QUASI-POSITIVITY AND RECOGNITION OF PRODUCTS OF CONJUGACY CLASSES IN FREE GROUPS

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Abstract. We formulate an algorithm for recognizing whether a given word in a free group \( F(X) \) is equal to a product of conjugates of positive powers of basis elements. Such a word is called quasi-positive. The study of quasi-positivity in braid groups has important connections to contact topology and knot theory. Our investigation is motivated by this problem. The algorithm we construct is based on a theorem of Gersten that gives a topological characterization of when, for a given finite sequence \( (w_1, \ldots, w_k) \) of words in \( F(X) \), there exist elements \( c_1, \ldots, c_k \in F(X) \) such that \( w_1^{c_1} \cdots w_k^{c_k} = 1 \).

1. Introduction

A natural topological question is whether a given link \( L \) in 3-dimensional Euclidean space is equivalent to the closure of a braid in which all of the strands cross in a positive sense. Such a braid is called a positive braid. The closure of the conjugate of a braid defines an equivalent link. Analogously, one might study when a link is equivalent to the closure of a product of conjugacy classes of positive braids; such a link is called quasi-positive (see, for instance, [4], [8], [15], [1], [5], [10], [11], [12], [9], [14], and [13]).

Quasi-positivity has a striking geometric interpretation: Rudolph proved that \( \mathbb{C} \)-transverse links, i.e. links which are the intersection of a complex curve and a sphere in \( \mathbb{C}^2 \) transverse to the given curve, are quasi-positive; and Boileau and Orekov proved the converse (see section 3.2.2 of [4] for references and further discussion).

Quasi-positivity is also natural from an algebraic perspective: an element of the braid group is quasi-positive if and only if it belongs to the monoid that is normally generated by the standard braid generators. The problem of deciding whether or not an element is quasi-positive is at least as challenging as the conjugacy decision problem. Public key sharing protocols have been proposed based on the difficulty of the conjugacy search problem [2], and so the study of quasi-positivity might have similar applications.

1.1. Quasi-positivity in abstract groups. Quasi-positivity is a condition on elements of a group \( G \) having a distinguished subset \( P \) consisting of a priori positive elements. An element \( g \in G \) is defined to be quasi-positive if it is equal in \( G \) to a product of conjugates of elements of \( P \). A natural starting point for this algebraic formulation of quasi-positivity is the case when the group is the free group \( F(X) \) with basis \( X \), and the basis \( X \) is the set of a priori positive elements. Hence, the question we pose is the following:

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Can one decide if a given word over $X$ represents a quasi-positive element of $F(X)$?

In this article, we give an algorithmic solution to this question. We hope that this investigation may be helpful to the study of quasi-positivity in braid groups.

1.2. Products of conjugacy classes in free groups. Gersten [6] studied the following problem:

Given $w_1, \ldots, w_k \in F(X)$, when do there exist $c_1, \ldots, c_k \in F(X)$ such that $w_1^{c_1} \cdots w_k^{c_k} = 1$?

Gersten reformulates this question in terms of the topology of surfaces as follows. Since the sequence $(w_1, \ldots, w_k)$, consisting of freely reduced words over $X$, involves only finitely many letters, we may suppose that $X$ is finite, say $X = \{x_1, \ldots, x_n\}$. Let $M$ be an oriented 2-sphere with $k$ mutually disjoint open disks removed. Let $\Gamma_1, \ldots, \Gamma_k$ be the oriented boundary circles. For each $i = 1, \ldots, k$, subdivide $\Gamma_i$ by choosing $|w_i|$ points as vertices, where $|a|$ denotes the length of a reduced word $a \in F(X)$. Choose one vertex to be the initial vertex on $\Gamma_i$, and then label the initial vertex by the first letter of $w_i$, the next (in the order determined by the orientation) vertex by the next letter of $w_i$, and so forth. For instance, if $w_1 = x_2x_1^{-1}x_2$, then $\Gamma_1$ is subdivided by choosing three vertices with labels $x_2$, $x_1^{-1}$, and $x_2$, respectively, in that order. Gersten’s theorem can now be stated as follows:

**Theorem** (Gersten, 1986, Theorem 5.2 in [6]). There exist $c_1, \ldots, c_k \in F(X)$ such that $w_1^{c_1} \cdots w_k^{c_k} = 1$ if and only if there exist pairwise disjoint properly embedded tame arcs in $M$ which join vertices labeled by inverse letters, i.e. one labeled by $x$ and one by $x^{-1}$ for some $x \in X$.

A spherical diagram of the type described in the theorem is called a cancellation diagram for $(w_1, \ldots, w_k)$ over $F(X)$. Gersten outlines a proof of this theorem and also states that this theorem also follows from work of Culler [3] and Goldstein–Turner [7].

We will show how to apply Gersten’s theorem to give an algorithmic solution to the problem of determining whether or not a given word is quasi-positive in $F(X)$.

2. Quasi-positivity Algorithms for Free Groups

Let $F(X)$ be the free group with basis $X$. Let $Y = X \cup X^{-1}$, and let $Y^*$ be the free monoid on $Y$. The natural surjection $Y^* \to F(X)$ is defined by freely reducing. The positive elements are elements of the positive monoid $F^+(X)$ consisting of freely reduced words with no occurrence of a letter in $X^{-1}$. By convention, $a^b = b^{-1}ab$. As in the introduction, we define an element $w \in F(X)$ to be quasi-positive if it admits a factorization of the form $w = w_1^{c_1} \cdots w_k^{c_k}$ such that every $w_i$ is positive. The starting point of our investigation is the following observation:

**Lemma 1.** If $w \in F(X)$ is quasi-positive, then $w = w_1^{c_1} \cdots w_k^{c_k}$, where each $w_i \in X$. Moreover, if $\Theta : F(X) \to \mathbb{Z}^X$ is the abelianization map, then the number of $w_i$ that are equal to some $x \in X$ is equal to the $x$-coordinate of $\Theta(w)$.

**Proof.** If $w$ is quasi-positive, then $w$ is equal to a word of the form $w_1^{c_1} \cdots w_k^{c_k} \in F(X)$ such that every $w_i \in F^+(X)$. If $w_i$ has more than one letter, then we can expand using the property that $(ab)^c = a^c b^c$. Assume that this has been done so that every $w_i \in X$. To see that the number of factors is computed by $\Theta$, observe that $\Theta(w_1^{c_1}) = \Theta(w_1)$ adds one to the $x$-coordinate of $\Theta(w)$ if and only if $w_1 = x$. \qed
We relate this to Gersten’s theorem as follows. Suppose that \( w \) is quasi-positive. By Lemma 1, there exist \( w_1, \ldots, w_k \in F^+(X) \) and \( c_1, \ldots, c_k \in F(X) \) such that
\[
w(w_1^{-1})^{c_1} \cdots (w_k^{-1})^{c_k} = 1.
\]
By Gersten’s theorem, there is a spherical diagram with \( k + 1 \) boundary circles: one circle is subdivided into \( |w| \) arcs and vertices and these vertices are labeled by the letters in \( w \), and the other \( k \) circles have a single vertex and are labeled by some \( w_i^{-1} \in X^{-1} \). We refer to the letters \( w_1, \ldots, w_k \in X \) in such a factorization as the base letters of \( w \). The multiset \( \{w_1, \ldots, w_k\} \) of base letters is an invariant of a quasi-positive element. We will see in subsection 2.2 that a quasi-positive element may admit more than one factorization as a product of conjugates of positive letters.

Conversely, we deduce from Gersten’s theorem and Lemma 1 that the question of whether a given \( w \in F(X) \) is quasi-positive is equivalent to whether there is a cancellation diagram for \( (w, x_1, \ldots, x_k) \) over \( F(X) \) where \( k = |\Theta(w)| \), the sum of the coordinates of \( \Theta(w) \), and the number of \( x_i \) equal to a given \( x \in X \) is the \( x \)-coordinate of \( \Theta(w) \).

It is not immediately clear how to decide for a given \( w \in F(X) \) whether or not such a cancellation diagram exists. The content of Algorithm 1 (TestQP) is an effective and algebraic procedure for deciding whether or not such a diagram exists. The content of Algorithm 2 (FactorQP) is an effective and algebraic procedure for writing a word \( w \) as a product of conjugates of positive letters if TestQP has determined that \( w \) is quasi-positive. The validity of both algorithms is established in Theorem 1.

2.1. First example: a quasi-positive element might be disguised. Let \( X = \{a, b\} \) and consider the following word over \( X \):
\[
b \cdot aba^{-1} \cdot a^{-2}b^{-1}a^{-1}baba^{-2}.
\]
We have chosen a word which represents a quasi-positive element of \( F(X) \). But, if we freely reduce this word and call the corresponding element \( w \), then the fact that \( w \in F(X) \) is quasi-positive may be less apparent:
\[
w = babab^{-1}a^{-1}baba^{-2}.
\]
Since \( \Theta(w) = (0, 3) \), any factorization of \( w \) as a product of conjugates of letters in \( X \) must consist of 3 base letters all of which are equal to \( b \). Topologically, there must exist a cancellation diagram for \( (w, b, b, b) \). We invite the reader to find such a diagram on sphere with 4 boundary circles, one labeled by \( w \) and the others each labeled by the letter \( b^{-1} \).

2.2. Second example: quasi-positive factorizations are not unique. Let \( X = \{a, b\} \) and consider the following word over \( X \):
\[
a \cdot (bab^{-1}) \cdot b \cdot (ba^{-1}bab^{-1}).
\]
If we freely reduce this word and call the corresponding element \( w \), then the fact that \( w \in F(X) \) is quasi-positive is more apparent than in the previous example:
\[
w = ababa^{-1}bab^{-1}.
\]
We find that \( \Theta(w) = (2, 2) \). So, any factorization of \( w \) as a product of conjugates of letters in \( X \) must consist of 4 base letters: two of which are \( a \) and two of which are \( b \). Topologically, there must exist a cancellation diagram for \( (w, a, a, b, b) \). We
again invite the reader to find such a diagram on a sphere with 5 boundary circles, one labeled by \( w \) and the others each labeled by the letters \( a^{-1}, a^{-1}, b^{-1}, \) and \( b^{-1} \), respectively. Note that a solution is not unique. Indeed, the following are different factorizations of \( w \) as a product of conjugates of positive letters:

\[
w = a \cdot b \cdot aba^{-1} \cdot bab^{-1} = a \cdot b \cdot ba^{-1}bab^{-1}.
\]

2.3. Third example: the abelianization is not a complete invariant. If \( w \in F(X) \) is quasi-positive, then \( \Theta(w) \) must have non-negative coordinates. If, additionally, \( w \) is non-trivial, then at least one coordinate must be positive. But the converse is not true, as the following example shows: \( w = [a, b] = a^{-1}b^{-1}abb \).

We have that \( \Theta(w) = (0, 1) \), but there is no spherical cancellation diagram for \( w \) with two boundary circles such that one is labeled by \( w \) and the other by \( b^{-1} \).

2.4. The Test for Quasi-Positivity algorithm. The algorithm \( \text{TestQP}(w) \) (described below in Algorithm 1) accepts a word \( w \) as input and returns true if \( w \) is quasi-positive and false if it is not. The reader is advised to read this algorithm in tandem with the example in subsection 2.5. The algorithm can be modified so that if \( w \) is quasi-positive, then it also returns a binary tree \( T \) that encodes how it was determined that \( w \) is quasi-positive. This tree can then be used as input for the algorithm \( \text{FactorQP}(T) \), and a factorization of \( w \) as a product of conjugates of positive letters will be returned. A detailed example of this procedure appears in subsection 2.6.

\begin{algorithm}
\caption{Test for Quasi-Positivity}
\begin{algorithmic}[1]
\Procedure{TestQP}{\( w \)} \Comment{The input word \( w = w_0 \cdots w_{n-1} \) is stored in a list so that \( w[i:j] = w_i \) and each \( w_i \in X \cup X^{-1} \). We use the convention that \( w[i:j] = [w_i, \ldots, w_{j-1}] \) for \( 0 \leq i < j \leq n \).}
\For{\( (i = 0; i < n; i++) \)} \Comment{Find negative letter, some \( x^{-1} \in X^{-1} \).}
\If{\( (w[i] \in X^{-1}) \)} \Comment{Double \( w \) to simulate a cyclic word.}
\State \( dw \gets ww \) \Comment{Search for the positive letter \( x \in X \).}
\EndIf
\EndFor
\If{\( (j = 1; j < n; j++) \)} \Comment{The pair \( x^{-1}, x \) subdivides \( w \).}
\If{\( (dw[i+j] = w[i]^{-1}) \)} \Comment{into two subwords:}
\State \( w_L \leftarrow dw[i+1 : i+j] \) \Comment{\( w_L \) and \( w_R \).}
\State \( w_R \leftarrow dw[i+j+1 : n+i] \)
\EndIf
\EndIf
\EndFor
\If{\( (\text{TestQP}(w_L) \text{ and TestQP}(w_R)) \)} \Comment{Recursively test \( w_L \) and \( w_R \).}
\State \text{return true}
\EndIf
\EndIf
\EndProcedure
\end{algorithmic}
\end{algorithm}
Algorithm 2 Factor Quasi-Positive element

1: procedure FactorQP(Tree)  \textsuperscript{\textcircled{\textgreater}} The input Tree is a rooted binary tree. Every degree one vertex contains a positive word. Every vertex of degree greater than one contains a quasi-positive word \(w\) and a positive letter \(x\) coming from good pair \(x^{-1}, x\) contained in \(w\). The two children of this vertex contain the words \(w_L\) and \(w_R\), respectively.
2: Number the vertices of Tree by 1, 2, \ldots using a Breadth First Search.
3: repeat
4: Find vertices (necessarily of degree one) with the two largest numbers. Let \(w_L\) and \(w_R\) be the words they contain.
5: Let \(w\) and \(x\) denote the word and positive letter, respectively, contained in the parent vertex of \(w_L\) and \(w_R\).
6: Since the tree was constructed from a call to TestQP, we know that \(w^{-1}w_Lxw_R\) are equal as cyclic words. Cyclically permute the letters of \(w^{-1}w_Lxw_R\) until it equals \(w\) letter for letter.
7: Using one of the cases of Lemma 3, write \(w\) as a product of conjugates of positive letters. Overwrite the contents of the parent vertex with this factorization.
8: Delete both children of the parent node.
9: Repeat using the next pair of vertices with the largest two numbers.
10: until all nodes visited
11: return the factorization of \(w\) contained in the root vertex
12: end procedure

2.5. Fourth example: the TestQP algorithm in practice. Consider the following element of \(F(X)\):
\[w = babab^{-1}a^{-1}baba^{-1}a^{-1} = w_0w_1 \ldots w_9w_{10},\]
where \(X = \{a, b\}\) and each \(w_i \in Y = X \cup X^{-1}\) (cf. subsection 2.4). Note that the initial letter of \(w\) is denoted by \(w_0\) to conform with the practice that lists and arrays are indexed starting with zero. We describe the steps that result from calling TestQP\((w)\):

1. The procedure (line 1) is called with input \([w_0, \ldots, w_{10}]\). The first negative letter \(w_4 = b^{-1}\) is found (line 2). The list \(dw = [w_0, \ldots, w_{10}, w_0, \ldots, w_{10}] = [dw_0, \ldots, dw_{21}]\) is constructed so that, effectively, \(w\) is a cyclic word (without need to speak of indices modulo \(|w|\)).
2. The next (after \(w_4\)) occurrence of the letter \(w_4^{-1}\) in the cyclic word is found, namely, \(w_0 = b\). Let \(w_L = a^{-1}\) and \(w_R = aba^{-1}a^{-1}baba\). The algorithm is called with input \(w_L\) (line 9, inside the if statement). This returns false since, although a negative letter, namely \(a^{-1}\) is found in \(w_L\), there is no occurrence of the positive letter \(a\) in \(w_L\). Since this first test fails, the algorithm is not called on \(w_R\) (in accordance with good compiler design).
3. The FOR loop (line 5) finds the next occurrence of \(b\), namely \(w_8\). Let \(w_L = a^{-1}ba\) and let \(w_R = a^{-1}a^{-1}baba\). The algorithm is called on \(w_L\). The letter \(a^{-1}\) is found, the positive letter \(a\) is found; we denote the two resulting subwords by \(w_{LL} = b\) and \(w_{RL} = 1\) (the empty word). Again, the
algorithm is called on \( w_{LL} \), and the algorithm returns true (line 16) since it contains no negative letters. The same happens for \( w_{RL} \).

(4) At this point, looking forward to constructing a factorization, the following information will be recorded in a rooted binary tree (see Figure 1). The root will contain the word \( w \) and the letter \( b \) (since \( b^{-1} \) was found in \( w \)); the left child of the root will contain the word \( w_L \) and \( a \) (since \( a^{-1} \) was found in \( w_L \)). The two children of \( w_L \) are vertices containing \( w_{LL} = b \) and \( w_{RL} = 1 \), respectively. These steps are not included in the algorithm so that the exposition is more clear.

(5) Next, the algorithm is called on \( w_R = aba^{-1}a^{-1}baba \). The first negative letter, \( a^{-1} \) is found. The next matching positive letter \( a \) is found; this results in the subwords \( w_{LR} = a^{-1}baba \) and \( w_{RR} = 1 \). But when the algorithm is called on \( w_{LR} \) it will find no positive letter to match \( a^{-1} \). Therefore, the algorithm will continue (line 5) to search \( w_R \) for a good match for the negative letter \( a^{-1} \). Indeed, the second \( a \) will work: \( w_{LR} = a^{-1}bab \) and \( w_{RR} = 1 \). The negative letter \( a^{-1} \) is found in \( w_{LR} \) and the subwords are \( w_{LLR} = b \) and \( w_{RLR} = b \). These are both positive. These two subwords and the letter \( a \) are recorded the binary tree.

(6) Finally, \( w_{RR} = 1 \) is recognized as positive (empty). Tracing back through the recursive calls, we find that both conditions in the test on line 9 are satisfied at each step. The result is that the algorithm will return true (line 10) for the input word \( w \).

2.6. **Fifth example: the FactorQP algorithm in practice.** We step through a call of FactorQP\((T)\), where \( T \) is the binary tree in Figure 1 corresponding to the the example in subsection 2.5

(1) The vertices of the tree \( T \) are numbered from 1 (the root) to 9 (the vertex containing \( w_{RLR} = b \) using a breadth first search (line 2).

(2) The vertices with the two largest numbers, 8 and 9, contain \( w_{LLR} = b \) and \( w_{RLR} = b \), respectively (line 4).

(3) The parent vertex (with number 6) contains the word \( w_{LR} = a^{-1}bab \) and the letter \( a \) (line 5).

(4) We construct \( a^{-1}w_{LLR}aw_{RLR} = a^{-1}bab \). It so happens that this is equal to \( w_{LR} \) and so no cycling is required (line 6).
We write the factorization $a^{-1}ba \cdot b$ of $w_{LR}$ to the vertex with the number 6 and delete the vertices with numbers 8 and 9 (lines 7 and 8). By abuse of notation, we denote this factorization by $w_{LR}$.

We repeat. The next two largest numbered vertices are vertex 6 and vertex 7 which contain $w_{LR} = a^{-1}ba \cdot b$ and $w_{RR} = 1$, respectively (line 5); their parent vertex contains $w_R$ and the letter $a$.

As we step through lines 6–9, we find that

$$a^{-1}w_{LR}aw_{RR} = a^{-1}(a^{-1}ba \cdot b)a \cdot 1$$

is equal to $w_R$ letter for letter (no cycling required); however expressing this as a product of conjugates of positive letters requires some care, as detailed in Lemma 3. The correct factorization is the following:

$$b^{a}a \cdot b = a^{-1}a^{-1}ba \cdot a^{-1}ba.$$ This factorization is written to vertex 3 and denoted by $w_{R}$.

The next two highest numbered vertices contain $w_{LL} = b$ and $w_{RL} = 1$, respectively. Using the letter $a$ of the parent vertex, we obtain the factorization $a^{-1}ba$ of $w_{L}$; no cycling is required.

Finally, we consider vertex 2 (containing the factorization $a^{-1}ba = w_{L}$) and vertex 3 (containing the factorization $a^{-1}a^{-1}baa \cdot a^{-1}ba = w_{R}$). The parent vertex (the root) contains the letter $b$ and so we try to match $w$ letter for letter with the word $b^{-1}w_{L}bw_{R}$. Cycling is required in this case. Moreover, expressing $w$ as a product of conjugates of positive letters requires some care. We have that

$$b^{-1}w_{L}bw_{R} = b^{-1}(a^{-1}ba)b \cdot (a^{-1}a^{-1}baa \cdot a^{-1}ba).$$

After freely reducing the above word, it is equal as a cyclic word to $w$. We cyclically permute the letters and obtain the following factorization as detailed in Lemma 3

$$b \cdot b^{a^{-1}} \cdot b^{aba^{-1}a^{-1}}.$$

The above factorization is the factorization of $w$ as a product of conjugates of positive letters that is returned by FactorQP($w$).

3. Validity of the algorithms

Lemma 2. Suppose $w = y_1 \cdots y_n$, where each $y_i \in X \cup X^{-1}$. If $w$ is quasi-positive and some $y_i \in X^{-1}$, then there exists a $y_j$ such that $y_j = y_i^{-1}$ and both $w_L$ and $w_R$ are quasi-positive, where

$$y_1 \cdots y_n y_1 \cdots y_{i-1} = y_i w_{L} y_j w_{R}.$$ 

Proof. If $w$ is quasi-positive and $w_i \in X^{-1}$, then by Gersten’s theorem there is a spherical cancelation diagram with an arc joining the vertex with label $y_i$ on the $w$-curve to a vertex with label $y_i^{-1}$ on the $w$-curve. Surgering this diagram shows that both $w_L$ and $w_R$ are quasi-positive. □

The significance of the above is that in Algorithm 1 (TestQP), if a letter in $X^{-1}$ is read from the input word $w$, then we can search for a matching inverse letter and run the algorithm again on the two subwords $w_L$ and $w_R$. If neither is found to be quasi-positive, then we proceed to the next inverse letter and repeat. If all matching inverse letters are exhausted and no pair of subword is quasi-positive,
Theorem 1.  
(1) The algorithm TestQP correctly determines whether or not an input word \( w \) is quasi-positive.

(2) Suppose that \( w \) is quasi-positive. Let \( T \) be the rooted binary tree encoding the good pairs of inverse letters and corresponding subwords returned by a modification of TestQP as described in subsections 2.6 and 2.7. Then FactorQP\( (T) \) will return a factorization of \( w \) as a product of conjugates of positive letters.

Proof. If \( w \) is quasi-positive, then there exists a cancellation diagram with one boundary cycle labeled by \( w \) and the remaining cycles each labeled by one letter in \( X^{-1} \). As described in Lemma 1, the number of these cycles and the number having a given negative letter is determined by the abelianization \( \Theta(w) \). We argue that the algorithm TestQP recognizes that \( w \) is quasi-positive and that the algorithm FactorQP returns a factorization as a product of conjugates of positive letters.

The argument is by induction on the length of \( w \). The statement is vacuously true for the trivial word. Suppose \( w \) has length \( L > 0 \). If the cancellation diagram has no arcs joining vertices of the boundary cycle labeled by \( w \) to itself, then \( w \) is positive and this is detected by line 16 of Algorithm 1. Otherwise there are vertices labeled by \( x \) and \( x^{-1} \) for some \( x \in X \), and there is an arc in the cancellation diagram joining \( x \) to \( x^{-1} \). The word \( w \) admits a factorization as \( w = tx^{-1}uxv \).

By surgering the diagram we obtain two cancellation diagrams, one for the word \( u \) and one for the word \( vt \). Since \( u \) and \( vt \) are shorter quasi-positive words (as in the proof of Lemma 2), by induction, the algorithms correctly identify them as quasi-positive and return factorizations of the desired type. Lemma 3 contains further explanation of how the factorization is obtained from the recursion.

Conversely, we argue that the algorithm TestQP can be used to construct a spherical cancellation diagram. Suppose that the algorithm returns true for an input word \( w \). If \( w \) is positive, it is clear that a spherical cancellation diagram exists. Otherwise, TestQP find a good pair of inverse letters \( x^{-1}, x \). Join the corresponding vertices by a proper tame arc on a sphere with \( |\Theta(w)| + 1 \) boundary circles of the correct type as discussed previously. By induction on the length of the input, spherical cancellation diagrams exist for the subwords \( w_L \) and \( w_R \) corresponding to this pair. These spherical diagrams can be glued to form a spherical cancellation diagram for \( w \). By Gersten’s theorem, \( w \) is quasi-positive. This proves that TestQP correctly identifies whether or not an input word is quasi-positive.

We saw in subsection 2.5 that finding a factorization of the input word as a product of conjugates of positive words, is delicate. The following lemma explains how Algorithm 2 (FactorQP) finds such a factorization of \( w \) given that \( w \) contains a good pair \( a, a^{-1} \) of inverse letters and quasi-positive subwords \( w_L \) and \( w_R \) with given factorizations as a product of conjugates of positive elements.

Lemma 3. Let \( F(X) \) be free with basis \( X \). Let \( w \in F(X) \). Let \( \bar{w} \) be the cyclic reduction of \( w \) so that \( w = v^{-1} \bar{w} v \) and \( 2|v| + |\bar{w}| = |w| \) and \( v \in F(X) \). Suppose that \( w_L \) and \( w_R \) are words over \( X \), each expressed as a product of conjugates of positive letters. Let \( a \in X \). Suppose that \( \bar{w} \) and \( w_L^a w_R \) are conjugate in \( F(X) \) via some \( u \in F(X) \):

\[
\bar{w} = (w_L^a w_R)^u.
\]
Then
\[ w = \bar{w}v = w_{L}^{a}w_{R}^{a}w_{L}^{a}w_{R}^{a} \]

is a factorization of \( w \) as a product of conjugates of positive letters. Moreover, elements \( u \) and \( v \) can be effectively determined from the words \( w \) and \( a^{-1}w_{L}aw_{R} \) and the assumption that these two words define conjugate elements in \( F(X) \).

**Proof.** The word \( v \) can be found in linear time by successively checking whether or not the initial and terminal letters of \( w \) are equal and \( |w| > 1 \). The element \( \bar{w} \) is defined as the freely reduced word obtained from \( vvv^{-1} \). Suppose that \( w \) and \( w_{L}^{a}w_{R} \) are conjugate. Freely and cyclically reduce \( w_{L}^{a}w_{R} \) to obtain a word \( z \). It follows that \( \bar{w} \) and \( z \) are related by a cyclic permutation. By the Knuth-Pratt-Morris string matching algorithm, we can find a subword \( u \) of \( \bar{w} \) such that \( \bar{w} = (w_{L}^{a}w_{R})^{u} \) in \( F(X) \). It then follows that
\[ w = \bar{w}v = (w_{L}^{a}w_{R})^{uv} = w_{L}^{a}w_{R}^{a}w_{L}^{a}w_{R}^{a}. \]

By distributing the exponents across the given factorizations of \( w_{L} \) and \( w_{R} \), a factorization of \( w \) as a product of conjugates of positive letters is obtained. \( \square \)

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