Fast and flexible preconditioners for solving multilinear systems

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Abstract

This paper investigates a type of fast and flexible preconditioners to solve multilinear system $A x^{m-1} = b$ with $\mathcal{M}$-tensor $A$ and obtains some important convergent theorems about preconditioned Jacobi, Gauss-Seidel and SOR type iterative methods. The main results theoretically prove that the preconditioners can accelerate the convergence of iterations. Numerical examples are presented to reverify the efficiency of the proposed preconditioned methods.

Keywords: Multilinear system, $\mathcal{M}$-tensor, Tensor splitting, Preconditioned methods.

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1. Introduction

In recent decades, Tensors or hypermatrices have been applied in many types of research and application areas such as data analysis, psychometrics, chemometrics, image processing, graph theory, Markov chains, hypergraphs, etc. [27]. Tensor equations (or multilinear systems [4]) involving the Einstein product have been discussed in [8], which has many applications in continuum physics, engineering, isotropic and anisotropic elastic models [14]. Wang and Xu presented some iterative methods for solving several kinds of tensor equations in [29], Huang and Ma, in [11], proposed the Krylov subspace methods to solve a class of tensor equations. In [12], Khosravi Dehdezi and Karimi proposed the extended conjugate gradient squared and conjugate residual squared methods for solving the generalized coupled Sylvester tensor equations

$$\sum_{j=1}^{n} X_j \times_1 A_{i1j} \times_2 A_{i2j} \times ... \times_d A_{ijd} = C_i, \quad i = 1, 2, ..., n,$$

where the matrices $A_{i1j} \in \mathbb{C}^{n_{i1} \times n_{1j}} (i = 1, 2, ..., n$ and $l = 1, 2, ..., d$), tensors $C_i \in \mathbb{C}^{n_{i1} \times ... \times n_{id}}(i = 1, 2, ..., n)$ are known, tensors $X_j \in \mathbb{C}^{n_{j1} \times ... \times n_{jd}}(j = 1, 2, ..., n)$ are unknown and $\times_j(j = 1, 2, ..., n)$ is the $j$-mode product. Also they proposed a fast and efficient Newton-Shultz-type iterative method for computing inverse and Moore-Penrose inverse of tensors in [13].

Very recently years, solving the following multilinear system has become a hot topic because of several applications such as data analysis, engineering and scientific computing [3, 4, 8]:

$$A x^{m-1} = b$$

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where $A = (a_{i_1 i_2 \ldots i_m})$ is an $m$ order $n$-dimensional tensor, $x$ and $b$ are vectors in $\mathbb{C}^n$. The $n$ dimensional vector $Ax^{m-1}$ is defined as:

$$
(Ax^{m-1})_i = \sum_{i_2=1}^{n} \ldots \sum_{i_m=1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n,
$$

(2)

and $x_i$ denotes the $i$-th component of $x$.

Many theoretical analyses and algorithms for solving (1) were also studied. Qi in [26] considered an $m$ order $n$-dimensional supersymmetric tensor and showed that when $m$ is even it has exactly $n(m - 1)^{n-1}$ eigenvalues, and the number of its E-eigenvalues is strictly less than $n(m - 1)^{n-1}$ when $m \geq 4$. Ding and Wei in [8] proved that a nonsingular $M$-equation with a positive right-hand side always has a unique positive solution. Also, they applied the $M$-equations to some nonlinear differential equations and the inverse iteration for spectral radii of nonnegative tensors. In [9], Han proposed a homotopy method for finding the unique positive solution to a multilinear system with a nonsingular $M$-tensor and a positive right side vector. Li et al., in [18] extended the Jacobi, Gauss-Seidel and successive over-relaxation (SOR) iterative methods to solve the tensor equation $Ax^{m-1} = b$, where $A$ is an $m$ order $n$-dimensional symmetric tensor. Under appropriate conditions, they showed that the proposed methods were globally convergent and locally $r$-linearly convergent. In [10], He et al. proved that solving multilinear systems with $M$-tensors is equivalent to solving nonlinear systems of equations where the involving functions are P-functions. Based on this result, they proposed a Newton-type method to solve multilinear systems with $M$-tensors. For a multilinear system with a nonsingular $M$-tensor and a positive right side vector, they showed that the sequence generated by the method converges to the unique solution of the multilinear system and the convergence rate is quadratic. For solving the multilinear systems, Liang et al. in [15], transformed equivalently the tensor equation into a consensus constrained optimization problem, and then proposed an ADMM type method for it. Also, they showed that each limit point of the sequences generated by the method satisfied the Karush-Kuhn-Tucker conditions. Liu et al., in [20], introduced the variant tensor splittings, and presented some equivalent conditions for a strong $M$-tensor based on the tensor splitting. Also, the existence and unique conditions of the solution for multi-linear systems were given. Besides, they proposed some tensor splitting algorithms for solving multi-linear systems with coefficient tensor being a strong $M$-tensor. As an application, a tensor splitting algorithm for solving the multi-linear model of higher-order Markov chains was proposed. Li et al., in [16] firstly derived a necessary and sufficient condition for an $M$-tensor equation to have nonnegative solutions. Secondly, developed a monotone iterative method to find a nonnegative solution to an $M$-tensor equation. Under appropriate conditions, they showed that the sequence of iterates generated by the method converges to a nonnegative solution of the $M$-tensor equation monotonically and linearly. Bai et al. in [3] proposed an algorithm that always preserves the nonnegativity of solutions of the multilinear system under consideration involves a nonsingular $M$-tensor and a nonnegative right-hand side vector. Also, they proved that the sequence generated by the proposed algorithm is a nonnegative componentwise nonincreasing sequence and converges to a nonnegative solution of the multilinear system. Cui et al. in [6]
intended to solve the multi-linear system by the preconditioned iterative method based on tensor splitting. For this purpose, they proposed the preconditioner $I + S_{\text{max}}$. Lv and Ma in [22] proposed a Levenberg-Marquardt (LM) method for solving tensor equations with semi-symmetric coefficient tensor and proved its global convergence and local quadratic convergence under the local error bound condition, which is weaker than non-singularity. As an application, they solved the H-eigenvalue of real semi-symmetric tensor by the LM method. Wang et al., in [28] proposed continuous-time neural network and modified continuous-time neural networks for solving a multi-linear system with $\mathcal{M}$-tensors. They proved that the presented neural networks are stable in the sense of Lyapunov stability theory. For solving the multilinear system $\mathbf{A} \mathbf{x}^{m-1} = \mathbf{b}$, where $\mathbf{A}$ is a symmetric $\mathcal{M}$-tensor, Xie et al. in [30] proposed some tensor methods based on the rank-1 approximation of the coefficient tensor. Li et al. in [17], considered tensor equations of 3 order whose solutions are the intersection of a group of quadrics from a geometric point of view. Inspired by the method of alternating projections for set intersection problems, they developed a hybrid alternating projection algorithm for solving these tensor equations. The local linear convergence of the alternating projection method was established under suitable conditions. Liu et al. in [21], presented a preconditioned SOR method for solving the multilinear systems whose coefficient tensor is an $\mathcal{M}$-tensor. Also, the corresponding comparison for spectral radii of iterative tensors was given. It is known that the preconditioning technique plays an important role in solving multilinear systems. In particular, when the coefficient tensor is an $\mathcal{M}$-tensor, there is little research on these techniques so far. By this motivation, we establish some effective preconditioners and give a theoretical analysis.

The rest of this paper is organized as follows. Section 2 is preliminary in which we introduce some related definitions and lemmas. In Section 3, new fast and flexible type preconditioners are proposed, and the corresponding theoretical analysis is given. In Section 4, numerical examples are given to show the efficiency of the proposed preconditioned iterative methods. Section 5 is the concluding remark and the final section is the future researches.

2. Preliminaries

In this section, we introduce some definitions, notations, and related properties which will be used in the following.

Let $\mathbf{0}$, $\mathbf{O}$ and $\mathcal{O}$ denote for null vector, null matrix and null tensor, respectively. Let $\mathbf{A}$ and $\mathbf{B}$ be a tensor (vector or matrix) with the same sign. The order $\mathbf{A} \geq \mathbf{B}(>\mathbf{B})$ means that each element of $\mathbf{A}$ is no less than (larger than) corresponding one of $\mathbf{B}$.

A tensor $\mathbf{A}$ consists of $n_1 \times \ldots \times n_m$ elements in the complex field $\mathbb{C}$:

$$\mathbf{A} = (a_{i_1i_2 \ldots i_m}), \quad a_{i_1i_2 \ldots i_m} \in \mathbb{C}, \quad 1 \leq i_j \leq n_j, j = 1, \ldots, m.$$  

When $m = 2$, $\mathbf{A}$ is an $n_1 \times n_2$ matrix. If $n_1 = \ldots = n_m = n$, $\mathbf{A}$ is called an $m$ order $n$-dimensional tensor. By $\mathbb{C}^{n_1 \times \ldots \times n_m}$ we denote all $m$ order tensors consist of $n_1 \times \ldots \times n_m$ entries and by $\mathbb{C}^{[m,n]}$ we denote the set of all $m$ order $n$-dimensional tensors. When $m = 1$, $\mathbb{C}^{[1,n]}$ is simplified as $\mathbb{C}^n$, which is the set of all
Let $A \in \mathbb{R}^{[m,n]}$. If each entry of $A$ is nonnegative, then $A$ is called a nonnegative tensor. The set of all $m$ order $n$-dimensional nonnegative tensors is denoted by $\mathbb{R}_+^{[m,n]}$. The $m$ order $n$-dimensional identity tensor, denoted by $I_m = (\delta_{i_1i_2...i_m}) \in \mathbb{R}^{[m,n]}$, is the tensor with entries:

$$
\delta_{i_1i_2...i_m} = \begin{cases} 
1, & i_1 = i_2 = ... = i_m \\
0, & \text{otherwise}.
\end{cases}
$$

When $m = 2$, the identity tensor reduces to identity matrix of size $n \times n$, denoted by $I$.

**Definition 1.** \cite{19} $A \in \mathbb{C}^{[m,n]}$ is called a reducible tensor if there exists a nonempty proper index subset $I \subseteq \{1,2,...,n\}$ such that $a_{i_1i_2...i_m} = 0$, $\forall i_1 \in I$, $\forall i_2...i_m \notin I$, else, we say that $A$ is irreducible.

**Definition 2.** \cite{31} A tensor $A \in \mathbb{R}^{[m,n]}$ is called a $Z$-tensor if its off-diagonal entries are non-positive. $A$ is an $M$-tensor if there exists a nonnegative tensor $B \in \mathbb{R}_+^{[m,n]}$ and a positive real number $\eta \geq \rho(B)$ such that $A = \eta I - B$. If $\eta > \rho(B)$, then $A$ is called a strong $M$-tensor.

**Definition 3.** \cite{20} Let $A \in \mathbb{R}^{[2,n]}$ ($A$ is an $n$-dimensional square matrix) and $B \in \mathbb{R}^{[m,n]}$. Then a product $C = AB \in \mathbb{R}^{[m,n]}$ is defined by

$$
c_{j_2...i_m} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2i_2...i_m},
$$

which can be written as follows

$$
C_{(1)} = (AB)_{(1)} = AB_{(1)},
$$

where $C_{(1)}$ and $B_{(1)}$ are the matrices obtained from $C$ and $B$ flattened along the first index, respectively.

**Definition 4.** \cite{25} Let $A \in \mathbb{R}^{[m,n]}$. The majorization matrix of $A$, denoted by $M(A)$, is defined as a square matrix of size $n \times n$ with its entries

$$
M(A)_{ij} = a_{ij...j}, \quad i, j = 1, 2, ..., n.
$$

If $M(A)$ is a nonsingular matrix and $A = M(A)I_m$, then $M(A)^{-1}$ is the order 2 left-inverse of $A$, i.e., $M(A)^{-1}A = I_m$, and then we call $A$ a left-invertible tensor or left-nonsingular tensor.

**Definition 5.** \cite{26} Let $A \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector(or simply eigenpair) of $A$ if they satisfy the equation

$$
A x^{m-1} = \lambda x^{m-1},
$$

where $x^{m-1} = (x_1^{m-1}, ..., x_n^{m-1})^T$. We call $(\lambda, x)$ an $H$-eigenpair if both $\lambda$ and $x$ are real.

Let $\rho(A) = \max\{|\lambda| | \lambda \in \sigma(A)|\}$ be the spectral radius of $A$, where $\sigma(A)$ is the set of all eigenvalues of $A$. 


Lemma 1. [24] If $A$ is a strong $M$-tensor, then $M(A)$ is a nonsingular $M$-matrix.

Lemma 2. [24] If $M(A)$ is an irreducible matrix, then $A$ is irreducible.

Definition 6. [25] Let $A, E, F \in \mathbb{R}^{[m,n]}$, $A = E - F$ is said to be a splitting of $A$ if $E$ is a left-nonsingular; a regular splitting of $A$ if $E$ is left-nonsingular with $M(E)^{-1} \geq O$ and $F \geq O$; a weak regular splitting of $A$ if $E$ is left-nonsingular with $M(E)^{-1} \geq O$ and $M(E)^{-1}F \geq O$; a convergent splitting if $\rho(M(E)^{-1}F) < 1$.

Lemma 3. [25] If $A$ is a $Z$-tensor, then the following conditions are equivalent

1. $A$ is a strong $M$-tensor.
2. $A$ has a convergent (weak) regular splitting.
3. All (weak) regular splittings of $A$ are convergent.
4. There exist a vector $x > 0$ such that $Ax^{m-1} > 0$.

Lemma 4. [25] If $A$ is a strong $M$-tensor, then for every positive vector $b$, the multilinear system $Ax^{m-1} = b$ has a unique positive solution.

Lemma 5. [13] Suppose that $A \in \mathbb{R}^{[m,n]}$. Let $A = E_1 - F_1 = E_2 - F_2$ be a weak regular splitting and a regular splitting, respectively, and $F_2 \leq F_1, F_2 \neq O$. One of the following statements holds.

1. $\rho(M(E_2)^{-1}F_2) \leq \rho(M(E_1)^{-1}F_1) < 1$.
2. $\rho(M(E_2)^{-1}F_2) \geq \rho(M(E_1)^{-1}F_1) \geq 1$.

If $F_2 < F_1, F_2 \neq O$ and $\rho(M(E_1)^{-1}F_1) > 1$, the first inequality in part (2) is strict.

Lemma 6. [25] Let $A$ be a strong $M$-tensor, and $A = E_1 - F_1 = E_2 - F_2$ be two weak regular splitting with $M(E_1)^{-1}F_1 \leq M(E_2)^{-1}F_2$. If the Perron vector $x$ of $M(E_2)^{-1}F_2$ satisfies $Ax^{m-1} \geq 0$ then $\rho(M(E_2)^{-1}F_2) \leq \rho(M(E_1)^{-1}F_1)$.

A general tensor splitting iterative method for solving (1) is

\[ x_{j+1} = (M(E)^{-1}F)x_j^{m-1} + (M(E)^{-1}b)^{\frac{1}{m-1}}, \quad j = 0, 1, \ldots. \]  

$M(E)^{-1}F$ is called the iterative tensor of the splitting method (1). Taking $A = D - L - F$, Liu et al. in [20], considered $E = D$, $E = D - L$ and $E = \frac{1}{2}(D - \tau L)$, the Jacobian, the Gauss-Seidel, and the SOR iterative methods, respectively, where $D = DL$ and $L = LL$. $D$, $L$ are the positive diagonal matrix and the strictly lower triangle nonnegative matrix, respectively. Without loss of generality, we always assume that $a_{ii,i} = 1$, $i = 1, 2, \ldots, n$. Consider the splitting of $A = I - L - F$, where $L = LL$ and $L$ is the strictly lower triangle part of $M(A)$.

Using iterative methods for solving (1) may have a poor convergence or even fail to converge. To overcome this problem, it is efficient to apply these methods which combine preconditioning techniques. These iterative methods usually involve some matrices that transform the iterative tensor $M(E)^{-1}F$ into
a favorable tensor. The transformation matrices are called preconditioners. Li et al. in [18], considered
the preconditioner $P_\alpha = I + S_\alpha$ for solving preconditioned multilinear system
\[ P_\alpha A x^{m-1} = P_\alpha b, \]
with
\[ S_\alpha = \begin{bmatrix}
0 & -\alpha_1 a_{12...2} & 0 & \ldots & 0 \\
0 & 0 & -\alpha_2 a_{23...3} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \alpha_{n-1} a_{n-1,n...n} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \]
firstly proposed for $M$-matrix systems and the authors extended the results for solving tensor case.
In [21], Liu et al. considered a new preconditioned SOR method for solving multilinear systems with
preconditioner $P_\beta = I + C_\beta$ where
\[ C_\beta = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
-\beta_1 a_{21...1} & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\beta_{n-2} a_{(n-1)1...1} & 0 & 0 & \ldots & 0 \\
-\beta_{n-1} a_{n1...1} & 0 & 0 & 0 & 0
\end{bmatrix}. \]
Here we consider the preconditioner $P_{\alpha\beta}(s,k) = D + S_\alpha^s + K_\beta^k$, where $1 \leq s, k \leq n - 1$, $D$ is the
diagonal part of majorization of $A$ (so herein $D = I$) and $S_\alpha^s$, $K_\beta^k$ are square matrices which all of their
elements are zeros except the $s$th upper and the $k$th lower diagonals, i.e.,
\[ S_\alpha^s = \begin{bmatrix}
0 & \ldots & 0 & -\alpha_1 a_{1(1+s)...(1+s)} & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & -\alpha_2 a_{2(2+s)...(2+s)} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & \alpha_{n-s} a_{n-s,n...n} \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}, \]
\[ K_\beta^k = \begin{bmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 \\
-\beta_{k+1} a_{(k+1)1...1} & 0 & \ldots & 0 & \ldots & 0 \\
0 & -\beta_{k+2} a_{(k+2)2...2} & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & -\beta_n a_{n(n-k+1)...(n-k+1)} & 0 & \ldots & 0
\end{bmatrix}. \]
Applying $P_{\alpha \beta}(s, k)$ on the left side of Eq. (1), we get a new preconditioned multi-linear system

$$A_{\alpha \beta}(s, k)x^{m-1} = b_{\alpha \beta}(s, k),$$

with $A_{\alpha \beta}(s, k) = P_{\alpha \beta}(s, k)A$ and $b_{\alpha \beta}(s, k) = P_{\alpha \beta}(s, k)b$.

**Proposition 1.** Let $A \in \mathbb{R}^{[m,n]}$ be a $\mathcal{M}$-tensor. If $A$ is a strong $\mathcal{M}$-tensor for any $\beta_j \in [0, 1]$, $j = k + 1, \ldots, n$ and $\alpha_i \in [0, 1]$, $i = 1, \ldots, n - s$, then $A_{\alpha \beta}(s, k)$ is a strong $\mathcal{M}$-tensor.

**Proof.** Without loss of generality, we assume that $k = s = 1$. Let $A_{\alpha \beta}(s, k) = P_{\alpha \beta}(s, k)A = (\hat{a}_{i_1i_2\ldots i_m})$. Then for $1 \leq i_2, \ldots, i_m \leq n$, we have

$$\hat{a}_{ji_2\ldots i_m} = \begin{cases} a_{i_1i_2\ldots i_m} - \alpha_1a_{i_2\ldots i_m}, & j = 1 \\ a_{ji_2\ldots i_m} - \beta_ja_{(j-1)i_2\ldots i_m} - \alpha_ja_{(j-1)i_2\ldots i_m} - \alpha_ja_{(j+1)i_2\ldots i_m}, & 2 \leq j \leq n - 1 \\ a_{ni_2\ldots i_m} - \beta_n\alpha(n-1)i_2\ldots i_m, & j = n. \end{cases}$$

For $(j, i_2, \ldots, i_m) \neq (j, j, \ldots, j)$ and $\alpha_i, \beta_j \in [0, 1]$, we have $\hat{a}_{ji_2\ldots i_m} \leq 0$, i.e., $A_{\alpha \beta}(s, k)$ is a $\mathcal{Z}$-tensor. According to Lemma [11] there exist a vector $x > 0$ such that $Ax^{m-1} > 0$. It follows from $P_{\alpha}(s) > O$ that $P_{\alpha}Ax^{m-1} > 0$. Thus there exists a vector $x > 0$ such that $A_{\alpha \beta}(s, k)x^{m-1} > 0$. Therefore, $A_{\alpha \beta}(s, k)$ is a strong $\mathcal{M}$-tensor.

**Proposition 2.** The preconditioned multi-linear system (6) has the same unique positive solution with multi-linear system (1).

**Proof.** Because $b_{\alpha \beta}(s, k) \geq b > 0$ for any $\beta_j \in [0, 1]$, $j = k + 1, \ldots, n$ and $\alpha_i \in [0, 1]$, $i = 1, \ldots, n - s$, by Lemma [10] and Proposition [11] it is obvious.

### 3. The preconditioned Jacobi, Gauss–Seidel and SOR type iteration schemes

#### 3.1. The preconditioned Jacobi type iterative scheme with the preconditioner $P_{\alpha \beta}(s, k)$

Let $A = I - L - F$. We consider the following five Jacobi type splittings:

$A_{\alpha \beta}(s, k) = P_{\alpha \beta}(s, k)A$

$$= P_{\alpha \beta}(s, k)I - P_{\alpha \beta}(s, k)(L + F) = E_1 - F_1.$$  

$A_{\alpha \beta}(s, k) = I - (P_{\alpha \beta}(s, k)(L + F) - (S_{\alpha}^s + K_{\beta}^s)I) = E_2 - F_2.$

$A_{\alpha}(s) = I - (P_{\alpha}(s)(L + F) - S_{\alpha}^s)I = E_3 - F_3.$

$A_{\beta}(k) = I - (P_{\beta}(L + F) - K_{\beta}^s)I = E_4 - F_4.$

**Remark 1.** The splitting $A_{\alpha}(s) = E_3 - F_3$, where $s = 1$, is the same as the splitting in [13].

**Remark 2.** When $s = k = 1$, we denote $K_{\beta}^1$ by $K_{\beta}$ and $S_{\alpha}^1$ by $S_{\alpha}$. Thus we have the following Jacobi type splitting:
Denote $K_\beta$ by $K$ and $S_\alpha$ by $S$ when all $\beta_j = 1$, $j = 2, 3, \ldots, n$ and $\alpha_i = 1$, $i = 1, 2, \ldots, n - 1$.

Let $\mathcal{L} = K \mathcal{T} + \mathcal{L}'$ and $\mathcal{F} = S \mathcal{T} + \mathcal{F}'$, then

$$A_{\alpha\beta}(s, k) = (I - S_\alpha K - K_\beta S) I - [\mathcal{L} + \mathcal{F} - (S_\alpha + K_\beta) I + S_\alpha (\mathcal{L}' + \mathcal{F}) + K_\beta (\mathcal{L} + \mathcal{F}')] = \mathcal{E}_5 - \mathcal{F}_5.$$

**Proposition 3.** Let $A \in \mathbb{R}^{[m, n]}$ be a strong $\mathcal{M}$-tensor for any $j \in [0, 1]$, $j = k + 1, \ldots, n$ and $\alpha_i \in [0, 1]$, $i = 1, \ldots, n - s$, then $A_{\alpha\beta}(s, k) = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$, $A_{\alpha}(s) = \mathcal{E}_3 - \mathcal{F}_3$ and $A_{\beta}(k) = \mathcal{E}_4 - \mathcal{F}_4$ are convergent. Moreover if

$$
0 < \alpha_1 a_1 \ldots a_d, 1 < 1, \\
0 < \alpha_i a_{(i+1)} \ldots a_{(i+1)} \ldots a_{(i+1)}, 1 < 1, i = 2, \ldots, n - 1, \\
0 < \beta_n a_{n-1} \ldots a_{n-1} \ldots a_{n-1} < 1,
$$

then the tensor splitting $A_{\alpha\beta}(s, k) = \mathcal{E}_5 - \mathcal{F}_5$ is convergent.

**Proof.** Suppose $A_{\alpha\beta}(s, k) = \mathcal{E}_1 - \mathcal{F}_1$. Since $A = I - \mathcal{L} - \mathcal{F}$ is a strong $\mathcal{M}$-tensor, $\rho(\mathcal{L} + \mathcal{F}) < 1$. Thus $\rho(M(\mathcal{E}_1)^{-1} \mathcal{F}_1) = \rho(\mathcal{L} + \mathcal{F}) < 1$. Hence $A_{\alpha\beta}(s, k) = \mathcal{E}_1 - \mathcal{F}_1$ is a convergent splitting.

Let $A_{\alpha\beta}(s, k) = \mathcal{E}_2 - \mathcal{F}_2$. We have $M(\mathcal{E}_i)^{-1} = I > O$ and since $\alpha_i, \beta_j \in [0, 1]$, it is easy to see that $\mathcal{F}_2 \geq O$. Thus $A_{\alpha\beta}(s, k) = \mathcal{E}_2 - \mathcal{F}_2$ is a regular splitting. By Proposition 1, $A_{\alpha\beta}(s, k)$ is a strong $\mathcal{M}$-tensor and using Lemma 3, $A_{\alpha\beta}(s, k) = \mathcal{E}_2 - \mathcal{F}_2$ is a convergent regular splitting.

When $A_{\alpha}(s) = \mathcal{E}_3 - \mathcal{F}_3$ and $A_{\beta}(k) = \mathcal{E}_4 - \mathcal{F}_4$, proof is similar to the proof of the case $A_{\alpha\beta}(s, k) = \mathcal{E}_2 - \mathcal{F}_2$. Suppose that $A_{\alpha\beta}(s, k) = \mathcal{E}_5 - \mathcal{F}_5$, and Eq. (7), holds. Thus $M(\mathcal{E}_5)^{-1}$ exists, and

$$
M(\mathcal{E}_5)^{-1} = \begin{cases}
\frac{1}{1 - \alpha_1 a_1 \ldots a_d, 1}, & i = 2, \ldots, n - 1, \\
\frac{1}{1 - \alpha_i a_{(i+1)} \ldots a_{(i+1)} \ldots a_{(i+1)}}, & i = 2, \ldots, n - 1, \\
\frac{1}{1 - \beta_n a_{n-1} \ldots a_{n-1} \ldots a_{n-1}},
\end{cases}
$$

which implies that $M(\mathcal{E}_5)^{-1} > O$. It is not difficult to see that $\mathcal{F}_5 = \mathcal{E}_5 - A_{\alpha\beta}(s, k) \geq O$. Using Proposition 1, $A_{\alpha\beta}(s, k)$ is a strong $\mathcal{M}$-tensor and from Lemma 3, $A_{\alpha\beta}(s, k) = \mathcal{E}_5 - \mathcal{F}_5$ is a convergent regular splitting.

**Proposition 4.** Let $A$ be a strong $\mathcal{M}$-tensor and Eq. (7) holds. There exist $x_1, x_2 \in \mathbb{R}^n_+$, such that

1. $(M(\mathcal{E}_2)^{-1} \mathcal{F}_2)_{\alpha\beta} x_1^{n-1} \leq (M(\mathcal{E}_1)^{-1} \mathcal{F}_1)_{\alpha\beta} x_1^{n-1}$.
2. $A_{\alpha\beta}(s, k) x_2^{n-1} \geq 0$.
3. $\rho((M(\mathcal{E}_3)^{-1} \mathcal{F}_3)_{\alpha\beta}) \leq \rho((M(\mathcal{E}_2)^{-1} \mathcal{F}_2)_{\alpha\beta}) \leq \rho((M(\mathcal{E}_1)^{-1} \mathcal{F}_1)_{\alpha\beta})$.

**Proof.**

1. Since $A = I - \mathcal{L} - \mathcal{F}$ is a strong $\mathcal{M}$-tensor, $\rho(M(\mathcal{E}_1)^{-1}(\mathcal{L} + \mathcal{F})) = \rho(\mathcal{L} + \mathcal{F}) < 1$. Thus, for the nonnegative Jacobi iteration tensor $M(\mathcal{E}_1)^{-1}(\mathcal{L} + \mathcal{F}) = \mathcal{L} + \mathcal{F}$, there exists a nonnegative vector $x_1$ such that $M(\mathcal{E}_1)^{-1}(\mathcal{L} + \mathcal{F}) x_1^{n-1} = \rho(\mathcal{L} + \mathcal{F}) x_1^{n-1}$ by the Perron–Frobenius theorem. Thus we have
Thus, for the nonnegative Jacobi iteration tensor $M$, by Proposition 3, we know that

$$\alpha' \leq 0,$$

and by Lemma 6, we have

$$\alpha' \leq 0.$$

It is easy to see that for every Perron vector of nonnegative Jacobi iteration tensor of convergence splitting method such as $x$, we have, $A_{\alpha\beta}(s,k)x^{m-1} \geq 0$.

3. Since $M(E_2)_{\alpha\beta} = I$ and $M(E_5)_{\alpha\beta} = (I - S_K - KGS)^{-1}$, thus $M(E_5)_{\alpha\beta} \geq M(E_2)_{\alpha\beta}$.

Let $(\rho((M(E_5)_{\alpha\beta})), x)$ be a Perron eigenpair of $(M(E_5)_{\alpha\beta})_{\alpha\beta}$, then by part 2, we have $A_{\alpha\beta}(s,k)x^{m-1} \geq 0$ and by Lemma 3, we have $\rho((M(E_5)_{\alpha\beta})), x)) \leq \rho((M(E_2)_{\alpha\beta})), x))$. Now suppose that $x$ is a nonnegative Perron vector of $(M(E_5)_{\alpha\beta})$, then by part 1, we have $(M(E_2)_{\alpha\beta}), x) \leq (M(E_5)_{\alpha\beta})_{\alpha\beta}x^{m-1} = \rho((M(E_2)_{\alpha\beta})), x))$. Since $(M(E_2)_{\alpha\beta}), x) \geq 0$, then $\rho((M(E_2)_{\alpha\beta})), x)) \leq \rho((M(E_5)_{\alpha\beta}))_{\alpha\beta}$. This completes the proof. 

**Remark 3.** It is easy to see that for every Perron vector of nonnegative Jacobi iteration tensor of convergence splitting method such as $x$, we have, $A_{\alpha\beta}(s,k)x^{m-1} \geq 0$.

**Proposition 5.** Let $A \in \mathbb{R}^{[m,n]}$ be a strong $M$-tensor. If Eq. (11) holds for any $\beta_{1,j}, \beta_{2,j} \in [0,1], j = k+1, \ldots, n$, $\alpha_{1,i}, \alpha_{2,i} \in [0,1], i = 1, \ldots, n-s, \alpha' \equiv (\alpha_{1,i}), \alpha'' \equiv (\alpha_{2,i}), \beta' \equiv (\beta_{1,j}), \beta'' \equiv (\beta_{2,j})$ and $\alpha' \geq \alpha'', \beta' \geq \beta''$, then we have

1. $\rho((M(E_1)_{\alpha'\beta'})) \leq \rho((M(E_1)_{\alpha''\beta''}))$.
2. $\rho((M(E_2)_{\alpha'\beta'})) \leq \rho((M(E_2)_{\alpha''\beta''}))$.
3. $\rho((M(E_3)_{\alpha'\beta'})) \leq \rho((M(E_3)_{\alpha''\beta''}))$.
4. $\rho((M(E_4)_{\alpha'\beta'})) \leq \rho((M(E_4)_{\alpha''\beta''}))$.
5. $\rho((M(E_5)_{\alpha'\beta'})) \leq \rho((M(E_5)_{\alpha''\beta''})).
3.2. Gauss-Seidel type iterative schemes with the tridiagonal preconditioner $P_{\alpha\beta}(s,k)$

We consider the following four Gauss-Seidel type splittings:

$$A_{\alpha\beta}(s,k) = P_{\alpha\beta}(s,k)A$$

$$= P_{\alpha\beta}(s,k)(I - L) - P_{\alpha\beta}(s,k)F = M_1 - N_1.$$  

$$A_{\alpha\beta}(s,k) = (I - L + K^k_{\alpha\beta}T - K^k_{\alpha\beta}L - D_{\alpha} - L_{\alpha} - D_{\beta} - L_{\beta}) - (F - S_{\alpha}^sT + S_{\alpha}^sF + F_{\alpha} + F_{\beta}) = M_2 - N_2.$$  

$$A_{\alpha}(s) = (I - L - D_{\alpha} - L_{\alpha}) - (F - S_{\alpha}^sT + S_{\alpha}^sF + F_{\alpha}) = M_3 - N_3.$$  

$$A_{\beta}(k) = ((I + K^k_{\beta}(I - L) - D_{\beta} - L_{\beta}) - (F + F_{\beta}) = M_4 - N_4.$$  

Where $D_{\alpha} = D_{\alpha}I$, $L_{\alpha} = L_{\alpha}I$, $D_{\beta} = D_{\beta}I$, $L_{\beta} = L_{\beta}I$, and $D_{\alpha}$, $D_{\beta}$, $L_{\alpha}$, $L_{\beta}$ are the diagonal parts and the strictly lower triangle parts of $M(S^s_{\alpha}L)$ and $M(K^k_{\beta}F)$, respectively, i.e.

$$S_{\alpha}^sL = D_{\alpha} + L_{\alpha} + F_{\alpha}, \quad K^k_{\beta}F = D_{\beta} + L_{\beta} + F_{\beta}.$$

Remark 4. The splitting $A_{\alpha}(s) = M_3 - N_3$, where $s = 1$, is the same as the splitting in [18].

Remark 5. If $k = l = 1$, similar to Remark 4 we have

$$A_{\alpha\beta}(s,k) = ((I + K^1_{\beta}(I - L) - S_{\alpha}L - K^1_{\alpha}ST) - ((I + S_{\alpha})F - S_{\alpha}T + K^1_{\alpha}F') = M_5 - N_5.$$  

Proposition 6. Let $A \in \mathbb{R}^{[m,n]}$ be a strong $M$-tensor for any $\beta_j \in [0,1]$, $j = k + 1, ..., n$ and $\alpha_i \in [0,1]$, $i = 1, ..., n - s$, then $A_{\alpha\beta}(s,k) = M_1 - N_1$ is convergent.

When $k < s$ if

$$0 < \alpha_i a_i(n-i)_{(n-i)}a_{(n-i)i}...i < 1, \quad i = 1, 2, ..., k,$$

$$0 < \alpha_i a_i(n-i)_{(n-i)}a_{(n-i)i}...i + \beta_i a_i(i-k)_{(i-k)}a_{(i-k)i}...i < 1, \quad i = k + 1, ..., s,$$  

$$0 < \beta_i a_i(i-k)_{(i-k)}a_{(i-k)i}...i < 1, \quad i = s + 1, ..., n.$$  

(9)

When $k > s$ if

$$0 < \alpha_i a_i(n-i)_{(n-i)}a_{(n-i)i}...i < 1, \quad i = 1, 2, ..., s,$$

$$0 < \alpha_i a_i(n-i)_{(n-i)}a_{(n-i)i}...i + \beta_i a_i(i-k)_{(i-k)}a_{(i-k)i}...i < 1, \quad i = s + 1, ..., k,$$  

$$0 < \beta_i a_i(i-k)_{(i-k)}a_{(i-k)i}...i < 1, \quad i = k + 1, ..., n.$$  

(10)

And when $k = s$ if

$$0 < \alpha_i a_i(i+k)_{(i+k)}a_{(i+k)i}...i < 1, \quad i = 1, 2, ..., k,$$

$$0 < \alpha_i a_i(i+k)_{(i+k)}a_{(i+k)i}...i + \beta_i a_i(i-k)_{(i-k)}a_{(i-k)i}...i < 1, \quad i = k + 1, ..., n - k,$$  

$$0 < \beta_i a_i(i-k)_{(i-k)}a_{(i-k)i}...i < 1, \quad i = n - k + 1, ..., n.$$  

(11)

Then the tensor splitting $A_{\alpha\beta}(s,k) = M_2 - N_2$, is convergent. If

$$0 < \alpha_i a_i(n-i)_{(n-i)}a_{(n-i)i}...i < 1, \quad i = 1, 2, ..., s.$$
Proposition 7. \( A_\alpha(s) = M_3 - N_3 \) is convergent. Finally, if
\[
0 < \beta_i a_{i(i-k)\ldots(i-k)a_{(i-k)i\ldots i}} < 1, \quad i = k + 1, \ldots, n,
\]
then \( A_\beta(k) = M_4 - N_4 \) is convergent.

**Proof.** Let \( A_{\alpha\beta}(s, k) = M_1 - N_1 \). Due to Proposition 11 \( A_{\alpha\beta}(s, k) \) is a strong \( M \)-tensor, and \( N_1 \geq O \). Since
\[
M(M_1)^{-1}N_1 = (I - L)^{-1}P_{\alpha\beta}(s, k)^{-1}P_{\alpha\beta}(s, k)N_1 \geq O,
\]
\( A_{\alpha\beta}(s, k) = M_1 - N_1 \) is a weak regular splitting and using Lemma 3 is convergent.

Suppose that \( A_{\alpha\beta}(s, k) = M_2 - N_2 \) and \( k = s \). Since \( M_2 = I - L + K_\beta^k I - K_\beta^k L - D_\alpha - L_\alpha - D_\beta - L_\beta \), then \( M(M_2) = I - D_\alpha - D_\beta - L + K_\beta^k - K_\beta^k L - L_\alpha - L_\beta \). Notice that \( D_\alpha \) and \( D_\beta \) are diagonal part of \( M(S_\alpha^k L) \) and \( M(K_\beta^k F) \), respectively. It is not difficult to see that
\[
(I - D_\alpha - D_\beta)_n = \begin{cases} 
1 - \alpha a_{i(i+k)\ldots(i+k)a_{(i+k)i\ldots i}}, & i = 1, 2, \ldots, k, \\
1 - \alpha a_{i(i+k)\ldots(i+k)a_{(i+k)i\ldots i}} - \beta a_{i(i-k)\ldots(i-k)a_{(i-k)i\ldots i}}, & i = k + 1, \ldots, n - k, \\
1 - \beta a_{i(i-k)\ldots(i-k)a_{(i-k)i\ldots i}}, & i = n - k + 1, \ldots, n.
\end{cases}
\]
(12)
Since Eq. (11) holds, \( I - D_\alpha - D_\beta \) exists, and \( I - (D_\alpha + D_\beta)^{-1} = I + (D_\alpha + D_\beta) + \ldots + (D_\alpha + D_\beta)^n \geq I \). Denote \( H := L + L_\alpha + L_\beta - K_\beta^k + K_\beta^k L \). \( H \) is a lower triangular matrix, to prove \( H \geq O \) it is sufficient to show \( L - K_\beta^k \geq 0 \) for any \( i = 1, \ldots, n - 1 \). Actually,
\[
(L - K_\beta^k)_{i+1,i} = -a_{i+1,i\ldots i} - (-\beta a_{i+1,i\ldots i}) = a_{i+1,i\ldots i}(\beta_{i+1} - 1) \geq 0.
\]
By the Neumann’s series \[23\), we have
\[
M(M_2)^{-1} = [I - (D_\alpha + D_\beta) - L]^{-1}
= [I + (I - (D_\alpha + D_\beta)^{-1}L)^{-1}H]^{-1}(I - (D_\alpha + D_\beta)^{-1})
= [I + (I - (D_\alpha + D_\beta)^{-1}L)^{-1}H + [(I - (D_\alpha + D_\beta)^{-1}L)^2 + \ldots +
+[(I - (D_\alpha + D_\beta)^{-1}L)^{n-1}] + (I - (D_\alpha + D_\beta)^{-1})]^{-1}
\geq O.
\]
Since \( N_2 \geq O \) (by the same discussion in proofing how \( H \geq O \), \( A_{\alpha\beta}(s, k) = M_2 - N_2 \) is a weak regular splitting and using Lemma 3 is convergent. For cases \( k < s \) and \( k > s \), similar discussion can be used for obtain desired results.

When \( A_\alpha(s) = M_3 - N_3 \) and \( A_\beta(k) = M_4 - N_4 \), proof is similar to the proof of the case \( A_{\alpha\beta}(s, k) = M_2 - N_2 \). \[23\]

**Proposition 7.** Let \( A \) be a strong \( M \)-tensor and Eqs. (10)-(11) hold. There exists \( x \in \mathbb{R}_+^n \), such that
1. \( A_{\alpha\beta}(s, k)x^{s-1} \geq 0 \).
2. \( \rho(M(M_2)^{-1}N_2)_{\alpha\beta} \leq \rho((M(M_3)^{-1}N_3)_{\alpha\beta}) \leq \rho(M(M_1)^{-1}N_1)_{\alpha\beta} < 1 \).
3. \( \rho(M(M_2)^{-1}N_2)_{\alpha\beta} \leq \rho((M(M_4)^{-1}N_4)_{\alpha\beta}) \leq \rho(M(M_1)^{-1}N_1)_{\alpha\beta} < 1 \).
3.3. The preconditioned SOR type method

In [20], the SOR type method for solving Eq. (11) is given by taking \( E = \frac{1}{\omega} (I - \omega L) \) and

\[
x_{j+1} = (M(I - \omega L))^{-1}(1 - \omega)I + \omega F)x_{j}^{-1} + \omega M(I - \omega L)^{-1}b\left(\frac{1}{m-1}\right).
\]

In this paper, we consider the following preconditioned SOR type method:

\[
x_{j+1} = (H_{\alpha\beta}(\omega)x_{j}^{-1} + h_{\alpha\beta}(\omega))\left(\frac{1}{m-1}\right),
\]

where

\[
H_{\alpha\beta}(\omega) = M(E_{\alpha\beta}(\omega))^{-1}F_{\alpha\beta}(\omega), \quad h_{\alpha\beta}(\omega) = M(E_{\alpha\beta}(\omega))^{-1}b_{\alpha\beta}(s, k),
\]

\[
E_{\alpha\beta}(\omega) = \frac{1}{\omega}(D_{\alpha\beta} - \omega L_{\alpha\beta}), \quad F_{\alpha\beta}(\omega) = \frac{1}{\omega}((1 - \omega)D_{\alpha\beta} + \omega F_{\alpha\beta}),
\]

and
\[D_{\alpha\beta} = I - D_{\alpha} - D_{\beta},\]
\[L_{\alpha\beta} = L - K_{\beta}^2 I + K_{\beta}^4 L + L_{\alpha} + L_{\beta},\]
\[F_{\alpha\beta} = F - S_{\alpha} I + S_{\alpha} F + F_{\alpha} + F_{\beta}.\]

**Remark 6.** When \(s = 1\) and \(k = 0\), the new preconditioned SOR method is similar to the preconditioned SOR method which is proposed in [15].

**Proposition 9.** Let \(A \in \mathbb{R}^{[m,n]}\) be a strong \(\mathcal{M}\)-tensor. If \(A = I - L - F\) and \(0 < \omega_1 < \omega_2 \leq 1\), then \(\rho(\mathcal{H}_{\alpha\beta}(\omega_2)) \leq \rho(\mathcal{H}_{\alpha\beta}(\omega_1)) < 1\).

**Proposition 10.** Let \(A \in \mathbb{R}^{[m,n]}\) be a strong \(\mathcal{M}\)-tensor. For any \(\omega \in (0,1]\), \(\rho(\Theta_{\alpha\beta}) \leq \rho(\mathcal{H}_{\alpha\beta}(\omega))\), where \(\Theta_{\alpha\beta}\) is the iteration tensor of the preconditioned Gauss-Seidel type methods.

**Proposition 11.** [21] Let \(A \in \mathbb{R}^{[m,n]}\) be a strong \(\mathcal{M}\)-tensor with \(a_{i(i+1) \ldots (i+1)i} > 0\), \(i = 1,2,\ldots,n-1\) and \(0 < a_{i1\ldots i1i} < 1\), \(i = 2,3,\ldots,n\). Then \(\mathcal{H}(\omega)\) is nonnegative and irreducible for \(0 < \omega < 1\).

**Proposition 12.** [21] Let \(A \in \mathbb{R}^{[m,n]}\) be a strong \(\mathcal{M}\)-tensor and \(\alpha_i, \beta_j \in [0,1]\), \(i = 1,2,\ldots,n-1\). Then if \(0 < \omega \leq 1\), \(a_{i(i+1)\ldots(i+1)i} > 0\), \(i = 1,2,\ldots,n-1\) and \(0 < a_{i1\ldots i1i} < 1\), \(i = 2,3,\ldots,n\), we have \(\rho(\mathcal{H}_{\alpha\beta}(\omega)) \leq \rho(\mathcal{H}(\omega)) < 1\).

**Proposition 13.** [21] Let \(A \in \mathbb{R}^{[m,n]}\) be a strong \(\mathcal{M}\)-tensor. If the conditions of the Proposition 11 hold and \(\beta_{1,j}, \beta_{2,j} \in [0,1]\), \(j = k+1,\ldots,n\), \(\alpha_{1,i}, \alpha_{2,i} \in [0,1]\), \(i = 1,\ldots,n-s\), \(\alpha' = (\alpha_{1,i})\), \(\alpha'' = (\alpha_{2,i})\), \(\beta' = (\beta_{1,j})\), \(\beta'' = (\beta_{2,j})\) and \(\alpha' \geq \alpha''\), \(\beta' \geq \beta''\), then \(\rho(\mathcal{H}_{\alpha'\beta'}(\omega)) \leq \rho(\mathcal{H}_{\alpha''\beta''}(\omega)) < 1\).

**Proposition 14.** [21] Let \(A \in \mathbb{R}^{[m,n]}\) be a strong \(\mathcal{M}\)-tensor. If \(0 < \omega_1 < \omega_2 \leq 1\), then
\[\rho(M(I - \omega_2 L)^{-1}(\omega_2 F + (1 - \omega_2)I)) \leq \rho(M(I - \omega_1 L)^{-1}(\omega_1 F + (1 - \omega_1)I)) < 1.\]

**Proposition 15.** [21] Let \(A \in \mathbb{R}^{[m,n]}\) be a strong \(\mathcal{M}\)-tensor. If \(0 < \omega_1 < \omega_2 \leq 1\) and \(\alpha_i, \beta_j \in [0,1]\), \(i = 1,\ldots,n-s\), \(j = k+1,\ldots,n\), then \(\rho(\mathcal{H}_{\alpha\beta}(\omega_2)) \leq \rho(\mathcal{H}_{\alpha\beta}(\omega_1)) < 1\).

### 4. Numerical Examples

In this section, we give some numerical examples to show the performance of the proposed algorithms. All tests were carried out in double precision with a MATLAB code, when the computer specifications are Microsoft Windows 10 Intel(R), Core(TM)i7-7500U, CPU 2.70 GHz, with 8 GB of RAM. All used codes came from the MATLAB tensor toolbox developed by Bader and Kolda [1, 2]. We use PJ, PGS
and PSOR to abbreviate the preconditioned Jacobi, Gauss-Seidel and SOR tensor splittings in \cite{32,19} and \cite{21}, respectively. In addition we use \( P_{\alpha\beta}E_2F_2, P_{\alpha\beta}E_3F_3, P_{\alpha\beta}M_2N_2, P_{\alpha\beta}M_5N_5 \) and \( P_{\alpha\beta}S\text{SOR} \) to abbreviate the preconditioned Jacobi, Gauss-Seidel and SOR type splittings methods, respectively, that are proposed in this paper. We use \( \text{Iter} \) and \( \text{Time} \) for denote the number of iterations and CPU Times in seconds, respectively that need to reach the desired solution. The stopping criterion is \( \|r_j\| < 10^{-12} \), where \( x_0 = 0, r_j = b - Ax_j^{m-1} \) is the \( j \)-th iteration residual, the right hand side vector \( b \) is \( 1 = (1,\ldots,1)^T \), if no other special illustration, and the maximum number of iterations is 2000. Also, we suppose that \( \beta = \beta 1 \) and \( \alpha = \alpha 1 \).

**Example 1.** Consider a strong \( \mathcal{M} \)-tensor \( A \in \mathbb{R}^{3 \times 3 \times 3} \), where

\[
A(:,:,1) = \begin{pmatrix}
1.00 & -0.01 & -0.02 \\
-0.02 & -0.03 & -0.04 \\
-0.04 & -0.05 & -0.06
\end{pmatrix}, \quad A(:,:,2) = \begin{pmatrix}
-0.06 & -0.07 & -0.08 \\
-0.08 & 1.00 & -0.09 \\
-0.01 & -0.02 & -0.03
\end{pmatrix}, \quad A(:,:,3) = \begin{pmatrix}
-0.03 & -0.04 & -0.05 \\
-0.05 & -0.06 & -0.07 \\
-0.07 & -0.08 & 1.00
\end{pmatrix}.
\]

We compare the mentioned methods, where the parameter \( \omega \) in the SOR method is chosen 1.2. We take \( \alpha = \beta \) in the interval \([0,10]\) with the step size 0.5 and \( s,k = 2 \). The comparison results are shown in Table 1. In addition, we take the \( \omega \) in the interval \([0.5,1.8]\) with the step size 0.1 and obtain the solution by using the proposed preconditioned SOR method for \( \alpha, \beta = 0.5 \) and \( s,k = 1,2 \). We depicted the results in Table 2. \( P_{\alpha}(s) = D + S_{\alpha}(s) \) and \( P_{\beta}(k) = D + K_{\beta}(k) \).

From Table 1 we find that all the preconditioned methods perform better in CPU Times and iteration numbers than the ones with unpreconditioned (\( \alpha = \beta = 0 \)). Also, the proposed preconditioned schemes of Jacobi, Gauss-Seidel and SOR methods are all better than the corresponding ones that are considered in this paper when the parameters \( \alpha \) and \( \beta \) can be taken suitably. The best answers in terms of CPU times and iteration numbers have bolded in Table 1. From Table 2 and for every choice of \( \omega \), we see that in most cases when \( \beta = 0 \) and \( s = 2 \), the best answers in terms of CPU Times and iteration numbers are obtained which are showed in bolded numbers.

**Example 2.** Let \( B \in \mathbb{R}^{[3,n]} \) be a nonnegative tensor with \( M(B) = \text{hilb}(n,n) \), where \( \text{hilb} \) is the function of MATLAB, for \( i = 2,3,\ldots,n, b_{ii-1} = b_{ii+1} = b_{i1+i3} = b_{ii+1} = \frac{1}{i} \) and other entries are zeros. Let \( A = n^2I - 0.01B \). We take \( \alpha = \beta = 1, s = k = n - 1 \) for \( P_{\alpha\beta}E_1F_1, P_{\alpha\beta}E_2F_2, P_{\alpha\beta}S\text{SOR} \). Also we obtained experimentally the optimal parameter \( \omega \) in the interval \([0,2]\). The numerical results are reported in Table 3 which illustrate that the proposed preconditioned methods perform better in CPU times than the ones with the others.

From Table 3 we find that when \( n \) increases, the CPU Times for obtaining the appropriate answer increase. Also, if the parameters \( \alpha, \beta \) and \( \omega \) can be taken suitably, the proposed preconditioned schemes of Jacobi, Gauss-Seidel and SOR methods are all better than the corresponding ones that are considered in this paper. The best answers in terms of CPU Times and iteration numbers for every \( n \) have bolded in Table 3 where shows that the proposed second scheme of the preconditioned Jacobi method is the best.

**Example 3.** Let \( B \in \mathbb{R}^{[3,10]} \) be a nonnegative tensor and \( b_{i1+i3} = |\tan(i_1 + i_2 + i_3)| \). It is not difficult \((24)\) to see that \( \rho(B) \approx 1450 \), thus \( A = 2000I - B \) is a strong \( \mathcal{M} \)-tensor. For mentioned methods, we obtained experimentally the optimal parameter \( \omega \) in the interval \([1,2]\), the values of \( \alpha, \beta \) from 0 to 30
and $s, k = 1$. The numerical results are reported in Table 4. We use $\dagger$ to indicate that there was no convergence up to 2000 iterations. Table 4 illustrates that the proposed preconditioned methods perform better in CPU times than the ones with the others.

As we see in Table 4, without preconditioners ($\alpha = \beta = 0$), the proposed preconditioned schemes of Jacobi, Gauss-Seidel and SOR methods obtained the same answers with the corresponding ones that are considered in this paper. When the parameters $\alpha$ and $\beta$ are considered as nonzero, we see that the PJ, PGS and PSOR methods are not convergent, but the proposed methods are convergent and improve the iteration numbers and CPU Times concerning unpreconditioned. The best answers in the iteration numbers and CPU Times are bolded in Table 4.

Table 1: Iteration number (Iter) and CPU Time (Time) for Example 1.

| $\alpha$ | PJ Iter | PJ Time | PGS Iter | PGS Time | PSOR Iter | PSOR Time | $P_{\alpha\beta}\mathcal{E}_2\mathcal{F}_2$ Iter | $P_{\alpha\beta}\mathcal{M}_2\mathcal{N}_2$ Iter | $P_{\alpha\beta}\text{SOR}$ Iter | $P_{\alpha\beta}\text{SOR}$ Time |
|----------|---------|---------|----------|----------|-----------|----------|---------------------------------|---------------------------------|----------------|----------------|
| 0.0      | 51      | 0.0066  | 50       | 0.0065   | 39        | 0.0100   | 51                               | 0.0048                          | 50                           | 0.0055         |
| 0.5      | 51      | 0.0046  | 49       | 0.0041   | 39        | 0.0044   | 50                               | 0.0018                          | 49                           | 0.0030         |
| 1.0      | 50      | 0.0044  | 47       | 0.0032   | 39        | 0.0033   | 49                               | 0.0028                          | 48                           | 0.0019         |
| 1.5      | 50      | 0.0039  | 46       | 0.0040   | 39        | 0.0024   | 48                               | 0.0017                          | 47                           | 0.0018         |
| 2.0      | 50      | 0.0040  | 45       | 0.0020   | 39        | 0.0021   | 46                               | 0.0017                          | 46                           | 0.0017         |
| 2.5      | 49      | 0.0040  | 44       | 0.0020   | 39        | 0.0021   | 45                               | 0.0017                          | 45                           | 0.0017         |
| 3.0      | 49      | 0.0030  | 43       | 0.0024   | 39        | 0.0020   | 44                               | 0.0016                          | 44                           | 0.0016         |
| 3.5      | 48      | 0.0028  | 43       | 0.0019   | 39        | 0.0019   | 43                               | 0.0015                          | 43                           | 0.0013         |
| 4.0      | 48      | 0.0032  | 42       | 0.0023   | 39        | 0.0027   | 42                               | 0.0022                          | 42                           | 0.0013         |
| 4.5      | 47      | 0.0021  | 43       | 0.0021   | 39        | 0.0019   | 41                               | 0.0015                          | 41                           | 0.0012         |
| 5.0      | 47      | 0.0023  | 43       | 0.0023   | 39        | 0.0028   | 40                               | 0.0016                          | 40                           | 0.0012         |
| 5.5      | 46      | 0.0023  | 44       | 0.0024   | 39        | 0.0021   | 39                               | 0.0021                          | 39                           | 0.0014         |
| 6.0      | 46      | 0.0026  | 44       | 0.0022   | 39        | 0.0019   | 38                               | 0.0022                          | 38                           | 0.0014         |
| 6.5      | 45      | 0.0024  | 45       | 0.0022   | 39        | 0.0020   | 37                               | 0.0016                          | 36                           | 0.0019         |
| 7.0      | 44      | 0.0022  | 46       | 0.0024   | 39        | 0.0020   | 35                               | 0.0020                          | 35                           | 0.0014         |
| 7.5      | 44      | 0.0027  | 48       | 0.0024   | 39        | 0.0019   | 34                               | 0.0017                          | 33                           | 0.0013         |
| 8.0      | 43      | 0.0023  | 49       | 0.0025   | 39        | 0.0021   | 29                               | 0.0014                          | 31                           | 0.0013         |
| 8.5      | 43      | 0.0029  | 50       | 0.0027   | 39        | 0.0021   | 31                               | 0.0016                          | 28                           | 0.0012         |
| 9.0      | 42      | 0.0026  | 52       | 0.0028   | 39        | 0.0022   | 33                               | 0.0018                          | 27                           | 0.0010         |
| 9.5      | 42      | 0.0024  | 53       | 0.0028   | 39        | 0.0022   | 34                               | 0.0017                          | 31                           | 0.0014         |
| 10.0     | 41      | 0.0024  | 55       | 0.0028   | 39        | 0.0022   | 34                               | 0.0017                          | 32                           | 0.0014         |


Table 2: Iteration numbers (Iter) and CPU Times (Time) for the preconditioned SOR type method.

| ω   | $P_{\alpha=5}^{(1)}$ Iter | $P_{\alpha=5}^{(2)}$ Time | $P_{5,5}^{(1,1)}$ Iter | $P_{5,5}^{(1,2)}$ Time | $P_{5,5}^{(2,1)}$ Iter | $P_{5,5}^{(2,2)}$ Time | $P_{5,5}^{(3,1)}$ Iter | $P_{5,5}^{(3,2)}$ Time |
|-----|---------------------------|-----------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 0.5 | 103 0.0206                | 95 0.0106                   | 105 0.0119             | **94 0.0133**          | 97 0.0128              | 96 0.0148              |                        |                        |
| 0.6 | 83 0.0089                 | 77 0.0025                   | 85 0.0036              | **76 0.0026**          | 79 0.0027              | 77 0.0027              |                        |                        |
| 0.7 | 69 0.0018                 | 63 0.0018                   | 70 0.0020              | 63 0.0018              | **60 0.0019**          | 64 0.0024              |                        |                        |
| 0.8 | 58 0.0017                 | **53 0.0015**               | 60 0.0022              | 53 0.0019              | 56 0.0017              | 54 0.0016              |                        |                        |
| 0.9 | 50 0.0014                 | **45 0.0014**               | 51 0.0018              | 46 0.0014              | 48 0.0014              | 46 0.0016              |                        |                        |
| 1.0 | 43 0.0012                 | **39 0.0012**               | 44 0.0013              | 40 0.0012              | 42 0.0014              | 40 0.0012              |                        |                        |
| 1.1 | 37 0.0012                 | **34 0.0007**               | 39 0.0007              | 35 0.0010              | 37 0.0011              | 35 0.0010              |                        |                        |
| 1.2 | 33 0.0009                 | **29 0.0007**               | 34 0.0010              | 30 0.0009              | 32 0.0019              | 30 0.0008              |                        |                        |
| 1.3 | 29 0.0008                 | **25 0.0007**               | 30 0.0008              | 26 0.0007              | 29 0.0008              | 26 0.0008              |                        |                        |
| 1.4 | 31 0.0009                 | **28 0.0008**               | 30 0.0009              | 29 0.0008              | 33 0.0011              | **28 0.0010**          |                        |                        |
| 1.5 | 40 0.0010                 | **35 0.0009**               | 39 0.0013              | 37 0.0010              | 44 0.0011              | **35 0.0010**          |                        |                        |
| 1.6 | 53 0.0014                 | **45 0.0012**               | 51 0.0013              | 47 0.0012              | 59 0.0017              | 46 0.0019              |                        |                        |
| 1.7 | 71 0.0028                 | **60 0.0022**               | 70 0.0020              | 64 0.0017              | 81 0.0013              | 61 0.0010              |                        |                        |
| 1.8 | 105 0.0015                | **84 0.0012**               | 101 0.0014             | 92 0.0013              | 135 0.0020             | 85 0.0019              |                        |                        |

5. Concluding remarks

In this paper, we proposed new types of flexible and fast preconditioners tensor splitting methods for solving multilinear system $A x^{m-1} = b$, when $A$ is a strong $M$-tensor. Some properties of convergent theorems about preconditioned Jacobi, Gauss-Seidel and SOR type iterative methods are obtained. Numerical examples are given to show the efficiency and superiority of the proposed methods.

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Table 3: Iteration number (Iter) and CPU Time (Time) for Example 2.

| n  | PJ Iter  | PJ Time  | PGS Iter  | PGS Time  | PSOR Iter  | PSOR Time  | $P_{\alpha\beta}E_2F_2$ | $P_{\alpha\beta}E_5F_5$ | $P_{\alpha\beta}M_2N_2$ | $P_{\alpha\beta}M_5N_5$ | $P_{\alpha\beta}SOR$ |
|----|----------|----------|-----------|-----------|------------|------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 30 | 4 0.0360 | 5 0.0169 | 3 0.0228  | 3 0.0151  | 3 0.0242   | 3 0.0196  | 3 0.0230        | 3 0.0240        |
| 40 | 4 0.0597 | 5 0.0273 | 3 0.0299  | 3 0.0174  | 3 0.0327   | 3 0.0233  | 3 0.0276        | 3 0.0243        |
| 50 | 4 0.0739 | 5 0.0293 | 3 0.0339  | 3 0.0201  | 3 0.0362   | 3 0.0257  | 3 0.0306        | 3 0.0288        |
| 60 | 4 0.0608 | 5 0.0443 | 3 0.0483  | 3 0.0292  | 3 0.0499   | 3 0.0417  | 3 0.0550        | 3 0.0425        |
| 70 | 4 0.0730 | 5 0.0558 | 3 0.0645  | 3 0.0416  | 3 0.0787   | 3 0.0575  | 3 0.0604        | 3 0.0657        |
| 80 | 4 0.0985 | 5 0.0829 | 3 0.0918  | 3 0.0497  | 3 0.1008   | 3 0.0733  | 3 0.0829        | 3 0.0929        |
| 90 | 4 0.1173 | 5 0.2064 | 3 0.1300  | 3 0.0687  | 3 0.1329   | 3 0.0859  | 3 0.0977        | 3 0.1179        |
| 100| 4 0.1480 | 5 0.1406 | 3 0.1735  | 3 0.1173  | 3 0.2311   | 3 0.1264  | 3 0.1373        | 3 0.1449        |
| 110| 4 0.2133 | 5 0.3544 | 3 0.1968  | 3 0.1314  | 3 0.2432   | 3 0.1525  | 3 0.1618        | 3 0.1790        |
| 120| 4 0.2250 | 5 0.2073 | 3 0.2406  | 3 0.1464  | 3 0.2773   | 3 0.1978  | 3 0.1967        | 3 0.2544        |

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Table 4: Iteration number (Iter) and CPU Time (Time) for Example 3.

| α  | β  | PJ Iter | PJ Time | PGS Iter | PGS Time | PSOR Iter | PSOR Time | $P_{\alpha \beta}F_2$ Iter | $P_{\alpha \beta}F_2$ Time | $P_{\alpha \beta}M_2N_2$ Iter | $P_{\alpha \beta}M_2N_2$ Time | $P_{\alpha \beta}SOR$ Iter | $P_{\alpha \beta}SOR$ Time |
|----|----|---------|---------|----------|---------|-----------|---------|----------------|-----------------|----------------|----------------|----------------|----------------|----------------|
| 0  | 0  | 91      | 0.0308  | 87       | 0.0181  | 69        | 0.0302  | 87              | 0.0168          | 69              | 0.0191          |               |               |
| 0.5| 0.5| † 0.1253| † 0.0831| † 0.0820 | † 0.0143| 90        | 0.0143  | 86              | 0.0184          | 68              | 0.0158          |               |               |
| 1  | 1  | † 0.1253| † 0.0831| † 0.0820 | 89      | 0.0147  | 85        | 0.0195          | 67              | 0.0227          |               |               |
| 2  | 2  | † 0.0960| † 0.0596| † 0.0853 | 87      | 0.0149  | 83        | 0.0241          | 65              | 0.0157          |               |               |
| 3  | 2  | † 0.1025| † 0.0807| † 0.0949 | 85      | 0.0169  | 81        | 0.0167          | 64              | 0.0152          |               |               |
| 4  | 2  | † 0.1051| † 0.0740| † 0.0788 | 83      | 0.0145  | 79        | 0.0204          | 62              | 0.0148          |               |               |
| 5  | 5  | † 0.1023| † 0.0745| † 0.0849 | 81      | 0.0167  | 77        | 0.0201          | 60              | 0.0160          |               |               |
| 7  | 5  | † 0.0953| † 0.0999| † 0.0831 | 77      | 0.0144  | 73        | 0.0185          | 57              | 0.0160          |               |               |
| 9  | 5  | † 0.0835| † 0.0822| † 0.0989 | 73      | 0.0142  | 69        | 0.0164          | 54              | 0.0150          |               |               |
| 10 | 8  | † 0.1347| † 0.1228| † 0.1262 | 71      | 0.0155  | 67        | 0.0193          | 52              | 0.0146          |               |               |
| 12 | 10 | † 0.0975| † 0.0848| † 0.0760 | 67      | 0.0159  | 64        | 0.0159          | 49              | 0.0145          |               |               |
| 15 | 12 | † 0.0934| † 0.0875| † 0.0987 | 61      | 0.0134  | 58        | 0.0167          | 44              | 0.0153          |               |               |
| 18 | 10 | † 0.0912| † 0.0985| † 0.1044 | 55      | 0.0166  | 52        | 0.0186          | 39              | 0.0150          |               |               |
| 20 | 15 | † 0.0987| † 0.0924| † 0.0901 | 49      | 0.0138  | 48        | 0.0157          | 38              | 0.0137          |               |               |
| 20 | 20 | † 0.0912| † 0.0914| † 0.0926 | 49      | 0.0144  | 48        | 0.0159          | 38              | 0.0182          |               |               |
| 25 | 20 | † 0.0989| † 0.0932| † 0.0911 | 40      | 0.0138  | 40        | 0.0151          | 38              | 0.0139          |               |               |
| 25 | 25 | † 0.0999| † 0.0924| † 0.0937 | 42      | 0.0139  | 40        | 0.0153          | 37              | 0.0137          |               |               |
| 30 | 20 | † 0.0974| † 0.0978| † 0.0945 | 47      | 0.0140  | 48        | 0.0166          | 45              | 0.0140          |               |               |
| 30 | 25 | † 0.0910| † 0.0934| † 0.0922 | 48      | 0.0139  | 48        | 0.0170          | 46              | 0.0140          |               |               |
| 30 | 30 | † 0.0900| † 0.0944| † 0.0891 | 48      | 0.0141  | 49        | 0.0158          | 46              | 0.0140          |               |               |

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