Elasticity bounds from Effective Field Theory

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Phonons in solid materials can be understood as the Goldstone bosons of the spontaneously broken spacetime symmetries. As such their low energy dynamics are greatly constrained and can be captured by standard effective field theory (EFT) methods. In particular, knowledge of the nonlinear stress-strain curves completely fixes the full effective Lagrangian at leading order in derivatives. We attempt to illustrate the potential of effective methods focusing on the so-called hyperelastic materials, which allow large elastic deformations. We find that the self-consistency of the EFT imposes a number of bounds on physical quantities, mainly on the maximum strain and maximum stress that can be supported by the medium. In particular, for stress-strain relations that at large deformations are characterized by a power-law behaviour $\sigma(\varepsilon) \sim \varepsilon^n$, the maximum strain exhibits a sharp correlation with the exponent $\nu$.

I. INTRODUCTION

Effective Field Theory (EFT) often emerges as a powerful tool to understand nature, especially when the microscopic degrees of freedom become strongly coupled at low energies. Classic examples of insightful EFTs range from the Ginzburg-Landau theory for the onset of superconductivity to the Fermi liquid theory, or the chiral perturbation theory as the low energy description of Quantum Chromodynamics.

Another notorious and early example is the theory of elasticity: the description of a solids mechanical response, including its sound wave excitations – the phonons [1, 2]. As in hydrodynamics, the elasticity theory is formulated in terms of an effective degree of freedom – the displacement vector of the solid elements with respect to their equilibrium configuration. Importantly, the classic elasticity theory can be promoted to the non-linear regime where the response to finite deformations is addressed [3,5]. Operationally, this is done by finding/measuring the stress-strain relations for both finite shear or bulk strain applied to the material. These diagrams contain a number of non-linear response parameters (such as the proportional limit or the failure point, see [5] for definitions) that are well defined material properties which go deep into the non-linear response regime. Typically, these parameters are difficult to compute from the microscopic constituents, so there is a chance that EFT methods may help in understanding some non-linear elasticity phenomena.

From the viewpoint of quantum field theory (QFT), it is clear that elasticity theory can be treated as a non-trivial (i.e., interacting) EFT. The way how this theory works as an EFT, however, is quite different from other well known examples; the main reason being that the symmetry breaking pattern behind it involves spacetime symmetries. The purpose of this work is to revisit finite elasticity theory from the viewpoint of QFT. We aim at clarifying a few points on how the EFT methodology works for broken spacetime symmetries and find novel relations between (and bounds on) various non-linear elasticity parameters.

II. FROM GOLDSTONES TO STRESS-STRAIN CURVES

We start by stating the precise QFT sense in which elasticity theory can be treated as an EFT. The first requirement is that the material must have a separation of scales: we shall consider only low frequency (acoustic) phonons; any other mode is considered as much heavier and integrated-out. (Materials displaying scale invariance violate this assumption and deserve a separate treatment. See more on this below.) Under this condition we can exploit the fact that the phonons can be viewed as
the Goldstone bosons of translational symmetry breaking [8, 9]. As such we obtain their fully non-linear effective action by the means of the standard coset construction [9]. For simplicity, we shall work in 2 + 1 spacetime dimensions, where the dynamical degrees of freedom are contained in two scalar fields \( \phi^I(x) \). The internal symmetry group is assumed to be the two-dimensional Euclidean group, ISO(2), acting like translations and rotations in the scalar fields space. The theory then must be shift invariant in the \( \phi^I \)'s implying that any field configuration that is linear in the spacetime coordinates will satisfy the equations of motion. The equilibrium configuration of an isotropic material is given by:

\[
\phi_{\text{eq}}^I = \delta_I^J x^J. \tag{1}
\]

This vacuum expectation value spontaneously breaks the symmetry group ISO(2) \( \times \) ISO(2, 1) down to the diagonal subgroup. Following the coset construction method, one concludes that the effective action at lowest order in derivatives takes the form

\[
S = - \int d^3x \sqrt{-g} V(X, Z), \tag{2}
\]

with \( X \) and \( Z \) defined in terms of the scalar fields matrix\(^1\)

\[
\mathcal{I}^{IJ} = g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J \quad \text{as} \quad X = \text{tr} \left( \mathcal{I}^{IJ} \right) = \mathcal{I}^{IJ} \delta_{IJ}, \quad Z = \text{det} \left( \mathcal{I}^{IJ} \right). \tag{3}
\]

The function \( V(X, Z) \) is ‘free’ and its form depends on the solid. In this language, the phonons \( \pi^I \) are identified as the small excitations around the equilibrium configuration defined through \( \phi^I = \phi^I_{\text{eq}} + \pi^I \). Plugging this decomposition into (2) one can find the phonon kinetic terms and their self-interactions \((\partial \pi)^2\)\(^2\). The leading phonon effective operators are determined by a few Wilson coefficients that are related to the lowest derivatives of \( V \) evaluated on the equilibrium configuration, see [10] for details. (Analogous results can be found in [11] for superconductors.) The effective action (2) also encodes the response to finite (large) deformations, and for that find the global form of \( V(X, Z) \) is needed.

The coset construction ends at Eq. (2) with an arbitrary function of two arguments\(\phi^I_{\text{str}}(X, Z)\). By symmetry considerations, then, one cannot restrict the Lagrangian any further. To identify what is the function \( V(X, Z) \) for a given material one needs more information, some kind of constitutive relation. Recalling the finite elasticity literature (see e.g., [3]), the reader will immediately realize that \( V(X, Z) \) is naturally recognized with the so-called strain-energy function. This is a function of the principal invariants characterizing the materials state of deformation. It encodes the full non-linear response for the so-called Cauchy (hyper-)elastic solids, for which the plastic and dissipative effects can be ignored [4].

The form of \( V \) can then be found from the stress-strain relations measured in both the shear and the bulk channels of real solids (see, e.g., [3, 4, 12]). More specifically, from the response of the material to constant and homogeneous deformations. These deformations reduce to configurations of the form

\[
\phi_{\text{str}}^I = O^I_{\text{str}} x^J, \quad O^I_{\text{str}} = \alpha \left( \sqrt{1 + \varepsilon^2/4} \frac{\varepsilon/2}{\sqrt{1 + \varepsilon^2/4}} \right), \tag{3}
\]

where \( \varepsilon \) and \( \alpha - 1 \) are the shear and the bulk strains respectively, and they induce constant but non-trivial values of \( X_{\text{str}} = \alpha^2 (2 + \varepsilon^2) \) and \( Z_{\text{str}} = \alpha^4 \). The amount of stress in the material generated by (or needed to support) such a configuration depends only on the strains \( \varepsilon \), \( \alpha \) and on the shape of \( V(X, Z) \), see e.g., Eq. (9). The upshot is that it is possible to reconstruct the full form of the effective Lagrangian (up to an irrelevant overall constant) by just measuring the stress-strain relations, that is, from the response to time-independent and homogeneous deformations\(\phi^I_{\text{str}}\). This already illustrates how the solid EFTs retain predictive power.

The next apparent challenge from the QFT viewpoint is that the real world stress-strain curves typically exhibit a dramatic feature: they terminate at some point, corresponding to the breaking (or elastic failure) of the material. Since we just concluded that the form of Lagrangian is directly related to the stress-strain curves, it is natural to ask how exactly is the breaking seen in the EFT. Must the function \( V(X, Z) \) be singular? Or does the breaking correspond to a dynamical process (e.g., an instability) that can be captured within the EFT even with a regular \( V(X, Z) \)? We argue below that the latter possibility can certainly arise. Moreover, in that case it is possible to extract relations between the parameters that control the large deformations.

The main task then is to analyze the stability properties of the strained configuration (4). This can be analyzed by setting \( \phi^I = \phi^I_{\text{str}} + \pi^I \) in (3) and expanding for ‘small’ \( \pi^I \). In doing so one easily finds that the phonon sound speeds depend on the applied strain \( O^I_{\text{str}} \). This is a long known phenomenon, the acoustoelastic effect, see e.g., [13, 14]. Still, we argue here that this can have a great impact on the stress-strain relations, eventually limiting the maximal stress that a material can withstand. The reason is that, very generically, increasing the strain results into increasing/decreasing of the

\(^1\)We have kept the curved spacetime metric \( g_{\mu\nu} \) only to make it clear how the energy-momentum tensor arises from this action. In practice we shall always work on the Minkowski background \( \eta_{\mu\nu} = \text{diag} (-1, +1, +1) \).

\(^2\)In 3 + 1 dimensions, there is one more scalar invariant, and the action involves a free function of three arguments.
various sound speeds, typically with a non-stop tendency (see (B15)). In particular, past some large enough strain, \( \varepsilon_{\text{max}} \) one is typically left with one of the following options: one of the sound speeds becomes either i) imaginary or ii) superluminal. Case i) clearly implies that the material develops a gradient instability, which we shall identify as the material breaking apart for simplicity (admittedly this is not the only option, see below). In any of the two cases, the stress-strain diagram certainly terminates at that maximum strain \( \varepsilon_{\text{max}} \).

Additionally, demanding that none of these pathologies occur for materials that we know admit large deformations (elastomers) significantly constrains the stress-strain curves and therefore the possible non-linear response of materials on quite general grounds. Below we illustrate the point by focusing on materials/EFTs which allow for large deformations and which realize stress-strain curves with a power-law scaling

\[
\sigma \sim \varepsilon^\nu \quad \text{for} \quad \varepsilon \gg 1 ,
\]

with \( \nu \) sometimes being called the strain hardening exponent. As we show below, both the maximum strain and the exponent \( \nu \) are bounded from above, and there is a general relation between the two. It is unclear to us to what extent these results were already known before. Nonetheless, our main goal is to show how the EFT perspective presented here brings some additional layer of understanding to these phenomena.

Let us summarize here a few important results that immediately follow from the effective action (2) still with \( \varepsilon \) the strain (8) as a function of \( \varepsilon \)

\[
\sigma \sim \varepsilon^\nu \quad \text{for} \quad \varepsilon \gg 1 ,
\]

with the sound speeds bearing a non-linear dependence on the strain parameters \( \alpha, \varepsilon \) given by Eq. (11). For the perturbative stability of the background (4) we require the absence of: i) modes with negative kinetic energy, i.e. ghosts; ii) negative sound speeds squared, i.e. gradient instability; iii) superluminal propagation.

### III. RESULTS IN A SCALING MODEL

For concreteness we shall focus on the simple potential

\[
V(X, Z) = \rho_{\text{eq}} X^A Z^{(B-A)/2} ,
\]

where \( \rho_{\text{eq}} \) is the dimensionful energy density set by the equilibrium configuration. Our only reason to choose this form is that it realizes a power-law scaling like \( Z \) at large deformations, \( \varepsilon \gg 1 \). This behavior is observed in hyperelastic rubber-like materials, and there are many phenomenological models \([4,12,19–24]\) that reduce to \( \varepsilon \) at large strains with various hardening exponents \( \nu \). Here we are only interested in characterizing how the stress-strain curves (and mainly the maximum stress and strain) depend on the parameters \( A, B \). Let us also note that there are two special ‘corners’ in parameter space: for \( A = 0 \), the benchmark potential reduces to \( V(X, Z) = V(Z) \) and hence describes a perfect fluid \([7,9]\); for \( A = 1 \), \( B = 1 \) the model reduces to two free scalar fields.

By using the expressions (A4) and (A5) we find that the linear elastic moduli for the potential (11) take the simple form

\[
G = \rho_{\text{eq}} 2^A A , \quad K = \rho_{\text{eq}} 2^A B (B-1) .
\]

FIG. 1. The non-linear shear stress-strain curve \( \sigma(\varepsilon) \) for the benchmark model \([11]\) for \( B = 1.6 \) and \( A \in [0,0.64] \) corresponding to the lines ordered from bottom to up. The black stars represent the ‘breaking’ points of the material that arise due to the onset of gradient instability; the red dot indicates the onset of superluminality.
They are both positive for $A > 0, B > 1$. For the full non-linear response to pure shear we get from Eq. (9):

$$\sigma(\varepsilon) = \rho_{eq} A \varepsilon \sqrt{\varepsilon^2 + 4 (\varepsilon^2 + 2)^{A-1}}.$$ (13)

This is shown in Fig. 1 for various values of $A$ and $B$. Notably, the stress-strain curves obtained from the benchmark models mimic a large variety of materials including fibers, glasses and elastomers [12]. Similarly the non-linear response to a pure compression, which we define as $\kappa \equiv \alpha - 1$, reads

$$\Delta T_{ii}(\kappa) = \rho_{eq} 2^{A+1} (B - 1)[(\kappa + 1)^2B - 1]$$ (14)

and it is shown in Fig. 3 in the Appendix. As per construction, at large strains, $\varepsilon, \kappa \gg 1$, the non-linear stresses display power-law scalings of the form:

$$\sigma(\varepsilon) \sim A \varepsilon^{2A}, \quad \Delta T_{ii}(\kappa) \sim (B - 1)^{2B},$$ (15)

from where we read off the bulk and shear hardening exponents as: $\nu_{\text{shear}} = 2A$ and $\nu_{\text{bulk}} = 2B$. Note that, as can be seen from Eq. (12), $A$ and $B$ also control the linear shear and bulk moduli. This is however a non-generic feature arising due to the simple power-law dependence on $X$ and $Z$ of the potential $V(X,Z)$. In particular, the direct correlation between the powers and the moduli can be broken by adding new terms to the potential (11). For the sake of simplicity we shall not do so here. Keeping this in mind, though, we shall stick to the physical meaning of $A$ and $B$ as the exponents in Eq. (15), because they characterize the large strain regime.

Combining the requirement of the absence of ghosts, gradient instabilities and superluminal propagation with the positivity of the elastic moduli, $K$ and $G$, constrains the allowed range of parameters. In the simple case of zero background shear strain, i.e. $\varepsilon = 0$, we obtain the following allowed region for the exponents $A, B$:

$$0 \leq A \leq 1 \quad \text{and} \quad 1 \leq B \leq \sqrt{1 - A} + 1.$$ (16)

The analysis can be extended to finite strain and leads us to another important result: the existence of a maximum strain $\varepsilon_{\text{max}}$ that can be supported by the system before the onset of one of the aforementioned pathologies. How $\varepsilon_{\text{max}}$ depends on the hardening exponents is shown in Fig. 2 (see Eqs. (B24) and (B26) for more details). We interpret $\varepsilon_{\text{max}}$ as the breaking (failure) point of the materials described by our benchmark models.\(^4\)

We note that the regions in $A - B$ space where large strains can be supported are precisely near the special points $A = 1, B = 1$ (free scalars) or $A = 0$ (fluid limit). Therefore we expect the real-world (non-relativistic) solids to lie near the $A = 0$ axis. In this limit, the maximum strain is set from the absence of gradient instability for almost all $B$.

Intriguingly, for $A \ll 1$ a number of ‘universal’ correlations appear. First, we find a universal scaling of the maximum strain

$$\varepsilon_{\text{max}} \sim \sqrt{\frac{B - 1}{A}}^{1/4}.$$ (17)

Next, inserting this in the expression (9) for the non-linear shear stress we obtain

$$\sigma_{\text{max}} \equiv \sigma(\varepsilon_{\text{max}}) = \rho_{eq} A.$$ (18)

This shows a linear dependence of the maximal stress supported by a material on the strain hardening exponent $A$, which in our simple model controls also the linear elastic modulus. Similar linear correlations between are observed experimentally in various materials [34–38]. Additionally, we also find a clear relation between the hardness and the maximum strain, $\sigma_{\text{max}} \sim \varepsilon_{\text{max}}^{-4}$. Interestingly enough, fits of the material stress strain curves to simple power laws [39] suggest that the strain hardening exponent obeys an upper bound $\nu \lesssim 0.5$ [40][41], which is roughly compatible with the one shown in Fig. 2. Still, let use emphasize that, whether the correlations that we find can be extrapolated to real world materials strongly depends on whether i) their stress-energy function $V$ behaves as a power law at large strain, and ii) that they can support large deformations.

Finally, let us note that within the benchmark model there are no constraints on the bulk strain $\kappa$ arising from the stability requirements. This is a consequence of being a monomial. For more general choices, additional bounds can arise. Let us also mention that for $B \in (0,1)$ it is possible to achieve a negative bulk modulus, $K < 0$, in a way that is perfectly consistent from the lower order Lagrangian given in [2]. If $\varepsilon_{\text{max}}$ is reached due to a gradient instability, the outcome also depends on the type of the tentative higher order stabilizing terms. In the case when they give rise to a positive $k^4$ correction to the phonon frequency squared, the instability will be slow and soft. One can speculate that this corresponds to the necking phenomenon – a decrease in the cross-sectional area of a material sample that is often seen under tensile stress. In that case, the onset of the instability does not really correspond to the material breaking apart. Still it certainly marks the limit of the elastic behaviour. See e.g. [26][33] for some references on how the failure of a solid proceeds via the so-called soft phonon instability. It appears to us, though, that this is not the only possibility – for different stabilizing terms the breaking instability could be a hard process. Since we leave the stabilizing terms unspecified, we shall abuse the language and identify the gradient instability seen in the EFT with the breaking of the material.

\(^4\) The subminimality condition differs from the others in the sense that it refers to the possibility of a Lorentz invariant UV completion, not about stability.\(^5\) Let us remark that, as can be inferred from Eq. (13), the power-law scaling can really be reached only for $\varepsilon \gtrsim 2$. Therefore, the limits shown in Fig. 2 can only be extended to a material following [15] at large strains in the bluish part of the diagram.\(^6\) The fate of the material past $\varepsilon_{\text{max}}$ is not entirely within the reach of the lowest order Lagrangian given in [2]. If $\varepsilon_{\text{max}}$ is reached due to a gradient instability, the outcome also depends on the type of the tentative higher order stabilizing terms. In the case when they give rise to a positive $k^4$ correction to the phonon frequency squared, the instability will be slow and soft. One can speculate that this corresponds to the necking phenomenon – a decrease in the cross-sectional area of a material sample that is often seen under tensile stress. In that case, the onset of the instability does not really correspond to the material breaking apart. Still it certainly marks the limit of the elastic behaviour. See e.g. [26][33] for some references on how the failure of a solid proceeds via the so-called soft phonon instability. It appears to us, though, that this is not the only possibility – for different stabilizing terms the breaking instability could be a hard process. Since we leave the stabilizing terms unspecified, we shall abuse the language and identify the gradient instability seen in the EFT with the breaking of the material.
FIG. 2. The allowed parameter region (16) for the benchmark model (11). The left, bottom and right edges are respectively given by: gradient instability, positivity of the bulk modulus, superluminality. The color indicates the maximum strain allowed $\varepsilon_{\text{max}}$. The red line (B25) separates the region where the maximum strain is due to the gradient instability (left) and the region where it is due to superluminality (right). The purple dashed are the isolines of constant maximum strain.

EFT perspective. In particular, as long as $K > -G$ the stability constraint $\varepsilon^2 > 0$ is still satisfied (see Appendix B). This has also been studied in four dimensions [42,43] and observed experimentally [44].

IV. DISCUSSION

In conclusion, let us emphasize that the EFT methods for solid materials allow to extract non-trivial information and bounds on their nonlinear elastic response. The list of observables that are fixed (to the leading order in the EFT) once the strain-energy function $V(X,Z)$ is known includes, for instance, all the $n$-point phonon correlation functions, the phonon-phonon self-interactions and, most remarkably, how these depend on the applied stresses – the first example of this being the acoustoelastic effect. This set of correlations is, of course, most directly relevant to materials that admit large deformations and where dissipative effects are unimportant.

As a specific application we have studied how the maximal strain supported by a given material is constrained by the consistency of the effective field theory. We focus in particular to the class of materials in which the stress-strain relations follow a power-law scaling at large strains, $\sigma \sim \varepsilon^\nu$. For such power law behaviour we find a universal linear relation between two intrinsically nonlinear response parameters, the maximum stress and the strain hardening exponent, which holds when both of them are small.

An interesting limiting case is what in [50] was dubbed conformal solids (see also [57,58]). This is realized by potentials of the form

$$V(X,Z) = X^{3/2} F \left( \frac{X}{Z^{1/2}} \right),$$

which indeed preserve scale invariance and imply $T_{\mu \nu} = 0$. Let us mention here that the particular form (19) also forces the bulk modulus to be directly proportional to the energy density $K = 3/4 \rho$, as observed in earlier holographic models [58]. It further implies the universal scaling properties $0 \leq 2A \leq 3/2$ and $2B = 3$, which are in good agreement with what is observed in the holographic realizations of critical materials [59]. Let us emphasize that the notion of a conformal solid, understood just as an EFT with Lagrangian (19), should be distinguished from a system whose low energy dynamics is controlled by a strongly coupled infrared fixed point. In that case, the standard EFT methods are not applicable. A study of the nonlinear elasticity for that case using holographic techniques will be the subject of a separate work [59].

It would be desirable to introduce dissipative and thermal effects within the EFT picture of condensed matter systems [72,60]. In this regard the holographic description could provide a valuable supplementary insight [57–59,61–63]. We hope to return to some of these questions in the near future.

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Appendix A: Linearized stress-strain relations

In this section we present the results for the linear elasticity theory, as derived from the effective action (4). We...
obtain the corresponding stress-energy tensor by varying the action with respect to the curved spacetime metric \( g_{\mu \nu} \) and evaluating it on the Minkowski background, \( g_{\mu \nu} = \eta_{\mu \nu} \):

\[
T_{\mu \nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu \nu}} \bigg|_{g=\eta} = - \eta_{\mu \nu} V + 2 \partial_{\mu} \phi^l \partial_{\nu} \phi_l V_X \\
+ 2 \left( \partial_{\mu} \phi^l \partial_{\nu} \phi_l \bigg( X - \partial_{\mu} \phi^l \partial_{\nu} \phi_l \bigg) \right) V_Z .
\] (A1)

Its components for any time independent scalar field configuration are given in the equations [4]-[8]. In the following we shall specifically consider the scalar field background configuration [4] on which \( X \) and \( Z \) take the values: \( X_{\text{str}} = \alpha^2 (2 + \varepsilon^2) \), \( Z_{\text{str}} = \alpha^4 \).

We already showed how the shear stress-strain curve deformations relates to \( V(X, Z) \) in [9]. The analogous stress-strain curve for pure bulk can also be found by expressing \( \Delta T_{xx} = T_{xx} - T_{xx}^{\text{eq}} \) as a function of the bulk strain, \( G \). It is clear then that from the knowledge (measurement) of both shear and bulk diagrams one can extract the shape of \( V(X, Z) \) – the full effective Lagrangian. For instance, under the assumption that the \( Z \)-dependence is negligible, then from a given \( \sigma(\varepsilon) \) shear strain stress curve one can extract

\[
V(X) \simeq \int_2^X dx \frac{\sigma(\sqrt{2} - 2)}{\sqrt{x^2/4 - 1}}.
\]

To make the connection to the linear elasticity theory explicit one considers small shear and bulk deformations i.e. small values of \( \varepsilon \) and \( \alpha - 1 \). Then, as usual, elastic deformations at linear level are described in terms of the displacement tensor

\[
\varepsilon_{ij} = \frac{1}{2} \left( \partial_i \delta_{ij} + \partial_j \delta_{ij} \right),
\] (A2)

where \( \delta \phi^l = \phi^l - \phi^l_0 \) is the displacement away from the equilibrium state \( \phi^l_0 \). A deformation of the body that changes its volume is given by the compression or bulk strain as \( \varepsilon_{ii} = \partial_i \delta \phi^i \). In turn, a deformation that only affects its shape – pure shear – is given by \( \varepsilon_{ik} = \frac{1}{2} \delta_{ik} \varepsilon_{jj} \).

Expanding both the stress-energy tensor components [0, 7] and the displacement tensor [A2] up to linear order in \( \varepsilon \) and \( \alpha - 1 \) one recovers the usual expression in 2+1 dimensions [12]:

\[
T_{ij}^{\text{lin}} = (p + K \varepsilon_{kk}) \delta_{ij} + 2 G \left( \varepsilon_{ij} - \frac{1}{2} \delta_{ij} \varepsilon_{kk} \right),
\] (A3)

where \( p \) is the equilibrium pressure and \( G, K \) are the shear and bulk elastic moduli.

In the case of a pure shear deformation this gives \( T_{xy} = 2G\varepsilon_{xy} + \ldots \) and we can read off the shear modulus \( G \) as

\[
G = V_X(2, 1).
\] (A4)

Similarly for the case of pure bulk stress (\( \varepsilon = 0 \)) we first note that the equation [7] holds at non-linear level, i.e. for arbitrarily large values of \( \alpha \).

In order to find the linear bulk modulus we expand both the bulk strain and the bulk stress \( \Delta T_{xx} \) around the equilibrium value \( \alpha = 1 \). For the stress this gives \( \Delta T_{ii} = 2p + 2K \varepsilon_{ii} + \ldots \) with the equilibrium pressure given in [7] and \( \varepsilon_{ii} = 2(\alpha - 1) \). The bulk modulus is then:

\[
K = 2ZV_Z + 4Z^2V_{ZZ} + 4XZV_{XZ} + X^2V_{XX} ,
\] (A5)

where all the quantities are evaluated at \( X = 2, Z = 1 \).

**Appendix B: Fluctuations and consistency**

In order to study the stability of perturbations around the background configuration [4] we expand the scalar fields as \( \phi^l = \hat{\phi^l} + \pi^l \). To identify the propagating degrees of freedom we perform the decomposition into longitudinal and transverse fluctuations by splitting \( \pi^l = \pi^l_L + \pi^l_T \), with \( \pi_{L/T} \) satisfying:

\[
O^l_K \partial_L \pi^K_L = 0, \quad \varepsilon^{IJ} O^l_J \partial_K \pi^K_J = 0 .
\] (B1)

This gives two dynamical scalar modes that can be defined through:

\[
\pi^l_L = O^{IK} \partial_K \pi^K_L, \quad \pi^l_T = \varepsilon^{IJ} O^l_J \partial_K \pi^K_T .
\] (B2)

Constraining the spatial dependence to \( \pi_{L/T}(t, x) \) and redefining \( \pi_{L/T} \to \pi_{L/T}/\sqrt{-\partial^2} \) we obtain the following quadratic action for the fluctuations

\[
\delta S_2 = \int d^4x \left[ N_T \dot{\pi}_T^2 + N_L \dot{\pi}_L^2 + 2N_{TL} \dot{\pi}_T \dot{\pi}_T - c^2_L (\partial_x \pi_T)^2 \right. \left. - 2c^2_T (\partial_x \pi_L)^2 - 2c^2_{TL} (\partial_x \pi_T \partial_x \pi_L) \right] ,
\] (B3)
where the parameters $N_T, N_L, N_{TL}$ and $c^2_T, c^2_L, c^2_{TL}$ depend on both the shear and bulk strains, i.e., they are functions of $\varepsilon$ and $\alpha$ introduced in \([4]\). The explicit expressions in terms of the derivatives of the function $V(X, Z)$ are found to be:

\[
N_T = \frac{1}{2} ((X^2 - 2Z)V_Z + XV_X), \tag{B4}
\]
\[
N_L = Z V_Z + \frac{X}{2} V_X, \tag{B5}
\]
\[
N_{TL} = \frac{1}{2} \sqrt{Z(X^2 - 4Z)}V_Z, \tag{B6}
\]
\[
c^2_T = Z (V_Z + 2ZV_Z) \tag{B7}
\]
\[
+ \frac{1}{2} X (V_X + 4ZV_XZ + XV_{XX}), \tag{B8}
\]
\[
c^2_L = \frac{1}{4} ((X^2 - 4Z)(V_Z + 2ZV_Z) + 2XV_X) \tag{B8}
\]
\[
c^2_{TL} = \frac{1}{2} \sqrt{Z(X^2 - 4Z)}(V_Z + 2ZV_Z + XV_{XX}), \tag{B9}
\]
with all the quantities evaluated on the scalar field background \([4]\).

Let us emphasize that for a non-diagonal matrix $O^f$, the transverse and longitudinal modes remain mixed both with respect to time and spatial derivatives. In order to study the stability of fluctuations we therefore first introduce the kinetic matrix as

\[
\mathcal{N} = \begin{pmatrix}
N_T & N_{TL} \\
N_{TL} & N_L
\end{pmatrix}. \tag{B10}
\]

The absence of ghost-like excitations then requires that the eigenvalues of the kinetic matrix, $\lambda_{\pm}$, are positive. This gives the first condition for stable propagation of the modes: $\lambda_{\pm} > 0$.

It is straightforward to determine the dynamical modes described by the action \([B3]\) by working at the level of the equations of motion of the mixed fields $\pi_{L/T}$. After Fourier transforming as $\pi_{L/T} = a_{L/T} e^{i\omega t - ikx}$ we can solve for the spectrum of perturbations to obtain

\[
\omega^2_{\pm} = c^2_{\pm}(\alpha, \varepsilon) k^2. \tag{B11}
\]

The other conditions for stability that we are going to impose are thus:

- $c^2_{\pm} \geq 0$, i.e. the absence of gradient instabilities;
- $c^2_{\pm} \leq 1$, i.e. the absence of superluminal modes.

The exact expressions of the kinetic eigenvalues can be put in the form

\[
\lambda_{\pm} = \frac{c}{2} \left[ 1 \pm \sqrt{1 - \frac{4d}{c^2}} \right], \tag{B12}
\]

with

\[
c = N_L + N_T, \tag{B13}
\]
\[
d = N_T N_L - N^2_{TL} = \det \mathcal{N}. \tag{B14}
\]

Similarly the sound speeds can be expressed as

\[
c^2_{\pm} = \frac{a}{2d} \left[ 1 \pm \sqrt{1 - \frac{4bd}{a^2}} \right] \tag{B15}
\]

with

\[
a = c^2_T N_L + c^2_L N_T - 2c^2_{TL} N_{TL}, \tag{B16}
\]
\[
b = c^2_T c^2_L - c^2_{TL}. \tag{B17}
\]

Let us point out that evaluating the sound speeds $c_{\pm}$ at $\alpha = 1$ and $\varepsilon = 0$ we find that the result coincides with the standard relationships obeyed by the transverse and longitudinal phonons of the equilibrium state \([1]\):

\[
c_T = \sqrt{\frac{G}{\rho + p}}, \quad c_L = \sqrt{\frac{K + G}{\rho + p}}, \tag{B18}
\]

where $\rho$ and $p$ are the equilibrium energy density and pressure, as in \([6\) and \([7\). The $K$ and $G$ refer to the linearized bulk and shear moduli given in \((A5)\) and \((A4)\).

The conditions necessary to ensure the positivity of $\lambda_{\pm}$ read:

\[
c > 0, \quad d > 0, \quad 1 - \frac{4d}{c^2} \geq 0. \tag{B19}
\]

The first two constraints above can be expressed as inequalities for quadratic polynomials in $\varepsilon^2$. For the benchmark model \([11]\) we find that upon setting

\[
A - B < 0, \quad A > 0 \tag{B20}
\]

these are satisfied for any choice of $\varepsilon$, while the last condition is fulfilled automatically for arbitrary choice of $A, B, \varepsilon$.

The conditions necessary for avoiding the gradient instability are in turn

\[
a > 0, \quad b > 0, \quad 1 - \frac{4bd}{a^2} \geq 0 \tag{B21}
\]

and are slightly harder to satisfy. It is easy to see that by setting

\[
A + B > 1 \tag{B22}
\]

and assuming that \((B20)\) holds the condition $a > 0$ can be satisfied for arbitrary values of $\varepsilon$. However, for these values of $A, B$ the equation $b = 0$ defines an inverse parabola in the $\varepsilon^2$ space with two real roots $c^2_{\pm}$ only when

\[
B - 1 > 0. \tag{B23}
\]

Hence the condition $b \geq 0$ is only satisfied for $\varepsilon^2 \in [\varepsilon^2_{\min}, \varepsilon^2_{\max}]$. Since we are only interested in positive values of $\varepsilon^2$ then we conclude that the condition $b \geq 0$ imposes a constraint on the maximal allowed strain applied to our system given by:

\[
\varepsilon^2_{\max} = 2\sqrt{2 + \frac{B - 1}{A} + \frac{A}{B - A} - 2}. \tag{B24}
\]
Analyzing the last condition in (B21) analytically becomes more involved. We find however that in the parameter region

\[ B \leq \frac{1}{2} \left( 2 - A + \sqrt{4 - 3A^2} \right) \]

the maximal strain is determined by the onset of the gradient instability and is thus given by (B24). Only in the region complementary to (B25) is the maximal strain fixed by requiring the absence of superluminal propagation, finding

\[ \epsilon^2_{\text{max}} = 2 \sqrt{\frac{A(A+B-2)}{A^2 + A(B-1) + (B-2)B}} - 2. \]

We present the full constraints on the parameter space obtained numerically in Fig. 2

Finally, let us quote our results for the simple case of zero background shear strain, i.e., \( \epsilon = 0 \). We obtain the following allowed region for the exponents \( A, B \):

\[ 0 \leq A \leq 1 \quad \text{and} \quad 1 \leq B \leq \sqrt{1-A+1}. \]

More specifically, the two kinetic eigenvalues in this case are equal and given by \( \lambda_{\pm} = 2^{-1+A-2B}B \) imposing the constraint \( B > 0 \). The sound speeds are in turn given by \( c^2_\alpha = \frac{A}{B} \) and \( c^2_\gamma = B - 1 + \frac{A}{B} \). The absence of gradient instabilities is thus setting \( A \geq 0 \) and \( B \geq 1 \geq -A/B \). The latter constraint can be made stronger by requiring the positivity of the bulk modulus, leading to \( B \geq 1 \); the positivity of the shear modulus gives again \( A \geq 0 \).

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