The Formulas for the Distribution of the
3-Smooth, 5-Smooth, 7-Smooth and all other
Smooth Numbers

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Abstract
In this paper we present and prove rapidly convergent formulas for the distribution of the 3-smooth, 5-smooth, 7-smooth and all other smooth numbers. One of these formulas is another version of a formula due to Hardy and Littlewood for the arithmetic function \( N_{a,b}(x) \), which counts the number of positive integers of the form \( a^p b^q \) less than or equal to \( x \).

1 Introduction
Let \( a \in \mathbb{N}_{\geq 2} \) be a fixed natural number.
Let \( N_a(x) \) denote the number of natural numbers of the form \( a^p \) which are smaller or equal to \( x \), where \( p \in \mathbb{N}_0 \).
By definition [1, 2], we have that the 2-smooth numbers are just the powers of 2, namely
\[
S_2 := \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots \}
\]
and that the formula for their distribution is
\[
N_2(x) = \frac{\log(x)}{\log(2)} + \frac{1}{2} - B_1 \left( \left\{ \frac{\log(x)}{\log(2)} \right\} \right).
\]
This follows directly from the more general formula
\[
N_a(x) = \frac{\log(x)}{\log(a)} + \frac{1}{2} - B_1 \left( \left\{ \frac{\log(x)}{\log(a)} \right\} \right).
\]
Numbers of the form \( 2^p 3^q \), where \( p \in \mathbb{N}_0 \) and \( q \in \mathbb{N}_0 \) are called 3-smooth numbers [1, 2, 3], because these numbers are exactly the numbers which have no prime factors larger than 3.
We will denote the sequence of 3-smooth numbers by \( S_{2,3} \). Thus, we have that
\[
S_{2,3} := \{2^p 3^q : p \in \mathbb{N}_0, q \in \mathbb{N}_0\} = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81, 96, 108, 128, 144, \ldots \}.
\]
More generally, let $a, b \in \mathbb{N}$ be fixed natural numbers such that $a < b$ and $\gcd(a, b) = 1$. Let $N_{a,b}(x)$ denote the number of natural numbers of the form $a^p b^q$ which are smaller or equal to $x$, where $p, q \in \mathbb{N}_0$. Furthermore, we denote by $\chi_{S_{a,b}}(x)$ the characteristic function of the natural numbers of the form $a^p b^q$.

In his first letter to Hardy [4, 5, 6, 7], Ramanujan gave the formula

$$N_{2,3}(x) \approx \frac{1}{2} \frac{\log(2x) \log(3x)}{\log(2) \log(3)} + \frac{1}{2} \chi_{S_{2,3}}(x),$$

which provides a very close approximation to the number $N_{2,3}(x)$ of 3-smooth numbers less than or equal to $x$.

In his notebooks [4, 7], Ramanujan later generalized this expression to all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, namely

$$N_{a,b}(x) \approx \frac{1}{2} \frac{\log(ax) \log(bx)}{\log(a) \log(b)} + \frac{1}{2} \chi_{S_{a,b}}(x),$$

which is again a very close approximation to $N_{a,b}(x)$.

The analog formula for $N_{a,b,c}(x)$ [8, 9, 10] is

$$N_{a,b,c}(x) \approx \frac{\log\left(\sqrt[3]{abc} x\right)^3}{6 \log(a) \log(b) \log(c)} + \frac{1}{2} \chi_{S_{a,b,c}}(x),$$

which is also a very good approximation to $N_{a,b,c}(x)$.

In the following sections, we present and prove rapidly convergent formulas for the functions $N_{a,b}(x)$ and $N_{2,3}(x)$, having the above Ramanujan approximations as their first term. These two formulas are other versions of a more rapidly convergent formula already found by Hardy and Littlewood around 1920 [11, 12], as it was communicated to us by Emanuele Tron [13]. We also prove very rapidly convergent formulas for the distribution of the 5-smooth, 7-smooth and all other smooth numbers.

At the end of the paper, we give an exact formula for the counting function of the natural numbers of the form $a^p b^q$.

We have searched all resulting formulas (which are given in theorems and corollaries) in the literature and on the internet, but we could only find the Hardy-Littlewood formula [11, 12]. Therefore, we believe that all other results are new.
2 The Formulas for \( N_{a,b}(x) \) and the 3-Smooth Numbers Counting Function \( N_{2,3}(x) \)

Let \( a, b \in \mathbb{N} \) such that \( a < b \) and \( \gcd(a, b) = 1 \).

For \( x \in \mathbb{R}^+ \), we define the function \( N_{a,b}(x) \) by

\[
N_{a,b}(x) := \sum_{p \in \mathbb{N}_0, q \in \mathbb{N}_0} \frac{1}{p^a q^b}.
\]

Moreover, we denote the set of natural numbers of the form \( a^p b^q \) by \( S_{a,b} \) and its characteristic function by \( \chi_{S_{a,b}}(x) \), that is

\[
S_{a,b} := \{a^p b^q : p \in \mathbb{N}_0, q \in \mathbb{N}_0\},
\]

\[
\chi_{S_{a,b}}(x) := \begin{cases} 1 & \text{if } x \in S_{a,b} \\ 0 & \text{if } x \notin S_{a,b} \end{cases}.
\]

We have that

\[
N_{a,b}(x) = 1 + \sum_{k=0}^{\lfloor \log_a(x) \rfloor} \left( \lfloor \log_{a^k}(x) \rfloor + \lfloor \log_{b^k}(x) \rfloor \right).
\]

**Theorem 1.** (Our Formula for \( N_{a,b}(x) \))

For every real number \( x > 1 \), we have that

\[
N_{a,b}(x) = \frac{\log(ax) \log(bx)}{2 \log(a) \log(b)} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} - \frac{1}{4} - \frac{1}{2} B_1^* \left( \left\lfloor \frac{\log(x)}{\log(a)} \right\rfloor \right) - \frac{1}{2} B_1^* \left( \left\lfloor \frac{\log(x)}{\log(b)} \right\rfloor \right) - \frac{\log(b)}{2 \log(a)} B_2 \left( \left\lfloor \frac{\log(x)}{\log(b)} \right\rfloor \right) - \frac{\log(a)}{2 \log(a)} B_2 \left( \left\lfloor \frac{\log(x)}{\log(a)} \right\rfloor \right) + \frac{\log(a) \log(b)}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \cos \left( \frac{2 \pi n \log(x)}{\log(a)} \right) - \cos \left( \frac{2 \pi m \log(x)}{\log(b)} \right) + \frac{1}{2} \chi_{S_{a,b}}(x),
\]

where

\[
B_1^*\left(\left\{x\right\}\right) := \begin{cases} \left\{x\right\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases},
\]

\[
B_2\left(\left\{x\right\}\right) := \left\{x\right\}^2 - \left\{x\right\} + \frac{1}{6} \quad \forall x \in \mathbb{R}.
\]

The above formula converges rapidly.

As usual we denote by \( \{x\} \) the fractional part of \( x \).

This is just another version of the following
Theorem 2. *(The Hardy-Littlewood formula for \( N_{a,b}(x) \))* [11, 12]
For every real number \( x \geq 1 \), we have that
\[
N_{a,b}(x) = \frac{1}{2} \log(\frac{ax}{\log(a)}) + \frac{1}{2} \log(\frac{bx}{\log(b)}) + \frac{1}{2} \log(\frac{b}{\log(b)}) - \frac{1}{4} - B_1^* \left( \frac{\log(x)}{\log(2)} \right) - B_1^* \left( \frac{\log(x)}{\log(3)} \right)
\]
\[- \frac{1}{2\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(b)} \right)}{k \sin \left( \pi k \frac{\log(b)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(a)} \right)}{k \sin \left( \pi k \frac{\log(a)}{\log(b)} \right)} \right) + \frac{1}{2} \delta_{a,b}(x),
\]
where the series is to be interpreted as meaning [12]
\[
\sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(a)} \right)}{k \sin \left( \pi k \frac{\log(b)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(b)} \right)}{k \sin \left( \pi k \frac{\log(a)}{\log(b)} \right)} \right) = \lim_{R \to \infty} \left( \sum_{k=1}^{\left\lfloor R \log(a) \right\rfloor} \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(a)} \right)}{k \sin \left( \pi k \frac{\log(b)}{\log(2)} \right)} + \sum_{k=1}^{\left\lfloor R \log(b) \right\rfloor} \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(b)} \right)}{k \sin \left( \pi k \frac{\log(a)}{\log(b)} \right)} \right),
\]
when \( R \to \infty \) in an appropriate manner.

This formula converges very rapidly.

Setting \( a = 2 \) and \( b = 3 \), we get immediately two formulas for the distribution of the 3-smooth numbers, namely

Corollary 3. *(Our Formula for the 3-Smooth Numbers Counting Function \( N_{2,3}(x) \)))*
For every real number \( x \geq 1 \), we have that
\[
N_{2,3}(x) = \frac{1}{2} \log(\frac{2x}{\log(2)}) + \frac{1}{2} \log(\frac{3x}{\log(3)}) + \frac{1}{2} \log(\frac{3}{\log(3)}) - \frac{1}{4} - B_1^* \left( \frac{\log(x)}{\log(2)} \right) - B_1^* \left( \frac{\log(x)}{\log(3)} \right)
\]
\[- \frac{1}{2\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left( \pi k \frac{\log(3)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left( \pi k \frac{\log(2)}{\log(3)} \right)} \right) + \frac{1}{2} \delta_{2,3}(x).
\]

Corollary 4. *(The Hardy-Littlewood formula for \( N_{2,3}(x) \)) [11, 12]*
For every real number \( x \geq 1 \), we have that
\[
N_{2,3}(x) = \frac{1}{2} \log(\frac{2x}{\log(2)}) + \frac{1}{2} \log(\frac{3x}{\log(3)}) + \frac{1}{2} \log(\frac{3}{\log(3)}) - \frac{1}{4} - B_1^* \left( \frac{\log(x)}{\log(2)} \right) - B_1^* \left( \frac{\log(x)}{\log(3)} \right)
\]
\[- \frac{1}{2\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left( \pi k \frac{\log(3)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left( \pi k \frac{\log(2)}{\log(3)} \right)} \right) + \frac{1}{2} \delta_{2,3}(x),
\]
where the series is to be interpreted as meaning $[12]$,

$$
\sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left( \frac{\pi k \log(2)}{\log(3)} \right)} \right) = \lim_{R \to \infty} \left( \sum_{k=1}^{[R \log(2)]} \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} + \sum_{k=1}^{[R \log(3)]} \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left( \frac{\pi k \log(2)}{\log(3)} \right)} \right),
$$

when $R \to \infty$ in an appropriate manner.

Using the computationally more efficient formula

$$
N_{2,3}(x) = 1 + \sum_{k=0}^{[\log_3(x)]} \left[ \log_2 \left( \frac{x}{3^k} \right) \right],
$$

we get the following two tables:

| $x$      | $N_{2,3}(x)$ | Our Formula for $N_{2,3}(x)$ | Number of terms $(n, m)$ |
|---------|-------------|------------------------------|-------------------------|
| 1       | 1           | 1.0510201857955517           | $(n, m) = (4, 4)$ at $x = 1.1$ |
| 10      | 7           | 7.0071497373839231           | $(n, m) = (22, 22)$     |
| $10^2$  | 20          | 20.0045160354084706         | $(n, m) = (10, 10)$     |
| $10^3$  | 40          | 40.0039084310672772         | $(n, m) = (12, 12)$     |
| $10^4$  | 67          | 67.0408408937206653         | $(n, m) = (20, 20)$     |
| $10^5$  | 101         | 101.0507215439969785        | $(n, m) = (28, 28)$     |
| $10^6$  | 142         | 142.01315000789587358       | $(n, m) = (70, 70)$     |
| $10^7$  | 190         | 190.007073892232323501      | $(n, m) = (110, 110)$   |
| $10^8$  | 244         | 244.0065991203209415        | $(n, m) = (140, 140)$   |
| $10^9$  | 306         | 306.00585869480145596       | $(n, m) = (160, 160)$   |
| $10^{10}$ | 376       | 376.02126583465866742       | $(n, m) = (170, 170)$   |
| $10^{10^2}$ | 35084     | 35084.056892623289416675    | $(n, m) = (2000, 2000)$ |
| $10^{10^3}$ | 3483931  | 3483931.035272714689991309386 | $(n, m) = (4000, 4000)$ |

Table 1: Values of $N_{2,3}(x)$
Corollary 5. (Modified version of our formula for $N_{a,b}(x)$)

For every real number $x > 1$, we have

$$N_{a,b}(x) = \frac{\log(x)^2}{2 \log(a) \log(b)} + \frac{\log(x)}{2 \log(a)} + \frac{\log(x)}{2 \log(b)} + \frac{1}{4} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} - \frac{1}{2} B_1^* \left( \left\{ \frac{\log(x)}{\log(a)} \right\} \right)$$

$$- \frac{1}{2} B_1^* \left( \left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{\log(a)}{2 \log(b)} B_2 \left( \left\{ \frac{\log(x)}{\log(a)} \right\} \right) - \frac{\log(b)}{2 \log(a)} B_2 \left( \left\{ \frac{\log(x)}{\log(b)} \right\} \right)$$

$$+ \frac{\log(a) \log(b)}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \cos \left( \frac{2\pi m \log(x)}{\log(a)} \right) - \cos \left( \frac{2\pi m \log(x)}{\log(b)} \right) \right) + \frac{1}{2} \chi_{a,b}(x).$$

Corollary 6. (Modified Hardy-Littlewood formula for $N_{a,b}(x)$)[11, 12]

For every real number $x \geq 1$, we have

$$N_{a,b}(x) = \frac{\log(x)^2}{2 \log(a) \log(b)} + \frac{\log(x)}{2 \log(a)} + \frac{\log(x)}{2 \log(b)} + \frac{1}{4} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} - B_1^* \left( \left\{ \frac{\log(x)}{\log(a)} \right\} \right)$$

$$- B_1^* \left( \left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{2} \log(a)}{\log(b)} \right) \right) + \cos \left( \frac{2\pi k \log(x) - \frac{1}{2} \log(b)}{\log(a)} \right) \right) + \frac{1}{2} \chi_{a,b}(x),$$

where the series is interpreted as mentioned above.
3 The Formula for the Distribution of the 5-Smooth Numbers

Let $a, b, c \in \mathbb{N}$ such that $a < b < c$ and $\gcd(a, b, c) = 1$. For $x \in \mathbb{R}_0^+$, we define the function $N_{a,b,c}(x)$ by

$$N_{a,b,c}(x) : = \sum_{a^p b^q c^l \leq x \atop p, q, l \in \mathbb{N}_0} 1.$$

We define also

$$S_{a,b,c} : = \{a^p b^q : p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0\},$$

$$\chi_{S_{a,b,c}}(x) : = \begin{cases} 1 & \text{if } x \in S_{a,b,c} \\ 0 & \text{if } x \notin S_{a,b,c}. \end{cases}$$

Thus, we have that

$$N_{a,b,c}(x) = \sum_{k=0}^{\lfloor \log_c(x) \rfloor} \sum_{l=0}^{\lfloor \log_a(x) \rfloor} \left( \left\lfloor \log_c \left( \frac{x}{a^k b^l} \right) \right\rfloor + 1 \right).$$

We have the following

**Theorem 7. (Formula for $N_{a,b,c}(x)$)**

For every real number $x \geq 1$, we have that

$$N_{a,b,c}(x) = \frac{\log(x)^3}{6 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{4 \log(a) \log(b)} + \frac{\log(x)^2}{4 \log(a) \log(c)} + \frac{\log(x)^2}{4 \log(b) \log(c)} + \frac{\log(x)}{4 \log(a)}
$$

$$+ \frac{\log(x)}{4 \log(b)} + \frac{\log(x)}{4 \log(c)} + \frac{\log(x)}{12 \log(b) \log(c)} + \frac{\log(x)}{12 \log(a) \log(c)} + \frac{\log(x)}{12 \log(a) \log(b)}
$$

$$+ \frac{\log(a)}{24 \log(b)} + \frac{\log(a)}{24 \log(c)} + \frac{\log(b)}{24 \log(a)} + \frac{\log(b)}{24 \log(c)} + \frac{\log(c)}{24 \log(a)} + \frac{\log(c)}{24 \log(b)} + \frac{1}{8}
$$

$$- B_1^* \left( \left\lfloor \log(x) / \log(a) \right\rfloor \right) - B_1^* \left( \left\lfloor \log(x) / \log(b) \right\rfloor \right) - B_1^* \left( \left\lfloor \log(x) / \log(c) \right\rfloor \right)
$$

$$- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(a)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(b)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(b)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(a)} \right)} \right)
$$

$$- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(c)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(b)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(c)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(a)} \right)} \right)
$$

$$- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(b)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(a)} \right)} \right).$$
This formula converges very rapidly.

In the above formula, the series are to be interpreted as meaning

\[
\sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) + \frac{1}{2} \log(b) - \frac{1}{2} \log(c)}{\log(a)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(a)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a) + \frac{1}{2} \log(c)}{\log(b)} \right)}{k \sin \left( \frac{\pi k \log(c)}{\log(b)} \right)} \right)
\]

\[
= \lim_{R \to \infty} \left( \sum_{k=1}^{\lfloor R \log(a) \rfloor} \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{2} \log(b) - \frac{1}{2} \log(c)}{\log(a)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(a)} \right)} + \frac{\sin \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a) + \frac{1}{2} \log(c)}{\log(b)} \right)}{k \sin \left( \frac{\pi k \log(c)}{\log(b)} \right)} \right)
\]

and

\[
\sum_{k=1}^{\infty} \left( \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{2} \log(b) - \frac{1}{2} \log(c)}{\log(a)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(a)} \right)} + \frac{\sin \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a) + \frac{1}{2} \log(c)}{\log(b)} \right)}{k \sin \left( \frac{\pi k \log(c)}{\log(b)} \right)} \right)
\]

\[
= \lim_{R \to \infty} \left( \sum_{k=1}^{\lfloor R \log(a) \rfloor} \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{2} \log(b) - \frac{1}{2} \log(c)}{\log(a)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(a)} \right)} + \frac{\sin \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(a) + \frac{1}{2} \log(c)}{\log(b)} \right)}{k \sin \left( \frac{\pi k \log(c)}{\log(b)} \right)} \right),
\]

when \( R \to \infty \) in an appropriate manner.

Setting \( a = 2, b = 3 \) and \( c = 5 \) and interpreting the series, like before, as meaning (for example)

\[
\sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(2) + \frac{1}{2} \log(3)}{\log(3)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(3)} \right)} \right)
\]

\[
= \lim_{R \to \infty} \left( \sum_{k=1}^{\lfloor R \log(2) \rfloor} \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{2} \log(2) + \frac{1}{2} \log(3)}{\log(3)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(3)} \right)} \right),
\]

8
and

\[
\sum_{k=1}^{\infty} \left( \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{3} \log(3) - \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} \right) + \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{3} \log(3) - \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(5)}{\log(2)} \right)}
\]

\[
= \lim_{R \to \infty} \left( \sum_{k=1}^{\lfloor R \log(2) \rfloor} \left( \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{3} \log(3) - \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} \right) + \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{3} \log(3) - \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(5)}{\log(2)} \right)} \right),
\]

when \( R \to \infty \) in an appropriate manner, we get for the sequence

\[
S_{2,3,5} : = \{2^p 3^q 5^l : p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0 \}
\]

= \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, \ldots \},

of 5-smooth numbers (regular numbers or Hamming numbers) \([9, 10]\), the following formula

**Corollary 8. (Formula for the 5-Smooth Numbers Counting Function \(N_{2,3,5}(x)\))**

For every real number \( x \geq 1 \), we have that

\[
N_{2,3,5}(x) = \frac{\log(x)^3}{6 \log(2) \log(3) \log(5)} + \frac{\log(x)^2}{4 \log(2) \log(3)} + \frac{\log(x)^2}{4 \log(2) \log(5)} + \frac{\log(x)^2}{4 \log(3) \log(5)} + \frac{\log(x)}{4 \log(2)}
\]

\[
+ \frac{\log(x)}{4 \log(3)} + \frac{\log(x)}{12 \log(3) \log(5)} + \frac{\log(x)}{12 \log(2) \log(5)} + \frac{\log(x)}{12 \log(2) \log(3)}
\]

\[
+ \frac{\log(2)}{24 \log(3)} + \frac{\log(2)}{24 \log(5)} + \frac{\log(2)}{24 \log(2)} + \frac{\log(5)}{24 \log(3)} + \frac{\log(5)}{24 \log(5)} + \frac{\log(5)}{24 \log(2)} + \frac{\log(5)}{8}
\]

\[
- B_1^* \left( \left\{ \frac{\log(x)}{\log(2)} \right\} \right) - B_1^* \left( \left\{ \frac{\log(x)}{\log(3)} \right\} \right) - B_1^* \left( \left\{ \frac{\log(x)}{\log(5)} \right\} \right)
\]

\[
- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{3} \log(3) + \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{3} \log(3) - \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(5)}{\log(2)} \right)} \right)
\]

\[
- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left( \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{3} \log(3) + \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(5)}{\log(2)} \right)} + \frac{\cos \left( 2\pi k \frac{\log(x) - \frac{1}{3} \log(3) - \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} \right)
\]

\[
- \frac{1}{8\pi} \sum_{k=1}^{\infty} \left( \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{3} \log(3) - \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} \right) + \frac{\sin \left( 2\pi k \frac{\log(x) - \frac{1}{3} \log(3) + \frac{1}{3} \log(5)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(5)}{\log(2)} \right)}
\]
This formula converges very rapidly. Using this formula for the 5-Smooth Numbers Counting Function \( N_{2,3,5}(x) \), we get the following table:

| \( x \) | \( N_{2,3,5}(x) \) | Formula for \( N_{2,3,5}(x) \) | Number of terms \( R \) |
|---|---|---|---|
| 1 | 1 | 1.0191146914343678209209456 | \( R = 3 \) |
| 10 | 9 | 9.0066388420020729763649195 | \( R = 11 \) |
| \( 10^2 \) | 34 | 34.01798108016701636663657078 | \( R = 32 \) |
| \( 10^4 \) | 86 | 86.01831146911104727455077198 | \( R = 40 \) |
| \( 10^4 \) | 175 | 175.01259815271196528318821070 | \( R = 52 \) |
| \( 10^6 \) | 313 | 313.01116052291470126065468770 | \( R = 100 \) |
| \( 10^6 \) | 507 | 507.04384962202822061525989835 | \( R = 104 \) |
| \( 10^7 \) | 768 | 768.05762686767314864195183397 | \( R = 110 \) |
| \( 10^8 \) | 1105 | 1105.00435666776355760375109758 | \( R = 260 \) |
| \( 10^9 \) | 1530 | 1530.0019878928910791841182114 | \( R = 300 \) |
| \( 10^{10} \) | 2053 | 2053.01709151724653660944693303 | \( R = 306 \) |
| \( 10^{10} \) | 1697191 | 1697191.10060827971167051326275935 | \( R = 20000 \) |

Table 3: Values of \( N_{2,3,5}(x) \)

4 The Formula for the Distribution of the 7-Smooth Numbers

Let \( a, b, c, d \in \mathbb{N} \) such that \( a < b < c < d \) and \( \text{gcd}(a, b, c, d) = 1 \). For \( x \in \mathbb{R}_0^+ \), we define the function \( N_{a,b,c,d}(x) \) by

\[
N_{a,b,c,d}(x) : = \sum_{\substack{a,b,c,d \leq x \\ p \in \mathbb{N}_0, q \in \mathbb{N}_0, \ell \in \mathbb{N}_0, f \in \mathbb{N}_0}} 1.
\]
We define also
\[ S_{a,b,c,d} : = \{a^p b^q c^l d^f : p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0, f \in \mathbb{N}_0 \}, \]
\[ \chi_{S_{a,b,c,d}}(x) : = \begin{cases} 1 & \text{if } x \in S_{a,b,c,d} \\ 0 & \text{if } x \notin S_{a,b,c,d} \end{cases}. \]

Thus, we have that
\[ N_{a,b,c,d}(x) = \sum_{k=0}^{\lfloor \log_a(x) \rfloor} \sum_{l=0}^{\lfloor \log_b(x) \rfloor} \sum_{m=0}^{\lfloor \log_c(x) \rfloor} \left( \lfloor \log_d \left( \frac{x}{a^k b^l c^m} \right) \rfloor + 1 \right). \]

We have the following

**Theorem 9. (Formula for \( N_{a,b,c,d}(x) \))**

For every real number \( x \geq 1 \), we have that
\[
N_{a,b,c,d}(x) = \frac{\log(x)^4}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(c)} + \frac{\log(x)^3}{12 \log(b) \log(c) \log(d)} \Bigg[ \frac{\log(x)^3}{24 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{24 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{24 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{24 \log(a) \log(b) \log(c)} \Bigg] \\
+ \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(b)} \\
+ \frac{\log(a) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(a) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} \\
+ \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b) \log(c) \log(d)} \\
+ \frac{1}{48 \log(b) \log(c) \log(d)} + \frac{1}{48 \log(b) \log(c) \log(d)} + \frac{1}{48 \log(b) \log(c) \log(d)} + \frac{1}{48 \log(b) \log(c) \log(d)} + \frac{1}{48 \log(b) \log(c) \log(d)} + \frac{1}{48 \log(b) \log(c) \log(d)} + \frac{1}{48 \log(b) \log(c) \log(d)} + \frac{1}{48 \log(b) \log(c) \log(d)} \\
+ \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(c)^3}{144 \log(a) \log(b) \log(c) \log(d)} \\
+ \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(b)^3}{720 \log(a) \log(b) \log(c) \log(d)} \Bigg].
\]
\[- \frac{7}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{7}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(c)} \right\} \right) - \frac{7}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(d)} \right\} \right) \]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(b)}{\log(a)} \right) \frac{\pi k \log(b)}{\log(a)} + \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(c)}{\log(b)} \right) \frac{\pi k \log(c)}{\log(b)} \]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(c)}{\log(a)} \right) \frac{\pi k \log(a)}{\log(b)} + \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(d)}{\log(c)} \right) \frac{\pi k \log(d)}{\log(c)} \]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(d)}{\log(a)} \right) \frac{\pi k \log(a)}{\log(c)} + \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(d)}{\log(b)} \right) \frac{\pi k \log(b)}{\log(d)} \]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(c)}{\log(a)} \right) \frac{\pi k \log(c)}{\log(b)} + \cos \left( 2\pi k \frac{\log(x) - \frac{1}{k} \log(c)}{\log(d)} \right) \frac{\pi k \log(c)}{\log(d)} \]

\[- \frac{1}{16\pi} \sum_{k=1}^{\infty} \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(b) - \frac{1}{k} \log(c)}{\log(a)} \right) \frac{\pi k \log(b)}{\log(a)} + \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(b) + \frac{1}{k} \log(c)}{\log(a)} \right) \frac{\pi k \log(b)}{\log(a)} \]

\[- \frac{1}{16\pi} \sum_{k=1}^{\infty} \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(c) - \frac{1}{k} \log(b)}{\log(a)} \right) \frac{\pi k \log(c)}{\log(b)} + \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(c) + \frac{1}{k} \log(b)}{\log(a)} \right) \frac{\pi k \log(c)}{\log(b)} \]

\[- \frac{1}{16\pi} \sum_{k=1}^{\infty} \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(b) + \frac{1}{k} \log(d)}{\log(a)} \right) \frac{\pi k \log(b)}{\log(a)} + \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(b) + \frac{1}{k} \log(d)}{\log(c)} \right) \frac{\pi k \log(b)}{\log(c)} \]

\[- \frac{1}{16\pi} \sum_{k=1}^{\infty} \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(b) + \frac{1}{k} \log(d)}{\log(c)} \right) \frac{\pi k \log(b)}{\log(c)} + \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(b) + \frac{1}{k} \log(d)}{\log(b)} \right) \frac{\pi k \log(b)}{\log(b)} \]

\[- \frac{1}{16\pi} \sum_{k=1}^{\infty} \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(c) - \frac{1}{k} \log(b)}{\log(b)} \right) \frac{\pi k \log(c)}{\log(b)} + \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(c) + \frac{1}{k} \log(b)}{\log(b)} \right) \frac{\pi k \log(c)}{\log(b)} \]

\[- \frac{1}{16\pi} \sum_{k=1}^{\infty} \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(c) - \frac{1}{k} \log(b)}{\log(b)} \right) \frac{\pi k \log(c)}{\log(b)} + \sin \left( 2\pi k \frac{\log(x) + \frac{1}{k} \log(c) + \frac{1}{k} \log(b)}{\log(b)} \right) \frac{\pi k \log(c)}{\log(b)} \]
\[-\frac{1}{16\pi} \sum_{k=1}^{\infty} \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{3} \log(a) - \frac{1}{3} \log(d)}{\log(b)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(b)} \right) \sin \left( \frac{\pi k \log(d)}{\log(b)} \right)} + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( \frac{\pi k \log(c)}{\log(a)} \right) \cos \left( \frac{\pi k \log(d)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(b)} \right) \sin \left( \frac{\pi k \log(c)}{\log(b)} \right) \sin \left( \frac{\pi k \log(d)}{\log(a)} \right)}\]

\[-\frac{1}{16\pi} \sum_{k=1}^{\infty} \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{5} \log(a) - \frac{1}{5} \log(b)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(c)} \right) \sin \left( \frac{\pi k \log(d)}{\log(c)} \right)} + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left( \frac{\pi k \log(a)}{\log(b)} \right) \cos \left( \frac{\pi k \log(c)}{\log(b)} \right) \cos \left( \frac{\pi k \log(d)}{\log(b)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(c)} \right) \sin \left( \frac{\pi k \log(c)}{\log(c)} \right) \sin \left( \frac{\pi k \log(d)}{\log(c)} \right)}\]

\[-\frac{1}{16\pi} \sum_{k=1}^{\infty} \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{7} \log(a) - \frac{1}{7} \log(c)}{\log(d)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(d)} \right) \sin \left( \frac{\pi k \log(c)}{\log(d)} \right) \sin \left( \frac{\pi k \log(d)}{\log(d)} \right)} + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left( \frac{\pi k \log(a)}{\log(c)} \right) \cos \left( \frac{\pi k \log(b)}{\log(c)} \right) \cos \left( \frac{\pi k \log(d)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(b)} \right) \sin \left( \frac{\pi k \log(b)}{\log(b)} \right) \sin \left( \frac{\pi k \log(d)}{\log(c)} \right)}\]

\[-\frac{1}{16\pi} \sum_{k=1}^{\infty} \frac{\sin \left( 2\pi k \frac{\log(x) + \frac{1}{9} \log(a) - \frac{1}{9} \log(c)}{\log(e)} \right)}{k \sin \left( \frac{\pi k \log(b)}{\log(e)} \right) \sin \left( \frac{\pi k \log(c)}{\log(e)} \right) \sin \left( \frac{\pi k \log(d)}{\log(e)} \right) \sin \left( \frac{\pi k \log(d)}{\log(e)} \right)} + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left( \frac{\pi k \log(a)}{\log(c)} \right) \cos \left( \frac{\pi k \log(b)}{\log(c)} \right) \cos \left( \frac{\pi k \log(d)}{\log(c)} \right) \cos \left( \frac{\pi k \log(e)}{\log(c)} \right)}{k \sin \left( \frac{\pi k \log(a)}{\log(b)} \right) \sin \left( \frac{\pi k \log(b)}{\log(b)} \right) \sin \left( \frac{\pi k \log(d)}{\log(c)} \right) \sin \left( \frac{\pi k \log(e)}{\log(c)} \right)}\]

This formula converges again very rapidly.
The series appearing in this formula are all interpreted like before.
Setting $a = 2$, $b = 3$, $c = 5$ and $d = 7$, we get for the sequence

\[ S_{2,3,5,7} = \{ 2^p 3^q 5^r 7^t : p \in \mathbb{N}_0, q \in \mathbb{N}_0, r \in \mathbb{N}_0, t \in \mathbb{N}_0 \} \]

\[ = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 35, 36, 40, 42, 45, 48, \ldots \} , \]

of 7-smooth numbers (Humble numbers or "highly composite numbers") \cite{1, 2, 14}, immediately the following

**Corollary 10. (Formula for the 7-Smooth Numbers Counting Function $N_{2,3,5,7}(x)$)**

For every real number $x \geq 1$, we have that

\[
N_{2,3,5,7}(x) = \frac{\log(x)^4}{24 \log(2) \log(3) \log(5) \log(7)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(5)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(7)} + \frac{\log(x)^3}{24 \log(2) \log(3) \log(7)} + \frac{\log(x)^2}{8 \log(2) \log(3)} + \frac{\log(x)^2}{8 \log(2) \log(5)} + \frac{\log(x)^2}{8 \log(2) \log(7)} + \frac{\log(x)^2}{24 \log(3) \log(5)} + \frac{\log(x)^2}{24 \log(3) \log(7)} + \frac{\log(x)^2}{24 \log(5) \log(7)} + \frac{\log(x)}{8 \log(2) \log(3)} + \frac{\log(x)}{8 \log(2) \log(5)} + \frac{\log(x)}{8 \log(2) \log(7)} + \frac{\log(x)}{24 \log(3) \log(5)} + \frac{\log(x)}{24 \log(3) \log(7)} + \frac{\log(x)}{24 \log(5) \log(7)} + \frac{\log(x)}{16} + \frac{\log(x)}{48 \log(3)} + \frac{\log(x)}{48 \log(5)} + \frac{\log(x)}{48 \log(7)} + \frac{\log(x)}{48 \log(3)} + \frac{\log(x)}{48 \log(5)} + \frac{\log(x)}{48 \log(7)} + \frac{\log(x)}{48 \log(3)} + \frac{\log(x)}{48 \log(5)} + \frac{\log(x)}{48 \log(7)} + \frac{\log(x)}{144 \log(5) \log(7)} + \frac{\log(x)}{144 \log(3) \log(7)} + \frac{\log(x)}{144 \log(5) \log(7)} + \frac{\log(x)}{144 \log(2) \log(5)} + \frac{\log(x)}{144 \log(2) \log(7)} + \frac{\log(x)}{720 \log(3) \log(5) \log(7)} + \frac{\log(x)}{720 \log(2) \log(5) \log(7)} - \frac{\log(3)^3}{720 \log(2) \log(3) \log(7)} - \frac{\log(3)^3}{720 \log(2) \log(3) \log(5)} - \frac{7}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(2)} \right\} \right) - \frac{7}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{7}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(5)} \right\} \right) - \frac{7}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{1}{8 \pi} \sum_{k=1}^{\infty} \frac{\cos \left( 2 \pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(2)} \right)}{k \sin \left( \frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left( 2 \pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left( \frac{\pi k \log(2)}{\log(3)} \right)} \right).
\]
\[ \sum_{k=1}^{\infty} \left( \frac{1}{8\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(2)}{\log(5)} \right) \right) + \frac{1}{8\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(5)} \right) \right) \right) \]

\[ \sum_{k=1}^{\infty} \left( \frac{1}{8\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(3)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(7)} \right) \right) + \frac{1}{8\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(3)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(7)} \right) \right) \right) \]

\[ \sum_{k=1}^{\infty} \left( \frac{1}{8\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(3)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(3)} \right) \right) + \frac{1}{8\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(3)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(3)} \right) \right) \right) \]

\[ \sum_{k=1}^{\infty} \left( \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) \right) + \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) \right) \right) \]

\[ \sum_{k=1}^{\infty} \left( \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) \right) + \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) \right) \right) \]

\[ \sum_{k=1}^{\infty} \left( \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) \right) + \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) \right) \right) \]

\[ \sum_{k=1}^{\infty} \left( \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(5)}{\log(2)} \right) \right) + \frac{1}{16\pi} \left( \cos \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) + \sin \left( \frac{2\pi k \log(x) - \frac{1}{7} \log(7)}{\log(2)} \right) \right) \right) \]
This formula converges very rapidly.
Every series is interpreted as mentioned above.
Using this formula for the 7-Smooth Numbers Counting Function $N_{2,3,5,7}(x)$, we get the following table:

| $x$ | $N_{2,3,5,7}(x)$ | Formula for $N_{2,3,5,7}(x)$ | Number of terms $R$ |
|-----|------------------|-------------------------------|---------------------|
| 1   | 1                | $1.030338881294024982617233653730019551$ | $R = 3$          |
| 10  | 10               | $10.01263249440259984789405319823431872565$ | $R = 3$          |
| 10$^2$ | 46               | $46.0366852149172637513023829386497852216$ | $R = 20$         |
| 10$^3$ | 141              | $141.01285390547424275647701138240776403195$ | $R = 80$         |
| 10$^4$ | 338              | $338.018699772052226169818534005048234745$ | $R = 80$         |
| 10$^5$ | 694              | $694.00540895426731024839939099335158382934$ | $R = 100$        |
| 10$^6$ | 1273             | $1273.0211557487663113791230619711970129327$ | $R = 1500$       |
| 10$^7$ | 2155             | $2155.0113332556847397569818080511876853632$ | $R = 1500$       |
| 10$^8$ | 3427             | $3427.01611847162744035197962908126411814549$ | $R = 1500$       |
| 10$^9$ | 5194             | $5194.0377142423207725446033555297308020543638$ | $R = 1600$       |
| 10$^{10}$ | 7575           | $7575.01767118495435682818874877606239707862$ | $R = 9000$       |

Table 4: Values of $N_{2,3,5,7}(x)$

5 The Formula for the Distribution of all Smooth Numbers

Let $a_1, a_2, a_3, \ldots, a_n \in \mathbb{N}$ such that $a_1 < a_2 < a_3 < \ldots < a_n$ and gcd($a_1, a_2, a_3, \ldots, a_n$) = 1. For $x \in \mathbb{R}_0^+$, we define the function $N_{a_1,a_2,a_3,\ldots,a_n}(x)$ by

$$N_{a_1,a_2,a_3,\ldots,a_n}(x) = \sum_{q_1 \in \mathbb{N}_0, q_2 \in \mathbb{N}_0, q_3 \in \mathbb{N}_0, \ldots, q_n \in \mathbb{N}_0} 1.$$

We define also

$$S_{a_1,a_2,a_3,\ldots,a_n} = \{ a_1^{q_1} a_2^{q_2} a_3^{q_3} \cdots a_n^{q_n} : q_1 \in \mathbb{N}_0, q_2 \in \mathbb{N}_0, q_3 \in \mathbb{N}_0, \ldots, q_n \in \mathbb{N}_0 \},$$

$$\chi_{S_{a_1,a_2,a_3,\ldots,a_n}}(x) = \begin{cases} 1 & \text{if } x \in S_{a_1,a_2,a_3,\ldots,a_n} \\ 0 & \text{if } x \notin S_{a_1,a_2,a_3,\ldots,a_n}. \end{cases}$$

Thus, we have that

$$N_{a_1,a_2,a_3,\ldots,a_n}(x) = \sum_{k_1=0}^{\lfloor \log_{a_1} x \rfloor} \sum_{k_2=0}^{\lfloor \log_{a_2} \left( \frac{x}{a_1^{k_1}} \right) \rfloor} \cdots \sum_{k_n-1=0}^{\lfloor \log_{a_n} \left( \frac{x}{a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-2}}} \right) \rfloor} \left( \left\lfloor \log_{a_n} \left( \frac{x}{a_1^{k_1} a_2^{k_2} a_3^{k_3} \cdots a_{n-1}^{k_{n-1}}} \right) \right\rfloor + 1 \right).$$
Expressions of this form for $N_{a_1, a_2, a_3, \ldots, a_n}(x)$ are called "Klauder-Ness Expressions" [15, 16]. We have the following

**Theorem 11.** *(Formula for $N_{a_1, a_2, a_3, \ldots, a_n}(x)$)*

For every real number $x \geq 1$, we have that

$$N_{a_1, a_2, a_3, \ldots, a_n}(x) = \text{Res}_{s=0} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k^s} \right)} \right) - \frac{1}{2n-1} \sum_{k=1}^{n} \sum_{m=1}^{n-1} B_1 \left( \left\{ \log(x) \right\} \right)$$

$$+ \frac{1}{2n-1} \pi \sum_{m=1}^{n} \sum_{r=1}^{n-1} \sum_{i_1 < i_2 < i_3 < \ldots < i_r \in \{i_1, i_2, i_3, \ldots, i_r\}} \sum_{k=1}^{\infty} \sin \left( \frac{2\pi k \log(x)}{\log(a_m)} - \frac{\pi r}{2} \right) \prod_{l=1}^{r} \cot \left( \frac{\pi k \log(i_l)}{\log(a_m)} \right)$$

$$+ \frac{1}{2} \chi_{S_{a_1, a_2, a_3, \ldots, a_n}}(x),$$

where the series are to be interpreted as meaning

$$\sum_{k=1}^{\infty} \sin \left( \frac{2\pi k \log(x)}{\log(a_m)} - \frac{\pi r}{2} \right) \prod_{l=1}^{r} \cot \left( \frac{\pi k \log(i_l)}{\log(a_m)} \right) = \lim_{R \to \infty} \left( \sum_{k=1}^{\lfloor R \log(a_m) \rfloor} \sin \left( \frac{2\pi k \log(x)}{\log(a_m)} - \frac{\pi r}{2} \right) \prod_{l=1}^{r} \cot \left( \frac{\pi k \log(i_l)}{\log(a_m)} \right) \right),$$

when $R \to \infty$ in an appropriate manner.

This formula converges again very rapidly.

**Proof.** We have that

$$\sum_{k=1}^{\infty} \chi_{S_{a_1, a_2, a_3, \ldots, a_n}}(k) = \prod_{k=1}^{n} \left( \sum_{m=0}^{\infty} \frac{1}{a_k^{ms}} \right)$$

$$= \prod_{k=1}^{n} \frac{1}{1 - e^{-\log(a_k)s}}.$$
Therefore, by Perron’s formula, we get that

\[ N_{a_1, a_2, a_3, \ldots, a_n}(x) = \frac{1}{2\pi i} \int_\gamma \left( \prod_{k=1}^{n} \frac{1}{1 - e^{-\log(a_k)s}} \right) \frac{x^s}{s} ds \]

\[ = \text{Res}_{s=0} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) \]

\[ + \sum_{m=1}^{n} \sum_{k=1}^{\infty} \left( \text{Res}_{s=\frac{2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) + \text{Res}_{s=\frac{-2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) \right) \]

\[ + \frac{1}{2} \chi_{a_1, a_2, a_3, \ldots, a_n}(x), \]

where \( \gamma = \) line from \( 1 - i \infty \) to \( 1 + i \infty \).

Using the relation

\[ \lim_{s \to \pm \frac{2\pi ik}{\log(a_m)}} \left( \frac{s \mp \frac{2\pi ik}{\log(a_m)}}{1 - e^{-\log(a_m)s}} \right) = \frac{1}{\log(a_m)} \quad \forall k \in \mathbb{N}, \]

we can compute the "\( a_m \)-Residues" to

\[ \text{Res}_{s=\frac{2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) = - \frac{i \prod_{l \neq m}^{n} a_l^{\frac{2\pi ik}{\log(a_m)}}}{2\pi k \prod_{l \neq m}^{n} \left( a_l^{\frac{2\pi ik}{\log(a_m)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N} \]

\[ \text{Res}_{s=\frac{-2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) = - \frac{(-1)^n i x^{\frac{2\pi ik}{\log(a_m)}}}{2\pi k \prod_{l \neq m}^{n} \left( a_l^{\frac{2\pi ik}{\log(a_m)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N}. \]

Using the relations

\[ \sin \left( \frac{2\pi k \log(x)}{\log(a_m)} \right) = \frac{1}{2} \frac{i x^{\frac{2\pi ik}{\log(a_m)}} - 1}{i^{\frac{2\pi ik}{\log(a_m)}}} \]

\[ \cos \left( \frac{2\pi k \log(x)}{\log(a_m)} \right) = \frac{1}{2} \frac{i x^{\frac{2\pi ik}{\log(a_m)}} + 1}{i^{\frac{2\pi ik}{\log(a_m)}}} \]

\[ \cot \left( \frac{\pi k \log(i)}{\log(a_m)} \right) = \frac{i^{\frac{2\pi ik}{\log(a_m)}} + 1}{i^{\frac{2\pi ik}{\log(a_m)}} - 1} \]

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we establish (by expanding everything out) that

\[
\sin \left( 2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2} \right) = \begin{cases} 
(-1)^{\frac{r}{2}} \sin \left( 2\pi k \frac{\log(x)}{\log(a_m)} \right), & \text{if } r \text{ is even} \\
(-1)^{\frac{r+1}{2}} \cos \left( 2\pi k \frac{\log(x)}{\log(a_m)} \right), & \text{if } r \text{ is odd}
\end{cases}
\]

\[
= \begin{cases} 
(-1)^{\frac{r}{2}} \left( \frac{1}{2} i x^2 \frac{2\pi k}{\log(a_m)} - \frac{1}{2} i x^2 \frac{2\pi k}{\log(a_m)} \right), & \text{if } r \text{ is even} \\
(-1)^{\frac{r+1}{2}} \left( \frac{1}{2} x^2 - \frac{2\pi k}{\log(a_m)} + \frac{1}{2} x^2 \frac{2\pi k}{\log(a_m)} \right), & \text{if } r \text{ is odd}
\end{cases}
\]

and

\[
\sin \left( 2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2} \right) = (-1)^{\frac{r}{2}(r \mod 4 + r \mod 2)} \left( \frac{1}{2} i^{r+1 \mod 2} x^2 \frac{2\pi k}{\log(a_m)} + (-1)^{r+1} \frac{1}{2} i^{r+1 \mod 2} x^2 \frac{2\pi k}{\log(a_m)} \right),
\]

we establish (by expanding everything out) that

\[
\sum_{r=1}^{n-1} \sum_{\{i_1 < i_2 < \cdots < i_r\} \subseteq \{a_1, a_2, a_3, \ldots, a_m\}} \sin \left( 2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2} \right) \prod_{l=1}^{r} \cot \left( \frac{\pi k \log(a_l)}{\log(a_m)} \right) + \sin \left( 2\pi k \frac{\log(x)}{\log(a_m)} \right)
\]

\[
= \sum_{r=1}^{n-1} \sum_{\{i_1 < i_2 < \cdots < i_r\} \subseteq \{a_1, a_2, a_3, \ldots, a_m\}} (-1)^{\frac{r}{2}(r \mod 4 + r \mod 2)} \left( \frac{1}{2} i^{r+1 \mod 2} x^2 \frac{2\pi k}{\log(a_m)} + (-1)^{r+1} \frac{1}{2} i^{r+1 \mod 2} x^2 \frac{2\pi k}{\log(a_m)} \right)
\]

\[
= \sum_{r=1}^{n-1} \sum_{\{i_1 < i_2 < \cdots < i_r\} \subseteq \{a_1, a_2, a_3, \ldots, a_m\}} (-1)^{\frac{r}{2}(r \mod 4 + r \mod 2)} \frac{1}{2} i^{r+1} x^2 \frac{2\pi k}{\log(a_m)} + (-1)^{r+1} \frac{1}{2} i^{r+1} x^2 \frac{2\pi k}{\log(a_m)}
\]

\[
+ \sum_{r=1}^{n-1} \sum_{\{i_1 < i_2 < \cdots < i_r\} \subseteq \{a_1, a_2, a_3, \ldots, a_m\}} (-1)^{r+1 + \frac{r}{2}(r \mod 4 + r \mod 2)} \frac{1}{2} i^{r+1} x^2 \frac{2\pi k}{\log(a_m)} - (-1)^{r+1} \frac{1}{2} i^{r+1} x^2 \frac{2\pi k}{\log(a_m)}
\]

\[
= (-1)^{n-1} \frac{1}{2} i^{2\pi k} x^2 \log(a_m) \left( \prod_{l=1}^{n} \frac{\epsilon_l a_l^{2\pi k} \log(a_m)}{a_l^{2\pi k} \log(a_m)} \right) \prod_{l=1}^{n} \frac{2\pi k}{\log(a_m)} + \epsilon_k \right)
\]

\[
= \frac{1}{2} \left( - \frac{2\pi k}{\log(a_m)} \right) \left( \prod_{l=1}^{n} \frac{a_l^{2\pi k} \log(a_m)}{a_l^{2\pi k} \log(a_m)} - 1 \right)
\]

\[
- \frac{1}{2} \left( - \frac{2\pi k}{\log(a_m)} \right) \left( \prod_{l=1}^{n} \frac{a_l^{2\pi k} \log(a_m)}{a_l^{2\pi k} \log(a_m)} - 1 \right)
\]

\[
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\]
In the above calculation, we have used the two algebraic identities

\[ S_1(x_1, x_2, x_3, \ldots, x_n) = 2^n, \]

\[ S_2(x_1, x_2, x_3, \ldots, x_n) = 2^n \prod_{k=1}^{n} x_k, \]

where

\[ S_1(x_1, x_2, x_3, \ldots, x_n) = \sum_{\{\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_n\} \subset \{\pm 1, \pm 1, \pm 1, \ldots, \pm 1\}} \prod_{k=1}^{n} \epsilon_k (x_k + \epsilon_k) \]

\[ S_2(x_1, x_2, x_3, \ldots, x_n) = \sum_{\{\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_n\} \subset \{\pm 1, \pm 1, \pm 1, \ldots, \pm 1\}} \prod_{k=1}^{n} (x_k + \epsilon_k). \]

These two identities follow by induction, because for \( n = 1 \), we have that

\[ S_1(x_1) = (x_1 + 1) - (x_1 - 1) = 2 \]

\[ S_2(x_1) = (x_1 + 1) + (x_1 - 1) = 2x_1. \]

These two identities are exactly the claimed formulas for \( S_1(x_1) \) and \( S_2(x_1) \).

Supposing now that the statement is also true for \( S_1(x_1, x_2, x_3, \ldots, x_{n-1}) \) and \( S_2(x_1, x_2, x_3, \ldots, x_{n-1}) \),
we prove by induction

\[ S_1(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = (x_n + 1)S_1(x_1, x_2, x_3, \ldots, x_{n-1}) - (x_n - 1)S_1(x_1, x_2, x_3, \ldots, x_{n-1}) \]
\[ = (x_n + 1 - x_n + 1)S_1(x_1, x_2, x_3, \ldots, x_{n-1}) \]
\[ = 2S_1(x_1, x_2, x_3, \ldots, x_{n-1}) \]
\[ = 2 \cdot 2^{n-1} \]
\[ = 2^n, \]

\[ S_2(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = (x_n + 1)S_2(x_1, x_2, x_3, \ldots, x_{n-1}) + (x_n - 1)S_2(x_1, x_2, x_3, \ldots, x_{n-1}) \]
\[ = (x_n + 1 + x_n - 1)S_2(x_1, x_2, x_3, \ldots, x_{n-1}) \]
\[ = 2x_nS_2(x_1, x_2, x_3, \ldots, x_{n-1}) \]
\[ = 2x_n2^{n-1} \prod_{k=1}^{n-1} x_k \]
\[ = 2^n \prod_{k=1}^{n} x_k. \]

These are the claimed statements for \( S_1(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \) and \( S_2(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \). Therefore, the inductive proof is finished.

The above established identity implies that

\[
\text{Res}_{s = \frac{2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) + \text{Res}_{s = -\frac{2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) + i \left( -\frac{x^s}{\prod_{k=1}^{n} \left( 1 - \frac{1}{a_k} \right)} \right) \]
\[ = (-1)^{n+1} \frac{2\pi k \prod_{l=1}^{n} \left( \frac{2\pi ik}{\log(a_m)} - a_l \frac{2\pi ik}{\log(a_m)} \right)}{2\pi k \prod_{l=1}^{n} \left( \frac{2\pi ik}{\log(a_m)} - 1 \right)} \]
\[ = \frac{1}{2^{n-1}\pi k} \left( \sum_{r=1}^{n-1} \sum_{i_1 < i_2 < \cdots < i_r} \text{Res}_{s = \frac{2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{l=1}^{n} \left( 1 - \frac{1}{a_l} \right)} \right) \right) \]
\[ = \frac{1}{2^{n-1}\pi k} \left( \sum_{r=1}^{n-1} \sum_{i_1 < i_2 < \cdots < i_r} \text{Res}_{s = \frac{2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{l=1}^{n} \left( 1 - \frac{1}{a_l} \right)} \right) \right) \]
\[ = \frac{1}{2^{n-1}\pi k} \left( \sum_{r=1}^{n-1} \sum_{i_1 < i_2 < \cdots < i_r} \text{Res}_{s = \frac{2\pi ik}{\log(a_m)}} \left( \frac{x^s}{s \prod_{l=1}^{n} \left( 1 - \frac{1}{a_l} \right)} \right) \right) \]

for all \( k \in \mathbb{N} \) and for all \( 1 \leq m \leq n \).

Summing up all the Residues, we get our formula for \( N_{a_1,a_2,a_3,\ldots,a_n}(x) \). \( \square \)

Remark 12. The first few identities of the family of identities, which we encountered in the
above proof, are

\[ S_1(x) = (x + 1) - (x - 1) = 2 \]

\[ S_2(x) = (x + 1) + (x - 1) = 2x \]

\[ S_1(x, x) = (x + 1)(x + 1) - (x - 1)(x + 1) - (x + 1)(x - 1) + (x - 1)(x - 1) = 4 \]

\[ S_2(x, x) = (x + 1)(x + 1) + (x - 1)(x + 1) + (x + 1)(x - 1) + (x - 1)(x - 1) = 4x_1x_2 \]

\[ S_1(x, x, x) = (x + 1)(x + 1)(x + 1) - (x - 1)(x + 1)(x + 1) - (x + 1)(x - 1)(x + 1) - (x + 1)(x - 1)(x - 1) \]

\[ + (x - 1)(x - 1)(x - 1) = 8 \]

\[ S_2(x, x, x, x) = (x + 1)(x + 1)(x + 1) + (x - 1)(x + 1)(x + 1) + (x + 1)(x - 1)(x + 1) + (x + 1)(x - 1)(x - 1) \]

\[ + (x - 1)(x - 1)(x - 1) = 8x_1x_2x_3. \]

And so on.

Setting \( a_1 = 2, a_2 = 3, a_3 = 5, a_4 = 7, \ldots, a_k = p_k = k\)-th prime number, \ldots, \( a_n = p_n = n\)-th prime number in the above theorem, we get for the sequence

\[ S_{2,3,5,7,\ldots,p_n} : = \left\{ 2^{q_1}3^{q_2}5^{q_3}7^{q_4}\ldots p_n^{q_n} : q_1 \in \mathbb{N}_0, q_2 \in \mathbb{N}_0, q_3 \in \mathbb{N}_0, \ldots, q_n \in \mathbb{N}_0 \right\}, \]

of \( p_n \)-smooth numbers \([1, 2]\), immediately the following

**Corollary 13. (Formula for the \( p_n \)-Smooth Numbers Counting Function \( N_{2,3,5,7,\ldots,p_n}(x) \))**

For every real number \( x \geq 1 \), we have that

\[
N_{2,3,5,7,\ldots,p_n}(x) = \text{Res}_{s=0} \left( \frac{s^x}{\prod_{k=1}^{n} \left( 1 - \frac{x}{p_k} \right)} \right) - \frac{1}{2^{n-1}} \sum_{k=1}^{n} B_{k}^* \left( \left\{ \log(x) \right\} / \log(p_k) \right)
\]

\[
+ \frac{1}{2^{n-1} \pi} \sum_{m=1}^{n} \sum_{r=1}^{n-1} \sum_{i_{1} < i_{2} < \ldots < i_r}^{\{1,2,3,\ldots,r\} \subset \{2,3,5,7,\ldots,p_n\}} \sum_{k=1}^{\infty} \frac{\sin \left( 2\pi k \log(x) \right)}{k} \prod_{l=1}^{r} \frac{\pi k \log(i_l)}{\log(p_k)}
\]

\[
+ \frac{1}{2} \chi_{S_{2,3,5,7,\ldots,p_n}}(x),
\]

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where the series are to be interpreted as meaning

\[
\sum_{k=1}^{\infty} \frac{\sin \left( \frac{2\pi k \log(x)}{\log(p_m)} - \frac{\pi r}{2} \right)}{k} \prod_{l=1}^{r} \cot \left( \frac{\pi k \log(i_l)}{\log(p_m)} \right)
\]

\[
= \lim_{R \to \infty} \left( \sum_{k=1}^{\lfloor R \log(p_m) \rfloor} \frac{\sin \left( \frac{2\pi k \log(x)}{\log(p_m)} - \frac{\pi r}{2} \right)}{k} \prod_{l=1}^{r} \cot \left( \frac{\pi k \log(i_l)}{\log(p_m)} \right) \right),
\]

when \( R \to \infty \) in an appropriate manner.

This formula converges also very rapidly.

Therefore, we have

**Corollary 14.** *(The Hardy-Littlewood formula for \( N_{a,b}(x) \) and \( N_{2,3}(x) \)) [11, 12]*

For every real number \( x \geq 1 \), we have that

\[
N_{a,b}(x) = \frac{\log(x)^2}{2 \log(a) \log(b)} + \frac{\log(x)}{2 \log(a)} + \frac{\log(x)}{2 \log(b)} + \frac{1}{4} + \frac{\log(a)}{12 \log(a)} + \frac{\log(b)}{12 \log(a)}
\]

\[- \frac{1}{2} B^*_1 \left( \left\{ \frac{\log(x)}{\log(a)} \right\} \right) - \frac{1}{2} B^*_1 \left( \left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(a)}{\log(b)} \right) \cos \left( \frac{2\pi k \log(x)}{\log(a)} \right) \sqrt{k} \sin \left( \frac{\pi k \log(a)}{\log(b)} \right) + \frac{1}{2} \chi_{S_{a,b}}(x)
\]

and

\[
N_{2,3}(x) = \frac{\log(x)^2}{2 \log(2) \log(3)} + \frac{\log(x)}{2 \log(2)} + \frac{\log(x)}{2 \log(3)} + \frac{1}{4} + \frac{\log(2)}{12 \log(3)} + \frac{\log(3)}{12 \log(2)}
\]

\[- \frac{1}{2} B^*_1 \left( \left\{ \frac{\log(x)}{\log(2)} \right\} \right) - \frac{1}{2} B^*_1 \left( \left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(3)} \right) \cos \left( \frac{2\pi k \log(x)}{\log(2)} \right) \sqrt{k} \sin \left( \frac{\pi k \log(2)}{\log(3)} \right) + \frac{1}{2} \chi_{S_{2,3}}(x).
\]

**Proof.** The proof that we give here is Hardy’s proof [11] of the formula for \( N_{a,b}(x) \).

We have that

\[
\sum_{k=1}^{\infty} \frac{\chi_{S_{a,b}}(k)}{k^s} = \left( \sum_{m_1=0}^{\infty} \frac{1}{q^{m_1 s}} \right) \left( \sum_{m_2=0}^{\infty} \frac{1}{b^{m_2 s}} \right) = \frac{1}{(1 - e^{-\log(a)s}) (1 - e^{-\log(b)s})}.
\]
Therefore, by Perron’s formula, we get that

\[
N_{a,b}(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s})} ds
\]

where \(\gamma = \text{line from } 1 - i\infty \text{ to } 1 + i\infty\).

Moreover, we have that

\[
\text{Res}_{s=0} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s})} \right) = \frac{\log(x)^2}{2 \log(a) \log(b)} + \frac{\log(x)}{2 \log(a)} + \frac{\log(x)}{2 \log(b)}
\]

and that

\[
\text{Res}_{s=2\pi ik/\log(a)} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s})} \right) = -\frac{ib}{2\pi k} \frac{2\pi ik}{\log(b)} \frac{x^{2\pi ik/\log(a)}}{ \log(a)}
\]

for all \(k \in \mathbb{N}\).
Using the relations
\[
\sin \left( \frac{2\pi k \log(x)}{\log(a)} \right) = \frac{1}{2} i x - \frac{2\pi k}{\log(a)} - \frac{1}{2} i x \frac{2\pi k}{\log(a)}
\]
\[
\cos \left( \frac{2\pi k \log(x)}{\log(a)} \right) = \frac{1}{2} x - \frac{2\pi k}{\log(a)} + \frac{1}{2} x \frac{2\pi k}{\log(a)}
\]
\[
\cot \left( \frac{\pi k \log(b)}{\log(a)} \right) = i \frac{b \frac{2\pi k}{\log(a)} + 1}{b \frac{2\pi k}{\log(a)} - 1},
\]
we establish (by expanding everything out) the following identity
\[
- \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( \frac{2\pi k \log(x)}{\log(a)} \right) + \sin \left( \frac{2\pi k \log(x)}{\log(a)} \right)
\]
\[
= - \left( \frac{b \frac{2\pi k}{\log(a)} + 1}{b \frac{2\pi k}{\log(a)} - 1} \right) \left( \frac{1}{2} x - \frac{2\pi k}{\log(a)} + \frac{1}{2} x \frac{2\pi k}{\log(a)} \right) + \left( \frac{1}{2} i x - \frac{2\pi k}{\log(a)} - \frac{1}{2} i x \frac{2\pi k}{\log(a)} \right)
\]
\[
= i \frac{x - \frac{2\pi k}{\log(a)} + \frac{2\pi k}{\log(a)} x \frac{2\pi k}{\log(a)}}{b \frac{2\pi k}{\log(a)} - 1}.
\]

Therefore, for the first Residues (the "a -Residues"), we have that
\[
\text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right)} \right) + \text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right)} \right)
\]
\[
= i \left( x - \frac{2\pi ik}{\log(a)} + a \frac{2\pi ik}{\log(b)} x \frac{2\pi ik}{\log(b)} \right)
\]
\[
= \frac{1}{2\pi k} \left( - \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( \frac{2\pi k \log(x)}{\log(a)} \right) + \sin \left( \frac{2\pi k \log(x)}{\log(a)} \right) \right) \quad \text{for all } k \in \mathbb{N}.
\]

Exchanging a and b ("permuting a and b"), we get also the other Residues (the "b -Residues"), namely
\[
\text{Res}_{s=\frac{2\pi ik}{\log(b)}} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right)} \right) + \text{Res}_{s=\frac{2\pi ik}{\log(b)}} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right)} \right)
\]
\[
= - \left( a \frac{2\pi ik}{\log(b)} - 1 \right) \left( x - \frac{2\pi ik}{\log(b)} + a \frac{2\pi ik}{\log(b)} x \frac{2\pi ik}{\log(b)} \right)
\]
\[
= \frac{1}{2\pi k} \left( - \cot \left( \frac{\pi k \log(a)}{\log(b)} \right) \cos \left( \frac{2\pi k \log(x)}{\log(b)} \right) + \sin \left( \frac{2\pi k \log(x)}{\log(b)} \right) \right) \quad \text{for all } k \in \mathbb{N}.
\]

Summing everything up, we get our formula for \( N_{a,b}(x) \). Setting \( a = 2 \) and \( b = 3 \), we get also the formula for \( N_{2,3}(x) \). \( \square \)
Corollary 15. (The Formulas for \(N_{a,b,c}(x)\) and \(N_{2,3,5}(x)\))

For every real number \(x \geq 1\), we have that

\[
N_{a,b,c}(x) = \frac{\log(x)^3}{6 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{4 \log(a) \log(b)} + \frac{\log(x)^2}{4 \log(b) \log(c)} + \frac{\log(x)^2}{4 \log(c) \log(a)} + \frac{\log(x)}{4 \log(b) + 4 \log(c)} + \frac{\log(x)}{24 \log(b) + 24 \log(c)} + \frac{\log(x)}{24 \log(c) + 24 \log(b) + \frac{1}{8}}
\]

\[-\frac{1}{4} B_1^* \left( \left\{ \frac{\log(x)}{\log(a)} \right\} \right) - \frac{1}{4} B_1^* \left( \left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{1}{4} B_1^* \left( \left\{ \frac{\log(x)}{\log(c)} \right\} \right)
\]

\[-\frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) - \frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(b)}{\log(c)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(c)} \right)
\]

\[-\frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(c)}{\log(b)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(b)} \right) - \frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(c)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right)
\]

\[-\frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(b)}{\log(a)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(a)} \right) - \frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(b)}{\log(c)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(c)} \right)
\]

\[-\frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(c)}{\log(b)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(b)} \right) - \frac{1}{4\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(c)}{\log(a)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(a)} \right) + \frac{1}{2} \chi_{\delta_{a,b,c}}(x)
\]

and
\[ N_{2,3,5}(x) = \frac{\log(x)^3}{6 \log(2) \log(3) \log(5)} + \frac{\log(x)^2}{4 \log(2) \log(3)} + \frac{\log(x)^2}{4 \log(2) \log(5)} + \frac{\log(x)^2}{4 \log(3) \log(5)} + \frac{\log(x)}{4 \log(2) \log(3) \log(5)} \\
+ \frac{\log(x)}{4 \log(2) \log(5)} + \frac{\log(x)}{4 \log(3) \log(5)} + \frac{12 \log(3) \log(5)}{24 \log(2) \log(3) \log(5)} + \frac{12 \log(2) \log(5)}{24 \log(2) \log(3) \log(5)} + \frac{12 \log(2) \log(3)}{24 \log(2) \log(3) \log(5)} + \frac{4 \log(2) \log(3)}{24 \log(2) \log(3) \log(5)} + \frac{24 \log(2)}{24 \log(2) \log(3) \log(5)} + \frac{24 \log(5)}{24 \log(2) \log(3) \log(5)} + \frac{1}{8} \]

- \frac{1}{4} B_1^* \left( \left\{ \frac{\log(x)}{\log(2)} \right\} \right) \cdot - \frac{1}{4} B_1^* \left( \left\{ \frac{\log(x)}{\log(3)} \right\} \right) \cdot - \frac{1}{4} B_1^* \left( \left\{ \frac{\log(x)}{\log(5)} \right\} \right)

Proof. We have that

\[ \sum_{k=1}^{\infty} \frac{\chi_{S_{a,b,c}}(k)}{k^s} = \left( \sum_{m_1=0}^{\infty} \frac{1}{e^{m_1 s}} \right) \left( \sum_{m_2=0}^{\infty} \frac{1}{e^{m_2 s}} \right) \left( \sum_{m_3=0}^{\infty} \frac{1}{e^{m_3 s}} \right) \cdot \frac{1}{\left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right) \left( 1 - e^{-\log(c)s} \right)} \cdot \]
Therefore, by Perron’s formula, we get that
\[
N_{a,b,c}(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \, ds
\]
\[
= \text{Res}_{s=0} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right)
\]
\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right)
\]
\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(b)}} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right)
\]
\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(c)}} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right)
\]
\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(c)}} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right)
\]
\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(c)}} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right)
\]
\[
+ \frac{1}{2} \chi_{S_{a,b,c}}(x),
\]
where $\gamma =$ line from $1 - i\infty$ to $1 + i\infty$.

Furthermore, we have that
\[
\text{Res}_{s=0} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right)
\]
\[
= \frac{\log(x)^3}{6 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{4 \log(a) \log(b)} + \frac{\log(x)^2}{4 \log(a) \log(c)} + \frac{\log(x)^2}{4 \log(b) \log(c)} + \frac{\log(x)}{4 \log(a)}
\]
\[
+ \frac{\log(x)}{4 \log(b)} + \frac{\log(x)}{4 \log(c)} + \frac{\log(x)}{12 \log(b) \log(c)} + \frac{\log(x)}{12 \log(a) \log(c)} + \frac{\log(x)}{12 \log(a) \log(b)}
\]
\[
+ \frac{\log(a)}{24 \log(b)} + \frac{\log(a)}{24 \log(c)} + \frac{\log(b)}{24 \log(a)} + \frac{\log(b)}{24 \log(c)} + \frac{\log(c)}{24 \log(a)} + \frac{\log(c)}{24 \log(b)} + \frac{1}{8}
\]
and that
\[
\text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right) = -\frac{\frac{2\pi ik}{\log(a)}}{\frac{2\pi ik}{\log(b)} - 1} \left( \frac{2\pi ik}{\log(c)} - 1 \right)
\]
\[
\text{Res}_{s=\frac{2\pi ik}{\log(b)}} \left( \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})} \right) = \frac{\frac{2\pi ik}{\log(a)}}{\frac{2\pi ik}{\log(b)} - 1} \left( \frac{2\pi ik}{\log(c)} - 1 \right)
\]
\[
\forall k \in \mathbb{N}.
\]
Exactly similar expressions hold also for the other Residues under exchanging \( a \) with \( b \), and \( a \) with \( c \) ("permuting \( a, b, c \)"). Using again the relations

\[
\sin \left( \frac{2\pi k \log(x)}{\log(a)} \right) = \frac{1}{2} ix^{\frac{-2\pi ik}{\log(a)}} - \frac{1}{2} ix^{\frac{2\pi ik}{\log(a)}}
\]

\[
\cos \left( \frac{2\pi k \log(x)}{\log(a)} \right) = \frac{1}{2} x^{\frac{-2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}}
\]

\[
\cot \left( \frac{\pi k \log(b)}{\log(a)} \right) = \frac{i b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1}
\]

we establish (by expanding everything out) the following identity

\[
- \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right) - \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) \\
- \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) + \sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right)
\]

\[
= - \left( i \frac{2\pi ik}{\log(a)} + 1 \right) \left( i \frac{2\pi ik}{\log(a)} - 1 \right) \left( \frac{1}{2} ix^{\frac{-2\pi ik}{\log(a)}} - \frac{1}{2} ix^{\frac{2\pi ik}{\log(a)}} \right) - \left( i \frac{2\pi ik}{\log(a)} + 1 \right) \left( \frac{1}{2} x^{\frac{-2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}} \right)
\]

\[
- \left( i \frac{2\pi ik}{\log(a)} - 1 \right) \left( \frac{1}{2} x^{\frac{-2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}} \right) + \left( \frac{1}{2} ix^{\frac{-2\pi ik}{\log(a)}} - \frac{1}{2} ix^{\frac{2\pi ik}{\log(a)}} \right)
\]

\[
= \frac{2i \left( x^{\frac{-2\pi ik}{\log(a)}} - b^{\frac{2\pi ik}{\log(a)}} \right) c^{\frac{2\pi ik}{\log(a)}}}{\left( b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left( c^{\frac{2\pi ik}{\log(a)}} - 1 \right)}
\]

Therefore, for the first Residues (the "\( a \)-Residues"), we have that

\[
\text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right) \left( 1 - e^{-\log(c)s} \right)} \right)
\]

\[
+ \text{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right) \left( 1 - e^{-\log(c)s} \right)} \right)
\]

\[
= \frac{i \left( x^{\frac{-2\pi ik}{\log(a)}} - b^{\frac{2\pi ik}{\log(a)}} \right) c^{\frac{2\pi ik}{\log(a)}}}{2\pi k \left( b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left( c^{\frac{2\pi ik}{\log(a)}} - 1 \right)}
\]

\[
= \frac{1}{4\pi k} \left( - \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right) - \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) \\
- \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) + \sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right) \right) \forall k \in \mathbb{N}
\]

Exchanging \( a \) with \( b \), and \( a \) with \( c \) ("permuting \( a, b, c \)", we get also the other Residues (the "\( b \)-Residues" and the "\( c \)-Residues"), which have exactly the same structure. Summing
everything up, we get the formula for $N_{a,b,c}(x)$. Setting $a = 2$, $b = 3$ and $c = 5$, we get also the formula for $N_{2,3,5}(x)$.

**Corollary 16.** (The Formulas for $N_{a,b,c,d}(x)$ and $N_{2,3,5,7}(x)$)
For every real number $x \geq 1$, we have that

$$N_{a,b,c,d}(x) = \frac{\log(x)^4}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(c)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(c)} + \frac{\log(x)^3}{24 \log(b) \log(c) \log(d)}$$

$$+ \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(c)} + \frac{\log(x)^2}{8 \log(a) \log(d)} + \frac{\log(x)^2}{8 \log(b) \log(c)} + \frac{\log(x)^2}{8 \log(b) \log(d)}$$

$$+ \frac{\log(x)^2}{8 \log(c) \log(d)} + \frac{\log(x)^2}{8 \log(a) \log(d)}$$

$$+ \frac{\log(a) \log(x)}{24 \log(b) \log(d)} + \frac{\log(b) \log(x)}{24 \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(c) \log(d)}$$

$$+ \frac{\log(d) \log(x)}{24 \log(b) \log(c)} + \frac{\log(d) \log(x)}{24 \log(b) \log(c)} + \frac{\log(d) \log(x)}{24 \log(b) \log(c)} + \frac{\log(d) \log(x)}{24 \log(b) \log(c)}$$

$$+ \frac{\log(b) \log(x)}{16} + \frac{\log(b) \log(x)}{48 \log(a)} + \frac{\log(b) \log(x)}{48 \log(c)} + \frac{\log(b) \log(x)}{48 \log(d)} + \frac{\log(b) \log(x)}{48 \log(a)} + \frac{\log(b) \log(x)}{48 \log(c)}$$

$$+ \frac{\log(a) \log(b)}{144 \log(c) \log(d)} + \frac{\log(a) \log(b)}{144 \log(c) \log(d)} + \frac{\log(a) \log(b)}{144 \log(c) \log(d)} + \frac{\log(a) \log(b)}{144 \log(c) \log(d)}$$

$$+ \frac{\log(d) \log(c)}{144 \log(a) \log(b)} + \frac{\log(d) \log(c)}{144 \log(a) \log(b)} + \frac{\log(d) \log(c)}{144 \log(a) \log(b)} + \frac{\log(d) \log(c)}{144 \log(a) \log(b)}$$

$$- \frac{\log(c)^3}{720 \log(a) \log(b) \log(d)} - \frac{\log(d)^3}{720 \log(a) \log(b) \log(c)} - \frac{1}{8} B_1 \left( \log(x) \log(b) \log(c) \log(d) \right)$$

$$- \frac{1}{8} B_1 \left( \log(x) \log(b) \log(c) \log(d) \right)$$

$$- \frac{1}{8} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( \frac{2 \pi k \log(x)}{\log(b)} \right)$$

$$- \frac{1}{8} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(c)}{\log(a)} \right) \cos \left( \frac{2 \pi k \log(x)}{\log(c)} \right)$$

$$- \frac{1}{8} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( \frac{2 \pi k \log(x)}{\log(b)} \right)$$

$$- \frac{1}{8} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(c)}{\log(a)} \right) \cos \left( \frac{2 \pi k \log(x)}{\log(c)} \right)$$
\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(d)}{\log(a)}\right) \cos\left(\frac{2\pi k \log(z)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(d)}{\log(a)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \cos\left(\frac{2\pi k \log(z)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \cos\left(\frac{2\pi k \log(z)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \cos\left(\frac{2\pi k \log(z)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \cos\left(\frac{2\pi k \log(z)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \cos\left(\frac{2\pi k \log(z)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(d)}{\log(a)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(d)}{\log(a)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(a)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(c)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(d)}{\log(a)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(d)}{\log(a)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(d)}{\log(a)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(d)}{\log(a)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(d)}{\log(a)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(d)}{\log(a)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right)} \]

\[-\frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} = \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \sin\left(\frac{2\pi k \log(z)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} \]
\[ -\frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(a)}{\log(c)} \right) \cos \left( \frac{\pi k \log(d)}{\log(c)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(c)} \right) \]
\[ + \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(b)}{\log(d)} \right) \cos \left( \frac{\pi k \log(c)}{\log(d)} \right) \cos \left( \frac{\pi k \log(d)}{\log(a)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(a)} \right) \]
\[ + \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(a)}{\log(c)} \right) \cos \left( \frac{\pi k \log(c)}{\log(b)} \right) \cos \left( \frac{\pi k \log(d)}{\log(b)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(b)} \right) \]
\[ + \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(a)}{\log(c)} \right) \cos \left( \frac{\pi k \log(b)}{\log(d)} \right) \cos \left( \frac{\pi k \log(c)}{\log(c)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(c)} \right) \]
\[ + \frac{1}{8\pi} \sum_{k=1}^{\infty} \sin \left( \frac{\pi k \log(a)}{\log(c)} \right) \sin \left( \frac{\pi k \log(b)}{\log(c)} \right) \sin \left( \frac{\pi k \log(c)}{\log(c)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(c)} \right) \]
\[ + \frac{1}{2} \chi_{a,b,c,d}(x) \]

and

\[ N_{2,3,5,7}(x) = \frac{\log(x)^4}{24 \log(2) \log(3) \log(5) \log(7)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(5)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(7)} + \frac{\log(x)^3}{24 \log(3) \log(5) \log(7)} \]
\[ + \frac{\log(x)^3}{8 \log(2) \log(3) + \log(x)^3} + \frac{\log(x)^2}{8 \log(2) \log(5) + \log(x)^2} + \frac{\log(x)^2}{8 \log(2) \log(7) + \log(x)^2} + \frac{\log(x)^2}{8 \log(3) \log(5) + \log(x)^2} + \frac{\log(x)^2}{8 \log(3) \log(7) + \log(x)^2} \]
\[ + \frac{\log(x)^2}{8 \log(5) \log(7) + \log(x)^2} + \frac{\log(x)^2}{8 \log(2) \log(3) + \log(x)^2} + \frac{\log(x)^2}{8 \log(3) \log(5) + \log(x)^2} + \frac{\log(x)^2}{8 \log(2) \log(7) + \log(x)^2} + \frac{\log(x)^2}{8 \log(5) \log(7) + \log(x)^2} \]
\[
\begin{align*}
&+ \frac{\log(2) \log(x)}{24 \log(3) \log(7)} + \frac{\log(2) \log(x)}{24 \log(5) \log(7)} + \frac{\log(3) \log(x)}{24 \log(2) \log(5)} + \frac{\log(3) \log(x)}{24 \log(2) \log(7)} + \frac{\log(3) \log(x)}{24 \log(5) \log(7)} \\
&+ \frac{\log(2) \log(x)}{24 \log(2) \log(3)} + \frac{\log(5) \log(x)}{24 \log(3) \log(7)} + \frac{\log(5) \log(x)}{24 \log(7) \log(3)} + \frac{\log(7) \log(x)}{24 \log(3) \log(5)} + \frac{\log(7) \log(x)}{24 \log(5) \log(3)} \\
&+ \frac{\log(3) \log(x)}{24 \log(2) \log(5)} + \frac{1}{16} + \frac{\log(2)}{48 \log(3)} + \frac{\log(5)}{48 \log(7)} + \frac{\log(7)}{48 \log(5)} \\
&+ \frac{\log(3)}{48 \log(7)} + \frac{\log(5)}{48 \log(2)} + \frac{\log(7)}{48 \log(3)} + \frac{\log(5)}{48 \log(7)} + \frac{\log(7)}{48 \log(5)} \\
&+ \frac{144 \log(5) \log(7)}{144 \log(2) \log(7)} + \frac{144 \log(5) \log(7)}{144 \log(2) \log(3)} + \frac{144 \log(3) \log(5)}{144 \log(2) \log(7)} + \frac{144 \log(3) \log(5)}{144 \log(2) \log(3)} \\
&+ \frac{\log(5)^3}{720 \log(2) \log(3) \log(7)} - \frac{\log(7)^3}{720 \log(2) \log(3) \log(5)} - \frac{1}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(2)} \right\} \right) \\
&- \frac{1}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{1}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(5)} \right\} \right) - \frac{1}{8} B_1^* \left( \left\{ \frac{\log(x)}{\log(7)} \right\} \right) \\
&- \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(2)} \right) \cos \left( 2 \pi k \frac{\log(x)}{\log(2)} \right) - \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(5)}{\log(2)} \right) \cos \left( 2 \pi k \frac{\log(x)}{\log(2)} \right) \\
&- \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(7)}{\log(2)} \right) \cos \left( 2 \pi k \frac{\log(x)}{\log(2)} \right) - \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(5)} \right) \cos \left( 2 \pi k \frac{\log(x)}{\log(5)} \right) \\
&- \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(7)} \right) \cos \left( 2 \pi k \frac{\log(x)}{\log(7)} \right) - \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(5)}{\log(7)} \right) \cos \left( 2 \pi k \frac{\log(x)}{\log(7)} \right) \\
&- \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(7)}{\log(5)} \right) \cos \left( 2 \pi k \frac{\log(x)}{\log(5)} \right) - \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(5)}{\log(2)} \right) \sin \left( \frac{\pi k \log(x)}{\log(2)} \right) \\
&- \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(2)} \right) \sin \left( \frac{\pi k \log(x)}{\log(2)} \right) \\
&- \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(5)}{\log(2)} \right) \sin \left( \frac{\pi k \log(x)}{\log(2)} \right) \\
&- \frac{1}{8 \pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(7)}{\log(2)} \right) \sin \left( \frac{\pi k \log(x)}{\log(2)} \right)
\end{align*}
\]
\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(5)}{\log(2)} \right) \cos \left( \frac{\pi k \log(7)}{\log(2)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(2)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(2)} \right) \cos \left( \frac{\pi k \log(7)}{\log(2)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(2)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(3)} \right) \cos \left( \frac{\pi k \log(5)}{\log(3)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(3)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(5)}{\log(3)} \right) \cos \left( \frac{\pi k \log(7)}{\log(3)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(3)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(3)} \right) \cos \left( \frac{\pi k \log(5)}{\log(3)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(3)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(5)} \right) \cos \left( \frac{\pi k \log(7)}{\log(5)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(5)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(5)} \right) \cos \left( \frac{\pi k \log(7)}{\log(5)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(5)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(5)} \right) \cos \left( \frac{\pi k \log(7)}{\log(5)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(5)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(3)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]

\[- \frac{1}{8\pi} \sum_{k=1}^{\infty} \cos \left( \frac{\pi k \log(2)}{\log(7)} \right) \cos \left( \frac{\pi k \log(7)}{\log(7)} \right) \sin \left( \frac{2\pi k \log(x)}{\log(7)} \right)\]
Therefore, by Perron’s formula, we get that

We have that

**Proof.** We have that

\[
\sum_{k=1}^{\infty} \frac{\chi_{S_{a,b,c,d}}(k)}{k^{s}} = \left( \sum_{m_1=0}^{\infty} \frac{1}{a^{m_1}s} \right) \left( \sum_{m_2=0}^{\infty} \frac{1}{b^{m_2}s} \right) \left( \sum_{m_3=0}^{\infty} \frac{1}{c^{m_3}s} \right) \left( \sum_{m_4=0}^{\infty} \frac{1}{d^{m_4}s} \right)
\]

Therefore, by Perron’s formula, we get that

\[
N_{a,b,c,d}(x) = \frac{1}{2\pi i} \int_{s} x^{s} \left( \frac{1}{1-e^{-\log(a)s}} \right) \left( \frac{1}{1-e^{-\log(b)s}} \right) \left( \frac{1}{1-e^{-\log(c)s}} \right) \left( \frac{1}{1-e^{-\log(d)s}} \right) ds
\]

\[
= \text{Res}_{s=0} \left( \frac{x^{s}}{s (1-e^{-\log(a)s}) (1-e^{-\log(b)s}) (1-e^{-\log(c)s}) (1-e^{-\log(d)s})} \right)
\]

\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left( \frac{x^{s}}{s (1-e^{-\log(a)s}) (1-e^{-\log(b)s}) (1-e^{-\log(c)s}) (1-e^{-\log(d)s})} \right)
\]

\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(b)}} \left( \frac{x^{s}}{s (1-e^{-\log(a)s}) (1-e^{-\log(b)s}) (1-e^{-\log(c)s}) (1-e^{-\log(d)s})} \right)
\]

\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(c)}} \left( \frac{x^{s}}{s (1-e^{-\log(a)s}) (1-e^{-\log(b)s}) (1-e^{-\log(c)s}) (1-e^{-\log(d)s})} \right)
\]

\[
+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(d)}} \left( \frac{x^{s}}{s (1-e^{-\log(a)s}) (1-e^{-\log(b)s}) (1-e^{-\log(c)s}) (1-e^{-\log(d)s})} \right)
\]

\[
= \frac{1}{2} \chi_{S_{2,3,5,7}}(x).
\]

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\[ + \sum_{k=1}^{\infty} \text{Res}_{s=0} \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right) \left( 1 - e^{-\log(c)s} \right) \left( 1 - e^{-\log(d)s} \right)} \]

+ \frac{1}{2} \chi_{s,a,b,c,d}(x),

where \( \gamma = \text{line from } 1 - i\infty \text{ to } 1 + i\infty \).

For the Residues, we have that

\[
\text{Res}_{s=0} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right) \left( 1 - e^{-\log(c)s} \right) \left( 1 - e^{-\log(d)s} \right)} \right)
\]

\[= \frac{24 \log(a) \log(b) \log(c) \log(d)}{\log(x)^3} + \frac{12 \log(a) \log(b) \log(c)}{\log(x)^3} + \frac{12 \log(a) \log(b) \log(d)}{\log(a) \log(x)^2} + \frac{24 \log(b) \log(c) \log(d)}{\log(d) \log(x)^2} + \frac{24 \log(a) \log(b) \log(c) \log(d)}{\log(x)^2} + \frac{8 \log(a) \log(b)}{\log(x)^2} + \frac{8 \log(a) \log(c)}{\log(x)} + \frac{8 \log(a) \log(d)}{\log(x)} + \frac{8 \log(b) \log(c)}{\log(x)} + \frac{8 \log(b) \log(d)}{\log(x)} + \frac{8 \log(c) \log(d)}{\log(x)} + \frac{8 \log(a) \log(c) \log(d)}{\log(x)} + \frac{8 \log(a) \log(b) \log(d)}{\log(x)} + \frac{8 \log(b) \log(c) \log(d)}{\log(x)} + \frac{8 \log(a) \log(b) \log(c) \log(d)}{\log(x)} + \frac{16 \log(b)}{48 \log(a)} + \frac{16 \log(c)}{48 \log(b)} + \frac{16 \log(d)}{48 \log(c)} + \frac{16 \log(a) \log(b) \log(c) \log(d)}{\log(b)^3}
\]

and that

\[
\text{Res}_{s=\frac{2\pi i k}{\log(a)}} \left( \frac{x^s}{s \left( 1 - e^{-\log(a)s} \right) \left( 1 - e^{-\log(b)s} \right) \left( 1 - e^{-\log(c)s} \right) \left( 1 - e^{-\log(d)s} \right)} \right)
\]

\[= -\frac{2\pi k}{\left( b_{\log(a)}^{2\pi i k} - 1 \right) \left( c_{\log(a)}^{2\pi i k} - 1 \right) \left( d_{\log(a)}^{2\pi i k} - 1 \right)} \quad \forall k \in \mathbb{N}
\]

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Exactly similar relations hold also for the other Residues under exchanging \( a \) with \( b \), \( a \) with \( c \), and \( a \) with \( d \) ("permuting \( a, b, c, d \)). Using the relations

\[
\sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right) = \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} \\
\cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) = \frac{1}{2} x \frac{2 \pi i k}{\log(a)} + \frac{1}{2} x \frac{2 \pi i k}{\log(a)} \\
cot \left( \pi k \frac{\log(b)}{\log(a)} \right) = i \frac{b^{2 \pi i k / \log(a)}}{b^{2 \pi i k / \log(a)} - 1} \\
\cot \left( \pi k \frac{\log(c)}{\log(a)} \right) = i \frac{c^{2 \pi i k / \log(a)}}{c^{2 \pi i k / \log(a)} - 1} \\
\cot \left( \pi k \frac{\log(d)}{\log(a)} \right) = i \frac{d^{2 \pi i k / \log(a)}}{d^{2 \pi i k / \log(a)} - 1}
\]

we establish (by expanding everything out) the following identity

\[
\begin{align*}
\cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \cot \left( \frac{\pi k \log(d)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) &= \\
- \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right) &= \\
- \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \cot \left( \frac{\pi k \log(d)}{\log(a)} \right) \sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right) &= \\
- \cot \left( \frac{\pi k \log(b)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) - \cot \left( \frac{\pi k \log(c)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) &= \\
- \cot \left( \frac{\pi k \log(d)}{\log(a)} \right) \cos \left( 2\pi k \frac{\log(x)}{\log(a)} \right) + \sin \left( 2\pi k \frac{\log(x)}{\log(a)} \right) &= \\
\left( \frac{i \frac{b^{2 \pi i k / \log(a)}}{b^{2 \pi i k / \log(a)} - 1} + 1}{i \frac{b^{2 \pi i k / \log(a)}}{b^{2 \pi i k / \log(a)} - 1} + 1} \right) \left( i \frac{c^{2 \pi i k / \log(a)}}{c^{2 \pi i k / \log(a)} - 1} + 1 \right) \left( i \frac{d^{2 \pi i k / \log(a)}}{d^{2 \pi i k / \log(a)} - 1} + 1 \right) &= \\
\left( \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} \right) \left( \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} \right) \left( \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} \right) &= \\
- \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} &= \\
\frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} &= \\
- \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} &= \\
\frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} &= \\
\frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} &= \\
\frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} &= \\
\frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)} &= \\
\frac{1}{2} i x \frac{2 \pi i k}{\log(a)} - \frac{1}{2} i x \frac{2 \pi i k}{\log(a)}
\end{align*}
\]
\[- \left( \frac{2\pi i k}{b \log(a)} + 1 \right) \left( \frac{1}{2} - \frac{2\pi i k}{\log(a)} + \frac{1}{2} \right) - \left( \frac{2\pi i k}{c \log(a)} + 1 \right) \left( \frac{1}{2} - \frac{2\pi i k}{\log(a)} + \frac{1}{2} \right) - \left( \frac{2\pi i k}{d \log(a)} + 1 \right) \left( \frac{1}{2} - \frac{2\pi i k}{\log(a)} + \frac{1}{2} \right) \]

\[- \left( \frac{2\pi i k}{d \log(a)} + 1 \right) \left( \frac{1}{2} - \frac{2\pi i k}{\log(a)} + \frac{1}{2} \right) + \left( \frac{1}{2} i x \frac{-2\pi i k}{\log(a)} - \frac{1}{2} i x \frac{2\pi i k}{\log(a)} \right) \]

\[= -4 i \left( x - \frac{2\pi i k}{\log(a)} + b \frac{2\pi i k}{c \log(a)} + c \frac{2\pi i k}{d \log(a)} + d \frac{2\pi i k}{x \log(a)} \right) \]

This shows that for the first Residues (the "a -Residues"), we have that

\[\text{Res}_{s= \frac{2\pi i k}{\log(a)}} \left( \frac{x^s}{s (1 - e^{-\log(a) s})(1 - e^{-\log(b) s})(1 - e^{-\log(c) s})(1 - e^{-\log(d) s})} \right)\]

\[+ \text{Res}_{s= -\frac{2\pi i k}{\log(a)}} \left( \frac{x^s}{s (1 - e^{-\log(a) s})(1 - e^{-\log(b) s})(1 - e^{-\log(c) s})(1 - e^{-\log(d) s})} \right)

\[= i \left( x - \frac{2\pi i k}{\log(a)} + b \frac{2\pi i k}{c \log(a)} + c \frac{2\pi i k}{d \log(a)} + d \frac{2\pi i k}{x \log(a)} \right) \]

\[= \frac{1}{2\pi k} \left( b \frac{2\pi i k}{\log(a)} - 1 \right) \left( c \frac{2\pi i k}{\log(a)} - 1 \right) \left( d \frac{2\pi i k}{\log(a)} - 1 \right)\]

By exchanging the variable a with all other variables b, c and d ("permuting a, b, c, d"), we get all four Residues (the "a, b, c, d -Residues"), which have all the same structure. Summing everything up, we get our formula for $N_{a,b,c,d}(x)$. Setting $a = 2$, $b = 3$, $c = 5$ and $d = 7$, we get also the formula for $N_{2,3,5,7}(x)$. 

And so on.

These formulas are exactly equivalent to the previous mentioned formulas.
Let \( a, b \in \mathbb{N} \) such that \( a < b \) and \( \gcd(a, b) = 1 \).

For \( x \in \mathbb{R}_+^* \), we define the function \( N_{a,b}^{(2)}(x) \) by

\[
N_{a,b}^{(2)}(x) := \sum_{a^{p^2}b^{q^2} \leq x \atop p \in \mathbb{N}_0, q \in \mathbb{N}_0} 1.
\]

Moreover, we define

\[
S_{a,b}^{(2)} := \left\{ a^{p^2}b^{q^2} : p \in \mathbb{N}_0, q \in \mathbb{N}_0 \right\},
\]

\[
\chi_{S_{a,b}^{(2)}}(x) := \begin{cases} 
1 & \text{if } x \in S_{a,b}^{(2)} \\
0 & \text{if } x \notin S_{a,b}^{(2)}.
\end{cases}
\]

We have that

\[
N_{a,b}^{(2)}(x) = 1 + \sum_{k=0}^{\left\lfloor \log_a(x) \right\rfloor} \left\lfloor \log_b \left( \frac{x}{b^{k^2}} \right) \right\rfloor + \left\lfloor \log_b(x) \right\rfloor.
\]

We have also the following

**Theorem 17. (Formula for \( N_{a,b}^{(2)}(x) \))**

For every real number \( x > 1 \), we have that

\[
N_{a,b}^{(2)}(x) = \frac{\pi \log(x)}{4 \sqrt{\log(a) \log(b)}} + \frac{1}{2} \sqrt{\frac{\log(x)}{\log(a)}} + \frac{1}{2} \sqrt{\frac{\log(x)}{\log(b)}} + \frac{1}{4} - \frac{1}{2} B_1^* \left( \left\lfloor \frac{\log(x)}{\log(a)} \right\rfloor \right)
\]

\[
- \frac{1}{2} B_1^* \left( \left\lfloor \frac{\log(x)}{\log(b)} \right\rfloor \right) + \sqrt{\log(x)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} J_1 \left( \frac{2\pi \sqrt{n^2 \log(a) + m^2 \log(b)}}{\log(a) \log(b)} \log(x) \right)
\]

\[
+ \frac{1}{2} \sqrt{\frac{\log(x)}{\log(a)}} \sum_{k=1}^{\infty} J_1 \left( \frac{2\pi k \log(x)}{k \log(a)} \right) + \frac{1}{2} \sqrt{\frac{\log(x)}{\log(b)}} \sum_{k=1}^{\infty} J_1 \left( \frac{2\pi k \log(x)}{k \log(b)} \right) + \frac{1}{2} \chi_{S_{a,b}^{(2)}}(x).
\]

This formula converges very rapidly.

Setting \( a = 2 \) and \( b = 3 \), we get
Corollary 18. (Formula for $N_{2,3}^{(2)}(x)$)

For every real number $x > 1$, we have that

$$N_{2,3}^{(2)}(x) = \frac{\pi \log(x)}{2\sqrt{\log(2) \log(3)}} + \frac{1}{2} \sqrt{\frac{\log(x)}{\log(2)}} + \frac{1}{2} \sqrt{\frac{\log(x)}{\log(3)}} + \frac{1}{4} - \frac{1}{2} B_1^* \left( \left\{ \sqrt{\frac{\log(x)}{\log(2)}} \right\} \right)$$

$$- \frac{1}{2} B_1^* \left( \left\{ \sqrt{\frac{\log(x)}{\log(3)}} \right\} \right) + \sqrt{\log(x)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} J_1 \left( \frac{2 \pi \sqrt{n^2 \log(2) + m^2 \log(3)}}{\log(x)} \right)$$

$$+ \frac{1}{2} \sqrt{\frac{\log(x)}{\log(2)}} \sum_{k=1}^{\infty} J_1 \left( \frac{2 \pi k \sqrt{\frac{\log(x)}{\log(3)}}}{k} \right) + \frac{1}{2} \sqrt{\frac{\log(x)}{\log(3)}} \sum_{k=1}^{\infty} J_1 \left( \frac{2 \pi k \sqrt{\frac{\log(x)}{\log(2)}}}{k} \right) + \frac{1}{2} \chi_{S_{2,3}}(x).$$

We have that

$$S_{2,3}^{(2)} := \left\{ 2^{p^2} 3^{q^2} : p \in \mathbb{N}_0, q \in \mathbb{N}_0 \right\}$$

$$= \{1, 2, 3, 6, 16, 48, 81, 162, \ldots \},$$

and therefore we get the following table:

| $x$   | $N_{2,3}^{(2)}(x)$ | Formula for $N_{2,3}^{(2)}(x)$ | Number of terms $(n, m)$ needed with $k = 400$ |
|-----------------|------------------|--------------------------------|-----------------------------------|
| 1               | 1                | 1.077194794603379               | $(n, m) = (1, 1)$ at $x = 1.1$    |
| 10              | 4                | 4.069103424005291               | $(n, m) = (1, 1)$                 |
| $10^2$          | 7                | 7.000949506610362               | $(n, m) = (5, 5)$                 |
| $10^3$          | 9                | 9.086395912838084               | $(n, m) = (3, 3)$                 |
| $10^4$          | 11               | 11.038613589820953              | $(n, m) = (5, 5)$                 |
| $10^5$          | 15               | 15.012706923272531              | $(n, m) = (5, 5)$                 |
| $10^6$          | 17               | 17.046462385363300              | $(n, m) = (5, 5)$                 |
| $10^7$          | 18               | 18.408421860888305              | $(n, m) = (9, 9)$                 |
| $10^8$          | 22               | 22.127760008955621              | $(n, m) = (6, 6)$                 |
| $10^9$          | 24               | 24.034210155019944              | $(n, m) = (8, 8)$                 |
| $10^{10}$       | 26               | 26.0098454154207983             | $(n, m) = (9, 9)$                 |
| $10^{10^2}$     | 226              | 226.001668111078420             | $(n, m) = (39, 39)$               |
| $10^{10^3}$     | 2122             | 2122.031291011313557            | $(n, m) = (168, 168)$             |
| $10^{10^4}$     | 20886            | 20886.032472386492101           | $(n, m) = (400, 400)$             |
| $10^{10^5}$     | 207756           | 207756.0303040763527672         | $(n, m) = (1000, 1000)$           |
| $10^{10^6}$     | 2074033          | 2074033.0733802760244109        | $(n, m) = (1400, 1400)$           |

Table 5: Values of $N_{2,3}^{(2)}(x)$
7 Conclusion

We have presented and proved the formulas for the distribution of every smooth number sequence. This article and the proofs of these formulas will soon be published in a Journal.

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