Curving the space by non-Hermiticity

Chenwei Lv\(^1,4\) , Ren Zhang\(^{1,2,4}\) , Zhengzheng Zhai\(^1\) & Qi Zhou\(^{1,3,✉}\)

Quantum systems are often classified into Hermitian and non-Hermitian ones. Extraordinary non-Hermitian phenomena, ranging from the non-Hermitian skin effect to the super-sensitivity to boundary conditions, have been widely explored. Whereas these intriguing phenomena have been considered peculiar to non-Hermitian systems, we show that they can be naturally explained by a duality between non-Hermitian models in flat spaces and their counterparts, which could be Hermitian, in curved spaces. For instance, prototypical one-dimensional (1D) chains with uniform chiral tunnelings are equivalent to their duals in two-dimensional (2D) hyperbolic spaces with or without magnetic fields, and non-uniform tunnelings could further tailor local curvatures. Such a duality unfolds deep geometric roots of non-Hermitian phenomena, delivers an unprecedented routine connecting Hermitian and non-Hermitian physics, and gives rise to a theoretical perspective reformulating our understandings of curvatures and distance. In practice, it provides experimentalists with a powerful two-fold application, using non-Hermiticity to engineer curvatures or implementing synthetic curved spaces to explore non-Hermitian quantum physics.

\(^1\)Department of Physics and Astronomy, Purdue University, West Lafayette, IN 47907, USA. \(^2\)School of Physics, Xi’an Jiaotong University, Xi’an, Shaanxi 710049, China. \(^3\)Purdue Quantum Science and Engineering Institute, Purdue University, West Lafayette, IN 47907, USA. \(^4\)These authors contributed equally: Chenwei Lv, Ren Zhang. ✉email: zhou753@purdue.edu
system-environment couplings lead to a plethora of intriguing non-Hermitian phenomena, such as non-orthogonal eigenstates, the non-Hermitian skin effect, real energy spectra in certain parameter regimes, and drastic responses to boundary conditions. While these phenomena have been extensively explored in quantum sciences and technologies, peculiar theoretical tools are often required to study non-Hermitian physics. In this work, we show a duality between non-Hermitian Hamiltonians in flat spaces and their counterparts in curved spaces, and discuss the existence of the PT symmetry does not guarantee a real energy spectrum and sophisticated mathematical techniques are required.

In practice, our duality has a two-fold implication. On the one hand, it establishes non-Hermiticity as a unique tool to simulate intriguing quantum systems in curved spaces. For instance, it offers an approach of using non-Hermitian systems in exploring the grand challenge of accessing gravitational responses of quantum Hall states (QHS) in curved spaces. On the other hand, the duality allows experimentalists to use curved spaces to explore non-Hermitian physics. Whereas a variety of non-Hermitian phenomena have been addressed in experiments, delicate designs of dissipations are often required. Our results show that curved spaces can serve as an unprecedented means to explore non-Hermitian Hamiltonians without resorting to dissipations.

Results

Hatano-Nelson model and hyperbolic surfaces. Our duality can be demonstrated using the celebrated Hatano-Nelson (HN) model, which reads,

$$-i_{\text{R}} \psi_{n-1} - i_{\text{L}} \psi_{n+1} = E \psi_n,$$

where $E$ is the eigenenergy, $E$ is the corresponding eigenenergy, and $i_{\text{L}}$ and $i_{\text{R}}$ are the nearest-neighbor tunneling amplitudes towards the left and the right, respectively. Under the open boundary condition (OBC), $\psi_0 = \psi_{N+1} = 0$, and $\psi_n = e^{i\gamma n} \sin(k_{\text{nn}} d) / (\sqrt{(N-1)/2})$, where $\gamma = i_{\text{L}} / i_{\text{R}}$ characterizes the strength of non-Hermiticity, $k_{\text{nn}} = m \pi / (N-1)d$, $m = 1, 2, \ldots, N-2$, and $d$ is the lattice constant. The eigenenergy reads $E_m = -2 \sqrt{i_{\text{L}} i_{\text{R}}} \cos(k_{\text{nn}} d)$. Our duality can restore orthogonality.

Similar to Hermitian lattice models, the effective theory of Eq. (1) in the continuum limit describes the motion of a non-relativistic (relativistic) particle at (away from) the band bottom and top, with a quadratic (linear) dispersion relation, as shown in Fig. 1a. At the band bottom, the effective theory is written as

$$- \frac{\hbar^2}{2M} \kappa \left( y^2 \partial_y^2 + 1 \right) \psi(y) = E \psi(y),$$

where $M = \hbar^2 / (2 \sqrt{i_{\text{L}} i_{\text{R}}} d^2)$ is the effective mass, and $\kappa = 4 \hbar^2 (|y|)^2 / d^2$. Solutions to Eq. (2), $y \sqrt{\kappa} \chi \kappa / \sqrt{\kappa}$, have the same energy, $\hbar^2 \kappa^2 / (2M)$. An eigenstate under OBC is their superposition,

$$\psi(y) = \sqrt{2 / \ln(y_{N-1} / y_0)} y(y_0) \sin \left[ k_0 \ln(y_{N-1} / y_0) / \sqrt{\kappa} \right],$$

with $k_0 = m \pi / (N-1)d$ and $y_0(y_{N-1})$ are the positions of the two edges. At the band top, we have $M = -M$. Eq. (2) is a dimension reduction of the Schrödinger equation on a Poincaré half-plane,

$$- \frac{\hbar^2}{2M} \kappa \left( y^2 \partial_x^2 + 1 \right) \psi(x, y) = E \Psi(x, y),$$

where $V^2 \equiv (\partial_x^2 + \partial_y^2)$, $\Psi(x, y) = e^{i\kappa_0 x} \psi(y)$, and $-\kappa$ is the curvature (Supplementary Material). Since $m$ is a good quantum number, Eq. (3) reduces to Eq. (2) when $k_0 = 0$. A finite $k_0$ adds an onsite potential to the HN model,

$$V_n \psi_n - t_0 \psi_{n-1} - t_1 \psi_{n+1} = E \psi_n,$$

where $V_n = a^2 \sqrt{i_{\text{L}} i_{\text{R}}} \chi \kappa$. The dimensionless quantity $a^2 = 4(\ln(\gamma)) \gamma \kappa_0^2$ characterizes the strength of $V_n$.

To derive the duality between the continuum limit of Eq. (1) at the band bottom and Eq. (2), we define $\psi_n \equiv \sqrt{\kappa} \psi(n)$ with $s_n = nd$, such that the eigenstate of the HN model, $\psi_n$, defined on discrete lattice sites is extended to $\psi(s)$ as a function of a continuous variable $s$. Since $\psi_n$ under OBC, includes a part that

---

**Fig. 1** The duality between the Hatano-Nelson (HN) model and a hyperbolic surface. **a** A HN chain and its energy spectrum as a function of $k$. Near a vanishing (finite) $K_0$, the effective theory in curved space is non-Hermitian. Eigenstates on the HN chain are localized at the edge, $|\psi_n|^2 \propto |y|^2$, $y > 0$. A HN chain is mapped to the shaded strip on the Poincaré half-plane, in which an eigenstate with $k_0 = 0$ satisfies $|\psi|^2 \propto y$. This shaded strip on the Poincaré half-plane with PBC in the $x$-direction is equivalent to a pseudosphere embedded in 3D Euclidean space. **b** The curvature and the inverse of the effective mass, as functions of $t_1$ for a fixed $t_0$. The units of $\kappa$ and $M^2$ are $1 / d^2$ and $2t_0 d^2 / (\hbar^2)$, respectively. (ii-iv) show the dual pseudospheres of the HN model at various $t_1 > 0$. A pseudosphere for $t_1 < 0$ is the same as that for $-t_1$. **c** The dimensionless effective mass $M$ as a function of $t_1 / t_0$. The bright band bottom and top, with a quadratic (linear) dispersion relation, as shown in Fig. 1a. At the band bottom, the effective theory is written as...

---

In practice, our duality has a two-fold implication. On the one hand, it establishes non-Hermiticity as a unique tool to simulate intriguing quantum systems in curved spaces. For instance, it offers an approach of using non-Hermitian systems in flat spaces to solve the grand challenge of accessing gravitational responses of quantum Hall states (QHS) in curved spaces. On the other hand, the duality allows experimentalists to use curved spaces to explore non-Hermitian physics. Whereas a variety of non-Hermitian phenomena have been addressed in experiments, delicate designs of dissipations are often required. Our results show that curved spaces can serve as an unprecedented means to explore non-Hermitian Hamiltonians without resorting to dissipations.
changes exponentially, i.e., \( e^{\text{const}} \), so does \( \psi \). We thus define \( \phi(s) \equiv \psi(s)e^{-\psi} \) with \( q = \ln(\gamma)/\ell = \frac{1}{2}\ln(\ell_0/\ell_t) \) determining the inverse of the localization length, and \( \phi \) varies slowly with changing \( t \). Then we have \( \psi_n = \sqrt{\Delta}(\phi(s))e^{\psi} \). Substituting \( \psi_n \) into Eq. (1) and using the Taylor expansion for \( \phi(s) \), \( \phi(s \pm \ell_t) = \phi(s) \pm \frac{1}{2}d_2^2 \psi(s) \), we obtain: 
\[
-\sqrt{\ell_0\ell_t}(2 + \ell_t^2)\phi = E\phi. 
\]
Consequently, \( \psi(s) \) satisfies 
\[
-\sqrt{\ell_0\ell_t}d_2^2(\ell_0^2 - 2q^2 + q^2 + j^2/d^2)\psi(s) = E\psi(s). 
\]
(5)

It describes a nonrelativistic particle subject to an imaginary vector potential, \( A_0 = \alpha_t \psi \). Unlike a real vector potential that amounts to a \( U(1) \) gauge field, here, an imaginary vector potential curves the space. Performing a coordinate transformation \( y/y_0 = e^{2\eta} \) and applying \( M = h^2/(2\sqrt{\ell_0\ell_t}K) \), \( K = 4n^2(\ln|\gamma|)/d^2 \), we obtain Eq. (2) up to a constant energy shift \( -2\sqrt{\ell_0\ell_t}\ell_R \). The mapping between these two models is summarized in Table 1, which provides a dictionary translating microscopic parameters between them. For instance, \( \psi(s) \), the wavefunction on the \( n \)-th lattice site of the NH model is identical to \( \psi(y_0) \), the wavefunction on the Poincaré half-plane evaluated at \( y = y_0\). The low-energy limit of the eigenenergy of Eq. (1) is also identical to the eigenenergy of Eq. (2) as shown by Table 1.

Away from the band bottom(top), similar calculations can be performed by defining \( \psi(s) = e^{-iK_1s}\rho^2(s) \) using Taylor expansions of the slowly varying \( \phi(s) \) and the same coordinate transformation \( y = y_0\). We obtain the effective theory near \( K_0 \neq 0, \pm \pi, \)
\[
\left( E(K_0) \pm i\sqrt{\hbar}\nu y_0(y_0 - 1/2(y_2)) \right) \psi(y) = E\psi(y), 
\]
(6)
where \( E(K_0) = -2\sqrt{\ell_0\ell_t}\cos(K_0) + K_0\sin(K_0) \), \( \nu = -2\sqrt{\ell_0\ell_t}\sin(K_0)/d_0 \), and \( \gamma \) corresponds to the left and right moving waves centered near \( \pm \pi, \) respectively. The previously defined \( K = 4n^2(\ln|\gamma|)/d^2 \) has been used. The eigenstate under OBC includes both the left and right moving waves and is written as \( \sqrt{2}\ln|\gamma_0|y_0(y_0 - 1/2(y_2)) \sin[K_0\ln|\gamma_0|]/\sqrt{\gamma} \) with eigenenergy of \( -2\sqrt{\ell_0\ell_t}\cos(K_0) + (K_0 - k_0)\sin(K_0) \), which recovers the results of the NH model near a finite \( K_0 \).

Geometric interpretations of non-Hermitian phenomena. Our duality provides a natural explanation of several peculiar non-Hermitian phenomena. Firstly, the orthonormal condition of effective theories in Eq. (2) and Eq. (6) reads
\[
\int \frac{dy}{\gamma^2} \psi(y; k)\psi(y; k') = N\delta_{k, k'}, 
\]
where the normalization constant \( N \) can be chosen freely. As a common feature of curved spaces, a finite curvature appears above the equilibrium. Considering a strip in the domain \( x_0 \leq x \leq x_0 + L \), its width in the \( x \)-direction depends on \( y, L(y) = \int_{x=x_0}^{x_0+L} dx/(\sqrt{\gamma y}) \). Thus, a wave packet traveling in the \( y \)-direction must include an extra factor \( \gamma^2 \) to guarantee the conservation of particle numbers. In the \( s \)-coordinate, Eq. (7) is written as
\[
\int dy (y_0) e^{-2\psi}(y) \psi(s, s) = N\delta_{k, k'}, 
\]
Discretizing this equation with \( N = (\sqrt{\gamma y_0})^{-1} \), and transforming it to the NH model, we obtain,
\[
\sum_j |y|^{-2n}y_n^*(k_m)\psi_j(k_m) = \delta_{k, k'}, 
\]
where \( |y|^{-2n} \) is precisely the difference between the left and right eigenvectors, or the metric operator. The mapping to a curved space thus establishes an explicit physical interpretation of orthonormal conditions in non-Hermitian systems.

Secondly, the duality allows us to equate the non-Hermitian skin effect to its counterpart on the Poincaré half-plane we found recently. This can be best visualized using the embedding of a hyperbolic surface in three-dimensional (3D) Euclidean space. We define \( y = y_0 \cosh(\gamma) \), \( x = r_0 \phi \), where \( r_0 \) is an arbitrary constant and \( \eta > 0 \), \( \varphi \in (-\pi, \pi) \). The embedding can then be written as
\[
(u, v, w) = \frac{1}{\sqrt{\gamma}} (\eta - \tanh(\eta)), \left( \cosh(\varphi) \frac{\sin(\varphi)}{\cosh(\eta)}, \frac{\cos(\varphi)}{\cosh(\eta)} \right). 
\]
(9)

This is a parameterization of a pseudosphere with a constant negative curvature and a radius of \( 1/\sqrt{\gamma} \), which satisfies \( (u + \text{arcsech}(v^2 + w^2))\gamma = \sqrt{-v^2 + w^2} = k^2 \). As shown in Fig. 1b, a pseudosphere features a funnel shape, since the circumferential of the circle with a fixed \( y(\eta) \) changes with changing \( y(\eta) \). As previously explained, a coordinate transformation \( y = y_0e^{2\eta} \) maps eigenstates on the hyperbolic surface, \( y_0^2 \gamma^{1/2}/\sqrt{\gamma} \), to \( \rho^2 \gamma^{1/2}/\sqrt{\gamma} \), which exponentially localizes near the funneling mouth, the smaller end.

Thirdly, the collapsed energy spectrum at EP of the NH models has a natural geometric interpretation. When \( t_0 = t_0 \), the pseudosphere reduces to a cylinder with a vanishing \( K \). For a given \( t_0 > t_1 \), \( K \) increases with decreasing \( t_1 \). Increasing the non-Hermiticity thus makes the space more curved, as shown by Fig. 1red. Approaching EP, \( t_1 \to 0, \) \( K \) diverges, and the localization length, \( 1/\ln|\gamma| \), vanishes, forcing all eigenstates to coalesce. As eigenenergies read \( E = h^2k_0^2/(2M) \) with divergent \( M \), eigenenergies collapse to zero with a massive degeneracy. Across EP, \( t_1k_0 = 0 \), and the effective mass becomes imaginary, all previous results of positive \( t_1k_0 \) still apply provided that \( M \to \pm iM \). Particles moving in hyperbolic spaces are thus dissipative, and stationary states no longer exist.

Lastly, similar to the NH model, changing OBC to PBC leads to drastic changes in the curved space. Eigenstates of Eq. (2) and Eq. (6) normalized to \( N \) become \( \sqrt{\gamma y_0} y_0^{-1}(y_{n-1} - y_{n-1})^{-1} y_{n-1}^2/(y_0)^2 \). As shown in the time-dependent Schrödinger equations. For instance, at the band bottom(top), we multiply \( \psi(y) \) to both sides of \( i\hbar\partial_t \psi = -\sqrt{\gamma y} \left( y^2 \right)^{1/2} + 1/4 \) \( \psi \), subtract from the resultant expression its complex conjugate, and integrate over \( y \) to \( y_{n-1} \). We find that the total particle number \( N_p = \int_{y_0}^{y_{n-1}} dy |\psi(y)|^2/(\gamma y_0) \) satisfies
\[
\partial_+ N_p = h\sqrt{\gamma} N k_0 / M, 
\]
(10)
which signifies the absence of a stationary state and explains complex eigenenergies under PBC. Using \( \partial_yN_p = \frac{\pi}{2} \text{Im}(E) N_p \), we find \( \text{Im}(E) = h^2 \sqrt{k_y}/(2M) \). This is distinct from the result for OBC, where \( \psi(y) \sim y^{1/2} \exp(r_y) \) such that \( \partial_yN_p = 0 \). Similar calculations can be performed for effective theories away from the band top (bottom), \( i\hbar \partial_y \psi = \left[ E(K_0) + \sqrt{k_y}v_F \psi \right] \psi \). Straightforward calculations show that \( \partial_yN_p = -\sqrt{k_y} N_p \), which explains the imaginary part of the eigenenergy, \( \text{Im}(E) = -\sqrt{k_y}/2 \).

Despite that \( y_0 \neq y_{N-1} \), these two edges of a hyperbolic surface can be identified in mathematics, since the solutions under PBC exist, as we previously discussed. In physical systems, such PBC can also be realized. In fact, the boundary condition can be continuously tuned. An onsite energy offset, \( V_L \geq 0 \), in one of the lattice sites of the HM model continuously changes PBC to OBC once \( V_L \) increases from 0 to \( \infty \). We consider a superlattice of a lattice spacing of \( Nd \), whose unit cell is a HM chain, as shown in Fig. 2a. Figure 2b shows eigenenergies as functions of \( V_L \). Similarly, an external potential can be added to the Poincaré half-plane, \( \Lambda \delta y V_\delta \sim 1/\sqrt{k} \) sets the energy scale of the potential, such that the larger the non-Hermiticity is, the more sensitive the system is to the boundary condition.

### Generalizations to long-range and non-uniform tunnelings

Whereas the HM model provides an illuminating example of the duality, applications of our approach to generic non-Hermitian models are straightforward. We consider

\[
-\sum_{m=1}^{M} t_{Rm} \psi_{n-m} + \sum_{m=1}^{M} t_{Lm} \psi_{n+m} = E \psi_n, 
\]

where \( t_{Rm} \) and \( t_{Lm} \) are tunneling amplitudes from the \((n+m)\)th to \(n-m\)th sites. An eigenstate under OBC in the bulk is written as \( \psi(kd+q0nd) \), where \( kd \in [0,2\pi] \) and \( q \) is real. Unlike the HM model, where \( q = \ln(t_{R}/t_{L})/(2d) \) is a constant, once beyond the nearest neighbor tunnelings exist, \( q \) becomes a function of \( k \) and defines the so-called generalized Brillouin zone (BZ) in the complex plane. Near any point in the generalized BZ specified by \( K_0d \in [0,2\pi] \), we define \( \psi(s) = e^{iK_0s} \psi(K_0s) \), where \( \psi(s) \) changes slowly as a function of \( s \), corresponding to small deviations of the momentum in the continuum limit. Similar to discussions about the HM model, the effective theory can be formulated straightforwardly using \( \psi(n) = \psi(t_{R}) + i t_{L} \phi + \frac{1}{2} d^2 \phi^2 \). The Schrödinger equation satisfied by \( \psi(s) \) is written as

\[
-\frac{B(K_0)}{2} \psi''(s) - A(K_0) \psi(s) = E \psi(s),
\]

which gives rise to inhomogeneous local curvatures. For slowly varying \( t_{R,n} \) and \( t_{L,n} \), we define \( t(s) \) and \( \gamma(s) \) such that \( t(\gamma,s) = t_{R,n} \gamma(N/2) \). We introduce a slowly changing function \( \psi(s) = e^{\gamma(s)/2} \psi(s) \) with \( \gamma(s) = \frac{3}{2} \int_0^s \text{ln}(\gamma(t')) dt' \). This is a generalization of the uniform case, where \( \gamma(s) \) reduces to a linear function of \( n \), i.e., the previously discussed \( n \ln(n) \) in the HM model. Using the same procedure, we obtain the effective theory of Eq. (14). For instance, the non-relativistic theory is written as

\[
\frac{\hbar^2}{2M} \left( -\frac{1}{\sqrt{g}} \partial_y g^{1/2} \partial_y g^{1/2} - \frac{K}{4} + V_c \right) \psi(x,y) = E \psi(x,y),
\]

where \( g_{xy} = g_{yx} = \sqrt{g} = i(s)e^{-\gamma(s)/2} \int_0^s \text{ln}(\gamma(t')) dt' \), \( g_{xy} = 0 \), \( V_c = \frac{h^2}{2M} \left( \frac{3}{2} \partial_y \text{ln} \gamma_N - 2 \right) \partial_y \), and the position-dependent curvature is written as \( \gamma(y) = \gamma(s) \left( 4 \text{ln} \gamma_N - 2d \partial_y \text{ln} \gamma_N \right) /d^2 \). In these expressions, \( s_y \) is obtained from \( y = y_0 + \int_0^s ds \gamma(s) \int_0^{s_y} \text{ln}(\gamma(t')) dt' / \gamma(t') \). The constant \( K \) of a hyperbolic surface is recovered when \( t_{R,n} \) and \( t_{L,n} \) are constants. Changing \( t_{R,n} \) and \( t_{L,n} \) then tunes local curvatures. For instance, when \( t_{R,n} = \frac{h^2}{2M} e^{-\Theta(n-N/2)}(n), t_{L,n} = \frac{h^2}{2M} e^{-\Theta(n-N/2)}(n) \), where \( \Theta(x) \) is the Heaviside step function, the curvature vanishes everywhere except at a particular location, i.e., \( K = \gamma(s) \gamma_N \), where \( y = y_0 + nNd \).

In addition to one dimension, many non-Hermitian models in higher dimensions can be constructed based on the HM model. For instance, 1D HM chains can be assembled to access higher dimensioned curved spaces. Whereas curved spaces in higher dimensions are, in general, more complex than those in two dimensions, inter-chain couplings can be engineered to access different higher dimensioned curved spaces (see Fig. S2 of Supplementary Materials).
Fig. 3 Non-Hermitian realization of QHS on hyperbolic surface. a A hyperbolic surface threaded by uniform magnetic fluxes. b An extra onsite potential in Eq. (4), where $\nu_{\text{(16)}}$ (Supplementary Materials). The counterpart of Eq. (18) in for a hyperbolic surface can be straightforwardly proved using Eq. (8).

Non-Hermitian realization of QHS in curved spaces. The duality we established has a wide range of profound applications. For instance, Fig. 3a shows a non-Hermitian realization of QHS in curved spaces. When a particle with a charge $-e$ is subjected to a uniform magnetic field, $\gamma^2 V^2$ in Eq. (3) is replaced by $\gamma^2 [(\partial_\nu - i \partial_x)^2 + \partial_y^2]$, where we have chosen the gauge with the vector potential $A = (-\nabla \psi, 0)$ such that $k_\nu$ is still a good quantum number. Wavefunctions of the lowest Landau level (LLL) are written as,

$$\psi_{\text{LLL}} = (2k_\nu)^{\frac{1}{2}} \frac{N_k}{(2 \pi^2 - 1)^{\frac{1}{2}}} e^{-k_\nu + k_\psi} \mathbb{1}_d, \quad (16)$$

whose eigenenergies, $E_{\text{LLL}} = \frac{k_\nu N_k}{\mathbb{1}_d + \frac{1}{2}}$ are independent of $k_\nu$ manifesting the degeneracy of the Landau levels. In the dual non-Hermitian systems, a finite magnetic field corresponds to an extra onsite potential in Eq. (4), $V_n \rightarrow V_n + V_{B,n}$ where $V_{B,n} = \gamma_b V_n^2 \sqrt{I_{\text{eff}}}$ as shown in Fig. 3b. The dimensionless $b$ characterizing the strength of $V_{B,n}$ relative to $V_n$ is written as

$$b = eBd^2/(h \ln |\gamma|), \quad (17)$$

where $\gamma_b k_\nu = a(2 \ln |\gamma|)^{-1}$ has been used. The magnetic flux density, $\rho_\phi = eB/(2\pi h) = \ln |\gamma|/(2\pi^2)$ is thus determined by the ratio of $V_{B,n}$ to $V_n$. For a given $\gamma$, a finite $k_\nu$ shifts the position of the minimum of the total onsite potential $V_n + V_{B,n}$ similar to the well-known result of flat spaces where $k_\nu$ determines the location of the minimum of the potential in the Landau gauge.

A complete description of QHS requires its gravitational responses in curved spaces. For instance, the particle density, $\rho$, depends on the local curvature $\kappa$

$$\rho = \nu \rho_\phi - \kappa/(4\pi), \quad (18)$$

where $\nu$ is the filling factor. For integer QHS, $\nu = 1$ and Eq. (18) for a hyperbolic surface can be straightforwardly proved using Eq. (16) (Supplementary Materials). The counterpart of Eq. (18) in the non-Hermitian lattice is

$$|\psi|^2 \left\{ \begin{array}{l} \text{da} N_n(a, b) = \ln|\gamma| - 2\ln^2|\gamma|, \quad (19) \end{array} \right.$$
18. Mostafazadeh, A. Pseudo-hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. J. Math. Phys. 43, 203 (2002).
19. Ashida, Y., Gong, Z. & Ueda, M. Non-Hermitian physics. Adv. Phys. 69, 249 (2020).
20. Bergholtz, E. J., Budich, J. C. & Kunst, F. K. Exceptional topology of non-Hermitian systems. Rev. Mod. Phys. 93, 015005 (2021).
21. Xiong, Y., Wannier, C. & Tator, R. Correspondence fail in some non-Hermitian topological models. J. Phys. Commun. 2, 035043 (2018).
22. Okuma, N. & Sato, M. Topological phase transition driven by infinitesimal instability: majorana fermions in non-Hermitian spintronics. Phys. Rev. Lett. 123, 097701 (2019).
23. Wiersig, J. Enhancing the sensitivity of frequency and energy splitting detection by using exceptional points: application to microcavity sensors for single-particle detection. Phys. Rev. Lett. 112, 203901 (2014).
24. Kawabata, K., Shiozaki, K., Ueda, M. & Sato, M. Symmetry and topology in non-Hermitian physics. Phys. Rev. X 9, 041015 (2019).
25. Lee, J.-Y., Ahn, J., Zhou, H. & Vishwanath, A. Topological correspondence between Hermitian and non-Hermitian systems: anomalous dynamics. Phys. Rev. Lett. 123, 206404 (2019).
26. Zhou, H. & Lee, J. Y. Periodic table for topological bands with non-Hermitian symmetries. Phys. Rev. B 99, 235112 (2019).
27. Xiao, L. et al. Non-Hermitian bulk–boundary correspondence in quantum dynamics. Nat. Phys. 16, 761 (2020).
28. Brody, D. C. Biorthogonal quantum mechanics. J. Phys. A: Math. Theoretical 47, 035305 (2013).
29. Scholtz, F., Geyer, H. & Hahne, F. Quasi-Hermitian representation of quantum mechanics. Int. J. Geometric Methods Modern Phys. 07, 1191 (2010).
30. Mostafazadeh, A. Pseudo-hermitian representation of quantum mechanics. Int. J. Geometric Methods Modern Phys. 07, 1191 (2010).
31. Kollár, A. J., Fitzpatrick, M. & Houck, A. A. Hyperbolic lattices in circuit quantum electrodynamics. Nature 571, 45 (2019).
32. Chen, N., Chalupnik, M., Can, T., Gromov, A. & Simon, J. Electromagnetic and gravitational responses of photonic Landau levels. Nature 565, 173 (2019).
33. Wen, X. G. & Zee, A. Shift and spin vector: new topological quantum numbers for the Hall fluids. Phys. Rev. Lett. 69, 953 (1992).
34. Avron, J. E., Seiler, R. & Zograf, P. G. Viscosity of quantum Hall fluids. Phys. Rev. Lett. 75, 697 (1995).
35. Can, T., Laskin, M. & Wiegmann, P. Fractional quantum Hall effect in a curved space: Gravitational anomaly and electromagnetic response. Phys. Rev. Lett. 120, 130402 (2018).
36. Schneeloch, J., Fitzpatrick, M. & Houck, A. A. Hyperbolic lattices in circuit quantum electrodynamics. Nature 571, 45 (2019).
37. Wen, X. G. & Zee, A. Shift and spin vector: new topological quantum numbers for the Hall fluids. Phys. Rev. Lett. 69, 953 (1992).
38. Avron, J. E., Seiler, R. Zograf, P. G. Viscosity of quantum Hall fluids. Phys. Rev. Lett. 75, 697 (1995).
39. Can, T., Laskin, M. & Wiegmann, P. Fractional quantum Hall effect in a curved space: Gravitational anomaly and electromagnetic response. Phys. Rev. Lett. 120, 130402 (2018).
40. Nelson, D. R. & Vinokur, V. M. Boson localization and correlated pinning of superconducting vortex arrays. Phys. Rev. B 48, 13060 (1993).
41. Amir, A., Hatano, N. & Nelson, D. R. Non-Hermitian localization in biological networks. Phys. Rev. E 93, 042310 (2016).
42. Lodahl, P. et al. Chiral quantum optics. Nature 541, 473 (2017).
43. Zhang, R., Lv, C., Yan, Y. & Zhou, Q. Efimov-like states and quantum tunneling effects on synthetic hyperbolic surfaces. Sci. Bull. 66, 1967 (2021).