JONES INDEX THEORY FOR HILBERT $C^*$–BIMODULES
AND ITS EQUIVALENCE WITH CONJUGATION THEORY

TSUYOSHI KAJIWARA
DEPARTMENT OF ENVIRONMENTAL AND MATHEMATICAL SCIENCES
OKAYAMA UNIVERSITY,
TSUSHIMA, 700-8530, JAPAN

CLAUDIA PINZARI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI ROMA LA SAPIENZA,
P.LE A. MORO, 2, 00185 ROMA, ITALY

YASUO WATATANI
DEPARTMENT OF MATHEMATICAL SCIENCES
KYUSHU UNIVERSITY, HAKOZAKI, 812-8581 JAPAN

Abstract

We introduce the notion of finite right (respectively left) numerical index on a $C^*$–bimodule $AX_B$ with a bi-Hilbertian structure. This notion is based on a Pimsner–Popa-type inequality. The right (respectively left) index element of $X$ can be constructed in the centre of the enveloping von Neumann algebra of $A$ (respectively $B$). $X$ is called of finite right index if the right index element lies in the multiplier algebra of $A$. In this case we can perform the Jones basic construction. Furthermore the $C^*$–algebra of bimodule mappings with a right adjoint is a continuous field of $C^*$–algebras over a compact Hausdorff space, whose fiber dimensions are bounded above by the index. We show that if $A$ is unital, the right index element belongs to $A$ if and only if $X$ is finitely generated as a right module. A finite index bimodule is a bi-Hilbertian $C^*$–bimodule which is at the same time of finite right and left index.

We study the relationship between the notion of finite index and the notion of conjugation in a tensor $2$-$C^*$–category, in the sense of Longo and Roberts. We show that bi-Hilbertian, finite index $C^*$–bimodules, when regarded as objects of the tensor $2$-$C^*$–category of right Hilbertian $C^*$–bimodules, are precisely those objects with a conjugate in that category.

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INTRODUCTION

The theory of conjugation in abstract tensor $C^\ast$–categories appeared in the algebraic formulation of Quantum Field Theory [H].

In this connection, Doplicher and Roberts showed in [DR1], [DR2] that any symmetric tensor $C^\ast$–category with conjugation can be embedded into a category of finite dimensional Hilbert spaces, and therefore the category is isomorphic to the representation category of a compact group.

However, some tensor $C^\ast$–categories with a unitary braiding, arising from low dimensional QFT, can not be embedded into categories of Hilbert spaces [LR].

R. Longo and J.E. Roberts recently studied in [LR] conjugation in general tensor $C^\ast$–categories, and they showed that this notion is closely related to the Jones index theory for subfactors [J].

A different, but related, approach to conjugation (or duality) in a tensor $C^\ast$–category has been recently studied by Yamagami in [Y].

An interesting open problem is to decide which tensor $C^\ast$–categories with conjugation can be embedded into categories of Hilbert $C^\ast$–bimodules. A related, easier, problem is to ask which sort of bimodules should appear.

In this paper we investigate the latter problem. We introduce Jones index theory for general Hilbert bimodules over pairs of $C^\ast$–algebras, and we show that bimodules of finite index are precisely those endowed with a conjugate object in the same category.

Therefore if a tensor $C^\ast$–category with conjugation can be embedded into a category of right Hilbert $C^\ast$–bimodules, these should be of finite Jones index.

In [KW1] the first and third–named authors studied Hilbert $C^\ast$–bimodules with finite Jones index in the case where the $C^\ast$–algebras are unital and the bimodules are finitely generated as right as well as left modules. It turns out that a bimodule of finite index over unital $C^\ast$–algebras, in the sense introduced in this paper (Def. 2.23), is automatically finitely generated as a bimodule, and therefore it is of the kind described in [KW1] (Cor. 2.25).

If $A$ and $B$ are $C^\ast$–algebras, an object of our category is a right Hilbert $B$–module $X$ with an action of $A$ on the left given by a nondegenerate $\ast$–homomorphism of $A$ into the $C^\ast$–algebra of $B$–module maps of $X$ into itself with adjoint. We will refer to such a bimodule as a right Hilbert $A\cdot B$ $C^\ast$–bimodule. The space of intertwiners from $AX_B$ to $AY_B$ is the set of adjointable bimodule maps.

Strictly speaking, the notion of tensor $C^\ast$–category is not correct for the category of right Hilbert $C^\ast$–bimodules. In fact, the set of objects is not a unital semigroup. We rather need the framework of tensor 2-$C^\ast$–categories, where a theory of conjugation can be developed without any substantial difficulty, as sketched in the appendix of [LR].

On the other hand, the notion of Jones index leads us to introduce bi-Hilbertian structures on $C^\ast$–bimodules. $AX_B$ is called bi-Hilbertian if it is at the same time a right and a left Hilbert $C^\ast$–bimodule in such a way that the two Banach space norms arising from the two inner products are equivalent. Examples are Rieffel’s imprimitivity bimodules [R1].

A bi-Hilbertian $C^\ast$–bimodule will be called of finite right (or left) numerical
index} if a suitable Pimsner–Popa-type inequality relating the two Banach space norms holds (see Definitions 2.8 and 2.9). A bimodule of finite numerical index is a bimodule which is at the same time of finite right and left numerical index.

If $X$ is of finite right (or left) numerical index, we construct the right (or left) index element of $X$ as a positive central element of $A''$ (or $B''$).

While the tensor product of two imprimitivity bimodules is still an imprimitivity bimodule, a tensor product of bi-Hilbertian bimodules can not be made, in general, into a bi-Hilbertian bimodule in the natural way. However, if $A X_B$ has finite right numerical index and $BY_C$ has finite left numerical index then the algebraic tensor product bimodule $X \odot_B Y$ can be completed into a bi-Hilbertian bimodule in the natural way (Prop. 2.13). One can not assume less: the left and right seminorms on $X \odot_B \ell^2(B)_{K \odot B}$ are equivalent if and only if $X$ is of finite right numerical index.

Typical examples of bimodules of finite right numerical index arise, of course, from conditional expectations between $C^*$-algebras satisfying a Pimsner–Popa inequality. The work of Frank and Kirchberg [FK] shows that under this only assumption the index element of the conditional expectation lies in the enveloping von Neumann algebra of the bigger algebra. This reflects the fact that the Jones basic construction is not always possible in the $C^*$-algebraic setting.

We introduce in our theory an extra requirement: the right index element of $A X_B$ should lie in the multiplier algebra of $A$, and therefore in its centre. When this assumption is satisfied, we say that $X$ is of finite right index.

We prove that this property is in fact equivalent to other properties which would seem stronger a priori, such as, e.g., the fact that the left action of $A$ on $X$ has range into the compacts $K(X_B)$ (Theorem 2.22). Finite bases are a useful tool in Jones index theory, as they lead to a simple formula for the index element. One of the problems in Jones index theory for $C^*$-algebras is to be able to establish existence of finite bases. In this direction, Izumi proved in [I] that a Pimsner–Popa conditional expectation from a simple, unital $C^*$-algebra admits a finite quasi–basis in the sense of [W].

We obtain, as a consequence of Theorem 2.22, the following more general result: if $X$ is a bi-Hilbertian $A$-$B$ bimodule of finite right numerical index and if $A$ is unital, the right index element of $X$ belongs $A$ if and only if $X$ is finitely generated as a right $B$-module.

More generally, we show that bimodules with finite right index over $\sigma$–unital $C^*$-algebras admit countable, unconditionally convergent, bases (see Prop. 1.6 and Cor. 2.24). Thus the the right index element of $X$ is the strict limit of the sum of left inner products with themeselves of elements of any countable right basis.

In the general case we shall deal with generalized bases in the sense of Def. 1.8, which always exist, as shown in Prop. 1.9.

Now the assumption that the right index element be a multiplier of $A$ guarantees the existence of the Jones basic construction (see Theorem 2.30), which takes the form of a positive, $A$–bilinear, strictly continuous map $F : K(X_B) \to A$ satisfying a Pimsner–Popa inequality. Since left $A$-action lies in $K(X_B)$, $F$ extends uniquely to a $A$-bilinear map $\tilde{F} : L(X_B) \to M(A)$ between the corresponding multiplier algebras. The right index element of $X$ coincides with $\tilde{F}(I)$ and it can be reached by the strict limit of the image under $F$ of an approximate
unit of $\mathcal{K}(X_B)$.

We illustrate our approach to index theory with a typical example of an inclusion of commutative unital $C^*$-algebras satisfying a Pimsner–Popa inequality, for which a finite quasi–basis in the sense of [W] does not exist. This class of examples arises from branched coverings, or orbifolds. It was first pointed out in [W], 2.8 and later analyzed by Frank and Kirchberg in [FK]. We show that this inclusion is in fact determined by a canonical nonunital subinclusion of finite right index in our sense (cf. Example 2.35).

Let us go back to our aim of comparing Jones index theory for Hilbert bimodules with conjugation theory. One of the main result of this paper is that these two approaches are equivalent (cf. Theorems 4.4 and 4.14). We show that a bi-Hilbertian $C^*$–bimodule has finite Jones index in our sense if and only if it has a conjugate object in the 2-$C^*$–category of right Hilbert $C^*$–bimodules with nondegenerate left actions. A choice of the operators $R$ and $\overline{R}$ satisfying the conjugate equations leads to a specification of a left $A$-valued inner product on $X$ making it into a finite index bi-Hilbertian bimodule. Conversely, a finite index bi-Hilbertian structure on $X$ yields a solution of the conjugate equations. The square of the Longo-Roberts dimension relative to a solution $(R, \overline{R})$ coincides with the corresponding numerical index of the bimodule.

Imprimitivity bimodules are finite index bimodules, with left and right index elements equal to the identities. They can be characterized, among general right Hilbert $C^*$–bimodules as those objects with trivial minimal dimension (Cor. 4.16).

We show two applications of our characterization theorem. The first one is that if $A X_B$ is of finite index, the set of Hilbert module mappings (i.e. those with an adjoint) on $X_B$ commuting with the left action coincides, as an algebra, with the set of Hilbert module mappings on $A X$ commuting with the right action (Cor. 4.6). Moreover, each one of these $C^*$–algebras is a continuous bundle of finite dimensional $C^*$–algebras over a compact space, in the sense of [KW] (Theorem 3.3).

As a second application, we show that the tensor product of two bi-Hilbertian bimodules of finite (resp. numerical) index is still of finite (resp. numerical) index (Theorem 5.2).

An index theory for Hilbert bimodules turns out to be more general than for conditional expectations in the case where the algebras are not $\sigma$-unital. In fact, it is known that if $E : B \to A$ is a conditional expectation satisfying a Pimsner-Popa inequality, an approximate unit of $A$ must be an approximate unit of $B$ as well. Therefore if $A$ is $\sigma$-unital, $B$ must be $\sigma$-unital as well. In particular, a unital $C^*$–algebra and a non-$\sigma$-unital one can not be linked by a Pimsner-Popa conditional expectation. However, a II$_1$ factor can be strongly Morita equivalent to a non-$\sigma$-unital $C^*$–algebra (see [BGR]).

In Section 6 we will discuss further examples of bimodules of finite index arising from locally finite directed graphs and topological correspondences.

As Franks pointed out to the third–named author, our notion of basis is not standard terminology in the theory of Banach spaces. Frames are replacement for bases in Hilbert spaces and naturally arise in wavelet theory. Franks and Larson have recently studied the concept of module frames for Hilbert $C^*$–modules in [FL1] and [FL2]. Module (quasi-)bases or frames are useful in Jones
index theory [BDH], [FK], [W]. Our concept of basis corresponds to their the notion of standard normalized tight frame. Since we will not consider orthogonal module bases in full generality, we will just use the term basis, for simplicity.

This paper is an extended version of an appendix contained in the draft of [KPW1].

1. Countable bases and generalized bases

Let $A$ be a $C^*$-algebra and $X = X_A$ a right Hilbert $C^*$-module over $A$. We denote by $\mathcal{L}(X_A)$ the $C^*$-algebra of $A$-module maps on $X$ with an adjoint.

A finite subset $\{u_i\}_i$ of $X$ is called a finite basis if $x = \sum_i u_i(x)A$ for $x \in X$. Our aim in this section is to generalize this notion to comprehend countable bases or, more generally, generalized bases in a sense that will be explained (see Definitions 1.1 and 1.8). These are infinite bases, and they will be a good substitute of finite bases in the case where the finite generation property does not hold. Indeed, we will show, generalizing slightly Dixmier’s proof of existence of approximate units in a $C^*$-algebra [Di], that a right Hilbert module $X$ always admits a generalized basis, and, in the case where $X$ is countably generated, it actually admits a countable basis.

We denote by $\theta^r_{x,y}$ the rank one operator on $X$ defined by $\theta^r_{x,y}(z) = x(y|z)A$. The linear span of rank one operators is denoted by $FR(X_A)$ and called the ideal of finite rank operators. Its norm closure, $\mathcal{K}(X_A)$, is the $C^*$-algebra of compact operators, which is a closed ideal in $\mathcal{L}(X_A)$.

For a left Hilbert $A$-module $X$, we define the rank one operators by $\theta^l_{x,y}(z) = A(z|x)y$, and the spaces of finite rank operators $FR(AX)$, compact operators $\mathcal{K}(AX)$ and adjointable left $A$-module maps $\mathcal{L}(AX)$ are defined similarly.

The right Hilbert module $X_A$ has a finite basis if and only if $\mathcal{L}(X_A) = \mathcal{K}(X_A)$. If in addition $A$ is unital, $\mathcal{L}(X_A) = \mathcal{K}(X_A)$ if and only if $X_A$ is finite projective as a right module.

We will often make use of the following formula, derived for example in Lemma 2.1 of [KPW1]. If $X_A$ is a right Hilbert $A$-module, for any $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in X_A$,

$$\left\| \sum_{i=1}^n \theta^r_{x_i,y_i} \right\| = \left\| ((x_i|x_j)A)_{ij}^{1/2} (y_i|y_j)A)_{ij}^{1/2} \right\|,$$

where the norm at the right hand side is evaluated in the matrix $C^*$-algebra $M_n(A)$.

1.1 Definition Let $X$ be a right Hilbert $A$-module. We say that a subset $\{u_i\}_{i \in \Lambda} \subset X$ is an (unconditionally convergent right) basis for $X$ if for any $x \in X$ and for any $\varepsilon > 0$ there exists a finite subset $F(\varepsilon, x)$ of $\Lambda$ such that for every finite subset $F'$ with $F' \supset F(\varepsilon, x)$, $\|x - \sum_{i \in F'} u_i(x)A\| < \varepsilon$.

In other words the net $F \in \{\text{finite subsets of $\Lambda$}\} \to \sum_{i \in F} u_i(x)A$ should converge $x$, for all $x \in X$.

One can easily show that if $\{u_i\}_{\Lambda}$ is an unconditionally convergent right basis then

$$\|u_i\| \leq 1, \quad i \in \Lambda$$
and
\[ \| \sum_{i \in F} u_i(x)A \| \leq \| x \| \]
for any finite subset $F$ of $\Lambda$. In fact, for $x \in X$,
\[ 0 \leq (x \sum_{i \in F} u_i(x)A)A \leq \varepsilon \sum_{i \in F \cup \{x\}} u_i(x)A \]
\[ \leq (xA + \varepsilon x), \]
therefore
\[ 0 \leq (x \sum_{i \in F} u_i(x)A)A \leq (xA), \]
and this shows that $0 \leq \sum_{i \in F} \theta_{u_i,u_i} \leq 1$, which implies
\[ (x)(\sum_{i \in F} \theta_{u_i,u_i})^2(x) \leq (xA). \]
Hence we have $\| \sum_{i \in F} u_i(x)A \| \leq \| x \|$. Moreover
\[ \| u_i \| = \| \theta_{u_i,u_i} \|^{1/2} \leq 1. \]
For any unconditionally convergent right basis $\{u_i\}_{i \in \Lambda} \subset X$, the net $F \to \sum_{i \in F} \theta_{u_i,u_i}$ is an approximate unit for $\mathcal{K}(XA)$. In fact, for any $x,y \in X$,
\[ \| (\sum_{i \in F} \theta_{u_i,u_i})\theta_{x,y} - \theta_{x,y} \| = \| \theta_{\sum_{i \in F} u_i(x)A-x,y} \| \leq \| \sum_{i \in F} u_i(x)A-x \| \| y \| \to 0. \]
This shows the claim as $\| \sum_{i \in F} \theta_{u_i,u_i} \| \leq 1$. For any fixed $n \in \mathbb{N}$, the $n \times n$ operator matrix $((u_i|u_j)A)_{i,j}$ is a positive contraction as $\| ((u_i|u_j)A)_{i,j} \| = \| \sum_{i=1}^{n} \theta_{u_i,u_i} \|$.
In this paper an unconditionally convergent right (or left) basis will be simply called a right (or left) basis.

A sequence $\{u_i\}_{i \in \mathbb{N}} \subset X$ is called a weak basis for $X$ if for any $x \in X$ we have $x = \lim_{n \to \infty} \sum_{i=1}^{n} u_i(x)A$. One can similarly show that $\| u_i \| \leq 1$ and $\| \sum_{i=1}^{n} u_i(x)A \| \leq \| x \|$ for all $n \in \mathbb{N}$. A countable subset $\{u_i\}_{i \in \Lambda} \subset X$ will also be called a weak basis if it is a weak basis with respect to some bijective correspondence identifying $\Lambda$ with $\mathbb{N}$.

One can easily show, using Kasparov’s stabilization trick, that any countably generated Hilbert $C^*$–module $X$ over a $\sigma$–unital $C^*$–algebra $A$ admits a weak basis. We shall show that that $X$ actually admits an unconditionally convergent countable basis.

1.2 Lemma Let $X$ be a right Hilbert $C^*$–module over $A$, and $\{u_i\}_{i \in \Lambda}$ a subset of $X$. If there exists a subset $X_0$ in $X$ such that $X_0A$ is total in $X$ and such that for every $x \in X_0$, $\sum_{i \in \Lambda} u_i(x)A = x$ then $\{u_i\}_{i \in \Lambda}$ is a right basis for $X$. Similarly, if $\Lambda = \mathbb{N}$ and if $\sum_{i} u_i(x)A = x$ for $x \in X_0$, then $\{u_i\}_{i \in \mathbb{N}}$ is a weak basis.
Proof By right linearity of the inner product, $x = \sum_{i \in \Lambda} u_i (u_i | x)_A$ for all $x$ in the linear span of $X_0 A$, which is dense in $X$. Let $F$ be a finite subset of $\Lambda$. We then have

$$ (x| \sum_{i \in F} \theta_{u_i, u_i} (x)_A ) \leq (x|x)_A $$

for $x$ in the linear span of $X_0 A$, therefore this inequality holds for all $x \in X$. This shows that $\| \sum_{i \in F} \theta_{u_i, u_i} \| \leq 1$. A 3$\varepsilon$ argument will conclude the proof.

Remark Any countable unconditionally convergent basis is a weak basis. We shall show that the converse is true under suitable conditions.

1.3 Definition We say that a weak basis $\{u_i\}_{i \in \Lambda}$ for $X$ satisfies the finite intersection property if for every $i$ there exists a finite subset $G_i \subset \Lambda$ such that for any $j \in G_i^*$ we have $(u_i | u_j)_A = 0$.

1.4 Lemma Let $\{u_i\}_{i \in \Lambda} \subset X$ be a countable weak basis for $X$. Suppose that either

1. $\{u_i\}_{i \in \Lambda}$ satisfies the finite intersection property,
2. The set $\{\theta_{u_i, u_i}, i \in \Lambda\}$ is commutative.

Then $\{u_i\}_{i \in \Lambda}$ is an unconditionally convergent basis.

Proof (1) Suppose that $\{u_i\}_{i}$ satisfies the finite intersection property. Then for every $i$ there exists a finite subset $G_i \subset \Lambda$ such that $u_i = \sum_{j \in G_i} u_j (u_j | u_i)_A$, therefore $u_i = \sum_{j \in \Lambda} u_j (u_j | u_i)_A$. We can now appeal to Lemma 1.2 with $X_0 = \{u_i, i \in \Lambda\}$.

(2) Suppose now that the rank one operators $\theta_{u_i, u_i}, i \in \Lambda$ commute pairwise. Let us fix an identification of $\Lambda$ with $\mathbb{N}$. For any $x \in X$ and for any $\varepsilon > 0$ there exists $N$ such that for any $n \geq N$,

$$ \|x - \sum_{k=1}^{n} u_k (u_k | x)_A\| < \varepsilon. $$

Setting $F_0 = \{1, 2, \cdots, N\}$, for any finite subset $F$ of $\Lambda$ with $F \supset F_0$, we have

$$ 0 \leq I - \sum_{k \in F} \theta_{u_k, u_k} \leq I - \sum_{k \in F_0} \theta_{u_k, u_k}. $$

Since $\{\theta_{u_k, u_k}\}$ commute each other, we have

$$ (I - \sum_{k \in F} \theta_{u_k, u_k})^2 \leq (I - \sum_{k \in F_0} \theta_{u_k, u_k})^2. $$

Thus we have $(x|(I - \sum_{k \in F} \theta_{u_k, u_k})^2 x)_A \leq (x|(I - \sum_{k \in F_0} \theta_{u_k, u_k})^2 x)_A$. This implies that

$$ \|x - \sum_{k \in F} u_k (u_k | x)_A\| \leq \|x - \sum_{k \in F_0} u_k (u_k | x)_A\| < \varepsilon. $$

Our next aim is to show that countable unconditionally convergent bases exist under some countability generation property.
1.5 Lemma Let $A$ be a $\sigma$–unital $C^*$–algebra. Then there exists a sequence $\{u_j\}_j$ of positive elements of $A$ such that for every $n \in \mathbb{N}$, $v_n := \sum_{j=1}^n u_j^2$ is a contraction, the sequence $\{v_n\}_n$ is a countable approximate unit of $A$, and $u_m u_n = 0$ for any pair of positive integers $m$ and $n$ with $|m - n| \geq 2$.

Proof Since $A$ is $\sigma$–unital, there exists a strictly positive contraction $h$ in $A$. For each positive integer $n$ we define a positive continuous function $f_n(x) \in C_0([0,1])$ as follows: For $n \geq 2$, let

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2^n}, \\ 2^n x - 1 & \frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}, \\ -2^{n-1} x + 2 & \frac{1}{2^{n-1}} \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases}$$

We set $f_1(x) = 0$ if $0 \leq x \leq \frac{1}{2}$ and $f_1(x) = 2x - 1$ if $\frac{1}{2} \leq x \leq 1$. Then we have $f_n f_m = 0$ if $|n - m| \geq 2$, $f_n(0) = 0$ and $\sum_{n=1}^\infty f_n(x) = 1$ for $x \in (0,1)$. Define $u_n = f_n(h)^{1/2}$, so that $v_n = \sum_{i=1}^n f_j(h)$. Then $u_n u_m = 0$ if $|n - m| \geq 2$. Moreover $\{v_n\}_n$ is a countable approximate unit of $A$ since the norm closure of $A h$ and $h A$ is equal to $A$ (see, e.g., Prop. 12.3.1 in [B]).

We are now ready to prove the existence of unconditionally convergent bases.

1.6 Proposition Let $X$ be a countably generated right Hilbert $C^*$–module over a $\sigma$–unital $C^*$–algebra $A$. Then $X$ has an unconditionally convergent right basis indexed by $\mathbb{N} = \{1, 2, \ldots\}$.

Proof As a first step, we consider the case where $X = A$ with the inner product $(x|y)_A = x^∗ y$ for $x, y \in A$. We choose a sequence $\{u_n\}_n$ of $A$ as in Lemma 1.5. Then $\sum_n u_n(u_n|x)_A = \sum_n (u_n^2 x) = \lim_n v_n x = x$. Thus $\{u_n\}_{n \in \mathbb{N}}$ is a weak basis with the finite intersection property. Hence $\{u_n\}_{n \in \mathbb{N}}$ is an unconditionally convergent right basis for $X = A$, by Lemma 1.4.

Next we consider the case where $X = \ell^2(A) := \ell^2(\mathbb{N}) \otimes_\mathbb{C} A$. Let $\{e_n\}_{n \in \mathbb{N}}$ be the canonical orthonormal basis for $\ell^2(\mathbb{N})$ and let $\varphi : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. Put $g_{i,j} = e_i \otimes u_j \in \ell^2(\mathbb{N}) \otimes_\mathbb{C} A$. Define $w_n = g_{i=n(n)} \in X$. Let $X_0 = \{(x_1, x_2, \ldots) \in X ; x_i = 0 \text{ except for finitely many } i\}$. Then $X_0$ is dense in $X$, and for any $x \in X_0$ we have $||x - \sum_{k=1}^n w_k(x|w_k)_A|| \to 0$ as $n \to \infty$. By Lemma 1.2, this holds for every $x \in X = \ell^2(A)$. Thus $\{w_n\}_{n \in \mathbb{N}}$ is a weak basis with finite intersection property and therefore a basis by Lemma 1.4.

In the general case $X \simeq p \ell^2(A)$ for some projection $p$ in $\mathcal{L}(\ell^2(A))$ by Kasparov’s stabilization theorem (see, e.g., [B]). We set $s_i = pw_i \in X$. For $x \in X \subseteq \ell^2(A)$, we have

$$x = \lim_{F \subseteq \Lambda \text{ finite}} \sum_{i \in F} w_i(x|w_i)_A,$$

thus

$$x = px = \lim_{F \subseteq \Lambda \text{ finite}} \sum_{i \in F} pw_i(x|pw_i)_A = \lim_{F \subseteq \Lambda \text{ finite}} \sum_{i \in F} s_i(x|s_i)_A$$

Therefore $\{s_i\}_{i \in \mathbb{N}}$ is an unconditionally convergent right basis for $X$. 

We do not know whether the tensor product of two bases is still a basis. But for some bases this is true, as the following proposition shows.

1.7 Proposition Let $A$ and $B$ be $\sigma$–unital $C^*$–algebras. Let $X$ and $Y$ be countably generated right Hilbert $C^*$–modules over $A$ and $B$ respectively. Let $\phi : A \to \mathcal{L}(Y_B)$ be a $^*$–homomorphism. Then there exist bases $\{a_i\}$ for $X$ and $\{t_j\}$ for $Y$ such that $\{a_i \otimes t_j\}_{i,j}$ is a basis for the right Hilbert $B$–module $X \otimes_A Y$.

Proof We first suppose that $X = \ell^2(A)$. Then, as in the proof of Prop. 1.6, we have a basis $\{w_i\}_{i \in \Lambda_1}$ for $\ell^2(A)$ with the finite intersection property, where $\Lambda_1 = \mathbb{N}$. We similarly have a basis $\{t_j\}_{j \in \Lambda_2}$ for $Y$.

Set $Z_0 = \{z \in X \otimes_A Y; z = \sum_{i=1}^m w_i a_i \otimes y_i, a_i \in A, y_i \in Y, m = 1, 2, \ldots\}$. Then $Z_0$ is dense in $X \otimes_A Y$. Using the finite intersection property one can show that for any $z = \sum_{i=1}^m w_i a_i \otimes y_i \in Z_0$ and any $\varepsilon > 0$ there exists a finite subset $F_0$ of $\Lambda_1 \times \Lambda_2$ such that for every finite subset $F$ containing $F_0$, we have

$$||z - \sum_{(k,j) \in F} w_k \otimes t_j(w_k \otimes t_j)B|| < \varepsilon.$$ 

In fact, for any $i = 1, \ldots, m$ there exists a finite subset $G_i \subset \Lambda_1$ such that for any $j \in G_i$ we have $(w_i | w_j)_A = 0$. Put $H_1 = \bigcup_{i=1}^m G_i$. Let $n$ be the cardinality of $H_1$. For $k \in G_i$ there exists a finite set $L_{i,k} \subset \Lambda_2$ such that for any finite subset $L$ of $\Lambda_2$ with $L_{i,k} \subset L$ we have

$$||\sum_{j \in L} t_j((w_k | w_i)_A a_i y_i)_B|| < \varepsilon/n.$$

Put $H_2 = \bigcup_{i=1}^m \bigcup_{k \in G_i} L_{i,k}$. Then it suffices to choose $F_0 = H_1 \times H_2$ By Lemma 1.2 $\{w_j \otimes t_j\}_{(j,k) \in \Lambda_1 \times \Lambda_2}$ is basis for $X \otimes_A Y$. In the general case there exists a projection $p \in \mathcal{L}(\ell^2(A)_A)$ such that $X \simeq p\ell^2(A)$. Define $s_i = pw_i \in X$. Then $\{s_i \otimes t_j\}_{i,j}$ is a basis for $X \otimes_A Y \simeq p\ell^2(A) \otimes_A Y$.

In the case where the right Hilbert $A$–module $X$ is not countably generated, unconditionally convergent countable bases will be replaced in the sequel by generalized bases, in the following sense.

1.8 Definition Consider a set $\Lambda$ and, for each finite subset $\mu \subset \Lambda$, let $u_\mu$ be a finite subset of $X$ with $|u_\mu| = |\mu|$. Let us endow the set of finite subsets of $\Lambda$ with the partial order defined by inclusion. The net $\mu \to u_\mu$ will be called a generalized (right) basis of $X$ if

1. for all $x \in X$, $\sum_{y \in u_\mu} (x|y)_A(y|x)_A \leq \sum_{y \in u_\nu} (x|y)_A(y|x)_A$, if $\mu \subset \nu$, 
2. $x = \lim_\mu \sum_{y \in u_\mu} (y|x)_A$.

Let $\Lambda$ be a set, and $\mu \subset \Lambda \to u_\mu \subset X$ a net with $|u_\mu| = |\mu|$. One can easily see that $\mu \to u_\mu$ is a generalized basis if and only if $\mu \to T_\mu := \sum_{y \in u_\mu} \theta_{y,y}$ is an increasing approximate unit of $\mathcal{K}(X_A)$ with norm $\leq 1$.

1.9 Proposition Any right (or left) Hilbert $C^*$–module $X$ admits a generalized right (or left) basis.
2.1 Definition  

Bimodules of finite right numerical index

includes the non unital case. B a right algebra of A concentrate on those bimodules for which the index element lies in the multiplier construction with nice properties. We also prove that in the case where unital, the index element belongs to A the complex algebras T the net A A if

\[ \sum_{n}^{\infty} \frac{a}{\sqrt{n}} \frac{a}{\sqrt{n}} \]

Thus, if X \( X \) be algebra \( \Lambda \)

\[ \sum_{n}^{\infty} \frac{a}{\sqrt{n}} \frac{a}{\sqrt{n}} \]

Therefore \( \phi : a \in A \rightarrow \phi(a) \in \mathcal{L}(X_B) \) is a \( * \)-homomorphism from A to the algebra \( \mathcal{L}(X_B) \) of right adjointable maps on \( X_B \). The map \( \phi \) will be referred to as the left action of A on X.

We introduce the notion of left Hilbert A–B bimodule in a similar manner. Thus, if X is a left Hilbert A–B bimodule, the map \( \psi : B \rightarrow \mathcal{L}(A_X) \), \( \psi(b) : x \in X \mapsto xb \in X \), for all \( b \in B \), and referred to as the right action of B on X, is a \( * \)-antihomomorphism from B to the algebra \( \mathcal{L}(A_X) \) of left adjointable maps on A.

Notice that left and right actions on a right (or left) Hilbert bimodule are not assumed to be faithful. In the following proposition we give a sufficient condition. Recall that a closed ideal J in a C*-algebra B is called essential if each nonzero closed ideal of B has a nonzero intersection with J (see 3.12.7 in [P]).

2.2 Proposition  

Let X be a right pre-Hilbert B–module (resp. left pre-Hilbert A–module). If the closed linear span in B (resp. A) of inner products \( (x|y)_B \)
(resp. $A(x|y)$) $x, y \in X$) is an essential ideal of $B$ (resp. $A$), the equation $Xb = 0$ for some $b \in B$ (resp. $aX = 0$ for some $a \in A$) implies $b = 0$ (resp. $a = 0$).

\textbf{Proof} Let $J$ denote the closed linear span in $B$ of right inner products. If $b \in B$ satisfies $yb = 0$ for all $y \in X$ then $(x|yb)_B = (x|y)_Bb = 0$ for all $x, y \in X$, hence $jb = 0$ for all $j \in J$. By Prop. 3.12.8 in [P] the natural injection of $J$ into its multiplier algebra $M(J)$ extends to an injection $B \to M(J)$. Thus reading the above equation in $M(J)$, we get $jb = 0$ for all $j \in M(J)$ and this implies that $b = 0$ since $M(J)$ is unital.

\textbf{2.3 Definition} A $A$--$B$ bimodule $AX_B$ will be called bi-Hilbertian if it is endowed with a right as well as a left Hilbert $A$--$B$ $C^*$--bimodule structure in such a way that the two Banach space norms arising from the two inner products are equivalent. In other words, $X$ is a left Hilbert module over $A$ and a right Hilbert module over $B$ such that the left $A$--action $\phi$ and the right $B$--action $\psi$ are $^*$--preserving maps into the algebras of right adjointable and left adjointable operators, respectively. Furthermore there should exist two constants $\lambda, \lambda' > 0$ such that, for $x \in X$, 
\[
  \lambda'\| (x|x)_B \| \leq \| A(x|x) \| \leq \lambda \| (x|x)_B \|.
\]

The inequality at the left hand side always extends to finite sums, in the sense of the following proposition.

\textbf{2.4 Proposition} Let $AX_B$ be a bi-Hilbertian $C^*$--bimodule, and let $\lambda' > 0$ satisfy $\lambda'\| (x|x)_B \| \leq \| A(x|x) \|$, $x \in X$. Then for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in X$ we have 
\[
  \lambda'\| \sum_{i=1}^{n} \theta_{x_i,x_i}^r \| \leq \| \sum_{i=1}^{n} A(x_i|x_i) \|.
\]

\textbf{Proof} Let $T \in M_n(B)$ be the positive matrix whose $(i,j)$-th entry is $(x_i|x_j)_B$. Notice that 
\[
  \| \sum_{i=1}^{n} \theta_{x_i,x_i}^r \| = \| T \| = \sup_{i,j=1} \sum_{i,j=1}^{n} b_i^* (x_i|x_j)_B b_j \]

where the supremum is taken over all the $n$-tuples $(b_1, \ldots, b_n)$ with elements in $B$ such that $\| \sum_{j} b_j^* b_j \| = 1$. Now the norm at the right hand side coincides with the norm of $(y|y)_B$, where $y = \sum_j x_j b_j$, therefore 
\[
  \lambda'\| (y|y)_B \| \leq \| A(y|y) \| = \| \sum_{i,j} A(x_i b_i b_j^* x_j) \| \leq \| \sum_{i} A(x_i|x_i) \|,
\]

and the proof is now complete.

On the contrary, there may exist no $\lambda > 0$ for which $\| \sum_{i=1}^{n} A(x_i|x_i) \| \leq \lambda\| \sum_{i=1}^{n} \theta_{x_i,x_i}^r \|$ for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$, as the following elementary example shows.
2.5 Example Let $A = B = \mathbb{C}$ and let $H = \ell^2(\mathbb{N})$ be an infinite dimensional Hilbert space, regarded as a bi-Hilbertian $\mathbb{C} \otimes \mathbb{C}$ bimodule in the natural way. Let $e_1, e_2, \ldots$ be a countable orthonormal subset of $H$. Then for all $n \in \mathbb{N}$, 
\[ \| \sum_{i=1}^{n} \theta e_i \| = 1, \text{ while } \| \sum_{i=1}^{n} c(e_i) e_i \| = n. \]

In fact, the existence of such a constant $\lambda$ will lead us to the notion of finite right numerical index of $X$. We anticipate a lemma.

2.6 Lemma Let $A X_B$ be a right Hilbert $C^*$–bimodule, and let $x, y \in X \rightarrow A(x|y)$ be an $A$–valued, biadditive, left $A$–linear, right $A$–antilinear form on $X$ such that $A(x|y)^* = A(y|x)$ and $A(x|x) \geq 0$ for all $x, y \in X$. If this form is continuous, in the sense that there is $\lambda > 0$ such that $\| A(x|x) \| \leq \lambda \| (x|x)_B \|$ for $x \in X$, and if the right $B$–action is adjointable with respect to this form (i.e. $A(x|yb) = A(x|yb^*)$, $x, y \in X$, $b \in B$), there exists a unique additive map $F : FR(X_B) \rightarrow A$ such that $F(\theta^{r,x}_{y,y}) = A(x|y)$. $F$ satisfies the following properties:

1. (positivity) $F(T^*T) \geq 0$, for $T \in FR(X_B)$,
2. (reality) $F(T) = F(T)^*$, for $T \in FR(X_B)$,
3. (A-bilinearity) $F(\phi(a)T) = aF(T)$, $F(T\phi(a)) = F(T)a$ for $a \in A$, $T \in FR(X_B)$,
4. (Pimsner-Popa inequality) if $X$ is bi-Hilbertian and if $\lambda' > 0$ satisfies $\lambda' \| (x|x)_B \| \leq \| A(x|x) \|$ for all $x \in X$ then $\| F(T) \| \geq \lambda' \| T \|$ for any $T \in FR(X)$ that can be written as a finite sum of operators of the form $\theta^{r,x}_{x,x}$.

Proof Uniqueness is obvious. Let $\mu \subset \Lambda \rightarrow u_{\mu}$ be a generalized right basis of the right Hilbert module $X_B$, which exists by Proposition 1.9. Consider the linear map $F_\mu : T \in \mathcal{L}(X_B) \rightarrow \sum_{y \in u_{\mu}} A(Ty|y) \in A$. Note that
\[ F_\mu(\theta^{r,x}_{y,y}) = \sum_{y \in u_{\mu}} A(x(z|y)_B|y) = A(x|z) \sum_{y \in u_{\mu}} y(y|z)_B. \]

Since $\lim_\mu \sum_{y \in u_{\mu}} y(y|z)_B = z$ in the norm $\| (\cdot|\cdot)_B \|^{1/2}$, and since $\| A(x|x) \| \leq \lambda \| (x|x)_B \|$, we also have that $\lim_\mu \sum_{y \in u_{\mu}} y(y|z)_B = z$ in the seminorm defined by the left inner product. Thus $\lim_\mu F_\mu(\theta^{r,x}_{y,y}) = A(x|z)$. Let us define $F$ as the pointwise norm limit of the net $\mu \rightarrow F_\mu$ on $FR(X_B)$. Obviously this limit does not depend on the generalized right basis. (1) follows from the fact that any element of the form $T^*T$, with $T \in FR(X)$, can be written as a finite sum of elements of the form $\theta^{r,x}_{y,y}$. Properties (2) and (3) are easy to check. (4) follows from Prop. 2.4.

The map $F$ will be referred to as the additive extension of the form $A(\cdot|\cdot)$ to the finite rank operators on $X_B$.

Notice that a bimodule satisfying the properties of the previous lemma is almost bi-Hilbertian. The only missing properties are the fact that the seminorm coming from the left-linear $A$–valued form is in fact a norm, and completeness of $X$ with respect to this norm.

2.7 Proposition Let $X$ be a right Hilbert $A$–$B$ $C^*$–bimodule and let $x, y \rightarrow A(x|y)$ be an $A$–valued form on $X$ satisfying the same properties as in the
already considered in the proof of Lemma 2.6. We claim that \( \|X\| \) module

\[
(1) \quad \text{There exists } \lambda > 0 \text{ such that for all } n \in \mathbb{N} \text{ and for all } x_1, \ldots, x_n \in X,
\]

\[
\| \sum_{i=1}^{n} A(x_i|x_i) \| \leq \lambda \| \sum_{i=1}^{n} \theta^r_{x_i,x_i} \|
\]

\( (2) \) there exists \( \lambda > 0 \) such that for all \( n \in \mathbb{N} \) and for all \( x_1, \ldots, x_n, y_1, \ldots, y_n \in X \),

\[
\| \sum_{i=1}^{n} A(x_i|y_i) \| \leq \lambda \| \sum_{i=1}^{n} \theta^r_{x_i,y_i} \|
\]

\( (3) \quad F(T) \geq 0 \) for any \( T \in FR(X_B) \cap \mathcal{K}(X_B)^+ \) and \( \sup_{\mu} \| F(\sum_{y \in u_{\mu}} \theta^r_{y,y}) \| \) is finite for some generalized right basis \( \mu \rightarrow u_{\mu} \) of \( X_B \).

If one of these conditions is satisfied, the smallest constants for which \( (1) \) and \( (2) \) hold, coincide and equal, in turn, \( \sup_{\mu} \| F(\sum_{y \in u_{\mu}} \theta^r_{y,y}) \| \). In particular, the latter does not depend on the generalized right basis.

**Proof** (1)⇒(2) Let \( \mu \subset \Lambda \rightarrow u_{\mu} \) be a generalized basis of the right Hilbert module \( X_B \). Consider the linear map \( F_{\mu} : T \in \mathcal{L}(X_B) \rightarrow \sum_{y \in u_{\mu}} A(Ty|y) \in A \), already considered in the proof of Lemma 2.6. We claim that \( \|F_{\mu}\| \leq \lambda \) for any \( \mu \). We show the claim. For any \( T \in \mathcal{L}(X_B) \),

\[
\|F_{\mu}(T)\| = \| \sum_{y \in u_{\mu}} A(Ty|y) \| \leq \| \sum_{y \in u_{\mu}} A(Ty|Ty) \|^{1/2} \| \sum_{y \in u_{\mu}} A(y|y) \|^{1/2}
\]

by the Cauchy–Schwarz inequality of the left inner product (see, e.g., [B] Prop. 13.1.3). Now by our assumption the last term is bounded above by

\[
\lambda \| \sum_{y \in u_{\mu}} \theta^r_{Ty,Ty} \|^{1/2} \| \sum_{y \in u_{\mu}} \theta^r_{y,y} \|^{1/2} =
\]

\[
\lambda \| T \sum_{y \in u_{\mu}} \theta^r_{y,y} T^* \|^{1/2} \| \sum_{y \in u_{\mu}} \theta^r_{y,y} \|^{1/2} \leq \lambda \| T \|.
\]

We have already seen that \( \lim_{\mu} F_{\mu}(\theta^r_{x,z}) = A(x|z) \). Since, for any \( x_1, \ldots, x_n, z_1, \ldots, z_n \in X \), \( \| F_{\mu}(\sum_1^n \theta^r_{x_i,z_i}) \| \leq \lambda \| \sum_1^n \theta^r_{x_i,z_i} \| \), the proof is completed taking the norm limit at the left hand side.

(2)⇒(3) We know that the linear extension \( F \) of the left inner product to the finite rank operators on \( X_B \) is positive on \( FR(X) \) (Lemma 2.6). We first show that \( F \) is still positive on \( FR(X) \cap \mathcal{K}(X_B)^+ \). By (2) \( F \) is norm continuous, therefore, if \( T \in FR(X) \cap \mathcal{K}(X_B)^+ \) and if \( \mu \rightarrow u_{\mu} \subset X \) is a generalized basis of \( X \), the net \( T^{1/2} \sum_{y \in u_{\mu}} \theta^r_{y,y} T^{1/2} \) converges to \( T \) in norm, therefore \( F(T) = \lim_{\mu} \sum_{y \in u_{\mu}} F(\theta^r_{T^{1/2}y,T^{1/2}y}) \in A^+ \). Furthermore for all \( \mu \), \( \| F(\sum_{y \in u_{\mu}} \theta^r_{y,y}) \| \leq \lambda \| \sum_{y \in u_{\mu}} \theta^r_{y,y} \| \leq \lambda \).
(3)⇒(1) Let \( x_1, \ldots, x_n \) be elements of \( X \), and set \( T = \sum_{i=1}^{n} \theta_{x_i,x_i}^r \). Let \( \mu \to u_\mu \) be a generalized right basis of \( X \). Since \( \sum_{y' \in u_\mu} \theta_{y',y'}^r \) is an approximate unit of \( K(X_B) \) and since the left inner product is continuous with respect to the right one, for all \( \mu \), the net \( \mu' \to \sum_{y \in u_\mu, y' \in u_{\mu'}} A(\theta_{y',y}^r T y|y) \) converges to \( \sum_{y \in u_\mu} A(T y|y) \) in norm. On the other hand this net coincides with \( F((\sum_{y' \in u_\mu} \theta_{y',y'}^r) T (\sum_{y \in u_\mu} \theta_{y,y}^r)) \). The form \( S, T \in FR(X_B) \to F(ST^*) \) is left \( A \)-linear, right \( A \)-antilinear, symmetric and positive and therefore it satisfies the Cauchy-Schwarz inequality \( ||F(ST^*)||^2 \leq ||F(SS^*)|| ||F(TT^*)|| \). It follows that
\[
\|F((\sum_{y' \in u_\mu} \theta_{y',y'}^r) T (\sum_{y \in u_\mu} \theta_{y,y}^r))\|^2 \leq \|F((\sum_{y \in u_\mu} \theta_{y,y}^r)^2)\|.
\]
Now
\[
(\sum_{y' \in u_\mu} \theta_{y',y'}^r) T T^* (\sum_{y' \in u_\mu} \theta_{y',y'}^r) \leq \|T\|^2 (\sum_{y' \in u_\mu} \theta_{y',y'}^r)
\]
and \( (\sum_{y \in u_\mu} \theta_{y,y}^r)^2 \leq (\sum_{y \in u_\mu} \theta_{y,y}^r) \), so, applying \( F \), we deduce that the above term is bounded above by \( \|T\|^2 \lambda_0^2 \) where \( \lambda_0 = \sup \|F(\sum_{y \in u_\mu} \theta_{y,y}^r)\| \). Passing first to the limit over \( \mu' \) and then over \( \mu \) we deduce that (1) holds with \( \lambda = \lambda_0 \).

It is now clear from the proof that if one of these three equivalent conditions holds, the best constants satisfying (1) and (2) coincide, and coincide in turn with \( \sup \|F(\sum_{y \in u_\mu} \theta_{y,y}^r)\| \).

### 2.8 Definition
A \( C^* \)-bimodule satisfying one of the equivalent properties described in the previous proposition will be called of **finite right numerical index**. The corresponding smallest positive constant will be called the **right numerical index** of \( X \), and denoted \( r - I[X] \).

Let \( X \) be an \( A \)-\( B \) bimodule. The **contragradient bimodule** of \( X \) is the \( B \)-\( A \) bimodule \( \overline{X} = \{ \overline{x} \mid x \in X \} \) with complex conjugate vector space structure and bimodule structure given by
\[
b \cdot \overline{x} = \overline{xb^*}, \quad \overline{x} \cdot a = \overline{a^*x}, \quad b \in B, a \in A.
\]
If \( X \) is a right (left) Hilbert \( A \)-\( B \) \( C^* \)-bimodule, \( \overline{X} \) becomes a left (right) Hilbert \( B \)-\( A \) \( C^* \)-bimodule with inner product given by:
\[
B(\overline{x}|y) = (x|y)_B
\]
\[
((x|y)_A = A(x|y)).
\]
Therefore if \( AX_B \) is bi-Hilbertian, \( B \overline{X}_A \) is bi-Hilbertian as well.

### 2.9 Definition
We will say that \( AX_B \) is of **finite left numerical index** if the contragradient bimodule \( B \overline{X}_A \) is of finite right numerical index. Its left numerical index is defined by \( \ell - I[X] := r - I[\overline{X}] \).
A bi-Hilbertian bimodule of finite left and right numerical indices will be simply called of finite numerical index. Its numerical index is defined by \( I[X] := (r - I[X])(\ell - I[X]) \).

2.10 Corollary Let \( A X_B \) be a bi-Hilbertian \( C^* \)-bimodule, and, for \( n \in \mathbb{N} \), let us consider \( Y_n := \oplus_1^n X \) as a \( M_n(A)B \) bimodule in the natural way. Endow \( Y_n \) with the following forms: \( M_n(A)(x,y) = (a(x_i|y_j)) \), \( (x,y)B = \sum_1^n (x_i|y_j)B \), where \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \).

1. If for some \( \lambda > 0 \), \( \|A(x|x)\| \leq \lambda \|x|x\| \) then also \( \|M_n(A)(x|x)\| \leq \lambda \|x|x\| \).

2. If \( X \) is of finite right numerical index, \( Y_n \) becomes a \( C^* \)-bimodule of finite right numerical index and \( r - I[Y_n] = r - I[X] \) for all \( n \in \mathbb{N} \).

3. If \( X \) is of finite numerical index, \( Y_n \) is bi-Hilbertian and of finite left numerical index (and hence of finite numerical index, by (1)): \( \|x|x\| \leq \ell - I[X] \|M_n(A)(x|x)\| \) and \( \ell - I[X] = \ell - I[Y_n] \), for all \( n \in \mathbb{N} \).

Remark Notice that the constants comparing the two norms on \( Y_n \) do not depend on \( n \).

Proof (1) It is easy to check that \( Y_n \) is a right Hilbert \( C^* \)-bimodule. Now \( \|M_n(A)(x|x)\| = \| \sum_1^n \theta_{i,i} \| \) and \( \|x|x\| = \| \sum_1^n (x_i|x_i) \| \). Therefore if for some \( \lambda > 0 \), \( \|A(x|x)\| \leq \lambda \|x|x\| \) then also \( \|M_n(A)(x|x)\| \leq \lambda \|x|x\| \) by Prop. 2.4 applied to the contragradient bimodule. The remaining properties of the left \( M_n(A) \)-valued form are easily checked.

(2) There is a natural identification of \( FR(Y_n) \) with \( M_n(\mathbb{C}) \otimes FR(X) \) taking \( \theta_{i,i} \) to the matrix \( (\theta_{i,i}) \). Under this identification, the map \( F_n : FR(Y_n) \to M_n(A) \) obtained extending additively the left form of \( Y_n \) identifies with \( id_{M_n} \otimes F \), where \( F \) is the additive extension of the left form of \( X \). Notice that \( FR(Y_n) \cap \mathcal{K}(X) \) identifies with \( (M_n \otimes FR(X))(\mathcal{K}(X)) \), so if \( T = (T_{i,j}) \in (M_n \otimes FR(X))(\mathcal{K}(X)) \), \( \sum_{i,j} a_{i,j}^* F(T_{i,j}) a_{i,j} = F(\sum_{i,j} \phi(a_{i,j})^* T_{i,j} \phi(a_{i,j})) \) for all \( a_{i,j} \in A \). Hence \( F_n \) takes positive values on \( FR(Y_n) \cap \mathcal{K}(X) \). If now \( \mu \to u_\mu \) is a generalized basis of \( X \), \( \mu' \to u_\mu' \) is a generalized basis of \( Y_n \) with corresponding approximate unit of \( \mathcal{K}(Y_n) \) given by the diagonal matrix of \( \sum_{y \in u_\mu} \theta_{y,y} \). Therefore the norm of the evaluation of \( F_n \) over this diagonal matrix coincides with \( \| \sum_{y \in u_\mu} A(y|y) \| \), and this concludes the proof of (2).

(3) Since \( \|M_n(A)(x|x)\| = \| \sum_1^n \theta_{i,i} \| \) and \( \|x|x\| = \| \sum_1^n (x_i|x_i) \| \), if \( X \) is of finite left numerical index, \( \|(x|x)\| \leq (\ell - I[X] \|M_n(A)(x|x)\| \) therefore, taking into account (1), \( Y_n \) is bi-Hilbertian. The compacts on \( Y_n \) with respect to the left inner product identify with \( \mathcal{K}(A)X \) via \( \theta_{i,i} \to \sum_1^n \theta_{i,i} \). This shows that

\[
(\ell - I[X]) \| \sum_{j=1}^m \theta_{j,j} \| = (\ell - I[X]) \| \sum_{j,k} \theta_{j,k} \| \geq \| \sum_{j,k} (x_j^* x_k^*) \| \geq \| \sum_{j,k} (x_j^* x_k^*) \|
\]

so \( Y_n \) has finite left numerical index and \( \ell - I[Y_n] = \ell - I[X] \).
The following result is the first step towards the Jones basic construction.

**2.11 Corollary** If $X$ is a bi-Hilbertian $A$-$B$ $C^*$–bimodule of finite right numerical index, the additive extension $F$ of the left inner product to $FR(X_B)$ extends uniquely to a norm continuous map $F : K(X_B) \to A$. One has: $\|F\| = r - I[X]$. This extension, still denoted by $F$, is positive, $A$-bilinear (in the sense that $F(\phi(a)T) = aF(T)$ and $F(T\phi(a)) = F(T)a$ for $a \in A, T \in K(X_B)$) and has range contained in the closed ideal of left inner products. Moreover one has $\phi(F(T)) \geq \lambda T$ for all $T \in K(X_B)^+$, where $\lambda$ is the best constant for which $\lambda\|(x|x)_B\| \leq \|A(x|x)\|.

**Proof** The only assertion that is not obvious yet is the inequality $\phi(F(T)) \geq \lambda T$ for $T \in K(X_B)^+$. Now part (4) in Prop. 2.6 implies that $\|F(T)\| \geq \lambda\|T\|$ for $T \in K(X_B)^+$. Left $A$–action is faithful on the norm closed ideal $\mathcal{J}$ generated by left inner products: $\phi(j) = 0$ for some $j \in \mathcal{J}^+$ implies $0 = A(x|y) = A(x)yj$ for $x, y \in X$ and therefore $j = 0$. Since $F(T) \in \mathcal{J}$ for all $T \in K(X_B)$, $\phi$ is isometric on $\mathcal{J}$, therefore for all $T \in K(X_B)^+$, $\|\phi(F(T))\| = \|F(T)\| \geq \lambda\|T\|$. Arguing as in [FK], with the map $\phi \circ F$ in place of a conditional expectation, we deduce the desired inequality.

Pimsner–Popa conditional expectations provide typical examples of bimodules of finite right numerical index.

**2.12 Proposition** Let $A \subset B$ be an inclusion of $C^*$–algebras and let $E : B \to A$ be a conditional expectation with fixed point set $A$. Assume that $\|E(b)\| \geq \lambda\|b\|$, for all positive elements $b \in B$ and for some $\lambda > 0$.

1. Consider $BX_A = B$ as a $B$–$A$ bimodule in the natural way, and with inner products $(x|y)_A = E(x^*y), B(x|y) = xy^*$. Since $\|(x|x)_A\| \leq \|B(x|x)\| \leq \lambda^{-1}\|(x|x)_A\|$, $X$ is bi-Hilbertian. By [FK], there is a constant $\lambda' > 0$ such that $E - \lambda'$ is completely positive. Let us choose the best such $\lambda'$. Then $X$ has finite right numerical index and $r - I[X] = \lambda'^{-1}$.

2. Consider now $AY_B = B$ as a $A$–$B$ bimodule with inner products $(x|y)_B = x^*y$ and $A(x|y) = E(xy^*)$. Then the $B$–$A$ antilinear map $X \to Y$ induced by the $^*$–involution of $B$ identifies $Y$ with the contragradient $X$ of $X$. Therefore $X$ is of finite left numerical index and $\ell - \text{Ind}[X] = 1$.

**Proof** (1) For all $n \in \mathbb{N}$, and all $x_1, \ldots, x_n \in X$,

$$\lambda'^{-1}\left\| \sum_1^n \theta_{x_1,x_2} \right\| = \lambda'^{-1}\left\| (E(x_i^*x_j))_{i,j} \right\|_{M_n(A)} \geq \left\| (x_i^*x_j)_{i,j} \right\|_{M_n(B)} = \sum_1^n B(x_i|x_i),$$

i.e. $X$ is of finite right numerical index in our sense and $r - I[X] \leq \lambda'^{-1}$. On the other hand, let $\mu \to u_\mu$ be a generalized basis of $X_A$, and set, for every $\mu$ and every $x \in X$, $x_\mu := \sum_{y \in u_\mu} yE(y^*x)$. Then

$$x_\mu^*x_\mu \leq \left\| \sum_{y \in u_\mu} y^*y \sum_{y \in u_\mu} E(x^*y)E(yx^*) = \right.$$
$E(x^* x_\mu) \| \sum_{y \in u_\mu} y y^* \| \leq \sup_{\mu} \| \sum_{y \in u_\mu} y y^* \| E(x^* x_\mu)$.

Taking the limit over $\mu$, we are led to the inequality $r - I[X] \geq \lambda^{-1}$. Consider now the inclusion $M_n \otimes A \subset M_n \otimes B$ and the conditional expectation $E_n := \text{id} \otimes E$, which satisfies $E_n(b) \geq \lambda b, B \in M_n(B)^\sim$. Cor. 2.10 shows that $\oplus n B$ is a $M_n(B)$-A bimodule with the same right index as $B$, hence, combining with the above argument, we deduce that $r - I[X] \geq \lambda^{-1}$.

The proof of part (2) is easy, therefore we omit it.

**Remark** If $B X_A$ and $A X_B$ arise from a Pimsner–Popa conditional expectation $E$, as the previous proposition, the corresponding map $F_Y$ constructed in Cor. 2.11 reduces to $E$ itself. More interestingly, $F_X : \mathcal{K}(X_A) \to B$ is related to the construction of the **dual** conditional expectation. However, if the index of $E$, as an element of $Z(B'')$ (cf. Def. 2.17), does not belong to the multiplier algebra of $B$, $F_X$ is not a multiple of a conditional expectation.

### 2.2 Tensoring bi-Hilbertian $C^*$–bimodules

In this subsection we analyze the behaviour of bi-Hilbertian bimodules under taking their tensor products. We show that the algebraic tensor product $X \otimes_B Y$ of bi-Hilbertian $C^*$–bimodules can be made into a bi-Hilbertian bimodule in a natural way if $X$ is of finite right numerical index and $Y$ is of finite left numerical index, and that this is also a necessary condition in general.

The problem of studying conditions under which $X \otimes_B Y$ is of finite index will be considered in section 5 (cf. Theorem 5.2).

Let $A X_B$ and $B Y_C$ be bi-Hilbertian $C^*$–bimodules. Then the algebraic tensor product $X \otimes_B Y$ is an $A C$ bimodule in a natural way, also endowed with a right and a left pre-bi-Hilbertian structure:

$$
(x_1 \otimes y_1 | x_2 \otimes y_2) C = (y_1 | x_1 | x_2) B y_2 C,
$$

$$
A (x_1 \otimes y_1 | x_2 \otimes y_2) = A (x_1 B (y_1) y_2 | x_2).
$$

Therefore $X \otimes_B Y$ can be made into a right Hilbert $C$–module $X \otimes_B^r Y$ completing with respect to the first inner product and also into a left Hilbert $A$–module $X \otimes_B^l Y$ completing with respect to the second inner product (always after dividing out by vectors of seminorm zero).

Under which conditions these two seminorms are equivalent on the algebraic tensor product $X \otimes_B Y$?

Choosing for $Y$ the strong Morita equivalence $B \ell^2(B) \otimes_B$ with inner products $B \langle b | b' \rangle = \sum_j b_j b'_j$, $\langle b | b' \rangle_{\mathcal{K} \otimes B} = \sum \delta_{i,j} \otimes b_i^* \langle b' \rangle$, $X \otimes_B \ell^2(B)$ identifies with $\ell^2(X)$ with inner products $A (x | x') = \sum_j A (x_j | x'_j)$, $\langle x | x' \rangle_{\mathcal{K} \otimes B} = \sum \delta_{i,j} \otimes (x_i | x'_j)_B$. Therefore the left and right seminorms on $X \otimes_B \ell^2(B)$ are equivalent if and only if $X$ is of finite right numerical index. Similarly, the left and right seminorms on $\ell^2(B) \otimes_B Y$, with $\ell^2(B)$ the inverse strong Morita equivalence, are equivalent if and only if $Y$ is of finite left numerical index. We show that these necessary conditions on $X$ and $Y$ are also sufficient.

**Proposition** Let $A X_B$ and $B Y_C$ be bi-Hilbertian $C^*$–bimodules. Assume that $X$ is of finite right numerical index and that $Y$ is of finite left numerical
index. Let \( F_X : \mathcal{K}(X_B) \to A, F_Y : \mathcal{K}(Y_B) \to C \) be the corresponding maps constructed in Cor. 2.11. Then

1. the two seminorms arising from the left and right inner products on \( X \otimes_B Y \) as above are equivalent. Therefore \( X \otimes_B Y = X \otimes_B Y (= X \otimes_B Y) \) and it is a bi-Hilbertian \( A \)-\( C \) bimodule.

2. Consider \( \mathcal{K}(Y_B, X_B) \) as a \( A \)-\( C \) bimodule with left and right inner products \( A(T|S) = F_X(TS^*) \) and \( (T|S)_C = F_Y(T^*S) \). Then \( \mathcal{K}(Y_B, X_B) \) is complete in any of the induced norms, and becomes in this way a bi-Hilbertian \( C^* \)-bimodule.

3. The map \( x \otimes y \in X \otimes Y \to \theta_{x,y}^r \in \mathcal{K}(Y_B, X_B) \) extends to a bijective \( A \)-\( C \) bimodule map \( U : A X_B \otimes_B Y_A \to \mathcal{K}(Y_B, X_B) \) preserving the left and right inner products.

Proof Consider \( X \) and \( Y \) as right Hilbert \( B \)-modules, and define the map \( U : X \otimes_B Y \to FR(\mathcal{Y}, X) \) associating \( \theta_{x,y}^r \) to the simple tensor \( x \otimes y \). \( U \) is a well defined \( A \)-\( C \)-bimodule map. For any \( x_1, x_2 \in X, y_1, y_2 \in Y \), we have

\[
(x_1 \otimes y_1|x_2 \otimes y_2)_C = (y_1|(x_1|x_2)y_2)_C = c(\overline{y_1}y_2(\overline{x_1}x_1)_B)
\]

and

\[
(\theta_{x_1,y_1}^r \theta_{x_2,y_2}^r)_C = F_Y(\theta_{x_1,y_1}^r \theta_{x_2,y_2}^r) = F_Y(\theta_{x_1,y_1}^r(x_1|x_2)b_{y_2}) = c(\overline{y_1}y_2(\overline{x_1}x_1)_B).
\]

Similarly we have

\[
A(x_1 \otimes y_1|x_2 \otimes y_2)_C = A(\theta_{x_1,y_1}^r \theta_{x_2,y_2}^r).
\]

Since, when \( X \) and \( Y \) are bi-Hilbertian, \( F_X \) and \( F_Y \) are faithful maps (see Cor. 2.11), the two seminorms have the same vectors of length zero (therefore \( U \) is an injective map). Furthermore the two norms \( ||F_Y(T^*T)||^{1/2} \) and \( ||F_X(TT^*)||^{1/2} \) on \( \mathcal{K}(Y_B, X_B) \) are both equivalent to the operator norm, still by Cor. 2.11, and therefore they are equivalent. We have thus shown that \( X \otimes_B Y \) and \( X \otimes_B Y \) are isomorphic as Banach spaces. It is now straightforward to check that right and left actions are adjointable, and therefore \( X \otimes_B Y \) is bi-Hilbertian. Since \( U \) is a bijective map which preserves both inner products, it extends to a bijective \( A \)-\( C \) bimodule map \( U : X \otimes_B Y \to \mathcal{K}(Y_B, X_B) \) still preserving the inner products, and the proof is now complete.

### 2.3 Nondegeneracy of the left action

The following nondegeneracy property will be relevant for our purposes.

**2.14 Definition** The left action \( \phi \) of a \( C^* \)-algebra \( A \) on a right Hilbert \( C^* \)-module \( X_B \) will be called nondegenerate if \( AX \) is total in \( X \).

We recall the following characterization of nondegeneracy, due essentially to Vallin [V], see also Prop. 2.5 in [L].
2.15 Proposition Let $A$ and $B$ be C$^*$–algebras and $X$ a right Hilbert $B$–module. For a $^*$–homomorphism $\phi : A \to \mathcal{L}(X_B)$ the following conditions are equivalent.

1. $\phi$ is nondegenerate,
2. $\phi$ is the restriction to $A$ of a unital $^*$–homomorphism $\hat{\phi} : M(A) \to \mathcal{L}(X_B)$, strictly continuous on the unit ball,
3. for some approximate unit $(u_\alpha)_\alpha$ of $A$, $(\phi(u_\alpha))_\alpha$ converges strictly to the identity map on $X$.

Note that if $\phi$ is nondegenerate, (3) must hold for all approximate units of $A$.

We show that the left action of a bi-Hilbertian C$^*$–bimodule is automatically nondegenerate.

2.16 Proposition Let $AX_B$ be a bi-Hilbertian $A$–$B$ C$^*$–bimodule. Then the left (right) action of $A$ ($B$) on the underlying right Hilbert C$^*$–module $X$ is nondegenerate.

Proof If $\{u_\alpha\}$ is an approximate unit of the closed ideal of $A$ generated by the left inner products, $u_\alpha x$ converges to $x$ for all $x \in X$, in the norm arising from the left inner product. Therefore $AX$ is total in $X$ with respect to the norm defined by the left inner product. Since the two norms on $X$ defined by the right and left inner product are equivalent, we also have that $AX$ is total with respect to the norm arising from the right inner product.

2.4 The index element and the Jones basic construction

If $X$ is bi-Hilbertian and of finite right numerical index, one can extend the maps $\phi : A \to \mathcal{L}(X_B)$, $F : K(X_B) \to A$ uniquely to normal positive maps $\phi'' : A'' \to K(X_B)''$, $F'' : K(X_B)'' \to A''$ between the corresponding enveloping von Neumann algebras. Since $\phi$ is nondegenerate, and the inclusion $M(A) \subset A''$ is unital, $\phi''$ is unital homomorphism. The same does not hold for $F'' : F''(I)''$ is, in general, neither the identity, nor invertible.

2.17 Definition If $AX_B$ is of finite right numerical index, the right index element of $AX_B$, denoted, $r - \text{Ind}[X]$ is the element $F''(I)$ of $A''$.

If in particular $B X_A$ is the bimodule arising from a conditional expectation $E : B \to A$ as in Prop. 2.12, the corresponding right index element will be denoted by $\text{Ind}[E]$. (We will give in Cor. 4.9 an alternative definition of $\text{Ind}[E]$.)

If $AX_B$ is of finite left numerical index, the left index element of $X$ is, of course, $\ell - \text{Ind}[X] := r - \text{Ind}[X]$.

Notice that the numerical indices and the index elements are related by

$$\|r - \text{Ind}[X]\| = r - I[X],$$

$$\|\ell - \text{Ind}[X]\| = \ell - I[X].$$

If one of $r - \text{Ind}[X]$ and $\ell - \text{Ind}[X]$ is a scalar, or if $A = B$, the index element of $X$ is $\text{Ind}[X] := (r - \text{Ind}[X])(\ell - \text{Ind}[X])$.

Our next aim is to define an index element $\text{Ind}[X]$ in the general case. We notice that for $c \in Z(B)$, the map $\psi(c) : x \in X \to xc \in X$ has the map
$x \in X \rightarrow xc^* \in X$ as an adjoint with respect to the right inner product of $X$. Furthermore $\psi(c)$ commutes obviously with all the elements of $\mathcal{L}(X_B)$, therefore $\psi(Z(B)) \subset Z(\mathcal{L}(X_B))$. On the other hand it is not difficult to see that $\psi(Z(B)) = Z(\mathcal{L}(X_B))$. We need to consider an extension of this right action of $Z(B)$ on $X$ to the centre of $B''$. Therefore we anticipate the following lemma.

**2.18 Lemma** Let $X$ be a right Hilbert $B$–module, and let $\psi : Z(B) \rightarrow Z(\mathcal{L}(X_B))$ denote the right action of $Z(B)$ on $X$. Then there is a canonical extension of $\psi$ to a unital surjective $^*$–homomorphism $\psi_0 : Z(B'') \rightarrow Z(\mathcal{K}(X_B)'')$ with $\ker \psi_0 = (1-q)Z(B'')$, where $q$ the central projection of $B''$ corresponding to the weak closure in $B''$ of the ideal generated by right inner products.

**Proof** Let $\pi$ denote a Hilbert space representation of $B$ on $H_\pi$. Consider the Stinespring induced representation $\bar{\pi}$ of $\mathcal{K}(X_B)$ on the Hilbert space $K_\pi := X_B \otimes_B H_\pi$, defined by $T \rightarrow T \otimes 1_{H_\pi}$. Since $B$ and $\mathcal{K}(X_B)$ are strongly Morita equivalent, it is well known that the map $\pi \rightarrow \bar{\pi}$ is a bijective correspondence between representations of $B$ and representations of $\mathcal{K}(X_B)$. Therefore the representation $\rho = \oplus_\pi \bar{\pi}'' : \mathcal{K}(X_B)'' \rightarrow \mathcal{L}(\oplus_\pi K_\pi)$ is a bijective correspondence between representations of $B$ and representations of $\mathcal{K}(X_B)$.

Let $\pi$ denote a Hilbert space representation of $B$ on $H_\pi$. Consider the Stinespring induced representation $\bar{\pi}$ of $\mathcal{K}(X_B)$ on the Hilbert space $K_\pi := X_B \otimes_B H_\pi$, defined by $T \rightarrow T \otimes 1_{H_\pi}$. Since $B$ and $\mathcal{K}(X_B)$ are strongly Morita equivalent, it is well known that the map $\pi \rightarrow \bar{\pi}$ is a bijective correspondence between representations of $B$ and representations of $\mathcal{K}(X_B)$. Therefore the representation $\rho = \oplus_\pi \bar{\pi}'' : \mathcal{K}(X_B)'' \rightarrow \mathcal{L}(\oplus_\pi K_\pi)$ is faithful and normal. It follows that $\rho(\mathcal{K}(X_B)''') = \rho(\mathcal{K}(X_B)'')$. On the other hand $Z(B'')$ acts on each $K_\pi$, and therefore on their direct sum, by $\psi'_0 : c \in Z(B'') \rightarrow 1_X \otimes \pi''(c) \in \mathcal{L}(K_\pi)$. Clearly, $\psi'_0(c) = \rho(\mathcal{K}(X_B)')$. If $A$ is a bounded operator on $\oplus_\pi K_\pi$ commuting elementwise with $\bar{\pi}(\mathcal{K}(X_B))$ then $A(xyz) = \theta_{x,y,z} \in 1_{\oplus_\pi H_\pi} A(z \otimes \xi)$ for all $x, y, z \in X, \xi \in H_\pi, \pi \in \text{Rep}(\mathcal{B})$. Choosing an approximate unit of the closed ideal of right inner products of $B$ constituted by finite sums of elements of the form $(y|y)_B$, we see that $A$ is of the form $1_X \otimes a$, with $a \in \mathcal{L}(\oplus_\pi H_\pi)$. Approximating now $\pi''(c)$ strongly with a norm bounded net in $\pi(B)$, shows that $T$ and $\psi'_0(c)$ commute. Therefore $\psi'_0(Z(B'')) \subset Z(\rho(\mathcal{K}(X_B)''))$. Notice that, $\psi'_0(c) = 0$ if and only if each $\pi''(c)$ annihilates the subspace $(y|y)_B$, or, equivalently $\pi''(c) = 0$ for all $\pi$, i.e. $c = 0$. On the other hand if $A \in Z(\rho(\mathcal{K}(X_B)''))$ and we write $A = 1_X \otimes a$, a one moment thought shows that $a \in Z(B'')$. The $^*$–homomorphism $\psi_0 := \rho^{-1}\psi'_0$ is then the desired extension of $\psi$.

**2.19 Proposition** Let $X$ be of finite right numerical index. Then for any generalized right basis $\mu \rightarrow u_\mu$ of $X$, the net $\mu \rightarrow \sum_{y \in u_\mu} A(y|y)$ is increasing and it converges strongly in $A''$ to $r-\text{Ind}[X]$. This limit is therefore independent on the choice of the basis, and belongs to the centre of $A''$. If in addition $X$ is bi-Hilbertian one has $\lambda r - \text{Ind}[X] \geq p$ where $p$ is the support projection of $r-\text{Ind}[X]$ in $A''$, and $\lambda$ is the best constant for which $\lambda ||A(x|x)|| \geq ||(x|x)_{B''}, x \in X$. Furthermore, if $\mathcal{I}$ denotes the weak closure in $A''$ of the span of left inner products $A(x|y), x, y \in X$, one has

1. $\ker \phi'' = (I-p)A''$,
2. $\mathcal{I} = pA''$,
3. the range of $F'' : \mathcal{K}(X)'' \rightarrow A''$ is $pA''$,
4. if $z'$ denotes the inverse of $r - \text{Ind}[X]$ in $pA''$, and $E'' := z'F'' : \mathcal{K}(X_B)'' \rightarrow pA''$. Then $\phi'' \circ E'' : \mathcal{K}(X_B)'' \rightarrow \phi''(A'')$ is a conditional expectation with range $\phi''(A'')$ satisfying

$$\lambda \phi''(r - \text{Ind}[X]E''(T)) \geq T, \quad T \in \mathcal{K}(X_B)''.$$
Proof The strong limit $z$ defined as in the statement is independent of the generalized basis since it coincides with $F''(I)$. Since $F''$ is still $A$-bilinear, $F(\phi(a)) = F''(I)a = aF''(I)$ for all $a \in A$, which shows that $F''(I)$ lies in the centre of $A''$. Let us assume $X$ bi-Hilbertian, and let $\lambda$ be as in the statement. 

The estimate

$$||T|| \leq \lambda ||F''(T)|| \leq r - I[X]||T||, \quad T \in \mathcal{K}(X)^{''},$$

still holds, therefore if $T = \phi''(a)$, with $a \in A''^+$,

$$||\phi''(a)|| \leq \lambda ||a^{-1}|| \leq r - I[X]||\phi''(a)||.$$

If we consider the restriction $\phi_0$ of $\phi''$ to the centre of $A''$, we deduce that $\ker \phi_0$ coincides with the weakly closed ideal generated by $I - p$. Therefore for a positive central element $a$ of $A''$, $||\phi_0(a)|| = ||pa|| \leq \lambda ||a^{-1}||$. Let us identify the centre of $A''$ with some $L^\infty(\Omega, \nu)$. We claim that for every $\epsilon > 0$, the function $z - (\lambda^{-1} - \epsilon)p$ can not take negative values on a measurable subset of $p\Omega$ with positive measure. Indeed, if $Y \subset \Omega$ where such a set, we would have, for some $\epsilon < \lambda^{-1}$,

$$\lambda^{-1} - \epsilon = (\lambda^{-1} - \epsilon)||\xi_Y|| \geq ||z\xi_Y|| \geq \lambda^{-1}||p\xi_Y|| = \lambda^{-1},$$

where $\xi_Y$ is the characteristic function of $Y$. Therefore $z \geq \lambda^{-1}p$. Now if $a \in A''$, $\phi''(a) = 0$ if and only if $\phi''(aa^*) = 0$ and this holds if and only if $pa = 0$, i.e. $pa = 0$, so $\ker \phi'' = (I - p)A''$, and (1) is proved. Let $\mathcal{I}$ be the weakly closed ideal of $A''$ defined as in the statement. Since the range of $F''$ is contained in $\mathcal{I}$ and since $F''(\phi''(I)) = z$, $p$ must belong to $\mathcal{I}$, and therefore $pA'' \subset \mathcal{I}$. Conversely, there exists an increasing, norm bounded net $\alpha \rightarrow \sum_{w \in \alpha} A(w|w)$, indexed by the set of finite subsets of $X$, which is a bounded approximate unit of the norm closed ideal generated by the left inner products. Its weak limit, say $q$, is the unit of $\mathcal{I}$. By Proposition 2.4 the net $\alpha \rightarrow \sum_{w \in \alpha} \theta_{w,w}^a$ is norm bounded. We have $\sum_{w \in \alpha} \phi(a)\theta_{w,w}^a \phi(a^*) \leq \lambda_0 \phi(aa^*)$ for some $\lambda_0 > 0$ and for all $a \in A$. Applying $\tilde{F}$ we obtain

$$\sum_{w \in \alpha} a_{A}(w|w)a^* \leq \lambda_0 a a^*.$$

Thus $\lambda_0^{-1}aq \leq az$. It follows that $\lambda_0^{-1}q \leq z$, hence $q \leq p$. Therefore $\mathcal{I} \subset pA''$, and the proof of (2) is complete. (3) We are left to show that $pA''$ is contained in the range of $F''$. Let $z'$ be the inverse of $r - \text{Ind}[X]$ in $pA''$. For $a \in A''$, $F''(\phi''(z'a)) = r - \text{Ind}[X]z'a = pa$. (4) It is now clear that $\phi''E''$ is a conditional expectation with range $\phi''(pA'') = \phi''(A)$. (5) Since $\lambda ||F''(T)|| \geq ||T||$ for $T$ positive is $\mathcal{K}(X_B)^{''}$ and since $\phi''$ is isometric on $\mathcal{I}$, and therefore on the range of $F''$, we see that $\lambda ||\phi''(r - \text{Ind}[X]E''(T))|| = \lambda ||\phi''F''(T)|| = \lambda ||F''(T)|| \geq ||T||$. Therefore $\lambda \phi''(r - \text{Ind}[X]E''(T)) \geq T$ (cf. [FK]).

2.20 Corollary Let $A X_B$ be a bi-Hilbertian $C^*$–bimodule of finite right numerical index. Then the following properties are equivalent.

(1) $r - \text{Ind}[X]$ is invertible,
(2) $\phi''$ is faithful,
(3) the linear span of the left inner products is weakly dense in $A''$. 

Remark Notice that if \( B \mathcal{X}_A \) is the bimodule arising from a Pimsner–Popa conditional expectation, as in Prop. 2.12, \( r - \text{Ind}[\mathcal{X}] = \text{Ind}[E] \) must be invertible since the left inner product is full.

2.21 Definition Notice that \( \phi''(r - \text{Ind}[\mathcal{X}]) \) is invertible in \( Z(\phi(A)' \cap \mathcal{K}(X_B)'') \) and \( \psi_0(\ell - \text{Ind}[\mathcal{X}]) \) is invertible in \( Z(\mathcal{K}(X_B)''') \). Therefore we define the index element of \( \mathcal{X} \) as an element of \( \mathcal{K}(X_B)''' \), in fact central in \( \phi(A)' \cap \mathcal{K}(X_B)''' \), by \( \text{Ind}[\mathcal{X}] := \psi_0(\ell - \text{Ind}[\mathcal{X}])\phi''(r - \text{Ind}[\mathcal{X}]) \).

2.5 On the condition \( r - \text{Ind}[\mathcal{X}] \in M(A) \) and existence of finite bases

Under which conditions the index element lies in \( M(A) \)? We first discuss this question in the case where \( A \) is commutative, and derive a general criterion afterwards.

Let \( A = C_0(\Omega) \) be a commutative \( C^* \)-algebra, and let \( \mathcal{X}_B \) be a bi-Hilbertian \( C^* \)-bimodule with finite right numerical index. We have seen in Prop. 2.12 that such bimodules arise, e.g., from conditional expectations \( E : A \to B \) satisfying a Pimsner-Popa inequality. The right index element, \( r - \text{Ind}[\mathcal{X}] \), lies in the commutative von Neumann algebra \( A'' \). However, being the strong limit in the universal representation of \( A \) of a net of continuous, vanishing at infinity, functions over \( \Omega \), \( r - \text{Ind}[\mathcal{X}] \) is the class function of a bounded, positive, lower semicontinuous function on \( \Omega \). By Dini’s theorem, \( r - \text{Ind}[\mathcal{X}] \) is continuous (or, in other words, lies in the multiplier algebra \( M(A) \)) if and only if the approximating net is uniformly convergent on compact subsets of \( \Omega \), that is, if and only if this net is strictly convergent in \( M(A) \).

The key idea, in the case where \( A \) is not commutative, is to replace the spectrum of \( A \) with the quasi-state space \( Q \) of \( A \). By Kadison’s function representation theorem (see, e.g., [P]), the real Banach space \( A_{sa} \) identifies isometrically with the real Banach space of continuous, vanishing at 0, affine functions on \( Q \). Now an analysis similar to the commutative situation leads to the following characterization of the property \( r - \text{Ind}[\mathcal{X}] \in M(A) \).

2.22 Theorem Let \( \mathcal{X}_B \) be a bi-Hilbertian \( C^* \)-bimodule of finite right numerical index. Then the following properties are equivalent:

1. \( r - \text{Ind}[\mathcal{X}] \in M(A) \) (and hence it is a central element of \( M(A) \)),
2. there is a generalized right basis \( \mu \to u_\mu \subset X \subset X \) such that the net \( \mu \to \sum_{y \in u_\mu} A(y|y) \) is strictly convergent in \( A \),
3. for any generalized right basis \( \mu \to u_\mu \subset X \subset X \) the net \( \mu \to \sum_{y \in u_\mu} A(y|y) \) is strictly convergent in \( A \),
4. the range \( \phi(A) \) of the left action is included in \( \mathcal{K}(X_B) \).

If one of these conditions is satisfied, \( r - \text{Ind}[\mathcal{X}] = \lim_\mu \sum_{y \in u_\mu} A(y|y) \) in the strict topology of \( A \).

Proof (3)\( \Rightarrow \) (2) is obvious. (4)\( \Rightarrow \) (3) Let \( \mu \to u_\mu \) be a generalized basis of \( X \). Then \( \sum_{y \in u_\mu} A(y|y) \) converges strictly if and only if for all \( a \in A \),

\[
\sum_{y \in u_\mu} a_A(y|y) = \sum_{y \in u_\mu} A(\phi(a)y|y) = F(\phi(a) \sum_{y \in u_\mu} \theta_{y,y})
\]
converges in norm. Therefore if (4) holds, convergence of the above net follows from norm continuity of \( F \) and the fact that \( \sum_{y \in u_{\mu}} \theta_{y,y}^{*} \) is an approximate unit of \( \mathcal{K}(X_{B}) \). (2) \( \Rightarrow \) (4) By Cor. 2.11, for any \( T \in \mathcal{K}(X_{B})^{+} \), \( \lambda \|T\| \leq \|F(T)\| \), where \( \lambda \) is the best positive constant for which \( \lambda \|\langle x|x\rangle_{B}\| \leq \|\langle x|x\rangle\| \). If now (2) holds for some generalized basis \( \mu \to u_{\mu} \subset X \), \( \phi(a)^{*} \sum_{y \in u_{\mu}} A(y)y\phi(a) = F(\phi(a)^{*}\sum_{y \in u_{\mu}} \theta_{y,y}^{*}\phi(a)) \) is an increasing, norm converging net, and therefore the net \( \phi(a)^{*}\sum_{y \in u_{\mu}} \theta_{y,y}^{*}\phi(a) \) is increasing and norm converging in \( \mathcal{K}(X_{B}) \). It follows that \( \phi(a^{*}a) \in \mathcal{K}(X_{B}) \) for all \( a \in A \). (2) \( \Rightarrow \) (1) is obvious, since \( r-\text{Ind}[X] \) is the strict limit of a strictly convergent net. (1) \( \Rightarrow \) (3) By Kadison’s function representation (see, e.g., [P] the selfadjoint part of \( A^{\prime}\prime \) identifies isometrically, as a real Banach space, with the real Banach space \( B_{0}(Q) \) of affine, bounded functions on the quasi-state space \( Q \) of \( A \) vanishing in \( 0 \). Under this identification, the selfadjoint elements of \( A \) correspond to the continuous functions. If \( r-\text{Ind}(A) \in M(A) \), for all \( a \in A \), \( a^{*}((r-\text{Ind}[X]) - \sum_{y \in u_{\mu}} A(y)y)a \in A \). On the other hand the net \( (r-\text{Ind}[X]) - \sum_{y \in u_{\mu}} A(y)y \) decreases weakly to \( 0 \) in \( A^{\prime}\prime \), therefore for any \( \phi \in Q \),

\[
\phi(a^{*}((r-\text{Ind}[X]) - \sum_{y \in u_{\mu}} A(y)y)a)
\]

decreases to 0. By Dini’s theorem, this net converges uniformly to 0 on \( Q \), and therefore \( \|(r-\text{Ind}[X]) - \sum_{y \in u_{\mu}} A(y)y\|^{1/2} \to 0 \), which implies \( \|(r-\text{Ind}[X]) - \sum_{y \in u_{\mu}} A(y)y\|^{2} \to 0 \) as the net \( \sum_{y \in u_{\mu}} A(y)y \) is norm bounded.

Assume now that one of these equivalent properties holds, and let \( z := \lim_{\mu} \sum_{y \in u_{\mu}} A(y)y \in M(A) \) for some generalized right basis \( \mu \to u_{\mu} \) of \( X \). Since, for all \( a \in A \), \( za = F(\phi(a)) \), \( z \) is independent of the choice of the basis.

2.23 Definition A bi-Hilbertian \( A-B \) \( C^{\ast} \)-bimodule \( X \) will be called of finite right index if

\[
\begin{align*}
(1) & \ X \text{ is of finite right numerical index}, \\
(2) & \ r-\text{Ind}[X] \in M(A) \ (\text{and hence } r-\text{Ind}[X] \in Z(M(A))).
\end{align*}
\]

Remark Notice that property (2) above can be replaced by any of the equivalent conditions in Theorem 2.22.

Similarly, \( X \) is of finite left index if the contragradient bimodule \( _{B}X_{A} \) is of finite right index.

\( _{A}X_{B} \) will be called of finite index if it is of finite right as well as left indices.

We study the special case where the \( C^{\ast} \)-algebras are \( \sigma \)-unital or unital.

2.24 Corollary Let \( _{A}X_{B} \) be a bi-Hilbertian \( C^{\ast} \)-bimodule of finite right index.

(1) If \( A \) is \( \sigma \)-unital, \( X \) is countably generated as a right Hilbert module.

(2) if \( A \) and \( B \) are \( \sigma \)-unital, \( X_{B} \) admits an unconditionally convergent countable right basis \( \{u_{i}\}_{i \in \mathbb{N}} \), therefore

\[
r-\text{Ind}[X] = \sum_{i \in \mathbb{N}} A(u_{i}|u_{i})
\]

in the strict topology of \( A \) and \( F(T) = \sum_{i \in \mathbb{N}} A(Tu_{i}|u_{i}), T \in \mathcal{K}(X_{B}) \) in norm.
Proof (1) Let \((u_i)_{i \in \mathbb{N}}\) be a countable approximate unit of \(A\). By nondegeneracy of the left action, and the fact that the left action has range in \(K\), this shows that \(\phi(u_i) \in K\) is a countable approximate unit for \(K\), so \(K\) is \(\sigma\)-unital and this shows that \(X\) is countably generated as a right Hilbert \(B\)-module (see, e.g., [B]). (2) If in addition \(B\) is \(\sigma\)-unital, \(X\) admits a countable right basis by Lemma 1.6, therefore the formulas for \(r - \text{Ind}[X]\) and for \(F\) follow.

2.25 Corollary Let \(_AX_B\) be a bi-Hilbertian bimodule of finite right numerical index, and let \(A\) be a unital \(C^*\)-algebra. The following are equivalent:

1. \(X\) admits a finite right basis,
2. \(r - \text{Ind}[X] \in A\).

Proof (1) ⇒ (2) follows from the definition of \(r - \text{Ind}[X]\). Conversely, assume that (2) holds. By nondegeneracy of the left action, \(\phi(I)\) must be the identity map on \(X\). Since, by Theorem 2.22, the range of \(\phi\) is included in the compacts, \(I \in K\), so \(X\) admits a finite basis.

2.26 Corollary Let \(_AX_B\) be a bi-Hilbertian \(C^*\)-bimodule with finite right numerical index. If \(A\) is simple then \(r - \text{Ind}[X]\) is a scalar, and therefore \(X\) is of finite right index.

Proof Since \(A\) is a simple \(C^*\)-algebra, the only positive elements of \(Z(A'')\) arising as strong limits of increasing nets in \(A\) must be scalar (see, e.g., Lemma 3.1 in [I]), so \(r - \text{Ind}[X]\) is a scalar, and therefore it belongs to \(M(A)\).

The following result has been obtained by Izumi [I] in the case where \(B\) is a simple \(C^*\)-algebra.

2.27 Corollary Let \(A \subset B\) be an inclusion of unital \(C^*\)-algebras, and let \(E : B \to A\) be a conditional expectation satisfying a Pimsner-Popa inequality. Then \(E\) admits a finite quasi–basis in the sense of [W] if and only if \(\text{Ind}[E] \in B\).

2.6 The Jones basic construction

2.28 Proposition Let \(_AX_B\) be a bi-Hilbertian \(C^*\)-bimodule of finite right index. Then

1. the map \(F : K(X_B) \to A\) extends uniquely to a strictly continuous map \(\hat{F} : \mathcal{L}(X_B) \to M(A)\). One has \(\hat{F}(I) = r - \text{Ind}[X]\), \(\|\hat{F}\| = r - I[X]\). \(\hat{F}\) is still positive, \(M(A)\)-bilinear and satisfies

\[
\lambda T \leq \hat{F}(T), \quad T \in \mathcal{L}(X_B)^+,
\]

where \(\lambda'\) is the best constant for which \(\lambda'\|\langle x|x\rangle_B\| \leq \|A(x|x)\|\), \(x \in X\),
2. the support projection \(p\) of \(r - \text{Ind}[X]\) in \(A''\) lies in fact the centre of \(M(A)\) and satisfies \(r - \text{Ind}[X] \geq \lambda' p\),
3. \(\ker \phi = (I - p)A\), \(\ker \hat{\phi} = (I - p)M(A)\),
4. the norm closed subspace of \(A\) generated by the left inner products coincides with \(pA\),
5. the range of \(F : K(X_B) \to A\) is \(pA\).
2.29 Corollary If $X$ is a bi-Hilbertian $A$-$B$ $C^*$-bimodule of finite right index, the following properties are equivalent.

(1) The left inner product is full,
(2) $r - \text{Ind}[X]$ is invertible,
(3) $\phi$ is faithful.

We next construct the analogue of the Jones basic construction in the $C^*$-algebra setting.

2.30 Corollary Let $A_XB$ be a bi-Hilbertian $C^*$-bimodule of finite right index. Consider the positive $A$-bilinear map $E : T \in \mathcal{K}(X_B) \to z'F(T) \in pA$, with $z'$ the inverse of $(r - \text{Ind}[X])p$ in $p\mathcal{Z}(M(A))$. Then $\phi E : \mathcal{K}(X_B) \to \phi(A)$ is a conditional expectation with range $\phi(A)$ which satisfies

$$\lambda \phi(r - \text{Ind}[X]E(T)) \geq T, \quad T \in \mathcal{K}(X_B)^+$$
2.11 and \( \phi \) is a conditional expectation. The remaining inequality follows from Cor. 2.11 and \( \phi \circ F = \phi(r - \text{Ind}[X]) \phi \circ E \).

2.7 Examples

We conclude this section with few examples of finite index bimodules already known in the literature. More examples will be discussed in section 6.

The first example arises from compact quantum groups, or, equivalently, conjugation in finite dimensional Hilbert spaces, [Wo], where the bi-Hilbertian structure is usually described in terms of antilinear invertible mappings between Hilbert spaces implementing the conjugation structure.

2.31 Example Let \( H = \mathbb{C}^n \) be a finite dimensional Hilbert space and \( \mathcal{T} \) be a positive invertible linear map on \( H \). \( H \) is an \( \mathbb{C} \)-\( \mathbb{C} \) bimodule in the obvious way. We endow \( H \) with a bi-Hilbertian bimodule structure by setting \( (x|y)_C = \sum_i x(i)y(i) \) and \( C(x|y) = \sum_i (Tx)(i)y(i) = (y|Tx)_C \). These inner products induce equivalent norms on \( H \) since \( \mathcal{T} \) is invertible, making \( H \) into a bi-Hilbertian bimodule. Since \( H_C \) and \( C H \) are finite dimensional Hilbert spaces, \( \mathcal{K}(H_C) = \mathcal{L}(H_C) \) and \( \mathcal{K}(C H) = \mathcal{L}(C H) \), therefore \( H \) will be of finite index if it is of finite left and right numerical indices. Let \( \text{Tr} \) be the nonnormalized trace on \( M_n(\mathbb{C}) = \mathcal{K}(H) \). Since \( \text{Tr}(\mathcal{T} \mathcal{C}|x,y) = \mathcal{C}(x|y) \), for \( x_1, \ldots, x_p \in H \),

\[
\| \sum_1^P C(x_j|x_j) \| \leq \text{Tr}(\mathcal{T}) \sum_1^P \theta_{x_j,y_j},
\]

therefore \( r - \text{Ind}[X] = \text{Tr}(\mathcal{T}) \). Since \( (x|y)_C = \mathcal{C}(T^{-1}y|x) \), we deduce that \( \ell - \text{Ind}[X] = \text{Tr}(T^{-1}) \).

2.32 Example More generally, let \( \Omega \) be a locally compact Hausdorff space, \( H = (\Omega, H(\omega)_{\omega \in \Omega}, \Gamma) \) a continuous field of Hilbert spaces, with \( \Gamma \) the space of continuous sections of \( H \). Let

\[
X = \{ x \in \Gamma : \omega \rightarrow \| x(\omega) \| \in C_0(\Omega) \}
\]

be the associated right Hilbert bimodule over \( C_0(\Omega) \). Let us consider a field \( \omega \rightarrow T(\omega) \in \mathcal{L}(H(\omega)) \) of positive, trace-class operators on each \( H(\omega) \) defining an element \( T \) of \( \mathcal{L}(X_{C_0(\Omega)}) \) (e.g. \( T \in FR(X) \)). We can then define a left inner product on \( X \), continuous with respect to the right one, by

\[
C_0(\Omega)(x|y)(\omega) := (y(\omega)|T(\omega)x(\omega)).
\]

Writing the left inner product in the form

\[
C_0(\Omega)(x|y)(\omega) = \text{Tr}T(\omega)\theta_{x(\omega),y(\omega)},
\]

shows that \( X \) is of finite right numerical index if and only if \( \sup_\omega \text{Tr}T(\omega) \) is finite. In this case, \( r - \text{Ind}[X](\omega) = \text{Tr}T(\omega) \). Therefore \( X \) is of finite index if and only
if $\omega \to \text{Tr} T(\omega)$ is a bounded, continuous function on $\Omega$ (e.g. $T \in FR(X) \cap K(X)^+$). Notice that the set of linear combinations of elements $T \in K(X)^+$ for which $T(\omega)$ is trace-class and $\omega \in \Omega \to \text{Tr} T(\omega)$ is continuous, is a $^*$-ideal of $K(X)$ by 4.5.2 in [Di], norm dense in $K(X)$. Assume from now on that $\sup \omega \dim H(\omega)$ is finite. Then $T = I$ defines a bi-Hilbertian bimodule of finite right (and left) numerical index, with $r - \text{Ind}[X](\omega) = \dim H(\omega)$. However, it is not of finite index, unless the dimension function is continuous. However, if $T \in K(X)^+$, $\omega \to \text{Tr} T(\omega)$ is always bounded and continuous by [F], and therefore $T$ does define a finite right index structure on $X$.

The next example concerns with conditional expectations between unital $C^*$-algebras, and was introduced in [W] as a $C^*$-algebraic analogue of Jones index theory for finite subfactors.

2.33 Example Let $A \subset B$ be an inclusion of unital $C^*$-algebras and let $E : B \to A$ be a conditional expectation of finite index in the sense of [W]. We thus have elements $\{u_1, ..., u_n\}$ in $B$ such that $x = \sum_{i=1}^n u_i E(u_i^* x)$ for any $x \in B$. Such elements were called a quasi-basis of $E$, and the index of $E$ was defined as $\text{Ind}[E] = \sum_{i=1}^n u_i u_i^*$ in [W]. Consider $X = B$ as a $B$–$A$ bimodule in the obvious way, and define on the left $B$-valued inner product $b(x|y) = xy^*$ and right $A$-valued inner product $(x|y)_A = E(x^*y)$. Since $\theta_{x,y}^i = y^* x$, $K(bX) = B$ acting on $X$ by right multiplication. Thus the right $A$-action has range in $K(bX)$. Proposition 2.6.2 in [W] shows that $E$ satisfies the inequality $E(x^* x) \geq ||\text{Ind}[E]||^{-1} x^* x$. Therefore $X$ is bi-Hilbertian and of finite left index: $\ell - \text{Ind}[X] = I_A$.

Its contragradient bimodule is $Y = A^*B$, as a $A$–$B$ bimodule, with $A$-valued inner product $A(x|y) = E(xy^*)$ and $B$-valued inner product $(x|y)_B = x^* y$. $Y$ is a bi-Hilbertian $B$–$A$ bimodule of finite left index. In fact the algebra $K_A(Y)$ is isomorphic to the $C^*$-basic construction $C^*(B, e_A)$ which contains the image of the left action of $B$ via $b = \sum_i \theta_{x_i}^* b u_i ...$. Therefore $X$ is of finite right index as well and $r - \text{Ind}[X] = \ell - \text{Ind}[Y] = \text{Ind}[E]$. Consider the dual conditional expectation $F : C^*(B, e_A) \to B$, $F(x e_A y) = (\text{Ind}[E])^{-1} xy$. The Pimsner-Popa inequality for $F$ shows that $Y$ is bi-Hilbertian, while the fact that $F$ is contractive gives

$$\| \sum_{i=1}^n (x_i|x_i)_B \| \leq \|\text{Ind}[E]\| \| \sum_{i=1}^n \theta_{x_i}^i \|.$$

Consider now the bimodule $Z = A^*B$ as a bi-Hilbertian $A$–$A$ Hilbert bimodule with right $A$-valued inner product $(x|y)_A = E(x^*y)$ and left $A$-valued inner product $A(x|y) = E(xy^*)$. Then $Z$ is isomorphic to $Y \otimes_B X$ so $Z$ is a Hilbert $A$–$A$ bimodule of finite right index.

The following example is a generalization of index theory to finitely generated Hilbert bimodules, studied in [KW1].

2.34 Example Let $A$ and $B$ be unital $C^*$-algebras and let $X$ be a Hilbert $A$–$B$ bimodule such that both left and right actions are unit preserving. Then $X$ is a bi-Hilbertian bimodule of finite index if and only if $X$ is of finite type in the sense of [KW1]. In fact, assume that $X$ is bi-Hilbertian and of finite
index, then $X$ is necessarily finitely generated projective as a right module (or as a left module), since $\mathcal{K}(X_B)$ (or $\mathcal{K}(A_X)$) contains the identity map. The two norms defined by the two inner products of $X$ are equivalent, thus $X$ is of finite type. Conversely, assume that $X$ is of finite type in the sense of [KW1]. Then it is clear that $X_B$ is bi-Hilbertian and that the left $A$-action on $X$ has range into $\mathcal{K}(X_B) = \mathcal{L}(X_B)$. Furthermore $X_B$ is of finite right numerical index by Lemma 1.26 in [KW1]. Thus $X$ is of finite right index. Similarly, $X$ is of finite left index and therefore of finite index.

We conclude this section with a discussion of a a Pimsner-Popa conditional expectation with no finite quasi–basis. This example was pointed out in [W]. Later it was considered also in in [FK]. We show that this inclusion is determined by a natural $\sigma$–unital subinclusion of finite index in the sense of Def. 2.23.

2.35 Example Consider the $C^*$-algebra $C([-1,1])$ of continuous functions over the interval $[-1,1]$ and the $C^*$-subalgebra $C([-1,1])_e = \{ f \in C([-1,1]) : f(-x) = f(x) \}$ of even functions. The conditional expectation $E : C([-1,1]) \to C([-1,1])_e$ associating to $f \in C([-1,1])$ the function $\frac{1}{2}(f(x)+f(-x))$ does not have a finite quasi–basis in the sense of [W] since $C([-1,1])$ is not a finite projective module over $C([-1,1])_e$. It follows that the bi–Hilbertian bimodule $C([-1,1])C([-1,1])_e$ with inner products

$$C([-1,1])(f)(g) = f \overline{g}, \quad (f)(g)c([-1,1])_e = E(\overline{f}g)$$

is not of finite right index in the sense of Def. 2.23 because the identity operator over $C([-1,1])C([-1,1])_e$ is not compact. However, the Pimsner–Popa inequality $E(f) \geq \frac{1}{2}f$ holds for any $f \in C([-1,1])^+$. One can treat this example by our methods passing to a subinclusion in the following way. Consider the $\sigma$–unital $C^*$–subalgebra $C_0([-1,1]) = \{ f \in C([-1,1]) : f(0) = 0 \}$. Then the restriction of $E$ still defines a conditional expectation $E : C_0([-1,1]) \to C_0([-1,1])_e$, where $C_0([-1,1])_e = C([-1,1])_e \cap C_0([-1,1])$, and therefore a bi–Hilbertian $C^*$–bimodule

$$X = C_0([-1,1])C_0([-1,1])_e C_0([-1,1])_e,$$

which we show to be of finite right index. We first show that the left action $C_0([-1,1])$ has range included in the compacts. Set, for $f \in C_0([-1,1])$,

$$f_\epsilon(x) = E(f)(x) = \frac{f(x)+f(-x)}{2} \in C_0([-1,1])_e,$$

$$f_o(x) = \frac{f(x)-f(-x)}{2} \in C_0([-1,1])_o \in \{ f \in C_0([-1,1]) : f(-x) = -f(x) \}.$$ 

Clearly $f = f_\epsilon + f_o$ and $E(\overline{f_\epsilon}g_o) = 0$, $f,g \in C_0([-1,1])$. Therefore the right Hilbert bimodule $X_B$ splits into the direct sum of the subspaces of even and odd functions: $X = X_e \oplus X_o$, $X_e := \{ f \in X : f(-x) = f(x) \}$, $X_o := \{ f \in X : f(-x) = -f(x) \}$. Similarly, as a vector space, $C_0([-1,1]) = \mathcal{C}_0([-1,1])_e \oplus \mathcal{C}_0([-1,1])_o$. For $f,g \in X = C_0([-1,1])$, $\theta_{f,g}(h_e + h_o) = f \overline{g} h_e + f \overline{g} h_o$. Therefore if, for $n \in \mathbb{N}$, $u_n$ is a positive continuous function in $C_0([-1,1])_e$ such that $u_n(x) = 1$ for $|x| \geq \frac{1}{n}$ and $u_n(x) = 0$ for $|x| \leq \frac{1}{n}$, the sequence $\theta_{f,u_n} + \theta_{f,f,u_n}$ is norm converging to the multiplication operator by $f$. Therefore by Theorem 2.22, (2) of Def. 2.23 holds. We are left to show that $X_B$ is of finite right numerical index, and this follows from the Pimsner–Popa inequality.
3. Continuous bundles of finite dimensional C*-algebras arising from bimodules of finite right index

Let $X$ be a right Hilbert $A$-$B$ bimodule with nondegenerate left action $\phi$, and let us consider the extension $\tilde{\phi} : M(A) \to \mathcal{L}(X_B)$ of $\phi$ to the multiplier algebra (see Prop. 2.15). Restricting $\tilde{\phi}$ to the centre $Z(M(A))$ of $M(A)$ yields a unital *-homomorphism $\tilde{\phi} : Z(M(A)) \to \mathcal{L}(X_B)$, still denoted $\tilde{\phi}$.

**3.1 Proposition** If $X$ is a right Hilbert $A$-$B$ bimodule with nondegenerate left action (this being the case if, e.g., $X$ is bi-Hilbertian, by Prop. 2.16), the range of $\tilde{\phi} : Z(M(A)) \to \mathcal{L}(X_B)$ is actually included in the centre of $\mathcal{A}\mathcal{L}(X_B)$, the algebra of right adjointable maps on $X_B$ commuting with the left action. Therefore $\mathcal{A}\mathcal{L}(X_B)$ becomes a $Z(M(A))$-algebra in the sense of [Ka].

Adopting a standard procedure we can represent $\mathcal{A}\mathcal{L}(X_B)$ as a semicontinuous field of C*-algebras $\omega \to \mathcal{L}_\omega$ over the spectrum $\Omega$ of $Z(M(A))$ in the sense of [Ka]. Let, for $\omega \in \Omega$, $J_\omega$ be the closed two-sided ideal of $\mathcal{A}\mathcal{L}(X_B)$ generated by the image under $\phi$ of $C_\omega(\Omega)$, the continuous functions on $\Omega$ vanishing at $\omega$. The fiber at $\omega$ is the quotient C*-algebra $\mathcal{L}_\omega := \mathcal{L}(X_B)/J_\omega$. We will show that the field $\omega \to \mathcal{L}_\omega$ is in fact continuous in the case where $X$ is bi-Hilbertian and of finite right index (see Theorem 2.3).

Let $\mathcal{A}X_B$ be bi-Hilbertian and of finite right index. In Proposition 2.28 we have constructed a $M(A)$-bilinear, positive, strictly continuous map $\tilde{F} : \mathcal{L}(X_B) \to M(A)$ satisfying a Pimsner-Popa inequality and with range the ideal $pM(A)$, with $p$ the support projection of $r - \text{Ind}[X]$. Restricting $\tilde{F}$ to the $C^*$-subalgebra $\mathcal{A}\mathcal{L}(X_B)$ yields a map, still denoted $\tilde{F}$, with the same properties, and with range the ideal $pZ(M(A))$ of the commutative $C^*$-algebra $Z(M(A))$. We write $\Omega = \Omega_0 \cup \Omega_1$, with $\Omega_0$ corresponding to the projection $p$ and $\Omega_1$ to $I - p$. The map $\tilde{F}$ makes $\mathcal{A}\mathcal{L}(X_B)$ into a right Hilbert $C(\Omega)$-module (in fact a Hilbert $C(\Omega_0)$-module) by $(S|T) = \tilde{F}(S^*T)$. Since $\tilde{F}$ is norm continuous and satisfies a Pimsner-Popa inequality, the operator norm and the Hilbert module norm are equivalent, therefore $\mathcal{A}\mathcal{L}(X_B)$ is complete in the Hilbert module norm.

Since the inner product is evaluated on a commutative $C^*$-algebra, we can represent $\mathcal{A}\mathcal{L}(X_B)$ as a continuous field of Hilbert spaces over $\Omega$ in the sense of [Di]. For each $\omega \in \Omega$, the fiber Hilbert space at $\omega$ is given by $H_\omega = \mathcal{A}(X_B)/M_\omega$, where $M_\omega$ is the norm closed subspace of $\mathcal{A}(X_B)$, in the Hilbert module norm, generated by $\mathcal{A}(X_B)\phi (C_\omega(\Omega))$. For each $\omega \in \Omega_1$, $M_\omega = \mathcal{A}(X_B)$ since $\phi$ annihilates $(I - p)Z(M(A)) = C(\Omega_1)$, therefore $M_\omega = 0$, as expected. Since the $C^*$-algebra norm and the Hilbert module norm are equivalent, $J_\omega = M_\omega$, as vector spaces, and they are isomorphic as Banach spaces. In particular, $\mathcal{L}_\omega = 0$ for $\omega \in \Omega_1$. Let $\pi_\omega : \mathcal{A}(X_B) \to \mathcal{L}_\omega$ and $p_\omega : \mathcal{A}(X_B) \to H_\omega$ denote the corresponding quotient maps in the $C^*$-algebraic and Banach space sense.

**3.2 Lemma** If $\mathcal{A}X_B$ is a bi-Hilbertian bimodule of finite right index, for all $T \in \mathcal{A}(X_B)$ and for all $\omega \in \Omega_0$,

$$\lambda^{1/2}\|\pi_\omega(T)\| \leq \|p_\omega(T)\| \leq (r - \text{Ind}[X])(\omega)^{1/2}\|\pi_\omega(T)\|,$$

where $\lambda$ is the best positive scalar for which $\|A(x|x)\| \geq \lambda\|\pi_\omega(T)\|$, $x \in X$. 
Proof The positive $M(A)$-bilinear map $\hat{F} : \mathcal{L}(X_B) \to M(A)$ satisfies $\hat{\phi}(F(T)) \geq \lambda'T$ for all $T \in \mathcal{L}(X_B)^+$, by Prop. 2.28. Therefore if $T \in A\mathcal{L}(X_B)^+$, evaluating $\pi_\omega$ on this estimate yields $\pi_\omega(\hat{\phi}(F(T)) \geq \lambda'\pi_\omega(T)$ which shows that

$$\|p_\omega(T)\|^2 = (p_\omega(T), p_\omega(T)) = \hat{F}(T^*T)(\omega) = \|\pi_\omega(\hat{F}(T^*T))\| = \lambda'\|\pi_\omega(T)\|^2.$$ 

Consider the map $G_\omega : \mathcal{L}_\omega \to \mathbb{C} = C(\Omega)/C_\omega(\Omega)$ associating $\hat{F}(T)(\omega)$ to $\pi_\omega(T)$. This map is well defined: $\pi_\omega(T) = 0$ implies that $T \in J_\omega$, therefore $\hat{F}(T)$ belongs to $F(J_\omega)$ which is contained in the closed linear span of $C_\omega(\Omega)F(A\mathcal{L}(X_B))$ in the $C^*$-algebra norm. Clearly the latter space is contained in $C_\omega(\Omega)$. Now $G_\omega$ is a positive functional on the $C^*$-algebra $\mathcal{L}_\omega$ taking the unit of $\mathcal{L}_\omega$ to $(r - \text{Ind}[X])(\omega)$, and therefore $\|G_\omega\| = (r - \text{Ind}[X])(\omega)$. Thus for all $T \in A\mathcal{L}(X_B)$,

$$\|p_\omega(T)\|^2 = |\hat{F}(T^*T)(\omega)| = \|G_\omega(\pi_\omega(T^*T))\| \leq (r - \text{Ind}[X])(\omega)\|\pi_\omega(T^*T)\| = (r - \text{Ind}[X])(\omega)\|\pi_\omega(T)\|^2.$$

We are now ready to show the following result.

3.3 Theorem Let $X$ be a bi-Hilbertian $A$-$B$ $C^*$–bimodule of finite right index, and let $\Omega$ be the spectrum of $Z(M(A))$. Then for each $\omega \in \Omega$, the quotient $C^*$-algebra $\mathcal{L}_\omega$ is finite dimensional, and

$$\text{dim}(\mathcal{L}_\omega) \leq [\lambda^{-1}(r - \text{Ind}[X])(\omega)]^2,$$

where $\lambda'$ is the best constant for which $\|_{A\mathcal{L}(X_B)}(x) \| \geq \lambda'\|x\|_B$ and $[\mu]$ denotes the integral part of the real number $\mu$. In particular, the fibers are trivial on $\Omega_1$. Furthermore the collection of epimorphisms $\pi_\omega : A\mathcal{L}(X_B) \to \mathcal{L}_\omega$, $\omega \in \Omega$, defines a continuous bundle of $C^*$–algebras in the sense of [KW].

Proof Let us consider the positive map $\hat{F} : \mathcal{L}(X_B) \to M(A)$, which satisfies $\hat{\phi}(\hat{F}(T)) \geq \lambda'T$ for $T \in \mathcal{L}(X_B)^+$ by Cor. 2.11. We restrict $\hat{\phi}$ to a map $A\mathcal{L}(X_B) \to \hat{\phi}(Z(M(A))$ satisfying a corresponding inequality. Evaluating $\pi_\omega$ on this inequality yields $\pi_\omega(\hat{\phi}(\hat{F}(T)) \geq \lambda\pi_\omega(T)$, $T \in A\mathcal{L}(X_B)^+$. On the other hand for each $\omega$ in the support projection of $r - \text{Ind}[X]$, $\pi_\omega(\hat{\phi}(\hat{F}(T)) = G_\omega(\pi_\omega(T))$, where $G_\omega$ is the positive functional of $\mathcal{L}_\omega$ defined as in the proof of the previous lemma: $G_\omega(\pi_\omega(T)) = \hat{F}(T)(\omega)$. Therefore $g_\omega := ((r - \text{Ind}[X])(\omega))^{-1}G_\omega$ is a state of $\mathcal{L}_\omega$ satisfying

$$(r - \text{Ind}[X])(\omega)g_\omega(\pi_\omega(T)) \geq \lambda'\pi_\omega(T)\pi_\omega(T), T \in A\mathcal{L}(X_B)^+.$$
all $\omega \in \Omega$ then $T \in J_\omega$ for all $\omega$ in $\Omega$ and therefore $F(T) = 0$ which implies $T = 0$ by the Pimsner-Popa inequality. This shows axiom (i). Axiom (ii) is obvious.

We are left to show that for all $T \in A\mathcal{L}(X_B)$, the function $\omega \in \Omega \rightarrow \|\pi_\omega(T)\|$ is continuous. We will appeal to the continuity criteria discussed in section 2 of [KW]. This function is upper semicontinuous by Lemma 2.3 in [KW] and it is lower semicontinuous by Lemma 2.2 in the same paper. Indeed, if $\Omega' \subset \Omega$ is a closed subset of $\Omega$ and $D \subset \Omega'$ is dense in $\Omega'$ then the condition $\pi_\omega(T) = 0$ for all $\omega \in D$ and some $T \in A\mathcal{L}(X_B)^+$ implies $T \in J_\omega$ for all $\omega \in D$, thus $F(T)(\omega) = 0$ for all $\omega \in D$ and therefore for all $\omega \in \Omega'$ by continuity of the function $F(T)$. Now evaluating $\pi_\omega$ on both sides of the inequality $\omega F(T) \geq \lambda^T$ shows that $\pi_\omega(T) = 0$ for all $\omega \in \Omega'$.

Remark Notice that the estimate given in Theorem 3.3 can not be improved in general. In fact, if $H$ is the finite index $\mathbb{C}$–$\mathbb{C}$ bimodule defined as in Example 2.31. Then $\Omega$ is a one point space, $c\mathcal{L}(H_\mathbb{C}) = M_n(\mathbb{C})$, which is the only fiber. In this case $\lambda^{-1} = \|T^{-1}\|$, so the corresponding estimate reduces to $n \leq \|T^{-1}\|\text{Tr}(T)$ which becomes an equality for $T = I$.

4. On the equivalence between finite index and conjugate equations

Our next aim is to show an equivalence between the notion of $C^*$–bimodule of finite index in the sense of Sect. 2 and Longo-Roberts conjugate object in the $C^*$–category of right Hilbert bimodules [LR].

4.1 The $C^*$–categories $\mathcal{H}_A$, $A\mathcal{H}_A$ and the $W^*$–categories $\mathcal{H}_A^w$, $A\mathcal{H}_A^w$

4.1 Definition Let $A$ be a fixed set of $C^*$–algebras. We will denote by $\mathcal{H}_A$ the category with objects and arrows defined as follows. Objects of $\mathcal{H}_A$ are right Hilbert $C^*$–bimodules $X$ over elements of $A$ for which the left action is nondegenerate. The set of arrows $(X, Y)$ in $\mathcal{H}_A$ between two objects $A X_B$ and $A Y_B$ is the set $(X, Y) := \mathcal{L}(X_B, Y_B)$ of (right) adjointable maps from $X$ to $Y$.

Given two objects $A X_B$ and $B Y_C$ of $\mathcal{H}_A$, their tensor product $X \otimes_B Y$ is still a nondegenerate right Hilbert $C^*$–bimodule, and therefore it is an object of $\mathcal{H}_A$. For any $T \in (X, Y)$, the map taking a simple tensor $x \otimes y \in X \otimes_B Y$ to $T(x) \otimes y$, and denoted $T \otimes I_Y$, extends to an adjointable map on $X \otimes_B Y$. For any $C^*$–algebra $A \in A$, let $\iota_A$ be $A$, regarded as a right Hilbert bimodule over $A$ itself, in the natural way. Since left action on $\iota_A$ is nondegenerate, $\iota_A$ is an object of $\mathcal{H}_A$. For any right Hilbert $A$–$B$ $C^*$–bimodule $X$, the tensor product Hilbert bimodule $X \otimes_B \iota_B$ identifies naturally with $X$. In general, $\iota_A \otimes_A X$ identifies with the right Hilbert sub-bimodule of $X$ generated by $A X$, which, in the case where the left action is nondegenerate, still coincides with $X$ (cf. Def. 2.14). Therefore $\{\iota_A, A \in A\}$ is the set of left and right units for the $\otimes$–product between objects. One can summarize the structure of $\mathcal{H}_A$, and say that $\mathcal{H}_A$ is a semitensor 2-$C^*$–category (in the sense of [DPZ]).

If we want a tensor 2-$C^*$–category we need to restrict the arrow spaces, and consider only bimodule maps. Namely, let $A\mathcal{H}_A$ be the subcategory of $\mathcal{H}_A$ with the same objects and arrows $(X, Y) := A\mathcal{L}(X_B, Y_B)$, the set of right adjointable maps from $X$ to $Y$ commuting with the left action. This is now a tensor 2-$C^*$–category.
In the sequel we will consider also the $W^*$-categories $\mathcal{H}_A^w$ with the same objects, and set of arrows between two objects $X$ and $Y$ obtained completing the corresponding arrow spaces of $\mathcal{H}_A$ and $\mathcal{A}_A$ in a suitable weak topology. Choose, for each unit object $\iota_B \in \mathcal{H}_A$, a state $\omega_B$ of $B$, and let us endow $X$ with the inner product

$$(x, x')_{\omega_B} = \omega_B((x|x')_B), x, x' \in X.$$ 

Completing $X$, after dividing out by vectors of seminorm zero, with respect to this inner product, yields a Hilbert space $H_{\omega_B}(X)$. For each $T \in \mathcal{L}(X_B, Y_B)$, let $F_\omega(T) \in \mathcal{B}(H_{\omega_B}(X), H_{\omega_B}(Y))$ be the operator which acts by left multiplication by $T$. We get in this way a $^*$-functor $F_\omega : \mathcal{H}_A \rightarrow \mathcal{H}$ to the category of Hilbert spaces. Consider now the universal $^*$-functor $F = \oplus_\omega F_\omega : \mathcal{H}_A \rightarrow \mathcal{H}$, where the direct sum is taken over all choice functions $\omega : B \in A \rightarrow \omega_B$. $F$ is faithful on arrows and strictly continuous on the unit ball of each arrow space.

Define $(X, Y)$ to be the completion of $F(K(X_B, Y_B))$ in the weak topology of the bounded operators from $\oplus \omega H_{\omega_B}(X)$ to $\oplus \omega H_{\omega_B}(Y)$, and let $\mathcal{H}_A^w$ be the $W^*$-category with arrows these $W^*$-closed subspaces. Since any operator in $\mathcal{L}(X_B, Y_B)$ is the strict limit of a norm bounded net from $K(X_B, Y_B)$, $F(L(X_B, Y_B)) \subset (X, Y)$, therefore $\mathcal{H}_A$ becomes a $C^*$-subcategory of $\mathcal{H}_A^w$ under $F$. The universal functor enjoys the following universality property.

**4.2 Proposition** A $^*$-functor $G : \mathcal{H}_A \rightarrow \mathcal{H}$ to the category of Hilbert spaces, strictly continuous on the unit ball of each arrow space of $\mathcal{H}_A$, extends uniquely to a $^*$-functor $G' : \mathcal{H}_A^w \rightarrow \mathcal{H}$, normal on the arrow spaces.

**Proof** Let us first assume that each Hilbert space $G(\iota_B)$ is cyclic for $G((\iota_B, \iota_B))$. Let $\xi_B$ be a normalized cyclic vector. Then, identifying $X$ with the subspace of intertwiners $K(\iota_B, X_B) \subset (\iota_B, X)$, $G(X)\xi_B$ is a subspace of the Hilbert space $G(X)$ associated to the object $X$. We claim that $G(X)\xi_B$ is the whole $G(X)$. Let $\eta \in G(X)$ be a vector orthogonal to $G(X)\xi_B$. For all $x \in X$, $G(x^*\eta)$ is orthogonal to $G((\iota_B, \iota_B))\xi_B$ and hence it is zero. Since $G$ is strictly continuous on the unit ball of $(X, X)$, we conclude that $\eta = 0$. We therefore have an identification of the Hilbert space $G(X)$ with $F_\omega(X)$ where $\omega : B \rightarrow \omega_B$, and also an identification of $G$ with $F_\omega$. Now every $^*$-functor $G : \mathcal{H}_A \rightarrow \mathcal{H}_C$ is the direct sum cyclic $^*$-functors, therefore $G$ is a direct sum of some $F_\omega$, and the rest now follows easily.

In particular, if $R_Y : \mathcal{H}_A \rightarrow \mathcal{H}_A$ is the $^*$-functor which tensors on the right by an object $Y \in \mathcal{H}_A$, the normal extension of $F \circ R_Y$, with $F$ the universal $^*$-functor, makes $\mathcal{H}_A^w$ into a semitensor $2-W^*$-category.

The subcategory $\mathcal{A}\mathcal{H}_A^w$ of $\mathcal{H}_A^w$ with the same objects and arrows

$$(X, Y) : = \{ T \in F(K(X_B, Y_B))'' : T \mathcal{F}(\phi(a)) = \mathcal{F}(\phi'(a))T, a \in A, x \in X \},$$

(where $\phi$ and $\phi'$ denote respectively the left actions of $A$ on $X$ and $Y$, and $\mathcal{F}$ is the universal $^*$-functor) is now a tensor $2-W^*$-category.

**Remark** The functor of $R_Y$ may not be injective on arrows in any of these categories. In other words, if $A_XB, A_X'B$ and $B_YC$ are right Hilbert $C^*$-bimodules, the natural $^*$-homomorphism

$$T \in \mathcal{L}(X_B, X_B') \rightarrow T \otimes I_Y \in \mathcal{L}((X \otimes_B Y)_C, (X' \otimes_B Y)_C)$$
may not be injective. In fact, if \( X = X' = \iota_B \) and \( b \in B \subset \mathcal{L}(\iota_B) = \mathcal{M}(B) \), under the identification of \( \iota_B \otimes_B Y \) with \( Y, b \otimes I_Y \) corresponds to the left action of \( B \) on \( Y \) evaluated in \( b \), which may vanish.

4.2 Conjugation in \( \mathcal{A}\mathcal{H}_A \) and \( \mathcal{A}\mathcal{H}_A^w \)

In the sequel \( \mathcal{T} \) will denote either \( \mathcal{A}\mathcal{H}_A \) or \( \mathcal{A}\mathcal{H}_A^w \). Following [LR], we can introduce the notion of conjugation in the tensor \( C^* \) (or \( W^* \)) category \( \mathcal{T} \).

4.3 Definition Let \( X = \mathcal{A}X_B \) be an object of \( \mathcal{T} \). An object \( Y = BY_A \) of \( \mathcal{T} \) is called a conjugate of \( X \) if there exist intertwiners \( R \in (\iota_B, Y \otimes_A X) \in \mathcal{T} \) and \( \overline{R} \in (\iota_A, X \otimes_B Y) \in \mathcal{T} \) such that

\[
\overline{R} \otimes I_X \circ I_X \otimes R = I_X \\
R^* \otimes I_Y \circ I_Y \otimes \overline{R} = I_Y.
\]

We adopt the convention that the \( \otimes \)–product is evaluated before \( \circ \)–product. We emphasize that, if \( \mathcal{T} = \mathcal{A}\mathcal{H}_A \), \( R \) and \( \overline{R} \) are \( C^* \)–bimodule maps, i.e. they commute with left as well as right actions of the appropriate \( C^* \)–algebras. Therefore in this case \( R^* R \) and \( \overline{R} \overline{R} \) are elements of \( \mathcal{B}\mathcal{L}(\iota_B) = \mathcal{Z}(\mathcal{M}(B)) \) and \( \mathcal{A}\mathcal{L}(\iota_A) = \mathcal{Z}(\mathcal{M}(A)) \) respectively. If, instead, \( \mathcal{T} = \mathcal{A}\mathcal{H}_A^w \), we can only conclude that \( R^* R \) and \( \overline{R} \overline{R} \) are central elements of \( \mathcal{B}^w \) and \( \mathcal{A}' \) respectively.

The above equations will be referred to as the conjugate equations. Clearly, if \( Y \) is a conjugate of \( X \) then \( X \) is a conjugate of \( Y \).

The dimension of \( X \) relative to the pair \((R, \overline{R})\) is defined by \( \dim_{R, \overline{R}} X = \| R \| \| \overline{R} \| \). The minimal dimension of \( X \), denoted \( \dim X \), is the infimum of all relative dimensions \( \dim_{R, \overline{R}} \).

Uniqueness of the conjugate object. Let \( Y \) be a conjugate object of \( X \) in \( \mathcal{T} \), and let \( R \) and \( \overline{R} \) solve the corresponding conjugate equations. Let \( U \in (Y, Y') \) be an invertible intertwiner in \( \mathcal{T} \). Set \( R' := U \otimes I_X \circ R \) and \( \overline{R}' := I_X \otimes U^{-1} \circ \overline{R} \). Then \((Y', R', \overline{R}')\) defines another conjugate of \( X \) and every conjugate of \( X \) arises in this way (see [LR]). In the case where \( U \) is a unitary, \( \overline{R}' \overline{R}' = R' \overline{R} \) and \( R'^* R' = R^* R \), so the dimension relative to this new pair of intertwiners does not change. In this situation we say that the conjugates \((Y, R, \overline{R})\) and \((Y', R', \overline{R}')\) are unitarily equivalent.

4.3 from finite index to conjugation

4.4 Theorem Let \( \mathcal{A}X_B \) be a bi-Hilbertian \( C^* \)–bimodule. Then left actions on the underlying right Hilbert \( C^* \)–bimodules \( X \) and \( \overline{X} \) are nondegenerate, and therefore these are objects of \( \mathcal{A}\mathcal{H}_A \) and \( \mathcal{A}\mathcal{H}_A^w \).

1. If \( X \) is of finite numerical index, \( \overline{X} \) is a conjugate of \( X \) in \( \mathcal{A}\mathcal{H}_A^w \). More specifically, if \( \{ u_\mu \}_\mu \) and \( \{ v_\nu \}_\nu \) are, respectively, a generalized right and left basis of \( X \), the nets \( \overline{R}_\mu := \sum_{\nu \in u_\mu} y \otimes \overline{\tau} \) and \( R_\nu := \sum_{\tau \in v_\nu} \overline{\tau} \otimes z \) converge strongly under the universal \(*\)-functor to intertwiners \( R \in (\iota_A, X \otimes_A \overline{X}) \) and \( \overline{R} \in (\iota_B, \overline{X} \otimes_B X) \) of \( \mathcal{A}\mathcal{H}_A^w \) which do not depend on
the choice of the bases, and solve the conjugate equations. The following relations also hold for \( x, x' \in X \),

\[
\mathcal{T}(\theta_{x,x'}^r \otimes I_X) = A(x|x'),
\]

\[
R^*(\theta_{x,x'}^r \otimes I_X)R = (x|x')_B,
\]

\[
R^*R = \ell - \text{Ind}[X],
\]

\[
\mathcal{T}^* \mathcal{T} = r - \text{Ind}[X].
\]

(2) If \( X \) is of finite index, \( R \) and \( \mathcal{T} \) belong to \( \mathcal{A} \). So the right Hilbert bimodule \( \mathcal{X} \) is a conjugate of \( X \) in \( \mathcal{A} \). Their right adjoint operators are given by:

\[
\mathcal{T}^* x \otimes x' = A(x|x'),
\]

\[
R^* \mathcal{T} \otimes x' = (x|x')_B.
\]

**Proof** By Prop. 2.16 the left (right) action on the right (left) Hilbert \( C^* \)-bimodule \( X \) is nondegenerate, therefore the right Hilbert \( C^* \)-bimodules \( X \) and \( \mathcal{X} \) are objects of \( \mathcal{A} \) and \( \mathcal{A}_\mathcal{A} \). We claim that, under the natural identifications of \( X \otimes_B \mathcal{X} \) with \( \mathcal{K}(\ell_A, X \otimes_B \mathcal{X}_A) \) and of \( \mathcal{X} \otimes_A X \) with \( \mathcal{K}(\ell_B, \mathcal{X} \otimes_A \mathcal{X}_B) \), the nets \( \mathcal{R}_\mu := \sum_{y \in \mu} y \otimes y \) and \( R_\nu := \sum_{z \in \nu} z \otimes z \) converge strongly in the universal \(*\)-functor to operators \( \mathcal{R} \) and \( R \) which do not depend on the choice of the bases. It suffices to show that the first net is strongly Cauchy, as, replacing \( X \) with \( \mathcal{X} \), \( \mu \rightarrow u_\mu \) changes to \( \nu \rightarrow v_\nu \). Now by Prop. 2.19, \( \sum_{y \in u_\mu} A(y|y) \) is a positive, increasing, norm bounded net, and it is strongly convergent in \( A'' \) to \( r - \text{Ind}[X] \). Since for \( \mu < \mu' \), \( \sum_{y \in u_\mu, \nu' \neq \mu' \atop \nu' \in \mu} \theta_{y,y}^r - \sum_{y \in u_\mu} \theta_{y,y}^r \) is a positive contraction, we have

\[
\mathcal{F}_X((\sum_{y \in u_\mu} \theta_{y,y}^r - \sum_{y \in u_\mu} \theta_{y,y}^r)^2) \leq \mathcal{F}_X(\sum_{y \in u_\mu} \theta_{y,y}^r - \sum_{y \in u_\mu} \theta_{y,y}^r) = \sum_{y \in u_\mu} A(y|y) - \sum_{y \in u_\mu} A(y|y).
\]

Therefore the net \( \mathcal{R}_\mu \in \mathcal{K}(\ell_A, X \otimes \mathcal{X}_A) \) is strongly convergent on a dense subspace of the underlying Hilbert space. We show that this net is norm bounded. We have, for \( a \in A \),

\[
\|\mathcal{R}_\mu(a)\mathcal{R}_\mu(a)\| = \|U(\sum_{y \in u_\mu} (y \otimes y)a)U(\sum_{y \in u_\mu} (y \otimes y)a)\| 
\]

\[
= \|F((\sum_{y \in u_\mu} \theta_{y,y}^r \phi(a))^*(\sum_{y \in u_\mu} \theta_{y,y}^r \phi(a))\| \leq \|F(\phi(a)^* \phi(a))\|
\]

\[
= \|(r - \text{Ind}[X])a^*a\|,
\]
where $U$ is the biunitary map defined in Prop. 2.13 (3). Hence
\[
\lVert \overline{T}_\mu \rVert \leq (\sup_{\omega \neq 0} \frac{\lVert (r - \text{Ind}[X])a^*a \rVert}{\lVert a^*a \rVert})^{1/2} = \lVert r - \text{Ind}[X] \rVert^{1/2}.
\]
It follows that $\overline{T}_\mu$ is strongly convergent to an operator $\overline{T} \in (\iota_A, X \otimes X) \subset \mathcal{H}^w_A$ with $\lVert \overline{T} \rVert \leq \lVert r - \text{Ind}[X] \rVert^{1/2}$. Similarly we define a map $R \in (\iota_B, X \otimes X) \subset \mathcal{H}^w_A$ as the strong limit of $\sum_{z \in v_\nu} (\tau \otimes z)$ such that $\lVert R \rVert \leq \lVert \ell - \text{Ind}[X] \rVert^{1/2}$. In order to show that $\overline{T}$ is independent on the basis, we compute its Hilbert space adjoint. Let $\omega : B \in A \to \omega_B$ be a choice of states of the $\mathcal{C}^*$-algebras of $A$, and let $\mathcal{F}_\omega : \mathcal{H}_A \to \mathcal{H}$ be the associated cyclic $^*$-functor to the category of Hilbert spaces. For $x, x' \in X$, $a \in A$,
\[
(a, \mathcal{F}_\omega(\overline{T}_\mu)^*x \otimes \overline{\tau} )_{\omega_A} = \sum_{y \in u_\mu} (y \otimes \overline{\tau} a, x \otimes \overline{\tau})_{\omega_A} = \sum_{y \in u_\mu} \omega_A((\overline{\tau}a)(y|x)B)_{A} = \sum_{y \in u_\mu} \omega_A(a^*(y|x')B) = \omega_A(a^*\sum_{y \in u_\mu} y(y|x)_{B, x'}),
\]
Therefore $\mathcal{F}_\omega(\overline{T}_\mu)^*x \otimes \overline{\tau}$ converges weakly to $A(x|x')$, regarded as an element of the Hilbert space $\mathcal{F}_\omega(\iota_A)$. It follows that $\overline{T}^*$, and hence $\overline{T}$, is independent of the generalized right basis. On the other hand the net $\overline{T}_\mu$, regarded as a net in the Hilbert space $\mathcal{F}_\omega(X \otimes X)$, has norm bounded above by $(r - I[X])^{1/2}$, therefore
\[
\lVert \overline{T} \rVert = \lVert \overline{T}^* \rVert \geq (r - I[X])^{-1/2} \lVert \overline{T} \sum_{y \in u_\mu} y \otimes \overline{\tau} \rVert = (r - I[X])^{-1/2} \sum_{y \in u_\mu} A(y|y),
\]
which shows that $\lVert R \rVert = (r - I[X])^{1/2}$. Let now $U \in M(A)$ be a unitary. For any generalized right basis $u_\mu$, $\mu \to \{U y, y \in u_\mu\}$ is still a generalized right basis, so $U \overline{T} U^* = \overline{T}$ by independence of the operator $\overline{T}$ on the basis. Hence $\overline{T} \in \mathcal{A} \mathcal{H}^w_A$.

We show that $\overline{R}$ and $\overline{T}$ solve the conjugate equations. For $x \in X$, $b \in B$, we have, in the Hilbert space associated to $X$ under the universal $^*$-functor:
\[
\overline{R} \otimes I_X \circ I_X \otimes R(xb) = \overline{R} \otimes I_X (x \otimes \lim_{z \in v_\nu} (\tau \otimes z)b) = \lim_{z \in v_\nu} A(x|z)zb = xb.
\]
Since $X \otimes_B \iota_B$ identifies with $X$ via the map $x \otimes b \mapsto xb$, we obtain the conjugate equation $\overline{R}^* \otimes I_Y \circ I_Y \otimes R = I_X$ in $\mathcal{A} \mathcal{H}^w_A$. Similarly we have $R^* \otimes I_Y \circ I_Y \otimes \overline{T} = I_Y$.

For any $a \in \mathcal{F}(\iota_A)$ we have
\[
\overline{R} \overline{T}(a) = \lim_{\mu} \overline{R} \left( \sum_{y \in u_\mu} y \otimes \overline{\tau} a \right) = \lim_{\mu} \sum_{y \in u_\mu} A(y|a^*) = \lim_{\mu} \sum_{y \in u_\mu} A(y|y)a,
\]
so $\overline{R}R = r - \text{Ind}[X]$ and $R^*R = \ell - \text{Ind}[X]$ as well.

We show that $\overline{R}(\theta_{x,z}^r \otimes I_Y)\overline{R} = A(x|z)$ (the similar equation relative to $R$ will follow replacing $X$ with $\overline{X}$). For $a \in A$,

$$\overline{R}(\theta_{x,z}^r \otimes I_Y)\overline{R}(a) = \overline{R}(\theta_{x,z}^r \otimes I_Y) \lim_{\mu} \left( \sum_{y \in u_\mu} (y \otimes \overline{y})a \right)$$

$$= \lim_{\mu} \overline{R}(\sum_{y \in u_\mu} \theta_{x,z}^r(y) \otimes \overline{y}) = \lim_{\mu} \sum_{y \in u_\mu} A(\theta_{x,z}^r(y)|a^*y)$$

$$= \lim_{\mu} \sum_{y \in u_\mu} A(x(z)y_B|y)a = A(x|z)a.$$  \hspace{\stretch{1}} (2)

In the case where $X$ is of finite index, the net $\overline{R}_\mu(a)$ converges in norm for all $a \in A$, therefore $R$ is actually mapping $A$ to $X \otimes \overline{X}$. Furthermore $\overline{R}$ is right adjointable, in fact its adjoint $\overline{R}^* : X \otimes_B \overline{X} \to A$ is defined by $\overline{R}^*(x \otimes \overline{x}) = A(x|x')$.

$$(\overline{R}(a)x \otimes \overline{x})_A = \lim_{\mu} \sum_{y \in u_\mu} a^*(y \otimes \overline{y}|x \otimes \overline{x})_A$$

$$= \lim_{\mu} \sum_{y \in u_\mu} a^*(\overline{R}(y|x)_B \overline{x})_A = \lim_{\mu} \sum_{y \in u_\mu} a^*_A(y|x'|y_B)$$

$$= a^*_A(\lim_{\mu} \sum_{y \in u_\mu} y(y|x)_B|x') = a^*_A(x|x').$$

A first consequence of the previous theorem is the fact that the left Hilbert bimodule structure on a finite index bimodule is unique up to equivalence.

**4.5 Corollary** Let $A \otimes_B$ be a bi-Hilbertian $C^*$–bimodule of finite index. Any other left inner product on the underlying right Hilbert bimodule $A \otimes_B$ making it into a finite index, bi-Hilbertian bimodule is of the form

$$A(x|y)' = A(Qx|y), x, y \in X,$$

where $Q$ is a positive invertible element of $\mathcal{L}_B(A \otimes_B)$.

**Proof** Consider another left inner product $A(\cdot|\cdot)'$ making $X$ into a bi-Hilbertian, finite index $C^*$–bimodule. Let $X'$ denote the left Hilbert bimodule structure over $X$ with inner product $A(\cdot|\cdot)'$. By part (2) of Theorem 4.4, we can find another solution $(\overline{X}', R', \overline{R})$ to the conjugate equations such that $A(x|x')' = \overline{R}'(\theta_{x,x'}^r \otimes U \overline{R})\overline{R}'$. By uniqueness of the conjugate object (cf. a remark following Definition 4.3) there is an invertible $U \in B\mathcal{L}(\overline{X}, \overline{X}')$ such that $\overline{R}' = I_X \otimes U \overline{R}$. Therefore $A(x|x')' = \overline{R}'(\theta_{x,x'}^r \otimes U^*U)\overline{R}'$. We just need to plug in the fact that $\overline{R}' = \lim_{\mu} \sum_{y \in u_\mu} y \otimes \overline{y}$ in the pointwise norm convergence topology and choose $Q := J_X^{-1}U^*J_X$, with $J_X : X_B \to H \overline{X}$ the natural conjugation map.

**4.4 On the equality $A \mathcal{L}(X_B) = \mathcal{L}_B(A \otimes_B)$ for finite index bimodules**
Let $AXB$ be a bi-Hilbertian $C^*$–bimodule. We can consider the $C^*$–algebra $\mathcal{A}\mathcal{L}(X_B)$ of right adjointable maps commuting with the left action, but also the $C^*$–algebra $\mathcal{L}_B(A_X)$ of left adjointable maps commuting with the right action. If $A$ and $B$ are unital, and $X$ is finitely generated, as a right and left module, any bimodule map on $X$ is right adjointable and left adjointable, therefore $\mathcal{A}\mathcal{L}(X_B) = \mathcal{L}_B(A_X) = A\text{End}_B(X)$. More generally, under which conditions $\mathcal{A}\mathcal{L}(X_B) = \mathcal{L}_B(A_X)$ as algebras? The following result provides an answer.

**4.6 Corollary** If $AXB$ is a bi-Hilbertian bimodule of finite index, any element of $\mathcal{A}\mathcal{L}(X_B)$ is adjointable with respect to the left inner product, and therefore it belongs to $\mathcal{L}_B(A_X)$. Similarly, any element of $\mathcal{L}_B(A_X)$ is adjointable with respect to the right inner product. Therefore $\mathcal{A}\mathcal{L}(X_B) = \mathcal{L}_B(A_X)$ as algebras.

**Proof** Let $R$ and $\overline{R}$ be the solution to the conjugate equations arising from the left and right inner products as in the proof of the previous theorem. By Frobenius reciprocity there is a linear isomorphism from $\mathcal{A}\mathcal{L}(X_B)$ to $B\mathcal{L}(\overline{I}_B, X \otimes X_B)$ given by $T \rightarrow I_{\overline{X}} \otimes T \circ R$ and an antilinear isomorphism from $B\mathcal{L}(I_B, \overline{X} \otimes X_B)$ to $B\mathcal{L}(X_A)$ given by $S \rightarrow S^* \otimes I_{\overline{X}} \circ \overline{R}$ (see [LR]). A straightforward computation shows that the composition of these maps is the map $T \in \mathcal{A}\mathcal{L}(X_B) \rightarrow JTJ^{-1} \in B\mathcal{L}(X_A)$ where $J : AX \rightarrow X_A$ is the conjugation map. Therefore the map $T \in \mathcal{A}\mathcal{L}(X_B) \rightarrow T \in \mathcal{L}_B(A_X)$ is a linear multiplicative isomorphism.

**Remark** In general the $^*$–involution of $\mathcal{A}\mathcal{L}(X_B)$ differs from that of $\mathcal{L}_B(A_X)$. We illustrate this phenomenon in the particular case where $X$ comes from a conditional expectation. Let $E : B \rightarrow A$ be a finite index conditional expectation in the sense of [W], between unital $C^*$–algebras. Set $BAX = B$ as a $B$–$A$ bimodule in the natural way, and with inner products $(x|y)_A = E(x^*y)$, $(x|y) = xyy^*$, where $q \in A' \cap B$ is a positive invertible element. Since $E$ satisfies a Pimsner-Popa inequality [W], there exists a positive scalar $\lambda$ such that $\lambda E - \text{id}$ is completely positive, by [FK]. Therefore

$$\lambda \| \sum_{i=1}^{n} \theta_{x_i,x_i}^* \| = \lambda \| (E(x_i^*x_i))_{i,j} \| \geq 0$$

$$\| \sum_{i=1}^{n} x_i x_i^* \| \geq \| q \|^{-1} \| \sum_{i=1}^{n} B(x_i|x_i) \|,$$

therefore $X$ is of finite right numerical index and also of finite index since $X_A = B$ is finitely generated over $A$. On the other hand, since $\theta_{x,x}^*(z) = zqx^*x$,

$$\| \sum_{i=1}^{n} \theta_{x_i,x_i}^* \| = \| q^{1/2} \sum_{i=1}^{n} x_i x_i^* q^{1/2} \| \geq 0$$

$$\| q^{-1/2} \| \sum_{i=1}^{n} x_i x_i^* \| \geq \| q^{-1} \|^{-1} \| \sum_{i=1}^{n} (x_i|x_i)_A \|,$$

which shows that $X$ is of finite left numerical index, and therefore of finite index since $BX$ is singly generated over $B$. By the previous corollary $B\mathcal{L}(X_A) = \mathcal{L}_A(BX)$ and the latter coincides with $A' \cap B$ acting on $X$ by right multiplication.
Therefore for any \( T \in A' \cap B \), the map \( x \in B \to xT \) is adjointable with respect to the right inner product. The \( \ast \)-involution of \( L_A(BX) \) (denoted by \( T \to \ast T \)) is defined by the equation
\[
B(x)^\ast T(y) = B(T(x))y = xTqy^\ast = xq(yqT^\ast q^{-1})^\ast,
\]
for \( T \in A' \cap B \). Therefore \( \ast T = qT^\ast q^{-1} \) where \( T \to T^\ast \) is the \( \ast \)-involution of \( B \). On the other hand the \( \tilde{\ast} \)-involution of \( B \mathcal{L}(X_A) \) is defined by
\[
(x|\tilde{T}y)_A = (T(x)|y)_A = (xT|y)_A = E(T^\ast x^\ast y).
\]

Since \( \tilde{T} \) acts as right multiplication by an element of \( A' \cap B \), it is determined by the equation
\[
E(b\tilde{T}) = E(T^\ast b), b \in B,
\]
which shows that \( \tilde{T} = \sum y_i E(T^\ast y_i^\ast) \), where \( \{y_i\} \) is a finite quasi–basis of \( E \).
Now if \( E \) was chosen to satisfy the equation
\[
E(bT) = E(Tb), b \in B, T \in A' \cap B,
\]
the \( \tilde{\ast} \)-involution (coming from the right inner product) and the original involution on \( A' \cap B \) coincide. This is possible if, e.g., \( Z(A) \) is finite dimensional. In fact, in this case \( A' \cap B \) is finite dimensional as well, and therefore \( E'(b) = E(\int_G ubu^\ast du) \), with \( G \) the unitary group of \( A' \cap B \), is still a conditional expectation from \( B \) onto \( A \) satisfying the required equation. However, the involution on \( A' \cap B \) coming from the left inner product differs from the original involution if \( q \) is not central in \( A' \cap B \).

4.5 Computing the left index element of \( X \)

4.7 Lemma Let \( X = A X_B \) and \( Y = B Y_A \) be nondegenerate right Hilbert \( C^* \)-bimodules, conjugate of each other as objects of \( A \mathcal{H}^w_A \), and let \( (R, \overline{R}) \) be a solution of the corresponding conjugate equations. Let us regard \( \mathcal{K}(X_B) \) as a \( C^* \)-subalgebra of the intertwiner space \( (X, X) \simeq \mathcal{K}(X_B)' \) of \( \mathcal{H}^w_A \). Then the map
\[
T \in \mathcal{K}(X_B) \to (\overline{R} \circ T \otimes I_Y \circ R) \otimes I_X \otimes R^* R - T \in \mathcal{K}(X_B)
\]
is completely positive.

Proof Let us take the adjoint of the first conjugate equation:
\[
I_X \otimes R^* \circ \overline{R} \otimes I_X = I_X,
\]
thus for all \( n \in \mathbb{N} \) and any positive \( T = (T_{ij}) \in M_n(\mathcal{K}(X_B)) \),
\[
T = (\overline{R} \otimes I_X \circ I_X \otimes RT_{ij} I_X \otimes R^* \circ \overline{R} \otimes I_X)_{i,j} = (\overline{R} \otimes I_X (T_{ij} \otimes R^* R) \overline{R} \otimes I_X)_{i,j} \leq ((\overline{R} \circ T_{ij} \otimes I_Y \circ R) \otimes I_X \otimes (R^* R))_{i,j}
\]
since \( RR^* \leq I_Y \otimes X \otimes (R^* R) \) by Lemma 2.7 in [LR].
Remark Choosing $T = I_X$, we obtain, in particular, $\dim_{R_r} X \geq 1$.

Combining the previous lemma with the main theorem of [FK], yields the following result.

4.8 Theorem Let $A \otimes_B \otimes$ be a bi–Hilbertian bimodule of finite right numerical index, and let $F : \mathcal{K}(X_B) \to A$ be the positive $A$–$A$ bimodule map constructed in Cor. 2.11. Then $X$ is also of finite left numerical index. Denoting by $\phi$ and $\psi$ the left and right actions of $A$ and $B$ on $X$ respectively, and by $q$ the support projection of the left index element in $B''$, $\ell - \text{Ind}[X]$ is the smallest central element $c$ of $qB''$ for which the map $\psi_0(c) \phi - \text{id} : \mathcal{K}(X_B) \to \mathcal{K}(X_B)^{\prime\prime}$ is completely positive. Here $\psi_0$ denotes the extension to $Z(B'')$ of the right action of $Z(B)$ on $X$ defined in Lemma 2.18.

Proof We claim that $X$ is of finite left numerical index if and only if there exists a positive real $c$ for which $c\phi \circ F - \text{id} : \mathcal{K}(X_B) \to \mathcal{L}(X_B)$ is completely positive. We show the claim. If $X_B$ has finite left numerical index, we can construct a solution $R$, $\mathcal{H}_R$, $\mathcal{X}$ to the conjugate equations as in the proof of Theorem 4.4. We have proved there that $R^* R = \ell - \text{Ind}[X]$ and that for $T \in \mathcal{K}(X_B)$, $\mathcal{R}(T \otimes I_R) R = F(T)$. So, recalling the definition of tensor products between operators in $\mathcal{A} \mathcal{H}_R$, with $\mathcal{A} = \{ A, B \}$, we see that

$$I_X \otimes R^* R = \psi_0(\ell - \text{Ind}[X])$$

and

$$(R^* (T \otimes I_R) R) \otimes I_X = \phi \circ F(T), \quad T \in \mathcal{K}(X_B).$$

Inserting these data in the conclusion of Lemma 4.7, we deduce that $\psi_0(\ell - \text{Ind}[X]) \phi \circ F - \text{id}$ is completely positive, as a map from $\mathcal{K}(X_B)$ to $\mathcal{K}(X_B)^{\prime\prime}$. Therefore, with $c = \| \ell - \text{Ind}[X] \|$, $c\phi \circ F - \text{id} : \mathcal{K}(X_B) \to \mathcal{L}(X_B)$ is completely positive. Conversely, if for some positive real $c$, $c\phi \circ F - \text{id}$ is completely positive on $\mathcal{K}(X_B)$, for $n \in \mathbb{N}$ and for $x_1, \ldots, x_n \in X$,

$$\| \sum_{i=1}^n (x_i | x_i)_B \| = \| (\theta_{x_i, x_j})_{ij} \| \leq c\| (\phi(A(x_i | x_j)))_{ij} \| = c\| \sum_{i=1}^n \theta_{x_i, x_i} \|,$$

so $X$ is of finite left numerical index. On the other hand in Prop. 2.19 we have constructed a surjective conditional expectation $\phi'' : \mathcal{K}(X_B)^{\prime\prime} \rightarrow \mathcal{K}(A'' \otimes B''$, which does satisfy $\mu \| \phi'' E''(T) \| \geq \| T \|$ for some positive real $\mu$ and all $T \in \mathcal{K}(X_B)^{\prime\prime}$. By the main result of [FK], $c\phi'' E'' - \text{id}$ is completely positive for some positive real $c$, and therefore $c\| \phi'' E''(r - \text{Ind}[X])^{-1} \| \phi'' F'' - \text{id} : \mathcal{K}(X_B)^{\prime\prime} \rightarrow \mathcal{K}(X_B)^{\prime\prime}$ is completely positive. Restricting this map to $\mathcal{K}(X_B)$ and combining with the claim, shows that $X$ is of finite left numerical index.

Let now $\nu \to \nu_0$ be a generalized left basis of $X$. Choosing $n = |\nu|$, $T = (\theta_{z_i, z_j}) \in M_n(\mathcal{K}(X))^+$, we see that, if $c$ is any central element of $qB''$ for which $T \in \mathcal{K}(X) \rightarrow \psi_0(c) \phi F(T) - T \in \mathcal{K}(X)^{\prime\prime}$ is completely positive then

$$(\psi_0(c) \phi(A(z_i | z_j))_{i,j} = (\psi_0(c) \phi F(\theta_{z_i, z_j}))_{i,j} \geq (\theta_{z_i, z_j})_{i,j},$$

which implies

$$\sum_{i,j} (z_i | A(z_i | z_j) z_j)_B \geq \sum_{i,j} (z_i | z_i(z_j | z_j)_B)_B,$$
or, in other words, \( R^*_u R_v c \geq (\sum_{z \in v} (z|z)_B)^2 \). Thus \( (\ell - \text{Ind}[X])c \geq (\ell - \text{Ind}[X])^2 \), so \( c \geq \ell - \text{Ind}[X] \).

4.9 Corollary Let \( A \subset B \) be an inclusion of \( C^* \)-algebras, and \( E : B \to A \) be a conditional expectation with range \( A \), for which there is \( \lambda > 0 \) such that \( \|E(bb^*)\| \geq \lambda \|bb^*\| \) for all \( b \in B \). Let \( \text{Ind}[E] \) be the index of \( E \) defined as in Def. 2.17. Then \( \text{Ind}[E] \) is the smallest central element \( c \) of \( B'' \) for which \( cE - \text{id} \) is completely positive.

Proof Let \( A_X B \) be the contragradient of the \( B \)-\( A \) bimodule \( X \) associated to \( E \) as part (2) of Prop. 2.12. Clearly \( X \) is of finite numerical index. Since \( \ell - \text{Ind}[X] = r - \text{Ind}[X] = \text{Ind}[E] \), and since \( K(X_B) = B \) and \( F_X = E \), \( \text{Ind}[E] \) is, by Cor. 4.9, the smallest central element \( c \) of \( B'' \) for which \( cE - \text{id} \) is completely positive (recall that \( r - \text{Ind}[X] \) is invertible by Cor. 2.20).

Remark If \( \phi E : K(X_B) \to \phi(A) \) is the faithful conditional expectation defined in Cor. 2.30 then

\[ \text{Ind}[X] \phi \circ E - \text{id} \]

is completely positive on \( K(X_B) \). In fortunate cases where \( \phi''(r - \text{Ind}[X]) \) is central in \( K(X_B)'' \) (e.g. either \( A \) is simple, cf. Cor. 2.26, or \( X \) arises from a conditional expectation, Prop. 2.12, or \( A = B \) is commutative and right action coincides with left action) then \( \text{Ind}[X] = \text{Ind}[\phi \circ E] \). This observation thus shows that the index element of a conditional expectation coincides with the index element of the dual conditional expectation.

4.6 From conjugation to finite index

Let \( X \) be a bi-Hilbertian bimodule with a conjugate in \( \mathcal{A} \mathcal{H}_A \). Our next aim is to construct a left inner product on \( X \) making it into a bi-Hilbertian bimodule of finite index.

4.10 Lemma Let \( Y_A \) be a conjugate object of \( A_X B \) in the tensor \( 2 \)-\( C^* \)-category \( \mathcal{A} \mathcal{H}_A \), and let \( R \) and \( \overline{R} \) be a pair of intertwiners solving the conjugate equations, in the sense of Def. 4.3. There exist unique positive semidefinite left inner products on \( X \) and \( Y \) such that

\[ A(x|a^* x') = \overline{R}(\theta^e_{x,x'} \otimes I_Y) \overline{R}(a) \in A \quad \text{for} \quad a \in A, x, x' \in X, \]

\[ b(y|b^* y') = R^*(\theta^e_{y,y'} \otimes I_X) R(b) \in B \quad \text{for} \quad b \in B, y, y' \in Y. \]

Proof For \( x, x' \in X \) we define an element \( A(x|x') \in M(A) = \mathcal{L}(\tau_A) \) by

\[ A(x|x')(a) = \overline{R}(\theta^e_{x,x'} \otimes I_Y) \overline{R}(a) \in A \quad \text{for} \quad a \in A. \]

Then \( x, x' \mapsto A(x|x') \) defines a continuous sesquilinear form on \( X \) with values in \( M(A) \). We claim that \( A(x|x') \in A \). Let \( \{u_\lambda\}_\lambda \) be a selfadjoint approximate unit of \( A \) with \( \|u_\lambda\| \leq 1 \). We show that \( \{\overline{R}(\theta^e_{x,x'} \otimes I_Y) \overline{R}(u_\lambda)\}_\lambda \) is a norm Cauchy net. First we assume that \( x' \) is of the form \( y = a^* x'' \) for some \( a \in A \) and \( x'' \in X \). Since

\[ \overline{R}(\theta^e_{x,a^* x''} \otimes I_Y) \overline{R}(u_\lambda) = \overline{R}(\theta^e_{x,x''} \otimes I_Y) \overline{R}(au_\lambda), \]
and \( au_i \to a \) in norm, \( \{ \overline{R} (\theta^r_{x,a} \otimes \iota Y) \overline{R}(u_i) \}_i \) is a Cauchy net in norm. For a general element \( x' \in X \), we choose \( \tilde{x} \in AX \) sufficiently close to \( x' \) (this being possible as left \( A \)-action is nondegenerate), so

\[
\| \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_i) - \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_j) \| \\
\leq \| \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_i) - \overline{R} (\theta^r_{x,\tilde{x}} \otimes I) \overline{R}(u_i) \| \\
+ \| \overline{R} (\theta^r_{x,\tilde{x}} \otimes \iota Y) \overline{R}(u_i) - \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_j) \| \\
+ \| \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_j) - \overline{R} (\theta^r_{x,\tilde{x}} \otimes \iota Y) \overline{R}(u_j) \| \\
\leq \| \overline{R} \| \| x' - \tilde{x} \| + \| \overline{R} \| \| x' - \tilde{x} \| \\
+ \| \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_i) - \overline{R} (\theta^r_{x,\tilde{x}} \otimes \iota Y) \overline{R}(u_j) \|.
\]

Thus \( \{ \overline{R} (\theta^r_{x,x'} \otimes I) \overline{R}(u_i) \}_i \) is still a Cauchy net in \( A \).

For \( a \in A \), we have

\[
\overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_i) a = \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_i a) \to \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(a)
\]

in norm. This shows that the limit of the Cauchy net

\[
\{ \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_i) \}_i
\]

in \( A \) coincides with \( A(x|x') \in A \) and does not depend on the choice of approximate unit \( \{ u_i \} \).

It is easy to see that \( (x,x') \mapsto A(x|x') \) is left \( A \)-linear and right conjugate \( A \)-linear. Since for \( b, c \in A \) and \( x, x' \in X \),

\[
((A(x|x'))^* b)^* c = b^* \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(c)
\]

\[
= b^* \lim_{i \to \infty} \overline{R} (\theta^r_{x,x'} \otimes \iota Y) \overline{R}(u_i) c = (A(x'|x)b)^* c,
\]

we have \( A(x|x')^* = A(x'|x) \). Since

\[
(A(x|x)(a) | a)_A = (\overline{R} (\theta^r_{x,x} \otimes \iota \overline{R}(a)) | a)_A = ((\theta^r_{x,x} \otimes \iota \overline{R}(a)) | \overline{R}(a))_A \geq 0
\]

in the canonical Hilbert \( A \)-module \( \iota A = A_A \) with \( (a|b)_A = a^* b \), we have \( A(x|x) \geq 0 \). Now exchanging the roles of \( X \) and \( Y \), and of \( R \) and \( \overline{R} \), and applying this argument to \( Y \), we deduce the existence of a left inner product on \( Y \) as well.

We next show that \( X \) and \( Y \) acquire a structure of left Hilbert modules. To do so, we construct isomorphisms with the contragradient left Hilbert bimodules \( X \) and \( \overline{X} \) respectively. For a \( A-B \) bimodule \( X \), we shall denote by \( J_X : X \to \overline{X} \) the map associating \( \overline{x} \) to \( x \), for any \( x \in X \). Clearly, \( J_X(ax) = J_X(x)a^* \) and \( J_X(xb) = b^* J_X(x) \) for \( a \in A, b \in B, x \in X \).

4.11 Lemma Let \( X \) and \( Y \) be conjugate objects of \( A \mathcal{H}_A \), and let us endow them with left inner products defined, as in the previous lemma, by a pair of intertwiners \( R \) and \( \overline{R} \) solving the conjugate equations. Then there exist natural
bimodule isomorphisms $U : \overline{Y} \to X$ and $V : \overline{X} \to Y$ from the contragradient bimodules, which preserve the corresponding left and right inner products and satisfy
\[ VJ_X = (UJ_Y)^{-1}. \]
In particular, $X$ and $Y$ become bi-Hilbertian $C^*$-bimodules.

Proof For $y \in Y$ we define \( l_y : X \to Y \otimes_A X \) by \( l_y(x) = y \otimes x \). Then we have \( l_y(y' \otimes x) = (y|y')_A x \). We notice that the set \( \{a^* x b | a \in A, x \in X, b \in B\} \) is total in $X$ since, by assumption, left action is nondegenerate.

By the first conjugate equation
\[ a^* x b = (\overline{R} \otimes I_X)(I_X \otimes R)(a^* x b) = (\overline{R} \otimes I_X)(a^* x \otimes R(b)). \]
For \( a \in A, x' \in X \)
\[
\begin{align*}
((\overline{R} \otimes I_X)(a^* x \otimes R(b))|\tilde{a} x')_B &= (a^* x \otimes R(b)|\overline{R}(\tilde{a}) \otimes x')_B \\
&= (x \otimes R(b)|\overline{R}(\tilde{a}) \otimes \tilde{a} x')_B \\
&= (R(b)|\overline{I}_x^* (\overline{R}(a)) \otimes \tilde{a} x')_B \\
&= (l_x^* (\overline{R}(a))|^* (R(b)))(\tilde{a} x')_B.
\end{align*}
\]
Thus for $a \in A$, $b \in B$ and \( x \in X \),
\[ a^* x b = l_x^* (\overline{R}(a))|^* (R(b)) \in X. \]
This shows that \( \{l_x^* (R(b))|y \in Y, b \in B\} \) is total in $X$. Similarly, for \( a \in A, b \in B \) and \( y \in Y \),
\[ b^* y a = l_y^* (R(b))^* (\overline{R}(a)) \in Y. \]
We next show that for \( y, y' \in Y, b, b' \in B \)
\[ A(l_y^* R(b')|l_y^* R(b)) = A(\overline{R}^{b'}|\overline{R}^b), \]
where the left hand side is computed with respect to the new inner product on $X$ introduced in Lemma 4.10 and the right hand side with respect to the left inner product on $Y$ defined in the paragraph following Def. 2.8. We start from the right hand side. For \( a, a' \in A \),
\[ A(l_y^{a^*} R(b')|l_y^{a^*} R(b)) = (b^{y'} a'^* | b^* y a^*)_A = (l_{y'}^{a'^*} R(b')|^* \overline{R}(a'^*))_A \\
= (l_{y'}^{a'^*} R(b')|^* R(b') \otimes I_Y)(\overline{R}(a'^*))_A \\
= (\overline{R}(\theta_{y'} | R(b)) | l_y^{a'^*} R(b'))_A = (A(l_y^{a'^*} R(b')|l_y^{a'^*} R(b'))|a'^*)_A \\
= (l_y^{a'^*} R(b')|l_y^{a'^*} R(b'))|a'^* = a'^* A(l_y^{a'} R(b')|l_y^{a'} R(b)) \overline{a'^*}. \]
Therefore \( U : \overline{Y} \to l_y^{a^*} R(b) \in X \) is well defined and extends to a left $A$-linear map from $Y$ to $X$ preserving left $A$-valued inner product. Since the right $A$-valued inner product of $Y$ is definite, the left $A$-valued inner product on $X$ constructed in Lemma 4.10 is also definite. Clearly this map is also right
$B$-linear. Since $\mathbb{Y}$ is a left Hilbert bimodule, so is $X$ with respect to its left inner product. Since this left inner product is continuous with respect to the right one, $X$ is bi-Hilbertian by general Banach space theory.

Similarly, $V : \mathbb{Y} \to X \ni l_x \to R(a) \in Y$ extends to a $B$–$A$ linear map preserving the left inner product from $X$ to $Y$ and making $Y$ into a left Hilbert bimodule. Now $UJ_Y$ takes $b^*y$ to $l_y R(b)$ and $VJ_X$ takes $a^*x$ to $l_x R(a)$. Therefore $UJ_Y V J_X$ takes $a^*xb$ to

$$UJ_Y(l_{xb}^* R(a)) = UJ_Y(b^* l_x^* R(a)) = l_{x^* R(a)}^* R(b)$$

which we have already shown to coincide with $a^*xb$. One similarly shows that $VJ_X UJ_Y = I_Y$. Since $U$ preserves the left inner products, $J_X U J_Y = V^{-1}$, and therefore $V$, preserves the right inner products. For the same reason, $U$ preserves the right inner products as well.

4.12 Lemma Let $AX_B$ and $BY_A$ be right Hilbert $C^*$–bimodules with nondegenerate left actions, conjugate of each other in $\mathcal{A} \mathcal{H}_A$. Consider $X = AX$ as a left Hilbert $C^*$–bimodule with left inner product defined by a solution $(R, \mathbb{R})$ of the conjugate equations as in Lemma 4.10 (cf. Lemma 4.11). Then the range of the right $B$–action on $X$ is contained in $\mathcal{K}(AX)$. Similarly, regarding $X = XB$ as a right Hilbert $C^*$–bimodule with its original right inner product, the range of the left $A$–action is included in $\mathcal{K}(XB)$.

Proof Each element of the form $R(b)$ can be approximated, in the norm induced by the right inner product of $Y \otimes A X$, by finite sums $\{(\sum_i y_i \otimes x_i)\}$. In turn, as seen in the course of the proof of Lemma 4.11, each element of $Y$ can be approximated by finite sums $\{(\sum_l l_x \otimes R(a_l))\}$. Thus for any $b \in B$ and any positive integer $n$, there exist $x_k^{(n)}$, $y_k^{(n)} \in X$ and $a_k^{(n)} \in A$ for $k = 1, \ldots, N_n$ such that

$$r_n := \sum_{k=1}^{N_n} l_{x_k^{(n)}} \otimes y_k^{(n)} \to R(b) \quad \text{as } n \to \infty.$$ 

By the first conjugate equation, for any $x \in X$

$$xb = (\mathbb{R}^* \otimes I_X)(I_X \otimes R)(xb) = (\mathbb{R}^* \otimes I_X)(x \otimes R(b)).$$

Define $Q(x) := xb \in X$ and $Q_n(x) := (\mathbb{R}^* \otimes I_X)(x \otimes r_n) \in X$. We claim that $Q_n$ is a finite rank operator on the left Hilbert module $AX$. We show the claim.

$$Q_n(x) = (\mathbb{R}^* \otimes I_X)(x \otimes r_n) = \sum_{k=1}^{N_n} \mathbb{R}^*(x \otimes l_{x_k^{(n)}} \otimes R(a_k^{(n)})) \otimes y_k^{(n)}$$

$$= \sum_{k=1}^{N_n} \mathbb{R}^* \left( (\mathbb{R}^* \otimes I_Y) R(a_k^{(n)}) \right) \otimes y_k^{(n)} = \sum_{k=1}^{N_n} A(x | x_k^{(n)}) a_k^{(n)} y_k^{(n)}$$

$$= \sum_{k=1}^{N_n} \theta l_{x_k^{(n)}} y_k^{(n)} x_k^{(n)}(x).$$
Using the norm on $X$ induced by the right inner product:

$$\|Q(x) - Q_n(x)\| \leq \|R^* \otimes I_X\| \|x\| \|R(b) - r_n\|.$$  

On the other hand, the norm on $X$ coming from the right inner product is equivalent to the norm coming from the left inner product, therefore $Q$ is the norm limit of $\{Q_n\}$ in $L(A X)$, and this shows that right action of $B$ on $X$ lies in $K(A X)$.

Replacing now $Y$ with $X$, we deduce that the right action of $A$ on $Y$ is compact with respect to the left inner product of $Y$. But by Lemma 4.11, the left Hilbert $C^*$–bimodule $Y$ identifies, through the map $V$, with the contragradient $X$ of the original right Hilbert $C^*$–bimodule $X$, therefore the left action of $A$ on $X$ is compact with respect to the original right inner product of $X$ itself.

4.13 Lemma Let $X$ be an object of $\mathcal{A}H_{\mathcal{A}}$ with a conjugate object $Y$ in $\mathcal{A}H_{\mathcal{A}}$, and let us make $X$ and $Y$ into bi-Hilbertian $C^*$–bimodules with left inner products defined, as in Lemma 4.10, by a solution $(R, \overline{R})$ of the conjugate equations. Let us identify $Y$, as a bi-Hilbertian $C^*$–bimodule, with $X$ via the biunitary map $V$: $X \rightarrow Y$ defined in Lemma 4.11. Then for any $x, x' \in X$,

$$R^*(x \otimes V x') = A(x|x')$$ and $$R^*(V x \otimes x') = (x|x')_B.$$  

Proof We shall show that

$$R^*(x \otimes V x') = A(x|x')$$ and $$R^*(V x \otimes x') = (x|x')_B.$$  

The first equation follows from

$$R^*(x \otimes V(x') = R^*(x \otimes l_x^* \overline{R}(a)) = R^*(\theta_{x,x'}^* \otimes I_Y) \overline{R}(a) = A(x|x')a.$$  

Similarly, we have

$$R^*(y \otimes U y') = B(y|y'),$$

where the operator $U$ is still defined in Lemma 4.11. Now writing $y = V x$ and $y' = V x'$, and using the relation $U J_Y V J_X = I_X$ obtained in Lemma 4.11, gives

$$R^* V x \otimes x' = B(V x|V x') = (x|x')_B.$$  

We are now ready to prove a converse of part (2) of Theorem 4.4.

4.14 Theorem Let $X$ be a right Hilbert $A$–$B C^*$–bimodule with a nondegenerate left action. If $X$ has a conjugate object in the $2$–$C^*$–category $\mathcal{A}H_{\mathcal{A}}$ of nondegenerate right Hilbert bimodules, it can be given a left $A$-valued inner product making it into a finite index bi-Hilbertian $C^*$–bimodule. More precisely, any solution $(Y, R, \overline{R})$ to the conjugate equations in $\mathcal{A}H_{\mathcal{A}}$ induces a left inner product defining a finite index bi-Hilbertian structure on $X$ by

$$A(x|ax') = R^* \theta_{x,x'}^* \otimes I_Y \overline{R}(a^*), \quad x, x' \in X, a \in A.$$  

(4.1)
It turns out that $Y$ is biunitarily equivalent, as a bi-Hilbertian bimodule, to $X$ and, under this identification, the intertwiners $R$ and $\overline{R}$ are defined by

$$\overline{R} x \otimes x = A(x|x'), \quad R^* x \otimes x' = (x|x')_B. \quad (4.2)$$

Proof Suppose that $X$ has a conjugate $Y$ defined by intertwiners $R$ and $\overline{R}$. So far we have proved that a solution $R$, $\overline{R}$ of the conjugate equations induces a bi-Hilbertian structure on $X$ (Lemma 4.11) in such a way that the left and right actions have range into the corresponding compact operators (Lemma 4.12). Also, we have been able to identify $Y$ biunitarily with $X$ (via the map $V$ defined in Lemma 4.11) with $R$ and $\overline{R}$ acting as in Lemma 4.13. Since $\theta^r_{x,x}$ is positive, we have

$$\|R\|^2 \theta^r_{x,x} \leq \overline{R} (\theta^r_{x,x} \otimes I) \overline{R} = A(x|x),$$

by Lemma 4.7. Therefore for any $x_1, \ldots, x_n \in X$,

$$\| \sum_{i=1}^n \theta^r_{x_i,x_i} \| \leq \|R\|^2 \| \sum_{i=1}^n A(x_i|x_i) \|.$$

On the other hand, since $\overline{R}$ is bounded, for any $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ we have

$$\| \sum_{i=1}^n A(x_i|y_i) \| = \| \overline{R} (\sum_{i=1}^n \theta^r_{x_i,y_i} \otimes I) \overline{R} \| \leq \|\overline{R}\|^2 \| \sum_{i=1}^n \theta^r_{x_i,y_i} \|.$$

Therefore, taking into account Theorem 2.22, all the assumptions of Definition 2.23 are satisfied, and this shows that $X$ is of finite right index. Similarly, $Y = \overline{X}$ is of finite right index, i.e. $X$ is of finite left index as well, and therefore of finite index.

The arguments of the proof show that the minimal dimension of a bimodule is the infimum of the square roots of the numerical indices.

4.15 Corollary

$$\dim X = \inf (r - [X])^{1/2} (\ell - I[X])^{1/2} = \inf I[X]^{1/2},$$

where the infimum is taken over all possible left inner products on the right Hilbert bimodule $X$ making it into a finite index bi-Hilbertian $C^*$–bimodule.

4.7 A characterization of strong Morita equivalences

We next characterize strong Morita equivalences among general right Hilbert $C^*$–bimodules as those objects with minimal dimension (or numerical index) equal to 1.

4.16 Corollary For a right Hilbert $C^*$–bimodule $AX_B$ the following properties are equivalent.

(1) $X_B$ is full and it can be given a full left inner product making it into finite index bi-Hilbertian bimodule with respect to which $I[X] = 1$, ...
(2) $X$ is an object of the category of nondegenerate full right Hilbert $C^*$-bimodules with a conjugate such that $\dim X = 1$.

(3) $X$ can be given a left inner product making it into finite index Hilbert bimodule with $r - \text{Ind}[X] = I_A$ and $\ell - \text{Ind}[X] = I_B$.

(4) $X$ can be given a left inner product making it into a strong Morita equivalence bimodule from $A$ to $B$.

Proof (1) ⇒ (2): This implication follows from the previous theorem. (2) ⇒ (4): Let $R$ and $\overline{R}$ satisfy the conjugate equations with $\parallel R\parallel \parallel \overline{R} \parallel < \sqrt{2}$. Since $X$ and its conjugate are full, the left and right indices of $X$ must be invertible by Cor. 2.29, and therefore so are $R^*R$ and $\overline{R}^*\overline{R}$. The operators $S := R(R^*R)^{-1/2}$ and $\overline{S} := \overline{R}^*(\overline{R}^*\overline{R})^{-1/2}$ are isometries whose ranges generate, as in [LR], projections satisfying the Jones relations with parameter $\beta$, where $\beta^{-1} = (\parallel R\parallel \parallel \overline{R} \parallel)^2 < 2$, thus $\beta = 1$ by Jones fundamental result [J]. This shows that the numerical index $I[X]$ of $X$ with respect to the original right inner product and the left inner product induced by this pair, is 1. Let $\phi E : \mathcal{K}(X_B) \rightarrow \phi(A)$ denote the conditional expectation defined in Cor. 2.30. Since, by Cor. 4.9, $I[X]\phi E(T) \geq T$ for any positive $T$ in $\mathcal{K}(X_B)$, and since $\phi E(\phi E(T) - T) = 0$, $\phi E$, being faithful, must be the identity map. Defining a new left inner product on $X$ by:

$$A(x|y)' := (\overline{R}^* \overline{R})^{-1} A(x|y) = \theta^{r}_{x,y},$$

makes $X$ into a strong Morita equivalence bimodule.

(4) ⇒ (3): It is easy to show that a strong Morita equivalence bimodule is full as a left as well as a right Hilbert module and has index 1. In fact, in this case $A = \mathcal{K}(X_B)$ and one has a bi-Hilbertian structure given by $A(x|x') = \theta^{r}_{x,x'}$, for $x, x' \in X$. Let $\{u_{\mu}\}_{\mu}$ be a generalized right basis for $X$. Since $A(x|x') = \theta^{r}_{x,x'}$, for $x, x' \in X$, we have

$$r - \text{Ind}[X] = \lim_{\mu} \sum_{y \in u_\mu} A(y|y) = \lim_{\mu} \sum_{y \in u_\mu} \theta^{r}_{y,y} = I_A.$$  

One similarly shows that $\ell - \text{Ind}[X] = I_B$.

(3) ⇒ (1): This implication is obvious.

5. Tensoring finite index bimodules

Let $A$, $B$ and $C$ be $\sigma$ unital $C^*$-algebras, $X$ a right Hilbert $A$–$B$ bimodule and $Y$ a right Hilbert $B$–$C$ bimodule. If $X$ and $Y$ are of finite index, is $X \otimes_{B} Y$ still of finite index? It does not seem to be easy to prove this property directly. However, one can obtain a proof from our characterization Theorems 4.4 and 4.14. We anticipate the following well known lemma.

5.1 Lemma Let $Y = _B Y_C$, $Y' = _B Y'_C$ be right Hilbert $C^*$-bimodules and $F \in _B \mathcal{L}(Y_C, Y'_C)$ be a $B$–$C$ bimodule homomorphism with adjoint. Then

(1) for any right Hilbert $A$–$B$ bimodule $X$, $\|I_X \otimes F\| \leq \|F\|$, and the equality holds if $X_B$ is full and the left action of $B$ on $Y$ is nondegenerate,

(2) for any right Hilbert $C$–$A$ bimodule $X$, $\|F \otimes I_X\| \leq \|F\|$ and the equality holds if the left $A$–action is faithful.
Proof (1) The operator \( I_X \otimes F \) is clearly well defined on the linear span of simple vectors \( x \otimes y, \ x \in X, \ y \in Y \), which is dense in \( X \otimes_B Y_C \). One can easily check that the norm of \( I_X \otimes F \) on that subspace is bounded above by \( \|F\| \), therefore \( I_X \otimes F \) extends to a bounded operator with the same norm on the completion \( X \otimes_B Y_C \) with adjoint \( I_X \otimes F^* \). It is also obvious that \( I_X \otimes F \) is a bimodule map. For the rest it suffices to assume \( Y = Y' \). The map \( F \in_B \mathcal{L}(Y_C) \rightarrow I_X \otimes F \in_A \mathcal{L}(X \otimes_B Y_C) \) is a \( * \)-homomorphism. \( I_X \otimes F = 0 \) implies
\[
(x \otimes F y | x \otimes F y)_C = (F y, (x | b) F y)_C = 0, \ x \in X, \ y \in Y,
\]
therefore \( F((x | b) y) = 0 \) for all \( x \in X, \ y \in Y \), which implies \( F = 0 \) since the right inner product of \( X \) is full and the left \( B \)-action on \( Y \) is nondegenerate. Therefore \( \|I_X \otimes F\| = \|F\| \). (2) The fact that \( F \otimes I_X \) is a well defined map on \( Y \otimes X \) with norm bounded above by \( \|F\| \) can be proved with arguments similar to those used above. Now \( F \otimes I_X = 0 \) implies \( (x | (F y) F y)_A = 0, \ x \in X, \ y \in Y \), so the left action of \( C \) on \( X \) evaluated on \( (F y) F y)_C \) vanish for all \( y \in Y \). If this action is faithful, \( F y = 0, \ y \in Y \) and therefore \( F = 0 \). This implies that \( \|F \otimes I_X\| = \|F\| \).

We have shown in Prop. 2.13 that if \( A X_B \) is of finite right numerical index and \( B Y_C \) is of finite left numerical index, the seminorms of \( X \otimes_B Y \) arising from the left and right inner products are equivalent, therefore we can form a unique bi-Hilbertian bimodule, \( X \otimes_B Y \) completing in any of these seminorms. We now show that this bimodule is of finite index if so are \( X \) and \( Y \).

5.2 Theorem Let \( A, B \) and \( C \) be \( C^* \)-algebras, and \( X = A X_B \) and \( Y = B Y_C \) be bi-Hilbertian \( C^* \)-bimodules. If \( A X_B \) and \( B Y_C \) have finite index (respectively, finite numerical index), then also \( X \otimes_B Y \) has finite index (respectively, finite numerical index) with respect to the bi-Hilbertian structure defined in Subsect. 2.2.

Proof Since \( X \) and \( Y \) are bi-Hilbertian and of finite numerical index \( X \otimes_B Y \) is bi-Hilbertian by Prop. 2.13, and therefore left and right actions are nondegenerate by Prop. 2.16. Since \( X \) and \( Y \) have finite numerical index, the contragredient of the corresponding underlying left Hilbert modules are their respective conjugates, by Theorem 4.4. Namely, there are intertwiners in \( \mathcal{A} \mathcal{H}^w_A \), with \( A = \{A, B, \}, R_1 \in (\iota A, X \otimes_B \overline{X}) , R_1 \in (\iota B, \overline{X} \otimes_A \overline{X}) \), \( R_2 \in (\iota B, Y \otimes C \overline{Y}) \), \( R_2 \in (\iota C, \overline{Y} \otimes_B \overline{Y}) \) solving the corresponding conjugate equations. We show that \( \overline{Y} \otimes_B \overline{X} \) is a conjugate of \( X \otimes_B Y \) in \( \mathcal{A} \mathcal{H}^w_A \). We define a map \( i(R_1) \) from \( \overline{Y} \otimes Y \simeq \overline{Y} \otimes \iota C \otimes Y \) to \( \overline{Y} \otimes \overline{X} \otimes Y \), by \( I_{\overline{Y}} \otimes R_1 \otimes I_Y \), and a map \( j(R_2) \) from \( X \otimes_B \overline{X} \simeq X \otimes \overline{X} \) to \( X \otimes Y \otimes \overline{Y} \otimes \overline{X} \) by \( I_X \otimes R_2 \otimes I_{\overline{Y}} \).

We also define a \( C \)-\( C \) bimodule homomorphism \( R \in \mathcal{C} \mathcal{L}(\iota C, (\overline{Y} \otimes \overline{X} \otimes X \otimes Y)) \) by \( R = i(R_1) \circ R_2 \), and \( A \rightarrow A \) bimodule homomorphism \( \overline{R} \in \mathcal{A} \mathcal{L}(\iota A, X \otimes Y \otimes \overline{Y} \otimes \overline{X}) \) by \( \overline{R} = j(R_2) \circ R_1 \). Then we have
\[
\overline{R} \otimes I_X \otimes Y \circ I_X \otimes Y \circ R = I_X \otimes Y
\]
\[
R^* \otimes I_{\overline{Y}} \otimes \overline{X} \circ I_{\overline{Y}} \otimes \overline{X} \circ \overline{R} = I_{\overline{Y}} \otimes \overline{X}
\]
We check the first relation:
\[
\mathcal{R} \otimes I_{X \otimes Y} \circ I_{X \otimes Y} \otimes R
= \mathcal{R}_1 i(\mathcal{R}_2) \otimes I_{X \otimes Y} \circ I_{X \otimes Y} \otimes i(R_1)R_2
= \mathcal{R}_1 \otimes I_X \otimes I_Y \otimes I_Y \circ I_X \otimes \mathcal{R}_2 \otimes I_Y \circ I_X \otimes I_Y \otimes R_2
= ((\mathcal{R}_1 \otimes I_X \circ I_X \otimes R_1) \otimes I_Y)(I_X \otimes (\mathcal{R}_2 \otimes I_Y \circ I_Y \otimes R_2))
= I_{X \otimes Y}.
\]

To show the second equation we proceed in a similar way and we use the following computation: for \( x \otimes y \otimes y' \otimes b \overrightarrow{x'} \in X \otimes Y \otimes Y' \otimes X' \) we have
\[
(I_X \otimes \overline{R}_2 \otimes I_X \otimes I_Y)(I_X \otimes I_Y \otimes I_Y' \otimes R_1 \otimes I_Y)(x \otimes y \otimes y' \otimes b \overrightarrow{x'})
= x \otimes B(y|y')R_1(b) \otimes \overrightarrow{x'} = x \otimes R_1(B(y|y')b) \otimes \overrightarrow{x'}
= x \otimes R_1(B(y|y')) \otimes b \overrightarrow{x'} = (I_X \otimes R_1 \otimes I_Y)(I_X \otimes \overline{R}_2 \otimes I_Y)(x \otimes y \otimes y' \otimes b \overrightarrow{x'}).
\]
One can easily check that the following relations:
\[
A(z|z') := \mathcal{R}(\theta_{z,z'}^{r} \otimes 1_{Y \otimes X})\mathcal{R},
(z|z')_C := R^*(\theta_{z',z}^{r} \otimes 1_{X \otimes Y})R.
\]
Here \( z \in X \otimes Y \to z \in Y \otimes X \) is the map taking the simple tensor \( x \otimes y \) to \( y \otimes x \).
(This map is a well defined, \( A \)-\( C \) antilinear and bi-antisymmetric with respect to the corresponding bi-Hilbertian structures.) For \( z_1, \ldots, z_n \in X \otimes B Y \),
\[
\| \sum_1^n A(z_i|z_i) \| = \| \mathcal{R} (\sum_1^n \theta_{z_i,z_i}^{r} \otimes 1_{Y \otimes X})\mathcal{R} \| \leq \| \mathcal{R} \|^2 \| \sum_1^n \theta_{z_i,z_i}^{r} \|,
\]
therefore \( X \otimes Y \) has finite right numerical index. With a similar argument, \( X \otimes Y \) has finite left numerical index. If \( X \) and \( Y \) have finite index, \( R \) and \( \overline{R} \) are intertwining of the \( C^* \)-category \( A_{HA} \), by part (2) of theorem 4.4, so \( X \otimes B Y \) has finite index by Theorem 4.14.

6. Examples

In this section we discuss examples of Hilbert \( C^* \) bimodules of finite index with countable bases.

6.1 Finite index bimodules generating Cuntz–Krieger algebras

In the next example we construct a Hilbert \( C^* \)-bimodule of finite index which generates a countably generated Cuntz–Krieger algebra, see [KPRR] and [KPW2].

Let \( \Sigma \) be a countable set, and let \( G = (G(i,j))_{i,j \in \Sigma} \) be an infinite matrix with entries in \( \{0,1\} \). We shall assume that no row and no column of \( G \) is identically zero. We associate to the matrix \( G \) the directed graph \( \mathcal{G} = (\Sigma, E, s, r) \), where \( \Sigma \) is the set of vertices and \( E = \{ (i,j) \in \Sigma \times \Sigma | G(i,j) = 1 \} \) is the set of edges. For an edge \( \gamma = (i,j) \in E \), the source \( s(\gamma) \) is \( i \) and the range \( r(\gamma) \) is \( j \). We
assume that \( \mathcal{G} \) is locally finite, that is, for any \( j \in \Sigma \), \( \{ i \in \Sigma | G(i, j) = 1 \} \) and, for any \( i \in \Sigma \), \( \{ j \in \Sigma | G(i, j) = 1 \} \) are finite.

Let \( A = c_0(\Sigma) \) be the C\(^*\)-algebra of the functions on \( \Sigma \) vanishing at infinity and let \( A_0 = c_{00}(\Sigma) \) be the dense \(*\)-subalgebra of functions with finite support. We denote by \( P_j \) the projection in \( A \) given by \( P_j(i) = \delta_{ij} \). Since the set of edges \( E \) is a subset of \( \Sigma \times \Sigma \), we may regard \( E \) as a set-theoretic correspondence. The vector space \( X_0 = c_{00}(E) \) of the function on \( E \) with finite support is an \( A-A \) bimodule by

\[
(a \cdot f \cdot b)(i, j) = a(i)f(i, j)b(j)
\]

for \( a, b \in A, f \in X_0 \) and \( (i, j) \in E \). We define an \( A \)-valued inner product on \( X_0 \) by

\[
(f|g)_{A}(j) = \sum_{\{i | (i, j) \in E\}} \overline{f(i, j)}g(i, j)
\]

for \( f, g \in X_0 \). \( X_0 \) becomes in this way a right pre-Hilbert \( A \)-module. We denote by \( X \) the completion of \( X_0 \). The left \( A \)-action on \( X_0 \) can be extended to an action \( \phi : A \rightarrow \mathcal{L}(X_A) \) on \( X \) by continuity. Since \( G \) is a row finite matrix, \( \phi(a) \subset \mathcal{K}(X_A) \). Since no column of \( G \) is zero, the range map \( r \) is onto. Thus \( X_A \) is full. Let \( \mathcal{O}_X = C^*\{S_\alpha | x \in X\} \) be the Pimsner algebra [Pim] generated by the bimodule \( X \). For \( \alpha \in E \), \( S_{\delta_\alpha} \) will be denoted by \( S_{\alpha} \).

Let \( F \) be the edge matrix defined by \( F(\alpha, \beta) = 1 \) if \( r(\alpha) = s(\beta) \) and \( F(\alpha, \beta) = 0 \) otherwise. Then the generators \( \{S_\alpha | \alpha \in E\} \) satisfy

\[
S_\alpha^*S_\alpha = \sum_{\beta} F(\alpha, \beta)S_\beta S_\beta^*.
\]

For \( i \in \Sigma \), we may define \( S_i = \sum_{s(\alpha) = i} S_\alpha \in \mathcal{O}_X \), because no row of \( G \) is zero and the source map \( s \) is onto. If \( \beta = (i, j) \in E \), then \( S_\beta = S_i P_j \). The \( C^* \)-algebra \( \mathcal{O}_X \) is also generated by \( \{S_i | i \in \Sigma\} \) satisfying the relations

\[
S_i^*S_i = \sum_j G(i, j)S_j S_j^*
\]

The \( C^* \)-algebra \( \mathcal{O}_X \) coincides with the countably generated Cuntz-Krieger algebra \( \mathcal{O}_G \).

We shall introduce an \( A \)-valued left inner product on \( X \). We need an additional datum. Assume that we are given a nonnegative matrix \( T = (T_{ij})_{ij} \) such that \( T_{ij} > 0 \) if and only if \((i, j)\) is an edge, i.e. \( G(i, j) = 1 \). We call such a matrix \( T \) a weight matrix for the graph \( \mathcal{G} \). K. Yonetani suggested that the weight matrix \( T \) gives an \( A \)-valued left inner product \( A(\cdot | \cdot) \) on \( X_0 \) by

\[
A(f|g)(i) = \sum_j T_{ij}f(i, j)\overline{g(i, j)}
\]

for \( f, g \in X_0 \). Then we have two associated norms

\[
A\|f\| = \sqrt{\sup_i \sum_j T_{ij}|f(i, j)|^2}
\]
and

\[ \| f \|_A = \sqrt{\sup_j \sum_i |f(i,j)|^2} \]

6.1 Definition A weight matrix \( T \) for the graph \( G \) is called of finite index if

\[ c_1 := \sup_i \sum_j T_{ij} < \infty \]

and

\[ c_2 := \sup_j \sum_i \frac{1}{T_{ij}} < \infty. \]

6.2 Example Let \( \Sigma = \mathbb{N} \) and \( G(i,j) = 1 \) if \( |i-j| = 1 \) and \( G(i,j) = 0 \) if \( |i-j| \neq 1 \). Consider a weight matrix \( T \) defined as follows:

\[ T_{12} = 1. \quad T_{ij} = \frac{1}{2} \text{ if } |i-j| = 1 \text{ and } (i,j) \neq (1,2). \quad T_{ij} = 0 \text{ if } |i-j| \neq 1. \]

Then \( c_1 = 1 \) and \( c_2 = 4 \). Thus \( T \) is of finite index.

6.3 Example Let \( \Sigma = \mathbb{Z} \) and \( G(i,j) = 1 \) if \( |i-j| = 1 \) and \( G(i,j) = 0 \) if \( |i-j| \neq 1 \). Consider a weight matrix \( T \) defined by

\[ T_{ij} = \frac{1}{2} \text{ if } |i-j| = 1 \text{ and } T_{ij} = 0 \text{ if } |i-j| \neq 1. \]

Then \( c_1 = 1 \) and \( c_2 = 4 \). Thus \( T \) is of finite index.

6.4 Example The homogeneous tree \( \text{Tree}(n) \) of degree \( n \) is the tree where all vertices have degree \( n \). For example \( \text{Tree}(2) \) is the graph above with \( \Sigma = \mathbb{Z} \) and \( G(i,j) = 1 \) if \( |i-j| = 1 \) and \( G(i,j) = 0 \) if \( |i-j| \neq 1 \). \( \text{Tree}(4) \) is the Cayley graph of the free group \( F_2 \) with respect to the generators. We define a weight matrix \( T \) for \( \text{Tree}(n) \) by associating the value \( 1/n \) with each edge. Then \( c_1 = 1 \) and \( c_2 = n^2 \). Thus \( T \) is of finite index.

6.5 Example A tree has a weight matrix of finite index if and only if it has bounded degree. In general a locally finite graph has a weight matrix of finite index if and only if both in- and out-degrees are bounded. In fact suppose that the in-degree is unbounded. We may assume that \( c_1 < \infty \). For any edge \((i,j) \in E, 0 < T_{ij} \leq \sup_i \sum_j T_{ij} = c_1 \). Then we have

\[ c_2 = \sup_j \sum_i \frac{1}{T_{ij}} \geq \sum_i \frac{1}{T_{ij}} \geq \sum_i \frac{1}{c_1}. \]

Since the in-degree is unbounded, the last term goes to \( \infty \). Therefore \( c_2 = \infty \). The rest may be similarly shown.

6.6 Lemma Let \( T = (T_{ij})_{ij} \) be a weight matrix for a graph \( G \). Then the following are equivalent:

1. \( T \) is of finite index.
2. The two norms \( \| \cdot \|_A \) and \( \| \cdot \|_A \) on \( X_0 \) are equivalent.

Proof (1) \( \Rightarrow \) (2): Suppose that \( T \) is of finite index. Then for any \( i \),

\[ \sum_j T_{ij}|f(i,j)|^2 \leq (\sum_j T_{ij})(\sup_j |f(i,j)|^2) \leq c_1(\sup_i \sum_j |f(i,j)|^2) = c_1\|f\|_A^2. \]
hence
\[ A\|f\| = \sqrt{\sup_i \sum_j T_{ij} |f(i,j)|^2} \leq c_1^{1/2} \|f\|_A. \]

Setting \( g(i, j) = \sqrt{T_{ij} f(i, j)} \), we have
\[ A\|f\| = \sqrt{\sup_i \sum_j |g(i, j)|^2}, \quad \text{and} \quad \|f\|_A = \sqrt{\sup_i \sum_j \frac{1}{T_{ij}} |g(i, j)|^2}. \]

Thus for any \( j \),
\[ \sum_i \frac{1}{T_{ij}} |g(i, j)|^2 \leq (\sum_i \frac{1}{T_{ij}})(\sup_j |g(i, j)|^2) \leq c_2 \sup_i \sum_j |g(i, j)|^2 = c_2 A\|f\|^2. \]

Hence
\[ \frac{1}{c_1^2} \|f\|_A = \frac{1}{c_2} \sqrt{\sup_i \sum_j \frac{1}{T_{ij}} |g(i, j)|^2} \leq A\|f\|. \]

(2) \(\Rightarrow\) (1): Suppose that \( T \) is not of finite type. Then \( c_1 = \infty \) or \( c_2 = \infty \). If \( c_1 = \infty \), then for any \( M > 0 \) there exist a positive integer \( k \) such that \( \sum_j T_{kj} \geq M \). Let \( f(i, j) = 1 \) if \( i = k, (k, j) \) is an edge, and \( f(i, j) = 0 \) otherwise. Then \( A\|f\|^2 \geq \sum_j T_{kj} \geq M \) and \( \|f\|_A = 1 \). Thus the two norms are not equivalent. If \( c_2 = \infty \), then for any \( M > 0 \) there exist a positive integer \( k \) such that \( \sum_i \frac{1}{T_{ij}} \geq M \). Let \( g(i, j) = \frac{1}{T_{ij}} \) if \( j = k, (i, k) \) is an edge, and \( g(i, j) = 0 \) otherwise. Then \( A\|g\| = 1 \) and \( \|g\|_A^2 \geq M \). Thus the two norms are not equivalent.

If \( T \) is of finite index, we can identify the two completions of \( X_0 \) with respect to the two norms above defined. We shall denote by \( X \) its completion. The left and right actions of \( A \) extend to injective *-homomorphisms \( \phi : A \to \mathcal{L}(X_A) \) and \( \psi : A \to \mathcal{L}(A X) \).

We need the following inequalities which are easily verified: Let \( Y \) be a normed space. Then for any \( y_1, \ldots, y_n \in Y \), positive numbers \( \lambda_1, \ldots, \lambda_n, c \) with \( \sum_i \lambda_i \leq c \), we have
\[ \| \sum_i \lambda_i y_i \| \leq c \sup_i \|y_i\|. \]

For any positive operators \( A, B, C, D \) with \( A \leq C, B \leq D \), we have
\[ \|A^{1/2} B^{1/2}\| \leq \|C^{1/2} D^{1/2}\|. \]

**6.7 Theorem** In the above situation, if a weighted matrix \( T \) is of finite index, then \( X \) is of finite index and
\[ \|r - \text{Ind}[X]\| = c_1 := \sup_i \sum_j T_{ij} \quad \text{and} \quad \|\ell - \text{Ind}[X]\| = c_2 := \sup_i \sum_j \frac{1}{T_{ij}}. \]

More precisely, we have
\[ r - \text{Ind}[X] = \left( \sum_j T_{ij} \right)_{ij} \in \ell^\infty(\Sigma) \quad \text{and} \quad \ell - \text{Ind}[X] = \left( \sum_i \frac{1}{T_{ij}} \right)_{ij} \in \ell^\infty(\Sigma). \]
Proof First we shall show that \( X \) is of finite right index. Since the graph is locally finite, we have \( \phi(A) \subset K(X_A) \) and \( \psi(A) \subset K(A^r X) \). In fact, we have
\[
\phi(P_i) = \sum_{\{j|s(j) = i\}} \theta_{a_j,\delta_j}^{i} \in K(X_A)
\]
and
\[
\psi(P_j) = \sum_{\{\gamma|r(\gamma) = j\}} \frac{1}{T_{1 \gamma}} \theta_{\delta_j,\delta_j}^{\gamma} \in K(A^r X).
\]

We are left to show the inequality described in part (2) of Prop. 2.7. (2) of Definition 2.3. For any \( f_1, ..., f_n \in X_0 \) and \( g_1, ..., g_n \in X_0 \), we shall show that
\[
\| \sum_{p=1}^{n} A(f_p | g_p) \| \leq c_1 \| \sum_{p=1}^{n} \theta_{f_p,g_p}^{r} \|.
\]
Since
\[
\| \sum_{p=1}^{n} \theta_{f_p,g_p}^{r} \| = \| ((f_p | f_q)_A)^{1/2}_pq ((g_p | g_q)_A)^{1/2}_pq \|
\]
\[
= \sup_{j} \| (\sum_{i} f_p(i,j)f_q(i,j))^{1/2}_pq (\sum_{i} g_p(i,j)g_q(i,j))^{1/2}_pq \|,
\]
we have that
\[
\| \sum_{p=1}^{n} A(f_p | g_p) \| = \sup_{i} \| \sum_{j} T_{ij} (\sum_{p=1}^{n} f_p(i,j)g_p(i,j)) \|
\]
\[
\leq \sup_{j} (\sum_{i} T_{ij} \sup_{i} \| \sum_{p=1}^{n} f_p(i,j)g_p(i,j) \|)
\]
\[
\leq (\sup_{i} \sum_{j} T_{ij}) (\sup_{i} \sup_{j} \| \sum_{p=1}^{n} f_p(i,j)g_p(i,j) \|)
\]
\[
= c_1 \sup_{j} \| \sum_{p=1}^{n} f_p(i,j)g_p(i,j) \|
\]
\[
= c_1 \sup_{j} \| ((f_p(i,j)f_q(i,j))^{1/2}_pq (g_p(i,j)g_q(i,j))^{1/2}_pq \|
\]
\[
\leq c_1 \sup_{j} \| (\sum_{i} f_p(i,j)f_q(i,j))^{1/2}_pq (\sum_{i} g_p(i,j)g_q(i,j))^{1/2}_pq \|
\]
\[
= c_1 \| \sum_{p=1}^{n} \theta_{f_p,g_p}^{r} \|.
\]

For any \( f_1, ..., f_n \in X_0 \), we shall show that
\[
\| \sum_{p=1}^{n} \theta_{f_p,f_p}^{r} \| \leq c_2 \| \sum_{p=1}^{n} A(f_p | f_p) \|.
\]
Put \( g_p(i, j) := \sqrt{T_{ij}f_p(i, j)} \). Then we have that

\[
\| \sum_{p=1}^{n} \theta_{f_p,f_p}^p \| = \| (f_p │ f_q)_{pq} \|
\]

\[
= \sup \| (\sum_{i} f_p(i, j) f_q(i, j))_{pq} \|
\]

\[
= \sup \| (\sum_{i} \frac{1}{T_{ij}} g_p(i, j) g_q(i, j))_{pq} \|
\]

\[
\leq \sup \| (\sum_{i} \frac{1}{T_{ij}} g_p(i, j) g_q(i, j))_{pq} \|
\]

\[
\leq \sup \| (\sum_{i} \frac{1}{T_{ij}} g_p(i, j) g_q(i, j))_{pq} \|
\]

\[
= c_2 \sup \sup_{i} \| (g_p(i, j) g_q(i, j))_{pq} \|
\]

\[
= c_2 \sup \sup_{i} \| (\sum_{p=1}^{n} g_p(i, j) g_q(i, j)) \|
\]

\[
\leq c_2 \sup \| (\sum_{j=1}^{n} \sum_{p=1}^{n} g_p(i, j) g_q(i, j)) \|
\]

\[
= c_2 \sup \| (\sum_{j=1}^{n} \sum_{p=1}^{n} T_{ij} f_p(i, j) f_q(i, j)) \|
\]

\[
= c_2 \| (\sum_{p=1}^{n} A(f_p│f_p)) \|
\]

We shall next show that \( X \) is of finite left index. We denote by \( Y_0 \) be the \( A-A \) bimodule \( c_{00}(E) \) with the following two-sided inner products: For \( f, g \in Y_0 \),

\[
\langle f, g \rangle_A(i) = \sum_{i} f(i, j) g(i, j)
\]

and

\[
\langle f, g \rangle_A(j) = \sum_{i} \frac{1}{T_{ij}} f(i, j) g(i, j).
\]

We denote by \( Y \) its completion. For \( f \in Y_0 \), define \( Uf \in Y_0 \) by \((Uf)(i, j) = \sqrt{T_{ij}f(i, j)}\). Then we have \( A(f│g) = A(Uf│Ug) \) and \( (f│g)_A = (Uf│Ug)_A \). The map \( U \) extends to a surjective isometry \( X \to Y \) with respect to the two-sided inner products. Combining the fact with the preceding argument, we see that \( X \) is of finite left index. Since \( \{δ_{(i,j)}\}_{(i,j)∈E} \) is a right basis for \( X \),

\[
r - \text{Ind}[X] = \sum_{(i,j)} A(δ_{(i,j)}│δ_{(i,j)}) = (\sum_{j} T_{ij})_i \in ℓ^∞(Σ).
\]
Since \( \{ \sqrt{E_{(i,j)}} \delta_{(i,j)} \}_{(i,j) \in E} \) is a left basis for \( X \) the formula for \( \ell - \text{Ind}[X] \) is obtained similarly. The rest is clear.

6.2 Crossed products of Hilbert \( C^* \)-bimodules by locally compact groups

In [K], the first-named author studied continuous crossed products of Hilbert \( C^* \)-bimodules by locally compact groups. Let \( B \) be a unital \( C^* \)-algebra, and \( A \) be a \( C^* \)-subalgebra of \( B \) with the same unit. Let \( E : B \to A \) be a conditional expectation of finite index in the sense of [W]. So there exists a finite basis \( \{ u_1, u_2, \ldots, u_n \} \) of \( B \) such that \( x = \sum_{i=1}^{n} E(xu_i^*)u_i \) for any \( x \in B \). Let \( X = A \overline{B_B} \) be a \( A-B \) bimodule with right \( B \)-valued inner product \( (x|y)_B = x^*y \) and left \( A \)-valued inner product \( _A(x|y) = E(xy^*) \). Then \( X \) is a Hilbert \( A-B \) bimodule of finite index.

Let \( G \) be a second countable locally compact group and \( \alpha \) a continuous homomorphism from \( G \) to the automorphism group of \( B \) such that \( E(\alpha_g(b)) = \alpha_g(E(b)) \) for every \( b \in B \) and every \( g \in G \). It can be shown that \( A \rtimes_\alpha G \) can be embedded as a \( C^* \)-algebra in \( B \rtimes_\alpha G \) in a natural way, and it can be shown that there exists a conditional expectation \( \hat{E} \) from \( B \rtimes_\alpha G \) to \( A \rtimes_\alpha G \) which extends \( E \). Put \( Y = B \rtimes_\alpha G \). We define a \( A \rtimes_\alpha G - B \rtimes_\alpha G \) bimodule structure on \( Y \) and a left inner product over \( A \rtimes_\alpha G \) and a right inner product over \( B \rtimes_\alpha G \) in the obvious way using \( \hat{E} \). Then it can be shown that \( Y \) is a countably generated Hilbert \( C^* \)-bimodule of finite index (see [K]). The left and right indices of \( Y \) are essentially the same as those of \( X \).

6.8 Correspondences

Let \( \Omega \) be a compact Hausdorff space. Most Hilbert \( C^* \)-bimodules over the commutative \( C^* \)-algebra \( A = C(\Omega) \) naturally arise from set-theoretical correspondences (i.e. closed subsets \( \mathcal{C} \) of \( \Omega \times \Omega \)) similar to the case of commutative von Neumann algebras as in [Co]. We say that a pair \( (\mathcal{C}, \mu) \) is a (multiplicity free) topological correspondence on \( \Omega \) if \( \mathcal{C} \) is a (closed) subset of \( \Omega \times \Omega \) and \( \mu = (\mu^y)_{y \in \Omega} \) is a family of finite regular Borel measure on \( \Omega \) satisfying the following conditions:

1. (faithfulness) the support \( \text{supp}\mu^y \) of the measure \( \mu^y \) is the \( y \)-section \( \mathcal{C}^y := \{ x \in \Omega \mid (x, y) \in \mathcal{C} \} \).
2. (continuity) for any \( f \in C(\mathcal{C}) \), the map \( y \in \Omega \to \int_{\mathcal{C}^y} f(x, y) \, d\mu^y(x) \in \mathbb{C} \) is continuous.

The vector space \( X_0 = C(\mathcal{C}) \) is an \( A-A \) bimodule by

\[ (a \cdot f \cdot b)(x, y) = a(x)f(x, y)b(y) \]

for \( a, b \in A \), \( f \in X_0 \) and \( (x, y) \in \mathcal{C} \). We define an \( A \)-valued inner product on \( X_0 \) by

\[ (f|g)_A(y) = \int_{\mathcal{C}^y} \overline{f(x, y)}g(x, y) \, d\mu^y(x) \]

for \( f, g \in X_0 \). Faithfulness and continuity of \( \mu \) imply that \( X_0 \) is a right pre-Hilbert \( A \)-module. We denote by \( X \) the completion \( X_0 \). The left \( A \)-action on
$X_0$ can be extended to a *-homomorphism $\phi : A \to \mathcal{L}_A(X_A)$. Thus we obtain a right Hilbert $A$-$A$ bimodule with right inner products from the correspondence $(C, \mu)$. See [D] and [KW1] for a more precise treatment.

We usually assume that for any $x \in \Omega$ there exists $y \in \Omega$ with $(x, y) \in C$. This condition implies that left action $\phi$ is faithful. We also assume that for any $y \in \Omega$ there exists $x \in \Omega$ with $(x, y) \in C$. The condition shows that right inner product on $X$ is full. In fact let $\omega(y) = \mu^y(C^y)$. Then $\omega \in A$ is invertible. For any $a \in A$, put $f(x, y) = a(y)$. Then $(I/f)_A = a\omega$. Hence the right inner product is full.

6.9 Example Let us assume that projection maps

$$r : (x, y) \in C \mapsto x \in \Omega \quad \text{and} \quad s : (x, y) \in C \mapsto y \in \Omega$$

are local homeomorphisms. For any $y \in \Omega$, let $\mu^y$ be the counting measure on $C^y$. Then $(C, \mu)$ is a topological correspondence on $\Omega$. We shall show $X$ has a finite basis. In fact, since $C$ is compact, and by our assumption, there exist a finite set $\{(x_1, y_1), \ldots, (x_n, y_n)\} \subset C$ and open neighborhoods $U_k$ of $(x_k, y_k)$ for $k = 1, \ldots, n$ such that the restrictions of the projection maps $r$ and $s$ to $U_k$ are local homeomorphisms and $C = \cup_{k=1}^n U_k$ is an open covering. Let $\{f_1, \ldots, f_n\} \subset C(\Omega)$ be a partition of unity for this open covering. Put $g_k = f_k^{1/2} \geq 0$. Then for any $(x_1, y), (x_2, y) \in C$, we have

$$\sum_{k=1}^n g_k(x_1, y)g_k(x_2, y) = \delta_{x_1,x_2}.$$ 

Using these equalities, for any $h \in C(\Omega)$, we have that $\sum_{k=1}^n g_k(g_k|h)_A = h$. Thus $\{g_1, \ldots, g_n\}$ is a finite basis for $X$.

As an example, put $\Omega = [0, 1]$. Let $h_1$ be a map on the interval $\Omega = [0, 1/2]$ given by $h_1(x) = 2x$ for $x \in [0, 1/2]$ and $h_2$ be a map on the interval $\Omega = [1/2, 1]$ given by $h_2(x) = 2x - 1$ for $x \in [1/2, 1]$. Let $C$ be the union of the graphs of $h_1$ and $h_2$. Then $A = C([0, 1])$ and the right inner product on $X_A = C(\Omega)$ is given by

$$(f|g)_A(y) = \overline{f(y/2, y)}g(y/2, y) + \overline{f(y/2 + 1/2, y)}g(y/2 + 1/2, y)$$

for $f, g \in X_A$ and $y \in [0, 1]$. Thus $X_A \cong A \oplus A$ as a right Hilbert $C^*$-module. The left action $\phi : A \to \mathcal{L}_A(X_A) \cong M_2(A) \cong C([0, 1], M_2(\mathbb{C}))$ is given by the diagonal matrices

$$(\phi(a))(x) = \text{diag}(a(x/2), a(x/2 + 1/2)).$$

The associated Pimsner algebra $\mathcal{O}_X$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$ and the fixed point algebra by the gauge action is isomorphic to a UHF algebra $M_{2^\infty}$. The left inner product on $X_A$ is similarly given by

$$A(f|g)(x) = \begin{cases} f(x, 2x)g(x, 2x) & \text{(if } 0 \leq x \leq 1/2) \\ f(x, 2x - 1)g(x, 2x - 1) & \text{(if } 1/2 \leq x \leq 1). \end{cases}$$
The right and the left norms on $X_A$ are given by
\[
\|f\|_A = \sup_{0 \leq y \leq 1} \sqrt{|f(y/2, y)|^2 + |f(y/2 + 1/2, y)|^2}
\]
and
\[
A\|f\| = \max\{ \sup_{0 \leq x \leq 1/2} |f(x, 2x)|, \sup_{1/2 \leq x \leq 1} |f(x, 2x - 1)| \},
\]
These two norms are equivalent:
\[
A\|f\| \leq \|f\|_A \leq \sqrt{2}A\|f\|.
\]
Thus the $A$–$A$ bimodule $X_A$ is of finite type in the sense of [KW1]. This implies that $X$ is of finite index as discussed in Example 2.34.

Let us replace $\Omega = [0, 1]$ by the circle $\Omega = \mathbb{T}$ and consider, similarly, the map $h$ on $\mathbb{T}$ such that $h(z) = z^2$ for $z \in \mathbb{T}$. Let $C$ be the graph of $h$ and consider the Hilbert $A$–$A$ bimodule $X$. Then the associated Pimsner algebra $O_X$ is isomorphic to the purely infinite simple $C^*$–algebra with $K$–theory: $K_0(O_X) \cong \mathbb{Z} \cong K_1(O_X)$. The fixed point algebra under the gauge action is isomorphic to the Bunce-Deddens algebra of type $2^\infty$. One can also show that $X$ is of finite index.

**6.10 Example** In general $X$ fails to have a finite basis. A typical example is supplied by a tent map $h$ on the unit interval $\Omega = [0, 1]$. This is essentially the same example as the one discussed in 2.35. The map $h$ is given by
\[
h(x) = 2x \quad \text{if} \quad 0 \leq x \leq 1/2 \quad \text{and} \quad h(x) = -2x + 2 \quad \text{if} \quad 1/2 \leq x \leq 1.
\]
Let $C$ be the graph of $h$. For any $y \in [0, 1)$, let $\mu^y$ be a counting measure on $\mathbb{C}^y$, that is, $\mu^y = \delta_{y/2} + \delta_{1-y/2}$. For $y = 1$, let $\mu^1 = 2\delta_{1/2}$. Then $(C, \mu)$ is a correspondence on $[0, 1]$. Let $X$ be the right Hilbert $A = C([0, 1])$–module obtained from the correspondence. The right Hilbert $A$–module $X$ does not have a finite basis. This fact is easily seen through a realization of it as an orbifold construction as follows: Let $B = C([0, 2])$ and let $\gamma$ be a homeomorphism on $[0, 2]$ with period two such that $\gamma(x) = 2 - x$ for $x \in [0, 2]$. Then $\gamma$ induces an automorphism $\alpha$ on $B$ such that $(\alpha(f))(x) = f(2 - x)$. The fixed point algebra $B^\alpha$ is isomorphic to $A = C([0, 1])$. We note that the action $\alpha$ is free except for $x = 1$. Consider the conditional expectation $E : B \to A$ defined by $E(f) = (f + \alpha(f))/2$. Let $Y = B_A$ be a right Hilbert $A$–module given by $(f|g)_A = E(f^*g)$. Then $X$ and $Y$ are isomorphic as right Hilbert $A$–modules: there is a unitary induced by the map $\phi : C \to [0, 2]$,
\[
\phi((x, 2x)) = 2x \quad \text{if} \quad 0 \leq x \leq 1/2
\]
\[
\phi((x, -2x + 2)) = 2x \quad \text{if} \quad 1/2 \leq x \leq 1.
\]
By a result in Prop. 2.8.2 in [W], $Y$ does not have a finite basis.

We shall construct a countable basis for $Y$ explicitly. Define $r_n \in C([0, 1])$ by
\[
r_n(x) = 1 \quad \text{if} \quad 0 \leq x \leq 1-1/n \quad \text{and} \quad r_n(x) = -n(x-1) \quad \text{if} \quad 1-1/n \leq x \leq 1.
\]
Put \( v_1 = r_1 \) and \( v_n = (r_n - r_{n-1})^{1/2} \in C([0, 1]) \) for \( n \geq 2 \). Then
\[
\sum_{i=1}^{\infty} |v_i(x)|^2 = \lim_{n \to \infty} r_n(x) = 1 \quad \text{for} \quad x \neq 1
\]
and \( v_1(1) = 0 \). Define \( u_0 = 1 \) and \( u_i \in C([0, 2]) \) by
\[
u_i(x) = v_i(x) \quad \text{(if} \quad 0 \leq x \leq 1) \quad \text{and} \quad u_i(x) = -v_i(2 - x) \quad \text{(if} \quad 1 \leq x \leq 2)
\]
for \( i = 1, 2, \ldots \), and \( u_i(1) = 0 \) for \( i \neq 0 \). Then for \( x \neq 1 \), we have \( \sum_{i=0}^{\infty} |u_i(x)|^2 = 2 \) and \( \sum_{i=0}^{\infty} u_i(x)u_i(2 - x) = 0 \). For \( x = 1 \), we have \( \sum_{i=0}^{\infty} |u_i(1)|^2 = 1 \). We claim that \((u_i)_{i=0,1,...} \) is a basis for \( Y \). For any \( f \in Y = C([0, 2]) \), we shall show that
\[
\lim_{F: \text{finite}} \sum_{i \in F} u_i(f)_A = f.
\]
Let \( F \) be a finite subset of \( \{0, 1, 2, \ldots \} \). Put \( S_F = \sum_{i \in F} u_i(f)_A \). Since \( \|g(g)_A\| = \|E(g^*g)\| \leq \|g\|_\infty^2 \) for \( g \in B \), it suffices to show that \( \lim_{F: \text{finite}} \|f - S_F\|_\infty = 0 \), where \( \|g\|_\infty \) is the sup norm. We may assume that \( 0 \in F \). For \( 0 \leq x \leq 1 \), we have
\[
f(x) - S_F(x) = f(x) - \frac{1}{2} \sum_{i \in F} |u_i(x)|^2 f(x) + \frac{1}{2} \sum_{i \in F} u_i(x)u_i(2 - x)f(2 - x)
\]
\[
eq f(x) - \left\{ \frac{1}{2}(1 + \sum_{i \in F \setminus \{0\}} v_i^2(x))f(x) + \frac{1}{2}(1 - \sum_{i \in F \setminus \{0\}} v_i^2(x))f(2 - x) \right\}
\]
\[
= \frac{1}{2}(1 - \sum_{i \in F \setminus \{0\}} v_i^2(x))(f(x) - f(2 - x)).
\]
For any \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that for any \( x \) satisfying \( 1 - \delta \leq x \leq 1 \), we have \( |f(x) - f(2 - x)| \leq \varepsilon \). Since \( |1 - \sum_{i \in F \setminus \{0\}} v_i^2(x)| \leq 1 \), we have
\[
|f(x) - S_F(x)| \leq \varepsilon \quad \text{for} \quad x \in [1 - \delta, 1].
\]
On the other hand, \( \sum_{i=1}^{\infty} v_i(x)^2 = 1 \) uniformly on \([0, 1 - \delta] \). We also have \( |f(x) - f(2 - x)| \leq 2\|f\|_\infty \). Therefore there exists a finite set \( F_0 \) such that for any finite subset \( F \supseteq F_0 \) and for any \( x \in [0, 1] \),
\[
|f(x) - S_F(x)| \leq \varepsilon.
\]
Similar arguments work for \( x \in [1, 2] \). Therefore \((u_i)_{i=0,1,...} \) is a countable basis for \( Y \). However, the range \( \phi(A) \) of the left action is not included in \( K(X_B) \), because the identity \( \phi(I_A) = I_{X_B} \) is not in \( K(X_B) \). Therefore the bimodule \( X \) is not of finite right index.
References

[ B] B. Blackadar, K-theory for operator algebras. Mathematical Sciences Research Institute Publications, 5 Cambridge University Press, Cambridge, 1998.
[ BDH] M. Ballet, Y. Denizeau, J.-F. Havet, Indice d’une esperance conditionelle, Compos. Math. 66(1988), 199-236.
[ BGR] L. G. Brown, P. Green, M.A. Rieffel, Stable isomorphism and strong Morita equivalence of $C^*$-algebras, Pacific J. Math. 71 (1977), 349–363.
[ D] V. Deaconu, Generalized solenoids and $C^*$-algebras, Pacific J.Math. 190 (1999), 247-260.
[ Di] J. Dixmier, Les $C^*$-algèbres et leurs représentations. Gauthier-Villars Éditeur, Paris 1969.
[ DPZ] S. Doplicher, C. Pinzari, R. Zuccante, The $C^*$-algebra of a Hilbert bimodule, Boll. Unione Mat. Ital. Sez. B, 1 (1998), 263–281.
[ DR1] S. Doplicher, J.E. Roberts, Endomorphisms of $C^*$-algebras, crossed products and duality for compact groups, Ann. Math. 130 (1989) 75–119.
[ F] J. M. G. Fell, The structure of algebras of operator fields, Acta Math. 106 (1961) 233–280.
[ DR2] S. Doplicher, J.E. Roberts, A new duality theory for compact groups Inventiones Math. 98 (1989) 157–218.
[ FK] M. Frank and E. Kirchberg, On conditional expectations of finite index, J. Operator Theory 40(1998), 87-111.
[ FL1] M. Frank and D. Larson, A module frame concept for Hilbert $C^*$-modules, Contemporary Mathematics 247(1999), 207-233.
[ FL2] M. Frank and D. Larson, Frames in Hilbert $C^*$-modules and $C^*$-algebras preprint.
[ H] R. Haag, Local quantum physics. Fields, particles, algebras. Second edition. Springer-Verlag, Berlin, 1996.
[ I] M. Izumi, Inclusions of simple $C^*$-algebras, J. Reine Angew. 547 (2002), 97–138.
[ J] V.F.R. Jones, Index for subfactors, Inv. Math. 72 (1983) 1-25.
[ K] T. Kajiwara, Continuous crossed products of Hilbert $C^*$-bimodules, Int. J. Math. 1 (2000), 969–981.
[ KW] E. Kirchberg, S. Wassermann, Operations on continuous bundles of $C^*$-algebras, Math. Ann. 303 (1995), 677–697.
[ KPRR] A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids and Cuntz-Krieger algebras, J. Funct. Anal. 184(1998), 161-174.
[ KPW1] T. Kajiwara, C. Pinzari and Y. Watatani Ideal structure and simplicity of the $C^*$-algebra generated by Hilbert bimodules, J. Funct. Anal. 159(1998), 295-322.
[ KPW2] T. Kajiwara, C. Pinzari and Y. Watatani Hilbert $C^*$-bimodules and countably generated Cuntz-Krieger algebras, J. Operator Theory 45(2001), 3-18.
[ KW1] T. Kajiwara and Y. Watatani Jones index theory by Hilbert $C^*$-bimodules and $K$-theory, Trans. Amer. Math. Soc. 352(2000), 3429-3472.
[ KW2] T. Kajiwara and W. Watatani Crossed product of Hilbert $C^*$-bimodules by countable discrete groups, Proc. Amer. Math. Soc. 126 (1998), 841-851.
[KW3] T.Kajiwara and Y. Watatani *Hilbert C∗-bimodules and continuous Cuntz-Krieger algebras*, J. Math. Soc. Japan. 54 (2002), 35-59.

[Ka] G.G. Kasparov, *Equivariant KK-theory and the Novikov Conjecture*, Invent. Math. 91 (1988), 147–201.

[L] E.C. Lance, *Hilbert C∗-modules: A toolkit for operator algebraists*, London Math. Soc. Lecture Note Ser. 210, Cambridge U.P., 1995

[LR] R. Longo, J.E. Roberts, *A theory of dimension*, K-theory 11(1997), 103-159.

[P] G.K. Pedersen, *C∗-algebras and their automorphism groups*, Academic Press 1979.

[Pim] M. Pimsner, *A class of C∗-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z*, in Voiculescu, D. (ed.) *Free probability theory*, AMS, 1997, 189-212.

[PiPo] M. Pimsner, S. Popa, *Entropy and index for subfactors*, Ann. Sci. Éc. Norm. Sup. 19 (1985), 57–106.

[R1] M. Rieffel, *Induced representations of C∗-algebras*, Adv. Math. 13 (1974), 176-257.

[Y] S. Yamagami *Tensor categories for operator algebras*, preprint.

[V] J.-M. Vallin, *C∗-algèbres de Hopf et C∗-algèbres de Kac*, Proc. London Math. Soc. 50 (1985), 131–174.

[W] Y. Watatani, *Index for C∗-subalgebras*, Memoir Amer. Math. Soc. 424 (1990).

[Wo] S. L. Woronowicz, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups*, Invent. Math. 93 (1988), 35-76.