A monoidal analogue of the 2-category anti-equivalence between \textbf{ABEX} and \textbf{DEF}

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Abstract

We prove that the 2-category of skeletally small abelian categories with exact monoidal structures is anti-equivalent to the 2-category of fp-hom-closed definable additive categories satisfying an exactness criterion.

For a fixed finitely accessible category \( C \) with products and a monoidal structure satisfying the appropriate assumptions, we provide bijections between the fp-hom-closed definable subcategories of \( C \), the Serre tensor-ideals of \( C^{fp}\)-mod and the closed subsets of a Ziegler-type topology.

For a skeletally small preadditive category \( A \) with an additive, symmetric, rigid monoidal structure we show that elementary duality induces a bijection between the fp-hom-closed definable subcategories of \( \text{Mod-} A \) and the definable tensor-ideals of \( A\text{-Mod} \).

1 Introduction

In \[11\], Prest and Rajani define an anti-equivalence between the 2-category \textbf{ABEX} with objects given by skeletally small abelian categories and morphisms given by additive exact functors and the 2-category \textbf{DEF} with objects given by definable categories and morphisms given by additive functors which commute with direct products and direct limits. In both cases the 2-morphisms are given by natural transformations. Under this equivalence, a finitely accessible category with products \( C \), which by definition is definable, corresponds to the module category \( C^{fp}\)-mod of finitely presented additive functors from the full subcategory of \( C \) given by the finitely presented objects, \( C^{fp} \), to the category of abelian groups, \( \textbf{Ab} \). In the case that \( C^{fp} \) has a monoidal structure, we may induce a monoidal structure on \( C^{fp}\)-mod via Day convolution product (see \[3\]). What’s more, the definable subcategories of \( C \) correspond to the Serre subcategories of \( C^{fp}\)-mod (e.g. see \[8\], Theorem 12.4.1 and Corollary 12.4.2) and the equivalence maps a definable subcategory \( D \subseteq C \) to the functor category \( \text{fun}(D) \cong C^{fp}\)-mod/S where \( S \) is the Serre subcategory corresponding (by annihilation) to \( D \) (e.g. see \[9\], Theorem 12.10). When the Serre subcategory \( S \) is a Serre tensor-ideal, the functor category \( \text{fun}(D) \cong C^{fp}\)-mod/S has a monoidal structure (see Definition \[5.10\]). Therefore, it is interesting to ask: Which definable subcategories of \( C \) correspond to Serre tensor-ideals of \( C^{fp}\)-mod? The answer is given in Theorem \[3.3\] More generally, we define 2-categories \textbf{ABEX} and \textbf{DEF} and give a 2-category anti-equivalence which can be viewed as a monoidal analogue of the 2-category anti-equivalence between \textbf{ABEX} and \textbf{DEF} (Section \[3\]).

In order for the 2-category anti-equivalence between \textbf{ABEX} and \textbf{DEF} to hold, we must take the objects of \textbf{DEF} to be the definable subcategories \( D \subseteq C \) which are fp-hom-closed and satisfy an exactness criterion. Here \( C \) is a finitely accessible category with products and an additive, symmetric, closed monoidal structure such that the subcategory of finitely presented objects, \( C^{fp} \), forms a symmetric monoidal subcategory. By the original anti-equivalence, \( D \) and
Ex(fun(D), Ab) are equivalent definable categories. However, the exactness criterion is necessary to ensure that the fp-hom-closed property is preserved (Theorem 3.12 and Proposition 3.12).

In practice, many fp-hom-closed definable subcategories do not satisfy the exactness criterion. Indeed, the exactness criterion for D implies that the monoidal structure on fun(D) is exact (Theorem 3.11), when in general this monoidal structure is only right exact. In Section 4, we discuss the relationship between definability and monoidal structures for fixed C without assuming the exactness criterion. We define a ‘coarser-version’ of the Ziegler spectrum (Section 4.1) and provide bijections between the fp-hom-closed definable subcategories of C, the Serre tensor-ideals of C^{op}-mod and the closed subsets of this Ziegler-type topology. We also consider what can be said under the additional assumption that C^{op} is rigid monoidal (Section 4.2). Here a definable subcategory is fp-hom-closed if and only if it is a tensor-ideal (Corollary 4.3). Furthermore, given a skeletally small preadditive category A with an additive, symmetric, rigid monoidal structure, we provide a bijection between the fp-hom-closed definable subcategories of Mod-A and the definable tensor-ideals of A-Mod (Section 4.3).

Finally, in Section 5 we will consider some examples. First we explore the examples given by the tensor product of R-modules for a commutative ring R and secondly we give an example where C^{op} is rigid.

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2 Preliminaries

2.1 Day convolution product

Given a finitely accessible category, C, with products and a monoidal structure, we will use Day convolution product to induce a monoidal structure on the associated functor category.

**Theorem 2.1.** ([13], Theorem 3.3 and Theorem 3.6) Given a complete and cocomplete closed symmetric monoidal category V, and a small (symmetric) monoidal V-enriched category C, the category of V-enriched functors from C to V, V[C, V], is a monoidal category admitting a (symmetric) closed monoidal structure.

**Notation 2.2.** Throughout our V (as above) will be the category of abelian groups, Ab. All our categories will be preadditive and all our functors will be additive. For a preadditive category C we denote by C(A, B) the abelian group of all morphisms in C from A to B. When the category is clear from context we will simply write (A, B). In addition, given preadditive categories A and B, we will denote by (A, B) the functor category of all additive functors from A to B. The functor category (A, Ab) will be denoted by A-Mod and the subcategory of all finitely presented objects with be denoted by A-mod := (A-Mod)^p. Similarly, we denote by Mod-A and mod-A the categories (A^{op}, Ab) and (A^{op}, Ab)^p respectively.

Given a symmetric monoidal structure (⊗, 1) on a small additive category A, we may refer to the Day convolution product on the functor category A-Mod := (A, Ab) as the ‘induced monoidal structure’ or ‘induced tensor product’ and denote the tensor product functor by ⊗. By Theorem 2.1 the induced monoidal structure on A-Mod is closed, that is, for every X ∈ A-Mod,
$X \otimes - : \mathcal{A}\text{-Mod} \to \mathcal{A}\text{-Mod}$ has a right adjoint functor which we will denote by $\text{hom}(X, -) : \mathcal{A}\text{-Mod} \to \mathcal{A}\text{-Mod}$ and call the internal hom-functor.

Since for each $F \in \mathcal{A}\text{-Mod}$, $F \otimes -$ is a left adjoint, it is right exact and commutes with direct limits. Furthermore, by definition of Day convolution product, given representable functors $(A, -)$ and $(B, -)$ in $\mathcal{A}\text{-Mod}$, we have $(A, -) \otimes (B, -) \cong (A \otimes B, -)$. Thus, by right exactness, if $F \in \mathcal{A}\text{-mod}$ has presentation $(B, -) \xrightarrow{(f, -)} (A, -) \to F \to 0$, with $f : A \to B$, then $(C, -) \otimes F$ has presentation $(C \otimes B, -) \xrightarrow{(\varepsilon \otimes f, -)} (C \otimes A, -) \to (C, -) \otimes F \to 0$.

**Notation 2.3.** Given an additive (skeletally) small category $\mathcal{A}$ every finitely presented module $F \in \mathcal{A}\text{-mod}$ has a presentation of the form

$$(B, -) \xrightarrow{(f, -)} (A, -) \xrightarrow{\pi_f} F \to 0,$$

with $f : A \to B$ in $\mathcal{A}$. We will denote such a functor by $F_f$.

By the above we have $(C, -) \otimes F_f = F_{C \otimes f}$. More generally, $F_f \otimes F_g = F_{(f \otimes U, A \otimes g)}$, where $f : A \to B$ and $g : U \to V$ and $(f \otimes U, A \otimes g) : A \otimes U \to (B \otimes U) \oplus (A \otimes V)$ is the canonical map.

Thus, Day convolution product restricts to a monoidal structure on the category of finitely presented additive functors $\mathcal{A}\text{-mod}$, which we may also refer to as the ‘induced monoidal structure’ or ‘induced tensor product’. This is exactly the tensor product given in ([10], Section 13.3) with $\mathcal{A} = R\text{-mod}$. Here we avoid the notation $(R\text{-mod})\text{-mod}$ in favour of $(R\text{-mod}, \text{Ab})^{fp}$.

### 2.2 Rigid monoidal categories

In this section we will outline the definition of a rigid monoidal category.

**Definition 2.4.** Let $\mathcal{C}$ be a symmetric monoidal category. $C' \in \mathcal{C}$ is dual to $C \in \mathcal{C}$ if there exist morphisms $\eta : 1 \to C' \otimes C$ and $\epsilon : C \otimes C' \to 1$ such that $(C' \otimes \epsilon) \circ (\eta \otimes C') = 1_{C'}$ and $(\epsilon \otimes C) \circ (C \otimes \eta) = 1_C$.

A closed symmetric monoidal category $\mathcal{C}$ is said to be rigid if every object of $\mathcal{C}$ has a dual.

An important consequence of the existence of dual objects is the following.

**Proposition 2.5.** (c.f. [5], Proposition 1.10.9) Let $\mathcal{C}$ be a symmetric monoidal category and suppose $C \in \mathcal{C}$ is rigid. Then $C' \otimes -$ is both left and right adjoint to $C \otimes -$.

**Corollary 2.6.** Let $\mathcal{C}$ be a closed symmetric monoidal category and suppose $C'$ is dual to $C$ in $\mathcal{C}$. There exists a natural isomorphism $\text{hom}(C, -) \cong C' \otimes -$.

**Corollary 2.7.** Let $\mathcal{A}$ be an abelian category with a closed symmetric monoidal structure and suppose $C \in \mathcal{A}$ has a dual. Then $C \otimes - : \mathcal{A} \to \mathcal{A}$ is exact.

**Definition 2.8.** Let $\mathcal{C}$ be a rigid symmetric monoidal category. Given any morphism, $f : A \to B$ in $\mathcal{C}$, there exists a dual morphism, $f' : B' \to A'$, in $\mathcal{C}$ given by the composition

$$B' \xrightarrow{\eta \otimes B'} A' \otimes A \otimes B' \xrightarrow{A' \otimes f \otimes A'} A' \otimes B \otimes B' \xrightarrow{\alpha' \otimes \varepsilon} A'.$$

### 2.3 Purity in finitely accessible categories

The results in this section will be stated without proof and we direct the reader to [8], [9] and [11] for more details. Throughout the paper, we use [8] and [9] as convenient secondary sources.

Let us recall the definition of a finitely accessible category.
Definition 2.9. A category $\mathcal{C}$ is said to be **finitely accessible**, if it has direct limits and there exists a set, $\mathcal{G}$, of finitely presentable objects of $\mathcal{C}$ such that for every $X \in \mathcal{C}$, we can write $X$ as a direct limit of objects of $\mathcal{G}$. That is, $X = \lim_{\to I} X_i$ where $I$ is some directed indexing set and each $X_i \in \mathcal{G}$. Note that in this case, the full subcategory of finitely presentable objects of $\mathcal{C}$, denoted by $\mathcal{C}^{fp}$, is skeletally small and we can take $\mathcal{G}$ to consist of a representative of each isomorphism class of $\mathcal{C}^{fp}$. For the purposes of this paper we will take ‘finitely accessible’ to mean additive and finitely accessible.

Next we give the definition of a definable subcategory of a finitely accessible category with products.

Definition 2.10. Let $\mathcal{C}$ be a finitely accessible category with products. A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is said to be **definable** if it is closed in $\mathcal{C}$ under products, direct limits and pure subobjects.

A definable category is a definable subcategory of some finitely accessible category with products.

We can use the definable subcategories of a finitely accessible category with product to define a topology called the Ziegler spectrum.

Definition 2.11. Let $\mathcal{C}$ be finitely accessible with products. A monomorphism $m : X \to Y$ in $\mathcal{C}$ is said to be a **pure monomorphism** if for every $f : A \to B$ in $\mathcal{C}^{fp}$ and for all morphisms $h : A \to X$ and $h' : B \to Y$ such that $h' \circ f = m \circ h$ there exist some $k : B \to X$ such that $k \circ f = h$.

Remark 2.12. If $\mathcal{C}$ is locally finitely presented (that is finitely accessible, complete and cocomplete), pure monomorphisms can be characterised as those monomorphism $m : X \to Y$ which fit into an exact sequence

$$0 \to X \xrightarrow{m} Y \xrightarrow{p} Z \to 0$$

such that for every $A \in \mathcal{C}^{fp}$,

$$0 \to (A, X) \xrightarrow{(A,m)} (A, Y) \xrightarrow{(A,p)} (A, Z) \to 0$$

is exact in $\text{Ab}$ (see [9], Theorem 5.2).

Definition 2.13. We say that an object $E \in \mathcal{C}$ is **pure-injective** if it is injective over pure monomorphisms, that is for every pure monomorphism $m : X \to Y$ in $\mathcal{C}$ and any morphism $k : X \to E$ there exists some $h : Y \to E$ such that $k = h \circ m$.

In fact, each finitely accessible category with products has, up to isomorphism, a set of indecomposable pure-injective objects and each definable subcategory is generated as such by its indecomposable pure-injectives. They form the underlying set of a topological space called the Ziegler spectrum.
Definition 2.14. We define the Ziegler spectrum of \( C \), denoted \( Zg(C) \), to have underlying set given by the set of isomorphism classes of indecomposable pure-injectives in \( C \), denoted \( \text{pinj}_C \), and closed subsets given by

\[ \{ [X] \in \text{pinj}_C : X \in \mathcal{D} \} \]

where \([X] \) denotes the isomorphism classes of the indecomposable pure-injective \( X \) and \( \mathcal{D} \) runs through the definable subcategories of \( C \).

Proposition 2.15. ([9], Theorem 14.1) Let \( C \) be a finitely accessible category with products. The closed subsets described above define a topology on \( \text{pinj}_C \).

Next we give another way to find the definable subcategories of \( C \).

Notation 2.16. Let \( C \) be a finitely accessible category. For \( F \in C^{fp}\text{-mod} \), denote by \( \overline{F} : C \to \text{Ab} \) the unique extension of \( F \) which commutes with direct limits (e.g. see [8], Proposition 10.2.41).

If \( C \) is additive finitely accessible with products, then a subcategory, \( D \subseteq C \) is definable if and only if there is a collection of finitely presented functors \( Y \subseteq C^{fp}\text{-mod} \) such that \( X \in D \) if and only if \( \overline{F}(X) = 0 \) for all \( F \in Y \). Furthermore, if \( D \subseteq C \) is definable then the set \( S = \{ F \in C^{fp}\text{-mod} : \overline{F}(X) = 0, \forall X \in D \} \) is a Serre subcategory. In fact, we have Theorem 2.17 below.

Theorem 2.17. ([9], Theorem 14.2) Let \( C \) be an additive finitely accessible category with products. There is a natural bijection between:

(i) the definable subcategories of \( C \),

(ii) the Serre subcategories of \( C^{fp}\text{-mod} \),

(iii) the closed subsets of the Ziegler Spectrum \( Zg(C) \).

Next let us describe elementary duality. Let \( A \) be a skeletally small preadditive category.

Definition 2.18. (see for example [9], Section 3) The tensor product of \( A \)-modules is given by a functor \( - \otimes_A - : \text{Mod-}A \times A\text{-Mod} \to \text{Ab} \) determined on objects (up to isomorphism) by the following two assertions. For every \( M \in \text{Mod-}A \),

(i) \( M \otimes_A (A, -) \cong M(A) \) for every \( A \in A \),

(ii) \( M \otimes_A - \) is right exact.

The functor is defined on morphisms in the obvious way.

We can now define a duality of functor categories as follows.

Theorem 2.19. ([9], Theorem 4.5) There is a duality \( \delta : (\text{mod-}A, \text{Ab})^{fp} \to (A\text{-mod, Ab})^{fp} \) given on objects by mapping \( F_f : \text{mod-}A \to \text{Ab} \), where \( f : A \to B \) in \( \text{mod-}A \), to \( \delta F : A\text{-mod} \to \text{Ab} \) where \( \delta F \) has copresentation

\[ 0 \to \delta F \to A \otimes_A - \xrightarrow{f \otimes_A -} B \otimes_A - \]

Next we note that \( \delta \) induces a bijection between definable subcategories.
Proposition 2.20. ([9], Theorem 8.1) The duality, $\delta$, of Theorem 2.19 maps Serre subcategories of $(\mathbf{mod-}A, \mathbf{Ab})^{\mathfrak{fp}}$ to Serre subcategories of $(A\mathbf{-mod}, \mathbf{Ab})^{\mathfrak{fp}}$ and therefore induces a bijection between the definable subcategories of $\mathbf{Mod-}A$ and those of $A\mathbf{-Mod}$.

Notation 2.21. Given a Serre subcategory $S \subseteq (\mathbf{mod-}A, \mathbf{Ab})^{\mathfrak{fp}}$ we will denote the dual Serre subcategory by $\delta S$, that is $\delta S = \{ \delta F : F \in S \} \subseteq (A\mathbf{-mod}, \mathbf{Ab})^{\mathfrak{fp}}$.

Similarly, given a definable subcategory $D \subseteq \mathbf{Mod-}A$ we will denote the dual definable subcategory, associated to $\delta S$ by annihilation, by $\delta D \subseteq A\mathbf{-Mod}$.

We will also use this $\delta$ notation for the inverse map. That is, if $S \subseteq (A\mathbf{-mod}, \mathbf{Ab})^{\mathfrak{fp}}$ is a Serre subcategory $\delta S \subseteq (\mathbf{mod-}A, \mathbf{Ab})^{\mathfrak{fp}}$ is the dual Serre subcategory and similarly for definable subcategories.

Below we give two key properties of the 2-category anti-equivalence between $\mathbf{ABEX}$ and $\mathbf{DEF}$.

Definition 2.22. Let $\mathbf{DEF}$ denote the 2-category with objects given by definable categories, morphisms given by additive functors which preserve direct product and direct limits and 2-morphisms given by natural transformations.

Let $\mathbf{ABEX}$ denote the 2-category with objects given by skeletally small abelian categories, morphisms given by additive exact functors and 2-morphisms given by natural transformations.

Theorem 2.23. ([11], Theorem 2.3) There exists a 2-category anti-equivalence between $\mathbf{ABEX}$ and $\mathbf{DEF}$ given on objects by $A \mapsto \text{Ex}(A, \mathbf{Ab})$ and $D \mapsto \text{fun}(D) := (D, \mathbf{Ab})^{-\text{H}}$, where $\text{Ex}(A, \mathbf{Ab})$ is the category of exact functors from $A$ to the category of abelian groups and $(D, \mathbf{Ab})^{-\text{H}}$ is the category of additive functors from $D$ to the category of abelian groups which commute with direct products and direct limits.

On morphisms the equivalence works in both directions by mapping an appropriate functor, say $F$, to precomposition by $F$, $- \circ F$, and on 2-morphisms it works in the obvious way.

Theorem 2.24. ([9], Theorem 12.10) Given a definable subcategory $D$ of a finitely accessible category $C$ with products, $\text{fun}(D) \simeq C^{\mathfrak{fp}}\mathbf{-mod}/S$ where $S \subseteq C^{\mathfrak{fp}}\mathbf{-mod}$ is the Serre subcategory corresponding to $D$ (as in Theorem 2.17).

Given a finitely presented functor $F \in C^{\mathfrak{fp}}\mathbf{-mod}$, the restriction to $D$ of its extension along direct limits, $(\overrightarrow{F})|_D : D \to \mathbf{Ab}$, commutes with direct product and direct limits and therefore is an object of $\text{fun}(D)$. Let $S \subseteq C^{\mathfrak{fp}}$ be the Serre subcategory corresponding to $D$ and recall that $C^{\mathfrak{fp}}\mathbf{-mod}/S$ is given by formally inverting the morphisms in $\Sigma_S = \{ \alpha \in \text{morph}(C^{\mathfrak{fp}}) : \ker(\alpha), \coker(\alpha) \in S \}$. Since every morphism in $\Sigma_S$ is an isomorphism when evaluated at any $D \in D$, by the universal property of the localisation, the functor $(\overrightarrow{-})|_D : C^{\mathfrak{fp}}\mathbf{-mod} \to \text{fun}(D)$ factors via the localisation $C^{\mathfrak{fp}}\mathbf{-mod}/S$. The equivalence in Theorem 2.24 is given by the exact functor $(\overrightarrow{-})|_D : C^{\mathfrak{fp}}\mathbf{-mod}/S \to \text{fun}(D)$ induced by this factorisation.

2.4 The 2-categories $\mathbf{DEF}^\otimes$ and $\mathbf{ABEX}^\otimes$

In this section we define the 2-categories $\mathbf{ABEX}^\otimes$ and $\mathbf{DEF}^\otimes$.

Notation 2.25. Every monoidal category is monoidally equivalent to a strict monoidal category ([6], Section XI, Subsection 3, Theorem 1). Therefore we are safe to suppress all unitors and associators, treating them as identities.
Definition 2.26. We will say that a functor $F : \mathcal{A} \to \mathcal{B}$ between monoidal categories $(\mathcal{A}, \otimes, 1_\mathcal{A})$ and $(\mathcal{B}, \otimes', 1_\mathcal{B})$ is monoidal if there exists an isomorphism in $\mathcal{B}$, $\epsilon : 1_\mathcal{B} \to F(1_\mathcal{A})$ and a natural isomorphism $\mu : (\otimes' \circ F \times F) \to (F \circ \otimes)$ satisfying the associativity condition $\mu_{X,Y,Z} \circ (\mu_{X,Y} \otimes' F(Z)) = \mu_{X,Y \otimes Z} \circ (F(\otimes') \circ Y, Z)$ and the unitality conditions, $\mu_{1_\mathcal{A}, X} \circ (\epsilon \otimes' F(X)) = id_{1_\mathcal{A} \otimes' F(X)}$ and $\mu_{X, 1_\mathcal{A}} \circ (F(X) \otimes' \epsilon) = id_{F(X) \otimes 1_\mathcal{A}}$.

Definition 2.27. Let $\text{ABEX}^\otimes$ denote the 2-category with objects given by skeletally small abelian categories equipped with an additive symmetric monoidal structure which is exact in each variable, morphisms given by additive exact monoidal functors and 2-morphisms given by natural transformations.

Notation 2.28. Given a category $\mathcal{C}$ and morphisms $f : A \to B$ and $k : A \to C$ in $\mathcal{C}$, we will write $f|k$ if there exists some morphism $k' : B \to C$ in $\mathcal{C}$ such that $k = k' \circ f$.

Definition 2.29. Let $\mathcal{C}$ be a finitely accessible category with products and an additive symmetric closed monoidal structure such that $\mathcal{C}^{\text{f.p.}}$ is a monoidal subcategory.

We say that a definable subcategory $\mathcal{D} \subseteq \mathcal{C}$ is fp-hom-closed if for every $A \in \mathcal{C}^{\text{f.p.}}$ and $X \in \mathcal{D}$, hom$(A,X) \in \mathcal{D}$, where hom denotes the internal hom-functor.

We say that a definable subcategory $\mathcal{D} \subseteq \mathcal{C}$ satisfies the exactness criterion if given morphisms $f : A \to B$ and $g : U \to V$ in $\mathcal{C}^{\text{f.p.}}$ and a morphism $h : A \otimes U \to X$ in $\mathcal{C}$ where $X \in \mathcal{D}$, if $(f \otimes U)|h$ and $(A \otimes g)|h$ then $(f \otimes g)|h$.

Definition 2.30. We define the 2-category $\text{DEF}^\otimes$ as follows. Let the objects of $\text{DEF}^\otimes$ be given by the triples $(\mathcal{D}, \mathcal{C}, \otimes)$ where $\mathcal{C}$ is a finitely accessible category with products, $\otimes$ is an additive symmetric closed monoidal structure on $\mathcal{C}$ such that $\mathcal{C}^{\text{f.p.}}$ is a monoidal subcategory and $\mathcal{D}$ is an fp-hom-closed definable subcategory of $\mathcal{C}$ satisfying the exactness criterion. Let the morphisms in $\text{DEF}^\otimes$ be given by the additive functors $I : \mathcal{D} \to \mathcal{D}'$ which commute with direct product and direct limits and such that the induced functor $I_0 : \text{fun}(\mathcal{D}') \to \text{fun}(\mathcal{D})$ (see [11], Theorem 2.3) is monoidal.

Remark 2.31. Notice that there exist forgetful 2-functors $\mathcal{F} : \text{ABEX}^\otimes \to \text{ABEX}$ and $\mathcal{F} : \text{DEF}^\otimes \to \text{DEF}$ which forget the monoidal structure.

3 The 2-category anti-equivalence

In this section we prove the main theorem of the paper.

Theorem 3.1. There exists a 2-category anti-equivalence between $\text{ABEX}^\otimes$ and $\text{DEF}^\otimes$ given on objects by $\mathcal{A} \mapsto ((\text{Ex}(\mathcal{A}, \text{Ab}), \mathcal{A}^{\text{Mod}}, \otimes))$ where the monoidal structure, $\otimes$, on $\mathcal{A}^{\text{Mod}}$ is induced by the monoidal structure on $\mathcal{A}$ via Day convolution product. Conversely, the anti-equivalence maps an object $(\mathcal{D}, \mathcal{C}, \otimes)$ in $\text{DEF}^\otimes$ to the skeletally small abelian category $\text{fun}(\mathcal{D}) = (\mathcal{D}, \text{Ab})^{\text{f.p.-mod}}$ with monoidal structure induced by Day convolution product on $\mathcal{C}^{\text{f.p.-mod}}$ (see Definition 2.10).

We prove Theorem 3.1 in several parts.

3.1 The 2-functor $\theta : \text{DEF}^\otimes \to \text{ABEX}^\otimes$

First let us define a 2-functor $\theta : \text{DEF}^\otimes \to \text{ABEX}^\otimes$. On objects, $\theta$ maps $(\mathcal{D}, \mathcal{C}, \otimes)$ in $\text{DEF}^\otimes$ to $\text{fun}(\mathcal{D})$. In Theorem 3.3 we show that the Serre subcategory of $\mathcal{C}^{\text{f.p.-mod}}$ corresponding to $\mathcal{D}$ is a Serre tensor-ideal. We use this in Definition 3.10 to define an additive symmetric monoidal
structure on \( \text{fun}(D) \). We then show that the monoidal structure is exact in each variable in Proposition \ref{prop:monoidal-structure-props}.

**Assumption 3.2.** Let \( C \) be an additive finitely accessible category with products. Suppose further that \( C \) has an additive closed symmetric monoidal structure such that \( C^{op} \) is a monoidal subcategory. Induce a monoidal structure on \( C^{op} \)-Mod via Day convolution product and note that this restricts to a monoidal structure on \( C^{op} \)-mod. We denote all tensor products by \( \otimes \). Note that the monoidal structures on \( C \) and \( C^{op} \)-Mod are assumed to be closed, and therefore in both cases the tensor product functor, \( \otimes \), is right exact in each variable. Furthermore, as \( C \) is an additive finitely accessible category with products, \( C^{op} \)-Mod is locally coherent (\cite{B}, Theorem 6.1) and therefore \( C^{op} \)-mod is an abelian subcategory of \( C^{op} \)-Mod (e.g. see \cite{S}, Theorem E.1.47).

Therefore, every exact sequence in \( C^{op} \)-mod is exact in \( C^{op} \)-Mod and consequently the restriction of Day convolution product to \( C^{op} \)-mod is also right exact in each variable.

Before we prove Theorem \ref{thm:monoidal-structure-props}, we prove some useful Lemmas. The first uses the tensor-hom adjunction of a closed monoidal category to find a natural isomorphism between two functors.

**Lemma 3.3.** Let \( C \) be as in Assumption \ref{assump:monoidal-category} and induce a monoidal structure on \( C^{op} \)-Mod via Day convolution product. Then for all \( F \in C^{op} \)-Mod and \( X \in C^{op} \), \( (X, -) \otimes F \) is naturally isomorphic to \( \overline{F} \circ \text{hom}_{C^{op}}(X, -) \), where \( \text{hom}(X, -) : C \to C \) denotes the internal hom-functor and \( \text{hom}(X, -)_{C^{op}} : C^{op} \to C \) is the restriction to finitely presented objects.

**Proof.** First suppose \( F \) is finitely presentable with presentation \( (B, -) \xrightarrow{(f, -)} (A, -) \xrightarrow{\pi} F \to 0 \), with \( f : A \to B \) in \( C^{op} \) and suppose \( X \in C^{op} \). Then \( (X, -) \otimes F \) has presentation \( (X \otimes B, -) \xrightarrow{(X \otimes f, -)} (X \otimes A, -) \xrightarrow{\pi_{X \otimes A}} (X, -) \otimes F \to 0 \), where \( X \otimes f : X \otimes A \to X \otimes B \) is in \( C^{op} \). For any \( Z \in C^{op} \) we have the following diagram in \( \text{Ab} \).

\[
\begin{array}{ccc}
(X \otimes B, Z) & \xrightarrow{(X \otimes f, -)_Z} & (X \otimes A, Z) \\
\downarrow{\alpha_B} & & \downarrow{\alpha_A} \\
(B, \text{hom}(X, Z)) & \xrightarrow{(f, \text{hom}(X, -))_Z} & (A, \text{hom}(X, Z))
\end{array}
\]

\[
\begin{array}{ccc}
& & (\pi_{X \otimes A})_Z \\
& & \downarrow{\eta_Z} \\
& & \overline{F} \circ \text{hom}(X, -)_{C^{op}}(Z)
\end{array}
\]

\[
\begin{array}{ccc}
& & 0 \\
& & \downarrow{\pi} \\
& & \overline{F} \circ \text{hom}(X, -)_{C^{op}}(Z)
\end{array}
\]

Since the isomorphisms \( \alpha_B \) and \( \alpha_A \) are natural in \( A \) and \( B \) respectively, the \( \eta_Z \) form the components of a natural isomorphism \( \eta : (X, -) \otimes F \to \overline{F} \circ \text{hom}(X, -)_{C^{op}} \).

If \( F : C^{op} \to \text{Ab} \) is any additive functor then \( F = \lim_{i \in I} F_i \) for some finitely presented functors \( F_i \). For each \( i \in I \), we have \( \eta_i : ((X, -) \otimes F_i) \to \overline{F}_i \circ \text{hom}(X, -)_{C^{op}} \), defined as above. Furthermore, for any natural transformation \( \lambda : F_i \to F_j \) the following diagram commutes, where \( \lambda : \overline{F}_i \to \overline{F}_j \) denotes the natural transformation with components given by the unique map between direct limits induced by \( \lambda \).

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Therefore, by the universal property of direct limits, the \( \eta_i \) for \( i \in I \) induce a unique natural isomorphism

\[
\lim_{\to i \in I} (X, -) \otimes F_i \to \lim_{\to i \in I} (\lambda \circ \text{hom}(X, -))_i.
\]

Since \( (X, -) \otimes - \) commutes with direct limits,

\[
\lim_{\to i \in I} (X, -) \otimes F_i \cong (X, -) \otimes \lim_{\to i \in I} F_i = (X, -) \otimes F.
\]

Therefore, we have a natural isomorphism \( \eta : (X, -) \otimes F \to \lim_{\to i \in I} \lambda \circ \text{hom}(X, -)_i \) as required. \( \square \)

**Lemma 3.4.** Let \( C \) be as in Assumption \([5,2]\). For every \( F \in \mathcal{C}^{fp}-\text{mod} \) and \( C \in \mathcal{C}^{fp} \),

\[
\lim_{\to i \in I} \lambda \circ \text{hom}(C, -) : \mathcal{C} \to \text{Ab}
\]

commutes with direct limits.

**Proof.** Suppose \( F = F_f \) has presentation \( (B, -) \xrightarrow{(f, -)} (A, -) \xrightarrow{\pi_f} F \to 0 \) and express \( Z \in \mathcal{C} \) as a direct limit of the directed diagram \( (\mathcal{C} \otimes A, Z_i) \), where each \( Z_i \) is finitely presented and the limits maps are denoted by \( \lambda_i : Z_i \to Z \). By \([5, \text{Corollary } 10.2.42]\), we have an exact sequence

\[
\lim_{\to i \in I} \lambda_i \circ \text{hom}(C, Z_i) \to \lim_{\to i \in I} \lambda_i \circ \text{hom}(C, Z_i) \to \lim_{\to i \in I} \lambda_i \circ \text{hom}(C, Z_i) \to 0.
\]

By the adjunction isomorphisms and the fact that \( A \otimes C \) is finitely presented we have isomorphisms

\[
(A, \text{hom}(C, Z)) \cong (C \otimes A, Z) = (C \otimes A, \lim_{\to i \in I} Z_i) \cong \lim_{\to i \in I} (C \otimes A, Z_i) = \lim_{\to i \in I} (A, \text{hom}(C, Z_i))
\]

and similarly with \( A \) replaced by \( B \). Furthermore these isomorphisms fit into the following commutative square.

\[
\begin{array}{ccc}
(B, \text{hom}(C, Z)) & (f, \text{hom}(C, Z)) & (A, \text{hom}(C, Z)) \\
\cong & \cong & \\
\lim_{\to i \in I} (B, \text{hom}(C, Z_i)) & \lim_{\to i \in I} (f, \text{hom}(C, Z_i)) & \lim_{\to i \in I} (A, \text{hom}(C, Z_i))
\end{array}
\]
Therefore the cokernels of the two horizontal maps, \( \overrightarrow{F}(\text{hom}(C, Z)) = (\overrightarrow{F} \circ \text{hom}(C, -))(\lim_{i \in I} Z_i) \) and \( \lim_{i \in I} \overrightarrow{F}(\text{hom}(C, Z_i)) \), are also isomorphic. \( \square \)

**Corollary 3.5.** Let \( C \) be as in Assumption \ref{3.5} For all \( C \in \mathcal{C}^{\text{fp}} \) and \( Z \in C \), if \( Z = \lim_{i \in I} Z_i \) then \( \text{hom}(C, Z) \) and \( \lim_{i \in I} \text{hom}(C, Z_i) \) belong to the same definable subcategories.

**Remark 3.6.** Note that if \( \mathcal{C}^{\text{fp}} \) is a rigid monoidal subcategory of \( C \), then for any \( C \in \mathcal{C}^{\text{fp}} \), \( \text{hom}(C, -) \cong \mathcal{C}^{\text{v}} \otimes - \) commutes with direct limits. Therefore for every \( Z \in C \) satisfying \( Z = \lim_{i \in I} Z_i \), \( \text{hom}(C, Z) \) and \( \lim_{i \in I} \text{hom}(C, Z_i) \) are isomorphic.

Next, we simplify the criteria for a Serre subcategory of \( \mathcal{C}^{\text{fp}} \)-mod to be a Serre tensor-ideal.

**Lemma 3.7.** Let \( C \) be as in Assumption \ref{3.5} and suppose \( S \) is a Serre subcategory of \( \mathcal{C}^{\text{fp}} \)-mod. Then \( S \) is a Serre tensor-ideal of \( \mathcal{C}^{\text{fp}} \)-mod if and only if for all \( C \in \mathcal{C}^{\text{fp}} \) and all \( F \in S \), 
\[
(C, -) \otimes F \in S.
\]

**Proof.** \((\implies)\) Holds by the definition of tensor-ideal.

\((\iff)\) Suppose that for all \( C \in \mathcal{C}^{\text{fp}} \) and all \( F \in S \), \((C, -) \otimes F \in S \). Let \( F_f \in S \) and \( F_g \in \mathcal{C}^{\text{fp}} \)-mod where \( F_g \) has projective resolution 
\[
(V, -) \xrightarrow{(g, -)} (U, -) \rightarrow F_g \rightarrow 0,
\]

for \( g : U \rightarrow V \) a morphism in \( \mathcal{C}^{\text{fp}} \).

By right exactness of the induced tensor product, we have the exact sequence 
\[
(V, -) \otimes F_f \rightarrow (U, -) \otimes F_f \rightarrow F_g \otimes F_f \rightarrow 0.
\]

By assumption, \((U, -) \otimes F_f \) is an object of \( S \). Therefore \( F_g \otimes F_f \in S \) as \( S \) is Serre. Hence \( S \) is tensor-closed as required. \( \square \)

Now we are ready to prove Theorem \ref{3.8}

**Theorem 3.8.** Let \( C \) be an additive finitely accessible category with products. Suppose further that \( C \) has an additive closed symmetric monoidal structure such that \( \mathcal{C}^{\text{fp}} \) is a monoidal subcategory.

Let \( D \) be a definable subcategory of \( C \) and \( S \subseteq \mathcal{C}^{\text{fp}} \)-mod be the corresponding Serre subcategory as in Theorem \ref{3.7}. Then \( S \) is a tensor-ideal of \( \mathcal{C}^{\text{fp}} \)-mod if and only if \( D \) is \( \text{fp-hom-closed} \).

**Proof.** Recall that the functors in \( S \) are exactly those whose unique extension along direct limits annihilates \( D \). Therefore \( D \) is \( \text{fp-hom-closed} \) if and only if for every \( A \in \mathcal{C}^{\text{fp}} \), \( X \in D \) and every \( F \in S \), \( \overrightarrow{F}(\text{hom}(A, X)) = 0 \). By Lemma \ref{3.4} \( \overrightarrow{F} \circ \text{hom}(A, -) \) commutes with direct limits and therefore \( \overrightarrow{F} \circ \text{hom}(A, -) = \overrightarrow{F} \circ \text{hom}(A, -)|_{\mathcal{C}^{\text{fp}}} \). Furthermore, by Lemma \ref{3.3} we have 
\[
(A, -) \otimes F \cong \overrightarrow{F} \circ \text{hom}(A, -)|_{\mathcal{C}^{\text{fp}}} : \mathcal{C}^{\text{fp}} \rightarrow \text{Ab}, \quad \text{so} \quad \overrightarrow{F} \circ \text{hom}(A, -) \cong (A, -) \otimes \overrightarrow{F}.
\]

Therefore, \( D \) is \( \text{fp-hom-closed} \) if and only if for every \( A \in \mathcal{C}^{\text{fp}} \), \( X \in D \) and \( F \in S \), 
\[
(A, -) \otimes \overrightarrow{F}(X) = 0, \quad \text{equivalently} \quad (A, -) \otimes F \in S.
\]

Finally note that \( S \) is a Serre tensor-ideal if and only if it is closed under tensoring with representable functors (see Lemma \ref{3.7}). \( \square \)

By Theorem \ref{3.8} if \( (D, C, \otimes) \) is an object of \( \text{DEF}^{\otimes} \), then the corresponding Serre subcategory of \( \mathcal{C}^{\text{fp}} \)-mod is a Serre tensor-ideal. Next we use this to define a monoidal structure on \( \text{fun}(D) \).

We prove the following lemma first.

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Lemma 3.9. Let $C$ be as in Assumption 3.2. Then $(C, -) \otimes - : \mathcal{C}^{fp}\text{-Mod} \to \mathcal{C}^{fp}\text{-Mod}$ is exact for all $C \in \mathcal{C}^{fp}$.

If we assume further that $C \in \mathcal{C}^{fp}$ is rigid then so is $(C, -)$ with dual given by $(C^\vee, -)$.

**Proof.** We already know that $(C, -) \otimes -$ is a left adjoint (3, Theorem 3.3 and Theorem 3.6) and therefore right exact. We show that $(C, -) \otimes -$ is also a right adjoint and therefore is an exact functor.

We first define the left adjoint $L_C : \mathcal{C}^{fp}\text{-Mod} \to \mathcal{C}^{fp}\text{-Mod}$ on finitely presented functors. Given $F_f \in \mathcal{C}^{fp}\text{-mod}$ with presentation $(B, -) \xrightarrow{(f, -)} (A, -) \to F_f \to 0$, denote by $L_C(F_f)$ the functor with presentation

$$(\text{hom}(C, B), -) \xrightarrow{\text{hom}(C, f), -} (\text{hom}(C, A), -) \to L_C(F_f) \to 0.$$

It can be checked that this definition does not depend on the choice of $f$.

Now, given another finitely presented functor $F_g$ with presentation $(V, -) \xrightarrow{(g, -)} (U, -) \to F_g \to 0$ and a morphism $\alpha : F_f \to F_g$, chose any $\alpha_1 : U \to A$ and $\alpha_2 : V \to B$ such that the following diagram commutes.

![Diagram](image)

Then define the morphism $L_C(\alpha) : L_C(F_f) \to L_C(F_g)$ by the unique map making the following diagram commute.
Note that this does not depend of the choice of \( \alpha_1 \) and \( \alpha_2 \). To define \( L_C \) (up to isomorphism) on any \( F \in \mathcal{C} \text{fp-Mod} \), we assert that \( L_C \) commutes with direct limits.

It is easy to check that \( L_C : \mathcal{C} \text{fp-Mod} \to \mathcal{C} \text{fp-Mod} \) defines (up to isomorphism) a functor, indeed this follows from the fact that \( \text{hom}(C, -) : \mathcal{C} \to \mathcal{C} \) is a functor. We claim that this functor is left adjoint to \( (C, -) \otimes - : \mathcal{C} \text{fp-Mod} \to \mathcal{C} \text{fp-Mod} \).

As \( (C, -) \otimes - \) and \( L_C \) commute with direct limits, it is enough to define the unit and counit of the adjunction on finitely presented functors. Indeed, any functor \( F \in \mathcal{C} \text{fp-Mod} \) can be expressed as a direct limit of finitely presented functors, say \( F = \varinjlim_{i \in I} F_i \) where each \( F_i \in \mathcal{C} \text{fp-mod} \). By the universal property of direct limits, the value of the unit, \( \eta_F : F \to ((C, -) \otimes L_C)(F) \), and the counit, \( \varepsilon_F : L_C((C, -) \otimes F) \to F \), at \( F \), is uniquely determined by the respective components at the \( F_i \).

The unit, \( \eta : \text{Id}_{\mathcal{C} \text{fp-Mod}} \to ((C, -) \otimes -) \circ L_C \), is defined on finitely presented functors as follows. For \( F_j \in \mathcal{C} \text{fp-mod} \) we define \( \eta_{F_j} : F_j \to (C, -) \otimes L_C(F_j) \) as the unique map such that the following diagram commutes, where \( \varepsilon_C : (C \otimes -) \circ \text{hom}(C, -) \to \text{Id}_\mathcal{C} \) is the counit of the adjunction between \( C \otimes - \) and \( \text{hom}(C, -) \).
Similarly we define the counit of the adjunction, \( \varepsilon : \mathbf{L}_C \circ (C, -) \otimes - \to \text{Id}_{C^\text{op}\text{-mod}} \), as follows. For \( F_f \in C^\text{op}\text{-mod} \) we define \( \varepsilon_{F_f} : \mathbf{L}_C((C, -)) \otimes F_f \to F_f \) as the unique map such that the following diagram commutes, where \( \eta_C : \text{Id}_C \to \text{hom}(C, -) \circ (C \otimes -) \) is the unit of the adjunction between \( C \otimes - \) and \( \text{hom}(C, -) \).

\[
\begin{array}{ccc}
0 & \xrightarrow{\varepsilon_{F_f}} & 0 \\
\mathbf{L}_C((C, -) \otimes F_f) & \xrightarrow{\pi} & F_f \\
\downarrow \pi_{\text{hom}(C, C \otimes f)} & & \downarrow \pi_f \\
(A, -) & \xrightarrow{(\eta_C)_A, -} & (C \otimes \text{hom}(C, A), -) \\
(hom(C, C \otimes f), -) & \xrightarrow{(\eta_C)_B, -} & (f, -) \\
(B, -) & \xrightarrow{(\eta_C)_B, -} & (C \otimes \text{hom}(C, B), -)
\end{array}
\]

It can be seen that \( \eta_{F_f} \) and \( \varepsilon_{F_f} \) don’t depend of the choice of presentation. It remains to check that the triangle identities hold. Again, it is enough to check when evaluating at finitely presented functors, but these follow easily from the triangle identities on the adjunction between \( C \otimes - \) and \( \text{hom}(C, -) \).

If, in addition, we assume that \( C \in C^\text{op} \) is rigid then \( \text{hom}(C, -) \cong C^\vee \otimes - \) and therefore \( L_C \cong (C^\vee, -) \otimes - \). Thus, \((C^\vee, -) \otimes - \) is left adjoint to \((C, -) \otimes - \) and \((C, -) \otimes - \cong ((C^\vee)^\vee, -) \otimes - \) is left adjoint to \((C^\vee, -) \otimes - \). Therefore, \((C, -) \otimes - \) is rigid with dual given by \((C^\vee, -) \otimes - \) as required. \( \square \)

Next we define an additive symmetric monoidal structure on \( \text{fun}(D) \).

**Definition 3.10.** Suppose \((D, C, \otimes) \in \text{DEF}^\otimes\) and let \( S \subseteq C^\text{op}\text{-mod} \) be the Serre subcategory corresponding to \( D \). By Theorem \[\text{Serre}\] \( S \) is a Serre tensor-ideal of \( C^\text{op}\text{-mod} \). First we define an additive symmetric monoidal structure on \( C^\text{op}\text{-mod}/S \).

By [Serre], if the multiplicative system \( \Sigma_5 \) of all the morphisms \( \alpha \) in \( C^\text{op}\text{-mod} \) such that \( \ker(\alpha) \), \( \text{coker}(\alpha) \in S \) is closed under tensoring with objects of \( C^\text{op}\text{-mod} \), then \( C^\text{op}\text{-mod}/S \) has a monoidal structure such that the localisation functor \( q : C^\text{op}\text{-mod} \to \text{fun}(D) = C^\text{op}\text{-mod}/S \) is universal among monoidal functors which map the morphisms in \( \Sigma_5 \) to isomorphisms.

By Lemma [Serre] for any \( C \in C^\text{op} \), \((C, -) \otimes - : C^\text{op}\text{-mod} \to C^\text{op}\text{-mod} \) is exact. Since, \( C^\text{op}\text{-mod} \) is an abelian subcategory of \( C^\text{op}\text{-Mod} \), \((C, -) \otimes - : C^\text{op}\text{-mod} \to C^\text{op}\text{-mod} \) is also exact. Consequently, for every \( \alpha : F \to G \) in \( C^\text{op}\text{-mod} \), \( \ker((C, -) \otimes \alpha) \cong (C, -) \otimes \ker(\alpha) \) and \( \text{coker}((C, -) \otimes \alpha) \cong (C, -) \otimes \text{coker}(\alpha) \). Therefore if \( S \) is a tensor-ideal and \( \alpha \in \Sigma_5 \) then \( \ker((C, -) \otimes \alpha) \), \( \text{coker}((C, -) \otimes \alpha) \in S \) so \((C, -) \otimes \alpha \in \Sigma_5\).

Now consider the morphism \( \alpha \otimes 1d_{F_f} \). We have the following commutative diagram with exact rows.
Since for every $D \in \mathcal{D}$, $(\alpha \otimes Id_{F_g})_D$ are isomorphisms, so is $(\alpha \otimes Id_{F_g})_D$. Hence ker$(\alpha \otimes Id_{F_g})$, coker$(\alpha \otimes Id_{F_g}) \in \mathcal{S}$ and $\alpha \otimes Id_{F_g} \in \Sigma_S$.

Applying (4, Corollary 1.4) we get an additive symmetric monoidal structure on $\mathcal{C}_{fp}-\text{mod}$.

We induce a monoidal structure on fun$(\mathcal{D}) = (\mathcal{D}, \text{Ab})^{I\rightarrow}$ via the equivalence given in (9, Theorem 12.10).

Next we show that the monoidal structure on fun$(\mathcal{D})$ is exact in each variable.

**Proposition 3.11.** Let $\mathcal{C}$ be as in Assumption 3.2. Suppose $\mathcal{D}$ is an fp-hom-closed definable subcategory and induce a monoidal structure on fun$(\mathcal{D})$ as in Remark 3.10.

If $\mathcal{D}$ satisfies the exactness criterion then the monoidal structure on fun$(\mathcal{D})$ is exact in each variable.

**Proof.** Suppose $\mathcal{D}$ satisfies the exactness criterion, i.e. for any $f : A \rightarrow B$ and $g : U \rightarrow V$ in $C_{fp}$ and for any $D \in \mathcal{D}$, if $h : A \otimes U \rightarrow D$ satisfies $(f \otimes U)|h$ and $(A \otimes g)|h$ then $(f \otimes g)|h$.

Suppose further that $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ is an exact sequence in fun$(\mathcal{D})$. It is (isomorphic to) the image of an exact sequence $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ in $C_{fp}$-mod (see [4, Lemma 11.1.6 and Corollary 11.1.42]). If $K \in \text{fun}(\mathcal{D})$ then $K$ is isomorphic to the image of $F_k$ for some $F_k \in C_{fp}$-mod. Therefore, since the localisation functor is monoidal, showing $0 \rightarrow K \otimes F \xrightarrow{K \otimes \alpha} K \otimes G \xrightarrow{K \otimes \beta} K \otimes H \rightarrow 0$ is a short exact sequence in fun$(\mathcal{D})$ is equivalent to showing that the image of the (not necessarily exact) sequence $0 \rightarrow F_k \otimes F_l \rightarrow F_k \otimes F_l \rightarrow F_k \otimes F_l \rightarrow 0$ under the localisation functor gives a short exact sequence. By (4, Theorem 12.10), this is equivalent to showing that $0 \rightarrow (F_k \otimes F_l)(D) \rightarrow (F_k \otimes F_l)(D) \rightarrow (F_k \otimes F_l)(D) \rightarrow 0$ is exact for all $D \in \mathcal{D}$.

Suppose $F_k$ has presentation $(T, -) \xrightarrow{(k, -)} (S, -) \rightarrow F_k \rightarrow 0$, where $k : S \rightarrow T$ is a morphism in $C_{fp}$, then we have the following commutative diagram in $C_{fp}$-mod.

\[
\begin{array}{cccccc}
F \otimes (V, -) & \xrightarrow{Id_F \otimes (g, -)} & F \otimes (U, -) & \xrightarrow{F \otimes F_g} & 0 \\
\downarrow \alpha \otimes Id_{F_g} & & \downarrow \alpha \otimes Id_{F_g} & & \\
G \otimes (V, -) & \xrightarrow{Id_G \otimes (g, -)} & G \otimes (U, -) & \xrightarrow{G \otimes F_g} & 0 \\
\end{array}
\]
Consider the following diagram where the second and fourth rows are exact by Lemma 3.9. We must show that \( (F_k \otimes \alpha)_D \) has zero kernel for all \( D \in \mathcal{D} \). Fix \( D \in \mathcal{D} \). To enhance readability, for a functor \( F : C \rightarrow \text{Ab} \) which commutes with direct limits, and simply use \( F \). For any \( w \in (F_k \otimes F_j)(D) \) there exists some \( u' \in ((S, -) \otimes F_j)(D) \) such that \( (\pi_k \otimes F_j)_D(u') = w \). If \( (F_k \otimes \alpha)_D(w) = 0 \) then there exists some \( z \in ((T, -) \otimes F_g)(D) \) such that \( ((k, -) \otimes F_g)_D(z) = ((S, -) \otimes \alpha)_D(u') \).

We will show that \( u' = ((k, -) \otimes F_j)_D(x) \) for some \( x \in ((T, -) \otimes F_j)(D) \), meaning \( w = (\pi_k \otimes F_j)(u') = 0 \).

Suppose that \( F_j \) has presentation \( (B, -) \xrightarrow{(f, -)} (A, -) \xrightarrow{\pi_j} F_j \rightarrow 0 \). Then there exists some morphism \( u'' : S \otimes A \rightarrow D \) such that \( ((S, -) \otimes \pi_j)_D(u'') = u' \). Similarly, if \( g : U \rightarrow V \) in \( \mathcal{C}^{op} \) then there exists some morphism \( z' : T \otimes U \rightarrow D \) such that \( ((T, -) \otimes \pi_g)_D(z') = z \). Therefore we have

\[
((k, -) \otimes F_g) \circ ((T, -) \otimes F_j)(D) = ((S, -) \otimes \alpha) \circ ((S, -) \otimes F_j)_D(w'').
\]

Consider the following diagram where \( z' \in (T \otimes U, D) \) and \( w'' \in (S \otimes A, D) \).

\[
\begin{array}{ccc}
(S \otimes B, D) & \xrightarrow{(S \otimes f, -)_D} & (S \otimes A, D) \\
\downarrow \quad (S \otimes \alpha_1, -)_D & & (S \otimes \alpha, -)_D \\
(S \otimes V, D) & \xrightarrow{(S \otimes g, -)_D} & (S \otimes U, D) \\
\downarrow \quad (k \otimes U, -)_D & & (k \otimes F_g)_D \\
(T \otimes V, D) & \xrightarrow{((T, -) \otimes \pi_g)_D} & (T \otimes F_g)(D) \\
\end{array}
\]

By projectivity of representables, there exists some \( \alpha_1 : U \rightarrow A \) such that \( \alpha \circ \pi_f = \pi_g \circ (\alpha_1, -) \).
Therefore, \((S, -) \otimes \alpha \circ ((S, -) \otimes \pi_f) = ((S, -) \otimes \pi_g) \circ (S \otimes \alpha_1, -)\). This gives us
\[
(((S, -) \otimes \pi_g) \circ (k \otimes U, -))D(z') = (((S, -) \otimes \pi_g) \circ (S \otimes \alpha_1, -))D(w'').
\]
That is \(((S, -) \otimes \pi_g)D(z' \circ (k \otimes U) - w'' \circ (S \otimes \alpha_1)) = 0\) and so there exists some \(y : S \otimes V \to D\) such that \(y \circ (S \otimes g) = z' \circ (k \otimes U) - w'' \circ (S \otimes \alpha_1)\) equivalently \(h = y \circ (S \otimes g) + w'' \circ (S \otimes \alpha_1) = z' \circ (k \otimes U)\).

Consider the morphism \((g, \alpha) : U \to V \otimes A\) such that \(p_1 \circ (g, \alpha_1) = g\) and \(p_2 \circ (g, \alpha_1) = \alpha_1\), where \(p_1\) and \(p_2\) denote the projection maps. Then \((S \otimes (g, \alpha_1))h\) and \((k \otimes U)h\) so since the exactness criterion holds for \(D\), \((k \otimes (g, \alpha_1))h\) and there exists some \(y' : T \otimes (V \otimes A) \to D\) such that \(y' \circ (k \otimes (g, \alpha_1)) = h\). Set \(y_1 = y' \circ (T \otimes i_1)\) and \(y_2 = y' \circ (T \otimes i_2)\), where \(i_1 : V \to V \otimes A\) and \(i_2 : A \to V \otimes A\) are the inclusion maps. Then we have
\[
y_1 \circ (k \otimes g) + y_2 \circ (k \otimes \alpha_1) = h = z' \circ (k \otimes U) = w'' \circ (S \otimes \alpha_1) + y \circ (S \otimes g).
\]
Applying \(((S, -) \otimes \pi_g)_D\) we get,
\[
(((S, -) \otimes \pi_g)_D(y_2 \circ (k \otimes \alpha_1)) = ((S, -) \otimes \pi_g)_D(h) = ((S, -) \otimes \pi_g)_D(w'' \circ (S \otimes \alpha_1)).
\]
As shown on the commutative diagram above, we have
\[
((S, -) \otimes \pi_g) \circ ((S, -) \circ (\alpha_1, -)) = ((S, -) \circ \alpha) \circ ((S, -) \otimes \pi_f).
\]
Therefore, as
\[
(((S, -) \otimes \pi_g)_D(y_2 \circ (k \otimes \alpha_1)) = [((S, -) \otimes \pi_g) \circ ((S, -) \circ (\alpha_1, -))]_D(y_2 \circ (k \otimes A))
\]
we have
\[
((S, -) \otimes \pi_g)_D(y_2 \circ (k \otimes \alpha_1)) = [((S, -) \otimes \pi_g) \circ ((S, -) \circ (\alpha_1, -))]_D(y_2 \circ (k \otimes A)).
\]
Similarly,
\[
(((S, -) \otimes \pi_g)_D(w'' \circ (S \otimes \alpha_1)) = [((S, -) \otimes \pi_g) \circ ((S, -) \circ (\alpha_1, -))]_D(w''),
\]
so we have
\[
(((S, -) \otimes \pi_g)_D(w'' \circ (S \otimes \alpha_1)) = [((S, -) \otimes \pi_g) \circ ((S, -) \circ (\alpha_1, -))]_D(w'').
\]
Therefore,
\[
[((S, -) \otimes \pi_g)_D(y_2 \circ (k \otimes A)) = [((S, -) \circ \alpha) \circ ((S, -) \circ \pi_f)]_D(w')
\]
and since \((S, -) \circ \alpha\) is a monomorphism we have
\[
((S, -) \circ \pi_f)_D(y_2 \circ (k \otimes A)) = ((S, -) \circ \pi_f)_D(w'') = w'.
\]
Finally note that
\[
((S, -) \circ \pi_f)_D(y_2 \circ (k \otimes A)) = [((S, -) \circ \pi_f) \circ ((k, -) \circ (A, -))]_D(y_2)
\]
and therefore, as
\[
(((S, -) \circ \pi_f) \circ ((k, -) \circ (A, -)) = ((k, -) \circ (F_f, -)) \circ ((T, -) \circ \pi_f)
\]
we get that \(w' = ((k, -) \circ (F_f, -))D(((T, -) \circ \pi_f)_D(y_2)) = (\pi_k \circ F)_D \circ ((k, -) \circ \pi_f)_D \circ ((T, -) \circ \pi_f)_D(y_2) = 0\). Therefore, \((F_k \circ \alpha)_D\) has zero kernel and is a monomorphism for all \(D \in \mathcal{D}\), as required. \(\square\)

In fact, the converse is also true.
Proposition 3.12. Let $\mathcal{C}$ be as in Assumption \ref{def}. Suppose $\mathcal{D}$ is an fp-hom-closed definable subcategory and induce a monoidal structure on $\text{fun}(\mathcal{D})$ as in Definition \ref{def}. If the monoidal structure on $\text{fun}(\mathcal{D})$ is exact in each variable then $\mathcal{D}$ satisfies the exactness criterion.

Proof. Suppose that the induced monoidal structure on $\text{fun}(\mathcal{D})$ is exact in each variable. Suppose $f : A \to B$ and $g : U \to V$ are morphisms in $\mathcal{C}^{fp}$. By \cite[Corollary 3.11]{fp}, $\mathcal{C}^{fp}$ has pseudocokernels. Let $g' : V \to W$ be a pseudocokernel of $g$.

Then $(W, -) \xrightarrow{(g', -)} (V, -) \xrightarrow{(g, -)} (U, -)$ is exact in $\mathcal{C}^{fp}$-mod, that is $\text{im}((g', -)) = \ker((g, -))$. Therefore, its image $(W, -)_5 \xrightarrow{(g', -)_5} (V, -)_5 \xrightarrow{(g, -)_5} (U, -)_5$ is exact in $\text{fun}(\mathcal{D})$ and by assumption, $(F_f)_5 \otimes (W, -)_5 \xrightarrow{(F_f)_5 \otimes (g', -)_5} (F_f)_5 \otimes (V, -)_5 \xrightarrow{(F_f)_5 \otimes (g, -)_5} (F_f)_5 \otimes (U, -)_5$ is also exact in $\text{fun}(\mathcal{D})$. As the localisation functor is monoidal, this is equivalent to, $(F_f \otimes (W, -))(D) \xrightarrow{(F_f \otimes (g', -))(D)} (F_f \otimes (V, -))(D)$ being exact in $\text{Ab}$ for all $D \in \mathcal{D}$, by \cite[Theorem 12.10]{fp}. Consider the diagram below.

\[
\begin{array}{ccc}
0 & \xrightarrow{F_f \otimes (W, -)} & 0 \\
\pi_f \otimes (W, -) & \xrightarrow{\pi_f \otimes (V, -)} & \pi_f \otimes (U, -) \\
(A \otimes W, -) & \xrightarrow{(A \otimes g', -)} & (A \otimes V, -) & \xrightarrow{(A \otimes g, -)} & (A \otimes U, -) \\
(f \otimes W, -) & \xrightarrow{(f \otimes V, -)} & (f \otimes U, -) \\
(B \otimes W, -) & \xrightarrow{(B \otimes g', -)} & (B \otimes V, -) & \xrightarrow{(B \otimes g, -)} & (B \otimes U, -) \\
\end{array}
\]

Given any $h : A \otimes U \to D$ such that $(f \otimes U)h$ and $(A \otimes g)h$ there exists $h_1 : B \otimes U \to D$ such that $h = h_1 \circ (f \otimes U)$ and $h_2 : A \otimes V \to D$ such that $h = h_2 \circ (A \otimes g)$. But this means that

\[
((F_f \otimes (g, -)) \circ (\pi_f \otimes (V, -)))_D(h_2) = (\pi_f \otimes (U, -))_D(h_2 \circ (A \otimes g)) = (\pi_f \otimes (U, -))_D(h_1 \circ (f \otimes U)) = 0.
\]

So $(\pi_f \otimes (V, -))_D(h_2)$ is in the kernel of $(F_f \otimes (g, -))_D$ which is equal to the image of $(F_f \otimes (k, -))_D$. Therefore there exists some $z \in (F_f \otimes (W, -))(D)$ such that $(F_f \otimes (k, -))(D)(z) = (\pi_f \otimes (V, -))(D)(h_2)$. But then, since $(\pi_f \otimes (W, -))_D$ is an epimorphism in $\text{Ab}$, there exists some morphism $z' : A \otimes W \to D$ such that $(\pi_f \otimes (W, -))(D)(z') = z$.

Next, notice that $(\pi_f \otimes (V, -))(D)(h_2 \circ (z' \circ (A \otimes k))) = 0$. So there exists some $y : B \otimes V \to D$ such that $y \circ (f \otimes V) = h_2 - (z' \circ (A \otimes k))$. But then $y \circ (f \otimes g) = y \circ (f \otimes V) \circ (A \otimes g) = h_2 \circ (A \otimes g) = h$. That is $(f \otimes g)h$, as required. \end{proof}

Proposition 3.13. There exists a 2-functor $\theta : \text{DEF}^{\otimes} \to \text{ABEX}^{\otimes}$ which maps $(\mathcal{D}, \mathcal{C}, \otimes) \in \text{DEF}^{\otimes}$ to $\text{fun}(\mathcal{D})$ with the monoidal structure given in Definition \ref{def}.14.
The following are equivalent:

Theorem 3.15. Let $A \to 0$ in each variable. Is exact in $\text{Ab}$ $F \to A$ evaluation at $X$ every $\text{Ab}$.

Proof. $\theta$ is well defined on objects by Theorem 3.8 and Proposition 3.11. On morphisms $\theta$ maps $I : D \to D'$ to $I_0 : \text{fun}(D') \to \text{fun}(D)$ as in the original anti-equivalence in [11]. $I_0$ is monoidal by definition of the morphisms $I$ in $\text{DEF}^\otimes$. On natural transformations $\theta$ acts as in the original anti-equivalence and therefore $\theta$ satisfies the necessary axioms to be a 2-functor. □

3.2 The 2-functor $\xi : \text{ABEX}^\otimes \to \text{DEF}^\otimes$

Next we define a 2-functor $\xi : \text{ABEX}^\otimes \to \text{DEF}^\otimes$ which maps a skeletally small abelian category $\mathcal{A}$ with an additive symmetric monoidal structure to $(\text{Ex}(\mathcal{A}, \text{Ab}), \mathcal{A}-\text{Mod}, \otimes)$, where $\otimes$ is induced by Day convolution product.

First we show that $(\text{Ex}(\mathcal{A}, \text{Ab}), \mathcal{A}-\text{Mod}, \otimes)$ is a well-defined object of $\text{DEF}^\otimes$ (see Theorem 3.15).

Lemma 3.14. Let $\mathcal{A}$ be an additive symmetric monoidal, skeletally small abelian category. Suppose that for every exact functor $E : \mathcal{A} \to \text{Ab}$, every $X \in \mathcal{A}$ and every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$,

$$0 \to E(X \otimes A) \to E(X \otimes B) \to E(X \otimes C) \to 0,$$

is exact in $\text{Ab}$. Suppose further that $\text{Ex}(\mathcal{A}, \text{Ab})$ is fp-hom-closed and induce a monoidal structure on $\text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ as in Definition 3.10. In addition, assume that the equivalence of categories $\mathcal{A} \cong \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ given by $A \mapsto \text{ev}_A$, where $\text{ev}_A : \text{Ex}(\mathcal{A}, \text{Ab}) \to \text{Ab}$ is given by ‘evaluation at $A$’ (see [11]), is monoidal.

Then the monoidal structure on $\mathcal{A}$ is exact in each variable.

Proof. Let $\mathcal{B} := \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$. As there exists a monoidal equivalence between $\mathcal{A}$ and $\mathcal{B}$, the exactness assumption on $\mathcal{A}$ carries over to $\mathcal{B}$, that is, for every exact functor $E : \mathcal{B} \to \text{Ab}$, every $X \in \mathcal{B}$ and every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{B}$,

$$0 \to E(X \otimes A) \to E(X \otimes B) \to E(X \otimes C) \to 0,$$

is exact in $\text{Ab}$.

Set $D := \text{Ex}(\mathcal{A}, \text{Ab})$ so $\mathcal{B} = \text{fun}(D)$. For every $D \in D$, $E = \text{ev}_D : \mathcal{B} = \text{fun}(D) \to \text{Ab}$ is exact, where $\text{ev}_D$ is given by ‘evaluation at $D$’. Therefore, for every exact sequence $0 \to F \to G \to H \to 0$ in $\text{fun}(D)$ and every $K \in \text{fun}(D)$, $0 \to (K \otimes F)(D) \to (K \otimes G)(D) \to (K \otimes H)(D) \to 0$ is exact for all $D \in D$. Since exactness in functor categories is pointwise, $0 \to K \otimes F \to K \otimes G \to K \otimes H \to 0$ is exact so the monoidal structure on $\mathcal{B} = \text{fun}(D)$ is exact in each variable.

As we are assuming the equivalence between $\mathcal{A}$ and $\mathcal{B} = \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ is monoidal, the monoidal structure on $\mathcal{A}$ is also exact in each variable, as required. □

Theorem 3.15. Let $\mathcal{A}$ be an additive symmetric monoidal, skeletally small abelian category. The following are equivalent:

(i) The definable subcategory $\text{Ex}(\mathcal{A}, \text{Ab}) \subseteq \mathcal{A}-\text{Mod}$ is fp-hom-closed (with respect to Day convolution product).

(ii) The Serre subcategory $S_{\text{Ex}} \subseteq (\mathcal{A}-\text{mod}, \text{Ab})^\text{op}$ of all functors $F$ such that $F^*(E) = 0$ for all $E \in \text{Ex}(\mathcal{A}, \text{Ab})$ is a tensor-ideal of $(\mathcal{A}-\text{mod}, \text{Ab})^\text{op}$ (with respect to day convolution product).

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(iii) The tensor product on $\mathcal{A}$ is exact in each variable.

**Proof.** (i) $\iff$ (ii) Follows directly from Theorem 3.8.

(iii) $\Rightarrow$ (i) Suppose the monoidal structure on $\mathcal{A}$ is exact in each variable. We first show that $\text{Ex}(\mathcal{A}, \mathcal{Ab})$ is closed under $\text{hom}(M, -)$ where $M \in \mathcal{A}$-mod is representable, say $M = (X, -)$. Indeed, in this case, for all $A \in \mathcal{A}$ and $E \in \text{Ex}(\mathcal{A}, \mathcal{Ab})$,

$\text{hom}((X, -), E)(A) \cong ((A, -), \text{hom}((X, -), E)) \cong ((A \otimes X, -), E) \cong E(A \otimes X)$,

by the Yoneda lemma and adjunction isomorphisms. What’s more, all these isomorphisms are natural in $A$. Therefore,

$0 \to \text{hom}((X, -), E)(A) \to \text{hom}((X, -), E)(B) \to \text{hom}((X, -), E)(C) \to 0$

is exact if and only if

$0 \to E(A \otimes X) \to E(B \otimes X) \to E(C \otimes X) \to 0$

is exact. But the latter statement holds by our assumption on $\mathcal{A}$, as $E$ is an exact functor. Therefore $\text{hom}((X, -), E)$ is exact as required.

Now we generalise to $F_f \in \mathcal{A}$-mod. We want to show that $\text{hom}(F_f, E) : \mathcal{A} \to \mathcal{Ab}$ is an exact functor.

First note that $(F_f, -)|_{\text{Ex}(\mathcal{A}, \mathcal{Ab})}$ commutes with direct products and direct limits and therefore is an object of $\text{fun}(\text{Ex}(\mathcal{A}, \mathcal{Ab}))$. By ([1], Theorem 2.2), there exists an equivalence $\mathcal{A} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{Ab}))$ given by $A \mapsto ev_A$, where $ev_A : \text{Ex}(\mathcal{A}, \mathcal{Ab}) \to \mathcal{Ab}$ maps an exact functor $E$ to $E(A)$. Therefore, there exists some $X_F \in \mathcal{A}$ such that $(F_f, -)|_{\text{Ex}(\mathcal{A}, \mathcal{Ab})} \cong ev_{X_F}$.

Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$. As $E$ is an exact functor and the monoidal structure on $\mathcal{A}$ is exact in each variable,

$0 \to E(A \otimes X_F) \to E(B \otimes X_F) \to E(C \otimes X_F) \to 0$,

is exact in $\mathcal{Ab}$. As a result, by the Yoneda lemma,

$0 \to ((A \otimes X_F, -), E) \to ((B \otimes X_F, -), E) \to ((C \otimes X_F, -), E) \to 0$,

is exact in $\mathcal{Ab}$ and by the adjunction isomorphism this gives the exact sequence

$0 \to ((X_F, -), \text{hom}((A, -), E)) \to ((X_F, -), \text{hom}((B, -), E)) \to ((X_F, -), \text{hom}((C, -), E)) \to 0$.

Applying the Yoneda lemma once more we have the exact sequence

$0 \to (\text{hom}((A, -), E))(X_F) \to (\text{hom}((B, -), E))(X_F) \to (\text{hom}((C, -), E))(X_F) \to 0$,

which is isomorphic to

$0 \to (F_f, \text{hom}((A, -), E))(F_f, \text{hom}((B, -), E))(F_f, \text{hom}((C, -), E)) \to 0$,

as we have already seen that $\text{hom}((A, -), E)$, $\text{hom}((B, -), E)$ and $\text{hom}((C, -), E)$ are exact functors and $(F_f, -)|_{\text{Ex}(\mathcal{A}, \mathcal{Ab})} \cong ev_{X_F}$.

Again, by the Yoneda lemma and adjunction isomorphisms we have for every $A \in \mathcal{A}$, $\text{hom}(F_f, E)(A) \cong ((A, -), \text{hom}(F_f, E)) \cong (F_f \otimes (A, -), E) \cong (F_f, \text{hom}((A, -), E))$. What’s more, all these isomorphisms are natural in $A$. Therefore

$0 \to \text{hom}(F_f, E)(A) \to \text{hom}(F_f, E)(B) \to \text{hom}(F_f, E)(C) \to 0,$
is exact in $\text{Ab}$ and $\text{hom}(F_I, E)$ is an exact functor as required.

((i) $\implies$ (iii)) Since we have shown that (i) $\iff$ (ii) we assume both hold. By (i) we have that $\text{hom}((X, -), E)$ is exact for all $X \in \mathcal{A}$. Therefore, for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$,

$$0 \to \text{hom}((X, -), E)(A) \to \text{hom}((X, -), E)(B) \to \text{hom}((X, -), E)(C) \to 0,$$

is exact. But we have isomorphisms $\text{hom}((X, -), E)(A) \cong ((A, -), \text{hom}((X, -), E)) \cong ((X, -) \otimes (A, -), E) = ((X \otimes A, -), E) \cong E(X \otimes A)$ which are natural in $A$. Therefore, $0 \to E(X \otimes A) \to E(X \otimes B) \to E(X \otimes C) \to 0$ is also exact.

So for any exact sequence, $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, $0 \to X \otimes A \to X \otimes B \to X \otimes C \to 0$ has exact image in $\text{Ab}$ under any exact functor $E : \mathcal{A} \to \text{Ab}$. By Lemma 3.14 it remains to show that the equivalence $\mathcal{A} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ is monoidal.

By (ii), $S_{\text{Ex}}$ is a tensor ideal of $(\mathcal{A} \text{-mod}, \text{Ab})^{fp}$. Therefore we can define a monoidal structure on $\text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ (as in Definition 3.10) such that the localisation functor $q : (\mathcal{A} \text{-mod}, \text{Ab})^{fp} \to (\mathcal{A} \text{-mod}, \text{Ab})^{fp}/S_{\text{Ex}} \cong \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ is a monoidal functor.

Note that the functor $Y^2 : \mathcal{A} \to (\mathcal{A} \text{-mod}, \text{Ab})^{fp}$ given by $A \mapsto ((A, -), -)$ is monoidal with respect to Day convolution product and the equivalence $\mathcal{A} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ from (II), Theorem 2.2) can be taken to be $q \circ Y^2$. Therefore, this equivalence is monoidal. Lemma 3.14 completes the proof. □

Remark 3.16. Recall that the objects of the 2-category $\text{ABEX}^\circ$ are skeletally small abelian categories with additive symmetric monoidal structures which are exact in each variable. However, in most examples (for instance $\mathcal{A} = R$-mod for $R$ a commutative ring) the monoidal structure is only right exact. Theorem 3.15 shows where the equivalence fails without the exactness assumption. Indeed, if we desire the equivalence $\mathcal{A} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ to be monoidal, the Serre subcategory $S_{\text{Ex}}$ must be a tensor-ideal, to induce a monoidal structure on $\text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$.

Proposition 3.17. There exists a 2-functor $\xi : \text{ABEX}^\circ \to \text{DEF}^\circ$ given on objects by $\mathcal{A} \mapsto (\text{Ex}(\mathcal{A}, \text{Ab}), \mathcal{A} \text{-Mod}, \otimes)$ where the monoidal structure on $\mathcal{A} \text{-Mod}$ is induced by Day convolution product.

Proof. By Theorem 3.15 $\text{Ex}(\mathcal{A}, \text{Ab})$ is an fp-hom-closed definable subcategory of $\mathcal{A} \text{-Mod}$. Furthermore, as noted in the proof of Theorem 3.16 $\mathcal{A} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ is a monoidal equivalence meaning the monoidal structure on $\text{fun}(\text{Ex}(\mathcal{A}, \text{Ab}))$ is exact. In turn this implies, by Proposition 3.11 that $\text{Ex}(\mathcal{A}, \text{Ab})$ satisfies the exactness criterion. Therefore $\xi$ is well defined on objects.

Next we need to show, given a morphism $E : \mathcal{A} \to \mathcal{B}$ in $\text{ABEX}^\circ, E^* : \text{Ex}(\mathcal{B}, \text{Ab}) \to \text{Ex}(\mathcal{A}, \text{Ab})$ is a morphism in $\text{DEF}^\circ$ that is $(E^*)_0 : \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab})) \to \text{fun}(\text{Ex}(\mathcal{B}, \text{Ab}))$ is monoidal.

By the original anti-equivalence in II, we have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\sim} & \text{fun}(\text{Ex}(\mathcal{A}, \text{Ab})) \\
E & \downarrow & (E^*)_0 \\
\mathcal{B} & \xrightarrow{\sim} & \text{fun}(\text{Ex}(\mathcal{B}, \text{Ab}))
\end{array}
\]
By the proof of Theorem 3.15, the equivalence given by the horizontal maps is monoidal. Therefore the inverse equivalence \( \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{B})) \rightarrow \mathcal{A} \) is also monoidal and \((E^*)_0\) is naturally isomorphic to a monoidal functor, hence monoidal.

Finally, \( \xi \) acts on natural transformations in the same way as the original anti-equivalence, (forgetting the monoidal structure) and therefore is a well-defined 2-functor. □

3.3 Completing the proof

It remains to prove the following proposition.

**Proposition 3.18.** For any \( \mathcal{A} \in \text{ABEX}^\circ \) the functor \( \epsilon_\mathcal{A} : \mathcal{A} \rightarrow \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{B})) \) given by \( \epsilon_\mathcal{A}(A) = \text{ev}_A \) is monoidal. Here \( \text{ev}_A : \text{Ex}(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{B} \) maps an exact functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) to \( F(A) \).

Similarly, for any \( (\mathcal{D}, \mathcal{C}, \otimes) \in \text{DEF}^\circ \) the functor \( \epsilon_\mathcal{D} : \mathcal{D} \rightarrow \text{Ex}(\text{fun}(\mathcal{D}), \mathcal{A}) \) (as in the proof of Theorem 2.3 in [1]) is a morphism in \( \text{DEF}^\circ \).

**Proof.** By ([9], Lemma 12.9 and Theorem 12.10) the functor 

\[
(\mathcal{A}\text{-mod}, \mathcal{B})^{fp} \rightarrow (\mathcal{A}\text{-mod}, \mathcal{B})^{fp}/\text{S}_{\text{Ex}} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{B})),
\]

maps a finitely presented functor \( F : \mathcal{A}\text{-mod} \rightarrow \mathcal{B} \) to \( \overline{F}|_D \) that is the restriction to \( D \) of the unique direct limit extension of \( F \). By the Yoneda lemma \( \epsilon_\mathcal{A} : \mathcal{A} \rightarrow \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{B})) \) is naturally equivalent to the functor

\[
\mathcal{A} \xrightarrow{\gamma^2} (\mathcal{A}\text{-mod}, \mathcal{B})^{fp} \rightarrow (\mathcal{A}\text{-mod}, \mathcal{B})^{fp}/\text{S}_{\text{Ex}} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{B})),
\]

where \( \gamma^2 : \mathcal{A} \rightarrow (\mathcal{A}\text{-mod}, \mathcal{B})^{fp} \) denotes the Yoneda embedding \( A \mapsto ((A, -), -) \). Therefore as the Yoneda embedding is monoidal with respect to Day convolution product and the monoidal structure on \( \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{B})) \) is defined such that the localisation functor \( g \) and the equivalence \( (\mathcal{A}\text{-mod}, \mathcal{B})^{fp}/\text{S}_{\text{Ex}} \simeq \text{fun}(\text{Ex}(\mathcal{A}, \mathcal{B})) \) are monoidal, \( \epsilon_\mathcal{A} \) is a monoidal functor.

Next we show that, for all \( (\mathcal{D}, \mathcal{C}, \otimes) \in \text{DEF}^\circ \), \( (\epsilon_\mathcal{D})_0 : \text{fun}(\text{Ex}(\text{fun}(\mathcal{D}), \mathcal{A})) \rightarrow \text{fun}(\mathcal{D}) \) is monoidal. By ([11], \( \epsilon_{\text{fun}(\mathcal{D})} : \text{fun}(\mathcal{D}) \rightarrow \text{fun}(\text{Ex}(\text{fun}(\mathcal{D}), \mathcal{A})) \) is an equivalence so we have a functor, \( \gamma : \text{fun}(\text{Ex}(\text{fun}(\mathcal{D}), \mathcal{A})) \rightarrow \text{fun}(\mathcal{D}) \), which is both right and left adjoint to \( \epsilon_{\text{fun}(\mathcal{D})} \). We show that \( (\epsilon_\mathcal{D})_0 \) is naturally isomorphic to \( \lambda \).

The unit of the adjunction \( \gamma^{-1} \epsilon_{\text{fun}(\mathcal{D})} \) gives a natural isomorphism \( \eta : \text{Id}_{\text{fun}(\text{Ex}(\text{fun}(\mathcal{D}), \mathcal{A}))} \cong (\epsilon_\mathcal{D})_0 \circ \gamma \).

Now, for \( X \in \mathcal{D} \) and \( F \in \text{fun}(\text{Ex}(\text{fun}(\mathcal{D}), \mathcal{A})) \), \( (\epsilon_\mathcal{D})_0((\epsilon_{\text{fun}(\mathcal{D})} \circ \gamma)(F))(X) = \text{ev}_{\gamma(F)}(F)(\text{ev}_X) = \text{ev}_X(\gamma(F)) = \gamma(F)(X) \), so \( (\epsilon_\mathcal{D})_0 \circ \epsilon_{\text{fun}(\mathcal{D})} \circ \gamma = \gamma \). Therefore the composition of the natural isomorphism \( \eta \) and the functor \( (\epsilon_\mathcal{D})_0 \) gives a natural isomorphism

\[
(\epsilon_\mathcal{D})_0 \eta : (\epsilon_\mathcal{D})_0 \rightarrow (\epsilon_\mathcal{D})_0 \circ \epsilon_{\text{fun}(\mathcal{D})} \circ \gamma = \gamma.
\]

We have already seen that \( \epsilon_{\text{fun}(\mathcal{D})} \) is monoidal and therefore we can take \( \gamma \) to also be monoidal (e.g. see [5], Remark 1.5.3).

Therefore \( (\epsilon_\mathcal{D})_0 \) is naturally isomorphic to a monoidal functor and so is itself a monoidal functor. Hence \( \epsilon_\mathcal{D} : \mathcal{D} \rightarrow \text{Ex}(\text{fun}(\mathcal{D}, \mathcal{A})) \) is a morphism in \( \text{DEF}^\circ \) as required. □

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Remark 3.19. The following diagram commutes, where the 2-functors denoted by \( \mathcal{F} \) are the forgetful 2-functors and the vertical maps are the 2-category anti-equivalences.

\[
\begin{array}{ccc}
\text{ABEX} \otimes & \xrightarrow{\mathcal{F}} & \text{ABEX} \\
\downarrow & & \downarrow \\
\text{DEF} \otimes & \xrightarrow{\mathcal{F}} & \text{DEF}
\end{array}
\]

4 Removing the exactness criterion

As noted in Remark 3.16, for our 2-category anti-equivalence to hold, we required the monoidal structure on the skeletally small abelian category to be exact in each variable. However, given any fp-hom-closed definable subcategory \( D \) of a finitely accessible category \( C \), which satisfies Assumption 3.2, we can induce a right exact monoidal structure on \( \text{fun}(D) \) as in Definition 3.10. In many cases, this monoidal structure on the functor category is not left exact. In this section we consider what can be said about the relationship between definability and the monoidal structure for fixed \( C \), when we remove the exactness assumption.

4.1 The Ziegler spectrum

In this section we define a coarser topology, \( \text{Zg}^{\hom}(C) \), on \( \text{pinj}_C \) such that the identity morphism \( \text{Zg}(C) \to \text{Zg}^{\hom}(C) \) is a continuous map.

Theorem 4.1. Setting the closed subcategories of \( \text{pinj}_C \) to be those given by the indecomposable pure-injectives contained in an fp-hom-closed definable subcategory of \( C \) defines a topology on \( \text{pinj}_C \) which we will call the fp-hom-closed Ziegler topology and denote by \( \text{Zg}(C)^{\hom} \).

Proof. We must show that a finite union and arbitrary intersection of closed subcategories is closed. Abusing notation slightly, we will write \( D \cap \text{pinj}_C \) for the isomorphism classes of indecomposable pure-injective objects contained in \( D \), that is the closed subset of the Ziegler spectrum corresponding to \( D \).

We know (since the Ziegler spectrum defines a topology e.g (9, Theorem 14.1)) that given two definable subcategories \( D \) and \( D' \), the definable subcategory generated by their union, \( \langle D \cup D' \rangle^{\text{def}} \), satisfies

\[
\langle D \cup D' \rangle^{\text{def}} \cap \text{pinj}_C = (D \cap \text{pinj}_C) \cup (D' \cap \text{pinj}_C).
\]

We must show that, if \( D \) and \( D' \) are fp-hom-closed, then so is \( \langle D \cup D' \rangle^{\text{def}} \). Notice that the Serre subcategory corresponding to \( \langle D \cup D' \rangle^{\text{def}} \) is given by the intersection of the Serre subcategories corresponding to \( D \) and \( D' \), say \( S_D \) and \( S_D' \) respectively. By Theorem 3.8 \( S_D \) and \( S_D' \) are tensor-ideals so \( S_D \cap S_D' \) must also be a tensor-ideal. Applying Theorem 3.8 again gives that \( \langle D \cup D' \rangle^{\text{def}} \) is fp-hom-closed. It is straightforward to see that the intersection of fp-hom-closed definable subcategories is fp-hom-closed and this completes the proof. \( \square \)

Thus we have the following tensor-analogue of Theorem 2.17.

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Corollary 4.2. Let $C$ be as in Assumption 3.2. There is a bijection between:

(i) the fp-hom-closed definable subcategories of $C$,

(ii) the Serre tensor-ideals of $C^\text{fp}$-mod,

(iii) the closed subsets of $Zg(C)\text{hom}^\text{fp}$.

4.2 A rigidity assumption

Next we move on to the context where $C^\text{fp}$ forms a rigid monoidal subcategory. In this setting, we get the following corollary to Theorem 3.8, giving a definable tensor-ideal/Serre tensor-ideal correspondence.

Corollary 4.3. Let $C$ be a finitely accessible category with products and suppose that $(C, \otimes, 1)$ is a closed symmetric monoidal category such that $C^\text{op}$ is a symmetric rigid monoidal subcategory. Let $S$ be a Serre subcategory of $C^\text{fp}$-mod and let $D$ be the corresponding definable subcategory of $C$ as in (Theorem 2.17).

Then, $S$ is a Serre tensor-ideal of $C^\text{fp}$-mod with respect to the induced tensor product if and only if $D$ is a definable tensor-ideal of $C$.

Proof. By Theorem 3.8 we have that $S$ is a Serre tensor-ideal if and only if $D$ is fp-hom-closed. By rigidity of $C^\text{fp}$, there exists a natural equivalence $\text{hom}(A, -) \cong A^\vee \otimes -$ for all every $A \in C^\text{fp}$, therefore $D$ is fp-hom-closed if and only if it is closed under tensoring with objects of $C^\text{fp}$. Suppose $X \in C$ and $D \in D$. As $C$ is finitely accessible we can write $X$ as a direct limit $X = \lim_{i \in I} X_i$ where the $X_i$ are finitely presented. Therefore, if $D$ is closed under tensoring with objects of $C^\text{fp}$, then $X \otimes D \cong (\lim_{i \in I} X_i) \otimes D \cong \lim_{i \in I} (X_i \otimes D) \in D$, as $- \otimes D$ commutes with direct limits and $D$ is closed under direct limits. □

4.3 Elementary duality

Throughout this section assume $A$ is a small preadditive category with an additive rigid monoidal structure. We show that elementary duality (see Proposition 2.20) maps fp-hom-closed definable subcategories of $\text{Mod-}A$ to definable tensor-ideals of $A\text{-Mod}$.

Notation 4.4. We will denote the monoidal structure on $A$ by $\otimes$, while $\otimes_A$ denotes the tensor product of $A$-modules given in Definition 2.18.

Definition 4.5. Given a finitely presented right $A$-module $M \in \text{mod-}A$ with presentation

$$(-, m_1) \xrightarrow{(\cdot, m)} (-, m_2) \rightarrow M \rightarrow 0$$

where $m : m_1 \rightarrow m_2$ is a morphism in $A$, define (up to isomorphism) the finitely presented left $A$-module $M^d \in \text{A-mod}$ to have presentation

$$(m_1^\vee, -) \xrightarrow{(m^\vee, \cdot)} (m_2^\vee, -) \rightarrow M^d \rightarrow 0,$$

where $m^\vee : m_2^\vee \rightarrow m_1^\vee$ is the dual morphism to $m$ in $A$.

Similarly, given a finitely presented left $A$-module $N \in A\text{-mod}$ with presentation

$$(n_2, -) \xrightarrow{(n, \cdot)} (n_1, -) \rightarrow N \rightarrow 0$$

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where \( n : n_1 \to n_2 \) is a morphism in \( \mathcal{A} \), define (up to isomorphism) the finitely presented right \( \mathcal{A} \)-module \( N^d \in \text{mod-} \mathcal{A} \) to have presentation

\[
(-, n^\vee_2) \xrightarrow{(-, n^\vee)} (-, n^\vee_1) \to N^d \to 0,
\]

where \( n^\vee : n_2^\vee \to n_1^\vee \) is the dual morphism to \( n \) in \( \mathcal{A} \).

**Proposition 4.6.** Let \( \mathcal{A} \) be a small preadditive category with an additive symmetric rigid monoidal structure and induce monoidal structures on \( \mathcal{A} \text{-Mod} \) and \( \text{Mod-} \mathcal{A} \) via Day convolution product.

The maps \((-)^d : \text{A-mod} \leftrightarrow \text{mod-} \mathcal{A} \) give an equivalence between \( \text{A-mod} \) and \( \text{mod-} \mathcal{A} \).

**Proof.** Fix a presentation for each \( N \in \mathcal{A} \text{-mod} \). First let us show that \((-)^d : \text{A-mod} \to \text{mod-} \mathcal{A} \) is functorial. Suppose \( h : N \to N' \) is a morphism in \( \mathcal{A} \text{-mod} \) where \( N \) and \( N' \) have presentations

\[
(n_2, -) \xrightarrow{(n_1, -)} N \to 0
\]

and

\[
(n'_2, -) \xrightarrow{(n'_1, -)} N' \to 0
\]

respectively.

By projectivity of representables we can choose \( h_1 : n'_1 \to n_1 \) and \( h_2 : n'_2 \to n_2 \) such that the following diagram commutes.

\[
\begin{array}{ccc}
(n_2, -) & \xrightarrow{(n_1, -)} & (n_1, -) \\
(h_2, -) & \downarrow & \downarrow h \\
(n'_2, -) & \xrightarrow{(n'_1, -)} & (n'_1, -) \\
& \downarrow & \\
& N & \longrightarrow 0 \\
\end{array}
\]

Thus, \( n \circ h_1 = h_2 \circ n' \) and dualising we get \( h_1^\vee \circ n^\vee = n'^\vee \circ h_2^\vee \). Therefore we have the following commutative diagram where the map \( h^d \) is uniquely determined.

\[
\begin{array}{ccc}
(-, n^\vee_2) & \xrightarrow{(-, n^\vee)} & (-, n^\vee_1) \\
(-, h_1^\vee) & \downarrow & \downarrow h^d \\
(-, n'^\vee_2) & \xrightarrow{(-, n'^\vee)} & (-, n'^\vee_1) \\
& \downarrow & \\
& N^d & \longrightarrow 0 \\
\end{array}
\]

It is straightforward to check that any choice of \( h_1 \) and \( h_2 \) induce the same map \( h^d \) and functoriality of \((-)^\vee : \mathcal{A} \to \mathcal{A} \) implies functoriality of \((-)^d \). So (given a choice of presentation for all \( N \in \mathcal{A} \text{-mod} \)) we have a well-defined functor, \((-)^d : \text{A-mod} \to \text{mod-} \mathcal{A} \). Furthermore, since we have a natural isomorphism \( 1_{\mathcal{A}} \to ((-)^\vee)^\vee \), by construction, the functor \((-)^d : \text{mod-} \mathcal{A} \to \text{A-mod} \) (fixing a presentation for each \( N \in \mathcal{A} \text{-mod} \)) clearly gives a quasi-inverse. \( \square \)

**Lemma 4.7.** For every \( M \in \text{mod-} \mathcal{A} \), \( L \in \text{Mod-} \mathcal{A} \) and \( N \in \mathcal{A} \text{-Mod} \), we have an isomorphism

\[
(L \otimes M) \otimes_{\mathcal{A}} N \cong L \otimes_{\mathcal{A}} (M^d \otimes N),
\]

natural in \( L \) and \( N \).
Proof. First let us prove that for every $a \in \mathcal{A}$, we have an isomorphism $(L \otimes (\cdot, a)) \otimes_{\mathcal{A}} N \cong L \otimes_{\mathcal{A}} ((a^\vee, \cdot) \otimes N)$ which is natural in $N$ and $L$. Recall that $- \otimes_{\mathcal{A}} N$ and $- \otimes (\cdot, a)$ are both right exact and therefore preserve direct limits. Therefore, as $\text{Mod-}\mathcal{A}$ is locally finitely presentable, it is sufficient to assume that $L$ is finitely presented. Suppose $L$ has presentation $(-, l_1) \xrightarrow{(-, l_0)} (-, l_2) \to L \to 0$. By right exactness of $- \otimes_{\mathcal{A}} N$ and $- \otimes (\cdot, a)$ we have an exact sequence

$$(-, l_1 \otimes a) \otimes_{\mathcal{A}} N \xrightarrow{(-, l_2 \otimes a)} (-, l_2 \otimes a) \otimes_{\mathcal{A}} N \to (L \otimes (\cdot, a)) \otimes_{\mathcal{A}} N \to 0.$$  

By definition of $\otimes_{\mathcal{A}}$, $(-, l \otimes a) \otimes_{\mathcal{A}} N : (-, l_1 \otimes a) \otimes_{\mathcal{A}} N \to (-, l_2 \otimes a) \otimes_{\mathcal{A}} N$ is given by $N(l \otimes a) : N(l_1 \otimes a) \to N(l_2 \otimes a)$. Thus by the Yoneda lemma we have the following commutative diagram in Ab.

$$(-, l_1 \otimes a) \otimes_{\mathcal{A}} N \xrightarrow{(-, l_2 \otimes a)} (-, l_2 \otimes a) \otimes_{\mathcal{A}} N \cong ((l_1 \otimes a, -), N) \xrightarrow{((l_2 \otimes a, -), N)} ((l_2 \otimes a, -), N)$$

By considering Lemma 3.9, we see that $(a^\vee, \cdot) \otimes - : \mathcal{A}-\text{Mod} \to \mathcal{A}-\text{Mod}$ is right adjoint to $(\cdot, \cdot) \otimes - : \mathcal{A}-\text{Mod} \to \mathcal{A}-\text{Mod}$. Thus we have the following commutative diagram where the first row of downwards arrows is given by the adjointness isomorphisms and the second row is given by the Yoneda lemma.

$$(l_1 \otimes a, -), N \xrightarrow{(l \otimes a, -)} ((l \otimes a, -), N) \cong (l_2 \otimes a, -), N$$

$$(l_1, -), (a^\vee, -) \otimes N \xrightarrow{(l, -), (a^\vee, -) \otimes N} ((l_2, -), (a^\vee, -) \otimes N) \cong ((a^\vee, -) \otimes N)(l_1) \xrightarrow{(a^\vee, -) \otimes N(l)} ((a^\vee, -) \otimes N)(l_2)$$

By the definition of $\otimes_{\mathcal{A}}$ we have $((a^\vee, -) \otimes N)(l) = (-, l \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N))$. Furthermore, by right exactness of $- \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N)$ we have exact sequence

$$(-, l_1) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \xrightarrow{(-, l_2) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N)} (-, l_2) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \to L \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \to 0.$$  

Thus we have an induced isomorphism $(L \otimes (\cdot, a)) \otimes_{\mathcal{A}} N \cong L \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N)$ as shown on the commutative diagram below.

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Proof.

By Theorem 3.8, we have that for every morphism $m : m_1 \rightarrow m_2$ in $\mathcal{A}$ the following square commutes for $i=1,2$:

\[
\begin{array}{ccc}
(l_i \otimes m_1, -) & \rightarrow & (l_i \otimes m_2, -) \\
(l_i \otimes m_1, -) & \rightarrow & (l_i \otimes m_2, -)
\end{array}
\]

As each of the isomorphisms in the first and second columns are natural in $l_i$ and $N$, the induced isomorphism is natural in $L$ and $N$. Furthermore, by properties of dual morphisms in $\mathcal{A}$ we have that for every morphism $m : m_1 \rightarrow m_2$ in $\mathcal{A}$ the following square commutes for $i=1,2$:

\[
\begin{array}{ccc}
(l_i \otimes m_1, -) & \rightarrow & (l_i \otimes m_2, -) \\
(l_i \otimes m_1, -) & \rightarrow & (l_i \otimes m_2, -)
\end{array}
\]

Therefore the induced isomorphisms $(L \otimes (-, m_i)) \otimes A N \cong L \otimes A ((m_i^\vee, -) \otimes N)$ for $i = 1, 2$ commute with any morphism $m : m_1 \rightarrow m_2$ in $\mathcal{A}$ in the following sense:

\[
\begin{array}{ccc}
(L \otimes (-, m_1)) \otimes A N & \rightarrow & (L \otimes (-, m_2)) \otimes A N \\
(L \otimes (-, m_1)) \otimes A N & \rightarrow & (L \otimes (-, m_2)) \otimes A N
\end{array}
\]

Hence the desired isomorphism $(L \otimes M) \otimes A N \cong L \otimes A (M^d \otimes N)$ is determined uniquely by the commutative diagram shown above. □

**Theorem 4.8.** Let $\mathcal{A}$ be a small preadditive category with an additive, symmetric, rigid, monoidal structure and induce monoidal structures on $\mathcal{A} \text{-Mod}$ and $\text{Mod-A}$ via Day convolution product.

A definable subcategory $\mathcal{D} \subseteq \text{Mod-A}$ is fp-hom-closed if and only if the dual definable subcategory $\delta \mathcal{D} \subseteq \mathcal{A} \text{-Mod}$ is a tensor-ideal.

**Proof.** By Theorem 3.8, $\mathcal{D} \subseteq \text{Mod-A}$ is an fp-hom-closed definable subcategory if and only if the corresponding Serre subcategory $\mathcal{S} \subseteq \text{mod-A, Ab}^{fp}$ is a tensor-ideal.

By [S, Proposition 10.3.5], every functor in the dual Serre subcategory $\delta \mathcal{S} \subseteq (\mathcal{A} \text{-mod, Ab})^{fp}$
has form $\delta F_f$ with presentation

$$0 \to \delta F \to A \otimes_A - \xrightarrow{f \otimes A} B \otimes_A -$$

for some $F_f \in S$. Therefore, $X \in \delta D$ if and only if for every $f : A \to B$ in $A$ such that $F_f \in S$, $\delta F_f(X) = 0$ equivalently, $f \otimes A X : A \otimes_A X \to B \otimes_A X$ is a monomorphism.

Thus $D$ is fp-hom-closed if and only if $\delta D$ satisfies the following. $X \in \delta D$ if and only if for every $F_f \in S$, and every $M \in \text{mod-}A$,

$$(M \otimes f) \otimes_A X : (M \otimes A) \otimes_A X \to (M \otimes B) \otimes_A X$$

is a monomorphism. But by Lemma 3.7

$$(M \otimes f) \otimes_A X : (M \otimes A) \otimes_A X \to (M \otimes B) \otimes_A X$$

is a monomorphism if and only if

$$f \otimes_A (M^d \otimes X) : A \otimes_A (M^d \otimes X) \to B \otimes_A (M^d \otimes X)$$

is a monomorphism. Therefore, $D$ is fp-hom-closed if and only if for every $X \in \delta D$ and all $M \in \text{mod-}A$, $M^d \otimes X \in \delta D$ or equivalently for all $N \in A$-mod, $N \otimes X \in \delta D$ as $(-)^d$ is an equivalence (see Proposition 4.6). That is, $D$ is a fp-hom-closed if and only if $\delta D$ is closed under tensoring with finitely presented left $A$-modules if and only if $\delta D$ is a tensor-ideal, as required.

5 Examples

5.1 Tensor product of $R$-modules

Let us consider the case where $C = R$-Mod for a commutative ring $R$. Here, $R$-Mod has a closed symmetric monoidal structure with tensor product given by $\otimes_R$. The tensor unit is $R$ and the internal hom-functor is given by the usual hom-set with $R$-module structure given by $(rf)(x) = rf(x) = f(rx)$ for all $x \in X$ where $f \in \text{hom}(X,Y) = \text{Hom}_R(X,Y)$ and $r \in R$. Note that $R$-mod is a monoidal subcategory.

The next result shows that if a functor $F \in (R$-mod, $\text{Ab})^f$ belongs to some Serre subcategory $S$, and if $F$ is ‘simple enough’ then $G \otimes_R F \in S$ for any finitely presented functor $G$.

Proposition 5.1. Let $R$ be a commutative ring, $S \subseteq (R$-mod, $\text{Ab})^f$ be a Serre subcategory and $F \in S$ satisfy $\text{pdim}(F) = 0$ or $\text{pdim}(F) = 1$. Then for any $G \in (R$-mod, $\text{Ab})^f$, $G \otimes F \in S$, where $\otimes$ denotes the tensor product induced by $\otimes_R$ on $R$-Mod.

Proof. By Lemma 4.7, we can take $G = (C, -)$. Throughout, let $D$ be the definable subcategory associated to $S$ as in Theorem 2.17.

Suppose $F \in S$ satisfies $\text{pdim}(F) = 0$. Then $F = (A, -)$ for some $A \in R$-mod. Therefore, for all $D \in D$, $(A, D) = 0$. For any $C \in R$-mod we have $(C, -) \otimes (A, -) = (C \otimes_R A, -)$. We want to show that for all $D \in D$, $(C \otimes_R A, D) = 0$. But by the adjunction isomorphism we have $(C \otimes_R A, D) \cong (C, \text{hom}(A,D)) = (C, (A, D)) = (C, 0) = 0$, so $(C, -) \otimes (A, -) \in S$, as required.

Now suppose $F \in S$ satisfies $\text{pdim}(F) = 1$. Then we have an exact sequence

$$0 \to (B, -) \xrightarrow{(f, -)} (A, -) \xrightarrow{\pi} F \to 0,$$
where the map \( f : A \to B \) is an epimorphism in \( R \)-mod. We want to show that for all \( C \in R \)-mod, 
\[(C, -) \otimes F \in \text{S} \]
the map \( (C \otimes_R B, D) \xrightarrow{(C \otimes_R f, D)} (C \otimes_R A, D) \) is an epimorphism for all \( D \in \mathcal{D} \).

As \( F \in \text{S} \), \( (B, D) \xrightarrow{(f, D)} (A, D) \) is an isomorphism, for every \( D \in \mathcal{D} \). Therefore for any \( C \in R \)-mod, the map \( (C, (B, D)) \xrightarrow{(C \otimes_R f, D)} (C, (A, D)) \) is an isomorphism and the tensor-hom adjunction gives the following commutative diagram.

\[
\begin{array}{ccc}
(C \otimes_R B, D) & \xrightarrow{(C \otimes_R f, D)} & (C \otimes_R A, D) \\
\downarrow & & \downarrow \\
(C, (B, D)) & \cong & (C, (A, D))
\end{array}
\]

Therefore for any \( C \in R \)-mod, the map \( (C \otimes_R B, D) \xrightarrow{(C \otimes_R f, D)} (C \otimes_R A, D) \) is an epimorphism for all \( D \in \mathcal{D} \) and \( (C, -) \otimes F \in \text{S} \) as required. \( \Box \)

Proposition 5.1 does not hold for \( \text{pdim}(F) = 2 \). Indeed, the Serre subcategory generated by \( T \) in Example 5.2 given below provides a counter example.

The following example is from (10), Section 13.

**Example 5.2.** (10), Section 13 Let \( R = k[\epsilon : \epsilon^2 = 0] \), where \( k \) is any field. We can define a monoidal structure on the category \( R \)-Mod with \( \otimes : R \text{-Mod} \times R \text{-Mod} \to R \text{-Mod} \) given by the usual tensor product of \( R \)-modules, \( \otimes = \otimes_R \). We extend this to a monoidal structure on \( (R \text{-mod}, \text{Ab})^{\text{fp}} \) using Day convolution product. First note that the only indecomposable \( R \)-modules are \( R \) and \( U = R/\text{rad}(R) \cong R/\langle \epsilon \rangle \cong k \). In fact every \( R \)-module is isomorphic to a direct sum of copies of these simple modules. We have \( R \otimes_R R \cong R \), \( R \otimes_R U \cong U \) and \( U \otimes_R U \cong U \).

Consider the exact sequence \( 0 \to \langle \epsilon \rangle \xrightarrow{j} R \xrightarrow{p} U \to 0 \). Let \( S \) and \( T \) be the functors such that we have exact sequences \( 0 \to (U, -) \xrightarrow{(p, -)} (R, -) \to S \to 0 \) and \( 0 \to (U, -) \xrightarrow{(p, -)} (R, -) \xrightarrow{(j, -)} ((\epsilon), -) \to T \to 0 \). By Section 13 of (10), the indecomposable functors in \( (R \text{-mod}, \text{Ab})^{\text{fp}} \) are

\[
S : M \mapsto \epsilon M,
\]

\[
T : M \mapsto \text{ann}_M(\epsilon)/\epsilon M,
\]

\[
(U, -) : M \mapsto \text{ann}_M(\epsilon),
\]

\[
W : M \mapsto M/\epsilon M,
\]

and

\[
(R, -) : M \mapsto M.
\]

The table below shows the action of the tensor product on \( (R \text{-mod}, \text{Ab})^{\text{fp}} \) (given in (10), Section 13.3).
Let us identify the definable subcategories of $R$-$\text{Mod}$ for $R = k[\epsilon : \epsilon^2 = 0]$. Recall that a module $M \in R$-$\text{Mod}$ has form $M = R^{(\kappa)} \oplus U^{(\lambda)}$ for some cardinals $\kappa$ and $\lambda$, (see [7], Section 6.8). If both $\kappa$ and $\lambda$ are non-zero, then $\langle M \rangle = R$-$\text{Mod}$. Therefore, the only non-trivial proper definable subcategories of $R$-$\text{Mod}$ are $\langle R \rangle = \{R^{(\kappa)} : \kappa \text{ a cardinal}\}$ and $\langle U \rangle = \{U^{(\lambda)} : \lambda \text{ a cardinal}\}$.

Since $\otimes_R$ commutes with direct sums it is easy to see that both $\langle R \rangle^{\text{def}}$ and $\langle U \rangle^{\text{def}}$ are closed under tensor product and $\langle U \rangle^{\text{def}}$ is even a tensor-ideal in $R$-$\text{Mod}$. Furthermore, we have $\text{hom}(R, -) = \text{Hom}_R(R, -) \cong \text{Id}_{R$-$\text{Mod}$}$. It can also be checked that $\text{hom}(U, R) \cong U$ meaning $\langle R \rangle^{\text{def}}$ is not fp-hom-closed.

Next let us consider the corresponding Serre subcategories. First take $\mathcal{D} = \langle U \rangle^{\text{def}}$. Then

$S(U) = \epsilon U = 0,$

$T(U) = \text{ann}_U(\epsilon)/\epsilon U = U/0 \cong U,$

$(U, -)(U) = \text{ann}_U(\epsilon) = U,$

$W(U) = U/\epsilon U = U/0 \cong U$

and

$(R, -)(U) = U.$

Therefore $S_\mathcal{D}$ is generated by the indecomposable functor $S$ and indeed consists just of direct sums of copies of $S$. As $\text{pdim}(S) = 1$, by Proposition [5.4], $G \otimes S \in S_\mathcal{D}$ for every finitely presented $G : R$-$\text{mod} \to \text{Ab}$. Therefore, $S_\mathcal{D}$ is a Serre tensor-ideal.

Now take $\mathcal{D} = \langle R \rangle^{\text{def}}$. Then

$S(R) = \epsilon R = U,$
\[T(R) = \text{ann}_R(\epsilon) / \epsilon R = U/U \cong 0,\]
\[(U, -)(R) = \text{ann}_R(\epsilon) = U,\]
\[W(R) = R/\epsilon R = U\]

and
\[(R, -)(R) = R.\]

Therefore \(S_D\) is generated by the indecomposable functor \(T\). As \(T \otimes T \cong (U, -)\) this Serre subcategory is not closed under tensor product.

In summary we get the following table, where \(\langle \cdot \rangle^{\text{def}}\) denotes ‘the definable subcategory generated by’ and \(\langle \cdot \rangle^5\) denotes ‘the Serre subcategory generated by’.

| Definable subcat. | Monoidal subcat. | fp-hom-closed | Tensor-ideal | Serre subcat. | Monoidal subcat. | Tensor-ideal |
|-------------------|------------------|---------------|--------------|---------------|------------------|--------------|
| 0                 | Yes              | Yes           | Yes          | \((R\text{-mod}, \mathbb{A}b)^{fp}\) | Yes          | Yes          |
| \(\langle U \rangle^{\text{def}}\) | Yes | Yes | Yes | \(\langle S \rangle^5\) | Yes | Yes |
| \(\langle R \rangle^{\text{def}}\) | Yes | No | No | \(\langle T \rangle^5\) | No | No |
| \(R\text{-Mod}\) | Yes | Yes | Yes | 0 | Yes | Yes |

5.1.1 Von Neumann regular rings

Let us consider the example of von Neumann regular rings.

**Definition 5.3.** A ring \(R\) is von Neumann regular if for every \(x \in R\) there exists some \(y \in R\) such that \(x = xyx\).

**Proposition 5.4.** Let \(R\) be a commutative von Neumann regular ring so the normal tensor product of rings, \(\otimes_R\), is a symmetric closed monoidal structure on \(R\text{-Mod}\). Every definable subcategory of \(R\text{-Mod}\) is fp-hom-closed.

**Proof.** By ([8], Proposition 10.2.20), the global dimension of \((R\text{-mod}, \mathbb{A}b)^{fp}\) is zero if and only if \(R\) is von Neumann regular. Thus by Lemma [5.1] for \(R\) von Neumann regular, every Serre subcategory of \((R\text{-mod}, \mathbb{A}b)^{fp}\) is a tensor-ideal and therefore, by Theorem [5.8] every definable subcategory is fp-hom-closed. \(\square\)

**Proposition 5.5.** Let \(R\) be a commutative von Neumann regular ring so the normal tensor product of rings, \(\otimes_R\), is a symmetric closed monoidal structure on \(R\text{-Mod}\). Every fp-hom-closed definable subcategory \(D\) of \(R\text{-Mod}\) satisfies the exactness criterion.
Proof. R is von Neumann regular if and only if every (left) R-module is flat, that is for every 
\( M \in \text{R-Mod}, M \otimes_R - : \text{R-Mod} \to \text{Ab} \) is exact (e.g. see [8], Theorem 2.3.22). Therefore, 
since R is commutative, we obtain a symmetric closed monoidal product on R-Mod which is 
each in each variable. Furthermore, by ([8], Proposition 10.2.38) we have \((\text{R-mod}, \text{Ab})^{\text{fp}} \simeq (\text{R-mod})^{\text{fp}} \sim \text{R-mod}\) where the direction \( R-\text{mod} \to (R-\text{mod}, \text{Ab})^{\text{fp}} \) is given by the Yoneda embedding. Therefore, this equivalence is monoidal with respect to Day convolution product. In 
other words, letting \( D = R-\text{Mod}, \text{fun}(D) = (R-\text{mod}, \text{Ab})^{\text{fp}} \sim \text{R-mod} \) has an additive symmetric 
monoidal structure which is exact in each variable. Thus, by Proposition 5.12 \( D = R-\text{Mod} \) satisfies the 
eaxtensus criterion. Consequently any fp-hom-closed definable subcategory of \( R-\text{Mod} \) also satisfies the exactness criterion. \( \square \)

Remark 5.6. By Proposition 5.11 for \( R \) von Neumann regular, \( Z_g(R-\text{Mod}) \) and \( Z^\text{hom}(R-\text{Mod}) \) 
are the same topology. Furthermore, by Proposition 5.5 and Proposition 5.11 for every definable 
subcategory \( D \subseteq R-\text{Mod} \), we can induce an exact, additive, closed, symmetric monoidal structure 
on the corresponding functor category \( \text{fun}(D) \).

By ([8], Proposition 3.4.30) the definable subcategory generated by \( M \in R-\text{Mod} \) is given by 
\( (R/\text{ann}_R(M)) \)-Mod viewed as a full subcategory of \( R-\text{Mod} \) via \( R \to R/\text{ann}_R(M) \). Therefore, it is easy to see directly that \( (M)^{\text{def}} \) is fp-hom-closed. Furthermore, the associated Serre subcategory of \( (R-\text{mod}, \text{Ab})^{\text{fp}} \sim R-\text{mod} \) is given by \( \{X \in R-\text{mod} : (X, M) = 0\} \).

5.1.2 Coherent rings

Now we consider the case where \( R \) is coherent.

Definition 5.7. A commutative ring \( R \) is coherent if every finitely generated ideal is finitely 
presented.

We will use the following properties.

Proposition 5.8. ([3], Corollary 2.3.18 and Proposition E.1.47) A (commutative) ring \( R \) is 
coherent if and only if \( \text{R-mod} \) is abelian.

Proposition 5.9. ([3], Theorem 3.4.24) A (commutative) ring \( R \) is coherent if and only if the 
subcategory \( \text{Abs-R} \subseteq \text{Mod-R} \) of absolutely pure modules is definable.

Lemma 5.10. Let \( C \) be as in Assumption 5.6 and suppose \( X \in C \) and \( U \in C^{\text{fp}} \) are such that 
\( \text{hom}(U, X) \) is absolutely pure. Let \( f : A \to B \) be a morphism in \( C^{\text{fp}} \). If \( f : A \to B \) is a 
monomorphism in \( C \), then every morphism \( h : A \otimes U \to X \) factors through \( f \otimes U \).

Proof. Via the tensor-hom adjunction there exists some \( h' : B \otimes U \to X \) such that \( h = h' \circ (f \otimes U) \) 
if and only if there exists some \( \tilde{h}' : B \to \text{hom}(U, X) \) such that \( \tilde{h} = \tilde{h}' \circ f \) where \( \tilde{h} : A \to \text{hom}(U, X) \) is the 
morphism corresponding to \( h \) via the adjunction isomorphism \( (A \otimes U, X) \cong (A, \text{hom}(U, X)) \). 
But \( \text{hom}(U, X) \) is absolutely pure and \( f : A \to B \) is a monomorphism with \( A, B \in C^{\text{fp}} \) so applying 
([8], Proposition 2.3.1) we get \( \tilde{h} \) factors via \( f \) as required. \( \square \)

Proposition 5.11. Let \( R \) be a commutative coherent ring. Any fp-hom-closed definable subcategory \( D \) of \( \text{Abs-R} \) satisfies the exactness criterion.

Proof. Suppose \( f : A \to B \) and \( g : U \to V \) are morphisms in \( \text{mod-R} \) and \( X \in D \). Suppose 
further that \( h : A \otimes U \to X \) satisfies \( h = h' \circ (f \otimes U) = h'' \circ (A \otimes g) \) for some \( h' : B \otimes U \to X \) 
and \( h'' : A \otimes V \to X \), that is the following diagram commutes.

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A ⊗ U \xrightarrow{f \otimes U} B \otimes U \\
A \otimes g \downarrow \quad h' \\
A \otimes V \xrightarrow{h''} X

As mod-R is abelian, we have exact sequences in mod-R, \( 0 \to A' \xrightarrow{k} A \xrightarrow{f} B \) and \( U \xrightarrow{g} V \xrightarrow{c} W \to 0 \) where \( k : A' \to A \) is the kernel of \( f \) and \( c : V \to W \) is the cokernel of \( g \). Furthermore, since \( A' \otimes - : \text{mod-} R \to \text{mod-} R \) is right exact we have an exact sequence

\[
A' \otimes U \xrightarrow{A' \otimes g} A' \otimes V \xrightarrow{A' \otimes c} A' \otimes W \to 0.
\]

Now \( h'' \circ (k \otimes V) \circ (A' \otimes g) = h'' \circ (A \otimes g) \circ (k \otimes U) = h'' \circ (f \otimes U) \circ (k \otimes U) = 0 \). Therefore, \( h'' \circ (k \otimes V) \) factors via \( A' \otimes c \), say \( h'' \circ (k \otimes V) = l \circ (A' \otimes c) \) for some \( l : A' \otimes W \to X \).

\[
\begin{array}{ccc}
A' \otimes U & \xrightarrow{A' \otimes g} & A' \otimes V \\
\downarrow{k \otimes V} & & \downarrow{k \otimes W} \\
A \otimes V & \xrightarrow{l} & A \otimes W \\
\downarrow{h''} & & \downarrow{W} \\
X & & X
\end{array}
\]

Note that \( D \subseteq \text{Abs-R} \) is fp-hom-closed so \( \text{hom}(W,X) \) is absolutely pure. In addition, \( k : A' \to A \) is a monomorphism therefore applying Lemma 5.10, \( l : A' \otimes W \to X \) factors via \( k \otimes W \), say \( l' : A \otimes W \to X \).

We have \( h'' \circ (k \otimes V) = l' \circ (A \otimes c) \circ (k \otimes V) \). Setting \( r = h'' - l' \circ (A \otimes c) : A \otimes V \to X \) we have \( r \circ (k \otimes V) = 0 \). Let \( r : A \to \text{hom}(V,X) \) be the morphism corresponding to \( r : A \otimes V \to X \) via the adjunction isomorphism. Then \( r \circ k = 0 \).

Now recall that mod-R is abelian and we have exact sequence \( 0 \to A' \xrightarrow{k} A \xrightarrow{f} B \). Therefore, \( \text{coker}(k) = \text{im}(f) \). Write \( f : A \to B \) as \( i_f \circ \pi_f \) where \( \pi_f : A \to \text{im}(f) \) is the cokernel of \( k \) and \( i_f : \text{im}(f) \to B \) is a monomorphism. Then \( r \) factors via \( \pi_f : A \to \text{im}(f) \), or equivalently, \( r \) factors via \( \pi_f \otimes V \), say \( r = r' \circ (\pi_f \otimes V) \). Noting that \( \text{hom}(V,X) \) is absolutely pure, we may apply Lemma 5.10 to get that \( r' = r'' \circ (i_f \otimes V) \) that is \( r \) factors via \( f \otimes V \).

Finally note that \( h = h'' \circ (A \otimes g) = r \circ (A \otimes g) = r'' \circ (f \otimes V) \circ (A \otimes g) = r'' \circ (f \otimes g) \).

Therefore we have shown that \( h \) factors via \( f \otimes g \) and the exactness criterion holds for \( D \).

\[\square\]

5.2 Examples satisfying the rigidity condition

Example 5.12 below gives a class of examples where the assumptions of Corollary 4.3 are satisfied.

**Example 5.12.** The category of left \( H \)-modules, \( H \text{-Mod} \), for some Hopf algebra \( H \) (e.g. a group algebra), has a closed symmetric monoidal structure by (11, Section 4.9). Furthermore,
the finitely presentable left $H$-modules, $H$-mod, form a symmetric rigid monoidal subcategory (see [23], Section 4.1).

Therefore, applying Corollary 4.3 for every Hopf algebra $H$, the definable tensor-ideals of $H$-Mod correspond bijectively with the Serre tensor-ideals of $(H$-mod, Ab)\text{fp}.

In particular let us consider an example from ([10], Section 13).

**Example 5.13.** ([10], Section 13) Consider $R = k[\epsilon : \epsilon^2 = 0]$ as in Examples 5.2 but suppose further that the field $k$ has characteristic 2. Then $R$ is a group ring. Indeed if we set $\epsilon + 1 = g$ and let $G = \langle g : g^2 = 1 \rangle \cong C_2$, then it is easy to see that $R \cong kG$ as rings. We can define a new tensor product $\otimes : R$-Mod $\times$ $R$-Mod $\rightarrow$ $R$-Mod given by $M \otimes N = M \otimes_k N$ and where the action of $R$ is determined by $g(M \otimes N) = gM \otimes gN$.

Note that here the unit object is given by $U$ and the tensor product satisfies $R \otimes_k R \cong R^2$. We will use the notation of the previous example. The table below shows how this tensor product extends to $(R$-mod, Ab)\text{fp}. (See Section 13.5 of [10] for details of the calculation.)

| $\otimes$ | $S$ | $T$ | $(U, -)$ | $W$ | $(R, -)$ |
|-----------|-----|-----|-----------|-----|----------|
| $S$       | $W$ | 0   | $S$       | $W$ | $(R, -)$ |
| $T$       | 0   | $T$ | $T$       | 0   | 0        |
| $(U, -)$  | $S$ | $T$ | $(U, -)$  | $W$ | $(R, -)$ |
| $W$       | 0   | $W$ | $W$       | $(R, -)$ |         |
| $(R, -)$  | $(R, -)$ | 0 | $(R, -)$ | $(R, -)$ | $(R, -)^2$ |

We get the following definable subcategory/Serre subcategory correspondence, where as required by Corollary 4.3 there is a one-to-one correspondence between the definable tensor-ideals of $R$-Mod and the Serre tensor-ideals of $(R$-mod, Ab)\text{fp}.

| Definable subcategory | Monoidal subcategory | Tensor-ideal | Serre subcategory | Monoidal subcategory | Tensor-ideal |
|-----------------------|----------------------|--------------|------------------|----------------------|--------------|
| 0                     | Yes                  | Yes          | $(R$-mod, Ab)\text{fp} | Yes                  | Yes          |
| $(U)^\text{def}$     | Yes                  | No           | $(S)^5$          | No                   | No           |
| $(R)^\text{def}$     | Yes                  | Yes          | $(T)^5$          | Yes                  | Yes          |
| $R$-Mod               | Yes                  | Yes          | 0                | Yes                  | Yes          |

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