Symmetric Spaces in Supergravity

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Abstract

We exploit the relation among irreducible Riemannian globally symmetric spaces (IRGS) and
supergravity theories in 3, 4 and 5 space-time dimensions. IRGS appear as scalar manifolds of the
theories, as well as moduli spaces of the various classes of solutions to the classical extremal black
hole Attractor Equations. Relations with Jordan algebras of degree three and four are also outlined.
1 Introduction

The aim of this contribution, devoted to the 70th birthday of Prof. R. Varadarajan, is to give some examples of interplay among some geometrical objects, Riemannian symmetric spaces, and physical theories such as the supersymmetric theories of gravitation, usually called supergravities.

Symmetric spaces occur as target spaces of the non-linear sigma models which encode the dynamics of scalar fields, related by supersymmetry to some spin-1/2 and spin-1/2 fermion fields, the latter called gravitinos, the gauge fields of local supersymmetry.

Many supergravities provide a unique (classical) extension of the Einstein-Hilbert action of General Relativity. By denoting with \( n \) the number of superfields (or equivalently the number of real components of suitably defined spinor supercharges), this holds for \( n > 16 \). In such a case, the non-linear sigma model of scalars is unique, and the dimension of the symmetric space \( \frac{n}{2} \) counts the number of scalar fields of the gravity multiplet. The isometry group \( H_R \) is nothing but the R-symmetric of the \( N \)-extended supersymmetry algebra, where \( N \) is the number of supercharges. The non-compact global isometry group \( G \) is uniquely determined by the number of scalar fields and by the fact that \( G \) is a non-compact, real form of a simple (finite-dimensional) Lie group \( G_C \), whose axion contains a compact subgroup (m, s, with m symmetric embedding, understood throughout) is \( H_R \). In \( d = 3, 4 \) and 5 space-time dimensions (which are the only ones we deal with in the present contribution) the \( G \)-symmetry is \( SO(N), U(N) \) and \( USp(N) \) respectively, depending on whether the spinors are real (R), complex (C) or quaternionic (H) [1]. For \( d = 3 \), \( m_{ax} = 16 \), whereas for \( d = 4 \) and \( 5 \), \( m_{ax} = 8 \) (the only even for \( d = 5 \)). In all cases the axion number of real components of the spinor supercharges is \( m_{ax} = 32 \) [2,3].

Thus, \( N \)-extended supergravity is unique in \( n = 32 \), while the uniqueness of the theory breaks down for \( n \leq 16 \). Nevertheless, for \( 8 < n < 16 \) the non-linear sigma model, also containing the scalars from the additional matter multiplets coupled to the supergravity one, are still described by symmetric spaces of the form \( \frac{m}{2} \sim \frac{m}{2} \), where \( H_M \) is a classical compact Lie group depending on the theory under consideration. Once again, the non-compact global isometry group \( G_M \) is uniquely fixed by the number of scalar fields and by the fact that \( G_M \) is a non-compact, real form of a simple (finite-dimensional) Lie group associated to \( G_C \) whose axions is \( H_M \) [4].

In all the aforementioned cases, the signature of the co-sets manifold is (negatively) Euclidean, i.e., we are dealing with Riemannian (globally) symmetric spaces [4,5].

The considered supergravity theories are invariant under \( G \) (or \( G_M \)) and contain a supergroup, as well as under general coordinate di electromagnets, as space-time. Fermion fields are assigned to a suitable representation of \( H_R \) (or \( H_M \)), while spin-1 vector fields are in a suitable representation of \( G_M \). Among the treated cases, if \( d = 3, 4 \), an exception is given by \( d = 4 \), in which case \( G_{M(1)} \) may mix electric and magnetic spin-1 fields strengths’ components, and the equations of matter – not the Lagrangian density – are invariant under \( G_M \), This phenomenon is nothing but the generalization of the electromagnetic duality of Maxwell equations, in which \( G = SL(2; R) \) (or \( SO(2; 1) \)) \( SU(1; 1) \) Spin (2; 1), with m = U (1), the electric field and the magnetic field transform as a real spinor (doublet) of \( G \).

2 Classificaton of Irreducible Riemannian Globally Symmetric Spaces

Irreducible Riemannian globally symmetric spaces (of the type I and type III in Helgason’s classification; see [4,5], denoted by the acronym IRS in the treatment given below, are those symmetric spaces with (strictly) negative definite metric metrics. They have the form \( \frac{m}{2} \), where \( G \) is a non-compact, real form of a simple (finite-dimensional) Lie group \( G_C \), and \( H \) is its mcs (with m symmetric embedding; \( H \) is also often referred to as the stabilizer of the coset). These are seven classical (finite) sequences, as well as twenty exceptional isolated cases (in which \( G_C \) is an exceptional Lie group).

Furthermore, another class of symmetric spaces exists, with the form \( \frac{m}{2} \), where \( G \) is any complex (non-compact) (semi-)simple Lie group regarded as a real group, and \( G_R \) is its real form, real form (m cs \( G_C = G_R \)). \( \frac{m}{2} \) is a Riemannian symmetric space with \( \dim_R = \dim_R(G_R) \), and rank \( = \text{rank}(G_R) \). A remarkable family of such spaces are provided by the manifolds \( SO(4,1), \) with \( G_R = SO(3) \sim SU(2) \) and \( G_C = SL(2; C) \). (Note, see e.g. [4]). Such a space is not quaternionic, despite having SU(2) as stabilizer; consistently, its real dimension is 3 (not a multiple of 4, as instead it holds for all
quaternionic manifolds; see below). On the other hand, as yielded by the treatment of Sect. 3, the unique example of such a class playing a role in supergravity theories is the IRG S $SU(3)|SU(3)$ (SU (3) = m cs (SL (3C)) $\mathbb{C}^3$ $\mathbb{C}^3$), which is both the real special symmetric vector multiplets’ scalar manifold in $N = 2, d = 5$ supergravity based on the Jordan algebra of degree three $J_3^g$, and the non-BPS Z 6 0 moduli space of the corresponding theory in $d = 4$, obtained by reduction along a spacelike direction (see Table 4).

Let us recall here that the symmetric nature of a coset (i.e., homomorphic) manifold can be deduced in purely algebraic terms through the so-called Cartan’s decom position of the Lie algebra $g$ of a Lie group $G$:

$$g = h \cdot k$$

where $h$ is the Lie algebra of a compact subgroup of $G$, and $k$ can be identified with the tangent space at the identity coset. The homogeneous space $\frac{g}{h}$ is symmetric if the three following properties hold (see e.g. [4,5,7]):

$$[h,h] = 0$$

The first property (from the left) holds by definition of subgroup. The second property holds in general in coset spaces, and it means that by the adjoint, $h$ acts on $k$ as a representation $R$ with $\dim_R (R) = \dim_h (h)$. The third property defines the symmetricity of the space under consideration, since in general $h$ is symmetric, and $k$ is symmetric.

All IRG S are Einstein spaces (see e.g. [8,9] and Refs. therein), thus with constant (negative) scalar curvature.

Moreover, one can deduce the rank of an IRG S is defined as the maximal dimension (in $R$) of a subgroup $\tilde{H}$ at (i.e. with vanishing Riemann tensor), totally geodesic submanifold of the IRG S itself (see e.g. $x_6$, page 209 of [3]).

In the following treatent Kähler [10], special Kähler [11,27], real special [12,13,24,28] and quaternionic [14,15,29,30,31,32,33,34,35] manifolds are denoted by $K$, $S U$, $R S$ and $H$, respectively. The role played by such spaces in supergravity is outlined in Sect. 4.

Tables 1 and 2 respectively list the seven classical in finite sequences and the twelve exceptional isolated cases (see e.g. Table II of [2]). Some observations are listed below (other properties are given in, or can be inferred from, Tables 3-11):

$\mathbb{I}$ is $S K$

$\mathbb{I}$ is not $H$, despite having SO (3) $\rightarrow$ SU (2) as stabilizer; consistently, its real dimension is 10 (not a multiple of 4, as instead it holds for all $H$ manifolds)

$\mathbb{I}^{I\beta}$ is both $H$ and $K$ (quaternionic Kähler). In particular, $\mathbb{I}^{I\beta} = \mathbb{I}^{I\beta}$, which is both $H$ and $K$, with $\dim_R = 4$, $\dim_h = 1$, and it is an example of Kähler space with self-dual Weyl curvature [36]

$\mathbb{I}^{I\beta} = \mathbb{I}^{I\beta}$ is $K$, but not $H$, despite having SO (3) $\rightarrow$ SU (2) $\rightarrow$ U (1) as stabilizer; consistently, its real dimension is 6 (not a multiple of 4)

$\mathbb{I}^{I\beta}$ is both $H$ and $K$ (quaternionic Kähler)

$\mathbb{I}^{I\beta}$ is $K$, but not $H$, despite having U (2) as stabilizer. Through the isomorphism SO (4) $\rightarrow$ SU (2) $\rightarrow$ SL (2R) [3], it holds that $\mathbb{I}^{I\beta} = \mathbb{I}^{I\beta}$, with real dimension 2 (not a multiple of 4)

$\mathbb{I}^{I\beta}$ is $K$, but not $H$, despite having U (2) as stabilizer. Through the isomorphism SO (3) $\rightarrow$ Sp (4R) [3], it holds that $\mathbb{I}^{I\beta} = \mathbb{I}^{I\beta}$

$\mathbb{I}^{I\beta}$ is both $H$ and $K$ (quaternionic Kähler)

$\mathbb{I}^{I\beta}$ is $K$, but not $H$, despite having U (2) as stabilizer. Through the isomorphism SO (3) $\rightarrow$ Sp (4R) [3], it holds that $\mathbb{I}^{I\beta} = \mathbb{I}^{I\beta}$

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3 Irreducible Riemannian Globally Symmetric Spaces in Supergravity

Supergravity is a theory which combines general covariance (dynamically induced supersymmetry) with local supersymmetric theory (superlocal induced supersymmetry). It contains a tetrad (VIeinstein) one-form e^a and a gravitino (spinor valued) one-form \( \lambda \) (\( \lambda = 1;\ldots; N \)), which for instance appear in the Einstein-Hilbert Lagrangian \( R \ e^a e^b \) and \( R \) respectively being the Levi-Civita and Riemann tensors, or in the Rarita-Schwinger Lagrangian \( \epsilon^{\alpha\beta\gamma} \ e^a \wedge \ e^b \wedge \ e^c \) (denoting the appropriate set of gamma matrices). The Lagrangians of the gauge fields are of the form \( (R e^a) F^a F^a \) and \( (\bar{\psi} e^a \psi) F^a \), where \( N \) is a complex symmetric kinetic vector manifold.

Symmetric spaces already occur in gravity, regardless of supersymmetric theory. A simple example is provided by the Kaluza-Klein reduction of D-dimensional gravity on a manifold

\[
M_D = M_d \ M_{D-d},
\]

where the internal manifold is here taken to be a d-dimensional torus (i.e., \( M_d = T^d \)) for simplicity's sake. For small size of \( T^d \), the Kaluza-Klein reduction of pure gravity as given by Eq. (3.1) yields a self-dual D-dimensional gravity coupled to \( \frac{1}{2} \) scalar fields and d Maxwell fields (gravitons). The scalar fields parametrize (as coordinates) the manifold \( \Sigma_0(2,1) \) odd; i.e., odd the overall size of \( T^d \), one obtains the IRG S (see Table 1), which is the simplest example of symmetric space occurring in gravity.

Supersymmetric theory restricts the holonomy group of \( \sigma \)-dimensional spaces which may occur in a given theory (see e.g., [3,4]). Let us consider for instance supergravity theories in \( d = 4 \) space-time dimensions. The geometry of the scalar manifold depends on the number \( N \) of supercharges: it is \( \mathbb{R}^N \) for \( N = 1 \), \( \mathbb{S}_{d} \) (for vector multiplets' scalars) or H \( [2,3] \), \( [3,4] \) (for hypermultiplets' scalars) for \( N = 2 \), and in general symmetric for \( N > 2 \). Concerning \( N = 2 \) supergravity in \( d = 5 \) and \( d = 3 \) space-time dimensions, the vector multiplets' scalar manifolds are endowed with RS \( [2,5] \), \( [3,5] \) and H \( [2,1], [2,3] \), \( [2,4] \), \( [3,1], [3,2] \), \( [3,3] \) geometric, respectively. The isolated cases of \( N = 1 \) symmetric (SK) manifolds are given by the so-called magic \( N = 2 \) supergravities (see Table 3). They are related to Freundenthal triple system \\(^{\prime}\), \( \mathbb{G}_2 \) (and their over the simple Euclidean Jordan algebras \( [2,3] \), \( [4,4] \), \( [4,2] \) of degree three with irreducible symmetric form, namely over the Jordan algebras \( J_3 \), \( J_3 \), and \( J_3 \) of magic 3 3 matrices over the four division algebras, i.e., respectively over the octonions (O), quaternions (H), complex numbers (C) and real numbers (R). Furthermore, they are all connected to the Magic Square of Freundenthal, Ruzenfeld and Tits (see also, for recent treatment, [35,36]). Jordan algebras were introduced and completely classified in [37] in an attempt to generalize quantum mechanics beyond the \( \mathbb{C} \) of complex number C.

The scalar manifold of \( N = 2 \) pure supergravities in \( d = 3,4,5 \) are all symmetric of the form \( \mathbb{C}^d \), where, as anticipated in the Introduction, \( \mathbb{A}_d \) is nothing but the automorphism group of the related \( N \)-extended, d-dimensional superalgebra, usually named R-symmetric group. As mentioned in the Introduction, in \( d = 3,4,5 \) the R-symmetric theory is SO(\( N \)), \( U(N) \) and USp(\( N \)) respectively, depending on whether the spinors are real, complex or quaternionic (see e.g., Table 2 of [38]). Since from group representation theory the number of scalar fields in the corresponding supergravity multiplet is known (being related to the relevant Clifford algebra — see e.g., [39]), the global symetny group \( \mathbb{A}_d \) is determined uniquely, at least locally.

A set of Tables shows the role played by IRG S in supergravities with \( N \) supercharges in \( d = 3,4,5 \) space-time dimensions.

Table 3 presents the relation among \( N = 2, d = 4 \) symmetric SK vector multiplets' scalar manifold and the symmetric H manifold of the corresponding \( d = 3 \) theory obtained by space-like dimensional reduction (or equivalently of the \( \mathbb{R}^{4} \)-hypermultiplets' scalar manifold), given by the so-called c-map [39]. The c-map of symmetric SK manifolds gives the whole set of symmetric H manifolds, the unique existence being the quaternionic projective spaces \( H^{n}/P \) introduced above: they are symmetric H manifolds which are not the c-map of any symmetric SK space. Furthermore, all symmetric SK manifolds but the complex projective spaces \( CP^{n} \) (and thus, through c-map, any other H manifolds exist, such as the homogenous non-symmetric ones studied in [39] and the (rather general, not necessarily homogenous) ones given by the c-map of general USK geometries (they are not complexly general, because they are endowed with \( 2n + 4 \) isom etries, if the corresponding SK geometry has \( \mathbb{H}^2_{n+1} \)). All H manifolds are Einstein, with constant (negative) scalar curvature (see e.g., [39,39]).
all symmetric H manifolds but HP^0) are related to a Jordan algebra of degree three. In Table 3 R

denotes the one-dimensional Jordan algebra, whereas m = n stands for the Jordan algebra of degree
two with a quadratic form of Lorentzian signature (m, n), which is nothing but the Clifford algebra
of (m, n) [59]. Further, one finds here worth pointing out that the theory with 8 supersymmetry
based on the Jordan algebra J^2 is dual to the supergravity with 24 supersymmetry, in d = 3 + 4; 5
dimensions: they share the same scalar manifold and the same number (and representation) of
vector fields (see e.g. [56, 73], and Refs. therein).

Table 4 lists the moduli spaces associated to non-degenerate non-BPS Z = 0 extremal black hole
attractors in N = 2, d = 4, 5 symmetric vector multiplets' scalar manifolds [60]. They are nothing
but the N = 2, d = 5 RS symmetric vector multiplets' scalar manifolds. Only another class of
N = 2, d = 5 RS symmetric vector multiplets' scalar manifolds exists, namely the infinite sequence
IV_{1,n} = \frac{1}{2\sin^{1/2} \pi n}; n \geq 2 N, usually denoted by L (1m = 2) in the classification of
homogeneous d-spaces [12]. It corresponds to homogeneous non-symmetric scalar manifolds in d = 4 (SK)
or 3 (H) space-time d dimensions (see e.g. Table 4 of [12]).

In general, an extremal black hole attractor is associated to a (stable) critical point of a suitably
defined black hole effective potential VB,H, and it describes a scalar con guration, stabilized purely in
term of the conserved electric and magnetic charges at the event horizon, regardless of the values of
the scalars at spatial in nity. This is due to the Attractor mechanism [61, 62], an important dynamical
phenomenon in the theory of gravitational objects, which naturally appears in modern theories of
gravity, such as supergravity, superstrings [65] or M-theory [69, 71].

In homogeneous (not necessarily symmetric) scalar manifolds \( \mathbb{R} \), the horizon attractor con gurations
of the scalar elds are supported by non-degenerate orbits (i.e. orbits with non-vanishing classical
entropy) of the representation of the charge vector in the group G, which can thus be used in order
to classify the various typologies of attractors. A complete classification of the (non-degenerate) charge
orbits \( \mathbb{O} \) has been performed for all supergravities based on symmetric scalar manifolds in d = 4 and 5
dimensions [44, 52, 60, 72] [77]. In such a framework, the charge orbits \( \mathbb{O} \) are homogeneously
(generally non-symmetric) manifolds (with Lorentzian signature) of the form \( \mathbb{H} \), where \( \mathbb{H} \) is some proper
subgroup of \( \mathbb{G} \). If \( \mathbb{H} \) is non-compact, then a moduli space can be associated to the charge orbit (and thus to the
appropriate class of attractors): it is an IRG of the form \( \mathbb{H} \), where \( \mathbb{H} = m c s (\mathbb{H}) \) (with symmetric
embedding) [60, 73, 79]. The moduli space \( \mathbb{H} \) is spanned by those scalar degrees of freedom which are not
stabilized in term of charges at the event horizon of the considered extremal black hole. In other words,
\( \mathbb{H} \) describes the at directions of the relevant VB,H at the considered class of non-degenerate attractors.
Within such a framework, the fact that in N = 2, d = 4, 5 supergravity the \( \frac{1}{2} \)-BPS attractors stabilize
all scalars at the event horizon can be traced back in the case of symmetric vector multiplets' scalar
manifolds, to the com pactness of the stabilizer \( \mathbb{H} = \frac{1}{2} \)-BPS of the corresponding \( \frac{1}{2} \)-BPS supporting charge
orbit \( \mathbb{O} = \frac{1}{2} \)-BPS = \( \mathbb{H} \).

Recent studies [83] suggest that the moduli spaces of non-degenerate attractors do not exist only
at the event horizon of the considered extremal black hole, but rather they can be extended (with no
changes) all along the corresponding attractor ow, i.e. all along the evolution dynamics of the scalar
elds (determined by the scalar equations of motion), from the spatial in nity \( r = 1 \) to the near-
horizon geometry (\( r = r_h \)); \( r \) and \( r_h \) being the radial coordinate and the radius of the event horizon,
respectively. However, such moduli spaces are not expected to survive the quantum corrections to the
classical geometry of the scalar manifolds, as con ned (at least in some black hole charge con gurations)
in [85].

Turning back to Table 4, \( \mathbb{H} \) denotes the non-commutative stabilizer of the corresponding
charge orbit \( \mathbb{O} = \frac{1}{2} \)-BPS = 0 [74], and \( \mathbb{H} \) is its m cs (with symmetric embedding).

Table 5 presents the moduli spaces of non-BPS Z = 0 critical points of VB,M,Z in N = 2, d = 4
 symmetric vector multiplets' scalar manifolds [60]. They are (non-special) Kaluza symmetric
manifolds. \( \mathbb{H} \) denotes the non-commutative stabilizer of the corresponding charge orbit
\( \mathbb{O} = \frac{1}{2} \)-BPS = 0 [74], and \( \mathbb{H} \) is its m cs (with symmetric embedding). Remarkably, \( \frac{1}{2} \)-BPS = \( \mathbb{H} \) is associ ated to M_{1,2} (O), which is another exceptional Jordan triple system, generated by 2, 1
\[
\text{H} \text{emitsian matries over the octonions } \mathcal{O}, \text{ found in } [33, 25]. \text{ Furthermone, } \frac{E(10)}{SO(10)}(1) \text{ is also the scalar manifold of } N = 10, d = 3 \text{ supergravity (see Table 11 below, and Table 2 of [35], as well).}
\]

Table 6 contains the scalar manifolds of \( N > 3 \)-extended, \( d = 4 \) supergravities. \( J_3^4 \) denotes the Jordan algebra of degree three over the split form \( \mathcal{O}_3 \) of the octonions (see e.g. [23] and Refs. therein for further, and recent, developments). Remkably, \( \mathcal{M}_{12}(0) \) is also associated to \( N = 5, d = 4 \) supergravity (see Table 2 of [35], and Refs. therein).

Table 7 lists the moduli spaces of non-degenerate extremal black hole attractors in \( 3 \leq N \leq 8, d = 4 \) supergravities \([60, 87], [76, 78]\). Here, \( H \) and \( \mathcal{H} \) respectively are the m cos’s (with symmetric embedding) of \( H \) and \( \mathcal{H} \), which in turn are the non-compact stabilizers of the corresponding supporting charge orbits \( O_{1-N, \text{BPS}}, O_{\text{non-BPS}} \) respectively \([44, 56, 60], [74, 78]\) (see Table 1 of [78]). It is here worth recalling that all non-degenerate \( \mathcal{H}^{-}\text{BPS} \) moduli spaces \( \mathcal{H}^{-}\) (see Table 7) and \( \mathcal{H}^{6,7}\) (see Table 10) of \( 8 > N > 2 \)-extended supergravities in \( d = 4,5 \) space-time dimensions are \( H \)-manifolds. This has a nice interpretations in terms of \( N \) ! 2 supersymmetry reduction : the flat directions of \( \mathcal{V}_{H,N} \) at the considered class of its (non-degenerate) critical points correspond to the would-be hyperkähler scalar degrees of freedom in the vector/hyper splitting determined by the \( N \) ! 2 supersymmetry reduction \([81, 89], [77, 60, 76]\).

Table 8 shows the moduli spaces of non-degenerate non-BPS (\( Z \neq 0 \)) critical points of \( \mathcal{V}_{H,N}^{-2} \) in \( N = 2, d = 5 \) RS symmetric vector multiplets’ scalar manifolds \([60]\). \( \mathcal{H}_5 \) stands for the non-compact stabilizer of the corresponding supporting charge orbit \( O_{\text{non-BPS}} \) and \( \mathcal{H}_2 \) is its \( m \) cos (with symmetric embedding).

Table 9 lists the scalar manifolds of \( N > 2 \)-extended, \( d = 5 \) supergravities.

Table 10 presents the moduli spaces of extremal black hole attractors with non-vanishing classical entropy in \( 4 \leq N \leq 8 \)-extended, \( d = 5 \) supergravities \([74, 60, 79]\). \( h_5 \) and \( H_5 \) respectively are the \( m \) cos’s (with symmetric embedding) of \( H_5 \) and \( \mathcal{H}_5 \), which in turn are the non-compact stabilizers of the corresponding supporting charge orbits \( O_{1-N, \text{BPS}} \) and \( O_{\text{non-BPS}} \) respectively \([44, 74, 56, 77, 60]\).

Finally, Table 11 contains the scalar manifolds of \( N > 5, d = 3 \) supergravities \([29]\).

As yielded by Tables 3-11, all topologies of IRG’s appear at least once in supergravity theories with \( N \) supercharges in \( d = 3; 4; 5 \) space-time dimensions (as scalar manifolds, or as moduli spaces associated to the various classes of extremal black hole attractors with non-vanishing classical entropy).

Let us now consider the supergravities with 8 supersymmetry metrics associated to the Jordan algebras of degree three \( J_3^4 \) over the four division algebras \( A = R, C, H \) and \( \mathcal{O} \), shortly called m agic supergravities, in \( d = 3, 4 \) and \( 5 \) space-time dimensions. By recalling the Tables 3, 4, 5 and 8 and recalling the definition \( \text{dim}_R (A) = 1; 2; 4; 8 \) (for \( A = R, C, H \) and \( \mathcal{O} \) respectively) (see Table 3), one gets that \([32]\)

\[
\begin{align*}
\text{dim}_R M_{J_3^4} & = 3A + 7 \quad (d) \quad (3.2) \\
\text{dim}_R F_{J_3^4} & = 2A \quad (d) \\
\text{dim}_d & = \text{dim}_R ; \text{dim}_C ; \text{dim}_H : \quad (d = 3) \quad (d = 4) \quad (d = 5) \\
\end{align*}
\]

In Eq. \((3.2)\), \( M_{J_3^4} \) denotes the scalar manifold of the supergravity theory with 8 supersymmetry associated to \( J_3^4 \) in \( d = 3; 4; 5 \) space-time dimensions. In Eq. \((3.3)\), \( F_{J_3^4} \) stands for the set of non-BPS \( Z = 0 \) moduli spaces of symmetrical \( J_3^4 \)-related SK manifolds (see Table 5), and \( F_{J_3^4} \) is the set of non-BPS \( Z = 0 \) moduli spaces of symmetric \( J_3^4 \)-related RS manifolds (see Table 8). Let us now consider the nine sequence (for \( A = R, C, H \)-related symmetric \( d = 4 \) SK manifolds \( \text{III}_1 \), \( \text{IV}_2 \), \( B_3 \) (Table 3), as well as its c-map sequence \( \text{IV}_4 \), \( B_3 \) (Table 3) and the
12 and 14, respectively to our knowledge.

It is interesting to notice that a constant, through a construction based on minimal Jordan orbitals and symplectic induction [97], related Jordan algebras of degree four to IGS $\tilde{R}$, in which $G$ is a particular
non-com pact real form of a simple exceptional (nine-di mensional) Lie group, and $K$ is its (sym metrically em bedded) m cs. The IRGS $G_2$ appearing in Kostant's construction (sum m arized by Table in page 422 of [23]), reported below in Table 15) are two H man ifolds, which are the c-map of the so-called t $^4$ m ode l ($G = G(2)$) and of the real magic $N = 2, d = 4$ supergravity ($G = F(4)$) [23], respectively based on the Jordan algebras $R$ (degree one) and $J_{10}^H$ (degree three), as well as the scalar man ifolds of mag ic supergravity in $d = 3j_3$ space-time di mensions ($G = E_8(8), E_7(7), E_6(6)$ respectively), based on $J_3^{0+}$. Through sym plectic induction [39], they are connected to som e com pact sym metric K ahler spaces $X = \frac{R^8}{H_K}$ being som e proper (sym metrically em bedded) com pact subgroup of $K$. $X$ is related to a Jordan algebra $J (X)$, with $\dim (X) = 2 \dim (J (X))$. For $G = G(2)$, this is a Jordan algebra of degree two, whereas in all other cases it has degree four. Consistently with previous notation, in Table 15 $J_4^H, J_6^H$ respectively denote the Jordan algebras of degree four with irreducible norm from $s, m$ ade by Hermitian 4 4 matrices over $R, C$ and $H$. It is worth rem arking here that $X$ has an associated (still K ahler) sym metric non-com pact form $X = \frac{R^8}{H_K}$, which is an IRGS, with $K = G$. Furtherm ore, $X$ is unique, because only one non-com pact, real form $K$ of $K$ exists, such that $K = G$ and $m$ cs ($K = H_K$ (see e.g. [5]). Notice also that rank ($X) = \text{rank } (X)$ is also the degree of the correspon ding $J (X)$. It is us ing to observe that $\dim (X) = 2 \dim (J (X))$ is also the real dimension of the representation of the type of the Abelian vector mx strengths (and of their dual) in $N = 2, d = 4$ magic supergravities over $O, R, C$ and $R$, as well as of the so-called t $^4$ m ode l [49, 50, 53, 44, 44]. It would be interesting to study further such a construction, and determine the origin of the IRGS $X$ in supergravity.

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References

[1] J. A. Strathe ice, Extended Poincare Supersymmetry, Int. J. Mod. Phys. A 2, 273 (1987); P. Deligne, Notes on Spinors, in : Quantum Fields and Strings: a Course for Mathematicians (American Mathematical Society, Providence, 1999); R. D'Auria, S. Ferrara, M. A. Lledo and V. S. Varadarajan, Spinor Algebra, J. Geom. Phys. 40, 101 (2001), hep-th/0010124.

[2] S. Ferrara, J. Scherk and B. Zumino, Supergravity and Local Extended Supersymmetry, Phys. Lett. B 66, 35 (1977).

[3] S. Ferrara, J. Scherk and B. Zumino, Algebraic Properties of Extended Supergravity Theories, Nucl. Phys. B 121, 393 (1977).

[4] S. Haldjason, Differential Geometry, Lie Groups and Sym metric Spaces (Academic Press, New York, 1978).

[5] R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications (Dover Publications, 2006).

[6] M. K. Gaillard and B. Zumino, Duality Rotations for Interacting Fields, Nucl. Phys. B 193, 221 (1981).

[7] R. Slansky, Group Theory for Unified Model Building, Phys. Rep. 79, 1 (1981).
(8) L. Castellani, R. D'Auria, P. Fre, "Supergravity and superstrings: A geometric perspective", 3 Vols. (World Scientific, Singapore), 1991.

(9) P. Fre and P. Sorani, "The N = 2 wonderland: From Calabi-Yau manifolds to topological fi brillations" (World Scientific, Singapore), 1995.

(10) B. Zumino, "Supersymmetry and Kähler manifolds, Phys. Lett. B 87, 203 (1979).

(11) E. Cremmer and A. Van Proeyen, "Classification of Kähler manifolds in N = 2 Vector Multiplet Supergravity Couplings, Class. Quant. Grav. 2, 445 (1985).

(12) B. de Wit and F. Vanderseypen and A. Van Proeyen, "Symmetry Structures of Special Geometries, Nucl. Phys. B 400, 463 (1993), hep-th/9210068.

(13) B. de Wit and A. Van Proeyen, "Special geometry, cubic polynomials and homogenous quaternionic spaces, Commun. Math. Phys. 149, 307 (1992), hep-th/9112027.

(14) S. Cecotti, "N = 2 Supergravity, Type IIB Superstrings and Algebraic Geometry, Commun. Math. Phys. 131, 517 (1990).

(15) L. Castellani, R. D'Auria and S. Ferrara, "Special Kähler Geometry: An Intrinsic Formulation From N = 2 Space-Time Supersymmetry, Phys. Lett. B 241, 57 (1990).

(16) L. Castellani, R. D'Auria and S. Ferrara, "Special geometry without special coordinates, Class. Quant. Grav. 7, 1767 (1990).

(17) R. D'Auria, S. Ferrara and P. Fre, "Special and quaternionic isom etries: General couplings in N = 2 supergravity and the scalar potential, Nucl. Phys. B 359, 705 (1991).

(18) B. de Wit and A. Van Proeyen, "Hidden symmetries, special geometry and quaternionic manifolds, Int. J. Mod. Phys. D 3, 31 (1994), hep-th/9310067.

(19) B. de Wit and A. Van Proeyen, "Special geometry and symplectic transformations, Nucl. Phys. Proc. Suppl. 45B, 196 (1996), hep-th/9510186.

(20) L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre, T. Magri, N = 2 Supergravity and N = 2 SuperYang-Mills Theory on General Scalar Manifolds: Symplectic Covariance, gaugings and the momentum map, J. Geom. Phys. 23, 111 (1997), hep-th/9605032.

(21) Daniel S. Freed, "Special Kähler manifolds, Commun. Math. Phys. 203, 31 (1999), hep-th/9712042.

(22) B. Craps, F. Roose, W. Troost and A. Van Proeyen, "What is special Kähler geometry?, Nucl. Phys. B 503, 565 (1997), hep-th/9703082.

(23) P. Fre, "Lectures on special Kähler geometry and electric -- magnetic duality rotations, Nucl. Phys. Proc. Suppl. 45B, 59 (1996), hep-th/9512043.

(24) Z. Lu, "A note on special Kähler manifolds, Math. Ann. 313, 711 (1999).

(25) A. Strominger, "Special Geometry, Commun. Math. Phys. 133, 163 (1990).

(26) B. de Wit and A. Van Proeyen, "Potentials and Symmetries of General Gauged N = 2 Supergravity: Yang-Mills Models, Nucl. Phys. B 245, 89 (1984).

(27) M. Lledo, "Special geometry of d = 4,5 supergravity, talk given at the Conference on Symmetry in Mathematics and Physics, IPAM, UCLA, Los Angeles, 18-20 Jan. 2008.

(28) B. de Wit and A. Van Proeyen, "Broken sigma models isometries in very special geometry, Phys. Lett. B 293, 94 (1992), hep-th/9207091.

(29) B. de Wit, A. K. Tolstsen and H. Nicolai, "Locally supersymmetric D = 3 nonlinear sigma models, Nucl. Phys. B 392, 3 (1993), hep-th/9208074.

(30) J. Bagger and E. Witten, "Matter Couplings in N = 2 Supergravity, Nucl. Phys. B 222, 1 (1983).
[31] C. K. Zachos, N = 2 Supergravity Theory With A Gauged Central Charge, Phys. Lett. B 76, 329 (1978).

[32] P. Breitenlohner and M. F. Sohnius, Super fields, Auxiliary Fields, And Tensor Calculus For N = 2 Extended Supergravity, Nucl. Phys. B 165, 483 (1980).

[33] S. Salam on, Invent. Math. 67, 143 (1982).

[34] J. A. Wolf, J. Math. Mech. 14, 1033 (1965).

[35] D. V. Alekseevskii, Math. USSR Izv. 9, 297 (1975).

[36] S. Ferrara and S. Sabharwal, Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces, Nucl. Phys. B 332, 317 (1990).

[37] S. Ishihara, Quaternion Kahlerian manifolds, J. Differential Geom. 9, 483 (1974).

[38] F. Gürsey and H. C. Tze, Complex And Quaternionic Analyticity In Chiral And Gauge Theories. Part 1, Annals Phys. 128, 29 (1980).

[39] M. Gunaydin, G. Sierra and P. K. Townsend, The Geometry of N = 2 Maxwell-Einstein Supergravity and Jordan Algebras, Nucl. Phys. B 242, 244 (1984).

[40] M. Gunaydin, G. Sierra and P. K. Townsend, Exceptional Supergravity Theories and the Magic Square, Phys. Lett. B 133, 72 (1983).

[41] H. Freudenthal, Proc. Konink. Ned. Akad. Wetenschap A 62, 447 (1959).

[42] H. Freudenthal, Adv. Math. 1, 145 (1964).

[43] M. Gunaydin, K. Koepsell and H. Nicolai, Conformal and quasiconformal realizations of exceptional Lie groups, Comm. Math. Phys. 221, 57 (2001), hep-th/0008063.

[44] S. Ferrara and M. Gunaydin, Orbits of Exceptional Groups, Duality and BPS States in String Theory, Int. J. Mod. Phys. A 13, 2075 (1998), hep-th/9708025.

[45] M. Gunaydin, Unitary realizations of U-duality groups as conformal and quasiconformal groups and extremal black holes of supergravity theories, AIP Conf. Proc. 767, 268 (2005), hep-th/0502235.

[46] M. Gunaydin and O. Pavlyk, Generalized spacetimes defined by cubic forms and the minimal unitary realizations of their quasiconformal groups, JHEP 0508, 101 (2005), hep-th/0506010.

[47] M. Gunaydin, G. Sierra and P. K. Townsend, Gauging the d = 5 Maxwell-Einstein Supergravity Theories: More on Jordan Algebras, Nucl. Phys. B 253, 573 (1985).

[48] M. Gunaydin, G. Sierra and P. K. Townsend, More on d = 5 Maxwell-Einstein Supergravity: Symmetric Space and Kinks, Class. Quant. Grav. 3, 763 (1986).

[49] P. Jordan, J. Von Neumann and E. Wigner, On an algebraic generalization of the quantum mechanical formalism, Ann. Math. 35, 29 (1934).

[50] N. Jacobson, Ann. Math. Soc. Coll. Publ. 39 (1968).

[51] M. Gunaydin, Exceptional Realizations of Lorentz Group: Supersymmetries and Leptons, Nuovo Cimento A 29, 467 (1975).

[52] M. Gunaydin, C. Piron and H. Ruegg, Moufang Plane and Octonionic Quantum Mechanics, Comm. Math. Phys. 61, 69 (1978).

[53] B. A. Rozhenfeld, Dokl. Akad. Nauk. SSSR 106, 600 (1956).

[54] J. Tits, Mem. Acad. Roy. Belg. Sci. 29, fasc. 3 (1955).
[55] B. Pioline, Lectures on black holes, topological strings and quantum attractors, Lectures delivered at the RTN W inter School on Strings, Supergravity and Gauge Theories, Geneva, Switzerland, 16-20 Jan 2006, Class. Quant. Grav. 23, S981 (2006), hep-th/0607227.

[56] S. Ferrara, E.G. Gimon and R. Kallosh, Mag ic supergravities, N = 8 and black hole com p osites, Phys. Rev. D 74, 125018 (2006), hep-th/0606211.

[57] M. Rio, Jordan Algebras and Extremal Black Holes, based on talk given at the 26th International Colloquium on Group Theoretical Methods in Physics (ICGMP26), New York, 26-30 June 2006, hep-th/0703238.

[58] K. Dasgupta, V. Hussain and A. Wissani, Quaternionic Kahler manifolds, Confined Instantons and the Magic Square, Int. J. Mod. Phys. B 793, 34 (2008), arXiv:0708.1023.

[59] S. Cecotti, S. Ferrara and L. Girardello, Geometry of Type II Superstrings and the M oduli of Superconformal Field Theories, Int. J. Mod. Phys. A 4, 2475 (1989).

[60] S. Ferrara and A. Marrani, On the M oduli Space of non-BPS Attractors for N = 2 Symmetric Manifolds, Phys. Lett. B 652, 111 (2007), arXiv:0706.1567.

[61] S. Ferrara, R. Kallosh, A. Strominger, N = 2 extremal black holes, Phys. Rev. D 52, 5412 (1995).

[62] S. Ferrara, R. Kallosh, Supersymmetric attractors, Phys. Rev. D 54, 1514 (1996); S. Ferrara, R. Kallosh, Universality of supersymmetric attractors, Phys. Rev. D 54, 1525 (1996).

[63] A. Strominger, M acroscopic entropy of N = 2 extremal black holes, Phys. Lett. B 383, 39 (1996).

[64] S. Ferrara, G. W. Gibbons and R. Kallosh, Black Holes and Critical Points in Moduli Space, Nucl. Phys. B 500, 75 (1997), hep-th/9702103.

[65] For reviews on black holes in superstring theory see e.g.: J. M. Maldacena, Black Holes in String Theory, hep-th/9607235; A. W. Peet, TASI lectures on black holes in string theory, arXiv:hep-th/0008241; A. Dabholkar, Black hole entropy and attractors, Class. Quant. Grav. 23, S957 (2006).

[66] For recent reviews see: J. H. Schwarz, Lectures on superstring and M-theory dualities, Nucl. Phys. Proc. Suppl. 55, 1 (1997); M. J. Du, M-theory (the theory formerly known as strings), Int. J. Mod. Phys. A 11, 5623 (1996); A. Sen, Unification of string dualities, Nucl. Phys. Proc. Suppl. 58, 5 (1997).

[67] J. H. Schwarz and A. Sen, Duality symmetries of 4D heterotic strings, Phys. Lett. B 312, 105 (1993); J. H. Schwarz and A. Sen, Duality Symmetries of M-theory, Nucl. Phys. B 411, 35 (1994).

[68] M. Gasperini, J. M. Maldacena and G. Veneziano, From trivial to non-trivial conformal string backgrounds via 0 (dR) transformations, Phys. Lett. B 272, 277 (1991); J. M. Maldacena and J. H. Schwarz, Nocom pact Symmetries in String Theory, Nucl. Phys. B 390, 3 (1993).

[69] E. Witten, String Theory Dynamics in Various Dimensions, Nucl. Phys. B 443, 85 (1995).

[70] J. H. Schwarz: M-theory extensions of T duality, arXiv:hep-th/9601077; C. Vafa, Evidence for F-theory, Nucl. Phys. B 469, 403 (1996).

[71] K. Becker, M. Becker and J. H. Schwarz, String theory and M-theory: A modern introduction", Cam bridge University Press (Cam bridge, UK), 2007.

[72] S. Ferrara and J. M. Maldacena, Branes, central charges and U-duality invariant BPS conditions, Class. Quant. Grav. 15, 749 (1998), hep-th/9706097.

[73] H. Lu, C. N. Pope and K. S. Stelle, Multiplet structures of BPS solitons, Class. Quant. Grav. 15, 537 (1998), hep-th/9708109.

[74] S. Bellucci, S. Ferrara, M. Gunaydin and A. Marrani, Charge orbits of symmetric special geometry and attractors, Int. J. Mod. Phys. A 21, 5043 (2006), hep-th/0606209.
[75] S. Ferrara and M. Gunaydin, Orbits and attractors for \( N = 2 \) Maxwell-Einstein supergravity theories in five dimensions, Nucl. Phys. B 759, 1 (2006), hep-th/0606108.

[76] L. Andrianopoli, R. D' Auria, S. Ferrara and M. Trigiante, Extremal black holes in supergravity, Lect. Notes Phys. 737, 661 (2008), hep-th/0611345.

[77] S. Ferrara and A. Marrani, \( N = 8 \) non-BPS attractors, Fixed Scalars and Magic Supergravities, Nucl. Phys. B 788, 63 (2008), arXiv:0705.3866.

[78] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, Extremal black hole and flux vacua attractors, contribution to the Proceedings of Winter School on Attractor Mechanism (SAM 2006), Frascati, Italy, 20-24 Mar 2006, [arXiv:0711.4547].

[79] L. Andrianopoli, S. Ferrara, A. Marrani and M. Trigiante, Non-BPS attractors in 5d and 6d Extended Supergravity, Nucl. Phys. B 795, 428 (2008), arXiv:0709.3488.

[80] R. Kallosh, N. Sivanandam and M. Soroush, Exact Attractive non-BPS STU black holes, Phys. Rev. D 74, 065008 (2006), hep-th/0606263.

[81] K. Hotta and T. Kubota, Exact Solutions and the Attractor Mechanism in Non-BPS Black Holes, Prog. Theor. Phys. 110, 5, 969 (2007), arXiv:0707.4554.

[82] E. G. Giannoni, F. Larsen and J. Simon, Black Holes in Supergravity: the non-BPS Branch, JHEP 0801, 040 (2008), arXiv:0710.4967.

[83] R. G. Cai and D. W. Pang, A Note on exact solutions and attractor mechanism for non-BPS black holes, JHEP 0801, 046 (2008), arXiv:0712.0217.

[84] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, Supergravities and Octonion Models, arXiv:0807.3503.

[85] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, Quantum Lift of Non-BPS Flat Directions, to appear.

[86] M. Gogberashvili, Rotations in the Space of Split Octonions, arXiv:0808.2496.

[87] L. Andrianopoli, R. D'Auria and S. Ferrara, U invariants, black hole entropy and fixed scalars, Phys. Lett. B 403 12 (1997), hep-th/9703156.

[88] L. Andrianopoli, R. D'Auria and S. Ferrara, Five-dimensional U duality, black hole entropy and topological invariants, Phys. Lett. B 411, 39 (1997), hep-th/9705024.

[89] L. Andrianopoli, R. D'Auria and S. Ferrara, Supersymmetry reduction of \( \mathcal{N} = 4 \) extended supergravities in four dimensions, JHEP 0203, 025 (2002), hep-th/0110277.

[90] M. Bianchi and S. Ferrara, Enriques and Octonionic Magic Supergravity Models, JHEP 0802, 054 (2008), arXiv:0712.2976.

[91] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Second quantized mirror symmetry, Phys. Lett. B 361, 59 (1995), hep-th/9505162.

[92] J. A. Harvey and G. W. Moore, Exact gravitational threshold correction in the FHSV model, Phys. Rev. D 57, 2329 (1998), hep-th/9611176.

[93] P. S. Aspinwall, An \( N = 2 \) Dual Pair and a Phase Transition, Nucl. Phys. B 460, 57 (1996), hep-th/9510142.

[94] A. Klemm and M. Mariño, Counting BPS states on the Enriques Calabi-Yau, Commun. Math. Phys. 280, 27 (2008), hep-th/0512227.

[95] J. R. David, On the dyon partition function in \( N = 2 \) theories, JHEP 0802, 025 (2008), arXiv:0711.1971.

[96] G. L. Cardoso, B. de Wit and S. Mahapatra, Subleading and non-holomorphc corrections to \( N = 2 \) BPS black hole entropy, arXiv:0808.2627.
[97] B. Kostant, Minimal Coadjoint Orbits and Symplectic Induction in: "The Breadth of Symplectic and Poisson Geometry", Prog. Math. 232, 391 (Birkhauser, Boston, 2003).
| IRGS Classical Sequence | rank | \( \dim_R \) |
|------------------------|------|----------------|
| \( I_n (A I) \) | \( \frac{SL(n, \mathbb{R})}{SO(n)} \) | \( n \) | \( \frac{1}{2}(n-1)(n+2) \) |
| \( II_n (A II) \) | \( \frac{SU(2n)}{USp(2n)} \) | \( n \) | \( (n-1)(2n+1) \) |
| \( III_{pR} (A III) \) | \( \frac{SU(p, q)}{USp(2n)} \) \( K \) | \( m \) \( \min \) (\( p, q \)) | \( 2pq \) |
| \( IV_{pR} (B D I) \) | \( \frac{SO(p, q)}{SO(p) \cdot SO(q)} \) \( K \) | \( m \) \( \min \) (\( p, q \)) | \( pq \) |
| \( V_n (D III) \) | \( \frac{SO(2n)}{SO(n)} \) \( K \) | \( \frac{1}{2} \) | \( n(n+1) \) |
| \( V_{I_n} (C I) \) | \( \frac{Sp(2n)}{SO(n)} \) \( K \) | \( n \) | \( n(n+1) \) |
| \( V_{II_{pR}} (C II) \) | \( \frac{USp(2p, 2q)}{USp(2p) \cdot USp(2q)} \) \( K \) | \( m \) \( \min \) (\( p, q \)) | \( 4pq \) |
| \( V_{III_G} \) (see text) | \( \frac{G}{G_R} \) | rank (\( G \)) | \( \dim_R (G) \) |

Table 1: Classical Infinite Sequences of Irreducible Riemann Globally Symmetric Spaces of type \( I \) and type \( III \) (IRGS) (see e.g. Table II of [4] and Table 9.3 of [5]). The notation of Helgason's classification [4] is reported in brackets in the \( n \)st column. Trivially, it holds that \( III_{pR} = III_{qR}, IV_{pR} = IV_{qR} \) and \( V_{II_{pR}} = V_{II_{qR}} \).
| Exceptional Case | rank | dim R |
|------------------|------|-------|
| 1 (E I)          | 6    | 42    |
| 2 (E II)         | 4    | 40    |
| 3 (E III)        | 2    | 32    |
| 4 (E IV)         | 2    | 26    |
| 5 (E V)          | 7    | 70    |
| 6 (E VI)         | 4    | 64    |
| 7 (E VII)        | 3    | 54    |
| 8 (E VIII)       | 8    | 128   |
| 9 (E IX)         | 4    | 112   |
| 10 (F I)         | 4    | 28    |
| 11 (F II)        | 1    | 16    |
| 12 (G)           | 2    | 8     |

Table 2: Exceptional Isolated Cases of IRGS (see e.g. Table II of [4] and Table 9.3 of [5]). The notation of Helgason’s classification [4] is reported in brackets in the first column. The subscript number in brackets denotes the character of the considered real form, defined as # non-compact generators # compact generators (see e.g. Eq. (1.29), p. 332, as well as Table 9.3, of [5]). Concerning the compact form of (finite-dimensional) exceptional Lie groups, the following alternative notations exist: $G_2$, $G_2(14)$, $F_4$, $F_4(52)$, $E_6$, $E_6(78)$, $E_7$, $E_7(133)$, and $E_8$, $E_8(248)$ (in other words, for a compact form $= \dim R$).
| Special Kähler Space | Quaternionic Space |
|----------------------|--------------------|
| III_{1,\mu} CP^n : SU(1,\mu) U(1) ; n 2 N | III_{2,\mu+1} : SU(2,\mu+1) SU(2) U(1) ; n 2 N (f0g) |
| III_{1,\mu} IV_{2,\mu} : SU(1,\mu) SO(2,\mu) U(1) ; n 2 N (R n 1\mu) | IV_{4,\mu+2} : SO(4,\mu+2) SO(4) SO(2) U(1) ; n 2 N (f0g; 1g 6R n 1\mu) |
| V I_3 : Sp(6) U(1) J_R^3 | 12 : Sp(6) SO(4) (R) |
| III_{1,\beta} : SU(4,\beta) U(1) J_C^3 | 10 : Sp(6) SU(2) J_R^3 |
| V_6 : SO(12) SU(6) U(1) J_H^3 ; N = 2 , N = 6 | 2 : E_{6(4)} SU(6) SO(4) J_C^3 |
| 7 : E_{6(2)} SU(2) J_3^O | 9 : E_{6(2)} SU(2) J_3^O |

Table 3: N = 2, d = 4 symmetric special Kähler vector multiplets' scalar manifolds and the corresponding symmetric quaternionic spaces, obtained through c-map [59]. In general, starting from a special Kähler geometry with dim C = n, the c-map generates a quaternionic manifold with dim H = n + 1 [59]. If any, the related Jordan algebras of degree three are reported in brackets throughout (the notation of [55] is used, see also Table 2 therein). By defining \( \dim R A = 1, 2, 4, 8 \) for \( A = \mathbb{R} \), \( C \), \( H \), \( O \) respectively), the complex dimension of the N = 2, d = 4 symmetric special Kähler manifolds based on \( J_3^C \) is 3A + 3 [80]. Thus, the quaternionic dimension of the corresponding N = 2, d = 4 symmertic quaternionic manifolds obtained through c-map is 3A + 4 [59, 90].
Table 4: Moduli spaces of non-BPS $Z_6=0$ critical points of $V_{BH}$ in $N=2$, $d=4$ special Kähler symmetric vector multiplets' scalar manifolds. They are nothing but the $N=2$, $d=5$ real symmetric vector multiplets' scalar manifolds. $b_H$ is the non-compact stabilizer of the corresponding supporting charge orbit $O_{non-BPS}Z_6=0$ [74], and $b_h$ is its maximal compact subgroup (with symmetric embedding). As observed in [68], the real dimension of $N=2$, $d=5$ real symmetric vector multiplet manifolds based on $J^h_3$ is $3A+2$. 

| Associated Jordan Algebra of degree three (in $d=5$) | $\frac{n}{n}$ |
|---------------------------------------------------|---------------|
| $R_{n \in 1: n 2 N}$ | SO(1;l) IV $\frac{1}{n 1} : SO(1;l)$ SO(1;m) SO(n-1) |
| $J^D_3$ | $4 : \frac{E_{6(6)}}{F_4}$ |
| $J^R_3$ | II $\frac{3}{2} : SU(2)$ |
| $J^C_3$ | V $\frac{3}{2} : \frac{SU(6)}{SO(3)}$ |
| $J^R_3$ | I$_3 : \frac{SL(3;R)}{SO(3)}$ |
Jordan Algebra of degree three (of the corresponding scalar manifold in $d = 4$)

\[
\begin{array}{|c|}
\hline
\text{Jordan Algebra} & \frac{\mathbb{H}}{k} = \frac{\mathbb{H}}{\mathbb{U}(1)} \\
\hline
\text{III}_{2\mu} : & SU(2\mu,1) \backslash SU(n,1); \text{SK} \ (H \ for \ n = 3) \\
\hline
R \quad n \geq 1, n > 3 & IV_{2\mu} : \frac{SO(2\mu,2)}{SO(2) \times SO(n,2)} \ (H \ for \ n = 6) \\
\hline
J_3^O & 3 : \frac{E_6(14)}{SO(10) \times U(1)} \\
\hline
J_3^E & \text{III}_{4\mu} : \frac{SU(4\mu)}{SU(4) \times SU(2) \times U(1)}; \ H \\
\hline
J_3^C & (\text{III}_{2\mu})^2 : \frac{SU(2\mu)}{SU(2) \times U(1)} \times \frac{SU(1\mu)}{SU(1) \times U(1)}; \text{SK} \ H \\
\hline
J_3^R & \text{III}_{2\mu} : \frac{SU(2\mu)}{SU(2) \times U(1)} \times \text{SK} \ H \\
\hline
\end{array}
\]

Table 5: Moduli spaces of non-BPS $Z = 0$ critical points of $V_{BPS} g_\mu$ in $N = 2, d = 4$ special Kähler symmetric vector multiplets’ scalar manifolds [60]. Unless otherwise noted, they are non-special Kähler symmetric manifolds. $\mathbb{H}$ is the non-compact stabilizer of the corresponding supporting charge orbit $O_{non-BPS} = 0$ [74], and $\mathbb{R}$ is its maximal compact subgroup (with symmetric embedding). As observed in [60], the complex dimension of the moduli spaces of non-BPS $Z = 0$ critical points of $V_{BPS} g_\mu$ in $N = 2, d = 4$ special Kähler symmetric manifolds based on $J_3^E$ is 2$A$. 

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| $N$ | $G_{N \mu} = H_{N \mu}$ |
|-----|------------------|
| 3   | III$_{3\mu} : \frac{SU(3)}{SU(1) \cdot U(1)} : n \geq 2$ N |
| 4   | III$_{4\mu}$ IV$_{6\mu} : \frac{SU(1\mu)}{U(1)} : \frac{SO(6\mu)}{SO(6) \cdot SO(2 \cdot n)} : n \geq 2$ N [ f0g (R $\times n \cdot 1\mu$) |
| 5   | III$_{5\mu} : \frac{SU(1\beta)}{SU(5) \cdot U(1)} (M = 2 \cdot 0)$ |
| 6   | V$_{6} : \frac{SO(12)}{SO(6) \cdot U(1)} J^{R}_{3}$ |
| 8   | 5 : $\frac{SU(8)}{SU(6) \times U(1)} J^{D}_{3}$ |

Table 6: Scalar manifolds of $N > 3$, $d = 4$ supergravities. Notice that the scalar manifold of $N = 6$ supergravity coincides with the one of $N = 2$ supergravity based on $J^{H}_{3}$ (see Table 3).
| N  | Moduli spaces of extremal black hole attractors with non-vanishing classical entropy in 36N8, d=4 supergravities |  |  |
|----|--------------------------------------------------------|----------------|----------------|
| 3  | III2µ : SU(2µ)                                        | non-BPS, ZAB = 0 | non-BPS, ZAB = 0 |
|    | n 2 N                                                  | m moduli space $\frac{h}{h}$ | m moduli space $\frac{h}{h}$ |
| 4  | IV 4µ : SO(4µ)                                        | SO(1;1)         | IV 5µ 1 : |
|    | n 2 N                                                  | SO(1;1)         | SO(1;1) |
|    |                                                        | SO(5)           | SO(5) |
|    |                                                        | SO(m)           | SO(m) |
|    |                                                        | m moduli space $\frac{h}{h}$ | m moduli space $\frac{h}{h}$ |
| 5  | III2µ : SU(2µ)                                        | SU(2µ)          | SU(2µ) |
| 6  | III4µ : SU(4µ)                                        | SU(4µ)          | SU(4µ) |
|    | n 2 N                                                  | SU(4µ)          | SU(4µ) |
| 8  | 2 : E6(2)                                              | 1 : E6(1)       | 1 : E6(1) |
|    | SU(6)                                                  | SU(3)           | SU(3) |
|    | SU(2)                                                  | SU(2)           | SU(2) |

Table 7: Moduli spaces of extremal black hole attractors with non-vanishing classical entropy in 36N8, d=4 supergravities [87, 76, 77, 60, 78]. (see Table 1 of [78]). h, $\mathbb{R}$, and $\mathbb{F}$ respectively are the maximal compact subgroups (with symmetric embedding) of H, $\mathbb{R}$, and $\mathbb{F}$, which in turn are the non-compact stabilizers of the corresponding supporting charge orbits $O_{1-N}$ BPS, $O_{\text{non-BPS}} = 0$ and $O_{\text{non-BPS}} = 0$, respectively [44, 74, 58, 76, 77, 63, 78] (see Table 1 of [78]).
Table 8: Moduli spaces of non-BPS \((\mathbb{Z} \neq 0)\) critical points of \(V_{BH} \neq -2\) in \(N = 2, d = 5\) real special symmetric vector multiplets’ scalar manifolds \([60]\). \(\mathbb{H}_5\) is the non-compact stabilizer of the corresponding supporting charge orbit \(O_{non-BPS} [60]\), and \(\mathbb{K}_5\) is its maximal compact subgroup (with symmetric embedding). As observed in \([60]\), the real dimension of the moduli spaces of non-BPS \((\mathbb{Z} \neq 0)\) critical points of \(V_{BH} \neq -2\) in \(N = 2, d = 5\) real symmetric manifolds based on \(J^A_3\) is \(2A\), and the stabilizer of such moduli spaces contains the group \(Spin(1 + A)\).

| \(N\) | \(G_{BH} = H_{N, \rho}\) |
|------|-----------------|
| 4    | \(SO(1;1)\) IV \(\rho = \{SO(1;1)\} \) | \(SO(1;1)\) IV \(\rho = \{SO(1;1)\} \) |
| 6    | \(\mathbb{II}_3 : \{SU(6)\} J^C_3\) |
| 8    | \(1 : \frac{E_{8(d)}}{Usp(8)} J^O_3\) |

Table 9: Scalar manifolds of \(N > 2, d = 5\) supergravities. Notice that, also for \(d = 5\), the scalar manifold of \(N = 6\) supergravity coincides with the one of \(N = 2\) supergravity based on \(J^O_3\) (see Table 4).
| N | \(\frac{1}{N}\)-BPS moduli space \(H_N^{1/2}\) | non-BPS \((Z_{AB} \neq 0)\) moduli space \(H_N^{1/2}\) |
|---|---|---|
| 4 | IV \(4\) : \(SO(4); SO(6)\); \(n > 2\) | IV \(5\) : \(SO(5); SO(6)\); \(n > 3\) |
| 6 | V\(II\) \(\beta\) : \(\frac{USp(4;2)}{USp(4) \times USp(2)}\) | |
| 8 | 10 : \(\frac{F_{4(4)}}{USp(6) \times USp(2)}\) | |

Table 10: Moduli spaces of extremal black hole attractors with non-vanishing classical entropy in 46 N = 8, d = 5 supergravities \([77,60,79]\). \(h_5\) and \(h_6\) respectively are the maximal compact subgroups (with symmetric embedding) of \(H_5\) and \(H_6\), which in turn are the non-compact stabilizers of the corresponding supporting charge orbits \(O_{1-N\ BPS}\) and \(O_{non\ BPS}\), respectively \([44, 15, 55, 77, 60, 79]\).

| N | \(G_{N \ 2} = H_{N \ 2}\) |
|---|---|
| 5 | V\(II\) \(\beta\) : \(\frac{USp(4;2)}{USp(4) \times USp(2)}\); \(n \geq 2\) |
| 6 | III\(I\) \(\beta\) : \(\frac{SU(4;2)}{SU(4) \times SU(2) \times U(1)}\); \(n \geq 2\) |
| 8 | IV \(8; n + 2\) : \(SO(8); SO(8)\); \(n \geq 2\) \(\{0; 1g; (R, n, 1)\}\) |
| 9 | 11 : \(\frac{F_{4(4)}}{SU(9)}\) |
| 10 | 3 : \(\frac{E_{6(6)}}{SO(10) \times SO(2)}\); \(M \ 1\ 2\ (O)\) |
| 12 | 6 : \(\frac{E_{7(7)}}{SU(12) \times SU(12)}\); \(J^H_3\) |
| 16 | 8 : \(\frac{E_{8(8)}}{SU(16) \times SU(16)}\); \(J^O_3\) |

Table 11: Scalar manifolds of \(N > 5, d = 3\) supergravities \([29]\). Notice that the scalar manifold of \(N = 12\) supergravity coincides with the one of \((N = 4)\) supergravity based on \(J^O_3\) (see Table 3).
Table 12: \( d = 5 \) Exceptional sequence [90]. Trivially, all manifolds of such a Table are real, and they also all are \( RS \) but the sequence \( F_{5,\mu_5^A}^{\lambda=R,\mathcal{C} \neq D} \), which is new.

| A   | \( M_{5,\mu_5^A} \)       | \( B_{5,\mu_5^A} \)       | \( F_{5,\mu_5^A} \)       |
|-----|--------------------------|--------------------------|--------------------------|
| O   | \( \frac{E_{6(14)}^{\pm}}{F_4} \) | \( SO(1;1) \) | \( SO(1;\beta) \) | \( \sigma_0 (\beta) \) |
| H   | \( SU(4) \) \( U_{sp}(6) \) | \( SO(1;1) \) | \( SO(1;\beta) \) | \( \sigma_0 (\beta) \) |
| C   | \( SU(4) \) \( U_{sp}(6) \) | \( SO(1;1) \) | \( SO(1;\beta) \) | \( \sigma_0 (\beta) \) |
| R   | \( SU(4) \) \( U_{sp}(6) \) | \( SO(1;1) \) | \( SO(1;\beta) \) | \( \sigma_0 (\beta) \) |

Table 13: \( d = 4 \) Exceptional sequence [90]. All manifolds of such a Table are \( K \), and they also all are \( SK \) but \( F_{4,\mu_4^A}^{\lambda=R,\mathcal{C} \neq D} \). The sequence \( F_{4,\mu_4^A}^{\lambda=R,\mathcal{C} \neq D} \) has been obtained in [58] through constrained instantons.
Table 14: $d = 3$ Exceptional sequence [90]. All manifolds of such a Table are $H$. The sequence $F_{3\mu}^\pm_{\lambda=\rho}$ is new.

|   | $M_{3\mu}^\pm$ | $B_{3\mu}$ | $F_{3\mu}^\pm$ |
|---|----------------|-----------|---------------|
| O | $E_{6,24}$ | $SO(4\mu)$ | $SO(12)$ $SU(4)$ | $6 : \frac{E_{7,51}}{SO(12)} SU(2)$ $H$ |
| H | $SO(12)$ $SU(2)$ | $SO(4\mu)$ | $SO(8)$ $SO(4)$ | $IV_{4\mu} : \frac{SO(4\mu)}{SO(8)} SU(4)$ $H$ |
| C | $SU(6)$ $SU(2)$ | $SO(4\mu)$ | $SO(6)$ $SO(4)$ | $III_{4\mu} : \frac{SU(4\mu)}{SU(6)} SU(2) U(1)$ $H$ |
| R | $USp(6)$ $SU(2)$ | $SO(4\mu)$ | $SU(6)$ $SO(4)$ | $V_{III_{4\mu}} H^R : \frac{USp(4\mu)}{USp(6)} USp(2)$ $H$ |

Table 15: Some particular IRG S $\frac{G}{H}$ and their associated compact spaces $X$ (along with their unique non-compact (IRG S $X$), and the corresponding Jordan algebra $J (X)$). The relation among $\frac{G}{H}$ and $X$ is based on minimal coadjoint orbits and symplectic induction, and it is due to Kostant [97].

| $\frac{G}{H}$ | $X$ | $X$ | $\dim R (X)$ | $\text{rank} (X)$ | $J (X)$ |
|---|---|---|---|---|---|
| $G_{2/1}$ | $SU(2)$ $SU(2)$ | $SU(2)$ | $SU(14)$ | 2 | 4 | 2 | $R$ $R$ |
| $F_{4/4}$ | $SU(2)$ $USp(6)$ | $SU(2)$ $USp(6)$ | $SU(14)$ | 14 | 3 | 3 | $R$ $J^R_3$ |
| $F_{4/4}$ | $USp(6)$ $USp(6)$ | $USp(6)$ $USp(6)$ | $SU(14)$ | 20 | 4 | 4 | $J^R_4$ |
| $E_{6/4}$ | $SU(2)$ $SU(4)$ $U(1)$ | $SU(2)$ $SU(4)$ $U(1)$ | $SU(14)$ | 32 | 4 | 4 | $J^C_4$ |
| $F_{4/4}$ | $SO(16)$ $SU(8)$ | $SO(16)$ $SU(8)$ | $SU(14)$ | 56 | 4 | 4 | $J^R_4$ |