HIGHER HASSE–WITT MATRICES

MASHA VLASENKO

Abstract. For a multivariate polynomial \( f(x) \) with coefficients in a ring \( R \) we construct a sequence of matrices with entries in \( R \) whose reductions modulo \( p \) give iterates of the Hasse–Witt operation for the hypersurface of zeroes of the reduction of \( f(x) \) modulo \( p \). We show that our matrices satisfy a system of congruences modulo powers of \( p \). If the Hasse–Witt operation is invertible these congruences yield \( p \)-adic limit formulas, which conjecturally describe the Gauss–Manin connection and the Frobenius operator on the attached unit-root F-crystal.

Contents

1. Introduction and main results 1
2. Lemmas on congruences for the powers of \( f(x) \) 5
3. Proof of Theorem 1 7
4. Constructing formal group laws from a Laurent polynomial 8
5. Atkin and Swinnerton-Dyer congruences 9
6. Unit-root formulas in families: an example 10
7. Variation of hypersurfaces and A-hypergeometric differential equations 11
References 13

1. Introduction and main results

Let \( X/F_q \) be a smooth projective variety of dimension \( n \) over a finite field \( F_q \), \( q = p^n \). The Katz congruence formula ([1]) states that modulo \( p \) the zeta function of \( X \) is described as

\[
Z(X/F_q; T) \equiv \prod_{i=0}^{n} \det(1 - T \cdot F^i | H^i(X, \mathcal{O}_X))^{(-1)^{i+1}} \mod p,
\]

where \( H^i(X, \mathcal{O}_X) \) is the Čech cohomology of \( X \) with the coefficients in the structure sheaf \( \mathcal{O}_X \) and \( F \) is the Frobenius map, the \( p \)-linear vector space map induced by \( h \mapsto h^p \) on the structure sheaf \( (p \text{-linear means } F(bs + ct) = b^p F(s) + c^p F(t) \text{ for } b, c \in F_q \text{ and } s, t \in H^i(X, \mathcal{O}_X)) \). When \( X \) is a complete intersection the only interesting term in formula (1.1) is given by \( H^n(X, \mathcal{O}_X) \). The action of \( F \) on this space is classically known as the Hasse–Witt operation.

The following algorithm (see [2 §7.10], [1] Corollary 6.1.13 or [3 §II.1]) can be used to compute the Hasse–Witt matrix of a hypersurface \( X \subset \mathbb{P}^{n+1} \) given by a homogeneous equation \( f(x_0, \ldots, x_{n+1}) = 0 \) of degree \( d > n + 2 \). We extend the Frobenius to a transformation of the exact sequence of sheaves on \( \mathbb{P}^{n+1} \):

\[
0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0
\]

\[
\downarrow f^{p-1} F \quad \downarrow F \quad \downarrow F
\]

\[
0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0.
\]

The coboundary in the resulting long exact cohomology sequence allows to identify

\[
H^n(X, \mathcal{O}_X) \cong H^{n+1}(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(-d)),
\]

so that the Frobenius \( F \) on \( H^n(X, \mathcal{O}_X) \) corresponds to the map on \( H^{n+1}(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(-d)) \) induced by

\[
0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^{n+1}}(-pd) \xrightarrow{f^{p-1}} \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to 0.
\]

Date: September 20, 2016.
Computing Čech cohomology we find that Laurent monomials \(x^{-u} = x_{0}^{-u_{0}} \ldots x_{n+1}^{-u_{n+1}}\) where \(u\) runs through the set

\[
U = \{u = (u_{0}, \ldots, u_{n+1}) : u_{i} \in \mathbb{Z}_{\geq 1}, \sum_{i=0}^{n+1} u_{i} = d\}
\]

form a basis in \(H^{n+1}(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(-d))\) and the Hasse–Witt matrix is given in this basis by

\[
F_{u,v \in U} = \text{the coefficient of } x^{p^{v-u}} \text{ in } f(x)^{p-1}.
\]

In this paper we study a sequence of matrices which generalize (1.3). Let \(R\) be a commutative ring with 1 and \(f \in R[\mathbb{x}_{1}^{\pm 1}, \ldots, \mathbb{x}_{N}^{\pm 1}]\) be a Laurent polynomial in \(N\) variables, which we will write as

\[
f(x) = \sum_{u} a_{u} x^{u}, \quad a_{u} \in R,
\]

where the summation runs over a finite set of vectors \(u \in \mathbb{Z}^{N}\). The support of \(f\) is the set of exponents of monomials in \(f\), which we denote by \(\text{supp}(f) = \{u : a_{u} \neq 0\}\). The Newton polytope \(\Delta(f) \subset \mathbb{R}^{N}\) is the convex hull of \(\text{supp}(f)\).

Consider the set of internal integral points \(J = \Delta(f)^{o} \cap \mathbb{Z}^{N}\), where \(\Delta(f)^{o}\) denotes the topological interior of the Newton polytope. Let \(g = \# J\) be the number of internal integral points in the Newton polytope, which we assume to be positive. Consider the following sequence of \(g \times g\) matrices \(\{\beta_{m}; m \geq 0\}\) with entries in \(R\) whose rows and columns are indexed by the elements of \(J\):

\[
\beta_{m}(u,v) = \text{the coefficient of } x^{(m+1)u-v} \text{ in } f(x)^{m}.
\]

By convention, \(\beta_{0}\) is the identity matrix.

Let us fix a prime number \(p\). We restrict our attention to the sub-sequence \(\{\alpha_{s} = \beta_{p^{-s} - 1}; s \geq 0\}\). The entries of these matrices are then given by

\[
(\alpha_{s})_{u,v \in J} = \text{the coefficient of } x^{p^{v-u}} \text{ in } f(x)^{p^{-s} - 1}.
\]

Notice that when \(R/pR\) is a finite field and \(f\) is a homogeneous polynomial of degree \(d\) such that its reduction modulo \(p\) defines a smooth hypersurface, then \(U\) in (1.2) coincides with \(J\) (with \(N = n + 2\)) and \(\alpha_{1} = \beta_{p-1}\) modulo \(p\) is the Hasse–Witt matrix.

We shall investigate \(p\)-adic properties of the sequence \(\{\alpha_{s}\}\). Assume that \(R\) is endowed with a \(p\)th power Frobenius endomorphism, that is a ring endomorphism \(\sigma : R \rightarrow R\) satisfying

\[
\sigma(a) \equiv a^{p} \mod p, \quad a \in R.
\]

Below we apply \(\sigma\) to \(g \times g\) matrices with entries in \(R\) entry-wise. This is an endomorphism of the ring of matrices but not a \(p\)th power Frobenius endomorphism: property (1.6) will not be satisfied in general for matrices of size \(g > 1\). In Sections 2 and 3 we prove the following

**Theorem 1.**

(i) For every \(s \geq 1\)

\[
\alpha_{s} \equiv \alpha_{1} \cdot \sigma(\alpha_{1}) \ldots \cdot \sigma^{s-1}(\alpha_{1}) \mod p.
\]

(ii) Assume that all \(\alpha_{s}\) are invertible over the \(p\)-adic closure \(\hat{R} = \lim_{\leftarrow i} R/p^{i}R\). Then

\[
\alpha_{s+1} \cdot \sigma(\alpha_{s})^{-1} \equiv \alpha_{s} \cdot \sigma(\alpha_{s-1})^{-1} \mod p^{s}
\]

for every \(s \geq 1\).

(iii) Under the condition of (ii), for any derivation \(D : R \rightarrow R\) one has

\[
D(\sigma^{m}(\alpha_{s+1})) \cdot \sigma^{m}(\alpha_{s+1})^{-1} \equiv D(\sigma^{m}(\alpha_{s})) \cdot \sigma^{m}(\alpha_{s})^{-1} \mod p^{s+m}
\]

for any \(s, m \geq 0\).

Note that (i) allows us to interpret \(\alpha_{s}\) mod \(p\) as the matrix of the \(s\)th iterate of the Hasse–Witt operation. In (iii) derivation \(D\) is applied to matrices entry-wise, and this way \(D\) is also a derivation of the ring of matrices \(\text{Mat}_{g \times g}(R)\). In (ii) and (iii) it is unessential that we multiply by the inverse matrices on the right. By the same arguments as in the proof given in Section 3 the respective congruences hold if one multiplies on the left, that is

\[
\sigma(\alpha_{s})^{-1} \cdot \alpha_{s+1} \equiv \sigma(\alpha_{s-1})^{-1} \cdot \alpha_{s} \mod p^{s},
\]

\[
\sigma^{m}(\alpha_{s+1})^{-1} \cdot D(\sigma^{m}(\alpha_{s+1})) \equiv \sigma^{m}(\alpha_{s})^{-1} \cdot D(\sigma^{m}(\alpha_{s})) \mod p^{s+m}.
\]
Theorem 1 implies existence of the $p$-adic limits
\begin{equation}
F = \lim_{s \to \infty} \alpha_{s+1} \cdot \sigma(\alpha_s)^{-1}
\end{equation}
and
\begin{equation}
\nabla_D = \lim_{s \to \infty} D(\alpha_s) \cdot \alpha_s^{-1}
\end{equation}
for every derivation $D \in \text{Der}(R)$.

These $g \times g$ matrices have entries in $\hat{R}$. Note that $F \equiv \alpha_1 \mod p$ is the Hasse–Witt matrix. It would be interesting to understand the meaning of the limiting matrices $F$ and $\nabla_D$ for $D \in \text{Der}(R)$. We will allow ourselves to formulate a conjecture based on the parallelism of Theorem 1 with the main result of [4].

To sketch the relation with [4], let $X$ be a smooth projective hypersurface over $S = \text{Spec}(R)$ of relative dimension $n$. Suppose we are also given an $S$-valued point $P \in X(S)$ and a set of formal coordinates $(T_1, \ldots, T_n)$ along $P$. For a differential form $\omega \in H^0(X, \Omega^n_{X/S})$ one considers the formal expansion
\[
\omega = \sum_{u \in \mathbb{Z}_p^n} a_u(\omega) T^n \prod_{i=1}^n \frac{dT_i}{T_i}.
\]
Assuming that $H^0(X, \Omega^n_{X/S})$ is a free $R$-module of rank $g$, one chooses any set of multi-indices $u_1, \ldots, u_g$ such that the map
\[
\omega \mapsto (a_{u_1}(\omega), \ldots, a_{u_g}(\omega))
\]
provides an isomorphism of $H^0(X, \Omega^n_{X/S})$ with $R^g$. Next, one chooses a basis of forms $\omega_1, \ldots, \omega_g$ so that $a_u(\omega_j) = \delta_{ij}$ for $0 \leq i, j \leq g$. After those choices are made, a sequence of $g \times g$ matrices $\{E_s; s \geq 0\}$ with entries in $R$ is defined as
\[
(E_s)_{ij} = a_{p^s u_i}(\omega_j)
\]
for $1 \leq i, j \leq g$.

Here $E_0$ is the identity matrix and $E_1 \mod p$ is the matrix of the Cartier operator on $H^0(X_0, \Omega^n_{X_0/S_0})$ where $S_0 = \text{Spec}(R/pR)$ and $X_0 = X \times_{\text{Spec}(S)} \text{Spec}(S_0)$ is the reduction of our hypersurface modulo $p$. According to [4, Theorem 6.2] under the condition of invertibility of the Cartier operator, these matrices satisfy congruences
\begin{equation}
\begin{aligned}
E_s &\equiv \sigma(s-1)(E_1) \cdot \ldots \cdot \sigma(E_1) \cdot E_1 \mod p

E_{s+1}^{-1} \cdot \sigma(E_s) &\equiv E_s^{-1} \cdot \sigma(E_{s-1}) \mod p^s

E_{s+1}^{-1} \cdot D(E_{s+1}) &\equiv E_s^{-1} \cdot D(E_s) \mod p^s
\end{aligned}
\end{equation}
for any $D \in \text{Der}(R)$ and the respective $p$-adic limits
\[
\lim_{s \to \infty} E_{s+1}^{-1} \cdot \sigma(E_s) \quad \text{and} \quad \lim_{s \to \infty} E_{s+1}^{-1} \cdot D(E_s)
\]
describe the Frobenius operator and the Gauss–Manin connection on the dual $U_0^\vee$ to the unit-root $F$-crystal $U_0 \subset H^n_{\text{cryst}}(X_0)$ in the basis given by the projections $p^s(\omega_i)$, $i = 1, \ldots, g$ of the chosen differential forms. Here crystalline cohomology $H^n_{\text{cryst}}(X_0)$ is identified with de Rham cohomology $H^n_{DR}(X/S) \otimes R$, where $U_0$ is a sub-module and $U_0^\vee$ is represented as a quotient of the latter module via the Poincaré duality (see [4, §1-2]).

The Cartier operator is dual to the Hasse–Witt operation, and our congruences from Theorem 1 resemble a transposed version of (1.9). Therefore we expect the $p$-adic limits (1.7) and (1.8) to have the following meaning.

**Conjecture.** Let $X \subset \mathbb{P}^{n+1}$ be the hypersurface of zeroes of a homogeneous polynomial $f \in R[x_0, \ldots, x_n]$. Assume that $R$, $f$, and $p$ satisfy the conditions under which the unit-root $F$-crystal $U_0 \subset H^n_{\text{cryst}}(X_0)$ is defined (see [4]; in particular it means that the Hasse–Witt matrix $\alpha_1 \mod p$ is invertible and therefore the limiting matrices in (1.7) and (1.8) exist). Then the $p$-adic limits $F$ and $\nabla_D$ for $D \in \text{Der}(R)$ describe the Frobenius operator and the Gauss–Manin connection on $U_0$ in the basis $\{\omega_u, u \in J = \Delta(f)^c \cap \mathbb{Z}^{n+1}\}$, where $\omega_u \in H^0(X, \Omega^n_{X/S})$ is represented by the form
\begin{equation}
\frac{x^n}{f(x)} \sum_{i=0}^{n+1} (-1)^i \frac{d x_0}{x_0} \wedge \frac{d x_i}{x_i} \wedge \ldots \wedge \frac{d x_{n+1}}{x_{n+1}} \in H^0(X, \Omega^n_{X/S})
\end{equation}
in the description of $H^n_{DR}(X)$ by Griffiths (see [5] and [6, §3.2]).
For example, let $X$ be an elliptic curve given in the Weierstrass form
\[ y^2 = x^3 + ax + b, \quad a, b \in \mathbb{R}. \]
Consider the parameter $T = -x/y$ at the origin. Then the coefficients in the formal expansion
\[ \frac{1}{2y} \frac{dx}{dy} = \left(1 + 2aT^4 + 3bT^6 + 6a^2T^8 + 20abT^{10} + (20a^3 + 15b^2)T^{12} + \ldots\right)dT = \sum_{m=0}^{\infty} c_m T^m dT \]
are given by
\[ c_m = \begin{cases} 
\text{the coefficient of } x^m \text{ in } (x^3 + ax + b)^{m/2}, & m \text{ even} \\
0, & m \text{ odd}
\end{cases} \]
(see [8] Example (0.4)). Our sequence $\{\beta_m\}$ is given by
\[ \beta_m = \text{the coefficient of } x^m y^m \text{ in } (x^3 + ax + b - y^2)^m = (-1)^{m/2} \left(\frac{m}{m/2}\right) c_m. \]
For each $p \neq 2$ the sequence of Katz (with the obvious choices) is given by $E_s = c_{p-1}$ and our sequence is given by $\alpha_s = (-1)^{(p-1)/2} \left(\frac{p-1}{2}\right) E_s$. They differ by a $p$-adic unit, and we have $D(\alpha_s) \alpha_{s-1} = D(E_s) E_{s-1}$ and $\alpha_{s} \cdot \sigma(\alpha_{s-1})^{-1} \equiv E_s \cdot \sigma(E_{s-1})^{-1} \bmod p^s$. Therefore in this simple example our $p$-adic limits coincide with those of Katz.

This paper is organized as follows. In Sections 2 and 3 we prove Theorem 1. Sections 5, 6 and 7 are devoted to various statements supporting the above conjecture. In Section 5 we show that when $R$ is the ring of integers of the unramified extension of $\mathbb{Q}_p$ of degree $a$ and the reduction of $X$ modulo $p$ is a smooth hypersurface $X_0$, then the eigenvalues of the matrix $\Phi = F \cdot \sigma(F) \cdot \ldots \cdot \sigma^{a-1}(F)$ with $F$ given by (1.4) are $p$-adic unit eigenvalues of the Frobenius operator on the middle crystalline cohomology of $X_0$. To prove this statement we combine Theorem 1 with the generalized Atkin and Swinnerton-Dyer congruences due to Stienstra. Matrices (1.4) showed up in Stienstra’s computation of the logarithms of certain coordinaizations of the Artin–Mazur formal groups (see [7]). The proof of the generalized Atkin and Swinnerton-Dyer congruences in [8] exploits the relation between the Artin–Mazur functors and crystalline cohomology. As the second main result of this paper we construct formal group laws over $R$ starting from a Laurent polynomial $f \in R[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$.

**Theorem 2.** Let $R$ be a ring of characteristic 0, that is the natural map $R \to R \otimes \mathbb{Q}$ is an embedding. Let $f \in R[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ be a Laurent polynomial. Let $J$ be either the set $\Delta(f) \cap \mathbb{Z}^N$ of all integral points in the Newton polytope of $f$ or the subset of integral points $\Delta(f)^c \cap \mathbb{Z}^N$. Assume that $J$ is non-empty and let $g = \# J$. Consider the sequence of matrices $\beta_m \in \text{Mat}_{g \times g}(R)$, $m \geq 0$ given by formula (1.4) and define a $g$-tuple of formal powers series $l(\tau) = (l_u(\tau))_{u \in J}$ in $g$ variables $\tau = (\tau_u)_{u \in J}$ as
\[ l(\tau) = \sum_{m=1}^{\infty} \frac{1}{m} \beta_{m-1} \tau^m. \]
Consider the $g$-dimensional formal group law $G_f(\tau, \tau') = l^{-1}(l(\tau) + l(\tau'))$ with coefficients in $R \otimes \mathbb{Q}$.

Let $p$ be a prime number. We denote by $R_{(p)} = R \otimes \mathbb{Z}(p)$ the subring of $R \otimes \mathbb{Q}$ formed by elements without $p$ in the denominator. Suppose that $R$ can be endowed with a $p$th power Frobenius endomorphism. Then $G_f \in R_{(p)}[[\tau, \tau']]$. We prove this theorem in Section 3. Similarly to Theorem 1, the proof uses combinatorial methods, and therefore we have little assumptions on the ring $R$ and polynomial $f$ in Theorem 2. Note that if one can define a Frobenius endomorphism on $R$ (or perhaps, on a larger ring) for every prime $p$ then Theorem 2 implies that $G_f \in R[[\tau, \tau']]$ because the subring $\cap_p R_{(p)} \subset R \otimes \mathbb{Q}$ coincides with $R$.

Sections 3 and 4 are devoted to the relation of our matrices $\alpha_s$ with the Gauss–Manin connection. In Section 5 we compute the limits (1.1) and (1.3) for the Legendre family of elliptic curves. In that section $R = \mathbb{Z}[t]$ and we observe that $\alpha_s(t)$ satisfies the Picard–Fuchs differential equation of this family modulo $p^s$. In Section 6 we develop this idea further and prove that for a family of smooth projective hypersurfaces of fixed degree each matrix entry $\alpha_s(u, v)$ is annihilated modulo $p^s$ by differential operators that annihilate the class (1.10) in the de Rham cohomology. We work in a setting where all coefficients of the hypersurface are variables. In this case the Gauss–Manin connection can be described by an $A$-hypergeometric system of differential equations (see [7]). One can obtain the same result for specific families performing the change of variables in our statement.
Acknowledgements. I am grateful to Alan Adolphson and Steven Sperber for numerous useful discussions which led to essential improvements in this paper. Particularly, Section appeared entirely as a result of our communication. I would like to thank John Voight and Susanne Müller for their remarks and questions.

2. Lemmas on congruences for the powers of \( f(x) \)

Let \( R \) be a commutative ring with 1 and \( p \) be a prime number. Assume that \( R \) is endowed with a \( p \)-th power Frobenius endomorphism, that is we have a ring endomorphism \( \sigma : R \rightarrow R \) such that \( \sigma(a) \equiv a^p \mod pR \) for every \( a \in R \).

**Lemma 3.** For \( a \in R \) we define a sequence of elements \( \delta_s = \delta_s(a) \in R \), \( s = 1, 2, \ldots \) by the recursive formula

\[
a^{p^s-1} = \delta_1(a) \cdot \sigma(a^{p^{s-1}-1}) + \delta_2(a) \cdot \sigma^2(a^{p^{s-2}-1}) + \ldots + \delta_s(a) .
\]

Then

(i) \( \delta_s(a) \in p^{s-1}R \) for every \( s \geq 1 \);
(ii) \( a^{mp^{s-1}} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{mp^{s-1}-1}) \equiv a^{p^s-1} \mod p^{s-1}R \)

Proof. For \( s = 1 \) we have \( \delta_1(a) = a^{p-1} \) and (i) holds trivially. For higher \( s \) we prove (i) by induction. We have

\[
\delta_s = a^{p^s-1} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{p^{s-1}-1}) = a^{p^s-1} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{p^{s-1}-1}) \cdot \sigma^i(a^{p^s-1}p^{s-1}) \equiv a^{p^s-1} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{p^{s-1}-1}) \cdot a^{p^s-p^{s-1}} \mod p^{s-1}R
\]

where the congruence in the middle row above holds for every \( i \) for the following reason. We notice that \( \sigma^i(a^{p^{s-1}}) \equiv a^{p^s} \mod pR \), then we raise this congruence \( s - 1 - i \) times to the power \( p \) and get

\[
\sigma^i(a^{p^{s-1}}p^{s-1-i}) \equiv a^{p^s-p^{s-1}} \mod p^{s-1}R .
\]

By induction assumption \( \delta_i \in p^{i-1}R \), and hence

\[
\delta_i \cdot \sigma^i(a^{p^{s-1}}p^{s-1-i}) \equiv \delta_i \cdot a^{p^{s-1}}p^{s-1} \mod p^{s-1}R .
\]

We prove (ii) in a similar manner:

\[
a^{mp^{s-1}} = \sum_{i=1}^{s} \delta_i(a) \cdot \sigma^i(a^{mp^{s-1}-1}) = a^{mp^{s-1}} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{mp^{s-1}-1}) - \delta_s(a) \cdot \sigma^s(a^{m-1}) \equiv a^{mp^{s-1}} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{mp^{s-1}-1}) - \delta_s(a) \cdot a^{(m-1)p^s} \]

( because \( \sigma^s(a^{m-1}) \equiv a^{(m-1)p^s} \mod pR \) and \( \delta_s(a) \in p^{s-1}R \) by (i) )

\[
= a^{mp^{s-1}} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{mp^{s-1}-1}) - a^{p^s-1} - \sum_{i=1}^{s-1} \delta_i(a) \cdot \sigma^i(a^{p^s-1-i}) \cdot a^{(m-1)p^s} \]

\[
= \sum_{i=1}^{s-1} \delta_i(a) \cdot \left( \sigma^i(a^{p^{s-1}-1}) \cdot a^{(m-1)p^s} - \sigma^i(a^{mp^{s-1}-1}) \right) \]

\[
\equiv \sum_{i=1}^{s-1} \delta_i(a) \cdot \left( \sigma^i(a^{p^{s-1}-1}) \cdot a^{(m-1)p^{s-1-i}} - \sigma^i(a^{mp^{s-1}-1}) \right) \]

( \( \sigma^i(a^{m-1}) \equiv a^{(m-1)p^s} \mod pR \Rightarrow \sigma^i(a^{m-1})p^{s-i} \equiv a^{(m-1)p^s} \mod p^{s-i+1}R \Rightarrow \delta_i(a) \cdot \sigma^i(a^{m-1})p^{s-i} \equiv \delta_i(a) \cdot a^{(m-1)p^s} \mod p^{s-i+1}R \) since \( \delta_i(a) \in p^{i-1}R \) )

\[
= \sum_{i=1}^{s-1} \delta_i(a) \cdot \left( \sigma^i(a^{p^{s-1}-1}) \cdot a^{p^{s-i}(m-1)} - \sigma^i(a^{mp^{s-1}-1}) \right) .
\]

\[\square\]
Let \( R' = \mathbb{R}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}] \) be the ring of Laurent polynomials in \( N \) variables. We extend \( \sigma \) to a Frobenius endomorphism of \( R' \) by assigning \( \sigma(x_i) = x_i^p \) for \( 1 \leq i \leq N \). Let \( f \in R' \) and \( \Delta(f) \subset \mathbb{R}^N \) is the Newton polytope of \( f \). Lemma 3 can be applied in the ring \( R' \) and we obtain a sequence of Laurent polynomials \( \{\delta_i(f); s \geq 1\} \). The following lemma gives an estimate for their Newton polytopes.

**Lemma 4.** \( \Delta(\delta_s(f)) \subset (p^s - 1)\Delta(f) \) for \( s \geq 1 \).

**Proof.** The statement obviously holds for \( s = 1 \) since \( \delta_1(f) = f^{p^{-1}}. \) For higher \( s \) it follows by induction because each term in the sum on the right in

\[
\delta_s(f) = f^{p^{-s}} - \sum_{i=1}^{s-1} \delta_i(f) \cdot \sigma^i(f^{p^{-i-1}})
\]

has its Newton polytope inside \( (p^s - 1)\Delta(f) \). Indeed, this is obvious for \( f^{p^{-s}} \) and for each \( 1 \leq i \leq s - 1 \) one has \( \Delta(\sigma^i(f^{p^{-i-1}})) \subset p^i \cdot (p^s - 1)\Delta(f) \) and

\[
\Delta(\delta_s(f) \cdot \sigma^i(f^{p^{-i-1}})) \subset (p^i - 1) + p^i \cdot (p^s - 1)\Delta(f) = (p^s - 1)\Delta(f).
\]

We assume that the set of integral points \( J = \Delta(f) \cap \mathbb{Z}^N \) is non-empty and let \( g = \#J \). The endomorphism \( \sigma \in \text{End}(R) \) naturally extends to an endomorphism of the ring of \( g \times g \) matrices with entries in \( R \) (we simply apply it to each matrix entry). We will denote this extension by the same letter \( \sigma \in \text{End}(\text{Mat}_{g \times g}(R)) \). However it is not a Frobenius endomorphism any more: the property \( \sigma(\alpha) \equiv \alpha^p \text{ mod } p \) will not hold in general for \( \alpha \in \text{Mat}_{g \times g}(R) \) when \( g > 1 \).

Recall that for \( m \geq 0 \) matrices

\[
(\beta_m)_{u,v \in J} = \text{the coefficient of } x^{(m+1)u-v} \text{ in } f(x)^m
\]

were defined in (1.4). By convention, \( \beta_0 \) is the identity matrix. We also use the notation \( \alpha_s = \beta_{p^{-s-1}}, s \geq 0 \) as in (1.5).

**Lemma 5.** For \( s \geq 1 \) consider \( g \times g \) matrices given by

\[
(\gamma_s)_{u,v \in J} = \text{the coefficient of } x^{p^sv-u} \text{ in } \delta_s(f).
\]

We have

(i) \( \gamma_s \in p^{s-1}\text{Mat}_{g \times g}(R) \) for \( s \geq 1 \);

(ii) \( \alpha_s = \gamma_1 \cdot \sigma(\alpha_{s-1}) + \gamma_2 \cdot \sigma^2(\alpha_{s-2}) + \ldots + \gamma_{s-1} \cdot \sigma^{s-1}(\alpha_1) + \gamma_s \) for \( s \geq 1 \);

(iii) for \( m, s \geq 1 \)

\[
\beta_{mp^{-s-1}} = \sum_{i=1}^{s} \gamma_i \cdot \sigma^i(\beta_{mp^{-i-1}}) \in p^s\text{Mat}_{g \times g}(R).
\]

**Proof.** (i) is clear since \( \delta^s(f) \in p^{s-1}R' \) by (i) in Lemma 3. To prove (ii) consider the identity

\[
(f^{p^{-s}} - 1)^s = \sum_{i=0}^{s} \delta_i(f) \cdot \sigma^i(f^{p^{-s-i-1}}).
\]

Let \( u, v \in J \). In order to compute the coefficient of \( x^{p^su-v} \) in \( \delta_i(f) \cdot \sigma^i(f^{p^{-s-i-1}}) \) we are interested in pairs of vectors \( w \in \text{supp}(\delta_i(f)) \) and \( \tau \in \text{supp}(f^{p^{-s-i-1}}) \) such that

\[
w + \tau = p^s(v-u).
\]

By Lemma 3 we have \( w \in (p^i - 1)\Delta(f) \), and since \( u \in \Delta(f)^o \) it follows that \( w + u \in p^i \Delta(f)^o \). Moreover, (2.3) implies that \( w + u \in p^s\mathbb{Z}^N \), and therefore \( \frac{1}{p}(w + u) \in \Delta(f)^o \cap \mathbb{Z}^N = J \). On the other hand, for every \( \mu \in J \) vectors

\[
w = p^i\mu - u, \quad \tau = p^{s-i}v - \mu
\]

satisfy (2.3). It follows that the coefficient of \( x^{p^su-v} \) in \( \delta_i(f) \cdot \sigma^i(f^{p^{-s-i-1}}) \) is equal to \( \sum_{\mu \in J}(\gamma_i)_{u,\mu} \sigma^i(\alpha_{s-i})_{\mu,v} \), and (ii) now follows from (2.2).

We prove (iii) in a similar vein. By (ii) in Lemma 3 we have

\[
(f^{mp^{-s-1}} - 1)^s = \sum_{i=0}^{s} \delta_i(f) \cdot \sigma^i(f^{mp^{-s-i-1}}) \in p^sR'.
\]

In order to compute the coefficient of \( x^{mp^s-u} \) in \( \delta_i(f) \cdot \sigma^i(f^{mp^{-s-i-1}}) \) we look at pairs of vectors \( w \in \text{supp}(\delta_i(f)) \) and \( \tau \in \text{supp}(f^{mp^{-s-i-1}}) \) such that \( w + \tau = mp^sv - u \). The same argument as above
shows that \( \mu = (w + u)/p^i \in \Delta(f)^\circ \cap \mathbb{Z}^N = J \). With this \( \mu \) we can rewrite \( w = p^i \mu - u, \tau = mp^{s-i}v - \mu \), and (iii) thus follows from (2.4).

**Remark 6.** It is clear from the proof of Lemma 5 that we could use a larger set \( \tilde{J} = \Delta(f) \cap \mathbb{Z}^N \) of all integral points in the Newton polytope of \( f \) instead of the set of internal integral points \( J = \Delta(f)^\circ \cap \mathbb{Z}^N \). The statement of Lemma 5 then holds for the sequences of larger matrices \( \{ \tilde{\beta}_m, m \geq 0 \}, \{ \tilde{\alpha}_s, s \geq 0 \} \) and \( \{ \tilde{\gamma}_s, s \geq 1 \} \) defined by formulas (1.4), (1.3) and (2.1) respectively with \( \tilde{J} \) instead of \( J \).

3. PROOF OF THEOREM 1

Our main tools for the proof will be Lemma 5, parts (i) and (ii). We will also need the following observation relating derivations and the Frobenius endomorphism.

**Lemma 7.** For any derivation \( D : R \to R \) one has \( D(\sigma^m(a)) \in p^mR \) for all \( a \in R \) and \( m \geq 1 \).

**Proof.** Since \( \sigma(a) = a^p + pb \) for some \( b \in R \), then
\[
D(\sigma(a)) = D(a^p) + D(b) = p(a^{p-1}D(a) + D(b)) \in pR,
\]
which proves the statement for \( m = 1 \). We will do induction on \( m \). If the statement holds for \( m - 1 \) then
\[
D(\sigma^m(a)) = D(\sigma^{m-1}(a^p + pb)) = D(\sigma^{m-1}(a)^p) + pD(\sigma^{m-1}(b)) = p\sigma^{m-1}(a)^{p-1}D(\sigma^{m-1}(r)) + pD(\sigma^{m-1}(b)) \in p^mR
\]
since both \( D(\sigma^{m-1}(a)) \) and \( D(\sigma^{m-1}(b)) \) belong to \( p^{m-1}R \).

**Proof of Theorem 7** By (i)–(ii) in Lemma 5 we have \( \alpha_s \equiv \gamma_1 \cdot \sigma(\alpha_{s-1}) \mod p \). Iteration yields
\[
\alpha_s \equiv \gamma_1 \cdot \sigma(\gamma_1) \cdot \ldots \cdot \sigma^{s-1}(\gamma_1) \mod p,
\]
and part (i) follows immediately since \( \alpha_1 = \gamma_1 \).

We will prove (ii) by induction on \( s \). We shall show that
\[
\alpha_{s+1} \cdot \sigma(\alpha_s)^{-1} \equiv \alpha_s \cdot \sigma(\alpha_{s-1})^{-1} \mod p^s.
\]
The case \( s = 1 \) follows from part (i). Let us substitute the recursive expressions for \( \alpha_s \) and \( \alpha_{s+1} \) from (ii) in Lemma 5 into the two sides of the desired congruence:
\[
\alpha_{s+1} \cdot \sigma(\alpha_s)^{-1} = \gamma_1 + \sum_{j=2}^{s+1} \gamma_j \cdot \sigma^j(\alpha_{s+1-j}) \cdot \sigma(\alpha_s)^{-1},
\]
\[
\alpha_s \cdot \sigma(\alpha_{s-1})^{-1} = \gamma_1 + \sum_{j=2}^{s} \gamma_j \cdot \sigma^j(\alpha_{s-j}) \cdot \sigma(\alpha_{s-1})^{-1}.
\]

Since we want to compare these two expressions modulo \( p^s \) and \( \gamma_{s+1} \equiv 0 \mod p^s \), the last term in the upper sum can be ignored. For every \( j = 2, \ldots, s \) we use the inductive assumption as follows:
\[
\alpha_s \sigma(\alpha_{s-1})^{-1} \equiv \alpha_{s-1} \sigma(\alpha_{s-2})^{-1} \mod p^{s-1}
\]
\[
\alpha_{s-1} \sigma(\alpha_{s-2})^{-1} \equiv \alpha_{s-2} \sigma(\alpha_{s-3})^{-1} \mod p^{s-2}
\]
\[
\vdots
\]
\[
\alpha_{s+2-j} \sigma(\alpha_{s+1-j})^{-1} \equiv \alpha_{s+1-j} \sigma(\alpha_{s-j})^{-1} \mod p^{s+1-j}
\]
We then apply the respective power of \( \sigma \) to each row and multiply these congruences out to get that modulo \( p^{s+1-j} \)
\[
\alpha_s \sigma^{j-1}(\alpha_{s+1-j})^{-1} \equiv \alpha_s \sigma(\alpha_{s-1})^{-1} \sigma(\alpha_{s-1}) \sigma^2(\alpha_{s-2})^{-1} \ldots \sigma^{j-1}(\alpha_{s+1-j})^{-1}
\]
\[
\equiv \alpha_{s-1} \sigma(\alpha_{s-2})^{-1} \sigma(\alpha_{s-2}) \sigma^2(\alpha_{s-3})^{-1} \ldots \sigma^{j-1}(\alpha_{s-j})^{-1} = \alpha_{s-1} \sigma^{j-1}(\alpha_{s-j})^{-1}.
\]
By our assumption, determinants of these matrices are units in \( \tilde{R} \). Hence we can invert them to get (3.1)
\[
\sigma^{j-1}(\alpha_{s+1-j}) \alpha_{s-1} \equiv \sigma^{j-1}(\alpha_{s-j}) \alpha_{s-1}^{-1} \mod p^{s+1-j}.
\]
Now we apply \( \sigma \) and multiply by \( \gamma_j \). Since \( \gamma_j \equiv 0 \mod p^{j-1} \) we get
\[
\gamma_j \sigma^j(\alpha_{s+1-j}) \sigma(\alpha_{s-1})^{-1} \equiv \gamma_j \sigma^j(\alpha_{s-j}) \sigma(\alpha_{s-1})^{-1} \mod p^s.
\]
Summation in \( j \) gives the desired result (ii).
For (iii) we shall show that

\[ D(\sigma^m(\alpha_{s+1})) \cdot \sigma^m(\alpha_{s+1})^{-1} \equiv D(\sigma^m(\alpha_s)) \cdot \sigma^m(\alpha_s)^{-1} \mod p^{s+m} \]

for every \( s, m \geq 0 \). For \( s = 0 \) the statement is true with any \( m \): the right-hand side vanishes since \( D(1) = 0 \) and the entries of the left-hand side belong to \( p^m R \) by Lemma [7]. We will now do induction on \( s \). Substituting the recursive expressions for \( \alpha_s \) and \( \alpha_{s+1} \) from (ii) in Lemma [5] we can write

\[
D(\sigma^m(\alpha_s))\sigma^m(\alpha_s)^{-1} = \sum_{i=1}^{s} D(\sigma^m(\gamma_i)) \sigma^m(\sigma^i(\alpha_{s-i})\alpha_{s-i}^{-1}) + \sum_{i=1}^{s} \sigma^m(\gamma_i) D(\sigma^{m+i}(\alpha_{s-i})) \sigma^m(\alpha_s)^{-1},
\]

(3.2)

\[
D(\sigma^m(\alpha_{s+1}))\sigma^m(\alpha_{s+1})^{-1} = \sum_{i=1}^{s+1} D(\sigma^m(\gamma_i)) \sigma^m(\sigma^i(\alpha_{s+1-i})\alpha_{s+1-i}^{-1}) + \sum_{i=1}^{s+1} \sigma^m(\gamma_i) D(\sigma^{m+i}(\alpha_{s+1-i})) \sigma^m(\alpha_{s+1})^{-1}.
\]

Consider the terms with \( i = s + 1 \) in the latter identity. In the first sum this term vanishes modulo \( p^{s+m} \) because the entries of \( \gamma_{s+1} \) are in \( p^s R \), thus Lemma [7] implies that the entries of \( D(\sigma^m(\gamma_{s+1})) \) belong to \( p^{s+m} R \). In the second sum this term vanishes since \( D(\sigma^{m+s+1}(\alpha_0)) = D(\alpha_0) = 0 \). Now take any \( 1 \leq i \leq s \). The respective terms in the first sums of both identities are equal modulo \( p^{s+m} \) because

\[ D(\sigma^m(\gamma_i)) \equiv 0 \mod p^{m+i-1} \]

by Lemmas [5] i) and 7 and

\[ \sigma^i(\alpha_{s-i})\alpha_{s-i}^{-1} \equiv \sigma^i(\alpha_{s+1-i})\alpha_{s+1-i}^{-1} \mod p^{s+i-1} \]

as a consequence of part (ii) of this theorem (e.g. take \( \gamma_{i+1} \) with \( i + 1 \) and \( s + 1 \) instead of \( j \) and \( s \) respectively). It remains to compare the terms with index \( i \) in the second sums of the two identities in (3.2). We have

\[
D(\sigma^{m+i}(\alpha_{s+1-i}))\sigma^{m+i}(\alpha_{s+1-i})^{-1} \equiv D(\sigma^{m+i}(\alpha_{s-i}))\sigma^{m+i}(\alpha_{s-i})^{-1} \mod p^{s+m}
\]

( both \( \equiv 0 \mod p^{m+i} \) by Lemma [7]),

\[
\sigma^{m+i}(\alpha_{s+1-i})\sigma^m(\alpha_{s+1})^{-1} \equiv \sigma^{m+i}(\alpha_{s-i})\sigma^m(\alpha_s)^{-1} \mod p^{s+i-1}.
\]

Here the first congruence follows from the inductional assumption and the last one follows from part (ii) of this theorem. Multiplying the above congruences we obtain

\[
D(\sigma^{m+i}(\alpha_{s+1-i}))\sigma^{m+i}(\alpha_{s+1})^{-1} \equiv D(\sigma^{m+i}(\alpha_{s-i}))\sigma^m(\alpha_s)^{-1} \mod p^{s+m}.
\]

Multiplying both sides by \( \sigma^m(\gamma_i) \) we see that the respective terms in (3.2) are congruent modulo \( p^{s+m+i-1} \) (which is even better than we need whenever \( i > 1 \)). This accomplishes the proof of the inductional step. \( \square \)

4. CONSTRUCTING FORMAL GROUP LAWS FROM A LAURENT POLYNOMIAL

In this section we prove Theorem [2]. The proof is based on Hazewinkel's functional equation lemma ([14] §10.2), the conditions of which are satisfied due to (iii) in Lemma [5].

Proof of Theorem [2] Recall that \( R \) is a characteristic zero ring, that is the natural map \( R \to R \otimes \mathbb{Q} \) is injective. We assume there is a \( p \)-th power Frobenius endomorphism \( \sigma : R \to R \), which we extend to \( R \otimes \mathbb{Q} \) by linearity. We consider the case \( J = \Delta(f) \cap \mathbb{Z}^N \) first. Let \( g = \#J \) and \( \{ \gamma_s; s \geq 1 \} \) is the sequence of \( g \times g \) matrices from Lemma [5]. Put \( \mu_s = \frac{1}{p^s} \gamma_s \). Then \( \mu_s \in \text{Mat}_{g \times g}(R) \) by (i) in Lemma [5].

We shall now check that each power series in the tuple

\[ h(\tau) = l(\tau) - \frac{1}{p} \sum_{s=1}^{\infty} \mu_s (\sigma^s l)(\tau) \]

has coefficients in \( R(p) \). Here \( \sigma \) extends to \( (R \otimes \mathbb{Q})[\tau] \) by assigning \( \sigma(\tau_u) = \tau_u^p \) for each \( u \in J \), and it then acts on tuples of power series coordinate-wise. For any \( k \geq 1 \) we write \( k = mp^r \) where \( (m, p) = 1 \).
Then
\[ \text{the coefficient of } \tau^k \text{ in } h_\alpha(\tau) = \frac{1}{k}(\beta_{k-1})_{u,v} - \frac{1}{p} \sum_{s=1}^{r} \sum_{w \in J} (\mu_s)_{u,w} \frac{1}{mp^r-s}(\beta_{mp^r-s-1})_{w,v} \]
\[ = \frac{1}{mp^r}(\beta_{mp^r-1})_{u,v} - \frac{1}{p} \sum_{s=1}^{r} \sum_{w \in J} (\gamma_s)_{u,w}(\beta_{mp^r-s-1})_{w,v} \]
\[ = \frac{1}{mp^r}(\beta_{mp^r-1} - \sum_{s=1}^{r} \gamma_s \cdot \beta_{mp^r-s-1})_{u,v} \in R(p) \]
by Lemma 5 (iii). Since \( h(\tau) \in (R(p)[\tau])^\theta \) and the Jacobian matrix of \( h(\tau) \) is the identity matrix, Hazewinkel’s functional equation lemma [10] \( \S 10.2(i) \) implies that \( G_f(\tau, \tau') = l^{-1}(l(\tau) + l(\tau')) \in R(p)[\tau, \tau'] \).

In the case \( J = \Delta(f) \cap \mathbb{Z}^N \) the proof is exactly the same but using bigger matrices \( \{\gamma_s; s \geq 1\} \). see Remark 6.

5. Atkin and Swinnerton-Dyer congruences

In this section we assume that \( R \) is the ring of integers of the unramified extension of \( \mathbb{Q}_p \) of degree \( a \). We then have \( R/pR = F_q, q = p^a \) and the Frobenius endomorphism \( \sigma: R \to R \) satisfies \( \sigma^a = Id \).

We also assume that the polynomial \( f \in R[x_1, \ldots, x_n] \) is homogeneous of degree \( d \geq N \). To distinguish this situation, we renumber our coordinates as \( x = (x_0, \ldots, x_{n+1}) \) where \( n = N - 2 \) in agreement with the beginning of Section 1.

If the projective hypersurface \( X = \{f(x) = 0\} \subset \mathbb{P}^{n+1} \) is smooth then the set \( J = \Delta(f)^c \cap \mathbb{Z}^{n+2} \) coincides with \((12)\) and \( g = \#J = \binom{d+1}{n+1} \). By \([1] \text{ Theorem 1}\) the formal group law constructed in \( \text{Theorem 2} \) using \( g \times g \) matrices \( \{\beta_m; m \geq 0\} \) is a coordinization of the Artin–Mazur formal group \( H^n(X, \mathbb{G}_m, X) \). Using the relation between Artin–Mazur functors and crystalline cohomology Stienstra proved in [8] the following generalized version of the Atkin and Swinnerton-Dyer congruences.

Suppose that the reduction \( X_0 = X \times_{\text{Spec} R} \text{Spec} (R/pR) \) is non-singular and consider the reciprocal characteristic polynomial of the \( q \)th power Frobenius operator on the middle crystalline cohomology of \( X_0 \)
\[ \det(1 - T \cdot F_q[H^{\dag}_{\text{crys}}(X_0)]) = 1 + c_1 T + \cdots + c_k T^k \in \mathbb{Z}[T]. \]
By [5] Theorem 0.1 and Remark 0.5 there exists a constant \( c \) such that
\[ \beta_{m-1} + c_1 \beta_{m/q-1} + c_2 \beta_{m/q^2-1} + \cdots + c_k \beta_{m/q^k-1} \equiv 0 \mod p^{\text{ord}_p(m)-c} \]
whenever \( \text{ord}_p(m) \) is sufficiently large.

Recall our notation \( \alpha_s = \beta_{p^s-1} \). Combined with Theorem 1 congruences (5.2) yield the following

**Proposition 8.** Suppose that the hypersurface \( X_0 = X \times_{\text{Spec} R} \text{Spec} (R/pR) \) is non-singular and the Hasse–Witt operation on \( H^n(X_0, \mathcal{O}_{X_0}) \) is an automorphism. Consider the \( g \times g \) matrix with entries in \( R \) given as the \( p \)-adic limit \( F = \lim_{s \to \infty} \alpha_s \cdot (\sigma(\alpha_s))^{-1} \), which exists by Theorem 4. Then the matrix
\[ \Phi = F \cdot \sigma(F) \cdot \ldots \cdot \sigma^{a-1}(F) \]
satisfies the characteristic equation
\[ \Phi^k + c_1 \Phi^{k-1} + \cdots + c_k = 0 \]
of the \( q \)th power Frobenius operator on the middle crystalline cohomology of \( X_0 \) defined in (5.1).

**Proof.** By (5.2) we have
\[ \alpha_s + c_1 \alpha_s-a + c_2 \alpha_{s-2a} + \ldots + c_k \alpha_{s-ka} \equiv 0 \mod p^{s-c} \]
for all sufficiently large \( s \). Multplying by \( \alpha_{s-ka}^{-1} \) on the right and letting \( s \to \infty \) we obtain the desired result because \( \Phi = \lim_{s \to \infty} \alpha_s \alpha_{s-ka}^{-1} \).

Note that under the conditions of the above proposition \( \det(\Phi) \) is a \( p \)-adic unit. Therefore the eigenvalues of \( \Phi \) are \( p \)-adic unit eigenvalues of the Frobenius operator on the middle crystalline cohomology of the hypersurface \( X_0 \).

By the Katz congruence formula [14] the number of \( p \)-adic unit Frobenius eigenvalues on the middle cohomology equals to the stable rank of the Hasse–Witt matrix. The Hasse–Witt matrix is invertible for a generic hypersurface (see [11] and [3]). Therefore for a generic hypersurface the number of \( p \)-adic
unit Frobenius eigenvalues is equal to the size of the matrix $\Phi$. The conjecture in Section 1 would imply that the multiplicities of eigenvalues should also coincide.

Let us give an example. Consider the hyperelliptic curve $C$ given by

$$y^2 = x^5 + 2x^2 + x + 1.$$  

Higher Hasse–Witt matrices here will have size $2 \times 2$:

$$\beta_n = \text{the coefficients of } \begin{pmatrix} x^n y^n & x^{2n+1} y^n \\ x^{n-1} y^n & x^{2n} y^n \end{pmatrix} \text{ in } (y^2 - x^5 - 2x^2 - x - 1)^n.$$  

For example, with $p = 11$ we have

| $s$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\alpha_s = \beta_{p^s - 1}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -81144 & -1260 \\ -81900 & -1260 \end{pmatrix}$ | ... | ... |
| $\text{tr}(\alpha_s \cdot \alpha_{s-1}^{-1}) \mod p^s$ | $8 + O(11)$ | $8 + 11 + O(11^2)$ | $8 + 11 + 11^2 + O(11^3)$ |
| $\det(\alpha_s \cdot \alpha_{s-1}^{-1}) \mod p^s$ | $7 + O(11)$ | $7 + 6 \cdot 11 + O(11^2)$ | $7 + 6 \cdot 11 + 3 \cdot 11^2 + O(11^3)$ |

Using Kedlaya’s algorithm we computed the reciprocal characteristic polynomial of the Frobenius on the first crystalline cohomology of the curve reduced modulo 11:

$$\det \left( 1 - T \cdot F_{11} \mid H^1_{\text{crys}}(C_{11}) \right) = 1 + 3T + 18T^2 + 3 \cdot 11T^3 + 11^2T^4 = (1 + 4T + 11T^2)(1 - T + 11T^2).$$

The eigenvalues of the Frobenius operator are

$$\lambda_{1,2} = -2 \pm \sqrt{-7}, \quad \lambda_{3,4} = \frac{1 \pm \sqrt{-43}}{2}.$$  

Both $-7$ and $-43$ are squares modulo 11, and 11-adic unit eigenvalues are

$$\lambda_1 = 7 + 2 \cdot 11 + 2 \cdot 11^2 + O(11^3)$$

$$\lambda_3 = 1 + 10 \cdot 11 + 9 \cdot 11^2 + O(11^3)$$

We see that in the above table traces and determinants converge to

$$\lambda_1 + \lambda_3 = 8 + 11 + 11^2 + O(11^3)$$

$$\lambda_1 \cdot \lambda_3 = 7 + 6 \cdot 11 + 3 \cdot 11^2 + O(11^3)$$

respectively.

In the case when $d = n + 2$ the geometric genus $g = \dim H^n(X, \mathcal{O}_X) = 1$, so we have at most one $p$-adic unit Frobenius eigenvalue and the equality of multiplicities follows automatically:

**Proposition 9 (p-adic unit-root formula).** Let $f \in R[x_0, \ldots, x_{n+1}]$ be a homogeneous polynomial of degree $n + 2$ such that the reduction of $f(x)$ modulo $p$ defines a smooth hypersurface $X_0$ over the finite field $\mathbb{F}_q$, $q = p^s$. For $s \geq 0$ let $\alpha_s \in R$ be the coefficient of $(x_0 \ldots x_{n+1})^{p^s - 1}$ in $f(x)^{p^s - 1}$. The Frobenius operator $F_0$ on the middle crystalline cohomology of $X_0$ has a $p$-adic unit eigenvalue $\lambda$ if and only if $\alpha_1 \not\equiv 0 \mod p$, and we have

$$\lambda \equiv \alpha_{s+a} \cdot \alpha_{s-1}^{-1} \mod p^{s+1}$$

for all $s \geq 0$.

6. UNIT-ROOT FORMULAS IN FAMILIES: AN EXAMPLE

Let us see how Theorem 1 works for the Legendre family of elliptic curves

$$y^2 = x(x - 1)(x - t).$$

This old example was a starting point for Dwork to suggest the relation between periods and local zeta functions (see [12 §5], [13 §8]). We know what to expect here, however we prefer to consider the Legendre family once again for the formulas are simple in this case and at the same time we will be able to see that our theorem brings some new features into the picture.
Let $R = \mathbb{Z}[t]$ and the Frobenius endomorphism is given by $\sigma(h(t)) = h(t^p)$. We have

$$\beta_m = \beta_m(t) = \begin{cases} 0, & m \text{ odd}, \\ (m/2) \cdot \sum_{k=0}^{(m/2)} (m/2)_k^2 t^k, & m \text{ even}. \end{cases}$$

For every prime $p \neq 2$ we see that $a_1(t) = \beta_{p-1}(t) = \big( \frac{p-1}{p-1/2} \big) + O(t)$ is invertible in $\hat{R} = \mathbb{Z}_p[t]$, and hence by Theorem 1 the $p$-adic limits $\nabla(t) = \lim_{s \to \infty} \alpha_s(t)/\alpha_s(t)$ and $F(t) = \lim_{s \to \infty} \alpha_s(t)/\alpha_{s-1}(t^p)$ must exist in $\mathbb{Z}_p[[t]]$. We claim that

$$\nabla(t) = \frac{g(t)'}{g(t)} \text{ and } F(t) = (-1)^{\frac{p-1}{2}} \frac{g(t)}{g(t^p)}$$

where

$$g(t) = 1 + \frac{1}{4} t + \frac{9}{64} t^2 + \ldots = \frac{1}{\pi} \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}} = \sum_{k=0}^{\infty} \left(\frac{2k}{k}\right)^2 \left(\frac{1}{16}\right)^k \in \mathbb{Z}_p[[t]]$$

is a period of the Legendre family. To prove our claim we notice that

$$\left(\frac{p-1/2}{k}\right) \equiv \left(\frac{-1/2}{k}\right) \equiv \left(\frac{2k}{4k}\right)^2 \mod p^{s-\text{ord}_p(k)}.$$

This congruence is easy to prove: since $a_k = k! \left(\frac{n-1}{k}\right)$ and $b_k = k! \left(\frac{-1}{k}\right)$ are $p$-adic integers satisfying $a_{k+1} = \left(\frac{p-1}{p-1/k}\right) a_k$ and $b_{k+1} = \left(\frac{-1}{k+1/2}\right) b_k$ respectively, it follows by induction on $k$ that $a_k \equiv b_k \mod p^s$. Since $\text{ord}_p(k!) \leq k/(p-1)$, we have

$$\sum_{k \geq 0} \left(\frac{p-1/2}{k}\right)^2 t^k \equiv g(t) \mod (t^s, t^{1/2+s}) \mathbb{Z}_p[[t]].$$

Now (6.1) follows from (6.2) and the fact that $\left(\frac{p-1/2}{k}\right)/\left(\frac{p-1-1/2}{k}\right) \equiv \left(\frac{p-1}{p-1/k}\right) \mod p^s$. Though we used rather weak congruences to compute the limits in (6.1), it now follows from Theorem 1 that actually

$$\alpha_s(t) \equiv \frac{g(t)'}{g(t)} \text{ and } \frac{\alpha_s(t)}{\alpha_{s-1}(t^p)} \equiv \left(\frac{-1}{p-1/2}\right) \frac{g(t)}{g(t^p)} \mod p^s \mathbb{Z}_p[[t]]$$

for every $s \geq 1$.

The congruence on the right in (6.3) gives a $p$-adic approximation of the function $\lambda(t) = \left(\frac{-1}{p-1/2}\right) g(t)/g(t^p)$ by rational functions. This approximation can be used for $p$-adic analytic continuation of $\lambda(t)$ beyond the interior of the $p$-adic unit disk (see [14] Theorem 3 and [15] Lemma 1.3) and by Proposition 3 the values of $\lambda(t)$ at the respective Teichmüller points give the $p$-adic unit roots for the members of the Legendre family.

Being a period of the Legendre family, function $g(t)$ is annihilated by the respective Picard–Fuchs differential operator

$$L = t(1-t) \frac{d^2}{dt^2} + (1-2t) \frac{d}{dt} - \frac{1}{4}.$$

An elementary check shows that $\alpha_s(t)$ is a mod-$p^s$-solution to the Picard–Fuchs differential equation, that is

$$L \alpha_s \equiv 0 \mod p^s \mathbb{Z}_p[[t]].$$

Indeed, $L \sum c_k t^k = \sum ((k+1)^2 c_{k+1} - (k+1/2)^2 c_k) t^k$ and with $c_k = \left(\frac{p-1}{k}\right)^2$ the expression in brackets is $O(p^s)$ for each $k$. This is a special feature of our $p$-adic approximation (6.3), and in the next section we prove it holds in general for the entries of matrices $\alpha_s$.

7. Variation of Hypersurfaces and $\Lambda$-Hypergeometric Differential Equations

Consider a homogeneous polynomial of degree $d$ in $n + 2$ variables with indeterminate coefficients

$$f_\Lambda(x_0, \ldots, x_{n+1}) = \sum_{k=1}^M \Lambda_k x^a_k.$$

We write $a_k = (a_{0k}, \ldots, a_{(n+1)k}) \in \mathbb{Z}^{n+2}$ with $\sum_{i=0}^{n+1} a_{ik} = d$. For each $k$ we denote

$$a_k^+ = (a_{0k}, \ldots, a_{(n+1)k}, 1) \in \mathbb{Z}^{n+3}.$$
We define the lattice of relations on the set $A = \{a_k^+\}_{k=1}^M$

\[(7.1) \quad L = \{l = (l_1, \ldots, l_M) \in \mathbb{Z}^M \mid \sum_{k=1}^M l_k a_k^+ = 0\}.\]

The $A$-hypergeometric system of partial differential equations associated to the set $A = \{a_k^+\}_{k=1}^M$ with the set of parameters $\mu = (\mu_0, \ldots, \mu_{n+2}) \in \mathbb{C}^{n+3}$ consists of box operators

\[(7.2) \quad \Box_l = \prod_{l_k > 0} \left( \frac{\partial}{\partial \Lambda_k} \right)^{l_k} - \prod_{l_k < 0} \left( \frac{\partial}{\partial \Lambda_k} \right)^{-l_k}, \quad l \in L\]

and Euler (or homogeneity) operators

\[(7.3) \quad Z_i = \sum_{k=1}^M a_k^+ \Lambda_k \frac{\partial}{\partial \Lambda_k} - \mu_i, \quad i = 0, \ldots, n + 2.\]

**Lemma 10.** Fix any pair of vectors $v, u \in \mathbb{Z}^{n+2}$ and an integer $m \geq 1$. Let $u^+ = (u, 1), v^+ = (v, 1) \in \mathbb{Z}^{n+3}$. The polynomial

\[h(\Lambda) = \text{the coefficient of } x^{mv-u} \text{ in } f_\Lambda(x)^{m-1}\]

is annihilated by the box operators \((7.2)\) for every $l \in L$ and by the Euler operators \((7.3)\) with the parameter $\mu = m v^+ - u^+$.

**Proof.** It is easy to see that

\[h(\Lambda) = \sum_{\sum_{e_k} a_k^+ = m v^+ - u^+} \frac{(m-1)!}{\prod_{k=1}^M e_k!} \Lambda_1^{e_1} \cdots \Lambda_M^{e_M}.\]

The assertion for the Euler operators follows from the fact that for every term in the above sum one has

\[\left( \sum_{k=1}^M a_{ik} \Lambda_k \frac{\partial}{\partial \Lambda_k} \right) \Lambda_1^{e_1} \cdots \Lambda_M^{e_M} = \left( \sum_{k=1}^M a_{ik} e_k \right) \Lambda_1^{e_1} \cdots \Lambda_M^{e_M} = \left( m v^+_i - u^+_i \right) \Lambda_1^{e_1} \cdots \Lambda_M^{e_M}.\]

Let us fix a nonzero $l \in L$ and $w = (w_1, \ldots, w_M) \in \mathbb{Z}_{\geq 0}^M$. For $1 \leq k \leq M$ we denote

\[e_k^+ = w_k + \max(l_k, 0), \quad e_k^- = w_k + \max(-l_k, 0).\]

Note that $e_k^+ - e_k^- = l_k$. The coefficient near $\Lambda^w$ in $\Box_l h$ is zero if the sum $\sum_{k=1}^M e_k^+ > \sum_{k=1}^M e_k^-$ exceeds $m - 1$ or if the sum $\sum_{k=1}^M e_k^+ \Lambda_k^j \sum_{k=1}^M e_k^- \Lambda_k^j$ is not equal to $m v^+ - u^+$. If both conditions are satisfied then the coefficient near $\Lambda^w$ is equal to

\[\frac{(m-1)!}{\prod_{k=1}^M e_k^+! \prod_{k.1k<0} w_k!} - \frac{(m-1)!}{\prod_{k=1}^M e_k^-! \prod_{k.1k<0} w_k!} = \frac{(m-1)!}{\prod_{k=1}^M w_k!} - \frac{(m-1)!}{\prod_{k=1}^M w_k!} = 0.\]

Here we used the observation that $e_k^+ = w_k$ whenever $l_k \leq 0$ and $e_k^- = w_k$ whenever $l_k \geq 0$ respectively. \(\square\)

Assume that the set of integral internal points of the polytope $\Delta = \text{span}\{a_1, \ldots, a_M\}$ coincides with

\[U = \{u = (u_0, \ldots, u_{n+1}) : u_i \in \mathbb{Z}_{\geq 1}, \quad \sum_{i=0}^{n+1} u_i = d\}.\]

Under this condition a generic hypersurface in our family $f_\Lambda(x) = 0$ is smooth. For a smooth projective hypersurface of dimension $n$ over a field of characteristic 0 given by a homogeneous equation $f(x_0, \ldots, x_{n+1}) = 0$ of degree $d$ Griffiths showed (see [5] and [6, §3.2]) that the primitive middle cohomology can be described as the quotient of the group of $(n+1)$-forms $g(x)\Omega/f(x)^m$ where $\Omega = \sum_{i=0}^{n+1} (-1)^i x_i dx_0 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_{n+1},$ $m \geq 0$ and $g(x)$ is a homogeneous polynomial of degree $dm - n - 2,$ by the subgroup generated by

\[(7.4) \quad \frac{\partial g(x)}{\partial x_i} \frac{\Omega}{f(x)^m} = m g(x) \frac{\partial f(x)}{\partial x_i} \frac{\Omega}{f(x)^{m+1}},\]
for all $m > 0$, all homogeneous polynomials $g(x)$ of degree $md - n - 1$ and all $i \in \{0, \ldots, n + 1\}$. Consider classes $[\omega_u]$ in the middle cohomology of fibres of our family represented by the forms

$$\omega_u = \frac{x^u}{f_{\Lambda}(x)} \sum_{i=0}^{n+1} (-1)^i \frac{dx_0}{x_0} \wedge \cdots \wedge \frac{dx_i}{x_i} \wedge \frac{dx_{n+1}}{x_{n+1}}, \quad u \in U.$$

These classes already appeared in (1.10) in Section 11. Differential operators $\partial/\partial A_i$ act on cohomology classes via the Gauss–Manin connection and it can be shown (see [16, Remark 2]) that $[\omega_u]$ satisfies the A-hypergeometric system associated to the set $\mathcal{A} = \{a_k^+ \}_{k=1}^M$ with the parameter $\mu = -u^+$. Indeed, for any box operator (72) of degree $r = \sum l_k > 0 l_k = -\sum l_k < 0 l_k$ we have

$$\square_l \omega_u = (-1)^r r! \left( \frac{x^{u+\sum l_k a_k}}{f_{\Lambda}(x)^{r+1}} - \frac{x^{u-\sum l_k a_k}}{f_{\Lambda}(x)^{r+1}} \right) \frac{\Omega}{x_0 \cdots x_{n+1}} = 0$$

since $\sum l_k > 0 l_k a_k = -\sum l_k < 0 l_k a_k$ for $l \in L$. For the Euler operators $Z_i$ with $i = 0, \ldots, n + 1$ one has

$$Z_i \omega_u = \left( -\sum_k a_k^+ A_k x^a_k + u_i^+ \right) \frac{x^u}{f_{\Lambda}(x)} \frac{\Omega}{x_0 \cdots x_{n+1}},$$

which is a coboundary (7.4) with $m = 1$ and $g(x) = x^u/(x_0 \cdots x_{n+1})$ when $i \leq n + 1$ and vanishes for $i = n + 2$.

**Proposition 11.** Let $u, v \in U$ and $s \geq 1$. The matrix entry $(\alpha_s)_{u,v} \in Z[\Lambda_1, \ldots, \Lambda_M]$ is a mod-$p^s$ solution to the $A$-hypergeometric system associated to the set $\mathcal{A} = \{a_k^+ \}_{k=1}^M$ with the parameter $\mu = -u^+$.\]

**Proof.** The statement follows from Lemma [11] because $(\alpha_s)_{u,v}$ is the coefficient of $x^v u^s$ in $f_{\Lambda}(x)^{p^s-1}$. □

This statement was proved in [10] for $s = 1$. As we just remarked, the $A$-hypergeometric system from the above proposition is the same one that annihilates the cohomology class $[\omega_u]$.

**References**

[1] N. Katz, *Une formule de congruence pour la fonction $\zeta$*, S.G.A. 7 II, Lecture Notes in Mathematics 340, Springer, 1973

[2] B. Dwork, *On the zeta function of a hypersurface II*, Annals of Math. 80 (1964), pp. 227–299

[3] N. Koblitz, *p-adic variation of the zeta function over families of varieties defined over finite fields*, Comp. Math. 31 (1975), f. 2, pp.119–218

[4] N. Katz, *Internal reconstruction of unit-root F-crystals via expansion-coefficients*, Annales scientifiques de l’ É.N.S. 4e série, tome 18, no. 2(1985), pp. 245–285

[5] Ph. Griffiths, *On the periods of certain rational integrals I, II*, Ann. of Math. 90 (1969), 496–541

[6] T. Abbott, K. Kedlaya and D. Roe, *Bounding Picard numbers of surfaces using p-adic cohomology*, Arithmetic, geometry, and coding theory (AGCT 2005), pp. 125–159, Sém. Congr., 21, Soc. Math. France, Paris, 2010

[7] J. Stienstra, *Formal group laws arising from algebraic varieties*, American Journal of Mathematics, 109, no. 5 (1987), pp. 907–925

[8] J. Stienstra, *Formal groups and congruences for L-functions*, American Journal of Mathematics, 109, no. 6 (1987), pp. 1111–1127

[9] A. Adolphson, S. Sperber, *Dwork cohomology, de Rham cohomology and hypergeometric functions*, American Journal of Mathematics, 122, no. 2 (2000), pp. 319–348

[10] M. Hazewinkel, *Formal groups and applications*, Academic Press, 1978

[11] I. Miller, *Uber gewöhnliche Hyperflächen I*, J. Reine Angew. Math. 282 (1976), pp. 96–113

[12] B. Dwork, *A deformation theory for the zeta function of a hypersurface*, in Proc. Internat. Congr. Math., Stockholm, 1963, pp. 247–259

[13] N. Katz, *Travaux de Dwork*, Sém. Bourbaki 409 (1972), pp. 167–200

[14] B. Dwork, *p-adic cycles*, Publ. Math. Inst. Hautes Études Sci. 37 (1969) 27–116

[15] A. Mellit, M. Vlasenko, *Dwork congruences for the constant terms of powers of a Laurent polynomial*, International Journal of Number Theory, vol. 12, no. 2 (2016), pp. 313–321

[16] A. Adolphson, S. Sperber, *A-hypergeometric series and the Hasse–Witt matrix of a hypersurface*, arXiv:1601.05127 (2016)

E-mail address: nasha.vlasenko@gmail.com

Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-656 Warsaw, Poland