Abstract—We study the traffic routing game among a large number of selfish drivers over a traffic network. We consider a specific scenario where the strategic drivers can be classified into teams, where drivers in the same team have identical payoff functions. An incentive mechanism is considered to mitigate congestion, where each driver is subject to dynamic tax penalties. We explore a special case in which the tax is affine in the logarithm of the number of drivers selecting the same route from each team. It is shown via a mean-field approximation that a Nash equilibrium in the limit of a large population can be found by linearly solvable algorithms.

I. INTRODUCTION

Transportation is a major energy consuming sector in the United States, accounting for 28% of total use and 26% of green-house emissions [1]. The economic loss due to traffic congestion is also significant; in 2014, it was estimated to account for $160 billion [2]. Recently, new stakeholders such as ride-hailing companies and mobile routing apps are reported to have substantial impacts on urban traffic networks [3]. Therefore, novel theoretical frameworks for traffic analysis and control in which ride-hailing companies act as decision-makers are urgently needed.

A traffic system can be analyzed by the theory of dynamic games, where the non-cooperative drivers compete over the shared network [4], [5]. Strategic behaviour of individuals often results in a game theoretic equilibrium that is not necessarily socially optimal. In the literature, the loss of optimality (efficiency loss) is commonly measured in terms of the price of anarchy (PoA) [6], [7]. PoA can be improved by an appropriate incentive design; [8], [9] considered dynamic toll mechanisms, which are assumed to be operated by a Traffic System Operator (TSO).

It should be noted that incentive mechanisms operated by TSOs are restricted by the information available to them. For example, [10], [11] consider fixed taxation for each route based on the full characterization of the network topology and user demands. In contrast, [12] proposed a tax structure which is independent of this information. A complete review of the design and evaluation of road network pricing schemes is provided in [13].

Traffic systems in reality typically consist of a large number of strategic drivers and their game-theoretic analysis is often computationally challenging. Therefore, an incentive mechanism should ideally be designed in such a way that (i) its implementation is computationally simple so that it is scalable to large networks, and (ii) there exists a simple computational procedure to find an equilibrium in the resulting dynamic game. If the later condition is not satisfied, it is questionable that drivers in reality can ever take the equilibrium strategy.

To simplify the analysis of routing games with a large number of drivers, a common approach is to adopt a macroscopic perspective, where the density of the vehicles is modeled rather than the dynamics of individual vehicles [14]. In a similar spirit, recently the framework of Mean Field Game (MFG) [15] was applied to compute the traffic density propagation induced from the interactions of many independent strategic drivers. Despite the substantial progress in MFGs, the number of papers utilize the MFG theory for macroscopic traffic modelling is still limited. Some recent studies have explored MFGs with multiple classes of drivers [16], which is appropriate to model traffic games. Additionally, some papers have considered scenarios with a major player and a large number of minor players [17].

In MFG setting, the traffic flow is modeled as a fluid whose behavior can be obtained by solving Hamilton-Jacobi-Bellman (HJB) equation coupled with the standard conservation law applied to the vehicles, refered to as Fokker-Planck-Kolmogrov (FPK) equation. In [18], an MFG approach is taken to address multi-lane traffic management, using a semi-Lagrangian scheme to solve an approximation of a coupled HJB-FPK.

In [19], [20], a discrete-time dynamic stochastic game over an urban network is studied, wherein at each intersection, each strategic driver selects one of the outgoing links randomly with respect to his/her mixed policy. A tax mechanism is offered which yields a linearly solvable game under the assumptions that all drivers have common origin and destination (O/D) and travel costs. Then, the backward HJB and forward FPK equations can be solved independently and thus a MFE is obtained through a linearly-solvable set of equations. This result [19] has an important computational advantage with respect to previous results [15] which dealt with a coupled set of HBJ-FPK equations. However, the simplified problem setting in [19] with shared pair of (O/D) and travel cost has a limited applicability to practical traffic routing problems. To address this gap, in this paper, we consider more realistic settings, in which there exist multiple teams with different (O/D) pairs and travel costs. As the main technical contribution of this paper, a Nash equilibrium (NE) in the limit of a large number of drivers (MFE) for the multi-team setting is given as the solution to a set of linearly

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solvable optimal control problems.

This paper is organized as follows: The multi-team road traffic game is set up in Section II, the behavior of the game in large population limit is studied in Section III-A. An auxiliary set of control problems is introduced in Section III-B which is used to solve the game in the large population limit in Section III-C. The game with finite number of drivers is considered in Section III-D. Numerical studies are summarized in Section IV before we conclude in Section V.

II. PROBLEM FORMULATION

We assume there is a large number of drivers traveling over a shared network $G = (\mathcal{V}, \mathcal{E})$, referred to as the traffic graph, where $\mathcal{V} = \{1, 2, \ldots, V\}$ is the set of nodes (intersections) and $\mathcal{E} = \{1, 2, \ldots, E\}$ is the set of directed edges (links). The set of nodes to which there exist a direct link from $i$ is labeled by $\mathcal{V}(i)$. We assume that drivers are categorized into L teams, and that drivers from the same team have the same (O/D) pair. The number of drivers in each team is denoted by $N_l$ $\forall l \in \{1, 2, \ldots, L\}$. At each time step $t \in T = \{0, 1, \ldots, T-1\}$, the node at which the $n$-th driver from the $l$-th team ($n \leq N_l$) is located is represented by $i_{n,l,t}$. For each $n$ and $l$, $i_{n,l,t}$ is a random process. Let $P_{n,l,t} = \{P_{i_{n,l,t}}\}_{i \in \mathcal{V}}$ be the probability distribution of the individual driver $n$ in the team $l$ and over the nodes of the traffic graph at time $t$. At $t = 0$, we assume all drivers in the same team will start from a common probability $P_{l,0}$; however, $i_{n,l,0}$ for each $n$ and $l$ is realized independently with respect to $P_{l,0}$.

For $(n_1, l_1) \neq (n_2, l_2)$, we assume $i_{n_1,l_1,t}$ and $i_{n_2,l_2,t}$ are independent random variables. At every time step, driver $(n, l)$ selects an action $j_{n,l,t} \in \mathcal{V}(i_{n,l,t})$, and moves to the node $j_{n,l,t}$ at time $t + 1$ (i.e. $i_{n,l,t+1} = j_{n,l,t}$). Each driver would follow a randomized policy (strategy) $Q_{i_{n,l,t}} = \{Q_{i_{n,l,t}}^{ij} \}_{j \in \mathcal{V}(i)}$, which represents the probability distribution according to which she chooses her next destination. Let $\Delta^\mathcal{V}$ be the $J$-dimensional probability simplex. Then for all drivers $(n, l)$ at time $t$, $Q_{n,l,t} = \{Q_{n,l,t}^{ij}\}_{j \in \mathcal{V}}$ belongs to the space of possible mixed strategies $\mathcal{Q}$, where

$$\mathcal{Q} = \{\{Q^{i}\}_{i \in \mathcal{V}} : Q^{i} \in \Delta^{\mathcal{V}(i)} - 1 \forall i \in \mathcal{V}\}.$$ 

We assume that each driver fixes her strategy $\{Q_{n,l,t}\}_{t \in T}$, at $t = 0$ based on the global knowledge the game’s parameters and she will not be able to update it during the game. In this setting, the location probability distribution for driver $(n, l)$ is computed recursively by

$$P_{n,l,t+1} = \sum_{i} P_{n,l,t} Q_{n,l,t}^{ij}, \quad (1)$$

with initial $P_{n,l,t} = P_{l,0}$. Further details of the game setup are given below. Note that they are natural generalizations of the setup studied in [19] to multi-team scenarios.

B. Cost Functions and Congestion-reducing Incentives

Each driver is subjected to two different categories of costs– travel cost and congestion cost.

1) Travel cost: Moving from node $i$ to $j$ requires a fixed amount of cost e.g., fuel cost. Let $C_{ij}^{l}$ be a given cost for drivers in team $l$ to take this action at time $t$.

2) Congestion cost: This is the tax cost imposed by the TSO to incentivize drivers to adopt a nominal routing policy, which we assume it is pre-specified. The nominal policy is denoted by $R_{ij}^{l}$, representing the probability of selecting the next destination $j$ at node $i$ at time step $t$. Namely, $R_{ij}^{l}$ satisfies $R_{ij}^{l} \geq 0$ and $\sum_{j} R_{ij}^{l} = 1$. To penalize the deviation from this reference distribution, TSO introduces the log-population tax mechanism in which $(n, l)$-th driver taking action $j$ at time $t$ from node $i$ will be charged the value of

$$\pi_{n,l,t}^{ij} = \sum_{m=1}^{L} a_{tm} \left( \log \frac{K_{m,t}^{ij}}{K_{m,t}^{ij}} - \log R_{ij}^{l} \right), \quad (2)$$

where $K_{m,t}^{ij}$ denotes the population of drivers (including driver $(n, l)$) in the case of $m = l$ from each team in node $i$ at time $t$. Similarly, $R_{m,t}^{ij}$ represents the number of drivers (from team $m$) who are at node $i$ at time $t$ and willing to take the action $j$. We consider the tax formula (2) mainly because (i) it is a natural extension of the mechanism in [19] and (ii) the resulting game is efficiently solvable, as we will see in Section III.

The tax mechanism (2) implies that the driver $(n, l)$ selecting action $j$ will be penalized if the fraction of drivers taking this action is greater than the desired value and it will be rewarded otherwise. The tax penalty (2) is the weighted sum of charges associated with individual teams’ deviation from the nominal routing policy. The parameter $a_{tm} > 0$ means that the drivers in team $l$ will be penalized by selecting action $j$ at node $i$ if the road from $i$ to $j$ is going to be overpopulated by the drivers from team $m$. Notice that $K_{m,t}^{ij} \geq R_{ij}^{l} \geq 0$ by construction. We adopt the convention that $\log(0) = 0$. In this setting, each driver is trying to minimize the expected value of the tax cost i.e., $\Pi_{n,l,t}^{ij} = E[\pi_{n,l,t}^{ij}|i_{n,l,t}, j_{n,l,t}]$ as well as fixed costs. In summary, each driver is willing to minimize her own charge selfishly by solving:

$$\min_{\{Q_{n,l,t}\}_{t = 0}^{T-1}} \sum_{t = 0}^{T-1} \sum_{i,j} P_{n,l,t} Q_{n,l,t}^{ij} C_{n,l,t}^{ij} + \Pi_{n,l,t}^{ij}$$

$$\text{s.t.} \sum_{j} Q_{n,l,t}^{ij} = 1 \forall i. \quad (3)$$

Notice that (3) is a game involving $\sum_{l=1}^{L} N_l$ players, since the value of $\Pi_{n,l,t}^{ij}$ depends on the strategies of all other drivers in the system. Analyzing this game is a nontrivial task. However, if the number of drivers is sufficiently large, the game is well-approximated by the one with infinitely many drivers. In the next section, we develop such an approximation via the MFG theory.

III. MAIN RESULTS

In this section, we study the MFG approximation of the game with a large number of drivers. Then, an efficient algorithm is provided to calculate the MFE of the game.
A. Large population limit

In [19], for the case of single team, an explicit expression for \( \lim_{N \to \infty} \Pi_{n,l} \) is obtained. Here, we provide a generalized alternative method to find an analogous limit for the multi-team case.

Due to the indistinguishability of drivers in a team, we can restrict ourselves to the symmetric setups where drivers in a team share a common policy \( Q_l \). For each \( l \), we can calculate the induced probability distribution \( P_l^* \), based on \( Q_l \). In what follows, we study the asymptotic behavior of the congestion charge in the large population limit (i.e., \( N = \sum_m N_m \to \infty \) for the fixed population ratios \( \frac{N_l}{N} \)).

**Lemma 1:** If \( P_l^* \) is the optimal response for the fixed population ratios \( N \) charge in the large population limit (i.e., \( \Pi_{n,l} \to \Pi_{n,l}^* \)), then

\[
\lim_{N \to \infty} \Pi_{n,l} = \sum_{m=1}^{L} a_{lm} \log \frac{Q_{m,l}^{ij}}{R_l^{ij}}.
\]

**Proof:** See appendix A.

Therefore, in the large population limit, the optimal response of \((n,l)\)-th driver is characterized by:

\[
\arg\min_{\{Q_{n,l},l \}_{l=0}^{T-1}} \sum_{t=0}^{T-1} \sum_{i,j} \Pi_{n,l,t} Q_l^{ij_l}(C_{l,t}^{ij} + \sum_{m=1}^{L} a_{lm} \log \frac{Q_{m,l}^{ij}}{R_l^{ij}})
\]

\[
\text{s.t} \quad \sum_{j} Q_l^{ij} = 1 \quad \forall i,
\]

where the probability distribution \( P_{n,l,t} \) propagates with respect to \( (1) \). In what follows, we show that \( Q^* \) is indeed an MFG of multi-team traffic routing game if the minimizer of (5) is again \( Q_{n,l}^* \) (That is, \( Q^* \) is a fixed point). In the next subsection, we study auxiliary \( L \)-player game by which such a fixed policy is found.

B. Auxiliary optimization problem

Consider an auxiliary \( L \)-player game in which each team tries to find her optimum policy \( Q_{l}^{ij} \), the solution to:

\[
\min_{\{Q_{l}^{ij} \}_{l=0}^{T-1}} \sum_{t=0}^{T-1} \sum_{i,j} \Pi_{l,t} Q_l^{ij}(C_{l,t}^{ij} + \sum_{m=1}^{L} a_{lm} \log \frac{Q_{m,l}^{ij}}{R_l^{ij}})
\]

\[
\text{s.t} \quad \sum_{j} Q_l^{ij} = 1 \quad \forall i,
\]

Notice that in (5) the logarithmic term is fixed, in contrast to (6) where it is an optimization variable. For each \( t \in T \), one can introduce value function associated with optimal control problem (6) as:

\[
V_{l,t}(P_{l,t}) = \min_{\{Q_l^{ij} \}_{l=0}^{T-1}} \sum_{t=0}^{T-1} \sum_{i,j} \Pi_{l,t} Q_l^{ij}(C_{l,t}^{ij} + \sum_{m=1}^{L} a_{lm} \log \frac{Q_{m,l}^{ij}}{R_l^{ij}}).
\]

The value function satisfies the following coupled Bellman equations:

\[
V_{l,t}(P_{l,t}) = \min_{Q_l^{ij}_{t+1}} \left\{ \sum_{i,j} \Pi_{l,t} Q_{l,t}^{ij}(C_{l,t}^{ij} + \sum_{m=1}^{L} a_{lm} \log \frac{Q_{m,l}^{ij}}{R_l^{ij}}) + V_{l,t+1}(P_{l,t+1}) \right\},
\]

with \( V_{l,T}(P_{l,T}) = 0 \) \( \forall l, P_{l,T} \). The first main result of this paper is summarized by the following Theorem 1. It states that this set of optimal control problems (6) is linearly solvable, which means the solution is obtained by solving a linear system.

**Theorem 1:** Let \( \{\lambda_{l,t}\}_{t \in T} \) be sequences of \( V \)-dimensional vectors, \( \{\phi_{l,t}\}_{t \in T} \) be sequences of \( V \times V \)-dimensional matrices, and \( A \) be an invertible matrix whose \((l,m)\)-th element is \( a_{lm} \). Then, optimal policies \( \{Q_{l,t}^{ij} \}_{t \in T} \) can be iteratively calculated by Algorithm 1. Moreover, the value functions for each \( t \in T \) are

\[
V_{l,t}(P_{l,t}) = \sum_{i} P_{l,t}^{i} (-a_{ll} - \lambda_{l,t}^{i}).
\]

**Proof:** See appendix B.

### Algorithm 1 Optimal Strategy Computation for T-stage Road Traffic Game

1. **Initialize** \( \phi_{l,T-1}^{ij} = 0 \) for all \( i, j \in V \)
2. **for** \( t = T - 1, T - 2, \ldots \) **do**
3. \( \text{for all } i, j \in V, \text{compute:} \)
   \[
   M_l^{ij} = \begin{bmatrix} M_l^{ij}_{i} \end{bmatrix} := A^{-1} \begin{bmatrix} -a_{11} - \phi_{l,t}^{ij} \\ \vdots \\ -a_{LL} - \phi_{l,t}^{ij} \end{bmatrix}
   \]
4. \( \text{for every } i \in V, \text{compute:} \)
   \[
   \Lambda_i^{l} = \begin{bmatrix} \lambda_i^{l,t} \\ \vdots \\ \lambda_i^{l,T} \end{bmatrix} := A \begin{bmatrix} \log(\sum_j R_l^{ij} \exp M_l^{ij}) \\ \vdots \\ \log(\sum_j R_l^{ij} \exp M_l^{ij}) \end{bmatrix}
   \]
5. \( \text{for every } i, j \in V, \text{compute the optimal policies:} \)
   \[
   Q_{l,t}^{ij} = R_l^{ij} \exp (M_l^{ij} - \sum_m A_l^{-1} \lambda_i^{l,m})
   \]
6. **end for**
7. **return** \( \{Q_{l,t}^{ij} \}_{t \in T}, \{\lambda_i^{l,t} \}_{t \in T} \) and \( \{\phi_{l,t}^{ij} \}_{t \in T} \).

We stress that Algorithm 1 is linear in \( \phi \) and \( \lambda \) and thus the optimal solution can be pre-computed backward in time.

C. Equilibrium in the large population limit

Here, we find the equilibrium point of the game with a large number of drivers. It is noteworthy that \( \Pi_{n,l,t}^* \), as proved in [19], is independent of the \((n,l)\)-th driver’s own strategy \( Q_{n,l,t}^* \), even for finite \( N \). In this game, each strategic driver wants to minimize her cost function, given by

\[
J_{(n,l)}(Q_{n,l}, Q_{-(n,l)}) = \sum_{t=0}^{T-1} \sum_{i,j} P_{n,l,t} Q_l^{ij}(C_{n,l,t}^{ij} + \Pi_{n,l,t}^{ij}),
\]

where \( Q_{n,l} = \{Q_{n,l,t} \}_{t \in T} \) denotes her policy and \( \{\Pi_{n,l,t} \}_{t \in T} \) is determined by other drivers’ policy \( Q_{-(n,l)} = \{Q_{n,l,t} \}_{t \in T, \forall \tilde{n}, l \neq (n,l)} \).
Definition 1: The set of strategies \( \{Q_{n,l}^{NE}\}_{l=1,\ldots,L,n=1,\ldots,N} \) is said to be a NE if the following inequalities hold.

\[
J_{(n,l)}(Q_{n,l},Q_{n,l}^{NE}) \geq J_{(n,l)}(Q_{n,l}^{NE},Q_{n,l}^{NE}) \quad \forall (n,l)
\]

As a candidate of NE in the limit of \( N \to \infty \), we consider \( Q_{n,l}^{ij*} \), the solution to (6). To validate this guess, let’s consider all drivers except \((n,l)\) adopt \( Q_{n,l}^{ij*} \) and see if it satisfies the condition in Definition 1. One can again apply dynamic programming and define the value function for (5) as:

\[
\tilde{V}_{n,l,t}(P_{n,l,t}) = \min_{\{Q_{n,l}\}_{l=1}^{L}} \left\{ \sum_{i,j} P_{n,l,t}^{i} Q_{n,l,t}^{ij} (C_{l,t}^{ij} + \sum_{m=1}^{L} a_{im} \log \frac{Q_{m,t}^{ij*}}{R_{t}^{ij*}}) \right\},
\]

and the associated Bellman equation will be

\[
\tilde{V}_{n,l,t}(P_{n,l,t}) = \min_{\{Q_{n,l}\}_{l=1}^{L}} \left\{ \sum_{i,j} P_{n,l,t}^{i} Q_{n,l,t}^{ij} (C_{l,t}^{ij} + \sum_{m=1}^{L} a_{im} \log \frac{Q_{m,t}^{ij*}}{R_{t}^{ij*}}) + \tilde{V}_{n,l,t+1}(P_{n,l,t+1}) \right\}. \tag{8}
\]

Note that \( \bar{V} \) and \( \tilde{V} \) are value functions associated with different optimal control problems.

Lemma 2: For all \( t \in \mathcal{T} \) and any given \( P_{n,l,t} \), \( \bar{V}_{n,l,t}(P_{n,l,t}) = V_{l,t}(P_{n,l,t}) \). Moreover, an arbitrary policy \( \{Q_{n,l}\}_{l=1}^{L} \in \mathcal{Q} \) is an optimal solution to (5).

Proof: Starting from \( t = T - 1 \), by substituting the explicit expression for \( Q_{n,l}^{ij*} \) from (14), we have:

\[
\bar{V}_{n,l,t}(P_{n,l,t}) = \min_{Q_{n,l,t-1}} \sum_{i,j} P_{n,l,t-1}^{ij} Q_{n,l,t-1}^{ij} \phi_{l,T-1}^{ij} + \sum_{m=1}^{L} a_{im} \log \frac{Q_{m,t}^{ij*}}{R_{t}^{ij*}}.
\]

Similarly, the proof can be repeated inductively for all \( t \). Since the final expression does not depend on \( Q_{n,l,t} \), any allowable policy is a minimizer.

This result (all feasible solutions are optimal) can be interpreted as the Wardrop’s first principle, which states that at equilibrium, “the journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route” [21].

D. Finite number of drivers

Here, we want to study the relation between NE of the game in the limit of infinite number of drivers and actual game with finite number of drivers.

Definition 2: A set of strategies \( \{Q_{n,l,t}^{MFE}\} \) is said to be an MFE if the following conditions are satisfied.

- They are symmetric i.e.,
  \[
  \{Q_{1,t}^{MFE}\} = \{Q_{2,t}^{MFE}\} = \cdots = \{Q_{N,t}^{MFE}\} \quad \forall t
  \]
- There exists a sequence \( \{\epsilon_{N_i}\} \) satisfying \( \epsilon_{N_i} \to 0 \) as \( N_i \to \infty \) with constant ratio of \( \frac{\epsilon_{N_i}}{N_i} \) such that for every possible set of \( (n,l) \) and any allowable \( Q_{n,l,t} \in \mathcal{Q} \),
  \[
  J_{(n,l)}(Q_{n,l,t},Q_{n,l}^{MFE}) + \epsilon_{N_i} \geq J_{(n,l)}(Q_{n,l,t},Q_{n,l}^{NE})
  \]

Theorem 2 summarizes our next main result which implies that the solution to (6) is indeed MFE of the actual game.

Theorem 2: The symmetric strategy profile \( Q_{n,l,t}^{ij*} = Q_{l,t}^{ij*} \) \( \forall n \leq N_i \) where \( Q_{l,t}^{ij*} \) are obtained by Algorithm 1, is the MFE of the road traffic game.

Proof: The proof is similar to the proof provided in [19, Theorem 2].

IV. NUMERICAL ILLUSTRATION

Here, we consider an example of routing game over a traffic network (a grid world with obstacles). Team 1 and 2 are concentrated in the origin cells (indicated by 1S and 2S) at \( t = 0 \). For \( t \in \mathcal{T} \), the travel cost for each driver is given by:

\[
C_{l,t}^{ij} = \begin{cases} C_{l,term} & \text{if } j = i \\ 1 + C_{l,term} & \text{if } j \notin \mathcal{V}(i) \text{ and } i \neq j \\ 100000 + C_{l,term} & \text{if } j \notin \mathcal{V}(i) \text{ or } j \text{ is an obstacle} \end{cases}
\]

where \( \mathcal{V}(i) \) contains the cell \( i \) itself and its north, east, south, and west neighboring cells. We introduce \( C_{term} = 0 \) if \( t \neq T - 1 \) and \( C_{term} = 10 \sqrt{\text{dist}(j,D)} \) if \( t = T - 1 \) to penalize drivers if they end up far from their targeted destination, where \( \text{dist}(j,D) \) is the Manhattan distance between the drivers’ final location \( j \) and the destination cell (indicated by 1E and 2E). As the desired distribution, we use \( R_{t}^{ij} = \frac{1}{|\mathcal{V}(j)|} \) (uniform distribution) for each \( i \in \mathcal{V} \) and \( t \in \mathcal{T} \) to incentivize drivers to avoid concentrations.
For various values of $A$, the optimal policy is computed based on Algorithm 1 for $T = 50$. Fig. 1 ($a_{11} = a_{22} = 3$ and $a_{12} = a_{21} = 2$) and Fig. 2 ($a_{11} = a_{22} = 0.06$ and $a_{12} = a_{21} = 0.04$) show the snapshot of population distributions of teams 1 and 2 (respectively in the red and blue) for intermediate time steps of $t = 0, 15, 27$, and 48. As expected, when $a_{ij}^s$ are small and the fixed cost dominates, drivers will stay concentrated with higher level of congestion. Conversely, when $a_{ij}^s$ are large and the congestion associated tax is dominant, they would prefer to spread more to lower the level of congestion.

V. CONCLUSION AND FUTURE WORK

In this paper, a log-population tax setting for the multi-team routing game is proposed. Under this toll mechanism, an approximation of the game in the large population limit is obtained. It is shown the NE of the approximated game can be efficiently computed by a linearly-solvable algorithm. Furthermore, it is shown that the NE of approximated game is indeed an MFE of a large but finite population game.

Providing a protocol to find the suitable values of mechanism’s parameters ($R^U_t$ and $a_{ij}$) is left unanswered, which is worth investigating in future work. Practical application of the results for congestion management in urban traffic networks also remains to be explored. We also plan to study the proposed framework from the viewpoint of mechanism design theory.

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APPENDIX

A. Proof of a Lemma 1

Consider a single term $m \neq l$. Then:

$$
E \left[ \log \left( \frac{K_{m,t}^{ij}}{K_{m,t}} \right) - \log R_{ij}^{l,t} \right] =
$$

$$
E \left[ \log \frac{K_{ij}^{l,t}}{N_m} \right] - E \left[ \log \frac{K_{l,t}}{N_m} \right] - \log R_{ij}^{l,t}.
$$

It is elementary to check that $E \left[ \frac{K_{m,t}^{ij}}{N_m} \right] = P_{m,t}^{ij}$ and $E \left[ \frac{K_{m,t}}{N_m} \right] = P_{m,t}^i$. However, since $\log(.)$ is an unbounded function, we cannot directly deploy continuous mapping theorem [22]. In the following, we show that if the general random variable $K_{ij}^{l,t} \overset{d}{\rightarrow} p$ has bounded variance $\sigma_N^2$, then

$$
\lim_{N \to \infty} E \log(\frac{K_{ij}^{l,t}}{N_m}) \rightarrow \log(p).
$$

Let $\delta$ be a number satisfying $0 < \delta < p$. Then, we can write $\log x = f_1(x) + f_2(x)$ where

$$f_1(x) = \max\{\log x, \log \delta\}
$$

and

$$f_2(x) = 1_{x \leq \delta} \log x.
$$

We first show $\lim_{N \to \infty} E f_1(\frac{K_{ij}^{l,t}}{N_m}) = \log p$. By the strong law of large numbers, we have $K_{ij}^{l,t} \overset{a.s.}{\to} p$. By the continuous mapping theorem, $f_1(\frac{K_{ij}^{l,t}}{N_m}) \overset{a.s.}{\to} f_1(p)$. By the bounded convergence theorem, we have

$$E f_1(\frac{K_{ij}^{l,t}}{N_m}) \to f_1(p) \quad \text{as} \quad N \to \infty.
$$

Next, we show $\lim_{N \to \infty} E f_2(\frac{K_{ij}^{l,t}}{N_m}) = 0$. Notice that $K_{ij}^{l,t}$ has mean $\mu_N = p$ and covariance $\sigma_N^2 = \frac{p(1-p)}{N}$. We have

$$E \left| f_2(\frac{K_{ij}^{l,t}}{N_m}) \right| \leq E \left( \frac{K_{ij}^{l,t}}{N_m} \leq \delta \right) \left| \log \left( \frac{K_{ij}^{l,t}}{N_m} \right) \right| + \log(\delta) \tag{10a}
$$

$$\leq \left( \frac{K_{ij}^{l,t}}{N_m} \leq \delta \right) \left( \log N_m + \log \delta \right) \tag{10b}
$$

and

$$\lim_{N \to \infty} E f_2(\frac{K_{ij}^{l,t}}{N_m}) = 0 \quad \text{as} \quad N \to \infty.
$$

From (10a) to (10b), we used the fact that $1 \leq K_{ij}^{l,t} \leq N$ and thus $\log \frac{K_{ij}^{l,t}}{N_m}$ is maximized when $K_{ij}^{l,t} = 1$. From (10c) to (10d), the Chebyshev inequality

$$\Pr(|K_{ij}^{l,t} - \mu_N| \geq \epsilon) \leq \frac{\sigma_N^2}{\epsilon}
$$

is used with $\epsilon = p - \delta$. Finally,

$$\lim_{N \to \infty} \frac{\log K_{ij}^{l,t}}{N} = \lim_{N \to \infty} \frac{\log f_1(\frac{K_{ij}^{l,t}}{N_m}) + \log f_2(\frac{K_{ij}^{l,t}}{N_m})}{N} = \log p.
$$

For $m = l$, the same method can be repeated by $\mu_N = p - \frac{1}{N}$ and $\sigma_N^2 = \frac{p(1-p)}{N-1}$.

B. Proof of Theorem 1

For the base step of backward induction, let’s define $C_{l,T-1} = C_{l,T-1}^{ij}$. The Bellman equation (7) for $t = T - 1$ reduces to:

$$
\begin{align*}
\min_{i,j} & \sum_{i,j} P_{i,T-1}^{ij} C_{i,T-1}^{ij} + \sum_{m=1}^{L} a_{tm} \log \frac{Q_{m,T-1}^{ij}}{R_{T-1}^{ij}} \\
\text{s.t} & \sum_{i,j} Q_{i,T-1}^{ij} = 1 \quad \forall i.
\end{align*}

The associated Lagrangian is:

$$
\begin{align*}
\mathcal{L}_{i,T-1}(Q_{i,T-1}^{ij}) &= \sum_{i,j} P_{i,T-1}^{ij} C_{i,T-1}^{ij} + \sum_{m=1}^{L} a_{tm} \log \frac{Q_{m,T-1}^{ij}}{R_{T-1}^{ij}} \\
&+ \sum_{i} P_{i,T-1}^{ij} \lambda_{i,T-1}^{ij} (\sum_{j} Q_{i,T-1}^{ij} - 1).
\end{align*}
$$

The optimal solution $Q^*$ satisfies the stationarity condition:

$$
\frac{\partial \mathcal{L}_{i,T-1}(Q^*)}{\partial Q_{i,T-1}^{ij}} |_{Q^*} = 0
$$

Equivalently:

$$
\log \left( \frac{Q_{i,T-1}^{ij}}{R_{T-1}^{ij}} \right) = A^{-1} \begin{pmatrix} M_{l,T-1}^{ij} \\ \vdots \\ \lambda_{l,T-1}^{ij} \end{pmatrix}.
$$

If we exponentiate both sides element-wise, we have:

$$Q_{i,T-1}^{ij} = R_{T-1}^{ij} \exp \left( \frac{M_{i,T-1}^{ij}}{A - 1} \right). \tag{13}
$$

We can use the fact that $\sum_{i,j} Q_{i,T-1}^{ij} = 1$ to see

$$[A^{-1} \lambda_{l,T-1}^{ij}]_i = \log \left( \sum_{j} R_{i,j,T-1}^{ij} \exp M_{i,T-1}^{ij} \right).
$$

By substituting $\lambda_{l,T-1}^{ij}$ in (13), $Q_{i,T-1}^{ij}$ is obtained explicitly. Furthermore, (12) implies that:

$$\phi_{i,T-1}^{ij} + \sum_{m=1}^{L} a_{tm} \log \frac{Q_{m,T-1}^{ij}}{R_{T-1}^{ij}} = -a_{ii} - \lambda_{i,T-1}^{ij}. \tag{14}
$$

Therefore, by definition:

$$V_{i,T-1}(P_{i,T-1})
$$

$$= \sum_{i,j} P_{i,T-1}^{ij} \phi_{i,T-1}^{ij} + \sum_{m=1}^{L} a_{tm} \log \frac{Q_{m,T-1}^{ij}}{R_{T-1}^{ij}}
$$

$$= \sum_{i,j} P_{i,T-1}^{ij} \phi_{i,T-1}^{ij} - a_{ii} - \lambda_{i,T-1}^{ij}
$$

$$= \sum_{i} P_{i,T-1}^{ij} (-a_{ii} - \lambda_{i,T-1}^{ij}) \sum_{j} Q_{i,T-1}^{ij}
$$

$$= \sum_{i} P_{i,T-1}^{ij} (-a_{ii} - \lambda_{i,T-1}^{ij}).
$$

Plugging this value in the Bellman equation for $t = T - 2$ leads to a similar optimization to (11) with $\phi_{i,T-2}^{ij} = C_{i,T-2}^{ij} - a_{ii} - \lambda_{i,T-1}^{ij}$ which completes the proof.