INFINITE NORM OF THE DERIVATIVE OF THE SOLUTION OPERATOR OF EULER EQUATIONS

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Abstract. Through a simple and elegant argument, we prove that the norm of the derivative of the solution operator of Euler equations posed in the Sobolev space \( H^n \), along any base solution that is in \( H^n \) but not in \( H^{n+1} \), is infinite. We also review the counterpart of this result for Navier-Stokes equations at high Reynolds number from the perspective of fully developed turbulence. Finally we present a few examples and numerical simulations to show a more complete picture of the so-called rough dependence upon initial data.

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1. Introduction

The solution operator of Euler equations of fluids is nowhere differentiable [2] [3]. This is what we called “rough dependence on initial data” for Euler equations. There are several ways for the solution operator to be non-differentiable. The most common way is that the norm of the derivative of the solution operator is infinite. Other ways include that the norm of the formal derivative of the solution operator is finite.

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but the Fréchet definition of derivative is violated. Our Main Theorem of this paper is:

**Theorem 1.1.** If the Euler equations are posed in the Sobolev space \( H^n \) and the base solution is in \( H^n \) but not in \( H^{n+1} \), then the norm of the formal derivative of the solution operator is infinite when \( t > 0 \).

If \( u_0 \) is an element in \( H^n \) but not in \( H^{n+1} \), then for any \( v_0 \) in \( H^{n+1} \), \( u_0 + v_0 \) is an element in \( H^n \) but not in \( H^{n+1} \). Thus there are more elements in \( H^n \) but not in \( H^{n+1} \) than in \( H^{n+1} \). When \( n > \frac{d}{2} + 1 \) (\( d \) is the spatial dimension), each initial element in \( H^n \) generates a local solution of the Euler equations in \( H^n \), and each initial element in \( H^{n+1} \) generates a local solution of the Euler equations in \( H^{n+1} \). Thus Theorem 1.1 is valid for a majority of base solutions in \( H^n \).

Even though it is everywhere differentiable, the solution operator of Navier-Stokes equations will somehow approach the solution operator of Euler equations in the infinite Reynolds number limit. This is the regime that we are most interested in from the perspective of fully developed turbulence. In this regime, we believe that the norm of the derivative of the solution operator along turbulent solutions of Navier-Stokes equations will approach infinity in the infinite Reynolds number limit. Since the norm of the derivative of the solution operator measures the maximal growth of perturbations, perturbations in fully developed turbulence grow superfast. This is what we called "rough dependence on initial data" for fully developed turbulence. Our theory is that fully developed turbulence is initiated and maintained by such superfast growth of ever existing perturbations. Such superfast growth can reach substantial scale even in short time, and leads to superfast nonlinear saturation and short term unpredictability of fully developed turbulence. The superfast growth of perturbations also implies that the turbulent solution of Navier-Stokes equations and the turbulent flow in reality (lab or nature) can be substantially different in short time, even though they have the same initial condition.

2. Basic formulation

The Navier-Stokes equations are given by

\[
\begin{align*}
\frac{d}{dt} u - \frac{1}{Re} \Delta u &= -\nabla p - u \cdot \nabla u, \\
\nabla \cdot u &= 0,
\end{align*}
\]

where \( u \) is the \( d \)-dimensional fluid velocity \((d = 2, 3)\), \( p \) is the fluid pressure, and \( Re \) is the Reynolds number. Setting the Reynolds number...
to infinity $Re = \infty$, the Navier-Stokes equations (2.1)-(2.2) reduce to the Euler equations

$u_t = -\nabla p - u \cdot \nabla u$, \hspace{1cm} (2.3)

$\nabla \cdot u = 0$. \hspace{1cm} (2.4)

For any $u \in H^n(\mathbb{R}^d)$ ($n > \frac{d}{2} + 1$), there is a neighborhood $B$ and a short time $T > 0$, such that for any $v \in B$ there exists a unique solution to the Navier-Stokes equations (2.1)-(2.2) in $C^0([0, T]; H^n(\mathbb{R}^d))$. As $Re \to \infty$, this solution converges to that of the Euler equations (2.3)-(2.4) in the same space [4] [5]. For any $t \in [0, T]$, let $S^t$ be the solution map:

$S^t : B \mapsto H^n(\mathbb{R}^d), \quad S^t(u(0)) = u(t)$, \hspace{1cm} (2.5)

i.e. the solution map maps the initial condition to the solution’s value at time $t$. The solution map is continuous for both Navier-Stokes equations (2.1)-(2.2) and Euler equations (2.3)-(2.4) [4] [5], but nowhere differentiable for Euler equations [2] [3]. Even though the derivative of the solution map for Navier-Stokes equations (2.1)-(2.2) exists, it is natural to conjecture that the norm of the derivative of the solution map along turbulent solutions approaches infinity as the Reynolds number approaches infinity. The following upper bound was obtained in [6].

$\|DS^t(u(0))\| = \sup_{du(0)} \|du(t)\|_n \leq e^{\sigma \sqrt{Re} \sqrt{T}} + \sigma_1 t$, \hspace{1cm} (2.6)

where $du(0)$ is any initial perturbation of $u(0)$, $\| \|_n$ represents the Sobolev $H^n$ norm, and

$\sigma = \frac{8c}{\sqrt{2e}} \max_{\tau \in [0, T]} \|u(\tau)\|_n$; \hspace{0.5cm} $\sigma_1 = \frac{\sqrt{2e}}{2} \sigma$,

where $c$ only depends on $n$ and the spatial domain. The above bound also applies to spatially periodic domain $\mathbb{T}^d$ instead of $\mathbb{R}^d$.

Sometimes, it is convenient to use the Leray projection of the Navier-Stokes equations. The Leray projection is an orthogonal projection in $L^2(\mathbb{R}^d)$, given by

$P g = g - \nabla \Delta^{-1} \nabla \cdot g$. \hspace{2cm}

Applying the Leray projection to the Navier-Stokes equations (2.1)-(2.2), one gets

$u_t + \frac{1}{Re} \Delta u = -P(u \cdot \nabla u)$, \hspace{1cm} (2.7)

and the corresponding Euler equations

$u_t = -P(u \cdot \nabla u)$. \hspace{1cm} (2.8)
3. PROOF OF THE MAIN THEOREM

Here we will present a simple and elegant proof of the Main Theorem \[.] We will present the periodic boundary condition case, of course the same proof applies to other boundary condition cases that allow translational invariance.

Proof. The Euler equations (2.3)-(2.4) are translationally invariant, i.e. if \( u(t, x) \) is a solution, then \( u(t, x - at) + a \) are also solutions for constant vectors \( a \). Using Fourier series, we have

\[
u(t, x) = \sum_{k \in \mathbb{Z}^d} u_k(t) e^{ik \cdot x},\]

\[
U(t, x, a) = u(t, x - at) + a = a + \sum_{k \in \mathbb{Z}^d} u_k(t) e^{ik \cdot (x - at)}.
\]

Let \( u(t, x) \) be any solution that is in \( H^n \) but not in \( H^{n+1} \), then \( U(t, x, a) \) is a family of solutions parametrized by \( a \), which have the same property, and \( U(t, x, 0) = u(t, x) \). Notice also that

\[
U(0, x, a) = u(0, x) + a.
\]

By varying \( a \) around \( a = 0 \), we can make a directional variation of the initial condition around \( u(0, x) \), which leads to a directional derivative of the solution operator:

\[
\frac{\partial}{\partial a_m} U(t, x, a)|_{a=0} = \frac{\partial}{\partial a_m} a + \sum_{k \in \mathbb{Z}^d} (-ik_m t)u_k(t)e^{ik \cdot x}, m = 1, \ldots, d.
\]

Thus

\[
\sum_{m=1}^d \| \frac{\partial}{\partial a_m} U(t, x, a)|_{a=0} \|_n^2
= d(2\pi)^d + t^2(2\pi)^d \sum_{k \in \mathbb{Z}^d} (|k|^2 + \cdots + |k|^{2(n+1)}) |u_k|^2
= d(2\pi)^d + t^2 \left( \|u(t, x)\|_{n+1}^2 - \|u(t, x)\|_0^2 \right).
\]

Since \( \|u(t, x)\|_{n+1} = \infty \),

\[
\| \frac{\partial}{\partial a_m} U(t, x, a)|_{a=0} \|_n = \infty, \quad \text{when } t > 0,
\]

for some \( m \). Since it is the supremum over the norms of all directional derivatives, the norm of the derivative of the solution operator along \( u(t, x) \) is infinite, and this completes the proof of the main theorem. \( \square \)
4. Example 1 - the trivial one

Under either periodic condition or decaying boundary condition in the whole space, the base solution is the trivial solution $u = 0 (p = 0)$. Then the corresponding linearized equations of (2.7) are given by

$$du_t - \frac{1}{Re} \Delta du = 0,$$

and the corresponding linearized equations of (2.8) are given by

$$du_t = 0.$$

Thus

(4.1) $$du(t) = e^{\frac{t}{Re} \Delta} du(0).$$

Starting from the same initial condition $\delta u(0) = du(0)$, the increment $\delta u(t)$ satisfies

$$\delta u_t - \frac{1}{Re} \Delta \delta u = -P(\delta u \cdot \nabla \delta u),$$

in the Navier-Stokes case, and

$$\delta u_t = -P(\delta u \cdot \nabla \delta u),$$

in the Euler case. By the method of variation of parameters,

(4.2) $$\delta u(t) = e^{\frac{t}{Re} \Delta} \delta u(0) - \int_0^t e^{\frac{\tau}{Re} \Delta} P(\delta u \cdot \nabla \delta u) d\tau,$$

in the Navier-Stokes case. Applying the inequality

$$\|e^{\frac{t}{Re} \Delta} u\|_n \leq \left( \frac{1}{\sqrt{2e}} \sqrt{\frac{Re}{t} + 1} \right) \|u\|_{n-1},$$

where $\| \|_n$ represents the Sobolev $H^n$ norm ($n > \frac{d}{2} + 1$),

$$\|\delta u(t)\|_n \leq \|\delta u(0)\|_n + 2c \int_0^t \left( \frac{1}{\sqrt{2e}} \sqrt{\frac{Re}{t - \tau} + 1} \right) \|\delta u(\tau)\|_n^2 d\tau,$$

where $c$ is a constant that only depends on $n$ and the spatial domain. Then

$$\max_{t \in [0, T]} \|\delta u(t)\|_n \leq \|\delta u(0)\|_n + 2 \max_{t \in [0, T]} \|\delta u(t)\|_n^2 \int_0^t \left( \frac{1}{\sqrt{2e}} \sqrt{\frac{Re}{t - \tau} + 1} \right) d\tau.$$

Thus

$$\max_{t \in [0, T]} \|\delta u(t)\|_n \sim \|\delta u(0)\|_n, \quad \text{as } \|\delta u(0)\|_n \to 0.$$
In view of the fact that $\delta u(0) = du(0)$, (4.2)-(4.1) leads to

$$\delta u(t) - du(t) = - \int_0^t e^{\frac{\tau}{Re}} \mathcal{P}(\delta u \cdot \nabla \delta u) d\tau,$$

which can be estimated as above,

$$\|\delta u(t) - du(t)\|_n \sim \|\delta u(0)\|_n^2, \quad \text{as} \quad \|\delta u(0)\|_n \to 0.$$ 

Thus when $Re < \infty$, the derivative of the solution operator at the trivial solution exists, and is given by (4.1). But in the Euler case,

$$\delta u(t) - du(t) = - \int_0^t \mathcal{P}(\delta u \cdot \nabla \delta u) d\tau,$$

and $\|\delta u(t) - du(t)\|_n$ is not of order $o(\|\delta u(0)\|_n)$, as $\|\delta u(0)\|_n \to 0$. Thus in the Euler case, the derivative of the solution operator at the trivial solution still does not exist, even though the norm of the formal derivative is bounded.

5. Example 2 - the simple one

The base solution is the 2D Couette linear shear $u = (x_2, 0)$. The boundary conditions are

$$u = (\pm 1, 0), \quad \text{at} \quad x_2 = \pm 1,$$

in the viscous case, no $u_1$ condition (slip) in the inviscid case, and periodic boundary condition along $x_1$-direction with period $2\pi$. It is more convenient to use the vorticity variable

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = -\Delta \psi,$$

where

$$u_1 = \frac{\partial \psi}{\partial x_2}, \quad u_2 = -\frac{\partial \psi}{\partial x_1}.$$

In terms of the vorticity variable, the 2D Navier-Stokes equations take the form

$$\omega_t - \frac{1}{Re} \Delta \omega = -u \cdot \nabla \omega.$$

The linearized 2D Navier-Stokes equation at the Couette linear shear is given by

$$d\omega_t - \frac{1}{Re} \Delta d\omega = -x_2 d\omega_{x_1}.$$

In terms of Fourier series

$$d\omega = \sum_{n=-\infty}^{+\infty} d\omega_n(t, x_2) e^{inx_1}, \quad d\omega_{-n} = \overline{d\omega_n},$$
we have
\[ \partial_t d\omega_n + i n x_2 d\omega_n = \frac{1}{Re} (\partial_{x_2}^2 d\omega_n - n^2 d\omega_n). \]
Let
\[ d\omega_n = d\Omega_n e^{-inx_2 t}, \quad d\Omega_{-n} = \overline{d\Omega_n}, \]
then
\[ \partial_t d\Omega_n = \frac{1}{Re} [\partial_{x_2}^2 d\Omega_n - i 2 n t \partial_{x_2} d\Omega_n - n^2 (t^2 + 1) d\Omega_n]. \]
When \( Re = \infty \), i.e. for inviscid linearized Couette flow,
\[ d\Omega_n(t, x_2) = d\Omega_n(0, x_2). \]
Thus \[ d\omega(t, x_1, x_2) = d\omega(0, x_1 - x_2 t, x_2). \]
When \( Re < \infty \), using Fourier transform
\[ d\Omega_n = \int_{-\infty}^{+\infty} d\Omega_{n\xi}(t) e^{i\xi x_2} d\xi, \quad d\Omega_{(-n)\xi} = \overline{d\Omega_{n\xi}}, \]
we have
\[ \partial_t d\Omega_{n\xi} = \frac{1}{Re} [-\xi^2 + 2 n t \xi - n^2 (t^2 + 1)] d\Omega_{n\xi}. \]
Thus
\[ d\Omega_{n\xi}(t) = d\Omega_{n\xi}(0) e^{-\frac{1}{Re} \left[ (\xi - \frac{1}{2} n t)^2 + n^2 (\frac{1}{4} t^2 + 1) \right]}, \]
\[ d\Omega_n(t, x_2) = \int_{-\infty}^{+\infty} d\Omega_{n\xi}(0) e^{-\frac{1}{Re} \left[ (\xi - \frac{1}{2} n t)^2 + n^2 (\frac{1}{4} t^2 + 1) \right]} e^{i\xi x_2} d\xi, \]
and
\[ d\omega = \sum_{n = -\infty}^{+\infty} e^{inx_1} e^{-inx_2 t} d\Omega_n(t, x_2) \]
\[ (5.1) = \sum_{n = -\infty}^{+\infty} e^{inx_1} e^{-inx_2 t} \int_{-\infty}^{+\infty} d\Omega_{n\xi}(0) e^{-\frac{1}{Re} \left[ (\xi - \frac{1}{2} n t)^2 + n^2 (\frac{1}{4} t^2 + 1) \right]} e^{i\xi x_2} d\xi. \]
Thus
\[ \|d\omega\|_{H^0}^2 = 4\pi^2 \sum_{n = -\infty}^{+\infty} \int_{-\infty}^{+\infty} |d\Omega_{n\xi}(0)|^2 e^{-\frac{1}{Re} \left[ (\xi - \frac{1}{2} n t)^2 + n^2 (\frac{1}{4} t^2 + 1) \right]} d\xi. \]
When \( Re = \infty \),
\[ \|d\omega(t)\|_{H^0}^2 = \|d\omega(0)\|_{H^0}^2. \]
When \( Re < \infty \),
\[ \|d\omega(t)\|_{H^0}^2 \leq \|d\omega(0)\|_{H^0}^2. \]
Based on calculations such as
\[ \partial_{xx}d\omega = \sum_{n=-\infty}^{\infty} [(-in\ell)d\Omega_n + \partial_{xx}d\Omega_n]e^{inx_1}e^{-inx_2t}, \]
one can see that when \( Re = \infty \), \( \|d\omega(t)\|_{H^k} \) is finite but grows in time as \( t^k \) for \( k = 1, 2, \cdots \). As in the case of Example 1, the norm of the formal derivative here is bounded, but the derivative does not exist due to the violation of the definition of Fréchet derivative when \( Re = \infty \). When \( Re \to \infty \), the norm of the derivative in the Navier-Stokes case approaches the norm of the formal derivative in the Euler case. Since the linear Couette shear is in \( C^{\infty} \), it is not a base solution that is in \( H^k \) but not in \( H^{k+1} \). Historically linear Couette shear was regarded as one the canonical flows for the study of transition to turbulence. Now we understand that it is not a good representative of transition to turbulence. Indeed, it is both linearly and nonlinearly stable for all values of Reynolds number including infinity [11]. On the other hand, transition to turbulence does happen in lab Couette flow. Our explanation of such a transition is that states arbitrarily close to the linear Couette shear are linearly unstable [10]. In a typical transition to turbulence, the rough dependence may play a significant role when the Reynolds number is large enough [8]. But for linear Couette flow, we believe that the transition is due to what we just mentioned [10]. The base solution of our next example does satisfy the criterion of being in \( H^k \) but not in \( H^{k+1} \).

6. Example 3 - Periodic Boundary Condition

For 2D Navier-Stokes equations under periodic boundary condition with period domain \([0, 2\pi] \times [0, 2\pi]\), we have the one-parameter family of exact solutions [7],
\[ u_1 = \sum_{n=1}^{\infty} \frac{1}{n^{3+\gamma}}e^{-\frac{n^2}{Re}t} \sin[n(x_2 - \sigma t)], \quad u_2 = \sigma, \]
which is a solution in the space \( C^0([0, \infty), H^3) \) for all values of the Reynolds number including infinity, \( \frac{1}{2} < \gamma \leq 1 \), and \( \sigma \) is the real parameter. The directional derivative in \( \sigma \) of the solution operator along the above exact solutions \( \partial_{\sigma}F^t \) is given by [7]
\[ \partial_{\sigma}u_1 = \sum_{n=1}^{\infty} \frac{-t}{n^{2+\gamma}}e^{-\frac{n^2}{Re}t} \cos[n(x_2 - \sigma t)], \quad \partial_{\sigma}u_2 = 1. \]
The norm of the derivative of the solution operator $\partial_\sigma F^t$ has the lower bound $[7]$

$$\|\partial_\sigma F^t\|_{H^3} > \sqrt{2\pi} + \frac{\pi}{\sqrt{e}} t^\gamma \left(\frac{\sqrt{7\sqrt{Re}}}{2\sqrt{2}}\right)^{1-\gamma}.$$  

As $Re \to \infty$,

$$\|\partial_\sigma F^t\|_{H^3} \to \infty,$$

thus

$$\|\nabla F^t\|_{H^3} \to \infty,$$

since $\|\nabla F^t\|_{H^3} \geq \|\partial_\sigma F^t\|_{H^3}$. When $Re = \infty$, direct calculation shows that indeed

$$\|\partial_\sigma F^t\|_{H^3} = \infty,$$

thus

$$\|\nabla F^t\|_{H^3} = \infty.$$

This example shows that the norm of the derivative of the solution operator along the family of exact solutions is infinite in the Euler case, and approaches infinity in the Navier-Stokes case as the Reynolds number approaches infinity.

7. The generic case - numerical simulations

We conducted extensive numerical simulations under periodic boundary condition $[1][9]$. At high Reynolds number (even moderate Reynolds number), along generic solutions, generic perturbations amplify super-fast (i.e. faster than exponential growth), typically as shown in Figure 1. In Figure 1(a), we plot the growth of the $H^3$ norm of the perturbation under the linearized Navier-Stokes dynamics at Reynolds number $Re = 1000$, and in Figure 1(b), we plot the same figure in vertical ln-scale. Clearly, the amplification of the perturbation is faster than exponential growth. We want to emphasize that such superfast amplification is observed even for base solutions in $H^4$ (in fact $C^\infty$), and for moderate Reynolds number.

In $[9]$, we conduct a large direct numerical simulation on the 3D Navier-Stokes equations with resolution up to $2048^3$ and Reynolds number up to 6210. We first run the numerical simulation for a long time until the dynamics reaches homogeneous and isotropic turbulence. The vorticity in this homogeneous and isotropic turbulence is very large, and we use $H^0$ norm to measure the perturbations. We are simulating the situation that the $H^1$ norm of the base solutions are very large, and we measure their $H^0$ norm. The square root nature of both time and Reynolds number in the exponent of (2.6) is verified. Figure 2 shows
the Reynolds number dependence of the amplifications of perturbations. Thus we numerically verified our theory that fully developed
turbulence is initiated, developed and maintained by such superfast growth of ever existing perturbations!

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