MORSE FLOWS WITH FIXED POINTS ON THE BOUNDARY OF 3-MANIFOLDS

A. O. Prishlyak\textsuperscript{1,2}, S. V. Bilun\textsuperscript{3}, and A. A. Prus\textsuperscript{4} 

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We investigate the topological properties, structures, and classifications of the Morse flows with fixed points on the boundary of three-dimensional manifolds. We construct a complete topological invariant of a Morse flow, namely, $Pr$-diagram, which is similar to the Heegaard diagram of a closed three-dimensional manifold.

Introduction

On each closed manifold, a vector field always generates a flow. In the case of a compact manifold with boundary, a vector field generates a flow if and only if it touches the boundary at every point of the boundary [1].

A vector field $X$ on a manifold $M$ is called structurally stable if, in the set of all vector fields on the manifold $M$, there exists a neighborhood $U$ such that any field $Y \in U$ is topologically equivalent to the field $X$.

On closed surfaces, structurally stable vector fields are Morse–Smale fields. For manifolds of higher dimensions, parallel with Morse–Smale vector fields, there exist other types of structurally stable vector fields. The property of structural stability of the vector fields on closed manifolds was investigated in [2–7]. For manifolds with boundary, an analog of Morse–Smale fields was described in [6, 8, 9].

In the case of dimension 2 (closed manifolds), Morse flows (Morse–Smale flows without closed trajectories) have three types of singularities: sources, saddles, and sinks. If we consider Morse flows with singularities on the boundary $\partial M$, then the following four types of singularities are possible on the surfaces: sources, sinks (Fig. 1), and two types of saddles (Fig. 2).

There are numerous works devoted to the investigation of the topological properties and construction of the topological classification of Morse–Smale flows on closed surfaces and their generalizations, namely, the flows with finitely many singular trajectories (fixed points, closed trajectories, and separatrices). Among these works, we especially mention [2, 10, 11]. In this case, the role of main invariant is played by the separatrix diagram of the flow. We also mention the work by V. V. Sharko and A. A. Oshemkov [12], which proposes new invariants and presents a survey of other invariants and the work [13], where the trajectory equivalence of the optimal Morse flows was investigated.

On closed three-dimensional manifolds, a topological classification of Morse and Morse–Smale fields with some restrictions was proposed by Ya. L. Umanskii [14] and A. O. Prishlyak [15–18].

The work [19] presents a topological classification of the $m$-fields on two- and three-dimensional manifolds with boundary. These fields generalize the notion of Morse fields; they are in the general position with boundary but do not generate flows.

In recent years, the topological properties of Morse flows on the surfaces with boundary have been extensively investigated, and their topological classification has been constructed.

\textsuperscript{1} T. Shevchenko Kyiv National University, Akad. Glushkov Ave., 4-E, Kyiv, 03022, Ukraine; e-mail: prishlyak@knua.ua.
\textsuperscript{2} Corresponding author.
\textsuperscript{3} Shevchenko Kyiv National University, Akad. Glushkov Ave., 4-E, Kyiv, 03022, Ukraine; e-mail: svbilun@knua.ua.
\textsuperscript{4} “Ostroh Academy” National University, Seminars’ka Str., 2, Ostroh, Rivne Oblast, 35800, Ukraine; e-mail: asp00pr@gmail.com.

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A. O. Prishlyak, S. V. Bilun, and A. A. Prus

Fig. 1. Source and sink on the boundary of a surface.

Fig. 2. Boundary saddles on the boundary of the surface.

In particular, in the works by M. V. Loseva and A. O. Prishlyak, topological classifications were proposed for the flows on a two-dimensional disk with singularities on the boundary [1], for the optimal flows on compact surfaces with boundary [20], and for the flows with mixed dynamics [21].

A complete topological classification of Morse flows on surfaces with boundary was obtained with the use of three-color graphs in [22].

The aim of the present work is to construct a complete topological invariant of Morse flows with fixed points on the boundary of a three-dimensional compact manifold, propose a topological classification of these flows with the help of the constructed invariant, and determine the number of topologically nonequivalent Morse flows on a three-dimensional disk and bodies with handles.

The present paper consists of four sections. In Section 1, we describe the process of construction of a complete topological invariant, a \( Pr \)-diagram, and a Morse flow with fixed points on the boundary of a tree-dimensional manifold and present examples.

In Section 2, we establish a criterion for the topological equivalence of flows via the isomorphism of their \( Pr \)-diagrams. Further, we describe the properties of \( Pr \)-diagrams, prove the theorem on realization, and explain the procedure of reconstruction of a flow on the boundary according to the \( Pr \)-diagram and clarify the possibility of making global the procedure of local continuation of the flow from the boundary onto the interior.

In Section 3, we consider the possibility of application of the obtained invariant to the evaluation of the number of topologically nonequivalent Morse flows on a three-dimensional disk with at most six fixed points on the boundary. As an example of application, we analyze the gravitation flow in the Sun–Earth system.

In Section 4, we describe all possible topological structures of the optimal Morse flows on a body with two handles and on a body with three handles and study the possibility of application of the obtained results to the analysis of water flows on a river with islands.
1. Pr-Diagram of a Morse Flow with Fixed Points on the Boundary of a Three-Dimensional Manifold

Definition 1. A flow $X$ on a manifold $M$ with boundary $\partial M$ is called a Morse flow if it satisfies the following conditions:

(i) the set of nonwandering points $\Omega(X)$ has finitely many points and all these points are of hyperbolic type;

(ii) if $u, v \in \Omega(X)$, then an unstable manifold $W^u(u)$ is transverse to a stable manifold $W^s(v)$ in $\text{Int} M$;

(iii) the restriction of $X$ to $\partial M$ is a Morse flow (stable and unstable manifolds have transverse intersections).

On the boundary of three-dimensional manifolds, there are six types of singularities (Fig. 3). They are determined by their indices.

A pair $(p, q)$ is called the index of a singular point; here, $p + q$ is equal to the dimension of a stable manifold $X$ and $p$ is the dimension of the stable manifold of the flow restricted to the boundary. On a three-dimensional manifold, we have $p = 0, 1, 2$ and $q = 0$ or $1$. For example, the source depicted in Fig. 3.1 has the index (0,0), and the sink in Fig. 3.6 has the index (2,1).

The other singular points in Fig. 3 have the following indices: (0, 1) in Fig.3.2, (1, 0) in Fig.3.3, (1, 1) in Fig.3.4, and (2, 0) in Fig.3.5. Every Morse flow has a source and a sink.

We now construct a Pr-diagram of the flow in the form of a surface with boundary and four collections of curves embedded in this boundary.

In this case, the surface decomposes the 3-manifold into two parts one of which contains the points with indices (0, 0), (0, 1), and (1, 0) and the other part contains the points with remaining indices. Moreover, every trajectory transversely crosses this surface at at most one point. These diagrams generalize the Heegaard diagrams.

This can be done in two ways:

(1) with the use of the axes and coaxes of decompositions into $m$-handles;

(2) with the use of the boundary of a neighborhood of the one-dimensional stable manifold and its intersection with two-dimensional stable and unstable manifolds.
The \( m \)-handles correspond to singular points [23]. Thus, there exist six types of handles (Fig. 4).

We now describe the construction of decomposition into handles. To obtain a decomposition into \( m \)-handles, we first consider the handles of type 1. Then all other handles are attached to these handles in the following way: The entire blue part of the boundary is attached to the union of the grey part and its intersections with the yellow part are attached to the boundary of the yellow zone. As a result of attachment of all \( m \)-handles, we get 3-manifolds with yellow edge. The surface \( F \) is the grey part of the boundary of the union of \( m \)-handles of types 1–3.

The green curves of the handles of type 3 form a \( u \)-system, and the green curves of the handles of type 2 form a \( U \)-system. The picture of red curves obtained as a result of attachment of the handles of type 4 forms a \( v \)-system, while the picture of red lines obtained for handles of type 5 forms a \( V \)-system.

If a handle of type 3 is attached to a handle of type 2, then we deform the green part of its section through the blue zone and the next grey zone. We can get the same result if we compress a half of the handle of type 3 to its middle part, which contains a green arc:

\[
D^2_+ \times [-1, 1] \to D^2_+ \times [-1, 0], \quad (x, t) \to (x, 0), \quad t \in [0, 1].
\]

As a result, \( u \) turns into a part of the cycle \( U \). Furthermore, we apply a similar procedure to the curves \( v \) and \( V \).

**Definition 2.** A quintuple \((F, u, U, v, V)\), where \( F \) is a grey part of the boundary of the union of handles of types 1, 2, and 3, and \( u, U, v, \) and \( V \) are the curves defined above, is called the \( Pr \)-diagram of the flow \( X \).

We now describe another method used for the construction of diagrams. The surface \( F \) is the closure of the intersection \( \text{Int} M \) with the boundary of a regular neighborhood of the union of the following integral manifolds:

1. sources, one-dimensional stable manifolds, and singular points of index \((1, 0)\) in \( \partial M \);
2. one-dimensional stable manifolds and singular points of index \((0, 1)\) in \( \text{Int} M \);

We also select the following sets of arcs and circles on the surface \( F \):

1. arcs \( u \) obtained as the intersections of unstable manifolds, singular points of index \((1, 0)\), and the surface \( F \);
2. arcs and circles \( U \) obtained as the intersections of stable manifolds, singular points of index \((0, 1)\), and the surface \( F \);
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Fig. 5. Optimal Morse flow on a solid torus.

Fig. 6. Flows on $D^3$.

(3) arcs $v$ obtained as the intersection of stable manifolds, singular points of index $(1, 0)$, and the surface $F$;

(4) arcs and circles $U$ obtained as the intersection of stable manifolds, singular points of index $(2, 0)$, and the surface $F$.

The set $(F, u, U, v, V)$ formed by the surface with edge and a set of circles and arcs is a $Pr$-diagram of the Morse flow on a three-dimensional manifold with boundary.

**Example.** We present an example of a $Pr$-diagram of Morse flow.

A gradient flow of the height function on a solid torus (Fig. 5).

The height function on a solid torus has four critical points in its restriction to the boundary. The corresponding singular points are observed for types 1, 3, 4, and 6.

Flows of this kind appear, e.g., in the process of pumping of a punctured wheel in the form of air flows inside the wheel.

In Fig. 6, we present the process of construction of $Pr$-diagrams for three flows in a three-dimensional disk with six fixed points.

Depending on the intersection of stable and unstable manifolds of dimension 2 inside a three-dimensional disk, the last $Pr$-diagram may have three or more points of intersection between the red and green curves.
2. Classification of Morse Flows According to the \(Pr\)-Diagrams

To classify Morse flows according to a \(Pr\)-diagram, we establish a criterion for the topological equivalence of flows, describe the topological properties of the diagrams, and determine the diagrams that can be realized by the Morse flows. Moreover, if the \(Pr\)-diagram specifies a flow on the three-dimensional manifold, then it also determines the flow obtained as a result of its restriction to the boundary. We establish the topological type of this flow. We also consider the inverse problem: Is it possible to turn the local continuation of the flow from the boundary into a global one?

2.1. Criterion for the Topological Equivalence of Flows. Two \(Pr\)-diagrams of a Morse flow are isomorphic if there exists a homeomorphism of the surface, which maps the sets of arcs and circles into the sets of arcs and circles of the same type.

**Theorem 1.** Two Morse–Smale flows on a three-dimensional manifold with edge are topologically trajectory-equivalent if and only if their \(Pr\)-diagrams are isomorphic.

**Proof.** *Necessity.* Let \(\varphi: M \rightarrow M\) be the topological equivalence of the flow generated by a field \(X\) to a flow generated by a field \(X'\). We construct a homeomorphism \(h: F \rightarrow F'\) between the surfaces of their \(Pr\)-diagrams. If \(\gamma_x\) is a trajectory that passes through the point \(x \in F\), then the desired homeomorphism is given by the following formula:

\[
h(x) = \varphi(\gamma_x) \cap F',
\]

*Sufficiency.* The proof is similar to the proofs used in the case of closed manifolds of dimensions 2 and 3 and, therefore, we present only a scheme of the proof. If the isomorphism \(h: F \rightarrow F'\) of \(Pr\)-diagrams is given, then it establishes the bijection between the trajectories of the vector field, namely, the trajectories crossing the surface \(F\) are associated with the trajectories passing through the images of intersections, and the trajectories that belong to one-dimensional stable or unstable manifolds are associated with the trajectories specified by the mappings of vicinities of the curves \(u, U, v,\) and \(V\).

The surface \(F\) decomposes the manifold \(M\) into two parts: the first part \(M_1\) contains semitrajectories entering the points in \(F\), whereas the second part \(M_2\) contains semitrajectories leaving the point \(F\). We construct the topological equivalence that maps \(M_1\) onto \(M'_1\). In these manifolds, we choose Riemannian metrics such that the corresponding trajectories and semitrajectories that belong to one-dimensional stable or unstable manifolds are associated with the trajectories specified by the mappings of vicinities of the curves \(u, U, v,\) and \(V\).

2.2. Topological Properties of the \(Pr\)-Diagrams of Morse Flows.

**Theorem 2.** The \(Pr\)-diagrams of Morse flows have the following properties:

(i) \(U_i, V_i \subset \partial M, \text{Int } u_i, \text{Int } v_i \subset \text{Int } M, \text{and } \partial u_i, \partial v_i \subset \partial M;\)

(ii) \(\partial U_j \subset \bigcup U_i \partial u_i \text{ and } \partial V_j \subset \bigcup V_i \partial v_i;\)

(iii) \(U_i \cap U_j = \emptyset \text{ if } i \neq j, u_i \cap u_j = \emptyset \text{ if } i \neq j, V_i \cap V_j = \emptyset \text{ if } i \neq j, \text{Int } u_i \cap \text{Int } U_j = \emptyset, \text{Int } v_i \cap \text{Int } V_j = \emptyset, \text{and } \partial u_i \cap \partial v_j = \emptyset.\)
(iv) \( U_k \) is either a closed curve or belongs to a cycle formed by \( U_i \) and \( u_j \) and such that, at the ends of the arcs \( u_j \), it always turns to the left; a similar property also holds for \( V_k \).

(v) if we cut the surface \( F \) along \( u_i \) and perform a spherical reconstruction in the \( U \)-cycles, then we get a union of two-dimensional disks; a similar procedure can be performed for \( v_i \) and \( V_j \).

**Remark.** The last two-dimensional disks (\( u \)-domain) correspond to sources (type 1), the \( U \)-cycles correspond to the points with index \( (0,1) \) (type 2), the \( u \)-curves correspond to the points with index \( (1,0) \) (type 3), the \( v \)-curves correspond to the points with index \( (1,1) \) (type 4), the \( V \)-cycles correspond to the points with index \( (2,0) \) (type 5), and the \( v \)-zones correspond to sinks (type 6).

**Proof.** (i) This follows from the fact that the corresponding trajectories of stable and unstable manifolds belong either to the interior or to the boundary of the manifold \( M \).

(ii) The boundaries of unstable manifolds with index \( (0,1) \) are formed by the curves \( U_i \) and intersections of unstable manifolds with \( \partial M \). Since the boundaries of these intersections coincide with \( \partial u_i \), we get the first inclusion. The second inclusion is proved similarly.

(iii) We proceed by contradiction. Assume that there exists a point, which belongs to these intersections. Then the trajectory that passes through this point must belong to two different stable (or unstable) manifolds of dimension 2, which is impossible.

(iv) These cycles correspond either to the boundaries of unstable manifolds with index \( (0,1) \) or to stable manifolds with index \( (2,0) \).

(v) The indicated disks correspond to unstable manifolds of the points with index \( (0,0) \).

2.3. **Theorem on Realization.**

**Theorem 3.** If the surface \( F \) with four sets of curves has properties (i)–(v), then it is a \( Pr \)-diagram of a Morse flow.

**Proof.** Since, according to property (iii), \( u_i \cap u_j = \emptyset \) for \( i \neq j \), we can choose their sufficiently small regular neighborhoods that do not intersect. We glue the handles of type 3 over the grey domains to these neighborhoods (in this case, the green curves are mapped onto the curves \( u_i \)). If these curves belong to \( U \)-cycles, then we realize the deformation of these cycles reverse to the deformation of \( U \)-cycles performed in the case of gluing a handle of type 3 to a handle of type 2.

Further, for these cycles, we choose regular neighborhoods in the union of the remaining part of the surface and the blue domains of glued handles. The grey domains of the handles of type 2 are glued to these neighborhoods. The remaining parts of the surface in the union with blue domains are glued with grey domains of the handles of type 1 [this can be done due to property (v)]. Similar constructions can be carried out for the \( v \)- and \( V \)-curves if we glue the handles of types 4, 5, and 6 to the surface. Inside each handle, we specify a standard vector field \( \{ \pm x, \pm y, \pm z \} \) as in Fig. 3. Smoothening these fields at the sites of gluing, we obtain the desired vector field.
2.4. Reconstruction of the Flow on the Boundary According to the Pr-Diagram. We now show how to determine the topological type of restriction of the flow to the boundary of a three-dimensional manifold by using a Pr-diagram. We represent the boundary of the three-dimensional manifold in the form of the union \( \partial M = F_u \cup F_v \), where

\[
\partial F_u = \partial F_v = F_u \cap F_v, 
\]

\( F_u \) contains fixed points of types 1–3 and the flow on it is determined according to \( F \) and the curves \( u_i \) and \( U_j \), whereas \( F_v \) contains fixed points of types 4–6 and the flow on it is determined according to \( F \) and the curves \( v_i \) and \( V_j \). On the common boundary, the flow is directed from \( F_u \) into \( F_v \). In order to construct \( F_u \), we consider the \( U \)-cycles on \( F \) and their regular neighborhoods. By \( U_1^i \) we denote the components of these neighborhoods that do not intersect with \( \partial M \). We perform spherical reconstructions along them, namely, we cut \( F \) along these components and glue up the obtained closed curves by two-dimensional disks. We denote the obtained surface by \( F_u \).

The center (arbitrary internal point) of each curve \( u_i \) is a saddle point of the flow and the curve itself is an unstable manifold for this point (the union of two separatrices). In each domain obtained as a result of decomposition of \( F_u \) by the curves \( u_i \), we choose a source and draw one separatix to each saddle that lies on the boundary of this domain. All other trajectories go from the sources to \( \partial F_u \). Similar constructions are performed for \( F \) and the curves \( v_i \) and \( V_j \). We obtain a surface \( F_v \) and a flow with saddle points and sinks. As a result, we get a flow on \( \partial M \) as the union of constructed flows. Smoothening the boundary \( \partial F_u \), we obtain the desired flow (smoothening is not necessary if a flow perpendicular to the boundary is constructed on each surface).

2.5. On the Continuation of Flow from the Neighborhood of the Boundary to the Interior of a Three-Dimensional Manifold. The topological type of continuation of the flow from the boundary of a three-dimensional manifold to a regular neighborhood of the boundary depends on the type of singular points (as points of a three-dimensional manifold) determined by the existence of a trajectory that belongs to the interior and enters a singular point. If this trajectory exists, then the second number in the index is equal to 1; at the same time, if it is absent (which is equivalent to the case of existence of a trajectory originating from the singular point), then the second number in the index is equal to zero. Moreover, the sum of Poincaré indices must be equal to zero (the Euler characteristic of doubling of the manifold).

The existence of global continuation to the entire three-dimensional manifold is equivalent to the existence of a flow diagram for which the procedure described in the previous subsection specifies the initial flow on the
surface. The construction of the surface $F$ and the curves $u_i$ and $U_j$ is similar to the procedure used for a given flow.

In this case, we can mention the arbitrariness of the choice of sources playing the role of origins for the trajectories passing to the points of the type $(0, 1)$. As a result of a similar procedure carried out for the curves $v_i$ and $V_j$, we obtain another surface with the same boundary. Then the problem of construction of the $Pr$-diagram is reduced to finding a homeomorphism of one surface on the other, which coincides with the identical map on the boundary. This is equivalent to the construction of the curves $v_i$ with their given points on the boundary (the curves $V_j$ already lie on the boundary) on the first surface containing the curves $u_i$ and $U_j$. In this case, the constructed $Pr$-diagram must specify a three-dimensional disk. This means that, as a result of gluing two-dimensional disks to the components of the boundary, the $U$- and $V$-cycles form a system of meridians of the Heegaard diagram of three-dimensional sphere.

In Fig. 7, we present an example in which the identical homeomorphism of the boundaries (three circles) cannot be extended to the homeomorphism of the surfaces (two-dimensional disk combined with a ring).

Thus, this flow in the neighborhood of a sphere cannot be extended to the three-dimensional disk.

3. Morse Flows on a Three-Dimensional Disk

In order to determine all flows with a given set of fixed points on the boundary of a three-dimensional disk, we analyze all possible continuations from the boundary. In this case, we assume that the flows do not have internal curvilinear biangles with green and red sides. Since the Euler characteristic of a closed surface is even, the number of fixed points on the boundary is also even. For two fixed points (a source and a sink), there exists a common flow. Its $Pr$-diagram is a two-dimensional disk without red and green curves.

3.1. Flows with Four Fixed Points on the Boundary. It follows from the definition of Morse flow that it has a source and a sink (singularities of types 1 and 6). The restriction of the flow to the sphere may have two sources, a saddle and a sink, or a source, saddle, and two sinks. These two flows can be obtained from each other by changing the directions of motion along all trajectories with an accuracy to within the topological equivalence. Therefore, in what follows, we consider solely the flows such that the number of sources in their restrictions to the boundary is not smaller than the number of sinks. There are two possible cases: (i) an internal trajectory originates from the saddle and, hence, from a source on the sphere; (ii) internal trajectories terminate at these points. The $Pr$-diagrams of these flows are depicted in Fig. 8.

Thus, there exist four topologically nonequivalent flows with four fixed points.

3.2. Flows with Six Fixed Points on the Boundary. In the case of six fixed points, the following flows are possible on a sphere: (i) three sources, two saddles, and a sink; (ii) two sources, two sinks, and two saddles; one of stable one-dimensional manifolds forms a loop; (iii) two sources, two sinks, and two saddles; stable one-dimensional manifolds do not form loops; (iv) one source, three sinks, and two saddles. The number of continuations of the flows in the fourth case is equal to the same number in the first case (because they are obtained
from each other by changing the directions of motion along the trajectories). Three flows of the first type have been described earlier (Fig. 6). If a trajectory goes from the right source into the middle source on the sphere, then the flow depicted in Fig. 9 is possible.

If a trajectory terminating at the middle fixed point originates from the left source (and all directions are the same as in Fig. 9), then we get the flow, which cannot be continued into the interior of the three-dimensional disk.

If only one source located on the sphere is a source on the three-dimensional disk, then the following two cases are possible: the indicated source is located either at the center (Fig. 10a) or on the side (Fig. 10b).

Thus, for the first type of flows on the sphere (and, hence, for the fourth type), there exist six nonequivalent continuations to the flow on the three-dimensional disk.

Further, we consider the second type of flows on the sphere. Two examples of continuation of these flows are presented in Fig. 11.

In view of the restriction imposed on the sum of indices, there are 14 possibilities of local continuation of this flow one of which cannot be realized globally (see Fig. 6). All other possibilities can be unambiguously realized and, therefore, we have 13 flows in this case.

Three more $Pr$-diagrams of these flows are depicted in Fig. 12. Further, two more $Pr$-diagrams can be obtained from the $Pr$-diagrams shown in Fig. 11 if we change the way of coloring of the boundary (red color or the absence of colors) into the opposite. Moreover, three more $Pr$-diagrams can be obtained from each of the first two $Pr$-diagrams depicted in Fig. 12 if we allow one to perform the following operations: (i) simultaneously exchange the red and green colors of all arcs and chords; (ii) change the red color of the boundary (colored or not colored).

Further, we consider the third type of flows on sphere. Flows of this kind can be represented on a unit sphere in the form symmetric about the coordinate planes. In view of their symmetric properties, these flows are determined by the number of global sources and sinks.
Thus, the following four situations are possible: (i) one source and one sink (Fig. 13); (ii) two sources and one sink (Fig. 14a); (iii) one source and two sinks (Fig. 14b), and (iv) two sources and two sinks (Fig. 14c).
Fig. 14. Three diagrams of the third type.

Fig. 15. Flow of the Sun–Earth system in $D^3$.

Summarizing all possible cases, we conclude that there are 29 flows with six singularities on the boundary $D^3$.

3.3. Gravitational Flow in the Sun–Earth System. In the Sun–Earth System, we consider points moving around the Sun (more precisely, around the center-of-mass $O$ of the Sun–Earth system) with an angular speed equal to Earth’s speed. There are three forces acting upon the points, namely, the gravity force of the Sun, the gravity force of the Earth, and the centrifugal force. The first two forces are inversely proportional to the moduli of the distances from the corresponding bodies, while the third force is proportional to the distance from $O$. In this case, the Earth and Sun centers are sinks. On the Sun–Earth straight line, there are three more fixed points (Lagrange points $L_1$, $L_2$, and $L_3$); which are saddles in the plane of rotation $S$. Moreover, this plane also contains two fixed points ($L_4$, $L_5$), which are sources.

The rotation of the plane $S$ unambiguously generates the rotation of the three-dimensional space about the axis perpendicular to $S$, which passes through $O$. Since the analyzed system is symmetric about this plane, it is described by a flow in one of the two half spaces, which is called the top half space. Consider a hemisphere of sufficiently large radius centered at $O$ and such that all fixed points lie inside this hemisphere in the union with a plane disk of the same radius. We perform the procedure of smoothening of this union at the intersection. Moreover, at the points of the obtained surface, we project the field onto the tangential plane and also perform its smoothening. This process reflects the fact that, for sufficiently large distances, we are interested not in the distance to the point but only in its location in the display of the telescope and the effect of smoothening at the points of intersection corresponds to the displacement of the observer into an internal point of the top half space.
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Fig. 16. Colored chord diagrams of the Morse flows of genus 2.

Fig. 17. Pr-diagrams of Morse flows on a body with two handles.

In this case, a source is located at the pole of the upper hemisphere and two more sinks and two saddles are added on the equator. This flow and its diagram are presented in Fig. 15. In the Pr-diagram, the green curves correspond to the points $L_4$ and $L_5$, the three red curves between them correspond to the points $L_1$, $L_2$, and $L_3$, while the curvilinear quadrangles formed by these curves correspond to the Sun and the Earth.

4. Optimal Morse Flows on a Body with Handles

**Definition 3.** A Morse flow with fixed points on the boundary of a body with handles $M$ is called optimal if it has the minimal number of singularities and saddle bundles among all flows of this kind on $M$.

It follows from the definition that the number of fixed points of the optimal flow is not smaller than the number of fixed points for the optimal flow on the boundary. Moreover, the Pr-diagram of the optimal flow has the smallest number of the points of intersection between the red and green meridians.

**Theorem 4.** On bodies with $g$ handles, the Morse flow is optimal if and only if it has one point of type 1, one point of type 6, $g$ curves $u$, and $g$ curves $v$ (fixed points of types 3 and 4) without points of intersection between them on the flow diagram.

**Proof.** Sufficiency. Since every Morse flow has a source and a sink, it has points of types 1 and 6. Since only the procedure of gluing a handle of type 3 increases the genus of the surface (of the body), the body with $g$ handles must have at least $g$ points of the third type (curves $u$). If we reverse the direction of motion along the flow, then, by using the same reasoning, we conclude that the number of points of type 4 is not smaller than $g$.

Necessity. We show that there exists a flow satisfying the conditions of the theorem. To do this, we present its diagram. This diagram is a two-dimensional disk with $g$ holes and each hole is connected with the boundary of the disk by a pair of parallel curves, one of which is red and the other is green. For $g = 2$, see Fig. 17a.

Since the optimal flows on connected closed oriented surfaces can be specified with the help of chord diagrams, in what follows, we show how they can be used to find the flows in the body with handles.
If we cut the surface $F$ of the $Pr$-diagram along the red curves, then we get a two-dimensional disk with green chords. We connect the middle points of the red sides by a red chord if it belongs to a single red curve. The obtained colored chord diagram is a complete topological invariant of the flow.

Thus, the colored chord diagram (with red and green chords) is a chord diagram of the optimal Morse flow if it is one-cycle, the number of red and green chords is equal, and there are no intersections between the green chords.

Note that if we remove coloring in the analyzed chord diagram (all chords have the same color), then we get a chord diagram of the flow of restriction to the boundary. Hence, in order to find the diagrams of all optimal flows on the body of genus $g$, it is necessary (in each diagram with $2g$ chords specifying the optimal flow on the surface) to select $g$ chords that do not intersect and make them green; at the same time, the remaining chords are made red.

4.1. Optimal Flows on a Solid Torus. On the surface of a torus, there exists a single optimal flow, and its chord diagram has two intersecting chords. One chord is colored in green and the other chord is colored in red. As a result, we get a colored chord diagram of the flow on solid torus (a body with one handle).

Thus, there exists a single optimal flow on the solid torus. Its $Pr$-diagram has been constructed earlier; see Fig. 5.

4.2. Optimal Flows on a Body with Two Handles. On a closed oriented surface of genus 2, there exist four topologically nonequivalent flows (see, e.g., [13]). If all chords intersect at the same point (center of the circle), then, in this collection of chords, it is impossible to choose two nonintersecting chords. For the remaining three diagrams, all possible ways of coloring are shown in Fig. 16.

The corresponding $Pr$-diagrams of the flows are shown in Fig. 17.

Thus, there exist five different structures of the optimal flows on the body with two handles.

4.3. Optimal Flows on a Body with Three Handles. In Fig. 18, we show 82 chord diagrams specifying the optimal flows on an oriented surface of genus 3 and indicate the number of ways in which it is possible to choose three disjoint chords.

Hence, there exist 177 different structures of the optimal flows on the body with three handles.

4.4. Water Flows in a River with Islands. In a river, the flow has zero speed at the points located on the coast and at the bottom. However, if we consider a body of certain diameter placed in water, then we are interested only in the analysis of points located beyond a certain small neighborhood of the bottom. This is why we omit this neighborhood and smooth the result on the boundary. The velocity vector on the obtained boundary surface is orthogonally projected onto the boundary and the vector field is smoothed in the vicinity of this boundary. Note that the regular trajectories of the boundary approach the coast at the bottom. At the same time, on the surface these trajectories are directed toward the center. If there are $n$ islands in the river, then we get a flow in the body with $n$ handles.

The optimal flow in the body with $n$ handles is called a river flow if it has the following properties: (i) the boundary surface can be separated into two homeomorphic surfaces, which are called top and bottom, so that all fixed points lie in the intersection of these two surfaces; (ii) one of the components of this intersection contains two fixed points, namely, a source and a sink, whereas every other component contains two saddles one of which is the end of a separatrix and the other saddle is the origin of a separatrix; both these separatrices belong to the top surface; (iii) the other separatrices belong to the bottom surface.

Theorem 5. A colored chord diagram is a diagram of a river flow provided that: (i) it has $n$ chords of each color; (ii) chords of the same color are not intersecting; (iii) in every red chord, it is possible to select one (top) end and arrange the chords so that the indicated ends turn into the successive points of the chord diagram (between these points, there are no other ends of the red or green chords).
**Fig. 18.** Optimal Morse flows in a body with handles of genus 3.

**Proof.** The top ends correspond to the separatrices passing from the source to the islands over the (top) surface of the river. The other separatrices originating from the source move over the bottom (bottom surface) and, hence, the points corresponding to these separatrices in the chord diagram are separated from the selected ends. The ends that are not top ends are called bottom ends. Thus, we can split all chords into pairs: each pair consists of one red and one green chord crossing the red chord at the point closest to the top end.
If we consider water flows in the river with two islands, then, among five diagrams depicted in Fig. 17, only diagrams 1 and 3 (islands close to each other and islands located one over the other, respectively) are possible.

For the flows with three islands, among 82 diagrams depicted in Fig. 18, there are several possible chord diagrams, namely, 5, 7, 11, 12, 19, 20, 38, and 42.

Conclusions

The constructed complete topological invariant of the Morse flow on oriented 3-manifolds with boundary, namely, the $Pr$-diagram of the flow, generalizes the notion of Heegaard diagrams for closed manifolds. The structures of flows in a three-dimensional disk and in bodies with handles determined with the help of this invariant reveal its high efficiency. It is also of interest to additionally investigate the following problems:

1. to construct the diagrams of optimal flows for some other three-dimensional manifolds (e.g., the complement to a neighborhood of a node in the three-dimensional sphere or the manifolds whose boundary is a sphere);
2. to construct the topological invariant of flows on nonoriented 3-manifolds;
3. to generalize these results to the case of Morse–Smale flows with closed orbits;
4. to study flows on the manifolds with corners;
5. to investigate the flows in which, parallel with fixed points on the boundary, there are internal fixed points.

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