THE COMBINATORICS OF BOREL COVERS

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Abstract. In this paper we extend previous studies of selection principles for families of open covers of sets of real numbers to also include families of countable Borel covers. The main results of the paper could be summarized as follows:

(1) Some of the classes which were different for open covers are equal for Borel covers – Section 1;
(2) Some Borel classes coincide with classes that have been studied under a different guise by other authors – Section 4.

1. Introduction

Let $X$ be a topological space. Let $\mathcal{O}$ denote the collection of all countable open covers of $X$. According to [6] an open cover $U$ of $X$ is said to be an $\omega$-cover if $X$ is not a member of $U$, but for each finite subset $F$ of $X$ there is a $U \in \mathcal{U}$ such that $F \subseteq U$. It is shown in [6] that every $\omega$-cover of $X$ has a countable subset which is an $\omega$-cover of $X$ if, and only if, all finite powers of $X$ have the Lindelöf property. All finite powers of sets of real numbers have the Lindelöf property. The symbol $\Omega$ denotes the collection of all countable $\omega$-covers of $X$. According to [9] and [19] an open cover of $X$ is said to be a $\gamma$-cover if it is infinite and each element of $X$ is a member of all but finitely many members of the cover. Since each infinite subset of a $\gamma$-cover is a $\gamma$-cover, each $\gamma$-cover has a countable subset which is a $\gamma$-cover. The symbol $\Gamma$ denotes the collection of all countable $\gamma$-covers of $X$.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of $X$. The following two selection hypotheses have a long history for the case when $\mathcal{A}$ and $\mathcal{B}$ are collections of topologically significant subsets of a space. Early instances of these can be found in [7] and [17]; many papers since then have studied these selection hypotheses in one form or another.

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$S_1(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of members of $\mathcal{A}$, there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of members of $\mathcal{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that each $B_n$ is a finite subset of $A_n$, and $\cup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

These selection hypotheses are monotonic in the second variable and anti-monotonic in the first. Moreover, each has a naturally associated game:

In the game $G_1(\mathcal{A}, \mathcal{B})$ ONE chooses in the $n$-th inning an element $O_n$ of $\mathcal{A}$ and then TWO responds by choosing $T_n \in O_n$. They play an inning per natural number. A play $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ is won by TWO if $\{T_n : n \in \mathbb{N}\}$ is a member of $\mathcal{B}$; otherwise, ONE wins. If ONE does not have a winning strategy in $G_1(\mathcal{A}, \mathcal{B})$, then $S_1(\mathcal{A}, \mathcal{B})$ holds. The converse is not always true; when it is true, the game is a powerful tool for studying the combinatorial properties of $\mathcal{A}$ and $\mathcal{B}$.

The game $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ is played similarly. In the $n$-th inning ONE chooses an element $O_n$ of $\mathcal{A}$ and TWO responds with a finite set $T_n \subseteq O_n$. A play $O_1, T_1, \ldots, O_n, T_n, \ldots$ is won by TWO if $\cup_{n \in \mathbb{N}} T_n$ is in $\mathcal{B}$; otherwise, ONE wins. As above: If ONE has no winning strategy in $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$, then $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ holds; when the converse is also true the game is a powerful tool for studying $\mathcal{A}$ and $\mathcal{B}$.

A third selection hypothesis, introduced by Hurewicz in [7], is as follows:

$U_{\text{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of members of $\mathcal{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n B_n$ is a finite subset of $A_n$, and either $\cup B_n = X$ for all but finitely many $n$, or else $\{\cup B_n : n \in \mathbb{N}\} \setminus \{X\} \in \mathcal{B}$.

The three classes of open covers above are related: $\Gamma \subseteq \Omega \subseteq \mathcal{O}$. This and the properties of the selection hypotheses lead to a complicated diagram depicting how the classes defined this way interrelate. However, only a few of these classes are really distinct, as was shown in [9] and [19]. Figure 1 (borrowed from [9]) contains the distinct ones among these classes (it is not known if the class $S_{\text{fin}}(\Gamma, \Omega)$ is $U_{\text{fin}}(\Gamma, \Omega)$, or if it contains $U_{\text{fin}}(\Gamma, \Gamma)$). In this diagram, as in the ones to follow, an arrow denotes implication.

Now we consider the following covers of $X$. The symbol $\mathcal{B}$ denotes the family of all countable covers of $X$ by Borel sets; call elements of $\mathcal{B}$ countable Borel covers of $X$. A countable Borel cover of $X$ is said to be a Borel $\omega$-cover of $X$ if $X$ is not a member of it but for each finite subset of $X$ there is a member of the cover which contains the
finite set. The symbol $\mathcal{B}_\Omega$ denotes the collection of Borel $\omega$-covers of $X$. A countable Borel cover of $X$ is said to be a Borel $\gamma$-cover of $X$ if it is infinite and each element of $X$ belongs to all but finitely many members of the cover. The symbol $\mathcal{B}_\Gamma$ denotes the collection of Borel $\gamma$-covers of $X$. It is evident that the following inclusions hold:

$$\mathcal{B}_\Gamma \subseteq \mathcal{B}_\Omega \subseteq \mathcal{B}; \quad \Gamma \subseteq \mathcal{B}_\Gamma; \quad \Omega \subseteq \mathcal{B}_\Omega \text{ and } \mathcal{O} \subseteq \mathcal{B}.$$

On account of these inclusions and monotonicity properties of the selection principles we have: $S_1(\mathcal{B}, \mathcal{B}) \subseteq S_1(\mathcal{O}, \mathcal{O}); \quad S_{\text{fin}}(\mathcal{B}, \mathcal{B}) \subseteq S_{\text{fin}}(\mathcal{O}, \mathcal{O}); \quad U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) \subseteq U_{\text{fin}}(\Gamma, \Gamma); \quad S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \subseteq S_1(\Omega, \Gamma); \quad \text{and so on.}$

The methods of [9] and [19] can be used to show that a diagram obtained from Figure 1 by substituting all the open classes by their corresponding Borel versions summarizes all the interrelationships among these.

But there are big differences about what is provable in these two situations. For example, it has been shown in [9] and [21] that there always is an uncountable set of real numbers in the class $S_1(\Gamma, \Gamma)$ and thus in $U_{\text{fin}}(\Gamma, \Gamma)$. According to a result of [10] it is consistent that no uncountable set of real numbers has property $U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. Thus it is consistent that some of the classes which provably do not coincide in the open covers diagram, do coincide in the Borel covers diagram.

It must be checked which, if any, of the classes in the Borel covers diagram are provably equal; this is our first task.
2. CHARACTERIZATIONS AND EQUIVALENCE OF PROPERTIES

In this section we give a number of characterizations for some of the Borel classes above. In particular, we get that some of the new properties are equivalent, even though their “open” versions are not provably equivalent.

The classes \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \) \( S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \) and \( U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma). \)

**Theorem 1.** For a set \( X \) of real numbers, the following are equivalent:

1. \( X \) has property \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \)
2. \( X \) has property \( S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \)
3. \( X \) has property \( U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \)
4. Every Borel image of \( X \) in \( {}^\mathbb{N}\mathbb{N} \) is bounded.

**Proof.** We must show that 3 \( \Rightarrow \) 4 and 4 \( \Rightarrow \) 1.

3 \( \Rightarrow \) 4: This is a Theorem of [2]. In short, note that the collections \( U_n = \{U^n_m : m \in \mathbb{N}\}, \) where \( U^n_m = \{f \in {}^\mathbb{N}\mathbb{N} : f(n) < m\}, \) are open \( \gamma \)-covers of \( {}^\mathbb{N}\mathbb{N}. \) Assume that \( \Psi \) is a Borel function from \( X \) to \( {}^\mathbb{N}\mathbb{N}. \) Then the collections \( B_n = \{\Psi^{-1}[U^n_m] : m \in \mathbb{N}\} \) are in \( \mathcal{B}_\Gamma \) for \( X. \) For all \( n, \) the sequence \( U^n_m \) is monotonically increasing with respect to \( m. \) We may assume that for each \( n, B_{n+1} \) refines \( B_n, \) so that we can use (1) instead of (3) to get a sequence \( \Psi^{-1}[U^n_m] \in B_n \) which is in \( \mathcal{B}_\Gamma \) for \( X. \) Then the sequence \( m_n \) bounds \( \Psi[X]. \)

4 \( \Rightarrow \) 1: Assume that \( B_n = \{B^n_k : k \in \mathbb{N}\}, \) are in \( \mathcal{B}_\Gamma \) for \( X. \) Define a function \( \Psi \) from \( X \) to \( {}^\mathbb{N}\mathbb{N} \) so that for each \( x \) and \( n: \)

\[ \Psi(x)(n) = \min\{k : (\forall m \geq k) \ x \in B^n_m\}. \]

Then \( \Psi \) is a Borel map, and so \( \Psi[X] \) is bounded, say by the sequence \( m_n. \) Then the sequence \( (B^n_{m_n} : n \in \mathbb{N}) \) is in \( \mathcal{B}_\Gamma \) for \( X. \)

**Corollary 2.** For a set \( X \) of real numbers, the following are equivalent:

1. \( X \) has property \( U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \)
2. Every Borel image of \( X \) has property \( U_{\text{fin}}(\Gamma, \Gamma). \)

**Proof.** An old Theorem of Hurewicz [8] asserts that \( X \) has property \( U_{\text{fin}}(\Gamma, \Gamma) \) if, and only if, every continuous image of \( X \) in \( {}^\mathbb{N}\mathbb{N} \) is bounded.

**Theorem 3.** For a set \( X \) of real numbers the following are equivalent:

1. \( X \) has property \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \)
2. Each subset of \( X \) has property \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma), \)
3. For each measure zero set \( N \) of real numbers, \( X \cap N \) has property \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma). \)
Proof. 1 ⇒ 2: This follows immediately from Theorem 1 and the fact that for sets of real numbers a function on a subspace which is Borel on the subspace, extends to one which is Borel on the whole space.

3 ⇒ 1: Let $X$ be as in 3, and let $\Psi$ be a Borel function from $X$ to $\mathbb{N}$. We may assume that $X$ is a subset of $[0,1]$, the unit interval (as was shown in [21], the property $S_1(\Gamma, \Gamma)$ is preserved by countable unions). Let $\Phi$ be a Borel function from $[0,1]$ to $\mathbb{N}$ whose restriction to $X$ is $\Psi$.

By Lusin’s Theorem choose for each $n$ a closed subset $C_n$ of the unit interval such that $\mu(C_n) \geq 1 - (\frac{1}{2})^n$, and such that $\Phi$ is continuous on $C_n$. Since $C_n$ is compact, the image of $\Phi$ on $C_n$ is bounded in $\mathbb{N}$, say by $h_n$. The set $N = [0,1] \setminus \bigcup_{n \in \mathbb{N}} C_n$ has measure zero, and so $X \cap N$ has property $S_1(B(\Gamma, B))$. It follows that the image under $\Psi$ of $X \cap N$ is bounded, say by $h$. Now let $f$ be a function which eventually dominates each $h_n$ and $h$. Then $f$ eventually dominates each member of $\Psi[X]$.

Since $\Psi$ was an arbitrary Borel function from $X$ to $\mathbb{N}$, Theorem 1 implies that $X$ has property $S_1(B(\Gamma, B))$. □

Proposition 4. If a set $X$ of real numbers has the $S_1(B(\Gamma, B))$ property, then it is a $\sigma$-set.

Proof. We show that each $G_\delta$-subset of $X$ is an $F_\sigma$-subset. Thus, let $A$ be a $G_\delta$-subset of $X$, say $A = \cap_{n \in \mathbb{N}} U_n$ where for all $n$ $U_n \supseteq U_{n+1}$ are open subsets of $X$. Since $X$ is metrizable, each $U_n$ is an $F_\sigma$-set. Write, for each $n$,

$$ U_n = \bigcup_{k \in \mathbb{N}} C_k^n $$

where for all $m$, $C_k^m \subseteq C_{k+1}^m$ are closed sets. Then for each $n B_n := (C_k^m : m \in \mathbb{N})$ is in $B(\Gamma, B)$. Since $S_1(B(\Gamma, B))$ is hereditary, $A$ has this property and we find for each $n$ an $m_n$ such that $(C_m^{n} : n \in \mathbb{N})$ is a $\gamma$-cover of $A$. For each $k$ define

$$ F_k := \cap_{n \geq k} C_m^n. $$

Then each $F_k$ is closed and $A = \bigcup_{k \in \mathbb{N}} F_k$. □

According to Besicovitch [4] a set $X$ of real numbers is concentrated on a set $Q$ if for every open set $U$ containing $Q$, the set $X \setminus U$ is countable.

Corollary 5. If an uncountable set of real numbers is concentrated on a countable subset of itself, then it does not have property $S_1(B(\Gamma, B))$.

The classes $S_1(B(\Gamma, B))$, $S_{\text{fin}}(B(\Gamma, B))$, and $U_{\text{fin}}(B(\Gamma, B))$.

Theorem 6. The following are equivalent:

1. $X$ has property $S_1(B(\Gamma, B))$. 

Proof. 1 ⇒ 2: This follows immediately from Theorem 1 and the fact that for sets of real numbers a function on a subspace which is Borel on the subspace, extends to one which is Borel on the whole space.

3 ⇒ 1: Let $X$ be as in 3, and let $\Psi$ be a Borel function from $X$ to $\mathbb{N}$. We may assume that $X$ is a subset of $[0,1]$, the unit interval (as was shown in [21], the property $S_1(\Gamma, \Gamma)$ is preserved by countable unions). Let $\Phi$ be a Borel function from $[0,1]$ to $\mathbb{N}$ whose restriction to $X$ is $\Psi$.

By Lusin’s Theorem choose for each $n$ a closed subset $C_n$ of the unit interval such that $\mu(C_n) \geq 1 - (\frac{1}{2})^n$, and such that $\Phi$ is continuous on $C_n$. Since $C_n$ is compact, the image of $\Phi$ on $C_n$ is bounded in $\mathbb{N}$, say by $h_n$. The set $N = [0,1] \setminus \bigcup_{n \in \mathbb{N}} C_n$ has measure zero, and so $X \cap N$ has property $S_1(B(\Gamma, B))$. It follows that the image under $\Psi$ of $X \cap N$ is bounded, say by $h$. Now let $f$ be a function which eventually dominates each $h_n$ and $h$. Then $f$ eventually dominates each member of $\Psi[X]$.

Since $\Psi$ was an arbitrary Borel function from $X$ to $\mathbb{N}$, Theorem 1 implies that $X$ has property $S_1(B(\Gamma, B))$. □

Proposition 4. If a set $X$ of real numbers has the $S_1(B(\Gamma, B))$ property, then it is a $\sigma$-set.

Proof. We show that each $G_\delta$-subset of $X$ is an $F_\sigma$-subset. Thus, let $A$ be a $G_\delta$-subset of $X$, say $A = \cap_{n \in \mathbb{N}} U_n$ where for all $n$ $U_n \supseteq U_{n+1}$ are open subsets of $X$. Since $X$ is metrizable, each $U_n$ is an $F_\sigma$-set. Write, for each $n$,

$$ U_n = \bigcup_{k \in \mathbb{N}} C_k^n $$

where for all $m$, $C_k^m \subseteq C_{k+1}^m$ are closed sets. Then for each $n B_n := (C_k^m : m \in \mathbb{N})$ is in $B(\Gamma, B)$. Since $S_1(B(\Gamma, B))$ is hereditary, $A$ has this property and we find for each $n$ an $m_n$ such that $(C_m^{n} : n \in \mathbb{N})$ is a $\gamma$-cover of $A$. For each $k$ define

$$ F_k := \cap_{n \geq k} C_m^n. $$

Then each $F_k$ is closed and $A = \bigcup_{k \in \mathbb{N}} F_k$. □

According to Besicovitch [4] a set $X$ of real numbers is concentrated on a set $Q$ if for every open set $U$ containing $Q$, the set $X \setminus U$ is countable.

Corollary 5. If an uncountable set of real numbers is concentrated on a countable subset of itself, then it does not have property $S_1(B(\Gamma, B))$.

The classes $S_1(B(\Gamma, B))$, $S_{\text{fin}}(B(\Gamma, B))$, and $U_{\text{fin}}(B(\Gamma, B))$.

Theorem 6. The following are equivalent:

1. $X$ has property $S_1(B(\Gamma, B))$. 

(2) \( X \) has property \( S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}) \).
(3) \( X \) has property \( U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}) \).
(4) No Borel image of \( X \) in \( {}^\mathbb{N}\mathbb{N} \) is dominating.

Proof. The proof is similar to that of Theorem 1.

3 \( \Rightarrow \) 4: Given a Borel function \( \Psi \) from \( X \) to \( {}^\mathbb{N}\mathbb{N} \), define \( B_n \) as in the proof of Theorem 1. Let \( A_k, k \in \mathbb{N} \), be a partition of \( \mathbb{N} \) into infinitely many infinite sets. From each sequence of covers \( B_n, n \in A_k \), we can extract by (1) a cover \( B_{mn}^n \) \( (n \in A_k) \). Taken together, \( B_{mn}^n \) \( (n \in \mathbb{N}) \) form a large cover of \( X \). Recalling that \( B_{mn}^n = \Psi^{-1}[U_{mn}^n] \), we get that the sequence \( m_n \) witnesses that \( \Psi[X] \) is not dominating.

4 \( \Rightarrow \) 1: With notation as in the proof of Theorem 1, we get that if \( m_n \) witnesses that \( \Psi[X] \) is not dominating, then \( (B_{mn}^n : n \in \mathbb{N}) \) is a (large) cover of \( X \). \( \square \)

Corollary 7. For a set \( X \) of real numbers, the following are equivalent:

(1) \( X \) has property \( U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}) \).
(2) Every Borel image of \( X \) in \( {}^\mathbb{N}\mathbb{N} \) has property \( U_{\text{fin}}(\Gamma, \mathcal{O}) \).

Proof. A Theorem of Hurewicz [8] asserts that a set \( X \) is \( U_{\text{fin}}(\Gamma, \mathcal{O}) \) if, and only if, every continuous image of \( X \) in \( {}^\mathbb{N}\mathbb{N} \) is not dominating. \( \square \)

The classes \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega), S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) \), and \( U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) \). The characterization of these classes is best stated in the language of filters. Let \( \mathcal{F} \) be a filter over \( \mathbb{N} \). An equivalence relation \( \sim_{\mathcal{F}} \) is defined on \( {}^\mathbb{N}\mathbb{N} \) by

\[
f \sim_{\mathcal{F}} g \iff \{n : f(n) = g(n)\} \in \mathcal{F}.
\]

The equivalence class of \( f \) is denoted \( [f]_{\mathcal{F}} \), and the set of these equivalence classes is denoted \( {}^\mathbb{N}\mathbb{N}/_{\mathcal{F}} \). Using this terminology, \( [f]_{\mathcal{F}} < [g]_{\mathcal{F}} \) means

\[
\{n : f(n) < g(n)\} \in \mathcal{F}.
\]

The following combinatorial notion and the accompanying Lemma 8 will be used to get a technical version of the filter-based characterization.

For a family \( Y \subseteq {}^\mathbb{N}\mathbb{N} \), define \( \text{maxfin}(Y) \) to be the set of elements \( f \) in \( {}^\mathbb{N}\mathbb{N} \) for which there is a finite set \( F \subseteq Y \) such that

\[
f(n) = \max\{h(n) : h \in F\}
\]

for all \( n \).

Lemma 8. Let \( Y \subseteq {}^\mathbb{N}\mathbb{N} \) be such that for each \( n \) the set \( \{h(n) : h \in Y\} \) is infinite. Then the following are equivalent:

(1) \( \text{maxfin}(Y) \) is not a dominating family.
(2) There is a non-principal filter $F$ on $\mathbb{N}$ such that the subset $\{ [f]_F : f \in Y \}$ of the reduced product $^{\mathbb{N}}\mathbb{N}/F$ is bounded.

Proof. 1 $\Rightarrow$ 2: Choose an $h \in ^{\mathbb{N}}\mathbb{N}$ which is strictly increasing, and which is not eventually dominated by any element of $\text{maxfin}(Y)$. For any finite subset $F$ of $Y$, put $f_F(n) = \max\{g(n) : g \in F\}$ for each $n$, and then define the set

$$A_F = \{ n \in \mathbb{N} : f_F(n) \leq h(n) \}.$$

Observe that for finite subsets $F$ and $G$ of $Y$, if $F \subseteq G$, then $A_G \subseteq A_F$. Thus, the family $\{ A_F : F \subseteq Y \text{ finite} \}$ is a basis for a filter $F$ on $\mathbb{N}$. By the hypothesis on $Y$ this filter is non-principal. It is evident that $[h]/F$ is an upper bound for $Y/F$.

2 $\Rightarrow$ 1: Let $F$ be a nonprincipal filter on $\mathbb{N}$ such that $Y/F$ is bounded, and choose a function $h$ in $^{\mathbb{N}}\mathbb{N}$ such that for each $f \in Y$ we have $[f]_F < [h]_F$. Then for each $f \in Y$ the set $\{ n : f(n) \leq h(n) \}$ is in $F$ and is infinite (since $F$ is non-principal). Since $F$ has the finite intersection property it follows that for each finite subset $F$ of $Y$ the set $S_F = \{ n : (\forall f \in F)(f(n) \leq h(n)) \}$ is in $F$. But then $h$ is not eventually dominated by any element of $\text{maxfin}(Y)$. $\square$

**Theorem 9.** For a set $X$ of real numbers, the following are equivalent:

1. $X$ has property $S_1(B_\Gamma, B_\Omega)$,
2. $X$ has property $S_{\text{fin}}(B_\Gamma, B_\Omega)$,
3. $X$ has property $U_{\text{fin}}(B_\Gamma, B_\Omega)$;
4. For each Borel function $\Psi$ from $X$ to $^{\mathbb{N}}\mathbb{N}$, $\text{maxfin}(\Psi[X])$ is not a dominating family;
5. For each Borel function $\Psi$ from $X$ to $^{\mathbb{N}}\mathbb{N}$, either there is a principal filter $G$ for which $\Psi[X]/G$ is finite, or else there is a nonprincipal filter $F$ on $\mathbb{N}$ such that the subset $\Psi[X]/F$ of the reduced product $^{\mathbb{N}}\mathbb{N}/F$ is bounded.

Proof. 1 $\Rightarrow$ 2 $\Rightarrow$ 3 are immediate. We will first show that 3 $\Rightarrow$ 4 $\Rightarrow$ 1, and then use Lemma 8 to establish the equivalence of 4 and 5. As in the previous proof, for any finite subset $F$ of $Y$, put $f_F(n) = \max\{g(n) : g \in F\}$ for each $n$.

3 $\Rightarrow$ 4: Let $Y = \Psi[X]$. By the upcoming Theorem 48, $Y$ has property $U_{\text{fin}}(B_\Gamma, B_\Omega)$. For each $n$ and each $k$, define $U^n_k := \{ f : f(n) < k \}$; then set $U_n := \{ U^n_k : k \in \mathbb{N} \}$. Each $U_n$ is a $\gamma$-cover of $^{\mathbb{N}}\mathbb{N}$ since for each $n$ and for $k < j$ we have $U^n_k \subseteq U^n_j$. Let $A_k, k \in \mathbb{N}$, be a partition of $\mathbb{N}$ into infinitely many infinite sets. From each sequence of $\gamma$-covers $U_n, n \in A_k$, we can use the $U_{\text{fin}}(B_\Gamma, B_\Omega)$ property of $Y$ to extract an $\omega$-cover $U^n_m : n \in A_k$. Then for each finite $F \subseteq X$, we have for each
\(k \in \mathbb{N}\) an \(n \in A_k\) such that \(\Psi[F] \subseteq U_n^m\), i.e., \(f_{\Psi[F]}(n) \leq m_n\). Thus, the sequence \(m_n\) witnesses that \(\text{maxfin}(\Psi[X])\) is not a dominating family.

4 \(\Rightarrow\) 1: Assume that \(B_n = \{B_n^m : m \in \mathbb{N}\}\) are in \(\mathcal{B}_\Gamma\) for \(X\). Define a Borel function \(\Psi\) from \(X\) to \(\mathbb{N}\mathbb{N}\) so that for each \(x\) and \(n\):

\[
\Psi(x)(n) = \min\{k : (\forall m \geq k) \ x \in B_n^m\}.
\]

Note that if \(F \subseteq X\) is finite, then for all \(m \geq f_{\Psi[F]}(n)\), \(F \subseteq B_n\) infinitely many times. That is, \((B_n^m : n \in \mathbb{N})\) is in \(\mathcal{B}_\Omega\) for \(X\).

4 \(\Rightarrow\) 5: There are two cases to consider:

Case 1: There is an \(n\) such that \(\{\Psi(x)(n) : x \in X\}\) is finite. Then the principal filter generated by \(\{n\}\) does the job.

Case 2: For each \(n\) the set \(\{\Psi(x)(n) : x \in X\}\) is infinite. Apply Lemma 8.

5 \(\Rightarrow\) 4: Again consider two cases, and apply Lemma 8.

\(\square\)

Remark 10. The implications 1 \(\Rightarrow\) 2 \(\Rightarrow\) 3 \(\Rightarrow\) 4 and 4 \(\Leftrightarrow\) 5 in Theorem 9 can be proved for the open version of these properties in a similar manner. The implication 3 \(\Rightarrow\) 2 in the open case is counter-exampled by the Cantor set \([9]\). We do not know whether the open version of 4 \(\Rightarrow\) 3 is true.

This gives the following characterization of \(\mathfrak{d}\):

Corollary 11. For an infinite cardinal number \(\kappa\) the following are equivalent:

1. \(\kappa < \mathfrak{d}\);
2. For each subset \(X\) of \(\mathbb{N}\mathbb{N}\) of cardinality at most \(\kappa\), there is a non-principal filter \(\mathcal{F}\) on \(\mathbb{N}\) such that in the reduced product \(\mathbb{N}\mathbb{N}/\mathcal{F}\) the set \(X/\mathcal{F}\) is bounded.

Proof. By Theorem 9, 2 implies 1. To see that 1 implies 2, consider an infinite \(\kappa < \mathfrak{d}\) and a subset \(X\) of \(\mathbb{N}\mathbb{N}\) which is of cardinality \(\kappa\). We may assume that \(y \in X\) whenever there is an \(x \in X\) such that \(y\) differs from \(x\) in only finitely many points. Then \(\text{maxfin}(X)\) also has cardinality \(\kappa\). By Lemma 8 there exists a nonprincipal filter \(\mathcal{F}\) on \(\mathbb{N}\) such that \(X/\mathcal{F}\) is bounded in \(\mathbb{N}\mathbb{N}/\mathcal{F}\). \(\square\)

Theorem 12. For a set \(X\) of real numbers, the following are equivalent:

1. \(X\) has property \(S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)\);
2. For each Borel mapping \(\Psi\) of \(X\) into \(\mathbb{N}\mathbb{Z}\) there is a nonprincipal filter \(\mathcal{F}\) such that the subring generated by \(\Psi[X]/\mathcal{F}\) in the reduced power \(\mathbb{N}\mathbb{Z}/\mathcal{F}\) is bounded below and above.
Proof. That 2 implies 1 is proved as before. Regarding 1 implies 2: It is evident that if we confine attention to the ring \(\mathbb{N}\) with pointwise operations, then a subset \(Y\) of it would have property \(S_1(B_\Gamma, B_\Omega)\) if, and only if, there is a nonprincipal filter \(\mathcal{F}\) such that \(Y/\mathcal{F}\) is bounded from below and from above in \(\mathbb{N}\). Let \(g\) be an element of \(\mathbb{N}\) such that \(\Psi[X]/\mathcal{F}\) is bounded by \([g]\). Since the set \(\{n \cdot g : n \in \mathbb{Z}\} \cup \{g^n : n \in \mathbb{N}\}\) is countable, we find a single \(h\) such that for all \(n\) \(h\) eventually dominates each of \(n \cdot g\) and \(g^n\). But then in the reduced power \(\mathbb{N}/\mathcal{F}\) the element \([-h]\) is a lower bound and the element \([h]\) is an upper bound for the ring generated by \(\Psi[X]/\mathcal{F}\). \(\square\)

The class \(S_1(B, B)\). The classes \(S_1(B, B)\) and \(S_1(B_\Gamma, B_\Gamma)\) appear to be each other’s “duals”.

**Theorem 13.** For a set \(X\) of real numbers, the following are equivalent:

1. \(X\) has property \(S_1(B, B)\);
2. Every subset of \(X\) has property \(S_1(B, B)\);
3. For each meager set \(M \subseteq \mathbb{R}\), \(X \cap M\) has property \(S_1(B, B)\).

**Proof.** We must show that 1 implies 2, and that 3 implies 1.

1 \(\Rightarrow\) 2: This is immediate from the equivalence of \(S_1(B, B)\) with another notion (see section 5). However, we give a direct proof.

Let \(M\) be a subset of \(X\), and assume that \(X\) has property \(S_1(B, B)\). For each \(n\) let \(U_n\) be a countable cover of \(M\) by Borel subsets of \(M\). For each \(U \in U_n\) let \(B_U\) be a Borel subset of \(X\) such that \(U = M \cap B_U\). Then \(X_n := \cup\{B_U : U \in U_n\}\) is a Borel subset of \(X\) since \(U_n\) is countable.

In turn, \(X := \cap_{n \in \mathbb{N}} X_n\) is a Borel subset of \(X\).

For each \(n\) let \(U_n\) be \(\{B_U : U \in U_n\} \cup \{X \setminus \bar{X}\}\). Then \((U_n : n \in \mathbb{N})\) is a sequence of countable Borel covers of \(X\). For each \(n\) choose a \(V_n \in \bar{U}_n\) such that \(\{V_n : n \in \mathbb{N}\}\) is a cover of \(X\). For each \(n\) for which \(V_n \neq \bar{X}\), choose \(U_n \in U_n\) such that \(V_n = B_{U_n}\); for other values of \(n\) let \(U_n\) be an arbitrary element of \(U_n\). Then \((U_n : n \in \mathbb{N})\) covers \(M\).

3 \(\Rightarrow\) 1: Let \((B_n : n \in \mathbb{N})\) be a sequence of countable Borel covers of \(X\); enumerate each \(B_n\) as \((B_{m}^n : m \in \mathbb{N})\).

Since Borel sets have the property of Baire we may choose for each \(B_m^n\) an open set \(O_m^n\) and a meager set \(M_m^n\) such that

\[B_m^n = (O_m^n \setminus M_m^n) \cup (M_m^n \setminus O_m^n).\]

Then \(A := \cup_{m,n \in \mathbb{N}} M_m^n\) is a meager set and so \(A \cap X\) has property \(S_1(B, B)\). For each \(n\) such that \(n \text{ mod } 3 = 0\), choose a \(B_{m_n}^n \in B_n\) such that \(A \cap X\) is covered by these.
For each $n$, $O_n$, defined to be $\{O^n_m : m \in \mathbb{N}\}$, is an open cover of $X \setminus A$. Let $Q$ be a countable dense subset of $X \setminus A$, and choose for each $n$ with $n \text{ mod } 3 = 1$ an $O^n_{m_n}$ such that these cover $Q$.

Then the set $B := X \setminus \bigcup \{O^n_{m_n} : n \text{ mod } 3 = 1\}$ is meager, and so has property $S_1(B, B)$. For each $n$ such that $n \text{ mod } 3 = 2$, choose an $O^n_{m_n} \in O_n$ such that these $O^n_{m_n}$’s cover $B$.

Then the sequence $(B^n_{m_n} : n \in \mathbb{N})$ covers $X$. □

Combining of a result from [1] and [12] with one from [2] yields the following characterization:

**Theorem 14.** For a set $X$ of real numbers, the following are equivalent:

1. $X$ has property $S_1(B, B)$.
2. Each Borel image of $X$ has the Rothberger property $S_1(O, O)$.

The selection property $S_1(O, O)$ manifests itself in several other interesting ways: these analogues hold also for $S_1(B, B)$.

**Theorem 15.** For a set $X$ of real numbers, the following are equivalent:

1. $S_1(B, B)$ holds.
2. ONE has no winning strategy in the game $G_1(B, B)$.

**Proof.** We must show that 1 ⇒ 2: Let $F$ be a strategy for ONE of the game $G_1(B, B)$. Using it, define the following array of Borel subsets of $X$: First, enumerate $F(\emptyset)$, ONE’s first move, as $(U_n : n \in \mathbb{N})$. For each response $U_n$ by TWO, enumerate ONE’s corresponding move $F(U_n)$ as $(U_{n_1,n} : n \in \mathbb{N})$. If TWO responds now with $U_{n_1,n_2}$, enumerate ONE’s corresponding move $F(U_{n_1,n_1}, U_{n_1,n_2})$ as $(U_{n_1,n_1,n_2,n} : n \in \mathbb{N})$, and so on.

The family $(U_\tau : \tau \in \wp^{\omega \mathbb{N}})$ has the property that for each $\tau$ the set $\{U_{\tau^{-n}} : n \in \mathbb{N}\}$ is a cover of $X$ by Borel subsets of $X$. Moreover, for each function $f$ in $\mathbb{N}^\mathbb{N}$, the sequence

$F(\emptyset), U_{f(1)}, F(U_{f(1)}), U_{f(1), f(2)}, F(U_{f(1), f(2)}), F(U_{f(1), f(2)}), \ldots$

is a play of $G_1(B, B)$ during which ONE used the strategy $F$. For each such $f$, define $S_f := \bigcup_{n \in \mathbb{N}} U_{f(1), \ldots, f(n)}$. (Thus, $S_f$ is the set of points covered by TWO during a play coded by $f$. We must show that for some such $f$ we have $S_f = X$.

Define the subset $D$ of $X \times \mathbb{N}^\mathbb{N}$ by

$D := \{(x, f) : x \not\in S_f\}$.

Then $D$ is a Borel subset of $X \times \mathbb{N}^\mathbb{N}$. Moreover, for each $x \in X$ the set $D_x = \{f : x \not\in S_f\}$ is nowhere dense. (To see this, let $[n_1, \ldots, n_k]$ be
a basic open subset of \( \mathbb{N} \). Since \( \{U_{n_1,\ldots,n_k,m} : m \in \mathbb{N}\} \) is a cover of \( X \) there is an \( n_{k+1} \) with \( x \in U_{n_1,\ldots,n_k,n_{k+1}} \). But then \( [(n_1,\ldots,n_k,n_{k+1})] \cap D_x = \emptyset \). Now recall from [2] that as \( X \) has property \( S_1(\mathcal{B},\mathcal{B}) \) it follows that \( \mathbb{N} \neq \cup_{x \in X} D_x \) (see section 5). Let \( f \) be a function not in \( \cup_{x \in X} D_x \). Then \( X = S_f \), and we have defeated ONE’s strategy \( F \). □

We next show that \( S_1(\mathcal{B},\mathcal{B}) \) is a Ramsey-theoretic property. First observe:

**Lemma 16.** For a set \( X \) of real numbers, the following are equivalent:

1. \( X \) has property \( S_1(\mathcal{B},\mathcal{B}) \).
2. \( X \) has property \( S_1(\mathcal{B}_\Omega,\mathcal{B}) \).

*Proof.* The proof for this is like that of Theorem 17 of [19]. □

The virtue of \( \mathcal{B}_\Omega \) for Ramsey-theoretic purposes is that if \( U \) is a member of \( \mathcal{B}_\Omega \), and if it is partitioned into finitely many pieces, then at least one of these pieces is a member of \( \mathcal{B}_\Omega \). This statement is denoted by the abbreviation:

\[
\text{for each } k, \mathcal{B}_\Omega \rightarrow (\mathcal{B}_\Omega)^1_k
\]

This is a special case of the more general notation

\[
\text{for all } n \text{ and } k, \mathcal{A} \rightarrow (\mathcal{C})^n_k,
\]

which denotes the statement:

For each \( n \) and \( k \), for each \( A \in \mathcal{A} \), and for each \( g : [A]^n \rightarrow \{1,\ldots,k\} \), there is a \( C \subseteq A \) such that \( C \in \mathcal{C} \) and \( g \) is constant on \([C]^n\).

**Theorem 17.** For a set \( X \) of real numbers the following are equivalent:

1. \( X \) has property \( S_1(\mathcal{B},\mathcal{B}) \).
2. \( X \) has the property that for all \( k \), \( \mathcal{B}_\Omega \rightarrow (\mathcal{B}_\Omega)^2_k \).

*Proof.* The proof of this is like that of Theorem 4 of [20]. □

**The class \( S_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega) \).** It is evident that unions of countably many spaces, each having property \( S_1(\mathcal{B},\mathcal{B}) \), have property \( S_1(\mathcal{B},\mathcal{B}) \).

**Theorem 18.** If all finite powers of \( X \) have property \( S_1(\mathcal{B},\mathcal{B}) \), then \( X \) has property \( S_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega) \).

*Proof.* The proof of this is a minor variation on the proof of (2) ⇒ (1) of Theorem 3.9 of [9]. □

**Problem 19.** Is it true that if \( X \) has property \( S_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega) \), then it has property \( S_1(\mathcal{B},\mathcal{B}) \) in all finite powers?
The class $S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. It is evident that unions of countably many spaces, each having property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$, have property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$.

**Theorem 20.** If all finite powers of $X$ have property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$, then $X$ has property $S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$.

**Proof.** Let $Y = \sum_{k \in \mathbb{N}} X^k$. Then by the assumption, $Y$ has property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$. Assume that $\mathcal{B}_n = \{B_m^n : m \in \mathbb{N}\}$ are in $\mathcal{B}_\Omega$ for $X$. Define a Borel function $\Psi$ from $Y$ to $\mathbb{N}^\mathbb{N}$ so that for all $k$, $x_0, \ldots, x_{k-1} \in X$, and $n$:

$$\Psi(x_0, \ldots, x_{k-1})(n) = \min\{k : (\forall m \geq k) \ x_0, \ldots, x_{k-1} \in B_m^n\}.$$ 

By Theorem 6, the image of $Y$ under $\Psi$ is not dominating. Choose a sequence $m_n$ witnessing this. For each $n$, set $\mathcal{W}_n := \{B_j^n : j \leq m_n\}$. Then each $\mathcal{W}_n$ is finite, and $\bigcup_{n \in \mathbb{N}} \mathcal{W}_n$ is in $\mathcal{B}_\Omega$ for $X$. □

**Problem 21.** Is it true that if $X$ has property $S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$, then it has property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$ in all finite powers?

The class $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$. A standard diagonalization trick gives the following.

**Lemma 22.** The following are equivalent:

1. $X$ has property $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.
2. Every Borel $\omega$-cover of $X$ contains a $\gamma$-cover of $X$.

**Proof.** The proof of this is like that of the corresponding result in [6]. □

For the next characterization we need some terminology and notation. For $a, b \subseteq \mathbb{N}$, $a \subseteq^* b$ if $a \setminus b$ is finite. Let $[\mathbb{N}]^\infty$ denote the set of infinite sets of natural numbers. $X \subseteq [\mathbb{N}]^\infty$ is centered if every finite $F \subseteq X$ has an infinite intersection. $a \in [\mathbb{N}]^\infty$ is a pseudo-intersection of $X$ if for all $b \in X$, $a \subseteq^* b$. $X \subseteq [\mathbb{N}]^\infty$ is a power if it is centered, but has no pseudo-intersection.

Every countable large Borel cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of $X$ is associated with a Borel function $h_\mathcal{U} : X \to [\mathbb{N}]^\infty$, defined by $h_\mathcal{U}(x) = \{n : x \in U_n\}$.

**Lemma 23 ([23]).** Assume that $\mathcal{U}$ is a cover of $X$. Then:

1. $\mathcal{U}$ is an $\omega$-cover of $X$ if, and only if, $h_\mathcal{U}(X)$ is centered;
2. $\mathcal{U}$ contains a $\gamma$-cover of $X$ if, and only if, $h_\mathcal{U}(X)$ has a pseudo-intersection.

**Lemma 24.** The following are equivalent:

1. Every Borel $\omega$-cover of $X$ contains a $\gamma$-cover of $X$. 

\( S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \rightarrow S_1(\mathcal{B}_\Omega, \mathcal{B}) \rightarrow S_1(\mathcal{B}_\Gamma, \mathcal{B}) \)

\( \downarrow \quad \downarrow \quad \downarrow \)

\( S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \)

\( \downarrow \quad \downarrow \quad \downarrow \)

\( S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \rightarrow S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \rightarrow S_1(\mathcal{B}, \mathcal{B}) \)

**Figure 2. The Surviving Borel Classes**

(2) No Borel image of \( X \) in \([\mathbb{N}]^\infty\) is a power.

**Proof.** 2 \( \Rightarrow \) 1: Follows from the preceding lemma.

1 \( \Rightarrow \) 2: Assume that \( f: X \rightarrow [\mathbb{N}]^\infty \) is Borel, such that \( f[X] \) is centered. Let \( O_n, n \in \mathbb{N} \), denote the clopen sets \( \{a : n \in a\} \). As \( f[X] \) is centered, \( \{O_n : n \in \mathbb{N}\} \) is an \( \omega \)-cover of \( f[X] \). Thus, \( U = \{f^{-1}(O_n) : n \in \mathbb{N}\} \) is a Borel \( \omega \)-cover of \( X \). But \( f = h_U \), so we can apply the preceding lemma. \( \square \)

We thus get the following characterization of \( S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \).

**Theorem 25.** For a set \( X \) of real numbers, the following are equivalent:

1. \( X \) has property \( S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \);
2. No Borel image of \( X \) in \([\mathbb{N}]^\infty\) is a power.

**Corollary 26.** For a set \( X \) of real numbers, the following are equivalent:

1. \( X \) has property \( S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \).
2. Every continuous image of \( X \) has property \( S_1(\Omega, \Gamma) \).

**Proof.** This follows from a Theorem of Recaw [15], asserting that \( X \) has property \( S_1(\Omega, \Gamma) \) if, and only if, no continuous image of \( X \) in \([\mathbb{N}]^\infty\) is a power. \( \square \)

Figure 2 summarizes the equivalences proved in this section.
3. Does Figure 2 contain all the provable information about these classes?

We now consider the question whether we have proved all the equalities that can be proved for these Borel cover classes. It will be seen that the answer is “Yes”; here is a brief outline of how this follows from the results of the present section:

(1) According to Corollary 41 it is consistent that there is a set of real numbers with property \( S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \), but not property \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) \). This means that none of the arrows from the left of Figure 2 to the middle is reversible.

(2) According to Theorem 32 it is consistent that there is a set of real numbers in \( S_1(\mathcal{B}, \mathcal{B}) \) which is not in \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) \). This means that none of the arrows from the middle of Figure 2 to the right is reversible.

(3) According to Theorem 43 it is consistent that there is a set of real numbers in \( S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) \) and not in either of \( S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \) or \( S_1(\mathcal{B}, \mathcal{B}) \). This implies that none of the arrows from the bottom of Figure 2 which terminates at the top is reversible.

(4) According to Theorem 27 the minimal cardinality of a set of real numbers not having property \( S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \) is \( \mathfrak{d} \), while the minimal cardinality of a set of real numbers not having property \( S_1(\mathcal{B}, \mathcal{B}) \) is \( \text{cov}(\mathcal{M}) \). Since it is consistent that \( \text{cov}(\mathcal{M}) < \mathfrak{d} \), none of the arrows starting at the bottom row of Figure 2 is reversible.

For a collection \( \mathcal{J} \) of separable metrizable spaces, let \( \text{non}(\mathcal{J}) \) denote the minimal cardinality for a separable metrizable space which is not a member of \( \mathcal{J} \). We also call \( \text{non}(\mathcal{J}) \) the critical cardinality for the class \( \mathcal{J} \).

**Theorem 27.**

\[
\begin{align*}
1 & \Rightarrow \text{non}(S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)) = p, \\
2 & \Rightarrow \text{non}(S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)) = b, \\
3 & \Rightarrow \text{non}(S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)) = \text{non}(S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)) = \text{non}(S_1(\mathcal{B}_\Gamma, \mathcal{B})) = \mathfrak{d}, \\
4 & \Rightarrow \text{non}(S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)) = \text{non}(S_1(\mathcal{B}, \mathcal{B})) = \text{cov}(\mathcal{M}).
\end{align*}
\]

**Proof.** 1 and 2 follow from Theorems 25 and 1, respectively. 3 follows from Theorems 6 and 20.

For 4, we need the following lemma.

**Lemma 28.** Let \( \mathcal{J}, \mathcal{S} \) be collections of separable metrizable spaces, such that \( X \in \mathcal{J} \) if and only if, every Borel image of \( X \) is in \( \mathcal{S} \). Then \( \text{non}(\mathcal{J}) = \text{non}(\mathcal{S}) \).
Proof. Since $\mathcal{J} \subseteq \mathcal{S}$, we have $\text{non}(\mathcal{J}) \leq \text{non}(\mathcal{S})$. Now, let $X$ witness $\text{non}(\mathcal{J})$. Then there is a Borel function $\Psi$ on $X$ such that $\Psi[X] \not\in \mathcal{S}$. As the cardinality of $\Psi[X]$ cannot be greater than the cardinality of $X$, we get that $\text{non}(\mathcal{J}) \geq \text{non}(\mathcal{S})$. □

Now, it is well known that $\text{non}(\mathcal{S}_1(\emptyset, \emptyset)) = \text{cov}(\mathcal{M})$. Therefore, by Theorem 14, $\text{non}(\mathcal{S}_1(\mathcal{B}, \mathcal{B})) = \text{cov}(\mathcal{M})$. Thus, by Theorem 18, $\text{non}(\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)) = \text{cov}(\mathcal{M})$ as well. □

Since it is consistent that $p < \text{cov}(\mathcal{M})$, it is consistent that $\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ is not equal to $\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. Similarly the consistency of the inequality $p < b$ implies that $\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ is not provably equal to $\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.

It is consistent that $b < \text{cov}(\mathcal{M})$, and so it is consistent that there is a set of real numbers which has property $\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ but which does not have property $\mathcal{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

Since it is consistent that $\text{cov}(\mathcal{M}) < d$, it is also not provable that $\mathcal{S}_{\text{fin}}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is equal to either of $\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ or $\mathcal{S}_1(\mathcal{B}, \mathcal{B})$.

What the cardinality results do not settle is whether $\mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ provably coincides with $\mathcal{S}_1(\mathcal{B}, \mathcal{B})$, or whether any of the three classes associated with the cardinal number $d$ coincides with another. They also do not give any indication as to what the interrelationships among two classes might be when their critical cardinals are equal. To treat these questions we now consider specific examples which could be constructed on the basis of a variety of axioms which are consistent. All of the axioms that we use have the form of equality between certain well known cardinal invariants. Readers who are not familiar with this type of axioms may assume the Continuum Hypothesis instead (in this case, all of the cardinal invariants become equal to $\aleph_1$).

**Special elements of $\mathcal{S}_1(\mathcal{B}, \mathcal{B})$.** A set of real numbers is a *Lusin* set if it is uncountable, but its intersection with each meager set of real numbers is countable. More generally, for a cardinal $\kappa$ an uncountable set $X \subseteq \mathbb{R}$ is said to be a $\kappa$-*Lusin* set if it has cardinality at least $\kappa$, but its intersection with each meager set is less than $\kappa$. It is evident that the smaller the value of $\kappa$, the harder it is for a set to be a $\kappa$-Lusin set. Towards the goal of using as weak hypotheses as possible, this means that we would be interested in $\kappa$-Lusin sets for as large a value of $\kappa$ that would allow the conclusion we are aiming at. We now work in the group $^\mathbb{N}\mathbb{Z}$ (which topologically is homeomorphic to the set of irrational numbers), and construct from weak axioms special elements of $\mathcal{S}_1(\mathcal{B}, \mathcal{B})$. 
Lemma 29. If $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M})$, and if $Y$ is a subset of $\mathbb{N}\mathbb{Z}$ of cardinality at most $\text{cof}(\mathcal{M})$, then there is a $\text{cov}(\mathcal{M})$-Lusin set $L \subseteq \mathbb{N}\mathbb{Z}$ such that $Y \subseteq L + L$.

Proof. Let $\{y_\alpha : \alpha < \text{cov}(\mathcal{M})\}$ enumerate $Y$. Let $\{M_\alpha : \alpha < \text{cov}(\mathcal{M})\}$ enumerate a cofinal family of meager sets, and construct $L$ recursively as follows: At stage $\alpha$ set $X_\alpha = \{a_i : i < \alpha\} \cup \{b_i : i < \alpha\} \cup \bigcup_{i<\alpha} M_i$. Then $(y_\alpha - X_\alpha) \cup X_\alpha$ is a union of fewer than $\text{cov}(\mathcal{M})$ meager sets. Choose an $a_\alpha \in \mathbb{N}\mathbb{Z} \setminus ((y_\alpha - X_\alpha) \cup X_\alpha)$. Evidently, $a_\alpha \in (y_\alpha - \mathbb{N}\mathbb{Z} \setminus X_\alpha) \cap (\mathbb{N}\mathbb{Z} \setminus X_\alpha)$. Thus, choose $b_\alpha \in \mathbb{N}\mathbb{Z} \setminus X_\alpha$ for which $y_\alpha - b_\alpha = a_\alpha$. Then we have $y_\alpha = a_\alpha + b_\alpha$.

Finally, set $L = \{a_\alpha : \alpha < \text{cov}(\mathcal{M})\} \cup \{b_\alpha : \alpha < \text{cov}(\mathcal{M})\}$. Then $L$ is a $\text{cov}(\mathcal{M})$-Lusin set and $L + L \supseteq Y$. \hfill $\square$

The next result is used to show that for $\kappa$ small enough, $\kappa$-Lusin sets are in $S_1(\mathcal{B}, \mathcal{B})$.

Corollary 30. If $X$ is a $\text{cov}(\mathcal{M})$-Lusin set, then it has property $S_1(\mathcal{B}, \mathcal{B})$.

Proof. If $M$ is any meager set, then $M \cap X$ has cardinality less than $\text{cov}(\mathcal{M})$, and thus is in $S_1(\mathcal{B}, \mathcal{B})$. Now apply Theorem 13. \hfill $\square$

The notion of a Lusin set (i.e., an \(\aleph_1\)-Lusin set in our current notation) was characterized as follows in [22]: For a topological space $X$ let $\mathcal{K}$ denote the collection of $\mathcal{U}$ such that $\mathcal{U}$ is a family of open subsets of $X$, and $X = \bigcup\{\overline{U} : U \in \mathcal{U}\}$. Then $X$ is a Lusin set if, and only if, it has property $S_1(\mathcal{K}, \mathcal{K})$.

Thus we have:

Corollary 31. If a set of real numbers has property $S_1(\mathcal{K}, \mathcal{K})$, then it has property $S_1(\mathcal{B}, \mathcal{B})$.

Theorem 32. If $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M})$, then there is a $\text{cov}(\mathcal{M})$-Lusin set in $S_1(\mathcal{B}, \mathcal{B})$ which is not in $\bigcup_{\text{fin}}(\Gamma, \Omega)$.

Proof. From the cardinality hypothesis and the fact that $\text{cov}(\mathcal{M}) \leq \vartheta \leq \text{cof}(\mathcal{M})$, we see that there is in $\mathbb{N}\mathbb{Z}$ a dominating family, say $Y$, of cardinality $\text{cov}(\mathcal{M})$. Let $L$ be a $\text{cov}(\mathcal{M})$-Lusin set as in Lemma 29, such that $L + L \supseteq Y$. As $\max\{|f(n)|, |g(n)|\} \geq \frac{|f(n)| + |g(n)|}{2}$, we see that for the identity mapping $\Psi$, $\text{maxfin}(\Psi[L])$ is dominating. Thus, by Remark 10, $L$ does not have property $\bigcup_{\text{fin}}(\Gamma, \Omega)$.

By Corollary 30 $L$ has property $S_1(\mathcal{B}, \mathcal{B})$. \hfill $\square$

This in particular implies that $S_1(\mathcal{B}_3, \mathcal{B}_\Omega)$ is not provably equivalent to $S_1(\mathcal{B}, \mathcal{B})$. 
Special elements of $S_1(B_1, B_1)$. Now that we have clarified most of the interrelationships among the Borel classes, we consider how the Borel classes are related to the classes in Figure 1. We have just seen that $S_1(B_1, B_1)$ need not be contained in $U_{fin}(\Gamma, \Omega)$, even when the critical cardinalities for sets not belonging to these classes are the same.

Next we treat $S_1(B_1, B_1)$ and $U_{fin}(\Gamma, \Gamma)$. We show how to use the Continuum Hypothesis to construct a Lusin set which has property $S_1(B_1, B_1)$. Since it is a Lusin set, it does not satisfy $U_{fin}(\Gamma, \Gamma)$.

In our construction we use the ad hoc concept of an $\omega$-fat collection of Borel sets. A collection $U$ of Borel sets is said to be fat if for each nonempty open interval $J$ and for each dense $G_\delta$-set $\mathcal{G}$ there is a $B \in U$ such that $B \cap G \cap J \neq \emptyset$. It is said to be $\omega$-fat if: for each dense $G_\delta$-set $G$ and for every finite family $\mathcal{F}$ of nonempty open sets there is a $B \in U$ such that for each $J \in \mathcal{F}$, $B \cap J \cap G$ is nonempty.

A number of facts about these $\omega$-fat families of Borel sets will play a crucial role in our construction. For ease of reference we state these as lemmas and give proofs where it seems necessary.

**Lemma 33.** Let $U$ be an $\omega$-fat family consisting of countably many Borel sets.

1. For each partition of $U$ into two pieces, at least one of the pieces is $\omega$-fat.
2. If $U$ is a Borel $\omega$-cover of the set $X$ and $F$ is a finite subset of $X$, then $\{U \in U : F \subseteq U\}$ is an $\omega$-fat Borel $\omega$-cover of $X$.

Added in proof. As stated, item (2) of Lemma 33 is wrong: Let $U = \{\mathbb{R} \setminus \mathbb{Z}\} \cup [\mathbb{Z}]^{<\omega}$. Then $U$ is an $\omega$-fat $\omega$-cover of $\mathbb{Z}$. But for any nonempty finite subset $F$ of $\mathbb{Z}$, the collection $\{U \in U : F \subset U\}$ is not $\omega$-fat. However, if $X$ is a Lusin set such that for each nonempty basic open set $G$, $X \cap G$ is uncountable, then (some minor modification of) item (2) of this Lemma holds. As the special set $X$ which we will construct is a Lusin set, we can easily make sure that it has the required property and the proof works. This idea is extended and explained further in [3, full version].

**Lemma 34.** If $\mathcal{B}$ is a countable fat Borel family, then there is a dense $G_\delta$-set contained in $\cup \mathcal{B}$.

**Proof.** Since $B = \cup \mathcal{B}$ is a Borel set, it has the property of Baire. Let $U$ be an open set such that $(U \setminus B) \cup (B \setminus U)$ is meager. Then $U$ is dense, for let $G$ be a dense $G_\delta$ disjoint from that meager set, and let $J$ be a nonempty open interval. Then $J \cap G \cap B$ is nonempty. But $B = (B \setminus U) \cup (B \cap U)$, so that $(B \cap U) \cap J$ is nonempty.
Now $\mathbb{R} \setminus U$ is nowhere dense, and we may assume that $G$ is also disjoint from this nowhere dense set. But then $G \subseteq B$. □

Lemma 35. If $U$ is a countable $\omega$-fat family of Borel sets and $F$ is a finite nonempty family of nonempty open intervals, then there are a $U \in U$ and for each $J \in F$ a nonempty open interval $I_{J} \subseteq J$ such that the set $U \cap I_{J}$ is comeager in $I_{J}$.

Proof. Towards proving the contrapositive, take a countable $\omega$-fat family $U$ of Borel sets, and a finite nonempty family $F$ of nonempty open intervals such that:

For each $U \in U$ there is a $J_{U} \in F$ such that for each nonempty open interval $I \subseteq J_{U}$ the set $U \cap I$ is not comeager in $I$. Fix such a $J_{U}$ for each $U \in U$.

Since $U \cap J_{U}$ is a Borel set, it has the property of Baire. Choose an open set $V \subseteq J_{U}$ such that $(V \setminus (U \cap J_{U})) \cup ((U \cap J_{U}) \setminus V)$ is meager. If $V$ is nonempty, then the meagerness of $V \setminus (U \cap J_{U})$ implies that $U \cap V$ is comeager in $V$, contradicting the choice of $U$ and $J_{U}$. Thus, $V$ is empty, and we find that $U \cap J_{U}$ is meager. Let $G_{U}$ be a dense $G_{\delta}$-set disjoint from $U \cap J_{U}$.

The set $G = \cap_{U \in U} G_{U}$ is an intersection of countably many dense $G_{\delta}$-sets, so is a dense $G_{\delta}$-set. But then $G$ and $F$ witness that $U$ is not $\omega$-fat. □

Lemma 36. Let $S$ be a countably infinite set and let $(F_{n} : n \in \mathbb{N})$ be an ascending sequence of finite sets with union equal to $S$. If $(U_{n} : n \in \mathbb{N})$ is a sequence of Borel $\omega$-covers of $S$ such that for each $n$ the set $\{U \in U_{n} : F_{n} \subseteq U\}$ is $\omega$-fat, then there is a sequence $(U_{n} : n \in \mathbb{N})$ such that for each $n$ $U_{n} \in U_{n}$, $\{U_{n} : n \in \mathbb{N}\}$ is a Borel $\gamma$-cover of $S$, and $\{U_{n} : n \in \mathbb{N}\}$ is $\omega$-fat.

Proof. Let $S$, the $F_{n}$’s, and the $U_{n}$’s be as in the hypotheses. We may assume for each $n$ that for all $U \in U_{n}$ we have $F_{n} \subseteq U$. Let $(J_{n} : n \in \mathbb{N})$ be an enumeration of the nonempty open intervals with rational endpoints.

Consider $n$. Since $U_{n}$ is $\omega$-fat, choose a $U_{n} \in U_{n}$ and for each $i \leq n$ an open nonempty interval $I_{n} \subseteq J_{i}$ such that $I_{n} \cap U_{n}$ is comeager in $I_{n}$.

Then the sequence $(U_{n} : n \in \mathbb{N})$ is as desired. To see this, let $G$ be any dense $G_{\delta}$-set and let $R_{1}, \ldots, R_{n}$ be nonempty open intervals. Choose $m$ so large that for each $i \leq n$ there is a $j \leq m$ with $J_{j} \subseteq R_{i}$. When we chose $U_{m}$ it was done so that for some open nonempty intervals $I_{j}$, $j \leq m$ we had $I_{j} \subseteq J_{i}$ and $U_{m} \cap I_{j}$ is comeager in $I_{j}$, whence $U_{m} \cap G \cap I_{j}$ is comeager in $I_{j}$. But then for each $r \leq n$, $U_{m} \cap G \cap I_{j}$ is nonempty. □
Lemma 37. If \( (\mathcal{U}_n : n \in \mathbb{N}) \) is a sequence of countable \( \omega \)-fat families of Borel sets such that for each \( n \mathcal{U}_{n+1} \subseteq \mathcal{U}_n \), then there is a countable \( \omega \)-fat family \( \{U_n : n \in \mathbb{N}\} \) of Borel sets such that for each \( n \), \( U_n \in \mathcal{U}_n \).

Proof. Let \( J_1, J_2, \ldots, J_n, \ldots \) be a bijective enumeration of a basis for the topology of \( \mathbb{R} \). Recursively choose for each \( n \) sequences \( (I^n_k : k \in \mathbb{N}) \) of nonempty open intervals, and for each \( n \) a \( U_n \in \mathcal{U}_n \) such that:

1. For \( k < n \) we have \( I^n_k = J_n \);
2. For \( k \geq n \) we have \( I^n_k \subseteq J_n \) and \( U_k \cap I^n_k \) is comeager in \( I^n_k \).

This is possible on account of Lemma 35. We claim that \( \mathcal{U} := \{U_n : n \in \mathbb{N}\} \) is \( \omega \)-fat.

For let \( G \) be a dense \( G_\delta \)-set and let \( R_1, \ldots, R_k \) be nonempty open intervals. Choose from the basis intervals \( J_{n_1}, \ldots, J_{n_k} \) such that \( n_1 < \cdots < n_k \) and for \( 1 \leq i \leq k \) we have \( J_{n_i} \subseteq R_i \). Let \( m \) be larger than \( n_k \). Then for \( 1 \leq i \leq k \) we have: \( U_m \cap I^n_{m_i} \) contains a dense \( G_\delta \)-subset of \( I^n_m \) and so has nonempty intersection with the dense \( G_\delta \)-set \( G \). Since for each \( i \) we have \( I^n_m \subseteq R_i \) we see that \( U \cap R_i \cap G \) is nonempty. \( \square \)

Lemma 38. Let \( G \) be a dense \( G_\delta \) set and let \( J \) be a nonempty open interval. If for each \( n \) \( \mathcal{U}_n \) is a countable \( \omega \)-fat family of Borel sets, then there is an \( x \in J \cap G \) such that for each \( n \) the set \( \{U \in \mathcal{U}_n : x \in U\} \) is \( \omega \)-fat.

Proof. For each \( n \) let \( \mathcal{U}_n \) be a countable \( \omega \)-fat family of Borel sets. Let \( J \) be a nonempty open interval, and let \( G \) be a dense \( G_\delta \)-set.

Let \( (J_n : n \in \mathbb{N}) \) bijectively enumerate a base for the topology of \( \mathbb{R} \), and write \( G = \cap_{n \in \mathbb{N}} V^1_n \), where \( V^1_1 \supseteq V^1_2 \supseteq \ldots \) are dense open sets. Also, write \( R_1 := J \). We may assume that the closure of \( J \) is compact.

Recursively construct four sequences \( (U^n_i : i \leq n) : n \in \mathbb{N} \), \( (I^n_i : i \leq n) : n \in \mathbb{N} \), \( (R^n_i : n \in \mathbb{N}) \) and \( (V^n_i : n \in \mathbb{N}) : i \in \mathbb{N} \), such that the following requirements are satisfied for each \( n \):

1. For all \( k \leq n \), \( U^n_k \subseteq U_k \setminus \{U^n_j : i, j < n\} \);
2. For each \( i \leq n \), \( I^n_i \subseteq J_i \) is a nonempty open interval such that \( I^n_i \cap (\cap_{j \leq n} U^n_j) \) is comeager in \( I^n_i \);
3. \( R^{n+1}_i \) is a nonempty open interval with closure contained in \((\cap_{i \leq n} V^n_{i+1}) \cap R^n_i \);
4. \( R^{n+1}_i \) is comeager in \( R^{n+1}_i \);
5. \( V^m_i \supseteq V^m_{i+1} \) for all \( m \) are dense open subsets of \( R^n_i \);
6. \( R^{n+1}_i \) is comeager in \( (\cap_{i \leq n} U^n_i) \supseteq \cap_{m \in \mathbb{N}} V^m_{n+1} \).

To see that this recursion can be carried out, first consider \( n = 1 \): Here we already have \( R_1 \) and each \( V^n_1 \) specified. Consider \( J_1 \) and \( R_1 \), and \( \mathcal{U}_1 \). Apply Lemma 35 to choose \( U^1_1 \in \mathcal{U}_1 \) and intervals \( I^1_1 \).
and \( R_2 \) such that \( \overline{R_2} \subseteq R_1 \cap \forall_1^1 \) and \( U_1^1 \cap R_2 \) is comeager in \( R_2 \) and \( U_1^1 \cap I_1^1 \) is comeager in \( I_1^1 \). Since \( U_1^1 \cap R_2 \) is comeager in \( R_2 \), choose a descending sequence \( \{V_n^2 : n \in \mathbb{N}\} \) of open dense subsets of \( R_2 \) such that \( R_2 \cap U_1^1 \supseteq \bigcap_{m \in \mathbb{N}} V_m^2 \). Thus for \( n = 1 \) sets as required by the five recursion specifications have been found.

Suppose now that \( n \geq 1 \) and that the recursion has been carried through for \( n \) steps. Consider \( R_n, J_1, \ldots, J_n, \) and \( U_1, \ldots, U_n \).

Choose for \( i \leq n+1 \) sets \( U_{n+1}^i \in U_i \setminus \{U_k^j : j, k \leq n\} \) and \( R_{n+1} \) an open nonempty interval with closure contained in \( R_n \cap (\cap_{i \leq n} V_{n+1}^i) \), as well as open nonempty intervals \( I_{n+1}^i, i \leq n+1 \), such that for each \( i \) \( I_{n+1}^i \subseteq J_i \), and \( \bigcap_{k \leq n+1} U_{n+1}^k \cap I_{n+1}^i \) is comeager in \( I_{n+1}^i \), and \( \bigcap_{k \leq n+1} U_{n+1}^k \cap R_{n+1} \) is comeager in \( R_{n+1} \). This can be done on account of Lemma 35. Then \( \{V_m^{n+1} : m \in \mathbb{N}\} \) be a descending sequence of sets open and dense in \( R_{n+1} \) such that \( R_{n+1} \cap (\bigcap_{k \leq n+1} U_{n+1}^k) \supseteq \bigcap_{m \in \mathbb{N}} V_m^{n+1} \).

This shows how to continue the recursion to the next step.

With the recursive procedure completed, for each \( n \) put \( \mathcal{V}_n = \{U_k^n : k \geq n\} \). By the compactness of \( \overline{R_1} \), and by specification 3 of the recursion, \( \bigcap_{n \in \mathbb{N}} R_n \) is nonempty. Let \( x \) be an element of this intersection.

We claim that each \( \mathcal{V}_n \) is a \( \omega \)-fat subset of \( \mathcal{U}_n \), and that for each \( V \in \mathcal{V}_n \), we have \( x \in V \cap J \cap G \).

To see that \( \mathcal{V}_n \) is \( \omega \)-fat, let a dense \( G_\delta \)-set \( H \) and a finite set \( \mathcal{F} \) of nonempty open intervals be given. Choose \( m \) so large that there is for each \( F \in \mathcal{F} \) a \( J_i \) with \( i \leq m \) such that \( J_i \subseteq F \). Then \( U_m^n \) was chosen so that for each of the nonempty open intervals \( I_m^i \subseteq J_i \), we have \( U_m^n \cap I_m^i \) comeager in \( I_m^i \). But then as \( H \) is a comeager set of reals, we have for each \( i \leq m \) that \( U_m^n \cap I_m^i \cap H \) is nonempty. This implies that for each \( F \in \mathcal{F} \), \( U_m^n \cap F \cap H \) is nonempty.

To see that \( x \) is a member of each element of \( \mathcal{V}_n \), consider a \( U_m^n \in \mathcal{V}_n \). We have \( U_m^n \cap R_m \supseteq \bigcap_{j \in \mathbb{N}} V_j^m \). But for each \( j \geq m+1 \) we have \( R_{j+1} \subseteq V_j^m \), and as \( x \) is in the intersection of the \( R_j \)'s, it is in the intersection of the \( V_j^m \)'s, so in \( U_m^n \).

\textbf{Lemma 39.} If \( \text{add}(\mathcal{M}) = c \), then there exists a family \( \{G_\alpha : \alpha < \aleph_1\} \) of dense \( G_\delta \)-sets of reals, such that:

- For each dense \( G_\delta \)-set \( G \) there is an \( \alpha \) with \( G_\alpha \subseteq G \);
- For \( \alpha < \beta < c \) we have \( G_\beta \subseteq G_\alpha \).

\textit{Proof.} Let \( \{M_\alpha : \alpha < c\} \) be a cofinal family of meager sets. We define by induction on \( \alpha < c \) a monotonically increasing sequence \( \{\hat{M}_\alpha : \alpha < c\} \) of of \( F_\sigma \) meager sets as follows: At stage \( \alpha \), let \( \hat{M}_\alpha = \cup_{\gamma < \alpha} M_\gamma \). As \( \alpha < \text{add}(\mathcal{M}) \), \( \hat{M}_\alpha \) is meager, so let \( \hat{M}_\alpha \) be an \( F_\sigma \) meager set containing \( \hat{M}_\alpha \).
By the Baire category Theorem, complements of meager sets in \( \mathbb{R} \) are dense. Thus, setting for each \( \alpha \) \( G_\alpha = \mathbb{R} \setminus M_\alpha \) yields the desired sequence. \( \square \)

**Theorem 40 (CH).** There is a \( c \)-Lusin set which has property \( S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \).

**Proof.** Let \((G_\alpha : \alpha < c)\) be as in Lemma 39. Let \((U^\alpha_n : n \in \mathbb{N}) : \alpha < c\) list all \( \omega \)-sequences where each term is an \( \omega \)-fat countable family of Borel sets. We shall now recursively construct the desired Lusin set \( X \) by choosing for each \( \alpha \) a countable dense set \( X_\alpha \) to satisfy certain requirements, and then setting \( X = \bigcup_{\alpha < c} X_\alpha \cup \mathbb{Q} \). Together with each \( X_\alpha \) we shall choose a sequence \((U^\alpha_n : n \in \mathbb{N})\) of Borel sets and a sequence \((S_\gamma^\alpha(\alpha) : \gamma < c)\) of infinite subsets of \( \mathbb{N} \) such that:

1. Whenever \( \gamma < \beta < c \), then \( S_\gamma(\beta) = \mathbb{N} \);
2. For each \( \beta < c \), for \( \gamma < \nu < c \) we have \( S_\nu(\beta) \subseteq S_\gamma(\beta) \);
3. For all \( \beta \) and \( \gamma \), \( \{U^\beta_n : n \in S_\gamma(\beta)\} \) is an \( \omega \)-fat \( \gamma \)-cover of \( \mathbb{Q} \cup (\bigcup_{\nu \leq \gamma} X_\nu) \);
4. For any \( \alpha \), if some \( U^\alpha_n \) is not an \( \omega \)-cover of \( \mathbb{Q} \cup (\bigcup_{\nu < c} X_\nu) \), then for each \( n \) we have \( U^\alpha_n = \mathbb{R} \);
5. If for each \( n \) \( U^\alpha_n \) is an \( \omega \)-cover of \( \mathbb{Q} \cup (\bigcup_{\nu < \alpha} X_\nu) \), then for each \( n \) we have \( U^\alpha_n \in \mathcal{U}_\alpha \), and \( \{U^\alpha_n : n \in \mathbb{N}\} \) is an \( \omega \)-fat \( \gamma \)-cover of \( \mathbb{Q} \cup (\bigcup_{\nu < c} X_\nu) \);
6. For each \( \alpha \), \( X_\alpha \subseteq G_\alpha \setminus (\mathbb{Q} \cup (\bigcup_{\nu < c} X_\nu)) \) is dense in \( \mathbb{R} \).

Before showing that this can be accomplished, we show that constructing \( X \) to satisfy these requirements is sufficient. Thus, let \( X \) be obtained like this. Let \((U_n : n \in \mathbb{N})\) be a sequence of countable Borel \( \omega \)-covers of \( X \). Since each \( X_\alpha \) is dense and contained in \( G_\alpha \) it follows that for each \( n \) \( U_n \) is \( \omega \)-fat. Thus, for some \( \beta \) we have \((U_n : n \in \mathbb{N}) = (U^\beta_n : n \in \mathbb{N})\). Since each \( U^\beta_n \) is an \( \omega \)-cover of \( X \), it is an \( \omega \)-cover of \( \mathbb{Q} \cup (\bigcup_{\gamma < \beta} X_\gamma) \), and thus as in 5. Let \( F \) be a finite subset of \( X \) and choose a \( \beta > \alpha \) such that \( F \subseteq \mathbb{Q} \cup (\bigcup_{\gamma < \beta} X_\gamma) \). By 3 \( \{U^\beta_n : n \in S_\beta(\alpha)\} \) is a \( \gamma \)-cover of \( \mathbb{Q} \cup (\bigcup_{\gamma < \beta} X_\gamma) \), whence for some \( n \) \( F \subseteq U^\beta_n \). It follows that \( \{U^\alpha_n : n \in \mathbb{N}\} \) is an \( \omega \)-cover of \( X \), as desired.

Now the recursive construction: Fix \( Q \), the set of rational numbers, and ask: Is \((U^0_n : n \in \mathbb{N})\) a sequence of \( \omega \)-covers of \( \mathbb{Q} ? \)

No: Then for each \( n \) set \( U^0_n = \mathbb{R} \), choose \( X_0 \subseteq G_0 \setminus \mathbb{Q} \) countable and dense, and put \( S_0(0) = \mathbb{N} \).

Yes: For each \( n \) choose a \( U^0_n \in U^0_n \) such that \( \{U^0_n : n \in \mathbb{N}\} \) is an \( \omega \)-fat \( \gamma \)-cover of \( \mathbb{Q} \). Repeatedly apply Lemma 38 to recursively choose numbers \( x_1 \in J_1 \cap G_0 \setminus \mathbb{Q} \) and \( x_{n+1} \in J_{n+1} \cap G_0 \setminus (\mathbb{Q} \cup \{x_1, \ldots, x_n\}) \) such that: \( V := \{U^0_m : x \in U^0_m\} \) is an \( \omega \)-fat family of Borel sets, and for each \( n \) \( V_{n+1} := \{U^0_m : x_{n+1} \in U^0_m\} \) is...
an \( \omega \)-fat family of Borel sets. In the end put \( X_0 = \{ x_n : n \in \mathbb{N} \} \), and choose by Lemma 37 a \( V \subseteq V_1 \) such that \( V \) is \( \omega \)-fat, and for each \( n \) also \( V \subseteq^* V_n \). Finally set \( S_0(0) = \{ n : U_n^0 \in V \} \). Observe that \( \{ U_n^0 : n \in S_0(0) \} \) is a \( \gamma \)-cover of \( \mathbb{Q} \cup X_0 \).

This shows that the six recursive requirements are satisfiable for \( \alpha = 0 \). Assume now that \( \alpha > 0 \) is given, and for each \( \beta < \alpha \) we already have \( X_\beta \) as well as the sequence \( (U_n^\beta : n \in \mathbb{N}) \) and \( (S_\gamma(\beta) : \gamma < \alpha) \) such that the six recursive requirements are satisfied. To verify that stage \( \alpha \) can then be carried out, do the following. First, for all \( \beta < \alpha \) define \( S_\beta(\alpha) = \mathbb{N} \). Also, using Lemma 37, choose for each \( \beta < \alpha \) an infinite set \( S_\beta \subseteq \mathbb{N} \) such that for all \( \gamma < \alpha \) we have \( S_\beta \subseteq^* S_\gamma(\beta) \), and such that \( \{ U_n^\beta : n \in S_\beta \} \) is an \( \omega \)-fat \( \gamma \)-cover of \( \cup_{\gamma < \alpha} X_\gamma \cup \mathbb{Q} \).

Consider \( (U_n^\alpha : n \in \mathbb{N}) \) and ask: Is each \( U_n^\alpha \) an \( \omega \)-cover of \( \cup_{\gamma < \alpha} X_\gamma \cup \mathbb{Q} \)?

No: Then for each \( n \) put \( U_n^\alpha = \mathbb{R} \), and declare \( S_\alpha(\alpha) = \mathbb{N} \). Next we choose \( X_\alpha \) recursively as follows from \( H_\alpha := G_\alpha \setminus (\cup_{\gamma < \alpha} X_\gamma \cup \mathbb{Q}) \):

By Lemma 38 choose an \( x_1 \in J \cap H_\alpha \) such that for each \( \beta < \alpha \) the set \( V_1^\beta = \{ U_n^\beta : n \in S_\beta \text{ and } x_1 \in U_n^\beta \} \) is an \( \omega \)-fat family. For each \( n \) choose \( x_{n+1} \in J_{n+1} \cap H_\alpha \setminus \{ x_1, \ldots, x_n \} \) such that \( V_{n+1}^\beta := \{ U_m^\beta : x_{n+1} \in U_m^\beta \} \) is an \( \omega \)-fat family. Finally apply Lemma 37 to choose for each \( \beta < \alpha \) an \( \omega \)-fat family \( V_\beta \subseteq \mathbb{V}_1^\beta \) such that for each \( n \) \( \mathcal{V}_n^\beta \subseteq^* \mathcal{V}_n^\gamma \), and set \( X_\alpha = \{ x_n : n \in \mathbb{N} \} \). Observe that each \( \mathcal{V}_n^\beta \) is a \( \gamma \)-cover of \( \cup_{\gamma < \alpha} X_\gamma \cup \mathbb{Q} \), and \( X_\alpha \) is a dense subset of \( \mathbb{R} \). For each \( \beta < \alpha \) define \( S_\alpha(\beta) := \{ m : U_m^\beta \in \mathcal{V}_n^\beta \} \).

Yes: Then first choose for each \( n \) a \( U_n^\alpha \in U_n^\alpha \) such that \( \{ U_n^\alpha : n \in \mathbb{N} \} \) is a \( \gamma \)-cover of \( \cup_{\gamma < \alpha} X_\gamma \cup \mathbb{Q} \). For each \( \beta < \alpha \) set \( S_\beta(\alpha) = \mathbb{N} \). Next we construct \( X_\alpha \). For convenience, put \( H_\alpha = G_\alpha \setminus (\cup_{\gamma < \alpha} X_\gamma \cup \mathbb{Q}) \). Applying Lemma 38 choose \( x_1 \in J_1 \cap H_\alpha \) such that for each \( \beta < \alpha \) the set \( U_1^\beta := \{ U_n^\beta : x_1 \in U_n^\beta \} \) is \( \omega \)-fat, and \( U_1^\alpha = \{ U_n^\alpha : x_1 \in U_n^\alpha \} \) is \( \omega \)-fat. For each \( n \) choose \( x_{n+1} \in J_{n+1} \cap H_\alpha \setminus \{ x_1, \ldots, x_n \} \) such that for \( \beta \leq \alpha \) we have \( \mathcal{V}_{n+1}^\beta = \{ U_m^\beta : x_{n+1} \in U_m^\beta \} \) is an \( \omega \)-fat family. Finally, by Lemma 37 choose for each \( \beta \) an \( \omega \)-fat family \( \mathcal{V}_n^\beta \) such that for all \( n \) \( \mathcal{V}_n^\beta \subseteq^* \mathcal{V}_n^\gamma \). Observe that each \( \mathcal{V}_n^\beta \) is a \( \gamma \)-cover of \( \cup_{\beta \leq \alpha} X_\beta \cup \mathbb{Q} \). For \( \beta \leq \alpha \) define: \( S_\alpha(\beta) = \{ n : U_n^\beta \in \mathcal{V}_n^\beta \} \).

In either case we succeeded in extending the satisfiability of the recursive requirements before stage \( \alpha \), to stage \( \alpha \). \( \square \)

**Corollary 41 (CH).** There is a set of real numbers with property \( S_1(\mathcal{B}_0, \mathcal{B}_1) \) which does not have property \( \bigcup \text{fin}(\Gamma, \Gamma) \).
Proof. We may think of having carried out the preceding construction in $\mathbb{N}^\mathbb{N}$; here, every set with property $U_{\text{fin}}(\Gamma, \Gamma)$ is bounded, and so meager. But a Lusin set is non-meager. □

Special elements of $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. Our next task is to determine the relationship of the top row of Figure 2 to the bottom rest of Figure 1. For this we compare $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ with $S_1(\mathcal{O}, \mathcal{O})$ and with $S_{\text{fin}}(\Omega, \Omega)$. A set $X$ of real numbers is said to be a Sierpiński set if it is uncountable, and its intersection with each Lebesgue measure zero set is countable.

More generally, for an uncountable cardinal number $\kappa$ a set of real numbers is a $\kappa$-Sierpiński set if it has cardinality at least $\kappa$, but its intersection with each set of Lebesgue measure zero is less than $\kappa$.

In Theorem 2.9 of [9] it was shown that all Sierpiński sets have the property $U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. This also follows easily from our characterization of $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ (Theorem 3), since each countable set has this property. Indeed, our characterization and the fact that every set of real numbers of cardinality less than $b$ has property $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ gives that every $b$-Sierpiński set has property $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. Since sets of real numbers having property $S_1(\mathcal{O}, \mathcal{O})$ have measure zero, no $b$-Sierpiński set has property $S_1(\mathcal{O}, \mathcal{O})$.

Let $\mathbb{P}$ denote the set of irrational numbers.

Lemma 42. If $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$, and if $Y \subseteq \mathbb{P}$ has cardinality at most $\text{cof}(\mathcal{N})$, then there is a $\text{cov}(\mathcal{N})$-Sierpiński set $S \subseteq \mathbb{P}$ such that $Y \subseteq S + S \subseteq \mathbb{P}$.

Proof. Let $\{y_\alpha : \alpha < \text{cov}(\mathcal{N})\}$ enumerate $Y$. Let $\{N_\alpha : \alpha < \text{cov}(\mathcal{N})\}$ enumerate a cofinal family of meager sets, and construct $S$ recursively as follows: At stage $\alpha$ set

$$X_\alpha = \bigcup_{i < \alpha} ((a_i, b_i) \cup (\mathbb{Q} - a_i) \cup (\mathbb{Q} - b_i) \cup N_i).$$

Note that for each $x \in \mathbb{P} \setminus X_\alpha$ and $i < \alpha$, $x + a_i$ and $x + b_i$ are irrational.

$X_\alpha$ is a union of fewer than $\text{cov}(\mathcal{N})$ measure zero sets. As in Lemma 29, we can choose $a_\alpha, b_\alpha \in \mathbb{P} \setminus X_\alpha$ such that $a_\alpha + b_\alpha = y_\alpha$. (Note that $y_\alpha \in \mathbb{P}$.)

Finally, set $S = \{a_\alpha : \alpha < \text{cov}(\mathcal{N})\} \cup \{b_\alpha : \alpha < \text{cov}(\mathcal{N})\}$. Then $S$ is a $\text{cov}(\mathcal{N})$-Sierpiński set and $Y \subseteq S + S \subseteq \mathbb{P}$. □

Theorem 43. If $b = \text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$, then there is a $b$-Sierpiński set of real numbers $S$ such that:

1. $S$ has property $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$,
2. $S$ does not have property $S_1(\mathcal{O}, \mathcal{O})$,
3. $S \times S$ does not have property $U_{\text{fin}}(\Gamma, \mathcal{O})$,.
(4) $S$ does not have property $S_{\text{fin}}(\Omega, \Omega)$.

**Proof.** Note that the hypothesis $b = \text{cof}(\mathcal{N})$ implies that $b = \frak{d}$. Let $\Psi$ be a homeomorphism from the irrationals onto $^{\mathbb{N}}\mathbb{N}$. Let $D \subseteq ^{\mathbb{N}}\mathbb{N}$ be a dominating family of size $\frak{d}$, and set $Y = \Psi^{-1}[D]$. Use Lemma 42 to construct a $b$-Sierpiński set $S \subseteq \mathcal{P}$ such that $Y \subseteq S + S \subseteq \mathcal{P}$. Now, define $f : S \times S \to ^{\mathbb{N}}\mathbb{N}$ by $f(x, y) = \Psi(x + y)$. Then $f$ is continuous, and $f[S \times S] = \Psi[S + S] \supseteq \Psi[X] = D$ is dominating. This makes 1, 2, and 3.

Now, in [9] it is proved that $S_{\text{fin}}(\Omega, \Omega)$ is closed under taking finite powers. Thus, 4 follows from 3. □

Thus, we have that $S_{1}(\mathcal{B}_{\Gamma}, \mathcal{B}_{\Gamma})$ is not provably contained in $S_{\text{fin}}(\Omega, \Omega)$. It follows that Figure 2 gives all the provable relations among the Borel covering classes.

In light of Theorem 6, the following Theorem of Reclaw [16] implies that none of the properties involving open classes implies any of the properties involving Borel classes. Reclaw’s proof assumes Martin’s axiom, but the partial order used is $\sigma$-centered so that in fact $\frak{p} = \frak{c}$ is enough.

**Theorem 44 ($\frak{p} = \frak{c}$).** There is a set having the $S_{1}(\Omega, \Gamma)$ property which can be mapped onto $^{\mathbb{N}}\mathbb{N}$ by a Borel function.

Figure 3 summarizes the relationships among the various classes considered so far in this paper and in [9], including the Borel classes. In this diagram there must also be a vector pointing from $S_{\text{fin}}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Omega})$ to $S_{\text{fin}}(\Omega, \Omega)$; we omitted this one for “aesthetic” reasons.

With this we have now shown that in Figure 3, no arrows can be added to, or removed from, the layer of Borel classes.

At present it is not known if there always is an uncountable set of real numbers which belongs to some class in Figure 2. In light of what we know about this diagram, the most modest form of this question is

**Problem 45.** Is there always an uncountable set of reals with property $S_{1}(\mathcal{B}_{\Gamma}, \mathcal{B})$?

while the boldest form would be

**Problem 46.** Is there always an uncountable set of real numbers with property $S_{\text{fin}}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Omega})$?

**Special elements of $S_{1}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Gamma})$.** It might be wondered whether any of our Borel notions trivializes to contain only sets of size smaller than the critical cardinality of that notion. With the knowledge obtained thus far, the only candidate to trivialize is $S_{1}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Gamma})$. A Theorem of Brendle [5] shows that this is not the case.
Theorem 47 (CH). There is a set of reals $X$ of size $\mathfrak{c}(=\aleph_1)$ which has property $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$. 

\textbf{Figure 3.} The Combined Diagram
4. Preservation of properties

The selection properties for open covers are preserved when taking continuous images or closed subsets [9]. We have the following analogue.

**Theorem 48.** Let $\Pi$ be one of $S_1$, $S_{\text{fin}}$, or $U_{\text{fin}}$ and let $\mathcal{U}$ and $\mathcal{V}$ range over the set $\{\mathcal{B}, \mathcal{B}_\Omega, \mathcal{B}_\Lambda, \mathcal{B}_\Gamma\}$. Assume that $X$ has property $\Pi(\mathcal{U}, \mathcal{V})$. Then:

1. If $Y$ is a Borel subset of $X$, then $Y$ has property $\Pi(\mathcal{U}, \mathcal{V})$;
2. If $f : X \to Y$ is Borel and onto, then $Y$ has property $\Pi(\mathcal{U}, \mathcal{V})$.

**Proof.** This proof is similar to the proof of Theorem 3.1 in [9].

In particular, if $\mathcal{U}$ and $\mathcal{V}$ are among $\{\mathcal{O}, \Omega, \Lambda, \Gamma\}$ for $X$, and $X$ has property $\Pi(\mathcal{B}_\mathcal{U}, \mathcal{B}_\mathcal{V})$ for some $\Pi$, then every Borel image of $X$ has property $\Pi(\mathcal{U}, \mathcal{V})$. This gives rise to the following question: Using the above notation, assume that every Borel image of $X$ has property $\Pi(\mathcal{U}, \mathcal{V})$. Does $X$ necessarily have the $\Pi(\mathcal{B}_\mathcal{U}, \mathcal{B}_\mathcal{V})$ property? For the following classes, a positive answer was given:

- $S_1(\mathcal{O}, \mathcal{O})$ – Theorem 14.
- $U_{\text{fin}}(\Gamma, \Gamma)$ – Theorem 2.
- $S_1(\Gamma, \Gamma)$ – this one follows from the preceding one, since $S_1(\Gamma, \Gamma)$ implies $U_{\text{fin}}(\Gamma, \Gamma)$, and $S_1(\mathcal{B}_\mathcal{R}, \mathcal{B}_\mathcal{R})$ is equivalent to $U_{\text{fin}}(\mathcal{B}_\mathcal{R}, \mathcal{B}_\mathcal{R})$ (Theorem 1).
- $U_{\text{fin}}(\Gamma, \mathcal{O})$ – Theorem 7.
- $S_1(\Gamma, \mathcal{O})$ – this one too follows from the preceding one, since $S_1(\Gamma, \mathcal{O})$ implies $U_{\text{fin}}(\Gamma, \mathcal{O})$, and $S_1(\mathcal{B}_\mathcal{R}, \mathcal{B})$ is equivalent to $U_{\text{fin}}(\mathcal{B}_\mathcal{R}, \mathcal{B})$ (Theorem 6).
- $S_1(\Omega, \Gamma)$ – Theorem 26.

For the following classes, the problem remains open:

- $S_1(\Gamma, \Omega)$, $S_{\text{fin}}(\Gamma, \Omega)$, and $U_{\text{fin}}(\Gamma, \Omega)$ – If 4 implies 3 were true in Remark 10, we could have added these classes to the positive list.
- $S_1(\Omega, \Omega)$.
- $S_{\text{fin}}(\Omega, \Omega)$.

**Finite powers.** $S_1(\mathcal{B}, \mathcal{B})$ is not provably closed under taking finite powers.

**Theorem 49.** If $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M})$, then there exists a set of reals $X$ such that $X$ has property $S_1(\mathcal{B}, \mathcal{B})$, and $X \times X$ does not have the property $U_{\text{fin}}(\Gamma, \mathcal{O})$. 


Proof. The \( \text{cov} (\mathcal{M}) \)-Lusin set \( L \) from Theorem 32 has the property that \( L + L \), a continuous image of \( L \times L \), is dominating. Thus, \( L \times L \) does not have the property \( U_{\text{fin}} (\Gamma, \mathcal{O}) \). \( \square \)

Dually, Theorem 43 shows that \( S_1 (\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) \) is not provably closed under taking finite powers.

**Problem 50.** Is any of the classes \( S_1 (\mathcal{B}_\Omega, \mathcal{B}_\Gamma), S_1 (\mathcal{B}_\Omega, \mathcal{B}_\Omega), \) and \( S_{\text{fin}} (\mathcal{B}_\Omega, \mathcal{B}_\Omega) \) closed under taking finite powers?

Note that a positive answer to Problem 19 would imply that \( S_1 (\mathcal{B}_\Omega, \mathcal{B}_\Omega) \) is closed under taking finite powers. Similarly, a positive answer to Problem 21 would imply that \( S_{\text{fin}} (\mathcal{B}_\Omega, \mathcal{B}_\Omega) \) is closed under taking finite powers.

5. Connections with other approaches to smallness properties

Three schemas for describing smallness of sets of real numbers have been developed over recent years. These have their roots in classical literature and can be described, broadly speaking, by:

- Properties of the vertical sections of a sufficiently describable planar set;
- Properties of the image in \( \mathbb{N} \mathbb{N} \) under a sufficiently describable function;
- Selection properties for sequences of sufficiently describable topologically significant families of subsets.

The vertical sections schema has been inspired by the papers [13], [14] and [15], and is as follows:

Let \( H \) be a subset of \( \mathbb{R} \times \mathbb{R} \) and let \( \mathcal{J} \) be a collection of subsets of \( \mathbb{R} \). For \( x \) and \( y \) real numbers, define

\[
H_x = \{ y \in \mathbb{R} : (x, y) \in H \};
\]

\[
H^y = \{ x \in \mathbb{R} : (x, y) \in H \}.
\]

A Borel set \( H \) is said to be a \( \mathcal{J} \)-set if for each \( x \) \( H_x \in \mathcal{J} \).

The following three collections of subsets of the real line have been defined in terms of properties of vertical sections – see [12]:

**ADD(\( \mathcal{J} \))**: The set of \( X \subseteq \mathbb{R} \) such that for each \( \mathcal{J} \)-set \( H \), \( \cup_{x \in X} H_x \in \mathcal{J} \);

**COV(\( \mathcal{J} \))**: The set of \( X \subseteq \mathbb{R} \) such that for each \( \mathcal{J} \)-set \( H \), \( \cup_{x \in X} H_x \neq \mathbb{R} \);

**COF(\( \mathcal{J} \))**: The set of \( X \subseteq \mathbb{R} \) such that \( \{ H_x : x \in X \} \) is not a cofinal subset of \( \mathcal{J} \).
The sets in $\text{COV}(\mathcal{M})$ have also been called $R^\mathcal{M}$-sets in [1]; in that paper it was shown that $X$ is an $R^\mathcal{M}$-set if, and only if, every Borel image of $X$ in $\mathbb{N}^\mathbb{N}$ has property $S_1(\mathcal{O}, \mathcal{O})$. It was shown in [2] that this class is also characterized by $S_1(\mathcal{B}, \mathcal{B})$.

The sets in $\text{ADD}(\mathcal{M})$ have also been called $SR^\mathcal{M}$-sets, and it has been shown in [1] that $X$ is in $\text{ADD}(\mathcal{M})$ if, and only if, every Borel image of $X$ in $\mathbb{N}^\mathbb{N}$ has both properties $S_1(\mathcal{O}, \mathcal{O})$ and $U_{\text{fin}}(\Gamma, \Gamma)$. Due to a result in [11], a set $X$ of real numbers has both properties $S_1(\mathcal{O}, \mathcal{O})$ and $U_{\text{fin}}(\Gamma, \Gamma)$ if, and only if, it has the property $(*)$ which was introduced in [6]. Using our results here and results of [11] one can show that a set of reals has property $\text{ADD}(\mathcal{M})$ if, and only if, it is a member of $S_1(\mathcal{B}, \mathcal{B})$ and $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

The “properties of the image” schema takes inspiration from three papers [8], [15] and [18] (Lemma 3). In each of these papers it is proven that a set of real numbers has a certain property of interest if, and only if, each of its continuous images (in some cases into a specific range space) has another property of interest.

The following four classes of sets were introduced in [12]:

**NON($\mathcal{J}$):** The set of $X \subseteq \mathbb{R}$ such that for every Borel function $f$ from $\mathbb{R}$ to $\mathbb{R}$, $f[X]$ is a member of $\mathcal{J}$;

**P:** The set of $X \subseteq \mathbb{R}$ such that for no Borel function $f$ from $\mathbb{R}$ to $\mathbb{N}^\mathbb{N}$, $f[X]$ is a power;

**B:** The set of $X \subseteq \mathbb{R}$ such that for every Borel function $f$ from $\mathbb{R}$ to $\mathbb{N}^\mathbb{N}$, $f[X]$ is bounded under eventual domination;

**D:** The set of $X \subseteq \mathbb{R}$ such that for every Borel function $f$ from $\mathbb{R}$ to $\mathbb{N}^\mathbb{N}$, $f[X]$ is not a dominating family.

The classes of sets defined by these two schemas are related for the special case where $\mathcal{J}$ is $\mathcal{M}$, the collection of meager sets of real numbers, or $\mathcal{N}$, the collection of measure zero subsets of the real line. The results from [12] regarding the interrelationships of these classes of sets are summarized in Figure 4.

The relationship between Figure 4 and the well-known Chichon diagram that expresses provable relationships among certain cardinal numbers is that a cardinal number in a particular position in Cichon’s diagram is actually the minimal cardinality for a set of real numbers not belonging to the class in the corresponding position in Figure 4.

Our results imply the following.

**Corollary 51.** $\text{COF}(\mathcal{M})$ contains a set of reals whose size is $\text{cov}(\mathcal{M})$.

**Proof.** If $\text{cov}(\mathcal{M}) < \text{cof}(\mathcal{M}) = \text{non}(\text{COF}(\mathcal{M}))$, then any set of size $\text{cov}(\mathcal{M})$ will do. Otherwise by Theorem 32 there exists a $\text{cov}(\mathcal{M})$-Lusin set in $S_1(\mathcal{B}, \mathcal{B})$, which is in $\text{COV}(\mathcal{M})$. □
In [8] Hurewicz characterized the covering properties $U_{\text{fin}}(\Gamma, \Gamma)$ and $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ in terms of properties of the continuous images in $\mathbb{N}$. In particular, Hurewicz showed that $X$ has property $U_{\text{fin}}(\Gamma, \Gamma)$ if, and only if, each continuous image of $X$ in $\mathbb{N}$ is bounded. He also showed that $X$ has property $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ if, and only if, each continuous image of $X$ into $\mathbb{N}$ is not a dominating family. The sets in $B$ have also been called $A$-sets in [2]; where they show that that $B = U_{\text{fin}}(B_{\Gamma}, B_{\Gamma})$, and $D = S_{\text{fin}}(B, B)$. By our results here we know $B = S_{1}(B_{\Gamma}, B_{\Gamma})$, and $D = S_{1}(B_{\Gamma}, B)$.

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