Harmonic oscillator states with integer and non-integer orbital angular momentum

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Abstract. We study the quantum mechanical harmonic oscillator in two and three dimensions, with particular attention to the solutions as basis states for representing their respective symmetry groups — O(2), O(1,1), O(3), and O(2,1). The goal of this study is to establish a correspondence between Hilbert space descriptions found by solving the Schrödinger equation in polar coordinates, and Fock space descriptions constructed by expressing the symmetry operators in terms of creation/annihilation operators. We obtain wavefunctions characterized by a principal quantum number, the group Casimir eigenvalue, and one group generator whose eigenvalue is \( m + s \), for integer \( m \) and real constant parameter \( s \). For the three groups that contain O(2), the solutions split into two inequivalent representations, one associated with \( s = 0 \), from which we recover the familiar description of the oscillator as a product of one-dimensional solutions, and the other with \( s > 0 \) (in three dimensions, solutions are found for \( s = 0 \) and \( s = 1/2 \)) whose solutions are non-separable in Cartesian coordinates, and are hence overlooked by the standard Fock space approach. The O(1,1) solutions are singlet states, restricted to zero eigenvalue of the symmetry operator, which represents the boost, not angular momentum. For O(2), a single set of creation and annihilation operators forms a ladder representation for the allowed oscillator states for any \( s \), and the degeneracy of energy states is always finite. However, in three dimensions, the integer and half-integer eigenstates are qualitatively different: the former can be expressed as finite dimensional irreducible tensors under O(3) or O(2,1) while the latter exhibit infinite degeneracy. Creation operators that produce the allowed integer states by acting on the non-degenerate ground state are constructed as irreducible tensor products of the fundamental vector representation. However, the half-integer eigenstates are infinite-dimensional, as expected for the non-compact O(2,1), and no Fock space description is found for this case. For all \( s \neq 0 \) solutions, the SU(N) symmetry of the harmonic oscillator Hamiltonian recently discovered by Bars is spontaneously broken by the ground state. The connection of this symmetry breaking to the non-separability into one-dimensional Cartesian solutions is demonstrated. We conclude with a discussion of the O(2,1) solutions, and the problem of timelike ghost excitations.

1. Introduction

1.1. Background

Along with its classical counterpart, the quantum harmonic oscillator is a well-studied model with exact solutions and connections to many physical systems for which it serves as foundation or approximation. Beyond its application to atomic and molecular spectra, statistical mechanics, and by way of various relativistic generalizations, to quark dynamics, several general techniques
associated with the harmonic oscillator, including Dirac’s factorization of the Hamiltonian into creation/annihilation operators and the resulting Fock space of uncoupled modes, serve as conceptual building blocks in areas ranging from blackbody radiation to canonical quantization and string theory. Yet, despite the subject’s long history, fundamental new insights continue to emerge.

In modeling the hadron mass spectrum on the excitation spectrum of quark bound states [1], Feynman, Kislinger, and Ravndal (FKR) stimulated interest in finding a consistent relativistic generalization of the quantum harmonic oscillator. This model, which the authors themselves characterized as “naive and obviously wrong in its simplicity”¹ raises a number of questions, including difficulty with ground state normalization, negative norm ghost states, and consequent problems with unitarity. Nevertheless, their generalization of the nonrelativistic Hamiltonian to a Lorentz-invariant Stueckelberg form [2], and the interpretation of its eigenvalues as the mass of a bound state, determined dynamically and not subject to a priori constraint, were not seen as shortcomings of the theory. In obtaining the relativistic path integral, Feynman had previously treated mass as a dynamical quantity that takes fixed eigenvalues for asymptotic particle states [3], but in that case argued that mass is conserved dynamically by the electromagnetic field (see also [4, 5] on this point). In the FKR model, the harmonic oscillator was taken as a phenomenological binding force, providing a Lorentz-invariant ground state and a positive definite spectrum. Because states with timelike excitations can have negative norm (and perhaps because the interpretation of such excitations had not been entirely clarified), only spacelike excitations are admitted in the model. This ad hoc exclusion of timelike modes can lead to violations of unitarity, because sums of states that omit negative norm contributions may exceed unity.

Seeking to address the difficulties in the FKR model, Kim and Noz (KN) proposed to replace the scalar ground state with a Lorentz covariant solution to the eigenvalue equation [6], and study excited states through the action of Lorentz boosts on this solution. Although this ground state is normalizable and all states have positive norm, timelike modes contribute negative energy. By dealing with boosts rather than excitation modes of the Hamiltonian, the model sidesteps the negative spectrum problem.

Another model for the relativistic oscillator was found by Horwitz and Arshansky [7] who applied a many-body extension [8] of Stueckelberg’s covariant Hamiltonian quantum mechanics to obtain relativistic generalizations of the classical central force bound state problems. According to a virial theorem, presented in appendix A, restriction to spacelike dynamics guarantees a positive spectrum, but since there is no obvious way to realize this nonholonomic constraint in Cartesian coordinates, the eigenvalue equation was posed in a hyperspherical parameterization. In order to obtain bound states solutions with the correct spectrum and symmetry properties, it was necessary to first solve the eigenvalue equation in an O(2,1)-invariant subspace of the spacelike sector, and then induce a representation of O(3,1) over the O(2,1)-covariant wavefunctions. The resulting wavefunctions are eigenstates of mass, the Casimir operators for O(3,1) and its O(2,1) and O(3) subgroups, and one angular momentum component. Under the raising/lowering combinations of the boost and angular momentum operators, the wavefunctions transform as O(3,1) tensors, providing an infinite dimensional representation of boosts, as expected for a non-compact group, and a finite, half-integer representation of angular momentum. Splitting the degeneracy through the Zeeman [9] and Stark [10] effects leads to finite, even multiplicity. The

¹ Such disclaimers appear in a number of Feynman’s papers, including his derivation of the relativistic path integral. He evidently found the technique useful in deflecting premature criticism of certain of his more original ideas.
half-integer representation seems to be related to a lowering of the ground state mass eigenvalue. In the FKR and KN models, despite the supression of timelike excitations, the ground state mass is found to be \(2\hbar \omega\) as one may expect for a relativistic system by associating \(\frac{1}{2} \hbar \omega\) with each of 4 degrees of freedom. The Horwitz-Arshansky solution, on the other hand, exhibits the lower ground state level \(\frac{3}{2} \hbar \omega\) one would like to obtain for an oscillator with 3 space dimensions. The common origin of the half-integer angular momentum and lowered ground state mass eigenvalue appears in the solution in hyperspherical coordinates. Kastrup [11] has shown that the choice of Cartesian coordinates overlooks the singularity at the origin of polar coordinates, implicitly choosing one solution from a family of harmonic oscillators with different ground state levels. Only in certain cases can solutions in hyperspherical coordinates be identified with the standard Cartesian wavefunctions.

In this paper, we study the harmonic oscillator in two and three dimensions, with emphasis on the solutions as basis states for representations of their respective symmetry groups: \(O(2)\), \(O(1,1)\), \(O(3)\), and \(O(2,1)\). The overall goal of this study was to solve the Schrödinger equation in polar coordinates, adapting the method of Horwitz and Arshansky to each group respectively, in order to establish a correspondence between the resulting Hilbert space descriptions and Fock space descriptions, constructed by writing the symmetry operators in terms of creation/annihilation operators. To provide a faithful representation, the states in both descriptions must be eigenstates of the Hamiltonian and a maximal set of mutually commuting symmetry operators. The states should also provide ladder representations — for the mode number in two and three dimensional Fock space, for the symmetry eigenvalues associated with the non-abelian groups in \(D = 3\), and for a unitary dynamical symmetry recently discovered by Bars [12]. The results of these attempts reflect physical differences among the oscillator models. Wavefunctions with integer and non-integer angular momentum levels were found that provide faithful representations of \(O(2)\), as well as the groups \(O(3)\) and \(O(2,1)\) that contain \(O(2)\) as a subgroup. Combinations of non-diagonal operators were found that raise and lower eigenvalues of mode number and angular momentum. On the other hand, the \(O(1,1)\) states are Lorentz invariant, rather than covariant, permitting only the zero eigenvalue for the symmetry generator, which in this case represents the boost, rather than angular momentum. These scalar states, which turn out to be non-normalizable for any mode number, effectively represent a one dimensional oscillator, providing an essentially trivial representation of the Lorentz symmetry.

For all groups, integer angular momentum wavefunctions can be expressed as linear combinations of the familiar Cartesian wavefunctions, while non-integer wavefunctions do not admit separation in Cartesian coordinates, and are therefore not unitarily connected to the familiar integer solutions. The non-integer solutions for \(O(2)\) are shown to be nonunitary on algebraic grounds. Nevertheless, complete solutions are obtained in Fock space, for the integer and non-integer representations of \(O(2)\), as well as the integer representations of \(O(3)\) and \(O(2,1)\), by positing states as representations of the polar ladder operators. However, the Dirac factorization of the Hamiltonian does not lead to a Fock space representation of any kind for the non-integer states of either group in \(D = 3\).

Generally, irreducible unitary representations of a compact group are finite dimensional, while unitary representations of a non-compact group are either singlet or infinite dimensional. As expected, the integer and non-integer representations of \(O(2)\) are finite dimensional, and the representation of \(O(1,1)\) is a singlet. While an infinite dimensional representation of \(O(1,1)\) may exist, it was not found either in Hilbert space or Fock space. On algebraic grounds, the infinite dimensional, non-integer representation of \(O(3)\) must be nonunitary, and a similar conclusion must hold for the finite dimensional, integer representation of \(O(2,1)\), which is equivalent to the FKR model.
Following a brief discussion of the oscillator formalism, we present the $D = 2$ representations in section 2 and the non-abelian $D = 3$ representations in section 3. Further approaches to a unitary, infinite dimensional Fock space representation of $O(2,1)$ will be discussed in a subsequent paper.

1.2. Cartesian formalism

We write a Schrödinger equation of the type

$$K \psi = i \partial_\tau \psi = \kappa \psi$$

with harmonic oscillator Hamiltonian

$$K = \frac{1}{2} \left( p^2 + \omega^2 x^2 \right) = \frac{1}{2} \eta_{\mu\nu} \left( p^\mu p^\nu + \omega^2 x^\mu x^\nu \right)$$

(2)

to describe either an $O(D)$ nonrelativistic oscillator with Euclidean metric

$$\eta_{\mu\nu} = \delta_{\mu\nu}, \ \mu, \nu = 1, \ldots, D$$

(3)
or an $O(D - 1, 1)$ relativistic oscillator with Lorentz metric

$$\eta_{\mu\nu} = \text{diag} \left( -1, 1, \ldots, 1 \right), \ \mu, \nu = 0, \ldots, D - 1.$$  

(4)

For the relativistic case, (1) is a Stueckelberg equation with Lorentz-invariant evolution parameter $\tau$, which as in the Euclidean case, admits a virial theorem (see appendix A) showing that restriction of the motion to spacelike dynamics guarantees a positive spectrum. The standard approach to Fock space proceeds by separation of Cartesian variables and subsequent application of Dirac’s factorization of the one-dimensional Hamiltonian for each degree of freedom. Assuming a product solution of one-dimensional oscillators

$$\psi (x) = \prod_\mu \psi (x^\mu) \quad \kappa = \sum_\mu n^\mu$$

(5)

the Hamiltonian separates into a sum of $D$ mode-number terms as

$$K = \omega \eta_{\mu\nu} \left( \bar{a}^\mu a^\nu + \frac{1}{2} \eta^{\mu\nu} \right) = \omega \sum_\mu \eta_{\mu\mu} \left( N^\mu + \frac{1}{2} \eta^{\mu\mu} \right)$$

(6)

with creation/annihilation operators

$$a^\mu = \frac{1}{\sqrt{2}} \left( x^\mu + ip^\mu \right) \quad \bar{a}^\mu = \frac{1}{\sqrt{2}} \left( x^\mu - ip^\mu \right)$$

(7)

that satisfy

$$[a^\mu, \bar{a}^\nu] = \eta^{\mu\nu}$$

(8)

and mode number operators $N^\mu = \bar{a}^\mu a^\mu$ (no summation) that satisfy

$$[N^\mu, \bar{a}^\nu] = \eta^{\mu\nu} \bar{a}^\nu \quad [N^\mu, a^\nu] = -\eta^{\mu\nu} a^\nu \quad [N^\mu, N^\nu] = 0.$$  

(9)

The products $|n\rangle = \prod_\mu |n^\mu\rangle$ of $N^\mu$ eigenstates form a Fock space of orthogonal oscillator modes with the ladder property

$$\bar{a}^\mu |n\rangle = e^{i\phi^\mu} \sqrt{n^\mu + \eta^{\mu\mu}} |n + \eta^{\mu\mu} e_\mu\rangle \quad a^\mu |n\rangle = e^{i\phi^\mu} \sqrt{n^\mu} |n - \eta^{\mu\mu} e_\mu\rangle,$$

(10)
where the $\mathbf{e}_\mu$ are unit vectors in the occupation number space

$$\langle \mathbf{e}_\mu \rangle^\lambda = \delta^\lambda_\mu$$

and the particular choice of phases $e^{i\phi^0_+}$ and $e^{i\phi^0_-}$, all taken to be 1 for the nonrelativistic Euclidean oscillator, has non-trivial consequences for the relativistic oscillator.

The FKR model [1] can be recovered by choosing the phases

$$e^{i\phi^0_+} = e^{i\phi^0_-} = i, \quad e^{i\phi^\mu_+} = e^{i\phi^\mu_-} = 1, \mu > 0,$$

which preserves the roles of $\bar{a}^0$ as creation operator and $a^0$ as annihilation operator for the timelike mode, under the requirement that $n^0 \leq 0$ so that

$$\bar{a}^0 |n\rangle = \sqrt{1 - n^0} \left| n - \mathbf{e}_0 \right\rangle, \quad a^0 |n\rangle = \sqrt{-n^0} \left| n + \mathbf{e}_0 \right\rangle.$$ (13)

Although these states have positive spectrum

$$\langle n^0 \mathbf{n} | K | n^0 \mathbf{n} \rangle = \omega \left( |n^0\rangle + \sum_{\mu > 0} n^\mu + \frac{1}{2} \eta_{\mu\nu} \eta^{\mu\nu} \right)$$

they have negative norm for odd values of $n^0$

$$\langle n^0 \mathbf{n} | n^0 \mathbf{n} \rangle = \frac{1}{(-n^0)!} (0 \langle a^0 \rangle^{-n^0} \langle \bar{a}^0 \rangle^{-n^0} |0\rangle = \langle 0 | \eta^{00} \rangle^{-n^0} |0\rangle = (-1)^{-n^0}$$ (15)

requiring that the negative norm states (ghosts) be suppressed. These authors chose to exclude all excited timelike modes by applying the covariant condition

$$(p \cdot a) \psi = 0$$ (16)

which forces $\psi$ into the ground state along the momentum $p$. Although this approach is formally comparable to Gupta-Bleuler quantization [13] of the electromagnetic field, where (16) expresses the Lorentz gauge condition as an operator equation and eliminates negative norm states along the field’s lightlike momentum, there is no general gauge freedom for the relativistic oscillator that justifies this procedure. The first order differential equations

$$a^\mu \psi_0 (x) = 0$$

lead to the ground state solution proposed in [1],

$$\psi_0 (x) = e^{-x^2/2} = e^{-(x^2 - t^2)/2}$$ (18)

which clearly requires some regularization procedure for normalization. In the absence of such a proper normalization, difficulties arise in the interpretation of all Fock space procedures, in particular the evaluation of the norm. To see this, we suppose that having no prior knowledge of Dirac factorization, we proceed naively from separation of Cartesian variables (5) to formulate the eigenvalue problem as

$$\frac{1}{2} (p^2 + x^2 - p_0^2 - t^2) \psi_n (x) \psi_{n_1} (t) = (\kappa_x + \kappa_y + \kappa_z + \kappa_t) \psi_n (x) \psi_{n_1} (t).$$ (19)
The space part has the familiar O(3) solution

\[ \psi_n (x) = A_n H_{n_x} (x) H_{n_y} (y) H_{n_z} (z) e^{-x^2/2} \]  

(20)

where the Hermite polynomials satisfy

\[ H'' (x) - 2xH' (x) + (\kappa - 1) H (x) = 0 \]
\[ H_n (x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \]  

(21)

while the time part is

\[ \psi_{n_1} (t) = A_{n_1} \tilde{H}_{n_1} (t) e^{t^2/2} \]  

(22)

with polynomials satisfying

\[ \tilde{H}'' (t) + 2t \tilde{H}' (t) - (\kappa - 1) \tilde{H} (t) = 0 \]
\[ \tilde{H}_n (t) = e^{-t^2} \frac{d^n}{dt^n} e^{t^2}. \]  

(23)

Writing the norm for these states as

\[ \| \psi_{n_1 n} (t, \mathbf{x}) \| = |A_n A_{n_1}|^2 \int d^3 x |H_{n_x} (x) H_{n_y} (y) H_{n_z} (z)|^2 e^{-x^2} \int dt |\tilde{H}_{n_1} (t)|^2 e^{t^2} \]  

(24)

the divergence of the second integral seems clear enough, but we would not suspect that the norm could be negative. Specifically, for the first excited timelike state, which by (15) should be negative, we might cleverly (but still naively) observe that the state may be written

\[ \psi_1 (t) = 2A_1 t e^{t^2} = A_1 (t + \partial_t) e^{t^2/2} \]  

(25)

so that integration by parts leads from

\[ \| \psi_{1 n} (t, \mathbf{x}) \| = 4 |A_1|^2 \int dt t^2 e^{t^2} = A_1^2 \int dt \left[ (t + \partial_t) e^{t^2/2} \right] (t + \partial_t) e^{t^2/2} \]  

(26)

to

\[ \| \psi_{1 n} (t, \mathbf{x}) \| = A_1^2 \int dt e^{t^2/2} (t - \partial_t) (t + \partial_t) e^{t^2/2} \]  

(27)

\[ = A_1^2 \int dt e^{t^2/2} \left\{ (t - \partial_t), (t + \partial_t) \right\} e^{t^2/2} \]  

(28)

\[ = -A_1^2 \int dt e^{t^2}. \]  

(29)

While this obvious error follows from the improper integration by parts for the divergent ground state, it is just the Hilbert space expression of the transposition of operators performed in (15). In the absence of a proper normalization for the ground state, the identification of negative normed states in this theory through (15) remains questionable, even when the states are restricted to spacelike support.

The KN model [6] can be recovered by choosing \( e^{i\phi_n} = e^{i\phi_0} = 1 \) for all phases, so it follows from (10) that \( \bar{a}^0 \) acts as an annihilation operator and \( a^0 \) is the creation operator for the timelike mode, such that the ground state mode must have \( n^0 \geq 1 \). This role reversal between \( \bar{a}^0 \) and \( a^0 \) insures that timelike excitations have positive norm,

\[ \langle n^0 n | n^0 n \rangle = \frac{1}{(n^0 - 1)!} \langle 1 0 | (\bar{a}^0)^{n^0-1} (a^0)^{n^0-1} | 1 0 \rangle = \langle 1 0 | (-\eta^{00})^{n^0-1} | 1 0 \rangle = 1 \]  

(30)
but leads to an indefinite spectrum

$$\langle n^0 | n^0 \rangle = \omega \left( - |n^0| + \sum_{\mu > 0} n^\mu + \frac{1}{2} \eta_{\mu\nu} \eta^{\mu\nu} \right). \quad (31)$$

The requirements on the ground state

$$\bar{a}^0 |1 0\rangle = 0 \quad a^\mu |1 0\rangle = 0, \mu > 0 \quad (32)$$

lead to a set of first order differential equations that reproduce the normalizable KN ground state solution

$$\psi_0(x) = A_0 e^{-t^2} e^{-x^2/2} = A_0 e^{-(t^2 + x^2)/2}. \quad (33)$$

Although this model suggests a negative dynamic contribution to the mass of the bound state, the states are normalizable and no ghosts are present.

In order to solve the Hamiltonian eigenvalue problem following the approach of Horwitz and Arshansky [7], we restrict the wavefunctions to spacelike support, imposed by solving the harmonic oscillator problem in polar coordinates of the generic form

$$x = \rho \hat{n} \quad (34)$$

where $\rho$ is the invariant length

$$\rho = \sqrt{\eta_{\mu\nu} x^\mu x^\nu} \quad (35)$$

$\hat{n}$ is a spacelike unit vector given for the respective symmetry groups by

$$\begin{align*}
O(2) & \\
\begin{bmatrix}
\hat{n}^1 \\
\hat{n}^2
\end{bmatrix} &= \begin{bmatrix}
\cos \phi \\
\sin \phi
\end{bmatrix} \\
\begin{bmatrix}
\hat{n}^0 \\
\hat{n}^1
\end{bmatrix} &= \begin{bmatrix}
\rho \\
|\rho| \cosh \beta
\end{bmatrix}
\end{align*} \quad (36)$$

$$\begin{align*}
O(3) & \\
\begin{bmatrix}
\hat{n}^1 \\
\hat{n}^2 \\
\hat{n}^3
\end{bmatrix} &= \begin{bmatrix}
\cos \phi \\
\sin \phi \\
\cos \theta
\end{bmatrix} \sin \theta \\
\begin{bmatrix}
\hat{n}^0 \\
\hat{n}^1 \\
\hat{n}^2
\end{bmatrix} &= \begin{bmatrix}
\sinh \beta \\
\cos \phi \\
\sin \phi \cosh \beta
\end{bmatrix}
\end{align*}$$

where for $O(1,1)$

$$-\infty < \rho < \infty \quad -\infty < \beta < \infty \quad (37)$$

and for $O(2), O(3), O(2,1)$

$$0 \leq \rho < \infty \quad -\infty < \beta < \infty. \quad (38)$$

The associated gradient operators are given in appendix B. In these coordinates, the Schrodinger equation takes the form

$$\begin{align*}
-\partial^2_{\rho} - \frac{D - 1}{\rho} \partial_{\rho} + \frac{1}{\rho^2} M^2 + \rho^2 - \varepsilon \psi &= 0 \quad (39)
\end{align*}$$

where the energy/mass eigenvalue is $\kappa = \frac{\omega}{2} \varepsilon$ and $M^2$ is the differential operator formed as the Casimir invariant from the generators

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \quad (40)$$

of the relevant symmetry group. Separation of radial and (hyper)angular variables leads to replacement of $M^2$ by its eigenvalue, so that the Schrodinger equation becomes a radial equation and the (hyper)angular equations have solutions which are generic representations of the symmetry group.
2. Harmonic oscillator in $D = 2$

2.1. Wavefunctions

In $D = 2$ with parameterizations (36), the Schrodinger equation becomes

$$
\left[ -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} M^2 + \rho^2 - \varepsilon \right] \psi = 0 \quad (41)
$$

and the group generator is

$$
M_{12}^{12} = x^1 p^2 - x^2 p^1 = -i \partial_\phi \quad \text{O(2)}
$$

$$
M_{01}^{01} = x^0 p^1 - x^1 p^0 = -i \epsilon (\rho) \partial_\beta \quad \text{O(1,1)}
$$

where

$$
\epsilon (\rho) = \frac{\rho}{|\rho|} \quad (43)
$$

and the Casimir operators

$$
M^2 = M_{12}^{12} M_{12} = \eta_{11} \eta_{22} (-i \partial_\phi)^2 = -\partial_\phi^2 \quad \text{O(2)}
$$

$$
M^2 = M_{01}^{01} M_{01} = \eta_{00} \eta_{11} (-i \partial_\phi)^2 = \partial_\beta^2 \quad \text{O(1,1)}
$$

are seen to have opposite sign. Separation of variables

$$
\psi (\rho, \phi) = R (\rho) \Phi (\phi) \quad \text{O(2)}
$$

$$
\psi (\rho, \phi) = \tilde{R} (\rho) \tilde{\Phi} (\beta) \quad \text{O(1,1)}
$$

leads to

$$
\frac{\rho^2}{R (\rho)} \left( -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \rho^2 - \varepsilon \right) R (\rho) = \frac{1}{\Phi (\phi)} \partial_\phi^2 \Phi (\phi) = -\Lambda^2 \quad \text{O(2)}
$$

$$
\frac{\rho^2}{\tilde{R} (\rho)} \left( -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \rho^2 - \varepsilon \right) \tilde{R} (\rho) = -\frac{1}{\tilde{\Phi} (\beta)} \partial_\beta^2 \tilde{\Phi} (\beta) = \tilde{\Lambda}^2 \quad \text{O(1,1)}
$$

where the signs of the separation constant are fixed by the requirement that the hermitian generators have real eigenvalues. The equations now take the form

$$
\left[ -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \Lambda^2 + \rho^2 - \varepsilon \right] R (\rho) = 0 \quad \Phi (\phi) = \exp (i \Lambda \phi) \quad \text{O(2)} \quad (47)
$$

$$
\left[ -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \tilde{\Lambda}^2 + \rho^2 - \varepsilon \right] R (\rho) = 0 \quad \tilde{\Phi} (\beta) = \exp (i \tilde{\Lambda} \beta) \quad \text{O(1,1)}.
$$

To solve the radial equation

$$
\left[ -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} Q + \rho^2 - \varepsilon \right] R (\rho) = 0 \quad \text{where} \quad Q = \begin{cases} 
\Lambda^2 & \text{O(2)} \\
-\tilde{\Lambda}^2 & \text{O(1,1)}.
\end{cases}
$$

(48)
we make the substitutions

\[ x = \rho^2 \]

\[
\begin{align*}
\partial_\rho &= 2x^{\frac{1}{2}} \partial_x \\
\partial^2_\rho &= 4x \partial_x^2 + 2 \partial_x
\end{align*}
\] (49)

and

\[ R(\rho) = x^{\frac{1}{2}} e^{-x/2} L(x) \]

leaving the constant \( A \) to be determined. The radial equation becomes

\[
4x \frac{d^2}{dx^2} L(x) + \left[ 4 + 8A - 4x \right] \frac{d}{dx} L(x) + \left[ \frac{4A^2}{x} - \frac{Q^2}{x} - 4A - 2 + \varepsilon \right] L(x) = 0
\] (50)

which by choosing

\[
A = \begin{cases} 
\frac{1}{2} \Lambda & \text{O}(2) \\
\frac{1}{2} \tilde{\Lambda} & \text{O}(1,1)
\end{cases}
\]

(51)

(52)

takes the form of the Laguerre equation

\[
x \frac{d^2}{dx^2} L(x) + \left[ \Lambda - x + 1 \right] \frac{d}{dx} L(x) + \frac{1}{2} \left[ \frac{1}{2} \varepsilon - \Lambda - 1 \right] L(x) = 0 \quad \text{O}(2)
\]

(53)

\[
x \frac{d^2}{dx^2} L(x) + \left[ i\tilde{\Lambda} - x + 1 \right] \frac{d}{dx} L(x) + \frac{1}{2} \left[ \frac{1}{2} \varepsilon - i\tilde{\Lambda} - 1 \right] L(x) = 0 \quad \text{O}(1,1).
\]

Now, the solution for the O(2) oscillator becomes

\[
\psi(\rho, \phi) = A e^{-\rho^2/2} \rho^L \tilde{L}_n^\Lambda \left( \rho^2 \right) e^{i\Lambda \phi}
\] (54)

with index

\[
n = \frac{1}{2} \left( \frac{1}{2} \varepsilon - \Lambda - 1 \right)
\] (55)

so that

\[
\kappa = \frac{\omega}{2} \varepsilon = \omega \left( 2n + \Lambda + 1 \right).
\] (56)

Thus far, the only restriction on the angular eigenvalue is \( 2n + \Lambda \geq 0 \), and so we put

\[
\Lambda = m + s \quad \kappa = \omega \left( 2n + m + s + 1 \right)
\] (57)

where \( s \) is a fixed whole number, and express the O(2) wavefunction as

\[
\psi_{nm}(\rho, \phi) = A_{nm} e^{-\rho^2/2} \rho^{m+s} L^{m+s}_n \left( \rho^2 \right) e^{i(m+s)\phi}.
\] (58)

The normalization is found from

\[
\int_0^{2\pi} d\phi \int_0^\infty \rho d\rho \left| \psi_{nm}(\rho, \phi) \right|^2 = |A_{nm}|^2 \pi \int_0^\infty dx x^{n+m+s} \left| L^{m+s}_n \left( x \right) \right|^2
\] (59)

\[
= |A_{nm}|^2 \pi \frac{\Gamma(n + m + s + 1)}{\Gamma(n + 1}).
\] (60)
Using the properties $L_\beta^\alpha = 0$ for $\beta < 0$ and $L_0^0 = P_0^0 = 1$, the ground state wavefunction is

$$\psi_0 (\rho, \phi) = A_0 e^{-\rho^2 / 2} (\rho e^{i\phi})^s = A_0 e^{-(x^2+y^2)^{s/2}} e^{is \arctan(y/x)} \tag{61}$$

and as expected, is separable in Cartesian coordinates only for $s = 0$, which recovers the product of familiar one dimensional oscillators.

For the O(1,1) solution, the requirement of real mass eigenvalue imposes $\tilde{\Lambda} = 0$ so that the O(1,1) wavefunction, also obtained by Bars [12], is

$$\psi_n (\rho, \beta) = A_n e^{-\rho^2 / 2} L_n^0 (\rho^2) \tag{62}$$

with

$$\kappa = \frac{\omega}{2} \varepsilon = \omega (2n + 1). \tag{63}$$

The ground state for this solution is the restriction to spacelike support of the $D = 2$ FKR ground state (18). Despite the absence of timelike excitations, as seen from

$$\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\rho \rho |\psi_n (\rho)|^2 = |A_n|^2 \int_{-\infty}^{\infty} d\beta \tag{64}$$

these states are non-normalizable as they stand.

2.2. Fock space

A number representation appropriate to the solutions (58) and (62) consists of polar creation/annihilation operators that act on eigenstates of the total mode number $N$ and the symmetry generator, $M^{12}$ or $M^{01}$, to produce new eigenstates. The resulting representation will be equivalent to the standard Cartesian Fock space if the polar eigenstates are unitarily connected to the Cartesian number states, in which case they can be found by expressing $N$ and $M^{12}$ or $M^{01}$ in terms of $\bar{a}^\mu$ and $a^\mu$ and diagonalizing the resulting operators. We consider the Cartesian multiplet $\varphi_1$ of first excited states as arising from the action of the vector multiplet of creation operators on the ground state $\varphi_0$. Then, by successive action of creation operators, we attempt to construct the full space of oscillator states.

2.2.1. O(2) oscillator  In the case of the O(2) oscillator, the vector operator multiplet takes $\varphi_0$ to the Cartesian multiplet $\varphi_1$ as

$$\varphi_1 = \begin{pmatrix} \varphi_{10} \\ \varphi_{01} \end{pmatrix} = \begin{pmatrix} \bar{a}^1 \varphi_0 \\ \bar{a}^2 \varphi_0 \end{pmatrix} = \begin{pmatrix} \bar{a}^1 \\ \bar{a}^2 \end{pmatrix} \varphi_0. \tag{65}$$

Using (7) to replace $x^\mu$ and $p^\mu$ with $\bar{a}^\mu$ and $a^\mu$, the angular momentum operator

$$M^{12} = x^1 p^2 - x^2 p^1 = -i (\bar{a}^1 a^2 - \bar{a}^2 a^1) \tag{66}$$

is seen to act on $\varphi_1$ as

$$M^{12} \varphi_1 = -i (\bar{a}^1 a^2 - \bar{a}^2 a^1) \begin{pmatrix} \bar{a}^1 \varphi_0 \\ \bar{a}^2 \varphi_0 \end{pmatrix} = \begin{pmatrix} i\bar{a}^2 \varphi_0 \\ -i\bar{a}^1 \varphi_0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \varphi_1 \tag{67}$$
and so has eigenvalues ±1 on eigenstates

\[ \tilde{\varphi}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_{10} + i\varphi_{01} \\ -\varphi_{10} + i\varphi_{01} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{a}_+ \\ -\tilde{a}_- \end{pmatrix} \varphi_0 \] (68)

where the polar creation/annihilation operators

\[ a_\pm = \frac{1}{\sqrt{2}} (a^1 \pm ia^2) \quad \tilde{a}_\pm = \frac{1}{\sqrt{2}} (\tilde{a}^1 \pm i\tilde{a}^2) \] (69)

commute among themselves except for

\[ [a_+, \tilde{a}_-] = [a_-, \tilde{a}_+] = 1. \] (70)

We note that the unitary transformation to polar operators

\[ \begin{pmatrix} \tilde{a}_+ \\ -\tilde{a}_- \end{pmatrix} \varphi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \tilde{a}^1 \\ \tilde{a}^2 \end{pmatrix} \varphi_0 \] (71)

takes the Cartesian vector wavefunction multiplet

\[ \varphi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} x^1 - \partial_1 \\ x^2 - \partial_2 \end{pmatrix} A_0e^{-x^2/2} = \sqrt{2} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} A_0e^{-x^2/2} \] (72)

to the polar multiplet

\[ \tilde{\varphi}_1 = \begin{pmatrix} x^1 + ix^2 \\ -x^1 + ix^2 \end{pmatrix} A_0e^{-x^2/2} \] (73)

familiar from helicity representations.

Because the Casimir invariant \( M^2 = (M^{12})^2 \) is automatically diagonal in the polar basis, operators (69) are sufficient to fully characterize the O(2) oscillator. From the four available products

\[ \tilde{a}_+ a_+ = \frac{1}{2} (\tilde{a}^1 + i\tilde{a}^2) (a^1 + ia^2) = \frac{1}{2} (N^1 - N^2 + i\tilde{a}^2 a^1 + i\tilde{a}^1 a^2) \] (74)

\[ \tilde{a}_- a_- = \frac{1}{2} (\tilde{a}^1 - i\tilde{a}^2) (a^1 - ia^2) = \frac{1}{2} (N^1 - N^2 - i\tilde{a}^2 a^1 - i\tilde{a}^1 a^2) \] (75)

\[ \tilde{a}_+ a_- = \frac{1}{2} (\tilde{a}^1 + i\tilde{a}^2) (a^1 - ia^2) = \frac{1}{2} (N^1 + N^2 + i\tilde{a}^2 a^1 - i\tilde{a}^1 a^2) \] (76)

\[ \tilde{a}_- a_+ = \frac{1}{2} (\tilde{a}^1 - i\tilde{a}^2) (a^1 + ia^2) = \frac{1}{2} (N^1 + N^2 - i\tilde{a}^2 a^1 + i\tilde{a}^1 a^2) \] (77)

we may form the symmetric Hermitian combinations

\[ N = \tilde{a}_+ a_- + \tilde{a}_- a_+ = N^1 + N^2 \] (78)

\[ \Delta = \tilde{a}_+ a_+ + \tilde{a}_- a_- = N^1 - N^2 = \tilde{a}^1 a^2 - \tilde{a}^2 a^1 \] (79)

and the antisymmetric Hermitian combinations

\[ M^{12} = \tilde{a}_+ a_- - \tilde{a}_- a_+ = -i (\tilde{a}^1 a^2 - \tilde{a}^2 a^1) \] (80)
In Cartesian coordinates, the maximal set of commuting operators is \( \{N, N^1, N^2\} \), and from these one may construct the Hamiltonian. Using

\[
[M^{12}, N^1] = -i \left[ a^1 a^2 - a^2 a^1, a^1 a^1 \right] = i (a^1 a^2 + a^2 a^1) = i Q
\]

\[
[M^{12}, N^2] = -i \left[ a^1 a^2 - a^2 a^1, a^2 a^2 \right] = -i (a^1 a^2 + a^2 a^1) - i Q
\]

we confirm that angular momentum commutes with the total mode number

\[
[M^{12}, N] = [M^{12}, N^1] + [M^{12}, N^2] = 0
\]

but, of course, \( N^1 \) and \( N^2 \) do not commute with \( M^{12} \) and are not separately observable in the polar representation.

From (6) the Hamiltonian in \( D = 2 \) is \( H = \omega (N + 1) \) so comparing with (57) we find

\[
N \psi_{nm} = (2n + m + s) \psi_{nm}
\]

and using the commutation relations

\[
[N, a_\pm] = -a_\pm, \quad [N, \bar{a}_\pm] = \bar{a}_\pm
\]

\[
[M^{12}, a_\pm] = \pm a_\pm, \quad [M^{12}, \bar{a}_\pm] = \pm \bar{a}_\pm
\]

we compare

\[
N a_+ \psi_{nm} = a_+ (N - 1) \psi_{nm} = (2n + m + s - 1) a_+ \psi_{nm}
\]

\[
M^{12} a_+ \psi_{nm} = (m + s + 1) a_+ \psi_{nm}
\]

with

\[
N \psi_{n-1,m+1} = (2n + m + s - 1) \psi_{n-1,m+1}
\]

\[
M^{12} \psi_{n-1,m+1} = (m + s + 1) \psi_{n-1,m+1}
\]

and conclude that

\[
a_+ \psi_{nm} = C^+_{nm} \psi_{n-1,m+1}
\]

where \( C^+_{nm} \) is a complex coefficient, with norm found from

\[
|a_+ \psi_{nm}|^2 = \langle \psi_{nm} | a_- a_+ | \psi_{nm} \rangle = |C^+_{nm}|^2 \langle \psi_{n-1,m+1} | \psi_{n-1,m+1} \rangle = |C^+_{nm}|^2.
\]

Similarly comparing,

\[
N a_- \psi_{nm} = (2n + m + s - 1) a_- \psi_{nm}
\]

\[
M^{12} a_- \psi_{nm} = (m + s - 1) a_- \psi_{nm}
\]

\[
N \bar{a}_+ \psi_{nm} = (2n + m + s + 1) a_+ \psi_{nm}
\]

\[
M^{12} \bar{a}_+ \psi_{nm} = (m + s + 1) \bar{a}_+ \psi_{nm}
\]

\[
N \bar{a}_- \psi_{nm} = (2n + m + s + 1) \bar{a}_- \psi_{nm}
\]

\[
M^{12} \bar{a}_- \psi_{nm} = (m + s - 1) \bar{a}_- \psi_{nm}
\]
with

\[ N\psi_{n,m-1} = [2n + (m - 1)] \psi_{n,m-1} = (2n + s + m - 1) \psi_{n,m-1} \] (100)

\[ M^{12}\psi_{n,m-1} = (m - 1) \psi_{n,m-1} \] (101)

\[ N\psi_{n,m+1} = [2n + s + (m + 1)] \psi_{n,m+1} = (2n + s + m + 1) \psi_{n,m+1} \] (102)

\[ M^{12}\psi_{n,m+1} = (m + s + 1) \psi_{n,m+1} \] (103)

\[ N\psi_{n+1,m-1} = [2(n + 1) + s + (m - 1)] \psi_{n+1,m-1} = (2n + s + m + 1) \psi_{n+1,m-1} \] (104)

\[ M^{12}\psi_{n+1,m-1} = (m + s - 1) \psi_{n+1,m-1} \] (105)

leads to

\[ a_-\psi_{nm} = C_{nm}^- \psi_{nm-1} \quad \bar{a}_+\psi_{nm} = \bar{C}_{nm}^+ \psi_{nm+1} \quad \bar{a}_-\psi_{nm} = \bar{C}_{nm}^- \psi_{n+1m-1}. \] (106)

We eliminate two coefficients

\[ |C_{nm}^-|^2 = |C_{nm}^+|^2 + 1 \quad |\bar{C}_{nm}^+|^2 = |C_{nm}^-|^2 + 1 \] (107)

using the commutation relations (70). Solving

\[ 2n + m + s = \langle nm | N | nm \rangle = \langle nm | \bar{a}_+ a_- + \bar{a}_- a_+ | nm \rangle = |C_{nm}^-|^2 + |C_{nm}^+|^2 \geq 0, \] (108)

together with

\[ m + s = \langle nm | M^{12} | nm \rangle = \langle nm | \bar{a}_+ a_- - \bar{a}_- a_+ | nm \rangle = |C_{nm}^-|^2 - |C_{nm}^+|^2 \] (109)

and taking the coefficients to be real leads to

\[ C_{nm}^+ = \sqrt{n} \quad C_{nm}^- = \sqrt{n + m + s} \] (110)

\[ \bar{C}_{nm}^+ = \sqrt{n + 1} \quad \bar{C}_{nm}^- = \sqrt{n + m + s + 1} \] (111)

and so we normalize the ladder operators as

\[ a_+\psi_{nm} = \sqrt{n} \psi_{n-1,m+1} \quad \bar{a}_-\psi_{nm} = \sqrt{n + 1} \psi_{n+1,m-1} \] (112)

\[ a_-\psi_{nm} = \sqrt{n + m + s} \psi_{n,m-1} \quad \bar{a}_+\psi_{nm} = \sqrt{n + m + s + 1} \psi_{n,m+1}. \] (113)

Special care must be taken with the ground state (61) because (112) and (113) lead to

\[ a_+\psi_0 = 0 \quad a_-\psi_0 = \sqrt{s} \psi_{0,-1} \] (114)

or equivalently

\[ a^1\psi_0 = \sqrt{\frac{s}{2}} \psi_{0,-1} \quad a^2\psi_0 = i \sqrt{\frac{s}{2}} \psi_{0,-1} \] (115)

suggesting a state with \( m = -1 \) and mode number \( N = s - 1 \). By restricting \( 0 \leq s < 1 \), the state \( \psi_{0,-1} \) is excluded from the Fock space. The action of \( a_- \) in (114) may then be understood
as taking the ground state to a non-observable function which must be taken into account in calculations such as

\[ N\psi_0 = (\bar{a}_+ a_- + \bar{a}_- a_+) \psi_0 = \sqrt{s} \bar{a}_+ \psi_{0,-1} = \sqrt{s} \sqrt{-1 + s + 1} \psi_0 = s \psi_0 \]  

but is effectively annihilated at the end of calculations. As can be seen from (58), allowing \( s = 1 \) merely reproduces the \( s = 0 \) states, shifted as \( m \to m - 1 \), with \( \psi_{0,-1} \) as the unique ground state.

Excited states may be constructed from the ground state as

\[ \zeta_{\alpha\beta} = \frac{1}{N_{\alpha\beta}} (\bar{a}_+)^\alpha (\bar{a}_-)^\beta \psi_0 \]  

with normalization coefficient \( 1/N_{\alpha\beta} \). It follows from (112) that

\[ (\bar{a}_-)^\beta \psi_0 = \sqrt{\beta!} \psi_{\beta,-\beta} \]  

and from (113) that

\[ (\bar{a}_+)^\alpha \psi_{\beta,-\beta} = \sqrt{\frac{\Gamma(s + \alpha + 1)}{\Gamma(s + 1)}} \psi_{\beta,-\beta+\alpha} \]  

and so we take

\[ N_{\alpha\beta} = \sqrt{\beta! \frac{\Gamma(s + \alpha + 1)}{\Gamma(s + 1)}} \]  

which reduces to \( \sqrt{\alpha! \beta!} \) in the case \( s = 0 \). Operating on these states with the total mode operator (78)

\[ N\zeta_{\alpha\beta} = \frac{1}{N_{\alpha\beta}} (\bar{a}_+ a_- + \bar{a}_- a_+) (\bar{a}_+)^\alpha (\bar{a}_-)^\beta \psi_0 \]  

and using the commutation relations (70) and the identity

\[ [B, A] = c, \text{ a c-number} \quad \Rightarrow \quad [B, A^n] = cnA^{n-1} \]  

we calculate

\[ \bar{a}_+ a_-(\bar{a}_+)^\alpha (\bar{a}_-)^\beta \psi_0 = \bar{a}_+ \left[ (\bar{a}_+)^\alpha a_- + \alpha (\bar{a}_+)^{\alpha-1} \right] (\bar{a}_-)^\beta \psi_0 \]  

\[ = \left[ (\bar{a}_+)^{\alpha+1} (\bar{a}_-)^\beta a_- + \alpha (\bar{a}_+)^\alpha (\bar{a}_-)^{\beta-1} \right] \psi_0 \]  

\[ = (\bar{a}_+)^\alpha (\bar{a}_-)^\beta (\bar{a}_+ a_- + \alpha) \psi_0 \]  

\[ \bar{a}_- a_+(\bar{a}_+)^\alpha (\bar{a}_-)^\beta \psi_0 = (\bar{a}_+)^\alpha \bar{a}_- a_+ (\bar{a}_-)^\beta \psi_0 \]  

\[ = (\bar{a}_+)^\alpha \bar{a}_- \left[ (\bar{a}_-)\beta a_+ + \beta (\bar{a}_-)^{\beta-1} \right] \psi_0 \]  

\[ = (\bar{a}_+)^\alpha (\bar{a}_-) (\bar{a}_- a_+ + \beta) \psi_0 \]

which combine to

\[ N\zeta_{\alpha\beta} = \frac{1}{N_{\alpha\beta}} (\bar{a}_+)^\alpha (\bar{a}_-) (\alpha + \beta + \bar{a}_+ a_- + \bar{a}_- a_+) \psi_0. \]  

\[ \]
Using (116) we obtain
\[ N\zeta_{\alpha\beta} = (\alpha + \beta + s) \zeta_{\alpha\beta} \] (130)
so that the states \( \zeta_{\alpha\beta} \) have total mode number given by
\[ N = \alpha + \beta + s = \alpha + \beta + N_{\text{ground state}}. \] (131)
A similar calculation using (80) leads to
\[ M_{12}^2 \zeta_{\alpha\beta} = (\alpha - \beta + s) \zeta_{\alpha\beta} \] (132)
so that the states \( \zeta_{\alpha\beta} \) have angular momentum
\[ M_{12} = \alpha - \beta + s = m + M_{12}^2_{\text{ground state}} \quad m = \alpha - \beta. \] (133)
Comparing (130) with (85) we see that
\[ \alpha + \beta = 2n + m \] (134)
which together with (133) and (131) provide expressions for the principal quantum number \( n \) and the integer part of the angular momentum \( m \)
\[ n = \frac{1}{2}(\alpha + \beta - m) = \beta \quad m = \alpha - \beta = N - s - 2\beta \] (135)
and fixes \( \alpha \) as
\[ \alpha = 2n + m - \beta = n + m = N - s - m. \] (136)
Thus, the states \( \zeta_{\alpha\beta} \) defined in (117) can be identified with explicit solutions \( \psi^{(N)}_{n,m} \) through
\[ \zeta_{\alpha\beta} = \psi^{(\alpha+\beta+s)}_{\beta,\alpha-\beta} \quad \psi^{(N)}_{n,m} = \zeta_{N-s-m,n} \] (137)
for which \( n = \beta = 0, 1, \ldots, N - s \) characterizes the \((N - s + 1)\)-fold multiplicity of states with mode number \( N \). Equivalently, the multiplicity can be enumerated by the angular momentum \( m \). For the case \( s = 0 \), this simple multiplicity structure is identical to the Cartesian picture in which there are \( N + 1 \) ways to build a state of total mode number \( N \) from a pair of one dimensional oscillators.

Bars has observed [12] that the harmonic oscillator Hamiltonian in \( D \) dimensional Fock space possesses a symmetry generated by the products \( \bar{a}^\mu a^\nu \) of the ladder operators. Because such products replace one \( \nu \)-mode of the oscillator with one \( \mu \)-mode, the total mode number, and therefore the total mass/energy, is conserved. The traceless part of the generators
\[ J^{\mu\nu} = \bar{a}^\mu a^\nu - \frac{1}{D} \eta^{\mu\nu} \eta_{\lambda\rho} \bar{a}^\lambda a^\rho \] (138)
generates an SU\((D - 1, 1)\) or SU\((D)\) dynamical symmetry of the Hamiltonian, while the trace
\[ \eta_{\lambda\rho} \bar{a}^\lambda a^\rho = \sum_{\mu} \eta_{\mu\mu} N^{\mu} = N \] (139)
is the total mode number and differs from the Hamiltonian by a c-number. The antisymmetric part of the generators
\[ \frac{1}{2} (J^{\mu\nu} - J^{\nu\mu}) = M^{\mu\nu} = \bar{a}^\mu a^\nu - \bar{a}^\nu a^\mu \] (140)
generates the SO\((D - 1, 1)\) or SO\((D)\) symmetry of the Hamiltonian. Bars argues that harmonic
oscillator states should belong to representations of the SU dynamical symmetry as well as to
representations of the SO symmetry, imposing additional constraints on admissible solutions.

For the O\((2)\) Cartesian ladder operators, the traceless operator is

\[
J = \begin{bmatrix}
\frac{1}{2} (\bar{a}^1 a^1 - \bar{a}^2 a^2) & \bar{a}^1 a^2 \\
\bar{a}^2 a^1 & -\frac{1}{2} (\bar{a}^1 a^1 - \bar{a}^2 a^2)
\end{bmatrix}
\]

(141)

with antisymmetric part equal to the angular momentum operator

\[
i M^{12} = \bar{a}^1 a^2 - \bar{a}^2 a^1
\]

(142)

and symmetric part given by

\[
S = \frac{1}{2} (J + J^T) = \frac{1}{2} \begin{bmatrix}
\bar{a}^1 a^1 - \bar{a}^2 a^2 & \bar{a}^1 a^2 + \bar{a}^2 a^1 \\
\bar{a}^2 a^1 & -\bar{a}^1 a^1 + \bar{a}^2 a^2
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\Delta & Q \\
Q & -\Delta
\end{bmatrix}
\]

(143)

where we use (79) and (81). Directly calculating

\[
\left[ \frac{1}{2} M^{12}, \frac{1}{2} \Delta \right] = \frac{1}{4} \left[ M^{12}, N^1 \right] - \frac{1}{4} \left[ M^{12}, N^2 \right] = i \frac{1}{2} Q
\]

(144)

\[
\left[ \frac{1}{2} \Delta, \frac{1}{2} Q \right] = \frac{1}{4} \left[ \bar{a}^1 a^1 - \bar{a}^2 a^2, \bar{a}^1 a^2 + \bar{a}^2 a^1 \right] = \frac{1}{2} (\bar{a}^1 a^2 - \bar{a}^2 a^1) = i \frac{1}{2} M^{12}
\]

(145)

\[
\left[ \frac{1}{2} Q, \frac{1}{2} M^{12} \right] = -i \frac{1}{4} \left[ \bar{a}^1 a^2 + \bar{a}^2 a^1, \bar{a}^1 a^2 - \bar{a}^2 a^1 \right] = \frac{1}{2} i (\bar{a}^1 a^1 - \bar{a}^2 a^2) = i \frac{1}{2} \Delta
\]

(146)

we verify that the three independent operators \(\{ \frac{1}{2} M^{12}, \frac{1}{2} \Delta, \frac{1}{2} Q \}\) satisfy the SU\((2)\) algebra. Equation (84) confirms that \(M\) commutes with total mode number \(N\); similarly,

\[
\begin{align}
[ N, \Delta ] &= [ N^1 + N^2, N^1 - N^2 ] = 0 \\
[ N, Q ] &= [ \bar{a}^1 a^1 + \bar{a}^2 a^2, \bar{a}^1 a^2 + \bar{a}^2 a^1 ] = 0
\end{align}
\]

(147)

(148)

so this SU\((2)\) is indeed a symmetry of the Hamiltonian. Of the three operators, \(\Delta\) is observable
in Cartesian coordinates, while \(M^{12}\) is observable in polar coordinates. However, since the
ground state satisfies \(M^{12} \psi_0 = s \psi_0\), the SU\((2)\) symmetry is spontaneous broken for \(s > 0\). From
(58), we see that the \(s > 0\) ground state is centered on a ring of radius \(r = \sqrt{s}\), so that the
operation \(e^{i \theta M^{12}}\) rotates the ground state on this ring. Nevertheless, this infinite degeneracy
merely contributes a phase.

Expanding the SU\((2)\) Casimir operator in terms of the creation and annihilation operators, (79)
— (81) lead to

\[
\left( \frac{1}{2} M \right)^2 + \left( \frac{1}{2} \Delta \right)^2 + \left( \frac{1}{2} Q \right)^2 = \frac{1}{4} N (N + 2) = \frac{1}{2} N \left( \frac{1}{2} N + 1 \right).
\]

(149)

Since the Casimir eigenvalue \(\frac{1}{2} N\) of a unitary representation of SU\((2)\) must be integral or half-
integer, the \(s \neq 0\) solutions must be non-unitarity representations of the SU\((2)\) dynamical
symmetry\(^2\). This result is unsurprising, because we have seen from the form of the wavefunctions
that for \(s \neq 0\) there is no unitary transformation from the unitary Cartesian solution to the polar
solution. The implications for these solutions as representations of the classical O\((2)\) symmetry

\(^2\) I am grateful to Itzhak Bars for this observation. See also [14]
are less clear. In appendix C, we verify that the solutions (58) satisfy the creation/annihilation ladder representation in polar coordinates. We may also construct new ladder operators

\[ J^+ = \frac{1}{2} (Q - i\Delta) = -i\bar{a}_+a_+ \quad J^- = \frac{1}{2} (Q + i\Delta) = i\bar{a}_-a_- \] (150)

which using commutation relations (144) — (145) satisfy

\[ \left[ \frac{1}{2} M^{12}, J^\pm \right] = \pm J^\pm. \] (151)

As is verified by comparison with (112) and (113), \( J^\pm \) raise/lower the \( M^{12} \) eigenvalue by 2, leaving the mode number \( N = 2n + m + s \) unchanged. The SU(2) symmetry thus enriches the description of the O(2) oscillator, by providing an algebraic transformation among the degenerate energy eigenstates.

2.2.2. O(1,1) oscillator

Unlike O(2), we cannot construct excited states for the O(1,1) oscillator by acting on the ground state with the Cartesian vector multiplet of creation operators. To see this, we recall that the eigenvalue of the Lorentz generator

\[ M^{01} = x^0p^1 - x^1p^0 = -i (\bar{a}^0a^1 - \bar{a}^1a^0) \] (152)
on the O(1,1) wavefunctions (62) is identically zero, and so (152) must annihilate these states. Writing the multiplet

\[ \varphi_1 = \begin{pmatrix} \varphi_{10} \\ \varphi_{01} \end{pmatrix} = \begin{pmatrix} \bar{a}^0\varphi_0 \\ \bar{a}^1\varphi_0 \end{pmatrix} = \begin{pmatrix} \bar{a}^0 \\ \bar{a}^1 \end{pmatrix} \varphi_0 \] (153)

we immediately see that on such states

\[ M^{01}\varphi_1 = -i (\bar{a}^0a^1 - \bar{a}^1a^0) \begin{pmatrix} \bar{a}^0\varphi_0 \\ \bar{a}^1\varphi_0 \end{pmatrix} = \begin{pmatrix} -i\bar{a}^1\varphi_0 \\ -i\bar{a}^0\varphi_0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \varphi_1 \] (154)

the Hermitian operator \( M^{01} \) would have eigenvalues \( \pm i \). The coordinate form of (153) is

\[ \varphi_1 = \frac{1}{\sqrt{2}} \rho \begin{pmatrix} \epsilon(\rho)\sinh\beta \\ \cosh\beta \end{pmatrix} e^{-\rho^2/2} = \frac{1}{\sqrt{2}} \epsilon(\rho) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} e^{-x^2/2} \] (155)

which, as in the O(2) case, represents a vector wavefunction multiplet, but is not consistent with the O(1,1) wavefunctions. The generators of the dynamical symmetry are

\[ J = \begin{pmatrix} \frac{1}{2} (-\bar{a}^0a^1 - \bar{a}^1a^0) & \bar{a}^0a^1 \\ \bar{a}^1a^0 & \frac{1}{2} (\bar{a}^0a^0 + \bar{a}^1a^1) \end{pmatrix} \] (156)

from which we identify the operators

\[ M^{01} = -i (\bar{a}^0a^1 - \bar{a}^1a^0) \quad \Delta^{01} = -\bar{a}^0a^0 - \bar{a}^1a^1 \quad Q^{01} = \bar{a}^0a^1 + \bar{a}^1a^0 \] (157)

that are seen to annihilate the ground state. One may verify that these operators satisfy the SU(1,1) algebra

\[ [K_1, K_2] = -iK_0 \quad [K_0, K_1] = iK_2 \quad [K_2, K_0] = iK_1 \] (158)
where

\[ K_0 = \frac{1}{2} Q^{01}, \quad K_1 = \frac{1}{2} M^{01}, \quad K_2 = \frac{1}{2} \Delta^{01}. \]  

(159)

We notice that \( Q^{01} \) rather than \( M^{01} \) acts as the rotation generator in this SU(1,1), but like \( M^{01} \), has eigenvalues \( \pm i \) on the vector multiplet, so that the implementation of the ladder operators

\[ [K_0, K^{\pm}] = \pm iK^{\pm} \quad K^{\pm} = K_1 \pm iK_2 = \frac{1}{2} M^{01} \pm i \frac{1}{2} \Delta^{01} \]  

(160)

provides no additional insight. Bars [12] has shown that since the ground state has zero \( M^{01} \) eigenvalue, the states (62) can be obtained through repeated application of the scalar operator

\[ \bar{a} \cdot a = - (\bar{a}^0)^2 + (\bar{a}^1)^2 = \frac{1}{2} \left( \rho^2 + \partial^2_\rho + \frac{1}{\rho^2} \partial_\rho - \frac{1}{\rho^2} \rho^2 M^2 - 2 \rho \partial_\rho - 2 \right) \]  

(161)

as

\[ \psi_n (\rho, \beta) = A_n e^{-\rho^2/2} L_n^0 (\rho^2) = \alpha_n (\bar{a} \cdot \bar{a})^n e^{-\rho^2/2} \]  

(162)

for appropriate \( \alpha_n \). He has argued that since all unitary representations of a non-compact group must be either a singlet or infinite dimensional, these states are naturally created by the singlet operator. Since

\[ [\bar{a} \cdot a, M^{01}] \psi_n (\rho, \beta) = 0 \]  

(163)

it follows that all states have zero \( M^{01} \) eigenvalue. Nevertheless, the states \( K_0 \psi_n (\rho, \beta) \) are not singlet states for \( n > 0 \) as can be seen from

\[ \frac{1}{2} Q^{01}, \bar{a} \cdot a \]  

(164)

so that, for example the state

\[ K_0 \psi_1 = K_0 \bar{a} \cdot \bar{a} \psi_0 = \bar{a} \cdot \bar{a} K_0 \psi_0 + (\bar{a}^0 \bar{a}^1 - \bar{a}^1 \bar{a}^0) \psi_0 = (\bar{a}^0 \bar{a}^1 - \bar{a}^1 \bar{a}^0) \psi_0 \]  

(165)

is not in the solution space. Therefore, the significance of the SU(1,1) symmetry is unclear for the wavefunction solutions, as well as for the vector multiplet.

The O(1,1) oscillator thus differs from O(2) in significant ways. For O(2), the properties of the creation/annihilation operators provide a solution equivalent to the explicit coordinate wavefunctions, in addition to a ladder representation of excited modes. Moreover, the states provide a non-trivial representation of the O(2) and SU(2) symmetries, with raising/lowering operators for angular momentum. In particular, the \( s = 0 \) polar solution is related to the Cartesian solution through the unitary transformation that diagonalizes the group generator. For O(1,1) however, all wavefunctions belong to the singlet representation of the Lorentz group, and the ladder representation consists only of the scalar combinations of creation/annihilation operators. In particular, the SU(1,1) symmetry does not lead to a ladder representation for these solutions. Defining creation and annihilation operators

\[ \bar{A} = \frac{1}{2} \bar{a} \cdot \bar{a} \quad A = \frac{1}{2} a \cdot a \]  

(166)

that satisfy

\[ [A, \bar{A}] = 1 + N \quad [N, \bar{A}] = 2\bar{A} \quad [N, A] = -2A \]  

(167)
the singlet representation is seen to be equivalent to a one dimensional oscillator. The number operator $N = \bar{A}A$ satisfies ladder relations

$$[N, \bar{A}] = \bar{A} (1 + N) \quad [N, A] = - (1 + N) A$$

(168)

and since

$$[N, N] = \bar{A} [A, N] + [\bar{A}, N] A = 0$$

(169)

it shares mode eigenstates with $N$. Using the general identity

$$[a, b^k] = \sum_{j=0}^{k-1} b^j [a, b] b^{k-(j+1)}$$

(170)

one finds

$$N \psi_k = \begin{cases} 0, & k = 0 \\ [k^2 + 2 (k - 1)] \psi_k, & k = 1, 2, 3, \ldots \end{cases}$$

(171)

where, as in (162), the states are generated from the ground state as

$$\psi_k = \alpha_k \bar{A}^k \psi_0$$

(172)

with normalizations $\alpha_k$ appropriate to the eigenvalues of $N$. While we have obtained an essentially one dimensional solution providing a unitary singlet representation of $O(1,1)$, we have not succeeded in identifying an infinite dimensional representation capable of providing a reasonable description of the infinite degeneracy expected in an invariant-mass system under boost. Such a representation would require either an infinitely degenerate ground state (such as we will obtain for the $O(2,1)$ oscillator) or an infinite dimensional multiplet of creation operators to act on a singlet ground state.

3. Harmonic oscillator in $D = 3$

3.1. Wavefunctions

In $D = 3$ with parameterizations (36), the Schrodinger equation becomes

$$\left[ -\partial_\rho^2 - \frac{2}{\rho} \partial_\rho + \frac{1}{\rho^2} M^2 + \rho^2 - \varepsilon \right] \psi = 0$$

(173)

with group generators

$$L^{\mu\nu} = x^{[\mu} p^{\nu]} - x^{[\nu} p^{\mu]}$$

(174)

which we notate

$$L^1 = x^2 p^3 - x^3 p^2 \quad L^2 = x^3 p^1 - x^1 p^3 \quad M = x^1 p^2 - x^2 p^1 \quad O(3)$$

$$K^1 = x^0 p^1 - x^1 p^0 \quad K^2 = x^0 p^2 - x^2 p^0 \quad M = x^1 p^2 - x^2 p^1 \quad O(2,1)$$

(175)

so that the parameterizations (36) diagonalize the $M$ angular momentum component

$$M = -i \partial_\phi.$$  

(176)
The Casimir operators in these coordinates are

\[ M^2 = \begin{cases} 
L_1^2 + L_2^2 + M^2 = -\partial^2_\theta - \frac{\cos \theta}{\sin \theta} \partial_\theta - \frac{1}{\sin^2 \theta} \partial^2_\phi & \text{O}(3) \\
M^2 - K_1^2 - K_2^2 = \partial^2_\beta + \frac{\sinh \beta}{\cosh \beta} \partial_\beta - \frac{1}{\cosh^2 \beta} \partial^2_\phi & \text{O}(2,1)
\end{cases} \]  

(177)

so writing a separation of variables

\[ \psi (\rho, \theta, \phi) = R(\rho) F(\theta) \Phi(\phi) \quad \text{O}(3) \]

\[ \psi (\rho, \beta, \phi) = R(\rho) G(\beta) \Phi(\phi) \quad \text{O}(2,1) \]  

(178)

leads to the common angular function

\[ \Phi(\phi) = e^{i\Lambda_1 \phi} \]  

(179)

allowing the replacement of \( M = -i\partial_\phi \) in (177) by its eigenvalue \( \Lambda_1 \). A second separation of variables, associated with the eigenvalue equation

\[ M^2 \psi = \Lambda_2 \psi \]  

(180)

for the Casimir operators, leads to

\[ (-M^2 + \Lambda_2) F(\theta) = \left( \partial^2_\theta + \frac{\cos \theta}{\sin \theta} \partial_\theta - \frac{\Lambda_1^2}{\sin^2 \theta} + \Lambda_2 \right) F(\theta) = 0 \]  

(181)

\[ (M^2 - \Lambda_2) G(\beta) = \left( \partial^2_\beta + \frac{\sinh \beta}{\cosh \beta} \partial_\beta + \frac{\Lambda_1^2}{\cosh^2 \beta} - \Lambda_2 \right) G(\beta) = 0 \]  

(182)

which may be approached in two inequivalent ways. Method I follows the approach generally applied to the classical central force problems, noting the form of the first order derivative terms to make the substitutions

\[ z = \cos \theta \quad \zeta = \sinh \beta \quad \Lambda_2 = l(l+1) \quad \Lambda_1 = m \]  

(183)

and

\[ F(\theta) \to P_l^m(z) \quad G(\beta) \to \hat{P}_l^m(\zeta) \]  

(184)

so that (181) and (182) become associated Legendre equations in the respective forms

\[ \left[ (1 - z^2) \partial^2_z - 2z \partial_z + l(l+1) - \frac{m^2}{1-z^2} \right] P_l^m(z) = 0 \]  

(185)

\[ \left[ (1 + \zeta^2) \partial^2_\zeta + 2\zeta \partial_\zeta - l(l+1) + \frac{m^2}{1+\zeta^2} \right] \hat{P}_l^m(\zeta) = 0. \]  

(186)

Notice that (186) can be obtained from (185) by putting \( z = i\zeta \). Method II produces an inequivalent set of solutions obtained through the substitution

\[ z = \frac{\cos \theta}{\sin \theta} \quad \zeta = \frac{\sinh \beta}{\cosh \beta} \]  

(187)

\[ F_l^m(z) = (1 + z^2)^{\frac{1}{2}} \hat{P}_l^m(z) \quad G_l^m(\zeta) = (1 - \zeta^2)^{\frac{1}{2}} P_l^m(\zeta) \]
leading to associated Legendre equations

\[
\begin{align*}
\left[ (1 + z^2) \frac{\partial^2}{\partial z^2} + 2z \frac{\partial}{\partial z} - m (m + 1) + \frac{l^2}{1 + z^2} \right] P^l_m (z) &= 0 \quad (188) \\
\left[ (1 - \zeta^2) \frac{\partial^2}{\partial \zeta^2} - 2\zeta \frac{\partial}{\partial \zeta} + m (m + 1) - \frac{l^2}{1 - \zeta^2} \right] P^l_m (\zeta) &= 0. \quad (189)
\end{align*}
\]

Notice that the constants \(l\) and \(m\) have reversed roles with respect to equations (185) and (186), having been introduced to satisfy

\[
m (m + 1) = \Lambda_1^2 - \frac{1}{4} \quad \rightarrow \quad \Lambda_1 = m + \frac{1}{2} \quad (190)
\]

\[
l^2 = \Lambda_2 + \frac{1}{4} \quad \rightarrow \quad \Lambda_2 = l^2 - \frac{1}{4}. \quad (191)
\]

Comparing (183) and (190), we write \(\Lambda_1 = m + s\), so that (179) becomes

\[\Phi (\phi) = e^{i \Lambda_1 \phi} = e^{i (m + s) \phi} \quad (192)\]

where the orbital angular momentum may be integral or half-integral as characterized by

\[s = \begin{cases} 0 & \text{method I} \\ 1/2 & \text{method II} \end{cases} \quad (193)\]

Replacing \(M^2\) with its eigenvalue in (173) leads to the radial equation

\[
\left[ -\frac{\partial^2}{\partial \rho^2} - \frac{2}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Lambda_2 + \rho^2 - \varepsilon \right] R (\rho) = 0 \quad (194)
\]

in which we again make the substitutions (49) and (50), and put

\[A = \sqrt{\frac{1}{4} + \Lambda_2 - \frac{1}{2} = l - s} \quad (195)\]

to obtain the Laguerre equation,

\[
\left[ x \frac{d^2}{dx^2} + \left( l - s - x + \frac{3}{2} \right) \frac{d}{dx} + \frac{1}{2} \left( \frac{1}{2} \varepsilon - (l - s) - \frac{3}{2} \right) \right] L^n_\alpha (x) = 0 \quad (196)
\]

with

\[\alpha = l - s + \frac{1}{2} \quad n = \frac{1}{2} \left( \frac{1}{2} \varepsilon - (l - s) - \frac{3}{2} \right). \quad (197)\]

From \(\kappa = \frac{\varepsilon}{2} \varepsilon\) the spectra are given by

\[\kappa = \omega (2n + l + 3/2 - s) \quad (198)\]
the O(3) wavefunctions $\psi_{nlm}$ and O(2,1) wavefunctions $\hat{\psi}_{nlm}$ up to normalization are shown in figure 1:

**Figure 1. Solutions to eigenvalue equation**

| $s = 0$ | $s = 1/2$ |
|---------|-----------|
| $e^{-\frac{\sigma^2}{4}} \rho^l L_n^{l+\frac{1}{2}} (\rho^2) \hat{\psi}_{lm}^m (\cos \theta)e^{im\phi}$ | $e^{-\frac{\sigma^2}{4}} \rho^l L_n^{l+\frac{1}{2}} (\rho^2) \hat{\psi}_{lm}^m (\sinh \beta)e^{im\phi}$ |
| $e^{-\frac{\sigma^2}{4}} \rho^l \frac{\rho^l}{\sqrt{\sin \theta}} \hat{\psi}_{lm}^m (\cot \theta)e^{(m+\frac{1}{2})\phi}$ | $e^{-\frac{\sigma^2}{4}} \rho^l \frac{1}{\sqrt{\cosh \beta}} \hat{\psi}_{lm}^m (\tanh \beta)e^{(m+\frac{1}{2})\phi}$ |

The ground states (taking $n = l = m = 0$) are

\[
\psi_0 (\rho, \theta, \phi) = A_0 e^{-\rho^2/2} \frac{e^{is\phi}}{(\rho|\sin \theta|)^s} = A_0 e^{-(x^2+y^2+z^2)/2} \frac{e^{is\arctan (\frac{y}{x})}}{(x^2+y^2)^{s/2}} \quad \text{O(3)}
\]

\[
\psi_0 (\rho, \beta, \phi) = A_0 e^{-\rho^2/2} \frac{e^{is\phi}}{(\rho \cosh \beta)^s} = A_0 e^{-(x^2+y^2-r^2)/2} \frac{e^{is\arctan (\frac{y}{x})}}{(x^2+y^2)^{s/2}} \quad \text{O(2,1)}
\]

For $s = 0$, the O(3) wavefunctions $\psi_{nlm}$ depend on $\theta$ and $\phi$ through the spherical harmonics

\[
Y_l^m (\theta, \phi) = C_{lm} P_l^m (\cos \theta)e^{im\phi}
\]

and thus provide the familiar $(2l + 1)$-dimensional representation of O(3) through

\[
M^2 Y_l^m (\theta, \phi) = l(l+1) Y_l^m (\theta, \phi) \quad \text{MY}^m_l (\theta, \phi) = m Y_l^m (\theta, \phi)
\]

\[
L^\pm Y_l^m (\theta, \phi) = \sqrt{(l\mp m)} (l \pm m + 1) Y_l^m (\theta, \phi)
\]

with allowed values

\[
l = 0, 1, ... \quad m = -l, -l+1, ..., l-1, l.
\]

Similarly, for $s = 0$ the O(2,1) wavefunctions $\hat{\psi}_{nlm}$ depend on $\beta$ and $\phi$ through the functions

\[
\hat{Y}_l^m (\beta, \phi) = C_{lm} \hat{P}_l^m (\sinh \beta)e^{im\phi}
\]

and it follows from (182) and (183) that

\[
M^2 \hat{\psi}_{nlm} = l(l+1) \hat{\psi}_{nlm} \quad \text{M}\hat{\psi}_{nlm} = m \hat{\psi}_{nlm}
\]

The remaining O(2,1) boost generators are

\[
K^\pm = K^1 \pm iK^0 = -ie^{\pm i\phi} \left( \partial_\beta \pm i \frac{\sinh \beta}{\cosh \beta} \partial_\phi \right)
\]
and for \( s = 0 \), where we set \( \zeta = \sinh \beta \), they take the form

\[
K^\pm = -i \frac{e^{\pm i \phi}}{(1 + \zeta^2)^{1/2}} \left[ (1 + \zeta^2) \partial_\zeta \pm i \zeta \partial_\phi \right]
\]

and

\[
K^\pm \hat{Y}_l^m (\beta, \phi) = -i C_{lm} \frac{e^{i(m+1)\phi}}{(1 + \zeta^2)^{1/2}} \left[ (1 + \zeta^2) \partial_\zeta \hat{P}_l^m (\zeta) \mp m \zeta \hat{P}_l^m (\zeta) \right].
\]

Using the identities [15]

\[
(1 + \zeta^2) \frac{d}{d \zeta} \hat{P}_\nu^\mu (\zeta) = \sqrt{1 + \zeta^2} \hat{P}_{\nu - 1}^{\mu + 1} (\zeta) + \mu \zeta \hat{P}_\nu^\mu (\zeta)
\]

and

\[
(1 + \zeta^2) \frac{d}{d \zeta} \hat{P}_\nu^\mu (\zeta) = (\nu - \mu + 1) (\nu + \mu) \sqrt{1 + \zeta^2} \hat{P}_{\nu - 1}^{\mu - 1} (\zeta) - \mu \zeta \hat{P}_\nu^\mu (\zeta)
\]

we find

\[
K^+ \hat{Y}_l^m = -i C_{lm} \frac{e^{i(m+1)\phi}}{(1 + \zeta^2)^{1/2}} \left( \sqrt{1 + \zeta^2} \hat{P}_{l + 1}^{m+1} (\zeta) + m \zeta \hat{P}_l^m (\zeta) - m \zeta \hat{P}_l^{m+1} (\zeta) \right)
\]

\[
= -i C_{lm} e^{i(m+1)\phi} \hat{P}_l^{m+1} (\zeta)
\]

\[
K^- \hat{Y}_l^m = -i C_{lm} \frac{e^{i(m-1)\phi}}{(1 + \zeta^2)^{1/2}} \left( (l - m + 1) (l + m) \sqrt{1 + \zeta^2} \hat{P}_{l - 1}^{m-1} (\zeta) - m \zeta \hat{P}_l^m (\zeta) \right)
\]

\[
+ m \zeta \hat{P}_l^{m-1} (\zeta)
\]

\[
= -i (l - m + 1) (l + m) C_{lm} e^{i(m-1)\phi} \hat{P}_l^{m-1} (\zeta)
\]

and so that the boost operators \( K^\pm \) raise and lower the \( m \) eigenvalue as

\[
K^\pm \hat{Y}_l^m (\beta, \phi) = \sqrt{(l \pm m)(l \pm m + 1)} \hat{Y}_l^{m \pm 1} (\beta, \phi)
\]

comparable to the action of \( L^\pm \) in (202). It follows from (205) and (215) that the hyperangular wavefunctions \( \hat{\psi}_{nlm} \) provide a \((2l + 1)\)-dimensional representation of \( O(2,1) \) with the same multiplicity structure found in the \( s = 0 \) solutions for \( O(3) \). Since the unitary representations of the non-compact group \( O(2,1) \) should be infinite-dimensional, the \( s = 0 \) solutions, which as noted previously are equivalent to the FKR solutions, must violate unitarity on algebraic grounds.

The \( s = 1/2 \) wavefunctions \( \psi_{nlm} \) for \( O(3) \) depend on \( \theta \) and \( \phi \) through the angular functions

\[
\chi^l_m (\theta, \phi) = F^l_m (z) e^{i(m+1/2)\phi} = C_{lm} (1 + z^2)^{1/2} \hat{P}_m^l (z) e^{i(m+1/2)\phi}
\]

where \( z = \cot \theta \), and it follows from (181), (190) and (191) that

\[
M^2 \chi^l_m (\theta, \phi) = (l^2 - 1/4) \chi^l_m (\theta, \phi) \quad M^l \chi^l_m (\theta, \phi) = (m + 1/2) \chi^l_m (\theta, \phi).
\]

Similarly, the \( s = 1/2 \) wavefunctions \( \hat{\psi}_{nlm} \) for \( O(2,1) \) depend on \( \beta \) and \( \phi \) through the hyperangular functions

\[
\chi^l_m (\beta, \phi) = F^l_m (\zeta) e^{i(m+1/2)\phi} = C_{lm} (1 - \zeta^2)^{1/2} \hat{P}_m^l (\zeta) e^{i(m+1/2)\phi}
\]
where $\zeta = \tanh \beta$ and
\[
\mathbf{M}^2 \chi^I_m (\beta, \phi) = (l^2 - 1/4) \chi^I_m (\beta, \phi) \quad M x^I_m (\beta, \phi) = (m + 1/2) \chi^I_m (\beta, \phi). \quad (219)
\]

In terms of the parameters (187) for $s = 1/2$, the non-diagonal operators for $O(3)$ and $O(2,1)$ take the forms
\[
L^\pm = L^1 \pm i L^2 = e^{\pm i \phi} \left[ \pm (1 + z^2) \partial_z - iz \partial_\phi \right] \quad O(3) \quad (220)
\]
\[
K^\pm = K^1 \pm i K^2 = e^{\pm i \phi} \left[ -i \left( 1 - \zeta^2 \right) \partial_\zeta \mp \zeta \partial_\phi \right] \quad O(2,1). \quad (221)
\]

For $O(3)$
\[
L^\pm \chi^I_m (\theta, \phi) = C_{lm} e^{i(m+1/2)\phi} \left[ \pm (1 + z^2) \partial_z + (m + 1/2) z \right] \left( 1 + z^2 \right)^{1/4} \hat{P}_m^l (z) \quad \quad (222)
\]
where
\[
(1 + z^2) \partial_z \left( 1 + z^2 \right)^{1/4} \hat{P}_m^l (z) = \left( 1 + z^2 \right)^{1/4} \left[ \frac{1}{2} z \hat{P}_m^l (z) + \left( 1 + z^2 \right) \frac{d}{dz} \hat{P}_m^l (z) \right] \quad (223)
\]
so that
\[
L^\pm \chi^I_m (\theta, \phi) = C_{lm} e^{i(m+1/2)\phi} \left( 1 + z^2 \right)^{1/4} \left[ \pm (1 + z^2) \frac{d}{dz} + (m + 1/2 \pm 1/2) z \right] \hat{P}_m^l (z) \quad (224)
\]
and using the identities [15]
\[
(1 + z^2) \frac{d}{dz} \hat{P}_l^\mu = -(\mu - \nu - 1) \hat{P}_l^\mu - (\nu + 1) z \hat{P}_l^\mu = (\mu + \nu) \hat{P}_l^\mu + \nu z \hat{P}_l^\mu \quad (225)
\]
one is led to
\[
L^+ \chi^I_m (\theta, \phi) = C_{lm} e^{i(m+1/2)\phi} \left( 1 + z^2 \right)^{1/4} \left[ - (l - m - 1) \hat{P}_m^l_{m+1} - (m + 1) z \hat{P}_m^l \right. \\
+ \left. (m + 1) z \hat{P}_m^l (z) \right] \quad (226)
\]
\[
= -(l - m - 1) C_{lm} e^{i(m+1/2)\phi} \left( 1 + z^2 \right)^{1/4} \hat{P}_m^l_{m+1} \quad (227)
\]
and
\[
L^- \chi^I_m (\theta, \phi) = C_{lm} e^{i(m+1/2)\phi} \left( 1 + z^2 \right)^{1/4} \left[ \left( l + m \right) \hat{P}_m^l_{m-1} + m z \hat{P}_m^l \right. \\
+ \left. m z \hat{P}_m^l (z) \right] \quad (228)
\]
\[
= -(l + m) C_{lm} e^{i(m+1/2)\phi} \left( 1 + z^2 \right)^{1/4} \hat{P}_m^l_{m-1}. \quad (229)
\]

The actions of $L^\pm$ and a similar calculation for $K^\pm$ using the identities [15]
\[
(1 - \zeta^2) \frac{d}{d\zeta} \hat{P}_l^\mu = (\mu - \nu - 1) \hat{P}_l^\mu_{\nu+1} + (\nu + 1) \zeta \hat{P}_l^\mu = (\mu + \nu) \hat{P}_l^\mu_{\nu-1} - \nu z \hat{P}_l^\mu \quad (230)
\]
leads to
\[
L^\pm \chi^I_m (\theta, \phi) = \hat{c} (l, m) \chi^I_{m\pm 1} (\theta, \phi) \quad K^\pm \chi^I_m (\beta, \phi) = c (l, m) \chi^I_{m\pm 1} (\beta, \phi) \quad (231)
\]
where \( \hat{c}(l, m) \) and \( c(l, m) \) are combinations of the eigenvalues. Since \( L^\pm \) and \( K^\pm \) act on the lower index (the associated Legendre functions \( P_n^\mu \) and \( \hat{P}_n^\mu \) are nonzero for \( \nu \geq 0 \) and \( \nu \geq |\mu| \)), there is no upper bound on the action of the raising operators \( L^+ \) and \( K^+ \), but from

\[
\hat{P}_n^m = \frac{(2n)!}{2^{2n}n!} (1 + z^2)^{n/2} \quad P_n^m(\zeta) = (-1)^n \frac{(2n)!}{2^{2n}n!} (1 + \zeta^2)^{n/2}
\]

we find the lower bounds

\[
L^- \hat{\chi}^m(\theta, \phi) = K^- \chi^m(\beta, \phi) = 0.
\]

The functions \( F \) and \( G \) therefore provide infinite-dimensional representations of \( O(3) \) and \( O(2,1) \), leading to mass/energy states of infinite degeneracy, similarly to the hyperangular functions found in [7]. Although the \( s = 1/2 \) wavefunctions \( \psi_{nlm} \) are thus unacceptable for \( O(3) \), the \( s = 1/2 \) wavefunctions \( \hat{\psi}_{nlm} \) for \( O(2,1) \) provide a faithful representation of the angular momentum and boost structure of the symmetry. Because \( O(2,1) \) is both non-compact and non-abelian, the infinite degeneracy is expressed through an infinite-dimensional group representation, and not merely absorbed into the phase of the wavefunction as was seen for \( O(1,1) \).

### 3.2. Fock space

To obtain a number representation in \( D = 3 \), we must simultaneously diagonalize the operators \( M^2 \) and \( M \) expressed in terms of creation/annihilation operators \( (\hat{a}^1, \hat{a}^2, \hat{a}^3) \) for \( O(3) \) and \( (\hat{a}^0, \hat{a}^1, \hat{a}^2) \) for \( O(2,1) \). Since

\[
M = -i (\hat{a}^1 a^2 - \hat{a}^2 a^1) \quad \Rightarrow \quad [M, \hat{a}^3] = [M, \hat{a}^0] = 0
\]

we regard the direction

\[
x^\parallel = \begin{cases} 
  x^3, & \text{O}(3) \\
  x^0, & \text{O}(2,1)
\end{cases}
\]

as longitudinal, orthogonal to the \( x^1 - x^2 \) subspace, chosen as the plane of observable angular momentum \( M \). Therefore, by extending the procedure in \( D = 2 \), the operator \( M \) is found to have eigenvalues \( \{0, \pm 1\} \) on first level excited eigenstates

\[
\varphi_1 = \frac{1}{\sqrt{2}} \left( a_+, \sqrt{2} a^\parallel, -a_- \right) \varphi_0
\]

where \( a_\pm \) are as defined in (69). However, the Casimir operators

\[
M^2 = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} = -\frac{1}{2} (\hat{a}^\mu a^\nu - \hat{a}^\nu a^\mu) (\hat{a}_\mu a_\nu - \hat{a}_\nu a_\mu) = N^2 + N - (\hat{a} \cdot \hat{a}) (a \cdot a)
\]

with total mode number

\[
N = \begin{cases} 
  \hat{a}^1 a^1 + \hat{a}^2 a^2 + \hat{a}^3 a^3 = \hat{a}_+ a_- + \hat{a}_- a_+ + \hat{a}^3 a^3, & \text{O}(3) \\
  \hat{a}^1 a^1 + \hat{a}^2 a^2 + \hat{a}^0 a^0 = \hat{a}_+ a_- + \hat{a}_- a_+ - \hat{a}^0 a^0, & \text{O}(2,1)
\end{cases}
\]

and scalar operators

\[
\hat{a} \cdot \hat{a} = \begin{cases} 
  2\hat{a}_+ \hat{a}_- + \hat{a}^3 a^3, & \text{O}(3) \\
  2\hat{a}_+ \hat{a}_- - \hat{a}^0 a^0, & \text{O}(2,1)
\end{cases}
\]
\[ a \cdot a = \begin{cases} 2a_+a_- + a^3a^3, & \text{O(3)} \\ 2a_+a_- - a^0a^0, & \text{O(2,1)} \end{cases} \]  

(240)

remain non-diagonal on the states (236). Nevertheless, \( \mathbf{M}^2 \) must be block diagonal with respect to \( N \) and \( M \) and so studying the expected multiplicity of the mass/energy states leads to a characterization of the oscillator states. Recall that despite the sign of the term \(-\bar{a}^0a^0\) in the second of (238), the total mass/energy of the O(2,1) oscillator was found in equation (198) to be positive definite; because the solutions in figure 1 are of the FKR type for which (13) requires \( n_0 \leq 0 \), the timelike modes contribute positive mass/energy. We therefore expect that the O(3) and O(2,1) will have similar multiplicity structure, which was seen to hold for the wavefunctions \( \psi_{nlm} \) and \( \hat{\psi}_{nlm} \), and we may study their common eigenvalue content. From

\[
\begin{bmatrix} M, N^1 \end{bmatrix} \neq 0 \quad \begin{bmatrix} M, N^2 \end{bmatrix} \neq 0 \quad \begin{bmatrix} N, N^\parallel \end{bmatrix} = \begin{bmatrix} M, N^\parallel \end{bmatrix} = 0, \tag{241}
\]

the matrix representation of \( \mathbf{M}^2 \) reduces to coherent subspaces labeled by eigenvalues \( N \) and \( n^\parallel \). A convenient parameterization of Cartesian states is

\[
\begin{pmatrix} n^1 \\ n^2 \\ n^\parallel \end{pmatrix} = \begin{pmatrix} k \\ N - n^\parallel - k \end{pmatrix} \quad \tag{242}
\]

with

\[
n^\parallel = 0, 1, \ldots, N, \quad k = 0, 1, \ldots, (N - n^\parallel). \tag{243}
\]

The number of states for given \( N \) and \( n^\parallel \) is therefore \( N - n^\parallel + 1 \), and the total number of states with mode number \( N \) is

\[
\sum_{n^\parallel=0}^{N} \left( N + 1 - n^\parallel \right) = (N + 1)(N + 1) - \frac{N(N + 1)}{2} = \frac{(N + 1)(N + 2)}{2}. \tag{244}
\]

For \( s = 0 \), we extend (117) and construct excited states through

\[
\zeta_{\alpha\beta\gamma} = \frac{1}{\sqrt{\alpha!\beta!\gamma!}} \left( \bar{a}_+ \right)^\alpha \left( \bar{a}_- \right)^\beta \left( \bar{a}^\parallel \right)^\gamma \psi_0 \quad \tag{245}
\]

which are eigenstates of \( N \) and \( M \) with

\[
N\zeta_{\alpha\beta\gamma} = (\alpha + \beta + \gamma) \zeta_{\alpha\beta\gamma} \quad M\zeta_{\alpha\beta\gamma} = (\alpha - \beta) \zeta_{\alpha\beta\gamma} \quad \tag{246}
\]

so that the states \( \zeta_{\alpha\beta\gamma} \) are precisely the states found by diagonalizing \( M \) in the Cartesian picture. Acting on (245) with (237) leads to

\[
\mathbf{M}^2\zeta_{\alpha\beta\gamma} = \left[ N(N + 1) - 4\alpha\beta - \gamma(\gamma - 1) \right] \zeta_{\alpha\beta\gamma} \\
- 2\sqrt{(\alpha + 1)(\beta + 1)\gamma(\gamma - 1)} \zeta_{(\alpha+1)(\beta+1)(\gamma-2)} \\
- 2\sqrt{\alpha\beta(\gamma + 2)(\gamma + 1)} \zeta_{(\alpha-1)(\beta-1)(\gamma+2)} \quad \tag{247}
\]

so that the states \( \zeta_{\alpha\beta\gamma} \) are not generally eigenstates of \( \mathbf{M}^2 \), but as expected are mixtures of states with \( (\alpha \pm 1, \beta \pm 1, \gamma \mp 2) \) and fixed \( M \) eigenvalue

\[
m = (\alpha \pm 1) - (\beta \pm 1) = \alpha - \beta. \quad \tag{248}
\]
It follows from (247) that
\[
M^2 \zeta_{N00} = N (N + 1) \zeta_{N00} \quad M \zeta_{N00} = N \zeta_{N00}
\]
and so the allowed eigenvalues of \(M\)
\[
m = \alpha - \beta = -l, -l + 1, \ldots, l - 1, l
\]
are consistent with the parameter range
\[
\alpha, \beta = 0, 1, \ldots, N.
\]
Generally, as demonstrated in appendix D by exploiting the invariance of \(\text{tr}(M^2)\) under unitary transformations, the Casimir content of the states \(\zeta_{\alpha\beta\gamma}\) is
\[
l = N, N - 2, \ldots, N - \text{int}(N/2)
\]
and since the multiplicity of \(l\)-states is \(2l + 1\), the multiplicity of states with total mode number \(N\) is
\[
\sum_{k=0}^{\text{int}(N/2)} 2(N-2k) + 1 = \frac{(N+1)(N+2)}{2}
\]
in agreement with (244). Since diagonalization of \(M^2\) does not mix states of different \(m\), the mode number \(N\) of states \(\psi_{lm}\) and \(\hat{\psi}_{lm}\) must depend on \(l\), with Casimir eigenvalues given in (253), but not on \(m\). The principal quantum number \(n\) complements the contribution of \(l\) to energy, incrementing by 2 when \(l\) is decremented by 1. Thus, the mode number is
\[
N = 2n + l, \quad n = 0, 1, 2, \ldots, N
\]
and the total mass/energy becomes
\[
\kappa = \omega \left(2n + l + \frac{3}{2}\right)
\]
in agreement with the solution (198) to the Schrodinger equation for \(s = 0\).

According to (245) — (247) the \(N = 1\) states constitute the \(l = 1\) vector multiplet
\[
\zeta^{(1)} = \begin{pmatrix} \zeta_{001} \\ \zeta_{010} \\ -\zeta_{100} \end{pmatrix} = \begin{pmatrix} \bar{a}_- \\ \bar{a}_\parallel \\ -\bar{a}_+ \end{pmatrix} \psi_0
\]
which we order according to the eigenvalues \(m = -1, 0, 1\) found by diagonalizing \(M\) on the \(N = 1\) multiplet of Cartesian states
\[
\varphi^{(1)} = \begin{pmatrix} \varphi_{100} \\ \varphi_{010} \\ \varphi_{001} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_\parallel \end{pmatrix} \varphi_0.
\]

Applying the creation/annihilation operators in polar parameterizations (36)
\[
\bar{a}_\pm = \frac{1}{2} e^{\pm i\phi} \left( \rho \sin \theta - \sin \theta \partial_\rho - \frac{\cos \theta}{\rho} \partial_\theta \mp \frac{i}{\rho \sin \theta} \partial_\phi \right)
\]
\[ a^3 = \frac{1}{\sqrt{2}} \left( \rho \cos \theta - \cos \theta \partial_\rho + \frac{\sin \theta}{\rho} \partial_\beta \right) \]  

for O(3), and

\[ \bar{a}_\pm = \frac{1}{2} e^{\pm i\phi} \left( \rho \cosh \beta - \cosh \beta \partial_\rho + \frac{\sinh \beta}{\rho} \partial_\beta \mp \frac{i}{\rho \cosh \beta} \partial_\phi \right) \]  

\[ \bar{a}^0 = \frac{1}{\sqrt{2}} \left( \rho \sinh \beta - \sinh \beta \partial_\rho + \cosh \beta \partial_\beta \right) \]  

for O(2,1), to the ground states

\[ \psi_0 = \hat{\psi}_0 = A_0 e^{-\rho^2/2} \]  

we obtain

\[ \zeta^{(1)} = A_0 \rho \begin{pmatrix} \sin \theta e^{-i\phi} \\ \sqrt{2} \cos \theta \\ -\sin \theta e^{i\phi} \end{pmatrix} e^{-\rho^2/2} \quad \text{O(3)} \]  

\[ \zeta^{(1)} = A_0 \rho \begin{pmatrix} \cosh \beta e^{-i\phi} \\ \sqrt{2} \sinh \beta \\ -\cosh \beta e^{i\phi} \end{pmatrix} e^{-\rho^2/2} \quad \text{O(2,1).} \]  

Using

\[ P_1^1(z) = -\sqrt{1-z^2} \quad P_0^0(z) = z \quad P_1^{-1}(z) = \sqrt{1-z^2} \]  

\[ \hat{P}_1^1(\zeta) = -\sqrt{1+\zeta^2} \quad \hat{P}_0^0(\zeta) = \zeta \quad \hat{P}_1^{-1}(\zeta) = \sqrt{1+\zeta^2} \]  

wavefunctions (264) and (265) are seen to agree with the \( l = 1 \) vector multiplet found for \( s = 0 \) in figure 1, corresponding to the well-known property of the spherical harmonics that the three components of \( Y_{l=1}(\theta, \phi) \) and \( \hat{Y}_{l=1}(\beta, \phi) \) form a unit vector.

The first level of excited states was found by acting on the ground state with the operator multiplet \( (\bar{a}_-, \bar{a}_1, -\bar{a}_+) \) which we regard as the fundamental representation of a set of irreducible tensor operators constructed successively by taking irreducible tensor products

\[ \bar{a}_{m}^{(j+1)} = \sum_{m_2=-1,0,1} \langle j ~ m-m_2 ~ 1 ~ m_2 | j ~ 1 ~ j \pm 1 ~ m \rangle \bar{a}_{m-m_2}^{(j)} \bar{a}_{m_2}^{(1)} \]  

where \( \bar{a}_{m}^{(j+1)} \) is an irreducible tensor operator of rank \( j+1 \), \( \bar{a}_{m_2}^{(1)} \) is the vector operator, and \( \langle j ~ m-m_2 ~ 1 ~ m_2 | j ~ 1 ~ j \pm 1 ~ m \rangle \) is the appropriate Clebsch-Gordan coefficient. Thus, according to (244), the \( N = 2 \) states have total multiplicity of 6, which by (253) must include the five \( l = 2 \) states and the \( l = 0 \) singlet state. The two irreducible tensor operators that can be constructed from the vector operator are the singlet (\( l = m = 0 \))

\[ \bar{a}_0^{(0)} = -\frac{1}{\sqrt{3}} (2\bar{a}_+\bar{a}_- + \bar{a}_3\bar{a}_3) = -\frac{1}{\sqrt{3}} \bar{a} \cdot \bar{a} \]  

(269)
and the \( l = 2 \) operators

\[
\begin{align*}
\hat{a}^{(2)}_{-2} &= \hat{a}^{(1)}_{-1} \hat{a}^{(1)}_{-1} = (\bar{a}_-)^2 \\
\hat{a}^{(2)}_{-1} &= \frac{1}{\sqrt{2}} \left( \hat{a}^{(1)}_{0} \hat{a}^{(1)}_{-1} + \hat{a}^{(1)}_{-1} \hat{a}^{(1)}_{0} \right) = -\sqrt{2}\bar{a}_- \bar{a}_3 \\
\hat{a}^{(2)}_{0} &= \frac{1}{\sqrt{6}} \left( \hat{a}^{(1)}_{1} \hat{a}^{(1)}_{-1} + 2\hat{a}^{(1)}_{0} \hat{a}^{(1)}_{0} + \hat{a}^{(1)}_{-1} \hat{a}^{(1)}_{1} \right) = -\frac{2}{\sqrt{6}} (\bar{a}_+ \bar{a}_- - \bar{a}_3 \bar{a}_3) \\
\hat{a}^{(2)}_{1} &= \frac{1}{\sqrt{2}} \left( \hat{a}^{(1)}_{1} \hat{a}^{(1)}_{0} + \hat{a}^{(1)}_{-1} \hat{a}^{(1)}_{-1} \right) = \sqrt{2}\bar{a}_+ \bar{a}_3 \\
\hat{a}^{(2)}_{2} &= \hat{a}^{(1)}_{1} \hat{a}^{(1)}_{1} = (\bar{a}_+)^2
\end{align*}
\]

which are precisely the operators found by diagonalizing the matrix representation of \( \mathbf{M}^2 \). In this way, the complete set of spherical polar harmonic oscillators in 3 dimensions can be constructed from the ground state.

For the \( s = 1/2 \) wave functions, the action of the operator multiplet \((\bar{a}_-, \bar{a}_0, -\bar{a}_+)\) on the ground state does not lead to excited wavefunctions. In particular, acting with (261) and (262) on the \( \text{O}(2,1) \) ground state in (199) one finds

\[
\begin{pmatrix}
\bar{a}_- \\
\bar{a}_0 \\
\bar{a}_+
\end{pmatrix} \psi_0 = \begin{pmatrix}
e^{-i\phi} \rho \cosh \beta \tilde{\psi}_0 \\
e^{i\phi} \left( \sqrt{2}\rho \sinh \beta \right) \\
e^{i\phi} \left( \rho \cosh \beta + \frac{1}{2\rho \cosh \beta} \right)
\end{pmatrix} \psi_0
\]

which is not an excited state and not even a vector in the polar representation. Expressing the singlet operator \( \bar{a} \cdot \bar{a} \) in the \( \text{O}(2,1) \) parameterization,

\[
\bar{a} \cdot \bar{a} = \frac{1}{2} \left( \rho^2 + \alpha^2 + \frac{2}{\rho} \partial_\rho - \frac{1}{\rho^2} \mathbf{M}^2 - 2\partial_\rho - 3 \right)
\]

and comparing with the Schrödinger equation (173) we may write

\[
(\bar{a} \cdot \bar{a}) \psi_0 = (2\rho^2 - 2\partial_\rho - \varepsilon_0 - 3) \psi_0 = \left( 2\rho^2 + \frac{1 + 2\rho^2}{\rho} - 5 \right) \psi_0
\]

which acts only on the radial coordinate. Examining the \( s = 1/2 \) wavefunctions for \( \text{O}(2,1) \) in figure 1, we see that (277) can only correspond to a state if the \( \rho \)-dependent factor on the RHS is equivalent to raising the value of \( l \). However the Legendre function \( P_l^m(\tanh \beta) \) must vanish for \( l > m \), and since the singlet operator does not contain \( \beta \) or \( \phi \), it cannot affect \( m \) and so cannot produce a wavefunction in the Hilbert space by operating on \( \psi_0 \). Apparently, the vector operator multiplet belongs only to the \( s = 0 \) vector representations of the symmetry groups, and not to the infinite-dimensional \( s = 1/2 \) representations. It may be possible to construct an appropriate ladder representation of creation/annihilation operators though a multipole expansion of the Hamiltonian, corresponding to an infinite summation of the associated Legendre functions. This will be discussed in a subsequent paper.

4. Conclusion

The goal of this study has been to explore polar solutions for the harmonic oscillator in two and three Euclidean and Lorentzian dimensions, in order to construct Fock space descriptions...
equivalent to the Hilbert space wavefunctions found by solution of the Schrodinger equation in (hyper)spherical coordinates. Although this work was initially motivated by particular interest in the infinite dimensional representation of $O(2,1)$ symmetric oscillator, we have seen that the methods which were successful for $O(2)$, $O(1,1)$, and the finite dimensional representation of $O(3)$, fail for the case of primary interest. Similarly, no infinite dimensional representation was found for $O(1,1)$ in either Hilbert space or Fock space.

The familiar treatment of the oscillator by separation into Cartesian modes, whether in Hilbert space or Fock space, does not directly address algebraic symmetries, but implicitly chooses as a maximal commuting set of operators the Hamiltonian and its orthogonal components $\{K, K_\mu\}$ where

$$\psi(x) = \prod_\mu \psi(x^\mu) \quad K = \sum_\mu K^\mu.$$  \hspace{1cm} (278)

As seen explicitly, for example in (269) — (274), the resulting solutions mix eigenstates of the Casimir invariant associated with the orthogonal symmetry, as well as a group generator that may be diagonalized in a polar representation. Although the degeneracy of the mass/energy is evident from the counting of total mode number, there is little motivation in the Cartesian approach to treat excited states as members of a multiplet.

In the polar approach to the Schrodinger equation, on the other hand, the orthogonal symmetry guides the separation of variables, and so the wavefunctions emerge naturally as irreducible representations of the group. Because the polar operators do not admit an obvious Dirac factorization, we built Fock space descriptions by decomposing the fundamental vector multiplet of Cartesian modes into eigenstates of the diagonal group generator and then constructing higher-level excited states as irreducible tensor products. The choice of the Cartesian vector multiplet as fundamental representation was thus made for reasons of convenience, and in retrospect, it is not entirely surprising that this finite-dimensional multiplet structure does not yield any of the desired infinite-dimensional representations. It is hoped that a different factorization of the Hamiltonian, such as a multipole expansion that naturally produces an infinite sum of terms, may lead to a faithful representation of the $O(2,1)$ symmetry equivalent to the wavefunctions found in section 3.

Aside from possible applications to relativistic quantum field theory and string theory, a principal motivation for developing a consistent Fock space approach to the $O(2,1)$ oscillator is to obtain a deeper understanding of the Horwitz-Arshansky bound states, and in particular, the question of ghost states. For $O(1,1)$, we demonstrated that the oscillator is equivalent to a one-dimensional system with commutation relations between the ladder operators $A$ and $\bar{A}$ that cannot produce ghosts. However, as was seen in section 1.2 for the FKR oscillator, the identification of ghost states is not straightforward even when the creation/annihilation operators are known. Except for the KN oscillator, whose states have positive definite norm by construction, the integration by parts implicit in the use of commutation relations to evaluate the norm of timelike modes presents difficulties that leave interpretation questionable. A detailed answer to these questions will require a Fock space description that has yet to be found.

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Appendix A. Virial theorem

As in nonrelativistic quantum mechanics, the Ehrenfest theorem for an observable \( \hat{O} \) and Hamiltonian

\[
K = \frac{1}{2} p^2 + V(x)
\]  

is

\[
\frac{d}{d\tau} \langle \psi | \hat{O} | \psi \rangle = i \langle \psi | [K, \hat{O}] | \psi \rangle + \langle \psi | \partial_\tau \hat{O} | \psi \rangle.
\]  

(A.2)

Generally then, we have

\[
\frac{d}{d\tau} \langle x_\mu p_\mu \rangle = i \langle \langle \frac{1}{2} p^2, x_\mu \rangle p_\mu \rangle + i \langle x_\mu [V(x), p_\mu] \rangle = \langle p^2 \rangle - \langle x_\mu \partial_\mu V(x) \rangle
\]  

(A.3)

and on eigenstates of the Hamiltonian,

\[
\frac{d}{d\tau} \langle x_\mu p_\mu \rangle = \frac{d}{d\tau} \langle \psi(0) | e^{-i\tau (x_\mu p_\mu)} e^{i\tau} | \psi(0) \rangle = \frac{d}{d\tau} \langle \psi(0) | x_\mu p_\mu | \psi(0) \rangle = 0
\]  

(A.4)

so that

\[
\langle p^2 \rangle = \langle x_\mu \partial_\mu V(x) \rangle.
\]  

(A.5)

For the harmonic oscillator

\[
V(x) = \frac{1}{2} \omega^2 x^2 \rightarrow x_\mu \partial_\mu V(x) = \omega^2 x^2
\]  

(A.6)

so that on states of spacelike support

\[
\langle p^2 \rangle = \langle \omega^2 x^2 \rangle > 0.
\]  

(A.7)

Appendix B. Differential operators in polar coordinates

\[
\begin{bmatrix}
\partial_1 \\
\partial_2
\end{bmatrix}
= \begin{bmatrix}
\cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta \\
\sin \theta \partial_r + \frac{1}{r} \cos \theta \partial_\theta
\end{bmatrix}
\]  

O(2)  

(B.1)

\[
\begin{bmatrix}
\partial_0 \\
\partial_1
\end{bmatrix}
= \begin{bmatrix}
-|\rho| \sinh \beta \partial_\rho + \frac{1}{|\rho|} \cosh \beta \partial_\beta \\
\cosh \beta \partial_\rho - \frac{1}{|\rho|} \sinh \beta \partial_\beta
\end{bmatrix}
\]  

O(1,1)  

(B.2)

\[
\begin{bmatrix}
\partial_1 \\
\partial_2 \\
\partial_3
\end{bmatrix}
= \begin{bmatrix}
\cos \phi \sin \theta \partial_r + \frac{\cos \theta \cos \phi}{r} \partial_\theta - \frac{\sin \phi}{r \sin \theta} \partial_\phi \\
\sin \theta \sin \phi \partial_r + \frac{\cos \theta \sin \phi}{r} \partial_\theta + \frac{\cos \phi}{r \sin \theta} \partial_\phi \\
\cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta
\end{bmatrix}
\]  

O(3)  

(B.3)
Applying the annihilation operators to the ground state (61), we recover (114) in the explicit so that (58), using the notation of (49)

\[ \begin{bmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \end{bmatrix} = \begin{bmatrix} -\sinh \beta \partial_\rho + \frac{\cosh \beta}{\rho} \partial_\beta \\ \cosh \beta \cos \phi \partial_\rho - \frac{\sin \beta \cos \phi}{\rho} \partial_\beta - \frac{\sin \phi}{\rho \cosh \beta} \partial_\phi \\ \cosh \beta \sin \phi \partial_\rho - \frac{\sin \beta \sin \phi}{\rho} \partial_\beta + \frac{\cos \phi}{\rho \cosh \beta} \partial_\phi \end{bmatrix} \quad \text{O(2,1)} \quad (B.4) \]

Appendix C. Coordinate ladder representation for O(2)

To verify that the solutions (58) form the basis for a representation of the operator algebra, we express the creation/annihilation operators in polar coordinates. Combining (7) and (69) as

\[ a_\pm = \frac{1}{2} [(x + \partial_x) \pm i (y + \partial_y)] = \frac{1}{2} [x \pm iy + (\partial_x \pm i \partial_y)] \quad (C.1) \]

\[ \bar{a}_\pm = \frac{1}{2} [(x - \partial_x) \pm i (y - \partial_y)] = \frac{1}{2} [x \pm iy - (\partial_x \pm i \partial_y)] \quad (C.2) \]

we obtain the polar expressions

\[ a_\pm = \frac{1}{2} e^{\pm i \phi} \left( \rho + \frac{\partial}{\partial \rho} \pm \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \quad \bar{a}_\pm = \frac{1}{2} e^{\pm i \phi} \left[ \rho - \left( \frac{\partial}{\partial \rho} \pm \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \right] \quad (C.3) \]

Applying the annihilation operators to the ground state (61), we recover (114) in the explicit form

\[ a_+ \psi_0 = \frac{1}{2} \sqrt{\frac{1}{\Gamma(s+1)}} (\rho + (s - \rho^2) \frac{1}{\rho} - 1) e^{-\rho^2/2} \rho^s e^{i \phi (s+1)} = 0 \quad (C.4) \]

\[ a_- \psi_0 = \sqrt{s} \frac{1}{\Gamma(s)} e^{-\rho^2/2} (\rho e^{\rho \phi})^{s-1} \quad (C.5) \]

where the result in (C.5) is formally equivalent to \( \sqrt{s} \psi_{0,-1} \) but as discussed above, does not correspond to any state in the Fock space, and we treat as annihilation. For the general state (58), using the notation of (49) \( x = \rho^2 \), the \( \rho \) derivative is

\[ \frac{\partial}{\partial \rho} \left[ e^{-\rho^2/2} L_n^{m+s} (\rho^2) \left( \rho e^{\rho \phi}\right)^{m+s} \right] = \frac{e^{-\rho^2/2} \left( \rho e^{\rho \phi}\right)^{m+s}}{\rho} \left[ -\rho^2 + m + s + 2x \frac{d}{dx} \right] L_n^{m+s} (x) \quad (C.6) \]

providing

\[ \frac{\partial}{\partial \rho} \psi_{nm} = \left( -\rho + \frac{m+s}{\rho} \right) \psi_{nm} + \frac{2}{\rho} A_{nm} e^{-\rho^2/2} \left( \rho e^{\rho \phi}\right)^{m+s} x \frac{d}{dx} L_n^{m+s} (x) \quad (C.7) \]

and the \( \phi \) derivative is

\[ \frac{i}{\rho} \frac{\partial}{\partial \phi} \psi_{nm} = -\frac{m+s}{\rho} \psi_{nm} \quad (C.8) \]

so that

\[ \left( \frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \psi_{nm} = -\rho \psi_{nm} + \frac{2}{\rho} A_{nm} e^{-\rho^2/2} \left( \rho e^{\rho \phi}\right)^{m+s} x \frac{d}{dx} L_n^{m+s} (x) \quad (C.9) \]

\[ \left( \frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \psi_{nm} = \left( -\rho + \frac{2m+s}{\rho} \right) \psi_{nm} + \frac{2}{\rho} A_{nm} e^{-\rho^2/2} \left( \rho e^{\rho \phi}\right)^{m+s} x \frac{d}{dx} L_n^{m+s} (x) \quad (C.10) \]
Then, using the identity [15]

\[
\frac{d}{dx} L_a^b(x) = -L_{a-1}^{b+1}(x)
\]  

(C.11)

for the Laguerre functions, we calculate

\[
a_+ \psi_{nm}(\rho, \phi) = \frac{1}{2} e^{i\phi} \left( \rho + \frac{\partial}{\partial \rho} + i \frac{\partial}{\rho \partial \phi} \right) \psi_{nm}
\]

\[
= e^{i\phi} \frac{1}{\rho} A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s} x \frac{d}{dx} L_n^{m+s}(x)
\]

\[
= A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s} e^{i\phi} \left( -L_{n-1}^{m+s+1}(x) \right), \ n > 0
\]

\[
= -A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s+1} L_{n-1}^{m+s+1}, \ n > 0.
\]

(C.13)

Using (60) for \( A_{nm} \) we obtain

\[
a_+ \psi_{nm} = (-1)^{n+1} \sqrt{\frac{\Gamma(n+1)}{\pi \Gamma(n+m+s+1)}} L_{n-1}^{m+s+1}(x) \left( \rho e^{i\phi} \right)^{m+s+1} e^{-\rho^2/2} = \sqrt{n} \psi_{n-1,m+1}
\]

(C.16)

as required by the first of (112). The second lowering operator acts as

\[
a_- \psi_{nm}(\rho, \phi) = \frac{1}{2} e^{-i\phi} \left( \rho + \frac{\partial}{\partial \rho} - i \frac{\partial}{\rho \partial \phi} \right) \psi_{nm}
\]

\[
= \frac{m+s}{\rho} e^{-i\phi} \psi_{nm} + e^{-i\phi} \frac{1}{\rho} A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s} x \frac{d}{dx} L_n^{m+s}(x)
\]

(C.17)

so the identities [15]

\[
x \frac{d}{dx} L_a^b(x) = a L_a^b(x) - (a+b) L_{a-1}^b(x)
\]

(C.19)

\[
L_{a-1}^b(x) = L_a^b(x) - L_{a-1}^b(x)
\]

(C.20)

lead to

\[
a_- \psi_{nm}(\rho, \phi) = A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s-1} \left[ (m+s) L_n^{m+s}(x) + n L_{n-1}^{m+s}(x) - (n+m+s) L_{n-1}^{m+s+1}(x) \right]
\]

\[
= (n+m+s) A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s-1} \left( L_n^{m+s}(x) - L_{n-1}^{m+s}(x) \right)
\]

\[
= (n+m+s) A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s-1} L_{n-1}^{m+s-1}(x)
\]

(C.21)

(C.22)

(C.23)

providing

\[
a_- \psi_{nm} = (-1)^n \sqrt{\frac{\Gamma(n+1)(m+s+n)^2}{\pi \Gamma(n+m+s+1)}} e^{-\rho^2/2} L_n^{m+s-1}(x) \left( \rho e^{i\phi} \right)^{m+s-1}
\]

\[
= \sqrt{n+m+s} \psi_{n,m-1}
\]

(C.24)
as required by the first of (113). The raising operator \( \bar{a}_+ \) acts as

\[
\bar{a}_+ \psi_{nm} (\rho, \phi) = \frac{1}{2} e^{i\phi} \left[ \rho - \left( \frac{\partial}{\partial \rho} + i \frac{\partial}{\partial \phi} \right) \right] \psi_{nm} (\rho, \phi)
\]

\[= \rho e^{i\phi} \psi_{nm} - e^{i\phi} A_{nm} e^{-\rho^2/2} \left( \frac{\rho e^{i\phi}}{\rho} \right)^{m+s} x \frac{d}{dx} L_n^{m+s} (x) \tag{C.25} \]

so applying identity (C.11) and the identities [15]

\[
L_a^b (x) = L_{a-1}^b (x) + L_{a-1}^{b-1} (x) \tag{C.27} \]

we calculate

\[
\bar{a}_+ \psi_{nm} (\rho, \phi) = \rho e^{i\phi} A_{nm} e^{-\rho^2/2} \left( \frac{\rho e^{i\phi}}{\rho} \right)^{m+s} \left( L_n^{m+s} (x) - \frac{d}{dx} L_n^{m+s} (x) \right) \tag{C.28} \]

\[
= A_{nm} e^{-\rho^2/2} \left( \frac{\rho e^{i\phi}}{\rho} \right)^{m+s+1} \left( L_n^{m+s} (x) + L_n^{m+s+1} (x) \right) \tag{C.29} \]

\[
= A_{nm} e^{-\rho^2/2} \left( \frac{\rho e^{i\phi}}{\rho} \right)^{m+s+1} L_n^{m+s+1} (x) \tag{C.30} \]

providing

\[
\bar{a}_+ \psi_{nm} = (-1)^n \sqrt{\frac{\Gamma (n+1)}{\pi \Gamma (n+m+s+1)}} e^{-\rho^2/2} \left( \frac{\rho e^{i\phi}}{\rho} \right)^{m+s+1} L_n^{m+s+1} (\rho^2) \tag{C.31} \]

\[
= \sqrt{n+m+s+1} \psi_{n,m+1} \tag{C.32} \]

confirming the second of (112). Finally, the action of the second raising operator \( \bar{a}_- \) is

\[
\bar{a}_- \psi_{nm} (\rho, \phi) = \frac{1}{2} e^{-i\phi} \left[ \rho - \left( \frac{\partial}{\partial \rho} - i \frac{\partial}{\partial \phi} \right) \right] \psi_{nm} (\rho, \phi)
\]

\[= e^{-i\phi} \left[ \left( \rho - \frac{m+s}{\rho} \right) \psi_{nm} - \frac{A_{nm}}{\rho} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s} x \frac{d}{dx} L_n^{m+s} (x) \right] \tag{C.34} \]

so using the identity [15]

\[
x \frac{d}{dx} L_a^b (x) = (a + 1) L_{a+1}^b (x) - (a + b + 1 - x) L_a^b (x) \tag{C.35} \]

and (C.11) leads to

\[
\bar{a}_- \psi_{nm} (\rho, \phi) = -A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s-1} \left[ (m+s-x) L_n^{m+s} (x) + x \frac{d}{dx} L_n^{m+s} (x) \right] \tag{C.36} \]

\[
= -A_{nm} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s-1} \left[ (n+1) L_{n+1}^{m+s+1} (x) \right] \tag{C.37} \]

we confirm

\[
\bar{a}_- \psi_{nm} = (-1)^n \sqrt{\frac{(n+1)^2 \Gamma (n+1)}{\pi \Gamma (n+m+s+1)}} e^{-\rho^2/2} \left( \rho e^{i\phi} \right)^{m+s-1} L_n^{m+s-1} (\rho^2) \tag{C.38} \]

\[
= \sqrt{n+1} \psi_{n+1,m-1} \tag{C.39} \]
so that the solutions (58) belong to the ladder representation for any value of \( s \).

Unlike the angular momentum \( M = -i \partial_\phi \), which is diagonal in polar coordinates, the remaining SU(2) generators are most conveniently expressed in Cartesian coordinates

\[
\Delta = \bar{a}_+ a_+ + \bar{a}_- a_- = N^1 - N^2 = \frac{1}{2} \left( x^2 - y^2 - \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

\[
Q = -i (\bar{a}_+ a_+ - \bar{a}_- a_-) = xy - \frac{\partial}{\partial x} \frac{\partial}{\partial y}
\]

from which it follows that

\[
M \psi_0 = s \psi_0
\]

\[
\Delta \psi_0 = s \left[ \frac{x^2 + y^2 - s + 1}{(x + iy)^2} \right] \psi_0
\]

\[
Q \psi_0 = i s \left[ \frac{x^2 + y^2 - s + 1}{(x + iy)^2} \right] \psi_0
\]

and again we see that the SU(2) symmetry of the Hamiltonian is spontaneously broken for \( s \neq 0 \).

Since the operator \( \Delta \) is diagonal in Cartesian coordinates, its action on the states is of special interest. From (117) and (120) and the commutator

\[
\left[ (\bar{a}_+ a_+ + \bar{a}_- a_-), (\bar{a}_+)^\alpha (-\bar{a}_-)^\beta \right] = \alpha \bar{a}_+ a_-^\alpha - 1 (\bar{a}_-)^\beta + 1 + \beta (\bar{a}_+)^\alpha + 1 \bar{a}_-^\alpha - 1
\]

we obtain

\[
\Delta \zeta_{\alpha \beta} = \alpha \sqrt{\frac{\beta + 1}{s + \alpha}} \zeta_{\alpha-1, \beta + 1} + \beta \sqrt{\frac{s + \alpha + 1}{\beta}} \zeta_{\alpha + 1, \beta - 1} + (\bar{a}_+)^\alpha (\bar{a}_-)^\beta \Delta \psi_0
\]

which cannot be diagonalized unless \( s = 0 \) because (C.43) shows that the ground state is not an eigenstate of \( \Delta \). Using (137) to construct the \( s = 0 \) multiplets of for given \( N \), it is easily shown, case by case, that diagonalization of \( \Delta \) recovers the standard Cartesian description of the oscillator. Since \( \Delta \) cannot be diagonalized on states with \( s \neq 0 \), there is no unitary combination of the spherical states \( \psi^{s l}_{nm} \) equivalent to the familiar Cartesian states of the harmonic oscillator.

**Appendix D. Multiplicity of \( D = 3 \) Casimir**

States \( \zeta_{\alpha \beta \gamma} \) have \( n = \alpha + \beta + \gamma \) and \( m = \alpha - \beta \) but are mixtures of \( l \)-states. The cases \((\alpha, \beta, \gamma) = (n, 0, 0) \) and \((\alpha, \beta, \gamma) = (0, n, 0) \) have eigenvalues

\[
M^2 \zeta_{n00} = n (n + 1) \zeta_{n00} \quad M \zeta_{n00} = n \zeta_{n00}
\]

\[
M^2 \zeta_{0n0} = n (n + 1) \zeta_{0n0} \quad M \zeta_{0n0} = -n \zeta_{n00}
\]

The allowed eigenvalues of \( M \)

\[
m = \alpha - \beta = -l, -l + 1, ..., l - 1, l
\]

are consistent with

\[
\alpha, \beta = 0, 1, ..., l_{\text{max}} = n.
\]
The Casimir $M^2$ mixes $(\alpha, \beta, \gamma)$-states with $(\alpha \pm 1, \beta \pm 1, \gamma \mp 2)$-states, but the mixed states have the same eigenvalues of $M$

$$m = \alpha - \beta = (\alpha \pm 1) - (\beta \pm 1). \quad \text{(D.5)}$$

The angular momentum content of the states $\zeta_{\alpha\beta\gamma}$ is

$$l = n, n - 2, n - 4, \ldots, n - 2 \cdot \text{int} \left( \frac{n}{2} \right) = \begin{cases} l_{\text{max}}, l_{\text{max}} - 2, \ldots, 0 & l_{\text{max}} \text{ even} \\ l_{\text{max}}, l_{\text{max}} - 2, \ldots, 1 & l_{\text{max}} \text{ odd} \end{cases}. \quad \text{(D.6)}$$

Since the multiplicity of $l$-states is $2l + 1$ the total multiplicity of $n$-states is

$$\text{int}(n/2) \sum_{k=0}^{\text{int}(n/2)} [2(2k) + 1] = \frac{(n + 1)(n + 2)}{2} \quad \text{(D.7)}$$

as required. For given values of $n = \alpha + \beta + \gamma$ and $m = \alpha - \beta$, the possible $(\alpha, \beta, \gamma)$-states are thus

$$\alpha = m + k, \quad \beta = k, \quad \gamma = n - m - 2k, \quad k = 0, 1, 2, \ldots, \text{int} \left( \frac{n - m}{2} \right) \quad \text{(D.8)}$$

and the on-diagonal elements of $M^2$ for given $n$ and $m$ are

$$M^2_{\text{on--diagonal}} = (2n - m) (m + 1) + 2k (2n - 4m - 1) - 8k^2. \quad \text{(D.9)}$$

Since $\text{tr} (M^2)$ is invariant under diagonalization, the sum of eigenvalues calculated from the diagonal elements is

$$\text{tr} (M^2) = \sum_{k=0}^{\text{int}(n-m)} M^2_{\text{on--diagonal}} = \begin{cases} \frac{1}{6} n^3 + \frac{3}{4} n^2 + \frac{5}{6} n - \frac{1}{6} m^3 + \frac{1}{4} m^2 + \frac{1}{6} m & , \ n - m \text{ even} \\ \frac{1}{6} n^3 + \frac{3}{4} n^2 + \frac{5}{6} n - \frac{1}{6} m^3 - \frac{1}{4} m^2 + \frac{1}{6} m + \frac{1}{4} & , \ n - m \text{ odd} \end{cases} \quad \text{(D.10)}$$

For given values of $n = \alpha + \beta + \gamma$ and $m = \alpha - \beta$, the possible $l$-states are

$$l = n - 2k \quad k = 0, 1, \ldots, \text{int} \left( \frac{n - m}{2} \right). \quad \text{(D.11)}$$

So, on the other hand, for given $n$ and $m$ the sum of eigenvalues is must be the sum of $l (l + 1)$

$$\text{tr} (M^2) = \sum_{k=0}^{\text{int}(n-m)} (n - 2k) (n - 2k + 1) = \begin{cases} \frac{1}{6} n^3 + \frac{3}{4} n^2 + \frac{5}{6} n - \frac{1}{6} m^3 + \frac{1}{4} m^2 + \frac{1}{6} m & , \ n - m \text{ even} \\ \frac{1}{6} n^3 + \frac{3}{4} n^2 + \frac{5}{6} n - \frac{1}{6} m^3 - \frac{1}{4} m^2 + \frac{1}{6} m + \frac{1}{4} & , \ n - m \text{ odd} \end{cases} \quad \text{(D.12)}$$

as required.
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