Finite-Size and Finite-Temperature Effects in the Conformally Invariant $O(N)$ Vector Model for $2 < d < 4$

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We study the operator product expansion (OPE) of the auxiliary scalar field $\lambda(x)$ with itself, in the conformally invariant $O(N)$ Vector Model for $2 < d < 4$, to leading order in $1/N$ in a strip-like geometry with one finite dimension of length $L$. We show that consistency of the finite-geometry OPE with bulk OPE calculations requires the physical conditions of, either finite-size scaling at criticality, or finite-temperature phase transition.

As a by-product, we verify some old results of Cardy on the finite-size scaling of the free-energy density.

I. INTRODUCTION

Euclidean conformal field theory (CFT) describes systems at second order phase transition points where the renormalised correlation length $\xi$ becomes infinite. For a theory in the bulk, such points, if they exist, are reached when an appropriate parameter (i.e. renormalised coupling) takes a “critical” value. The same theory defined in finite-geometry, i.e. a hyper-strip with one finite dimension of length $L$, can also exhibit critical behaviour when the renormalised coupling retains its bulk critical value. In such a case, the correlation length is not infinite but it scales with $L$ and the system exhibits finite-size scaling at criticality. It may exist, however, another possible “critical state” for the same theory in finite-geometry. This is reached whenever it is possible to tune the length scale $L$ (which is now interpreted as inverse temperature $1/T$), to a “critical” value related to the renormalised coupling, such that the renormalised correlation length diverges. Such a case would then describe a finite-temperature phase transition.

In the present note we quantify the above general remarks by studying the conformally invariant $O(N)$ vector model in $2 < d < 4$, in a hyper-strip geometry with one finite dimension of length $L$ and periodic boundary conditions. Our tool for studying finite-geometry critical systems is the OPE which, by probing short distances is insensitive to the macroscopic size of the system. In the case of the conformally invariant $O(N)$ vector model, the OPE of the auxiliary field $\lambda(x)$ with itself in the finite-geometry, should be consistent with the corresponding OPE results in the bulk, which were studied in \[\text{1}\]. We show that, consistency of the finite-geometry OPE with the OPE of the theory in the bulk can be achieved in two different ways which correspond to two distinct physical states of the system: a) finite-size scaling at criticality and b) finite-temperature phase transition. Our main result is a unified approach to finite-size scaling and finite-temperature phase transitions based on CFT and OPEs.

II. THE CONFORMALLY INVARIANT $O(N)$ VECTOR MODEL IN $2 < D < 4$

This model is one of the most extensively used testing grounds for ideas related to CFT and phase transitions. Its “effective” partition function, obtained after integrating out the fundamental $O(d)$-scalar and $O(N)$-vector fields $\phi^\alpha(x)$ with $\alpha = 1, 2, \ldots, N$, reads

$$Z = \int (D\sigma) \exp \left[ -\frac{N}{2} S_{_{\text{eff}}} (\sigma, g) \right],$$

(1)

$$S_{_{\text{eff}}} (\sigma, g) = \text{Tr} \left[ \ln(-\partial^2 + \sigma) \right] - \frac{1}{g} \int d^d x \sigma(x),$$

(2)

where, $\sigma(x)$ is the auxiliary $O(d)$-scalar field and $g$ the coupling. Setting $\sigma(x) = m^2 + (i/\sqrt{N})\lambda(x)$, \[\text{1}\] can be calculated in a $1/N$ expansion provided that the gap equation

$$\frac{1}{g} = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2},$$

(3)

is satisfied. The leading-$N$ inverse propagator of $\lambda(x)$ is

$$\Pi^{-1}(p^2) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)((q + p)^2 + m^2)}.$$  

(4)

A non-trivial CFT, to any fixed order in $1/N$, is obtained for $2 < d < 4$ by tuning the coupling to the critical value $1/g = 1/g_* = (2\pi)^{-d} \int d^d p/p^2$. Then, the mass or inverse correlation length $m = 1/\xi = 0$.

Putting the model in a hyper-strip geometry with one finite dimension of length $L$ and periodic boundary conditions, does not affect its renormalisability in the $1/N$
the theory in the bulk, has been studied in the $1/N$ expansion. For example, the gap equation and inverse invariant OPE of the auxiliary field $\lambda(x)$ is a mass or inverse correlation length parameter.

**III. THE CFT RESULTS**

Before studying the possible critical points in finite-geometry, we recall the CFT results. The conformally invariant OPE of the auxiliary field $\lambda(x)$ with itself for the theory in the bulk, has been studied in the $1/N$ expansion in a number of articles [2, 4]. Here, we confine ourselves to the leading order in $1/N$ where the OPE reads

$$\lambda(x)\lambda(0) = \frac{1}{x^4} + g_\lambda \frac{1}{x^2} \lambda(0) + \frac{d}{\pi^{d/2} (d-1) C_T} \frac{x_\mu x_\nu}{x^2} T_{\mu\nu}(0) + \cdots. \quad (7)$$

Only the most singular terms as $x \to 0$ are displayed on the r.h.s. of (7) which, in general, receives contributions from an infinity of operators [2, 4]. $C_T$ is the normalisation coefficient of the two-point function of the energy momentum tensor $T_{\mu\nu}(x)$ and $g_\lambda$ is the coupling constant of the three-point function $\langle \lambda \lambda \lambda \rangle$. Note that to leading-N the dimension of $\lambda(x)$ is 2.

As mentioned in the introduction, the finite-geometry OPE of $\lambda(x)$ with itself is expected to retain the bulk form (f), when the system is at a conformally invariant point. Then, finite-geometry corrections to the bulk form $1/x^4$ of the $\lambda(x)$ propagator are obtained from (f), through non-vanishing expectation values of the operators in the r.h.s. In general, these expectation values are unknown. However, using only the conformal invariance of the critical theory, Cardy [4] showed that there exists a relation between the expectation value of the diagonal elements of $T_{\mu\nu}(x)$ and the finite-size scaling correction to the free-energy density $f = -\ln Z/V$ of the system at criticality. Namely,

$$f_\infty - f_L = \tilde{c} \frac{\Gamma\left(\frac{d}{2}\right) \zeta(d)}{\pi^{d/2} L^d} = \frac{1}{d-1} \langle T_{11} \rangle = -\langle T_{11} \rangle. \quad (8)$$

Here, $\tilde{c}$ is a parameter which characterises the specific CFT model under consideration. It has been proposed [10, 11] to be a possible generalisation of the central charge in $d > 2$.

Upon transforming the expectation value of (f) to momentum space [12] and inverting, we obtain by virtue of (f) the inverse propagator of $\lambda(x)$ in finite geometry as

$$\Pi_{\lambda}^{-1}(p^2, \omega^2) = \frac{1}{2L} \sum_{m=\infty}^{\infty} \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \left[ \frac{1}{q^2 + \omega_n^2 + M_L^2} \right] \times \frac{1}{(p+q)^2 + (\omega_n^2 + \omega_m^2 + M_L^2)} \delta(p^2 - q^2), \quad (6)$$

where $M_L$ is a mass or inverse correlation length parameter.

**IV. FINITE-SIZE SCALING**

One way to obtain a critical theory in finite-geometry is by tuning the coupling $1/g$ in (f) to its bulk critical value $1/g_*$. Then, for the whole range $2 < d < 4$, the (renormalised) gap equation (f) has a finite solution which gives the finite-size scaling of the inverse correlation length $M_L \equiv M_* \sim 1/L$ as shown in Fig.1 (see also [2, 4]).

![FIG. 1. The finite-size scaling of $M_*$](image-url)
Expression (10) can be evaluated for a general $M_L$, but for $M_L = M_c$ it takes the simple form

$$\Pi_L^{-1}(p^2, \omega_n^2) = \frac{K_d}{(p^2 + \omega_n^2)^{2-d}} \left\{ 1 + \frac{M_c^2}{p^2 + \omega_n^2} + \frac{\gamma^2 d (d - \frac{1}{2}) C_{d-1}^d (y) M_c^d}{\sqrt{\pi} \Gamma(d) \Gamma(2 - \frac{1}{2} d) (p^2 + \omega_n^2)^{\frac{1}{2} d}} \mathcal{F}_d (LM_c) + \ldots \right\},$$

(10)

$$\mathcal{F}_d (LM_c) = \frac{1 - \frac{d}{2} \mathcal{I}_0 - \mathcal{I}_1}{d},$$

(11)

$$T_n = \int_1^\infty dt \left\{ (t^2 - 1)^{\frac{1}{2} (d-3)+n} \right\},$$

(12)

Consistency of (10) with (8) requires

$$\langle \lambda \rangle = - \frac{M_c^2 (d - 3)}{2g_3 (d - 2)^2},$$

(13)

$$f_\infty - f_L = N \frac{M_c^2 \Gamma(\frac{d}{2})}{\pi \Gamma(d)} \left\{ \frac{d}{2} L_0 - \frac{d}{2} \mathcal{I}_0 \right\}.$$  

(14)

To obtain (14) we have used (8). One can show [13] that (13) is consistent with known CFT calculations of the coupling constant $g_3$ and the OPE of the fundamental $O(N)$-vector field $\phi^a(x)$ with itself [14]. On the other hand, (14) is exactly what it is obtained from a direct calculation of the finite-size scaling correction to the free-energy density at criticality [15]. In particular, for $d = 3$ (14) reproduces the intriguing result $\bar{c}/N = 4/5$ [15,16], while for $2 < d < 4$ a plot of $\bar{c}/N$ is depicted in Fig. 2.

V. FINITE-TEMPERATURE TRANSITION

The second way to obtain a finite-geometry critical theory is by tuning the renormalised coupling to a certain value such that the inverse correlation length $M_L \equiv M_c = 0$. In the present case, this can only be achieved for $3 < d < 4$, since for $2 < d < 3$ the gap equation (8) is divergent when $M_L = 0$ [17], implying that the massless phase of the theory is saturated. In other words, there is no finite critical temperature for the $O(N) \rightarrow O(N - 1)$ phase transition, in agreement with the Mermin-Wagner-Coleman theorem [18,19].

When $3 < d < 4$, the (renormalised) gap equation (8) is satisfied for $M_L \equiv M_c = 0$ if the renormalised coupling

$$\frac{1}{g} = \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} m_s^{d-2},$$

(15)

is such that

$$\frac{1}{L_s^{d-2}} = T_{cr}^{d-2} = \frac{\Gamma(1 - \frac{d}{2})}{2^{d-1} \Gamma(\frac{d}{2} - 1)} \zeta(d - 2) m_s^{d-2},$$

(16)

for some “critical temperature” $T_{cr}$. The mass parameter $m_s$ is related to the mass of the fundamental fields $\phi^a(x)$ of the bulk theory [18]. This agrees with [19] for $3 < d < 4$ and reproduces the leading-$N$ critical temperature for the $O(N) \phi^4$ theory in $d = 4$ up to an appropriate normalisation of the $\phi^4$ coupling. Hence, the critical theory now corresponds to a finite-temperature phase transition.

Furthermore, the critical theory is also conformally invariant since for $M_L \equiv M_c = 0$, the inverse propagator (11) has the large momentum expansion

$$\Pi_L^{-1}(p^2, \omega_n^2) = \frac{K_d}{(p^2 + \omega_n^2)^{2-d}} \left\{ 1 + \frac{\gamma^2 d (d - \frac{1}{2}) C_{d-1}^d (y) M_c^d}{\sqrt{\pi} \Gamma(d) \Gamma(2 - \frac{1}{2} d) (p^2 + \omega_n^2)^{\frac{1}{2} d}} \mathcal{F}_d (LM_c) + \ldots \right\}.$$

(17)

Indeed, (17) is similar to the conformal OPE (11). Consistency then of (17) and (11) requires $\bar{c}/N = 1$. This value of $\bar{c}$ corresponds to a free CFT of $N$ massless scalars. However, the critical theory under study is not trivial since it is expected that $\bar{c}$ receives $1/N$ corrections which will alter the above leading-$N$ value.

An important difference between (17) and (11) is that the field $\lambda(x)$ does not show up in (17) - instead the “shadow field” of $\lambda(x)$ appears, which has leading-$N$ dimension $d - 2$. To this end, it is worth pointing out that scalar representations of the conformal algebra with dimensions $\eta$ and $d - \eta$ are equivalent in $d > 2$ [17].

![FIG. 2. The parameter $\bar{c}/N$ for general $2 < d < 4$.](image-url)
VI. RESULTS AND OUTLOOK

In this note, we investigated the intimate relationship connecting finite-size scaling at criticality and conformal OPEs and we presented a method for calculating the finite-size scaling correction to the free-energy density at criticality from two-point functions. Also, we discussed a critical point of the $O(N)$ vector model in $3 < d < 4$, where the renormalised mass or inverse correlation length is zero and argued that it corresponds to the physical situation of a finite-temperature phase transition. It may be interesting to point out that the physical situation of finite-temperature phase transition corresponds effectively to a CFT in $d - 1$ dimensions, since critical fluctuations with infinite correlation length do not “see” the finite dimension.

Possible extensions of our work would be the study of other conformally invariant models in finite geometries for $2 < d < 4$, such as the Gross-Neveu or $CP^{N-1}$ models. One could also consider other finite geometries i.e. with two finite dimensions, where there is evidence that there exist similarities with two-dimensional CFT, at least for antiperiodic boundary conditions.

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