The Faraday effect revisited: Thermodynamic limit

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Abstract

This paper is the second in a series revisiting the (effect of) Faraday rotation. We formulate and prove the thermodynamic limit for the transverse electric conductivity of Bloch electrons, as well as for the Verdet constant. The main mathematical tool is a regularized magnetic and geometric perturbation theory combined with elliptic regularity and Agmon-Combes-Thomas uniform exponential decay estimates.

Contents

1 Introduction
   1.1 Generalities
   1.2 The main result
   1.3 A short description of the proof strategy

2 Proof of i.

3 Proof of ii.

4 Proof of iii.
   4.1 Partitions and cut-offs
   4.2 Proof of (1.12)
   4.3 Proof of (1.13)
      4.3.1 Only $U_L(B)$ counts.
      4.3.2 The boundary terms
      4.3.3 The bulk contribution

5 Appendix: Uniform exponential decay

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1 Introduction

1.1 Generalities

The rotation of the polarization of a plane-polarized electromagnetic wave passing through a material immersed in a homogeneous magnetic field oriented parallel to the direction of propagation, is known in physics as the Faraday effect (called sometimes also Faraday rotation). The experiment consists in sending a monochromatic light wave, parallel to the 0z direction and linearly polarized in the plane x0z. When the light enters the sample, the polarization plane starts rotating. A simple argument based on (classical) Maxwell equations shows that there exists a linear relation between the angle $\theta$ of rotation of the plane of polarization per unit length and the transverse component of the conductivity tensor of the material (see e.g. formula (1) in [29]). For most materials - and we will restrict ourselves to this case - the transverse component of the conductivity tensor vanishes when the magnetic field is absent and is no longer zero when the magnetic field is turned on. Under the proviso that the dependence of the conductivity tensor upon the strength $B$ of the magnetic field is smooth, for weak fields one expands the conductivity tensor to the first order and neglect the higher terms. The coefficient of the linear term is known as the Verdet constant of the corresponding material.

It follows that the basic object is the conductivity tensor and the main goal of the theory (classical or quantum) is to provide a workable formula for it, in particular for the Verdet constant. The problem has a long and distinguished history in solid state physics theory and the spectrum of possible applications ranges from astrophysics to optics and general quantum mechanics (see e.g. [29, 28, 33, 13, 23, 16, 22] and references therein).

Using quantum theory in the setting in which the sample is modeled by a system of independent electrons subjected to a periodic electric potential, Laura Roth [29] obtained (albeit only at a formal level) a formula for the Verdet constant in full generality and applied it to metals as well as semiconductors. Roth’s method is based on an effective Hamiltonian approach for Bloch electrons in the presence of a weak constant magnetic field (see [31] and references therein) which in turn is based on a (proto) magnetic pseudodifferential calculus (for recent mathematical developments see [21] and references therein).

But Roth’s theory is far from being free of difficulties. Due to her highly formal way of doing computations, it seems almost hopeless - even with present day mathematical tools - to control the errors or push the computations to higher orders in $B$ except maybe the case of simple bands. Even more, the final formula contains terms which are singular at the crossings of Bloch bands. Accordingly, in spite of the fact that it has been considered a landmark of the subject, it came as no surprise that at the practical level this theory only met a moderate success and a multitude of unrelated, simplified models have been tailored for specific cases.

Our paper is the second in a series aiming at a complete, unitary and mathematically sound theory of Faraday effect having the same generality as Roth’s theory (i.e. a theory of the conductivity tensor for electrons subjected to a periodic electric potential and to a constant magnetic field in the linear response approximation), but free of its shortcomings. More precisely in the first paper [13] we started by a rigorous derivation of the transverse component of the conductivity tensor in the linear response regime for a finite sample. It is given as the formula 1.10 below. To proceed further, we employ a method going back at least Sondheimer and Wilson [32] and which has been also used in the rigorous study of the Landau magnetism [1, 10, 5, 6, 7]. The basic idea is that the traces involved in computing various physical quantities can be written as integrals involving Green functions (i.e. integral kernels in the configuration space of either the resolvent or the semi-group of the Hamiltonian of the system), which are more robust and easier to control.

As it stands, the conductivity tensor depends upon the shape of the sample and of boundary conditions which define the quantum Hamiltonian. The physical idea of the thermodynamic limit for an intensive physical quantity is that in the limit of large samples it approaches a limit which is independent upon the shape of the sample, boundary conditions etc. The existence of the thermodynamic limit is one of the basic (and far from trivial) problem of statistical mechanics (see e.g. [30, 4]). In [13] we took for granted that the thermodynamic limit of the the transverse
component of the conductivity tensor exists and the limit is smooth as a function of the magnetic field strength \( B \). Moreover, we assumed that the thermodynamic limit commuted with taking the derivative with respect to \( B \). Under these assumptions, we gave - among other things - explicit formulas for the Verdet constant in terms of zero magnetic field Green functions, free of any divergences.

The proof of the thermodynamic limit, which from the mathematical point of view is the most delicate part of the theory of the Faraday effect, was left aside in [13], and is the content of the current paper. The mathematical problem behind it is hard due to the singularity induced by the long range magnetic perturbation. Even for a simpler problem involving constant magnetic field - namely the Landau diamagnetism of free electrons - the existence of the thermodynamic limit leading to a correct thermodynamic behavior was a long standing problem. Naive computations led to unphysical and contradictory results (see [1] for historical remarks). Accordingly, the first rigorous results came as late as 1975 [1] and were based on various identities expressing the gauge invariance which was crucial in dealing with the singular terms appearing in the thermodynamic limit. Even though the importance of gauge invariance was already highlighted in [1], an efficient way to implement this idea at a technical level was still lacking. Only recently a regularized magnetic perturbation theory based on factorizing the (singular in the thermodynamic limit) magnetic phase factor has been fully developed in [10, 11, 12, 27]. This regularized magnetic perturbation theory has been already used in [10, 5, 6, 7] in order to prove far reaching generalizations of the results in [1].

Coming back to the Faraday effect, we would like to stress that the object at hand is much more singular than the one encountered in the Landau diamagnetism. This adds an order of magnitude to the mathematical difficulty and requires an elaborate and tedious combination of regularized magnetic perturbation theory with techniques like Combes-Thomas exponential decay, trace norm estimates and elliptic regularity.

We expect that the method developed here in order to control the thermodynamic limit in the presence of an extended magnetic field to be useful in related problems, e.g. to obtain an elegant and complete study of the diamagnetism and de Haas-van Alphen effect for electrons in metals.

The content of the paper is as follows. In the rest of this Introduction we state the mathematical problem, give the main result in Theorem 1.1, and since the proof is quite long and technical we will briefly describe the main points. The other sections are devoted to the proof of our main theorem. The core of the proof heavily involving regularized magnetic perturbation theory is contained in Section 4. Some technical estimates about exponential decay with uniform control in the spectral parameter are given as an Appendix.

1.2 The main result

Consider a simply connected open and bounded set \( \Lambda_1 \subset \mathbb{R}^3 \), which contains the origin. We assume that the boundary \( \partial \Lambda_1 \) is smooth. Consider a family of scaled domains

\[
\Lambda_L = \{ x \in \mathbb{R}^3 : x/L \in \Lambda_1 \}, \quad L > 1.
\] (1.1)

We have the estimates

\[
\text{Vol}(\Lambda_L) \sim L^3, \quad \text{Area}(\partial \Lambda_L) \sim L^2.
\] (1.2)

We are interested in the thermodynamic limit, which will mean \( L \to \infty \), that is \( \Lambda_L \) will fill out the whole space. The one particle Hilbert space is \( \mathcal{H}_L := L^2(\Lambda_L) \). Note that we include the case \( L = \infty \).

The one body Hamiltonian of a non-confined particle, subjected to a constant magnetic field \((0, 0, B)\), in an external potential \( V \), formally looks like this:

\[
H_\infty(B) = \mathbf{P}^2(B) + V,
\] (1.3)

with

\[
\mathbf{P}(B) = -i \nabla + B a = \mathbf{P}(0) + B a.
\] (1.4)
Let us explain the various terms. Here $a(x)$ is a smooth magnetic vector potential which generates a magnetic field of intensity $B = 1$ i.e. $\nabla \wedge a(x) = (0, 0, 1)$. A frequently used magnetic vector potential is the symmetric gauge:

$$A(x) = \frac{1}{2} \mathbf{n}_3 \wedge x = (-x_2/2, x_1/2, 0), \quad (1.5)$$

where $\mathbf{n}_3$ is the unit vector along $z$ axis. We neglect the spin structure since it only complicates the notation and does not influence the mathematical problem.

On components, (1.4) reads as:

$$P_j(B) = D_j + Ba_j =: P_j(0) + Ba_j, \quad j \in \{1, 2, 3\}. \quad (1.6)$$

We will from now on assume that $V$ is a $C^\infty(\mathbb{R}^3)$ function, periodic with respect to the lattice $\mathbb{Z}^3$. Standard arguments then show that $H_\infty(B)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$.

When $L < \infty$ we need to specify a boundary condition. We will only consider Dirichlet boundary conditions, that is we start with the same expression as in (1.3), defined on $C_0^\infty(\Lambda_L)$, and we define $H_L(B)$ to be the Friedrichs extension of it. This is indeed possible, because our operator can be written as $-\Delta_D + W$, where $\Delta_D$ is the Dirichlet Laplacian and $W$ is a first order differential operator, relatively bounded to $-\Delta_D$ (remember that $L < \infty$). The form domain of $H_L(B)$ is the Sobolev space $H^1_0(\Lambda_L)$, while the operator domain (use the estimates in section 10.5, Lemma 10.5.1, in [19]) is $H^2(\Lambda_L) \cap H^1_0(\Lambda_L)$. Moreover, $H_L(B)$ is essentially self-adjoint on $C_0^\infty(\overline{\Lambda_L})$, i.e. functions with support in $\overline{\Lambda_L}$ and indefinitely differentiable in $\Lambda_L$ up to the boundary.

Another important operator is $(-i\nabla + Ba)^2$, i.e. the usual free magnetic Schrödinger operator defined with Dirichlet boundary conditions. We know that its spectrum is non-negative for all $L > 1$. By adding a positive constant, we can always assume that the spectrum of $H_L(B)$ is non-negative, uniformly in $L > 1$.

Let us now introduce the physical quantity we want to study. Consider $\omega \in \mathbb{C}$ and $\Im(\omega) < 0$. For some fixed $\mu \in \mathbb{R}$ and $\beta > 0$, define the Fermi-Dirac function on its maximal domain of analyticity:

$$f_{FD}(z) = \frac{1}{e^{\beta(z-\mu)} + 1}. \quad (1.7)$$

Define

$$d := \min \left\{ \frac{\pi}{2\beta}, \frac{|\Im(\omega)|}{2} \right\}, \quad (1.8)$$

and introduce a counter-clockwise oriented contour given by

$$\Gamma_\omega = \{ x \pm id : a \leq x < \infty \} \bigcup \{ a + iy : -d \leq y \leq d \} \quad (1.9)$$

where $a + 1$ lies below the spectrum of $H_L(B)$. By adding a positive constant to $V$, we can take $a = -1$ uniformly in $L \geq 1$ and $B \in [0, 1]$.

We introduce the transverse component of the conductivity tensor (see [29, 13]) as

$$\sigma_L(B) = -\frac{1}{\text{Vol}(\Lambda_L)} \quad (1.10)$$

$$\cdot \text{Tr} \int_{\Gamma_\omega} f_{FD}(z) \left\{ P_1(B)(H_L(B) - z)^{-1}P_2(B)(H_L(B) - z - \omega)^{-1} + P_1(B)(H_L(B) - z + \omega)^{-1}P_2(B)(H_L(B) - z)^{-1} \right\} dz.$$

Here $\text{Tr}$ assumes that the integral is a trace-class operator. Now we are prepared to formulate our main result.

**Theorem 1.1.** The above defined transverse component of the conductivity tensor admits the thermodynamic limit; more precisely:
Moreover, the mapping $B \mapsto s_B(B)$ is jointly continuous on its variables. Moreover, the continuous function defined by $F(x, x') := F(x, x') \in \mathbb{R}$ is periodic with respect to $\mathbb{Z}^3$.

iii. Denote by $\Omega$ the unit cube in $\mathbb{R}^3$. The thermodynamic limit exists:

$$\sigma_\infty(B) := \lim_{L \to \infty} \sigma_L(B) = - \int_{\Omega} s_B(x)dx.$$  \hspace{1cm} (1.12)

Moreover, the mapping $B \to s_B \in L^\infty(\Omega)$ is differentiable at $B = 0$ and:

$$\partial_B \sigma_\infty(0) = - \int_{\Omega} \partial_B s_B \bigg|_{B=0} (x)dx = \lim_{L \to \infty} \partial_B \sigma_L(0).$$  \hspace{1cm} (1.13)

**Remark 1.** The formula (1.12) is only the starting point in the study of the Faraday rotation. A related problem is the diamagnetism of Bloch electrons, where the main object is the integrated density of states of magnetic Schrödinger operators (see [17, 15, 24]). For a systematic treatment of magnetic pseudo-differential operators which generalizes our magnetic perturbation theory, see [21, 24, 25, 26].

**Remark 2.** The Dirichlet boundary conditions are important for us. Even though we suspect that our main result should also hold for Neumann conditions and for less regular domains (see [2, 14]), we do not see an easy way to prove it.

**Remark 3.** We believe that the method we use in the proof of (1.13) can be used in order to obtain a stronger result: the mapping $B \to s_B \in L^\infty(\Omega)$ is smooth and for any $n \geq 1$:

$$\partial_B^n \sigma_\infty(B) = - \int_{\Omega} \partial_B^n s_B(x)dx = \lim_{L \to \infty} \partial_B^n \sigma_L(B).$$  \hspace{1cm} (1.14)

We leave this statement as an open problem. In the rest of the paper we give the proof of Theorem 1.1.

### 1.3 A short description of the proof strategy

Since the proof is rather long, we list here the main steps and ideas. Let us start with some general considerations about the thermodynamic limit.

If we are interested in the thermodynamic limit of a quantum physical quantity, the object we need to control is the trace of the operator representing the corresponding quantity. The basic idea is to write this trace as an integral of the diagonal value of the operator's integral kernel (Schwartz kernel) over the confining box. This procedure makes the quantum thermodynamic limit look very similar to what happens in classical statistical mechanics. More precisely, we need to show that the difference between the integral kernel for the finite box and the one for the entire space decays sufficiently rapidly with the distance from the boundary of the box, so that the replacement of the integral kernel for the box with the one corresponding to the entire space gives an error term increasing slower than the volume, which then disappears in the limit.

It turns out that for the transverse conductivity this kernel is far more complicated than say the heat kernel - whose behavior has been extensively studied in the literature. Notice for example, that the integrand in (1.10) contains two resolvents sandwiched with magnetic momentum operators.
Thus we need a good control of their integral kernels, in particular when the distance between their arguments increases to infinity, and all that uniformly in the spectral parameter \( z \). Since a constant magnetic field is present, the biggest difficulty is to deal with the linear growth of the vector potential. Here, the use of gauge covariance is crucial.

Now let us list the main ideas of the proof. For the first statement of the theorem one simply uses integration by parts with respect to \( z \) in order to transform the integrand into a product of Hilbert-Schmidt operators.

The proof of the second statement is based on elliptic regularity. The main technical difficulty is to control the \( z \) behavior of all our bounds, especially the exponential localization of the magnetic resolvents sandwiched with momentum operators. We also have to control the linear growth of the magnetic potential. We turn the operator norm bounds which we obtain in the Appendix into pointwise bounds for certain integral kernels. It is a long road using magnetic perturbation theory, but nevertheless, we use nothing more than well-known Combes-Thomas exponential bounds, local gauge covariance, the Cauchy-Schwarz inequality, and integration by parts.

The third statement of Theorem 1.1 contains the main result and is proved in Section 4. We start with a bit strange three-layered partition of unity defined in (4.1)-(4.12). This idea goes back at least to [4, 5]. The main effort consists in isolating the bulk of \( \Lambda_L \) - where only operators defined in the whole space will act - from the region close to the boundary. Note that in the absence of the magnetic field, it would be enough to work with only two cut-off functions: one isolating the bulk from the boundary, and the other one supported in a tubular neighborhood of the boundary. When long-range magnetic fields are present, this is not enough. The tubular neighborhood needs to be chopped up in many small pieces, in order to apply local gauge transformations (see below why we need them).

The central idea in proving (1.12) is to show that the contribution to the total trace of the region close to the boundary grows slower than the volume. Technically, this is obtained by approximating the true resolvent \( (H_L(B) - z)^{-1} \) with an operator \( U_L(B, z) \) given in (4.10). \( U_L(B, z) \) contains the bulk term, plus a boundary contribution which consists from a sum of terms each locally approximating \( (H_L(B) - z)^{-1} \) and containing a specially tailored local gauge given in (4.13). These locally defined vector potentials are made globally bounded with the help of our third layer of cut-off functions \( \tilde{g}_\gamma \)'s. The switch to the local gauge is performed through the central identity (3.18). An explanation of why \( U_L(B, z) \) is a convenient approximation for the full resolvent can be found right after (4.18), and there is the place where we first fully use the exponential estimates of the Appendix.

In Proposition 4.1 we prove that we can replace \( (H_L(B) - z)^{-1} \) with \( U_L(B, z) \) in the conductivity formula, without changing the value of the limit. In Proposition 4.1 we show that only the bulk term from \( U_L(B, z) \) will contribute. In Proposition 4.6 we show that removing the cut-offs only gives a surface contribution.

The proof of (1.13) is heavily based on magnetic perturbation theory. Although the technical estimates are considerably more involved than at the previous point, the main idea is the same: the boundary terms can be discarded. The full power of the magnetic phases is used in Lemma 4.10; an heuristic explanation of how and why they manage to kill the linear growth of the magnetic potential is given right after (4.73).

## 2 Proof of i.

The integrand in (1.11) is a bounded operator, with an \( L^2 \) norm bounded by a constant times \( |\Re(z)|^2 \), uniformly in \( L \), as we can see from (1.6) and (5.2). Because \( f_{FD} \) has an exponential decay in \( |\Re(z)| \), the integral converges and defines a bounded operator. Let us note that the integrand is not a trace class operator under our conditions. But the total integral is a different matter. The point is that we can integrate by parts with respect to \( z \) by using anti-derivatives of \( f_{FD} \) which are still decaying exponentially at infinity. By doing this at least four times, we obtain integrals of the form
Proposition 3.1. Fix $\langle|\rangle$ has a smooth kernel which does not grow too fast with 
where the above constant does not depend on $L$ where $x$ and by using (5.7) with 
The main point is that $(H_L(B) - z)^{-1}$ is Hilbert-Schmidt since we can write 
$$(H_L(B) - z)^{-1} = [(-i\nabla + B\alpha)^2 + 1]^{-1} \{[(-i\nabla + B\alpha)^2 + 1](H_L(B) - z)^{-1}\},$$
and by using (5.7) with $\delta = 0$, and (5.13), we obtain 
$$\| (H_L(B) - z)^{-1} \|_{B_2} \leq \text{const} \cdot \sqrt{\text{Vol}(\Lambda_L)} \| R(z) \|,$$
where the above constant does not depend on $L$ and $z$. Thus $(H_L(B) - z)^{-2}$ is trace class, and 
the trace norm of the integrand in (2.2) is bounded by 
$$\text{const} \cdot |f(z)| \cdot (R(z))^4 \cdot \text{Vol}(\Lambda_L)$$
where again the constant is uniform in $L$ and $z$. This now is integrable on the contour, thus the 
integral defines a trace class operator. Moreover, its trace grows at most like the volume of $\Lambda_L$, 
hence $\limsup_{L \to \infty} |\sigma_L| < \infty$.

Remark. The same type of argument may be used to show that $\sigma_L(B)$ is smooth in $B$, by 
repeatedly using the formal identity 
$$\partial_B(H_L(B) - z)^{-1} = -(H_L(B) - z)^{-1} \{\partial_B H_L(B)\} (H_L(B) - z)^{-1}$$
in the sense of bounded operators. Note the important fact that $\partial_B H_L(B)$ will generate some 
linear growing terms coming from the magnetic vector potential $a(x)$, therefore the trace norm 
of the new integrand will grow like $L^4$. We therefore cannot conclude here that the derivatives 
$|\partial_B^m \sigma_L(B)|$ will admit a finite lim sup when $L \to \infty$.

3 Proof of ii.

We are going to prove the regularity statement for the kernel without using the periodicity of $V$, 
only the fact that the potential is smooth and bounded on $\mathbb{R}^3$ together with all its derivatives.

The strategy consists in integrating by parts with respect to $z$ many times, such that we obtain 
high powers of the resolvent $(H_\infty(B) - \zeta)^{-N}$. Then we will prove that $P_j(B)(H_\infty(B) - \zeta)^{-N} P_k(B)$ 
has a smooth kernel which does not grow too fast with $|\zeta|$.

Let us now be more precise and start with some technical results.

Proposition 3.1. Fix $0 < \eta < 1$ and choose $z \in \mathbb{C}$ with $\text{dist}\{z, [0, \infty]\} = \eta > 0$. Let $r = \|R(z)\|$. 
Then the operator $(H_\infty(0) - z)^{-1}$ has an integral kernel $G_1(x, x'; z)$ which is smooth away from 
the diagonal $x = x'$. There exists $\delta > 0$ and some $M \geq 1$ such that for any multi-index $\alpha \in \mathbb{N}^3$ 
with $|\alpha| \leq 1$ we have the estimate 
$$\sup_{x \neq x' \in \mathbb{R}^3} |x - x'|^{\alpha + 1} e^{-\frac{|x - x'|}{\eta}} |D_2^\alpha G_1(x, x'; z)| = C_1(\alpha, \eta) r^M < \infty.$$
Proof. The result without the exponential decay is essentially contained in [15]. The symbol of $H_{\infty}(0)$, denoted by $h_0(\mathbf{x}, \xi)$ belongs to $S^2_{1,0}(\mathbb{R}^3 \times \mathbb{R}^3)$, $H_{\infty}(0) \in L^2_{1,0}(\mathbb{R}^3)$ (see Example 3.1 in [15]), and is uniformly elliptic.

Fix $\lambda > 0$ large enough. We can apply Theorem 4.1 in [15] and construct a parametrix for $H_{\infty}(0) + \lambda$ starting from the symbol $q_0(\mathbf{x}, \xi) := 1/(h(\mathbf{x}, \xi) + \lambda) \in S^{-3}_{1,0}(\mathbb{R}^3 \times \mathbb{R}^3)$. The symbol giving the parametrix is an asymptotic sum of symbols, starting with $q_0$, then the next one is in $S^{-3}_{1,0}$ and so on. Each term gives a contribution to the integral kernel of the parametrix. The most singular contribution is (in the sense of oscillatory integrals):

$$
\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i(\mathbf{x} - \mathbf{x}') \cdot \xi} q_0(\mathbf{x}, \xi) d\xi.
$$

We only have to consider a few terms besides this one, since symbols in $S^{-N}_{1,0}$ for large $N$ generate more and more regular kernels at the diagonal. By standard “integration by parts” tricks, and using the facts that we work in three dimensions, one can prove the estimate

$$
\sup_{\mathbf{x} \neq \mathbf{x}' \in \mathbb{R}^3} |\mathbf{x} - \mathbf{x}'|^{3/2} |D^2_{\mathbf{x},\mathbf{x}'} G_1(\mathbf{x}, \mathbf{x}'; -\lambda)| = \text{const}(\alpha, \lambda) < \infty.
$$

(3.2)

In fact, outside the region $|\mathbf{x} - \mathbf{x}'| \geq 1$ we can integrate by parts several times with respect to $\xi$ and prove that $G_1(\mathbf{x}, \mathbf{x}'; -\lambda)$ decays faster than any power of $|\mathbf{x} - \mathbf{x}'|$. But the Combettes-Thomas method will give a better, exponential localization.

The important thing is that the $L^2$ estimates from the Combettes-Thomas argument can be transferred into point-wise estimates for the kernel. Let us now prove this.

Using [5.13] at $L = \infty$ and $B = 0$, together with the triangle and Cauchy-Schwarz inequalities, we get that $(-\Delta + \lambda)^{-1}$ with exponential weights maps $L^2$ into $L^\infty$. The key estimate is (0 < $c < 1$)

$$
e^{-cv\sqrt{\lambda}|\mathbf{x}-\mathbf{x}_0|}|K_{\infty}(\mathbf{x},\mathbf{x}')|e^{c \sqrt{\lambda}|\mathbf{x}'-\mathbf{x}_0|} \leq \frac{e^{-(1-c)v\sqrt{\lambda}|\mathbf{x}-\mathbf{x}'|}}{4\pi |\mathbf{x} - \mathbf{x}'|}.
$$

Since we can write:

$$
(H_{\infty}(0) + \lambda)^{-1} = (-\Delta + \lambda)^{-1}(-\Delta + \lambda)(H_{\infty}(0) + \lambda)^{-1},
$$

(3.3)

using [5.12] (at $L = \infty$ and $B = 0$), it follows that this resolvent with exponential weights is a bounded map from $L^2$ into $L^\infty$. More precisely, for any $\mathbf{x}_0 \in \mathbb{R}^3$, there exists $0 < c < 1$ small enough such that:

$$
\sup_{\mathbf{x}_0} ||e^{-c \sqrt{\lambda}(-\mathbf{x}_0)}(H_{\infty}(0) + \lambda)^{-1} e^{c \sqrt{\lambda}(-\mathbf{x}_0)}||_{B(L^2, L^\infty)} \leq \text{const}, \quad \lambda \geq \lambda_0.
$$

(4.3)

Now if we look at the map

$$
C_0^\infty(\mathbb{R}^3) \ni \Psi \rightarrow \int_{\mathbb{R}^3} G_1(\mathbf{x}_0, \mathbf{x}; -\lambda)e^{\delta_{\lambda}(\mathbf{x}-\mathbf{x}_0)}\Psi(\mathbf{x}) d\mathbf{x},
$$

(it makes sense to fix $\mathbf{x}_0$ since the resolvent maps smooth functions into smooth functions), we see that by using [5.14] we can extend this map to a linear and bounded functional on $L^2$. Riesz’ representation theorem then gives:

$$
\sup_{\mathbf{x}_0 \in \mathbb{R}^3} ||e^{c \sqrt{\lambda}(-\mathbf{x}_0)} G_1(\mathbf{x}_0, \cdot; -\lambda)||_{L^2} = \sup_{\mathbf{x}_0 \in \mathbb{R}^3} ||e^{c \sqrt{\lambda}(-\mathbf{x}_0)} G_1(\cdot, \mathbf{x}_0; -\lambda)||_{L^2} \leq \text{const},
$$

(5.3)

uniformly in $\lambda \geq \lambda_0$. Using this, together with the Cauchy-Schwarz and the triangle inequality, we get that the integral kernel $G_2(\mathbf{x}, \mathbf{x}'; -\lambda)$ of $(H_{\infty}(0) + \lambda)^{-2}$ obeys uniformly in $\lambda \geq \lambda_0$:

$$
\sup_{\mathbf{x}, \mathbf{x}'} e^{\sqrt{\lambda}||\mathbf{x} - \mathbf{x}'||} |G_2(\mathbf{x}, \mathbf{x}'; -\lambda)|
$$

$$
\leq \sup_{\mathbf{x}, \mathbf{x}'} \int e^{\sqrt{\lambda}||\mathbf{x} - \mathbf{x}'||} G_1(\mathbf{x}, \mathbf{x}''; -\lambda)e^{c \sqrt{\lambda}||\mathbf{x}'' - \mathbf{x}||} G_1(\mathbf{x}'', \mathbf{x}'; -\lambda) d\mathbf{x}'' \leq \text{const}.
$$

(6.3)
Now if $|x - x'| \geq 1$, write
\[
G_1(x, x'; -\lambda) = \int_\lambda^\infty G_2(x, x'; -\lambda_1)d\lambda_1,
\]
which together with (3.6) and the integrability of $e^{-\frac{1}{2}\sqrt{x}}$ imply that
\[
\sup_{|x-x'| \geq 1} e^{\frac{1}{2}\sqrt{|x-x'|}} |G_1(x, x'; -\lambda)| \leq \text{const.}
\]

We can also deal with derivatives with respect to $x$, by showing that the operator $D_j(H_\infty(0) + \lambda)^{-N}$ ($N$ large enough) has an integral kernel $D_jG_N(x, x'; -\lambda)$ obeying the same type of estimate as in (3.6). This is done by commuting $D_j$ several times with a few resolvents; let us see how it works.

First, by commuting we have:
\[
D_j(H_\infty(0) + \lambda)^{-1} = (H_\infty(0) + \lambda)^{-1}D_j + (H_\infty(0) + \lambda)^{-2}T_1,
\]
where $T_1$ is bounded due to the fact that $[H_\infty(0), D_j]$ is bounded, while $[H_\infty(0), [H_\infty(0), D_j]]$ is relatively bounded with respect to $H_\infty(0)$.

Second, by commuting twice we have:
\[
D_j(H_\infty(0) + \lambda)^{-2} = (H_\infty(0) + \lambda)^{-2}D_j + (H_\infty(0) + \lambda)^{-3}T_1 + (H_\infty(0) + \lambda)^{-2}T_1(H_\infty(0) + \lambda)^{-1},
\]

This identity allows us to write:
\[
D_j(H_\infty(0) + \lambda)^{-N+2} = (H_\infty(0) + \lambda)^{-2}T(H_\infty(0) + \lambda)^{-N+5},
\]
where $T$ with exponential weights is bounded from $L^2$ to $L^2$. Then we prove that the integral kernel of $D_j(H_\infty(0) + \lambda)^{-N+2}$ obeys an $L^2$ estimate like in (3.5), then from the identity
\[
D_j(H_\infty(0) + \lambda)^{-N} = D_j(H_\infty(0) + \lambda)^{-N+2}(H_\infty(0) + \lambda)^{-2}
\]
we get the needed $L^\infty$ estimate by mimicking (3.6).

Then we write
\[
D_jG_1(x, x'; -\lambda) = \frac{(-1)^N}{(N-1)!} \int_\lambda^\infty d\lambda_1 \int_\lambda^\infty d\lambda_2 \ldots \int_\lambda^\infty d\lambda_N D_j G_N(x, x', -\lambda')
\]
and propagate the exponential decay over the integrals in $\lambda$.

Therefore we can state the first result regarding the exponential localization. For $\lambda$ large enough we have:
\[
\sup_{x \neq x' \in \mathbb{R}^3} |x - x'|^{[\alpha+1]} e^{\frac{1}{2}\sqrt{|x-x'|}} |D_x^\alpha G_1(x, x'; -\lambda)| = \text{const}(\alpha, \lambda) < \infty.
\]

Now let us investigate the $z$ dependence. Let us apply the resolvent identity several times and get ($N \geq 2$):
\[
(H_\infty(0) - z)^{-1} = (H_\infty(0) + \lambda)^{-1} + (z + \lambda)(H_\infty(0) + \lambda)^{-2} + \ldots + (z + \lambda)^N (H_\infty(0) + \lambda)^{-N}(H_\infty(0) - z)^{-1}.
\]

The idea is to keep the $z$ dependence to the right in the last term, and to keep a regular kernel to the left. We start with a norm estimate. From the usual resolvent identity:
\[
(H_\infty(0) - z)^{-1} = (H_\infty(0) + \lambda)^{-1} + (z + \lambda)(H_\infty(0) + \lambda)^{-1}(H_\infty(0) - z)^{-1},
\]
the use (3.14) and (3.15) (with $L = \infty$) provides us with some $N_1 > N$ such that:

$$
\sup_{x_0} \| e^{L(x_0)} (H_\infty(0) - z)^{-1} e^{L(x_0)} \|_{L^2(L^\infty)} \leq \text{const}(\eta) \cdot r^{N_1}.
$$

(3.14)

This estimate implies that the map (initially defined on compactly supported functions)

$$
L^2(\mathbb{R}^3) \ni \Psi \to \int_{\mathbb{R}^3} G_1(x_0, x; \tau) e^{L(x-x_0)} \Psi(x) dx \in \mathbb{C},
$$

is a bounded linear functional. Riesz’ representation theorem leads us to:

$$
\sup_{x_0 \in \mathbb{R}^3} \| e^{L(x_0)} G_1(x_0, \cdot; \tau) \|_{L^2} = \sup_{x_0 \in \mathbb{R}^3} \| e^{L(x_0)} G_1(\cdot, x_0; \eta) \|_{L^2} \leq \text{const}(\eta) \cdot r^{N_1}.
$$

(3.15)

We are only left with the case in which we have a derivative on the left. Using (3.15) in (3.13) and the other results we have obtained for the kernel where $z = \lambda$, it is not hard to obtain the exponential decay claimed in (3.1).

Proposition 3.2. Assume that $\alpha \in \{0, 1\}$ and $0 < \eta < 1$. Let $\text{dist}\{z, [0, \infty)\} = \eta$ and $N$ is large enough. Then the operator $P^\alpha(B)(H_\infty(B) - z)^{-\eta} P^{1-\alpha}_L(B)$ has a jointly continuous integral kernel $K_{N,B}(x, x'; z)$, and there exists $\delta > 0$ small enough and $M$ large enough such that:

$$
\sup_{x, x' \in \mathbb{R}^3} e^{\delta \delta} \| \rho(x-x') \|_{K_{N,B}(x, x'; z)} \leq \text{const}(N, B, \eta) \cdot \langle r \rangle^M.
$$

(3.16)

Proof. Although this particular result might be obtained with other methods, we will employ the magnetic perturbation theory as developed in [11, 5, 27]. For different approaches involving magnetic pseudo-differential calculus, see [17, 13, 24, 25, 21].

We can assume that the magnetic vector potential is expressed in the transverse gauge, given in (1.5). Define the antisymmetric magnetic gauge phase

$$
\varphi_0(x, y) := -A(y) \cdot x = \frac{1}{2} (y_2 x_1 - y_1 x_2) = \frac{1}{2} e_3 \cdot (x \wedge y),
$$

(3.17)

where $e_3$ denotes the unit vector $(0, 0, 1) \in \mathbb{R}^3$. Then we have the following identity (true for instance on Schwartz functions), valid for every vector $y$ kept fixed in $\mathbb{R}^3$:

$$
\{ P_x(0) + B A(x) \} e^{iB \varphi_0(x, y)} = e^{iB \varphi_0(x, y)} \{ P_x(0) + B A(x - y) \}.
$$

(3.18)

For every $z \in \mathbb{C} \setminus [0, \infty)$ define

$$
S_B(x, x'; z) := e^{iB \varphi_0(x, x')} G_1(x, x'; z).
$$

(3.19)

This integral kernel generates an $L^2$ bounded operator (use the estimate (3.1) and employ the Schur-Holmgren criterion). We denote this operator by $S_B(z)$. There are two important facts related to this kernel. First, $G_1(x, x'; z)$ solves the distributional equation

$$
(P^2_0(0) + V(x) - z) G_1(x, x'; z) = \delta(x - x').
$$

Second, one can prove that $S_B(z)$ maps Schwartz functions into Schwartz functions. Moreover, employing (3.18) and integrating by parts, we can establish an identity which holds at first in the weak sense on Schwartz functions:

$$
((H_\infty(B) - \tau) \Psi, S_B(z) \Xi) = \langle \Psi, (1 + B T_B(z)) \Xi \rangle,
$$

(3.20)

where $T_B(z)$ is the operator generated by the following integral kernel:

$$
T_B(x, x'; z) := e^{iB \varphi_0(x, x')} \cdot \big\{ -2iA(x - x') \cdot \nabla_x G_1(x, x'; z) + B|A(x - x')|^2 G_1(x, x'; z) \big\}.
$$

(3.21)
Using (3.1), and the fact that \(|A(x - x')| \leq |x - x'|\), we obtain the following point-wise estimate true for all \(x \neq x'\):

\[
\max[|S_B(x, x'; z)|, |T_B(x, x'; z)|] \leq \text{const} \frac{e^{-\frac{1}{
abla_1^2}|x-x'|}}{|x-x'|} \cdot \langle r \rangle^M. \tag{3.22}
\]

Clearly, \(T_B(z)\) can be extended to a bounded operator.

Now let us prove that (3.20) also holds true in the strong sense. Because \(H_\infty(B)\) is essentially self-adjoint on the set of Schwartz functions, (3.20) can be extended to any \(\Xi \in \mathcal{L}^2\) and any \(\Psi \in \text{Dom}(H_\infty(B))\). It means that the range of \(S_B(z)\) belongs to the domain of \(H_\infty(B)\) and

\[
(H_\infty(B) - z)S_B(z) = 1 + B T_B(z). \tag{3.23}
\]

At this point we can establish the following identity, valid for all \(z\) of interest:

\[
(H_\infty(B) - z)^{-1} = S_B(z) - B(H_\infty(B) - z)^{-1}T_B(z). \tag{3.24}
\]

Denote by \(\hat{T}_B(z) := T_B(z)\) the bounded operator generated by the kernel

\[
\hat{T}_B(x, x'; z) := \overline{T_B(x', x; \overline{z})}.
\]

Now replace \(z\) with \(\overline{z}\) in (3.24) and then take the adjoint. We obtain:

\[
(H_\infty(B) - z)^{-1} = S_B(z) - B\hat{T}_B(z)(H_\infty(B) - z)^{-1}. \tag{3.25}
\]

Denote by \(K_{1, B}(x, x'; z)\) the integral kernel of \((H_\infty(B) - z)^{-1}\). With (3.25) as starting point, together with (3.26), we can use the same argument as in the zero magnetic field case, in order to show that the resolvent (with exponential weights) maps \(L^2\) into \(L^\infty\). Thus its kernel obeys an estimate like in (3.15). Moreover, (3.25) implies via the Cauchy-Schwarz inequality that

\[
|K_{1, B}(x, x'; z)| \leq \text{const} \frac{e^{-\frac{1}{
abla_1^2}|x-x'|}}{|x-x'|} \cdot \langle r \rangle^M. \tag{3.26}
\]

Moreover, we can repeat the arguments from the proof of Proposition 3.1 and get the boundedness and joint continuity for the kernel of \(P_j(B)(H_\infty(B) - z)^{-N}\) for large \(N\), since all we have to do is to change \(D_j\) with \(P_j(B)\) and to notice that the formal commutator \([P_j(B), H_\infty(B)]\) is only linear in \(P_k(B)\)'s, therefore \(P_j(B)\) "commutes well" with \((H_\infty(B) - z)^{-1}\).

Now let us show that \((H_\infty(B) - z)^{-1}\) maps \(L^2\) into Hölder continuous functions. In fact, one can prove the following estimate:

**Lemma 3.3.** Fix a compact set \(U \subset \mathbb{R}^3\), and fix \(\beta \in (0, 1/2)\). Take \(\psi\) with \(||\psi||_{L^2} = 1\). Then there exist two positive constants \(C\) and \(M\) such that

\[
\sup_{z \in \Gamma_\nu} \langle r \rangle^{-M} |\{(H_\infty(B) - z)^{-1}\psi}(x) - \{(H_\infty(B) - z)^{-1}\psi}(y)| \leq C \cdot |x - y|^{\beta}, \tag{3.27}
\]

for any \(x, y \in U\). The same estimate holds true for \(S_B(z)\).

**Proof.** The domain of \(H_\infty(B)\) is locally \(H^2\), hence for some positive \(\lambda\), the function \((H_\infty(B)_+ \lambda)^{-1}\psi\) is locally \(H^2\). From the Sobolev embedding lemma, we obtain that \((H_\infty(B) + \lambda)^{-1}\psi\) is \(\beta\)-Hölder continuous for every \(\beta \in (0, 1/2)\). The estimate (3.27) follows from the resolvent identity

\[
(H_\infty(B) - z)^{-1} = (H_\infty(B) + \lambda)^{-1} + (z + \lambda)(H_\infty(B) + \lambda)^{-1}(H_\infty(B) - z)^{-1}.
\]

The same result for \(S_B(z)\) follows from (3.24). \(\square\)
We now can start the actual proof of (ii). We integrate by parts $N$ times with respect to $z$ in the expression of $F_{\infty}$, and $N$ is supposed to be large. The terms we obtain in the integrand will look like this one:

\[
P_j(B)(H_\infty(B) - z_1)^{-N_1}P_k(B)(H_\infty(B) - z_2)^{-N_2},
\]

where $N_1 + N_2 = N + 2$, and either $N_1$ or $N_2$ is large. Here $z_1$ and $z_2$ are complex numbers like $z \in \Gamma_{\omega}$ or $z \pm \omega$.

By repeated commutations, we can always write this operator as $(H_\infty(B) - z_1)^{-1}W(H_\infty(B) - z_2)^{-1}$, where $W$ is a sum of terms like this one:

\[
W_1 := (H_\infty(B) - z_1)^{-n_1}v_{a_1}P_{a_1}(B)(H_\infty(B) - z_1)^{-n_2}v_{a_2}P_{a_2}(B)(H_\infty(B) - z_1)^{-n_3}.
\]

Here $n_1, n_2, n_3 \geq 1$, $v_{a_i}$'s are smooth and uniformly bounded functions. Note that such a term already starts and ends with a resolvent. Using the fact that $P_{a_1}(B)(H_\infty(B) - z_1)^{-n_2}P_{a_2}(B)$ is a bounded operator, then we can always write $W$ as the product of the form $(H_\infty(B) - z_1)^{-1}W$ where $W$ is a bounded operator whose norm increases at most polynomially in $r$. Thus $e^{-\langle \cdot \rangle \delta/r}W$ is a Hilbert-Schmidt operator, with a H-S norm which increases polynomially in $r$. Hence it is an integral operator which obeys the estimate:

\[
\int \int e^{-2\langle y, y' \rangle / r} |W(y, y'; z)|^2 dy \, dy' \leq \text{const} \, r^M,
\]

where the constant and $M$ are independent of $r$. Thus we have:

\[
P_j(B)(H_\infty(B) - z_1)^{-N_1}P_k(B)(H_\infty(B) - z_2)^{-N_2} = (H_\infty(B) - z_1)^{-1}e^{\langle \cdot \rangle \delta/r}e^{-\langle \cdot \rangle \delta/r}W(H_\infty(B) - z_2)^{-1}.
\]

Using an identity like the one in (5.8), we can rewrite the above operator as:

\[
e^{\langle \cdot \rangle \delta/r}W(H_\infty(B) - z_1)^{-1}Te^{-\langle \cdot \rangle \delta/r}W(H_\infty(B) - z_2)^{-1},
\]

where $T$ is a bounded operator uniformly in $r$ if $\delta$ is small enough. Thus $W' := Te^{-\langle \cdot \rangle \delta/r}W$ is Hilbert-Schmidt, and has an integral kernel whose norm in $L^2(\mathbb{R}^d)$ is polynomially bounded in $r$.

Therefore we are left with the investigation of the joint continuity in $x$ and $x'$ of the integral kernel defined by:

\[
f_\infty(x, x') := \int \int K_{1, B}(x, y; z_1)W'(y, y')K_{1, B}(y', x'; z_2) dy \, dy',
\]

where $\int \int |W'(y, y')|^2 dy \, dy' \leq \text{const} \, r^M$. Using this, (3.20), and the Cauchy-Schwarz inequality, we obtain $|f(x, x')| \leq \text{const} \, r^{M_1}||W'||_{L^2}$, uniformly in $x, x' \in U$. Since $W'$ is Hilbert-Schmidt, we can approximate $W'(y, y')$ with a finite sum of the type $\sum g_j(y)h_j(y')$ where $g$'s and $h$'s are $L^2$. Thus we can (uniformly in $x, x' \in U$) approximate the function $f_\infty$ with functions of the type

\[
\sum_{j=1}^n (H_\infty(B) - z_1)^{-1}g_j(x)(H_\infty(B) - z_2)^{-1}h_j(x'),
\]

which from Lemma 3.3 we know they are continuous on compacts. Hence $f_\infty$ is jointly continuous on its variables. As for the integral kernel of $F_{\infty}$, we see that it can be written as the integral with respect to $z$ of a finite number of kernels of the same type as $f_\infty$. Due to the exponential decay of $f_{F_D}$ (see (1.7)), we see that we can approximate $F_{\infty}(x, x')$ uniformly on compacts with continuous functions, and we are done.

Therefore the function $s_B$ as defined in Theorem 1.1 is a continuous function. If $V$ is periodic with respect to $Z^d$, then $H_\infty(B)$ commutes with the magnetic translations, defined for every $\gamma \in Z^d$ as (see also (3.18)):

\[
[M_\gamma \psi](x) := e^{iB\varphi_\psi(x, \gamma)}\psi(x - \gamma), \quad M_{-\gamma}M_\gamma = 1.
\]
Hence we have that as operators, $M_{-\gamma} F_\alpha M_\gamma = F_\alpha$, which for kernels gives

$$e^{-iB\varphi_0(x,\gamma)}F(x + \gamma, x' + \gamma)e^{iB\varphi_0(x' + \gamma, \gamma)}F(x, x').$$

(3.31)

Now since $\varphi_0(\gamma, \gamma) = 0$, when we put $x = x'$ we get $s_B(x + \gamma) = s_B(x)$ and we are done with (ii).

4 Proof of iii.

We start by proving the thermodynamic limit for the conductivity, that is (1.12). We need to introduce a special partition of unity in $\Lambda_L$.

4.1 Partitions and cut-offs

Fix $0 < \alpha < 1$ (small enough, to be chosen later) and define for $t > 0$:

$$\Xi_L(t) := \{ x \in \Lambda_L : \text{dist}(x, \partial\Lambda_L) \leq tL^\alpha \}.$$  

(4.1)

This models a "thin" compact subset of $\Lambda_L$, near the boundary, with a volume of order $tL^{2+\alpha}$. Because we assumed that the boundary $\partial\Lambda_1$ was smooth, all points of $\Xi_L(t)$ have unique projections on $\partial\Lambda_L$, if $L$ is large enough.

Then if $t_1 < t_2$ we have $\Xi_L(t_1) \subset \Xi_L(t_2)$ and:

$$\text{dist}(\Xi_L(t_1), \Lambda_L \setminus \Xi_L(t_2)) \geq (t_2 - t_1)L^\alpha.$$  

(4.2)

The subset $\Lambda_L \setminus \Xi_L(1)$ models the "bulk region" of $\Lambda_L$, which is still "far-away" from the boundary.

Now consider the inclusion of $\Xi_L(2)$ in the dilated lattice $L^\alpha \mathbb{Z}^3$. That is we cover $\Xi_L(2)$ with disjoint closed cubic boxes parallel to the coordinate axis, centered at points in $L^\alpha \mathbb{Z}^3$, of side length $L^\alpha$. Denote by $E \subset L^\alpha \mathbb{Z}^3$ the set of centers of those cubes which have common points with $\Xi_L(2)$. Clearly, due to volume considerations, $\# E \sim L^{2-2\alpha}$.

In order to fix notation, let us denote by $K(\gamma, s)$ the cube centered at $\gamma \in E$, with side length equal to $s \geq L^\alpha$. Moreover, denote by $\tilde{E} := E \cup \{(0,0,0)\}$ (note that the origin cannot belong to $E$ if $L$ is large enough).

Now choose a partition of unity $\{g_\gamma\}_{\gamma \in \tilde{E}}$ of $\Lambda_L$ which has the following properties:

$$\text{supp}(g_\gamma) \subset \Lambda_L \setminus \Xi_L(1);$$  

(4.3)

$$\text{supp}(g_\gamma) \subset K(\gamma, 2L^\alpha), \gamma \in E;$$  

(4.4)

$$0 \leq g_\gamma \leq 1, \quad \sum_{\gamma \in \tilde{E}} g_\gamma(x) = 1, \quad \forall x \in \Lambda_L;$$  

(4.5)

$$||D^3 g_\gamma||_\infty \sim L^{-\alpha|\beta|}, \quad \forall \beta \in \mathbb{N}^3, \quad \forall \gamma \in \tilde{E}.$$  

(4.6)

This partition has the property that if we restrict ourselves to $E$, then uniformly in $L$, the number of $g_\gamma$'s which are not zero at the same time is bounded by a constant. Only $g_0$ has $\sim L^{2-2\alpha}$ neighbors whose supports have common points with $\text{supp}(g_0)$.

Now we choose another set of functions, $\{\tilde{g}_\gamma\}_{\gamma \in \tilde{E}}$ having the following properties:

$$\text{supp}(\tilde{g}_0) \subset \Lambda_L \setminus \Xi_L(1/4); \quad \tilde{g}_0(x) = 1 \text{ if } x \in \Lambda_L \setminus \Xi_L(1/2);$$  

(4.7)

$$\text{supp}(\tilde{g}_\gamma) \subset K(\gamma, 10L^\alpha), \gamma \in E; \quad \tilde{g}_\gamma(x) = 1 \text{ if } x \in K(\gamma, 9L^\alpha)$$  

(4.8)

$$||D^3 \tilde{g}_\gamma||_\infty \sim L^{-\alpha|\beta|}, \quad \forall \beta \in \mathbb{N}^3, \quad \forall \gamma \in \tilde{E}.$$  

(4.9)

These functions were chosen "wider" than the $g_\gamma$'s, and obey

$$\tilde{g}_\gamma g_\gamma = g_\gamma, \quad \text{dist}(\text{supp}(D\tilde{g}_\gamma), \text{supp}(g_\gamma)) \sim L^\alpha, \quad \gamma \in \tilde{E}.$$  

(4.10)
By $D\tilde{g}$, we mean that we take at least one derivative of $\tilde{g}_\gamma$.

Now let us define the third type of cut-offs, $\{\tilde{g}_\gamma\}_{\gamma \in E}$:

\begin{equation}
\text{supp}(\tilde{g}_\gamma) \subset K(\gamma, 12L^\alpha), \gamma \in E; \quad \tilde{g}_\gamma(x) = 1 \text{ if } x \in K(\gamma, 11L^\alpha) \quad (4.11)
\end{equation}

\begin{equation}
\|D^\beta\tilde{g}_\gamma\|_\infty \sim L^{-\alpha|\beta|}, \quad \forall \beta \in \mathbb{N}^3, \quad \forall \gamma \in E. \quad (4.12)
\end{equation}

Note that we have $\tilde{g}_\gamma \tilde{g}_\gamma = \tilde{g}_\gamma$ (the origin is not considered here).

### 4.2 Proof of $(1.12)$

Define for every $\gamma \in E$:

\begin{equation}
A_\gamma(x) := \tilde{g}_\gamma(x) A(x - \gamma). \quad (4.13)
\end{equation}

Due to the support properties of our cut-off functions, we have the estimates

\begin{equation}
\begin{aligned}
\tilde{g}_\gamma A_\gamma &= \tilde{g}_\gamma A(x - \gamma), \\
\|A_\gamma\|_{C^1(\mathbb{R}^3)} &\leq \text{const} \cdot L^\alpha.
\end{aligned} \quad (4.14)
\end{equation}

Define for every $\gamma \in E$ (see also $(4.3)$ and $(4.4)$):

\begin{equation}
P_\gamma(B) := P(0) + BA_\gamma, \quad H_L(B, \gamma) := P_\gamma(B)^2 + V, \quad (4.15)
\end{equation}

where the Hamiltonian is defined with Dirichlet boundary conditions. Note that $H_L(B, \gamma) - H_L(0)$ is a relatively bounded perturbation of $H_L(0)$.

Define the operator

\begin{equation}
U_L(B, z) := \tilde{g}_0(H_\infty(B) - z)^{-1}\tilde{g}_0 + \sum_{\gamma \in E} e^{iB_\phi(\cdot, \gamma)} \tilde{g}_\gamma(H_L(B, \gamma) - z)^{-1}e^{-iB_\phi(\cdot, \gamma)} \tilde{g}_\gamma, \quad (4.16)
\end{equation}

One can prove that the range of $U_L(B, z)$ is in the domain of $H_L(B)$ and we have:

\begin{equation}
\begin{aligned}
(H_L(B) - z)U_L(B, z) &= 1 + V_L(B, z), \\
V_L(B, z) &:= \{-2i(\nabla \tilde{g}_0) \cdot P(0) - (\Delta \tilde{g}_0)(H_\infty(B) - z)^{-1} \tilde{g}_0 \}
\end{aligned} \quad (4.17)
\end{equation}

\begin{equation}
+ \sum_{\gamma \in E} e^{iB_\phi(\cdot, \gamma)} \{-2i(\nabla \tilde{g}_\gamma) \cdot P_\gamma(B) - (\Delta \tilde{g}_\gamma)(H_L(B, \gamma) - z)^{-1}e^{-iB_\phi(\cdot, \gamma)} \tilde{g}_\gamma.
\end{equation}

In order to obtain this equality we used the locality of our operators and various support properties of our cut-off functions, the identity $(4.3)$, definition $(4.4)$, and $(4.5)$.

Then we can write

\begin{equation}
(H_L(B) - z)^{-1} = U_L(B, z) + (H_L(B) - z)^{-1}V_L(B, z). \quad (4.18)
\end{equation}

The good thing about $V_L(B, z)$ is that its operator norm is exponentially small in $L^\alpha$. This is because we have the boundedness from $(4.3)$ and $(4.6)$ (valid also for $H_L(B, \gamma)$, as can easily be seen from the proofs), and because of the estimate in $(4.11)$. Indeed, for terms involving $\gamma \neq 0$, put $x_0 = \gamma$ in the two exponential estimates, and take $+\delta$ on the left and $-\delta$ on the right. Then we gain an overall decaying term from the left as $(4.11)$ implies:

\begin{equation}
\sup_{x \in \text{supp}(D\tilde{g}_\gamma)} \sup_{x' \in \text{supp}(\tilde{g}_\gamma)} e^{-\frac{\delta}{|\gamma|}((x - \gamma) - (x' - \gamma))} \leq e^{-\frac{\delta_1}{|\gamma|}L^\alpha}, \quad \gamma \neq 0, \quad (4.19)
\end{equation}

where $\delta_1 > 0$ is small enough and $L$ is larger than some $L_0$. 

14
For $\gamma = 0$ the situation is slightly different, because we did not assume convexity for $\Lambda_L$. But one of the terms whose norm we need to estimate is (see (4.17))

$$(\nabla \hat{g}_0) \cdot P(B)(H_{\infty}(B) - z)^{-1}g_0.$$  

From (4.1) follows that the integral kernel of this operator is bounded by

$$||(\nabla \hat{g}_0) \cdot P(B)(H_{\infty}(B) - z)^{-1}g_0||_{\infty}(x, x') \leq \text{const}(\eta)\langle r \rangle^{M} e^{-\frac{\alpha}{2\gamma}||x-x'||}.$$  

(4.20)

Because $x$ and $x'$ are always separated by $\sim L^\alpha$ (see the support properties for our cut-offs), we can write:

$$||(\nabla \hat{g}_0) \cdot P(B)(H_{\infty}(B) - z)^{-1}g_0||_{\infty}(x, x') \leq \text{const}(\eta)\langle r \rangle^{M} e^{-\frac{\delta_1}{2\gamma}||x-x'||}.$$  

(4.21)

where $\delta_1$ and $\delta_2$ are smaller than $\delta$.

Therefore we can write for all $N \geq 1$:

$$||V_L(B, z)|| \leq \text{const}(\eta) \cdot L^{2-2\alpha} \langle r \rangle^{M} e^{-\frac{\delta_1}{2\gamma}L^\alpha} \leq \text{const}(\eta, \alpha, N) \cdot L^{-N} \langle r \rangle^{M},$$  

(4.22)

where we have to remember that we have $\sim L^{2-2\alpha}$ of $g_j$’s. The second estimate says that the norm decays faster than any power of $L$, with the price of a higher power in $\langle r \rangle$.

We now want to show that $V_L(B, z)$ does not contribute to the thermodynamic limit of $\sigma_L(B)$. We have the following result:

**Proposition 4.1.**

$$\sigma_L(B) = -\frac{1}{\text{Vol}(\Lambda_L)} \cdot \text{Tr} \int_{\Gamma_\omega} f_{FD}(z) \{P_1(B)U_L(B, z)P_2(B)U_L(B, z + \omega)$$

$$+ P_1(B)U_L(B, z - \omega)P_2(B)U_L(B, z)\} \, dz + O(L^{-\infty}).$$  

(4.23)

**Proof.** The main idea is to show that when we replace $(H_L(B) - z)^{-1}$ in $F_L$ with the right hand side of (4.18), all terms containing $V_L(B)$ will generate (after integrating by parts with respect to $z$) some operators which in the trace norm will decay faster than any power of $L$.

If the differentiation does not act on $V_L(B)$ but on the other resolvents, then it is enough to know that in the norm of $B(L^2)$ it goes to zero faster than any power in $L$, as we saw in (4.22).

If the differentiation with respect to $z$ acts on $V_L(B)$, then there will be a few terms which must be separately considered, and prove their smallness in the trace norm.

To give an example, after differentiating $N - 1$ times with respect to $z$ ($N$ large), we obtain a term containing a factor like:

$$\sum_{j=1}^{3} (\partial_j \hat{g}_0)P_j(B)(H_{\infty}(B) - z)^{-N}g_0.$$  

(4.24)

We will prove (in the trace norm) that it decays faster than any power of $L$, times some polynomially bounded factor in $\langle r \rangle$. Note that all the other factors multiplying the above operator are bounded operators, with a norm which is polynomially bounded in $\langle r \rangle$.

Let us start with a technical result:

**Lemma 4.2.** Let $Q_1$ and $Q_2$ be two compact unit cubes such that $\text{dist}(Q_1, Q_2) = d > 1$, and let $\chi_1$, $\chi_2$ denote their characteristic functions. Let $\alpha \in \{0, 1\}$ and $j \in \{1, 2, 3\}$. Then if $N$ is large enough, there exist three constants $\delta_2 > 0$, $N_1 > 1$ and $C > 0$, all three independent of $z \in \Gamma_\omega$, $d$, $\alpha$, $j$ and $Q$’s such that

$$||\chi_1 P_j^\alpha(B)(H_{\infty}(B) - z)^{-N} \chi_2||_{B_1} \leq C r^{N_1} \exp\{-d\delta_2/r\}.$$  

(4.25)
\[ \chi_1(H_\infty(B) - z)^{-1} T_j(H_\infty(B) - z)^{-N+1} \chi_2, \]

where \( T_j \) is a bounded operator which contains factors like \( P_k(B)(H_\infty(B) - z)^{-1} \).

Denote by \( x_2 \) an arbitrary point in the support of \( \chi_2 \). We insert some exponentials in the following way:

\[ \chi_1 e^{\frac{d_2}{r}(-x_2)} \chi_2 \quad \text{and} \quad \chi_1 e^{\frac{d_2}{r}(-x_2)}(H_\infty(B) - z)^{-1} e^{-\frac{d_2}{r}(-x_2)} \chi_2. \]

If \( \delta_2 < \delta \) we can write:

\[ e^{-\frac{d_2}{r}(x-x_2)} e^{-\frac{d_2}{r}(x-x')} e^{\frac{d_2}{r}(x'-x_2)} \leq \text{const} \ e^{-\frac{\delta - d_2}{r}|x-x'|}. \]

Now the two factors containing resolvents in (4.26) are Hilbert-Schmidt due to the presence of the cut-offs \( \chi \) and the exponential decay of our kernels (Proposition 5.2 and (3.1)). Their Hilbert-Schmidt norm will grow polynomially with \( r \), but be independent of \( d \). The choice of \( x_2 \) provides the decaying exponential factor on the right hand side of (4.25).

Let us go back to (4.24) and try to use the previous lemma. We will show that in the trace norm, this operator decays exponentially in \( L^\alpha \). Consider only one \( j \). Cover both \( \text{supp}(\partial_j g_0) \) and \( \text{supp}(g_0) \) with disjoint cubes centered at points in \( \mathbb{Z}^3 \) and side length equal to 1; we only use cubes which have common points with the respective supports. We then have

\[ (\partial_j g_0) P_j(B)(H_\infty(B) - z)^{-N} g_0 = \sum_{s,s'} \chi_s(\partial_j g_0) P_j(B)(H_\infty(B) - z)^{-N} g_0 \chi_{s'}, \]

where \( \chi_s \) and respectively \( \chi_s' \) denote the characteristic function of such unit cubes which cover \( \text{supp}(\partial_j g_0) \) and respectively \( \text{supp}(g_0) \). The number of cubes needed to cover the support of \( g_0 \) is of order \( L^3 \), while for the other one is of order \( L^3 \alpha \); hence we have about \( L^{3+3\alpha} \) terms in the above double sum.

But each operator of the form

\[ \tilde{\chi}_s(\partial_j g_0) P_j(B)(H_\infty(B) - z)^{-N} g_0 \chi_{s'}, \]

is exponentially small in the trace norm due to Lemma 4.2 since the distance between any two supports of \( \tilde{\chi}_s \) and \( \chi_{s'} \) is of order \( L^\alpha \). Hence the entire sum in (4.27) will be \( (r \text{ dependent}) \) exponentially small in \( L^\alpha \). But then we can trade off the fading exponential decay with a polynomial decay in \( L \) and a polynomial growth in \( r \) as we did in (4.22). So this term is under control.

Now let us go back to the beginning of the proof of Proposition 4.1. Other “bad” terms from the remainder in (4.28) after differentiation with respect to \( z \) will contain powers of \((H_L(B, \gamma) - z)^{-1}\), like for example

\[ (\partial_j g_0) P_j,\gamma(B)(H_L(B, \gamma) - z)^{-N} g_\gamma. \]

Here we cannot easily commute with \( P \)'s due to various boundary terms. But we do not need to do that. Look at (5.8), where we put \( s = \frac{1}{r \gamma} \) and \( x_0 = \gamma \). Because \( A_\gamma \) is bounded from above by \( L^\alpha \) (see (4.13)), it follows:

\[ ||(-\Delta_D + 1)(H_L(B, \gamma) - z)^{-1}|| \leq \langle r \rangle^M L^\alpha. \]

It means that for small \( \delta_0 \), the resolvent \( e^{\frac{d_0}{r}(\cdots)}(H_L(B, \gamma) - z)^{-1} e^{-\frac{d_0}{r}(\cdots)} \) sandwiched with exponentials remains Hilbert-Schmidt (with a norm which does not grow faster than the \( B_2 \) norm.
of \((-\Delta_D + 1)^{-1}\) times \(L^\alpha\) and some polynomial in \((r')\). Since we have \(N\) resolvents, the product of two of them will give a trace class operator. Now we can repeat the insertion of exponentials as we did in (4.20), and use (4.19) for getting the exponential decay in \(L^\alpha\). 

\[
\lim_{L \to \infty} \left\{ \sigma_L(B) + \frac{1}{\text{Vol}(\Lambda_L)} \cdot \text{Tr} \int_{\Gamma_r} f_{FD}(z) \left[ P_1(B)\tilde{\gamma}_0(H_\infty(B) - z)^{-1}g_0P_2(B)\tilde{\gamma}_0(H_\infty(B) - z - \omega)^{-1}g_0 
+ P_1(B)\tilde{\gamma}_0(H_\infty(B) - z + \omega)^{-1}g_0P_2(B)\tilde{\gamma}_0(H_\infty(B) - z)^{-1}g_0 \right] dz \right\} = 0. \tag{4.30}
\]

**Proof.** If we compare this with (4.16), we see that we need to show that all terms containing factors localized near boundary will converge to zero. Let us look at one such term, and prove the next lemma:

**Lemma 4.4.** Assume that \(0 < \alpha < 1/3\). Then we have that

\[
\lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \cdot \sum_{\gamma \in E} \text{Tr} \int_{\Gamma_r} f_{FD}(z) P_1(B)g_0(H_\infty(B) - z)^{-1}g_0 
\cdot P_2(B) \sum_{\gamma \in E} e^{iB\phi_0(\gamma)} \tilde{\gamma}_\gamma(H_L(B, \gamma) - z - \omega)^{-1}e^{-iB\phi_0(\gamma)} g_\gamma dz = 0. \tag{4.31}
\]

**Proof.** Using (3.18), and the fact that \(\tilde{z} \cdot \tilde{g}_\gamma = \tilde{g}_\gamma\), we can rewrite the above term as

\[
\frac{1}{\text{Vol}(\Lambda_L)} \cdot \sum_{\gamma \in E} \text{Tr} \int_{\Gamma_r} f_{FD}(z) P_1(B)g_0(H_\infty(B) - z)^{-1}g_0 \tilde{z}_\gamma 
\cdot e^{iB\phi_0(\gamma)} P_2(\gamma/B)g_\gamma(H_L(B, \gamma) - z - \omega)^{-1}e^{-iB\phi_0(\gamma)} g_\gamma dz. \tag{4.32}
\]

After integrating by parts \(N - 1\) times with respect to \(z\), we have to deal with several situations. Let us take one resulting term (just the operator in the integrand):

\[
P_1(B)g_0(H_\infty(B) - z)^{-1}g_0 \tilde{z}_\gamma e^{iB\phi_0(\gamma)} P_2(\gamma/B)g_\gamma(H_L(B, \gamma) - z)^{-N}e^{-iB\phi_0(\gamma)} g_\gamma, \tag{4.33}
\]

and let us estimate its trace norm. The factor containing \(H_\infty(B)\) is just bounded, and we cannot use it as a Hilbert-Schmidt factor. What we do is to commute \(\tilde{g}_\gamma\) over one resolvent to the right and get the identity:

\[
P_2(\gamma/B)g_\gamma(H_L(B, \gamma) - z - \omega)^{-N}g_\gamma \tag{4.34}
\]

\[
= P_2(\gamma/B)(H_L(B, \gamma) - z - \omega)^{-1}\tilde{g}_\gamma(H_L(B, \gamma) - z)^{-N+1}g_\gamma 
+ P_2(\gamma/B)(H_L(B, \gamma) - z - \omega)^{-1}H_L(B, \gamma), \tilde{g}_\gamma(H_L(B, \gamma) - z - \omega)^{-N}g_\gamma.
\]

Now the second term will again contain at least one derivative of \(\tilde{g}_\gamma\) and reasoning as we did for (4.28) we can show it will be exponentially small. Let us look at the first term. The operator \(P_2(\gamma/B)(H_L(B, \gamma) - \zeta)^{-1}\) is only bounded. But \(\tilde{g}_\gamma(H_L(B, \gamma) - \zeta)^{-N+1}g_\gamma\) is trace class and

\[
\|\tilde{g}_\gamma(H_L(B, \gamma) - \zeta)^{-N+1}g_\gamma\|_{B_1} \leq \frac{1}{\eta^{N-2}} \|\tilde{g}_\gamma(H_L(B, \gamma) - \zeta)^{-1}g_\gamma\|_{B_2} \cdot \|(H_L(B, \gamma) - \zeta)^{-1}g_\gamma\|_{B_2}. \tag{4.35}
\]
Hence using (4.24) we have
\[ \| \tilde{g}_\gamma (H_L(B, \gamma) - \zeta)^{-1} \|_{B_2} \leq \text{const} \langle r \rangle^M \| \tilde{g}_\gamma (-\Delta_D + 1)^{-1} \|_{B_2} \cdot L^\alpha. \]

But the Hilbert-Schmidt norm of \( \tilde{g}_\gamma (-\Delta_D + 1)^{-1} \) is of order the square root of the support of \( \tilde{g}_\gamma \), that is \( L^{3\alpha/2} \) (use here (5.13) with \( B = 0 \)). The other factor comes with a similar contribution, hence we can write
\[ \| \tilde{g}_\gamma (H_L(B, \gamma) - z - \omega)^{-N+1} g_\gamma \|_{B_1} \leq \text{const} \langle r \rangle^M \cdot L^{5\alpha}. \]

Now using this in the integral with respect to \( z \), this particular term will give a contribution of \( L^{5\alpha} \) for each \( \gamma \). Since we have \( \sim L^{2-2\alpha} \) different \( \gamma \)'s, the total contribution will be bounded by \( L^{2+3\alpha} \). But if \( \alpha < 1/3 \), after we divide with the volume of \( \Lambda_L \) it will converge to zero.

Now let us go back to (4.32), and see that after integration by parts we can get a term like
\[ P_1(B) \tilde{g}_0(H_\infty(B) - z)^{-N} g_0 \tilde{g}_\gamma, \quad e^{iB\phi_0(\cdot, \gamma')} P_2(\gamma) \tilde{g}_\gamma(H_L(B, \gamma) - z - \omega)^{-1} g_\gamma. \]  

Here the operator \( P_2(\gamma) \tilde{g}_\gamma(H_L(B, \gamma) - z)^{-1} \) is not Hilbert-Schmidt, so we have to look at the first factor.

We can write
\[
\begin{align*}
P_1(B) \tilde{g}_0(H_\infty(B) - z)^{-N} g_0 \tilde{g}_\gamma \\
= P_1(B) \tilde{g}_0(H_\infty(B) - z)^{-N+1} \tilde{g}_\gamma (H_\infty(B) - z)^{-1} g_0 \\
+ P_1(B) \tilde{g}_0(H_\infty(B) - z)^{-N} [H_\infty(B), \tilde{g}_\gamma] (H_\infty(B) - z)^{-1} g_0.
\end{align*}
\]

In the first term, the function \( \tilde{g}_\gamma \) makes the two resolvents next to it become Hilbert-Schmidt, each having a norm proportional with \( L^{3\alpha/2} \) and some power of \( \langle r \rangle \) (use the exponential decay of the kernels). So this term is "good", considering that we have to divide with \( L^3 \) in the end. The second term contains at least one derivative of \( \tilde{g}_\gamma \) (here \( [H_\infty(B), \tilde{g}_\gamma] \) is linear in \( P_j(B) \)'s), together with factors like
\[
\{ P_1(B) \tilde{g}_0(H_\infty(B) - z)^{-N} P_j(\gamma) \} (\partial_\gamma \tilde{g}_\gamma)(H_\infty(B) - z)^{-1} g_0.
\]

Now here we can use the estimate from Proposition 5.2 and see that we again have a product of two Hilbert-Schmidt operators: the first Hilbert-Schmidt norm will be proportional with \( L^{3/2} \), while the other one will behave like \( L^{3\alpha/2} \). When we divide by \( L^3 \), this contribution will go to zero.

All the other terms resulting from integrating by parts with respect to \( z \) can be treated in a similar way. \( \Box \)

Now let us go back to (4.23) and analyze another boundary term:

**Lemma 4.5.** For every \( 0 < \alpha < 1/3 \) we have:
\[
\lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \cdot \text{Tr} \int_{\Gamma_{\infty}} f_{FD}(z) \cdot \left( P_1(B) \sum_{\gamma' \in E} e^{iB\phi_0(\cdot, \gamma')} \tilde{g}_\gamma'(H_L(B, \gamma') - z)^{-1} e^{-iB\phi_0(\cdot, \gamma')} g_{\gamma'} \right. \\
\left. \quad + P_2(B) \sum_{\gamma' \in E} e^{iB\phi_0(\cdot, \gamma')} \tilde{g}_\gamma'(H_L(B, \gamma) - z - \omega)^{-1} e^{-iB\phi_0(\cdot, \gamma')} g_{\gamma'} \right) dz = 0. \]  

**Proof.** Let us note that when keeping \( \gamma \) fixed, only a finite number (\( L \)-independent) of \( \gamma' \)'s will have an overlapping support. This means that the above double sum will only contain around
$L^{2-2\alpha}$ non-zero terms. Now use again Propositions 3.1 and 3.2, and integration by parts with respect to $z$. Each non-zero term in the double sum will be a product of two Hilbert-Schmidt operators, each with a Hilbert-Schmidt norm of the order of $L^{5\alpha/2}$. The total trace norm will grow at most like $L^{2+3\alpha}$, hence if $\alpha < 1/3$ this term will not contribute. We do not give more details.

The last ingredient in proving Proposition 1.12 is contained in the following result (see also Proposition 1.30):

**Proposition 4.6.**

$$\lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \cdot \text{Tr} \int_{\Gamma_\omega} f_{FD}(z) \left\{ P_1(B)\tilde{g}_0(H_\infty(B) - z)^{-1}g_0P_2(B)\tilde{g}_0(H_\infty(B) - z - \omega)^{-1}g_0 \right\} dz = -\int_{\Omega} s_B(x)dx. \quad (4.39)$$

**Proof.** First, due to support properties, we have

$$g_0P_2(B)\tilde{g}_0 = g_0P_2(B) = P_2(B) - (1 - g_0)P_2(B). \quad (4.40)$$

Because $1 - g_0$ is supported outside of a thin region around the boundary of $\Lambda_L$, all terms generated by $(1 - g_0)P_2(B)$ will converge to zero; let us prove this. After integrating $N - 1$ by parts with respect to $z$, we obtain several terms from the integrand which look like this one: $(N_1 + N_2 \geq N \geq 10)$:

$$P_1(B)\tilde{g}_0(H_\infty(B) - z)^{-N_1}(1 - g_0)P_2(B)(H_\infty(B) - z - \omega)^{-N_2}g_0. \quad (4.41)$$

Due to the symmetry of this term, assume without loss of generality that $N_1 \geq 5$. Then by writing $1 - g_0 = (1 - g_0)(1 - \chi_{\Lambda_L})$, we get two types of contributions. The one coming from $(1 - g_0)(1 - \chi_{\Lambda_L}) = (1 - \chi_{\Lambda_L})$ is localized outside $\Lambda_L$, and its trace norm will be exponentially small in $L^\alpha$; we can apply Lemma 1.2 since the distance between $x$ and $\tilde{g}_0$ and $g_0$ is of order $L^\alpha$. The number of needed covering unit cubes is polynomially bounded in $L$.

Thus the only contribution from (4.41) can come from:

$$P_1(B)\tilde{g}_0(H_\infty(B) - z)^{-N_1}(1 - g_0)\chi_{\Lambda_L}P_2(B)(H_\infty(B) - z - \omega)^{-N_2}g_0. \quad (4.42)$$

Here we can write:

$$P_1(B)\tilde{g}_0(H_\infty(B) - z)^{-N_1}(1 - g_0)\chi_{\Lambda_L}$$

\{\{P_1(B)\tilde{g}_0(H_\infty(B) - z)^{-2}\cdot\{(H_\infty(B) - z)^{-N_1 + 2}(1 - g_0)\chi_{\Lambda_L}\},

where both factors are Hilbert-Schmidt, with kernels exponentially localized near diagonal as in Propositions 3.1 and 3.2. The Hilbert-Schmidt norm of the first factor is bounded by $L^{3/2}$, while for the second one we have a bound of $L^{1+\alpha/2}$ (square roots of certain volumes). The trace norm of the product is thus bounded by $L^{5/2+\alpha/2}$ and some polynomial in $L$. After integrating with respect to $z$, and dividing with $L^3$ (the volume of $\Lambda_L$), this term will converge to zero provided $\alpha < 1$.

Now we can go back to (4.40) and analyze the term generated by $P_2(B)$. Because there are no other cut-offs in the middle, and because the commutator $\{P_1(B), \tilde{g}_0\}$ will generate another fast decaying term, we see that we have just proved the following identity (see Theorem 1.1ii for the definition of $F_\infty$):

$$\lim_{L \to \infty} \left( \sigma_L(B) + \frac{1}{\text{Vol}(\Lambda_L)} \text{Tr}\{\tilde{g}_0F_\infty g_0\} \right). \quad (4.44)$$

But the operator $\tilde{g}_0F_\infty g_0$ is trace class, with a jointly continuous kernel, hence (see 4.1, 4.3 and 4.7)

$$\text{Tr}\{\tilde{g}_0F_\infty g_0\} = \int_{\text{supp}(\tilde{g}_0)} s_B(x, x)g_0(x)dx.$$
Using the periodicity of $s_B$ with respect to $\mathbb{Z}^3$ and the support properties of $g_0$, we finally get:

$$
\lim_{L \to \infty} \sigma_L(B) = - \int_{\Omega} s_B(x,x) dx
$$

and the proof of (1.12) is over. \qed

4.3 Proof of (1.13).

We start by investigating $\partial_B \sigma_L(0)$ and try to put it in a form which is better suited for the thermodynamic limit.

Note that we have already argued that $\sigma_L$ was smooth in $B$ near zero (see the remark around (2.5)). The hard part is to show that the polynomial growth in $L$ of the trace norm does not appear in the actual trace. Similar difficulties involving magnetic semi-groups were encountered in [1, 10, 6, 7].

4.3.1 Only $U_L(B)$ counts.

**Notation.** In order to shorten our formulas, we will sometimes write $(z \to z - \omega)$, which means that we have to repeat the previously appearing product of operators in which we have to change $z$ with $z - \omega$.

First, let us prove that even if we differentiate with respect to $B$, we still have a result similar to Proposition 4.1.

**Proposition 4.7.** At $B = 0$ we have:

$$
\partial_B \left\{ \sigma_L(B) + \frac{1}{\text{Vol}(\Lambda_L)} \cdot \text{Tr} \int_{\mathbb{R}^3} f_{FD}(z) \{ P_1(B) U_L(B, z) P_2(B) U_L(B, z + \omega) \right. \\
+ \left. z \to z - \omega \} \right\} = O(L^{-\infty}). \quad (4.46)
$$

**Proof.** As usual, before differentiating with respect to $B$ one has to perform a certain number of integrations by parts with respect to $z$. There will appear many terms in the remainder containing $V_L(B, z)$ (see (4.17)), but all of them will have the distinctive feature of containing pairs of cut-off functions whose supports are at a distance $\sim L^\alpha$ one from another.

We do not want to treat all the situations which can appear here, and instead we will only prove a typical estimate needed for the result.

**Lemma 4.8.** Assume that $N$ is large enough, and choose $z$ such that $\text{dist}\{z, [0, \infty)\} = \eta > 0$, $r = \Re(z)$. Then the map

$$
(-1, 1) \ni B \to (\Delta \tilde{g}_0)(H_{\infty}(B) - z)^{-N} g_0 \in B_1(L^2)
$$

is differentiable in the trace norm, and there exists $\delta_1$ small enough and $M$ large enough such that

$$
\| \{ \partial_B (\Delta \tilde{g}_0)(H_{\infty}(B) - z)^{-N} g_0 \}_{B=0} \|_1 \leq \text{const}(\eta, N) e^{-\frac{\delta_1}{\sqrt{\pi}}} L^\alpha (r)^M. \quad (4.47)
$$

Moreover,

$$
\| (\Delta \tilde{g}_0)(H_{\infty}(B) - z)^{-N} g_0 - (\Delta \tilde{g}_0)(H_{\infty}(0) - z)^{-N} g_0 \\
- B \{ \partial_B (\Delta \tilde{g}_0)(H_{\infty}(B) - z)^{-N} g_0 \}_{B=0} \|_{B_1} \leq \text{const}(\eta, N) e^{-\frac{\delta_1}{\sqrt{\pi}}} L^\alpha (r)^M. \quad (4.48)
$$
Proof of the lemma. We only prove differentiability at $B = 0$. The key ingredient is to give a proper sense to the formal identity:

$$
(H_\infty(B) - z)^{-1} - (H_\infty(0) - z)^{-1}
= -(H_\infty(B) - z)^{-1} \{ H_\infty(B) - H_\infty(0) \} (H_\infty(0) - z)^{-1}.
$$

(4.49)

As it stands, the right hand side makes no sense because $H_\infty(B) - H_\infty(0)$ contains terms like $-2iB\mathbf{A}(x) \cdot \nabla_x$ which are not relatively bounded to the free Laplacian due to the linear growth of our magnetic potential. For $0 < \delta_2 < \delta_3$ and $x_0 \in \mathbb{R}^3$, we can write (still formally):

$$
e^\frac{\delta_3}{\|r\|}(-x_0) \{ (H_\infty(B) - z)^{-1} - (H_\infty(0) - z)^{-1} \} e^{-\frac{\delta_3}{\|r\|}(-x_0)}
= -e^{\frac{\delta_3}{\|r\|}(-x_0)} (H_\infty(B) - z)^{-1} \{ H_\infty(B) - H_\infty(0) \} (H_\infty(0) - z)^{-1} e^{-\frac{\delta_3}{\|r\|}(-x_0)}.
$$

(4.50)

Note the fact that the growing exponential is weaker that the decaying one. This identity now holds in the Hilbert-Schmidt class; in order to see this, introduce the notation

$$1 < \alpha_1 < \alpha_2 < \frac{\delta_3}{\delta_2},$$

(4.51)

Then we can write the above identity as

$$
e^{-\frac{\delta_3}{\|r\|}(-x_0)} \left\{ \left( H_\infty(B) - z \right)^{-1} - (H_\infty(0) - z)^{-1} \right\} e^{-\frac{\delta_3}{\|r\|}(-x_0)}
= -e^{-\frac{\alpha_1-\alpha_2}{\|r\|}(-x_0)} \left\{ \left( H_\infty(B) - z \right)^{-1} e^{-\frac{\alpha_1}{\|r\|}(-x_0)} \right\}
\cdot \left\{ (H_\infty(B) - H_\infty(0)) e^{-\frac{\alpha_2}{\|r\|}(-x_0)} \right\}
\cdot \left\{ e^{\frac{\alpha_1-\alpha_2}{\|r\|}(-x_0)} (H_\infty(0) - z)^{-1} e^{-\frac{\alpha_1}{\|r\|}(-x_0)} \right\}
\cdot e^{-\frac{\delta_3}{\|r\|}(-x_0)}.
$$

(4.52)

Now $H_\infty(B) - H_\infty(0) = 2B\mathbf{A} \cdot \mathbf{P}(0) + B^2\mathbf{A}^2$ is proportional with $B$, and the linear growth of $\mathbf{A}$ is compensated by the higher exponential decay on the right hand side. Hence (5.5) and (5.9) (at $B = 0$) imply that if $\delta_2$ is small enough, we have:

$$
\left\| e^{\frac{\alpha_1-\alpha_2}{\|r\|}(-x_0)} \left\{ (H_\infty(B) - H_\infty(0)) (H_\infty(0) - z)^{-1} e^{-\frac{\alpha_2}{\|r\|}(-x_0)} \right\} \right\|_{B_2} \leq \text{const}(x_0)^2 \cdot |B| \cdot \langle r \rangle^M.
$$

(4.53)

The estimate (3.26) (valid uniformly in $B \in [-1, 1]$ and the $\delta$ there should be chosen slightly larger than the $\delta_3$) tells us that:

$$
\left\| e^{-\frac{\alpha_1-\alpha_2}{\|r\|}(-x_0)} \left\{ \left( H_\infty(B) - z \right)^{-1} e^{-\frac{\alpha_1}{\|r\|}(-x_0)} \right\} \right\|_{B_2} \leq \text{const} \langle r \rangle^M.
$$

(4.54)

Hence we have proved the estimate

$$
\left\| e^{\frac{\delta_3}{\|r\|}(-x_0)} \left\{ (H_\infty(B) - z)^{-1} - (H_\infty(0) - z)^{-1} \right\} e^{-\frac{\delta_3}{\|r\|}(-x_0)} \right\|_{B_2} \leq \text{const} |B| \langle x_0 \rangle^2 \langle r \rangle^M.
$$

(4.55)

Now go back to (4.50) and isolate the linear term in $B$:

$$
e^{\frac{\delta_3}{\|r\|}(-x_0)} \left\{ (H_\infty(B) - z)^{-1} - (H_\infty(0) - z)^{-1} \right\} e^{-\frac{\delta_3}{\|r\|}(-x_0)}
= -Be^{\frac{\delta_3}{\|r\|}(-x_0)} (H_\infty(0) - z)^{-1} \{ 2\mathbf{A} \cdot \mathbf{P}(0) \} (H_\infty(0) - z)^{-1} e^{-\frac{\delta_3}{\|r\|}(-x_0)}
- B^2 e^{\frac{\delta_3}{\|r\|}(-x_0)} (H_\infty(0) - z)^{-1} \{ \mathbf{A}^2 \} (H_\infty(0) - z)^{-1} e^{-\frac{\delta_3}{\|r\|}(-x_0)}
- e^{\frac{\delta_3}{\|r\|}(-x_0)} \left\{ (H_\infty(B) - z)^{-1} - (H_\infty(0) - z)^{-1} \right\} e^{-\frac{\delta_3}{\|r\|}(-x_0)}
\cdot e^{\frac{\delta_3}{\|r\|}(-x_0)} \left\{ (H_\infty(B) - H_\infty(0)) (H_\infty(0) - z)^{-1} e^{-\frac{\delta_3}{\|r\|}(-x_0)} \right\}.
$$

(4.56)
where in the last term we inserted an exponential with \( \delta_2 < \tilde{\delta} < \delta_3 \). Now it is easy to get the estimate:

\[
\left| e^{\frac{\tilde{\delta}}{\sigma} (-x_0)} \{ (H_\infty(B) - z)^{-1} - (H_\infty(0) - z)^{-1} \} e^{-\frac{\tilde{\delta}}{\sigma} (-x_0)} + B e^{\frac{\tilde{\delta}}{\sigma} (-x_0)} (H_\infty(0) - z)^{-1} \{ 2A \cdot P(0) \} (H_\infty(0) - z)^{-1} e^{-\frac{\tilde{\delta}}{\sigma} (-x_0)} \right|_{B_2} = \text{const} |B|^2 (x_0)^4 \langle r \rangle^M.
\] (4.57)

This is enough to prove that if \( N \geq 2 \), the mapping in Lemma 4.8 is differentiable in the trace norm sense at \( B = 0 \). Indeed, proceeding as we did for (4.27), we can insert many cut-off functions and cover the supports of \( g_0 \) and \( \tilde{g}_0 \). Using the same notations, we have that for some \( \tilde{\delta}_2 < \tilde{\delta}_3 < \tilde{\delta}_4 \) we have

\[
\tilde{\chi}_s (\Delta \tilde{g}_0) (H_\infty(B) - z)^{-N} g_0 \chi_{s'}
\]

(4.58)

Now we can differentiate with respect to \( B \) in the trace norm-sense, using the result for the Hilbert-Schmidt norm and the fact that we have at least two factors in \( B_2 \) (adapt (4.54)). At last we again use the fact that the distance between the supports of \( \chi \)'s is \( \sim L^n \), and that all growing factors are just polynomials in \( L \). We consider Lemma 4.8 proved.

Now we can use (4.48) in all the terms on the right hand side of (4.56) which contain operators of the type treated in Lemma 4.8. The exponential decay of \( f_{FD} \) can be used to obtain a decay faster than any power of \( L \).

All other terms from the remainder can be treated in a similar manner, and we consider Proposition 4.7 as proved.

In the remaining part of our paper we will investigate the thermodynamic limit of the main contribution to \( \partial_B \sigma_L(B) \), given by

\[
\partial_B \text{Tr} \int_{\Gamma_w} f_{FD}(z) \{ P_1(B) U_L(B,z) P_2(B) U_L(B,z+\omega) + (z \to z-\omega) \} \, dz.
\] (4.59)

4.3.2 The boundary terms

Here the magnetic perturbation theory will play a crucial role. There are several terms in the definition of \( U_L \) (see (4.16)), and when we insert them into (4.59) they will generate even more terms.

We will now prove that only the term which contains two resolvents with \( H_\infty \) will contribute at the thermodynamic limit. The main result can be stated in the following way:

**Proposition 4.9.** At \( B = 0 \) and \( \alpha \) sufficiently small we have:

\[
\lim_{L \to \infty} \partial_B \left\{ \sigma_L(B) + \frac{1}{\text{Vol}(\Lambda_L)} \cdot \text{Tr} \int_{\Gamma_w} f_{FD}(z) \cdot \{ P_1(B) \tilde{g}_0 (H_\infty(B) - z)^{-1} g_0 P_2(B) \tilde{g}_0 (H_\infty(B) - z - \omega)^{-1} g_0 
+ z \to z-\omega \} \, dz \right\} = 0.
\] (4.60)

**Proof.** Let us start with the following boundary term:
\[
X_1(B, L) := \text{Tr} \int_{\Gamma_{\omega}} f_{FD}(z) \left\{ P_1(B)\hat{g}_0(H_{\infty}(B) - z)^{-1}g_0 \cdot P_2(B) \sum_{\gamma \in E} e^{iB\varphi_0(\cdot, \gamma)} \hat{g}_\gamma(H_L(B, \gamma) - z - \omega)^{-1}e^{-iB\varphi_0(\cdot, \gamma)} g_\gamma + (z \to z - \omega) \right\} dz. \tag{4.61}
\]

We have already seen in Lemma 4.4 that \(X_1(B, L)\) divided by the volume of \(\Lambda_L\) converges to zero when \(L\) converges to infinity. Now we would like to show that \(\{\partial_B X_1\}(0, L)\) has the same property.

Proceeding as in (4.32), we can rewrite \(X_1(B, L)\) as

\[
X_1(B, L) = \sum_{\gamma \in E} X_1(B, L, \gamma), \tag{4.62}
\]

\[
X_1(B, L, \gamma) := \text{Tr} \int_{\Gamma_{\omega}} f_{FD}(z) \left\{ P_1(B)\hat{g}_0(H_{\infty}(B) - z)^{-1}g_0 \cdot e^{iB\varphi_0(\cdot, \gamma)} P_2(B)\hat{g}_\gamma(H_L(B, \gamma) - z - \omega)^{-1}e^{-iB\varphi_0(\cdot, \gamma)} g_\gamma + (z \to z - \omega) \right\} dz. \tag{4.63}
\]

We now prove the following estimate:

**Lemma 4.10.** For every \(L \geq 1\) we have

\[
\sup_{\gamma \in E} \sup_{0 < |\gamma|} \left| \frac{1}{B} \left\{ X_1(B, L, \gamma) - X_1(0, L, \gamma) \right\} \right| \leq const(\alpha) \cdot L^{5\alpha}. \tag{4.64}
\]

**Remark.** Before starting the proof of this lemma, let us note that it immediately implies that

\[
\lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \{\partial_B X_1\}(0, L) = 0. \tag{4.65}
\]

This is so because we know that \(\{\partial_B X_1\}(0, L)\) exists, and moreover, it must be bounded from above by the right hand side of (4.64) times the number of \(\gamma\)'s in \(E\), i.e. \(\sim L^{2\alpha}\). Then if \(\alpha\) is chosen small enough, after dividing by \(\sim L^3\) we get something converging to zero.

**Proof of Lemma 4.10.** Define the function

\[
\hat{X}_1(B, L, \gamma) := \text{Tr} \int_{\Gamma_{\omega}} f_{FD}(z) \left\{ P_1(B)\hat{g}_0S_B(z)g_0 \cdot e^{iB\varphi_0(\cdot, \gamma)} P_2(0)\hat{g}_\gamma(H_L(0) - z - \omega)^{-1}e^{-iB\varphi_0(\cdot, \gamma)} g_\gamma + (z \to z - \omega) \right\} dz. \tag{4.66}
\]

The proof of this lemma has two parts. The first one will state that

\[
\sup_{\gamma \in E} \sup_{0 < |\gamma|} \left| \frac{1}{B} \left\{ X_1(B, L, \gamma) - \hat{X}_1(B, L, \gamma) \right\} \right| \leq const(\alpha) \cdot L^{5\alpha}, \tag{4.67}
\]

while the second one is

\[
\sup_{\gamma \in E} \sup_{0 < |\gamma|} \left| \frac{1}{B} \hat{X}_1(B, L, \gamma) \right| \leq const(\alpha) \cdot L^{5\alpha}. \tag{4.68}
\]

**Part one.** The first estimate is not very much different from what we have already done until now. One uses repeated integration by parts with respect to \(z\) in \(X_1(B, L, \gamma)\) and then we expand each
resolvent \((H_{\infty}(B) - \zeta)^{-1}\) using the first equality in (3.22), and each resolvent \((H_L(B, \gamma) - \zeta)^{-1}\) as perturbation of \((H_L(0) - \zeta)^{-1}\). Note that due to (1.11), the growth induced by \(A_{\gamma}(x)\) does not exceed \(L^*\). As for the \(H_{\infty}(B)\), commuting \(P_j(B)\) with the magnetic phases will always transform \(A(x)\) into \(A(x - x')\), whose growth will now be tempered by the exponential decay from (3.22).

The general strategy for all terms arising from integration by parts with respect to \(G\), remember that the trace norm of a product is bounded by the Hilbert-Schmidt norm of two factors. Several other estimates are needed in order to prove (4.67). Remember that \(G_N(x, x'; z)\) is the integral kernel of \((H_{\infty}(0) - z)^{-N}, N \geq 1\). Denote by \(S_B^{(N)}(z)\) the operator corresponding to the integral kernel \(e^{iB\varphi_0(x, x')}G_N(x, x'; z)\). If \(N = 1\) they coincide with the operators \(S_B(z)\) defined in (3.19).

Then we have:

i. For every \(B \in [-1, 1]\), and \(N \geq 1\)

\[
\|S_B^{(N)}(z)\| \leq \text{const}(N)\langle r \rangle^M. \tag{4.69}
\]

ii. For every \(B \in [-1, 1]\), \(N \geq 1\) and \(Q \subset \mathbb{R}^3\) a compact set:

\[
\|\chi_Q S_B^{(N)}(z)\|_{B_2} \leq \text{const}(N)\sqrt{\text{Vol}(Q)}\langle r \rangle^M. \tag{4.70}
\]

iii. For every \(B \in [-1, 1]\),

\[
\|(H_{\infty}(B) - z)^{-1} - S_B(z) + BS_B(z)T_B(z)\| \leq \text{const}|B|^2\langle r \rangle^M. \tag{4.71}
\]

iv. For every \(B \in [-1, 1]\), \(N \geq 2\) and \(Q_{1,2} \subset \mathbb{R}^3\) two compact sets:

\[
\|\chi_{Q_1} S_B^{(N)}(z)\chi_{Q_2}\|_{B_1} \leq \text{const}(N)\sqrt{\text{Vol}(Q_1)\sqrt{\text{Vol}(Q_2)}}\langle r \rangle^M. \tag{4.72}
\]

The first and the third ones are easy consequences of (3.22). For the second and fourth ones we have to differentiate \(N - 1\) times in (3.22) and write:

\[
S_B^{(N)}(z) = (-1)^{N-1}(H_{\infty}(B) - z)^{-N}
- B \sum_{k=0}^{N-1} (-1)^{N-1-k}(H_{\infty}(B) - z)^{-N+k}T_B^{(k)}(z). \tag{4.73}
\]

**Part two.** We will now concentrate on (4.69), which needs a new idea.

An heuristic argument. First we perform some formal computations, in order to illustrate how magnetic phases will transform the trace into a more regular object. Assume that the operator under the trace in (4.66) has a jointly continuous integral kernel; remember that the operator \(S_B(z)\) defined in (3.19) had a magnetic phase. Commute this phase with \(P_1(B)\) as in (3.18), and write the following formal expression for the integral kernel of the operator whose trace we want to estimate:

\[
\int_{\Gamma_{\omega}} d\tilde{z} f_{FD}(\tilde{z}) \left\{ \int_{\Delta_{L}} dx' e^{i B \varphi_0(x, x')} [P_1(x)(0) + BA_1(x - x')] \tilde{g}_0(x) G_1(x, x', z) g_0(x')
+ e^{i B \varphi_0(x', \gamma)} P_2(x', 0) \tilde{g}_2(x') (H_L(0) - z - \omega)^{-1}(x', x'') e^{-i B \varphi_0(x'', \gamma)} g_2(x'') \right\}.
\tag{4.74}
\]

The above expression gives an integral kernel \(I(x, x'')\). If we could prove joint continuity, then we could write

\[
\tilde{X}_1(B, L, \gamma) = \int_{\Delta_{L}} I(x, x) dx.
\tag{4.75}
\]
Now let us see what happens with the phases in (4.72) when we put \( x = x'' \). We have the identity:

\[
fl(x, x', \gamma) := \frac{1}{2} e_3 \cdot [(x' - x) \land (\gamma - x')],
\]

\[
\varphi_0(x, x') + \varphi_0(x', \gamma) = \varphi_0(x, \gamma) + fl(x, x', \gamma).
\]

The crucial thing is that when \( x = x'' \) in (4.74), the magnetic phases cancel each other and only the flux \( fl \) remains.

Thus the operator given by the following integral kernel

\[
\int_{\Gamma_{\omega}} dz f_{FD}(z) \left\{ \int_{\Lambda_L} dx' \, e^{iBf_l(x, x', \gamma)} [P_1(x) + B A_1(x, x')] g_0(x) G_1(x, x', z) g_0(x') \right. \\
\left. \cdot P_{2, x'}(0) \tilde{g}_\gamma(x') (H_L(0) - z - \omega)^{-1} (x', x'') g_\gamma(x'') + (z \to z - \omega) \right\},
\]

must have the same trace since its kernel has the same diagonal value. Remember that this is just an heuristic argument, precise details are given in the next paragraph.

The rigorous argument. We now start the rigorous proof of (4.68). We integrate by parts with respect to \( z \) in (4.66), and let us first focus on one term, namely the one obtained when all derivatives act on the resolvent in the middle:

\[
R(B, L, \gamma) := \text{Tr} \int_{\Gamma_{\omega}} \tilde{f}_{FD}(z) \left\{ P_1(B) g_0 S_B(z) g_0 \ e^{iB\varphi_0(\cdot, \gamma)} P_2(0) \tilde{g}_\gamma (H_L(0) - z - \omega)^{-N} e^{-iB\varphi_0(\cdot, \gamma)} g_\gamma \right. \\
\left. + (z \to z - \omega) \right\} dz, \quad N \geq 3.
\]

Denote by \( \{\psi_k\}_{k \geq 1} \) and \( \{\lambda_k\}_{k \geq 1} \) the eigenfunctions and eigenvalues of \( H_L(0) \) respectively. Then the operator:

\[
R_K := \sum_{j=1}^{K} \int_{\Gamma_{\omega}} \tilde{f}_{FD}(z) \left\{ P_1(B) g_0 S_B(z) g_0 \ e^{iB\varphi_0(\cdot, \gamma)} P_2(0) \tilde{g}_\gamma |\psi_k\rangle \langle \psi_k| \frac{1}{(\lambda_k - z - \omega)^N} e^{-iB\varphi_0(\cdot, \gamma)} g_\gamma \\
+ (z \to z - \omega) \right\} dz
\]

is trace class. Using

\[
\text{Tr}(R_K) - R(B, L, \gamma) \text{Tr} \int_{\Gamma_{\omega}} \tilde{f}_{FD}(z) \left\{ P_1(B) g_0 S_B(z) g_0 \ e^{iB\varphi_0(\cdot, \gamma)} P_2(0) \tilde{g}_\gamma (H_L(0) - z - \omega)^{-1} \\
\sum_{j > K} |\psi_k\rangle \langle \psi_k| \frac{1}{(\lambda_k - z - \omega)^{N-1}} e^{-iB\varphi_0(\cdot, \gamma)} g_\gamma + (z \to z - \omega) \right\} dz
\]

we obtain

\[
\lim_{K \to \infty} \text{Tr}(R_K) = R(B, L, \gamma),
\]

where we use the fact that the square of the resolvent is trace class (here \( N - 1 \geq 2 \)), together with the boundedness of the rest of the factors, with norms polynomially bounded in \( \langle r \rangle \).

Now let us show that \( R_K \) has an integral kernel \( R_K(x, x') \) which is jointly continuous on \( \Lambda_L \times \Lambda_L \). For, choose a point \( (x_0, x'_0) \in \Lambda_L \times \Lambda_L \) and let us prove continuity there. We know by elliptic regularity that \( \psi_k \)'s are smooth in \( \Lambda_L \). Then the function \( P_2(0) \tilde{g}_\gamma \psi_k \) is smooth in \( \Lambda_L \). Denote by \( \chi_0 \) a smoothed-out characteristic function of a small ball around \( x_0 \), whose support is included in \( \Lambda_L \). The operator \( g_0 S_B(z) g_0 \chi_0 P_1(0) \tilde{g}_\gamma \psi_k \) is smooth in \( \Lambda_L \). Then since the integral kernel of \( S_B \) is smooth outside its diagonal, it means that \( g_0 S_B(z) g_0 (1 - \chi_0) P_1(0) \tilde{g}_\gamma \psi_k \) is smooth at \( x_0 \), and remains so even after applying \( P_1(B) \). All bounds are growing at most like a polynomial in \( \langle r \rangle \), hence the integral in \( z \) preserves the continuity. The variable \( x'_0 \) only meets smooth functions in \( \Lambda_L \), and we are done.
We thus conclude that \( R_K \) is trace class and has a continuous integral kernel. For every increasing sequence of compacts \( \Omega_s \) such that \( \Omega_s \not\subset \Lambda_L \) (in the sense that \( \Omega_s \subset \Lambda_L \) and \( \lim_{s \to \infty} \text{Vol}(\Lambda_L \setminus \Omega_s) = 0 \), we can write:

\[
\text{Tr}(R_K) = \lim_{s \to \infty} \int_{\Omega_s} R_K(x, x)dx. \tag{4.82}
\]

Let us denote by \( Q_\gamma(B, z) \) the operator given by the integral kernel

\[
Q_\gamma(x, x'; B, z) := e^{iBf(x, x', \gamma)}[P_1, x(0) + B \lambda_1(x - x')]\tilde{g}_0(x)G_1(x, x', z)g_0(x')\tilde{g}_\gamma(x'), \tag{4.83}
\]

At this point we can get rid of the magnetic phases using (4.76), and using the notation from (4.80), then (4.81) implies that:

\[
\lim_{\gamma \to \infty} q_{\lambda, x} = 0, \tag{4.82}
\]

\[
\text{Tr}(R_K) = \sum_{j=1}^K \text{Tr} \int_{\Gamma_{\nu}} \tilde{f}_{FD}(z) \left\{ Q_\gamma(B, z) P_2(0)\tilde{g}_\gamma(H_L(0) - z - \omega)^{-n}g_\gamma 
+ (z \to z - \omega) \right\} dz. \tag{4.84}
\]

Now \( Q_\gamma \) is a nice bounded operator, and we can let \( K \to \infty \), because the integral will converge in the trace class in the same way as in (4.80). Then (4.81) implies that:

\[
R(B, L, \gamma)\text{Tr} \int_{\Gamma_{\nu}} \tilde{f}_{FD}(z) \left\{ Q_\gamma(B, z) P_2(0)\tilde{g}_\gamma(H_L(0) - z - \omega)^{-n}g_\gamma 
+ (z \to z - \omega) \right\} dz. \tag{4.85}
\]

Denote by \( \tilde{Q}_\gamma(B, z) \) the operator given by the integral kernel

\[
\tilde{Q}_\gamma(x, x'; B, z) := \frac{1}{B}(e^{iBf(x, x', \gamma)} - 1)P_1, x(0)\tilde{g}_0(x)G_1(x, x', z)g_0(x')\tilde{g}_\gamma(x') + e^{iBf(x, x', \gamma)}A_1(x - x')\tilde{g}_0(x)G_1(x, x', z)g_0(x')\tilde{g}_\gamma(x'). \tag{4.86}
\]

We can write:

\[
R(B, L, \gamma) - R(0, L, \gamma) = B \text{Tr} \int_{\Gamma_{\nu}} \tilde{f}_{FD}(z) \left\{ \tilde{Q}_\gamma P_2(0)\tilde{g}_\gamma(H_L(0) - z - \omega)^{-n}g_\gamma 
+ (z \to z - \omega) \right\} dz. \tag{4.87}
\]

We also note the estimates:

\[
|f \lambda, x, x', \gamma| \leq \frac{1}{2} |x - x'| |x' - \gamma|, \quad |e^{iBf(x, x', \gamma)} - 1| \leq B|f \lambda, x, x', \gamma|. \tag{4.88}
\]

Now it is easy to see from (4.86), (4.88) and (3.1) that \( \tilde{Q}_\gamma \) belongs to \( B(L^2) \), with a norm bounded by \( \langle r \rangle^M L^\alpha \). By writing

\[
(H_L(0) - z - \omega)^{-2}g_\gamma = (H_L(0) - z - \omega)^{-1}\tilde{g}_\gamma(H_L(0) - z - \omega)^{-1}g_\gamma 
+ (H_L(0) - z - \omega)^{-1}(1 - \tilde{g}_\gamma)(H_L(0) - z - \omega)^{-1}g_\gamma, \tag{4.89}
\]

we can see that the operator \((H_L(0) - z - \omega)^{-2}g_\gamma\) is trace class and

\[
\|(H_L(0) - z - \omega)^{-2}g_\gamma\|_{B_1} \leq \text{const} \cdot \langle r \rangle^M L^{3\alpha}, \tag{4.90}
\]

after estimating the Hilbert-Schmidt norm of each factor in the two terms. The second one will be exponentially small due to the support properties of \( g_\gamma \)'s.
We have thus proved
\[
\sup_{\gamma \in E} \sup_{0 < |B| \leq 1} \left| \frac{1}{B} \left( R(B, L, \gamma) - R(0, L, \gamma) \right) \right| \leq \text{const} \cdot L^{4\alpha}. \tag{4.91}
\]

After summation over \( \gamma \), the bound is like \( L^{2+2\alpha} \), and if \( \alpha < 1/2 \) it will not contribute to the thermodynamic limit.

Remember that this was just one possible term arising after integrating by parts \( N - 1 \) times in (4.65). All other terms having sufficiently many derivatives acting on the resolvent, can be treated in a similar way. A different class of terms is represented by the one in which all derivatives act on \( S_B(z) \). Let us define:
\[
R_1(B, L, \gamma) := \text{Tr} \int_\Gamma \hat{f}_{FD}(z) \left\{ P_1(B) \tilde{g}_0 S_B^{(N)}(z) g_0 e^{iB\varphi_0(\cdot, \cdot)} P_2(0) \tilde{g}_\gamma (H_L(0) - z - \omega)^{1} e^{-iB\varphi_0(\cdot, \cdot)} g_\gamma 
+ (z \to z - \omega) \right\} dz. \tag{4.92}
\]

We commute back \( P_2(0) \) over the phase at its left and write:
\[
Q_2(B, z, \gamma) := P_1(B) \tilde{g}_0 S_B^{(N)}(z) g_0 P_2(B) \tilde{g}_\gamma , \tag{4.93}
\]
\[
R_1(B, L, \gamma) = \text{Tr} \int_\Gamma \hat{f}_{FD}(z) \left\{ Q_2(B, z, \gamma) e^{iB\varphi_0(\cdot, \cdot)} \tilde{g}_\gamma (H_L(0) - z - \omega)^{1} e^{-iB\varphi_0(\cdot, \cdot)} g_\gamma 
+ (z \to z - \omega) \right\} dz.
\]

Now if \( N \) is large enough, by using (3.18), the regularity of \( G_N(x, x'; z) \), and the exponential decay of the kernels, one can prove an estimate (\( N \) large enough):
\[
|Q_2(B, z, \gamma)(x, x')| \leq \text{const}(N) \cdot (r)^{M} e^{-\frac{5}{8} |x-x'|} \tilde{g}_\gamma (x'). \tag{4.94}
\]

It means that the integrand in \( R_3 \) is a product of two Hilbert-Schmidt operators. We can again introduce the cut-off with the spectral projection of \( H_L(0) \), get rid of the magnetic phases and introduce the more regular phases, and so on. We consider that Lemma 4.10 is proved. \( \square \)

Besides \( X_1 \) treated in the previous lemma, there is only one other boundary term which needs special attention. This term is the one containing a double boundary sum:
\[
X_2(B, L) = \sum_{\gamma, \gamma' \in E} X_2(B, L, \gamma, \gamma'), \tag{4.95}
\]
\[
X_2(B, L, \gamma, \gamma') := \text{Tr} \int_\Gamma \hat{f}_{FD}(z) 
\cdot \left\{ e^{iB\varphi_0(\cdot, \cdot)} P_1(\gamma') P_1(\gamma) \tilde{g}_\gamma (H_L(B, \gamma') - z)^{-1} e^{-iB\varphi_0(\cdot, \cdot)} g_\gamma 
+ (z \to z - \omega) \right\} dz. \tag{4.96}
\]

and we want to prove that for every \( L \geq 1 \) we have
\[
\sup_{\gamma, \gamma' \in E} \sup_{0 < |B| \leq 1} \left| \frac{1}{B} \left( X_2(B, L, \gamma, \gamma') - X_2(0, L, \gamma, \gamma') \right) \right| \leq \text{const}(\alpha) \cdot L^{5\alpha}. \tag{4.97}
\]

Note that this would again imply something like (4.65) but for \( X_2 \), because there is a finite, \( L \)-independent number of \( \gamma \)'s and \( \gamma' \)'s with joint support.
The strategy is the same. We define
\[
\tilde{X}_2(B, L, \gamma, \gamma') := \text{Tr} \int_{\Gamma_B} f_{FD}(z) \left\{ e^{iB\varphi_0(\cdot, \gamma')} P_1(0) \tilde{g}_\gamma(H_L(0) - z) - e^{-iB\varphi_0(\cdot, \gamma')} g_{\gamma'} \\
+ (z \to z - \omega) \right\} dz.
\]
and we want to prove two analogues of (4.67) and (4.68). The analogue of (4.67) is “easy”, but the analogue of (4.68) again requires a limiting procedure which would allow us to write the \(\tilde{X}_2(B, L, \gamma, \gamma')\) as the trace of a more regular object in \(B\). The main point is that this new object will be given by the composition of those four magnetic phases present in \(\tilde{X}_2\). Namely, let us notice the following identity:
\[
\varphi_0(\mathbf{x}, \gamma') + \varphi_0(\gamma', \mathbf{x'}) + \varphi_0(\mathbf{x'}, \gamma) + \varphi_0(\gamma, \mathbf{x}) = f l(\mathbf{x}, \gamma', \mathbf{x'}) + f l(\mathbf{x'}, \gamma, \mathbf{x}) = f l(\mathbf{x}, \gamma', \gamma).
\]
Now (4.100) allows us to write:
\[
\text{Tr} \int_{\Gamma_B} f_{FD}(z) \left\{ e^{iBfl(\cdot, \gamma')} P_1(0) \tilde{g}_\gamma(H_L(0) - z) - e^{-iBfl(\cdot, \gamma')} g_{\gamma'} \\
+ (z \to z - \omega) \right\} dz.
\]
The good thing about this formula is that on the supports of \(g_{\gamma}\)'s, these fluxes are at most of order \(L^{2n}\), being bounded from above by \(|\mathbf{x} - \gamma| \cdot |\gamma - \gamma'|\). Remember that the non-zero terms must have \(|\gamma - \gamma'| \leq \text{const} \cdot L^{n}\). Now we can expand the exponentials and prove the analogue of (4.68). Thus Proposition 4.9 is proved.

\[\Box\]

### 4.3.3 The bulk contribution
At this point we are left with the contribution coming from terms only containing \(H_\infty(B)\). Define:
\[
X_0(B, L) := \text{Tr} \int_{\Gamma_B} f_{FD}(z) \left\{ P_1(B) \tilde{g}_0(H_\infty(B) - z) - e^{iBfl(\cdot, \gamma')} P_1(0) \tilde{g}_\gamma(H_L(0) - z) - e^{-iBfl(\cdot, \gamma')} g_{\gamma'} \\
+ (z \to z - \omega) \right\} dz.
\]
We will compute \(\frac{1}{\text{Vol}(A_L)} \partial_B X_0(0, L)\) and show that it converges to \(\{\partial_B \sigma_\infty\}(0)\).
Define:
\[
\tilde{X}_0(B, L) := \text{Tr} \int_{\Gamma_B} f_{FD}(z) \left\{ P_1(B) \tilde{g}_0 S_B(z) g_0 P_2(B) \tilde{g}_0 S_B(z + \omega) g_0 + (z \to z - \omega) \right\} dz.
\]
Now we can prove the last technical result:

**Proposition 4.11.** The following two double limits exist:
\[
\sigma_1 := \lim_{L \to \infty} \frac{1}{\text{Vol}(A_L)} \lim_{B \to 0} \frac{1}{B} \{X_0(B, L) - \tilde{X}_0(B, L)\},
\]
\[
\sigma_2 := \lim_{L \to \infty} \frac{1}{\text{Vol}(A_L)} \lim_{B \to 0} \frac{1}{B} \{\tilde{X}_0(B, L) - X_0(0, L)\}.
\]
Moreover, the mapping $s_B$ defined in Theorem 1.1 is differentiable at $B = 0$ and

\[
\lim_{L \to \infty} \{\partial_B \sigma_L\}(0) \lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \{\partial_B X_0\}(0, L) = \sigma_1 + \sigma_2 = \{\partial_B \sigma_\infty\}(0) = -\int_\Omega \{\partial_B s_B\}_{B=0}(x) dx. \quad (4.106)
\]

**Proof.** Let us start with (4.104). Using (3.24) one can show the following identity:

\[
\lim_{B \to 0} \frac{1}{B} \{X_0(B, L) - \hat{X}_0(B, L)\} = -\text{Tr} \int_{\Gamma_\omega} f_{FD}(z) \cdot \left\{ \begin{array}{l}
\tilde{P}_1(0) \tilde{g}_0(H_\infty(0) - z)^{-1} T_0(z) g_0 P_2(0) \tilde{g}_0 \{H_\infty(0) - z - \omega\}^{-1} g_0 \\
+ z \to z - \omega \end{array} \right\} dz - \lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \text{Tr} \chi_{\Lambda_L} \int_{\Gamma_\omega} f_{FD}(z) \cdot \left\{ \begin{array}{l}
\tilde{P}_1(0) \tilde{g}_0 (H_\infty(0) - z)^{-1} g_0 P_2(0) \tilde{g}_0 \{H_\infty(0) - z - \omega\}^{-1} T_0(z + \omega) g_0 \\
+ z \to z - \omega \end{array} \right\} dz. \quad (4.107)
\]

Then by integrating many times by parts, the integrand will become trace class, and we can get rid of the cut-off functions $\tilde{g}_0$ and $g_0$ since their removal will only contribute with a surface correction. Hence we can write:

\[
\sigma_1 = -\lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \text{Tr} \chi_{\Lambda_L} \int_{\Gamma_\omega} f_{FD}(z) \cdot \left\{ \begin{array}{l}
\tilde{P}_1(0) \tilde{g}_0 (H_\infty(0) - z)^{-1} T_0(z) g_0 P_2(0) \tilde{g}_0 \{H_\infty(0) - z - \omega\}^{-1} \\
+ z \to z - \omega \end{array} \right\} dz - \lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \text{Tr} \chi_{\Lambda_L} \int_{\Gamma_\omega} f_{FD}(z) \cdot \left\{ \begin{array}{l}
\tilde{P}_1(0) \tilde{g}_0 (H_\infty(0) - z)^{-1} g_0 P_2(0) \tilde{g}_0 \{H_\infty(0) - z - \omega\}^{-1} T_0(z + \omega) \\
+ z \to z - \omega \end{array} \right\} dz. \quad (4.108)
\]

Now one can prove (as we did for $F_\infty$) that the two operators defined above by integrals over $\Gamma_\omega$ have jointly continuous integral kernels, whose diagonal values are $\mathbb{Z}^3$-periodic. It means that the limit exists and equals the integral of the kernels’ diagonal value over the unit cube in $\mathbb{R}^3$.

Now let us continue with the proof of (4.105). First, we can get rid of $\hat{g}_0$ because $P_j(B)$ is local. Let us integrate by parts $N$ times with respect to $z$, $N$ large. Then a typical term in the integrand defining $\hat{X}_0(B, L)$ will be:

\[
P_1(B) S_B^{(N+1-k)}(z) g_0 P_2(B) S_B^{(k+1)}(z + \omega) g_0, \quad k \in \{0, \ldots, N\},
\]

where as before $S_B^{(N)}(z)$ has the integral kernel $e^{iB \varphi_0(x, x') G_N(x, x'; z)}$.

This operator will have an integral kernel given by:

\[
\int_{\mathbb{R}^3} P_1(x)(B) e^{i B \varphi_0(x, y)} G_{N+1-k}(x, y; z) g_0(y) \cdot P_2(y)(B) e^{i B \varphi_0(y, x')} G_{k+1}(y, x'; z + \omega) g_0(x') dy. \quad (4.109)
\]

We commute the momenta with the magnetic phases and obtain:

\[
e^{i B \varphi_0(x, x')} P_1(x)(0) + B A_1(x - y)) G_{N+1-k}(x, y; z) g_0(y) \\
e^{i B \varphi_0(y, x')} P_2(y)(0) + B A_2(y - x')) G_{k+1}(y, x'; z + \omega) g_0(x') dy. \quad (4.110)
\]
This integral is absolutely convergent and defines a continuous function in \( x \) and \( x' \) (we can see this from the regularity and exponential localization of \( G_N(x, x'; z) \) and its first order derivatives). In order to perform the trace of this operator we put \( x = x' \). The two magnetic phases will disappear, thus we get:

\[
\int_{\mathbb{R}^3} \{ P_{1,x}(0) + BA_1(x - y) \} G_{N+1-k}(x, y; z) g_0(y) \\
\{ P_{2,y}(0) + BA_2(y - x) \} G_{k+1}(y, x; z + \omega) g_0(x) dy.
\] (4.111)

The contribution to \( \lim_{B \to 0} \frac{1}{B} (\mathcal{X}_B(B) - \mathcal{X}_0(0, L)) \) coming from this term will be:

\[
R_L(x) := g_0(x) \int_{\mathbb{R}^3} A_1(x - y) G_{N+1-k}(x, y; z) g_0(y) A_2(y - x) G_{k+1}(y, x; z + \omega) dy.
\] (4.112)

Now we have to investigate the existence of the limit:

\[
\lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \int_{\Lambda_L} R_L(x) dx.
\] (4.113)

Let us first note that due to the exponential localization of \( G_k \)'s (see (3.1)) we have the following uniform estimate:

\[
\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} |A_1(x - y)| |G_{N+1-k}(x, y; z)||A_2(y - x)||G_{k+1}(y, x; z + \omega)| dy \\
\leq \text{const} \cdot \rho^M.
\] (4.114)

If we look back at the definition of \( g_0 \), we see that it equals 1 on the complementary in \( \Lambda_L \) of a boundary neighborhood like \( \Xi_L(t_0) \) with \( t_0 > 1 \) (see (4.1)). Denote by \( \chi_L \) the characteristic function of \( \Lambda_L \setminus \Xi_L(2t_0) \). Thus we have

\[
\chi_L \ g_0 = \chi_L, \quad \text{dist}\{\text{supp}(1 - g_0), \text{supp}(\chi_L)\} \geq t_0 L^\alpha.
\] (4.115)

Because of the uniform estimate (4.114), the limit in (4.113) exists if and only if the following one exists:

\[
\lim_{L \to \infty} \frac{1}{\text{Vol}(\Lambda_L)} \int_{\Lambda_L} \chi_L(x) R_L(x) dx,
\] (4.116)

because the difference between integrands only gives a surface contribution. Let us now define:

\[
R_{\infty}(x) := \int_{\mathbb{R}^3} A_1(x - y) G_{N+1-k}(x, y; z) A_2(y - x) G_{k+1}(y, x; z + \omega) dy.
\] (4.117)

The difference between \( \chi_L R_L \) and \( \chi_L R_{\infty} \) comes from the integration over the support of \( 1 - g_0 \). But due to (4.115) and the exponential decay of \( G_k \)'s, this difference is of order \( e^{-\delta L^7/\rho} \), thus will not contribute to the limit. Moreover, \( R_{\infty} \) is \( \mathbb{Z}^3 \)-periodic, therefore we can write:

\[
\lim_{L \to \infty} \sup_{z \in \Gamma_\omega} \langle r \rangle^{-M} \left| \frac{1}{\text{Vol}(\Lambda_L)} \int_{\Lambda_L} R_L(x) dx \right| = \left| \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} R_{\infty}(x) dx \right| = 0,
\] (4.118)

where \( M \) is some large enough positive number. Then the exponential decay of \( f_{FD} \) will assure the convergence of the \( \Gamma_\omega \)-integrals, thus (4.110) is proved.

The last ingredient in the proof of Proposition 4.14 is the computation of \( \partial_B \sigma_\infty(0) \) and the comparison with \( \sigma_1 + \sigma_2 \). But the steps are very similar to those we have already done in order to compute \( \sigma_1 \) and \( \sigma_2 \). First, one integrates by parts many times with respect to \( z \) in order to
obtain a “nice” form for $F_\infty$. Second, using the magnetic perturbation theory one writes down a Taylor expansion in $B$ of $s_B(x)$ at $B = 0$ which only contains “regularized” terms and where we can interchange the expansion in $B$ with the thermodynamic limit $L \to \infty$. This strategy has been already used in [13] for the Faraday effect (including the spin contribution, neglected here), and in [7] for generalized susceptibilities.

5 Appendix: Uniform exponential decay

The following proposition contains two key estimates which we are going to use throughout this paper.

**Proposition 5.1.** Assume that $z \in \mathbb{C}$ and $\text{dist}(z, [0, \infty)) = \eta > 0$. Then for any $\alpha \in \{1, 2, 3\}$ we have

$$
\sup_{L > 1} ||| [D_\alpha + B_0^\alpha] \{(-i\nabla + Ba)^2_D - z\}^{-1} |||
\leq \sqrt{1/\eta + \max\{\Re(z), 0\}/\eta^2}.
$$

Moreover, there exists a constant $C$ such that:

$$
\sup_{L > 1} \sup_{\Re(z) \geq 0} \langle |z|^{-1} ||[D_\alpha + B_0^\alpha](H_L(B) - z)^{-1}|| \rangle \leq C/\eta.
$$

**Proof.** The estimate (5.1) is an easy consequence of the following trivial identity, valid for every $\psi \in L^2(\Lambda_L)$:

$$
\sum_{\alpha=1}^{3} ||| [D_\alpha + B_0^\alpha] \{(-i\nabla + Ba)^2_D - z\}^{-1} \psi |||^2
= \Re \langle \{((-i\nabla + Ba)^2_D - z)^{-1} \psi, \psi\} \rangle + \Re(z) \langle \{(-i\nabla + Ba)^2_D - z\}^{-1} \psi, \psi \rangle.
$$

The estimate (5.2) is a bit more involved. From (5.1) we have that for every $\lambda > 1$:

$$
\sup_{L > 1} ||| [D_\alpha + B_0^\alpha] \{(-i\nabla + Ba)^2_D - i\lambda\}^{-1} ||| \leq \frac{C}{\sqrt{\lambda}}.
$$

Since $V$ is bounded we have:

$$
\sup_{L > 1} ||| V \{(-i\nabla + Ba)^2_D - i\lambda\}^{-1} ||| \leq \frac{C}{\lambda}.
$$

Choosing a $\lambda_0$ large enough and using the Neumann series in $V$ for the resolvent we have

$$
\sup_{L > 1} \{||| V(H_L(B) - i\lambda_0)^{-1} ||| + |||((-i\nabla + Ba)^2_D - i\lambda_0)(H_L(B) - i\lambda_0)^{-1} ||| \} \leq 1/2.
$$

Using the resolvent identity we obtain:

$$
\sup_{L > 1} |||((-i\nabla + Ba)^2_D - i\lambda_0)(H_L(B) - z)^{-1} ||| \leq C |z|/\eta.
$$

Hence writing

$$
[D_\alpha + B_0^\alpha](H_L(B) - z)^{-1} = [D_\alpha + B_0^\alpha]((-i\nabla + Ba)^2_D - i\lambda_0)^{-1}
\cdot ((-i\nabla + Ba)^2_D - i\lambda_0)(H_L(B) - z)^{-1}
$$

we obtain the result.
We will need a certain type of uniform exponential localization, stated in the proposition below. If \( x_0 \) is some point in \( \Lambda_L \) and \( \alpha \in \mathbb{R} \), then let \( e^{\alpha (-x_0)} \) denote the multiplication operator with the function \( e^{\alpha \sqrt{|x-x_0|^2+1}} \). Note that multiplication with the exponential weight is a bounded operator if \( L < \infty \) and leaves invariant the domain of \( H_L(B) \).

**Proposition 5.2.** Fix \( x_0 \in \Lambda_L \) and \( \text{dist}\{z, [0, \infty)\} = \eta > 0 \). Denote by \( r := \Re (z) \). Then there exists a \( \delta_0 > 0 \) and a constant \( C \) such that for every \( 0 \leq \delta \leq \delta_0 \) we have

\[
\sup_{L>1} \sup_{r \in \mathbb{R}} \sup_{x_0 \in \Lambda} \left\| e^{\alpha (-x_0)} \frac{\partial}{\partial r} (H_L(B) - z) e^{\alpha (-x_0)} \right\| \leq C, \tag{5.5}
\]

\[
\sup_{L>1} \sup_{r \in \mathbb{R}} \sup_{x_0 \in \Lambda} \left\{ \langle r \rangle^{-1} \left\| (D_0 + Ba_0) e^{\alpha (-x_0)} \frac{\partial}{\partial r} (H_L(B) - z)^{-1} e^{\alpha (-x_0)} \right\| \right\} \leq C, \tag{5.6}
\]

and

\[
\sup_{L>1} \sup_{r \in \mathbb{R}} \sup_{x_0 \in \Lambda} \left\{ \langle r \rangle^{-1} \left\| e^{\alpha (-x_0)} \frac{\partial}{\partial r} \left[ (-i \nabla + Ba)^2_D + 1 \right] (H_L(B) - z)^{-1} e^{\alpha (-x_0)} \right\| \right\} \leq C. \tag{5.7}
\]

**Proof.** An heuristic explanation of (5.5) is the following: if we apply the resolvent on a function which is exponentially localized near \( x_0 \) and decays like \( e^{-\frac{\alpha}{\delta_0} (x-x_0)} \), then we do not lose exponential decay. Moreover, the \( L^2 \) bounds are uniform in \( z, L \) and the location of \( x_0 \).

That for \( s \in \mathbb{R} \), the well-known Combes-Thomas rotation \[\] gives:

\[
e^{s(-x_0)} (H_L(B) - z) e^{-s(-x_0)} = H_L(B) - z + s \sum_{j=1}^{3} w_j [D_j + Ba_j] + sV_1 + s^2V_2,
\]

where \( w_j, V_1, V_2 \) are bounded functions, uniformly in \( L \) and \( x_0 \).

Now put \( s = \delta / \langle r \rangle \), and use (5.2). If \( \delta \) is small enough, we get that uniformly in \( L, x_0 \) and \( r \) we have

\[
\| s \sum_{j=1}^{3} w_j [D_j + Ba_j] + sV_1 + s^2V_2 \| (H_L(B) - z)^{-1} \| \leq 1/2,
\]

which gives

\[
e^{s(-x_0)} (H_L(B) - z)^{-1} e^{-s(-x_0)} = (H_L(B) - z)^{-1}
\]

\[
\times \left\{ 1 + s \sum_{j=1}^{3} w_j [D_j + Ba_j] + sV_1 + s^2V_2 \right\} (H_L(B) - z)^{-1} \cdot \tag{5.8}
\]

This implies (5.5), and together with (5.2) we also get (5.6).

Let us now concentrate ourselves on the last estimate (5.7). Up to a commutation, (5.6) gives

\[
\sup_{L>1} \sup_{r \in \mathbb{R}} \sup_{x_0 \in \Lambda} \left\{ \langle r \rangle^{-1} \left\| (D_0 + Ba_0) e^{\alpha (-x_0)} \frac{\partial}{\partial r} (H_L(B) - z)^{-1} e^{\alpha (-x_0)} \right\| \right\} \leq C. \tag{5.9}
\]

Thus again up to a commutation, (5.7) follows if we can prove

\[
\sup_{L>1} \sup_{r \in \mathbb{R}} \sup_{x_0 \in \Lambda} \left\{ \langle r \rangle^{-1} \left\| (-i \nabla + Ba)^2_D + 1 \right\| (H_L(B) - z)^{-1} e^{\alpha (-x_0)} \right\| \right\} \leq C. \tag{5.10}
\]

But this estimate follows from (5.8) and (5.4).
Corollary 5.3. Let $\lambda \geq \lambda_0 > 0$. Then there exists $c > 0$ such that
\[
\sup_{L > 1} \sup_{\lambda \geq \lambda_0} \sup_{x_0 \in \Lambda} \left\| e^{\pm c(-x_0)\sqrt{\lambda}} (H_L(B) + \lambda)^{-1} e^{\mp c(-x_0)\sqrt{\lambda}} \right\| \leq \frac{\text{const}(\lambda_0)}{\lambda}. \tag{5.11}
\]

Proof. We use the key estimate (5.1) for the case when $\eta \geq \lambda$ and $\Re(z) = -\lambda < 0$. This gives us
\[
\left\| (D_\alpha + Ba_\alpha)(-i\nabla + Ba)^2_D + \lambda \right\| \leq \text{const}/\sqrt{\lambda},
\]
and hence
\[
\left\| (D_\alpha + Ba_\alpha)(H_L(B) + \lambda)^{-1} \right\| \leq \text{const}/\sqrt{\lambda}.
\]
Now we proceed as in (5.8) and we get the result. Finally, note that by repeating the argument of Proposition 5.2 we can obtain a uniform estimate in $\lambda$, $L$, and $x_0$:
\[
\left\| e^{\pm c(-x_0)\sqrt{\lambda}} (-i\nabla + Ba)^2_D + \lambda (H_L(B) + \lambda)^{-1} e^{\mp c(-x_0)\sqrt{\lambda}} \right\| \leq \text{const}. \tag{5.12}
\]

Proposition 5.4. The operator $\left( (-i\nabla + Ba)^2_D + \lambda \right)^{-1}$ has an integral kernel $K_L(x, x')$ which is jointly continuous away from the diagonal $x = x'$, and obeys the estimate
\[
|K_L(x, x')| \leq \frac{e^{-\sqrt{\lambda}|x-x'|}}{4\pi|x-x'|}, \tag{5.13}
\]
for every $x \neq x'$ in $\Lambda_L$.

Proof. The argument is based on several well known results. First, one uses the Feynman-Kac-Itô representation for the kernel of the semi-group $e^{-t(-i\nabla + Ba)_D^2}$, $t > 0$ (see [8]) and obtains a diamagnetic inequality in $\Lambda_L$:
\[
\left| e^{-t(-i\nabla + Ba)_D^2}(x, x') \right| \leq e^{t\Delta_D}(x, x') \leq (4\pi t)^{-3/2} e^{-\frac{|x-x'|^2}{4t}}, \quad x, x' \in \Lambda_L.
\]
Second, we perform a Laplace transform and obtain the result.

Proposition 5.5. All the results in this section are also valid if the operators are defined on the whole space $\mathbb{R}^3$ (formally $L = \infty$).

Proof. The argument relies on various standard limiting and cut-off arguments, which are necessary because the exponential growing factors do not invariate the operator domain of $H_{\infty}(B)$. The most important ingredient (uniformity in $L > 1$ of all our previous estimates) has been already proved.

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References
[1] Angelescu, N., Bundaru, M., Nenciu, G.: On the Landau diamagnetism. Comm. Math. Phys. 42, 9–28 (1975)
[2] Bonnaillie-Noël, V., Fournais, S.: Superconductivity in domains with corners. Rev. Math. Phys. 19 no. 6, 607–637 (2007)
[3] Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics. 2. (Equilibrium states. Models in quantum statistical mechanics.) Texts and Monographs in Physics. Springer-Verlag, Berlin, (1997)
[4] Briet, P.; Combes, J.M.; Duclos, P.: Spectral stability under tunneling. *Comm. Math. Phys.* **126** no. 1, 133-156 (1989)

[5] Briet, P., Cornean, H.D.: Locating the spectrum for magnetic Schrödinger and Dirac operators. *Comm. Partial Differential Equations* **27** no. 5-6, 1079–1101 (2002)

[6] Briet, P.; Cornean, H. D.; Louis, D.: Generalized susceptibilities for a perfect quantum gas. *Markov Process. Related Fields* **11** no. 2, 177–188 (2005)

[7] Briet, P.; Cornean, H. D.; Louis, D.: Diamagnetic expansions for perfect quantum gases. *J. Math. Phys.* **47** no. 8, 083511 (2006)

[8] Broderix, K., Hundertmark, D., Leschke, H.: Continuity properties of Schrödinger semigroups with magnetic fields. *Rev. Math. Phys.* **12** no. 2, 181–225 (2000)

[9] Combes, J. M.; Thomas, L. Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. *Comm. Math. Phys.* **34**, 251–270 (1973)

[10] Cornean, H.D.: On the magnetization of a charged Bose gas in the canonical ensemble. *Comm. Math. Phys.* **212** no.1, 1–27 (2000)

[11] Cornean, H.D., Nenciu, G.: On eigenfunction decay for two-dimensional magnetic Schrödinger operators. *Comm. Math. Phys.* **198**, no. 3, 671–685 (1998)

[12] Cornean, H.D., Nenciu, G.: Two-dimensional magnetic Schrödinger operators: width of mini bands in the tight binding approximation. *Ann. Henri Poincaré* **1** (2), 203–222 (2000)

[13] Cornean, H.D., Nenciu, G., Pedersen, T. G.: The Faraday effect revisited: general theory. *J. Math. Phys.* **47** no. 1, 013511 (2006)

[14] Fournais, S., Helffer, B.: On the third critical field in Ginzburg-Landau theory. *Commun. Math. Phys.* **266** no. 1, 153–196 (2006)

[15] Grigis, A., Sjöstrand, J.: *Microlocal analysis for differential operators. An introduction.* London Mathematical Society Lecture Note Series, 196. Cambridge University Press, Cambridge, 1994.

[16] Hacyan, S., Jauregui, R.: Faraday effect and Bessel beams in a magneto-optic medium. *Journal of Physics B. (Molecular and Optical Physics)* **41** no. 1, 015402 (2008)

[17] Helffer, B., Sjöstrand, J.: On diamagnetism and de Haas-van Alphen effect. *Ann. Inst. H. Poincaré Phys. Théor.* **52** no. 4, 303–375 (1990)

[18] Helffer, B., Sjöstrand, J.: Équation de Schrödinger avec champ magnétique et équation de Harper. *Lecture Notes in Phys.* **345**, 118–197, Springer, Berlin 1989.

[19] Hörmander, L.: *Linear partial differential operators. I.* Die Grundlehren der mathematischen Wissenschaften, Bd. 116 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg 1963.

[20] Iftimie, V.: Uniqueness and existence of the integrated density of states for Schrödinger operators with magnetic field and electric potential with singular negative part. *Publications of the Research Institute for Mathematical Sciences* **41** no. 2, 307-327 (2005)

[21] Iftimie, V., Măntoiu, M., Purice, R.: Magnetic Pseudodifferential Operators. *Publications of the Research Institute for Mathematical Sciences* **43** no. 3, 585-623 (2007)

[22] Kastrup, H.A.: Quantization of the canonically conjugate pair angle and orbital angular momentum. *Phys. Rev. A.* **73**, 052104 (2006)
[23] Martínez, D., Plechaty, C., Presura, R.: Magnetic fields for the laboratory simulation of astrophysical objects. *Astrophysics and Space Science* **307** no. 1–3, 109-114 (2007)

[24] Măntoiu, M., Purice, R.: Strict deformation quantization for a particle in a magnetic field. *J. Math. Phys.* **46** no. 5, 052105 (2005)

[25] Măntoiu, M., Purice, R.: The magnetic Weyl calculus. *J. Math. Phys.* **45** no. 4, 1394–1417 (2004)

[26] Măntoiu, M., Purice, R., Richard, S.: Spectral and propagation results for magnetic Schrödinger operators; A $C^*$-algebraic framework. *J. Funct. Anal.* **250** no. 1, 42-67 (2007)

[27] Nenciu, G.: On asymptotic perturbation theory for quantum mechanics: almost invariant subspaces and gauge invariant magnetic perturbation theory. *J. Math. Phys.* **43**, no. 3, 1273–1298 (2002)

[28] Pedersen, T.G.: Tight-binding theory of Faraday rotation in graphite. *Phys. Rev. B* **68**, 245104 (2003)

[29] Roth, L.M.: Theory of the Faraday effect in solids. *Phys. Rev.* **133**, A542–A553 (1964)

[30] Statistical mechanics: Rigorous results. World Scientific Publishing Co., Inc., River Edge, NJ; Imperial College Press, London (1999)

[31] Roth, L.M.: Theory of Bloch electrons in a magnetic field. *Journal of Physics and Chemistry of Solids* **23**, 433 (1962)

[32] Sondheimer, E.H., Wilson, A.H.: The theory of the magneto-resistance effects in metals. *Proceedings of the Royal Society of London Series A - Mathematical and Physical Sciences* **190** (1023), 435-455 (1947)

[33] Zarifi, A., Pedersen, T.G.: Theoretical analysis of the Faraday effect in zigzag carbon nanotubes. *Phys. Rev. B* **77**, 085409 (2008)