Temperley–Lieb $K$-matrices

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Abstract. This work concerns studies of boundary integrability of the vertex models from representations of the Temperley–Lieb algebra associated with the quantum group $U_q[X_n]$ for the affine Lie algebras $X_n = A_{1}^{(1)}, B_{n}^{(1)}, C_{n}^{(1)}$ and $D_{n}^{(1)}$.

A systematic computation method is used to construct solutions of the boundary Yang–Baxter equations. We find a $2n^2 + 1$ free parameter solution for $A_{1}^{(1)}$ spin-$n - 1/2$ and $C_{n}^{(1)}$ vertex models. It turns out that for $A_{1}^{(1)}$ spin-$n$, $B_{n}^{(1)}$ and $D_{n}^{(1)}$ vertex models, the solution has $2n^2 + 2n + 1$ free parameters.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), solvable lattice models, symmetries of integrable models
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1. Introduction

The search for integrable models through solutions of the Yang–Baxter equation \[1\]–\[3\]
\[
R_{12}(u-v)R_{13}(u) = R_{23}(v)R_{13}(u)R_{12}(u-v)
\]
has been performed by the quantum group approach in \[4\]. In this way, the \(R\)-matrices corresponding to vector representations of all non-exceptional affine Lie algebras have been determined in \[5\].

A similar approach is desirable for finding solutions of the boundary Yang–Baxter equation \[6, 7\] where the boundary weights follow from \(K\)-matrices which satisfy a pair of equations, namely the reflection equation
\[
R_{12}(u-v)K_1^-(u)R_{12}(u+v)K_1^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{12}(u-v) \tag{1.2}
\]
and the dual reflection equation
\[
R_{12}(-u+v) (K_2^+)^t_1 (u) M_1^{-1} R_{12}(-u-v-2\rho) M_1 (K_2^+)^t_2 (v) = (K_2^+)^t_2 (v) M_1 R_{12}(-u-v-2\rho) M_1^{-1} (K_1^-)^t_1 (u) R_{12}(u+v). \tag{1.3}
\]
In this case duality supplies a relation between \(K^- \) and \(K^+ \) \[8\]:
\[
K^+(u) = K^-(-u-\rho)^t M, \quad M = V^t V. \tag{1.4}
\]
Here \(t\) denotes transposition and \(t_i\) denotes transposition in the \(i\)th space. \(V\) is the crossing matrix and \(\rho\) the crossing parameter, both being specific to each model \[9\].

With this goal in mind, the study of boundary quantum groups was initiated in \[10\]. This study has been used to determine \(A_1^{(1)}\) reflection matrices for arbitrary spin \[11\], and

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the $A_2^{(2)}$ and some $A_n^{(1)}$ reflection matrices were derived again in [12]. Reflection solutions from $\mathcal{R}$-matrices corresponding to vector representations of Yangians and super-Yangians were presented in [13]. However, as observed in [12], an independent systematic method of constructing the boundary quantum group generators is not yet available. In contrast to the bulk case [5], one cannot exploit boundary affine Toda field theory, since appropriate classical integrable boundary conditions are not yet known [14].

However, the algebraic structures related to reflection equations are well known [15] and a boundary quantum group approach was recently used in [16] to derive the classification of the constant $K$-matrix (without spectral parameter) solutions for the Temperley–Lieb (TL) models. The main result, already found in [17], is that for a given constant TL $\mathcal{R}$-matrix, the corresponding constant $K$-matrix satisfies a quadratic relation:

$$qK^2 + c_1K + (q + q^{-1})^{-1}(c_1^2 + qc_2)I = 0$$

(1.5)

with appropriate central elements $c_1$ and $c_2$.

From this result, the Yang-Baxterization procedure, as used in [18]–[20], allows one to obtain spectral parameter dependent reflection matrices:

$$K(u) = u^2K - \frac{1}{u^2}K^{-1} + cI$$

(1.6)

with an arbitrary central element $c$.

Independently, there has been an increasing amount of effort towards the understanding of two-dimensional integrable theories with boundaries via solutions of the functional equation (1.2). In field theory, attention is focused on the boundary $S$ matrix [21, 23]. In statistical mechanics, the emphasis has been laid on deriving all solutions of (1.2) because different $K$-matrices lead to different universality classes of surface critical behavior [24] and allow the calculation of various surface critical phenomena, both at and away from criticality [25].

Although a hard task, direct computation has been used to solve (1.2). For instance, we mention the solutions with $\mathcal{R}$ matrix based on non-exceptional Lie algebras [22, 26] and superalgebras [27, 28]. The regular $K$-matrices for the exceptional $\mathcal{U}_q[G_2]$ vertex model were obtained in [29]. Many diagonal solutions for face and vertex models associated with affine Lie algebras were presented in [25]. For A–D–E interaction-round face (IRF) models, diagonal and some non-diagonal solutions were presented in [30]. Reflection matrices for Andrews–Baxter–Forrester models in the RSOS/SOS representation were presented in [31]. Apart from these $c$-number solutions of the reflection equations there must also exist nontrivial solutions that include boundary degree of freedom as were derived for the sine–Gordon theory in [32] and the projected $K$-matrices [33].

Motivated by the results presented in [34] we again touch on this issue in order to include once more the TL lattice models [35] arising from the quantum group $\mathcal{U}_q[X_n]$ for $X_n = A_1^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ [36].

The TL algebra is very useful in the study of two-dimensional lattice statistical mechanics. It provided an algebraic framework for constructing and analyzing different types of integrable lattice models, such as the $Q$-state Potts model, IRF model, $O(n)$ loop model, six-vertex model, etc [37].
We have organized this paper as follows. In section 2 the model is presented, in section 3 we choose the reflection equations and their solutions. Section 4 is reserved for the conclusion.

2. The model

From the representation of the TL algebra, one can build solvable vertex models with the \( R \) operator defined by

\[
R(u) = x_1(u)I + x_2(u)U,
\]

where \( I \) is the identity operator and \( U \) is the TL projector. Here \( u \) is the spectral parameter and the anisotropic parameter \( \eta \) is chosen so that

\[
x_1(u) = \frac{\sinh(\eta - u)}{\sinh \eta}, \quad x_2(u) = \frac{\sinh u}{\sinh \eta},
\]

\[
2 \cosh \eta = \text{Tr} U.
\]

Setting

\[
R_j(u) = 1 \otimes \cdots \otimes R(u) \otimes 1 \otimes \cdots \otimes 1
\]

one can show that the Yang–Baxter equation

\[
R_{j+1}(u)R_j(u + v)R_{j+1}(v) = R_j(v)R_{j+1}(u + v)R_j(u)
\]

is valid due to the definition relations of the TL algebra

\[
U^2 = 2 \cosh \eta U
\]

\[
U_iU_j = U_jU_i \quad |i - j| > 1.
\]

For the affine Lie algebras \( A_1^{(1)}, B_n^{(1)}, C_n^{(1)} \) and \( D_n^{(1)} \) i.e., the \( q \)-deformations of the spin-s representation of \( sl(2) \) and the vector representation of \( so(2n + 1), sp(2n) \) and \( so(2n) \), the corresponding TL projector, using the notation and results of [24], has the form

\[
U = \sum_{i,j=1}^{N} \varepsilon(i) \varepsilon(j) \ q^{-(\hat{\epsilon}_i + \hat{\epsilon}_j)}e_{i,j} \otimes e_{i',j'},
\]

where \( e_{i,j} \) is the matrix unit \((e_{i,j}v_k = \delta_{j,k}v_i)\) and we have used the conjugated index \( a' = N + 1 - a \).

Here, one has to take into account the set of orthonormal vectors \( \langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j} \), the sign \( \varepsilon(i) \) and \( \hat{p} \), the half-sum of positive roots of the \( q \)-deformed affine Lie algebras, in order to write explicitly the TL projector for each model:

• \( A_1^{(1)} \): the \( U_q[sl(2)] \) spin-s Temperley–Lieb model

\[
U = \sum_{i=1}^{N} \sum_{j=1}^{N} (-1)^{i+j} q^{i+j-N-1}e_{i,j} \otimes e_{i',j'},
\]

\[
2 \cosh \eta = [2s + 1],
\]
where $N = 2s + 1$ $(s = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots)$. Note the use of the quantum number notation $[n] = (q^n - q^{-n})/(q - q^{-1})$ in the trace of the $U$ projectors.

- $B_n^{(1)}(n \geq 2)$: the $U_q[so(2n+1)]$ Temperley–Lieb model

\[
U = \sum_{i=1}^{n} \sum_{j=1}^{n} q^{i+j-2n-1} e_{i,j} \otimes e_{i',j'} - \sum_{i=1}^{n} q^{i-n-1/2} e_{i,n+1} \otimes e_{i',n+1} + \sum_{i=1}^{n} \sum_{j=n+2}^{2n+1} q^{i+j-2n-2} e_{i,j} \otimes e_{i',j'} - \sum_{i=1}^{n} q^{i-j-1/2} e_{n+1,j} \otimes e_{n+1,j'} + e_{n+1,n+1} \otimes e_{n+1,n+1} - \sum_{j=n+2}^{2n+1} q^{j-n-3/2} e_{n+1,j} \otimes e_{n+1,j'} \\
+ \sum_{i=n+2}^{2n+1} \sum_{j=1}^{n} q^{i+j-2n-2} e_{i,j} \otimes e_{i',j'} - \sum_{i=n+2}^{2n+1} q^{i-j-3/2} e_{i,n+1} \otimes e_{i',n+1} + \sum_{i=n+2}^{2n+1} \sum_{j=n+2}^{2n+1} q^{i+j-2n-3} e_{i,j} \otimes e_{i',j'}
\]

\[
2 \cosh \eta = \frac{[2n-1][n+1/2]}{[n-1/2]}.
\]

- $C_n^{(1)}(n \geq 1)$: the $U_q[sp(2n)]$ Temperley–Lieb model

\[
U = \sum_{i=1}^{n} \sum_{j=1}^{n} q^{i+j-2n-2} e_{i,j} \otimes e_{i',j'} - \sum_{i=1}^{n} q^{i+j-2n-1} e_{i,j} \otimes e_{i',j'} - \sum_{i=1}^{n} \sum_{j=n+1}^{2n} q^{i-j-2n-1} e_{i,j} \otimes e_{i',j'} + \sum_{i=n+1}^{2n} \sum_{j=n+1}^{2n} q^{i+j-2n} e_{i,j} \otimes e_{i',j'}
\]

\[
2 \cosh \eta = \frac{[n][2n+2]}{[n+1]}.
\]

- $D_n^{(1)}(n \geq 3)$: the $U_q[so(2n)]$ Temperley–Lieb model

\[
U = \sum_{i=1}^{n} \sum_{j=1}^{n} q^{i+j-2n} e_{i,j} \otimes e_{i',j'} + \sum_{i=1}^{n} \sum_{j=n+1}^{2n} q^{i+j-2n-1} e_{i,j} \otimes e_{i',j'} + \sum_{i=n+1}^{2n} \sum_{j=n+1}^{2n} q^{i+j-2n-2} e_{i,j} \otimes e_{i',j'}
\]

\[
2 \cosh \eta = \frac{[n][2n-2]}{[n-1]}.
\]

We also have to consider the permuted operator $\mathcal{R} = PR$ which is regular satisfying PT-symmetry, unitarity and crossing symmetry:

\[
\mathcal{R}_{12}(0) = P, \\
\mathcal{R}_{12}^{t_{12}}(u) = P \mathcal{R}_{12}(u) P = \mathcal{R}_{21}(u), \\
\mathcal{R}_{12}(u) \mathcal{R}_{21}(-u) = x_1(u) x_1(-u) I, \\
\mathcal{R}_{21}(u) = \kappa(V \otimes 1) \mathcal{R}_{12}^{t_{12}}(-u - \rho)(V \otimes 1)^{-1},
\]

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where $\rho = -\eta$ is the crossing parameter, $\kappa = (-1)^{2s}$ for $A_1^{(1)}$, $\kappa = -1$ for $C_n^{(1)}$ and $\kappa = 1$ for $B_n^{(1)}$ and $D_n^{(1)}$. $P$ is the permutation operator: $P(a \otimes b) = b \otimes a$ for any vectors $a, b$.

The crossing matrices $V$ for the TL models are specified in [24] by

$$V_{i,j} = \varepsilon(i) q^{-\epsilon(i)\rho} \delta_{i,j}. \quad (2.12)$$

However, for the isomorphism (1.4) we only need the diagonal matrix $M = V^t V$ for each model:

$$M_{i,i} = q^{2i-2n-2} \quad i = 1, \ldots, 2s + 1 \quad \text{for } A_1^{(1)} \text{ spin } s \quad (2.13)$$

$$M_{i,i} = \begin{cases} q^{2i-2n-1} & i = 1, \ldots, n \\ 1 & i = n + 1 \\ q^{2i-2n-3} & i = n + 2, \ldots, 2n + 1 \end{cases} \quad \text{for } B_n^{(1)} \quad (2.14)$$

$$M_{i,i} = \begin{cases} q^{2i-2n-2} & i = 1, \ldots, n \\ q^{2i-2n} & i = n + 1, \ldots, 2n \end{cases} \quad \text{for } C_n^{(1)} \quad (2.15)$$

and

$$M_{i,i} = \begin{cases} q^{2i-2n} & i = 1, \ldots, n \\ 1 & i = n + 1, n + 2 \\ q^{2i-2n-2} & i = n + 3, \ldots, 2n + 1 \end{cases} \quad \text{for } D_n^{(1)} \quad (2.16)$$

The Hamiltonian limit

$$R(u) = I + u(\alpha^{-1} H + \beta I) \quad (2.17)$$

with $\alpha = \sinh \eta$, $\beta = -\coth \eta$ leads to the quantum spin chains

$$H = \sum_{k=1}^{N-1} U_{k,k+1} + bt, \quad (2.18)$$

where, instead of a periodic boundary condition, we are taking into account the existence of integrable boundary terms $bt$ [7], derived from the $K^-$ and $K^+$ matrices presented in the next sections.

3. The reflection matrices

In the reflection equation (1.2) we used the notation $K_1^+ = K^- \otimes I$, $K_2^- = I \otimes K^-$, $R_{12} = R$ and $R_{12}^{t12} = PRP$. For a given $R$-matrix the unknown is the $N \times N$ matrix $K^{-}(u)$ satisfying the normal condition $K^{-}(0) = I$. The dimension $N$ is equal to $2s + 1, 2n + 1, 2n$ and $2n + 1$ for $A_1^{(1)}, B_n^{(1)}, C_n^{(1)}$ and $D_n^{(1)}$, respectively.

Substituting

$$K^{-}(u) = \sum_{i,j=1}^{N} k_{i,j}(u)e_{i,j} \quad (3.1)$$

and $R(u) = P[x_1(u)I + x_2(u)U]$ into (1.2), we have $N^4$ functional equations for the $k_{i,j}$ elements, many of them not independent equations. In order to solve these functional equations, we proceed as follows. First we consider the $(i, j)$ component of the matrix...
equation (1.2). By differentiating it with respect to $v$ and taking $v = 0$, we get algebraic equations involving the single variable $u$ and $N^2$ parameters

\[ \beta_{i,j} = \frac{d k_{i,j}(v)}{dv} \bigg|_{v=0}, \quad i, j = 1, 2, \ldots, N. \quad (3.2) \]

Analyzing the reflection equations one can see that they possess a special structure. Several equations exist involving only two non-diagonal elements. They can be solved by the relations

\[ k_{i,j}(u) = \frac{\beta_{i,j}}{\beta_{1,N}} k_{1,N}(u) \quad i \neq j = \{1, 2, \ldots, N\}. \quad (3.3) \]

We are thus left with several equations involving two diagonal elements and $k_{1,N}(u)$. Such equations are solved by the relations

\[ k_{i,i} = k_{1,1}(u) + (\beta_{i,i} - \beta_{1,1}) \frac{k_{1,N}(u)}{\beta_{1,N}} \quad i = 2, 3, \ldots, N. \quad (3.4) \]

Finally, we can use the equation for $(1, N)$ in order to find the element $k_{1,1}(u)$:

\[ k_{1,1}(u) = \frac{k_{1,N}(u)}{\beta_{1,N} x_2(u) \cosh \eta + x_1(u)} \left\{ x_1(u) x_2'(u) - x_1'(u) x_2(u) \right\} \]

\[ - \frac{1}{2} x_1(u)(\beta_{N,N} - \beta_{1,1} + \Psi_{1,N}) - \frac{1}{2} x_2(u) \sum_{j=2}^{N} (\beta_{j,j} - \beta_{1,1}) M_{j,j} \right\}, \quad (3.5) \]

where $M_{j,j}$ are given in (2.13)–(2.16) and $\Psi_{1,N}$ belongs to the set of new relations of the parameters $\beta_{i,j}$ defined by

\[ \Psi_{i,j} = \frac{1}{\beta_{i,j}} \sum_{k=2}^{N-1} \beta_{i,k} \beta_{k,j} \quad i \neq j = 1, \ldots, N. \quad (3.6) \]

The prime in the Boltzmann weights $x_i(u)$ means its first derivative with respect to $u$.

Now, substituting these expressions into the remaining equations $(i, j)$, we are left with factored equations of the form

\[ F_a(\beta_{i,j}) q^a x_1(u) x_2(u) k_{1,N}(u) = 0, \quad (3.7) \]

where each factor $F_a(\beta_{i,j})$ does not depend on the weights $x_i(u)$ nor on the corresponding quantum group $q$-parameter. This means that they are the same in all four models we consider. Therefore the computation used in [34] gives us a general procedure.

First, we collect all matrix element $(i, j)$ of (1.2) in blocks of four equations [38]

\[ B[i, j] = \{(i, j), (j, i), (i'', j''), (j'', i'')\} \quad i = 1, \ldots, N, \quad j = i, \ldots, i''. \quad (3.8) \]

where $a'' = N^2 + 1 - a$.

From the first equation of the blocks $B[j, N - 1], j = 2, \ldots, N - 2$ one can fix $N - 1$ diagonal parameters

\[ \beta_{j,j} = \beta_{1,1} + \Psi_{1,N} - \Psi_{j,N} \quad j = 2, 3, \ldots, N - 1 \quad (3.9) \]

and the first equation of the block $B[N, N + 1]$ fixes the parameter $\beta_{N,N}$:

\[ \beta_{N,N} = \beta_{1,1} + \Psi_{1,N-1} - \Psi_{N-1,N}. \quad (3.10) \]
All equations from the block $B[1,k]$ to the block $B[N-1,k]$ are now substituted by $N(N-1)/2$ symmetric relations

$$\Psi_{j,i} = \Psi_{i,j}, \quad j > i$$ (3.11)

and $2(N-3)$ relations involving four $\Psi_{i,j}$ functions:

$$\Psi_{2,j} = \Psi_{2,3} + \Psi_{1,j} - \Psi_{1,3}, \quad \Psi_{3,j} = \Psi_{2,3} + \Psi_{1,j} - \Psi_{1,2}, \quad j = 4, \ldots, N.$$ (3.12)

The remaining equations contained in the block $B[N,k]$ are rewritten by $(N-3)(N-4)/2$ relations involving six $\Psi_{i,j}$ functions,

$$\Psi_{i,j} = \Psi_{1,i} + \Psi_{1,j} + \Psi_{2,3} - \Psi_{1,2} - \Psi_{1,3}, \quad i = 4, \ldots, N-1, \quad j = i + 1, \ldots, N$$ (3.13)

and $2N-3$ relations involving the diagonal $\beta_{k,k}$ parameters, $\Psi_{1,N}$ and a new function $\Theta_{j,j}$,

$$\Theta_{j,j} = \Theta_{N,N} + (\beta_{N,N} - \beta_{j,j})(\beta_{j,j} - \beta_{1,1} - \Psi_{1,N}), \quad j = 2, 3, \ldots, N-1,$$

$$\Theta_{j',j'} = \Theta_{1,1} + (\beta_{j',j'} - \beta_{N,N} - \Psi_{1,N}), \quad j = 2, 3, \ldots, N-1,$$

$$\Theta_{N,N} = \Theta_{1,1} - (\beta_{1,1} - \beta_{N,N})\Psi_{1,N},$$

where $j' = N + 1 - j$ and

$$\Theta_{j,j} = \sum_{k \neq j} \beta_{j,k}\beta_{k,j}.$$ (3.15)

From (3.11)–(3.14) one can account for $N^2 - 3$ constraint equations but, after substitution of the relations (3.9) and (3.10) into (3.14), we only need to look at the symmetric relations (3.11). However, for computational convenience, we add all relations with four $\Psi_{i,j}$ functions (3.12). Therefore, our final task is to look for the solutions of $N(N-2)$ constraint equations.

From these relations we have fixed $N(N-2)/2 - 1/2$ parameters $\beta$ for the TL models with $N$ odd and $N(N-2)/2 - 1$ parameters for the TL models with $N$ even.

Taking into account that the parameter $\beta_{1,1}$ is determined by the normal condition, we end the calculus with $K$-matrix solutions of (1.2) with $N^2/2 + 1/2$ free parameters $\beta_{i,j}$ if $N$ is odd and with $N^2/2 + 1$ free parameters if $N$ is even.

Now, let us describe the corresponding diagonal $K$-matrix solutions.

### 3.1. The diagonal solutions

Taking into account only the diagonal entries, the reflection equations are solved when we find all matrix elements $k_{j,j}(u)$, $j = 2, \ldots, N$, as a function of $k_{1,1}(u)$, provided that the diagonal parameters $\beta_{j,j}$ satisfy $(N-1)(N-2)/2$ constraint equations of the type

$$(\beta_{N,N} - \beta_{i,i})(\beta_{N,N} - \beta_{j,j})(\beta_{j,j} - \beta_{i,i}) = 0 \quad (i \neq j \neq N).$$ (3.16)

From (3.16) we find solutions with only two types of entries. Let us normalize one of them to be equal to 1 such that the other one has the form

$$k_{p,p}(u) = -\frac{\beta_{p,p}x_2(u)[\Delta_1x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{p,p}x_2(u)[\Delta_2x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}$$ (3.17)

with $\Delta_1 + \Delta_2 = 2\cosh \eta$.

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Identifying the diagonal positions of the $K$-matrix with the matrix elements of the $M$-matrix (2.13)–(2.16), (1, 2, ..., $N$) $\cong (M_{1,1}, M_{2,2}, \ldots, M_{N,N})$ one can see that $\Delta_1$ is the sum of the $M_{i,j}$ corresponding to the positions of the entries 1, and $\Delta_2$ is the sum of the $M_{j,i}$ corresponding to the positions of the entries $k_{p,p}(u)$.

We denote the diagonal solutions by $K_{a}^{[r]}$ where $a = (a_1, a_2, \ldots, a_N)$ with $a_i = 0$ if $k_{i,i}(u) = 1$ or $a_i = 1$ if $k_{i,i}(u) = k_{pp}(u)$, with $r$ being the number of the entries $k_{p,p}(u)$ distributed on diagonal positions, where $p$ is the first position with entry different from 1. Thus, we count

$$Z = \sum_{r=1}^{N-1} \frac{N!}{r!(N-r)!}$$

(3.18)

for the number of diagonal $K^-$ matrix solutions with one free parameter.

The dual equation (1.3) is solved by the $K^+$ matrices via the isomorphism (1.4) with $\rho = -\eta$ and the matrix $M$ specified by (2.13)–(2.16). Here we note that the trace of each diagonal $M$-matrix is equal to $2 \cosh \eta$.

Now, we explicitly show these computations for the first models. Here, we can use the identity

$$x_2(u)[x_2(u) \cosh \eta + x_1(u)] = \sinh(u) \cosh(u)$$

(3.19)

in order to simplify our presentation. From the solution ((3.3) to (3.5)) one can see $k_{1,N}(u)$ as an arbitrary function satisfying the normal condition. Therefore, the choice

$$k_{1,N}(u) = \frac{1}{2} \beta_{1,N} \sinh(2u)$$

(3.20)

do not imply any restriction as compared to the general case.

### 3.2. The $A_1^{(1)}$ spin-$\frac{1}{2}$ and $C_1^{(1)}$ Temperley–Lieb $K$-matrices

For these models we have the well-known three free parameter solution for the $U_q[sl(2)]$ spin-$\frac{1}{2}$ model \[21, 22\]

$$K^-(u) = \begin{pmatrix} k_{1,1}(u) & \frac{1}{2} \beta_{1,2} \sinh(2u) \\ \frac{1}{2} \beta_{2,1} \sinh(2u) & k_{1,1}(u) + \frac{1}{2}(\beta_{2,2} - \beta_{1,1}) \sinh(2u) \end{pmatrix}.$$  

(3.21)

Using the identity (3.19) and (3.20), the expression for $k_{1,1}(u)$ (3.5) has a simplified form

$$k_{1,1}(u) = 1 - \frac{1}{2}(\beta_{2,2} - \beta_{1,1}) \left[x_1(u) + q \ x_2(u)\right] x_2(u) \sinh \eta$$

(3.22)

where $\beta_{1,2}$, $\beta_{2,1}$ and $\beta_{2,2}$ are the free parameters and $2 \cosh \eta = q + q^{-1}$. Moreover, we find that the $U_q[sp(2)]$ TL model has the same $K$-matrix form but with

$$k_{1,1}(u) = 1 - \frac{1}{2}(\beta_{2,2} - \beta_{1,1}) \left[x_1(u) + q^2 \ x_2(u)\right] x_2(u) \sinh \eta$$

(3.23)

since now $2 \cosh \eta = q^2 + q^{-2}$. 

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The entries of the diagonal solutions \( k_{1,1}(u) \) and \( k_{2,2}(u) \) are given by (3.17) and we have two solutions for each model:

\[
K^{[1]}_{(1,0)} = \begin{pmatrix} k_{1,1}(u) & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta_1 = M_{2,2}, \quad \Delta_2 = M_{1,1}
\]

\[
K^{[1]}_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & k_{2,2}(u) \end{pmatrix}, \quad \Delta_1 = M_{1,1}, \quad \Delta_2 = M_{2,2},
\] (3.24)

where \( M_{1,1} = q^{-1}, \ M_{2,2} = q \) for \( U_q[sl(2)] \) spin-\( \frac{1}{2} \) model and \( M_{1,1} = q^{-2}, \ M_{2,2} = q^{2} \) for \( U_q[sp(2)] \) model. Of course, in both models \( K^{[1]}_{(1,0)} \) and \( K^{[1]}_{(0,1)} \) are equivalent by the exchange \( q \leftrightarrow q^{-1} \).

### 3.3. The \( A_1^{(1)} \) spin-1 Temperley–Lieb K-matrices

For the biquadratic model [39, 40], it follows from (3.3) and (3.4) that

\[
K^{-}(u) = \begin{pmatrix} k_{1,1}(u) & \frac{1}{2} \beta_{1,2} \sinh(2u) & \frac{1}{2} \beta_{1,3} \sinh(2u) \\ \frac{1}{2} \beta_{2,1} \sinh(2u) & k_{1,1}(u) + \frac{1}{2} (\beta_{2,2} - \beta_{1,1}) \sinh(2u) & \frac{1}{2} \beta_{2,3} \sinh(2u) \\ \frac{1}{2} \beta_{3,1} \sinh(2u) & \frac{1}{2} \beta_{3,2} \sinh(2u) & k_{1,1}(u) + \frac{1}{2} (\beta_{3,3} - \beta_{1,1}) \sinh(2u) \end{pmatrix}
\] (3.25)

where \( k_{1,1}(u) \) is given by (3.8):

\[
k_{1,1}(u) = 1 - \frac{1}{2} \{ (\beta_{3,3} - \beta_{1,1}) [x_1(1) + M_{3,3} x_2(u)] + x_1(u) \Psi_{1,3} \\
+ (\beta_{2,2} - \beta_{1,1}) M_{2,2} x_2(u) \} x_2(u) \sinh \eta.
\] (3.26)

The diagonal parameters are fixed by the constraint equations (3.9) and (3.10):

\[
\beta_{2,2} = \beta_{1,1} + \Psi_{1,3} - \Psi_{2,3} = \beta_{1,1} + \frac{\beta_{1,2} \beta_{2,3}}{\beta_{1,3}} - \frac{\beta_{2,1} \beta_{1,3}}{\beta_{2,3}}
\] (3.27)

\[
\beta_{3,3} = \beta_{1,1} + \Psi_{2,1} - \Psi_{2,3} = \beta_{1,1} + \frac{\beta_{1,3} \beta_{3,2}}{\beta_{1,2}} - \frac{\beta_{2,1} \beta_{1,3}}{\beta_{2,3}}
\] (3.28)

and \( \beta_{11} \) is fixed by the normal condition. Moreover, all remaining constraint equations are solved by the relation

\[
\beta_{3,1} = \beta_{1,3} \frac{\beta_{3,2} \beta_{2,1}}{\beta_{1,2} \beta_{2,3}} \quad \text{or} \quad \Psi_{3,1} = \Psi_{1,3}
\] (3.29)

and we have derived a five free parameter solution.

Here \( 2 \cosh \eta = q^{-2} + 1 + q^{2} \). This means that \( M_{11} = q^{-2}, \ M_{22} = 1 \) and \( M_{33} = q^{2} \). Among the several possibilities, we made the choice \( \beta_{1,2}, \beta_{1,3}, \beta_{2,1}, \beta_{2,3} \) and \( \beta_{3,2} \) for the free parameters.

There are six corresponding diagonal solutions, half of them with one entry different from 1:

\[
K^{[1]}_{(1,0,0)} = \begin{pmatrix} k_{1,1}(u) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_1 = M_{2,2} + M_{3,3}, \quad \Delta_2 = M_{1,1}
\] (3.30)
and three further diagonal solutions with two equal entries different from unity:

\[
\begin{align*}
K_{(0,1,0)}^{[1]} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{2,2}(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \Delta_1 = M_{1,1} + M_{3,3}, & \Delta_2 = M_{2,2} \quad (3.31) \\
K_{(0,0,1)}^{[1]} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k_{3,3}(u) \end{pmatrix}, & \Delta_1 = M_{1,1} + M_{2,2}, & \Delta_2 = M_{3,3} \quad (3.32)
\end{align*}
\]

and three further diagonal solutions with two equal entries different from unity:

\[
\begin{align*}
K_{(1,1,0)}^{[2]} &= \begin{pmatrix} k_{1,1}(u) & 0 & 0 \\ 0 & k_{1,1}(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \Delta_1 = M_{3,3}, & \Delta_2 = M_{1,1} + M_{2,2} \quad (3.33) \\
K_{(1,0,1)}^{[2]} &= \begin{pmatrix} k_{1,1}(u) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k_{1,1}(u) \end{pmatrix}, & \Delta_1 = M_{2,2}, & \Delta_2 = M_{1,1} + M_{3,3} \quad (3.34) \\
K_{(0,1,1)}^{[2]} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{2,2}(u) & 0 \\ 0 & 0 & k_{2,2}(u) \end{pmatrix}, & \Delta_1 = M_{1,1}, & \Delta_2 = M_{2,2} + M_{3,3}, \quad (3.35)
\end{align*}
\]

where the entries \(k_{p,p}(u)\) are given by (3.17).

Here we notice again that the difference between the diagonal entries comes from the partitions of \(\Delta_1 + \Delta_2 = q^{-2} + 1 + q^2\) and the equivalence between them due to the symmetry \(q \leftrightarrow q^{-1}\).

We also note that if \(q^2\) is replaced by \(q\), the equivalence \(B_1^{[1]} \simeq A_1^{[1]}\) spin-1 is manifested in all expressions above, since \(2 \cosh \eta = q^{-1} + 1 + q\) for \(B_1^{[1]}\).

These diagonal solutions were recently used in [42] to study the spectrum of the spin-1 TL spin chain with integrable open boundary conditions.

### 3.4. The \(A_1^{(1)}\) spin-\(\frac{3}{2}\) and \(C_2^{(1)}\) Temperley–Lieb K-matrices

For \(\mathcal{U}_q[sl(2)]\) spin-\(\frac{3}{2}\) and \(\mathcal{U}_q[sp(4)]\) models, we have from (3.3) to (3.8) the following non-diagonal entries:

\[
k_{i,j}(u) = \frac{1}{2} \beta_{i,j} \sinh(2u), \quad (i \neq j = 1, 2, 3, 4) \quad (3.36)
\]

and the diagonal one

\[
k_{i,i}(u) = k_{1,1}(u) + \frac{1}{2}(\beta_{i,i} - \beta_{1,1}) \sinh(2u), \quad (i = 2, 3, 4) \quad (3.37)
\]

with

\[
k_{1,1}(u) = 1 - \frac{1}{2} \left\{ (\beta_{4,4} - \beta_{1,1}) [x_1(u) + M_{4,4}x_2(u)] + x_1(u)\Psi_{1,4} \\
+ ((\beta_{2,2} - \beta_{1,1})M_{2,2} + (\beta_{3,3} - \beta_{1,1})M_{3,3})x_2(u) \right\} x_2(u) \sinh \eta, \quad (3.38)
\]

where \(M_{1,1} = q^{-3}, M_{2,2} = q^{-1}, M_{3,3} = q\) and \(M_{4,4} = q^3\) for the \(\mathcal{U}_q[sl(2)]\) spin-\(\frac{3}{2}\) model and \(M_{1,1} = q^{-4}, M_{2,2} = q^{-2}, M_{3,3} = q^2\) and \(M_{4,4} = q^4\) for the \(\mathcal{U}_q[sp(4)]\) model.
From equations (3.9) and (3.10) we choose to fix the following diagonal parameters:

\[
\beta_{2,2} = \beta_{1,1} + \Psi_{1,4} - \Psi_{2,4} = \beta_{1,1} + \frac{\beta_{1,2}\beta_{2,4} + \beta_{1,3}\beta_{3,4}}{\beta_{1,4}} - \frac{\beta_{2,1}\beta_{1,4} + \beta_{2,3}\beta_{3,4}}{\beta_{2,4}},
\]

\[
\beta_{3,3} = \beta_{1,1} + \Psi_{1,4} - \Psi_{3,4} = \beta_{1,1} + \frac{\beta_{1,2}\beta_{2,4} + \beta_{1,3}\beta_{3,4}}{\beta_{1,4}} - \frac{\beta_{3,1}\beta_{1,4} + \beta_{3,2}\beta_{2,4}}{\beta_{3,4}},
\]

\[
\beta_{4,4} = \beta_{1,1} + \Psi_{1,3} - \Psi_{3,4} = \beta_{1,1} + \frac{\beta_{1,2}\beta_{2,3} + \beta_{1,4}\beta_{4,3}}{\beta_{1,3}} - \frac{\beta_{3,1}\beta_{1,4} + \beta_{3,2}\beta_{2,4}}{\beta_{3,4}}.
\]

(3.39)

All remaining constraint equations are solved by the choice

\[
\beta_{4,1} = \frac{\beta_{4,2}\beta_{2,1} + \beta_{4,3}\beta_{3,1}}{\beta_{1,2}\beta_{2,4} + \beta_{1,3}\beta_{3,4}} \beta_{1,4} \quad \text{or} \quad \Psi_{4,1} = \Psi_{1,4}
\]

\[
\beta_{3,2} = -\frac{\beta_{3,1}\beta_{1,2} + \beta_{3,4}\beta_{4,2}}{\beta_{1,2}\beta_{2,4} + \beta_{1,3}\beta_{3,4}} \beta_{1,4} \quad \text{or} \quad \Psi_{3,2} = -\Psi_{1,4}
\]

\[
\beta_{2,3} = -\frac{\beta_{2,1}\beta_{1,3} + \beta_{2,4}\beta_{4,3}}{\beta_{1,2}\beta_{2,4} + \beta_{1,3}\beta_{3,4}} \beta_{1,4} \quad \text{or} \quad \Psi_{2,3} = -\Psi_{1,4}.
\]

(3.40)

This means that we have found a $K^-$ matrix with nine free parameters.

There are four diagonal solutions with one entry $k_{p,p}(u)$ and three equal to unity:

\[
K^{[1]}_{(1,0,0,0)}: \Delta_1 = M_{1,1}, \quad \Delta_2 = M_{2,2} + M_{3,3} + M_{4,4}
\]

\[
: \quad \Delta_1 = M_{4,4}, \quad \Delta_2 = M_{1,1} + M_{2,2} + M_{3,3}.
\]

There are six diagonal solutions with two entries $k_{p,p}(u)$ and two equal to unity,

\[
K^{[2]}_{(1,1,0,0)}: \Delta_1 = M_{1,1} + M_{2,2}, \quad \Delta_2 = M_{3,3} + M_{4,4}
\]

\[
: \quad \Delta_1 = M_{3,3} + M_{4,4}, \quad \Delta_2 = M_{1,1} + M_{2,2}
\]

and four more solutions with three entries $k_{p,p}(u)$ and one equal to unity:

\[
K^{[3]}_{(1,1,1,0)}: \Delta_1 = M_{1,1} + M_{2,2} + M_{3,3}, \quad \Delta_2 = M_{4,4}
\]

\[
: \quad \Delta_1 = M_{2,2} + M_{3,3} + M_{4,4}, \quad \Delta_2 = M_{1,1}.
\]

(3.43)

Remember that in these 14 diagonal solutions we have $\Delta_1 + \Delta_2 = q^{-3} + q^{-1} + q + q^3$ for the $A_1^{(1)}$ spin-$\frac{3}{2}$ model and $\Delta_1 + \Delta_2 = q^{-4} + q^{-2} + q^2 + q^4$ for the $C_2^{(1)}$ model.

3.5. The $A_1^{(1)}$ spin-2 and $B_2^{(1)}$ Temperley–Lieb K-matrices

For $N = 5$, the matrix elements are

\[
k_{i,j}(u) = \frac{1}{2}\beta_{i,j} \sinh(2u), \quad i \neq j = 1, \ldots, 5
\]

(3.44)

and

\[
k_{i,i}(u) = k_{1,1}(u) + \frac{1}{2}(\beta_{i,i} - \beta_{1,1}) \sinh(2u), \quad i = 2, \ldots, 5
\]

(3.45)
where
\[ k_{1,1}(u) = 1 - \frac{1}{2} \{ (\beta_{5,5} - \beta_{1,1}) [x_1(u) + M_{5,5}x_2(u)] + \Psi_{1,5}x_1(u) + [(\beta_{2,2} - \beta_{1,1})M_{2,2} + (\beta_{3,3} - \beta_{1,1})M_{3,3} + (\beta_{4,4} - \beta_{1,1})M_{4,4}]x_2(u) \} x_2(u) \sinh \eta. \] (3.46)

The diagonal parameters are given by (3.9),
\[ \beta_{2,2} = \beta_{1,1} + \Psi_{1,5} - \Psi_{2,5}, \quad \beta_{3,3} = \beta_{1,1} + \Psi_{1,5} - \Psi_{3,5} \] (3.47)
and by (3.10)
\[ \beta_{5,5} = \beta_{1,1} + \Psi_{1,4} - \Psi_{4,5}. \] (3.48)
Moreover, we have \( 5(5 - 1)/2 = 10 \) symmetric relations
\[ \Psi_{j,i} = \Psi_{i,j} \quad (j > i), \] (3.49)
2(5 - 3) = 4 relations with four \( \Psi_{i,j} \),
\[ \begin{align*}
\Psi_{2,4} &= \Psi_{2,3} + \Psi_{1,4} - \Psi_{1,3}, \\
\Psi_{2,5} &= \Psi_{2,3} + \Psi_{1,5} - \Psi_{1,3}, \\
\Psi_{3,4} &= \Psi_{2,3} + \Psi_{1,4} - \Psi_{1,2}, \\
\Psi_{3,5} &= \Psi_{2,3} + \Psi_{1,5} - \Psi_{1,2},
\end{align*} \] (3.50)
and (5 - 3)(5 - 4)/2 = 1 relation with six \( \Psi_{i,j} \),
\[ \Psi_{4,5} = \Psi_{1,5} + \Psi_{1,4} + \Psi_{2,3} - \Psi_{1,2} - \Psi_{1,3}. \] (3.51)
As mentioned above, these 5(5 - 2) = 15 relations are enough to fix the seven remaining parameters:
\[ \beta_{2,1} = \frac{\beta_{3,2}\beta_{4,3}\beta_{5,3}}{\beta_{1,3}^2} - \frac{\beta_{2,3}\beta_{3,5}}{\beta_{1,5}} - \frac{\beta_{2,5}\beta_{5,3}}{\beta_{1,3}} - \frac{\beta_{2,3}\beta_{4,5}(\beta_{1,4}\beta_{2,5} - \beta_{1,5}\beta_{2,4})}{\beta_{1,5}(\beta_{1,3}\beta_{2,5} - \beta_{1,5}\beta_{2,3})} \]
\[ - \frac{\beta_{2,5}\beta_{4,3}(\beta_{1,3}\beta_{2,4} - \beta_{1,4}\beta_{2,3})}{\beta_{1,3}(\beta_{1,3}\beta_{2,5} - \beta_{1,5}\beta_{2,3})} \] (3.52)
from which we can find \( \beta_{3,1}, \beta_{4,1} \) and \( \beta_{5,1} \), replacing the indices (2 \( \leftrightarrow \) 3), (2 \( \leftrightarrow \) 4) and (2 \( \leftrightarrow \) 5), respectively. From the parameter
\[ \beta_{3,2} = \frac{\beta_{1,2}\beta_{1,4}\beta_{4,5}}{\beta_{1,3}\beta_{1,5}} + \frac{\beta_{1,2}\beta_{3,5}}{\beta_{1,5}} - \frac{\beta_{1,4}\beta_{4,2}}{\beta_{1,3}} + \frac{(\beta_{1,3}\beta_{2,5} - \beta_{1,5}\beta_{2,3})(\beta_{1,2}\beta_{4,5} - \beta_{1,5}\beta_{4,2})}{(\beta_{1,3}\beta_{4,5} - \beta_{1,5}\beta_{4,3})} \]
\[ \times \left\{ \begin{array}{l}
\beta_{1,2} + \beta_{1,4}\beta_{5,4} - \beta_{1,4}\beta_{5,3} \\
\beta_{1,3}\beta_{2,4} - \beta_{1,4}\beta_{2,3}
\end{array} \right\} \] (3.53)
we can also find \( \beta_{5,2} \), replacing the indices (3 \( \leftrightarrow \) 5). The last parameter is
\[ \beta_{3,4} = \frac{\beta_{1,4}\beta_{3,5}}{\beta_{1,5}} + \frac{\beta_{1,4}\beta_{2,4} - \beta_{1,5}\beta_{2,4}}{\beta_{1,3}\beta_{1,5}} \left( \begin{array}{c}
\beta_{1,4}(\beta_{1,3}\beta_{4,5} - \beta_{1,5}\beta_{4,3}) \\
\beta_{1,3}\beta_{4,5} - \beta_{1,5}\beta_{4,3}
\end{array} \right) \]
\[ + \beta_{1,2} \] (3.54)
These seven parameters plus the five diagonal parameters \( \beta_{i,i} \) give us a \( 5 \times 5 \) reflection matrix solution with 13 free parameters. Similarly, the corresponding 30 diagonal solutions can be written for both models.

\[ \text{doi:10.1088/1742-5468/2013/10/P10021} \]
In the sequence \((N \geq 6)\), the expressions of the fixed parameters are too large and cumbersome.

### 3.6. A reduced solution

An important characteristic of the TL boundary solutions is their large number of free parameters. This means that we have many different reduced solutions for a given \(R\)-matrix. In particular, choosing the free parameters one can get an appropriate \(K\)-matrix solution. For instance, if we consider all \(\beta_{i,j} = \beta\) \((i \neq j)\) and all \(\beta_{i,i} = \alpha\), in (3.5), we will get a one-parameter solution of the form

\[
K(u) = f_1(u)I + f_2(u)G, \tag{3.55}
\]

where

\[
f_1(u) = 1 - \frac{N - 2}{2} \beta x_1(u)x_2(u) \sinh(\eta) \tag{3.56}
\]

\[
f_2(u) = \frac{1}{2} \beta \sinh(2u)
\]

and \(G\) is a \(N \times N\) matrix with entries

\[
G_{i,j} = \begin{cases} 
0 & \text{if } i = j \\
1 & \text{if } i \neq j. 
\end{cases} \tag{3.57}
\]

Although \(G\) satisfies a quadratic relation as (1.5),

\[
G^2 - (N - 2)G - (N - 1)I = 0, \tag{3.58}
\]

we do not know whether the solution (3.55) can be fitted with (1.6). But, certainly, the infinity spectral parameter limit of the solutions presented above will solve the constant reflection equations [15].

Many other reduced solutions can be derived in a similar way. See, for instance, the cases presented in [34] for the \(U_q[sl(2)]\) model.

### 4. Conclusion

In this work we have presented solutions of the reflection equation for the TL vertex models. Our findings can be summarized into two classes of solutions depending on \(N\)-parity. A large number of free parameters is an important characteristic of these solutions.

In analogy with the \(R\)-matrices form (2.1), the \(K\)-matrices have the same form as (3.3) and (3.4), with all quantum group dependence in the diagonal entries through the matrix elements of \(M\).

These results pave the way to construct, solve and study the physical properties of the underlying quantum spin chains with open boundaries, generalizing the previous efforts made for the case of periodic boundary conditions [41, 43].

We expect that the coordinate Bethe ansatz for all diagonal solutions presented here can be obtained by adapting the results of [42, 44] and that its generalization, as in [45], may be a possibility to treat the non-diagonal solutions. We also expect that these results will give rise to further developments of the subject of integrable open boundaries for the TL vertex models based on superalgebras [46].
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