SEVERAL SERIES EXPANSIONS FOR REAL POWERS AND
SEVERAL CLOSED-FORM FORMULAS FOR PARTIAL BELL
POLYNOMIALS WITH RELATION TO THE SINC AND SINHC
FUNCTIONS IN TERMS OF CENTRAL FACTORIAL NUMBERS
AND STIRLING NUMBERS OF THE SECOND KIND

FENG QI

Abstract. In the paper, with the help of the Faá di Bruno formula, in terms of central factorial numbers of the second kind, and with the terminology of the Stirling numbers of the second kind, the author derives several series expansions for any positive integer powers of the sinc and sinhc functions, discovers several closed-form formulas for partial Bell polynomials with relation to derivatives of the sinc function, establishes several series expansions for any real powers of the sinc and sinhc functions, and presents several identities for central factorial numbers of the second kind and for the Stirling numbers of the second kind.

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1. Motivations

According to common knowledge in complex analysis, the principal value of the number $\alpha^\beta$ for $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ is defined by $\alpha^\beta = e^{\beta \ln \alpha}$, where $\ln \alpha = $
\[ \ln|\alpha| + \text{i} \arg \alpha \text{ and } \arg \alpha \text{ are principal values of the logarithm and argument of } \alpha \neq 0 \text{ respectively. In what follows, we always consider principal values of real or complex functions discussed in this paper.} \]

In mathematical sciences, one usually and most possibly consider elementary functions
\[ e^z, \ln(1 + z), \sin z, \csc z, \cos z, \sec z, \tan z, \cot z, \arcsin z, \arccos z, \arctan z, \sinh z, \csc h z, \cosh z, \sech z, \tanh z, \coth z; \]
and their series expansions at the point \( z = 0 \). Their series expansions can be found in mathematical handbooks such as [1, 14, 25].

What are series expansions at \( x = 0 \) of positive integer powers or real powers of these functions?

It is combinatorial knowledge [10, 13] that coefficients of the series expansion of the power function \((e^z - 1)^k\) for \( k \in \mathbb{N} = \{1, 2, \ldots \} \) are the Stirling numbers of the second kind, while coefficients of the series expansion of the power function \([\ln(1 + z)]^k\) for \( k \in \mathbb{N} \) are the Stirling numbers of the first kind. In other words, the power functions \((e^z - 1)^k\) and \([\ln(1 + z)]^k\) for \( k \in \mathbb{N} \) are generating functions of the Stirling numbers of the first and second kinds.

In the paper [9], among other things, Carlitz introduced the notion of weighted Stirling numbers of the second kind \( R(n, k, r) \). Carlitz also proved in [9] that the numbers \( R(n, k, r) \) can be generated by
\[ \frac{(e^z - 1)^k}{k!} e^{rz} = \sum_{n=k}^{\infty} R(n, k, r) \frac{z^n}{n!} \]  
and can be explicitly expressed by
\[ R(n, k, r) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (r+j)^n \]  
for \( r \in \mathbb{R} \) and \( n \geq k \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \). Specially, when \( r = 0 \), the quantities \( R(n, k, 0) \) become the Stirling numbers of the second kind \( S(n, k) \). By the way, the notion
\[ \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r = R(n - r, k - r, r) \]
is called the \( r \)-Stirling numbers of the second kind in [6] by Broder.

The central factorial numbers of the second kind \( T(n, \ell) \) for \( n \geq \ell \in \mathbb{N}_0 \) can be generated [8, 21] by
\[ \frac{1}{\ell!} \left( \frac{2 \sinh \frac{z}{2}}{2} \right)^\ell = \sum_{n=\ell}^{\infty} T(n, \ell) \frac{z^n}{n!}. \]  
In [37, Chapter 6, Eq. (26)], it was established that
\[ T(n, \ell) = \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \left( \frac{\ell}{2} - j \right)^n. \]  
Note that \( T(0, 0) = 1 \) and \( T(n, 0) = 0 \) for \( n \in \mathbb{N} \). See also [8, Proposition 2.4, (xii)] and [31, 35]. Comparing (1.3) with (1.1) or comparing (1.4) with (1.2) figures out
the relation

\[ R(n, \ell, -\frac{\ell}{2}) = T(n, \ell) \]

between weighted Stirling numbers of the second kind and central factorial numbers of the second kind. See also \cite[Theorem 3.1]{35}.

In the handbook \cite{14}, series expansions at \( z = 0 \) of the functions \( \arcsin^2 z \), \( \arcsin^3 z \), \( \sin^2 z \), \( \cos^2 z \), \( \sin^3 z \), and \( \cos^3 z \) are collected.

In the papers \cite{5, 15, 16, 22, 30, 34} and plenty of references collected therein, the series expansions at \( z = 0 \) of the functions \( \arcsin^m z \), \( \arcsinh^m z \), \( \arctan^m z \), \( \arctanh^m z \) for \( m \in \mathbb{N} \) have been established, applied, reviewed, and surveyed.

In the papers \cite{7, 26}, explicit series expansions at \( z = 0 \) of the functions \( \tan^2 z \), \( \cot^2 z \), \( \cot^3 z \), \( \sin^m z \), \( \cos^m z \) for \( m \in \mathbb{N} \) were written down.

In the papers \cite{2, 3, 18, 19, 23, 39, 40}, series expansions of the functions \( \arcsin z \), \( \arcsinh z \), \( \arctan z \), \( \arctanh z \) were explicitly written out, while the series expansion of the power function \( [I_r(z)]^r \) for \( r \in \mathbb{C} \setminus \{-1, -2, \ldots\} \) and \( r, z \in \mathbb{C} \) was recursively formulated, where \( I_r(z) \) denotes modified Bessel functions of the first kind.

In the paper \cite{27}, series expansions at \( z = 0 \) of the functions \( (\arccos z)^r \) and \( (\arcsin z)^r \) were established for real \( r \in \mathbb{R} \). In \cite{30}, a series expansion at \( z = 1 \) of the function \( \left(\frac{(\arccos z)^2}{2(1-z)}\right)^r \) was invented for real \( r \in \mathbb{R} \).

For \( z \in \mathbb{C} \), the functions

\[
\text{sinc } z = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}
\]

and

\[
\text{sinhc } z = \begin{cases} \frac{\sinh z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}
\]

are called the sinc function and hyperbolic sinc function respectively. The function \( \text{sinc } z \) is also called the sine cardinal or sampling function, as well as the function \( \text{sinhc } z \) is also called hyperbolic sine cardinal, see \cite{38}. The sinc function \( \text{sinc } z \) arises frequently in signal processing, the theory of the Fourier transforms, and other areas in mathematics, physics, and engineering. It is easy to see that these two functions \( \text{sinc } z \) and \( \text{sinhc } z \) are analytic on \( \mathbb{C} \), that is, they are entire functions.

In \cite[Theorem 11.4]{10} and \cite[p. 139, Theorem C]{13}, the Faà di Bruno formula is given for \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \) by

\[
\frac{d^n}{dz^n} f \circ h(z) = \sum_{k=1}^{n} f^{(k)}(h(z)) B_{n,k}(h'(z), h''(z), \ldots, h^{(n-k+1)}(z)), \quad (1.5)
\]

where partial Bell polynomials \( B_{n,k} \) are defined for \( n \geq k \in \mathbb{N}_0 \) by

\[
B_{n,k}(z_1, z_2, \ldots, z_{n-k+1}) = \sum_{\sum_{i=1}^{k-1} \ell_i = n-k+1, \sum_{i=1}^{k} \ell_i = n} \frac{n!}{\prod_{i=1}^{k-1} \ell_i! \cdot \prod_{i=1}^{k} (z_i)_{\ell_i}}
\]

in \cite[Definition 11.2]{10} and \cite[p. 134, Theorem A]{13}.

In this paper, with the help of the Faà di Bruno formula (1.5), in terms of central factorial numbers of the second kind \( T(n, k) \), and with the terminology...
of the Stirling numbers of the second kind $S(n, k)$, we will derive several series expansions at $z = 0$ of the positive integer power functions $\text{sinc}^\ell z$ and $\text{sinhc}^\ell z$ for $\ell \in \mathbb{N}$ and $z \in \mathbb{C}$, we will deduce several closed-form formulas for central factorial numbers of the second kind $T(j + \ell, \ell)$ with $j, \ell \in \mathbb{N}$ in terms of the Stirling numbers of the second kind $S(n, k)$, we will discover several closed-form formulas of specific partial Bell polynomials

$$B_{n,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, (-1)^{n-k} \frac{1}{n-k+2} \sin \left( \frac{(n-k)\pi}{2} \right) \right)$$

for $n \geq k \in \mathbb{N}$, we will establish series expansions at $z = 0$ of the real power functions $\text{sinc}^\ell z$ and $\text{sinhc}^\ell z$ for $z \in \mathbb{C}$ and $r \in \mathbb{R}$, and we will present several identities for central factorial numbers of the second kind $T(n, k)$ and for the Stirling numbers of the second kind $S(n, k)$.

2. Several series expansions of positive integer powers

In this section, we derive several series expansions at $z = 0$ of the positive integer power functions $\text{sinc}^\ell z$ and $\text{sinhc}^\ell z$ for $\ell \in \mathbb{N}$ and $z \in \mathbb{C}$ in terms of central factorial numbers of the second kind $T(n, k)$ and the Stirling numbers of the second kind $S(n, k)$, we deduce several closed-form formulas of $T(j + \ell, \ell)$ for $j \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ in terms of the Stirling numbers of the second kind $S(n, k)$, and we present several identities for central factorial numbers of the second kind $T(n, k)$.

**Theorem 2.1.** For $\ell \in \mathbb{N}_0$ and $z \in \mathbb{C}$, we have

$$\text{sinc}^\ell z = 1 + \sum_{j=1}^\infty (-1)^j \frac{2^j}{(2j)!} \frac{T(\ell + 2j, \ell) (2z)^{2j}}{(2j)!}. \quad (2.1)$$

**Proof.** For $\ell \in \mathbb{N}$, the formula

$$\text{sin}^\ell z = \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^\ell (-1)^q \left( \frac{\ell}{q} \right) \cos \left[ (2q - \ell)z - \frac{\ell\pi}{2} \right] \quad (2.2)$$

is given in [17, Corollary 2.1]. Applying the identity

$$\cos(z - y) = \cos z \cos y + \sin z \sin y$$

to the formula (2.2) leads to

$$\text{sin}^\ell z = \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^\ell (-1)^q \left( \frac{\ell}{q} \right) \left[ \cos((2q - \ell)z) \cos \frac{\ell\pi}{2} + \sin((2q - \ell)z) \sin \frac{\ell\pi}{2} \right]$$

$$= \frac{(-1)^\ell}{2^\ell} \cos \frac{\ell\pi}{2} \sum_{q=0}^\ell (-1)^q \left( \frac{\ell}{q} \right) \left[ 1 + \sum_{j=1}^\infty (-1)^j (2q - \ell)^{2j} \frac{z^{2j}}{(2j)!} \right]$$

$$+ \frac{(-1)^\ell}{2^\ell} \sin \frac{\ell\pi}{2} \sum_{q=0}^\ell (-1)^q \left( \frac{\ell}{q} \right) \sum_{j=0}^\infty (-1)^j (2q - \ell)^{2j+1} \frac{z^{2j+1}}{(2j+1)!}$$

$$= \frac{(-1)^\ell}{2^\ell} \cos \frac{\ell\pi}{2} \sum_{j=1}^\infty (-1)^j \left[ \sum_{q=0}^\ell (-1)^q \left( \frac{\ell}{q} \right) (2q - \ell)^{2j} \right] \frac{z^{2j}}{(2j)!}$$

$$+ \frac{(-1)^\ell}{2^\ell} \sin \frac{\ell\pi}{2} \sum_{j=0}^\infty (-1)^j \left[ \sum_{q=0}^\ell (-1)^q \left( \frac{\ell}{q} \right) (2q - \ell)^{2j+1} \right] \frac{z^{2j+1}}{(2j+1)!}.$$
Replacing \( \ell \) by \( 2\ell - 1 \) and by \( 2\ell \) and simplifying result in the series expansions
\[
\sin^{2\ell-1} z = \frac{(-1)^\ell}{2^{2\ell-1}} \sum_{j=\ell}^\infty (-1)^{j-1} \left[ \sum_{q=0}^{2\ell-1} (-1)^q \binom{2\ell-1}{q} (2q - 2\ell + 1)^{2j-1} \right] \frac{z^{2j-1}}{(2j-1)!}
\]
and
\[
\sin^{2\ell} z = \frac{(-1)^\ell}{2^{2\ell}} \sum_{j=\ell}^\infty (-1)^j 2^{2j} \left[ \sum_{q=0}^{2\ell} (-1)^q \binom{2\ell}{q} (q - \ell)^{2j} \right] \frac{z^{2j}}{(2j)!}.
\]
The series expansions (2.3) and (2.4) can be reformulated as
\[
\text{sinc}^{2\ell-1} z = \frac{1}{2^{2\ell-1}} \sum_{j=0}^\infty \frac{(-1)^{j-1}}{(2\ell + 2j - 1)!} \left[ \sum_{q=0}^{2\ell-1} (-1)^q \binom{2\ell-1}{q} (2q - 2\ell + 1)^{2j-1} \right] \frac{z^j}{2^j}
\]
and
\[
\text{sinc}^{2\ell} z = \frac{1}{2^{2\ell}} \sum_{j=0}^\infty \frac{(-1)^j}{(2\ell + 2j)!} \left[ \sum_{q=0}^{2\ell} (-1)^q \binom{2\ell}{q} (2q - 2\ell)^{2j+2} \right] \frac{z^j}{2^j}.
\]
for \( \ell \in \mathbb{N} \) and \( z \in \mathbb{C} \). These two series expansions can be unified and rearranged as the series expansion (2.1). Theorem 2.1 is thus proved.

**Corollary 2.1.** For \( \ell \in \mathbb{N} \), we have
\[
T(2j-1, 2\ell-1) = \begin{cases} 
0, & 1 \leq j \leq \ell-1 \\ 
1, & j = \ell
\end{cases}
\]
and
\[
T(2j, 2\ell) = \begin{cases} 
0, & 1 \leq j \leq \ell-1 \\ 
1, & j = \ell
\end{cases}
\]

**Proof.** This follows from Theorem 2.1 and its proof.

**Theorem 2.2.** For \( j, \ell \in \mathbb{N}_0 \), we have
\[
T(2j + \ell + 1, \ell) = 0. \tag{2.5}
\]

For \( \ell \in \mathbb{N}_0 \) and \( z \in \mathbb{C} \), the series expansions
\[
\text{sinhc}^\ell z = 1 + \sum_{j=1}^\infty \frac{T(2j + \ell, \ell) \, (2z)^{2j}}{(2j)!} \tag{2.6}
\]
and (2.1) are valid.

**Proof.** Replacing \( z \) by \( 2z \) in (1.3) and rearranging yield
\[
\left( \frac{\sinh z}{z} \right)^\ell = 1 + \sum_{n=1}^\infty \frac{T(n + \ell, \ell) \, (2z)^n}{n!}.
\]
Considering that the function \( \frac{\sinh z}{z} \) is even on \( \mathbb{R} \), we conclude that the identity (2.5) and the series (2.6) are valid.

Substituting \( z \) for \( z \) in (2.6) and employing the relation \( \sinh(z) = i \sin z \) give the series expansion (2.1) in Theorem 2.1.
**Theorem 2.3.** For \(j, \ell \in \mathbb{N}_0\) and \(z \in \mathbb{C}\), we have

\[
\sum_{k=0}^{2j+1} (-1)^k {2j+1 \choose k} \left(\frac{2}{\ell}\right)^k S(k + \ell, \ell) = 0 \tag{2.7}
\]

and

\[
\sin^\ell z = 1 + \sum_{j=1}^\infty (-1)^j \left[ \sum_{k=0}^{2j} (-1)^k {2j \choose k} \left(\frac{2}{\ell}\right)^k S(k + \ell, \ell) \right] (\ell z)^{2j} (2j)! \tag{2.8}
\]

**Proof.** Taking \(r = 0\) in (1.1) and reformulating give

\[
\left(\frac{e^{z} - 1}{z}\right)^{k} = \sum_{n=0}^{\infty} S(n + k, k) \frac{z^n}{(n+k)!} \tag{2.9}
\]

Since

\[
\sin z = \frac{e^{z}i - e^{-z}i}{2i} = \frac{e^{z}i - 1}{2i} e^{-z}i,
\]

by (2.9) and the Cauchy product of two series, we acquire

\[
\sin^\ell z = \left(\frac{\sin z}{z}\right)^\ell = \left(\frac{e^{z}i - 1}{2z i}\right)^\ell e^{-\ell z}i
\]

\[
= \left[ \sum_{n=0}^{\infty} \frac{S(n + \ell, \ell) (2z i)^n}{(n+\ell)!} \right] \left[ \sum_{n=0}^{\infty} \frac{(-\ell z)^n}{n!} \right] z^j
\]

\[
= \sum_{j=0}^{\infty} \left[ \frac{j!}{\ell^j} \sum_{k=0}^{2j} (-1)^k {2j \choose k} \left(\frac{2}{\ell}\right)^k S(k + \ell, \ell) \right] (\cos \frac{j\pi}{2} + i \sin \frac{j\pi}{2}) z^j
\]

\[
= \sum_{j=0}^{\infty} \left[ \frac{(2j)!}{j!} \sum_{k=0}^{2j-1} (-1)^k {2j-1 \choose k} \left(\frac{2}{\ell}\right)^k S(k + \ell, \ell) \right] z^{2j-1} + 1 \sum_{j=1}^{\infty} \frac{(2j-1)}{(2j)!} \left[ \sum_{k=0}^{2j-1} (-1)^k {2j-1 \choose k} \left(\frac{2}{\ell}\right)^k S(k + \ell, \ell) \right] z^{2j-1}
\]

\[
= \sum_{j=0}^{\infty} \left[ \frac{(2j)!}{j!} \sum_{k=0}^{2j} (-1)^k {2j \choose k} \left(\frac{2}{\ell}\right)^k S(k + \ell, \ell) \right] z^{2j}.
\]

The proof of Theorem 2.3 is complete. \(\square\)

**Corollary 2.2.** For \(j \in \mathbb{N}_0\) and \(\ell \in \mathbb{N}\), we have

\[
\frac{T(2j + \ell, \ell)}{(2j+\ell)!} = \sum_{m=0}^{2j} (-1)^m {2j \choose m} \left(\frac{\ell}{2}\right)^m \frac{S(2j + \ell - m, \ell)}{(2j+\ell-m)!} \tag{2.10}
\]

**Proof.** This follows from comparing the series expansion (2.1) in Theorem 2.1 with the series expansion (2.8) in Theorem 2.3 and simplifying. \(\square\)
Corollary 2.3. For \( j \in \mathbb{N}_0 \) and \( \ell \in \mathbb{N} \), we have
\[
\frac{T(j + \ell, \ell)}{(j+\ell)^m} = \sum_{m=0}^{\infty} (-1)^m \binom{j}{m} \binom{\ell}{m} S(j + \ell - m, \ell) \left( \frac{j}{2} \right)^m . \tag{2.11}
\]

Proof. This follows from combining the identities (2.5), (2.7), and (2.10). \( \square \)

3. Closed-form formulas for specific partial Bell polynomials

In this section, with the help of Theorem 2.1 and other results in the above section, we establish several closed-form formulas for specific partial Bell polynomials \( B_{n,k} \) with relation to all derivatives at \( z = 0 \) of the sinc function sinc \( z \).

Theorem 3.1. For \( n \geq k \geq 1 \) and \( m \in \mathbb{N} \), partial Bell polynomials \( B_{n,k} \) satisfy
\[
B_{2m-1,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{(-1)^m}{2m-k+1} \cos \frac{k\pi}{2} \right) = 0
\]
and
\[
B_{2m,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{(-1)^m}{2m-k+2} \sin \frac{k\pi}{2} \right)
= (-1)^{m+k} \frac{2^{2m}}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} T(2m+j, j) .
\]

Proof. From
\[
sinc z = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \frac{z^{2j}}{(2j)!} , \quad z \in \mathbb{C},
\]
it follows that
\[
(sinc z)^{(2j)} \big|_{z=0} = \left( \frac{(-1)^j}{2j+1} \right) \quad \text{and} \quad (sinc z)^{(2j-1)} \big|_{z=0} = 0 \tag{3.1}
\]
for \( j \in \mathbb{N} \).

On [13, p. 133], the identity
\[
\frac{1}{m!} \left( \sum_{\ell=1}^{\infty} \frac{z\ell \ell!}{\ell!} \right)^m = \sum_{n=m}^{\infty} B_{n,m} (z_1, z_2, \ldots, z_{n-m+1}) \frac{t^n}{n!} \tag{3.2}
\]
is given for \( m \in \mathbb{N}_0 \). The formula (3.2) implies that
\[
B_{n+k,k} (z_1, z_2, \ldots, z_{n+1}) = \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \sum_{\ell=0}^{\infty} \frac{z_{\ell+1}}{(\ell+1)!} t^{\ell} \right]^{k} \tag{3.3}
\]
for \( n \geq k \in \mathbb{N}_0 \). Substituting \( z_{2j} = \frac{(-1)^j}{2j+1} \) and \( z_{2j-1} = 0 \), that is, \( z_j = \frac{1}{j+1} \cos \left( \frac{j\pi}{2} \right) \), for \( j \in \mathbb{N} \) into (3.3) results in
\[
B_{n+k,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{n+2} \cos \left( \frac{n+1}{2} \pi \right) \right)
= \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \sum_{\ell=0}^{\infty} \frac{1}{(\ell+2)!} \cos \left( \frac{\ell+1}{2} \pi \right) t^{\ell} \right]^{k}
= \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left( \frac{sinc t - 1}{t} \right) .
\]
Corollary 3.1. For $k \geq 2$ and $1 \leq \ell \leq k - 1$, we have
\[
\sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2\ell + j, j)}{\binom{2\ell + j}{j}} = 0
\]  
(3.4)

and
\[
\sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{2\ell}{m} \sum_{m=0}^{\ell} (-1)^m \binom{2\ell}{m} \left(\frac{j}{2}\right)^m \frac{S(2\ell + j - m, j)}{\binom{2\ell + j - m}{j}} = 0.
\]

Proof: This follows from the proof of Theorem 3.1 and further making use of the formula (2.10). \qed

Corollary 3.2. For $n \geq k \geq 1$, partial Bell polynomials $B_{n,k}$ satisfy
\[
B_{n,k}\left(0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{n-k+2}, \cos\left(\frac{n-k+1}{2} \pi\right)\right)
= (-1)^k \cos\left(\frac{n\pi}{2}\right)^{2n} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(n + j, j)}{\binom{n+j}{j}}.
\]
and

\[ B_{n,k}\left(0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{n-k+2} \cos \left(\frac{n-k+1}{2} \pi \right) \right) \]
\[ = (-1)^k \cos \left(\frac{n \pi}{2}\right) \sum_{j=1}^{k} \sum_{\ell=1}^{j} (-1)^\ell \binom{j}{\ell} T(2k + \ell, \ell) \int \frac{S(n+j-m, j)}{j^{n+j-m}}. \]

Proof. This follows from combining the identity (2.5) with Theorem 3.1 and the formula (2.11).

Corollary 3.3. For \( z \in \mathbb{C} \), we have

\[ e^{\text{sinc} z^{-1}} = 1 + \sum_{k=1}^{\infty} (-1)^k \left[ \sum_{j=1}^{2k} \left(\frac{1}{j!} \sum_{\ell=1}^{j} (-1)^\ell \binom{j}{\ell} T(2k + \ell, \ell) \int \frac{S(2k + \ell - m, \ell)}{\ell^{2k+\ell-m}} \right) (2z)^{2k} \right] \frac{1}{(2k)!}. \]

and

\[ e^{\text{sinc} z^{-1}} = 1 + \sum_{k=1}^{\infty} (-1)^k \left[ \sum_{j=1}^{2k} \left(\frac{1}{j!} \sum_{\ell=1}^{j} (-1)^\ell \binom{j}{\ell} T(2k + \ell, \ell) \int \frac{S(2k + \ell - m, \ell)}{\ell^{2k+\ell-m}} \right) (2z)^{2k} \right] \frac{1}{(2k)!}. \]

Proof. Making use of the Fa\’a di Bruno formula (1.5), the derivatives in (3.1), and Theorem 3.1, we acquire

\[ e^{\text{sinc} z} = \sum_{k=0}^{\infty} \left( \lim_{z \to 0} \frac{d^k e^{\text{sinc} z}}{dz^k} \right) \frac{z^k}{k!} \]
\[ = e + \sum_{k=1}^{\infty} \left[ \lim_{z \to 0} \sum_{j=1}^{k} e^{\text{sinc} z} B_{k,j} (\text{sinc} z)^j, (\text{sinc} z)^{j'}, \ldots, (\text{sinc} z)^{(k-j+1)} \right] \frac{z^k}{k!} \]
\[ = e + e \sum_{k=1}^{\infty} \left[ \lim_{z \to 0} \sum_{j=1}^{k} B_{k,j} (\text{sinc} z)^j \big|_{z=0}, (\text{sinc} z)^{j'} \big|_{z=0}, \ldots, (\text{sinc} z)^{(k-j+1)} \big|_{z=0} \right] \frac{z^k}{k!} \]
\[ = e + e \sum_{k=1}^{\infty} \left[ \lim_{z \to 0} \sum_{j=1}^{k} B_{k,j} \left(0, -\frac{1}{3}, \ldots, \frac{1}{k-j+2} \cos \left(\frac{k-j+1}{2} \pi \right) \right) \right] \frac{z^k}{k!} \]
\[ = e + e \sum_{k=1}^{\infty} \left[ \lim_{z \to 0} \sum_{j=1}^{2k} B_{2k,j} \left(0, -\frac{1}{3}, \ldots, \frac{1}{2k-j+2} \cos \left(\frac{2k-j+1}{2} \pi \right) \right) \right] \frac{z^{2k}}{(2k)!}. \]

Further considering (2.10), we prove Corollary 3.3.
4. Series expansions of real powers of sinc and sinhc functions

In this section, with the aid of Theorem 3.1 and other results in the above sections, we establish series expansions at the point \( z = 0 \) of the power functions \( \text{sinc}^r z \) and \( \text{sinh}^r z \) for real \( r \in \mathbb{R} \).

**Theorem 4.1.** When \( r \geq 0 \), the series expansions

\[
\text{sinc}^r z = 1 + \sum_{q=1}^{\infty} (-1)^q \left[ \sum_{k=1}^{2q} \frac{(-r)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} T(2q+j,j) \left( \frac{2z}{(2q+j)!} \right)^{2q} \right] (2q)! (4.1)
\]

and

\[
\text{sinc}^r z = 1 + \sum_{q=1}^{\infty} (-1)^q \left[ \sum_{k=1}^{2q} \frac{(-r)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \right] \times \sum_{m=0}^{2q} (-1)^m \left( \frac{2q}{m} \right) \left( \frac{1}{2} \right)^m \left( \frac{S(2q+j-m,j)}{(2q+j-m)!} \right) \right] (2q)! (4.2)
\]

are convergent in \( z \in \mathbb{C} \), where the rising factorial \( (r)_k \) is defined by

\[
(r)_k = \prod_{\ell=0}^{k-1} (r + \ell) = \begin{cases} r(r+1) \cdots (r+k-1), & k \geq 1; \\ 1, & k = 0. \end{cases}
\]

When \( r < 0 \), the series expansions (4.1) and (4.2) are convergent in \( |z| < \pi \).

**Proof.** By virtue of the Faà di Bruno formula (1.5), we obtain

\[
\frac{d^j}{dz^j} (\text{sinc}^r z) = \sum_{k=1}^{j} \frac{d^k u^r}{d u^k} \text{B}_{j,k}( (\text{sinc} z)'(\text{sinc} z)'(\text{sinc} z)' \cdots (\text{sinc} z)'(\text{sinc} z)^{(j-k+1)})
\]

\[
= \sum_{k=1}^{j} (r)_k \text{sinc}^{r-k} z \text{B}_{j,k}( (\text{sinc} z)'(\text{sinc} z)'(\text{sinc} z)' \cdots (\text{sinc} z)'(\text{sinc} z)^{(j-k+1)})
\]

\[
\rightarrow \sum_{k=1}^{j} (r)_k \text{B}_{j,k}(0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{j-k+2} \sin \frac{(j-k)\pi}{2}), \quad z \to 0
\]

\[
= \begin{cases} 0, & j = 2m - 1 \\ \sum_{k=1}^{2m} (r)_k \text{B}_{2m,k}(0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{j-k+2} \sin \frac{(2m-k)\pi}{2}), & j = 2m \end{cases}
\]

for \( m \in \mathbb{N} \), where \( u = u(z) = \text{sinc} z \), the notation

\[
\langle r \rangle_k = \prod_{k=0}^{k-1} (r-k) = \begin{cases} r(r-1) \cdots (r-k+1), & k \geq 1 \\ 1, & k = 0 \end{cases}
\]

for \( r \in \mathbb{R} \) is called the falling factorial, and we used derivatives in (3.1). Therefore, with the help of Theorem 3.1, we arrive at

\[
\text{sinc}^r z = 1 + \sum_{j=1}^{\infty} \left[ \lim_{z \to 0} \frac{d^j}{dz^j} (\text{sinc}^r z) \right] \frac{z^j}{j!}
\]

\[
= 1 + \sum_{m=1}^{\infty} \left[ \lim_{z \to 0} \frac{d^{2m}}{dz^{2m}} (\text{sinc}^r z) \right] \frac{z^{2m}}{(2m)!}
\]
= 1 + \sum_{m=1}^{\infty} \sum_{k=1}^{2m} \frac{(\frac{r}{2})^k}{k!} \frac{1}{2^{m+j}} \frac{1}{(2m+j)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} T(2m+j, j) \frac{(2z)^{2m+j}}{(2m)!} \]

By virtue of (2.10), the proof of Theorem 4.1 is thus complete. □

**Corollary 4.1.** For \( r \in \mathbb{R} \) and \( z \in \mathbb{C} \), we have

\[
\sinh c^r z = 1 + \sum_{q=1}^{\infty} \sum_{k=1}^{2q} \frac{(-r)^k}{k!} \sum_{j=1}^{k} (-1)^j \frac{1}{2^q} \binom{k}{j} T(2q+j, j) \frac{(2z)^{2q}}{(2q)!} \tag{4.3}
\]

and

\[
\sinh c^r z = 1 + \sum_{q=1}^{\infty} \sum_{k=1}^{2q} \frac{(-r)^k}{k!} \sum_{j=1}^{k} (-1)^j \frac{1}{2^q} \binom{k}{j} \times \sum_{m=0}^{2q} \frac{(-1)^m}{m!} \binom{2q}{m} \frac{(2q)^{m}}{(2q)!} \frac{S(2q+j-m, j)}{(2^q+j-m)} \frac{(2z)^{2q}}{(2q)!} \tag{4.4}
\]

**Proof.** The series expansions (4.3) and (4.4) follow from replacing \( \text{sinc} z \) by \( \sinh c(z) \) in (4.1) and (4.2) and then substituting \( z \) for \( z \).

\[\square\]

5. **Remarks**

Finally we list several remarks about our main results and related things.

**Remark 5.1.** The formulation of the series expansions (2.1) and (2.8) in Theorems 2.1 and 2.3 are better and simpler than corresponding ones in [7, pp. 798–799].

The formula (2.2) can also be found at https://math.stackexchange.com/a/4331451/ and https://math.stackexchange.com/a/4332549/.

**Remark 5.2.** After reading the preprint [29] of this paper, Dr. Jacques Gélinas, a retired mathematician at Ottawa in Canada, pointed out that the series expansion (2.1) in Theorem 2.1, or say, the series expansion (2.8) in Theorem 2.3, has been considered by John Blissard in [4, pp. 50–51] with different and old notations.

**Remark 5.3.** Among other things, the identity (2.5) and the identity (3.4) were ever posted at https://mathoverflow.net/q/420121/ as a part of guesses. At the site https://mathoverflow.net/a/420309/, Peter Taylor (https://mathoverflow.net/users/46140/peter-taylor, pjt33@cantab.net), an independent researcher, combinatorially described confirmative answers to the identity (2.5) and (3.4).

**Remark 5.4.** The series expansion (2.1) in Theorem 2.1 has been applied to answer questions at the sites https://math.stackexchange.com/a/4429078/, https://math.stackexchange.com/a/4331451/.
The series expansion (2.1) in Theorem 2.1 or the series expansion (4.1) in Theorem 4.1 can be used to answer questions at https://math.stackexchange.com/q/2267836/ and https://math.stackexchange.com/q/3673133/

The series expansion (4.1) in Theorem 4.1 has been employed to answer questions at the websites https://math.stackexchange.com/a/4427504/, https://math.stackexchange.com/a/4426821/, and https://math.stackexchange.com/a/4428010/.

The series expansion (4.1) in Theorem 4.1 has been utilized in [12, Theorem 3] to derive two closed-form formulas for the Bernoulli numbers $B_{2m}$ in terms of central factorial numbers of the second kind $T_{2m+j, j}$.

Remark 5.5. The first identity in Theorem 3.1 is a special case of the following general conclusion in [16, Theorem 1.1].

For $k, n \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $x_m \in \mathbb{C}$, we have

$$B_{2n+1, k}(0, x_2, 0, x_4, \ldots, \frac{1+(-1)^k}{2}x_{2n-k+2}) = 0.$$ 

Remark 5.6. As done in Corollary 3.3, as long as the function $f(u)$ is infinitely differentiable at the point $u = 1$, Theorem 3.1 can be utilized to compute series expansions at $x = 0$ of the functions $f(sinc x)$ and $f(sinhc x)$.

Remark 5.7. Let $r > 0$ and $k \in \mathbb{N}_0$. Making use of the Faà di Bruno formula (1.5) and employing the formula

$$B_{n,k}(x, 1, 0, \ldots, 0) = \frac{1}{2^{n-k}} \frac{n!}{(n-k)!} \left( \frac{k}{n-k} \right) x^{2k-n}$$

collected in [33, Section 1.4], we obtain

$$\left[ \frac{1}{(1 + x^2)^r} \right]^{(k)} = \sum_{j=0}^{k} \frac{d^j}{du^j} \left( \frac{1}{u^r} \right) B_{k,j}(2x, 2, 0, \ldots, 0)$$

$$= \sum_{j=0}^{k} \frac{(-r)_j}{u^{r+j}} 2^j B_{k,j}(x, 1, 0, \ldots, 0)$$

$$= \sum_{j=0}^{k} \frac{(-r)_j}{(1 + x^2)^{r+j}} 2^j \frac{k!}{2^{k-j} j!} \binom{j}{k-j} x^{2j-k}$$

$$= \frac{k!}{2^k x^k (1 + x^2)^r} \sum_{j=0}^{k} \frac{(-r)_j}{j!} 2^{2j} \binom{j}{k-j} \frac{x^{2j}}{(1 + x^2)^j},$$

where $u = u(x) = 1 + x^2$. See also texts at the site https://math.stackexchange.com/a/4418636/.

Remark 5.8. We would like to mention the papers [11, 36, 41], in which the power function sinc$^r z$ for some specific ranges of $r, x \in \mathbb{R}$ is bounded from both sides, and to mention the papers [20, 24, 32], in which many bounds of the sinc function sinc $x$ for $x \in (0, \frac{\pi}{2})$ are established, reviewed, and surveyed.

Remark 5.9. This paper is a revised version of the electronic preprints [28, 29].
6. Declarations

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Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China
School of Mathematical Sciences, TianGong University, Tianjin 300387, China
Email address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: https://qifeng618.wordpress.com, https://orcid.org/0000-0001-6239-2968