THE LOG-CONCAVITY CONJECTURE FOR THE
DUISTERMAAT-HECKMAN MEASURE REVISITED

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ABSTRACT. Karshon constructed the first counterexample to the
log-concavity conjecture for the Duistermaat-Heckman measure:
a Hamiltonian six manifold whose fixed points set is the disjoint
union of two copies of $T^4$. In this article, for any closed symplectic
four manifold $N$ with $b^+ > 1$, we show that there is a Hamiltonian
six manifold $M$ such that its fixed points set is the disjoint union of
two copies of $N$ and such that its Duistermaat-Heckman function
is not log-concave.

On the other hand, we prove that if there is a torus action of
complexity two such that all the symplectic reduced spaces taken
at regular values satisfy the condition $b^+ = 1$, then its Duistermaat-
Heckman function has to be log-concave. As a consequence, we
prove the log-concavity conjecture for Hamiltonian circle actions
on six manifolds such that the fixed points sets have no four di-

1. INTRODUCTION

Consider the effective Hamiltonian action of a torus $T$ on a 2n-
dimensional connected symplectic manifold $(M, \sigma)$ with a proper
moment map $\Phi : M \to t^*$, where $t = \text{Lie}(T)$. The Duistermaat-
Heckman measure $[DH82]$ on $t^*$ is the push-forward of the Liouville
measure $|\beta|$, the one defined by the symplectic volume form $\frac{1}{n!} \omega^n$,
via the momentum map $\Phi$. The Duistermaat-Heckman measure is
absolutely continuous with respect to the Lebesgue measure, and
its density function, which is well defined once the normalization
of the Lebesgue measure is declared, is said to be the Duistermaat-
Heckman function.

More generally, consider the Hamiltonian action of a compact con-
ected Lie group $G$ on the symplectic manifold $(M, \omega)$ with a proper
moment map $\Phi : M \to g^*$, where $g = \text{Lie}(G)$. Let $T$ be the maximal
torus of $G$ with Lie algebra $t$ and $W$ the Weyl group of $G$. Choose
a $W$-invariant inner product on $t$ so as to identify it with $t^*$. Then we define measure $\nu$ on the positive Weyl chamber $h_+ \subset t^*$ by letting $\frac{1}{|W|} \nu$ be the pushforward of the measure $|\beta|$ via the composition $M \xrightarrow{\Phi} g^* \rightarrow h_+ = g^*/G$.

A measure defined on $\mathbb{R}^k$ is said to be log-concave if it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^k$ and if the logarithm of its density function is a concave function. The log-concavity of the Duistermaat-Heckman measure, and, more generally, of the measure $\nu$ we described in the previous paragraph, has been established for circle actions on four manifolds by Y. Karshon [Ka94, Remark 2.19], for torus actions on compact Kähler manifolds by W. Graham [Gr96], and for compact connected group actions on projective varieties, possibly singular, by A. Okounkov [Ok97], [Ok96], [OK98]. In particular, the result of [Ok97] led V. Ginzburg to conjecture that the Duistermaat-Heckman measure is log-concave for any Hamiltonian torus action on a compact symplectic manifold $M$.

The same conjecture, independently of [Ok96], was proposed by A. Knutson. In view of the above positive results, this conjecture seems very plausible in early nineties. However, motivated by an example of McDuff [MD88], Karshon [Ka96] constructed a Hamiltonian circle action on a compact six manifold for which the Duistermaat-Heckman measure is non-log-concave, which provides the first counterexample. For more background materials and motivations of the log-concavity properties of the Duistermaat-Heckman measure, the interested readers are strongly encouraged to read [OK00] for an excellent expository account.

In a different direction, inspired by [Ka96] and [Yan96], the author [Lin07] constructed the first counterexamples that the Hard Lefschetz property does not survive the symplectic reduction. These examples provide us with an infinite class of six dimensional Hamiltonian circle manifolds which do not admit a Kähler structure. Naturally, the author was led to the question of whether the construction used in [Lin07] can be adapted to produce general examples of non-Kähler Hamiltonian manifolds with non-log-concave Duistermaat-Heckman functions so as to offer a better understanding why the log-concavity property fails in the general symplectic category.

In this article we will present a rather satisfactory answer to the above question. First, we prove that for any closed symplectic four manifold $N$ with $b^+ > 1$, there exists a symplectic six manifold fibred over $N$ such that there is a Hamiltonian $S^1$ action on $M$ for which the Duistermaat-Heckman function is non-log-concave. This
provides us with a huge class of Hamiltonian manifolds with a non-log-concave Duistermaat-Heckman function, since there are many symplectic manifolds which satisfy $b^+ > 1$, c.f., [Gom95] and [PS00]. As an application, we construct simply-connected six dimensional Hamiltonian circle manifolds which satisfy the Hard Lefschetz property and which have a non-log-concave Duistermaat-Heckman function. In particular, this shows that the Hard Lefschetz property, unlike that of invariant Kähler condition, does not imply the log-concavity conjecture.

Second, we give a useful cohomological condition which ensures the Duistermaat-Heckman function of a complexity two Hamiltonian torus action to be log-concave. More precisely, we prove that if the symplectic quotients taken at any regular value have $b^+ = 1$, then the Duistermaat-Heckman function must be log-concave. As a result, we establish the log-concavity conjecture for circle actions on six manifolds such that the fixed points sets have no four dimensional components, or have only four dimensional pieces with $b^+ = 1$.

Indeed, given a circle action on a six manifold which satisfies the above assumptions, when the action is semi-free, i.e., it is free on the complement of the fixed point set, the fact that the symplectic quotients taken at regular values have $b^+ = 1$ can be seen by the following observations. First, applying the equivariant Darboux theorem to an invariant open neighborhood of the minimal critical submanifold, one checks easily that $b^+ = 1$ for symplectic quotients taken at a regular value sufficiently close to the minimum. By the Duistermaat-Heckman theorem [DH82], in the same connected component of the regular values of the moment map, the diffeotype of symplectic quotients does not change. When passing a critical level of the moment map, the diffeotype of symplectic quotients changes by a blow up followed by a blow down [GS89]. Since the symplectic quotients under consideration here are all four dimensional, blowing up along a symplectic submanifold of dimension two does not change the diffeotype, while blowing up at a point gives us an exceptional divisor of self-intersection number $-1$. So $b^+ = 1$ for all symplectic quotients taken at a regular value.

However, when the action is not semi-free, there is a glitch in the above argument since in this case the symplectic quotients taken at regular values are orbifolds, which causes some technical difficulties. One might want to use the results established in [Go00] and compute

\[1\] Details are given in the proof of Theorem 5.3.
the change in $b^+$ when passing a critical value bare-handedly. However, in this paper, we circumvent this by resorting to the wall crossing formula for the signature of symplectic quotients developed by Metzler [Me00], which holds for Hamiltonian torus actions in general.

This paper is organized as follows. Section 2 reviews some basic concepts and results in symplectic geometry to set up the stage. Section 3 proves for any closed symplectic four manifold $N$ with $b^+ > 1$, there exists a Hamiltonian manifold fibred over $N$ such that the Duistermaat-Heckman function is non-log-concave. Section 4 applies these results to construct simply connected examples with the Hard Lefschetz property. Section 5 proves the log-concavity conjecture for Hamiltonian circle actions on six manifolds whose fixed points sets are either of codimension at most two or only having four dimensional components with $b^+ = 1$.

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2. PRELIMINARIES

2.1. Intersection form of $4n$ dimensional symplectic manifolds. For a compact orientable manifold $N$ of dimension $4n$, the intersection form $Q$ on the $2n$-th integer cohomology of $N$ is a symmetric bilinear form defined by:

$$Q : H^{2n}(N, \mathbb{Z}) \times H^{2n}(N, \mathbb{Z}) \to \mathbb{Z}, \quad Q([\alpha], [\beta]) = \langle [\alpha] \cup [\beta], [N] \rangle,$$

where $[N]$ is the fundamental class of the manifold $N$. Using the De Rham model, the corresponding form on the real cohomology can be defined by:

$$Q : H^{2n}(N, \mathbb{R}) \times H^{2n}(N, \mathbb{R}) \to \mathbb{R}, \quad Q([\alpha], [\beta]) = \int_N \alpha \wedge \beta.$$
By the Poincaré duality, this is a non-degenerate symmetric bilinear form. We define $b^+$ and $b^-$ to be the dimensions of maximal positive and negative subspaces of the form, and define the signature of the manifold $N$ to be $\sigma(N) = b^+ - b^-$. Note that when $N$ is a symplectic manifold, we will assume that the orientation on $N$ is the one induced by the symplectic form.

The following simple looking lemma is actually a key point for our construction of Hamiltonian manifolds with non-log-concave Duistermaat-Heckman measure in Theorem 3.1.

**Lemma 2.1.** Let $(N, \omega_0)$ be a compact symplectic four manifold such that $b^+ > 1$ and such that $[\omega_0]$ is a rational cohomology class in $H^2(N)$. And let $Q$ be the intersection form of $(N, \omega_0)$. Then there exists an integral cohomology class $[c] \in H^2(N, \mathbb{R})$ such that

$$Q([c], [c]) > 0 \quad \text{and} \quad Q([c], [\omega_0]) = 0.$$

**Proof.** Write $\alpha_1 = [\omega_0]$. Since $b^+ > 1$ and since $Q(\alpha_1, \alpha_1) > 0$, over the field of rational numbers there exists a basis $\alpha_1, \alpha_2, \cdots, \alpha_r$ of $H^2(N, Q)$ such that

a) if $1 \leq i = j \leq b^+$, then $Q(\alpha_i, \alpha_j) > 0$;

b) if $b^+ < i = j \leq r$, then $Q(\alpha_i, \alpha_j) < 0$;

c) if $i \neq j$, then $Q(\alpha_i, \alpha_j) = 0$.

Choose $[c] = \alpha_2 \in H^2(N, Q)$. Then it is easy to see that $Q([c], [c]) > 0$ and $Q([c], [\omega_0]) = 0$. Replace $[c]$ by $[nc]$ for some appropriate positive integer if necessary, we get an integral class $[c]$ such that the condition (2.1) holds.

The following non-trivial fact was proved by Baldridge [Ba04] using Seiberg-Witten invariants and provides a useful criterion when a symplectic four manifold must have $b^+ = 1$.

**Theorem 2.2.** ([Ba04]) A symplectic 4-manifold which admits a circle action with fixed points must have $b^+ = 1$.

2.2. Duistermaat-Heckman function. Consider the Hamiltonian action of a torus $T$ on a symplectic manifold $(M, \omega)$. Let $a \in t^*$ be a regular value of the moment map $\Phi : M \to t^*$. When the action of $T$ on $M$ is not quasi-free\footnote{An action of a group $G$ on a manifold $M$ is called quasi-free if the stabilizers of the points are connected.}, the quotient $M_a = \Phi^{-1}(a)/T$ taken at a regular value of the moment map is not a smooth manifold in general. However, the singularity is mild and $M_a$ does admit an orbifold
structure in the sense of Satake [S56], [S57]. Orbifolds, although not necessarily smooth, do carry differential structures such as differential forms, fiber bundles, etc. So the usual definition of symplectic structures extends to the orbifold case. In particular, the restriction of the symplectic form $\omega$ to the level set $\Phi^{-1}(a)$ descends to a symplectic form $\omega_a$ on the reduced space $M_a$ [W77]. For our purpose, it is also important to note that any orbifold is a rational homology manifold [Ful93] and does satisfy the Poincaré duality. Thus any orbifold has a well defined signature just as in the manifold case.

The theorems that we are going to state in the rest of this section hold for general torus actions which are not necessarily semisimple. Their statements actually involve some basic orbifold related notions, such as a principal bundle over an orbifold and a diffeomorphism of orbifolds. For basic notions in orbifold theory, we refer to [R01], [CR01] and [ALR2007]. For a modern treatment of orbifolds from the viewpoint of Lie groupoids, we refer to [MO02]. For the foundation of Hamiltonian actions on symplectic orbifolds, the interested reader may consult [LeTo97].

The following Duistermaat and Heckman theorem [DH82] is a fundamental result in symplectic geometry.

**Theorem 2.3.** ([DH82]) Consider the effective Hamiltonian action of a $k$ dimensional torus $\mathbb{T}$ on a connected compact $2n$ dimensional symplectic manifold $M$ with moment map $\Phi: M \to \mathfrak{t}^*$. We have that

a) at a regular value $a \in \mathfrak{t}^*$ of $\Phi$, the Duistermaat-Heckman function $f$ is computed by the following formula:

$$f(a) = \int_{M_a} \frac{\omega_0^{n-k}}{(n-k)!},$$

where $M_a = \Phi^{-1}(a)/\mathbb{T}$ is the symplectic quotient, $\omega_0$ is the corresponding reduced symplectic form, and $M_a$ has been given the orientation of $\omega_0^{n-k}$.

b) if $a, a_0 \in \mathfrak{t}^*$ lie in the same connected component $C$ of the regular values of the moment map $\Phi$, then the reduced space $M_a = \Phi^{-1}(a)/\mathbb{T}$ is diffeomorphic to $M_{a_0} = \Phi^{-1}(a_0)/\mathbb{T}$; furthermore, let $\Gamma$ be the finite subgroup of $\mathbb{T}$ generated by all the finite stabilizer groups $\mathbb{T}_z$, where $z \in \Phi^{-1}(a_0)$, and let $Z_0 = \Phi^{-1}(a_0)/\Gamma$, then using the diffeomorphism $M_a \to M_{a_0}$, the reduced symplectic form on $M_a$ can be identified with

$$\omega_a = \omega_{a_0} + <c, a - a_0>, \quad (2.2)$$
where \( c \in \Omega^2(M, t^*) \) is a closed \( t \)-valued two form which represents the Chern class of the principal torus \( T/\Gamma \)-bundle \( \pi : Z_0 \to M_{ae} \).

By the Atiyah-Guillemin-Sternberg convexity theorem (cf. \cite{A82} and \cite{GS82}), the image of the moment map \( \Delta = \Phi(M) \) is a convex polytope. In fact, \( \Delta \) is a union of subpolytopes with the property that the interiors of the subpolytopes are disjoint and constitute the set of regular values of \( \Phi \). \cite{GLS88} gave an explicit formula computing the jump in the Duistermaat-Heckman function \( f \) across the wall of \( \Phi(M) \). Making use of it, Graham established the following result, c.f., \cite{Gr96} Section 3.

**Proposition 2.4.** (\cite{Gr96}) Suppose \( \Phi : M \to t^* \) is the moment map of the effective Hamiltonian action of torus \( T \) on a connected compact symplectic manifold \( M \). Let \( a \) be a point on a codimension one interior wall of \( \Phi(M) \), and let \( v \in t^* \) be such that the line segment \( \{a + tv\} \) is transverse to the wall. For \( t \) in a small open interval near 0, write \( g(t) = f(a + tv) \), where \( f \) is the Duistermaat-Heckman function. Then we have \( g'_+(0) \leq g'_-(0) \).

**2.3. The wall crossing formula for the signature of symplectic quotients.** Consider the Hamiltonian action of \( S^1 \) on a compact symplectic manifold \( M \) with moment map \( \Phi : M \to \mathbb{R} \). Let \( a < a_1 \) be two points in the image of the moment map such that \( a_0 \) is the unique critical value between \( a \) and \( a_1 \). Let \( X \) be the set of the critical points of the moment map \( \Phi \) which lies inside \( \Phi^{-1}(a_0) \). Then each connected component of \( X \) is a submanifold of \( M \), which we will call critical submanifolds of \( M \). Let \( X_1, X_2, \ldots, X_k \) be all the critical submanifolds sitting inside \( \Phi^{-1}(a_0) \), and let \( E_i \to X_i \) be the symplectic normal bundle of \( X_i \) in \( M \). Then for each \( 1 \leq i \leq k \) the Hessian of \( \Phi \) gives us a splitting

\[
E_i = E_i^+ \oplus E_i^-
\]

of \( E_i \) into a direct sum of positive and negative normal bundles. We denote by \( 2b_i \) and \( 2f_i \) the real dimensions of \( E_i^- \) and \( E_i^+ \) respectively. The following Theorem 2.5 of Metzler \cite{Me00} computes the change in the signature and Poincaré polynomial of symplectic quotients across the critical value \( a_0 \). By the way, given a topological space \( Y \), throughout this paper we will always denote by \( P(Y)(t) \) its Poincaré polynomial.
Theorem 2.5. ([Me00] pp. 3502, 3518) Denote the half rank of the symplectic normal bundle $E_i$ by $q_i$. Then

$$\sigma(M_{a_1}) - \sigma(M_a) = \sum_{1 \leq i \leq k, q_i \text{ odd}} (-1)^{b_i} \sigma(X_i),$$

$$P(M_{a_1})(t) - P(M_a)(t) = \sum_{i=1}^{k} P(X_i)(t) \frac{t^{2b_i} - t^{2f_i}}{1 - t^2},$$

where $M_{a_1}$ and $M_a$ denote the symplectic quotients of the Hamiltonian $S^1$ action taken at $a_1$ and $a$ respectively.

We will also need the following result [Me00] Thm. 2.8 which is the orbifold version of a result of Chern, Hirzebruch, and Serre [CHS57].

Theorem 2.6. Let $P \to B$ be a fibre bundle over $B$ with fibre $F$ such that

1) $P, B, F$ are compact connected oriented orbifolds;
2) the structure group of $P$ is compact and connected.

If $P, B, F$ are oriented coherently, then $\sigma(P) = \sigma(B)\sigma(F)$.

3. Main construction

A measure defined on $\mathbb{R}^k$ is said to be strictly non-log-concave if it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^k$ and if the logarithm of its density function is a strictly convex function.

Theorem 3.1. Assume that $N$ is a closed symplectic four manifold with $b^+ > 1$, then there exists a sphere bundle $\pi : M \to N$ such that there is a symplectic form $\omega$ on $M$ and a Hamiltonian $S^1$ action on $(M, \omega)$ for which the Duistermaat-Heckman measure is strictly non-log-concave.

Proof. Note that any closed symplectic manifold admits another symplectic form whose cohomology class is integral, see for e.g. [Gom95, pp.561]. Without the loss of generality, henceforth we will assume that $N$ is equipped with a symplectic form $\omega_0$ such that $[\omega_0]$ lies in the image of $H^2(N, \mathbb{Z})$ in $H^2(N, \mathbb{R})$.

By Lemma 2.1, there exists an integral class $[c] \in H^2(N, \mathbb{R})$ which satisfies the condition 2.1. Let $\pi_P : P \to N$ be the principal $S^1$ bundle with Euler class $[c]$, let $\Theta$ be the connection 1-form such that $d\Theta = \pi_P^* c$, let $S^1$ act on $S^2$ by rotation and let $M$ be the associated

\[3\] Strictly speaking, the density function is well defined only if we declare the normalization of the Lebesgue measure.
bundle \( P \times_{S^1} S^2 \). Note that the action of \( S^1 \) on \( S^2 \) preserves the standard symplectic form, i.e., the area form, on \( S^2 \), and is Hamiltonian. Indeed, in cylindrical polar coordinates \((\theta, h)\) away from the poles, \(0 \leq \theta < 2\pi, -1 \leq h \leq 1\), the area form \( \sigma \) on \( S^2 \) can be written as \( d\theta \wedge dh \) and the moment map for the rotating action of \( S^1 \) is just the height function \( \mu = h \). Since \( S^1 \) is the structure group, \( \pi : M \to N \) is a symplectic fibration over the compact symplectic four-manifold \( N \). The symplectic form \( \sigma \) on \( S^2 \) gives rise to a symplectic form \( \sigma_x \) on each fibre \( \pi^{-1}(x), x \in N \); moreover, the \( S^1 \)-action on \( S^2 \) induces a fibrewise \( S^1 \) action on \( M \).

Next, we resort to minimal coupling construction to get a closed two form \( \eta \) on \( M \) which restricts to the forms \( \sigma_x \) on the fibres. Let us give a sketch of this construction here and refer to [W78] and [GS84] for technical details.

Consider the closed two form \( -d(x\Theta) = -xd\Theta - dx \wedge \Theta \) defined on \( P \times \mathbb{R} \), where \( x \) is the linear coordinate on \( \mathbb{R} \). It is easy to see the \( S^1 \) action on \( P \times \mathbb{R} \) given by \( \lambda(p, x) = (\lambda p, x) \) is Hamiltonian with moment map \( x \). Thus the diagonal action of \( S^1 \) on \( (P \times \mathbb{R}) \times S^2 \) is also Hamiltonian, and \( M \) is just the reduced space of \( (P \times \mathbb{R}) \times S^2 \) at the zero level. Moreover, the closed two form \( (-d(x\Theta) + \sigma) \big|_{\text{zero level}} \) descends to a closed two form \( \eta \) on \( M \) with the desired property; whereas \( x \big|_{\text{zero level}} \) descends to a globally defined function \( H \) on \( M \) whose restriction to each fibre \( S^2 \) is just the height function \( h \).

It is useful to have the following explicit description of \( \eta \). Observe that \( d\theta - \Theta \) is a basic form on \( (P \times \mathbb{R}) \times S^2 \). Its restriction to the zero level of \( (P \times \mathbb{R}) \times S^2 \) descends to a one form \( \tilde{\theta} \) on \( M \) whose restriction to each fibre \( S^2 \) is just \( d\theta \). It is easy to see that on the associated bundle \( P \times_{S^1} (S^2 - \{\text{two poles}\}) \) we actually have \( \eta = H\pi_M^*\omega + \tilde{\theta} \wedge dH \).

Note that the restriction of \( \eta \) to each fiber are symplectic forms \( \sigma_x \). Thus by a famous argument due to Thurston [MS98], for sufficiently small constant \( \epsilon > 0 \) the form \( \pi^*_M(\omega_0 + \epsilon \eta) \) is symplectic. Without the loss of generality, we will assume that \( \epsilon \) is so small that

\[
Q([\omega_0], [\omega_0]) > \epsilon^2 Q([c], [c]).
\]

Having chosen such a symplectic form \( \pi^*\omega_0 + \epsilon \eta \) on \( M \), a simple calculation shows that the fibrewise \( S^1 \)-action on \( (M, \omega) \) is Hamiltonian with the moment map \( t := \epsilon H : M \to \mathbb{R} \).

Now let us compute the Duistermaat-Heckman function \( f \). Observe that at the level set \( -\epsilon < t < \epsilon \), the symplectic quotient is just
N with the reduced symplectic form $\omega_0 + tc$. Therefore by Proposition 2.3 the Duistermaat-Heckman function

$$f(t) = \frac{1}{N} \int_N (\omega_0 + tc)^2$$

$$= \frac{1}{2} \left( Q([c], [c]) t^2 + 2Q([c], [\omega_0]) t + Q([\omega_0], [\omega_0]) \right)$$

First note that $\ln f$ is strictly non-log-concave if and only if $f''f - (f')^2 > 0$. However, a simple calculation shows that

$$2 \left( f''f - (f')^2 \right) = \left( Q([c], [c]) Q([\omega_0], [\omega_0]) - 2Q^2([c], [\omega_0]) \right)$$

$$- \left( Q^2([c], [c]) t^2 + 2Q([c], [c]) Q([c], [\omega_0]) t \right).$$

Since by construction $[c]$ satisfies the condition (2.1), we have

$$2 \left( f''f - (f')^2 \right) = Q([c], [c]) Q([\omega_0], [\omega_0]) - Q^2([c], [c]) t^2.$$  

Now it follows easily from the inequality (3.1) and $-\epsilon < t < \epsilon$ that $f''f - (f')^2 > 0$. Hence the Duistermaat-Heckman function $f$ is strictly non-log-concave. This finishes the proof of Theorem 3.1.

□

**Remark 3.2.** For instance, K3 surfaces and complex tori are Kähler surfaces with $b^+ = 3$. Applying Theorem 3.1 to the case $N = T^4$, we recover Karshon’s example of a non-log-concave Duistermaat-Heckman measure. In the general symplectic category, we note that there are many examples of symplectic four manifolds with $b^+ > 1$, c.f., [Com95] and [PS00]. Thus Theorem 3.1 provides us with a large family of Hamiltonian manifolds with a non-log concave Duistermaat-Heckman measure.

4. SIMPLY CONNECTED EXAMPLES WITH THE HARD LEFSCHETZ PROPERTY

A compact symplectic manifold $(M, \omega)$ of dimension $2m$ is said to have the Hard Lefschetz property or equivalently to be a Lefschetz manifold if and only if for any $0 \leq k \leq m$, the Lefschetz type map

$$L^k_{[\omega]} : H^{m-k}(M, \mathbb{R}) \to H^{m+k}(M, \mathbb{R}), \ [\alpha] \to [\alpha \wedge \omega^k]$$

is an isomorphism.

The follow result allows us to construct simply connected Hamiltonian manifolds with the Hard Lefschetz property which have a strictly non-log-concave Duistermaat-Heckman measure.
Theorem 4.1. Assume that \( N \) is a closed symplectic four manifold which satisfies the Hard Lefschetz property and \( b^+ > 1 \). Then there exists a sphere bundle \( \pi : M \to N \) such that there is a symplectic form \( \omega \) on \( M \) and a Hamiltonian \( S^1 \) action on \( (M, \omega) \) for which the Duistermaat-Heckman function is strictly non-log-concave; moreover, \( (M, \omega) \) is a compact symplectic manifold which satisfies the Hard Lefschetz property.

Proof. Let \( (M, \omega) \) be the symplectic six manifold constructed in the proof of Theorem 3.1. It suffices to show that in the proof of Theorem 3.1 for sufficiently small constant \( \epsilon > 0 \), \((M, \pi^* \omega_0 + \epsilon \eta)\) satisfies the Hard Lefschetz property. The proof is very similar to the one given in [Lin07, Prop. 4.2], though the assumption here is slightly different from the one used in [Lin07, Prop. 4.2].

Note that the restriction of the cohomology class of \( \eta \) to each fibre \( S^2 \) generates the second cohomology group \( H^2(S^2, \mathbb{R}) \). By the Leray-Hirsch theorem (see e.g., [BT82, Thm. 5.11]), the pullback map \( \pi^* : H^*(N) \to H^*(M) \) embeds \( H^*(N) \) into \( H^*(M) \), and additively \( H^*(M, \mathbb{R}) \) is a free module generated by 1 and \([\eta]\). It follows that
\[
[\eta^2] = [\pi^* \beta_2 \wedge \eta] + [\pi^* \beta_4]
\]
for some closed forms \( \beta_2 \) and \( \beta_4 \) on \( N \) of degree two and four respectively.

Choose an \( \epsilon > 0 \) which is sufficiently small such that the determinant of the linear map \( L_{2\omega_0 + \epsilon\beta_2} : H^1(N, \mathbb{R}) \to H^3(N, \mathbb{R}) \) is non-zero and such that
\[
[\omega_0]^2 \neq -\epsilon^2[\beta_4] + \epsilon[\omega_0 \wedge \beta_2]. \tag{4.2}
\]

We claim for the \( \epsilon \) chosen above, the symplectic manifold \((M, \pi^* \omega_0 + \epsilon \eta)\) satisfy the Hard Lefschetz property. By the Poincaré duality it suffices to show the two Lefschetz maps
\[
L^2_{[\omega]} : H^1(M, \mathbb{R}) \to H^5(M, \mathbb{R}) \tag{4.3}
\]
and
\[
L_{[\omega]} : H^2(M, \mathbb{R}) \to H^4(M, \mathbb{R}) \tag{4.4}
\]
are injective. We will give a proof in two steps below.

(i) Since by the Leray-Hirsch theorem \( H^1(N) \xrightarrow{\pi^*} H^1(M) \), to show Map (4.3) is injective we need only to show that for any \([\lambda] \in H^1(N, \mathbb{R})\) if \( L^2_{[\omega]}(\pi^*[\lambda]) = 0 \), then \([\lambda] = 0\). A straightforward calculation shows that
\[
0 = L^2_{[\omega]}(\pi^*[\lambda]) = \pi^*(2\epsilon[\omega_0] + \epsilon^2[\beta_2]) \wedge [\pi^*\lambda] \wedge [\eta]. \tag{4.5}
\]
Since $H^*(M)$ is free over 1 and $[\eta]$, $0 = \pi^* \left( ([2\epsilon \omega_0 + \epsilon^2 \beta_2]) \wedge [\lambda] \right)$. By our choice of $\epsilon$, the determinant of the linear map $L_{[2\epsilon \omega_0 + \epsilon^2 \beta_2]} : H^1(N) \to H^3(N, \mathbb{R})$ is non-zero and so we have $[\lambda] = 0$.

(ii) To show that Map (4.4) is injective it suffices to show that if $L_{\omega}(\pi^* [\varphi] + k[\eta]) = 0$ for arbitrarily chosen scalar $k$ and second cohomology class $[\varphi] \in H^2(N, \mathbb{R})$, then we have $[\varphi] = 0$ and $k = 0$. Since $\omega = \pi^* \omega_0 + \epsilon \eta$ and $[\eta^2] = [\pi^* \beta_2 \wedge \eta] + [\pi^* \beta_4]$, we have

\begin{equation}
0 = L_{\omega}(\pi^* [\varphi] + k[\eta])
= (\pi^* [\omega_0 \wedge \varphi] + \epsilon k \pi^* [\beta_4]) + (k \pi^* [\omega_0] + \epsilon \pi^* [\varphi] + \epsilon k \pi^* [\beta_2]) \wedge \eta
\end{equation}

Since $H(M)$ is a free module over 1 and $[\eta]$, we have that

\begin{equation}
\pi^* [\omega_0 \wedge \varphi] + \epsilon k \pi^* [\beta_4] = 0
\end{equation}

\begin{equation}
k \pi^* [\omega_0] + \epsilon \pi^* [\varphi] + \epsilon k \pi^* [\beta_2] = 0
\end{equation}

If $k = 0$, it follows easily from Equation (4.8) that $[\varphi] = 0$. Assume $k \neq 0$, substitute $\pi^* [\varphi] = -\frac{1}{\epsilon} k \pi^* [\omega_0] - k \pi^* [\beta_2]$ into Equation (4.7). As a result,

\begin{align*}
\pi^* [\omega_0] \wedge ( - k \pi^* [\omega_0] - \epsilon k \pi^* [\beta_2] ) + \epsilon^2 k \pi^* [\beta_4] = 0
\end{align*}

For $k \neq 0$, we get

\begin{align*}
\pi^* ([\omega_0]^2) = -\epsilon^2 \pi^* [\beta_4] + \epsilon \pi^* ([\omega_0] \wedge \beta_2),
\end{align*}

which clearly contradicts Equation (4.2).

Example 4.2. Choose any simply connected compact symplectic four manifold $N$ such that $b^+ > 1$. (Examples of such symplectic manifolds are abundant. For instance, choose $N$ to be $3\mathbb{CP}^2 \# 19 \mathbb{C}P^2$, c.f., [Gom95].) Note that by the Poincaré duality any simply connected compact symplectic four manifold satisfies the Hard Lefschetz property. Applying Theorem 4.1 we get a six dimensional Hamiltonian $S^1$ manifold $(M, \omega)$ which satisfies the Hard Lefschetz property and which has a strictly non-log-concave Duistermaat-Heckman function. It then follows easily from the long exact sequence of homotopy groups for an $S^2$ fibration that $M$ is simply connected as well. The Hamiltonian manifold $(M, \omega)$ does not admit an $S^1$ invariant Kähler structure since its Duistermaat-Heckman function is non-log-concave, c.f., [Gr96].
5. THE LOG-CONCAVITY FOR TORUS ACTIONS OF COMPLEXITY TWO

The Hamiltonian action of a k-dimensional torus on a 2n-dimensional symplectic manifold is said to be of complexity two if \( n - k = 2 \). Theorem 5.1 gives a useful criterion to ensure the log concavity of the Duistermaat-Heckman measure for a Hamiltonian torus action of complexity two.

**Theorem 5.1.** Assume that the action of a torus \( T \) on a connected compact symplectic manifold \( M \) is an effective Hamiltonian action of complexity two with moment map \( \Phi : M \to t^* \). And assume that for any regular value \( \xi \in t^* \) of \( \Phi \), the symplectic reduced space \( M_\xi = \Phi^{-1}(\xi)/T \) has that \( b^+ = 1 \). Then the Duistermaat-Heckman function \( f \) is log-concave.

**Proof.** In view of Proposition 2.4 to establish the log concavity of \( f \) on \( \Phi(M) \), it suffices to show that the restriction of \( \ln f \) to each connected component of the set of regular values of \( \Phi \) is concave. Let \( C \) be such a component, let \( v \in t^* \), and let \( \{a + tv\} \) be a line segment in \( C \) passing through a point \( a \in C \), where the parameter \( t \) lies in some small interval containing 0. We need to show that \( g(t) := f(a + tv) \) is log-concave, or equivalently, \( g''g - (g')^2 \leq 0 \).

It follows from Theorem 2.3 that the Duistermaat-Heckman function \( f \) is computed by

\[
\begin{align*}
f(a + tv) &= \frac{1}{2} \int_{M_a} (\omega_a + tc)^2 \\
&= \frac{1}{2} \left( Q([c], [c])t^2 + 2Q([c], [\omega_a])t + Q([\omega_a], [\omega_a]) \right),
\end{align*}
\]

(5.1)

where \( M_a = \Phi^{-1}(a)/T \) is the reduced space at \( a \in C \), \( \omega_a \) is the reduced symplectic form on it, and \( c \in \Omega^2(M_a) \) is a closed two form depending only on \( v \) in \( C \). Consequently,

\[
2(g''g - (g')^2) = (Q([c], [c])Q([\omega_a], [\omega_a]) - 2Q^2([c], [\omega_a])) - (Q^2([c], [c])t^2 + 2Q([c], [c])Q([c], [\omega_a])t)
\]

Since \( b^+(M_a) = 1 \), there exists a real basis \( \alpha_1, \alpha_2, \ldots, \alpha_k \) of \( H^2(N, \mathbb{R}) \) such that \( [\omega_a] = r\alpha_1 \) for some positive constant \( r \) and such that

\[
Q(\alpha_i, \alpha_j) = \begin{cases} 
1, & \text{if } i = j = 1 \\
-1, & \text{if } 2 \leq i = j \leq k \\
0, & \text{otherwise.}
\end{cases}
\]
Write $[c] = \sum_{i=1}^{k} \lambda_i \alpha_i$ for some real scalars $\lambda_i$. Then we have

$$2(g'' g - (g')^2) = -((\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)t^2 + 2(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)\lambda_1 rt) + ((\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)t^2 - 2\lambda_1^2 r^2)$$

$$= -(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)t^2 - 2(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)\lambda_1 rt - (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_k^2) r^2.$$  

If the leading coefficient $-(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)^2$ of the above polynomial equals zero, then obviously we have $g'' g - (g')^2 \leq 0$. Otherwise, $2(g'' g - (g')^2)$ is a quadratic polynomial with a negative leading coefficient. Furthermore, the discriminant of this quadratic polynomial is

$$\Delta = 4(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)^2 \lambda_1^2 r^2 - 4(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_k^2)r^2$$

$$= 4(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)^2 (\lambda_1^2 r^2 - (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_k^2)r^2)$$

$$= -4(\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2)^2 (\lambda_2^2 + \cdots + \lambda_k^2)r^2$$

which is clearly non-positive. Thus $2(g'' g - (g')^2)$ has to be negative for all $t \in I$. This finishes the proof of the theorem.

\[\square\]

Theorem 5.3 and Theorem 5.5 below give a rather satisfactory answer to the question when the log-concavity conjecture holds for $S^1$ actions on six manifolds. To prove them, we need to establish the following key lemma.

**Lemma 5.2.** Suppose the action of $S^1$ on a connected compact symplectic six manifold $M$ is Hamiltonian. Let $\Phi : M \to \mathbb{R}$ be the moment map, and let $M_\alpha := \Phi^{-1}(\alpha)/S^1$ be the symplectic quotient taken at the regular value $\alpha$ of $\Phi$. Then $b^+(M_\alpha)$ remains constant as $\alpha$ runs through all the regular values of $\Phi$.

**Proof.** Let $a_0 = \min < a_1 < \cdots < a_k = \max$ be all the critical values of $\Phi$.

By Theorem 2.3 the diffeotype of $M_\alpha$ remains unchanged on each open interval $(a_{i-1}, a_i)$, $1 \leq i \leq k$. Thus $b^+(M_\alpha)$ is a constant on each open interval $(a_{i-1}, a_i)$. Next we note that for dimension reasons, if a critical submanifold $X$ is neither the minimum nor the maximum submanifold, the signature of the Hessian of $\Phi$ at $X$ can only be of the form $(2, 2p)$ or $(2, 2q)$ for some integers $p, q > 0$. It follows that any such critical submanifold can be of dimension at most 2. By the way, when the signature of the Hessian of $\Phi$ at a critical submanifold
X is of the form \((2p, 2q)\), we will say that the critical submanifold \(X\) is of type \((2p, 2q)\).

Now let \(X_i\) be all the critical submanifolds sitting inside \(\Phi^{-1}(a_i)\), and let \((2f_i, 2b_i)\) be the signature of the Hessian of \(\Phi\) at \(X_i\). Then by Theorem 2.5, the change in the signature of symplectic quotients when passing the critical value \(a_i\) is computed by

\[
(5.2) \quad \sum_{1 \leq i \leq k, q_i \text{ odd}} (-1)^{b_i} \sigma(X_i),
\]

while the change in the Poincaré polynomial is computed by

\[
(5.3) \quad \sum_{i=1}^{k} P(X_i)(t) \frac{t^{2b_i} - t^{2f_i}}{1 - t^2}.
\]

Note that if the dimension of \(X_i\) is two, then the signature of the Hessian of \(\Phi\) at \(X_i\) is \((2, 2)\) and \(\sigma(X_i) = 0\). So \(X_i\) does not have any contribution in either Equation (5.2) or Equation (5.3).

Let \(N_1\) be the number of the type \((2, 4)\) isolated fixed points sitting inside \(\Phi^{-1}(a_i)\), and let \(N_2\) be the number of the type \((4, 2)\) isolated fixed points sitting inside \(\Phi^{-1}(a_i)\). Then when passing the critical value \(a_i\), the change in the signature of symplectic quotients equals \(N_1 - N_2\), whereas the change in the second Betti number equals \(N_2 - N_1\). Therefore the change in the sum \(\sigma + b_2\) is null. Note that for any four manifold \(b^+ = \frac{1}{2}(\sigma + b_2)\). So the change in \(b^+(M_a)\) when \(a\) passes through the critical level \(a_i\) is also null. This finishes the proof of the lemma.

\[\square\]

**Theorem 5.3.** Let \((M, \omega)\) be a compact connected symplectic six manifold equipped with a Hamiltonian \(S^1\) action whose fixed points set has codimension greater than or equal to four. Then the Duistermaat-Heckman function of \(M\) is log-concave.

**Proof.** Let \(\Phi\) be the moment map of the \(S^1\) action on \(M\) such that \(a_0 \in \mathbb{R}\) is the minimum value, let \(F\) be the unique local minimum fixed points submanifold in \(\Phi^{-1}(a_0)\) which is of codimension \(k\), and let \(E\) be the symplectic normal bundle of \(F\) in \(M\). Choose an \(S^1\) invariant Hermitian inner product on \(E\) such that \(E\) becomes a Hermitian vector bundle. Denote by \(P\) the principal \(U(k)\)-bundle, i.e., the unitary frame bundle, associated to \(E\) and choose a connection on it.

\(^4\)The standard terminology is to say that the critical submanifold is of signature \((2p, 2q)\). Because in our paper the word “signature” has been reserved to refer to something else, we use the word “type” here to avoid the confusion.
This gives a projection map $TP \to VP$, where $TP$ is the tangent bundle of $P$ and $VP$ is the bundle of vertical tangent vectors. Dually, we have an embedding $i : V^*P \to T^*P$. Let $\omega_P$ be the standard symplectic form on the cotangent bundle $T^*P$. Then the $U(n)$ action on $P$ lifts to an action on $V^*P$ which is Hamiltonian with respect to the two form $i^*\omega_P$ on $V^*P$.

Consider the diagonal Hamiltonian action of $U(k)$ on $V^*F \times C^k$ and perform reduction at the zero level. Then we get a closed two form which is non-degenerate on a tubular neighborhood $E_\delta$ of $F$. Since the standard $S^1$ action on $C^n$ commutes with the $U(k)$-action, it descends to a Hamiltonian action on $E_\delta$. By the equivariant Darboux theorem, we can identify the above Hamiltonian $S^1$ manifold $E_\delta$ with an $S^1$ invariant open neighborhood of $F$ in $M$. Then by a reduction by stage argument, it is easy to see that the for any $a > a_0$ sufficiently close to the minimum value $a_0$, as a topological space the symplectic reduced space $M_a = \Phi^{-1}(a)/S^1$ is just a weighted $CP^k$-bundle over $F$. Indeed, when $k = 6$, topologically $M_a$ is a weighted projective space $CP^2$, and when $k = 4$, is a weighted $CP^1$-bundle over the surface $F$. We claim that in both cases $b^+ = 1$. In the former case, since the rational cohomology of the weighted projective space $CP^2$ is isomorphic to that of the ordinary projective space $CP^2$ (c.f., [Me00, pp. 3500]) , we have that $b^+ = 1$. In the latter case, the restriction of the reduced symplectic form $\omega_a$ on $M_a$ to each fiber, a weighted projective space $CP^1$, generates its second cohomology which is one dimensional. So it follows easily from the Leray-Hirsch theorem\(^5\) that $H^2(M_a, \mathbb{R})$ is two dimensional. Beside, it is easy to see from Theorem 2.6 that the signature of $M_a$ is zero. Therefore we have $b^+(M_a) = 1$. Applying Lemma 5.2 we have that all the symplectic quotients taken at regular values of $\Phi$ satisfy $b^+ = 1$. By Theorem 5.1 the Duistermaat-Heckman function of $M$ has to be log-concave. This finishes the proof of the theorem.

\(^5\)The spectral sequence argument given in [BT82 pp.170] can be easily adapted to show that the Leray-Hirsch theorem does extend to this case.
Theorem 5.5. Let $M$ be a compact connected symplectic six manifold equipped with a Hamiltonian $S^1$ action whose fixed points set has components of dimension four. Then we have that

a) there are only two such components of dimension four: the unique minimum submanifold and the unique maximum submanifold;
b) $b^+(\text{minimum}) = b^+(\text{maximum})$.

If in addition, we assume $b^+(\text{minimum}) = b^+(\text{maximum}) = 1$, then the Duistermaat-Heckman function of the Hamiltonian manifold $M$ is log-concave.

Proof. It is a well known result that the any level set of the moment map is connected, c.f., [A82]. In particular, $M$ has a unique local minimum and a unique local maximum. Thus if a critical submanifold $F$ is neither minimum nor maximum, then it must be of signature $(2p, 2q)$ for some integers $p, q > 0$. Now Assertion (a) follows easily from this observation. Next using the equivariant Darboux theorem, it is easy to see that for a regular value $a$ sufficiently close to the minimum value of the moment map $\Phi$, as a topological space the symplectic quotient $M_a = \Phi^{-1}(a)/S^1$ can be identified with the minimum submanifold. Then it follows from Lemma 5.2 that $b^+(M_a) = b^+(\text{minimum}) = b^+(\text{maximum})$ for any regular value $a$ of $\Phi$. This proves Assertion (b). The last assertion in the theorem now follows easily from Theorem 5.1. □

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