On Characteristic Polynomials of the Family of Cobweb Posets

Ewa Krot-Sieniawska

Institute of Computer Science, Białystok University
PL-15-887 Białystok, ul.Sosnowa 64, POLAND
e-mail: ewakrot@wp.pl, ewakrot@ii.uwb.edu.pl

Abstract
This note is a response to one of problems posed by A.K. Kwaśniewski in [7]. Namely with \( \{P_n\}_{n \geq 0} \) being the sequence of finite cobweb subposets, the looked for explicit formulas for corresponding sequence \( \{\chi_n(t)\}_{n \geq 0} \) of \( P_n \)'s characteristic polynomials are discovered and delivered here. The recurrence relation defining arbitrary family \( \{\chi_n(t)\}_{n \geq 0} \) is also derived.

KEY WORDS: cobweb poset, the Möbius function of a poset, Whitney numbers, characteristic polynomials.
AMS Classification numbers: 06A06, 06A07, 06A11, 11C08, 11B37

Presented at Gian-Carlo Rota Polish Seminar: [http://ii.uwb.edu.pl/akk/sem/semrota.htm](http://ii.uwb.edu.pl/akk/sem/semrota.htm)

1 Cobweb posets

The family of the so called cobweb posets \( \Pi \) has been invented by A.K. Kwaśniewski few years ago (for references see: [5][6]). These structures are such a generalization of the Fibonacci tree growth that allows joint combinatorial interpretation for all of them under the admissibility condition (see [7][8]).

Let \( \{F_n\}_{n \geq 0} \) be a natural numbers valued sequence with \( F_0 = 1 \) (with \( F_0 = 0 \) being exceptional as in case of Fibonacci numbers). Any sequence satisfying this property uniquely designates cobweb poset defined as follows.

For \( s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) let us to define levels of \( \Pi \):

\[
\Phi_s = \{(j, s), \ 1 \leq j \leq F_s\},
\]

(in case of \( F_0 = 0 \) level \( \Phi_0 \) corresponds to the empty root \( \{\emptyset\} \). )

Then
Definition 1.1. Corresponding cobweb poset is an infinite partially ordered set $\Pi = (V, \leq)$, where
\[ V = \bigcup_{0 \leq s} \Phi_s \]
are the elements (vertices) of $\Pi$ and the partial order relation $\leq$ on $V$ for $x = \langle s, t \rangle, y = \langle u, v \rangle$ being elements of cobweb poset $\Pi$ is defined by formula
\[ (x \leq_P y) \iff [(t < v) \lor (t = v \land s = u)]. \]
Obviously any cobweb poset can be represented, via its Hasse diagram, as infinite directed graf $\Pi = (V, E)$, where set $V$ of its vertices is defined as above and
\[ E = \{(j, p), (q, (p + 1))\} \cup \{(1, 0), (1, 1)\}, \]
where $1 \leq j \leq F_p$ and $1 \leq q \leq F_{(p+1)}$ stays for set of (directed) edges.
For example the Hasse diagram of Fibonacci cobweb poset designated by the famous Fibonacci sequence is presented below.

\[ \text{Fig. 1. The construction of the Fibonacci cobweb poset} \]
The Kwasniewski cobweb posets under consideration represented by graphs are examples of orderable directed acyclic graphs (oDAG) which we start to call from now in brief: KoDAGs. These are structures of universal importance for the whole of mathematics - in particular for discrete “mathemagics” [http://ii.uwb.edu.pl/akk/] and computer sciences in general (quotation from [7, 8]):

For any given natural numbers valued sequence the graded (layered) cobweb posets’ DAGs are equivalently representations of a chain of binary relations. Every relation of the cobweb poset chain is biunivocally represented by the uniquely designated complete bipartite digraph-a digraph which is a di-biclique designated by the very given sequence. The cobweb poset is then to be identified with a chain of di-bicliques i.e. by definition - a chain of complete bipartite one direction digraphs. Any chain of relations is therefore obtainable from the cobweb poset chain of complete relations via deleting arcs (arrows) in di-bicliques. Let us underline it again: any chain of relations is obtainable from the cobweb poset chain of complete relations via deleting arcs in di-bicliques of the complete relations chain. For that to see note that any relation \( R_k \) as a subset of \( A_k \times A_{k+1} \) is represented by a one-direction bipartite digraph \( D_k \). A "complete relation" \( C_k \) by definition is identified with its one direction di-biclique graph \( d - B_k \). Any \( R_k \) is a subset of \( C_k \). Correspondingly one direction digraph \( D_k \) is a subgraph of an one direction digraph of \( d - B_k \).

The one direction digraph of \( d - B_k \) is called since now on the di-biclique i.e. by definition - a complete bipartite one direction digraph. Another words: cobweb poset defining di-bicliques are links of a complete relations’ chain.

According to the definition above arbitrary cobweb poset \( \Pi = (V, \leq) \) is a graded poset (ranked poset) and for \( s \in \mathbb{N}_0 \):

\[
x \in \Phi_s \quad \rightarrow \quad r(x) = s,
\]

where \( r: \Pi \rightarrow \mathbb{N}_0 \) is a rank function on \( \Pi \).

Let us then define Kwasniewski finite cobweb sub-posets as follows

**Definition 1.2.** Let \( P_n = (V_n, \leq) \), \((n \geq 0)\), for \( V_n = \bigcup_{0 \leq s \leq n} \Phi_s \) and \( \leq \) being the induced partial order relation on \( \Pi \).

It’s easy to see that \( P_n \) is ranked poset with rank function \( r \) as above. \( P_n \) has a unique minimal element \( 0 = (1,0) \) ( with \( r(0) = 0 \)). Moreover \( \Pi \) and all \( P_n \)s satisfy the Jordan chain condition and the length of \( P_n \) is \( l(P_n) = r(P_n) = n \) for \( n \geq 0 \).

For finite graded poset \( P \) one can define (see [1]) Whitney numbers of the first and second kind \( w_k(P) \) and \( W_k(P) \) respectively as follows

\[
w_k(P) = \sum_{x \in P, r(x) = k} \mu(0, x),
\]
\[ W_k(P) = \sum_{x \in P, r(x) = k} 1 = |\{x \in P : r(x) = k\}|, \]

where \( \mu \) stays for Möbius function of \( P \) indispensable in numerous inversion type formulas of countless applications (see \([1, 9, 11, 12, 13]\)).

Then the characteristic polynomial of \( P \) \([9, 11, 12, 13]\) is the polynomial

\[ \chi_P(t) = \sum_{x \in P} \mu(0, x)t^{n-r(x)} = \sum_{k=0}^{n} w_k(P)t^{n-k}, \]

where \( n = l(P) \).

Here next we answer the question posed by A.K. Kwaśniewski in the source paper for the problem in question \([7]\).

Let \( \{P_n\}_{n \geq 0} \) be the sequence of finite cobweb subposets \((\ldots)\). What is the form and properties of \( \{P_n\}_{n \geq 0} \)’s characteristic polynomials \( \{\rho_n(\lambda)\}_{n \geq 0} \)? \((\ldots)\) What are recurrence relations defining the family \( \{\rho_n(\lambda)\}_{n \geq 0} \)?

2 Whitney numbers of cobweb posets

Obviously for arbitrary cobweb poset \( \Pi \) and for all its finite subposets \( P_n, (n \geq 0) \) one has:

\[ W_k(\Pi) = F_k, \quad k \geq 0, \quad (1) \]

where \( \{F_n\}_{n \geq 0} \) is a natural numbers valued sequence uniquely designating \( \Pi \).

Now let us consider the corresponding numbers \( w_k(\Pi) \). The explicit formula for Möbius function of the Fibonacci cobweb poset uniquely designated by the Fibonacci sequence was derived by the present author in \([2, 3]\). It can be easily extend to the whole family of cobweb posets and their finite subposets \( P_n, (n \geq 0), [4] \). Moreover, by the use of notion of the standard reduced incidence algebra \( R(\Pi) \), (see \([4]\)) one can show, that for \( x \in \Pi \) the value \( \mu(0, x) \) depends on \( r(x) \) only. So for \( x \) as above we have:

\[ \mu(0, x) = \mu(r(0), r(x)) = \mu(0, r(x)) = \mu(r(x)). \quad (2) \]

Moreover

\[ \mu(0, x) = \mu(r(x)) = (-1)^{r(x)} \prod_{i=1}^{r(x)-1} (F_i - 1). \quad (3) \]

Then

**Proposition 2.1.** For arbitrary cobweb poset \( \Pi \) and for all its finite subposets \( P_n, (n \geq 0) \) corresponding Whitney numbers of the first kind are given by the formulas:
for $k > 0$

$$w_k(\Pi) = \sum_{\{x \in \Pi : \rho(x) = k\}} \mu(0, x) = F_k \cdot \mu(0, x)$$

$$= F_k \cdot (-1)^k \cdot \prod_{i=1}^{k-1} (F_i - 1)$$

(4)

and

$$w_0(\Pi) = 1.$$  

(5)

3 The characteristic polynomials of finite cobweb posets

The knowledge of Whitney numbers $w_k(P_n)$, enables us to construct the characteristic polynomials for all $P_n$, ($n \geq 0$). Let us recall the formula defining $\chi_n(t)$:

$$\chi_{P_n}(t) = \sum_{x \in P_n} \mu(0, x) t^{n-\rho(x)} = \sum_{k=0}^{n} w_k(P_n) t^{n-k}.$$  

Using the above formulas one has

**Theorem 3.1.** The characteristic polynomials $\chi_{P_n}(t)$, ($n \geq 0$) are given by the following explicit formula:

$$\chi_{P_n}(t) = \chi_n(t) = x^n + \sum_{k=1}^{n} (-1)^k F_k \cdot \prod_{i=1}^{k-1} (F_i - 1) x^{n-k}.$$  

(6)

Moreover, as in case of Fibonacci cobweb poset, the following holds:

**Corollary 3.1.** Let $\{F_n\}_{n \geq 0}$ be the sequence designating the cobweb poset $\Pi$ (and all corresponding sub-posets $P_n$). In the case $F_1 = 1$ (or equivalently $|\Phi_1| = 1$) one has

$$\chi_n(t) = t^n - t^{n-1}$$  

(7)

for $n \geq 1$ and

$$\chi_0(t) = 1.$$

(8)

**Corollary 3.2.** Let $\{F_n\}_{n \geq 0}$ be the sequence designating the cobweb poset $\Pi$ (and all corresponding sub-posets $P_n$). Then the sequence $\{\chi_n(t)\}_{n \geq 0}$ of $\{P_n\}_{n \geq 0}$’s characteristic polynomials is defined by the following recurrence relation

$$\chi_0(t) = 1, \quad \chi_1(t) = t - F_1$$

$$\chi_n(t) = t\chi_{n-1}(t) + (-1)^n F_n(F_{n-1} - 1)(F_{n-2} - 1)...(F_1 - 1), \quad n \geq 2.$$  

(9)

(10)
Example 3.1. Let the sequence of finite cobweb posets \( \{ P_n \}_{n \geq 0} \) be designated by the sequence \( \{ F_n \}_{n \geq 0} \) such that \( F_n = n + 1 \) (i.e. by the sequence \( N \) of natural numbers). The examples of corresponding characteristic polynomials are:

\[
\begin{align*}
\chi_0(t) & = 1, \\
\chi_1(t) & = t - 2, \\
\chi_2(t) & = t^2 - 2t + 4, \\
\chi_3(t) & = t^3 - 2t^2 + 4t - 18, \\
\chi_4(t) & = t^4 - 2t^3 + 4t^2 - 18t + 120, \\
\chi_5(t) & = t^5 - 2t^4 + 4t^3 - 18t^2 + 120t - 1050, \\
\chi_6(t) & = t^6 - 2t^5 + 4t^4 - 18t^3 + 120t^2 - 1050t + 11340.
\end{align*}
\]

Example 3.2. Let the sequence of finite cobweb posets \( \{ P_n \}_{n \geq 0} \) be designated by the sequence \( \{ F_n \}_{n \geq 0} \) such that \( F_1 = 1 \) and \( F_n = 2n + 1 \) for \( n \geq 1 \). The examples of corresponding characteristic polynomials are:

\[
\begin{align*}
\chi_0(t) & = 1, \\
\chi_1(t) & = t - 3, \\
\chi_2(t) & = t^2 - 3t + 10, \\
\chi_3(t) & = t^3 - 3t^2 + 10t - 56, \\
\chi_4(t) & = t^4 - 3t^3 + 10t^2 - 56t + 432, \\
\chi_5(t) & = t^5 - 3t^4 + 10t^3 - 56t^2 + 432t - 4224,
\end{align*}
\]

Example 3.3. Let the sequence of finite cobweb posets \( \{ P_n \}_{n \geq 0} \) be designated by the sequence \( \{ F_n \}_{n \geq 0} \) such that \( F_1 = 1 \) and \( F_n = k \) for \( n \geq 1 \) and for some \( k > 1 \). The examples of corresponding characteristic polynomials are:

\[
\begin{align*}
\chi_0(t) & = 1, \\
\chi_1(t) & = t - k, \\
\chi_2(t) & = t^2 - kt + k(k - 1), \\
\chi_3(t) & = t^3 - kt^2 + k(k - 1)t - k(k - 1)^2, \\
\chi_4(t) & = t^4 - kt^3 + k(k - 1)t^2 - k(k - 1)^2t + k(k - 1)^3.
\end{align*}
\]

In general one has

\[
\chi_n(t) = t^n - kt^{n-1} + k(k-1)t^{n-2} + \ldots + (-1)^nk(k-1)^{n-1}, \quad n \geq 1.
\]

Acknowledgements

Discussions with Participants of Gian-Carlo Rota Polish Seminar, [http://ii.uwb.edu.pl/akk/sem/sem_rota.htm](http://ii.uwb.edu.pl/akk/sem/sem_rota.htm) are highly appreciated.

References

[1] Joni S.A., Rota. G.-C., Sagan B.: From Sets to Functions: Three Elementary Examples, Discreta Mathematics 37 (1981), p.193-202.
[2] Krot E.: A note on Mobius function and Mobius inversion formula of Fibonacci Cobweb Poset, Bulletin de la Societe des Sciences et des Lettres de d (54), Serie: Recherches sur les Deformations Vol. 44, s.39-44, ArXiv: math.CO/040415, cs.DM http://arxiv.org/abs/math/0404158

[3] Krot E.: The first ascent into the Fibonacci Cobweb Poset, Advanced Studies in Conterporary Mathematics 11 (2005), No. 2, p.179-184, ArXiv: math.CO/0411007, cs.DM http://arxiv.org/abs/math/0411007

[4] Krot-Sieniawska E.: On incidence algebras of cobweb posets, being in preparation

[5] Kwaśniewski A.K.: Cobweb posets as noncommutative prefabs, Adv. Stud. Contemp. Math. 14, 1 (2007), s. 37-47, ArXiv:math/0503286, cs.DM http://arxiv.org/abs/math/0503286

[6] A.K.Kwaśniewski: First observations on Prefab posets' Whitney numbers, Advances in Applied Clifford Algebras Volume 18, Number 1 / February, 2008, p. 57-73, ONLINE FIRST, Springer Link Date, August 10, 2007, arXiv:0802.1696, cs.DM http://arxiv.org/abs/0802.1696

[7] Kwaśniewski A.K.: On cobweb posets and their combinatorially admissible sequences, ArXiv:math.Co/0512578v4 21 Oct 2007, submitted to Graphs and Combinatorics; Japan, cs.DM http://arxiv.org/abs/math/0512578

[8] Kwaśniewski A.K., Dziemiańczuk M.: Cobweb posets - Recent Results, IS-RAMA 2007, December 1-17 2007 Kolkata, INDIA, arXiv:0801.3985, cs.DM http://arxiv.org/abs/0801.3985

[9] Rota G.-C.: On the Foundations of Combinatorial Theory: I. Theory of Möbius Functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, vol.2, 1964, p.340-368.

[10] Sagan B.: Why the characteristic polynomial factors, Bull. Amer. Math. Soc. 36 (1999), p. 113134.

[11] Sagan B.: Möbius Functions of Posets IV: Why the Characteristic Polynomial Factors, www.math.msu.edu/ sagan/Slides/mfp4.pdf

[12] Spiegel E., O'Donnell Ch.J.: Incidence algebras, Marcel Dekker, Inc. Basel 1997

[13] Stanley R.P.: Enumerative Combinatorics, Volume I, Wadsworth& Brooks/Cole Advanced Books & Software, Monterey California, 1986.