CLOSED MEAN CURVATURE SELF-SHRINKING SURFACES OF GENERALIZED ROTATIONAL TYPE

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Abstract. For each $n \geq 2$ we construct a new closed embedded mean curvature self-shrinking hypersurface in $\mathbb{R}^{2n}$. These self-shrinkers are diffeomorphic to $S^{n-1} \times S^{n-1} \times S^1$ and are $SO(n) \times SO(n)$ invariant. The method is inspired by constructions of Hsiang and these surfaces generalize self-shrinking “tori” diffeomorphic to $S^{n-1} \times S^3$ constructed by Angenent.

1. Introduction

In the study of mean curvature flow, self-similar and in particular self-shrinking solutions arise naturally as separable solutions for the mean curvature flow PDE.

An $n$-dimensional hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is called a mean curvature self-shrinker (hereafter referred to as a self-shrinker) if it satisfies the equation

\[ H = \frac{\langle \vec{X}, \vec{v} \rangle}{2} \]

where $\vec{X}$ is the position vector, and $\vec{v}$ is the unit normal such that $\vec{H} = -\vec{v}$. Under this normalization, $\Sigma$ shrinks to a point at $t = 1$.

Self-shrinking solutions are important in the study of singularities in mean curvature flow as a whole, as Huisken [7] has shown that the formation of a singularity in mean curvature flow resembles a self-shrinking solution after a sequence of appropriate rescalings.

Despite the importance of self-shrinkers in the study of singularities, the list of rigorously constructed examples of closed self-shrinkers is quite short. In 1989, Angenent [1] proved for each $n \geq 1$ the existence of an embedded mean curvature self-shrinker diffeomorphic to $S^{n-1} \times S^1$. To the author’s knowledge, the only known closed, embedded self-shrinkers other than the sphere and those constructed by Angenent were constructed by Møller in [16]. In the latter paper, the author constructs self-shrinkers with genus $g = 2k$ for each large enough $k \in \mathbb{N}$ in $\mathbb{R}^3$. His method involves desingularizing the intersection of a sphere and Angenent’s torus in $\mathbb{R}^3$.

Within the class of rotationally invariant surfaces, the mean curvature flow equation reduces to a second-order nonlinear ODE on the space of orbits. Kleene and Møller [15] conducted an analysis of the rotationally invariant case which resulted in a classification of complete embedded shrinkers in $\mathbb{R}^{n+1}$ invariant under $O(n)$. We study mean curvature shrinkers with a more general rotational type, namely surfaces invariant under $O(m) \times O(n)$ for $m, n > 1$. A primary motivation is to construct new examples of closed, embedded self-shrinkers.

The ansatz of rotational invariance as a mechanism for constructing objects satisfying some geometric property is not new. Consider for example the constant
mean curvature constructed by Delaunay in 1841 [4]. The Delaunay surfaces were the first nontrivial examples of constant mean curvature surfaces and have been used as building blocks for more recent constructions [13]. Beginning in the 1960s, Hsiang began a systematic study of manifolds invariant under general Lie groups. In a joint work with Lawson [11], the authors classify minimal surfaces in $S^n$ with certain invariance groups of “low cohomogeneity.” Subsequently Hsiang used these methods which he described as “equivariant differential geometry” to prove various results, in particular the existence of minimal hyper-spheres in $S^n$ not congruent to the equator for various $n \geq 4$ (the so-called “spherical Bernstein problem” [3]), and the existence of infinitely many noncongruent, closed, embedded minimal surfaces in $S^n$ for $n \geq 3$ [10].

In the spirit of Angenent’s construction of self-shrinking surfaces diffeomorphic to $S^{n-1} \times S^1$ and invariant under $SO(n)$, we more generally prove

**Theorem 1.** For each integer $n \geq 2$, there is an embedded mean curvature self-shrinker $\Sigma^{2n-1} \subset \mathbb{R}^{2n}$ diffeomorphic to $S^{n-1} \times S^{n-1} \times S^1$ and invariant under the action of $SO(n) \times SO(n)$ on $\mathbb{R}^{2n}$.

The self-shrinkers constructed in Theorem 1 are the first rigorously constructed family of odd-dimensional closed embedded self-shrinkers since Angenent’s examples from 1989.

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2. Notation and Terminology

Let $O(m) \times O(n)$ act on

$$\mathbb{R}^{m+n} = \mathbb{R}^m \oplus \mathbb{R}^n = \{(\vec{x}, \vec{y}) : \vec{x} \in \mathbb{R}^m, \vec{y} \in \mathbb{R}^n\}$$

with the usual product action. Let $i : \Sigma^{m+n-1} \hookrightarrow \mathbb{R}^{m+n}$ be an immersed hypersurface. We will abuse notation by identifying $\Sigma$ with its image in $\mathbb{R}^{m+n}$. We say a hypersurface $\Sigma^{m+n-1}$ is *invariant* under the action of $O(m) \times O(n)$ if the action of $O(m) \times O(n)$ on $\mathbb{R}^{m+n}$ preserves $\Sigma$, so in particular $\Sigma$ has a foliation by copies of $S^{m-1} \times S^{n-1}$ of varying radii. We identify the orbit space $\mathbb{R}^m \times \mathbb{R}^n / (O(m) \times O(n))$ with the closed first quadrant $Q = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ under the projection map $\Pi : \mathbb{R}^{m+n} \to Q$ defined by

$$\Pi(\vec{x}, \vec{y}) = (|\vec{x}|, |\vec{y}|) := (x, y).$$

We call the image $\Pi(\Sigma)$ in the orbit space of an $O(m) \times O(n)$-invariant self-shrinker $\Sigma$ the associated profile curve $\gamma_\Sigma$. Up to isometries, there is a one to one correspondence between smooth $O(m) \times O(n)$ invariant hypersurfaces in $\mathbb{R}^{m+n}$ and smooth curves in the interior of $Q$. We denote $S^k(r) = \{|x| \in \mathbb{R}^{k+1} : |x| = r\}$ the sphere of radius $r$ in $\mathbb{R}^k$. For a submanifold $\Sigma \subset S^{k-1}(1) \subset \mathbb{R}^k$ the cone over $\Sigma$ is the set $C(\Sigma) = \{p \in \mathbb{R}^k : p/|p| \in \Sigma\}$.

Since the class of self-shrinkers in $\mathbb{R}^k$ are minimal surfaces with respect to the metric

$$e^{-\frac{|\vec{x}|^2}{2m}} \sum_{i=1}^{k} (dx^i)^2$$
(where $X$ denotes the position vector in $\mathbb{R}^k$) which is conformal to the standard metric by a Gaussian factor [1], it follows that the profile curve associated to an $O(m) \times O(n)$ invariant shrinker is a geodesic with respect to the metric

$$g = x^{2(m-1)} y^{2(n-1)} e^{-\frac{2(x^2+y^2)}{x^2+y^2}} (dx^2 + dy^2)$$

which degenerates along the $x$ and $y$ axes of $Q$. The Euler-Lagrange equation for the length functional corresponding to this metric is an ODE whose solution curves correspond to self-shrinkers, but we choose to derive this ODE by directly computing the principle curvatures below.

Let $\Sigma$ be an $O(m) \times O(n)$ invariant hypersurface with profile curve $\gamma_\Sigma(t) = (x(t), y(t))$, where $\gamma_\Sigma$ is parametrized with respect to Euclidean arc-length. By the rotational invariance, one finds $\Sigma$ has $m-1$ principle curvatures equal to

$$\frac{y'(t)}{x(t)(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}$$

$n-1$ principle curvatures equal to

$$\frac{x'(t)}{y(t)(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}$$

and one principle curvature equal to

$$\frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

The unit normal $\nu(t)$ to $\gamma(t)$ is

$$\frac{(-y'(t), x'(t))}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}$$

hence

$$\frac{\langle \overrightarrow{X}, \overrightarrow{\nu} \rangle}{2} = \frac{1}{2} \frac{y(t)x'(t) - y'(t)x(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

Therefore the self-shrinker Equation (1) reduces in this case to

$$-x''(t)y'(t) + x'(t)y''(t) = \frac{x(t)y'(t) - x'(t)y(t)}{2} + \frac{(n-1)x'(t)}{y(t)} - \frac{(m-1)y'(t)}{x(t)} \left( x'(t)^2 + y'(t)^2 \right).$$

If we assume $\gamma_\Sigma$ is locally graphical over the $x$-axis, $y = u(x)$, then Equation (3) reduces to

$$u''(x) = \left( \frac{2xu'(x) - u(x)}{2} + \frac{(n-1)}{u(x)} - \frac{(m-1)u'(x)}{x} \right) \left( 1 + (u'(x))^2 \right).$$

If we introduce $\theta(t) = \arctan \left( \frac{y'(t)}{x'(t)} \right)$ and compute $\theta'(t)$ via Equation (3), we get the system
\[
\begin{cases}
  \dot{x} = \cos \theta \\
  \dot{y} = \sin \theta \\
  \dot{\theta} = \left( \frac{x}{\sqrt{2}} - \frac{m-1}{x} \right) \sin \theta + \left( \frac{n-1}{y} - \frac{y}{\sqrt{2}} \right) \cos \theta
\end{cases}
\]

which is a flow on the unit tangent bundle of \( Q \). Observe that equation \([5]\) remains true even at places where \( x'(t) = 0 \). This system will be useful in describing the local behavior of solutions of Equation \([3]\).

Below, we will say that a curve \( \gamma(t) \) is a geodesic or a solution if it solves equation \([3]\). With an appropriate parametrization, such a curve is actually a geodesic with respect to the metric given in \([2]\). If \( \gamma \) is further unit speed parametrized (with respect to the usual Euclidean metric on \( Q \)), we will also sometimes call \( \gamma(t) \) a geodesic or a solution if the triple \( (x(t), y(t), \theta(t)) \) satisfies Equation \([5]\).

We will denote the graph of the line \( y = \sqrt{\frac{n-1}{m-1}} x \) by \( \ell \), and let \( \ell^+ \) and \( \ell^- \) be the regions in \( Q \) where \( y > \sqrt{\frac{n-1}{m-1}} \) and \( y < \sqrt{\frac{n-1}{m-1}} \) respectively.

3. ODE Analysis

We first catalogue some trivially verified solutions to equation \([3]\) which will be useful in the proofs of Theorem 1.

**Proposition 3.1.** The following curves are solutions to \((1)\):

1. \( y = \sqrt{\frac{n-1}{m-1}} x \)
2. \( x^2 + y^2 = 2(m + n - 1) \)
3. \( x = \sqrt{2(m - 1)}; x = 0 \)
4. \( y = \sqrt{2(n - 1)}; y = 0 \).

Moreover, these are the unique solutions among the following classes of curves: lines through the origin, circles centered at the origin, vertical lines, horizontal lines.

Geometrically, the self-shrinker corresponding to the line \( y = \sqrt{\frac{n-1}{m-1}} x \) is the cone over the product \( S^{m-1} \left( \sqrt{\frac{m-1}{m+n-2}} \right) \times S^{n-1} \left( \sqrt{\frac{n-1}{m+n-2}} \right) \). It is straightforward to see that \( S^{m-1} \left( \sqrt{\frac{m-1}{m+n-2}} \right) \times S^{n-1} \left( \sqrt{\frac{n-1}{m+n-2}} \right) \) is minimal in \( S^{m+n-1} \), and we more generally have the following trivial result.

**Proposition 3.2.** Suppose \( \Sigma^{k-2} \subset S^{k-1} \) is a minimal surface. Then \( C(\Sigma) \), the cone over \( \Sigma \), satisfies the self-shrinker equation in \( \mathbb{R}^k \).

**Proof.** It is a well known fact that \( C(\Sigma) \subset \mathbb{R}^k \) is minimal if \( \Sigma \subset S^{k-1} \) is minimal, so \( H(C(\Sigma)) = 0 \). Since \( C(\Sigma) \) is a cone, \( \langle \vec{X}, \vec{\nu} \rangle = 0 \). \( \square \)

The portion of the circle \( x^2 + y^2 = 2(m + n - 1) \) in \( Q \) corresponds to the sphere \( S^{m+n-1} \left( \sqrt{2(m + n - 1)} \right) \) as foliated by copies of \( S^{m-1} \times S^{n-1} \). The lines \( x = \sqrt{2(m - 1)} \) and \( y = \sqrt{2(n - 1)} \) correspond to the products \( S^{m-1} \left( \sqrt{2(m - 1)} \right) \times C(S^{n-1}) \) and \( C(S^{m-1}) \times S^{n-1} \left( \sqrt{2(n - 1)} \right) \). Finally, the axes \( x = 0 \) and \( y = 0 \) correspond to the cones \( C(S^{n-1}) \) and \( C(S^{m-1}) \). These are not hypersurfaces, hence will not be discussed further.
Lemma 1. Suppose $\gamma_i : (a,b), i = 1, 2$ are solutions of equation (3) and

1. $\gamma_1$ is a graph over the $x$-axis and $\lim_{t \to b} = (x_b, 0)$ where $x_b > 0$.
2. $\gamma_2$ is a graph over the $y$-axis and $\lim_{t \to b} = (0, y_b)$ where $y_b > 0$.

Then $\gamma_i$ extends smoothly to $(a, b)$ and $\gamma_i$ intersects the corresponding axis orthogonally.

The following lemma places some coarse restrictions on the behavior of geodesics of Equation (3). Part (2) is analogous to lemma 8 of [15].

Lemma 2. Let $\gamma(t) = (x(t), y(t))$ be a solution of (3).

1. Unless $\gamma$ is either $x = \sqrt{2(m - 1)}$ or $y = \sqrt{2(n - 1)}$, any critical point of $x(t)$ or $y(t)$ is either a strict local minimum or maximum.
2. The functions $y(t) - \sqrt{2(n - 1)}$ and $x(t) - \sqrt{2(m - 1)}$ have neither positive minima nor negative maxima, and these functions have different signs at successive critical points.
3. Suppose $\gamma(t_0) \in \ell^-$ and $\dot{\theta}(t_0) > 0, \dot{x}(t_0) > 0, \dot{y}(t_0) > 0$. For any $t$ in the maximal interval containing $t_0$ on which $\gamma$ lies in $\ell^-$ and $x(t), y(t)$ remain monotone, $\dot{\theta}(t) > 0$. An analogous statement is true for $\ell^+$.  

Proof. The first two statements follow from inspecting the system

\[
\begin{align*}
\dot{x} &= \cos \theta \\
\dot{y} &= \sin \theta \\
\dot{\theta} &= \left(\frac{x^2 - 2(m - 1)}{2x}\right) \sin \theta + \left(\frac{2(n-1)y^2}{2y}\right) \cos \theta.
\end{align*}
\]

For the third, we compute from the above system

\[
\ddot{\theta} = \dot{x} \dot{y} \left(\frac{m-1}{x^2} - \frac{n-1}{y(t)^2}\right) + \dot{x} \left(\frac{x^2 - 2(m - 1)}{2x}\right) \cos \theta + \dot{y} \left(\frac{2(n-1)y^2}{2y}\right) \sin \theta.
\]

In particular, when $\dot{\theta} = 0, \ddot{\theta} = \dot{x} \dot{y} \left(\frac{m-1}{x^2} - \frac{n-1}{y(t)^2}\right)$. Hence if $\gamma(t) \in \ell^-$ and $\dot{\theta}(t) = 0$, then $\ddot{\theta}(t) > 0$. This implies (3). \(\Box\)

When $m = n$ will be convenient to consider geodesics that can be written locally as normal graphs over the line $\ell$ and so we introduce the following rotated coordinates.

\[
r(t) = \frac{1}{\sqrt{2}} (x(t) + y(t)), \quad s(t) = \frac{1}{\sqrt{2}} (x(t) - y(t)).
\]

Then defining $\phi = \arctan \left(\frac{x'(t)}{y'(t)}\right)$, we find (when $m = n$) the flow on the unit tangent bundle in these coordinates becomes
Let \( \gamma \) actually use) and leave the routine modifications for the general case to the reader.

Any solution of Equation (3) which intersects \((0, 0)\) is the line \( y = \sqrt{\frac{n-1}{m-1}} x \).

**Proposition 3.3.** Any solution of Equation (3) which intersects \((0,0)\) is the line \( y = \sqrt{\frac{n-1}{m-1}} x \).

**Proof.** We prove the proof in the case that \( m = n \) (which will be the only case we actually use) and leave the routine modifications for the general case to the reader.

Let \( \gamma(t) : [0, t_m] \) be a geodesic parametrized such that \( \gamma(t_m) = (0, 0) \). By Lemma 2, when \( 0 \leq x < \sqrt{2(n-1)} \) and \( 0 \leq y < \sqrt{2(n-1)} \) the only critical points \( x(t) \) and \( y(t) \) may have are minima. Moreover, since \( \gamma(t_m) = (0, 0) \), it follows that \( \dot{x}(t) < 0, \dot{y}(t) < 0 \) for \( t \) sufficiently close enough to \( t_m \), and we assume without loss of generality that \( \dot{x}(t), \dot{y}(t) < 0 \) for all \( t \in [0, t_m] \). It follows that \( \theta(t) \in (-\pi/2, \pi) \) for all \( t \in [0, t_m] \). These facts imply \( \gamma \) is graphical over the \( x \) and \( y \) axes, and since both \( x(t) \) and \( y(t) \) are monotonically decreasing, \( \gamma \) can also be written as a normal graph \( s = f(r) \) over the line \( \ell \). We will show that \( \lim_{\ell \setminus t_m} \theta(t) = -\frac{3\pi}{4} \), so that by uniqueness \( \gamma \) coincides with \( \ell \). We split the remainder of the argument into two cases.

Case 1: \( \gamma(t) \) eventually remains on one side of \( \ell \). Without loss of generality, we suppose that \( x(t) \geq y(t) \) for \( t \in [0, t_m] \). By inspection of the system (7), we conclude that \( f \) has at most one critical point which must be a maximum. By Lemma 2 part (3), \( \theta(t) \) is eventually monotonic and so \( \lim_{\ell \setminus t_m} \theta(t) \) exists. If \( \theta(t_m) = -\frac{3\pi}{4} \), then \( \gamma \) is the line \( \ell \), so first suppose \( \theta(t_m) < -\frac{3\pi}{4} \) and \( \dot{\theta} < 0 \), so \( \gamma \) lies below \( \ell \). Then \( y(t) \leq \tan(\theta(t_m)) x(t) \) and so

\[
\dot{\theta}(t) = \frac{x(t)}{2} \sin \theta(t) - \frac{y(t)}{2} \cos \theta(t) + (n-1) \left( \frac{1}{y(t)} - \frac{1}{x(t)} \right)
\]

By assumption, \( 1 - \frac{1}{\tan(\theta(t_m))} > 0 \), so after using the fact that \( |\dot{x}|, |\dot{y}| \leq 1 \) and integrating this inequality, we conclude that for any \( t_2 > t_1 \), we have

\[
\theta(t_2) - \theta(t_1) > O(1) + (n-1) \log \left[ \frac{1}{2} \left( \frac{1}{\tan(\theta(t_m))} - 1 \right) \frac{x(t_1)}{x(t_2)} \right].
\]

But since \( \dot{\theta}(t_2) - \dot{\theta}(t_1) \) is bounded, it follows that \( x(t_2) \) is bounded away from 0 as \( t_2 \to t_m \), a contradiction. The other cases, where \( \dot{\theta} > 0 \) and \( \gamma \) is above \( \ell \), are similar. Hence, it must be that \( \lim_{\ell \setminus t_m} \theta(t) = -\frac{3\pi}{4} \).

Case 2: \( \gamma(t) \) intersects \( \ell \) infinitely many times. By compactness, there is a convergent sequence \( t_k \to t_\infty \) of places where \( \gamma(t_k) \) lies on \( \ell \). If \( t_\infty < t_m \), then because \( \gamma \) is analytic where it is smooth, it is immediate that \( \gamma \) coincides with \( \ell \). Hence we may suppose that \( t_k \to t_m \). We claim that \( \lim_{k \to \infty} \theta(t_k) = -\frac{3\pi}{4} \). To see this, observe that

\[
\dot{\theta}(t_k) = \frac{n-1}{x(t_k)} (\sin \theta(t_k) - \cos \theta(t_k)) + O(x).
\]
If it is not the case that \( \theta(t_k) \to -\frac{3\pi}{4} \), the preceding equation implies \( \dot{\theta}(t_k) \) becomes unbounded as \( k \to \infty \). In this case, it is straightforward to see from the system \([5]\) that for a sufficiently large \( K \), \( \gamma \) will fail to be graphical over \( \ell \) near \( t_K \), a contradiction. Hence \( \lim_{k \to \infty} \theta(t_k) = -\frac{3\pi}{4} \) and so the image of \( \gamma \) is contained in \( \ell \). \( \square \)

4. Construction of a closed embedded geodesic when \( m = n \)

In this section, we prove Theorem 1. Throughout, we assume that \( m = n > 1 \). When \( m = n \), the metric

\[
(8) \quad g = x^{2(n-1)} y^{2(n-1)} e^{-\frac{(x^2+y^2)}{2}} \{dx^2 + dy^2\}
\]

is preserved by reflection through the line \( \ell \). If follows from this that the set of geodesics of \( g \) is also preserved under reflection through \( \ell \).

We will prove the existence of a geodesic \( \gamma_{R_m} \) with an embedded segment which lies on one side of \( \ell \) and intersects \( \ell \) orthogonally two times. To do this, we will adapt the argument of Angenent [1] to this setting. Reflecting the said geodesic segment through \( \ell \) shows that \( \gamma_{R_m} \) is a closed embedded geodesic in \( Q \). Under the identification between geodesics of the metric \((8)\) and \( O(n) \times O(n) \) invariant self-shrinkers described in Section 1, \( \gamma_{R_m} \) corresponds to a closed embedded \( O(n) \times O(n) \) invariant self-shrinker and Theorem 1 will follow.

We require that \( m = n \) so we may use the preceding reflection argument. It is probable that when \( m \neq n \) a closed embedded geodesic intersecting the line \( \ell \) exists, although a different method would be needed for the proof.

Proof. (of Theorem 1).

We shall only consider parts of geodesics \( \gamma(t) = (r(t), s(t)) \) for which \( s(t) \geq 0 \), in other words, parts which lie below \( \ell \). Define \( \gamma_R = (r_R(t), s_R(t)) \) to be the solution of Equation \((7)\) with initial conditions \( \gamma_R(0) = (0, R) \) and \( \phi_R(0) = 0 \). Inspection of \((7)\) shows that \( \phi_R'(0) < 0 \), so near 0, \( \gamma_R \) can be written as a graph \( f_R : [r_R(t_m(R)), R] \to [0, \infty) \) of a non-negative function over a maximal connected interval \( [r_R(t_m(R)), R] \).

In particular, \( t_m(R) \) satisfies either \( \phi_R(t_m(R)) = 0, \phi_R(t_m(R)) = -\pi, s_R(t_m) = 0 \) or \( s_R(t_m) = 0 \). At a critical point of \( f_R \), \( \phi_R \) is \(-\pi/2\), so Equation \((7)\) implies that \( \phi_R' < 0 \) there. Then by definition, \( f_R \) has at most one critical point, which (if it exists) must be a local maximum.

Lemma 3. For \( R \) sufficiently large, there is \( s_0(R) = R - O(\frac{1}{R}) \) so that \( f_R \) attains a maximum at \( s_m \). Furthermore, \( f_R(s_0) = R - O(\frac{1}{R}) \).

Proof. Define a rescaled time variable \( \tau \) by \( \tau = Rt \), so in particular \( \frac{dt}{d\tau} = \frac{1}{R} \). Then from Equation \((7)\) we see

\[
(9) \quad \frac{d\phi}{d\tau} = \frac{1}{R} \left( \frac{s(\tau)}{2} + \frac{s(\tau)}{r^2(\tau) - s^2(\tau)} \right) \sin(\phi(\tau)) - \frac{r}{R} \left( \frac{1}{2} - \frac{1}{r^2(\tau) - s^2(\tau)} \right) \cos(\phi(\tau)).
\]

Given any \( \epsilon > 0 \) and \( C > 0 \) and \( 0 < \tau < C \), we can pick \( R \) large enough that \( \frac{r(\tau)}{R} > (1 - \epsilon) \). Then by estimating \((9)\) it follows that for such \( \tau \)

\[
\frac{d\phi}{d\tau} < -\frac{1 - \epsilon}{2} \cos(\phi(\tau)).
\]
The equation
\[ \frac{d\phi}{d\tau} = -\frac{1 - \epsilon}{2} \cos(\phi(\tau)) \]
has explicit solution
\[ \phi(\tau) = -2 \arctan\left( \tanh\left( \frac{(1 - \epsilon)\tau}{4} \right) \right) \]
and it is straightforward to see that
\[ -2 \arctan\left( \tanh\left( \frac{(1 - \epsilon)\tau}{4} \right) \right) + \frac{\pi}{2} = O(e^{-\frac{(1-\epsilon)\tau}{2}}). \]
For some fixed small \( \gamma_0 > 0 \) it is not hard to see that as long as \( \frac{d\phi}{d\tau} < 0 \) and \( \tau > \gamma_0 \), there is a constant \( c_1 \) such that \( s(\tau) > \frac{c_1}{\tau} \). Then combining (9) and (10), it follows that there is a \( \tau < C \) such that \( \phi(\tau) = -\frac{\pi}{2} \).

By combining Lemma 2 part (3) and Lemma 3, we conclude

**Lemma 4.** For large \( R \), \( \phi_R \) is monotonic at least until \( \gamma \) crosses \( \ell \) or \( \theta_R(t) = -\pi \).

**Lemma 5.** Let \( \delta_n \searrow 0 \). There is either a sufficiently large \( R \) so that \( f_R(r_R(t_m)) = 0 \) or there is a sequence \( R_n \nearrow \infty \) such that \( f_{R_n} \) is defined on an interval which contains \((\delta_n, R)\).

**Proof.** First we show \( f_R \) cannot end on the \( x \)-axis. By Lemma 1 if \( \gamma_R(t_1) \) lies on the \( x \)-axis, \( \theta_R(t_1) = \frac{-\pi}{2} \). This is not possible since we know that \( \theta_R \) is decreasing at least until a critical point of \( y_R \), where \( \theta_R = -\pi \). By continuity, if there were a later time when \( \theta_R = -\frac{\pi}{2} \), there would be a second critical point of \( f_R \), which is impossible. By Lemma 4, \( \phi_R < 0 \) as long as \( \phi_R > -\frac{3\pi}{4} \). Next we show that for some fixed large \( R_0 \), which depends only on \( n \), we have
\[
\begin{align*}
(1) \quad f_R(R_0) &= O\left(\frac{1}{R}\right) \\
(2) \quad \phi_R(R_0) &= O\left(\log\frac{R_0}{R}\right).
\end{align*}
\]
The first item follows trivially since \( f_R = O\left(\frac{1}{R}\right) \) at its maximum. Now define \( \alpha_R(t) = \phi_R(t) + \frac{\pi}{2} \). Then since \( \cot \phi = -\tan \alpha \),
\[
\frac{d\alpha}{dr} = \frac{d\phi}{dt} \frac{dt}{dr} = \frac{\frac{2}{2} + \frac{n-1}{r^2-s^2} s}{\sin \phi} \sin \phi + \frac{n-1}{r^2-s^2} \frac{r-s}{r-s} \cos \phi
= \left( \frac{s}{2} + \frac{n-1}{r^2-s^2} s \right) + \left( \frac{r}{2} - \frac{n-1}{r^2-s^2} r \right) \tan \alpha.
\]
Thus when
\[ \alpha = \arctan\left( \frac{\frac{2}{2} + \frac{(n-1)s}{r^2-s^2}}{\frac{n-1}{2} r} \right) = O\left( \frac{1}{R^2} \right) \]
when \( r \) is large and \( s \) is small) one has \( \frac{d\alpha}{dr} = \frac{d\phi}{dt} = 0 \). However, by the above, we know that \( \frac{d\phi}{dt} < 0 \) as long as \( \phi \) is not too small, so for large \( r \), \( \frac{d\phi}{dt} \) is \( O\left(\frac{1}{r^2} \right) \).

By integrating, the second claim follows. From the two claims and the smooth dependence of ODE solutions on initial conditions, \( f_R \) converges to \( \ell \) in \( C^1 \) on compact subsets. Hence given \( \delta_n \), there is an \( R_n \) such that \( \gamma_{R_n} \) remains graphical over \( \ell \) at least until \( \delta_n \). Hence, if \( \gamma_{R_n} \) does not cross \( \ell \), \( f_{R_n} \) is defined on an interval containing \([\delta_n, R] \).
Lemma 6. For sufficiently large $R$, $s_R(t_1) = 0$ and $r_R(t_1) \to 0$.

Proof. Suppose according to the conclusion of lemma [3] that there are sequences $\delta_n \nearrow 0$ and $R_n \nearrow \infty$ and functions $0 < f_{R_n}(r) < \frac{C}{R_n}$ defined on $(\delta_n, R_n)$ where each $f_{R_n}$ satisfies the equation

$$\frac{f''}{1 + f'^2} = \left(\frac{(n-1)r}{r^2 - f(r)^2} - \frac{r}{2}\right) f'(r) + \left(\frac{1}{2} + \frac{n-1}{r^2 - f(r)^2}\right) f = 0. \tag{11}$$

By Lemma [5] it follows that $f_{R_n}(r)$ and $f'_{R_n}(r) \to 0$ uniformly on compact sets as $n \to \infty$. We now note that there is a constant $C > 0$ such that $|f'_{R_n}(1)| \leq C|f_{R_n}(1)|$ for every $n$. Indeed, no such $C$ exists, since $f'_{R_n}(1) > 0$, the Mean Value Theorem $f_{R_n}$ must intersect the $r$ axis or become nongraphical for some $r > \frac{1}{R_n}$, a contradiction. Define $g_{R_n}(r)$ to be the rescaling

$$g_{R_n}(r) = \frac{f_{R_n}(r)}{f_{R_n}(1)}. \tag{1}$$

By combining Equation (11) and the bound $|f'_{R_n}(1)| \leq C|f_{R_n}(1)|$, the $f_{R_n}$ have uniform $C^2$ bounds on compact subintervals. Therefore, by the Ascoli-Arzela Theorem, there is a subsequence of the $g_{R_n}$ (which for convenience of notation, we take to be the original sequence) such that $g_{R_n}$ converges to $g$ in $C^2$ on compact subsets of $(0, \infty)$. Since $f_{R_n} \to 0$, the limit $g$ of the rescalings is a solution of the linearization of Equation (11) about the zero solution. Since $g_{R_n}(1) = 1$ and $g_{R_n} > 0$ for each $n$, $g$ is a positive solution of

$$g''(r) = \left(\frac{r-1}{r} - \frac{r}{2}\right) g' + \left(\frac{1}{2} + \frac{n-1}{r^2}\right) g = 0. \tag{12}$$

Furthermore, by Lemmas [3], [4] and [5], $g'(r) \geq 0$ for all $x \in (0, \infty)$, so $\lim_{r \to 0} g(r)$ exists and is finite. Set $h(r) = e^{-\frac{r^2}{2}} g(r)$. $h(r)$ is also positive on $(0, \infty)$, $\lim_{r \to 0} h(r) = \lim_{r \to 0} g(r)$ exists, and moreover $h$ satisfies the equation

$$h'' + \frac{n-1}{r} h' + \left(\frac{n}{4} - \frac{r^2}{16} + \frac{1}{2} + \frac{n-1}{r^2}\right) h = 0. \tag{12}$$

This equation has a regular singularity at $r = 0$, so using a method of Frobenius (see for instance [2]), one can write the solution space of (12) as the span of two solutions $\{x^{\alpha_1}A(x), x^{\alpha_2}B(x)\}$ for some $\alpha_1, \alpha_2 \neq 0$ and $A(x), B(x)$ analytic. More specifically, the $\alpha_i$ are the roots of $x^2 + (n-2)x + (n-1) = 0$, that is

$$\alpha_i = \frac{-(n-2) \pm \sqrt{(n-2)^2 - 4(n-1)}}{2}. \tag{12}$$

It is easy to see that $\sqrt{(n-2)^2 - 4(n-1)}$ is never an integer when $n \in \mathbb{N}$ is greater than 1, so the solution set is spanned by $\{r^{\alpha_1}A(r), r^{\alpha_2}B(r)\}$ for some $A(r), B(r)$ analytic in a neighborhood of 0.

Thus, when $2 \leq n < 7$, solutions to the linearized equation have an oscillatory behavior near 0 and hence fail to be strictly positive. When $n \geq 7$, any nonzero solution of (12) has a singularity like $x^{-\frac{n-2}{2}}$ near 0. Hence $\lim_{r \to 0} h(r)$ fails to be finite, a contradiction. \[\square\]
Figure 1. The line $\ell$ and pieces of curves $\gamma_R$ for three different values of $R$ (when $m = n = 4$.) For large $R$, $\gamma_R$ behaves as in Lemma 3. For $R = R_*$, $\gamma_R$ intersects $\ell$ orthogonally twice, and for $R = \sqrt{2(2n-1)}$, $\gamma_R$ is the arc of a circle and intersects the $x$-axis.

By lemma (6), $\gamma_R(t_1)$ lies on $\ell$ when $R$ is large enough. However, by Proposition (3.1), when $R = \sqrt{2(2n-1)} f_R$ begins and ends on the $x$-axis. Hence $R_* = \inf \{ R > 0 : f_R(r_m(t_m)) = 0 \text{ for all } R > R \}$ is well defined and greater than 0.

Lemma 7. $R_*$ satisfies
1. $\liminf_{R \searrow R_*} r_R(t_m) > 0$.
2. $\liminf_{R \searrow R_*} y_R(t_m) > 0$.

Proof. We prove both statements by contradiction. For (1), suppose there is a sequence $R_n \searrow R_*$ with $r_{R_n}(t_1(R_n)) \to 0$. It follows that $\gamma_{R_n}$ converges in $C^2$ on compact sets to a geodesic $\gamma$ passing through the origin. By proposition 3.3, $\gamma$ is the line $\ell$. But $\gamma_{R_n}$ does not converge in $C^2$ to $\ell$, since in particular $\theta_{R_n}(0) = -\frac{\pi}{4}$.

For (2), if there were a sequence $R_n \searrow R_*$ and times $t_{R_n}$ with $y_{R_n}(t_{R_n}) \to 0$, by compactness, we could find a subsequence (which we assume to be the original sequence) such that $\gamma_{R_n}(t_{R_n}) \to (x_*, 0)$ for some $x_* \geq 0$. By part (1) of this lemma, we may assume that $x_* > 0$. Then $\gamma_{R_n}$ converges to a geodesic $\gamma$ which intersects the $x$-axis at $(x_*, 0)$. By Lemma 1, $\gamma$ intersects the $x$-axis orthogonally. Since $\gamma_{R_n}$ smoothly converges to $\gamma$ on compact sets away from $(x_*, 0)$, there are $\delta_n, \epsilon \to 0$ and times $t_n$ such that $\gamma_{R_n}(t_n) = (x_* + \delta_n, \epsilon_n)$ and $\theta_{R_n}(t_n) = -\frac{\pi}{4}$. Since $\dot{\theta}(t_n) = -O(x_*) > 0$ independent of $n$. Hence, shortly after $t_n$, $\dot{\theta} = -O(x_*) - O(\frac{1}{\epsilon_n})$. Hence, for large $n$, near $(x_*, 0)$, $\gamma_{R_n}$ travels nearly vertically downward, makes a sharp bend near $(x_*, 0)$, and then travels nearly vertically upward. In particular, for sufficiently large $n$, $\gamma_{R_n}$ fails to be a normal graph over $\ell$ on an interval strictly smaller than $[0, t_{R_n}]$. This contradicts the definition of $t_{R_n}$. □

Proposition 4.1. $\gamma_{R_*}$ begins and ends on $\ell$. Moreover $\phi_{R_*}(t_m) = -\pi$. 
Figure 2. Plot in Mathematica of a geodesic which is likely closed and immersed (when $m = n = 4$).

Proof. The preceding lemma shows that as $R \searrow R_*$ the $\gamma_R$ are contained in a compact subset of $Q$ which is disjoint from the $x$ and $y$ axes. Hence, smooth dependence on initial conditions shows that $\gamma_{R_*}$ starts and ends on $\ell$. By the definition of $R_*$, it follows that $\theta_{R_*} \geq -\pi$. Since the set $\{R : \phi_R(t_m) > -\pi\}$ is open by smoothness on initial conditions, it must be that $\phi_{R_*}(t_m) = -\pi$ for otherwise, we would contradict the minimality of $R_*$. □

Since $\gamma_{R_*}$ intersects $\ell$ orthogonally, the union of $\gamma_{R_*}$ with its reflection it through $\ell$ is a smooth embedded closed geodesic $\gamma$. Under the identification between geodesics of (8) and $O(n) \times O(m)$ invariant self-shrinkers in Section 2, $\gamma$ corresponds to an embedded closed $O(n) \times O(n)$ shrinker and Theorem 1 follows. □

5. Final Remarks

In [6], the authors construct a large number of immersed closed self-shrinkers with a single rotational symmetry. It seems likely analogous techniques apply to the setting of this paper. It is easy to find numerical evidence for such closed immersed examples; see Figure 2.

We conclude the paper with an observation (compare with Theorem 4, part (1) in [15]) which places some restrictions on the behavior of embedded geodesics.

Proposition 5.1. Any embedded closed solution of (3) intersects $\ell$ at least twice.

Proof. We argue by contradiction. There is clearly no closed geodesic which intersects $\ell$ exactly once, since the intersection would have to be tangential which would contradict uniqueness. Hence we may suppose $\gamma$ is a closed geodesic lying below $\ell$. Compactness implies the distance between $\gamma$ and $\ell$ is greater than 0, hence there is a smallest $c > 0$ such that the curve $\hat{\gamma} := \gamma(t) - ce_1$ (where $e_1$ is the standard basis vector in $\mathbb{R}^2$ pointing in the positive $x$ direction) makes its first point of contact $p := (x_0, y_0)$ with $\ell$ at a time $t_0$. Near $p$, if $\hat{\gamma}(t) = (x_{\hat{\gamma}}(t), y_{\hat{\gamma}}(t), \theta_{\hat{\gamma}}(t))$
is parametrized so that \( \dot{x}_3(t) > 0 \), one has \( \dot{\gamma}_3(t_0) \leq 0 \), since otherwise \( \dot{\gamma} \) would go above \( \ell \). On the other hand, by using Equation (5), we have that

\[
-\dot{\theta}_1(t_0) = \dot{\theta}_3(t_0) - \dot{\theta}_3(t_0) = \left( \frac{x_0}{2} - \frac{n-1}{x_0} \right) - \left( \frac{x_1}{2} - \frac{n-1}{x_1} \right) \sin \theta(t_0)
\]

where \( x_1 = x_0 + c \).

It is easily checked that \( h \) is monotone increasing. Since \( \sin \theta > 0 \) and \( x_0 \) and \( x_1 \) are the \( x \) coordinates of the points on \( p \) and \( p + ce_1 \), this implies the right hand side of the above equation is \(< 0\), which implies \( \dot{\theta}_3(t_0) > 0 \), which is a contradiction. □

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