Abstract—This paper studies the benefit of time-varying actuator scheduling on the controllability of undirected complex networks with linear dynamics. We define a new notion of nodal communicability, termed $2k$-communicability, and unveil the role that this centrality measure plays in selecting which nodes to actuate over time to optimize network controllability, as measured by the trace of the controllability Gramian. We identify various conditions on the network structure that determine whether it benefits or not from time-varying actuator scheduling, and quantify the sub-optimality gap of time-invariant policies. Intuitively, our analysis shows that time-varying actuator policies should be used when the network has multiple heterogeneous central nodes, as measured by $2k$-communicability. We illustrate our results with examples of deterministic and random networks.

I. INTRODUCTION

Many natural and man-made systems, ranging from the nervous system to power and transportation grids and the Internet, exhibit dynamic behaviors that evolve over a sparse and complex interconnection structure. The ability to control network dynamics is not only a theoretically challenging problem but, more importantly, a barrier to fundamental breakthroughs across engineering, social sciences, and neuroscience. While multiple studies have addressed various aspects of this problem, several fundamental questions remain unanswered, including to what extent the capability of actuating a different set of nodes over time can improve controllability of large-scale, time-varying network systems.

In this paper we study controllability of networks with fixed topology and weights where, however, the set of control nodes can be selected over time. Intuitively, the ability to actuate different nodes at different times allows for targeted interventions at different network locations, and can ultimately decrease the control effort to accomplish a desired task. Yet, from a practical standpoint, the implementation of time-varying control schemes requires the ability to geographically relocate actuators or the presence of actuation mechanisms at different, possibly all, network nodes, and more sophisticated control policies. To justify the additional implementation costs and control complexity, we seek to characterize network topologies and dynamics that benefit from time-varying actuator schedules, and quantify the associated control improvement. We consider time-varying control policies where a small subset of available actuators is used at any given time. While this may seem at odds with the objective of reducing the overall control effort, this scenario is relevant when, for instance, (i) actuators exhibit nonlinear dead-zone behaviors, so that each one requires a sizable activation energy, (ii) actuators are controlled via communication channels with limited capacity, so that only a small number of devices can be simultaneously operated, (iii) actuators are geographically disperse so that precise coordination becomes difficult or time-consuming, and (iv) simultaneous actuation of proximal nodes results in actuator interference. Examples include suppression of cascading failures in power networks, tracking in wireless sensor/actuator networks, and correction of neural disorders via external stimulation.

Literature review. Complex networks have long been the subject of active research, see e.g. [2]–[5] and references therein. Our work builds on the growing literature on controllability of linear complex networks, which seeks to address two fundamental questions: how network controllability relates to macroscopic properties such as size and degree distribution [6]–[9], and how to choose an optimal set of control nodes to maximize certain controllability measures [10]–[13]. In addressing these questions, a number of works employ binary controllability measures, i.e., measures that only determine whether a system is fully controllable or not. The work [6] uses structural controllability to formulate a graph-based approach to identify the minimum number of leader nodes required for network controllability, and conjectures that this number is determined by network degree distribution. Instead, [7] emphasizes the role played by nodal dynamics in determining this minimum number. The work [10] proposes a heuristic greedy approach to approximate the NP-hard problem of finding the smallest set of nodes to ensure network controllability. The fact that binary controllability measures are oblivious to the energy cost of steering the network state has motivated the introduction of several controllability metrics to quantify the required control effort, including the smallest eigenvalue, determinant, and trace of the controllability Gramian [14]. The work [8] studies the scaling laws between the control horizon and network controllability, measured by the best and worst-case energy needed to steer the network state in all directions. The works [11], [13] analyze the problem of finding the smallest set of control nodes that maximize average network controllability (using the metric of the trace of the Gramian and trace of Gramian inverse, resp.), while [9] analyzes how the worst-case controllability scales with network size. With the exception of [12], that proposes methods to select different control nodes over time to maximize worst-case controllability, the literature above relies on the implicit assumption that the set of control nodes is fixed over time. Depending on the specific network structure, this assumption may come at the expense of a significant limitation on its controllability, especially for large-scale systems. Time-varying scheduling is also employed in the design of (sub)optimal actuator/sensor scheduling algorithms [15]–[17], where periodic schedules can

A preliminary version will appear at ACC’17 as [1].

E. Nozari and J. Cortés are with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, {enozari,cortes}@ucsd.edu. Fabio Pasqualetti is with the Department of Mechanical Engineering, University of California, Riverside, fabiopas@engr.ucr.edu.
approximate the optimal performance arbitrarily well. Yet, little is known about the network properties that make time-varying schedules beneficial, which is our focus here.

**Statement of contributions.** We consider networks described by discrete-time linear state-space models over a time-invariant undirected graph. We employ the trace of the controllability Gramian, which is a measure of average network controllability. The contributions of the paper are threefold. First, we introduce a novel notion of nodal communicability, termed 2k-communicability, characterize its properties, and show how it naturally arises in time-varying actuator scheduling; the optimal schedule consists of actuating, at each time index, the set of nodes with the highest communicability. Based on this connection, we show that the optimal schedule consists of only a finite number of switches in the set of actuation nodes irrespective of the time horizon (in contrast, for instance, to the universal approximation property of periodic schedules in optimal sensor scheduling). Second, we provide three conditions on the network topology that guarantee optimality of time-invariant actuator schedules, and we identify networks that satisfy these conditions including uniform line, ring, and star networks without self-loops. We also show that Barabási-Albert scale-free random networks have time-invariant optimal actuator schedules with high probability. These results and examples reveal that time-invariant actuator schedules are optimal when the network has a single authority, that is, a single node with distinctly higher influence on the dynamics. Third, we characterize the class of networks for which the optimal actuator schedule is time-varying. In addition to a main sufficient condition based on 2k-communicability, we show that this class includes networks with small but “powerful” (in terms of local edge weights) subnetworks, as well as Watts-Strogatz small-world networks (with high probability). Erdős-Rényi random networks, instead, do not require a time-varying actuator schedule due to the lack of leader heterogeneity, except when they are moderate-sized and sparse (with high probability). Finally, we examine the sub-optimality gap of time-invariant actuator schedules for star, Watts-Strogatz, and Erdős-Rényi networks. For ease of exposition, we state our results for single-input networks and discuss the generalization to multiple-input networks in various remarks along the paper.

II. PRELIMINARIES

Here, we introduce notational conventions and review basic concepts on graph theory and network centrality.

1) Notation: We use \( \mathbb{R}, \mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{N}, \) and \( \mathbb{E} \) to denote the set of reals, integers, non-negative integers, positive integers, and positive even integers, respectively. For \( a, b \in \mathbb{Z}, a|b \) denotes that \( a \) divides \( b \). The \( n \)-vector of all ones is denoted by \( \mathbf{1}_n \) and \( \{e_1, \ldots, e_n\} \) stands for the standard basis of \( \mathbb{R}^n \). Given \( x \in \mathbb{R}^n \), \( x_i \) and \( (x)_i \) refer to its \( i \)th component. Similarly, \( a_{ij} \) and \( (A)_{ij} \) refer to the \((i,j)\)th entry of \( A \), and \( a_i \) refers to its \( i \)th column. The notation \( \text{diag}(\sigma_1, \ldots, \sigma_n) \) denotes the diagonal matrix with \( \sigma_1, \ldots, \sigma_n \in \mathbb{R} \) in its diagonal. For \( \lambda \in \mathbb{R}^n \) and \( \ell \in \mathbb{Z}_{\geq 0}, \lambda^\ell \triangleq [\lambda_1^\ell \cdots \lambda_n^\ell]^T \) and \( |\lambda| \triangleq [|\lambda_1| \cdots |\lambda_n|]^T \). We use bold face letters for finite sequences of the form \( u_k \triangleq (u(k))_{k=0}^{K-1}, \) and use \( \|u_k\|_p \) for \( (\sum_{k=0}^{K-1} |u(k)|^p)^{1/p} \). Given a matrix \( M \in \mathbb{R}^{m \times n}, \) its trace, determinant, rank, and eigenvalue with smallest magnitude are denoted by \( \text{tr}(M), |M|, \text{rank}(M), \) and \( \lambda_{\text{min}}(M) \), resp. For two functions \( f, g : \mathbb{N} \rightarrow \mathbb{R}, f(n) = O(g(n)) \) (resp. \( f = \Omega(g) \)) if there exist \( C \geq 0 \) and \( N \in \mathbb{N} \) such that \( f(n) \leq Cg(n) \) (resp. \( f(n) \geq Cg(n) \)) for \( n \geq N, \) and \( f(n) = \Theta(g(n)) \) if it is both \( O(g(n)) \) and \( \Omega(g(n)) \). A matrix \( V \) is orthogonal if \( V^{-1} = VT \). A nonnegative matrix is doubly-stochastic if its rows and columns sum up to one. For a random variable \( X, \text{Var}(X) \) and \( \text{std}(X) \) denote its mean and standard deviation.

2) Graph theory: A weighted undirected graph \( G = (V, E, A) \) consists of a vertex set \( V = \{1, \ldots, n\}, \) an edge set \( E = \{(i, j) \mid i \text{ is connected to } j\}, \) and an adjacency matrix \( A \in \mathbb{R}^{n \times k} \) where, for any \( i, j \in V, a_{ij} \geq 0 \) is the weight of the edge between nodes \( i \) and \( j. \) A path in \( G \) from node \( i \) to \( j \) is a finite sequence \( \ell_0, \ell_1, \ldots, \ell_p \) of nodes where \( \ell_0 = i, \ell_p = j, \) and \( \ell_m \in E \) for \( \ell \in \{1, \ldots, p\}. \) A cycle is a path with \( \ell_0 = \ell_p. \) For \( k \geq 1, (A^k)_{ij} \) gives the (weighted) number of paths of length \( k \) between nodes \( i \) and \( j. \) A regular graph of degree \( k \) is a graph where all the vertices have \( k \) neighbors. A strongly regular graph with parameters \( (n, k, \lambda, \mu) \) is a regular graph of \( n \) nodes with degree \( k \) where any two adjacent vertices have \( \lambda \) common neighbors and any pair of non-adjacent vertices have \( \mu \) neighbors in common. Given a network \( G \) with \( n \) nodes, a cone on \( G \) is a network with \( n + 1 \) nodes where the last one is connected to all others.

3) Network centrality: We briefly review here three centrality measures with spectral characterizations. Consider a network of size \( n \) represented by the adjacency matrix \( A. \)

a) Eigenvector centrality \([18, 19]\): Let \( v_i \in \mathbb{R}^{\geq 0} \) denote the centrality value of node \( i \in \{1, \ldots, n\}. \) Eigenvector centrality is based on the idea that the influential nodes are the ones that are connected to other influential nodes. In other words, \( v_i \propto \sum_{j \in N_i} a_{ij}v_j \) for all \( i \) (where \( N_i \) is the set of neighbors of \( i \)). This requires the existence of a constant \( \lambda > 0 \) such that \( \lambda v_i = \sum_{j \in N_i} a_{ij}v_j \) for all \( i. \) In matrix notation, \( v = [v_1 \cdots v_n]^T, \) this becomes \( Av = \lambda v, \) which is an eigenvalue problem. Since \( A \) is non-negative, by the Perron-Frobenius Theorem \([20, \text{Fact 4.11.4}], \) there always exists a pair \((\lambda, v) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\leq 0} \) such that \( Av = \lambda v. \) This vector \( v \) is thus defined as the vector of eigenvector centralities. Throughout the paper, unless otherwise noted, centrality refers to eigenvector centrality.

b) Exponential and resolvent communicability \([21, 22]\): The communicability of a node measures its ability to communicate with the rest of the network. Different notions of communicability have been proposed for complex networks. For a given node \( i, \) these include exponential communicability \((e^{\beta A})_{ii}\) and the resolvent communicability \(((I - \beta A)^{-1})_{ii}, \) resp., where \( \beta > 0. \) From the power series expansion of \( e^{\beta A} \) and \((I - \beta A)^{-1}, \) it follows that the exponential and resolvent communicabilities count the total number of cycles that pass through node \( i, \) weighting the “importance” of cycles of length \( k \) by \( \beta^k/k! \) and \( \beta^k, \) resp. Thus, the role of \( \beta \) is to determine how local/global these measures are: increasing \( \beta \) increases the weights of longer cycles. One can show \([22]\) that in the extreme cases of \( \beta \to \infty \) for the exponential and \( \beta \to 1/\lambda_{\text{min}}(A) \) in the resolvent case, both notions result in the same rankings of nodes as eigenvector centrality.

c) Degree centrality: The degree centrality of node \( i \) is the sum of the \( i \)-th row (or column) of \( A \) and provides a measure of the immediate influence of node \( i \) on its neighbors.
III. PROBLEM STATEMENT

We consider a network of \( n \) nodes that communicate, in discrete time, over a weighted undirected graph \( G \) with adjacency matrix \( A \). We assume each node has linear and time-invariant dynamics and that, at each time, one node can be controlled exogenously. The overall network dynamics are

\[
x(k + 1) = Ax(k) + b(k)u(k), \quad k \in \mathbb{Z}_{\geq 0}. \tag{1}
\]

Here, \( x_i(k) \in \mathbb{R} \) is the state of node \( i \) for \( i \in \{1, \ldots, n\} \), \( u(k) \in \mathbb{R} \) is the control input, and \( b(k) \in \mathbb{R}^n \) is the time-varying input vector, all at time \( k \). Since we can always normalize \( b(k) \) and include its magnitude in \( u(k) \), we assume \( \|b(k)\| = 1 \) so that \( b(k) \in \{e_1, \ldots, e_n\} \). Given a time horizon \( K \in \mathbb{N} \), \( \mathcal{F}_{TV} \triangleq \{e_1, \ldots, e_n\}^K \) is the set of time-varying input selections over the time horizon \( K \).

Controllability of (1) at time \( K \in \mathbb{N} \) is defined as the possibility of steering the network from any initial state \( x(0) \) to any desired state \( x(K) \) using the input \( u_K \). It is well-known [23] that this is equivalent to \( |\mathcal{W}_K| \neq 0 \), where

\[
\mathcal{W}_K \triangleq \sum_{k=0}^{K-1} A^k b(K-1-k)b(K-1-k)^T (A^T)^k,
\tag{2}
\]

is called the controllability Gramian. It is also well-known [23, Thm 6.1] that when \( |\mathcal{W}_K| \neq 0 \), among all the controls \( u_K \) that can steer the network from the origin to an arbitrary \( x_f \) at time \( K \), the one with minimum energy \( \|u_K\|_F \) is given by

\[
u^*(k) = b(k)^T (A^T)^{K-k} W_K^{-1} x_f, \quad k \in \{0, \ldots, K-1\}.
\]

It is immediate to verify that \( \|u^*_K\|_F^2 = x_f^T W_K^{-1} x_f \). It is thus desirable to have \( \mathcal{W}_K \) as "large" as possible. To quantify how large the Gramian is, several spectral measures have been proposed in the literature [9], [14], [24], including \( \lambda_{\min}(\mathcal{W}_K), |\mathcal{W}_K|, \text{tr}(\mathcal{W}_K^{-1}), \text{tr}(\mathcal{W}_K), \) and \( \text{rank}(\mathcal{W}_K) \).

While each metric has its own benefits and limitations, we focus here on the trace of the Gramian motivated by the following considerations. First, it is straightforward to show that \( \text{tr}(\mathcal{W}_K^{-1}) \) is the expected value of \( \|u^*_K\|_F^2 \) over unit-variance random \( x_f \). Using the relationship

\[
\frac{n}{\text{tr}(\mathcal{W}_K^{-1})} \leq \frac{\text{tr}(\mathcal{W}_K)}{n},
\]

maximizing \( \text{tr}(\mathcal{W}_K) \) is a necessary (though not sufficient) condition for minimizing \( \text{tr}(\mathcal{W}_K^{-1}) \). In other words, any property that must be satisfied by the optimizers of \( \text{tr}(\mathcal{W}_K) \) is also valid for the optimizers of \( \text{tr}(\mathcal{W}_K^{-1}) \). Given the intractability of \( \text{tr}(\mathcal{W}_K^{-1}) \), cf. [14], [25], maximizing \( \text{tr}(\mathcal{W}_K) \) is a widely used proxy for the minimization of \( \text{tr}(\mathcal{W}_K^{-1}) \). Second, although in some extreme cases (e.g., fully decoupled networks), the maximizer of \( \text{tr}(\mathcal{W}_K) \) does not guarantee controllability, the maximizers of \( \text{tr}(\mathcal{W}_K), \text{tr}(\mathcal{W}_K^{-1}), \text{log det}(\mathcal{W}_K), \) and \( \lambda_{\min}(\mathcal{W}_K) \) have been shown to be close in several real-world applications [14], [26]. Third, the maximization of \( \text{tr}(\mathcal{W}_K^{-1}), \text{log det}(\mathcal{W}_K), \) or \( \lambda_{\min}(\mathcal{W}_K) \) may result in unrealistic over-conservatism that is not present in the maximization of \( \text{tr}(\mathcal{W}_K) \) [27]. These considerations have motivated the wide use of \( \text{tr}(\mathcal{W}_K) \) in the literature of controllability of networked systems, see, e.g., [11], [14], [27]–[33].

Given this metric of network controllability, we are interested in the solution of the following optimization problem:

\[
b^*_K = \arg \max_{b_K \in \mathcal{F}_{TV}} \text{tr}(\mathcal{W}_K). \tag{3}
\]

Using the definition (2) and the invariance property of trace under cyclic permutations, we can write

\[
\text{tr}(\mathcal{W}_K) = \sum_{k=0}^{K-1} b(K-1-k)^T A^{2k} b(K-1-k).
\]

Note that \( b(K-1-k)^T A^{2k} b(K-1-k) = (A^{2k})_{i_k} \) is the \( i_k \)-th diagonal entry of \( A^{2k} \), where \( i_k \) is the index of the node to which \( u(K-1-k) \) is applied, i.e., \( b(K-1-k) = e_{i_k} \). Therefore, the optimization (3) at each time \( K - 1 - k \) boils down to finding the largest diagonal element of \( A^{2k} \). The computation of the exact solution to (3) is feasible (with polynomial time complexity) for large networks since the optimization in (3) is completely decoupled over time. This is because the trace of the Gramian is a modular function over the set \( \mathcal{F}_{TV} \).

One might also constrain the actuator schedule to be time-invariant. To this end, let \( \mathcal{F}_{TI} = \{b_K \in \mathcal{F}_{TV} \mid b(1) = b(2) = \cdots = b(K-1) \} \) and, instead of (3), consider

\[
\hat{b}_K^* = \arg \max_{b_K \in \mathcal{F}_{TI}} \text{tr}(\mathcal{W}_K).
\]

Since \( \mathcal{F}_{TI} \subset \mathcal{F}_{TV} \), the solution of this problem will in general be sub-optimal with respect to (3). However, time-varying actuator scheduling is more difficult to compute and more expensive to implement in practice, as it requires an actuator to be connected to possibly all nodes in the network. Our problem of interest is then to determine under what conditions, for which networks, and by how much an optimal time-varying actuator schedule outperforms an optimal time-invariant one.

Remark III.1. (Networks with multiple inputs). Our exposition above considers the case of a single input applied to the network at each time \( k \). In the ensuing discussion, we also provide extensions of our results for the general case where \( m > 1 \) inputs are selected at each time. For time-varying actuator scheduling to be beneficial for a network with \( m \) inputs, any of the first \( m \) largest of \( \{(A^{2k})_{i_k}\}_{i_k=1}^{m} \) can change over time, which is only implied by (but does not imply) a change in \( \arg \max_{i \in \{1, \ldots, n\}} (A^{2k})_{ii} \). Therefore, the results of the single-input case cannot be readily transcribed to the \( m \)-input case, and some care is required to generalize them.

IV. 2k-COMMUNICABILITY IN DYNAMIC NETWORKS

Here, we introduce the notion of 2k-communicability and explain its connection with the optimal actuator scheduling problem of Section III. We also discuss its similarities and differences with the existing notions of communicability as well as its limiting scenarios (\( k = 1 \) and \( k \to \infty \)). From the discussion of Section III, the optimal input selection in (3) is

\[
b^*(K-1-k) = e_{\arg \max_{i \leq \leq n}(A^{2k})_{ii}}, \tag{4}
\]

Interestingly, our forthcoming results on the optimality of time-varying versus time-invariant schedules remain unchanged when considering linear network models with lossy channels in the actuation path, as studied in [34].
for all $k \in \{0, \ldots, K-1\}$. For each $i \in \{1, \ldots, n\}$, $(A^{2k})_{ii}$ is a convex sum of $n$ exponential functions of the variable $k$. Precisely, let $A = V \Lambda V^T$ be the eigen-decomposition of $A$, where $V = [v_{ij}]_{n \times n}$ is orthogonal and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. For convenience, we define $\lambda = [\lambda_1 \cdots \lambda_n]^T$. Let $W = [w_{ij}]_{n \times n}$ be the doubly stochastic matrix such that $w_{ij} = v_{ij}^2$ for all $i, j \in \{1, \ldots, n\}$. Then, after some algebraic manipulations, one can show that

$$(A^{2k})_{ii} = \sum_{j=1}^{n} v_{ij}^2 \lambda_j^{2k} = (W \lambda^{2k})_{ii}. \quad (5)$$

According to (4), the one function among $(A^{2k})_{11}, \ldots, (A^{2k})_{nn}$ which is on top at time $k$ determines $b^*(K-1-k)$.

Thus, all we need to know is the number of times that the maximizer of these $n$ sums of $n$ exponentials changes over $\{0, \ldots, K-1\}$. If the maximizer is constant, a time-varying input allocation is not beneficial and vice versa. Motivated by these considerations, we introduce the following definition.

**Definition IV.1. (2k-communicability).** Consider a dynamic network of $n$ nodes defined by the adjacency matrix $A$. For any $k \in \mathbb{N}$, the $2k$-communicability of each node $i \in \{1, \ldots, n\}$ is $R_i(k) = (A^{2k})_{ii}$.

The $2k$-communicability of a node $i$ counts the (weighted) number of cycles of length $2k$ that pass through node $i$ (unlike the exponential and resolvent communicabilities described in Section II that count a weighted number of the total number of cycles of all lengths that pass through each node). The advantage of this notion is its direct connection with optimal actuator scheduling in discrete-time networks (c.f. (4)), the same role that is played by exponential communicability in continuous-time networks (with $\beta$ playing the “time” role of $k$). Also, a different but related notion of centrality, called “Average Energy Controllability Centrality”, is proposed in [11] in order to measure the average ability of any individual node to move the network in all directions in the state space.

To study the number of changes in $\arg \max_i R_i(k)$ over $\{1, \ldots, K-1\}$, it is convenient to extend the domain of definition of $\{R_i\}_{i=1}^n$ to $\mathbb{R}_{\geq 0}$. For consistency, define $R_i(t) = (W \lambda^{2t})_{ii}$, for $t \in \mathbb{R}_{\geq 0}$ and $i \in \{1, \ldots, n\}$. The following result, whose proof is straightforward, summarizes basic properties of this function.

**Lemma IV.2. (Basic properties of $R_i$).** For $i \in \{1, \ldots, n\}$, $R_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is smooth and strictly convex, satisfies $R_i(0) = 1$, and is monotonically decreasing if $\lambda_1 \leq 1$.

Figure 1 shows a small network of $n = 6$ nodes (without self-loops) as well as the evolution of $\{R_i(t)\}_{i=1}^n$, where the optimal control node switches $n - 2 = 4$ times. For general networks, the following result provides an upper bound on the possible number of switches of $\arg \max_i R_i(t)$.

**Lemma IV.3. (Bound on the number of optimal control node switches).** The maximum number of switches in $\arg \max_{1 \leq i \leq n} R_i(t)$ over $t \in \mathbb{R}_{\geq 0}$ is $O(n^3)$, where $n$ is the network dimension.

**Proof:** According to [35, Thm 1], an arbitrary sum of $n$ (distinct) exponential functions has at most $n - 1$ zeros. Therefore, any pair $R_i$ and $R_j$ can swap orders at most $n - 2$ times (given that $R_i(0) = 1$ for all $i \in \{1, \ldots, n\}$). Since there are $\binom{n}{2}$ such pairs, the total number of control node switches cannot be more than $\binom{n}{2}(n - 2)$, which is $O(n^3)$.

Since the number of switches in $\arg \max_i R_i(k)$ over $\{1, \ldots, K-1\}$ is upper bounded by the number of switches in $\arg \max_i R_i(t)$ over $\mathbb{R}_{\geq 0}$, Lemma IV.3 rules out the possibility of arbitrarily large number of optimal control node switches. As Figure 1 and Examples V.4 and VI.3 later show, even the bound $O(n^3)$ is quite conservative. In fact, we conjecture that the number of switches in $\arg \max_i R_i(t)$ over $\mathbb{R}_{\geq 0}$ is at most $n - 1$. The rationale for this conjecture is twofold. First, since $\{R_i(t)\}_{i=1}^n$ are convex combinations of the same $n$ functions $\{\lambda^{2t}\}_{i=1}^n$ and these functions decay exponentially fast with respect to each other, the (at most $n - 2$) positive roots of any difference of the form $R_i(t) - R_i(t), i_1 \neq i_2$ occur roughly at the times $\{\tau_j\}_{j=1}^{n-1}$ where $\tau_j$ is the (first) time such that $\lambda_j^{2\tau_j} \ll \lambda_j^{2\tau_{j+1}}$. Thus, we expect to have at most one switch close to each $\tau_j$, giving a total of $n - 1$. Second, our extensive simulations over various network architectures never yielded more than $n - 1$ switches in $\arg \max_i R_i(t)$, supporting our conjecture.

Lemma IV.3 also highlights how the optimal actuator schedule is determined by the dependence of the nodes’ 2k-communicabilities on the time index $k$. For small $k$, this notion depends on the local network structure, and incorporates more global information as $k$ grows. In particular,

(i) the 2-communicability of a node is closely related to its (weighted) degree centrality. To see this, note that
If the \( R_t(1) = (A^2)_{ik} = \sum_{j=1}^{n} a_{ij}^2 \), so \( R_t(1) \) is equal to the degree in unweighted networks. In the case of weighted networks, the 2-communicability and degree centrality become more different as the network weights become more heterogeneous (see e.g., Figure 1, where the vector of degree centralities is \((1.06, 0.81, 0.99, 1.01, 0.22, 1.166)\)).

(ii) the \( \infty \)-communicability of a node, i.e., \((A^{2k})_{ii}\), as \( k \to \infty \), results in the same ordering of the nodes as the eigenvector centrality (assuming that \( |\lambda_1| > |\lambda_2| \)). If \( \lambda_1 = 1 \), this immediately follows from \( \lim_{k \to \infty} A^{2k} = v_1 v_1^T\), with \( v_1 \) being the vector of centralities. If \( \lambda_1 \neq 1 \), one can normalize \( A \) by \( \lambda_1 \) and then take the limit \( \lim_{k \to \infty} A^{2k} \) (which does not affect the order of the nodes).

**Remark IV.4. (Beyond undirected networks).** The extension of our results to normal networks (i.e., directed networks with a normal adjacency matrix) is straightforward. In this case, using \( AA^T = A^T A \), we have

\[
\text{tr}(W_K) = \frac{K-1}{2} \sum_{k=0}^{K-1} b(K-1-k)^2 (A^T A)^k b(K-1-k)
\]

where \( b = \text{diag}(b_1, \ldots, b_n) \) and the unitary matrix \( V \) contains the singular values and singular vectors of \( A \), resp. Defining \( w = \lceil |v_{ij}|^2 \rceil_{n \times n} \) and \( \sigma = [\sigma_1 \cdots \sigma_n]^T \), we have similar to (5),

\[
b(K-1-k)^2 V \Sigma^2 V^* b(K-1-k) = (W \sigma^{2k})_{ik},
\]

where \( i_k \) is the index of the node to which \( u(K-1-k) \) is applied. Therefore, defining the \( 2k \)-communicability of node \( i \) as \( R_t(k) = (A^T A)^{k} \), all the ensuing results for the undirected case are generalizable to normal networks. In this case, for large \( k \), the \( 2k \)-communicabilities of a node gives the same ranking as its HITS hub or HITS authority ranking [36], rather than eigenvector centrality. The extension of the results to non-normal networks remains an open problem.

In the remainder of the paper, we assume for simplicity that the largest element of the first column of \( W \) is \( w_{11} \), i.e.,

\[
w_{11} = \max_{1 \leq i \leq n} w_{1i}.
\]

This can always be realized by a permutation of the rows of \( W \) achieved by relabeling the node with the largest centrality as node 1 (note that relabeling the nodes only permutes the rows of \( W \) while the order of its columns is arbitrary and corresponds to the order of the diagonal elements of \( \Lambda \)).

In the following, we provide sufficient (c.f. Section VI) and necessary (c.f. Section V) conditions for the optimality of time-invariant actuator schedules in terms of the network topology. We should note that these conditions are not tight. The reason for it is the inherent trade-off between having accurate necessary and sufficient conditions in algebraic terms, with little insight about the structure of the networks that satisfy them, and the identification of insightful topological properties. In algebraic terms, the trivial condition that is both necessary and sufficient for the optimality of time-invariant schedules is

\[
\sum_{j=1}^{n} w_{1j} \lambda_{ij}^{2k} \geq \sum_{j=1}^{n} w_{ij} \lambda_{ij}^{2k} \quad \forall i \in \{2, \ldots, n\}.
\]

In contrast, our aim is to gain qualitative insights about the structural properties of the network topology that lead to optimality of time-invariant schedules.

**V. NETWORKS WITH TIME-INVARIANT OPTIMAL ACTUATOR SCHEDULES**

In this section we give conditions and examples of networks that do not benefit from time-varying actuator schedules. Equivalently, according to the discussion in Section IV, we seek to characterize networks for which there are no changes in \( \arg \max R_t(k) \). The following result characterizes three such cases.

**Theorem V.1. (Networks with a single extreme authority).** Consider the optimal node selection problem (3) over a time horizon \( K \) for a network of \( n \) nodes with dynamics described by (1). Let \( A = V \text{diag}(\lambda_1, \ldots, \lambda_n) V^T \) be the eigen-decomposition of the adjacency matrix, with \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \). Define \( W = [w_{ij}]_{n \times n} \) by \( w_{ij} = v_{ij}^2 \) for \( i, j \in \{1, \ldots, n\} \) and let (6) hold. If any of the following conditions holds:

(i) \( \frac{1}{w_{11}} - \frac{1}{w_{12}} \leq \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| - |\lambda_2|} \),

(ii) \( w_{11} + w_{12} = 1 \),

(iii) the network has three or fewer nonzero eigenvalues with different absolute values and \( 1 \in \arg \max_{1 \leq i \leq n} R_t(k) \),

then selecting the node with the largest eigenvector centrality at every time step is the solution to (3), i.e.,

\[
1 \in \arg \max_{1 \leq i \leq n} R_t(k), \quad \forall k \in \{0, \ldots, K-1\}.
\]

**Proof:** The time \( k = 0 \) is trivial in all cases. Under condition (i), for \( k = 1 \), we have

\[
\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 - \lambda_2} \geq \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| - |\lambda_2|} \geq 1 - \frac{w_{11}}{w_{12}}.
\]

For \( k \geq 2 \), using the above inequality, we have

\[
\frac{\lambda_1^{2k} - \lambda_2^{2k}}{\lambda_1^2 - \lambda_2^2} \geq \frac{\lambda_1^{2k} - \lambda_2^{2k}}{\lambda_1^2 - \lambda_2^2} \geq 1 - \frac{w_{11}}{w_{12}}.
\]

Thus, the result follows from Lemma A.1. Under condition (ii), for any \( k \geq 1 \),

\[
1 \in \arg \max_{1 \leq i \leq n} R_t(k) \iff \sum_{j=1}^{n} w_{1j} \lambda_{ij}^{2k} \geq \sum_{j=1}^{n} w_{ij} \lambda_{ij}^{2k} \iff w_{11} \lambda_1^{2k} + (1 - w_{11}) \lambda_2^{2k} \geq \sum_{j=1}^{n} w_{ij} \lambda_{ij}^{2k} \iff w_{11} \lambda_1^{2k} + (1 - w_{11}) \lambda_2^{2k} \geq \lambda_1^{2k} + (1 - w_{11}) \lambda_2^{2k} \iff (w_{11} - w_{11})(\lambda_1^{2k} - \lambda_2^{2k}) \geq 0,
\]

where the last inequality is always true.

Finally, under condition (iii), first consider the case when \( |\lambda_1| > |\lambda_2| \). By contradiction, assume \( R_t(k) > R_1(k) \) for some \( i \in \{2, \ldots, n\} \) and \( k \geq 2 \). Since \( |\lambda_1| > |\lambda_2| \) for all \( i \in \{2, \ldots, n\} \), there exists a sufficiently large \( k \) where \( R_1(k) > \)}
with different absolute values and of all nodes is determined by the weight of the link to the node equally have the largest self-loops, any of them would make an optimal solution for (3)).

We next interpret the conditions in Theorem V.1:

(i) Condition (i) holds for networks where there is a sufficiently distinct authority, in the sense of eigenvector centrality, and the network dynamics is dominated by the largest eigenvalue. Note that an extreme case of such networks is a totally disconnected network where \( W = I \) and the highest authority is the node with the largest self-loop (clearly, if more than one node equally have the largest self-loops, any of them would make an optimal solution for (3)).

(ii) Condition (ii) holds for networks where the centrality of all nodes is determined by the weight of the link to the most central node. To see this, note that we have \( w_{ij} = 0 \) for \( j \geq 3 \), implying \( v_{ij} = 0, j \geq 3 \). Since the rows of \( V \) are orthogonal, we deduce \( v_{i2} = \alpha v_{i1} \) for all \( i \geq 2 \), where \( \alpha = -v_{11}/v_{12} \) is constant. Using \( A = VA^T \), we have

\[
a_{i1} = \lambda_1 v_{i1} v_{i1} + \lambda_2 v_{i2} v_{i2} = (v_{11} \lambda_1 + \alpha v_{12} \lambda_2) v_{i1},
\]

so \( v_{i1} \propto a_{i1} \) for all \( i \geq 2 \). Examples are star networks with no (or small-weight) self-loops, as we show in Example V.3.

(iii) Regarding condition (iii), the most well-known families of networks with three distinct eigenvalues are the complete bipartite networks and connected strongly regular networks. Moreover, cones on \( (n, k, \lambda, \mu) \)-strongly regular graphs satisfying \( \lambda_{\min}(A)(\lambda_{\min}(A) - k) = n \) are also known to have three distinct eigenvalues [37]. The other condition 1 \( \in \arg\max_{1 \leq k \leq n} R_i(k) \) holds when the node with the largest eigenvector centrality has also the largest 2-communicability (c.f. the correlation between 2-communicability and degree in Section IV). The simplest example of a network with these properties is the star network (with no or equal self-loops).

**Remark V.2. (Networks with multiple inputs – cont’d.)**

In the case of a network with \( m \) inputs, having only \( 1 \in \arg\max_{1 \leq k \leq n} R_i(k) \) is not enough to conclude that time-invariant input allocation is optimal. Instead, we need that all the nodes with the \( m \) largest 2k-communicabilities remain the same over time. To guarantee this property, the three conditions in Theorem V.1 can be generalized as follows:

(i) For all \( i \in \{1, \ldots, m\} \),

\[
\frac{1 - w_{i1}}{\sum_{i' \neq i} w_{i1}} \leq \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| - |\lambda_n|}.
\]

This condition can be simplified, at the expense of being more conservative, to \( \frac{1 - w_{i1}}{\sum_{i' \neq i} w_{i1}} \leq \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| - |\lambda_n|} \), for all \( i \in \{1, \ldots, m\} \),

(ii) for all \( i \in \{1, \ldots, m\} \), \( w_{i2} = 1 - w_{i1} \),

(iii) the network has three or fewer nonzero eigenvalues with different absolute values and \n
\[
R_1(1) \geq R_2(1) \geq \cdots \geq R_m(1) \geq R_i(1),
\]

for all \( i \in \{m + 1, \ldots, n\} \).

\[\text{Fig. 2: Uniform networks in Example V.3 with edge weight} \ a \in \mathbb{R}_{\geq 0}: \text{ (a) line network, (b) ring network, and (c) star network.}\]

The proof of this result follows the same argumentation as the proof of Theorem V.1, and it is omitted here.

In the following example, we introduce additional classes of networks, namely, the uniform line and ring networks, for which time-invariant actuator scheduling is optimal (though they do not satisfy the hypothesis of Theorem V.1). We also use Theorem V.1(iii) to establish the optimality of time-invariant actuator schedules for star networks.

**Example V.3. (Uniform line, ring, and star networks).**

Explicit expressions of 2k-communicability can be obtained in the case of uniform line, ring, and star networks (see Figure 2) as given in Propositions B.1-B.3 in Appendix B. In these cases, we assume uniform edge and self-loop weights across the network (edge and self-loop weights need not be equal).

**Line networks:** in the case of no self-loops, the value of \( R_i(k) \) increases with \( i \) until \( i = \left\lfloor \frac{n}{2} \right\rfloor \) (i.e., the middle node) for \( k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \) (this can be observed from the expression (23)). For general \( k \), it can be shown that the value of the sum in (22) for \( R_i(k) \) is strongly dominated by the summand corresponding to the index \( p = 0 \), which increases with \( i \) until \( i = \left\lfloor \frac{n}{2} \right\rfloor \) and decreases afterwards. Thus, the optimal actuator scheduling is always to (one of the) center node(s), i.e., \( b^*(k) = e_{\left\lfloor \frac{n}{2} \right\rfloor} \) for all \( k \). If nodes have uniform self-loops (i.e., self-loops all with the same weight), \( R_i(k) \) can no longer be computed analytically but simulations show the exact same behavior;

**Ring networks:** without self-loops, the value of \( R_i(k) \) is independent of \( i \) (as shown by (24)), so the optimal actuator scheduling is arbitrary for all \( k \). Similar result can be proved analytically if the nodes have uniform self-loops.

**Star networks:** if all self-loop weights are the same (\( l_c = l_p \) in (25)), then \( R_1(1) > R_i(1) \) for all \( i \geq 2 \) from (15). Therefore Theorem V.1(iii) implies that the center node is the optimal control node at all times.

**Example V.4. (Role of leader multiplicity and heterogeneity: Barabási-Albert scale-free networks).** The result in Theorem V.1 can be interpreted as stating that time-invariant actuator scheduling is optimal for networks with sufficiently distinct authorities. By construction, scale-free networks generated by the Barabási-Albert (B-A) preferential attachment rule [38] have this property with high probability.\(^2\) Thus, we expect these networks to have optimal time-invariant actuator

\(^2\)Recall that in Barabási-Albert preferential attachment, the network starts growing from an initial network, say with one node, in a way that at each growth step a single node is added to the network and stochastically connected to \( m_0 \) existing nodes, where the probability of connecting to any existing node is proportional to the current degree of that node.
schedules with high probability as well. Let \( p_{TV} \) be the probability of having at least one change in the optimizer of \( \arg \max_{1 \leq i \leq n} R_i(k) \) as \( k \) grows. Figure 3 shows a heatmap of \( p_{TV} \) as a function of the network size \( n \) and the number of links \( m_a \) added at each stage of the growth process. For linear preferential attachment (\( m_a = 1 \)), \( p_{TV} \) is almost zero irrespective of \( n \), confirming the intuition above. For larger \( m_a \), we observe a slow increase of \( p_{TV} \) as the network size grows. This is also in line with the intuition expressed above because for \( m_a \geq 2 \), more than one node receive a new link at each stage of the network growth, thus contributing to the formation of multiple central nodes. However, this does not imply that these nodes will also be heterogeneous, explaining why \( p_{TV} \) does not grow significantly with \( n \). □

**Fig. 3**: Heat map of the probability of having at least one switch in \( \arg \max_i R_i(k) \) as a function of network size \( n \) and link attachment rate \( m_a \) for Barabási-Albert scale-free networks.

VI. NETWORKS WITH TIME-VARYING OPTIMAL ACTUATOR SCHEDULES

In this section we give results and examples of networks where any optimal actuator schedule involves at least one change in the control node and we study the sub-optimality gap of time-invariant schedules. Equivalently, according to the discussion in Section IV, we seek to characterize networks for which there are some changes in \( \arg \max_i R_i(k) \). Note that while the results of Section V provide sufficient conditions for the optimality of time-invariant schedules, the conditions of this section (when negated) serve as necessary conditions for it.

A. Optimality of Time-Varying Actuator Scheduling

The following result characterizes a class of networks where no time-invariant actuator schedule is optimal.

**Theorem VI.1. (Networks with heterogeneous authorities).** Consider the optimal node selection problem (3) for a network of \( n \) nodes with dynamics described by (1) over a time horizon \( K \). Let \( A = V \text{diag}(\lambda_1, \ldots, \lambda_n)V^T \) be the eigendecomposition of \( A \), with \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \). Define \( W = [w_{ij}]_{n \times n} \) by \( w_{ij} = v_{ij}^2 \) for \( i, j \in \{1, \ldots, n\} \) and let (6) hold. Then, if \( |\lambda_1| > |\lambda_2| \) and

\[
R_i(1) > R_1(1),
\]

for some \( i \in \{2, \ldots, n\} \) that has \( w_{i1} < w_{11} \), then the optimal actuator scheduling involves more than one node when the time horizon satisfies \( K > K = \frac{\log(2/(w_{11} - w_{i1}))}{\log(|\lambda_1|/|\lambda_2|)} \).

**Proof:** Recall from (6) that node 1 is the node with the highest eigenvector centrality, but other nodes with the same centrality may in general exist. The condition \( w_{i1} < w_{11} \) of the statement requires that the node \( i \) has strictly smaller centrality than node 1. Then, at time \( k = K - 1 \geq K \), we have

\[
w_{11} - w_{i1} > \left( \frac{\lambda_2}{\lambda_1} \right)^{2k} 2 > \left( \frac{\lambda_2}{\lambda_1} \right)^{2k} \sum_{j=2}^{n} (w_{ij} + w_{1j})
\]

\[
\geq \left( \frac{\lambda_2}{\lambda_1} \right)^{2k} \sum_{j=2}^{n} |w_{ij} - w_{1j}| \geq \sum_{j=2}^{n} \left( \frac{\lambda_j}{\lambda_1} \right)^{2k} |w_{ij} - w_{1j}|
\]

\[
\Rightarrow (w_{11} - w_{i1})\lambda_1^{2k} > \sum_{j=2}^{n} (w_{ij} - w_{1j})\lambda_j^{2k} \Rightarrow R_1(k) > R_i(k).
\]

Thus, the sequence of optimal control nodes involves at least one change from node \( i \) to node 1, as claimed. □

The condition \( |\lambda_1| > |\lambda_2| \) is not restrictive because, by the Perron-Frobenius theorem [20, Fact 4.11.4], it is satisfied by all connected and aperiodic networks (aperiodicity is in particular satisfied by the presence of any self-loop [39]). As we discussed in Section IV, 2-communicability of a node is closely related to its degree centrality if all edges have similar weight. In this case, the condition \( R_i(1) > R_1(1) \) requires that the nodes with highest eigenvector and degree centralities do not coincide, preventing the existence of extreme authorities. We elaborate more on this distinction between eigenvector and degree centralities in the sequel.

**Remark VI.2. (Networks with multiple inputs – cont’d).** In the case of networks with multiple inputs, cf. Remark III.1, the condition of Theorem VI.1 is still sufficient to conclude that any optimal actuator schedule is not constant over time. However, this condition can be relaxed by requiring that the \( m \) nodes with the largest 2-communicabilities be different from those with the largest centralities. Formally, the generalization of Theorem VI.1 is as follows: let \( \{r_j^d \in \mathbb{R}^n \mid j \in \mathcal{J}^d \} \) be the set of rankings of nodes based on their 2-communicabilities, where the index set \( \mathcal{J}^d \) accounts for different choices of rankings if there are nodes with equal 2-communicabilities. Similarly, let \( \{r_j^c \in \mathbb{R}^n \mid j \in \mathcal{J}^c \} \) be the set of rankings based on node centralities. Then, if \( |\lambda_1| > |\lambda_2| \) and, \( r_{j1}^d \neq r_{j2}^c \) for any \( (j_1, j_2) \in \mathcal{J}^d \times \mathcal{J}^c \), time-invariant actuator selections are not optimal when the horizon \( K \) is sufficiently large. □

An important take-away message from Theorem VI.1 is that for a change to occur in \( \arg \max_i R_i(k) \), besides the existence of multiple leaders, heterogeneity of leaders is also necessary (a property that, e.g., a uniform ring network lacks). In other words, for time-varying actuator scheduling to be beneficial, some node(s) should have the most local significance (to maximize \( R_i(k) \) for small \( k \)) while different node(s) have global centrality (to maximize \( R_i(k) \) for large \( k \)). The following example illustrates this point.

**Example VI.3. (Role of leader multiplicity and heterogeneity, cont’d: Erdős-Rényi and Watts-Strogatz networks).** Figure 4 shows a heatmap of the probability \( p_{TV} \) of having at least one change in \( \arg \max_i R_i(k) \) for Erdős-Rényi (E-R) random [40] and Watts-Strogatz (W-S) small-world [41] networks (the latter with mean degree 4), as a function of size
By construction, all the nodes in an E-R random network are treated uniformly and randomly, resulting in a low probability that the network has a single distinct authority (unlike the B-A networks considered in Example V.4). However, there is most often no significant difference between the nodes, and this lack of heterogeneity prevents $p_{TV}$ to grow beyond $\sim 0.35$, and makes $p_{TV} \to 0$ as $n \to \infty$. On the contrary, W-S networks have both leader multiplicity and heterogeneity when the network size is large and the rewiring probability $\beta$ is around 0.3. For smaller or larger $\beta$, the network approaches a ring or an E-R random network, resp., which have low $p_{TV}$. With $\beta \sim 0.3$, there is a sufficiently high probability of having multiple nodes that are close to many rewired links (thus more “central”), yielding leader multiplicity. Yet, there is a low probability that these nodes, and the nodes close to them, are rewired all alike, resulting in heterogeneity of leaders. □

**Remark VI.4. (Networks with multiple inputs – cont’d).** Our discussion suggests that increasing the number of inputs makes time-varying actuator scheduling more likely to be beneficial. However, the network structure also plays an important role in quantifying this effect. Figure 5 shows how the probability $p_{TV}$ of having at least one change in the optimal set of $m$ control nodes varies with $m$ for the three classes of random networks discussed above. The common factor among all plots is the dramatic increase in $p_{TV}$ towards 1 as we increase $m$ (except for small E-R networks). We further observe that for all values of $m$, $p_{TV}(n)$ approaches 1 in W-S, saturates to some value in B-A, and decays to zero in E-R networks as $n$ grows. Also, note that in B-A networks, increasing $m$ has less effect on $p_{TV}$ if we simultaneously increase $m_a$. The reason is that the B-A preferential attachment algorithm generates approximately $m_a$ distinctly central nodes. Thus, a network with $m = m_a > 1$ has larger $p_{TV}$ than a network with the same number of nodes but $m > m_a = 1$.

Fig. 4: Heat map of the probability of having at least one switch in $\arg \max R_i(k)$ as a function of network size $n$ and, (a) wiring probability $p$ for Erdős-Rényi random networks, (b) re-wiring probability $\beta$ for Watts-Strogatz networks.

Fig. 5: Effect of increasing the number of input nodes on the optimality of time-varying actuator scheduling. The sample probability of having at least one change in the optimal set of $m$ input nodes is plotted together with its best cubic smoothing spline fit as a function of network size $n$ for different values of $m$ for (a) E-R networks with $p = 0.1$, (b) W-S networks with $k = 4$, $\beta = 0.3$, (c) B-A networks with $m_a = 1$, (d) B-A networks with $m_a = m$.

Following up on the role of leader heterogeneity, another class of networks for which optimal actuator schedules are time-varying are those where the nodes with small centrality belong to a subnetwork with strong connections. The next result formalizes this statement by showing that increasing the local weights of nodes with small centrality can turn them into globally optimal control nodes at some (or even all) times.

**Theorem VI.5. (Empowerment of subnetworks).** Consider the optimal node selection problem (3) over a time hori-
zon $K$. Given a network of $n$ nodes with adjacency matrix $A_0 \in \mathbb{R}^{n \times n}$, let $E \in \mathbb{R}^{n \times n}$ be a symmetric nonnegative matrix of the form

$$E = \begin{bmatrix} n - n_1 & n_1 \\ n_1 & 0 \\ 0 & * \end{bmatrix}^{n-n_1},$$

corresponding to a subnetwork involving the last $n_1 < n$ nodes (this is without loss of generality, since nodes can be renumbered). Let $i^* \in \{n-n_1+1, \ldots, n\}$ be the most central node in $E$ and consider the dynamic network described by (1) with adjacency matrix $A = A_0 + \alpha E$, where $\alpha > 0$. Then, there exists $\alpha^* > 0$ such that for $\alpha > \alpha^*$,

$$R_i^*(k) > R_i(k),$$

for all $i \in \{1, \ldots, n\}$ and all $k \geq 1$.

**Proof:** Let $\gamma \in \mathbb{R}_{\geq 0}$ be such that $\gamma_1 \geq \cdots \geq \gamma_n$. Fix $i \in \{1, \ldots, n\}$ arbitrarily. Let $r \leq n_1$ be the rank of $E$. Using the inequalities

$$\sum_{j=1}^n w_{ij} \gamma_j \geq w_{i^*1} \gamma_1, \quad \sum_{j=1}^r w_{ij} \gamma_j \leq 1 \gamma_1 \sum_{j=1}^r w_{ij},$$

it follows that $(W \gamma)^{i^*} > (W \gamma)^i$, if

$$w_{i^*1} \gamma_1 > \gamma_1 \sum_{j=1}^r w_{ij} + \gamma_{r+1}.$$ \hfill (8)

Note that if (8) holds for $\gamma = |\lambda|$, then it holds for $\gamma = \lambda^2$ for $k \geq 1$. This is because

$$w_{i^*1} \lambda_1^{2k} = |\lambda_1|^{2k-1} w_{i^*1} \lambda_1 > |\lambda_1|^{2k-1} \left( \left| \lambda_1 \right| \sum_{j=1}^r w_{ij} + |\lambda|_{r+1} \right) > \lambda_1^{2k} \sum_{j=1}^r w_{ij} + \lambda_1^{2k+1}.$$ \hfill (9)

Our proof strategy is then to find $\tilde{\pi}$ such that (8) holds for $\gamma = |\lambda|$, if $\alpha > \tilde{\pi}$. To this end, let $\tilde{\lambda} = [\lambda_1 \ldots \lambda_n]^T \in \mathbb{R}^n$ and $V \in \mathbb{R}^{n \times n}$ be the vector of eigenvalues (with $|\lambda_1| \geq \cdots \geq |\lambda_n|$) and the matrix of eigenvectors of $E$, resp., and $W$ be the element-wise square of $V$. Note that $W$ has the structure

$$W = \begin{bmatrix} n_1 & n-n_1 \\ n-n_1 & 0 \\ 0 & * \end{bmatrix}^{n-n_1}.$$ \hfill (10)

In the following, we bound $\lambda$ and $V$ using perturbation theory of eigenvalues and eigenvectors. For simplicity of exposition, we only deal with the case where the $r$ nonzero eigenvalues of $E$ are all distinct (the proof for the general case proceeds along the same lines but is more involved).

To bound the eigenvalues in $\lambda$, let $\pi_A : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ be a permutation that re-orders the eigenvalues of $A$ based on their signed value, i.e., $\lambda_{A(1)} \geq \lambda_{A(2)} \geq \cdots \geq \lambda_{A(n)}$. Define $\pi_E$ similarly for $E$ (i.e., such that $\lambda_{\pi_E(1)} \geq \lambda_{\pi_E(2)} \geq \cdots \geq \lambda_{\pi_E(n)}$). By Weyl’s Theorem [42, Thm 4.3.1],

$$|\lambda_{\pi_A(j)} - \alpha \lambda_{\pi_E(j)}| \leq \rho(A_0),$$

for all $j \in \{1, \ldots, n\}$. We know from the Perron-Frobenius theorem [20, Fact 4.11.4] for nonnegative matrices that $\pi_A(1) = \pi_E(1) = 1$. Therefore, (10) implies that

$$\alpha \rho(E) - \rho(A_0) \leq \lambda_1 \leq \alpha \rho(E) + \rho(A_0).$$ \hfill (11a)

Moreover, since $E$ has $n - r$ zero eigenvalues, (10) implies that $A$ has at least $n - r$ eigenvalues with absolute value less than or equal to $\rho(A_0)$, i.e.,

$$|\lambda_{r+1}| \leq \rho(A_0).$$ \hfill (11b)

Next, we bound the eigenvectors in $V$. Define

$$\delta_E = \min \{ \lambda_{\pi_E(j)-1} - \lambda_{\pi_E(j)} | \lambda_{\pi_E(j)} - \lambda_{\pi_E(j)+1} > 0, j \in \{1, \ldots, n-1\} \}.$$ \hfill (12)

Using [43, Cor. 1], we have

$$\|v_{\pi_E(j)} - \hat{v}_{\pi_E(j)}\| \leq \frac{2^{\frac{3}{2}} \|A_0\|}{\alpha \delta_E},$$

for $j \in \pi_E^{-1}\{1, \ldots, r\}$.\hfill (13)

Using $\pi_A(1) = \pi_E(1) = 1$ and (13), we get

$$|w_{i^*1} - \hat{w}_{i^*1}| = |v_{i^*1} - \hat{v}_{i^*1}| \leq 2 |v_{i^*1} - \hat{v}_{i^*1}| \leq 2 |v_{i^*1} - \hat{v}_{i^*1}| \leq 2 \frac{2^{\frac{3}{2}} \|A_0\|}{\alpha \delta_E},$$

which together with $\hat{w}_{i^*1} \geq \frac{1}{n_1}$ gives

$$w_{i^*1} \geq \frac{1}{n_1} - \frac{2^{\frac{3}{2}} \|A_0\|}{\alpha \delta_E},$$ \hfill (14a)

To derive similar bounds on $w_{ij}, j \in \{1, \ldots, r\}$ (recall that we fixed $i \in \{1, \ldots, n\}$ arbitrarily at the beginning of the proof), we need to choose $\alpha > \frac{2 \rho(A_0)}{|\lambda_1|}$. This choice of $\alpha$ guarantees that $\pi_A(j) \in \{1, \ldots, r\}$ for all $j \in \pi_E^{-1}\{1, \ldots, r\}$. Therefore, using (12) and (9) and following the same steps as in (13), we get

$$w_{ij} \leq \frac{2^{\frac{3}{2}} \|A_0\|}{\alpha \delta_E}, \quad j \in \{1, \ldots, r\}.$$ \hfill (14b)

Now, using (11) and (14), (8) holds with $\gamma = |\lambda|$ if

$$\left( \frac{1}{n_1} - \frac{2^{\frac{3}{2}} \|A_0\|}{\alpha \delta_E} \right) \left( \alpha \lambda_1 - \rho(A_0) \right) \geq \left( \alpha \lambda_1 + \rho(A_0) \right) \frac{r 2^{\frac{3}{2}} \|A_0\|}{\alpha \delta_E} + \rho(A_0),$$

which itself holds if $\alpha > \tilde{\pi}$, where $\tilde{\pi} \triangleq \max \left\{1, \frac{2 \rho(A_0)}{|\lambda_r|}, \frac{8 \|A_0\|}{\delta_E} \left( 1 + \frac{\rho(A_0)}{\rho(E)} \right) n_1^2 + 2 \frac{\rho(A_0)}{\rho(E)} n_1 \right\}$, completing the proof.

The significance of Theorem VI.5 is twofold. First, it ensures that locally central nodes can turn into globally central ones provided that their local subnetwork becomes sufficiently

\footnote{To see this, set $\Sigma = \alpha E$ and $\Sigma = A_0$ in [43, Cor. 1]. This is the only place where we need the nonzero eigenvalues of $E$ to be distinct. If $E$ has a repeated nonzero eigenvalue, then the corresponding eigenvectors are no longer unique, i.e., one has to study the perturbation of eigenspaces rather than eigenvectors. Therefore, one can no longer use the simplified variant [43, Cor. 1] of the Davis-Kahan Theorem but the original result itself, which provides essentially the same result but is more technically involved.}
strong. Second, and more importantly, it suggests that, for some $0 < \alpha < \pi$, the $2\alpha$-communicabilities of nodes $i^*$ and 1 are comparable, potentially leading to a time-dependent $\arg \max_i R_i(k)$. This further shows the “type” of heterogeneity of leaders that results in time-varying optimal schedules: local ($i = i^*$) vs. global ($i = 1$) leaders. In some cases, we can even compute the critical values of $\alpha$ analytically, as in the next example for star networks.

Example VI.6. (Star networks with varying self-loop weights). Consider the star network (25) with $a = a_{cp}1_{n-1}$, where $a_{cp} > 0$ is the link weight between the center node and any peripheral node. It follows from (26) that

$$R_1(1) - R_i(1) = l_c^2 - l_p^2 + ||a||^2 - a_{cp}^2.$$ 

(15)

Therefore, if $l_p \leq l_c$, then $R_i(k) > R_1(k)$ for all $k \geq 1$. It is straightforward to see that under this condition, the optimal time-invariant actuator placement is the node 1. Therefore, the sub-optimality gap of time-invariant actuator scheduling is

$$\Gamma(k) = \frac{1}{\sum_{k=1}^{K-1}} \max_{i \in \{1, \ldots, n\}} R_i(k) - \max_{i \in \{1, \ldots, n\}} R_1(k).$$

In this case, it is sometimes more tractable to study the lower bound on the gap $\Gamma$ given by

$$\Gamma_1 \triangleq \sum_{k=1}^{t_s} R_{i_1}(k) - R_1(k),$$

(17)

where the switch time $t_s$ is the smallest root of $R_i(t) - R_1(t)$ greater than $t = 1$. In the following, we characterize $\Gamma$ analytically for star networks with varying self-loop weights and empirically for W-S and E-R random networks.

1) Star networks with different self-loops: The simplest deterministic and uniform networks that exhibit a switch in the optimal control node are the star networks of Example VI.6. Recall that in these networks, a switch occurs if the peripheral self-loop weights $l_p$ satisfy

$$\Theta(n^2) = \sqrt{l_c^2 + (n-2)a_{cp}^2} < l_p < l_c + (n-2)a_{cp} = \Theta(n).$$

We next show that for these networks the sub-optimality gap increases at least super-polynomially in the number of nodes.

Proposition VI.7. (Sub-optimality gap for uniform star networks with varying self-loops). Consider the class of star networks with varying self-loops and let

$$l_p = l_p(n) = \Theta(n^a), \quad a \in (1/2, 1).$$

Then, $\Gamma = \Gamma_1 > l_p^{\frac{1}{2(2a+1)}}$, for $n \gg 1$, which grows super-polynomially in $n$.

Proof: First, note that since the network has 3 distinct eigenvalues and $W$ has only 2 distinct rows, there can be only a single switch in $\arg \max_i R_i(k)$ (c.f. [35, Thm 1]), so $\Gamma = \Gamma_1$. To lower bound $\Gamma_1$, pick any $i \in \{2, \ldots, n\}$. Using (26) and simplifying, we get

$$R_i(k) - R_1(k) = -C_1 \lambda_1^{2k} - C_2 \lambda_2^{2k} + C_3 \lambda_3^{2k},$$

where the coefficients are $C_1 = \frac{a_{cp}^2(n-1)^2-(l_p-a_{cp})^2}{a_{cp}^2(n-1)^2-(l_p-a_{cp})^2}$, $C_2 = \frac{(l_p-a_{cp})^2-a_{cp}^2}{(l_p-a_{cp})^2-a_{cp}^2}$, and $C_3 = \frac{n-2}{n-2}$. Note that $C_1, C_2, C_3 \in (0, 1), C_1 \rightarrow 0$ and $C_2, C_3 \rightarrow 1$ as $n \rightarrow \infty$. On the other hand, we can show that $\lambda_3 > \lambda_2 > \lambda_1 > \lambda_p > 0$, and $\lambda_2 < \lambda_p < \lambda_1$. It is straightforward to show that $\lambda_1 < l_p + a_{cp} \sqrt{n}$. Thus,

$$R_i(k) - R_1(k) > \frac{1}{2} \frac{l_p^{2k}}{l_p^{2k}} - \frac{na_{cp}^2}{l_p^{2k}} \lambda_1^{2k} - \lambda_2^{2k}$$

$$> \frac{n-1}{2} \frac{l_p^{2k}}{l_p^{2k}},$$

$$> \frac{n-1}{2} \frac{l_p^{2k}}{l_p^{2k}} \lambda_1^{2k} - \lambda_2^{2k}.$$

It is straightforward to show that $\lambda_1 < l_p + a_{cp} \sqrt{n}$. Thus,

$$R_i(k) - R_1(k) > \frac{1}{2} \frac{l_p^{2k}}{l_p^{2k}} - \frac{2na_{cp}^2}{l_p^{2k}} (l_p + a_{cp} \sqrt{n})^{2k}.$$

(19)

Solving the RHS for $k$, we get

$$\tilde{t}_s > \frac{\log \frac{l_p^{2k}}{2a_{cp} \sqrt{n}}}{2 \log(1 + \frac{a_{cp} \sqrt{n}}{t_p})} \frac{l_p^{2k}}{l_p^{2k}} \log \frac{l_p^{2k}}{2a_{cp} \sqrt{n}} \frac{n-1}{2} \frac{l_p^{2k}}{l_p^{2k}} \lambda_1^{2k} \lambda_2^{2k} \lambda_3^{2k} \lambda_4^{2k}. $$

$$\tilde{t}_s \triangleq \tilde{t}_s.$$

$$\tilde{t}_s > \frac{\log \frac{l_p^{2k}}{2a_{cp} \sqrt{n}}}{2 \log(1 + \frac{a_{cp} \sqrt{n}}{t_p})} \frac{l_p^{2k}}{l_p^{2k}} \log \frac{l_p^{2k}}{2a_{cp} \sqrt{n}} \frac{n-1}{2} \frac{l_p^{2k}}{l_p^{2k}} \lambda_1^{2k} \lambda_2^{2k} \lambda_3^{2k} \lambda_4^{2k}. $$

$$\tilde{t}_s \triangleq \tilde{t}_s.$$
where we used \( \log(1 + x) < x \) in \((a) \). To obtain the result, we substitute \( l_s \) and \((19) \) into \((17) \) which, after summing the geometric series, yields

\[
\Gamma_1 = \frac{1}{2} \left( l_p (2^{(i+1)} - 1) - \frac{2na^2}{t_p^2} \right)
\]

\[
> \frac{1}{2} \left( l_p (2^{(i+1)} - 1) - \frac{2na^2}{t_p^2} \right).
\]

\[
n^{\geq 1} \geq \frac{2a^2}{t_p^2} \left( \frac{1}{3} - \frac{na^2}{t_p^2} \right)^{2i}.
\]

It is straightforward to check that \( l_p^{\frac{a^2}{t_p^2}} \) grows faster than any polynomial, and this completes the proof. ■

Figure 6 shows the exact value of \( \Gamma \) as a function of \( n \) for the star networks of Example VI.6 with the choice (18), the lower bound of Proposition VI.7, and a super-polynomial fit of the exact values. As the plot shows, the lower bound of Proposition VI.7 is conservative but still correctly identifies the super-polynomial and sub-exponential growth rate of \( \Gamma \).

2) Watts-Strogatz small-world networks: We consider the W-S networks of Example VI.3 with \( \beta \simeq 0.3 \) since this value maximizes \( p_{TV} \). Little is known about the spectrum of W-S networks. The work [44, §7.4] shows numerically that the distribution of the eigenvalues of a W-S network moves from the discrete eigenvalue distribution of (a deterministic) regular ring lattice to the Wigner semicircle distribution of a random E-R network as \( \beta \) increases. Yet, no analytic characterizations are currently available.

Figure 7 shows the sample mean of \( \Gamma(n) \) as a function of the network size \( n \), with over \( 10^5 \) iterations per value, together with its best cubic-spline fit. By analyzing the slope of this curve, we see that the growth of \( \Gamma(n) \) is polynomial with an exponent that is close to 1 (i.e., almost linear growth).

3) Erdős-Rényi random networks: We consider the E-R networks of Example VI.3 with \( p = 0.1 \). Since the entries of the adjacency matrix \( A \) are i.i.d. (except for the symmetry condition), the eigenvalues in \( \lambda \) can be accurately characterized with high probability for large \( n \) [45]. However, not enough is known about the eigenvectors of E-R networks and, in particular, about the difference between the largest and the second largest element of the dominant eigenvector. In the following, we use the theoretical results on the eigenvalues together with empirical results on the eigenvectors to conjecture an important property of E-R networks.

Conjecture VI.8. (E-R networks can only have a switch at \( k = 1 \) with high probability.) For E-R random networks with \( n \gg 1 \), we have

\[ \arg\max_{1 \leq i \leq n} R_i(k) = 1, \]

for any \( k \geq 2 \) with probability at least \( 1 - \frac{1}{n} \).

Rationale: For E-R networks, one has (i) \( |\lambda_j| < (2 + \epsilon) \sqrt{p(1-p)n^{1/2}} \) for any \( \epsilon > 0 \) and all \( j \in \{2, \ldots, n\} \) if \( n \gg 1 \) (where larger \( n \) is required for smaller \( \epsilon \), c.f. [45, Thm 2.13]), and (ii) \( \lambda_1 > pn \) if \( n \gg 1 \), c.f. [45, Thm 6.2], both with probabilities at least \( P(n) = 1 - e^{-\tau(n\log n)^{2\log\log n}} \), for some \( \nu, \alpha > 0 \). Therefore, for any \( i \in \{2, \ldots, n\} \) and \( k \geq 2 \),

\[
R_i(k) = (w_{11} - w_{i1})2^{k} + \sum_{j=2}^{n} w_{1j} \lambda_j 2^{k} - \sum_{j=2}^{n} w_{ij} \lambda_j 2^{k} > (w_{11} - w_{i1}) 2^{k} n^{2k} - 2(2 + \epsilon) 2^{k} p(1-p)^k n^{2k}
\]

\[
> g(n)p^{2k} n^{2k} - 2(2 + \epsilon)^{2k} p^{k} (1-p)^k n^{2k},
\]

with probability at least \( P(n) \) (possibly with other \( \nu \) or \( \alpha \)), and

\[ g = g(n) \triangleq w_{11} - \max_{2 \leq i \leq n} w_{i1}. \]

Figure 8 shows the sample mean and standard deviation of \( g \) as a function of the network size \( n \), which are both \( \Theta(n^{-\alpha}) \) with \( \alpha \simeq 1.85 \) for \( n \gg 1 \). Therefore, it follows from Chebyshev’s inequality [46, Thm 1.6.4] with \( a = n \cdot \text{std}(g) \) that \( g = \Theta(n^{-\alpha + 1}) \) with probability at least \( 1 - \frac{1}{n} \). Substituting this into (20), we get

\[ R_i(k) - R_i(k) > C p^{2k} n^{2k + 1 - \alpha} - 2(2 + \epsilon)^{2k} p^{k} (1-p)^k n^{2k} \]

\[ > 0, \]

for some constant \( C \), with probability at least

\[ 1 - \frac{1}{n^2} - P(n) + \frac{P(n)}{n^2} n^{\geq 1} > 1 - \frac{1}{n}, \]
leading to the conjecture.

In addition to its general significance in understanding optimal time-varying actuator schedules in E-R networks, Conjecture VI.8 suggests that $\Gamma = \Gamma_1$ with probability at least $\frac{1}{n}$ when $n \gg 1$. Figure 9 shows the sample mean of $\Gamma(n)$ as a function of $n$. We see that, unlike the star and W-S networks, the value of $\Gamma$ decreases polynomially to zero as $n$ grows, an observation which is in line with the results of Example VI.3. That is, optimal actuator schedules are time-varying when both leader multiplicity and heterogeneity are present.

VII. CONCLUSIONS AND FUTURE WORK

We have studied when and how much time-varying actuator scheduling is beneficial for dynamic complex networks that evolve linearly in discrete time. We have shown how optimal time-varying actuator schedules consist of selecting, at each time $K-1 - k$, the nodes with highest $2k$-communicability, a novel notion of nodal communicability. This characterization has allowed us to identify conditions on the network topology that determine whether or not time-invariant actuator scheduling is optimal. Our main conclusion is that networks with a single distinct authority (central node) do not benefit from time-varying actuator schedules while networks with many comparable, yet heterogeneous authorities do. Examples from deterministic and random networks have illustrated our results. When time-varying actuation is optimal, we have also studied the sub-optimality gap of time-invariant actuator schedules. Numerous questions remain open for future research, including the analysis of controllability metrics beyond the trace of the Gramian, the development of tighter bounds on the number of optimal control node switches, the use of state feedback, in particular, via distributed agent interactions, in finding optimal schedules for controllability, and the generalization of our results to non-normal, nonlinear, and time-varying networks.

REFERENCES

[1] E. Nozari, F. Pasqualetti, and J. Cortés, “Time-invariant versus time-varying actuator scheduling in complex networks,” in American Control Conference, Seattle, WA, May 2017, pp. 4995–5000.
[2] M. Newman, A. L. Barabási, and D. J. Watts, The Structure and Dynamics of Networks: (Princeton Studies in Complexity). Princeton, NJ, USA: Princeton University Press, 2006.
[3] G. Chen, X. Wang, and X. Li, Fundamentals of Complex Networks: Models, Structures and Dynamics, 1st ed. Wiley, 2015.
[4] M. Newman, Networks: An Introduction. New York, NY, USA: Oxford University Press, Inc., 2010.
[5] O. Sporns, Networks of the Brain, 1st ed. The MIT Press, 2010.
[6] Y. Y. Liu, J. J. Slotine, and A. L. Barabási, “Controllability of complex networks,” Nature, vol. 473, no. 7346, pp. 167–173, 2011.
[7] N. J. Cowan, E. J. Chastain, D. A. Vilhena, D. S. Freudenberg, and C. T. Bergstrom, “Nodal dynamics, not degree distributions, determine the structural controllability of complex networks,” PLoS ONE, vol. 7, no. 6, pp. 1–5, 06 2012.
[8] G. Yan, J. Ren, Y. Lai, C. Lai, and B. Li, “Controlling complex networks: How much energy is needed?” Physical Review Letters, vol. 108, no. 21, p. 218703, 2012.
[9] F. Pasqualetti, S. Zampieri, and F. Bullo, “Controllability metrics, limitations and algorithms for complex networks,” IEEE Transactions on Control of Network Systems, vol. 1, no. 1, pp. 40–52, 2014.
[10] A. Olshevsky, “Minimal controllability problems,” IEEE Transactions on Control of Network Systems, vol. 1, no. 4, pp. 249–258, 2014.
[11] T. H. Summers and J. Lygeros, “Optimal sensor and actuator placement in complex dynamical networks,” in IFAC World Congress, Cape Town, South Africa, 2014, pp. 3784–3789.
[12] Y. Zhao, F. Pasqualetti, and J. Cortés, “Scheduling of control nodes for improved network controllability,” in IEEE Conf. on Decision and Control, Las Vegas, NV, 2016, pp. 1859–1864.
[13] V. Tzoumas, M. A. Rahimian, G. J. Pappas, and A. Jadbabaie, “Minimal actuator placement with bounds on control effort,” IEEE Transactions on Control of Network Systems, vol. 3, pp. 67–78, 2016.
[14] T. H. Summers, F. L. Cortesi, and J. Lygeros, “On submodularity and controllability in complex dynamical networks,” IEEE Transactions on Control of Network Systems, vol. 3, no. 1, pp. 91–101, 2016.
[15] L. Zhao, W. Zhang, J. Hu, A. Abate, and C. J. Tomlin, “On the optimal solutions of the infinite-horizon linear sensor scheduling problem,” IEEE Transactions on Automatic Control, vol. 59, no. 10, pp. 2825–2830, 2014.
[16] S. T. Jwaid and S. L. Smith, “Submodularity and greedy algorithms in sensor scheduling for linear dynamical systems,” Automatica, vol. 61, pp. 282–288, 2015.
[17] D. Han, J. Wu, H. Zhang, and L. Shi, “Optimal sensor scheduling for multiple linear dynamical systems,” Automatica, vol. 75, pp. 260–270, 2017.
[18] P. Bonacich, “Some unique properties of eigenvector centrality,” Social Networks, vol. 29, no. 4, pp. 555–564, 2007.
[19] ——, “Factoring and weighting approaches to status scores and clique identification,” Journal of Mathematical Sociology, vol. 2, no. 1, pp. 113–120, 1972.
[20] D. S. Bernstein, Matrix Mathematics, 2nd ed. Princeton University Press, 2009.
[21] E. Estrada and N. Hatano, “Communicability in complex networks,” Physical Review E, vol. 77, p. 036111, 2008.
[22] C. Klymko, “Centrality and communicability measures in complex network: Analysis and algorithms;” Ph.D. dissertation, Emory University, 2013.
[23] C. T. Chen, Linear System Theory and Design, 3rd ed. New York, NY, USA: Oxford University Press, Inc., 1998.
[24] P. C. Müller and H. I. Weber, “Analysis and optimization of certain qualities of controllability and observability for linear dynamical systems,” Automatica, vol. 8, no. 3, pp. 237–246, 1972.
[25] A. Olshevsky, “On (non)supermodularity of average control energy,” preprint. Available at https://arxiv.org/abs/1609.08706.
[26] J. F. Goncalves, D. M. D. Leon, and J. Fonseca, “A comparison study on actuator topology optimization designs using the controllability graminian,” in XXXVI Ibero-Latin American Congress on Computational Methods in Engineering, Rio de Janeiro, 11 2015.
[27] S. Gu, F. Pasqualetti, M. Cieslak, Q. K. Telesford, A. B. Yu, A. E. Kahn, I. D. Medaglia, J. M. Vettel, M. B. Miller, S. T. Grafton, and D. S. Bassett, “Controllability of structural brain networks,” Nature Communications, vol. 6, pp. 8414 EP–, 10 2015.

Fig. 8: Sample mean (left) and standard deviation (right) of the quantity $g$ in (21) as a function of the network size $n$.

Fig. 9: Sample mean of $\Gamma(n)$ and its best cubic-spline fit as a function of the network size $n$ for random Erdős-Rényi networks. The sample size is $10^4$ for each value of $n$. Both axes have logarithmic scaling.
which, given that $\gamma_n \leq \gamma_j \leq \gamma_2$ for all $j \in \{2, \ldots, n\}$, implies
\[
 w_{11} \gamma_1 + \sum_{j=2}^{n} w_{1j} \gamma_j \geq w_{11} \gamma_1 + \sum_{j=2}^{n} w_{ij} \gamma_j,
\]
for any $i \in \{2, \ldots, n\}$. This can be equivalently written as
\[
\sum_{j=1}^{n} w_{ij} \gamma_j \geq \sum_{j=1}^{n} w_{ij} \gamma_j \iff (W \gamma)_1 \geq (W \gamma)_i,
\]
completing the proof.

\section*{APPENDIX B}

\textbf{2k-Communicabilities of Simple Networks}

Here we provide analytic expressions for the node communicabilities in the line, ring, and star networks.\footnote{We omit the proofs in the revised version because paperplaza has a 13-page hard limit on submissions. If the paper is accepted, we will put the proofs back, as the hard limit for final versions is 15 pages. The proofs are also available in the single-column supplementary manuscript accompanying this paper.}

\textbf{Proposition B.1. (2k-communicabilities of line networks).} Consider a line network of $n$ nodes with uniform link weights $a$ (and no self-loops). Then, for $i \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$,
\[
 R_i(k) = a^{2k} \sum_{p \in I} \left( \frac{2k}{p + n + 1} \right) - \left( \frac{2k}{p + n + 1 - i} \right),
\]
where $I = \{-\lceil \frac{k}{n+1} \rceil, \ldots, \lceil \frac{k}{n+1} \rceil \}$ and $\left( \frac{i}{k} \right) = 0$ if $k \notin \{0, \ldots, n\}$. In particular, if $i \leq \lceil \frac{n}{2} \rceil$ and $k \leq \lceil \frac{n}{2} \rceil - 1$,
\[
 R_i(k) = a^{2k} \left( \frac{2k}{k} - \left( \frac{2k}{k - i} \right) \right).
\]

\textbf{Proposition B.2. (2k-communicabilities of ring networks).} Consider a ring network of $n$ nodes and uniform link weights $a$ (with no self-loops). Then, for $i \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$,
\[
 R_i(k) = \left( \frac{(2a)^{2k}}{n} \right) \left[ 1 + 2 \sum_{j=1}^{\lceil \frac{a}{2} \rceil - 1} \cos \left( \frac{2j \pi}{n} \right) + \delta_n^k \right],
\]
where $\delta_n^k$ equals one if $n$ is even and zero otherwise.

\textbf{Proposition B.3. (2k-communicabilities of star networks).} Consider a star network given by
\[
 A = \left[ \begin{array}{cc} l_c & a^T \\ a & l_p n_{-1} \end{array} \right],
\]

where $a \in \mathbb{R}^{n-1}$ contains the link weights between the center node and peripheral nodes. Then
\[
 R_i(k) = \frac{(\lambda_1 - l_p)^2}{(\lambda_1 - l_p)^2 + ||a||^2 \lambda_1^{2k}} + \frac{(l_p - \lambda_2)^2}{(l_p - \lambda_2)^2 + ||a||^2 \lambda_2^{2k}},
\]
\[
 R_i(k) = a^{2k-1} \left( \frac{\lambda_1 - l_p)^2}{(\lambda_1 - l_p)^2 + ||a||^2 \lambda_1^{2k}} + \frac{a^{2k-1}}{(l_p - \lambda_2)^2 + ||a||^2 \lambda_2^{2k}} \right.
\]
\[
 + \left. \frac{\lambda_1 - l_p)^2}{(l_p - \lambda_2)^2 + ||a||^2 \lambda_2^{2k}} \right),
\]
for all $k \in \mathbb{Z}_{\geq 0}$ and $i \in \{2, \ldots, n\}$, where
\[
 \lambda_{1,2} = \frac{l_c + l_p \pm \sqrt{(l_c - l_p)^2 + 4 ||a||^2}}{2}.
\]