Inflationary signatures of single-field models beyond slow-roll

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Abstract. If the expansion of the early Universe was not purely de Sitter, the statistical imprints of the primordial density perturbation on the cosmic microwave background can be quite different from those following slow-roll inflation. In this paper we study the inflationary signatures of all single-field models not plagued by ghost-like instabilities. We assume small deviations from exact scale-invariance, as supported by current cosmological constraints, allow for a rapid change of the Hubble parameter and the phase speed of scalar fluctuations. We obtain the propagator of scalar fluctuations and compute the bispectrum, keeping next-order corrections proportional to the deviation of the spectral index from unity. These theories offer an explicit example where the shape and scale dependences of the bispectrum are highly non-trivial for reasonable breaking of slow-roll.

Keywords: inflation, cosmology of the very early universe, cosmological perturbation theory, non-gaussianity
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1 Introduction

In recent years our understanding of the early Universe has become much clearer as we gain access to precise measurements of the cosmic microwave background radiation (CMBR) \cite{1,2,3}. It is widely believed that in the early Universe a period of inflationary expansion took place with scales being pushed outside the observable horizon \cite{4}. With this simple idea the Universe becomes very smooth, every number density in relics is diluted away \cite{5}, and the standard cosmological problems are ameliorated. At the same time, distinctive signatures are imprinted in the CMBR anisotropies which can be observed by high resolution surveyors, such as the Planck satellite.

From CMB and large scale structure (LSS) data \cite{6} we know that the spectrum of scalar perturbations is quasi-scale invariant, with the power being only slightly stronger at larger scales (according to the most recent WMAP7 data analysis, the spectral index of density perturbations evaluated at $k = 0.002\text{Mpc}^{-1}$ is $n_s = 0.969 \pm 0.012$ \cite{2}). However, a scale invariant spectrum is a model-independent feature of inflation. To look for distinctive signatures of different models it is necessary to study higher-point statistics of the primordial curvature perturbation, known as non-gaussianities \cite{7}, which are a result of the inflaton self-interactions and couplings to the gravity sector.

Amongst the most important of these statistics and potentially easiest to observe is the bispectrum or, in quantum field theory language, the three-point correlation function. The first studies on the bispectrum started with Maldacena’s pioneering work \cite{8} devoted to canonical single field-models, later generalized by Seery & Lidsey \cite{9} and Chen et al. \cite{10} for theories with a varying speed of sound for scalar fluctuations.\footnote{For simplicity we refer to the phase speed of perturbations \cite{11} as the speed of sound.} The latter become known as $P(X, \phi)$ models, because the Lagrangian is an arbitrary function, $P$, of the field profile, $\phi$, and its first derivatives through $X \equiv - (\nabla \phi)^2$. These models have the property of allowing for large non-gaussianities whenever the speed of sound for perturbations is small. On the other hand, models with canonical kinetic terms, even if including higher powers of single-derivative operators (suppressed by powers of some high energy physics scale), predict observationally small non-gaussianities \cite{12}. For more recent works see, for example, Refs. \cite{13–21}. To understand better the microphysics processes which operated in the early Universe one needs to fully understand these non-gaussian signatures, since ultimately they have the power to eliminate models against observations. With Planck data soon to become available our best hope to achieve this lies within a non-zero measurement of the bispectrum, which will be the object of interest in this paper.

The simplest inflationary models are those where the scalar field theory contains only one scalar degree of freedom responsible for sourcing inflation and seeding perturbations: the inflaton field. In spite of their simplicity, these models accommodate quite a remarkable variety of interesting scenarios: from Dirac–Born–Infeld (DBI) inflation \cite{22,23} and galileon inflation \cite{17}, which are the only known radiatively stable theories, to models with canonical kinetic terms. Recently a number of authors have also studied models where the galilean symmetry could be relaxed, and focused on the much milder requirement that the Lagrangian preserved unitarity, resulting in a sensible quantum field theory \cite{24–28}. This way the Lagrangian operators give rise to equations of motion for the scalar field which are at most second-order in derivatives. Shortly after, these theories were shown to be equivalent to Horndeski
models [26, 28]. First derived over thirty years ago [29], Horndeski theories are described by the most
generic action of single-field models which do not contain ghost-like instabilities, hence encapsulating
all the single-field models of interest.

Until recently the traditional methodology to compute the bispectrum started from the action for
the comoving curvature perturbation, $\zeta$ [30, 31], for each model independently. This perturbation is
the appropriate quantity to study since it is conserved on super-horizon scales, if isocurvature modes are
absent. However, recent developments have simplified this procedure. First, Gao & Steer [28] and De
Felice & Tsujikawa [32] (see also Ref. [33]) obtained the universal action for perturbations in stable
single-field models involving what we call Horndeski operators (cubic operators in $\zeta$). The only model
dependent features in the action would reside in different coefficients of each operator. Second, it was
shown in Ref. [33] that the cubic action for $\zeta$ has a minimal representation in terms of only five of these
Horndeski operators, initially derived in Ref. [34].

With these recent developments we arrive at a universal methodology to compute the bispectrum
of single-field models, with the cubic action for $\zeta$ being always of the form

$$S^{(3)} = \int d^3x \, d\tau \left\{ a\Lambda_1 \zeta^2 + a^2\Lambda_2 \zeta^2 i \zeta \right\}, \quad (1.1)$$

where $a$ is the scale factor, dotted quantities are differentiated with respect to conformal (not cosmologi-
cal) time and $\partial$ denotes spatial partial derivative. This action appeared in this form in Ref. [34] although
with different coefficients $\Lambda_i$.\textsuperscript{2} As we mentioned before, the model-dependent imprints will be encoded
in each of the five coefficients $\Lambda_i$ of the Horndeski operators. There is a priori no hierarchy between
these coefficients, although specialization to different models can impose specific ratios between $\Lambda_i$ (DBI
inflation being one example). The action above will be our starting point in computing the bispectrum.

Many authors have focused on the study of the action (1.1) under the assumption of slow-roll
regime in which the inflationary expansion is quasi-de Sitter, whilst allowing for small variation of the
sound speed of perturbations. In this approximation $\epsilon \equiv -d \ln H / dN$, $\eta \equiv d \ln \epsilon / dN$ and $s \equiv d \ln c_s / dN$
all obey $\epsilon, |\eta|, |s| \ll 1$, where $N$ stands for the number of e-folds. Examples of such works include, for
example, Refs.[9, 10, 17, 21, 34]. But, what if inflation was not almost de Sitter, and the parameters
above are not perturbative? Khoury & Piazza [35] showed that this scenario would still produce a
scale-invariant spectrum, in fair agreement with observations [2], provided the relation $s = -2\epsilon$ was
satisfied.\textsuperscript{3} We know that both $\epsilon$ and $|\eta|$ must be small to allow for a successful period of inflation.
However, how small are they required to be? If $\epsilon$ is small but not necessarily much smaller than 1,
then a calculation beyond slow-roll is technically required especially to fit in the current era of precision
Cosmology. Previous works which have attempted to study correlations beyond the slow-roll regime
include corrections to the power spectrum in canonical models worked out by Steward & Lyth [39], and
the work by Gong & Stewart [40] (see also Ref. [41]) who set up a formalism using the Green’s function

\textsuperscript{2}This action had appeared first in Refs. [9, 10] specialized for $P(x, \phi)$ models. There the action involved a larger number
of cubic operators and also a field redefinition. The equivalence of the actions therein and in Eq. (1.1) was explained in Ref.
[34].

\textsuperscript{3}These authors have also considered solutions within the ekpyrotic mechanism (see Ref. [36] for a review) – see also Refs.
[37, 38]. In this paper however we focus on the inflationary scenario.
method to obtain the propagator of scalar fluctuations to next-next-order in slow-roll. This was later generalized by Wei et al. [42] who considered models with a varying speed of sound of perturbations. See also Ref. [43] for a calculation report by Bartolo et al. on the power spectrum beyond leading order in the context of effective field theories of inflation [44, 45].

In this paper we will be interested in models where the spectrum is almost scale-invariant, so that the condition found by Khoury & Piazza is mildly broken and becomes \( s = -2\epsilon + \delta \). This requires working to all orders in \( \epsilon \) and \( s \), but perturbatively in \( \delta \), in deviations of the spectral index from unity, \( n_s - 1 \ll 1 \), and time variations of \( \epsilon \) and \( s \). This last assumption ensures that the conditions \( \epsilon, |s| < 1 \) are preserved for a sufficiently large number of e-folds, which is a primary requisite for inflation to last at least 60 e-folds.\(^4\)

In this sense, this is a calculation beyond the (strong) slow-roll approximation. We refer to it as a scale invariant approximation. Starting from the action (1.1) the calculation is immediately applicable to all Horndeski models. For this reason we will work with arbitrary interaction coefficients \( \Lambda_i \) – assignments to these coefficients will correspond to specific models. We organise the calculation in increasing powers in the hierarchy of slow-variation parameters, focusing on leading and next-order contributions. This organisation scheme follows the one used throughout Refs. [10, 17, 34] and generalizes their results. We explicitly obtain the scale dependence of the bispectrum, which appears more intricate than the one found with the assumptions of (strong) slow-roll. In particular, the bispectrum exhibits a strong, power-law scale dependence, accompanied by the traditional weak, logarithmic dependence.

Outline.— This paper is organised as follows. In §2 we obtain the dynamical behaviour of the scale factor and the other relevant background quantities. The dynamical behaviour is exact in \( \epsilon \) and \( s \), and perturbative in the time variation of these parameters and \( n_s - 1 \). In §2.1 we obtain the power spectrum for scalar fluctuations in the quasi scale invariant approximation, and we derive in §2.2 formulas for the elementary wavefunctions from which the scalar propagator is built to next-order in scale-invariance. We use the results obtained in the previous sections to calculate the bispectrum of perturbations in §3 and comment on the differences when using the slow-roll approximation. We conclude in §4. The appendices collect useful formulas and detailed expressions to recurring integrals necessary to produced closed-form bispectra.

Notation.— We choose natural units in which \( c = \hbar = 1 \) and we take the reduced Planck mass, \( M_p = (8\pi G)^{-1/2} \), to be unity. The metric signature is \((-+,+,+),\) and purely spatial indices are denoted by Latin letters \( \{i,j,\ldots\} \), which are also occasionally used to refer to each Horndeski operator. Derivatives with respect to conformal time, \( \tau = \int_\infty^t dt/a(t) \), will be denoted by dots.

2 Background evolution beyond exact scale invariance

We are interested in quasi-scale invariant models which fall under the class of Horndeski theories. As argued in Ref. [35] a scale invariant spectrum for perturbations can be accommodated by a background

\(^4\)This work explicitly excludes models with features in the potentials which can trigger the slow-roll parameters to temporarily grow during inflation (see for example the review by Chen [46].
where both the Hubble parameter and the speed of sound of perturbations vary significantly in time provided they obey the exact relation \( s = -2\epsilon \).

We start by considering an arbitrary Horndeski theory for a homogeneous scalar field, on which small, inhomogeneous perturbations, \( \delta \phi \), develop. All we require is the spectrum of perturbations to be quasi-scale invariant. The comoving curvature perturbation is related to these perturbations via \( \zeta = aH\delta \phi / \dot{\phi} \). Often, one computes the correlations functions in the comoving or \( \zeta \)–gauge rather than in the uniform-density or \( \delta \phi \)–gauge. This has an important advantage: the correlation functions of \( \zeta \) will be time-independent whereas the ones for \( \delta \phi \) will not. For this reason it is more convenient to work with the action (1.1) since the primordial comoving perturbation, \( \zeta \). We will come back to this point shortly.

For all inflation models involving one single clock in the Universe, the quadratic action for the primordial comoving perturbation, \( \zeta \), can be written as

\[
S^{(2)} = \frac{1}{2} \int d^3x d\tau a^2 z \left[ \zeta^2 - \zeta_s^2 (\dot{\zeta})^2 \right],
\]

(2.1)

where \( z \) is required to be a well defined, differentiable function, but otherwise arbitrary.\(^5\) The time evolution of \( z(y) \) will be parametrized by \( w \equiv d\ln z / dN \). For consistency, we also work to all orders in \( w \). For an arbitrary \( w \), Khoury & Piazza’s relation between \( \epsilon \) and \( s \) required for scale invariance, is generalized to\(^6\)

\[ 3s = -2\epsilon - w. \]

This formula reduces to \( s = -2\epsilon \) for \( P(X, \phi) \) models. It is more convenient to study the dynamics of background quantities as a function of \( y \), satisfying \( dy = c_s d\tau \), rather than conformal time, \( \tau \).\(^7\) In this new time-coordinate, the action (2.1) then becomes

\[
S^{(2)} = \frac{1}{2} \int d^3x dy q^2 \left[ \zeta''^2 - (\partial \zeta)^2 \right],
\]

(2.2)

with \( q = a \sqrt{\dot{z}c_s} \) and where the prime denotes derivative with respect to \( y \). Introducing the canonically normalized field \( v_k = q\zeta_k \), where the subscript \( k \) indicates the Fourier mode, and changing variables to \( v_k = A_k \sqrt{-ky} \), we find that the modes \( A_k \) obey a Bessel equation of the form

\[
A''_k + \frac{1}{y} A'_k + A_k \left[ k^2 - \left( \frac{q''}{q} + \frac{1}{4y^2} \right) \right] = 0.
\]

The solution to this equation is a linear combination of Hankel functions, \( H^{(1)}_\nu(-ky) \) and \( H^{(2)}_\nu(-ky) \), with order

\[ \nu = \frac{3}{2} - \frac{n_s - 1}{2}. \]

\(^5\)For \( P(X, \phi) \) models, \( z = \epsilon / c_s^2 \) [34], but the same need not be true for other models. That is the case for galileon inflation theories Ref. [17], for example. We will come back to this point in §2.2. We note that \( z \) here is not the usual \( z \) for \( k \)-inflation.

\(^6\)We refer the reader to Eq. (A.3) in appendix A for details.

\(^7\)The reasons for this choice will be clarified in appendix B.
For perfectly scale invariant spectrum of perturbations then $\nu_0 = 3/2$ and the spectral index is unity. This results in the power being constant in all scales. See appendix A for details of how the spectral index relates to the background quantities – the precise formula will be unimportant in what follows.

Agreement with the appropriate normalization of the propagator gives the overall evolution of the elementary wavefunction as follows

$$\zeta_k = \sqrt{\frac{\pi}{8k}} \frac{\sqrt{-ky}}{a\sqrt{2\zeta}}, H_v^{(1)}(-ky) . \quad (2.3)$$

Finally, the power spectrum of perturbations can be obtained by evaluating the scalar propagator at equal times, yielding

$$\langle \zeta(k_1)\zeta(k_2) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2) \frac{\pi}{8} \frac{(-y)}{a^2(y)\zeta(y)c_\eta(y)} \left| H_v^{(1)}(-ky) \right|^2 . \quad (2.4)$$

To further simplify this expression we will require the evolution of the scale factor $a$ in the $y$ coordinates. In order to do so, we specify the parameters in the slow-variation catalogue, which will allow us to perform a uniform expansion up to next-order terms in the quasi scale invariant approximation.

**Slow-variation catalogue.**— To obtain the dynamical evolution of the scale factor in a background with $\varepsilon$ and $s$ smaller than 1 (but not necessarily much smaller than 1), we introduce the following slow-variation parameters:

$$\eta \equiv \frac{d\ln \varepsilon}{dN} \quad \text{and} \quad t \equiv \frac{d\ln s}{dN} . \quad (2.5)$$

We assume these parameters satisfy $|\eta|, |t| \ll 1$.

To determine the behaviour of the scale factor, we start by integrating $c_s \, d\tau$ to compute $y$. Working perturbatively in $\eta$ and $t$ but to all orders in $\varepsilon$ and $s$, we find

$$y = \frac{c_s}{aH} \sum_{m=0}^{+\infty} \left\{ (\varepsilon + s)^m + (\varepsilon + s)^{m-1} \sum_{k=0}^{m} k(ts + \varepsilon\eta) \right\} .$$

These converge and give the dynamical behaviour of $a$ in $y$-time as follows

$$a(y) = \frac{c_s}{H y (1 - \varepsilon - s)} \left[ 1 + \frac{\varepsilon\eta + ts}{(1 - \varepsilon - s)^2} \right] . \quad (2.6)$$

This result was first obtained by Khoury & Piazza in Ref. [35]. Using Eq.(2.6) we can further simplify the two-point correlator (2.4) in $y$-space, which becomes

$$\langle \zeta(k_1)\zeta(k_2) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2) \frac{\pi}{8} \frac{H^2(y)(-y)^3(1 - \varepsilon - s)^2}{z(y)c_\eta^2(y)} \left[ 1 - \frac{2\varepsilon\eta + ts}{(1 - \varepsilon - s)^2} \right] \left| H_v^{(1)}(-ky) \right|^2 . \quad (2.7)$$

In a generic model, the two-point correlator will evolve in time. However, for a single-field model, by virtue of conservation of the primordial curvature perturbation on super-horizon scales [47, 48], it follows that the power spectrum will rapidly converge to its asymptotic value.\footnote{See Ref.[49] for comments on the number of e-folds necessary for such asymptotic behaviour to be reached after horizon crossing. There the authors study a two-field inflation model, but the same conclusions still apply in single-field models.} Therefore, we can
focus our calculation a few e-folds after horizon crossing for a given scale, \( k_* \). This has become the standard approach when calculating the asymptotics of the power spectrum. This remains true even in multi-field inflation, where in this case this procedure sets the appropriate initial conditions for evolving correlation functions after horizon crossing, using techniques such as \( \delta N \) \cite{50}, transport equations \cite{51,52} or transfer matrices \cite{53} (see, for example, \cite{54–59}). The advantages of applying this approach are twofold: on one hand, at this point in the evolution the elementary wavefunction is completely characterized by the growing mode, as the decaying mode has become negligible; on the other hand, the power spectrum obtained in this way is already the asymptotic value, which corresponds to the late times regime. In other words, at this point of evaluation the primordial curvature perturbation has already classicalized.

With this methodology in mind, and working perturbatively in \( \eta \) and \( t \) we find that

\[
\varepsilon \approx \varepsilon_* \left[ 1 - \frac{\eta_*}{1 - \varepsilon_* - s_*} \ln(-k_*y) \right] \quad (2.8a)
\]

\[
s \approx s_* \left[ 1 - \frac{t_*}{1 - \varepsilon_* - s_*} \ln(-k_*y) \right] \quad (2.8b)
\]

to next-order in our approximation. In the remainder of this paper we employ the same notation as in Refs. \cite{17,34} where starred quantities are to be evaluated when some scale, \( k_* \), has exited the \( y \)-horizon, that is, when \( k_*y_* = -1 \). It is important to leave this scale arbitrary since this will produce the scale dependence of the correlation functions. The appearance of the logarithmic term corresponds precisely to the number of e-folds, \( N_* \), in the \( y \)-coordinates, which have elapsed since the scale \( k_* \) has exited the horizon.\(^9\) The expansion in Eqs. (2.8) is only valid up to a few e-folds outside the horizon, when one can trust the Taylor series expansion to first order. We note that quasi scale invariance demands \( \eta_* = t_* \), up to next-next-order corrections, which are beyond the scope of this paper.

Given the individual dynamics of \( c_c(y) \) and \( H(y) \) deduced in Eqs. (C.3) and (C.5) of the appendix C, we can replace for the evolution of the scale factor in Eq. (2.6). We find:

\[
a(y) = \frac{c_c(-k_*y)^{-1/\varepsilon_* - s_*}}{H_*(1 - \varepsilon_* - s_*)(-y) \left\{ 1 + \beta_* - \frac{\beta_*}{1 - \varepsilon_* - s_*} \ln(-k_*y) + \frac{\alpha_*}{2(1 - \varepsilon_* - s_*)} (\ln(-k_*y))^2 \right\}} \quad (2.9)
\]

where the next-order parameters \( \alpha \) and \( \beta \) are defined in Eqs. (C.2). Despite its complicated structure, it correctly reproduces the results of Refs. \cite{34,35} when we take the limit of exact scale invariance, for which \( \eta_* = t_* = 0 \). The extra terms in Eq. (2.9) are precisely the corrections to a pure power-law evolution, since they correspond to the perturbative expansion of the dominant power law behaviour.

### 2.1 The scalar power spectrum

As mentioned above, the power spectrum, defined as the propagator evaluated a common time

\[
\langle \zeta(k_1)\zeta(k_2) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2)P(k) \quad (2.10)
\]

\(^{9}\)The notation for the number of e-folds as a measure of expansion was initially defined by Sasaki and Tanaka in Ref.\cite{60}.
where \( k = |k| = |k_1| = |k_2| \), will in general depend on time. Taking the super-horizon limit, \(|k y| \to 0\), we find that the dominant contribution arising from the Hankel function in Eq. (2.7) is given by

\[
|H^{(1)}_{1}(-k y)| \to -\frac{2}{\pi k y^3} \left\{ 1 + (n_s - 1) \left[ -2 + \gamma_E + \ln(-2k y) \right] \right\},
\]

where \( \gamma_E \approx 0.577 \) is the Euler-Mascheroni constant. Whatever \( y \)-evolution the remaining terms in Eq. (2.7) have in this limit, they should precisely cancel the \( y \) dependence of the Hankel function. This is a requirement imposed by conservation of \( \zeta \) on super-horizon scales in single-field models.\(^{10}\) This allows us to write the power spectrum as

\[
P(k) = \frac{H^2}{4 \varepsilon_* (k \varepsilon)^3} \left\{ 1 + 2E_* + (n_s - 1) \ln(k/k_*) \right\},
\]

(2.11)

where we have defined

\[
E_* = -\beta_* + \frac{n_s - 1}{2} \left[ -2 + \gamma_E + \ln 2 \right].
\]

(2.12)

This is the power spectrum for a quasi-scale invariant Horndeski theory. To recover the perfectly scale invariant formula obtained by Khoury & Piazza \([35]\) we simply set \( E_* \) and \( n_s - 1 \) to vanish, for these enter the power spectrum as next-order corrections only. Whilst being time-independent, Eq. (2.11) correctly reproduces the expected logarithmic scale-dependence in the strong slow-roll regime, in which \( \epsilon \) and \( s \) are taken to be slow-roll parameters. Indeed, we can check that defining the dimensionless version of the power spectrum using the usual rule \( \mathcal{P} = k^3 P(k)/2\pi^2 \), this formula agrees with

\[
n_s - 1 = \frac{d \ln \mathcal{P}}{d \ln k}.
\]

Again, the explicit expression for \( n_s - 1 \) will not be required in any stage of the calculation. It suffices to treat it as an arbitrary next-order quantity.

2.2 Obtaining next-order corrections to the wavefunctions, \( \zeta_k \)

We are interested now in obtaining the bispectrum of perturbations starting from the action Eq. (1.1). We will apply the same procedure defined in the previous sections and rely on the slow-variation catalogue introduced there. Since we will be keeping up to next-order corrections in the slow-variation parameters, we also require corrections to the elementary wavefunctions up to the same order. It is easier to think about these corrections by locating them in a Feynman diagram corresponding to the three-point function in question. The sources of next-order corrections are therefore quite well-defined and we explore them in what follows. Next-order corrections were first enumerated by Chen et al. \([10]\) for \( P(X, \phi) \) models.

External legs.— The corrections to the external lines of Feynman diagram correspond to evaluating the propagator in the asymptotic regime, when \(|k y| \to 0\). From Eq. (2.11) we can read off the external legs corrections and conclude that the propagator becomes

\[
\zeta_k(y)^{\text{external}} = \frac{i}{2} \frac{H_* (1 - \epsilon_* - s_*)}{\sqrt{\varepsilon_* (k \varepsilon)^{3/2}}} \left\{ 1 + E_* + \frac{n_s - 1}{2} \ln(k/k_*) \right\}.
\]

(2.13)\(^{10}\)

The time-independence of the power spectrum in single-field inflation was carefully explained in Ref. \([34]\) to which we refer the reader for more details.
As argued in §2.1 the time independence of this value is guaranteed because the primordial perturbation \( \zeta \) is conserved on super-horizon scales.

From the requirement of time independence of the power spectrum it also follows that whatever function we assign to \( z(y) \) it must generically behave, at leading order, as

\[
z(y) \sim (-k_s y)^{2s + \ln k_s / \epsilon_s}. \tag{2.14}
\]

This represents a mild requirement on an otherwise arbitrary but smooth function \( z(y) \). We check that for \( P(X, \phi) \) models, where the formula for \( z \) is well-defined, \( z = \epsilon / \zeta_s^2 \), this behaviour is obeyed by virtue of Eqs. (2.8) and (C.3), together with the condition \( s_s = -2 \epsilon_s \) (which holds at leading order in our approximation scheme).

**Internal legs.**— This correction is a result of the spectrum being slightly tilted. This manifests directly in Eq. (2.3) for the elementary wavefunction which, using Eq. (2.6), we rewrite for clarity as

\[
\zeta_k(y) = \sqrt{\frac{-\pi}{8}} \frac{H(y)(-y)^{3/2}(1 - \epsilon - s)}{\sqrt{z(y)} c_s^{3/2}(y)} \left\{ 1 - \frac{\epsilon \eta + ts}{(1 - \epsilon - s)^2} \right\} H_v^{(1)}(-ky) . \tag{2.15}
\]

Since the order of the Hankel function, \( \nu \), differs from \( 3/2 \) by next-order corrections, these originate in turn corrections to \( \zeta_k \). To evaluate these we start by Taylor expanding the Hankel function around order \( \nu_0 = 3/2 \). Following Refs. [10, 17], we find that the background evolution of the wavefunctions is given by

\[
\zeta_k(y) = \frac{iH_s(1 - \epsilon_s - s_s)}{2 \sqrt{\nu}(k c_s)^{3/2}(1 - i ky)} e^{i ky} , \tag{2.16}
\]

which agrees with the leading order behaviour of Eq. (2.13) in the limit \( |ky| \to 0 \). The next-order corrections arising from \( \delta \nu = \nu - 3/2 = -(n_s - 1)/2 \) and the slow-variation of the remaining background quantities in (2.15) organise themselves in what we label as internal leg corrections as follows:

\[
\delta \zeta_k(y)^{\text{internal}} = \frac{iH_s(1 - \epsilon_s - s_s)}{2 \sqrt{\nu}(k c_s)^{3/2}} \left\{ -e^{ik y}(1 - i ky) \left[ \beta_s + \frac{n_s - 1}{2} \ln(-k_s y) \right] \right.
\]

\[
+ \frac{n_s - 1}{2} \left[ e^{-ik y}(1 + i ky) \int_{-\infty}^{y} \frac{e^{2ik \xi}}{\xi} d\xi - 2e^{ik y} - i\frac{\pi}{2} e^{ik y} (1 - iky) \right] \right\} . \tag{2.17}
\]

The integral representation written above corresponds to the exponential integral function, \( \text{Ei}(-ky) \), defined for real, non-zero argument:

\[
\text{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt .
\]

The time dependence of the internal leg corrections is fairly intricate and verifying that there are no divergences in Eq. (2.17) in the late time limit represents a minimal check on this result. We indeed
confirm this is the case. Expressions for time derivatives of the wavefunctions can be found in appendix C.

**Corrections arising from interaction vertices.**— The interaction coefficients $\Lambda_i$ can generically evolve very fast, and one expects

$$\Lambda_i \simeq \Lambda_i^\star (-ky)^n \left\{ 1 + \tilde{a} \ln(-k_i y) + \tilde{b} \left( \ln(-k_i y) \right)^2 \right\}, \quad (2.18)$$

with the power $n$ depending on the exact expression of $\Lambda_i^\star$ and $\tilde{a}$ and $\tilde{b}$ being next-order terms. This power law dependence needs to be taken into account for consistency with the scale invariant approximation, and we will study an explicit example in §3.2. There we explain in detail that this power law behaviour can potentially bring problems to the convergence of the integral. In particular, we might have to require that only some values of $\varepsilon$ are allowed in order that the three-point correlator does not evolve in time. Mainly to simplify our results, we assume the interaction vertices to have a very smooth and slow evolution in the $y$-coordinates. To this end we introduce a supplementary slow-variation parameter

$$h_i \equiv \frac{1}{\Lambda_i H} \frac{d\Lambda_i}{dt} \quad (2.19)$$

which satisfies $|h_i| \ll 1$. We expect this approximation to be valid in all Horndeski models where the interaction vertices are slowly varying in time, even though the respective assignments for $\Lambda_i$ can be rather intricate. As described in §3.2 the methods of this paper can still be used to compute the $n$-point correlation functions in models where this assumption fails. We conclude that, if $\Lambda_i$ is slowly varying, then each interaction vertex evolves as

$$\Lambda_i \simeq \Lambda_i^\star \left[ 1 - \frac{h_i^\star}{1 - \varepsilon^\star - s^\star} \ln(-k_i y) \right]. \quad (2.20)$$

This means that for simplicity in the remainder of the paper we will work with $n = 0$ in Eq. (2.18), but we will show how to deal with $n \neq 0$ in one simple example. This assumption concludes the presentation of the slow-variation catalogue.

We note that a further simplification to the background dynamics follows from inspection of Eqs. (2.13) and (2.17). The variable $w$ that parametrizes the variation of $z_i$ does not appear in our formulas. Indeed such term is only present implicitly since it is encapsulated in the spectral index $n_i - 1$. Therefore the calculation is independent of our choice of parameter $w$. As a consequence we can effectively reduce the relation we found between $\varepsilon, s$ and $w$, to one which only involves the first two variables. To this end, we take the relation between $\varepsilon$ and $s$ to be

$$s = -2\varepsilon + \delta$$

where $\delta$ is a next-order parameter. We emphasize that this procedure in no way lacks generality: the calculation is still non-perturbative in $\varepsilon$ and $s$. The formula above generalizes the regime studied in Ref. [35] by considering a quasi-scale invariant spectrum. This also implies that $t = \eta$, which is in agreement with the observations in §2. This procedure dramatically simplifies our formulas and reduces
the slow-variation catalogue to a minimum of parameters, which we choose to be $\delta$ and $\eta$. The quantum field theory calculation we will present is thus only non-perturbative in the $\epsilon$ parameter.\textsuperscript{11} We note that this will imply some conceptual restructuring of our formulas. In particular, Eq. (2.16) now contains background and next-order contributions in $\delta_\star$, which can now be interpreted as an extra contribution to the external legs at next-order in Eq. (2.13).

Eqs. (2.13), (2.17), and (2.20) combine to produce the overall corrections which are required to produce the bispectrum at next-order. They naturally combine to produce two separate contributions to the bispectrum: the background contributions and next-order corrections arising from the vertex and external legs; and the next-order contributions arising from the internal legs of the propagator. We will use this way of partitioning the bispectrum when presenting the results for the bispectrum in §3.1 by labelling the first contributions as type a bispectrum and type b bispectrum for the second.

3 Non-gaussianity in the bispectrum

Non-gaussian signatures encoded in the CMBR work as a powerful discriminant between inflationary models. Quantum field theory correlation functions were initially studied by Schwinger [61] and Keldysh [62] in the 60s, and later applied to Cosmology by Jordan [63] and Calzetta & Hu [64]. But it was only with Maldacena’s publication in 2002 [8] that the applications of the in-in formalism to non-gaussianity were made more clear, together with a pair of papers by Weinberg [65, 66]. This formalism is the appropriate construction to compute expectation values, and we briefly summarize its basic ideas (there are a number of papers which review the in-in formalism – see, for example, Refs. [67, 68]).

In this paper, we will be interested in computing three-point correlation functions, $B$, at tree-level in the interactions of the comoving curvature perturbation, $\zeta$. In terms of the Argand diagram in complex time $y$, these correlations are obtained by performing a path integral from the true vacuum of the theory, at $t \to -\infty$, to the time of interest when we compute the expectation value, added by the path integral performed backwards to $-\infty$. Schematically, this can be translated into

$$
\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle = \int \left[ d\zeta_+ d\zeta_- \right] \zeta_+(k_1)\zeta_+(k_2)\zeta_+(k_3) e^{iS[\zeta_+]} e^{-iS[\zeta_-]} 
$$

(3.1)

where the forward path integral is labelled by the fields $\zeta_+$ whereas the backwards path integral is labelled by the fields $\zeta_-$.\textsuperscript{12} These fields are constrained to agree at any one time later than that of the observation, such that the path integrals translate into a contour of integration turning and crossing the real $y$ axis. From this construction we see that we need the cubic action, $S$, for the perturbation $\zeta$, written in Eq. (1.1), to obtain the lowest order non-vanishing Wick contractions between the fields. This is because at leading order in the interaction, the first non-vanishing contribution arises from contracting

\textsuperscript{11} Our calculation is non-perturbative in $\epsilon$ and $s$. By realizing that we can write $s = -2\epsilon + \delta$, this allows us to replace the non-perturbative parameter $s$ by the perturbative parameter $\delta$. In the remainder of the text, the dependence in $s$ is absorbed into $\delta$.

\textsuperscript{12} In practice, to project onto the true vacuum of the interacting theory, one needs to translate the integration contour to slightly above and below the negative real axis of conformal time, respectively. This prescription is in many ways similar to the $i\epsilon$ trick recurrent in quantum field theory and ensures the convergence of the integral.
the three fields inside the correlator (essentially three copies of \( \zeta \)) with the three fields in the cubic action. As the action is composed by five Horndeski operators, the total bispectrum will be given by the sum of the corresponding five (individual) bispectra. The form of Eq. (3.1) also ensures that the contribution from the backwards path integral is precisely the complex conjugate of that of the forward integral.\(^{13}\) Therefore, it suffices to compute one of these path integrals.

There are several ways of presenting the bispectra. Some authors prefer to present their answers in terms of a rescaled bispectrum traditionally defined as \([50, 69]\)

\[
f_{\text{NL}} = \frac{5}{6} \frac{B(k_1, k_2, k_3)}{P(k_1)P(k_2) + \text{cyclic permutations}}
\]  

(3.2)

Cosmological constraints for \(f_{\text{NL}}\) are quoted in Refs. \([70, 71]\). When calculating \(f_{\text{NL}}\) we need to add two symmetrized copies of the power spectrum – this is what we mean by “cyclic permutations”. To place constraints in model building, it is common to write \(f_{\text{NL}}\) evaluated in some specific momentum configuration, such as the equilateral limit. We then compare this estimate with the amplitude of the bispectrum in the equilateral template used in CMB analysis. However, simply because some bispectrum shape peaks in the equilateral limit, does not mean it perfectly matches the bispectrum equilateral template. Hence, in order to place constraints on the parameters of some theory we ought to further require the evaluation of the correlation between the bispectrum we are studying and its closest template, since it is desirable that the error bar associated with the constraints is comparable to the resolution of the data.

Others have observed that the entire bispectrum shape can be used as the individual signature of each inflationary model \([72, 73]\). Decoding the entire bispectrum shape into primitive or fundamental harmonic shapes by performing a partial-wave expansion acts in the same way as the decomposition of the angular power spectrum in terms of \(C_\ell\), once we identify the number of the harmonic with the multipole moment, \(\ell\). Whichever final bispectrum shape is produced it will be a linear superposition of five individual bispectra, with weights given by the respective interaction coefficients, \(\Lambda_i\). This has a major significance for the overall bispectrum shape: it can be viewed as a unique fingerprint of the Lagrangian structure of the theory. In this perspective, there is no real advantage in specifying the results for \(f_{\text{NL}}\), when one can use the bispectrum shape as the most sensitive and discriminating degree of freedom. Which of these methods will prove to be more efficient in data analysis is still not absolutely clear, although the modal decomposition seems to be quite promising (see, for example, Ref. \([74]\) for comments on the efficiency of this algorithm). In particular, it is possible to derive consistency relations between the amplitudes of each harmonic, which make it easy to rule out classes of inflation models \([21]\) whenever these relations are not supported by data.

In this paper we shall adopt this last philosophy and present the general formulas for the bispectrum. The study of shapes is however beyond the scope of this paper, and we leave it to forthcoming work. We specify estimates of \(f_{\text{NL}}\) in specific applications only.

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\(^{13}\)This is only true at tree-level for diagrams which involve one interaction vertex only.
3.1 Bispectrum of Horndeski theories

As argued before, to compute the non-gaussianities it will be convenient to write the action for perturbations in \( y \) coordinates. From Eq. (1.1) this is given by

\[
S^{(3)} = \int \frac{d^3x}{a} \frac{d^3y}{a^2} \left\{ \frac{c_\delta^2}{\Lambda_1} \delta^3 + c_\delta \Lambda_2 \delta^3 \zeta^2 + \frac{1}{c_s} \Lambda_3 \zeta (\partial \zeta)^2 + c_\delta \Lambda_4 \delta_i \zeta \partial^i (\partial^{-2} \zeta') + c_\delta \Lambda_5 \partial^2 \zeta (\partial \zeta^{-2} \zeta')^2 \right\}.
\]

Our calculation is organised as follows:

**Bispectrum type a.**— The background contributions to the bispectrum and the corrections arising from the slow-variation of the interaction vertices and external legs can be written in the form

\[
B^a = \frac{\Lambda_i}{2 \epsilon_i (1 + \epsilon_i)^4} \Lambda^a \left( \frac{1}{k_1^2} \right) \sum_{i} \Re \left( P^a(k_i) \tilde{J}_i \right) + \Re \left( Q^a(k_i) \tilde{J}_{i+1} \right) + T^a(k_i) + \text{cyclic permutations},
\]

where the addition of cyclic permutations entails exchange of momenta \( k_1 \to k_2 \to k_3 \). The functions \( N^a(k_1), P^a(k_1), Q^a(k_1) \) and \( T^a(k_1) \) are listed on table 1 and table 2 contains the assignments for the functions \( \tilde{J}_i \).

We note that the coefficients on table 2 for the operators \( \zeta \zeta'^2, \zeta' \partial^2 \zeta \partial^{-2} \zeta' \), and \( \partial^2 \zeta (\partial \zeta^{-2} \zeta')^2 \) are equivalent. Having done the calculation independently for each operator, this is a minimal check of the correctness of our results. This is because these operators have the same time derivative structure, and differ only in the arrangement of spatial derivatives, and therefore, in the momentum dependence.

**Bispectrum type b.**— Likewise, the contributions from the propagator in the internal lines of the Feynman diagram can be consolidated for all operators in the form

\[
B^b = \frac{\Lambda_i}{2 \epsilon_i (1 + \epsilon_i)^4} \Lambda^b \left( \frac{1}{k_1^2} \right) \sum_{i} \left[ \Re \left( P^b(k_i) \tilde{I}_i \right) + \Re \left( Q^b(k_i) \tilde{I}_{i+1} \right) \right]
\]

\[
+ \Re \left( R^b(k_i) \tilde{J}_i \right) + \Re \left( S^b(k_i) \tilde{J}_{i+1} \right) + (n_3 - 1) T^b(k_i) + \text{cyclic permutations},
\]

where the addition of cyclic permutations entails exchange of momenta \( k_1 \to k_2 \to k_3 \). The functions \( N^b(k_1), P^b(k_1), Q^b(k_1), R^b(k_1), S^b(k_1) \) and \( T^b(k_1) \) are listed on tables 3 and 4. Table 5 contains the assignments for the functions \( \tilde{I}_i \) and \( \tilde{J}_i \).

Both types of corrections are computed using the same \textit{in–in} formalism rules. The third Horndeski operator, \( \zeta (\partial \zeta)^2 \), however, presents an additional degree of complexity (which also had to be dealt with in Ref. [34]). We briefly review it here. We start by noting that this operator is undifferentiated (with respect to time), and therefore the integrand will be, at least, power-law divergent. This is because there will be insufficient powers of \( y \), to counteract the powers in denominator owing to the presence
of \(a(y)\) in the integrand. As a result the integrand behaves, at leading order, as \(a^2/c_s \sim 1/y^2\). The next-order corrections will contribute with logarithmic terms. This has serious repercussions in the final result because it can potentially lead to power law and logarithmic divergences in \(y\) when one takes the limit \(|y| \to 0\) – these are dangerous interactions, as named by Weinberg [65]. In order to ensure that the real part of the correlation function converges in the the asymptotic limit, one needs to carefully isolate the primitively divergent contributions: either power laws like \(|y|^{-\alpha}\), with \(\alpha > 0\), or logarithmic terms, \(\ln(-y)\). This is done by integrating all divergent integrals by parts. Only two situations occur as a result:

i. the isolated divergent terms contribute with a purely imaginary part to the correlation function. In this case, this divergence becomes irrelevant since the total correlation function is given by the sum of two path integrals which are complex conjugates.

ii. the isolated divergent contributions are real and contribute to the final answer. However when we sum type a and type b bispectra we verify that these divergent contributions cancel out amongst themselves. The final result contains only finite contributions.

It is crucial to take into account these two possibilities to obtain a correct result: first, to ensure that the correlation functions do not diverge in their asymptotic limit; and second, to guarantee that the calculation contains all the relevant convergent contributions to the overall bispectrum. This is particularly important when checking whether our results are consistent with Maldacena’s factorization theorem [8, 75] – this asserts that the bispectrum, in the limit when one of the momenta is small, should factorize into two copies of the power spectrum multiplied by \(n_s - 1\); in other words, in this limit, two and three-point correlators talk to each other.

3.2 Features of the bispectrum of Horndeski theories

These results extend those obtained following a strong slow-roll inflationary phase in the early Universe. Some observations following our formulas share the same ideas as in previous works in the literature, and we summarize them here for completeness:

Enhancement of non-gaussianities.— It is apparent from our formulas that the enhancement of the bispectrum can be very different from the one found in models under the slow-roll approximation. This strictly depends on the coefficients \(\Lambda_i\). Just like observed in Ref. [17], if the interaction vertices in the action (1.1) do not contain enough powers in the speed of sound \(c_s\) to counteract those in the denominator of the bispectrum, then the overall dependence can be stronger than that commonly associated with DBI models \(f_{\text{NL}} \sim c_s^{-2}\), corresponding to a bispectrum behaving as \(B \sim c_s^{-4}\). Depending on \(z\), more exotic models could reproduce such behaviour. To gauge whether this scenario would be permissible could involve applying the partial wave decomposition method to the Horndeski overall bispectrum shape. Five measurements would be required to fix each of the \(\Lambda_i\) interactions, and a number of additional measurements to break the degeneracy in other parameters, such as \(\varepsilon\) and \(c_s\), in \(\Lambda_i\). Ultimately this would

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[14] The calculation of the bispectrum of this operator is very similar to the one presented in Ref. [34], which was restricted to the slow-roll approximation. The reason this happens is because, contrary to the remaining operators, the constant \(\gamma\) in the integrals in Eqs. (D.1) and (D.7) is an integer, which allows us to write compact results in tables 1 and 3.
allow us to place constraints on all the parameters of the theory. Only then would we be able to conclude whether there would be enhancement of non-gaussianities in these models.

**Presence of logarithms.**— Our formulas contain logarithms of momenta, encoded, for example, in the master integral \( J_\epsilon \) (defined in Eq. (D.7)). There are two varieties of logarithms as noted in Ref. [34]: those which depend on the reference scale, \( \ln(k_*/k_t) \) and those which depend on the perimeter momentum scale, \( \ln(k_t/k_t) \). The first type of logarithms are clearly of the same nature as the ones identified in the power spectrum (2.11): they encode the scale dependence of the quantities they multiply. One can choose the reference scale, \( k_* \), to minimize these logs; alternatively, one can use this degree of freedom to measure primordial non-gaussianities on different scales. The last type of logarithms are shape dependent. To better understand this, one can write one of the three momenta, \( k_i \), in terms of the perimeter, \( k_t \), and two additional coordinates, describing the angular dependence. This implies that \( \ln(k_*/k_t) \) is effectively only a function of the angular coordinates, and it is therefore responsible for the shape dependence to the bispectrum.

**Away from (strong) slow-roll.**— If the inflationary background is almost de Sitter, it is easy to see that the power law behaviour \( k_*^{-\frac{\alpha}{1+\varepsilon_*}} \), where \( \alpha \) is some integer, can be written as a next-order logarithmic correction by performing a Taylor series expansion. This can be checked by taking the limit of very small \( \varepsilon \) and comparing our results to those of Ref. [34]. Away from the strong slow-regime, the dependence on the reference scale, \( k_* \), arises from non-expandable power laws in addition to logarithmic contributions. For comparison, let us write the formula of the bispectrum type \( a \) for one operator, say \( \zeta^3 \). We find that:

\[
B^{(a)}_{\zeta^3} = \frac{12H^5(1 + \varepsilon_*)^5 A_{1s}}{2^{1+\varepsilon_*} c_{\zeta}^3 k_t^2 \prod_i k_i} \left( \frac{k_*}{k_t} \right)^{\frac{5\varepsilon_*}{1+\varepsilon_*}} \left\{ \cos(\xi)\Gamma(\xi) \left[ A_* + \ln \frac{k_*}{k_t} \left( B_* + C_* \ln \frac{k_*}{k_t} \right) \right. \right.

\left. + \psi^{(0)}(3) \left( B_* + 2C_* \ln \frac{k_*}{k_t} + C_* \psi^{(0)}(\xi) \right) + C_* \psi^{(1)}(\xi) \right] \right.

\left. \right. \left. - \sin(\xi)\Gamma(\xi) \frac{\pi}{2} \left[ B_* + C_* \ln \frac{k_*}{k_t} + \ln \frac{k_*}{k_t} + 2C_* \psi^{(0)}(\xi) \right] \right\} + \text{cyclic permutations}
\]

(3.6)

with \( \xi \equiv 3 + \frac{5\varepsilon_*}{1+\varepsilon_*} \) and \( \zeta \equiv \frac{5\pi\varepsilon_*}{2(1+\varepsilon_*)} \). This expression exhibits an unusual power law dependence on the reference scale, \( k_* \), which is absent from previous studies, where only the logarithmic contributions \( \ln(k_*/k_t) \) were known [34]. By keeping the reference scale \( k_* \) arbitrary in our calculation, the scale dependence of the bispectrum can be calculated. This will include the contribution of the power law scaling as \( k_t^{-3-\frac{3\varepsilon_*}{1+\varepsilon_*}} \), which could receive large corrections from \( \varepsilon_* \). In Ref. [35] Khoury & Piazza chose \( k_* = k_t \), which masks the power law effect.

Considering a slow-roll expansion, by which we take \( \varepsilon_* \) to be a slow-roll parameter (treating it on
equal footing as $\delta$ and $n_s - 1$), Eq. (3.6) simplifies to

$$B^{(a)}_{\zeta\zeta'\zeta} \simeq \frac{3\Lambda_1 M_p^2}{8\epsilon^3 C_s^6} \prod_i k_i k_i' \left\{ \frac{h_{1s}}{2} (-3 + 2\gamma_E) + \frac{3(n_s - 1)}{2} (-2 + \gamma_E) + \frac{\delta_s}{2} (-13 + 6\gamma_E) + \frac{\epsilon_s}{2} (25 - 10\gamma_E) \right. \\
+ \left. \frac{3(n_s - 1)}{2} \ln \left( \frac{2k_1 k_2 k_3}{k_s} \right) - (h_{1s} + 3\delta_s - 5\epsilon_s) \ln \frac{k_s}{k_{1s}} \right\}$$

which resembles the bispectrum obtained in Ref. [34].\textsuperscript{15} We note that in Eq. (3.7) the dependence on $k_s$ appears through logarithms, and the power law behaviour has explicitly disappeared from the result. This is because by taking $\epsilon$ to be a small parameter, the power law is well described, at next-order, by a logarithmic contribution. We also observe that the reference scale, $k_s$, appears in the form of two logarithmic functions: $\ln (k_i/k_s)$ and $\ln (k_s/k_i)$. These species of logarithms were thoroughly studied in Ref. [34] to which we refer the reader for details.

The behaviour described above shows that whereas the power spectrum has a universal weak logarithmic scale-dependence, cf. Eq. (2.11), the bispectrum of these theories beyond the slow-roll regime can have a much stronger scale-dependence. In principle, this could be used to distinguish, on observational grounds, between these theories: from CMB ($k \lesssim 0.5h\text{Mpc}^{-1}$) to cluster scales ($k \gtrsim 0.5h\text{Mpc}^{-1}$), interpolating with scales probed by the galaxy bispectrum. A number of authors have studied constraints on non-gaussianity arising from galaxy surveys [76, 77], including its relation with biasing [78]. The scale dependence of the bispectrum was initially studied by Chen in an infrared model of DBI inflation [79], and later investigated in $P(X, \phi)$ models (a subset of the Horndeski class) by LoVerde et al. [80]. It is not the aim of this paper to present a detailed analysis of the scale dependence of the bispectrum. We rather want to offer an explicit example of theories which inherit a highly intricate scale dependence from the background dynamics.

\textbf{Comparison with previous results}

In certain limits, some of our results overlap with others in the literature. Here we compare them.

\textit{Khoury & Piazza.}— First, we recall that Khoury & Piazza [35] have calculated the bispectrum of Lagrangians involving only the inflaton and its first derivative in an exactly scale-invariant background, in which $s_* = -2\epsilon_*$. In this paper we have extended this study in two ways: by performing the calculation perturbatively away from the exact scale-invariance regime and applying it to the Horndeski class of models.

In particular let us take a specific example for comparison. Focusing on the operator $\zeta\zeta'^2$ and in the action studied by Khoury & Piazza we take

$$\Lambda_2(y) = \frac{\epsilon}{C_s^4} \left( \epsilon - 3 + 3C_s^2 \right).$$

\textsuperscript{15}To be more precise, we can indeed show that our results for the overall bispectrum agree (summing type a and type b), including the explicit dependence on the reference scale, $k_*$. 

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We note that $\Lambda_2(y)$ is rapidly varying in time and should therefore be placed inside the integral to compute the bispectrum:

$$B_{\zeta'\zeta'\zeta} \sim \int_{-\infty}^{0} dy \Lambda_2(y) c_s(y) a^2(y) y^2 (1 - iky) ,$$

where the factor $y^2$ comes from the two time derivatives of the wavefunctions and the last one from the undifferentiated wavefunction. For comparison purposes, we only retain the contributions at leading order in scale-invariance, which means setting all the corrections we have focused on this paper to zero. Therefore, selecting the leading order contribution to our bispectrum in Eq. (3.4), we find

$$B^{(\text{leading})}_{\zeta'\zeta'\zeta} \approx \frac{H^4 (1 + \epsilon_s)^4 k^2 k_t^2}{16 c_s^4 \epsilon_s^2} \left[ 3c_s^2 (k_1 + k_t) + \right.$$

$$\left. + \left( \epsilon_s - 3 \right) \left( \frac{k_t}{k_s} \right)^{4\epsilon_s} \Gamma \left( 1 - \frac{4\epsilon_s}{1 + \epsilon_s} \right) \cos \left( \frac{2\pi \epsilon_s}{1 + \epsilon_s} \right) (k_1 + k_t + \epsilon_s (k_t - 3k_1)) \right]$$

$$+ \text{cyclic permutations} ,$$

(3.8)

where we have retained the scale dependence through $k_s$. This formula reproduces the results of Ref. [35] provided we choose the reference scale to satisfy $k_s = k_t$.

Whenever comparison was possible our results reproduce those of Ref. [35].

Noller & Magueijo.— Second, in Ref. [81] Noller & Magueijo estimated the magnitude of the next-order corrections to the bispectrum of models whose Lagrangian was a polynomial function of the field profile and its first derivatives – the so-called $P(X, \phi)$ models. In comparison, our calculation extends to the larger Horndeski scalar field theories. Their estimate focused on an approximation of a subset of next-order corrections important when $|k y| \to 0$. However, there are other contributions which contribute equally to the bispectrum and are sensitive to the dynamics around the horizon crossing, when the approximation $|k y| \to 0$ fails. As we have shown these corrections can be obtained by evaluating the change in the Hankel function when we take the order to be $\nu = 3/2 - (n_s - 1)/2$, rather than at the exact scale invariant choice, $\nu = 3/2$.

Moreover, in our calculation we have performed a uniform expansion in next-order corrections in the quasi-scale invariance approximation, using in particular Eq. (2.9) for evolution of the scale factor, $a$, and Eq. (C.3) for the evolution of the sound speed of perturbations, $c_s$. Ref. [81] assumed a perfectly scale-invariant background, on top of which perturbations would develop. This amounts to setting $\alpha_* = \beta_* = 0$ in our Eq. (2.9), even when their contribution is as relevant as terms linear in $n_s - 1$, as we have previously argued. As we have seen, the scalar fluctuations are sensitive to the background dynamics, and in particular to the next-order corrections we have calculated.

For completeness, we will investigate one particular limit studied in Ref. [81] when the speed of sound of fluctuations is small, $c_s \ll 1$, in $P(X, \phi)$ theories (these include DBI inflation). This regime is phenomenologically interesting since it is known to be related with large non-gaussianities [23].
make this application more explicit we apply the notation of Ref. [34] (see also Ref. [10]), and we observe that the dominant operator in this regime is $\zeta'^3$

$$S^{(3)} \supset \int d^3x \, dy \alpha \Lambda_1 c_s^2 \xi'^3,$$

with

$$\Lambda_1 = \frac{\epsilon}{Hc_s^4} \left( 1 - c_s^2 - 2 c_s^2 \frac{\lambda}{\Sigma} \right),$$

where

$$\frac{\lambda}{\Sigma} = \frac{1}{6} \left( \frac{2f_X + 1}{c_s^2} - 1 \right).$$

$f_X$ is taken to be a constant.\footnote{In DBI models, $f_X = 1 - c_s^2$ and therefore one sees that we cannot technically require $f_X$ to be constant given that we are interested precisely in the regime when the speed of sound is rapidly varying. However, for the purposes of making this comparison more transparent, we assume this is the case, similarly to what was done in Refs. [34, 35, 81]. Technically, a more precise estimate would have to involve the time dependence of $f_X$ in the bispectrum.}

We can see that taking the limit of small $c_s$, corresponds to considering $|\frac{\lambda}{\Sigma}| \gg 1$, with which the interaction vertex is well approximated by

$$\Lambda_1(y) \approx -\frac{2}{3} \frac{\epsilon f_X}{Hc_s^4}.$$

Inspection of this formula reveals that this interaction vertex is rapidly varying, by virtue of its dependence in $\epsilon$, $H$ and $c_s$, of which only $\epsilon$ is slowly varying.

Our formulas are easily adapted to the case of rapidly varying $\Lambda_i$. For $\Lambda_1(y)$ we find

$$\Lambda_1(y) = \Lambda_1,(-k_s y)^{-\frac{\epsilon f_X}{1 + \epsilon_s}} \left\{ 1 - \frac{1}{1 + \epsilon_s} \left[ \eta_s + 9 \epsilon_s \delta_s + \frac{\epsilon_s \delta_s}{1 + \epsilon_s} + \frac{4 \epsilon_s \delta_s (\epsilon_s - 1)}{1 + \epsilon_s} \right] \ln(-k_s y) 
+ \frac{9}{2} \frac{\alpha_s \epsilon_s}{1 + \epsilon_s} \ln(-k_s y) \right\}^{(3.9)}$$

up to next-order corrections in the quasi-scale invariant approximation, following what was described in Eq. (2.18). These corrections can be absorbed into the coefficients $A_\star$, $B_\star$ and $C_\star$ in table 2. This generalizes Eq. (2.20) to rapidly varying interaction vertices. Since (3.9) adds power law contributions to the integral, one might worry that not all the previously allowed values of $\epsilon$ will allow convergence of the final result – we recall that the overall power law needs to decay faster than $(-y)^{-1}$ for convergence criteria to be met. In the event this does not happen then one is required to perform integration by parts an appropriate number of times so as to isolate the primitively divergent contributions, in a completely analogous way as we have dealt with the operator $\zeta(\partial \zeta)^2$ operator. A convergent answer should similarly place constraints on the allowed values of $\epsilon$ such that the correlators do not evolve in time, since this could signal a spurious divergence.

However, for this operator the integral (D.8) has $\gamma = 2 - 4 \epsilon_s/(1 + \epsilon_s)$ and is therefore always convergent since the condition $\text{Re} \left[ \epsilon_s/(1 + \epsilon_s) \right] < 3/4$ is satisfied for all the range of $0 < \epsilon_s < 1$. The
calculation is carried out in a similar fashion to what we described in 3.1, and which we take as a basis. Using the definition of \( f_{\text{NL}} \) following Eq. (3.2) we find that at leading order in our approximation

\[
f_{nL}^{(\text{leading, equilateral})} = \frac{5f_{X}}{c_s^2} \frac{k^2}{\sum k_i^2} (k) \frac{(1 + \epsilon_*)}{s^{\gamma}} \Gamma(\gamma) \cos \left( \frac{\pi(1 - \epsilon_*)}{1 + \epsilon_*} \right),
\]

(10.3)

where \( \gamma = \frac{3 - \epsilon_*}{1 + \epsilon_*} \) is a non-negative constant. This result gives the possibility of \( f_{\text{NL}} \) changing sign if \( \epsilon_* > 1/3 \), when the argument of the trigonometric function changes from the first to the second quadrant. Including the next-order corrections, the result becomes substantially more complicated and in an attempt to simplify these expressions as much as possible we evaluate \( f_{nL} \) in the so called equilateral limit \( k_1 = k_2 = k_3 = k \) when \( k_* = k \).

We organize the correction to the leading order non-gaussianity, \( \delta f_{nL} \), in terms proportional to the various slow-variation parameters, as follows:

\[
\delta f_{nL}^{(\text{equilateral})} = (n_\epsilon - 1)f_{nL}^{-1} + \delta f_{nL}^5 + \eta \delta f_{\eta},
\]

(11.1)

with

\[
\begin{align*}
f_{nL}^{-1} &= -\frac{5f_{X}k^{2+4\gamma}}{2c_s^2(1 + \epsilon_*)} \left( 3^{\frac{3 - \epsilon_*}{1 + \epsilon_*}} \cos \left( \frac{2\pi}{1 + \epsilon_*} \right) k^{\frac{2(3 + \epsilon_*)}{1 + \epsilon_*}} (1 + \epsilon_*)^3 \Gamma(\gamma) H_{\sigma} \right. \\
&\quad - (1 + \epsilon_*)^3 k^{\frac{4\epsilon_*/(1 + \epsilon_*)}{1 + \epsilon_*}} \sin \left( \frac{2\pi\epsilon_*}{1 + \epsilon_*} \right) \left. \right) \\
\delta f_{nL}^5 &= -\frac{5}{2c_s^2(1 + \epsilon_*)} f_{X} \Gamma(\gamma) \left\{ \cos \left( \frac{\pi}{2} \right) \pi(3 + \epsilon_*(-5 + 4\epsilon_*)) + \sin \left( \frac{\pi}{2} \right) \left[ 2 \left[ 1 + \epsilon_* + \ln 3 \left( -3 + \epsilon_*(5 - 4\epsilon_*) \right) \right] + 2(3 + \epsilon_*(-5 + 4\epsilon_*))\psi(0)(\gamma) \right\} \\
\delta f_{\eta} &= \frac{5}{2c_s^2(1 + \epsilon_*)^2} f_{X} \Gamma(\gamma) \left\{ \cos \left( \frac{\pi \psi}{2} \right) k^{-\frac{2(3 + \epsilon_*)}{1 + \epsilon_*}} \left[ -2\epsilon_*(1 + \epsilon_*) + \ln 3(2 + 2\epsilon_*(1 + \epsilon_*)^3) \right] - (2(1 + \epsilon_*) + 8\epsilon_*\ln 3 - 4\epsilon_*\psi(0)(\gamma) + 4\epsilon_*\psi(1)(\gamma) \right] \\
&- 8 \cos \left( \frac{\pi \psi}{2} \right) k^{-\frac{3 - \epsilon_*}{1 + \epsilon_*}} \epsilon_*(1 + \epsilon_*) \left[ \pi + 4\epsilon_* \pi - 8\epsilon_* \pi \psi(0)(\gamma) \right] \right\}.
\end{align*}
\]

(12.1)

where \( \sigma = 2(1 - \epsilon_*)/(1 + \epsilon_*) \). We note the dependence on the scale through \( k \), expected whenever the slow-roll approximation breaks down. The polygamma functions of order zero, \( \psi(0) \), were introduced in Eq. (D.4) in appendix D. Also, \( H_m \) denotes the \( m \)-th harmonic number which relates to \( \psi(0) \) via \( H_m = \psi(0)(m) + \gamma_{\epsilon} \) and \( \psi \) satisfies

\[
\psi = \Gamma(\gamma) \left[ \gamma_{\epsilon} + \ln 2 + \psi(0)(\gamma) \right] - 2\Gamma \left( \frac{4}{1 + \epsilon_*} \right) \Gamma \left( \frac{1}{1 + \epsilon_*} \right) \psi_{2} \left( \left\{ 1, 1, \frac{4}{1 + \epsilon_*} \right\}, \{2, 2\}, -2 \right),
\]

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where we have used the results derived in appendix D. Eq. (3.11) is exact at next-order in the quasi scale invariant approximation, and each contribution in Eqs. (3.12) to the value of $f_{\text{NL}}$ is as important as the other. Eqs. (3.10) and (3.11) are to be compared to Eq. (3.15) of Ref. [81]. As pointed out in Ref. [34] there is no reason to believe these corrections will be negligible, but their magnitude will depend on the values of the parameters $\epsilon, \eta, \delta,$ and $n_s - 1$. We do not attempt to produce an order of magnitude of these corrections here, as this is beyond the scope of this paper. We also notice that the dependence on the scale $k$ vanishes in the slow-roll limit, when $\epsilon_s$ is taken to be a perturbative parameter. This observation is in line with our previous comments on the strong scale dependence whenever there was an appreciable deviation from the slow-roll regime.

Burrage et al.— Finally, as previously discussed, our results extend what was obtained by Burrage et al. [34] and also Chen et al. [10] who treated the speed of sound of scalar fluctuations, $s$, and the expansion rate, $\epsilon$, as slowly varying in the $P(X, \phi)$ class of models. Taking the limit of small $\epsilon$ and restoring $s$ by setting $\delta = s + 2\epsilon$, we find perfect agreement between our results to next-order in slow-roll, operator by operator (which is the only sensible way of partitioning the comparison), including the logarithmic corrections previously discussed. As emphasized before, these corrections are important to correctly evaluate the scale and shape dependence of the bispectrum, as well as in obtaining an estimate of non-gaussianities with a certainty level comparable to the resolution of Planck data. As a consequence, our results obey Maldacena’s factorization theorem [8, 75] in this limit, which represents a non trivial consistency check of our calculation. The results presented in Ref. [34] regarding the scale and shape dependence of the bispectrum apply to our analysis if the slow-roll approximation is valid. Whenever there is significant breaking of the slow-roll approximation, our results diverge.

4 Conclusions and outlook

In the present era of precision Cosmology it is crucial to be able to use the experimental data soon to be delivered by Planck efficiently. At the same time, the theoretical instruments at our disposal should be able to compete with this level of accuracy. The most important of these are the search for non-gaussian signatures imprinted in the CMBR which act as a model differentiator. To this end, a primary object of study has been the bispectrum of primordial perturbations.

Whereas most studies of the bispectrum assume strong slow-roll conditions, cosmological constraints still allow for a deviation from this regime, whilst being compatible with inflationary scenarios [35]. In this paper we have focused on the phenomenology of inflationary backgrounds where $\epsilon$ and $s$ are not perturbative parameters, and satisfy the mild requirement that $\epsilon, |s| < 1$, but combined to produce a quasi scale invariant spectrum of scalar perturbations. Under these assumptions we have calculated the bispectrum for single-field Horndeski models perturbatively in $n_s - 1$ but to all orders in $\epsilon$ and $s$. We have found that both the scale as well as the shape dependence of the bispectrum is encapsulated not only in a logarithmic [34], but in a stronger, power-law form. The power-law behaviour is more relevant if the breaking from strong slow-roll is stronger.
In an optimistic scenario, such behaviour can be used in our attempt to constraint the parameters of a given theory more tightly, and potentially eliminate it from the list of sensible models against observations. It is quite likely that with Planck the amount of information one is able to extract from the CMBR about the early Universe will come to a halt. It is therefore important to be able to retain the scale dependence of the bispectrum in our theoretical computations and estimations, which may enable results from smaller scales (LSS) [82–85]. In this paper we have focused on obtaining the general expression for the bispectrum in Horndeski theories with fixed interaction coefficients, $\Lambda_i$. We have also shown how to generalize our formulas to include a rapidly varying $\Lambda_i$. This work should be regarded as a first step towards estimating observables on a background that does not obey the strong slow-roll regime. Because these theories have a strong scale dependence this increases our chances to assess the feasibility of these theories as working models of the early Universe, by using data from different scales.

We leave the study of decomposing bispectrum shapes into fundamental harmonics (basis shapes) for future work. This partial-wave decomposition is very important in distinguishing between models as discussed in Ref. [21]. Also, it will be interesting to discuss the subsequent implications in the consistency relations between the amplitudes of each harmonic, which can ultimately allow us to project out this class of models based on a dynamics beyond slow-roll. If single field models are excluded by observations, we will have to embark on a multi-field inflation exploration, where the interplay between curvature and isocurvature modes can be quite intricate (see, for example, Refs. [86–88]).

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Note.— This paper is dedicated to the memory of Martinho Henriques.
Table 1. Coefficients of the functions appearing in the $B^a$ bispectrum. For simplicity of notation, we define the quantities $\bar{k}$, $\Omega_1$ and $\Omega_2$ in table 7 of appendix E.

| operator | contributions to $B^a$ |
|----------|------------------------|
| $\zeta^3$ | $12H_x(1 + \epsilon_x) \prod_i k_i^2 k_i^{-\epsilon_x}$ | $-i \left( \frac{1}{i k_1} \right)^{3 + \frac{\epsilon_x}{1 + \epsilon_x}}$ |
| $\zeta \zeta^2$ | $4k_2^2 k_3^2 k_i^{-\epsilon_x}$ | $i \left( \frac{1}{i k_1} \right)^{1 + \frac{\epsilon_x}{1 + \epsilon_x}}$ | $-k_1 \left( \frac{1}{i k_1} \right)^{2 + \frac{\epsilon_x}{1 + \epsilon_x}}$ |
| $\zeta(\partial \zeta)^2$ | $\frac{4(k_2 \cdot k_3)}{c_s^2}$ | $\Omega_1, \bar{k}$ | $+\Omega_2, -k_1 + \frac{k_1 k_2 k_3}{k_1^2}$ | $-\Omega_2, \gamma \ln \frac{k_1}{k_1^2} \bar{k}$ |
| $\partial \zeta \partial \zeta \partial^{-2} \zeta' \partial^{-2}$ | $2k_2^2 (k_2 \cdot k_3) k_i^{-\epsilon_x}$ | $2i \left( \frac{1}{i k_1} \right)^{1 + \frac{\epsilon_x}{1 + \epsilon_x}}$ | $-(k_2 + k_3) \left( \frac{1}{i k_1} \right)^{2 + \frac{\epsilon_x}{1 + \epsilon_x}}$ |
| $\partial^2 \zeta (\partial \partial^{-2} \zeta')^2$ | $4k_2^2 (k_2 \cdot k_3) k_i^{-\epsilon_x}$ | $i \left( \frac{1}{i k_1} \right)^{1 + \frac{\epsilon_x}{1 + \epsilon_x}}$ | $-k_1 \left( \frac{1}{i k_1} \right)^{2 + \frac{\epsilon_x}{1 + \epsilon_x}}$ |

Table 2. Coefficients of the functions $\bar{J}_G$ appearing in the $B^a$ bispectrum, where $\Omega_{1*}$ and $\Omega_{3*}$ are defined in table 7 of appendix E.
| function | \( \zeta^3 \) | \( \zeta \zeta^2 \) | \( \zeta (\partial \zeta)^2 \) |
|-----------|---------------|----------------|-----------------|
| \( N^b(k_i) \) | \( 4H_s(1 + \epsilon_s) \prod_i k_i^2 k_{s_i}^{2 \epsilon_s} \) | \( 4 k_i^2 k_{s_i}^{2 \epsilon_s} \) | \( \frac{2(k_2 \cdot k_3)}{c_i^2} \) |
| \( P^b_1(k_i) \) | \( \frac{1}{2} \left( \frac{1}{\theta_1 k_i} \right)^{3+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{5\pi \epsilon_s}{2(1 + \epsilon_s)} \right] \) | \( \frac{1}{2} \left( \frac{1}{\theta_1 k_i} \right)^{1+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{2\pi \epsilon_s}{1 + \epsilon_s} \right] \) | \( -\frac{1}{2k_i} (k_1 k_2 + k_1 k_3 - k_2 k_3) \) |
| \( P^b_2(k_i) \) | \( \frac{1}{2} \left( \frac{1}{\theta_2 k_i} \right)^{3+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{5\pi \epsilon_s}{2(1 + \epsilon_s)} \right] \) | \( \frac{1}{2} \left( \frac{1}{\theta_2 k_i} \right)^{1+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{2\pi \epsilon_s}{1 + \epsilon_s} \right] \) | \( -\frac{1}{2k_i} (k_1 k_2 + k_2 k_3 - k_1 k_3) \) |
| \( P^b_3(k_i) \) | \( \frac{1}{2} \left( \frac{1}{\theta_1 k_i} \right)^{3+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{5\pi \epsilon_s}{2(1 + \epsilon_s)} \right] \) | \( \frac{1}{2} \left( \frac{1}{\theta_1 k_i} \right)^{1+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{2\pi \epsilon_s}{1 + \epsilon_s} \right] \) | \( -\frac{1}{2k_i} (k_1 k_3 + k_2 k_3 - k_1 k_2) \) |
| \( Q^b_1(k_i) \) | \( -\frac{k_1}{2} \left( \frac{1}{\theta_1 k_i} \right)^{2+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{2\pi \epsilon_s}{1 + \epsilon_s} \right] \) | \( -\frac{k_1 k_2 k_3}{k_i^3} \) |
| \( Q^b_2(k_i) \) | \( \frac{k_1}{2} \left( \frac{1}{\theta_2 k_i} \right)^{2+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{2\pi \epsilon_s}{1 + \epsilon_s} \right] \) | \( -\frac{k_1 k_3 k_3}{k_i^3} \) |
| \( Q^b_3(k_i) \) | \( \frac{k_1}{2} \left( \frac{1}{\theta_1 k_i} \right)^{2+\frac{\epsilon_s}{1+\epsilon_s}} \cos \left[ \frac{2\pi \epsilon_s}{1 + \epsilon_s} \right] \) | \( -\frac{k_1 k_3 k_3}{k_i^3} \) |
| \( R^b(k_i) \) | \( 3i \left( \frac{1}{ik_i} \right)^{3+\frac{\epsilon_s}{1+\epsilon_s}} \) | \( i \left( \frac{1}{ik_i} \right)^{1+\frac{\epsilon_s}{1+\epsilon_s}} \) | \( \Xi \) |
| \( S^b(k_i) \) | \( -k_1 \left( \frac{1}{ik_i} \right)^{2+\frac{\epsilon_s}{1+\epsilon_s}} \) | \( \Xi \) | \( f(k_1, k_2, k_3) \) |
| \( T^b(k_i) \) | \( -\left( \frac{1}{ik_i} \right)^{1+\frac{\epsilon_s}{1+\epsilon_s}} \Xi \) | \( f(k_1, k_2, k_3) \) |

Table 3. Coefficients of the functions appearing in the \( B^b \) bispectrum for the first three operators, where \( \theta_i = \frac{1}{k_i} (k_i - 2k_i) \), and \( \Xi \) and \( f(k_1, k_2, k_3) \) are defined in table 7 in appendix E.
| function | operator |
|----------|----------|
| $\zeta' \partial_i \zeta \partial_j \partial^{-2} \zeta'$ | $\partial^2 \zeta(\partial_i \partial^{-2} \zeta')^2$ |
| $N^b$ | $2k_i^2(k_2 \cdot k_3)k_i^{\frac{2+4i}{1+e_*}}$ |
| | $4k_i^2(k_2 \cdot k_3)k_i^{\frac{2+4i}{1+e_*}}$ |
| $p^b_1$ | $\left( \frac{1}{\theta_1k_i} \right)^{1+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| | $\frac{1}{2} \left( \frac{1}{\theta_1k_i} \right)^{1+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| $p^b_2$ | $\left( \frac{1}{\theta_2k_i} \right)^{1+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| | $\frac{1}{2} \left( \frac{1}{\theta_2k_i} \right)^{1+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| $p^b_3$ | $\left( \frac{1}{\theta_3k_i} \right)^{1+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| | $\frac{1}{2} \left( \frac{1}{\theta_3k_i} \right)^{1+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| $Q^b_1$ | $\frac{k_2+k_3}{2} \left( \frac{1}{\theta_1k_i} \right)^{2+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| | $k_i \left( \frac{1}{\theta_1k_i} \right)^{2+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| $Q^b_2$ | $\frac{k_3-k_2}{2} \left( \frac{1}{\theta_2k_i} \right)^{2+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| | $-k_i \left( \frac{1}{\theta_2k_i} \right)^{2+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| $Q^b_3$ | $\frac{k_2-k_3}{2} \left( \frac{1}{\theta_3k_i} \right)^{2+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| | $-k_i \left( \frac{1}{\theta_3k_i} \right)^{2+\frac{4i}{1+e_*}} \cos \left[ \frac{2\pi e_*}{1+e_*} \right]$ |
| $R^b$ | $-2i \left( \frac{1}{ik_i} \right)^{1+\frac{4i}{1+e_*}}$ |
| | $-i \left( \frac{1}{ik_i} \right)^{1+\frac{4i}{1+e_*}}$ |
| $S^b$ | $(k_2+k_3) \left( \frac{1}{ik_i} \right)^{2+\frac{4i}{1+e_*}}$ |
| | $k_i \left( \frac{1}{ik_i} \right)^{2+\frac{4i}{1+e_*}}$ |
| $T^b$ | $-2 \left( \frac{1}{k_i} \right)^{1+\frac{4i}{1+e_*}} \Xi$ |
| | $- \left( \frac{1}{k_i} \right)^{1+\frac{4i}{1+e_*}} \Xi$ |

Table 4. Coefficients of the functions appearing in the $B^b$ bispectrum for the last two operators, where $\theta_i = \frac{1}{k_i}(k_i - 2k_i)$. 

---
| operator | assignments to $\tilde{J}_\gamma$ in $B^b$ |
|----------|------------------------------------------|
| $\zeta^3$ | $\frac{2 + \frac{5\epsilon_s}{1 + \epsilon_s} \delta_s - \frac{\epsilon_s \eta_s}{(1 + \epsilon_s)^2} + \frac{i\pi}{4}(n_s - 1)}{1 + \epsilon_s} \frac{n_s - 1}{2}$ |
| $\zeta^\prime$ | $\frac{4\epsilon_s}{1 + \epsilon_s} - \frac{3\delta_s}{1 + \epsilon_s} + \frac{3\epsilon_s \eta_s}{(1 + \epsilon_s)^2} - \frac{3\pi i}{4}(n_s - 1) \frac{-3}{2}(n_s - 1)$ |
| $\zeta(\partial \zeta)^2$ | 0 | not applicable |
| $\zeta^\prime \partial \zeta \partial \zeta^\prime$ | $\frac{4\epsilon_s}{1 + \epsilon_s} - \frac{3\delta_s}{1 + \epsilon_s} + \frac{3\epsilon_s \eta_s}{(1 + \epsilon_s)^2} + \frac{3\pi i}{4}(n_s - 1) \frac{3}{2}(n_s - 1)$ |
| $\partial^2 \zeta(\partial \zeta^\prime)^2$ | $\frac{4\epsilon_s}{1 + \epsilon_s} - \frac{3\delta_s}{1 + \epsilon_s} + \frac{3\epsilon_s \eta_s}{(1 + \epsilon_s)^2} + \frac{3\pi i}{4}(n_s - 1) \frac{3}{2}(n_s - 1)$ |

Table 5. Coefficients of $\tilde{J}_\gamma$ appearing in the $B^b$ bispectrum.
Appendices

A The spectral index beyond exact scale invariance

In our main formulas we use \( n_s - 1 \) as a perturbative parameter, but never its explicit formula, since this is never necessary throughout the calculation. We have surpassed the need to know this because of the special properties of the two-point correlator of single-field models, in particular that it should be time independent on super-horizon scales. In this short appendix we present the formula for \( n_s - 1 \) for completeness. In Eq. (2.2) the variable \( q = a \sqrt{2 \epsilon \zeta_a} \) obeys the following differential equation

\[
\frac{d \ln q}{d \ln y} = -\frac{1}{1 - \epsilon - s} \left[ 1 + \frac{w}{2} + \frac{s}{2} \right] \left[ 1 + \frac{\epsilon \eta + ts}{(1 - \epsilon - s)^2} \right].
\]

From this it follows that

\[
\frac{q''}{q} = \frac{1}{2y^2 1 - \epsilon - s} + \frac{1}{2y^2 (1 - \epsilon - s)^2} \left[ \frac{\rho^2}{2} + wx + ts + \frac{\epsilon \eta + ts}{(1 - \epsilon - s)^2} \rho (4 - 2\epsilon - s + w) \right],
\]

where \( \rho \equiv 2 + w + s \) and \( x = d\ln w/dN \) (which contributes at next-order only in the quasi scale-invariant approximation). We conclude that the spectral index in these theories satisfies

\[
n_s - 1 = 3 - \sqrt{1 + \frac{2\rho}{1 - \epsilon - s} + \frac{2}{(1 - \epsilon - s)^2} \left[ \frac{\rho^2}{2} + wx + ts + \frac{\rho (\epsilon \eta + ts)}{(1 - \epsilon - s)^2} (4 - 2\epsilon - s + w) \right]}.
\]

To leading order in the quasi-scale invariant approximation, we find

\[
n_s - 1 = \frac{\eta + \delta - 2\epsilon - 3s - w}{1 - \epsilon - s},
\]

in agreement with Ref. [34]. If we instead assume that \( \epsilon \) and \( s \) are strictly constant and consider only \( P(X, \phi) \) models, we recover the formula deduced in Ref. [35] that

\[
n_s - 1 = -\frac{2\epsilon + s}{1 - \epsilon - s},
\]

which reproduces the exact scale-invariance relation \( s = -2\epsilon \). In the slow-roll approximation this reduces to

\[
n_s - 1 \approx -2\epsilon - s
\]

at leading order in the approximation, which agrees with the results from Ref. [34].

In our approximation scheme \( \delta \) is a next-order quantity and the only requirement we use in the text is that it is defined in such a way that \( \delta = s + 2\epsilon \), in \( P(X, \phi) \) models. The general formula for \( \delta \) in all the Horndeski models will depend on the form of \( w \), and therefore \( x \). Using (A.2) we conclude that

\[
n_s - 1 = \frac{\eta + \delta - 2\epsilon}{1 + \epsilon} - \frac{2}{3(1 + \epsilon)^2} \left\{ 2\epsilon x + 4(\epsilon \eta + ts) - \epsilon t \right\}.
\]

This formula agrees with our expectation that \( \delta \) is indeed a parameter contributing at next-order only.
\section*{B Dynamics in } y \text{ – why?}

In this appendix we motivate the use of the time coordinate \( y \) instead of the conformal time variable, \( \tau \). Why is this choice of space-time foliation, which defines a hypersurface at each \( y = \text{constant} \), preferred compared to the more traditional conformal time? Were we to solve for the perturbations starting from Eq. (2.1) in \( \tau \) coordinates, we would obtain a formula for the propagator for scalar perturbations, \( G_k \), which corresponds to the two-point correlator:

\[
\langle \zeta(x, \tau) \zeta(y, \tilde{\tau}) \rangle = G(x, \tau; y, \tilde{\tau})
\]  

\text{(B.1)}

We proceed to solve the Green’s function equation for the propagator, in Fourier space, which reads

\[
\left\{ \frac{d^2}{d\tau^2} + \left( \frac{1}{z} \frac{dz}{d\tau} + \frac{2}{a} \frac{da}{d\tau} \right) \frac{d}{d\tau} + k^2 c_s^2 \right\} G_k(\tau; \tilde{\tau}) = -\frac{i}{2a^2 z} \delta(\tau - \tilde{\tau}) ,
\]

\text{(B.2)}

where \( k \) is the comoving wavenumber and the Dirac delta enforces evaluation at \( \tau = \tilde{\tau} \). Because scalar fluctuations will propagate at a phase velocity \( c_s \) generically different from that of the light, one usually performs a change of variables such that the propagator is a function of the sound horizon \( z = -k c_s / \tau \).

To get the equation for the evolution of \( G_k \) in \( z \) will demand inverting \( dz/d\tau \), which without the premiss of working with perturbative \( s \) becomes algebraically challenging when plugging into the equations of motion since \( s \equiv dc_s/dN \) cannot be treated perturbatively. It turns out that expressing the \( y \) evolution of background quantities allows to naturally accommodate a rapidly changing speed of sound for the scalar perturbations, and therefore large \( s \), avoiding the difficulty just described. In other words, the \( y \) variable allows to sum all the powers in \( s \).

Moreover, writing the quadratic action for the fluctuations in the form of Eq. (2.2) makes the reproduction of Bessel’s equation more transparent, without any need for perturbative expansions. For these reasons, our dynamical analysis is presented in \( y \)-time, and follows the same lines of analysis of Khoury & Piazza in Ref. [35].

\section*{C Next-order corrections – useful formulas}

When applying the Schwinger-Keldysh formalism we will summon formulas for the corrections of the elementary wavefunctions for \( \zeta \) at next order in the quasi-scale invariant approximation scheme. This is because one should perform a uniform expansion in the slow-variation parameters which include the interaction vertices, but also the propagator for scalar perturbations. In this appendix we collect some of the formulas necessary for this expansion.

\textit{Background evolution.---} As explained in the main text, we wish to study the evolution of the background quantities in the time coordinate \( y \). To do so, we make a Taylor expansion around the time \( y \) of horizon crossing of some reference scale \( k_\star \) and make a uniform expansion for small \( \eta_\star \) and \( t_\star \). Such expansion is well defined provided we restrict our analysis to a few e-folds after horizon crossing. This ensures that \( \varepsilon \) and \( s \) have not varied significantly up to that point, such that we can treat \( \eta \) and \( t \) as perturbative parameters in the dynamics. As justified in the main text, this is indeed a good approximation.
Again, recasting this result in terms of only one non-perturbative parameter, we find
\[ \frac{1}{1 - \epsilon - s} \left\{ 1 + \frac{\epsilon \eta + ts}{1 - \epsilon - s} \right\} \ln(-k_s y) \].

To avoid cluttering the notation, we introduce the following parameters which contribute at next-order only in our approximation
\[ \alpha = \frac{t}{1 - \epsilon - s} + \frac{\epsilon \eta + ts}{(1 - \epsilon - s)^2} \]
\[ \beta = \frac{\epsilon \eta + ts}{(1 - \epsilon - s)^2} \].

Upon integration, we find
\[ c_i = c_i(-k_s y)^{-\frac{\epsilon_s}{1 - \epsilon - s}} \left\{ 1 + \frac{2\epsilon_s}{1 + \epsilon_s} \left[ \beta_s + \frac{\delta_s(\epsilon_s - 1)}{2\epsilon_s(1 + \epsilon_s)} \right] \ln(-k_s y) - \frac{\alpha_s}{2} \left( \ln(-k_s y) \right)^2 \right\} \].

This becomes more simplified if we write \( s_* = -2\epsilon_s + \delta_s \) and work perturbatively in \( \delta_s \) (that is, assuming \( O(\delta_s) = O(\eta_*^2) = O(t_s^2) \)), as follows
\[ c_i = c_i(-k_s y)^{-\frac{\epsilon_s}{1 - \epsilon - s}} \left\{ 1 + \frac{2\epsilon_s}{1 + \epsilon_s} \left[ \beta_s + \frac{\delta_s(\epsilon_s - 1)}{2\epsilon_s(1 + \epsilon_s)} \right] \ln(-k_s y) - \frac{\alpha_s}{2} \left( \ln(-k_s y) \right)^2 \right\} \].

In the limit when \( \epsilon \) and \( s \) are both strictly constant, thereby resulting in vanishing \( \alpha_s \) and \( \beta_s \), we reproduce the results in Ref. [35]:
\[ c_i \sim (-k_s y)^{-\frac{\epsilon_s}{1 - \epsilon - s}} \],
where we have temporarily restored the dependence in \( s \) through \( \delta_s \). The additional contributions appearing in Eq. (C.4) are precisely the corrections to this purely power-law behaviour and are relevant whenever \( \epsilon \) and \( s \) are slowly-varying.

Proceeding similarly to get the dynamical evolution of the Hubble parameter, we obtain
\[ H = H_s(-k_s y)^{-\frac{\epsilon_s}{1 - \epsilon - s}} \left\{ 1 + \frac{\epsilon_s}{1 + \epsilon_s} \left[ \beta_s + \frac{\delta_s}{1 + \epsilon_s} \right] \ln(-k_s y) - \frac{\alpha_s}{2} \left( \ln(-k_s y) \right)^2 \right\} \],
which in the limit of constant \( \epsilon \) and \( s \) reduces to simply
\[ H \sim (-k_s y)^{-\frac{\epsilon_s}{1 - \epsilon - s}} \].

Again, recasting this result in terms of only one non-perturbative parameter, we find
\[ H = H_s(-k_s y)^{-\frac{\epsilon_s}{1 - \epsilon - s}} \left\{ 1 + \frac{\epsilon_s}{1 + \epsilon_s} \left[ \beta_s + \frac{\delta_s}{1 + \epsilon_s} \right] \ln(-k_s y) - \frac{\alpha_s}{2} \left( \ln(-k_s y) \right)^2 \right\} \],

The explicit formulas in Eqs. (C.4) and (C.6) are relevant when replacing in Eq. (2.6) for the scale factor, \( a(y) \).

Derivatives of the elementary wavefunctions.— When calculating the bispectra of Horndeski operators which are at least once \( y \)-differentiated we will require the derivatives of the elementary wavefunctions. We find the evolution of the background primordial perturbation is given by
\[ \frac{d}{dy} \zeta_k^{(\text{background})} = \frac{iH_s(1 + \epsilon_s)}{2\sqrt{\epsilon_s}(k c_s)^{3/2}} k^2 y e^{iky} \],

\[ -27 - \]
whereas the wavefunction corrections to the internal lines obey
\[
\frac{d}{dy} \xi_k^{(\text{internal})} = \frac{i H_s(1 + \varepsilon_s)}{2 \sqrt{\pi} (k c_s)^{3/2}} k^2(-y) \left\{ -\frac{n_s - 1}{2} e^{-iky} \int_{-\infty}^{y} \frac{e^{2ik\xi}}{\xi} d\xi + e^{iky} \left[ \frac{\delta_s}{1 + \varepsilon_s} - \frac{\varepsilon_s \eta_s}{(1 + \varepsilon_s)^2} + i \frac{\pi n_s}{4} (n_s - 1) + \frac{n_s - 1}{2} \ln(-k, y) \right] \right\}.
\]
(C.8)

In obtaining these expressions we have explicitly eliminated the non-perturbative dependence in the variation of the speed of sound, parametrized by \(s\), in favour of the perturbative parameter, \(\delta\).

The time variation of the wavefunctions is relevant in correctly reproducing the time independence of the correlation functions for \(\zeta\).

D Useful integrals

In this appendix we list the integrals which occur when computing the three-point correlators. They fall in essentially two different varieties: integrals involving the \(\text{Ei}(\xi)\) function (which first appeared in Eq. (2.17)) and those which involve power laws and logarithmic integrand functions. We will identify master integrals for each of these families and analyse them separately.

D.1 Integrals involving the \(\text{Ei}(\xi)\) function

These integrals appear in the form
\[
I_{\gamma}(k_3) = \int_{-\infty}^{0} dy \left\{ (-y)^{\gamma} e^{i(k_1 + k_2 - k_3)y} \int_{-\infty}^{y} \frac{d\xi}{\xi} e^{2ik_3\xi} \right\}
\]
(D.1)

where we have explicitly written \(I\) as a function of the asymmetric momentum (in this case \(k_3\)), even though it shall be a function of the three momenta, or equivalently, \(k_t\) (perimeter in momentum space, \(k_t = k_1 + k_2 + k_3\)). The constant \(\gamma\) need not be an integer, and this is where the algebra differs from that presented in Refs. [17, 34]. In what follows we consider positive, arbitrary \(\gamma\). To simplify this integral, one follows the algorithm developed in these references: we apply a transformation of variables to convert this integral into its dimensionless version (defining \(x = i(k_t - 2k_3)y\)) and we make a rotation in the complex plane (using \(w = -ik_3\xi\)). Applying the Cauchy integral theorem we arrive at
\[
I_{\gamma}(k_3) = \left( \frac{1}{i \vartheta_3 k_t} \right) \int_{0}^{\infty} dx \left\{ x^{\gamma} e^{-x} \int_{0}^{\vartheta_3 x} \frac{d\theta_3}{\theta_3} e^{-2\theta_3 x} \right\}
\]
(D.2)

In this notation
\[
\vartheta_3 = \frac{k_3}{k_t - 2k_3} \quad \text{and} \quad \vartheta_3 = \frac{1}{k_t}(k_t - 2k_3).
\]

We will make use of these variables in §3.1. Focusing on the inner integral in (D.2) we note that upon integrating by parts and using the expansion series of the exponential function, it has a convergent series representation as follows
\[
\int_{0}^{\vartheta_3 x} \frac{d\theta_3}{\theta_3} e^{-2\theta_3 x} = \gamma_E + \ln(2\vartheta_3 x) + \frac{2}{2} \int_{1}^{\infty} \left( \frac{\theta_3 x}{n!} \right) \frac{e^{-n}}{n}.
\]

(D.3)
In obtaining this result we used the fact that \( \text{Re}(\gamma) > -1 \) for the first two contributions, whereas \( \text{Re}(\gamma) > -2 \) for the last term. This will indeed be true for all the integrals we will analyse for \( \gamma \) is strictly non-negative. Finally, we can perform the last sum in Eq. (D.4) by observing that it converges for \( \text{Re}(\theta_3) \in ] -1/2; 1/2 [ \). We conclude that the integral (D.1) has a closed-form representation, given by

\[
I_\gamma(k_3) = \left(\frac{1}{i\theta_3 k_i}\right)^{\gamma+1} \left\{ \Gamma(1 + \gamma) \left[ \gamma_E + \ln(2\theta_3) + \psi^{(0)}(1 + \gamma) \right] + \sum_{n=1}^{\infty} \frac{(-2\theta_3)^n(\gamma+n)!}{n!} \right\},
\]

in which \( \psi^{(0)}(z) \) denotes the polygamma function of order zero and argument \( z \).\(^{17}\) In obtaining this result we can perform the last sum in Eq. (D.1) by observing that it converges for \( \text{Re}(\theta_3) \in ] -1/2; 1/2 [ \). We conclude that the integral (D.1) has a closed-form representation, given by

\[
I_\gamma(k_3) = \left(\frac{1}{i\theta_3 k_i}\right)^{\gamma+1} \left\{ \Gamma(1 + \gamma) \left[ \gamma_E + \ln(2\theta_3) + \psi^{(0)}(1 + \gamma) \right] - 2\theta_3 \Gamma(2 + \gamma) _3F_2\left(\{1, 1, 2 + \gamma\}, \{2, 2\}, -2\theta_3\right) \right\},
\]

where \( F \) is a (convergent) generalized hypergeometric function, HypergeometricPFQ \( [89] \). In the main text of this paper we will use the following abbreviated notation

\[
I_\gamma(k_3) = \left(\frac{1}{i\theta_3 k_i}\right)^{\gamma+1} \tilde{I}_\gamma(k_3),
\]

which, given that \( \tilde{I}_\gamma \) is a real-valued function, makes the identification of the real part of the result of the integral more transparent. Explicit results for positive integers values of \( \gamma = 0, 1, 2 \) are listed in table 6.

### D.2 Other integrals

The other family of integrals which arises in the calculation of the bispectrum is of the form

\[
J_\gamma \equiv \int_{-\infty}^{0} \text{d}y e^{ik_i y} (-y)^\gamma \left\{ A_\gamma + B_\gamma \ln(-k_i y) + C_\gamma (\ln(-k_i y))^2 \right\},
\]

\(^{17}\)The polygamma function of order \( m \) is the \( (m+1)^{th} \) derivative of the logarithm of the gamma function, \( \Gamma \). We will only require polygamma functions of order zero and one in our formulas.
where the constant coefficients $A_\star$, $B_\star$ and $C_\star$ contain, in general, leading and next-order contributions in the quasi-scale invariance approximation. Provided $\text{Re}(\gamma) > -1$, which is indeed the case for all the integrals which will arise in our calculation, this integral converges and gives

\[
J_\gamma = \left( \frac{1}{ik_t} \right)^{1+\gamma} \Gamma(1+\gamma) \left\{ A_\star + \left[ \ln(k_*/k_t) - i \frac{\pi}{2} \right] \left[ B_\star + C_\star \left( \ln(k_*/k_t) - i \frac{\pi}{2} \right) \right] + \psi^{(0)}(1+\gamma) \left[ B_\star + 2C_\star \left( \ln(k_*/k_t) - i \frac{\pi}{2} \right) + C_\star \psi^{(0)}(1+\gamma) \right] + \left. \psi^{(1)}(1+\gamma) \right\}.
\]

Again, to simplify the notation we will refer to this integral in the form

\[
J_\gamma \equiv \left( \frac{1}{ik_t} \right)^{1+\gamma} \tilde{J}_\gamma
\]

so as to identify the real contribution to the result. We note that $\tilde{J}_\gamma$ is a complex-valued function, but given we only require the real part of $J_\gamma$ we need to use Euler’s function and write

\[
\left( \frac{1}{ik_t} \right)^{1+\gamma} = \left( \frac{1}{k_t} \right)^{1+\gamma} \left\{ \cos \left[ \frac{\pi}{2} (1+\gamma) \right] - i \sin \left[ \frac{\pi}{2} (1+\gamma) \right] \right\}.
\]

This explains the presence of trigonometric functions in tables 3 and 4. As a final remark, we note that these integrals involve logarithmic contributions of the form $\ln(k_*/k_t)$, which are responsible for scale dependence, precisely in the same way the power spectrum has a weak logarithmic scale dependence, cf. Eq. (2.11).

### E Listing variables

In the main tables of §3.1, to simplify the formulas, we have introduced a compact notation for combinations of momenta and slow-variation parameters. This is summarised in the following table.
| variable | mathematical expression |
|----------|------------------------|
| \( \bar{k} \) | \(-k_1 + k_2 + k_3/k^2_1 \) |
| \( \kappa^2 \) | \(k_1k_2 + k_1k_3 + k_2k_3\) |
| \( \Omega_{1*} \) | \(1 + 3\bar{E}_* + 2\frac{\kappa}{1+\kappa_0} \left( \frac{h_0}{k^2} \right) + \frac{2\kappa}{1+\kappa_0} - \frac{2\kappa_0}{1+\kappa_0} \) |
| \( \Omega_{2*} \) | \(-\frac{\kappa}{1+\kappa_0} + \frac{2\kappa_0}{1+\kappa_0} \) |
| \( \Omega_{3*} \) | \(\Omega_{1*} - \frac{\kappa}{1+\kappa_0} + \frac{2\kappa_0}{1+\kappa_0} \) |

\[ f(k_1,k_2,k_3) = 3\bar{k} \left( \frac{h_0}{k^2} - \ln k_0/k_1 \right) \cdot 3\bar{k} \left( \ln k_1/k_2 + 2k_2/k_1 \right) \cdot 3\bar{k} \left( \ln k_2/k_3 + 2k_3/k_2 \right) \cdot 3\bar{k} \left( \ln k_3/k_1 + 2k_1/k_3 \right) \]

\[ \Xi = \Gamma \left(1 + \frac{4\kappa}{1+\kappa_0}\right) \cos \left( \frac{2\pi x}{1+\kappa_0} \right) \]

Table 7. This table collects all the variables used in the tables included in the main text for simplicity of notation.

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