Analysis of the Dynamics and Topology Dependencies of Small Perturbations in Electric Transmission Grids

Luiscarlos A. Torres-Sánchez,1‡ Giuseppe T. Freitas de Abreu,2,3‡ and Stefan Kettemann4,5,†

1Department of Energy Division, deon international GmbH, 61352 Bad Homburg, Germany.
2Department of Computer Science and Electrical Engineering, Jacobs University Bremen, 28759 Bremen, Germany.
3Department of Electrical and Electronic Engineering, Ritsumeikan University, Kusatsu 525-8577, Japan.
4Department of Physics and Earth Science, Jacobs University Bremen, 28759 Bremen, Germany.
5Division of Advanced Materials Science, Pohang University of Science and Technology, Pohang 790-784, South Korea.

(Dated: March 12, 2022)

Through an eigenanalysis of small perturbations, as typically done in small-signal stability studies, we intend to discover the underlying reasons that make those perturbations propagate in some way or another in the grid. To this end, we establish connections between the perturbations time-scale and topological metrics. Namely, the algebraic connectivity and the Fiedler vector of a generalized/weighted Laplacian matrix that depends on the stationary phase solutions of the system and is thereby inherently conditioned by the topology and the power distribution. Then, we aim to find out the isolated influence of topology on the perturbations when the network interacting agents have, in principle, opposite behaviors (i.e. producers and consumers). To do so, we study three networks: Small-world, Random, German grid. Furthermore, we tackle the effect of machine clustering on small perturbations and the influence of the network’s average clustering coefficient on the intensity localization of the generalized Fiedler vector. Finally, we propose ways in which future (dynamic topology control) and existing (power system stabilizer) grid control strategies can adapt their response to comprehensively consider the topology and remote signals in the system.

### Nomenclature

| Symbol | Description |
|--------|-------------|
| $\alpha_i$ | Rotor angle perturbation at node $i$ in rad |
| $\epsilon_k$ | Real eigenfrequency $k$ in Hz |
| $\Gamma$ | Perturbation’s relaxation/damping rate in Hz |
| $\gamma$ | Damping coefficient in Nms |
| $\kappa$ | Number of edges in the network |
| $l_{ij}$ | Shortest distance between nodes $i$ and $j$ |
| $\omega$ | Grid angular frequency in rad/s |
| $\Omega_k$ | Complex eigenfrequency $k$ in Hz |
| $\phi_i$ | Rotor angle at node $i$ in rad |
| $A$ | Unweighted Graph Adjacency matrix |
| $a(G)$ | Algebraic connectivity of graph $G$ |
| $B$ | Oriented Incidence matrix |
| $b_{k\sigma_k}$ | Fourier series expansion coefficients |
| $C_i$ | Clustering coefficient of node $i$ |
| $c_{ik}$ | Element $i$ of eigenvector $k$ |
| $D$ | Node-degree matrix |
| $d_i$ | Degree of node $i$ |
| $E$ | Coupling matrix, Generalized Laplacian matrix |
| $h_i$ | Number of links shared among the $d_i$ neighbors of $i$ |
| $J$ | Moment of inertia in $kgm^2$ |
| $K_{ij}$ | Power line capacity between nodes $i$ and $j$ in W |
| $L$ | Unnormalized Unweighted Graph Laplacian matrix |

I. Introduction

Electric power grids provide a highly reliable electrical service to billions of customers. In fact, the average outage time experienced by a consumer has kept decreasing in recent years, reaching a record low of 12.5 minutes in Germany, in 2014 [1]. However, the energy transition from a centralized power production with unilateral power flow towards an increased supply of decentralized and more volatile renewable energy resources with bidirectional flow, might become harmful for the stability of electricity grids in the future. In the currently existing grids, the synchronous generators and synchronous motors provide, with their rotating masses, high inertia to the system, which automatically reacts to disturbances. For instance, an abrupt increase in load demand can be momentarily balanced by a change of the kinetic energy of rotating synchronous generators, causing some generators to slow down and deviate from the grid frequency, but ensuring the overall stability of the network. With an increasing share of renewable energy, this buffer for the electrical energy is expected to decrease since solar cells and conventional wind turbines do not provide such inertia to the system [2]. Therefore, it will be increasingly important to obtain a deeper understanding of
how fast disturbances decay and spread in the grid and 
how this depends on the topological connectivity and the 
system parameters, in order to maintain a reliable control 
of the network.

Many authors have studied the role of system topol-
gy for the robustness of power grids against large dis-
turbances, such as intentional and random removals of 
nodes and edges [6, 10]. For small disturbances, on the 
other hand, the small-disturbance rotor angle stability 
has been properly defined [3] and thoroughly studied by 
assessing the solution of the system swing equations and 
its conditions of stability. In fact, extensive attention 
has been given to the eigenvalues and eigenvectors of 
the stability matrix of multiple-machine systems [6, 8, 9] to, 
for instance, optimize the parameters and grid location 
of Power System Stabilizers (PSS) [8]. Nonetheless, lit-
tle attention has been given to the propagation of small 
disturbances and how the latter depends on grid topol-
y and the distribution of system parameters. In order 
to study the decay and propagation of disturbances, we 
implement a hybrid approach to combine graph theory 
tools with electric parameters of inductive grids [3] and 
consider only undirected graphs to depict the smart grid 
concept, according to which consumers could rapidly be-
come producers and exchange the existing hierarchical 
power transmission into a bidirectional system.

In this article, we first introduce the mathematical 
model for generators, loads and perturbations. Secondly, 
we explain the construction of three networks (Small-
world, Random, German grid) and how their prop-
ties relate to the perturbations dynamics. Then, we 
perform a spectral analysis of the perturbations eigen-
frequency distributions and eigenvectors localization in 
these topologies. Finally, in an attempt to highlight the 
crucial connection to topology, we analyze the responses 
of the dynamic topology control strategy and the power 
system stabilizers.

II. Phase Dynamics Analysis

A. Mathematical Model

Phase dynamics in AC electricity grids have been 
modeled by active power balance equations with addi-
tional terms describing the dynamics of rotating ma-
tines [6, 10–14]. We specifically assume loads to be syn-
chronous motors whose 
chronous motors whose 
chronous motors whose 
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inertia, damping and power line capacity 
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This control response can be simply modeled as an addi-
tional damping term [6]. Here, we consider fixed voltages 
(i.e. at $V_i = 1\ p.u.$), which eliminates dynamic terms in 
the reactive power balance equation as they only appear 
in higher order when voltage dynamics, in addition to 
phase dynamics, are considered [6, 12, 14].

The rotor angle is expressed as $\varphi_i(t) = \omega t + \theta_i(t)$. By 
assuming that $\theta_i << \omega$ and that the rate at which energy 
is stored in the kinetic term is much less than the rate at 
which energy is dissipated by friction (i.e. $|J\dot{\theta}_i| << 2\gamma\omega$), 
Eq. (2) can be simplified as [11]:

$$P_i = J\omega\dot{\theta}_i + 2\gamma\omega\dot{\theta}_i + K \sum_j A_{ij} \sin(\theta_i - \theta_j).$$

B. Dynamics of Disturbances in the Grid:

In order to study the propagation of disturbances, we 
set $\varphi_i(t) = \omega t + \theta^0_i + \alpha_i(t)$ with steady state phases $\theta^0_i$; the 
solutions of Eq. (2). The dynamics of $\alpha_i(t)$ are governed by:

$$\partial^2_t \alpha_i + 2\Gamma \dot{\alpha}_i + \frac{P^i}{J\omega} - \sum_j \frac{K}{J\omega} A_{ij} \sin(\theta^0_i - \theta^0_j + \alpha_i - \alpha_j),$$

where $\Gamma = \gamma/J$. Since the steady-state natural or 
inherent stability of a system can be analyzed via a lin-
erized unregulated condition [3], we consider small pertur-
bations from the stationary state, as typically done 
in small-signal stability analyses, and expand Eq. (3) in 
($\alpha_i - \alpha_j$), which yields linear equations on the grid [17]:

$$\partial^2_t \alpha_i + 2\Gamma \dot{\alpha}_i = - \sum_j t_{ij} (\alpha_i - \alpha_j),$$

with $t_{ij} = \frac{K}{J\omega} A_{ij} \cos(\theta^0_i - \theta^0_j)$ [17]. We let $E$ to be 
formed as $E_{ii} = \sum_j t_{ij}$, and $E_{ij} = -t_{ij}$. This matrix is 
a weighted Laplacian and has been previously identified 
in synchronization studies of coupled-oscillator networks,
with possibly including ohmic losses \[13\], as well as in linear stability studies for purely inductive grids \[9\], under the name of stability matrix.

We express the perturbation at node \(i\) as a complex Fourier series, \(\alpha_i(t) = \sum_{k=1,\sigma_2=\pm}^{N} b_{k\sigma_2} c_i \exp(-j\Omega_{k\sigma_2} t)\), to then plug it into Eq. (4) to obtain:

\[
(\Omega_{k\sigma_2} + j 2\Gamma \Omega_{k\sigma_2}) c_i = E \tilde{c}_k. \tag{5}
\]

The stationary solution of the perturbation is \(c_i^* = \frac{1}{\sqrt{N}} 1\) and \(\epsilon_j = 0\). This holds for both unweighted and weighted Laplacian matrices due to the linear dependence of the diagonal on the off-diagonal elements. For \(\Gamma = 0\), we find from the eigenvalue equation \(E \tilde{c}_k = \Lambda_k \tilde{c}_k\), the eigenvectors \(\tilde{c}_k\) and the eigenvalues \(\Lambda_k\) of the coupling matrix, related to the eigenfrequencies by \(\epsilon_k^2 = \Lambda_k\). From the real symmetry of \(E\), the eigenvalues and eigenvectors are real. Furthermore, since we assume the same \(\Gamma\) at every node, we obtain for \(\Gamma \neq 0\), the same eigenmodes \(\tilde{c}_k\) with two complex eigenfrequencies \(\Omega_{k\sigma_2} = -j \Omega + \sigma_j \sqrt{T^2 - \epsilon_k^2}\). For \(\epsilon_k \geq \Gamma\), \(\Im(\Omega_k) = -\Gamma\). For \(\epsilon_k < \Gamma\), we obtain \(\Im(\Omega_k) < -\Gamma\), which produces the fastest amplitude decay and \(\Im(\Omega_k) > -\Gamma\), which produces the slowest amplitude decay, creating long-lasting perturbations. Since slowly decaying modes may increase the impact of disturbances on the power system stability, it is highly important to find out the topological and system conditions for such slow amplitude decays.

On the other hand, the stationary state of Eq. (2) can also account for the grid topology when written in matrix form,

\[
P = KB \sin(B^T \theta). \tag{6}
\]

A DC approximation of the angular differences (i.e. \(B^T \theta \ll \frac{\pi}{2}\)) considerably reduces the computational time \[13\] when compared to other more accurate methods such as solving the nonlinear swing equations Eq. (2) or solving Eq. (3) via a root-finding algorithm. Since \(B\), of size \((N, \kappa)\), is related to \(L = B B^T\) (and \(L = D - A\), Eq. (4) can be expressed as \(P = K B B^T \theta\) or \(P = K L \theta\) in a DC approximation. For a fully connected graph, \(L\) is non-invertible (it possess one zero eigenvalue). Therefore, the steady state phases are obtained from the Moore-Penrose pseudoinverse:

\[
\theta = \frac{1}{K} L^+ P, \tag{7}
\]

where \(L^+ = (L^T L)^{-1} L^T\). This approximation is accurate enough as long as \(P_i \ll d_i K\). The reason is that \(P_{avg} = \frac{P_i}{d_i}\) is the average mechanical power generated or consumed at node \(i\), which leaves or enters the node in the form of electric power through the transmission lines that are connected to it. If we consider that the electric power in each of these lines does not deviate much from the average, then \(P_{avg} \approx K \sin(\theta_i - \theta_j)\), and since the condition \(|\theta_i - \theta_j| << \frac{\pi}{2}\) is needed for linearization, then \(P_{avg} \ll 1\) must be fulfilled. The statement \(P_i \ll d_i K\) follows.

III. Electric Power Transmission Grid Models

Authors in \[20\] proposed a model that interpolates between a lattice and a random graph based on \(p\). For a certain range of \(p\), there is a coexistence of small average Path Length, \(l_{avg}(p)\), and high average Clustering Coefficient, \(C_{avg}(p)\), forming the Small-world network, which mimics many real-world networks that contain small average path lengths, but also have unusually large clustering coefficients \[21\]. The average Path Length is defined as \(l_{avg}(p) = \frac{1}{N(N-1)} \sum_{i,j} l_{ij}\). The Clustering Coefficient is a ratio between the actual number of edges among the neighbors of node \(i\) and the number of edges that would exist if those neighbors were fully connected among themselves. Mathematically, \(C_i = \frac{1}{\binom{d_i}{2} (d_i - 1)} \sum_j c_{ij}\). The average Clustering Coefficient is simply \(C_{avg}(p) = \frac{1}{N} \sum_i C_i\).

A \(p = 1\) generates a Random network, which may not be necessarily similar to the Erdős-Rényi random network commonly referred to in the literature. To be more precise, despite similar clustering coefficients and average path lengths, a Watts-Strogatz network with \(p = 1\) is not identical to an Erdős-Rényi random network with same size and same \(d_{avg}\), since for example, the Watts-Strogatz algorithm does not allow nodes to exist with degree smaller than \(\frac{d_{avg}}{2}\), whereas Erdős-Rényi does \[22\].

Here, we study Small-world and Random networks. Firstly, there is a considerable amount of transmission grids that present similar characteristics to the former: Sweden, Finland, Norway, part of Denmark, U.S. Western States, Shanghai, Italy, France, Spain and Northern China \[23\]. Secondly, Small-world networks have economical and structural feasible features for electricity distribution in smart grids, as proven by using real data from the Dutch power grid \[24\]. On the other hand, Random networks are proven to be more robust than multiple networks against intentional attacks \[3\], which makes their inclusion also important for our study.

To create our grids, we select \(N = 500\) and \(d_{avg} = 10\). For the Small-world network we fulfill the condition \(N >> d_{avg} >> \ln(N) >> 1\), to have a sparse but connected graph \[21\]. Then, we find \(l_{avg}(p)\) and \(C_{avg}(p)\), for different values of \(p\). For each \(p\), we average \(l_{avg}(p)\) and \(C_{avg}(p)\) over 25 realizations. We compare both normalized parameters in Fig. \(1\), and then retrieve the \(p\) that yields the greatest difference between them. This rewiring probability is \(p = 3.42 \times 10^{-2}\), with normalized parameters \(l_{avg}(p)/l_{avg}(0) = 0.198\) and \(C_{avg}(p)/C_{avg}(0) = 0.923\), which results in the Small-world network shown in Fig. \(20\). For the Random network, we set \(p = 1\), which results in the grid shown in Fig. \(3\), with \(l_{avg}(p)/l_{avg}(0) = 0.1154\) and \(C_{avg}(p)/C_{avg}(0) = 0.0239\). The parameters for \(p = 0\) are
function, is given by the power-angle curve of a synchronous generator connected to an infinite busbar. The smallest nonzero eigenvalue of the Laplacian matrix is called the algebraic connectivity, and its corresponding eigenvector, the Fiedler vector. Since the coupling matrix depends on the angular differences, which are inherently related to the power distribution, it would be highly convenient to know lower and upper bounds for its respective generalized algebraic connectivity $a_E(G)$ to avoid performing an eigenvalue decomposition every time $P$ changes. The $a_E(G)$ of the matrix $L$ can be set as the upper bound for $a_E(G)$. The smaller the angular differences, the more the coupling matrix approaches the scaled Laplacian, which can be considered as nothing else than the Laplacian matrix of a positively weighted graph. Besides, all properties of $L$ apply to $E$, including the decomposition into a product of oriented incidence matrices, i.e., $E = Q Q^T$, where $Q$ is a normalized incidence matrix. This implies that $a_E(G)$ gets to $a_L(G)$. The matrix $L$ corresponds to the coupling matrix of a network with $P = 0$, where the only power in the grid is that of the perturbation itself. It would also be very useful to provide lower bounds for the generalized algebraic connectivity of $E$. Some lower bounds have been derived for weighted graph Laplacian matrices, but the inclusion of, for instance, the weighted isoperimetric number makes their calculation computationally expensive in comparison to the explicit solution of the eigenvalue problem, which we perform in the following section.

V. Simulation

A. Selection of Grid Parameters

So far we have described multiple ways in which system variables can be directly related to topology. A strong emphasis has been given to $E$ as it not only contains information about the topology but also about the system operating state. In real power systems, those variables (node power, power line capacity, inertia, etc.) change depending on, for instance, the location of power generating sources and loads. These can exert different effects on mode distributions when small perturbations occur; thus, results from one system may not precisely apply to another one. Therefore, as previously mentioned, we prioritize parameter homogeneity to capture the influence that topology may have when the system in-
(a) Generators are represented as circles and consumers as crosses. Power line capacity of the network, $K_{\text{sworld}} \approx 5.24 \text{ GW}$. Smallest nonzero eigenfrequency, $\epsilon_2 = 5.5470 \text{ Hz}$.

(b) Randomization of generator and consumer clusters. The squares are those machines that have switched to the opposite power in comparison to Fig. (2a). Power line capacity of the network, $K_{\text{sworld}} \approx 1.12 \text{ GW}$. Smallest nonzero eigenfrequency, $\epsilon_2 \approx 2.7198 \text{ Hz}$.

(c) Bars size, 0.125 Hz. Smallest value of $\epsilon_2$ found after 1500 iterations, $\epsilon_2 = 2.3764 \text{ Hz}$.

(d) Intensity of the generalized Fiedler vector components (i.e. $|c_i|^2/\max_i |c_i|^2$) for the grid in Fig. (2b).

FIG. 2. Small-world Network from the Watts-Strogatz Model with Parameters: $N = 500$, $d_{\text{avg}} = 10$, $p = 3.42 \times 10^{-2}$, $l_{\text{avg}}(p)/l_{\text{avg}}(0) = 0.198$, $C_{\text{avg}}(p)/C_{\text{avg}}(0) = 0.923$, $l_{\text{avg}}(0) = 25.40$ and $C(0) \approx 0.67$.

Terminating agents have opposite behaviors (i.e. producers and consumers).

To be more specific, we assign values of power to each node from a bipolar distribution, i.e $P_i = \pm P$ in Watts. $P > 0$ for generators and $P < 0$ for motors (consumers). Eq. (2) synchronizes at a frequency $\theta_{\text{synch}} = \sum_{i=1}^{N} \frac{P_i}{J}$, which implies that the condition $\sum_{i=1}^{N} P_i = 0$ must be fulfilled at all times for the system to reach steady state. This is a realistic consideration since power generation must constantly match load demand. We take as reference the German installed capacity of 199.2 GW as per November 10th, 2015 [31], and consider half of the nodes for an even number- to be generators and the remaining half to be consumers. For the 500-node complex networks (Small-world and Random) this results in $P_i \approx \pm 796.80 \text{ MW}$. For the 489-node German grid, we have on average $P_i \approx \pm 814.72 \text{ MW}$. We choose the grid angular frequency $\omega = 2\pi(50 \text{ Hz})$ and moment of inertia $J = 10^5 \text{ kgm}^2$. This $J$ is, for instance, for a generator working at $\omega$, with
(a) Generators are represented as circles and consumers as crosses. Power line capacity of the network, $K_{rand} \approx 335.14$ MW. Smallest nonzero eigenfrequency, $\epsilon_2 = 5.8986$ Hz.

(b) Randomization of generator and consumer clusters. The squares are those machines that have switched to the opposite power in comparison to Fig. (3a). Power line capacity of the network, $K_{rand} \approx 309.34$ MW. Smallest nonzero eigenfrequency, $\epsilon_2 \approx 5.6250$ Hz.

(c) Bars size, 0.125 Hz. Smallest value of $\epsilon_2$ found after 1500 iterations, $\epsilon_2 = 5.4880$ Hz.

FIG. 3. Random Network from the Watts-Strogatz Model with Parameters: $N = 500$, $d_{avg} = 10$, $p = 1$, $l_{avg}(p)/l_{avg}(0) = 0.1154$, $C_{avg}(p)/C_{avg}(0) = 0.0239$, $l_{avg}(0) = 25.40$ and $C_{avg}(0) \approx 0.67$.

(d) Intensity of the generalized Fiedler vector components (i.e $|c_2|^2 / \max_i |c_i|^2$) for the grid in Fig. (3a).

B. Density of Eigenfrequencies

We study the eigenfrequency density $\rho_m(\epsilon)$ for different arrangements of generators and consumers by randomizing, within the bipolar distribution, the $P_i$ of each node. We perform $R = 1500$ iterations to obtain the average density, $\rho(\epsilon) = \frac{1}{R} \sum_{m=1}^{R} \rho_m(\epsilon)$. The results are shown in Figs. (2c, 3c, 4c). The stationary solution corresponds to $\epsilon_1 = 0$ Hz and it is not shown.

Due to the parameter homogeneity (precisely, $J$ and $K$) and the very small angular differences, it is easily perceivable that the eigenfrequencies plots are very close to the stationary solution.
(a) Generators are represented as circles and consumers as crosses. Power line capacity of the network, $K_G = 10$ GW.

(b) Intensity of the generalized Fiedler vector components (i.e. $|c_i|^2 / \max_i |c_i|^2$) for the grid in Fig. 4.

(c) Bars size, 0.5 Hz. Smallest value of $\epsilon_2$ found after 1500 iterations, $\epsilon_2 = 1.7343$ Hz. Peak density around $\rho_{peak} = \sqrt{\frac{K_G}{J_0}} = 17.84$ Hz.

FIG. 4. Extra-high-AC Voltage (380 kV and 220 kV) German Transmission Grid with Parameters: $N = 489$, $d_{avg} = 2.71$, $l_{avg} = 9.9384$, $C_{avg} = 0.2021$.

to the scaled versions- by a factor of $\frac{K}{J_0}$ - of their corresponding L-spectra. This means that they represent the influence of topology alone when the power vector is binarily distributed throughout the grid. In such circumstances, we observe that:

- The nonzero eigenfrequencies for all networks exceed, for the chosen parameters, the damping rate $\Gamma$ (i.e. $\epsilon > \Gamma$), so that disturbances decay exponentially fast with relaxation rate $\Gamma$. We currently know that under high integration of renewable energy, the system inertia will be considerably reduced [4]. With homogeneous parameters, a decrease of $J$ increases $\Gamma$ much more than $\epsilon$ since $\Gamma \propto \frac{1}{J}$ whereas $\epsilon \propto \frac{1}{\sqrt{J}}$. A considerable decrease of inertia shall produce at least one long-lasting per-
turbation mode (i.e. $\Gamma > \epsilon_2$) in the system.

- Although highly distributed, a significant peak of the German grid eigenfrequency density is located around $\sqrt{\frac{K_{sw}}{2\epsilon}} \approx 17.84$ Hz. This could be attributed to the influence of the 149 one-degree nodes in the grid whose diagonal entries in $E$ are, with very small angular differences in the system, $\sqrt{\frac{K_{sw}}{2\epsilon}} \cos(\theta_i - \theta_j) \approx \sqrt{\frac{K_{sw}}{2\epsilon}}$.

- The average eigenfrequency density of the Random network resembles the Marchenko-Pastur distribution, expected for uncorrelated random matrices.

### C. The Effect of Clustering

The values $K_{sworld}$ and $K_{rand}$ were obtained for the vectors $P_{sworld}$ and $P_{rand}$ assigned to Figs. 2a,3a, in which there are visible clusters of generators and consumers. If $P$ is randomized, reducing the size of the clusters, smaller values of $K$ can be found. This effect was studied in [34] for a bipolar distribution of frequencies (power in our case), and it was shown that synchronization is enhanced when adjacent nodes have opposite frequencies, resulting in a diminished frequency similarity throughout the grid. This simply means that synchronization is enhanced when generators are surrounded by consumers and vice versa. Moreover, critical effects, such as cascading failures, are less likely to be triggered if a greater frequency dissimilarity prevails. This was demonstrated statistically in [35], where authors claim that the existence of large clusters of generators and consumers turns the grid vulnerable against cascading failures, since the likelihood for a whole cluster to disconnect at once appears to increase with increasing cluster size.

Figs. 2b,3b provide an insight into the effect of randomization. The squares are those machines that have appeared to increase with increasing cluster size. It is clear that clusters are reduced, resulting in smaller power line capacities (i.e $K_{sworld} \approx 1.12$ GW and $K_{rand} \approx 309.34$ MW), but also in smaller nonzero eigenfrequencies. For the Small-world network, we obtained for Fig. 2d, $\epsilon_2 = 5.5470$ Hz, whereas for Fig. 2b, $\epsilon_2 = 2.7198$ Hz. For the Random network, we obtained for Fig. 3a, $\epsilon_2 = 5.8986$ Hz, whereas for Fig. 3d, $\epsilon_2 = 5.6250$ Hz. In fact, out of the 1500 iterations for each complex network, no single value of $\epsilon_2$ was greater than the ones from Figs. 2a,3a. This implies that whereas clustering is detrimental to grid stability and to cascading outage prevention, the larger power capacity needed to ensure stability in the presence of clusters results in an increment of the smaller nonzero eigenfrequency, leading in fact to faster mode damping rates and thereby to a greater resilience of the power system to small perturbations.

### D. Spatial Distribution of the Generalized Fiedler Vector Intensity

In Figs. 2d,3d,4b, we show the intensity, $|\epsilon_2|^2$, of the generalized Fiedler vector; the eigenmode corresponding to the smallest nonzero eigenfrequency. The intensity at each node is divided by max$_i |\epsilon_2|^2$. We observe that in the Random network, Fig. 3d, the eigenmode is strongly localized with most of its intensity on a single node. In the Small-world network, Fig. 2d, the eigenmode intensity is spread over many nodes, which are far away from each other. In the German transmission grid, the intensity is spread over most of the grid, with greater intensity in the Southern and Northwestern part of the system, see Fig. 4b. To understand this behavior, we can refer to the fact that the discrete wave equation Eq. 4 was first derived for the problem of randomly coupled atoms in harmonic approximation and has been intensively studied for various random distributions of the coupling $t_j$ [34,37]. For nonzero eigenfrequency $\epsilon_k$, the eigenmodes were found, for a random chain of nodes, to be localized with localization length $\xi(\epsilon_k) \sim 1/\epsilon_k$ [34,38], due to the random scattering of waves along the chain. This is an example of the so-called Anderson localization, which is enhanced when the amplitude of randomness is increased [39]. In grids with higher $d_{avg}$, the localization length is typically found to be larger. Moreover, the localization length is typically smallest in tree-like grids, whereas it becomes larger the more meshed the grid is; in which case the eigenmodes can become even delocalized [37,38]. On the other hand, $C_{avg}$ is a measure of how strongly meshed a grid is. We observe that the Random network, Fig. 3d, has a very small average clustering coefficient $C_{avg} = 0.016$, explaining the fact that its eigenmode is strongly localized, whereas the German transmission grid, shown in Fig. 4d, is meshed with $C_{avg} = 0.2021$ and the Small-world network in Fig. 2d is more strongly meshed with $C_{avg} = 0.61841$, explaining that the eigenmode intensity in these grids is more delocalized and spread over many nodes. The observation that the eigenmode is strongly localized in the Random network may have important consequences for the design of stable electricity grids: if the phase perturbation is initially in a state localized around a node with localization length $\xi_k$, then that disturbance remains localized there and it decays exponentially in time [17]. Thus, lesser meshed grids may help to localize disturbances more strongly.

### VI. Potential Contributions to Grid Control Strategies

**Dynamic Transmission Topology Control (TC):** Although scalable practical solutions have not been yet achieved, there is an ongoing interest in the research community for this emerging control technology [10] for its potential to manage the uncertainty of power sources and flow patterns on a grid with high penetration of...
renewable energy [41]. The control strategy consists of switching lines on or off to relieve voltage and line flow violations [41], with a considerable impact on small-signal and transient stability [41]. Based on our analysis, any TC action that modifies the structure of the grid should firstly guarantee phase-cohesiveness (e.g. through grid mechanical power tuning [42] as an optimization problem) and secondly that \(\sqrt{a(L')}\), where \(L'\) is the Laplacian matrix of the modified topology, is never less than the damping rate of the system; otherwise, there will surely exist at least one mode that decays slowly in time.

Power System Stabilizer: Here, we highlight the benefit of providing a mode-related input to a PSS, which is the main grid control device to guard the system against small-signal instability. PSS are off-line tuned generator controllers for which significant disadvantages have been found, mainly due to being local devices that do not use remote signal inputs and therefore do not adaptively change their setpoints according to the power system operating conditions [43].

Formerly, PSS used a measurement of the speed deviation of a number of points along the generator’s shaft to then calculate the average speed deviation. For long shafts prone to torsional oscillations, this method turned out to be troublesome [8]. It was later found that the need to measure the speed deviation at a number of points along the shaft can be avoided by calculating the average speed deviation from measured electrical quantities. This method indirectly calculates the equivalent speed deviation \(\Delta \omega_i^{eq}(t)\) from the integral of the accelerating power at machine \(i\) [8]:

\[
\Delta \omega_i^{eq}(t) = \frac{1}{J\omega} \int_{-\infty}^{t} (\Delta P_i(t') - \Delta P_e(t')) \, dt',
\]

where \(\Delta P_i\) and \(\Delta P_e\) are power changes at node \(i\). The former can be retrieved from the angular frequency measured by the end-of-shaft speed sensing system [8].

Thus, this PSS requires two local input signals. Nonetheless, we subsequently show that \(\Delta \omega_i^{eq}(t)\) can account for the topology and system state once the eigenmodes of the coupling matrix are known. From Eq. (4), we note that \((\Delta P_i(t) - \Delta P_e(t))/(J\omega) = \partial_i^2 \alpha_i + 2\Gamma \frac{\partial_i}{\partial t} \alpha_i\). Therefore, we find that \(\Delta \omega_i^{eq}(t) = \frac{\partial_i}{\partial t} \alpha_i(t) + 2\Gamma \alpha_i(t)\). Next, we can insert the expansion of the phase deviations in terms of the eigenvectors \(\vec{c}_k\) of the coupling matrix, \(\alpha_i(t) = \sum_{k=1,\sigma=\pm}^N b_{k\sigma} c_{ik} \exp(-j\Omega_{k\sigma} t)\). Thereby, we find:

\[
\Delta \omega_i^{eq}(t) = \sum_{k,\sigma} (-j\Omega_{k\sigma} + 2\Gamma)c_{ik} b_{k\sigma} \exp(-j\Omega_{k\sigma} t).
\]

Then, it remains to find the expansion coefficients \(b_{k\sigma}\) in response to changes of electric power. This has been recently obtained as a spectral representation of the linear response to changes in \(P_e\) [44] and in terms of a weighted integral over time of the change in \(P_e\) [45]. Employing these results, we finally get:

\[
\Delta \omega_i^{eq}(t) = \sum_{k,\sigma_k} \frac{\sigma_k(-j\Omega_{k\sigma_k} + 2\Gamma)c_{ik}}{\sqrt{1 - \tau^2\epsilon_k^2}} \times \\
\int_{-\infty}^{t} \frac{dt'}{\tau} \exp(-j\Omega_{k\sigma_k}(t - t')) \sum_v c_{vk} \Delta P_{v}(t'),
\]

where \(\tau = 1/\Gamma\). As noted, only \(\Delta P_{v}(t')\) is now needed as an input signal. Furthermore, knowing the eigenvectors \(\vec{c}_k\) allows to reduce the number of nodes \(v\) from which the input signal \(\Delta P_{v}(t')\) is needed. This is because, since the generalized Fiedler vector has the largest impact on grid stability, the PSS may only need to take signals from those nodes with greatest eigenmode intensity.

A proper design for a PSS to implement Eq. (11) in order to produce a leading voltage signal to control perturbation modes would then be a topic of further discussion.

VII. Conclusion

This paper, far from delving into the already thorough studies of small-signal stability, aims to identify the causes that make perturbations propagate in some way or another. For this purpose, we perform a similar eigenanalysis but establish a direct relationship between the dynamics and the network structure. We first focus on eigenfrequency plots to show that the German grid or other real networks (e.g. Italy, France, etc.) with Small-world traits may produce long-tailed distributions. Of course, in real power grids, system parameters exert a strong influence on the distributions, but in order to isolate the influence of topology, we have only assigned binary powers and homogeneous parameters. Moreover, with these plots, we have shown that although clusters are detrimental to the control of critical effects (i.e cascading failures, synchronization), they make small perturbations fall faster due to increased power line capacities and therefore increased damping rates.

In power system planning, the addition of one node or one edge shall take into account the network’s average clustering coefficient in order to improve system controllability, as we have found strong indications that the degree of localization of the generalized Fiedler eigenmode intensity tends to increase with a decrease of \(C_{avg}\). It is, at the same time, crucial to implement strategies to keep track of the generalized algebraic connectivity to avoid having long-lasting perturbations in the system.

Finally, we have proposed ways in which future and existing grid control strategies can consider the influence of topology in their designs and actions. Therefore, we expect that, having addressed multiple topological aspects in relation to perturbations dynamics, this study serves as an insightful source for upcoming research on power system planning and control.
VIII. Acknowledgment

We gratefully acknowledge the support of BMBF CoNDyNet FK. 03SF0472A.
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