IMAGES OF COMMUTING DIFFERENTIAL OPERATORS OF ORDER ONE WITH CONSTANT LEADING COEFFICIENTS

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Abstract. We first study some properties of images of commuting differential operators of polynomial algebras of order one with constant leading coefficients. We then propose what we call the image conjecture on these differential operators and show that the Jacobian conjecture [BCW], [E], [Bo] (hence also the Dixmier conjecture [D]) and the vanishing conjecture [Z3] of differential operators with constant coefficients are actually equivalent to certain special cases of the image conjecture. A connection of the image conjecture, and hence also the Jacobian conjecture, with multidimensional Laplace transformations of polynomials is also discussed.

1. Introduction

1.1. Background and Motivation. Let $z = (z_1, z_2, \ldots, z_n)$ be $n$ commutative free variables and $\mathbb{C}[z]$ the polynomial algebra in $z$ over $\mathbb{C}$. Recall that the Jacobian conjecture (JC) proposed by O. H. Keller [Ke] in 1939 claims that any polynomial map $F$ of $\mathbb{C}^n$ with Jacobian $j(F) \equiv 1$ must be an automorphism of $\mathbb{C}^n$. Despite intense study from mathematicians in the last seventy years, the conjecture is still open even for the case $n = 2$. In 1998, the Fields Medalist and also the Wolf Prize Winner S. Smale [S] included the Jacobian conjecture in his list of 18 fundamental mathematical problems for the 21st century. For more history and known results on the Jacobian conjecture, see [BCW], [E], [Bo] and references therein.

Recently, the equivalence of the JC with the Dixmier conjecture proposed by J. Dixmier [D] in 1968 has been established first by Y. Tsuchimoto [T] in 2005 and later by A. Belov and M. Kontsevich [BK] and P. K. Adjamagbo and A. van den Essen [AE] in 2007. The implication of the Jacobian conjecture from the Dixmier conjecture was

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actually proved much earlier by V. Kac (unpublished but see \cite{BCW}) in 1982.

The Dixmier conjecture claims that *any homomorphism of the Weyl algebras must be an automorphism of the Weyl algebras*. Note that in \cite{AE} P. K. Adjamagbo and A. van den Essen also showed that the JC is also equivalent to what they called the Poisson conjecture on Poisson algebras.

This paper is mainly motivated by the connections discussed in \cite{Z2} and \cite{Z3} of the JC with the vanishing conjecture (VC) on differential operators (of any order) with constant coefficients of the polynomial algebras $\mathbb{C}[z]$.

First let us recall the vanishing conjecture (VC) proposed in \cite{Z2} and \cite{Z3}. For later purposes, here we put it in a more general form.

**Conjecture 1.1. (The Vanishing Conjecture)** Let $P(z), Q(z) \in \mathbb{C}[z]$ and $\Lambda$ be a differential operator of $\mathbb{C}[z]$ with constant coefficients. Assume that $\Lambda^m(P^n) = 0$ for each $m \geq 1$. Then we have $\Lambda^m(P^nQ) = 0$ when $m \gg 0$.

The motivation for the conjecture above is the following theorem proved in \cite{Z2}.

**Theorem 1.2.** Let $\Delta = \sum_{i=1}^n \partial^2/\partial z_i^2$, i.e. the Laplace operator of $\mathbb{C}[z]$. Then the JC holds for all $n \geq 1$ if and only if the VC holds for all $n \geq 1$ with $\Lambda = \Delta$ and $P(z) = Q(z)$, where $P(z)$ is any homogeneous polynomial in $z$ of degree 4.

The proof of the theorem above is based on the classical celebrated homogeneous reduction of the JC achieved by H. Bass, E. Connell, D. Wright \cite{BCW} and A. V. Yagzhev \cite{Y} and also based on the remarkable symmetric reduction on the JC achieved independently by M. de Bondt and A. van den Essen \cite{BE} and G. Meng \cite{Me}. For more details of the proof, see \cite{Z1} and \cite{Z2}.

Note that it has been shown in \cite{EZ} that Theorem 1.2 also holds without the condition $P(z) = Q(z)$ and in \cite{Z3} that it also holds if the Laplace operators $\Delta$ are replaced by any sequence of quadratic homogeneous differential operators $\{\Lambda_n(\partial) \mid n \in \mathbb{N}\}$ with the rank of the quadratic form $\Lambda_n(\xi)$ goes to $\infty$ as $n \to \infty$, where $\Lambda_n(\xi)$ is the symbol of the differential operator $\Lambda_n(\partial)$ ($n \in \mathbb{N}$).

In this paper, we will show that the JC (hence also the Dixmier conjecture \cite{D}) and the Poisson conjecture in \cite{AE}) and the VC above are actually equivalent to certain special cases of what we call the *image conjecture* (See Conjecture 1.3 below) of commuting differential operators of $\mathbb{C}[z]$ of order one with constant leading coefficients. See
Theorems 3.5, 3.6 and 3.7 for the precise statements. To formulate the image conjecture (IC), let us first fix some notations that will be used throughout this paper.

Let \( \mathcal{A} \) be any commutative algebra over a field of characteristic zero and \( \mathcal{A}[z] \) the polynomial algebra of \( z \) over \( \mathcal{A} \). For any \( 1 \leq i \leq n \), we set \( \partial_i := \partial / \partial z_i \) and \( \partial := (\partial_1, \partial_2, ..., \partial_n) \). We denote by \( \mathbb{D}_\mathcal{A}[z] \) the Weyl algebra of differential operators of \( \mathcal{A}[z] \), and \( \mathbb{D}_\mathcal{A}[z] \) the subspace of differential operators of order one with constant leading coefficients, i.e. the differential operators of the form \( h(z) + \sum_{i=1}^{n} c_i \partial_i \) for some \( h(z) \in \mathcal{A}[z] \) and \( c_i \in \mathcal{A} \) (\( 1 \leq i \leq n \)).

For any \( \mathcal{C} \subset \mathbb{D}_\mathcal{A}[z] \), we define the image, denoted by \( \text{Im} \mathcal{C} \), of \( \mathcal{C} \) to be \( \sum_{\Phi \in \mathcal{C}} \Phi(\mathcal{A}[z]) \). Furthermore, we say \( \mathcal{C} \) is commuting if any two differential operators in \( \mathcal{C} \) commute with each other.

**Conjecture 1.3. (The Image Conjecture)** Let \( \mathcal{A} \) be any commutative algebra over a field of characteristic zero and \( \mathcal{C} \) a commuting subset of differential operators of \( \mathcal{A}[z] \) of order one with constant leading coefficients. Then for any \( f, g \in \mathcal{A}[z] \) with \( f^m \in \text{Im} \mathcal{C} \) for each \( m \geq 1 \), we have \( f^m g \in \text{Im} \mathcal{C} \) when \( m \gg 0 \).

Note that the statement of the image conjecture (IC) above is similar to the statement of the Mathieu conjecture proposed by O. Mathieu [Ma] in 1995.

**Conjecture 1.4. (The Mathieu Conjecture)** Let \( G \) be a compact connected real Lie group with the Haar measure \( d\sigma \). Let \( f \) be a complex-valued \( G \)-finite function over \( G \) such that \( \int_{G} f^m d\sigma = 0 \) for each \( m \geq 1 \). Then for any \( G \)-finite function \( g \) over \( G \), \( \int_{G} f^m g d\sigma = 0 \) when \( m \gg 0 \).

In [Ma] Mathieu also proved that his conjecture implies the JC. Later J. Duistermaat and W. van der Kallen [DK] proved the Mathieu conjecture for the case of tori, which can be re-stated as follows.

**Theorem 1.5. (Duistermaat and van der Kallen)** Let \( \mathbb{C}[z^{-1}, z] \) be the algebra of Laurent polynomials over \( \mathbb{C} \), and \( \mathcal{M} \subset \mathbb{C}[z^{-1}, z] \) the subspace of all Laurent polynomials in \( z \) with no constant terms. Then for any \( f(z), g(z) \in \mathbb{C}[z^{-1}, z] \) with \( f^m(z) \in \mathcal{M} \) for each \( m \geq 1 \), we have \( f^m(z)g(z) \in \mathcal{M} \) when \( m \gg 0 \).

Actually, if we consider certain generalizations of the IC for differential operators of some localizations of \( \mathbb{C}[z] \), especially, for those commuting differential operators of order one with leading coefficients related with classical orthogonal polynomials, the IC is indeed connected with the Mathieu conjecture and the Duistermaat-van der Kallen theorem. For example, let \( \Phi_i = \partial_i - z^{-1}_i \) (\( 1 \leq i \leq n \)) and \( \mathcal{C} = \{ \Phi_1, \Phi_2, ..., \Phi_n \} \).
Note that \( \mathcal{C} \) is a commuting subset of differential operators of the Laurent polynomial algebra \( \mathbb{C}[z^{-1}, z] \). Then one can show that the image 
\[ \text{Im} \, \mathcal{C} = \sum_{i=1}^{n} \Phi_i(\mathbb{C}[z^{-1}, z]) \] 
is exactly the subspace \( M \) in Theorem 1.5. Hence, the IC for \( \mathbb{C}[z^{-1}, z] \) and the differential operators \( \mathcal{C} \) above is equivalent to the Duistermaat-van der Kallen theorem. But, unfortunately, the straightforward generalization of the IC to \( \mathbb{C}[z^{-1}, z] \) does not always hold. For a modified generalization of the IC and their connections with the Mathieu conjecture and the Duistermaat-van der Kallen theorem, see [Z4].

1.2. Arrangement. In section 2, we assume that \( \mathbb{A} \) is a commutative algebra over a field \( \mathbb{K} \) of characteristic zero and prove some general properties of the images of commuting differential operators of order one with constant leading coefficients in \( \mathbb{K} \). We show in Theorem 2.5 that in this case, the IC can be reduced to the case that \( \mathcal{C} = \{ \Phi_1, \Phi_2, \ldots, \Phi_n \} \), where \( \Phi_i = \partial_i - \partial_i(q(z)) \) for some \( q(z) \in \mathbb{A}[z] \). We then assume that \( \mathbb{A} = \mathbb{K} \) and \( \mathcal{C} \) as above except with \( n \) replaced by any \( 1 \leq k \leq n \), and show in Theorem 2.10 that with a properly defined \( \mathcal{D} \)-module structure on \( \mathbb{K}[z] \), \( \text{Im} \, \mathcal{C} \) forms a \( \mathcal{D} \)-submodule of \( \mathbb{K}[z] \), and the quotient \( \mathbb{K}[z]/\text{Im} \, \mathcal{C} \) is a holonomic \( \mathcal{D} \)-module. Consequently, we have that, when \( k = n \), \( \text{Im} \, \mathcal{C} \) is a finite co-dimensional subspace of \( \mathbb{K}[z] \) (See Corollary 2.11).

In Section 3, we mainly consider the relations of the VC and the JC with the IC. In Subsection 3.1, we study the image of the commuting differential operators \( \Theta_i := \xi_i - \partial_i \) \((1 \leq i \leq n)\) of \( \mathbb{C}[\xi, z] \), where \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \) are another \( n \) free commutative variables which also commute with \( z \). Let \( \Theta = \{ \Theta_1, \Theta_2, \ldots, \Theta_n \} \). The main results of this subsection are Theorems 3.1 and 3.2 which identify \( \text{Im} \, \Theta \) with the kernel of another linear map \( \mathcal{E} : \mathbb{C}[\xi, z] \to \mathbb{C}[z] \) (See Eq. (3.1)). The kernel of \( \mathcal{E} \) captures the condition that \( \Lambda^m(f^m) = 0 \) \((m \geq 1)\) as well as the claim \( \Lambda^m(f^mg) = 0 \) \((m \gg 0)\) in the VC. Therefore Theorems 3.1 and 3.2 provide a bridge connecting the VC, hence also the JC, the Dixmier Conjecture and the Poisson Conjecture, with the IC.

In Subsection 3.2, by using the results obtained in Subsection 3.1, we show that the VC and the JC are equivalent to certain special cases of the IC for the polynomial algebra \( \mathbb{C}[\xi, z] \) and the commuting differential operators \( \Theta \) defined above. The main results of this subsection are Theorems 3.5, 3.6 and 3.7.

Finally, in Subsection 3.3, we give a different description for the image \( \text{Im} \, \Theta \) of the differential operators \( \Theta \) above. The main result of this subsection is Theorem 3.9 for which we give two proofs. From the second proof, we will see that the IC is actually also related with the
multidimensional Laplace transformations of polynomials (See Corollary 3.10 and Conjecture 3.11).

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2. The Image Conjecture of Commuting Differential Operators of Order One with Constant Leading Coefficients

The most interesting case of the image conjecture (IC), Conjecture 1.3 probably is the case when the commutative algebra $A$ is a field $K$ of characteristics zero. For example, as we will show later in Subsection 3.2, the Jacobian conjecture is actually equivalent to a very special case of the IC with $A = \mathbb{C}$. But, in order to get a good reduction for the IC, we need to consider a slightly more general case. Namely, in this section we consider the case of the IC under the following two assumptions.

(C1) the commutative algebra $A$ is a commutative algebra over a field $K$ of characteristics zero;

(C2) the commuting differential operators in $\mathcal{C}$ have the form $h(z) + \sum_{i=1}^{n} c_i \partial_i$ with $h(z) \in A[z]$ and the leading coefficients $c_i \in K$ (not just in $A$) for any $1 \leq i \leq n$.

Throughout this section, $K$ always denotes a field of characteristic zero and $A$ a commutative algebra over $K$. We denote by $D_K[z]$ the set of differential operators of $A[z]$ of the form $h(z) + \sum_{i=1}^{n} c_i \partial_i$, where $h(z) \in A[z]$ and the leading coefficients $c_i \in K$ ($1 \leq i \leq n$). Furthermore, all the terminologies and notations introduced in the Introduction will also be in force throughout this paper.

We start with the following simple lemma.

Lemma 2.1. Let $1 \leq k \leq n$ and $\Phi_i = \partial_i - h_i(z)$ ($1 \leq i \leq k$) with $h_i(z) \in A[z]$. Then the differential operators $\Phi_i$ ($1 \leq i \leq k$) commute with one another iff there exists a $q(z) \in A[z]$ such that $h_i = \partial_i(q)$ for all $1 \leq i \leq k$.

Proof: First, for any $1 \leq i, j \leq k$, we have the following identity for the commutator of $\Phi_i$ and $\Phi_j$:

$$[\partial_i - h_i, \partial_j - h_j] = \partial_j(h_i) - \partial_i(h_j).$$
Therefore, the differential operators $\Phi_i$’s commute with one another iff, for any $1 \leq i, j \leq k$,

\[(2.1) \quad \partial_j(h_i) = \partial_i(h_j). \]

Since $\mathcal{A}$ is an $K$-algebra, hence also an $\mathbb{Q}$-algebra, by Poincaré’s Lemma, we know that Eq. \[(2.1)\] holds for all $1 \leq i, j \leq k$ iff there exists a $q(z) \in K[z]$ such that $h_i(z) = \partial_i(q)$ for all $1 \leq i \leq k$. \qed

Note that $\mathbb{D}_K[z]$ is a $K$-vector space and, for any subset $\mathcal{C} \subset \mathbb{D}_K[z]$, it is easy to check that $\text{Im} \mathcal{C} = \text{Im} V$, where $V$ is the $K$-subspace of $\mathbb{D}_K[z]$ spanned by elements of $\mathcal{C}$ over $K$. Therefore, without losing any generality, we may freely replace the commuting subsets in the $\mathbf{IC}$ by commuting $K$-subspaces of $\mathbb{D}_K[z]$.

**Lemma 2.2.** Let $V$ be a commuting $K$-subspace of $\mathbb{D}_K[z]$ that contains at least one (nonzero) differential operator of order one. Then up to an automorphism of $\mathcal{A}[z]$, there exist $1 \leq k \leq n$, $q(z) \in \mathcal{A}[z]$, and finitely many polynomials $g_i(z) \in \mathcal{A}[z]$ $(i \in I)$ such that

1. $g_i(z)$ $(i \in I)$ does not involve $z_j$ for any $1 \leq j \leq k$, i.e. $\partial_j g_i(z) = 0$ for all $i \in I$ and $1 \leq j \leq k$;

2. $\text{Im} V$ is the same as the image of the collections of the differential operators $\partial_j - \partial_j(q(z))$ $(1 \leq j \leq k)$ and the multiplication operators by $g_i(z)$ $(i \in I)$.

**Proof:** First, for any $u = (a_1, a_2, \ldots, a_n) \in K^n$, set $u\partial := \sum_{i=1}^{n} a_i \partial_i$. Let $U = \{ u \in K^n \mid u\partial + h(z) \in V \text{ for some } h(z) \in \mathcal{A}[z] \}$ and $V_0$ the set of all differential operators in $V$ of order zero, i.e. the multiplication operators by elements of $\mathcal{A}[z]$. Then by the fact that $V$ is a vector space over $K$ (actually an $K$-subspace of $\mathbb{D}_K[z]$), it is easy to check that both $U$ and $V_0$ are also vector spaces over $K$.

Furthermore, by the assumption on $V$ in the lemma, we have $U \neq 0$. Let $u_1, u_2, \ldots, u_k$ be a basis of $U$. Then up to a changing of coordinates, we may assume that $u_i = e_i$ $(1 \leq i \leq k)$, where the $e_i$’s are the standard basis vectors of $K^n$. Under this assumption, we have $\partial_i - h_i(z) \in V$ $(1 \leq i \leq k)$ for some $h_i(z) \in \mathcal{A}[z]$. Since $V$ is a commuting subset of $\mathbb{D}_K[z]$, by Lemma \[2.1\] there exists a $q(z) \in \mathcal{A}[z]$ such that $h_i(z) = \partial_i(q)$ for all $1 \leq i \leq k$. We denote by $V_1$ the $K$-subspace of $V$ spanned by $\partial_i - \partial_i(q)$ $(1 \leq i \leq k)$ over $K$.

Next we consider $V_0$, which can be viewed as a $K$-subspace of $\mathcal{A}[z]$. Let $g_i(z) \in \mathcal{A}[z]$ $(i \in J)$ be a $K$-basis of $V_0$. Then the polynomials $g_i(z) \in \mathcal{A}[z]$ $(i \in J)$ also generate the ideal $V_0 K[z]$ of $K[z]$. On the other hand, by Hilbert’s theorem, we know that $K[z]$ is Noetherian.
Therefore, there exists a finite subset \( I \subset J \) such that \( g_i(z) \in \mathcal{A}[z] \) \((i \in I)\) also generate the ideal \( V_0K[z] \).

Now, by the fact that \( \text{Im} V = \text{Im} V_1 + \text{Im} V_0 = \text{Im} V_1 + V_0K[z] \), it is easy to check that \( \text{Im} V \) is the same as the image of the collections of the differential operators \( \partial_j - \partial_j(q(z)) \) \((1 \leq j \leq k)\) and the multiplication operators by \( g_i(z) \) \((i \in I)\).

Finally, for any \( i \in I \) and \( 1 \leq j \leq k \), since \( V \) is commuting, we have
\[
[\partial_j - h_j, g_i(z)] = \partial_j(g_i) = 0.
\]
Therefore, \( g_i(z) \) \((i \in I)\) does not involve \( z_j \) for any \( 1 \leq j \leq k \).

Note that, if a commuting \( K \)-subspace \( V \subset \mathbb{D}_K[z] \) does not contain any nonzero differential operators of order one, i.e. \( V \) contains only multiplication operators by elements of \( K[z] \), we can identify \( V \) with a \( K \)-subspace of \( K[z] \). Then it is easy to see that \( \text{Im} V \) is the same as the ideal \( VK[z] \) of \( K[z] \) generated by elements of \( V \). In this case, the IC holds trivially.

Therefore, we may assume that \( V \) contains at least one nonzero differential operators of order one. Furthermore, by Lemma 2.2, without changing the image, we may also assume that \( V \) is linearly spanned over \( K \) by the differential operators \( \partial_j - \partial_j(q(z)) \) \((1 \leq j \leq k)\) for a \( q(z) \in K[z] \) and finitely many multiplication operators by \( g_i(z) \in \mathcal{A}[z] \) \((i \in I)\) such that the polynomials \( g_i(z) \) \((i \in I)\) do not involve the variables \( z_j \) \((1 \leq j \leq k)\).

Throughout the rest of this section, we will fix a commuting subspace \( V \subset \mathbb{D}_K[z] \) as above. Set \( z' := (z_1, \ldots, z_k) \) and \( z'' := (z_{k+1}, z_{k+2}, \ldots, z_n) \). Then we have \( g_i(z) \in \mathcal{A}[z''] \) \((i \in I)\). Let \( \mathcal{I}_1 \) (resp. \( \mathcal{I}_2 \)) be the ideal of \( \mathcal{A}[z''] \) (resp. \( \mathcal{A}[z] \)) generated by \( g_i(z) \) \((i \in I)\). Set \( \mathcal{B} := \mathcal{A}[z'']/\mathcal{I}_1 \) and denote by \( \pi_1 : \mathcal{A}[z''] \rightarrow \mathcal{B} \) the quotient map.

Since \( g_i(z) \in \mathcal{A}[z''] \) \((i \in I)\), the quotient space \( \mathcal{A}[z]/\mathcal{I}_2 \) can (and will) be naturally identified with \( \mathcal{B}[z'] \), and the quotient map
\[
\pi_2 : \mathcal{A}[z] = \mathcal{A}[z''][z'] \rightarrow \mathcal{A}[z]/\mathcal{I}_2 \simeq \mathcal{B}[z']
\]
can be viewed as the linear extension of the quotient map \( \pi_1 : \mathcal{A}[z''] \rightarrow \mathcal{B} \) to the polynomial algebras in \( z' \) over \( \mathcal{A}[z'] \) and \( \mathcal{B} \).

For convenience, we introduce the following shorter notations. For any \( u(z) \in \mathcal{A}[z] \), we denote by \( \bar{u}(z) \) the image of \( u(z) \) under the quotient map \( \pi_2 \), i.e. \( \bar{u}(z) := \pi_2(u(z)) \). For any \( 1 \leq i \leq k \), we set \( h_i(z) := \partial_i(q(z)) \); \( \Phi_i := \partial_i - h_i(z) \) and \( \Psi_i := \partial_i - \bar{h}_i(z) \). Note that \( \Psi_i \) \((1 \leq i \leq k)\) are commuting differential operators of \( \mathcal{B}[z'] \) of order one with leading coefficients lying in the base field \( K \). Finally, set \( \nabla := \text{Span}_K\{\Psi_j | 1 \leq j \leq k\} \).

**Lemma 2.3.** With the notations fixed as above, we have...
(a) for any \(1 \leq j \leq k\) and \(f(z) \in \mathcal{A}[z]\),

\[
\tag{2.2}
\pi_2(\Phi_j f) = \Psi_j \bar{f}.
\]

(b)

\[
\tag{2.3}
\text{Im} V = \pi_2^{-1}(\text{Im} \nabla).
\]

Proof: (a) Note first that, since \(g_i(z) (i \in I)\) are independent on \(z'\), the ideal \(I_2 \subset \mathcal{A}[z]\) is preserved by the derivations \(\partial_j (1 \leq j \leq k)\). Consequently, for any \(1 \leq j \leq k\), we have

\[
\tag{2.4}
\pi_2 \circ \partial_j = \partial_j \circ \pi_2.
\]

For any \(1 \leq j \leq k\), by Eq. (2.4) and the fact that \(\pi_2\) is a homomorphism of algebras, we have

\[
\pi_2(\Phi_j f) = \pi_2(\partial_j f - h_j f) = \pi_2(\partial_j f) - \pi_2(h_j f)
\]

\[
= \partial_j \pi_2(f) - \pi_2(h_j) \pi_2(f) = \partial_j \bar{f} - \bar{h}_j \bar{f}
\]

\[
= (\partial_j - \bar{h}_j) \bar{f} = \Psi_j \bar{f}.
\]

Therefore, we have (a).

To show (b), let \(f \in \text{Im} V\) and write it as \(f = \sum_{j=1}^n \Phi_j a_j + \sum_{i \in I} g_i b_i\) for some \(a_j, b_i \in \mathcal{A}[z] (i \in I)\). Then by Eq. (2.2) and the fact that \(\pi_2(g_i) = 0\) for any \(i \in I\), we have \(\pi_2(f) = \sum_{j=1}^n \Psi_j \bar{a}_j \in \text{Im} \nabla\) and \(f \in \pi_2^{-1}(\text{Im} \nabla)\).

Conversely, let \(f \in \pi_2^{-1}(\text{Im} \nabla)\), i.e. \(\bar{f} \in \text{Im} \nabla\). Since \(\pi_2\) is surjective, we may write \(\bar{f} = \sum_{j=1}^n \Psi_j \bar{a}_j\) for some \(a_j \in \mathcal{A}[z] (1 \leq j \leq k)\). Then by Eq. (2.2), we have \(\bar{f} = \sum_{j=1}^n \pi_2(\Phi_j a_j)\). Hence \(f = \sum_{j=1}^n \Phi_j a_j + u\) for some \(u \in \mathcal{J}_2 = \text{Ker} \pi_2\). Since \(\mathcal{J}_2\) is the ideal of \(\mathcal{A}[z]\) generated by \(g_i(z) (i \in I)\), we have \(\mathcal{J}_2 \subset \text{Im} V\). Therefore, \(u \in \text{Im} V\). Hence we also have \(f \in \text{Im} V\). □

Lemma 2.4. With \(V\) and related notations fixed as above, we have that the \textbf{IC} holds for \(\mathcal{A}[z]\) and the commuting differential operators \(V\), iff the \textbf{IC} holds for \(\mathcal{B}[z']\) and the commuting differential operators \(\{\Psi_j | 1 \leq j \leq k\}\).

Proof: (⇒) Assume that the \textbf{IC} holds for \(\mathcal{A}[z]\) and the commuting differential operators \(V\). Take any \(\bar{f}, \bar{g} \in \mathcal{B}[z']\) with \(\bar{f}^m \in \text{Im} \{\Psi_j | 1 \leq j \leq k\} = \text{Im} \nabla\) for all \(m \geq 1\). By Eq. (2.3), we have \(f^m \in \text{Im} V\) for all \(m \geq 1\). Therefore, we have \(f^m g \in \text{Im} V\) when \(m \gg 0\). By Eq. (2.3) again, \(f^m \bar{g} = \pi_2(f^m g) \in \text{Im} \nabla = \text{Im} \{\Psi_j | 1 \leq j \leq k\}\) when \(m \gg 0\).

(⇐) Assume that the \textbf{IC} holds for \(\mathcal{B}[z']\) and the commuting differential operators \(\{\Psi_j | 1 \leq j \leq k\}\), hence also for \(\nabla\). For any \(f, g \in \mathcal{A}[z]\)
with \( f^m \in \text{Im} V \) for all \( m \geq 1 \), by Eq. (2.3) we have \( \bar{f}^m \in \text{Im} \bar{V} \) for any \( m \geq 1 \). Therefore, \( \pi_2(f^m g) = \bar{f}^m \bar{g} \in \text{Im} \bar{V} \) when \( m \gg 0 \). By Eq. (2.3) again, we have \( f^m g \in \text{Im} V \) when \( m \gg 0 \).

By Lemmas 2.1–2.4, it is easy to see that we have the following reduction on the IC under the conditions (C_1) and (C_2) on page 5.

**Theorem 2.5.** To prove or disprove the IC under the conditions (C_1) and (C_2), it is enough to consider the IC for polynomial algebras \( A[z] \) (in \( n \) variables) for the commuting differential operators \( \mathcal{C} = \{ \partial_i - \partial_i(q(z)) \mid 1 \leq i \leq n \} \) with \( q(z) \in A[z] \).

Two remarks on the IC are as follows. First, it does not hold in general for commuting differential operators of order one with non-constant leading coefficients.

**Example 2.6.** Consider the polynomial algebra \( \mathbb{C}[t] \) in one variable \( t \). Let \( \Phi := td/dt - 1 \). Then it is easy to check that for any \( f(t) \in \mathbb{C}[t] \) \( f(t) \in \text{Im} \Phi \) iff \( f'(0) = 0 \), i.e. \( f(t) \) has no degree-one term.

Now let \( f(t) = 1 + t^2 \). Then \( f^m(t) \in \text{Im} \Phi \) for each \( m \geq 1 \). But \( tf^m(t) \notin \text{Im} \Phi \) for each \( m \geq 1 \) since the degree-one terms of \( tf^m(t) \) are always \( t \). Therefore, \( \text{Im} \Phi \) is not a Mathieu subspace of \( \mathbb{C}[t] \) and the IC fails in this case.

Second, the IC is also false in general when the base algebra \( A \) is an algebra over a field of characteristic \( p > 0 \).

**Example 2.7.** Let \( K \) be any field of characteristic \( p > 0 \) and \( x \) a free variable. Consider the differential operator \( \Lambda := d/dx \) of the polynomial algebra \( K[x] \). Then it is easy to check that \( \text{Im} \Lambda \) is linearly spanned over \( K \) by the monomials \( x^m \) (\( m \in \mathbb{N} \) and \( m \neq -1 \) (mod \( p \))). In particular, the constant polynomial 1 \( \in \text{Im} \Lambda \). Thus by taking \( f = 1 \) and \( g = x^{p-1} \), we see that the IC fails for the positive characteristic case.

Next we further assume \( A = K \) and consider some \( \mathcal{D} \)-module structures on \( \text{Im} \mathcal{C}' \) and \( K[z]/\text{Im} \mathcal{C}' \) for any subset \( \mathcal{C}' \) of the commuting differential operators \( \mathcal{C} \) in Theorem 2.5.

We believe that most of the results to be discussed below, if not all, are known in the theory of \( \mathcal{D} \)-modules. But, for the sake of completeness and also due to lack of more specific references in the literature, we give a more detailed account here. It is a little surprising to see that the problem raised by the IC has not gotten any attention from the point of view of \( \mathcal{D} \)-modules.

Note first that up to a permutation of the variables \( z_i \) we may assume \( \mathcal{C}' = \{ \Phi_i \mid 1 \leq i \leq k \} \) for some \( 1 \leq k \leq n \). For the simplicity of
notation, throughout the rest of this subsection, we fix an $1 \leq k \leq n$ and still denote $C'$ by $C$.

Let $D[z]$ be the Weyl algebra of the polynomial algebra $K[z]$, i.e. the subalgebra of $\text{End}_K(K[z])$ (the algebra of all $K$-linear maps from $K[z]$ to $K[z]$) generated by the derivations $\partial_i$ ($1 \leq i \leq n$) and the multiplication operators by $z_i$ ($1 \leq i \leq n$). We first define an action of $D[z]$ on $K[z]$ by setting

\begin{align}
(2.5) & \quad z_i \ast f := z_i f, \\
(2.6) & \quad \partial_i \ast f := \Phi_i f = \partial_i f - f \partial_i q
\end{align}

for all $1 \leq i \leq n$ and $f \in K[z]$.

Since $\Phi_i$ ($1 \leq i \leq n$) commute with one another, it is easy to see that $K[z]$ with the actions defined above forms a module of the Weyl algebra $D[z]$. We denote this $D[z]$-module by $M$.

The $D[z]$-module $M$ can also be constructed as follows. First let $M_q := K[z]e^{-q(z)}$. We may formally view elements of $K[z]e^{-q(z)}$ as formal power series in $z$ over $K$. Then the standard action of $D[z]$ on the formal power series algebra $K[[z]]$ induces an action of $D[z]$ on $M_q$. More precisely, for any $1 \leq i \leq n$ and $f(z) \in K[z]$, we have

\begin{align}
(2.7) & \quad z_i \cdot (f(z)e^{-q(z)}) = (z_i f(z))e^{-q(z)}, \\
(2.8) & \quad \partial_i \cdot (f(z)e^{-q(z)}) = e^{-q(z)}(\partial_i f - f \partial_i q) = e^{-q(z)}(\Phi_i f).
\end{align}

Since $M_q$ is obviously closed under the actions above, it is also a $D[z]$-module. Furthermore, let $E : M \to M_q$ be the $K$-linear map that maps any $f(z) \in K[z]$ to $f(z)e^{-q(z)} \in M_q$. Then by Eqs. (2.5)–(2.8), it is straightforward to check that the following lemma holds.

**Lemma 2.8.** (a) The map $E$ is an isomorphism of $D[z]$-modules.
(b) For any $1 \leq i \leq n$, we have

\begin{align}
(2.9) & \quad E(\partial_i \ast M) = \partial_i \cdot M_q,
\end{align}

where $\partial_i \ast M$ and $\partial_i \cdot M_q$ are the images of $M$ and $M_q$ under the actions of $\partial_i \ast$ and $\partial_i \cdot$, respectively.

(c) Let $\mathcal{C} = \{\Phi_i \mid 1 \leq i \leq k\}$ as fixed before. Then viewing $\text{Im} \mathcal{C}$ as a subspace of $M$, we have $\text{Im} \mathcal{C} = \sum_{i=1}^{k} \partial_i \ast M$.

**Lemma 2.9.** $M$ is a holonomic module of the Weyl algebra $D[z]$ with multiplicity $e(M) \leq \max\{1, d^m\}$, where $d := \deg q(z)$.

Since $M \simeq M_q$ as $D[z]$-modules, it is enough to show the lemma for $M_q$. First, it is easy to just show that $M_q$ is holonomic as follows.
First proof: Let $I_q$ be the left ideal of $\mathcal{D}[z]$ consisting of differential operators that annihilate $e^{-q(z)}$. Then it is easy to see that $\mathcal{M}_q \simeq \mathcal{D}[z]/I_q$ as $\mathcal{D}[z]$-modules. Since $e^{-q(z)}$ is an (hyper)exponential function, by Theorem 2.6 in [C] (p. 183) we obtain that $e^{-q(z)}$ is a holonomic function, i.e. $\mathcal{D}[z]/I_q$ is a holonomic $\mathcal{D}[z]$-module. Hence so is $\mathcal{M}_q$. □

A more straightforward proof of Lemma 2.8 which also gives the information on the multiplicity $e(\mathcal{M})$ can be given as follows.

Second Proof: First, if $d = 0$, then $q(z)$ is a constant and $\mathcal{M}_q$ is isomorphic to the standard $\mathcal{D}[z]$-module $K[z]$. It is well-known that $K[z]$ is holonomic with multiplicity $e(K[z]) = 1$. Hence the lemma holds in this case.

Assume $d \geq 1$ and introduce a filtration on $\mathcal{M}_q$ by setting, for any $m \geq 0$,

$$\Gamma_m := \{u(z)e^{-q(z)} \mid \deg u(z) \leq md\}$$

(2.10)

First, it is easy to see that $\bigcup_{m \geq 0} \Gamma_m = K[z]e^{-q(z)} = \mathcal{M}_q$ since $md \to \infty$ as $m \to \infty$.

Second, for any $m \geq 0$, $f(z)e^{-q(z)} \in \Gamma_m$ and $1 \leq i \leq n$, by Eqs. (2.7)-(2.8), we have $z_i \cdot (f(z)e^{-q(z)}) = (z_if(z))e^{-q(z)} \in \Gamma_{m+1}$ since 

$$\deg(z_if(z)) = \deg f + 1 \leq md + 1 \leq (m + 1)d,$$

and

$$\partial_i \cdot (f(z)e^{-q(z)}) = e^{-q(z)}(\partial_if - f\partial_iq) \in \Gamma_{m+1}$$

since $\deg(\partial_i f - f\partial_iq) \leq \deg f + d - 1 \leq md + d - 1 < d(m + 1)$.

Third, for any fixed $m \geq 0$, $\dim \Gamma_m$ is the same as the dimension of the subspace of $\mathbb{C}[z]$ consisting of the polynomials of degree less or equal to $md$. Therefore, for any $r \in \mathbb{R}$ such that $r \geq \frac{1}{m}(n)\binom{n}{k}n^{n-k}$ for all $0 \leq k \leq n - 1$, we have

$$\dim \Gamma_m = \binom{n + md}{n} \leq \frac{(n + md)^n}{n!}$$

(2.11)

$$= \frac{(md)^n}{n!} + \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} n^{n-k} (md)^k$$

$$\leq \frac{(md)^n}{n!} + r n (md)^{n-1} \leq \frac{d^n m^n}{n!} + c(m + 1)^{n-1},$$

where $c = rnd^{m-1}$. 


Then by Theorem 5.4 (p. 12) in [Bj] or Lemma 3.1 (p. 91) in [C], $M_q$ is a holonomic $D_K$-module with multiplicity $e(M_q) \leq d^n$.  

Now let $1 \leq k \leq n$ and $C$ as fixed before. Set $z' := (z_1, ..., z_k)$ and $z'' := (z_{k+1}, ..., z_n)$. Denote by $D[z'']$ the Weyl algebra of the polynomial algebra $K[z'']$. Note that the $D[z]$-module $M$ is also a $D[z'']$-module since $D[z'']$ is a subalgebra of the Weyl algebra $D[z]$.

**Theorem 2.10.** (a) $\text{Im} C$ as a subspace of $M$ is a $D[z'']$-submodule of the $D[z'']$-module $M$.

(b) The quotient $M/\text{Im} C$ is a holonomic $D[z'']$-module.

**Proof:** (a) Note that for any $k+1 \leq j \leq n$, the differential operator $\Phi_j$ and the multiplication operator by $z_j$ commute with $\Phi_i$ ($1 \leq i \leq k$). Therefore, $\text{Im} C$ is closed under the actions of $\Phi_j$ and $z_j$ ($k+1 \leq j \leq n$) and forms a $D[z'']$-submodule of $M$.

(b) By Lemma 2.9, we know that $M$ is a holonomic $D[z]$-module. Then it follows from Theorem 6.2 (p. 16) in [Bj] that the quotient $M/(\sum_{i=1}^{k} \partial_i * M)$ is a holonomic $D[z'']$-module. Combining this fact with Lemma 2.8 (c), we see that the statement (b) in the theorem holds.

**Corollary 2.11.** When $k = n$, $\text{Im} C$ is a finite co-dimensional $K$-subspace of $K[z]$, i.e. the $K$-vector space $K[z]/\text{Im} C$ is of finite dimension.

**Proof:** Note that when $k = n$, we have $D[z''] = K$. Then by Theorem 2.10 (b), we know that $K[z]/\text{Im} C$ is a holonomic $K$-module. Hence, $K[z]/\text{Im} C$ must be finite dimensional over $K$.

3. The Vanishing Conjecture and the Jacobian Conjecture in Terms of the Image Conjecture

In this section, we study the relations of the vanishing conjecture (VC), Conjecture 1.1 and the Jacobian conjecture (JC) with the image conjecture (IC), Conjecture 1.3. But first, we need to fix more notations.

Let $\xi = (\xi_1, \xi_2, ..., \xi_n)$ be another $n$ commutative free variables which also commute with the free variables $z = (z_1, z_2, ..., z_n)$. For any $1 \leq i \leq n$, set $\Theta_i := \xi_i - \partial_i$ and $\Theta := \{\Theta_i | 1 \leq i \leq n\}$. Note that $\Theta$ is a commuting set of differential operators of order one with constant leading coefficients of the polynomial algebra $\mathbb{C}[\xi, z]$. 
In Subsection 3.2, we show that the VC and the JC are equivalent to some special cases of the IC for the polynomial algebra \( \mathbb{C}[\xi, z] \) and the commuting differential operators \( \Theta \). The main results are given in Theorems 3.5, 3.6 and 3.7.

One crucial result needed for the proofs of the theorems above is Theorem 3.1. So we devote Subsection 3.1 to prove a stronger version, Theorem 3.2, of this result. We believe that these two results are interesting and important on their own rights, so we have formulated them as theorems.

In Subsection 3.3 we give another description for \( \text{Im} \Theta \) which does not involve differential operators (See Theorem 3.9). From the second proof of the theorem, we will see that the IC is also connected with integrals of polynomials such as the multidimensional Laplace transformations.

3.1. Images of Commuting Differential Operators \( \Theta \). First, let us fix the following conventions and notations that will be used throughout the rest of this paper.

1. For any \( n \geq 1 \), we define a partial order on \( \mathbb{N}^n \) by setting, for any \( \alpha, \beta \in \mathbb{N}^n \), \( \alpha \geq \beta \) if, for each \( 1 \leq i \leq n \), the \( i \)th component of \( \alpha \) is greater or equal to the \( i \)th component of \( \beta \).
2. We will freely use some multi-index notations. For instance, for any \( \alpha = (k_1, k_2, ..., k_n) \in \mathbb{N}^n \), we set
   \[
   \alpha! = k_1!k_2! \cdots k_n!.
   \]
   \[
   z^\alpha = z_1^{k_1}z_2^{k_2} \cdots z_n^{k_n}.
   \]
   \[
   \partial^\alpha = \partial_1^{k_1}\partial_2^{k_2} \cdots \partial_n^{k_n}.
   \]

Next, we define a linear map \( E : \mathbb{C}[\xi, z] \rightarrow \mathbb{C}[z] \) by setting, for any \( g(\xi) \in \mathbb{C}[\xi] \) and \( h(z) \in \mathbb{C}[z] \),

\[
E(g(\xi)h(z)) := g(\partial)h(z).
\] (3.1)

In other words, for any \( f(z, \xi) \in \mathbb{C}[z, \xi] \), \( E(f(z, \xi)) \in \mathbb{C}[z] \) is obtained by putting the free variables \( \xi_i \)'s on the most left in each monomial of \( f(z, \xi) \) and then replacing \( \xi_i \) by \( \partial_i \) and applying them to \( z_i \)'s on the right. For example, we have

1. \( E(1) = 1; \)
2. \( E(z_1^n\xi_1^2) = \partial_1^2(z_1^n) = n(n-1)z_1^{n-2} \) for each \( n \geq 2; \)
3. \( E(a(z)) = a(z) \) for each \( a(z) \in \mathbb{C}[z]; \)
4. \( E(\xi^\alpha) = \partial^\alpha(1) = 0 \) for each \( 0 \neq \alpha \in \mathbb{N}^n. \)

Next, we are going to derive another description of the subspace \( \text{Ker} \, E \subset \mathbb{C}[\xi, z] \) by using the differential operators \( \Theta = \{\Theta_i = \xi_i - \partial_i \mid 1 \leq i \leq n\} \) (as fixed in the first paragraph of this section) of the polynomial
algebra \( \mathbb{C}[\xi, z] \). But, for convenience, in the rest of this paper, we also use \( \Theta \) to denote the ordered \( n \)-tuple \((\Theta_1, \Theta_2, ..., \Theta_n)\), so the notations such as \( \Theta^\alpha \) \((\alpha \in \mathbb{N}^n)\) will make perfect sense.

**Theorem 3.1.** \( \text{Ker } \mathcal{E} = \text{Im } \Theta \).

Actually, a much more explicit theorem, Theorem 3.2, can be formulated and proved. Theorem 3.1 follows immediately from Theorem 3.2 and lemma 3.3 below. Indeed, assume \( f \in \text{Ker } \mathcal{E} \). Then by Theorem 3.2, \( a_0(z) = 0 \) and thus \( f \in \text{Im } \Theta \).

**Theorem 3.2.** (1) Any \( f(\xi, z) \in \mathbb{C}[\xi, z] \) can be written uniquely as a sum of the form

\[
(3.2) \quad f(\xi, z) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \Theta^\alpha a_\alpha(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (\xi - \partial)^\alpha a_\alpha(z)
\]

for some \( a_\alpha(z) \in \mathbb{C}[z] \ (\alpha \in \mathbb{N}^n) \).

(2) With the notation above, for any \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n \),

\[
(3.3) \quad a_\alpha(z) = \mathcal{E}(\partial_{\xi}^\alpha f),
\]

where \( \partial_{\xi}^\alpha = \prod_{i=1}^n \partial_{\xi_i}^{\alpha_i} \).

In particular, we have \( a_0(z) = \mathcal{E}(f) \).

To prove Theorem 3.2, we first need to show the following lemma which is actually the easy part of Theorem 3.1.

**Lemma 3.3.** \( \text{Im } \Theta \subseteq \text{Ker } \mathcal{E} \).

**Proof:** For any \( f(\xi, z) \in \text{Im } \Theta \), we need to show \( \mathcal{E}(f) = 0 \). By the linearity, we may assume \( f(\xi, z) = \Theta_i(a(\xi)b(z)) \) for some \( 1 \leq i \leq n \), \( a(\xi) \in \mathbb{C}[\xi] \) and \( b(z) \in \mathbb{C}[z] \). Then by Eq. (3.1), we have

\[
\mathcal{E}(\Theta_i(a(\xi)b(z))) = \mathcal{E}(\xi_i a(\xi)b(z) - \partial_i (a(\xi)b(z)))
\]

\[
= \mathcal{E}(a(\xi)\xi_i b(z) - a(\xi)\partial_i b(z))
\]

\[
= a(\partial)\partial_i b(z) - a(\partial)\partial_i b(z) = 0.
\]

Therefore, we have \( f(\xi, z) \in \text{Ker } \mathcal{E} \) and \( \text{Im } \Theta \subseteq \text{Ker } \mathcal{E} \). \( \square \)

**Proof of Theorem 3.2.** First, we show that Eq. (3.2) does hold for some \( a_\alpha(z) \in \mathbb{C}[z] \ (\alpha \in \mathbb{N}^n) \). By the linearity, we may assume that \( f(\xi, z) = \xi^\beta h(z) \) for some \( \beta \in \mathbb{N}^n \) and \( h(z) \in \mathbb{C}[z] \).

Consider

\[
(3.4) \quad \xi^\beta h(z) = (\xi - \partial + \partial)^\beta h(z) = (\Theta + \partial)^\beta h(z)
\]

\[
= \sum_{\alpha \in \mathbb{N}^n, \alpha \leq \beta} \frac{\beta!}{\alpha!(\beta - \alpha)!} \Theta^\alpha \partial^{\beta - \alpha} h(z),
\]

which is of the desired form of Eq. \((3.2)\) for \(f(\xi, z) = \xi^\beta h(z)\) with
\[
a_\alpha(z) = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} \partial^{\beta-\alpha} h(z) & \text{if } \alpha \leq \beta; \\ 0 & \text{otherwise.} \end{cases}
\]

Next, we show the uniqueness of Eq. \((3.2)\) as follows.

Note first that for any \(\alpha, \beta \in \mathbb{N}^n\), we have
\[
\partial^\beta \xi \left( \frac{1}{\alpha!} \Theta^\alpha a_\alpha(z) \right) = \begin{cases} \frac{1}{(\alpha-\beta)!} \Theta^{\alpha-\beta} a_\alpha(z) & \text{if } \beta \leq \alpha; \\ 0 & \text{otherwise.} \end{cases}
\]

Then by applying \(\partial^\beta \xi\) to Eq. \((3.2)\), we get
\[
\partial^\beta \xi f(\xi, z) = \sum_{\gamma \in \mathbb{N}^n} \frac{1}{\gamma!} \Theta^\gamma (a_{\gamma+\beta}(z)).
\]

Applying \(\mathcal{E}\) to the equation above, by Lemma \(3.3\) we see that \(a_\beta(z) = \mathcal{E}(\partial^\beta \xi f(\xi, z))\) for each \(\beta \in \mathbb{N}^n\), which is exactly Eq. \((3.3)\) with the index \(\alpha\) replaced by \(\beta\). Hence, the theorem follows. \(\Box\)

**Remark 3.4.** A more conceptual proof of Theorem \(3.2\) will be given in [Z5]. Eq. \((3.2)\) actually corresponds to the Taylor series in a deformation of the polynomial algebra \(\mathbb{C}[\xi, z]\). It can also be viewed as a twisted version of the usual Taylor series of \(f(\xi, z) \in \mathbb{C}[\xi, z]\) by the commuting differential operators \(\Theta\).

### 3.2. The Vanishing Conjecture and the Jacobian Conjecture in Terms of the Image Conjecture

Now we are ready to show that some relations of the VC and the JC with the IC. First, the relation between the VC and the IC is given as follows.

**Theorem 3.5.** For any \(\Lambda(\xi) \in \mathbb{C}[\xi]\) and \(P(z) \in \mathbb{C}[z]\), the following two statements are equivalent:

(a) the VC holds for \(\Lambda = \Lambda(\partial)\) and \(P(z)\), \(g(z) \in \mathbb{C}[z]\);

(b) the IC holds for \(f(\xi, z) := \Lambda(\xi) P(z)\) and \(g(z)\) in \(\mathbb{C}[\xi, z]\).

**Proof:** For any \(m \geq 1\), by Eq. \((3.1)\) and Theorem \(3.1\) we have that \(\Lambda(\partial)^m(P^m(z)) = 0\) iff \(\mathcal{E}(\Lambda^m(\xi) P^m(z)) = \mathcal{E}(f^m(\xi, z)) = 0\) iff \(f^m(\xi, z) \in \text{Ker } (\mathcal{E}) = \text{Im } \Theta\). Similarly, we also have, \(\Lambda^m(P^m(z))g(z) = 0\) iff \(f^m(\xi, z)g(z) \in \text{Im } \Theta\). From these two results, it is easy to see that the statements (a) and (b) in the theorem are equivalent to each other. \(\Box\)

Next we give two relations between the JC and the IC. The first one is as follows.
**Theorem 3.6.** The following statements are equivalent to each other.

(a) The JC holds for all \( n \geq 1 \).

(b) For any \( n \geq 1 \) and homogeneous \( P(z) \in \mathbb{C}[z] \) of degree 4, the IC holds for \( f(\xi, z) = (\sum_{i=1}^{n} \xi_i^2)P(z) \) and \( g(z) = P(z) \).

**Proof:** By Theorem 1.2 we know that the JC holds for all \( n \geq 1 \) iff for any \( n \geq 1 \) the VC holds for the Laplace operator \( \Delta = \sum_{i=1}^{n} \partial_i^2 \) and any homogeneous \( P(z) = g(z) \in \mathbb{C}[z] \) of degree 4. Therefore, the equivalence of (a) and (b) in the theorem follows directly from this fact and Theorem 3.5 above. \( \square \)

The second relation between the JC and the IC is given by the following theorem.

**Theorem 3.7.** The following statements are equivalent to each other.

(a) The JC holds for all \( n \geq 1 \).

(b) For any \( n \geq 1 \) and homogeneous \( H_i(z) \in \mathbb{C}[z] \) (\( 1 \leq i \leq n \)) of degree 3, the IC holds for \( f(\xi, z) = \sum_{i=1}^{n} \xi_i H_i(z) \) and \( g(z) = z_i \) for each \( 1 \leq i \leq n \).

**Proof:** First, by the classical homogeneous reduction (See [BCW] and [Y]) on the JC, to prove or disprove the JC holds, it is enough to consider polynomial maps \( F(z) \) of the form \( F(z) = z - H(z) \) with \( H(z) \in \mathbb{C}[z]^{\times n} \) homogeneous of degree 3.

We fix a polynomial map \( F(z) \) as above and denote by \( G(z) \) the formal inverse map of \( F(z) \). Then by the Abhyankar-Gurjar inversion formula [A] (See also [BCW] and [Z6]), we have

\[
\sum_{m \geq 0} \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z) j(F)(z) g(z)) = g(G(z))
\]

for any formal power series \( g(z) \in \mathbb{C}[[z]] \), where \( j(F) \) denotes the Jacobian of the polynomial map \( F \), i.e. the determinant of the Jacobian matrix of \( F \).

In particular, choose \( g(z) = j(F)^{-1} \in \mathbb{C}[[z]] \). Since \( j(F)(G)j(G) \equiv 1 \), we get

\[
\sum_{m \geq 0} \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z)) = j(F)^{-1}(G(z)) = j(G(z)).
\]
Since additionally $j(G)(F)j(F) \equiv 1$, we have that $j(F)(z) \equiv 1$ iff $j(G)(z) \equiv 1$. So by the equation above, we have

$$ j(F)(z) \equiv 1 \iff \sum_{m \geq 0} \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z)) \equiv 1. $$

(3.8)

Since $\deg \partial^\alpha (H^\alpha(z)) = 2|\alpha|$ for any $\alpha \in \mathbb{N}^n$, the equation above is equivalent to

$$ \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z)) = 0 $$

(3.9)

for any $m \geq 1$.

Let $f(\xi, z) := \sum_{i=1}^n \xi_i H_i(z)$. Then for any $m \geq 1$, we have

$$ \mathcal{E}(f^m(\xi, z)) = \mathcal{E}\left(\left(\sum_{i=1}^n \xi_i H_i(z)\right)^m\right) = \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{m!}{\alpha!} \mathcal{E}(\xi^\alpha H^\alpha(z)). $$

Applying Eq. (3.1) gives:

$$ \mathcal{E}(f^m(\xi, z)) = m! \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z)). $$

(3.10)

By Eqs. (3.8) - (3.10) and also Theorem 3.1, we see that

$$ j(F)(z) \equiv 1 \iff f^m(\xi, z) \in \text{Im } \Theta \text{ for each } m \geq 1. $$

(3.11)

Now assume $j(F) \equiv 1$. By applying Eq. (3.6) to $g(z) = z$, we get

$$ \sum_{m \geq 0} \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z) z) = G(z). $$

(3.12)

Note that, for any $m \geq 1$ and any $1 \leq i \leq n$, we have

$$ \deg \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z) z_i) = 2m + 1. $$

Therefore, from Eq. (3.12), we see that

$$ \text{the JC holds for } F(z) \iff \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z) z) = 0 \text{ when } m \gg 0. $$

(3.13)
But, on the other hand, by a similar argument as for Eq. (3.10), for any \( m \geq 1 \) and \( 1 \leq i \leq n \), we also have

\[
E(f^m(\xi, z_i)) = m! \sum_{\alpha \in \mathbb{N}^n, |\alpha| = m} \frac{1}{\alpha!} \partial^\alpha (H^\alpha(z) z_i).
\]

Therefore, by Eqs. (3.13), (3.14) and Theorem 3.1, we have that

\[
\text{the JC holds for } F(z)
\]

\[
\iff f^m(\xi, z_i) \in \text{Im } \Theta \text{ for each } 1 \leq i \leq n \text{ when } m \gg 0.
\]

Finally, from Eqs. (3.11) and (3.15), it is easy to see that the statements \((a)\) and \((b)\) in the theorem are indeed equivalent to each other.

From the proofs of Theorems 3.6 and 3.7, we have the following “non-trivial” families of polynomials all whose powers lie in \( \text{Im } \Theta \).

**Corollary 3.8.** (a) For any Hessian nilpotent polynomial \( P(z) \in \mathbb{C}[z] \), i.e. the Hessian matrix \( \text{Hes} (P) = \left( \frac{\partial^2 P}{\partial z_i \partial z_j} \right) \) is nilpotent, set \( f(\xi, z) := \left( \sum_{i=1}^{n} \xi_i^2 \right) P(z) \). Then for any \( m \geq 1 \), we have \( f^m(\xi, z) \in \text{Im } \Theta \).

(b) For any homogeneous \( H = (H_1, H_2, ..., H_n) \in \mathbb{C}[z]^n \) of degree \( d \geq 2 \) with the Jacobian matrix \( \text{JH} \) nilpotent, set \( f(\xi, z) := \sum_{i=1}^{n} \xi_i H_i(z) \). Then for any \( m \geq 1 \), we have \( f^m(\xi, z) \in \text{Im } \Theta \).

**Proof:** (a) First, from the proof of Theorem 3.6 we see that for any \( m \geq 1 \), \( f^m(\xi, z) \in \text{Im } \Theta \) iff \( \Delta^m(P^m(z)) = 0 \). Second, by Theorem 4.3 in [Z2], we know that \( P(z) \) is Hessian nilpotent iff \( \Delta^m(P^m(z)) = 0 \) for all \( m \geq 1 \). Hence \((a)\) follows.

(b) From the proof of Theorem 3.7 we see that the statement holds if \( H \) is homogeneous of degree 3. But the argument there does not depend on the specific degree of \( H \) but the homogeneity of \( H \), as long as \( d \geq 2 \). Therefore, \((b)\) also follows from the same argument.

### 3.3. Another Description of \( \text{Im } \Theta \)

In this subsection, we give a new description (see Theorem 3.9) for the image of the differential operators \( \Theta = (\Theta_1, \Theta_2, ..., \Theta_n) \) with \( \Theta_i = \xi_i - \partial_i \) as before. The advantage of this description is that it does not involve any differential operators. We will give two different proofs for this result. While the first one is more straightforward and shorter, the second one is more conceptual. From the second proof, we will also see that the \( \text{IC} \) is actually connected with integrals of polynomials such as multidimensional Laplace transformations of polynomials. For more discussions on the connections of the \( \text{IC} \) with integrals of polynomials over open subsets of \( \mathbb{R}^n \), see [Z4].
First, let us fix the following notation and terminology.

For any Laurent polynomial \( q(z) \in \mathbb{C}[z^{-1}, z] \) and \( \alpha \in \mathbb{Z}^n \), we denote by \( [z^\alpha]q(z) \) the coefficient of \( z^\alpha \) of \( q(z) \). For any subspace \( V \subset \mathbb{C}[z^{-1}, z] \), we say the \( V \)-part of \( q(z) \) is zero if, for any \( \alpha \in \mathbb{Z}^n \) with \( z^\alpha \in V \), we have \( [z^\alpha]q(z) = 0 \). In particular, we also say that the holomorphic part of \( q(z) \) is zero if the \( \mathbb{C}[z] \)-part of \( q(z) \) is zero.

We define a linear map \( Z : \mathbb{C}[\xi, z] \to \mathbb{C}[z^{-1}, z] \) by setting
\[
Z(g(\xi)z^\beta) := \beta! g(z^{-1}) z^\beta
\]
for each \( g(\xi) \in \mathbb{C}[\xi] \) and \( \beta \in \mathbb{N}^n \).

The main result of this subsection is the following theorem.

**Theorem 3.9.** For any \( f(\xi, z) \in \mathbb{C}[\xi, z], f(\xi, z) \in \text{Im } \Theta \) iff the holomorphic part of the Laurent polynomial \( Z(f(\xi, z)) \) is equal to zero.

**First Proof:** We write \( f(\xi, z) = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} \xi^\alpha z^\beta \) for some \( c_{\alpha, \beta} \in \mathbb{C} \). By Eq. (3.1), we have
\[
E(f(\xi, z)) = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} \partial^\alpha z^\beta = \sum_{\alpha, \beta \in \mathbb{N}^n} \frac{\beta!}{(\beta - \alpha)!} c_{\alpha, \beta} z^{\beta - \alpha}
\]
\[
= \sum_{\gamma \in \mathbb{N}^n} \sum_{\alpha, \beta \in \mathbb{N}^n} \frac{\beta!}{\gamma!} c_{\alpha, \beta} z^\gamma
\]

On the other hand, by Eq. (3.16), we have
\[
Z(f(\xi, z)) = \sum_{\alpha, \beta \in \mathbb{N}^n} \beta! c_{\alpha, \beta} z^{\beta - \alpha} = \sum_{\gamma \in \mathbb{N}^n} \sum_{\alpha, \beta \in \mathbb{N}^n} \beta! c_{\alpha, \beta} z^\gamma.
\]

By the two equations above, we see that for any \( \gamma \in \mathbb{N}^n \), we have
\[
\gamma! [z^\gamma]E(f(\xi, z)) = [z^\gamma]Z(f(\xi, z)).
\]

Therefore, we have that \( f(\xi, z) \in \text{Ker } E \) iff the holomorphic part of the Laurent polynomials \( Z(f(\xi, z)) \) is equal to zero. By Theorem 3.1, \( \text{Im } \Theta = \text{Ker } E \), hence the theorem follows. \( \square \)

Next we give another proof for Theorem 3.9 by using the multidimensional Laplace transformations. In order to do that, we first need to fix the following notation.

Let \( \mathbb{R}^+ \) denote the set of all positive real numbers. By the restriction, we can and will view \( f(\xi, z) \) as a family of \( \mathbb{C} \)-valued functions in \( z \in (\mathbb{R}^+)^n \) parameterized by \( \xi \in (\mathbb{R}^+)^n \), which we will still denote by \( f(\xi, z) \).
Now we consider the following integral:

\[ \mathcal{L}(f)(\xi) := \int_{(\mathbb{R}^+)^n} f(\xi, z) e^{-\xi z} \, dz, \]

where \( \xi z = \sum_{i=1}^{n} \xi_i z_i \).

Note that when \( f(\xi, z) = h(z) \) for some \( h(z) \in \mathbb{C}[z] \), i.e. \( f(\xi, z) \) is independent on \( \xi \), \( \mathcal{L}(f) \) is nothing but the multidimensional Laplace transformation of the polynomial \( h(z) \). In general, the integral in Eq. (3.18) is the base-field extension of the multidimensional Laplace transformation from \( \mathbb{C}[z] \) to \( \mathbb{C}[\xi][z] = \mathbb{C}[\xi, z] \). So we will still refer \( \mathcal{L}(f) \) as the (multidimensional) Laplace transformation of \( f(\xi, z) \in \mathbb{C}[\xi, z] \).

**Second Proof of Theorem 3.9.** We start with the following two observations on the Laplace transformation defined in Eq. (3.18).

First, for any \( \beta \in \mathbb{N}^n \), we have

\[
\mathcal{L}(z^\beta)(\xi) = \int_{(\mathbb{R}^+)^n} z^\beta e^{-\xi z} \, dz = \int_{(\mathbb{R}^+)^n} (-\partial_\xi)^\beta e^{-\xi z} \, dz
\]

\[
= (-\partial_\xi)^\beta \int_{(\mathbb{R}^+)^n} e^{-\xi z} \, dz = (-\partial_\xi)^\beta (\xi^{-1})
\]

\[
= \beta! \xi^{-\beta} \xi^{-[1]},
\]

where \( \xi^{-[1]} := \prod_{i=1}^{n} \xi_i^{-1} \).

Therefore, for any polynomial \( h(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha \) with \( c_\alpha \in \mathbb{C} \), we have

\[
\mathcal{L}(h)(\xi) = \xi^{-[1]} \sum_{\alpha \in \mathbb{N}^n} \alpha! c_\alpha \xi^{-\alpha} \in \xi^{-[1]} \mathbb{C}[\xi^{-1}].
\]

Also, from the equation above, we see that \( h(z) \neq 0 \) iff \( \mathcal{L}(h)(\xi) \neq 0 \).

Second, for any \( 1 \leq i \leq n \) and \( h(\xi, z) \in \mathbb{C}[\xi, z] \), we have

\[
\int_0^\infty e^{-\xi z} (\Theta_i h(\xi, z)) \, dz_i = - \int_0^\infty e^{-\xi z} \left( (\partial_\xi - \xi_i) h(\xi, z) \right) \, dz_i
\]

\[
= - \int_0^\infty \partial_\xi \left( h(\xi, z) e^{-\xi z} \right) \, dz_i
\]

\[
= - (h(\xi, z) e^{-\xi z}) \bigg|_{z_i=0}^{+\infty} = (h(\xi, z) e^{-\xi z}) \bigg|_{z_i=0}.
\]

Hence we also have

\[
\mathcal{L}(\Theta_i h)(\xi) = \int_{(\mathbb{R}^+)^{n-1}} (h(\xi, z) e^{-\xi z}) \bigg|_{z_i=0} \, dz_i \cdots dz_{i-1} \, dz_{i+1} \cdots dz_n.
\]

From the equation above, it is easy to see that \( \mathcal{L}(\Theta_i h)(\xi) \) can not have any \( \xi^\alpha \)-term with the \( i^{th} \) component of \( \alpha \) strictly less than 0.
Therefore, for any \( g(\xi, z) \in \text{Im } \Theta \), its Laplace transformation \( \mathcal{L}(g)(\xi) \) can not have any non-zero \( \xi^{-1}C[\xi^{-1}] \) part.

Now, for any \( f(\xi, z) \in C[\xi, z] \), by Theorem 3.2 we may write \( f(\xi, z) \) uniquely as \( f(\xi, z) = a(z) + g(\xi, z) \) with \( a(z) \in C[z] \) and \( g(\xi, z) \in \text{Im } \Theta \). Then by the observations above, we see that the \( \xi^{-1}C[\xi^{-1}] \)-part of \( \mathcal{L}(f)(\xi) \) is equal to \( \mathcal{L}(a)(\xi) \). Therefore, we have that \( f(\xi, z) \in \text{Im } \Theta \) iff \( a(z) = 0 \) (by Theorem 3.2) iff \( \mathcal{L}(a) = 0 \) iff the \( \xi^{-1}C[\xi^{-1}] \)-part of \( \mathcal{L}(f)(\xi) \) is equal to zero.

On the other hand, if we write \( f(\xi, z) = \sum_{\beta \in \mathbb{N}^n} b_\beta(\xi) z^\beta \) with \( b_\beta(\xi) \in C[\xi] \). Then by Eqs. (3.19) and (3.16), we have

\[
\mathcal{L}(f)(\xi) = \sum_{\beta \in \mathbb{N}^n} b_\beta(\xi) \int_{(\mathbb{R}^+)^n} z^\beta e^{-\xi z} dz = \xi^{-1} \sum_{\beta \in \mathbb{N}^n} \beta! b_\beta(\xi) \xi^{-\beta} = \xi^{-1} \mathcal{Z}(f)(z)|_{z=\xi^{-1}}.
\]

Therefore, we have that the \( \xi^{-1}C[\xi^{-1}] \)-part of \( \mathcal{L}(f)(\xi) \) is equal to zero iff the \( C[\xi^{-1}] \)-part of \( \mathcal{Z}(f)(z)|_{z=\xi^{-1}} \) is equal to zero iff the holomorphic part of \( \mathcal{Z}(f)(z) \in C[z^{-1}, z] \) is equal to zero. Hence the theorem follows.

From the second proof above, it is easy to see that Theorem 3.9 can be re-stated in terms of multi-dimensional Laplace transformations as follows.

**Corollary 3.10.** For any \( f(\xi, z) \in C[\xi, z] \), \( f(\xi, z) \in \text{Im } \Theta \) iff the \( \xi^{-1}C[\xi^{-1}] \)-part of its Laplace transformation \( \mathcal{L}(f)(\xi) \) is equal to zero.

By Theorem 3.9 and Corollary 3.10 above, we see that the IC can be re-stated as follows.

**Conjecture 3.11.** Let \( \mathcal{M} \) be the subspace of polynomials \( f(\xi, z) \in C[\xi, z] \) such that \( \mathcal{Z}(f) \) has no holomorphic part, or equivalently, the Laplace transformation \( \mathcal{L}(f)(\xi) \) has no \( \xi^{-1}C[\xi^{-1}] \)-part. Then for any \( f(z), g(z) \in C[\xi, z] \) with \( f^m \in \mathcal{M} \) for each \( m \geq 1 \), we have \( f^m g \in \mathcal{M} \) when \( m \gg 0 \).

For comparison, let us point out that the following theorem first conjectured in [Z3] has recently been proved in [EZW] by using the Duistermaat-van der Kallen theorem, Theorem 1.3.

**Theorem 3.12.** Let \( \mathcal{M} \) be the subspace of Laurent polynomials \( f(z) \in C[z^{-1}, z] \) such that \( f(z) \) has no holomorphic part. Then for any \( f, g \in C[z^{-1}, z] \) with \( f^m \in \mathcal{M} \) for each \( m \geq 1 \), we have \( f^m g \in \mathcal{M} \) when \( m \gg 0 \).
Actually, as pointed out in [Z3] and [EWZ], the theorem above implies that the VC holds when either Λ is a monomial of \( \partial \) or \( P(z) \) is a monomial of \( z \) (See [Z3] and [EWZ] for more details). Consequently, by Theorem 3.5 we see that the IC holds when \( f(z, \xi) \in \mathbb{C}[\xi, z] \) has the form \( \xi^\alpha P(z) \) or \( \Lambda(\xi)z^\alpha \) with \( \Lambda(\xi) \in \mathbb{C}[\xi] \), \( P(z) \in \mathbb{C}[z] \) and \( \alpha \in \mathbb{N}^n \).

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