Separability Problems in Creative Telescoping*

Shaoshi Chen\textsuperscript{a,b}, Ruyong Feng\textsuperscript{a,b}, Pingchuan Ma\textsuperscript{a,b}, and Michael F. Singer\textsuperscript{c}

\textsuperscript{a}KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China  
\textsuperscript{b}School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China  
\textsuperscript{c}Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA  
schen@amss.ac.cn, ryyfeng@amss.ac.cn, mapingchuan15@mails.ucas.ac.cn, singer@math.ncsu.edu

ABSTRACT

For given multivariate functions specified by algebraic, differential or difference equations, the separability problem is to decide whether they satisfy linear differential or difference equations in one variable. In this paper, we will explain how separability problems arise naturally in creative telescoping and present some criteria for testing the separability for several classes of special functions, including rational functions, hyperexponential functions, hypergeometric terms, and algebraic functions.

CCS CONCEPTS

- Computing methodologies → Algebraic algorithms.

KEYWORDS

Creative telescoping, Separable functions, Separation of variables, Zeilberger’s algorithm

ACM Reference Format:
Shaoshi Chen\textsuperscript{a,b}, Ruyong Feng\textsuperscript{a,b}, Pingchuan Ma\textsuperscript{a,b}, and Michael F. Singer\textsuperscript{c}. 2021. Separability Problems in Creative Telescoping. In Proceedings of the 2021 International Symposium on Symbolic and Algebraic Computation (ISSAC ’21), July 18–23, 2021, Virtual Event, Russian Federation. ACM, New York, NY, USA, 8 pages. https://doi.org/10.1145/3452143.3465514

1 INTRODUCTION

The method of separation of variables has been used widely in solving differential equations [23]. In order to solve the one-dimensional heat equation

$$\frac{\partial y}{\partial t} - c \frac{\partial^2 y}{\partial x^2} = 0,$$

together with the boundary conditions \(y(t, 0) = y(t, L) = 0\), where \(L \neq 0\), one can try to find a nonzero solution of the form

$$y = u(t)v(x),$$

and then substitute this form into the equation to get

$$\frac{\partial u(t)}{\partial t} = c \frac{\partial^2 v(x)}{\partial x^2}.$$ 

Since both sides only depend on one variable, there exits some constant \(\lambda \in \mathbb{C}\) such that

$$\frac{\partial u}{\partial t} - \lambda u = 0 \quad \text{and} \quad c \frac{\partial^2 v}{\partial x^2} - \lambda v = 0.$$ 

Note that the above two equations are also satisfied by \(y = u(t)v(x)\), which are linear differential equations in only one variable. After solving these special equations by taking the boundary conditions into account, a special solution of the heat equation can be given as

$$y(t, x) = \sum_{n=1}^{\infty} d_n \sin \left( \frac{n\pi x}{L} \right) \exp \left( -\frac{n^2 \pi^2 c t}{L^2} \right),$$

where \(d_n \in \mathbb{C}\) are coefficients determined by the initial conditions. Motivated by this example, one can ask the following natural question.

**Problem 1.1 (Separability Problem).** Given a multivariate function specified by certain equations (e.g. algebraic, differential or difference equations), decide whether this function satisfies linear differential or difference equations in one of the arising variables.

To make the problem more tractable, we will consider some special classes of functions, such as rational functions, algebraic functions, hyperexponential functions and hypergeometric terms. Our main contributions are separability criteria for these classes of functions, especially the algebraic case is new. We will also show the close connection between the separability problem and Zeilberger’s method of creative telescoping [29, 30].

The remainder of this paper is organized as follows. We specify the separability problem and the existence problem of telescopes precisely in Section 2 together with the definition of orders and (local) dispersions of rational functions. After this, we explain how the separability problems arise naturally in creative telescoping for rational functions in Section 3, hyperexponential functions and hypergeometric terms in Section 4, and for algebraic functions in Section 5. Separability criteria will be given for these classes of special functions. We then conclude our paper with some comments on the separability problem on D-finite functions and P-recursive sequences.

---

* S. Chen was partially supported by the NSFC grants 11871067, 11688101, the Fund of the Youth Innovation Promotion Association, CAS, and the National Key Research and Development Project 2020YFA0712300. R. Feng was partially supported by the NSFC grants 11771433, 11688101, and Beijing Natural Science Foundation under Grant Z190004. P. Ma was partially supported by the NSFC grants 11871067. M.F. Singer was partially supported by a grant from the Simons Foundation (No. 349357, Michael Singer).

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than the author(s) must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.

ISSAC ‘21, July 18–23, 2021, Virtual Event, Russian Federation
© 2021 Copyright held by the owner/author(s). Publication rights licensed to ACM.
ACM ISBN 978-1-4503-8382-0/21/07... $15.00
https://doi.org/10.1145/3452143.3465514
2 PRELIMINARIES

Let $F$ be a field of characteristic zero and let $E = F(t, x)$ be the field of rational functions in $t$ and $x = (x_1, \ldots, x_m)$ over $F$. Let $\delta_t, \delta_x$, be the usual partial derivations $\partial/\partial t, \partial/\partial x$ with $x \in \{x_1, \ldots, x_m\}$, respectively. The shift operators $\sigma_t$ and $\sigma_x$ on $E$ are defined as the $F$-automorphisms such that for any $f \in E$, $\sigma_t(f(t, x)) = f(t + 1, x)$ and $\sigma_x(f(t, x)) = f(t, x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_m)$.

The ring of linear functional operators in $t$ and $x$ over $E$ is denoted by $E(\delta_t, \delta_x)$, where $\delta_x = (\delta_{x_1}, \ldots, \delta_{x_m})$ and $\delta_t$ with $t \in \{t, x\}$ is either the derivation $D_t$, such that $D_tf = fD_t + \delta_t(f)$ or the shift operator $S_t$, such that $S_tf = \sigma_t(f)S_t$, for any $f \in E$, and $\delta_t$ and $\delta_{x_i}$ commute. For $t \in \{t, x\}$, we let $\Delta_t$ denote the difference operator $S_t - 1$, where $1$ stands for the identity map on $E$. Abusing notation, we let $\Delta_x$ and $\sigma_x$ denote arbitrary extensions of $\delta_x$ and $\sigma_x$ to derivation and $F$-automorphism of $E$, the algebraic closure of $E$. The functions we consider will be in one common differential or difference extension of $E$, which is also an $E(\delta_t, \delta_x)$-module via the action defined by simply interpreting $D_t, S_t$ by $\delta_t, \sigma_x$, respectively, for $t \in \{t, x\}$. The ring $E(t)(\delta_t)$ is a subring of $E(\delta_t, \delta_x)$ that is also a left Euclidean domain. Efficient algorithms for basic operations in $E(t)(\delta_t)$, such as computing the least common left multiple (LCLM) of operators, have been developed in [6, 10].

Definition 2.1 (Separable functions). Let $M$ be an $E(\delta_t, \delta_x)$-module and $f \in M$. We say that $f(t, x)$ is $\delta_t$-separable if there exists a nonzero $L \in E(t)(\delta_t)$ such that $L(f) = 0$.

As an example, the special solution (1.1) of the one-dimensional heat equation is both $D_t$-separable and $D_x$-separable. We should mention that the notion of separable functions in Definition 2.1 and that of separable polynomials and extensions in Galois theory [20, Chapter V] are not connected. The problem of separating variables in bivariate polynomial ideals in [11] is somehow also irrelevant to the separability problem studied in this paper. Note that $\delta_t$-separable functions are just the D-finite functions in the differential case and the P-recursive sequences in the shift case, which are both introduced in [28]. By the closure properties of D-finite functions and P-recursive sequences, we have the same closure properties for $\delta_t$-separable functions.

Proposition 2.2. Let $M$ be an $E(\delta_t, \delta_x)$-module. If $f, g \in M$ are $\delta_t$-separable, so are $f + g, f \cdot g$, and $a \cdot f$ for all $a \in E(t)$.

We will focus on the separability problem on functions in an $E(\delta_t, \delta_x)$-module.

Definition 2.3 (Creative telescoping). Let $M$ be an $E(\delta_t, \delta_x)$-module and $f \in M$. A nonzero operator $L \in E(t)(\delta_t)$ is called a telescoper of type $(\delta_t, \delta_x)$ for $f$ if there exist $Q_1, \ldots, Q_m \in E(\delta_t, \delta_x)$ such that

$$L(t, \delta_t)(f) = \delta_{x_1}(Q_1(f)) + \cdots + \delta_{x_m}(Q_m(f)),$$

where $\delta_t \in \{D_t, S_t\}$ and $\delta_{x_i} \in \{D_{x_i}, \Delta_{x_i}\}$.

The central problem in the Wilf-Zeilberger theory of automatic proving of special-function identities is related to the existence and the computation of telescopers for special functions. In the next sections, we will show that this central problem of creative telescoping is closely connected to the separability problem on the corresponding class of special functions.

Let $V = (V_1, \ldots, V_r)$ be any set partition of the variables $v = (t, x_1, \ldots, x_m)$. A rational function $f \in F(t, x)$ is called a split function with respect to the partition $V$ if $f = f_1 \cdots f_r$ with $f_i \in F(V_i)$ and a semi-split function with respect to $V$ if there are split functions $q_i \in F(t, x)$ such that $f = q_1 \cdots + q_n$. By definition, we have $f = p/q$ with $p, q \in F[t, x]$ and gcd$(p, q) = 1$ is semi-split with respect to the partition $V$ if and only if the denominator $q$ is a split polynomial with respect to the partition $V$. Split rational functions will be used to describe the separability of given functions.

Let $K = F(x)$ and $p \in K[t]$ be an irreducible polynomial in $t$. For any $f \in K(t)$, we can write $f = p^na/b$, where $m \in a, b \in K[t]$ with gcd$(a, b) = 1$ and $p \notin ab$. The integer $m$ is called the order of $f$ at $p$, denoted by $v_p(f)$. Conventionally, we set $v_p(0) = +\infty$. We collect some basic properties of valuations as follows and refer to [9, Chapter 4] for their proofs.

Proposition 2.4. Let $f, g \in K(t)$ and $p \in K[t]$ be an irreducible polynomial. Then,

(i) $v_p(fg) = v_p(f) + v_p(g)$.

(ii) $v_p(f + g) \geq \min\{v_p(f), v_p(g)\}$ and equality holds if $v_p(f) \neq v_p(g)$.

(iii) If $v_p(f) \neq 0$, then $v_p(D_t(f)) = v_p(f) - 1$. In particular, for any $i \in \mathbb{N}$, $v_p(D^i_t(f)) = v_p(f) - i$ if $v_p(f) < 0$.

The dispersion introduced by Abramov in [1] can be viewed as a shift analogue of the order. For any polynomial $u \in K[t]$ with $\deg_t(u) \geq 1$, the dispersion $\delta u$ of $u$, denoted by $\delta(u)$, is defined as $\max\{k \in \mathbb{N} : \gcd(u, \sigma_t^k(u)) = 1\}$, which is the maximal integer root-distance $|\alpha - \beta|$ with $\alpha, \beta$ being roots of $u$ in $K$. Define $\delta(u) = 0$ if $u \in K - \{0\}$ and $\delta(u) = +\infty$. For a rational function $f = a/b \in K(t)$ with $a, b \in K[t]$ and gcd$(a, b) = 1$, define $\delta(f) = \delta(b)$. For later use, we introduce a local version of Abramov’s dispersion. Let $p \in K[t]$ be an irreducible polynomial. If $\sigma_t^k(p) \mid u$ for some $k \in \mathbb{Z}$, the local dispersion of $u$ at $p$, denoted by $\delta_p(u)$, is defined as the maximal integer distance $|i - j|$ with $i, j \in \mathbb{Z}$ satisfying $\sigma_t^k(p) \mid u$ and $\sigma_t^j(p) \mid u$; otherwise we define $\delta_p(u) = 0$. Conventionally, we set $\delta_p(0) = +\infty$. For a rational function $f = a/b \in K(t)$ with $a, b \in K[t]$ and gcd$(a, b) = 1$, we also define $\delta_p(f) = \delta_p(b)$. By definition, we have

$$\delta(u) = \max\{\delta_p(u) : p \text{ is an irreducible factor of } u\}.$$

The set $\{\sigma_t^k(p) \mid i \in \mathbb{Z}\}$ is called the $\sigma_t$-orbit at $p$, denoted by $[p]_{\sigma_t}$. Note that $\delta_p(u) = \delta_p(u)_{\sigma_t} = 0$, and we can define the local dispersion of a rational function $f$ at a $\sigma_t$-orbit at $p$, denoted by $\delta_{[p]_{\sigma_t}}(f)$.

Example 2.5. Let $u = x(x + 1)(x - 5)(x^2 + 1)(x^2 + 4x + 5) \in \mathbb{Q}[x]$. Then we have $\delta_{[0]}(u) = 6$ and $\delta_{[0]}(u) = 2$. Abramov’s dispersion of $u$ is then equal to 6.

We now show how the local dispersions change under the action of linear recurrence operators, which was first proved for Abramov’s dispersion in [1, 2] and [25, Section 3.1].

Lemma 2.6. Let $f = a/b \in K(t)$ with $a, b \in K[t]$ and gcd$(a, b) = 1$ and let $p \in K[t]$ be an irreducible factor of $b$. Let $L = \sum_{\ell \geq 0} L_{\ell}^p \in K[t]S_1$ be such that $\ell_pL_{\ell}^p \neq 0$ and $\sigma_t^k(p)$ does not divide $\ell_pL_{\ell}^p$ for any
\[ i \in \mathbb{Z}. \text{ Then } \text{disp}(L(f)) = \text{disp}(f) + p. \text{ In particular, } \text{disp}(\Delta_t(f)) = \text{disp}(f) + 1. \]

**Proof.** Let \( d = \text{disp}(b). \) Without loss of generality, we may assume that \( p \mid b \) but \( \sigma_t^p(p) \nmid b \) for any \( i < 0. \) Since \( \gcd(a, b) = 1 \) and \( \sigma_t \) is a \( \mathbb{K} \)-automorphism of \( \mathbb{K}[t] \), we have \( \gcd(\sigma_t^i(a), \sigma_t^i(b)) = 1 \) for any \( i \in \mathbb{Z}. \) Applying \( L \) to \( f \)

\[
L(f) = \sum_{i=0}^{\rho} \ell_i(\sigma_t^i(a))^u_i, 
\]

where \( u = b\sigma_t(\cdots \sigma_t^p(b) \cdots \) and \( u_i = u_i/\sigma_t^i(b). \) Write \( L(f) = A/B \) with \( A, B \in \mathbb{K}[t] \) and \( \gcd(A, B) = 1. \) Then \( B \mid u \) and \( \text{disp}(L(f)) = \text{disp}(B) \) by definition. Since \( \sigma_t^i(p) \nmid \ell_0 \) and \( \sigma_t^i(p) \nmid \ell_p \) for any \( i \in \mathbb{Z}, \) we have both \( p \) and \( \sigma_t^{d+p}(p) \) do not divide the sum \( \sum_{i=0}^{\rho} \ell_i(\sigma_t^i(a))u_i, \) but they divide \( u. \) So \( p \mid B \) and \( \sigma_t^{d+p}(p) \mid B, \) which implies that \( \text{disp}(B) \geq d + p. \) Since \( B \mid u, \) we have \( \text{disp}(B) \leq \text{disp}(u) = d + p. \) Therefore, \( \text{disp}(L(f)) = d + p. \]

3 THE RATIONAL CASE

We first explain how the existence problem of telescopes for rational functions is naturally connected to the separability problem on this class of functions. Let \( f(t, x) \) be a bivariate rational function in \( \mathbb{F}(t, x). \) By the Ostrogradsky-Hermite reduction [19, 24], we can decompose \( f \) into the form

\[
f = D_x(g) + a/b,
\]

where \( g \in \mathbb{F}(t, x) \) and \( a, b \in \mathbb{F}(t)[x] \) with \( \gcd(a, b) = 1, \) \( \deg_x(a) < \deg_x(b) \) and \( b \) being squarefree in \( \mathbb{F}(t) \) for all \( i \in \mathbb{N} \) and \( \deg_x(a(t + i, x)) < \deg_x(b(t + i, x)). \) So \( \deg_x(p) < \deg_x(q) \) and \( q \) is also squarefree in \( \mathbb{F}(t) \) for all \( i \in \mathbb{N} \). This implies that the operator \( L \) is a telescope of type \( (S_t, X_t, S_t^\perp) \) for \( a/b, \) i.e., \( L(a/b) = D_x(g) \) for some \( g \in \mathbb{F}(t, x) \) if and only if \( p = 0, i.e., L(a/b) = 0. \) Therefore, we conclude that \( f \) has a telescope of type \( (S_t, X_t) \) if and only if \( a/b \) is \( S_t \)-separable.

We can also consider telescopes of type \( (D_t, X_t, S_t) \). By Abramov's reduction [3, 4], we can decompose \( f \in \mathbb{F}(t, x) \) into the form

\[
f = \Delta_t(g) + a/b,
\]

where \( g \in \mathbb{F}(t, x) \) and \( a, b \in \mathbb{F}(t)[x] \) with \( \gcd(a, b) = 1, \) \( \deg_x(a) < \deg_x(b) \) and \( b \) being shift-free in \( \mathbb{F}(t) \) for all \( i \in \mathbb{N} \). If telescopes of type \( (S_t^\perp, X_t) \) exist for \( H, \) Abramov's algorithm which computes telescopes for hypergeometric terms (see [26, Chapter 6]).

Let \( H(t, x) \) be a nonzero hyperexponential term over \( \mathbb{F}(t, x) \), i.e., both \( \sigma_t(H)/H \) and \( \sigma_x(H)/H \) are in \( \mathbb{F}(t, x). \) If telescopes of type \( (S_t^\perp, X_t) \) exist for \( H, \) Abramov's algorithm starts from an ansatz: for fixed \( p \in \mathbb{N}, \) let \( S = \sum_{i=0}^{\rho} \ell_i(t)^x \in \mathbb{F}(t)(S_t^\perp) \) with the \( \ell_i \)'s being undetermined coefficients. Applying \( L \) to \( H \) yields

\[
T := L(H) = \sum_{i=0}^{\rho} \ell_i(t)^x H = \sum_{i=0}^{\rho} \ell_i(t)^x a_i H = \sum_{i=0}^{\rho} \ell_i(t)^x a_i H = \sum_{i=0}^{\rho} \ell_i(t)^x a_i H = \sum_{i=0}^{\rho} \ell_i(t)^x a_i H.
\]
where \( a_i = \sigma^i(\mathcal{H})/\mathcal{H} = p_i/q \in \mathbb{F}(t, x) \) with \( p_i, q \in \mathbb{F}[t, x] \). The second step of Zeilberger’s algorithm is computing the Gosper form of \( \mathcal{L}(\mathcal{H}) \) that gives

\[
\frac{\sigma_x(\mathcal{L}(\mathcal{H}))}{\mathcal{L}(\mathcal{H})} = \frac{\sigma_x \left( \sum_{i=0}^{\rho} \ell_i p_i \right)}{\sum_{i=0}^{\rho} \ell_i p_i} \frac{\sigma_x(p)}{p} \frac{1}{r},
\]

where \((p, q, r) \in \mathbb{F}(t)[x]^3\) is a Gosper form of the rational function \( Q_{\sigma_x}(\mathcal{H})/(\sigma_x(\mathcal{Q})) \) satisfying that \( \gcd(q, \sigma_x^i(r)) = 1 \) for all \( i \in \mathbb{N} \). The last step is finding \( \delta_0, \ldots, \delta_{\rho} \in \mathbb{F}(t) \), not all zero, such that the equation

\[
\left( \sum_{i=0}^{\rho} \ell_i p_i \right) p = q \sigma_x(z) - \frac{1}{\sigma_x(r)}(z).
\]

has a polynomial solution in \( \mathbb{F}(t)[x] \). If so, then \( \mathcal{L} = \sum_{i=0}^{\rho} \ell_i S^i \) is a telescoper for \( \mathcal{H} \). It may happen that the final choice of the \( \ell_i \)'s satisfies that \( \sum_{i=0}^{\rho} \ell_i p_i = 0 \). This means division by zero may happen in the second step. To avoid this, we should first detect whether \( \mathcal{L}(\mathcal{H}) = 0 \) for some \( \mathcal{L} \in \mathbb{F}(t)(S_t) \), i.e., the separability problem on hypergeometric terms. In this special situation, Zeilberger’s algorithm still works and returns \( z = 0 \).

The following theorem characterizes all possible separable hyperexponential functions and hypergeometric terms, whose proof was given in [21, Lemma 4] or in [14, Proposition 10].

**Theorem 4.1.** Let \( \mathcal{M} \) be an \( \mathbb{E}(\partial_x, \partial_y) \)-module and let \( \mathcal{H} \in \mathcal{M} \) be such that

\[ \partial_t(\mathcal{H}) = aH \text{ and } \partial_x(\mathcal{H}) = b_t H \text{ with } a, b_t \in \mathbb{F}(t, x). \]

Then the following holds,

(i) Hyperexponential case: \( \mathcal{H} \) is \( D_t \)-separable if and only if there exist \( p \in \mathbb{F}(x)[t] \) and \( r \in \mathbb{F}(t) \) such that

\[ a = \frac{\delta_t(p)}{p} + r. \]

(ii) Hypergeometric case: \( \mathcal{H} \) is \( S_t \)-separable if and only if there exist \( p \in \mathbb{F}(x)[t] \) and \( r \in \mathbb{F}(t) \) such that

\[ a = \frac{\sigma_x(p)}{p} \cdot r. \]

**Remark 4.2.** The above form for \( \partial_t(\mathcal{H})/\mathcal{H} \) can be detected by algorithms for computing the Gosper form and its differential analogue in [7, 18].

### 5 THE ALGEBRAIC CASE

In this section, we solve the separability problem on algebraic functions. We first explain the connection between this problem and the following existence problem of telescopers for rational functions in three variables.

**Problem 5.1.** Given \( f \in \mathbb{F}(t, x, y) \), decide whether there exists a nonzero operator \( L \in \mathbb{F}(t)(D_t) \) such that \( L(f) = \Lambda_x(g) + D_y(h) \) for some \( g, h \in \mathbb{F}(t, x, y) \).

By applying first the Ostrogradsky-Hermite reduction in \( y \) and then Abramov’s reduction in \( x \) to \( f \in \mathbb{F}(t, x, y) \), we get

\[ f = \Lambda_x(u) + D_y(v) + r \text{ with } r = \frac{1}{y - \beta_i}. \]

where \( u, v, r \in \mathbb{F}(t, x, y), \alpha_i, \beta_i \in \overline{\mathbb{F}(t, x)} \) and the \( \beta_i \)'s are in distinct \( \sigma_x \)-orbits. Then \( f \) has a telescoper of type \((D_t, S_y, D_y)\) if and only if \( r = 0 \). By Theorem 4.21 in [13] or Theorem 4.43 in [12], we have that \( f \) has a telescoper of type \((D_t, S_x, D_y)\) if and only if for each \( i \) \( \alpha_i \) is \( D_t \)-separable in \( \mathbb{F}(t, x) \) or \( \beta_i \in \mathbb{F}(t) \) and \( \alpha_i \in \mathbb{F}(t, x)(\beta_i) \) has a telescoper of type \((D_t, S_x)\).

The existence problem of telescopers of type \((D_t, S_x) \in \mathbb{F}(t,x)(\beta) \) with \( \beta \in \mathbb{F}(t) \) has been solved in [16]. To completely solve Problem 5.1, it remains to solve the following separability problem.

**Problem 5.2.** Given an algebraic function \( f(t, x) \) over \( \mathbb{F}(t, x) \), decide whether \( f(t, x) \) is \( D_t \)-separable.

We assume that \( \mathbb{F} \) is an algebraically closed and computable subfield of \( \mathbb{C} \) in the remaining part of this section.

#### 5.1 A descent theorem

We first recall some basic notions and results from the theory of algebraic functions of one variable [17]. Let \( k \) be a field of characteristic zero and \( k(x, y) \) be an algebraic function field of one variable over \( k \), i.e., the transcendence degree of \( k(x, y) \) over \( k \) is one. This means there exists a nonzero polynomial \( f \in k[X, Y] \) such that \( f(x, y) = 0 \). The field of constants of \( k(x, y) \) is defined as the set of elements of \( k(x, y) \) which are algebraic over \( k \). A subring \( R \) of \( k(x, y) \) is called a valuation ring if \( k \subset R \subset k(x, y) \) and for any \( a \in k(x, y) \) nonzero, either \( a \in R \) or \( a^{-1} \in R \). Any valuation ring \( R \) of \( k(x, y) \) is a local ring, whose unique maximal ideal \( p \) is called a place of \( k(x, y) \) and the quotient field \( R/p \) is called the residue field of the place \( p \), denoted by \( \Sigma_p \).

**Lemma 5.3.** Let \( k(x, y) \) and \( f \in k[X, Y] \) be as above. Assume that \((\bar{x}, \bar{y}) \in k^2 \) satisfies that \( f(\bar{x}, \bar{y}) = 0 \) and \( \frac{\partial f}{\partial Y}(\bar{x}, \bar{y}) \neq 0 \). Then there is a unique place \( p \) of \( k(x, y) \) containing \( x-\bar{x} \) and \( y-\bar{y} \). Furthermore, the residue field \( \Sigma_p \) of \( p \) is isomorphic to \( k \) and \( k \) is the field of constants of \( k(x, y) \).

**Proof.** By Corollary 2 of [17, page 8], there is a place of \( k(x, y) \) containing \( x-\bar{x} \) and \( y-\bar{y} \), say \( p \). Let \( a \) be the discrete valuation ring (DVR) with respect to \( p \). It is easy to see that the ring \( k[\bar{x}, \bar{y}] \) is contained in \( a \). Let \( m \) be the ideal in \( k[\bar{x}, \bar{y}] \) generated by \( x-\bar{x} \) and \( y-\bar{y} \). Then \( m \) is a maximal ideal. Denote by \( R \) the localization of \( k[\bar{x}, \bar{y}] \) at \( m \) and we still use \( m \) to denote the unique maximal ideal of \( R \). Rewriting \( f(X, Y) \) as a polynomial in \( X - \bar{x}, Y - \bar{y} \) yields that

\[
\frac{\partial f}{\partial Y}(\bar{x}, \bar{y}) + (Y - \bar{y})A(X - \bar{x}, Y - \bar{y}) + (X - \bar{x})B
\]

for some \( A, B \in k[X - \bar{x}, Y - \bar{y}] \). Since \( \frac{\partial f}{\partial Y}(\bar{x}, \bar{y}) \neq 0 \), one has that \( \frac{\partial f}{\partial Y}(\bar{x}, \bar{y}) + (Y - \bar{y})A(X - \bar{x}, Y - \bar{y}) \) is invertible in \( R \) and so \( y - \bar{y} \in (x - \bar{x})R \). It implies that \( R \) is a regular local ring, i.e., a DVR. Therefore \( R = a \), since \( R \subset a \). This concludes that \( p \) is unique.

We have that \( \Sigma_p = R/m = k[x, y]/m \cong k \). Since the field of constants of \( k(x, y) \) is a subfield of \( \Sigma_p \) under the natural homomorphism, it coincides with \( k \).

**Remark 5.4.** Let \( k(x, y) \) and \( (\bar{x}, \bar{y}) \) be as in Lemma 5.3. The above proof implies that \( k(x, y) \) can be embedded into the field of formal Laurent series \( k((x-\bar{x})) \).
Theorem 5.5. Let \( F \subseteq k \subseteq C \) be fields with \( F \) being algebraically closed. Let \( f(t, Y) \) be an irreducible polynomial in \( k[t, Y] \). Let \( k(t, y) \) be the quotient field of \( k[t, Y]/(f) \). Assume that

1. the places of \( k(t) \) that ramify in \( k(t, y) \) are defined over \( F \), i.e., their uniformizing parameters can be chosen to be \( 1/t \) or \( t-c \) with \( c \in F \).
2. there exists a solution \((a, \alpha)\) of the system

\[
\begin{align*}
 f(a, \alpha) &= 0, \\
 \frac{\partial f}{\partial Y}(a, \alpha) &\neq 0,
\end{align*}
\]

where \( a \in F \) and \( \alpha \in k \).

Then there exists \( \beta \in \overline{F(t)} \) such that \( k(t, y) = k(t, \beta) \).

Proof. Since \((a, \alpha)\) is a simple point of \( f(t, Y) = 0 \) in \( k^2 \), by [27], \( f(t, Y) \) is absolutely irreducible over \( k \). This implies that \( f \) is irreducible over \( C \), i.e., \( C[t, Y]/(f) \) is an integral domain. Let \( C(t, y) \) be the quotient field of \( C(t, Y)/f \). Then \( k(t, y) \) can be considered as a subfield of \( C(t, y) \) under the natural homomorphism. From Theorem 3 in [17, page 92], none of places of \( C(t, y) \) is ramified with respect to \( k(t, y) \). Therefore the condition 1 holds for \( C(t, y) \).

Proposition 2.1 in [22, page 10] states that there is \( \beta \in \overline{F(t)} \) such that \( C(t, y) = C(t, \beta) \). Now there are \( g_0(t), \ldots, g_{n-1}(t) \in C(t) \) such that

\[
\beta = \sum_{i=0}^{n-1} g_i(t)\alpha^i,
\]

(5.1)

where \( n = [C(t, y) : C(t)] \). For each \( i \), let \( g_i = q_i/q \) with \( q_i, q \in C[t] \) and let \( s = \max \{ \deg q_i, \deg \alpha \} \). Equation (5.1) implies that \( q\beta = \sum_{i=0}^{s} q_i t^{d_i} \) and therefore the set

\[
\{ t^i \beta, t^j \alpha \} \quad j=0, \ldots, s, i=0, \ldots, n-1
\]

is linearly dependent over \( C \). This set lies in \( k(t, y) \) and, since it is linearly dependent over \( D_t \)-constants in a larger differential field, it is linearly dependent over \( D_t \)-constants in \( k(t, y, \beta) \). Denote by \( k \) the set of \( D_t \)-constants of \( k(t, y, \beta) \). If \( k = k(t) \), then \( \beta \in k(t, y) \), which will conclude the proposition. Therefore it suffices to prove that \( k = k(t) \). It is easy to verify that \( k \) coincides with the field of constants of \( k(t, y, \beta) \). In the following, we will show that the field of constants of \( k(t, y, \beta) \) is equal to \( k(t) \).

From Remark 5.4, \( k(t, y) \) and \( C(t, y) \) can be embedded into \( k((t-a)) \) and \( C((t-a)) \) respectively. We will consider them as the subfields of \( k((t-a)) \) and \( C((t-a)) \) respectively. Since \( \beta \in C(t, y) \cap \overline{F(t)} \), \( F \) is algebraically closed and \( a \in F, \beta \in F((t-a)) \). Therefore, \( k(t, y, \beta) \subseteq k((t-a)) \). Since \( k \) is algebraically closed in \( k((t-a)) \), the field of constants of \( k(t, y, \beta) \) is equal to \( k(t) \). This completes the proof.

5.2 Separability criteria

Throughout this section, assume that \( F \subseteq C \) with \( F \) being algebraically closed. Let \( P = \sum_{i=0}^n A_i Y^i \in F(t, x) \). Let \( A_n(\alpha, \beta) \neq 0 \), \( P(x, a, \alpha) = 0 \) and \( \frac{\partial P}{\partial Y}(x, a, \alpha) \neq 0 \).

Let \( K = F(x, a) \) and \( \ell = [K(t, y) : K(t)] \).

Lemma 5.1. Assume that \( z \in \overline{F(t, x)} \) satisfies \( P(z) = 0 \). Then \( y \) is \( D_t \)-separable if and only if \( z \) is also \( D_t \)-separable.

Proof. Note that \( z \) and \( y \) are conjugated over \( F(t, x) \). By Theorem 3.2.4 in [9], any field automorphism of the splitting field of \( P \) commutes with the derivation \( D_t \). So for any \( L \in F(t)(D_t) \), \( L(z) = 0 \) if and only if \( L(y) = 0 \).

The above lemma implies that to detect if there is a nonzero \( L \in F(t)(D_t) \) such that \( L(y) = 0 \), it suffices to detect if there exists such operator for \( z \). In the following, we will characterize all possible \( D_t \)-separable algebraic functions. Let us firstly prove that the function field over \( F(x, a, t) \) generated by a \( D_t \)-separable algebraic function satisfies the condition 1 of Theorem 5.5.

Proposition 5.6. Let \( K \) be as above. If \( y \) is \( D_t \)-separable then \( K(t, y) \) satisfies the condition 1 of Theorem 5.5.

Proof. Assume that \( y \) is \( D_t \)-separable, i.e., there exists a nonzero \( P(x, y) \in F(t)(D_t) \) such that \( L(y) = 0 \). Let \( \nu \) be a place of \( K(t, y) \) and let \( \nu(a) \) be a place of \( K(t, y) \) that is ramified with respect to \( \nu \). Suppose that \( p \) and \( q \) are uniformizing parameters of \( \nu \) and \( \nu(a) \) respectively, and \( e \) is the corresponding ramification index. Then \( p = aq^e \) for some invertible \( a \) in the DVR with respect to \( \nu \). If \( p = c/\nu \) for some \( c \in K \) then there is nothing to prove. In the following, assume that \( p \) is an irreducible polynomial in \( K[t] \). Let \( \nu(a) \) be a place of \( C(t, y) \), lying above \( \nu \). Then by Theorem 3 in [17, page 92], \( \nu \) is not ramified with respect to \( \nu(a) \) and so \( q \) is a uniformizing parameter of \( \nu \). Since \( p \neq \nu(a) \), the uniformizing parameter of \( \nu(a) \cap C(t) \) can be selected as a factor of \( p \), say \( t - c \) for some \( c \in C \). It is easy to see that \( p/(t-c) \) is an invertible element in the DVR with respect to \( \nu \). It implies that \( t - c = aq^e \) for some invertible element \( a \) and thus \( K(t, y) \) can be embedded into \( C((t-c)^{1/e}) \). Therefore \( y \in C((t-c)^{1/e}) \) and \( c \) is a singular point of \( L \). Note that the singular points of \( L \) lie in the algebraically closed field \( F \). So \( c \in F \) and then \( p = b(t-c) \) for some \( b \in K \). In other words, \( t - c \) is a uniformizing parameter of \( \nu \). Hence \( K(t, y) \) satisfies the condition 1 of Theorem 5.5.

By Proposition 5.6 and Theorem 5.5, if \( y \) is \( D_t \)-separable, then there is \( \beta \in \overline{F(t)} \) such that \( K(t, y) = k(t, \beta) \). We now characterize separable algebraic functions as follows.

Proposition 5.7. Let \( P = \sum_{i=0}^n A_i Y^i \in F(t, x) \) with \( A_n \neq 0 \) be the minimal polynomial of \( y \in \overline{F(t, x)} \). Suppose that \( y \) is \( D_t \)-separable. Let \( K = F(x, a) \) with \( a \in \overline{F(x)} \) be as in (5.2) and \( \beta \in \overline{F(t)} \) be such that \( K(t, y) = k(t, \beta) \). Then

1. \( A_n(x, t) \) is split, i.e., \( A_n(x, t) = a(x)b(t) \), where \( a(x) \in F[x], b(t) \in F[t] \), and
2. there are \( a_i(t) \in K[t] \) such that

\[
y = \frac{1}{b(t)} \left( \sum_{i=0}^{\ell-1} a_i(t) \beta^i \right),
\]

(5.3)

where \( b \) is as in (1), \( \ell = [K(t, y) : K(t)] \) and \( q(t) \) is the discriminant of the base \( \{1, \beta, \ldots, \beta^{\ell-1}\} \).

Proof. Let \( r_i = A_i/A_n = p_i/q_i \in \overline{F(t, x)} \) with \( 0 \leq i \leq n \), \( p_i, q_i \in F[t, x] \) and \( \gcd(p_i, q_i) = 1 \). Since \( y \) is \( D_t \)-separable, so are all of the conjugate roots of \( P(Y) = 0 \). By Vieta’s formulas, the
$r_i$’s are polynomials of these roots, which therefore are also $D_i$-separable by Proposition 2.2. By Theorem 3.1, $g_i$ is split for all $i$ with $0 \leq i \leq n$. Since $A_n$ is the LCM of the $g_i$’s, we have $A_n(x, t)$ is also split.

Let $S$ be the integral closure of $K[t]$ in $K(t, y)$. Then $A_n(x, t) y \in S$. Since $\{1, \beta, \cdots, \beta^{q-1}\}$ is a base of $K(t, y)$ over $K(t)$, one has that

$$A_n(x, t) y = \frac{1}{q(t)} \sum_{i=0}^{q-1} g_i(t) \beta^i,$$

where $g_i(t) \in K[t]$. Setting $a_i(t) = g_i(t)/a(x)$, we obtain the required expression for $y$.

Recall that $K = \mathbb{F}(x, \alpha)$ and $\ell = \left[ K((t)) : K(t) \right]$. Since the derivative of $y$ is also in $K(t, y)$ for any $i \in \mathbb{N}$, we have that $Y = (1, y, y^2, \cdots, y^{\ell-1})^t$ satisfies a linear differential system of the form

$$Y' = AY, \quad \text{where } A \in \text{Mat}_\ell(K(t)). \quad (5.4)$$

We will call (5.4) the associated differential equation of $y$ over $K(t)$. The following proposition will allow us to design an algorithm for testing the separability of algebraic functions.

**Proposition 5.8.** Let $y$ and $K$ be as above. Assume that (5.4) is the associated differential equation of $y$ over $K(t)$. Then $y$ is $D_i$-separable if and only if there is an invertible matrix $G$ with entries in $K[t]$ such that

$$G^{-1}G' - G^{-1}AG \in \text{Mat}_\ell(\mathbb{F}(t)).$$

Furthermore, if $y$ is $D_i$-separable then

$$G^{-1}G' - G^{-1}AG = \frac{(b^{\ell-1}q)' - b^{\ell-1}qG^{-1}G + b^{\ell-1}qG^{-1}AG}{b^{\ell-1}q} = B$$

where $b, q, \ell$ are as in (5.3) and $B$ is the associated differential equation of $\beta$ over $\mathbb{F}(t)$ with $\beta$ being as in (5.3).

**Proof.** Assume that there exists a nonzero $L \in \mathbb{F}(t)(D_i)$ such that $L(y) = 0$. Then by Proposition 5.7, $y$ has the form (5.3). Let $E$ be the Galois closure of $K(t, \beta)$ over $K(t)$. Let $\beta_1 = \beta, \beta_2, \cdots, \beta_\ell$ be the conjugates of $\beta$ and $\sigma_i \in \text{Gal}(E/K(t))$ such that $\sigma_i(\beta) = \beta_i$. Then $\sigma_1(y), \cdots, \sigma_\ell(y)$ are all zeroes of $P(x, t, y)$. We will denote the Vandermonde matrix generated by $\sigma_1(y), \cdots, \sigma_\ell(y)$ by $U(y)$ and the one generated by $\beta_1, \cdots, \beta_\ell$ by $U(\beta)$. Then $U(y)$ is a fundamental matrix of the system (5.4) and $U(\beta)$ is a fundamental matrix of a system $Y' = BY$ with $B \in \text{Mat}_\ell(\mathbb{F}(t))$. Using the argument similar to that in the proof of Proposition 5.8, we have that for all $j$ with $1 \leq j \leq \ell - 1$,

$$y'^j = \frac{1}{b(t)^{j}q(t)} \sum_{i=0}^{\ell-1} a_{i,j}(t) \beta^i,$$

where $a_{i,j}(t) \in K[t]$ and $b(t), q(t)$ are as in Proposition 5.7. Applying $\sigma_1$ to both sides of the equalities (5.5) implies that

$$\sigma_1(y)^j = \frac{1}{b(t)^{j}q(t)} \sum_{i=0}^{\ell-1} a_{i,j}(t) \beta_1^i,$$

where $j = 1, \cdots, \ell - 1, l = 1, \cdots, \ell$. Let $\tilde{a}_{i,j} = a_{i,j}b^{\ell-1-j}$ and

$$G = \begin{pmatrix} b^{\ell-1}q^{-1}(t) & 0 & \cdots & 0 \\ \tilde{a}_{0,1}(t) & \tilde{a}_{1,1}(t) & \cdots & \tilde{a}_{1,\ell-1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{0,\ell-1}(t) & \tilde{a}_{1,\ell-1}(t) & \cdots & \tilde{a}_{\ell-1,\ell-1}(t) \end{pmatrix}$$

that is an element in $\text{Mat}_\ell(K[t])$. Then the equations (5.6) can be rewritten as $U(y) = (G(\beta))/(b(t)^{\ell-1}q(t))$. Hence $G$ is invertible and an easy calculation yields that

$$U(\beta)' = \left(\frac{b^{\ell-1}qG^{-1}U(y)}{b^{\ell-1}q}\right)' = \left(\frac{(b^{\ell-1}q)' - b^{\ell-1}qG^{-1}G + b^{\ell-1}qG^{-1}AG}{b^{\ell-1}q}\right)G^{-1}U(y).$$

This implies that

$$G^{-1}AG - G^{-1}G' = B - \frac{(b^{\ell-1}q)'}{b^{\ell-1}q} \in \text{Mat}_\ell(\mathbb{F}(t)).$$

Now we prove the converse. Assume that there is an invertible matrix $G \in \text{Mat}_\ell(K[t])$ such that

$$\tilde{B} = G^{-1}AG - G^{-1}G' \in \text{Mat}_\ell(\mathbb{F}(t)).$$

Then $U(y) = GF$, where $F$ is a fundamental matrix of $Y' = \tilde{B}Y$ with entries in some differential extension field of $K(t)$. Obviously, the entries of both $G$ and $F$ are annihilated by nonzero operators in $\mathbb{F}(t)(D_i)$ and thus so are the sum of products of entries of $G$ and $F$, in particular, so is $y$.

**Remark 5.9.** Once $\beta$ is computed, one can obtain the linear differential equations $Y' = BY$ satisfied by $U(\beta)$.

### 5.3 An algorithm for testing separability

All arguments presented in this section are summarized in Algorithm 5.11 which decides whether a given algebraic function $y \in \mathbb{F}(t, x)$ is $D_i$-separable or not. For the sake of simplicity, we may take $\mathbb{F} = \mathbb{Q}$, the field of all algebraic numbers over $\mathbb{Q}$. Let $P = \sum_{i=0}^{n} A_i Y^i \in \mathbb{F}(t, x)[Y]$ be the minimal polynomial of $y$. We may assume that $A_n$ is split, otherwise we can conclude with Proposition 5.7 that $y$ is not $D_i$-separable. Under this assumption, $y$ is $D_i$-separable if and only if $A_n y$ is $D_i$-separable. Therefore without loss of generality, we may assume that

$$P(x, t, y) = Y^n + A_{n-1}(x, t)Y^{n-1} + \cdots + A_0(x, t),$$

where $A_i \in \mathbb{F}[x, t]$. Let $(a, \alpha) \in \mathbb{F} \times \mathbb{F}(x)$ satisfy

$$P(x, a, \alpha) = 0, \quad \frac{\partial P}{\partial y}(x, a, \alpha) \neq 0,$$

and let $K = \mathbb{F}(x, \alpha)$. Then $P(x, t, y)$ may be factorized into a product of irreducible polynomials in $K[t, Y]$. There is a unique factor of $P(x, t, Y)$ in $K[t, Y]$ vanishing at $(a, \alpha)$, denoted by $P(x, \alpha, t, Y)$. Let $K(t, Y)$ be the quotient field of $K[t, Y]/(P(x, \alpha, t, Y))$. Now suppose that $y$ is $D_i$-separable. Then Proposition 5.6 and Theorem 5.5 imply that there is $\beta \in \mathbb{F}(t)$ such that $K(t, y) = K(t, \beta)$. We shall show how to find such $\beta$.

Let $R = \mathbb{F}(t)[x]$ and $S$ the integral closure of $R$ in $K(t, y)$. Then $\alpha, y \in S$. Suppose that

$$P(x, \alpha, t, Y) = B_{\ell}Y^\ell + B_{\ell-1}y^{\ell-1} + \cdots + B_0,$$
where $B_i \in \mathbb{F}[x]$, $B_i \in \mathbb{F}[x, \alpha, t]$ with $i = 0, \cdots, \ell - 1$. Note that

$$[K(t, y) : \mathbb{F}(x, t)] = [K(t, y) : K(t)][K(t) : \mathbb{F}(x)]$$

$$= [K(t, y) : K(t)][K : \mathbb{F}(x)] = \ell[K : \mathbb{F}(x)].$$

The set

$$\{a^j y^i | i = 0, \cdots, [K : \mathbb{F}(x)] - 1, j = 0, \cdots, \ell - 1 \}$$

is a base of $K(t, y)$ over $\mathbb{F}(x, t)$. Let $D(t, x)$ be the discriminant of the above base and let $F(x, y)$ be an irreducible polynomial in $\mathbb{F}[x, y]$ such that $F(x, \alpha) = 0$. Then we have

**Lemma 5.10.** Let $(c, b) \in \mathbb{F}^{m+1}$ be such that $F(c, b) = 0$ and $D(t, x)B_t(c) \neq 0$. Then $P(c, b, t, y)$ is irreducible in $\mathbb{F}[t, y]$ and for any root $y' = y$ of $P(c, b, t, y) = 0$, we have that $K(t, y)$ is isomorphic over $K(t)$ to $K(t, y)$.

**Proof.** Let $\beta \in K(t, y)$ be such that $K(t, y) = K(t, \beta)$. Since $\beta$ is algebraic over $\mathbb{F}(t)$ we have that $\beta$ is integral over $R = \mathbb{F}(t)[x]$. Therefore we may write

$$\beta = \frac{1}{D(x, t)} \sum b_{i,j} a^j y^i,$$

where the $b_{i,j} \in R$. Let $(c, b)$ satisfy the hypothesis of the lemma and consider the ideal

$$\mathfrak{p} = (x_1 - c_1, \ldots, x_m - c_m, \alpha - b) \triangleleft \mathbb{R}[\alpha].$$

Note that $\mathfrak{p}$ is a maximal ideal. The Going Up Theorem implies that there is a maximal ideal $\mathfrak{q} \triangleleft \mathfrak{p}$ such that $\mathfrak{q} \cap \mathbb{R}[\alpha] = \mathfrak{p}$. In particular, $D(x, t) \notin \mathfrak{q}$. There is a natural map $\phi : S \rightarrow S/\mathfrak{q}$. We will let $M$ denote the field $S/\mathfrak{q}$. The element $y = \phi(y)$ is a root of $P(c, b, t, y) = 0$. Since the minimal polynomial $Q(t, y)$ of $\beta$ lies in $\mathbb{F}(t, y)$, it remains unchanged when we apply $\phi$ to its coefficients. Therefore $\phi(\beta)$ satisfies $Q(t, \phi(\beta)) = 0$. In particular, the degree of $\phi(\beta)$ over $\mathbb{F}(t)$ is equal to $\ell$, the degree of $K(t, \beta)$ over $K(t)$. Since

$$\phi(\beta) = \frac{1}{D(x, t)} \sum \phi(b_{i,j}) \phi(\alpha)^i \phi(y)^j$$

we have that $\phi(\beta) \in \mathbb{F}(t)(y)$. Note that $P(c, b, t, y) \neq 0$. The element $y$ satisfies $P(c, b, t, y) = 0$ and so it has degree at most $\ell$ over $\mathbb{F}(t)$. Since $\phi(\beta) \in \mathbb{F}(t)$, we have that

$$\ell \geq [\mathbb{F}(t, y) : \mathbb{F}(t)] \geq [\mathbb{F}(t, \phi(\beta)) : \mathbb{F}(t)] = [K(t, \beta) : K(t)]$$

$$= [K(t, y) : K(t)] = \ell$$

and so $[\mathbb{F}(t, y) : \mathbb{F}(t)] = \ell$. Therefore $P(c, b, t, y)$ is irreducible. Furthermore $\mathbb{F}(t, \beta)$ is isomorphic over $\mathbb{F}(t)$ to $\mathbb{F}(t, \phi(\beta)) = \mathbb{F}(t, y)$. This implies that $K(t, y)$ is isomorphic over $K(t)$ to $K(t, y)$.

Let $P(x, \alpha, t, y)$ be as above. Lemma 5.10 implies that if $y$ is $D_t$-separable then one can compute $(c, b) \in \mathbb{F}^{m+1}$ such that $P(c, b, t, y)$ is irreducible over $\mathbb{F}(t)$. Furthermore, $\beta$ can be taken to be a zero of $P(c, b, t, y)$. To see this, let $\phi$ be the isomorphism map over $K(t)$ from $K(t, y)$ to $K(t)$ where $y$ is a zero of $P(c, b, t, y) = 0$ in $\mathbb{F}(t)$. Set $\beta = \phi(y)$. Then one has that $K(t, \beta) = K(t, y)$ and $P(c, b, t, \beta) = 0$. From $P(c, b, t, y)$, we can construct the associated differential equation of $\beta$ over $\mathbb{F}(t)$. Denote this associated differential equation by $Y' = BY$ with $B \in \text{Mat}_\ell(\mathbb{F}(t))$. The proof of Proposition 5.8 implies that if $y$ is $D_t$-separable then there is an invertible matrix $G$ with entries in $K[t]$ such that

$$G' = AG - G \left( B - \frac{q'(t)}{q(t)} \right),$$

where $q(t)$ is the discriminant of $(1, \beta, \cdots, \beta^{\ell-1})$ and $Y' = AY$ is the associated differential equation of $y$ over $K(t)$. Here the polynomial $b(t)$ in (5.5) disappears because we assume that $P$ is monic in $Y$. Note that $G$ is a polynomial solution of the linear differential equation $Y' = AY - (B - q(t)/q(t))$, which can be computed by algorithms developed in [5, 8].

We summarize the above results as the following algorithm.

**Algorithm 5.11.** Input: An irreducible polynomial

$$P(t, x, Y) = A_n Y^n + A_{n-1} Y^{n-1} + \cdots + A_0 \in \mathbb{F}[t, x, Y].$$

Output: "Yes" if $y$ is $D_t$-separable, otherwise "No", where $y \in \mathbb{F}(x, t)$ is a root of $P(Y) = 0$.

1. If $A_n$ is not split, then $y$ is not $D_t$-separable and return "No".
2. Transform $P(t, x, Y)$ into a monic polynomial by replacing $Y$ by $Y/A_n$ and clear the denominators.
3. Compute $\beta$:
   3.1. Find $(a, \alpha) \in \mathbb{F} \times \mathbb{F}(x)$ satisfying the conditions (5.8).
   3.2. Decompose $P$ into a product of irreducible polynomials over $\mathbb{F}(x, \alpha)$. Let $P(x, \alpha, t, Y)$ be the irreducible factor satisfying $\bar{P}(x, \alpha, a, a) = 0$.
   3.3. Compute $D(x, t)$, the discriminant of the base $(a^j y^i)$, where $\bar{y}$ is a zero of $\bar{P}(x, \alpha, a, a, Y)$ in $\mathbb{F}(t, x)$.
   3.4. Compute a point $(c, b) \in \mathbb{F}^{m+1}$ such that

$$D(c, t)B_t(c) \neq 0 \text{ and } P(c, b, t) = 0,$$

where $F$ is the minimal polynomial of $\alpha$ over $\mathbb{F}(x)$ and $B_t(x)$ is the leading coefficient of $P(x, \alpha, t, Y)$.

4. Compute $G$:
   4.1. Compute $q(t)$, the discriminant of the base $(b^j | j = 0, \cdots, \ell - 1)$ and compute the associated differential equations of $y$ and $\beta$, which are denoted by $Y' = \bar{A}Y$ and $Y' = BY$ respectively.
   4.2. By algorithms developed in [5, 8], compute a base of polynomial solutions of $Z' = AZ - Z(B - q(t)/q(t))$, where $Z = (z_j)$ with indeterminate entries, say $(Q_1, \ldots, Q_s)$.
   4.3. Compute $C = \text{det}(z_1 Q_1 + \cdots + z_s Q_s)$ with $z_1, \ldots, z_s$ being indeterminates. If $C = 0$ then return "No", otherwise return "Yes".

We now show an example to illustrate the main steps of the above algorithm.

**Example 5.12.** Let $E = \mathbb{Q}(t, x)$ and $y$ be the algebraic function over $E$ defined by

$$P(x, t, y) := Y^2 - 2(\alpha x + 1)Y + (\alpha x + 1)^2 - t.$$

We are going to decide whether $y$ is $D_t$-separable or not. We will follow the above algorithm step by step. Since $P(x, t, Y)$ is monic in $Y$. We begin with the third step, i.e., computing $\beta$. We have

$$(3) \text{ Compute } \beta = \sqrt{t} + 1,$$
We present a connection between the separability problems and the
variables for partial differential equations as in [23].

6 CONCLUSION AND FUTURE WORK

We present a connection between the separability problems and the existence problems in creative telescoping. Separability criteria are given for rational functions, hyperexponential functions, hypergeometric terms and algebraic functions. Some results in the algebraic case have been generalized to the case of D-finite functions whose annihilating operators of minimal order are completely reducible in [15]. Moreover in [15], for a D-finite function y, a nonzero operator $L \in \mathcal{P}(y)(\partial_y)$ such that $L(y) = 0$ is also computed if it exists. The existence problems of telescopes for rational functions in three variables are now completely settled by combining the results in [12] with the separability criteria in this paper.

In terms of future research, the first natural direction is to design efficient algorithms for computing minimal annihilators of separable functions. The second direction is to solve the separability problem for P-recusive sequences, which may have applications in solving the general termination problem of Zeilberger’s algorithms beyond the hypergeometric case. We can also try to develop more symbolic computational tools for the method of separation of variables for partial differential equations as in [23].

Acknowledgement. The authors thank the anonymous referees for their constructive and helpful comments which help us improve the writing of our paper significantly.

REFERENCES

[1] Sergei A. Abramov. The summation of rational functions. Z. Vyzis. Mat i Mat. Fiz., 11(1071–1075, 1971).

[2] Sergei A. Abramov. Solution of linear finite-difference equations with constant coefficients in the field of rational functions. USSR Computational Mathematics and Mathematical Physics, 14(4):247 – 251, 1974.

[3] Sergei A. Abramov. The rational component of the solution of a first order linear recurrence relation with rational right hand side. Z. Vyzis. Mat i Mat. Fiz., 15(4):1035–1039, 1090, 1975.

[4] Sergei A. Abramov. Indefinite sums of rational functions. In ISSAC ’95: Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation, pages 303–308, New York, NY, USA, 1995. ACM.

[5] Sergei A. Abramov and Manuel Bronstein. On solutions of linear functional systems. In ISSAC ’91: Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation, pages 1–6, New York, NY, USA, 2001. ACM.

[6] Sergei A. Abramov, Ha Quang Le, and Ziming Li. Univariate Ore polynomial rings in computer algebra. J. of Mathematical Sci., 131(5):5885–5903, 2005.

[7] Gert Almkvist and Doron Zeilberger. The method of differentiating under the integral sign. J. Symbolic Compt., 10:571–591, 1990.

[8] Moulay A. Barkatou. On rational solutions of linear differential equations. J. Symbolic Compt., 28(4-5):547–567, 1999. Differential algebra and differential equations.

[9] Manuel Bronstein. Symbolic Integration I: Transcendental Functions, volume 1 of Algorithms and Computation in Mathematics. Springer-Verlag, Berlin, second edition, 2005.

[10] Manuel Bronstein and Marko Petkovšek. An introduction to pseudo-linear algebra. Theoret. Comput. Sci., 157:3–33, 1996.

[11] Manfred Buchacher, Manuel Kauers, and Gleb Pogudin. Separating variables in bivariate polynomial ideals. In ISSAC’20: Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation, pages 54–61, New York, NY, USA, 2020. Association for Computing Machinery.

[12] Shaoshi Chen and Michael F. Singer. Residues and telescopers for bivariate polynomial ideals. In ISSAC'19: Proceedings of the 2019 on International Symposium on Symbolic and Algebraic Computation, pages 82–89, New York, NY, USA, 2019. ACM.

[13] Shaoshi Chen, Ruyong Feng, Ziming Li, and Michael F. Singer. Parallel telescoping and parameterized Picard–Vessiot theory. In ISSAC ’14: Proceedings of the 2014 International Symposium on Symbolic and Algebraic Computation, pages 99–106, New York, NY, USA, 2014. ACM.

[14] Shaoshi Chen, Ruyong Feng, Ziming Li, Michael F. Singer, and Stephen Watt. Telescopes for differential forms with one parameter. 2021. arXiv: 2101.06576.

[15] Shaoshi Chen and Michael F. Singer. Residues and telescopes for bivariate rational functions. Adv. Appl. Math., 49(2):113–133, August 2012.

[16] Claude Chevalley. Introduction to the Theory of Algebraic Functions of One Variable. Mathematical Surveys, No. VI. American Mathematical Society, New York, 1951.

[17] Ralph William Gosper, Jr. Decision procedure for indefinite hypergeometric summation. Proc. Nat. Acad. Sci. U.S.A., 75(1):40–42, 1978.

[18] Charles Hermite. Sur l’intégration des fractions rationnelles. Ann. Sci. École Norm. Sup. (2), 1:215–218, 1872.

[19] Kevin Lang. Algebra, volume 211 of Graduate Texts in Mathematics. Springer-Verlag, New York, third edition, 2002.

[20] Ha Quang Le and Ziming Li. On a class of hyperexponential elements and the fast versions of Zeilberger’s algorithm, 2004. MM-Res. Preprints (2004) No. 23, 136-150.

[21] Gunter Malle and B. Heinrich Matzat. Inverse Galois theory. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[22] Willard Miller, Jr. Symmetry and Separation of Variables. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1977. With a foreword by Richard Askey. Encyclopedia of Mathematics and its Applications, Vol. 4.

[23] Mikhail Vasil’evich Ostrogradskiĭ. De l’intégration des fractions rationnelles. Bull. de la classe physico-mathematique de l’Academie Impériale des Sciences de Saint-Pétersbourg, 4:145–167, 1860–1861.

[24] Jean-François Ragot. Probabilistic absolute irreducibility test for polynomials. J. Pure Appl. Algebra, 172(1):87–107, 2002.

[25] Richard P. Stanley. Differentiably finite power series. European J. Combin., 1(2):175–188, 1980.

[26] Herbert S. Wilf and Doron Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and “q”) multisum/integral identities. Invent. Math., 108(3):575–633, 1992.

[27] Doron Zeilberger. The method of creative telescoping. J. Symbolic Compt., 11(3):195–204, 1991.