Mesoscopic Physics of Granular Flows

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Dynamics of granular materials have recently attracted substantial attention of physicists [1]. Despite the numerous experimental and numerical results, there is still very little established theoretical framework for description of granular flows. One of the dominant approaches is based on the principles of the kinetic theory of liquids [2]. Within this paradigm, it is believed that the dissipation in a granular system does not alter the basic structure of conventional hydrodynamics: the motion is expressed in terms of local velocity, density and (granular) temperature. The dissipation can be accounted for by adding an appropriate dissipative term to the energy and momentum balance equation. Below, we argue that such an approach can hardly be valid for many granular systems, and propose an alternative approach to granular dynamics.

Implicitly, kinetic approach assumes that energy and momentum in the system is transferred through the collisions between particles. In the case of hard-core interactions, these collision events take essentially no time compared to the free flight of the grains. However, this picture does not survive in a general case of a dense dissipative granular system. It is confronted by the numerically-discovered phenomenon known as inelastic collapse [3]: if we start with an arbitrary distribution of grain velocities, the system of inelastic grains undergoes infinite number of collisions within a finite time and a large part of it virtually stops and creates dense clusters.

If the system were sheared, the collapsed structures would provide a skeleton for the solid-like momentum transfer trough inter-grain mechanical contacts. This mechanism is supported by direct observation of force-carrying mesoscopic structures in the sheared granular system [4]. These string-like structures closely resemble the force chains responsible for stress transmission through static granular material [5], and they are obviously inconsistent with the collisional picture for the momentum transfer. Our argument does not rule out the applicability of the kinetic theory to, for example, sufficiently dilute granular systems with weak dissipation, or to granular materials subjected to the bulk pumping of energy.

In this letter, we are interested in physics of steady granular flow in the limit of high dissipation. As it was pointed out, in this case the fluid-like motion appears to be consistent with solid-like mechanism for stress transmission. The fundamental reason for this coexistence is that even if the contact network becomes percolating and can transfer the stress across the system, its connectivity may still be insufficient for mechanical stability (i.e. it may be below the rigidity percolation threshold). Following this observation, we can distinguish between fluid and solid states of granular matter, based on its local connectivity. If the connectivity is high, the granular matter will be “jammed”: a bulk of particles is only able to move as a solid, no motion of granules with respect to each other is possible. If, on the other hand, the connectivity of the network is low, grains can move with respect to each other.

Thus, we can introduce a local “order parameter” \( \Psi \), as a local fraction of particles belonging to a given solid cluster. The motion of particle belonging to the same solid cluster is coherent, which makes the \( (\Psi = 1) \)-phase similar to more conventional correlated states known in condensed matter physics, such as superconducting or superfluid condensates. In the following, it will be more convenient for us to use mobility parameter \( \phi \equiv 1 - \Psi \), which is the local fraction of the fluid component. Similar ideas of characterization of the granular system with the amount of fluidized component have been previously successfully used within BCRE model of surface avalanches [6]. The distinctive feature of our approach is that we are interested in the behavior of the system on the length scale of several beads, rather than in its macroscopic phenomenology.

We limit ourselves to the generic case of planar geometry, in which all principle variables, such as time-averaged velocity or the order parameter, vary only in the direction \( z \) normal to that of motion (\( x \) is directed along the flow). We shall consider two examples: flow of granules down an inclined plane and a shear flow of granular matter in Couette-type experiment. In the first case, the motion occurs under a given local stress, so the principle equations will include forces. In the second
case, the velocity on the inner wall is imposed, which is a
kinematic condition, so the equations are also kinematic.

**Completely fluidized case: \( \phi = 1 \).** We consider the
motion of the granular material down an inclined plane,
whose tilt is large enough for the system to be completely
fluidized. In this state, the force chains transmitting an
external stress are unstable with respect to *buckling*. In
other words, the difference between a static force chain
and one in the flow is that the latter has "weak bonds"
to be broken due to external forcing (see Figure 1).

\[
\begin{align*}
\text{"fluid"} & \quad \stackrel{\rightarrow}{v} \quad \xi \\
\text{"solid"} & \quad \stackrel{\rightarrow}{v}
\end{align*}
\]

**FIG. 1.** Illustration to our discussion of flow mechanisms
in “fluid” (a) and “solid” (b) phases.

The network rearrangement associated with any of the
buckling modes takes a finite time, which can be estimated
by noticing that if \( f \) is the typical force transmitted
by a force chain, the unbalanced torque at a particle
of diameter \( d \) should be of order of \( fd \). Over the buckling
time, this torque should rotate the particle (of mass \( m \))
by an angle of order of unity, until a new contact is
created. If \( \tau_b \) is the buckling time, this gives \( md^2/\tau_b^2 \simeq fd \)
or
\[
\tau_b \simeq \sqrt{\frac{md}{f}}. \quad (1)
\]

The above estimate yields the time scale over which a
single "weak" bond is destroyed. In a steady flow there
is certain equilibrium fraction of the "weak" bonds (their
number is equal to the number of independent "zero mode"
deformations permitted by the contact network).
This fraction can be expressed as \( \alpha \equiv d/\xi \), where \( \xi \)
is the typical length of a stable chain segment between two
unstable ("weak") points. Within our model, these stable
segments move like a solid, while the shear is concen-
trated near the "weak" bonds. Therefore, \( \xi \) can be
interpreted as *coherence length* associated with the order
parameter parameter \( \Psi \). The shear rate \( \partial_z v \), is the time
scale over which any particle will change its neighbors.
We can estimate this rate as \( \alpha/\tau \). Now the shear rate can
be related to the local stress by: \( \partial_z v \simeq \alpha \sqrt{\sigma d/m} \), where
\( \sigma \) is the typical value of the stress. In the spirit of recent
ideas on the constitutive law in granular solids [11], we
assume that the stress tensor is completely determined by
its \( \sigma_{zz} \) and \( \sigma_{xx} \) components (given that \( \sigma_{yy} = 0 \)). Now,
we can rewrite the earlier result in the following form:

\[
|\partial_z v| \frac{\partial_z v}{\partial_z v} = \frac{\sigma_{xx} d^2}{m \xi (\sigma_{xx}/\sigma_{zz})}. \quad (2)
\]

This equation of velocity has to be accompanied by an
equation for the parameter \( \alpha \) or, equivalently, coherence
length \( \xi \). It is a natural assumption that \( \xi \) is a decreas-
ing function of the ratio of the tangential and normal
stresses. In fact, as this stress ratio decreases, the num-
ber of free modes goes to zero, and \( \xi \) should diverge at
certain critical point, \( \sigma_{xx}/\sigma_{zz} = \mu_0 \). After this jamming
transition, the granular system becomes solid, i.e. \( \alpha = 0 \),
and \( \xi = \infty \).

If we start with the solid phase at the jamming point \( \mu_0 \)
and gradually increase the ratio \( \sigma_{xx}/\sigma_{zz} \), e.g. by chang-
ing the slope of the free interface with respect to the
gravity direction, the solid will remain stable until an-
other critical point, \( \mu_1 > \mu_0 \). This means that \( \xi \) is a
two-branch function of the stress ratio: between \( \mu_0 \) and
\( \mu_1 \), \( \xi = \infty \) for \( \phi = 0 \), and \( \xi \) is finite for \( \phi = 1 \). This
non-uniqueness and the associated hysteresis, result in
the stick-slip response to certain class of external driv-
ing, observed experimentally [11].

We now apply (2) to the granular flow induced by an
external bulk force (e.g. gravitational), tilted by an-
gle \( \theta \) with respect to the \( z \)-direction. The breakage
of bonds is dependent on the stress ratio \( \sigma_{xx}/\sigma_{zz} \), which
remains constant everywhere and equals to \( \tan \theta \). Let the
height of the moving granular layer be \( h \), with \( z = 0 \) cor-
responding to its bottom and \( z = h \) being its top. The
component \( \sigma_{xx} \) of the stress is essentially given by the
"hydrostatic" pressure: \( \sigma_{xx} = \rho g (h - z) \sin \theta \). Thus, (2)
results in the following velocity field:

\[
v(z) \simeq \sqrt{\frac{g \sin(\theta)}{\xi (\tan \theta)}} \left( h^{3/2} - (h - z)^{3/2} \right) \quad (3)
\]

We conclude that the typical velocity of a thick developed
avalanche scales as \( h^{3/2} \) with its height, and that the
dependencies of the average flow velocity on the layer
height collapse onto a single master curve

\[
v \simeq \sqrt{gh^{3/2}/\xi}, \quad (4)
\]

for a variety of physical parameters (i.e., slope, friction,
etc.). The scaling law \( v \sim h^{3/2} \) have indeed been ob-
served experimentally by Azanza et al [11]. Moreover,
Pouliquen [12] in recent study has demonstrated the va-
didity of above master relationship, Eq(4). In the ex-
periment, the fundamental length scale \( \xi \) reveals itself as the
minimal layer thickness at which the material flows for
a given slope, which is completely consistent with our
interpretation of \( \xi \). The experiments also support our
observation that \( \xi \) diverges as the slope approaches the
critical value, \( \tan^{-1} \mu_0 \).
Partially–fluidized case ($\phi \simeq 0$). Our further discussion is devoted to the shear dynamics of the solid phase perturbed by a moving wall, as in Couette flow experiment. Since granular system tends to create collapsed solid phase, the shear motion normally occurs only within a mesoscopic layer (several bead diameters wide) near the moving interface. Attempts to construct continuous description may be inadequate on this length scale. That is why we make $z$ a discreet variable, e.g. imagine that the beads near the surface are organized in single-bead-wide layers (which in reality are reasonably well-defined). The layer at $z = 0$ belongs to the moving wall, and has velocity $v_0$; in the limit $z = \infty$ all beads belong to a single coherent solid cluster and have zero velocity.

Consider $n$-th layer. The typical rate with which a particle belonging to it is being hit by a particle at the adjoint layer, $n - 1$, is $v_{n-1}/d$. Any time when this happens, there is a chance that the bead starts moving (gets “excited”). The particle will move by a certain distance $\delta$ (its free path) until being stopped by another particle at the same layer, as shown on Figure 1(b) (one can easily modify the model to account for the possibility of “secondary excitations” to be born). As a result, the average velocity of the $n-th$ layer is

$$v_n = \frac{\sigma(v_{n-1} + v_{n+1})}{d} \langle \delta \rangle_n,$$

Here $\sigma$ is the coupling coefficient which measures the probability for a particle to be moved after being hit by a neighbor. We made sure that $v_{n-1}$ and $v_{n+1}$ appear at the above equation in a symmetric manner. The asymmetry comes in boundary conditions: once the position of the source of the excitations (the moving wall) is specified, the velocities decay very rapidly with $n$ and one of the velocities becomes negligible compared to the other.

The obtained equation does not give a complete description of penetration of the motion inside the solid phase, because nothing was said about the typical interparticle gap $\langle \delta \rangle_n$. It would be reasonable to assume that these gaps make a leading contribution to the free volume of the system, and therefore, $\langle \delta \rangle$ can be related to the deviation of the particle density $\rho_n$ from its random–closed–packing value $\rho_{cp}$:

$$\langle \delta \rangle_n \simeq \frac{\rho_{cp} - \rho_n}{\rho_{cp}}d.$$

Eqs. (3) and (5) provide and opportunity to relate density and velocity profiles obtained experimentally. Note that constant $\langle \delta \rangle_n$ would result in exponential dependence of the velocity. It is more natural to expect the available free volume to decrease with depth $n$, thus resulting in a stronger than exponential velocity decay.

We present one possible way of modeling the depth dependence of $\delta$. We shall call this treatment the Quantized Free Volume approach. It is based on the observation that after a particle jumps and gets stopped by its neighbors, the free volume available to it becomes zero. As a result, we expect the free path parameter $\delta$ to be either zero (for particles in contact with their front neighbors) or of order of $d$ (for movable particles), as shown on Figure 2(b). In the spirit of our earlier discussion, we interpret those movable particle as fluid component, and identify their local fraction with $\phi$. If $\delta_n \sim d$ is a mean free path for the fluid sub-system, then

$$\langle \delta \rangle_n = \delta_n \phi_n,$$

In order to find a missing equation for $\delta$, or $\phi$, we consider an exchange of free volume between adjoint layers. In the limit of $\phi \ll 1$, our assumption that the free volume is quantized, has an important implication. Since inter-layer free volume exchange may occur only near movable particles (free volume quanta), its flux is linear with their concentration. The natural time scale of the volume exchange between $n$-th and $n+1$-th layers is given by $v_{n-1}/d$ (the rate of change of the nearest neighborhood). Hence, the general expression for the corresponding free volume flux is

$$J_{n,n+1} = \frac{v_{n-1}}{d} (\sigma_+ \phi_n - \sigma_- \phi_{n+1}) +$$

Here $v_{n+2}$-term is added in order to make the structure of the model consistent with the inversion symmetry. In reality, after the boundary conditions are chosen, one of the two terms can be neglected, since velocity decays very rapidly with $n$. In the case of shear flow, we can neglect $v_{n+2} \ll v_{n-1}$. In the steady state, the flux given by (8) should vanish. The condition that $J_{n,n+1} = 0$ provides the desired closure for (3). The result is an exponentially-decaying $\phi$–profile:

$$\phi_n = \phi_0 \exp \left(-\frac{n}{\lambda}\right).$$

Here $\lambda = 1/(\log(\sigma_-/\sigma_+))$ is the penetration depth of the fluid component into the solid bulk. We assume $\sigma_+ < \sigma_-$, since otherwise the solution is clearly unphysical. Parameter $\phi_0 \simeq 1$ depends on the detail of interactions between the moving wall an the first layer. Given the solution (9), we can determine velocity field from (8)

$$v_n = V_0 \exp \left[-\frac{1}{2\lambda} \left(n + 1 + \frac{\lambda}{\lambda} \right)\right].$$

Here $\lambda_\ast = 1/(\log(d/\delta_n \phi_0 \sigma))$ is another length scale in the problem. Its variability from one system to another is expected mostly through the coupling coefficient $\sigma$. In the limit of very weak inter-layer coupling $\sigma$ (which corresponds to very smooth particles), $\lambda$ may become much less than $\lambda$, which would result in exponential velocity
profile: $v_\alpha \sim \exp(-n/\lambda_*).$ If the coupling is strong, both lengths $\lambda$ and $\lambda_*$ are expected to be of the same order, thus making the velocity profile Gaussian-shaped. The analytic and numerical results of our model are presented on Figure 2, both for strong and weak coupling regimes.

Existing experiments do show super-exponential decay of the velocity profiles, in some systems it is remarkably close to Gaussian $8,9$. Our model suggests the transition from exponential to Gaussian velocity profile depends on the strength of inter-layer coupling $\sigma$. Note that this result differs from the conclusions of recent Debregeas–Josserand model $10,11$, which attributes the change in velocity profile to the effect of dimensionality.

In conclusion, we have presented a description of granular flows based on the notion that dissipation results in a strong correlation of particles’ motion. An extreme case of such correlated motion is a coherent solid phase ($\phi = 0$). We have constructed a model for the partial surface fluidization of this phase near a moving wall. The basic structure of our model employs some concepts similar to those of the conventional condensed matter theory: the motion is associated with discrete excitations (movable particles) “penetrating” into the coherent solid phase within a mesoscopic surface layer. We made a quantitative prediction relating velocity and density profiles $12,13$, which can be checked experimentally. One more check of our assumptions is possible by studying the velocity distribution function with high temporal resolution. If our approach is valid, one should be able to identify the condensed and fluidized sub-systems with such kind of experiment. In the case of completely fluidized state, $\phi = 1$, the collective behavior reveals itself through existence of short-living force chains. Once again, it is instructive to focus on the mesoscopic scales, set by the typical length of a stable force chain segment (coherence length $\zeta$). Our approach gives several important results, which include $v(h)$ scaling law for thick avalanches, as well as velocity and density profiles in granular Couette–flow experiment. Our conclusions are in a good agreement with existing experiment, and some of them are yet to be tested.

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FIG. 2. Velocity profiles (circles) and $\phi$-field (diamonds) in partially-fluidized system. White symbols correspond to the weak coupling regime ($\lambda_* = 0.4$), and black ones correspond to the strong coupling ($\lambda_* = 4.5$). One can clearly see the crossover from exponential to Gaussian velocity profile. Curves and data points represent analytic and numeric results, respectively. $\lambda = 2$ for all curves.

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