Chaos in the one-dimensional gravitational three-body problem

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Abstract

We have investigated the appearance of chaos in the 1-dimensional Newtonian gravitational three-body system (three masses on a line with $-1/r$ pairwise potential). In the center of mass coordinates this system has two degrees of freedom and can be conveniently studied using Poincaré sections. We have concentrated in particular on how the behavior changes when the relative masses of the three bodies change. We consider only the physically more interesting case of negative total energy. For two mass choices we have calculated 18000 full orbits (with initial states on a $100 \times 180$ lattice on the Poincaré section) and obtained dwell time distributions. For 105 mass choices we have calculated Poincaré maps for $10 \times 18$ starting points. Our results show that the Poincaré section (and hence the phase space) divides into three well defined regions with orbits of different characteristics: 1) There is a region of fast scattering, with a minimum of pairwise collisions. This region consists of ‘scallops’ bordering the $E = 0$ line, within a scallop the orbits vary smoothly. The number of the scallops increases as the mass of the central particle decreases. 2) In the chaotic scattering region the interaction times are longer, and both the interaction time and the final state depend sensitively on the starting point on the Poincaré section. For both 1) and 2) the initial and final states consists of a binary + single particle. 3) The third region consists of quasiperiodic orbits where the three masses are bound together forever. At the center of the quasiperiodic region there is a periodic orbit discovered (numerically) by Schubart in 1956. The stability of the Schubart orbit turns out to correlate strongly with the global behavior.

I. INTRODUCTION

The three-body problem is one of the fundamental problems of dynamical systems. It has a long tradition, even in connection with chaos research: This is the problem that led Poincaré to make his famous observations of chaos.

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Most of the chaotic systems studied during the past twenty years have been compact systems, where a good understanding of e.g. the way chaos enters has been obtained. Recently more interest has been paid to scattering problems. One popular problem is the scattering of a particle on the plane from a fixed localized object [1,2]. More closely related to the present topic is the work on reactive scattering in chemistry [3].

The gravitational Newtonian three body scattering problem is interesting and special due to the following three properties: It is noncompact, the interaction is long range, and it is not obtainable as a small perturbation of an integrable system. These properties mean that many of the important theorems characterizing chaos are not directly applicable to this system. [In the present gravitational case the interaction is attractive, and our results cannot be translated to the corresponding electrostatic case where some objects interact repulsively.]

The general three dimensional three-body problem has been studied extensively by numerical simulations [4] and by analytical studies [5,6] (for a review see [7]). However, in full generality this problem is still too complicated for a systematic analysis. This has led to the study of various simplifications or restrictions of it; these include:

1. The restricted three-body problem: Three dimensional, but one of the masses is assumed to be zero. The motion of the two massive bodies is integrable, but the motion of the third body in the field of the other two is not.

2. Three dimensional with finite nonzero masses but with special symmetries. [E.g. two masses symmetrically on the \((x, y)\)-plane with the third moving on the \(z\)-axis [8].]

3. The planar three-body problem: motion is restricted to a plane, masses are free.

4. The rectilinear three-body problem: motion is restricted to a line, masses are free.

The last, simplest case, which is the topic of this article, is still complicated enough to show a rich variety of phenomena. It also has a long history [9-11] but it has still not received a definitive analysis.

Recently the present authors made a comprehensive study of the rectilinear three-body system when the masses are equal [12,13] (hereafter referred to as papers I and II). We have also obtained some results for arbitrary masses [14] (paper III). In the present paper we review the previously obtained data and then go on to discuss in detail the situation where the masses are arbitrary. We will mainly consider the more interesting negative energy case, the dynamics with nonnegative energies and equal masses was discussed in paper II.

There are various points of view that one can use when studying a dynamical system. In the present case the system can be reduced to a conservative system with two degrees-of-freedom and therefore we can look at it through conventional method of Poincaré sections. Since the system allows scattering we have two other quantities that can be used to characterize the dynamics. First we can look at the dwell time, i.e. the time all particles stay together and interact strongly before the triplet again breaks up. Secondly we can characterize the scattering process by the types of initial and final states. All of these methods are used here.

We start in Sec. 2 by presenting the system and showing how the initial data/Poincaré section is defined so that it is possible to include all orbit types. In Sec. 3 we get a preview
of the richness of the system by looking at a set of typical trajectories. In Sec. 4 we take
the first global view by looking for two particular mass choices how the dwell time depends
on the initial values. In the remaining Sections we consider the full mass freedom, first in
Sec. 5 through the stability of the periodic Schubart orbit. In Sec. 6 we present a set of
105 Poincaré sections computed for a lattice of mass values. The Poincaré sections show
interesting transitions as the masses change. In Sec. 7 we discuss how the stability of the
periodic Schubart orbit influences the initial and final state types.

II. THE SYSTEM

In this section we first describe the system and its reduction to two degrees of freedom.
Then the definition of the Poincaré section (and initial values for numerical computations)
is discussed in detail. Finally we describe how the Hamiltonian is regularized into a form
that is useful in numerical integrations and discuss the accuracy of the computations.

A. The Hamiltonian

The mass configuration is given in Fig. 1. We have three masses, labeled $i = 1$, $0$, $2$ from
left to right, with masses $m_i$ at positions $x_i$. The Hamiltonian is

$$H = \frac{1}{2} \sum_{i=0}^{2} \frac{w_i^2}{m_i} - \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|},$$  \hspace{1cm} (1)

where $w_i$ are the momenta canonical to $x_i$ and we have normalized the system so that the
gravitational constant $G = 1$.

The attractive nature of the interaction leads to collisions, which seem to be singular.
For a pairwise collision the singularity is not essential and there is a standard method by
which the singularity can be regularized [15,16]. Intuitively the singularity can be opened
if we consider colliding masses with a small but nonzero impact parameter. When they are
close they will not collide head on but revolve rapidly around each other and emerge from
the collision region with momenta reversed in their center of mass frame. Thus when we let
the impact parameter approach zero we get an attractive potential with a reflecting hard
core. As a result we have a ‘reflection’ in which the bodies cannot change their order on the
line. The rectilinear two-body collision can also be solved analytically when the distance to
the third body is much larger than that between the two colliding bodies. [In that case we
get two approximate two-body systems: one for the two colliding pair, and one for the third
body and the center of mass of the colliding pair.] The triple collision, however, is a genuine
singularity and cannot be regularized for generic masses [17].

Since we have only pairwise interactions we can go to the center-of-mass frame and
eliminate one degree of freedom. The canonical transformation from $(x_i, w_i)$ to $(q_i, p_i)$ is
defined as follows [$M = m_1 + m_0 + m_2$]:

$$q_1 = x_0 - x_1, \quad q_2 = x_2 - x_0, \quad q_0 = (m_1 x_1 + m_2 x_2 + m_0 x_0),$$  \hspace{1cm} (2)

$$w_1 = -p_1 + \frac{m_1}{M} p_0, \quad w_2 = p_2 + \frac{m_2}{M} p_0, \quad w_0 = p_1 - p_2 + \frac{m_0}{M} p_0,$$  \hspace{1cm} (3)
and after omitting the center of mass term the new Hamiltonian becomes

$$H = \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_0} \right) p_1^2 - \frac{p_1 p_2}{m_0} + \frac{1}{2} \left( \frac{1}{m_2} + \frac{1}{m_0} \right) p_2^2 - \frac{m_0 m_1}{q_1} - \frac{m_0 m_2}{q_2} - \frac{m_1 m_2}{q_1 + q_2}. \quad (4)$$

As shown before the particles preserve their order on the line and therefore we always have $q_1 \geq 0$, $q_2 \geq 0$. The motion takes place in the first quadrant of the $(q_1, q_2)$-plane with attracting walls at coordinate axes, see Fig. 2. There is also attraction from the nonphysical wall at $q_1 + q_2 = 0$, which can be reached only at the origin which is an essential singularity. In Fig. 2 the Poincaré section (defined in Sec. 2.2) is given by the line $q_1 = q_2$, i.e. when the system crosses this line it crosses the surface of section. The motion illustrated in Fig. 2 is a typical fast scattering orbit. The arrows show the direction of motion of the point $(q_1, q_2)$ (since the system is time reversal invariant the direction of the arrows can be reversed). The trajectory is unbounded in both directions of time, which is the typical situation. There are three crossings of the Poincaré section in this example, two of which are very close to each other (on the surface of section) but not equal.

Let us note that there is one special solution, the so called homographic solution [18], for which the ratio

$$\tau = \frac{q_1}{q_2}, \quad (5)$$

is time independent. Clearly this leads to a triple collision. To find this solution and the particular ratio, $\tau = \tau_h$, one substitutes $q_2 = q_1/\tau_h$ and $p_2 = \beta p_1$ into the equations of motion obtained from (4) and requires that the two pairs of equations for $q_1$ and $p_1$ are consistent. [In fact both pairs lead to the one-dimensional two-body problem $\ddot{q} = \tilde{M} q^{-2}$, which is analytically solvable, and yields a periodic orbit. Thus this triple collision orbit is regular, but any nearby orbit is completely different.] The consistency condition yields two equations for $\tau_h$, $\beta$ and the masses $m_i$. After eliminating $\beta$ and parametrizing

$$\tau_h = \frac{1 + z}{1 - z}. \quad (6)$$

one can write the final condition in the nice form

$$\frac{-1 \cdot m_1 + z \cdot m_0 + 1 \cdot m_2}{m_1 + m_0 + m_2} = \frac{z^5 - 2z^3 + 17z}{z^4 - 10z^2 - 7}. \quad (7)$$

where the LHS gives the position of the center of mass when the particles are located at positions $-1$, $z$, $+1$. Although (7) is a quintic equation for $z$ it has only one solution in the physically allowed interval $-1 < z < 1$ and thus $\tau_h$ is unique. In particular, if $m_1 = m_2$ we have $\tau_h = 1$.

B. The Poincaré section

For a comprehensive study of system (1) we must define the initial values (which define the Poincaré surface of section) so that in principle all orbits can be included. For a compact system it is easy to choose a suitable section, but when open trajectories are possible the
situation can be more involved. Note also that since our system is open many trajectories will hit the Poincaré section only a few times, while in a closed system this will happen repeatedly.

The location of the section is defined by the particular value of the ratio $\tau$ defined in (5) above. As coordinates on this section we introduce the mean value $R$ of the distances $q_1$, $q_2$, and an angle $\theta$ specifying the distribution of kinetic energy between the particles. We will now discuss this in detail.

The value of distance ratio $\tau$ that defines the surface of section must be chosen in a particular way in order to ensure that every trajectory crosses the surface. Due to the attractive nature of the interaction every trajectory (with the exception of the above mentioned homographic solution) will have pairwise collisions between particles 1 and 0 and also between particles 0 and 2 (in fact for most orbits there will be several such points). Thus at a certain time we will have $q_1 = 0$ and another time $q_2 = 0$, and between these times the ratio (5) will assume all values between 0 and $\infty$. Only for the homographic solution this ratio will have a fixed value $\tau_h$. Thus to make sure all trajectories pierce the Poincaré section at least once we will define the location of the Poincaré section by the condition $\tau = \tau_h$.

As the first coordinate on the section we take half the distance between the outermost particles

$$R = \frac{1}{2}(q_1 + q_2)|_{\tau = \tau_h}.$$  

(8)

When $\tau = \tau_h$ the energy is $E = T - \frac{1}{R} \left[ \frac{m_0 m_1}{1+z} + \frac{m_0 m_2}{1-z} + \frac{m_1 m_2}{2} \right]$, and for a fixed negative energy the maximum value of $R$ is obtained from this expression when the kinetic energy $T$ vanishes:

$$R_{\text{max}} = \frac{1}{|E|} \left[ \frac{m_0 m_1}{1+z} + \frac{m_0 m_2}{1-z} + \frac{m_1 m_2}{2} \right].$$  

(9)

For the remaining coordinate on the Poincaré section we must choose something related to the momenta, but we do not want the kinetic energy depend on this quantity. Thus we diagonalize the kinetic part $T$ of $H$ in (7) by writing it as a sum of squares $T = (a_1 p_1 + b p_2)^2 + (c_1 p_1 + d p_2)^2$. We can then parametrize $p_i$ in the required way as

$$a_1 p_1 + b p_2 = \sqrt{T} \cos(\theta), \quad c_1 p_1 + d p_2 = \sqrt{T} \sin(\theta),$$  

(10)

where $\theta$ is the remaining coordinate of the Poincaré section. The constants $a, b, c, d$ are not defined uniquely, but there is a one parameter family of ways to write $T$ as a sum of squares. To fix this final free parameter we impose one more constraint: the above mentioned homographic solution should be located on the line $\theta = 0$. This fixes the diagonalization.

In practice the computation of the constants $a, b, c, d$ proceeds as follows: If we use $\dot{q}_i$ instead of momenta, we may write

$$a_{11} \dot{q}_1 + a_{12} \dot{q}_2 = \sqrt{T} \sin(\theta), \quad a_{21} \dot{q}_1 + a_{22} \dot{q}_2 = \sqrt{T} \cos(\theta)$$  

(11)

and choose the constants $a_{ij}$ such that for $\theta = 0$ we have $\dot{q}_1/\dot{q}_2 = q_1/q_2 = \tau_h$. When we also require that the kinetic energy is independent of $\theta$, a straightforward calculation gives the following set of formulae from which $a_{ij}$ can be determined:
\[ r_1 = 1 + z, \quad r_2 = 1 - z \]
\[ A = \frac{\frac{1}{2} m_1 (m_0 + m_2)}{m_1 + m_0 + m_2}, \quad B = \frac{\frac{1}{2} m_2 (m_0 + m_1)}{m_1 + m_0 + m_2}, \quad C = \frac{m_1 m_2}{m_1 + m_0 + m_2} \]
\[ c_1 = \frac{1}{4} (4AB - C^2)/(Ar_1^2 + Br_2^2 + Cr_1r_2) \]
\[ a_{11} = + r_2 \sqrt{c_1}, \quad a_{12} = - r_1 \sqrt{c_1}, \quad a_{21} = \sqrt{A - c_1 r_2^2}, \quad a_{22} = \sqrt{B - c_1 r_1^2}. \]  

(12)

Since the derivatives \( \dot{q}_i \) and the momenta \( p_i \) are related by
\[ p_1 = 2A \dot{q}_1 + C \dot{q}_2, \quad p_2 = 2B \dot{q}_2 + C \dot{q}_1, \]
the constants \( a, b, c, d \) are finally determined by comparing (11) and (10).

If \( m_1 = m_2 (=: m_u) \) the formulae simplify considerably and we have
\[ \tau_h = 1, \quad R = q_1 |_{\tau=1}, \]
\[ \frac{1}{2} \sqrt{\frac{1}{m_u} + \frac{1}{m_0}} (p_1 - p_2) = \sqrt{T} \sin(\theta), \quad \frac{1}{2} \sqrt{\frac{1}{m_u} (p_1 + p_2)} = \sqrt{T} \cos(\theta), \]

(14)

and if \( E < 0 \) we have also \( 0 < R < \frac{5}{2m_0} \). For this mass relation it is sufficient to take \( 0 < \theta < \pi \), since \( \theta \to 2\pi - \theta \) implies \( 1 \leftrightarrow 2 \), which is a symmetry on the Poincaré section. Furthermore, if we change \( \theta \to \pi - \theta \) we get the same trajectory with momenta reversed and indices changed \( 1 \leftrightarrow 2 \). Since \( q_1 = q_2 \) on the section these \( \theta \)-values do in fact correspond to the future and past of the same trajectory. For other mass values these symmetries do not exist and we must compute twice as many trajectories.

In summary, by using the scaling invariances of the Hamiltonian we can fix the energy and the sum of masses, which leaves two essential constants in the Hamiltonian. By the canonical change of coordinates (2,3) we can eliminate the center of mass motion and reduce the system to a two dimensional one (4). The Poincaré section is then defined by the values of \( R \) and \( \theta \) at \( \tau = \tau_h \) as defined in (5-8,10-14).

C. The numerical method

The Hamiltonian (4) is regularized by the Aarseth-Zare [16] method. The one-dimensional form of this method consists simply of the point transformation \( q_i = Q_i^2 \), which gives the new canonical momenta \( P_i = 2Q_i q_i \). Substituting this together with the time transformation \( dt/ds = q_i q_2 \), where \( s \) is a new independent variable (note that \( dt/ds \geq 0 \)), and applying Poincaré’s transformation we have the regularized Hamiltonian \( \Gamma = q_1 q_2 (H - E) \) in the form
\[ \Gamma = \frac{1}{8} \left\{ \left( \frac{1}{m_1} + \frac{1}{m_0} \right) P_1^2 Q_2^2 + \left( \frac{1}{m_2} + \frac{1}{m_0} \right) P_2^2 Q_1^2 - \frac{2}{m_0} P_1 P_2 Q_1 Q_2 \right\} - m_0 m_2 Q_1^2 - m_0 m_1 Q_2^2 - m_1 m_2 \frac{Q_1^2 Q_2^2}{Q_1^2 + Q_2^2} - Q_1^2 Q_2^2 E. \]  

(15)

Here \( E \) is the constant numerical value of the energy, calculated from initial values.

The equations of motion derived from (15) are \( Q_1' = \partial \Gamma / \partial P_1, \quad P_1' = -\partial \Gamma / \partial Q_1, \quad t' = Q_2^2 \), where differentiation with respect to the new independent variable \( s \) has been denoted by
a prime and an equation for the time has been added. With these equations the numerical integration of pairwise collisions is no more difficult than that of the harmonic oscillator (into which equations of motion of the colliding pair reduce asymptotically).

The initial values in our calculations are chosen on the surface of section discussed in Sec. 2.2. When we integrated numerically the evolution of a particular orbit we checked for the changes of the sign of the quantity $q_1 - \tau h q_2$ and used it to determine accurately the point on the surface of section.

The calculations were done using the Bulirsch-Stoer method [19], which uses rational function extrapolation to zero steplength from results obtained by the midpoint rule with several different (sub-)steps. The method estimates its error from the various extrapolation outcomes and adjusts the stepsize by comparing the error estimates with a specified tolerance for the one-step error. The tolerance we used for relative precision was $10^{-12}$. The value of the quantity $\delta E/L$, where $\delta E$ is the error in energy and $L$ is the Lagrangian (sum of the absolute values of kinetic and potential energy), was less than $10^{-12}$ for two thirds of the computed orbits, while the maximal values were between $10^{-9}$ and $10^{-10}$. The fraction of cases in this worst interval was less than one out of 2000. Thus we conclude that a high precision was ensured by the used error tolerance.

### III. TYPES OF MOTION

In this section we first discuss the types of motion at a general level and then survey a selection of typical three-body orbits.

#### A. General taxonomy

For a first rough classification of orbit types of the rectilinear three-body systems one can distinguish between those that stay bound and those that break up. If the total energy is negative the breakup result is always a bound binary and a single unbound particle. A connection between the past and future was provided by Hopf [20], who showed that the system actually stays bound forever or breaks up in both directions of time, with the exception of a set of measure zero. (Since the exceptional set has measure zero it cannot appear in a numerical study, even though it is not empty [21].) In a system with positive total energy also a total breakup into three unbounded particles is possible (but not necessary). In a more detailed classification one must separate the ‘bound forever’ set into periodic and quasi-periodic orbits, and the ‘breakup’ set into fast scattering (often called ‘non resonant’ scattering [4]) and long interplay (‘resonant interactions’).

Among the bounded orbits there are periodic orbits, of which a special one is known as the Schubart orbit. This orbit is the periodic orbit which has the shortest period and it is thus the simplest periodic orbit in the system under consideration. It was found numerically by Schubart [10] in 1956 for the case of equal masses. Later the corresponding orbits for other masses were studied by Hénon [11] and by us (Paper III). If the Schubart orbit is linearly stable (stability is discussed further in Sec. 5) then it is usually (in the absence of destructive resonances) surrounded by a finite region of quasi-periodic orbits. This follows from the KAM-theory.
Since the system is two dimensional the quasiperiodic motion has two basic frequencies. When the ratio of these frequencies is a rational number the motions resonate. We will later see that especially the 1:2 and 1:3 resonances have strong influence on the dynamics.

If the orbit is not bounded for all times the triplet will break up in the past and in the future. The initial configuration of such an orbit can be characterized by the label of the free particle and by the binding energy of the other two particles, similarly for the final configuration. We can have either an exchange interaction (where the free particle is different) or backscattering (where the free particle is the same one in both directions of time). Furthermore, in the latter case the free particle may be the lighter or the heavier one of the outside particles. Interesting features will be shown to be related to this classification.

In addition to characterizing the initial and final state constituents a scattering system can be characterized by the ‘dwell time’ (also called ‘interplay time’), which is defined as the time during which all the three particles stay ‘close together’. To make this vague term more precise we need a working definition for the moments of breakup in the future and in the past and then define dwell time as their difference. In Paper I we defined the moment of breakup as the formal pericenter time for the asymptotic two-body orbit of the escaping (entering) particle with respect to the center of mass of the remaining binary. Another possible definition would be the time of last (first) crossing of the Poincaré section. In practice these definitions work much the same way and their numerical values are close to each other.

As will be shown later, the scattering orbits can be classified into fast scattering and chaotic scattering. For fast scattering the dwell-time defined by difference of pericenter times is nearly zero and sometimes even negative, while the difference of last and first crossing of Poincaré section is always a small positive number. We will often refer to the fast scattering orbits as ‘orbits of zero interaction time’. For fast scattering the dwell time and the initial and final states depend smoothly on the initial point. For chaotic orbits the dwell time can be arbitrarily large, and it as well as the final state depend sensitively on the starting point on the Poincaré section.

One can also arrive at essentially the same classification to fast and chaotic scattering by counting the number of Poincare map points produced by the orbit. Each full orbit has a first and a last Poincaré map point, which may be called the entry point and the exit point. As was shown in Paper I, these entry and exit points are located in clearly defined regions of the Poincaré section. For each mass configuration there is a minimum number of crossings, which is larger for smaller center mass. For chaotic orbits the number of sections is characteristically much larger than the minimum and it depends sensitively on the starting point.

We will show later that the different types of orbits, fast scattering, chaotic scattering, and quasi-periodic, are located in separate regions in phase-space (and on the Poincare map).

B. Typical trajectories

To get the first impression of the different orbit types we look now in more detail at a selection of individual orbits given in Fig. 3. The orbits are arranged in increasing dwell time. With the exception of the quasiperiodic orbits g) and h) these illustrations are complete in
the sense that at both ends the final states are shown and the parts will just fly apart without any further threebody interaction.

In Fig. 3a we have an orbit with three isolated particles at both \( t = -\infty \) and \( t = +\infty \). Here the energy is positive (as it always is for systems which break up into three unbounded particles) and the individual trajectories are almost straight lines outside the interaction region. During interaction there is some energy transfer. The trajectories seem to be translated back somewhat, indicating a small negative dwell time, which is natural for an attractive interaction.

Figs. 3b and c illustrate fast scattering with a binary and a single particle at both \( t = -\infty \) and \( t = +\infty \). This is the typical situation for fast scattering at negative energy but is also possible for positive energy. In Fig. 3b the single particle is different at \(-\infty\) and \(+\infty\), in Fig. 3c it is the same. In both cases some energy transfer takes place, as can be seen from the changing oscillation time of the binary. [For positive energy it is also possible to have a binary + single particle in one direction of time and three separated particles in the other, c.f. paper II for further illustrations.]

Figs. 3d-f show typical chaotic orbits for various mass choices. The sensitivity to initial values in Fig. 3d is particularly illuminating: When the single particle makes a long detour and thereafter again interacts with the binary the outcome of that interaction depends on the relative phase of the binary oscillation. The phase in turn depends sensitively on how long the detour took, i.e. how fast the binary and the single particle started to separate. In Fig. 3e the three body system breaks up after an almost three-body collision resulting with a very tight binary plus an equally energetic single particle. This outcome would change drastically with even a small change in the starting configuration.

Finally in Figs. 3g and h we have typical quasiperiodic orbits. In Fig. 3g the masses are equal. The orbits seems to be near a 1:2 resonance, at which the Schubart orbit becomes unstable (see Sec. 5). The configuration in Fig. 3h seems to be close to a 1:5 resonance.

**IV. DWELL TIME CHARTS**

In order to get a comprehensive overall picture of the possible motions it is not sufficient to look at particular orbits. In Papers I and II, where we studied the equal mass case in detail, we computed the orbits for a rather dense lattice of starting points on the \((R, \theta)\)-plane. Here we use a \(100 \times 360\) linear lattice for the initial values on the Poincaré section

\[
R_\nu = (\nu - \frac{1}{2}) \times 0.025, \quad \nu = 1, \ldots, 100; \quad \theta_\mu = (\mu - \frac{1}{2}) \times 1^\circ, \quad \mu = 1, \ldots, 360.
\]

[If the outside masses are equal (papers I and II) there is a reflection symmetry (see Sec. 2.3) and only the range \( \mu = 1, \ldots, 180 \) needs to be studied.] We calculated the orbits starting at these points \((R_\nu, \theta_\mu)\) until the final type of motion was evident.

Now with 36000 orbits calculated one must worry about data presentation. In I and II we looked at how the trajectory type and the dwell time depended on the initial values. The data was presented by drawing a box around the initial point on the Poincaré section and coloring it with a shade of gray according to how the orbit behaved. Here we use the same method and first recall the dwell time data for the equal mass case of Paper I and for comparison present similar new data for a nearby mass configuration.
Fig. 4a shows the dwell time for each of the initial values (16) [Energy has been scaled to $-1$]. The darker the small square is the longer it takes for the orbit starting at the center of the square to break up (since energy is negative the break up result can only be a single particle + a binary). The figure is symmetric across 180°, because $m_1 = m_2$. The ‘one directional’ dwell time shown in Fig. 4a means the time starting from an initial value (16) until the system breaks up. The two directional dwell time of Fig. 4b is the sum of the two dwell times calculated from the same starting point into both directions of time, i.e. this is a total time for the triple interplay for the orbit that at some time passes through the center of the square. This figure is symmetric across 90°.

The most prominent feature of Figs. 4 is the way the Poincaré section is divided into three well defined regions:

1) Around $R = 0.8, \theta = 90°, 270°$ there is a black region where the orbits are quasiperiodic and the three-body system stays together forever. We call this the Schubart region, because at its center there is the periodic Schubart orbit. The discretization does not reveal the nature of the region’s boundary, i.e. whether it is fractal of smooth. We have studied this in more detail but did not find any signs of fractality. [The situation is probably different for other mass values.]

2) The Schubart region is surrounded by a grayish area that extends to the boundaries of the section. This region seems to have no apparent structure, i.e. there no correlation between the dwell times of neighboring initial values. We call this region chaotic, because the trajectories depend sensitively on the initial values, which is one signature of chaos. The origin of this chaotic behavior was discussed in connection with Fig. 3d.

The statistical distribution of dwell times (paper I, Fig. 9) shows an excess of long time orbits when compared to the usually obtained exponential decay [1]. This is due to the long range interaction, which makes arbitrarily long detours quite common; such detours are completely absent if the interaction has finite range.

3) The third region consists of the white ‘scallops’ at the lower part of the figure. When the initial values are chosen from this region the system breaks up with at most 2 further collisions. The leftmost scallop turns out to be the exit region, i.e. if the initial values are chosen from it the system breaks up with no further crossing of the Poincaré section. [The shape of the exit region can be approximated analytically [22].]

Even for a chaotic scattering orbit it is necessary that its last point in the Poincaré section is also in the leftmost scallop, which is the exit region discussed above. [This is because the exit point has zero dwell time in the future and long dwell time in the past.] Indeed, in the folded Fig. 4b there is a gray spike inside the leftmost scallop. This is where the chaotic orbits pierce the Poincaré section the last time before they break up, the rest is for orbits whose total interaction time is practically zero.

In Fig. 5 we have charted the dwell time in the same way for the mass choice (0.9, 1, 1.1). The forward orbit chart in Fig. 5a is no more symmetric and neither is the full orbit chart Fig. 5b. The masses are almost equal but the mass configuration is nevertheless such that the Schubart orbit is unstable (see next section). Indeed, there is no black Schubart region left. Its remnants show up as darker gray, indicating that the orbits there do break up, but not so fast. These changes look rather drastic given that the masses do not differ much from the equal mass case. In Fig. 5a the chaotic region is still for the most part without structure, but there also seem to be some continuous lighter regions. The fast scattering scallops are
still there, but clearly modified from the case of equal masses.

In papers I and II we used the same kind of charting technique to show the type of the initial and final state (Fig. 2b in paper I, and Figs. 1-3 in paper II) and their binding energy (Fig. 3 in I). For negative energy all but the quasiperiodic orbits have initial and final states consisting of a single particle and a binary. The orbits were then classified according to whether it was particle 1 or 2 that was the single particle. In the negative energy region the figures show a structure similar to Fig. 4: In the chaotic region the final state depends sensitively on the initial conditions, while within each scalloped fast scattering region it does not change.

Let us also discuss here how the transition to chaos takes place. First consider the case when the initial point moves from the fast scattering region to the chaotic scattering region. As we move inside a scallop towards its border the asymptotic speed of the escaping particle changes smoothly and approaches zero when the initial point approaches the boundary. Thus at the boundary we have an (asymptotically) parabolic escape. Just at the other side of the boundary the escape is replaced by an ‘ejection without escape’, i.e. the nearly escaping particle completes a very long elliptic orbit before returning to strong interaction with the other particles. While this ejected particle is making its journey the binary completes a large number of periods and the nature of the subsequent threebody interaction depends sensitively on the escape velocity. This implies in fact that on the chaotic side of the line of parabolic escape there is a clustering of singular triple collisions.

The transition from quasiperiodic bounded system to chaotic scattering is different in nature. If we start from a quasi-periodic motion and move the initial point towards the boundary of the Schubart region we arrive at a ‘broken’ KAM-torus and eventually the particle finds its way out of the quasi-periodic behavior. Near the border some quasi-periodic looking behavior is still observable. For some mass values the Poincaré sections discussed later show unmistakably the cross sections of broken tori around the Schubart orbit.

V. THE MASS TRIANGLE AND THE STABILITY OF THE SCHUBART ORBIT

In the following sections we will look more closely on how the previously observed characteristics change when the masses are arbitrary. We will see that the three basic regions identified above will for the most part remain, although in a distorted form, but for some mass values the Schubart region vanishes completely. This is determined by the stability of the Schubart orbit, which we will discuss next.

Because of scaling invariance we can normalize the masses so that

\[ m_1 + m_0 + m_2 = 3, \]  

and then parametrize the mass plane by \( a \) and \( b \) as follows:

\[ m_1 = 1 - a - b, \ m_0 = 1 + 2a, \ m_2 = 1 - a + b, \]  

where \(-\frac{1}{2} \leq a \leq 1, \ a - 1 \leq b \leq 1 - a.\)

The mass triangle is given in Fig. It. The equal mass case discussed in Papers I and II is marked with a cross. For computations (Paper III) we discretized the mass triangle by
\[ a = 0.01 \cdot I_a, \quad b = 0.01 \cdot I_b, \quad I_i \text{ integers.} \quad (19) \]

For compatibility with our earlier published work we use this same notation here, although we mostly discuss the case were the mass indices \( I_i \) are multiples of ten.

One of the most prominent features in the equal mass case was the Schubart region. As the masses change we would expect this region to change and perhaps even vanish. [A preview of this is provided by comparing Figs. 4 and 5.] At the center of the Schubart region is the periodic Schubart orbit, which presumably would be the last orbit to destabilize.

In Paper III we analyzed the Schubart orbits by finding numerically the periodic solution (it exists for all mass values) and calculating its stability. The stability is related to the eigenvalues of the transition matrix \( A \) of the Poincaré map; \( A \) gives the difference in the next Poincaré maps point as a function of the difference in the initial point: \( dY = AdY_0 \), where \( Y = (R, \theta) \). Since the system is Hamiltonian the matrix \( A \) is symplectic and the product of its eigenvalues is \( = 1 \). If the eigenvalues are real, one of them has magnitude \( > 1 \) and the orbit is unstable. If the eigenvalues are complex conjugates, they are both on the unit circle and the Schubart orbit is linearly stable. Furthermore, if the eigenvalues have the form \( \lambda = \exp(i\theta) \), with \( \theta = 2\pi/m \), \( m \) an integer, then we have a \( 1: m \) resonance in the two periodicities associated with the motion (in the immediate neighborhood of the Schubart orbit).

In Fig. 6 we have given as gray those regions for which the Schubart orbit is stable under perturbations that nevertheless keep the system rectilinear (longitudinal perturbation) [11, III]. (For a discussion of perturbations that allow the masses to move on a plane (transverse perturbations), see [11, III] .) It is interesting to note that the equal mass case is quite close to the fairly large unstable region.

In Fig. 6 we have also given one eigenvalue of the map \( A \) at selected points using a dial. As expected the region of instability is bounded by the 1:2 resonance where both eigenvalues reach the point \(-1\) on the real line (‘reading of the dial’ is 180°). Thus the destabilization is a typical one: the eigenvalues move from the unit circle through the point \(-1\) to the real axis. The figure shows nicely how in the stable region the eigenvalues turn smoothly as the masses change.

We will later show that the mass regions where the Schubart orbit has longitudinal stability correlate also with a certain characteristic of the final states of chaotic orbits.

VI. POINCARÉ SECTIONS FOR VARIOUS MASSES

To get a more detailed look at the changes due to mass variation we have computed the Poincaré sections for 105 mass values

\[ I_a = -40, -30, \ldots , 90, \quad I_b = 0, 10, \ldots , 90 - I_a. \quad (20) \]

In each case we used 180 initial values

\[ \theta = 5, 15, \ldots , 175, \quad R/R_{max} = 0.05, 0.15, \ldots , 0.95, \quad (21) \]

and integrated the trajectories in both directions of time (this is equivalent to computing trajectories forward only, but allowing \( \theta \) to range from \( 0^\circ \) to \( 360^\circ \)). [Note that for general mass values the many useful symmetries present in the equal mass case no longer exist.]
The 105 Poincaré sections are arranged in the form of the mass triangle as shown in Fig. 7. This shows the contents of the more detailed Figs. 8 a-g, but it does also give an overall picture of the basic features discussed in the previous two sections.

1) Let us start from the Schubart region. In general its size increases as the central mass decreases (going down in the mass triangle) and it gets tilted when the ratio of the outside masses change. In the mass region where the Schubart orbit is longitudinally unstable (c.f. Fig. 6) the whole Schubart region has vanished, as expected. There are also other mass values with vanishing Schubart regions. Some of them are at the 1:3 resonance which is located near $I_a = 40$ line illustrated in Figs. 8 e,f (the position of the 1:3 resonance depends weakly on $b$ as well). There is also another curve in the linear stability region where the Schubart region vanishes, it starts from the lower right hand corner and passes close to points $(I_a, I_b) = (-40, 110), (-30, 60)$ [Figs. 8 d,c, respectively.] This corresponds also to a 1:3 resonance. However, this latter curve of vanishing Schubart regions does not seem to extend all the way to $I_b = 0$. Fig. 9 shows a magnified Poincaré section for the mass parameters $(40, 0)$. There is a tiny Schubart region, and around it trajectories that diverge with period three.

2) The Schubart orbit is surrounded by the chaotic scattering region, even when the Schubart region vanishes (either by linear instability or the 1:3 resonance). The boundary between the chaotic scattering region and Schubart region (when the latter exists) is not always so clear as it was in the equal mass case. As is expected from general theory, the tori surrounding the Schubart orbit do sometimes break up and form islands in the sea of chaotic scattering. Rather large islands appear e.g. in $(-20, 40)$ [Fig. 8 b].

Let us look more closely how these islands appear as the mass parameter $I_b$ changes in the $I_a = -40$ region [bottom of Fig. 8 b,c]. A detailed illustration of this is provided in Fig. 10. In Fig. 10a ($I_b = 59$) the 9 islands are well inside a clear torus, then the islands seem to move away from the center and finally in Fig. 10c ($I_b = 53$) the islands are inside the chaotic sea. The same process takes place when we approach the above mentioned $(-20, 40)$ from $(-20, 30)$ or $(−30, 40)$ [Fig. 8 b].

The dark crescent in $(20, 60)$ [Fig. 8 g] is intriguing. It is located in the region of unstable Schubart orbit, but nevertheless there are some very long-lived orbits, whose fate we could not determine. It seems that the angle of homoclinic intersection is here close to zero. This region needs further investigation.

3) Finally, let us look at the scallops of fast scattering. The overall behavior is clear from Fig. 4. The scallops at the bottom of each Poincaré section get smaller and more numerous as the central mass decreases going down in Fig. 4, and when we go towards right (towards more lopsided mass configurations) the scallops increase in size and get more asymmetric.

The number of scallops indicates the minimum number of collisions needed in the fast region. This can be seen as follows: For $\theta$-values near $\pi$ the masses 1 and 2 are moving towards mass 0 ($\dot{q}_1 \sim \dot{q}_2 < 0$) and thus the rightmost scallop is the entry region. With each collision the speed of approach gets smaller and the corresponding points in the Poincaré section move left. Eventually the masses 1 and 2 start to recede and the exit takes place from the leftmost scallop where $\theta$ is near 0 ($\dot{q}_1 \sim \dot{q}_2 > 0$). The number of scallops increases as the center mass decreases, because when we have a smaller intermediary mass it must make more collisions with the other two in order to transfer enough momentum to make the approaching masses recede.
Between the scallops the spikes of chaotic scattering region reach the $R = 0$ line, as was observed before. What was not visible in the dwell-time charts is that the spikes seem to continue inside the chaotic region as slightly darker lines. These darker lines seem to reach from the $R = 0$ line the all the way to the corners of the Schubart region. This is clear e.g. for $I$ values $(-20, 40), (-20, 50)$ [Fig 8b], but can also be seen elsewhere. This suggest that the stability of the Schubart orbit has global influence, another manifestation of this is discussed next.

VII. DISTRIBUTION OF THE TRAJECTORY TYPES

From the previous it seems that the Schubart region is a dominating feature of the system. At the end of the previous section we noted that there are faint lines that extend from the corners of the Schubart region all the way to the $R = 0$ line. In this section we present another puzzling observation about the global influence of the stability of the Schubart orbit.

In the Figs. 11 a-e we have plotted the initial and final states types using the same method as in Figs. 4 and 5. That is, at the starting point of an orbit we have drawn a mark according to the fate of the orbit in the future and in the past. We have used only a $10 \times 18$ initial value grid for each mass configuration so that we could combine the individual charts as in Fig. 7. For each property we have constructed a different overview chart. In each figure we have also drawn borders around the region where the Schubart orbit is unstable under longitudinal perturbation, see Fig. 6. [In a very few cases the system was still evolving when the computations were stopped, these cases are omitted.]

Fig. 11a we have marked the quasiperiodic orbits with a small triangle while all other orbits were left blank. The region where the Schubart orbit is unstable is of course empty, except in the subchart $(I_a, I_b) = (20, 60)$, where an orbit survived as a bound system for more than 40000 Poincaré section crossings (at which point the computation was halted). This orbit is from the crescent shown in Fig. 8g. This particular mass triplet is very close to the border of the Schubart orbit instability region, but a careful analysis shows that it is inside. The row at $I_a = 40$, which is near 1:3 resonance, is also clearly visible as an empty region showing that the resonance has destroyed the KAM-tori entirely. On some subcharts there are also isolated points which corresponding to the separate islands; they are more clearly visible in the Fig. 8 (e.g. Fig. 8b $(-20, 40)$).

Fig. 11b shows the fast scattering orbits, all others were left blank. There seems to be no particularly interesting structure here. [For this figure we used a working definition by which the scattering was termed fast if it had at most 4 points on the Poincaré section. This is not a good definition since the minimum number depends on the center mass, as was discussed before, and causes the apparent absence of fast scattering at the bottom of the figure where the center mass is small.] In Fig. 11c we have plotted those chaotic scattering orbits for which the single particle is light in both past and future, there is no interesting structure here either. This interaction type is obviously the most common one. At the leftmost column the outside particles have equal masses and we have used particle 1 as the lighter one, as it is elsewhere.

However, when we plot the initial conditions leading to a heavier single particle, either in the past or in the future (Fig. 11d) or both (Fig. 11e) some interesting structure emerges:
We see that such scattering orbits seem to be curiously absent when the Schubart orbit is unstable. Thus the stability of the Schubart orbit seems to radiate its effect throughout the chaotic region, correlating strongly with the ability of the heavier particle to escape. We observed the same phenomenon also when we were calculating the results illustrated in Fig. 6. In this much denser grid the rule also worked without exception: In Fig. 6 the mass configuration was such that the Schubart orbit was unstable and for the chaotic scattering orbits we found indeed that the escaper was always the lighter body.

From this numerically observed rule we can also get the following predictive forms (in each case we must of course assume that the mass configuration is such that the Schubart orbit is unstable and the total energy of the system is negative): 1) When a binary and a single particle collide the scattering will take place in minimum time if the incoming single particle is the heavier one of the outer particles. 2) If the scattering process looks chaotic the lighter one of the outside particles will eventually be ejected.

This numerically observed correlation with the stability of the Schubart orbit and the initial and final state type is still waiting for an analytical explanation.

VIII. CONCLUSIONS

In this paper we have presented data characterizing the behavior of the rectilinear gravitational three-body motion. This, possibly the simplest nonintegrable three-body case, nevertheless shows a rich variety of phenomena which is not yet thoroughly understood.

Scattering systems have been studied actively in recent years, but the usual three-disk case is quite different from the system studied here and as a consequence shows different behavior. This is not unexpected, since when the present system is looked as a 2-dimensional system it is described by a particle moving in the field of three attracting walls (Fig. 2), furthermore the interaction is long-range.

Perhaps the most characteristic feature of the present system is the division of the Poincaré section into three regions: The quasiperiodic Schubart region where the system stays bounded for all times, the chaotic scattering region with finite nonzero dwell times that depend sensitively on the initial values, and the ‘scalloped’ fast scattering region. The three regions can be seen clearly in the dwell time figures, in the Poincaré sections, and in the final state distributions.

An interesting finding is the way the Schubart region influences the global picture. In Figs. 8 we saw the darker lines extending from the bottom of the Poincare section to the corners of the Schubart region. In Fig. 11 we saw how certain types of scattering processes are completely absent if the periodic Schubart orbit is unstable. It is of great interest to find an analytical explanation for these numerical observations.

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FIGURES

FIG. 1. The mass configuration. The three masses, labeled $i = 1, 0, 2$ from left to right (masses $m_i$ at positions $x_i$) attract each other with the Newtonian force $m_im_j|x_i - x_j|^{-2}$.

FIG. 2. A typical motion in the $(q_1, q_2)$-plane. The motion is restricted to the first quadrant. In this representation the coordinate axes and the line $q_1 + q_2 = 0$ are attracting walls. The line $q_1 = q_2$ corresponds to the Poincaré section. The plotted trajectory represents a fast exchange scattering process with equal masses. The arrows show the direction of motion (reverse motion is also possible).

FIG. 3. Typical examples of the basic orbit types. The coordinates of the particles (vertical axis) are plotted as functions of time (horizontal axis). The lower curve corresponds to the motion of the particle number 1 and the upper curve to particle 2. The orbit of the middle body has been plotted using dashed curve. a) A high energy orbit. $[E = 10, R = 1, \theta = 10^\circ, m = (0.9, 1, 1.1)]$. b) A fast scattering orbit. This is an exchange interaction in which the free particle is different at $t \to -\infty$ and $t \to \infty$. $[E = 0, R = 1, \theta = 45^\circ, m = (0.9, 1, 1.1)]$. c) A fast back-scattering orbit. The free particle is the same at both directions of time. $[E = 0, R = 1, \theta = 10^\circ, m = (0.9, 1, 1.1)]$. d) A chaotic orbit with almost equal masses. $[E = -1, R = 1, \theta = 10^o, m = (0.9, 1, 1.1)]$. e) A chaotic orbit with a large central mass. $[E = -1.1766, R = 1, \theta = 90^\circ, m = (0.2, 2.4, 0.4)]$. f) A chaotic orbit with a lopsided mass choice. $[E = -1.7456, R = 1, \theta = 90^\circ, m = (0.4, 1, 1.6)]$. g) A quasi-periodic orbit in the equal-mass case. $[E = -2.000, R = 1, \theta = 75^\circ, m = (1, 1, 1)]$. h) A quasi-periodic orbit with a small middle mass. $[E = -1.2138, R = 1, \theta = 45^\circ, m = (1.2, 0.2, 1.6)]$.

FIG. 4. Map of the dwell times for the equal mass system as a function of initial values, a) one directional dwell times, b) total dwell times for the full orbit. The shade of gray corresponds to the length of the triple interaction for the orbit which started at the center of each box. In both illustration the integer part of $\ln(1+ t)$ is used to determine the shade of gray: we used white when the integer part of $\ln(1+ t) = 0$, and black when it was $\geq 10$. Note the three different regions: black quasiperiodic region around the Schubart orbit, the grayish chaotic region, and the white fast scattering region.

FIG. 5. Map of the dwell times for the mass choice $(0.9, 1, 1.1)$, as a function of initial values, a) one directional dwell times. b) two directional dwell times. The grayness scale is the same as in the previous figure. Note that the Schubart region has disappeared although some traces are still visible as a darker distorted area.

FIG. 6. The mass triangle. The mass indices $I_a$, $I_b$ are, as introduced in the text, 100-fold values of the mass-triangle coordinates $a$, $b$. In the gray region the Schubart orbit is linearly stable. A dial is placed at certain values of the mass indices, it shows one of the eigenvalues of the Poincaré map at the Schubart orbit of that particular mass configuration. If the eigenvalue has unit magnitude we plot the one with positive imaginary part with a dial ending with a black dot. This happens in the grey stability region. When the eigenvalues are on the real line we plot the one with the smaller magnitude with a cross. This happens in the white instability region.
FIG. 7. A global overview of the Poincaré maps for various mass choices. This also indicates how the various parts of the next figure are to be combined.

FIG. 8. Poincaré maps for various mass values. For each map we used a $10 \times 18$ lattice of initial values. The indices (e.g. $(-10, 0)$) in the lower left corner of each figure indicate the mass-indices $(I_a, I_b)$.

FIG. 9. A detail of the Poincaré section for $m = (0.6, 1.8, 0.6)$ near the 1:3 resonance of the Poincaré map. The initial values were chosen from a box around the point $\theta = 90^\circ$, $R = 0.82$.

FIG. 10. A detailed look on the behavior of the islands as the masses change, the masses are $(0.81, 0.2, 1.99)$ for a), $(0.84, 0.2, 1.96)$ for b), and $(0.87, 0.2, 1.93)$ for c).

FIG. 11. Characterization of orbits according to the type of motion. a) Quasi-periodic: The initial value points in the $10 \times 18$ $(R, \theta)$-grid that gave rise to quasi-periodic motion. The region where the Schubart orbit is unstable is empty, as is the row at $I_a = 40$ near the 1:3 resonance. b) Fast: Here we have marked fast trajectories, which were here defined as having at most 4 points on the surface of section for the entire orbit from $-\infty \leq t \leq +\infty$. c) Light→Light: Here we plot the starting points of those orbits which were classified as chaotic and for which free particle is the lighter one of the two possible escapers (=outside particles) in both direction of time. d) Heavy→Light & Light-Heavy: This is the exchange interaction. Observe the complete absence of this orbit type in the region where the Schubart orbit is linearly unstable. e) Heavy→Heavy: One notes that transitions from heavy free particle back to the same heavy free particle are in general rather rare and completely impossible when the Schubart orbit is unstable.