Noncommutative Classical Dynamics on Velocity Phase Space and Souriau Formalism

José F. Cariñena*, Héctor Figueroa† and Partha Guha‡ §

* Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain
† Departamento de Matemáticas, Universidad de Costa Rica, 2060 San Pedro, Costa Rica
‡ Institut des Hautes Études Scientifiques Le Bois-Marie 35, route de Chartres 91440 Bures-sur-Yvette France
§ S.N. Bose National Centre for Basic Sciences JD Block, Sector-3, Salt Lake Calcutta-700098, India

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Abstract

We consider Feynman-Dyson’s proof of Maxwell’s equations using the Jacobi identities on the velocity phase space. In this paper we generalize the Feynman-Dyson’s scheme by incorporating the non-commutativity between various spatial coordinates along with the velocity coordinates. This allows us to study a generalized class of Hamiltonian systems. We explore various dynamical flows associated to the Souriau form associated to this generalized Feynman-Dyson’s scheme. Moreover, using the Souriau form we show that these new classes of generalized systems are volume preserving mechanical systems.

Keywords: Feynman problem, Souriau form, noncommutativity, generalized Hamiltonian dynamics.

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1 Introduction

The study of exotic particle models with non-commutative position coordinates was started in the last decade. There are several physical phenomena appearing in condensed matter physics, namely
semiclassical Bloch electron phenomena, fractional quantum Hall effect, double special relativity models, etc., that exhibit such feature. All these models share the somewhat unusual feature that the Poisson brackets of the planar coordinates do not vanish. This class of dynamical structures has appeared in geometric mechanics and geometric control theory [1, 2, 3, 4]. In her thesis, Sánchez de Álvaro [5] indicates a characterization of the Poisson structure in terms of Poisson brackets of particular functions on the tangent bundle $TP$ of a Poisson manifold $P$, and discusses its functorial properties.

A very noble derivation of a pair of Maxwell equations was originally proposed by Feynman, but the exact details of his argument came to the scientific community from the work of Freeman Dyson [6]. According to Dyson, Feynman showed him the construction and examples of the Lorentz force law and the homogeneous Maxwell equations in 1948. A derivation of a pair of Maxwell equations and the Lorentz force is based on the commutation relations between positions and velocities for a single non-relativistic particle. In general the locality property that different coordinates commute is assumed. Due to increasing interest in non-commutative field theories, it is worthwhile to consider the non-commutative analogue of Feynman approach. This destroys the axiom of locality, which, according to Dyson, was the original aim of Feynman. Tanimura [7] gave both a special relativistic and a general relativistic versions of Feynman’s derivation. Land et al. [8] examined Tanimura’s derivation in the framework of the proper time method in relativistic mechanics and showed that Tanimura’s result then corresponds to the five-dimensional electromagnetic theory previously derived from a Stueckelberg-type quantum theory in which one gauges the invariant parameter in the proper time method. An extension of Tanimura’s method has been achieved [9] by using the Hodge duality to derive the two groups of Maxwell’s equations with a magnetic monopole in flat and in curved spaces. A rigorous mathematical description of Feynman’s derivation connected to the inverse problem for Poisson geometry has been formulated in [10] (see also [11]). Hughes [12] considered Feynman’s derivation in the framework of the Helmholtz inverse problem for the calculus of variations (see also [13] and [14]).

In fact, it was pointed out by Jackiw that Heisenberg suggested in a letter to Peierls that spatial coordinates may not commute, Peierls communicated the same idea to Pauli, who informed it to Openheimer; eventually the idea arrived to Snyder [15, 16] who wrote the first paper on the subject. Nowadays the physics in non-commutative planes is relevant not only in string theory but also in condensed matters physics [17]. In the context of the Feynman’s derivation of electrodynamics, it has been shown that non-commutativity allows other particle dynamics than the standard formalism of electrodynamics [18]. Noncommutative quantum mechanics is recently the subject of a wide range of works from particle physics to condensed matter physics. This has also been studied from the point of view of Feynman’s formalism in [19].

The examples of exotic mechanics started to appear around 1995. Physicists obtained various models such that the Poisson brackets of the planar coordinates do not vanish. Souriau’s orbit method [20, 21, 22] was used to construct a classical mechanics associated with Lévy-Leblond’s exotic Galilean symmetry. In terms of the Souriau 2-form a wide set of Hamiltonian dynamical systems have been described in [24, 25, 26, 27]. Lévy-Leblond [28] has realized that due to the commutativity of the rotation group $O(2, \mathbb{R})$, the Lie algebra of the Galilei group in the plane admits a second exotic extension defined by

$$[K_1, K_2] = i \kappa I,$$
where $\kappa$ is the new extension parameter. For a free particle the usual equations of motions are unchanged and $\kappa$ only contributes to the conserved quantities. It yields the non-commutativity of the position coordinates.

Feynman procedure to obtain Maxwell’s equation in electrodynamics has been reviewed under different kind of settings, and several nontrivial and interesting generalizations are possible, see for instance [29, 30, 31, 32, 33, 34, 35, 36]. Recently, Duval and Horváthy [29] successfully applied the techniques of Souriau’s orbit method [37, 38, 39] to various models. Incidentally, one of these models can be viewed as the non-relativistic counterpart of the relativistic anyon considered before by Jackiw and Nair [32]. Mathematically, the ‘exotic’ model arises due to the particular properties of the plane. A wide set of dynamical systems can be derived from the Lagrange-Souriau 2-form approach in three dimensions and the generalizations to higher number of degrees of freedom have been outlined in [26].

Wong’s equations describe the interaction between the Yang-Mills field and an isotopic-spin carrying particle in the classical limit. Feynman-Dyson’s proof offers a way to check the consistency of these equations [40]. See also [41] for a very recent paper.

In a slightly different context Kauffman [42] introduced discrete physics based on a non-commutative calculus of finite differences. This gives a context for the Feynman–Dyson derivation of non-commutative electromagnetism. More recently, Kauffman [43] found an interesting way to describe mechanics in a curved background interacting with gauge fields in such a way that the physical equations of motion emerge automatically from underlying algebraic relations in a non-commutative geometry and this construction depends largely on the Feynman-Dyson construction. In an interesting paper, Cortese and García [44] studied a variational principle for noncommutative dynamical systems in the configuration space. In particular they showed that the non-commutative consistency conditions (NCCC), that come from the analysis of the dynamical compatibility, are not the Helmholtz conditions of the generalized inverse problem of the calculus of variations. It has been shown that the $\theta$-deformed Helmholtz conditions are connected to a third-order time derivative system of differential equations. Noncommutative phase spaces have been introduced by minimal couplings in [45] and then some of them are realized as coadjoint orbits of the anisotropic Newton-Hooke groups in two- and three-dimensional spaces. This has been further generalized to realize noncommutative phase spaces as coadjoint orbit extensions of the Aristotle group in a two dimensional space [46].

In this article we apply Souriau’s orbit method to study exotic mechanics on the tangent bundle or velocity phase space. Souriau first unified both the symplectic structure and the Hamiltonian into a single two-form. It has an exotic symplectic form and a free Hamiltonian and yields a generalized Hamiltonian mechanics. Duval and Horváthy used Souriau’s orbit method to construct a classical planar system associated with Lévy-Leblond’s two-fold extended Galilean symmetry. The four dimensional phase space is endowed with the following exotic form

$$\Omega = dp_i \wedge dq_i + \frac{\theta}{2} \epsilon_{ij} dp_i \wedge dp_j,$$

where summation on repeated indices is understood. The exotic term in the symplectic form only exists in the plane. Following [47] [48] we also explore a volume-preserving flow on a symplectic manifold from the Souriau form associated with the velocity phase space.

Many authors [29, 30, 49] have generalized this modification of the symplectic form by intro-
ducing the so-called dual magnetic field such that
\[ \Omega = dp_i \wedge dq_i + \frac{1}{2} g_{ij} dp_i \wedge dp_j + \frac{1}{2} f_{ij} dq_i \wedge dq_j. \]

The coefficients \( g_{ij} \) and \( f_{ij} \) are responsible for the noncommutativity of momenta and positions, respectively. The classical dynamics in noncommutative space leads to noncommutative Newton’s second law \[50, 51\]. This generalization can be studied in various types of noncommutative spacetimes; for instance Harikumar and Kapur studied in \[52\] the modification to Newton’s second law due to the kappa-deformation. In a very recent paper the modification of integrable models in the kappa-deformed scheme is analyzed \[53\] and kappa-Minkowski space-time through exotic oscillator is studied in \[54\]. Zhang et al. \[55, 56\] studied the 3D mechanics with non-commutativity, where the potential may also depend on the momentum. They obtained the conserved quantities by using van Holten’s covariant framework. It is known that the Snyder model has the remarkable property of leaving the Lorentz invariance intact. Recently, motivated from loop quantum gravity an idea has been proposed to extend the Snyder model \[15\] to space-times of constant curvature, by introducing a new fundamental constant whose inverse is proportional to the inverse of the cosmological constant. More recently, classical dynamics on Snyder space-times has drawn a lot of attention to physicists \[57, 58, 59, 60\]. Moreover Snyder dynamics in curved space-time has been extended by Mignemi et al. in \[61, 62\]. See also \[63\].

The main theme of our paper is to show that non-commutativity between coordinates allows us to construct various other generalized classes of dynamical systems. Tools of non-commutative geometry often appear in quantum gravity. Using a differential geometric theory on non-commutative space-time Aschieri et al. \[64\] defined \( \theta \)-deformed Einstein-Hilbert action, and by means of their technique of the deformation of the algebra of diffeomorphisms one can derive \( \ast \)-deformed integrable systems \[65\] and Newtonian mechanics \[66\]. Today we find non-commutativity in various fields of modern physics such as, graphene, Hydrogen atom spectrum, etc. \[67, 68, 69\].

This paper is organized as follows: in order to the paper be self-contained, Section 2 is devoted to a review of multivector fields, Poisson bivector, Schouten–Nijenhuis bracket and various other geometrical tools. We give a brief geometrical description of Poisson manifolds in Section 3. Section 4 is devoted to Souriau’s formalism of generalized symplectic forms. We illustrate Souriau’s construction through examples. Section 5 is focused on the construction of Feynman-Dyson’s scheme and its connection to Souriau’s method. Section 6 relates volume preserving mechanical systems and Souriau’s form. We finish our paper with an outlook in Section 7.

## 2 Geometrical background

Let \( \mathcal{F}(M) \) be the algebra of \( C^\infty \)-class functions (the algebra of classical observables) on a manifold \( M \) (the classical state space). We denote by \( \Omega^p(M) \) the space of \( C^\infty \)-class differentiable \( p \)-forms, and by \( A^p(M) \) the space of \( C^\infty \)-class skew-symmetric contravariant tensor fields of order \( p \), often called \( p \)-vectors. By convention we set \( A^0(M) = \Omega^0(M) = \mathcal{F}(M) \) and \( A^p(M) = \Omega^p(M) = 0 \), when \( p < 0 \). Then,
\[
\Omega(M) = \bigoplus_{p \in \mathbb{Z}} \Omega^p(M) \quad \text{and} \quad A(M) = \bigoplus_{p \in \mathbb{Z}} A^p(M),
\]
are \( \mathbb{Z} \)-graded algebras under their exterior products; moreover both are anticommutative, so, for instance, if \( P \in A^p(M) \) and \( Q \in A^q(M) \)

\[
P \wedge Q = -(-1)^{pq} Q \wedge P.
\]

When \( \alpha \) is a 1-form and \( X \) is a vector field the \( C^\infty(M) \)-class function \( \langle \alpha, X \rangle \) given by

\[
\langle \alpha, X \rangle(x) := \langle \alpha(x), X(x) \rangle := \alpha(x)(X(x)), \quad \forall x \in M,
\]

defines a pairing between \( \Omega^1(M) \) and \( A^1(M) \). More generally when \( \eta \) in \( \Omega^q(M) \) and \( P \) in \( A^p(M) \)

are decomposable, so \( \eta = \alpha_1 \wedge \cdots \wedge \alpha_q \) and \( P = X_1 \wedge \cdots \wedge X_p \), for \( \alpha_i \) in \( \Omega^1(M) \) and \( X_j \) in \( A^1(M) \), we set

\[
\langle \eta, P \rangle := \langle \alpha_1 \wedge \cdots \wedge \alpha_q, X_1 \wedge \cdots \wedge X_p \rangle = \begin{cases} 
0 & \text{if } p \neq q, \\
\det \langle \alpha_i, X_j \rangle & \text{if } p = q.
\end{cases}
\]

Since the value of \( \langle \eta, P \rangle \) at a point only depends on the value of \( \eta \) and \( P \) at this point, we can extend by bilinearity, in a unique way, this pairing to arbitrary elements \( \eta \) in \( \Omega(M) \) and \( P \) in \( A(M) \). Furthermore, it is easy to check that if \( \eta \) is in \( \Omega^p(M) \), then

\[
\langle \eta, X_1 \wedge \cdots \wedge X_p \rangle = \eta(X_1, \ldots, X_p).
\]

If \( X \) is a vector field on \( M \), the inner product \( i(X) \) is a derivation of degree \(-1\) on the graded algebra \( \Omega(M) \) and since the exterior derivative \( d \) is a derivation on \( \Omega(M) \) of degree \(1\), it follows that the Lie derivative with respect to \( X \), given by Cartan’s formula:

\[
\mathcal{L}_X := [i(X), d] = i(X) \circ d + d \circ i(X),
\]

where \([\cdot, \cdot]\) means graded commutator, is a graded derivation of degree \(0\) on \( \Omega(M) \). \( \mathcal{L}_X \) can also be defined on \( A(M) \) as the unique derivation of degree \(0\) such that, for \( f \) in \( A^0(M) \) and \( Y \) in \( A^1(M) \),

\[
\mathcal{L}_X f = X(f) \quad \text{and} \quad \mathcal{L}_X Y = [X, Y],
\]

where \([X, Y]\) is the usual Lie bracket on vector fields. Furthermore the Schouten–Nijenhuis bracket is defined as a natural extension of the Lie derivative with respect to a vector field on \( A(M) \). More specifically, it is defined as the unique bilinear map \( [\cdot, \cdot]_{SN} : A(M) \times A(M) \to A(M) \) such that, for \( f \) and \( g \) in \( A^0(M) = \mathcal{F}(M) \), \( X \in A^1(M) \), \( P \in A^p(M) \), \( Q \in A^q(M) \) and \( R \in A^r(M) \),

\[
\begin{align*}
\text{a-)} & \quad [f, g]_{SN} = 0 \\
\text{b-)} & \quad [X, Q]_{SN} = \mathcal{L}_X Q \\
\text{c-)} & \quad [P, Q]_{SN} = -(-1)^{(p-1)(q-1)}[Q, P]_{SN} \\
\text{d-)} & \quad [P, Q \wedge R]_{SN} = [P, Q]_{SN} \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]_{SN}
\end{align*}
\]
From these properties it readily follows that $[P, Q]_{SN}$ belongs to $A^{p+q-1}(M)$, therefore the last property means that the endomorphism $d_P : A(M) \to A(M)$ given by

$$d_P Q := [P, Q]_{SN},$$

(1)

is a derivation of $A(M)$ of degree $p - 1$. A somewhat long, but otherwise easy, induction, based on the defining properties, gives

$$(-1)^{(p-1)(r-1)} [P, [Q, R]_{SN}]_{SN} + (-1)^{(q-1)(p-1)} [Q, [R, P]_{SN}]_{SN} + (-1)^{(r-1)(q-1)} [R, [P, Q]_{SN}]_{SN} = 0,$$

(2)

which is called graded Jacobi identity.

This, together with bilinearity, c-), and the fact that $[P, Q]_{SN}$ belongs to $A^{p+q-1}(M)$ means that $A(M)$, equipped with the Schouten–Nijenhuis bracket, is a graded Lie algebra when the degree of $P$ in $A^p(M)$ is declared to be $p - 1$, not $p$. So, for instance, vector fields would be the homogeneous elements of degree 0 under this new grading of $A(M)$. To perform computations with the Schouten–Nijenhuis bracket it is convenient to extend the definition of the interior product. If $\eta$ is in $\Omega(M)$, $f$ is a function and $X_1, \ldots, X_p$ are vector fields we set

$$i(f)\eta := f\eta \quad \text{and} \quad i(X_1 \wedge \cdots \wedge X_p)\eta := i(X_1) \circ \cdots \circ i(X_p)\eta.$$

In general, $i(P)$ is defined in such a way that the map $i(\cdot) : A(M) \to \mathcal{E}(\Omega(M))$, where $\mathcal{E}(\Omega(M))$ is the space of endomorphism of $\Omega(M)$, is $\mathcal{F}(M)$-linear. In particular, $i(P \wedge Q)\eta = i(P)(i(Q)\eta)$, for all $P$ and $Q$ in $A(M)$. Furthermore, when $\eta$ is a $p$-form

$$i(X_1 \wedge \cdots \wedge X_p)\eta = i(X_1) \circ \cdots \circ i(X_p)\eta = \eta(X_p, \ldots, X_1) = (-1)^{(p-1)p/2} \eta(X_1, \ldots, X_p),$$

therefore for any $P \in A^p(M)$

$$i(P)\eta = (-1)^{(p-1)p/2} \eta, P).$$

(3)

Unfortunately $i(P)$, in general, is not a derivation, which complicates computations. Nevertheless, another simple induction gives

$$i([P, Q]_{SN}) = [[i(P), d], i(Q)],$$

(4)

where the brackets on the right are the usual brackets on the algebra of endomorphisms of $A(M)$. Notice that when $P = X$ is a vector field this reduces to the well-known relation among interior products and Lie derivatives: $i(L_X Q) = [L_X, i(Q)]$. 


If $P$ is a $p$-vector and $\eta$ is a $(p - 1)$-form, then $i(P)\eta = 0$ and $i(P) \circ i(f)\eta = i(P)(f\eta) = 0$. These, together with \((3)\) and \((4)\) entail
\[
\langle \eta, [P, f]_{SN}\rangle = (-1)^{(p-2)(p-1)/2} i([P, f]_{SN})\eta \\
= (-1)^{(p-2)(p-1)/2} [[i(P), d], i(Q)]\eta \\
= (-1)^{(p-2)(p-1)/2} \left( i(P) \circ d \circ i(f)\eta - (-1)^p d \circ i(P) \circ i(f)\eta \right. \\
\left. - i(f) \circ i(P) \circ d\eta - (-1)^p i(f) \circ d \circ i(P)\eta \right) \\
= (-1)^{(p-2)(p-1)/2} \left( i(P) \circ d \circ i(f)\eta - i(f) \circ i(P) \circ d\eta \right) \\
= (-1)^{(p-2)(p-1)/2} i(P)(df \wedge \eta) \\
= (-1)^{(p-1)^2} \langle df \wedge \eta, P \rangle \\
= (-1)^{(p-1)(p-2)} \langle \eta \wedge df, P \rangle \\
= \langle \eta \wedge df, P \rangle.
\]
Repited use of this gives
\[
\langle df_1 \wedge \cdots \wedge df_p, P \rangle = \langle df_1 \wedge \cdots \wedge df_{p-1}, [P, f_p]_{SN} \rangle \\
= \cdots = \left[ \cdots \left( [P, f_p]_{SN}, f_{p-1} \right)_{SN} \right]_{SN} \\
\]
\[\text{(5)}\]

3 Poisson Manifolds

A Poisson structure on $M$ is a skew-symmetric $\mathbb{R}$-bilinear map $\{\cdot, \cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ satisfying the Jacobi identity:
\[
\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \quad \forall f, g, h \in \mathcal{F}(M),
\]
and such that the map $X_f = \{\cdot, f\}$ is a derivation of the associative and commutative algebra $\mathcal{F}(M)$, for each $f \in \mathcal{F}(M)$, or in other words, $X_f$ is a vector field, usually called a Hamiltonian vector field, and $f$ is said to be the Hamiltonian of $X_f$. This property characterizing derivations of the associative and commutative algebra $\mathcal{F}(M)$, $\{g_1, g_2, f\} = g_1\{g_2, f\} + g_2\{g_1, f\}$, called Leibniz’ rule, is very important and gives a compatibility condition of the associative and commutative algebra structure in $\mathcal{F}(M)$ with the Lie algebra given in $\mathcal{F}(M)$ by the Poisson bracket.

To construct Poisson structures let $\Lambda$ be an element of $A^2(M)$, if $f \in A^0(M)$ and $g \in A^0(M)$ are two functions, using \((5)\), we define a third function by
\[
\{f, g\} := \Lambda(df, dg) = -\Lambda(dg, df) = -\langle df \wedge df, \Lambda \rangle = -[[\Lambda, f]_{SN}, g]_{SN}.
\]
\[\text{(6)}\]
By construction $X_f := [\Lambda, f]_{SN}$ is a vector field, and the defining property \(b)\) entails
\[
X_f(g) = \mathcal{L}_X g = [X_f, g]_{SN} = -\{f, g\} = \{g, f\}.
\]
\[\text{(7)}\]
In particular
\[
\{g, \{h, f\}\} = \left[ [\Lambda, g]_{SN}, \left[ [\Lambda, h]_{SN}, f \right]_{SN} \right]_{SN} = [X_g, [X_h, f]_{SN}]_{SN} = \mathcal{L}_{X_g} \circ \mathcal{L}_{X_h} f.
\]

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satisfies the cocycle condition. In other words, for Poisson tensors, the derivation

\[ \{ f, \{ g, h \} \} = - \{ \{ g, f \}, h \} = - \mathcal{L}_{X_g} \circ \mathcal{L}_{X_h} f. \]

On the other hand, the graded Jacobi identity (2), the defining property for bivector fields \( \Lambda \), and (7) entail

\[
\{ f, \{ g, h \} \} = - \{ \{ g, f \}, h \} = \left[ [X_g, h]_{SN}, f \right]_{SN} - \left[ [h, X_g]_{SN}, f \right]_{SN} = - \left[ [X_g, X_h]_{SN}, f \right]_{SN} - \left[ [h, [X_g]_{SN}]_{SN}, f \right]_{SN} = - \left[ [h, [X_g]_{SN}]_{SN}, f \right]_{SN}.
\]

Altogether gives

\[
\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = \left( \mathcal{L}_{X_g} \circ \mathcal{L}_{X_h} - \mathcal{L}_{X_h} \circ \mathcal{L}_{X_g} - \mathcal{L}_{[X_g, X_h]} \right) f
= - \left[ [h, [X_g]_{SN}]_{SN}, f \right]_{SN}
= - \left[ [h, [X_g]_{SN}]_{SN}, f \right]_{SN}.
\]

Furthermore, from the graded Jacobi identity

\[
0 = [\Lambda, [\Lambda, g]_{SN}]_{SN} + [\Lambda, [\Lambda, g]_{SN}]_{SN} + [g, [\Lambda, \Lambda]_{SN}]_{SN}
= 2[\Lambda, [\Lambda, g]_{SN}]_{SN} + [g, [\Lambda, \Lambda]_{SN}]_{SN}
= 2[\Lambda, [g, [\Lambda, \Lambda]_{SN}]_{SN}, f]_{SN},
\]

therefore, by (5)

\[
\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = \frac{1}{2} \left[ [h, [g, [\Lambda, \Lambda]_{SN}]_{SN}]_{SN}, f \right]_{SN}
= \frac{1}{2} \left[ [h, [[\Lambda, \Lambda]_{SN}, g]_{SN}]_{SN}, f \right]_{SN}
= - \frac{1}{2} \left[ [[\Lambda, \Lambda]_{SN}, g]_{SN}, h \right]_{SN}, f \right]_{SN}
= - \frac{1}{2} (df \wedge dh \wedge dg, [\Lambda, \Lambda]_{SN}).
\]

It follows that the bracket defined via a bivector field \( \Lambda \) is a Poisson structure exactly when

\[ [\Lambda, \Lambda]_{SN} = 0, \]

and when this happens we say that \( \Lambda \) is a Poisson tensor. This elementary, but clever, computation was first performed by Lichnerowicz in [70], who also realized that, when \( \Lambda \) is a Poisson tensor and \( P \) is in \( A^p(M) \), the graded Jacobi identity implies

\[
0 = (-1)^{p-1} [\Lambda, [\Lambda, P]_{SN}]_{SN} - [\Lambda, [P, \Lambda]_{SN}]_{SN} + (-1)^{p-1} [P, [\Lambda, \Lambda]_{SN}]_{SN}
= 2(-1)^{p-1} \Lambda, [\Lambda, P]_{SN}]_{SN}.
\]

In other words, for Poisson tensors, the derivation \( d_\Lambda \), defined as in (1) by

\[ d_\Lambda P = [\Lambda, P]_{SN}, \]

satisfies the cocycle condition

\[ d_\Lambda \circ d_\Lambda = 0. \]
On the other hand, from (8) we see that, for Poisson tensors, \([X_f, \Lambda]_{SN} = 0\), this together with the graded Jacobi identity and (7) give
\[
[X_f, X_g] = [X_f, [\Lambda, g]_{SN}]_{SN} = -[\Lambda, [g, X_f]_{SN}]_{SN} + [g, [X_f, \Lambda]_{SN}]_{SN} = [\Lambda, [X_f, g]_{SN}]_{SN} = -[\Lambda, \{f, g\}] = X_{\{f, g\}}. \tag{10}
\]

The converse is also true: a Poisson structure determines a Poisson tensor. To see this let \(\xi_a\) denote a set of local coordinates on \(M\), then, using the summation index convention,
\[
X_f = X_f(\xi_a) \frac{\partial}{\partial \xi_a} = \{\xi_a, f\} \frac{\partial}{\partial \xi_a},
\]

hence
\[
\{f, g\} = X_g(f) = \{\xi_a, g\} \frac{\partial f}{\partial \xi_a}.
\]

Thus,
\[
\{\xi_a, g\} = -\{g, \xi_a\} = -\{\xi_b, \xi_a\} \frac{\partial g}{\partial \xi_b} = \{\xi_a, \xi_b\} \frac{\partial g}{\partial \xi_b}, \tag{11}
\]

and the local coordinate expression of the Poisson Bracket becomes
\[
\{f, g\} = \{\xi_a, \xi_b\} \frac{\partial g}{\partial \xi_b} \frac{\partial f}{\partial \xi_a}. \tag{12}
\]

Therefore to compute the Poisson bracket of any pair of functions is enough to know the fundamental Poisson brackets
\[
\Lambda_{ab} = \{\xi_a, \xi_b\}.
\]

Moreover, the value of \(\{f, g\}\) at a point \(m \in M\) does not depend neither on \(f\) nor on \(g\) but only on \(df\) and \(dg\), as explicitly shown in (12), hence from the Poisson structure we get a twice contravariant skew-symmetric tensor
\[
\Lambda(df, dg) := \{f, g\}.
\]

Indeed, the local coordinate expression of \(\Lambda\) is
\[
\Lambda = \Lambda_{ab} \frac{\partial}{\partial \xi_a} \wedge \frac{\partial}{\partial \xi_b},
\]

and if \(\tilde{\xi} = \phi(\xi)\) is another set of local coordinates on \(M\), then,
\[
\tilde{\Lambda}_{ab} = \{\tilde{\xi}_a, \tilde{\xi}_b\} = \{\phi_a, \phi_b\} = \{\xi_c, \xi_d\} \frac{\partial \phi_a}{\partial \xi_c} \frac{\partial \phi_b}{\partial \xi_d} = \Lambda_{cd} \frac{\partial \phi_a}{\partial \xi_c} \frac{\partial \phi_b}{\partial \xi_d},
\]

so the components of \(\Lambda\) do change like the local coordinates of a skew-symmetric twice contravariant tensor which, by (2), is a Poisson tensor. We are using the convention that in the local expression of the wedge product only summands whose subindex on the left hand side term is smaller than the subindex on the right hand side term do appear.
For any function $h \in \mathcal{F}(M)$ the integral curves of the dynamical vector field $X_h$ are precisely determined by the solutions of the system of differential equations

$$\frac{d\xi_a}{dt} = \{\xi_a, h\},$$

and the dynamical evolution of a function $f$ on $M$ is given by

$$\frac{df}{dt} = \{f, h\},$$

or in local coordinates

$$\frac{df}{dt} = \Lambda_{ab} \frac{\partial f}{\partial \xi_a} \frac{\partial h}{\partial \xi_b}.$$

Interesting examples of Poisson manifolds are symplectic manifolds. A symplectic form $\omega$ on $M$ determines a bundle map $\omega^\flat : TM \to T^*M$ over the identity, which gives rise to the corresponding linear map between their spaces of sections, defined by

$$(\omega^\flat(X))(Y) := \langle \omega^\flat(X), Y \rangle = \omega(X, Y).$$

Since $\omega$ is non-degenerate $\omega^\flat$ is actually an isomorphism; we denote $\omega^\sharp$ the inverse map and define a bivector $\Lambda$ by

$$\Lambda(\alpha, \beta) := \omega(\omega^\sharp(\alpha), \omega^\sharp(\beta)),$$

if $\alpha$ and $\beta$ are 1-forms. When $\alpha = df$ is exact the corresponding vector field is denoted by $X_f := \omega^\sharp(df)$, and we say $X_f$ is the vector field associated to $f$ with respect to $\omega$. It is actually defined by the equation $i(X_f)\omega = df$. Furthermore, let $\{\cdot, \cdot\}$ be the bracket associated to $\Lambda$ via (6). Then the vector field associated to $f$ with respect to $\omega$ is also the Hamiltonian vector field given in (7), explaining why we use the same notation. If $\xi_a, a = 1, \ldots, m$, denote local coordinates and $\partial/\partial \xi_1, \ldots, \partial/\partial \xi_m$, and $d\xi_1, \ldots, d\xi_m$ are, respectively, the local basis of $\Lambda^1(M)$ and $\Omega^1(M)$ associated to $\xi_a$, let $B = (B_{ab})$ be the matrix of the linear map $\omega^\flat$ relative to these bases, and $\omega = \omega_{ab} d\xi_a \wedge d\xi_b$ the local expression of the symplectic form, then

$$\omega_{ab} = \omega \left( \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b} \right) = \left( \omega^\flat \left( \frac{\partial}{\partial \xi_a} \right) \right) \left( \frac{\partial}{\partial \xi_b} \right) = \left( B_{ac} d\xi_c \right) \left( \frac{\partial}{\partial \xi_b} \right) = B_{ab}.$$

Thus, $B = (\omega_{ab})$, and the matrix of $\omega^\sharp$ associated to these bases is the inverse of $B$. On the other hand,

$$d\omega(X_f, X_g, X_h) = X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g)) - \omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g).$$

Nevertheless, $\omega(X_g, X_h) = \Lambda(dg, dh) = \{g, h\}$, so (7) entails

$$X_f(\omega(X_g, X_h)) = X_f(\{g, h\}) = -\{f, \{g, h\}\}.$$

Also, by (10), $[X_g, X_h] = [X_g, X_h]_{SN} = X_{-(g,h)}$, hence

$$\omega([X_g, X_h], X_f) = \omega(X_{-(g,h)}, X_f) = -\{g, h\}, f\} = \{f, \{g, h\}\}.$$
It follows that

\[ 0 = d\omega(X_f, X_g, X_h) = -2(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}), \]

so \(\Lambda\) is a Poisson tensor.

Reciprocally, from a Poisson tensor \(\Lambda\) we get a bundle map \(\Lambda^\sharp: T^*M \to TM\), defined by

\[ \langle \alpha, \Lambda^\sharp(\beta) \rangle := \Lambda(\alpha, \beta). \]

In general \(\Lambda^\sharp\) is not a bundle isomorphism. We say the Poisson structure is regular when that is the case, and then we denote the inverse map by \(\Lambda^\flat\). Thus we have an identification of \(T_xM\) and \(T^*_xM\), for each point \(x\) of \(M\) and, therefore, an identification of higher order contravariant and covariant tensors. In particular, we define a 2-form \(\omega\) by

\[ \omega(X, Y) := \Lambda(\Lambda^\flat(X), \Lambda^\flat(Y)). \]

Notice that, from this point of view, \(X_f = \Lambda^\sharp(df)\). Indeed, from (7)

\[ \langle dg, X_f \rangle = X_f(g) = \{g, f\} = \Lambda(dg, df) = \langle dg, \Lambda^\sharp(df) \rangle. \]

Therefore,

\[ \omega(X_f, X_g) = \Lambda(\Lambda^\flat(df), \Lambda^\flat(dg)) = \Lambda(df, dg). \]

In particular the brackets associated to \(\omega\) and \(\Lambda\) coincide. Since the Poisson bracket satisfies the Jacobi identity, (14) entails, \(d\omega(X_f, X_g, X_h) = 0\). Given that locally one can consider a basis consisting of Hamiltonian vector fields, we conclude that \(\omega\) is a closed 2-form, moreover by definition it is non-degenerate, hence \(\omega\), so defined, is indeed a symplectic form. Using local coordinates as above, we see that the matrix of \(\Lambda^\sharp\) with respect to the standard bases is \((\Lambda_{ab})\), where \(\Lambda_{ab}\frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}\) is the local expression of \(\Lambda\).

## 4 Souriau’s prescription

As far as we know, Souriau [37, 38, 39] was the first to realize that since a dynamical system has two pieces, the symplectic form and the Hamiltonian, the equations of motion can be described by different data, modifying one or the other component. Thus, a perturbed dynamical system can be described starting from the free case by modifying the Hamiltonian, as was classically done, or simply by changing the symplectic form (see also [71]). This idea of adding an extra term to the symplectic form was successfully exploited by Souriau in his study of the orbit method, and it is what is behind the exotic mechanics, and several other models where non-commutativity of the variables is employed. But before we tackle that, let us consider a more down to earth example.

The classical method to derive the Lorentz equations in a relativistically invariant way is to use the so called minimal coupling, which consists in making the substitution \(p \mapsto p - eA\) inside the free Hamiltonian, where \(A\) is the vector potential of the electromagnetic field and \(e\) is the electric charge. Thus, the starting point is the cotangent bundle \(T^*M\), of a manifold \(M\), endowed with its canonical symplectic form \(\omega_0 = d\theta_0\), where \(\theta_0\) is the canonical 1-form given, in local cotangent bundle coordinates \((q_i, p_i)\), induced from local coordinates \((q_i)\) on \(M\), by \(\theta_0 = p_i dq_i\), together with
a Hamiltonian $H: T^*M \to \mathbb{R}$, which one replaces by $H_A = H \circ \phi^{-1}_A$, where $\phi_A: T^*M \to T^*M$ is the bundle map over the identity given by $\phi_A(q, p) = (q, p + eA(q))$, and $A = A_i(q) dq_i$ is a basic 1-form on $T^*M$. The Hamiltonian vector field $X_{H_A}$ associated to this new Hamiltonian $H_A = (\phi^{-1}_A)^*H$, that leads to the equation of motion, is given by

$$i(X_{H_A})d\theta_0 = dH_A = (\phi^{-1}_A)^*(dH).$$

On the other hand, and with an abuse of notation, we denote $\sigma$ both the 1-form on $M$ defined by $\sigma = eA^i(q) dq_i$, as well as the basic 1-form on $T^*M$ obtained by pulling back $\sigma$ by the canonical projection $\pi: T^*M \to M$. Then,

$$\phi^*_A(d\theta_0) = \phi^*_A(dp_i \wedge dq_i) = d\phi^*_A(p_i) \wedge d\phi^*_A(q_i) = d(p_i + eA_i(q)) \wedge dq_i = d\theta_0 + d\sigma,$$

and since $\phi$ is a diffeomorphism,

$$i(\phi_{A*}(X_{H_A}))(d\theta_0 + d\sigma) = i(\phi_{A*}(X_{H_A}))(\phi^*_A d\theta_0) = \phi^*_A(i(X_{H_A})d\theta_0) = \phi^*_A((\phi^{-1}_A)^*(dH)) = dH.$$

In other words, by adding the extra term $d\sigma$ to the symplectic form, which, by the way, it is a basic 2-form (i.e. locally it is of the form $g_{ij}(q) dq_i \wedge dq_j$, so it does not involve the $p$’s), we see that the vector field $\phi_{A*}(X_{H_A})$ is the Hamiltonian vector field associated to the original Hamiltonian $H$ with respect to this new symplectic form, and we obtain the same equations of motion.

If $\omega = \omega_0 + \frac{1}{2}g_{ij}(q) dq_i \wedge dq_j$, where $g_{ij}(q, p)$ is a skew-symmetric matrix, and the Hamiltonian vector field $X_H$ is $X_H = V_i \partial_\xi_i + W_i \partial_p$, then

$$i(X_H)\omega = (i(X_H) dq_i) \wedge dp_i - dq_i \wedge (i(X_H) dp_i) + \frac{1}{2}g_{ij}(i(X_H) dq_i) \wedge dq_j$$

$$= -W_i \partial_\xi_i + V_i \partial_p + g_{ij} V_j \partial_\xi_i$$

The equation $i(X_H)\omega = dH$, entails

$$V_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad W_i = -\frac{\partial H}{\partial q_i} + g_{ki} \frac{\partial H}{\partial p_k},$$

therefore by (7) the Poisson bracket associated to $\omega$ is given by

$$\{F, H\} = \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} + g_{ij} \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial p_j} = \{F, H\}_0 + g_{ij} \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial p_j} \equiv \mathcal{X}_H(F),$$

where $\{\cdot, \cdot\}_0$ stands for the Poisson bracket corresponding to $\omega_0$. It follows that the generalized (Hamiltonian) vector field is

$$\mathcal{X}_H = X_H + g_{ij} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial p_j}.$$
Our construction can be extended easily to a more general Souriau form
\[ \omega = \omega_0 + \frac{1}{2} g_{ij} \, dq_i \wedge dq_j + \frac{1}{2} f_{ij} \, dp_i \wedge dp_j, \]
and the equations of motion are then given by
\[ \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} + f_{ki} \frac{\partial H}{\partial q_i}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} + g_{ik} \frac{p_i}{m}. \]
Then if we assume the Hamiltonian is of the form
\[ H(q, p) = \frac{\delta_{ij} p_i p_j}{2m} + V(q), \]
with the potential energy \( V \) depending only on the configuration coordinates \( q_i \), the equations of motion are
\[ \frac{dq_k}{dt} = \frac{p_k}{m} + f_{ki} \frac{\partial V}{\partial q_i}, \quad \frac{dp_k}{dt} = -\frac{\partial V}{\partial q_k} + g_{ik} \frac{p_i}{m}. \]
These are equivalent to the modified Newton’s second law [30, 45, 50, 51]
\[ m \frac{d^2 q_k}{dt^2} = -\frac{\partial V}{\partial q_k} + g_{ik} \frac{p_i}{m} + m \frac{d}{dt} \left( f_{ki} \frac{\partial V}{\partial q_i} \right). \] (15)
The second term of equation (15) is a correction due to the noncommutativity of momenta and the third term is that of noncommutativity of coordinates. It follows that even for the case \( V = 0 \) the particle accelerates because of the noncommutativity of momenta.

The second procedure has the advantage that it works even when only the 2-form is globally defined, with no reference to the 1-form \( \theta_0 \) made. In [72] this idea was generalized to the case of a classical particle in the presence of a Yang-Mills field. When the Poisson manifold \( M \) is the tangent bundle \( TQ \) of a \( n \)-dimensional manifold, so \( M \) is known as the velocity phase space, Souriau also proposed to describe the dynamics not on phase space but in what he called evolution space, with coordinates \((x_i, \dot{x}_j, t)\). His idea was to join the symplectic form \( \omega \) on phase space with the Hamiltonian by considering the two-form \( \omega - dH \wedge dt \) on the evolution space, and then perform the minimal coupling recipe. This allows to recover the Euler-Lagrange equations, and it is equivalent to Faddeev-Jackiw construction [29, 73]. Recently, Bolsinov and Jovanović [74] considered \( G \)-invariant magnetic geodesic flows on coadjoint orbits of a compact Lie group \( G \), where \( \sigma \) is the Kirillov-Kostant two-form.

### 4.1 Souriau’s formalism and exotic mechanics

For concreteness let us ponder the ‘exotic’ plane studied by Horváthy in [21]. Thus, we consider the dynamical system \((T^*\mathbb{R}^2, \omega_\vartheta, H_0)\), where \( \vartheta \in \mathbb{R} \),
\[ \omega_\vartheta = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - \vartheta \, dp_1 \wedge dp_2 \quad \text{and} \quad H_0 = \frac{p_1^2 + p_2^2}{2m}. \]
The 2-form $\omega_\vartheta$ is not only closed but exact, and as the associated map $\omega_\vartheta^\flat: A^1(T^*\mathbb{R}^2) \to \Omega^1(T^*\mathbb{R}^2)$ is given by the matrix

$$\omega_\vartheta^\flat = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \vartheta \\ 0 & 1 & -\vartheta & 0 \end{pmatrix},$$

which is regular for any value of $\vartheta$, the 2-form $\omega_\vartheta$ is symplectic. The inverse matrix is

$$\omega_\vartheta^\sharp = \begin{pmatrix} 0 & \vartheta & 1 & 0 \\ -\vartheta & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

which corresponds to the Poisson structure associated to the bi-vector

$$\Lambda_\vartheta = \vartheta \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2},$$

so the fundamental Poisson commutators are

$$\{q_1, q_2\} = \vartheta, \quad \{q_1, p_1\} = \{q_2, p_2\} = 1, \quad \{q_1, p_2\} = \{q_2, p_1\} = 0, \quad \{p_1, p_2\} = 0,$$

and the dynamical vector field is given by

$$\dot{q}_1 = \{q_1, H_0\} = \frac{p_1}{m}, \quad \dot{q}_2 = \{q_2, H_0\} = \frac{p_2}{m}, \quad \dot{p}_1 = \{p_1, H_0\} = 0, \quad \dot{p}_2 = \{p_2, H_0\} = 0.$$

In this way we obtain a 1-parameter family of symplectic structures for the free particle.

These symplectic structures are the sum of two symplectic structures that are invariant under rotations in the plane, one of them being the canonical symplectic structure on $T^*\mathbb{R}^2$. The generating function of such 1-parameter group will be the function $f$ satisfying

$$i(X)\omega_\vartheta = df,$$

where $X$ is the vector field that is the cotangent lift of the rotation generator in configuration space,

$$X = q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1}.$$ 

Since

$$i(X)\omega_\vartheta = p_2 dq_1 - p_1 dq_2 - q_2 dp_1 + q_1 dp_2 + \vartheta(p_1 dp_1 + p_2 dp_2),$$

we find that the generating function is given by

$$f(q_1, q_2, p_1, p_2) = q_1 p_2 - q_2 p_1 + \frac{\vartheta}{2} \left(p_1^2 + p_2^2\right).$$

We now apply Souriau’s minimal coupling procedure, so we introduce a basic 2-form $\sigma = B(q_1, q_2) dq_1 \wedge dq_2$, which is closed and can be interpreted as a magnetic field, and consider the closed 2-form

$$\omega_{\vartheta, \sigma} := \omega_\vartheta - \pi^* \sigma.$$
The corresponding linear map \( \omega \flat_{\vartheta,\sigma} : A^1(T^*\mathbb{R}^2) \to \Omega^1(T^*\mathbb{R}^2) \) is represented by the matrix

\[
\omega \flat_{\vartheta,\sigma} = \begin{pmatrix}
0 & B & -1 & 0 \\
-B & 0 & 0 & -1 \\
1 & 0 & 0 & \vartheta \\
0 & 1 & -\vartheta & 0
\end{pmatrix},
\]

whose determinant is \((1 - \vartheta B)^2\), therefore \( \omega_{\vartheta,\sigma} \) is regular in the points where \( B \vartheta \neq 1 \); the inverse matrix being given by

\[
\omega ^\sharp_{\vartheta,\sigma} = \frac{1}{1 - \vartheta B} \begin{pmatrix}
0 & \vartheta & 1 & 0 \\
-\vartheta & 0 & 0 & 1 \\
-1 & 0 & 0 & B \\
0 & -1 & -B & 0
\end{pmatrix},
\]

which corresponds to the Poisson structure defined by the bi-vector

\[
\Lambda_{\vartheta,\sigma} = \frac{1}{1 - \vartheta B} \left( \vartheta \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2} + B \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2} \right).
\]

The corresponding fundamental Poisson brackets are

\[
\{q_1, q_2\} = \frac{\vartheta}{1 - \vartheta B}, \quad \{q_1, p_1\} = \{q_2, p_2\} = \frac{1}{1 - \vartheta B}, \quad \{p_1, p_2\} = \frac{B}{1 - \vartheta B}, \quad \{q_1, p_2\} = \{q_2, p_1\} = 0.
\]

When the Hamiltonian is

\[
H = \frac{p^2}{2m} + V(q_1, q_2),
\]

using (11), (13) and (16), we see that the time evolution is given by

\[
\dot{q}_1 = \frac{\vartheta}{1 - \vartheta B} \frac{\partial V}{\partial q_2} + \frac{p_1}{m(1 - \vartheta B)}, \quad \dot{q}_2 = -\frac{\vartheta}{1 - \vartheta B} \frac{\partial V}{\partial q_1} + \frac{p_2}{m(1 - \vartheta B)},
\]

\[
\dot{p}_1 = -\frac{1}{1 - \vartheta B} \frac{\partial V}{\partial q_1} + \frac{Bp_2}{m(1 - \vartheta B)}, \quad \dot{p}_2 = -\frac{1}{1 - \vartheta B} \frac{\partial V}{\partial q_2} + \frac{Bp_1}{m(1 - \vartheta B)}.
\]

The system is still invariant under rotations if \( B \) is a rotationally invariant function, i.e. \( B \) is a function of \( q_1^2 + q_2^2 \), \( B = B(q_1^2 + q_2^2) \), and the generating function for the infinitesimal generator of rotations is

\[
f(q_1, q_2, p_1, p_2) = q_1 p_2 - q_2 p_1 + \frac{\vartheta}{2}(p_1^2 + p_2^2) + \frac{1}{2} B(q_1^2 + q_2^2).
\]

On the other hand, when \( B \) is constant, and \( B \vartheta = 1 \), the determinant of \( \omega \flat_{\vartheta,\sigma} \) is zero and the rank of \( \omega \flat_{\vartheta,\sigma} \) is two, the kernel of the 2-form \( \omega_{\vartheta,\sigma} \) being generated by the vector fields

\[
X_1 = \vartheta \frac{\partial}{\partial q_2} + \frac{\partial}{\partial p_1}, \quad \text{and} \quad X_2 = -\vartheta \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2}.
\]

The solutions of \( X_1 F = X_2 F = 0 \) are to be found from the method of characteristics and turn out to be the functions which depend on \( \xi_1 = q_1 + \vartheta p_2 \) and \( \xi_2 = q_2 - \vartheta p_1 \). This suggests the change of
variables \((q_1, q_2, p_1, p_2) \mapsto (\xi_1, \xi_2, p_1, p_2)\), i.e. \(q_1 = \xi_1 - \vartheta p_2, q_2 = \xi_2 + \vartheta p_1\). In such coordinates, \(X_1 = \partial/\partial p_1, X_2 = \partial/\partial p_2\) and \(\omega_{\vartheta, \sigma}\) becomes
\[
\omega_{\vartheta, \sigma} = -B \, d\xi_1 \wedge d\xi_2.
\]
This means that the quotient manifold \(T^*\mathbb{R}^2/\ker\omega_{\vartheta, \sigma}\) is parametrized by \(\xi_1\) and \(\xi_2\) which, moreover, are Darboux coordinates for such 2-dimensional symplectic manifold.

Using the same idea with commutators Nair and Polychronakos [32] described quantum mechanics for both the non-commutative plane and the non-commutative sphere, and proved that the Landau problem for the non-commutative plane can be recovered as the limit of large radius of the Landau problem for the non-commutative sphere.

4.2 Nonrelativistic anyon model in Souriau formalism

Let \((q_1, q_2)\) be orthogonal Cartesian coordinates in \(Q = \mathbb{R}^2\), and consider the Lagrangian \(L_0\) in \(\mathcal{F}(TQ)\) of the free particle
\[
L_0(q_1, q_2, v_1, v_2) = \frac{1}{2} m (v_1^2 + v_2^2).
\]

Let \(M\) be the graph of the corresponding Legendre transformation. This is the submanifold of the Pontryagin bundle \(TQ \oplus T^*Q\) given by the constraint functions
\[
\lambda_i(q_1, q_2, v_1, v_2, p_1, p_2) = p_i - m v_i.
\]

Let \(\kappa \in \mathbb{R}\) be a constant and consider \(TQ\) endowed with the exact 2-form \(\omega_1\) defined by
\[
\omega_1(q_1, q_2, v_1, v_2) := \kappa \, dv_1 \wedge dv_2.
\]

In the spirit of Souriau’s idea, we consider the closed 2-form on \(TQ \oplus T^*Q\)
\[
\omega := -\text{pr}_1^* \omega_1 + \text{pr}_2^* \omega_0,
\]
where \(\text{pr}_1\) and \(\text{pr}_2\) are the natural projections \(\text{pr}_1 : TQ \oplus T^*Q \to TQ\) and \(\text{pr}_2 : TQ \oplus T^*Q \to T^*Q\), \(\omega_0\) is the canonical symplectic structure in \(T^*Q\) and \(\omega_1 \in \Omega^2(TQ)\) is as before. The corresponding map \(\omega^\flat : A^1(TQ \oplus T^*Q) \to \Omega^1(TQ \oplus T^*Q)\) is represented by the matrix
\[
\omega^\flat = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & \kappa & 0 & 0 & 0 \\
0 & 0 & -\kappa & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
which is regular for \(\kappa \neq 0\). In this case, the 2-form \(\omega\) is symplectic, and since the inverse matrix is
\[
\omega^\sharp = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1/\kappa & 0 & 0 & 0 \\
0 & 0 & 1/\kappa & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
we obtain the bi-vector field on $TQ \oplus T^*Q$

$$\Lambda = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2} - \frac{1}{\kappa} \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2},$$

and the corresponding Poisson structure is defined by the following fundamental relations:

$$\{q_1, p_1\} = \{q_2, p_2\} = 1, \quad \{q_1, q_2\} = \{p_1, p_2\} = \{q_1, p_2\} = \{p_1, q_2\} = 0,$$

$$\{v_1, v_2\} = -\frac{1}{\kappa}, \quad \{v_1, p_2\} = \{v_2, p_1\} = 0.$$

The two constraint functions $\lambda_1$ and $\lambda_2$ are second class constraints, because

$$\{\lambda_1, \lambda_2\} = \{p_1 - mv_1, p_2 - mv_2\} = m^2 \{v_1, v_2\} = -\frac{m^2}{\kappa}.$$

They define a four dimensional symplectic manifold.

5 **Feynman–Dyson’s method and non-commutativity**

In this section we review the Feynman’s derivation of Maxwell’s equations [6], in the framework of a tangent bundle, so the Poisson manifold $M$ is the tangent bundle $TQ$ of a configuration space $Q$. In terms of local tangent bundle coordinates in $TQ$ induced from local coordinates in $Q$, denoted $x_i$ and $\dot{x}_i$, a general Poisson bracket on $TQ$ is locally given by

$$\{f, g\} = \{x_a, x_b\} \frac{\partial g}{\partial x_a} \frac{\partial f}{\partial x_a} + \{x_a, \dot{x}_b\} \frac{\partial g}{\partial \dot{x}_b} \frac{\partial f}{\partial x_a} + \{\dot{x}_a, x_b\} \frac{\partial g}{\partial x_b} \frac{\partial f}{\partial \dot{x}_a} + \{\dot{x}_a, \dot{x}_b\} \frac{\partial g}{\partial \dot{x}_b} \frac{\partial f}{\partial \dot{x}_a}. \quad (17)$$

The assumptions in [6] are Newton’s equations of motion

$$m\ddot{x}_j = F_j(x, \dot{x}),$$

i.e. the dynamics is given by the second order differential equation vector field

$$\Gamma = \dot{x}_j \frac{\partial}{\partial x_i} + F_i(x, \dot{x}) \frac{\partial}{\partial \dot{x}_i},$$

together with the fundamental brackets

$$\{x_i, x_j\} = 0 \quad \text{and} \quad m\{x_i, \dot{x}_j\} = \delta_{ij}. \quad (18)$$

The goal is to determine the other fundamental Poisson brackets, and as $\{\dot{x}_i, \dot{x}_j\}$ must be skew symmetric it can be written as

$$\{\dot{x}_i, \dot{x}_j\} = \frac{1}{m^2} \varepsilon_{ijk} B_k(x, \dot{x}),$$

where $\varepsilon_{ijk}$ denotes the fully skew-symmetric Levi–Civita tensor and $B$ is defined as the magnetic field. Now (18) implies that

$$\{x_i, F_j\} = \frac{1}{m} \frac{\partial F_j}{\partial \dot{x}_i},$$
and using the derivation property for the time derivative of the second equation in (18), i.e. assuming that the vector field $\Gamma$ is a derivation of the Poisson structure, we get

$$\{\dot{x}_i, \dot{x}_j\} = -\frac{1}{m} \{x_i, F_j\} = \frac{1}{m^2} \varepsilon_{ijk} B_k(x, \dot{x}),$$

(19)

and the Jacobi identity for the functions $x_i, \dot{x}_j, \dot{x}_k$,

$$\{x_i, \{\dot{x}_j, \dot{x}_k\}\} + \{\dot{x}_k, \{x_i, \dot{x}_j\}\} + \{\dot{x}_j, \{\dot{x}_k, x_i\}\} = 0$$

entails

$$0 = \{x_r, B_s\} = \frac{1}{m} \frac{\partial B_s}{\partial \dot{x}_r}.$$ 

In particular $B$ does not depend on the dotted variables. Moreover from the Jacobi identity with three different velocities we obtain

$$\text{div } B = 0,$$

which reveals that the flux of the field $B$ through a closed surface is zero, and that magnetic monopoles do not exist!

On the other hand, since $B$ does not depend on $\dot{x}_i$, equations (17) and (19) entail that $F$ is at most linear in such variables, therefore we can define another field $E$, called the electric field, by $E_j = F_j - \varepsilon_{jkl} \dot{x}_k B_l$. Using repeatedly all the equations above, one arrive to Maxwell’s equation corresponding to Faraday’s law of electrodynamics, in the setting suggested at the beginning of this section, namely $\text{rot } E = 0$ in the autonomous case, or in general

$$\frac{\partial B}{\partial t} + \text{rot } E = 0,$$

a magnetic field that is changing in time produces a non-conservative electric field. We refer the reader to [33, 34] for details.

To obtain a dynamic different from the standard formalism of electrodynamics, one needs to modify the fundamental brackets in (18). The first idea is to use Souriau’s technique, namely to replace the first fundamental bracket by

$$\{x_i, x_j\} = g_{ij}(x),$$

where $g_{ij}$ is an arbitrary skew-symmetric matrix of functions, fulfilling the constraints that the Poisson bracket properties impose, and keep the other assumptions. In particular, the Jacobi identity

$$\{x_i, \{x_j, \dot{x}_k\}\} + \{\dot{x}_k, \{x_i, x_j\}\} + \{x_j, \{\dot{x}_k, x_i\}\} = 0,$$

entails

$$0 = \{\dot{x}_k, g_{ij}\} = \{\dot{x}_k, x_l\} \frac{\partial g_{ij}}{\partial x_l} + \{\dot{x}_k, \dot{x}_l\} \frac{\partial g_{ij}}{\partial \dot{x}_l} = -\frac{1}{m} \frac{\partial g_{ij}}{\partial x_k},$$

(20)

Then the matrix $g_{ij}$ is a constant skew-symmetric $3 \times 3$ matrix, which is an interesting, but somewhat restrictive, case. We then modify Souriau’s idea and settle for

$$\{x_i, x_j\} = g_{ij}(x, \dot{x}).$$
Accordingly, (20) becomes
\[ 0 = \{ \dot{x}_k, g_{ij} \} = -\frac{1}{m} \frac{\partial g_{ij}}{\partial x_k} + \{ \dot{x}_k, \dot{x}_l \} \frac{\partial g_{ij}}{\partial \dot{x}_l}. \]

This equation clearly relates the part of the Poisson structure on the base (the positions) with the part of the Poisson structure on the fibre (the velocities). Hence if the Poisson structure on the base is known one can compute the fundamental brackets on the fibre.

On the other hand, from the Jacobi identity among \((x_i, x_j, x_k)\) we obtain
\[ \{ x_i, g_{jk} \} + \{ x_k, g_{ij} \} + \{ x_j, g_{ki} \} = 0, \]
and this leads to another constraint:
\[ 0 = g_{il} \frac{\partial g_{jk}}{\partial x_l} + g_{kl} \frac{\partial g_{ij}}{\partial x_l} + g_{jl} \frac{\partial g_{ki}}{\partial x_l} + \frac{1}{m} \left( \frac{\partial g_{jk}}{\partial \dot{x}_i} + \frac{\partial g_{ij}}{\partial \dot{x}_k} + \frac{\partial g_{ki}}{\partial \dot{x}_j} \right). \quad (21) \]

Note that if \(d_x\) denotes the exterior derivative on the vector space \(T_xQ\), for \(x\) in \(Q\), then the term inside the parenthesis in (21) are the local coordinates of \(d_x\bar{\omega}\), where \(\bar{\omega}_x\) is the 2-form in \(\Omega^2(T_xQ)\) defined by \(\bar{\omega}_x = g_{ij}(x, \dot{x}) d\dot{x}_i \wedge d\dot{x}_j\). On the other hand, if \(\bar{\Lambda}\) is the bivector in \(\Lambda^2(TQ)\) given by \(\bar{\Lambda} = g_{ij}(x, \dot{x}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\) the terms outside the parenthesis in (21) are the local coordinates of \([\bar{\Lambda}, \bar{\Lambda}]_{SN}\). Therefore equation (21) is fulfilled when
\[ d_x\bar{\omega} = 0 \quad \text{and} \quad d_x\bar{\Lambda} = [\bar{\Lambda}, \bar{\Lambda}]_{SN} = 0, \]
that is, when \(\bar{\omega}_x\) is closed and \(\bar{\Lambda}\) is a Poisson tensor.

Furthermore, similar ideas as in the commutative case, using the other Jacobi identities, leads, see [18], to the modified Gauss law
\[ \text{div} B = -\frac{1}{m} B \cdot \hat{\nabla} \times B, \]
where \(\hat{\nabla} = (\frac{\partial}{\partial \dot{x}_1}, \frac{\partial}{\partial \dot{x}_2}, \frac{\partial}{\partial \dot{x}_3})\), and also to
\[ (\text{rot} E)_k + \frac{1}{m} \left( (E \cdot \hat{\nabla}) B_k + B \cdot \frac{\partial E}{\partial \dot{x}_k} - (\hat{\nabla} \cdot E) B_k \right) = 0, \]
which is what replaces the Maxwell equation corresponding to Faraday’s law.

### 5.1 Generalized Lorentz force equations

Consider the Hamiltonian dynamical system on \(T\mathbb{R}^3\), where the Hamiltonian and the symplectic form are given respectively by
\[ H = \frac{1}{2m} \delta_{ij} \dot{x}_i \dot{x}_j + \phi(x), \]
and
\[ \omega = \frac{1}{m} dx_i \wedge d\dot{x}_i + B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2 + \frac{1}{2} g_{ij} d\dot{x}_i \wedge d\dot{x}_j. \]
Assume that, in local coordinates, the Hamiltonian vector field is written as \( \dot{X}_H = S_i \partial x_i + R_i \dot{x}_i \).

The equation \( i(\dot{X}_H)\Omega = dH \) becomes

\[
\begin{align*}
\frac{1}{m} \dot{x}_1 &= \frac{1}{m} S_1 + g_{21} R_2 + g_{31} R_3, \\
\frac{1}{m} \dot{x}_2 &= \frac{1}{m} S_2 + g_{12} R_1 + g_{32} R_3, \\
\frac{1}{m} \dot{x}_3 &= \frac{1}{m} S_3 + g_{13} R_1 + g_{23} R_2,
\end{align*}
\]

Therefore if we assume that \( \dot{X}_H \) is a second order differential equation, namely that \( S_i = \dot{x}_i \), then

\[
\begin{align*}
\frac{d^2 x_1}{dt^2} &= -\frac{\partial \phi}{\partial x_1} + \dot{x}_3 B_2 - \dot{x}_2 B_3, \\
\frac{d^2 x_2}{dt^2} &= -\frac{\partial \phi}{\partial x_2} + \dot{x}_1 B_3 - \dot{x}_3 B_1, \\
\frac{d^2 x_3}{dt^2} &= -\frac{\partial \phi}{\partial x_3} + \dot{x}_2 B_1 - \dot{x}_1 B_2,
\end{align*}
\]

whereas the left column provides the constraints

\[
\begin{align*}
0 &= mg_{21} \left( \frac{\partial \phi}{\partial x_2} - \dot{x}_1 B_3 + \dot{x}_2 B_1 \right) + mg_{31} \left( \frac{\partial \phi}{\partial x_3} - \dot{x}_2 B_1 + \dot{x}_1 B_2 \right), \\
0 &= mg_{12} \left( \frac{\partial \phi}{\partial x_1} - \dot{x}_3 B_2 + \dot{x}_2 B_3 \right) + mg_{32} \left( \frac{\partial \phi}{\partial x_3} - \dot{x}_2 B_1 + \dot{x}_1 B_2 \right), \\
0 &= mg_{13} \left( \frac{\partial \phi}{\partial x_1} - \dot{x}_3 B_2 + \dot{x}_2 B_3 \right) + mg_{23} \left( \frac{\partial \phi}{\partial x_2} - \dot{x}_1 B_3 + \dot{x}_3 B_1 \right).
\end{align*}
\]

In particular when \( \nabla \phi = -eE \), (23) is a generalized Lorentz force: a force experienced by a charged particle moving in an electromagnetic field, subject to a system of constraints.

Another interesting class of systems can be studied via the “generalized Souriau form”

\[
\tilde{\omega}_0 = d\dot{x}_i \wedge dx_i + g_{ij} d\dot{x}_i \otimes d\dot{x}_j.
\]

It is a mixture of a symplectic and a gradient structure, known as a metriplectic system. The symmetric bracket associated to the metric tensor incorporates the dissipative structure of the system. The Leibniz vector field \( X_h \) associated to a function \( h \in C^\infty(M) \) satisfies \( X_h = \nabla h \), i.e. \( X_h \) generates a gradient dynamical system. In local coordinates the vector field \( X_h \) is given by

\[
X_h = g_{ij} \frac{\partial h}{\partial x_j} \frac{\partial}{\partial x_i},
\]

and the corresponding bracket in this context is called a Leibniz bracket.
Another interesting class of dynamical systems that generalize the Hamiltonian systems, where noncommutativity is also possible, was introduced in [47, 48]. Let \((M, \omega_0)\) be a \(2n\)-dimensional symplectic manifold, a vector field \(X\) is said to be symplectic if \(L_X\omega_0 = 0\), from the Cartan identity, this is equivalent to \(i(X)\omega_0\) being closed, in particular every Hamiltonian vector field is symplectic. On the other hand, we say a vector field \(X\) preserves the volume if \(L_X\omega_0^n = 0\); here and in what follows powers are meant with respect to the wedge product. Since
\[
\frac{d}{ds} \Phi^*_t \omega_0^n \bigg|_{s=0} = \Phi^*_t d \Phi^*_s \omega_0^n \bigg|_{s=0} = \Phi^*_t L_X\omega_0^n = 0,
\]
where \(\Phi_t\) is the flow of \(X\), it follows that \(\Phi_t\) does preserves the volume form \(\omega_0^n\). Furthermore, a simple induction gives
\[
L_X\omega_0^k = k L_X\omega_0 \wedge \omega_0^{k-1}.
\]
In particular, we see that every symplectic vector field preserves the volume, but the converse is not true in general. The divergence of a vector field \(X\) is defined as the unique function \(\text{div} X\) in \(C^\infty(M)\) such that
\[
L_X\omega_0^n = \text{div} X \omega_0^n.
\]
Therefore \(X\) preserve the volume if, and only if, it is divergence free. Let \((x_1, \ldots, x_{2n})\) be Darboux coordinates, then \(\omega_0 = dx_i \wedge dx_{n+i}\), and if \(X = \sum_{i=1}^{2n} X_i \partial_{x_i}\) it is easy to check that
\[
\text{div} X = \sum_{i=1}^{2n} \frac{\partial X_i}{\partial x_i}.
\]

We now describe a procedure that produces dynamical systems that preserves the volume. First consider the map \(F: A^1(M) \to \Omega^{2n-1}(M)\) given by \(F(X) := i(X)\omega_0^n\). Using Darboux coordinates, simple combinatorial arguments entail
\[
\omega^k = (-1)^{\frac{k(k-1)}{2} k!} \sum_{1 \leq i_1 < \cdots < i_{n-k} \leq n} \left( dx_1 \wedge dx_{n+1} \wedge \cdots \wedge \hat{dx}_{i_1} \wedge \hat{dx}_{n+i_1} \wedge \cdots \right.
\]
\[
\left. \wedge \hat{dx}_{i_{n-k}} \wedge \hat{dx}_{n+i_{n-k}} \wedge \cdots \wedge dx_n \wedge dx_{2n} \right),
\]
(24)
where as usual the hat means that the term is to be deleted, in particular
\[
i(X)\omega_0^n = (-1)^{\frac{n(n-1)}{2} n!} i(X)dx_1 \wedge \cdots \wedge dx_{2n}
\]
\[
= n! (-1)^{\frac{n(n-1)}{2} + i-1} X_i dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_{2n},
\]
(25)
therefore \(F\) is surjective. Moreover, if \(X(p) \neq 0\) and \(F(X) = 0\), we can locally find a basis \(X_1, \ldots, X_{2n}\), of vector fields such that \(X_1 = X\), then
\[
\omega_0^n (X_1, \ldots, X_{2n}) = i(X)\omega_0^n (X_2, \ldots, X_{2n}) = 0,
\]
which is absurd since $\omega_0$ is nondegenerate, hence $F$ is injective and therefore a linear isomorphism. Since $L_X\omega^n_0 = di(X)\omega^n_0$, under $F$, the space of volume preserving vector fields corresponds to the space of closed $(2n - 1)$-forms. Thus, if $\eta$ is a 2-form

$$X_\eta := F^{-1}(d(\eta \wedge \omega^{n-2}_0))$$

is a volume preserving vector field. In particular, if $\omega$ is the Souriau form, written in Darboux coordinates as

$$\omega = dx_{n+i} \wedge dx_i + \frac{1}{2}g_{ij}dx_{n+i} \wedge dx_{n+j} + \frac{1}{2}B_{ij}dx_i \wedge dx_j,$$

then

$$d(\omega \wedge \omega^{n-2}_0) = d\omega \wedge \omega^{n-2}_0 = \frac{1}{2} \frac{\partial g_{ij}}{\partial x_k} dx_k \wedge dx_{n+i} \wedge dx_{n+j} \wedge \omega^{n-2}_0 + \frac{1}{2} \frac{\partial B_{ij}}{\partial x_{n+k}} dx_{n+k} \wedge dx_i \wedge dx_j \wedge \omega^{n-2}_0.$$ 

Now, from (24)

$$dx_k \wedge dx_{n+i} \wedge dx_{n+j} \wedge \omega^{n-2}_0 = \delta_{ki} dx_{n+j} \wedge dx_k \wedge dx_{n+i} \wedge \omega^{n-2}_0 - \delta_{kj} dx_{n+i} \wedge dx_k \wedge dx_{n+j} \wedge \omega^{n-2}_0 = (-1)^n \frac{n}{n-1} \left( \delta_{ki} dx_{n+j} \wedge \omega^{n-1}_0 - \delta_{kj} dx_{n+i} \wedge \omega^{n-1}_0 \right).$$

Similarly

$$dx_{n+k} \wedge dx_i \wedge dx_j \wedge \omega^{n-2}_0 = \frac{(-1)^n}{n-1} \left( \delta_{kj} dx_i \wedge \omega^{n-1}_0 - \delta_{ki} dx_j \wedge \omega^{n-1}_0 \right).$$

Thus

$$d(\omega \wedge \omega^{n-2}_0) = \frac{(-1)^n}{n-1} \left( \frac{\partial g_{ij}}{\partial x_i} dx_{n+j} \wedge \omega^{n-1}_0 - \frac{\partial B_{ij}}{\partial x_{n+k}} dx_j \wedge \omega^{n-1}_0 \right).$$

Let $X$ be the vector field

$$X = c_n \frac{\partial g_{kl}}{\partial x_k} \frac{\partial}{\partial x_l} - c_n \frac{\partial B_{kl}}{\partial x_{n+k}} \frac{\partial}{\partial x_{n+l}},$$

with $c_n = \frac{(-1)^{n(n-1)}}{n(n-1)}$, then, using (25) and (24)

$$i(X)\omega^n_0 = n!(-1)^{\frac{n(n-1)}{2} + i - 1} c_n \frac{\partial g_{ki}}{\partial x_k} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{2n}$$

$$- n!(-1)^{\frac{n(n-1)}{2} + n + i - 1} c_n \frac{\partial B_{ki}}{\partial x_{n+k}} dx_1 \wedge \cdots \wedge \widehat{dx_{n+i}} \wedge \cdots \wedge dx_{2n}$$

$$= (n-2)!(-1)^n \frac{\partial g_{ki}}{\partial x_k} dx_{n+i} \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+i} \wedge \cdots \wedge dx_{2n}$$

$$- (n-2)!(-1)^n \frac{\partial B_{ki}}{\partial x_{n+k}} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+i} \wedge \cdots \wedge dx_{2n}$$

$$= (n-2)!(-1)^{\frac{n+1}{2} + (n-1)(n-2)} \left( \frac{\partial g_{ki}}{\partial x_k} dx_{n+i} - \frac{\partial B_{ki}}{\partial x_{n+k}} dx_i \right)$$

$$\wedge dx_1 \wedge dx_{n+1} \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+i} \wedge \cdots \wedge dx_n \wedge dx_{2n}$$

$$= d(\omega \wedge \omega^{n-2}_0).$$
In other words, $X_\omega = X$, so we can associate a volume preserving flow with the Souriau’s form, and the equations of motion are given by

$$\frac{dx_i}{dt} = (-1)^n \frac{\partial g_{ki}}{\partial x_k},$$
$$\frac{dx_{n+i}}{dt} = -\frac{(-1)^n}{n-1} \frac{\partial B_{ki}}{\partial x_{n+k}}.$$

A Nambu-Poisson system is a volume preserving flow, on a Nambu-Poisson manifold of order $2n$, determined by $(2n - 1)$ Hamiltonian functions $H_1, \ldots, H_{2n-1} \in C^\infty(M)$

$$\frac{dx_i}{dt} = X_{H_1, \ldots, H_{2n-1}}(x_i) = \{H_1, \ldots, H_{2n-1}, x_i\}.$$

In general, $\mathcal{L}_{X_{H_1, \ldots, H_{2n-1}}} f = \{H_1, \ldots, H_{2n-1}, f\}$ if $X_{H_1, \ldots, H_{2n-1}}$ is a Nambu-Hamiltonian vector field. If $\eta$ is a Nambu-Poisson tensor, then Takhtajan [75] proved $\mathcal{L}_{X_{H_1, \ldots, H_{2n-1}}} \eta = 0$.

### 7 Conclusion and Outlook

In this paper we have studied the classical non-commutative mechanical systems using Souriau’s method of generalized symplectic form. In particular we have explored a large class of non-commutative flows which includes the non-commutative magnetic geodesic flows, non-relativistic anyon model [76, 77, 78], generalized Lorentz force equation, etc. Souriau’s formalism allows us to study geometrically all these non-commutative dynamical systems in an unified manner. The dynamics of these systems boil down to generalized Hamiltonian dynamics where the Poisson structure can be complicated functions of phase space coordinates and momenta. However, some questions should be addressed in near future.

At first we must consider the quantization of these classical non-commutative system. There is an interesting paper [79] which addresses the connection between non-commutative quantum mechanics and Feynman–Dyson’s method. Actually, the generalization of Feynman–Dyson’s idea to the quantum world would be an interesting subject to be studied. This would lead to unveil the close relation existing between the non-commutative geometry and the geometric phases. Quantization of these models could give rise to new physics at some very high energy scale [80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91].

This method can be extended to other directions also. We can simply generalize this construction to supersymmetric non-commutative systems [93, 94]. In other words, we can try to generalize the Feynman–Dyson’s scheme to supersymmetric framework. This would certainly yield a generalization of supersymmetric generalized Hamiltonian dynamics. There is a recent upsurge of interest in $(2+1)$-dimensional model [92, 95] with a kind of nonstandard noncommutativity, where both coordinates and momenta get deformed commutators. So far most of the papers concern time-independent systems, it would be rather challenging to extend this framework to time-dependent systems. In a recent note Liang and Jiang [96] studied the time-dependent harmonic oscillator in a background of time-dependent electric and magnetic fields. Recently noncommutative quantum mechanics [79, 97, 98, 99, 100] is becoming a exciting topic to study, it would be interesting for us to investigate this subject using geometrical methods of quantizing noncommutative phase space mechanics.
Possibly one can study the Helmholtz condition analogous to standard classical dynamics. Then the corresponding variational formulation can be used to construct Noether symmetries and conserved densities. It is known [44] that the Helmholtz condition connected to \( \theta \)-deformed Poisson system is a third-order time derivative equation. One should analyse carefully all these new aspects.

We can also study the field theoretic Poisson brackets on jet space. This will yield an interesting class of partial differential equations. One must try to explore its connection to other branches of mechanics and geometry, namely, non-holonomic systems, control theory, Finslerian mechanics, Lie algebroid theory, etc.

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