Linear to quadratic crossover of Cooper-pair dispersion relation

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Abstract

Cooper pairing is studied in three dimensions to determine its binding energy for all coupling using a general separable interfermion interaction. Also considered are Cooper pairs (CPs) with nonzero center-of-mass momentum (CMM). A coupling-independent linear term in the CMM dominates the pair excitation energy in weak coupling and/or high fermion density, while the more familiar quadratic term prevails only in the extreme low-density (i.e., vacuum) limit for any nonzero coupling. The linear-to-quadratic crossover of the CP dispersion relation is analyzed numerically, and is expected to play a central role in a model of superconductivity (and superfluidity) simultaneously accommodating a BCS condensate as well as a Bose-Einstein condensate of CP bosons.

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I. INTRODUCTION

The large-momentum divergence in the Cooper pair (CP) problem [1] was originally eliminated by a momentum-space cutoff introduced in what is now known as the Bardeen-Cooper-Schrieffer (BCS) model interaction which was also successfully used by BCS in the study of conventional low-temperature superconductors [2]. On the other hand, the short coherence length of some high-$T_c$ superconductors [3] possibly implies a stronger interfermion interaction. In the limit of very strong coupling one gets well-isolated “diatomic molecules” or dimers of fermions, as opposed to strongly overlapping CPs in the weak-coupling limit of BCS superconductivity [4–6]. In strong coupling these dimers can conceivably undergo Bose-Einstein (BE) condensation (BEC). Although there is considerable controversy over the precise pairing dynamics in so-called “exotic” [7] superconductors, tracking this many-body problem from strong to weak coupling—known as the BCS-Bose crossover—has promoted the understanding of various properties of exotic materials [4–6,8–17].

Here we focus on how Cooper pairing itself evolves from weak to strong coupling. We also study the excitation of CPs with nonzero center-of-mass momentum (CMM), which should play an important role in a superconducting or superfluid transition of the many-fermion system simultaneously exhibiting both BCS and BE condensates as in the formulation by Friedberg and Lee [18] and more generally by Tolmachev [19].

The BCS model interaction simulates a phonon-mediated force which is effective in explaining the isotopic effect by appropriate momentum-space cutoffs placed symmetrically on either side of the Fermi surface [20]. However, exotic materials do not exhibit such systematic isotopic effects. Also, as coupling increases the chemical potential $\mu$ decreases in value from the (positive) Fermi energy $E_F$ (its value at zero temperature and zero interaction), and even turns negative as the so-called Bose regime is entered [4–6]. The Fermi surface then washes out, eliminating any physical motivation for the BCS model interaction with Fermi-surface dependent cutoffs. Alternatively, in the BCS-Bose crossover a renormalization procedure [21,22] may be used to handle the large-momentum divergences. This leads to a renormalized dynamical model expressible in terms of physical observables of the system rather than ad hoc cutoffs.

Here we derive renormalized $t$-matrix and Cooper equations for a pair of fermions which move in vacuum and in the Fermi sea, respectively. In three dimensions (3D) Cooper binding is expressed in terms of the two-fermion scattering length in vacuum. For a CP with a nonzero CMM we define a pair excitation energy as the (positive) difference between the CP binding energy at zero and at a finite CMM. For high fermion density and any coupling only a linear term in CMM dominates [23,24] the CP excitation, which was in fact mentioned as far back as
1964 (Ref. [20], p. 33). At any coupling and for vanishing fermion-number density a quadratic term dominates which is just the kinetic energy of the composite pair. The crossover from a linear to a quadratic dispersion for the pair excitation energy is then illustrated via numerical calculations.

The CP dispersion relation enters the BE distribution function of the boson number equation from which $T_c$ is extracted [25]. The linear CP dispersion relation for weak coupling leads to novel phase-transition properties in either a heuristic [26] or a first-principles [27] BEC picture of superconductivity as BE-condensing CPs. It is common knowledge that BEC is possible only for dimensions $d > 2$ for bosons with quadratic dispersion; this limitation reappears in virtually all BEC schemes thus far applied to explain superconductivity [12,17–19]. But for bosons with a linear dispersion relation found here in weak and medium coupling, BEC can now occur for all $d > 1$. This should be relevant in models of superconductivity encompassing both BCS and BE condensates [18,19].

In Sec. II the two-body problem is formulated in vacuum for a short-range, separable interaction. In Sec. III the renormalized CP equation is derived for nonzero CMM. In Sec. IV the CP dispersion relation in CMM is obtained numerically. Finally, Sec. V offers discussion and Sec. VI conclusions.

II. TWO-BODY PROBLEM IN VACUUM

Consider $N$ fermions in a box with sides of length $L$ that interact via an S-wave short-range, attractive (rank-one) separable potential in 3D of the form [17]

$$V_{pq} = -(v_0/L^3)g_pg_q,$$

where $v_0 \geq 0$ is the interaction strength and $g_p$ are dimensionless form factors $g_p = (1 + p^2/p_0^2)^{-1/2}$ [17], where the parameter $p_0$ is the inverse range of the potential. Such an interaction model may mimic a wide variety of short-range effective interactions: a force mediated by phonons, plasmons, excitons, magnons, etc. or even a purely electronic interaction. Here $p_0 \rightarrow \infty$ implies $g_p = 1$ and corresponds to a zero-range potential. The advantage of potential (1) is that many problems then yield analytic solutions. The BCS model interaction is a special case of (1) when $g_p$ is constant in the interval $E_F - \hbar\omega_D < \hbar^2 p^2/2m < E_F + \hbar\omega_D$ and zero otherwise, where $\omega_D$ is the Debye frequency. More realistic potentials can be approximated by a rank-N separable potential [28].

The Lippmann-Schwinger equation for the $t$-matrix with potential (1) between two fermions each of mass $m$ in free space is
$$t_{pq}(E) = V_{pq} + \sum_k V_{pk} \frac{1}{E - \hbar^2 k^2 / m + i0} t_{kq}(E),$$  \hspace{1cm} (2)$$

where $E$ is the two-particle energy. For potential (1) the solution of Eq. (2) is [28]

$$t_{pq}(E) = \frac{g_p g_q}{-\frac{L^3}{v_0} - \sum_k \frac{g_k^2}{E - \hbar^2 k^2 / m + i0}}.$$  \hspace{1cm} (3)$$

In the limit $L \to \infty$, the momentum sum may be replaced by an integral

$$\sum_k \to \nu \frac{L^3}{(2\pi)^3} \int d^3 k,$$  \hspace{1cm} (4)$$

where $\nu$ is the spin degeneracy. Using (4), the zero-energy, on-shell $t$-matrix is given in terms of the (S-wave) scattering length $a$ by [28]

$$t_{00}(0) = \frac{4\pi \hbar^2 a}{m\nu L^3},$$  \hspace{1cm} (5)$$

Eq. (3) for $E = 0$ then becomes

$$\frac{\nu \frac{mL^3}{4\pi \hbar^2 a}}{-\frac{L^3}{v_0} + \sum_k \frac{g_k^2}{\hbar^2 k^2 / m}},$$  \hspace{1cm} (6)$$

since $g_0 \equiv 1$, where the $i0$ term in the denominator is unnecessary as the sum no longer diverges in the small $k$ limit.

### III. TWO-BODY (COOPER) PROBLEM IN FERMI SEA

The CP equation for two fermions above the Fermi sea with momentum wavevectors $k_1$ and $k_2$ is given by

$$\left[ \frac{\hbar^2 k^2}{m} - E_K + \frac{\hbar^2 K^2}{4m} \right] C_k = -\sum_q' V_{kq} C_q,$$  \hspace{1cm} (7)$$

where $k \equiv \frac{1}{2}(k_1 - k_2)$ is the relative, and $K \equiv k_1 + k_2$ the center-of-mass, momentum wavevectors, $E_K \equiv 2E_F - \Delta_K$ the total pair energy, $\Delta_K \geq 0$ the CP binding energy, $C_q$ its momentum-space wave function, and the prime on the sum implies restriction to states above the Fermi surface: viz., $|k \pm K/2| > k_F$, where $k_F$ is the Fermi wave number. For potential (1), Eq. (7) can be solved and $\Delta_K$ determined from
\[
\sum_k' \frac{g_k^2}{\hbar^2 k^2 / m + \Delta_K - 2E_F + \hbar^2 K^2 / 4m} = \frac{L^3}{v_0}.
\] (8)

Although the summand in Eq. \((8)\) is angle-independent, the restriction on the sum arising from the full Fermi sea is a function of the relative wave vector \(\mathbf{k}\), and therefore angle-dependent. The interaction strength \(v_0\) in Eq. \((8)\) may then be eliminated by combining with Eq. \((6)\).

Thus, in terms of the scattering length \(a\), the renormalized CP equation is

\[
\sum_k' \frac{g_k^2}{\hbar^2 k^2 / m + \Delta_K - 2E_F + \hbar^2 K^2 / 4m} - \sum_k \frac{g_k^2}{\hbar^2 k^2 / m} = -\frac{m\nu L^3}{4\pi\hbar^2 a}.
\] (9)

For an attractive interaction with \(g_k = 1\) the sums in Eq. \((9)\) have large-momentum divergences. However, each of the sums of Eq. \((9)\) diverges in the same fashion so that the difference is finite. A similar renormalization of a general scattering equation can be performed [21,22]. Also, renormalized BCS gap and number equations have been used in the BCS-Bose crossover problem [4–6,8–17]. We shall determine the binding energy \(\Delta_K\) with Eq. \((9)\) employing \(a\) as interaction coupling parameter instead of the potential parameter \(v_0\). In what follows variables are dimensionless and expressed in terms of \(k_F\) or \(E_F \equiv \hbar^2 k_F^2 / 2m\), viz., \(\xi \equiv k / k_F, \tilde{K} \equiv K / k_F, \tilde{\Delta}_K \equiv \Delta_K / E_F, \tilde{a} \equiv ak_F\), etc.

The sums in Eq. \((9)\) are transformed into integrals using Eq. \((4)\). The restriction in the first term of Eq. \((9)\) arising from the full Fermi sea leads to two expressions depending on whether \(\tilde{K} \equiv K / k_F\) is \(< 2\) or \(> 2\), as discussed in the Appendix,

\[
\int_0^{\pi/2} d\theta \sin \theta \left[ \int_0^\Lambda d\xi g_\xi^2 - \int_0^\Lambda d\xi \xi^2 g_\xi^2 \frac{\xi^2}{\xi^2 - \alpha_K^2} \right] = \frac{\pi}{2\tilde{a}}, \quad \tilde{K} < 2,
\] (10)

\[
\int_0^{\pi/2} d\theta \sin \theta \left[ \int_0^\Lambda d\xi g_\xi^2 - \int_0^\Lambda d\xi \xi^2 g_\xi^2 \frac{\xi^2}{\xi^2 + \beta_K^2} \right] + \int_0^{\theta_0} \sin \theta d\theta \int_{\xi_0(\theta)}^{\xi_0(\theta)} d\xi \xi^2 g_\xi^2 \frac{\xi^2}{\xi^2 + \beta_K^2} = \frac{\pi}{2\tilde{a}}, \quad \tilde{K} > 2,
\] (11)

where \(\alpha_K^2 \equiv 1 - \Delta_K / 2 - \tilde{K}^2 / 4, \beta_K^2 = -\alpha_K^2, \xi_0(\theta) \equiv \sqrt{1 - \tilde{K}^2 \sin^2 \theta / 4 + \tilde{K} \cos \theta / 2}, \xi_0'(\theta) \equiv -\sqrt{1 - \tilde{K}^2 \sin^2 \theta / 4 + \tilde{K} \cos \theta / 2}, \theta_0 = \arcsin(2 / \tilde{K}) < \pi / 2\), with \(\theta\) the angle between \(\mathbf{k}\) and \(\mathbf{K}\). To deal with the large-momentum divergences we have introduced a finite upper limit \(\Lambda\) and eventually let \(\Lambda \to \infty\). The momentum-space integrals in Eqs. \((10)\) and \((11)\) are easily performed for a zero-range interaction, i.e., with \(g_k = 1\). For CPs with \(\tilde{K} = 0\), only the first equation is of concern. Since in this case \(\xi_0(\theta) = 1\) and \(\alpha_K^2 \equiv \alpha_0^2 \equiv -\beta_0^2 = 1 - \Delta_0 / 2\), one
arrives at
\[ 1 + \frac{\alpha_0}{2} \ln \left| \frac{1 - \alpha_0}{1 + \alpha_0} \right| = \frac{\pi}{2\tilde{a}}, \quad \alpha_0^2 > 0, \quad (12) \]
\[ \frac{\pi}{2} \beta_0 + 1 - \beta_0 \arctan \frac{1}{\beta_0} = \frac{\pi}{2\tilde{a}}, \quad \alpha_0^2 < 0. \quad (13) \]
These two equations relate the dimensionless scattering length \(-\infty < \tilde{a} \equiv k_F a < +\infty\) to the dimensionless CP binding energy for zero CMM \(\tilde{\Delta}_0 \equiv \Delta_0 / E_F\) for all coupling. Transcendental equation (12) can be solved for CP binding for weak coupling. In this limit the argument of the logarithm in Eq. (12) reduces to \(\tilde{\Delta}_0 / 8\), and one obtains
\[ \tilde{\Delta}_0 \to (8/e^2) \exp(-\pi/|\tilde{a}|) \quad \text{(weak coupling)}, \quad (14) \]
a limit first reported by Van Hove [29]. One can also find \(\tilde{\Delta}_0\) for strong coupling or for \(\tilde{\Delta}_0 \to \infty\). Since in this limit \(\beta_0 \arctan(1/\beta_0) \approx 1\), we obtain from Eq. (13)
\[ \tilde{\Delta}_0 \to 2/\tilde{a}^2 \quad \text{(strong coupling)}. \quad (15) \]
Equations (12) and (13) were solved numerically to obtain the exact functional dependence of \(\tilde{\Delta}_0\) on \(1/\tilde{a}\), and this is compared with asymptotic forms Eqs. (14) and (15). In Fig. 1 we plot \(\tilde{\Delta}_K\) vs \(1/\tilde{a}\) spanning weak to strong coupling. One sees how the asymptotic form given by Eq. (14) (short-dashed line) coincides with the exact \(\tilde{K} = 0\) result in weak coupling, whereas Eq. (15) (long-dashed curve) is also quite accurate over a sizeable region for strong coupling.

**IV. COOPER PAIR DISPERSION CURVES**

Equations (10) and (11) are valid for all \(K \geq 0\) and all coupling. They can be solved numerically for CP binding \(\Delta_K\) for any \(K\). Before discussing numerical results we derive analytically the small-CMM behavior for zero range using \(g_k = 1\) for weak coupling in Eq. (10) which we take both for a small but non-zero \(\tilde{K}\) and for \(\tilde{K} = 0\), and then subtract one equation from the other. A small-CMM expansion of the resultant equation leads to the weak coupling expression
\[ \lim_{\tilde{\Delta}_0 \ll 1} \varepsilon_K = \frac{1}{2} hv_F K + O(K^2) + \ldots, \quad (16) \]
where a positive CP excitation energy \(\varepsilon_K \equiv (\tilde{\Delta}_0 - \Delta_K)\) has been defined, and the Fermi velocity \(v_F\) is given by \(E_F/k_F = hv_F/2\). The coefficient of the linear term depends only on properties of
the Fermi sea and not on any parameters of the potential. In contrast, the complete excitation energy does depend on the coupling parameter \( \tilde{a} \equiv k_F a \).

It is this excitation energy that must enter the BE distribution function in determining the critical temperature in a picture of superconductivity as a BEC of CPs \([17–19, 21]\). The leading term in (10) is linear in the CMM, followed by a quadratic term. But it is only for sufficiently small fermion density, i.e., when \( k_F \) or \( E_F \to 0 \), and for any nonzero coupling, that the quadratic term dominates, viz.,

\[
\varepsilon_K \to \frac{\hbar^2 K^2}{2(2m)},
\]

the familiar nonrelativistic kinetic energy of the composite pair of mass \( 2m \) and CMM \( \hbar K \). As mentioned, it is this dispersion relation that has been assumed in virtually all BEC studies of superconductivity \([12, 17–19]\). However, recent calculations of root-mean-square radii in two (2D) and three (3D) dimensions in the BCS-Bose crossover scheme, when compared with experimental coherence lengths of several typical 2D cuprates \([15]\) as well as of 3D materials \([16]\), suggest that they are describable well within the BCS (weak-coupling) regime and away from the Bose (strong-coupling) one. This implies that the linear approximation to the dispersion relation would be relevant in these cases, and that perhaps a more general description of the BEC of CPs for all coupling might require the exact dispersion relation.

In Fig. 2 we display the reduced CP excitation energy \( \varepsilon_K/\Delta_0 \) as a function of reduced CMM \( K/k_F \) for zero- and finite-range potentials. Note that the CPs break up when \( \varepsilon_K/\Delta_0 \equiv (\Delta_0 - \Delta_K)/\Delta_0 = 1 \), i.e., when \( \Delta_K \) vanishes and turns negative. These points are marked by dots in Fig. 2. For zero range we solve Eqs. (10) and (11) for \( g_\xi = 1 \) and for typical values of \( \Delta_0/E_F \) spanning weak to strong coupling. For finite range we display results using \( g_p = (1 + p^2/p_0^2)^{-1/2} \) with \( p_0 = k_F \) (i.e., range \( 1/p_0 \) of the order of the average interferron spacing \( \sim 1/k_F \)). Also shown in Fig. 2 is the quadratic approximation in \( K \) as given by Eq. (17). We have labeled the curves by \( \Delta_0/E_F \) as we found that the zero-range curves are closer to the corresponding finite-range ones than if they are labeled by \( 1/k_F a \) as in Fig. 1. The linear approximation Eq. (16) is valid only in the very weak-coupling and/or high fermion density limit. The quadratic term dominates only at vanishing density, for any nonzero coupling. For finite-range interaction the crossover in Fig. 2 is characterized by an inflection point with positive slope while for zero-range there is no such inflection point.
V. DISCUSSION

The single-CP problem treated here may appear academic at first. However, it has serious consequences. Our CPs are taken as “bosonic” even though they do not obey (Ref. [20] p. 38) Bose commutation relations. This is because for a given $K$ they have indefinite occupation number since for fixed $K$ there are (in the thermodynamic limit) an indefinitely large number of allowed (relative wavenumber) $k$ values. Hence, for any coupling—and thus any degree of overlap between them—CPs do in fact obey the Bose-Einstein distribution from which BEC is determined. There have been attempts [18,19] to formulate the superfluid and superconducting transition problem in a many-fermion system by accommodating both BE and BCS condensed phases. In these studies, the BCS (BE) condensed phase dominates for weak (strong) coupling. For intermediate coupling one could have both types of condensation with a certain density of CP bosons (fermions) available for BE (BCS) condensation. However, the full boson dispersion relation should be used to calculate the BEC transition temperature.

The linear dispersion relation of a CP should not be confused with the linear dispersion of Anderson-Bogoliubov-Higgs (ABH) many-body excitation phonon-like modes. Collective modes in a superconductor were studied since the late 1950’s by several workers. A more recent treatment for 1D, 2D and 3D is available [30] which confirms the linear ABH form $\hbar v_F K / \sqrt{d}$ for $d = 1, 2$ or $3$ in the zero-coupling limit. ABH phonons (like photons or plasmons, etc.) cannot suffer a BEC as their number is always indefinite. The number of CPs, on the other hand, is fixed, say, at half the number of (pairable) fermions if all of these are imagined paired.

The above crossover of linear to quadratic forms of the CP dispersion relation was also found in 2D [24,31] so that we include both 2D and 3D cases in the following discussion of the BEC transition temperature $T_c$. The 2D case is specially interesting as $T_c$ is zero for the usual quadratic dispersion relation of CP bosons but nonzero for linear dispersion.

The general BEC $T_c$ formula for bosons in any dimension $d$ and with a general boson dispersion relation $\varepsilon_K = C_s K^s$, for $s > 0$, is given [26] for $d > s$ by

$$T_c = \frac{C_s}{k_B} \left[ \frac{s \Gamma(d/2) (2\pi)^d n_B}{2 \pi^{d/2} \Gamma(d/s) \zeta(d/s)} \right]^{s/d},$$

(18)

but vanishes for $d \leq s$. Here $n_B$ is the number density of bosons of mass $m_B$, $k_B$ the Boltzmann constant, and $\zeta(d/s)$ the Riemann Zeta function. For quadratic dispersion $s = 2$, $C_s = \hbar^2 / 2m_B$ and in 3D $\zeta(3/2) \simeq 2.612$ [18] leads to the familiar $T_c$ formula $T_c \simeq 3.31 \hbar^2 n_B^{2/3} / m_B k_B$ [23] and to the fact that $T_c = 0$ for all $d \leq 2$. For the linear dispersion case $s = 1$, and consequently
\( T_c = 0 \) for all \( d \leq 1 \) and \( T_c > 0 \) for all \( d > 1 \). The latter is precisely the range of effective dimensionalities for all known superconductors if one includes the quasi-1D organo-metallic Bechgaard salts \([32,33]\).

Before discussing the consequences of the \( T_c \) formula \((18)\) with \( s = 1 \) in superconductivity, we stress its limitations. Firstly, this \( T_c \) formula with \( s = 1 \) is derived with the linear dispersion relation predominant for weak to moderate coupling, while the full correct dispersion relation should be used in general. Secondly, in deriving this formula we have taken the full momentum space of CP bosons so that the momentum integrals run from 0 to \( \infty \), whereas we have seen that the CPs break up above some specific momentum value so the integrals should run only from 0 to the breakup \( K_0 \). Thirdly, the effect of unpaired fermions in the background is ignored. Nevertheless, preliminary study shows \([26]\) that once we remove these three limitations the result \((18)\) for \( s = 1 \) does not change drastically.

If one assumes that all fermions are paired into CP bosons so that the boson density \( n_B \) is \( n/2 \) with \( n \equiv N/L^3 \) the fermion density in the normal state, Eq. \((18)\) with \( s = 1 \) leads to huge values of \( T_c \sim 10^3 \) K for weak to intermediate coupling—the region of interest in superconductivity even though \( T_c \) empirically is at most about \( \sim 100 \) K. However, the number of paired fermions vulnerable to BEC is strongly coupling-dependent and generally \([27]\) is only a small fraction of all the fermions so that \( n_B \) in Eq. \((18)\) is effectively much smaller than \( n/2 \). Thus, a realistic \( T_c \) is certainly feasible. In the extreme weak coupling limit \( n_B \to 0 \), driving the BEC \( T_c \) to zero and allowing the BCS theory to be recovered from analyses as in Refs. \([18,19]\). For higher coupling \( n_B \) increases so that one accommodates both BE and BCS condensates. One can surmise that in a realistic theory of superconductivity BE and BCS condensates play their respective roles. Elaborate calculations must still be performed with the exact CP dispersion in order to find a more accurate \( T_c \) for this many-body system.

**VI. CONCLUSIONS**

The single CP problem with non-zero CMM is studied as it evolves (or crosses over) by varying the interfermion short-range pair interaction from weak to strong. The CP excitation energy is exhibited as a function of its CMM. For weak coupling the correct excitation energy is a linear dispersion relation in the CMM, which changes gradually to a quadratic relation as coupling increases and/or density is reduced to the vacuum limit. These results will play a critical role in a model of superconductivity that includes both BCS and BE condensates \([18,19]\). With a quadratic dispersion the BEC \( T_c = 0 \) in 2D, from which one might infer that BEC is irrelevant for quasi-2D cuprate superconductivity. However, even in 2D, nonzero BEC
transition temperatures emerge for weak and medium coupling where the linear dispersion relation is found to dominate \[31,24\], thus vindicating the relevance of BEC for such materials. The pioneering attempts \[18,19\] developing a model of superconductivity accommodating BE and BCS condensates both assumed the quadratic dispersion relation for unbreakable CPs for all coupling. It would be interesting to reformulate those studies using the proper dispersion relation with a \textit{finite} breakup momentum. Lastly, although our study is based on a separable potential we expect our conclusions on the CP dispersion to be valid more generally, and applicable to superconductivity, and to neutral-atom superfluidity such as in liquid \(^3\)He and in trapped Fermi gases \[34\].

\section*{VII. APPENDIX}

The restriction that both fermions lie above the Fermi sea in Eq. (9) can be written as

\begin{equation}
(k/k_F \pm K/2k_F)^2 - 1 = \xi^2 \pm \xi \bar{K} \cos \theta + \bar{K}^2/4 - 1 \geq 0,
\end{equation}

where \(\xi \equiv k/k_F\) and \(\bar{K} \equiv K/k_F\). The equality leads to two pairs of roots in \(\xi\), say \(\xi_{1,2} = -a \pm b\) and \(\xi_{3,4} = a \pm b\), where \(a \equiv (\bar{K}/2) \cos \theta\), \(b \equiv \sqrt{1 - (\bar{K}^2/4) \sin^2 \theta}\), and \(\theta\) is the angle between \(\mathbf{k}\) and \(\mathbf{K}\).

For \(\bar{K} < 2\), \(b > a\), one root of the two pairs is positive and the other negative. Thus, Eq. (A.1) can be satisfied provided that \(\xi > \xi_1, \xi_2, \xi_3, \xi_4\), or specifically, if \(\xi > \xi_0(\theta) \equiv a + b\). For \(\bar{K} > 2\) and \(\theta < \theta_0 \equiv \arcsin(2/\bar{K})\), \(b\) becomes imaginary and Eq. (A.1) is satisfied for all \(\xi\). Therefore, there is no restriction in the integration over \(\xi\). However, for \(\bar{K} > 2\) and \(\theta < \theta_0\), \(b < a\) the pair of roots \(\xi_{1,2}\) are both negative while the pair \(\xi_{3,4}\) are both positive (with \(\xi_3 > \xi_4\)). Consequently, in both cases Eq. (A.1) is satisfied only if \(\xi\) is in the interval \([0, \xi_0(\theta)] \equiv [0, a - b]\), and in the interval \([\xi_0(\theta), \infty]\), respectively. Equation (9) is evaluated using these restrictions on the \(\xi\) integration.

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IX. FIGURE CAPTIONS

1. Dimensionless CP binding energy $\Delta_K/E_F$ calculated for zero-range interaction using Eqs. (10) and (11) for different CMM momentum $K$ ($K/k_F = 0, 1, \text{and} 3$) versus the inverse dimensionless scattering length $1/k_F a$. Short-dashed straight line (14) holds for weak coupling while long-dashed quadratic one (15) is valid for strong coupling.

2. Reduced CP excitation energy $\varepsilon_K/\Delta_0$ versus $K/k_F$ for different reduced couplings measured in terms of $\Delta_0/E_F$ calculated from Eqs. (10) and (11) for zero-range (full curves) and finite-range potential with $p_0 = k_F$ (long-dashed). Short-dashed curves are the quadratic term of Eq. (17). Dots denote CMM wavenumbers $K_0$ where the CPs break up.
Fig. 1

$\Delta_K / E_F$ vs. $1/k_F a$

$K / k_F = 3$
Fig. 2

![Graph](graph.png)

The graph shows the relationship between $\varepsilon_K / \Delta_0$ and $K / k_F$ for various values of $\Delta_0 / E_F$ and different $p_0$.

Key:
- $\Delta_0 / E_F = 20$
- $p_0 = k_F$
- $p_0 = \infty$

The graph includes a quadratic fit for comparison.