Asymptotic expansions of several series 
and their application

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Abstract. Asymptotic expansions of series \( \sum_{k=0}^{\infty} \varepsilon^k (k+a)^\gamma e^{-(k+a)^\alpha x} \) and \( \sum_{k=0}^{\infty} \frac{\varepsilon^k (k+a)^\gamma}{(k+a)^{\alpha x}} \) in powers of \( x \) as \( x \to +0 \) are found, where \( \varepsilon = 1 \) or \( \varepsilon = -1 \). These expansions are applied to obtain precise inequalities for Mathieu series.

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1 Introduction and formulation of main results

One of the purposes of this paper is to investigate functional series of the form

\[
    f(x, a, \gamma, \alpha) := \sum_{k=0}^{\infty} (k + a)^\gamma e^{-(k+a)^\alpha x}, \quad x > 0, \tag{1.1}
\]

\[
    \tilde{f}(x, a, \gamma, \alpha) := \sum_{k=0}^{\infty} (-1)^k (k + a)^\gamma e^{-(k+a)^\alpha x}, \quad x > 0, \tag{1.2}
\]

with parameters \( a > 0, \gamma \in \mathbb{R} \) and \( \alpha > 0 \). Series (1.1) and (1.2) appear in many problems of the analysis. In particular, for \( x = \ln \frac{1}{\rho}, \quad a = \frac{1}{2}, \quad \alpha = 1, \gamma = -r - 1 \) and \( r \in \mathbb{N} \) series (1.1) and (1.2) appeared in the paper due to A. F. Timan [1] in 1950. He proved that these series give an exact value of the remainder when periodic differentiable functions are approximated by Poisson integrals. Finding a complete asymptotic representation was the aim of the papers due to L. V. Malei [2], É. L. Shtark [3], V. A. Baskakov [4], and K. M. Zhigallo and Yu. I. Kharkevich [5]. A complete solution to this problem was obtained in the author’s paper [6], where expansions in series in powers of \( x \) were found in the explicit form.
for the functions (1.1) and (1.2) with $a > 0$, $\alpha = 1$, $\gamma = -r - 1$ and $r \in \mathbb{Z}_+$ if $0 < x < 2\pi$ and $0 < x < \pi$ respectively.

Applying the residue theory, Gel’fond [7, §4.3] in 1966 found an asymptotic expansion in powers of $x^k$, $k \in \mathbb{Z}_+$ as $x \to +0$ for the function (1.1) with $a = 1$, $-\frac{\gamma + 1}{\alpha} \notin \mathbb{Z}_+$. Here $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ stands for the set of all nonnegative integers. In the case $a = 1$, $-\frac{\gamma + 1}{\alpha} \in \mathbb{Z}_+$ he pointed out only that the sum in an asymptotic expansion has to be changed in an appropriate way. In [7, §4.3], it was also claimed that for $a = 1$ an asymptotic expansion of the function (1.2) in powers of $x^k$, $k \in \mathbb{Z}_+$ as $x \to +0$ can be obtained similarly. In 2008, using the Euler–Maclaurin formula it was found by the author [8] the asymptotic expansion of the functions (1.1) and (1.2) as $x \to +0$ for any $a > 0$ and $\gamma \in \mathbb{Z}_+$, $\alpha \in \mathbb{N}$ (note that, in examples on pp. 56–57 of [8], the term $(-1)^{\alpha k + \gamma}$ is missing under the sum sign in the right-hand side of asymptotic expansions).

In theorems 1.1 and 1.2 here the asymptotic expansions of the functions (1.1) and (1.2) as $x \to +0$ are given for all admissible parameters. Coefficients of these expansions are expressed by the Hurwitz function $\zeta(s,a)$ and the function $\tilde{\zeta}(s,a)$ respectively. The last ones are defined by formulas

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \text{ Re } s > 1;$$

$$\tilde{\zeta}(s,a) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s}, \text{ Re } s > 0,$$

for a fixed $a > 0$.

Using the Hermite formula one can continue the Hurwitz function analytically to $\mathbb{C} \setminus \{1\}$. Moreover, the point $s = 1$ stands for its first-order pole, and for $a > 0$ the following relations hold (see [9] [10]):

$$\lim_{s \to 1} \left( \zeta(s,a) - \frac{1}{s - 1} \right) = -\frac{\Gamma'(a)}{\Gamma(a)},$$

$$\lim_{s \to 1} \left( \zeta(s,a) - 2^{1-s} \zeta \left( s, \frac{a + 1}{2} \right) \right) = -\frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma' \left( \frac{a + 1}{2} \right)}{\Gamma \left( \frac{a + 1}{2} \right)} + \ln 2.$$  

(1.4)

Here $\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} \, dt$, Re $s > 0$ is referred to as the Euler gamma-function. The relation

$$\tilde{\zeta}(s,a) = \zeta(s,a) - 2^{1-s} \zeta \left( s, \frac{a + 1}{2} \right), \text{ Re } s > 1,$$

(1.5)
implies that the function $\tilde{\zeta}(s, a)$ is analytically continued to $\mathbb{C}$, and

$$\tilde{\zeta}(1, a) = -\frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(\frac{a+1}{2})}{\Gamma(\frac{a+1}{2})} + \ln 2, \ a > 0. \quad (1.6)$$

**Theorem 1.1.** Suppose that $a > 0$, $\gamma \in \mathbb{R}$, and $\alpha > 0$. Then the following asymptotic expansions hold:

$$f(x, a, \gamma, \alpha) \sim \frac{1}{\alpha} \left(\frac{\gamma + 1}{\alpha}\right) x^{-\frac{\gamma + 1}{\alpha}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(-\alpha k - \gamma, a) x^k, \quad (1.7)$$

$$f(x, a, \gamma, \alpha) \sim \left(\frac{\ln x}{\alpha} + \frac{\Gamma'(r + 1)}{r + 1} - \frac{\Gamma'(a)}{\Gamma(a)}\right) + \sum_{k=0, k \neq r}^{\infty} \frac{(-1)^k}{k!} \zeta(-\alpha k - \gamma, a) x^k, \quad (1.8)$$

if $-\gamma + 1 \notin \mathbb{Z}$ and $-\gamma + 1 = r \in \mathbb{Z}$ respectively. If $0 < \alpha < 1$, then (1.7) and (1.8) turn into equalities whenever $x > 0$. If $\alpha = 1$, then (1.7) and (1.8) are equalities whenever $x \in (0, 2\pi)$.

The fact that the relations (1.7) are equalities for all $x > 0$ if $a = 1$, $0 < \alpha < 1$, was mentioned without proof in [7].

**Theorem 1.2.** Let $a > 0$, $\gamma \in \mathbb{R}$, and let $\alpha > 0$. Then the following asymptotic expansion holds:

$$\tilde{\tilde{f}}(x, a, \gamma, \alpha) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \tilde{\zeta}(-\alpha k - \gamma, a) x^k. \quad (1.9)$$

If $0 < \alpha < 1$, then the (1.9) turns into an equality for all $x > 0$. If $\alpha = 1$, then (1.9) is an equality for all $x \in (0, \pi)$.

In what follows we will consider the functional series of the form

$$g(x, a, \gamma, \alpha, \mu) := \sum_{k=0}^{\infty} \frac{(k + a)^\gamma}{(x(k + a) + 1)^\alpha} \cdot \mu > \max\left\{\frac{\gamma + 1}{\alpha}; 0\right\}, \ x > 0, \quad (1.10)$$

$$\tilde{g}(x, a, \gamma, \alpha, \mu) := \sum_{k=0}^{\infty} \frac{(-1)^k(k + a)^\gamma}{(x(k + a) + 1)^\alpha} \cdot \mu > \max\left\{\frac{\gamma}{\alpha}; 0\right\}, \ x > 0. \quad (1.11)$$

with parameters $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$ and $\mu > 0$. 
Theorem 1.3. Suppose that \( a > 0, \gamma \in \mathbb{R}, \alpha > 0 \) and \( \mu > \max \left\{ \frac{\gamma + 1}{\alpha}; 0 \right\} \). Then the following asymptotic expansions hold:

\[
g(x,a,\gamma,\alpha,\mu) \sim_{x \to +0} \Gamma \left( \frac{\gamma + 1}{\alpha} \right) \Gamma \left( \frac{\mu + \gamma + 1}{\alpha} \right) x^{-\frac{\gamma + 1}{\alpha}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\mu + k)}{\Gamma(\mu)} \zeta(-\alpha k - \gamma, a) x^k,
\]

\( (1.12) \)

\[
g(x,a,\gamma,\alpha,\mu) \sim_{x \to +0} \frac{\Gamma(\mu + r)(-1)^r x^r}{\Gamma(\mu)\Gamma(r + 1)} \left( -\frac{\ln x}{\alpha} + \frac{\Gamma'(r + 1)}{\alpha \Gamma(r + 1)} - \frac{\Gamma'(a)}{\alpha \Gamma(a)} - \frac{\Gamma'(\mu + r)}{\alpha \Gamma(\mu + r)} \right) + \sum_{k=0, k \neq r}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\mu + k)}{\Gamma(\mu)} \zeta(-\alpha k - \gamma, a) x^k,
\]

\( (1.13) \)

for \( -\frac{\gamma + 1}{\alpha} \notin \mathbb{Z}_+ \) and \( -\frac{\gamma + 1}{\alpha} = r \in \mathbb{Z}_+ \) respectively.

Theorem 1.4. Let \( a > 0, \gamma \in \mathbb{R}, \alpha > 0 \), and let \( \mu > \max \left\{ \frac{\gamma}{\alpha}; 0 \right\} \). Then the following asymptotic expansion holds:

\[
\tilde{g}(x,a,\gamma,\alpha,\mu) \sim_{x \to +0} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\mu + k)}{\Gamma(\mu)} \tilde{\zeta}(-\alpha k - \gamma, a) x^k.
\]

\( (1.14) \)

Note that Theorems 1.1, 1.2, 1.3 and 1.4 are proved by using the technique of [7]. In § 2 these theorems are applied to obtain precise inequalities for Mathieu series.

2 Precise inequalities for Mathieu series

Consider the following functional series with parameters \( a > 0, \gamma \in \mathbb{R}, \alpha > 0 \) and \( \mu > 0 \):

\[
S(x,a,\gamma,\alpha,\mu) := \sum_{k=0}^{\infty} \frac{(k + a)^{\gamma}}{(k + a)^{\alpha} + x^{\mu}}, \mu > \max \left\{ \frac{\gamma + 1}{\alpha}; 0 \right\}, x \geq 0,
\]

\( (2.1) \)

\[
\tilde{S}(x,a,\gamma,\alpha,\mu) := \sum_{k=0}^{\infty} \frac{(-1)^k(k + a)^{\gamma}}{(k + a)^{\alpha} + x^{\mu}}, \mu > \max \left\{ \frac{\gamma}{\alpha}; 0 \right\}, x \geq 0.
\]

\( (2.2) \)
As is customary in recent years, (2.1) and (2.2) are said to be a generalized Mathieu series and a generalized alternating Mathieu series respectively. In 1890, Emile Leonard Mathieu [13] introduced the hypothesis on the validity of the following inequality:

\[ S(x, 1, 1, 2, 2) = \sum_{k=0}^{\infty} \frac{k+1}{((k+1)^2 + x)^2} < \frac{1}{2x}, \quad x > 0. \quad (2.3) \]

Different proofs of the inequality (2.3) were published in the papers due to Berg [14], van der Corput, Heflinger [15] and Makai [16] in 1952–1957. In the Makai’s paper [16] there were proved the inequalities

\[ \frac{1}{2(q+x)} < \sum_{k=0}^{\infty} \frac{k+1}{((k+1)^2 + x)^2} < \frac{1}{2(p+x)}, \quad x > 0, \quad (2.4) \]

where \( q = \frac{1}{2} \) and \( p = 0 \). Thus, the following natural problem arises: to find a maximal possible \( p \) and a minimal possible \( q \) satisfying the inequality (2.4).

In 1982, Elbert [17] conjectured that one can take \( q = \frac{1}{2\zeta(3)} \) and \( p = \frac{1}{6} \) are sharp constants in the inequality (2.4).

In 1998, Alzer, Brenner and Ruehr [18] proved that \( q = \frac{1}{2\zeta(3)} \) and \( p = \frac{1}{6} \) are sharp constants in the inequality (2.4).

In 2008, it was proved by the author [19] that for any \( \mu > 1 \) and \( a \geq 1 \) there exist positive constants \( m(\mu, a) \) and \( M(\mu, a) \) such that the inequality

\[ \frac{1}{2(\mu-1)(q+x)^{\mu-1}} \leq \sum_{k=0}^{\infty} \frac{(k+a)}{((k+a)^2 + x)^\mu} \leq \frac{1}{2(\mu-1)(p+x)^{\mu-1}} \quad (2.5) \]

is fulfilled for every \( x > 0 \) if and only if \( 0 \leq p \leq m(\mu, a) \) and \( q \geq M(\mu, a) \). In this case, for any fixed \( a \geq 1 \) the functions \( m(\mu, a) \) and \( M(\mu, a) \) decrease and increase respectively on \( \mu \in (1, +\infty) \), and for all \( a \geq 1 \), \( \mu > 1 \) the following inequalities hold:

\[ a^2 - a < m(\infty, a) \leq m(\mu, a) \leq a^2 - a + \frac{1}{6} \]

\[ a^2 - a + \frac{1}{4} < M(\mu, a) < M(\infty, a) = a^2. \]

It was also proved that \( m(\mu, 1) = \frac{1}{6}, \mu \in (1, 3) \). Thus, if \( a \geq 1 \) then inequality (2.5) is valid for all \( \mu > 1 \) if and only if \( 0 \leq p \leq m(\infty, a) \) and \( q \geq M(\infty, a) = a^2 \). The right-hand side inequality in (2.5) was proved for \( a = 1, p = 0 \) and \( \mu > 1 \) by Diananda [20] in 1980. A big list related to this matter can be found in [21].
If $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$ and $\mu > \max \left\{ \frac{\gamma+1}{\alpha}; 0 \right\}$, then by Theorem 1.3 we have

\[
S(x, a, \gamma, \alpha, \mu) \sim \begin{cases} 
\frac{\Gamma\left(\frac{\gamma+1}{\alpha}\right) \cdot \Gamma\left(\frac{\gamma+1}{\alpha}\right) \cdot x^{\frac{\gamma+1}{\alpha}-\mu}}{\Gamma(\mu)} \cdot x^{-\mu} & , \quad \gamma + 1 > 0 , \\
\frac{1}{\alpha} \cdot x^{-\mu} \ln x & , \quad \gamma + 1 = 0 , \\
\zeta(-\gamma, a) x^{-\mu} & , \quad \gamma + 1 < 0 .
\end{cases}
\]

Hence the following problem is natural. Assume that $a > 0$, $\gamma + 1 > 0$, $\alpha > 0$ and $\mu_0 \geq \frac{\gamma+1}{\alpha}$. For which $q \geq 0$, $p \geq 0$, $A \in \mathbb{R}$ and $B > 0$ the inequality

\[
\frac{B \cdot \Gamma\left(\mu - \frac{\gamma+1}{\alpha}\right)}{\Gamma(\mu)(q+x)^{\mu-\frac{\gamma+1}{\alpha}}} \leq S(x, a, \gamma, \alpha, \mu) \leq \frac{A \cdot \Gamma\left(\mu - \frac{\gamma+1}{\alpha}\right)}{\Gamma(\mu)(p+x)^{\mu-\frac{\gamma+1}{\alpha}}}
\]

holds for any $\mu > \mu_0$ and $x > 0$? This problem is completely solved in Theorem 2.1 and an analogous problem is solved in Theorem 2.2 in the case of $\gamma + 1 < 0$.

**Theorem 2.1.** Suppose that $a > 0$, $\gamma + 1 > 0$, $\alpha > 0$, $\mu_0 \geq \frac{\gamma+1}{\alpha}$, $q \geq 0$, $p \geq 0$, $A \in \mathbb{R}$ and $B > 0$. Then the inequality (2.7) is valid for all $\mu > \mu_0$ and $x > 0$ if and only if $0 < p < a^\alpha \leq q$, $A \geq A_p(a, \gamma, \alpha)$, $0 < B \leq B_q(a, \gamma, \alpha)$, where

\[
\begin{align*}
A_p(a, \gamma, \alpha) & := \sup_{x>0} e^{px} x^{\frac{\gamma+1}{\alpha}} f(x, a, \gamma, \alpha) , \\
B_q(a, \gamma, \alpha) & := \inf_{x>0} e^{qx} x^{\frac{\gamma+1}{\alpha}} f(x, a, \gamma, \alpha) .
\end{align*}
\]

In this case, the inequality (2.7) is strict for all $x > 0$. If $p > 0$, it is also strict at $x = 0$. Moreover, $A_p(a, \gamma, \alpha) < +\infty$ if and only if $p < a^\alpha$, and $B_q(a, \gamma, \alpha) > 0$ if and only if $q \geq a^\alpha$. If $a \geq 1$ then $A_p(a, 1, 2) = B_q(a, 1, 2) = \frac{1}{2}$ for all $q \geq a^2$ and $0 \leq p \leq m(\infty, a)$; in particular, this is valid for all $p \in [0, a^2 - a]$.

**Theorem 2.2.** Assume that $a > 0$, $\gamma + 1 < 0$, $\alpha > 0$, $\mu_0 \geq 0$, $q \geq 0$, $p \geq 0$, $D \in \mathbb{R}$ and $E > 0$. Then the inequality

\[
\frac{E}{(q+x)^{\mu}} \leq S(x, a, \gamma, \alpha, \mu) \leq \frac{D}{(p+x)^{\mu}}
\]

holds for every $\mu > \mu_0$ and $x > 0$ if and only if $0 \leq p < a^\alpha \leq q$, $D \geq D_p(a, \gamma, \alpha)$, $0 < E \leq E_q(a, \gamma, \alpha)$, where

\[
\begin{align*}
D_p(a, \gamma, \alpha) & := \sup_{x>0} e^{px} f(x, a, \gamma, \alpha) , \\
E_q(a, \gamma, \alpha) & := \inf_{x>0} e^{qx} f(x, a, \gamma, \alpha) .
\end{align*}
\]
In this case, inequality (2.9) is strict for all $x > 0$. If $p > 0$, it is also strict at $x = 0$. Moreover, $D_p(a, \gamma, \alpha) < +\infty$ if and only if $p \leq a^{\alpha}$, and $E_q(a, \gamma, \alpha) > 0$ if and only if $q \geq a^{\alpha}$. Besides, $D_p(a, \gamma, \alpha) = \zeta(-\gamma, a)$ for all $p \leq a^{\alpha}$ and $E_q(a, \gamma, \alpha) = a^{\gamma}$ at $q = a^{\alpha}$.

If $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$ and $\mu > \max\{\frac{\gamma}{\alpha}; 0\}$, then Theorem 1.4 yields that

$$\tilde{S}(x, a, \gamma, \alpha, \mu) = x^{-\mu} \left( \zeta(-\gamma, a) + o(1) \right), \quad x \to +\infty. \quad (2.11)$$

Therefore the following problem is natural for series (2.2). Let $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$, and let $\mu_0 \geq \max\{\frac{\gamma}{\alpha}; 0\}$. For which $q \geq 0$, $p \geq 0$ and $C, F \in \mathbb{R}$ the inequality

$$\frac{F}{(q + x)^\mu} \leq \tilde{S}(x, a, \gamma, \alpha, \mu) \leq \frac{C}{(p + x)^\mu} \quad (2.12)$$

holds for any $\mu > \mu_0$ and $x > 0$? In the paper due to Tomovski and Hilfer [22], it is claimed that this is satisfied in the case $a = 1$, $\gamma > 0$ if we take $p = C = 1$ and $\mu_0 = \frac{\gamma}{\alpha}$ in the right-hand side of the (2.12). A mistake in the proof of this Tomovski and Hilfer’s assertion was indicated by the author in [23]. In the same paper [23], it was proved that for $m, \alpha \in \mathbb{N}$, $\gamma = 4m + 5$, $\alpha \mu - \gamma > 0$ the right-hand side inequality of the (2.12) with $a = p = C = 1$ is impossible for large $x > 0$. Theorem 2.3 presents all the solutions to this problem.

**Theorem 2.3.** Suppose that $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$, $\mu_0 \geq \max\{\frac{\gamma}{\alpha}; 0\}$, $q \geq 0$, $p \geq 0$ and $C, F \in \mathbb{R}$. Then the inequality (2.12) is satisfied for every $\mu > \mu_0$ and $x > 0$ if and only if $0 \leq p \leq a^{\alpha}$, $C \geq C_p(a, \gamma, \alpha)$, $F \leq F_q(a, \gamma, \alpha)$, where

$$C_p(a, \gamma, \alpha) := \sup_{x > 0} e^{p x} \tilde{f}(x, a, \gamma, \alpha),$$

$$F_q(a, \gamma, \alpha) := \inf_{x > 0} e^{q x} \tilde{f}(x, a, \gamma, \alpha). \quad (2.13)$$

In this case, the inequality (2.12) is strict for $x > 0$. If $q, p > 0$, it is also strict at $x = 0$. Moreover, we have $0 < C_p(a, \gamma, \alpha) < +\infty$ for $p \leq a^{\alpha}$ and $C_p(a, \gamma, \alpha) = +\infty$ for $p > a^{\alpha}$.

### 3 Preliminaries

#### 3.1 Euler gamma-function

The function $\Gamma(s)$ is analytically continued to the whole plane $\mathbb{C}$ except to the points $s = -k$, $k \in \mathbb{Z}_+$ in which it has simple poles. Moreover,
the following relations

\[ \Gamma(s + 1) = s \Gamma(s) \ , \ \Gamma(s) \Gamma(1 - s) = \frac{\pi}{\sin \pi s} \] \hspace{1cm} (3.1)

hold for all admissible \( s \in \mathbb{C} \). If \( s = \sigma + it = |s|e^{i\varphi} \), where \( \sigma, t \in \mathbb{R} \) and \( \varphi = \varphi(s) = \arg s \in (-\pi, \pi) \), we have

\[
\begin{align*}
\Gamma(s + 1) &= \sqrt{2\pi s^{\sigma + \frac{1}{2}} e^{-s} e^{R(s)}} , \quad |R(s)| \leq \frac{1}{12|s| \cos^2 \frac{\pi}{2}} , \\
|s^{\sigma + \frac{1}{2}}| &= |e^{\varphi(s)|t|} , \\
|\Gamma(s + 1)| &= \sqrt{2\pi |s|^{\sigma + \frac{1}{2}} e^{-|\varphi(s)||t|} |e^{R(s)}|} \\
\end{align*}
\] \hspace{1cm} (3.2)

(see [III §1.5.1]). If, in addition, \( \Re s > 0 \), then \( |\varphi(s)| = \arctg \frac{|t|}{\sigma} \), and hence

\[
\begin{align*}
|\Gamma(s + 1)| &= \sqrt{2\pi |s|^{\sigma + \frac{1}{2}} e^{-|\varphi(s)||t|} e^{R(s)}} , \quad |R(s)| \leq \frac{1}{6|s|} , \\
|\Gamma(s + 1)| &\leq \sqrt{2\pi |s|^{\sigma + \frac{1}{2}} e^{-\frac{\pi}{2}t} e^{\frac{1}{2\pi\sin t}}} , \\
\frac{1}{|\Gamma(s + 1)|} &\leq (2\pi)^{-\frac{1}{2}} |s|^{-\frac{\pi}{2}} e^{\frac{1}{2e^{\frac{1}{2\pi\sin t}}}} .
\end{align*}
\] \hspace{1cm} (3.3)

Here we take the inequalities \( |e^w| \leq e^{|w|} \), \( w \in \mathbb{C} \) and \( 0 < \frac{\pi}{2} - \arctg u < \frac{1}{u}, u > 0 \) into account.

Let \( 0 < \delta \leq \frac{\pi}{2}, |\arg s| \leq \pi - \delta \), and let \( \Re s = \sigma \). Considering the cases \( \sigma > 0 \) and \( \sigma \leq 0 \) (here \( |\arg s| \geq \frac{\pi}{2} \)) separately we derive from (3.2) that

\[
|\Gamma(s + 1)| \leq \sqrt{2\pi |s|^{\sigma + \frac{1}{2}} e^{\frac{|s| - \sigma}{2}} e^{-\frac{\pi}{2}t} e^{\frac{1}{12|s| \sin^2 \frac{\pi}{2}}} , \quad |\arg s| \leq \pi - \delta .
\] \hspace{1cm} (3.4)

### 3.2 Hurwitz function

If the case of \( a = p + a_0 \), where \( p \in \mathbb{N}, 0 < a_0 \leq 1 \), we have

\[
\zeta(s, a) = \zeta(s, a_0) - \sum_{k=0}^{p-1} \frac{1}{(k + a_0)^s} , \quad s \neq 1 .
\] \hspace{1cm} (3.5)

The relation (3.5) is obvious whenever \( \Re s > 1 \), and for the remaining \( s \neq 1 \) it is implied by the uniqueness theorem for analytic functions. The following formula is due to Hurwitz:

\[
\zeta(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{k=1}^{\infty} \sin \left( \frac{2\pi ak + \frac{\pi}{2}s}{k^{1-s}} \right) , \quad \Re s < 0 , \quad 0 < a \leq 1 .
\] \hspace{1cm} (3.6)
If $0 < a \leq 1$ then it follows from [9, §13.51] the existence of positive constants $c(a) > 0$ and $t(a) > 1$ such that the inequality

$$|\zeta(\sigma + it, a)| \leq c(a)|t|^\tau(\sigma) \ln |t| , \ |t| \geq t(a) ,$$

where

$$\tau(\sigma) := \begin{cases} \frac{1}{2} - \sigma , & \sigma \leq 0 , \\ \frac{1}{2} , & 0 \leq \sigma \leq \frac{1}{2} , \\ 1 - \sigma , & \frac{1}{2} \leq \sigma \leq 1 , \\ 0 , & \sigma \geq 1 \end{cases}$$

is fulfilled for $\sigma, t \in \mathbb{R}$.

### 3.3 Mellin transform

If $x^{\sigma-1} f(x) \in L(0, +\infty)$ for some $\sigma \in \mathbb{R}$, then the Mellin transform of a function $f$ is defined by

$$g(s) = \int_0^{+\infty} x^{\sigma-1} f(x) \, dx \ , \ s = \sigma + it \ , \ t \in \mathbb{R} \ .$$

If, in addition, the function $f$ is of bounded variation in a neighborhood of $x > 0$, then the following inversion formula holds:

$$\frac{f(x + 0) + f(x - 0)}{2} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} g(s) x^{-s} \, ds$$

(see [12, §1.29]), where the integral is treated in the sense of Cauchy principal value. If we take $f(x) = e^{-x}$ and $f(x) = (x + 1)^{-\mu}$, $\mu > 0$, we find $g(s) = \Gamma(s)$, $\Re s > 0$, and $g(s) = \frac{\Gamma(\mu - s)\Gamma(s)}{\Gamma(\mu)}$ if $0 < \Re s < \mu$, respectively. Therefore for any $x > 0$ the following two relations are met:

$$e^{-x} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s) x^{-s} \, ds \ , \ \sigma > 0 \ , \quad (3.8)$$

$$\frac{1}{(x + 1)^\mu} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(\mu - s)\Gamma(s)}{\Gamma(\mu)} x^{-s} \, ds \ , \ 0 < \sigma < \mu \ . \quad (3.9)$$

### 4 Proofs of Theorems 1.1, 1.2, 1.3 and 1.4

#### Proof of asymptotic expansions in Theorem 1.1

Take $\sigma = \beta > \max \left\{ 0, \frac{\gamma + 1}{\alpha} \right\}$ in (3.8), and replace $x$ by $(k+a)^{\alpha} x$, $a > 0$, $k \in \mathbb{Z}_+, \alpha > 0$, $x > 0$. Then we summarize the obtained inequalities

$$(k+a)^{\gamma} e^{-(k+a)^{\alpha} x} = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Gamma(s) x^{-s} (k+a)^{-\alpha s + \gamma} \, ds$$
over all \( k \in \mathbb{Z}_+ \). In the left-hand side, we obtain \( f(x, a, \gamma, \alpha) \). In the right-hand side, we interchange the sum and the integral signs (this is well defined in view of (3.2), (3.3) and \( \alpha \beta > 1 \)). We obtain that for any \( a > 0, \gamma \in \mathbb{R}, \alpha > 0, x > 0 \) and \( \beta > \max \left\{ 0, \frac{\gamma + 1}{\alpha} \right\} \) the following relation holds:

\[
f(x, a, \gamma, \alpha) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} F(s) \, ds, \quad F(s) = \Gamma(s)x^{-s}\zeta(\alpha s - \gamma, a). \quad (4.1)
\]

The function \( F(s) \) is analytic on the whole plane except in the poles \( s = -k, k \in \mathbb{Z}_+ \) and \( s = \frac{\gamma + 1}{\alpha} \). If \( -\gamma + 1 \frac{1}{\alpha} \notin \mathbb{Z}_+ \) then these poles are different and simple. Take \( \sigma_n = n + \frac{1}{2}, n \in \mathbb{Z}_+ \), and consider the rectangle

\[
K_{n,m} := \{ s \in \mathbb{C} : |\text{Im} \, s| \leq m, -\sigma_n \leq \text{Re} \, s \leq \beta \}, \quad m \in \mathbb{N}, \, n \in \mathbb{Z}_+.
\]

If \( \sigma_n \neq -\gamma + 1 \frac{1}{\alpha} \) then by the residue theorem we have

\[
\frac{1}{2\pi i} \oint_{\partial K_{n,m}} F(s) \, ds = \Sigma_n, \quad (4.2)
\]

where \( \Sigma_n \) stands for the sum of residues of the function \( F \) in poles lying on the interval \( (-\sigma_n, \beta) \). If \( n > -\frac{1}{2} - \frac{\gamma + 1}{\alpha} \), then the interval \( (-\sigma_n, \beta) \) contains only poles \( s = -k, k = 0, \ldots, n \), and \( s = \frac{\gamma + 1}{\alpha} \). Both estimates (3.4), (3.7) and relation (3.5) yield (in the case of \( a > 1 \)) that the integrals over horizontal segments \( s = \sigma \pm im, -\sigma_n \leq \sigma \leq \beta \) tend to zero as \( m \to +\infty \) in the left-hand side of (4.2). Therefore both (4.1) and (4.2) imply the following relation:

\[
f(x, a, \gamma, \alpha) = \Sigma_n + I_n, \quad I_n = \frac{1}{2\pi i} \int_{-\sigma_n - i\infty}^{-\sigma_n + i\infty} F(s) \, ds, \quad \sigma_n \neq -\gamma + \frac{1}{\alpha}. \quad (4.3)
\]

First, let us calculate \( \Sigma_n \). It follows from relation (1.4) that the expansion of the function \( \zeta(\alpha s - \gamma, a) \) in a Laurent series in a neighborhood of the pole \( s = \frac{\gamma + 1}{\alpha} \) can be written as

\[
\zeta(\alpha s - \gamma, a) = \frac{c_{-1}}{s - \frac{\gamma + 1}{\alpha}} + c_0 + c_1 \left( s - \frac{\gamma + 1}{\alpha} \right) + \ldots, \quad (4.4)
\]

\[c_{-1} = \frac{1}{\alpha}, \quad c_0 = -\frac{\Gamma'(a)}{\Gamma(a)}.\]

Complement formula (3.1) implies that the expansion of the \( \Gamma \)-function in a Laurent series in a neighborhood of the pole \( s = -k, k \in \mathbb{Z}_+ \) is of
the form
\[
\Gamma(s) = \frac{a_{-1}}{s+k} + a_0 + a_1 (s+k) + \ldots ,
\]
\[
a_{-1} = \frac{(-1)^k}{\Gamma(k+1)} , \quad a_0 = \frac{(-1)^k \Gamma'(k+1)}{\Gamma^2(k+1)} .
\]

Thus if \(-\gamma + 1/\alpha \not\in \mathbb{Z}_+\) then for \(n > -1/2 - \gamma + 1/\alpha\), \(n \in \mathbb{Z}_+\), the following equality holds:
\[
\Sigma_n = \frac{1}{\alpha} \Gamma\left(\frac{\gamma + 1}{\alpha}\right) x^{-\gamma + 1/\alpha} + \sum_{k=0}^{n} \frac{(-1)^k}{k!} \zeta(-\alpha k - \gamma, a) x^k .
\]  

If \(-\gamma + 1/\alpha = r \in \mathbb{Z}_+\) then for \(n > -1/2 - \gamma + 1/\alpha\), \(n \in \mathbb{Z}_+\), the relation
\[
\Sigma_n = \text{res}_s F(s) + \sum_{k=0, k \neq r}^{n} \text{res}_s F(s) ,
\]
is met, where the residues at the points \(s = -k, k \in \mathbb{Z}_+, k \neq r\), are calculated as above. To calculate the residue of the function \(F\) at the point \(s = -r\) one should take account of the following expansion of the function \(x^{-s}\) in a Taylor series in a neighborhood of the point \(s = -r\):
\[
x^{-s} = b_0 + b_1 (s + r) + \ldots , \quad b_0 = x^r , \quad b_1 = -x^r \ln x .
\]

In view of relations \((4.4)\) and \((4.5)\) for \(-\gamma + 1/\alpha = -r\) and \(k = r\) respectively, we obtain the following expansion of the function \(F\) in a Laurent series in a neighborhood of the pole \(s = -r\):
\[
F(s) = B_{-2} \frac{1}{(s+r)^2} + B_{-1} \frac{1}{(s+r)} + B_0 + B_1 (s+r) + \ldots ,
\]
\[
\begin{aligned}
B_{-2} &= c_{-1} a_{-1} b_0 = \frac{1}{\alpha} \cdot \frac{(-1)^r}{\Gamma(r+1)} x^r , \\
B_{-1} &= c_{-1} a_{-1} b_1 + c_{-1} a_0 b_0 + c_0 a_{-1} b_0 .
\end{aligned}
\]  

Therefore,
\[
\text{res}_{s=-r} F(s) = B_{-1} = \frac{(-1)^r x^r}{\Gamma(r+1)} \left( -\frac{\ln x}{\alpha} + \frac{\Gamma'(r+1)}{\Gamma(r+1)} \frac{1}{\alpha} - \frac{\Gamma'(a)}{\Gamma(a)} \right) .
\]  

Now let us find estimate for the integral \(I_n\) in \((4.3)\). If \(\sigma_n \neq -\gamma + 1/\alpha\) we have:
\[
|I_n| \leq \frac{x^{\sigma_n}}{2\pi} \int_{-\infty}^{+\infty} |\Gamma(-\sigma_n + it)||\zeta(-\alpha \sigma_n - \gamma + i\alpha t, a)| dt , \quad n \in \mathbb{Z}_+ , \quad x > 0 .
\]  

(4.10)
The convergence of the integral in (4.10) is implied by both the relation
\[ |\Gamma(-\sigma_n + it)| = \frac{\pi}{\text{ch}(\pi t)|\Gamma(1 + \sigma_n - it)|} \]
and estimates (3.3) and (3.7) (in the case of \( a > 1 \) it should also be considered the relation (3.5)). Thus asymptotic expansions (1.7) and (1.8) are proved.

**Case of** \( 0 < \alpha \leq 1 \) **in Theorem 1.1**

For fixed \( 0 < \alpha \leq 1 \) and \( \gamma \in \mathbb{R} \), and an arbitrary \( \varepsilon > 0 \) we put:

\[ n(\varepsilon, \gamma, \alpha) := \max \left\{ \frac{1 - \gamma}{\alpha} - \frac{1}{2}, \frac{\varepsilon}{\alpha} - \frac{1}{2}; 1 \right\} \]

Then, for all positive integers \( n \geq n(\varepsilon, \gamma, \alpha) \), the following inequalities hold:

\[ \alpha \sigma_n + \gamma \geq 1 ; \quad |\gamma| \leq \varepsilon \sigma_n \leq \varepsilon |\sigma_n - it|, \quad t \in \mathbb{R} ; \quad \sigma_n > 1 . \]

If \( 0 < a \leq 1 \) then we conclude from (3.6) and (3.3) that, for all positive integers \( n \geq n(\varepsilon, \gamma, \alpha) \) and \( t \in \mathbb{R} \), there hold inequalities (it should also be taken into account that \( |\sin w| \leq e^{\Im w}, w \in \mathbb{C} \)):

\[ |\zeta(-\alpha \sigma_n - \gamma + i\alpha t, a)| \leq \frac{2|\Gamma(1 + \alpha \sigma_n + \gamma - i\alpha t)|}{(2\pi)^{1+\alpha \sigma_n + \gamma}} e^{-\frac{\alpha |t|}{2}} \zeta(1 + \alpha \sigma_n + \gamma) \leq \frac{C(\gamma)}{(2\pi)^{\alpha \sigma_n}} e^{\frac{\alpha |t|}{2}} |\alpha \sigma_n + \gamma - i\alpha t|^\alpha e^{-\alpha |t| \arctg \frac{|t|}{\alpha \sigma_n}} \leq \frac{C(\gamma)}{(2\pi)^{\alpha \sigma_n}} e^{\frac{\alpha |t| - \alpha |t| \arctg \frac{|t|}{\alpha \sigma_n}}{\alpha \sigma_n + \gamma}} e^{-\alpha |t| (\alpha + \varepsilon)^\alpha \sigma_n + \gamma + \frac{1}{2} |\sigma_n - it|^\alpha} \sigma_n + \gamma + \frac{1}{2} \]

where \( C(\gamma) = \frac{2\zeta(2) e^{\frac{\pi}{12}}}{(2\pi)^{\gamma + 1/2}} \), and \( \zeta(s) := \zeta(s, 1) \) stands for the Riemann zeta-function. Here we use the inequalities

\[ \zeta(1 + \alpha \sigma_n + \gamma) \leq \zeta(2), \]

\[ |\alpha \sigma_n + \gamma - i\alpha t| \leq \alpha |\sigma_n - it| + |\gamma| \leq (\alpha + \varepsilon)|\sigma_n - it| . \]

Since for all \( n \geq n(\varepsilon, \gamma, \alpha) \) and \( t \in \mathbb{R} \) the inequality \( |\sigma_n - it| \geq \sigma_n \geq 1 \) is satisfied, we have for those \( n \) and \( t \) that

\[ |\sigma_n - it|^\gamma \leq |\sigma_n - it|^{\gamma|t|} \leq |\sigma_n|^{\gamma|t|} \left( 1 + \frac{|t|}{\sigma_n} \right)^{|\gamma|} \leq |\sigma_n|^{\gamma|t|} (1 + |t|)^{|\gamma|} . \]
Finally, we obtain that for any $n \geq n(\varepsilon, \gamma, \alpha)$ and $t \in \mathbb{R}$ there holds the inequality:

\[
|\zeta(-\alpha \sigma_n - \gamma + i\alpha t, a)| \leq \frac{C(\gamma)}{(2\pi)^{\alpha \sigma_n}} e^{\frac{\pi}{2} |\alpha| |t| - \alpha |t| \arctg \frac{\alpha|t|}{\alpha \sigma_n + \gamma}} e^{-\alpha \sigma_n} \times (\alpha + \varepsilon)^{\alpha \sigma_n + \gamma + \frac{1}{2}} |\sigma_n - it|^{\alpha \sigma_n + \frac{1}{2} \sigma_n |\gamma|} (1 + |t|)^{\gamma} .
\]

(4.11)

The relation (4.11) implies the validity of the following inequality:

\[
|\Gamma(-\sigma_n + it)| = \frac{\pi}{\text{ch}(\pi t)|\Gamma(1 + \sigma_n - it)|} \leq (2\pi)^{\frac{1}{2} \gamma} e^{\frac{-\pi}{2} |\sigma_n - it|^{\sigma_n - \frac{1}{2}}} e^{\frac{\pi}{2} |\gamma|}.
\]

for all $n \geq n(\varepsilon, \gamma, \alpha)$ and $t \in \mathbb{R}$. Both the relation (4.11) and the last inequality yield that, for any $n \geq n(\varepsilon, \gamma, \alpha)$, $t \in \mathbb{R}$ and $0 < a \leq 1$, the inequality

\[
|\zeta(-\alpha \sigma_n - \gamma + i\alpha t, a) \Gamma(-\sigma_n + it)| \leq \frac{C_1(\gamma)}{(2\pi)^{\alpha \sigma_n}} e^{\psi(t)+(1-\alpha)\sigma_n (\alpha + \varepsilon)^{\alpha \sigma_n + \gamma + \frac{1}{2}} |\sigma_n - it|^{(\alpha-1)\sigma_n |\gamma|} (1 + |t|)^{\gamma} |}
\]

(4.12)

is fulfilled, where $C_1(\gamma) = (2\pi)^{\frac{1}{2}} e^{\pi} C(\gamma)$ and

\[
\psi(t) = -\pi |t| + |t| \arctg \frac{|t|}{\sigma_n} + \frac{\pi}{2} \alpha |t| - \alpha |t| \arctg \frac{\alpha|t|}{\alpha \sigma_n + \gamma} = \frac{\pi}{2} |t| - (1 - \alpha) |t| \left( \frac{\pi}{2} - \arctg \frac{|t|}{\sigma_n} \right) + \alpha |t| \left( \arctg \frac{|t|}{\sigma_n} - \arctg \frac{\alpha|t|}{\alpha \sigma_n + \gamma} \right) \leq - \frac{\pi}{2} |t| + |\gamma| .
\]

Here we take account of $0 < \alpha \leq 1$ and use the inequality

\[
|\arctg v - \arctg u| = \left| \int_u^v \frac{dx}{x^2 + 1} \right| \leq \left| \int_u^v \frac{dx}{x^2} \right| = \left| \frac{1}{u} - \frac{1}{v} \right| , u, v > 0 .
\]

Combining the inequality $|\sigma_n - it|^{(\alpha-1)\sigma_n} \leq \sigma_n^{(\alpha-1)\sigma_n}$ in (4.12) with the estimate for $\psi(t)$ yields the validity of the relation:

\[
|\zeta(-\alpha \sigma_n - \gamma + i\alpha t, a) \Gamma(-\sigma_n + it)| \leq \frac{C_1(\gamma)e^{\frac{\gamma}{2} |t|}}{(2\pi)^{\alpha \sigma_n}} e^{-\frac{\pi}{2} |\sigma_n + (1-\alpha)\sigma_n (\alpha + \varepsilon)^{\alpha \sigma_n + \gamma + \frac{1}{2}} |\sigma_n |\gamma| (1 + |t|)^{\gamma|}}.
\]
for every $n \geq n(\varepsilon, \gamma, \alpha)$, $t \in \mathbb{R}$ and $0 < a \leq 1$. Applying this inequality to (4.10) we obtain the following estimate for $I_n$ in (4.3) in the case of $n \geq n(\varepsilon, \gamma, \alpha)$:

$$|I_n| \leq \frac{x^{\sigma_n}e^{(1-\alpha)\sigma_n}(\alpha+\varepsilon)^{\alpha\sigma_n+\gamma+\frac{1}{2}\sigma_n(a-1)\sigma_n+\gamma}|}{(2\pi)^{\alpha\sigma_n+1}}I(\gamma),$$

$$I(\gamma) = C_1(\gamma)e^{\gamma_\alpha|} \int_{-\infty}^{+\infty} e^{-\frac{\pi}{2}t|} (1 + |t|)^{\gamma_\alpha} dt.$$  

(4.13)

If $0 < \alpha < 1$ then (4.13) implies that $\lim_{n \to \infty} I_n = 0$ for any $x > 0$. If $\alpha = 1$ then $\lim_{n \to \infty} I_n = 0$ for every $x \in (0, \frac{2\pi}{1+\varepsilon})$ and hence for every $x \in (0, 2\pi)$. Thus the second part of Theorem 1.1 is proved in the case $0 < a \leq 1$.

If $a > 1$ then $a = p + a_0$, where $p \in \mathbb{N}$, $0 < a_0 \leq 1$, and we find:

$$|\zeta(-\alpha\sigma_n - \gamma + iat, a)| \leq |\zeta(-\alpha\sigma_n - \gamma + iat, a_0)| + a(a-1)^{\alpha\sigma_n+\gamma}$$

(see (3.3)). In this case the right-hand side of the inequality (4.13) contains one more summand:

$$\frac{x^{\sigma_n}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\pi|t|} |\sigma_n - it|^{-\alpha\sigma_n - \gamma - \frac{1}{2}} e^{\sigma_n + \frac{1}{2}\sigma_n(a-1)\sigma_n+\gamma} dt \leq$$

$$\frac{x^{\sigma_n}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\pi|t|} |\sigma_n - it|^{-\alpha\sigma_n - \gamma - \frac{1}{2}} a(a-1)^{\alpha\sigma_n+\gamma} dt, n \geq n(\varepsilon, \gamma, \alpha).$$

(4.14)

Here we use the inequality $|\sigma_n - it|^{-\alpha\sigma_n - \gamma - \frac{1}{2}} \leq \sigma_n^{-\alpha\sigma_n - \gamma - \frac{1}{2}}$. The right-hand side of the inequality (4.14) tends to 0 as $n \to \infty$ for any $x > 0$. Theorem 1.1 is complete.

**Proof of Theorem 1.2**

The proof follows from both Theorem 1.1 and the obvious relation

$$\tilde{f}(x, a, \gamma, \alpha) = f(x, a, \gamma, \alpha) - 2^{\gamma+1} f \left(2^\alpha x, \frac{a + 1}{2}, \gamma, \alpha \right), x > 0.$$

**Proof of Theorem 1.3**

Take $\sigma = \beta \in \left(\max \left\{0, \frac{\gamma+1}{\alpha} \right\}, \mu \right)$ in (3.4), and replace $x$ by $(k + a)\alpha x$, $a > 0$, $k \in \mathbb{Z}_+$, $\alpha > 0$, $x > 0$. Then summarize the obtained relations

$$\frac{(k + a)^\gamma}{(x(k + a)^\alpha + 1)^\mu} = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Gamma(\mu - s)\Gamma(s) \Gamma(\mu) x^{-s} (k + a)^{-\alpha s + \gamma} ds$$

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over all $k \in \mathbb{Z}_+$. In the left-hand side, we obtain $g(x, a, \gamma, \alpha, \mu)$. In the right-hand side, we interchange the sum and integral signs (this is well defined in view of (3.2), (3.3) and $a > 1$). It follows that, for any $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$, $\mu > \max \{0, \frac{\gamma + 1}{\alpha}\}$, $x > 0$ and max $\{0, \frac{\gamma + 1}{\alpha}\} < \beta < \mu$, the following relation holds:

$$g(x, a, \gamma, \alpha, \mu) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} G(s) \, ds,$$

where $G(s) = \frac{\Gamma(\mu-s)\Gamma(s)}{\Gamma(\mu)} x^{-s} \zeta(\alpha s - \gamma, a) = \frac{\Gamma(\mu-s)}{\Gamma(\mu)} F(s)$ and the function $F$ from (4.11). Singular points of the functions $G$ and $F$ coincide on the half-plane $\text{Re} \, s < \mu$. Consider $\sigma_n = n + \frac{1}{\alpha}$, $n \in \mathbb{Z}_+$. As in the proof of Theorem 1.1, taking account of (3.4), (3.7) and the equality (3.5) (for $a > 1$), in the case of $\sigma_n \neq -\frac{\gamma + 1}{\alpha}$ we obtain the following relation:

$$g(x, a, \gamma, \alpha, \mu) = \Sigma_n + I_n, \quad I_n = \frac{1}{2\pi i} \int_{-\sigma_n + i\infty}^{-\sigma_n - i\infty} G(s) \, ds,$$

where $\Sigma_n$ stands for the sum of residues of the function $G$ at poles lying on the interval $(-\sigma_n, \beta)$. If $n > -\frac{1}{2} - \frac{\gamma + 1}{\alpha}$, $n \in \mathbb{Z}_+$, then the interval $(-\sigma_n, \beta)$ contains only the poles $s = -k$, $k = 0, \ldots, n$, and $s = \frac{\gamma + 1}{\alpha}$. If $-\frac{\gamma + 1}{\alpha} \notin \mathbb{Z}_+$ then these poles are simple, and we have:

$$\text{res}_{s = -\frac{\gamma + 1}{\alpha}} G(s) = \frac{\Gamma(\mu - \frac{\gamma + 1}{\alpha})}{\Gamma(\mu)} \text{res}_{s = -\frac{\gamma + 1}{\alpha}} F(s) = \frac{\Gamma(\mu - \frac{\gamma + 1}{\alpha})}{\Gamma(\mu)} \frac{\Gamma\left(\frac{\gamma + 1}{\alpha}\right)}{\Gamma(\alpha)} x^{-\frac{\gamma + 1}{\alpha}},$$

$$\text{res}_{s = -k} G(s) = \frac{\Gamma(\mu + k)}{\Gamma(\mu)} \frac{(-1)^k}{k!} \zeta(-\alpha k - \gamma, a) x^{k}, \quad k \in \mathbb{Z}_+.$$

For $-\frac{\gamma + 1}{\alpha} = r \in \mathbb{Z}_+$, the residues at the points $s = -k$, $k \in \mathbb{Z}_+$, $k \neq r$, can be calculated as above. To calculate the residue of the function $G$ at the point $s = -r$ it should be considered (4.8), (4.9) as well as the following expansion of the function $\frac{\Gamma(\mu-s)}{\Gamma(\mu)}$ in a Taylor series in a neighborhood of the point $s = -r$:

$$\frac{\Gamma(\mu-s)}{\Gamma(\mu)} = A_0 + A_1(s + r) + \ldots, \quad A_0 = \frac{\Gamma(\mu + r)}{\Gamma(\mu)}, \quad A_1 = -\frac{\Gamma'(\mu + r)}{\Gamma(\mu)}.$$

$$\text{res}_{s = -r} G(s) = B_{-1}A_0 + B_{-2}A_1 = \frac{\Gamma(\mu + r)(-1)^r x^r}{\Gamma(\mu)\Gamma(r + 1)} \left( \frac{-\ln x}{\alpha} + \frac{\Gamma'(r + 1)}{\alpha \Gamma(r + 1)} - \frac{\Gamma'(a)}{\alpha \Gamma(a)} - \frac{\Gamma'(\mu + r)}{\alpha \Gamma(\mu + r)} \right).$$

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Let us find estimate for the integral $I_n$ in (4.15). If $n \in \mathbb{Z}_+$, $\sigma_n \neq -\frac{\gamma+1}{\alpha}$ and $x > 0$, we have:

\[
|I_n| \leq \frac{x^{\sigma_n}}{2\pi \Gamma(\mu)} \int_{-\infty}^{+\infty} |\Gamma(\mu+\sigma_n-it)||\Gamma(-\sigma_n+it)||\zeta(-\alpha \sigma_n-\gamma+i\alpha t,a)| dt .
\]  

(4.16)

The convergence of the integral in (4.16) is proved exactly in the same way as that for the integral in (4.10). Theorem 1.3 is complete.

Proof of Theorem 1.4

If $\mu > \max \left\{0, \frac{\gamma+1}{\alpha} \right\}$ then the proof is implied both by Theorem 1.3 and the relation

\[
\tilde{g}(x,a,\gamma,\alpha,\mu) = g(x,a,\gamma,\alpha,\mu) - 2^{\gamma+1} g \left(2^\alpha x, \frac{a+1}{2}, \gamma, \alpha, \mu \right), \quad x > 0.
\]

If $\mu > \max \left\{0, \frac{\gamma}{\alpha} \right\}$, the proof is exactly the same as that of Theorem 1.3. It should be noted that in this case, for any $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$, $\mu > \max \left\{0, \frac{\gamma}{\alpha} \right\}$, $x > 0$ and $\mu > \max \left\{0, \frac{\gamma}{\alpha} \right\}$, $\beta < \mu$ the relation

\[
\tilde{g}(x,a,\gamma,\alpha,\mu) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \tilde{G}(s) ds
\]

holds, where $\tilde{G}(s) = \frac{\Gamma(\mu-s) \Gamma(\gamma+s)}{\Gamma(\mu)} x^{-s} \tilde{\zeta}(\alpha s - \gamma, a)$.

5 Proofs of Theorems 2.1, 2.2 and 2.3

Lemma 5.1. Suppose that $a > 0$, $\gamma \in \mathbb{R}$ and $\alpha > 0$. Then there are no constants $p, \beta, c \in \mathbb{R}$ such that one of the identities $x^\beta e^{px} f(x,a,\gamma,\alpha) \equiv c$ or $x^\beta e^{px} \tilde{f}(x,a,\gamma,\alpha) \equiv c$ holds for $x > 0$.

Proof. Assume that $x^\beta e^{px} f(x,a,\gamma,\alpha) \equiv c$, $x > 0$. Then $c > 0$, and it follows from the asymptotics $f(x,a,\gamma,\alpha) \sim a^\gamma e^{-a^\alpha x}$, $x \to +\infty$ that $p = a^\alpha$ (if $p > a^\alpha$ or $p < a^\alpha$ then $c = +\infty$ or $c = 0$ respectively, which is impossible). Hence $\beta = 0$ (if $\beta > 0$ or $\beta < 0$ then $c = +\infty$ or $c = 0$, which is impossible) and $c = a^\gamma$. Therefore $f(x,a,\gamma,\alpha) \equiv a^\gamma e^{-a^\alpha x}$, $x > 0$ but $f(x,a,\gamma,\alpha) > a^\gamma e^{-a^\alpha x}$ for every $x > 0$.

Assume that $x^\beta e^{px} \tilde{f}(x,a,\gamma,\alpha) \equiv c$, $x > 0$. It follows from the asymptotics $\tilde{f}(x,a,\gamma,\alpha) \sim a^\gamma e^{-a^\alpha x}$, $x \to +\infty$ that $c > 0$. As above we obtain similarly that $\tilde{f}(x,a,\gamma,\alpha) \equiv a^\gamma e^{-a^\alpha x}$, $x > 0$. Then $\tilde{f}(x,a+1,\gamma,\alpha) \equiv a^\gamma e^{-a^\alpha x} - \tilde{f}(x,a,\gamma,\alpha) \equiv 0$, $x > 0$ but $\tilde{f}(x,a+1,\gamma,\alpha) > 0$ for large $x > 0$. Lemma 5.1 is proved.  

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Proof of Theorem 2.1

In the case of \( a > 0, \gamma + 1 > 0, \alpha > 0, \mu > \frac{\gamma + 1}{\alpha}, p \geq 0 \) and \( c \in \mathbb{R} \), define the following function of the variable \( x > 0 \):

\[
\varphi(x, a, \gamma, \alpha, \mu, c, p) := \frac{c}{(p + x)^{\mu - \frac{\gamma + 1}{\alpha}}} \cdot \frac{\Gamma\left(\mu - \frac{\gamma + 1}{\alpha}\right)}{\Gamma(\mu)} - S(x, a, \gamma, \alpha, \mu).
\]

It is easy to check that for any \( x > 0 \) and \( k \in \mathbb{Z}_+ \) there hold the relations:

\[
(-1)^k \frac{d^k}{dx^k} \{\varphi(x, a, \gamma, \alpha, \mu, c, p)\} = \frac{\Gamma(\mu + k)}{\Gamma(\mu)} \varphi(x, a, \gamma, \alpha, \mu + k, c, p),
\]

\[
\varphi(x, a, \gamma, \alpha, \mu, c, p) = \frac{1}{\Gamma(\mu)} \int_0^{+\infty} e^{-xt} t^{\mu - \frac{\gamma + 1}{\alpha} - 1} \left( c e^{-pt} - i \frac{\gamma + 1}{\alpha} f(t, a, \gamma, \alpha) \right) dt.
\]  

(5.1)

The integral representation in (5.1) follows from both the inequalities \( \mu > \frac{\gamma + 1}{\alpha}, p \geq 0 \) and the asymptotics \( f(t, a, \gamma, \alpha) \sim a^\gamma e^{-a^\alpha t}, t \to +\infty \) and \( f(t, a, \gamma, \alpha) \sim \frac{1}{\alpha} \Gamma\left(\frac{\gamma + 1}{\alpha}\right) t^{-\frac{\gamma + 1}{\alpha}}, t \to +0 \) (see Theorem 1.1). These asymptotics imply also that \( A_p(a, \gamma, \alpha) < +\infty \) if and only if \( p < a^\alpha \), and \( B_q(a, \gamma, \alpha) > 0 \) if and only if \( q \geq a^\alpha \). We will need the following theorem in what follows (see [24, 25, 26]).

Theorem 5.1 (Bernstein–Hausdorff–Widder). The following two conditions are equivalent:

1. A function \( f \in C^\infty(0, +\infty) \), and the inequality \((-1)^k f^{(k)}(x) \geq 0 \) holds for all \( k \in \mathbb{Z}_+, x > 0 \).

2. We have \( f(x) = \int_0^{+\infty} e^{-xt} d\mu(t), x > 0 \), where \( \mu \) is a nonnegative Borel measure on \( [0, +\infty) \) such that the integral converges for every \( x > 0 \). In this case the measure \( \mu \) is finite on \( [0, +\infty) \) if and only if \( f(+0) < +\infty \).

Both the Bernstein–Hausdorff–Widder theorem and relations (5.1) yield that the validity of inequalities (2.7) for every \( \mu > \mu_0 \) and \( x > 0 \) is equivalent to that for the inequalities \( A e^{-pt} - \frac{\gamma + 1}{\alpha} f(t, a, \gamma, \alpha) \geq 0 \) and \( B e^{-qt} - t^{-\frac{\gamma + 1}{\alpha}} f(t, a, \gamma, \alpha) \leq 0 \) for any \( t > 0 \). The last ones are equivalent to the inequalities \( A \geq A_p(a, \gamma, \alpha) \) and \( B \leq B_q(a, \gamma, \alpha) \) respectively.

The relation (1.7) implies that \( B_q(a, \gamma, \alpha) \leq \frac{\Gamma\left(\frac{\gamma + 1}{\alpha}\right)}{\alpha} \leq A_p(a, \gamma, \alpha) \) for \( 0 \leq p < a^2 \leq q \). If \( a \geq 1 \) (see the inequality (2.5) and the text below it), we have \( \frac{1}{2} \leq B_q(a, 1, 2), A_p(a, 1, 2) \leq \frac{1}{2} \) for \( 0 \leq p \leq m(\infty, a) \),
\( q \geq a^2 \), and hence \( B_q(a, 1, 2) = A_p(a, 1, 2) = \frac{1}{2} \) for any \( a \geq 1 \) and for the same \( p \) and \( q \); in particular, this is satisfied for \( p \in [0, a^2 - a] \) (since \( a^2 - a < m(\infty, a) \)).

Let \( 0 \leq p < a^\alpha \), \( A \geq A_p(a, \gamma, \alpha) \), and let \( q \geq a^\alpha \), \( B \leq B_q(a, \gamma, \alpha) \). If either the right-hand side or the left-hand side inequality in (2.7) turns into an equality for some \( x \geq 0 \) (or for \( x > 0 \) if \( p = 0 \)) then it follows from the integral representation (5.1) that either \( A \equiv e^{pt} f(t, a, \gamma, \alpha) \) or \( B \equiv e^{qt} f(t, a, \gamma, \alpha) \) for \( t > 0 \), which contradicts Lemma 5.1.

Theorem 2.2 is complete.

**Proof of Theorem 2.2**

For \( a > 0 \), \( \gamma + 1 < 0 \), \( \alpha > 0 \), \( \mu > 0 \), \( p \geq 0 \) and \( c \in \mathbb{R} \), define the function

\[
\psi(x, a, \gamma, \alpha, \mu, c, p) := \frac{c}{(p + x)^\mu} - S(x, a, \gamma, \alpha, \mu), \quad x > 0.
\]

It is easy to check that for any \( x > 0 \) and \( k \in \mathbb{Z}_+ \) there hold the inequalities:

\[
(-1)^k \frac{d^k}{dx^k} \{\psi(x, a, \gamma, \alpha, \mu, c, p)\} = \frac{\Gamma(\mu + k)}{\Gamma(\mu)} \psi(x, a, \gamma, \alpha, \mu + k, c, p),
\]

\[
\begin{align*}
\psi(x, a, \gamma, \alpha, \mu, c, p) &= \frac{1}{\Gamma(\mu)} \int_0^{+\infty} e^{-xt} t^{\mu-1} \left(ce^{-pt} - f(t, a, \gamma, \alpha)\right) \, dt.
\end{align*}
\]

The integral representation in (5.2) is implied by both the inequalities \( \mu > 0 \), \( p \geq 0 \), the asymptotics \( f(t, a, \gamma, \alpha) \sim a^\gamma e^{-a^\gamma t} \), \( t \to +\infty \) as well as by the equality \( f(+0, a, \gamma, \alpha) = \zeta(-\gamma, a) > 0 \) (see Theorem 1.1). These relations yield also that \( D_{p}(a, \gamma, \alpha) < +\infty \) if and only if \( p \leq a^\alpha \), and \( E_q(a, \gamma, \alpha) > 0 \) if and only if \( q \geq a^\alpha \). If \( p \leq a^\alpha \) then the function \( e^{pt} f(t, a, \gamma, \alpha) \) strictly decreases with respect to \( t > 0 \). Therefore \( D_p(a, \gamma, \alpha) = \zeta(-\gamma, a) \) for all \( p \leq a^\alpha \) and \( E_{a^{\alpha}}(a, \gamma, \alpha) = a^\gamma \).

Both the Bernstein–Hausdorff–Widder theorem and the relations (5.2) imply that the validity of inequalities (2.9) for every \( \mu > \mu_0 \) and \( x > 0 \) is equivalent to that for the inequalities \( De^{-pt} - f(t, a, \gamma, \alpha) \geq 0 \) and \( Ee^{-qt} - f(t, a, \gamma, \alpha) \leq 0 \) for any \( t > 0 \). The last ones are equivalent to the inequalities \( D \geq D_p(a, \gamma, \alpha) \) and \( E \leq E_q(a, \gamma, \alpha) \) respectively.

Let \( 0 \leq p \leq a^\alpha \), \( D \geq D_p(a, \gamma, \alpha) \), and let \( q \geq a^\alpha \), \( E \leq E_q(a, \gamma, \alpha) \). If either the right-hand side or the left-hand side inequality in (2.9) turns into an equality for some \( x \geq 0 \) (or for \( x > 0 \) if \( p = 0 \)), then the integral representation (5.2) yields that either \( D \equiv e^{pt} f(t, a, \gamma, \alpha) \) or \( E \equiv e^{qt} f(t, a, \gamma, \alpha) \) for \( t > 0 \), which is impossible (see Lemma 5.1). Theorem 2.2 is complete.
Proof of Theorem 2.3

For $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$, $\mu > \max\{\frac{\gamma}{\alpha}; 0\}$, $p \geq 0$ and $c \in \mathbb{R}$ define the function

$$\tilde{\psi}(x,a,\gamma,\alpha,\mu,c,p) := \frac{c}{(p+x)^\mu} - \tilde{S}(x,a,\gamma,\alpha,\mu), \quad x > 0.$$ 

It is easy to check that for any $x > 0$ and $k \in \mathbb{Z}_+$ there hold the relations:

$$(-1)^k \frac{d^k}{dx^k} \left\{ \tilde{\psi}(x,a,\gamma,\alpha,\mu,c,p) \right\} = \frac{\Gamma(\mu+k)}{\Gamma(\mu)} \tilde{\psi}(x,a,\gamma,\alpha,\mu+k,c,p),$$

$$\tilde{\psi}(x,a,\gamma,\alpha,\mu,c,p) = \frac{1}{\Gamma(\mu)} \int_{+\infty}^{+\infty} e^{-xt} t^{\mu-1} \left( c e^{-pt} - \tilde{f}(t,a,\gamma,\alpha) \right) dt.$$  

(5.3)

The integral representation in (5.3) follows from both the inequalities $\mu > \max\{\frac{\gamma}{\alpha}; 0\}$, $p \geq 0$, the asymptotics $\tilde{f}(t,a,\gamma,\alpha) \sim a^\gamma e^{-a^\alpha t}$, $t \to +\infty$ as well as from the equality $\tilde{f}(+0,a,\gamma,\alpha) = \tilde{\zeta}(\gamma,a)$ (see Theorem 1.2). These relations yield also that $0 < C_p(a,\gamma,\alpha) < +\infty$ for $p \leq a^\alpha$ and $C_p(a,\gamma,\alpha) = +\infty$ for $p > a^\alpha$, and that $F_q(a,\gamma,\alpha) > -\infty$ as well.

Both the Bernstein–Hausdorff–Widder theorem and the equalities (5.3) imply that the validity of inequalities (2.12) for any $\mu > \mu_0$ and $x > 0$ is equivalent to that for the inequalities $Ce^{-pt} - \tilde{f}(t,a,\gamma,\alpha) \geq 0$ and $Fe^{-qt} - \tilde{f}(t,a,\gamma,\alpha) \leq 0$ for any $t > 0$. The last ones are equivalent to the inequalities $C \geq C_p(a,\gamma,\alpha)$ and $F \leq F_q(a,\gamma,\alpha)$ respectively.

Let $0 \leq p \leq a^\alpha$, $C \geq C_p(a,\gamma,\alpha)$, and let $q \geq 0$, $F \leq F_q(a,\gamma,\alpha)$. If either the right-hand side or the left-hand side inequality in (2.12) becomes the equality for some $x \geq 0$ (or for $x > 0$ if $p = 0$ or $q = 0$) then the representation (5.3) implies that either $C \equiv e^{pt} \tilde{f}(t,a,\gamma,\alpha)$ or $F \equiv e^{qt} \tilde{f}(t,a,\gamma,\alpha)$ for $t > 0$, which is impossible due to Lemma 5.1.

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