 1 \right) < B_\delta(z) \quad (z \in \Delta),$$

where $B_\delta(z)$ defined in (1.10). The class $M(\delta)$ introduced by Kargar et al. [12]. It is necessary to mention that functions mapping onto the strip connected with $M(\delta)$ was studied by Dorff [2] (see also [4], [13], [14]). However, Dorff was interested in harmonic mappings which map the unit disk $\Delta$ onto vertical strip.

In this paper, some subordination relations among the classes $S(\alpha,\beta)$ and $M(\delta)$ are presented. These relations are then used to obtain sharp estimates for sums involving their logarithmic coefficients. Also, the estimate of initial logarithmic coefficients for functions belonging to these subclasses are determined.

### 2. Main Results

Let’s start by recalling the definition of Hadamard product. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their Hadamard product (or convolution) is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Many of convolution problems were studied by St. Ruscheweyh in [26] and have found many applications in various fields. One of them is the following lemma due to St. Ruscheweyh and J. Stankiewicz which will be useful in this paper.
Lemma 2.1. (see [29]) Let $\phi, \varphi \in H$ be any convex univalent functions in $\Delta$. If $f(z) \prec \phi(z)$ and $g(z) \prec \varphi(z)$, then
\begin{equation}
 f(z) * g(z) \prec \phi(z) * \varphi(z) \quad (z \in \Delta).
\end{equation}

One of the aims of this article is the following. The following theorem will be useful in order to estimate of sums involving logarithmic coefficients of functions in the class $S(\alpha, \beta)$.

Theorem 2.1. Let $f(z) \in A$, $\alpha < 1$ and $\beta > 1$. Also let $P_{\alpha, \beta}(z)$ be defined by (1.6). If $f(z) \in S(\alpha, \beta)$, then
\begin{equation}
 \log \left\{ \frac{f(z)}{z} \right\} \prec \hat{P}_{\alpha, \beta}(z),
\end{equation}
where
\begin{equation}
 \hat{P}_{\alpha, \beta}(z) := \int_0^z \frac{P_{\alpha, \beta}(t) - 1}{t} \, dt,
\end{equation}
and $\hat{P}_{\alpha, \beta}$ is convex univalent.

Proof. Let $f(z) \in A$. If we define $p(z) := f(z)/z$, then $p(z)$ is analytic in $\Delta$ and $p(0) = 1$. Also, since $f(z) \in S(\alpha, \beta)$, therefore by (1.15), we have
\begin{equation}
 \frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)} - 1 \prec P_{\alpha, \beta}(z) - 1 \quad (z \in \Delta),
\end{equation}
where $P_{\alpha, \beta}$ is of the form (1.6). On the other hand, it is well-known that (see [24]) the function
\begin{equation}
 \tilde{h}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}
\end{equation}
is convex univalent in $\Delta$ and
\begin{equation}
 \psi(z) * \tilde{h}(z) = \int_0^z \frac{\psi(t)}{t} \, dt \quad (\psi \in H).
\end{equation}
Now by Lemma 2.1 and from (2.4) we get
\begin{equation}
 \frac{zp'(z)}{p(z)} \ast \tilde{h}(z) \prec (P_{\alpha, \beta}(z) - 1) \ast \tilde{h}(z) \quad (z \in \Delta).
\end{equation}
Moreover by (2.7), we can obtain (2.2). On the other hand, since $P_{\alpha, \beta}(z)$ and $\tilde{h}(z)$ are convex univalent functions, by the Pólya–Schoenberg conjecture (this conjecture states that the class of convex univalent functions is preserved under the convolution) that is proved by Ruscheweyh and Sheil–Small (see [28]), the function $\hat{P}_{\alpha, \beta}(z)$ is convex univalent, too. \qed

Because $\hat{P}_{\alpha, \beta}(z)$ is convex univalent, thus we get.

Corollary 2.1. Let $f(z) \in S(\alpha, \beta)$. Then
\begin{equation}
 \frac{f(z)}{z} \prec \exp \hat{P}_{\alpha, \beta}(z) \quad (z \in \Delta),
\end{equation}
where $\hat{P}_{\alpha, \beta}(z)$ given by (2.3).

Theorem 2.2. For $\alpha < 1$ and $\beta > 1$, the logarithmic coefficients of $f \in S(\alpha, \beta)$ satisfy the following inequality
\begin{equation}
 \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{(\beta - \alpha)^2}{4\pi^2} \left( \frac{\pi^4}{45} - \text{Li}_4 \left( e^{-2\pi i \frac{\beta-\alpha}{\beta+\alpha}} \right) - \text{Li}_4 \left( e^{2\pi i \frac{\beta-\alpha}{\beta+\alpha}} \right) \right),
\end{equation}
where $\text{Li}_4$: is the polylogarithm function.
where \( Li_4 \) is defined as following

\[
Li_4(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^4} = -\frac{1}{2} \int_0^1 \frac{\log^2(1/t) \log(1-tz)}{t} \, dt.
\]

The result is sharp.

**Proof.** Let \( f \in S(\alpha, \beta) \). Then by Theorem 2.1, we have

\[
\log \left\{ \frac{f(z)}{z} \right\} \prec \hat{P}_{\alpha, \beta}(z),
\]

where \( \hat{P}_{\alpha, \beta}(z) \) defined in (2.3). By using (1.7) and (1.8), one can rewrite \( \hat{P}_{\alpha, \beta}(z) \) as the following

\[
\hat{P}_{\alpha, \beta}(z) = \sum_{n=1}^{\infty} \frac{\beta - \alpha}{\pi n^2} i \left( 1 - e^{2\pi ni \frac{1-\alpha}{\beta-\alpha}} \right) z^n.
\]

With placement of (1.1) and (2.11) into (2.10), we get

\[
\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{\beta - \alpha}{\pi n^2} i \left( 1 - e^{2\pi ni \frac{1-\alpha}{\beta-\alpha}} \right) z^n.
\]

Applying Rogosinski’s theorem (see [24] or [5, Sec. 6.2]), we obtain

\[
4 \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \left( \frac{\beta - \alpha}{\pi n^2} \right)^2 \left( 1 - e^{2\pi ni \frac{1-\alpha}{\beta-\alpha}} \right)^2
\]

and we get the inequality (2.8). The inequality is sharp for the logarithmic coefficients of the function

\[
\tilde{\delta}_{\alpha, \beta}(z) = z \exp \hat{P}_{\alpha, \beta}(z),
\]

where \( \hat{P}_{\alpha, \beta}(z) \) given by (2.3). A simple check gives us

\[
\gamma_n(\tilde{\delta}_{\alpha, \beta}(z)) = \frac{\beta - \alpha}{2\pi n^2} i \left( 1 - e^{2\pi ni \frac{1-\alpha}{\beta-\alpha}} \right)
\]

and concluding the proof. \( \square \)

**Theorem 2.3.** Let \( f \in A \) belongs to the class \( S(\alpha, \beta) \) and \( \gamma_n \) be the logarithmic coefficients of \( f \). Then

\[
|\gamma_n| \leq \frac{\beta - \alpha}{n \pi} \left| \sin \left( \frac{\pi(1-\alpha)}{\beta - \alpha} \right) \right| (n \geq 1, \alpha < 1 < \beta).
\]

**Proof.** Assume that \( f \in A \) belongs to the class \( S(\alpha, \beta) \). Then by (1.13) we have

\[
z \frac{f'(z)}{f(z)} - 1 = z \left( \log \left\{ \frac{f(z)}{z} \right\} \right)' \prec P_{\alpha, \beta}(z) - 1,
\]

where \( P_{\alpha, \beta} \) defined in (1.3). Moreover, in terms of the logarithmic coefficients \( \gamma_n \) of \( f \) defined by (1.1) and (1.7), is equivalent to

\[
\sum_{n=1}^{\infty} 2n \gamma_n z^n \prec \sum_{n=1}^{\infty} B_n z^n.
\]

Now by Rogosinski’s theorem, we get the inequality (2.14). This completes the proof. \( \square \)
It is clear that if $\beta \to +\infty$, then $S(\alpha, \beta) \to S^*(\alpha)$ (the class of starlike functions of order $0 \leq \alpha < 1$). Thus we have the following result.

**Corollary 2.2.** If $f \in S(\alpha, \beta)$ when $\beta \to +\infty$, then

$$|\gamma_n| \leq \frac{\beta - \alpha}{\pi} \left| \sin \left( \frac{(1 - \alpha) \beta}{\beta - \alpha} \right) \right| \leq \frac{\beta - \alpha}{\pi n} \times \frac{\pi (1 - \alpha)}{\beta - \alpha} = \frac{1 - \alpha}{n} \quad (n \geq 1).$$

Indeed, if $f \in S^*(\alpha)$ ($0 \leq \alpha < 1$), then the corresponding logarithmic coefficients satisfy the inequality $|\gamma_n| \leq (1 - \alpha)/n$ for $n \geq 1$.

Next, we have the following.

**Theorem 2.4.** Let $\pi/2 \leq \delta < \pi$. Also let $B_\delta(z)$ and $A_n$ be defined by (1.10) and (1.13), respectively. If $f(z) \in M(\delta)$, then

$$\log \left\{ \frac{f(z)}{z} \right\} \prec \int_0^z \frac{B_\delta(t)}{t} \, dt.$$

Moreover,

$$\tilde{B}_\delta(z) := \int_0^z \frac{B_\delta(t)}{t} \, dt = \sum_{n=1}^{\infty} \frac{A_n}{n} z^n$$

is a convex univalent function.

**Proof.** The proof is similar to the proof of the Theorem 2.1, and thus we omit the details. □

Since $\tilde{B}_\delta(z)$ is a convex univalent function, thus we have.

**Corollary 2.3.** If $f(z) \in M(\delta)$, then

$$\frac{f(z)}{z} \prec \exp \tilde{B}_\delta(z) \quad (z \in \Delta),$$

where $\tilde{B}_\delta(z)$ is of the form (2.16).

**Theorem 2.5.** Let $f \in A$ belongs to the class $M(\delta)$ and $\pi/2 \leq \delta < \pi$. Then the logarithmic coefficients of $f$ satisfy the inequality

$$(2.17) \quad \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{16 \sin^2 \delta} \left[ \frac{\pi^4}{45} - \text{Li}_4 \left( e^{-2i\delta} \right) - \text{Li}_4 \left( e^{2i\delta} \right) \right],$$

where $\text{Li}_4$ defined in (2.9). The result is sharp.

**Proof.** Let $f \in M(\delta)$. Then by Theorem 2.4, we have

$$\log \left\{ \frac{f(z)}{z} \right\} \prec \tilde{B}_\delta(z) \quad (z \in \Delta).$$

By using (1.1) and (1.12), the relation (2.18) implies that

$$\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} A_n z^n \quad (z \in \Delta).$$

Now by Rogosinski's theorem, we get

$$4 \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} |A_n|^2$$

$$= \frac{1}{\sin^2 \delta} \sum_{n=1}^{\infty} \frac{\sin^2 \pi \delta}{n^4}$$

$$= \frac{1}{\sin^2 \delta} \sum_{n=1}^{\infty} \left( \frac{1}{180} \left[ \pi^4 - 45 \text{Li}_4 \left( e^{-2i\delta} \right) - 45 \text{Li}_4 \left( e^{2i\delta} \right) \right] \right),$$

where $\text{Li}_4$ defined in (2.9). The result is sharp.
where $L_i$ is defined by (2.9). Therefore the desired inequality (2.17) follows. For the sharpness of (2.17), consider
\begin{equation}
F_\delta(z) = z \exp \tilde{B}_\delta(z),
\end{equation}
where $\tilde{B}_\delta(z)$ is defined by (2.16). It is easy to see that $F_\delta(z) \in M(\delta)$ and $\gamma_n(F_\delta) = A_n/2n$, where $A_n$ is given by (1.13). Therefore, we have the equality in (2.17). This is the end of proof.

**Theorem 2.6.** Let $\pi/2 \leq \delta < \pi$. If $f \in A$ belongs to the class $M(\delta)$, then the logarithmic coefficients of $f$ satisfy
\[ |\gamma_n| \leq \frac{1}{2^n} \quad (n \geq 1). \]
The result is sharp.

**Proof.** The proof is similar to the proof of the Theorem 2.3, and thus the details are omitted.

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Independent researcher, without dependence on the organization.
E-mail address: rkargar1983@gmail.com