Linear response theory for
diffeomorphisms with tangencies of stable
and unstable manifolds—a contribution to
the Gallavotti–Cohen chaotic hypothesis

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Abstract
This note presents a non-rigorous study of the linear response for an SRB (or ‘natural physical’) measure ρ of a diffeomorphism f in the presence of tangencies of the stable and unstable manifolds of ρ. We propose that generically, if ρ has no zero Lyapunov exponent, if its stable dimension is sufficiently large (greater than 1/2 or perhaps 3/2) and if it is exponentially mixing in a suitable sense, then the following formal expression for the first derivative of ρ(φ) with respect to f along X is convergent:

\[ \Psi(z) = \sum_{n=0}^{\infty} z^n \int \rho(dx) X(x) \cdot \nabla_x (\phi \circ f^n) \quad \text{for} \quad z = 1. \]

This suggests that an SRB measure may exist for small perturbations of f, with weak differentiability.

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1. Introduction

Let \( f \) be a diffeomorphism of the compact manifold \( M \) and \( \ell \) a probability measure with smooth density. Assume that \((f^n)^* \) applied to \( \ell \) has a weak limit \( \rho \) when \( n \to +\infty \), so that \( \rho \) is a `natural physical’ or SRB measure. If \( f \) and \( \rho \) depend on a real parameter \( \alpha \), we say that we have linear response if \( \alpha \to \rho(\phi) \) is \( C^1 \) when \( \phi : M \to \mathbb{R} \) is smooth. This is a physically significant fact which is known to hold if \( f_\alpha \) is uniformly hyperbolic (specifically if \( \rho_\alpha \) is an SRB measure on a mixing Axiom A attractor for \( f_\alpha \)).

Uniform hyperbolicity is uncommon in physical `chaotic’ systems (those for which \( \rho \) has a Lyapunov exponent > 0). Usually chaos follows if \( f \) effects some folding in the phase space \( M \) and this folding results in tangencies of the stable and unstable manifolds of \( \rho \), so that uniform hyperbolicity does not hold. The conditional measures of \( \rho \) on unstable manifolds have a density with a projection along the stable direction which (generically) has \( 1/\text{square root} \) singularity corresponding to tangencies.

If \( f_\alpha \) and \( \rho_\alpha \) depend on \( \alpha \), linear response means thus that \( \rho \) has a (weak) derivative with respect to \( \alpha \). This derivative can be formally computed, and the formal result may diverge because of the \( 1/\text{square root} \) singularities mentioned above. We shall however argue that if the unstable manifolds are piled up sufficiently densely, corresponding to a large stable dimension \( d_S \) for \( \rho_\alpha \) (\( d_S > 1/2 \) or perhaps >3/2) this beats the \( 1/\text{square root} \) singularities and linear response may hold. We follow here the ideas of an earlier paper [15] with more precise calculations.

Note that \( \rho \) is more likely to have a large stable dimension \( d_S \) when \( \dim M \) increases (systems with many degrees of freedom).

The arguments presented here definitely do not constitute a rigorous proof but support a physically relevant conjecture in section 6 below. This conjecture agrees in particular with the fact (observed in computer experiments) that many nontrivial time evolutions behave as if they corresponded to uniformly hyperbolic dynamics (this is the chaotic hypothesis of Gallavotti–Cohen [2, 3]). (A weaker and vaguer form of the conjecture is that a change of regime of linear response occurs when the unstable dimension of \( \rho \) increases beyond 1/2).

We assume sufficient differentiability of \( f \), say \( C^3 \). Flows are not discussed in the present paper.

2. Formal linear response formula

Let \( M \) be a smooth compact manifold and \( f : M \to M \) a diffeomorphism. We denote by \( \rho \) a `natural physical’ measure on \( M \). We may take this to mean that \( \rho \) is a weak limit

\[
\rho = \lim_{n \to \infty} (f^n)^* \ell
\]

where \( \ell \) is a probability measure on \( M \) absolutely continuous with respect to Lebesgue (i.e. with respect to a Riemann volume). In good cases (1) means that \( \rho \) is an SRB measure.

SRB measures (Sinai, Ruelle, Bowen measures, see for instance [18]) were first defined for uniformly hyperbolic diffeomorphisms. A modern mathematical definition of an SRB measure \( \rho \) is that it is an \( f \)-invariant probability measure which satisfies either of the following equivalent conditions (Ledrappier–Strelcyn [5], Ledrappier–Young [6]):

(i) the (Kolmogorov–Sinai) entropy of \( (\rho, f) \) is the sum of the positive Lyapunov exponents,
(ii) the conditional measures of \( \rho \) on local unstable manifolds of \( (\rho, f) \) are absolutely continuous with respect to Lebesgue on the unstable manifold.
(We may take \( \rho \) to be ergodic. We shall define Lyapunov exponents and unstable manifolds in section 4.)

If \( X \) is a vector field on \( M \), which we may think of as infinitesimal, we perturb \( f : x \to fx \) to \( \tilde{f} : x \to fx + X(fx) \). We have to first order in \( X \):

\[
\tilde{f}^k x = f^k x + \sum_{j=1}^k (T_{f^j} f^{k-j})X(f^j x)
\]

where \( T_x f \) denotes the tangent map to \( f \) at the point \( x \).

If the perturbation \( \tilde{f} \) of \( f \) replaces \( \rho \) by an SRB measure \( \rho + \delta \rho \) for \( \tilde{f} \) we call \( \delta \rho \) the linear response. If \( \phi \) is a smooth function on \( M \) we have

\[
\phi(\tilde{f}^k x) = \phi(f^k x) + \nabla_{f^k x}(\phi \circ f^{k-j})
\]

hence

\[
\delta \rho(\phi) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \sum_{j=1}^k \int \ell(dx) X(f^j x) \cdot \nabla_{f^j x}(\phi \circ f^{k-j})
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \sum_{j=1}^k \int ((f^j)^* \ell(dx)) X(x) \cdot \nabla_x(\phi \circ f^{k-j})
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-i} \sum_{j=1}^i \int ((f^j)^* \ell(dx)) X(x) \cdot \nabla_x(\phi \circ f^i).
\]

If we interchange in the right-hand side \( \lim_{n \to \infty} \) and \( \sum_{i=0}^{n-i} \) (without a good mathematical justification), and use \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-i} (f^j)^* \ell = \rho \) we obtain the formal linear response formula

\[
\delta \rho(\phi) = \sum_{n=0}^{\infty} \rho(dx) X(x) \cdot \nabla_x(\phi \circ f^n) = \Psi(1)
\]

where we have defined the susceptibility function

\[
\Psi(z) = \sum_{n=0}^{\infty} z^n \int \rho(dx) X(x) \cdot \nabla_x(\phi \circ f^n).
\]

Suppose that \( f = f_{\alpha}|_{\alpha=0} \) where \( \alpha \) is in a neighborhood of 0 in \( \mathbb{R} \) and \( f_{\alpha} \) has SRB measure \( \rho_{\alpha} \), we would like to know if

\[
\left[ \frac{d}{d\alpha} \int \rho_{\alpha}(dx) \phi(x) \right]_{\alpha=0} = \Psi(1)
\]

where \( \Psi \) is defined with \( X(fx) = d f_{\alpha} x / d\alpha |_{\alpha=0} \). In fact we shall be mostly concerned with the preliminary step of studying the convergence of \( \Psi(1) \).
3. Results on linear response

The main positive result is that if \( \alpha \to f_\alpha \) is \( C^3 \) from a neighborhood of 0 in \( \mathbb{R} \) to \( C^3 \) diffeomorphisms of \( M \), and \( \rho_\alpha \) is SRB on a mixing Axiom A attractor for \( f_\alpha \), then (4) holds if \( \phi \) is \( C^2 \). (For this result see Ruelle [16], and also Gouëzel-Liverani [4]). We may here assume that \( \rho_\alpha \) is defined by (1) with \( f_\alpha \) replacing \( f \) and \( \ell \) has its support in a neighborhood of the compact attractor \( A_\alpha \) such that \( A_\alpha = \lim_{n \to \infty} f_\alpha^n \operatorname{supp} \rho \), where \( f_\alpha \) satisfies uniform hyperbolicity assumptions on \( A_\alpha \).

The above positive result is based on uniform hyperbolicity: the tangent bundle \( T_\alpha M \) over the attractor \( A_\alpha \) is the continuous direct sum of a contracting vector bundle \( E^s \) and an expanding vector bundle \( E^u \). Write \( X = X^s + X^u \) with \( X^u, X^s \in E^u \), then (3) becomes

\[
\Psi(z) = \sum_{n=0}^{\infty} z^n \int \rho(dx) X^s(x) \cdot \nabla_x (\phi \circ f^n) - \sum_{n=0}^{\infty} z^n \int \rho(dx) (\operatorname{div} X^u(x)) (\phi(f^n x))
\]

(5)

where \( \operatorname{div} X^u \) is the divergence of \( X^u \) along a piece of unstable manifold containing \( x \), with its natural measure. The contracting term \( \int \rho(dx) X^s(x) \cdot \nabla_x (\phi \circ f^n) \) tends to 0 exponentially with respect to \( n \). The expanding term \( \int \rho(dx) (\operatorname{div} X^u(x)) (\phi(f^n x)) \) is a correlation function of \( \operatorname{div} X^u(x) \) and \( \phi \) which also tends to 0 exponentially with respect to \( n \). Therefore the radius of convergence of \( \Psi \) is \( > 1 \) and \( \Psi(1) \) is well defined.

Another result holds if \( f \) preserves a smooth volume \( \rho \) on \( M \) and if \( (f, \rho) \) is exponentially mixing. Then (3) can be rewritten as

\[
\Psi(z) = -\sum_{n=0}^{\infty} z^n \int \rho(dx) (\operatorname{div} X(x)) (\phi(f^n x))
\]

and the radius of convergence of \( \Psi \) is \( > 1 \).

In computer studies, nontrivial dynamics of moderate dimension can sometimes lead to very long characteristic times, or sometimes to systems that behave like uniformly hyperbolic systems (chaotic hypothesis [2, 3], which remains to be explained).

There are a number of other results, positive and negative, for which we refer to Baladi [1]. In particular for interval maps it is known that \( \alpha \to \tilde{\rho}_\alpha \) is often discontinuous [17]; in fact \( X^s(x) = X(x) \) often has singularities \( \sim 1/\sqrt{\pm (x - x_j)} \) so that \( \operatorname{div} X(x) \) does not make sense as a function\(^1\).

There is an important literature on stable–unstable tangencies typified by Newhouse’s infinitely many sinks (see [8–11]). This literature refers to dynamically unstable situations. Here we try to investigate the opposite stable situation where SRB measures change differentiably under a small change of the diffeomorphism \( f \).

4. Stable and unstable manifolds

Let \( \rho \) be an ergodic measure on an attractor \( A \) for the diffeomorphism \( f \) of the manifold \( M \). In this Section \( \rho \) need not be SRB but we assume that \( \rho \) is hyperbolic namely that all Lyapunov exponents (defined below) of \( \rho \) are different from 0. We describe now briefly the theory of stable and unstable manifolds (Pesin [12, 13] in a version due to Ruelle [14]).

Define \( T^s_x = T_x f^s \), where \( T_x f : T_x M \to T_x M \) is the tangent map to \( f \) at \( x \in M \). For \( \rho \)-almost all \( x \) one can show that the following limit exists:

\(^1\) We use \( \sim \) to denote proportionality, while \( \approx \) will denote approximate equality.
\[
\lim_{n \to \infty} \left( T_x^\mu T_x^\nu \right)^{1/2n} = \Lambda_x
\]
and its eigenvalues \( \exp \lambda^{(1)} < \cdots < \exp \lambda^{(k)} \) with multiplicities \( m^{(1)}, \ldots, m^{(k)} \) are \( x \)-independent for \( \rho \)-almost all \( x \) (note that we have \( k \leq \dim M \)). The \( \lambda^{(i)} \) are called the Lyapunov exponents of \( \rho \). Let \( U_x^{(1)}, \ldots, U_x^{(k)} \) be the eigenspaces of \( \Lambda_x \). Writing \( V_x^{(0)} = \{0\} \) and \( V_x^{(i)} = U_x^{(1)} \oplus \cdots \oplus U_x^{(i)} \), we have
\[
\frac{1}{n} \log \| T_x^n u \| = \lambda^{(i)} \quad \text{when} \quad u \in V_x^{(i)} \backslash V_x^{(i-1)}.
\]
If \( \lambda^{(i)} \) is the largest negative Lyapunov exponent the space \( V_x^{(i)} = E_x^s \) is the stable subspace at \( x \) and \( (E_x^s)_{\rho \in A} = E^s \) the contracting vector bundle. Replacing \( f \) by \( f^{-1} \) replaces \( E_x^s \) by the unstable subspace \( E_x^u \) and \( (E_x^u)_{\rho \in A} = E^u \) is the expanding vector bundle.

If \( 0 < a < -\lambda^{(i)} \) and \( b > 0 \) we define a local exponential stable manifold
\[
V_x^a = \{ y \in M : d(f^n y, f^n x) < be^{-na} \quad \text{for all} \quad n \geq 0 \}.
\]
This stable manifold is a nonlinear version of the stable subspace \( E_x^s \) as follows from (7). The definition (8) implies that \( T_{f^{i-1}x} f \) is exponentially close to \( T_{f^{i-1}x} f \) for \( n \to \infty \). From this one obtains a perturbation theorem: if \( b \) is sufficiently small and \( y \in V_x^a \), then \( \Lambda_x \) has the same eigenvalues as \( \Lambda_y \) including multiplicity and there is \( B > 0 \) such that \( || V_y^{(i)} - V_y^{(i)} || \leq Bd(y, x) \).

In fact \( E_x^s \) is the tangent space to \( V_x^a \) at \( y \in V_x^a \) and depends continuously on \( y \). Therefore \( V_x^a \) is a differentiable manifold. One can deal with higher derivatives as with \( T_x f \) so that the stable manifolds are as smooth as \( f \).

Similarly for the unstable manifolds \( V_x^u \). Note furthermore that these unstable manifolds are contained in the attractor \( A \).

5. Stable-unstable tangencies

A simple feature of the diffeomorphism \( f \) which can lead to positive Lyapunov exponents (chaotic behavior) and exponential decay of correlations is when \( f \) folds the attractor \( A \). This folding causes tangencies of the stable and unstable manifolds, which we now want to study. The set of points of tangency of a stable and an unstable manifold has \( \rho \)-measure 0 in \( M \) since \( \rho \) has no zero Lyapunov exponent. One can nevertheless deal with stable–unstable tangencies as follows.

In the framework of section 4 the stable manifolds \( V_x^s \) and the unstable manifolds \( V_x^u \) defined \( \rho \)-almost everywhere, depend measurably on \( x \) together with their tangential derivatives. Let the open neighborhoods \( N \) of a point of \( M \) have measure \( \rho(N) > 0 \) and write again \( V_x^s \) and \( V_x^u \) for the restrictions of these manifolds to \( N \). Using Luzin’s theorem we can choose \( N \) and a family \( (V_y^s)_{y \in S} \) with compact \( S \subset N \), where each \( V_y^s \) contains only one \( x \in S \), such that the \( V_y^s \) and tangential derivatives depend continuously on \( x \) and \( \rho(N \setminus \cup_{y \in S} V_y^s) < \epsilon \) for small \( \epsilon \). We can also choose a similar family \( (V_y^u)_{y \in U} \) of unstable manifolds.

In brief we can, locally and up to a set of small \( \rho \)-measure, consider that the stable manifolds form a family (a pile) \( (V_x^s)_{x \in S} \) continuous together with their tangential derivatives. A pile \( (V_y^u)_{y \in U} \) is similarly defined for unstable manifolds. Since we have control of the second derivatives of the stable and unstable manifolds we can impose that the intersections \( V_x^s \cap V_y^s \) are interior to \( V_x^s \) and \( V_y^s \) and each is either empty, or consists of two points, or of one regular tangency point of \( V_x^s \), \( V_y^u \).
We shall take coordinates such that the stable manifolds \((V^s_x)_{x \in \mathcal{S}}\) are roughly parallel and the unstable manifolds \((V^u_y)_{y \in \mathcal{U}}\) are folded so that the intersections \(V^s_x \cap V^u_y\) are as above. Note that the folding of unstable manifolds which gives rise to tangencies gives folds dense in the support of \(\rho\), and this explains why part of \(\mathcal{N}\) has to be excluded from the pile \((V^u_y)_{y \in \mathcal{U}}\). Similarly for stable manifolds.

We choose now a manifold \(\mathcal{W}\) with \(\dim \mathcal{W} = \dim \mathcal{V}^u\) such that \(\mathcal{W}\) is ‘parallel to the unstable fold’ namely transversal to the \(\mathcal{V}^s_x\), and we project \(\mathcal{N}\) on \(\mathcal{W}\) along the \(\mathcal{V}^u_y\). This projection \(\varpi\) is ‘almost’ absolutely continuous from each \(\mathcal{V}^u_y\) with its natural measure to \(\mathcal{W}\) with Lebesgue measure (some Riemann measure). The corresponding Jacobian \(J(x)\) is ‘almost’ Hölder continuous with Hölder exponent close to 1 except along the fold (the stable–unstable tangency points of \(V^s_x\)). The Jacobian (generically) has a \(1/\text{square root}\) singularity transversal to the projection of the fold on \(\mathcal{W}\).

6. A conjecture on linear response in the presence of stable–unstable tangencies

In the presence of regular tangencies of stable–unstable manifolds (a condition on second-order derivatives for the piles \((V^s_x)_{x \in \mathcal{S}}\) and \((V^u_y)_{y \in \mathcal{U}}\), and with a genericity assumption on the projection along the stable direction, we expect the following:

If the SRB measure \(\rho\) for the diffeomorphism \(f\) has no zero Lyapunov exponent, if its stable dimension is sufficiently large (greater than 1/2 or perhaps 3/2) and if it is exponentially mixing in a suitable sense, then the following formal expression for the first derivative of SRB with respect to \(f\) along \(X\) evaluated at \(\phi\) is convergent:

\[
\Psi(z) = \sum_{n=0}^{\infty} z^n \int \rho(\mathrm{d}x) X(x) \cdot \nabla x (\phi \circ f^n) \quad \text{for} \quad z = 1.
\]

The nonrigorous nature of the following study of \(\Psi\) results in part from the fact that we shall (in the spirit of [15]) ignore the small measure sets \(\mathcal{N} \setminus \cup_{x \in \mathcal{S}} \mathcal{V}^s_x\) and \(\mathcal{N} \setminus \cup_{y \in \mathcal{U}} \mathcal{V}^u_y\).

7. Estimating the radius of convergence of \(\Psi\)

We consider the contribution \(\Psi_{\mathcal{N}}(z)\) of \(\rho|\mathcal{N}\) to the susceptibility:

\[
\Psi_{\mathcal{N}}(z) = \sum_{n=0}^{\infty} z^n \int_{\mathcal{N}} \rho(\mathrm{d}x) X(x) \cdot \nabla x (\phi \circ f^n)
\]

in order to estimate how the fold in \(\mathcal{N}\) influences the radius of convergence of \(\Psi\).

Write \(\rho(\mathrm{d}x) = \int \sigma(\mathrm{d}x) \rho_x(\mathrm{d}x)\) where \(\rho_x\) is the natural measure on the unstable manifold \(\mathcal{V}^u_x\) and \(\sigma(\mathrm{d}x)\) is a ‘stable’ measure carried by a set of Hausdorff dimension \(d_\Sigma\). The projection \(\varpi[X(\xi)\rho_x(\mathrm{d}\xi)]\) has a ‘almost’ Hölder density on \(\mathcal{W}\) (Hölder exponent close to 1 if \(\mathcal{N}\) is small) with a \(1/\text{square root}\) singularity transversal to the projection of the fold of \(\mathcal{V}^u_x\) on \(\mathcal{W}\).

We have used here the philosophy that we can ignore the small-measure sets \(\mathcal{N} \setminus \cup_{x \in \mathcal{S}} \mathcal{V}^s_x\) and \(\mathcal{N} \setminus \cup_{y \in \mathcal{U}} \mathcal{V}^u_y\).

We parametrize \(\mathcal{W}\) by variables \(w, \theta\) with \(\theta \in \mathbb{R}\) such that the lines \(\ell_w\) given by \(w = \text{constant} (\text{and parametrized by} \theta)\) are transversal to the folds.

The projection \(\varpi[X(\xi)\rho(\mathrm{d}\xi)]\) restricted to \(\ell_w\) contains a factor due to \(X(\xi)\) and we can estimate its density with respect to \(\theta\) as
\[ \Delta(\theta) = \int \sigma(dx) \Delta_\theta(\theta). \] (10)

Here \( \Delta_\theta(\theta) \) is the projection \( \pi[X(\xi \rho_x(d\xi))] \) restricted to \( \ell_\omega \) so that \( \Delta_\theta(\theta) \sim 1/\sqrt{\pm(\theta - \theta)} \) where \( \theta \) depends on \( \omega \) (the presence of the square root rather than another power is a genericity assumption as mentioned in section 6).

Define \( \psi(\tau) = \int_{\theta < \tau} \sigma(dx) \) or \( \psi(\tau) = -\int_{\theta > \tau} \sigma(dx) \). The measure \( d(\psi(\tau)) \) is a 1-dimensional projection of the \( d_\xi \)-dimensional measure \( \sigma(dx) \). Using Marstrand’s theorem [7] we assume that \( d(\psi(\tau)) \) has dimension \( d_\xi = \min\{1, d_\xi\} \) (this is again a genericity assumption as mentioned in section 6). Therefore \( |\psi(\theta) - \psi(\tau)| < |\theta - \tau|^{d_\xi} \) up to a multiplicative constant. We can estimate \( \Delta(\theta) \) in terms of

\[ \int \sigma(dx) \Delta_\theta(\theta) \sim \int_{\tau < \theta} \frac{d(\psi(\tau))}{\sqrt{\theta - \tau}} + \int_{\tau > \theta} \frac{d(\psi(\tau))}{\sqrt{\tau - \theta}} \leq 2d_\xi \int_0^T \rho_\xi^{-3/2} \, dt \]

with \( T \) depending on the support of \( \rho \).

If \( d_\xi > 1/2 \) we show now that \( \Delta \) is Hölder continuous. This will replace the assumptions and loose calculations in section 4.A,B of [15]. For \( \delta > 0 \) we can estimate \( \Delta(\theta + \delta) - \Delta(\theta) \) in terms of

\[ \int_{-T}^{\theta + \delta} \frac{d(\psi(\tau))}{\sqrt{\theta + \delta - \tau}} - \int_{-T}^{\theta} \frac{d(\psi(\tau))}{\sqrt{\theta - \tau}} \quad \text{and} \quad \int_{-T}^{\theta} \frac{d(\psi(\tau))}{\sqrt{\tau - \theta}} - \int_{-T}^{\theta + \delta} \frac{d(\psi(\tau))}{\sqrt{\tau - \delta - \theta}} \] (11)

The first term of (11) is

\[ \int_{-T}^{\theta + \delta} \frac{d(\psi(\tau))}{\sqrt{\theta + \delta - \tau}} + \int_{-T}^{\theta} \frac{d(\psi(\tau))}{\sqrt{\theta - \tau}} - \int_{-T}^{\theta + \delta} \frac{d(\psi(\tau))}{\sqrt{\theta - \tau}} \]

\[ + \int_{-T}^{\theta + \delta} \left( \frac{1}{\sqrt{\theta + \delta - \tau}} - \frac{1}{\sqrt{\theta - \tau}} \right) d(\psi(\tau)). \]

With \( t = \theta + \delta - \tau \) we have

\[ \left| \int_{-T}^{\theta + \delta} \frac{d(\psi(\tau))}{\sqrt{\theta + \delta - \tau}} - \int_{-T}^{\theta + \delta} \frac{d(\psi(\tau))}{\sqrt{\theta - \tau}} \right| \sim \int_0^T \left| \frac{d(\psi(\tau))}{\sqrt{\tau}} \right| < d_\xi \int_0^T \rho_\xi \, dt = \left( \frac{d_\xi}{d_\xi - 1/2} \right) \delta_\xi^{d_\xi - 1/2} \] (13)

and similar estimates for the next two terms of (12). Furthermore

\[ \left| \int_{-T}^{\theta + \delta} \left( \frac{1}{\sqrt{\theta + \delta - \tau}} - \frac{1}{\sqrt{\theta - \tau}} \right) d(\psi(\tau)) \right| < \frac{\delta}{2} \int_{-T}^{\theta + \delta} \left| \frac{d(\psi(\tau))}{\sqrt{\tau}} \right| (\theta - \tau)^{-3/2} \]

\[ < d_\xi \frac{\delta}{2} \int_{-T}^{\theta + \delta} (\theta - \delta - \tau)^{d_\xi - 1} (\theta - \tau)^{-3/2} \, d\tau \approx d_\xi \frac{3\delta}{4} \int_{-T}^{\theta + \delta} (\theta - \delta - \tau)^{d_\xi - 1} (\theta - \tau)^{-5/2} \, d\tau \]

\[ < d_\xi \frac{3\delta}{4} \int_{-T}^{\theta + \delta} (t - \delta)^{d_\xi - 5/2} \, dt \leq d_\xi \frac{3\delta}{4} \int_{-T}^{\theta + \delta} t^{d_\xi - 5/2} \, dt \sim \delta_\xi |\theta - \delta|^{d_\xi - 3/2} |\theta + T|^{d_\xi + T} \approx \delta_\xi^{d_\xi - 1/2}. \] (14)

From (12)–(14) and similar results for the second term of (11) we see that \( \Delta \) is \( \langle d_\xi - 1/2 \rangle \)-Hölder continuous.

We note that the Hölder continuity of \( \Delta \) implies that \( \int_X \rho(d\xi) X(\xi) : \nabla_\xi (\phi \circ f^\rho) \to 0 \) exponentially when \( n \to \infty \) if the correlations for \( \rho \) decay exponentially in a suitable sense. In this case \( \Psi^N(1) \) is thus well defined.
Writing $d_\delta$ as a sum of partial dimensions one may conjecture that $\psi$ is $d_\delta$-Hölder, and therefore differentiable if $d_\delta > 1$. The derivative $\Delta'$ of $\Delta$ satisfies then

$$\Delta' (\theta) \sim \int_{-\tau}^{\theta} \frac{d \psi' (\tau)}{\sqrt{\theta - \tau}} + \int_{\theta}^{T} \frac{d \psi' (\tau)}{\sqrt{\tau - \theta}}$$

The same argument as above shows that $\Delta'$ is $(\min(2, d_\delta) - 3/2)$-Hölder if $d_\delta > 3/2$. The divergence $\mathcal{D}(\eta)$ of the projection $\varpi[X(\xi) \rho(d\xi)]$ on $\mathcal{W}$ is thus also $(\min(2, d_\delta) - 3/2)$-Hölder continuous (this results from the arbitrariness of the choice of the line $\ell_\delta$ if $\dim \mathcal{W} > 1$).

Returning to (9) we see that

$$\int_{\mathcal{B}} \rho(d\xi) X(\xi) \cdot \nabla \xi (\phi \circ f^n) \sim - \int \rho(\eta) \nabla \eta$$

is a correlation function of Hölder functions which we assume to decay exponentially. Therefore the radius of convergence of $\Psi(\alpha)(z)$ is $> 1$. Since $\Psi(\alpha)(z)$ is the most singular part of $\Psi(z)$, we find that the radius of convergence of $\Psi(z)$ is $> 1$ and that the formal expression of the derivative (4) converges.

8. Conclusions

To summarize we have shown that if the stable dimension $d_\delta$ of $\rho$ is $> 1/2$ then $\varpi[X(\xi) \rho(d\xi)]$ is ‘approximately’ Hölder, and if $d_\delta > 3/2$ then $\varpi[\nabla \xi \cdot X(\xi) \rho(d\xi)]$ is ‘approximately’ Hölder. These Hölder conditions and a suitable property of exponential decay of correlations for $\rho$ imply that $\Psi(\alpha)(1)$ and $\Psi(1)$ are well defined. Therefore the derivative w.r.t. $f$ of the SRB measure $\rho$ makes sense generically in the presence of stable–unstable tangencies if $d_\delta$ is sufficiently large and there is a suitable exponential decay of correlations.

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