Why should one compute periods of algebraic cycles?

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Abstract

In this article we show how the data of integrals of algebraic differential forms over algebraic cycles can be used in order to prove that algebraic and Hodge cycle deformations of a given algebraic cycle are equivalent. As an example, we prove that most of the Hodge and algebraic cycles of the Fermat sextic fourfold cannot be deformed in the underlying parameter space. We then take a difference of two linear cycles inside the Fermat variety with intersection of codimension two in both cycles, and gather evidences that the Hodge locus corresponding to this is smooth and reduced. This implies the existence of new algebraic cycles in the Fermat variety whose existence is predicted by the Hodge conjecture for all hypersurfaces, but not the Fermat variety itself.

1 Introduction

A quick answer to the question of the title is the following: if we compute such numbers, put them inside a certain matrix and compute its rank, then either we will be able to verify the Hodge conjecture for deformed Hodge cycles, or more interestingly, we will find a right place to look for counterexamples for the Hodge conjecture. In direction of the second situation, we collect evidences to Conjecture 1, and for the first situation we prove Theorem 1. In the present text all homologies with \( \mathbb{Z} \) coefficients are up to torsion and all varieties are defined over complex numbers. Let \( n \) be an even number. For an integer \( -1 \leq m \leq \frac{n}{2} \) let \( \mathbb{P}^\frac{n}{2} \subset \mathbb{P}^{n+1} \) be projective spaces given by:

\[
\begin{align*}
\mathbb{P}^\frac{n}{2} : & \quad \begin{cases} 
x_0 - \zeta_{2d}x_1 = 0, \\
x_2 - \zeta_{2d}x_3 = 0, \\
x_4 - \zeta_{2d}x_5 = 0, \\
\vdots \\
x_n - \zeta_{2d}x_{n+1} = 0. 
\end{cases} \\
\bar{\mathbb{P}}^\frac{n}{2} : & \quad \begin{cases} 
x_0 - \zeta_{2d}x_1 = 0, \\
\vdots \\
x_{2m} - \zeta_{2d}x_{2m+1} = 0, \\
x_{2m+2} - \zeta_{2d}^3x_{2m+3} = 0, \\
\vdots \\
x_n - \zeta_{2d}^3x_{n+1} = 0. 
\end{cases}
\end{align*}
\]

where \( \zeta_{2d} := e^{\frac{2\pi i}{2d}} \). These are linear algebraic cycles in the Fermat variety \( X^d_n \subset \mathbb{P}^{n+1} \) given by the homogeneous polynomial \( x_0^d + x_1^d + \cdots + x_{n+1}^d = 0 \), and satisfy \( \mathbb{P}^\frac{n}{2} \cap \bar{\mathbb{P}}^\frac{n}{2} = \mathbb{P}^m \). By convention \( \mathbb{P}^{-1} \) means the empty set. In general we can take arbitrary linear cycles in the Fermat variety, see [19].

Conjecture 1. Let \( n \geq 6 \) be an even number, \( m := \frac{n}{2} - 2 \) and let \( \mathbb{P}^\frac{n}{2} \) and \( \bar{\mathbb{P}}^\frac{n}{2} \) be two linear cycles with \( \mathbb{P}^\frac{n}{2} \cap \bar{\mathbb{P}}^\frac{n}{2} = \mathbb{P}^m \) inside the Fermat variety of degree \( d > \frac{2(n+1)}{n+2} \), and let \( Z_\infty \) be the intersection of a linear \( \mathbb{P}^{\frac{n}{2}+1} \subset \mathbb{P}^{n+1} \) with \( X^d_n \). There is a finite, nonempty set of pairs \((r, \bar{r})\) of coprime integers with the following property: there exists a semi-irreducible algebraic cycle \( Z \) of dimension \( \frac{n}{2} \) in \( X^d_n \) such that

1. For some \( a, b \in \mathbb{Z} \), \( a \neq 0 \), the algebraic cycle \( Z \) is homologous to \( a(r \mathbb{P}^\frac{n}{2} + \bar{r}\bar{\mathbb{P}}^\frac{n}{2}) + bZ_\infty \).
2. The deformation space of the pair \((X_d^n, Z)\), as an analytic variety, contains the intersection of deformation spaces of \((X_d^n, \mathbb{P}^\mathbb{Z})\) and \((X_d^n, \mathbb{P}^\mathbb{Z})\) as a proper subset.

An algebraic cycle \(Z = \sum_{i=1}^r n_i Z_i\), \(n_i \in \mathbb{Z}\) in a smooth projective variety \(X\) is called semi-irreducible if the pair \((X, Z)\) can be deformed into \((X_t, Z_t)\) with \(Z_t\) irreducible, for a precise definition see \([10]\). Note that \(Z, a, b\) in the above conjecture depend on \(r\) and \(\tilde{r}\). If \(d\) is a prime number or \(d = 4\) or \(d = 6\) is relatively prime with \((n + 1)\) then the Hodge conjecture for the Fermat variety \(X_d^n\) can be proved using only linear cycles, see \([10,8]\) and \([11,9]\). Therefore, the existence of the algebraic cycle \(Z\) in Conjecture \([1]\) is not predicted by the Hodge conjecture for \(X_d^n\). We have derived it assuming the Hodge conjecture for all smooth hypersurfaces of degree \(d\) and dimension \(n\) and few other conjectures with some computational evidences (Conjectures \([8]\) Conjecture \([10]\) and Conjecture \([11]\)). The number \(a\) is equal to 1 if the integral Hodge conjecture is true and the term \(b\mathbb{Z}\) pops up because the relevant computations are done in primitive (co)homologies. Since the algebraic cycle \(Z\) is numerically equivalent to \(a(r\mathbb{P}^2 + \tilde{r}\mathbb{P}^2) + b\mathbb{Z}\), this might be used to investigate its (non-)existence, at least for Fermat cubic tenfold. Our computations in this article suggest that \((r, \tilde{r}) = (1, -1)\) satisfies the property in Conjecture \([1]\).

Let \(\mathbb{C}[x]_d = \mathbb{C}[x_0, x_1, \ldots, x_{n+1}]_d\) be the set of homogeneous polynomials of degree \(d\) in \(n + 2\) variables, and let \(T\) be the open subset of \(\mathbb{C}[x]_d\) parameterizing smooth hypersurfaces \(X\) of degree \(d\) and \(T_1 \subset T\) be its subset parameterizing those with a linear \(\mathbb{P}^2\) inside \(X\). We use the notation \(X_t, t \in T\) and denote by \(0 \in T\) the point corresponding to the Fermat variety, and so, \(X_0 = X_d^n\). The algebraic variety \(T_1\) is irreducible, however, as an analytic variety in a neighborhood (usual topology) of \(0 \in T\) it has many irreducible components corresponding to deformations of a linear cycle inside \(X_d^n\). Let us denote by \(V_{\mathbb{P}^2}\) the local branch of \(T_1\) parameterizing deformations of the pair \(\mathbb{P}^2 \subset X_d^n\). In general, for a Hodge cycle in \(H_n(X_d^n, \mathbb{Z})\) we define the Hodge locus \(V_{\mathbb{H}_0} \subset (T, 0)\) which is an analytic scheme and its underlying analytic variety consists of points \(t \in (T, 0)\) such that the monodromy \(\delta_t \in H_n(X_t, \mathbb{Z})\) of \(\delta_0\) along a path in \((T, 0)\) is still Hodge, see \([4]\). For \([\mathbb{P}^2]\) in \(H_n(X_d^n, \mathbb{Z})\) we know that \(V_{[\mathbb{P}^2]}\) as analytic scheme is smooth and reduced and moreover \(V_{\mathbb{P}^2} = V_{[\mathbb{P}^2]}\), see the discussion after Theorem \([6]\). This is not true for an arbitrary Hodge cycle. Conjecture \([1]\) says that \(V_{\mathbb{P}^2} \cap V_{\mathbb{P}^2}\) is a proper subset of the Hodge locus \(V_{\mathbb{P}^2} \cap V_{\mathbb{P}^2}\), see Figure \([1]\). In Conjecture \([1]\) the case \(m = \frac{n}{2} - 1\) and \(r = \tilde{r} = 1\) is excluded, as the pair \((X_d^n, \mathbb{P}^2 + \tilde{r}\mathbb{P}^2)\) can be deformed into a hypersurface containing a complete intersection of type \(1, 1, \cdots, 1, 2\). For small \(m\)’s, the situation is not also strange.

**Theorem 1** \([10,17]\). Let \((n, d, m)\) be one of the following triples

\[
(2, d, -1), \ 5 \leq d \leq 14,
\]
\[
(4, 4, -1), (4, 5, -1), (4, 6, -1), (4, 5, 0), (4, 6, 0),
\]
\[
(6, 3, -1), (6, 4, -1), (6, 4, 0),
\]
\[
(8, 3, -1), (8, 3, 0),
\]
\[
(10, 3, -1), (10, 3, 0), (10, 3, 1),
\]

and \(\mathbb{P}^2\) and \(\mathbb{P}^2\) be linear cycles in \([1]\). The Hodge locus passing through the Fermat point \(0 \in T\) and corresponding to deformations of the Hodge cycle \(\mathbb{P}^2 + \tilde{r}\mathbb{P}^2\) is \(H_n(X_d^n, \mathbb{Z})\) with \(\mathbb{P}^2 \cap \mathbb{P}^2 = \mathbb{P}^m\) and \(r, \tilde{r} \in \mathbb{Z}\), \(r \neq 0, \tilde{r} \neq 0\) is smooth and reduced. Moreover, its underlying analytic variety is simply the intersection \(V_{\mathbb{P}^2} \cap V_{\mathbb{P}^2}\).

The cases \((n, d) = (2, 4), (4, 3)\) are the only cases such that the \((\frac{n}{2} + 1, \frac{n}{2} - 1)\) Hodge number of \(X_d^n\) is equal to one, and these are out of our discussion as all Hodge loci \(V_{\mathbb{H}_0}\) are of codimension one, smooth and reduced. For the discussion of these cases and a baby version of Conjecture
We conjecture that for a fixed $n \geq 6$ and $d > \frac{2(n+1)}{n-2}$, there is $0 \leq M_{n,d} < \frac{n}{2} - 2$ depending only on $n$ and $d$ such that for $m \leq M_{n,d}$, respectively $M_{n,d} < m < \frac{n}{2} - 1$, we have similar statements as in Theorem 1, respectively Conjecture 1. We do not have any idea how to describe $M_{n,d}$ in general. We expect that Theorem 1 for $m = -1$ is always true. In this case $\mathbb{P}_2^n$ and $\hat{\mathbb{P}}_2^n$ do not intersect each other. The restriction on $n$ and $d$ in Theorem 1 is due to the fact that our proof is computer assisted, and upon a better computer programing and a better device, it might be improved. For now, the author does not see any theoretical proof. The first evidence for Conjecture 1 is the fact that for many examples of $n$ and $d$, the codimension of the Zariski tangent space of the analytic scheme $V_{r[\mathbb{P}_2^n] + \tilde{r}\hat{\mathbb{P}}_2^n}$ is strictly smaller than the codimension of $V_{\mathbb{P}_2^n} \cap V_{\hat{\mathbb{P}}_2^n}$ which is smooth. In order to be able to investigate the smoothness and reducedness of this analytic scheme, we have worked out Theorem 14 which is just computing a Taylor series. Its importance must not be underestimated. The linear part of such Taylor series encode the whole data of infinitesimal variation of Hodge structures (IVHS) introduced by Griffiths and his coauthors in 1980’s, and from this one can derive most of the applications of IVHS, such as global Torelli problem, see [CG80]. In particular, the proof of Theorem 1 uses just such linear parts. In a personal communication C. Voisin pointed out the difficulties on higher order approximation of the Noether-Lefschetz locus. This motivated the author to elaborate some of his old ideas in [Mov11] and develop it into Theorem 14. The second order approximations in cohomological terms (similar to IVHS), has been formulated in [Mac05], however it is not enough for the investigation of Conjecture 1, see Theorem 2 and it turns out one has to deal with third and fourth order approximations, see Theorem 3. We use Theorem 14 to check reducedness and smoothness of components of the Hodge loci. We break the property of being reduced and smooth into $N$-smooth for all $N \in \mathbb{N}$, see [8] and prove the following theorem which is not covered in Theorem 1.

**Theorem 2.** Let $(n, d, m)$ be one of the triples

1. $(6, 3, 1), (6, 3, 0), (8, 3, 1)$
2. $(4, 4, 0), (8, 3, 2), (8, 3, 1), (10, 3, 3), (10, 3, 2)$

and $\mathbb{P}_2^n$ and $\hat{\mathbb{P}}_2^n$ be linear cycles in (1). For all $r, \tilde{r} \in \mathbb{Z}$ with $1 \leq |r| \leq |\tilde{r}| \leq 10$ the analytic scheme $\mathbb{V}_{r[\mathbb{P}_2^n] + \tilde{r}\hat{\mathbb{P}}_2^n}$ with $\mathbb{P}_2^n \cap \hat{\mathbb{P}}_2^n = \mathbb{P}^m$ is 2-smooth. It is 3-smooth in the cases (2) and for
$(n, d, m, r, \tilde{r}) = (4, 4, 0, 1, -1)$. It is 4-smooth in the case $(n, d, m, r, \tilde{r}) = (6, 3, 1, 1, -1)$ and $(n, d, m) = (6, 3, 0)$.

Note that the triples in Theorem 3 are not covered in Theorem 1 and we do not know the corresponding Hodge locus. In order to solve Conjecture 1 we will need to identify non-reduced Hodge loci. We prove that:

**Theorem 3.** Let $\mathbb{P}^n_2$ and $\mathbb{P}^n_2$ be linear cycles in $\mathbb{P}^n_2$ with $\mathbb{P}^n_2 \cap \mathbb{P}^n_2 = \mathbb{P}^m$. The analytic scheme $V_{r[\mathbb{P}^n_2]+r[\mathbb{P}^n_2]}$ is either singular at the Fermat point 0 or it is non-reduced, in the following cases:

1. For all $r, \tilde{r} \in \mathbb{Z}$, $1 \leq |r| \leq |\tilde{r}| \leq 10$, $r \neq \tilde{r}$, $m = \frac{n}{2} - 1$ and $(n, d)$ in the list

   (4)
   (5)

2. For all $r, \tilde{r} \in \mathbb{Z}$, $1 \leq |r| \leq |\tilde{r}| \leq 10$, $r \neq -\tilde{r}$ and $(n, d, m)$ in the list

   $(4, 4, 0), (6, 3, 1), (8, 3, 2)$

The upper bounds for $|r|$ and $|\tilde{r}|$ is due to our computational methods, and it would not be difficult to remove this hypothesis. The verification of the case $(n, d, m) = (8, 3, 2)$ in the second item by a computer takes more than 14 days! Theorem 3 in the case $(n, d, m) = (2, 5, 0)$ and without the upper bound on $|r|$, $|\tilde{r}|$ follows from a theorem of Voisin in [Voi89], see Exercise 2, page 154 [Voi03] and its reproduction in [Mov17a] Exercise 16.9. Based on Theorems 2 and 3 we may conjecture that for $(n, d, m, r, \tilde{r}) = (4, 4, 0, 1, -1), (6, 3, 1, 1, -1)$, the analytic scheme $V_{r[\mathbb{P}^n_2]+r[\mathbb{P}^n_2]}$ is smooth and reduced. If this is the case, its underlying analytic variety is bigger than $V_{[\mathbb{P}^n_2] \cap [\mathbb{P}^n_2]}$ (see [6]), and so, we may try to formulate similar statements as in Conjecture 1 in these cases. However, one of the main ingredients of Conjecture 1 fails to be true in lower degrees, see Conjecture 8 and comments after this.

The present article together with the book [Mov17a] is written during the years 2014-2017. One of the main aims of the book [Mov17a] has been to focus on computational aspects of Hodge theory. From this book we have just collected few results relevant to the content of this article, and in particular the study of the components of the Hodge locus passing through the Fermat point.

The proof of Theorem 5, Theorem 6, Theorem 7, Theorem 11, Theorem 12 and Theorem 14 are theoretical, whereas the proof of Theorem 1, Theorem 2, Theorem 3, Theorem 7, Theorem 8, Theorem 10 are computer assisted. These are partial verifications of many conjectures, for which we have to work with particular examples of $d$ and $n$. In many cases we have just mentioned these as comments after each conjecture and have avoided producing more theorem-style statements. An undergraduate student in mathematics interested in challenging problems is invited to read conjectures in [9]. We have to confess that we have not done our best to verify such conjectures as much as the computer performs the computations, and have contented ourselves to few special cases. There are few other results in the book [Mov17a] which are not announced here, and they might be useful for the investigation of Conjecture 1.

The computer codes used in the present text are written as procedures in the library foliation.lib (version 2.20) of SINGULAR, see [GPS01]. The reader who wants to get used to them is referred to [Mov17a] Chapter 18. This is mainly for codes used until [6]. From this section on, the name of procedures appears in the foot note of the pages where they are used. A different computer implementation of the proofs would be essential for two main reasons: first, it will be another confirmation of the results of the present paper, second, it will produce more results that the author was not able to obtain by his own primitive codes. This may produce precise conjectures for arbitrary dimension $n$ and degree $d$. 

\[ \text{(version 2.20) of foliation.lib} \]
The organization of the text is as follows. Sections 2, 3, 4, and 5 are essentially the first version of the article which appeared in the Arxiv in 2015. These are the announcement of some of the author’s results in the book [Mov17a]. In §2 we reformulate the Hodge conjecture using integrals. In §3 we introduce an alternative Hodge conjecture. This compares the deformation space of both algebraic and Hodge cycles. In §4 we recall the missing ingredient in the formulation of infinitesimal variation of Hodge structures. This is namely periods of Hodge/algebraic cycles. We then relate it to the alternative Hodge conjecture. In §5 we focus on Hodge cycles in the Fermat variety which cannot be deformed to nearby hypersurfaces. We then present the formula of periods of linear cycles inside the Fermat variety. From §6 we start to examine Conjecture 1. In this section we also prove Theorem 1. We first observe that the Zariski tangent space of the Hodge locus corresponding to the Hodge cycle \([\mathbb{P}^n + \mathbb{P}^n]\) has codimension strictly less than the codimension of the locus corresponding to deformations of the algebraic cycle \(\mathbb{P}^n + \mathbb{P}^n\). This indicates the existence of a strange component of the Hodge locus provided that such a component is smooth and reduced. For this reason in §7 we introduce Conjecture 8 which ensures us that such components exists for certain linear combination of \(\mathbb{P}^n\) and \(\mathbb{P}^n\). In order to investigate this conjecture, in §8 we announce our main result on the full power series expansion of periods. This might be used in order to investigate the smoothness and reducedness of the components of the Hodge loci. In this section we also prove Theorem 2 and Theorem 3. In §9 we introduce few other conjectures purely of linear algebraic nature. These are the last missing pieces in the proof of Conjecture 1. Finally, in §11 we explain how to handle Conjecture 1.

My heartfelt thanks go to P. Deligne for all his emails in January and February 2016 which motivated me and gave me more courage and inspiration to work on my book [Mov17a] and the present article. This was in a time I was getting many disappointments and complains. I would like to thank C. Voisin for her comments on higher order approximation of Noether-Lefschetz locus. This research has not been possible without the excellent ambient of my home institute IMPA in Rio de Janeiro and the hospitality of MPIM at Bonn during many short visits. My sincere thanks go to both institutes. The last version of the article was written during a visit of Paris VII. I would like to thank H. Mourtada and F. El Zein for the invitation and CNRS for financial support. Finally, I would like to dedicate this article to two women, one in my memories and the other by my side: Rogayeh Mollayipour, my mother, who thought me lessons of life no other could do it, Sara Ochoa, my wife, whose contribution to the existence of this article is not less than mine.

2 Hodge conjecture

For a complex smooth projective variety \(X\), an even number \(n\), an element \(\omega\) of the algebraic de Rham cohomology \(\omega \in H^n_{\text{dR}}(X)\) and an irreducible subvariety \(Z\) of dimension \(\frac{n}{2}\) in \(X\), by a period of \(Z\) we simply mean

\[
\frac{1}{(2\pi \sqrt{-1})^{\frac{n}{2}}} \int_{[Z]} \omega,
\]

where \([Z] \in H_n(X,\mathbb{Z})\) is the topological class induced by \(Z\). All the homologies with integer coefficients are modulo torsions, and hence they are free \(\mathbb{Z}\)-modules. We have to use a canonical isomorphism between the algebraic de Rham cohomology and the usual one defined by \(C^\infty\)-forms in order to say that the integration makes sense, see Grothendieck’s article [Gro66]. However, this does not give any clue how to compute such an integral. In general, integrals are transcendental numbers, however, in our particular case if \(X, Z, \omega\) are defined over a subfield \(k\) of complex numbers then (6) is also in \(k\), see Proposition 1.5 in Deligne’s lecture notes in [DMOS82], and
so it must be computable. In the $C^\infty$ context many of integrals (6) are automatically zero. This is the main content of the celebrated Hodge conjecture:

**Conjecture 2** (Hodge Conjecture). Let $X$ be a smooth projective variety of even dimension $n$ and $\delta \in H_n(X, \mathbb{Z})$ be a Hodge cycle, that is,

$$\int \delta \omega = 0,$$

for all closed $(p, q)$-form in $X$ with $p > \frac{n}{2}$, $p + q = n$.

Then there is an algebraic cycle

$$\sum_{i=1}^{s} n_i Z_i, \quad n_i \in \mathbb{Z}, \quad \dim(Z_i) = \frac{n}{2}$$

and a natural number $a \in \mathbb{N}$ such that $a \cdot \delta = \sum n_i[Z_i]$.

Using Poincaré duality our version of the Hodge conjecture is equivalent to the official one, see for instance Deligne’s announcement of the Hodge conjecture [Del06], however, we wrote it in this format in order to point out that the Hodge decomposition is not needed in its announcement and bring it to its origin which is the study of integrals due to Abel, Poincaré, Picard among many others. For a prehistory of the Hodge conjecture see [Mov17a], Chapters 2 and 3.

### 3 An alternative conjecture

The Hodge conjecture does not give any information about non-vanishing integrals (6). In this article we show that explicit computations of (6) lead us to verifications of the following alternative for the Hodge conjecture:

**Conjecture 3** (Alternative Hodge Conjecture). Let $\{X_t\}_{t \in T}$ be a family of complex smooth projective varieties of even dimension $n$, and let $Z_0$ be a fixed irreducible algebraic cycle of dimension $\frac{n}{2}$ in $X_0$ for $0 \in T$. There is an open neighborhood $U$ of $0$ in $T$ (in the usual topology) such that for all $t \in U$ if the monodromy $\delta_t \in H_n(X_t, \mathbb{Z})$ of $\delta_0 = [Z_0]$ is a Hodge cycle, then there is an algebraic deformation $Z_t \subset X_t$ of $Z_0 \subset X_0$ such that $\delta_t = [Z_t]$. In other words, deformations of $Z_0$ as a Hodge cycle and as an algebraic cycle are the same.

Before explaining the relation of this conjecture with integrals (6), we say few words about the importance of Conjecture 3. First of all, Conjecture 3 might be false in general, therefore, it might be called a property of $Z_0$. P. Deligne pointed out that there are additional obstructions to the hope that algebraic cycles could be constructed by deformation (personal communication, 31 January 2016). For instance, the dimension of the intermediate Jacobian coming from the largest sub Hodge structure of $H^{n-1}_d(X_0, \mathbb{Q}) \cap (H^{n-1}_{dR}(X_0, \mathbb{Q}))$ might jump down by deformation. This observation does not apply to a smooth hypersurface, for which only the middle cohomology is non-trivial. We are interested in cases in which Conjecture 3 is true, see Theorem 4 below. Both Hodge conjecture and Conjecture 3 claim that a given Hodge cycle must be algebraic, however, note that Conjecture 3 provides a candidate for such an algebraic cycle, whereas the Hodge conjecture doesn’t, and so, it must be easier than the Hodge conjecture. Verifications of Conjecture 3 support the Hodge conjecture, however, a counterexample to Conjecture 3 might not be a counterexample to the Hodge conjecture, because one may have an algebraic cycle homologous to, but different from, the given one in Conjecture 3.

In [Gro66] page 103 Grothendieck states a conjecture which is as follows: let $X \to S$ be a smooth morphism of schemes and let $S$ be connected and reduced. A global section $\alpha$ of $H^{2p}_{dR}(X/S)$ is algebraic at every fiber $s \in S$ if and only if it is a flat section with respect to the...
Gauss-Manin connection and it is algebraic for one point \( s \in S \). Conjecture \( \mathcal{H} \) for instance for complete intersections inside hypersurfaces, implies this conjecture in the same context, however the vice versa is not true. The variety \( T_d \) defined in \( \mathcal{H} \) might be a proper subset of a component of the Hodge locus. This would imply that \( Z \) is homologous to another algebraic cycle with a bigger deformation space. This cannot happen for the linear case \( d = (1, 1, \ldots, 1) \), see Theorem \( \mathcal{H} \) below, and many examples of \( n \) and \( d \) and \( d \), see [MV17]. The article [Blo72] is built upon the Grothendieck’s conjecture explained above and it considers semi-regular algebraic cycles, that is, the semi-regularity map \( \pi : H^1(Z, N_{X/Z}) \to H^2_{\mathbb{Q}}(X, \Omega^1_{\mathbb{Q}}) \) is injective. The semi-regularity is a very strong condition. For instance, for curves inside surfaces, [Blo72] only considers the semi-regular curves with \( H^1(Z, N_{X/Z}) = 0 \). Using Serre duality, one can easily see that this is not satisfied for curves with self intersection less than \( 2g - 2 \), where \( g \) is the genus of \( Z \). A simple application of adjunction formula shows that apart from few cases, complete intersection curves inside surfaces do not satisfy this condition.

In situations where the Hodge conjecture is true, for instance for surfaces, Conjecture \( \mathcal{H} \) is still a non-trivial statement. For a smooth hypersurface \( X \subset \mathbb{P}^3 \) of degree \( d \geq 4 \) and a line \( \mathbb{P}^1 \subset X \), deformations of \( \mathbb{P}^1 \) as a Hodge cycle and as an algebraic curve are the same. This follows from classical IVHS techniques introduced in [CGH83]. In [Gre88, Gre89, Voi88], Green and Voisin prove a stronger statement which says that the space of surfaces \( X \subset \mathbb{P}^3 \) containing a line \( \mathbb{P}^1 \) is the only component of the Noether-Lefschetz locus of minimum codimension \( d - 3 \). In order to reproduce the full statement of Green and Voisin’s results in our context and in a neighborhood of the Fermat point, see Conjecture 9 and the comments after. In a similar way some other results of Voisin on Noether-Lefschetz loci, see [Voi90], fit into the framework of Conjecture \( \mathcal{H} \). A weaker version of the mentioned statement in higher dimensions is generalized in the following way:

**Theorem 4 ([Mov17b] Theorem 2).** For any smooth hypersurface \( X \) of degree \( d \) and dimension \( n \) in a Zariski neighborhood of the Fermat variety with \( d \geq 2 + \frac{4}{n} \) and a linear projective space \( \mathbb{P}^d_{\mathbb{Q}} \subset X \), deformations of \( \mathbb{P}^d_{\mathbb{Q}} \) as an algebraic cycle and Hodge cycle are the same.

### 4 Infinitesimal variation of Hodge structures for Fermat variety

The relation between integrals \( (6) \) and Conjecture \( \mathcal{H} \) is established through the so-called infinitesimal variation of Hodge structures developed in [CGH83]. This is explained in [Mov17b], where the author has tried to keep the classical language of IVHS, and so we do not reproduce it here. The main application is going to be on Hodge and Noether-Lefschetz loci. The reader is referred to Voisin’s expository article [Voi13] which contains a full exposition and main references on this topic.

In order to keep the content of this text elementary, we explain this for complete intersection algebraic cycles inside hypersurfaces, and in particular, the Fermat variety. Let \( T \) be the parameter space of smooth hypersurfaces of degree \( d \) in \( \mathbb{P}^{n+1} \). A hypersurface \( X = X_t, t \in T \) is given by the projectivization of \( f(x_0, x_1, \ldots, x_{n+1}) = 0 \), where \( f \) is a homogeneous polynomial of degree \( d \). Fix integers \( 1 \leq d_1, d_2, \ldots, d_{\frac{n}{2}+1} \leq d \) and \( d := (d_1, d_2, \ldots, d_{\frac{n}{2}+1}) \). Let \( T_{\mathbb{Q}} \subset T \) be the parameter space of smooth hypersurfaces with

\[
f = f_1f_{\frac{n}{2}+2} + \cdots + f_{\frac{n}{2}+1}f_{n+2}, \quad \deg(f_i) = d_i, \quad \deg(f_{\frac{n}{2}+1+i}) = d - d_i,
\]

where \( f_i \)'s are homogeneous polynomials. The algebraic cycle

\[
Z := \mathbb{P}\{f_1 = f_2 = \cdots = f_{\frac{n}{2}+1} = 0\} \subset X
\]
is called a complete intersection (of type \(d\)) in \(X\). Note that this cycle is a complete intersection in \(\mathbb{P}^{n+1}\) and it is not a complete intersection of \(X\) with other hypersurfaces. Let

\[
\omega_i := \text{Res}_i \left( \frac{x^i \cdot \sum_{j=0}^{n+1} (-1)^j x_j \ dx_0 \wedge \cdots \wedge \tilde{dx}_j \wedge \cdots \wedge dx_{n+1}}{f^k} \right)
\]

with \(k := \frac{n+2+\sum_{d=0}^{n+1} i_e}{d}\), where \(\text{Res}_i : H^{n+1}_{\text{dR}}(\mathbb{P}^{n+1} - X) \to H^n_{\text{dR}}(X)\) is the residue map and \(x^i = x_0^i \cdots x_n^{i+1}\). After Griffiths \cite{Gri69}, we know that \(\delta \in H_n(X, \mathbb{Z})\) is a Hodge cycle if and only if

\[
\int_\delta \omega_i = 0, \quad \forall i \quad \text{with} \quad \frac{n+2+\sum_{d=0}^{n+1} i_e}{d} \leq \frac{n}{2}.
\]

A cycle \(\delta \in H_n(X, \mathbb{Z})\) is called primitive if its intersection with \([Z_\infty]\) is zero. Recall that \(Z_\infty\) is the intersection of a linear \(\mathbb{P}^{2n+1}_z \subset \mathbb{P}^{n+1}\) with \(X\). The \(\mathbb{Z}\)-module \(H_n(X, \mathbb{Z})_0\) by definition is the set of primitive cycles. We denote by \(\text{Hodge}_n(X, \mathbb{Z}) \subset H_n(X, \mathbb{Z})\) the \(\mathbb{Z}\)-modules of \(n\)-dimensional Hodge cycles in \(X\), and by \(\text{Hodge}_n(X, \mathbb{Z})_0\) its submodule consisting of primitive cycles. All the \(\mathbb{Z}\)-modules in this text are up to torsions, and hence they are free.

Let us now focus on the Fermat variety \(X^n_d\) which is obtained by the projectivization of

\[
X^n_d : \ x_0^d + x_1^d + \cdots + x_{n+1}^d = 0.
\]

We denote by \(0 \in T\) the point corresponding to \(X^n_d\), that is, \(X_0 = X^n_d\). Hodge cycles of the Fermat variety have been extensively studied by Shioda in his seminal works \cite{Shi79a, Shi79b, Shi81}. We are mainly interested in the Hodge cycles \(K_n\) of \(X^n_d\), where \(K\) is a complete intersection of type \(d\) in \(\mathbb{P}^{n+1}\) which lies in \(X^n_d\). This is because all the examples of \(n\) and \(d\) in which the Hodge conjecture is known for \(X^n_d\), one has only used this type of algebraic cycles, see \cite{Mov17} Chapter 17. The periods of a Hodge cycle \(\delta \in \text{Hodge}_n(X^n_d, \mathbb{Z})\) are defined in the following way

\[
p_i = p_i(\delta) := \frac{1}{(2\pi i)^{n+1}} \int_\delta \omega_i,
\]

\[
\sum_{e=0}^{n+1} i_e = \frac{n}{2} + 1 - d - (n+2).
\]

Using Deligne’s result in \cite{DMOSS2} Proposition 1.5, we know that \(p_i\)’s are in an abelian extension of \(\mathbb{Q}(\zeta_d)\). If \(p_i\)’s are all zero then \(\delta\) is necessarily in the one dimension \(\mathbb{Q}\)-vector space generated by \([Z_\infty]\). We are going to explain the role of these numbers in the deformation of Hodge cycles.

**Definition 1.** For natural numbers \(N, n\) and \(d\) let us define

\[
I_N := \{(i_0, i_1, \ldots, i_{n+1}) \in \mathbb{Z}^{n+2} \mid 0 \leq i_e \leq d - 2, \ i_0 + i_1 + \cdots + i_{n+1} = N\}
\]

Assume that \(n\) is even and \(d \geq 2 + \frac{4}{n}\). Consider complex numbers \(p_i\) indexed by \(i \in I_{\frac{n}{2}+1}^{d-n-2}\). For any other \(i\) which is not in the set \(I_{\frac{n}{2}+1}^{d-n-2}\), we define \(p_i\) to be zero. Let \([P_{i+j}]\) be a matrix whose rows and columns are indexed by \(i \in I_{\frac{n}{2}+d-n-2}\) and \(j \in I_d\), respectively, and in its \((i, j)\) entry we have \(p_{i+j}\).

The numbers \(\#I_{d}, \#I_{\frac{n}{2}+d-n-2}, \ #I_{\frac{n}{2}+1}^{d-n-2}\) are respectively, the dimension of the moduli space, \((\frac{n}{2}+1, \frac{n}{2} - 1)\) Hodge number and \((\frac{n}{2}, \frac{n}{2})\) Hodge number minus one, of smooth hypersurfaces of dimension \(n\) and degree \(d\). The following theorem justifies the importance of the algebraic numbers \(p_i\)’s in \((11)\).
Theorem 5. Let $X^d_n$ be the Fermat variety of dimension $n$ and degree $d$ parameterized by the point $0 \in T$. Let also $\delta_0 \in H^0(X^d_n, \mathbb{Z})$ be a Hodge cycle. The kernel of the matrix $[p_{i+j}]$ is canonically identified with the Zariski tangent space of the Hodge locus $V_{\delta_0}$ passing through $0 \in T$ and corresponding to $\delta_0$.

The Hodge locus mentioned in the above theorem is actually the analytic scheme defined by

$$O_{V_{\delta_0}} := O_{T,0}/\left\langle \int_{\delta_t} \omega_1, \int_{\delta_t} \omega_2, \cdots, \int_{\delta_t} \omega_a \right\rangle$$

where $\omega_1, \omega_2, \cdots, \omega_a$ are reindexed $\omega_i$’s in (9). These are sections of the cohomology bundle $H^d_{\text{dR}}(X_t)$, $t \in (T,0)$ such that for $t \in (T,0)$ they form a basis of $F^{n+1}H^d_{\text{dR}}(X_t)$, where $F^i$’s are the pieces of the Hodge filtration of $H^d_{\text{dR}}(X_t)$. Its points are all $t$ in a small neighborhood of $0$ such that the monodromy $\delta_t \in H^d_n(X_t, \mathbb{Z})$ of $\delta_0$ is a Hodge cycle, or equivalently, $\int_{\delta_t} \omega_1 = \int_{\delta_t} \omega_2 = \cdots = \int_{\delta_t} \omega_a = 0$. This is a local analytic subset of $T$ and by a deep theorem of Cattani-Deligne-Kaplan in [CDK95] we know that it is algebraic. This together with the fact that Hodge cycles of the Fermat variety are absolute and Deligne’s Principle B in [DMO92] implies that such an algebraic set is defined over $\mathbb{Q}$, for details see [Vo13] Proposition 5.7. The Hodge locus in $T$ is the union of all such local loci defined as before for all $t \in T$ (one might take different $\omega_i$’s as in (9)). Theorem 5 follows from Voisin’s result [Vo03] 5.3.3 on the Zariski tangent space of the Hodge locus and the computations of the infinitesimal variation of Hodge structures for the Fermat variety in [Mov17b]. An alternative proof using some ideas of holomorphic foliations is given in the later reference.

Theorem 6. Let $X^d_n$ be the Fermat variety $[10]$ and let $Z$ be a complete intersection of type $d$ inside $X^d_n$. Let also $p_i$ be the periods of $\delta = [Z]$ defined in (11). If

$$\text{rank}([p_{i+j}]) = \binom{n+1+d}{n+1} - \sum_{k=1}^{n+2} (-1)^{k-1} \sum_{a_1+a_2+\cdots+a_k \leq d} \binom{n+1+d-a_1-a_2-\cdots-a_k}{n+1},$$

where $(a_1, a_2, \ldots, a_{n+2}) = (d_1, d_2, \ldots, d_{n+1}, d-d_1, d-d_2, \ldots, d-d_{n+1})$ and the second sum runs through all $k$ elements (without order) of $a_i$, $i = 1, 2, \ldots, n+2$, then $T_d$ is a component of the Hodge locus. In particular Conjecture 3 is true for smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ containing a complete intersection of type $d$, and in a non-empty Zariski open subset of $T_d$.

The number in the right hand side of (14) is actually the codimension of $T_d$ in $T$, see [Mov17a] Proposition 17.5, and so Theorem 6 is a consequence of this fact and Theorem 5 see [Mov17a] Theorem 17.6. For arbitrary $d$ and $n$ the hypothesis of Theorem 6 is verified for projective spaces $Z = \mathbb{P}^{n+1} \subset X$, that is, for the case $d = (1, 1, \ldots, 1)$. This is

$$\text{rank}([p_{i+j}]) = \binom{n/2 + d}{d} - \binom{n/2 + 1}{d+1}^2,$$

see [Mov17b]. In this way we have derived Theorem 6. For this particular class of algebraic cycles, it is possible to prove the identity (15) without computing $p_i$’s. We may expect or conjecture that the equality (15) is always true. This is the case for many examples of complete intersection algebraic cycles worked out in [MV17]. This includes the author’s favorite example $(n, d) = (4, 6)$, that is, the sextic Fermat fourfold:

$$X^6_4 : x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 = 0.$$
The Hodge numbers of the fourth cohomology of a smooth sextic fourfold is 1,426,1752, 426,1 and the Fermat sextic fourfold has a very peculiar property that the $\mathbb{Q}$-vector space of its Hodge cycles has the maximum dimension which is 1752. In this case the matrix $[p_{i+j}]$ is a quadratic $426 \times 426$ matrix. We have only 10 possibilities for the locus $T_d$ of hypersurfaces with a complete intersection algebraic cycle. The corresponding data are listed in the table below:

| $(d_1, d_2, d_3)$ | codim$_T(T_d)$ | $(d_1, d_2, d_3)$ | codim$_T(T_d)$ |
|------------------|----------------|------------------|----------------|
| (1,1,1)          | 19             | (1,3,3)          | 71             |
| (1,1,2)          | 32             | (2,2,2)          | 92             |
| (1,1,3)          | 37             | (2,2,3)          | 106            |
| (1,2,2)          | 54             | (2,3,3)          | 122            |
| (1,2,3)          | 62             | (3,3,3)          | 141            |

The Fermat cubic tenfold

$$X_{10}^3: x_0^3 + x_1^3 + \cdots + x_{11}^3 = 0$$

has the Hodge numbers 0,0,0,1,220,925,220,1,0,0,0 and the $\mathbb{Q}$-vector space of its Hodge cycles has the maximum dimension which is 925. In this case the Hodge conjecture can be verified using linear cycles $\mathbb{P}^5$, see Theorem 9. We have only one possibility for $T_d$. This is namely $d = (1,1,1,1,1,1)$. Its codimension is 20.

5 General Hodge cycles for Fermat variety

We say that a Hodge cycle $\delta \in \text{Hodge}_n(X_n^d, \mathbb{Q})$ is general if $\text{rank}[p_{i+j}]$ attains the maximal rank, that is,

$$\text{(17)} \quad \text{rank}[p_{i+j}] = \text{minimum}\{#I_d, #I_{2d-n-2}\}. $$

Note that $#I_{2d-n-2}$ (resp, $#I_d$) is the number of rows (resp. columns) of $[p_{i+j}]$. If there exists a general Hodge cycle then the subvariety of $\text{Hodge}_n(X_n^d, \mathbb{Q})$ given by $\text{rank}[p_{i+j}] < \text{min}\{#I_d, #I_{2d-n-2}\}$ is proper and so there is a Zariski open subset $U$ of $\text{Hodge}_n(X_n^d, \mathbb{Q})$ such that all $\delta \in U$ are general. This will hopefully justify the name. Moreover, Theorem 5 implies that for a general Hodge cycle $\delta$ the Hodge locus $V_\delta$ is always smooth and reduced.

**Conjecture 4.** *The Fermat variety $X_n^d$ for $d \geq 2 + \frac{4}{n}$ has always a general Hodge cycle.*

Let us discuss two extreme cases in the above conjecture. First, if $n = 2$ then a general Hodge cycle has $\text{rank}[p_{i+j}] = \binom{d-1}{3}$. Conjecture 4 and Theorem 5 imply that there are infinite number of components of the Noether-Lefschetz locus of codimension $(d-1)$ passing through the Fermat point $0 \in T$, provided that there are infinite number of general Hodge cycles with different $\text{ker}[p_{i+j}]$. This is compatible with the result in [CHMSS] that the components of the Noether-Lefschetz locus with the maximal codimension are dense in $T$, both in the Zariski and usual topology. Second, if $d > \frac{2(n+1)}{n-2}$ then the right hand side of (17) is $#I_d$ which is also the dimension of the moduli of hypersurfaces of degree $d$ and dimension $n$. Therefore, Conjecture 4 in this case implies that general Hodge cycles of the Fermat variety cannot be deformed in the moduli space of hypersurfaces of degree $d$ and dimension $n$, in other words, any deformation of a general Hodge cycle of the Fermat variety to a nearby hypersurface $X \subset \mathbb{P}^{n+1}$ implies that $X$ is obtained from $X_n^d$ by a linear transformation of $\mathbb{P}^{n+1}$.

For a moment assume that we have a collection of algebraic cycles $Z_i, i = 1,2, \ldots, s$ such that $[Z_i]$’s generate the $\mathbb{Q}$-vector space $\text{Hodge}_n(X_n^d, \mathbb{Q})$ of Hodge cycles, and so, we know the Hodge conjecture for $X_n^d$ is valid. This together with Conjecture 4 implies that a general algebraic
cycle \( \sum_{i=1}^{s} n_i Z_i \), \( n_i \in \mathbb{Z} \) has a deformation space of the expected codimension which is the right hand side of (17). In particular, for \( d > \frac{2(n+1)}{n-2} \) such an algebraic cycle cannot be deformed at all if we consider the parameter space \( T \) parameterizing the homogeneous polynomials of the type

\[
(18) \quad f := x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{j \in I_d} t_j x^j.
\]

Any smooth hypersurface in a Zariski open neighborhood of the Fermat point \( 0 \) after a linear transformation of \( \mathbb{P}^{n+1} \) can be written as the zero set of some \( f \) in this format.

The equalities (14) and (17) can be checked computationally, as far as, we take particular examples of the degree \( d \) and the dimension \( n \), compute the periods \( p_i \) and the rank of \( [p_{i+j}] \). Here, is the result

**Theorem 7.** The Fermat surface \( X^d_2 \), \( 4 \leq d \leq 8 \) has a general Hodge cycle. The Fermat fourfold \( X^d_4 \), \( 3 \leq d \leq 6 \) has also a general Hodge cycle.

The upper bound on \( d \) is just due to the limitation of our computer and it might be improved if one uses a better computing machine. Theorem 7 for \( n = 4, d = 6 \) says the following:

**Theorem 8.** A general Hodge cycle \( \delta_0 \in H^4_4(X^d_4, \mathbb{Q}) \) is not deformable, that is, the monodromy \( \delta_t \in H^4_4(X_t, \mathbb{Z}) \), \( t \in (T, 0) \) of \( \delta_0 \) to \( X_t \) is no more a Hodge cycle.

Note that we are using the parameter space in (18), otherwise, we should have stated that \( X_t \) is obtained by a linear transformation of \( X^d_4 \). For the computations of periods of Hodge cycles and proof of Theorem 7 and Theorem 8 see [Mov17a] Chapter 15 and 16. See also §18.8 for details of the computer codes used for the proofs. The same codes for the Fermat cubic tenfold runs out of memory. In this case one might use Theorem 11.

**Theorem 9** ([Ran81], [Shi79a], [ASS3]). Suppose that either \( d \) is a prime number or \( d = 4 \) or \( d \) is relatively prime with \( (n+1)! \). Then \( H^4_n(X^d_n, \mathbb{Q}) \) is generated by the homology classes of the linear cycles \( \mathbb{P}^2 \), and in particular, the Hodge conjecture for \( X^d_n \) is true.

This theorem is the outcome of many efforts in order to prove the Hodge conjecture for \( X^d_n \) using linear projective cycles. The cases \((n, d) = (2, 6), (4, 6)\) are not covered by this theorem because such algebraic cycles are not enough in these cases. N. Aoki in [Aok87], inspired by his work with Shioda [ASS3], has introduced more algebraic cycles and in this way he has been able to verify the Hodge conjecture for many other Fermat varieties, and in particular for the sextic Fermat fourfold. In this case we can determine the homology classes of linear cycles \( \mathbb{P}^2 \) explicitly, [Mov17a] Section 16.7. This together with Theorem 8 gives us:

**Theorem 10.** A general \( \mathbb{Z} \)-linear combination of projective linear cycles \( \mathbb{P}^2 \) is not deformable in the moduli space of degree 6 hypersurfaces in \( \mathbb{P}^5 \).

We propose two different methods in order to compute integrals (6). The first method is purely topological and it is based on the computation of the intersection numbers of algebraic cycles with vanishing cycles. In the case of the Fermat variety, we are able to write down vanishing cycles explicitly, however, they are singular, even though they are homeomorphic to spheres, and many interesting algebraic cycles of the Fermat variety intersect them in their singular points. This makes the computation of intersection numbers harder. The second method is purely algebraic and it is a generalization of Carlson-Griffiths computations in [CG80]. One has to compute the restriction of differential \( n \)-forms in \( X \) to the top cohomology of \( Z \), and then, one has to compute the so-called trace map. The second method is the main topic of the
Ph.D. thesis of R. Villaflor, see \cite{Vil18}. For \(a = (a_1, a_3, \ldots, a_{n+1}) \in \{0, 1, 2, \ldots, d - 1\}^{\frac{n}{2} + 1}\) and a permutation \(b = (b_0, b_1, \ldots, b_{n+1})\) of \(\{0, 1, 2, \ldots, n + 1\}\) let

\[
\mathbb{P}^{\frac{n}{2}}_{a, b} : \left\{ \begin{array}{l}
x_{b_0} - \zeta_{2d}^{1+2a_1}x_{b_1} = 0, \\
x_{b_2} - \zeta_{2d}^{1+2a_3}x_{b_3} = 0, \\
x_{b_4} - \zeta_{2d}^{1+2a_5}x_{b_5} = 0, \\
\vdots \\
x_{b_{n+1}} - \zeta_{2d}^{1+2a_{n+1}}x_{b_{n+1}} = 0.
\end{array} \right.
\]

We call it a linear cycle inside the Fermat variety. Hopefully, this \(a, b\) notation will not be confused with the integers \(a, b\) in Conjecture \([1]\). In order to avoid repetitions, we may assume that \(b_0 = 0\) and for \(i\) an even number \(b_i\) is the smallest number in \(\{0, 1, 2, \ldots, n + 1\}\) \(\backslash\{b_0, b_1, b_2, \ldots, b_{i-1}\}\). In this way the number of linear cycles is

\[
(n + 1) \cdot (n - 1) \cdots 3 \cdot 1 \cdot d^{\frac{n}{2} + 1}.
\]

For linear cycles the computation of periods is a direct consequence of a theorem of Carlson and Griffiths in \cite{CG80}:

**Theorem 11.** For \(i \in I_{\left(\frac{n}{2} + 1\right) d - n - 2}\) we have

\[
\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_{\mathbb{P}^{\frac{n}{2}}_{a,b}} \omega_i = \begin{cases} 
\frac{\text{sign}(b) \cdot (-1)^{\frac{n}{2}}}{d^{\frac{n}{2} + 1} \cdot \frac{n}{2}!} \zeta_{2d}^\epsilon & \text{if } \ i_{2e-2} + i_{2e-1} = d - 2, \ \forall e = 1, \ldots, \frac{n}{2} + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

where \(\zeta_{2d}\) is the \(2d\)-th primitive root of unity and

\[
\epsilon = \sum_{e=0}^{\frac{n}{2}} (i_{2e} + 1) \cdot (1 + 2a_{2e+1}).
\]

This is done in \cite{MV17} and it is the main ingredient of Theorem \([1]\). Theorem \([4]\) follows from the verification of the equality \((15)\) for periods of linear cycles computed in Theorem \([11]\). This verification turns out to be an elementary problem. Using Theorem \([11]\) we can make Theorem \([10]\) more concrete.

### 6 Sum of two linear cycles

Let \(\mathbb{P}^{\frac{n}{2}}, \bar{\mathbb{P}}^{\frac{n}{2}}\) be two linear algebraic cycles in the Fermat variety. We define

\[
H^d_n(m) := \text{rank} \left( \left[ p_{i+j} \left( \left[ \mathbb{P}^{\frac{n}{2}} \right] + \left[ \bar{\mathbb{P}}^{\frac{n}{2}} \right] \right) \right] \right), \text{ where } \mathbb{P}^{\frac{n}{2}} \cap \bar{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m.
\]

**Conjecture 5.** The number \(H^d_n(m)\) depends only on \(d, n, m\) and not on the choice of \(\mathbb{P}^{\frac{n}{2}}, \bar{\mathbb{P}}^{\frac{n}{2}}\). We have verified the conjecture for \((n, d)\) in \((2, d), \ 5 \leq d \leq 8, \ (4, 4), (6, 3)\).\footnote{The procedure \texttt{nadm} is used for this purpose.}
We can use the automorphism group $G^d_\mathbb{F}$ of the Fermat variety and we can assume that $\mathbb{P}^2\mathbb{F}$ is (19) with $a = (0,0,\cdots,0)$ and $b = (0,1,\cdots,n+1)$. In order to avoid Conjecture 5 we will fix our choice of linear cycles:

$$\mathbb{P}^2_a = \mathbb{P}^2_{a,b} \text{ with } a = (0,0,\cdots,0), \ b = (0,1,\cdots,n+1)$$

$$\mathbb{P}^2_\mathbb{F} = \mathbb{P}^2_{a,b} \text{ with } a = (0,0,\cdots,0,1,\cdots,1), \ b = (0,1,\cdots,n+1)$$

which are those used in Introduction. For examples of $H^d_n(m)$ see Table 1. For a sequence of natural numbers $a = (a_1,\ldots,a_s)$ let us define

$$C_a = \binom{n + 1 + d}{n + 1} - \sum_{k=1}^{s} (-1)^{k-1} \sum_{a_1 + a_2 + \cdots + a_k \leq d} \binom{n + 1 + d - a_1 - a_2 - \cdots - a_k}{n + 1},$$

where the second sum runs through all $k$ elements (without order) of $a_i$, $i = 1,2,\ldots,s$. By our convention, the projective space $\mathbb{P}^{n-1}$ means the empty set. By abuse of notation we write

$$a^b := a_1, a_2, \cdots, a_s.$$

Hopefully, there will be no confusion with the exponential $a^b$.

**Theorem 12.** Let $\mathbb{P}^2_\mathbb{F}, \mathbb{P}^2_{\mathbb{F}}$ be two linear algebraic cycles in a smooth hypersurface of dimension $n$ and degree $d$ and with the intersection $\mathbb{P}^m$. We have

$$K^d_n(m) := \text{codim}(V_{\mathbb{P}^2_\mathbb{F}} \cap V_{\mathbb{P}^2_{\mathbb{F}}}) = 2C_{1,2,d+1,(d-1)^2+1} - C_{1,n-m+1,(d-1)^{m+1}}.$$

In particular, if $\mathbb{P}^2_{\mathbb{F}}$ does not intersect $\mathbb{P}^2_{\mathbb{F}}$ then $V_{\mathbb{P}^2_{\mathbb{F}}}$ intersects $V_{\mathbb{P}^2_{\mathbb{F}}}$ transversely.

The proof is a simple application of Koszul complex and can be found in Section 17.9 of [Mov17a].

We are now going to analyze the number $H^d_n(m)$ for $m = \frac{n}{2}, \frac{n}{2} - 1, \ldots$. Let $\mathbb{P}^2_\mathbb{F}$ and $\mathbb{P}^2_{\mathbb{F}}$ be as in Introduction. Let us first consider the case $m = \frac{n}{2}$. For the proof of Theorem 4 we have verified the first equality in

$$H^d_n\left(\frac{n}{2}\right) = K^d_n\left(\frac{n}{2}\right) = C_{1,2,d,(d-1)^2},$$

(the second equality follows from Theorem 12). One of the by-products of the proof is that $V_{\mathbb{P}^2_{\mathbb{F}}}$ as an analytic scheme is smooth and reduced. For $m = \frac{n}{2} - \frac{1}{2}$, we have

$$H^d_n\left(\frac{n}{2} - 1\right) = C_{1,2,2,(d-1)^2,d-2} \leq K^d_n\left(\frac{n}{2} - 1\right) = 2C_{1,2,d+1,(d-1)^2+1} - C_{1,2,d+2,(d-1)^{2+1}}.$$
Conjecture 6. Let $\mathbb{P}^3_2, \mathbb{P}^6_2$ be two linear algebraic cycles in the Fermat variety and with no common point. The only deformations of $\mathbb{P}^3_2 + \mathbb{P}^6_2$ as an algebraic or Hodge cycle is again a sum of two linear cycles.

Particular cases of this conjecture has been announced in Theorem 1 (those with $m = -1$). It might happen that in (24) we have a strict inequality, see for instance Table 1.

Conjecture 7. For $n \geq 6$ we have

$$H^d_n \left( \frac{n}{2} - 2 \right) < K^d_n \left( \frac{n}{2} - 2 \right).$$

Our favorite examples for verifying Conjecture 7 are cubic Fermat varieties, that is $d = 3$. For $n \geq 4$ we have the following range:

$$\left( \frac{n}{2} + 1 \right) \leq \text{rank}([p_{i+j}]) \leq \left( \min \left\{ 3, \frac{n}{2} - 2 \right\} \right)$$

and in Table 1 we have computed $H^d_n(m)$ for $4 \leq n \leq 10$ and $-1 \leq m \leq \frac{n}{2}$. The following table is the main evidence for Conjecture 7.

| $n \backslash (\frac{n}{2} - m)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------------------------------|---|---|---|---|---|---|---|---|
| 4                             | (1, 1) | (1, 2) | (1, 2) | (1, 2) |
| 6                             | (4, 4) | (4, 7) | (6, 8) | (7, 8) | (8, 8) |
| 8                             | (10, 10) | (10, 16) | (16, 19) | (19, 20) | (20, 20) |
| 10                            | (20, 20) | (20, 30) | (32, 36) | (38, 39) | (40, 40) | (40, 40) |
| 12                            | (35, 35) | (35, 50) | (55, 60) | (65, 66) | (69, 69) | (70, 70) | (70, 70) |

Table 1: The numbers $(H^d_n(m), K^d_n(m))$.

We were also able to compute the five-tuples $(n, d, m|H^d_n(m), K^d_n(m))$ in the list below:

$$(4, 4, 0|11, 12), (4, 4, -1|12, 12),$$
$$(4, 5, 0|24, 24), (4, 5, -1|24, 24),$$
$$(4, 6, 0|38, 38), (4, 6, -1|38, 38),$$
$$(6, 4, 1|36, 37), (6, 4, 0|38, 38), (6, 4, -1|38, 38).$$

We were not able to compute more data such as $\ ?$ in $(4, 7, 0?), 54)$. For $n = 2$ and $4 \leq d \leq 14$ we were also able to check Conjecture 6. Note that for the quartic Fermat fourfold we have the range $6 \leq \text{rank}([p_{i+j}]) \leq 21$ and $T_{1,1,2}$ has codimension 8.

Proof of Theorem 1 for $r = \tilde{r} = 1$. This is just the outcome of above computations in which $H^d_n(m) = K^d_n(m)$. The full proof will be given after Theorem 13. For $r = \tilde{r} = 1$ we have Theorem 1 for $(n, d, m)$ in

$$(27) \quad (12, 3, -1), (12, 3, 0), (12, 3, 1), (12, 3, 2),$$

however, we were not able to verify Theorem 13 in these cases. \[\triangleq \]

\[\triangleq \text{For the computations of } H^d_n(m) \text{ and } K^d_n(m) \text{ we have used the procedures SumTwoLinearCycle and Codim, respectively.} \]
For the convenience of the reader we have also computed the table of Hodge numbers for cubic Fermat varieties. Note that for \( d = 3 \), \( n = 4 \) the Hodge conjecture is well-known, see [Zuc77].

| \( n \) | \( (\frac{3 + n + 1}{3}) - (n + 2)^2 \) | \( (\frac{2 + 1}{3}), (\frac{n + 2}{\min(3, \frac{2}{d}) - 2}) \) | Hodge numbers |
|-------|---------------------------------|---------------------------------|-------------|
| 4     | 20                              | 1,1                             | 0, 1, 21, 1, 0 |
| 6     | 56                              | 4,8                             | 0, 0, 8, 71, 8, 0, 0 |
| 8     | 120                             | 10, 45                          | 0, 0, 0, 45, 253, 45, 0, 0, 0 |
| 10    | 220                             | 20, 220                         | 0, 0, 0, 1, 220, 925, 220, 1, 0, 0, 0 |
| 12    | 364                             | 35, 1001                        | 0, 0, 0, 0, 14, 1001, 3432, 1001, 14, 0, 0, 0, 0 |

Table 2: Hodge numbers

**Theorem 13.** For all pairs \((n, d)\) in Theorem 7 with arbitrary \(-1 \leq m \leq \frac{n}{2}\) and all \( x \in \mathbb{Q} \) with \( x \neq 0 \), we have

\[
\text{Proof of Theorem 13: For all the cases in Theorem 7, we have}
\]

\[
(28) \quad \text{rank}(\langle [p_{i+j}(\mathbb{P}^2_x + x\mathbb{P}^2_x)] \rangle) = \text{rank}(\langle [p_{i+j}(\mathbb{P}^2_x + \mathbb{P}^2_x)] \rangle)
\]

and so this number does only depend on \((n, d, m)\) and not on \( x \).

**Proof.** Let \( a := H^d_R(m) \) be the number in the right hand side of (28) and let \( A(x) := [p_{i+j}(\mathbb{P}^2_x + x\mathbb{P}^2_x)] \). Except for a finite number of \( x \in \mathbb{Q} \), we have \( \text{rank}(A(x)) \geq a \) and in order to prove the equality, it is enough to check it for \( a + 2 \) distinct values of \( x \). This is because if \( \text{rank}(A(x)) > a \) then we have a \((a + 1) \times (a + 1)\) minor of \( A(x) \) whose determinant is not zero. This is a polynomial of degree at most \( a + 1 \) in \( x \), and it has \((a + 2)\) roots which leads to a contradiction. This argument implies that except for a finite number of values for \( x \) we have \( \text{rank}(A(x)) = a \). These are the roots of \( \det(B(x)) = 0 \), where \( B \) is any \( a \times a \) minor of \( A(x) \) such that \( P(x) := \det(B(x)) \) is not identically zero. We find such a minor and compute \( \text{rank}(A(x)) \) for all rational roots of \( P(x) \) and prove that this is \( a \) except for \( x = 0 \). It seems interesting that only for \((n, d, m) = (6, 3, 1), (6, 3, 0), (8, 3, 2), (8, 3, 1), (6, 4, 1), (10, 3, 3), (10, 3, 2)\) we find a rational root of \( P(x) \), and in all these cases it is \( x = 1 \). This seems to have some relation with Conjecture [1] for \((r, \tilde{r}) = (1, -1)\).

**Proof of Theorem 7:** For all the cases in Theorem 1

\[
\text{rank}\left(\left[ r[p_{i+j}(\mathbb{P}^2_x) + \tilde{r}[p_{i+j}(\mathbb{P}^2_x)] \right]\right) = \text{rank}\left(\left[ p_{i+j}(\mathbb{P}^2_x + \mathbb{P}^2_x) \right]\right) = K^d(m),
\]

where for the first equality we have used Theorem 13. We know that \( V_{\mathbb{P}^2_x} \cap V_{\tilde{r}[\mathbb{P}^2_x]} \) is the subset of the analytic variety underlying \( V_{[\mathbb{P}^2_x + \mathbb{P}^2_x]} \) and its codimension is \( K^d(m) \). This proves the theorem.

**7 Smooth and reduced Hodge loci**

Based on the computation in [6] we have formulated Conjecture 7 and we further claim that:

**Conjecture 8.** Let \( \mathbb{P}^2_x \) and \( \mathbb{P}^2_{\tilde{r}} \) be two linear cycles in the Fermat variety \( X_d^n \) with \( d \geq 2 + \frac{4}{n} \) and \( \mathbb{P}^2_x \cap \mathbb{P}^2_{\tilde{r}} = \mathbb{P}^m \) with \(-1 \leq m \leq \frac{n}{2} - 1\) and \( H^d_R(m) < K^d(m) \). There is a finite number of coprime non-zero integers \( r, \tilde{r} \) such that the analytic scheme \( V_{[\mathbb{P}^2_x + r[\mathbb{P}^2_{\tilde{r}}]} \) is smooth and reduced.

See GoodMinor and ConstantRank.
If \( r = 0 \) and \( r = 1 \) then we have the Hodge locus \( V_{[\mathbb{P}^n]_2} \) which is smooth and reduced by Theorem 6. Conjecture 8 is true in the following case: \( m = \frac{n}{2} - 1 \) and

\[
(n, d) = (2, d), \quad 4 \leq d \leq 15 \quad (4, d), \quad d = 3, 5, 6 \quad (6, d), \quad d = 3, 4.
\]

In this case, the Hodge locus \( V_{[\mathbb{P}^n]_2} \) is smooth and reduced at 0 and it parameterizes hypersurfaces with a complete intersection of type \((1^2, 2)\), see the comments before Theorem 1. The proof can be found in [MV17]. The analytic scheme \( V_{[\mathbb{P}^n]_2} \) is non-reduced or singular at 0 in the cases covered in Theorem 3. Other evidences to Conjecture 8 are listed in Theorem 2 and Theorem 3.

Assuming the Hodge conjecture, the points of the Hodge locus \( V_{[\mathbb{P}^n]_2} \) parametrizes hypersurfaces with certain algebraic cycles. We do not have any idea how such algebraic cycles look like. In order to verify Conjecture 8 without constructing algebraic cycles, we have to analyze the the generators \( f_\delta \), \( \omega \) of the defining ideal of the Hodge locus in [13]. These are integrals depending on the parameter \( t \in T \) and their linear part is gathered in the matrix \([p_{t+j}]\). If Conjecture 8 is true in these cases then we have discovered a new Hodge locus, different from \( V_{[\mathbb{P}^n]_2}, V_{[\mathbb{P}^n]_2} \) and their intersection. The whole discussion of [8] has the goal to provide tools to analyze Conjecture 8.

8 The creation of a formula

In this section we compute the Taylor series of the integration of differential forms over monodromies of the algebraic cycle \( \mathbb{P}^n_{a,b} \) inside the Fermat variety. Let us consider the hypersurface \( X_t \) in the projective space \( \mathbb{P}^{n+1} \) given by the homogeneous polynomial:

\[
(29) \quad f_t := x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{\alpha} t_\alpha x^\alpha = 0,
\]

where \( \alpha \) runs through a finite subset \( I \) of \( \mathbb{N}_{0}^{n+2} \) with \( \sum_{i=0}^{n+1} \alpha_i = d \). In practice, we will take the set \( I \) of all such \( \alpha \) with the additional constrain \( 0 \leq \alpha_i \leq d - 2 \). For a rational number \( r \) let \( \lfloor r \rfloor \) be the integer part of \( r \), that is \( \lfloor r \rfloor \leq r < \lfloor r \rfloor + 1 \), and \( \{r\} := r - \lfloor r \rfloor \). Let also \( (x)_y := (x+1)(x+2)\cdots(x+y-1) \), \( (x)_0 := 1 \) be the Pochhammer symbol. For \( \beta \in \mathbb{N}_{0}^{n+2} \), \( \beta \in \mathbb{N}_{0}^{n+2} \) is defined by the rules:

\[
0 \leq \beta_i \leq d - 1, \quad \beta_i \equiv d \beta_i.
\]

**Theorem 14.** Let \( \delta_t \in H_n(X_t, \mathbb{Z}) \), \( t \in (T, 0) \) be the monodromy (parallel transport) of the cycle \( \delta_0 := [\mathbb{P}^n_{a,b}] \in H_n(X_0, \mathbb{Z}) \) along a path which connects 0 to \( t \). For a monomial \( x^\beta = x_0^{\beta_0} x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}} \) with \( k := \sum_{i=0}^{n+1} \frac{\beta_i + 1}{d} \in \mathbb{N} \) we have

\[
(30) \quad \frac{C}{(2\pi \sqrt{-1})^{\frac{n}{2}}} \int_{\delta_t} \text{Res} \left( \frac{x^\beta \Omega}{f_t^k} \right) = \sum_{a: I \to \mathbb{N}_0} \left( \frac{1}{a!} D_{\beta+a^*} e^{\pi \sqrt{-1} E_{\beta+a^*}} \right) \cdot t^a,
\]

where the sum runs through all \#I-tuples \( a = (a_\alpha, \alpha \in I) \) of non-negative integers such that for \( \bar{\beta} := \beta + a^* \) we have

\[
(31) \quad \left\{ \frac{\beta_{2e+1} + 1}{d} \right\} + \left\{ \frac{\beta_{2e+1} + 1}{d} \right\} = 1, \quad \forall e = 0, \ldots, \frac{n}{2},
\]
and

\[ C := \sign(b) \cdot (-1)^{\frac{n}{2}} \cdot d^{\frac{n}{2}+1} \cdot (k-1)!, \]
\[ t^a := \prod_{\alpha \in I} t^a_{\alpha}, \quad |a| := \sum_{\alpha \in I} a_{\alpha}, \]
\[ a! := \prod_{\alpha \in I} a_{\alpha}!, \quad a^* := \sum_{\alpha} a_{\alpha} \cdot \alpha, \]
\[ D_{\beta} := \prod_{\beta = 0}^{n+1} \left( \left\{ \frac{\beta_i + 1}{d} \right\} \right)^{\beta_i+1}, \]
\[ E_{\beta} := \sum_{e = 0}^{n+1} \left\{ \frac{\beta_{2e} + 1}{d} \right\} \cdot (1 + 2a_{2e+1}). \]

This theorem is the outcome of many computations in [Mov17a]. Its proof is obtained after a careful analysis of the Gauss-Manin connection of the full family of hypersurfaces around the Fermat point \( 0 \in T \). For thus see Sections 13.9, 13.10, 17.11 of this book. In the next paragraph we are going to explain how to use Theorem 14 and give evidences for Conjecture 8.

Recall the definition of the Hodge locus as an scheme in (13). Let \( f_1, f_2, \ldots, f_a \in O_{T,0} \) be the integrals such that \( f_1 = f_2 = \cdots = f_a = 0 \) is the underlying analytic variety of the Hodge locus \( V_{\delta_0} \). We take \( f_1, f_2, \ldots, f_k, \ k \leq a \) such that the linear part of \( f_1, f_2, \ldots, f_k \) form a basis of the vector space generated by the linear part of all \( f_1, f_2, \ldots, f_a \). By Griffiths transversality those of \( f_i \) which come from \( F_{n+2}^2 H^2_{dR}(X_0) \) have zero linear part and so only \( F_{n+2}^1/F_{n+2}^2 \) part of the cohomology contribute to the mentioned vector space, see for instance [Mov17a] Section 16.5.

The Hodge locus \( V_{\delta_0} \) is smooth and reduced if and only if the two ideals \( \langle f_1, f_2, \ldots, f_k \rangle \) and \( \langle f_1, f_2, \ldots, f_a \rangle \) in \( O_{T,0} \) are the same. For this we have to check

\[ f \in \langle f_1, f_2, \ldots, f_k, \rangle \quad \text{for} \quad f = f_i, \ i = k+1, \ldots, a, \]

or equivalently

\[ f = \sum_{i=1}^{k} f_i g_i, \ g_i \in O_{T,0}. \]

Let \( f = \sum_{i=1}^{\infty} f_i, \ f_i = \sum_{j=1}^{\infty} f_{i,j} g_i = \sum_{j=0}^{\infty} g_{i,j} \) be the homogeneous decomposition of \( f, f_i \) and \( g_i \), respectively. The identity (33) reduces to infinite number of polynomial identities:

\[ f_1 = \sum_{i=1}^{k} f_{i,1} g_{i,0}, \]
\[ f_2 = \sum_{i=1}^{k} f_{i,2} g_{i,0} + \sum_{i=1}^{k} f_{i,1} g_{i,1}, \]
\[ \vdots \]
\[ f_j = \sum_{i=1}^{k} f_{i,j} g_{i,0} + \sum_{i=1}^{k} f_{i,j-1} g_{i,1} + \cdots + \sum_{i=1}^{k} f_{i,1} g_{i,j-1}. \]

**Definition 2.** For a Hodge locus \( V_{\delta_0} \) as in (13) and \( N \in \mathbb{N} \) we say that it is \( N \)-smooth if the first \( N \) equations in (34) holds for all \( f = f_i, i = k+1, k+2, \ldots, a \). In other words (33) holds up to monomials of degree \( \geq N + 1 \).
By definition a Hodge locus $V_{\Ho}$ is 1-smooth. Theorems 2 and Theorem 3, and in particular their computational proof, must be considered our strongest evidence to Conjecture 8.

**Proof of Theorem 2 and Theorem 3.** The proof is done using a computer implementation of the Taylor series [30]. In order to be sure that this Taylor series and its computer implementation are mistake-free we have also checked many $N$-smoothness property which are already proved in Theorem 1. In Theorem 3 Item 1 we have proved that the corresponding Hodge locus is not 2-smooth except in the following case which we highlight it. Let $\mathbb{P}^1$ and $\mathbb{P}^1$ be two lines in the Fermat quintic surface intersecting each other in a point. The Hodge locus $V_{r\mathbb{P}^1+p\mathbb{P}^1}$ for all $r, \tilde{r} \in \mathbb{Z}$ is 2-smooth. Moreover it is not 3-smooth for $0 < |r| < |\tilde{r}| \leq 10$. In Theorem 3 Item 2 (resp. 3) we have proved that the corresponding Hodge locus is not 3-smooth (resp. 4-smooth).

The property of being $N$-smooth for larger $N$’s is out of the capacity of my computer codes, see [12] for some comments.

9 Uniqueness of components of the Hodge locus

A Hodge cycle $\delta \in H_n(X_d, \mathbb{Z})$ is uniquely determined by its periods $p_i(\delta)$. This data gives the Poincaré dual of $\delta$ in cohomology, and hence, the classical Hodge class in the literature. Let $\Ho^d_n$ be the $\mathbb{Z}$-module of period vectors $p$ of Hodge cycles. We will also use its projectivization $\mathbb{P}\Ho^d_n$ (two elements $p$ and $\tilde{p}$ in the $\mathbb{Z}$-module are the same if there are non-zero integers $a$ and $\tilde{a}$ such that $ap = \tilde{a}\tilde{p}$). This $\mathbb{Z}$-module can be described in an elementary linear algebra context without referring to advanced topics, such as homology and algebraic de Rham cohomology, see Chapter 16 of [Mov17a]. Therefore, the conjectures of the present section can be understood by any undergraduate mathematics student! If either $d$ is a prime number or $d = 4$ or $d$ is relatively prime with $(n + 1)!$ then we may redefine $\Ho^d_n$ the $\mathbb{Z}$-modules generated by $p^{a,b}$, where

$$p_{i}^{a,b} := \begin{cases} \sum_{2d}^{n} (ib_{2e-1}+1)(1+2a_{2e+1}) & \text{if } ib_{2e-2} + ib_{2e-1} = d - 2, \forall e = 1, \ldots, \frac{n}{2} + 1, \\ 0 & \text{otherwise.} \end{cases}$$

and $a$ and $b$ are as in [19]. By Theorem 11 and Theorem 9 this will be a sub $\mathbb{Z}$-module of the $\Ho^d_n$ defined earlier. This will not modify our discussion below. Recall the matrix $[p_{i,j}]$ in Definition 1 and the map $\mathbb{P}\Ho^d_n \to \mathbb{N}, p \mapsto \text{rank}([p_{i,j}])$. If $p \neq 0$ then

$$\left(\frac{n}{2} + d\right) - \left(\frac{n}{2} + 1\right)^2 \leq \text{rank}([p_{i,j}]) \leq \begin{cases} \frac{n+1}{d+1} & \text{if } d < \frac{2(n+1)}{n-2}, \\ \frac{n}{d+1} - (n+2) & \text{if } d = \frac{2(n+1)}{n-2}, \\ \frac{n+1}{d+1} - (n+2)^2 & \text{if } d > \frac{2(n+1)}{n-2}. \end{cases}$$

Before stating our main conjecture in this section, let us state a simpler one.

**Conjecture 9.** Let $n \geq 2$ be an even number and $d \geq 3$ an integer with $(n,d) \neq (2,4), (4,3)$. Let also $p \in \Ho^d_n$ such that

$$\text{rank}([p_{i,j}]) = \left(\frac{n}{2} + d\right) - \left(\frac{n}{2} + 1\right)^2.\quad (36)$$

Then $p$, up to multiplication by a rational number, is necessarily of the form $p^{a,b}$.

---

[3] See SmoothReduced and TaylorSeries.

[6] The list of $p^{a,b}$’s is implemented in the procedure ListPeriodLinearCycle.
One can also formulate a similar conjecture for the next admissible rank. For \( n = 2 \) Voisin’s result in \[\text{Voi88}\] tells us that this must be \( 2d - 7 : = \text{codim}(T_{1,2}) \). For further discussion on this topic see \[\text{Mov17a}\] Chapter 19. It might happen that in Conjecture 10 one must exclude more examples of \((n,d)\). Note that for \((n,d) = (2,4), (4,3)\) both sides of \[\text{(43)}\] are equal to one for all non-zero \( p \).

We need to write down in an elementary language when the linear cycles \( \mathbb{P}^\frac{2}{d} \) and \( \mathbb{P}^\frac{2}{d}_j \) underlying two period vectors \( p^i, i = (a, b) \) and \( p^j, j = (\tilde{a}, \tilde{b}) \), respectively, have the intersection \( \mathbb{P}^m \). This is as follows: A bicycle attached to the permutations \( b \) and \( \tilde{b} \) is a sequence \( (c_1c_2 \ldots c_r) \) with \( c_i \in \{0,1,2,\ldots,n + 1\} \) and such that if we define \( c_{i+1} = c_i \) then for \( 1 \leq i \leq r \) odd (resp. even) there is an even number \( k \) with \( 0 \leq k \leq n + 1 \) such that \( \{c_i, c_{i+1}\} = \{b_k, b_{k+1}\} \) (resp. \( \{c_i, c_{i+1}\} = \{b_k, b_{k+1}\} \)) and there is no repetition among \( c_i \)’s. By definition there is a sequence of even numbers \( k_1, k_2, \ldots \) such that

\[
\{c_1, c_2\} = \{b_{k_1}, b_{k_1+1}\}, \quad \{c_2, c_3\} = \{b_{k_2}, b_{k_2+1}\}, \quad \{c_3, c_4\} = \{b_{k_3}, b_{k_3+1}\}, \ldots .
\]

Bicycles are defined up to twice shifting \( c_i \)’s, that is, \((c_1c_2c_3 \cdots c_r) = (c_3 \cdots c_1c_2)\) etc., and the involution \((c_1c_2c_3 \cdots c_{r-1}c_r) = (c_rc_{r-1} \cdots c_2c_1)\). For example, for the permutations

\[
b = (0,1,2,3,4,5), \quad \tilde{b} = (1,0,5,3,4,2)
\]

we have in total two bicycles \((01), (2354)\). Note that bicycles give us in a natural way a partition of \( \{0,1,\ldots,n + 1\} \). For such a bicycle we define its conductor to be the sum over \( k \), as before, of the following elements: if \( c_i = b_k \) and \( c_{i+1} = b_{k+1} \) (resp. \( c_i = \tilde{b}_k \) and \( c_{i+1} = \tilde{b}_{k+1} \)) then the element \( 1 + 2a_{k+1} \) (resp. \( 1 + 2\tilde{a}_{k+1} \)), and if \( c_i = b_{k+1} \) and \( c_{i+1} = b_k \) (resp. \( c_i = \tilde{b}_{k+1} \) and \( c_{i+1} = \tilde{b}_k \)) then \(-1 - 2a_{k+1} \) (resp. \(-1 - 2\tilde{a}_{k+1} \)). Because of the involution, the conductor is defined up to sign. In our example, the conductor of \((01)\) and \((2354)\) are respectively given by

\[
1 + 2a_1 + 1 + 2\tilde{a}_1, \quad 1 + 2a_3 - 1 - 2\tilde{a}_3 - 1 - 2a_5 + 1 + 2\tilde{a}_5.
\]

A bicycle is called new if \( 2d \) divides its conductor, and is called old otherwise. Let \( m_{ij} \) be the number of new bicycles attached to \((i,j)\) minus one.

**Conjecture 10.** Let \( n \geq 4 \) and \( d > \frac{2(n+1)}{n-2} \). If for some \( p \in \text{Ho}^d_n \), \( p \neq 0 \) we have

\[
\text{rank}([p_{i+j}]) \leq H^d_n\left(\frac{n}{2}-2\right),
\]

then \( p \) after multiplication with a natural number is in the set

1. \( Zp^a \) and so \( \text{rank}([p_{i+j}]) = \left(\frac{a^2+d}{d}\right) - \left(\frac{a}{2} + 1\right)^2 \).

2. \( Zp^a + Zp^{\tilde{a},b} \) with \( m_{a,b,\tilde{a},\tilde{b}} = \frac{n}{2} - 1 \) and so \( \text{rank}([p_{i+j}]) = C_1^{\frac{n}{2},2, (d-1)\frac{n}{2}, d-2} \).

3. \( Zp^a + Zp^{\tilde{a},b} \) with \( m_{a,b,\tilde{a},\tilde{b}} = \frac{n}{2} - 2 \) and so \( \text{rank}([p_{i+j}]) = H^d_n\left(\frac{n}{2}-2\right) \).

A complete analysis of Conjecture 10 would require an intensive search for the elements \( p \in \text{Ho}^d_n \) of low rank([p_{i+j}]). It might be true for \( n = 2 \) and large \( d \)’s, and this has to do with the Harris-Voisin conjecture, see \[\text{Mov17b}\], and will be discussed somewhere else. Note that the numbers in items 1,2,3 of Conjecture 10 for \( n = 2 \) are respectively \( d - 3, 2d - 7 \) and \( 2d - 6 \) (for the last one see Conjecture 6). We just content ourselves with the following strategy for confirming Conjecture 10. Let \( p^i, i = 1,2,3 \) be three distinct vectors of the form \( p^{a,b} \). We claim that for \( d > 3 \) we have

\[
\text{rank}([p_{i+j}]) > H^d_n\left(\frac{n}{2}-2\right), \quad \text{where} \ p = p^1 + p^2 + p^3.
\]
The number $H_4^d((\frac{n}{2} - 2)$ is computed in [6] and so we check in total $(\frac{n}{2})$ inequalities (37), where $N$ is the number of $p^{a,b}$'s in (20). This is too many computations and we have checked (37) for samples of $p$'s for $(n,d) = (4,6)$. In this way we have also observed that the lower bound for $d$ is necessary as (37) is not true for our favorite examples $(n,d) = (4,4), (6,3)$. For $d = 3$, the vector $p$ in (37) can be zero.\footnote{For this computations we have used the procedure SumThreeLinearCycle.}

The final ingredient of Conjecture [1] is the following. In virtue of Theorem [5] it compares the Zariski tangent spaces of components of the Hodge locus passing through the Fermat point.

**Conjecture 11.** Let $n \geq 6$ and $d \geq 3$. There is no inclusion between any two vector spaces of the form

$$\ker([p_{i+1} + r\tilde{p}_{i+1}])$$

where $p$ and $\tilde{p}$ ranges in the set of all $p^{a,b}$ with $m_{a,b,\hat{a},\hat{b}} = \frac{n}{2} - 1, \frac{n}{2} - 2$, $r, \tilde{r} \in \mathbb{Z}$ coprime and $m_{a,b,\hat{a},\hat{b}} = \frac{n}{2}, r = 1, \tilde{r} = 0$.\footnote{For this proof we have used DistinctHodgeLocus.}

Let $\delta, \tilde{\delta} \in H_n(X^d_n, \mathbb{Q})$ be two Hodge cycles with

$$\ker([p_{i+1} + r\hat{p}_{i+1}]) \subset \ker([p_{i+1} + r\tilde{p}_{i+1}])$$

that is, the Zariski tangent space of $V_\delta$ is contained in the Zariski tangent space of $V_{\tilde{\delta}}$. The first trivial example to this situation is when $\delta$ is a rational multiple of $[Z_{\infty}]$ for which we have $p(\delta) = 0$ and $V_\delta = (T, 0)$. Let us assume that none of $\delta$ and $\tilde{\delta}$ is a rational multiple of $[Z_{\infty}]$. Next examples for this situation are in Theorem [1]. In this theorem the Zariski tangent space of the Hodge locus $V_{r[p_{\frac{n}{2}} + r\tilde{p}_{\frac{n}{2}}]} = V_{\frac{n}{2}} \cap V_{\frac{n}{2}^2}$, with $F_{\frac{n}{2}} \cap F_{\frac{n}{2}^2} = P^m$ and $r, \tilde{r} \in \mathbb{Z}$, $r \neq 0$, $\tilde{r} \neq 0$, at the Fermat point does not depend on $r, \tilde{r}$. For larger $m$'s such as $\frac{n}{2} - 1$, the Zariski tangent spaces of $V_{r[p_{\frac{n}{2}} + r\tilde{p}_{\frac{n}{2}}]}$ at the Fermat point form a pencil of linear spaces and so there is no inclusion among its members. For $(n, d) = (2, 4), (4, 3)$, $V_{\delta}$'s are of codimension one, smooth and reduced, and so, any inclusion (39) will be an equality and it implies that the period vectors of $\delta, \tilde{\delta}$ are the same. This implies that $\delta = a\delta + b[Z_{\infty}]$ for some $a, b \in \mathbb{Q}$, and so, $V_\delta = V_{\delta}$.\footnote{For this proof we have used DistinctHodgeLocus.}

We can verify Conjecture [11] in the following way. For simplicity we restrict ourselves to the pairs $(n, d)$ in Theorem [1] and $r = \tilde{r} = 1$. Let us take two matrices $A$ and $B$ as inside kernel in (38). Let also $A * B$ be the concatenation of $A$ and $B$ by putting the rows of $A$ and $B$ as the rows of $A * B$. Therefore, $A * B$ is a $(2\#I_{\frac{n}{2} - n} - 2) \times (\#I_{d})$ matrix. In order to prove that there is no inclusion between $\ker(A)$ and $\ker(B)$ it is enough to prove that

$$\text{rank}(A * B) > \text{rank}(A), \text{rank}(B).$$

The number of verifications (40) is approximately $N^4$, where $N$ is the number of linear cycles given in (20). This is a huge number even for small values of $n$ and $d$. Note that the vector space in (38) for $(n, d, m)$'s in Theorem [1] is equal to the Zariski tangent space of $V_{\frac{n}{2}} \cap V_{\frac{n}{2}^2}$ at the Fermat point, and hence it does not depend on $r$ and $\tilde{r}$. This is the main reason why we restrict ourselves to the cases in Conjecture [11].

### 10 Semi-irreducible algebraic cycles

Let $X$ be a smooth projective variety and $Z = \sum_{i=1}^{r} n_i Z_i, \quad n_i \in \mathbb{Z}$ be an algebraic cycle in $X$, with $Z_i$ an irreducible subvariety of codimension $\frac{n}{2}$ in $X$. The following definition is done using analytic deformations and it would not be hard to state it in the algebraic context.
Definition 3. We say that \( Z = \sum_{i=1}^{r} n_{i}Z_{i}, \quad n_{i} \in \mathbb{Z} \) is semi-irreducible if there is a smooth analytic variety \( X \), an irreducible subvariety \( Z \subset X \) of codimension \( \frac{n}{2} \) (possibly singular), a holomorphic map \( f : X \to (\mathbb{C}, 0) \) such that

1. \( f \) is smooth and proper over \((\mathbb{C}, 0)\) with \( X \) as a fiber over \( 0 \). Therefore, all the fibers \( X_{t} \) of \( f \) are \( C^{\infty} \) isomorphic to \( X \).

2. The fiber \( Z_{t} \) of \( f \) over \( t \neq 0 \) is irreducible and \( Z_{0} = \bigcup_{i=1}^{r} Z_{i} \).

3. The homological cycle \( [Z] := \sum_{i=1}^{r} n_{i}[Z_{i}] \in H_{\ast}(X, \mathbb{Z}) \) is the monodromy of \([Z_{t}] \in H_{\ast}(X_{t}, \mathbb{Z})\).

It is reasonable to expect that Item 3 is equivalent to a geometric phenomena, purely expressible in terms of degeneration of algebraic varieties. For instance, one might expect that \( n_{i} \) layers of the algebraic cycle \( Z_{i} \) accumulate on \( Z_{t} \), and hence semi-irreducibility implies the positivity of \( n_{i} \)'s. Moreover, for distinct \( Z_{i} \) and \( Z_{j} \), the intersection \( Z_{i} \cap Z_{j} \) is of codimension one in both \( Z_{i} \) and \( Z_{j} \), because \( Z_{i} \)'s are irreducible and of codimension one in \( Z \). In particular, the algebraic cycle \( r\mathbb{P}^{\frac{n}{2}} + r\mathbb{P}^{\frac{n}{2}} \) with \( \mathbb{P}^{\frac{n}{2}} \cap \mathbb{P}^{\frac{n}{2}} = \mathbb{P}^{m}, \quad m \leq \frac{n}{2} - 2 \) in Conjecture 1 is not semi-irreducible.

A smooth hypersurface of degree \( d \) and dimension \( n \) has the Hodge numbers \( h^{0,0} = h^{n-1,1} = \ldots = h^{\frac{n}{2}+2, \frac{n}{2}-2} = 0, \quad h^{\frac{n}{2}+1, \frac{n}{2}-1} = 1 \) if and only if \((n, d) = (2, 4), (4, 3)\). Recall that \( Z_{\infty} \) is the intersection of a linear \( \mathbb{P}^{\frac{n}{2}+1} \) with \( X_{n}^{d} \). The following theorem can be considered as a counterpart of Conjecture 1.

Theorem 15. Let \((n, d) = (2, 4), (4, 3)\) and let \( Z \) be an algebraic cycle of dimension \( \frac{n}{2} \) and with integer coefficients, in a smooth hypersurface of dimension \( n \) and degree \( d \). If \([Z] \in H_{n}(X_{n}^{d}, \mathbb{Q})\) is not a rational multiple of \([Z_{\infty}]\) then there is a semi-irreducible algebraic cycle \( \tilde{Z} \) of dimension \( \frac{n}{2} \) in \( X_{n}^{d} \) such that \( a\tilde{Z} + bZ + cZ_{\infty} \) is homologous to zero for some \( a, b, c \in \mathbb{Z} \) with \( a, b \neq 0 \).

Proof. The algebraic cycle \( Z \) induces a homology class \( \delta_{0} = [Z] \in H_{n}(X_{n}^{d}, \mathbb{Z}) \) and the Hodge locus \( V_{\delta_{0}} \) is given by the zero locus of a single integral \( f(t) := \int_{t} \omega_{0} \), where \( \omega_{0} \) is given by \( [8] \) for \( i = (0, 0, \ldots, 0) \). By our hypothesis on \( Z, f \) is not identically zero and since \( \delta_{0} = [Z] \) it vanishes at \( t = 0 \). We show that \( V_{\delta_{0}} \) is smooth and reduced, and for this it is enough to show that the linear part of \( f \) is not identically zero. This follows from \( \nabla \omega_{0} = \omega_{i}, \quad i \in I_{d}, \quad d = (\frac{n}{2}+1)d_{n}-d-2 \) and the fact that \( \omega_{0}, \omega_{i}, \quad i \in I_{d} \) form a basis of \( F^{1} H_{n, d}(X) \). Here, \( \nabla \) is the Gauss-Manin connection of the family of hypersurfaces given by \([29]\). The Hodge conjecture in both cases is well-known. In the first case it is the Lefschetz (1, 1) theorem and in the second case it is a result of Zucker \([Zuc77]\). This implies that \( \delta_{i} = [Z_{i}] \), where \( Z_{i} := \sum_{i=1}^{r} n_{i}Z_{i,t}, \quad Z_{i,t} \subset X_{t}, \quad t \in V_{\delta_{0}} \), \( \dim(Z_{i,t}) = \frac{n}{2}, \quad n_{i} \in \mathbb{Q} \) and for generic \( t, \quad Z_{i,t} \) is irreducible. Since \( V_{\delta_{0}} \subset V_{[Z_{i,0}]} \), we conclude that \([Z_{i,0}] = a_{i}[Z] + b_{i}[Z_{\infty}]\) for some \( a_{i}, b_{i} \in \mathbb{Q} \). By our hypothesis on \( Z \), one of \( a_{i}'s \) is not zero, let us call it \( a_{1} \). We get \([Z] = a_{1}^{-1}[Z_{1,0}] - b_{1}a_{1}^{-1}[Z_{\infty}]\). \( \square \)

In Theorem 15 let us assume that \( Z \) is a sum of linear cycles. It would be useful to see whether the algebraic cycle \( \tilde{Z} \) is a sum of linear cycles. One might start with the sum of two lines in the Fermat surface \( X_{2}^{4} \) without any common points (the case \((n, d, m) = (2, 4, -1)\)).

11 How to to deal with Conjecture 1?

In this section we sketch a strategy to prove Conjecture 1 which follows the same guideline as of the proof of Theorem 15. Let \( \delta_{0} := r[p^{\frac{n}{2}}] + r[p^{\frac{n}{2}}] \in H_{n}(X_{n}^{d}, \mathbb{Z}) \) with \( \mathbb{P}^{\frac{n}{2}} \cap \mathbb{P}^{\frac{n}{2}} = \mathbb{P}^{m}, \quad m = \frac{n}{2} - 2 \) and \( H^{d}_{n}(m) < K^{d}_{n}(m) \). Let also \( \delta_{i} \in H_{n}(X_{i}, \mathbb{Z}), \quad t \in (T, 0) \) be its monodromy to nearby fibers. Conjecture 8 implies that the intersection of \( V_{p^{\frac{n}{2}}} \) and \( \mathbb{P}^{\frac{n}{2}} \) is a proper subset of the underlying
analytic variety of $V_{\delta_0}$. If the Hodge conjecture is true then there is an algebraic family of algebraic cycles

\begin{equation}
Z_t := \sum_{k=1}^{r} n_k Z_{k,t}, \quad Z_{k,t} \subset X_t, \quad t \in V_{\delta_0},
\end{equation}

\begin{equation}
\text{dim}(Z_{k,t}) = \frac{n}{2}, \quad n_k \in \mathbb{Z},
\end{equation}

such that $Z_{k,t}$ is irreducible for generic $t$ and $Z_t$ is homologous to a non-zero integral multiple of $\delta_t$, see Figure 2. By Conjecture 8 we know that $V_{\delta_0}$ is smooth and reduced, and so, we have the inclusion of analytic schemes

\begin{equation}
V_{\delta_0} \subset V[Z_{k,0}], \quad k = 1, 2, \ldots, r
\end{equation}

which implies that

\begin{equation}
\ker[p_{i+j}(\delta_0)] \subset \ker[p_{i+j}(Z_{k,0})], \quad \text{and so rank}([p_{i+j}(Z_{k,0})]) \leq \text{rank}[p_{i+j}(\delta_0)].
\end{equation}

In order to proceed, we consider the cases of Fermat varieties such that linear cycles generates the the space of Hodge cycles over rational numbers (these are the cases in Theorem 9), or we assume Conjecture 10 for $\text{Ho}_n^d$ being the the lattice of periods of all Hodge cycles and not just linear cycles. We apply Conjecture 10 and we conclude that for some linear cycles $\mathbb{P}_k^n, \tilde{\mathbb{P}}_k^n$ in $X_n^d$ with $\mathbb{P}_k^n \cap \tilde{\mathbb{P}}_k^n = \mathbb{P}^{m_k}$, $m_k \geq \frac{n}{2} - 2$ and $r_k, \tilde{r}_k, b_k, c_k \in \mathbb{Z}$, $c_k \neq 0$ we have

\begin{equation}
r_k \mathbb{P}_k^n + \tilde{r}_k \tilde{\mathbb{P}}_k^n + b_k Z_{\infty} + c_k Z_{k,0} \sim 0,
\end{equation}

where $\sim$ means homologous. The inclusion in (43) and (44) imply

\begin{equation}
\ker[p_{i+j}(r \mathbb{P}_k^n + \tilde{r} \tilde{\mathbb{P}}_k^n)] \subset \ker[p_{i+j}(r_k \mathbb{P}_k^n + \tilde{r}_k \tilde{\mathbb{P}}_k^n)],
\end{equation}
Now, Conjecture 11 and the fact that \( r \) and \( \check{r} \) are coprime imply that for some non-zero integer \( a \) we have \( r k \mathbb{P}^n_k + \check{r} k \mathbb{P}^n_k = a(r \mathbb{P}^n \check{P} + \check{r} \mathbb{P} \check{P}) \), as an equality of algebraic cycles, and hence \( \{ \mathbb{P}^n, \check{P} \} = \{ \mathbb{P}^n_k, \check{P}^n_k \} \). This means that in (41) we can assume that \( Z_t \) is irreducible for generic \( t \) and so we get

\[
(46) \quad a(r \mathbb{P}^n + \check{r} \mathbb{P} \check{P}) + b Z_\infty + c Z \sim 0, \quad a, c \neq 0, a, b, c \in \mathbb{Z},
\]

where \( Z = Z_0 \). Taking the intersection of (46) with any third linear cycle \( \check{P} \mathbb{P} \) with \( \check{P} \mathbb{P} \cdot \mathbb{P} \mathbb{P} = \check{P} \mathbb{P} \cdot \mathbb{P} \mathbb{P} = 0 \) we get \( c \mid b \). Moreover, taking the intersection of (46) with any third linear cycle \( \check{P} \mathbb{P} \) with \( \check{P} \mathbb{P} \cdot \mathbb{P} \mathbb{P} = 1 \) and \( \check{P} \mathbb{P} \cdot \check{P} \check{P} = 0 \) we get \( c \mid (a r + b) \). In a similar way, we have \( c \mid (a \check{r} + b) \). Since \( r \) and \( \check{r} \) are coprime we conclude that \( c \mid a \) and, therefore, in (46) we can assume that \( c = -1 \).

One of the most important information about the algebraic cycle \( Z \subset X^n_d \) is the data of its intersection numbers with other algebraic cycles of the Fermat variety, and in particular all linear cycles. Recall that \( Z_\infty \cdot Z_\infty = d \), for a linear cycle \( \mathbb{P}^n \subset X^n_d \) we have \( Z_\infty \cdot \mathbb{P} \mathbb{P} = 1 \), and for two linear cycles \( \mathbb{P} \mathbb{P} \) and \( \check{P} \mathbb{P} \) with \( \mathbb{P} \mathbb{P} \cap \check{P} \mathbb{P} = \check{P} \mathbb{P} \) we have

\[
(47) \quad \mathbb{P} \mathbb{P} \cdot \check{P} \mathbb{P} = \frac{1 - (-d + 1)^{m+1}}{d}
\]

This follows from the adjunction formula, see for instance [Mov17a], Section 17.6. Using this we know \( a \) and \( b \):

\[
(48) \quad b = Z \cdot \check{P} \mathbb{P}, \quad \text{where} \quad \check{P} \mathbb{P} \cdot \mathbb{P} \mathbb{P} = \check{P} \mathbb{P} \cdot \check{P} \mathbb{P} = 0,
\]

and

\[
\deg(Z) = a \cdot (r + \check{r}) + b \cdot d.
\]

In particular, if \( (r, \check{r}) = (1, -1) \) then the degree \( d \) of the Fermat variety divides the degree of \( Z \). Another important information about the algebraic cycle \( Z \) is a lower bound of the dimension of the Hilbert scheme parameterizing deformations of the pair \( (X^n_d, Z) \). One may look for the classification of the components of the Hilbert schemes of projective varieties in order to see whether such a \( Z \) exists or not. For instance, we know that if \( Z \subset \mathbb{P}^{n+1} \) is an irreducible reduced projective variety of dimension \( \frac{n}{2} \) and degree \( 2 \) then it is necessarily a complete intersection of type \( 1^2, 2 \), see [EH87]. One might look for generalizations of this kind of results.

12 Final comments

One of the main difficulties in generalizing our main theorems in Introduction for other cases is that the moduli of hypersurfaces of dimension \( n \) and degree \( d \) is of dimension \( \#I_d = \binom{d+n+1}{n+1} - \binom{n+2}{2} \) which is two big even for small values of \( n \) and \( d \). One has to prepare similar tables as in Table 1 with smaller number of parameters and then start to analyze \( N \)-smoothness. For some suggestions see [Mov17a] Exercises 15.13, 15.16, 15.17. The author has analyzed statements similar to Theorem 2 and Theorem 3 for hypersurfaces given by homogeneous polynomials of the form

\[
(49) \quad f := A(x_0, x_2, \ldots, x_n) + B(x_1, x_3, \ldots, x_{n+1}).
\]

The moduli of such hypersurfaces is of dimension \( 2 \cdot \binom{d+2}{2} - 2(\frac{n}{2} + 1)^2 \) and this makes the computations much faster. Here are some sample results mainly in direction of Theorem 3. The Hodge locus \( V_{r \mathbb{P}^n \mathbb{P} + \check{r} \mathbb{P} \check{P}} \) for \( r, \check{r} \) coprime non-zero integers and \( |r|, |\check{r}| \leq 10 \) is 7-smooth and 4-smooth for \( (n, d, m) = (6, 3, 1) \) and \( (4, 4, 0) \), respectively. Therefore, it seems that we are in situations similar to Theorem 1. For \( (n, d, m) = (8, 3, 2), (10, 3, 3) \) the situation is similar to
Theorem 2 and Theorem 3. Such a Hodge locus is not 3-smooth except for $(r, \tilde{r}) = (1, \pm 1)$ for which we have even 4-smoothness in the case $(8, 3, 2)$. The coefficients of the Taylor series in Theorem 14 seem to be defined in a reasonable ring, for instance, for $(n, d) = (4, 3), (6, 3)$ and some sample truncated Taylor series, the ring of coefficients is $\mathbb{Z}[\zeta_d]$. If so, one may consider them modulo prime ideals, and in this way, study many related conjectures. The tools introduced in this article can be used in order to answer the following question which produces an explicit counterexample to a conjecture of J. Harris: determine the integer $d$ (conjecturally less than 10) such that the Noether-Lefschetz locus of surfaces of degree $d$ (resp. degree $< d$) has infinite (resp. finite) number of special components crossing the Fermat point. Notice that Voisin’s counterexample in [Voi91] is for a very big $d$. This problem will be studied in subsequent articles. For this and its generalization to higher dimensions one needs to classify linear combination of linear cycles in the Fermat variety which are semi-irreducible. The combinatorics of arrangement of linear cycles seems to play some role in this question. The author’s favorite examples in this article have been cubic varieties, see Manin’s book [Man86] for an overview of some results and techniques. Cubic surfaces carry the famous 27-lines which is exactly the number $20$ of linear cycles for the Fermat cubic surface. Hodge conjecture is known for cubic fourfolds (see [Zuc77]), and for a restricted class of cubic 8-folds the Hodge conjecture is also known (see [Ter90]). In general the Hodge conjecture remains open for cubic hypersurfaces of dimension $n \geq 6$. Conjecture 4 makes sense starting from cubic tenfolds whose moduli is 220-dimensional. It might be useful to review all the results in this case and to see what one can say more about the algebraic cycle $Z$ in this conjecture.

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