Convergence analysis for Lasserre’s measure–based hierarchy of upper bounds for polynomial optimization

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Abstract We consider the problem of minimizing a continuous function $f$ over a compact set $K$. We analyze a hierarchy of upper bounds proposed by Lasserre in [SIAM J. Optim. 21(3) (2011), pp. 864 – 885], obtained by searching for an optimal probability density function $h$ on $K$ which is a sum of squares of polynomials, so that the expectation $\int_K f(x)h(x)dx$ is minimized. We show that the rate of convergence is $O(1/\sqrt{r})$, where $2r$ is the degree bound on the density function. This analysis applies to the case when $f$ is Lipschitz continuous and $K$ is a full-dimensional compact set satisfying some boundary condition (which is satisfied, e.g., for polytopes and the Euclidean ball). The $r$th upper bound in the hierarchy may be computed using semidefinite programming if $f$ is a polynomial of degree $d$, and if all moments of order up to $2r + d$ of the Lebesgue measure on $K$ are known, which holds for example if $K$ is a simplex, hypercube, or a Euclidean ball.

Keywords Polynomial optimization · Semidefinite optimization · Lasserre hierarchy

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1 Introduction and Preliminaries

1.1 Background

We consider the problem of minimizing a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ over a compact set $K \subseteq \mathbb{R}^n$. That is, we consider the problem of computing the parameter:

$$f_{\text{min}, K} := \min_{x \in K} f(x).$$

Our main interest will be in the case where $f$ is a polynomial, and $K$ is defined by polynomial inequalities and equations. For such problems, active research has been done in recent years to construct tractable hierarchies of (upper and lower) bounds for $f_{\text{min}, K}$, based on using sums of squares of polynomials and semidefinite programming (SDP). The starting point is to reformulate $f_{\text{min}, K}$ as the problem of finding the largest scalar $\lambda$ for which the polynomial $f - \lambda$ is nonnegative over $K$ and then to replace the hard positivity condition by a suitable sum of squares decomposition. Alternatively, one may reformulate $f_{\text{min}, K}$ as the problem of finding a probability measure $\mu$ on $K$ minimizing the integral $\int_K f d\mu$. These two dual points of view form the basis of the approach developed by Lasserre [16] for building hierarchies of semidefinite programming based lower bounds for $f_{\text{min}, K}$ (see also [17,20] for an overview). Asymptotic convergence to $f_{\text{min}, K}$ holds (under some mild conditions on the set $K$). Moreover, error estimates have been shown in [25] when $K$ is a general basic closed semi-algebraic set, and in [4,5,6,7,9,11,26] for simpler sets like the standard simplex, the hypercube and the unit sphere. In particular, [25] shows that the rate of convergence of the hierarchy of lower bounds based on Schmüdgen’s Positivstellensatz is in the order $O(1/\sqrt{2r})$, while [24] shows a convergence rate in $O(1/\sqrt{\log(2r/c')})$ for the (weaker) hierarchy of bounds based on Putinar’s Positivstellensatz. Here, $c,c'$ are constants depending only on $K$ and $2r$ is the selected degree bound. For the case of the hypercube, [4] shows a convergence rate in $O(1/r)$ using Bernstein approximations.

On the other hand, by selecting suitable probability measures on $K$, one obtains upper bounds for $f_{\text{min}, K}$. This approach has been investigated, in particular, for minimization over the standard simplex and when selecting some discrete distributions over the grid points in the simplex. The multinomial distribution is used in [23,24] to show convergence in $O(1/r)$ and the multivariate hypergeometric distribution is used in [7] to show convergence in $O(1/r^2)$ for quadratic minimization over the simplex (and in the general case assuming a rational minimizer exists).

Additionally, Lasserre [18] shows that, if we fix any measure $\mu$ on $K$, then it suffices to search for a polynomial density function $h$ which is a sum of squares and minimizes the integral $\int_K fh d\mu$ in order to compute the minimum $f_{\text{min}, K}$ over $K$ (see Theorem 1 below). By adding degree constraints on the polynomial density $h$ we get a hierarchy of upper bounds for $f_{\text{min,K}}$ and our main objective in this paper is to analyze the quality of this hierarchy of upper bounds for $f_{\text{min,K}}$. Next we will recall this result of Lasserre [18] and then we describe our main results.

1.2 Lasserre’s hierarchy of upper bounds

Throughout, $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ is the set of polynomials in $n$ variables with real coefficients, and $\mathbb{R}[x]_r$ is the set of polynomials with degree at most $r$. $\Sigma[x]$ is the set of sums of squares of polynomials, and $\Sigma[x]_r = \Sigma[x] \cap \mathbb{R}[x]_r$ consists of all sums of squares of polynomials with degree at
most $2r$. We now recall the result of Lasserre [18], which is based on the following characterization for nonnegative continuous functions on a compact set $K$.

**Theorem 1** [18, Theorem 3.2] Let $K \subseteq \mathbb{R}^n$ be compact, let $\mu$ be an arbitrary, fixed, finite Borel measure supported by $K$, and let $f$ be a continuous function on $\mathbb{R}^n$. Then, $f$ is nonnegative on $K$ if and only if

$$\int_K g^2 \, d\mu \geq 0 \quad \forall g \in \mathbb{R}[x].$$

Therefore, the minimum of $f$ over $K$ can be expressed as

$$f_{\min,K} = \inf_{h \in \Sigma[x]} \int_K h(f - \lambda) \, d\mu \quad \text{s.t.} \quad \int_K h \, d\mu = 1. \quad (1)$$

Note that formula (1) does not appear explicitly in [18, Theorem 3.2], but one can derive it easily from it. Indeed, one can write

$$f_{\min,K} = \sup \{ \lambda : f(x) - \lambda \geq 0 \text{ over } K \}.$$ As $\int_K h(f - \lambda) \, d\mu = \int_K h(f - \lambda) \, d\mu$, after normalizing $\int_K h \, d\mu = 1$, we can conclude (1).

If we select the measure $\mu$ to be the Lebesgue measure in Theorem 1, then we obtain the following reformulation for $f_{\min,K}$, which we will consider in this paper:

$$f_{\min,K} = \inf_{h \in \Sigma[x]} \int_K h(x)f(x) \, dx \quad \text{s.t.} \quad \int_K h(x) \, dx = 1.$$ By bounding the degree of the polynomial $h \in \Sigma[x]$ by $2r$, we can define the parameter:

$$f^{(r)} := \inf_{h \in \Sigma[x]} \int_K h(x)f(x) \, dx \quad \text{s.t.} \quad \int_K h(x) \, dx = 1. \quad (2)$$

Clearly, the inequality $f_{\min,K} \leq f^{(r)}$ holds for any $r \in \mathbb{N}$. Lasserre [18] gives conditions under which the infimum is attained in the program (2).

**Theorem 2** [18, Theorems 4.1 and 4.2] Assume $K \subseteq \mathbb{R}^n$ is compact and has nonempty interior and let $f$ be a polynomial. Then, the program (2) has an optimal solution for every $r \in \mathbb{N}$ and

$$\lim_{r \to \infty} f^{(r)} = f_{\min,K}.$$ We now recall how to compute the parameter $f^{(r)}$ in terms of the moments $m_\alpha(K)$ of the Lebesgue measure on $K$, where

$$m_\alpha(K) := \int_K x^\alpha \, dx \quad \text{for } \alpha \in \mathbb{N}^n,$$

and $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$. 
Let \( N(n, r) := \{ \alpha \in \mathbb{N}^n : \sum_{i=1}^n \alpha_i \leq r \} \), and suppose \( f(x) = \sum_{\beta \in N(n,d)} f_\beta x^\beta \) has degree \( d \). If we write \( h \in \Sigma[x]_r \) as \( h(x) = \sum_{\alpha \in N(n,2r)} h_\alpha x^\alpha \), then the parameter \( f^{(r)}_K \) from (2) can be reformulated as follows:

\[
 f^{(r)}_K = \min_{\beta \in N(n,d)} \sum_{\alpha \in N(n,2r)} f_\beta \sum_{\alpha \in N(n,2r)} h_\alpha m_{\alpha + \beta}(K) \\
 \text{s.t.} \sum_{\alpha \in N(n,2r)} h_\alpha m_{\alpha}(K) = 1, \\
 \sum_{\alpha \in N(n,2r)} h_\alpha x^\alpha \in \Sigma[x]_r.
\]  

(3)

Hence, if we know the moments \( m_\alpha(K) \) for any \( \alpha \in \mathbb{N}^n \) with \( |\alpha| := \sum_{i=1}^n \alpha_i \leq d + 2r \), then we can compute the parameter \( f^{(r)}_K \) by solving the semidefinite program (3) which involves an LMI of size \( (\binom{d+2r}{2r}) \).

When \( K \) is the standard simplex \( \Delta_n = \{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq 1 \} \), the unit hypercube \( Q_n = [0,1]^n \), or the unit ball \( B_1(0) = \{ x \in \mathbb{R}^n : \| x \| \leq 1 \} \), there exist explicit formulas for the moments \( m_\alpha(K) \). Namely, for the standard simplex, we have

\[
m_\alpha(\Delta_n) = \prod_{i=1}^n \frac{\alpha_i!}{(|\alpha| + n)!},
\]

(4)

see e.g., [15, equation (2.4)] or [14, equation (2.2)]. From this one can easily calculate the moments for the hypercube \( Q_n \):

\[
m_\alpha(Q_n) = \int_{Q_n} x^\alpha dx = \prod_{i=1}^n \int_0^1 x_i^{\alpha_i} dx_i = \prod_{i=1}^n \frac{1}{\alpha_i + 1}.
\]

To state the moments for the unit Euclidean ball, we will use the notation \([n] := \{1, \ldots, n\}\), the Euler gamma function \( \Gamma(\cdot) \), and the notation for the double factorial of an integer \( k \):

\[
k!! = \begin{cases} k \cdot (k-2) \cdots 3 \cdot 1, & \text{if } k > 0 \text{ is odd}, \\ k \cdot (k-2) \cdots 4 \cdot 2, & \text{if } k > 0 \text{ is even}, \\ 1 & \text{if } k = 0 \text{ or } k = -1. \end{cases}
\]

In terms of this notation, the moments for the unit Euclidean ball are given by:

\[
m_\alpha(B_1(0)) = \begin{cases} \frac{\pi^{n/2} \prod_{i=1}^n (\alpha_i - 1)!!}{\Gamma(1 + (1 + \frac{1}{2} + \frac{n+1/2}{2}) \prod_{i=1}^n (\alpha_i - 1)!!)} & \text{if } \alpha_i \text{ is even for all } i \in [n], \\ \int_{B_1(0)} x^\alpha dx & \text{otherwise}, \end{cases}
\]

(5)

One may prove relation (5) using

\[
\int_{B_1(0)} x^\alpha dx = \frac{1}{\Gamma(1 + (n + |\alpha|)/2)} \int_{\mathbb{R}^n} x^\alpha \exp \left( -\| x \|^2 \right) dx
\]

(see e.g. [19, Theorem 2.1]), together with the fact (see e.g., page 872 in [18]) that

\[
\int_{-\infty}^{+\infty} t^p \exp \left( -t^2/2 \right) dt = \begin{cases} \sqrt{2\pi} (p-1)!! & \text{if } p \text{ is even}, \\ 0 & \text{if } p \text{ is odd}, \end{cases}
\]
and the identity $\Gamma(1 + \frac{k}{2}) = \frac{k!}{2^{k+1/2}} \sqrt{\pi}$ for any integer $k \in \mathbb{N}$ (see e.g., [1, Section 6.1.12]).

For a general polytope $K \subseteq \mathbb{R}^n$, it is a hard problem to compute the moments $m_\alpha(K)$. In fact, the problem of computing the volume of polytopes of varying dimensions is already #P-hard [10]. On the other hand, any polytope $K \subseteq \mathbb{R}^n$ can be triangulated into finitely many simplices (see e.g., [8]) so that one could use (4) to obtain the moments $m_\alpha(K)$ of $K$. The complexity of this method depends on the number of simplices in the triangulation. However, this number can be exponentially large (e.g., for the hypercube) and the problem of finding the smallest possible triangulation of a polytope is NP-hard, even in fixed dimension $n = 3$ (see e.g., [8]).

**Example**

Consider the minimization of the Motzkin polynomial $f(x_1, x_2) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$ over the hypercube $K = [-2, 2]^2$, which has four global minimizers at the points $(\pm 1, \pm 1)$, and $f_{\min,K} = 0$. Figure 1 shows the computed optimal sum of squares density function $h^*$, for $r = 12$, corresponding to $f^{(12)}(K) = 0.406076$. We observe that the optimal density $h^*$ shows four peaks at the four global minimizers and thus, it appears to approximate the density of a convex combination of the Dirac measures at the four minimizers.

We will present several more numerical examples in Section 4.

1.3 Our main results

In this paper we analyze the quality of the upper bounds $f^{(r)}_{\max,K}$ from (2) for the minimum $f_{\min,K}$ of $f$ over $K$. Our main result is an upper bound for the range $f^{(r)}_{\max,K} - f_{\min,K}$, which applies to the case when $f$ is Lipschitz continuous on $K$ and when $K$ is a full-dimensional compact set satisfying the additional condition from Assumption 1, see Theorem 3 below. We will use throughout the following notation about the set $K$.

We let $D(K) = \max_{x, y \in K} \|x - y\|^2$ denote the diameter of the set $K$, where $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ is the $\ell_2$-norm. Moreover, $w_{\min}(K)$ is the minimal width of $K$, which is the minimum distance between
two distinct parallel supporting hyperplanes of $K$. Throughout, $B_\epsilon(a) := \{ x \in \mathbb{R}^n : \| x - a \| \leq \epsilon \}$ denotes the Euclidean ball centered at $a \in \mathbb{R}^n$ and with radius $\epsilon > 0$. With $\gamma_n$ denoting the volume of the $n$-dimensional unit ball, the volume of the ball $B_\epsilon(a)$ is given by $\text{vol}B_\epsilon(a) = \epsilon^n \gamma_n$.

**Assumption 1** There exist constants $\eta_K > 0$ and $\epsilon_K > 0$ such that, for any point $a \in K$,

$$\text{vol}(B_\epsilon(a) \cap K) \geq \eta_K \text{vol}B_\epsilon(a) = \eta_K \epsilon^n \gamma_n, \quad \text{for all } 0 < \epsilon \leq \epsilon_K. \quad (6)$$

For instance, full-dimensional polytopes and the Euclidean balls satisfy Assumption 1; see Section 5.1 for details. Moreover, for any compact set $K \subseteq \mathbb{R}^n$ satisfying Assumption 1, define

$$r_K := \max \left\{ \frac{D(K)e}{2 \epsilon_K}, n \right\} \quad \text{if } \epsilon_K \leq 1 \quad \text{and} \quad r_K := \frac{D(K)e}{2} \quad \text{if } \epsilon_K \geq 1. \quad (7)$$

We can now present our main result.

**Theorem 3** Assume that $K \subseteq \mathbb{R}^n$ is compact and satisfies Assumption 1. Then there exists a constant $\zeta(K)$ (depending only on $K$) such that, for any Lipschitz continuous function $f$ with Lipschitz constant $M_f$ on $K$, the following inequality holds:

$$f^{(r)}_K - f_{\text{min},K} \leq \frac{\zeta(K)M_f}{\sqrt{r}} \quad \text{for any } r \geq r_K. \quad (8)$$

Moreover, if $f$ is a polynomial of degree $d$ and $K$ is a convex body, then

$$f^{(r)}_K - f_{\text{min},K} \leq \frac{2d^2 \zeta(K) \sup_{x \in K} |f(x)|}{w_{\text{min},K}\sqrt{r}} \quad \text{for any } r \geq r_K. \quad (9)$$

The key idea to show this result is to select suitable sums of squares densities which we are able to analyse. For this, we will select a global minimizer $a$ of $f$ over $K$ and consider the Gaussian distribution with mean $a$ and, as sums of squares densities, we will select the polynomials $H_{r,a}$ obtained by truncating the Taylor series expansion of the Gaussian distribution, see relation (14).

### 1.4 Contents of the paper

Our paper is organized as follows. In Section 2 we give a constructive proof for our main result Theorem 3. In Section 3 we show how to obtain feasible points in $K$ that correspond to the bounds $f^{(r)}_K$ through sampling. This is followed by a section with numerical examples (Section 4). Finally, in the concluding remarks (Section 5), we revisit Assumption 1 and discuss computational perspectives of the approach studied here.
2 Proof of our main result in Theorem 3

In this section we prove our main result in Theorem 3. Our analysis will hold for Lipschitz continuous $f$, so we will start by reviewing some relevant properties in Section 2.1. In the next step we indicate in Section 2.2 how to select the polynomial density function $h$ as a special sum of squares that we will be able to analyze. Namely, we let $a$ denote a global minimizer of the function $f$ over the set $K \subseteq \mathbb{R}^n$. Then we consider the density function $G_a$ in (12) of the Gaussian distribution with mean $a$ and the polynomial $H_{r,a}$ in (14), which is obtained from the truncation at degree $2r$ of the Taylor series expansion of the Gaussian density function $G_a$. The final step will be to analyze the quality of the bound obtained by selecting the polynomial $H_{r,a}$ and this will be the most technical part of the proof, carried out in Section 2.3.

2.1 Lipschitz continuous functions

A function $f$ is said to be Lipschitz continuous on $K$, with Lipschitz constant $M_f$, if it satisfies:

$$|f(y) - f(x)| \leq M_f \|y - x\| \quad \text{for all} \quad x, y \in K.$$  \hspace{1cm} (10)

If $f$ is continuous and differentiable on $K$, then $f$ is Lipschitz continuous on $K$ with respect to the constant

$$M_f = \max_{x \in K} \|\nabla f(x)\|.$$  \hspace{1cm} (11)

Furthermore, if $f$ is an $n$-variate polynomial with degree $d$, then the Markov inequality for $f$ on a convex body $K$ reads as

$$\max_{x \in K} \|\nabla f(x)\| \leq \frac{2d^2}{w_{\min}(K)} \sup_{x \in K} |f(x)|,$$

see e.g., [3, relation (8)]. Thus, together with (10), we have that $f$ is Lipschitz continuous on $K$ with respect to the constant

$$M_f \leq \frac{2d^2}{w_{\min}(K)} \sup_{x \in K} |f(x)|.$$  \hspace{1cm} (11)

2.2 Choosing the polynomial density function $H_{r,a}$

Consider the function

$$G_a(x) := \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left( -\frac{\|x - a\|^2}{2\sigma^2} \right),$$  \hspace{1cm} (12)

which is the probability density function of the Gaussian distribution with mean $a$ and standard variance $\sigma$ (whose value will be defined later). Let the constant $C_{K,a}$ be defined by

$$\int_K C_{K,a}G_a(x)dx = 1.$$  \hspace{1cm} (13)

Observe that $G_a(x)$ is equal to the function $\frac{1}{(2\pi\sigma^2)^{n/2}} e^{-t}$ evaluated at the point $t = \frac{\|x-a\|^2}{2\sigma^2}$. 


Denote by $H_{r,a}$ the Taylor series expansion of $G_a$ truncated at the order $2r$. That is,

$$H_{r,a}(x) = \frac{1}{(2\pi\sigma^2)^{n/2}} \sum_{k=0}^{2r} \frac{1}{k!} \left(-\frac{\|x-a\|^2}{2\sigma^2}\right)^k. \tag{14}$$

Moreover consider the constant $c_{K,a}^r$, defined by

$$\int_K c_{K,a}^r H_{r,a}(x)\,dx = 1. \tag{15}$$

The next step is to show that $H_{r,a}$ is a sum of squares of polynomials and thus $H_{r,a} \in \Sigma[x]_{2r}$. This follows from the next lemma.

**Lemma 1** Let $\phi_{2r}(t)$ denote the (univariate) polynomial of degree $2r$ obtained by truncating the Taylor series expansion of $e^{-t}$ at the order $2r$. That is,

$$\phi_{2r}(t) := \sum_{k=0}^{2r} \frac{(-t)^k}{k!}. \tag{16}$$

Then $\phi_{2r}$ is a sum of squares of polynomials. Moreover, we have

$$0 \leq \phi_{2r}(t) - e^{-t} \leq \frac{2^{2r+1}}{(2r+1)!} \text{ for all } t \geq 0. \tag{17}$$

**Proof** First, we show that $\phi_{2r}$ is a sum of squares. As $\phi_{2r}$ is a univariate polynomial, by Hilbert’s Theorem (see e.g., [20, Theorem 3.4]), it suffices to show that $\phi_{2r}(t) \geq 0$ for any $t \in \mathbb{R}$. As $\phi_{2r}(-\infty) = \phi_{2r}(+\infty) = +\infty$, it suffices to show that $\phi_{2r}(t) \geq 0$ at all the stationary points $t$ where $\phi'_{2r}(t) = 0$. For this, observe that $\phi'_{2r}(t) = \sum_{k=1}^{2r} (-1)^k \frac{t^{k-1}}{(k-1)!}$, so that it can be written as $\phi'_{2r}(t) = -\phi_{2r}(t) + \frac{t^2}{2r!}$. Hence, for any $t$ with $\phi'_{2r}(t) = 0$, we have $\phi_{2r}(t) = \frac{t^2}{(2r)!} \geq 0$.

Next, we show that $\phi'_{2r}(t) \geq e^{-t}$ for all $t \geq 0$. Fix $t \geq 0$. Then, by Taylor Theorem (see e.g., [30]), one has $e^{-t} = \phi_{2r}(t) + \frac{t^{2r+1}(\xi(t))^{2r+1}}{(2r+1)!}$ for some $\xi \in [0, t]$. As $\phi'(2r+1)(\xi) = -e^{-\xi}$, one can conclude that $e^{-t} - \phi_{2r}(t) = -\frac{e^{-\xi}t^{2r+1}}{(2r+1)!} \leq 0$ and $e^{-t} - \phi_{2r}(t) \geq -\frac{t^{2r+1}}{(2r+1)!}$. \hfill $\Box$

We now consider the parameter $f_{K,a}^{(r)}$ defined as

$$f_{K,a}^{(r)} := \int_K f(x) c_{K,a}^r H_{r,a}(x)\,dx. \tag{17}$$

Our main technical result is the following upper bound for the range $f_{K,a}^{(r)} - f_{\min,K}$, whose proof is given in Section 2.3 below. Theorem 4 follows then as a direct application of Theorem 4.

**Theorem 4** Assume $K \subseteq \mathbb{R}^n$ is compact and satisfies Assumption 4 and consider the parameter $r_K$ from (7). Then there exists a constant $\zeta(K)$ (depending only on $K$) such that, for any Lipschitz continuous function $f$ with Lipschitz constant $M_f$ on $K$, the following inequality holds:

$$f_{K,a}^{(r)} - f_{\min,K} \leq \frac{\zeta(K)M_f}{\sqrt{2r+1}}, \text{ for any } r \geq r_K. \tag{18}$$
Moreover, if \( f \) is a polynomial of degree \( d \) and \( K \) is a convex body, then
\[
f_{K,a}^{(r)} - f_{\min,K} \leq \frac{2d^2 \zeta(K) \sup_{x \in K} |f(x)|}{u_{\min}(K) \sqrt{2r + 1}}, \quad \text{for any } r \geq \frac{r_K}{2}. \tag{19}
\]

**Proof (of Theorem 3)** Assume \( f \) is Lipschitz continuous with Lipschitz constant \( M_f \) on \( K \) and \( a \) is a minimizer of \( f \) over the set \( K \). Using the definitions (2) and (17) of the parameters and the fact that \( H_{r,a} \) is a sum of squares with degree \( 4r \), it follows that
\[
f_{K,a}^{(2r+1)} \leq f_{K,a}^{(2r)} \leq f_{K,a}^{(r)}, \quad \text{for any } r \in \mathbb{N}.
\]
Then, from inequality (18) in Theorem 4, one obtains
\[
f_{K,a}^{(2r+1)} - f_{\min,K} \leq f_{K,a}^{(2r)} - f_{\min,K} \leq f_{K,a}^{(r)} - f_{\min,K} \leq \frac{\zeta(K) M_f}{\sqrt{2r + 1}} \quad \text{for any } r \geq \frac{r_K}{2}.
\]
Hence, for any \( r \geq r_K \),
\[
f_{K,a}^{(r)} - f_{\min,K} \leq \frac{\zeta(K) M_f}{\sqrt{r + 1}} \quad \text{for even } r,
\]
\[
f_{K,a}^{(r)} - f_{\min,K} \leq \frac{\zeta(K) M_f}{\sqrt{r}} \quad \text{for odd } r.
\]
This concludes the proof for relation (8), and relation (9) follows from (19) in an analogous way. This finishes the proof of Theorem 3. \( \square \)

### 2.3 Analyzing the polynomial density function \( H_{r,a} \)

In this section we prove the result of Theorem 4. Recall that \( a \) is a global minimizer of \( f \) over \( K \). For the proof, we will need the following four technical lemmas.

**Lemma 2** Assume \( K \subseteq \mathbb{R}^n \) is compact and satisfies Assumption 4. Then, for any \( 0 < \epsilon \leq c_K \) and \( r \in \mathbb{N} \), we have:
\[
c_{r,K,a}^\epsilon \leq C_{K,a} \leq \frac{(2\pi \sigma^2)^{n/2} \exp \left( \frac{\epsilon^2}{2\sigma^2} \right)}{\eta_K c_n \gamma_n}. \tag{20}
\]

**Proof** By Lemma 1, \( \phi_{2r}(t) \geq e^{-t} \) for all \( t \geq 0 \), which implies \( H_{r,a}(x) \geq G_a(x) \) for all \( x \in \mathbb{R}^n \). Together with the relations (13) and (15) defining the constants \( C_{K,a} \) and \( c_{r,K,a}^\epsilon \), we deduce that \( c_{r,K,a}^\epsilon \leq C_{K,a} \). Moreover, by the definition (13) of the constant \( C_{K,a} \), one has
\[
\frac{1}{C_{r,K,a}^\epsilon} = \int_K G_a(x) dx = \int_K \frac{1}{(2\pi \sigma^2)^{n/2}} \exp \left( -\frac{\|x - a\|^2}{2\sigma^2} \right) dx
\]
\[
\geq \int_{K \cap B_r(a)} \frac{1}{(2\pi \sigma^2)^{n/2}} \exp \left( -\frac{\|x - a\|^2}{2\sigma^2} \right) dx
\]
\[
\geq \frac{1}{(2\pi \sigma^2)^{n/2}} \exp \left( -\frac{\epsilon^2}{2\sigma^2} \right) \text{vol}(K \cap B_r(a)).
\]
We now use relation (6) from Assumption 1 in order to conclude that $\text{vol}(K \cap B_r(a)) \geq \eta_k e^n \gamma_n$, which gives the desired upper bound on $C_{K,a}$.

Lemma 3 Given $\tilde{x} \in \mathbb{R}^n$ and a function $F : \mathbb{R}^+ \rightarrow \mathbb{R}$, define the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(x) = F(\|x - \tilde{x}\|)$ for any $x \in \mathbb{R}^n$. Then, for any $\rho_2 \geq \rho_1 \geq 0$, one has

$$\int_{B_{\rho_2}(\tilde{x}) \setminus B_{\rho_1}(\tilde{x})} f(x)dx = n \gamma_n \int_{\rho_1}^{\rho_2} z^{n-1}F(z)dz,$$

where $\gamma_n = \frac{2^{(n-1)/2} \Gamma(n/2)}{n!!}$ is the volume of the unit Euclidean ball in $\mathbb{R}^n$.

Proof Apply a change of variables using spherical coordinates as explained, e.g., in [2].

Lemma 4 For any positive integers $r$ and $n$, one has

$$\left(\frac{1}{2r+1}\right)^{\frac{n}{2}} < 6n.$$

Proof Let $n \in \mathbb{N}$ be given. Denote

$$g(r) := \left(\frac{1}{2r+1}\right)^{\frac{n}{2r+1}} = (2r+1)^{\frac{-n}{2r+1}} \quad (r \geq 0).$$

Observe that, $g(0) = 1$, $g(r) > 0$ for all $r \geq 0$, $\ln(g(r)) = \frac{n}{8r+4+2n} \ln(2r+1)$, and thus $\lim_{r \to \infty} g(r) = 1$. It suffices to show $g(r^*) < 6$ for any stationary point $r^*$. Since

$$\frac{d\ln(g(r))}{dr} = -\frac{8n}{(8r+4+2n)^2} \left[\frac{2n}{2n+1} + \frac{1}{2n+1}\right],$$

and $g'(r) = \frac{1}{g(r)} \frac{d\ln(g(r))}{dr}$, any stationary point $r^*$ satisfies

$$\frac{d\ln(g(r^*))}{dr} = 0 \iff (2r^*+1) [\ln(2r^*+1) - 1] = \frac{n}{2}.$$

Since

$$(2r^*+1)(\ln(3) - 1) \leq (2r^*+1) [\ln(2r^*+1) - 1] = \frac{n}{2},$$

one has $2r^*+1 \leq \frac{n}{2(\ln(3) - 1)} < 6n$. Since $g(r) \leq 2r+1$ for all $r \geq 0$, one has $g(r^*) \leq 2r^*+1 < 6n$.

Lemma 5 Assume $K \subseteq \mathbb{R}^n$ is compact and satisfies Assumption 1. Then, for any $0 < \epsilon \leq \epsilon_k$, one has

$$\int_K \mathcal{C}_{K,a} \|x-a\|G_a(x)dx \leq \epsilon + \frac{n\sigma^{n+1}p(n)}{\epsilon^n \eta_k} e^{\frac{\epsilon^2}{8\pi^2}},$$

where $p(n) := \int_0^{+\infty} t^n e^{-t^2/2} dt$ is a constant depending on $n$, given by

$$p(n) = \begin{cases} 1 & \text{if } n = 1, \\ \sqrt{\frac{n}{2}} \prod_{j=1}^{k} (2j-1) & \text{if } n = 2k \text{ and } k \geq 1, \\ \prod_{j=1}^{k} (2j) & \text{if } n = 2k+1 \text{ and } k \geq 1. \end{cases} \quad (21)$$
Proof Let \( \varphi := \int_K C_{K,a} \|x-a\| G_a(x) \, dx \) denote the integral that we need to upper bound. We split the integral \( \varphi \) as \( \varphi = \varphi_1 + \varphi_2 \), depending on whether \( x \) lies in the ball \( B_\epsilon(a) \) or not.

First, we upper bound the term \( \varphi_1 \) as

\[
\varphi_1 := \int_{K \cap B_\epsilon(a)} \|x-a\| C_{K,a} G_a(x) \, dx \leq \epsilon \int_{K \cap B_\epsilon(a)} C_{K,a} G_a(x) \, dx \leq \epsilon \int_K C_{K,a} G_a(x) \, dx = \epsilon.
\]

Second, we bound the integral

\[
\varphi_2 := C_{K,a} \int_{K \setminus B_\epsilon(a)} \|x-a\| G_a(x) \, dx.
\]

Since \( K \subseteq B_{\sqrt{D(K)}}(a) \), one has

\[
\varphi_2 \leq C_{K,a} \int_{B_{\sqrt{D(K)}}(a) \setminus B_\epsilon(a)} \|x-a\| G_a(x) \, dx,
\]

where the right hand side, by Lemma 3, is equal to

\[
\frac{C_{K,a} n \gamma_n}{(2\pi \sigma^2)^{n/2}} \int_0^{\sqrt{D(K)}} z^n \exp \left( -\frac{z^2}{2\sigma^2} \right) \, dz.
\]

By a change of variable \( t = \frac{z}{\sigma} \), one obtains

\[
\varphi_2 \leq \frac{C_{K,a} n \gamma_n \sigma}{(2\pi)^{n/2}} \int_{\epsilon/\sigma}^{\sqrt{D(K)}/\sigma} t^n \exp \left( -\frac{t^2}{2} \right) \, dt,
\]

and thus

\[
\varphi_2 \leq \frac{C_{K,a} n \gamma_n \sigma}{(2\pi)^{n/2}} \int_0^{+\infty} t^n \exp \left( -\frac{t^2}{2} \right) \, dt = \frac{C_{K,a} n \gamma_n \sigma}{(2\pi)^{n/2}} p(n).
\]

Here we have set \( p(n) := \int_0^{+\infty} t^n e^{-\frac{t^2}{2}} \, dt \) which can be checked to be given by (21) (e.g., using induction on \( n \)). Now, combining with the upper bound for \( C_{K,a} \) from (20), we obtain

\[
\varphi_2 \leq \frac{n \sigma^{n+1} p(n)}{\epsilon^n \eta_K} e^{\frac{x^2}{2\sigma^2}}.
\]

Therefore, we have shown:

\[
\varphi = \varphi_1 + \varphi_2 \leq \epsilon + \frac{n \sigma^{n+1} p(n)}{\epsilon^n \eta_K} e^{\frac{x^2}{2\sigma^2}},
\]

which shows the lemma.

\( \square \)

We are now ready to prove Theorem 4.
Proof (of Theorem 4). Observe that, if \( f \) is a polynomial, then we can use the upper bound (11) for its Lipschitz constant and thus the inequality (19) follows as a direct consequence of the inequality (18). Therefore, it suffices to show the relation (18).

Recall that \( a \) is a minimizer of \( f \) over \( K \). As \( f \) is Lipschitz continuous with Lipschitz constant \( M_f \), on \( K \), we have

\[
f(x) - f(a) \leq M_f \|x - a\| \quad \forall x \in K.
\]

This implies

\[
f^{(r)}_{K,a} - f_{\min, K} = \int_K c_{K,a}^r H_{r,a}(x)(f(x) - f(a))dx \leq M_f \int_K \|x - a\| c_{K,a}^r H_{r,a}(x)dx.
\]

Our objective is now to show the existence of a constant \( \zeta(K) \) such that

\[
\psi := \int_K c_{K,a}^r \|x - a\| H_{r,a}(x)dx \leq \frac{c(K)}{\sqrt{2r + 1}} \quad \text{for any } r \geq r_K, \text{ (see (7))}
\]

by which we can then conclude the proof for (18).

For this, we split the integral \( \psi \) as the sum of two terms:

\[
\psi = \int_K c_{K,a}^r \|x - a\| G_a(x)dx + \int_K c_{K,a}^r \|x - a\| (H_{r,a}(x) - G_a(x))dx.
\]

First, we upper bound the term \( \psi_1 \). As \( c_{K,a}^r \leq C_{K,a} \) (by (20)), we can use Lemma 5 to conclude that, for any \( 0 < \epsilon \leq \epsilon_K \),

\[
\psi_1 \leq \int_K C_{K,a} \|x - a\| G_a(x)dx \leq \epsilon + \frac{n\sigma^{n+1}p(n)}{\epsilon \eta K} e^{\frac{\sigma^2}{2\sigma^2}} = \epsilon \left[ 1 + \frac{n\sigma^{n+1}p(n)}{\epsilon \eta K} e^{\frac{\sigma^2}{2\sigma^2}} \right] = \epsilon \mu_1. \tag{22}
\]

Second we bound the integral

\[
\psi_2 = \int_K c_{K,a}^r \|x - a\| (H_{r,a}(x) - G_a(x))dx.
\]

We can upper bound the function \( H_{r,a}(x) - G_a(x) \) using the estimate from (16) and we get

\[
H_{r,a}(x) - G_a(x) \leq \frac{1}{(2\pi \sigma^2)^{n/2} (2\sigma^2)^{2r+1}(2r + 1)!} \|x - a\|^{4r+2}.
\]

Then we have

\[
\psi_2 \leq \frac{1}{(2\pi \sigma^2)^{n/2}} \int_K c_{K,a}^r \frac{\|x - a\|^{4r+3}}{(2\sigma^2)^{2r+1}(2r + 1)!}dx = \frac{c_{K,a}^r}{(2\pi \sigma^2)^{n/2} (2\sigma^2)^{2r+1}(2r + 1)!} \int_K \|x - a\|^{4r+3}dx.
\]
Now we upper bound the integral \( \int_K \|x - a\|^{4r+3}dx \). Since \( K \subseteq B_{\sqrt{D(K)}}(a) \), one has
\[
\int_K \|x - a\|^{4r+3}dx \leq \int_{B_{\sqrt{D(K)}}(a)} \|x - a\|^{4r+3}dx,
\]
where the right hand side, by Lemma 3, is equal to
\[
n\gamma_n \int_0^{\sqrt{D(K)}} z^{4r+n+2}dz = \frac{n\gamma_n D(K)^{4r+n+3}}{4r+n+3} \leq n\gamma_n D(K)^{4r+n+3}.
\]
Thus, we obtain
\[
\psi_2 \leq \frac{1}{(2\pi\sigma^2)^{n/2}} \left( \frac{c_{K,a}}{(2\sigma^2)^{2r+1}(2r+1)!} \right) n\gamma_n D(K)^{4r+n+3}.
\]
We now use the upper bound for \( c_{K,a} \) from (20):
\[
c_{K,a}^r \leq \frac{(2\pi\sigma^2)^{n/2} \exp \left( \frac{\sigma^2}{2\sigma^2} \right)}{\eta_K \epsilon^n \gamma_n}
\]
and we obtain
\[
\psi_2 \leq \frac{n \exp \left( \frac{x^2}{2\sigma^2} \right) D(K) \frac{4r+n+3}{2}}{\eta_K \epsilon^n (2r+1)! (2\sigma^2)^{2r+1}}.
\]
Finally we use the Stirling’s inequality:
\[
(2r+1)! \geq \sqrt{2\pi(2r+1)} \left( \frac{2r+1}{e} \right)^{2r+1},
\]
and obtain
\[
\psi_2 \leq \frac{n \exp \left( \frac{x^2}{2\sigma^2} \right) D(K) \frac{4r+n+3}{2}}{\eta_K \epsilon^n (2r+1)! (2\sigma^2)^{2r+1}} \left( \frac{D(K)e}{2\sigma^2 e^{n/(2r+1)}(2r+1)} \right)^{2r+1} \leq \frac{1}{\sqrt{2\pi(2r+1)}} \left( \frac{D(K)e}{2\sigma^2 e^{n/(2r+1)}(2r+1)} \right)^{2r+1}.
\]
We can now upper bound the quantity \( \psi = \psi_1 + \psi_2 \), by combining the upper bound for \( \psi_1 \) in (22) with the above upper bound (23) for \( \psi_2 \). That is,
\[
\psi \leq \epsilon \mu_1 + \frac{\mu_2}{\sqrt{2\pi(2r+1)}} \left( \frac{D(K)e}{2\sigma^2 e^{n/(2r+1)}(2r+1)} \right)^{2r+1}.
\]
We now indicate how to select the parameters \( \epsilon \) and \( \sigma \).
First we select \( \sigma = \epsilon \), so that both parameters \( \mu_1 \) and \( \mu_2 \) appearing in (22) and (23) are constants depending on \( n \) and \( K \), namely
Note that in the proof of Theorem 4, we use Assumption 1 only for the selected minimizer.

Remark 1

Since we have assumed $\varepsilon_3$ holds when selecting $\eta$ together with the two inequalities

Finally, by defining the constant $\zeta(K) := 6n \left( \mu_1 \max \left\{ 1, \sqrt{\frac{D(K)e}{2}} \right\} + \frac{\mu_2}{\sqrt{2\pi}} \right)$,

which indeed depends only on $K$ and its dimension $n$, we can conclude the proof for (15). \qed

Remark 1 Note that in the proof of Theorem 4, we use Assumption 1 only for the selected minimizer $a \in K$ (and we use it only in the proof of Lemma 2). Hence, if the selected point $a$ lies in the interior of $K$, i.e., if there exists $\delta > 0$ such that $B_\delta(a) \subseteq K$, then the result of Theorem 4 (and thus Theorem 3) holds when selecting $\eta_K = 1$ and $\varepsilon_K = \delta$.

Our results extend also to unconstrained global minimization:

$$f^* := \min_{x \in \mathbb{R}^n} f(x),$$

if we know that $f$ has a global minimizer $a$ and we know a ball $B_\delta(0)$ containing $a$. We can then indeed minimize $f$ over a compact set $K$, which can be chosen to be the ball $B_\delta(0)$ or a suitable hypercube containing $a$.  

$$\mu_1 = 1 + \frac{np(n)e^{1/2}}{\eta_K} \quad \text{and} \quad \mu_2 = \frac{ne^{1/2}D(K)^{\frac{3+1}{2}}}{\eta_K}.$$ 

Next we select $\varepsilon$ so that $\frac{D(K)e}{2r^{2+n/[(2r+1)(2r+1)+1]}} = 1$, i.e.,

$$\varepsilon = \left( \frac{D(K)e}{2} \right)^{\frac{2r+1}{2r^{2+n/[(2r+1)(2r+1)+1]}}} = \left( \frac{D(K)e}{2} \right)^{\frac{2r+1}{2r^{2+n/[(2r+1)(2r+1)+1]}}} \left( \frac{1}{2r+1} \right)^{\frac{1}{2}-n/[(2r+1)+2n]}.$$ 

Summarizing, we have shown that

$$\psi \leq \left( \frac{1}{2r+1} \right)^{\frac{1}{2}-n/[(2r+1)+2n]} \left[ \left( \frac{D(K)e}{2} \right)^{\frac{2r+1}{2r^{2+n/[(2r+1)(2r+1)+1]}}} \mu_1 + \frac{\mu_2}{\sqrt{2\pi}} \left( \frac{1}{2r+1} \right)^{\frac{1}{2}-n/[(2r+1)+2n]} \right]$$

To obtain the last inequality (24), we use the inequality $\left( \frac{1}{2r+1} \right)^{\frac{1}{2}-n/[(2r+1)+2n]} < 6n$ (recall Lemma 4), together with the two inequalities $\left( \frac{D(K)e}{2} \right)^{\frac{2r+1}{2r^{2+n/[(2r+1)(2r+1)+1]}}} \leq \max \left\{ 1, \sqrt{\frac{D(K)e}{2}} \right\}$ and $\left( \frac{1}{2r+1} \right)^{\frac{1}{2}-n/[(2r+1)+2n]} \leq 1$.

Since we have assumed $\varepsilon \leq \varepsilon_K$ (recall Lemma 2), this implies the condition $r \geq \frac{D(K)e}{4\varepsilon_K^{-(2+\frac{n}{2r+1})}} - 1$, i.e., the inequality (24) holds for any $r \geq \frac{D(K)e}{4\varepsilon_K^{-(2+\frac{n}{2r+1})}} - 1$. If $\varepsilon_K \leq 1$ and $r \geq n/2$, then we have $\varepsilon_K^{-(2+\frac{n}{2r+1})} \leq \varepsilon_K^{-3}$ and thus the inequality (24) holds for any $r \geq \max \left\{ \frac{D(K)e}{4\varepsilon_K^{-(2+\frac{n}{2r+1})}}, \frac{n}{2} \right\}$. If $\varepsilon_K \geq 1$ then $\varepsilon_K^{-(2+\frac{n}{2r+1})} \leq 1$ and thus (24) holds for any integer $r \geq \frac{D(K)e}{4\varepsilon_K}$.

Hence, the inequality (24) holds for any $r \geq r_K/2$, where $r_K$ is as defined in (7).

Finally, by defining the constant $\zeta(K) := 6n \left( \mu_1 \max \left\{ 1, \sqrt{\frac{D(K)e}{2}} \right\} + \frac{\mu_2}{\sqrt{2\pi}} \right)$, we can conclude the proof for (15). \qed
3 Obtaining feasible solutions through sampling

In this section we indicate how to sample feasible points in the set \( K \) from the optimal density function obtained by solving the semidefinite program \( \mathcal{P} \).

Let \( f \in \mathbb{R}[x] \) be a polynomial. Suppose \( h^*(x) \in \Sigma[x] \) is an optimal solution of the program \( \mathcal{P} \), i.e., \( \mathcal{P}(r) = \int_K f(x)h^*(x)dx \) and \( \int_K h^*(x)dx = 1 \). Then \( h^* \) can be seen as the probability density function of a probability distribution on \( K \), denoted as \( \mathcal{P}_K \) and, for any random vector \( X = (X_1, \ldots, X_n) \sim \mathcal{P}_K \), the expectation of \( f(X) \) is given by:

\[
\mathbb{E}[f(X)] = \int_K f(x)h^*(x)dx = \int_K f(r).
\] (25)

As we now recall one can generate random samples \( x \in \mathbb{K} \) from the distribution \( \mathcal{P}_K \) using the well known method of conditional distributions (see e.g., [21, Section 8.5.1]). Then we will observe that with high probability one of these sample points satisfies (roughly) the inequality \( f(x) \leq f(r(K)) \) (see Theorem 3 for details).

In order to sample a random vector \( X = (X_1, \ldots, X_n) \sim \mathcal{P}_K \), we assume that, for each \( i = 2, \ldots, n \), we know the cumulative conditional distribution of \( X_i \) given that \( X_j = x_j \) for \( j = 1, \ldots, i-1 \), defined in terms of probabilities as

\[
F_i(x_i \mid x_1, \ldots, x_{i-1}) := \Pr[X_i \leq x_i \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}].
\]

Additionally, we assume that we know the cumulative marginal distribution function of \( X_i \), defined as:

\[
F_i(x_i) := \Pr[X_i \leq x_i].
\]

Then one can generate a random sample \( x = (x_1, \ldots, x_n) \in \mathbb{K} \) from the distribution \( \mathcal{P}_K \) by the following algorithm:

- Generate \( x_1 \) with cumulative distribution function \( F_1(\cdot) \).
- Generate \( x_2 \) with cumulative distribution function \( F_2(\cdot \mid x_1) \).
- \( \vdots \)
- Generate \( x_n \) with cumulative distribution function \( F_n(\cdot \mid x_1, \ldots, x_{n-1}) \).

Then return \( x = (x_1, x_2, \ldots, x_n)^T \).

There remains to explain how to generate a (univariate) sample point \( x \) with a given cumulative distribution function \( F(\cdot) \), since this operation is carried out at each of the \( n \) steps of the above algorithm. For this one can use the classical inverse-transform method (see e.g., [21, Section 8.2.1]), which reduces to sampling from the uniform distribution on \([0,1]\) and can be described as follows:

- Generate a sample \( u \) from the uniform distribution over \([0,1]\).
- Return \( x = F^{-1}(u) \) (if \( F \) is strictly monotone increasing, or \( x = \min\{y : F(y) \geq u\} \) otherwise).

As an illustration, we now indicate how to compute the cumulative marginal and conditional distributions \( F_i(\cdot) \) and \( F_i(\cdot \mid x_1 \ldots x_{i-1}) \) for the case of the hypercube \( Q_n = [0,1]^n \). We will then apply this method to several examples of polynomial minimization over the hypercube \( Q_n \) in the
next section. As before we are given a sum of squares density function \( h^*(x) \) on \( K = [0, 1]^n \). For \( i = 1, \ldots, n \), define the function \( f_{1 \ldots i} \in \mathbb{R}[x_1, \ldots, x_i] \) by

\[
f_{1 \ldots i}(x_1, \ldots, x_i) = \int_0^1 \cdots \int_0^1 h^*(x_1, \ldots, x_n) \, dx_{i+1} \cdots dx_n.
\]

Then the cumulative marginal distribution function \( F_1(\cdot) \) is given by

\[
F_1(x_1) = \int_0^{x_1} f_1(y) \, dy
\]

and, for \( i = 2, \ldots, n \), the cumulative conditional distribution function \( F_i(\cdot \mid x_1 \ldots x_{i-1}) \) is given by

\[
F_i(x_i \mid x_1 \ldots x_{i-1}) = \frac{\int_0^{x_i} f_{1 \ldots i}(x_1, \ldots, x_{i-1}, y) \, dy}{f_{1 \ldots (i-1)}(x_1, \ldots, x_{i-1})}.
\]

We now observe that if we generate sufficiently many samples from the distribution \( T_K \) then, with high probability, one of these samples is a point \( x \in K \) satisfying (roughly) \( f(x) \leq f^{(r)}_K \).

**Theorem 5** Let \( X \sim T_K \). For any \( \epsilon > 0 \),

\[
\Pr \left[ f(X) > f^{(r)}_K + \epsilon \left( f^{(r)}_K - f_{\text{min}, K} \right) \right] < \frac{1}{1 + \epsilon}.
\]

**Proof** Let \( X \sim T_K \) so that \( \mathbb{E}[f(X)] = f^{(r)}_K \). Define the nonnegative random variable

\[
Y := f(X) - f_{\text{min}, K}.
\]

Then, one has \( \mathbb{E}[Y] = f^{(r)}_K - f_{\text{min}, K} \). Given \( \epsilon > 0 \), the Markov Inequality (see e.g., [22 Theorem 3.2]) implies

\[
\Pr [Y \geq (1 + \epsilon)\mathbb{E}[Y]] \leq \frac{1}{1 + \epsilon}.
\]

This completes the proof. \( \Box \)

For given \( \epsilon > 0 \), if one samples \( N \) times independently from \( T_K \), one therefore obtains an \( x \in K \) such that

\[
f(x) \leq f^{(r)}_K + \epsilon \left( f^{(r)}_K - f_{\text{min}, K} \right)
\]

with probability at least \( 1 - \left( \frac{1}{1 + \epsilon} \right)^N \). For example, if \( N \geq 1 + \frac{1}{\epsilon} \) then this probability is at least \( 1 - 1/e \).
Table 1 Test functions

| Name                        | Formula                                                                 | Minimum \((f_{\min,K})\)                      | Search domain \((K)\)   |
|-----------------------------|-------------------------------------------------------------------------|-----------------------------------------------|------------------------|
| Booth Function              | \(f = (x_1+2x_2-7)^2 + (2x_1 + x_2 - 5)^2\)                            | \(f(1, 3) = 0\)                              | \([-10, 10]^2\)         |
| Matyas Function             | \(f' = 0.26(x_1^2 + x_2^2) - 0.48x_1x_2\)                             | \(f(0, 0) = 0\)                              | \([-10, 10]^2\)         |
| Three–Hump Camel Function   | \(f = 2x_1^2 - 1.05x_1^4 + \frac{x_1^6}{4} + x_1x_2 + x_2^2\)         | \(f(0, 0) = 0\)                              | \([-5, 5]^2\)           |
| Motzkin Polynomial          | \(f = x_1^2x_2^2 + x_2^2x_3 - 3x_1x_2x_3 + 1\)                       | \(f(\pm 1, \pm 1) = 0\)                      | \([-2, 2]^n\)           |
| Styblinski–Tang Function (w/ \(n = 2, 3, 4\)) | \(f = \sum_{i=1}^{n} x_i^2 - 8x_i^2 + \frac{2}{3}x_i\) | \(f(-2.093534, \ldots, -2.093534) = -39.16599n\) | \([-5, 5]^n\)           |
| Rosenbrock Function (w/ \(n = 2, 3, 4\)) | \(f = \sum_{i=1}^{n} 100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2\) | \(f(1, \ldots, 1) = 0\)                      | \([-2.048, 2.048]^n\)   |

Table 2 \(f^{(r)}_K\) for Booth, Matyas, Three–Hump Camel and Motzkin Functions

| \(r\) | Booth Function | Matyas Function | Three–Hump Camel Function | Motzkin Polynomial |
|-------|----------------|----------------|---------------------------|-------------------|
|       | Value (sec.)   | Value (sec.)   | Value (sec.)              | Value (sec.)      |
| 1     | 244.680        | 8.26667        | 265.774                   | 4.2               |
| 2     | 162.486        | 5.32223        | 29.0005                   | 0.38              |
| 3     | 118.383        | 4.28172        | 29.0005                   | 0.31              |
| 4     | 97.6473        | 3.894127       | 9.58064                   | 0.39              |
| 5     | 69.8174        | 3.68942        | 9.58064                   | 0.55              |
| ...   |                |                |                           |                   |
| 10    | 28.7248        | 1.47840        | 1.71275                   | 0.84              |
| 11    | 25.6050        | 1.37644        | 1.71275                   | 0.84              |
| 12    | 21.1869        | 1.11785        | 1.27749                   | 1.11              |

Table 3 \(f^{(r)}_K\) for Styblinski–Tang and Rosenbrock Functions (w/ \(n = 2, 3\))

| \(r\) | Sty.-Tang (\(n = 2\)) | Rosenb. (\(n = 2\)) | Sty.-Tang (\(n = 3\)) | Rosenb. (\(n = 3\)) |
|-------|------------------------|----------------------|------------------------|----------------------|
|       | Value (sec.)           | Value (sec.)         | Value (sec.)           | Value (sec.)         |
| 1     | -12.9249               | 214.648              | -18.8832               | 629.086              |
| 2     | -25.7727               | 152.310              | -36.0339               | 394.187              |
| 3     | -34.4030               | 104.859              | -44.9525               | 295.811              |
| 4     | -41.4436               | 75.6010              | -54.4244               | 206.903              |
| 5     | -45.1032               | 51.5037              | -60.5823               | 168.135              |
| 6     | -50.0509               | 41.7878              | -67.6027               | 121.558              |
| 7     | -56.3050               | 30.1392              | -74.5791               | 101.953              |
| 8     | -58.6004               | 25.8329              | -79.1261               | 77.479              |
| 9     | -60.7968               | 19.4972              | -82.9681               | 66.606              |
| 10    | -64.0147               | 17.3999              | -87.6127               | 53.036              |
| 11    | -65.7111               | 13.6289              | -91.0233               | 46.587              |
| 12    | -66.5532               | 12.5024              | -93.2038               | 38.428              |
### Table 4 $f_{K}^{(r)}$ for Styblinski–Tang and Rosenbrock Functions (with $n = 4$)

| $r$ | Sty.–Tang ($n = 4$) | Rosenb. ($n = 4$) |
|-----|---------------------|--------------------|
|     | Value               | Time (sec.)        | Value               | Time (sec.)        |
| 1   | $-24.8541$          | 0.25               | $1048.19$           | 0.34               |
| 2   | $-45.5192$          | 0.34               | $690.332$           | 0.42               |
| 3   | $-55.0577$          | 0.61               | $536.367$           | 0.48               |
| 4   | $-66.8202$          | 0.78               | $382.729$           | 0.72               |
| 5   | $-74.7215$          | 1.37               | $314.758$           | 1.39               |
| 6   | $-82.8699$          | 3.09               | $256.709$           | 3.09               |
| 7   | $-90.8863$          | 9.98               | $202.674$           | 6.61               |
| 8   | $-102.387$          | 28.64              | $156.205$           | 19.62              |
| 9   | $-102.387$          | 83.01              | $137.015$           | 60.59              |

#### 4 Numerical examples

In this section, we consider several well-known polynomial test functions from global optimization that are listed in Table 1.

For these functions, we calculate $f_{K}^{(r)}$ by solving the SDP (3) for increasing $r$.

We performed the computation on a PC with AMD Phenom(tm) 9600B Quad-Core CPU (2.30 GHz) and with 4 GB RAM. Moreover, we use CVX [12,13] in MATLAB, selecting SDPT3 [27,28] as the SDP solver.

We record the values $f_{K}^{(r)}$ as well as the CPU times (needed to solve the SDP) in Tables 2, 3 and 4.

Furthermore, for each order $r$, we use the method described in Section 3 to generate samples that are feasible solutions of (2), for the bivariate Rosenbrock and the Three–Hump Camel function in Table 1. For each order, the sample sizes 20 and 1000 are used. We also generate samples uniformly from the feasible set, for comparison. We give the results in Tables 5 and 6, where we record the mean, variance and the minimum value of these samples together with $f_{K}^{(r)}$ (which equals the sample mean by (25)).

Note that the average of the sample function values approximate $f_{K}^{(r)}$ reasonably well for sample size 1000, but poorly for sample size 20. Moreover, the average sample function value for uniform sampling from $K$ is much higher than $f_{K}^{(r)}$. Also, the minimum function value for sampling from $\mathcal{F}_K$ is significantly lower than the minimum function value obtained by uniform sampling for most values of $r$. In terms of generating “good” feasible solutions, sampling from $\mathcal{F}_K$ therefore outperforms uniform sampling from $K$ for these examples, as one would expect.

#### 5 Concluding remarks

We conclude with some additional remarks on Assumption 1, and some discussion on the computation perspectives of the approach studied here for global optimization.
Table 5 Sampling results for the Rosenbrock Function ($n = 2$)

| r  | $f_K^{(r)}$ | Mean     | Variance | Minimum | Sample Size |
|----|-------------|----------|----------|---------|-------------|
| 1  | 214.648     | 121.125  | 14065.5  | 0.00451826 | 20          |
|    |             | 209.9    | 80699.0  | 0.0008754 | 1000        |
| 2  | 152.310     | 341.496  | 75427.9  | 4.942956   | 20          |
|    |             | 149.6    | 04355.0  | 0.028854   | 1000        |
| 3  | 104.889     | 146.018  | 64611.2  | 0.0113339  | 20          |
|    |             | 110.1    | 26022.0  | 0.0665     | 1000        |
| 4  | 75.6010     | 62.4961  | 5833.21  | 0.0542813  | 20          |
|    |             | 75.05    | 3777.0   | 0.007285   | 1000        |
| 5  | 51.5037     | 68.4032  | 4397.0   | 0.668679   | 20          |
|    |             | 50.64    | 6285.0   | 0.01382    | 1000        |
| 6  | 41.7878     | 35.4183  | 2936.24  | 1.1054     | 20          |
|    |             | 37.04    | 3097.0   | 0.06188    | 1000        |
| 7  | 30.1392     | 29.8045  | 1022.2   | 1.95813    | 20          |
|    |             | 27.11    | 1332.0   | 0.02044    | 1000        |
| 8  | 25.8329     | 19.5392  | 301.334  | 0.505628   | 20          |
|    |             | 34.32    | 4106.0   | 0.074      | 1000        |
| 9  | 19.4972     | 20.8082  | 328.475  | 0.564992   | 20          |
|    |             | 18.65    | 593.6    | 0.07951    | 1000        |
| 10 | 17.3999     | 9.37959  | 146.496  | 0.562473   | 20          |
|    |             | 15.33    | 685.7    | 0.1448     | 1000        |
| 11 | 13.6289     | 8.74923  | 52.1436  | 0.75774    | 20          |
|    |             | 15.1     | 7495.0   | 0.1719     | 1000        |
| 12 | 12.5024     | 5.43151  | 66.901   | 0.438172   | 20          |
|    |             | 12.7     | 764.7    | 0.0945     | 1000        |
| Uniform Sample | 488.722 | 433549.0 | 9.0764 | 20        |
|    |             | 465.729  | 361150.0 | 0.0771463 | 1000        |

5.1 Revisiting Assumption 1

In this section we consider in more detail our Assumption 1. First we recall another condition, known as the interior cone condition, which is classically used in approximation theory (see, e.g., Wendland [29]).

**Definition 1** [29, Definition 3.1] A set $K \subseteq \mathbb{R}^n$ is said to satisfy an interior cone condition if there exist an angle $\theta \in (0, \pi/2)$ and a radius $\rho > 0$ such that, for every $x \in K$, a unit vector $\xi(x)$ exists such that the set

$$
C(x, \xi(x), \theta, \rho) := \{x + \lambda y : y \in \mathbb{R}^n, \|y\| = 1, y^T \xi(x) \geq \cos \theta, \lambda \in [0, \rho]\}
$$

(26)
is contained in $K$.

In fact, one can show that any set satisfying an interior cone condition also satisfies Assumption 1.

**Lemma 6** If a set $K \subseteq \mathbb{R}^n$ satisfies the interior cone condition (26) then $K$ also satisfies Assumption 1, where we set

$$
\eta_K = \left[\frac{\sin \theta}{1 + \sin \theta}\right]^n \text{ and } \epsilon_K = \rho.
$$
Table 6 Sampling results for the Three–Hump Camel Function

| r     | Mean  | Variance | Minimum | Sample Size |
|-------|-------|----------|---------|-------------|
| 1     | 216.773 | 177142.0 | 0.106854 | 20          |
|       | 281.239 | 193466.9 | 0.117069 | 1000        |
| 2     | 28.6344  | 2964.85  | 1.17187 | 20          |
|       | 27.712   | 6712.8   | 0.014255 | 1000        |
| 3     | 14.9951  | 523.904  | 0.452655 | 20          |
|       | 32.363   | 16681.0  | 0.0088426 | 1000       |
| 4     | 2.99756  | 14.1201  | 0.175016 | 20          |
|       | 10.364   | 1944.0   | 0.010013 | 1000        |
| 5     | 4.41907  | 14.358   | 0.419394 | 20          |
|       | 9.1658   | 643.88   | 0.0015924 | 1000       |
| 6     | 7.98481  | 245.089  | 0.126147 | 20          |
|       | 4.5191   | 493.12   | 0.0035581 | 1000       |
| 7     | 3.90711  | 90193.4  | 0.2060341 | 20          |
|       | 3.7911   | 94.587   | 0.0076111 | 1000        |
| 8     | 2.18925  | 3.87943  | 0.0310113 | 20          |
|       | 2.2302   | 8.3767   | 0.0028817 | 1000        |
| 9     | 1.38102  | 2.27333  | 0.138641 | 20          |
|       | 3.2217   | 812.18   | 0.00014804 | 1000      |
| 10    | 1.03179  | 0.992636 | 0.0645815 | 20          |
|       | 1.5069   | 3.9581   | 0.0014225 | 1000        |
| 11    | 1.30757  | 1.90985  | 0.0320489 | 20          |
|       | 1.6379   | 7.2518   | 0.0021144 | 1000        |
| 12    | 0.84194  | 0.91454  | 0.0369565 | 20          |
|       | 1.2105   | 2.5      | 0.0005154 | 1000        |
| Uniform Sample | 304.632  | 163021.0 | 1.6885 | 20          |
|        | 243.216  | 183724.0 | 0.00975034 | 1000      |

Proof Assume that \( K \) satisfies the interior cone condition \( (26) \). Then, using \( [29] \) Lemma 3.7, we know that, for every \( x \in K \) and \( h \leq \rho/(1 + \sin \theta) \), the closed ball \( B_{h\sin \theta}(x + h \xi(x)) \) is contained in \( C(x, \xi(x), \theta, \rho) \) and thus in \( K \). Then, for any \( x_n \in K \) and \( \epsilon \in (0, \rho) \), after setting \( h = \epsilon/(1 + \sin \theta) \), one can obtain

\[
\frac{\text{vol}(B_{\epsilon}(x_0) \cap K)}{\text{vol}B_{\epsilon}(x_0)} \geq \frac{\text{vol}C(x_0, \xi(x_0), \theta, \epsilon)}{\text{vol}B_{\epsilon}(x_0)} \geq \frac{\text{vol}B_{h\sin \theta}(x_0 + h \xi(x_0))}{\text{vol}B_{\epsilon}(x_0)} = \left[ \frac{\sin \theta}{1 + \sin \theta} \right]^n.
\]

Thus, Assumption 1 holds after setting \( \gamma_K = \left[ \frac{\sin \theta}{1 + \sin \theta} \right]^n \) and \( \epsilon_K = \rho \).

For instance, every Euclidean ball with radius \( \epsilon > 0 \) satisfies an interior cone condition, with radius \( \epsilon \) and angle \( \theta = \pi/3 \), see e.g., \( [29] \) Lemma 3.10. Moreover we now show that any full-dimensional polytope satisfies an interior cone condition.

**Theorem 6** Any full-dimensional polytope satisfies an interior cone condition.

Proof Let \( K \subset \mathbb{R}^n \) be a full-dimensional polytope with set of vertices \( \{u_1, \ldots, u_N\} \). Since \( K \) is full-dimensional then, for any vertex \( u_i \ (i \in [N]) \), there exist a unit vector \( \xi_i \), an angle \( \theta_i \), and a radius \( \rho_i \) such that \( C(u_i, \xi_i, \theta_i, \rho_i) \subset K \). Set \( \theta := \min_{i \in [N]} \theta_i \) and \( \rho := \min_{i \in [N]} \rho_i \). Then, for any vertex \( u_i \ (i \in [N]) \), one has \( C(u_i, \xi_i, \theta, \rho) \subset K \).

We now claim that, for any \( x \in K \), a unit vector \( \xi(x) \) exists such that \( C(x, \xi(x), \theta, \frac{\rho}{\sqrt{n+1}}) \subset K \).
We may assume w.l.o.g. that \( x = \sum_{i=1}^{n+1} \alpha_i u_i \) with \( \alpha_i \geq 0 \) for any \( i \in [n+1] \) and \( \sum_{i=1}^{n+1} \alpha_i = 1 \). One can easily see that there exists \( j \in [n+1] \) such that \( \alpha_j \geq \frac{1}{n+1} \), and we can assume w.l.o.g. that \( j = 1 \), that is, \( \alpha_1 \geq \frac{1}{n+1} \).

From the fact that \( C(u_1, \xi_1, \theta, \rho) \subseteq K \), we can obtain that, for any unit vector \( y \in \mathbb{R}^n \) with \( y^T \xi_1 \geq \cos \theta \) and for any \( 0 \leq r \leq \rho \), then \( u_1 + ry \in K \) holds.

We now consider \( x + \lambda y \), where \( 0 \leq \lambda \leq \frac{\rho}{n+1} \). We have

\[
x + \lambda y = \alpha_1 u_1 + \lambda y + \sum_{i=2}^{n+1} \alpha_i u_i = \alpha_1 \left( u_1 + \frac{\lambda}{\alpha_1} y \right) + \sum_{i=2}^{n+1} \alpha_i u_i.
\]

As \( \alpha_1 \geq \frac{1}{n+1} \) and \( 0 \leq \lambda \leq \frac{\rho}{n+1} \), we deduce that \( 0 \leq \frac{\lambda}{\alpha_1} \leq \rho \) and thus \( u_1 + \frac{\lambda}{\alpha_1} y \in K \).

Hence, \( x + \lambda y \in K \) and thus \( C(x, \xi_1, \theta, \frac{\rho}{n+1}) \subseteq K \). \( \square \)

Combining with Lemma \( \text{[6]} \) we get the following corollary.

**Corollary 1** Full-dimensional polytopes and Euclidean balls satisfy Assumption \( \text{[1]} \).

For example, as the hypercube \( Q_n \) is a full-dimensional polytope, it satisfies an interior cone condition and thus Assumption \( \text{[1]} \). This can also be seen directly. Set

\[
\theta = \arcsin \frac{1}{\sqrt{n}}, \quad \rho = 1/2, \quad \text{and} \quad \xi(x) = -\frac{s(x - \frac{1}{2} e)}{\|x - \frac{1}{2} e\|},
\]

where \( s(x) \) denotes the sign vector of \( x \) for any \( x \in \mathbb{R}^n \). Then, one can check that

\[
C(x, \xi(x), \theta, \rho) \subseteq Q_n \quad \text{for any} \quad x \in Q_n.
\]

Furthermore, one can also easily check that \( Q_n \) satisfies Assumption \( \text{[1]} \) with the constants

\[
\eta_{Q_n} = \frac{1}{2^n} \quad \text{and} \quad \epsilon_{Q_n} = \frac{1}{2}.
\]

### 5.2 Computational perspectives for global optimization

Recall that the computation of the upper bound \( f^{(r)} \) by solving the semidefinite programs \( \text{[9]} \) involve matrix variables of order \( \binom{n+2r}{2r} \). Thus one is limited to relatively small values of \( n \) and \( r \), when using interior point SDP solvers.

Having said that, the sampling approach of Section \( \text{[3]} \) often provides good feasible solutions for the examples in Section \( \text{[4]} \) even for small values of \( r \). One may therefore explore using the sampling technique (for small \( r \)) as a way of generating starting points for multi-start global optimization algorithms.

Another possibility to enhance computation would be to investigate more general sufficient conditions for nonnegativity of \( h \) on \( K \), than the sum-of-squares condition studied here. This may result in a faster rate of convergence than for \( f^{(r)} \).
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