Slow quenches in XXZ spin-chains – the role of Galilean invariance breaking

Piotr Chudzinski

Institute for Theoretical Physics, Center for Extreme Matter and Emergent Phenomena, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands
(Dated: January 18, 2018)

We study a XXZ spin-chain in a gapless Tomonaga-Luttinger liquid (TLL) phase with time dependent anisotropy of spin exchange interactions. To begin we focus on a linear ramp of $J_z$, starting at XX point and slowly increasing towards the anti-ferromagnetic Heisenberg point. Although the problem of a linear ramp in the TLL has been recently under intense scrutiny in a perturbative $g$-ology framework, an aspect that has been overlooked so far is the role of the Galilean invariance breaking. We find that, although the differential equation that needs to be solved to find time evolution of the system is substantially more complicated, in some cases exact analytic solutions can be given. We obtain them for the linear ramp in the limit of small $J_z$ as well as $J_z \to 1$, and for such protocols that are tailored to keep the Galilean invariance breaking term constant for every $J_z$. We point out the features of dynamics during the quench that stays unaltered, and those that need to be taken with care when Galilean invariance breaking is present. We are able to deduce that it is the shape of the propagating front that is affected in the most pronounced way.

PACS numbers:

I. INTRODUCTION

In recent years a non-equilibrium dynamics of a quantum system subjected to a non-adiabatic variation of its parameters, has grown into a quite important field of research\textsuperscript{1,2}. This is thanks to ground-breaking advances in simulations and experiments\textsuperscript{3,4}, which has allowed us to probe\textsuperscript{5} intriguing, non-perturbative phenomenology of the quenched system\textsuperscript{6,7}. As it frequently happens, the 1D systems plays a prominent role\textsuperscript{8,9}, because exact analytical solutions are available for also for a strongly correlated regime\textsuperscript{10}. It should be emphasized that any realistic experimental implementation of the quench has to be extended over some finite time, say from $t = 0$ to $t = \tau$, which has raised an interest in a dynamics of a quantum system \textit{during} the quench\textsuperscript{11}. This is a subject of our study.

The common description\textsuperscript{11} of a dynamics of a 1D system that has undergone a quench is given in terms of propagating front. That is entangled particles propagate throughout the system and carry information about the change of the hamiltonian. It clearly manifests in correlation functions of the system where, in experimental\textsuperscript{12,13} and numerical\textsuperscript{14,15} studies, one clearly observes different behaviour inside and outside the light cone, with a front that is moving with an instantaneous velocity of collective eigen-particles of the system. This intuitive picture has a hidden assumption that the physics obeyed by the moving particles that propagate the signal do not depend on their movement, it is an assumption of the Galilean invariance. It is then crucial to ask the following question: up to what extend this simple picture is valid when Galilean invariance is broken?

The question we have raised is not of a pure academic interest. One of the most important models in 1D physics, the spin XXZ model does not obey Galilean invariance. Its high prominence in the field of low dimensional systems is because it is solvable (using the Bethe-ansatz techniques) hence the TLL parameter $K$ and velocity $u$ are known exactly. Also numerical methods to study this model have a long history and are very well developed\textsuperscript{16,17}. Other models have been mapped on XXZ model, to mention two leg ladder with strong on-rung coupling as an example\textsuperscript{18}. Finally, several experimental realization of this model has been proposed, mostly based on 1D Mott insulators\textsuperscript{19,20}, but also in cold atoms systems\textsuperscript{21}. The problem under consideration has been already under scrutiny, in Ref.\textsuperscript{22}, where a combination of numerical and perturbative ($g$-ology) method was used. A systematic discrepancy between analytical solution and numerics was found, especially in the area of the propagating front, when the final $J_z$ ($\equiv \Delta$) was increasing. We hope to be able to understand this effect with our exact analytic solutions.

Gapless phases are probably the most interesting from the point of view of time-dependent quench dynamics because there is no gap that would inhibit propagation of the lowest energy excitations appearing upon quenching\textsuperscript{7,22}. A range of parameters where XXZ model is in the gapless TLL phase is quite broad, hence from the experimental/numerical viewpoints there is a space to vary the model’s parameters. It should be emphasized that from the exact analytic solution we know not only the precise values of $K, u$ but also non-universal amplitudes of each correlation term\textsuperscript{22,23}. If we assume that this functional dependence stays the same also during the quench, then this paves the way for a parameter free comparison with high precision numerical and experimental studies\textsuperscript{24}. This could in principle allow to investigate deviations\textsuperscript{25} from non-equilibrium TLL predictions, for instance due to irrelevant operators that may introduce thermalization, or help to resolve recently raised issue of the validity of generalized Gibbs ensemble in this model\textsuperscript{15,25}. Hence, the existence of an exact analytical
solution is of an uttermost importance. Recently, analytical solutions for several various quenching protocols have been found in Refs. 14, 14. All were for the case when the Galilean invariance is obeyed, which obviously raises a question if analogous results may be provided when the Galilean invariance is broken.

II. MODEL

A. Time dependent TLL

We study a 1D system in a gapless Tomonaga-Luttinger liquid (TLL) phase with a time dependent parameters. The TLL hamiltonian reads26,27,

\[ H = \int \frac{dx}{2\pi} \left[ \partial_x \phi(x,t) \right]^2 + \left( \frac{u(t)}{K(t)} \right) \left[ \partial_x \phi(x,t) \right]^2 \]

where \( u(t), K(t) \) are velocity and TLL parameter of collective bosonic mode, these depend on underlying theory. The Eq (1) is written in terms of density fields \( \phi(x) \) and canonically conjugate fields \( \theta(x) \), with \( \Pi(x) = \partial_x \theta(x) \). To define them, first one considers a 1D theory of spinless fermions \( c^\dagger \) and extracts their long wavelength behavior around the Fermi points given by the fields \( \psi(x) = \exp(i k_F x) \psi_L^\dagger(x) + \exp(-i k_F x) \psi_R^\dagger(x) \). Then the bosonic fields, the collective modes, \( \psi_{L,R}^\dagger(x) = \kappa_{R,L} \frac{1}{2\pi a} \exp(i \sum (\phi_{L,R}(x) \pm \theta_{L,R}(x))) \) (where \( \kappa_{R,L} \) is a constant operator, Majorana fermion, introduced to ensure proper anti-commutation relations).

In our problem the bosonic fields are time dependent and obey the following differential equation which describes the time evolution of the system:

\[ \frac{d\phi(q,t)}{dt} = u(t)K(t)q\theta(q,t), \quad \frac{d\theta(q,t)}{dt} = -\frac{u(t)}{K(t)} \phi(q,t) \]

Following Refs. 26-29 we introduce an auxiliary function \( F(t) \) which is implicitly defined in the following way:

\[ \phi(q,t) = 2 \sqrt{\pi K_0} |q| (a_q F^*(t) + a_q^\dagger F(t)) \]

\[ \theta(q,t) = \frac{1}{qu(t)K(t)} \sqrt{\frac{\pi K_0}{2|q|}} (a_q \partial_t F^*(t) + a_q^\dagger \partial_t F(t)) \]

where \( a_q \) are bosonic operators that diagonalize the hamiltonian at \( t = 0 \). The time evolution of the system is encapsulated inside \( F(t) \). We write down a single ODE which is solved by \( F(t) \):

\[ \frac{d^2}{dt^2} F(q,t) + \partial_t \log[u(t)K(t)] \frac{d}{dt} F(q,t) + u^2(t)q^2 F(q,t) = 0 \]

At this moment we have generalized the result of Ref. 28. When Galilean invariance is preserved, as it was assumed in Ref. 28, then \( uK = cste \) and the second term in Eq (5) drops. We abandon this assumption and arrived at a more general differential equation in a form \( \frac{d^2}{dt^2} F(q,t) + \theta_1 q \frac{d}{dt} F(q,t) + \theta_2 q^2 F(q,t) = 0 \) which, is much more difficult to solve when \( \theta_1 \neq 0 \). In the following we shall point out a few cases for which the solution \( F(t) \) in a closed analytic form exists.

B. XXZ model

As already mentioned in the introduction, one prominent example where the Galilean invariance is not preserved is the XXZ spin-chain defined by the following hamiltonian:

\[ H_{XXZ}(t) = J \sum_i \left[ S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta(t) S_i^z S_{i+1}^z \right] \]

The Jordan-Wigner transformation allows to re-express the spin problem in terms of the spinless fermion fields \( \psi(x) \). For \( |\Delta| < 1 \) the spectrum of the system is gapless and its low energy dynamics can be expressed using the TLL hamiltonian, Eq (1). The \( \Delta = 0 \) is a point where fermions are non-interacting and \( K = 1 \), while at \( \Delta = 1 \) (and \( J > 0 \)) a gap opens in the spectrum that corresponds to an onset of an anti-ferromagnetic order. Below we are going to take initial \( \Delta(t = 0) = 0 \), although our results can be straightforwardly generalized to any initial/final \( |\Delta(t)| < 1 \). For XXZ model the TLL parameters are non-perturbative and known exactly:

\[ u = J \frac{\pi^{\frac{1}{2}}}{\arccos(\Delta(t))} \]

\[ K = \frac{\pi}{2 \pi - \arccos(\Delta(t))} \]

one directly checks that \( uK \neq cste \). In the following we consider a time dependent variation \( \Delta(t) \). By substituting Eq's (3) to Eq (5) this translates to the following temporal variations of the new coefficient of the differential equation:

\[ \theta_1 = \partial_t \log[u(t)K(t)] = \left[ \frac{\pi \Delta}{(\Delta^2 - 1)(\pi - \arccos(\Delta)) \arccos(\Delta)} \right] \partial_t \Delta(t) \]

and \( \theta_2 \) is simply a square of Eq (7). These coefficients, given by a complicated formula involving inverse trigonometric functions, do not allow for any further analytic treatment. We need to look for a good approximate formulas. It turns out that \( u(\Delta) \) can be quite well approximated by \( J\sqrt{1+\Delta} \), a formula that is nearly the same as the one for a velocity in a TLL with the Galilean invariance preserved, where \( u = V_F \sqrt{1+2g} \) (albeit factor two
of a difference). We then conclude that the position of the front during the quench should be the same in both cases. As for the other term given by Eq. [9] the factor in front of a derivative $\partial_t \Delta(t)$, has a singularity as we approach the point $\Delta = 1$, where a transition of XXZ model to a gapped phase takes place. We can then immediately distinguish two substantially different cases: A) a derivative $\partial_t \Delta(t)$ stays finite during the quench B) a quench is continuous at $t = \tau$, hence a derivative $\partial_t \Delta(t) = 0$ at $t = \tau$ (when quenched is finished). In the case A), the new term in Eq.[9] can grow up to very large values and dominate. In the case B) this term either stays constant or goes down to zero even if the system reaches the point $|\Delta| = 1$. Finally, we note that the above described distinct behaviour of $u(\Delta)$, that suggest $g_2 - g_1 \to 0$, and the divergent $\theta_1$ term, which is proportional to $g_2 - g_1$ in the perturbative g-ology language, naively poses a paradox. However this unusual combination of coefficients is simply a manifestation of the fact that the perturbative g-ology language and cannot be used for spin chains at strong $\Delta$. In particular perturbative framework cannot be used for time dependent problems when this issue explicitly enters into coefficients of the central ODE, Eq.[10].

### III. LINEAR RAMP

#### A. Small $\Delta$

For small $\Delta$ we find that the $\theta_1$ term depends linearly on $\Delta$, that is $\log\left\{ u(t)K(t) \right\} \approx \left( \frac{\Delta}{\tau} - \frac{1}{2} \right) \Delta(t)$, where we took a Taylor series expansion of logarithm function around $\log(1) = 0$. If we further assume the simplest time-resolved protocol, the linear ramp: $\Delta(t) = \Delta t / \tau$ then the differential equation reads:

$$F(t) - \theta \frac{\Delta t}{\tau} F'(t) + F''(t) = 0$$

where $\theta = \left( -\frac{1}{2\tau} + \frac{1}{2} \right)$. It is possible to give an analytic solution for such equation:

$$F(t) = e^{\frac{3}{2} \theta^2 \tau^2} \left( c_1 H_{J^2 \tau^2 \gamma \Delta^2} \left( \kappa_1 \right) + c_2 \frac{1}{2} F_1 \left[ -\frac{J^2 q^2 \tau (J^2 q^2 + \theta^2)}{\Delta \theta^3}, \frac{1}{2} \kappa_1^2 \right] \right).$$

where the variable $\kappa_1 = \sqrt{\Delta \theta^2 - \frac{\Delta^2 q^2}{\tau^2}}$. The $H$ is the Hermite polynomial and $F_1$ is the hypergeometric function. When the sign of $\Delta(t)$ does not change during the quench, this solution can be also expressed in terms of the Bessel functions:

$$F(t) = \left[ C_1 \sqrt{3} J_{1/3} \left( \frac{J^2 q^2 \tau^2 \gamma \Delta}{12 J^2 q^2 \gamma^2 \frac{\Delta^3}{\tau^3}} \right) + C_2 \sqrt{3} J_{-1/3} \left( \frac{J^2 q^2 \tau^2 \gamma \Delta}{12 J^2 q^2 \gamma^2 \frac{\Delta^3}{\tau^3}} \right) \right]$$

where, this time the variable $\gamma_2 = 4\Delta J^2 q^2 \tau + 4 J^2 q^2 \gamma^2 - \theta^2 \Delta^2$ was chosen. It is indeed quite easy to compare Eq.[12] with the solution given by Ref[28]. We immediately see two effects caused by the new term: an exponential damping factor in front of $F(t)$ and a shift of an argument. The first effect is straightforward to interpret as it resembles a damping of a harmonic oscillator. All correlation functions, which are proportional to momentum integral of the $F(q,t)$, will acquire this extra exponential. Although for spin-chain, where $\theta$ coefficient is tiny, this factor stays close to one, it is still remarkable as it manifest an emergence of a characteristic time-scale present in a time dependent problem. Moreover the time and space are not any longer equivalent. Furthermore, for the correlation functions of $\theta(x)$ field that are proportional to $(\partial_t F(q,t))^2$, this new time dependent factor shall lead to new terms contributing to $I(x,t)$ (see App[13] which are proportional to $(\theta^2 \Delta^2)$. Thanks to the fact that in our case $\theta \Delta \ll 1$ we can neglect them. However these extra terms has to be kept for larger $\Delta$, to ensure SU(2) invariance[28] at the Heisenberg point. [note that the singularity of $\theta_1(\Delta)$ when $\Delta \to 1$ can be interpreted as an effective increase of the $\theta$ coefficient].

The shift of the argument of all functions by $-\theta \Delta)^2$ has more non-trivial implications (we note that a full analytical solution with the given boundary conditions does exist and also have an argument shifted in the same way). Any observable, any correlation function $I(x,t)$, is a functional of a momentum integral of $F(q,t)$ as outlined in App[13] A standard procedure, implemented e.g. in Ref[28], is to divide the whole range of integration into distinct ranges where the Bessel functions are either monotonous (for argument $< 2/3$) or oscillating (for argument $> 2/3$). Shifting argument of Bessel function effectively (up to terms of order $(\theta \Delta)^2$) shifts the ranges of integrals. This is particularly important for the intermediate range of momenta which describes intermediate distances, where the moving front is located. Instead of ranges defined as $[(2/3) \Delta^{-3/2}, 2/3]$ now we have $[(2/3 + \theta \Delta)^{-3/2}, 2/3 + \theta \Delta]$. The peculiar stretched exponential shape of the front remains but, an amplitude of the intermediate regime is alerted. In general one can de-
duce that it is the front that will be the most susceptible to Galilean invariance breaking.

For larger values of $\Delta$, for $\Delta > 0.5$, the linear approximation fails and one has to resort to another approximation for the logarithmic derivative of $u(t)K(t)$.

**B. Large $\Delta$**

Probably the most interesting aspect is what happens with the mode occupation of TLL when, while quenching, we approach the Heisenberg point where a phase transition to the gapped phase takes place. It is reflected in Eq.15 by the fact that the $\Delta_1$ term diverges. One can try to write down an approximate ODE which would capture this effect and at the same time give a linear increase of velocity. One way is to express Eq.5 as a direct generalization of the Euler equation:

$$\frac{d}{dt}(1-gt)^2a_n(t)+(1-gt)k_n\frac{d}{dt}a_n(t)+(1-g^2t^2)^2k_n^2a_n(t) = 0,$$

(13)

Another is to take the approximation $\theta_1(t) = \Delta(t)/(1 - \Delta(t))$. Both cases are analytically solvable and the solution is given in terms of the HeunB functions. These novel functions are very hard to operate with on the computational side, so we refrain from elaborating on the exact expressions that are rather lengthy. Nevertheless, it is interesting to note that this class of functions is actually relevant as a solution of a well known problem in 1D physics.

To get an analytic insight into this critical regime one can try even stronger approximation, with $\theta_2 = cste$, then the ODE is:

$$(\pi/2Jq)^2F(t) - \frac{\theta}{1 - \frac{\Delta}{\Delta}}F'(t) + F''(t) = 0$$

(14)

The main advantage is that its solution can be again expressed in the form of a Bessel function which allows for a direct comparison with Ref.28. The solution reads:

$$\Delta(t) = \Delta(-\frac{t}{\tau})^2 - (t/\tau),$$

and is discussed in the App.\[A\].

**B. Constant $\theta_1$**

In order to ensure that at every time of the quench the Galilean invariance breaking $\theta_1$ term in Eq.5 is time independent (which greatly simplifies the ODE) we need to solve a following equation:

$$\Delta'(t) = \frac{c(\Delta(t) - \Delta(t)^3)}{\Delta(t)^3 + 1}$$

(16)

where $c$ is the constant value of $\theta_1$ term which we arbitrarily set. The Eq.16 has a solution that reads:

$$\Delta(t) = \Delta_0 \frac{1}{2}e^{-ct} \left( \sqrt{4e^{2ct} + a^2} - a \right)$$

(17)

where we have an arbitrary choice of $c_0$ and $a$ to build quenches of various profiles. It is possible to fix their ratio in a way such that $\Delta(t \to 0) \to 0$, but we keep it unconstrained, just demand $\Delta(t \to 0) \ll J$.

We have an ODE equation with constant $\theta_1$ and exponentially increasing velocity. The protocol described by general Eq.17 is not analytically solvable, however it is enough to demand $c \gg a$, complete the square under the square-root and perform an appropriate Taylor
expansion to arrive at a simpler exponential increase of velocity:

\[ (Jq)^2 (1 + \Delta(1/2 + a e^{-ct/\tau})) F(t) - c \cdot F'(t) + F''(t) = 0 \]

\[ (18) \]

\[ F(t) = \left( \tau c Jq \sqrt{a \Delta e^{-\frac{t}{\tau}}} \right)^{-\frac{3}{2}} \left( c_2 \Gamma (\nu_3 + 1) J_{\nu_3} \left( \frac{2 \sqrt{\Delta}}{c} Jq \sqrt{\Delta} \right) + c_1 \Gamma (1 - \nu_3) J_{-\nu_3} \left( \frac{2 \sqrt{\Delta}}{c} Jq \sqrt{\Delta} \right) \right) \]

where a Bessel function of the first kind with an index \( \nu_3 = \sqrt{c_2 - 1/4} q^2 (1/2 + \Delta) \). The solution Eq. (19) with an interaction dependent index, also confirms the approximation we made to obtain the solution in the Eq. (15) where a similar expression was found. It should be noted that, contrary to the previous case, now the index of Bessel function does depend on \( q \), which makes an integration over \( q \) necessary to obtain correlation functions (see App. B), a much more difficult task.

\section*{V. CONCLUSIONS}

In conclusion, we have explored how the Galilean invariance breaking enters the dynamics of XXZ spin chain during the quench of its anisotropy \( \Delta(t) \). We are able to obtain exact analytical solutions in several limiting cases. For the case of small \( \Delta \) we can make a direct comparison of full analytic solutions with and without the term breaking the Galilean invariance. We noticed that the key differences are: an extra exponential factor in correlation functions and a modified amplitude of the front. Upon increasing \( \Delta \), hence approaching the Heisenberg point, another relatively simple analytical solution can be derived. Here we see that the quenching rate changes the shape of the front: it modifies the power of the stretched exponential that describes the front. Furthermore, we notice that by changing the quenching protocol, the time dependence \( \Delta(t) \), we are able to suppress the term breaking the Galilean invariance, also in the vicinity of the Heisenberg point. This changes the bosonic modes occupation at intermediate momenta, which poses an intriguing question whether by changing the quenching protocol one may be able to influence the Renormalization Group flow and hence the physics of the non-equilibrium phase transition.

\section*{Acknowledgments}

I would like to thank Dirk Schuricht for useful discussions in a very early stage of this work.

This equation is solvable:

\[ F(t) = e^{\dot{\nu} \cdot c_1 \cdot H_{-\nu} (\Upsilon_3)} + c_2 \cdot F_1 \left( \frac{\dot{\nu}}{2}, \frac{1}{2}, (\Upsilon_3) \right) \]

\[ (A1) \]

\section*{Appendix A: Approximate solution for intermediate \( \Delta \)}

Instead of linear ramp we now take the protocol for which \( d\Delta(t)/dt = 0 \) at \( t = \tau \). An obvious choice is \( \Delta(t) = \Delta(-t^2/2 + \text{t}) \), because then the coefficient of the Galilean invariance breaking \( \theta_1 \) term in Eq. (5) is approximately constant for \( \Delta \) that is sufficiently large (but not at Heisenberg point). This gives the following ODE: which is fortunately solvable. The solution reads:

\[ F(t) = e^{\dot{\nu} \cdot c_1 \cdot H_{-\nu} (\Upsilon_3)} + c_2 \cdot \frac{1}{2} \left( \frac{\dot{\nu}}{2}, \frac{1}{2}, (\Upsilon_3) \right) \]

\[ (B1) \]

\section*{Appendix B: Shape of the front}

To compute a correlation function e.g. \( \langle \theta(x, t) \theta(0, t) \rangle \) we need to evaluate the following integral:

\[ I(x, t) = \langle \theta(x, t) \theta(0, t) \rangle = \int \frac{dq}{q^2} |F(t)|^2 \]

\[ (B1) \]

where \( \alpha \) is a UV cut-off of the theory. In case of Eq. (12) or Eq. (15) the formula Eq. (B1) boils down to an integral over a combination of Bessel functions. Following standard procedure we divide Bessel function into a monotonous part power law \( J_\nu(z) \sim 1/z^\nu \) (for small argument \( z < 2/3 \)) and an oscillating part (for large argument \( z > 2/3 \)). The most important range of integration, that determines the shape of propagating front is the intermediate range of
momenta defined in Ref.\textsuperscript{28}, time dependent Bessel functions are monotonous while the time-independent amplitudes (set by the boundary conditions) Bessel functions oscillates.

1. Bessel function $J_{\pm 1/3}(q,t)$

In this case the most divergent term of Eq.\textsuperscript{B1} in \( q \rightarrow 0 \) limit takes the following form in this regime:

\[
I_{f_{ra}}(x,t) \approx \int_{b_1}^{b_2} dh_q \left( \frac{\sin(h_q \dot{t}) \sin(qx/2)}{h_q^{3/3}} \right)^2 (B2)
\]

where, from Eq.\textsuperscript{B2} \( \dot{t}_q = \gamma_2/q^2 \) and \( h_q = q/(12J^2\pi^2\Delta) \). The first sine stems from the (oscillating) approximation of Bessel function and the second is from the Fourier transform. The limits of integral are: \( b_1 = 2/3\gamma_q^{−3/2} \) and \( b_2 = 2/3 \). The very existence of this intermediate regime, \( b_1 < b_2 \) is related to \( \dot{t}_q > 1 \). This is possible only for substantial amplitude of a quench, which clearly indicates that we study effects inaccessible by any perturbative approach. The integration in Eq.\textsuperscript{B2} can be performed in a closed analytical form, it leads to:

\[
I_{f_{ra}}(x,t) = z^{1/3}_q \Gamma(2/3, z_0) \big|_{b_1}^{b_2} (B3)
\]

where \( \Gamma(2/3, z) \) is the incomplete Gamma function and \( z_0 = 2q(qt − x) \). We first estimate an upper limit term, \( \Gamma(2/3, 2tb_2(qt − x)) \). We use the identities \( \Gamma(s, z) = (1 − s)\Gamma(s − 1, z) + \exp(z)z^{−s−1} \) and \( \Gamma(s, z) = z^s E_{1−s}(z) \) where \( E_{1−s}(z) \) is the generalized exponential function to obtain

\[
\Gamma(2/3, z) = z^{−1/3} \left( -\frac{1}{3} E_{4/3}(z) + \exp(iz) \right)
\]

and the \( z^{−1/3} \) above cancels out with \( z^{1/3} \) in Eq.\textsuperscript{B2} and one is left with the \( E_{4/3}(z) \) which in turn for large \( (qt−x) \) \( (q = b_2 \) is kept constant) can be approximated well by the Log\([z]\) times an oscillatory function (extracting the oscillating part leads to the Triconi confluent hypergeometric function \( U(4/3, 4/3, z) \)). We have arrived at the desired \( I_{f_{ra}}(x,t) \approx \text{Log}(x) \) behaviour. We see that for large \( (qt−x) \) we make a smooth cross-over to a power law behaviour expected deep inside the light cone (it matches well with the adiabatic range).

For intermediate \( z \), that is small to moderate \( (qt−x) \), there exist a range where \( \Gamma(s, z) \approx 0 \) thus a term originating from upper integration limit is negligible and it is the term originating from lower integration limit \( b_1 > b_2 \) that dominates. When \( b_1 \) goes to zero the incomplete Gamma function remains finite (actually it even increases) and it equals to the complete Gamma function \( \Gamma(2/3) \). The \( z^{1/3} \) does not drop out and this is the source of stretched exponential behavior discovered in Ref.\textsuperscript{28}: \( I_{f_{ra}}(x,t) \approx \Gamma(2/3) \exp[x^{1/3}] \).

2. Bessel function with arbitrary index

Once we have shown how to re-obtain results of Ref.\textsuperscript{28} we can generalize them to Bessel function with an arbitrary, but momentum independent index. What is changing is the exponent in the denominator of Eq.\textsuperscript{B2} which follows the law \( 1 + a = 2(1 − \nu \alpha) \). This implies the following functional form of the result (analogue of Eq.\textsuperscript{B3} for general \( \nu \)):

\[
I_{f_{ra}}(x,t) \approx \int_{b_1}^{b_2} dq \left( \frac{\sin(h_q \tilde{t}) \sin(qx)}{q^{1+a}} \right)^2 = z_0^a \Gamma(1 − a, z_0) \big|_{b_1}^{b_2} (B4)
\]

where now \( h_q = J\pi/2q \) and \( \tilde{t} = t − \frac{x^2}{2} \). One can perform exactly the same manipulations like before to arrive at the stretched exponential shape of the front, but now the exponent is modified \( I_{f_{ra}}(x,t) \approx \exp[x^a] \). For \( \epsilon \ll 1 \) the \( \Gamma(−a, \epsilon) \) is an increasing function of \( \alpha \). As a result the amplitude of the stretched exponential region becomes larger as \( \nu \) decreases. For a special case \( \alpha = 1/2 \) the result can be also rewritten in terms of Fresnel integral \( C(z) \).

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\textsuperscript{*} Electronic address: P.M.Chudzinski@uu.nl

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