Brownian motion approach to the ideal gas of relativistic particles

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The relativistic generalization of a free Brownian motion theory is presented. The global characteristics of the relaxation are explicitly found for the velocity and momentum (stochastic) kinetics. It is shown that the thermal corrections, to the both relaxation times $T$ (of stationary autocorrelations) and transient relaxation time of momentum, appear slowing down the processes. The transient relaxation time of the velocity does not depend explicitly on temperature, $T(v_0) = m(v_0)/\gamma \equiv \varepsilon_0/\gamma c^2$, and it is proportional to the initial energy of a relativistic Brownian particle.

During the last twenty years the Brownian motion approach has become the most important tool for discovering and studying the spectacular phenomena, the critical slowing down [1], noise–induced transitions [2], stochastic resonance [3], ratchet dynamics [4], and resonant activation [5], in (nonlinear) nonequilibrium systems. In the most of thermodynamical applications the nonlinear Langevin equation appears as an overdamped limit of a dissipative dynamic in a certain external potential [6]. The important conclusion of the present paper is that, within the Markovian diffusion theory, the nonlinear stochastic kinetic results already on the basic level of (spatially homogeneous) ideal gases description if the relativistic particles are considered. Particularly the relativistic free Brownian motion theory based on identical and non-covariant formula $\varepsilon W(p) = \varepsilon W(p')$, where $\varepsilon$ and $p$ should be expressed as functions of $v'$ and $p'$, according to the Lorentz transformation [9]. Because the time increments (counted from the initial preparation of the system) are given by $t'$ and $t = t'/(1 - V^2)$, respectively, the relation

\[ P'(p', t') = \frac{1 + V v'}{1 - V^2} P\left(\frac{p' + Ve'}{\sqrt{1 - V^2}}, \frac{t'}{\sqrt{1 - V^2}}\right) \]

is required. One has $v = \sqrt{v' + V}/(1 + V v')$, $\xi_t = (1 - V^2)^{1/4} \xi_t$, and, using the Stratonovich calculus, $dp/dt = (1 + V v') dp'/dt'$, thus the transformed stochastic kinetic (1) takes the form

\[ \frac{dp'}{dt'} = -\gamma v' + V \left(1 + V v'\right)^{1/4} \xi_t, \]

where $v' = p'/\sqrt{p'^2 + m^2}$ is used throughout the paper. $p = mv/\sqrt{1 - v^2}$ is a relativistic momentum of a Brownian particle, $v = p/\sqrt{p^2 + m^2}$ is constant both with the thermodynamical requirements of the equilibrium Gibbs ensemble theory and with the special relativity requirements of transformational properties of the momentum probability density distribution.

First note that the general Kramers–Fokker–Planck kinetic equation (see, e.g., the equation (26.26) of the Ref. [7]) of a Markovian diffusion description simplifies to the form

\[ \partial_t P = \partial_p L_{22}[(\partial_p H_1) P + k_B T \partial_p P] \]

for the spatially homogeneous system of free noninteracting particles of the Hamiltonian $H_1(p)$. The kinetic coefficient $L_{22} = L_{22}(p, T)$ is in principle determined by the properties of the interaction with the thermal bath and do not depend on $p$ for the potential interaction. For the latter case (of the state independent diffusion) the Langevin description Eq. (1), and the corresponding Fokker–Planck one [8], Eq. (2), are equivalent in view of the identification $L_{22} = \gamma = \gamma(T)$, $D = \gamma k_B T$ (the Einstein relation, or the fluctuation–dissipation theorem), and $\partial_p H_1 = v$. The Eq. (2) is the most general form of the diffusion in the momentum space leading to the Gibbs–Boltzmann stationary state $W(p) \propto e^{-\beta H_1(p)}$ of thermal equilibrium ($\beta = 1/k_B T$). The corresponding (nonlinearly coupled) Langevin equation (in the Stratonovich interpretation [8]) is

\[ \dot{v} = -g^2 v + D gg' + g \xi_t, \]

where $L_{22} = \gamma g^2$ and $g' = \partial g/\partial p$.

The corresponding probability density distributions $W(p)$ and $W'(p')$ in a resting and moving (with a constant velocity $V$) reference frame are related by non-covariant formula $\varepsilon W(p) = \varepsilon W'(p')$, where $\varepsilon$ and $p$ should be expressed as functions of $v'$ and $p'$, according to the Lorentz transformation [9]. Because the time increments (counted from the initial preparation of the system) are given by $t'$ and $t = t'/(1 - V^2)$, respectively, the relation

\[ P'(p', t') = \frac{1 + V v'}{1 - V^2} P\left(\frac{p' + Ve'}{\sqrt{1 - V^2}}, \frac{t'}{\sqrt{1 - V^2}}\right) \]

is required. One has $v = (v' + V)/(1 + V v')$, $\xi_t = (1 - V^2)^{1/4} \xi_t$, and, using the Stratonovich calculus, $dp/dt = (1 + V v') dp'/dt'$, thus the transformed stochastic kinetic (1) takes the form

\[ \frac{dp'}{dt'} = -\gamma v' + V \left(1 + V v'\right)^{1/4} \xi_t, \]

where $v' = p'/\sqrt{p'^2 + m^2}$. It is easy to verify that the r.h.s. of (4) solves the Fokker–Planck equation corresponding to the Eq. (5) if (and only if) $P(p, t)$ satisfies the proper Eq. (2) related to the kinetic (1). The same conclusion applies also to the general case (3).

The Eq. (1), $\dot{v} = -p/\sqrt{p^2 + m^2} + \xi_t$, for the momentum, or, equivalently, the (Stratonovich) equation for the velocity,

\[ \dot{v} = -\gamma v(1 - v^2)^{3/2}/m + \xi_t(1 - v^2)^{3/2}/m, \]

thus provides the relativistic generalization of the diffusion Markovian description of equilibration for noninteracting free particles system [10]. The normalized stationary distributions are

\[ W(p) = \frac{1}{2mK_1(m\gamma/D)} e^{-(\gamma/D)\sqrt{p^2 + m^2}} \]
and \[ W(v) = \frac{(1 - v^2)^{-3/2}}{2K_1(m\gamma/D)}e^{-v(D/\gamma)m/\sqrt{1-v^2}}, \] where \( \gamma = D\beta \), \( K_n \) is a modified Bessel function. The second moments read
\[ \langle p^2 \rangle = \frac{m K_2(m\beta)}{K_1(m\beta)}, \quad \langle v^2 \rangle = 1 - \int_{m\beta}^\infty dz K_0(z)/K_1(m\beta), \] so the (even) moments of the velocity, as incomplete integrals of Bessel functions, are not given in closed analytical form.

The nonlinear kinetic (1) or (6) does not belong to the class of the known solvable models and the time-dependent solution is not known. Nevertheless, there are still the quantities, useful to characterize the relaxation of the nonlinear systems, which can be expressed by quadratures of the (necessarily autonomous) drift and diffusion coefficients, without solving the kinetic equation. These are the complete (over time) integrals of the moments equal to the nonhomogeneous term (Tryagin equation for the mean first passage time [13] in nonstationary behavior, the formulae for the quantities of the ordinary Brownian motion are simply equal to the inverse of the relaxation rate of a linear model (1), \( T_v = T_p = 1 \), and do not depend explicitly on temperature [16]. In contrast both \( T_v \) or \( T_p \) increases with temperature, see Fig. 1, approaching the asymptotic \( T_v(\tau) \approx \tau + \pi/2 \) or \( T_p(\tau) \approx (5/2)\tau \), respectively. The latter asymptotic law follows from representation (9) [the complete integral of \( K_0 \) is equal to \( \pi/2 \)], the former asymptotic follows from the alternative form of the nonstationary autocorrelations in a relativistic case proceed slower in higher temperatures, and slower than for the ordinary Brownian motion. We want to stress however that, in the case of nonlinear system, the relaxation time \( T_x \) should not be interpreted as a characteristic of the equilibration process from a certain nonstationary state. One has in particular \( T_x(\tau = 0) = 1 \), which would be misunderstood that the (deterministic) evolution of the relativistic and classical system proceeds with the same (or at least similar) speed. Meanwhile it may be generally proved that at the limit \( \tau \to 0 \) only the lowest order terms of the drift and diffusion coefficients contribute to \( T_x \). Thus, such a conclusion applies at most to the relaxation from the close to the equilibrium states.

In order to compare the nonstationary behavior let us define the transient relaxation time
\[ T_\tau(x_0) = x_0^{-1} \int_0^x \rho(x)dx, \] which is related to the quantity \( X \), Eq. (10), via \( x_0T_\tau(x_0) = \langle x \rangle(x_0) \). Then \( P_\tau(p_0) \) satisfies
\[ -p_0P_\tau'/\sqrt{1 + p_0^2 + \tau^2} = -p_0. \]

The particular solution of the nonhomogeneous equation is \( P_\tau' = \sqrt{1 + p_0^2 + \tau^2} \), whereas the (general) solution of...
the homogeneous equation is singular at \( \tau = 0 \). Thus,
\[
T_\tau (p_0) = \left[ p_0 \sqrt{1 + p_0^2 + \log(p_0 + \sqrt{1 + p_0^2})}/2p_0 + \tau \right].
\] (16)

Similarly,
\[
\int_0^\infty dx(t)/x_0 = \int_0^\infty dx(t)/x_0 \text{ yields } T_\tau (x_0) \text{ as given by Eq. (16) or (17).}
\]

The deterministic (generic) Stokes equations, \( \dot{p} = -p/p^2 + 1 \) and \( \dot{v} = -v(1-v^2)^{3/2}, \) are solvable by quadratures. The solution, with the initial condition \( x_0 \), is
\[
t(x) = t_x(x) - t_x(x_0), \] (18)

where \( t_\rho(p) = -\sqrt{1 + p^2 + \log(1 + \sqrt{1 + p^2})}/p \), and \( t_\rho(v) = -(1 - v^2)^{-1/2} + \log(1 + \sqrt{1 - v^2})/v \), respectively. One can verify that the direct calculation of \( \int_0^\infty dx(t)/x_0 \) and \( \int_0^\infty dx(t)/x_0 \) yields \( T_\tau (x_0) \) as given by Eq. (16) or (17). The previous results (11) and (12) are also recovered when \( T_\tau (x_0) \) is integrated with \( x_0^2 W(x_0) \), where \( W \) is a stationary probability density distribution (8) or (7), respectively. Both (dimensionless) transient relaxation times for nonrelativistic Brownian motion are equal to unity. The relativistic relaxation of momentum depends both on the temperature and the initial state. \( T_\tau (p_0) \), Eq. (16), is an increasing function of both arguments and has the following asymptotic \( T_\tau (p_0) \approx 1 + \tau + p_0^2/6 \), for \( p_0 \to 0 \), and \( T_\tau (p_0) \approx |p_0|/2 + \tau \), for \( p_0 \to \infty \). The result for the velocity is somehow unexpected. The Eq. (17) shows that indeed the relativistic relaxation proceeds slower, however does not depend on temperature. It turns out that with the increase of temperature the early stage of evolution becomes faster in such a way that compensates the subsequent slower long-time behavior, see Fig. 3. The transient relaxation time \( T_\tau (v_0) \) is equal to the (dimensionless) initial energy of the relativistic Brownian particle \( \varepsilon_0 \). The plots of \( T_\tau (x_0) \) are shown in Fig. 2.

The equation (1) (in a generic form) has been solved numerically by the Runge–Kutta method for \( 10^3 \) sample realizations of the white Gaussian noise, for different initial conditions and temperatures. \( \langle p_t \rangle \) and \( \langle v_t \rangle \) have then been computed as the arithmetical average of appropriate values obtained for different trajectories of the noise. The typical curves are plotted in Figs. 3 and 4, together with the corresponding deterministic results (18). The solution of the relativistic Stokes equation for velocity changes qualitatively, when \( v_0 \) exceeds \( 1/2 \). In fact below this value of the initial state the function \( v(t) \) is convex down for all \( t \) [as well as (always) \( p(t) \)]. However for \( v_0 > 1/2 \) at the early stage of evolution the \( v(t) \) appears convex up [in order to achieve the required asymptote \( v(t) = 1 - c \) for \( v_0 \to 1 \)]. The former subrelativistic case, with particular \( v_0 = 2/5 \) (\( p_0 \approx 0.436 \)), and the latter, with \( p_0 = 4 \) (\( v_0 \approx 0.970 \)), are presented.

The results of the paper may be summarized as follows. We have shown that using the relations of the relativistic dynamics we can generalize the Brownian motion theory in a way consistent with thermodynamical requirements. The general solvable equation (10) for the quantities simply related to the transient relaxation times (15) has been found. These, as well the stationary relaxation times \( T \), have been found analytically for the relativistic momentum and velocity (stochastic) kinetics, Eqs. (13, 14, 16, 17). All these quantities (considered as dimensionless) are equal unity for the ordinary Brownian motion. The relaxation times \( T_\tau (\tau) \), by the normalization equal unity in the deterministic limit \( \tau \to 0 \), exhibit positive first order corrections for small temperatures, becoming just proportional to \( \tau \) in a high temperature asymptotic. Thus the decay of correlations in a stationary state proceed slower than for the nonrelativistic Brownian motion. The transient relaxation time (describing the relaxation from a given initial state \( x_0 \), and by the normalization equal unity for the deterministic system set initially at the equilibrium, \( T_0(0) = 1 \)) of momentum \( T_\tau (p_0) \) is a random function of temperature. The asymptotic behavior for the large argument(s) reads \( T_\tau (p_0) \approx |p_0|/2 + \tau \). The transient relaxation time for the velocity (17) does not depend on temperature, and it is equal to the dimensionless initial energy of the relativistic Brownian particle. The thermal corrections to the considered global characteristics of the relaxation are linear with the slope of the unity order and appear (even) for the noninteracting free relativistic Brownian particles system. The corrections can be observed for sufficiently high physical temperatures. For, e.g., hydrogen particle (proton) the dimensionless quantity, which characterize the deviation from the ordinary nonrelativistic behavior, \( \tau = k_B T/mc^2 \approx 10^{-13} T[K] \). So for the temperature \( T \sim 1 \text{MeV} \sim 10^{16} \text{K} \) the relative correction is of the order 0.1%. The correction to transient relaxation time of the velocity has the kinematic origin only and for \( T_\tau (p_0) \) both kinds of corrections are decoupled. Note that the general dependence on temperature in physical units, introduced (multiplicatively) by the coefficient \( \gamma(T) \), should be counted separately [16].

The three-dimensional generalization of the proposed diffusion description is obvious, however in contrast to the ordinary Brownian motion theory it leads to the set of the three coupled \( \dot{\vec{v}} = \vec{p}/\varepsilon = \vec{p}/m + \vec{p}/\varepsilon \) Langevin equations (with statistically independent noises in each direction) or equivalently to the (forward) Fokker–Planck description [8] \( D \nabla^2 \) the required form of the (Gibbs-Boltzmann) stationary distribution is still recovered, however the calculations of relaxation times cannot be analytically completed in this case.
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FIG. 1: Plots of $T_p$ [Eq. (14), upper curve] and $T_v$ [Eq. (13)] vs $\tau$ (dimensionless temperature). The latter has been computed numerically by the Romberg method.

FIG. 2: Plots of $T_0(p_0)$ [Eq. (16), $\tau = 0$, upper curve] vs $p_0$ and $T_r(v_0)$ [Eq. (17)] vs $10 \times v_0$.

FIG. 3: Plots of $\langle v_t \rangle v_0 / v_0$ vs $t$. The upper and lower curves correspond to $v_0 \approx 0.970$ and $v_0 = 2/5$; respectively. The deterministic solutions, Eq. (18), are marked by 0 ($\tau = 0$). The dotted curve, $e^{-t}$, corresponds to the nonrelativistic universal result. The remaining curves have been obtained numerically for $\tau = 0.2$ and 0.6, respectively.

FIG. 4: The same for $\langle p_t \rangle p_0 / p_0$ vs $t$. 
