FREE ENERGY OF THE CAUCHY DIRECTED POLYMER MODEL AT HIGH TEMPERATURE

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Abstract. We study the Cauchy directed polymer model on $\mathbb{Z}^{1+1}$ with a trivial weak disorder regime. Under some assumptions, we prove that for this model, very strong disorder holds at any inverse temperature $\beta > 0$, and we can also give a sharp asymptotics on the free energy by

$$\lim_{\beta \to 0} \beta^2 \log(-p(\beta)) = -c.$$ 

1. Introduction

In this paper, we study a specific long-range directed polymer model, that is, the Cauchy directed polymer model on $\mathbb{Z}^{1+1}$. The long-range directed polymer model is an extension of the classic nearest-neighbor directed polymer model. For details about the nearest-neighbor model, we refer to [8, 10, 13]; for details about the long-range model, we refer to [7, 18].

1.1. The model. We now introduce the Cauchy directed polymer model on $\mathbb{Z}^{1+1}$. The model consists of a random field and a heavy-tailed random walk on $\mathbb{Z}$, whose increment distribution is in the domain of attraction of the Cauchy law. The random field models the random environment and the random walk models the polymer chain. When the polymer chain stretches in the random environment, there is an interaction between them. We want to investigate whether this interaction significantly influences the behavior of the polymer chain compared to the case with no random environment.

To be precise, we denote the random walk, its probability and expectation by $S = (S_n)_{n \geq 0}$, $P$, and $E$ respectively. The random walk $S$ has i.i.d. increment satisfying

$$\begin{cases} P(S_1 - S_0 > k)/P(|S_1 - S_0| > k) \sim p, \\ P(|S_1 - S_0| > k) \sim k^{-1}L(k), \end{cases}$$

for some $p \in (0, 1)$ as $k \to \infty$,

where $L(\cdot)$ is a function slowly varying at infinity, i.e., $L(\cdot) : (0, \infty) \to (0, \infty)$ and for any $a > 0$, $\lim_{t \to \infty} L(at)/L(t) = 1$. The condition is necessary and sufficient for $S_1$ to belong to the domain of attraction of the Cauchy law, i.e., we can find a positive sequence $(a_n)_{n \geq 1}$ and a real sequence $(b_n)_{n \geq 1}$, such that

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} G, \quad \text{as } n \to \infty,$$

where $\xrightarrow{d}$ stands for weak convergence and $G$ is some Cauchy random variable. Convergence is a well-known result and one may refer to [10, Chapter 9]. We should mention that $a_n$ can be chosen by $n\varphi(n)$ for some function $\varphi(\cdot)$ slowly varying at infinity determined by

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the law of $S$. For $b_n$, however, the expression is usually not as simple as $a_n$ and we will briefly discuss it when we state our main results later.

The random field, its probability and expectation are denoted by $\omega := (\omega_{n,x})_{n \in \mathbb{N}, x \in \mathbb{Z}}, \mathbb{P}$ and $\mathbb{E}$ respectively. Here $\omega$ is a family of i.i.d. random variables independent of the random walk $S$. We assume that $\omega$ has finite logarithmic moment generating function, at least in a small neighborhood of the origin, which we denote by

$$\lambda(\beta) := \log \mathbb{E}[\exp(\beta \omega_{n,x})] < \infty, \quad \forall \beta \in (-c, c),$$

where $\beta$ is the inverse temperature and $c$ is some positive constant. By (1.3), we may also assume that

$$\mathbb{E}[\omega_{n,x}] = 0 \quad \text{and} \quad \mathbb{E}[(\omega_{n,x})^2] = 1,$$

which does not lose much generality.

The interaction between the polymer chain and the random environment up to time $N$ is modeled via a Gibbs transformation, defined by

$$d\mathbb{P}_{N,\beta}^\omega := \frac{1}{Z_{N,\beta}^\omega} \exp \left( \sum_{n=1}^{N} \beta \omega_{n,S_n} \right),$$

where

$$Z_{N,\beta}^\omega = \mathbb{P} \left[ \exp \left( \sum_{n=1}^{N} \beta \omega_{n,S_n} \right) \right]$$

is the partition function such that $\mathbb{P}_{N,\beta}^\omega$ is a probability measure on the space of random walk paths.

It turns out that $Z_{N,\beta}^\omega$ plays a key role in the study of the directed polymer model. In [5], Bolthausen first showed that the normalized partition function

$$\hat{Z}_{N,\beta}^\omega := \exp(-N \lambda(\beta)) Z_{N,\beta}^\omega$$

converges to a limit $\hat{Z}_{\infty,\beta}^\omega$ almost surely with either $\mathbb{P}(\hat{Z}_{\infty,\beta}^\omega = 0) = 0$ or $\mathbb{P}(\hat{Z}_{\infty,\beta}^\omega = 1) = 1$ (may depend on $\beta$). The range of $\beta$ satisfying the former is called the weak disorder regime and the range of $\beta$ satisfying the latter is called the strong disorder regime. It has been shown (cf. [12, 18]) that in the weak disorder regime, the polymer chain still fluctuates on scale $a_n$, similar to the case with no random environment, i.e. $\beta = 0$. This phenomenon is called delocalization. It is believed that in the strong disorder regime, there should be a narrow corridor in space-time with distance to the origin much larger than $a_n$ at time $n$, to which the polymer chain is attracted with high probability. This phenomenon is called localization.

There actually exists a stronger condition than strong disorder, which we now introduce. As in the physics literature, we define the free energy of the system by

$$p(\beta) := \lim_{N \to \infty} \frac{1}{N} \log \hat{Z}_{N,\beta}^\omega.$$ 

Celebrated results like [9, Proposition 2.5] and [7, Proposition 3.1] show that the limit in (1.8) exists almost surely and

$$p(\beta) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\log \hat{Z}_{N,\beta}^\omega]$$
is non-random. By Jensen’s inequality, we have a trivial bound \( p(\beta) \leq 0 \). It is easy to see that if \( p(\beta) < 0 \), then \( \hat{Z}_{N,\beta} \) decays exponentially fast and thus strong disorder holds. Therefore, we call the range of \( \beta \) with \( p(\beta) < 0 \) the very strong disorder regime.

It has been shown in [12, Theorem 3.2] and [7, Theorem 6.1] that as \( \beta \) increases, there is a phase transition from the weak disorder regime, through the strong disorder regime, to the very strong disorder regime, which we summarize in the following.

**Theorem 1.1.** There exist \( 0 \leq \beta_1 \leq \beta_2 \leq \infty \), such that weak disorder holds for \( \beta \in \{0\} \cup (0, \beta_1) \); strong disorder holds for \( \beta \in (\beta_1, \infty) \); and very strong disorder holds for \( \beta \in (\beta_2, \infty) \).

In [18, Proposition 1.13], the author showed that for the Cauchy directed polymer, \( \beta_1 = 0 \) if and only if the random walk \( S - \tilde{S} \) is recurrent, where \( \tilde{S} \) is an independent copy of \( S \). When \( \beta_1 = 0 \), the model is called disorder relevant, since even for arbitrarily small \( \beta > 0 \), disorder modifies the large scale behavior of the underlying random walk.

It is conjectured that \( \beta_1 = \beta_2 \), i.e., the strong disorder regime coincides with the very strong disorder regime (excluding the critical \( \beta \)). So far, the conjecture has only been proved for the nearest-neighbor directed polymer on \( \mathbb{Z}^d \) with \( d = 1 \) in [11] and 2 in [17], and for the long-range directed polymer with heavy-tailed random walks with stable exponent \( \alpha \in (1, 2] \) in [18].

The main purpose of this paper is to show that for disorder relevant Cauchy directed polymer model, under some extra mild assumptions, \( \beta_1 = \beta_2 \), i.e., \( \beta_2 = 0 \). We are going to give full details in the following subsection.

### 1.2. Main results

Recall that \( S \) is the random walk defined in (1.1) and \( \tilde{S} \) is its independent copy. We introduce the expectation of local time for \( S - \tilde{S} \) up to time \( N \) by

\[
D(N) := \sum_{n=1}^{N} P^{\otimes 2}(S_n = \tilde{S}_n),
\]

where \( P^{\otimes 2} \) is the probability on product space. The quantity \( D(\cdot) \) is crucial in our analysis.

Note that \( S - \tilde{S} \) is symmetric, and by [14, Page 271],

\[
\frac{S_n - \tilde{S}_n}{a_n} \xrightarrow{d} H, \quad \text{as } n \to \infty,
\]

where \( a_n \) is the same as in (1.2) and \( H \) is some symmetric Cauchy random variable. Hence, by Gnedenko’s local limit theorem (cf. [4, Theorem 8.6.3]),

\[
P^{\otimes 2}(S_n = \tilde{S}_n) \sim \frac{C}{a_n}.
\]

For some constant \( C > 0 \). Hence, \( S - \tilde{S} \) is recurrent if and only if \( \sum_{n=1}^{\infty} a_n^{-1} = \infty \). Therefore, for disorder relevant Cauchy directed polymer model, \( D(N) \) tends to infinity as \( N \) tends to infinity.

We mention that in [13, Proposition 3.1], the author showed that

\[
\sum_{n=1}^{\infty} \frac{1}{nL(n)} = \infty \iff \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty,
\]

where \( L(\cdot) \) was introduced in (1.1). Since it is usually not easy to obtain an explicit form for \( a_n \), checking the recurrence by the distribution of \( S \) is easier in application.
Before we state our main results, we shortly revisit the centering constant $b_n$ in \eqref{1.2}. When the mean of $S_n - S_{n-1}$ does not exist, $b_n$ may not be proportional to $n$, which can cause some technical difficulty. Therefore, we sometimes just assume $b_n = 0$ for simplicity. We will emphasize this when we state our theorems.

To prove $\beta_2 = \beta_1 = 0$ in this paper, we need an extra assumption on the distribution of $S$, which is
\begin{equation}
\label{1.14}
\sup_{k \in \mathbb{N}} k\mathbb{P}(S_1 - S_0 = k) < \infty \quad \text{and} \quad \sup_{k \in \mathbb{N}} k\mathbb{P}(S_1 - S_0 = -k) < \infty.
\end{equation}

The condition \eqref{1.14} does not allow too many high spikes in the distribution of $S$. Generally speaking, if $S$ is regular enough, for example $\mathbb{P}(S_n - S_{n-1} = k) \leq Ck^{-2}L(k)$ with the same $L(\cdot)$ as in \eqref{1.1}, then \eqref{1.14} is satisfied. The reason that we assume \eqref{1.14} is that we want to have a good control of the local behavior of $S$. The following result is proved by Berger \cite{Berger} and will be used in our proof.

**Theorem 1.2.** For heavy-tailed random walk $S$ defined in \eqref{1.1} and satisfying \eqref{1.14}, there exist positive constants $c_1$ and $c_2$, such that for $|k| \geq c_1a_n$,
\begin{equation}
\label{1.15}
P(S_n - b_n = k) \leq c_2 a_n L(k)k^{-2} = c_2 \frac{a_n L(k)}{k^2 L(a_n)},
\end{equation}
where $a_n$ and $b_n$ were introduced in \eqref{1.2}, and we can modified $b_n$ by $\lfloor b_n \rfloor$ such that $S_n - b_n$ is an integer and the convergence \eqref{1.2} still holds.

Since \cite{Berger} has not been published yet, we cite a proof of Theorem 1.2 given by Berger in the Appendix, where we will only focus on symmetric $S$, which is enough for the completeness of this paper.

Now we can state our main results. Throughout the rest of this paper, we assume that $S - \bar{S}$ is recurrent, where $S$ is the underlying random walk of the Cauchy directed polymer and $\bar{S}$ is an independent copy of $S$. By the statement right below Theorem 1.1, this is an equivalent condition for $\beta_1 = 0$, i.e., the model is disorder relevant. Meanwhile, by \eqref{1.12}, $D(N)$ defined in \eqref{1.10} tends to infinity as $N$ tends to infinity.

We first prove that with some extra assumptions on the underlying random walk $S$, $\beta_2 = \beta_1 = 0$, i.e., in the entire strong disorder regime, very strong disorder holds and there is a trivial weak disorder regime $\beta = \{0\}$. This will be done by giving negative upper bounds for $p(\beta)$ at any $\beta > 0$. We have

**Theorem 1.3.** Let the Cauchy directed polymer model be defined as in subsection 1.7. We assume that the underlying random walk $S$ satisfies \eqref{1.14} and $S - \bar{S}$ is recurrent. Denote $D^{-1}(\cdot)$ by
\begin{equation}
\label{1.16}
D^{-1}(x) := \max\{N : D(N) \leq x\},
\end{equation}
and recall from \eqref{1.2} that the scaling constant $a_n = n\varphi(n)$. Then we have

(i) If $\varphi(\cdot)$ is non-decreasing and $S$ is symmetric, then for arbitrarily small $\epsilon > 0$, there exists a $\beta^{(1)} > 0$, such that for $\beta \in (0, \beta^{(1)})$,
\begin{equation}
\label{1.17}
p(\beta) \leq -D^{-1}((1 + \epsilon)\beta^{-2})^{1+\epsilon}.
\end{equation}

(ii) If $\varphi(\cdot)$ is bounded from above, then there exist a $\beta^{(2)} > 0$ and positive constants $C_1$ and $C_2$, such that for $\beta \in (0, \beta^{(2)})$,
\begin{equation}
\label{1.18}
p(\beta) \leq -C_1 D^{-1}(C_2\beta^{-4}).
\end{equation}
Theorem 1.3 shows that except for a non-monotone $\varphi(\cdot)$ with $\lim_{n \to \infty} \varphi(n) = \infty$, the disorder relevant Cauchy directed polymer satisfies $\beta_2 = 0$. The constraints on $\varphi(\cdot)$ should not be essential for our results to hold, but we cannot overcome some technical difficulties in proof if we drop those extra assumptions. The non-decreasing property of $\varphi(\cdot)$ in Theorem 1.3 (i) may seem weird and we can actually replace it by assuming that $L(\cdot)$ in (1.1) is non-decreasing, which is much easier to check. The reason is that we can choose the scaling constant $a_n$ by $n \mathbb{P}(|S_1| > a_n) \sim 1$, i.e. $L(a_n) \sim \varphi(n)$, which implies that $\varphi(\cdot)$ is non-decreasing. We will discuss how could we release the assumption on $\varphi(\cdot)$ right after the proof of Theorem 1.3.

Note that to obtain (1.17), we assume that $S$ is symmetric, which automatically implies $b_n \equiv 0$, while to obtain (1.18), we only assume that $b_n \equiv 0$, which is weaker than the symmetry of $S$.

We can also give a lower bound for the free energy, and the lower bound is valid under fairly general conditions.

Theorem 1.4. Let the Cauchy directed polymer model be defined as in subsection 1.1. If $S - \bar{S}$ is recurrent, where $S$ is the underlying random walk, then for arbitrarily small $\epsilon > 0$, there exists a $\beta(3) > 0$, such that for $\beta \in (0, \beta(3))$,

$$p(\beta) \geq -D^{-1}((1 - \epsilon)\beta^2) - (1 - \epsilon).$$

Note that we do not need any further assumption on $S$ in Theorem 1.4. Even (1.14) and $b_n \equiv 0$ are not needed.

As an example, if the underlying random walk $S$ is symmetric with

$$\mathbb{P}(S_1 - S_0 = -k) = \mathbb{P}(S_1 - S_0 = k) \sim Ck^{-2}, \quad \text{as } k \to \infty,$$

then it satisfies the condition (1.13) and the slowly varying function $\varphi(\cdot)$ in the scaling constant $a_n$ can be chosen by some constant, which satisfies the condition in Theorem 1.3 (i). Note that in this case $D(N) \sim C \log N$, so $D^{-1}(x) \sim \exp(x/C)$ and thus we have

Corollary 1.5. Let the Cauchy directed polymer model be defined as in subsection 1.1 with (1.20). Denote $C = g(0)\varphi(1)^{-1}$, where $g(\cdot)$ is the density function of $H$ in (1.11), and then we can show that

$$\lim_{\beta \to 0} \beta^2 \log(-p(\beta)) = -\frac{1}{C}.$$

We believe that the bounds (1.17) and (1.19) is sharp while the bound (1.18) is not sharp. We conjecture that the sharp upper bound should be (1.17) as long as $\lim_{N \to \infty} D(N) = \infty$ without any further assumption on model, since by Corollary 1.5 the upper bound (1.17) and the lower bound (1.19) match.

1.3. Organization of the paper. Theorem 1.3 (i) will be proved in Section 2 by a now classic fractional-moment/coarse-graining/change-of-measure procedure. We will adapt the approaches developed in [2, 3].

Theorem 1.3 (ii) can be proved by the process in [17] Section 4. We only need to replace the log $N$ and $1/(i - j)$ in [17] by $D(N)$ and $1/a_{i - j}$ respectively, since log $N$ and $1/(i - j)$ are related to the local time of 2-dimensional simple random walk. The adaption is quite straightforward, so we omit the proof in this paper.

Theorem 1.4 will be proved in Section 3 by a second moment computation introduced in [3] and a usage of a concentration inequality, which was invented in [6].
Although our proofs are adaptation of known methods, new subtle and careful treatments are needed since the random walk in the Cauchy domain is much harder to deal with than 2-dimensional random walk.

We mention that for some technical reason, it seems not very possible to prove the upper bound (1.17) under the general condition in Theorem 1.4 by the fractional-moment/coarse-graining/change-of-measure procedure. One may have to invent a totally new approach to deal with this problem.

2. Proof of Theorem 1.3 (i)

We start with the fractional-moment method. For any \( \theta \in (0,1) \),

\[
E[\log \hat{Z}_{N,\beta}] = \frac{1}{\theta} E[\log(\hat{Z}_{N,\beta})^\theta] \leq \frac{1}{\theta} \log E[(\hat{Z}_{N,\beta})^\theta].
\]

Hence, by (1.9),

\[
p(\beta) = \lim_{N \to \infty} \frac{1}{N} E[\log \hat{Z}_{N,\beta}] \leq \lim_{N \to \infty} \frac{1}{\theta N} \log E[(\hat{Z}_{N,\beta})^\theta].
\]

In this proof, \( \theta \) cannot be chosen arbitrarily. In fact, we will see later that \( \theta \) should be larger than \( 1/2 \). Then our strategy is to chose a coarse-graining length \( l = l(\beta) \), write \( N = ml \), and let \( m \) tend to infinity. Along the subsequence \( N = ml \), we have

\[
p(\beta) \leq \lim_{m \to \infty} \frac{1}{ml \theta} \log E[(\hat{Z}_{ml,\beta})^\theta].
\]

If we can prove

\[
E[(\hat{Z}_{ml,\beta})^\theta] \leq 2^{-m},
\]

then we obtain \( p(\beta) < 0 \). In order to further prove the bound (1.17), by noting that \( D^{-1}(\cdot) \) is increasing to infinity, one appropriate choice of \( l \) is

\[
l = l(\beta) := \inf \{ n \in \mathbb{N} : D([n^{1-\epsilon^2}]) \geq (1 + \epsilon)\beta^{-2} \}.
\]

Now we introduce the coarse-graining method. First, we partition the integer set \( \mathbb{Z} \) according to \( l \) by denoting

\[
I_g := ya_l + (-a_l/2, a_l/2], \quad \forall y \in \mathbb{Z},
\]

where \( a_l \) is the scaling constant in (1.2). We can actually choose the whole scaling sequence \( (a_n)_{n \geq 1} \) by integers to avoid using many \( \lfloor \cdot \rfloor \)'s. Note that \( (I_y)_{y \in \mathbb{Z}} \) is a disjoint family and \( \bigcup_{y \in \mathbb{Z}} I_y = \mathbb{Z} \). Next, for any \( \mathcal{Y} = (y_1, \cdots, y_m) \), define

\[
T_{\mathcal{Y}} = \{ S_i \in I_{y_i}, \text{for } 1 \leq i \leq m \},
\]

and we say \( \mathcal{Y} \) is a coarse-grained trajectory for \( S \in T_{\mathcal{Y}} \). We can now decompose the partition function \( \hat{Z}_{ml,\beta}^\omega \) in terms of different coarse-grained trajectories by

\[
\hat{Z}_{ml,\beta}^\omega = \sum_{\mathcal{Y} \in \mathbb{Z}^m} E \left[ \exp \left( \sum_{n=1}^{m} (\hat{\beta}_{\omega_n, S_n} - \lambda(\beta)) \right) \mathbb{1}_{\{ S \in T_{\mathcal{Y}} \}} \right] := \sum_{\mathcal{Y} \in \mathbb{Z}^m} Z_{\mathcal{Y}}.
\]

By inequality \( (\sum_n a_n)^\theta \leq \sum_n a_n^\theta \) for \( \theta \in (0,1] \),

\[
E[(\hat{Z}_{ml,\beta}^\omega)^\theta] \leq \sum_{\mathcal{Y} \in \mathbb{Z}^m} E[(Z_{\mathcal{Y}})^\theta].
\]
Therefore, to prove (2.4), we only need to prove

**Proposition 2.1.** If \( l \) is sufficiently large, then uniformly in \( m \in \mathbb{N} \), we have

\[
\sum_{Y \in \mathbb{Z}^m} \mathbb{E}[(Z_\beta)_{\theta}^Y] \leq 2^{-m}
\]

To prove Proposition 2.1, we need a change-of-measure argument. For any \( Y \in \mathbb{Z}^m \), we introduce a positive random variable \( g_Y(\omega) \), which can be considered as a probability density after scaling. Then by Hölder’s inequality,

\[
\mathbb{E}[(Z_\beta)_{\theta}] = \mathbb{E}[g_Y^{\theta} (g_Y Z_\beta)] \leq \left( \mathbb{E}[g_Y^{-\theta/(1-\theta)}] \right)^{1-\theta} \left( \mathbb{E}[g_Y Z_\beta] \right)^{\theta}.
\]

Here \( M_{g_Y}(\cdot) := \mathbb{E}[g_Y \mathbb{1}_{\{z\}}] \) can be considered as a new measure. We hope that the expectation of \( Z_\beta \) under this new measure is significantly smaller than that under the original measure \( \mathbb{E} \), and the cost of change of measure, the term \( \mathbb{E}[g_Y^{-\theta/(1-\theta)}] \), is not too large.

To choose \( g_Y \), we need some preliminary. We can first choose an integer \( R \) (not depend on \( \beta \)) and then define space-time blocks (with the convention \( y_0 = 0 \))

\[
B_{i,y_{i-1}} := [(i-1)l + 1, \ldots, il] \times I_{y_{i-1}}, \text{ for } i = 1, \ldots, m,
\]

where

\[
I_y = y a_l + (-R a_l, R a_l).
\]

Since \( S \) is attracted to a 1-stable Lévy process, the graph of \( (S_{(i-1)l} + k)_k \) with \( S_{(i-1)l} = y_{i-1} \) is contained within \( B_{i,y_{i-1}} \) with probability close to 1 when \( R \) is large enough. Therefore, it suffices to perform change of measure on \( B = \cup_{i=1}^m B_{i,y_{i-1}} \). Furthermore, since each block \( B_{i,y_{i-1}} \) plays an equivalent role in \( B \), it is natural to choose

\[
g_Y(\omega) = \prod_{i=1}^m g_{i,y_{i-1}}(\omega)
\]

such that each \( g_{i,y_{i-1}} \) depends only on \( \omega \) in \( B_{i,y_{i-1}} \).

To make \( \mathbb{E}[g_Y Z_\beta] \) small, we can construct \( g_Y \) according to the following heuristics: we first set a threshold. For any block \( B_{i,y} \), if the environment in \( B_{i,y} \) is very large so that the mass of the partition function \( Z_{N,\beta}^\omega \) gained from \( B_{i,y} \) is larger than the threshold, then we choose \( g_{i,y} \) to be small. If the environment in \( B_{i,y} \) is not too large and the contribution to the mass of \( Z_{N,\beta}^\omega \) from \( B_{i,y} \) is less than the threshold, then we simply set \( g_{i,y} \) to be 1. To construct \( g_Y \), we still need to define some auxiliary quantities.

For any arbitrarily small \( \epsilon > 0 \), we introduce

\[
u = u(l) := [l^{1-\epsilon^2}] \quad \text{and} \quad q = q(l) := \frac{1}{\epsilon^2} \max \left\{ \log \left( \sqrt{\varphi(l)} \right), \log D(l) \right\},
\]

where \( \varphi(\cdot) \) is the slowly varying function in the scaling factor \( a_n \), and then we define \( X(\omega) \) depending on \( \omega \) in \( B_{1,0} \) by

\[
X(\omega) := \frac{1}{\sqrt{2R \log(D(u))^{\epsilon^2}}} \sum_{z \in (l_0)^{q+1}} P(\ell_z \omega | \xi) \omega \xi,
\]

where

\[
x := (x_0, \ldots, x_q) \quad \text{and} \quad t := (t_0, \ldots, t_q),
\]

\[
J_{l,u} := \{ (l \xi) : 1 \leq t_0 < \cdots < t_q \leq l, t_i - t_{i-1} \leq u, \forall j = 1, \ldots, q \},
\]

\[
J_{l,u} := \{ (l \xi) : 1 \leq t_0 < \cdots < t_q \leq l, t_i - t_{i-1} \leq u, \forall j = 1, \ldots, q \},
\]

\[
J_{l,u} := \{ (l \xi) : 1 \leq t_0 < \cdots < t_q \leq l, t_i - t_{i-1} \leq u, \forall j = 1, \ldots, q \},
\]
Then, by translation invariance, for the contribution from \( \omega \)
\[ \frac{(2.23)}{2} \]
this kind of approximation by \( \frac{(2.24)}{2} \)
g where \( \theta \)
\[ \frac{(2.22)}{2} \]
we will use \( \frac{(2.21)}{2} \) many times. It is not hard to check that by symmetry of \( \frac{(2.22)}{2} \) if we choose \( K \)
\[ \frac{(2.26)}{2} \]
function directly. One may refer to \[ 2, \text{Section 4.2} \] for more about the discussion of \( X(\omega) \).

Next, we turn to analyze the term \( E \)
\[ \frac{(2.27)}{2} \]
and \( q \) both tend to infinity as \( \beta \) tends to 0, and the definitions of \( q \) and \( u \) make sure that
\[ \frac{(2.21)}{2} \]
We will use \( \frac{(2.21)}{2} \) many times. It is not hard to check that by symmetry of \( S \),
\[ \frac{(2.22)}{2} \]
we can define this kind of approximation by
\[ \frac{(2.23)}{2} \]
where \( \theta_i \)
\[ \frac{(2.24)}{2} \]
where \( K \) is a fixed constant independent of any other parameter. We then have
\[ \frac{(2.25)}{2} \]
by Chebyshev’s inequality and \( \frac{(2.22)}{2} \) if we choose \( K \) large enough. Since \( g_{i,j} \) and \( g_{j,j} \) are defined on disjoint blocks \( B_{i,j} \) and \( B_{j,j} \) for \( i \neq j \), by independence of \( \omega \) in \( B_{i,j} \) and \( B_{j,j} \),
\[ \frac{(2.26)}{2} \]
Next, we turn to analyze the term \( E[g_{y}Z_{y}] \). We can rewrite it as
\[ \frac{(2.27)}{2} \]
For any given trajectory of \( S \), we define a change of measure by
\[ \frac{(2.28)}{2} \]
We can check that \( P^S \) is a probability measure, and \( \omega \) is still a family of independent random variables under \( P^S \), but the distribution of \( \omega_{n,x} \) is tilted and

\[
E^S[\omega_{n,x}] = \lambda(\beta) 1_{\{S_n=x\}} \quad \text{and} \quad \text{Var}^S(\omega_{n,x}) = 1 + (\lambda''(\beta) - 1) 1_{\{S_n=x\}}.
\]

One can check that

\[
\lim_{\beta \to 0} \frac{\lambda(\beta)}{\beta} = 1 \quad \text{and} \quad \lim_{\beta \to 0} \lambda''(\beta) = 1.
\]

Hence, for the \( \epsilon \) given above, when \( \beta \) is sufficiently small, we have

\[
\left| \frac{\lambda(\beta)}{\beta} - 1 \right| \leq \epsilon^3 \quad \text{and} \quad \left| \lambda''(\beta) - 1 \right| \leq 1 + \frac{\epsilon^3}{2}.
\]

By independence of \( \omega \), \( \text{2.21} \) can be further rewritten as

\[
E[g_y Z_Y] = E \left[ E^S[g_Y] 1_{\{S \in \mathcal{T}_Y\}} \right] = E \left[ \prod_{i=1}^{m} E^S[g_{i,y_{i-1}}] 1_{\{S_{i-1} \in I_y\}} \right].
\]

Thanks to the Markov property, \( \text{2.21} \) can be bounded above by

\[
\prod_{i=1}^{m} \max_{x \in I_{Y_{i-1}}} E \left[ E^S[g_{i,y_{i-1}}] 1_{\{S_{i} \in I_{y_{i}}\}} | S_{i-1} = x \right]
\]

By translation invariance \( \text{2.24} \), and summing over \( Y \in \mathbb{Z}^m \), we obtain

\[
\sum_{Y \in \mathbb{Z}^m} (E[g_y Z_Y])^\theta \leq \left( \sum_{y \in I_0} \max_{x \in I_0} \left( E^x \left[ E^S[g_{1,0}] 1_{\{S_{1} \in I_y\}} \right] \right) \right)^\theta,
\]

where \( E^x \) is the expectation with respect to \( P^x \), which is the probability measure for random walk \( S \) staring at \( x \).

Now by \( \text{2.11}, \text{2.26} \) and \( \text{2.34} \), to prove Proposition \( \text{2.1} \), we only need to show

**Proposition 2.2.** For small enough \( \beta > 0 \),

\[
\sum_{y \in I_0} \max_{x \in I_0} \left( E^x \left[ E^S[g_{1,0}] 1_{\{S_{1} \in I_y\}} \right] \right) \leq \frac{1}{4}
\]

To prove Proposition \( \text{2.2} \), we split the summation in \( \text{2.35} \) into two part. Firstly,

\[
\sum_{|y| \geq M} \max_{x \in I_0} \left( E^x \left[ E^S[g_{1,0}] 1_{\{S_{1} \in I_y\}} \right] \right) \leq \sum_{|y| \geq M} \max_{x \in I_0} P^x(S_t \in I_y)^\theta,
\]

since \( g_{1,0} \leq 1 \). By Theorem \( \text{1.2} \) when \( M \) is large enough and fixed, for any \( k \geq M \) and \( j \in \{1, \ldots, a_l - 1\} \),

\[
P(S_t = ka_l + j) \leq C \frac{a_l L(ka_l + j)}{(ka_l + j)^2 L(a_l)} \leq C \frac{L(ka_l + j)}{k^2 a_l L(a_l)}.
\]

Then by Potter bounds (cf. \( \text{4.1 Theorem 1.5.6} \)), for any \( \gamma > 0 \), there exist some constant \( C \), such that for \( k \) and \( j \),

\[
\frac{L(ka_l + j)}{L(a_l)} \leq C k^\gamma \quad \text{uniformly}.
\]
Hence, the summand in (2.36) can be uniformly bounded from above by $Ck^\gamma^{-2}$. Therefore, when $\gamma < 1$, we can choose $\theta$ close to 1 enough such that $\theta(\gamma - 2) < -1$ and then (2.36) can be bounded from above by $1/8$ for sufficiently large $M$.

Next, we turn to the control of the summand in (2.35) for $|y| \leq M$. We can first apply a trivial bound

$$E^x [E^S[g_{1,0}] \mathbb{1}_{\{S \in I_0\}}] \leq E^x [E^S[g_{1,0}]].$$

Then we want to show

**Lemma 2.3.** For any $\eta > 0$, we can choose an appropriate $K$ in (2.24), which only depends on $\eta$, such that for small enough $\beta > 0$, we have

$$\max_{x \in I_0} E^x [E^S[g_{1,0}]] \leq \eta$$

By (2.39) and Lemma 2.3, if we choose $\eta = (16M)^{-1/\theta}$, then

$$\sum_{|y| \leq M} \max_{x \in I_0} \left( E^x [E^S[g_{1,0}] \mathbb{1}_{\{S \in I_0\}}] \right)^{\theta} \leq \frac{1}{8}$$

Combine (2.41) and the bound on (2.36), we conclude Proposition 2.2. Therefore, it only remains to prove Lemma 2.3.

Indeed, Lemma 2.3 can be established by the following two lemmas.

**Lemma 2.4.** For any $\delta > 0$, we can choose a large enough $R$ in (2.13), which only depends on $\delta$, such that for small enough $\beta > 0$, for any $x \in I_0$, we have

$$P^x(P^S(X(\omega) \leq \exp(K^2)) \leq (1 + \epsilon^2)^q \geq 1 - \delta.)$$

**Lemma 2.5.** If $\beta$ is positive and sufficiently small, then for any trajectory $S$ of the underlying random walk, we have

$$\forall \alpha > 0, \text{ small}, \text{ then for any trajectory } S \text{ of the underlying random walk, we have}$$

$$\sum_{|y| \leq M} \max_{x \in I_0} \left( E^x [E^S[g_{1,0}] \mathbb{1}_{\{S \in I_0\}}] \right)^{\theta} \leq \frac{1}{8}$$

We postpone the proof of Lemma 2.4 and Lemma 2.5 and we prove Lemma 2.3 first.

**Proof of Lemma 2.3.** By the definition of $g_{1,0}$, for any trajectory $S$, we have the following trivial bound

$$E^S[g_{1,0}] \leq \exp(-K) + E^S(X(\omega) \leq \exp(K^2))$$

By Chebyshev’s inequality,

$$P^S(X(\omega) \leq \exp(K^2)) \leq (\exp(K^2) - E^S[X])^{-2} \text{Var}^S(X).$$

We denote $A = \{E^S[X] \geq (1 + \epsilon^2)^q\}$. For any $x \in I_0$, by (2.45), Lemma 2.4 and Lemma 2.5 we then have

$$E^x \left[ P^S(X(\omega) \leq \exp(K^2)) \right] \leq P^x(A^c) + E^x \left[ P^S(X(\omega) \leq \exp(K^2)) \mathbb{1}_{A} \right]$$

where we use the fact that $(1 + \epsilon^2)^q - \exp(K^2) \geq \sqrt{2}(1 + \epsilon^2)^q$ to obtain the last line, since $q$ can be made arbitrarily large by choosing $\beta$ close enough to 0.
Now we first take $E^x$-expectation on the both sides of (2.44). Then, we choose $K$ large enough such that $\exp(-K) < \eta/3$. Next, we let $\beta$ tend to 0 so that the last line of (2.44) is smaller than $2\eta/3$ and finally Lemma 2.3 is proved.

The proofs of Lemma 2.4 and Lemma 2.5 are quite long and some computations in them are very complicated and tedious. We will write some intermediate steps as lemmas and try to make things as clear as possible. First, let us prove Lemma 2.4.

**Proof of Lemma 2.4.** First, we recall the definition (2.16) of $X$. Note that $\omega$ is an independent family under the probability $P^S$ and $E^S[\omega_{n,x}] = 0$ if $\omega_{n,x}$ is not on the path of $S$. Hence, for any trajectory of $S$, we have

$$E^S[X] = \frac{(\lambda(\beta))^{q+1}}{\sqrt{2R|\alpha_1 D(u)|^{q/2}}} \sum_{l \in J_1, u} P(l, S(U)) 1_{\{S_{k- \in J_0, \forall k \in \{0, \ldots, q\}\}}$$

(2.47)

$$\geq \frac{(\lambda(\beta))^{q+1}}{\sqrt{2R|\alpha_1 D(u)|^{q/2}}} \sum_{l \in J_1, u} P(l, S(U)) 1_{\{\max_{1 \leq l \leq t} |S_l| \leq Ra\}}$$

where

(2.48)

$$S(U) := (S_0, \ldots, S_t)$$

and we will use notation (2.48) many times. Readers should keep in mind that in (2.47), $S(U)$ is substituted into the $x$ in (2.14) and not mix it up with the $S$ there.

Note that for any $x \in \tilde{I}_0 = (-l/2, a_1/2],$

(2.49)

$$P^x \left( \max_{1 \leq l \leq t} |S_l| > Ra \right) \leq P \left( \max_{1 \leq l \leq t} |S_l| > (R - 1)a \right).$$

Since $S$ is attracted to some 1-stable Lévy process, for any $\delta > 0$, we can choose $R$ large enough such that uniformly in $l$, the probability in (2.49) is smaller than $\delta/2$.

On the event $\{\max_{1 \leq l \leq t} |S_l| \leq Ra\}$, by (2.41), (2.42), and (2.46), we have

$$E^S[X] \geq \frac{\beta}{\sqrt{2R}} \exp(-\epsilon^2 q)(1 - \epsilon^3)^{q+1}(\beta^2 D(u))^{q/2} \frac{1}{lD(u)^q} \sum_{l \in J_1, u} P(l, S(U))$$

(2.50)

$$\geq \frac{\beta}{\sqrt{2R}} \exp(-\epsilon^2 q)(1 + \epsilon)^{q/2} \frac{1}{lD(u)^q} \sum_{l \in J_1, u} P(l, S(U)).$$

Note that for $\epsilon$ small enough, by (2.15),

$$\frac{\beta(1 - \epsilon^3)^{q+1}(1 + \epsilon)^{q/2}}{(1 + \epsilon^2)^2 \exp(\epsilon^2 q)} \geq \beta \left(1 + \frac{\epsilon}{20}\right)^q$$

(2.51)

$$\geq \beta \left(1 + \frac{\epsilon}{20}\right) \frac{\log D(l)}{D(l)} \gg \beta \exp(\log D(u)) \geq \frac{1 + \epsilon}{\beta} \gg 1.$$

Hence,

$$\frac{\beta}{\sqrt{2R}} \frac{(1 - \epsilon^3)^{q+1}(1 + \epsilon)^{q/2}}{\exp(\epsilon^2 q)} \geq (1 + \epsilon)^{2q}$$

(2.52)
and

\[(2.53) \implies \mathbb{E}^S[X] \geq (1 + \epsilon^2)^{2q} \frac{1}{lD(u)^q} \sum_{t \in J_{l,u}} P(t, S^{(U)}) .\]

Recall that the probability in \((2.49)\) is smaller than \(\delta/2\) and by \((2.53)\) on \(\{\max_1 \leq t \leq |S_t| \leq Ra\}\), we have

\[(2.54) \quad P_x(\mathbb{E}^S[X] < (1 + \epsilon^2)^q) \leq \frac{\delta}{2} + P^x \left( \frac{1}{lD(u)^q} \sum_{t \in J_{l,u}} P(t, S^{(U)}) < \frac{1}{(1 + \epsilon^2)^q} \right) .\]

To bound the probability on the right-hand side of \((2.54)\), we introduce a random variable

\[(2.55) \quad W_l := \frac{1}{lD(u)^q} \sum_{t \in J'_{l,u}} P(t, S^{(U)}) ,\]

where

\[(2.56) \quad J'_{l,u} = \{ t \in J_{l,u} : 1 \leq t_0 \leq l/2 \} .\]

Since \(J'_{l,u} \subset J_{l,u}\), it suffices to prove

\[(2.57) \quad P_x \left( W_l < \frac{1}{(1 + \epsilon^2)^q} \right) \leq \frac{\delta}{2} .\]

Note that by the definition of \(P(t, S^{(U)})\), the law of \(W_l\) does not depend on the starting point \(S_0 = x\). Hence, during the rest of the proof, we can simply use \(P\) instead of \(P^x\) for short. Our strategy to prove \((2.57)\) is to show that the mean of \(W_l\) is not too small and the variance of \(W_l\) can be controlled.

First, by recalling the definition of \(l, u,\) and \(q\), when \(\beta\) is small enough, \(l/2 + qu < l\). Since the value of \(P(t, S^{(U)})\) does not depend on the position of \(S_{t_0}\), we have

\[(2.58) \quad \mathbb{E} \left[ \sum_{t \in J'_{l,u}} P(t, S^{(U)}) \right] = \frac{l}{2} \sum_{t \in J'_{l,u}} \mathbb{E} \left[ P(t, S^{(U)}) \right] = \frac{l}{2} \left( \sum_{t=1}^u P(S_{2t} = 0; |S_t| \leq Ra_t) \right)^q .\]

By Gnedenko’s local limit theorem, there exists a constant \(C_1\), such that uniformly for \(t \in \mathbb{N}\) and \(x \in \mathbb{Z}\),

\[(2.59) \quad P(S_t = x) \leq \frac{C_1}{a_t} .\]

Therefore,

\[(2.60) \quad P(S_{2t} = 0; |S_t| > Ra_t) = \sum_{|x| > Ra_t} P(S_t = x)^2 \leq \frac{C_1}{a_t} P(|S_t| > Ra_t) \leq \frac{C_1C_2}{a_t} ,\]

where \(C_2\) can be made sufficiently small by choosing \(R\) large enough such that the inequality holds for \(t\) uniformly. Again, by Gnedenko’s local limit theorem,

\[(2.61) \quad P(S_{2t} = 0) \sim \frac{C}{a_{2t}} \sim \frac{C}{2a_t}, \quad \text{as } t \to \infty .\]
Hence,
\begin{equation}
(2.62) \quad \mathbf{P}(S_{2t} = 0; |S_t| \leq Ra_t) \geq (1 - \epsilon^4)\mathbf{P}(S_{2t} = 0).
\end{equation}
provided \( t \) is large enough. Then for sufficiently large \( u \), we have
\begin{equation}
(2.63) \quad (1 - \epsilon^3)D(u) \leq \hat{D}(u) := \sum_{i=1}^{u} \mathbf{P}(S_{2t} = 0; |S_t| \leq Ra_t) \leq D(u).
\end{equation}

Therefore,
\begin{equation}
(2.64) \quad \frac{1}{2}(1 - \epsilon^3)^q \leq \mathbf{E}[W_i] \leq \frac{1}{2}
\end{equation}
and we can bound (2.57) by
\begin{equation}
(2.65) \quad \mathbf{P}\left(W_i - \mathbf{E}[W_i] < \frac{1}{(1 + \epsilon^2)^q} - \mathbf{E}[W_i]\right) \\
\leq \mathbf{P}\left(W_i - \mathbf{E}[W_i] < \frac{1}{(1 + \epsilon^2)^q} - \frac{1}{2}(1 - \epsilon^3)^q\right) \\
\leq (1 + \epsilon^2)^2q\mathbf{Var}(W_i),
\end{equation}

since \((1 - \epsilon^3)^q \geq 4(1 + \epsilon^2)^{-q}\) when \( \epsilon \) is small enough.

It remains to control the variance of \( W_i \). We define
\begin{equation}
(2.66) \quad Y_j = \frac{1}{D(u)^q} \sum_{\xi \in J_{t,u}(j)} \mathbf{P}(t, S^{(u)}) \leq \frac{1}{2} \left(\frac{\hat{D}(u)}{D(u)}\right)^q,
\end{equation}

where \( J_{t,u}(j) = \{ t \in J_{t,u} : t_0 = j \} \). By (2.58), it is obvious that \( W_i - \mathbf{E}[W_i] = \left(\sum_{j=1}^{l/2} Y_j\right)/l \)
and \( \mathbf{E}[Y_j] = 0 \). Then we have
\begin{equation}
(2.67) \quad \mathbf{Var}(W_i) = \frac{1}{l^2} \sum_{j_1,j_2=1}^{l/2} \mathbf{E}[Y_{j_1}Y_{j_2}].
\end{equation}

By (2.69) and (2.61)
\begin{equation}
(2.68) \quad Y_j = \frac{1}{D(u)^q} \sum_{\xi \in J_{t,u}(j)} \mathbf{P}(t, S^{(u)}) \leq \frac{1}{D(u)^q} \left(\sum_{i=1}^{u} \frac{C_i}{a_i}\right)^q \leq (C_2)^q.
\end{equation}

Then, we will show that most summands in (2.67) are zero. Note that for \( j \in \{1, \ldots, l/2\} \),
\( t_q - t_0 \leq qu \) for \( t_0, t_q \in J_{t,u}(j) \). If we denote the increment of \( S \) by \( (X_n)_{n \geq 1} \),
then \( Y_j \) only depends on \( (X_{j+1}, \ldots, X_{j+qu}) \). Therefore, for \( |j_1 - j_2| > qu \), \( Y_{j_1} \) and \( Y_{j_2} \) are independent
and \( \mathbf{E}[Y_{j_1}Y_{j_2}] = \mathbf{E}[Y_{j_1}]\mathbf{E}[Y_{j_2}] = 0 \). By (2.68),
\begin{equation}
(2.69) \quad \mathbf{Var}(W_i) \leq \frac{qu}{l}(C_2)^{2q} \leq q(C_2)^{2q} \epsilon^2.
\end{equation}

Then (2.65) is bounded above by \((C_3)^{q(\beta - 2/3 + \epsilon)})l^{-2/3+\epsilon}\), which tends to infinity as \( \beta \) tends to 0 by
the definition of \( q \) and \( l \) and we complete the proof of Lemma 2.4. \( \square \)

At the end of this section, we prove Lemma 2.5. We will use \( C \) to represent the constants
in the proof and it could change from line to line without emphasis.
Proof of Lemma 2.5. For any trajectory of \( S \), we shift the environment by
\[
(2.70) \quad \hat{\omega}_{n,x} := \omega_{n,x} - \lambda'(\beta) \mathbb{1}_{\{s_n = x\}}.
\]
It is not hard to check that under \( \mathbb{P}^S \), \( \hat{\omega} \) is a family of independent random variables with expectation 0. Besides, when \( \beta \) is small enough, the variance of \( \hat{\omega}_{n,x} \) can be bounded by \( 1 + (\epsilon^3/2) \).

To compute the \( \mathbb{V} \text{ar}(X) \), we start by observing that
\[
(2.71) \quad \mathbb{E}^S[X^2] = \frac{1}{2R\text{la}_1D(u)^q} \mathbb{E}^S \left[ \left( \sum_{\tilde{t} \in (I_0)^{q+1} \tilde{A} \subset J_{t,u}} \mathbb{P}(\tilde{t}, \tilde{\omega}) \prod_{i=0}^{q} \left( \hat{\omega}_{t_j, x_j} + \lambda(\beta) \mathbb{1}_{\{s_j = x_j\}} \right) \right)^2 \right].
\]
A simple expansion shows that
\[
(2.72) \quad \prod_{i=0}^{q} \left( \hat{\omega}_{t_j, x_j} + \lambda(\beta) \mathbb{1}_{\{s_j = x_j\}} \right) = \sum_{r=0}^{q+1} \left( \lambda(\beta)^r \sum_{A \subset \{0, \ldots, q\}} \prod_{k \in A} \mathbb{1}_{\{s_k = x_k\}} \prod_{j \in \{0, \ldots, q\} \setminus A} \hat{\omega}_{t_j, x_j} \right).
\]
Therefore, the square term in \( \mathbb{E}^S \) in (2.71) is the summation over \( \tilde{t}, \tilde{\omega} \in (I_0)^{q+1}, \tilde{t}', \tilde{\omega}' \in J_{t,u} \) of \( \mathbb{P}(\tilde{t}, \tilde{\omega})\mathbb{P}(\tilde{t}', \tilde{\omega}') \) times
\[
(2.73) \quad \sum_{r=0}^{q+1} \sum_{r'=0}^{q+1} \lambda(\beta)^{r+r'} \sum_{A \subset \{0, \ldots, q\}, |A|=r \ B \subset \{0, \ldots, q\}, |B|=r' \ B'=B} \prod_{k \in A} \mathbb{1}_{\{s_k = x_k\}} \prod_{j \in \{0, \ldots, q\} \setminus B} \hat{\omega}_{t_j, x_j} \hat{\omega}_{t'_j, x'_j}.
\]
Note that since \( \hat{\omega} \) is an independent and mean-0 family, when taking \( \mathbb{P}^S \)-expectation, the summation is nonzero if and only if \( r = r' \) and
\[
(2.74) \quad \{(t_j, x_j) \mid j \in \{0, \ldots, q\} \setminus A\} = \{(t'_j, x'_j) \mid j \in \{0, \ldots, q\} \setminus B\}.
\]
Hence, we can rewrite the \( \mathbb{P}^S \)-expectation of (2.73) by first fixing \( (t_j, x_j) \) for \( j \in \{0, \ldots, q\} \setminus A \).

We define a set of \( (q - r + 1) \)-tuples by
\[
(2.75) \quad S_{q-r} := \{ \underline{s} := (s_0, \ldots, s_q) : 1 \leq s_0 < \cdots < s_{q-r} \leq l, s_{q-r} - s_0 \leq qu \}.
\]
For any given \( \underline{s} \in S_{q-r} \), we further define a related set of \( r \)-tuples by
\[
(2.76) \quad T_r(\underline{s}) := \{ \underline{t} = (t_1, \cdots, t_r) : 1 \leq t_1 < \cdots < t_r \leq l, \underline{s} \cdot \underline{t} \in J_{t,u} \},
\]
where \( \underline{s} \cdot \underline{t} \) is a \( (q + 1) \)-tuple, which contains all the entries of \( \underline{s} \) and \( \underline{t} \) and the entries are ordered from small to large.

Now we can have a nicer form for \( \mathbb{V} \text{ar}^S(X) \). Note that the \( \mathbb{P}^S \)-expectation of the term \( r = r' = q + 1 \) in (2.73) is exactly the term \( \mathbb{E}^S[X]^2 \), so we can subtract it on both of (2.71)
and by recalling that $E^S[\tilde{\omega}_{n,x}]^2 \leq (1 + e^3/2)^2 \leq 2$, we obtain
\[
\text{Var}^S(X) \leq \frac{(1 + e^3/2)^{q+1}}{2Rl(D(u))^q} \sum_{x \in (I_0)^r+1, \ell \in J_{t,u}} P(\ell, x)^2 \\
+ \frac{1}{2Rl(D(u))^q} \sum_{r=1}^{q} \sigma_{l}(\beta))^{2r} q^{q+1-r} \sum_{x \in (I_0)^r+1, \ell' \in T_r(\ell)} \sum_{x \in (I_0)^r+1} P((x \cdot \ell), (x, S^{(q)})) P((x \cdot \ell'), (x, S^{(q)})), \tag{2.77}
\]
where the first term on the right-hand side of (2.77) corresponds to $r = 0$, and it is actually equal to $(1 + e^3/2)^q \sum_{x \in (I_0)^r+1} X$ and bounded above by $(1 + e^3/2)^{q+1}$. For the $(q+1)$-tuple $(x, S^{(q)})$ in the last summation, its $i$-th element is $x_j$ if and only if the $i$-th element in $s \cdot \ell$ is $s_j$, while it is $S_{t_j}$ if and only if the $i$-th element in $s \cdot \ell$ is $t_j$.

Finally, we will estimate
\[
\sum_{x \in (I_0)^r+1, \ell' \in T_r(\ell)} \sum_{x \in (I_0)^r+1} \sum_{x \in (I_0)^r+1} P((x \cdot \ell), (x, S^{(q)})) P((x \cdot \ell'), (x, S^{(q)})) \\
= \sum_{x \in (I_0)^r+1, \ell' \in T_r(\ell)} \left( \sum_{x \in (I_0)^r+1} P((x \cdot \ell), (x, S^{(q)})) \right)^2,
\]
which would be very lengthy.

First, for convention, we can denote $s_{-1} := 0$ and $s_{q-r+1} = l$. We can split the summation $\sum_{x \in (I_0)^r+1, \ell' \in T_r(\ell)} \sum_{x \in (I_0)^r+1} P((x \cdot \ell), (x, S^{(q)}))$ according to the position of $t_1$. We have
\[
\sum_{x \in (I_0)^r+1, \ell' \in T_r(\ell)} \sum_{x \in (I_0)^r+1} P((x \cdot \ell), (x, S^{(q)})) = \sum_{k=0}^{q+1-r} \sum_{x \in (I_0)^r+1, t_1 \in (s_{k-1}, s_k)} P((x \cdot \ell), (x, S^{(q)})). \tag{2.79}
\]
We observe that
\[
\sum_{x \in (I_0)^r+1, t_1 \in (s_{k-1}, s_k)} P((x \cdot \ell), (x, S^{(q)})) \leq \prod_{i=1}^{q-r} \sum_{s_{k-1} < t_{m_i+1} < \cdots < t_{m_i} < s_i} \sum_{0 < t_1 < \cdots < t_{m_0} < s_0} P((s_{i-1}, L_{m_i+1}, \cdots, t_{m_i}, s_i), (x_{i-1}, S_{t_{m_i+1}}, \cdots, S_{t_{m_i}}, x_i)) \times \sum_{s_{q-r} < t_{m_{q-r}+1} < \cdots < t_{r} < t} P((s_{q-r}, t_{m_{q-r}+1}, \cdots, t_{r}), (x_{q-r}, S_{t_{m_{q-r}+1}}, \cdots, S_{t_{r}})). \tag{2.80}
\]
Here $m_i$ denotes the number of $t$-indices before $s_i$. If $m_0 = 0$, then the third line of (2.80) is simply 1 and so is the fourth line of (2.80) if $m_{q-r} = r$.

We can bound the factor in the second line of (2.80) for any $i \in \{1, \cdots, q-r\}$ according to the following lemma.
Lemma 2.6. For any \( j \in \mathbb{N} \), if \( 1 \leq t_i - t_{i-1} \leq u \) for all \( i \in \{1, \ldots, j + 1\} \), where \( t_0 = 0 \), \( t_{j+1} = s \) for convention, then there exists a uniform constant \( C \), such that

\[
(2.81) \quad \sum_{0 < t_1 < \cdots < t_j < s} P((0, t_1, \ldots, t_j, s), (0, z_1, \ldots, z_j, x)) \leq (C D((j+1)u))^j p_s(0, x),
\]

where we use the notation

\[
(2.82) \quad p_t(x, y) = P(S_t = y - x)
\]

for any \( t \geq 1 \) and \( y, x \in \mathbb{Z} \).

Proof of Lemma 2.6. Recall the definition (2.19) for \( P(t, x) \) and note that the first two factors of \( P((0, t_1, \ldots, t_j, s), (0, z_1, \ldots, z_j, x)) \) is

\[
(2.83) \quad P(S_{t_1} = z_1)P(S_{t_2} - S_{t_1} = z_2 - z_1)1_{\{|z_1| \leq R(t_1), |z_2 - z_1| \leq R(t_2 - t_1)\}}.
\]

Since we have assumed that \( \varphi(\cdot) \) is non-decreasing, we have

\[
(2.84) \quad |z_2| \leq Ra_1 + Ra_2 - t_1 \leq Ra_2.
\]

By Gnedenko’s local limit theorem, for any \( \Delta > 0 \), there exists an \( N = N(\Delta) \), such that for any \( n > N \),

\[
(2.85) \quad P(S_n = x) \geq \frac{g(x/a_n) - \Delta}{a_n}
\]

for all \( x \in \mathbb{Z} \), where \( g(\cdot) \) is the density function of the limiting Cauchy distribution. Then for \( |x| \leq Ra_n \), since \( g(t) \) is symmetric and decreasing for \( t > 0 \), we have

\[
(2.86) \quad P(S_n = x) \geq \frac{g(R) - \Delta}{a_n}.
\]

Note that \( R \) only depends on the \( \epsilon \) throughout the paper and the \( \delta \) in Lemma 2.4. We can choose \( \Delta = g(R)/2 \) and then for \( n \geq N(\Delta) \) and \( |x| \leq Ra_n \),

\[
(2.87) \quad P(S_n = x) \geq \frac{g(R)}{2a_n}.
\]

Therefore, there exists a constant \( C = C(R) \), such that for any \( n \geq 1 \) and \( |x| \leq Ra_n \),

\[
(2.88) \quad P(S_n = x) \geq \frac{C}{a_n}.
\]

Then by (2.84) and (2.59),

\[
(2.89) \quad \leq p_t(0, z_1) p_{t_2 - t_1}(z_1, z_2)1_{\{|z_2| \leq Ra_2\}} \leq \frac{C}{a_1 a_{t_2 - t_1}} 1_{\{|z_2| \leq Ra_2\}}.
\]

Since

\[
(2.90) \quad \frac{1}{2} \leq \frac{t_2}{t_1 \vee (t_2 - t_1)} \leq 1,
\]

by (2.88) and Potter bounds (cf. [4] Theorem 1.5.6),

\[
(2.91) \quad \frac{1}{a_1 a_{t_2 - t_1}} 1_{\{|z_2| \leq Ra_2\}} \leq \frac{P(S_{t_2} = z_2) a_{t_2}}{C a_1 a_{t_2 - t_1}} 1_{\{|z_2| \leq Ra_2\}} \leq \frac{C}{a_1 a_{t_2 - t_1}} P(S_{t_2} = z_2) 1_{\{|z_2| \leq Ra_2\}}.
\]
Then, by (2.83), (2.89) and (2.91), we have
\[ \sum_{0 < t_1 < \cdots < t_j < s} C \mathcal{P}((0, t_1, \cdots, t_j, s), (0, z_1, \cdots, z_j, x)) \leq \sum_{0 < t_1 < \cdots < t_j < s} \mathcal{P}((0, t_2, \cdots, t_j, s), (0, z_2, \cdots, z_j, x)) \]

(2.92)

By induction, we then prove (2.81).

□

D differentiable by [4, Theorem 1.8.2]. Then by definition of \( t, s \) we apply Lemma 2.6 for all terms in (2.80) with \( t, s \)-indices larger than \( s_k \) to obtain an upper bound

(2.93)

\[ \sum_{0 = m_0 = \cdots = m_{k-1}} m_k \leq m_{k+1} \leq \cdots \leq m_{q-r} \leq r \]

\[ \times \prod_{i=1}^{k-1} p_{s_i-s_{i-1}}(x_{i-1}, x_i) \]

\[ \times \mathcal{P}((t_1, \cdots, t_{m_0}, s_0), (S_1, \cdots, S_{m_0}, x_0)) \]

\[ \times \sum_{s_k-1 < t_{m_k-1}+1 < \cdots < t_{m_q} < s_k} \mathcal{P}((x_{k-1}, S_{t_{m_k-1}+1}, \cdots, S_{m_k}, x_k)) \]

\[ \times (CD(qu))^{m_q-r-m_k} \prod_{i=k+1}^{q-r} p_{s_i-s_{i-1}}(x_{i-1}, x_i) \]

\[ \times \sum_{s_q-r < t_{m_q-r+1} < \cdots < t_l} \mathcal{P}((s_{q-r}, t_{m_q-r+1}, \cdots, t_l), (x_{q-r}, S_{t_{m_q-r+1}}, \cdots, S_{t_l})). \]

Recall that the factor in the first line of (2.93) is 1 if \( m_0 = 0 \) and note that if \( m_{q-r} < r \), i.e. \( t_1 < s_{q-r} \), we should further bound the last line in (2.93) from above by \( (CD(u))^{r-m_{q-r}} \), which can be established by the argument in (2.68).

Next, we show that the term \( D(qu) \) in (2.93) can be replaced by \( D(u) \).

**Lemma 2.7.**

(2.94) \[ \lim_{u \to \infty} \frac{D(qu)}{D(u)} = 1. \]

**Proof of Lemma 2.7.** Without loss of generality, we may assume that \( D(\cdot) \) and \( \varphi(\cdot) \) are differentiable by [4, Theorem 1.8.2]. Then by definition of \( D(\cdot) \), we have \( D'(u) \sim (u\varphi(u))^{-1} \).
Then we estimate
\[
(2.95) \quad \frac{uD'(u) \log q}{D(u)} \leq C \max \left\{ \frac{\log \log \left( \frac{\sqrt{D(l)}}{\varphi(u)D(u)} \right)}{\varphi(u)D(u)}, \frac{\log \log D(l)}{\varphi(u)D(u)} \right\}.
\]
Since \( u^4 \geq l \), we can further bound (2.95) from above by
\[
(2.96) \quad C \max \left\{ \frac{\log \log \varphi(u^4)}{\varphi(u)D(u)}, \frac{\log \log D(u^4)}{\varphi(u)D(u)} \right\}.
\]
Then by L’Hospital rule,
\[
(2.97) \quad \lim_{u \to \infty} \frac{\log \log \varphi(u^4)}{\varphi(u)D(u)} = \lim_{u \to \infty} \frac{1}{\log \varphi(u^4)} \frac{1}{\varphi(u^4)D(u) + \varphi(u)D'(u)}
\]
\[
= \lim_{u \to \infty} \frac{1}{\log \varphi(u^4)} \frac{1}{\varphi(u^4) u \varphi'(u)D(u) + 1} = 0,
\]
since \( \lim_{x \to \infty} (x \varphi'(x)/\varphi(x)) = 0 \) by \[4\], Section 1.8. By the same computation,
\[
(2.98) \quad \lim_{u \to \infty} \frac{\log \log D(u^4)}{\varphi(u)D(u)} = 0.
\]
Hence, (2.97) tends to 0 as \( u \) tends to infinity. By \[4\], Proposition 2.3.2 and \[4\], Theorem 2.3.1, (2.94) is proved. \( \square \)

Note that the number of possible interlacements of \( 0 \leq m_0 \leq \cdots \leq m_{q-r} \leq r \) is not larger than \( 2^q \). Hence, according to the value of \( k \), (2.95) can be bounded above by
\[
(2.99) \quad J_0 = 2^q \sum_{m_k = 1}^r (CD(u))^{r-m_k} \times \sum_{0<t_1<\cdots<t_{m_0}<s_0} P((t_1, \cdots, t_{m_0}, s_0), (S_{t_1}, \cdots, S_{t_{m_0}}, x_0)) \prod_{i=1}^{q-r} p_{s_i-s_{i-1}}(x_{i-1}, x_i)
\]
if \( k = 0 \);
\[
(2.100) \quad J_k = 2^q \sum_{m_k = 1}^r (CD(u))^{r-m_k} \prod_{i=1}^{k-1} p_{s_i-s_{i-1}}(x_{i-1} - x_i) \times \sum_{s_{k-1} < t_1 < \cdots < t_{m_k} < s_k} P((s_{k-1}, t_{m_k-1}+1, \cdots, t_{m_k}, s_k)) \times \prod_{i=k+1}^{q-r} p_{s_i-s_{i-1}}(x_{i-1}, x_i)
\]
if \( 1 \leq k \leq q-r \); and
\[
(2.101) \quad J_{q+1-r} = 2^q \sum_{m_k = 1}^r p_{s_i-s_{i-1}}(x_{i-1}, x_i) \times \sum_{s_{q-r} < t_1 < \cdots < t_r \leq l} P((s_{q-r}, t_1, \cdots, t_r), (x_{q-r}, S_{t_1}, \cdots, S_{t_r}))
\]
if $k = q + 1 - r$.

The we can expand the square term in (2.78) according to the upper bounds above. We obtain the summation $\sum_{k \in S_{q-r}} \sum_{x \in (I_0)_{g+1-r}} \sum_{m,k} = 1$ of the product $J_k J_k'$ between two of (2.99), (2.100) and (2.101). We will use different summing strategies for different summands.

As a beginning, we prove a common result that is valid for all cases.

**Lemma 2.8.** For any $n \geq 1$,

$$\sum_{s=1}^{n} \sum_{x \in I_0} p_s(0, x)^2 \leq D(n) \quad (2.102)$$

**Proof of Lemma 2.8.** By symmetry of $S$,

$$\sum_{s=1}^{n} \sum_{x \in I_0} p_s(0, x)^2 \leq \sum_{s=1}^{n} \mathbf{P}^{2}(S_t = \hat{S}_t) = D(n). \quad (2.103)$$

Now, we can bound (2.78) according to (2.99), (2.100), (2.101). First, we consider the case $k = k'$.

If $k = k' = 0$, then we can first fix the position of $s_0$, which has at most $l$ choices. Note that we have the term $\prod_{i=1}^{n-r} \sum_{x \in (I_0)_{g+1-r}} (p_{s_i-s_{i-1}}(x_{i-1}, x_i))^2$. Hence, for any $x_0$, we can sum over $s_1, \cdots, s_{q-r}$ and $x_1, \cdots, x_{q-r}$ by Lemma 2.8 which gives $(D(qu))^q-r$. Next, we use the trivial bound

$$\sum_{0 < t_1' < \cdots < t_{m_0}' < s_0} \mathbf{P}((t_1', \cdots, t_{m_0}'), (S_{t_1'}, \cdots, S_{t_{m_0}'}, x_0)) \leq (CD(u))^{m_0} \quad (2.104)$$

and then sum over $s_0 - t_{m_0}$ and $x_0$ by

$$\sum_{s_0 - t_{m_0} = 1}^{u} \sum_{x_0 \in I_0} p_{s_0 - t_{m_0}}(S_{t_1}, x_0) \leq \sum_{t_1}^{1} = u. \quad (2.105)$$

At last, we use the trivial bound

$$\sum_{0 < t_1 < \cdots < t_{m_0}} \mathbf{P}((t_1, \cdots, t_{m_0}), (S_{t_1}, \cdots, S_{t_{m_0}})) \leq (CD(u))^{m_0-1}. \quad (2.106)$$

Now we obtain that for any $m_0$ and $m_0'$,

$$\sum_{x \in S_{q-r}} \sum_{x \in (I_0)_{g+1-r}} (J_0)^2 \leq C^{q} \mathcal{U} D(u)^{q+r-1}. \quad (2.107)$$

If $k = k' = q + 1 - r$, then we can first fix the position of $s_{q-r}$ and then apply the strategy above to obtain that

$$\sum_{x \in S_{q-r}} \sum_{x \in (I_0)_{g+1-r}} (J_{q+1-r})^2 \leq C^{q} \mathcal{U} D(u)^{q+r-1}. \quad (2.108)$$

If $1 \leq k = k' \leq q - r$, then we can first fix the position of $s_{k-1}$, which has at most $l$ choices. Note that we have the term $\prod_{i=1}^{k-1} \sum_{x \in (I_0)_{g+1-r}} (p_{s_i-s_{i-1}}(x_{i-1}, x_i))^2$. Hence, for any $x_{k-1}$, we can sum over $s_0, \cdots, s_{k-2}$ and $x_0, \cdots, x_{k-2}$ by Lemma 2.8 (hold
Lemma 2.9. We need the following lemma.

Proof of Lemma 2.9. Suppose (2.111) is valid for $k$. Then the induction is completed and the lemma has been proved.

To deal with (2.110), the first step is to sum over $x_{k-1}$ and $x_k$ by

$$
\sum_{x_{k-1}, x_k \in I_0} p_{x_{k-1}}(x_{k-1}, S_{t_1}) p_{x_k}(x_k) p_{x_{k-1}}(S_{t_{m_k}}, x_k)
$$

(2.114)

$$
\leq p_{x_{k-1}+s_k-s_{k-1}+s_k-t_{m_k}}(S_{t_1}, S_{t_{m_k}}) \leq \frac{C}{a_{t_1-s_{k-1}+s_k-t_{m_k}+s_k-s_{k-1}}}.
$$
where we use the symmetry of $S$. Then, by Lemma 2.9

\[ \sum_{s_k=s_{k-1}=1}^{m_k u} \sum_{s_{k-1} < t_1 < \cdots < t_m < s_k} \frac{1}{a_{t_1-s_{k-1}+s_k-t_m+s_{k-1}-s_k-1}} \leq C^{m_k} (m_k u)^{1+\epsilon} (D(m_k u))^{m_k-1} \leq C^{m_k} (qu)^{1+\epsilon} (D(qu))^{m_k-1}. \]  

\[ \sum_{s_k=s_{k-1}=1}^{m_k u} \sum_{s_{k-1} < t_1 < \cdots < t_m < s_k} \frac{1}{a_{t_1-s_{k-1}+s_k-t_m+s_{k-1}-s_k-1}} \leq C^{m_k} (m_k u)^{1+\epsilon} (D(m_k u))^{m_k-1} \leq C^{m_k} (qu)^{1+\epsilon} (D(qu))^{m_k-1}. \]

Now we obtain that for any $m_k$ and $m'_k$,

\[ \sum_{x \in S_{q-r}} \sum_{x \notin (I_0)^{q+r}} (J_k)^2 \leq C^q (qu)^{1+\epsilon} \ln D(u) q^{q+r-1}. \]

Next, we consider the case $k \neq k'$. We may just assume that $k < k'$. First, we can fix the position of $s_{k-1}$, which has at most $l$ choices. Then, if $k' = q + 1 - r$, then we just use the trivial bound

\[ \sum_{s_k=s_{k-1}=1}^{m_k u} \sum_{s_{k-1} < t_1 < \cdots < t_m < s_k} \frac{1}{a_{t_1-s_{k-1}+s_k-t_m+s_{k-1}-s_k-1}} \leq C^{m_k} (m_k u)^{1+\epsilon} (D(m_k u))^{m_k-1} \leq C^{m_k} (qu)^{1+\epsilon} (D(qu))^{m_k-1}. \]

\[ \sum_{s_k=s_{k-1}=1}^{m_k u} \sum_{s_{k-1} < t_1 < \cdots < t_m < s_k} \frac{1}{a_{t_1-s_{k-1}+s_k-t_m+s_{k-1}-s_k-1}} \leq C^{m_k} (m_k u)^{1+\epsilon} (D(m_k u))^{m_k-1} \leq C^{m_k} (qu)^{1+\epsilon} (D(qu))^{m_k-1}. \]

If $k = 0$, then we have the term \( \prod_{i=1}^{q-r} (r_i \leq \ln D(u) q^{q+r-1}) \) and for any $x_0$, we can sum over $s_1, \cdots, s_{q-r}$ and $s_1, \cdots, x_{q-r}$ by Lemma 2.8 to obtain an upper bound $(D(qu))^{q-r}$. Then we can complete the estimate by

\[ \sum_{s_0-t_0 = 1}^{u} \sum_{x_0 \in I_0} p_{s_0-t_0} (t_m, x_0) \leq u \]

and

\[ \sum_{0 < t_1 < \cdots < t_m} \sum_{x_0 \in I_0} p_{s_0-t_0} (t_m, x_0) \leq (CD(u))^{m_0-1}. \]

If $k > 0$, then we have $\prod_{i=1}^{1} (r_i \leq \ln D(u) q^{q+r-1})$ and for any $x_0$, we can sum over $s_0, \cdots, s_{k-2}$ and $x_0, \cdots, x_{k-2}$ by Lemma 2.8 (hold for the moment), which gives $(D(qu))^{k-1}$. For the same reason, then we can sum over $s_{k+1}, \cdots, s_{q-r}$ and $x_{k+1}, \cdots, x_{q-r}$ (hold for the moment), which gives $(D(qu))^{q-r-k}$. These summations and products together give $(D(qu))^{q-r-1}$, and then we can complete all the estimate by bounding

\[ \sum_{s_k=s_{k-1}=1}^{m_k u} \sum_{s_{k-1} < t_1 < \cdots < t_m < s_k} \sum_{x_k \in I_0} p_{s_k-s_{k-1}} (x_k, x_k) \sum_{t_m \leq s_k} \sum_{x_k} (s_k, t_k, \cdots, t_m, s_k, (x_k-1, S_{t_m}, x_k)). \]

\[ \sum_{s_k=s_{k-1}=1}^{m_k u} \sum_{s_{k-1} < t_1 < \cdots < t_m < s_k} \sum_{x_k \in I_0} p_{s_k-s_{k-1}} (x_k, x_k) \sum_{t_m \leq s_k} \sum_{x_k} (s_k, t_k, \cdots, t_m, s_k, (x_k-1, S_{t_m}, x_k)). \]

In any case, we can have an uniform upper bound $C^q (qu)^{1+\epsilon} \ln D(u) q^{q+r}$. Note that we still need to sum over $k, k', m_k, m'_k$, which can be simply bounded by $q^4$. Since $q^0 \ll C^q$,
for $1 \leq r \leq q - 1$, we have bound (2.78) by $C q u^{1+\epsilon} l D(u)^{q+r}$, where $C$ does not depend on $\beta$.

It still remains to bound the case $r = q$, where $s = \{s_0\}$. This is relatively simple. We use the expression in the first line of (2.78). Suppose that the $t$-index right beside $s_0$ is $t_j$. Without loss of generality, we may assume $s_0 < t_j$. Then we have

\begin{equation}
\sum_{t_j - s_0 = 1}^{u} \sum_{x_0 \in \tilde{I}_0} p_{t_j - s_0}(x_0, S_{t_j}) \leq u.
\end{equation}

For the other $t, t'$-indices, we just use the trivial bound

\begin{equation}
\sum_{t=1}^{u} p_t(0, S_t) \leq D(u)
\end{equation}

and then we obtain an upper bound $C q u^{1+\epsilon} l D(u)^{q+r}$ for the case $r = q$ in (2.78).

Finally, we substitute everything into (2.77) and by recalling $\lambda'(\beta) \sim \beta^2$, we have

\begin{equation}
\Var^S(X) \leq (1 + \epsilon^3/2)^{q+1} + \frac{C q u^{1+\epsilon} l}{2 R a_t} \sum_{r=1}^{q} (1 + 2\epsilon)^r
\end{equation}

\begin{equation}
\leq (1 + \epsilon^3/2)^{q+1} + \frac{q(2C)q^q l^{-\epsilon^3}}{2 R} + 1 \leq (1 + \epsilon^3)^q
\end{equation}

and we conclude Lemma 2.5. \qed

**Remark 2.1.** The condition (1.14) is used to prove Proposition 2.2. Without (1.14), the proposition may fail. We use the non-decreasing property of $\varphi(\cdot)$ in two places: (i) showing $D(qu)/D(u) \to 1$, and (ii) showing $a_t + a_s \leq a_{t+s}$ for any $t, s \in \mathbb{N}$. For (i), readers may check that if we simply apply the Potter’s bound here, then $D(qu) \leq q D(u)$ and we will obtain an extra term $q^q l$. Under some fairly mild condition, for example, $(\log x)^{-k} \leq L(x) \leq (\log x)^{k}$ with some $k > 0$, we have $q^q \leq l^k$, which is good enough. For (ii), it is a sufficient condition such that (2.88) holds. Unfortunately, it seems impossible to deduce (2.88) by the distribution of $S$ directly.

### 3. Proof of Theorem 1.4

In this proof, for any given $\beta$ and $\epsilon$, we will estimate the partition function at a special time $N$, defined by

\begin{equation}
N_{\beta, \epsilon} := \max_n \{D(n) \leq (1 - \epsilon)/\beta^2\}.
\end{equation}

By [9, Proposition 2.5], we have

\begin{equation}
p(\beta) = \sup_N \frac{1}{N} E[\log \hat{Z}_N^\omega] \geq \frac{1}{N_{\beta, \epsilon}} E[\log \hat{Z}_{N_{\beta, \epsilon}, \beta}^\omega].
\end{equation}

To simplify the notation, we will use $N$ as $N_{\beta, \epsilon}$ in the following without any ambiguity. We may emphasize several times that the choice of $N$ satisfies (3.1).

To show (1.19), we need to bound $E[\log \hat{Z}_{N_{\beta, \epsilon}}^\omega]$ appropriately. The key ingredient of the proof is the following result proved in [6]. Here we cite a version stated in [3].
Proposition 3.1 ([3, Proposition 4.3]). For any $m \in \mathbb{N}$ and any random vector $\eta = (\eta_1, \cdots, \eta_m)$ which satisfies the property that there exists a constant $K > 0$ such that
\[ \mathbb{P}(|\eta| \leq K) = 1. \]

Then for any convex function $f$, we can find a constant $C_1$ uniformly for $m, \eta$ and $f$, such that for any $a, M$ and any positive $t > 0$, the inequality
\[ \mathbb{P}(f(\eta) \geq a, |\nabla f(\eta)| \leq M) \mathbb{P}(f(\eta) \leq a - t) \leq 2 \exp\left(-\frac{t^2}{C_1 K^2 M^2}\right) \]
holds, where $\nabla f := \sqrt{m \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\right)^2}$ is the gradient of $f$.

We will apply Proposition 3.1 to $\hat{Z}_{\omega_{N,\beta}}$ and the environment $\omega$. However, this proposition is only valid for bounded and finite-dimension random vector. Since $\hat{Z}_{\omega_{N,\beta}}$ is a function of countable-dimension random field and $\omega$ may not be bounded, we need to restrict the range of the random walk $S$ so that $\hat{Z}_{\omega_{N,\beta}}$ is determined by finite many $\omega_{i,x}$’s and respectively, truncate $\omega$ so that it is finite.

First, we define a subset of $\mathbb{N} \times \mathbb{Z}$ by
\[ \mathcal{T} = \mathcal{T}_N := \{(n, x) : 1 \leq n \leq N, |x - b_N| \leq R a_N\}, \]
where $R$ is a constant that will be determined later and $a_N, b_N$ has been introduced in (1.2). We will choose $R$ large enough so that the trajectory of $S$ up to time $N$ entirely falls in $\mathcal{T}$ with probability close to 1 for any $N = N_{\beta, \epsilon}$. We can also assume that $a_N$ is an integer without loss of generality.

Then we define
\[ \bar{Z}_{\omega_{N,\beta}} := \mathbb{E}\left[\exp\left(\beta \sum_{n=1}^{N} \omega_{n,S_n} - N \lambda(\beta)\right) \mathbb{1}_{\{S \in \mathcal{T}\}}\right], \]
where $\mathbb{1}_{\{S \in \mathcal{T}\}} := \{S : (n, S_n) \in \mathcal{T}, \forall 1 \leq n \leq N\}$. Note that $\bar{Z}_{\omega_{N,\beta}} \leq \hat{Z}_{\omega_{N,\beta}}$. Readers may check that log $\hat{Z}_{\omega_{N,\beta}}$ is indeed a finite-dimension convex function and hence, we can apply Proposition 3.1 to log $\hat{Z}_{\omega_{N,\beta}}$. Since our goal is to find a lower bound for $\mathbb{E}[\log \hat{Z}_{\omega_{N,\beta}}]$, we can first estimate the left tail of log $\hat{Z}_{\omega_{N,\beta}}$, which can be done by bounding the first probability on the right-hand side of (3.4) from below.

We show the following result.

Lemma 3.2. For arbitrarily small $\epsilon > 0$, there exist $\beta_{\epsilon}$ and $M = M_{\epsilon}$, such that for any $\beta \in (0, \beta_{\epsilon})$, it follows that
\[ \mathbb{P}\left(\bar{Z}_{\omega_{N,\beta}} \geq \frac{1}{2}, |\nabla \log \bar{Z}_{\omega_{N,\beta}}| \leq M\right) \geq \frac{\epsilon}{100}. \]

To prove Lemma 3.2, we need a result from [2], which we state as

Lemma 3.3 ([2, Lemma 6.4]). For any $\epsilon > 0$, if $\beta$ is sufficiently small such that $N = N_{\beta, \epsilon}$ is large enough, then
\[ \mathbb{E}[(\bar{Z}_{\omega_{N,\beta}})^2] \leq \frac{10}{\epsilon}. \]
Proof of Lemma 3.2. By Lemma 3.3 and the fact \( \tilde{Z}_{N,\beta} \leq \check{Z}_{N,\beta} \),

\[
\mathbb{E}[(\tilde{Z}_{N,\beta})^2] \leq \frac{10}{\epsilon}.
\]

Then by Paley-Zygmund inequality, we have

\[
\mathbb{P}\left( \tilde{Z}_{N,\beta} \geq \frac{1}{2} \right) \geq \frac{\left( \mathbb{P}(S \in T) - \frac{1}{2} \right)^2}{\mathbb{E}[\tilde{Z}_{N,\beta}^2]} \geq \frac{\epsilon}{50},
\]

where the last inequality holds by choosing \( R \) large enough in \( T \).

By using notation

\[
f(\omega) := \log \tilde{Z}_{N,\beta},
\]

we have

\[
\mathbb{P}\left( \tilde{Z}_{N,\beta} \geq \frac{1}{2}, |\nabla f(\omega)| \leq M \right) = \mathbb{P}\left( \tilde{Z}_{N,\beta} \geq \frac{1}{2} \right) - \mathbb{P}\left( \tilde{Z}_{N,\beta} \geq \frac{1}{2} \right. |\nabla f(\omega)| > M \)
\]
\[
\geq \frac{\epsilon}{50} - \frac{1}{M^2} \mathbb{E} \left[ |\nabla f(\omega)|^2 \mathbb{1}_{\{\tilde{Z}_{N,\beta} > \frac{1}{2}\}} \right].
\]

To compute \( \nabla f(\omega) \), we find that

\[
\frac{\partial}{\partial \omega_{k,x}} \log \tilde{Z}_{N,\beta} = \frac{\beta}{\tilde{Z}_{N,\beta}} \mathbb{E} \left[ \exp \left( \beta \sum_{n=1}^{N} \omega_{n,S_n} - N\lambda(\beta) \right) \mathbb{1}_{\{S_k=x, S \in T\}} \right]
\]
\[
\leq \frac{\beta}{\tilde{Z}_{N,\beta}} \mathbb{E} \left[ \exp \left( \beta \sum_{n=1}^{N} \omega_{n,S_n} - N\lambda(\beta) \right) \mathbb{1}_{\{S_k=x\}} \right].
\]

Then

\[
|\nabla f(\omega)|^2 = \sum_{(k,x) \in T} \left| \frac{\partial}{\partial \omega_{k,x}} \log \tilde{Z}_{N,\beta} \right|^2
\]
\[
\leq \frac{\beta^2}{(\tilde{Z}_{N,\beta})^2} \sum_{k=1}^{N} \sum_{x \in T} \mathbb{E} \left[ \exp \left( \beta \sum_{n=1}^{N} \omega_{n,S_n} - N\lambda(\beta) \right) \mathbb{1}_{\{S_k=x\}} \right]^2.
\]

Note that

\[
\left( \mathbb{E} \left[ \exp \left( \beta \sum_{n=1}^{N} \omega_{n,S_n} - N\lambda(\beta) \right) \mathbb{1}_{\{S_k=x\}} \right] \right)^2
\]
\[
= \mathbb{E} \otimes \mathbb{E} \left[ \exp \left( \beta \sum_{n=1}^{N} (\omega_{n,S_n} + \omega_{n,\tilde{S}_n}) - 2N\lambda(\beta) \right) \mathbb{1}_{\{S_k=\tilde{S}_k=x\}} \right].
\]

Therefore,

\[
|\nabla f(\omega)|^2 \leq \frac{\beta^2}{(\tilde{Z}_{N,\beta})^2} \mathbb{E} \otimes \mathbb{E} \left[ \sum_{k=1}^{N} \mathbb{1}_{\{S_k=S_k\}} \exp \left( \beta \sum_{n=1}^{N} (\omega_{n,S_n} + \omega_{n,\tilde{S}_n}) - 2N\lambda(\beta) \right) \mathbb{1}_{\{S_k=\tilde{S}_k=x\}} \right].
\]

Then we have

\[
\mathbb{E} \left[ |\nabla f(\omega)|^2 \mathbb{1}_{\{\tilde{Z}_{N,\beta} > \frac{1}{2}\}} \right] \leq 4 \mathbb{E} \otimes \mathbb{E} \left[ \beta^2 \sum_{k=1}^{N} \mathbb{1}_{\{S_k=S_k\}} \exp \left( \gamma(\beta) \sum_{n=1}^{N} \mathbb{1}_{\{S_n=\tilde{S}_n\}} \right) \right].
\]
where
\[(3.18)\quad \gamma(\beta) := \lambda(2\beta) - 2\lambda(\beta).\]

We denote
\[(3.19)\quad Y := \sum_{n=1}^{N} \mathbb{1}_{\{S_n = \tilde{S}_n\}}\]
for short. It is not hard to check that
\[(3.20)\quad \lambda(2\beta) - 2\lambda(\beta) \sim \beta^2, \quad \text{as } \beta \to 0.\]

Hence, when \(\beta\) is sufficiently small, we have
\[(3.21)\quad E_\otimes \left[ \beta^2 N \sum_{k=1}^{N} \mathbb{1}_{\{S_k = \tilde{S}_k\}} \exp \left( \gamma(\beta) \sum_{n=1}^{N} \mathbb{1}_{\{S_n = \tilde{S}_n\}} \right) \right] \leq \frac{10C_\epsilon}{\epsilon}.
\]

We can choose \(M = M_\epsilon = 20\sqrt{10C_\epsilon/\epsilon^2}\) and then combine (3.12), (3.17), (3.21), (3.23), we then conclude Lemma 3.2. \(\square\)

Finally, we can now prove Theorem 1.4. Readers should keep in mind that \(N = N_{\beta,\epsilon}\).

**Proof of Theorem 1.4.** Because the environment \(\omega\) has a finite moment generating function, we can find some positive constants \(C_2\) and \(C_3\), such that
\[(3.24)\quad P(|\omega_{1,0}| \geq t) \leq C_2 \exp(-C_3 t).\]

Note that we will focus on the environment with index in \(T\). We can estimate that
\[(3.25)\quad P \left( \max_{(n,x) \in T} |\omega_{n,x}| \geq t \right) \leq C_4 N a_N \exp(-C_3 t).\]

Note that
\[(3.26)\quad \left\{ \max_{(n,x) \in T} |\omega_{n,x}| < t \right\} \subset \{ \omega_{n,x} > -t, \quad \forall (n,x) \in T \}\]
and recall the definition of \(\hat{Z}_{N,\beta}^\omega\) from (3.6), then we obtain a rough bound
\[(3.27)\quad P( \log \hat{Z}_{N,\beta}^\omega < -(\beta t + \lambda(\beta))N ) \leq C_4 N a_N \exp(-C_3 t).\]

We will use (3.27) later to bound the left tail of \(\log \hat{Z}_{N,\beta}^\omega\) for large \(t\).

In order to apply Proposition 3.1 we need to truncate the environment appropriately. We set \(\tilde{\omega}_{n,x} := \omega_{n,x} \mathbb{1}_{\{|\omega_{n,x}| \leq (\log N)^2\}}\) and define
\[(3.28)\quad f(\tilde{\omega}) := \log E_\otimes \left[ \exp \left( \beta \sum_{n=1}^{N} \tilde{\omega}_{n,S_n} - N\lambda(\beta) \right) \mathbb{1}_{\{S \in T\}} \right].\]
Then
\[(3.29)\]
\[
\mathbb{P}\left( \tilde{Z}_{N}^\omega, \beta \geq \frac{1}{2}, |\nabla \log \tilde{Z}_{N}^\omega, \beta| \leq M \right)
\]  
\[= \mathbb{P}\left( \tilde{Z}_{N}^\omega, \beta \geq \frac{1}{2}, |\nabla \log \tilde{Z}_{N}^\omega, \beta| \leq M, \tilde{\omega} = \omega \right) + \mathbb{P}\left( \tilde{Z}_{N}^\omega, \beta \geq \frac{1}{2}, |\nabla \log \tilde{Z}_{N}^\omega, \beta| \leq M, \tilde{\omega} \neq \omega \right)
\]  
\[\leq \mathbb{P}(f(\tilde{\omega}) \geq -\log 2, |\nabla f(\tilde{\omega})| \leq M) + \mathbb{P}(\tilde{\omega} \neq \omega)
\]

By Lemma 3.2 and (3.25),
\[(3.30)\]
\[
\mathbb{P}(f(\tilde{\omega}) \geq -\log 2, |\nabla f(\tilde{\omega})| \leq M)
\]  
\[\geq \mathbb{P}\left( \tilde{Z}_{N}^\omega, \beta \geq \frac{1}{2}, |\nabla \log \tilde{Z}_{N}^\omega, \beta| \leq M \right) - \mathbb{P}(\tilde{\omega} \neq \omega)
\]  
\[\geq \frac{\epsilon}{100} - C_4 N a N \exp(-C_3 (\log N)^2) \geq \frac{\epsilon}{200},
\]
where the last inequality holds for large \(N\), i.e., for small \(\beta\). Now we apply Proposition 3.1 to \(f(\tilde{\omega})\) and we obtain
\[(3.31)\]
\[
\mathbb{P}(f(\tilde{\omega}) \leq -\log 2 - t) \leq \frac{400}{\epsilon} \exp \left( -\frac{t^2}{C_1 (\log N)^4 M^2} \right).
\]

Finally,
\[(3.32)\]
\[
\mathbb{P}(\log \tilde{Z}_{N}^\omega, \beta \leq -\log 2 - t)
\]  
\[= \mathbb{P}(\log \tilde{Z}_{N}^\omega, \beta \leq -\log 2 - t, \tilde{\omega} = \omega) + \mathbb{P}(\log \tilde{Z}_{N}^\omega, \beta \leq -\log 2 - t, \tilde{\omega} \neq \omega)
\]  
\[\leq \mathbb{P}(f(\tilde{\omega}) \leq -\log 2 - t) + \mathbb{P}(\tilde{\omega} \neq \omega)
\]  
\[\leq \frac{400}{\epsilon} \exp \left( -\frac{t^2}{C_1 (\log N)^4 M^2} \right) + C_4 N a N \exp(-C_3 (\log N)^2).
\]

We can now bounded the left tail of \(\log \tilde{Z}_{N}^\omega, \beta\). Since it is larger than \(\log \tilde{Z}_{N}^\omega, \beta\), we can rewrite (3.27) and (3.32) as
\[(3.33)\]
\[
\mathbb{P}\left( \log \tilde{Z}_{N}^\omega, \beta < -((\beta t + \lambda(\beta)) N) \right) \leq C_4 N a N \exp(-C_3 t)
\]

and respectively,
\[(3.34)\]
\[
\mathbb{P}\left( \log \tilde{Z}_{N}^\omega, \beta \leq -\log 2 - t \right)
\]  
\[\leq \frac{400}{\epsilon} \exp \left( -\frac{t^2}{C_1 (\log N)^4 M^2} \right) + C_4 N a N \exp(-C_3 (\log N)^2).
\]

For \(\log \tilde{Z}_{N, \beta}^\omega\) with large negative value (for example, it is less than \(-N^2\)), we use the bound (3.33), which shows that the mass of \(\log \tilde{Z}_{N, \beta}^\omega\) on \((-N^2, -\infty)\) can be bounded below by some constant \(-C\). For \(\log \tilde{Z}_{N, \beta}^\omega\) with small negative value, we use the bound (3.34), which shows that the mass of \(\log \tilde{Z}_{N, \beta}^\omega\) is bounded below by \(-\tilde{C}_\epsilon (\log N)^2\) with some constant \(\tilde{C}_\epsilon\). Therefore, we obtain
\[(3.35)\]
\[
p(\beta) \geq \frac{1}{N} E \left[ \log \tilde{Z}_{N, \beta}^\omega \right] \geq \frac{C_5 \epsilon (\log N)^2}{N} \geq -\frac{1}{D^{-1} ((1 - \epsilon)/\beta^2)^{1-\epsilon}}
\]
for $\beta$ small enough, where the last inequality is due to the definition of $N = N_{\beta, \epsilon}$.

4. Appendix

The following proof is given in Berger’s not-yet-published paper [1].

Proof of Theorem 1.2 for symmetric $S$. Let

$$M_n = \max_{1 \leq i \leq n} \{S_i - S_{i-1}\}.$$  

By setting $t = 2$ in the proof of [15, Theorem 1.2] and the trick for symmetric $S$ in [15, Page 654-655], for $x \geq y > 0$, we have

$$P(S_n \geq x, M_n \leq y) \leq \left[\frac{1}{e} \left(1 + \frac{x}{cnL(y)}\right)\right]^{-\frac{y}{\beta}},$$  

since $E[|S_1|^2 \mathbb{1}_{\{|S_1| \leq y\}}] \sim yL(y)$ by [16, Theorem 3.2].

Then

$$P(S_n = x, M_n \leq y) \leq P(S_n = x, S_{[n/2]} \geq x/2, M_{[n/2]} \leq y)$$

$$+ P\left(S_n = x, S_n - S_{[n/2]} \geq x/2, \max_{[n/2] < i \leq n} \{S_i - S_{i-1}\} \leq y\right).$$

Since the two terms on the right-hand side of (4.3) are similar, we just bound the first term. The treatment for the second term is the same.

$$P(S_n = x, S_{[n/2]} \geq x/2, M_{[n/2]} \leq y)$$

$$= \sum_{z=x/2}^{\infty} P(S_{[n/2]} = z, M_{[n/2]} \leq y)P(S_n - S_{[n/2]} = x - z)$$

$$\leq \frac{C}{a_n} \sum_{z=x/2}^{\infty} P(S_{[n/2]} = z, M_{[n/2]} \leq y) = \frac{C}{a_n} P(S_{[n/2]} \geq x/2, M_{[n/2]} \leq y)$$

$$\leq \frac{C}{a_n} \left[\frac{1}{e} \left(1 + \frac{x}{cnL(y)}\right)\right]^{-\frac{x}{\beta}},$$

where the first inequality is due to Gnedenko’s local limit theorem and the last inequality is due to (1.2). Hence,

$$P(S_n = x, M_n \leq y) \leq \frac{C}{a_n} \left[\frac{1}{e} \left(1 + \frac{x}{cnL(y)}\right)\right]^{-\frac{x}{\beta}}.$$  

Finally,

$$P(S_n = x) \leq P(S_n = x, M_n \geq x/8) + P(S_n = x, M_n \leq x/8).$$

For the first term on the right-hand side of (4.6),

$$P(S_n = x, M_n \geq x/8)$$

$$= \sum_{y=x/8}^{\infty} P(S_n = x, M_n = y) \leq \sum_{y=x/8}^{\infty} nP(S_1 = y)P(S_{n-1} = x - y, M_{n-1} \leq y)$$

$$\leq \frac{CnL(x)}{x^2} \sum_{y=x/8}^{\infty} P(S_{n-1} = x - y, M_{n-1} \leq y) \leq \frac{CnL(x)}{x^2},$$
where in the last line, we use condition \( (1.14) \).

For the second term on the right-hand side of (4.4), when \( x \geq ca_n \), we have

\[
P(S_n = x, M_n \leq x/8) = P(S_n = x, M_n \leq a_n) + P(S_n = x, M_n \in (a_n, x/8]).
\]

For the first term on the right-hand side of (4.8), we apply (4.5) to obtain

\[
P(S_n = x, M_n \leq a_n) \leq \frac{C}{a_n} \exp(-x/2a_n) \ll \frac{nL(x)}{x^2},
\]

where in the first inequality, we use the property that \( nL(a_n) \sim a_n \) and the last inequality is due to the choice \( x \geq ca_n \).

For the second term,

\[
P(S_n = x, M_n \in (a_n, x/8]) = \sum_{j=4}^{[\log_2(x/a_n)]} P(S_n = x, M_n \in (2^{-j}, 2^{-(j-1)}]x)
\]

\[
\leq \sum_{j=4}^{[\log_2(x/a_n)]} \sum_{y \in (2^{-j}, 2^{-(j-1)}]x} nP(S_1 = y)P(S_{n-1} = x - y, M_{n-1} \leq y)
\]

\[
\leq \frac{Cn}{x^2} \sum_{j=4}^{[\log_2(x/a_n)]} \frac{L(2^{-j}x)}{(2^{-j}x)^2} P(S_{n-1} \geq x/2, M_{n-1} \leq 2^{-(j-1)}x)
\]

\[
\leq \frac{CnL(x)}{x^2} \sum_{j=4}^{[\log_2(x/a_n)]} 2^j P(S_{n-1} \geq x/2, M_{n-1} \leq 2^{-(j-1)}x),
\]

where in the last line, we use Potter’s bound [4, Theorem 1.5.6].

Note that by our choice of \( j \), \( 2^{-(j-1)}x \geq a_n \), and by (4.2), we have

\[
P(S_{n-1} \geq x/2, M_n \leq 2^{-(j-1)}x)
\]

\[
\leq \left[ \frac{1}{e} \left( 1 + \frac{x}{2cnL(2^{-j-1}x)} \right) \right]^{-2^{j-2}} = \left[ \frac{1}{e} \left( 1 + \frac{2^{j-2}}{cnP(S_1 > 2^{-(j-1)}x)} \right) \right]^{-2^{j-2}}
\]

\[
\leq \left[ \frac{1}{e} \left( 1 + \frac{2^{j-2}}{C} \right) \right]^{-2^{j-2}} \leq \left( \frac{2}{3} \right)^{2^{j-2}},
\]

where in the last line, we use the property that \( nP(S_1 > a_n) \leq C' \) and the last inequality holds provided \( j \) is large.

By (4.11),

\[
\sum_{j=4}^{[\log_2(x/a_n)]} 2^j P(S_{n-1} \geq x/2, M_{n-1} \leq 2^{-(j-1)}x) \leq \tilde{C}
\]

and the proof is completed. \( \square \)

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