END REDUCTIONS AND COVERING TRANSLATIONS OF CONTRACTIBLE OPEN 3-MANIFOLDS

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Abstract. This paper uses Brin and Thickstun’s theory of end reductions of non-compact 3-manifolds to study groups of covering translations of irreducible contractible open 3-manifolds $W$ which are not homeomorphic to $\mathbb{R}^3$. We associate to $W$ an object $S(W)$ called the simplicial complex of minimal $\mathbb{R}^2$-irreducible end reductions of $W$. Whenever $W$ covers another 3-manifold the group of covering translations is isomorphic to a fixed point free group of automorphisms of $S(W)$. We apply this result to give uncountably many examples of $\mathbb{R}^2$-irreducible such $W$ which cover 3-manifolds with infinite cyclic fundamental group and non-trivially cover only 3-manifolds with infinite cyclic fundamental group. We also give uncountably many examples of $\mathbb{R}^2$-irreducible $W$ which are not eventually end irreducible and do not non-trivially cover any 3-manifold.

1. Introduction

The universal covering conjecture states that whenever $M$ is a closed, connected, orientable, irreducible 3-manifold with infinite cyclic fundamental group, then its universal covering space must be homeomorphic to $\mathbb{R}^3$. Valentin Poénaru has announced a proof of this conjecture [17]. His proof is the outgrowth of a massive project which investigates the structure of simply connected 3-manifolds. He translates the problem into another problem in higher dimensions, solves that problem, and then uses the result to solve the original 3-dimensional problem. His methods are highly original and perhaps not that familiar to most 3-manifold topologists. (See [F] for an introduction to some of Poénaru’s techniques.)

The present paper continues an investigation of an alternative approach to the universal covering conjecture which uses more traditional 3-manifold techniques. An irreducible contractible open 3-manifold which is not homeomorphic to $\mathbb{R}^3$ is called a Whitehead manifold. Thus one wants to show that Whitehead manifolds cannot cover compact 3-manifolds. In [13] I showed that genus one Whitehead manifolds (those which are monotone unions of solid tori) cannot non-trivially cover any 3-manifold. Wright then showed in [23] that the same result is true for the much larger class of eventually end irreducible Whitehead manifolds; his proof was based on the interplay

1991 Mathematics Subject Classification. Primary: 57M10, 57N10.

Key words and phrases. 3-manifold, covering space, end reduction.

This research was supported in part by NSF grant DMS-0072429.
of two new ideas: the “Ratchet Lemma” (which uses the assumption of eventual end irreducibility) and the “Orbit Lemma” (which does not). Tinsley and Wright \[20\] then extended this result to certain examples of Whitehead manifolds which are not eventually end irreducible; these manifolds contain a family of disjoint proper planes which split them into an infinite collection of certain genus one Whitehead manifolds. Their proof that these Whitehead manifolds cannot non-trivially cover any other 3-manifolds is based on the Orbit Lemma and an extension of the Ratchet Lemma called the Special Ratchet Lemma which applies to Whitehead manifolds which split along proper planes into eventually end irreducible Whitehead manifolds. They also gave other examples of manifolds of this type which could non-trivially cover other non-compact 3-manifolds; however, their methods did not rule out their covering compact 3-manifolds. In \[15\] I used the Special Ratchet Lemma and the Orbit Lemma to show that certain Whitehead manifolds which covered non-compact 3-manifolds with free fundamental groups could cover only 3-manifolds with free fundamental groups, and so could not cover compact 3-manifolds. These examples are different from those of \[20\] but like them they can be split along a family of proper planes into eventually end irreducible Whitehead manifolds. One can associate to this decomposition a tree whose vertices correspond to the eventually end irreducible Whitehead manifolds and whose edges correspond to the planes. For these examples I showed that the decomposition is unique up to ambient isotopy. Thus the isotopy class of any self-homeomorphism induces an automorphism of the tree. When the self-homeomorphism is a non-trivial covering translation the Special Ratchet Lemma and the Orbit Lemma can be used to show that the automorphism does not fix any points of the tree. It follows that the group of covering translations is isomorphic to the corresponding group of automorphisms and that this group is free.

This paper explores the situation when the Whitehead manifold is not eventually end irreducible and cannot be split into eventually end irreducible pieces by a collection of proper planes. It turns out that even in this situation the manifold still contains some very useful open submanifolds which are eventually end irreducible. They are called end reductions. They were developed by Brin and Thickstun \[1\] as a tool in their study of the ends of non-compact 3-manifolds. An elementary example of an end reduction is one of the eventually end irreducible pieces into which the collection of planes breaks a Whitehead manifold of the type described above; such an example has topological boundary consisting of the planes adjacent to it. In general, however, an end reduction is embedded in a very contorted manner in the manifold containing it and has a very complicated topological boundary.

In this paper an extension of the Special Ratchet Lemma called the “End Reduction Ratchet Lemma” is proven. This result and the Orbit Lemma are used to prove that a non-trivial covering translation of a Whitehead manifold cannot fix the isotopy class of an end reduction. The isotopy in this statement need not be an ambient isotopy; it can refer to a non-ambient isotopy of embedding maps.
In order to apply this theorem one needs some structure on the set of isotopy classes of end reductions. We define an object called the simplicial complex of minimal $R^2$-irreducible end reductions of a Whitehead manifold $W$, denoted $S(W)$. An end reduction $V$ of $W$ is minimal if the only end reductions of $W$ that it contains are isotopic to it. It is $R^2$-irreducible if it does not contain any proper planes which split it in a non-trivial fashion. Two isotopy classes of end reductions are joined by an edge if there is an isotopically unique $R^2$-irreducible end reduction $E$ containing representatives $V_0$ and $V_1$ of the vertices such that every end reduction of $W$ which is contained in $E$ is isotopic to $V_0$, $V_1$, or $E$. Simplices of higher dimension are defined by an inductive extension of this definition.

In [19] Scott and Tucker gave an example of a Whitehead manifold which is not eventually end irreducible, which is an infinite cyclic covering space of a non-compact 3-manifold, and which was purported to be $R^2$-irreducible. Unfortunately this example contained a mistake which resulted in its not being $R^2$-irreducible. In [14] I showed how to modify this example to make it $R^2$-irreducible. In the present paper I determine $S(W)$ for this modified example and uncountably many non-homeomorphic variations of it. In all cases $S(W)$ is a triangulation of the real line. It follows that any 3-manifold which it non-trivially covers must have infinite cyclic fundamental group. A further variation produces uncountably many examples which cannot non-trivially cover any 3-manifold.

The paper is organized as follows. In section 2 we review Brin and Thickstun’s theory of end reductions and prove that any end reduction of a Whitehead manifold is a Whitehead manifold. End reductions are associated to certain compact subsets; we speak of an end reduction at the compact set. In this section we also prove the technically convenient result that any end reduction can be chosen to be an end reduction at a knot. In section 3 we prove the End Reduction Ratchet Lemma. In section 4 we apply it and the Orbit Lemma to show that a non-trivial covering translation cannot fix the isotopy class of an end reduction. In the next two sections some further results about end reductions are proven which will be needed later. In section 5 it is shown that the end reductions associated to nested compact sets have representatives which are themselves nested; moreover, the smaller end reduction is an end reduction of the larger end reduction. In section 6 it is shown that any $R^2$-irreducible end reduction can be isotoped off any non-trivial plane. Section 7 collects some standard facts about gluing together 3-manifolds in such a way as to achieve such properties as being incompressible, irreducible, $\partial$-irreducible, anannular, or atoroidal. Section 8 defines certain compact 3-manifolds which will be used in our construction and proves that they have certain properties we will need. In section 9 we construct the family $\mathcal{F}$ of Whitehead manifolds $W$ we will be considering. In section 10 we prove that these manifolds are $R^2$-irreducible. We also associate to each finite set of integers $P$ a certain open subset $V^P$ of $W$. It is a Whitehead manifold. We prove that it is $R^2$-irreducible if and only if the integers are consecutive. In section 11 we prove that $V^P$ is a genus one Whitehead manifold if and only if $P$ has a single
element. In section 12 we prove that each $V^P$ is an end reduction of $W$. In section 13 we prove that each $R^2$-irreducible end reduction of $W$ is isotopic to one of the $V^P$. In section 14 we prove that $V^P$ and $V^Q$ are isotopic if and only if $P = Q$. In section 15 we introduce a variation into our construction and classify the resulting uncountably many variations of those $V^P$ of genus one up to homeomorphism. In section 16 we determine $S(W)$ for the family of Whitehead manifolds we have constructed. The vertices are the isotopy classes of those $V^P$ for which $P$ has a single element $p$. The edges are the isotopy classes of those $V^P$ for which $P = \{p, p+1\}$. It follows that $S(W)$ is a triangulation of $R^3$. In section 17 we apply these results to show that uncountably many of these manifolds cover only 3-manifolds with infinite cyclic fundamental group and uncountably many of them do not non-trivially cover any 3-manifold.

2. End reductions

In this section we review the Brin-Thickstun theory of end reductions of irreducible open 3-manifolds developed in [1]. We work in somewhat less generality by restricting attention to connected, orientable, irreducible open 3-manifolds $W$ which are not homeomorphic to $R^3$. Each such $W$ is contained in the class $\mathcal{Z}$ of 3-manifolds considered in [1]. A compact, connected 3-manifold $J \subseteq W$ is regular in $W$ if $W - J$ is irreducible and has no component with compact closure. Note that in our situation the first condition is satisfied if and only if $J$ is not contained in a 3-ball in $W$; if, in addition, $W$ has one end and trivial second homology, then the second condition is satisfied if and only if $\partial J$ is connected.

Let $J$ be a regular 3-manifold in $W$, and let $V$ be a connected open subset of $W$ which contains $J$. We say that $V$ is end irreducible rel $J$ in $W$ if for each compact, connected 3-manifold $K$ with $J \subseteq K \subseteq V$ there is a compact, connected 3-manifold $L$ with $K \subseteq L \subseteq V$ so that any loop in $V - L$ which is null homotopic in $W - J$ must be null homotopic in $V - K$. We say that $W$ is end irreducible rel $J$ if $W$ is end irreducible rel $J$ in $W$. We say that $W$ is eventually end irreducible if it is end irreducible rel $J$ for some $J$. (In [1] neither $K$ nor $L$ is required to be connected, but since $V$ is connected one can enlarge either of these sets to a connected set in $V$, so the two definitions are equivalent.)

It will be useful to have the following alternative characterization in terms of exhaustions. An exhaustion of a connected open 3-manifold $U$ is a sequence $\{C_n\}_{n \geq 0}$ of compact, connected 3-manifolds in $U$ such that $C_n \subseteq \text{int } C_{n+1}$ for each $n \geq 0$ and $U = \bigcup_{n=0}^{\infty} C_n$. It is a regular exhaustion if each $C_n$ is regular in $U$.

Lemma 2.1 (Brin-Thickstun). Let $J$ be a regular 3-manifold in $W$, and let $V$ be a connected open subset of $W$ which contains $J$. Then $V$ is end irreducible rel $J$ in $W$ if and only if $V$ has a regular exhaustion $\{C_n\}_{n \geq 0}$ such that $C_0 = J$ and $\partial C_n$ is incompressible in $W - J$ for $n \geq 1$.

Proof. This is part of Lemma 2.2 of [1].
Let $X$ and $Y$ be manifolds and $h : X \times [0, 1] \to Y$ a map; we also use the notation $h_t : X \to Y$, where $h_t(x) = h(x, t)$, to describe this map. This is an isotopy if for each $t$ we have that $h_t$ is an embedding. It is an ambient isotopy if each $h_t$ is a homeomorphism. If $X$ and $X'$ are subsets of $Y$, then they are isotopic if there is an isotopy $h_t : X \to Y$ such that $h_0$ is the inclusion map of $X$ into $Y$ and $h_1(X) = X'$; if $A$ is a subset of $X$ such that the restriction of each $h_t$ to $A$ is the identity of $A$, then we say that $X$ and $X'$ are isotopic rel $A$. If $h_t$ is the restriction of an ambient isotopy $g_t : Y \to Y$ with $g_0$ the identity of $Y$, then we say that they are ambient isotopic; if each $g_t$ is the identity on $A$ we say that they are ambient isotopic rel $A$. The closure of the set of all points $x$ such that for some $t$ one has $g_t(x) \neq x$ is called the support of $g$.

Note that if $X$ and $X'$ are isotopic via the isotopy $h_t : X \to Y$, then $X'$ and $X$ are isotopic via the isotopy $p_t : X' \to Y$ given by $p_t(x') = h_{1-t}(h_{-t}^{-1}(x'))$.

Let $J$ be a regular 3-manifold in $W$, and let $V$ a connected open subset of $W$ containing $J$. We say that $V$ is an end reduction of $W$ at $J$ if $V - J$ is irreducible, $W - V$ has no components with compact closure, $V$ is end irreducible rel $J$ in $W$, and whenever $N$ is a regular 3-manifold in $W$ such that $J \subseteq \text{int} N$ and $\partial N$ is incompressible in $W - J$ it must be the case that for some compact subset $C$ of $W - J$ one has that $V$ is ambient isotopic rel $W - C$ to a subset $V'$ of $W$ such that $N \subseteq V'$. The last property is called the weak engulfing property.

**Theorem 2.2** (Brin-Thickstun). Let $W$ be a connected, orientable, irreducible, open 3-manifold. Let $J$ be a regular 3-manifold in $W$. Then an end reduction $V$ of $W$ at $J$ exists. If $V$ and $V'$ are end reductions of $W$ at $J$, then they are isotopic rel $J$.

**Proof.** Existence and uniqueness are Theorems 2.1 and 2.3, respectively, of [1]. □

We remark, as pointed out in [1], that two end reductions of $W$ at $J$ need not be ambient isotopic.

**Lemma 2.3** (Brin-Thickstun). Let $V$ be an end reduction of $W$ at $J$. Suppose $J'$ is a regular 3-manifold in $V$ which contains $J$ such that $\partial J'$ is incompressible in $W - J$. Then $V$ is an end reduction of $W$ at $J'$.

**Proof.** This is Corollary 2.2.1 of [1]. □

Suppose $\kappa$ is a knot in $W$ which does not lie in a 3-ball in $W$. We say that $V$ is an end reduction of $W$ at $\kappa$ if $V$ is an end reduction of $W$ at a regular neighborhood of $\kappa$ in $W$.

**Lemma 2.4.** Let $V$ be an end reduction of $W$ at $J$. Then there is a knot $\kappa$ in $\text{int} J$ such that $V$ is an end reduction of $W$ at $\kappa$.

**Proof.** Since $J$ is regular in $W$ no component of $\partial J$ is a 2-sphere. Therefore by [2] there is a knot $\kappa$ in $\text{int} J$ such that $\partial J$ is incompressible in $J - \kappa$. The result then follows from Lemma 2.3. □
In dealing with the examples to be presented later we will need a means of recognizing an end reduction $V$ of a specific $W$ at a specific $J$. In their proof of existence Brin and Thickstun provide a procedure for constructing $V$, which we now briefly describe.

Let $X$ and $Y$ be 3-manifolds, with $X \subseteq \text{int} \ Y$. Suppose $\partial X$ has a component which is compressible in $Y$ and is not a 2-sphere. Then there is a disk $D$ in $Y$ such that $D \cap \partial X$ is a non null homotopic simple closed curve in $\partial X$. We can construct a new 3-manifold $X'$ by doing surgery on $\partial X$ along $D$. If $D \subseteq Y - \text{int} \ X$, then we obtain $X'$ by adding a 2-handle $H$ to $X$. If $D \subseteq X$ then we obtain $Y - \text{int} \ X'$ by adding a 2-handle $H$ to $Y - \text{int} \ X$, i.e. by removing a 1-handle from $X$. In the two cases the handle $H$ is a regular neighborhood $D \times [-1, 1]$ of $D$ in $Y - \text{int} \ X$ or $X$ respectively. We say that a 3-manifold $X^*$ in $Y$ is obtained from $X$ by completely compressing $\partial X$ in $Y$ if there is a sequence $X = X_0, X_1, \ldots, X_k = X^*$ of 3-manifolds in $Y$ such that $X_{i+1}$ is obtained by adding or removing a handle $H_i$ along a compressing disk $D_i$ for $\partial X_i$ in $Y$, the $D_i$ are pairwise disjoint, the attaching/removing annuli $\partial D_i \times [-1, 1]$ are pairwise disjoint, and each component of $\partial X^*$ which is not a 2-sphere is incompressible in $Y$. If, in addition, there is a 3-manifold $A$ in $Y$ which contains $\partial X$ and all the $H_i$, then we say that the compressions are confined to $A$.

For future reference we note that if $X^*$ is obtained from $X$ by completely compressing $\partial X$ in $Y$ with the compressions confined to $A$, then by general position one may arrange for the following property to hold. Suppose $i < j$. Then $H_i$ and $H_j$ are either disjoint or meet in a finite union of disjoint 3-balls each of which has the form $D \times [a, b], -1 < a < b < 1$, when considered as a subset of $H_i$ and the form $E \times [-1, 1]$, $E$ a disk in $\text{int} D$, when considered as a subset of $H_j$.

After a 2-handle is added it may subsequently have “sub-handles” removed by the removal of 1-handles and added by addition of 2-handles, etc., so that the net result within the 2-handle is the addition of several smaller parallel copies of itself. Similarly after a 1-handle has been removed it may subsequently have sub-handles added by the addition of 2-handles and sub-handles removed by the removal of 1-handles, and so on, so that the net result within the 1-handle is the removal of several smaller parallel copies of itself.

Note also that since the removing/attaching annulus for $H_j$ is not null homotopic in $\partial X_j$ it cannot lie in the interior of any of the disks $D \times \{ \pm 1 \}$ belonging to any of the handles $H_i$ with $i < j$. Since the entire set of annuli is pairwise disjoint this implies that it must lie in the original $\partial X$.

Now suppose that $J$ is a regular 3-manifold in $W$. It follows from our assumptions on $W$ that it has a regular exhaustion $\{ C_n \}$ with $C_0 = J$. By passing to a subsequence we may assume that each $\partial C_n$, for $n \geq 1$, can be completely compressed in $W - C_0$ with the compressions confined to $C_{n+1} - C_0$. We obtain $C^*_1$ from $C_1$ by completely compressing $\partial C_1$ in $W - C_0$ so that $C^*_1 \subseteq C_2$. Now $\partial C_2$ can be completely compressed in $W - C_0$ with the compressions confined to $C_3 - C_0$. The components of $\partial C^*_1$ which are not 2-spheres are incompressible in $W - C_0$ and are therefore incompressible in
the smaller set $C_3 - C_0$. Any compressing disk for $\partial C_2$ can be isotoped in $C_3 - C_0$ so as to lie in $C_3 - C^*_1$. The same is true for the compressing disks for the intermediate surfaces obtained in the process of completely compressing $\partial C_2$ in $W - C_0$. It follows that $\partial C_2$ can be completely compressed in $W - C^*_1$ with the compressions confined to $C_3 - C^*_1$, so that we obtain $C^*_1 \subseteq C^*_2 \subseteq C_3$. We continue in this fashion, and set $C^*_0 = C_0$ to get a sequence $\{C^*_n\}$.

The union $V^*$ of the sequence $\{C^*_n\}$ is an open subset of $W$ which is called the constructed end reduction of $W$ at $J$.

**Theorem 2.5** (Brin-Thickstun). *The component $V$ of $V^*$ containing $J$ is an end reduction of $W$ at $J$.*

**Proof.** The proof is that of Theorem 2.1 of [1].\qed

Note that $C_n^*$ may have several components and that $W - C_n^*$ may have components with compact closure. Suppose $Z$ is a compact, connected 3-manifold in $W$ of one of these two types. Since the handles used to compress $\partial C_{n+1}$ miss $C_n^*$ they also miss $Z$. It follows that $Z \subseteq \text{int} C_{n+1}^*$ and that either $Z$ is in $V$ or $Z$ is in $W - V$. By choosing an appropriate union of those $Z$ which are in $V$ we can produce a regular 3-manifold $C''_n$ in $W$ so that $\{C''_n\}$ is an exhaustion of $V$ satisfying Lemma 2.1.

We recall that a *Whitehead manifold* is an irreducible, contractible open 3-manifold which is not homeomorphic to $\mathbb{R}^3$.

**Lemma 2.6.** *Every end reduction of a Whitehead manifold is a Whitehead manifold.*

**Proof.** Let $W$ be a Whitehead manifold and $V$ an end reduction of $W$ at a regular 3-manifold $J$.

We first show that $V$ is simply connected. We may assume that $V$ is the component of a constructed end reduction $V^*$ of $W$ at $J$ which contains $J$. Let $\{C^*_n\}_{n \geq 0}$ be the exhaustion of $W$ used in the construction of $V^*$. Let $\alpha$ be a loop in $V$. Then $\alpha \subseteq C^*_p$ for some $p \geq 1$, where $C^*_p$ is obtained from $C_p$ by completely compressing $\partial C_p$ in $W - J$, where the compressions are confined to $C^*_{p+1} - C_0$. Recall that there is a sequence $X_0, \ldots, X_k$ such that $X_0 = C_p, X_k = C^*_p$, and $X_{i+1}$ is obtained from $X_i$ by either adding a 2-handle or removing a 1-handle.

We first claim that $\alpha$ can be homotoped in $C^*_p$ so that it lies in $C_p \cap C^*_p$. Suppose, by induction, that we have homotoped $\alpha$ in $C^*_p$ so that it lies in $X_{i+1} \cap C^*_p$. If $X_{i+1}$ is obtained from $X_i$ by removing a 1-handle, then $\alpha$ automatically lies in $X_i \cap C^*_p$. So assume that $X_{i+1}$ is obtained from $X_i$ by adding a 2-handle $H_i$. As noted above $H_i \cap C^*_p$ consists of either $H_i$ itself or several parallel copies of itself. In any case $\alpha$ can be homotoped out of each such component by pushing it into $X_i$ across the intersection of that component with the attaching annulus for $H_i$. This homotopy takes place in $C^*_p$.

Since $W$ is simply connected there is a map $f : B \to W$, where $B$ is a disk and $f(\partial B) = \alpha$. There is a $q > p$ such that $f(B) \subseteq C^*_q$. $C^*_q$ is obtained from
$C_q$ by completely compressing $\partial C_q$ in $W - C_q$ with the compressions confined to $C_{q+1} - C_q \subseteq C_{q+1} - C_q$. There is a sequence $Y_0, \ldots, Y_m$ such that $Y_0 = C_q, Y_m = C_q$, and $Y_{i+1}$ is obtained from $Y_i$ by either adding a 2-handle or removing a 1-handle. Note that $C_q^{\ast} \subseteq C_q$. We claim that there is a map $f_m : B \rightarrow C_q^{\ast}$ such that $f_m$ agrees with $f$ on $\partial B$. Let $f_0 = f$. Suppose, by induction, that we have an $f_i : B \rightarrow Y_i$ which agrees with $f$ on $\partial B$. If $Y_{i+1}$ is obtained by attaching a 2-handle to $Y_i$, then $f_i(B)$ is automatically in $Y_{i+1}$, so we let $f_{i+1} = f_i$. So assume that $Y_{i+1}$ is obtained by removing a 1-handle $H_i$ from $Y_i$. Then $H_i = D \times [-1, 1]$ for a disk $D$. Since $H_i \cap C_q^{\ast} = \emptyset$ we may assume that $f_i^{-1}(D \times \{0\})$ consists of simple closed curves in int $B$. We may then redefine $f_i$ in a neighborhood of the outermost disks bounded by these curves to obtain the desired map $f_{i+1}$. This finishes the proof of simple connectivity.

We next show that $V$ is irreducible. Let $S$ be a 2-sphere in $V$. Since $W$ is irreducible, $S$ bounds a 3-ball $B$ in $W$. Since $W - V$ has no components with compact closure we must have that $B \subseteq V$.

Since $V$ is orientable it then follows from the sphere theorem that $\pi_2(V) = 0$. Since $V$ is non-compact $H_3(V) = 0$. It then follows from the Hurewicz theorem that $\pi_i(V) = 0$ for all $i \geq 3$. By the Whitehead theorem $V$ is contractible.

Since $J$ is a regular 3-manifold in $W$ it cannot lie in a 3-ball in $W$, hence cannot lie in a 3-ball in $V$. Therefore $V$ is not homeomorphic to $\mathbb{R}^3$. □

3. The End Reduction Ratchet Lemma

In this section we prove a technical result, called the End Reduction Ratchet Lemma, which generalizes earlier work of Wright [23] and of Tinsley and Wright [20]. The Ratchet Lemma of [23] is basically the special case $W = V$ of our result. The Special Ratchet Lemma of [20] is the special case where $V$ is an eventually end irreducible open subset of $W$ bounded by proper planes.

**Theorem 3.1** (End Reduction Ratchet Lemma). Let $W$ be a connected, orientable, irreducible open 3-manifold which is not homeomorphic to $\mathbb{R}^3$. Let $J$ be a regular 3-manifold in $W$. Suppose $V$ is an end reduction of $W$ at $J$. Let $g$ be a homeomorphism of $W$ onto itself such that each of $g(J)$ and $g^{-1}(J)$ can be isotoped into $V$. Then there is a compact 3-manifold $R$ in $W$ containing $J$ such that a loop in $W - \bigcup_{i=\pm\infty}^\infty g^i(R)$ is null homotopic in $W - J$ if and only if it is null homotopic in $W - g^i(J)$ for each $i \in \mathbb{Z}$.

**Proof.** $g(J)$ and $g^{-1}(J)$ are isotopic to subsets $J^+$ and $J^-$ of $V$, respectively. Since $J$ is compact the covering isotopy theorem [2, 3] implies that there are ambient isotopies $k^\pm_0 : W \rightarrow W$ with $k^\pm_0$ the identity of $W$ and $k^\pm_1(g(\pm 1)(J)) = J^\pm$; moreover there are compact sets $T^\pm$ such that $k^\pm_t(g(\pm 1)(J)) \subseteq T^\pm$ for all $t \in [0, 1]$ and $k_t(x) = x$ for all $x \in W - T^\pm$ and $t \in [0, 1]$. Let $K$ be a compact, connected 3-manifold in $V$ which contains $J \cup J^+ \cup J^-$. Since $V$ is end irreducible rel $J$ in $W$ there is a compact,
connected 3-manifold $L$ in $V$ such that $K \subseteq L$ and any loop in $V - L$ which is null homotopic in $V - J$ is null homotopic in $V - K$. Let $R$ be a compact, connected 3-manifold in $W$ which contains $T^+ \cup T^- \cup L$.

Suppose $\gamma$ is a loop in $W - \cup_{i=-\infty}^{\infty} g^i(R)$ which is null homotopic in $W - J$. Let $D$ be a disk and $f : D \to W - J$ a map which is transverse to $\partial L$ with $f(\partial D) = \gamma$. There is a collection (possibly empty) of disjoint disks $E_1, \ldots, E_p$ in $\text{int} \, D$ whose boundaries are in $f^{-1}(\partial L)$ and whose interiors contain all other points of $f^{-1}(L)$. Each loop $f(\partial E_j)$ lies in $\partial L$ and is null homotopic in $W - J$; it follows that it is null homotopic in $V - K$ and hence is null homotopic in $V - J^+$. We may therefore modify $f$ so that $f(D) \subseteq W - J^+$. Since $\gamma \subseteq W - R \subseteq W - T^+$, we have that $\gamma = (k_1^+)^{-1}(\gamma)$ and so bounds the singular disk $(k_1^+)^{-1}(f(D))$ in $W - g(J)$. Thus $\gamma$ is null homotopic in $W - g(J)$.

A similar argument shows that a loop in $W - \cup_{i=-\infty}^{\infty} g^i(R)$ which is null homotopic in $W - J$ must be null homotopic in $W - g^{-1}(J)$. Translation by appropriate powers of $g$ then completes the proof. $\square$

4. THE MAIN THEOREM

**Theorem 4.1.** Let $W$ be an irreducible contractible open 3-manifold which is not homeomorphic to $\mathbb{R}^3$. Suppose $W$ is a non-trivial covering space of a 3-manifold $M$. Let $G \cong \pi_1(M)$ be the group of covering translations. Let $J$ be a regular 3-manifold in $W$, and let $V$ be an end reduction of $W$ at $J$. If $g$ is an element of $G$ such that $g(J)$ and $g^{-1}(J)$ can each be isotoped into $V$, then $g$ is the identity of $W$.

A group $G$ acting on a set $X$ acts without fixed points if the only element of $G$ which fixes a point is the identity.

**Corollary 4.2.** $G$ acts without fixed points on the set of isotopy classes of end reductions of $W$ and on the set of ambient isotopy classes of end reductions of $W$.

**Proof of Corollary 4.2.** Suppose $g \in G$ fixes the isotopy class of the end reduction $V$ of $W$. Thus there is an isotopy $h_t : V \to W$ such that $h_0$ is the inclusion map of $V$ into $W$ and $h_1(V) = g(V)$. Let $J$ be a regular 3-manifold in $W$ at which $V$ is the end reduction of $W$. We will construct isotopies $f^+_t : g^+1(J) \to W$ such that $f^+_0$ is the inclusion map of $g^+1(J)$ into $W$ and $f^+_1(g^+1(J)) \subseteq V$.

We define $f^-_t : g^{-1}(J) \to W$ by $f^-_t(x) = g^{-1}(h_t(g(x)))$. If $x \in g^{-1}(J)$, then $g(x) \in J \subseteq V$, so we have $f^-_0(x) = g^{-1}(h_0(g(x))) = g^{-1}(g(x)) = x$. On the other hand $f^-_1(g^{-1}(J)) = g^{-1}(h_1(g(g^{-1}(J)))) = g^{-1}(h_1(J)) \subseteq g^{-1}(h_1(V)) = g^{-1}(g(V)) = V$.

Recall that $g(V)$ is isotopic to $V$ via the isotopy $p_t(x) = h_{1-t}(h_1^{-1}(x))$. Thus $p_0$ is the inclusion map of $g(V)$ into $W$, and $p_1(g(V)) = V$. Let $f^+_t$ be the restriction of $p_t$ to $g(J)$. If $x \in g(J)$, then $f^+_0(x) = p_0(x) = x$. On the other hand $f^+_1(g(J)) = p_1^+(g(V)) \subseteq p_1^+(g(V)) = V$.

The case of an ambient isotopy clearly follows from the case just done. $\square$
Proof of Theorem 4.1. By Lemma 2.5 $V$ is a Whitehead manifold. We will show that if $g$ is not the identity, then $V$ must be homeomorphic to $\mathbb{R}^3$, thereby contradicting this statement. The argument follows the tradition of [23], [20], and [15], in combining the Orbit Lemma of Wright with an appropriate generalization of his Ratchet Lemma, in this case the End Reduction Ratchet Lemma.

A group $G$ acting on an $n$-manifold $W$ acts totally discontinuously if for every compact subset $C$ of $W$ one has that $g(C) \cap C = \emptyset$ for all but finitely many elements of $G$. $G$ acts without fixed points and totally discontinuously on $W$ if and only if the projection to the orbit space of the action is a regular covering map with group of covering translations $G$ and the orbit space is an $n$-manifold [10]. In this case if $W$ is contractible, then $G$ is torsion free.

Proposition 4.3 (Orbit Lemma (Wright)). Let $W$ be a contractible, open $n$-manifold, $n \geq 3$. Let $g$ be a non-trivial homeomorphism of $W$ onto itself such that the group $\langle g \rangle$ of homeomorphisms generated by $g$ acts without fixed points and totally discontinuously on $W$. Given compact subsets $B$ and $Q$ of $W$, there is a compact subset $C$ of $W$ containing $B$ such that every loop in $W - C$ is homotopic in $W - B$ to a loop in $W - \bigcup_{i=-\infty}^{\infty} g^i(Q)$.

Proof. Except for the statement that $C$ contains $B$ this is Lemma 4.1 of [23]; we can clearly enlarge the $C$ of that result to satisfy this requirement. A somewhat shorter alternative proof for the special case in which $W$ is an irreducible contractible open 3-manifold is given in [15].

Returning to the proof of Theorem 4.1, we assume that $g$ is not the identity. It then satisfies the hypotheses of the Orbit Lemma. We shall prove that $V$ is $\pi_1$-trivial at $\infty$, i.e. for every compact subset $A$ of $V$ there is a compact subset $A'$ of $V$ containing $A$ such that every loop in $V - A'$ is null-homotopic in $V - A$. By a result of C. H. Edwards [1] and C. T. C. Wall [22] every irreducible, contractible, open 3-manifold which is $\pi_1$-trivial at $\infty$ must be homeomorphic to $\mathbb{R}^3$.

We may assume that $V$ is the component of the constructed end reduction $V^*$ containing $J$. Thus we have a regular exhaustion $\{C_n\}$ of $W$ with $C_0 = J$ from which we obtain $\{C'_n\}$ with union $V^*$ and $\{C''_n\}$ with union $V$.

Let $A$ be a compact subset of $V$. Let $K$ be a compact 3-manifold in $V$ which contains $J \cup A$. Since $V$ is end irreducible rel $J$ in $W$, there is a compact 3-manifold $L$ in $V$ such that every loop in $V - L$ which is null homotopic in $W - J$ is null homotopic in $V - K$. By replacing $L$ by a $C''_n$ containing $L$ we may assume that $\partial L$ is incompressible in $W - J$. Then by Lemma 2.3 $V$ is an end reduction of $W$ at $L$.

By the End Reduction Ratchet Lemma there is a compact 3-manifold $R$ in $W$ containing $J$ such that a loop in $W - \bigcup_{i=-\infty}^{\infty} g^i(R)$ is null-homotopic in $W - J$ if and only if it is null-homotopic in $W - g^i(J)$ for all $i \in \mathbb{Z}$. Apply the Orbit Lemma with $B = L$ and $Q = R$ to get a compact subset $C$ of $W$ containing $L$ such that every
loop in $W - C$ is homotopic in $W - L$ to a loop in $W - \bigcup_{i=-\infty}^{\infty} g^i(R)$. By enlarging $C$ we may assume that it is some $C_p$, $p > n$.

Now $C_p^*$ is the 3-manifold obtained by completely compressing $\partial C_p$ in $W - C_p^*$. Let $A' = C_p^* \cap V$. We must show that every loop $\gamma$ in $V - A'$ is null-homotopic in $V - A'$.

We first show that $\gamma$ is homotopic in $W - L$ to a loop $\gamma'$ in $W - C_p$. Since $\gamma$ is in $V$ it cannot meet any component of $C_p^*$ which is not in $V$. Therefore $\gamma \cap C_p$ must lie in the union of the 1-handles which were removed in the process of completely compressing $\partial C_p$. These handles miss $C_p^*$ and hence miss $C_n^* = L$, so we can homotop $\gamma$ in $W - L$ to a $\gamma'$ in $W - C_p$.

Now $\gamma'$ is homotopic in $W - L$ to a loop $\gamma''$ in $W - \bigcup_{i=-\infty}^{\infty} g^i(R)$. Since $W$ is contractible $\gamma''$ is null-homotopic in $W$. Since $< g >$ is totally discontinuous $\gamma''$ is null-homotopic in $W - g^i(J)$ for some $i$. Since $\gamma''$ lies in $W - L$ the End Reduction Ratchet Lemma implies that $\gamma''$ is null-homotopic in $W - J$. Since $J \subseteq L \subseteq A'$ we have that $\gamma$ is null-homotopic in $W - J$. Since $\gamma$ lies in $V - L$ and $V$ is end irreducible rel $J$ in $W$ we have that $\gamma$ is null-homotopic in $V - J$. Thus $\gamma$ is null-homotopic in $V - K \subseteq V - A$, as required.

5. Nested end reductions

Theorem 5.1. Let $W$ be a Whitehead manifold. Let $J$ and $K$ be regular 3-manifolds in $W$ such that $J \subseteq \text{int } K$. Let $U$ and $V$ be end reductions of $W$ at $J$ and $K$, respectively. Then $U$ is isotopic rel $J$ to $U'$ such that $U' \subseteq V$. Moreover $U'$ is an end reduction of $V$ at $J$.

Proof. Let $\{C_n\}_{n \geq 0}$ be a regular exhaustion for $W$ with $C_0 = J$ and $C_1 = K$. We may assume that $V$ is the component of the constructed end reduction $V^*$ of $W$ at $K$ associated to the exhaustion $\{C_n\}_{n \geq 1}$ which contains $K$. Let $C_n^*$ be the 3-manifold arising in the construction of $V^*$. Recall that $C_n^*$ is obtained from $C_n$ by completely compressing $\partial C_n$ in $W - K$ with the compressions confined to $C_{n+1}^* - C_{n-1}^*$.

It suffices to show that after passing to a subsequence the constructed end reduction $U^*$ of $W$ at $J$ associated to the exhaustion $\{C_n\}_{n \geq 0}$ can be built in such a way that the 3-manifold $C_n^*$ obtained by completely compressing $C_n$ in $W - J$ with the compressions confined to $C_{n+1}^* - C_{n-1}^*$ can be obtained by first constructing $C_n^*$ as above and then completely compressing $\partial C_n^*$ in $W - J$ with the compressions confined to $C_{n+1}^* - C_{n-1}^*$.

We first completely compress $\partial C_1$ in $W - J$. Let $E_1, \ldots, E_p$ be the sequence of compressing disks and $H_1, \ldots, H_p$ the associated sequence of 1-handles and 2-handles. We require the handles to intersect as described in Section 2. Let $C_1 = Q_0, \ldots, Q_p$ be the associated sequence of 3-manifolds. We may assume that the compressions are confined to $C_2 - J$.

Let $D_1, \ldots, D_q$ be the sequence of compressing disks involved in the complete compression of $\partial C_2$ in $W - K$. Let $C_2 = X_0, X_1, \ldots, X_q = C_2^*$ be the associated
sequence of 3-manifolds. We may assume that the compressions are confined to $C_3 - K$.

We will show that by replacing $E_1, \ldots, E_p$ by a new set of disks we can completely compress $\partial C_1$ in $W - J$ with the compressions confined to $C_2 - J$. The resulting 3-manifold will be our new $C_1^\#$. This will be done in $q$ steps. After each step we will rename our new disks and 3-manifolds with the names of their predecessors.

Assume as the inductive hypothesis that $\partial C_1$ can be completely compressed in $W - J$ with the compressions confined to $X_{i-1} - J$. If $X_i$ is obtained by adding a 2-handle to $X_{i-1}$, then there is nothing to do. So assume that it is obtained by removing a 1-handle from $X_{i-1}$ associated to the compressing disk $D_i$ for $\partial C_2$ in $C_2 - K$. Let $E = E_1 \cup \cdots \cup E_p$. Put $E$ in general position with respect to $D_i$.

Let $\{\alpha_1, \ldots, \alpha_m\}$ be the set of components of $E \cap D_i$ which are outermost on $E$. So $\alpha_s = \partial \Delta_s$ for a disk $\Delta_s$ in $E$, the $\Delta_s$ are disjoint, and this union contains $E \cap D_i$. Let $F = E - (\Delta_1 \cup \cdots \cup \Delta_m)$. It is a disk with $m$ holes. Now $\alpha_s = \partial \Delta'_s$ for a disk $\Delta'_s$ in $D_i$. The $\Delta'_s$ need not be disjoint. If two $\Delta'_s$ intersect, then one is contained in the interior of the other.

We may assume that $\Delta'_1$ is innermost on $D_i$ among the $\Delta'_s$. Let $F_1 = F \cup \Delta'_1$. Isotop $F_1$ by moving $\Delta'_1$ to a parallel disk which misses $D_i$ in such a way that now $F_1 \cap D_i = \alpha_2 \cup \cdots \cup \alpha_m$. We may now assume that $\Delta'_2$ is innermost on $D_i$ among the remaining $\Delta'_s$. Let $F_2 = F_1 \cup \Delta'_2$. Isotop $F_2$ by moving $\Delta'_2$ to a parallel disk which misses $D_i$ in such a way that now $F_2 \cap D_i = \alpha_3 \cup \cdots \cup \alpha_m$. We continue in this fashion until we get a surface $E' = F_m$ which is a union of $q$ disks which are disjoint from each other and from $D_i$. Let $E'_j$ be the component of $E'$ with $\partial E'_j = \partial E_j$.

Let $A_j$ be the removing/attaching annulus for the handle $H_j$. Let $G_j = \partial C_1 - \text{int} (A_1 \cup \cdots \cup A_j)$. By construction $E'_j \cap G_{j-1} = \partial E'_j = \partial E_j$, which is a core of $A_j$. Since $E'_j \cap E'_k = \emptyset$ for $k < j$ and $Q'_j \cap Q'_{j-1}$ is the union of $G_j$ and two parallel copies of each $E'_k$ with $k < j$ we have that $E'_j \cap Q'_{j-1} = \partial E'_j$. It follows that $Q'_j$ is obtained from $Q'_{j-1}$ by removing a 1-handle or adding a 2-handle along $E'_j$. Since $\partial Q'_{j-1}$ and $\partial Q'_{j-1}$ are homeomorphic by a homeomorphism which is the identity on $G_{j-1}$ and $\partial E_j$ does not bound a disk in $\partial Q_{j-1}$ we have that $\partial E'_j$ does not bound a disk in $\partial Q'_{j-1}$. Thus $E'_j$ is a compressing disk.

Now suppose that $D'$ is a compressing disk for $\partial Q'_p$ in $W - J$. Since $\partial Q'_p - \text{int} G_p$ consists of disks we can isotop $D'$ so that $\partial D'$ is in $G_p$. If $D' \cap \partial Q_p \neq \partial D'$, then the excess intersections must lie in the interior of the disks comprising $\partial Q_p - \text{int} G$. We do surgery on $D'$ along these intersections to get a disk $D$ with $\partial D = \partial D'$ and $D \cap \partial Q_p = \partial D$. Since $Q_p = C_1^\#$ and $C_1^\#$ is incompressible in $W - J$ we have that $\partial D'$ bounds a disk in $\partial Q_p$ and hence bounds a disk in $\partial Q'_p$.

Thus $Q'_p$ has been obtained from $C_1$ by completely compressing $\partial C_1$ in $W$ with the compressions confined to $X_i - J$. We then rename $Q'_p$ as $C_1^\#$ and continue the induction on $i$ to finally get a $C_1^\#$ that is obtained from $C_1$ by completely compressing
∂C₁ in W − J with the compressions confined to C₂* − J. Note that ∂C₁ has also been completely compressed in U − J.

We next completely compress ∂C₂* in W − J to obtain C₂#. We may assume that the compressions are confined to C₃ − C₁#. Recycling our notation we let E₁, . . . , Eₚ be the sequence of compressing disks and H₁, . . . , Hₚ the sequence of handles. C₂* = Q₀, Q₁, . . . , Qₚ = C₂# is the sequence of 3-manifolds.

We now let D₁, . . . , Dₙ be the sequence of compressing disks involved in the complete compression of ∂C₃ in W − K and let C₃ = X₀, X₁, . . . , Xₙ = C₃* be the associated sequence of 3-manifolds. We may assume that the compressions are confined to C₄ − C₂*.

As in the previous step we replace the E_j by disks E_j' with ∂E_j' = ∂E_j by doing surgery on the E_j along their intersections with the D_i. This again results in the complete compression of ∂C₂* in W − J with the compressions confined to C₃* − C₁#.

It follows that we have completely compressed ∂C₂ in W − J and ∂C₂* in U − J.

We then continue this process to complete the proof. □

6. End reductions and planes

Let W be a Whitehead manifold. An embedded proper plane Π in W is trivial if some component of W − Π has closure R² × [0, ∞). W is R²-irreducible if every proper plane in W is trivial.

Theorem 6.1. Let W be a Whitehead manifold. Suppose Π is a non-trivial plane in W and J is a regular submanifold of W such that J ∩ Π = ∅. Then any open subset V of W which is end irreducible rel J in W can be isotoped so that V ∩ Π = ∅. In particular, any end reduction of W at J can be isotoped off Π.

Proof. By Lemma 2.1 there is a regular exhaustion {Cₙ} of V such that C₀ = J and ∂Cₙ is incompressible in W − C₀ for all n > 0. Let Wₙ = W − int Cₙ.

We first isotop ∂C₁ in W₀ so that it is in minimal general position with respect to Π. This is to be done via an ambient isotopy which is fixed on ∂W₀. Since ∂C₁ is incompressible in W₀ we must now have that Π ∩ ∂C₁ = ∅. Note that our isotopy may have moved all the Cₙ and Wₙ for n > 0.

We next isotop our new ∂C₂ in our new W₁ so that it is in minimal general position with respect to Π. This is to be done by an ambient isotopy which is fixed on ∂W₁. Since ∂C₂ is incompressible in W₁ we must now have that Π ∩ ∂C₂ = ∅. Again our isotopy has possibly moved all the Cₙ and Wₙ for n > 1.

We continue in this fashion to move each ∂Cₙ in turn off of Π. The points of each Cₙ are moved a finite number of times and thereafter remain fixed. Thus the restriction of this series of ambient isotopies to V converges to an isotopy of V in W which moves V off Π. (Note that since we have established no control over the points of W − V we cannot guarantee that this isotopy is ambient.) □
Theorem 6.2. Let $W$ be a Whitehead manifold. Suppose $\Pi$ is a non-trivial plane in $W$. Then any $\mathbb{R}^2$-irreducible open subset $V$ of $W$ which is end irreducible rel $J$ in $W$ for some regular submanifold $J$ of $W$ can be isotoped so that $V \cap \Pi = \emptyset$. In particular, any $\mathbb{R}^2$-irreducible end reduction of $W$ can be isotoped off $\Pi$.

Proof. By Lemma 2.4 $V$ is an end reduction of $W$ at a knot $\kappa$. Put $\kappa$ in minimal general position with respect to $\Pi$. If $\kappa \cap \Pi = \emptyset$, then we are done by Theorem 5.1. So assume that $\kappa \cap \Pi$ has $k > 0$ components. We will show that $V$ contains a non-trivial plane.

By Lemma 2.1 $V$ has a regular exhaustion $\{C_n\}$ such that $C_0$ is a regular neighborhood of $\kappa$ and $\partial C_n$ is incompressible in $W - C_0$ for all $n > 0$. Let $W_n = W - \text{int} C_n$.

$\Pi \cap C_0$ consists of $k$ disks. Let $\Pi_0 = \Pi \cap W_0$. Isotop $\Pi_0$ in $W_0$ rel $\partial \Pi_0$ so that $\Pi_0$ is in minimal general position with respect to $\partial C_1$. Then no component of $\Pi_0 \cap \partial C_1$ bounds a disk in $\Pi_0$. Every component of $\Pi_0 \cap \partial C_1$ which is innermost on $\Pi$ bounds a disk in $\Pi$ which lies in $C_1$ and contains at least one component of $\Pi \cap C_0$. Thus there are at most $k$ such components.

Let $\Pi_1 = \Pi_0 \cap W_1$. Isotop $\Pi_1$ in $W_1$ rel $\partial \Pi_1$ so that $\Pi_1$ is in minimal general position with respect to $\partial C_2$. Then no component of $\Pi_1 \cap \partial C_2$ bounds a disk in $\Pi_1$. Every component of $\Pi_1 \cap \partial C_2$ which is innermost on $\Pi$ bounds a disk in $\Pi$ which lies in $C_2$ and contains at least one of the disks bounded by an innermost component of $\Pi_0 \cap \partial C_1$. Thus there are at most the same number of innermost components.

We continue in this fashion by suitably isotoping $\Pi_{n+1} = \Pi_n \cap W_{n+1}$. There is an $n_0$ such that for all $n \geq n_0$ we have that $\Pi_n \cap \partial C_{n+1}$ and $\Pi_{n+1} \cap \partial C_{n+2}$ have the same number of components which are innermost on $\Pi$. Then the disk bounded by an innermost component of $\Pi_{n+1} \cap \partial C_{n+2}$ contains exactly one disk bounded by an innermost component of $\Pi_n \cap \partial C_{n+1}$. The union of each such family of concentric disks is a plane which by construction is proper in $V$. It is non-trivial in $V$ since otherwise one could use the copy of $\mathbb{R}^2 \times [0, \infty)$ which it splits off in $V$ to isotop $\kappa$ so as to reduce the number of components of $\kappa \cap \Pi$, a contradiction. \qed

7. Some gluing lemmas

Let $Q$ be a compact 3-manifold. Let $F$ be a compact surface in $\partial Q$, and let $G$ be a union of components of $\partial Q - \text{int} F$. The triple $(Q, F, G)$ has the halfdisk property if whenever $D$ is a proper disk in $Q$ such that $D \cap F$ and $D \cap G$ are each arcs and the union of these arcs is $\partial D$, then $\partial D = \partial D'$ for some disk $D'$ in $\partial Q$. The ordered triple $(Q, F, G)$ has the band property if whenever $D$ is a proper disk in $Q$ such that $D \cap F$ consists of two disjoint arcs which lie in the same component of $F$ and the remainder of $\partial D$ lies in $G$, then $\partial D = \partial D'$ for some disk $D'$ in $\partial Q$. Note that if $Q$ is $\partial$-irreducible, then $(Q, F, G)$ automatically has both properties.

Lemma 7.1. Let $Y$ be a compact 3-manifold. Let $S$ be a compact, 2-sided, proper surface in $Y$. Let $Y'$ be the 3-manifold obtained by splitting $Y$ along $S$. Let $S'$ be the surface in $\partial Y'$ homeomorphic to two copies of $S$ which are identified to obtain $Y$. Let
Z be a union of components of $\partial Y$ and $Z'$ the surface in $\partial Y'$ obtained by splitting $Z$ along $Z \cap S$. Suppose that

1. $Y'$ is irreducible,
2. $S'$ and $Z'$ are incompressible in $Y'$, and
3. $(Y', S', Z')$ has the halfdisk property.

Then

(a) $Y$ is irreducible, and
(b) $Z$ in incompressible in $Y$.

Proof. This is a standard argument which will be left to the reader. □

Lemma 7.2. Suppose $Y$, $S$, and $Z$ satisfy all the hypotheses of Lemma 5.1. Let $Z_0$ be a union of components of $Z$ and $Z_0'$ the surface in $\partial Y'$ obtained by splitting $Z_0'$ along $Z_0 \cap S$. Suppose that

4. every proper incompressible annulus in $Y'$ whose boundary lies in $(\text{int } S') \cup (\text{int } Z')$ either is parallel to an annulus in $\partial Y'$ or cobounds a compact submanifold of $Y'$ with an annulus in $\partial Y'$ which meets $Z_0'$.
5. $(Y', S', Z')$ has the band property,
6. each component of $S$ separates $Y$, and
7. no component of $S$ is an annulus.

Then

(c) every proper incompressible annulus in $Y$ whose boundary lies in $Z$ either is parallel to an annulus in $\partial Y'$ or cobounds a compact submanifold of $Y$ with an annulus in $Z_0$.

Proof. Let $A$ be a proper incompressible annulus in $Y$ such that $\partial A$ lies in $Z$. We assume that $A$ is in general position with respect to $S$ and that $A \cap S$ has a minimal number of components. We may assume that no component is a simple closed curve which bounds a disk in $A$ or is an arc whose boundary lies in one component of $\partial A$.

Suppose $A \cap S = \emptyset$. Then $A$ lies in $Y'$. If $A$ is parallel in $Y'$ to an annulus $A'$ in $\partial A'$, then $A'$ must lie in $\partial Y'$ since otherwise it would contain an annulus component of $S'$, and thus $S$ would have an annulus component. If $A \cup A' = \partial Q$, where $Q$ is a compact submanifold of $Y'$ and $A'$ is an annulus in $\partial Y'$ which meets $Z_0$, then $A'$ must lie in $Z_0'$ since otherwise it would contain an annulus component of $S'$. Thus $Q$ must lie in $Y$.

Suppose $A \cap S$ contains a simple closed curve. Then there is such a curve $\alpha$ which cobounds an annulus $A_0$ in $A$ with a component $\beta$ of $\partial A$. No component of $A \cap S$ is an arc. We have that $\alpha$ lies in $\text{int } S'$ and $\beta$ lies in $\text{int } Z'$, so $A_0$ cobounds a compact submanifold $Q$ of $Y'$ with an annulus $A_0'$ in $\partial Y'$. Since $S$ has no annulus components $A_0'$ must be the union of an annulus $A_0''$ in $Z'$ and an annulus $A_1$ in $S'$ which meet in a component $\beta'$ of $Z' \cap S'$. If $A_0$ is parallel to $A_0''$ across $Q$, then an isotopy of $A$ in $Y$ which moves $A_0$ it across $Q$ and then past $A_1$ removes at least $\alpha$ from the intersection, thereby contradicting minimality. If $A_0$ is not parallel to $A_0''$ across $Q$, then $A_0''$ must lie in $Z_0'$. $A_1$ meets $A$ in $\alpha$ and possibly other components.
Suppose first that \( A_1 \cap A = \alpha \). Now \( \alpha \) splits \( A \) into two subannuli, one of which is \( A_0 \). Let \( A' \) be the union of the other subannulus and \( A_1 \). \( A' \) is a proper incompressible annulus in \( Y \) which can be isotoped so that it meets \( S \) in one fewer component than \( A \) did. Apply induction to the number of components of the intersection. Then \( A' \) cobounds a compact submanifold \( Q' \) of \( Y \) with an annulus \( A'' \) in \( \partial Y \). If \( A'' \) does not contain \( A_0'' \), then \( Q \cap Q' = A_1 \) and \( A'' \) lies in \( Z_0 \). In this case \( Q \cup Q' \) is the desired compact submanifold of \( Y \) cobounded by \( A \) and the annulus \( A_0'' \cup A'' \). If \( A'' \) contains \( A_0'' \), then \( Q' \) contains \( Q \). In this case the desired compact submanifold is the closure of \( Y' \) of \( Q' - Q \). It is cobounded by \( A \) and the annulus in \( Z_0 \) which is the closure of \( A'' - A_0'' \).

Suppose now that \( A_1 \cap A \) has some components other than \( \alpha \). Then there must be some annulus components of \( A \cap Q \) other than \( A_0 \). Suppose there is such a component \( A_2 \) with \( \partial A_2 \) in \( \text{int} \ A_1 \). Then \( A_2 \) is a proper incompressible annulus in \( Y' \) with \( \partial A_2 \) in \( \text{int} \ S' \). So \( A_2 \) cobounds a compact submanifold \( Q' \) of \( Y' \) with an annulus \( A_2'' \) in \( \partial Y \). \( A_2'' \) must lie in \( A_1 \), for otherwise \( S \) must have an annulus component. Thus \( A_2'' \) cannot meet \( Z_0' \), and so it is parallel to \( A_2 \) across \( Q' \). Hence an isotopy of \( A \) could be performed which would reduce the number of intersection curves, contradicting minimality. It follows that there is only one component \( A_2 \) of \( A \cap Q \). It joins a curve \( \alpha' \) in \( \text{int} \ A_1 \) to a curve \( \beta'' \) in \( \text{int} A_0' \). There is a subannulus \( A_2'' \) of \( A_1 \) with \( \partial A_2'' = \beta'' \cup \alpha' \). \( A_2'' \cap A = \alpha' \). Now \( \alpha' \) splits \( A \) into two subannuli, one of which is \( A_2' \). Let \( A' \) be the union of \( A_2' \) and the other subannulus. \( A' \) is a proper incompressible annulus in \( Y \) with \( \partial A' = \beta \cup \beta' \), which lies in \( Z_0 \). \( A' \cap S \) has one fewer component than did \( A \cap S \). We apply induction to conclude that \( A' \) satisfies (7). Let \( Q' \) be the compact submanifold of \( Y \) cobounded by \( A' \) and an annulus \( A'' \) in \( \partial Y \). Now \( A'' \) either equals \( A_0' \) or the component of \( Z_0 \) containing it is a torus which is the union of these two annuli. In the first case \( Q' \) contains \( Q \), and \( A_2 \) is a proper incompressible annulus in \( Q' \) which splits \( Q' \) into two submanifolds, one of which is the desired submanifold of \( Y \) bounded by \( A \cup A_0' \). In the second case \( Q' \) meets \( Q \) in \( A_0' \cup A_2' \). \( A_2 \) splits \( Q \) into two components. The union of \( Q' \) with the component which meets it in \( A_2' \) is the desired compact submanifold.

Suppose some component \( \alpha \) of \( A \cap S \) is an arc. Then it must meet both components of \( \partial A \). By (6) \( A \cap S \) must have an even number of components, and there must be a component \( \Delta \) of \( A \cap Y' \) which meets some component of \( S' \) in two disjoint arcs and has the remainder of its boundary in \( Z' \). By (5) \( \partial \Delta = \partial \Delta' \) for some disk \( \Delta' \) in \( \partial Y \). \( \Delta \cup \Delta' \) bounds a 3-ball in \( Y' \), and an isotopy of \( \Delta \) across this 3-ball reduces the number of components, again contradicting minimality. So this case cannot occur. □

**Lemma 7.3.** Suppose \( Y, S, Z, \) and \( Z_0 \) satisfy all the hypotheses of Lemmas 7.1 and 7.2. Suppose that

8. every incompressible torus in \( Y' \) bounds a compact submanifold of \( Y' \), and
9. no component of \( \partial Y \) is a torus.

Then
(d) every incompressible torus in $Y$ is bounds a compact submanifold of $Y$.

Proof. Let $T$ be an incompressible torus in $Y$. Put $T$ in general position with respect to $S$ so that $T \cap S$ has a minimal number of components. We may assume that no component bounds a disk in either surface.

Suppose $T \cap S = \emptyset$. Then $T$ lies in $Y'$, so $T$ bounds a compact submanifold $Q$ of $Y'$. Since $Q$ lies in $Y$ we are done.

Suppose $T \cap S \neq \emptyset$. Then $S$ splits $T$ into annuli which lie in $Y'$. Since each component of $S$ separates $Y$ there is such an annulus $A$ with $\partial A$ in a single component of $S'$. Then there is an annulus $A'$ in $\partial Y'$ such that $A \cup A' = \partial Q$ for a compact submanifold $Q$ of $Y'$.

Assume that $A'$ lies in $S'$. Then $A'$ does not meet $Z_0$, and so $A$ is parallel across $Q$ to $A'$. Thus there is an isotopy of $T$ in $Y$ which removes at least $\partial A$ from the intersection, contradicting minimality. Hence $A'$ does not lie in $S'$. Since $S$ has no annulus components $A'$ must consist of an annulus $A_1$ in $\partial Y$ together with a collar $C$ in $S'$ on two components of $\partial S'$. Let $A_2$ be the union of $C$ and $T-\text{int} A$. Note that $\partial A_2 = \partial A_1$. Now $A_2$ is a proper incompressible annulus in $Y$ which can be isotoped to have boundary disjoint from $\partial S$. Therefore $A_2$ cobounds a submanifold $Q'$ of $Y$ with an annulus $A'_2$ in $\partial Y$. If $A'_2 = A_1$, then $Q'$ contains $Q$ and $A$ is a proper annulus in $Q'$ which splits it into two components, one of which is the desired submanifold having boundary $T$. If $A'_2 \neq A_1$, then the union of these two annuli is a torus component of $\partial Y'$ contrary to the hypotheses on $Y$. \hfill $\square$

8. Some building blocks

In this section we define certain compact 3-manifolds which will be used to construct our examples and establish some of their properties which will be used in conjunction with the gluing lemmas of the previous section.

We first introduce some general notation. Let $\Gamma$ be a graph in a 3-manifold $Q$. We assume that $\Gamma \cup \partial Q$ is either empty or consists of vertices of order one. We denote a regular neighborhood of $\Gamma$ in $Q$ by $N(\Gamma, Q)$. The exterior of $\Gamma$ in $Q$ is the closure of the complement of $N(\Gamma, Q)$ in $Q$ and is denoted by $X(\Gamma, Q)$. The lateral surface determined by $\Gamma$ is the set $S(\Gamma, Q) = N(\Gamma, Q) \cap X(\Gamma, Q)$. The $Q$ will be suppressed when its identity is clear from the context. If $\gamma$ is an edge of $\Gamma$, then a simple closed curve in $S(\Gamma, Q)$ which bounds a disk in $N(\Gamma, Q)$ which meets $\gamma$ transversely in a single point is called a meridian of $\gamma$.

Let $P = D \times [0, 2]$, where $D$ is a closed disk. Set $L = D \times [0, 1]$, $R = D \times [1, 2]$, and $D_j = D \times \{j\}$ for $j = 0, 1, 2$. Attach a 1-handle $H$ to $L$ along $\partial D \times (0, 1)$ to obtain a solid torus $L \cup H$. Set $G = \partial (L \cup H) - \text{int} (D_0 \cup D_1)$ and $O = \partial R - \text{int} (D_1 \cup D_2)$. Let $J = P \cup H$. Let $J$ be the genus two handlebody obtained from $J$ by identifying $D_0$ and $D_2$. Let $P^\#$ be the solid torus in $J^\#$ which is the image of $P$ under the identification. With the exception of $J$ and $J^\#$ and of $P$ and $P^\#$ we will usually use
the same notation for subsets of $J$ and their images in $J^\#$, relying on the context for which is meant. Thus we regard $J^\#$ as $P^\# \cup H$.

We next let $\beta$ and $\varepsilon$ be the two disjoint arcs in the solid torus $L \cup H$ shown in Figure 1. Each of them runs from $\text{int} D_0$ to $\text{int} D_1$. We have that $\beta$ meets $H$ in a Whitehead clasp while $\varepsilon$ is a product arc in $L$. This configuration is called an eyebolt.

We then let $\alpha$, $\gamma$, $\delta$, $\zeta$, and $\rho$ be the arcs in the 3-ball $R$ shown in Figure 2. We require that $\gamma \cap D_1 = \beta \cap D_1$, that $\delta \cap D_1 = \varepsilon \cap D_1$, and that $\alpha \cap D_2$ and $\zeta \cap D_2$, respectively, are identified with $\beta \cap D_0$ and $\varepsilon \cap D_0$ in $J^\#$. We have that $\alpha$, $\zeta$, and $\rho$ meet in a common endpoint while $\gamma$, $\delta$, and $\rho$ also meet in a common endpoint. Except as indicated all the arcs are disjoint. We have that $\alpha \cup \zeta$ and $\gamma \cup \delta$ are linked trefoil knotted arcs whose union is a tangle $\tau$ in $R$. We have that $\delta \cup \rho \cup \zeta$ is a product arc in $R$. Let $\rho' = \rho \cap X(\tau, R)$. We may assume that $\rho$ is the union of $\rho'$ with two disjoint arcs in $N(\tau, R)$. This configuration is called a textitjunction.

In $J^\#$ set $\lambda = \delta \cup \varepsilon \cup \zeta$, $\eta = \alpha \cup \beta \cup \gamma$, $\kappa = \lambda \cup \eta$, and $\mu = \kappa \cup \rho$ as in Figure 3.

**Lemma 8.1.** Let $Y = X(\beta \cup \varepsilon, L \cup H)$, and let $F_i = Y \cap D_i$ for $i = 0, 1$. Then

1. $Y$ is irreducible,
2. $F_i$ is incompressible in $Y$,
3. $S(\varepsilon, L \cup H)$ is incompressible in $Y$,
4. $S(\beta, L \cup H)$ is incompressible in $Y$,
5. $(Y, F_0 \cup F_1, \partial Y - \text{int} (F_0 \cup F_1))$ has the halfdisk property,
6. $(Y, F_0 \cup F_1, \partial Y - \text{int} (F_0 \cup F_1))$ has the band property.
7. Every proper incompressible annulus in $Y$ whose boundary lies in $(\text{int} (F_0 \cup F_1) \cup (\partial Y - (F_0 \cup F_1))$ is parallel to an annulus in $\partial Y$, and
8. $Y$ contains no incompressible tori.
Proof. For \( i = 0,1 \) attach 3-balls \( B_i \) to \( L \cup H \) so that \( B_i \cap (L \cup H) = D_i \); let \( D_i' = \partial B_i - \text{int} D_i \). Let \( \omega_i \) be a proper unknotted arc in \( B_i \) joining the points of \( (\beta \cup \varepsilon) \cap D_i \). Then \( \beta \cup \varepsilon \cup \omega_0 \cup \omega_1 \) is a Whitehead curve in the solid torus \( K = L \cup H \cup B_0 \cup B_1 \). Let \( X \) be the exterior of this curve in this solid torus. \( X \) is irreducible, \( \partial \)-irreducible, anannular, and atoroidal \([11]\).

(0) and (8) hold because \( Y \) is a handlebody.

(1) holds for homological reasons.

(2) Suppose \( D \) is a compressing disk for \( G \) in \( Y \). \( \partial D = \partial D' \) for a disk \( D' \) on \( \partial K \). If \( D' \) contains, say, \( D_0' \) but not \( D_1' \), then \( \partial D \) is parallel in \( G \) to \( \partial D_0 \), and so \( F_0 \) is compressible in \( Y \), contradicting (1). Assume \( D' \) contains \( D_0' \) and \( D_1' \). Then \( D \) splits \( K \) into a solid torus and a 3-ball \( B' \) such that \( B_0 \) and \( B_1 \) lie in \( B' \). Thus \( \beta \cup \varepsilon \cup \omega_0 \cup \omega_1 \) lies in \( B' \), contradicting the irreducibility of \( X \). Thus we have that \( D' \) lies in \( G \).

(3) \( S(\varepsilon, L \cup H) \) is incompressible for homological reasons.

(4) \( S(\beta, L \cup H) \) is incompressible for homological reasons.

(5) Suppose \( D \) is a proper disk in \( Y \) such that \( D \cap (F_0 \cup F_1) \) is an arc \( \theta \) in, say, \( F_0 \), and \( \partial D - \text{int} \theta = D \cap (\partial Y - \text{int} (F_0 \cup F_1)) \) is an arc \( \varphi \). If \( \varphi \) lies in one of the annuli \( S(\beta, L \cup H) \) or \( S(\varepsilon, L \cup H) \), then \( D \) can be isotoped so that \( \partial D \) lies in \( F_0 \), and so the result follows from (1). So assume that \( \varphi \) lies in \( G \). By (2) we may assume that \( \theta \) separates \( \beta \cap D_0 \) from \( \varepsilon \cap D_0 \). There is a disk \( E \) in \( B_0 \) which meets \( \omega_0 \) transversely in a single point and meets \( D_0 \) in \( \theta \). Then \( D \cup E \) is a proper disk in the solid torus \( K \) which meets the Whitehead curve transversely in a single point, thereby contradicting the anannularity of \( X \).
(6) Suppose $D$ is a proper disk in $Y$ such that $D \cap (F_0 \cup F_1)$ consists of two arcs $\theta'$, $\theta''$ in, say, $F_0$ and such that $\partial D - \text{int} (\theta' \cup \theta'') = D \cap (\partial Y - \text{int} (F_0 \cup F_1))$ consists of the two arcs $\varphi'$ and $\varphi''$. If $\theta'$ or $\theta''$ is $\partial$-parallel in $F_0$ or $\varphi'$ or $\varphi''$ is $\partial$-parallel in $\partial Y - \text{int} (F_0 \cup F_1)$, then there is an isotopy of $D$ after which it meets $F_0$ and $\partial Y - \text{int} (F_0 \cup F_1)$ each in a single arc, and the union of these arcs is $\partial D$. Then by (5) we are done. So we may assume that none of $\theta'$, $\theta''$, $\varphi'$, or $\varphi''$ are $\partial$-parallel. In particular this implies that $\varphi' \cup \varphi''$ lies in $G$ and that $\theta'$ and $\theta''$ are parallel arcs in $F_0$ which join $\partial D_0$ to itself and separate $\beta \cap D_0$ from $\varepsilon \cap D_0$.

Put $D$ in general position with respect to $L \cap H \cap Y$ so that the intersection has a minimal number of components. $L \cap H \cap Y$ is incompressible in $X(\beta \cap H, H)$ by [11] and is incompressible in $X((\beta \cup \varepsilon) \cap L, L)$ for homological reasons. These two spaces are irreducible since they are handlebodies. Thus $D \cap L \cap H \cap Y$ has no simple closed curve components.
If $\nu$ is an outermost arc on $D$ of this intersection such that $\partial \nu$ lies in, say, $\varphi'$, then $\partial \nu = \partial \nu'$ for an arc $\nu'$ in $\text{int} \varphi'$ such that $\nu \cup \nu' = \partial \Delta$ for a disk $\Delta$ in $D$.

Suppose $\nu'$ lies in the annulus $G \cap H$. Then it is $\partial$-parallel in $G \cap H$, and so by the incompressibility of $L \cap H \cap Y$ in $X(\beta \cap H, H)$ we have that $\nu$ is $\partial$-parallel in $L \cap H \cap Y$. It follows that $\nu$ could have been removed by an isotopy of $D$, thereby contradicting minimality.

Suppose $\nu'$ lies in the disk with three holes $G \cap L$. Since $\Delta$ misses $\varepsilon$ we must have that $\partial \Delta = \partial \Delta'$ for a disk $\Delta'$ in the annulus $\partial L - \text{int} (D_0 \cup D_1)$. Let $\Delta''$ be the component of $H \cap L$ which does not contain $\nu$. Then $\Delta'$ either contains $\Delta''$ or is disjoint from it. In the first case $\Delta$ would separate $\Delta''$ from $D_0 \cup D_1$, which is impossible since they are joined by a subarc of $\beta$. Thus $\Delta'$ is disjoint from $\Delta''$.

It follows that $\nu'$ is $\partial$-parallel in $G \cap L$. By the incompressibility of $L \cap H \cap Y$ in $X((\beta \cup \varepsilon) \cap L, L)$ we have that $\nu$ is $\partial$-parallel in $L \cap H \cap Y$. Again $\nu$ could have been removed by an isotopy of $D$, contradicting minimality.

So every component of $D \cap L \cap H \cap Y$ must join $\varphi'$ to $\varphi''$. Since $L \cap H \cap Y$ separates $Y$ there must be an even number of such components. Let $\nu_0$ and $\nu_1$ be adjacent such components which are parallel in $D$ across a disk $\Delta$ which meets $\varphi'$ and $\varphi''$ in arcs $\nu'$ and $\nu''$, respectively. If $D \cap H \neq \emptyset$, then there must be such a $\Delta$ lying in $X(\beta \cap H, H)$. Suppose that $\nu_0$ and $\nu_1$ lie in the same component of $L \cap H \cap Y$, and so $\nu'$ and $\nu''$ are $\partial$-parallel in $G \cup H$. It follows from the incompressibility of $L \cap H \cap Y$ and the irreducibility of $X(\beta \cap H, H)$ that there is an isotopy of $D$ which removes $\nu_0 \cup \nu_1$ from the intersection, again contradicting minimality. Therefore for every such $\Delta$ we have that $\nu_0$ and $\nu_1$ lie in different components of $L \cap H \cap Y$. By $\Delta$ is $\partial$-parallel in this space. Let $\Delta_0$ be the component of $D \cap L$ containing $\theta'$. There is a $\Delta$ as above which meets $\Delta_0$ in the arc $\nu_0$. Since $\nu_0$ is $\partial$-parallel in $L \cap H \cap Y$ we can isotop $\Delta_0$ in $L \cap Y$ to a disk $\Delta'_0$ which meets $D_0$ in $\theta'$ and is disjoint from $L \cap H$ and $D_1$. $\Delta'_0$ splits $L$ into two 3-balls $B'$ and $B''$ with $\beta \cap D_0$ contained in, say, $B'$ and $\varepsilon$ contained in $B''$. The component of $\beta \cap L$ meeting $D_0$ must lie in $B'$. Thus the component of $L \cap H$ meeting this arc must lie in $B'$. Hence so must the component of $\beta \cap L$ joining the two components of $L \cap H$. Hence so must the other component of $L \cap H$. Hence so must $D_1$. Hence so must $\varepsilon$, a contradiction.

Thus we now have that $D \cap H = \emptyset$. Now $\partial D = \partial D' = \partial D''$, where $D'$ and $D''$ are disks on $\partial L$ whose union is $\partial L$. We may assume that $D''$ contains $D_0 \cap (\beta \cup \varepsilon)$. Then for homological reasons we must have that $D''$ also contains $\beta \cap L \cap H$ and $D_1$. Thus $D'$ lies in $\partial Y$, and we are done.

(7) Let $A$ be an annulus with $\partial A$ in $(\text{int} (F_0 \cup F_1) \cup (\partial Y - (F_0 \cup F_1)))$.

Note that $H_1(G)$ has basis consisting of classes $d$, $\ell$, and $m$ represented by, respectively, $\partial D_0$, and a longitude and a meridian of the solid torus $L \cup H$. $H_1(Y)$ has basis consisting of classes $b$, $e$, and $\ell_0$ represented by, respectively, by meridians of $\beta$ and $\varepsilon$ and a longitude of $L \cup H$. These curves and their orientations can be chosen so that under the inclusion induced map $d \to b + e$, $\ell \to \ell_0$, and $m \to 0$. It follows that if
\( \theta' \) is a meridian of \( \beta \) or \( \epsilon \), respectively, then \( \theta'' \) must be also. Hence in this case we may assume that \( \partial A \) lies in a single component of \( S(\beta \cup \epsilon, L \cup H) \). Since \( \partial A \) consists of meridians of the Whitehead curve \( A \) is incompressible in \( X \) and thus is parallel to an annulus \( A' \) in \( \partial X \). Thus \( A \) splits \( X \) into a solid torus containing \( A' \) and another 3-manifold containing \( \partial K \). It follows that \( A' \) and the solid torus lie in \( Y \), so we are done in this case.

Now suppose that \( \partial A \) lies in \( G \).

Let \( F \) be the disk in Figure 1. Let \( F' = F \cap Y \). Put \( A \) in general position with respect to \( F' \) so that \( A \cap F' \) has a minimal number of components. Then this intersection has no simple closed curve components.

Suppose some component \( \nu \) of \( A \cap F' \) is \( \partial \)-parallel in \( A \). We may assume that \( \nu \) cuts off an outermost disk \( D \) on \( A \) and that \( \partial D = \nu \cup \nu' \) for an arc \( \nu' \) in \( \theta' \). Then \( \nu \) cuts off a disk \( D' \) in \( F' \). Since \( G \) is incompressible in \( Y \) we have that \( \partial(D \cup D') = \partial D'' \) for a disk \( D'' \) in \( G \). An isotopy of \( A \) which moves \( D \) across the 3-ball bounded by \( D \cup D' \cup D'' \) and then past \( D' \) removes at least \( \nu \) from \( A \cap F' \), thereby contradicting minimality.

Thus every component \( \nu \) of \( A \cap F' \) is a spanning arc on \( A \). Suppose \( \nu \) is outermost on \( F' \), cutting off an outermost disk \( D \) on \( F' \). Performing a boundary compression on \( A \) along \( D \) yields a disk \( D' \) with \( \partial D' \) in \( G \). Then \( \partial D' = \partial D'' \) for a disk \( D'' \) in \( G \). Let \( B \) be the 3-ball in \( Y \) bounded by \( D' \cup D'' \). \( B \) cannot contain \( D \) since this would make \( A \) compressible in \( Y \). It follows that the union of \( B \) and an appropriate regular neighborhood of \( D \) is a solid torus across which \( A \) is parallel to an annulus in \( \partial Y \).

Thus we may assume that \( A \cap F' = \emptyset \).

Suppose \( A \) is compressible in \( X \). Since \( X \) is \( \partial \)-irreducible each component of \( \partial A \) must bound a disk in \( \partial X \), say \( \theta' = \partial \Delta' \) and \( \theta'' = \partial \Delta'' \). Since \( G \) is incompressible in \( Y \) we must have that \( \Delta' \) and \( \Delta'' \) each contains at least one of the disks \( D_0' \) and \( D_1' \). We may assume that \( \Delta' \) contains \( D_0' \) and that \( \Delta'' \) either contains \( \Delta' \) or is disjoint from it. If \( \Delta' \) does not contain \( D_1' \), then \( \Delta' \cap F' \neq \emptyset \), hence \( A \cap F' \neq \emptyset \), a contradiction. So \( (D_0' \cup D_1') \subseteq \Delta' \subseteq \Delta'' \). Therefore \( \Delta' \) and \( \Delta'' \) are isotopic to concentric regular neighborhoods of \( D_0' \cup D_1' \cup (F \cap G) \) in \( \partial K \).

Put \( A \) in general position with respect to \( L \cap H \cap Y \) so that the intersection has a minimal number of components among annuli in \( Y \) with \( \partial A = \partial \Delta' \cup \Delta'' \). Then each component is a simple closed curve which bounds a disk on neither \( A \) nor \( L \cap H \cap Y \). So there is an annulus component \( A' \) of \( A \cap Y \cap L \) with \( \partial A' = \theta' \cup \theta \), where \( \theta \) is a curve in \( L \cap H \cap Y \). It is easily checked that this is homologically impossible.

Thus we have that \( A \) lies in \( Y \cap L \). Let \( C \) be the disk in Figure 1. Let \( C' = C \cap Y \cap L \). \( A \) must meet \( C' \). Assume, as usual, general position and minimality. Each component is an arc. Suppose \( \nu \) is an outermost \( \partial \)-parallel arc on \( A \) cutting off a disk \( D \) on \( A \). Then \( \nu \) cuts off a disk \( D' \) on \( C' \). Since \( G \) is incompressible \( \partial(D \cup D') = \partial D'' \) for a disk \( D'' \) on \( G \). An isotopy of \( D \) across the 3-ball bounded by \( D \cup D' \cup D'' \) and past \( D' \) removes at least \( \nu \) from the intersection. Hence we may assume that all components
are spanning arcs of \( A \). Let \( \nu \) be a component which is outermost on \( C' \), cutting off a disk \( D \) on \( C' \) with \( \partial D = \nu \cup \nu' \) for an arc \( \nu' \) in \( C' \cap G \). Performing a boundary compression on \( A \) along \( D \) yields a disk \( D' \) with \( \partial D' \) in \( G \). \( \partial D' = \partial D'' \) for a disk \( D'' \) in \( G \). The 3-ball \( B \) bounded by \( D' \cup D'' \) cannot contain \( A \) as this would make \( A \) compressible in \( Y \). So the union of \( B \) and an appropriate regular neighborhood of \( D \) is a solid torus in \( Y \) across which \( A \) is parallel to an annulus in \( \partial Y \).

Finally, suppose that \( A \) is incompressible in \( Y \). Let \( A' \) be the annulus in \( \partial X \) to which \( A \) is parallel. Suppose, say, \( D_0' \) lies in \( A' \). Since \( A \cap D_0 = \emptyset \) we have that \( B_0 \) lies in the solid torus bounded by \( A \cup A' \). It follows that \( \partial X - \partial K \) lies in this solid torus, which is impossible. \( \square \)

The configuration of \( \beta \cup \gamma \cup \delta \cup \epsilon \) in \( J \) is called a right hitch. There is an obvious twin configuration called a left hitch.

**Lemma 8.2.** Let \( Y = X(\beta \cup \gamma \cup \delta \cup \epsilon, J) \). Then

1. \( Y \) is irreducible,
2. \( F_0 \) is incompressible in \( Y \),
3. \( \partial J - \text{int} D_0 \) is incompressible in \( Y \),
4. \( S(\beta \cup \gamma \cup \delta \cup \epsilon, J) \) is incompressible in \( Y \),
5. \( (Y, F_0, \partial Y - \text{int} F_0) \) has the halfdisk property,
6. \( (Y, F_0, \partial Y - \text{int} F_0) \) has the band property,
7. every proper incompressible annulus in \( Y \) whose boundary misses \( \partial F_0 \) is either \( \partial \)-parallel or is isotopic to \( F_1 \cup O \cup D_2 \), and
8. every incompressible torus in \( Y \) bounds a compact submanifold of \( Y \).

**Proof.** (0) Attach a 3-ball \( B_0 \) and unknotted arc \( \omega_0 \) to \( P \cup H \) along \( D_0 \) as in the proof of Lemma 6.1. The result is a solid torus \( K \) containing a Whitehead curve in which one has locally tied a trefoil knot. The exterior \( X \) of this curve is homeomorphic to the union of the Whitehead link exterior and a trefoil knot exterior along an annulus which is incompressible in both. This implies that \( X \) is irreducible and \( \partial \)-irreducible. So any 2-sphere in \( Y \) must bound a 3-ball in \( X \). Since \( X(\omega_0, B_0) \) has torus boundary this 3-ball must lie in \( Y \).

(1) Let \( D \) be a compressing disk for \( F_0 \). For homological reasons \( \partial D \) must be isotopic in \( F_0 \) to \( \partial D_0 \). Since an arc of \( \beta \cap L \) joins \( D_0 \) to \( L \cap H \) we must have that \( D \) intersects \( H \). Suppose \( \theta \) is a component of \( D \cap L \cap H \cap Y \) which bounds an innermost disk \( D' \) on \( D \). For homological reasons \( D' \) cannot lie in \( P \cap Y \). So \( D' \) lies in \( H \cap Y \). By [8] it is parallel to a disk in \( L \cap H \cap Y \), so can be removed by an isotopy. Continuing in this fashion we can remove all intersections, contradicting the fact that \( D \) must meet \( H \).

(2) Let \( D \) be a compressing disk for \( \partial J - \text{int} D_0 \). Since \( X \) is \( \partial \)-irreducible \( \partial D = \partial D' \) for a disk \( D' \) in \( \partial X \). Let \( D_0' = \partial B_0 - \text{int} D_0 \). If \( D' \) contains \( D_0' \), then \( \partial D \) is isotopic in \( \partial Y \) to \( \partial D_0 \), contradicting (1). Thus \( D' \) lies in \( \partial Y \), and we are done.

(3) This holds for homological reasons.
(4) Suppose $D$ is a proper disk in $Y$ which meets $F_0$ in an arc $\theta$ and $\partial Y - \mathrm{int} F_0$ in an arc $\varphi$. By (1), (2), and (3) we may assume that neither arc is $\partial$-parallel. It follows that either $\theta$ joins $\partial D_0$ to itself and separates $\beta \cap D_0$ from $\epsilon \cap D_0$ or $\theta$ joins the two components of $\partial F_0 - \partial D_0$.

In the first case attach $B_0$ and $\omega_0$ as in (2). As in the proof of Lemma 7.1 (5) we choose a disk $E$ in $B_0$ which meets $\omega_0$ transversely in a single point and meets $D_0$ in $\theta$. Then $D \cup E$ is a proper disk in the solid torus $B_0 \cup J$ which meets a (locally knotted) Whitehead curve transversely in a single point, contradicting the fact that the corresponding link has linking number zero.

In the second case we note that $\partial D$ can be regarded as a knot in $\mathbb{R}^3$ which is both a trivial knot and a trefoil knot, so this cannot occur.

(5) Suppose $D$ is a proper disk in $Y$ such that $D \cap F_0$ consists of two arcs $\theta'$ and $\theta''$. Let $\varphi'$ and $\varphi''$ be the components of $\partial D - \mathrm{int} (\theta' \cup \theta'')$. By (4) we may assume that none of these arcs are $\partial$-parallel. Then $\theta'$ and $\theta''$ must be parallel, and they either join $\partial D_0$ to itself, separating $\beta \cap D_1$ from $\epsilon \cap D_0$, or they join the two components of $\partial F_0 - \partial D_0$.

In the first case put $D$ in general position with respect to $L \cap H \cap Y$ so that the intersection has a minimal number of components. $L \cap H \cap Y$ is incompressible in $H \cap Y$ by [11] and in $P \cap Y$ for homological reasons. $H \cap Y$ is irreducible since it is a handlebody. $P \cap Y$ is irreducible since it is the union of a handlebody and a knot exterior along an annulus which is incompressible in both. Thus the intersection has no simple closed curve components.

Let $\nu$ be an outermost arc on $D$ cutting off a disk $\Delta$ with $\partial \Delta = \nu \cup \nu'$, where $\nu'$ is an arc in, say, $\varphi'$. If $\nu'$ lies in $G \cap H$, then $D$ can be isotoped to remove $\nu$ as in the proof of Lemma 7.1 (6), contradicting minimality. So $\nu'$ must lie in the disk with two holes $\partial P - \mathrm{int} (D_0 \cup (L \cap H))$. Suppose $\nu'$ separates $\partial D_0$ from the boundary component of this surface which does not meet $\nu'$. Then $\Delta$ separates $D_0$ in $P$ from the corresponding component of $L \cap H$. But this is impossible since $D_0$ is connected to each component of $L \cap H$ by arcs in $P$ which miss $\Delta$. Thus $\nu'$ must be $\partial$-parallel in our disk with two holes. $D$ can therefore be isotoped to remove $\nu$, a contradiction.

So, each component of the intersection must join $\varphi'$ to $\varphi''$. Let $\nu_0$ and $\nu_1$ be adjacent such components which are parallel in $D$ across a disk $\Delta$ which meets $\varphi'$ and $\varphi''$ in arcs $\nu'$ and $\nu''$, respectively. If $D \cap H \neq \emptyset$, then there must be such a disk $\Delta$ lying in $H \cap Y$. If $\nu_0$ and $\nu_1$ lie in the same component of $L \cap H \cap Y$, then as in the proof of Lemma 7.1 (6) we can isotop $D$ to remove $\nu_0 \cup \nu_1$, contradicting minimality. Thus $\nu_0$ and $\nu_1$ must lie in different components, and so by [11] $\Delta$ is $\partial$-parallel in $H \cap Y$. We let $\Delta_0$ be the component of $D \cap P$ containing $\theta'$ and $\Delta$ the component of $D \cap H$ meeting $\Delta_0$ in the arc $\nu_0$. Since $\nu_0$ is $\partial$-parallel in $L \cap H \cap Y$ we can isotop $\Delta_0$ to a disk $\Delta'_0$ which meets $F_0$ in $\theta'$ and is disjoint from $L \cap H$. $\Delta'_0$ splits $P$ into two 3-balls $B'$ and $B''$ with $\beta \cap D_0$ in $B'$ and $\epsilon \cap D_0$ in $B''$. The component of $L \cap H$ joined to $\beta \cap D_0$ by a subarc of $\beta$ must lie in $B'$. The other component of $L \cap H$ is joined to
this one by another subarc of $\beta$, so it must lie in $B'$. This is joined to $\epsilon \cap D_0$ by the union of a third subarc of $\beta$ with $\gamma$, $\delta$, and $\epsilon$, so $\epsilon \cap D_0$ must lie in $B'$, a contradiction.

Thus we have that $D \cap H = \emptyset$. Now $\partial D - \partial D' = \partial D''$ for disks $D'$ and $D''$ on $\partial P$ whose union is $\partial P$. We may assume that $D''$ contains $D_0 \cap (\beta \cup \epsilon)$. By following subarcs of $\beta$ as above we get that $D''$ also contains $L \cap H$. Thus $D'$ lies in $\partial Y$, and we are done in this case.

Recall that in the second case $\theta'$ and $\theta''$ join the two components of $\partial F_0 - \partial D_0$. Then $\partial D$ can be regarded as a knot in $\mathbb{R}^3$ which is both a trivial knot and a 2-strand cable of a trefoil knot, so this case cannot occur.

(6) Let $A$ be a proper incompressible annulus in $Y$ such that $(\partial A) \cap (\partial F_0) = \emptyset$. Denote the components of $\partial A$ by $\theta'$ and $\theta''$. Isotop $A$ so that $(\partial A) \cap F_0 = \emptyset$. Attach $B_0$ and $\omega'$ as in (2) to obtain a locally unknotted Whitehead curve in a solid torus $K$. Let $X$ be the exterior of this curve in $K$.

Suppose $A$ is compressible in $X$. Since $X$ is $\partial$-irreducible each component of $\partial A$ bounds a disk on $\partial X$. Since $A$ is incompressible in $Y$ these disks must contain $D'_0$. Thus each component of $\partial A$ is isotopic in $\partial Y$ to $\partial D_0$. Isotop $A$ so that $\partial A$ lies in $F_0$.

Put $A$ in general position with respect to $L \cap H \cap Y'$ so that the intersection has a minimal number of components. By the incompressibility of $L \cap H \cap Y$ in $Y$ each component is a simple closed curve which does not bound a disk on either surface. If $A \cap L \cap H \cap Y' \neq \emptyset$, then there is a component $\theta$ of the intersection such that $\theta \cup \theta' = \partial A_0$ for a component $A_0$ of $A \cap P \cap Y$. However, it is easily checked that this is homologically impossible. Thus $A \cap H = \emptyset$.

Recall the disk $C$ in Figure 1. It intersects $Y$ in a disk $C'$. Put $A$ in general position with respect to $C'$ so that $A \cap C'$ has a minimal number of components. We must have that $A \cap C'$ is non-empty and contains no simple closed curves. Suppose $\nu$ is a component of this intersection which is an outermost $\partial$-parallel arc on $A$, cutting off a disk $D$ with $\partial D = \nu \cup \nu'$, where $\nu'$ is an arc in, say, $\theta'$. There is an arc $\nu''$ in $F_0$ such that $\nu \cup \nu''$ bounds a disk $D'$ in $C'$. Since $F_0$ is incompressible and $P \cap Y$ is irreducible $\partial(D \cup D') = \partial D''$ for a disk $D''$ in $F_0$, and $D \cup D' \cup D''$ bounds a 3-ball $B$ in $P \cap Y$. An isotopy of $A$ which moves $D$ across $B$ and then past $D'$ removes at least $\nu$ from the intersection, thereby contradicting minimality. So, every component of $A \cap C'$ is a spanning arc on $A$. Assume now that $\nu$ is such a component which is outermost on $C'$, cutting off a disk $D'$ with $\partial D' = \nu \cup \nu'$ for an arc $\nu'$ in $F_0$. The result of performing a boundary compression on $A$ along $D'$ is a disk $A'$ in $F_0$, $\partial A' = \partial D''$ for a disk $D''$ in $F_0$. The 2-sphere $A' \cup D''$ bounds a 3-ball $B$ in $P \cap Y$. $B$ cannot contain $D'$ as that would imply that $A$ is compressible in $Y$. It follows that the union of $B$ an an appropriate regular neighborhood of $D'$ is a solid torus in $Y$ across which $A$ is parallel to an annulus in $\partial Y$. So we are done in this case.

Now suppose that $A$ is incompressible in $X$. Let $(\partial R) \times [0,1]$ be a collar on $\partial R$ in $R$ with $\partial R = (\partial R) \times \{0\}$. We may assume that $\gamma$ and $\delta$ meet this collar in product arcs. Let $Q = Y \cap (R - ((\partial R) \times [0,1]))$. Then $Q$ is the exterior of a trefoil knot, and $Y \cap \partial Q$ is a proper annulus $A_Q$ in $Y$ which is isotopic to $F_1 \cup \partial \mathbb{R} \cup D_2$. $A_Q$
splits $X$ into $Q$ and a space $X'$ homeomorphic to the Whitehead link exterior. $A_Q$ is incompressible in both spaces. We may assume that $(\partial A) \cap (\partial Q) = \emptyset$.

First suppose that $A \cap Q = \emptyset$. Then $A$ is an incompressible annulus in $X'$ and hence must be parallel in $X'$ to an annulus $A'$ in $\partial X'$. Assume $A'$ lies in $\partial K$. Since $A \cap (\partial X' - \partial K) = \emptyset$ the solid torus in $X'$ bounded by $A \cup A'$ must lie in $Y$, and we are done. Assume $A'$ lies in $\partial X' - \partial K$. If $A'$ contains the annulus $S(\omega_0, B_0)$, then the solid torus in $X'$ bounded by $A \cup A'$ must contain $X(\omega_0, B_0)$, which is impossible since this space contains $D_0$, which lies in $\partial K$. Thus $A'$ does not contain $S(\omega_0, B_0)$. It follows that $A'$ either must lie in $\partial Y \cap L$, in which case it is $\partial$-parallel in $Y$, or must contain $A_Q$, in which case $A$ is parallel in $Y$ to $A_Q$.

Now suppose that $A \cap Q \neq \emptyset$. Subject to the requirement that $(\partial A) \cap (\partial Q) = \emptyset$ put $A$ in general position with respect to $A_Q$ so that the intersection has a minimal number of components. Then each component is a simple closed curve which does not bound a disk on either surface. Suppose $A_1$ is a component of $A \cap Q$. Then $A_1$ is an incompressible annulus in $Q$ with meridian boundary components. Since a trefoil knot is prime this implies that $A_1$ is parallel in $Q$ to an annulus in $\partial Q$ [3]. If that annulus lies in $A_Q$, then we can isotop $A$ to remove at least $\partial A_1$ from the intersection, thereby contradicting minimality. Thus each component $A_1$ of $A \cap Q$ is parallel in $Q$ to $\partial Q - \text{int} A_Q$. More precisely there is an embedding of $U_1 = S^1 \times [0, 1] \times [0, 1]$ in $Q$ with $S^1 \times [0, 1] \times \{0\} = \partial Q - \text{int} A_Q$, $S^1 \times [0, 1] \times \{1\} = A_1$, and $S^1 \times [0, 1] \times [0, 1]$ a collar on $\partial A_Q$ in $A_Q$. Let $A_0$ be the component of $A \cap X'$ which contains $\theta'$. The other component $\theta$ of $\partial A_0$ lies in $\text{int} A_Q$. We may assume that $A_1$ is the component of $A \cap Q$ containing $\theta$ and that $S^1 \times \{0\} \times \{1\} = \theta$. Now $A_0$ is $\partial$-parallel in $X'$. More precisely there is a solid torus $U_0$ in $X'$ with $\partial U_0 = A_0 \cup A_0'$, where $A_0'$ lies in $\partial X'$, $A_0 \cap A_0' = \partial A_0 \cap \partial A_0'$, and these annuli are longitudinal in $U_0$. Let $A_0' = A_0' \cap A_0$ and $A_0'' = A_0' \cap \partial Y$. Thus $A_0'' = A_0'' \cup A_0'''$.

There are two possible configurations of $U_0$ and $U_1$ with respect to each other. In the first case $U_0 \cap U_1 = A_0' = S^1 \times \{0\} \times [0, 1]$. Then $U_0 \cup U_1$ is a solid torus across which $A_0 \cup A_1$ is parallel to $A_0' \cup (\partial Q - \text{int} A_Q) \cup (S^1 \times \{1\} \times [0, 1])$. We can then isotop $A$ so as to move $A_0 \cup A_1$ across $U_0 \cup U_1$ to $S^1 \times \{1\} \times [0, 1]$ and then past this annulus into $X'$. This removes at least $\partial A_1$ from $A \cap A_Q$, thereby contradicting minimality.

In the second case $U_0 \cap U_1 = \theta \cup (S^1 \times \{1\} \times [0, 1])$ and $A_0'$ properly contains $S^1 \times \{1\} \times [0, 1]$, i.e. $A_0$ is “folded back over $A_Q$.” Let $A_2$ be the component of $A \cap X'$ other than $A_0$ which meets $A_1$. Let $\psi = A_1 \cap A_2$ and $\psi' = \partial A_2 - \psi$. Now $A_2$ lies in $U_0$ and is parallel to both of the annuli into which $\partial A_2$ splits $\partial U_0$. If $\psi'$ lies in $A_0'$, then as in the first case we can isotopy $A$ so as to move $A_1 \cup A_2$ to $S^1 \times \{0\} \times [0, 1]$ and then past this annulus into $X'$ to remove $\partial A_1$ from $A \cap A_Q$ (and in fact to conclude that $A$ is isotopic to $A_Q$.) If $\psi'$ lies in $A_Q$, then $\psi'' = A_2 \cap A_3$ for a component $A_3$ of $A \cap Q$ which is parallel in $Q$ to $A_1$. We can then isotop $A$ to move $A_1 \cup A_2 \cup A_3$ to
an annulus in $A_Q$ and then past that annulus into $X'$ to remove at least $\partial A_1 \cup \partial A_3$ from $A \cap A_Q$, again contradicting minimality.

(7) Let $T$ be an incompressible torus in $Y$. Let $Q$ and $A_Q$ be as in (6). Let $Y'$ be the closure of $Y - Q$. $Y'$ is a handlebody of genus two. There is a proper disk $D$ in $Y'$ which meets $A_Q$ in a spanning arc. Put $T$ in general position with respect to $A_Q$ so that the intersection has a minimal number of components. As usual no component is a simple closed curve which bounds a disk on both surfaces.

Suppose $T \cap A_Q \neq \emptyset$. Let $A$ be a component of $T \cap Y'$. Put $A$ in general position with respect to $D$ so that $A \cap D$ has a minimal number of components. This intersection must be non-empty. If there is a component which is $\partial$-parallel on $A$, then by the irreducibility of $Y'$ and the incompressibility of $A_Q$ there is an isotopy of an outermost disk on $A$ past the corresponding disk on $D$ which reduces the intersection, contradicting minimality. So every arc of the intersection is a spanning arc on $A$. There is such an arc $\nu$ of $A \cap D$ which cobounds an outermost disk $\Delta$ on $D$ with an arc $\nu'$ in $D \cap A_Q$. Performing a boundary compression on $A$ along $\Delta$ gives a proper disk $A'$ in $Y'$ which cobounds a 3-ball $B$ with a disk in $A_Q$. $\Delta$ cannot lie in $B$ since this would make $A$ compressible in $Y'$. It follows that the union of $B$ and an appropriate regular neighborhood of $\Delta$ is a solid torus across which $A$ is parallel to an annulus in $A_Q$. There is then an isotopy of $T$ in $Y$ which removes at least $\partial A$ from the intersection, contradicting minimality. Thus $T \cap A_Q = \emptyset$. $T$ cannot lie in the handlebody $Y'$ since then it would be compressible. So $T$ must lie in $Q$. Since this is a trefoil knot exterior it must be parallel to $\partial Q$ \[\Box\], and we are done.

\begin{lemma}
Let $Y = X(\tau \cup \rho, R)$, and let $F_i = Y \cap D_i$ for $i = 1, 2$. Then

(0) $Y$ is irreducible,
(1) $F_i$ is incompressible in $Y$,
(2) $O$ is incompressible in $Y$,
(3) $S(\tau \cup \rho, R)$ is incompressible in $Y$,
(4) $(Y, F_1 \cup F_2, \partial Y - \text{int} (F_1 \cup F_2))$ has the halfdisk property,
(5) $(Y, F_1 \cup F_2, \partial Y - \text{int} (F_1 \cup F_2))$ has the band property,
(6) every proper incompressible annulus in $Y$ whose boundary lies in $(\text{int} (F_1 \cup F_2) \cup (\partial Y - (F_1 \cup F_2)))$ is parallel to an annulus in $\partial Y$, and
(7) $Y$ contains no incompressible tori.
\end{lemma}

\begin{proof}
The tangle $\tau$ is \textit{excellent} in the sense that $X(\tau, R)$ is irreducible, $\partial$-irreducible, anannular, and atoroidal \[\Box\]. (This is not the same tangle as in \[\Box\], but the same proof works.)

(0) $Y$ is homeomorphic to the result of attaching a 1-handle to $X(\tau, R)$ and is therefore irreducible.

$F_i$ and $O$ are each incompressible in $X(\tau, R)$ so they each must be incompressible in the smaller space $X(\tau \cup \rho, R)$. This establishes (1) and (2).

To prove (3), (4), and (5) it suffices to prove that $\partial Y - \text{int} (F_1 \cup O)$ is incompressible in $Y$. Suppose $D$ is a compressing disk for this surface. Let $E$ be the disk shown
in Figure 2. Then $E \cap Y$ is a proper disk $E'$ in $Y$ which splits $Y$ into a space $Y'$ homeomorphic to $X(\tau,R)$. Let $E_1$ and $E_2$ be the two copies of $E'$ in $\partial Y'$ which are identified to obtain $E'$ in $Y$. Let $F_1'$ and $F_2'$ be the annuli in $\partial Y'$ into which $\partial E'$ splits $F_1$ and $F_2$, respectively. Put $D$ in general position with respect to $E'$. We assume that $D \cap E'$ has a minimal number of components.

First assume that $D \cap E' = \emptyset$. Then $D$ lies in $Y'$, $D \cap (E_1 \cup E_2) = \emptyset$, and $\partial D = \partial D'$ for a disk $D'$ in $\partial Y'$. If $D'$ lies in $\partial Y$, then we are done. If $D'$ does not lie in $\partial Y$, then $\text{int} \, D'$ must contain $E_1$ or $E_2$ (or both). We may assume that it contains $E_1$. Since $E_1 \cap F_1' \neq \emptyset$ and $F_1'$ is connected and disjoint from $\partial D'$ we must have that $F_1'$ lies in $\text{int} \, D'$. It follows that a meridian of $\gamma$ bounds a disk in $\partial Y'$, which is clearly not the case.

Now assume that $D \cap E' \neq \emptyset$. By minimality and irreducibility each component of the intersection is an arc. Let $\xi$ be an outermost such arc on $D$, and let $D_0$ be the outermost disk which it cuts off on $D$. We may assume that $\partial \xi$ lies in $\text{int} \, F_2$. We may regard $D_0$ as a disk in $Y'$ with $\xi$ lying in, say, $E_1$. Then $\partial D_0 = \partial D_0'$ for a disk $D_0'$ in $\partial Y'$. Note that $F_1' \cap \partial D_0' = \emptyset$. If $F_1'$ lies in $\text{int} \, D_0'$, then a meridian of $\gamma$ again bounds a disk in $\partial Y'$, which is not the case. Thus $D_0'$ misses $F_1'$. Hence the disk $E_0$ on $E$ bounded by the the union of $\xi$ and an arc in $\partial F_2$ must lie in $D_0'$. Let $B$ be the 3-ball in $Y'$ bounded by $D_0 \cup D_0'$. We may regard $B$ and $E_0$ as lying in $Y$. An isotopy of $D$ which moves $D_0$ across $B$ and then past $E_0$ removes at least $\xi$ from $D \cap E$ while keeping $\partial D$ in our surface, thereby contradicting minimality.

Next we prove (6). Suppose $A$ is a proper, incompressible annulus in $Y$ whose boundary misses $\partial (F_1 \cup F_2)$. Since every simple closed curve in $F_1 \cup F_2$ is $\partial$-parallel, we may isotop $A$ so that it misses $F_1 \cup F_2$. Thus $\partial A$ lies in $S(\tau \cup \rho,R) \cup O$, which is incompressible in $Y$. Put $A$ in general position with respect to $E'$ so that $A \cap E'$ has a minimal number of components.

Assume that $A \cap E' = \emptyset$. Then we may regard $A$ as lying in $Y'$, and so it is parallel in $Y'$ to an annulus $A'$ in $\partial Y'$. If $A'$ lies in $\partial Y$, then we are done. If this is not the case, then $A'$ must contain $E_1$ or $E_2$ (or both). We may assume that it contains $E_1$. Since $O - E_1$ is simply connected $\partial A'$ must lie in the disk with two holes $S'$ obtained by splitting $S(\tau \cup \rho,R)$ along its intersection with $\partial E'$. Inspection of $\partial Y'$ shows that there is no annulus $A'$ satisfying all these requirements.

Now assume that $A \cap E' \neq \emptyset$. Since $A$ is incompressible the components of the intersection are either $\partial$-parallel arcs in $A$ or spanning arcs of $A$.

Suppose there is a $\partial$-parallel arc. Let $\xi$ be an outermost such arc on $A$ which cuts off an outermost disk $D_0$ from $A$. We may regard $D_0$ as a disk in $Y'$ which meets $E_1$. Let $\xi' = (\partial D_0) - \text{int} \, \xi$. Now $\xi'$ must lie in either $S'$ or $O'$, where $O'$ is the disk obtained by splitting $O$ along its intersection with $\partial E'$. In either case $\partial D_0 = \partial D_0'$ for a disk $D_0'$ in $\partial Y'$ which lies in the union of $E_1$ and either $S'$ or $O'$. Let $E_0 = D_0' \cap E_1$. Let $B$ be the 3-ball in $Y'$ bounded by $D_0 \cup D_0'$. We may regard $B$ as being embedded in $Y$. An isotopy of $D$ which moves $D_0$ across $B$ and then past $E_0$ removes at least $\xi$
from the intersection thereby contradicting minimality. Thus there are no arcs which are \( \partial \)-parallel in \( A \).

Now suppose that there is a spanning arc \( \xi \). We may assume that it is outermost on \( E \). Let \( E_0 \) be an outermost disk on \( E \) cut off by \( \xi \). We may assume that \( (\partial E_0) - \text{int} \xi \) lies in the surface \( S(\tau \cup \rho) \cup F_2 \cup \partial Y \), which is incompressible in \( Y \). Let \( D \) be the disk obtained by boundary compressing \( A \) along \( E_0 \). Then \( \partial D = \partial D' \) for a disk \( D' \) in \( S(\tau \cup \rho) \cup F_2 \cup \partial Y \). Let \( B \) be the 3-ball in \( Y \) bounded by \( D_0 \cup D_0' \). Then the union of \( B \) and an appropriate regular neighborhood of \( E_0 \) is a solid torus across which \( A \) is parallel to an annulus in \( \partial Y \).

Finally we prove (7). Any incompressible torus \( T \) in \( Y \) can be isotoped off the disk \( E' \), and so can be moved into a space homeomorphic to \( X(\tau,R) \). Since this space is atoroidal \( T \) must be boundary parallel in it. But this is impossible since the space has no torus boundary component. \( \square \)

**Lemma 8.4.** Let \( Y = X(\beta \cup \gamma \cup \delta \cup \epsilon \cup \rho \cup \zeta, J) \) and \( A = Y \cap D_2 \). Then

1. \( Y \) is irreducible,
2. \( F_0 \cup A \) is incompressible in \( Y \),
3. \( \partial Y - \text{int} (F_0 \cup A) \) is incompressible in \( Y \), and
4. \( (Y, F_0, \partial Y - \text{int} F_0) \) has the halfdisk property.

**Proof.** (1) \( Y \) is obtained from the space of Lemma 8.2, call it \( \hat{Y} \), by drilling out a regular neighborhood of the arc \((\rho \cup \zeta) \cap \hat{Y}\). Any 2-sphere in \( Y \) must bound a ball in \( \hat{Y} \); since \( \partial Y \) is connected this ball must lie in \( Y \).

(2) \( F_0 \) is incompressible in \( \hat{Y} \) and so must be incompressible in the smaller space \( Y \). \( A \) is incompressible in \( Y \) for homological reasons.

(3) Suppose \( D \) is a compressing disk for \( G \cup O \) in \( Y \). Since \( G \cup O \cup D_2 \) is incompressible in \( \hat{Y} \), \( \partial D \) must bound a disk in \( G \cup O \cup D_2 \) which contains \( D_2 \). But this implies that \( A \) is compressible in \( Y \), contradicting (2).

Suppose \( D \) is a compressing disk for the other component \( S \) of \( \partial Y - \text{int} (F_0 \cup A) \). It follows from Lemma 6.2 that \( \partial D \) must be parallel in \( S \) to \( S \cup A \), which again contradicts (2).

(4) Now suppose that \( D \) is a proper disk in \( Y \) which meets \( F_0 \cup A \) in an arc \( \theta \) and \( \partial Y - \text{int} (F_0 \cup A) \) in an arc \( \varphi \). We may assume that neither arc is \( \partial \)-parallel. This implies that \( \theta \) must lie in \( F_0 \).

Assume that \( \varphi \) lies in \( G \cup O \). Then by Lemma 6.2 \( \partial D = \partial D' \) for a disk \( D' \) in \( \partial \hat{Y} \). If \( D' \) does not lie in \( \partial Y \), then it must contain \( D_2 \). It follows that \( A \) is compressible in \( Y \), again contradicting (2).

Assume that \( \varphi \) lies in \( S \). Let \( \hat{S} \) be the surface obtained by removing from \( S \) a collar on \( S \cup A \) in \( S \) and replacing it by a meridinal disk for \( \zeta \). We may assume that \( \varphi \) lies in \( \hat{S} \) and misses the meridinal disk. Hence we may assume that it lies in \( \partial \hat{S} \). By Lemma 6.2 \( \partial D = \partial D' \) for a disk \( D' \) in \( \partial \hat{S} \). If \( D' \) does not lie in \( \partial Y \), then it must
contain the meridional disk. It follows that $A$ is compressible in $Y$, contradicting (2) yet a final time.

Let $\bar{R}$ be a copy of $R$. Given any subset $\Sigma$ of $R$ denote the corresponding copy by $\bar{\Sigma}$. In the following two lemmas $\bar{R}$ is glued to $J$ by identifying $\bar{D}_2$ with $D_0$ so that $\bar{\alpha} \cap \bar{D}_2 = \beta \cap D_0$ and $\bar{\zeta} \cap \bar{D}_2 = \varepsilon \cap D_0$.

**Lemma 8.5.** Let $Y = X(\delta \cup \bar{\rho} \cup \bar{\zeta} \cup \bar{\alpha} \cup \beta \cup \epsilon \cup \gamma \cup \delta \cup \rho \cup \zeta, \bar{R} \cup J)$. Let $A_1 = \bar{D}_1 \cap Y$ and $A_2 = D_2 \cap Y$. Then

1. $Y$ is irreducible,
2. $A_1 \cup A_2$ is incompressible in $Y$,
3. $\partial Y - \text{int} (A_1 \cup A_2)$ is incompressible in $Y$, and
4. $(Y, A_1 \cup A_2, \partial Y - \text{int} (A_1 \cup A_2))$ has the halfdisk property.

**Proof.** Let $\tilde{Y} = X(\bar{\zeta} \cup \bar{\alpha} \cup \beta \cup \epsilon \cup \gamma \cup \delta, \bar{R} \cup J)$. This space is the exterior of a Whitehead link in which one component has acquired a local granny knot. It is therefore irreducible and $\partial$-irreducible. $Y$ is homeomorphic to the space obtained from $\tilde{Y}$ by drilling out regular neighborhoods of the arcs $(\delta \cup \bar{\rho}) \cap Y$ and $(\rho \cup \zeta) \cap Y$.

1. Any two sphere in $Y$ must bound a 3-ball in $\tilde{Y}$. Since $\partial \tilde{Y}$ is connected this 3-ball must lie in $Y$.
2. $A_1 \cup A_2$ is incompressible for homological reasons.
3. Suppose $D$ is a compressing disk for $\bar{O} \cup G \cup O$. We may assume that $\partial D$ lies in $G$. Since $G$ is incompressible in $X(\beta \cup \epsilon, L \cup H)$ we have that $D$ must intersect $F_0 \cup F_1$. We may assume that $D$ is in minimal general position with respect to this surface. The boundary of an innermost disk on $D$ would be $\partial$-parallel in $F_0$. This is homologically impossible.

Now suppose that $D$ is a compressing disk for the other component $S$ of $\partial Y - \text{int} (A_1 \cup A_2)$. We may assume that $\partial D$ lies in $\partial \tilde{Y}$. Then $\partial D = \partial D'$ for a disk $D'$ in $\partial \tilde{Y}$ which must contain one or both of the disks in $\partial \tilde{Y}$ whose boundaries are meridians of $\bar{\rho}$ and $\rho$. But this is homologically impossible.

4. Suppose $D$ is a proper disk in $Y$ which meets $A_1 \cup A_2$ in an arc $\theta$ and $\partial Y - \text{int} (A_1 \cup A_2)$ in an arc $\varphi$ such that $\theta \cup \varphi = \partial D$. The only way this can occur is for $\theta$ to be $\partial$-parallel. The result then follows from (3). \hfill $\square$

**Lemma 8.6.** Let $Y = X(\delta \cup \bar{\rho} \cup \bar{\zeta} \cup \bar{\alpha} \cup \beta \cup \epsilon \cup \gamma, \bar{R} \cup J)$. Let $A_1 = \bar{D}_1 \cap Y$. Then

1. $Y$ is irreducible,
2. $A_1$ is incompressible in $Y$,
3. $\partial Y - \text{int} A_1$ is incompressible in $Y$, and
4. $(Y, A_1, \partial Y - \text{int} A_1)$ has the halfdisk property.

**Proof.** Let $\tilde{Y} = X(\bar{\zeta} \cup \bar{\alpha} \cup \beta \cup \epsilon \cup \gamma \cup \delta, \bar{R} \cup J)$. This space is the exterior of a Whitehead link in which one component has acquired a local granny knot. It is therefore irreducible and $\partial$-irreducible. $Y$ is homeomorphic to the space obtained from $\tilde{Y}$ by drilling out a regular neighborhood of the arc $(\delta \cup \bar{\rho}) \cap Y$.
(1) Any 2-sphere in \( Y \) must bound a 3-ball in \( \hat{Y} \). Since \( \partial \hat{Y} \) is connected this 3-ball must lie in \( Y \).

(2) Suppose \( D \) is a compressing disk for \( A_1 \) in \( Y \). Put \( D \) in minimal general position with respect to \( F_0 \). An innermost disk on \( D \) cannot lie in \( J \) by Lemma 8.4 and cannot lie in \( \hat{R} \) for homological reasons. So \( A_1 \) is incompressible in \( Y \).

(3) Suppose \( D \) is a compressing disk for \( \bar{O} \cup G \cup O \cup D_2 \). We may assume that \( \partial D \) lies in \( G \). Put \( D \) in minimal general position with respect to \( F_0 \). An innermost disk on \( D \) with boundary in \( D \cap F_0 \) cannot lie in \( J \) by Lemma 8.4 and cannot lie in \( \hat{R} \) for homological reasons. Thus \( D \) must lie in \( J \) and so by Lemma 8.4 \( \partial D = \partial D' \) for a disk \( D' \) in \( \partial(Y \cap J) \). For homological reasons \( D' \) cannot contain \( F_0 \), and so must lie in \( G \cup D_2 \). Thus \( \bar{O} \cup G \cup D_2 \) is incompressible in \( Y \).

Now suppose that \( D \) is a compressing disk for the other component \( S \) of \( \partial Y - \text{int} A_1 \). We may assume that \( \partial D \) lies in \( \hat{Y} \). Then \( \partial D = \partial D' \) for a disk \( D' \) in \( \partial \hat{Y} \) which must contain the disk in \( \partial \hat{Y} \) whose boundary is a meridian of \( \bar{p} \). This contradicts (2), so \( S \) is incompressible in \( Y \).

(4) Suppose \( D \) is a proper disk in \( Y \) which meets \( A_1 \) in an arc \( \theta \) and \( \partial Y - \text{int} A_1 \) in an arc \( \varphi \) such that \( \theta \cup \varphi = \partial D \). The only way this can occur is for \( \theta \) to be \( \partial \)-parallel. The result then follows from (3).

\[ \square \]

9. The construction of \( W \)

In this section we construct a Whitehead manifold \( W \) which is an infinite cyclic covering space of another 3-manifold \( W^\# \). In the sections that follow it will be shown that \( W \) is \( \mathbf{R}^2 \)-irreducible and that whenever it non-trivially covers a 3-manifold the group of covering translations must be infinite cyclic. Later we will modify the construction to obtain uncountably many such examples as well as examples which cannot non-trivially cover any 3-manifold.

We will also define certain open subsets of \( W \) which we will later show to be a complete set of representatives for the isotopy classes of end reductions of \( W \).

For each integer \( n \geq 0 \) take a copy of each of the objects defined in section 8. Denote the \( n \)th copy of \( D_j \) by \( D_{n,j} \), that of each of the other objects by a subscript \( n \).

We regard the arcs with subscript \( n \) as embedded in the 3-manifolds with subscript \( n + 1 \).

We embed \( J^\#_n \) in \( \text{int} J^\#_{n+1} \) as \( N(\mu_n, J^\#_{n+1}) \) in the following manner. \( L_n \) is sent to \( N(\lambda_n, J^\#_{n+1}) \). We assume that this set is the image in \( J^\#_{n+1} \) of \( N(\varepsilon_n, L_{n+1}) \cup N(\delta_n \cup \zeta_n, R_{n+1}) \), that \( \rho_n \) meets \( N(\lambda_n, J^\#_{n+1}) \) in \( \rho_n - \text{int} \rho'_n \), and that each of \( \alpha_n \) and \( \gamma_n \) meet \( N(\lambda_n, J^\#_{n+1}) \) in an arc. Let \( \alpha'_n \) and \( \gamma'_n \), respectively, be the intersections of \( \alpha_n \) and \( \gamma_n \) with \( J^\#_{n+1} - \text{int} N(\lambda_n, J^\#_{n+1}) \), and let \( \eta'_n = \alpha'_n \cup \beta_n \cup \gamma'_n \). Send \( H_n \) to \( N(\eta'_n, J^\#_{n+1} - \text{int} (\lambda_n, J^\#_{n+1})) \). We assume that this set is the image in \( J^\#_{n+1} \) of \( N(\beta_n, L_{n+1}) \cup N(\alpha'_n \cup \gamma'_n, R_{n+1} - \text{int} N(\delta_n \cup \zeta_n)) \). Thus \( L_n \cup H_n \) is sent to \( N(\kappa_n, J^\#_{n+1}) \).
We assume that the intersection of this set with $R_{n+1}$ is $N(\tau_n, R_{n+1})$. Finally we send $R_n$ to $N(\rho_n, X(\tau_n, R_{n+1}))$. The result is shown in Figure 4.

Now let $W^\#$ be the direct limit of the $J_n^\#$, and let $p : W \to W^\#$ be the universal covering map. Then $\pi_1(W^\#)$ is infinite cyclic. Let $h : W \to W$ be a generator of the group of covering translations. We regard $p^{-1}(P_n^\#)$ as $D_n \times \mathbb{R}$ with $P_{n,j} = D_n \times [2j, 2j+2]$, $L_{n,j} = D_n \times [2j, 2j+1]$, and $R_{n,j} = D_n \times [2j+1, 2j+2]$. We have that $p^{-1}(H_n)$ is a disjoint union of 1-handles $H_{n,j}$, where $H_{n,j}$ is attached to $\partial D_n \times (2j, 2j+1)$, thereby yielding a copy $J_{n,j} = P_{n,j} \cup H_{n,j}$ of $J_n$. Set $D_{n,k} = D_n \times \{k\}$ for $k \in \mathbb{Z}$. For all the objects with subscript $n$ contained in $L_{n+1,j} \cup H_{n+1,j}$ or contained in $R_{n+1,j}$ denote the preimage contained in $L_{n+1,j} \cup H_{n+1,j}$ or in $R_{n+1,j}$ by using the subscripts $n,j$. We assume that $h$ is chosen so that $h(D_{n,k}) = D_{n,k+2}$ and the image under $h$ of any object with subscripts $n,j$ has subscripts $n,j+1$. The embedding of $p^{-1}(J_n^\#)$ in $p^{-1}(J_{n+1}^\#)$ is shown in Figure 5.
Let $\mathcal{P} = \{p_1, p_2, \cdots, p_m\}$ be a finite non-empty set of distinct integers ordered so that $p_1 < p_2 < \cdots < p_m$. We say that $\mathcal{P}$ is good if $p_{i+1} = p_i + 1$ for $1 \leq i \leq m - 1$. Otherwise $\mathcal{P}$ is bad. Note that if $m = 1$, then $\mathcal{P}$ is automatically good.

For $n \geq 0$ we let $C^\mathcal{P}_n$ be the union of those $R_{n,j}$ with $p_1 - 1 \leq j \leq p_m$, those $L_{n,j}$ with $p_1 \leq j \leq p_m$, and those $H_{n,p}$ with $p \in \mathcal{P}$. Each $C^\mathcal{P}_n$ is a cube with $m$ handles. We have that $C^\mathcal{P}_n \subseteq \text{int} C^\mathcal{P}_{n+1}$. This embedding is illustrated in Figure 6 for $\mathcal{P} = \{0, 1, 3\}$.

The sequence $\{C^\mathcal{P}_n\}_{n \geq 0}$ is denoted by $C^\mathcal{P}$; its union is denoted by $V^\mathcal{P}$. Whenever $\mathcal{P}$ is good and $m > 1$ we denote $V^\mathcal{P}$ by $V^{p,q}$, where $p = p_1$ and $q = p_m$. When $\mathcal{P} = \{p\}$ we use the notation $V^p$. The expressions $C^{p,q}_n$, $C^p_n$, $C^{p,q}$, and $C^p$ are defined similarly.

**10. $\mathbf{R^2}$-irreducibility**

In this section we show that the manifold $W$ constructed in section 9 is $\mathbf{R^2}$-irreducible. In the process we will also show that certain important open submanifolds of $W$ are $\mathbf{R^2}$-irreducible.

We begin by recalling a criterion due to Scott and Tucker [19] for a $\mathbf{P^2}$-irreducible open 3-manifold to be $\mathbf{R^2}$-irreducible. A reformulation of this criterion was used in [14]; we will need a slightly more general reformulation here.
A proper plane $\Pi$ in an open 3-manifold $U$ is homotopically trivial if for every compact subset $C$ of $U$ the inclusion map of $\Pi$ into $U$ is end-properly homotopic to a map whose image is disjoint from $C$.

**Lemma 10.1.** Let $U$ be an irreducible, open 3-manifold, and let $\Pi$ be a proper plane in $U$. If $\Pi$ is homotopically trivial, then $\Pi$ is trivial.

*Proof.* This is Lemma 4.1 of [19]. \(\square\)

**Lemma 10.2.** Let $U$ be an irreducible, open 3-manifold, and let $\{C_n\}_{n \geq 1}$ be a sequence of compact 3-dimensional submanifolds of $U$ such that $C_n \subseteq \text{int} \, C_{n+1}$ and

1. each $C_n$ is irreducible,
2. each $\partial C_n$ is incompressible in $U - \text{int} \, C_n$, and
3. if $D$ is a proper disk in $C_{n+1}$ which is in general position with respect to $\partial C_n$ such that $\partial D$ is not null-homotopic in $\partial C_{n+1}$, then $D \cap \partial C_n$ has at least two components which are not null-homotopic in $\partial C_n$ and bound disjoint disks in $D$.

Then any proper plane in $U$ can be end-properly homotoped off $C_n$ for any $n$.

*Proof.* This is Lemma 4.2 of [19]. \(\square\)

The precise criterion we shall use is the following slightly stronger version of Lemma 3.3 of [14]. Define a 3-manifold $Y$ to be weakly anannular if every proper incompressible annulus in $Y$ has both its boundary components in the same component of $\partial Y$.

**Lemma 10.3.** Let $U$ be a connected, irreducible, open 3-manifold. Suppose that for each compact subset $K$ of $U$ there is a sequence $\{C_n\}_{n \geq 1}$ of compact 3-dimensional submanifolds of $U$ such that $C_n \subseteq \text{int} \, C_{n+1}$ and

1. each $C_n$ is irreducible,
2. each $\partial C_n$ is incompressible in $U - \text{int} \, C_n$,
3. each $C_{n+1} - \text{int} \, C_n$ is irreducible, $\partial$-irreducible, and weakly anannular, and
4. $K \subseteq C_1$.

Then $U$ is $\mathbb{R}^2$-irreducible.

*Proof.* Let $D$ be a proper disk in $C_{n+1}$ which is in general position with respect to $\partial C_n$ such that $\partial D$ is not null-homotopic in $\partial C_{n+1}$. If some component of $D \cap \partial C_n$ bounds a disk on $\partial C_n$ then there is such a disk whose interior misses $D$. Surgery on $D$ along this disk gives a disk $D'$ whose intersection with $\partial C_n$ is contained in the intersection of $D$ with $\partial C_n$ but has fewer components. By repeating this procedure, if necessary, we may assume that no component of $D \cap \partial C_n$ is null-homotopic in $\partial C_n$. If $D \cap \partial C_n = \emptyset$, then $\partial C_{n+1}$ is compressible in $C_{n+1} - \text{int} \, C_n$, thereby contradicting the $\partial$-irreducibility of $C_{n+1} - \text{int} \, C_n$. If $D \cap \partial C_n$ has only one component, then it and $\partial D$ together bound an incompressible annulus in $C_{n+1} - \text{int} \, C_n$ joining $\partial C_{n+1}$ and $\partial C_n$, thereby contradicting the weak anannularity of $C_{n+1} - \text{int} \, C_n$. If $D \cap \partial C_n$ has more than one component and no two of them bound disjoint disks in $D$, then the components must be nested on $D$. Then $\partial D$ and an outermost such component
on $D$ again together bound an incompressible annulus in $C_{n+1} - \text{int} C_n$ which joins $\partial C_{n+1}$ and $\partial C_n$. Thus there must be two components which bound disjoint disks in $D$. Now apply Lemma 10.2 and then Lemma 10.1. □

Let $C = \{C_n\}$ be a sequence of compact, connected, 3-dimensional submanifolds of an irreducible, open 3-manifold $U$ such that $C_n \subseteq \text{int} C_{n+1}$ and $U - \text{int} C_n$ has no compact components. This will be called a quasi-exhaustion for $U$. Note that a quasi-exhaustion for $U$ whose union is $U$ is an exhaustion for $U$. A quasi-exhaustion is good if it satisfies conditions (1)--(3) of Lemma 10.3. Thus that lemma can be rephrased by saying that if every compact subset of $U$ is contained in the first term of a good quasi-exhaustion, then $U$ is $\mathbb{R}^2$-irreducible.

Recall the quasi-exhaustion $C_P$ defined in section 9, where $P = \{p_1, p_2, \ldots, p_m\}$. The embedding of $C_P^n$ in $C_P^{n+1}$ is shown in Figure 7 for the case $m = 1$, $p = p_1$ and in Figure 8 for the case $P = \{0, 1, 2\}$.

**Proposition 10.4.** If $P$ is good, then $C_P$ is good.

**Corollary 10.5.** $W$ is $\mathbb{R}^2$-irreducible. Each $V^P$ with $P$ good is $\mathbb{R}^2$-irreducible.

**Proof of Corollary 10.5.** A given compact subset $K$ of $W$ lies in some $p^{-1}(J^n)$ and thus in a finite union of $J_{n,j}$ and hence in some $C_P^{n,q}$, which can be renumbered as the first term of the good quasi-exhaustion $C_P$. Thus by Lemma 10.3 $W$ is $\mathbb{R}^2$-irreducible.

$V^P$ is $\mathbb{R}^2$-irreducible because it has the good exhaustion $C_P$. □
Proof of Proposition 10.4. We have that $C^m_n \subseteq \text{int} C^m_{n+1}$, and $C^m_n \subseteq C^m_{n+1}$. A given compact set $K$ of $W$ lies in some $p^{-1}(J^n_w)$ and thus in a finite union of $J_{n,j}$ and hence in some $C^m_n \subseteq C^n_q$, where $q = \max\{m, n\}$. Thus $\{C^m_n\}$ is an exhaustion for $W$.

$C^m_n$ is a cube with $2m + 1$ handles and so is irreducible. Let $Y = C^m_{n+1} - \text{int} C^m_n$.

By Lemma 8.3 it suffices to prove the following. \hfill \Box

Lemma 10.6. $Y$ is irreducible, $\partial$-irreducible, and weakly annular.

Proof. To simplify the notation we may assume that $p = 0$.

We first consider the case $m = 1$. Then the embedding of $C^0_n$ into $C^0_{n+1}$ is shown in Figure 7.

$Y$ is homeomorphic to the union of the Whitehead link exterior $Y_0$ and two trefoil knot exteriors $Y_1$ and $Y_2$. Thus each $Y_i$ is irreducible and $\partial$-irreducible. $Y_1 \cap Y_2 = \emptyset$.

For $i = 1, 2$ we have that $Y_i \cap Y_0 = \partial Y_i \cap \partial Y_0 = A_i$, an annulus in the component $T$ of $\partial Y_0$ which is not equal to $\partial C_{n+1}$. Each $A_i$ is a regular neighborhood of a simple closed curve which is a meridian in both $Y_0$ and $Y_i$. Thus $A_i$ incompressible in both $Y_i$ and $T - \text{int}(A_1 \cup A_2)$ is incompressible in $Y_0$. We apply Lemma 7.1 with $S = A_1 \cup A_2$ and $Z = \partial Y$ to conclude that $Y$ is irreducible and $\partial$-irreducible.

Every proper incompressible annulus in $Y_0$ is $\partial$-parallel \[\Box\]. For $i = 1, 2$ each proper incompressible annulus in $Y_i$ with meridian boundary components is $\partial$-parallel \[\Box\]. We apply Lemma 7.2 with $S = A_1 \cup A_2$ and $Z = \partial C_{n+1}$ to conclude that $Y$ is weakly annular.

We now consider the case $m > 1$. The embedding of $C^0_{n,q}$ in $C^0_{n+1}$ is shown in Figure 8 for $q = 2$.

We split $Y$ along the surface $F = \bigcup_{i=1}^{2d} F_{n+1,i}$ into a 3-manifold $Y'$. Let $F'$ be the surface in $\partial Y'$ whose image in $Y$ is $F$. Let $Z = \partial Y'$ and $Z_0 = \partial C^m_n$. Let $Z'$ and $Z'_0$ be the surfaces into which $Z$ and $Z_0$, respectively, are split by their intersections with $F$.

Each component of $Y'$ is homeomorphic to $X(\beta \cup \epsilon, L \cup H)$, $X(\beta \cup \gamma \cup \delta \cup \epsilon, J)$, or $X(\tau \cup \rho, R)$. It follows from Lemmas 8.1, 8.2, and 8.3 that $Y'$ is irreducible, $F'$ and $Z'$ are incompressible in $Y'$, and $(Y', F', Z')$ has the halfdisk property. So by Lemma 7.1 $Y$ is irreducible and $\partial$-irreducible.

By Lemmas 8.1, 8.2, and 8.3 $(Y', F', Z')$ has the band property and every proper incompressible annulus in $Y'$ which misses $\partial F'$ is either $\partial$-parallel in $Y'$ or has boundary which is parallel in $\partial Y'$ to a pair of simple closed curves in $Z'_0$. Therefore by Lemma 7.2 every proper incompressible annulus in $Y$ is either $\partial$-parallel in $Y$ or has its boundary in $Z_0$. In particular $Y$ is weakly annular. \hfill \Box

Proposition 10.7. If $P$ is bad, then $V^P$ is not $\mathbb{R}^2$-irreducible.

Proof. Suppose $P = \{p_1, p_2, \ldots, p_m\}$. There is an $i$, $1 \leq i \leq m - 1$, such that $p_{i+1} > p_i + 1$. Set $d = p_i + 1$. Let $\Delta_0$ be a proper disk in $C_0^P$ which lies in the interior of $L_{1,d}$ and whose boundary is a meridian of $\varepsilon_{1,d}$. There is a proper annulus $\Omega_1$ in $C_1^P - \text{int} C_0^P$ which lies in the interior of $L_{2,d}$ and whose boundary is the union of $\partial \Delta_0$
and a meridian of \( \varepsilon_{1,d} \). Let \( \Delta_1 = \Delta_0 \cup \Omega_1 \). Continuing in this fashion we build for each \( n > 1 \) a proper annulus \( \Omega_n \) in \( C_n^P - \text{int} C_{n-1}^P \) which lies in the interior of \( L_{n+1,d} \) and whose boundary is the union of \( \partial \Delta_{n-1} \) and a meridian of \( \varepsilon_{n,d} \). Then \( \Delta_n = \Delta_{n-1} \cup \Omega_n \) is a proper disk in \( C_n^P \) such that \( \partial \Delta_n \) is a meridian of \( \varepsilon_{n,d} \). The union \( \Pi \) of the \( \Delta_n \) is a plane which is proper in \( V^P \). Its complement in \( V^P \) has two components. One contains \( V^{p_1} \); the other contains \( V^{p_m} \). By [6] none of the \( V^p \) embeds in \( \mathbb{R}^3 \). It follows that \( \Pi \) is non-trivial in \( V^P \). \( \square \)

11. Genus

Suppose \( \{C_n\} \) is an exhaustion for a Whitehead manifold \( V \). The \textit{genus} of \( \{C_n\} \) is defined to be the maximum of the genera of the surfaces \( \partial C_n \) if this set of numbers is bounded and to be \( \infty \) if it is unbounded. The \textit{genus} of \( V \) is defined to be the minimum of the genera of those exhaustions for \( V \) which have finite genus if such exhaustions exist and to be \( \infty \) if all exhaustions for \( V \) have infinite genus.

\textbf{Proposition 11.1.} \( V^P \) has finite genus. It has genus one if and only if \( P \) has exactly one element.

\textit{Proof.} Suppose \( P = \{p_1, \ldots, p_m\} \). Then \( V^P \) has an exhaustion by cubes with \( m \) handles, so it has genus at most \( m \). Since \( V^P \) contains the manifold \( V^{p_1} \) which does not embed in \( \mathbb{R}^3 \) we have that \( V^P \) is not homeomorphic to \( \mathbb{R}^3 \) and so has genus at least one. So if \( m = 1 \), then \( V^P \) has genus one.

Suppose \( m > 1 \). If \( P \) is bad, then by Proposition 10.7 \( V^P \) is not \( \mathbb{R}^2 \)-irreducible. Since every genus one Whitehead manifold is \( \mathbb{R}^2 \)-irreducible we the genus of \( V^P \) must be greater than one.

Now suppose that \( P \) is good. As in the \( m > 1 \) case of the proof of Lemma 10.6 we let \( Y = C_{n+1}^{[1,m]} - \text{int} C_n^{[1,m]} \), etc. By Lemmas 8.1, 8.2, and 8.3 all the hypotheses of Lemmas 7.1, 7.2, and 7.3 are satisfied. Thus every incompressible torus in \( Y \) bounds a compact submanifold of \( Y \) and every proper incompressible annulus in \( Y \) is either \( \partial \)-parallel in \( Y \) or cobounds a compact submanifold of \( Y \) with an annulus in \( \partial C_n^{[1,m]} \).

Suppose \( V^P \) has an exhaustion \( \{K_n\} \) of genus one. We may assume that each \( K_n \) has genus one, that each \( \partial K_n \) is incompressible in \( V^P - \text{int} K_0 \), and that \( C_0 \subseteq \text{int} K_0 \). Choose \( q > 0 \) such that \( K_1 \subseteq \text{int} C_q \). Let \( T = \partial K_1 \). Then \( T \) is an incompressible torus in \( C_q - \text{int} C_0 \) which separates \( \partial C_q \) from \( \partial C_0 \). Hence \( T \) cannot bound a compact submanifold of this space.

On the other hand \( C_q - \text{int} C_0 \) is the union of spaces \( Y_j = C_j - \text{int} C_{j-1} \) for \( j = 1, \ldots, q \) which satisfy the hypotheses of Lemma 7.3. Hence every incompressible torus in this space must bound a compact submanifold of the space. This contradiction shows that \( V^P \) must have genus greater than one. \( \square \)
12. Every $V^P$ is an end reduction of $W$.

Let $\mathcal{P} = \{p_1, \ldots, p_m\}$. Recall that $V^P$ is defined as the union of the quasi-exhaustion $C^P$, where for each $n \geq 0$ $C^P_n$ is the union of those $R_{n,j}$ with $p_1 - 1 - n \leq j \leq p_m - n$, those $L_{n,j}$ with $p_1 - 1 \leq j \leq p_m$, and those $H_{n,p}$ with $p \in \mathcal{P}$.

**Proposition 12.1.** For each $k \geq 0$ $V^P$ is an end reduction of $W$ at $C^P_k$.

**Proof.** We may assume that $k = 0$. We will show that $V^P$ is the constructed end reduction of $W$ associated to a certain exhaustion $\{K_n\}$ of $W$. For each $n \geq 0$ let $K^+_n$ be the union of those $R_{n,j}$ with $p_1 - 1 - n \leq j \leq p_m + n$, and those $L_{n,j}$ and $H_{n,j}$ with $p_1 - n \leq j \leq p_m + n$. Each $K^+_n$ is a cube with $1 + 2n + p_m - p_1$ handles. Let $K_n$ be $K^+_n$ for $n > 0$ and $C^P_0$ for $n = 0$.

Note that for each $n > 0$ we have that $C^P_n$ is obtained by compressing $\partial K_n$ in $W - C^P_0$. More specifically we remove those 1-handles $H_{n,j}$ of $K_n$ for which $j \notin \mathcal{P}$. Note that the compressions are confined to $K_{n+1} - \text{int} C^P_{n-1}$. It suffices to show that each $\partial C^P_m$, $m > 0$, is incompressible in $W - \text{int} C^P_0$.

First note that for each $n \geq 0$ we have by Lemmas 7.1, 8.1, 8.2, 8.3, 8.4, and 8.5 that $C^P_{n+1} - \text{int} C^P_n$ is $\partial$-irreducible. It follows that for any $m > 0$ we have that $C^P_m - \text{int} C^P_0$ is $\partial$-irreducible and that for all $q > 0$ we have that $C^P_{m+q} - \text{int} C^P_m$ is $\partial$-irreducible. Now $K_{m+q} - \text{int} C^P_m$ is obtained by attaching 1-handles to this space, so $\partial C^P_m$ is incompressible in this new space. Since $K$ is an exhaustion for $W$ it follows that $\partial C^P_m$ is incompressible in $W - \text{int} C^P_0$. Hence we have that $\partial C^P_m$ is incompressible in $W - \text{int} C^P_0$. $\square$

13. Every $\mathbb{R}^2$-irreducible end reduction of $W$ is a $V^P$.

**Proposition 13.1.** Every $\mathbb{R}^2$-irreducible end reduction of $W$ is isotopic to some $V^P$. More precisely, if $J$ is a regular submanifold of $W$, $V$ is an $\mathbb{R}^2$-irreducible end reduction of $W$ at $J$, and $J \subseteq \text{int} C^Q$, then there is a good subset $\mathcal{P}$ of $C^Q$ such that $V$ is isotopic in $W$ to $V^P$. If $V$ is already contained in $V^Q$ then the isotopy can be chosen to lie in $V^Q$.

**Proof.** By Lemma 2.4 we may assume that $V$ is an end reduction of $W$ at a knot $\kappa \subseteq \text{int} J$. By Theorem 5.1 we can isotop $V$ rel $\kappa$ so that $V$ lies in $V^Q$ and is an end reduction of $V^Q$ at $\kappa$.

Let $\mathcal{P}$ be a minimal subset of $Q$ such that $V$ can be isotoped in $V^Q$ so that it lies in $V^P$.

If $\mathcal{P}$ is bad, then $V^P$ contains a non-trivial plane $\Pi$ which splits it into components which are isotopic in $V^P$ to $V^R$ and $V^S$, where $R$ and $S$ form a non-trivial partition of $\mathcal{P}$. By Theorem 6.2 $V$ can be isotoped in $V^P$ into one of these two components. This contradicts the minimality of $\mathcal{P}$, and so $\mathcal{P}$ must be good.

Now $\kappa$ lies in the interior of some $C^P_n$. We say that $\kappa$ is $\mathcal{P}$ busting in $C^P_n$ if for each $p \in \mathcal{P}$ the annulus $H_{n,p} \cap \partial C^P_n$ is incompressible in $C^P_n - \kappa$. 

If $\kappa$ is not $\mathcal{P}$ busting in $C_n^\mathcal{P}$, then for some $p \in \mathcal{P}$ the annulus $H_{n,p} \cap \partial C_n^\mathcal{P}$ is compressible in $C_n^\mathcal{P} - \kappa$. It follows that $\kappa$ can be isotoped in $\text{int} \, C_n^\mathcal{P}$ so that it lies in $C_n^\mathcal{P} - \{p\}$. This implies by Theorem X that $V$ can be isotoped in $V^\mathcal{P}$ so as to lie in $V^\mathcal{P} - \{p\}$, contradicting the minimality of $\mathcal{P}$. Thus $\kappa$ is $\mathcal{P}$ busting in $C_n^\mathcal{P}$.

Whenever $\kappa$ lies in the interior of a handlebody $K$ we say that $\kappa$ is disk busting in $K$ if $\partial K$ is incompressible in $K - \kappa$.

**Lemma 13.2.** $\kappa$ is disk busting in $C_{n+2}^\mathcal{P}$.

The proof of this lemma appears below. We note that $\kappa$ is not disk busting in $C_{n+1}^\mathcal{P}$. By the proof of Proposition 10.1 we know that $\partial C_{n+2}^\mathcal{P}$ is incompressible in $V^\mathcal{Q} - \text{int} \, C_{n+2}^\mathcal{P}$. Thus $\partial C_{n+2}^\mathcal{P}$ is incompressible in $V^\mathcal{Q} - \kappa$. By Lemma 2.3 we have that $V$ is an end reduction of $V^\mathcal{Q}$ at $C_{n+2}^\mathcal{P}$ and therefore is isotopic to $V^\mathcal{P}$. This concludes the proof of Proposition 11.1.

**Proof of Lemma 13.2.** We first consider the case $m = 1$. Then $\kappa$ lies in the solid torus $C_n^\mathcal{P}$. Any compressing disk for $\partial C_n^\mathcal{P}$ must be isotopic to a compressing disk for $H_{n,p} \cap \partial C_n^\mathcal{P}$. So $\partial C_n^\mathcal{P}$ is incompressible in $C_n^\mathcal{P} - \kappa$. Since $C_{n+1}^\mathcal{P} = \text{int} \, C_{n+1}^\mathcal{P}$ and $C_{n+2}^\mathcal{P} = \text{int} \, C_{n+2}^\mathcal{P}$ are $\partial$-irreducible we have that $\partial C_{n+2}^\mathcal{P}$ is incompressible in $C_{n+2}^\mathcal{P} - \kappa$.

We now assume that $m > 1$. Isotop $\kappa$ in $C_n^\mathcal{P}$ so that it is in minimal general position with respect to the union $\mathcal{D}$ of the disks $D_{n+2,i} \cap C_n^\mathcal{P}$, where $2p_i + 1 \leq i \leq 2p_m$. Note that since $\kappa$ is $\mathcal{P}$ busting in $C_n^\mathcal{P}$ it must meet those 3-balls in $C_n^\mathcal{P}$ which are regular neighborhoods of the arcs $\beta_{n,p}$.

Recall that the configuration consisting of $C_{n+1}^\mathcal{P}$ and the graph of which $C_n^\mathcal{P}$ is a regular neighborhood is split by certain of the disks $D_{n+1,i}$ into a left hitch, junctions, eyebolts, and a right hitch as in the proof of Proposition 12.1. $C_n^\mathcal{P}$ meets each of these configurations in a collection of 3-balls which contain proper subarcs of $\kappa$. Our strategy will be to apply Lemma 7.1 to the case of $Y = X(\kappa, C_{n+2}^\mathcal{P})$ with $S$ equal to the union of those $Y \cap D_{n+2,i}$ which arise in the above decomposition, and with $Z = \partial C_{n+2}^\mathcal{P}$. The proof of Lemma 13.2 will then follow from the following sequence of lemmas.

**Lemma 13.3.** Let $(L_{n+2,p} \cup H_{n+2,p} \cup \varepsilon_{n+1+p})$ be an eyebolt. Let $\kappa'$ be the intersection of $\kappa$ with $L_{n+2,p} \cup H_{n+2,p}$. Let $X' = X(\kappa', L_{n+2,p} \cup H_{n+2,p})$ and $F'_{n+2,i} = X' \cap D_{n+2,i}, \, i = 2p, 2p + 1$. Then

1. $X'$ is irreducible,
2. $F'_{n+2,2p}, \, F'_{n+2,2p+1}, \, \text{and } G_{n+2,p}$ are incompressible in $X'$, and
3. $(X', F'_{n+2,2p} \cup F'_{n+2,2p+1} \cup G_{n+2,p})$ has the halfdisk property.

**Proof.** Let $N_{n+1} = C_{n+1}^\mathcal{P} \cap (L_{n+2,p} \cup H_{n+2,p})$. It has two components, namely the regular neighborhoods $N_{n+1}^+$ of $\beta_{n+1,p}$ and $N_{n+1}^-$ of $\varepsilon_{n+1+p}$ in $L_{n+2,p} \cup H_{n+2,p}$. Let $X_{n+2}$ be the exterior in $L_{n+2,p} \cup H_{n+2,p}$ of the union of these arcs, so we have $L_{n+2,p} \cup H_{n+2,p} = X_{n+2} \cup N_{n+1}$. Let $A_{n+1}^\pm$ be the annuli $N_{n+1}^\pm \cap X_{n+2}$. By Lemma 6.1 we have that $X_{n+2}$ is irreducible, $F_{n+2,2p} \cup F_{n+2,2p+1}, \, A_{n+1}^\pm$, and $G_{n+2,p}$ are incompressible in $X_{n+2}$, and $X_{n+2}, \, F_{n+2,2p} \cup F_{n+2,2p+1}, \, \partial X_{n+2} = \text{int} \, (F_{n+2,2p} \cup F_{n+2,2p+1})$ has the halfdisk property.


Next let \( N_n = C^p_n \cap (L_{n+2,p} \cup H_{n+2,p}) \). It has four components, namely the two components of the regular neighborhood \( N^+_n \) of the union of two arcs \( \beta_{n,p} \cap N^+_{n+1} \) in \( N^+_{n+1} \), the regular neighborhood \( N^-_n \) of \( \beta_{n,p} \cap N^-_{n+1} \in N^-_{n+1} \), and the regular neighborhood \( N^0_n \) of \( \varepsilon_{n,p} \cap N^-_{n+1} \) in \( N^-_{n+1} \). Let \( X_{n+1} \) be the exterior in \( N_{n+1} \) of the union of these four arcs. Let \( A^+ \) be the union of two annuli \( N^+_n \cap X_{n+1}, A^- \) the annulus \( N^-_n \cap X_{n+1}, \) and \( A^0_n \) the annulus \( N^0_n \cap X_{n+1} \).

\( X_{n+1} \) has two components. The component \( X^+_{n+1} \) lying in \( N^+_{n+1} \) is the exterior of a Whitehead clasp. By [1] we have that \( X^+_{n+1} \) is irreducible, \( X^-_{n+1} \cap (D_{n+2,2p} \cup D_{n+2,2p+1}), A^+ \) and \( A^- \) are incompressible in \( X^-_{n+1} \), and \( A^0 \) are incompressible in \( X^0_{n+1} \), and \( (X^-_{n+1}, X_{n+1} \cap (D_{n+2,2p} \cup D_{n+2,2p+1})), \partial N^+_n \cup \partial N^+_{n+1} \) has the halfdisk property. The component \( X^-_{n+1} \) lying in \( N^-_{n+1} \) is the product of a disk with two holes and a closed interval. So it is irreducible, \( X^-_{n+1} \cap (D_{n+2,2p} \cup D_{n+2,2p+1}), A^+_{n+1}, A^+ \), and \( A^0 \) are incompressible in \( X^-_{n+1} \), and \( (X^-_{n+1}, X_{n+1} \cap (D_{n+2,2p} \cup D_{n+2,2p+1})), A^-_{n+1} \cup A^- \cup A^0 \) has the halfdisk property.

Let \( N' \) be a regular neighborhood of \( \kappa' \) in \( N_n \). Let \( X_n \) be the exterior of \( \kappa' \) in \( N_n \). Since \( \kappa' \) consists of proper arcs in a collection of 3-balls it follows that \( X_n \) is irreducible. Denote the components of \( X_n \) contained in \( N^+_n \), \( N^-_n \), and \( N^0_n \) by \( X^+_n \), \( X^-_n \), and \( X^0_n \), respectively.

We have that \( X' = X_{n+2} \cup X_{n+1} \cup X_n \).

Since \( \kappa \) must join a left hitch in \( C_n^p \) to a right hitch and \( p \in P \), we must have that \( \kappa \) meets each component of \( N^+_n \cup N^-_n \). It follows from the minimality of \( \kappa \cap D \) that \( A^+_n \cup A^-_n \) must be incompressible in \( X_n \). Now \( \kappa' \) may or may not meet \( N^0_n \). If it does meet \( N^0_n \), then as above we have that \( A^0_n \) is incompressible in \( X_n \). In this case it follows that \( X' \) is irreducible and that \( G_{n+2,p} \) is incompressible in \( X' \). If \( \kappa' \) does not meet \( N^0_n \), then since \( N^0_n \) is a product 3-ball in the product 3-ball \( N^-_{n+1} \) we have that \( A^-_n \) is parallel in \( X' \) to \( A^-_{n+1} \). Hence we again get that \( X' \) is irreducible and \( G_{n+2} \) is incompressible in \( X' \). Thus we have (1) and a part of (2).

Suppose \( D \) is a compressing disk for, say, \( F^+_{n+2,2p} \) in \( X' \). Assume first that \( \kappa \) meets \( N^0_n \). Put \( D \) in minimal general position with respect to \( A^+_n \cup A^-_n \cup A^+ \cup A^- \cup A^0_n \). There are no simple closed curve intersections. If \( \nu \) is an outermost arc on \( D \) cutting off an outermost disk \( \Delta \), then the arc \( \nu' = \partial \Delta - int \nu \) is \( \partial \)-parallel in the annulus containing it. Note that this uses the halfdisk property for the exterior of the Whitehead clasp \( X^+_{n+1,p} \). It follows from the incompressibility of \( F^+_{n+2,2p} \) in \( X_{n+2} \), of \( X_{n+1} \cap D_{n+2,2p} \) in \( X_{n+1} \), and of \( X_n \cap D_{n+2,2p} \) in \( X_n \) that \( \nu \) can be removed by an isotopy of \( D \). Assume next that \( \kappa' \) misses \( N^0_n \). Put \( D \) in minimal general position with respect to \( A^+_n \cup A^-_n \cup A^+ \cup A^- \). Since \( A^-_n \) and \( A^-_{n+1} \) are parallel a similar argument shows that we can remove all intersections of \( D \) with these annuli. Thus \( D \) must lie in \( X_{n+2}, X_{n+1} \), or \( X_n \) and \( \partial D \) must lie in the intersection of this manifold with \( D_{n+2,2p} \), which is incompressible. This completes the proof of (2).

Now suppose that \( D \) is a proper disk in \( X' \) with, say, \( D \cap F^+_{n+2,2p} \) an arc \( \theta \) and with \( \partial D - int \theta \) an arc \( \theta' \) in \( G_{n+2,p} \). As in the previous paragraph we put \( D \) in minimal general position with respect to the appropriate collection of annuli and use
minimality to remove all the intersections. \( D \) must then lie in \( X_{n+2} \), and we use Lemma 6.1 to complete the proof of (3).

Lemma 13.4. Let \( (J_{n+2,p}; \beta_{n+2,p} \cup \gamma_{n+2,p} \cup \delta_{n+2,p} \cup \varepsilon_{n+2,p}) \) be a right hitch. Let \( \kappa' = \kappa \cap J_{n+2,p} \). Let \( X' = X(\kappa', J_{n+2,p}) \). Let \( F'_{n+2,2j} = D_{n+2,2j} \cap X' \). Then

1. \( X' \) is irreducible,
2. \( F'_{n+2,2j} \) and \( D_{n+2,2j} \) are incompressible in \( X' \), and
3. \( X', F'_{n+2,2j}, D_{n+2,2j} \) has the halfdisk property.

Proof. Let \( N_{n+1} = C_{n+1}^P \cap J_{n+2,p} \). It is a regular neighborhood of the arc \( \beta_{n+1,p} \cup \gamma_{n+1,p} \cup \delta_{n+1,p} \) in \( J_{n+2,p} \). Note that it includes \( R_{n+1,p} \). Let \( X_{n+2} \) be the exterior in \( J_{n+2,p} \) of this arc, so we have \( J_{n+2,p} = X_{n+2} \cup N_{n+1} \). Let \( A_{n+1} \) be the annulus \( N_{n+1} \cap X_{n+2} \).

By Lemma 8.2 we have that \( X_{n+2} \) is irreducible, \( F_{n+2,2j}, D_{n+2,2j}, A_{n+1} \) are incompressible in \( X_{n+2} \), and \( (X_{n+2}, F_{n+2,2j}, D_{n+2,2j}) \) has the halfdisk property.

Next let \( N_n = C_n^P \cap J_{n+2,p} \). It has two components \( N_n^+ \) and \( N_n^- \). \( N_n^+ \) is a regular neighborhood in \( N_{n+1} \) of the component of \( \beta_{n,p} \cap J_{n+2,p} \) which lies entirely within \( L_{n+2} \). \( N_n^- \) is a regular neighborhood in \( N_{n+1} \) of the other component of \( (\beta_{n,p} \cup \gamma_{n,p} \cup \delta_{n,p} \cup \varepsilon_{n,j}) \cap J_{n+2,p} \). Let \( X_{n+1} \) be the exterior of \( N_n \) in \( N_{n+1} \). Let \( A_n^\pm \) be the annulus \( N_n^\pm \cap X_{n+1} \). \( N_n \) sits in \( N_{n+1} \) as a regular neighborhood of a Whitehead clasp in which a trefoil knot has been tied in one component. From the properties of the unknotted Whitehead clasp \( \] \) it is easily checked that \( X_{n+1} \) is irreducible, \( X_{n+1} \cap D_{n+2,2j}, A_{n+1} \) and \( A_n^\pm \) are incompressible in \( X_{n+1} \), and \( (X_{n+1}, X_{n+1} \cap D_{n+2,2j}, A_{n+1} \cup A_n^+ \cup A_n^-) \) has the halfdisk property.

Let \( N \) be a regular neighborhood of \( \kappa' \) in \( N_n \). Let \( X_n \) be the exterior of \( \kappa' \) in \( N_n \). Then \( X' = X_{n+2} \cup X_{n+1} \cup X_n \). Since \( p \in \mathcal{P} \) we must have that \( \kappa' \) meets both components of \( N_n \). From the minimality of \( \kappa \cap \mathcal{D} \) we have that \( A_n^+ \cup A_n^- \) is incompressible in \( X_n \). We now put a compressing disk or halfdisk in minimal general position with respect to \( A_{n+1} \cup A_n^+ \cup A_n^- \) and argue as in Lemma 13.3 to complete the proof.

We warn the reader that the proof of the next lemma, which occupies the rest of this section, is a very lengthy checking of special cases, subcases, and subsubcases. It may be advisable to skip it on a first reading.

Lemma 13.5. Let \( (R_{n+2,p}; \gamma_{n+1,p} \cup \delta_{n+1,p} \cup \rho_{n+1,p} \cup \alpha_{n+1,p} \cup \zeta_{n+1,p}) \) be a junction, where \( p, p+1 \in \mathcal{P} \). Let \( \kappa' = \kappa \cap R_{n+2,p} \). Let \( X' = X(\kappa', R_{n+2,p}) \). Let \( F'_{n+2,i} = D_{n+2,i} \cap X' \), \( i = 2p + 1, 2p + 2 \). Then

1. \( X' \) is irreducible,
2. \( F'_{n+2,i} \) and \( O_p \) are incompressible in \( X' \), and
3. \( X', F'_{n+2,2p+2}, O_p \) has the halfdisk property.

Proof. Let \( N_{n+1} = C_{n+1}^P \cap R_{n+2,p} \). It is a 3-ball which is a regular neighborhood in \( R_{n+2,p} \) of \( \gamma_{n+1,p} \cup \delta_{n+1,p} \cup \rho_{n+1,p} \cup \alpha_{n+1,p} \cup \zeta_{n+1,p} \). Let \( X_{n+2} \) be the exterior in
$R_{n+2,p}$ of this union of arcs, so we have $R_{n+2,p} = X_{n+2} \cup N_{n+1}$. By Lemma 8.3 we have that $X_{n+2}$ is irreducible, $F_{n+2,2p+1}$, $F_{n+2,2p+2}$, $O_p$, and the lateral surface $S_{n+1} = S(\gamma_{n+1,p} \cup \delta_{n+1,p} \cup \rho_{n+1,p} \cup \alpha_{n+1,p} \cup \zeta_{n+1,p}, R_{n+2,p}$ are incompressible in $X_{n+2}$, and $(X_{n+2}, F_{n+2,2p+1} \cup F_{n+2,2p+2}, \partial X_{n+2} - \text{int} (F_{n+2,2p+1} \cup F_{n+2,2p+2}))$ has the halfdisk property.

Next let $N_n = C_n^p \cap R_{n+2,p}$. It has three components $N_n^-$, $N_n^+$, and $N_n^0$. $N_n^-$ is a regular neighborhood in $N_{n+1}$ of that component of $\beta_{n,p} \cap N_{n+1}$ which joins one component of $N_{n+1} \cap D_{n+2,2p+1}$ to the other. This arc is $\partial$-parallel in $N_{n+1}$ via a disk which misses the other two components of $N_n$. Similarly $N_n^+$ is a regular neighborhood in $N_{n+1}$ of that component of $\beta_{n,p+1} \cap N_{n+1}$ which joins one component of $N_{n+1} \cap D_{n+2,2p+2}$ to the other. This arc is also $\partial$-parallel in $N_{n+1}$ via a disk which misses the other two components of $N_n$. $N_n^0$ is a 3-ball which sits in $N_{n+1}$ in the same fashion that $N_{n+1}$ sits in $R_{n+2,p}$. Let $X_{n+1}$ be the exterior of $N_n$ in $N_{n+1}$. It is homeomorphic to the space obtained from $X_{n+2}$ by attaching a 1-handle to $F_{n+2,2p+1}$ and a 1-handle to $F_{n+2,2p+2}$. It is therefore irreducible. We can also regard $X_{n+1}$ as being the space obtained by drilling out two $\partial$-parallel arcs from a homeomorphic copy $\hat{X}_{n+1}$ of $X_{n+2}$, where each arc has its endpoints in the surface corresponding to $F_{n+2,2p+1}$ or $F_{n+2,2p+2}$. Since $O_p$ is incompressible in $X_{n+2}$, we have that $S_{n+1}$ is incompressible in the smaller space $X_{n+1}$. Likewise the surface $S_n = N_n^0 \cap X_{n+1}$ is incompressible in $X_{n+1}$. The surface $X_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ has four components each of which is a disk with two holes; it is incompressible in $X_{n+1}$ for homological reasons. Let $A_n^\pm = N_n^\pm \cap X_{n+1}$. Each of these annuli is incompressible in $X_{n+1}$ for homological reasons. Suppose $D$ is a halfdisk in $X_{n+1}$ with one boundary arc $\theta$ in, say, $X_{n+1} \cap D_{n+2,2p+1}$. Then we may assume that the other boundary arc $\theta'$ misses $A_n^-$. By Lemma 8.3 we have that $\partial D = \partial D'$ for a disk $D'$ in $\partial \hat{X}_{n+1}$. Now there is a proper arc $\nu$ in $D'$ which lies in $\partial S_{n+1} \cup \partial S_n$ and splits $D'$ into a disk $\Delta$ with $\partial \Delta = \theta \cup \nu$ and a disk $\Delta'$ with $\partial \Delta' = \theta' \cup \nu$. $\Delta$ lies in a component of $\hat{X}_{n+1} \cap D_{n+2,2p+1}$, and $\Delta'$ lies in either $S_n$ or $S_{n+1}$. $\Delta$ must in fact lie in $X_{n+1} \cap D_{n+2,2p+1}$, for otherwise we could find a compressing disk for $A_n^-$ in $X_{n+1}$. Thus $(X_{n+1}, X_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_n \cup S_{n+1} \cup A_n^- \cup A_n^+)$ has the halfdisk property.

Let $N'$ be a regular neighborhood of $\kappa'$ in $N_n$. Let $X_n$ be the exterior of $\kappa'$ in $N_n$. Since $\kappa'$ consists of proper arcs in 3-balls we have that $X_n$ is irreducible. Denote the components of $X_n$ contained in $N_n^-$, $N_n^+$, and $N_n^0$ by $X_n^-$, $X_n^+$, and $X_n^0$, respectively.

Since $p, p+1 \in P$ we must have that $\kappa'$ meets all three components of $N_n$. It follows from the minimality of $\kappa \cap D$ that $X_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ is incompressible in $X_n$ and hence that $A_n^\pm$ is incompressible in $X_n^\pm$, and that $(X_n^-, X_n^- \cap D_{n+2,2p+1}, A_n^-)$ and $(X_n^+, X_n^+ \cap D_{n+2,2p+2}, A_n^+)$ have the halfdisk property. Note, however, that $S_n$ may or may not be incompressible in $X_n^0$ and that $(X_n^0, X_n^0 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_n)$ may or may not have the halfdisk property.
First suppose that both of these properties are satisfied. Then we regard $X'$ as $X_{n+2} \cup X_{n+1} \cup X_n$ and apply the techniques of the previous two lemmas to obtain the result.

Now suppose that at least one of these properties is not satisfied. We will show how to reorganize $X'$ as $X'_{n+2} \cup X'_{n+1} \cup X'_n \cup X^+_n \cup X^-_n$ in such a way as to obtain the result. We will replace $\kappa'_n$ by a 3-manifold $\kappa'_n$ and let $X'_n$ be the exterior of $\kappa' \cap N'_1$ in $N'_n$. We will replace $N_{n+1}$ by a 3-manifold $N'_{n+1}$ and let $X'_{n+1}$ be the exterior of $N'_n \cup N^-_n \cup N^+_n$ in $N'_{n+1}$. Then we will let $X'_{n+2}$ be the exterior of $N'_{n+1}$ in $R_{n+2,p}$. There are several possibilities and possibly several steps in constructing these manifolds.

Case 1: $S_n$ is compressible in $X'_n$ via a compressing disk $D$ such that $\partial D$ is not $\partial$-parallel in $S_n$.

The surface $S^1_n$ resulting from the compression consists of two annuli. Since $p, p + 1 \in P$ each of these annuli must join $D_{n+2,2p+1}$ to $D_{n+2,2p+2}$. The compression splits $N'_n$ into a 3-manifold $N^1_n$ which consists of two 3-balls each of which meets both $D_{n+2,2p+1}$ and $D_{n+2,2p+2}$. Let $X^1_n$ be the exterior of $\kappa' \cap N^1_n$ in $N^1_n$.

Subcase (a): $S^1_n$ is incompressible in $X^1_n$.

By the minimality of $\kappa \cap D$ we have that $X^1_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ is incompressible in $X^1_n$. Since $S^1_n$ consists of annuli we then have that $(X^1_n, X^2_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S^1_n)$ has the halfdisk property.

We now have $N^1_n \cup N^-_n \cup N^+_n$ sitting in $N_{n+1}$ with exterior $X^1_{n+1}$. Clearly $X^1_{n+1}$ is irreducible. $X^1_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ is incompressible in $X^1_{n+1}$ for homological reasons.

Thus the boundary of any compressing disk for $S^1_{n+1}$ in $X^1_{n+1}$ would have to separate two of the components of $\partial S^1_{n+1}$ from the other two. Hence the compressing disk would have to separate two of the disks in $(N^1_n \cup N^-_n \cup N^+_n) \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ from the other two. Since the components of this intersection which lie in $D_{n+2,2p+1}$ are joined by $N^-_n$ and those which lie in $D_{n+2,2p+2}$ are joined by $N^+_n$ we must have that the compressing disk separates the pair of disks in $D_{n+2,2p+1}$ from the pair in $D_{n+2,2p+2}$. But this is impossible since each component of $N^1_n$ joins $D_{n+2,2p+1}$ to $D_{n+2,2p+2}$. Thus $S^1_{n+1}$ is incompressible in $X^1_{n+1}$.

If $(X^1_{n+1}, X^1_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S^1_{n+1} \cup S^1_n \cup A^-_n \cup A^+_n)$ does not have the halfdisk property, then there is a proper disk $\Delta$ in $X^1_{n+1}$ which meets, say, $D_{n+2,2p+1}$ in an arc $\theta$ and $S^1_{n+1}$ in an arc $\theta'$ such that $\partial \Delta = \theta \cup \theta'$, and neither arc is $\partial$-parallel. $\Delta$ then separates $N^-_n$ from a component of $N^1_n$. It must therefore separate the two components of $N^1_n$ from each other. But this is impossible since they meet disks which are joined by $N^+_n$. Thus this triple has the halfdisk property.

We now let $N'_n = N^1_n$, $X'_n = X^1_n$, $N'_{n+1} = N^1_{n+1}$, and $X'_{n+2} = X_{n+2}$. Then we apply the usual methods to complete the proof.

Subcase (b): $S^1_n$ is compressible in $X^1_n$.

One component $S^2_n$ of $S^1_n$ must be incompressible in $X^1_n$. Since $p, p + 1 \in P$ it must join a meridian of $\gamma_{n,p}$ to a meridian of $\alpha_{n,p}$. We let $N^2_n$ and $X^2_n$ be the
components of $N^1_n$ and $X'_{n}$ which meet $S^2_n$. By the arguments given in (a) we have that $X^2_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ is incompressible in $X^2_n$ and that $(X^2_n, X^2_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S^2_n)$ has the halfdisk property.

We let $X^2_{n+1}$ be the exterior of $N^2_n \cup N_1^-$ in $N^2_{n+1}$. As in (a) we have that $X^2_{n+1}$ is irreducible and that $S_{n+1}$ and $X^2_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ are incompressible in $X^2_{n+1}$.

Note that $(X^2_{n+1}, X^2_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_{n+1} \cup S^2_n \cup A^-_n \cup A^+_n)$ does not have the halfdisk property. There are disjoint proper disks $\Delta^2_i$ and $\Delta^2_j$ in $X^2_{n+2}$ such that, for $i = 1, 2$, $\Delta^2_i$ meets $D_{n+2,2p+1}$ in an arc $\theta_i$ and $S_{n+1}$ in an arc $\theta'_i$ such that $\partial \Delta^2_i = \theta_i \cup \theta'_i$ and neither arc is $\partial$-parallel. $\Delta^2_i \cup \Delta^2_j$ splits $N^2_{n+1}$ into a 3-manifold $N^3_{n+1}$ consisting of three 3-balls $B^-_{n+1}, B^0_{n+1}$, and $B_{n+1}^+$. $B_{n+1}^+$ contains $N^-_n$. $B^0_{n+1}$ contains $N^2_n$. We let $S^3_{n+1}$ be the result of boundary compressing $S_{n+1}$ along $\Delta^2_i \cup \Delta^2_j$. It consists of three annuli. Let $Q^2_{n+1}$ be the exterior of $N^-_n$ in $B^0_{n+1}$. Let $Q^0_{n+1}$ be the exterior of $N^+_n$ in $B^0_{n+1}$. We let $Q^1_{n+1}$ be the result of boundary compressing $S_{n+1}$ along $\Delta^2_i \cup \Delta^2_j$.

We now consider the exterior $X^3_{n+2}$ of $N^3_{n+1}$ in $R_{n+2, p}$. It is irreducible. $O_p$ and $S^3_{n+1}$ are incompressible in $X^3_{n+2}$ for homological reasons. The boundary of any compressing disk for $X^3_{n+2} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ in $X^3_{n+2}$ would, for homological reasons, bound a disk in, say, $D_{n+2,2p+1}$ which contains $B^-_{n+1} \cap D_{n+2,2p+1}$ but does not contain $B^0_{n+1} \cap D_{n+2,2p+1}$. The compressing disk could be isotoped in the exterior of $B^-_{n+1} \cup B^+_{n+1}$ in $R_{n+2, p}$ to a compressing disk for $O_p$ in this space. But this space is homeomorphic to the exterior of the tangle $\tau_{n+1,p}$ in $R_{n+2, p}$, which is $\partial$-irreducible. Thus $X^3_{n+2} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ is incompressible in $X^3_{n+2}$.

Suppose $(X^3_{n+2}, X^3_{n+2} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), O_p \cup S^3_{n+2})$ does not have the halfdisk property. Then there is a proper disk $\Delta$ in $X^3_{n+2}$ with, say, $\Delta \cap D_{n+2,2p+1}$ an arc $\theta$ and $\Delta \cap S^3_{n+1}$ an arc $\theta'$ such that $\theta \cup \theta' = \partial \Delta$ and neither arc is $\partial$-parallel. $\theta'$ must lie in the component $A^-_{n+1}$ of $S^3_{n+1}$ which joins $D_{n+2,2p+1}$ to itself and must join the two components of $\partial A^3_{n+1}$. It follows that $\theta \cup \theta'$ must be a trefoil knot in $R_{n+2, p}$ and so cannot bound a disk $\Delta$.

Thus our triple has the halfdisk property, and so we set $X^3_i = X^3_i$ for $n \leq i \leq n+2$ and complete the proof as usual.

**Case 2:** $S_n$ is compressible in $X^0_n$ via a compressing disk $D$ such that $\partial D$ is $\partial$-parallel in $S_n$.

By Case 1 we may assume that there is no compressing disk for $S_n$ in $X^0_n$ whose boundary is not $\partial$-parallel in $S_n$. The surface resulting from the compression along $D$ consists of a disk with two holes $S^1_n$ and a disk $S'$. We may assume that $\partial S'$ lies in $D_{n+2,2p+1}$. $D$ splits $X^0_n$ into a 3-manifold consisting of 3-balls $N^1_n$ and $B'$ such that $\kappa' \cap B' = \emptyset$ and $S'$ is $\partial$-parallel across $B'$. Since $p, p+1 \in P$ we have that $\partial S'$ is a meridian of $\zeta_{n,p}$. Let $X^1_{n}$ be the exterior of $\kappa' \cap N^1_n$ in $N^1_n$. Because $p, p+1 \in P$
any compressing disk for $S_n^1$ in $X_n^1$ would have boundary a meridian of $\zeta_{n,p}$. But then one could form a band sum along an arc in $S$ of this compressing disk and $D$ which would be a compressing disk for $S_n$ in $X_n^0$ whose boundary is not $\partial$-parallel in $S_n$. Therefore $S_n^1$ is incompressible in $X_n^1$.

Subcase (a): $(X_n^1, X_n^1 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_n^1)$ has the halfdisk property.

Let $X_{n+1}$ be the exterior of $N_n^1 \cup N_n^- \cup N_n^+$ in $N_{n+1}$. $X_{n+1} \cup (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ and $S_n^1$ are incompressible in $X_{n+1}$ for homological reasons. The boundary of a compressing disk for $S_{n+1}$ in $X_{n+1}$ would have to separate two components of $\partial S_{n+1}$ from the other two. Thus in $N_{n+1}$ the compressing disk would have to separate two disks of $N_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ from the other while missing $N_n^1 \cup N_n^- \cup N_n^+$. This is impossible, and so $S_{n+1}$ is incompressible in $X_{n+1}$.

Note that $(X_{n+1}^1, X_{n+1}^1 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_{n+1} \cup S_n^1 \cup A_n^- \cup A_n^+)$ does not have the halfdisk property. There is a proper disk $\Delta^1$ in $X_{n+1}^1$ such that $\Delta^1$ meets $D_{n+2,2p+1}$ in an arc $\theta_1$ which lies in the non-annulus component of $X_{n+1}^1 \cap D_{n+2,2p+1}$. It meets $S_{n+1}$ in an arc $\theta'_1$ such that $\partial \Delta^1 = \theta_1 \cup \theta'_1$ and neither arc is $\partial$-parallel. Moreover $\Delta^1$ splits $N_{n+1}$ into 3-balls $B_{n+1}$ and $B_n^-$ such that $B_{n+1}$ contains $N_n^-$ and $B_n^-$ contains $N_n^+ \cup N_n^1$. Let $N_{n+1}^2 = B_{n+1}^+ \cup B_{n+1}^-$. We let $S_{n+1}^2$ be the result of boundary compressing $S_{n+1}$ along $\Delta^1$. Let $Q_{n+1}^−$ be the exterior of $N_n^1 \cup \Sigma_{n+1}^−$. Let $Q_{n+1}^+$ be the exterior of $N_n^1 \cup N_n^1$ in $B_{n+1}^+$. Let $X_{n+1} = Q_{n+1}^− \cup Q_{n+1}^+$. It is irreducible.

$S_{n+1}^2 \cap Q_{n+1}^−$ is an annulus $A_{n+1}^2$ which is parallel across $Q_{n+1}^−$ to $A_n^−$, and so $A_{n+1}^-, A_n^-$, and $Q_{n+1}^- \cap D_{n+2,2p+1}$ are all incompressible in $Q_{n+1}^−$, and $(Q_{n+1}^−, Q_{n+1}^- \cap D_{n+2,2p+1}, A_n^- \cup A_{n+1}^2)$ has the halfdisk property.

$S_{n+1}^2 \cap Q_{n+1}^+$ is a disk with two holes $\Sigma_{n+1}^2$, $S_{n+1}^2$, $\Sigma_{n+1}^2$, and $Q_{n+1}^+ \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ are incompressible in $Q_{n+1}^+$ for homological reasons.

Suppose $(Q_{n+1}^-, Q_{n+1}^+ \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), (\Sigma_{n+1}^2 \cup S_n^1 \cup A_n^+)$ does not have the halfdisk property. Then there is a proper disk $\Delta$ in $Q_{n+1}^-$ such that $\Delta$ meets $D_{n+2,2p+2}$ in an arc $\theta$ and $\Sigma_{n+1}^2 \cup S_n^1$ in an arc $\theta'$ such that $\partial \Delta = \theta \cup \theta'$ and neither arc is $\partial$-parallel. First assume that $\theta'$ lies in $S_{n+1}^1$. Then $\Delta$ separates $B_{n+1}^+ \cap D_{n+2,2p+1}$ from one of the components of $B_{n+1}^+ \cap D_{n+2,2p+2}$ in $B_{n+1}^+$ while missing $N_n^1 \cap N_n^+$. This is impossible since $N_n^1$ meets all three components of $B_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$. So $\theta'$ lies in $S_{n+1}^1$. $\partial \theta'$ must lie in a single component of $\partial S_n^1$ and separate the other two components from each other. $\theta$ must lie in $D_{n+2,2p+2}$. Then in the exterior of $N_n^1$ in $B_{n+1}^+$ we can isotope $\theta \cup \theta'$ to a non-trivial simple closed curve on $S_n^1$. This contradicts the fact that $S_n^1$ is incompressible in this space for homological reasons. Thus our triple must have the halfdisk property.

We now consider the exterior $X_{n+2}^2$ of $N_{n+1}^2$ in $R_{n+2,2p}$. It is irreducible. $O_n$ and $S_{n+1}^2$ are incompressible in $X_{n+2}^2$ for homological reasons. The boundary of a compressing disk for $X_{n+2}^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ in $X_{n+2}^2$ would, for homological reasons, bound a disk in $D_{n+2,2p+1}$ which contains $B_{n+1} \cap D_{n+2,2p+1}$ but does not contain $B_{n+1} \cap D_{n+2,2p+2}$. The compressing disk could be isotooped in the exterior of $N_n^- \cup N_n^+$.
in $R_{n+2,p}$ to give a compressing disk for $O_p$ in this space. However this space is homeomorphic to the exterior of the tangle $\tau_{n+1,p}$ in $R_{n+2,p}$, which is $\partial$-irreducible \cite{N}. Thus $X_{n+2}^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ is incompressible in $X_{n+2}$.

Suppose $(X_{n+2}^2, X_{n+2}^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), O_p \cup S_{n+1}^2)$ does not have the halfdisk property. Then there is a proper disk $\Delta$ in $X_{n+2}^2$ with $\Delta \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ an arc $\alpha$ and $\Delta \cap S_{n+1}^2$ an arc $\alpha'$ with $\partial \Delta = \theta \cup \theta'$ and neither arc is $\partial$-parallel. Recall that $S_{n+1}^2 = A_{n+1}^2 \cup \Sigma_{n+1}$, where $A_{n+1}^2$ is an annulus with $\partial A_{n+1}^2$ in $D_{n+2,2p+1}$ and $\Sigma_{n+1}$ is a disk with two holes having one boundary component in $D_{n+2,2p+1}$ and the other two in $D_{n+2,2p+2}$.

Assume $\theta$ lies in $D_{n+2,2p+1}$. Suppose $\theta'$ lies in $A_{n+1}^2$. Then $\theta \cup \theta'$ is a trefoil knot in $R_{n+2,p}$ and so cannot bound a disk $\Delta$. So $\theta'$ must lie in $\Sigma_{n+1}^2$ and separate the components of $\Sigma_{n+1}^2 \cap D_{n+2,2p+2}$. But this implies that a meridian of the arc $\alpha_{n+1,p}$ is compressible in the exterior of the arc $\alpha_{n+1,p} \cup \zeta_{n+1,p}$ in $R_{n+2,p}$, which is homologically impossible.

So $\theta$ must lie in $D_{n+2,2p+2}$ and $\theta'$ in $\Sigma_{n+1}^2$. Suppose $\theta'$ joins different components of $\partial \Sigma_{n+1}^2$, then $\theta \cup \theta'$ is a trefoil knot in $R_{n+2,p}$ and so cannot bound a disk $\Delta$. So $\theta'$ must join a component of $\partial \Sigma_{n+1}^2$ to itself and separate the other two boundary components. But this implies that either a meridian of the arc $\alpha_{n+1,p}$ is compressible in the exterior of the arc $\alpha_{n+1,p} \cup \rho_{n+1,p} \cup \delta_{n+1,p}$ in $R_{n+2,p}$ or that a meridian of the arc $\zeta_{n+1,p}$ is compressible in the exterior of the arc $\zeta_{n+1,p} \cup \rho_{n+1,p} \cup \delta_{n+1,p}$ in $R_{n+2,p}$, both of which are homologically impossible.

Thus our triple has the halfdisk property. Letting $X_i' = X_i^2$ we complete the proof in the usual manner.

Subcase (b): $(X_1^2, X_1^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_1^2)$ does not have the halfdisk property.

There is a proper disk $\Delta^1$ in $X_1^2$ such that $\Delta^1$ meets $D_{n+2,2p+1} \cup D_{n+2,2p+2}$ in an arc $\theta_1$ and $S_1^2$ in an arc $\theta_1'$. $\partial \theta_1'$ must lie in one component of $\partial S_1^2$ and separate the other two components. Let $N_n^2$ be the 3-manifold obtained by splitting $N_n^1$ along $\Delta^1$. Let $S_2^2$ be the result of performing the corresponding boundary compression on $S_1^2$ along $\Delta^1$. $S_2^2$ consists of a pair of annuli and $N_n^2$ a pair of 3-balls. There are three possibilities for how $N_n^2$ joins the components of $N_n^1 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$, depending on which component of this set meets both components of $N_n^2$.

Let $X_2^2$ be the exterior of $\kappa \cap N_n^2$ in $N_n^2$. $X_2^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ and $S_2^2$ are incompressible in $X_2^2$. Since $S_2^2$ consists of annuli $(X_2^2, X_2^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_2^2)$ has the halfdisk property.

Let $X_{n+1}^2$ be the exterior of $N_n^2 \cup N_n^- \cup N_n^+$ in $N_{n+1}$. $X_{n+1}^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ and $S_n^2$ are incompressible in $X_{n+1}^2$ for homological reasons. The boundary of a compressing disk for $S_{n+1}^2$ in $X_{n+1}^2$ would have to separate two components of $\partial S_{n+1}^2$ from the other two. Thus in $N_{n+1}$ the compressing disk would have to separate two disks of $N_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ from the other two while missing $N_n^2 \cup N_n^- \cup N_n^+$. This is impossible, and so $S_{n+1}$ is incompressible in $X_{n+1}^2$. 

Note that \((X_{n+1}^2, X_{n+1}^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_{n+1} \cup S_n^2 \cup A_n^- \cup A_n^+)\) does not have the halfdisk property. There is a proper disk \(\Delta^2\) in \(X_{n+1}^2\) such that \(\Delta^2\) meets \(D_{n+2,2p+1}\) in an arc \(\theta_2\) which lies in the non-annulus component of \(X_{n+1}^2 \cap D_{n+2,2p+1}\). It meets \(S_{n+1}\) in an arc \(\theta'_2\) such that \(\partial \Delta^2 = \theta_2 \cup \theta'_2\) and neither arc is \(\partial\)-parallel. Moreover \(\Delta^2\) splits \(N_{n+1}\) into 3-balls \(B_{n+1}^-\) and \(B_{n+1}^+\) such that \(B_{n+1}^-\) contains \(N_n^+\) and \(B_{n+1}^+\) contains \(N_n^- \cup N_n^2\). Let \(N_{n+1}^3 = B_{n+1}^- \cup B_{n+1}^+\). We let \(S_{n+1}\) be the result of boundary compressing \(S_{n+1}\) along \(\Delta^2\). Let \(Q_{n+1}^-\) be the exterior of \(N_n^-\) in \(B_{n+1}^-\). Let \(Q_{n+1}^+\) be the exterior of \(N_n^+ \cup N_n^2\) in \(B_{n+1}^+\). Let \(X_{n+1}^3 = Q_{n+1}^- \cup Q_{n+1}^+\). It is irreducible. \(S_{n+1}^3 \cap Q_{n+1}^+\) is an annulus \(A_{n+1}^3\) which is parallel across \(Q_{n+1}^-\) to \(A_n^3\), and so \(A_{n+1}^3\) is homeomorphic to the exterior of \(S_{n+1}^3\). Let \(Q_{n+1}^+ \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})\) be the exterior of \(N_{n+1}^+ \cup N_{n+1}^2\) for homological reasons.

Suppose \((Q_{n+1}^+, Q_{n+1}^+ \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_{n+1}^3 \cup S_n^2 \cup A_n^+\) does not have the halfdisk property. Then there is a proper disk \(\Delta\) in \(Q_{n+1}^+\) such that \(\Delta\) meets \(S_{n+1}^3\) in an arc \(\theta'\) such that \(\partial \theta'\) lies in a single component of \(\partial S_{n+1}^3\) and \(\theta'\) separates the other two components. \(\Delta\) meets \(Q_{n+1}^+ \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})\) in an arc \(\theta\) such that \(\partial \Delta = \theta \cup \theta'\). Neither arc is \(\partial\)-parallel. \(\Delta\) separates the two disks of \(B_{n+1}^+ \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})\) which do not contain \(\theta\) from each other and does not meet \(N_n^+ \cup N_n^2\). It follows that one component \(C^+\) of \(N_n^2\) joins the two components of \(B_{n+1}^+ \cap D_{n+2,2p+2}\) while the other component \(C\) of \(N_n^2\) joins \(B_{n+1}^+ \cap D_{n+2,2p+2}\) to a component of \(B_{n+1}^+ \cap D_{n+2,2p+2}\). Let \(\hat{Q}_{n+1}^+ = Q_{n+1}^+ \cup N_n^+.\) Then \(C \cup C^+\) is a regular neighborhood of a tangle \(\hat{\tau} = \sigma \cup \sigma^+\) in \(B_{n+1}^+\).

There are two possibilities for \(\hat{\tau}\).

If \(\theta'\) lies in the component of \(B_{n+1}^+ \cap D_{n+2,2p+2}\) which meets \(\alpha_{n,p}\), then \(\sigma\) is isotopic to \(\gamma_{n,p} \cup \rho_{n,p} \cup \alpha_{n,p}\) and \(\sigma^+\) is isotopic to a parallel copy of \(\alpha_{n,p} \cup \zeta_{n,p}\). In this case one can isotop \(\sigma^+\) by moving one of its endpoints along \(\sigma\) so as to obtain the graph \(\gamma_{n,p} \cup \rho_{n,p} \cup \alpha_{n,p} \cup \zeta_{n,p}\). By isotoping \(\rho_{n,p}\) so that \(\rho_{n,p} \cap \zeta_{n,p}\) is moved along \(\zeta_{n,p}\), we obtain a tangle in \(B_{n+1}^+\) which is equivalent to the tangle \(\tau_{n,p}\) in \(R_{n+1,p}\). Thus \(\hat{Q}_{n+1}^+\) is homeomorphic to the exterior of \(\tau_{n,p}\) in \(R_{n+1,p}\) and so is \(\partial\)-irreducible. Thus \(\partial \Delta = \partial \Delta'\) for a disk \(\Delta'\) in \(\partial \hat{Q}_{n+1}^+\). If \(\Delta'\) does not lie in \(\partial \hat{Q}_{n+1}^+\), then for homological reasons it must contain both components of \(N_{n+1}^+ \cap D_{n+2,2p+2}\). But this is impossible since then \(\Delta'\) would also contain the component of \(B_{n+1}^+ \cap D_{n+2,2p+2}\) which meets \(\zeta_{n,p}\), and so \(\Delta'\) could not lie in \(\partial \hat{Q}_{n+1}^+\). Thus \(\Delta'\) lies in \(\partial \hat{Q}_{n+1}^+\).

If \(\theta'\) lies in the component of \(B_{n+1}^+ \cap D_{n+2,2p+2}\) which meets \(\zeta_{n,p}\), then \(\sigma\) is isotopic to \(\gamma_{n,p} \cup \rho_{n,p} \cup \zeta_{n,p}\) and \(\sigma^+\) is isotopic to a parallel copy of \(\alpha_{n,p} \cup \zeta_{n,p}\). Then \(\hat{\tau}\) is equivalent to the tangle \(\tau_{n,p}\) in \(R_{n+1,p}\), and so the exterior of \(\hat{\tau}\) in \(B_{n+1}^+\) is \(\partial\)-irreducible. Thus \(\partial \Delta = \partial \Delta'\) for a disk \(\Delta'\) in \(\partial \hat{Q}_{n+1}^+\). If \(\Delta'\) does not lie in \(\partial \hat{Q}_{n+1}^+\), then for homological reasons it must contain both components of \(N_{n+1}^+ \cap D_{n+2,2p+2}\). This is impossible since
then $\Delta'$ would also contain the component of $B_{n+1}^+ \cap D_{n+2,2p+2}$ which meets $\alpha_{n,p}$, and so $\Delta'$ could not lie in $\partial Q_{n+1}^+$. Thus $\Delta'$ lies in $\partial Q_{n+1}^+$.

We have thus shown that our triple has the halfdisk property.

We now consider the exterior $X_{n+2}^3$ of $N_{n+1}^3$ in $R_{n+2,p}$. This is the same as the exterior $X_{n+2}^3$ of $N_{n+1}^2$ in $R_{n+2,p}$ that we had in subcase (a). Thus $X_{n+2}^3$ is irreducible, $O_p$, $S_{n+1}^3$, and $X_{n+2}^3 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ are incompressible in $X_{n+2}^3$, and $(X_{n+2}^3, X_{n+2}^3 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), O_p \cup S_{n+1}^3)$ has the halfdisk property. We let $X_1 = X_{n+2}^3$ and complete the proof as usual.

Case 3: $S_n$ is incompressible in $X_0^n$, but $(X_0^n, X_0^n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_n)$ does not have the halfdisk property.

There is a proper disk $\Delta^1$ in $X_0^n$ such that, say, $\Delta^1 \cap D_{n+2,2p+1}$ is an arc $\theta_1$ and $\Delta^1 \cap S_n$ is an arc $\theta_2'$ such that $\partial \Delta^1 = \theta_1 \cup \theta_2'$ and neither arc is $\partial$-parallel. $\Delta^1$ splits $N^0_n$ not a 3-manifold consisting of two 3-balls $B_n^+$ and $B_n^-$. Let $S^1_n$ be the surface obtained by boundary compressing $S_n$ along $\Delta^1$. It has two components $\Sigma_n^-$ and $\Sigma_n$, where $\Sigma_n^-$ is an annulus with $\partial \Sigma_n^-$ in $D_{n+2,2p+1}$ and $\Sigma_n$ is a disk with two holes having one boundary component in $D_{n+2,2p+1}$ and the other two in $D_{n+2,2p+2}$. Let $X^1_{n+1}$ be the exterior of $N^1_n \cup N^-_n \cup N^+_n$ in $N_{n+1}$. We choose the notation so that $\Sigma_n^+ = B_n^- \cap X^1_{n+1}$ and $\Sigma_n = B_n \cap X^1_{n+1}$. Let $Q^-_n$ be the exterior of $\kappa' \cap F^-_n$ in $B_n^-$, and let $Q_n$ be the exterior of $\kappa' \cap B_n$ in $B_n$. Let $X_1 = Q^-_n \cup F^+_n$.

We have that $\Sigma_n^-$ and $Q_n \cap D_{n+2,2p+1}$ are incompressible in $Q^-_n$ and that $(Q^-_n, Q_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), \Sigma_n)$ has the halfdisk property. Also $\Sigma_n$ and $Q_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ are incompressible in $Q_n$.

Subcase (a): $(Q_n, Q_n \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), \Sigma_n)$ has the halfdisk property.

Consider $X^1_{n+1} = A_n^+ \cup A_n^- \cup \Sigma_n^+ \cup \Sigma_n^-$ incompressible in $X^1_{n+1}$ for homological reasons, as is $X^1_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$. The boundary of any compressing disk for $S_{n+1}$ in $X^1_{n+1}$ thus has to separate two of the components of $\partial S_{n+1}$ from the other two. So in $N_{n+1}$ the compressing disk must separate two of the components of $N_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ from the other two components. But this is impossible since each pair of these disks is joined by some component of $N^-_n \cup N^+_n \cup N^0_n$. So $S_{n+1}$ is incompressible in $X^1_{n+1}$.

Suppose $(X^1_{n+1}, X^1_{n+1} \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), S_{n+1} \cup A_n^- \cup A_n^+ \cup \Sigma_n^- \cup \Sigma_n)$ does not have the halfdisk property. Then there is a proper disk $\Delta$ in $X^1_{n+1}$ with $\Delta \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ an arc $\theta$ and $\Delta \cap (S_{n+1} \cup \Sigma_n)$ an arc $\theta'$ such that $\theta \cup \theta' = \partial \Sigma$ and neither arc is $\partial$-parallel.

Assume $\theta'$ lies in $\Sigma_n$. Then $\partial \theta'$ lies in one component of $\partial \Sigma_n$ and $\theta'$ separates the other two components. Then in the 3-manifold $\tilde{X}_{n+1}$ obtained by adjoining $N^-_n \cup N^+_n \cup B_n^-$ to $X^1_{n+1}$ we have that $\partial \Delta$ is isotopic to one of these two components of $\partial \Sigma_n$. But this is impossible since $\Sigma$ is incompressible in $\tilde{X}_{n+1}$ for homological reasons.

Thus $\theta'$ lies in $S_{n+1}$. $\theta$ lies in the unique component of $N_{n+1} \cap D_{n+2,2p+1}$ which meets both $B_n^-$ and $B_n$. $\Delta$ separates $N^-_n \cup B_n^-$ from $N^+_n \cup B_n$ in $N_{n+1}$. But $B_n^-$
contains the arc $\gamma_{n,p} \cup \delta_{n,p}$, and $B_n$ contains the arc $\alpha_{n,p} \cup \zeta_{n,p}$. Thus the tangle $\tau_{n,p}$ in $N_{n+1}$ is splittable, contradicting the fact that its exterior is $\partial$-irreducible.

Hence our triple has the halfdisk property.

Let $X^1_{n+2}$ be the exterior of $N_{n+1}$ in $R_{n+2,p}$. We already know that it is irreducible, $X^1_{n+2} \cap (D_{n+2,p} \cup D_{n+2,p+2})$ and $S_{n+1}$ are incompressible in $X^1_{n+2}$ and $(D_{n+2,p} \cup D_{n+2,p+2}) \cap (D_{n+2,p} \cup D_{n+2,p+2}), O_p \cup S_{n+1}$) has the halfdisk property. We let $X'_i = X^i_{n+2}$ and complete the proof as usual.

**Subcase (b):** $(Q_n, Q_n \cap (D_{n+2,p} \cup D_{n+2,p+2}), \Sigma_n)$ does not have the halfdisk property.

There is a proper disk $\Delta^2$ in $Q_n$ with $\Delta^2 \cap (D_{n+2,p} \cup D_{n+2,p+2})$ an arc $\theta_2$ and $\Delta^2 \cap \Sigma_n$ an arc $\theta'_2$ such that $\partial \Delta^2 = \theta_2 \cup \theta'_2$ and neither arc is $\partial$-parallel. $\Delta$ splits $B_n$ into a pair of 3-balls $B^0_n$ and $B^+$. Let $N^+_n = B^-_n \cup O_n \cup B^+_n$. Let $X^2_{n+1}$ be the exterior of $N^-_n \cup N^- \cup N^+_n$ in $N_{n+1}$. Let $S^0_n$ be the surface obtained by boundary compressing $S^1$ along $\Delta^2$. It consists of three annuli $\Sigma^-_n$, $\Sigma^-_n$, and $\Sigma^+_n$ with $\Sigma^+_n = B^+_n \cap X^2_{n+1}$ and $\Sigma^-_n = B^-_n \cap X^2_{n+1}$. Let $Q^+_n$ and $Q^-_n$ be the exteriors of $\kappa' \cap B^+_n$ and $\kappa' \cap B^-_n$, respectively. Let $X^2 = Q^- \cup Q^0 \cup Q^+.

$X^2_n$ is irreducible. $X^2_n \cap (D_{n+2,p} \cup D_{n+2,p+2})$ and $S^2_n$ are incompressible in $X^2_n$. $(X^2_n, (X^2_n \cap (D_{n+2,p} \cup D_{n+2,p+2}), \Sigma^2_n)$ has the halfdisk property.

Consider $X^2_{n+1}$. It is irreducible. $A^-_n$, $A^+_n$, $\Sigma^-_n$, $\Sigma^+_n$, and $\Sigma^0_n$ are incompressible in $X^2_{n+1}$ for homological reasons, as is $X^2_{n+1} \cap (D_{n+2,p} \cup D_{n+2,p+2})$. Thus the boundary of any compressing disk for $S_{n+1}$ in $X^2_{n+1}$ must separate two of the components of $\partial S_{n+1}$ from the other two. Hence the compressing disk must separate two of the components of $N_{n+1} \cap (D_{n+2,p} \cup D_{n+2,p+2})$ from the other two in $N_{n+1}$. But this is impossible since the union of these four disks with $N^2_n$ is connected. Thus $S_{n+1}$ is incompressible in $X^2_{n+1}$. Suppose $(X^2_{n+1}, X^2_{n+1} \cap (D_{n+2,p} \cup D_{n+2,p+2}), S_{n+1} \cup A^-_n \cup A^+_n \cup \Sigma^-_n \cup \Sigma^0_n \cup \Sigma^+_n)$ does not have the halfdisk property. Then there is a proper disk $\Delta$ in $X^2_{n+1}$ which meets $D_{n+2,p} \cup D_{n+2,p+2}$ in an arc $\theta$ and $S_{n+1}$ in an arc $\theta'$ such that $\theta \cup \theta' = \partial \Delta$ and neither arc is $\partial$-parallel. Let $N^\pm_{n+1}$ be the two 3-balls into which $N_{n+1}$ is split by $\Delta$.

There are six patterns in which $B^-_n$, $B^0_n$, and $B^+_n$ can connect the four components of $N_{n+1} \cap (D_{n+2,p} \cup D_{n+2,p+2})$. These patterns can be described as follows. Note that $C^p_n \cap N_{n+1}$ is a regular neighborhood of the union of a set of arcs $\alpha$, $\gamma$, $\delta$, $\zeta$, $\rho$, $\beta^-$, and $\beta^+$, where $(N_{n+1}, \gamma \cup \delta \cup \rho \cup \alpha \cup \zeta)$ is equivalent to $(R_{n+1,p}, \gamma_{n,p} \cup \delta_{n,p} \cup \rho_{n,p} \cup \alpha_{n,p} \cup \zeta_{n,p})$ and $\beta^-$ and $\beta^+$ are proper $\partial$-parallel arcs in the complement of this graph. The 3-balls $B^-_n$, $B^0_n$, and $B^+_n$ are regular neighborhoods of arcs which are certain unions of the $\alpha$, $\gamma$, $\delta$, $\zeta$, and $\rho$ in $N_{n+1}$, where two such arcs are pushed apart slightly to make the 3-balls disjoint.

(i) $B^-_n$, $B^0_n$, and $B^+_n$ are regular neighborhoods of $\gamma \cup \delta$, $\gamma \cup \rho \cup \alpha$, and $\gamma \cup \rho \cup \zeta$, respectively.
Then $N_n^- \cup B_n^-$ lies in $N_n^{-1}$, and $N_n^+ \cup B_n^0 \cup B_n^+$ lies in $N_n^{+1}$. Thus $\Delta$ separates $\gamma \cup \rho \cup \alpha$ and a copy of $\gamma \cup \delta$ in $N_{n+1}$. The exterior of this tangle is homeomorphic to the exterior of the graph $\gamma \cup \delta \cup \rho \cup \alpha$ in $N_{n+1}$ which in turn is homeomorphic to the exterior of the tangle $\tau = (\gamma \cup \delta) \cup (\alpha \cup \zeta)$ in $N_{n+1}$ and is therefore $\partial$-irreducible. So $\partial \Delta = \partial \Delta'$ for a disk $\Delta'$ in the boundary of the exterior of $B_n^- \cup B_n^0$ in $N_{n+1}$. But this is impossible since $\partial \Delta$ splits the boundary of this space into a pair of punctured tori. So this case cannot occur.

(ii) $B_n^-$, $B_n^0$, and $B_n^+$ are regular neighborhoods of $\gamma \cup \delta$, $\delta \cup \rho \cup \zeta$, and $\delta \cup \rho \cup \alpha$, respectively.

Then $N_n^- \cup B_n^-$ lies in $N_n^{-1}$, and $N_n^+ \cup B_n^0 \cup B_n^+$ lies in $N_n^{+1}$. Thus $\Delta$ separates $\delta \cup \rho \cup \alpha$ and a copy of $\gamma \cup \delta$ in $N_{n+1}$. The exterior of this tangle is homeomorphic to the exterior of the tangle $\tau = (\gamma \cup \delta) \cup (\alpha \cup \zeta)$ in $N_{n+1}$ and so is $\partial$-irreducible. So this case cannot occur.

(iii) $B_n^-$, $B_n^0$, and $B_n^+$ are regular neighborhoods of $\gamma \cup \delta$, $\gamma \cup \rho \cup \alpha$, and $\alpha \cup \zeta$, respectively.

There are then two possibilities. The first is that $N_n^- \cup B_n^-$ lies in $N_n^{-1}$ and $N_n^+ \cup B_n^0 \cup B_n^+$ lies in $N_n^{+1}$. Thus $\Delta$ separates $\gamma \cup \delta$ and $\alpha \cup \zeta$ in $N_{n+1}$. This is impossible since the exterior of this tangle is $\partial$-irreducible. The second is that $N_n^- \cup B_n^- \cup B_n^0$ lies in $N_{n+1}$ and $N_n^+ \cup B_n^+$ lies in $N_{n+1}$. Again $\Delta$ separates $\gamma \cup \delta$ and $\alpha \cup \zeta$ in $N_{n+1}$, so this case cannot occur.

(iv) $B_n^-$, $B_n^0$, and $B_n^+$ are regular neighborhoods of $\gamma \cup \delta$, $\delta \cup \rho \cup \zeta$, and $\alpha \cup \zeta$, respectively.

The first possibility is that $N_n^- \cup B_n^- \cup B_n^0$ lies in $N_n^{-1}$ and $N_n^+ \cup B_n^0 \cup B_n^+$ lies in $N_n^{+1}$. The second is that $N_n^- \cup B_n^- \cup B_n^0$ lies in $N_{n+1}$ and $N_n^+ \cup B_n^+$ lies in $N_{n+1}$. As in (iii) either of these implies that $\Delta$ separates $\gamma \cup \delta$ and $\alpha \cup \zeta$ in $N_{n+1}$, so this case cannot occur.

(v) $B_n^-$, $B_n^0$, and $B_n^+$ are regular neighborhoods of $\gamma \cup \delta$, $\gamma \cup \rho \cup \zeta$, and $\alpha \cup \zeta$, respectively.

The first possibility is that $N_n^- \cup B_n^- \cup B_n^0$ lies in $N_n^{-1}$ and $N_n^+ \cup B_n^0 \cup B_n^+$ lies in $N_n^{+1}$. The second is that $N_n^- \cup B_n^- \cup B_n^0$ lies in $N_{n+1}$ and $N_n^+ \cup B_n^+$ lies in $N_{n+1}$. Thus $\Delta$ separates $\gamma \cup \delta$ and $\alpha \cup \zeta$ in $N_{n+1}$, so this case cannot occur.

(vi) $B_n^-$, $B_n^0$, and $B_n^+$ are regular neighborhoods of $\gamma \cup \delta$, $\delta \cup \rho \cup \gamma$, and $\alpha \cup \zeta$, respectively.

The first possibility is that $N_n^- \cup B_n^- \cup B_n^0$ lies in $N_n^{-1}$ and $N_n^+ \cup B_n^0 \cup B_n^+$ lies in $N_n^{+1}$. The second is that $N_n^- \cup B_n^- \cup B_n^0$ lies in $N_{n+1}$ and $N_n^+ \cup B_n^+$ lies in $N_{n+1}$. Again $\Delta$ separates $\gamma \cup \delta$ and $\alpha \cup \zeta$ in $N_{n+1}$, so this case cannot occur.

So our triple has the halfdisk property.

Let $X_{n+2}$ be the exterior of $N_{n+1}$ in $R_{n+2,p}$. We already know that it is irreducible, $X_{n+2} ^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2})$ and $S_{n+1}$ are incompressible in $X_{n+2}^2$ and $(X_{n+2}^2, X_{n+2}^2 \cap (D_{n+2,2p+1} \cup D_{n+2,2p+2}), O_p \cup S_{n+1})$ has the halfdisk property. We let $X_i = X_i^2$ and complete the proof as usual.
14. Isotopy classification of the $V^P$

Proposition 14.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be finite non-empty sets of integers. $V^P$ and $V^Q$ are isotopic if and only if $\mathcal{P} = \mathcal{Q}$.

Proof. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ and $\mathcal{Q} = \{q_1, \ldots, q_n\}$ with their natural orderings. Suppose $h_t : V^P \to W$ is an isotopy with $h_0$ the inclusion map of $V^P$ into $W$ and $h_1(V^P) = V^Q$. Suppose that there is a $p \in \mathcal{P}$ such that $p \notin \mathcal{Q}$. We may assume that $V^p$ is an end reduction of $W$ at a knot $\kappa$. Then $h_1(V^p)$ is an end reduction of $W$ at $h_1(\kappa)$, and $h_1(\kappa) \subseteq h_1(V^p) \subseteq V^Q$.

First suppose that $q_1 \leq p \leq q_n$. Then as in the proof of Theorem 10.7 there is a plane $\Pi$ in $V^Q$ which is proper and non-trivial in $V^Q$ such that $V^Q - \Pi$ has components isotopic to $V^R$ and $V^S$, where $R = Q \cap [q_1, p]$ and $S = Q \cap [p, q_n]$. Since $h_1(V^p)$ is end irreducible relative to $h_1(\kappa)$ in $W$ we have that $h_1(V^p)$ is end irreducible relative to $h_1(\kappa)$ in the smaller set $V^Q$. Thus by Theorem 6.2 $h_1(V^p)$ can be isotoped to lie in $V^R$ or $V^S$. Hence we may assume that $Q$ has the property that either $q < p$ for all $q \in Q$ or $q > p$ for all $q \in Q$.

There is an $i \geq 0$ such that $h_1(\kappa) \subseteq \text{int} C_i^Q$. Since $V^p$ has genus one and is therefore $R^2$-irreducible we have by Theorem 13.1 that $h_1(V^p)$ is isotopic to $V^R$ for some good subset of $Q$. Since $h_1(V^p)$ has genus one we have by Proposition 11.1 that $R$ has a single element. Thus it suffices to prove the following result.

□

Lemma 14.2. If $p \neq q$, then $V^p$ is not isotopic to $V^q$.

Proof. We may assume that $p < q$ and that $V^p$ is an end reduction of $W$ at a knot $\kappa$ in $V^p$. Suppose $h_t : V^P \to W$ is an isotopy with $h_0$ the inclusion map of $V^P$ into $W$ and $h_1(V^P) = V^Q$.

Let $T = \cup_{t \in [0, 1]} h_t(\kappa)$ be the track of $\kappa$ under this isotopy. There exist integers $n$ and $r \leq s$ such that $T \subseteq \text{int} C_{n}^{[r, s]}$. Note that $r \leq p$ and $q \leq s$. By the Covering Isotopy Theorem [2, 3] there is an ambient isotopy $g_t : W \to W$ such that $g_0$ is the identity of $W$, $g_t(x) = h_t(x)$ for all $x \in \kappa$ and $t \in [0, 1]$, and $g_t(x) = x$ for all $x \in W - \text{int} C_{n}^{[r, s]}$ and $t \in [0, 1]$.

Recall that $C_{n}^{[r, s]}$ is the union of all those $R_{n,j}$ with $r - 1 \leq j \leq s$ and all those $L_{n,j}$ and $H_{n,j}$ with $r \leq j \leq s$. Now $C_{n}^p = R_{n,p-1} \cup L_{n,p} \cup H_{n,p} \cup R_{n,p}$, and $C_{n}^q = R_{n,q-1} \cup L_{n,q} \cup H_{n,q} \cup R_{n,q}$. We have that $\kappa \subseteq \text{int} C_{n}^p$, and the annulus $\partial H_{n,p} - \text{int} (H_{n,p} \cap L_{n,p})$ is incompressible in $C_{n}^p - \kappa$. Since $C_{n}^p$ meets the rest of $C_{n}^{[r, s]}$ in disjoint disks we have that this annulus is incompressible in $C_{n}^{[r, s]} - \kappa$. Since $g_1$ is the identity on $\partial C_{n}^{[r, s]}$, we have that the annulus is incompressible in $C_{n}^{[r, s]} - g_1(\kappa)$. But $g_1(\kappa) = h_1(\kappa) \subseteq V^q$, and so $g_1(\kappa)$ does not meet $H_{n,p}$, from which it follows that the annulus must be compressible in $C_{n}^{[r, s]} - g_1(\kappa)$. This contradiction completes the proof. □
Figure 9. Granny Whitehead solid tori $C^+$ and $C^-$

15. Homeomorphism classification of the $V^p$

Until now the sense of the Whitehead clasp in the 1-handle $H_{n,p}$ in our construction of $W$ has been immaterial. We will now modify our construction by choosing different senses for the clasp depending on $n$ and $p$. This will be used to construct uncountably many pairwise non-homeomorphic $W$ which cover 3-manifolds with infinite cyclic fundamental group such that the only 3-manifolds non-trivially covered by them must have infinite cyclic fundamental group. This modification will also be used to construct uncountably many pairwise non-homeomorphic $W$ which cannot non-trivially cover any 3-manifold.

Let $C$ be a solid torus with oriented meridian $m$ and longitude $\ell$ on $\partial C$ as shown in Figure 9. It is regarded as embedded in $\mathbb{R}^3 \subseteq S^3$ in the manner shown. We will blithely confuse an oriented simple closed curve on $\partial C$ and its homology class in $H_1(\partial C)$. Let $C^+$ and $C^-$ be the solid tori in the interior of $C$ as indicated in Figures 9(a) and 9(b), respectively.

**Lemma 15.1.** There is no homeomorphism $h : C \rightarrow C$ such that $h(C^+) = C^-$ and $h(\ell) = \pm \ell$.

**Proof.** Since $h(\ell) = \pm \ell$ we have that $h$ extends from $C$ to $S^3$. It must be orientation preserving since otherwise the granny knot is invariant under an orientation reversing homeomorphism of $S^3$. This cannot occur because the granny knot has signature $\pm 4$ and is therefore non-amphicheiral. (See Rolfsen [18].) Thus $h(m) = \pm m$, with the same sign as $h(\ell) = \pm \ell$. 

```text
(9)
```

(a)

(b)
Let \( t : C \to C \) be the homeomorphism obtained by cutting \( C \) along a meridinal disk, twisting, and regluing so that \( t(m) = m \) and \( t(\ell) = \ell + m \). The results of applying \( t \) to \( C^+ \) and \( C^- \) are shown in Figure 10.

Now \( t^{-1}(\ell) = \ell - m \), and so \( th^{-1}(\ell) = th(\ell - m) = t(\pm \ell \mp m) = \pm (\ell + m) \mp m = \pm \ell \pm m \mp m = \pm \ell \). Thus \( th^{-1} \) extends to a homeomorphism of \( S^3 \) carrying \( t(C^+) \) to \( t(C^-) \). But this is impossible since the core of \( t(C^+) \) is the sum of a granny knot and a figure eight knot while the core of \( t(C^-) \) is the sum of a granny knot and a trefoil knot. These knots can be distinguished by, for example, their Alexander polynomials.

We remark that with slightly more work the condition \( h(\ell) = \pm \ell \) can be dropped from the hypotheses of this lemma. However, we will not need this stronger result.

Now let \( \varphi : \mathbb{N} \to \{\pm 1\} \) be a function. We construct a contractible open 3-manifold \( V[\varphi] \) by specifying an exhaustion \( \{C_n\}_{n \geq 0} \) where the model for the pair \((C_{n+1}, C_n)\) is \((C, C^\pm)\) for \( \varphi(n) = \pm 1 \). Note that \( V[\varphi] \) is a modified version of \( V^p \).

**Proposition 15.2.** \( V[\varphi] \) and \( V[\varphi'] \) are homeomorphic if and only if there are integers \( m_0, n_0 \geq 0 \) such that \( \varphi(m_0 + i) = \varphi'(n_0 + i) \) for all \( i \geq 0 \).

**Proof.** Denote \( V[\varphi] \) and \( V[\varphi'] \) by \( V \) and \( V' \), respectively. Distinguish their defining exhaustions by \( \{C_n\} \) and \( \{C'_n\} \), respectively. Clearly the condition is sufficient, so assume that \( h : V \to V' \) is a homeomorphism.

Choose \( p \geq 0 \) such that \( h(C_0) \subseteq int C'_p \). Then choose \( q \geq 1 \) such that \( C'_p \subseteq int h(C_q) \). Now \( h(\partial C_q) \) is incompressible in \( V' - h(C_0) \) and so is incompressible in
the smaller space \( V' - C'_p \). Put \( h(\partial C_q) \) in minimal general position with respect to \( \cup_{n>p} \partial C'_n \). Then no component of the intersection bounds a disk on either surface. If the intersection is non-empty, then let \( n \) be the smallest \( n > p \) such that \( h(\partial C_q) \cap \partial C'_n \neq \emptyset \). There is then a component \( A \) of \( h(\partial C_q) \cap C'_n \) which is an annulus lying in \( Y' = C'_n - \text{int} C'_{n-1} \). Recall that \( Y' \) is the union of a Whitehead link exterior \( Y'_0 \) and two disjoint trefoil knot exteriors \( Y'_1 \) and \( Y'_2 \) with \( Y'_0 \cap Y'_j = A'_j \) an annulus, \( j = 1, 2 \).

As in the proof of Lemma 9.6 we have that \( A \) is parallel in \( Y' \) to an annulus in \( \partial C'_n \), contradicting minimality. Thus we have that the intersection is empty.

So \( h(\partial C_q) \) lies in \( Y' = C'_n - \text{int} C'_{n-1} \) for some \( n > p \). Put it in minimal general position with respect to \( A'_1 \cup A'_2 \). As usual there are no trivial intersection curves.

**Case 1:** The intersection is non-empty. Then there is an annulus component \( A \) of, say, \( h(\partial C_q) \cap Y'_1 \). Since it has meridian boundary components on \( \partial Y'_1 \) it must be parallel in \( Y'_1 \) to an annulus \( A' \) in \( \partial Y'_1 \). By minimality \( A' \) cannot lie in \( A'_1 \), so it must contain \( Y'_1 \cap \partial C'_{n-1} \). By minimality we may assume that no component of \( h(\partial C_q) \cap Y'_1 \) lies between \( A \) and \( Y'_1 \cap \partial C'_{n-1} \) in the product \( I \) bundle joining \( A \) and \( Y'_1 \cap \partial C'_{n-1} \).

Consider the annuli \( A_0 = h(\partial C_q) - \text{int} A \) and \( A'_0 = (Y'_1 \cap \partial C'_{n-1}) \cup (A' \cap A'_1) \). \( A'_0 \) is a proper incompressible annulus in \( h(C_q - \text{int} C_0) \) with \( \partial A'_0 \) in \( h(\partial C_q) \). It follows from the annularity of the Whitehead link exterior \( \{1\} \) that \( A'_0 \) is \( \partial \)-parallel in this space. It cannot be parallel to \( A \) since \( A \cup A'_0 \) bounds a solid torus containing \( C'_{n-1} \) and hence \( h(C_0) \). Therefore \( A'_0 \) is parallel to \( A_0 \). It follows that \( h(\partial C_q) \) is parallel in \( Y' \) to \( \partial C'_n \). We then isotop \( h \) so that \( h(C_q) = C'_{n-1} \). In this case we let \( m_0 = q \) and \( n_0 = n - 1 \).

**Case 2:** \( h(\partial C_q) \cap (A'_1 \cup A'_2) = \emptyset \). Then \( h(\partial C_q) \) lies in \( Y'_0 \). Since \( Y'_0 \) is atoroidal \( h(\partial C_q) \) is \( \partial \)-parallel in \( Y'_0 \). We isotop \( h \) so that \( h(\partial C_q) \) is a component of \( \partial Y'_0 \).

Suppose \( h(\partial C_q) = \partial Y'_0 - \partial C'_m \). Then \( A'_1 \) is a proper incompressible annulus in \( h(C_q - \text{int} C_0) \) with \( \partial A'_1 \) in \( h(\partial C_q) \), so it must be \( \partial \)-parallel in this space. But this is impossible since \( A'_1 \) splits this space into the trefoil knot space \( Y'_1 \) and a space with two boundary components. So this situation cannot occur.

Thus \( h(\partial C_q) = \partial C'_n \). In this case we let \( m_0 = q \) and \( n_0 = n \).

We now have that \( h(C_{m_0}) = C'_{m_0} \). Using the methods just employed we isotop \( h \) rel \( C_{m_0} \) so that \( h(C_{m_0+1}) = C'_{m_0+1} \). By Lemma 15.1 we have that \( \varphi(m_0) = \varphi'(n_0) \). We then isotop \( h \) rel \( C_{m_0+1} \) so that \( h(C_{m_0+2}) = C'_{m_0+2} \) and conclude that \( \varphi(m_0 + 1) = \varphi'(n_0 + 1) \). We then continue this process to get that \( \varphi(m_0 + i) = \varphi'(n_0 + i) \) for all \( i \geq 0 \).

\[ \square \]

16. The Complex of End Reductions

Let \( W \) be a Whitehead manifold. An end reduction \( V \) of \( W \) is **minimal** if whenever \( U \) is an end reduction of \( W \) which is contained in \( V \) we have that \( U \) is isotopic to \( V \).

**Theorem 16.1.** Genus one end reductions of \( W \) are minimal and \( \mathbb{R}^2 \)-irreducible.

**Proof.** Suppose \( V \) is a genus one end reduction of \( W \). Then \( V \) has an exhaustion \( \{C_n\}_{n \geq 0} \) such that each \( \partial C_n \) is a torus. \( \partial C_n \) is incompressible in \( W - \text{int} C_0 \). By
Lemma \( V \) is a Whitehead manifold, and so \( \partial C_n \) must be compressible in \( C_n \). Since \( V \) is irreducible and \( V - \text{int} C_n \) is non-compact \( C_n \) must be a solid torus.

Let \( J \) be a regular submanifold of \( W \) such that \( J \subseteq V \). Then \( J \subseteq \text{int} C_n \) for some \( n \). If \( \partial C_n \) is compressible in \( C_n - J \), then \( J \) lies in a 3-ball, contradicting the fact that it is regular. Therefore \( \partial C_n \) is incompressible in \( C_n - J \) and hence in \( W - J \). By Lemma \( X \) we then have that any end reduction of \( W \) at \( J \) is isotopic to \( V \).

The \( R^2 \)-irreducibility of genus one Whitehead manifolds was proven by Kinsoshita [1].

We remark that \( R^2 \)-irreducible end reductions need not be minimal (any \( V^p \) with \( P \) a good set having more than one element is an example), and minimal end reductions need not be \( R^2 \)-irreducible (the double of the Tucker manifold [21] can be shown to be an example).

We now define the simplicial complex of minimal, \( R^2 \)-irreducible end reductions of \( W \), denoted \( S(W) \).

The vertices of \( S(W) \) are the isotopy classes \( [V] \) of minimal, \( R^2 \)-irreducible end reductions of \( W \).

Two distinct vertices \([V_0] \) and \([V_1] \) are joined by an edge if there is an \( R^2 \)-irreducible end reduction \( E_{0,1} \) of \( W \) having the following properties: (1) \( E_{0,1} \) contains representatives of \([V_0] \) and \([V_1] \). (2) Every \( R^2 \)-irreducible end reduction of \( W \) which is contained in \( E_{0,1} \) is isotopic to \( V_0 \), \( V_1 \), or \( E_{0,1} \). (3) Among \( R^2 \)-irreducible end reductions of \( W \) one has that \( E_{0,1} \) is unique up to isotopy with respect to properties (1) and (2).

Three distinct vertices \([V_0], [V_1], \) and \([V_2] \) span a 2-simplex if each pair of vertices is joined by an edge and there is an \( R^2 \)-irreducible end reduction \( T_{0,1,2} \) of \( W \) having the following properties: (1) \( T_{0,1,2} \) contains representatives of \([V_0], [V_1], [V_2], [E_{0,1}], [E_{1,2}], \) and \([E_{2,0}] \). (2) Every \( R^2 \)-irreducible end reduction of \( W \) which is contained in \( T_{0,1,2} \) is isotopic to one of these six end reductions or \( T_{0,1,2} \). (3) Among \( R^2 \)-irreducible end reductions \( T_{0,1,2} \) is unique with respect to properties (1) and (2).

There is an obvious generalization of these definitions which inductively defines simplices of higher dimension.

**Theorem 16.2.** Let \( W \) be a member of the family \( F \) of Whitehead manifolds constructed in section 9. Then \( S(W) \) is isomorphic to a triangulation of \( R \).

**Proof.** Each \( V^p \) has genus one and so by Theorem 16.1 is minimal and \( R^2 \)-irreducible. By Theorem 14.1 \( V^p \) and \( V^q \) are isotopic if and only if \( p = q \).

Suppose \( V \) is \( R^2 \)-irreducible and minimal. By Theorem 13.1 \( V \) is isotopic to \( V^p \) for some good set \( P \). If \( P \) has more than one element, then for \( p \in P \) we have by Theorem 11.1 that \( V^p \) is not isotopic to \( V^p \), and so \( V \) is not minimal. This contradiction implies that \( V \) must be isotopic to some \( V^p \). We thus can associate the vertices of \( S(W) \) with the integers.

Given \( p \), let \( R = \{p, p + 1\} \). Then \( V^p \) and \( V^{p+1} \) are contained in \( V^R \). By Corollary 10.5 \( V^R \) is \( R^2 \)-irreducible. Let \( V \) be an \( R^2 \)-irreducible end reduction of \( W \) which
is contained in \( V^\mathcal{R} \). By Theorem 13.1 \( V \) is isotopic to \( V^\mathcal{Q} \) for some subset \( \mathcal{Q} \) of \( \mathcal{R} \). Hence \( V \) is isotopic to \( V^p, V^{p+1} \), or \( V^\mathcal{R} \). We will see in the next paragraph that \( V^\mathcal{R} \) is unique up to isotopy with respect to these properties, and thus the vertices corresponding to \( p \) and \( p + 1 \) are joined by an edge in \( \mathcal{S}(W) \).

Consider integers \( p < q \). Suppose \( E \) is an \( \mathbb{R}^2 \)-irreducible end reduction which contains representatives of \([V^p]\) and \([V^q]\) and has the property that every end reduction of \( W \) which is contained in \( E \) is isotopic to \( V^p, V^q \), or \( E \). Theorem 13.1 says that \( E \) must be isotopic to \( V^\mathcal{R} \) for some good set \( \mathcal{R} \). Thus \( V^p \) and \( V^q \) must be isotopic to subsets of \( V^\mathcal{R} \). Since they are \( \mathbb{R}^2 \)-irreducible by Theorem 13.1 they are isotopic to \( V^\mathcal{P} \) and \( V^\mathcal{Q} \), respectively, where \( \mathcal{P} \) and \( \mathcal{Q} \) are good subsets of \( \mathcal{R} \). Since they have genus one these sets must each have one element. By Corollary 10.5 we then have that \( \mathcal{P} = \{p\} \) and \( \mathcal{Q} = \{q\} \). Thus \( p, q \in \mathcal{R} \). If \( \mathcal{R} \neq \{p, q\} \) then choose a third element \( r \in \mathcal{R} \). By Lemma 14.2 \( V^r \) is not isotopic to \( V^p \) or \( V^q \). Since \( V^r \) has genus one and \( V^\mathcal{R} \) does not we have that \( V^r \) is not isotopic to \( V^\mathcal{R} \). This contradicts a property of \( E \). Thus \( \mathcal{R} = \{p, q\} \). If \( q \neq p + 1 \), then by Theorem 10.7 \( V^\mathcal{R} \) is not \( \mathbb{R}^2 \)-irreducible. This contradicts another property of \( E \). Thus \( q = p + 1 \), \( \mathcal{R} = \{p, p + 1\} \) and \( E \) is isotopic to \( V^\mathcal{R} \).

Thus two vertices are joined by an edge if and only if they correspond to consecutive integers. It follows that there are no higher dimensional simplices, and so \( \mathcal{S}(W) \) is isomorphic to a triangulation of \( \mathcal{R} \). \( \square \)

17. Covering translations

**Theorem 17.1.** Let \( W \) be a Whitehead manifold, and let \( \mathcal{S}(W) \) be the simplicial complex of minimal, \( \mathbb{R}^2 \)-irreducible end reductions of \( W \). Suppose \( W \) is a covering space of a 3-manifold \( M \). Then \( \pi_1(M) \) is isomorphic to a fixed point free, torsion free group of simplicial automorphisms of \( \mathcal{S}(W) \).

**Proof.** Since \( W \) is contractible \( M \) is a finite dimensional \( K(\pi, 1) \), implying that \( \pi_1(M) \) has finite cohomological dimension. Since groups with torsion have infinite cohomological dimension \( \square \) have that \( \pi_1(M) \) is torsion free.

\( \pi_1(M) \) is isomorphic to the group of covering translations of the covering map \( p : W \rightarrow M \). From the definition of \( \mathcal{S}(W) \) it is immediate that any group of homeomorphisms of \( W \) induces a group of simplicial automorphisms. If some point of \( \mathcal{S}(W) \) were fixed by the automorphism \( \gamma \) associated to a non-trivial covering translation \( g \), then the simplex of minimum dimension containing that point would be carried to itself by \( \gamma \). In particular the vertices of the simplex would be permuted by \( \gamma \). So some power \( \gamma^k, k > 0 \), would fix a vertex \([V]\). This would imply that \( g^k(V) \) is isotopic to \( V \). By the Main Theorem X, \( g^k \) would be the identity. Since \( \pi_1(M) \) is torsion free this would imply that \( g \) is trivial, a contradiction. \( \square \)

**Theorem 17.2.** There are uncountably many pairwise non-homeomorphic \( \mathbb{R}^2 \)-irreducible Whitehead manifolds \( W \) which are covering spaces of 3-manifolds \( W^\# \) with
\( \pi_1(W^\#) \) infinite cyclic and have the property that whenever \( W \) non-trivially covers a 3-manifold \( M \) one has that \( \pi_1(M) \) is infinite cyclic.

**Proof.** Choose a function \( \varphi : \mathbb{N} \to \{\pm 1\} \). We let \( W^\#[\varphi] \) be the 3-manifold with \( \pi_1(W^\#[\varphi]) \) infinite cyclic which was constructed in Section 9, where the sense of the clasp in the 1-handle \( H_n \) is determined by \( \varphi(n) \) as explained in Section 15. We let \( W[\varphi] \) be its universal covering space. All of the \( V^\rho \) in \( W[\varphi] \) are homeomorphic to the genus one Whitehead manifold \( V[\varphi] \) defined in Section 15. If \( W[\varphi] \) and \( W[\varphi'] \) are homeomorphic, then the minimal \( \mathbb{R}^2 \)-irreducible end reductions of one must be homeomorphic to those of the other, and so we have that \( V[\varphi] \) and \( V[\varphi'] \) are homeomorphic. By Theorem 15.2 we have that there exist \( m_0 \) and \( n_0 \) such that for all \( i \geq 0 \) one has that \( \varphi(m_0 + i) = \varphi'(n_0 + i) \). There are uncountably many equivalence classes of functions \( \varphi \) under this relation and so uncountably many pairwise non-homeomorphic \( W[\varphi] \).

(More concretely, given any sequence \( \{s_k\} \) of 1’s and -1’s construct the sequence \( \{t_n\} \) for which \( t_0 = s_0, t_1 \) and \( t_2 \) equal \( s_1, t_3 \) through \( t_6 \) equal \( s_2, t_7 \) through \( t_{14} \) equal \( s_3 \), etc. For each \( k \) one has a block of \( 2^k \) copies of \( s_k \). Then let \( \varphi(n) = t_n \). Two sequences \( \{s_k\} \) and \( \{s'_k\} \) are equal if and only if the corresponding sequences \( \{t_n\} \) and \( \{t'_n\} \) are equal and thus are equal if and only if the functions \( \varphi \) and \( \varphi' \) are equal. But \( \varphi(m_0 + i) = \varphi'(n_0 + i) \) if and only if \( \varphi = \varphi' \).

Now suppose that one of these \( W \) non-trivially covers a 3-manifold \( M \). By the previous theorem \( \pi_1(M) \) is isomorphic to a subgroup of the simplicial automorphism group of \( \mathcal{S}(W) \). Since \( \mathcal{S}(W) \) is isomorphic to a triangulation of \( \mathbb{R} \), its automorphism group is isomorphic to the infinite dihedral group, whose only non-trivial torsion free subgroups are infinite cyclic. \( \square \)

**Theorem 17.3.** There are uncountably many pairwise non-homeomorphic \( \mathbb{R}^2 \)-irreducible, non-eventually end-irreducible Whitehead manifolds \( W \) which cannot non-trivially cover any 3-manifold.

**Proof.** We modify the construction of \( W[\varphi] \) as follows. Choose a function \( \psi : \mathbb{N} \to \pm 1 \) such that \( V[\varphi] \) is not homeomorphic to \( V[\psi] \). Replace the original manifold \( V^0 \) which was homeomorphic to \( V[\varphi] \) by a new \( V^0 \) homeomorphic to \( V[\psi] \). This can be done by changing the claspings in the 1-handles \( H_{n,0} \) in the expression of \( W[\varphi] \) as a monotone union of infinite genus handlebodies. Call the new manifold \( W \).

Any homeomorphism \( g \) of \( W \) must induce an automorphism \( \gamma \) of \( \mathcal{S}(W) \) which fixes \( [V^0] \). Thus \( g(V^0) \) is isotopic to \( V^0 \). If \( g \) were a covering translation it would therefore have to be the identity. \( \square \)

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