A study on the fractional Gruschin type process

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Abstract. In this article, we first establish derivative formulae for fractional Gruschin type process, which generalize the result of Wang (J Theor Probab 27:80–95, Theorem 1.1, 2012). Since we work on a non-Markovian context, some technical difficulties appear in the study. Then, using the fractional calculus technique, we also derive the gradient estimate.

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1 Introduction

For the usual $(m + d)$-dimensional Brownian motion $(B, \tilde{B})$ and $\sigma \in C^1(\mathbb{R}^m; \mathbb{R}^d \times \mathbb{R}^d)$ that might be degenerate, consider the following stochastic differential equation (SDE) on $\mathbb{R}^{m+d}$:

$$\begin{cases}
    dX_t = dB_t, \\
    dY_t = \sigma(X_t) d\tilde{B}_t.
\end{cases}$$

(1.1)

It is easy to see that the associated generator is the following Gruschin type operator on $\mathbb{R}^{m+d}$:

$$L(x, y) = \frac{1}{2} \left[ \sum_{i=1}^m \partial^2_{x_i} + \sum_{j,k=1}^d (\sigma(x)\sigma(x)^*)_{jk} \partial_{y_j} \partial_{y_k} \right] \text{ with } (x, y) \in \mathbb{R}^{m+d}.$$

If $m = d = 1$ and $\sigma(x) = x$, it reduces to the Gruschin operator. In [34] Bismut type derivative formula and gradient estimates were established for the semigroup generated by the operator $L$ i.e., the Gruschin type semigroup. The argument consists of using Malliavin calculus techniques together with solving a control problem. Remark that, as pointed out by the author in [34], it seems hard to adopt the same arguments developed for Heisenberg group and subelliptic operators satisfying the generalized curvature.

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In probability theory, the gradient estimate problem is an interesting research topic and has been extensively studied. Among many probabilistic methods, the derivative formula, which is called the Bismut formula or the Bismut-Elworthy-Li formula due to [8, 15], is known to be quite effective. As far as we are aware of, in the Markovian setting there exist many results on the derivative formulae. Here we would like to mention a few of them. Based upon martingale method, coupling argument, or Malliavin calculus, the derivative formulae are derived in diffusion processes [35], degenerate SDEs [3, 17, 31, 38, 42], and stochastic partial differential equations (SPDEs) [4, 14, 36]. Besides, in [37] log-Harnack inequality of the Gruschin type semigroup is established by using coupling by change of measure. Afterwards, Deng and Zhang [12] extended the results to the Markov semigroups generated by non-local Gruschin type operators via coupling in two steps together with the regularization approximations of the underlying subordinators.

Recently, the theory of SDE driven by fractional Brownian motion was developed. As a result, an analogous gradient estimate problem is gather attention. The SDE above is usually understood in the sense of the pathwise Riemann-Stieltjes integration (or Young integration) originally due to [40] and developed in [41], and the rough path introduced in [22]. See e.g. [3, 10, 13, 16, 25, 29] for the study of the existence and uniqueness of solutions to such stochastic equations. For other interesting results involved with the distributional and paths regularities, one may refer to [2, 5, 21, 27, 33, 39] and references within. In the previous papers [18] and [19, 20], we studied the derivative formulae and the Harnack type inequalities for SDEs with additive fractional noises for $H < 1/2$ and $H > 1/2$, respectively.

The main purpose of this article is to provide the derivative formulae and the gradient estimate for multidimensional SDE driven by fractional Brownian motion, which is a “fractional version” of the Gruschin type process and allows the coefficient to be degenerate. This can be regarded as an attempt toward the solving of the derivative formulae for more general SDEs with multiplicative fractional noise. Although the basic strategy of the proof relies on Malliavin calculus which is similar to the case in [34], the proof gets much more technically difficult since we work on a non-Markovian context. We will carry it out using fractional integral and derivative operators combined with the transfer principle.

The structure of this paper is organized in the following way. Section 2 includes some basic results about fractional calculus and fractional Brownian motion. In Section 3, we state the derivative formulae and then present the gradient estimate for multidimensional SDE with fractional noise which is a “fractional version” of the Gruschin type process via Malliavin calculus. Section 4 contains the study of the derivative formula of a more general model.

Throughout the paper, we will denote by $C$ a positive constant that may vary from one formula to another.

2 Preliminaries

2.1 Fractional calculus

For later use, we introduce some basic definitions and results about fractional calculus. An exhaustive survey can be found in [32].

Let $a, b \in \mathbb{R}$ with $a < b$. For $f \in L^1(a, b)$ and $\alpha > 0$, the left-sided (resp. right-sided)
The corresponding Weyl representation reads as follows
\[
I_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} dy, \quad \text{(resp. } I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(1-\alpha)} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} dy )
\]
where \((-1)^{-\alpha} = e^{-i\alpha \pi}, \Gamma\) stands for the Euler function. The above integral extends the usual n-order iterated integrals of \(f\) for \(n = n \in \mathbb{N}\).

The fractional derivative may be introduced as an inverse operation. Let \(\alpha \in (0,1)\) and \(p \geq 1\). If \(f \in I_{a+}^\alpha(L^p([a,b], \mathbb{R}))\) (resp. \(I_{b-}^\alpha(L^p([a,b], \mathbb{R})))\), the function \(\phi\) satisfying \(f = I_{a+}^\alpha \phi\) (resp. \(f = I_{b-}^\alpha \phi\)) is unique in \(L^p([a,b], \mathbb{R})\) and it agrees with the left-sided (resp. right-sided) Riemann-Liouville derivative of \(f\) of order \(\alpha\) given by
\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} dy \quad \text{(resp. } D_{b-}^\alpha f(x) = \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} dy )
\]
The corresponding Weyl representation reads as follows
\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right)
\]
\[
\text{(resp. } D_{b-}^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right) ),
\]
where the convergence of the integrals at the singularity \(y = x\) holds pointwise for almost all \(x\) if \(p = 1\) and in \(L^p\) sense if \(1 < p < \infty\).

Suppose that \(f \in C^\lambda(a,b)\) (the set of \(\lambda\)-Hölder continuous functions on \([a,b]\)) and \(g \in C^\mu(a,b)\) with \(\lambda + \mu > 1\). The Riemann-Stieltjes integral \(\int_a^b f dg\) exists by the results of Young [40]. In [41], Zähle presents an explicit expression for the integral \(\int_a^b f dg\) in terms of fractional derivatives. Let \(\lambda > \alpha\) and \(\mu > 1 - \alpha\). Then the Riemann-Stieltjes integral has the following representation:
\[
\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt,
\]
where \(g_{b-}(t) = g(t) - g(b)\). This can be regarded as fractional integration by parts formula.

### 2.2 Fractional Brownian motion

In this part, we shall recall some important definitions and results concerning the fractional Brownian motion. For a deeper discussion, we refer the reader to [11] [17] [24] [28].

The \(d\)-dimensional fractional Brownian motion with Hurst parameter \(H \in (0,1)\) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) can be defined as the centered Gaussian process \(B^H = \{B^H_t, t \in [0,T]\}\) with covariance function \(\mathbb{E}(B^H_t, B^H_s) = R_H(t,s) \delta_{i,j}\), where
\[
R_H(t,s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
\]
If \(H = 1/2\), the process \(B^H\) is a standard \(d\)-dimensional Brownian motion. By the above covariance function, one can show that \(\mathbb{E}|B^H_t, B^H_s|^p = C(p)|t-s|^{pH}, \forall p \geq 1\). Then, by
the Kolmogorov continuity criterion $B^{H,i}$ have $(H - \epsilon)$-order Hölder continuous paths for all $\epsilon > 0$, $i = 1, \cdots, d$.

For each $t \in [0, T]$, we denote by $\mathcal{F}_t$ the $\sigma$-field generated by the random variables $\{B^{H,i}_s : s \in [0, t]\}$ and the sets of probability zero.

We denote by $\mathcal{E}$ the set of step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle (I_{[0,t_1]} \cdots, I_{[0,t_d]}), (I_{[0,s_1]} \cdots, I_{[0,s_d]}) \rangle_{\mathcal{H}} = \sum_{i=1}^d R_H(t_i, s_i).$$

By bounded linear transform theorem, the mapping $(I_{[0,t_1]} \cdots, I_{[0,t_d]}) \mapsto \sum_{i=1}^d B^{H,i}_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$ associated with $B^H$. We denote this isometry by $\phi \mapsto B^H(\phi)$.

We recall that by (11) the covariance kernel $R_H(t, s)$ can be written as

$$R_H(t, s) = \int_0^{\wedge t, s} K_H(t, r)K_H(s, r)dr,$$

where $K_H$ is a square integrable kernel given by

$$K_H(t, s) = \Gamma \left( H + \frac{1}{2} \right)^{-1} (t - s)^{H-\frac{1}{2}} F \left( H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s} \right),$$

in which $F(\cdot, \cdot, \cdot, \cdot)$ is the Gauss hypergeometric function (for details see (11) or (26)).

Now, consider the linear operator $K^*_H : \mathcal{E} \to L^2([0, T], \mathbb{R}^d)$ defined by

$$(K^*_H \phi)(s) = K_H(T, s)\phi(s) + \int_s^T (\phi(r) - \phi(s)) \frac{\partial K_H}{\partial r}(r, s)dr.$$

By integration by parts, it is easy to see that when $H > 1/2$, the above relation can be rewritten as

$$(K^*_H \phi)(s) = \int_s^T \phi(r) \frac{\partial K_H}{\partial r}(r, s)dr. \tag{2.1}$$

Due to (11), for any $\phi, \psi \in \mathcal{E}$, there holds $\langle K^*_H \phi, K^*_H \psi \rangle_{L^2([0,T],\mathbb{R}^d)} = \langle \phi, \psi \rangle_{\mathcal{H}}$ and then $K^*_H$ can be extended to an isometry between $\mathcal{H}$ and $L^2([0, T], \mathbb{R}^d)$. So, owing to (11) again, the process $\{W_t = B^H((K^*_H)^{-1}1_{[0,t]}), t \in [0, T]\}$ is a Wiener process, and $B^H$ has Volterra’s representation of the form

$$(2.2) B^H_t = \int_0^t K_H(t, s)dW_s.$$

On the other hand, define the operator $K_H : L^2([0, T], \mathbb{R}^d) \to L^{H+1/2}([0, T], \mathbb{R}^d)$ associated with the kernel $K_H(\cdot, \cdot)$ by

$$(K_H f)(t) = \int_0^t K_H(t, s)f(s)ds, \quad i = 1, \cdots, d.$$
By [11], it is an isomorphism and can be expressed in terms of fractional integrals as follows:

\[(K_H f)(s) = I_{0+}^{1/2} I_{0+}^{H-1/2} I_{0+}^{H-1/2} s^{1/2-H} f, \quad H \geq 1/2,\]
\[(K_H f)(s) = I_{0+}^{H} I_{0+}^{H-1/2} I_{0+}^{H-1/2} s^{H-1/2} f, \quad H \leq 1/2.\]

where \(f \in L^2([0,T],[\mathbb{R}^d])\). Consequently, for every \(h \in L^2_0([0,T],[\mathbb{R}^d])\), the inverse operator \(K_H^{-1}\) is of the following form

\[(K_H^{-1} h)(s) = s^{H-1/2} D_{0+}^{H-1/2} s^{1/2-H} h', \quad H \geq 1/2,\]  
\[(K_H^{-1} h)(s) = s^{1/2-H} D_{0+}^{1/2-H} s^{H-1/2} D_{0+}^{2H} h, \quad H \leq 1/2.\]  

(2.3)  

(2.4)

The remaining part will be devoted to the Malliavin calculus of fractional Brownian motion.

Let \(\Omega\) be the canonical probability space \(C_0([0,T],[\mathbb{R}^d])\), the set of continuous functions, null at time 0, equipped with the supremum norm. Let \(\mathbb{P}\) be the unique probability measure on \(\Omega\) such that the canonical process \(\{B_t^H; t \in [0,T]\}\) is a \(d\)-dimensional fractional Brownian motion with Hurst parameter \(H\). Then, the injection \(R_H = K_H \circ K_H^* : \mathcal{H} \to \Omega\) embeds \(\mathcal{H}\) densely into \(\Omega\) and \((\Omega, \mathcal{H}, \mathbb{P})\) is an abstract Wiener space in the sense of Gross. In the sequel we will make this assumption on the underlying probability space.

We denote by \(\mathcal{S}\) the set of smooth and cylindrical random variables of the form

\[F = f(B^H(\phi_1), \ldots, B^H(\phi_n)),\]

where \(n \geq 1, f \in C_0^\infty(\mathbb{R}^n)\), the set of \(f\) and all its partial derivatives are bounded, \(\phi_i \in \mathcal{H}, 1 \leq i \leq n\). The Malliavin derivative of \(F\), denoted by \(\mathbb{D} F\), is defined as the \(\mathcal{H}\)-valued random variable

\[\mathbb{D} F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (B^H(\phi_1), \ldots, B^H(\phi_n)) \phi_i.\]

For any \(p \geq 1\), we define the Sobolev space \(\mathbb{D}^{1,p}\) as the completion of \(\mathcal{S}\) w.r.t. the norm

\[\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|\mathbb{D} F\|_{\mathcal{H}}^p.\]

While we will denote by \(\delta\) and \(\text{Dom}\delta\) the divergence operator of \(\mathbb{D}\) and its domain. We conclude this section by giving a transfer principle that connects the derivative and divergence operators of both processes \(B^H\) and \(W\).

**Proposition 2.1** [28, Proposition 5.2.1] For any \(F \in \mathbb{D}^{1,2}_W = \mathbb{D}^{1,2}\), there holds

\[K_H^* \mathbb{D} F = \mathbb{D} W F,\]

where \(\mathbb{D} W\) denotes the derivative operator w.r.t. \(W\), and \(\mathbb{D}^{1,2}_W\) the corresponding Sobolev space.

**Proposition 2.2** [28, Proposition 5.2.2] \(\text{Dom}\delta = (K_H^*)^{-1}(\text{Dom}\delta_W)\), and for any \(\mathcal{H}\)-valued random variable \(u\) in \(\text{Dom}\delta\) we have \(\delta(u) = \delta_W(K_H^* u)\), where \(\delta_W\) denotes the divergence operator w.r.t. \(W\).
Lemma 3.1

Owing to [29, Theorem 2.1], the condition (H1) ensures that there exists a unique solution with the initial value $(x, y) \in \mathbb{R}^{d_1+d_2}$. In this part, we aim to establish the Bismut type derivative formulae for the associated family of operators $(P_t)_{0 \leq t \leq T}$:

$$P_t f(x, y) = \mathbb{E}^{x,y} f(X_t, Y_t), \quad (x, y) \in \mathbb{R}^{d_1+d_2}, f \in \mathcal{B}_b(\mathbb{R}^{d_1+d_2}),$$

where $\mathcal{B}_b(\mathbb{R}^{d_1+d_2})$ is the set of all bounded measurable functions on $\mathbb{R}^{d_1+d_2}$. Besides, for $f \in C^\alpha([0, T]; \mathbb{R}^m)$, set

$$\|f\|_\infty := \sup_{t \in [0, T]} |f(t)|, \quad \|f\|_\alpha := \sup_{s \neq t, s, t \in [0, T]} \frac{|f(t) - f(s)|}{|t - s|^{\alpha}}.$$
where \( \{e^i\}_{i=1}^m \) is the canonical ONB on \( \mathbb{R}^m \).

Then, we have, for each \( h \in \mathcal{H} \),

\[
\langle \mathbb{D}Z^i_t, h \rangle_{\mathcal{H}} = \sum_{l=1}^m \langle (\bar{\sigma}_{il}(Z)I_{[0,t]}e^i, h \rangle_{\mathcal{H}} + \sum_{k=1}^d \int_0^t \partial_k \bar{b}_i(Z_u) \langle \mathbb{D}Z^k_u, h \rangle_{\mathcal{H}} du
\]

\[
+ \sum_{k=1}^d \sum_{l=1}^m \int_0^t \partial_k \bar{\sigma}_{il}(Z_u) \mathbb{D}Z^k_u, h \rangle_{\mathcal{H}} \bar{B}^l_u.
\tag{3.4}
\]

Observe that, for each \( i = 1, \ldots, d, l = 1, \ldots, m \), by (2.1) we get

\[
\langle (\bar{\sigma}_{il}(Z)I_{[0,t]}e^i, h \rangle_{\mathcal{H}} = \langle K^*_H((\bar{\sigma}_{il}(Z)I_{[0,t]}e^i), K^*_H h \rangle_{L^2([0,T],\mathbb{R}^d)}
\]

\[
= \sum_{j=1}^d \int_0^T \left( K^*_H((\bar{\sigma}_{il}(Z)I_{[0,t]}e^j)) \right) (s)(K^*_H h)(s) ds
\]

\[
= \int_0^T \left( K^*_H(\bar{\sigma}_{il}(Z)I_{[0,t]}) \right) (s)(K^*_H h)^j(s) ds
\]

\[
= \int_0^T \int_s^T \bar{\sigma}_{il}(Z_r) I_{[0,t]}(r) \frac{\partial K^*_H}{\partial r}(r,s)(K^*_H h)^j(s) dr ds
\]

\[
= \int_0^T \bar{\sigma}_{il}(Z_r) \int_0^r \frac{\partial K^*_H}{\partial r}(r,s)(K^*_H h)^j(s) ds dr
\]

\[
= \int_0^T \bar{\sigma}_{il}(Z_r) d(R_H h)^j(r),
\tag{3.5}
\]

where the last equality is due to the fact: \( R_H = K_H \circ K^*_H \).

Next, by [30] page 400] or following the arguments of [19] Lemma 3.1 and Proposition 3.3], we deduce that \( \mathbb{D}_{R_H}Z^i_t = \frac{d}{dt} \big|_{t=0} Z^i_t(w + \epsilon R_H h), \ h \in \mathcal{H} \), satisfies

\[
\mathbb{D}_{R_H}Z^i_t = \langle \mathbb{D}Z^i_t, h \rangle_{\mathcal{H}}, \quad 1 \leq i \leq d.
\tag{3.6}
\]

So, plugging (3.5) and (3.6) into (3.4) yields

\[
\mathbb{D}_{R_H}Z^i_t = \sum_{l=1}^m \int_0^t \bar{\sigma}_{il}(Z_u) d(R_H h)^j(u) + \sum_{k=1}^d \int_0^t \partial_k \bar{b}_i(Z_u) \mathbb{D}_{R_H}Z^k_u du
\]

\[
+ \sum_{k=1}^d \sum_{l=1}^m \int_0^t \partial_k \bar{\sigma}_{il}(Z_u) \mathbb{D}_{R_H}Z^k_u \bar{B}^l_u.
\]

Then we obtain

\[
\mathbb{D}_{R_H}Z_t := \begin{pmatrix}
\mathbb{D}_{R_H}Z^1_t \\
\vdots \\
\mathbb{D}_{R_H}Z^d_t
\end{pmatrix}
\]

\[
= \int_0^t \bar{\sigma}(Z_u) d(R_H h)(u) + \int_0^t \nabla \bar{b}(Z_u) \mathbb{D}_{R_H}Z_u du + \int_0^t \nabla \bar{\sigma}(Z_u) \mathbb{D}_{R_H}Z_u d\bar{B}_u.
\]
Proposition 3.2 Suppose that
\[ R_d f \]
Consequently, for any \( R \) being in (3.2), we have, for each \( h \) such that
\[ \text{dom} \sigma = \{ (x,y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \} \]
holds. We will show that \( h \) satisfies (3.7) on condition that it is in \( \text{dom} \delta \) with
\[ (R_H h_1)(0) = 0, \quad (R_H h_1)(T) = v_1 \]
and
\[ \int_0^T \sigma(X_u)(R_H h_2)'(u)du + \int_0^T \nabla \sigma(X_u)((R_H h_1)(u) - v_1)d\tilde{B}_u^H - v_2 = 0. \]  

Proof. According to Lemma 3.1 for \( d = m = d_1 + d_2 \), \( Z_t = (X_t, Y_t)^* \), \( B^H = (B^H, \tilde{B}_u^H)^* \), \( b = 0 \) and
\[ \sigma(x,y) = \begin{pmatrix} I_{d_1 \times d_1} & 0 \\ 0 & \sigma(x) \end{pmatrix} \]
being in (3.2), we have, for each \( h = (h_1, h_2) \in \mathcal{H} \),
\[ \begin{cases} \mathbb{D}_{R_H h} X_t = (R_H h_1)(t) - (R_H h_1)(0), \\ \mathbb{D}_{R_H h} Y_t = \int_0^t \nabla \sigma(X_u)(R_H h_2)'(u)du + \int_0^t \nabla \sigma(X_u)d\mathbb{D}_{R_H h} X_u d\tilde{B}_u^H. \end{cases} \]
This, together with \( (R_H h_1)(0) = 0 \), leads to
\[ \begin{cases} \mathbb{D}_{R_H h} X_t = (R_H h_1)(t), \\ \mathbb{D}_{R_H h} Y_t = \int_0^t \nabla \sigma(X_u)(R_H h_2)'(u)du + \int_0^t \nabla \sigma(X_u)(R_H h_1)(u)d\tilde{B}_u^H. \tag{3.10} \end{cases} \]
On the other hand, we easily know that the directional derivative processes satisfy that, for any \( (v_1, v_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \),
\[ \begin{cases} \nabla_v X_t = v_1, \\ \nabla_v Y_t = v_2 + \int_0^t \nabla \sigma(X_u)\nabla_v X_u d\tilde{B}_u^H. \end{cases} \]
Hence, we get
\[ \begin{cases} \nabla_v X_t = v_1, \\ \nabla_v Y_t = v_2 + \int_0^t \nabla \sigma(X_u)v_1 d\tilde{B}_u^H. \tag{3.11} \end{cases} \]
In view of the control condition (3.9) and \( (R_H h_1)(T) = v_1 \), (3.10) and (3.11) imply
\[ (\mathbb{D}_{R_H h} X_T, \mathbb{D}_{R_H h} Y_T) = (\nabla_v X_T, \nabla_v Y_T). \]
Consequently, for any \( f \in C_b^1(\mathbb{R}^{d_1 + d_2}) \), we conclude
\[ \nabla_v P_T f(x,y) = \mathbb{E}^{x,y} \nabla_v f(X_T, Y_T) = \mathbb{E}^{x,y}(\nabla f)(X_T, Y_T)(\nabla_v X_T, \nabla_v Y_T) \]
\[ = \mathbb{E}^{x,y}(\nabla f)(X_T, Y_T)d\mathbb{D}_{R_H h} X_T, \mathbb{D}_{R_H h} Y_T) = \mathbb{E}^{x,y}d\mathbb{D}_{R_H h} f(X_T, Y_T) \]
\[ = \mathbb{E}^{x,y}(\nabla f)(X_T, Y_T), h)_{\mathcal{H}} = \mathbb{E}^{x,y}(f(X_T, Y_T)\delta(h)). \]
With the above proposition in hand, to establish explicit derivative formula, we need to calculate \( \delta(h) \) for \( h \) which will be given a concrete choice satisfying (3.8) and (3.9). To this end, we assume (H2):

\( \sigma \) is differentiable with bounded derivative, and for any \( x \in \mathbb{R}^{d_1}, \int_0^T (\sigma \sigma^*)(x + B_u^H) \, du \) is invertible such that

\[
\mathbb{E} \left\| \left( \int_0^T (\sigma \sigma^*) (x + B_u^H) \, du \right)^{-1} \right\|^{2+\epsilon_0} < \infty,
\]

where \( \epsilon_0 > 0 \) is a constant.

**Theorem 3.3** Assume (H2) and let \( v = (v_1, v_2) \in \mathbb{R}^{d_1+d_2} \). Then

\[
\nabla_v P_T f(x, y) = \mathbb{E}^{x,y} [f(X_T, Y_T)M_T], \quad f \in C^1_b(\mathbb{R}^{d_1+d_2}),
\]

(3.13)

holds for

\[
M_T = \int_0^T \left( K_H^{-1} \left( \frac{\cdot}{T} \right) (t) v_1, dW_t \right) + \left< \vartheta(T), \int_0^T \left( K_H^{-1} \left( \int_0^T \sigma^* (x + B_u^H) \, du \right) \right)^* (t) d\tilde{W}_t \right> - Tr \left( \left( \int_0^T (\sigma \sigma^*) (x + B_u^H) \, du \right)^{-1} \int_0^T \frac{T-u}{T} \nabla \sigma (x + B_u^H) v_1 \sigma^* (x + B_u^H) \, du \right)
\]

(3.14)

and

\[
\vartheta(T) = \left( \int_0^T (\sigma \sigma^*) (x + B_u^H) \, du \right)^{-1} \left( v_2 + \int_0^T \frac{T-u}{T} \nabla \sigma (x + B_u^H) v_1 d\tilde{B}_u^H \right).
\]

(3.15)

where \( \tilde{W} := (W, \tilde{W}) \) is the underlying Wiener process w.r.t. \((B, \tilde{B})\) defined in (2.2).

**Proof.** We first take \( h = (h_1, h_2) \) as follows: for \( t \in [0, T] \),

\[
h_1(t) = R_H^{-1} \left( \frac{\cdot}{T} v_1 \right) (t)
\]

(3.16)

and

\[
h_2(t) = R_H^{-1} \left( \left( \int_0^T \sigma^* (X_u) \, du \right) \left( \int_0^T (\sigma \sigma^*)(X_u) \, du \right)^{-1} \left( v_2 + \int_0^T \frac{T-u}{T} \nabla \sigma (X_u) v_1 d\tilde{B}_u^H \right) \right) (t).
\]

(3.17)

It is easy to check that the above \( h = (h_1, h_2) \) satisfies (3.8) and (3.9). Now, to prove (3.13)-(3.15), according to Proposition 3.2 it remains to verify \( h = (h_1, h_2) \in \text{Dom} \vartheta \) and then calculate \( \delta(h) \) which will be fulfilled via Proposition 2.1 and Proposition 2.2 as well as (28, Proposition 1.3.3). Notice that, by (3.17) we know that

\[
(R_H h_2)(t) = \left( \int_0^t \sigma^* (X_u) \, du \right) \left( \int_0^T (\sigma \sigma^*)(X_u) \, du \right)^{-1} \left( v_2 + \int_0^T \frac{T-u}{T} \nabla \sigma (X_u) v_1 d\tilde{B}_u^H \right)
\]

=: \int_0^t \sigma^* (X_u) \, du \cdot \vartheta(T)
\]

(3.18)
is not adapted and then \((K^*_H h_2)(t) = K^{-1}_H \left( \int_0^t \sigma^*(X_u) du \cdot \vartheta(T) \right) (t)\) is so. To circumvent the impediment, let \(\{e_i\}_{i=1}^{d_2}\) be the canonical ONB on \(\mathbb{R}^{d_2}\), then we have

\[
(R_H h_2)(t) = \int_0^t \sigma^*(X_u) du \left( \sum_{i=1}^{d_2} \langle \vartheta(T), e_i \rangle e_i \right) = \sum_{i=1}^{d_2} \langle \vartheta(T), e_i \rangle \int_0^t \sigma^*(X_u) e_i du.
\]

So, combining this with (3.16) yields

\[
(R_H h)(t) = ((R_H h_1)(t), (R_H h_2)(t)) = ((R_H h_1)(t), 0) + (0, (R_H h_2)(t))
\]

\[
= \left( \frac{t}{T} v_1, 0 \right) + \sum_{i=1}^{d_2} \langle \vartheta(T), e_i \rangle \left( 0, \int_0^t \sigma^*(X_u) e_i du \right).
\]

As a consequence, we obtain

\[
(K^*_H h)(t) = ((K^*_H h_1)(t), 0) + (0, (K^*_H h_2)(t))
\]

\[
= \left( K^{-1}_H \left( \frac{t}{T} \right) v_1, 0 \right) + \sum_{i=1}^{d_2} \langle \vartheta(T), e_i \rangle \left( 0, K^{-1}_H \left( \int_0^t \sigma^*(X_u) e_i du \right) (t) \right).
\]

Obviously, we get

\[
\delta_W (K^*_H h_1, 0) = \delta_W \left( K^{-1}_H \left( \frac{t}{T} \right) v_1, 0 \right) = \int_0^T \left( \left( K^{-1}_H \left( \frac{t}{T} \right) (t)v_1, dW_t \right) \right).
\]

Next, we shall focus on the second term of the right hand of (3.19). By (2.8), we first obtain

\[
K^{-1}_H \left( \int_0^t \sigma^*(X_u) e_i du \right) (t) = t^{H - \frac{1}{2}} D_{0+}^{\frac{1}{2} - H} \left( \cdot H \sigma^*(X_u) e_i \right) (t)
\]

\[
= \frac{1}{\Gamma \left( \frac{3}{2} - H \right)} \left[ t^{\frac{1}{2} - H} \sigma^*(X_t) e_i + \left( H - \frac{1}{2} \right) \sigma^*(X_t) e_i t^{H - \frac{1}{2}} \int_0^t t^{\frac{1}{2} - H} - r^{\frac{1}{2} - H} \right] \left[ r^{\frac{1}{2} + H} dr \right]
\]

\[
= \frac{1}{\Gamma \left( \frac{3}{2} - H \right)} \left[ \alpha_1(t) + \alpha_2(t) + \alpha_3(t) \right].
\]

Using the estimate

\[
|\sigma^*(X_t) e_i| = |\sigma^*(x + B^H_t) e_i| \leq \|\sigma(x)\| + C |B^H_t|
\]

and the relation

\[
\int_0^t t^{\frac{1}{2} - H} - t^{\frac{1}{2} - H} \right] \left[ r^{\frac{1}{2} + H} dr \right] = \int_0^1 \theta^{1/2 - H} - 1 \frac{1}{(1 - \theta)^{1/2 + H}} d\theta \cdot t^{1 - 2H} =: C_0 t^{1 - 2H}
\]

with some positive constant \(C_0\), we get

\[
|\alpha_1(t) + \alpha_2(t)| \leq Ct^{\frac{1}{2} - H} (\|\sigma(x)\| + |B^H_t|) \leq Ct^{\frac{1}{2} - H} \left( 1 + \|B^H\|_\infty \right).
\]
As for $\alpha_3(t)$, from (H2) it follows that

$$|\alpha_3(t)| \leq C t^{H-\frac{1}{2}} \int_0^t |B_t^{H} - B_r^{H}| \frac{1}{(t-r)^{\frac{1}{2}+H}} dr \leq C t^{\frac{1}{2} - \epsilon} \|B^H\|_{H-\epsilon}, \quad (3.24)$$

where $\epsilon$ is fixed satisfying $0 < \epsilon < \frac{1}{4}$.

Then, by (3.21) and the Fernique theorem (see, for instance, [4]), we derive that $K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) \in L^2([0,T] \times \Omega, \mathbb{R}^d)$. Moreover, it is clear that $K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right)$ is adapted due to the fact that the operator $K_H^{-1}$ preserves the adaptability property. So, we have

$$\delta_W\left(0, K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right)\right) = \int_0^T \left(K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) (t), d\tilde{W}_t\right). \quad (3.25)$$

Now, let $\mathcal{F}_T = \sigma(B_t^H : 0 \leq t \leq T)$. By (3.25), the definition of $\vartheta(T)$, the $C_r$-inequality and the Hölder inequality, we obtain

$$\mathbb{E} \left(\langle \vartheta(T), e^i \rangle \cdot \delta_W\left(0, K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right)\right)^2\right)$$

$$= \mathbb{E} \left[ \mathbb{E} \left(\langle \vartheta(T), e^i \rangle \cdot \int_0^T \left(K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) (t), d\tilde{W}_t\right)\right)^2 \left| \mathcal{F}_T\right.\right]$$

$$\leq 2 \mathbb{E} \left\{ \left\| \left(\int_0^T (\sigma^*)(X_u) du\right)^{-1} v_2, e^i \right\|^2 \cdot \mathbb{E} \left(\left(\int_0^T \left(K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) (t), d\tilde{W}_t\right)\right)^2 \left| \mathcal{F}_T\right.\right) \right\}$$

$$= 2 \mathbb{E} \left[ \mathbb{E} \left(\left(\int_0^T (\sigma^*)(X_u) du\right)^{-1} \int_0^T \frac{T-u}{T} \nabla \sigma(X_u) v_1 d\tilde{B}^H_u, e^i\right)\right]^\frac{1}{2}$$

$$\times \mathbb{E} \left(\left| \int_0^T \left(K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) (t), d\tilde{W}_t\right)\right|^2 \left| \mathcal{F}_T\right.\right)$$

$$\leq C \mathbb{E} \left\{ \left\| \left(\int_0^T (\sigma^*)(X_u) du\right)^{-1} \right\|^2 \cdot \int_0^T \left| \int_0^\infty \sigma^*(X_u) e^i du\right| (t)^2 dt \right\}$$

$$+ C \mathbb{E} \left[ \mathbb{E} \left(\left(\int_0^T (\sigma^*)(X_u) du\right)^{-1} \nabla \sigma(X_u) v_1 \right)^* e^i, d\tilde{B}^H_u\right|^4 \left| \mathcal{F}_T\right.\right]$$

$$\times \left| \int_0^T \left(K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) (t), dt\right\|^\frac{1}{2} \right\}$$

$$\leq C \mathbb{E} \left\{ \left\| \left(\int_0^T (\sigma^*)(X_u) du\right)^{-1} \right\|^2 \cdot \int_0^T \left| \int_0^\infty \sigma^*(X_u) e^i du\right| (t)^2 dt \right\}$$

$$+ C \mathbb{E} \left[ \left(\int_0^T (\sigma^*)(X_u) du\right)^{-1} \nabla \sigma(X_u) v_1 \right)^* e^i \left| \int_0^T \left(K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) (t), dt\right\|^\frac{1}{2} \right\}$$

$$\times \left| \int_0^T \left(K_H^{-1}\left(\int_0^\infty \sigma^*(X_u) e^i du\right) (t), dt\right\|^\frac{1}{2} \right\} \right. \right), \quad (3.26)$$
where the last inequality is due to [28, page 292-293]. Then, by \([3.21]-[3.24]\) again and (H2), we deduce

\[
\begin{align*}
\mathbb{E} \left( (\vartheta(T), e^i) \cdot \delta_{\bar{W}} \left( 0, K^{-1}_H \left( \int_0^T \sigma^*(X_u)e^i du \right) \right) \right)^2 & \leq C \mathbb{E} \left\{ \left| \left( \int_0^T (\sigma^*)(X_u) du \right)^{-1} \right|^2 \left( \int_0^T \left| K^{-1}_H \left( \int_0^T \sigma^*(X_u) e^i du \right) (t) \right|^2 dt \right. \\
& \quad \left. + \int_0^T \left| K^{-1}_H \left( \int_0^T \sigma^*(X_u) e^i du \right) (t) \right|^4 dt \right. \\
& \quad \left. + \int_0^T \left\| \nabla \sigma(X_u) \right\|^4 dt \right\} \right. \\
& \leq C \mathbb{E} \left\{ \left( \int_0^T (\sigma^*)(X_u) du \right)^{-1} \right\|^2 \left( 1 + \| B^H \|_\infty^4 + \| B^H \|_{H^{-\epsilon}} \right) \right. \\
& \leq C \left[ \mathbb{E} \left\{ \left( \int_0^T (\sigma^*)(X_u) du \right)^{-1} \right\}^{2+\gamma_0} \cdot \left( 1 + \| B^H \|_\infty^4 + \| B^H \|_{H^{-\epsilon}} \right)^{\frac{2+\gamma_0}{\gamma_0}} \right] \left[ \int_0^T \left\| \nabla \sigma(X_u) \right\|^4 dt \right] \\
& < \infty. \quad (3.27)
\end{align*}
\]

That is, \( (\vartheta(T), e^i) \cdot \delta_{\bar{W}} \left( 0, K^{-1}_H \left( \int_0^T \sigma^*(X_u)e^i du \right) \right) \in L^2(\mathbb{P}), \ i = 1, \cdots, d_2. \)

On the other hand, by [28, Example 1.2.1] we know that in the case of a classical Brownian motion, \( K^{-1}_H \) is the identity map on \( L^2([0,T], \mathbb{R}^{d_1+d_2}) \) which means \( W = \mathcal{H} = L^2([0,T], \mathbb{R}^{d_1+d_2}) \), and \( K_H(t,s) \) equal to \( I_{[0,t]}(s) \) which implies that \( \mathcal{H}_H \) is the space of absolutely continuous functions, vanishing at zero, with a square integrable derivative. Then, using Proposition [2.1] and the fact that \( K^{-1}_H \) is an isometry between \( \mathcal{H} \) and \( L^2([0,T], \mathbb{R}^{d_1+d_2}) \), we obtain

\[
\begin{align*}
\langle \mathbb{D}_{\bar{W}} (\vartheta(T), e^i), \left( 0, K^{-1}_H \left( \int_0^T \sigma^*(X_u) e^i du \right) \right) \rangle & \in \mathcal{H}_W \\
= & \langle \mathbb{D}_{\bar{W}} (\vartheta(T), e^i), \left( 0, K^{-1}_H \left( \int_0^T \sigma^*(X_u) e^i du \right) \right) \rangle_{L^2([0,T], \mathbb{R}^{d_1+d_2})} \\
= & \langle K^{-1}_H \mathbb{D} (\vartheta(T), e^i), K^{-1}_H \left( 0, \sigma^*(X_u) e^i du \right) \rangle_{L^2([0,T], \mathbb{R}^{d_1+d_2})} \\
= & \mathbb{D} \langle (\vartheta(T), e^i), 0, R^{-1}_H \left( \int_0^T \sigma^*(X_u) e^i du \right) \rangle_{\mathcal{H}_W} \\
= & \mathbb{D} \langle (\vartheta(T), e^i), 0, \left( \int_0^T \sigma^*(X_u) e^i du \right) \rangle_{\mathcal{H}_W} \\
= & \mathbb{D} \langle \int_0^T (\sigma^*)(X_u) du \rangle \langle (\vartheta(T), e^i) \rangle_{\mathcal{H}_W} \langle \int_0^T \frac{T-u}{T} \nabla \sigma(X_u) v_1 \sigma^*(X_u) e^i du, e^i \rangle. \quad (3.28)
\end{align*}
\]

So, by (H2) it follows that the left hand of (3.28) is square integrable. Combining this with (3.27), by [28, Proposition 1.3.3] we show that, for each \( i = 1, \cdots, d_2 \),

\[
\langle \vartheta(T), e^i \rangle \left( 0, K^{-1}_H \left( \int_0^T \sigma^*(X_u) e^i du \right) \right) (t) \in \text{Dom} \delta_W, \quad (3.29)
\]

and moreover,

\[
\delta_W \left( \langle \vartheta(T), e^i \rangle \left( 0, K^{-1}_H \left( \int_0^T \sigma^*(X_u) e^i du \right) \right) \right)
\]
where the last relation is due to (3.25) and (3.28).
Consequently, we derive that $K^*_h h \in \text{Dom} \delta_{\bar{W}}$ and by (3.19), (3.20) and (3.30),
\[
\delta_{\bar{W}}(K^*_h h) = \delta_{\bar{W}}(K^*_h h_1, 0) + \delta_{\bar{W}}(0, K^*_h h_2)
\]
\[
= \int_0^T \left\langle K^{-1}_H \left( \frac{\bar{t}}{T} \right) (t)v_1, dW_t \right\rangle + \sum_{i=1}^{d_2} \left\langle \vartheta(T), e^i \right\rangle \int_0^T \left\langle K^{-1}_H \left( \int_0^t \sigma^*(X_u)e^i du \right) (t), d\tilde{W}_t \right\rangle
\]
\[-\sum_{i=1}^{d_2} \left( \int_0^T (\sigma\sigma^*)(X_u)du \right)^{-1} \int_0^T \frac{T-u}{T} \nabla \sigma(X_u)v_1 \sigma^*(X_u)e^i du, e^i \right\rangle
\]
\[
= \int_0^T \left\langle K^{-1}_H \left( \frac{\bar{t}}{T} \right) (t)v_1, dW_t \right\rangle + \sum_{i=1}^{d_2} \left\langle \vartheta(T), e^i \right\rangle \left\langle \int_0^T \left( K^{-1}_H \left( \int_0^t \sigma^*(X_u)du \right) \right) (t)v_1, d\tilde{W}_t \right\rangle
\]
\[-\sum_{i=1}^{d_2} \left( \int_0^T (\sigma\sigma^*)(X_u)du \right)^{-1} \int_0^T \frac{T-u}{T} \nabla \sigma(X_u)v_1 \sigma^*(X_u)e^i du, e^i \right\rangle
\]
\[
= \int_0^T \left\langle K^{-1}_H \left( \frac{\bar{t}}{T} \right) (t)v_1, dW_t \right\rangle + \left\langle \vartheta(T), \int_0^T \left( K^{-1}_H \left( \int_0^t \sigma^*(X_u)du \right) \right)^* (t)d\tilde{W}_t \right\rangle
\]
\[-Tr \left( \left( \int_0^T (\sigma\sigma^*)(X_u)du \right)^{-1} \int_0^T \frac{T-u}{T} \nabla \sigma(X_u)v_1 \sigma^*(X_u)du \right) .
\]

Then it follows by Proposition 2.2 that $h \in \text{Dom} \delta$ and $\delta(h) = \delta_{\bar{W}}(K^*_h h)$. The proof is finished.

**Remark 3.4** (i) If $H = \frac{1}{2}$, the inverse operator $K^{-1}_H$ appeared in Theorem 3.3 can be viewed as the derivative operator. By simple calculus, we know that our result covers that of [14, Theorem 1.1].

(ii) By (3.3), it is not difficult to verify that the right side of (3.14) equals to the following relation
\[
M_T = c_1 \int_0^T \left\langle \frac{t^{1-H}}{T} v_1, dW_t \right\rangle + c_2 \left\langle \vartheta(T), \int_0^T t^{1-H} \sigma(x+B^H_t) d\tilde{W}_t \right\rangle
\]
\[+ c_3 \left\langle \vartheta(T), \int_0^T t^{1-H} \int_0^t \sigma(x+B^H_t) - \sigma(x+B^H_r) \frac{t-r}{(t-r)^{1+H}} \right\rangle \int_0^T t^{1-H} dr d\tilde{W}_t \right\rangle
\]
\[-Tr \left( \left( \int_0^T (\sigma\sigma^*)(x+B^H_u) du \right)^{-1} \int_0^T \frac{T-u}{T} \nabla \sigma(x+B^H_u) v_1 \sigma^*(x+B^H_u) du \right),
\]
where $c_i, i = 1, 2, 3$ are three constants.
Let us go back to the above proof and note that, for (3.17), the choice of $h_2$ is not unique. Now, we begin with the assumption (H3): 

$\sigma$ is differentiable with bounded derivative, and $\sigma\sigma^*$ is invertible such that $(\sigma\sigma^*)^{-1}$ is bounded and Hölder continuous of order $\gamma \in (1 - 1/(2H), 1]$:

$$
\|(\sigma\sigma^*)^{-1}(z_1) - (\sigma\sigma^*)^{-1}(z_2)\| \leq K|z_1 - z_2|^{\gamma}, \ \forall z_1, z_2 \in \mathbb{R}^{d_1},
$$

where $K$ is a positive constant.

Set

$$
h_2(t) = R_H^{-1}\left(\int_0^T (\sigma^*(\sigma\sigma^*)^{-1})(X_u)du \cdot \frac{1}{T}\left(v_2 + \int_0^T \frac{T-u}{T} \nabla \sigma(X_u)v_1 d\tilde{B}_u^H\right)\right)(t). \quad (3.32)
$$

It is not hard to verify that $h_1$ defined as (3.16) before and the above $h_2$ also satisfy (3.8) and (3.9). In the spirit of the proof of Theorem 3.3, we have the following derivative formula for the equation (3.1):

**Theorem 3.5** Assume (H3) and let $v = (v_1, v_2) \in \mathbb{R}^{d_1+d_2}$. Then we have

$$
\nabla_v P_T f(x, y) = \mathbb{E}^{x,y}\left[f(X_T, Y_T)\tilde{M}_T\right], \ f \in C_0^1(\mathbb{R}^{d_1+d_2}), \quad (3.33)
$$

where

$$
\tilde{M}_T = \int_0^T \left(K_H^{-1}\left(\frac{t}{T}\right) v_1, dW_t\right) + \left(\tilde{\vartheta}(T), \int_0^T \left(K_H^{-1}\left(\int_0^T (\sigma^*(\sigma\sigma^*)^{-1})(x + B^H_u)du\right)\right)(t)dW_t\right)
$$

$$
- \text{Tr}\left(\int_0^T \frac{T-u}{T^2} \nabla \sigma(x + B^H_u) v_1 (\sigma^*(\sigma\sigma^*)^{-1})(x + B^H_u)du\right) \quad (3.34)
$$

and

$$
\tilde{\vartheta}(T) = \frac{1}{T}\left(v_2 + \int_0^T \frac{T-u}{T} \nabla \sigma(x + B^H_u) v_1 d\tilde{B}_u^H\right). \quad (3.35)
$$

**Remark 3.6** Theorem 3.3 and Theorem 3.5 are established under the conditions (H2) and (H3), respectively. Observe that these two assumptions are not compatible. For instance, setting

$$
\zeta(t) = \left(\begin{array}{c} 1 \\ t \\ t^2 \end{array}\right),
$$

it is easy to see that $\int_0^T \zeta(t)dt$ is invertible, yet $\zeta(t)$ is degenerate. On the other hand, taking

$$
\eta(t) = \left(\begin{array}{cc} \sin t & \cos t \\ \cos t & -\sin t \end{array}\right),
$$

it is not hard to verify that $\eta(t)$ is invertible, while $\int_0^T \eta(t)dt$ is degenerate.

The next result states that with the help of Theorem 3.5, an explicit gradient estimate can be derived.
Corollary 3.7 Assume (H3). Then for each $p > 1/ (\frac{3}{2} - H)$ there exists a constant $C(p, H) > 0$ such that

$$|\nabla_v P_T f(x, y)| \leq C(p, H) (P_T |f|^p)^\frac{1}{p} (x, y) \times \left[ v_1 \left( \frac{1}{TH} + 1 + T^{\gamma(H-\tilde{c})} + T^{H-\tilde{c}} \right) + |v_2| \left( \frac{1}{TH} + \frac{1}{T^H - \gamma(H-\tilde{c})} \right) \right]$$

holds for $v = (v_1, v_2) \in \mathbb{R}^{d_1+d_2}$ and $(x, y) \in \mathbb{R}^{d_1+d_2}$, where $\tilde{c}$ is fixed with $\tilde{c} < H - \frac{1}{\gamma} (H - \frac{1}{2})$.

Before proving Corollary 3.7 we will first present a technical lemma used to calculate the moment of stochastic integral w.r.t. a Brownian motion (see [34 Lemma 2.2]).

Lemma 3.8 Let $\varrho(t)$ be a predictable process on $\mathbb{R}^d$ with $\mathbb{E} \int_0^T |\varrho(t)|^q dt < \infty$ for some $q \geq 2$. Then there holds

$$\mathbb{E} \left| \int_0^T \langle \varrho(t), dW_t \rangle \right|^q \leq C T^{\frac{q-2}{2}} \int_0^T \mathbb{E} |\varrho(t)|^q dt,$$

where $W_t$ is a Brownian motion on $\mathbb{R}^d$.

Proof of Corollary 3.7. We first note that by the Hölder inequality, it suffices to prove for $p \in (1, 2]$. According to Theorem 3.5 and the Hölder inequality again, we have

$$|\nabla_v P_T f(x, y)| = \left| \mathbb{E}^{x,y} \left[ f(X_T, Y_T) M_T \right] \right| \leq (P_T |f|^p)^\frac{1}{p} (x, y) \cdot \left( \mathbb{E} |\bar{M}_T|^q \right)^\frac{1}{q},$$

where $q = \frac{p}{p-1}$.

Now, we shall bound the term $\mathbb{E} |\bar{M}_T|^q$. For the convenience of the notations, let

$$\psi(z) = (\sigma^* (\sigma^*)^{-1})(z), \quad z \in \mathbb{R}^{d_1}.$$

By (2.33), we first obtain

$$K_H^{-1} \left( \frac{1}{T} \right) (t) = t^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} \left( \frac{1}{T} \right) (t) = \frac{1}{\Gamma \left( \frac{3}{2} - H \right)} - \frac{1}{T} \left[ t^{\frac{1}{2}-H} - \frac{1}{2} t^{-\frac{1}{2}} \int_0^t \left( \frac{1}{t} - \frac{1}{r} \right)^{\frac{1}{2}-H} dr \right]$$

and

$$K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) du \right) (t) = t^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} \left( \frac{1}{T} \right) (t) = \frac{1}{\Gamma \left( \frac{3}{2} - H \right)} \left[ t^{\frac{1}{2}-H} \psi(x + B_t^H) + \left( H - \frac{1}{2} \right) t^{H-\frac{1}{2}} \int_0^t \left( \frac{1}{t} - \frac{1}{r} \right)^{\frac{1}{2}-H} dr \right]$$

$$+ \left( H - \frac{1}{2} \right) t^{H-\frac{1}{2}} \int_0^t \psi \left( x + B_u^H \right) - \psi \left( x + B_{r+}^H \right) \left( r^{\frac{1}{2}-H} \right) dr$$

$$= \frac{1}{\Gamma \left( \frac{3}{2} - H \right)} \left( \frac{1}{T} \right) (t).$$
\[ + \frac{H - \frac{1}{2}}{\Gamma\left(\frac{1}{2} - H\right)} t^{H - \frac{1}{2}} \int_0^t \frac{\psi(x + B_t^H) - \psi(x + B_r^H)}{(t - r)^{\frac{1}{2} + H}} r^{\frac{1}{2} - H} \, dr. \tag{3.38} \]

Recall that the last equalities of (3.37) and (3.38) are due to (3.22). Combining (3.37) with Lemma 3.8 yields

\[
E \left[ \int_0^T \left( K_H^{-1} \left( \frac{t}{T} \right) (t)v_1, dW_t \right)^q \right] = CE \left[ \int_0^T \left( \frac{t^{\frac{1}{2} - H}}{T} v_1, dW_t \right)^q \right] \leq C|v_1|^q \frac{1}{T^{Hq}}. \tag{3.39} \]

Next, we are to estimate the \( q \)th moment of the second term of the right side of (3.34). By (3.38) and (H3), we first get, for each \( 1 < \alpha < 1/(H - \frac{1}{2}) \),

\[
\int_0^T \left| K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) \, du \right) (t) \right|^\alpha \, dt \\
\leq C \left( \int_0^T t^{(\frac{1}{2} - H)\alpha} \left( \psi(x + B_t^H) \right)^\alpha \, dt + \int_0^T t^{H - \frac{1}{2}} \int_0^t \frac{\psi(x + B_t^H) - \psi(x + B_r^H)}{(t - r)^{\frac{1}{2} + H}} r^{\frac{1}{2} - H} \, dr \right)^\alpha \, dt \\
\leq C \left( T^{(\frac{1}{2} - H)\alpha + 1} + \int_0^T t^{H - \frac{1}{2}} \int_0^t \frac{B_t^H - B_r^H}{(t - r)^{\frac{1}{2} + H}} \, dt \right)^\alpha \\
\leq C \left( T^{(\frac{1}{2} - H)\alpha + 1}\|B_t^H\|_{H^{\frac{1}{2}}} + \int_0^T t^{(\frac{1}{2} - \frac{1}{2})\alpha} \, dt \cdot \|B_t^H\|_{H^{\frac{1}{2}}} \right) \\
= C \left( T^{(\frac{1}{2} - H)\alpha + 1}\|B_t^H\|_{H^{\frac{1}{2}}} + T^{(\frac{1}{2} - \frac{1}{2})\alpha + 1} \cdot \|B_t^H\|_{H^{\frac{1}{2}}} \right),
\]

where \( \tilde{\alpha} \) is fixed with \( \tilde{\alpha} < H - \frac{1}{2} \) or \( p > 1/(\frac{3}{2} - H) \).

Then, using the fact that \( x + B_t^H \) is measurable w.r.t. \( F_T \) while \( \tilde{W} \) is independent of \( F_T \), Lemma 3.8 and [23, Theorem 1.1] along with (H3), we derive that, for each \( 1 < q < 1/(\frac{3}{2} - H) \) or \( p > 1/(\frac{3}{2} - H) \),

\[
E \left[ \left( \int_0^T \frac{v_2}{T} \int_0^T \left( K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) \, du \right) \right)^* \, dt \, dW_t \right)^q \mid F_T \right] \\
\leq C \left( \frac{|v_2|^q}{T^{\frac{3}{2} + 1}} \int_0^T \left| K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) \, du \right) (t) \right|^q \, dt \right) \\
\leq C \left( \frac{|v_2|^q}{T^{\frac{3}{2} + 1}} \left( T^{(\frac{1}{2} - H)q + 1} + T^{(\gamma(H - \frac{1}{2}) + \frac{1}{2} - H)q + 1} \|B_t^H\|^q_{H^{\frac{1}{2}}} + T^{(\frac{1}{2} - \frac{1}{2})q + 1} \cdot \|B_t^H\|^q_{H^{\frac{1}{2}}} \right) \right) \\
= C \left( \frac{|v_2|^q}{T^{Hq}} + \frac{\|B_t^H\|^q_{H^{\frac{1}{2}}} + T^{(\gamma(H - \frac{1}{2})q + 1)}}{T^{\frac{3}{2} + 1}} \right)
\]

and

\[
E \left[ \left( \int_0^T \frac{T - u}{T^2} \nabla \sigma \left( x + B_u^H \right) v_1 \, d\tilde{B}_u^H \right) \int_0^T \left( K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) \, du \right) \right)^* \, dt \, dW_t \right] \mid F_T \right] \\
\leq \left[ E \left( \left( \int_0^T \frac{T - u}{T^2} \nabla \sigma \left( x + B_u^H \right) v_1 \, d\tilde{B}_u^H \right)^{2q} \mid F_T \right) \right]^{\frac{1}{2}} \cdot \left[ E \left( \left( \int_0^T \left( K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) \, du \right) \right)^* \, dt \right)^{2q} \mid F_T \right) \right]^{\frac{1}{2}} 
\]

16
\[ C \left( \int_0^T \left| \frac{T-u}{T^2} \nabla \sigma \left( x + B_u^H \right) v_1 \right|^{\frac{1}{q}} du \right)^{Hq} \cdot \left( T^{q-1} \int_0^T \left| K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) du \right) \right| * (t)^{2q} dt \right)^{\frac{1}{2}} \leq C |v_1|^q \frac{1}{T^{Hq}} \left( T^{(\frac{1-H}{2})q + \frac{1}{2}} + T^{(\gamma(H-\tilde{\gamma}) + (\frac{1-H}{2})q + \frac{1}{2})} \| B_H^H \|^q_{H^{-\tilde{\gamma}}} + T^{(\gamma(H-\tilde{\gamma}) + (\frac{1-H}{2})q + \frac{1}{2})} \cdot \| B_H^H \|^q_{H^{-\tilde{\gamma}}} \right). \]

Consequently, these, together with (3.35), lead to

\[
E \left| \tilde{\partial}(T), \int_0^T \left( K_H^{-1} \left( \int_0^t (\sigma^*(\sigma^*)^{-1}) \left( x + B_u^H \right) du \right) \right) * (t) d\tilde{W}_t \right| ^q \leq C E \left[ \left| \tilde{\partial}(T), \int_0^T \left( K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) du \right) \right) * (t) d\tilde{W}_t \right| ^q |F_T \right] + C E \left[ \left| \tilde{\partial}(T), \int_0^T \left( K_H^{-1} \left( \int_0^t \psi \left( x + B_u^H \right) du \right) \right) * (t) d\tilde{W}_t \right| ^q |F_T \right] \leq C |v_2|^q \left( \frac{1}{T^{Hq}} + \frac{1}{T^{(H-\gamma(H-\tilde{\gamma}))q}} \right) + C |v_1|^q \left( 1 + T^{(\gamma(H-\tilde{\gamma}) + T^{(H-\tilde{\gamma})q}} \right). \tag{3.40} \]

As for the third term of the right side of (3.34), by (H3) it is easy to see that

\[
E \left| \text{Tr} \left( \int_0^T \frac{T-u}{T^2} \nabla \sigma \left( x + B_u^H \right) v_1 (\sigma^*(\sigma^*)^{-1}) \left( x + B_u^H \right) du \right) \right| ^q \leq C \frac{|v_1|^q}{T^q} E \left( \int_0^T \left| \nabla \sigma \left( x + B_u^H \right) du \right| ^q \right) \leq C |v_1|^q. \tag{3.41} \]

So, by (3.39)-(3.41) we get

\[
E |\tilde{M}_T|^q \leq C E \left[ \int_0^T \left| K_H^{-1} \left( \frac{1}{T} \right) (t) v_1, dW_t \right| ^q \right] + C |v_1|^q \frac{1}{T^{Hq}} + C |v_2|^q \left( \frac{1}{T^{Hq}} + \frac{1}{T^{(H-\gamma(H-\tilde{\gamma}))q}} \right) \leq C |v_1|^q \left( \frac{1}{T^{Hq}} + 1 + T^{(\gamma(H-\tilde{\gamma}) + T^{(H-\tilde{\gamma})q}} \right) + C |v_2|^q \left( \frac{1}{T^{Hq}} + \frac{1}{T^{(H-\gamma(H-\tilde{\gamma}))q}} \right). \]

Then, the desired assertion follows from (3.36).

\[ \square \]

**Remark 3.9** If we use the approach as in the proof of Corollary 3.7 to obtain gradient estimate from Theorem 3.3, there will appear the following term

\[ \left\| \left( \int_0^T (\sigma^*) \left( x + B_u^H \right) du \right)^{-1} \right\| \]

which seems to be very difficult to deal with. So, compared with Theorem 3.3, the advantage of Theorem 3.5 is that an explicit gradient estimate is obtained with simpler proofs, while the drawback is the restriction to non-degenerate noise situation.
4 An extension to a more general model

This section is devoted to extend Theorem 3.3 to a more general model. That is,

\[
\begin{align*}
    dX_t &= b_1(X_t)dt + \sigma_1 dB^H_t, & X_0 &= x \in \mathbb{R}^{d_1}, \\
    dY_t &= b_2(X_t)dt + \sigma_2(X_t) d\tilde{B}^H_t, & Y_0 &= y \in \mathbb{R}^{d_2},
\end{align*}
\]

(4.1)

where \( b_1 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}, b_2 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}, \sigma_1 \in \mathbb{R}^{d_1 \times \mathbb{R}^{d_1}} \) is invertible, \( \sigma_2 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2 \times \mathbb{R}^{d_1}} \) and \( (B^H_t, \tilde{B}^H_t) \) is a fractional Brownian motion on \( \mathbb{R}^{d_1+1} \). In this setting, one can allow \( \sigma_2 \) to be degenerate. Let \( P_t \) be the associated operators. To establish the derivative formula for \( P_t \), we shall make use of the following assumption.

(H4) \( b, i = 1, 2 \) and \( \sigma_2 \) are differentiable with bounded derivatives, and \( \int_0^T (\sigma_2^* (X_u)) du \) is invertible such that

\[
\mathbb{E} \left\| \left( \int_0^T (\sigma_2^* (X_u)) du \right)^{-1} \right\| ^{2+\epsilon_0} < \infty,
\]

where \( \epsilon_0 > 0 \) is a fixed constant.

Obviously, by (H4), it follows from [29, Theorem 2.1] that (4.1) has a unique solution.

The main result of this part is the following.

**Theorem 4.1** Assume (H4) and let \( v = (v_1, v_2) \in \mathbb{R}^{d_1+d_2} \). Then

\[
\nabla_v P_T f(x, y) = \mathbb{E}^{x,y} [f(X_T, Y_T) N_T], \quad f \in C^1_b(\mathbb{R}^{d_1+d_2}),
\]

(4.2)

where

\[
N_T = \int_0^T \left\langle \sigma_1^{-1} K_H^{-1} \left( \frac{T-u}{T} \nabla b_1(X_u) + \frac{1}{T} \right), v_1 \right\rangle dt + \int_0^T \left\langle \chi(T), \int_0^T K_H^{-1} \left( \int_0^T \sigma_2^*(X_u) du \right) \right\rangle \sigma_1^{-1} \nabla b_1(X_u) dt + \int_0^T \left( \int_0^T \sigma_2^*(X_u) du \right) \left( \int_0^T \nabla \sigma_2(X_u) v_1 \sigma_2^*(X_u) du \right)^{-1} \right\rangle dt + \int_0^T \nabla \sigma_2(X_u) v_1 \sigma_2^*(X_u) du
\]

(4.3)

and

\[
\chi(T) = \left( \int_0^T (\sigma_2^* (X_u)) du \right)^{-1} \left( v_2 + \int_0^T \nabla b_2(X_u) v_1 du + \int_0^T \nabla \sigma_2(X_u) v_1 dB^H_u \right).
\]

(4.4)

**Proof.** Let \( h = (h_1, h_2) \) be as follows: for \( t \in [0, T] \),

\[
h_1(t) = R_H^{-1} \left( \int_0^T \sigma_1^{-1} \left( \frac{T-u}{T} \nabla b_1(X_u) + \frac{1}{T} \right) v_1 du \right) (t)
\]

(4.5)

and

\[
h_2(t) = R_H^{-1} \left( \int_0^T \sigma_2^*(X_u) du \cdot \left( \int_0^T (\sigma_2^* (X_u)) du \right)^{-1} \right.
\]

\[
\left. \int_0^T \nabla \sigma_2(X_u) v_1 dB^H_u \right).
\]

18
\[ \left( v_2 + \int_0^T \frac{T-u}{T} \nabla b_2(X_u) v_1 du + \int_0^T \frac{T-u}{T} \nabla \sigma_2(X_u) v_1 d\tilde{B}_u^H \right)(t). \]

(4.6)

Following the argument of Theorem 3.3 by (H4) we show that the above \( h \in \Dom \delta \) and then \( \delta(h) = N_T \). Hence, if one has \( (\mathbb{D}_{R,h} X_T, \mathbb{D}_{R,h} Y_T) = (\nabla_v X_T, \nabla_v Y_T) \), then by (3.12) we get the desired assertion.

Indeed, by Lemma 3.1 and the definition of \( h_1 \) defined as (4.5) we have

\[ \mathbb{D}_{R,h} X_t = \int_0^t \nabla b_1(X_u) \mathbb{D}_{R,h} X_u du + \int_0^t \sigma_1(R_h h_1)'(t) du \]

\[ = \int_0^t \nabla b_1(X_u) \mathbb{D}_{R,h} X_u du + \int_0^t \left( \frac{T-u}{T} \nabla b_1(X_u) + \frac{1}{T} \right) v_1 du. \]

This leads to

\[ \mathbb{D}_{R,h} X_t + \frac{T-t}{T} v_1 = v_1 + \int_0^t \nabla b_1(X_u) \left( \mathbb{D}_{R,h} X_u + \frac{T-u}{T} v_1 \right) du. \]

Observe that \( \nabla_v X_t \) satisfies the same equation, i.e.

\[ \nabla_v X_t = v_1 + \int_0^t \nabla b_1(X_u) \nabla_v X_u du. \]

Then by the uniqueness of the ODE we obtain

\[ \mathbb{D}_{R,h} X_t + \frac{T-t}{T} v_1 = \nabla_v X_t, \quad t \in [0, T]. \]

(4.7)

In particular, there holds \( \mathbb{D}_{R,h} X_T = \nabla_v X_T \).

As for \( \mathbb{D}_{R,h} Y_t \) and \( \nabla_v Y_t \), we have

\[ \mathbb{D}_{R,h} Y_t = \int_0^t \nabla b_2(X_u) \mathbb{D}_{R,h} X_u du + \int_0^t \nabla \sigma_2(X_u) \mathbb{D}_{R,h} X_u d\tilde{B}_u^H + \int_0^t \sigma_2(X_u)(R_h h_2)'(u) du \]

and

\[ \nabla_v Y_t = v_2 + \int_0^t \nabla b_2(X_u) \nabla_v X_u du + \int_0^t \nabla \sigma_2(X_u) \nabla_v X_u d\tilde{B}_u^H. \]

Consequently, by (4.7) we get

\[ \mathbb{D}_{R,h} Y_T - \nabla_v Y_T = -v_2 + \int_0^T \nabla b_2(X_u) (\mathbb{D}_{R,h} X_u - \nabla_v X_u) du \]

\[ + \int_0^T \nabla \sigma_2(X_u) (\mathbb{D}_{R,h} X_u - \nabla_v X_u) d\tilde{B}_u^H + \int_0^T \sigma_2(X_u)(R_h h_2)'(u) du \]

\[ = -v_2 - \int_0^T \frac{T-u}{T} \nabla b_2(X_u) v_1 du \]

\[ - \int_0^T \frac{T-u}{T} \nabla \sigma_2(X_u) v_1 d\tilde{B}_u^H + \int_0^T \sigma_2(X_u)(R_h h_2)'(u) du. \]

Therefore, it follows from (4.6) that \( \mathbb{D}_{R,h} Y_T = \nabla_v Y_T \). This completes the proof. \( \square \)
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