ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES

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Abstract. Let

\[ M_1 f(x, y) : = \frac{3}{4} f(x + y) - \frac{1}{4} f(-x - y) + \frac{1}{4} f(x - y) + \frac{1}{4} f(y - x) - f(x) - f(y), \]
\[ M_2 f(x, y) : = 2 f \left( \frac{x + y}{2} \right) + f \left( \frac{x - y}{2} \right) + f \left( \frac{y - x}{2} \right) - f(x) - f(y). \]

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities

\[ N (M_1 f(x, y) - \rho M_2 f(x, y), t) \geq \frac{t}{t + \varphi(x, y)} \] (0.1)

and

\[ N (M_2 f(x, y) - \rho M_1 f(x, y), t) \geq \frac{t}{t + \varphi(x, y)} \] (0.2)

in fuzzy Banach spaces, where ρ is a fixed real number with ρ ≠ 1.

1. Introduction and Preliminaries

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 16, 37]. In particular, Bag and Samanta [3], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 19, 20] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1 ([3, 19, 20, 21]). Let X be a real vector space. A function \( N : X \times \mathbb{R} \to [0, 1] \) is called a fuzzy norm on X if for all \( x, y \in X \) and all \( s, t \in \mathbb{R} \),

\[ N (x, y) = \frac{t}{t + \varphi(x, y)} \]

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(N_1) \(N(x, t) = 0\) for \(t \leq 0\);

(\text{N}_2) \(x = 0\) if and only if \(N(x, t) = 1\) for all \(t > 0\);

(\text{N}_3) \(N(cx, t) = N(x, \frac{t}{|c|})\) if \(c \neq 0\);

(\text{N}_4) \(N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}\);

(\text{N}_5) \(N(x, \cdot)\) is a non-decreasing function of \(\mathbb{R}\) and \(\lim_{t \to \infty} N(x, t) = 1\).

(\text{N}_6) for \(x \neq 0\), \(N(x, \cdot)\) is continuous on \(\mathbb{R}\).

The pair \((X, N)\) is called a \textit{fuzzy normed vector space}.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [19, 20].

**Definition 1.2** ([3, 19, 20, 21]). Let \((X, N)\) be a fuzzy normed vector space. A sequence \(\{x_n\}\) in \(X\) is said to be \textit{convergent} or \textit{converge} if there exists an \(x \in X\) such that \(\lim_{n \to \infty} N(x_n - x, t) = 1\) for all \(t > 0\). In this case, \(x\) is called the \textit{limit} of the sequence \(\{x_n\}\) and we denote it by \(N\text{-lim}_{n \to \infty} x_n = x\).

**Definition 1.3** ([3, 19, 20, 21]). Let \((X, N)\) be a fuzzy normed vector space. A sequence \(\{x_n\}\) in \(X\) is called \textit{Cauchy} if for each \(\varepsilon > 0\) and each \(t > 0\) there exists an \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) and all \(p > 0\), we have \(N(x_{n+p} - x_n, t) > 1 - \varepsilon\).

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be \textit{complete} and the fuzzy normed vector space is called a \textit{fuzzy Banach space}.

We say that a mapping \(f : X \to Y\) between fuzzy normed vector spaces \(X\) and \(Y\) is continuous at a point \(x_0 \in X\) if for each sequence \(\{x_n\}\) converging to \(x_0\) in \(X\), then the sequence \(\{f(x_n)\}\) converges to \(f(x_0)\). If \(f : X \to Y\) is continuous at each \(x \in X\), then \(f : X \to Y\) is said to be \textit{continuous} on \(X\) (see [4]).

The stability problem of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms.

The functional equation \(f(x + y) = f(x) + f(y)\) is called the \textit{Cauchy equation}. In particular, every solution of the Cauchy equation is said to be an \textit{additive mapping}. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [2] for additive mappings and by Rassias [28] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.
The functional equation \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The stability of quadratic functional equation was proved by Skof [35] for mappings \( f : E_1 \rightarrow E_2 \), where \( E_1 \) is a normed space and \( E_2 \) is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain \( E_1 \) is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 5, 6, 7, 10, 17, 18, 22, 25, 26, 27, 29, 30, 31, 32, 33, 34, 38, 39]).

Park [23, 24] defined additive \( \rho \)-functional inequalities and proved the Hyers-Ulam stability of the additive \( \rho \)-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that \( X \) is a real vector space and \( (Y, N) \) is a fuzzy Banach space. Let \( \rho \) be a real number with \( \rho \neq 1 \).

2. ADDITIVE-QUADRATIC \( \rho \)-FUNCTIONAL INEQUALITY (0.1)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.1) in fuzzy Banach spaces.

We need the following lemma to prove the main results.

**Lemma 2.1.**

(i) If an odd mapping \( f : X \rightarrow Y \) satisfies

\[
M_1 f(x, y) = \rho M_2 f(x, y)
\]

for all \( x, y \in X \), then \( f \) is the Cauchy additive mapping.

(ii) If an even mapping \( f : X \rightarrow Y \) satisfies \( f(0) = 0 \) and (2.1), then \( f \) is the quadratic mapping.

**Proof.** (i) Letting \( y = x \) in (2.1), we get \( f(2x) - 2f(x) = 0 \) and so \( f(2x) = 2f(x) \) for all \( x \in X \). Thus

\[
f \left( \frac{x}{2} \right) = \frac{1}{2} f(x)
\]

for all \( x \in X \).
It follows from (2.1) and (2.2) that
\[ f(x + y) - f(x) - f(y) = \rho \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right) \]
\[ = \rho (f(x + y) - f(x) - f(y)) \]
and so
\[ f(x + y) = f(x) + f(y) \]
for all \( x, y \in X \).

(ii) Letting \( y = x \) in (2.1), we get
\[ \frac{1}{2} f(2x) - 2f(x) = 0 \]
and so \( f(2x) = 4f(x) \) for all \( x \in X \). Thus
\[
(2.3) \quad f \left( \frac{x}{2} \right) = \frac{1}{4} f(x)
\]
for all \( x \in X \).

It follows from (2.1) and (2.3) that
\[
\frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y)
\]
\[ = \rho \left( 2f \left( \frac{x + y}{2} \right) + 2f \left( \frac{x - y}{2} \right) - f(x) - f(y) \right) \]
\[ = \rho \left( \frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y) \right) \]
and so
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]
for all \( x, y \in X \).

\[ \square \]

**Theorem 2.2.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that
\[
\sum_{j=1}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) < \infty
\]
for all \( x, y \in X \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying
\[
N \left( M_1 f(x, y) - \rho M_2 f(x, y), t \right) \geq \frac{t}{t + \varphi(x, y)}
\]
for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[
N \left( f(x) - A(x), t \right) \geq \frac{t}{t + \frac{1}{2} \Psi(x, x)}
\]
for all \( x \in X \) and all \( t > 0 \), where \( \Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \).
(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (2.5). Then \( Q(x) := N \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[
N \left( f(x) - Q(x), t \right) \geq \frac{t}{t + \frac{1}{2} \Phi(x, x)}
\]

for all \( x \in X \) and all \( t > 0 \), where \( \Phi(x, y) := \sum_{j=1}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \) for all \( x, y \in X \).

Proof. (i) Letting \( y = x \) in (2.5), we get

\[
N \left( f(2x) - 2f(x), t \right) \geq \frac{t}{t + \varphi(x, x)}
\]

and so

\[
N \left( f(x) - 2f\left( \frac{x}{2} \right), t \right) \geq \frac{t}{t + \varphi\left( \frac{x}{2}, \frac{x}{2} \right)}
\]

for all \( x \in X \). Hence

\[
N \left( 2^l f \left( \frac{x}{2^l} \right) - 2^m f \left( \frac{x}{2^m} \right), t \right)
\]

\[
\geq \min \left\{ N \left( 2^l f \left( \frac{x}{2^l} \right) - 2^l f \left( \frac{x}{2^{l+1}} \right), t \right), \cdots, \right. \]

\[
N \left( 2^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 2^m f \left( \frac{x}{2^m} \right), t \right) \left. \right\}
\]

\[
= \min \left\{ N \left( f \left( \frac{x}{2^l} \right) - 2 f \left( \frac{x}{2^{l+1}} \right), t \right), \cdots, N \left( f \left( \frac{x}{2^{m-1}} \right) - 2 f \left( \frac{x}{2^m} \right), t \right) \right\}
\]

\[
\geq \min \left\{ \frac{t}{2^l + \varphi \left( \frac{x}{2^l}, \frac{x}{2^{l+1}} \right)}, \cdots, \frac{t}{2^{m-1} + \varphi \left( \frac{x}{2^{m-1}}, \frac{x}{2^m} \right)} \right\}
\]

\[
= \min \left\{ \frac{t}{t + 2^l \varphi \left( \frac{x}{2^{l+1}}, \frac{x}{2^{l+2}} \right)}, \cdots, \frac{t}{t + 2^{m-1} \varphi \left( \frac{x}{2^m}, \frac{x}{2^{m+1}} \right)} \right\}
\]

\[
\geq \frac{t}{t + \frac{1}{2} \sum_{j=l+1}^{m} 2^j \varphi \left( \frac{x}{2^j}, \frac{x}{2^{j+1}} \right)}
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \) and all \( t > 0 \). It follows from (2.4) and (2.9) that the sequence \( \{2^n f \left( \frac{x}{2^n} \right)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^n f \left( \frac{x}{2^n} \right)\} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := N \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
\]
for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.6).

By (2.5),

$$N \left( 2^n \left( f \left( \frac{x+y}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right) - \rho \left( 2^{n+1} f \left( \frac{x+y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right) \right), 2^n t \right) \geq \frac{t}{t + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}$$

for all $x, y \in X, \forall t > 0$ and all $n \in \mathbb{N}$. So

$$N \left( 2^n \left( f \left( \frac{x+y}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right) - \rho \left( 2^{n+1} f \left( \frac{x+y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right) \right), t \right) \geq \frac{t}{\frac{t}{2^n} + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)} = \frac{t}{t + 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}$$

for all $x, y \in X, \forall t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n \to \infty} \frac{t}{t + 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)} = 1$ for all $x, y \in X$ and all $t > 0,$

$$A(x+y) - A(x) - A(y) = \rho \left( 2A \left( \frac{x+y}{2} \right) - A(x) - A(y) \right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive.

(ii) Letting $y = x$ in (2.5), we get

$$N \left( \frac{1}{2} f(2x) - 2f(x), t \right) \geq \frac{t}{t + \varphi(x,x)}$$

(2.10)

and so

$$N \left( f(x) - 4f \left( \frac{x}{2} \right), t \right) \geq \frac{\frac{t}{2}}{\frac{t}{2} + \varphi \left( \frac{x}{2}, \frac{x}{2} \right)} = \frac{t}{t + 2\varphi \left( \frac{x}{2}, \frac{x}{2} \right)}$$

for all $x \in X$. Hence

$$N \left( 4^n f \left( \frac{x}{2^n} \right) - 4^m f \left( \frac{x}{2^m} \right), t \right) \geq \min \left\{ N \left( 4^nf \left( \frac{x}{2^n} \right) - 4^{n+1} f \left( \frac{x}{2^{n+1}} \right), t \right), \ldots, \right\}$$

(2.11)
\[
= \min \left\{ N \left( f \left( \frac{x}{2^l} \right) - 4f \left( \frac{x}{2^{l+1}} \right) \right), \ldots, N \left( f \left( \frac{x}{2^{m-1}} \right) - 4f \left( \frac{x}{2^m} \right) \right) \right\}
\geq \min \left\{ \frac{t}{4^l} + 2\varphi \left( \frac{x}{2^l}, \frac{x}{2^{l+1}} \right), \ldots, \frac{t}{4^m} + 2\varphi \left( \frac{x}{2^m}, \frac{x}{2^{m-1}} \right) \right\}
= \min \left\{ \frac{t}{t+2 \cdot 4^l \varphi \left( \frac{x}{2^l}, \frac{x}{2^{l+1}} \right), \ldots, \frac{t}{t+2 \cdot 4^m \varphi \left( \frac{x}{2^m}, \frac{x}{2^{m-1}} \right)} \right\}
\geq \frac{t}{t + \frac{1}{2} \sum_{j=l}^{m} 4^j \varphi \left( \frac{x}{2^j}, \frac{x}{2^{j+1}} \right)}
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \) and all \( t > 0 \). It follows from (2.4) and (2.11) that the sequence \( \{ 4^n f \left( \frac{x}{2^n} \right) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ 4^n f \left( \frac{x}{2^n} \right) \} \) converges. So one can define the mapping \( Q : X \to Y \) by

\[
Q(x) := \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right)
\]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.11), we get (2.7).

The rest of the proof is similar to the above additive case. \( \square \)

**Corollary 2.3.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 2 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying

\[
N \left( M_1 f(x, y) - \rho M_2 f(x, y), t \right) \geq \frac{t}{t + \theta \left( \| x \|^p + \| y \|^p \right)}
\]
for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
N \left( f(x) - A(x), t \right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta \| x \|^p}
\]
for all \( x \in X \) and all \( t > 0 \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (2.12). Then \( Q(x) := \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[
N \left( f(x) - Q(x), t \right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 4\theta \| x \|^p}
\]
for all \( x \in X \) and all \( t > 0 \).
Proof. The proof follows from Theorem 2.2 by taking \( \varphi(x, y) = \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \), as desired. \( \square \)

**Theorem 2.4.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that
\[
\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty
\]
for all \( x, y \in X \).
(i) Let \( f : X \to Y \) be an odd mapping satisfying (2.5). Then \( A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[
N(f(x) - A(x), t) \geq \frac{t}{t + \frac{1}{2} \Phi(x, x)}
\]
for all \( x \in X \) and all \( t > 0 \), where \( \Phi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) \) for all \( x, y \in X \).
(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (2.5). Then \( Q(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that
\[
N(f(x) - Q(x), t) \geq \frac{t}{t + \frac{1}{2} \Psi(x, x)}
\]
for all \( x \in X \) and all \( t > 0 \), where \( \Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y) \) for all \( x, y \in X \).

Proof. (i) It follows from (2.8) that
\[
N\left( f(x) - \frac{1}{2} f(2x), \frac{1}{2} t \right) \geq \frac{t}{t + \varphi(x, x)}
\]
and so
\[
N\left( f(x) - \frac{1}{2} f(2x), t \right) \geq \frac{2t}{2t + \varphi(x, x)} = \frac{t}{t + \frac{1}{2} \varphi(x, x)}
\]
for all \( x \in X \) and all \( t > 0 \).
(ii) It follows from (2.10) that
\[
N\left( f(x) - \frac{1}{4} f(2x), \frac{1}{2} t \right) \geq \frac{t}{t + \varphi(x, x)}
\]
and so
\[
N\left( f(x) - \frac{1}{4} f(2x), t \right) \geq \frac{2t}{2t + \varphi(x, x)} = \frac{t}{t + \frac{1}{2} \varphi(x, x)}
\]
for all \( x \in X \) and all \( t > 0 \).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

**Corollary 2.5.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( 0 < p < 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \).
(i) Let \( f : X \to Y \) be an odd mapping satisfying (2.12). Then \( A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
N (f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2\theta \|x\|^p}
\]

for all \( x \in X \) and all \( t > 0 \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (2.12). Then \( Q(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[
N (f(x) - Q(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 4\theta \|x\|^p}
\]

for all \( x \in X \) and all \( t > 0 \).

Proof. The proof follows from Theorem 2.4 by taking \( \varphi(x, y) := \theta (\|x\|^p + \|y\|^p) \) for all \( x, y \in X \), as desired. \( \square \)

3. Additive-Quadratic \( \rho \)-Functional Inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic \( \rho \)-functional inequality (0.2) in fuzzy Banach spaces.

Lemma 3.1.

(i) If an odd mapping \( f : X \to Y \) satisfies

\[
M_2 f(x, y) = \rho M_1 f(x, y)
\]

for all \( x, y \in X \), then \( f \) is the Cauchy additive mapping.

(ii) If an even mapping \( f : X \to Y \) satisfies \( f(0) = 0 \) and (3.1), then \( f \) is the quadratic mapping.

Proof. (i) Letting \( y = 0 \) in (3.1), we get

\[
f \left( \frac{x}{2} \right) = \frac{1}{2} f(x)
\]

for all \( x \in X \).

It follows from (3.1) and (3.2) that

\[
f(x + y) - f(x) - f(y) = 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) = \rho (f(x + y) - f(x) - f(y))
\]
and so
\[ f(x + y) = f(x) + f(y) \]
for all \( x, y \in X \).

(ii) Letting \( y = 0 \) in (3.1), we get
\[ f \left( \frac{x}{2} \right) = \frac{1}{4} f(x) \]
for all \( x \in X \).

It follows from (3.1) and (3.3) that
\[
\frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y) \\
= 2f \left( \frac{x + y}{2} \right) + 2f \left( \frac{x - y}{2} \right) - f(x) - f(y) \\
= \rho \left( \frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y) \right)
\]
and so
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]
for all \( x, y \in X \). \( \square \)

**Theorem 3.2.** Let \( \varphi : X^2 \to \mathbb{R} \) be a function such that
\[ \sum_{j=0}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) < \infty \]
for all \( x, y \in X \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying
\[ N \left( M_2 f(x, y) - \rho M_1 f(x, y), t \right) \geq \frac{t}{t + \varphi(x, y)} \]
for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := N\lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[ N \left( f(x) - A(x), t \right) \geq \frac{t}{t + \Phi(x, 0)} \]
for all \( x \in X \) and all \( t > 0 \), where \( \Phi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \) for all \( x, y \in X \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (3.5). Then \( Q(x) := N\lim_{n \to \infty} 4^nf \left( \frac{x}{2^n} \right) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that
\[ N \left( f(x) - Q(x), t \right) \geq \frac{t}{t + \Psi(x, 0)} \]
for all \( x \in X \) and all \( t > 0 \), where \( \Psi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \) for all \( x, y \in X \).
Proof. (i) Letting \( y = 0 \) in (3.5), we get

\[
N \left( f(x) - 2f \left( \frac{x}{2} \right), t \right) = N \left( 2f \left( \frac{x}{2} \right) - f(x), t \right) \geq \frac{t}{t + \varphi(x,0)}
\]

for all \( x \in X \). Hence

\[
N \left( 2^n f \left( \frac{x}{2^n} \right) - 2^m f \left( \frac{x}{2^m} \right), t \right)
\]

\[
\geq \min \left\{ N \left( f \left( \frac{x}{2^n} \right) - 2^{n+1} f \left( \frac{x}{2^{n+1}} \right), t \right), \ldots, N \left( 2^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 2^m f \left( \frac{x}{2^m} \right), t \right) \right\}
\]

\[
= \min \left\{ N \left( f \left( \frac{x}{2^n} \right) - 2f \left( \frac{x}{2^{n+1}} \right), t \right), \ldots, N \left( f \left( \frac{x}{2^{m-1}} \right) - 2f \left( \frac{x}{2^m} \right), t \right) \right\}
\]

\[
\geq \min \left\{ \frac{t}{t + \varphi \left( \frac{x}{2^n}, 0 \right)}, \ldots, \frac{t}{t + \varphi \left( \frac{x}{2^{m-1}}, 0 \right)} \right\}
\]

\[
= \min \left\{ \frac{t}{t + 2^n \varphi \left( \frac{x}{2^n}, 0 \right)}, \ldots, \frac{t}{t + 2^{m-1} \varphi \left( \frac{x}{2^{m-1}}, 0 \right)} \right\}
\]

\[
\geq \frac{t}{t + \sum_{j=0}^{m-1} 2^j \varphi \left( \frac{x}{2^j}, 0 \right)}
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \) and all \( t > 0 \). It follows from (3.4) and (3.9) that the sequence \( \{2^n f(\frac{x}{2^n})\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^n f(\frac{x}{2^n})\} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := N_\infty \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.9), we get (3.6).

By (3.5),

\[
N \left( 2^n f \left( \frac{x + y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right), \rho \left( 2^n f \left( \frac{x + y}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) \right)
\]

\[
\geq \frac{t}{t + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}
\]
for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So
\[
N \left( 2^{n+1} f \left( \frac{x+y}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) - 2^n f \left( \frac{y}{2^n} \right) \right)
\geq \frac{\frac{t}{2^n} + \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}{t + 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)}
\]
for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n \to \infty} \frac{t}{t + 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right)} = 1$ for all $x, y \in X$ and all $t > 0$,
\[
2A \left( \frac{x+y}{2} \right) - A(x) - A(y) = \rho \left( A(x+y) - A(x) - A(y) \right)
\]
for all $x, y \in X$. By Lemma 3.1, the mapping $A : X \to Y$ is Cauchy additive.

(ii) Letting $y = 0$ in (3.5), we get
\[
N \left( f(x) - 4f \left( \frac{x}{2} \right), t \right) = N \left( 4f \left( \frac{x}{2} \right) - f(x), t \right) \geq \frac{t}{t + \varphi(x,0)}
\]
for all $x \in X$. Hence
\[
N \left( 4^l f \left( \frac{x}{2^l} \right) - 4^m f \left( \frac{x}{2^m} \right), t \right)
\geq \min \left\{ N \left( 4^l f \left( \frac{x}{2^l} \right) - 4^{l+1} f \left( \frac{x}{2^{l+1}} \right), t \right), \ldots, \right. \]
\[
\left. N \left( 4^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 4^m f \left( \frac{x}{2^m} \right), t \right) \right\}
\]
\[
= \min \left\{ N \left( f \left( \frac{x}{2^l} \right) - 4f \left( \frac{x}{2^{l+1}} \right), \frac{t}{4^l} \right), \ldots, N \left( f \left( \frac{x}{2^{m-1}} \right) - 4f \left( \frac{x}{2^m} \right), \frac{t}{4^{m-1}} \right) \right\}
\]
\[
\geq \min \left\{ \frac{t}{4^l + \varphi \left( \frac{x}{2^l}, 0 \right)}, \ldots, \frac{t}{4^{m-1} + \varphi \left( \frac{x}{2^{m-1}}, 0 \right)} \right\}
\]
\[
= \min \left\{ \frac{t}{t + 4^l \varphi \left( \frac{x}{2^l}, 0 \right)}, \ldots, \frac{t}{t + 4^{m-1} \varphi \left( \frac{x}{2^{m-1}}, 0 \right)} \right\}
\]
\[
\geq \frac{t}{t + \sum_{j=l}^{m-1} 4^j \varphi \left( \frac{x}{2^j}, 0 \right)}
\]
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.4) and (3.11) that the sequence $\{4^n f \left( \frac{x}{2^n} \right) \}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f \left( \frac{x}{2^n} \right) \}$ converges. So one can
define the mapping \( Q : X \to Y \) by
\[
Q(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right)
\]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.11), we get (3.7).

The rest of the proof is similar to the above additive case. \( \square \)

**Corollary 3.3.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 2 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying
\[
N\left(M_2 f(x, y) - \rho M_1 f(x, y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]
for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[
N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p \theta \|x\|^p}
\]
for all \( x \in X \) and all \( t > 0 \).

(ii) Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (3.12). Then \( Q(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) \) exists for each \( x \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that
\[
N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p \theta \|x\|^p}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.2 by taking \( \varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \), as desired. \( \square \)

**Theorem 3.4.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that
\[
\sum_{j=1}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y\right) < \infty
\]
for all \( x, y \in X \).

(i) Let \( f : X \to Y \) be an odd mapping satisfying (3.5). Then \( A(x) := \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right) \) exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[
N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x, 0)}
\]
for all \( x \in X \) and all \( t > 0 \), where \( \Phi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y\right) \) for all \( x, y \in X \).
(ii) Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then $Q(x) := N\lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{t}{\frac{1}{2} \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$, where $\varphi(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y)$ for all $x, y \in X$.

Proof. (i) It follows from (3.8) that

$$N \left( f(x) - \frac{1}{2} f(2x), \frac{t}{2} \right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N \left( f(x) - \frac{1}{2} f(2x), t \right) \geq \frac{2t}{2t + \varphi(2x, 0)} = \frac{t}{t + \frac{1}{2} \varphi(2x, 0)}$$

for all $x \in X$ and all $t > 0$.

(ii) It follows from (3.10) that

$$N \left( f(x) - \frac{1}{4} f(2x), \frac{t}{4} \right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N \left( f(x) - \frac{1}{4} f(2x), t \right) \geq \frac{4t}{4t + \varphi(2x, 0)} = \frac{t}{t + \frac{1}{4} \varphi(2x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

**Corollary 3.5.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\| \cdot \|$. (i) Let $f : X \to Y$ be an odd mapping satisfying (3.12). Then $A(x) := N\lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p \theta \|x\|^p}$$

for all $x \in X$.

(ii) Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.12). Then $Q(x) := N\lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p \theta \|x\|^p}$$

for all $x \in X$. 

Proof. The proof follows from Theorem 3.4 by taking \( \varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \), as desired. \( \square \)

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