Low-temperature anomaly in disordered superconductors near $B_{c2}$ as a vortex-glass property

Benjamin Sacépé1*, Johanna Seidemann1, Frédéric Gay1, Kevin Davenport2, Andrey Rogachev2, Maoz Ovadia3, Karen Michaeli3 and Mikhail V. Feigel’man4,5

Strongly disordered superconductors in a magnetic field exhibit many characteristic properties of type-II superconductivity—except at low temperatures, where an anomalous linear temperature dependence of the resistive critical field $B_{c2}$ is routinely observed. This behaviour violates the conventional theory of superconductivity, and its origin has posed a long-standing puzzle. Here we report systematic measurements of the critical magnetic field and current on amorphous indium oxide films with various levels of disorder. Surprisingly, our measurements show that the $B_{c2}$ anomaly is accompanied by mean-field-like scaling of the critical current. Based on a comprehensive theoretical study we argue that these observations are a consequence of the vortex-glass ground state and its thermal fluctuations. Our theory further predicts that the linear-temperature anomaly occurs more generally in both films and disordered bulk superconductors, with a slope that depends on the normal-state sheet resistance, which we confirm experimentally.

The magnetic-field-tuned transition of disordered superconductors continues to surprise as well as to pose intriguing and challenging puzzles. A wealth of experimental results obtained over decades of study still defies current theoretical understanding. The anomalous temperature dependence of the resistive critical field $B_{c2}(T)$ near the $T = 0$ quantum critical point (QCP) between superconductor and normal metal is a well-known example. Within the conventional Bardeen–Cooper–Schrieffer theory, $B_{c2}(T)$ is expected to saturate at low temperatures. In contrast, a strong upturn of $B_{c2}$ with a linear temperature dependence as $T \to 0$ has been observed in numerous disordered superconductors. These systems range from alloys and oxides, both in thin films and bulk, to boron-doped diamond as well as gallium monolayers.

Substantial theoretical efforts have been unable to fully resolve the origin of this anomalous behaviour. The main challenge lies in the complexity of these systems and the subtle interplay between strong fluctuations, disorder and vortex physics. The prevailing explanation for the low-$T$ anomaly of $B_{c2}(T)$ is based on mesoscopic fluctuations, which result in a spatially inhomogeneous superconducting order parameter. A recent alternative interpretation invokes a quantum Griffiths singularity to account for the upturn in $B_{c2}(T)$ observed in ultra-thin gallium films. Although these theoretical approaches are generally plausible, they predict an exponential increase of $B_{c2}(T)$ at very low $T$, which manifestly does not capture the specific linear dependence measured in disordered superconductors.

To gain new insights into the underlying physical mechanism it is desirable to not only study $B_{c2}(T)$, but to also extract information on additional characteristic quantities such as the superfluid stiffness. We therefore conducted systematic measurements of both $B_{c2}$ and the critical current $j_c$ in films of amorphous indium oxide (a:InO), a prototypical disordered superconductor. In the absence of magnetic fields or when vortices are strongly pinned by disorder (that is, form a vortex glass) the superfluid stiffness can be directly related to the critical current. Since this is expected to apply to all materials that exhibit the low-temperature $B_{c2}$ anomaly, measurements of $j_c(B,T \approx 0)$ provide access to the critical behaviour of the superfluid density $\rho_s(B)$ near the QCP $B_{c2}(0)$.

The key experimental finding of this work is that the linear temperature dependence of $B_{c2}(T)$ at low temperatures is accompanied by a power-law dependence of the critical current density on $B$—namely, $j_c(B) \sim (B - B_{c2})^\nu$, with $\nu \approx 1.6$. As explained below, this is consistent with the mean-field (MF) value $\nu = 3/2$ (but not with the mesoscopic fluctuation scenario, which predicts an exponential dependence). Our unexpected finding has direct implications for the critical behaviour of $\rho_s(B)$, and demands a revised theory of disordered superconductors in the presence of magnetic fields. We therefore complement our experimental work with a comprehensive theoretical study, which identifies the key to understanding the low-$T$ anomaly in the vortex glass. When vortices are strongly pinned by impurities, their presence only weakly affects the $T = 0$ limit of superfluid stiffness and critical current. As a result, both exhibit a MF-like dependence on magnetic field. In contrast, the temperature variation of the superfluid stiffness is strongly affected by thermal fluctuations of the vortex glass. This gives rise to the observed linear temperature dependence of $B_{c2}(T)$ near the QCP. Moreover, we predict a strong dependence of the slope $\partial B_{c2}(T)/\partial T|_{T=0}$ on the sheet resistance, which we confirm experimentally.

Low-temperature anomaly near $B_{c2}(0)$

In this study we focus on a series of a:InO samples, which exhibit critical temperatures $T_c = 2.4–3.5$ K and normal-state sheet resistances $R = 0.5–3.5$ kΩ (see Supplementary Table 1). Those samples are far from the disorder-tuned superconductor–insulator transition and behave in many ways as standard dirty superconductors. Moreover, to demonstrate the universality of the low-$T$ anomaly, we extended our measurements to amorphous molybdenum–germanium films (MoGe) (for sample characterization see Supplementary Information).

1Université Grenoble Alpes, CNRS, Grenoble INP, Institut Néel, Grenoble, France. 2Department of Physics and Astronomy, University of Utah, Salt Lake City, UT, USA. 3Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot, Israel. 4L. D. Landau Institute for Theoretical Physics, Chernogolovka, Russia. 5Skolkovo Institute of Science and Technology, Moscow, Russia. *e-mail: benjamin.sacepe@neel.cnrs.fr
Figure 1a shows the magneto-resistance isotherms of sample J033 measured down to 0.03 K. We define the critical magnetic field $B_{c2}$ through the resistive transition—that is, the onset of superconducting phase coherence. This critical field in general differs from the onset of pairing. To determine $B_{c2}$, we used three different criteria: 1, 10 and 50% of the high-field normal-state resistance, as indicated by open circles on magneto-resistance isotherms. The resulting $B_{c2}$ versus $T$ curves are shown in Fig. 1b together with a fit (solid line) of the high-temperature data according to the theory for dirty superconductors. We see that $B_{c2}(T)$ deviates below $\lesssim 1$ K from the fit and increases linearly with decreasing $T$ down to our base temperature of 0.03 K (see Supplementary Information). This deviation, which is the focus of this work, is independent of the criterion used to determine $B_{c2}$. Notice that the linear dependence of $B_{c2}(T)$ persists down to our lowest $T$ approaching the quantum phase transition. This is a crucial point in our analysis below.

The critical current density
We turn to the study of the evolution of the critical current density with $B$ at our lowest temperature and focus on three samples: J033, ITb1 and J038. We systematically measured the differential resistance $dV/dI$ versus current bias $I$ at fixed $B$ values. As shown in Fig. 2a, on increasing $I$, a sudden, non-hysteretic jump occurs in the $dV/dI$ curve, indicating the critical current. The resulting $j_c$ of the samples is plotted versus $B$ in Fig. 2b. Interestingly, the continuous suppression of $j_c$ with increasing $B$ tails off before vanishing at $B_{c1} = 12.8$ T, 12.1 T and 11.9 T for samples J033, ITb1 and J038, respectively. Such a resilience of $j_c$ to the applied field when approaching $B_{c2}(0)$ is reminiscent of the anomalous upturn of the $B_{c2}(T)$ line at low $T$. We also notice that the value of $B_{c1}$ at which $j_c$ vanishes differs slightly from $B_{c2}(0)$ obtained in Fig. 1 ($B_{c2}(0) = 13.3$ T, 12.4 T and 12.6 T determined with the 50% criterion for samples J033, ITb1 and J038, respectively). This stems from the finite-resistance criterion used to determine $B_{c2}(T)$.
Fig. 3 | Scaling of the critical current density with magnetic field, $j_c$ versus $|B|^2 - B$. The $B_{c2}^0$ values are adjusted to obtain straight lines that are emphasized by black solid lines. Inset: the dark grey and light grey curves are the data of sample J033 plotted with $B_{c2}^0 \pm \delta$, where $\delta = 0.05$ T.

The key result of this work is shown in Fig. 3, where $j_c$ is plotted on a logarithmic scale as a function of $|B|^2 - B$. By adjusting the value of $B_{c2}^0$ to 12.8, 12.1 and 11.9 T for samples J033, ITb1 and J038, respectively, one obtains clear straight lines that reveal a scaling relation of the form:

$$j_c(B) = j_c(0) \left(1 - \frac{B}{B_{c2}^0}\right)^\nu$$

where the fitted values for the exponent are $\nu_{J033} = 1.62 \pm 0.02$, $\nu_{ITb1} = 1.67 \pm 0.02$ and $\nu_{J038} = 1.65 \pm 0.02$. A similar critical exponent has been obtained for the MoGe sample (see Supplementary Information). The prefactor $j_c(0)$ falls in the range (2.5–4) x 10^3 A cm$^{-2}$ for all three samples. The inset of Fig. 3 shows the sensitivity of the straight line to $B_{c2}^0$, where a small variation of 0.05 T yields a noticeable deviation from linearity. It is noteworthy that the values of the exponent are within 10% of the MF value (3/2) of the classical temperature-dependent critical current. Mean-field theory predicts $j_{c\text{GL}} = \rho_0 / \xi_{\text{GL}}$, with the Ginzburg–Landau superconducting coherence length $\xi_{\text{GL}} \propto (T_c - T)^{-\nu_{\text{GL}}}$ and $\rho_0 \propto (T_c - T)$. This striking similarity suggests that both the scaling of $B_{c2}(T)$ and $j_c(B)$ at low temperature may be captured by MF theory of the bulk material.

Interpretation of experimental results within MF theory

The MF critical exponent of $j_c(B)$ at $T = 0$ near $B_{c2}(0)$ can be extracted from the Ginzburg–Landau free energy:

$$F = \alpha |\Delta(r)|^2 + \beta |\Delta(r)|^4 + \gamma \left[-i\nabla - \frac{2e}{\hbar c} A(r)\right] \Delta(r),$$

where $\Delta(r)$ is the superconducting order parameter, $A(r)$ is the vector potential, and $\alpha$, $\beta$ and $\gamma$ are the coefficients of the Ginzburg–Landau expansion. Within MF theory, the coefficient $\alpha$ depends strongly on temperature and magnetic field:

$$\alpha = \nu \left(\ln \frac{T}{T_c} + \psi \left(\frac{1}{2}\right) + \frac{eDB}{2\pi cT} - \psi \left(\frac{1}{2}\right)\right),$$

whereas $\beta$ and $\gamma$ depend only weakly on either. Here, $D$ and $\nu$ are the electron diffusion coefficient and density of states, respectively, and $\psi(x)$ is the digamma function. $B$ in equation (3) is the magnetic field penetrating the superconductor. Although for type-II superconductors such as our a:InO films this magnetic field can be non-uniform at low $B$, close to $B_c$, the spatial fluctuations of $B$ are negligible, and the average magnetic field is equal to the externally applied field. Consequently, equation (2) captures the two effects of magnetic fields on superconductors: the suppression of the transition point due to pair-breaking (through the parameter $\alpha$), and the diamagnetic response captured by the vector potential $A$. Note that the expression for $\nu$ in equation (3) is typical for superconductors in the presence of a pair-breaking mechanism$^{24}$. The MF treatment is performed under the assumption that vortices are strongly pinned. As a result, the presence of magnetic-field-induced vortices can be neglected, and $j_c$ is proportional to the depairing current. A detailed explanation of the origin and the consequences of strong pinning is given in the next section.

The $T = 0$ limit of equation (2) yields the magnetic-field dependence of the order parameter and the coherence length near the QCP, $|\Delta(B)| \sim |B - B_{c2}(0)|$ and $\xi_{\text{GL}}(B) \approx 1 - B/B_{c2}(0)^{1/2}$ (where $\xi_{c2}(0) \approx 5$ nm in our a:InO samples). The latter, together with $\rho_0$, determines the MF value of the critical de-pairing current: $j_{c\text{GL}} \propto \rho_0 / \xi_{\text{GL}}$. To find the superfluid stiffness, we match the superconducting current extracted from the free energy, $j = -c\partial F / \partial A$, with the London equation, $j = -4\rho_0 eA / h\xi_{\text{GL}}^2$. We find that

$$\rho_0(B) = \frac{12}{\pi^2} \rho_0 \left(1 - \frac{B}{B_{c2}(0)}\right)$$

and the relation between the superfluid stiffness and the critical current yields

$$j_{c\text{GL}}(B) = \frac{4\rho_0(B)}{3\pi h}\frac{1}{\xi_{\text{GL}}(B)} \left(\frac{B}{B_{c2}(0)}\right)^{3/2}$$

The corresponding critical exponent $\nu = 3/2$ is in excellent agreement with our experimental findings. The prefactor $j_{c\text{GL}}(0)$ can be estimated using available experimental data$^{21}$, where the superfluid stiffness of 20-nm-thick a:InO films was measured. From their experimental results at low magnetic field, we estimate the critical current to be $j_{c\text{GL}}(0) \sim 10^3 A$ cm$^{-2}$—larger than our experimentally observed values by only a factor of $\sim 4$. This is a non-trivial observation which is in contrast to a weakly pinned vortex state, where the critical current is set by the depinning current $j_{c\text{dep}} \ll j_{c\text{GL}}$. It also provides an important hint to the origin of the anomalous critical magnetic field and, as we show below, follows naturally from our theory.

To complete the comparison between the experimental results and MF theory, we extract the low-temperature critical field from the condition $\alpha(B, T) = 0$. Within MF theory, the resistive critical magnetic field coincides with the onset of pairing, and $B_{c2}(0) = B_{c2}(T) \sim T$. This is inconsistent with the experimentally observed linear dependence $B_{c2}(0) = B_{c2}(T) \sim T$ (see Fig. 2). Power-laws with exponents smaller than two are known to arise in strongly correlated superconductors$^{22-26}$. In the context of high-$T_c$ cuprates, the deviation from the MF result has been attributed$^{24}$ to an extended region of strong fluctuations around $B_{c2}(T)$. However, in conventional superconductors, such as the a:InO films studied here, the Ginzburg–Landau theory is expected to describe the onset of pairing at all temperatures and magnetic fields$^{27}$. Indeed, MF theory captures the scaling of the critical temperature with magnetic field at low $B$. This indicates that to understand the low-temperature behaviour of $B_{c2}(T)$ other effects beyond mean field should be
considered. In particular, thermal fluctuations of the vortex glass, which are essential in the finite-temperature transition to the normal state. As we show below, these fluctuations are the key ingredient to understanding the linear temperature dependence of $B_{c2}$; however, they do not change the scaling behaviour of $j_c(B, 0)$.

Vortex-glass fluctuations

In low-dimensional superconductors the pairing instability is known to differ from the onset of phase coherence. A prominent example is the Berezinskii–Kosterlitz–Thouless (BKT) transition in thin films$^{30,34}$. Similar decoupling occurs in moderately disordered type-II superconductors near $B_{c2}$ where the magnetic field gives rise to the formation of a weakly pinned vortex lattice$^{35,36}$. There, the superconducting state becomes resistive when the force applied on the vortex lattice by the current exceeds the pinning forces. The corresponding value of $j^\text{depin}$ is significantly lower than the pair-breaking critical current extracted from the Ginzburg–Landau theory, $j^\text{GL}$, and is not expected to obey simple scaling behaviour$^{30,31}$ close to $B_{c2}$.

In contrast, in highly disordered superconductors such as our aInO films, we expect a strongly pinned vortex glass to form$^{32,33}$. This is caused by large spatial fluctuations of the order parameter$^{30,34}$, which are predicted by the theory of ‘fractal’ superconductors$^{35}$. According to this theory$^{36}$, in the absence of a magnetic field, the superconducting condensation energy $E_s$ fluctuates strongly in space, $\delta E_s \sim E_s$, over distances comparable to the coherence length $\xi$. Correspondingly, core energies of vortices, which are induced by an applied magnetic field, exhibit similar fluctuations, and hence become strongly pinned. In fact, vortex pinning in such systems resembles that found in models of columnar defects$^{37}$. On applying a current, vortices de-pin only when the superconducting order parameter is sufficiently reduced—that is, within MF theory $j^\text{depin}$ scales like the Ginzburg–Landau de-pairing current. Thus, in our systems $j^\text{depin} = j^\text{GL}$, where $Y < 1$ (for example, in the model of ref. $^{36}$, $Y \approx 1/3$), and the $I$–$V$ curves are expected to follow those studied theoretically in ref. $^{37}$. Consequently, equation (3) still applies even in the presence of vortices.

The above analysis implies that the MF values of the critical field and current in highly disordered superconductors are modified primarily by fluctuations of the vortex glass$^{32}$. As we show in detail in the Methods, this results in renormalization of the superfluid stiffness, which at low temperatures takes the form

$$\delta \rho_s(T, B) = \rho_s(T, B) - \rho_s(0, B) = -C \frac{\sigma_0}{\epsilon^2} \frac{T^2}{3 \pi \rho_s(B)a_0}$$

(6)

Here, $\sigma_0$ is the normal-state conductance, $a_0$ is the inter-vortex spacing, and $C$ is a material-specific coefficient of order unity. The last equality holds in the low-temperature limit $T \ll \pi \rho_s(B)a_0$; in the opposite limit one recovers the classical result $\delta \rho_s(T, B) \propto T$.

Thermal fluctuation corrections to the critical current

To substantiate the correction to the superfluid stiffness given in equation (6), we conducted additional measurements of $j_c(T)$ in the vicinity of $B_{c1}(0)$. The differential resistance $dV/dI$ as a function of current measured on sample IT1b at various temperatures and fixed $B = 11.25$ T is shown in Fig. 4a. A clear jump in the resistance, similar to that in Fig. 1a, develops at ultra-low temperatures and indicates the position of $j_c$. At higher temperatures, $j_c$ decreases and a non-zero resistance is found already before the jump. This resistance rises above the noise level for $T > 0.05$ K, and exhibits a clear exponential increase with the current, which is highlighted by the black dashed line in this semi-log plot. Such resistance curves are expected to be observed when the vortex glass is strongly pinned: The resistance at low current is a typical signature of vortex creep, where the Lorentz force induced by the current reduces the barrier. Above $j_c$, the current–voltage characteristics show an excess current$^{30,31}$ (see Supplementary Information), which implies that thermal creep persists there, in agreement with recent strong-pinning theory$^{38,39}$. At temperatures above 0.07 K, strong thermal fluctuations cause the sharp jump to be replaced by a smooth crossover$^{40}$. Moreover, the resistance jump that is seen here only at very low $T$ points to a collective de-pinning of the vortex glass. Notice that Joule overheating is in play here, but mainly above the critical current (see detailed analysis in Supplementary Information).

As we showed before, the (zero-temperature) $B$-dependence of the critical current scales like $j^\text{GL}$ near $B_{c1}(0)$, indicating that $j^\text{depin} \propto j^\text{GL}$. Correspondingly, we expect the temperature dependence of the critical current to be determined by the thermal corrections to the superfluid stiffness

$$\delta j^\text{GL}_c(T, B) \propto \frac{\delta \rho_s(T, B)}{\xi^\text{GL}} \propto \frac{T^2}{B_{c2}(0)-B}$$

(7)

To test this predicted scaling of $j_c$, we measured additional resistance curves at different magnetic fields (see Supplementary Information). The critical current near $B_{c1}(0)$ extracted from the jump in the resistance is plotted as a function of temperature in Fig. 4b. The dashed lines are fits of the $j_c(T,B)$ data to equation (7), which were performed by setting $B_{c1}(0) = 12.1$ T (deduced from Fig. 3), by adjusting the $T = 0$ value $j_c(0, B)$ for each $B$ and finding one global prefactor. The fit reproduces remarkably well the temperature dependence of the data for $B = 10.5$, 1.75 and 11 T, confirming the $T^2$ correction to the critical current as well as its dependence on $B$. Deviations from the fit occur for the highest-temperature data points as well as in the immediate vicinity of $B_{c1}(0)$. This is not surprising, since our theoretical derivation of the correction to the superfluid stiffness given in equation (6) is valid so long as the fluctuations are small $\delta \rho_s(B, T) \ll \rho_s(B)$. The thermal fluctuations, however, become strong at lower $T$ as the magnetic field is increased.

The excellent agreement between theory and measurement has important implications: together with the observation of the MF scaling of the critical current shown in Fig. 3, it confirms that the vortex depinning current, which causes the jump in the
resistance, is indeed proportional to the de-pairing critical current $I_0^{cl}$. Moreover, this result validates our prediction for renormalization of the superfluid stiffness by thermal fluctuations of the vortex glass, and suggests that this should affect other observables such as magnetotransport near the QCP. In the following we show these fluctuations can account for the linear upturn of $B_c(T)$ at low temperature.

**Theory for the low-temperature anomaly**

The boundary between superconductor and normal state follows from $\rho_s(T, B) \approx 0$ in bulk systems, or from the BKT formula in films. As we show in the Methods, combining these conditions with the renormalized $\rho_s$ given by equation (6) yields the scaling of the transition temperature with magnetic field as a function of thickness

$$1 - \frac{B_c(T)}{B_c(0)} = 1 + \frac{C_2^2}{\epsilon a_0^2} \frac{d^2}{24\rho_0 d}$$

where $C_2^2 = 4\sqrt{\sigma_0} a_0 / 3\pi e^2$ and $\epsilon$ are numerical coefficients of order unity. $d$ is the film thickness and $\chi^{-1} \approx 2$. We thus obtain a linear temperature dependence of $B_c(T)$ at low $T$, which well describes the experimental data shown in Fig. 1b. In addition, this expression agrees with numerous experiments in films\(^5\)–\(^9\) and bulk\(^10\),\(^11\) disordered superconductors. Equations (6) and (8) are the main theoretical results of this work.

Finally, we provide a quantitative comparison between the theoretical prediction and experimental data from eight samples of various thicknesses and resistances. To eliminate the non-universal dependence of $T_c$ and $B_c(0)$ on disorder, in Fig. 5 we plot $b(t) = B_c(T) / (\xi T, d_B/B_c(T))$ versus the reduced temperature $t = T/T_c$ as well as the theoretical MF curve (solid line). We see that, for $t \geq 0.2$–0.3, all high-temperature data collapse on the theoretical curve. At lower $t$, the low-temperature anomaly of $B_c$ develops as a linear deviation from the MF curve. Our theory explains the anomalous slope in this regime and, as we show below, captures the dependence on the sample parameters.

To compare the plot in Fig. 5 with our theory, we extract $b(t)$ from equation (8) and find:

$$-\frac{db}{dt} = \left( \frac{K R_C}{R_Q} \right) / \frac{1}{\sigma R_Q a_0} + K \frac{R_C}{R_Q}$$

where $K = h/4e^2$ is the quantum of resistance for electron pairs (for details see the Methods). Neglecting higher-order corrections to $\rho_s$, beyond equation (8), and setting $\rho_s/\rho_0 = 1.75$, $T_c = 0.4T_s$, we find $K \approx 0.1$ and $\xi T_c \approx 0.05C$. The slopes $db/dt|_{t=0}$ of four samples: 30-nm-, 50-nm- and two 60-nm-thick films are shown in the inset of Fig. 5. We clearly see that the slope of the linear-temperature anomaly increases with sheet resistance as predicted by equation (9), demonstrating the consistency of our theoretical description. Furthermore, the linear dependence of $db/dt|_{t=0}$ on the sheet resistance does not extrapolate to zero in the bulk limit ($d \rightarrow \infty$), confirming again our theoretical result in equation (9). However, fitting the data to equation (9) gives $-db/dt = 0.4\rho_0 R_C / R_Q + 0.44$ (that is, $\xi T_c \approx 0.45$ and $\xi T_c \approx 0.4$). These values exceed the ones found via our simplified approximation by a factor of about four to ten. In part, this might be due to the overestimation of the ratio $\rho_s/\rho_0$ (see Methods). Obtaining the correct numerical coefficients presumably requires including corrections neglected above, which is beyond the scope of this work. A fully quantitative analysis should be based on extension of the theory of classical gauge glasses\(^13\)–\(^16\) to the quantum limit.

In conclusion, we have conducted a systematic study of the critical current near $B_c(T=0)$ in disordered a-InO and MoGe films, and uncovered a power-law dependence of $j_c$ on magnetic field. We have shown theoretically that the behaviour of both $B_c(T)$ and $j_c(B)$ can be attributed to properties of the vortex glass, which is the characteristic state of disordered films in the presence of a magnetic field. Although this mechanism is generally applicable for any disordered superconductor, both two- and three-dimensional, the magnitude of the effect is appreciable for superconductors with low superfluid density, $\rho_s \ll T_c$, where phase fluctuations are strong\(^14\). Our analysis provides sharp predictions for $B_c(T)$ and $j_c(B)$, which allow a clear distinction between the physical mechanisms in play. We would be very interesting to look for similar scaling in other superconducting films where a similar $B_c$ anomaly has been observed\(^17\)–\(^20\). Moreover, the theory developed here relates the anomaly to the absence of any vortex lattice order. Decay of vortex correlations over a very short length may have already been experimentally observed in amorphous tungsten-based thin films\(^21\); however, the connection to the quantum critical behaviour needs to be further investigated.

**Online content**

Any methods, additional references, Nature Research reporting summaries, source data, statements of data availability and associated accession codes are available at https://doi.org/10.1038/s41567-018-0294-6.

Received: 7 November 2017; Accepted: 28 August 2018; Published online: 8 October 2018

**References**

1. Abrikosov, A. A. & Gor’kov, L. P. Contribution to the theory of superconducting alloys with paramagnetic impurities. Zh. Eksp. Teor. Fiz. 39, 1781 (1960). (Sov. Phys. JETF 12, 1243 (1961)).

2. Maki, K. Critical fluctuation of the order parameter in a superconductor. I. Prog. Theor. Phys. 40, 193–200 (1968).

3. Tenhover, M., Johnson, W. L. & Tsuei, C. C. Upper critical fields of amorphous transition metal based alloys. Solid State Commun. 38, 53–57 (1981).

4. Okuma, S., Komori, F., Ootuka, Y. & Kobayashi, S.-I. Superconducting properties of disordered films of Zn. Phys. Soc. Jpn. 52, 2639–2641 (1983).

5. Hember, A. F. & Paalanen, M. A. Pair-breaking model for disorder in two-dimensional superconductors. Phys. Rev. B 30, 4063–4066 (1984).
6. Graybeal, J. M. & Beasley, M. R. Localization and interaction effects in ultrathin amorphous superconducting films. *Phys. Rev. B* **29**, 4167–4169 (1984).

7. Furubayashi, T., Nishida, N., Yamaguchi, M., Morigaki, K. & Ishimoto, H. Superconducting properties of amorphous Si$_x$As$_{1-x}$ near metal-insulator transition. *Solid State Commun.* **55**, 513–516 (1985).

8. Nordström, A., Dahlborg, U. & Rapp, O. Variation of disorder in superconducting glassy metals. *Phys. Rev. B* **48**, 12866–12873 (1993).

9. Sacépé, B. et al. High-field termination of a Cooper-pair insulator. *Phys. Rev. B* **91**, 220508(R) (2015).

10. Ren, Z. et al. Anomalous metallic state above the upper critical field of the conventional three-dimensional superconductor Ag$_2$Sn, with strong intrinsic disorder. *Phys. Rev. B* **87**, 064512 (2013).

11. Bustarret, E. et al. Dependence of the superconducting transition temperature on the doping level in single-crystalline diamond films. *Phys. Rev. Lett.* **93**, 237005 (2004).

12. Xing, Y. et al. Quantum Griffiths singularity of superconductor–metal transition in Ga thin films. *Science* **350**, 542–545 (2014).

13. Spivak, B. & Zhou, F. Mesoscopic effects in disordered superconductors near $H_c$. *Phys. Rev. Lett.* **74**, 2800–2803 (1995).

14. Galitski, V. M. & Larkin, A. I. Disorder and quantum fluctuations in superconducting films in strong magnetic fields. *Phys. Rev. Lett.* **87**, 087001 (2001).

15. Coffey, L., Levin, K. & Mutual, K. A. Upper critical field of strongly disordered three-dimensional superconductors: Localization effects. *Phys. Rev. B* **32**, 4382–4391 (1985).

16. Sadovskii, M. V. Superconductivity and localization. *Phys. Rep.* **328**, 225–348 (1997).

17. Smith, R. A., Handy, B. S. & Ambegaokar, V. Upper critical field in superconductors. *Phys. Rev. B* **35**, 311–318 (1987).

18. Vinokur, V. M., Ioffe, L. B., Larkin, A. I. & Feigel’man, M. V. System of Josephson junctions as a model of a spin glass. *Sov. Phys. JETP* **66**, 198–210 (1987).

19. Feigel’man, M. V. & Ioffe, L. B. Theory of diamagnetism in granular superconductors. *Phys. Rev. Lett.* **74**, 3447–3450 (1995).

20. Emery, V. J. & Kivelson, S. A. Importance of phase fluctuations in superconductors with small superfluid density. *Nature* **374**, 434–437 (1995).

21. Guilmém, I. et al. Enhancement of long-range correlations in a 2D vortex lattice by an incommensurate 1D disorder potential. *Nature* **10**, 851–856 (2014).

Acknowledgements
We are grateful to V. Geshkenbein, L. Ioffe, T. Klein and M. Skvortsov for useful discussions. We thank I. Tamir and D. Shahar for providing sample 1T8B1. B.S. and J.S. acknowledge support from the LANEF framework (ANR-10-LABX-51-01) and the H2020 ERC grant QUEST no. 637815 K.D. and A.B. acknowledge support from NSF grant no. DMR 1611423. The research of K.M. was supported by the Israel Science Foundation grant no. 1889/16. The research of M.V.F. was partially supported by a Skoltech NGP grant.

Author contributions
J.S., K.D. and B.S. fabricated the samples. F.G. provided technical support for low-temperature set-ups and measurements. B.S., J.S. and M.O. performed the measurements. B.S. and J.S. carried out data analysis. K.M. and M.F. developed the theory. B.S., K.M. and M.F. wrote the manuscript. All authors discussed the results and commented on the manuscript.

Competing interests
The authors declare no competing interests.

Additional information
Supplementary information is available for this paper at https://doi.org/10.1038/nphys4548.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher’s note: Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
Methods
Sample fabrication and measurement set-up. Disordered a:InO films were prepared by electron-gun evaporation of 99.99% In2O3 pellets on a Si/SiO2 substrate in a high-vacuum chamber with a controlled O2 partial pressure. Films were patterned into a Hall bar geometry (100 μm wide for all samples except T1B1, which is 20 μm wide) by optical lithography, enabling four-terminal transport measurements using standard low-frequency lock-in amplifier and d.c. techniques. Measurements were performed in dilution refrigerators equipped with superconducting solenoids. Multi-stage filters, including feed-through z-filters at room temperature, highly dissipative shielded-stainless-steel twisted pairs down to the mixing chamber, copper-powder filters at the mixing chamber stage, and 47 nF capacitors to ground on the sample holder, were installed on each d.c. line of the fridge. This careful filtering of low- and high-frequency noise was crucial to measure well-defined critical currents as low as ~30 nA (see Fig. 4), which would have been otherwise disguised by spurious noise. Furthermore, a calibrated RuO2 thermometer was installed directly on the sample holder to precisely monitor the sample temperature. This accurate thermometer eliminates small temperature gradients below 0.1 K.

Derivation of the renormalized superfluid stiffness. To estimate the renormalization of the superfluid stiffness, ρs, we focus exclusively on phase fluctuations of the order parameter—that is, Δ(r)→|Δ(r)|e^{iφ(r)}). Inserting this into the free energy (equation (2)) yields F(θ)→ρs∇θ(r)·∇A(r)² /2. It is convenient to further separate θ(r) into smooth phase fluctuations (the superfluid mode), ω(r), with V×ω=0, and fluctuations of the vortex glass, φ(r), with φ(r)=θ(r)+ω(r). We determine the renormalized superfluid stiffness via the static current–current correlation function ψ(0,0)=∫ dΦ dω(0) A(0)(0) A(0)(0), where as before ξ(r)=−i∂t A(r). Since ρs is determined by the long-wavelength properties, it is sufficient to focus on length scales larger than the (typical) inter-vortex spacing a0. Moreover, within such a coarse-grained description, the coupling between superfluid and vortex fluctuations is local.

We note that a charge encircling a vortex acquires phases from both the external and the vortex field, ∫ dθ ψ(r)/2π = h/2e. In a vortex glass this is not cancelled. Still, near Bc2 and at the length scales of η=√(π/2κ)≡ζ/2κ, the density of vortices is nearly uniform. We introduce the field VR(r)=R(r), which describes the deviation of the vortex positions from uniformity. R thus encodes the particular realization of the vortex glass, and, in the absence of forces, its spatial average vanishes, V→∫ dR ρs(R)=0. It is thus appropriate to expand VR(r)=2πρs/a0, which is the dimensionless displacement field. It follows that the leading coupling terms in the free energy between φ(r) and ω(r) are

\[ \delta F = \rho_s \omega \cdot [\nabla \psi(0) \times \nabla \psi(0)] - C_{15} \nabla^2 \psi(0) \psi(0) \]

where C is a material-specific coefficient of order unity. Note that here we assume isotropy in the xy plane.

The first term in (11) corresponds to the lowest-order expansion with respect to gradients in the Ginzburg–Landau free energy (equation (2)). It gives rise to a temperature- and magnetic-field-independent correction to the superfluid stiffness that depends on the realization of the vortex glass. This reduction enters the measurable quantity ρs, which we treat as a phenomenological parameter. As we show below, higher-order gradients (like the second term in equation (11)) become important at non-zero temperature. The leading contribution of such terms to ρs is given by

\[ \delta \rho_s = C_{15} \int \frac{d^2 r}{a_0^2} \int d^2 u \psi^2(0) \psi(0,0) \]

(12)

It remains to evaluate the ω–ω correlation function. In the strong-pinning regime, a restoring force acts to keep the vortex structure near its low temperature minimum. Consequently, we can further reduce the equation of motion to its local form for ω(r, t), which describes individual vortices, and in addition neglect spatial gradients of the ω(r) field

\[ \rho_s \omega_t(r) + \kappa \nabla \cdot \left( \nabla \omega(r) \right) = f(r) \equiv \frac{h}{2e} \times \frac{2\pi}{a_0} \]

(13)

where ω0(r) is the static displacement at zero current. The right-hand side of equation (13) is the Lorentz force acting on a segment of a vortex of length a0 in the presence of a supercurrent J. We emphasize the absence of gradients of ω in the equation of motion—this manifests the locality of the vortex dynamics. Effectively, the vortex fluctuations in the glass state at low temperature resemble a (damped) optical phonon mode. The parameter κ=⟨(h/2e)ν⟩/2κ is a dimensionless factor that reflects the presence of normal electrons in the vortex cores, whose resistance σc increases from the vortex motion (note that we defined friction coefficient ν with respect to dynamics of the dimensionless coordinate u=2πR/(a0)). For strongly disordered materials σc→e²/h and thus κ→h/a0. This overdamped character of the vortex motion leads us to neglect the inertial term ρsω_t with respect to friction. Likewise, we neglect the contribution to the Lorentz force due to vortex velocity since it does not affect the relevant current–current correlation function. The parameter κ can be determined similar to the penetration length in pinned vortex systems, also known as the Campbell length (66) (for a recent review see ref. 67) according to equation (13), the shift of a due to current j is equal to u(r)−u_j(r)=h/a0(2π×j/2κ).

The corresponding shift of the vortex magnetic flux can be expressed through δΦ = −h/2e f(r). Using the London relation, we get κ=πF. From equation (13) we obtain the Matsubara Green’s function ψ(ω,0)=(1+2κω)−1.

We thus arrive at the reduction of the superfluid stiffness due to thermal fluctuations of the vortex glass. Combining equations (12) and (11) yields the following correction to the superfluid stiffness δρs(T,B)=ρs(T,B)−ρs(0,B):

\[ \delta \rho_s(T,B) = -i \int \frac{d\omega}{2\pi} \frac{1}{1+2\kappa \omega} \]

(15)

The last equality holds in the low-temperature limit T ≪ h/2eκ/Ω. In the regime of superconductors, in which the fluctuations at T>0 are controlled by the dissipation in the gapless vortex cores, and it provides the key to understanding the low-temperature linear upturn of δρs.

Derivation of the critical field at low temperature. The superconducting transition in the bulk limit can be estimated from the condition ρs(T,B)|Bc|=0, or equivalently when δρs(T,B)=ρs(0,B), with B being a number of the order unity (similar to the Lindemann criterion for melting of solids). Under this condition and using equations (4) and (6) from the main text, we find

\[ B_{c}(T) = B_{c}(0) \frac{\sqrt{\frac{\kappa T}{2\pi}}}{\kappa T} \]

(16)

where C_F=4Ch/a0/3πe² is of order unity for our a:InO films. In films, the transition temperature is set instead by the BKT condition:

\[ \rho_s(0, B_{KT}) = \frac{\kappa T}{d} \]

(17)

d is the film thickness and g⁻¹ was numerically found (19) to be between 1.5 to 2.2 (this holds for any value of δ; refs. 10–13). Moreover, kinetic inductance measurements of thin a:InO films have observed a universal jump in the two-dimensional superfluid density per square even in the presence of a magnetic field, indicating that the BKT transition persists near the quantum critical point (1). We note that equation (17) in the limit d→0 reproduces the condition for the bulk transition [ρs(0, B)=0], and thus this equation describes the scaling of Bc(T) for any d. Thus, combining the BKT condition with the renormalized ρs given by equation (6) yields equation (8).

We emphasize that the linear temperature dependence of Bc(T) does not depend on the sample dimensionality and, in particular, remains valid for bulk superconductors. For films, |h0(B,T)| is determined by the film thickness d; however, it remains much below unity so long as d<<a0 (as is the case for previously reported experiments) (14). In this thin limit our theory predicts that the slope −dBc(T)/dT grows linearly with (δ/δ)−1. For thicker films, however, |h0(B,T)| increases and higher-order corrections to ρs(B) becomes important. Still, dBc(T)/dT decreases maintains the structure (ρs(a0))−1×gF(0)/a0/δ, with numerical coefficients gF, that (may) deviate from those given in equation (8). To conclude, although our analysis provides an exact expression for the slope only in the thin-film limit, it predicts a distinct d-dependence that holds for any thickness.

Derivation of the normalized slope dBc(T)/dT. Here we present the transformation between equations (8) and (9). Inserting equation (8) into b(t)=Bc(T)/−dTdBc(T)/dT yields
\[
\frac{b(0)-b(t)}{t} = \left(1 + \frac{C_{\Delta}^2 t^2}{\sigma_s^2} \right) \left(\frac{\pi \nu_{\Delta}}{24 \sigma_s d} \left(\frac{R_{c2}(0)}{dT} / dT\right)\right) \gamma \nu_{\Delta}
\]

(18)

It remains only to determine \(R_{c2}\) in two limits, \(T \rightarrow 0\) and \(T \rightarrow T_c\), starting with the latter. As discussed in the main text, the transition in the films is of the BKT type with the condition \(\rho_c(T, B) = x T / d\). Near \(T_c\), the superfluid stiffness of a disordered superconductor is given by\(^54\):

\[
\rho_c(T, B) = \frac{\sigma_s h}{8 \pi e^2} |\Delta(T, B)|^2
\]

(19)

The gap \(\Delta(T, B)\) near \(T_c\) can be found by minimizing the free energy in equation (2), without the gradient term: \(|d = \mu^2(1/2)(32 \pi^2 T) |\psi(s) = d^s s(x) / dx^2\) is the second derivative of the digamma function. Thus, in this limit the superfluid stiffness as a function of temperature and magnetic field is

\[
\rho_c(T, B) = \frac{2 \pi^2 T_\text{coh} h}{e \psi^{(1/2)}} \ln \left(\frac{T}{T_{\text{coh}}}\right) + \frac{1}{2} \epsilon B D
\]

(20)

Inserting this expression into the BKT condition and taking a derivative with respect to the temperature yields

\[
\frac{dR_{c2}(T)}{dT} \bigg|_{T_{\text{coh}}} = \frac{2 \pi n h c}{e \psi^{(1/2)}(1/2)}
\]

(21)

The critical magnetic field at zero temperature, in contrast, is determined by the MF condition \(\alpha = 0\)

\[
R_{c2}(T = 0) = \frac{2 \pi n h c}{e \psi^{(1/2)(1/2)}}
\]

(22)

Using the expressions in equations (21) and (22), we can write equation (18) as

\[
\frac{b(0) - b(t)}{t} = \left(1 + \frac{C_{\Delta}^2 t^2}{\sigma_s^2} \right) \left(\frac{\pi \nu_{\Delta}}{24 \sigma_s d} \left(\frac{R_{c2}(0)}{dT} / dT\right)\right) \gamma \nu_{\Delta}
\]

(23)

The final step in estimating \(b(t)\) is to find the ratio of \(\rho_c\) to the MF transition temperature \(T_{\text{coh}}\). For moderately disordered superconductors, in the absence of any pair-breaking mechanism, semiclassical theory yields\(^54\):

\[
\rho_{\text{coh}} = \frac{\pi h n c}{4 e^2} = 1.76 n h \sigma_{\text{coh}} / 4 e^2
\]

(24)

We emphasize that the ratio \(\rho_{\text{coh}} / T_c\) is expected to be reduced for strongly disordered superconductors\(^53,54\) with respect to the semiclassical formula in equation (24).

For ultra-thin films with \(d \ll a_0\) we find, using equations (23) and (24),

\[
\frac{b(0) - b(t)}{t} = 0.8 \frac{e^2}{h \sigma d} = 0.4 \pi \sigma_{\text{coh}} / R_{c2} \equiv \frac{K R_{c2}}{R_{c2}}
\]

(25)

Here \(R_{c2} = k/4e^2\) is the quantum of resistance for electron pairs and we used \(x = 2\pi\), as appropriate for the two-dimensional limit. Since this result relies on equation (24), which overestimates the ratio \(\rho_d / T_c\), we expect to experimentally observe larger values for \(K\). For thick films with \(d \gg a_0\), equations (23) and (24) imply that \(\left[d b / d t\right]\) grows linearly with \(R_{c2} / R_{c2}\), that is,

\[
\frac{b(0) - b(t)}{t} = \frac{\pi h n c}{e \psi^{(1/2)}} \left(\frac{R_{c2}}{R_{c2}}\right)
\]

(26)

where \(\frac{\psi^{(1/2)}}{2} \approx 0.05 \pi T_{\text{coh}}\), and \(R_{c2} / R_{c2} \approx 0.1\). Note that \(\left[d b / d t\right]\) does not extrapolate to zero as \(R_{c2} = 1 / \sigma_{\text{coh}} d \rightarrow 0\). Similar to the result in the thin-film limit, the true numerical values of both \(\frac{\psi^{(1/2)}}{2}\) and \(R_{c2} / R_{c2}\) are expected to be larger due to the lower value of \(\rho_{\text{coh}} / T_{\text{coh}}\). Moreover, as explained in the previous section, the coefficients \(\frac{\psi^{(1/2)}}{2}\) and \(R_{c2} / R_{c2}\) can deviate from the values given by straightforward expansion of equation (23) for large \(d / a_0\). Such a deviation would reflect higher-order corrections to \(\rho_c(T, B)\), which may become important in thick films.

**Data availability**

The data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

**References**

46. Campbell, A. M. The response of pinned flux vortices to low-frequency fields. *J. Phys. C* 2, 1492–1501 (1969).

47. Campbell, A. M. The interaction distance between flux lines and pinning centres. *J. Phys. C* 4, 3186–3198 (1971).

48. Coffey, M. W. & Clem, J. R. Unified theory of effects of vortex pinning and flux creep upon the rf surface impedance of type-II superconductors. *Phys. Rev. Lett.* 67, 386–389 (1991).

49. Willa, R., Geshkenbein, V. B. & Blatter, G. Probing the pinning landscape in type-II superconductors via Campbell penetration depth. *Phys. Rev. B* 93, 194515 (2016).

50. Schneider, T. & Schmidt, A. Dimensional crossover scaling in the layered xy-model and He films. *J. Phys. Soc. Jpn.* 61, 2169–2172 (1992).

51. Ambegaokar, V., Halperin, B. I., Nelson, D. R. & Siggia, E. D. Dynamics of small-aperture arrays of flux lines in superfluid films. *Phys. Rev. B* 21, 1806–1826 (1980).

52. Williams, G. A. Dimensionality crossover of the He superfluid transition in a slab geometry. *J. Low Temp. Phys.* 101, 415–420 (1995).

53. Schultka, N. & Manousakis, E. Crossover from two- to three-dimensional behavior in superfluids. *Phys. Rev. B* 51, 11712–11720 (1995).

54. Tinkham, M. *Introduction to Superconductivity* (Dover, Mineola, 1996).

55. Feigel’man, M. V. & Ioffe, L. B. Superfluid density of a pseudogapped superconductor near the superconductor–insulator transition. *Phys. Rev. B* 92, 100509(R) (2015).