Deducing Physical Properties of Weakly Bound States from Low-Energy Scattering Data

Application to $^{16}O \leftrightarrow ^{12}C + \alpha$

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1. Bound ↔ scattering states: $^{16}$O ↔ $^{12}$C+$\alpha$ example

2. Bound ↔ scattering states: potential model
   - using several partial waves
   - using a single partial wave

3. Bound ↔ scattering states: effective-range expansion (ERE)
   - Test on $^{16}$O+N potential models
   - Test on $^{12}$C+$\alpha$ potential model
   - Analysis of $^{12}$C+$\alpha$ experimental scattering data

4. Conclusions and perspectives
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4 Conclusions and perspectives
$^{16}\text{O}$ subthreshold bound states $\leftrightarrow ^{12}\text{C}+\alpha$ low-energy scattering

- Two weakly bound states below $^{12}\text{C}+\alpha$ threshold ($\hbar = 2\mu = e = 1$)
  - $1^-$: $E_b = -\kappa_b^2 = -45$ keV
  - $2^+$: $E_b = -\kappa_b^2 = -245$ keV, $\in \alpha$-cluster $0^+_2$ rotational band
- Slowly decreasing $^{12}\text{C}+\alpha$ channel radial wave function
  \[ R_b(r) \sim r \rightarrow \infty C_b \exp(-\kappa_br) / r|\eta_b| + 1 \]
  - $|\eta_b| = Z_1Z_2/2\kappa_b$
  - Asymptotic Normalization Constant (ANC) $C_b$
- Astrophysical interest: impact on $^{12}\text{C} (\alpha, \gamma) ^{16}\text{O}$ at 300 keV
- Indirect access to ANC: radiative cascade, $\alpha$ transfer, $^{16}\text{N} \beta$ decay...
- Here: $^{12}\text{C}+\alpha$ elastic-scattering phase shifts $\delta_l(E)$ for $l = 1, 2$
1. **Bound ↔ scattering states: \(^{16}\text{O} \leftrightarrow ^{12}\text{C} + \alpha\) example**

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3. **Bound ↔ scattering states: effective-range expansion (ERE)**
   - Test on \(^{16}\text{O} + \text{N}\) potential models
   - Test on \(^{12}\text{C} + \alpha\) potential model
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4. **Conclusions and perspectives**
$^{12}\text{C} + \alpha \ l = 2$ potential models from phase-shift inversion (I)

- One phase shift...
  - fitted on experimental data
    - [Tischhauser et al., PRL 2002]
    - [Buchmann, private com. 2004]
  - with two resonances removed

- An infinity of potentials...
  - arbitrary binding energies; here: $E_b = -245 \text{ keV}$ (experiment)
  - arbitrary ANCs; here:
    - $C_b = 20, 200, 2000 \times 10^3 \text{ fm}^{-1/2}$

- All potentials are equal, but some potentials are more equal than others:
  - only one fits the $0^+, 2^+, 4^+$ \(\alpha\)-cluster rotational band
    \[ C_b \approx 145(10) \times 10^3 \text{ fm}^{-1/2} \]

[Sparenberg, PRC 2004]
$^{12}\text{C} + \alpha \ l = 2$ potential models from phase-shift inversion (II)

- All potentials are equal, but some potentials are more equal than others:
  - only one has a shorter nuclear range
  \[ C_b \approx 190(10) \times 10^3 \text{ fm}^{-1/2} \] (preliminary)

- For a potential decreasing faster than \( \exp(-2\kappa_b r) \), both \( E_b = -\kappa_b^2 \) and \( C_b \) can be deduced from the corresponding partial-wave scattering matrix
  
  \[ \text{[Blokhintsev et al., PAN 2008]} \]

- Possibility to avoid potential model (cf $^{16}\text{O} \ 1^-\text{ state}$)?
Bound ↔ scattering states: effective-range expansion (ERE)

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Definitions

- Scattering matrix (charged and neutral cases, partial wave $l$):
  - phase shifts ($E = k^2 > 0$): $S_l(k) = e^{2i\sigma_l} e^{2i\delta_l}$
  - bound states ($E_b = -\kappa_b^2 < 0$): $S_l(k) \sim_{k \to i\kappa_b} (-1)^{l+1} i e^{-\pi \eta_b} \frac{|C_b|^2}{k - i\kappa_b}$

- Effective-range function
  
  $K_l(k^2) = F^{-1}_l(k^2) + \frac{2w_l(\eta^2)}{l!2a_B^{2l+1}} h(\eta^2)$,
  
  $K_l(k^2) = F^{-1}_l(k^2) + ik^{2l+1}$

  - nuclear Bohr radius $a_B = 2/Z_1 Z_2$
  - Sommerfeld parameter $\eta = 1/a_B k$
  - $F_l(k^2) \propto e^{2i\delta_l} - 1 \Rightarrow$ phase shift: $K_l(k^2) = k^{2l+1} \cot \delta_l(k) (\eta = 0)$
  - analytic $\Rightarrow$ Taylor expansion: $K_l(k^2) = -\frac{1}{a_l} + \frac{r_l}{2} k^2 - P_l r_l^3 k^4 + O(k^6)$
    (scattering length, effective range, shape parameter...)

- $F_l(k^2) \propto e^{2i\delta_l} - 1 \Rightarrow$ poles of $S_l$ and $F_l$ coincide

  - condition on bound-state energy: $F^{-1}_l(-\kappa_b^2) = 0$
  - condition on bound-state ANC [Iwinski et al., PRC 1984]
    
    $|C_b| = \kappa_b^l \frac{\Gamma(l + 1 + |\eta_b|)}{l!} \left[ -\frac{dF^{-1}_l}{dk^2} \right]_{\kappa^2 = -\kappa_b^2}^{-\frac{1}{2}}$
$^{16}\text{O}+\text{N}$ Woods-Saxon potential: diffuseness $a$, variable depth

- **Bound-state energy (first order ERE)**
  \[
  \frac{1}{a_l} \approx \left( -\frac{r_l}{2} + \frac{1}{6a_B^{2l-1}l!^2} \right) \kappa_b^2
  \]

- **Bound-state ANC (first order ERE)**
  \[
  |C_b| \frac{l!}{\Gamma(l+1+|\eta_b|)} \approx \kappa_b^{l+1} \sqrt{a_l} \propto \kappa_b^l
  \]

- **Application:** $^{17}\text{F} \ s_{1/2}$ state at $-105$ keV
  - 5% error on $a_0$
    - (0.4% with second order)
  - 0.1% error on ANC (from $a_0$)

- **Conclusion:** weakly-bound-state ANC can be deduced from ERE

[Sparenberg et al., PRC 2010]
Test on $^{12}\text{C}+\alpha \ l = 2$ potential model: $V_{\text{nuc}}(r) = -112.332e^{-(r/2.8)^2}$

[cf Buck and Rubio, JPG 1985]

- Quick convergence of ER function on [0-5] MeV and of bound-state properties: **third order** sufficient
- ANC very sensitive to slope

| ERE order | $C_b \ (10^3 \ \text{fm}^{-1/2})$ |
|-----------|----------------------------------|
| 0         | 10.9                             |
| 1         | imaginary!                       |
| 2         | 89.7                             |
| 3         | 138.2                            |
| 4         | 138.5                            |
| exact     | 138.4                            |

- Phase shifts very sensitive: requires **fifth order** above 2 MeV
Analysis of $^{12}$C+$\alpha$ experimental scattering data

**Fit of experimental data [Tischhauser et al., PRL 2002, PRC 2009]**

- Narrow 2.7 MeV resonance removed
- Polynomial fit of ERF $K_2$ on [1.95-3.1] MeV (92 points)
- Starting values of parameters: Gaussian-potential ERF
- Bound-state constraint: $E_b = -245$ keV
  $\Rightarrow$ one parameter ($a_2$) less
- No strong constraint on ANC

| order | ERF $\chi^2$ | $C_b \left(10^3 \text{ fm}^{-1/2}\right)$ |
|-------|--------------|------------------------------------------|
| 2     | 7.0          | 19.7                                     |
| 3     | 4.1          | 31.0                                     |
| 4     | 4.1          | 179.2                                    |
| 5     | 4.1          | 151.2                                    |
Conclusions

- Bound-state ANC can be deduced from single-partial-wave scattering phase shift but only for weak binding energy
- $^{12}\text{C} + \alpha$ $2^+$ subthreshold bound state ANC
  - third-order effective-range expansion sufficient, in principle
  - no strong constraint (yet?) from $l = 2$ phase shift only
  - rather well constrained by rotational-band potential model:
    \[ C_b \approx 145(10) \times 10^3 \text{ fm}^{-1/2} \]

Perspectives

- Use ERE as background phase shift in R-matrix fit
- Use Effective-Range Padé Expansion (instead of Taylor) $\Rightarrow$ resonances
- $^{12}\text{C} + \alpha$ $1^-$ subthreshold bound state ANC