Stochastic stability at the boundary of expanding maps

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Abstract
We consider endomorphisms of a compact manifold which are expanding except for a finite number of points and prove the existence and uniqueness of a physical measure and its stochastic stability. We also characterize the zero-noise limit measures for a model of the intermittent map and obtain stochastic stability for some values of the parameter. The physical measures are obtained as zero-noise limits which are shown to satisfy Pesin’s entropy formula.

Dedicated to C Gutierrez on the occasion of his 60th birthday

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1. Introduction

After the long and deep developments in the last decade on the structural stability theory of dynamical systems, we know that this form of stability is too strong to be a generic property (see e.g. [1, 13, 24, 27, 30, 33–35] and references therein). Recently there has been some emphasis on the study of stochastic stability of dynamical systems, among other forms of stability (for an up-to-date survey see, e.g., [10]).

On the one hand, one of the challenging problems of smooth ergodic theory is to prove the existence of ‘nice’ invariant measures called physical measures or sometimes SRB (Sinai–Ruelle–Bowen) measures. On the other hand, a natural formulation of stochastic stability of dynamical systems assumes the existence of physical measures. However, the characterization of zero-noise limit measures involved in the study of stochastic stability may provide ways to construct physical measures. In this work the study of zero-noise limit measures for endomorphisms which are expanding except at a finite number of points yields a construction of physical measures and also their stochastic stability.
Let $M$ be a compact and connected Riemannian manifold and $T := C^{1+\alpha}(M, M)$ be the space of $C^{1+\alpha}$ maps of $M$ where $\alpha > 0$. We write $m$ for some fixed measure induced by a normalized volume form on $M$ that we call Lebesgue measure, dist for the Riemannian distance on $M$ and $\|\cdot\|$ for the induced Riemannian norm on $TM$.

We recall that an invariant probability measure $\mu$ for a transformation $T : M \to M$ on a manifold $M$ is physical if the ergodic basin
\[
B(\mu) = \left\{ x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x)) \to \int \phi \, d\mu \text{ for all continuous } \phi : M \to \mathbb{R} \right\}
\]
has positive Lebesgue measure.

Let $(\theta_{\epsilon})_{\epsilon > 0}$ be a family of Borel probability measure on $(T, B(T))$, where we write $B(X)$ for the Borel $\sigma$-algebra of a topological space $X$. We deal with random dynamical systems generated by independent and identically distributed maps of $T$, and $\theta_{\epsilon}$ will be the common probability distribution driving the choice of the maps at each iteration.

We say that a probability measure $\mu^\epsilon$ on $M$ is stationary for the random system $(\hat{T}, \theta_{\epsilon})$ if
\[
\int \psi(T(x)) d\mu^\epsilon(x) d\theta_{\epsilon}(T) = \int \psi \, d\mu \text{ for all continuous } \psi : M \to \mathbb{R}. \tag{1.1}
\]

We assume that the support $\text{supp}(\theta_{\epsilon})$ of $\theta_{\epsilon}$ shrinks to a given transformation $T$, i.e., $\text{supp}(\theta_{\epsilon}) \to \{T\}$ when $\epsilon \to 0$ in a suitable topology. A classical result in random dynamical systems (see [5] or [17]) implies that every weak accumulation point of the stationary measures $(\mu^\epsilon)_{\epsilon > 0}$ when $\epsilon \to 0$ is a $T$-invariant probability measure, which is called a zero-noise limit measure. This naturally leads to the study of the kind of zero-noise limits that can arise and to the notion of stochastic stability.

**Definition 1.** A map $T$ is stochastically stable (under the random perturbation $(\hat{T}, \theta_{\epsilon})_{\epsilon > 0}$) if every weak accumulation point $\mu$ of the family of stationary measures $(\mu^\epsilon)_{\epsilon > 0}$, when $\epsilon \to 0$, is a linear convex combination of the physical measures of $T$.

Uniformly expanding maps and uniform hyperbolic systems are known to be stochastically stable [16, 17, 38, 39]. Some non-uniform hyperbolic systems, like quadratic maps, Hénon maps and Viana maps, were recently shown to be much more stochastically stable [3, 7, 8]. These systems either exhibit expansion/contraction everywhere or are expanding/contracting away from a critical region and, moreover, the orbits of most points have a small frequency of visits to this region. This is enough to show that the visits to a neighbourhood of the critical region are negligible on average, and also that this behaviour persists under small random perturbations.

It is not obvious how to apply the standard techniques to systems whose typical orbits do not have a slow recurrence rate of visits to the non-hyperbolic regions. This is the case of the intermittent maps [25]. These applications are expanding, except at a neutral fixed point. The local behaviour near this neutral point is responsible for various phenomena.

Consider $\alpha > 0$ and the map $T : [0, 1] \to [0, 1]$ defined as follows:
\[
T(x) = \begin{cases} 
  x + 2^\alpha x^{1+\alpha} & x \in [0, \frac{1}{2}), \\
  x - 2^\alpha (1-x)^{1+\alpha} & x \in [\frac{1}{2}, 1].
\end{cases} \tag{1.2}
\]

This map defines a $C^{1+\alpha}$ map of the unit circle $\mathbb{S} := [0, 1]/\{0 \sim 1\}$ into itself. The unique fixed point is 0 and $DT(0) = 1$. The above family of maps provides many interesting results in ergodic theory. If $\alpha \geq 1$, i.e. if the order of tangency at zero is high enough, then the Dirac mass at zero $\delta_0$ is the unique physical probability measure and so the Lyapunov exponent of Lebesgue almost all points vanishes [37]. The situation is completely different for $0 < \alpha < 1$: in this
case there exists a unique absolutely continuous invariant probability measure $\mu_{SRB}$, which is therefore a physical measure and whose basin has full Lebesgue measure in $\mathbb{S}$ [36]. Moreover these maps provide examples of dynamical systems with polynomial decay of correlations. Even sub-polynomial rates of mixing have been obtained by modifying these intermittent maps [14]. In particular it is well known that $\mu_{SRB}$ is always mixing, when it exists.

1.1. Statement of the results

We consider additive noise applied to a map $T$ of $\mathbb{S}$ with an indifferent fixed point at 0 and expanding everywhere else, as in example (1.2). Let $\alpha > 0$ be fixed and consider $T_t := T + t$ for $|t| \leq \varepsilon$. Then $\hat{T} : [-\frac{1}{2}, \frac{1}{2}] \to C^{1+\alpha}(\mathbb{S}, \mathbb{S})$, $t \mapsto T_t$ is a (smooth) family of $C^{1+\alpha}$ maps of $\mathbb{S}$.

Let $\theta_0$ be an absolutely continuous probability measure, with respect to the Lebesgue measure $\mu$ on $\mathbb{S}$, whose support is contained in $[-\varepsilon, \varepsilon]$ (e.g. $\theta_0 = (2\varepsilon)^{-1} m$ | $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$). This naturally induces a probability measure on $T = \{T_t, t \in [-\frac{1}{2}, \frac{1}{2}]\}$ which we denote by the same symbol $\theta_0$ (the meaning being clear from the context).

In this setting it is well known that there always exist a stationary probability measure $\mu^\varepsilon$ for all $\varepsilon > 0$. Moreover this measure is ergodic and is the unique absolutely continuous stationary measure for $(\hat{T}, \theta_0)$ (see section 2.2).

Let us now fix $\alpha \in (0, 1)$ and let

$$E = \{t \delta_0 + (1 - t) \mu_{SRB} : 0 \leq t \leq 1\}$$

be the set of linear convex combinations of the Dirac mass at 0 with the unique absolutely continuous invariant probability measure for these maps.

**Theorem A.** Let $\mu_0$ be a weak* accumulation point of the stationary measures $(\mu^\varepsilon)_{\varepsilon > 0}$ when $\varepsilon \to 0$ for the random perturbation $(\hat{T}, \theta_0)_{\varepsilon > 0}$ with $\alpha \in (0, 1)$. Then $\mu_0 \in E$.

In the case $\alpha \geq 1$ there does not exist any absolutely continuous invariant probability measure for $T$. However, the Dirac measure $\delta_0$ is the unique physical measure. In this case we are able to deduce stochastic stability.

**Theorem B.** Let $\alpha \geq 1$ in (1.2) and let $(\mu^\varepsilon)_{\varepsilon > 0}$ be the family of stationary measures for the random perturbation $(\hat{T}, \theta_0)_{\varepsilon > 0}$. Then $\mu^\varepsilon \to \delta_0$ when $\varepsilon \to 0$ in the weak* topology.

However, taking a different family $f_\varepsilon$ unfolding the saddle-node at 0, e.g.

$$f_\varepsilon(x) = \begin{cases} tx + 2^\alpha (2 - t)x^{1+\alpha} & x \in [0, \frac{1}{2}), \\ 1 - t (1 - x) - 2^\alpha (2 - t) (1 - x)^{1+\alpha} & x \in [\frac{1}{2}, 1] \end{cases} \quad (1.3)$$

with $\alpha \in (0, 1)$ we obtain an example of non-stochastic stability. In fact, since $f_\varepsilon'(0) = t$ then for $t < 1$ the fixed point 0 is a sink for $f_\varepsilon$ (see figure 1) and we prove that the physical measure for the random system is always $\delta_0$ for specific choices of the probability measures $\theta_\varepsilon$. Here we take $T = \{f_\varepsilon, t \in [\frac{1}{2}, \frac{1}{2}]\}$ and $\hat{T} : [\frac{1}{2}, \frac{1}{2}] \to C^{1+\alpha}(\mathbb{S}, \mathbb{S})$, $t \mapsto f_\varepsilon$.

**Theorem C.** For every small enough $\varepsilon > 0$ there are $0 < a(\varepsilon) < b(\varepsilon) < 1$ such that $a(\varepsilon) \to 1$ when $\varepsilon \to 0$ and, for any given probability measure $\theta_\varepsilon$ supported in $[a(\varepsilon), b(\varepsilon)]$, the unique stationary measure $\mu^\varepsilon$ for the random system $(\hat{T}, \theta_\varepsilon)$ equals $\delta_0$.

Since $f_1 = T$ admits an absolutely continuous invariant measure $\mu_{SRB}$ and clearly $\delta_0$ cannot converge to this physical measure, we have an example of a stochastically unstable system (under this kind of perturbation).

Using the same kind of additive perturbation considered in theorems A and B, our methods provide the following results for maps in higher dimensions.
Theorem D. Let \( f : M \to M \) be a \( C^{1+\alpha} \) local diffeomorphism such that

1. \( \|Df(x)^{-1}\| \leq 1 \) for all \( x \in M \);
2. \( K = \{ x \in M : \|Df(x)^{-1}\| = 1 \} \) is finite and \( |\det Df(x)| > 1 \) for every \( x \in K \).

Then, for any non-degenerate random perturbation \( (\hat{f}, \theta_{\varepsilon})_{\varepsilon > 0} \), there exists a unique ergodic stationary probability measure \( \mu_{\varepsilon} \) for all \( \varepsilon > 0 \). Moreover \( \mu_{\varepsilon} \) converges, in the weak* topology when \( \varepsilon \to 0 \), to a unique absolutely continuous \( f \)-invariant probability measure \( \mu_0 \) whose basin has full Lebesgue measure, and \( f \) is stochastically stable.

Here we will assume that \( M \) is a \( n \)-dimensional torus since the maps \( f \) satisfying the conditions on theorem D are at the boundary of expanding maps, which can only exist on infranilmanifolds [12, 32], the best known example being the torus. This is no restriction since infranilmanifolds are homogeneous spaces, thus in particular are parallelizable, and we can define additive perturbations just as we did on the circle. If \( \mathbb{T}^n \) is a \( n \)-dimensional torus, then the tangent bundle is globally isomorphic to \( \mathbb{R}^n \), \( T\mathbb{T}^n \simeq \mathbb{R}^n \) and \( \hat{f} : B \subset \mathbb{R}^n \to C^{1+\alpha}(M, M), v \to f + v \) (where \( B \) is a ball around the origin of \( \mathbb{R}^n \) and \( f + v \) is the map \( x \mapsto f(x) + v \)) is the kind of additive perturbation we will consider, together with a family \( (\theta_{\varepsilon})_{\varepsilon > 0} \) of absolutely continuous probability measures on \( B \). (See also section 2.3 for the definition of non-degenerate random perturbation.)

These results will be derived from the following theorem which is not only more technical but also interesting in itself.

Theorem E. Let \( f : M \to M \) be a \( C^{1+\alpha} \) local diffeomorphism such that

1. \( \|Df(x)^{-1}\| \leq 1 \) for all \( x \in M \);
2. \( K = \{ x \in M : \|Df(x)^{-1}\| = 1 \} \) is finite.

Then for any non-degenerate random perturbation \( (\hat{f}, \theta_{\varepsilon})_{\varepsilon > 0} \), every weak* accumulation point \( \mu \) of the sequence \( (\mu_{\varepsilon})_{\varepsilon > 0} \), when \( \varepsilon \to 0 \), is an equilibrium state for the potential \(-\log |\det Df(x)|\), i.e.

\[
h_{\mu}(f) = \int \log |\det Df(x)| \, d\mu(x). \tag{1.4}\]

Moreover every equilibrium state \( \mu \) as above is a convex linear combination of an absolutely continuous invariant probability measure with finitely many Dirac measures concentrated on periodic orbits whose Jacobian equals 1.
Cowieson and Young have presented results similar to ours for $C^2$ or $C^\infty$ diffeomorphisms. However, their assumptions are on the convergence of the sum of the positive Lyapunov exponents of the random maps to the same sum for the original map, and they obtain SRB measures, not necessarily physical ones (see [11] for more details). We make much stronger assumptions on both the kinds of maps being perturbed (expanding except at finitely many points) and the kinds of perturbations used (additive besides being absolutely continuous), and we obtain physical measures for $C^{1+\alpha}$ endomorphisms.

In what follows, we first present some examples of applications and then general results about random dynamical systems (section 2) to be used to prove theorem E (section 3). At this point we are ready to obtain theorem D (section 3.4). Finally we apply the ideas to the specific case of the intermittent maps (section 4), completing the proof of theorems A, B and C.

1.2. Examples

In what follows we write $T$ for $S \times S$. We assume that these spaces are endowed with the metrics induced by the standard Euclidean metric. The Lebesgue measure on these spaces will be denoted by $m$ (area) on $T$ and $m_1$ (length) on $S$. An extra example is the intermittent map itself which we consider in more detail in section 4.

1.2.1. Direct product ‘intermittent $\times$ expanding’. Let $f : T \to T$, $(x, y) \mapsto (T_\alpha(x), g(y))$, where $T_\alpha$ is defined in the introduction with $\alpha > 0$, and $g : S \to S$ is $C^{1+\alpha}$, admit a fixed point $g(0) = 0$ and $g' > Dg > 1$.

Since $f$ is a direct product, if $\alpha \in (0, 1)$, then $f$ admits an invariant probability measure $\nu = \mu_\alpha \times \lambda$, where $\mu_\alpha$ is the unique absolutely continuous invariant measure for $T_\alpha$ and $\lambda$ is the unique absolutely continuous invariant measure for $g$, i.e., $\mu_\alpha \ll m_1$ and $\lambda \ll m_1$. Hence the product measure is absolutely continuous: $\nu \ll m = m_1 \times m_1$. These measures are ergodic and also mixing, and the basins of $\mu_\alpha$ and $\lambda$ equal $S$, $m_1 \text{ mod } 0$. Thus their direct product $\nu$ is ergodic and so $B(\nu) = T, m \text{ mod } 0$.

If $\alpha \geq 1$, then $\nu = \delta_0 \times \lambda$ is again an ergodic invariant probability measure for $f$ with $B(\nu) = T, m \text{ mod } 0$, since $\lambda$ is the same as before and so is mixing for $g$, and $\delta_0$ is $T_0$-ergodic, with the basin of both measures equal to $S$.

Here $K = \{0\} \times S$ (the definition of $K$ is given in the statement of theorem D) is not finite, and the conclusion of theorem D does not hold when $\alpha \geq 1$: we have a physical measure which is not absolutely continuous with respect to $m$. Note that clearly $\|(Df)^{-1}\| \leq 1$ everywhere and since $K$ contains fixed (and periodic) points, $f$ is not uniformly expanding.

1.2.2. Direct product ‘intermittent $\times$ intermittent’. Let $f : T \to T$, $(x, y) \mapsto (T_\alpha(x), T_\beta(y))$ where $\alpha, \beta > 0$. Now $K = \{0\} \times S \cup S \times \{0\}$ and, by the same reasoning as for the previous example, the probability measure $\nu = \mu_\alpha \times \mu_\beta$ is the unique physical measure for $f$. Moreover $B(\nu) = T$ as before. However, $\nu$ is absolutely continuous with respect to $m$ if, and only if, $\alpha, \beta \in (0, 1)$.

1.2.3. Skew-product ‘intermittent $\times$ expanding’. Let $f : T \to T$, $(x, y) \mapsto (T_\alpha(x) + \eta \cdot y, g(y))$, for $\alpha > 0$, $\eta \in (0, 1)$ and $g : S \to S$ as in example 1.2.1.

In this case we easily calculate $Df = \begin{pmatrix} DT_\alpha & \eta \\ 0 & Dg \end{pmatrix}$ and so $K = \{0, 0\}$. Clearly $\|(Df)^{-1}\| \leq 1$ everywhere and since $K$ is a fixed point the map $f$ is not uniformly expanding. Applying theorem D we get an absolutely continuous invariant probability measure...
\( \mu \) for \( f \) with \( \int \log \| (Df)^{-1} \| \, d\mu < 0 \). Hence the Lyapunov exponents for Lebesgue almost every point on the basin of \( \mu \) are all positive, so \( f \) is a non-uniformly expanding transformation.

This map is stochastically stable, since every weak* accumulation point of \((\mu^\varepsilon)_{\varepsilon>0}\) when \( \varepsilon \to 0 \) equals \( \mu \) by the uniqueness part of theorem D. We stress that since the value of \( \alpha \) played no role in the arguments, these conclusions hold for any \( \alpha > 0 \).

1.2.4. Several indifferent directions. It is straightforward to adapt example 1.2.3 in order to have an example with an indifferent fixed point having several indifferent directions. It is enough to take \( \alpha, \beta > 0, \eta, \tau, \xi, \zeta \in (0, 1) \) and consider the following map on \( \mathbb{T} \times \mathbb{S} \)

\[
(x, y, z) \mapsto (T_\omega(x) + \xi \cdot y + \eta \cdot z, \xi \cdot x + T_\beta(y) + \tau \cdot z, g(z)).
\]

By an analysis similar to that in example 1.2.3 we see that \((0, 0, 0)\) is an indifferent fixed point and that the derivative of the map has eigenvalues 1 and \( g'(0) > 1 \), but the eigenspace corresponding to the eigenvalue 1 has multiplicity 2.

1.2.5. Several indifferent fixed points. We can easily adapt the map \( T \) given in (1.2) in order to get an example \( f \) of a \( C^{1+\alpha} \) map expanding everywhere except at two (or any finite number of) indifferent fixed points. The previous examples can trivially be manufactured with this map \( f \). For example, letting \( \alpha \) be a positive constant we may consider

\[
f(x) = \begin{cases} 
  x + 3 \cdot 4^\alpha \cdot x^{1+\alpha} & \text{if } x \in [0, \frac{1}{2}), \\
  x + 4^{3+2\alpha} \cdot x^{3(1+\alpha)} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}), \\
  x - 3 \cdot 4^\alpha \cdot (1 - x)^{1+\alpha} & \text{if } x \in (\frac{3}{4}, 1].
\end{cases}
\]

which has 0 and \( \frac{1}{2} \) as two indifferent fixed points.

2. Preliminary results

Throughout this section we outline some general results about random dynamical systems to be used in what follows.

Having a parametrized family of maps \( \hat{T} : X \to T, t \mapsto T_t \), where \( X \) is some connected compact metric space, enables us to identify a sequence \( T_{1}, T_{2}, \ldots \) of maps in \( T \) with a sequence \( \omega_{1}, \omega_{2}, \ldots \) of parameters in \( X \). The probability measure \( \theta_t \) can then be assumed to be supported in \( X \).

We set \( \Omega = X^\mathbb{N} \), the space of sequences \( \omega = (\omega_i)_{i \geq 1} \) with elements in \( X \). Then we endow \( \Omega \) with the standard infinite product topology, which makes \( \Omega \) a compact metric space, with distance given by \( d(\omega_0, \omega') = \sum_{j \geq 1} 2^{-j} d_X(\omega_j, \omega'_j) \), where \( d_X \) is the distance on \( X \). We also consider the standard product probability measure \( \theta^\varepsilon = \theta^\varepsilon_\mathbb{N} \) on \((\Omega, \mathcal{B}), \) which makes \((\Omega, \mathcal{B}, \theta^\varepsilon)\) a probability space. Here \( \mathcal{B} = \mathcal{B}(\Omega) \) is the \( \sigma \)-algebra generated by cylinder sets, that is, the minimal \( \sigma \)-algebra of subsets of \( \Omega \) containing all sets of the form \( \{ \omega \in \Omega : \omega_i \in A_1, \omega_2 \in A_2, \ldots, \omega_l \in A_l \} \) for any sequence of Borel subsets \( A_i \subset X, i = 1, \ldots, l \) and \( l \geq 1 \).

The following skew-product map provides the natural setting for many definitions connecting random with standard dynamical systems

\[
S : \Omega \times M \to \Omega \times M, \quad (\omega, x) \mapsto (\sigma(\omega), T_{\omega_0}(x)),
\]

where \( \sigma \) is the left shift on \( \Omega \), defined as \( (\sigma(\omega))_n = \omega_{n+1} \) for all \( n \geq 1 \). It is an exercise to check that \( \mu^\varepsilon \) is a stationary measure for the random system \((\hat{T}, \theta_\varepsilon)\) (i.e. satisfying 1.1) if, and
only if, \( \theta^\varepsilon \times \mu^\varepsilon \) on \( \Omega \times M \) is invariant by \( S \). Ergodicity of stationary measures is defined in a natural way. A Borel set \( A \subset M \) is invariant if for \( \mu^\varepsilon \)-almost every point \( x \in M \)
\[
x \in A \Rightarrow T_t(x) \in A \quad \text{for} \ \theta^\varepsilon \text{-almost every} \ t \in X
\]
and
\[
x \in A^c \Rightarrow T_t(x) \in A^c \quad \text{for} \ \theta^\varepsilon \text{-almost every} \ t \in X.
\]

**Definition 2.** A stationary measure \( \mu^\varepsilon \) is said to be ergodic if every Borel invariant set has either \( \mu^\varepsilon \)-measure zero or one.

It is not difficult to prove that \( \mu^\varepsilon \) is ergodic if and only if \( \theta^\varepsilon \times \mu^\varepsilon \) is an ergodic measure for \( S \) (see, e.g., [22]).

2.1. Metric entropy of random dynamical systems

The notion of metric entropy can be defined for random dynamical systems in different ways. We point out two definitions which will be used in this paper and relate them. The following results can be found in the book of Kifer [15, section II].

Let \( \mu \) be a stationary measure for the random system \( (\hat{T}, \theta^\varepsilon) \). Given two finite measurable partitions \( \zeta \) and \( \xi \) of \( M \) it is standard to write \( \zeta \vee \xi \) for the partition whose elements are the non-empty intersections of the elements of \( \zeta \) with those of \( \xi \), and analogously for any finite number of partitions.

**Theorem 2.1** ([15, theorem 1.3]). For any finite measurable partition \( \xi \) of \( M \) the limit
\[
h_{\mu^\varepsilon}((\hat{T}, \theta^\varepsilon), \xi) = \lim_{n \to \infty} \frac{1}{n} \int H_{\mu^\varepsilon} \left( \bigvee_{k=0}^{n-1} (T^k_\omega)^{-1} \xi \right) d\theta^\varepsilon(\omega)
\]
exists. This limit is called the entropy of the random dynamical system with respect to \( \xi \) and to \( \mu^\varepsilon \).

**Remark 2.2.** As in the deterministic case the above limit can be replaced by the infimum.

**Definition 3.** The metric entropy of the random dynamical system \( (\hat{T}, \theta^\varepsilon) \) is given by
\[
h_{\mu^\varepsilon}(\hat{T}, \theta^\varepsilon) = \sup_{\xi} h_{\mu^\varepsilon}((\hat{T}, \theta^\varepsilon), \xi), \text{where the supremum is taken over all measurable partitions.}
\]

It seems natural to define the entropy of a random system by \( h_{\theta^\varepsilon \times \mu^\varepsilon}(S) \) where \( S \) is the corresponding skew-product map. Kifer [15, theorem 1.2] shows that this definition is not very convenient: under some mild conditions the entropy of \( S \) is infinite. However, considering an appropriate \( \sigma \)-algebra of \( \Omega \times M \), the conditional entropy of \( \theta^\varepsilon \times \mu^\varepsilon \) with respect to this \( \sigma \)-algebra coincides with the entropy defined in definition 3.

Let \( B \times M \) denote the minimal \( \sigma \)-algebra containing all products of the form \( A \times M \) with \( A \in B \). In what follows we denote by \( h_{\theta^\varepsilon \times \mu^\varepsilon}^{B \times M}(S) \) the conditional metric entropy of \( S \) with respect to the \( \sigma \)-algebra \( B \times M \) (see, e.g., [9] for the definition and properties of conditional entropy).

**Theorem 2.3** ([15, theorem 1.4]). Let \( \mu^\varepsilon \) be a stationary probability measure for the random system \( (\hat{T}, \theta^\varepsilon) \). Then
\[
h_{\mu^\varepsilon}(\hat{T}, \theta^\varepsilon) = h_{\theta^\varepsilon \times \mu^\varepsilon}^{B \times M}(S).
\]

The useful Kolmogorov–Sinai result about generating partitions is also available in a random version. We denote \( A = B(M) \), the Borel \( \sigma \)-algebra of \( M \).
Theorem 2.4 ([15, corollary 1.2]). If $\xi$ is a random generating partition for $A$, that is, $\xi$ is a finite partition of $M$ such that

$$
\lim_{k \to \infty} \frac{1}{T_k} \xi = h \quad \text{for } \theta^f\text{-almost all } \omega \in \Omega,
$$

then $h_{\mu_\omega}(\tilde{T}, \theta) = h_{\mu_\omega}((\tilde{T}, \theta), \xi)$. 

### 2.2. Topologically mixing property

Here we show that in the setting of theorems D and E it is always the case that the transformation $T$ is topologically mixing. This ensures uniqueness of stationary measures under non-degenerate random perturbations, as we shall see.

Let $T : M \to M$ be a local diffeomorphism on a compact Riemannian manifold in the setting of theorem E and let $K = \{ x \in M : \|Df(x)\| = 1 \}$. Then there is a positive number $\rho$ such that $T | B(x, \rho)$ is a diffeomorphism onto its image and $B(x, \rho)$ is a convex neighbourhood for every $x \in M$, i.e. for every pair of points $y, z$ in $B(x, \rho)$ there exists a smooth geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = y$, $\gamma(1) = z$ and, moreover, $\gamma$ is the curve of minimal length between any pair of points $\gamma(s), \gamma(t)$ with $s < t, t \in [0, 1]$.

In what follows we fix a smooth curve $\gamma : [a, b] \to M$ we write $\ell(\gamma)$ for the length of $\gamma$, i.e. $\ell(\gamma) = \int_a^b \|\dot{\gamma}\|$. We also write $B(K, \delta_0) := \bigcup_{z \in K} B(z, \delta_0)$ for the $\delta_0$-neighbourhood of $K$.

**Lemma 2.5.** Let $T : M \to M$ satisfy the conditions of theorem E. Then for every open subset $U$ there exists $n \geq 1$ such that $T^n(U) = M$.

**Proof.** Arguing by contradiction, let us suppose that for some $x \in M$ and small $r > 0$ there exists a ball $B(x, r)$ such that $T^n(B(x, r)) \neq M$ for every $n > 1$. It is no restriction to assume that $r < \rho$.

In what follows we fix $n > 1$ such that $T^k(B(x, r)) \neq M$ for all $k = 1, \ldots, n$. Then there is $y \in M \setminus T^n(B(x, r))$ and a geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = T^n(x)$, $\gamma(1) = y$ and whose length is bounded by $\kappa = \text{diam}(M) + 1$ (which is finite, because $M$ is compact).

We observe that given $z \in M$ and a geodesic $\hat{\gamma} : [0, 1] \to B(z, r)$ with $\hat{\gamma}(0) = z$ and $\hat{\gamma}(1) \in \partial B(z, r)$, then $\ell(\hat{\gamma}) < \ell(\gamma)$ if $\delta < (0, r)$ is small enough. Moreover, since every such geodesic is determined by a vector in $\{ v \in T_zM : \|v\| = 1 \}$, which is a compact set, and because $M$ is also compact, for every given $\delta > 0$ small there exists

$$
\xi = \xi(r, \delta) = \sup_{z: \hat{\gamma}} \left( \frac{\ell(\hat{\gamma})}{\ell(\gamma)} \right) \in (0, 1),
$$

where the supremum is taken over every geodesic from a point $z \in M$ to the boundary of the $r$-ball $B(z, r)$ around $z$.

Now we fix $\delta \in (0, r/100)$ and $\delta' \in (0, \delta)$ small enough such that $T(B(z, \delta')) \subset B(T(z), \delta)$ for all $x \in M$. Moreover we set

$$
\lambda = \max\{\|DT(z)^{-1}\| : z \in M \setminus B(K, \delta')\} \in (0, 1)
$$

and write $\gamma_0$ for the unique smooth curve such that $T \circ \gamma_0 = \gamma$ and $\gamma_0(0) = T^{n-1}(x)$, whose existence is a consequence of the local diffeomorphism assumption on $T$.

Now we carefully analyse the length of various pieces of $\gamma$ and $\gamma_0$. For that we define

1. $\gamma_1 := \gamma | \gamma^{-1}(T^n(x), r)$, the piece of $\gamma$ inside $B(T^n(x), r)$;
2. $\gamma_2 := \gamma - \gamma_1$, the rest of $\gamma$;
• \( \gamma_0 \) and \( \gamma_2 \) are pieces of \( \gamma \) such that \( \gamma_0 = \gamma_0 + \gamma_2, T \circ \gamma_0 = \gamma_1 \) and \( T \circ \gamma_2 = \gamma_2 \);
• \( \overline{\gamma}_0 := \gamma_0 \\mid_{B(K, \delta')} \), the piece of \( \gamma_0 \) inside \( B(K, \delta') \);
• \( \overline{\gamma}_1 := T \circ \overline{\gamma}_0 \), the piece of \( \gamma_1 \) which is the image of \( \overline{\gamma}_0 \) by \( T \).

Assumption (1) in the statement of theorem E enables us to write
\[
\ell(\gamma_0) = \ell(\gamma y_0) + \ell(\gamma_0) = \ell(\overline{\gamma}_0) + \ell(\gamma_0) \leq \ell(\overline{\gamma}_1) + \ell(\gamma_0) + \ell(\gamma_2).
\]
The definitions of \( \lambda \) and \( \xi \) together with the choices of \( \delta \) and \( \delta' \) provide the following bounds:
\[
\ell(\overline{\gamma}_1) + \ell(\gamma_0) \leq \ell(\overline{\gamma}_1) + \lambda \cdot \ell(\gamma_1) = \ell(\gamma_1) \cdot (1 - \lambda) \cdot \ell(\overline{\gamma}_1) \leq \ell(\gamma_1) \cdot (1 - \lambda) \cdot \xi.
\]
Thus
\[
\ell(\gamma_0) \leq \ell(\gamma_1) \cdot (1 - \lambda) \cdot \xi + \ell(\gamma_2) = \ell(\gamma_1) + (1 - \lambda - (1 - \lambda) \cdot \xi) \cdot \frac{\ell(\gamma_1)}{\ell(\gamma_1)} \leq \lambda_0 \cdot \ell(\gamma_1) \leq \lambda_0 \cdot \kappa.
\]
where \( \lambda_0 \in (0, 1) \), as long as \( \ell(\gamma_1) \geq r \).

We observe that \( \ell(\gamma_0) \geq r \), for otherwise we would have \( y = T(\gamma_0(1)) \) with
\[
\text{dist}(T^{n-1}(x), \gamma_0(1)) = \text{dist}(\gamma_0(0), \gamma_0(1)) \leq \ell(\gamma_0) < r.
\]
However by the non-contracting assumption on \( DT \), if \( \gamma_\ast \) is the unique smooth curve satisfying
\( T^{n-1} \circ \gamma_\ast = \gamma_0 \) and \( \gamma_\ast(0) = x \), then
\[
\text{dist}(x, \gamma_\ast(1)) = \text{dist}(\gamma_\ast(0), \gamma_\ast(1)) \leq \ell(\gamma_\ast) \leq \ell(\gamma_0) < r \quad \text{thus } \gamma_\ast(1) \in B(x, r)
\]
and \( y = T(\gamma_0(x)) = T^{n}(\gamma_\ast(1)) \), a contradiction with the choice of \( y \).

Hence we can find a new geodesic \( \gamma : [0, 1] \to M \) such that
\( \gamma(0) = T^{n-1}(x), \gamma(1) = \gamma_0(1) \) and
\[
r \leq \text{dist}(T^{n-1}(x), \gamma_0(1)) \leq \ell(\gamma) \leq \ell(\gamma_0) \leq \lambda_0 \cdot \kappa.
\]
Repeating the previous arguments we get the same bounds (since the bounds do not depend on \( n \)) and obtain by induction
\[
r \leq \lambda_0^k \cdot \kappa \quad \text{for } k = 1, \ldots, n.
\]

However this cannot be true for arbitrarily large values of \( n \), since \( r, \kappa > 0 \) are fixed and \( \lambda_0 \in (0, 1) \). This shows that for every \( x \in M \) and \( r > 0 \) there is \( n \) such that \( T^n(B(x, r)) = M \), which concludes the proof of the lemma.

\[\boxed{\text{Proposition 2.6.}} \text{ Let } \hat{T} : B \subset \mathbb{R}^n \to C^{1+\alpha}(M, M), v \to T + v, \text{ with } B \text{ a ball around the origin of } \mathbb{R}^n, \text{ as defined in the introduction, where } T : M \to M \text{ satisfies the conditions of theorem E. Then for every } v \in B, \text{ all } x \in M \text{ and every given } \varepsilon > 0, \text{ there exists } n \in \mathbb{N} \text{ such that } T^n(B(x, \varepsilon)) = M.}\]

\[\boxed{\text{Proof.} \text{ We just have to note that the subset } K \text{ does not depend on } v \text{ for the maps } T_v \text{ since } DT_v = DT. \text{ Hence if } T \text{ satisfies the conditions of theorem E, then every } T_v \text{ also does. Thus we can use lemma 2.5 with } T_v \text{ in the place of } T.}\]
2.3. Non-degenerate random perturbations and uniqueness of stationary measures

Here we recall the setting of non-degenerate random perturbations as defined in [5]. (For a more complete list of properties and proofs see [2].) We assume that the family \((\theta_\varepsilon)_{\varepsilon>0}\) of probability measures on \(X\) is such that \(\text{supp}(\theta_\varepsilon)\) have non-empty interior and 

\[ \text{supp}(\theta_\varepsilon) \rightarrow \{t_0\} \quad \text{when} \ \varepsilon \rightarrow 0, \]

where \(t_0 \in X\) is such that \(T_{t_0} = T\).

For \(\omega = (\omega_1, \omega_2, \ldots) \in \Omega\) and for \(n \geq 1\) we set 

\[ T_\omega^n = T_{\omega_n} \circ \cdots \circ T_{\omega_1}. \]

Given \(x \in M\) and \(\omega \in \Omega\) we call the sequence \((T_\omega^n(x))_{n \geq 1}\) a random orbit of \(x\).

In what follows we need the map \(\tau_x : X \rightarrow M, \tau_x(t) = T_t(x)\).

**Definition 4.** We say that \((\hat{T}, \theta_\varepsilon)_{\varepsilon>0}\) is a non-degenerate random perturbation of \(T\) if, for every small enough \(\varepsilon\) and fixed \(t^*\) in the interior of \(\text{supp}(\theta_\varepsilon)\), there is \(\delta_1 = \delta_1(\varepsilon, t^*) > 0\) such that for all \(x \in M\):

1. \(\{T_t(x) : t \in \text{supp}(\theta_\varepsilon)\}\) contains a ball of radius \(\delta_1\) around \(T_{t^*}(x)\);
2. \((\tau_x)_* \theta_\varepsilon\) is absolutely continuous with respect to \(m\).

**Remark 2.7.** We note that \(\theta_\varepsilon\) cannot have atoms because of the non-degeneracy condition (2) above.

We outline some interesting consequences of the non-degeneracy conditions, see [5] for a proof.

- Any stationary measure \(\mu_\varepsilon\) is absolutely continuous with respect to \(m\).
- \(\text{supp}(\mu_\varepsilon)\) has non-empty interior and \(T_t(\text{supp}(\mu_\varepsilon)) \subseteq \text{supp}(\mu_\varepsilon)\) for any \(t \in \text{supp}(\theta_\varepsilon)\).
- \(\text{supp}(\mu_\varepsilon) \subseteq B(\mu_\varepsilon)\).

Here \(B(\mu_\varepsilon)\) is the ergodic basin of \(\mu_\varepsilon\)

\[ B(\mu_\varepsilon) = \left\{ x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T_j^\varepsilon(x)) \rightarrow \int \varphi \, d\mu \text{ for all } \varphi \in C(M, \mathbb{R}) \text{ and } \theta^\varepsilon\text{-a.e. } \omega \in \Omega \right\}, \]

which by the above properties has positive Lebesgue measure in \(M\).

If \(T\) is in the setting of theorem E, since the support of a stationary measure \(\mu_\varepsilon\) has non-empty interior and is forward invariant by \(T_t\) for any \(t \in \text{supp} \theta_\varepsilon\), then the support must contain \(M\), by proposition 2.6. Hence there exists only one physical measure \(\mu_\varepsilon\) for all \(\varepsilon > 0\), because the support is contained in the basin \(m \mod 0\).

These non-degeneracy conditions are not too restrictive since there always exists a non-degenerate random perturbation of any differentiable map of a finite dimensional compact manifold, where \(X\) is the closed ball of radius 1 around the origin of a Euclidean space, see [5].

In the setting \(M = \mathbb{T}^m\) with additive noise, as explained in the introduction, we also have that \(DT_t\) can be identified with \(DT\), so that \(\|DT_t^{-1}\| = \|DT^{-1}\|, t \in X\), which is very important in our arguments.

3. Zero-noise limits are equilibrium measures

Here we present a proof of theorem E. Let \(f : M \rightarrow M\) be a local diffeomorphism on a manifold \(M\) satisfying the conditions stated in theorem E. Let also \(\hat{f} : X \rightarrow C^{1+\alpha}(M, M), t \mapsto f_t\) be a continuous family of maps, where \(X\) is a metric space with \(f_{t_0} \equiv f\) for some
fixed $t_0 \in X$, and $(\theta_{\varepsilon})_{\varepsilon > 0}$ be a family of probability measures on $X$ such that $(\tilde{f}, (\theta_{\varepsilon}))_{\varepsilon > 0}$ is a non-degenerate random perturbation of $f$.

Our strategy is to find a fixed random generating partition for the system $(\tilde{f}, \theta)$ for every small $\varepsilon > 0$ and use the absolute continuity of the stationary measure $\mu^\varepsilon$, together with the 'non-contractive' conditions on $f$, to obtain a semi-continuity property for entropy on zero-noise limits.

**Theorem 3.1.** Let us assume that there exists a finite partition $\xi$ of $M$ (Lebesgue modulo zero) which is generating for random orbits, for every small enough $\varepsilon > 0$.

Let $\mu^0$ be a weak$^*$ accumulation point of $(\mu^\varepsilon)_{\varepsilon > 0}$ when $\varepsilon \to 0$. If $\mu^{\varepsilon_0} \to \mu^0$ for some $\varepsilon_0 \to 0$ when $k \to \infty$, then

$$\limsup_{k \to \infty} h_{\mu^{\varepsilon_k}}(\tilde{f}, \theta_{\varepsilon_k}, \xi) \leq h_{\mu^0}(f, \xi).$$

The absolute continuity of $\mu^\varepsilon$ and the quasi-expansion enable us to use a random version of the entropy formula for endomorphisms (for a more general setting see [6, 19]).

**Theorem 3.2.** If an ergodic stationary measure $\mu^\varepsilon$, for a $C^1$ random perturbation $(\tilde{f}, \theta)$, is absolutely continuous and $\int \log \|Df_{\mu^0}(x)^{-1}\| \, d\theta(t) \, d\mu^\varepsilon(x) < 0$ for a given $\varepsilon > 0$, then

$$h_{\mu^\varepsilon}(\tilde{f}, \theta) = \int \int \log |\det Df_{\mu^\varepsilon}(x)| \, d\mu^\varepsilon(x) \, d\theta(t).$$

Putting theorems 3.1 and 3.2 together shows that $h_{\mu^\varepsilon}(f) \geq \int \log |\det Df(x)| \, d\mu^{\varepsilon_0}(x)$, since $\theta_0 \to \delta_0$ in the weak$^*$ topology when $\varepsilon \to 0$, by the assumptions on the support of $\theta_0$ in section 2.3. Since the reverse inequality holds in general (Ruelle’s inequality [31]), theorem 3.2 is proved.

### 3.1. Random entropy formula

Now we explain the meaning of theorem 3.2.

Let $\varepsilon > 0$ be fixed in what follows. The Lyapunov exponents $\lim_{n \to \infty} -n^{-1} \log \|Df^{\mu^\varepsilon}_{\omega}(x) \cdot v\|$ exist for $\theta^\varepsilon \times \mu^\varepsilon$-almost every $(\omega, x)$ and every $v \in T_x M \setminus \{0\}$, and are always positive in this setting. In fact, since the random perturbations are additive we have $D\tilde{f} = Df$ and, moreover, $\mu^\varepsilon$ is ergodic, absolutely continuous and $\mu^\varepsilon(K) = 0$, so for $\theta^\varepsilon \times \mu^\varepsilon$-almost every $(\omega, x)$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\omega(j+1)}(f^j_{\omega}(x))^{-1}\| = \int \log \|Df_{\mu^\varepsilon}(x)^{-1}\| \, d\mu^\varepsilon(x) \, d\theta(t) < 0.$$

(Setting $\psi(t, x) = \log \|Df_t(x)^{-1}\|$, this is just the ergodic theorem applied to $S : \Omega \times M \to \Omega \times M$ with $\psi = \psi \circ \pi$, where $\pi : \Omega \times M \to X \times M$, $(\omega, x) \mapsto (\omega(1), x)$.) This ensures that the Lyapunov exponent is positive in every direction under random perturbations, because

$$\log \|Df^{\mu^\varepsilon}_{\omega}(x)^{-1}\| \leq \sum_{j=0}^{n-1} \log \|Df_{\omega(j+1)}(f^j_{\omega}(x))^{-1}\|.$$

According to the multiplicative ergodic theorem (Oseledets [26]) the sum of the Lyapunov exponents (with multiplicities) equals the following limit $\theta^\varepsilon \times \mu^\varepsilon$-almost everywhere

$$\lim_{n \to \infty} \frac{1}{n} \log |\det Df_{\mu^\varepsilon}(x)| = \int \int \log |\det Df_{\tilde{f}}(x)| \, d\mu^\varepsilon(x) \, d\theta(t) > 0.$$

Cowieson–Young [11] obtained the same result for random diffeomorphisms without assuming the existence of a uniform generating partition but needed either a local entropy condition or that the maps $\tilde{f}$ involved be of class $C^\infty$. 
and the identity above follows from the ergodic theorem, since the value of the limit is $\mathcal{S}$-invariant, thus constant.

Now Pesin’s entropy formula states that for $C^1$ maps, $\alpha > 0$, with positive Lyapunov exponents everywhere, as in our setting, the metric entropy with respect to an invariant measure $\mu^\varepsilon$ satisfies the relation in theorem 3.2 if, and only if, $\mu^\varepsilon$ is absolutely continuous. In general we integrate the sum of the positive Lyapunov exponents (see Liu [21] for a proof in the $C^2$ setting). In our setting, the proof that $\mu^\varepsilon \ll m$, together with positive Lyapunov exponents in every direction for $\mu$-almost every point, implies that the entropy formula is an exercise using the bounded distortion provided by the Hölder condition on the derivative.

### 3.2. Random generating partition

Here we construct the uniform random generating partition assumed in the statement of theorem 3.1. In what follows we fix a weak* accumulation point $\mu^0$ of $\mu^\varepsilon$ when $\varepsilon \to 0$, so there exists $\varepsilon_k \to 0$ when $k \to \infty$ such that $\mu = \lim_{k} \mu^{\varepsilon_k}$.

To understand how to obtain a generating partition, we need a preliminary result.

For the following lemma, we recall that $K = \{ x \in M : \| Df(x) \| = 1 \}$ and set $\rho > 0$ such that $f \mid B(x, \rho)$ is a diffeomorphism onto its image and $B(x, \rho)$ is a convex neighbourhood for every $x \in M$, as in section 2.2. Using uniform continuity, we set $\rho_0 > 0$ such that for every $x, y \in M$ and $t \in X$, if $\text{dist}(x, y) < \rho_0$, then $\text{dist}(f_t(x), f_t(y)) < \rho$.

**Lemma 3.3.** Let $(f_t)_{t \in X}$ be a family of maps as in the definition of a non-degenerate additive random perturbation. For any given $\delta_0 \in (0, \rho_0)$ there exists $\beta > 0$ such that if $x \in M$ and $y \in M \setminus B(K, \delta_0)$ are such that $\delta_0 \leq \text{dist}(x, y) < \rho_0$, then $\text{dist}(f_t(x), f_t(y)) \geq \text{dist}(x, y) + \beta$ for every $t \in X$.

**Proof.** Let us assume that $x \in K$, let $y \in M \setminus B(K, \delta_0)$ be such that $\text{dist}(x, y) \in [\delta_0, \rho_0]$ and let $t \in X$ be fixed. By the choice of $\rho$ there is a smooth geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = f_t(x)$ and $\gamma(1) = f_t(y)$ and $\text{dist}(f_t(x), f_t(y)) = \int_0^1 \| \gamma'(s) \| \, ds < \rho$. In addition, there exists a unique smooth curve $\gamma_0 : [0, 1] \to M$ such that $f \circ \gamma_0 = \gamma$, $\gamma_0(0) = x$ and $\gamma_0(1) = y$.

Let us set $b = \| Df_t(y)^{-1} \| = \| Df(y)^{-1} \| < 1$ and $K(a) = \{ z \in M : \| Df_t(z)^{-1} \| \geq a \}$ for $a \in (0, 1)$. Then there must be $b_1, b_2 \in (b, 1)$ with $b_1 < b_2$ such that $K(b_1)$ (a compact set) is in the interior of $K(b_2)$ (recall that $z \mapsto \| Df_t(z)^{-1} \|$ is continuous and we are assuming that $x \in K$, that is, $\| (Df_t)^{-1} \|$ assumes the value 1).

We note that $\| Df_t(z)^{-1} \| < b_1$ for all $z \in K(b_2) \setminus K(b_1)$ and, moreover, that $\Gamma = \gamma^{-1}(K(b_2) \setminus K(b_1))$ has non-empty interior, thus positive Lebesgue measure on $[0, 1]$. Then

$$\text{dist}(f_t(x), f_t(y)) = \int_0^1 \| \gamma'(s) \| \, ds = \int_0^1 \| Df_t(\gamma(s)) \cdot Df_t(\gamma(s))^{-1} \cdot \dot{\gamma}(s) \| \, ds$$

$$\geq \frac{1}{b_1} \int_\Gamma \| Df_t(\gamma(s))^{-1} \cdot \dot{\gamma}(s) \| \, ds + \int_{[0,1]\setminus\Gamma} \| Df_t(\gamma(s))^{-1} \cdot \dot{\gamma}(s) \| \, ds$$

$$> \int_0^1 \| \gamma_0(s) \| \, ds \geq \text{dist}(x, y).$$

If $x \in M \setminus K$, then there exists $b \in (0, 1)$ such that both $x, y \in M \setminus K(b)$ and thus we may take $\Gamma = [0, 1]$ in the calculations above, arriving at the same sharp inequality.
Remark 3.4. Stochastic stability at the boundary of expanding maps

Proof. Let us argue by contradiction assuming that there are two points $x, y$ such that $\rho$ is a probability measure, we may assume that $\mu^0(\delta \xi) = 0$, for otherwise we can replace each ball by $B(x_i, \rho_0^2)$, for some $\gamma \in (1, \frac{1}{2})$ and for all $i = 1, \ldots, \ell$. Now let $\xi$ be the finest partition of $M$ obtained through all possible intersections of these balls:

$$\xi = \left\{ B \left( x_1, \frac{\rho_0}{2} \right), M \setminus B \left( x_1, \frac{\rho_0}{2} \right) \right\} \cup \cdots \cup \left\{ B \left( x_\ell, \frac{\rho_0}{2} \right), M \setminus B \left( x_\ell, \frac{\rho_0}{2} \right) \right\}.$$

In the following lemma we let $\rho$ stand for this new radius.

**Lemma 3.5.** $\sqrt{\frac{\rho}{\rho_0}}(f^{j\rho_0}_{\rho_0}(f^{j\rho_0}_{\rho_0})^{-1}\xi = A$ when $n \to +\infty$ for each $\omega \in \Omega$.

**Proof.** Let us argue by contradiction assuming that there are two points $x, y$ such that for some $\delta_0 > 0$ and $\omega \in \Omega$ fixed: $\text{dist}(f^{\rho_0}_\rho(x), f^{\rho_0}_\rho(y)) \in [\delta_0, \rho]$ and $y \in (\sqrt{\frac{\rho}{\rho_0}}(f^{j\rho_0}_{\rho_0})^{-1}(\xi)(x)$ for every $n \geq 1$.

Let $\delta = \min\{\text{dist}(z_1, z_2) : z_1, z_2 \in K, z_1 \neq z_2\}$ be the minimum separation between points in $K$ (we recall that $K$ is finite) and take $V = B(K, \min[\delta, \delta_0]/4)$. Then it is not possible that both $f^{\rho_0}_\rho(x), f^{\rho_0}_\rho(y)$ are in the same connected component of $V$. Using the fact that every $\rho$-neighbourhood is a convex neighbourhood and expressing $\text{dist}(f^{\rho_0}_\rho(x), f^{\rho_0}_\rho(y))$ through the length of a geodesic, we get a point $z \in M \setminus V$ and $\beta = \beta(\delta_0, \delta) > 0$ such that

$$\text{dist}(f^{\rho_0}_\rho(x), f^{\rho_0}_\rho(y)) = \text{dist}(f^{\rho_0}_\rho(x), f^{\rho_0}_\rho(z)) + \text{dist}(f^{\rho_0}_\rho(z), f^{\rho_0}_\rho(y))$$

$$\geq \text{dist}(f^{j\rho_0}_\rho(x), f^{j\rho_0}_\rho(z)) + \text{dist}(f^{j\rho_0}_\rho(z), f^{j\rho_0}_\rho(y)) + 2\beta$$

$$\geq \text{dist}(f^{j\rho_0}_\rho(x), f^{j\rho_0}_\rho(y)) + 2\beta$$

for every $j > 0$, applying lemma 3.3 twice. But then the upper bound $\rho$ for the distance between iterates of $x$ and $y$ cannot hold for all $j \geq 1$. This shows that $\delta_0$ cannot be positive, hence the diameter of the atoms of the refined partitions tends to zero. This is enough to conclude the statement of the lemma. □

Lemma 3.5 shows that $\xi$ is a random generating partition as in the statement of the random Kolmogorov–Sinai theorem 2.4. Hence we conclude that $h_{\mu_\xi}(\hat{f}, \theta_\xi, \xi) = h_{\mu_\xi}(\hat{f}, \theta_\xi)$ for all $k \geq 1$.

3.3. Semi-continuity of entropy on zero-noise

Now we start the proof of theorem 3.1. We need to construct a sequence of partitions of $\Omega$ according to the following result. For a partition $\mathcal{P}$ of $\Omega$ and $\omega \in \Omega$ we denote by $\mathcal{P}(\omega)$ the element (atom) of $\mathcal{P}$ containing $\omega$. We set $\omega_0 = (\ell_0, \ell_0, \ell_0, \ldots) \in \Omega$ in what follows and write $\text{int}(A)$ for the topological interior of the set $A \subset \Omega$. 
Lemma 3.6. There exists an increasing sequence of measurable partitions \((B_n)_{n \geq 1}\) of \(\Omega\) such that

1. \(\omega_0 \in \text{int} B_0(\omega_0)\) for all \(n \geq 1\);
2. \(B_n \not\subseteq B, \theta^\omega \equiv 0\) for all \(k \geq 1\) when \(n \to \infty\);
3. \(\lim_{n \to \infty} \mathcal{H}_\rho(\xi | B_n \times M) = \mathcal{H}_\rho(\xi | B \times M)\) for every measurable finite partition \(\xi\) of \(\Omega \times M\) and any \(\mathcal{S}\)-invariant probability measure \(\rho\).

**Proof.** For the first two items we let \(C_n\) be a finite \(\theta^\omega\) mod 0 partition of \(X\) such that \(t_0 \in \text{int} C_n(\omega_0)\) with \(\text{diam} C_n \to 0\) when \(n \to \infty\). As an example, take a cover \((B(t, 1/n))_{t \in X}\) of \(X\) by \(1/n\)-balls and take a sub-cover \(U_1, \ldots, U_k\) of \(X \setminus B(t_0, 2/n)\) together with \(U_0 = B(t_0, 3/n)\); then let

\[
\mathcal{C}_n = \{U_0, M \setminus U_0\} \cup \cdots \cup \{U_k, M \setminus U_k\}.
\]

We observe that we may assume the boundary of these balls has null \(\theta^\omega\)-measure for all \(k \geq 1\), since \((\theta^\omega)_{n \geq 1}\) is a denumerable family of non-atomic probability measures on \(X\) (see remark 2.7). Now we set

\[
B_n = C_n \times \cdots \times C_n \times \Omega \quad \text{for all } n \geq 1.
\]

Then since \(\text{diam} C_n \leq 2/n\) for all \(n \geq 1\) we also have that \(\text{diam} B_n \leq 2/n\) and so tends to zero when \(n \to \infty\). Clearly \(B_n\) is an increasing sequence of partitions. Hence \(\bigvee_{n \geq 1} B_n\) generates the \(\sigma\)-algebra \(\mathcal{B}, \theta^\omega\) mod 0 (see, e.g., [9, lemma 3, chapter 2]) for all \(k \geq 1\). This proves items (1) and (2). Item (3) of the statement of the lemma is theorem 12.1 of Billingsley [9].

Now we use some properties of conditional entropy to obtain the right inequalities. We start with

\[
\mathcal{H}_{\mu^\omega}(\hat{f}, \theta^\omega) = \mathcal{H}_{\mu^\omega}(\hat{f}, \theta^\omega, \xi) = \mathcal{H}_{\mu^\omega \times \mu^\omega}(S, \Omega \times \xi)
\]

\[
= \inf \frac{1}{n} \mathcal{H}_{\theta^\omega \times \mu^\omega} \left( \bigvee_{j=0}^{n-1} (S^j)^{-1}(\Omega \times \xi) \mid B \times M \right),
\]

where the first equality comes from section 3.2 and the second one can be found in Kifer [15, theorem 1.4, chapter II], with \(\Omega \times \xi = \{\Omega \times A : A \in \xi\}\). Hence for arbitrary fixed \(N \geq 1\) and for any \(m \geq 1\)

\[
\mathcal{H}_{\mu^\omega}(\hat{f}, \theta^\omega) \leq \frac{1}{N} \mathcal{H}_{\theta^\omega \times \mu^\omega} \left( \bigvee_{j=0}^{N-1} (S^j)^{-1}(\Omega \times \xi) \mid B \times M \right)
\]

\[
\leq \frac{1}{N} \mathcal{H}_{\theta^\omega \times \mu^\omega} \left( \bigvee_{j=0}^{N-1} (S^j)^{-1}(\Omega \times \xi) \mid B_m \times M \right).
\]

because \(B_m \times M \subseteq B \times M\). Now we fix \(N\) and \(m\), let \(k \to \infty\) and note that since \(\mu^0(\partial \xi) = 0 = \delta_{\omega_0}(\partial B_m)\) it must be that

\[
(\delta_{\omega_0} \times \mu^0)(\partial (B_i \times \xi_j)) = 0 \quad \text{for all } B_i \in B_m \text{ and } \xi_j \in \xi,
\]

where \(\delta_{\omega_0}\) is the Dirac mass concentrated at \(\omega_0\) \(\in \Omega\). Thus by weak* convergence of \(\theta^\omega \times \mu^\omega\) to \(\delta_{\omega_0} \times \mu^0\) when \(k \to \infty\) we get

\[
\limsup_{k \to \infty} \mathcal{H}_{\mu^\omega}(\hat{f}, \theta^\omega) \leq \frac{1}{N} \mathcal{H}_{\omega_0} \left( \bigvee_{j=0}^{N-1} (S^j)^{-1}(\Omega \times \xi) \mid B_m \times M \right) = \frac{1}{N} \mathcal{H}_{\mu^0} \left( \bigvee_{j=0}^{N-1} f^{-j}(\xi) \right).
\]

(3.1)
Here it is easy to see that the middle conditional entropy of (3.1) (involving only finite partitions)
equals $N^{-1} \sum_i \mu_i^0(P_i) \log \mu_i^0(P_i)$, with $P_i = \xi_n \cap f^{-1} \xi_{n-1} \cap \cdots \cap f^{-(N-1)} \xi_{N-1}$ ranging over every possible sequence of $\xi_n, \ldots, \xi_{N-1} \in \xi$.

Finally, since $N$ was an arbitrary integer, theorem 3.1 follows from (3.1). We have completed the proof of the first part of theorem E.

3.4. Existence of absolutely continuous measure and stochastic stability

Let $f : M \to M$ be as in the statement of theorem E. The assumptions on $f$ ensure that for every $x \in M$ and all $v \in T_x M \setminus \{0\}$ we have

$$\liminf_{n \to \infty} \frac{1}{n} \log \|Df^n(x) \cdot v\| \geq 0.$$  

Thus the Lyapunov exponents for any given $f$-invariant probability measure $\mu$ are non-negative. Hence the sum $\chi(x)$ of the positive Lyapunov exponents of a $\mu$-generic point $x$ is such that

$$\chi(x) = \lim_{n \to \infty} \frac{1}{n} \log |\det Df^n(x)| \quad \text{and} \quad \int \chi \, d\mu = \int \log |\det Df| \, d\mu$$

by the multiplicative ergodic theorem and the standard ergodic theorem.

We know that there is only one stationary measure for every $\varepsilon > 0$, by section 2.3. By section 3.3, every weak$^*$ accumulation point $\mu$ of the stationary measures $(\mu_\varepsilon)_{\varepsilon > 0}$, when $\varepsilon \to 0$, is an equilibrium state for $-\log |\det Df|$, that is 1.4 holds. We may and will assume that $\mu$ is ergodic due to the following:

**Lemma 3.7.** Almost every ergodic component of an equilibrium state for $-\log |\det Df|$ is itself an equilibrium state for the same function.

**Proof.** Let $\mu$ be an $f$-invariant measure satisfying $h_\mu(f) = \int \log |\det Df| \, d\mu$. On the one hand, the ergodic decomposition theorem (see, e.g., Mañé [23]) ensures that

$$\int \log |\det Df| \, d\mu = \int \int \log |\det Df| \, d\mu_z \, d\mu(z) \quad \text{and} \quad h_\mu(f) = \int h_{\mu_z}(f) \, d\mu(z).$$

(3.3)

On the other hand, Ruelle’s inequality guarantees that for a $\mu$-generic $z$ (recall 3.2)

$$h_{\mu_z}(f) \leq \int \log |\det Df| \, d\mu_z.$$  

(3.4)

By (3.3) and (3.4), and because $\mu$ is an equilibrium state (1.4), we conclude that we have an identity in (3.4) for $\mu$-almost every $z$. □

Now we note that since $K$ is finite, if $\mu(K) > 0$, then $\mu$ (which is ergodic) is concentrated on a periodic orbit. Hence $h_\mu(f) = 0$ and so by the entropy formula these orbits are non-volume-expanding (the Jacobian equals 1).

Finally, for an ergodic equilibrium state $\mu$ with $\mu(K) = 0$, we must have

$$\lim_{n \to \infty} \frac{1}{n} \log \| (Df^n(x))^{-1} \| \leq \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df(f^j(x))^{-1} \| = \int \log \| (Df)^{-1} \| \, d\mu < 0,$$

$\mu$-almost everywhere. This means that the Lyapunov exponents of $\mu$ are strictly positive, where $f$ is a $C^{1+\alpha}$ endomorphism, $\alpha > 0$.

Now the extension of the entropy formula for endomorphisms, see Liu [20] and Qian–Zhu [29] (in the $C^2$ setting, but the distortion estimates need only a Hölder condition on
the derivative), ensures that an equilibrium state whose Lyapunov exponents are all positive must be absolutely continuous with respect to the Lebesgue measure. Hence $\mu \ll m$.

The previous discussion shows that $f$ is non-uniformly expanding in the sense of Alves–Bonatti–Viana [4]. We obtain that

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df(f^j(x))^{-1} \| < 0, \quad \mu\text{-almost every } x. \quad (3.5)$$

These authors showed that any absolutely continuous invariant measure $\mu$, in this setting, has an ergodic basin $B(\mu)$ containing an open subset $U$ Lebesgue modulo 0. By the topological mixing property of $f$ (lemma 2.5) and because $f$ is a regular map, we deduce that $B(\mu)$ must contain all of $M$, Lebesgue modulo 0, and is thus unique.

These arguments can be carried out for every ergodic component of any weak* accumulation point of stationary measures when $\varepsilon \to 0$, thus every such accumulation point is a linear convex combination of an absolutely continuous invariant measure with finitely many Dirac masses concentrated on the non-volume-expanding orbits. This ends the proof of theorem E.

### 3.5. Stochastic stability

Now we assume that $f : M \to M$ satisfies all the conditions of the statement of theorem D. We arrive at the same conclusion as theorem E but since we assume that $|\det Df| > 1$ on $K$, we would arrive at a contradiction if $\mu$ is an equilibrium state with $\mu(K) > 0$, that is, the expanding volume condition avoids Dirac masses on periodic orbits as ergodic components of zero-noise limit measures. Thus, in this setting, we must have $\mu(K) = 0$ for all ergodic equilibrium states. Hence there is only one equilibrium state: the unique absolutely continuous invariant probability measure $\mu$ of $f$.

Therefore every weak* accumulation point of the stationary measures, when $\varepsilon \to 0$, must be equal to $\mu$. This shows the stochastic stability of $\mu$ and concludes the proof of theorem D.

### 4. Random perturbations of the intermittent map

Let $T$ be defined as in the introduction and consider a non-degenerate random perturbation of $T$. For every small $\varepsilon > 0$ we know that there exists a single stationary measure $\mu^\varepsilon$, see section 2.2.

#### 4.1. Characterization of zero-noise measures

Let $T$ be as in the introduction for $\alpha \in (0, 1)$. Here we prove theorem A in two steps. Let $\mu$ be a weak* accumulation point of $\mu^\varepsilon$, when $\varepsilon \to 0$. We show that $\mu \in \mathcal{E} = \{t \delta_0 + (1-t)\mu_{SRB} : 0 \leq t \leq 1\}$.

First we prove that $h_\mu(T) = \int \log DT \, d\mu$ and in the following we deduce that any $T$-invariant measure satisfying the entropy formula as above should belong to $\mathcal{E}$. The latter is also proved in [28] by different methods.

As we are considering additive random perturbations of $T$, we can apply theorem E and conclude that $\mu$ is an equilibrium state of $-\log DT$. As in section 3.4 we may and will assume that $\mu$ is ergodic by lemma 3.7.

Now we consider two cases: either $\mu(\{0\}) > 0$, or $\mu(\{0\}) = 0$.

In the first case, the ergodicity of $\mu$ ensures that $\mu = \delta_0$, since 0 is a fixed point for $T$.  

For the second case, since $\log DT > 0$ for all points of $S$ except 0, we have that

$$h_\mu(T) = \int \log DT \, d\mu > 0$$

and the ergodic theorem together with the fact that $S$ is one dimensional guarantees that the Lyapunov exponent of $\mu$ is positive. Thus $\mu$ is an ergodic probability measure with positive Lyapunov exponent and positive entropy which satisfies the entropy formula. Now we apply a version of Pesin’s entropy formula obtained by Ledrappier [18], which holds for $C^{1+\alpha}$ endomorphisms of $S^1$, to conclude that $\mu$ must be absolutely continuous with respect to the Lebesgue measure.

The above arguments show that any typical ergodic component of every zero-noise limit measure $\mu$ equals either $\delta_0$ or $\mu_{SRB}$. Hence a straightforward application of the ergodic decomposition theorem to $\mu$ concludes the proof of theorem A.

### 4.2. Stochastic stability without absolutely continuous measure

Now we prove theorem B. After theorem E every zero-noise limit measure $\mu$ for the additive random perturbation of the intermittent map $T$ is an equilibrium state for $-\log DT$. For $\omega \geq 1$ this is enough to deduce stochastic stability of $T = T_\omega$.

Indeed, let $\mu$ be a weak$^*$ accumulation point of $\mu^\varepsilon$ when $\varepsilon \to 0$. As in the proof of theorem A (in the previous section), we consider the ergodic decomposition of $\mu$.

We claim that almost all ergodic components of $\mu$ equal the Dirac measure $\delta_0$ concentrated on 0. Arguing by contradiction, we suppose that for some ergodic component $\eta$ of $\mu$ we have $\eta(\{0\}) = 0$. Thus by the same arguments in section 4.1 (using positive Lyapunov exponents and the entropy formula for one-dimensional maps), this implies that $\eta$ is an absolutely continuous invariant probability measure for $T$.

However, because the intermittent map $T$ is $C^2$ for $\omega \geq 1$, it is well known that $T$ does not admit any absolutely continuous invariant probability measure in this setting, see, e.g., [38] for a proof of this fact.

Hence if some ergodic component $\eta$ of $\mu$ is such that $\eta(\{0\}) = 0$ we arrive at a contradiction. Thus $\eta(\{0\}) > 0$ and $\eta = \delta_0$ by ergodicity, for every ergodic component of $\mu$. Therefore $\mu = \delta_0$.

This proves the **stochastic stability of the intermittent map when it does not admit absolutely continuous invariant probability measures** and ends the proof of theorem B.

### 4.3. Stochastically unstable random perturbation

Here we prove theorem C. Let $f_t$ be the family (1.3) and let $0 < s < 1$. Then there exists a unique fixed source

$$p_t = \frac{1}{2} \left( \frac{s - 1}{s - 0} \right)^{1/\alpha} \in (0, \frac{1}{2}) \quad \text{ such that } f_s'(p_t) = 1 + \alpha(1 - s) > 1.$$

Now we choose $u \in (s, 1)$ such that $f_t'_s \mid [p_u, p_s] > 1$ for all $t \in [s, u]$. For this we just have to take $u$ close enough to $s$.

Clearly $f_t^n_u(x) \to 0$ for all $x \in S \setminus [p_u, 1 - p_u]$ when $n \to \infty$, see figure 1 and recall that the maps $f_t$ are symmetric ($f_t(x) = 1 - f_t(1 - x)$) on $S = [0, 1]/0 \sim 1$.

**Lemma 4.1.** For every $x \in S \setminus [p_u, 1 - p_u]$ and every sequence $t_n \in [s, u]^\mathbb{N}$ we have that $f_{t_n}^n(x) \to 0$ when $n \to \infty$. 


Proof. It is straightforward to check that the graph of \( f_1 \mid [0, \frac{1}{2}] \) is below the graph of \( f_1 \) and above the graph of \( f_2 \) for every \( t \in (s, u) \). Hence \( f_1^n(x) \leq f_2^n(x) \to 0 \) when \( n \to \infty \) for every \( x \in (0, p_u) \). Using the symmetry we arrive at \( f_2^n(x) \to 0 \sim 1 \) when \( n \to \infty \) for all \( x \in (1 - p_u, 1) \). \( \square \)

Now we let \( \theta \) be any probability measure with support contained in \([s, u]\) and set \( \theta_0 = \theta^N \).

Proposition 4.2. For \( \theta_0 \times m \)-almost every \((t, x) \in [s, u]^N \times [p_u, 1 - p_u]\) there exists \( n \geq 1 \) such that \( f_2^n(x) \not\in [p_u, 1 - p_u] \).

Combining the two results above we conclude that for \( \theta_0 \times m \)-almost every \((t, x) \in [s, u]^N \times S\) we have that

\[
\frac{1}{n} \sum_{j=0}^{n-1} \delta \frac{f_2^j(x)}{f_1^j(x)} \to \delta_0 \quad \text{in the weak* topology} \quad \text{when} \quad n \to \infty,
\]

finishing the proof of theorem C.

To prove proposition 4.2 we need the following result whose proof follows standard steps, using the uniform expansion and the \( C^1\)-condition on every \( f_t \). Let us fix \( t \in [s, u]^N \) and a point \( 0 < r < p_u \) such that \( f_t'(r) > 1 \) for all \( t \in [s, u] \). Let also \( \beta_1 = \min \{ f_t'(x) : x \in [r, 1 - r], t \in [s, u] \} > 1 \) and \( \beta_2 = \max \{ f_t'(x) : x \in S, t \in [s, u] \} > 1 \).

Lemma 4.3. There exists \( C > 1 \) such that for any interval \( I \subset [r, 1 - r] \), all \( t \in [s, u]^N \) and \( k \geq 1 \) such that \( f_2^k(I) \subset [r, 1 - r] \) for every \( j = 0, \ldots, k - 1 \), we have

\[
1 \leq \frac{C}{f_2^k(I)} \leq \frac{C}{f_1^k(I)} \leq C
\]

for all \( s, y \in I \).

Proof of the proposition. For \( t \in [s, u]^N \) we define \( E_k(t) = (f_1^k)^{-1}([p_u, 1 - p_u]) \) for \( k \geq 1 \). We will show that \( \bigcap_{k \geq 1} E_k(t) \) has zero Lebesgue measure for any \( t \), which is enough to conclude the statement of the lemma. In fact, this means that \( n(t, x) = \min \{ k \geq 1 : f_2^k(x) \not\in [p_u, 1 - p_u] \} \) is finite for every \( x \) in a set \( X(t) \) with \( m(X(t)) = 1 \), for every given \( t \in [s, u]^N \).

Thus \( \Delta = \bigcup_{t \in [s, u]^N} [t] \times X(t) \) is measurable and \( \theta_0 \times m)(\Delta) = 1 \).

Let us fix \( t \in [s, u]^N \), take a non-empty interval \( I \subset [p_u, 1 - p_u] \) and show that \( I \cap \bigcap_{k \geq 1} E_k(t) \) has zero Lebesgue measure.

Let \( k > 1 \) be the first time such that \( f_2^k(I) \not\subset [r, 1 - r] \). There exists such \( k \) since by uniform expansion \( m(f_2^k(I)) \geq \beta_1^k m(I) \) whenever \( f_2^j(I) \subset [r, 1 - r] \) for \( j = 0, \ldots, k - 1 \). Now there are two possibilities: either \( f_2^k(I) \subset S \setminus [p_u, 1 - p_u] \) or we have \( f_2^k(I) \cap [p_u, 1 - p_u] \neq \emptyset \neq [1 - r, r] \cap f_2^k(I) \).

In the former case we conclude that \( I \cap \bigcap_{k \geq 1} E_k(t) = \emptyset \), and the argument ends.

In the latter case, we let \( F = f_2^k(I) \cap S \setminus [p_u, 1 - p_u] \) and observe that either \( F \supset [s, p_u] \) or \( F \supset [1 - p_u, 1 - s] \), so \( m(F) \geq p_u - s \). Since \( f_2^{k-1}(I) \subset [r, 1 - r] \) we have \( m(f_2^{k-1}(I)) \leq \beta_2(1 - 2r) \) and hence

\[
\frac{m(F)}{m(I)} \leq \frac{C}{m(f_2^k(I))} \leq \frac{C}{\beta_2(1 - 2r)} \leq \frac{p_u - r}{\beta_2(1 - 2r)},
\]

where \( G = (f_2^k(I))^{-1}(\text{closure}(F)) \) and \( C > 0 \) is a bounded distortion constant from lemma 4.3.
This shows that 
\[ m(I \setminus G) \leq (1 - C(p_u - r)/(\beta_2(1 - 2r)))m(I). \]
We may take \( r \) so close to \( p_u \) that
\[ 0 < \gamma = 1 - C \frac{p_u - r}{\beta_2(1 - 2r)} < 1. \]
If \( m(I \setminus G) = 0 \), we are done. Otherwise we apply the same argument to each connected component of \( I_1 = I \setminus G \) inductively, as follows:

Let \( I_k \subset I \) be a compact set formed by finitely many pairwise disjoint closed intervals \( I_k = I_{k,1} \cup \cdots \cup I_{k,i_k} \) such that for each \( j \in \{1, \ldots, i_k\} \) there is a maximal iterate \( n_j \) so that
\[ f^n_{k,j}(I_{k,j}) \subset [r, 1 - r] \text{ for all } n = 1, \ldots, n_j - 1. \]

We observe that \( \gamma \) does not depend on the number of iterates of the first exit. Then either \( f^{n_j}_{k,j}(I_{k,j}) \subset S \setminus [p_u, 1 - p_u] \), or there exists \( G_{k,j} \subset I_{k,j} \) maximal such that \( f^{n_j}_{k,j}(G_{k,j}) \subset S \setminus [p_u, 1 - p_u] \) and
\[ m(I_{k,j} \setminus G_{k,j}) \leq \gamma \cdot m(I_{k,j}) \]

In the former case we delete \( I_{k,j} \) from \( I_{k+1} \). In the latter case, we add the connected components \( I' \) of \( I_{k,j} \setminus G_{k,j} \) to \( I_{k+1} \) and associate with each of them the maximal number of iterates \( n' \) such that \( f^{n'_j}(I') \subset [r, 1 - r] \text{ for } j = 1, \ldots, n' - 1. \) Then \( m(I_{k,j}) \leq \gamma \cdot m(I_k) \). This shows that \( m(I_k) \leq \gamma^k m(I) \to 0 \text{ when } k \to \infty. \) Since by construction \( \bigcap_{k \geq 1} I_k \) contains \( I \cap \bigcap_{k \geq 1} E_k(t) \), the proof is complete. \( \Box \)

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