The role of torsion in $f(R)$ gravity is considered in the framework of metric-affine formalism. We discuss the field equations in empty space and in the presence of perfect fluid matter, taking into account the analogy with the Palatini formalism. As a result, the extra curvature and torsion degrees of freedom can be dealt as an effective scalar field of a fully geometric origin. From a cosmological point of view, such a geometric description could account for the whole dark side of the universe.

PACS numbers: 04.20.Cv, 04.20+Fy, 04.20.Gz, 98.80.–k

1. Introduction

In the last 30 years, some shortcomings appeared in the Einstein general relativity (GR) and several investigations began to study if alternative approaches to gravitational interaction are possible and self-consistent. Such issues come from cosmology and quantum field theory. In the first case, the presence of big bang singularity, flatness and horizon problems [1] led to the result that the standard cosmological model [2], based on GR and the standard model of particle physics, is inadequate to describe the universe at extreme regimes. On the other hand, GR does not work for a quantum description of spacetime. Due to these facts and to the lack of a quantum gravity theory, alternative theories of gravity have been pursued in order to attempt, at least, a semi-classical scheme where GR and its positive results could be recovered.

A fruitful approach has been that of extended theories of gravity (ETG), which have become a sort of paradigm in the study of gravitational interaction. They are essentially based on corrections and enlargements of the Einstein theory. The paradigm consists of adding higher order curvature invariants and non-minimally coupled scalar fields into dynamics resulting from the effective action of quantum gravity [3, 4].

Other motivations for modifying GR come from the issue of recovering Mach’s principle [5]. This principle states that the local inertial frame is determined by some average of the motion of distant astronomical objects [6], so that gravitational coupling can be scale
dependent and related to some scalar field. This viewpoint leads us to assume a varying gravitational coupling. As a consequence, the concepts of ‘inertia’ and equivalence principle have to be revised [5, 7–9].

Furthermore, every unification scheme such as superstrings, supergravity or grand unified theories takes into account effective actions where non-minimal couplings to the geometry or higher order terms in the curvature invariants appear. Such contributions are due to one-loop or higher loop corrections in the high-curvature regimes. In particular, this scheme has been adopted in order to deal with quantization in curved spacetimes and, as a result, the interactions between quantum scalar fields, the gravitational self-interactions and the background geometry yield correction terms in the Hilbert–Einstein Lagrangian [10].

Moreover, it has been realized that such terms are inescapable if we want to obtain the effective action of quantum gravity on scales close to the Planck length [11]. Higher order terms in the curvature invariants, such as \( R^2, R^{ij} R_{ij}, R^{ijkl} R_{ijkl}, R \Box R, R \Box^2 R \), or non-minimally coupled terms between scalar fields and geometry, such as \( \phi^2 R \), have to be added to the effective Lagrangian of the gravitational field when quantum corrections are considered. For example, one has to stress that such terms occur in the effective Lagrangian of strings or in Kaluza–Klein theories, when the mechanism of dimensional reduction is used [12].

From a conceptual point of view, there would be no a priori reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar \( R \), minimally coupled with matter [13]. The idea that there are no ‘exact’ laws of physics but that the Lagrangians of physical interactions are ‘stochastic’ functions—with the property that local gauge invariances (i.e. conservation laws) are well approximated in the low energy limit and that physical constants can vary—has been taken into serious consideration; see [14] and references therein.

Besides fundamental physics motivations, all these theories have acquired a huge interest in cosmology due to the fact that they ‘naturally’ exhibit inflationary behaviors able to overcome the shortcomings of the standard cosmological model (based on GR). The related cosmological models seem very realistic and capable of matching with the observations [15–17].

Furthermore, it is possible to show that, via conformal transformations, the higher order and non-minimally coupled terms always correspond to Einstein gravity plus one or more than one minimally coupled scalar fields [18–22]. More precisely, higher order terms always appear as a contribution of order 2 in the equations of motion. For example, a term like \( R^2 \) gives fourth-order equations [23], \( R \Box R \) gives sixth-order equations [21, 24], \( R \Box^2 R \) gives eighth-order equations [25] and so on. By a conformal transformation, any second-order of derivation corresponds to a scalar field: for example, fourth-order gravity gives Einstein plus one scalar field, sixth-order gravity gives Einstein plus two scalar fields and so on [21, 26]. This features results are very interesting if we want to obtain multiple inflationary events since a former early stage could select ‘very’ large-scale structures (clusters of galaxies today), while a latter stage could select ‘small’ large-scale structures (galaxies today) [24]. The philosophy is that each inflationary era is connected with the dynamics of a scalar field. Furthermore, these extended schemes could naturally solve the problem of ‘graceful exit’, bypassing the shortcomings of former inflationary models [16, 27].

Recently, ETG also played an interesting role in describing the observed universe. In fact, the amount of good quality data of the last decade has made it possible to shed new light on the effective picture of the universe. Type Ia supernovae (SNeIa) [28], anisotropies in the cosmic microwave background radiation (CMBR) [29] and matter power spectrum inferred from large galaxy surveys [30] represent the strongest evidence for a radical revision of the cosmological standard model also at recent epochs. In particular, the concordance \( \Lambda \) cold dark matter (CDM) model predicts that baryons contribute only \( \sim 4\% \) of the total matter-energy budget,
while the exotic CDM represents the bulk of the matter content (∼25%) and the cosmological constant Λ plays the role of the so-called ‘dark energy’ (∼70%) [31]. Although being the best fit to a wide range of data [32], the ΛCDM model is severely affected by strong theoretical shortcomings [33] that have motivated the search for alternative models [34, 35].

Dark energy models mainly rely on the implicit assumption that Einstein’s GR is indeed the correct theory of gravity. Nevertheless, its validity at the larger astrophysical and cosmological scales has never been tested [36], and it is therefore conceivable that both cosmic speed-up and dark matter represent signals of a breakdown in our understanding of gravitation law so that one should consider the possibility that the Hilbert–Einstein Lagrangian, linear in the Ricci scalar $R$, should be generalized.

Following this line of thinking, the choice of a generic function $f(R)$ can be derived by matching the data and by the ‘economic’ requirement that no exotic ingredients have to be added. This is the underlying philosophy of what is referred to as $f(R)$ gravity [37–47]. In this context, the same cosmological constant could be removed as an ingredient of the cosmic pie being nothing else but a particular eigenvalue of a general class of theories [48].

However, $f(R)$ gravity can be encompassed in ETG being a ‘minimal’ extension of GR where (analytical) functions of Ricci scalar are taken into account.

Although higher order gravity theories have received much attention in cosmology, since they are naturally able to give rise to accelerating expansions (both in the late and in the early universe) and systematic studies of the phase space of solutions are in progress [49–53], it is possible to demonstrate that $f(R)$ theories can also play a major role at astrophysical scales. In fact, modifying the gravity Lagrangian can affect the gravitational potential in the low energy limit. Provided that the modified potential reduces to the Newtonian one on the solar system scale, this implication could represent an intriguing opportunity rather than a shortcoming for $f(R)$ theories (see, for example, [54–58]).

Furthermore, a corrected gravitational potential could offer the possibility of fitting galaxy rotation curves without the need of dark matter [59–62]. In addition, it is possible to work out a formal analogy between the corrections to the Newtonian potential and the usually adopted dark matter models. In general, any relativistic theory of gravitation can yield corrections to the Newton potential [63] which, in the post-Newtonian (PPN) formalism, could give rise to tests for the same theory [36, 64–66].

In this paper, we want to face the problem of studying $f(R)$ gravity also considering torsion. Torsion theories have been taken into account firstly by Cartan and were then introduced by Sciama and Kibble in order to deal with spin in general relativity (see [67] for a review). Being the spin as fundamental as the mass of the particles, torsion was introduced in order to complete the following scheme: the mass (energy) as the source of curvature and the spin as the source of torsion.

Up until some time ago, torsion did not seem to produce models with observable effects since phenomena implying spin and gravity were considered to be significant only in the very early universe. Afterwards, it has been proven that spin is not the only source of torsion. As a matter of fact, the torsion field can be decomposed in three irreducible tensors, with different properties. In [68], a systematic classification of these different types of torsion and their possible sources is discussed. This means that a wide class of torsion models could be investigated independently of spin as their source.

In principle, torsion could be constrained at every astrophysical scale and, as recently discussed, data coming from gravity probe B could also contribute to this goal at the solar system level [69].

In [70, 71], a systematic discussion of metric-affine $f(R)$ gravity has been pursued. In particular, the role of connection in the presence of matter has been studied considering the
various possible matter actions depending on connection. The main result of these papers has been the evidence that matter can tell spacetime how to curve as well as how to twirl.

In this paper, following the same philosophy, we want to show that, starting from a generic \( f(R) \) theory, the curvature and the torsion can give rise to an effective curvature-torsion stress-energy tensor capable, in principle, of addressing the problem of the dark side of the universe in a very general geometric scheme. We do not consider the possible microscopic distribution of spin, but a general torsion vector field in \( f(R) \) gravity.

The layout of the paper is as follows. In sections 2 and 3, we derive the metric-affine field equations of \( f(R) \) gravity with torsion in empty space and in the presence of matter, respectively. Section 4 is devoted to the discussion of the formal equivalence with scalar-tensor theories, while applications to the Friedmann–Robertson–Walker (FRW) cosmology are discussed in section 5. Summary and conclusions are drawn in section 6.

2. Field equations in empty space

Let us discuss the main features of an \( f(R) \) gravity considering the most general case in which torsion is present in a \( U_4 \) manifold. In a metric-affine formulation, the metric \( g \) and the connection \( \Gamma \) can be, in general, considered independent fields. More precisely, the dynamical fields are pairs \( (g, \Gamma) \) consisting of a pseudo-Riemannian metric \( g \) and a metric compatible linear connection \( \Gamma \) on the spacetime manifold \( M \). The corresponding field equations are derived by varying separately with respect to the metric and the connection the action functional

\[
A(g, \Gamma) = \int \sqrt{|g|} f(R) \, ds,
\]

where \( f \) is a real function, \( R(g, \Gamma) = g^{ij} R_{ij} \) (with \( R_{ij} := R_{i\,b\,j}^b \)) is the scalar curvature associated with the connection \( \Gamma \) and \( ds := dx^1 \wedge \cdots \wedge dx^4 \). Throughout the paper, we use the index notation

\[
R_{\,ki\,}^j = \frac{\partial \Gamma_{jk}^h}{\partial x^i} - \frac{\partial \Gamma_{ik}^h}{\partial x^j} + \Gamma_{ip}^h \Gamma_{jk}^p - \Gamma_{jp}^h \Gamma_{ik}^p
\]

(2)

for the curvature tensor and

\[
\nabla \frac{\partial}{\partial x^i} = \Gamma_{ij}^h \frac{\partial}{\partial x^h}
\]

(3)

for the connection coefficients.

In order to evaluate the variation \( \delta A \) under arbitrary deformations of the connection, we recall that, given a metric tensor \( g_{ij} \), every metric connection \( \Gamma \) may be expressed as

\[
\Gamma_{ij}^h = \tilde{\Gamma}_{ij}^h - K_{ij}^h,
\]

(4)

where (in the holonomic basis \( \{ \frac{\partial}{\partial x^i}, dx^i \} \) \( \tilde{\Gamma}_{ij}^h \) denote the coefficients of the Levi-Civita connection associated with the metric \( g_{ij} \), while \( K_{ij}^h \) indicate the components of a tensor satisfying the antisymmetry property \( K_{ij}^h = -K_{ji}^h \). This last condition ensures the metric compatibility of the connection \( \Gamma \).

In view of this, we can identify the actual degrees of freedom of the theory with the (independent) components of the metric \( g \) and the tensor \( K \). Moreover, it is easily seen that the curvature and the contracted curvature tensors associated with every connection (4) can be expressed respectively as

\[
R_{\,i\,j\,}^h = \tilde{R}_{\,i\,j\,}^h + \tilde{\nabla}_j K_{qi}^h - \tilde{\nabla}_q K_{ji}^h + K_{ji}^p K_{qp}^h - K_{qi}^p K_{jp}^h
\]

(5a)

and

\[
R_{ij} = \tilde{R}_{ij} + \tilde{\nabla}_j K_{hi}^h - \tilde{\nabla}_h K_{ji}^h + K_{ji}^p K_{hp}^h - K_{hi}^p K_{jp}^h,
\]

(5b)
where $\hat{R}^h_{ijl}$ and $\hat{R}_{ij} = \hat{R}^h_{ihj}$ are respectively the Riemann and the Ricci tensors of the Levi-Civita connection $\Gamma$ associated with the given metric $g$, and $\hat{\nabla}$ indicates the Levi-Civita covariant derivative.

Making use of the identities (5b), the action functional (1) can be written in the equivalent form

$$A(g, \Gamma) = \int \sqrt{|g|} f\left(g^{ij}\left(\hat{R}_{ij} - \hat{\nabla}_i \hat{\nabla}_j K + K_{ji}^p K_{hp} - K_{hi}^p K_{jp}^h\right) + \hat{\nabla}_i \delta K_{hp}^j - \hat{\nabla}_j \delta K_{hp}^i\right)\right) ds,$$

more suitable for variations in the connection. Taking the metric $g$ fixed, we have the identifications $\delta \Gamma^j_{ih} = \delta K_{ij}^h$ and then the variation

$$\delta A = \int \sqrt{|g|} f'(R) g^{ij} \left[\hat{\nabla}_i \delta K_{hi}^j - \hat{\nabla}_j \delta K_{hi}^i + \delta K_{ji}^p K_{hp} + K_{ji}^p \delta K_{hp}^i - \delta K_{hi}^p K_{jp}^h\right] \delta K_{hij} ds.$$

Using the divergence theorem, taking the antisymmetry properties of $K$ into account and finally renaming some indexes, we get the expression

$$\delta A = \int \sqrt{|g|} \left[-\frac{\partial f'}{\partial x^i} \delta_i^h + \frac{\partial f'}{\partial x^j} \delta_j^h + f' K_{pj}^p \delta_i^h - f' K_{pi}^p \delta_j^h - f' K_{ij}^h + f' K_{ji}^h\right] \delta K_{hij} ds.$$

(8)

The requirement $\delta A = 0$ yields, therefore, a first set of field equations given by

$$K_{pj}^p \delta_i^h - K_{pi}^p \delta_j^h - K_{ij}^h + K_{ji}^h = \frac{1}{f'} \frac{\partial f'}{\partial x^p} \left(\delta_i^p \delta_j^h - \delta_j^p \delta_i^h\right).$$

(9)

Considering that the torsion coefficients of the connection $\Gamma$ are $T_{ij}^h := \Gamma_{ij}^h - \Gamma_{ji}^h = -K_{ij}^h + K_{ji}^h$ and thus (due to antisymmetry) $T_{pi}^p = -K_{pi}^p$, equation (9) can be rewritten as

$$T_{ij}^h + T_{jp}^p \delta_i^h - T_{ip}^p \delta_j^h = \frac{1}{f'} \frac{\partial f'}{\partial x^p} \left(\delta_i^p \delta_j^h - \delta_j^p \delta_i^h\right).$$

(10)

or, equivalently, as

$$T_{ij}^h = -\frac{1}{2} \frac{\partial f'}{\partial x^p} \left(\delta_i^p \delta_j^h - \delta_j^p \delta_i^h\right).$$

(11)

In order to study the variation $\delta A$ under arbitrary deformations of the metric, it is convenient to resort to the representation (1). Indeed, from the latter, we directly have

$$\delta A = \int \sqrt{|g|} \left[f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij}\right] \delta g^{ij} ds,$$

thus getting the second set of field equations

$$f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} = 0.$$

(13)

Of course, one can obtain the same equations (13) starting from the representation (6) instead of (1). In that case, the calculations are just longer.

As a remark concerning equations (13), it is worth noting that any connection satisfying equations (4) and (11) gives rise to a contracted curvature tensor $R_{ij}$ automatically symmetric. Indeed, since the tensor $K$ coincides necessarily with the contorsion tensor, namely

$$K_{ij}^h = \frac{1}{2} \left(-T_{ij}^h + T_{ji}^h - T_{hij}\right),$$

(14)

from equations (11) we have

$$K_{ij}^h = \frac{1}{2} \left(T_{ij}^h - T_{pi}^p g_{pi} g_{ij}\right).$$

(15)
being
\[ T_i := T_{ih} = - \frac{3}{2f'} \frac{\partial f'}{\partial x^i}. \] (16)

Inserting equation (15) in equation (5b), the contracted curvature tensor can be represented as
\[ R_{ij} = \tilde{R}_{ij} + \frac{2}{9} \tilde{\nabla}_j T_i + \frac{1}{3} \tilde{\nabla}_h T^{h} T_{ij} + \frac{1}{9} T_i T_j - \frac{1}{3} T_h T^{h} g_{ij}. \] (17)

The last expression, together with equations (16), entails the symmetry of the indexes \(i\) and \(j\). Therefore, in equation (13) we can omit the symmetrization symbol and write
\[ f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} = 0. \] (18)

Now, considering the trace of equation (18), we get
\[ f'(R) R - 2 f(R) = 0. \] (19)

The latter is an identity automatically satisfied by all possible values of \( R \) only in the special case \( f(R) = \alpha R^2 \). In all other cases, equation (19) represents a constraint on the scalar curvature \( R \).

As a conclusion it follows that, if \( f(R) \neq \alpha R^2 \), the scalar curvature \( R \) has to be constant (at least on connected domains) and coincides with a given solution value of (19). Under such a circumstance, equation (11) implies that the torsion \( T_{ih} \) has to be zero and the theory reduces to a \( f(R) \) theory without torsion. In particular, we note that in the case \( f(R) = R \), equation (19) yields \( R = 0 \) and therefore equation (18) is equivalent to Einstein’s equations in empty space \( R_{ij} = 0 \). On the other hand, if we assume \( f(R) = \alpha R^2 \), we can have non-vanishing torsion. In this case, by replacing equation (19) in equations (11) and (18), we obtain field equations of the form
\[ R_{ij} - \frac{1}{4} R g_{ij} = 0 \] (20a)
\[ T_{ij}^{h} = - \frac{1}{2R} \frac{\partial R}{\partial x^i} b_j^h + \frac{1}{2R} \frac{\partial R}{\partial x^j} b_i^h. \] (20b)

Finally, making use of equation (17) and the consequent relation
\[ R = \tilde{R} + 2 \tilde{\nabla}_h T^h - \frac{2}{3} T_h T^h \] (21)
in equations (20), we can separately point out the contribution due to the metric and that due to the torsion. In fact, directly from equation (20a) we have
\[ \tilde{R}_{ij} - \frac{1}{3} \tilde{\nabla}_j T_i + \frac{1}{3} \tilde{\nabla}_h T^h g_{ij} = \tilde{T}_i T_j + \frac{1}{3} T_h T^{h} g_{ij}, \] (22)
while from the ‘trace’ \( T_i := T_{ih} = - \frac{3}{2R} \frac{\partial R}{\partial x^i} \) of equation (20b) we derive
\[ \frac{\partial}{\partial x^i} \left( \tilde{R} + 2 \tilde{\nabla}_h T^h - \frac{2}{3} T_h T^h \right) = - \frac{2}{3} \left( \tilde{R} + 2 \tilde{\nabla}_h T^h - \frac{2}{3} T_h T^h \right) T_i. \] (23)

Equations (22) and (23) are the coupled field equations in vacuum for metric and torsion in the \( f(R) = \alpha R^2 \) gravitational theory.
3. Field equations in the presence of matter

The presence of matter is embodied in the action functional (1) by adding to the gravitational Lagrangian a suitable material Lagrangian density $L_m$, namely

$$A(g, \Gamma) = \int (\sqrt{|g|} f(R) + L_m) \, ds. \tag{24}$$

Throughout the paper, we shall consider the material Lagrangian density $L_m$ not containing terms depending on torsion degrees of freedom as in [71]. The physical meaning of this assumption will be discussed later. In this case, the field equations take the form

$$f'(R)R_{ij} - \frac{1}{2} f(R)g_{ij} = \Sigma_{ij} \tag{25a}$$

and

$$T^h_{ij} = -\frac{1}{2f'(R)} \frac{\partial f'(R)}{\partial x^p} (\delta^p_i \delta^h_j - \delta^p_j \delta^h_i), \tag{25b}$$

where $\Sigma_{ij} := -\frac{1}{\sqrt{|g|}} \frac{\delta L_m}{\delta g_{ij}}$ plays the role of the energy–momentum tensor. From the trace of equation (25a), we obtain a fundamental relation between the curvature scalar $R$ and the trace $\Sigma := g^{ij} \Sigma_{ij}$, which is

$$f'(R)R - 2f(R) = \Sigma \tag{26}$$

(see also [72] and references therein). In what follows, we shall systematically suppose that relation (26) is invertible and that $\Sigma \neq \text{const}$, thus allowing us to express the curvature scalar $R$ as a suitable function of $\Sigma$, namely

$$R = F(\Sigma). \tag{27}$$

With this assumption in mind, using equations (26) and (27) we can rewrite equations (25a) and (25b) in the form

$$R_{ij} - \frac{1}{2} Rg_{ij} = \frac{1}{f'(F(\Sigma))} \left( \Sigma_{ij} - \frac{1}{4} \Sigma g_{ij} \right) - \frac{1}{4} F(\Sigma) g_{ij} \tag{28a}$$

and

$$T^h_{ij} = -\frac{1}{2f'(F(\Sigma))} \frac{\partial f'(F(\Sigma))}{\partial x^p} (\delta^p_i \delta^h_j - \delta^p_j \delta^h_i). \tag{28b}$$

Moreover, making use of equations (17) and (21), in equation (28a) we can decompose the contracted curvature tensor and the curvature scalar in their Christoffel and torsion-dependent terms, thus getting an Einstein-like equation of the form

$$\tilde{R}_{ij} - \frac{1}{2} \tilde{R}g_{ij} = \frac{1}{f'(F(\Sigma))} \left( \Sigma_{ij} - \frac{1}{4} \Sigma g_{ij} \right) - \frac{1}{4} F(\Sigma) g_{ij} - \frac{2}{3} \tilde{\nabla}^l T_l T_{ij} = \frac{2}{9} T_i T_j - \frac{1}{9} T_h g_{ij} \tag{29}$$

Now, setting

$$\varphi := f'(F(\Sigma)) \tag{30}$$

from the trace of equations (28b), we obtain

$$T_i := T^h_{ih} = -\frac{3}{2\varphi} \frac{\partial \varphi}{\partial x^i}. \tag{31}$$
Therefore, substituting in equations (29), we end up with the final equations

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\phi} \tilde{\Sigma}_{ij} + \frac{1}{\phi^2} \left( -\frac{3}{2} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} + \phi \tilde{\nabla}_j \frac{\partial \phi}{\partial x^i} + \frac{3}{4} \frac{\partial \phi}{\partial x^h} \frac{\partial \phi}{\partial x^k} \tilde{g}^{hk} g_{ij} \right.
\]

\[
- \phi \tilde{\nabla}^h \frac{\partial \phi}{\partial x^h} g_{ij} - V(\phi) g_{ij},
\]

(32)

where we defined the effective potential

\[
V(\phi) := \frac{1}{4} \left[ \phi F^{-1}(f')^{-1}(\phi)) + \phi^2 (f')^{-1}(\phi) \right].
\]

(33)

Equation (32) may be difficult to solve; nevertheless, we can simplify this task finding solutions for a conformally related metric. Indeed, performing a conformal transformation of the kind \( \tilde{g}_{ij} = \phi g_{ij} \), equations (32) may be rewritten in the easier form (see, for example, [72, 75, 76])

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} \tilde{g}_{ij} = \frac{1}{\phi} \tilde{\Sigma}_{ij} - \frac{1}{\phi^3} V(\phi) \tilde{g}_{ij},
\]

(34)

where \( \tilde{R}_{ij} \) and \( \tilde{R} \) are respectively the Ricci tensor and the Ricci scalar curvature associated with the conformal metric \( \tilde{g}_{ij} \).

Concerning the connection \( \Gamma \), solution of the variational problem \( \delta A = 0 \), from equations (4), (15) and (31), one gets the explicit expression

\[
\Gamma_{ij}^h = \Gamma_{ij}^h + \frac{1}{2\phi} \frac{\partial \phi}{\partial x^i} \delta_i^h - \frac{1}{2\phi} \frac{\partial \phi}{\partial x^i} \phi^{ph} g_{ij}.
\]

(35)

We can now compare our results with those obtained for \( f(R) \) theories in the Palatini formalism [44, 70, 71, 74, 75, 77, 78]. If both the theories (with torsion and Palatini-like) are considered as ‘metric’, in the sense that the dynamical connection \( \Gamma \) is not coupled with matter \( \delta L_m / \delta \Gamma = 0 \) and it does not define parallel transport and covariant derivative in spacetime, then the two approaches are completely equivalent. Indeed, in the ‘metric’ framework, the true connection of spacetime is the Levi-Civita one associated with the metric \( g \), and the role played by the dynamical connection \( \Gamma \) is just to generate the right Einstein-like equations of the theory. Now, surprisingly enough, our field equations (32) are identical to the Einstein-like equations derived within the Palatini formalism [75].

On the other hand, if the theories are genuinely metric-affine, then they are different even though the condition \( \delta L_m / \delta \Gamma = 0 \) holds. In order to stress this point, we recall that in a metric-affine theory the role of dynamical connection \( \Gamma \) is not only that of generating Einstein-like field equations but also defining parallel transport and covariant derivative in spacetime. Therefore, different connections imply different spacetime properties. This means that the geodesic structure and the causal structures could not obviously coincide. For a discussion on this point, see [78]. Furthermore, it can be easily shown that the dynamical connection (35) differs from that derived within the Palatini formalism. Indeed the latter results are the Levi-Civita connection \( \Gamma \) associated with the conformal metric \( \tilde{g} = \phi g \) [74, 75], while (35) is clearly not. More precisely, (35) is related to \( \tilde{\Gamma} \) by the projective transformation

\[
\tilde{\Gamma}_{ij}^h = \Gamma_{ij}^h + \frac{1}{2\phi} \frac{\partial \phi}{\partial x^i} \delta_i^h,
\]

(36)

which is not allowed in the present theory because, for a fixed metric \( g \), the connection (36) is no longer metric compatible.

To conclude, we note that equation (34) is deducible from an Einstein–Hilbert-like action functional only under restrictive conditions. More precisely, let us suppose that the material Lagrangian depends only on the components of the metric and not on its derivatives as well
as that the trace \( \Sigma = \Sigma_{ij} g^{ij} \) is independent of the metric and its derivatives. Then, from the identities

\[
\sqrt{|g|} = \varphi^2 \sqrt{|\bar{g}|}, \quad \frac{\partial}{\partial g^{ij}} = \frac{1}{\varphi} \frac{\partial}{\partial \bar{g}^{ij}} \quad \text{and} \quad \Sigma_{ij} = -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}} = -\frac{1}{\sqrt{|\bar{g}|}} \frac{\delta \mathcal{L}_m}{\delta \bar{g}^{ij}},
\]

(37)

we have the following relation:

\[
\Sigma_{ij} = -\varphi \frac{1}{\sqrt{|\bar{g}|}} \frac{\partial \mathcal{L}_m}{\partial \bar{g}^{ij}} := \varphi \Sigma_{ij}.
\]

(38)

In view of this, and being \( \varphi = \varphi(\Sigma) \), it is easily seen that equations (34) may be derived by varying with respect to \( \bar{g}^{ij} \) the action functional

\[
\mathcal{A}(\bar{g}) = \int \left[ \sqrt{|\bar{g}|} \left( \bar{R} - \frac{2}{\varphi^3} V(\varphi) \right) + \mathcal{L}_m \right] ds.
\]

(39)

Therefore, under the stated assumptions, \( f(R) \) gravity with torsion in the metric framework is conformally equivalent to an Einstein–Hilbert-like theory.

### 4. Equivalence with scalar–tensor theories

The above considerations directly lead us to study the relations between \( f(R) \) gravity with torsion and scalar–tensor theories with the aim to investigate their possible equivalence. To this end, we recall that the action functional of a (purely metric) scalar–tensor theory is

\[
\mathcal{A}(g, \varphi) = \int \left[ \sqrt{|g|} \left( \varphi \tilde{R} - \frac{\alpha_0}{\varphi} \varphi \varphi^i \varphi_i - U(\varphi) \right) + \mathcal{L}_m \right] ds,
\]

(40)

where \( \varphi \) is the scalar field which, depending on the sign of the kinetic term, could also assume the role of a phantom field [73], \( \varphi := \frac{\psi}{\psi'} \) and \( U(\varphi) \) is the potential of \( \varphi \). For \( U(\varphi) = 0 \), such a theory reduces to the standard Brans–Dicke theory [5]. The matter Lagrangian \( \mathcal{L}_m(g_{ij}, \psi) \) is a function of the metric and some matter fields \( \psi \); \( \alpha_0 \) is the so-called Brans–Dicke parameter.

The field equations derived by varying with respect to the metric and the scalar field are

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{\alpha_0}{\varphi^2} \left( \varphi \varphi_j - \frac{1}{2} \varphi_h \varphi^h g_{ij} \right) + \frac{1}{\varphi} (\tilde{\nabla}_j \varphi_i - \tilde{\nabla}_i \varphi_j) - \frac{U'}{2 \varphi^2} g_{ij}
\]

(41)

and

\[
\frac{2 \alpha_0}{\varphi} \tilde{\nabla}_h \psi^h + \tilde{\nabla} - \frac{\alpha_0}{\varphi^2} \varphi_h \psi^h - U' = 0,
\]

(42)

where \( \Sigma_{ij} := -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g_{ij}} \) and \( U' := \frac{dU}{d\varphi} \).

Taking the trace of equation (41) and using it to replace \( \tilde{R} \) in equation (42), one obtains the equation

\[
(2 \alpha_0 + 3) \tilde{\nabla}_h \psi^h = \Sigma + \varphi U' - 2U.
\]

(43)

By a direct comparison, it is immediately seen that for \( \alpha_0 = -\frac{3}{2} \) and \( U(\varphi) = \frac{2}{3} V(\varphi) \) (where \( V(\varphi) \) is defined as in equation (33)), equation (41) becomes formally identical to the Einstein-like equation (32) for a \( f(R) \) theory with torsion. Moreover, under such a circumstance, equation (43) reduces to the algebraic equation

\[
\Sigma + \varphi U' - 2U = 0
\]

(44)
relating the matter trace $\Sigma$ to the scalar field $\varphi$, exactly as it happens for $f(R)$ gravity. In particular, it is a straightforward matter to verify that (under the condition $f'' \neq 0$) equation (44) expresses exactly the inverse relation of (30), namely

$$\Sigma = F^{-1}((f')^{-1}(\varphi)) \iff \varphi = f'(F(\Sigma))$$

(45)

being $F^{-1}(x) = f'(x)X - 2f(x)$. In fact, we have

$$U(\varphi) = \frac{2}{\varphi} V(\varphi) = \frac{1}{2} \left[F^{-1}((f')^{-1}(\varphi)) + \varphi(f')^{-1}(\varphi)\right] = \left[\varphi((f')^{-1}(\varphi)) - f((f')^{-1}(\varphi))\right]$$

(46)

so that

$$U'(\varphi) = (f')^{-1}(\varphi) + \frac{\varphi}{f''((f')^{-1}(\varphi))} - \frac{\varphi}{f''(f')^{-1}(\varphi)} = (f')^{-1}(\varphi)$$

(47)

and then

$$\Sigma = -\varphi U'(\varphi) + 2U(\varphi) = f'((f')^{-1}(\varphi))(f')^{-1}(\varphi) - 2f((f')^{-1}(\varphi)) = F^{-1}((f')^{-1}(\varphi))$$

(48)

As a conclusion it follows that in the ‘metric’ interpretation, $f(R)$ theories with torsion are equivalent to $\omega_0 = -\frac{2}{3}$ Brans–Dicke theories.

Of course, the above statement is not true if we regard $f(R)$ theories as genuinely metric-affine ones. Nevertheless, in this case it is also possible to prove the equivalence between $f(R)$ theories with torsion and a certain class of Brans–Dicke theories, namely $\omega_0 = 0$ Brans–Dicke theories with torsion [76].

In this regard, let us consider the action functional

$$A(g, \Gamma, \varphi) = \int \left[\sqrt{|g|} (\varphi R - U(\varphi)) + L_m\right] ds,$$

(49)

where the dynamical fields are a metric $g_{ij}$, a metric connection $\Gamma^h_{ij}$ and a scalar field $\varphi$. As mentioned above, the action (49) describes a Brans–Dicke theory with torsion and parameter $\omega_0 = 0$.

The variation with respect to $\varphi$ yields the field equation

$$R = U'(\varphi).$$

(50)

To evaluate the variations with respect to the metric and the connection, we may repeat exactly the same arguments stated in the previous discussion for $f(R)$ gravity. Omitting for brevity the straightforward details, the resulting field equations are

$$T^h_{ij} = -\frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^p} \left(\delta^h_p g^h - \delta^h_p g^h_i \right)$$

(51)

and

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{2\varphi} U(\varphi) g_{ij}.$$  

(52)

Inserting the content of equation (50) in the trace of equation (52)

$$\frac{1}{\varphi} \Sigma - \frac{2}{\varphi} U(\varphi) + R = 0,$$

(53)

we again obtain an algebraic relation between $\Sigma$ and $\varphi$ identical to equation (44).

Therefore, choosing as above the potential $U(\varphi) = \frac{2}{\varphi} V(\varphi)$, from (44) we get $\varphi = f'(F(\Sigma))$. In view of this, decomposing $R_{ij}$ and $R$ in their Christoffel and torsion-dependent terms, equations (51) and (52) become identical to equation (31) and (32) respectively. As mentioned previously, this fact shows the equivalence between $f(R)$ theories and $\omega_0 = 0$ Brans–Dicke theories with torsion, in the metric–affine framework. These considerations can be extremely useful in order to give a geometrical characterization to the Brans–Dicke scalar field.
5. Applications to FRW cosmology

We have seen that the field equation (32) may be recast in the form (34) by performing a conformal transformation $\bar{g}_{ij} = \varphi g_{ij}$. In order to apply the above considerations to FRW cosmological models, let us suppose that $\Sigma_{ij}$ is the energy–momentum tensor of a cosmological perfect fluid with a negligible pressure and energy density $\rho$ (dust case), namely

$$\Sigma_{ij} = \rho U^i U^j,$$  \hspace{1cm} (54)

where $\rho = \rho(\tau)$ only depends on the cosmic time and $U^i$ is the 4-velocity of the fluid satisfying the condition

$$g_{ij} U^i U^j = -1.$$  \hspace{1cm} (55)

From now on, we shall suppose $\varphi > 0$ (a sufficient condition for this is $f' > 0$) so that the vector field $\bar{U}^i := \frac{U^i}{\varphi}$ represents the 4-velocity of the fluid with respect to the conformal metric $\bar{g}_{ij}$, while $\bar{U}_i := \bar{U}^j \bar{g}_{ji} = \sqrt{\varphi} U_i$ denotes the corresponding dual relation. In view of this, the identity

$$\frac{1}{\varphi} \Sigma_{ij} = \frac{\partial}{\partial \tau} U^i U_j = \frac{1}{\varphi^2} \bar{\Sigma}_{ij}$$  \hspace{1cm} (56)

holds, where we have defined $\bar{\Sigma}_{ij} = \rho \bar{U}_i \bar{U}_j$. Consequently, equation (34) may be rewritten as

$$\bar{G}_{ij} = \frac{1}{\varphi^2} \left( \bar{\Sigma}_{ij} - \frac{1}{\varphi} V(\varphi) \bar{g}_{ij} \right),$$  \hspace{1cm} (57)

where $\bar{G}_{ij}$ is the Einstein tensor in the barred metric. We look for an FRW solution $\bar{g}_{ij}$ of (57), being

$$d\bar{s}^2 = -d\tau^2 + A^2(\tau) \left[ d\psi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2 \right].$$  \hspace{1cm} (58)

Therefore, once a solution $\bar{g}_{ij}$ is found, the conformal metric $g_{ij} = \frac{1}{\varphi} \bar{g}_{ij}$ (solution of the starting equations (32)) will also be of the FRW form. Indeed, the line element associated with $g_{ij}$ is

$$d\bar{s}^2 = -\frac{1}{\varphi(t)} d\tau^2 + \frac{a^2(t)}{\varphi(t)} \left[ d\psi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2 \right]$$  \hspace{1cm} (59)

so that by performing the time variable transformation

$$d\tau := \frac{1}{\sqrt{\varphi(t)}} dt,$$  \hspace{1cm} (60)

it may be expressed as

$$d\bar{s}^2 = -d\tau^2 + A^2(\tau) \left[ d\psi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2 \right]$$  \hspace{1cm} (61)

with $A := \frac{a}{\sqrt{\varphi}}$. The field equation (57) reduces to the Friedmann equations

$$3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} = \frac{\rho}{\varphi^2} + \frac{V(\varphi)}{\varphi^3}$$  \hspace{1cm} (62a)

and

$$2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{V(\varphi)}{\varphi^3},$$  \hspace{1cm} (62b)

which are the cosmological equations arising from our theory.
For the sake of completeness, let us derive the conservation laws of the theory. The Bianchi identities of equation (57) give

$$\bar{\nabla}_i \left( \frac{1}{\varphi^2} \Sigma^{ij} - \frac{1}{\varphi^3} V(\varphi) \bar{g}^{ij} \right) = 0,$$

(63)

where $\bar{\nabla}$ denotes the covariant derivative with respect to the Levi-Civita connection associated with $\bar{g}_{ij}$. In the FRW metric, equations (63) reduce to the continuity equation

$$\frac{d}{dt} \left( \rho \varphi^2 a^3 \right) + a^3 \frac{d}{dt} \left( \frac{V(\varphi)}{\varphi^3} \right) = 0,$$

(64)

which completes the cosmological dynamical system.

Finally, an important point deserves further discussion in relation to the above results. Let us consider the cosmological equation (62a). In the lhs, it is clear that standard matter $\rho$ and the effective cosmological constant $\Lambda_{\text{eff}} = \frac{V(\varphi)}{\varphi^3}$ play two distinct roles in the dynamics: their evolution is 'tuned' by the scalar field $\varphi$ (i.e. $f'(R)$). The first term could be relevant for large-scale clustered structures (always involving baryonic matter and dark matter) and the second term can be read as dark energy. If at the present epoch they are $\rho \varphi^2 \approx \frac{V(\varphi)}{\varphi^3}$, this reveals a simple mechanism to explain why today we are observing $\Omega_M \simeq 0.3$ and $\Omega_{\Lambda} \simeq 0.7$. Furthermore, if the field $\varphi$ at the denominator is small (that is, $f'(R)$ is small), then this could be the reason why the amount of dark energy and dark matter results are huge.

As a toy model, let us take into account the well-known $f(R) = R + \alpha R^2$ theory where, obviously, $f'(R) = 1 + 2\alpha R$. As above, the matter stress–energy tensor is $\Sigma_{ij} = \rho U_i U_j$ and then equation (26) becomes

$$(1 + 2\alpha R)R - 2R - 2\alpha R^2 = -\rho \iff R = \rho.$$  

(65)

We have

$$\varphi(\rho) = f'(R(\rho)) = 1 + 2\alpha \rho$$

(66)

and then the term $\frac{\rho}{\varphi^2}$ becomes

$$\frac{\rho}{(1 + 2\alpha \rho)^2}.$$  

(67)

Let us now consider the potential term

$$V(\varphi) = \frac{1}{2} \left[ \varphi F^{-1}(f')^{-1}(\varphi) + \varphi^2 (f')^{-1}(\varphi) \right].$$

(68)

Being $(f')^{-1}(\varphi) = \rho$, one has

$$\frac{1}{4} \varphi^2 (f')^{-1}(\varphi) = \frac{1}{4} (1 + 2\alpha \rho)^2 \frac{1}{2\alpha}(1 + 2\alpha \rho - 1) = \frac{1}{4} (1 + 2\alpha \rho)^2 \rho,$$

(69)

and considering the relation $F^{-1}(Y) = f'(Y) K - 2f(Y)$, it is

$$\frac{1}{4} F^{-1}(f')^{-1}(\varphi)) = \frac{1}{4} F^{-1}(\rho) = -\rho.$$  

(70)

We also have

$$\frac{1}{4} \varphi F^{-1}(f')^{-1}(\varphi) = \frac{- (1 + 2\alpha \rho) \rho}{4},$$

(71)

and then we conclude that

$$V(\varphi(\rho)) = \frac{\alpha \rho^2 (1 + 2\alpha \rho)}{2}.$$  

(72)

and

$$\frac{V(\varphi(\rho))}{\varphi^3} = \frac{\alpha \rho^2}{2(1 + 2\alpha \rho)^2}.$$  

(73)

These arguments show that the condition $\frac{\rho}{\varphi^2} \approx \frac{V(\varphi)}{\varphi^3}$ can be simply achieved leading to comparable values of $\Omega_M$ and $\Omega_{\Lambda}$. A detailed discussion of these topics, also in relation to data, will be the argument of a forthcoming paper.
6. Discussion and conclusions

$f(R)$ gravity seems a viable approach for solving some shortcomings coming from GR, in particular problems related to quantization on curved spacetime and cosmological issues related to early universe (inflation) and late-time dark components. Besides, the scheme of GR is fully preserved and $f(R)$ can be considered a straightforward extension where the gravitational action does not have to be necessarily linear in the Ricci scalar $R$.

In this paper, we have discussed the possibility that the torsion field could also play an important role in the dynamics, the $U_4$ manifolds being the straightforward generalization of the pseudo-Riemannian manifolds $V_4$ (torsionless) usually adopted in GR.

As discussed above, the torsion field, in the metric-affine formalism, plays a fundamental role in clarifying the relations between the Palatini and the metric approaches: it gives further degrees of freedom which contribute, together with curvature degrees of freedom, to the dynamics. The aim is to achieve a self-consistent theory where unknown ingredients such as dark energy and dark matter (until now not detected at a fundamental level) could be completely ‘geometrized’. The torsion field assumes a relevant role in the presence of standard matter since it allows us to establish a definite equivalence between scalar–tensor theories and $f(R)$ gravity, also in relation to conformal transformations.

Furthermore, from a cosmological viewpoint, as shown in the toy model of the previous section, the torsion field could dynamically trigger the amount of dark components giving a straightforward explanation of the coincidence problem.

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