On Estimation and Inference of Large Approximate Dynamic Factor Models via Principal Component Analysis

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This version: July 20, 2023

Abstract

We provide an alternative derivation of the asymptotic results for the Principal Components estimator of a large approximate factor model. Results are derived under a minimal set of assumptions and, in particular, we require only the existence of 4th order moments. A special focus is given to the time series setting, a case considered in almost all recent econometric applications of factor models. Hence, estimation is based on the classical $n \times n$ sample covariance matrix and not on a $T \times T$ covariance matrix often considered in the literature. Indeed, despite the two approaches being asymptotically equivalent, the former is more coherent with a time series setting and it immediately allows us to write more intuitive asymptotic expansions for the Principal Component estimators showing that they are equivalent to OLS as long as $\sqrt{n}/T \to 0$ and $\sqrt{T}/n \to 0$, that is the loadings are estimated in a time series regression as if the factors were known, while the factors are estimated in a cross-sectional regression as if the loadings were known. Finally, we give some alternative sets of primitive sufficient conditions for mean-squared consistency of the sample covariance matrix of the factors, of the idiosyncratic components, and of the observed time series, which is the starting point for Principal Component Analysis.

Keywords: Approximate Dynamic Factor Model; Principal Component Analysis; Inference

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1 Model and Assumptions

We adopt the following approach taken from time series analysis. Frst, we define the model for an infinite dimensional process \( \{x_{it}, i \in \mathbb{N}, t \in \mathbb{Z}\} \). Second, the properties and the assumptions of the model are defined for the \( n \)-dimensional sub-process \( \{x_{it}, i = 1, \ldots, n, t \in \mathbb{Z}\} \) in the limit \( n \to \infty \). Third, the properties of the estimators are derived for a given \( nT \)-dimensional realization \( \{x_{it}, i = 1, \ldots, n, t = 1, \ldots, T\} \) in the limit \( n, T \to \infty \).

We define a factor model driven by \( r \) factors as

\[
x_{it} = \lambda_i' F_t + \xi_{it}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}.
\]

where \( \lambda_i = (\lambda_{i1} \cdots \lambda_{ir})' \) and \( F_t = (F_{t1} \cdots F_{tr})' \) are the \( r \)-dimensional vectors of loadings for series \( i \) and factors, respectively. We call \( \xi_{it} \) the idiosyncratic component and \( \chi_{it} = \lambda_i' F_t \) the common component.

Model (1) in vector notation reads

\[
x_t = \Lambda F_t + \xi_t, \quad t \in \mathbb{Z},
\]

where \( x_t = (x_{1t} \cdots x_{nt})' \) and \( \xi_t = (\xi_{1t} \cdots \xi_{nt})' \) are \( n \)-dimensional vectors of observables and idiosyncratic components, respectively, and \( \Lambda = (\Lambda_1 \cdots \Lambda_n)' \) is the \( n \times r \) matrix of factor loadings. We call \( \chi_t = \Lambda F_t \) the vector of common components.

Given an observed sample of \( T \) observations, then we can write model (1) also in matrix notation as

\[
X = F \Lambda' + \Xi,
\]

where \( X = (x_1 \cdots x_T)' \) and \( \Xi = (\xi_1 \cdots \xi_T)' \) are \( T \times n \) matrices of observables and idiosyncratic components, and \( F = (F_1 \cdots F_T)' \) is the \( T \times r \) matrix of factors. This latter notation is used for estimation mainly but not for describing the model in population where we deal with stochastic processes and so \( t \in \mathbb{Z} \).

We characterize the common component by means of the following assumption.

**Assumption 1 (COMMON COMPONENT).**

(a) \( \lim_{n \to \infty} ||n^{-1} \Lambda' \Lambda - 1 || = 0 \), where \( \Sigma_\Lambda \) is \( r \times r \) positive deinite, and, for all \( i \in \mathbb{N}, ||\lambda_i|| \leq M_\Lambda \) for some fnite positive real \( M_\Lambda \) independent of \( i \).

(b) For all \( t \in \mathbb{Z}, \mathbb{E}[F_t] = 0_r \) and \( \Gamma^F = \mathbb{E}[F_t F_t'] \) is \( r \times r \) positive deinite and \( ||\Gamma^F|| \leq M_F \) for some fnite positive real \( M_F \) independent of \( t \).

(c) (i) For all \( t \in \mathbb{Z}, \mathbb{E}[||F_t||^4] \leq K_F \) for some fnite positive real \( K_F \) independent of \( t \);

(ii) \( \mathfrak{p}\lim_{n \to \infty} ||T^{-1} F_t F_t' - \Gamma^F|| = 0 \).

(d) There exists an integer \( N \) such that for all \( n > N \), \( r \) is a fnite positive integer, independent of \( n \).

Parts (a) and (b), are also assumed in Bai (2003, Assumptions A and B) for PC estimation. They imply that, the loadings matrix has asymptotically maximum column rank \( r \) (part (a)) and the factors have a fnite full-rank covariance matrix (part (b)). Moreover, because of part (a), for any given \( n \in \mathbb{N} \), all the factors have a fnite contribution to each series (upper bound on ||\( \lambda_i || ||). Because of Assumption 1(a) we have that the eigenvalues of \( \frac{\Lambda_i \Lambda_i}{n} \), denoted as \( \mu_j(n^{-1} \Lambda' \Lambda) \), \( j = 1, \ldots, r \) are such that (see also
Lemma 3(iv))

\[ m_{\Lambda}^2 \leq \mu_j\left(\frac{\Lambda'\Lambda}{n}\right) \leq M_{\Lambda}^2, \]

for some finite positive real \( m_{\Lambda} \) independent of \( n \) and where also \( M_{\Lambda} \) is independent of \( n \). Therefore, the eigenvalues of \( \Sigma_{\Lambda} \), denoted as \( \mu_j(\Sigma_{\Lambda}) \), \( j = 1, \ldots, r \) are such that:

\[ m_{\Lambda}^2 \leq \mu_j(\Sigma_{\Lambda}) \leq M_{\Lambda}^2. \] (2)

Similarly, because of Assumption 1(b) we have that the eigenvalues of \( F'TF \), denoted as \( \mu_j\left(\frac{F'F}{T}\right) \), \( j = 1, \ldots, r \) are such that (this is because of stationarity, see also Lemma 3(v))

\[ m_{F} \leq \mu_j\left(\frac{F'F}{T}\right) \leq M_{F}, \]

for some finite positive real \( m_{F} \) independent of \( T \) and where also \( M_{F} \) is independent of \( T \). Therefore, because of Assumption 1(c), the eigenvalues of \( \Gamma F \), denoted as \( \mu_j(\Gamma F) \), \( j = 1, \ldots, r \) are such that:

\[ m_{F} \leq \mu_j(\Gamma F) \leq M_{F}. \] (3)

Part (c) is made also in Bai (2003, Assumption A). In part (c-i) we assume finite 4th order moments of the factors, indeed

\[ E[\|F_t\|^4] = \sum_{j=1}^{r} \sum_{k=1}^{r} E[F_{jt}^2 F_{kt}^2]. \]

Part (c-ii) is very general, it simply says that the sample covariance matrix of the factors is a consistent estimator of its population counterpart \( \Gamma^F \). In principle, we do not have to require for stationarity in part (b) and we can have \( \Gamma^F \) that depends on \( t \) as long as (3) still holds. For this reason we stress that \( M_{F} \) in part (b) is independent of \( t \). This is, for example the case when we are in presence of structural breaks/change-points (see, e.g., Barigozzi et al., 2018 and Duan et al., 2022) or regime shifts (Massacci, 2017; Barigozzi and Massacci, 2022). Unit roots are however excluded since in that case \( \{F_t\} \) does not have a finite covariance, and everything that is done in this paper would have to be applied to properly differenced data. Still the assumption requires introducing the sample size \( T \) and does not provide a rate, which should be \( \sqrt{T} \). If we assume stationarity, then, a discussion on the primitive conditions the process \( \{F_t\} \) needs to satisfy is given in Section 3 where assumptions alternative to part (c) are proposed.

We only consider non-random factor loadings for simplicity. As in Bai (2003), the generalization to the case of random loadings is possible provided that: we assume convergence in probability in part (a), for all \( i \in \mathbb{N} \), \( E[\|\lambda_i\|^4] \leq K_\Lambda \), for some finite positive real \( K_\Lambda \) independent of \( i \), and \( \{\lambda_i, i \in \mathbb{N}\} \) is an independent sequence.

Part (d) implies the existence of a finite number of factors. In particular, the numbers of common factors, \( r \), is identified only for \( n \to \infty \). Here \( N \) is the minimum number of series we need to be able to identify \( r \) so that \( r \leq N \). Without loss of generality hereafter when we say “for all \( n \in \mathbb{N} \)” we always mean that \( n > N \) so that \( r \) can be identified. In practice we must always work with \( n \) such that \( r < n \). Moreover, because PCA is based on eigenvalues of a matrix \( n \times n \) estimated using \( T \) observations then we must also have samples of size \( T \) such that \( r < T \). Therefore, sometimes it is directly assumed that
r < \min(n, T).

Let \( \Gamma^\chi = E[x_i x_i'] = \Lambda \Gamma^F \Lambda' \) with \( r \) largest eigenvalues \( \mu_j^L \), \( j = 1, \ldots, r \), collected in the \( r \times r \) diagonal matrix \( \Lambda^\chi \) (sorted in descending order) and with corresponding normalized eigenvectors which are the columns of the \( n \times r \) matrix \( V^\chi \). In Lemma 1(iv) we prove that (see also (2) and (3)) for all \( j = 1, \ldots, r \),

\[
m^2 \leq C_j \leq \lim\inf_{n \to \infty} \frac{\mu_j^L}{n} \leq \lim\sup_{n \to \infty} \frac{\mu_j^L}{n} \leq M_j M_F.
\]

This means we consider only pervasive strong factors.

To characterize the idiosyncratic component, we make the following assumptions.

**Assumption 2 (Idiosyncratic Component).**

(a) For all \( i \in \mathbb{N} \) and all \( t \in \mathbb{Z} \), \( E[\xi_{it}] = 0 \) and \( \sigma_i^2 := E[\xi_{it}^2] \geq C_\xi \) for some finite positive real \( C_\xi \) for some finite positive real \( i \) independent of \( i \) and \( t \).

(b) For all \( i, j \in \mathbb{N} \), all \( t \in \mathbb{Z} \), and all \( k \in \mathbb{Z} \), \( |E[\xi_{it} \xi_{jt+k}]| \leq \rho |M_{ij} | \) for some finite positive real \( \rho \) and \( M_{ij} \) are finite positive reals independent of \( t \) such that \( 0 \leq \rho < 1 \), \( \sum_{j=1}^n M_{ij} \leq M_\xi \), and \( \sum_{i=1}^n M_{ij} \leq M \) for some finite positive real \( \rho \) independent of \( i, j, \) and \( n \).

(c) (i) For all \( i = 1, \ldots, n \), all \( t = 1, \ldots, T \), and all \( n, T \in \mathbb{N} \), \( E[\xi_{it}^4] \leq Q_\xi \) for some finite positive real \( Q_\xi \) independent of \( i \) and \( t \);

(ii) for all \( j = 1, \ldots, n \), all \( s = 1, \ldots, T \), and all \( n, T \in \mathbb{N} \),

\[
E \left[ \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (\xi_{is}^2 - E[\xi_{is}^2]) \right] \leq K_\xi
\]

for some finite positive real \( K_\xi \) independent of \( j, s, n, \) and \( T \).

By part (a), we have that the idiosyncratic components have zero mean. This, jointly with Assumption 1(b) by which \( E[F_{it}] = 0_{r} \), implies that we are implicitly assuming that the data has zero mean, i.e., \( E[x_i] = 0 \) for all \( n \in \mathbb{N} \). Obviously, we could modify (1) to allow for non-zero mean in the data, by just adding a constant term. Since there are no additional difficulties in doing so and all results that follow would still hold once we pre-center the data, we do not consider this case, to keep the notation simple.

Part (b) has a twofold purpose. First, it limits the degree of serial correlation of the idiosyncratic components, implying weak stationarity of idiosyncratic components. Second, it also limits the degree of cross-sectional correlation between idiosyncratic components, which is usually assumed in approximate DFM. Moreover, part (b) implies the usual conditions required by Bai (2003, Assumptions C.2, C.3, and C.4) for PC estimation (see Lemma 1(i)-(iii)), i.e.,

\[
\sup_{n, T \in \mathbb{N}} \frac{1}{nT} \sum_{i,j=1}^n \sum_{t,s=1}^T |E[\xi_{it} \xi_{jt}]| \leq \frac{M_\xi (1+\rho)}{1-\rho},
\]

\[
\sup_{n, T \in \mathbb{N}} \max_{t=1,\ldots,T} \frac{1}{n} \sum_{i,j=1}^n |E[\xi_{it} \xi_{jt}]| \leq M_\xi,
\]

\[
\sup_{n, T \in \mathbb{N}} \max_{i=1,\ldots,n} \frac{1}{T} \sum_{t,s=1}^T |E[\xi_{it} \xi_{is}]| \leq \frac{M_\xi (1+\rho)}{1-\rho},
\]

\[
\sup_{n, T \in \mathbb{N}} \max_{i=1,\ldots,n} \frac{1}{T} \sum_{t,s=1}^T |E[\xi_{it} \xi_{is}]| \leq \frac{M_\xi (1+\rho)}{1-\rho}.
\]
where $M_\xi$ and $\rho$ are defined in Assumption 2(b). In fact Bai (2003) directly assumes conditions as those in (5) without asking for stationarity. Notice that the formulation of part (b) is inspired by Forni et al. (2017) and it implicitly bounds the L1 norm of $\Gamma^\xi$ as in Fan et al. (2013). Furthermore in Lemma 1(v) we prove that the largest eigenvalue of $\Gamma^\xi$ is such that

$$\sup_{n \in \mathbb{N}} \Lambda^\xi_1 \leq M_\xi,$$

(6)

where $M_\xi$ is defined in Assumption 2(b). Condition (6) was originally assumed by Chamberlain and Rothschild (1983). Finally, by setting $k = 0$ in Assumption 2(b), it follows also that for all $i = 1, \ldots, N$ and all $N \in \mathbb{N}$, $\sigma_i^2 \leq M_\xi$. Thus, all idiosyncratic components have finite variance.

Part (c-i) assumes finite 4th order moments of the idiosyncratic components, it is weaker than what assumed by Bai (2003, Assumption C.1) where finite 8th order moments are required. Part (c-ii) gives summability conditions across the cross-section and time dimensions for the 4th order cumulants of $\{\xi_{it}\}$, indeed, it can be equivalently written as:

$$E \left[ \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \{\xi_{is}\xi_{jt} - E[\xi_{is}\xi_{jt}]\} \right]^2 = \frac{1}{nT} \sum_{i,k=1}^{n} E \left[ \left( \sum_{t=1}^{T} \{\xi_{is}\xi_{jt} - E[\xi_{is}\xi_{jt}]\} \right) \left( \sum_{u=1}^{T} \{\xi_{ks}\xi_{ju} - E[\xi_{ks}\xi_{ju}]\} \right) \right]$$

$$= \frac{1}{nT} \sum_{i,k=1}^{n} \sum_{t,u=1}^{T} \{E[\xi_{is}\xi_{jt}\xi_{ks}\xi_{ju}] - E[\xi_{is}\xi_{jt}]E[\xi_{ks}\xi_{ju}]\} \leq K_\xi.$$ 

Then, if we choose $s = t$ we get

$$E \left[ \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \{\xi_{it}\xi_{jt} - E[\xi_{it}\xi_{jt}]\} \right]^2 = \frac{1}{nT} \sum_{i,k=1}^{n} \sum_{t,u=1}^{T} \{E[\xi_{it}\xi_{jt}\xi_{ku}\xi_{ju}] - E[\xi_{it}\xi_{jt}]E[\xi_{ku}\xi_{ju}]\} \leq K_\xi,$$

which is assumed by Bai and Li (2016). And if we choose $j = i$ we get

$$E \left[ \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \{\xi_{it}\xi_{is} - E[\xi_{it}\xi_{is}]\} \right]^2 = \frac{1}{nT} \sum_{t,u=1}^{T} \sum_{i,k=1}^{n} \{E[\xi_{it}\xi_{is}\xi_{ku}\xi_{ks}] - E[\xi_{it}\xi_{is}]E[\xi_{ku}\xi_{ks}]\} \leq K_\xi,$$

which is similar to what is assumed by Bai (2003, Assumption C.5), where, however, $T = 1$, and 8th cross-moments are bounded.

Part (c-ii) has some important consequences. First, it follows that (see Lemma 3(iii) and (A81) in the proof of Proposition 6):

$$E \left\| \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} \{\Lambda^i\xi_{it} - \Lambda^i E[\xi_{it}]\} \right\|_F^2 \leq r M_\Lambda K_\xi,$$

$$E \left\| \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \{\xi_{it} - \xi_{it} E[\xi_{it}]\} \right\|_F^2 \leq r M_\xi K_\xi,$$

where the second inequality is analogous to Bai (2003, Assumption F.1). Second, part (c-ii) implies also
that (setting \( n = 1 \) therein):

\[
E \left[ \frac{1}{T} \sum_{t=1}^{T} \{ \xi_{is} \xi_{jt} - E[\xi_{is} \xi_{jt}] \} \right]^2 = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{u=1}^{T} \{E[\xi_{is} \xi_{jt} \xi_{iu} \xi_{ju}] - E[\xi_{is} \xi_{jt}] E[\xi_{iu} \xi_{ju}] \} \leq \frac{K_{\xi}}{T},
\]

which means that the sample (auto)covariances between \( \{\xi_{it}\} \) and \( \{\xi_{jt}\} \) are \( \sqrt{T} \)-consistent estimators of their population counterparts. In particular, choosing \( s = t \), we see that we can consistently estimate the \((i,j)\)th entry of the idiosyncratic covariance matrix \( \Gamma^\xi \). Again this assumption requires introducing the sample size \( T \) and it is stronger that what we ask for the factors in Assumption 1(c), which must hold only in the limit \( T \to \infty \), while here it holds for all \( T \in \mathbb{N} \). As for the factors, a discussion on the primitive conditions the process \( \{\xi_{it}\} \) needs to satisfy is postponed in Section 3 where alternative approaches to directly assuming part (c) are proposed.

We conclude with two identifying assumptions.

**Assumption 3 (Distinct Eigenvalues).** For all \( n \in \mathbb{N} \) and all \( i = 1, \ldots, r, \mu_i^\xi > \mu_{i+1}^\xi \).

Note that this implies that \( \bar{C}_j < C_{j-1}, j = 2, \ldots, r, \) in (4) and that the eigenvalues of \( \Sigma_A \Gamma^F \) are distinct, as also required in Bai (2003, Assumption G). Furthermore, letting \( V_0 \) be the \( r \times r \) diagonal matrix of non-zero eigenvalues of \( \{\Sigma_A\}^{1/2} \Gamma^F \{\Sigma_A\}^{1/2} \) or equivalently of \( \{\Gamma^F\}^{1/2} \Sigma_A \{\Gamma^F\}^{1/2} \), it follows that these are also distinct because of Assumption 3.\(^1\) This is needed in order to identify the eigenvectors when doing PCA.

**Assumption 4 (Independence).** The processes \( \{\xi_{it}, i \in \mathbb{N}, t \in \mathbb{Z}\} \) and \( \{F_{jt}, j = 1, \ldots, r, t \in \mathbb{Z}\} \) are mutually independent.

This assumption obviously implies that the factors and, therefore, the common components are independent of the idiosyncratic components at all leads and lags and across all units. This is compatible with a structural macroeconomic interpretation of factor models, according to which the factors driving the common component are independent of the idiosyncratic components representing measurement errors or local dynamics. Assumption 4 implies (see the proof of Lemma 3(i))

\[
E \left[ \left\| \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} F_t \xi_t' \right\|^2 \right] \leq M_{F\xi},
\]

for some finite positive real \( M_{F\xi} \) independent of \( n \) and \( T \), and also (see (A16), (A22) and (A23) in the proof of Proposition 2)

\[
E \left[ \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_t \right\|^2 \right] \leq M_{F\xi}, \quad E \left[ \left\| \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} \sum_{j=1}^{n} F_t \xi_{jt} \right\|^2 \right] \leq M_{F\xi},
\]

\(^1\)In Lemma 8 we also show that the diagonal entries \( V_0, j, j = 1, \ldots, r, \) of \( V_0 \) are such that \( V_0, j, j = p\text{-lim}_{n,T \to \infty} \mu_j \left( \frac{2x}{j} \right) = p\text{-lim}_{n,T \to \infty} \mu_j \left( \frac{2x}{j} \right) \).
which are similar but weaker than what assumed by Bai (2003, Assumption D), i.e.,

$$
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right\|^2 \right] \leq M_{F \xi}.
$$

(7)

Notice that all instances where we use Assumption 4 would follow if we used (7) directly. Note that if we assumed random loadings then here we should add that also \{\lambda_{ij}, i \in \mathbb{N}, j = 1, \ldots, r\} is independent of the factor and idiosyncratic processes.

Because of Weyl’s inequality (6) and (4) and Assumption 4 imply the eigengap in the eigenvalues \(\mu_{xj}, j = 1, \ldots, n\), of \(\Gamma_x = \mathbb{E}[x_t x_t'] = \Gamma_x + \Gamma_\xi\) (see Lemma 1(vi)):

$$
C_r \leq \lim \inf_{n \to \infty} \frac{\mu_{xj}}{n} \leq \lim \sup_{n \to \infty} \frac{\mu_{xj}}{n} \leq C_r,
$$

$$
\sup_{n \in \mathbb{N}} \mu_{xj+1} \leq M_{\xi},
$$

where \(M_{\xi}\) is defined in Assumption 2(b).

Assumption 5 (Central limit theorems).

(a) For all \(i \in \mathbb{N}\), as \(T \to \infty\),

$$
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \to_d N(0_r, \Phi_i),
$$

where \(\Phi_i = \lim_{T \to \infty} \frac{1}{T} \sum_{t,s=1}^{T} \mathbb{E}[F_t F_s' \xi_{it} \xi_{is}]\).

(b) For all \(t \in \mathbb{Z}\), as \(n \to \infty\),

$$
\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \lambda_{ij} \xi_{it} \to_d N(0_r, \Gamma_t),
$$

where \(\Gamma_t = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \lambda_{ij} \mathbb{E}[\xi_{it} \xi_{is}]\).

(c) \(\sqrt{T}/n \to 0\) and \(\sqrt{n}/T \to 0\), as \(n, T \to \infty\).

Part (a) can be derived from more primitive assumptions. For example, we could assume strong mixing factors and idiosyncratic components and such that, for all \(t \in \mathbb{Z}\) and all \(i \in \mathbb{N}\), we strengthen Assumptions 1(c-i) and 2(c-i) to: \(\mathbb{E}[\|F_t\|^4 + \epsilon] \leq K_F\) and \(\mathbb{E}[|\xi_{it}|^4 + \epsilon] \leq Q_{\xi}\), for some \(\epsilon > 0\). Then, \(\{F_t \xi_{it}\}\) is also strong mixing because of Bradley (2005, Theorem 5.1.a) and with finite 4th order moments because of Assumption 4 and part (a) would follow from Ibragimov (1962, Theorem 1.4). For other assumptions see also Section 3. Notice also that under Assumption 4 in part (a) we actually have

$$
\Phi_i = \lim_{T \to \infty} \frac{1}{T} \sum_{t,s=1}^{T} \mathbb{E}[F_t F_s' \xi_{it} \xi_{is}]\).
$$

Part (b) of course holds if we assumed cross-sectionally uncorrelated idiosyncratic components, i.e., if \(\Gamma_\xi\) were diagonal as in an exact factor model. In general, to derive part (b) from primitive assumptions we could either introduce ordering of the \(n\) cross-sectional items and a related notion of spatial dependence, or we could derived it from the properties of stationary mixing random fields (Bolthausen, 1982) or of cross-sectional martingale difference sequences (Kuersteiner and Prucha, 2013), or we could apply results on exchangeable sequences, which are instead independent of the ordering, and are in turn obtained by virtue of the Hewitt-Savage-de Finetti theorem (Austern and Orbanz, 2022, Theorem 4, and Bolthausen, 1984). In any case we would need also to strengthen Assumption 2(c-i)
by asking that for all \( t \in \mathbb{Z}, \) all \( i \in \mathbb{N}, \) 
\( \mathbb{E}[|\xi_{it}|^{4+\epsilon}] \leq Q_{\epsilon}, \)
for some \( \epsilon > 0. \) In the case of random 
loadings we should also require that for all \( i \in \mathbb{N}, \) 
\( \mathbb{E}[||\lambda_i||^{4+\epsilon}] \leq K_{\epsilon}, \)
for some \( \epsilon > 0, \) and we would have: 
\[ \Gamma_t = \lim_{n \to \infty} \frac{1}{n} \sum_{i,j=1}^{n} \mathbb{E}[|\xi_{it}|^2 | \xi_{jt}] \]

2 Estimation by means of Principal Component Analysis

All estimated quantities are denoted with \( \hat{\cdot} \) or \( \tilde{\cdot} \) and depend implicitly on \( n \) and \( T. \)

Consider the \( n \times n \) sample covariance matrix (note that data has zero mean by assumption) 
\[ \hat{\Gamma}^x = \frac{1}{T} \sum_{t=1}^{T} x_t x_t' = \frac{X'X}{T}, \]
with \( r \) largest eigenvalues in the \( r \times r \) diagonal matrix \( \hat{\mathbf{M}}^x \) (sorted in descending order) and corresponding normalized eigenvectors as columns of the \( n \times r \) matrix \( \hat{\mathbf{V}}^x. \)

Consider also the \( T \times T \) sample covariance matrix (note that data has zero mean by assumption) 
\[ \tilde{\Gamma}^x = \frac{1}{n} \sum_{i=1}^{n} x_i x_i' = \frac{XX'}{n}, \]
with \( r \) largest eigenvalues in the \( r \times r \) diagonal matrix \( \tilde{\mathbf{M}}^x \) (sorted in descending order) and corresponding normalized eigenvectors as columns of the \( T \times r \) matrix \( \tilde{\mathbf{V}}^x. \)

The PC estimators of \( \Lambda \) and \( F \) are the solutions of the following minimization:

\[
\min_{\Lambda, F} \frac{1}{nT} \sum_{t=1}^{nT} \sum_{i=1}^{T} \left( x_{it} - \bar{X} \right)^2 = \min_{\Lambda, F} \frac{1}{nT} \text{tr} \left\{ (X - F\Lambda)' (X - F\Lambda) \right\} = \min_{\Lambda, F} \frac{1}{nT} \text{tr} \left\{ (X - F\Lambda)' (X - F\Lambda) \right\}.
\]

The idea of PCs as the \( r \) directions of best fit in an \( n \)-dimensional space is originally due to Pearson (1901). There are four equivalent ways to solve the problem in (8).

2.1 Approach A by Forni et al. (2009)

We solve for \( \Lambda \) such that \( \frac{\Lambda'\Lambda}{n} \) is diagonal and then solve for \( F \) by linear projection. So substitute 
\( F = X\Lambda(\Lambda'\Lambda)^{-1} \) in (8) and we have 
\[
\min_{\Lambda} \frac{1}{nT} \text{tr} \left\{ X (I_n - \Lambda(\Lambda'\Lambda)^{-1} \Lambda') X' \right\}
\]
which is equivalent to

\[
\max_{\Lambda} \frac{1}{nT} \text{tr} \left\{ X'X(\Lambda'\Lambda)^{-1} \Lambda' \right\} = \max_{\Lambda} \frac{1}{nT} \text{tr} \left\{ (\Lambda'\Lambda)^{-1/2} \Lambda' \frac{X'X}{T} (\Lambda'\Lambda)^{-1/2} \right\}
\]

Now since by construction each column of \( \Lambda(\Lambda'\Lambda)^{-1/2} \) is normalized (since we assumed \( \frac{\Lambda'\Lambda}{n} \) to be diagonal), then the above maximization must give as maximum the \( r \) largest eigenvalues of \( \Gamma^x = \frac{X'X}{T} \) divided by \( n, \) i.e., it must give \( \frac{M^x}{n}. \) In other words, our estimator \( \hat{\Lambda} \) must be such that \( \hat{\Lambda}(\hat{\Lambda}'\hat{\Lambda})^{-1/2} \) is
the matrix of normalized eigenvectors corresponding the $r$ largest eigenvalues of $\frac{X'X}{nT}$, i.e., such that:

$$\left(\hat{\Lambda}'\hat{\Lambda}\right)^{-1/2}\hat{\Lambda}'\frac{X'X}{nT}\hat{\Lambda}(\hat{\Lambda}'\hat{\Lambda})^{-1/2} = \frac{\tilde{M}^x}{n}$$

(11)

but also

$$\hat{\nu}'\frac{X'X}{nT}\hat{\nu} = \frac{\tilde{M}^x}{n}$$

(12)

therefore

$$\hat{\Lambda} = \hat{\nu}^x(\tilde{M}^x)^{1/2}$$

(13)

and

$$\hat{F} = X\hat{\Lambda}(\hat{\Lambda}'\hat{\Lambda})^{-1} = X\hat{\nu}^x(\tilde{M}^x)^{-1/2}$$

(14)

which are the normalized PCs of $X$ and such that $\frac{\hat{F}'\hat{F}}{T} = I_r$. This is the approach followed in the rest of the paper. It is the classical and original definition of PC estimators dating back to Hotelling (1933) (see also Lawley and Maxwell, 1971, Chapter 4, Mardia et al., 1979, Chapter 9.3, Jolliffe, 2002, Chapter 7.2).

2.2 Approach B by Bai (2003)

We solve (8) for $F$ such that $\frac{F'F}{T} = I_r$ and then solve for $\Lambda$ by linear projection. So substitute $\Lambda = X'E(F'E)^{-1} = \frac{X'E}{T}$ in (8) and we have

$$\min_F \frac{1}{nT} \text{tr} \left\{ X' \left( I_T - \frac{F'F}{T} \right) X \right\},$$

which is equivalent to

$$\max_F \frac{1}{nT} \text{tr} \left\{ XX'\frac{F'F}{T} \right\} = \max_F \frac{1}{T} \text{tr} \left\{ \frac{F'X'X'E}{\sqrt{T} \frac{n}{\sqrt{n}}} \right\}.$$

Now since by construction each column of $\frac{F}{\sqrt{T}}$ is normalized (since we assumed $\frac{F'F}{T} = I_r$), then the above maximization must give as maximum the $r$ largest eigenvalues of $\tilde{\Gamma}^x = \frac{XX'}{nT}$ divided by $T$, i.e., it must give $\frac{\tilde{M}^x}{T}$. In other words, our estimator $\tilde{F}$ must be such that $\frac{\tilde{F}}{\sqrt{T}}$ is the matrix of normalized eigenvectors corresponding the $r$ largest eigenvalues of $\frac{XX'}{nT}$, i.e., such that:

$$\frac{\tilde{F}'X'X'\tilde{F}}{T \sqrt{T} \frac{n}{\sqrt{n}}} = \tilde{M}^x$$

but also

$$\hat{\nu}'\frac{XX'}{nT}\hat{\nu} = \frac{\tilde{M}^x}{T},$$

therefore

$$\tilde{F} = \tilde{\nu}^x \sqrt{T}$$

and

$$\tilde{\Lambda} = \frac{X'\tilde{F}}{T} = \frac{X'\tilde{\nu}^x}{\sqrt{T}}.$$
which are such that \( \frac{\Lambda'\Lambda}{n} = \frac{M^z}{T} \) is diagonal.

2.3 Approach C by Stock and Watson (2002)

We solve (8) for \( \Lambda \) such that \( \frac{\Lambda'\Lambda}{n} = \mathbf{I}_r \) and then solve for \( F \) by linear projection. So substitute \( F = X\Lambda(\Lambda'\Lambda)^{-1} = \frac{X\Lambda}{n} \) in (8) and we have

\[
\min_{\Lambda} \frac{1}{n} \text{tr} \left\{ X \left( \mathbf{I}_n - \frac{\Lambda'\Lambda}{n} \right) X' \right\},
\]

which is equivalent to

\[
\max_{\Lambda} \frac{1}{n} \text{tr} \left\{ X'X \frac{\Lambda'\Lambda}{n} \right\} = \max_{\Lambda} \frac{1}{n} \text{tr} \left\{ \frac{\Lambda'X'X\Lambda}{\sqrt{n}} \right\}.
\]

Now since by construction each column of \( \frac{\Lambda}{\sqrt{n}} \) is normalized (since we assumed \( \frac{\Lambda'\Lambda}{n} = \mathbf{I}_r \)), then the above maximization must give as maximum the \( r \) largest eigenvalues of \( \hat{\Gamma}^x = \frac{XX'}{n} \) divided by \( n \), i.e., it must give \( \frac{M^z}{n} \). In other words, our estimator \( \hat{\Lambda} \) must be such that \( \frac{\hat{\Lambda}}{\sqrt{n}} \) is the matrix of normalized eigenvectors corresponding the \( r \) largest eigenvalues of \( \frac{XX'}{n} \), i.e., such that:

\[
\frac{\hat{\Lambda}'X'X\hat{\Lambda}}{\sqrt{n}n} = \frac{M^z}{n},
\]

but also

\[
\hat{V}^x\frac{X'X}{n}\hat{V}^x = \frac{M^z}{n}.
\]

Therefore,

\[
\hat{\Lambda} = \hat{V}^x\sqrt{n},
\]

and

\[
\hat{F} = \frac{XX\hat{\Lambda}}{n} = \frac{XX\hat{V}^x}{\sqrt{n}},
\]

which are such that \( \hat{F}'\hat{F} = \frac{M^z}{n} \) is diagonal.

2.4 Approach D

We solve for \( F \) such that \( \frac{F'F}{T} \) is diagonal and then solve for \( \Lambda \) by linear projection. So substitute \( \Lambda = X'F(F'F)^{-1} \) in (8) and we have

\[
\min_{F} \frac{1}{n} \text{tr} \left\{ X' \left( \mathbf{I}_T - F(F'F)^{-1}F' \right) X \right\},
\]

which is equivalent to

\[
\max_{F} \frac{1}{n} \text{tr} \left\{ XX'F(F'F)^{-1}F' \right\} = \max_{F} \frac{1}{T} \text{tr} \left\{ (F'F)^{-1/2}F'XX'F(F'F)^{-1/2} \right\}.
\]

Now since by construction each column of \( (F'F)^{-1/2} \) is normalized (since we assumed \( \frac{F'F}{T} \) to be diagonal), then the above maximization must give as maximum the \( r \) largest eigenvalues of \( \Gamma^x = \frac{XX'}{n} \).
divided by $T$, i.e., it must give $\frac{\tilde{M}^x}{T}$. In other words, our estimator $\tilde{F}$ must be such that $\tilde{F}'(\tilde{F}' \tilde{F})^{-1/2}$ is the matrix of normalized eigenvectors corresponding the $r$ largest eigenvalues of $\frac{XX'}{nT}$, i.e., such that:

$$(\tilde{F}' \tilde{F})^{-1/2} \frac{XX'}{nT} \tilde{F}'(\tilde{F}' \tilde{F})^{-1/2} = \frac{\tilde{M}^x}{T}$$

but also

$$\tilde{\Lambda} = X' \frac{XX'}{nT} \tilde{F}'(\tilde{F}' \tilde{F})^{-1} = X' \frac{XX'}{nT} \tilde{F} \tilde{F}^{-1}$$

which are the normalized PCs of $X'$ and such that $\frac{\tilde{\Lambda} \tilde{\Lambda}}{n} = I_r$.

### 2.5 Comparison of the four approaches

The four methods are asymptotically equivalent. Indeed, as long as $r < \min(n, T)$ the $r$ largest eigenvalues of $X'X$ and those of $XX'$ coincide. More precisely, by Weyl’s inequality, in case A and C, for $j = 1, \ldots, r$,

$$\mu_j \left( \frac{XX'}{nT} \right) + \mu_n \left( \frac{XX'}{nT} \right) \leq \mu_j \left( \frac{XX'}{nT} \right) \leq \mu_j \left( \frac{XX'}{nT} \right) + \mu_1 \left( \frac{XX'}{nT} \right)$$

and since, by continuity of eigenvalues and Assumptions 1(a) and 1(c),

$$\lim_{n,T \to \infty} \mu_j \left( \frac{XX'}{nT} \right) = \lim_{n,T \to \infty} \mu_j \left( \frac{XX'}{nT} \right) = \mu_j \left( \frac{XX'}{nT} \right)$$

and in Lemma 3 we prove $\frac{\|XX\|}{nT} = O_p \left( \frac{1}{\sqrt{nT}} \right)$, then

$$\left| \mu_j \left( \frac{XX'}{nT} \right) - \mu_j \left( \frac{XX'}{nT} \right) \right| = o_p(1).$$

So letting $V_0$ be the $r \times r$ diagonal matrix with entries the eigenvalues $\mu_j \left( \frac{XX'}{nT} \right)$, we have proved that in case A

$$\left\| \frac{\tilde{\Lambda} \tilde{\Lambda}}{n} - V_0 \right\| = \left\| \frac{\tilde{M}^x}{n} - V_0 \right\| = o_p(1),$$

and in case C

$$\left\| \frac{XX'}{nT} \right\| = \left\| \frac{XX'}{nT} \right\| = o_p(1),$$

Similarly, in case B and D

$$\left| \mu_j \left( \frac{XX'}{nT} \right) - \mu_j \left( \frac{XX'}{nT} \right) \right| = o_p(1).$$

So letting $V_0$ be the $r \times r$ diagonal matrix with entries the eigenvalues of $\Gamma^F \Sigma_A$, which are the same as
those of $\Sigma_\Lambda \Gamma^F$, we have proved that in case B
\[
\left\| \frac{\tilde{\Lambda}'\tilde{\Lambda}}{n} - V_0 \right\| = \left\| \frac{\hat{M}^2}{T} - V_0 \right\| = o_p(1),
\]
and in case D
\[
\left\| \frac{\hat{F}'\hat{F}}{T} - V_0 \right\| = \left\| \frac{\hat{M}^2}{T} - V_0 \right\| = o_p(1).
\]

3 On the estimation of the covariance matrix

To be able to apply PCA we need a consistent estimator of the covariance matrix of \{\boldsymbol{x}_t\}. In fact, we just need consistency of the estimators of each entry of the matrix. Indeed, for the scope of factor analysis, we do not need a consistency result for the whole matrix, but we just need to ensure that
\[
\frac{1}{n} \left\| \hat{\Gamma}^x - \Gamma^x \right\| = O_p \left( \frac{1}{\sqrt{T}} \right), \tag{15}
\]
So even if when $n > T$ the sample covariance matrix is not positive definite, this does not affect any of the methods presented in the previous section nor the results in the next sections. The proof of (15) is in Lemma 5(i), and it relies on Assumption 1(c) for estimation of the covariance of the factors and on Assumption 2(c) for estimation of the covariance of the idiosyncratic components. Both these assumptions are high-level ones and it is natural to ask when are they satisfied. In this section, we introduce more primitive conditions on the data generating process of \{\boldsymbol{F}_t\} and \{\xi_{it}\} which ensure that (15) holds, or even that:
\[
\frac{1}{n} \left\| \hat{\Gamma}^x - \Gamma^x \right\| = O_{ms} \left( \frac{1}{\sqrt{T}} \right). \tag{16}
\]
The proof of (16) under the assumptions given below is in Lemma 5(ii).

3.1 Consistent estimation of the covariance of the factors

We consider four different ways to state primitive conditions such that Assumption 1(c) holds.

3.1.1 Linear representation

Assume that \{\boldsymbol{F}_t\} admits the following linear representation:
\[
\boldsymbol{F}_t = \sum_{j=1}^{r} \sum_{k=0}^{\infty} c_{kj} u_{jt} = C(L)\boldsymbol{u}_t, \tag{17}
\]
where $C(L) = \sum_{k=0}^{\infty} C_k L^k$ is an $r \times r$ matrix of polynomials with coefficients $C_k = (c_{k1} \cdots c_{kr})'$. We characterize (17) by means of the following assumption.

Assumption 6.

(W1) $C(z) = \sum_{k=0}^{\infty} C_k z^k$, where $C_k$ are $r \times r$ and $\sum_{k=0}^{\infty} \|C_k\| \leq M_C$ for some finite positive real $M_C$.
(W2) For all $t \in \mathbb{Z}$, $E[\boldsymbol{u}_t] = 0_r$ and $E[\boldsymbol{u}_t \boldsymbol{u}_t'] = I_r$.
(W3) For all $t \in \mathbb{Z}$ and all $k \in \mathbb{Z}$ with $k \neq 0$, $\boldsymbol{u}_t$ is independent of $\boldsymbol{u}_{t-k}$.
(W4) For all $t \in \mathbb{Z}$, $\mathbb{E}[\|\mathbf{u}_i\|^4] \leq K_u$ for some finite positive real $K_u$ independent of $t$.

Because of Assumption 6(W1)-6(W4) and by using explicitly (17), we can prove that (see Lemma 4(i))

$$
\left\| \frac{1}{T} \sum_{t=1}^{T} \{ \mathbf{F}_t \mathbf{F}_t' - \Gamma^F \} \right\| = O_{\text{ms}} \left( \frac{1}{\sqrt{T}} \right). \tag{18}
$$

This is the approach adopted for example by Barigozzi et al. (2021) when considering $\{\mathbf{F}_t\}$ as a cointegrated $I(1)$ process and representation (17) is assumed for the differenced process $\{\Delta \mathbf{F}_t\}$, where then $\mathbf{C}(1)$ has reduced rank equal to the number of common trends.

An implicit argument leading to consistency of the sample covariance matrix under Assumption 6(W1)-6(W4) is that $\{\mathbf{F}_t\}$ is ergodic with finite 4th order moments, therefore $\{\mathbf{F}_t \mathbf{F}_t'\}$ is also ergodic. However, this would lead to convergence in probability only (see below).

Independence of the innovations in Assumption 6(W3) might be too much to ask, and, by relaxing it we could allow, for example, for conditional heteroskedasticity in $\{\mathbf{F}_t\}$. This can be done by asking that for all $j_1, j_2, j_3, j_4 = 1, \ldots, r$,

$$
\frac{1}{T} \sum_{t,s=1}^{T} |\mathbb{E}[u_{j_1t} u_{j_2t} u_{j_3s} u_{j_4s}]| \leq C_u, \quad \frac{1}{T} \sum_{t,s=1}^{T} |\mathbb{E}[u_{j_1t} u_{j_2t}] \mathbb{E}[u_{j_3s} u_{j_4s}]| \leq C_u, \tag{19}
$$

for some finite positive real $C_u$ independent of $t, s, j_1, j_2, j_3,$ and $j_4$. Notice that the second condition in (19) holds trivially if $\{\mathbf{u}_t\}$ is a white noise process. In this case, however, we should change Assumption 6(W4) by asking that for all $j_1, j_2, j_3, j_4 = 1, \ldots, r$, all $t_1, t_2, t_3, t_4 = 1, \ldots, T$ and all $T \in \mathbb{N}$, $\mathbb{E}[u_{j_1t_1} u_{j_2t_2} u_{j_3t_3} u_{j_4t_4}] \leq K_u$ for some finite positive real $K_u$ independent of $j_1, j_2, j_3, j_4, t_1, t_2, t_3, t_4,$ and $T$. These changes would still ensure that $T^{-1} \sum_{t=1}^{T} \mathbf{u}_t \mathbf{u}_t'$ is a consistent estimator of the covariance matrix of $\{\mathbf{u}_t\}$ (see the arguments in the next section).

### 3.1.2 Fourth order cumulants

Alternatively, we could directly assume for all $i, j = 1, \ldots, r$,

**Assumption 7.**

(H1) $\frac{1}{T} \sum_{t,s=1}^{T} |\mathbb{E}[F_{it} F_{jt} F_{is} F_{js}]| \leq C_F$;

(H2) $\frac{1}{T} \sum_{t,s=1}^{T} |\mathbb{E}[F_{it} F_{jt}] \mathbb{E}[F_{is} F_{js}]| \leq C_F$;

(H3) $\frac{1}{T} \sum_{t,s=1}^{T} |\mathbb{E}[F_{it} F_{is}] \mathbb{E}[F_{jt} F_{js}]| \leq C_F$;

(H4) $\frac{1}{T} \sum_{t,s=1}^{T} |\mathbb{E}[F_{it} F_{js}] \mathbb{E}[F_{is} F_{jt}]| \leq C_F$;

for some finite positive real $C_F$ independent of $i$ and $j$.

Then, it is straightforward to see that (18) holds (see Lemma 4(i)). Notice that, since $\mathbb{E}[F_{it}] = 0$ by Assumption 1(b), then

$$
\text{cum}_4(F_{it}, F_{jt}, F_{is}, F_{js}) = \mathbb{E}[F_{it} F_{jt} F_{is} F_{js}] - \mathbb{E}[F_{it} F_{jt}] \mathbb{E}[F_{is} F_{js}] - \mathbb{E}[F_{it} F_{is}] \mathbb{E}[F_{jt} F_{js}] - \mathbb{E}[F_{it} F_{js}] \mathbb{E}[F_{jt} F_{is}],
$$

and we are in fact assuming summability of the 4th order cumulants, which is necessary and sufficient condition for (18) to hold (Hannan, 1970, pp. 209-211). This approach is a classical one and it is high-level in that it does not make any specific assumption on the dynamics of $\{\mathbf{F}_t\}$. On this approach see also Forni et al. (2009, Assumption 8 and the related comments).
3.1.3 Mixing

Alternatively we can assume that:

(M1) \( \{ F_t \} \) is strong mixing, i.e., \( \alpha \)-mixing, with exponentially decaying mixing coefficients \( \alpha(T) \leq \exp(-c_1 T^{c_2}) \), for some finite positive reals \( c_1 \) and \( c_2 \) independent of \( T \).

(M2) One of the following two:

(M2a) \( \mathbb{E}[\| F_t \|^4] \leq K_F \) for some finite positive real \( K_F \) independent of \( t \).

(M2b) For all \( s > 0 \) and \( j = 1, \ldots, r \), \( \mathbb{P}(| F_{jt} | > s) \leq \exp(-bs^k) \) for some finite positive reals \( b \) and \( k \) independent of \( t \) and \( j \).

Notice that (M2b) is much stronger than (M2a), indeed it is equivalent to the Cramér condition: \( \sup_{m \geq 1} r^{-1/k} (\mathbb{E}[| F_{jt}|^m])^{1/m} \leq K_F' \), for some finite positive real \( K_F' \) independent of \( t \) and \( j \) (Kuchibhotla and Chakrabarti 2020, Section 2), which implies (M2a). The approach based on (M1) and (M2b) is considered in Fan et al. (2013).

Then, \( \{ F_t \} \) is ergodic (White, 2001, Proposition 3.44, and Rosenblatt, 1972), and therefore \( \{ F_t F'_t \} \) is also ergodic (White, 2001, Theorem 3.35, and Stout, 1974, pp. 170, 182), so that

\[
\left\| \frac{1}{T} \sum_{t=1}^{T} \{ F_t F'_t - \Gamma^F \} \right\| \leq O_P \left( \frac{1}{\sqrt{T}} \right).
\]

In the same spirit of (M1)-(M2) we could assume \( \{ F_t \} \) to admit the linear representation (17), with the coefficients \( \{ C_k \} \) satisfying Assumption 6(W1) and \( \{ u_t \} \) satisfying Assumptions 6(W2)-6(W3), plus the integral Lipschitz condition:

(M3) for all \( t \in \mathbb{Z} \), \( \{ u_t \} \) has pdf \( f_{u_t}(u) \) such that \( \int_{\mathbb{R}^r} | f_{u_t}(u + v) - f_{u_t}(v) | \leq C_f \| v \| \), for any \( v \in \mathbb{R}^r \) and for some finite positive real \( C_f \) independent of \( t \).

Then, Assumptions 6(W1)-6(W3) and (M3) imply that \( \{ F_t \} \) is strong mixing with exponentially decaying coefficients, i.e., (M1) holds (see Pham and Tran, 1985, Theorem 3.1). Note that independence in Assumption 6(W3) is not strictly necessary for having (M1) to hold, as we could allow for GARCH effects by assuming geometric ergodicity of \( \{ u_t \} \) instead. Indeed, geometric ergodicity implies \( \beta \)-mixing, which implies strong mixing (Francq and Zakoïan, 2006).

3.1.4 Physical or functional dependence

Let \( \mathcal{F}_{\nu,t} \) be the \( \sigma \)-field generated by \( \{ \nu_s, s \leq t \} \) where \( \nu_t = (\nu_{1t} \cdots \nu_{rt})' \). Let also \( g_{j,u}(\cdot) \) be a real-valued measurable function of \( \mathcal{F}_{\nu,t} \) and define \( G_u(\cdot) = (g_{1,u}(\cdot) \cdots g_{r,u}(\cdot))' \). Then, we assume that \( \{ F_t \} \) admits the following representation:

\[
F_t = \sum_{j=1}^{r} \sum_{k=0}^{\infty} c^j_{k} u_{jt} = C(L) u_t, \quad (20)
\]

\[
u_t = G_u(\mathcal{F}_{\nu,t}). \quad (21)
\]
For any $\ell = 1, \ldots, r$ and $t \in \mathbb{Z}$, define the element-wise physical or functional dependence measure as

$$
\delta_{u,d,\ell}^u = \left\{ \mathbb{E} \left[ \| g_{\ell,u}(F_{\nu,t}) - g_{\ell,u}(F_{\nu,t,(0)}) \|_d \right] \right\}^{1/d},
$$

where $F_{\nu,t,(0)}$ is the $\sigma$-algebra generated by $\{\nu_1, \ldots, \nu_t, \nu_0, \nu_1, \nu_2, \ldots\}$ where $\nu_0$ is an i.i.d. copy of $\nu_0$. Finally, define the uniform dependence adjusted norm as:

$$
\Phi_{d,\alpha}^u = \max_{\ell=1,\ldots,r} \sup_{k \geq 0} \sum_{t=k}^{\infty} \delta_{t,d,\ell}^u.
$$

(22)

We characterize (20)-(21) by means of the following assumption.

**Assumption 8.**

(P1) $C(z) = \sum_{k=0}^{\infty} C_k z^k$, where $C_k$ are $r \times r$ and $\sum_{k=0}^{\infty} \|C_k\| \leq M_C$ for some finite positive real $M_C$.

(P2) For all $t \in \mathbb{Z}$, $\mathbb{E}[\nu_t] = 0$, $\mathbb{E}[\nu_t \nu_t'] = I_r$, $\mathbb{E}[u_t] = 0_r$, and $\Gamma^u = \mathbb{E}[u_t u_t']$ is $r \times r$ positive definite and such that $\|\Gamma^u\| \leq M_u$ for some finite positive real $M_u$ independent of $t$.

(P3) For all $t \in \mathbb{Z}$ and all $k \in \mathbb{Z}$ with $k \neq 0$, $\nu_t$ is independent of $\nu_{t-k}$ and $\mathbb{E}[u_t u_{t-k}'] = 0_{r \times r}$.

(P4) For all $t \in \mathbb{Z}$ and all $\ell = 1, \ldots, r$, $\mathbb{E}[\{g_{\ell,u}(F_{\nu,t})\}] \leq K_u$ for some $q \geq 4$ and some finite positive real $K_u$ independent of $\ell$ and $t$.

(P5) For all $d \leq q$ with $q > 4$, $\Phi_{d,\alpha}^u \leq M_{u,d}$ for some finite positive real $M_{u,d}$ independent of $\alpha$.

We can then prove that, under Assumption 8, (18) still holds (see Lemma 4(i)).

Parts (P1)-(P3) are simply defining the Wold representation of $\{F_t\}$ as being driven by a weak white noise $\{u_t\}$ which in turn can be a function of an i.i.d. process $\{\nu_t\}$. Moreover, from (P4) and (P5) it follows that we must have $\mathbb{E}[|u_{t\ell}|^{4+\epsilon}] \leq K_u$ which is slightly stronger than Assumption 6(W4).

Finally (P5) deserves two comments. First, as seen from the definition of physical dependence (22), for (P5) to hold we must choose an appropriate value of $\alpha$, which, in principle, is constrained by the degree of serial dependence in $\{u_t\}$. Now, since in (P3) we make the natural assumption of $\{u_t\}$ being a weak white noise, then no constraint on $\alpha$ is needed for (P5) to hold. And the same is true if we assumed a geometric decay of the lag-$k$ autocovariance of $\{u_t\}$, i.e., $\rho^k$ for some $0 < \rho < 1$. In general, we could even assume an hyperbolic decay of the lag-$k$ autocovariance of $\{u_t\}$, i.e., as $(1 + k)^{-\varsigma}$, with $\varsigma > 2$ and in that case (P5) would hold as long as we choose $\alpha \leq \varsigma - 1$ (see Barigozzi et al., 2022, Lemma C.1). In all cases, since $q > 4$, we can choose $\alpha > \frac{1}{2} - \frac{2}{q}$, which is the condition needed to apply the results of Zhang and Wu (2021).

Second, part (P5) is very general. It includes non linear models as threshold autoregressive models, exponential autoregressive models, and conditionally heteroskedastic models, e.g., univariate stationary GARCH(1,1) models, i.e., when $g_{\ell,u}(F_{\nu,t}) = (\omega_{\ell} + \alpha_{\ell} u_{t-1}^2 + \beta_{\ell} \sigma_{\ell-1}^2)^{1/2} \nu_{t\ell}$ with $\omega_{\ell} > 0, \alpha_{\ell}, \beta_{\ell} \geq 0$ and $\alpha_{\ell} + \beta_{\ell} < 1$ for all $\ell = 1, \ldots, r$. For all these cases, it is shown in Wu (2005, page 14152) that Assumption 8(P5) is satisfied. Obviously, the linear representation in (17) is a special case of (20)-(21) when $\nu_t \equiv u_t$ for all $t \in \mathbb{Z}$ and $g_{\ell,u}(F_{\nu,t}) = u_{t\ell}$ for all $\ell = 1, \ldots, r$. Then, it is easy to see that Assumption 8(P5) holds if we make Assumption 6(W4), i.e., finite 4th moments of innovations. This can be shown using the same arguments in Forni et al. (2017, Proposition 5) but applied to $\{F_t\}$.
3.2 Consistent estimation of the covariance of the idiosyncratic components

Assumption 2(c) already implies, as a special case, a condition on summability of 4th order moments along the time dimension as assumed in Assumption 7 for the factors. Hence, it immediately follows that, for any $i, j = 1, \ldots, n$ (see Lemma 4(ii))

$$\left| \frac{1}{T} \sum_{t=1}^{T} \{ \xi_{it} \xi_{jt} - \mathbb{E}[\xi_{it} \xi_{jt}] \} \right| = O_{\text{ms}} \left( \frac{1}{\sqrt{T}} \right).$$

(23)

Analogously to what discussed for the factors, in place of Assumption 2(c) we could make more primitive assumptions.

First, we could assume a linear representation:

$$\xi_{it} = \sum_{j=1}^{n} \sum_{k=0}^{\infty} \beta_{k,ij} \eta_{jt-k}.$$

Then, letting $\eta = (\eta_{t} \cdots \eta_{nt})'$, we can assume the analogous of Assumptions 6(W1)-6(W4).

(W1) For all $k \in \mathbb{Z}^+$ and all $i, j \in \mathbb{N}$, $|\beta_{k,ij}| \leq \rho^k B_{ij}$ with $0 \leq \rho < 1$, $\sum_{i=1}^{n} B_{ij} \leq B$ and $\sum_{j=1}^{n} B_{ij} \leq B$ for some finite positive real $B$ independent of $i$ and $j$.

(W2) For all $t \in \mathbb{Z}$ and all $n \in \mathbb{N}$ and $\mathbb{E}[\eta_{t}] = 0_n$, $\mathbb{E}[\eta_{t} \eta_{t}'] = I_n$.

(W3) For all $t \in \mathbb{Z}$, all $n \in \mathbb{N}$, and all $k \in \mathbb{Z}$ with $k \neq 0$, $\eta_t$ is independent of $\eta_{t-k}$.

(W4) For all $t \in \mathbb{Z}$ and all $j \in \mathbb{N}$, $\mathbb{E}[|\eta_{jt}|^4] \leq K_\eta$ for some finite positive real $K_\eta$ independent of $t$ and $j$.

Under (W1)-(W4) we can still prove (23) (see, e.g., Lemma 11 in Forni et al., 2017 for a similar setting and proof).

Second, we could assume $\{\xi_{it}\}$ to be strong mixing, i.e., $\alpha$-mixing, with exponentially decaying mixing coefficients and that $\mathbb{E}[|\xi_{it}|^4] \leq K_\xi$ for some finite positive real $K_\xi$ independent of $t$ and $i$. Or we could even assume exponentially decaying tails as in (M2b). This would imply ergodicity for $\{\xi_{it}\}$, and, therefore,

$$\left| \frac{1}{T} \sum_{t=1}^{T} \{ \xi_{it} \xi_{jt} - \mathbb{E}[\xi_{it} \xi_{jt}] \} \right| = O_{\text{P}} \left( \frac{1}{\sqrt{T}} \right).$$

This latter approach is in Fan et al. (2013).

Third, we could instead introduce an $n$-dimensional process $\{\omega_t\}$ such that $\{\xi_{it}\}$ follows the representation:

$$\xi_{it} = \sum_{j=1}^{n} \sum_{k=0}^{\infty} \beta_{k,ij} \eta_{jt-k},$$

$$\eta_{jt} = g_{j,\eta}(\mathcal{F}_{\omega,t}),$$

where $\mathcal{F}_{\omega,t}$ is the $\sigma$-field generated by $\{\omega_s, s \leq t\}$, and $g_{j,\eta}(\cdot)$ is a real-valued measurable function of $\mathcal{F}_{\omega,t}$. Define the uniform dependence adjusted norm as:

$$\Phi_{d,\alpha} = \max_{\ell=1, \ldots, n} \sup_{k \geq 0} (1 + k)\alpha \sum_{t=k}^{\infty} \left\{ \mathbb{E} \left[ \left\| g_{\ell,\eta}(\mathcal{F}_{\omega,t}) - g_{\ell,\eta}(\mathcal{F}_{\omega,t,0}) \right\|^d \right] \right\}^{1/d},$$

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where $\mathcal{F}_{\omega,t}\{0\}$ is the $\sigma$-algebra generated by $\{\omega_t, \ldots, \omega_1, \omega_0^*, \omega_{-1}, \ldots\}$ where $\omega_0^*$ is an i.i.d. copy of $\omega_0$.

Then, we could assume the analogous of Assumption 8(P1)-8(P5).

(P11) For all $k \in \mathbb{Z}^+$, all $i, j = 1, \ldots, n$, and all $n \in \mathbb{N}$, $|\beta_{k,ij}| \leq \rho^kB_{ij}$ with $0 \leq \rho < 1$, $\sum_{i=1}^n B_{ij} \leq B$ and $\sum_{j=1}^n B_{ij} \leq B$ for some finite positive real $B$ independent of $i$ and $j$.

(P12) For all $t \in \mathbb{Z}$ and all $n \in \mathbb{N}$, $\mathbb{E}[\omega_t] = 0_n$, $\mathbb{E}[\omega_t^2] = \mathbf{I}_n$, $\mathbb{E}[\eta_t] = 0_n$, and $\Gamma^n = \mathbb{E}[\eta_t\eta_t']$ is positive definite and such that $||\Gamma^n|| \leq M_n$ for some finite positive real $M_n$ independent of $t$.

(P13) For all $t \in \mathbb{Z}$, all $n \in \mathbb{N}$, and all $k \in \mathbb{Z}$ with $k \neq 0$, $\omega_t$ is independent of $\omega_{t-k}$ and $\mathbb{E}[\eta_t\eta_{t-k}'] = 0_{n \times n}$.

(P14) For all $t \in \mathbb{Z}$, all $j = 1, \ldots, n$, and all $n \in \mathbb{N}$, $\mathbb{E}[\{g_{j,\eta}(\mathcal{F}_{\omega,t})\}^q] \leq K_q$ for some $q \geq 4$ and some finite positive real $K_q$ independent of $t$ and $j$.

(P15) For all $d \leq q$ with $q > 4$, $\Phi_{d,\alpha}^4 \leq M_{q,d}$ for some finite positive real $M_{q,d}$ independent of $\alpha$.

Under (P11)-(P15) we can prove that (23) still holds. The same comments made for Assumption 8(P5) in the case of the factors, apply also here, in particular notice that (P14)-(P15) imply $\mathbb{E}[|\eta_t|^{4+\epsilon}] \leq K_{\eta}$ which is slightly stronger than with (W14).

### 3.3 Consistent estimation of the covariance between factors and idiosyncratic components

From Assumption 4 it follows that for all $t \in \mathbb{Z}$, all $i = 1, \ldots, n$, and all $n \in \mathbb{N}$, $\mathbb{E}[\mathbf{F}_{\xi,itm}] = \mathbf{0}_r$. Moreover, as a consequence of Assumption 5(a) we have:

$$\frac{1}{T} \sum_{t=1}^T \mathbf{F}_{\xi,itm} = O_p \left( \frac{1}{\sqrt{T}} \right).$$

(24)

It is then natural to ask if we can find primitive assumptions such that (24) or even Assumption 5(a) holds.

Now, if we assume that one of the following holds:

(C1) Assumption 6(W1)-6(W4) and (W11)-(W14) hold with $\{\mathbf{u}_t\}$ and $\{\eta_t\}$ independent.

(C2) Assumption 8(P1)-8(P5) and (P11)-(P15) hold with $\{\mathbf{v}_t\}$ and $\{\omega_t\}$ independent.

Then it is immediate to show that under (C1) (24) holds, even in mean-square, that is

$$\frac{1}{T} \sum_{t=1}^T \mathbf{F}_{\xi,itm} = O_{ms} \left( \frac{1}{\sqrt{T}} \right).$$

(25)

Under (C2) the same holds, just notice that because $\{\mathbf{v}_t\}$ and $\{\omega_t\}$ are independent then also $\{\mathbf{u}_t\}$ and $\{\eta_t\}$ are independent and the uniform dependence adjusted norm of the process $\{(\mathbf{u}_t', \eta_t')\}$ is such that $\Phi_{d,\alpha}^{4u} \leq \Phi_{d,\alpha}^4 \Phi_{d,\alpha}^{4u} \leq M_{u,d}M_{q,d}$. Then, the results by Zhang and Wu (2021) can be applied to the estimation of the covariance of the process $\{\mathbf{F}_{\xi,itm}\}$.

If we assume that $\{\mathbf{F}_t\}$ and $\{\xi_{it}\}$ are both strongly mixing with exponentially decaying coefficients and have exponentially decaying tails, then $\{\mathbf{F}_{\xi,itm}\}$ is also strongly mixing (see, e.g., Bradley, 2005, Theorem 5.1.a) with exponentially decaying coefficients and has exponentially decaying tails (see, e.g., Fan et al., 2011, Lemma A2), and by Ibragimov (1962, Theorem 1.4) the CLT for strongly mixing processes as in Assumption 5(a) holds and so (24) holds.
4 Consistency of loadings and factors space

The following result provides preliminary bounds derived using only properties of sample covariance and its eigenvectors. It is based on Fan et al. (2013) and used for example in Barigozzi et al. (2018, 2021). Part (a) is the only one we need for the subsequent results and it is tighter than what Bai (2003) and Bai and Ng (2020) get.

Proposition 1. Under Assumptions 1 through 4, as \( n, T \to \infty \)

(a) \( \min(n, \sqrt{T}) \left\| \frac{\Lambda - \Lambda n}{\sqrt{n}} \right\| = O_P(1) \);

(b) \( \min(\sqrt{n}, \sqrt{T}) \left\| X_i^T - X_i^T \Lambda \right\| = O_P(1) \), uniformly in \( i \);

(c) \( \min(\sqrt{n}, \sqrt{T}) \left\| F - F \hat{\Lambda}^{-1/2} \right\| = O_P(1) \);

(d) \( \min(\sqrt{n}, \sqrt{T}) \left\| \hat{F}_t - \mathcal{H}^{-1}F_t \right\| = O_P(1) \), uniformly in \( t \);

where \( \mathcal{H} = (\Lambda'\Lambda)^{-1}\Lambda'\mathbf{V}^\top(\mathbf{M}^\top)^{1/2}\mathbf{J} \) and \( \mathbf{J} \) is an \( r \times r \) diagonal matrix with entries \( \pm 1 \).

Notice that, as \( n \to \infty \), \( \mathcal{H} \) is finite and positive definite because of Lemma 10, thus \( \mathcal{H}^{-1} \) is well defined.\(^2\)

5 Asymptotics for the loadings

From (11) and (12), since by (13) we have \( \hat{\Lambda}' = \hat{\mathbf{M}}^\top \) and \( \hat{\mathbf{V}}^\top = \hat{\Lambda}(\hat{\mathbf{M}}^\top)^{-1/2} \) which is well defined because of Lemma 7

\[
\frac{X'X}{nT} \hat{\Lambda} = \frac{\hat{\mathbf{M}}^\top}{n}.
\]

Then, substituting \( X'X = (\Lambda F' + \Xi)'(F \Lambda' + \Xi) \) into (26)

\[
\frac{\Lambda F'FA'\hat{\Lambda}}{nT} + \frac{\Lambda F'\Xi\hat{\Lambda}}{nT} + \frac{\Xi'F\Lambda'\hat{\Lambda}}{nT} + \frac{\Xi'\Xi\hat{\Lambda}}{nT} = \frac{\hat{\mathbf{M}}^\top}{n}.
\]

Define

\[
\hat{\mathcal{H}} = \left( \frac{F'F}{T} \right) \left( \frac{\Lambda' \hat{\Lambda}}{n} \right) \left( \frac{\hat{\mathbf{M}}^\top}{n} \right)^{-1},
\]

\(2\)By looking at the proof of Proposition 1 we see that to prove mean-squared consistency we would need to prove results as

\[
E \left[ \left\| \left( \hat{\mathbf{V}}^\top - \mathbf{V}^\top \right)(\frac{1}{\sqrt{n}} \left( (\hat{\mathbf{M}}^\top)^{1/2} - (\mathbf{M}^\top)^{1/2} \right)) \right\|^2 \right]
\leq \sqrt{E \left[ \left\| \left( \hat{\mathbf{V}}^\top - \mathbf{V}^\top \right) \right\|^4 \right]} \sqrt{E \left[ \left\| (\frac{1}{\sqrt{n}} \left( (\hat{\mathbf{M}}^\top)^{1/2} - (\mathbf{M}^\top)^{1/2} \right)) \right\|^4 \right]} = o(1),
\]

because of Cauchy-Schwarz inequality. This would follow if we could show in Lemma 5 that

\[
E \left[ \left\| \frac{1}{n} \left( \mathbf{F}^\top - \mathbf{F}^\top \right) \right\|^4 \right] = o(1),
\]

which requires stronger assumptions as

\[
E \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T (\xi_t \xi_t - E[\xi_t \xi_t]) \right]^4 \leq K \xi
\]

as Assumption C.5 in Bai (2003), which amounts to assume finite and summable 8th order cumulants.
which, as \( n,T \to \infty \), is finite because of Lemma 11(i).

Then, from (27) and (28)

\[
\tilde{\lambda} - \Lambda \tilde{H} = \left( \frac{\Lambda F' \Xi \hat{\lambda} + \Xi' F' \Lambda' \hat{\lambda} + \Xi' \Xi' \hat{\lambda}}{nT} \right) \left( \frac{M^x}{n} \right)^{-1} \\
= \left( \frac{\Lambda F' \Xi \hat{\lambda} + \Xi' F' \Lambda' \hat{\lambda} + \Xi' \Xi' \hat{\lambda}}{nT} \right) \mathcal{H} \left( \frac{M^x}{n} \right)^{-1} \\
+ \left( \frac{\Lambda F' \Xi \hat{\lambda} + \Xi' F' \Lambda' \hat{\lambda} + \Xi' \Xi' \hat{\lambda}}{nT} \right) (\hat{\lambda} - \Lambda \mathcal{H}) \left( \frac{M^x}{n} \right)^{-1}.
\]

Taking the \( i \)th row of (29)

\[
\hat{\lambda}'_i - \lambda'_i \hat{H} = \left( \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} F_t \xi_{jt} \lambda'_i \right) + \left( \frac{1}{nT} \sum_{t=1}^{T} \xi_{it} F'_t \sum_{j=1}^{n} \lambda_j \lambda'_i \right) + \left( \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{it} \xi_{jt} \lambda'_i \right) \\
+ \left( \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{jt} (\hat{\lambda}'_i - \lambda'_i \mathcal{H}) \right) + \left( \frac{1}{nT} \sum_{t=1}^{T} \xi_{it} F'_t \sum_{j=1}^{n} \lambda_j (\hat{\lambda}'_i - \lambda'_i \mathcal{H}) \right) \\
+ \left( \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{jt} (\hat{\lambda}'_i - \lambda'_i \mathcal{H}) \right) \left( \frac{M^x}{n} \right)^{-1}.
\]

**Proposition 2.** Under Assumptions 1 through 4, as \( n,T \to \infty \)

(a) \( \sqrt{nT} \| (1.a) \| = O_P(1) \);

(b) \( \sqrt{T} \| (1.b) \| = O_P(1) \);

(c) \( \min(n, \sqrt{nT}) \| (1.c) \| = O_P(1) \);

(d) \( \min(\sqrt{nT}, T) \| (1.d) \| = O_P(1) \);

(e) \( \min(\sqrt{nT}, T) \| (1.e) \| = O_P(1) \);

(f) \( \min(n, \sqrt{nT}, T) \| (1.f) \| = O_P(1) \);

uniformly in \( i \).

From (30) and Proposition 2, for any \( i = 1, \ldots, n \), we get

\[
\hat{\lambda}_i - \hat{H}' \lambda_i = \left( \frac{M^x}{n} \right)^{-1} \mathcal{H}' \left( \frac{\Lambda' \Lambda}{n} \right) \left( \frac{1}{T} \sum_{t=1}^{T} F_t \xi_{it} \right) + O_P \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{nT}} \right) \right).
\]

Consistency follows immediately since the first term in (31) is \( O_P \left( \frac{1}{\sqrt{T}} \right) \).

**Proposition 3** (Consistency of loadings). Under Assumptions 1 through 4, as \( n,T \to \infty \)
(a) \( \min(n, \sqrt{nT}, \sqrt{T}) \left\| \lambda_i - \hat{H}' \lambda_i \right\| = O_p(1) \), uniformly in \( i \);
(b) \( \min(n, \sqrt{nT}, \sqrt{T}) \left\| \frac{\hat{\Lambda} - \hat{H} \hat{\Lambda}}{\sqrt{n}} \right\| = O_p(1) \);
(c) \( \min(n, \sqrt{nT}, \sqrt{T}) \left\| \frac{\hat{\Lambda}' \hat{A} - \hat{H}' \hat{A} \hat{H} \hat{A}}{n} \right\| = O_p(1) \).

Part (a) refines the result in Proposition 1(b). Note that for part (b) Bai and Ng (2020) get a rate \( \min(\sqrt{n}, \sqrt{T}) \).

Proposition 4. Under Assumptions 1 through 4,
(a) \( \left\| H' \left( \frac{\hat{\Lambda} \hat{A}}{n} \right) - Q_0 \right\| = o(1) \), as \( n \to \infty \);
(b) \( \left\| H' \left( \frac{\hat{\Lambda} \hat{A}}{n} \right) - \left( \frac{\hat{\Lambda} \hat{A}}{n} \right) \right\| = o_p(1) \), as \( n, T \to \infty \);
(c) \( \left\| \left( \frac{\hat{\Lambda} \hat{A}}{n} \right) - Q_0 \right\| = o_p(1) \), as \( n, T \to \infty \);

where \( Q_0 = V_0 J_0 Y_0' (\Gamma^F)^{-1/2} \), and where \( J_0 \) is an \( r \times r \) diagonal matrix with entries \( \pm 1 \), \( Y_0 \) is the \( r \times r \) matrix having as columns the normalized eigenvectors of \( (\Gamma^F)^{1/2} \Sigma_{\hat{A}} (\Gamma^F)^{1/2} \), and \( V_0 \) is the \( r \times r \) matrix of corresponding eigenvectors sorted in descending order.

Notice that rate is not needed in Proposition 4. To prove the properties of the loadings it is enough to use statement (a), and then (b) and (c) follow directly from Proposition 1(a). Still, we prove (c) also more explicitly by analogy with Proposition 1 in Bai (2003) (see Section A.5). Although such complex proof is not needed here.

Hereafter, let
\[
V_{nT} = \frac{\hat{M}^2}{n},
\]
which is the \( r \times r \) diagonal matrix with as entries the \( r \) largest non-zero eigenvalues of \( X' \hat{X} \) in decreasing order.

Theorem 1 (Asymptotic Normality of loadings). Under Assumptions 1 through 5, as \( n, T \to \infty \), for any given \( i = 1, \ldots, n \),
\[
\sqrt{T} (\hat{\lambda}_i - \hat{H}' \lambda_i) = V_{nT}^{-1} \left( \frac{\hat{\Lambda}' \hat{A}}{n} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) \to_d N \left( 0_r, \Theta_i \right),
\]
where \( \Theta_i = V_0^{-1} Q_0 \hat{\Phi}_i Q_0' V_0^{-1} \).

Proof of Theorem 1. If \( \sqrt{T}/n \to 0 \) as \( n, T \to \infty \), then, from (31), by Proposition 4(b):
\[
\sqrt{T} (\hat{\lambda}_i - \hat{H}' \lambda_i) = V_{nT}^{-1} H' \left( \frac{\hat{\Lambda}' \hat{A}}{n} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1)
\]
\[
= V_{nT}^{-1} \left( \frac{\hat{\Lambda}' \hat{A}}{n} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1). \tag{32}
\]
The proof follows from Slutsky’s theorem and (32), and because of Proposition 4(c), Lemma 8(iv) Assumption 5(a). □
6 Asymptotics for the factors

From (13) and (14)

\[
\hat{F} = X\hat{\Lambda}(\hat{\Lambda}'\hat{\Lambda})^{-1} = F\Lambda'(\Lambda'\Lambda)^{-1} + \Xi\hat{\Lambda}(\hat{\Lambda}'\hat{\Lambda})^{-1} = \frac{FA'\hat{\Lambda}}{n}\left(\frac{M^x}{n}\right)^{-1} + \frac{\Xi\hat{\Lambda}}{n}\left(\frac{M^x}{n}\right)^{-1}. \tag{33}
\]

Then,

\[
\hat{\Lambda} = \hat{\Lambda} - \Lambda\hat{H} + \Lambda\hat{H}
\]

and

\[
\Lambda = (\Lambda - \hat{\Lambda}\hat{H}^{-1}) + \hat{\Lambda}\hat{H}^{-1}, \tag{35}
\]

where

\[
\hat{H}^{-1} = \left(\frac{\tilde{M}^x}{n}\right)\left(\frac{\Lambda'\hat{\Lambda}}{n}\right)^{-1}\left(\frac{F'F}{T}\right)^{-1}, \tag{36}
\]

which, as \(n, T \to \infty\), is finite because of Lemma 11(ii).

Substituting (34) and (35) into (33):

\[
\hat{F} = \frac{F(\Lambda - \hat{\Lambda}\hat{H}^{-1})'\hat{\Lambda}}{n}\left(\frac{\tilde{M}^x}{n}\right)^{-1} + F(\hat{H}^{-1})'\left(\frac{\Lambda'\hat{\Lambda}}{n}\right)^{-1}\left(\frac{\tilde{M}^x}{n}\right)^{-1} + \frac{\Xi\hat{\Lambda}}{n}\left(\frac{\tilde{M}^x}{n}\right)^{-1}
\]

\[
= F(\hat{H}^{-1})' + \frac{F(\Lambda - \hat{\Lambda}\hat{H}^{-1})'\hat{\Lambda}}{n}\left(\frac{\tilde{M}^x}{n}\right)^{-1} + \frac{\Xi(\hat{\Lambda} - \Lambda\hat{H})}{n}\left(\frac{\tilde{M}^x}{n}\right)^{-1} + \frac{\Xi\Lambda\hat{H}}{n}\left(\frac{\tilde{M}^x}{n}\right)^{-1}, \tag{37}
\]

which is equivalent to

\[
\hat{F} - F(\hat{H}^{-1})' = \left(\frac{F(\Lambda - \hat{\Lambda}\hat{H}^{-1})'\hat{\Lambda}}{n} + \frac{\Xi(\hat{\Lambda} - \Lambda\hat{H})}{n} + \frac{\Xi\Lambda\hat{H}}{n}\right)\left(\frac{\tilde{M}^x}{n}\right)^{-1}. \tag{38}
\]

Taking the \(t\)th row of (38)

\[
\hat{F}_t - F(\hat{H}^{-1})_t' = \left(\frac{F(\Lambda - \hat{\Lambda}\hat{H}^{-1})'\hat{\Lambda}}{n} \tag{2.a} + \xi(\hat{\Lambda} - \Lambda\hat{H}) \tag{2.b} + \xi\Lambda\hat{H} \tag{2.c}\right)\left(\frac{\tilde{M}^x}{n}\right)^{-1}. \tag{39}
\]

**Proposition 5.** Under Assumptions 1 through 4, as \(n, T \to \infty\)

(a) \(\min(n, \sqrt{nT}, T)\|a\| = O_P(1)\);
(b) \(\min(n, \sqrt{nT}, T)\|b\| = O_P(1)\);
(c) \(\sqrt{n}\|c\| = O_P(1)\);

uniformly in \(t\).

Note that the equivalent of part (a) is in Bai (2003, Lemma B2) where the bound is \(O_P\left(\max\left(\frac{1}{n}, \frac{1}{T}, \frac{1}{\sqrt{nT}}\right)\right)\) which is \(O_P\left(\max\left(\frac{1}{\sqrt{n}}, \frac{1}{T}\right)\right)\) since \(\sqrt{nT} = \min(\sqrt{n}, \sqrt{T})\max(\sqrt{n}, \sqrt{T}) > [\min(\sqrt{n}, \sqrt{T})]^2\).
From (39) and Proposition 5, for any \( t = 1, \ldots, T \), we get

\[
\hat{F}_t - \hat{H}^{-1}F_t = \left( \frac{M^n}{n} \right)^{-1} \hat{H}' \left( \frac{1}{n} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + O_P \left( \frac{1}{T}, \frac{1}{\sqrt{nT}} \right).
\] (40)

Consistency follows immediately since the first term in (40) is \( O_P \left( \frac{1}{\sqrt{n}} \right) \).

**Proposition 6** (Consistency of factors). Under Assumptions 1 through 4, as \( n, T \to \infty \)

(a) \( \min(\sqrt{n}, \sqrt{nT}, T) \| \hat{F}_t - \hat{H}^{-1}F_t \| = O_P(1) \), uniformly in \( t \);

(b) \( \min(\sqrt{n}, \sqrt{nT}, T) \left\| \hat{F} - F(\hat{H}^{-1})' \right\| = O_P(1) \);

(c) \( \min(\sqrt{n}, \sqrt{nT}, T) \left\| \hat{F}^{-1} \hat{F}T - \hat{H}^{-1}F(\hat{H}^{-1})' \right\| = O_P(1) \).

This refines the result in Proposition 1(c). Note that Bai and Ng (2020) get a rate \( \min(\sqrt{n}, \sqrt{T}) \).

**Theorem 2** (Asymptotic Normality of factors). Under Assumptions 1 through 5, as \( n, T \to \infty \), for any given \( t = 1, \ldots, T \),

\[
\sqrt{n}(\hat{F}_t - \hat{H}^{-1}F_t) = V_{nT}^{-2} \left( A' \Lambda \right) \left( \frac{F'F}{T} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + o_P(1) \to_d \mathcal{N}(0_r, \Pi_t),
\]

where \( \Pi_t = (Q_0^{-1})' \Gamma_t Q_0^{-1} \).

**Proof of Theorem 2.** If \( \sqrt{n}/T \to 0 \) as \( n, T \to \infty \), by substituting in (40) the definition of \( \hat{H} \) in (28)

\[
\sqrt{n}(\hat{F}_t - \hat{H}^{-1}F_t) = V_{nT}^{-2} \hat{H}' \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + o_P(1)
= V_{nT}^{-2} \left( A' \Lambda \right) \left( \frac{F'F}{T} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + o_P(1). \] (41)

The proof follows from Slutsky’s theorem and because of Assumptions 1(c), 5(b), Lemma 8(iv), and recalling from Proposition 4 that \( Q_0 = V_0 \mathcal{J}_0 \chi_0 (\Gamma^F)^{-1/2} \) and, thus, \( Q_0^{-1} = (\Gamma^F)^{1/2} \chi_0 \mathcal{J}_0 V_0^{-1} \). \( \square \)

**7 Discussion**

**7.1 Relation to OLS**

Here we derive more intuitive asymptotic expansions which show that PCA is indeed equivalent to OLS.

By using the definition of \( \hat{H} \) in (28) and of \( \hat{F} \) in (14), and by Proposition 6(a) and 6(c), another
useful expansion of (32) is
\[
\sqrt{T(\hat{\lambda}_i - \hat{H}'\lambda_i)} = V_{nT}^{-1} \left( \frac{\hat{\Lambda}'\Lambda}{n} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1) \quad \text{from Theorem 1(a)}
\]
\[
= \hat{H}' \left( \frac{F'F}{T} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1) \quad \text{from def. of } \hat{H} \text{ in (28)}
\]
\[
= \left( \frac{\hat{H}^{-1}F'F(\hat{H}^{-1})'}{T} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{H}^{-1}F_t \xi_{it} \right) + o_p(1) \quad \text{move } \hat{H}' \text{ inside}
\]
\[
= \left( \frac{\hat{F}'\hat{F}}{T} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{F}_t \xi_{it} \right) + o_p(1) \quad \text{from consistency Prop. 6}
\]
\[
= \hat{H}^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1) \quad \text{from def. of } \hat{F} \text{ in (14)}
\]
where, for ease of comparison, in the first line we write again the second line of (32) used to derive Theorem 1. This is OLS when, for a fixed i, we regress \(x_{it}\) onto \(\hat{H}^{-1}F_t\) or, equivalently, onto \(\hat{F}_t\), which is normalized such that \(\hat{F}'\hat{F} = I_r\).

By using the definition of \(\hat{H}\) in (28) and of \(\hat{\Lambda}\) in (13), and by Proposition 3(a) and 3(c), another useful expansion of (41) is
\[
\sqrt{n}(\hat{F}_t - \hat{H}^{-1}F_t) = V_{nT}^{-2} \left( \frac{\hat{\Lambda}'\Lambda}{n} \right) \left( \frac{F'F}{T} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + o_p(1) \quad \text{from Theorem 2(a)}
\]
\[
= V_{nT}^{-1} \hat{H}' \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + o_p(1) \quad \text{from def. of } \hat{H} \text{ in (28)}
\]
\[
= \left( \frac{\hat{\Lambda}'\Lambda}{n} \right)^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \hat{\lambda}_i \xi_{it} \right) + o_p(1) \quad \text{from def. of } \hat{\Lambda} \text{ in (13)}
\]
\[
= \left( \frac{\hat{H}'\Lambda'\hat{\Lambda}}{n} \right)^{-1} \hat{H}' \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + o_p(1) \quad \text{from consistency Prop. 3}
\]
\[
= \hat{H}^{-1} \left( \frac{\hat{\Lambda}'\Lambda}{n} \right)^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) + o_p(1) \quad \text{move } H \text{ outside},
\]
which is OLS when, for a fixed t, we regress \(x_{it}\) onto \(\hat{H}'\lambda_i\) or, equivalently, onto \(\hat{\lambda}_i\).

7.2 Comparison with Approach B by Bai (2003)

Define
\[
\tilde{H} = \left( \frac{\Lambda'\Lambda}{n} \right) \left( \frac{F'\hat{F}}{T} \right) V_{nT}^{-1}.
\]
where
\[
V_{nT} = \frac{\hat{M}^2}{T},
\]
which is the $r \times r$ diagonal matrix with as entries the $r$ largest non-zero eigenvalues of $\frac{XX'}{nT}$ in decreasing order, which coincide with those of $\frac{XX'}{nT}$.

Now, it is shown that

$$
\left\| \hat{F}_t - \tilde{H}'F_t \right\| = O_P \left( \max \left( \frac{1}{\sqrt{n}}, \frac{1}{T} \right) \right),
$$

(45)

$$
\left\| \hat{\lambda}_i - \tilde{H}^{-1}\lambda_i \right\| = O_P \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{T}} \right) \right).
$$

(46)

Theorem 1 in Bai (2003) for the factors says,

$$
\sqrt{n}(\hat{F}_t - \tilde{H}'F_t) = V_{nt}^{-1} \left( \frac{\hat{F}'F}{T} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_it \right) + o_P(1).
$$

This is the only expression reported by Bai (2003) and it is the analogous of the expression we derived for the loadings in the first line of (42). Using the definition of $\tilde{H}$ in (44) we also have

$$
\sqrt{n}(\hat{F}_t - \tilde{H}'F_t) = \tilde{H}' \left( \frac{\Lambda'\Lambda}{n} \right)^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_it \right) + o_P(1).
$$

(47)

This coincides with the last line of the expression we derived for the factors in (43), thus it is OLS when, for a fixed $t$, we regress $x_{it}$ onto $\tilde{H}^{-1}\lambda_i$ or, equivalently, onto $\hat{\lambda}_i$ because of (46).

Theorem 2 in Bai (2003) for the loadings says,

$$
\sqrt{T}(\hat{\lambda}_i - \tilde{H}^{-1}\lambda_i) = V_{nt}^{-1} \left( \frac{\hat{F}'F}{T} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_it \right) + o_P(1).
$$

This is the only expression reported by Bai (2003) and it is analogous to the expression we derived for the loadings in the first line of (42), up to a factor $V_{nt}^{-1}$.

$$
\sqrt{T}(\hat{\lambda}_i - \tilde{H}^{-1}\lambda_i) = \tilde{H}' \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_it \right) + o_P(1).
$$

(48)

This coincides with the last line of the expression we derived for the factors in (42), thus it is OLS when, for a fixed $i$, we regress $x_{it}$ onto $\tilde{H}'F_t$ or, equivalently, onto $\tilde{F}_t$ because of (45), which is normalized such that $\frac{\hat{F}'F}{T} = I_r$.

Moreover, let $Q_1 = V_0^{1/2}Y_1^{'}(\Sigma_\lambda)^{-1/2}$, where $Y_1$ is the $r \times r$ matrix having as columns the normalized eigenvectors of $(\Sigma_\lambda)^{1/2}F^T(\Sigma_\lambda)^{1/2}$, and $V_0$ is the $r \times r$ matrix of corresponding eigenvalues sorted in descending order. Then, it is shown that

$$
\sqrt{T}(\hat{\lambda}_i - \tilde{H}^{-1}\lambda_i) \rightarrow_d N \left( 0_r, \Theta_i^B \right),
$$

(49)

where $\Theta_i^B = (Q_1^{-1})'\Phi_iQ_1^{-1}$. And

$$
\sqrt{n}(\tilde{F}_t - \tilde{H}'F_t) \rightarrow_d N \left( 0_r, \Pi_t^B \right),
$$

(50)
where $\Pi_t^B = V_0^{-1}Q_1\Gamma_tQ_1'V_0^{-1}$.

Now note that, since $\Upsilon_0$ are normalized eigenvectors of $\Gamma_t^F \Sigma_{\Lambda}^1 \Gamma_t^F$, then $\Upsilon_0'\Upsilon_0 = \Upsilon_0'\Upsilon_0 = I_r$

and

$\Upsilon_0 V_0^{-1} = (\Gamma_t^F)^{-1/2} \Sigma_{\Lambda}^{-1}(\Gamma_t^F)^{-1/2} \Upsilon_0$, \hspace{1cm} (51)

and, since $\Upsilon_1$ are normalized eigenvectors of $\Sigma_{\Lambda}^{-1/2} \Gamma_t^F \Sigma_{\Lambda}^{-1/2}$, then $\Upsilon_1'\Upsilon_1 = \Upsilon_1'\Upsilon_1 = I_r$ and

$V_0^{-1} \Upsilon_1' = \Upsilon_1' \Sigma_{\Lambda}^{-1/2}(\Gamma_t^F)^{-1} \Sigma_{\Lambda}^{-1/2}$. \hspace{1cm} (52)

Let us compare the asymptotic covariances in Theorems 1 and 2 with the expressions by Bai (2003).

From Theorem 1(a) and using the definition of $Q_0$

$$\text{tr} (\Theta_t) = \text{tr} (V_0^{-1}Q_0'\Phi_0 Q_0^{-1}V_0^{-1})$$
$$= \text{tr} \left( \Upsilon_0'(\Gamma_t^F)^{-1/2} \Phi_0 (\Gamma_t^F)^{-1/2} \Upsilon_0 \right)$$
$$= \text{tr} \left( (\Gamma_t^F)^{-1/2} \Phi_0 (\Gamma_t^F)^{-1/2} \right)$$
$$= \text{tr} \left( (\Gamma_t^F)^{-1} \Phi_0 \right) \hspace{1cm} (53)$$

From (49) and using the definition of $Q_1$ and (52)

$$\text{tr} (\Theta_t^F) = \text{tr} \left( (Q_1^{-1})'\Phi_1 Q_1^{-1} \right)$$
$$= \text{tr} \left( V_0^{-1/2} \Upsilon_1' \Sigma_{\Lambda}^{-1/2} \Phi_1 \Sigma_{\Lambda}^{-1/2} \Upsilon_1 V_0^{-1/2} \right)$$
$$= \text{tr} \left( V_0^{-1} \Upsilon_1' \Sigma_{\Lambda}^{-1/2} \Phi_1 \Sigma_{\Lambda}^{-1/2} \Upsilon_1 \right)$$
$$= \text{tr} \left( \Upsilon_1' \Sigma_{\Lambda}^{-1/2} (\Gamma_t^F)^{-1} \Phi_1 \Sigma_{\Lambda}^{-1/2} \Upsilon_1 \right)$$
$$= \text{tr} \left( \Sigma_{\Lambda}^{-1/2} (\Gamma_t^F)^{-1} \Phi_1 \Sigma_{\Lambda}^{-1/2} \right)$$
$$= \text{tr} \left( (\Gamma_t^F)^{-1} \Phi_1 \right) \hspace{1cm} (54)$$

From Theorem 2(a) and using the definition of $Q_0$ and (51)

$$\text{tr} (\Pi_t) = \text{tr} \left( (Q_0^{-1})'\Gamma_t Q_0^{-1} \right)$$
$$= \text{tr} \left( V_0^{-1} \Upsilon_0' (\Gamma_t^F)^{1/2} \Gamma_t (\Gamma_t^F)^{1/2} \Upsilon_0 V_0^{-1} \right)$$
$$= \text{tr} \left( \Upsilon_0' (\Gamma_t^F)^{-1/2} \Sigma_{\Lambda}^{-1/2} \Gamma_t \Sigma_{\Lambda}^{-1} (\Gamma_t^F)^{-1/2} \Upsilon_0 \right)$$
$$= \text{tr} \left( \Gamma_t^{-1/2} \Sigma_{\Lambda}^{-1} \Gamma_t \Sigma_{\Lambda}^{-1} (\Gamma_t^F)^{-1/2} \right)$$
$$= \text{tr} \left( (\Gamma_t^F)^{-1} \Sigma_{\Lambda}^{-1} \Gamma_t \Sigma_{\Lambda}^{-1} \right) \hspace{1cm} (55)$$
From (50) and using the definition of $Q_1$ and (52)

$$
\text{tr} (\Pi_t^B) = \text{tr} (V_0^{-1} Q_1 \Gamma_i' Q_1' V_0^{-1}) \\
= \text{tr} \left( V_0^{-1/2} \chi_i' \Sigma_A^{-1/2} \Gamma_i \Sigma_A^{-1/2} \chi_i V_0^{-1/2} \right) \\
= \text{tr} (V_0^{-1} \chi_i' \Sigma_A^{-1/2} \Gamma_i \Sigma_A^{-1/2} \chi_i) \\
= \text{tr} \left( \chi_i' \Sigma_A^{-1/2} (\Gamma^F)^{-1} \Sigma_A^{-1/2} \Gamma_i \right) \\
= \text{tr} \left( (\Gamma^F)^{-1} \Sigma_A^{-1} \Gamma_i \Sigma_A^{-1} \right) .
$$

(56)

Thus,

$$
\text{tr} (\Theta_t) = \text{tr} (\Theta_t^B) , \quad \text{tr} (\Pi_t) = \text{tr} (\Pi_t^B) .
$$

8 Asymptotics for the common and idiosyncratic components

For any given $i = 1, \ldots, n, t = 1, \ldots, T$, the estimator of the common component is

$$
\hat{\lambda}_it = \hat{\lambda}' \hat{F}_t ,
$$

(57)

**Theorem 3** (Asymptotic Normality of common component). Under Assumptions 1 through 5(a) and 5(b) as $n, T \to \infty$, for any given $i = 1, \ldots, n$ and $t = 1, \ldots, T$,

(a) $\min(\sqrt{n}, \sqrt{T})|\hat{\lambda}_it - \lambda_it| = \text{Op}(1)$;

(b) $(\frac{1}{n} V_T + \frac{1}{T} W_T)^{-1/2}(\hat{\lambda}_it - \lambda_it) \to_d N (0, 1)$, where $V_T = \chi_i' \Sigma_A^{-1} \Gamma_i \Sigma_A^{-1} \lambda_i$ and $W_T = F_t'(\Gamma^F)^{-1} \Phi_t (\Gamma^F)^{-1} F_t$.

**Proof of Theorem 3.** From definition (57), Propositions 3(a) and 6(a), have

$$
\hat{\lambda}_it - \lambda_it = \hat{\lambda}' \hat{F}_t - \chi_i' \hat{H} \hat{H}^{-1} F_t \\
= \lambda_i' \hat{H}(\hat{F}_t - \hat{H}^{-1} F_t) + (\hat{\lambda}_i - \hat{H}' \lambda_i)' \hat{F}_t \\
= \lambda_i' \hat{H}(\hat{F}_t - \hat{H}^{-1} F_t) + (\hat{\lambda}_i - \hat{H}' \lambda_i)' \hat{H}^{-1} F_t + (\hat{\lambda}_i - \hat{H}' \lambda_i)'(\hat{F}_t - \hat{H}^{-1} F_t) \\
= \lambda_i' \hat{H}(\hat{F}_t - \hat{H}^{-1} F_t) + (\hat{\lambda}_i - \hat{H}' \lambda_i)' \hat{H}^{-1} F_t + \text{Op} \left( \max \left( \frac{1}{n}, \frac{1}{T} \right) \right) \\
= \lambda_i' \hat{H}(\hat{F}_t - \hat{H}^{-1} F_t) + F_t'(\hat{H}^{-1})'(\hat{\lambda}_i - \hat{H}' \lambda_i) + \text{Op} \left( \max \left( \frac{1}{n}, \frac{1}{T} \right) \right) .
$$

(58)

Part (a) follows from Propositions 3(a) and 6(a).

For part (b), by using (31) and the second line of (42)

$$
\min(\sqrt{n}, \sqrt{T}) F_t'(\hat{H}^{-1})'(\hat{\lambda}_i - \hat{H}' \lambda_i) = \frac{\min(\sqrt{n}, \sqrt{T})}{\sqrt{T}} F_t'(\hat{H}^{-1})' \hat{H}^{-1} F_t \left( \frac{\sum_{t=1}^T F_t \xi_{it}}{T} \right) \\
+ \min(\sqrt{n}, \sqrt{T}) \text{Op} \left( \max \left( \frac{1}{T}, \frac{1}{\sqrt{nT}} \right) \right) \\
= \frac{\min(\sqrt{n}, \sqrt{T})}{\sqrt{T}} F_t' \left( \frac{\sum_{t=1}^T F_t \xi_{it}}{T} \right) + \text{Op} \left( \frac{1}{\sqrt{T}} \right) .
$$

(59)
Similarly, by using (40) and the last line of (43)

\[
\min(\sqrt{n}, \sqrt{T}) \lambda'_i \hat{H}(\hat{F}_t - \hat{H}^{-1}F_t) = \frac{\min(n, \sqrt{T})}{\sqrt{n}} \lambda'_i \hat{H}^{-1} \left( \frac{\Lambda' \Lambda}{n} \right)^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) \\
+ \frac{\min(n, \sqrt{T})}{\sqrt{T}} \max \left( \frac{1}{n}, \frac{1}{\sqrt{nT}} \right) + O_p \left( \frac{1}{\sqrt{n}} \right). \tag{60}
\]

By using (59) and (60) into (58),

\[
\min(n, \sqrt{T})(\hat{\chi}_{it} - \chi_{it}) = \frac{\min(n, \sqrt{T})}{\sqrt{n}} \lambda'_i \left( \frac{\Lambda' \Lambda}{n} \right)^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) \\
+ \frac{\min(n, \sqrt{T})}{\sqrt{T}} \max \left( \frac{1}{n}, \frac{1}{\sqrt{T}} \right) + O_p \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{T}} \right) \right). \tag{61}
\]

Now, by Assumptions 1(a) and 5(b), as \( n \to \infty \),

\[
\eta_{it, nT} = \lambda'_i \left( \frac{\Lambda' \Lambda}{n} \right)^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i \xi_{it} \right) \to_d \mathcal{N}(0, \lambda'_i \Sigma^{-1}_n \Gamma \Sigma^{-1}_n \lambda_i). \tag{62}
\]

Similarly, by Assumptions 1(c) and 5(a), as \( T \to \infty \),

\[
\theta_{it, nT} = F'_t \left( \frac{F'F}{T} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{i=1}^{T} F_i \xi_{it} \right) \to_d \mathcal{N}(0, F'_t \left( \Gamma^F \right)^{-1} \Phi \left( \Gamma^F \right)^{-1} F_t). \tag{63}
\]

Moreover, \( \eta_{it, nT} \) and \( \theta_{it, nT} \) are asymptotically independent because the former is the sum of cross-section random variables at time \( t \) and the latter is the sum of the \( i \)th time series.

Let \( a_{nT} = \frac{\min(n, \sqrt{T})}{\sqrt{n}} \) and \( b_{nT} = \frac{\min(n, \sqrt{T})}{\sqrt{T}} \). Then, from (61),

\[
\min(n, \sqrt{T})(\hat{\chi}_{it} - \chi_{it}) = a_{nT} \eta_{it, nT} + b_{nT} \theta_{it, nT} + O_p \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{T}} \right) \right). \tag{64}
\]

Following the same arguments in the proof of Theorem 3 in Bai (2003), because of (62) and (63), as \( n, T \to \infty \), we have

\[
a_{nT} \eta_{it, nT} + b_{nT} \theta_{it, nT} \to_d \mathcal{N}(0, a^2_{nT} V_{it} + b^2_{nT} W_{it}) \tag{65}
\]

and because of (64) and (65), as \( n, T \to \infty \),

\[
\min(n, \sqrt{T})(\hat{\chi}_{it} - \chi_{it}) \to_d \mathcal{N}(0, a^2_{nT} V_{it} + b^2_{nT} W_{it}),
\]

29
which is equivalent to
\[
\frac{\min(\sqrt{n}, \sqrt{T})(\hat{x}_{it} - x_{it})}{\sqrt{a_{nT}^2 V_{it} + b_{nT}^2 W_{it}}} \xrightarrow{d} N(0,1).
\] (66)

Now,
\[
\frac{\min(\sqrt{n}, \sqrt{T})}{\sqrt{a_{nT}^2 V_{it} + b_{nT}^2 W_{it}}} = \frac{\min(\sqrt{n}, \sqrt{T})}{\sqrt{\frac{1}{n} V_{it} + \frac{1}{T} W_{it}}} = \frac{1}{\sqrt{\frac{1}{n} V_{it} + \frac{1}{T} W_{it}}}
\]

So, from (66), as \( n, T \to \infty \),
\[
\frac{\min(\sqrt{n}, \sqrt{T})(\hat{x}_{it} - x_{it})}{\sqrt{a_{nT}^2 V_{it} + b_{nT}^2 W_{it}}} = \frac{(\hat{x}_{it} - x_{it})}{\sqrt{\frac{1}{n} V_{it} + \frac{1}{T} W_{it}}} \xrightarrow{d} N(0,1).
\]

This completes the proof. □

The estimator of the idiosyncratic component is:
\[
\hat{\xi}_{it} = x_{it} - \hat{x}_{it},
\]

Clearly, then
\[
\hat{\xi}_{it} - \xi_{it} = (x_{it} - \hat{x}_{it}) - (x_{it} - \xi_{it}) = \xi_{it} - \hat{\xi}_{it},
\] (67)
and we have \( \min(\sqrt{n}, \sqrt{T}) \)-consistency. Notice that the estimator of the idiosyncratic covariance matrix given by the sample covariance is
\[
\hat{\Gamma} = \frac{1}{T} \sum_{t=1}^{T} \xi_{it} \hat{\xi}_{it} = \hat{W}^x \hat{D}^x \hat{W}^{xt},
\] (68)
where \( \hat{D}^x \) is the \((n-r) \times (n-r)\) diagonal matrix with diagonal entries \( \hat{\mu}^x_j \), \( j = r + 1, \ldots, n \), and \( \hat{W}^x \) is the \( n \times (n-r) \) matrix having as columns the corresponding normalized eigenvectors. Therefore, such estimator has rank \( \min(n-r,T) \) and even if \( n < T \) it can never be a consistent estimators. An alternative based on shrinking is given in Fan et al. (2013).

9 Asymptotics under identifying constraints

Let us make the classical identifying assumptions used for exploratory factor analysis.

Assumption 9.
(a) For all \( n \in \mathbb{N} \), \( \Lambda_n^A \Lambda_n^A \) diagonal.
(b) For all \( T \in \mathbb{N} \), \( F_t F_t' = I_r \).

This assumption has two main implications.

Proposition 7. Under Assumptions 1 and 9
(a) \( \Lambda_n^A = \Lambda_n^M \), for all \( n \in \mathbb{N} \);
(b) \( \Lambda = V^x (M^x)^{1/2} \), for all \( n \in \mathbb{N} \);
(c) \( F_t = (M^x)^{-1/2} V^x \chi_t \) for all \( t = 1, \ldots, T \) and all \( n \in \mathbb{N} \);
\((d)\) \(\Sigma_\lambda = V_0;\)
\((e)\) \(\Gamma^F = I_r.\)

**Proposition 8.** Under Assumptions 1 through 4, and Assumptions 5(c) and 9, as \(n, T \to \infty\)
\((a)\) \(\min(\sqrt{n}, \sqrt{T})\|\hat{H} - J\| = o_p(1);\)
\((b)\) \(\min(\sqrt{n}, \sqrt{T})\|\hat{H}^{-1} - J\| = o_p(1);\)
where \(J\) is a diagonal \(r \times r\) matrix with entries \(\pm 1.\)

Note that Bai and Ng (2013) prove this with a rate \(\min(n, \sqrt{T}) = \min(n, T)\) based on Lemma B2 of Bai (2003). Here we prove it with a rate \(\min(\sqrt{n}, \sqrt{T})\) (we do not have the term \(n\)). The only thing that matters is that the rate is smaller than \(\min(\sqrt{n}, \sqrt{T})\). We now show that, in this case, PCA is equivalent to OLS, i.e. as if, when estimating the loadings, the factors were known, and, when estimating the factors, the loadings were known.

For the loadings, using Proposition 8(a) in the second line of (42), we get (note that \(\|\lambda_i\| = O(1)\) by Assumption 1(a))
\[
\sqrt{T}(\hat{\lambda}_i - J\lambda_i) = \sqrt{T}(\hat{\lambda}_i - \hat{H}'\lambda_i) + \sqrt{T}(\hat{H}' - J)\lambda_i + o_p(1)
\]
\[
= \hat{H}' \left( \frac{F'F}{T} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1)
\]
\[
= J \left( \frac{F'F}{T} \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1)
\]
\[
= J \left( \frac{1}{T} \sum_{t=1}^{T} F_t F_t' \right)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t \xi_{it} \right) + o_p(1)
\]
\[
\to_d N(0_r, (\Gamma^F)^{-1} \Phi_t (\Gamma^F)^{-1}). \quad (69)
\]

Notice that, \(J\) plays no role in the covariance since \(J\) is diagonal and \(J^2 = I_r\), and, in fact, here \(\frac{F'F}{T} = I_r\) and \(\Gamma^F = I_r\) so both could be omitted in (69).

From (69) it is clear that the PC estimator is asymptotically equivalent to the OLS estimator, \(\lambda_i^{\text{OLS}},\) we get when, for fixed \(i\), we regress \(x_{it}\) onto \(JF_t\). Indeed,
\[
\lambda_i^{\text{OLS}} - J\lambda_i = \left( \frac{1}{T} \sum_{t=1}^{T} JF_t F_t' J \right)^{-1} \left( \frac{1}{T} \sum_{t=1}^{T} JF_t x_{it} \right) - J\lambda_i
\]
\[
= \left( \frac{1}{T} \sum_{t=1}^{T} JF_t F_t' J \right)^{-1} \left( \frac{1}{T} \sum_{t=1}^{T} JF_t (F_t' J\lambda_i + \xi_{it}) \right) - J\lambda_i
\]
\[
= \left( \frac{1}{T} \sum_{t=1}^{T} JF_t F_t' J \right)^{-1} \left( \frac{1}{T} \sum_{t=1}^{T} JF_t \xi_{it} \right)
\]
\[
= J \left( \frac{1}{T} \sum_{t=1}^{T} F_t F_t' \right)^{-1} \left( \frac{1}{T} \sum_{t=1}^{T} F_t \xi_{it} \right). \quad (70)
\]

For the factors using Proposition 8(b) in the last line of (43), we get (note that \(\|F_t\| = O_p(1)\) by
Lemma 2(ii))

\[
\sqrt{n}(\hat{F}_t - JF_t) = \sqrt{n}(\hat{F}_t - \hat{H}^{-1}F_t) + \sqrt{n}(\hat{H}^{-1} - J)F_t + o_p(1)
\]

\[
= \hat{H}^{-1}\left(\frac{N'_N}{n}\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \lambda_i \xi_{it}\right) + o_p(1)
\]

\[
= J\left(\frac{N'_N}{n}\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \lambda_i \xi_{it}\right) + o_p(1)
\]

\[
= J\left(\frac{1}{n}\sum_{i=1}^{n} \lambda_i \lambda_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \lambda_i \xi_{it}\right) + o_p(1)
\]

\[
\Rightarrow_d N\left(0_r, (\Sigma'_\Lambda)^{-1}\Gamma_t(\Sigma'_\Lambda)^{-1}\right).
\] (71)

Notice that, \(J\) plays no role in the covariance since \(J\) is diagonal and \(J^2 = I_r\). From (71) it is clear that the PC estimator is asymptotically equivalent to the OLS estimator, \(F_t^{OLS}\), we get when, for fixed \(t\), we regress \(x_{it}\) onto \(J\lambda_i\). Indeed,

\[
F_t^{OLS} - JF_t = \left(\frac{1}{n}\sum_{i=1}^{n} J\lambda_i \lambda_i'J\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} J\lambda_i x_{it}\right) - JF_t
\]

\[
= \left(\frac{1}{n}\sum_{i=1}^{n} J\lambda_i \lambda_i'J\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} J\lambda_i (\lambda_i'JF_t + \xi_{it})\right) - JF_t
\]

\[
= \left(\frac{1}{n}\sum_{i=1}^{n} J\lambda_i \lambda_i'J\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} J\lambda_i \xi_{it}\right)
\]

\[
= J\left(\frac{1}{n}\sum_{i=1}^{n} \lambda_i \lambda_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \lambda_i \xi_{it}\right).
\] (72)
A Proofs of of main results

A.1 Proof of Proposition 1

First notice that \( \text{rk} \left( \frac{A}{n} \right) = r \) for all \( n \), since \( \text{rk}(\Gamma^F) = r \) by Assumption 1(b) and \( \text{rk} \left( \frac{M}{n} \right) = r \) by Lemma 1(iv). Indeed, \( \text{rk} \left( \frac{M}{n} \right) \leq \min(\text{rk}(\Gamma^F), \text{rk} \left( \frac{A}{n} \right)) \). This holds for all \( n > N \) and since eigenvalues are an increasing sequence in \( n \). Therefore, \( (\frac{A}{n})^{-1} \) is well defined for all \( n \) and \( A \) admits a left inverse.

Second, since

\[
\frac{\Gamma}{n} = V \frac{M}{n} V' = \frac{\Lambda}{\sqrt{n}} \frac{\Gamma^F}{\sqrt{n}} \Lambda',
\]

(A1)

the columns of \( \frac{V \frac{M}{n}}{\sqrt{n}} \) and the columns of \( \frac{\Lambda \frac{\Gamma^F}{\sqrt{n}}}{\sqrt{n}} \) must span the same space.

So there exists an \( r \times r \) invertible matrix \( K \) such that

\[
A(\Gamma^F)^{1/2} K = V \frac{M}{n} (M^n)^{1/2}
\]

(A2)

Therefore, from (A2)

\[
K = (\Gamma^F)^{-1/2} (\Lambda' \Lambda)^{-1} \Lambda' V \frac{M}{n} (M^n)^{1/2}
\]

(A3)

which is also obtained by linear projection, and also

\[
K^{-1} = (M^n)^{-1/2} V \frac{M}{n} \frac{\Lambda}{\sqrt{n}} \frac{\Gamma^F}{\sqrt{n}}
\]

(A4)

which are both finite and positive definite because of Lemma 9.

Now, from (A2) and (A3)

\[
V \frac{M}{n} (M^n)^{1/2} = A(\Gamma^F)^{1/2} K = A(\Lambda' \Lambda)^{-1} \Lambda' V \frac{M}{n} (M^n)^{1/2}
\]

(A5)

Let

\[
\mathcal{H} = (\Lambda' \Lambda)^{-1} \Lambda' V \frac{M}{n} (M^n)^{1/2} J,
\]

(A6)

which is finite and positive definite because of Lemma 10.

Now, because of Lemmas 5(iii), 5(iv), 7(i), using (13), (A5), and (A6),

\[
\left\| \frac{\hat{A} - \Lambda \mathcal{H}}{\sqrt{n}} \right\| = \left\| \hat{V}^x \left( \frac{M^x}{n} \right)^{1/2} - V \frac{M}{n} \frac{M^n}{n}^{1/2} J \right\| = \left\| \hat{V}^x \left( \frac{M^x}{n} \right)^{1/2} - V \frac{M}{n} \frac{M^n}{n}^{1/2} J \right\|
\]

\[
\leq \left\| \hat{V}^x - V \frac{M}{n} \right\| \left\| \frac{M}{n} \right\| + \frac{1}{\sqrt{n}} \left\{ \left( \frac{M^x}{n} \right)^{1/2} - (M^n)^{1/2} \right\} \left\| V \frac{M}{n} \right\|
\]

\[
+ \left\| \hat{V}^x - V \frac{M}{n} \right\| \left\{ \frac{1}{\sqrt{n}} \left( \frac{M^x}{n} \right)^{1/2} - (M^n)^{1/2} \right\}
\]

\[
= O_p \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{n}} \right) \right) + O_p \left( \max \left( \frac{1}{n^2}, \frac{1}{n^3} \right) \right),
\]

since \( \|V\| = 1 \) because eigenvectors are normalized. This proves part (a).

For part (b), since from (13), for any \( i = 1, \ldots, n \), we have

\[
\hat{X}_i = \hat{V}_i^x \left( \frac{M}{n} \right)^{1/2},
\]

(A7)
then, because of Lemma 5(iii), 6(ii), and 6(iii), using (A5) and (A7),

\[
\|\hat{X} - X^\prime\| = \left\| \sqrt{n}\tilde{v}_1' \left( \frac{\tilde{M}^x}{n} \right)^{1/2} - \sqrt{n}\tilde{v}_1' \left( \frac{M^x}{n} \right)^{1/2} \right\|
\]

\[
\leq \left\| \sqrt{n}\tilde{v}_1' - \sqrt{n}\tilde{v}_1' \right\| \left\| \frac{M^x}{n} \right\| + \frac{1}{\sqrt{n}} \left\{ \left( \frac{\tilde{M}^x}{n} \right)^{1/2} - \left( \frac{M^x}{n} \right)^{1/2} \right\} \left\| \sqrt{n}\tilde{v}_1' \right\|
\]

\[
+ \left\| \sqrt{n}\tilde{v}_1' - \sqrt{n}\tilde{v}_1' \right\| \left\{ \frac{1}{\sqrt{n}} \left\{ \left( \frac{\tilde{M}^x}{n} \right)^{1/2} - \left( \frac{M^x}{n} \right)^{1/2} \right\} \right\|
\]

\[
= O_p \left( \max \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}} \right) \right) + O_p \left( \max \left( \frac{1}{n}, \frac{1}{T} \right) \right) .
\]

This proves part (b).

For part (c), using (14)

\[
\tilde{F} = X\hat{\Lambda}(\hat{\Lambda}\hat{\Lambda})^{-1} = \frac{X\hat{\Lambda}}{n} \left( \frac{\hat{M}^x}{n} \right)^{-1} = \frac{X(\hat{\Lambda} - \Lambda\mathcal{H} + \Lambda\mathcal{H})}{n} \left( \frac{\tilde{M}^x}{n} \right)^{\prime -1}
\]

\[
= \frac{F\Lambda'\Lambda\mathcal{H}}{n} \left( \frac{\tilde{M}^x}{n} \right)^{-1} + \left\{ \frac{X(\hat{\Lambda} - \Lambda\mathcal{H})}{n} + \frac{\Xi\Lambda\mathcal{H}}{n} \right\} \left( \frac{\tilde{M}^x}{n} \right)^{-1} . \tag{A8}
\]

Now, from (A5) and (A6)

\[
\mathcal{H} = (\Gamma^F)^{1/2}KJ , \tag{A9}
\]

and so from (A9)

\[
\mathcal{H}^{-1} = JK^{-1}(\Gamma^F)^{-1/2} . \tag{A10}
\]

which is finite and positive definite because of Lemma 10(ii). From (A10) and (A4):

\[
\frac{\Lambda'\Lambda\mathcal{H}}{n} \left( \frac{\tilde{M}^x}{n} \right)^{-1} = \frac{\Lambda'\Lambda}{n} (\Lambda'\Lambda)^{-1} \Lambda'\mathcal{H}(\Lambda'\Lambda)^{1/2} \left( \frac{\tilde{M}^x}{n} \right)^{-1}
\]

\[
= \Lambda'\mathcal{H}(\Lambda'\Lambda)^{-1/2}J = (\Gamma^F)^{-1/2}(\mathcal{K}^{-1})'J = (\mathcal{H}^{-1})' . \tag{A11}
\]

Then, because of Lemma 5(iii), 2(ii), 7(ii), 7(iv), and 10(i), by using (A11),

\[
\left\| \frac{\Lambda'\Lambda\mathcal{H}}{n} \left( \frac{\tilde{M}^x}{n} \right)^{-1} - (\mathcal{H}^{-1})' \right\| = \left\| \frac{\Lambda'\Lambda\mathcal{H}}{n} \left( \frac{\tilde{M}^x}{n} \right)^{-1} - \frac{\Lambda'\Lambda\mathcal{H}}{n} \left( \frac{M^x}{n} \right)^{-1} \right\|
\]

\[
\leq \left\| \frac{\Lambda'\Lambda}{n} \right\| \left\| \mathcal{H} \right\| \left\| \frac{\tilde{M}^x}{n} \right\|^{-1} - \left\| \frac{\tilde{M}^x}{n} \right\|^{-1} \right\|
\]

\[
\leq \left\| \frac{\Lambda'\Lambda}{n} \right\| \left\| \mathcal{H} \right\| \left\| \frac{\tilde{M}^x}{n} \right\|^{-1} \left\| \tilde{M}^x - M^x \right\| \left\| \frac{M^x}{n} \right\|^{-1} \right\|
\]

\[
= O_p \left( \max \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}} \right) \right) . \tag{A12}
\]
By using (A12) in (A8), because of part (a), Lemma 2(ii), 2(iv), 3(ii), and Lemma 7(iv), and 10(i)

\[
\left\| \frac{\hat{F} - F(\mathcal{H}^{-1})'}{\sqrt{T}} \right\| \leq \left\| \frac{\Lambda' \mathcal{H} n}{\sqrt{\Lambda}} \left( \frac{\hat{M}}{n} \right)^{-1} - (\mathcal{H}^{-1})' \right\| \left\| F \right\| \quad + \left\{ \left\| \frac{(\hat{\Lambda} - \Lambda \mathcal{H})}{\sqrt{n}} \right\| \left\| X \right\| + \left\| \frac{\Lambda \mathcal{H}}{n \sqrt{T}} \right\| \right\} \left\| \hat{F} \right\| \quad + \left\{ \left\| \frac{\Lambda}{n \sqrt{T}} \right\| \right\} \left\| \hat{F} \right\| \\
= \text{OP} \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{T}} \right) \right) + \text{OP} \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{T}} \right) \right) + \text{OP} \left( \frac{1}{\sqrt{n}} \right).
\]

This proves part (c).

Similarly, for part (d), from (14) and (A8), for any \( t = 1, \ldots, T \), we have

\[
\hat{F}_t = (\hat{\Lambda}' \hat{\Lambda})^{-1} \hat{\Lambda}' x_t = \left( \frac{\hat{M}}{n} \right)^{-1} \frac{\mathcal{H}' \Lambda' \Lambda F}{n} + \left( \frac{\hat{M}}{n} \right)^{-1} \left\{ (\hat{\Lambda} - \Lambda \mathcal{H}) x_t + \mathcal{H}' \Lambda' \epsilon_t \right\}.
\]

(A13)

Because of Assumption 1(a) and Lemma 1(ii), we have

\[
\max_{t=1, \ldots, T} E \left\{ \left\| \frac{\Lambda' \epsilon_t}{n} \right\|^2 \right\} = \max_{t=1, \ldots, T} \frac{1}{n^2} \sum_{k=1}^{r} \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} \mathcal{E} (\hat{\lambda}_k \lambda_{ij}) \right] = \max_{t=1, \ldots, T} \frac{1}{n^2} \sum_{k=1}^{r} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{E}[\lambda_{ik} \lambda_{ij}] \leq \frac{r M^2_\lambda}{n^2} \max_{t=1, \ldots, T} \frac{1}{n} \sum_{k=1}^{r} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{E}[\epsilon_{it} \epsilon_{jt}] \leq \frac{r M^2_\lambda}{n^2} \max_{t=1, \ldots, T} \frac{1}{n} \sum_{k=1}^{r} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{E}[\epsilon_{it} \epsilon_{jt}] \leq \frac{r M^2_\lambda}{n^2},
\]

(A14)

since \( M_\lambda \) is independent of \( t \). This is a special case of Lemma 3(ii).

Therefore, by using (A12) and (A14) in (A13), because of part (a), Lemma 2(ii), 2(iv), 7(iv), 10(i)

\[
\left\| \hat{F}_t - \mathcal{H}^{-1} F_t \right\| \leq \left( \frac{\hat{M}}{n} \right)^{-1} \frac{\mathcal{H}' \Lambda' \Lambda F}{n} + \left( \frac{\hat{M}}{n} \right)^{-1} \left\{ (\hat{\Lambda} - \Lambda \mathcal{H}) x_t + \mathcal{H}' \Lambda' \epsilon_t \right\}.
\]

This proves part (d) and it completes the proof. \( \square \)

A.2 Proof of Proposition 2

For part (a), for any \( i = 1, \ldots, n \), by Assumption 1(a),

\[
\left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \mathcal{F}_t \epsilon_{jt} \mathcal{X}_t \right\| \leq \left\| \lambda_i \right\| \left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \mathcal{F}_t \epsilon_{jt} \mathcal{X}_t \right\| \leq M_\lambda \left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \mathcal{F}_t \epsilon_{jt} \mathcal{X}_t \right\|.
\]

(A15)
Then, by Assumptions 1(a) and 4

\[
E \left[ \left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} F_t \xi_{jt} \mathbf{X}_j \right\|^2 \right] \leq E \left[ \left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} F_t \xi_{jt} \mathbf{X}_j \right\|^2 \right] = \frac{1}{n^2T^2} \sum_{k=1}^{r} \sum_{h=1}^{r} E \left[ \left( \sum_{t=1}^{T} F_{kt} \sum_{j=1}^{n} \xi_{jt} [A_{jh}] \right)^2 \right]
\]

\[
\leq \frac{r^2}{n^2T^2} \max_{h,k=1,\ldots,r} \sum_{t=1}^{T} \sum_{s=1}^{T} E \left[ F_{kt} F_{ks} \right] \left( \sum_{j=1}^{n} \sum_{t=1}^{T} E \left[ \xi_{jt} \xi_{ts} \right] [A_{jh}] [A_{th}] \right)
\]

\[
= \frac{r^2}{n^2T^2} \max_{k=1,\ldots,r} \sum_{t=1}^{T} \sum_{s=1}^{T} E \left[ F_{kt} F_{ks} \right] \left( \sum_{j=1}^{n} \sum_{t=1}^{T} E \left[ \xi_{jt} \xi_{ts} \right] \right) \leq \left\{ \frac{r^2}{T} \max_{k=1,\ldots,r} \sum_{t=1}^{T} \sum_{s=1}^{T} E \left[ F_{kt} F_{ks} \right] \right\} \left( \sum_{t,s=1}^{T} \sum_{j=1}^{n} \sum_{\ell=1}^{T} E \left[ \xi_{jt} \xi_{\ell s} \right] \right) . \tag{A16}
\]

Now, by Cauchy-Schwarz inequality, for any \( k = 1, \ldots, r \),

\[
\left\| \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} F_{kt} F_{ks} \right\| \leq \left( \frac{1}{T} \sum_{t=1}^{T} F_{kt}^2 \right)^{1/2} \left( \frac{1}{T} \sum_{s=1}^{T} F_{ks}^2 \right)^{1/2} . \tag{A17}
\]

and, by Assumption 1(b), using (A17) and again Cauchy-Schwarz inequality,

\[
\max_{k=1,\ldots,r} \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E \left[ F_{kt} F_{ks} \right] = \max_{k=1,\ldots,r} \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E \left[ F_{kt} F_{ks} \right] \leq \max_{k=1,\ldots,r} \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E \left[ F_{kt}^2 \right] \left( \frac{1}{T} \sum_{s=1}^{T} E \left[ F_{ks}^2 \right] \right)^{1/2}
\]

\[
\leq \max_{k=1,\ldots,r} \left( \frac{1}{T} \sum_{t=1}^{T} E \left[ F_{kt}^2 \right] \right)^{1/2} \left( \frac{1}{T} \sum_{s=1}^{T} E \left[ F_{ks}^2 \right] \right)^{1/2}
\]

\[
\leq \max_{k=1,\ldots,r} \left( \frac{1}{T} \sum_{t=1}^{T} E \left[ F_{kt}^2 \right] \right)^{1/2} \max_{k=1,\ldots,r} \left( \frac{1}{T} \sum_{s=1}^{T} E \left[ F_{ks}^2 \right] \right)^{1/2}
\]

\[
= \max_{k=1,\ldots,r} \frac{1}{T} \sum_{t=1}^{T} E \left[ F_{kt}^2 \right] \leq \max_{k=1,\ldots,r} \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E \left[ \xi_{jt} \xi_{ts} \right] \leq \frac{M_F}{T} \tag{A18}
\]

since \( M_F \) is independent of \( t \) and where \( \mathbf{\eta}_k \) is an \( r \)-dimensional vector with one in the \( k \)th entry and zero elsewhere. And, because of Lemma 1(ii)

\[
\max_{t,s=1,\ldots,T} \frac{1}{nT} \sum_{j=1}^{n} \sum_{t=1}^{T} E \left[ \xi_{jt} \xi_{ts} \right] \leq \frac{M_{\xi}}{nT} . \tag{A19}
\]

since \( M_{\xi} \) is independent of \( n \) and \( t \). By substituting (A18) and (A19) into (A16),

\[
E \left[ \left\| \sum_{t=1}^{T} \sum_{j=1}^{n} F_t \xi_{jt} \mathbf{X}_j \right\|^2 \right] \leq \frac{r^2 M_F M_{\xi}^2 M_{\xi}}{nT} . \tag{A20}
\]

By substituting (A20) into (A15), we prove part (a).
For part (b), for any $i = 1, \ldots, n$, because of Lemma 2(i),
\[
\left\| \frac{1}{nT} \sum_{t=1}^{T} \xi_{it} F'_t x_j \right\| \leq \left\| \frac{1}{nT} \sum_{t=1}^{T} \xi_{it} F'_t (A' A) \right\| \leq \left\| \frac{1}{T} \sum_{t=1}^{T} \xi_{it} F'_t \right\| \left\| \frac{A}{\sqrt{n}} \right\| ^2 \leq \left\| \frac{1}{T} \sum_{t=1}^{T} \xi_{it} F'_t \right\| M_A^2. \tag{A21}
\]

Then, by Assumptions 4 and 2(b) and using (A18)
\[
E \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} \xi_{it} F'_t \right\| ^2 \right] = \frac{1}{T^2} \sum_{j=1}^{r} E \left[ \left( \sum_{t=1}^{T} \xi_{jt} F'_j \right) ^2 \right] \leq \frac{r}{T^2} \max_{j=1, \ldots, r} \sum_{t=1}^{T} \sum_{s=1}^{T} E[\xi_{jt} F'_j \xi_{is} F'_s] = \frac{r}{T^2} \max_{j=1, \ldots, r} \sum_{t=1}^{T} \sum_{s=1}^{T} E[F'_j F'_s] E[\xi_{jt} \xi_{is}] \leq \left\{ \frac{1}{T} \max_{j=1, \ldots, r} \sum_{t=1}^{T} \sum_{s=1}^{T} E[F'_j F'_s] \right\} \left\{ \frac{1}{T} \max_{t, s=1, \ldots, n} \left| E[\xi_{jt} \xi_{is}] \right| \right\} \leq r M_F \frac{M_\xi}{T}, \tag{A22}
\]

since $M_\xi$ is independent of $i$. Or, equivalently, by Lemma 1(iii) and Cauchy-Schwarz inequality
\[
E \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} \xi_{it} F'_t \right\| ^2 \right] \leq \frac{r}{T^2} \max_{j=1, \ldots, r} \sum_{t=1}^{T} \sum_{s=1}^{T} E[F'_j F'_s] E[\xi_{jt} \xi_{is}] \leq \left\{ \frac{1}{T} \max_{j=1, \ldots, r} \sum_{t=1}^{T} \sum_{s=1}^{T} E[F'_j F'_s] \right\} \left\{ \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E[\xi_{jt} \xi_{is}] \right\} \leq \frac{r M_F M_\xi}{T} \frac{M_\xi(1 + \rho)}{1 - \rho} \leq \frac{r M_F M_\xi}{T}, \tag{A23}
\]

since $M_F$ is independent of $t$ and $M_\xi$ is independent of $i$. Notice that $M_\xi \leq M_\xi$. Notice that (A23) is a special case of Lemma 3(i). By substituting (A22), or (A23), into (A21), we prove part (b).

For part (c), for any $i = 1, \ldots, n$, because of Assumption 1(a),
\[
\left\| \frac{1}{nT} \sum_{t=1}^{T} \xi_{it} F'_t x_j \right\| = \left\{ \sum_{k=1}^{r} \left( \frac{1}{nT} \sum_{t=1}^{T} \xi_{it} \xi_{jt} \lambda_{jk} \right) \right\}^{1/2} \leq \sqrt{r} M_A \left\{ \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{it} \xi_{jt} \right\} \leq \sqrt{r} M_A \left\{ \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{it} \xi_{jt} \right\} + \left\{ \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} E[\xi_{jt} \xi_{jt}] \right\}. \tag{A24}
\]

Then, by Assumption 2(b),
\[
\left\| \frac{1}{nT} \sum_{t=1}^{T} \xi_{it} F'_t \right\| \leq \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \left| E[\xi_{jt} \xi_{jt}] \right| \leq \max_{j=1, \ldots, r} \frac{1}{n} \sum_{j=1}^{n} \left| E[\xi_{jt} \xi_{jt}] \right| \leq \frac{1}{n} \sum_{j=1}^{n} M_{ij} \leq \frac{M_\xi}{n}. \tag{A25}
\]

since $M_\xi$ is independent of $i$ and $t$. Moreover, by Assumption 2(c),
\[
E \left[ \left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{it} \xi_{jt} \right\| ^2 \right] \leq K_\xi \frac{n}{T}. \tag{A26}
\]

By substituting (A25) and (A26) into (A24), we prove part (c).
For part (d), for any \( i = 1, \ldots, n \), because of Assumption 1(a)

\[
\left\| \frac{1}{nT} \lambda_i \sum_{t=1}^{T} \sum_{j=1}^{n} F_t \xi_{jt} (\hat{\lambda}_j - \lambda_j) \right\| \leq M_\lambda \left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} F_t \xi_{jt} (\hat{\lambda}_j - \lambda_j) \right\|
\]

\[
= M_\lambda \left\| \frac{F' \Xi (\hat{\Lambda} - \Lambda) H}{nT} \right\| \leq M_\lambda \left\| \frac{\hat{\Lambda} - \Lambda H}{\sqrt{n}} \right\| .
\]  \quad (A27)

Then, by using Lemma 3(i) and Proposition 1(a) in (A27), we prove part (d).

For part (e), for any \( i = 1, \ldots, n \),

\[
\left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{it} F'_t \lambda_j (\hat{\lambda}_j - \lambda_j) \right\| \leq \left\| \frac{1}{T} \sum_{t=1}^{T} \xi_{it} F'_t \right\| \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \frac{\hat{\Lambda} - \Lambda H}{\sqrt{n}} \right\| .
\]

(A28)

By substituting part (ii), Lemma 2(i), and part (a) of Proposition 1 into (A28), we prove part (e).

Finally, for part (f), for any \( i = 1, \ldots, n \), let \( \zeta_i = (\xi_{i1} \cdots \xi_{iT})' \), then

\[
\left\| \frac{1}{nT} \sum_{t=1}^{T} \sum_{j=1}^{n} \xi_{it} \xi_{jt} (\hat{\lambda}_j - \lambda_j) \right\| = \left\| \frac{\zeta_i' \Xi (\hat{\Lambda} - \Lambda) H}{nT} \right\| \leq \left\| \frac{\zeta_i' \Xi}{\sqrt{nT}} \right\| \left\| \frac{\hat{\Lambda} - \Lambda H}{\sqrt{n}} \right\|. \quad (A29)
\]

Then, by the \( C_r \)-inequality with \( r = 2 \),

\[
\left\| \frac{\zeta_i' \Xi}{\sqrt{nT}} \right\|^2 = \left\| \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} \xi_{it} \xi_{jt} \right\|^2 = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{T} \sum_{t=1}^{T} \xi_{it} \xi_{jt} \right)^2
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{T} \sum_{t=1}^{T} [\xi_{it} \xi_{jt} - \mathbb{E}[\xi_{it} \xi_{jt}]] + \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\xi_{it} \xi_{jt}] \right)^2
\]

\[
\leq \frac{2}{n} \sum_{j=1}^{n} \left\{ \left( \frac{1}{T} \sum_{t=1}^{T} [\xi_{it} \xi_{jt} - \mathbb{E}[\xi_{it} \xi_{jt}]] \right)^2 + \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\xi_{it} \xi_{jt}] \right)^2 \right\} \quad (A30)
\]

By taking the expectation of (A30), and because of Assumption 2(c) and Lemma 1(v),

\[
\mathbb{E} \left[ \left\| \frac{\zeta_i' \Xi}{\sqrt{nT}} \right\|^2 \right] = \frac{2}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \left( \frac{1}{T} \sum_{t=1}^{T} [\xi_{it} \xi_{jt} - \mathbb{E}[\xi_{it} \xi_{jt}]] \right)^2 \right] + \frac{2}{n} \sum_{j=1}^{n} \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\xi_{it} \xi_{jt}] \right)^2
\]

\[
\leq 2 \max_{j=1, \ldots, n} \mathbb{E} \left[ \left( \frac{1}{T} \sum_{t=1}^{T} [\xi_{it} \xi_{jt} - \mathbb{E}[\xi_{it} \xi_{jt}]] \right)^2 \right] + \frac{2}{n} \sum_{j=1}^{n} \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\xi_{it} \xi_{jt}] \right)^2
\]

\[
\leq \frac{2K_\xi}{T} + \frac{2}{n} \sum_{t=1}^{T} \sum_{s=1}^{T} \mathbb{E}[\xi_{it} \xi_{jt}] \sum_{s=1}^{T} \mathbb{E}[\xi_{is} \xi_{js}]
\]

\[
\leq \frac{2K_\xi}{T} + \max_{t=1, \ldots, n} \frac{2}{n} \sum_{s=1}^{T} \max_{s=1, \ldots, T, j=1, \ldots, n} \mathbb{E}[\xi_{is} \xi_{js}]
\]

\[
\leq \frac{2K_\xi}{T} + \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \max_{i,j=1, \ldots, n} \mathbb{E}[\xi_{ij}] \leq \frac{2K_\xi}{T} + \frac{2M_\xi}{n} ||\xi|| \leq \frac{2K_\xi}{T} + \frac{2M_\xi^2}{n}, \quad (A31)
\]

since \( K_\xi \) is independent of \( j \) and \( M_\xi \) is independent of \( i, j, t, \) and \( s \) and where \( \xi \) is an \( n \)-dimensional vector.
with one in the $i$th entry and zero elsewhere. By substituting (A31) and part (a) of Proposition 1 into (A29), we prove part (f). This completes the proof. □

### A.3 Proof of Proposition 3

Part (a), is proved using Proposition 2 in (30) and since $\left\| \left( \frac{\hat{M}_x}{n} \right)^{-1} H \right\| = O_p(1)$ because of Lemma 7(iv) and 10(i).

For part (b), from (29)

$$
\left\| \frac{\hat{\Lambda} - \hat{\Lambda} \hat{H}}{\sqrt{n}} \right\| \leq \left( \left\| \frac{F \varepsilon}{\sqrt{nT}} \right\| \frac{\Lambda}{\sqrt{n}} \right)^2 + \left( \left\| \frac{\varepsilon F}{\sqrt{nT}} \right\| \frac{\Lambda}{\sqrt{n}} \right) \left\| H \right\| \left\| \frac{\hat{M}_x}{n} \right\|^{-1} \left\| \frac{\hat{M}_x}{n} \right\|^{-1}
$$

and the proof of part (b) follows from Proposition 1(a) and Lemma 2(i), 3(i), 3(iii), 3(iv), 7(iv), 10(i).

For part (c), because of part (b), and Lemma 2(i) and 11(i), we have

$$
\left\| \frac{\hat{\Lambda} \Lambda}{n} - \frac{\hat{H} \Lambda \hat{H}}{n} \right\| = \left\| \left( \hat{\Lambda}' - \hat{H} \Lambda + \hat{H} \Lambda \right) \left( \hat{\Lambda} - \hat{\Lambda} \hat{H} + \hat{\Lambda} \hat{H} \right) - \frac{\hat{H} \Lambda \hat{H}}{n} \right\|
$$

$$
\leq 2 \left\| \frac{\hat{\Lambda} - \hat{\Lambda} \hat{H}}{\sqrt{n}} \right\| \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \hat{H} \right\| + \left\| \frac{\hat{\Lambda} - \hat{\Lambda} \hat{H}}{\sqrt{n}} \right\|^2
$$

$$
= O_p \left( \max \left( \frac{1}{n}, \frac{1}{\sqrt{nT}} \right) \right).
$$

This proves part (c) and completes the proof. □

### A.4 Proof of Proposition 4

Start with part (a). From (A6) and (A4)

$$
H' \left( \frac{\Lambda \Lambda}{n} \right) = J \left( M \right)^{1/2} V^\top \Lambda \left( \Lambda' \Lambda \right)^{-1} \left( \frac{\Lambda' \Lambda}{n} \right) = J \left( \frac{M}{n} \right)^{1/2} V^\top \Lambda \left( \Lambda' \Lambda \right)^{-1} \left( \frac{M}{n} \right) K^{-1} \left( \Gamma^F \right)^{-1/2}.
$$

(A32)

Then, from (A3)

$$
V^\top = \Lambda \left( \Gamma^F \right)^{1/2} K \left( M \right)^{-1/2}
$$

(A33)

thus, from (A33) and (A4)

$$
J \left( \frac{M}{n} \right)^{-1} K = J \left( \frac{M}{n} \right)^{-1/2} V^\top \Lambda \left( \Gamma^F \right)^{1/2}
$$

$$
= J \left( \frac{M}{n} \right)^{-1} K' \left( \Gamma^F \right)^{1/2} \Lambda' \Lambda \left( \Gamma^F \right)^{1/2}
$$

$$
= J K' \left( \Gamma^F \right)^{1/2} \Lambda' \Lambda \left( \Gamma^F \right)^{1/2}.
$$

(A34)
And, from (A3)

\[ \text{JK} \text{K}'J = J(\Gamma^F)^{-1/2}(n'\Lambda^{-1}A'V)(\Gamma^F)^{-1/2}(M'\alpha)(\Gamma^F)^{-1/2}V^\Lambda_{\alpha}(\Gamma^F)^{-1/2}J \]
\[ = J(\Gamma^F)^{-1/2}(n'\Lambda^{-1}A'V)(\Gamma^F)^{-1/2}V^\Lambda_{\alpha}(\Gamma^F)^{-1/2}J \]
\[ = J(\Gamma^F)^{-1/2}(n'\Lambda^{-1}A'V)(\Gamma^F)^{-1/2}V^\Lambda_{\alpha}(\Gamma^F)^{-1/2}J \]
\[ = J(\Gamma^F)^{-1/2}n(\Gamma^F)^{-1/2}J = I_r. \]  

(A35)

Therefore, the columns of \( \text{JK} \) are the normalized eigenvectors of \((\Gamma^F)^{1/2}(n'\Lambda^{-1}A'V)(\Gamma^F)^{1/2} \) with eigenvalues \( \frac{m_n^x}{n} \) (notice that \( J \left( \frac{M_n}{n} \right) = \left( \frac{M_n}{n} \right) J \)). Moreover, by Assumption 1(a)

\[ \lim_{n \to \infty} \left\| (\Gamma^F)^{1/2} \frac{n'\Lambda^{-1}A'V}{n}(\Gamma^F)^{-1/2} - (\Gamma^F)^{-1/2}n(\Gamma^F)^{-1/2} \right\| = 0. \]  

(A36)

Letting, \( V_0 \) be the matrix of eigenvalues of \((\Gamma^F)^{1/2}n(\Gamma^F)^{-1/2} \) sorted in descending order, from (A36) we also have (this is proved also in Lemma 8(i))

\[ \lim_{n \to \infty} \left\| \frac{M_n}{n} - V_0 \right\| = 0. \]  

(A37)

Let \( \Upsilon_0 \) be the normalized eigenvectors of \((\Gamma^F)^{1/2}n(\Gamma^F)^{-1/2} \). Hence, by continuity of eigenvectors, and since the eigenvalues \( \frac{m_n^x}{n} \) are distinct because of Assumption 3, from Theorem 2 in Yu et al. (2015) by Lemma 8(iii) and using (A36) it follows that

\[ \lim_{n \to \infty} \left\| \text{JK} - \Upsilon_0 \text{F} \right\| \leq \lim_{n \to \infty} \frac{2^{3/2} \sqrt{r}}{\mu_r(V_0)} \left\| (\Gamma^F)^{-1/2}n(\Gamma^F)^{1/2} - (\Gamma^F)^{-1/2}n(\Gamma^F)^{1/2} \right\| \leq 0, \]  

(A38)

where \( F \) is an \( r \times r \) diagonal matrix with entries \( \pm 1 \), which is in general different from \( J \). Finally, since \( \Upsilon_0 \) is an orthogonal matrix, we have \( F = F' = 0 \), so from (A38)

\[ \lim_{n \to \infty} \left\| K^{-1}J - F \right\| = 0. \]  

(A39)

By using (A39) and (A37) into (A32) and since \( K^{-1}J = JK^{-1} \), we have

\[ \lim_{n \to \infty} \left\| \text{H} \left( \frac{n'\Lambda^{-1}A'}{n} \right) - V_0 F \Upsilon_0 (\Gamma^F)^{-1/2} \right\| = 0. \]

By defining \( Q_0 = V_0 F \Upsilon_0 (\Gamma^F)^{-1/2} \), we prove part (a).

Part (b) follows from Proposition 1(a) and Lemma 2(i), and since

\[ \left\| \hat{\Lambda}'n - \frac{\text{H}' \hat{\Lambda}'n}{\sqrt{n}} \right\| \leq \left\| \hat{\Lambda}' - \text{H}' \hat{\Lambda}' \right\| \frac{\sqrt{n}}{\sqrt{n}} = a_p(1). \]  

(A40)

Part (c) follows from parts (a) and (b) and Proposition 1(a) and since

\[ \left\| \hat{\Lambda}'n - Q_0 \right\| \leq \left\| \hat{\Lambda}'n - \frac{\text{H}' \hat{\Lambda}'n}{\sqrt{n}} \right\| + \left\| \text{H}' \hat{\Lambda}'n - Q_0 \right\| = a_p(1). \]  

(A41)

This completes the proof. \( \square \)
A.5 Alternative proof of Proposition 4(c)

Recall (26), i.e.,

\[
\frac{X'X}{nT} \frac{\Lambda}{n} = \frac{\Lambda \tilde{M}^x}{n}. \tag{A42}
\]

Then, from (A42) we have

\[
\frac{1}{n} \left( \frac{F'F}{T} \right)^{1/2} \frac{X'X}{nT} \frac{\Lambda}{n} = \left( \frac{F'F}{T} \right)^{1/2} \frac{\Lambda \tilde{M}^x}{n}. \tag{A43}
\]

Then, substituting \( X'X = (\Lambda F' + \Xi') (F' \Lambda + \Xi) \) into (A43)

\[
\left( \frac{F'F}{T} \right)^{1/2} \frac{\Lambda}{n} \left( \frac{F'F}{T} \right) \frac{\Lambda}{n} + D_{nT} = \left( \frac{F'F}{T} \right)^{1/2} \frac{\Lambda \tilde{M}^x}{n}, \tag{A44}
\]

where

\[
D_{nT} = \left( \frac{F'F}{T} \right)^{1/2} \left\{ \frac{\Lambda' \Lambda' \Xi \Xi}{nT} + \frac{\Lambda' \Xi' F \Lambda' \Xi}{nT} + \frac{1}{nT} \frac{\Lambda' \Xi' \Xi' \Xi}{n} \right\} \nonumber
\]

\[
= \left( \frac{F'F}{T} \right)^{1/2} \left\{ \frac{\Lambda' \Lambda' \Xi \Xi}{nT} + \frac{\Lambda' \Xi' F \Lambda' \Xi}{nT} + \frac{1}{nT} \frac{\Lambda' \Xi' \Xi' \Xi}{n} \right\} \mathcal{H} \nonumber
\]

\[
+ \left( \frac{F'F}{T} \right)^{1/2} \left\{ \frac{\Lambda' \Lambda' \Xi (\hat{\Lambda} - \Lambda \mathcal{H})}{nT} + \frac{\Lambda' \Xi' F (\hat{\Lambda} - \Lambda \mathcal{H})}{nT} + \frac{1}{nT} \frac{\Lambda' \Xi' (\hat{\Lambda} - \Lambda \mathcal{H})}{n} \right\}. \tag{A45}
\]

Let,

\[
B_{nT} = \left( \frac{F'F}{T} \right)^{1/2} \left\{ \frac{\Lambda' \Lambda \Xi }{nT} \right\} \left( \frac{F'F}{T} \right)^{1/2}, \quad R_{nT} = \left( \frac{F'F}{T} \right)^{1/2} \left\{ \frac{\Lambda' \hat{\Lambda} }{n} \right\}. \tag{A46}
\]

Then, notice that because of Proposition 1(a) and Lemma 2(i) and 3(v)

\[
\left\| R_{nT} - \left( \frac{F'F}{T} \right)^{1/2} \left\{ \frac{\Lambda' \Lambda \mathcal{H} }{nT} \right\} \right\| \leq \left\| \frac{F'F}{T} \right\|^{1/2} \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \frac{\hat{\Lambda} - \Lambda \mathcal{H} }{\sqrt{n}} \right\| = O_p \left( \max \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}} \right) \right).
\]

Moreover, \( \| F^F - \Gamma^F \| = O_p(1) \) because of Lemma 4(i), with \( \Gamma^F \) positive definite because of Assumption 1(b), and \( \left\{ \frac{\Lambda' \Lambda \mathcal{H} }{nT} \right\} - \Sigma \mathcal{H} = o(1) \) because of Assumption 1(a), with \( \Sigma \mathcal{H} \) positive definite because of Assumption 1(a) and Lemma 10(ii). Then, \( \| R_{nT} \| = O_p(1) \). Therefore, (A44) can be written as

\[
\left( B_{nT} + D_{nT} \right) R_{nT} = R_{nT} \left( \frac{\tilde{M}^x}{n} \right). \tag{A47}
\]

So the columns of \( R_{nT} \) are the non-normalized eigenvectors of \( \left( B_{nT} + D_{nT} \right) \), which has as normalized eigenvectors the columns of \( \Upsilon_{nT} \) so that we can write

\[
R_{nT} = \Upsilon_{nT} (V_{nT})^{1/2}. \tag{A48}
\]

where \( V_{nT} \) is a diagonal matrix with entries the diagonal elements of \( R_{nT} R_{nT}' \). Thus, (A47) can be written as:

\[
\left( B_{nT} + D_{nT} \right) \Upsilon_{nT} = \Upsilon_{nT} \left( \frac{\tilde{M}^x}{n} \right). \nonumber
\]
Now, from (A45), because of Lemma 2, 3, 10, and Proposition 1(a),

$$\|D_{nT}\| \leq \left\| \left( \frac{F'F}{T} \right)^{1/2} \left\{ \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{F'F}{\sqrt{\psi nT}} \right) \left( \frac{F'F}{\sqrt{T}} \right) \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{\Lambda^\prime \Lambda}{n} \right) \right\} \right\| \|H\| \right.$$

$$+ \left\| \left( \frac{F'F}{T} \right)^{1/2} \left\{ \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{F'F}{\sqrt{\psi nT}} \right) \left( \frac{F'F}{\sqrt{T}} \right) \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{\Lambda^\prime \Lambda}{n} \right) \right\} \right\| \|\Lambda - \Lambda H\| \sqrt{n}$$

$$= O_p \left( \max \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{nT}} \right) \right) = o_p(1). \hspace{2cm} (A49)$$

While because of Lemma 4(i) and Assumption 1(a),

$$\left\| B_{nT} - (\Gamma^F)^{1/2} \Sigma_{\Lambda}(\Gamma^F)^{1/2} \right\| = o_p(1). \hspace{2cm} (A50)$$

Hence, by continuity of eigenvectors, and since the eigenvalues $\tilde{M}_r$ are distinct because of Assumption 3 and Lemma 5(iii), from Theorem 2 in Yu et al. (2015) by Lemma 8(ii) and 8(iii) and using (A36) it follows that

$$\left\| Y_{nT} - Y_0 \mathcal{J}_0 \right\| \leq \frac{2^{3/2} \sqrt{T}}{\mu_v(V_0)} \left\| B_{nT} - (\Gamma^F)^{1/2} \Sigma_{\Lambda}(\Gamma^F)^{1/2} \right\| = o_p(1), \hspace{2cm} (A51)$$

where $\mathcal{J}_0$ is an $r \times r$ diagonal matrix with entries $\pm 1$.

Furthermore, from (A46) and (A48)

$$\left( \frac{\Lambda^\prime \Lambda}{n} \right) = \left( \frac{F'F}{T} \right)^{-1/2} R_{nT} = \left( \frac{F'F}{T} \right)^{-1/2} Y_{nT}(V_{nT})^{1/2}. \hspace{2cm} (A52)$$

Notice that $\left\| \left( \frac{F'F}{T} \right)^{-1/2} \right\| = O_p(1)$ because of Lemma 4(i) and Assumption 1(b).

Now, by Proposition 1(a), Lemma 2(i) and 3(v), and using (A6)

$$R_{nT}R'_{nT} = \left( \frac{F'F}{T} \right)^{1/2} \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{F'F}{T} \right)^{1/2}$$

$$= \left( \frac{F'F}{T} \right)^{1/2} \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{\Lambda^\prime \Lambda}{n} \right) \left( \frac{F'F}{T} \right)^{1/2} + o_p(1) \hspace{2cm} (A53)$$

because of Lemma 4(i) and Assumption 1(a). Since the eigenvalues of $(\Gamma^F)^{1/2} \Sigma_{\Lambda}\Gamma^F \Sigma_{\Lambda}(\Gamma^F)^{1/2}$ are the same as those of $(\Gamma^F)^{1/2} \Sigma_{\Lambda}(\Gamma^F)^{1/2}$, then, from (A53) we have that

$$\left\| V_{nT} - V_0^2 \right\| = o_p(1). \hspace{2cm} (A54)$$
Finally, from (A52), because of Lemma 4(i) and using (A51) and (A54), we have:

\[ \left\| \frac{\Lambda' \hat{\Lambda}}{n} - (\Gamma^F)^{1/2} Y_0 V_0 J_0 \right\| = \left\| \frac{\Lambda' \hat{\Lambda}}{n} - Q_0 \right\| = o_p(1). \]

This proves Proposition 4(c).

### A.6 Proof of Proposition 5

For part (a):

\[
\frac{\Lambda'(\Lambda - \Lambda \hat{H}^{-1})F}{n} = \frac{\Lambda'(\Lambda - \Lambda \hat{H}) \hat{H}^{-1} F}{n} 
= \left\{ \frac{\Lambda - \Lambda \hat{H}'}{n} \hat{\Lambda} + \frac{\Lambda - \Lambda \hat{H}'}{n} \hat{\Lambda} \right\} \hat{H}^{-1} F, 
\]

Then, from (A55)

\[
\left\| \frac{\Lambda'(\Lambda - \Lambda \hat{H}^{-1})F}{n} \right\| \leq \left\{ \left\| \frac{\Lambda - \Lambda \hat{H}'}{n} \hat{\Lambda} \right\| + \left\| \frac{\Lambda - \Lambda \hat{H}'}{n} \hat{\Lambda} \right\| \right\} \left\| \hat{H}^{-1} \right\| \left\| F \right\|, 
\]

First, consider I in (A56). From (29)

\[
I = \frac{\Lambda - \Lambda \hat{H}'}{n} \hat{\Lambda} 
= \left( \frac{\bar{M}^c}{n} \right)^{-1} \Lambda' \Lambda F' \Sigma \Lambda \hat{H} 
+ \left( \frac{\bar{M}^c}{n} \right)^{-1} \Lambda' \Lambda F' \Lambda \hat{H} 
+ \left( \frac{\bar{M}^c}{n} \right)^{-1} \Lambda' \Lambda F' \Sigma \Lambda \hat{H} 
+ \left( \frac{\bar{M}^c}{n} \right)^{-1} \Lambda' \Lambda F' \Lambda \hat{H} 
\]

Then, because of (A16) and (A20) in the proof of Proposition 2(a),

\[
E \left[ \left\| \frac{F' \Sigma \Lambda}{n T} \right\|^2 \right] = \frac{1}{n^2 T^2} \sum_{k=1}^{r} \sum_{h=1}^{r} E \left[ \sum_{i=1}^{T} F_{ki} \xi_{hi} \sum_{j=1}^{n} \lambda_{jh} \right]^2 \leq \frac{r^2 M_F M^2 M^2}{n T}. 
\]

Therefore, by Assumption 1(a), Lemma (7)(iv), 11(i), 10(i), and using (A58), we get

\[
\| I_a \| \leq \left\| \left( \frac{\bar{M}^c}{n} \right)^{-1} \right\| \left\| \Lambda' \Lambda \right\| \left\| \frac{F' \Sigma \Lambda}{n T} \right\| \left\| \hat{H} \right\| = o_p \left( \frac{1}{\sqrt{n T}} \right), 
\]

\[
\| I_b \| \leq \left\| \left( \frac{\bar{M}^c}{n} \right)^{-1} \right\| \left\| \Lambda' \Lambda \right\| \left\| \frac{F' \Sigma \Lambda}{n T} \right\| \left\| \hat{H} \right\| = o_p \left( \frac{1}{\sqrt{n T}} \right). 
\]
Moreover, because of Lemma 2(i), 3(iii), 7(iv), 11(i), and 10(i),

\[
\|I_c\| \leq \left\| \left( \frac{\hat{M}^x}{n} \right)^{-1} \| \mathcal{H} \| \left\| \frac{\Lambda \hat{\Xi} \Xi}{n^{3/2}T} \right\| \| \hat{\mathbf{H}} \| = O_p \left( \max \left( \frac{1}{n}, \frac{1}{n \sqrt{T}} \right) \right). \tag{A61}
\]

Similarly, because of Proposition 1(a), Lemma 2(i), 3(i), 3(iv), 7(iv), and 11(i),

\[
\|I_d\| \leq \left\| \left( \frac{\hat{M}^x}{n} \right)^{-1} \right\| \left\| \frac{\hat{\Lambda} - \Lambda \mathcal{H}}{\sqrt{n}} \right\| \left\| \frac{\Xi' F' \Lambda' \mathcal{H}}{n^2 T} \right\| \left\| \hat{\mathbf{H}} \| = O_p \left( \max \left( \frac{1}{n \sqrt{T}}, \frac{1}{T} \right) \right) , \tag{A62}
\]

\[
\|I_e\| \leq \left\| \left( \frac{\hat{M}^x}{n} \right)^{-1} \right\| \left\| \frac{\hat{\Lambda} - \Lambda \mathcal{H}}{\sqrt{n}} \right\| \left\| \frac{\Xi' \Xi}{\sqrt{n^2 T}} \right\| \left\| \frac{\Lambda \mathbf{H}}{\sqrt{n}} \right\| = O_p \left( \max \left( \frac{1}{n^2}, \frac{1}{n \sqrt{T}}, \frac{1}{T} \right) \right) . \tag{A63}
\]

and, because of Proposition 1(a), Lemma 2(i), 3(iii), 7(iv), and 11(i),

\[
\|I_f\| \leq \left\| \left( \frac{\hat{M}^x}{n} \right)^{-1} \right\| \left\| \frac{\hat{\Lambda} - \Lambda \mathcal{H}}{\sqrt{n}} \right\| \left\| \frac{\Xi' \Xi}{\sqrt{n^2 T}} \right\| \left\| \frac{\Lambda \mathbf{H}}{\sqrt{n}} \right\| = O_p \left( \max \left( \frac{1}{n^2}, \frac{1}{n \sqrt{T}}, \frac{1}{T} \right) \right) . \tag{A64}
\]

By using (A59), (A60), (A61), (A62), (A63), and (A64) into (A57)

\[
\|I\| = O_p \left( \max \left( \frac{1}{n}, \frac{1}{n \sqrt{T}}, \frac{1}{T} \right) \right) . \tag{A65}
\]

Second, consider \(II\) in (A56). From Proposition 3(b),

\[
\|II\| \leq \frac{1}{n} \left\| \hat{\Lambda} - \Lambda \mathcal{H} \right\|^2 = O_p \left( \max \left( \frac{1}{n}, \frac{1}{T} \right) \right) . \tag{A66}
\]

And, by using (A65) and (A66) in (A56), because of Lemma 2(ii), and Lemma 11(i), we prove part (a).

For part (b), from (30) we have

\[
\frac{\xi_i (\hat{\Lambda} - \Lambda \mathcal{H})}{n} = \frac{\xi_i \Lambda \mathcal{H}}{n^2 T} \hat{\mathcal{H}} \left( \frac{\hat{M}^x}{n} \right)^{-1} + \frac{\xi_i \Xi' F' \Lambda' \mathcal{H}}{n^2 T} \hat{\mathcal{H}} \left( \frac{\hat{M}^x}{n} \right)^{-1} + \frac{\xi_i \Xi' \Xi \mathcal{H}}{n^2 T} \hat{\mathbf{H}} \left( \frac{\hat{M}^x}{n} \right)^{-1}
\]

\[
+ \frac{\xi_i \hat{\Lambda} \mathcal{H}}{n^2 T} \left( \frac{\hat{M}^x}{n} \right)^{-1} + \frac{\xi_i \Xi' F' \Lambda' \mathcal{H}}{n^2 T} \left( \frac{\hat{M}^x}{n} \right)^{-1} + \frac{\xi_i \Xi' \Xi \mathcal{H}}{n^2 T} \left( \frac{\hat{M}^x}{n} \right)^{-1}
\]

\[
= III_a + III_b + III_c + III_d + III_e + III_f, \text{ say.} \tag{A67}
\]

Then, because of Lemma 2(i), 2(iii), and Lemma 7(iv), 10(i), and using (A57) in the proof of part (a)

\[
\|III_a\| \leq \left\| \frac{\xi_i}{\sqrt{n}} \right\| \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \frac{F' \Xi \Lambda}{n^2 T} \right\| \left\| \mathcal{H} \right\| \left\| \left( \frac{\hat{M}^x}{n} \right)^{-1} \right\| = O_p \left( \frac{1}{\sqrt{n T}} \right) . \tag{A68}
\]
Now, by Assumptions 1(a), 1(b), 2(c), 4, and Lemma 1(i)

\[
\mathbb{E} \left[ \left\| \frac{\xi_k^T \mathbf{F}}{nT} \right\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{nT} \sum_{t=1}^n \sum_{s=1}^T \xi_{it} \xi_{is} \mathbf{F} \right\|^2 \right] = \frac{1}{n^2T^2} \sum_{k=1}^r \max_{i_1=1} \ldots \max_{i_2=1} \mathbb{E} \left[ \xi_{i_1t} \xi_{i_2t} \xi_{i_1s} \xi_{i_2s} \mathbf{F}_{ks_1} \mathbf{F}_{ks_2} \right]
\]

\[
\leq \frac{r}{n^2T^2} \sum_{k=1}^r \max_{i_1=1} \ldots \max_{i_2=1} \mathbb{E} \left[ \xi_{i_1t} \xi_{i_2t} \xi_{i_1s} \xi_{i_2s} \mathbf{F}_{ks_1} \mathbf{F}_{ks_2} \right]
\]

\[
\leq r \max_{k=1} \ldots \max_{s, 1^2} \mathbb{E} \left[ \xi_{i_1t} \xi_{i_2t} \xi_{i_1s} \xi_{i_2s} \mathbf{F}_{ks_1} \mathbf{F}_{ks_2} \right]
\]

\[
+ \frac{1}{n^2T^2} \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{s_1=1}^s \sum_{s_2=1}^s \mathbb{E} \left[ \xi_{i_1t} \xi_{i_2t} \xi_{i_1s} \xi_{i_2s} \mathbf{F}_{ks_1} \mathbf{F}_{ks_2} \right]
\]

\[
\leq r \max_{k=1} \ldots \max_{s, 1^2} \mathbb{E} \left[ \xi_{i_1t} \xi_{i_2t} \xi_{i_1s} \xi_{i_2s} \mathbf{F}_{ks_1} \mathbf{F}_{ks_2} \right]
\]

\[
\sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{s_1=1}^s \sum_{s_2=1}^s \mathbb{E} \left[ \xi_{i_1t} \xi_{i_2t} \xi_{i_1s} \xi_{i_2s} \mathbf{F}_{ks_1} \mathbf{F}_{ks_2} \right]
\]

\[
\leq \frac{r M_F K_\xi}{nT} + \frac{r M_F M_\xi M_{3\xi}}{nT}, \tag{A69}
\]

since \(K_\xi, M_\xi, \) and \(M_{3\xi}\) are independent of \(i_1\) and \(i_2,\) and \(M_F\) is independent of \(s,\) and where \(\eta_k\) is an \(r\)-dimensional vector with one in the \(k\)th entry and zero elsewhere. Notice that

\[
\frac{1}{n^2T^2} \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{s_1=1}^s \sum_{s_2=1}^s \mathbb{E} \left[ \xi_{i_1t} \xi_{i_2t} \xi_{i_1s} \xi_{i_2s} \mathbf{F}_{ks_1} \mathbf{F}_{ks_2} \right] = \mathbb{E} \left[ \frac{1}{nT} \sum_{i=1}^n \sum_{s=1}^s \sum_{t=1}^t \left( \xi_{it} \xi_{is} - \mathbb{E} [\xi_{it} \xi_{is}] \right) \right]^2.
\]

So by Lemma 3(iv), 7(iv), 10(i), and using (A69), we get

\[
\|III_6\| \leq \frac{\|\xi_k^T \mathbf{F}\|}{\sqrt{n}} \|A A^T\| \|H\| \left( \frac{\tilde{M}_n}{n} \right)^{-1} = O_P \left( \frac{1}{\sqrt{nT}} \right). \tag{A70}
\]

Similarly, because of Lemma 2(ii), 2(iii), and Lemma 3(iii), 7(iv), and 10(i),

\[
\|III_c\| \leq \frac{\|\xi_k^T \mathbf{F}\|}{\sqrt{n}} \|\xi_k^T \mathbf{F}\| \|H\| \left( \frac{\tilde{M}_n}{n} \right)^{-1} = O_P \left( \max \left( \frac{1}{nT} \right) \right). \tag{A71}
\]

Then, notice that, because of Assumption 1(a), Lemma 1(ii),

\[
\mathbb{E} \left[ \frac{\|\xi_k^T \mathbf{F}\|^2}{\sqrt{n}} \right] = \frac{1}{r} \sum_{k=1}^r \mathbb{E} \left[ \sum_{i=1}^n \xi_{it} \lambda_{ik} \right]^2 \leq \frac{1}{n} \max_{k=1} \ldots \max_{s, 1^2} \sum_{i=1}^n \sum_{j=1}^j \mathbb{E} |\xi_{it} \xi_{jt}| \left| \lambda_{ik} \lambda_{jk} \right|
\]

\[
\leq r \frac{M_\xi^2}{n} \sum_{i=1}^n \sum_{j=1}^j \mathbb{E} |\xi_{it} \xi_{jt}| \leq r M_\xi^2 M_\xi, \tag{A72}
\]

since \(M_\xi\) does not depend on \(t.\) Notice that (A72) is proved also in (A14) in the proof of Proposition 1(d), but
we repeat it here for convenience. Therefore, by Proposition 1(a), Lemma 3(i), 7(iv), and 10(i), and using (A72)

\[ \|III_d\| \leq \left\| \frac{\xi \Lambda}{\sqrt{n}} \right\| \left\| \frac{F'\Xi}{\sqrt{n}T} \right\| \left\| \frac{\hat{\Lambda} - \Lambda \mathcal{H}}{\sqrt{n}} \right\| \left\| \hat{\mathcal{H}} \right\| \left\| \left( \frac{\hat{M}^2}{n} \right)^{-1} \right\| = O_P \left( \max \left( \frac{1}{n \sqrt{T}}, \frac{1}{T} \right) \right). \]  

(Similarly, by Proposition 1(a), Lemma 2(i), 7(iv), and 10(i), and using (A69),

\[ \|III_l\| \leq \left\| \frac{\xi \Xi F}{nT} \right\| \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \frac{\hat{\Lambda} - \Lambda \mathcal{H}}{\sqrt{n}} \right\| \left\| \hat{\mathcal{H}} \right\| \left\| \left( \frac{\hat{M}^2}{n} \right)^{-1} \right\| = O_P \left( \max \left( \frac{1}{n^{3/2} \sqrt{T}}, \frac{1}{nT} \right) \right). \]  

And, finally, by Proposition 1(a), Lemma 2(iii), and Lemma 3(iii), 7(iv), and 10(i),

\[ \|III_f\| \leq \left\| \frac{\xi \Xi F}{nT} \right\| \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \frac{\hat{\Lambda} - \Lambda \mathcal{H}}{\sqrt{n}} \right\| \left\| \hat{\mathcal{H}} \right\| \left\| \left( \frac{\hat{M}^2}{n} \right)^{-1} \right\| = O_P \left( \max \left( \frac{1}{n^2}, \frac{1}{n \sqrt{T}}, \frac{1}{T} \right) \right). \]  

Hence, by using (A68), (A70), (A71), (A73), (A74), and (A75) in (A67), we prove part (b).

Last, for part (c), by Lemma 11(i), using (A72) in the proof of part (b),

\[ \left\| \frac{\xi \Lambda \hat{H}}{n} \right\| \leq \left\| \frac{\xi \Lambda}{n} \right\| \left\| \hat{H} \right\| = O_P \left( \frac{1}{\sqrt{n}} \right). \]  

Notice that this is a special case of Lemma 3(ii). This completes the proof. □

A.7 Proof of Proposition 6

Part (a) is proved by using Proposition 5 in (39) and since \( \left\| \left( \frac{\hat{M}^2}{n} \right)^{-1} \right\| = O_P(1) \) because of Lemma 7(iv).

For part (b), from (38) and using (A56) in the proof of Proposition 5(a)

\[ \left\| \frac{\hat{F} - F(\hat{H}^{-1})}{\sqrt{T}} \right\| \leq \left( \left\| \frac{\Xi F}{\sqrt{T}} \right\| \left\| \frac{\Lambda - \hat{\Lambda} \hat{H}^{-1} \Lambda}{n} \right\| + \left\| \Xi(\hat{\Lambda} - \Lambda \mathcal{H}) \right\| \right) \left\| \hat{\mathcal{H}} \right\| \left\| \left( \frac{\hat{M}^2}{n} \right)^{-1} \right\|. \]  

Then, from the proof of Proposition 5(a), we have

\[ \left\| \frac{\Lambda - \hat{\Lambda} \hat{H}^{-1} \Lambda}{n} \right\| = O_P \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right). \]  

Moreover, from (A67) in the proof of Proposition 5(b)

\[ \Xi(\hat{\Lambda} - \Lambda \mathcal{H}) = \frac{\Xi \Lambda F' \Xi \Lambda}{n^2 T^{3/2}} \mathcal{H} \left( \frac{\hat{M}^2}{n} \right)^{-1} + \frac{\Xi \Xi F' \Lambda \Lambda \mathcal{H}}{n^2 T^{3/2}} \left( \frac{\hat{M}^2}{n} \right)^{-1} + \frac{\Xi \Xi \Lambda \Xi \Lambda \mathcal{H}}{n^2 T^{3/2}} \left( \frac{\hat{M}^2}{n} \right)^{-1} \]

\[ + \frac{\Xi \Lambda F' \Xi}{n^2 T^{3/2}} (\hat{\Lambda} - \Lambda \mathcal{H}) \left( \frac{\hat{M}^2}{n} \right)^{-1} + \frac{\Xi \Xi F' \Lambda}{n^2 T^{3/2}} (\hat{\Lambda} - \Lambda \mathcal{H}) \left( \frac{\hat{M}^2}{n} \right)^{-1} + \frac{\Xi \Xi \Xi \Xi}{n^2 T^{3/2}} (\hat{\Lambda} - \Lambda \mathcal{H}) \left( \frac{\hat{M}^2}{n} \right)^{-1} \]

\[ = IV_a + IV_b + IV_c + IV_d + IV_e + IV_f, \text{ say.} \]  

(A79)
Then, because of Lemma 2(i), 2(iii), and Lemma 7(iv), 10(i), and using (A58) in the proof of Proposition 5(a)
\[
\|IV_a\| \leq \left\| \frac{\Xi}{\sqrt{nT}} \right\| \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| F' \Xi \Lambda \right\| \|\mathcal{H}\| \left( \frac{\tilde{M}^x}{n} \right)^{-1} = Op \left( \frac{1}{\sqrt{nT}} \right).
\]
(A80)

Now, by Assumptions 1(a), 1(b), 2(c), 4, and Lemma 1(i) (see also (A69) in the proof of Proposition 5(b)), letting
\[
\xi = (\xi_1 \cdots \xi_t)',
\]
\[
E \left[ \left\| \frac{\Xi' F}{nT^{3/2}} \right\|^2 \right] = E \left[ \left\| \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \xi_{it} F'_t \right\|^2 \right] = \frac{1}{n^2 T^3} \sum_{t=1}^{T} \sum_{k=1}^{r} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{s_1=1}^{T} \sum_{s_2=1}^{T} E \left[ \xi_{i_1 t} \xi_{i_2 t} \xi_{i_1 s_1} \xi_{i_2 s_2} F'_{ks_1} F_{ks_2} \right]
\]
\[
\leq \frac{r}{nT^3} \max_{k=1, \ldots, r} \max_{i_1=1}^{n} \max_{i_2=1}^{n} \max_{s_1=1}^{T} \max_{s_2=1}^{T} E \left[ \xi_{i_1 t} \xi_{i_2 t} \xi_{i_1 s_1} \xi_{i_2 s_2} \right] E \left[ F_{ks_1} F_{ks_2} \right]
\]
\[
\leq \frac{r}{nT^3} \max_{k=1, \ldots, r} \max_{s_1=1}^{n} \max_{s_2=1}^{n} \max_{t=1}^{T} E \left[ F_{ks_1} F_{ks_2} \right]
\]
\[
\leq r \left\{ \max_{k=1, \ldots, r} \max_{s_1=1}^{n} \max_{s_2=1}^{n} \max_{t=1}^{T} E \left[ F_{ks_1} F_{ks_2} \right] \right\}
\]
\[
\leq \frac{r}{nT^3} \max_{k=1, \ldots, r} \max_{i_1=1}^{n} \max_{i_2=1}^{n} \max_{s_1=1}^{T} \max_{s_2=1}^{T} E \left[ \xi_{i_1 t} \xi_{i_2 t} \xi_{i_1 s_1} \xi_{i_2 s_2} \right]
\]
\[
+ \frac{1}{n^2 T^2} \max_{t=1}^{T} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{s_1=1}^{T} \sum_{s_2=1}^{T} E \left[ \xi_{i_1 t} \xi_{i_2 t} \right] E \left[ \xi_{i_1 s_1} \xi_{i_2 s_2} \right]
\]
\[
\leq \frac{r M_F K_s}{nT} + \frac{r M_F M_F M_F}{nT},
\]
(A81)
since $K_s$, $M_F$, and $M_F$ are independent of $i_1$, $i_2$ and $t$, and $M_F$ is independent of $s$, and where $\eta_k$ is an $r$-dimensional vector with one in the $k$th entry and zero elsewhere. So by Lemma 3(iv), 7(iv), 10(i), and using (A81), we get
\[
\|IV_b\| \leq \left\| \frac{\Xi' F}{nT^{3/2}} \right\| \left\| \frac{\Lambda'}{\sqrt{n}} \right\| \|\mathcal{H}\| \left( \frac{\tilde{M}^x}{n} \right)^{-1} = Op \left( \frac{1}{\sqrt{nT}} \right).
\]
(A82)

Similarly, because of Lemma 2(i), 2(iii), 3(iii), 7(iv), and 10(i),
\[
\|IV_c\| \leq \left\| \frac{\Xi}{\sqrt{nT}} \right\| \left\| \frac{\Xi' \Lambda}{n^{3/2} T} \right\| \|\mathcal{H}\| \left( \frac{\tilde{M}^x}{n} \right)^{-1} = Op \left( \max \left( \frac{1}{n}, \frac{1}{nT} \right) \right).
\]
(A83)

Moreover, by Proposition 1(a), Lemma 3(i), 3(ii), and Lemma 7(iv) and 10(i)
\[
\|IV_d\| \leq \left\| \frac{\Xi' \Xi}{\sqrt{nT}} \right\| \left\| \frac{F' \Xi}{\sqrt{n}} \right\| \|\tilde{\Lambda} - \Lambda \mathcal{H}\| \|\mathcal{H}\| \left( \frac{\tilde{M}^x}{n} \right)^{-1} = Op \left( \max \left( \frac{1}{n}, \frac{1}{nT} \right) \right).
\]
(A84)
Similarly, by Proposition 1(a), Lemma 2(i), 7(iv), and 10(i), and using (A81),
\[
\|IV\r_\leq \left(\frac{\Xi^\prime F}{nT^3/2}\right) \left(\frac{\lambda}{\sqrt{n}}\right) \left(\frac{\hat{\lambda} - \hat{\Lambda}\hat{\mathcal{H}}}{\sqrt{n}}\right) \left(\|\mathcal{H}\|\right) \left(\frac{\hat{M}^2}{n}\right)^{-1} = O_p\left(\max\left(\frac{1}{n^{3/2}T}, \frac{1}{\sqrt{nT}}\right)\right). \tag{A85}
\]
And, finally, by Proposition 1(a), Lemma 2(iii), and Lemma 3(iii), 7(iv), and 10(i),
\[
\|IV\r_\leq \left(\frac{\Xi}{\sqrt{nT}}\right) \left(\frac{\Xi'}{nT}\right) \left(\frac{\hat{\lambda} - \hat{\Lambda}\hat{\mathcal{H}}}{\sqrt{n}}\right) \left(\|\mathcal{H}\|\right) \left(\frac{\hat{M}^2}{n}\right)^{-1} = O_p\left(\max\left(\frac{1}{n^2}, \frac{1}{nT}, \frac{1}{T}\right)\right). \tag{A86}
\]
By using (A80), (A82), (A83), (A84), (A85), and (A86) in (A79) we get
\[
\left|\frac{\Xi(\hat{\lambda} - \hat{\Lambda}\hat{\mathcal{H}})}{n\sqrt{T}}\right| = O_p\left(\max\left(\frac{1}{n}, \frac{1}{\sqrt{nT}}, \frac{1}{T}\right)\right). \tag{A87}
\]
The proof of part (b) follows from Lemma 2(ii), 3(ii), 11(i), and 7(iv), and using (A78) and (A87) in (A77).

For part (c), because of part (b), and Lemma 2(ii) and 11(ii), we have
\[
\left\|\hat{F}'\hat{F}' \frac{\hat{F}'\hat{F}(\hat{\mathcal{H}})^{-1}}{\hat{\mathcal{H}}^{-1}}\right\| = \left\|\left(\hat{F}' - \hat{\mathcal{H}}^{-1}\hat{F}' + \hat{\mathcal{H}}^{-1}\hat{F}'\right)\left(\hat{F}' - \hat{\mathcal{H}}^{-1}\hat{F}'\right) - \hat{\mathcal{H}}^{-1}\hat{F}'\hat{F}(\hat{\mathcal{H}})^{-1}\right\|
\leq 2\left\|\frac{\hat{F}' - \hat{\mathcal{H}}^{-1}\hat{F}'}{\sqrt{T}}\right\| \left\|\hat{F}'\hat{F}\hat{\mathcal{H}}^{-1}\right\| + \left\|\frac{\hat{F}' - \hat{\mathcal{H}}^{-1}\hat{F}'}{\sqrt{T}}\right\|^2
= O_p\left(\max\left(\frac{1}{\sqrt{n}}, \frac{1}{T}, \frac{1}{\sqrt{nT}}\right)\right).
\]
This proves part (c) and completes the proof. □

A.8 Proof of Proposition 7

For part (a), Assumption 9(b) implies $\Gamma^\times = \Lambda\Lambda'$. Therefore, since the non-zero eigenvalues of $\Gamma^\times_n$ are the same as the $r$ eigenvalues of $\Lambda\Lambda'\times_n$, which is diagonal by Assumption 9(a). Then, we must have, for all $n \in \mathbb{N}$,
\[
\frac{\Lambda\Lambda'}{n} = \frac{M^\times}{n}.
\]
This proves part (a).

For part (b), since $\Gamma^\times = V^\times M^\times V^\times'$, it must be that
\[
\Lambda K_* = V^\times (M^\times)^{1/2}, \tag{A88}
\]
for some $r \times r$ invertible $K_*$. Notice that since $K_*$ is a special case of the matrix $K$ defined in (A3) in the proof of Proposition 1(a), then $K_*$ is finite and positive definite because of Lemma 9.

Now, since $\text{rk}(\frac{\Lambda\Lambda'}{n}) = r$ for all $n > N$ (see the proof of Proposition 1(a)):
\[
K_* = (\Lambda\Lambda')^{-1}\Lambda'V^\times(M^\times)^{1/2} = (M^\times)^{-1}\Lambda'V^\times(M^\times)^{1/2}, \tag{A89}
\]
which is also obtained by linear projection, and
\[
K_*^{-1} = (M^\times)^{-1/2}V^\times A. \tag{A90}
\]
Moreover, from (A89), because of Assumption 9(b) and part (a):
\[
K, K' = (M^\delta)^{-1} \Lambda' \Lambda M^\delta V^\delta V' \Lambda (M^\delta)^{-1} \\
= (M^\delta)^{-1} \Lambda' \Lambda M^\delta (M^\delta)^{-1} \\
= (M^\delta)^{-1} \Lambda' \Lambda M^\delta (M^\delta)^{-1} \\
= I_r. 
\]
(A91)

So because of (A91), we have that \( K \) is an orthogonal matrix, i.e., \( K = K^{-1} \). Finally, by (A89) we also have
\[
V^\delta = \Lambda K, (M^\delta)^{-1/2} 
\]
(A92)

and by substituting (A92) into (A90), because of (A91),
\[
K^{-1} = (M^\delta)^{-1} K' \Lambda' \Lambda = (M^\delta)^{-1} K^{-1} \Lambda' \Lambda, 
\]
which is equivalent to:
\[
K^{-1} \Lambda' \Lambda K = M^\delta, 
\]
and by part (a), we must have \( K = I_r \). Alternatively, by multiplying on the right both sides of (A88) by their transposed:
\[
K' \Lambda' \Lambda K = M^\delta, 
\]
(A94)

since eigenvectors are normalized, and again by part (a), we must have \( K = I_r \). So from (A88) or (A94) we prove part (b).

For part (c), since \( \chi_t = \Lambda F_t \), then by linear projection of \( \Lambda \) onto \( \chi_t \), and using parts (a) and (b), for all \( t = 1, \ldots, T \),
\[
F_t = (\Lambda' \Lambda)^{-1} \Lambda' \chi_t = (M^\delta)^{-1/2} V^\delta \chi_t. 
\]

This proves part (c).

For part (d), from Assumption 1(a) and part (a), as \( n \to \infty \),
\[
\| M^\delta - \Sigma \| \left\| \frac{\Lambda' \Lambda}{n} - \Sigma \right\| = o_p(1). 
\]
and from Lemma 8(i), as \( n \to \infty \),
\[
\left\| \frac{M^\delta}{n} - V_0 \right\| = o(1). 
\]

By uniqueness of the limit we prove part (d).

Part (e) follows directly from Assumptions 1(c) and 9(b). This completes the proof. \( \square \)

A.9 Proof of Proposition 8

Because of (A56) in the proof of Proposition 5(a), using (13)
\[
\frac{\hat{\Lambda}' \Lambda \hat{H}}{n} = \frac{\hat{\Lambda}'(\Lambda \hat{H} - \hat{\Lambda} + \hat{\Lambda})}{n} = \frac{\hat{M}^\delta}{n} + O_p \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right). 
\]
(A95)

Or, equivalently,
\[
\left( \frac{\hat{M}^\delta}{n} \right)^{-1} \frac{\hat{\Lambda} \Lambda}{n} = \hat{H}^{-1} + O_p \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right). 
\]
(A96)
Moreover, by Assumption 9 and using (A96) in (28) we have

\[
\hat{\mathbf{H}}' = \left( \frac{\hat{\mathbf{M}}'}{n} \right)^{-1} \frac{\hat{\Lambda}' \Lambda}{n} = \hat{\mathbf{H}}^{-1} + O_p \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right). \tag{A97}
\]

Therefore, because of (A97), as \( n, T \to \infty \), \( \hat{\mathbf{H}} \) is an \( r \times r \) orthogonal matrix thus it has eigenvalues \( \pm 1 \).

Moreover, because of (A57) in the proof of Proposition 5(a)

\[
\frac{\hat{\Lambda}' \Lambda}{n} = \frac{\hat{\Lambda} - \hat{\Lambda} \hat{\mathbf{H}} + \hat{\Lambda} \hat{\mathbf{H}}' \Lambda}{n} = \frac{\hat{\mathbf{H}}' \Lambda' \Lambda}{n} + O_p \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right). \tag{A98}
\]

Thus, from (A97) and (A98) and by Lemma 7(iv),

\[
\hat{\mathbf{H}}' = \left( \frac{\hat{\mathbf{M}}'}{n} \right)^{-1} \frac{\hat{\Lambda}' \Lambda}{n} = \left( \frac{\hat{\mathbf{M}}'}{n} \right)^{-1} \frac{\hat{\mathbf{H}}' \Lambda' \Lambda}{n} + O_p \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right). \tag{A99}
\]

And from (A99) it follows that

\[
\left( \frac{\hat{\mathbf{M}}'}{n} \right) \hat{\mathbf{H}}' = \hat{\mathbf{H}}' \Lambda' \Lambda + O_p \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right). \tag{A100}
\]

So, because of (A100), as \( n, T \to \infty \), the columns of \( \hat{\mathbf{H}} \) are the eigenvectors of \( \frac{\hat{\Lambda}' \Lambda}{n} \) with eigenvalues \( \frac{\hat{\mathbf{M}}'}{n} \).

The eigenvectors are normalized since \( \mathbf{H} \) is orthogonal, as \( n, T \to \infty \). Moreover, under Assumption 9, \( \frac{\hat{\Lambda}' \Lambda}{n} \) is diagonal, so, as \( n, T \to \infty \), \( \hat{\mathbf{H}} \) must be diagonal with eigenvalues \( \pm 1 \). By noticing that, under Assumption 5(c),

\[\max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) = o_p \left( \max \left( \frac{1}{\sqrt{nT}}, \frac{1}{T} \right) \right), \]

we prove part (a).

Part (b) follows from the fact that \( \hat{\mathbf{H}} \) is orthogonal, as \( n, T \to \infty \). This completes the proof. \( \square \)

## B Auxiliary results

### B.1 Major auxiliary lemmata

**Lemma 1.** Under Assumptions 1 and 2:

(i) for all \( n \in \mathbb{N} \) and \( T \in \mathbb{N} \), \( \frac{1}{nT} \sum_{i,j=1}^{n} \sum_{t,s=1}^{T} |E[\xi_{it}\xi_{js}]| \leq M_{1\xi} \), for some finite positive real \( M_{1\xi} \) independent of \( n \) and \( T \);

(ii) for all \( n \in \mathbb{N} \) and \( t \in \mathbb{Z} \), \( \frac{1}{n} \sum_{i,j=1}^{n} |E[\xi_{it}\xi_{jt}]| \leq M_{2\xi} \), for some finite positive real \( M_{2\xi} \) independent of \( n \) and \( t \);

(iii) for all \( i \in \mathbb{N} \) and \( T \in \mathbb{N} \), \( \frac{1}{T} \sum_{t,s=1}^{T} |E[\xi_{it}\xi_{is}]| \leq M_{3\xi} \), for some finite positive real \( M_{3\xi} \) independent of \( i \) and \( T \);

(iv) for all \( j = 1, \ldots, r \), \( C_{j} \leq \liminf_{n \to \infty} \frac{\mu_{j}}{n} \leq \limsup_{n \to \infty} \frac{\mu_{j}}{n} \leq C_{j} \), for some finite positive reals \( C_{j} \) and \( C_{j} \);

(v) for all \( n \in \mathbb{N} \), \( \mu_{j} \leq M_{\xi} \), where \( M_{\xi} \) is defined in Assumption 2(b);

(vi) for all \( j = 1, \ldots, r \), \( C_{j} \leq \liminf_{n \to \infty} \frac{\nu_{j}}{n} \leq \limsup_{n \to \infty} \frac{\nu_{j}}{n} \leq C_{j} \), and for all \( n \in \mathbb{N} \), \( \mu_{j+1} \leq M_{\xi} \), where \( M_{\xi} \) is defined in Assumption 2(b).

**Proof.** Using Assumption 2(b), we have:

\[
\frac{1}{nT} \sum_{i,j=1}^{n} \sum_{t,s=1}^{T} |E[\xi_{it}\xi_{js}]| = \frac{1}{n} \sum_{i,j=1}^{n} \sum_{k=-(T-1)}^{T-1} \left( 1 - \frac{|k|}{T} \right) |E[\xi_{it}\xi_{j,t-k}]| \leq \max_{i=1,\ldots,n} \sum_{j=1}^{n} \sum_{k=-\infty}^{\infty} \rho^{k} |M_{ij} | \leq \frac{M_{\xi}(1+\rho)}{1-\rho}.
\]

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Similarly,

\[
\frac{1}{n} \sum_{i,j=1}^{n} |\mathbb{E}[\xi_{it}\xi_{jt}]| \leq \max_{i=1,\ldots,n} \sum_{j=1}^{n} M_{ij} \leq M_{\xi},
\]

and

\[
\frac{1}{T} \sum_{t,s=1}^{T} |\mathbb{E}[\xi_{it}\xi_{is}]| = \frac{T-1}{1-(T-1)} \sum_{k=-\infty}^{\infty} \rho^{|k|} M_{it} \leq \frac{1+\rho}{1-\rho} \sum_{i=1}^{n} M_{it} \leq \frac{M_{\xi}(1+\rho)}{1-\rho}.
\]

Defining, \(M_{1\xi} = M_{3\xi} = M_{\xi}(1+\rho)\) and \(M_{2\xi} = M_{\xi}\), we prove parts (i), (ii), and (iii).

For part (iv), by Merikoski and Kumar (2004, Theorem 7), for all \(j = 1, \ldots, r\), we have

\[
\frac{\mu_r(\Lambda'\Lambda)}{n} \leq \frac{\mu_j(\Lambda'\Lambda)}{n} \leq \mu_1(\Lambda'\Lambda)\]

and the proof follows from Assumption 1(a) which, by continuity of eigenvalues, implies that, for any \(j = 1, \ldots, r\), as \(n \to \infty\)

\[
\lim_{n \to \infty} \frac{\mu_j(\Lambda'\Lambda)}{n} = \mu_j(\Sigma_{\lambda})
\]

with (see also (2) and Lemma 3(iv))

\[
0 < m_{\lambda}^2 \leq \mu_r(\Sigma_{\lambda}) \leq \mu_1(\Sigma_{\lambda}) \leq M_{\lambda}^2 < \infty,
\]

and by Assumption 1(b) and Assumption 1(c) which imply (see also (3) and Lemma 3(v))

\[
0 < m_F \leq \mu_r(\Gamma^F) \leq \mu_1(\Gamma^F) \leq M_F < \infty.
\]

For part (v), by Assumption 2(b):

\[
\|\Gamma^\xi\| \leq \max_{i=1,\ldots,n} \sum_{j=1}^{n} |\mathbb{E}[\xi_{it}\xi_{jt}]| \leq \max_{i=1,\ldots,n} \sum_{j=1}^{n} M_{ij} \leq M_{\xi}.
\]

Part (vi) follows from parts (iv) and (v) and Weyl’s inequality. This completes the proof. □

Lemma 2. Under Assumptions 1 through 4, for all \(t = 1, \ldots, T\) and all \(n, T \in \mathbb{N}\)

(i) \(\frac{\Lambda}{\sqrt{n}}\) = \(O(1)\);

(ii) \(\|\tilde{F}_t\| = O_{\text{ms}}(1)\) and \(\|\tilde{F}_t\| = O_{\text{ns}}(1)\);

(iii) \(\|\tilde{F}_t\| = O_{\text{ms}}(1)\) and \(\|\tilde{F}_t\| = O_{\text{ns}}(1)\);

(iv) \(\|\tilde{F}_t\| = O_{\text{ns}}(1)\) and \(\|\tilde{F}_t\| = O_{\text{ns}}(1)\).

Proof. By Assumption 1(a), which holds for all \(n \in \mathbb{N}\),

\[
\sup_{n \in \mathbb{N}} \frac{\Lambda}{\sqrt{n}} \leq \sup_{n \in \mathbb{N}} \frac{\Lambda}{\sqrt{n}} = \sup_{n \in \mathbb{N}} \frac{\Lambda}{\sqrt{n}} = \sup_{n \in \mathbb{N}} \frac{\Lambda}{\sqrt{n}} \leq \sup_{n \in \mathbb{N}} \frac{\Lambda}{\sqrt{n}} \leq M_{\Lambda}^2,
\]

since \(M_{\Lambda}\) is independent of \(i\). This proves part (i).
By Assumption 1(b), which holds for all \( T \in \mathbb{N} \) because of stationarity (see also part (i) of Lemma 5),

\[
\sup_{T \in \mathbb{N}} \max_{t=1,\ldots,T} \mathbb{E}[\|F_t\|^2] = \sup_{T \in \mathbb{N}} \max_{t=1,\ldots,T} \sum_{j=1}^r \mathbb{E}[F_{jt}^2] \leq r \sup_{T \in \mathbb{N}} \max_{t=1,\ldots,T} \mathbb{E}[F_{jt}^2] 
\leq r \sup_{T \in \mathbb{N}} \max_{t=1,\ldots,T} \eta_j^T \Gamma^F \eta_j \leq r \|\Gamma^F\| \leq r M_F, \tag{B2}
\]

since \( M_F \) is independent of \( t \) and where \( \eta_j \) is an \( r \)-dimensional vector with one in the \( j \)th entry and zero elsewhere. Therefore, from (B2):

\[
\sup_{T \in \mathbb{N}} \mathbb{E} \left[ \left\| \frac{F}{\sqrt{T}} \right\|^2 \right] \leq \sup_{T \in \mathbb{N}} \mathbb{E} \left[ \left\| \frac{F}{\sqrt{T}} \right\|^2 \right] = \sup_{T \in \mathbb{N}} \frac{1}{T} \sum_{t=1}^T \sum_{j=1}^r \mathbb{E}[F_{jt}^2] \leq \sup_{T \in \mathbb{N}} \max_{t=1,\ldots,T} \mathbb{E}[\|F_t\|^2] \leq r M_F.
\]

This proves part (ii).

Then, because of Lemma 1(v), which holds for all \( n \in \mathbb{N} \) and holds also for all \( T \in \mathbb{N} \) by stationarity, which is implied by Assumption 2(b),

\[
\sup_{n,T \in \mathbb{N}} \max_{t=1,\ldots,T} \mathbb{E} \left[ \left\| \frac{\xi_t}{\sqrt{n}} \right\|^2 \right] \leq \sup_{n,T \in \mathbb{N}} \max_{t=1,\ldots,T} \sum_{i=1}^n \mathbb{E}[\xi_i^2] \leq \sup_{n,T \in \mathbb{N}} \max_{t=1,\ldots,T} \max_{i=1,\ldots,n} \mathbb{E}[\xi_i^2] 
= \sup_{n \in \mathbb{N}} \max_{t=1,\ldots,T} \mathbb{E}[\xi^T \xi] \leq \sup_{n \in \mathbb{N}} \max_{t=1,\ldots,T} \mathbb{E}[\xi^T \xi] \leq M_\xi, \tag{B3}
\]

since \( M_\xi \) is independent of \( i \) and \( t \) and where \( \varepsilon_i \) is an \( n \)-dimensional vector with one in the \( i \)th entry and zero elsewhere. Therefore, from (B3)

\[
\sup_{n,T \in \mathbb{N}} \mathbb{E} \left[ \left\| \frac{\Xi}{\sqrt{nT}} \right\|^2 \right] \leq \sup_{n,T \in \mathbb{N}} \mathbb{E} \left[ \left\| \frac{\Xi}{\sqrt{nT}} \right\|^2 \right] = \sup_{n,T \in \mathbb{N}} \frac{1}{nT} \sum_{j=1}^T \sum_{i=1}^n \mathbb{E}[\Xi_i^2] \leq \sup_{n,T \in \mathbb{N}} \max_{t=1,\ldots,T} \max_{i=1,\ldots,n} \mathbb{E}[\xi_i^2] \leq M_\xi. 
\]

This proves part (iii).

Finally, as a consequence of parts (ii) and (iii) and Assumptions 1(a) and 4:

\[
\sup_{n,T \in \mathbb{N}} \max_{t=1,\ldots,T} \mathbb{E} \left[ \left\| \frac{X_t}{\sqrt{n\sqrt{T}}} \right\|^2 \right] \leq \sup_{n,T \in \mathbb{N}} \max_{t=1,\ldots,T} \max_{i=1,\ldots,n} \mathbb{E}[x_{it}^2] \leq M_X, 
\]

where \( M_X \) is independent of \( i, t, n, \) and \( T \). Therefore, from part (i) and (B4):

\[
\sup_{n,T \in \mathbb{N}} \mathbb{E} \left[ \left\| \frac{X}{\sqrt{n\sqrt{T}}} \right\|^2 \right] \leq \sup_{n,T \in \mathbb{N}} \max_{t=1,\ldots,T} \max_{i=1,\ldots,n} \mathbb{E}[x_{it}^2] \leq M_X. 
\]

This proves part (iv) and it completes the proof. \( \square \)

**Lemma 3.** Under Assumptions 1 through 4, for all \( n,T \in \mathbb{N} \)

\begin{enumerate}
\item \( \sqrt{T} \left\| \frac{F}{\sqrt{T}} \right\| = O_{\text{ms}}(1) \);
\item \( \sqrt{n} \left\| \frac{\Xi}{\sqrt{nT}} \right\| = O_{\text{ms}}(1) \);
\item \( \min(n, \sqrt{T}) \left\| \frac{\Xi}{\sqrt{n}} \right\| = O_{\text{ms}}(1) \) and \( \min(n, \sqrt{nT}) \left\| \frac{\Lambda \Xi}{\sqrt{nT}} \right\| = O_{\text{ms}}(1) \);
\end{enumerate}
(iv) \( \frac{A' \Lambda}{n} = O(1) \);
(v) \( \frac{F' F}{T} = O_{\text{ms}}(1) \).

**Proof.** For part (i), by Assumption 4, Lemma 1(iii) and Cauchy-Schwarz inequality

\[
\mathbb{E} \left[ \left\| \frac{F' \Xi}{\sqrt{nT}} \right\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} F_t \xi_t \right\|^2 \right] \leq \mathbb{E} \left[ \left\| \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} F_t \xi_t \right\|^2 \right] = \frac{1}{nT^2} \sum_{j=1}^{r} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \sum_{t=1}^{T} F_{jt} \xi_{jt} \right)^2 \right] \leq \frac{r}{T^2} \max_{j=1, \ldots, r} \max_{t=1, \ldots, n} \sum_{i=1}^{n} \sum_{s=1}^{T} \mathbb{E} \left[ F_{jt} F_{js} \right] \mathbb{E} \left[ \xi_{jt} \xi_{js} \right] \leq \left\{ \frac{r}{T} \max_{j=1, \ldots, r} \max_{t, s=1, \ldots, n} \mathbb{E} \left[ F_{jt} F_{js} \right] \right\} \left\{ \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \mathbb{E} \left[ \xi_{jt} \xi_{js} \right] \right\} \leq \frac{r}{T} \max_{j=1, \ldots, r} \max_{t, s=1, \ldots, n} \mathbb{E} \left[ F_{jt}^2 \right] \frac{M_F(t+1)}{1 - \rho} \leq \frac{rM_F M_{\Xi}}{T},
\]

since \( M_F \) is independent of \( t \) and \( s \) and \( M_{\Xi} \) is independent of \( i \). This proves part (i).

For part (ii), let \( \zeta_i = (\xi_{it} \cdots \xi_{iT})' \), then by

\[
\mathbb{E} \left[ \left\| \frac{\Xi \Lambda}{n \sqrt{T}} \right\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{n \sqrt{T}} \sum_{i=1}^{n} \lambda_i \xi_i \right\|^2 \right] \leq \mathbb{E} \left[ \left\| \frac{1}{n \sqrt{T}} \sum_{i=1}^{n} \lambda_i \xi_i \right\|^2 \right] = \frac{1}{n^2 T} \sum_{j=1}^{r} \sum_{t=1}^{T} \mathbb{E} \left[ \sum_{i=1}^{n} \lambda_{ij} \xi_{jt} \right] \leq \frac{r}{n^2} \max_{j=1, \ldots, r} \max_{t=1, \ldots, T} \sum_{i=1}^{n} \lambda_{ij} \sum_{k=1}^{n} \lambda_{ik} \mathbb{E} \left[ \xi_{it} \xi_{kt} \right] \leq \frac{r M_F^2}{n^2} \max_{j=1, \ldots, r} \max_{t=1, \ldots, T} \sum_{i=1}^{n} \sum_{k=1}^{n} \mathbb{E} \left[ \xi_{it} \xi_{kt} \right] \leq \frac{r M_F^2 M_{\Xi}}{n},
\]

since \( M_{\Lambda} \) is independent of \( i \) and \( k \) and \( M_{\Xi} \) is independent of \( t \). This proves part (ii).

For part (iii), first notice that, by Lemmas 1(v) and 4(ii)

\[
\left\| \frac{\Xi' \Xi}{n T} \right\| \leq \left\| \frac{\Xi' \Xi}{n T} - \frac{\Gamma \xi}{n} \right\| + \left\| \frac{\Gamma \xi}{n} \right\| = \left\| \frac{\Xi' \Xi}{n T} - \frac{\Gamma \xi}{n} \right\| + \left\| \frac{\Gamma \xi}{n} \right\| = O_P \left( \frac{1}{\sqrt{T}} \right) + O \left( \frac{1}{n} \right). \tag{B5}
\]

Similarly, by Lemmas 2(i) and 1(v)

\[
\left\| \frac{\Lambda' \Xi' \Xi}{n^{3/2} T} \right\| \leq \left\| \frac{\Lambda' \Xi' \Xi}{n^{3/2} T} - \frac{\Lambda' \Gamma \xi}{n^{3/2} T} \right\| + \left\| \frac{\Lambda' \Gamma \xi}{n^{3/2} T} \right\| = \left\| \frac{\Lambda' \Xi' \Xi}{n^{3/2} T} - \frac{\Lambda' \Gamma \xi}{n^{3/2} T} \right\| + O \left( \frac{1}{n} \right). \tag{B6}
\]
Then, because of Assumption 2(c),

\[
\mathbb{E} \left[ \frac{\Lambda' \Xi' \Xi - \Lambda' T \xi}{n^{3/2} T} \right]^2 \leq \mathbb{E} \left[ \frac{\Lambda' \Xi' \Xi - \Lambda' T \xi}{n^{3/2} T} \right]^2_F
\]

\[
= \sum_{k=1}^{r} \sum_{j=1}^{n} \mathbb{E} \left[ \frac{1}{n^{3/2} T} \sum_{i=1}^{n} \sum_{t=1}^{T} \{ \lambda_{ik} \xi_{it} \xi_{jt} - \lambda_{ik} \mathbb{E}[\xi_{it} \xi_{jt}] \}^2 \right]
\]

\[
\leq \frac{r M_{\Lambda} n}{n^{3/2} T} \max_{j=1,...,n} \mathbb{E} \left[ \frac{1}{n^{3/2} T} \sum_{i=1}^{n} \sum_{t=1}^{T} \{ \xi_{it} \xi_{jt} - \mathbb{E}[\xi_{it} \xi_{jt}] \}^2 \right] \leq \frac{r M_{\Lambda} K_{\xi}}{n T}, \quad (B7)
\]

since \( K_{\xi} \) is independent of \( j \). By using (B7) into (B6), we prove part (iii).

Part (iv) follows from Lemma 2(i) since

\[
\left\| \frac{\Lambda' \Lambda}{n} \right\| \leq \left\| \frac{\Lambda}{\sqrt{n}} \right\|^2 \leq M_{\Lambda}^2.
\]

Likewise, part (v) follows from Lemma 2(ii) since

\[
\left\| \frac{F' F}{T} \right\| \leq \left\| \frac{F}{\sqrt{T}} \right\|^2 \leq r M_{F}.
\]

This completes the proof. □

Lemma 4.

(i) Under either Assumption 6, or 7, or 8, for all \( T \in \mathbb{N} \), \( \sqrt{T} \left\| \frac{F' F}{T} - \Gamma^F \right\| = O_{\text{mas}}(1) \);

(ii) Under Assumption 2, for all \( n, T \in \mathbb{N} \), \( \sqrt{T} \left\| \frac{\Xi' \Xi - \xi' \xi}{n} \right\| = O_{\text{mas}}(1) \).

Proof. Because of Assumption 6, for any \( i, j = 1, \ldots, r \), letting \( \gamma^F_{ij} \) be the \((i, j)\)th entry of \( \Gamma^F \),

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t,s=1}^{T} \{ F_{it} F_{jt} - \gamma^F_{ij} \} \right]^2 = \frac{1}{T^2} \sum_{t,s=1}^{T} \{ \mathbb{E}[F_{it} F_{jt}] - \mathbb{E}[F_{it} F_{jt}]^2 \}
\]

\[
= \frac{1}{T^2} \sum_{t,s=1}^{T} \left\{ \sum_{k_1 \leq k_2 \leq k_3 \leq k_4=0} C_{k_1,i,h_1} C_{k_2,j,h_2} C_{k_3,i,h_3} C_{k_4,j,h_4} \mathbb{E}[u_{h_1,t-k_1} u_{h_2,t-k_2} u_{h_3,i-k_3} u_{h_4,i-k_4}] - \left( \sum_{k_1 \leq k_2=0} C_{k_1,i,h_1} C_{k_2,j,h_2} \mathbb{E}[u_{h_1,t-k_1} u_{h_2,t-k_2}] \right)^2 \right\}
\]

\[
\leq \frac{M_{C}^2}{T^2} \left\{ \sum_{t=1}^{T} \sum_{h=1}^{r} \mathbb{E}[u_{h,t}]^2 + \sum_{t,s=1}^{T} \sum_{h=1}^{r} \mathbb{E}[u_{h,t}] \sum_{h=1}^{r} \mathbb{E}[u_{h,s}] - \sum_{t,s=1}^{T} \left( \sum_{h=1}^{r} \mathbb{E}[u_{h,t}]^2 \right) \right\} \leq \frac{M_{C}^2 r K_{u}}{T}.
\]

Then,

\[
\mathbb{E} \left[ \left\| \frac{F' F}{T} - \Gamma^F \right\| \right]^2 = \mathbb{E} \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} F_{it} F_{jt} - \Gamma^F \right\| \right]^2 = \sum_{i,j=1}^{r} \mathbb{E} \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} \{ F_{it} F_{jt} - \gamma^F_{ij} \} \right\| \right]^2 \leq \frac{r^2 M_{C}^2 K_{u}}{T},
\]

since \( M_{C} \) and \( K_{u} \) do not depend on \( i \) and \( j \). This proves part (i) under Assumption 6.
Alternatively, because of Assumption 7,

\[
E \left[ \left\| \frac{\mathbf{F}^t \mathbf{F}_t - \mathbf{F}^T}{T} \right\|_F^2 \right] = E \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} (\mathbf{F}_t \mathbf{F}_t' - \mathbf{F}^T) \right\|_F^2 \right] \leq E \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} \{\mathbf{F}_t \mathbf{F}_t' - \mathbf{F}^T\} \right\|_F^2 \right]
\]

\[
= \frac{1}{T^2} \sum_{j=1}^{r} \sum_{k=1}^{r} E \left[ \left( \sum_{t=1}^{T} \{F_{jt} F_{kt} - E[F_{jt} F_{kt}]\} \right)^2 \right]
\]

\[
= \frac{r^2}{T^2} \max_{j,k=1,\ldots,r} \sum_{t=1}^{T} \sum_{s=1}^{T} \{E[F_{jt}F_{ks}] - E[F_{jt}F_{kt}]\}
\]

\[
\leq \frac{r^2}{T^2} \max_{j,k=1,\ldots,r} \sum_{t=1}^{T} \sum_{s=1}^{T} E[F_{jt}F_{ks}]
\]

since \(C_F\) is independent of \(j\) and \(k\). This proves part (i) under Assumption 7.

Let us now consider the case in which we make Assumption 8. For any \(i,j = 1, \ldots, r\) we have

\[
\frac{1}{T} \sum_{t=1}^{T} \{F_{it} F_{jt} - \gamma^2_{ij}\} = \sum_{k,k'=0}^{r} \sum_{\ell,\ell'=1}^{r} C_{kk',\ell} C_{k',\ell'} \left( \frac{1}{T} \sum_{t=1}^{T} \{u_{t,\ell-k} u_{t,\ell'-k} - E[u_{t,\ell-k} u_{t,\ell'-k}]\} \right). \tag{B8}
\]

For simplicity let

\[
C_{kk',\ell,\ell'} = \frac{1}{T} \sum_{t=1}^{T} \{u_{t,\ell-k} u_{t,\ell'-k} - E[u_{t,\ell-k} u_{t,\ell'-k}]\}
\]

Then, from (B8), because of Assumption 8(P1), and Minkowsky inequality

\[
\left\{ E \left[ \left\| \frac{1}{T} \sum_{t=1}^{T} \{F_{it} F_{jt} - \gamma^2_{ij}\} \right\|_F^2 \right] \right\}^{1/2} = \left\{ E \left[ \left\| \sum_{k,k'=0}^{r} \sum_{\ell,\ell'=1}^{r} C_{kk',\ell} C_{k',\ell'} \left( \frac{1}{T} \sum_{t=1}^{T} \{u_{t,\ell-k} u_{t,\ell'-k} - E[u_{t,\ell-k} u_{t,\ell'-k}]\} \right) \right\|_F^2 \right] \right\}^{1/2}
\]

\[
\leq \sum_{k,k'=0}^{r} \sum_{\ell,\ell'=1}^{r} \left\{ E \left[ \left\| C_{kk',\ell} C_{k',\ell'} T \right\|_F^2 \right] \right\}^{1/2}
\]

\[
= \sum_{k,k'=0}^{r} \sum_{\ell,\ell'=1}^{r} |C_{kk',\ell}| |C_{k',\ell'}| \left\{ E \left[ \left\| C_{kk',\ell,\ell'} \right\|_F^2 \right] \right\}^{1/2}
\]

\[
\leq M^2_{\ell,\ell'} \sup_{k,k' \in \mathbb{Z}^+} \max_{\ell,\ell'=1,\ldots,r} \left\{ E \left[ \left\| C_{kk',\ell,\ell'} \right\|_F^2 \right] \right\}^{1/2}. \tag{B9}
\]

Now, since as explained in the text we can always choose \(\alpha > \frac{1}{2} - \frac{3}{q}\), then, from Zhang and Wu (2021, Proposition 3.3), for any \(\ell,\ell'=1,\ldots,r\), \(k,k' \in \mathbb{Z}^+\), and \(z > 0\), (see also Barigozzi et al., 2022, Lemma C.4 and Remark C.1)

\[
P \left( |C_{kk',\ell,\ell'}| > z \right) \leq \frac{C_{q,\alpha} T M_{q}^{3/2} \Phi_{q}^{4/3}}{(Tz)^{q/2}} + C_{q,\alpha} \exp \left( -\frac{Tz^2}{C_{q,\alpha} \Phi_{q}^{4/3}} \right)
\]

\[
\leq \frac{C_{q,\alpha} T M_{q}^{3/2} \Phi_{q}^{4/3}}{(Tz)^{q/2}} + C_{q,\alpha} \exp \left( -\frac{Tz^2}{C_{q,\alpha} M_{q,\alpha}^{3/2}} \right)
\]

\[
= \frac{C_{q,\alpha}}{z^{q/2} T^{q/2-1}} + C_{q,\alpha} \exp \left( -\frac{Tz^2}{C_{q,\alpha} M_{q,\alpha}^{3/2}} \right). \tag{B10}
\]
where \( q > 4 \), \( C \) is a finite positive real independent of \( T, i, j, \alpha, \) and \( q \), while \( M_{u,q}, C_{q,\alpha}, C_{\alpha}, C_{1,q,\alpha}, \) and \( C_{2,\alpha} \) are finite positive reals depending only on their subscripts. So, as expected, from (B10) we have that \( |C_{kk',\ell'\ell,T}| = O_p \left( \max(T^{-1/2}, T^{2q^{-1}}) \right) = O_p(T^{-1/2}) \) since \( q > 4 \).

Moreover, from (B10) it follows that

\[
E \left[ |C_{kk',\ell'\ell,T}|^2 \right] = E \left[ \int_0^\infty \mathbb{I} \left( |C_{kk',\ell'\ell,T}|^2 > y \right) dy \right] = \int_0^\infty P \left( |C_{kk',\ell'\ell,T}|^2 > y \right) dy \\
= \int_0^{T_T} P \left( |C_{kk',\ell'\ell,T}|^2 > y \right) dy + \int_{T_T}^{\infty} P \left( |C_{kk',\ell'\ell,T}|^2 > y \right) dy \\
\leq r_T + \frac{C_{1,q,\alpha}}{(q/4-1)T^{q/2-1}} \int_{r_T}^{\infty} \frac{1}{y^{q/4}} dy + C_0 \int_{r_T}^{\infty} \exp \left( -\frac{T y}{C_{2,\alpha}} \right) dy \\
= r_T + \frac{C_{1,q,\alpha}}{(q/4-1)T^{q/2-1}} \frac{1}{y^{q/4}} \bigg|_{r_T}^{\infty} - C_0 \frac{C_{2,\alpha}}{T} \exp \left( -\frac{T r_T}{C_{2,\alpha}} \right) \\
= r_T + \frac{C_{1,q,\alpha}}{(q/4-1)T^{q/2-1}} \frac{1}{y^{q/4}} + C_0 \frac{C_{2,\alpha}}{T} \exp \left( -\frac{T r_T}{C_{2,\alpha}} \right). \tag{B11}
\]

Then, by choosing \( r_T = \frac{1}{T} \) from (B11) and since \( q > 4 \), we get

\[
E \left[ |C_{kk',\ell'\ell,T}|^2 \right] \leq \frac{1}{T} + \frac{C_{1,q,\alpha}}{(q/4-1)T^{q/2-1}T^{1-q/4}} + \frac{C_0 C_{2,\alpha}}{T} \exp \left( -\frac{1}{C_{2,\alpha}} \right) \\
= \frac{1}{T} \left( 1 + C_0 C_{2,\alpha} \exp \left( -\frac{1}{C_{2,\alpha}} \right) \right) + \frac{C_{1,q,\alpha}}{(q/4-1)T^{q/4}} \\
\leq \frac{1}{T} \left( 1 + C_0 C_{2,\alpha} \exp \left( -\frac{1}{C_{2,\alpha}} \right) \right) + \frac{C_{1,q,\alpha}}{(q/4-1)} \\
\leq \frac{C_{u,\alpha,q}}{T}, \tag{B12}
\]

where \( C_{u,\alpha,q} \) is independent of \( k, k', \ell, \) and \( \ell' \).

By substituting (B12) into (B9), we get

\[
E \left[ \left\| \frac{1}{T} \sum_{t=1}^T \left\{ F_{it} F_{jt} - \gamma_{ij}^F \right\} \right\|^2 \right] \leq \frac{M^{r^4} C_{u,\alpha,q}}{T}. \tag{B13}
\]

Finally, from (B13)

\[
E \left[ \left\| \frac{F'F}{T} - \Gamma^F \right\|^2 \right] = E \left[ \left\| \frac{1}{T} \sum_{t=1}^T \left( F_t F_t' - \Gamma^F \right) \right\|^2 \right] = \sum_{i,j=1}^r E \left[ \left\| \frac{1}{T} \sum_{t=1}^T \left( F_{it} F_{jt} - \gamma_{ij}^F \right) \right\|^2 \right] \\
\leq r^2 \max_{i,j=1,...,r} E \left[ \left\| \frac{1}{T} \sum_{t=1}^T \left( F_{it} F_{jt} - \gamma_{ij}^F \right) \right\|^2 \right] \leq \frac{M^{r^4} C_{u,\alpha,q}}{T},
\]

since \( M_T \) and \( C_{u,\alpha,q} \) do not depend on \( i \) and \( j \). This proves part (i) under Assumption 8.
For part (ii), by Assumption 2(c), letting $\gamma_{ij}^\xi$ be the $(i,j)$th entry of $\Gamma^\xi$, we have

$$
\mathbb{E} \left[ \left\| \frac{\hat{\Gamma}^x - \Gamma^x}{nT} \right\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{nT} \sum_{t=1}^T \xi_t \xi_t' - \frac{\hat{\Gamma}^x}{n} \right\|^2 \right] \leq \mathbb{E} \left[ \left\| \frac{1}{nT} \sum_{t=1}^T \xi_t \xi_t' - \Gamma^\xi \right\|^2 \right] = \frac{1}{n^2 T^2} \sum_{i,j=1}^n \mathbb{E} \left[ \left( \sum_{t=1}^T \{ \xi_t \xi_t' - \gamma_{ij}^\xi \} \right) \right]^2 \leq \max_{i,j=1,\ldots,n} \frac{1}{T^2} \mathbb{E} \left[ \left( \sum_{t=1}^T \{ \xi_t \xi_t' - \gamma_{ij}^\xi \} \right) \right]^2 \leq \frac{K^\xi}{T},
$$

since $K^\xi$ is independent of $i$ and $j$. This completes the proof. □

**Lemma 5.** Under Assumptions 1 through 4, for all $n, T \in \mathbb{N}$

(i-a) $\sqrt{n} \left\| \hat{\Gamma}^x - \Gamma^x \right\| = O_P(1)$;

(i-b) if also Assumption 6 or 7 or 8 hold, then $\sqrt{n} \left\| \hat{\Gamma}^x - \Gamma^x \right\| = O_m(1)$;

(ii) $\left\| \min(n, \sqrt{T}) \left( \hat{\Gamma}^x - \Gamma^x \right) \right\| = O_m(1)$;

(iii) $\left\| \min(n, \sqrt{T}) \left( \hat{M}^x - M^x \right) \right\| = O_m(1)$;

(iv) as $n \to \infty$, $\min(n, \sqrt{T}) \| \hat{V}^x - V^x J \| = O_m(1)$, where $J$ is $r \times r$ diagonal with entries $\pm 1$.

**Proof.** For part (i-a), from Assumption 1(c), and Lemmas 3(i), 4(ii), and 2(i):

$$
\left\| \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right\| \leq \left\| \frac{1}{n} \left\{ \Lambda \left( \frac{1}{T} \sum_{t=1}^T F_t F_t' - \Gamma^F \right) \Lambda' + \frac{1}{T} \sum_{t=1}^T \xi_t \xi_t' - \frac{\hat{\Gamma}^x}{\sqrt{n}} \right\} \right\| + \left\| \frac{2}{nT} \sum_{t=1}^T \xi_t \xi_t' \right\|
$$

$$
= \left\| \frac{\Lambda}{\sqrt{n}} \right\|^2 \left\| \frac{1}{T} \sum_{t=1}^T F_t F_t' \right\| + \left\| \frac{1}{n} \left( \frac{1}{T} \sum_{t=1}^T \xi_t \xi_t' - \Gamma^\xi \right) \right\| + \left\| \frac{2}{nT} \sum_{t=1}^T \xi_t \xi_t' \right\|
$$

$$
= O_P \left( \frac{1}{\sqrt{T}} \right) + O_P \left( \frac{1}{\sqrt{nT}} \right).
$$

For part (i-b), if we make either Assumption 6 or 7 or 8, from Lemma 4(i), 4(ii) and 2(i):

$$
\mathbb{E} \left[ \left\| \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{n} \left\{ \Lambda \left( \frac{1}{T} \sum_{t=1}^T F_t F_t' - \Gamma^F \right) \Lambda' + \frac{1}{T} \sum_{t=1}^T \xi_t \xi_t' - \Gamma^\xi \right\} \right\|^2 \right] \leq \mathbb{E} \left[ \left\| \frac{1}{n} \left\{ \Lambda \left( \frac{1}{T} \sum_{t=1}^T F_t F_t' - \Gamma^F \right) \Lambda' \right\} \right\|^2 \right] + \mathbb{E} \left[ \left\| \frac{1}{n} \left( \frac{1}{T} \sum_{t=1}^T \xi_t \xi_t' - \Gamma^\xi \right) \right\|^2 \right]
$$

$$
\leq \left\| \frac{\Lambda}{\sqrt{n}} \right\|^4 \mathbb{E} \left[ \left\| \frac{1}{T} \sum_{t=1}^T F_t F_t' - \Gamma^F \right\|^2 \right] + \mathbb{E} \left[ \left\| \frac{1}{n} \left( \frac{1}{T} \sum_{t=1}^T \xi_t \xi_t' - \Gamma^\xi \right) \right\|^2 \right]
$$

$$
\leq M_{14}^2 K_F + K_F \xi T,
$$

where $K_F = r^2 M_{14}^2 K_a$ under Assumption 6, or $K_F = r^2 C_F$ under Assumption 7, or $K_F = M_{14}^2 r^6 C_{u,a,q}$ under Assumption 8. This proves part (i).
Part (ii), follows from

\[
\mathbb{E} \left[ \left\| \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right\|^2 \right] \leq \mathbb{E} \left[ \left( \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right) + \left\| \Gamma^x \right\|^2 \right] \leq \frac{M_1 \epsilon K_F + K_\epsilon}{T} + \frac{\mu_3^x}{n^2} \leq \frac{M_1 \epsilon K_F + K_\epsilon}{T} + \frac{M_1}{n^2}
\]

because of part (i) and Lemma 1(v) and where \(M_1\) is a finite positive real independent of \(n\) and \(T\).

For part (iii), for any \(j = 1, \ldots, r\), because of Weyl’s inequality and part (ii), it holds that

\[
|\mu_j^x - \mu_j^y| \leq \mu_1(\hat{\Gamma}^x - \Gamma^x) = \left\| \hat{\Gamma}^x - \Gamma^x \right\|.
\]  

(B14)

Hence, from (B14) and part (ii),

\[
\mathbb{E} \left[ \left( \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right) \left( \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right) \right] \leq \mathbb{E} \left[ \left( \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right) \left( \frac{1}{n} \left( \hat{\Gamma}^x - \Gamma^x \right) \right) \right] \leq \frac{rM_1 \epsilon K_F + K_\epsilon}{T} + \frac{rM_1}{n^2}
\]

(B15)

This proves part (iii).

For part (iv), because of Theorem 2 in Yu et al. (2015), which is a special case of Davis Kahn Theorem, there exists an \(r \times r\) diagonal matrix \(J\) with entries \(\pm 1\) such that

\[
\left\| \hat{\Gamma}^x - \Gamma^x J \right\| \leq 2^{3/2} \sqrt{T} \left\| \hat{\Gamma}^x - \Gamma^x \right\|
\]

(B16)

where \(\mu_0^x = \infty\). This holds provided the eigenvalues \(\mu_j^x\) are distinct as required by Assumption 3. Therefore, from (B16), part (ii) and Lemma 1(iv), and since \(\mu_{r+1}^x = 0\), as \(n \to \infty\)

\[
\min(n^2, T) \mathbb{E} \left[ \left\| \hat{\Gamma}^x - \Gamma^x J \right\|^2 \right] \leq \frac{\min(n^2, T) 2^{3/2} \sqrt{T} \mathbb{E} \left[ \left\| \hat{\Gamma}^x - \Gamma^x \right\|^2 \right]}{\text{min}([\mu_0^x - \mu_1^x], [\mu_2^x - \mu_{r+1}^x])} \leq \frac{8rM_1}{C^2} = M_2, \text{ say},
\]

where \(M_2\) is a finite positive real independent of \(n\) and \(T\). This proves part (iv). This completes the proof. \(\Box\)

**Lemma 6.** Let \(x_i\) be the \(n\)-dimensional vector with one in entry \(i\) and zero elsewhere. Under Assumptions 1 through 4, for all \(i = 1, \ldots, n\) and \(T \in \mathbb{N}\) and as \(n \to \infty\)

(i) \(\min(n^{1/2}, \sqrt{T}) \|x_i(\hat{\Gamma}^x - \Gamma^x)\| = O_{\text{max}}(1)\);

(ii) \(\sqrt{n} \|\upsilon_i^x\| = O(1)\);

(iii) \(\min(n^{1/2}, \sqrt{T}) \sqrt{n} \|\upsilon_i^{x'} - \upsilon^y \| = O_{\text{max}}(1)\).

**Proof.** First notice that

\[
\max_{i=1, \ldots, n} \mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} x_i' \left( \frac{1}{T} \sum_{t=1}^T \xi_{it} \xi_i - [\Gamma^x]_{ij} \right) \right\|^2 \right] \leq \max_{i,j=1, \ldots, n} \mathbb{E} \left[ \left( \frac{1}{T} \sum_{t=1}^T \xi_{it} \xi_j - [\Gamma^x]_{ij} \right)^2 \right] \leq \frac{K_\epsilon}{T}.
\]

(B17)
since $K_\xi$ is independent of $i$ and $j$. Therefore, from Lemma 4(i) and Lemma 2(i), and using (B17),

\[
\max_{i=1,\ldots,n} \mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} \varepsilon_i^t \left( \hat{\Gamma}^x - \Gamma^x \right) \right\|^2 \right] = \max_{i=1,\ldots,n} \mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} \left( A_i' \left( \frac{1}{T} \sum_{j=1}^T F_i F_j' - \Gamma^F \right) + \varepsilon_i^t \left( \frac{1}{T} \sum_{j=1}^T \xi_j \xi_j' - \Gamma^\xi \right) \right) \right\|^2 \right]
\]

\[
\leq \max_{i=1,\ldots,n} \| \lambda_i \|^2 \left\| \frac{A}{\sqrt{n}} \right\|^2 \mathbb{E} \left[ \left\| \frac{1}{T} \sum_{j=1}^T F_i F_j' - \Gamma^F \right\|^2 \right] + \max_{i=1,\ldots,n} \mathbb{E} \left[ \left\| \varepsilon_i^t \left( \frac{1}{T} \sum_{j=1}^T \xi_j \xi_j' - \Gamma^\xi \right) \right\|^2 \right]
\]

\[
\leq \frac{M_1^2 K_F + K_\xi}{T}.
\]

(B18)

since $M_1$ and $K_F$ are independent of $i$. Then, following the same arguments as Lemma 5(ii), because of (B18) and Lemma 1(v):

\[
\max_{i=1,\ldots,n} \mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} \varepsilon_i^t \left( \hat{\Gamma}^x - \Gamma^x \right) \right\|^2 \right] \leq \max_{i=1,\ldots,n} \mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} \left( \lambda_i \right)^2 \right\|^2 \right] + \max_{i=1,\ldots,n} \mathbb{E} \left[ \left\| \varepsilon_i^t \frac{\Gamma^\xi}{\sqrt{n}} \right\|^2 \right]
\]

\[
\leq \frac{M_1^2 K_F + K_\xi}{T} + \max_{i=1,\ldots,n} \| \varepsilon_i \|^2 \left\| \frac{\Gamma^\xi}{\sqrt{n}} \right\|^2 \leq \frac{M_1^2 K_F + K_\xi}{T} + \frac{M_1}{n} \leq M_1 \max \left( \frac{1}{T}, \frac{1}{n} \right),
\]

since $\| \varepsilon_i \| = 1$ and where $M_1$ is a finite positive real independent of $n$ and $T$ defined in Lemma 5(ii). This proves part (i).

For part (ii), notice that for all $i = 1, \ldots, n$ we must have:

\[
\var(\chi_{it}) = \lambda_i' \Gamma^F \lambda_i \leq \| \lambda_i \|^2 \left\| \Gamma^F \right\| \leq M_3^2 M_F,
\]

which is finite for all $i$ and $t$. So, since by Lemma 1(iv)

\[
\lim_{n \to \infty} \max_{i=1,\ldots,n} \var(\chi_{it}) = \lim_{n \to \infty} \max_{i=1,\ldots,n} \sum_{j=1}^r \mu_j^t \| \chi_{ij} \|^2 \geq \lim_{n \to \infty} \max_{i=1,\ldots,n} \sum_{j=1}^r \| \chi_{ij} \|^2 \geq C_r \max_{i=1,\ldots,n} \| \chi_{ij} \|^2,
\]

then, because of (B19), we must have, as $n \to \infty$,

\[
C_r \max_{i=1,\ldots,n} \| \chi_{ij} \|^2 \leq M_3^2 M_F
\]

which implies that, as $n \to \infty$,

\[
n \max_{i=1,\ldots,n} \| \chi_{ij} \|^2 \leq M_V,
\]

for some finite positive real $M_V$ independent of $n$. This proves part (ii).

Finally, using the same arguments in Lemma 5(iv), from (B16), part (i) and Lemma 1(iv), and since $\mu_0 = \infty$
and \( \mu^{r+1}_{\chi} = 0 \), as \( n \to \infty \)

\[
\max_{i=1,\ldots,n} \min(n, T) \mathbb{E} \left[ \| \sqrt{n} (\hat{V}_i^{x_0} - V^x J) \|^2 \right] = \max_{i=1,\ldots,n} \min(n, T) \mathbb{E} \left[ \| \sqrt{n} e_i' (\hat{V}^x - V^x J) \|^2 \right]
\]

\[
\leq \max_{i=1,\ldots,n} \frac{\min(n, T) 2^3 \frac{1}{n^2} n \mathbb{E} \left[ \| e_i' (\hat{\Gamma}^x - \Gamma^\chi) \|^2 \right]}{\frac{1}{n^2} \left\{ \min(\|\mu^0_\chi - \mu^1_\chi\|, |\mu^0_\chi - \mu^{r+1}_\chi|) \right\}^2}
\]

\[
= \max_{i=1,\ldots,n} \frac{\min(n, T) 2^3 \frac{1}{n^2} \mathbb{E} \left[ \| e_i' (\hat{\Gamma}^x - \Gamma^\chi) \|^2 \right]}{\frac{1}{n^2} \left\{ \min(\|\mu^0_\chi - \mu^1_\chi\|, |\mu^0_\chi - \mu^{r+1}_\chi|) \right\}^2}
\]

\[
\leq \frac{8r M_4 C^2}{C^2} = M_2,
\]

where \( M_2 \) is a finite positive real independent of \( n \) and \( T \) defined in Lemma 5(iv). This proves part (iii) and completes the proof. \( \square \)
B.2 Minor auxiliary lemmata

Lemma 7. Under Assumptions 1 through 4, for all \(n,T \in \mathbb{N}\),

(i) \(\left\| \frac{M^X}{n} \right\| = O(1)\);
(ii) \(\left\| (\frac{M^X}{n})^{-1} \right\| = O(1)\);
(iii) \(\left\| \frac{M^X}{n} \right\| = O_P(1)\);
(iv) \(\left\| (\frac{M^X}{n})^{-1} \right\| = O_P(1)\).

Proof. Parts (i) and (ii) follow directly from Lemma 1(iv), indeed,

\[
\left\| \frac{M^X}{n} \right\| = \frac{\mu^X}{n} \leq C_1,
\]

and

\[
\left\| \left(\frac{M^X}{n}\right)^{-1} \right\| = \frac{n}{\mu^X} \leq \frac{1}{C}.
\]

Both statements hold for all \(n \in \mathbb{N}\) since the eigenvalues are an increasing sequence in \(n\).

For part (iii), because of part (i) and Lemma 5(iii),

\[
\left\| \frac{\tilde{M}^x}{n} \right\| \leq \left\| \frac{M^X}{n} \right\| + \left\| \frac{M^x}{n} - \frac{M^X}{n} \right\| \leq C_1 + O_p \left( \max \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}} \right) \right).
\]

For part (iv) just notice that, because of Lemma 5(iii) and part (ii), then \(\frac{\tilde{M}^x}{n}\) is positive definite with probability tending to one as \(n, T \to \infty\). This completes the proof. \(\square\)

Lemma 8. Under Assumptions 1 through 4,

(i) \(\left\| \frac{M^x}{n} - V_0 \right\| = o(1)\), as \(n \to \infty\) and \(\left\| V_0 \right\| = O(1)\);
(ii) \(\left\| \frac{M^x}{n} - V_0 \right\| = o_{as}(1)\), as \(n, T \to \infty\);
(iii) \(\left\| \left(\frac{M^x}{n}\right)^{-1} - V_0^{-1} \right\| = o(1)\), as \(n \to \infty\) and \(\left\| V_0^{-1} \right\| = O(1)\)
(iv) \(\left\| \left(\frac{M^x}{n}\right)^{-1} - V_0^{-1} \right\| = o_{as}(1)\), as \(n, T \to \infty\),

where \(V_0\) is \(r \times r\) diagonal with entries the eigenvalues of \(\Sigma_\lambda \Gamma^F\) sorted in descending order.

Proof. For part (i), first notice that the \(r\) non-zero eigenvalues of \(\frac{P^T}{n}\) are also the \(r\) eigenvalues of \((\Gamma^F)^{1/2} \frac{\Sigma_\lambda}{n} (\Gamma^F)^{1/2}\) which in turn are also the entries of \(V_0\). The proof follows from continuity of eigenvalues and since, because of Assumption 1(a), as \(n \to \infty\),

\[
\left\| (\Gamma^F)^{1/2} \frac{\Sigma_\lambda}{n} (\Gamma^F)^{1/2} - (\Gamma^F)^{1/2} \Sigma_\lambda (\Gamma^F)^{1/2} \right\| = o(1).
\]

Part (ii) is a consequence of part (i) and Lemma 7(iii). For part (iii) notice that

\[
\left\| \left(\frac{M^x}{n}\right)^{-1} - V_0^{-1} \right\| \leq \left\| \left(\frac{M^x}{n}\right)^{-1} \right\| \left\| \frac{M^x}{n} - V_0 \right\| \left\| V_0^{-1} \right\|,
\]

then the proof follows from part (i), Lemma 7(ii), and since \(V_0\) is positive definite since \(\Sigma_\lambda\) and \(\Gamma^F\) are positive definite by Assumptions 1(a) and 1(b), respectively. This completes the proof. \(\square\)

Lemma 9. Under Assumptions 1 through 4, as \(n \to \infty\),

(i) \(\left\| K \right\| = O(1)\);
(ii) \(\left\| K^{-1} \right\| = O(1)\).
Proof. For part (a), from (A38) in the proof of Proposition 4(a) it follows that
\[ \|K - JY_0\| = o(1), \]  
(B20)
where \( Y_0 \) is the \( r \times r \) matrix of normalized eigenvectors of \( (\Gamma F)^{1/2}\Sigma_{\Lambda}(\Gamma F)^{1/2} \) and \( J \) is a diagonal matrix with entries \( \pm 1 \). Part (i) follows from the fact that \( \|JY_0\| = O(1) \), since \( (\Gamma F)^{1/2}\Sigma_{\Lambda}(\Gamma F)^{1/2} \) is finite. Likewise part (ii) follows from the fact that \( J \) is obviously positive definite and \( Y_0 \) is also positive definite because the eigenvalues of \( (\Gamma F)^{1/2}\Sigma_{\Lambda}(\Gamma F)^{1/2} \) are distinct by Assumption 3. This completes the proof. \( \square \)

**Lemma 10.** Under Assumptions 1 through 4, as \( n \to \infty \),

(i) \( \|H\| = O(1) \);

(ii) \( \|H^{-1}\| = O(1) \).

Proof. From (A9) in the proof of Proposition 1(c) \( H = (\Gamma F)^{1/2}KJ \). Then, part (i) and (ii) follow immediately from Assumption 1(b), Lemma 9(i), 9(ii), and since \( J \) is obviously finite and positive definite. This completes the proof. \( \square \)

**Lemma 11.** Under Assumptions 1 through 4, as \( n,T \to \infty \),

(i) \( \|\hat{H}\| = O_P(1) \);

(ii) \( \|\hat{H}^{-1}\| = O_P(1) \).

Proof. From (28), by Proposition 1(a), Lemma 3(v), 7(iv) 10(i), we have
\[
\|\hat{H}\| \leq \|F'F\| \left\| \frac{\Lambda'\hat{\Lambda}}{n} \right\| \left\| \left( \frac{M^*}{n} \right)^{-1} \right\| \leq \|F'F\| \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \frac{\hat{\Lambda}}{\sqrt{n}} \right\| \left\| \left( \frac{M^*}{n} \right)^{-1} \right\| \leq \frac{F}{\sqrt{T}} \left\| \frac{\Lambda}{\sqrt{n}} \right\| \|H\| \left\| \left( \frac{M^*}{n} \right)^{-1} \right\| + \frac{F}{\sqrt{T}} \left\| \frac{\Lambda}{\sqrt{n}} \right\| \left\| \frac{\hat{\Lambda} - \Lambda H}{\sqrt{n}} \right\| \left\| \left( \frac{M^*}{n} \right)^{-1} \right\| = O_P(1) + O_P \left( \max \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}} \right) \right),
\]
which proves part (i).

For part (ii), we have
\[
\|\hat{H}^{-1}\| \leq \left\| \frac{M^*}{n} \right\| \left\| \left( \frac{\Lambda'\hat{\Lambda}}{n} \right)^{-1} \right\| \left\| \left( \frac{F'F}{T} \right)^{-1} \right\| \]  
(B21)
Now, because of Proposition 1(a), Lemma 2(i), 10(i), and Assumption 1(a),
\[
\left\| \frac{\Lambda'\hat{\Lambda}}{n} - \Sigma_{\Lambda}H \right\| \leq \left\| \frac{\Lambda'\hat{\Lambda}}{n} - \frac{\Lambda'\Lambda H}{n} \right\| + \left\| \frac{\Lambda'\Lambda H}{n} - \Sigma_{\Lambda}H \right\| = o_P(1).
\]
And since by Assumption 1(a) and Lemma 10(ii), \( \Sigma_{\Lambda}H \) is positive definite, then \( \frac{\Lambda'\hat{\Lambda}}{n} \) is positive definite with probability tending to one, as \( n,T \to \infty \), i.e.,
\[
\left\| \left( \frac{\Lambda'\hat{\Lambda}}{n} \right)^{-1} \right\| = O_P(1).
\]  
(B22)
Moreover, because of Lemma 4 and Assumption 1(b), \( F'F \) is positive definite with probability tending to one,
as $T \to \infty$, i.e.,

$$\left\| \left( \frac{F'F}{T} \right)^{-1} \right\| = O_p(1).$$

(B23)

Then, because of Lemma 7(iii), using (B22) and (B23) in (B21), we prove part (ii). This completes the proof. □
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