Loops and legs master integrals for splitting functions from differential equations in QCD

Oleksandr Gituliar
oleksandr.gituliar@ifj.edu.pl

Institute of Nuclear Physics
Polish Academy of Sciences
Cracow, Poland

Loops and Legs in Quantum Field Theory
Leipzig (Germany)
26 April 2016
Splitting functions in perturbative QCD

- process independent quantities
- govern collinear evolution of hard scattering processes with hadrons
  - \textit{parton distribution functions} in the \textit{initial state}
    (a \textit{space-like} hard scale \(Q^2 = -q^2 > 0\))
    \[
    \frac{d}{d \ln q^2} f_a^h(x, q^2) = \int_x^1 \frac{dz}{z} P_{ab}^S (z, \alpha_s(q^2)) f_b^h \left( \frac{x}{z}, q^2 \right)
    \]
  - \textit{fragmentation functions} in the \textit{final state}
    (a \textit{time-like} hard scale \(Q^2 = q^2 > 0\))
    \[
    \frac{d}{d \ln q^2} D_a^h(x, q^2) = \int_x^1 \frac{dz}{z} P_{ba}^T (z, \alpha_s(q^2)) D_b^h \left( \frac{x}{z}, q^2 \right)
    \]
- expanded in perturbation theory, e.g.,
  \[
  P_{ab}^T (x, \alpha_s) = \frac{\alpha_s}{4\pi} P_{ab}^{T(0)}(x) + \left( \frac{\alpha_s}{4\pi} \right)^2 P_{ab}^{T(1)}(x) + \left( \frac{\alpha_s}{4\pi} \right)^3 P_{ab}^{T(2)}(x) + \ldots
  \]

Analytical expression for \(P_{qg}^{T(2)}\) is known with a \textit{small uncertainty}. 
| Space-like | Time-like |
|-----------|-----------|
| • Axial gauge | • Axial gauge |
| – Principal Value | – Principal Value |
| Curci Furmanski Petronzio '80 | Furmanski Petronzio '80 |
| Ellis Vogelsang '98 | |
| – Mandelstam-Leibbrandt | • Feynman gauge |
| Bassetto Heinrich Kunszt Vogelsang '98 | – x space |
| – New Principal Value | Floratos Kounnas Lacaze '81 |
| OG Jadach Skrzypek Kusina '14 | Rijken, van Neerven '96 |
| • Feynman gauge | – Mellin space Mitov Moch '06 |
| – x space | • analytic cont. $-q^2 \rightarrow +q^2$ |
| Floratos Kounnas Lacaze '81 | Stratmann Vogelsang '96 |
| – Mellin space Moch Vermaseren '99 | Blumlein, Ravindran, van Neerven '00 |
| | Moch Vogt '07 |
| | – Drell-Yan-Levy relation |
| | Drell Levy Yan '70 |
| | – Gribov-Lipatov relation |
| | Gribov Lipatov '72 |
Splitting functions at NNLO

**Space-like**
- Axial gauge*
- Feynman gauge
  - Mellin space
  
  Moch Vermaseren Vogt '04

**Time-like**
- Axial gauge*
- analytic continuation
  \[ P_{ab}^{S(2)} \rightarrow P_{ba}^{T(2)} \]
  - NNLO non-singlet
    
    Mitov Moch Vogt '06
  - NNLO singlet \( P_{qq}^{T(2)} \) and \( P_{gg}^{T(2)} \)
    
    Moch Vogt '07
  - NNLO singlet \( P_{gq}^{T(2)} \) and \( P_{qg}^{T(2)} \)
    
    Almasy Moch Vogt '11

* Spurious poles of axial gauge are too complex at NNLO.

— How can we find a missing contribution to \( P_{qg}^{T(2)} \) splitting function?
— Well, let us try to calculate it explicitly!
In this talk

• I will briefly discuss how to extract $P_{qg}^{T(NNLO)}$ from $e^+e^- \rightarrow 3$ jets at NNLO.
• We will see that to solve this (and many other problems in QFT) we need tools for automatic calculation of master integrals.
• I will introduce Fuchsia — program for reducing differential equations for master integrals to canonical form based on the Lee algorithm.
• We will consider $P_{gq}^{T(NLO)}$ as a demonstration example.
• At the very end I will discuss current status of the Fuchsia project.
\[ \frac{d^2 \sigma}{dx \, d \cos \theta} = \frac{3}{8} (1 + \cos^2 \theta) F_T(x, \epsilon) + \frac{3}{4} \sin^2 \theta F_L(x, \epsilon) + \frac{3}{4} \cos \theta F_A(x, \epsilon) \]

- Transverse fragmentation functions

\[ F_T(x, \epsilon) \simeq (x^2 g^{\mu \nu} + 4 k_0^\mu k_0^\nu) W_{\mu \nu}(x, \epsilon), \quad x = 2q \cdot k_0 \]

- Hadronic tensor

\[ W_{\mu \nu}(x, \epsilon) \simeq \int d^m \text{PS}^{(n)} M_{\mu}^{(n)} M_{\nu}^{(n)*} \]

where \( d^m \text{PS}^{(n)} \) is \( n \)-particle phase-space in \( m = 4 - 2\epsilon \) dimensions and \( M_{\mu}^{(n)} \) is amplitude of the process

Example: LO contribution

\[ F_T^{(1)}(x, \epsilon) \equiv \quad \simeq \quad \left( x^2 g^{\mu \nu} + 4 k_0^\mu k_0^\nu \right) \int d^m \text{PS}^{(3)} \quad \left( \quad \frac{q}{k_0} + \frac{q}{k_0} \quad \right)^2 \]

\[ d^m \text{PS}^{(3)} = d^m k_0 \delta^+(k_0^2) \quad d^m k_1 \delta^+(k_1^2) \quad d^m k_2 \delta^+(k_2^2) \delta (x - 2q \cdot k_0) \delta^m(q - k_0 - k_1 - k_2) \]
We can extract splitting functions on the rhs \( (P_{gq}^{(0)}, P_{gq}^{(1)}, P_{gq}^{(2)}) \) when we know the lhs of the following expressions:

- \( \mathcal{F}_T^{(1)}(x, \epsilon) = \frac{1}{\epsilon} P_{gq}^{(0)}(x) + c_{T,g}^{(1)}(x) + \epsilon a_{T,g}^{(1)}(x) + \epsilon^2 b_{T,g}^{(1)}(x) \)

- \( \mathcal{F}_T^{(2)}(x, \epsilon) = \frac{1}{\epsilon^2} \left\{ \frac{1}{2} P_{gi}^{(0)} P_{iq}^{(0)} + \frac{1}{2} \beta_0 P_{gq}^{(0)} \right\} - \frac{1}{\epsilon} \left\{ \frac{1}{2} P_{gq}^{(1)} + P_{gi}^{(0)} c_i^{(1)} \right\} + \left\{ c_g^{(2)} - P_{gi}^{(0)} a_i^{(1)} \right\} + \epsilon \left\{ a_g^{(2)} - P_{gi}^{(0)} b_i^{(1)} \right\} \)

- \( \mathcal{F}_T^{(3)}(x, \epsilon) = \frac{1}{\epsilon^3} \left\{ \frac{1}{6} P_{gi}^{(0)} P_{ij}^{(0)} P_{jq}^{(0)} + \frac{1}{2} \beta_0 P_{gi}^{(0)} P_{ij}^{(0)} + \frac{1}{3} \beta_0^2 P_{gq}^{(0)} \right\} + \frac{1}{\epsilon^2} \left\{ \frac{1}{6} P_{gi}^{(1)} P_{iq}^{(1)} + \frac{1}{3} P_{gi}^{(0)} P_{ij}^{(0)} + \frac{1}{3} \beta_1 P_{gq}^{(0)} + \frac{1}{2} P_{gi}^{(0)} P_{ij}^{(0)} c_j^{(1)} + \beta_0 \left( \frac{1}{3} P_{gq}^{(0)} + \frac{1}{2} P_{gi}^{(0)} c_i^{(1)} \right) \right\} - \frac{1}{\epsilon} \left\{ \frac{1}{3} P_{gq}^{(2)} + \frac{1}{2} P_{gi}^{(1)} c_i^{(1)} + P_{gi}^{(0)} c_i^{(2)} - \frac{1}{2} P_{gi}^{(0)} P_{ij}^{(0)} a_j^{(1)} - \frac{1}{2} \beta_0 P_{gq}^{(0)} a_i^{(1)} \right\} + \left\{ c_g^{(3)} - P_{gi}^{(0)} a_i^{(2)} - \frac{1}{2} P_{gi}^{(1)} a_i^{(1)} + \frac{1}{2} P_{gi}^{(0)} P_{ij}^{(0)} b_j^{(1)} + \frac{1}{2} \beta_0 P_{gq}^{(0)} b_i^{(1)} \right\} \)
Feynman integrals for $\mathcal{F}_T(x, \epsilon)$

\[
\mathcal{F}_T^{(\text{NNLO})}(x, \epsilon) = \underbrace{\ldots}_{\text{RR}} + \underbrace{\ldots}_{\text{RV}} + \underbrace{\ldots}_{\text{VV}}
\]

\[
d^m\text{PS}^{VV} = d^m l_1 \quad d^m l_2 \quad d^m k_0 \delta(k_0^2) \quad d^m k_1 \delta(k_1^2) \quad d^m k_2 \delta(k_2^2)
\]

\[
\times \delta^m(q - k_0 - k_1 - k_2) \delta(x - 2q \cdot k_0)
\]

Perfectly suits for IBP reduction:

- Loops and Legs integrals are reduced simultaneously
- $\delta(x - 2q \cdot k_0)$ and $\delta(k_i^2)$ are replaced by cut propagators, i.e., according to Cutkosky’s rules
- LiteRed and Reduze2 have support for cut propagators
- define system of differential equations in $x$-space
Fuchsia is a program for reducing differential equations for master integrals to the canonical form Henn '13:

- based on the Lee algorithm Lee '14
- open-source and free (no proprietary software dependencies)
- implemented in SageMath (Python, Maxima, GiNaC)

The idea is to find a rational transformation in three reduction steps:

1. **Fuchsification** decrease Poincaré rank to 0 at all singular points (i.e. get rid of irregular singularities)
2. **Normalization**: balance eigenvalues to $n\epsilon$ form
3. **Factorization**: reduce to canonical form
Let us consider a system of ODEs

$$\frac{d\bar{f}}{dx} = \mathbb{A}(x, \epsilon) \bar{f},$$

where $\bar{f}(x, \epsilon)$ is a vector of $n$ unknown functions (e.g., master integrals).

For any system we can define an integer number

$$m_{x=x_k}(\mathbb{A}) = p_k \geq 0$$

as a Poincaré rank of $\mathbb{A}$ at $x = x_k$.

For $\mathbb{B} = \begin{pmatrix} \frac{\epsilon}{x} & 0 \\ \frac{\epsilon}{x^2} & \frac{\epsilon}{1+x} \end{pmatrix}$ we have $m_{x=0} = 1$ and $m_{x=-1} = 0$. 
We say that such matrix has **Fuchsian form**\(^1\) if its Poincaré rank is 0 at every singular point (including \(\infty\)).

For example, we can transform
\[
\mathcal{B} = \begin{pmatrix}
\frac{\epsilon}{x} & 0 \\
-\frac{\epsilon}{x^2} & \frac{\epsilon}{1+x}
\end{pmatrix}
to \begin{pmatrix}
\frac{\epsilon}{x} & 0 \\
-\frac{1}{x} & \frac{\epsilon}{1+x} + \frac{1}{x}
\end{pmatrix}
\]
with \(m(\mathcal{B}) = 0\) at any point.

Not every system can be transformed to the Fuchsian form, however
- due to the analyticity of S-matrix; and
- structure of the Feynman integrals

**every system for master integrals** should, in principle, be reducible to Fuchsian form.

**Fuchsia** finds Fuchian form and transformation matrix of rational functions by analyzing Jordan form of the input matrix.

---

\(^1\)after German mathematician Lazarus Fuchs (1833–1902)
Example: differential equations for $P_{gq}^T(1)$ at NLO

\[
\begin{align*}
\text{Input} & \\
\begin{pmatrix}
\frac{(2\epsilon - 1)(2x - 1)}{x(1-x)} & -\frac{1-3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{x(1-x)}{3\epsilon - 2} & \frac{1-6\epsilon}{x+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{x(1-x)}{(2\epsilon - 1)} & 0 & \frac{2}{x+1} & 0 & 0 & 0 & 0 & 0 \\
\frac{\epsilon^2 x(x+1)(1-x)}{(2\epsilon - 1)} & -\frac{2(\epsilon - 1)(3\epsilon - 1)}{x^2} & \frac{2(6\epsilon - 1)}{x(x+1)} & \frac{2\epsilon (x^2 + 3x - 2)}{(1-x)x(x+1)} & 0 & 0 & 0 & 0 \\
\frac{2(x^2 + 4x + 1)}{\epsilon^2 (1-x)^3 x^3 (x+1)^3} & \frac{2(2\epsilon - 1)(x-1)}{\epsilon^2 x^2 (x+1)^2} & \frac{2(6\epsilon - 1)(x-1)}{x^2 (x+1)^3} & \frac{4(x^2 + 1)}{x^2 (x+1)^3} & \frac{(2\epsilon + 1)(2x+1)}{x(x+1)} & 0 & 0 & 0 \\
-\frac{4}{\epsilon^2 (1-x)^3 x^3 (x+1)} & -\frac{2(2\epsilon - 1)(x-2)}{\epsilon(1-x)^2 x^3} & -\frac{2(6\epsilon - 1)}{x^2 (1-x)(x+1)} & \frac{4(x^2 + 1)}{(1-x)^2 x^2 (x+1)} & 0 & -\frac{4\epsilon}{(1-x)^2 x} & \frac{(2\epsilon + 1)(2x+1)}{(1-x)x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon + 1)(2x+1)}{(1-x)x}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{Output} & \\
\begin{pmatrix}
\frac{(2\epsilon - 1)(2x - 1)}{x(1-x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x(1-x)} & \frac{1-3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{1+(2\epsilon - 1)x}{x(1-x)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(1+2\epsilon)(2x-1)}{x(1-x)} & 0 & 0 & 0 & 0 \\
\frac{\epsilon}{x(1+x)} & \frac{\epsilon^2}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2(6\epsilon - 1)}{(1-x)(1+x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{14\epsilon - 3) x + 22\epsilon - 3}{2(1-x)(1+x)} & \frac{\epsilon x - 5\epsilon + 2}{x(1-x)} & \frac{4\epsilon^2}{x} & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{x(1-x)(1+x)} & \frac{4\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\epsilon (1-x)(1+x)}{1-x} & \frac{\epsilon (1-x)(1+x)}{4(6\epsilon - 1) x^2 + 6\epsilon x - 4\epsilon + 1} & \frac{1+x}{x(1-x)(x+1)} & 0 & 0 & 0 & 0 & 0 \\
\frac{1-2(3\epsilon - 1)x}{x(1-x)(1+x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(2\epsilon - 3)x^2 - 18\epsilon + 8x + 11}{\epsilon x(1-x)(1+x)} & \frac{(4 \epsilon - 3)x^2 - 2(2\epsilon - 3)x + 9}{4\epsilon x(1-x)(1+x)} & \frac{(4 \epsilon - 1)x - 2\epsilon}{x(1-x)} & 0 & 0 & 0 & 0 & 0 \\
\frac{8(\epsilon^2 + 2x - 1)}{x(1-x)(1+x)} & \frac{x + 3}{x(1+x)} & 0 & 0 & 0 & \frac{4\epsilon x + 3}{2(1-x)(1+x)} & 0 & 0 \\
\frac{4}{2(1-x)(1+x)} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon + 1)(2x+1)}{(1-x)x}
\end{pmatrix}
\end{align*}
\]
Step II: Normalization

We say that matrix $A(x, \epsilon)$ is normalized if eigenvalues of all its residues have form $m\epsilon$, where $m$ is some number.

We assume that initial eigenvalues have form $n + m\epsilon$, where $n$ is integer.

A key idea for Lee’s normalization algorithm is a balance transformation between two points $x_1$ and $x_2$

$$T(x) = B(P, x_1, x_2; x) = 1 - P(\epsilon) + \frac{x - x_2}{x - x_1}P(\epsilon)$$

where $P(\epsilon)$ is some projector matrix, i.e. $P^2 = P$.

We choose

- points $x_1$ and $x_2$ by analyzing eigenvalues
- projector $P(\epsilon)$ by analyzing eigenvectors

Example: eigenvalues for $x_1$ and $x_2$:

$x_1 = 0$:

$[1 - 4\epsilon, 1 - 3\epsilon, -1 - 2\epsilon, -2\epsilon, 1 - 2\epsilon, 1 - 2\epsilon, 1, 1]$

$x_2 = 1$:

$[1 - 2\epsilon, -2\epsilon, -2\epsilon, 1 - 2\epsilon, 1 - 2\epsilon, 0, 0, 0]$
### Input

\[
\begin{pmatrix}
\frac{(2\epsilon - 1)(2x - 1)}{x(1-x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x(1-x)} & \frac{1 - 3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{1 + (2\epsilon - 1)x}{x(1-x)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1 + 2\epsilon)(2x - 1) & 0 & 0 & 0 & 0 \\
\frac{\epsilon}{x(1+x)} & \frac{\epsilon^2}{x} & 0 & 0 & \frac{(2\epsilon - 1)x^2 + 6\epsilon x - 4\epsilon + 1}{x(1-x)(x+1)} & 0 & 0 & 0 \\
\frac{2(6\epsilon - 1)}{(1-x)(1+x)} & 0 & 0 & 0 & \frac{1 - 2(3\epsilon - 1)x}{x(1+x)} & 0 & 0 & 0 \\
\frac{(14\epsilon - 3)x + 22\epsilon - 3}{2(1-x)x(1+x)} & \frac{\epsilon x - 5\epsilon + 2}{x(1-x)} & \frac{4\epsilon^2}{x} & 0 & \frac{(2\epsilon - 3)x^2 - 18\epsilon x + 81 + 11}{\epsilon x(1-x)(1+x)} & 0 & 0 & 0 \\
\frac{4}{x(1-x)(1+x)} & \frac{4\epsilon}{x} & 0 & 0 & \frac{8(x^2 + 2x - 1)}{x(1-x)(1+x)} & 0 & 0 & \frac{4\epsilon + x - 3}{2x(1-x)} \\
\end{pmatrix}
\]

### Output

\[
\begin{pmatrix}
\frac{2\epsilon}{1-x} - \frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x} - \frac{3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{2\epsilon}{1-x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2\epsilon}{1-x} - \frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{x} & \frac{\epsilon}{x} & 0 & 0 & \frac{12\epsilon^2}{1-x} + \frac{12\epsilon^2}{1+x} + \frac{32\epsilon^2}{x} & \frac{-1}{x} & 0 & 0 \\
\frac{6\epsilon}{1+x} + \frac{6\epsilon}{1-x} + \frac{16\epsilon}{x} & \frac{8\epsilon^2}{x} - \frac{4\epsilon^2}{1+x} & 0 & 0 & \frac{16\epsilon}{x} - \frac{4\epsilon}{1-x} - \frac{16\epsilon}{x} & \frac{-3}{x} & \frac{2\epsilon}{1-x} - \frac{2\epsilon}{x} & 0 \\
\frac{4}{x} - \frac{2}{1-x} & \frac{4\epsilon}{1-x} - \frac{8\epsilon}{x} & \frac{4\epsilon^2}{1-x} - \frac{4\epsilon^2}{x} & 0 & \frac{8\epsilon}{1+x} - \frac{4\epsilon}{1-x} - \frac{16\epsilon}{x} & \frac{3}{x} + \frac{2}{1+x} & 0 & \frac{-2\epsilon}{1+x} - \frac{2\epsilon}{x} \\
\frac{6}{1+x} - \frac{8}{x} & \frac{4\epsilon}{1-x} - \frac{8\epsilon}{x} & \frac{4\epsilon^2}{1-x} - \frac{4\epsilon^2}{x} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Eigenvalues before and after normalization:

x = 0:

\[
\begin{bmatrix}
1-4\epsilon, & 1-3\epsilon, & 1-2\epsilon, & 1-2\epsilon, & 1-2\epsilon, & 1, & 1
\end{bmatrix} \rightarrow
\begin{bmatrix}
-4\epsilon, & -3\epsilon, & -2\epsilon, & -2\epsilon, & -2\epsilon, & -2\epsilon, & 0, & 0
\end{bmatrix}
\]

x = 1:

\[
\begin{bmatrix}
1-2\epsilon, & -2\epsilon, & -2\epsilon, & 1-2\epsilon, & 1-2\epsilon, & 0, & 0, & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
-2\epsilon, & -2\epsilon, & -2\epsilon, & -2\epsilon, & -2\epsilon, & 0, & 0, & 0
\end{bmatrix}
\]

x = -1:

\[
\begin{bmatrix}
1-2\epsilon, & 1-2\epsilon, & 0, & 0, & 0, & 0, & 0, & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
-2\epsilon, & -2\epsilon, & 0, & 0, & 0, & 0, & 0, & 0
\end{bmatrix}
\]

x = \infty:

\[
\begin{bmatrix}
1+2\epsilon, & 1+2\epsilon, & 1+3\epsilon, & 2+4\epsilon, & -1+4\epsilon, & 1+4\epsilon, & -2+6\epsilon
\end{bmatrix} \rightarrow
\begin{bmatrix}
2\epsilon, & 2\epsilon, & 3\epsilon, & 4\epsilon, & 4\epsilon, & 4\epsilon, & 4\epsilon, & 6\epsilon
\end{bmatrix}
\]
Step III: Factorization

Now we can find an $x$-independent transformation $\mathbb{T}(\epsilon)$ for any point $x = x_k$ such that

$$\mathbb{T}^{-1}(\epsilon)A_k(\epsilon)\mathbb{T}(\epsilon) = \epsilon C_k$$

Since matrix $C_k$ is constant, for every residue of $A$ we write

$$\frac{\mathbb{T}^{-1}(\epsilon)A_k(\epsilon)\mathbb{T}(\epsilon)}{\epsilon} = C_k = \frac{\mathbb{T}^{-1}(\mu)A_k(\mu)\mathbb{T}(\mu)}{\mu}, \quad \mu = \text{any number}$$

We treat components of $\mathbb{T}$ as unknown variables and solve linear system of equations for them. That gives unknown transformation $\mathbb{T}(\epsilon)$. 
\[
\begin{pmatrix}
\frac{2\epsilon}{1-x} - \frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -\frac{3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{2\epsilon}{1-x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2\epsilon}{1-x} - \frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 \\
\frac{6\epsilon}{1+x} + \frac{6\epsilon}{1-x} + \frac{16\epsilon}{x} & \frac{8\epsilon^2}{x} - \frac{4\epsilon^2}{1+x} & 0 & 0 & \frac{12\epsilon^2}{1+x} + \frac{12\epsilon^2}{1-x} + \frac{32\epsilon^2}{x} & -\frac{1}{x} & 0 & 0 \\
\frac{4}{x} - \frac{2}{1-x} & \frac{16\epsilon}{1-x} - \frac{8\epsilon}{1+x} & \frac{4\epsilon^2}{1-x} - \frac{4\epsilon^2}{x} & 0 & 0 & \frac{3}{x} + \frac{2}{1+x} & 0 & -\frac{2\epsilon}{1+x} - \frac{2\epsilon}{x} \\
\frac{6}{1+x} - \frac{8}{x} & \frac{4\epsilon}{1-x} - \frac{4\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{19\epsilon}{1-x} - \frac{4}{1-x} & \frac{4\epsilon}{1-x} + \frac{8}{x} & \frac{76}{x} - \frac{76}{1-x} & \frac{4}{19(1-x)} - \frac{16}{19x} & 0 & 0 & 0 & 0 \\
\frac{8}{19x} - \frac{6}{19(1+x)} & \frac{4}{1-x} - \frac{4}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
**Summary**

**Fuchsia** — a tool for reducing differential equations for master integrals to canonical form

- all main algorithms from Lee ’14
- open-source and free (SageMath: Python, Maxima, GiNaC)

A little bit of optimization and we are ready for release! Please, send us some examples of your systems.

Stay tuned we will release soon

Thank you!