Phase Transition of Laminated Models at Any Temperature

Eugene Pechersky,* Elena Petrova† and Sergey Pirogov‡
Institute for Information Transmission Problems

Abstract

The standard Pirogov – Sinai theory is generalized to the class of models with two modes of interaction: longitudinal and transversal. Under rather general assumptions about the longitudinal interaction and for one specific form of the transversal interaction it is proved that such system has a variety of phase transitions at any temperature: the parameter which plays the role of inverse temperature is the strength of the transversal interaction. The concrete examples of such systems are (1 + 1)-dimensional models.

1 Introduction

This work is devoted to an extension of the results of the primary paper [7] concerning phase transitions in lattice models which gave rise to a great amount of papers on the subject that nowadays is being referred to as the Pirogov–Sinai theory (PS-theory). In [7] the authors considered a class of Gibbs lattice models with a finite spin space. The main condition for phase transitions to take place is the existence of a finite number of periodic ground states satisfying the Peierls condition. The authors constructed the complete phase diagram for low temperatures in the space of parameters of the model. The Peierls condition guarantees the existence of energetic barriers between the ground states, and every temperature phase falls in a domain of influence of one of the ground states.

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In this paper we address the problem whether there can be several phases at any fixed temperature. Our idea is to stack infinitely many identical $d$-dimensional models, each with the interaction that satisfies the Peierls condition (we call these models the horizontal layers or the horizontal models) making them interact with each other with a Potts energy (we call this a vertical interaction or a vertical model). We prove that such a model (which we call a laminated model) has a phase transition for any given positive temperature if the parameter of the Potts interaction $\lambda$ is sufficiently large (depending on the temperature).

In Section 2 we introduce the model. Section 3 contains the formulation of the main result. Section 4 is devoted to the proof. Section 5 studies the $(1 + 1)$-dimensional case. The phase diagram is obtained for any $(1 + 1)$-dimensional laminated model with finite number of periodic ground states.

2 Definitions

Let $\mathbb{Z}^{d+1}$ be a $(d + 1)$-dimensional integer lattice. We define the distance on the lattice as follows: for $i = (i_1, \ldots, i_{d+1}), j = (j_1, \ldots, j_{d+1}) \in \mathbb{Z}^{d+1}$

$$d(i, j) = \max_{1 \leq k \leq d+1} |i_k - j_k|. \quad (1)$$

Two sets $K, L \subseteq \mathbb{Z}^{d+1}$ are called distant if $d(K, L) > 1$. For any finite $B \subset \mathbb{Z}^{d+1}$ denote by $|B|$ the number of sites $i \in B$. We say that a set $B \subset \mathbb{Z}^{d+1}$ is connected if for any $i, j \in B$ there exists a sequence of sites $i_1, \ldots, i_n \in B$ such that $d(i, i_1) = 1, d(i_1, i_2) = 1, \ldots, d(i_n, j) = 1$.

Let $S$ be a finite set (the spin space) and $S = S^{\mathbb{Z}^{d+1}}$ be the set of all configurations, i.e. all maps $s : \mathbb{Z}^{d+1} \rightarrow S$. Denote by $pr(s, V)$ a restriction of a configuration $s$ to the set $V \subset \mathbb{Z}^{d+1}$.

For any $t \in \mathbb{Z}$, the set $\mathbb{Z}^d \times \{t\} \subset \mathbb{Z}^{d+1}$ is called a horizontal layer and is denoted by $\mathbb{Z}^d_t$.

Definition 1. We say that two configurations $s', s''$ are equal almost everywhere if the set \{\(i \in \mathbb{Z}^{d+1} : s'(i) \neq s''(i)\}\) is finite.

The interaction in the laminated model is a composition of a ‘horizontal’ $d$-dimensional interaction which involves the sites within a horizontal layer (with the same Hamiltonian $H_g$ for each layer) and a one-dimensional ‘vertical’ interaction which acts along the $d + 1$st direction (with the same Hamiltonian $H_v$ for any ‘vertical axis’ \{(i, t), i \in \mathbb{Z}^d \text{ fixed}, t \in \mathbb{Z}\}). Now we are going to define the horizontal and vertical Hamiltonians, $H_g$ and $H_v$.

2.1 Horizontal model

The horizontal model is defined on a $d$-dimensional lattice $\mathbb{Z}^d$. Throughout the paper we will use the same notation $s$ for a configuration on $\mathbb{Z}^d$ if it does not cause misunderstanding.
The Hamiltonian of the horizontal model, $H_g$, is defined by a collection of finite range potential functions $\Phi^g_i(s) = \{\Phi^g_i(s(j), |j - i| \leq R)\}$. The function $\Phi^g_i$ determines the energy of interaction between the site $i \in \mathbb{Z}^d$ with all the other sites. We suppose that the interaction is of a finite range, and denote by $R$ the radius of interaction. We consider the case when $\Phi^g_i$ are periodical functions, i.e. invariant with respect to the action of some subgroup $\hat{\mathbb{Z}} \subset \mathbb{Z}^d$ of finite index. Thus the energy $H_g(s)$ of a configuration $s$ is formally defined by

$$H_g(s) = \sum_{i \in \mathbb{Z}^d} \Phi^g_i(s).$$

(2)

However, the difference

$$H_g(s', s'') = H_g(s') - H_g(s'') = \sum_{i \in \mathbb{Z}^d} \Phi^g_i(s') - \Phi^g_i(s''),$$

which is called a relative Hamiltonian is well defined if the configurations $s', s''$ are equal almost everywhere.

**Definition 2.** A configuration $s \in S^{\mathbb{Z}^d}$ is called periodic if it is invariant with respect to a subgroup $\hat{\mathbb{Z}} \subset \mathbb{Z}^d$ of a finite index.

**Definition 3.** A periodical configuration $a$ is called a ground state for Hamiltonian $H_g$ if for any configuration $s$ that is almost everywhere equal to $a$ it holds

$$H_g(s, a) \geq 0.$$  

(3)

**Definition 4.** Assume that for the Hamiltonian $H_g$ there exist a finite number of periodical ground states $S(H) = \{s_1, \ldots, s_r\}$. Choose a subgroup $\hat{\mathbb{Z}} \subset \mathbb{Z}^d$ of a finite index such that all the ground states are invariant with respect to $\hat{\mathbb{Z}}$. Let $N = (\mathbb{Z}^d : \hat{\mathbb{Z}})$. We denote by $W_N(i)$ a cube of a size $2N$ centered at $i$, $W_N(i) = \{j: d(i, j) \leq N\}$.

A cube $W_N(i)$ is called frustrated for a configuration $s$ if $\text{pr}(s, W_N(i)) \neq \text{pr}(s_q, W_N(i))$ for all $q = 1, \ldots, r$. A union of all frustrated cubes $B_g(s)$ for $s$ is called the horizontal boundary of configuration $s$.

**Definition 5.** We say that a horizontal Hamiltonian $H_g$ that has a finite number of the ground states, $s_1, \ldots, s_r$, satisfies the Peierls condition if there exists a positive constant $\theta$ such that for any $q = 1, \ldots, r$, for any configuration $s$ that is almost everywhere equal to $s_q$ it holds

$$H_g(s, s_q) \geq \theta |B_g(s)|.$$  

(4)
2.2 Vertical model

The vertical model is defined on a one-dimensional lattice \( \mathbb{Z} \) with the same spin space \( S \). The Hamiltonian of the model is defined by the following potential function

\[
\Phi^v(s_1, s_2) = \lambda (1 - \delta(s_1, s_2)),
\]

where \( s_1, s_2 \in S \), \( \lambda > 0 \) and \( \delta(s_1, s_2) \) is the Kronecker symbol. The formal vertical Hamiltonian is defined by a nearest neighbour interaction:

\[
H_v(s) = \frac{1}{2} \sum_{t \in \mathbb{Z}} [\Phi^v(s(t), s(t + 1)) + \Phi^v(s(t - 1), s(t))].
\]

2.3 Laminated model

The model of our studies is defined on \( \mathbb{Z}^{d+1} = \mathbb{Z}^d \times \mathbb{Z} \), with the configuration space \( S = \mathbb{Z}^{d+1} \) and with anisotropic potential

\[
\Phi_{i,t}(s) = \Phi^g_{i}(\text{pr}(s, \mathbb{Z}^d)) + \frac{1}{2} [\Phi^v(s(i, t), s(i, t + 1)) + \Phi^v(s(i, t), s(i, t - 1))], \quad (i, t) \in \mathbb{Z}^{d+1}.
\]

Thus the formal Hamiltonian of the laminated model is

\[
H(s) = \sum_{(i,t) \in \mathbb{Z}^{d+1}} \Phi_{i,t}(s).
\]

The relative Hamiltonian is defined as

\[
H(s, s') = \sum_{(i,t) \in \mathbb{Z}^{d+1}} [\Phi_{i,t}(s) - \Phi_{i,t}(s')].
\]

The following obvious proposition describes the properties of the Hamiltonian of a laminated model.

**Proposition 1.** Let a horizontal Hamiltonian \( H_g \) have a finite number of periodical ground states, \( s_1, \ldots, s_r \) on \( \mathbb{Z}^d \). Then the Hamiltonian of a laminated model \( H \) has the same number of periodical ground states, \( \bar{s}_1, \ldots, \bar{s}_r \) on \( \mathbb{Z}^{d+1} \), and for any \( q = 1, \ldots, r \), for any \( t \in \mathbb{Z} \), \( \bar{s}_q(i,t) = s_q(i) \). If all the ground states of \( H_g \) are invariant with respect to a subgroup \( \hat{\mathbb{Z}} \), then all the ground states of \( H \) are invariant with respect to \( \hat{\mathbb{Z}} \times \mathbb{Z} \).

Further on we drop the bar in the notations of the ground states of the laminated model: \( s_1, \ldots, s_r \).

We recall the definition of a Gibbs state.
Definition 6. Given a finite volume $V \subset \mathbb{Z}^{d+1}$ and a configuration $\sigma \in \mathcal{S}$, consider a set of configurations $\mathcal{S}(\sigma, V) \subset \mathcal{S}$ consisting of all configurations $s' \in \mathcal{S}$ such that $s'(i) = \sigma(i)$ for all $i \notin V$. Given a Hamiltonian $H$, a probability distribution $P_{\sigma, V}$ on $\mathcal{S}(\sigma, V)$ is called a Gibbs distribution in volume $V$ with boundary conditions $\sigma$ if for any two configurations $s', s'' \in \mathcal{S}(\sigma, V)$

$$
\frac{P_{\sigma, V}(s')}{P_{\sigma, V}(s'')} = \exp(-\beta H(s', s'')),
$$

where $\beta > 0$ is the parameter of the model (inverse temperature).

Definition 7. A probability distribution $P$ on $\mathcal{S}$ is called a limiting Gibbs distribution for a Hamiltonian $H$ and parameter $\beta$ if for any finite $V \subset \mathbb{Z}^{d+1}$ its conditional probabilities

$$
P_V(s(i) \mid i \in V \mid s(i) = \sigma(i), i \notin V)
$$

are equal to $P_{\sigma, V}(s(i), i \in V)$ $P$-a.s.

A limiting Gibbs distribution is called a pure thermodynamical phase if it is periodic and is an extreme point of the convex set of all periodic limiting Gibbs distributions.

3 Main result

From now on we fix some horizontal Hamiltonian $H^0$ that satisfies the Peierls condition. Let $\Phi^0$ be the corresponding potential. Let us consider the family of horizontal Hamiltonians given by

$$
H_g = H^0 + \sum_{k=1}^{r-1} \mu_k H_k,
$$

where the Hamiltonians $H_1, \ldots, H_{r-1}$ are of a finite range with the same radius $R$ and $\mu_k \in \mathbb{R}$ are the parameters of the model. Throughout the paper we assume that $\mu_k$ are sufficiently small in absolute value. The exact condition will be given later (see (17) and (18)).

We assume that the family (11) is non-degenerate in the sense that the matrix

$$
\begin{pmatrix}
  e_1^1 & \cdots & e_1^r \\
  \vdots & \ddots & \vdots \\
  e_{r-1}^1 & \cdots & e_{r-1}^r \\
  1 & \cdots & 1
\end{pmatrix}
$$

of the specific energies $e_k^q = \lim_{V \to \infty} H_k,V(s_q)/|V|$ completed by the constant row is non-degenerate. Here $H_k,V(s)$ is the energy of configuration $s$ in a finite volume $V$ corresponding to Hamiltonian $H_k$,

$$
H_k,V(s) = \sum_{i \in V} \Phi_i^{(k)}(s).
$$
From now on we consider the laminated model with the Hamiltonian \( H(\mu) \) which is determined by (7), where
\[
\Phi^g = \Phi^0 + \sum_{k=1}^{r-1} \mu_k \Phi^{(k)}
\] (12)
and \( \Phi^v \) is given by (5). We will say that the Hamiltonian \( H(\mu) \) is generated by the horizontal Hamiltonian (11) and the vertical Hamiltonian (6).

Fixing \( \beta \) we will vary the vertical interaction \( \lambda \).

Our main result is the following theorem. Define by
\[
O_r = \{ a = (a_1, \ldots, a_r) : \min a_q = 0 \}
\]
the non-negative octant in \( \mathbb{R}^r \).

**Theorem 8.** Consider a family of Hamiltonians \( H(\mu) \) generated by a horizontal Hamiltonian (11) and the vertical Hamiltonian (6), where \( H^0 \) is a horizontal Hamiltonian (2) with \( r \) ground states, \( s_1, \ldots, s_r \), which satisfies the Peierls condition (4), \( H_1, \ldots, H_{r-1} \) is a collection of horizontal Hamiltonians such that the family (11) is non-degenerate. Then for any \( \beta > 0 \) there exists \( \lambda_0(\beta) > 0 \) such that for any \( \lambda > \lambda_0(\beta) \) there is a neighbourhood \( U \) of the origin in the space \( \mathbb{R}^{r-1} \) of parameters \( \mu = (\mu_1, \ldots, \mu_{r-1}) \) and a homeomorphism \( J(\beta, \lambda) : U \to A \) of \( U \) onto a neighbourhood \( A \) of the origin in non-negative octant \( O_r \subset \mathbb{R}^r \), such that for the Hamiltonian \( H(\mu) \) with \( \mu = (\mu_1, \ldots, \mu_{r-1}) \in U \), there exist different pure thermodynamical phases (for given \( \beta \)), each phase corresponds to that \( q \) for which \( a_q = 0 \), where \( a = (a_1, \ldots, a_r) = J(\beta, \lambda)\mu \).

### 4 Proof of Theorem 8

In [7], a result similar to Theorem 8 was proved for the case of large \( \beta \). Here we prove the statement for arbitrary \( \beta > 0 \) but \( \lambda \) sufficiently large depending on \( \beta \). To extend the contour method of [7] to the laminated model with arbitrary temperature we need auxiliary construction that we call a vertical aggregation of \( \mathbb{Z}^{d+1} \). For any given temperature we can choose the aggregation size \( l \) and the value of the vertical interaction \( \lambda \) in such a way that the resulting model is effectively low temperature.

#### 4.1 Contours

Recall that \( N \) is the index of the maximal subgroup \( \hat{\mathbb{Z}} \subset \mathbb{Z}^d \) such that the ground states \( s_1, \ldots, s_r \) of \( H_g \) are invariant with respect to it. Choose a real \( \bar{R} \) such that

1) \( \bar{R} > R \), where \( R \) is the interaction radius of \( H^0, H_1, \ldots, H_{r-1} \);
2) \( R > N \).

Let an integer \( l = l(\beta) \) be given, which we call an aggregation size and which we will choose later. We divide \( \mathbb{Z}^{d+1} \) into “columns” \( C_{i,k} = \{(i,t) : kl \leq t < (k+1)l\} \).

**Definition 9.** We say that a column \( C_{i,k} \) is variable with respect to configuration \( s \) if \( s \) is not constant on \( C_{i,k} \). Otherwise \( C_{i,k} \) is called invariable.

Define \( U_R(i,k) = \bigcup_{j:|j-i| \leq R} C_{j,k} \). (13)

**Definition 10.**

a) A column \( C_{i,k} \) is called \( q \)-regular with respect to configuration \( s \) if

\[ \text{pr}(s, U_R(i,k)) = \text{pr}(s, U_R(i,k)) \]

for all \( q = 1, \ldots, r \).

b) A column \( C_{i,k} \) is called frustrated with respect to configuration \( s \) if there exists a variable column \( C_{j,k} \subset U_R(i,k) \).

c) A column \( C_{i,k} \) is called defective with respect to configuration \( s \) if all columns \( C_{j,k} \subset U_R(i,k) \) are invariable but

\[ \text{pr}(s, U_R(i,k)) \neq \text{pr}(s, U_R(i,k)) \]

for all \( q = 1, \ldots, r \).

**Remark 11.** If \( C_{i,k} \) is \( q \)-regular and \( C_{i',k} \) is \( q' \)-regular with respect to the same configuration \( s \) and if \( |i - i'| = 1 \), then \( q = q' \). However, it can happen that \( C_{i,k} \) is \( q \)-regular and \( C_{i,k+1} \) is \( q' \)-regular with \( q \neq q' \).

**Definition 12.** A pair of \( q \)-regular and \( q' \)-regular columns, \( C_{i,k} \) and \( C_{i,k+1} \) respectively, is called a faced pair if \( q \neq q' \). Each of these columns is called a faced column as well.

**Definition 13.** The boundary \( B(s) \) of a configuration \( s \) is a union of sites \( (i,t) \) such that the column \( C_{i,k} \supset (i,t) \) is either frustrated or defective or faced.

**Remark 14.** Note that all sites of \( C_{i,k} \) belong to \( B(s) \) as soon as one of them belongs to \( B(s) \). A site \( (i,t) \in C_{i,k} \) is called frustrated, defective or faced if so is the column \( C_{i,k} \). Thus all the sites of \( C_{i,k} \) are of the same type.

**Notation 15.** As we have just seen, the boundary \( B(s) \) of a configuration \( s \) is a union of columns. Let \( N_d(s) \) be the number of defective columns, \( N_c(s) \) be the number of frustrated columns and \( N_b(s) \) be the number of faced columns in \( B(s) \).

Next we introduce a notion of a contour.
Definition 16. A contour generated by configuration \( s \) is a pair \( \Gamma = \Gamma(s) = (M, \text{pr}(s, M)) \), where \( M \) is a connected component of the boundary \( B(s) \) and \( \text{pr}(s, M) \) is the restriction of configuration \( s \) on \( M \). The set \( M \) is called the support of the contour, \( M = \text{supp} \Gamma \). Denote by 
\[
\|M\| = \frac{|M|}{l}
\]
the number of columns in \( M \).

Consider a contour \( \Gamma = (M, \text{pr}(s, M)) \). The set \( M^c = \mathbb{Z}^{d+1} \setminus M \) parts into a number of maximal connected components, \( A_\alpha \). Every set \( A_\alpha \) is a union of columns as well. Define 
\[
\partial A_\alpha = \{(i, t) \in A_\alpha : \text{d}((i, t), M) = 1\},
\]
where \( A_\alpha \) are the parts into a number of maximal connected components, \( A_\alpha \). Every set \( A_\alpha \) is a union of columns as well. Define 
\[
\partial A_\alpha = \{(i, t) \in A_\alpha : \text{d}((i, t), M) = 1\}.
\]

Further we consider only such configurations that each of the contours is of a finite support. Let \( \Gamma = (M, \text{pr}(s, M)) \) be a contour with \( |M| < \infty \). Then, since \( d + 1 \geq 2 \), all maximal connected components \( A_\alpha \) but one are finite. Any finite component \( A_\alpha \) is said to belong to the interior of contour \( \Gamma \). A unique infinite component is called the exterior of contour \( \Gamma \) and is denoted by \( \text{Ext} \Gamma \).

For any \( (i, t) \in \partial A_\alpha \), there exists \( q = q(i, t) \) such that the column \( C_{i, k} \ni (i', t') \) for any \( (i', t') \in A_\alpha \). Denote the common value of \( q \) of all the sites of \( \partial A_\alpha \) by \( q(A_\alpha) \).

For any \( m \in \{1, \ldots, r\} \) (one of the ground states of \( H^0 \)) we define the \( m \)-interior of a contour \( \Gamma \) as the union of internal parts \( A_\alpha \) of \( M^c \) with \( q(A_\alpha) = m \):
\[
\text{Int}_m \Gamma = \bigcup_{\alpha : q(A_\alpha) = m, |A_\alpha| < \infty} A_\alpha.
\]

We define the interior of a contour \( \Gamma \) as the union of its \( m \)-interiors:
\[
\text{Int} \Gamma = \bigcup_m \text{Int}_m \Gamma.
\]

We denote by \( q = q(\text{Ext} \Gamma) \) the common value \( q(i, t) \) for \( (i, t) \in \text{Ext} \Gamma \). We say that the contour \( \Gamma \) is a contour with a boundary condition \( s_q \) and denote it by \( \Gamma^q \) if \( q(\text{Ext} \Gamma) = q \). We say that a contour \( \Gamma \) is an external contour of configuration \( s \) if \( \Gamma \) is not contained in the interior of any other contour of configuration \( s \).

Definition 17. Given a contour \( \Gamma^q = (M, \text{pr}(s, M)) \) define a configuration \( s_{\Gamma^q} \) to be equal to \( s \) on \( M \), \( s_q \) on \( \text{Ext} \Gamma \) and \( s_m \) on \( \text{Int}_m \Gamma \).

We use the following notation for the number of columns in the interior of a contour \( \Gamma \):
\[
\begin{align*}
V_m(\Gamma^q) &= \|\text{Int}_m \Gamma^q\|, \\
V(\Gamma^q) &= \|\text{Int} \Gamma^q\|.
\end{align*}
\]
Further we sometimes use the notion of a contour just as a pair \((M, s_M)\) where \(M\) is a finite connected subset of \(\mathbb{Z}^{d+1}\) and \(s_M\) is a configuration on \(M\), without fixing a configuration outside \(M\). Remark that configuration \(s_M\) determines the values of \(q(i, t)\) for \((i, t)\) such that \(d((i, t), M) = 1\). We write \(\Gamma^q = (M, s_M)\) to indicate that the configuration on the external boundary of \(\Gamma\) coincides with \(s_q\). For a contour \(\Gamma^q\) we denote by \(\mathcal{L}(\Gamma^q)\) the set of configurations \(s'\) that are equal to \(s_q\) almost everywhere and, moreover, \(\Gamma^q\) is their unique external contour.

For any finite (not necessarily simply connected) volume \(V \subset \mathbb{Z}^{d+1}\) we denote by \(\mathcal{R}_q(V)\) the set of all configurations \(s\) such that

1. \(s = s_q\) out of \(V\),
2. \(B(s)\) and \(\mathbb{Z}^{d+1} \setminus V\) are distant,
3. for any external contour \(\Gamma^q\) of \(s\) we have \(\text{Int} \Gamma^q \subset V\).

Let \(\Gamma^q = (M, s_M)\) be a contour. We denote by \(\Lambda(\Gamma^q)\) the set of all contours with the same support \(M\) and the same external condition \(q\).

### 4.2 Contour functionals

Following [7] we introduce a notion of a contour functional \(F_q\) as a real function on contours \(\Gamma^q\). A relative Hamiltonian \(H(s_{\Gamma^q}, s_q)\) [9] is an example of a contour functional. We denote it by \(H(\Gamma^q)\).

**Definition 18.** A contour functional \(F_q\) is called a \(\pi\tau\)-functional if there exists \(\tau > 0\) such that

\[
\sum_{\tilde{\Gamma}^q \in \Lambda(\Gamma^q)} \exp(-F_q(\tilde{\Gamma}^q)) \leq \exp(-\tau \|\text{supp} \Gamma^q\|). \tag{14}
\]

The definition analogous to (14) but without a summation was introduced in [4, 5, 6], the functionals being called \(\tau\)-functionals. We are going to extend the theory of contour models for \(\tau\)-functionals developed in these papers to contour models for \(\pi\tau\)-functionals.

**Definition 19.** A contour functional \(F_q\) is called uv-functional if there exist \(u, v > 0\) such that for any \(\Gamma^q\)

\[
F_q(\Gamma^q) \geq u\|\text{supp} \Gamma^q\| + vlN_c(\Gamma^q), \tag{15}
\]

where \(N_c(\Gamma^q) \equiv N_c(s_{\Gamma^q})\) is the number of frustrated columns of the configuration \(s_{\Gamma^q}\) (see Definitions [17, 18] and Notation [13], \(l\) is the aggregation size of the laminated model.

**Proposition 2.** For any \(\tau > 0\) there exist positive constants \(u\) and \(v\) (depending on \(\tau\)) such that if a contour functional \(F_q\) is uv-functional, then it is \(\pi\tau\)-functional.
Proof. The proof follows from a direct substitution.

Following [7] we use the representation

\[ H(\Gamma^q) = \Psi(\Gamma^q) + \sum_m (\tilde{h}_m - \tilde{h}_q) V_m(\Gamma^q), \]  

(16)

where \( \tilde{h}_q = l h_q, \) \( h_q = \sum e^q_k \mu_k. \) The functional \( \Psi \) represents the energy of the 'boundary' and the second term represents the 'volume' energies of the ground states. For the laminated model the representation (16) can be specified for horizontal and vertical parts of the model separately. Namely, the functional \( \Psi \) is a sum of 'horizontal' and 'vertical' components, that is \( \Psi = \Psi_g + \Psi_v. \) Let \( t \) be the value of the vertical coordinate and let \( H^t_g \) be the energy along the horizontal layer \( Z^d_t. \) Denote by \( \Gamma^q_t \) the intersection of contour \( \Gamma^q \) with the layer \( Z^d_t. \) Then

\[ H^t_g(\Gamma^q_t) = \Psi^t_g(\Gamma^q_t) + \sum_m (h_m - h_q) V^t_m(\Gamma^q_t), \]

where \( V^t_m \) is the volume of a part of \( \text{Int}_m(\Gamma^q) \) that lies in the layer \( Z^d_t. \) Denote by \( N_d(\Gamma^q, t) \) the number of defective columns of \( \Gamma^q \) intersecting the layer \( Z^d_t. \) We assume the constants \( \mu_i \) in (11) to be sufficiently small so that the inequality

\[ \Psi^t_g(\Gamma^q) \geq \rho N_d(\Gamma^q, t), \]

(17)

is satisfied for some \( \rho > 0, \) given the Peierls condition for the horizontal Hamiltonian \( H^0 \) (4) is fulfilled. Let us define \( U \) as a domain in the space of parameters \( \mu \) for which (17) is satisfied.

Since any column intersects \( l \) layers, summing over \( t \) we obtain

\[ \Psi_g(\Gamma^q) \geq \rho l N_d(\Gamma^q), \]  

(18)

where \( N_d(\Gamma^q) \equiv N_d(s_{\Gamma^q}) \) is the number of defective columns of the configuration \( s_{\Gamma^q} \) (see Definitions 10, 17 and Notation 15).

For the vertical interaction we have

\[ \Psi_v(\Gamma^q) \geq \lambda N_b(\Gamma^q) + \lambda N_c(\Gamma^q), \]  

(19)

where \( N_b(\Gamma^q) \) is the number of faced columns and \( N_c(\Gamma^q) \) is the number of frustrated columns of the configuration \( s_{\Gamma^q}. \) Hence for \( \Psi = \Psi_g + \Psi_v \) and for \( \beta > 0 \) the estimate

\[ \beta \Psi(\Gamma^q) \geq \beta \rho l N_d(\Gamma^q) + \beta \lambda (N_b(\Gamma^q) + N_c(\Gamma^q)) \]

(20)
holds. If
\[ \beta \rho l \geq u, \]
\[ \beta \lambda \geq u + v \cdot l, \]
then \( \beta \Psi \) is a \( uv \)-functional. Therefore, if \( \beta \) and \( \rho \) are fixed then we have to choose \( l \) sufficiently large so that (21) is satisfied and to take \( \lambda \) so large that (22) is satisfied. We can change slightly \( l \) and \( \lambda \) so that \( \beta \Psi \) becomes a \((u + 1)v\)-functional, that is,
\[ \beta \Psi(\Gamma^q) \geq (u + 1)\|\Gamma^q\| + vlN_c(\Gamma^q), \]
where
\[ \|\Gamma^q\| =: N_b(\Gamma^q) + N_c(\Gamma^q) + N_d(\Gamma^q) = \|\text{supp}\Gamma^q\|. \]

4.3 Partition functions

Our aim now is to compare partition functions of the laminated model with partition functions of contour models with specially adjusted contour functionals \( F_q \).

Let \( V \) be a finite volume. Following [7] we introduce the notion of a \((V, q)\)-partition function, which is the partition function of the laminated model in the volume \( V \) with boundary conditions \( s_q^q \):
\[ \Xi^q(V|\beta H) = \sum_{s \in \mathcal{R}_q(V)} \exp(-\beta H(s, s_q)). \]

For a given contour \( \Gamma^q \) let us define the contour partition function as follows:
\[ \Xi(\Gamma^q|\beta H) = \sum_{s \in \mathcal{L}(\Gamma^q)} \exp(-\beta H(s, s_q)). \]

A relation between these partition functions is established in the next

**Lemma 20.** For any finite volume \( V \subset \mathbb{Z}^{d+1} \)
\[ \Xi^q(V|\beta H) = \sum \prod_{i} \Xi(\Gamma_i^q|\beta H), \]
where the sum is taken over all collections \( \{\Gamma_1^q, \ldots, \Gamma_n^q\} \) of external contours (including an empty collection) with pairwise distant supports such that the following conditions are fulfilled:
\[ \text{supp } \Gamma_i^q \subset V, \]
\[ d(\text{supp } \Gamma_i^q, \mathbb{Z}^d \setminus V) > 1, \]
\[ \text{Int } \Gamma_i^q \subset V. \]
The product runs over all contours in the collection. For the empty collection of contours the weight 1 is assigned.

For any contour $\Gamma^q$, 
\[
\Xi(\Gamma^q|\beta H) = \exp(-\beta H(\Gamma^q)) \prod_m \Xi_m(\text{Int}_m \Gamma^q|\beta H). \tag{29}
\]

Remark 21. Assume that there is a collection of numbers $\Xi(\Gamma^q)$ indexed by contours $\Gamma^q$ and a collection of numbers $\Xi(V)$ defined for every finite volume $V \subset \mathbb{Z}^{d+1}$ which are related by (27), (29). Then $\Xi(\Gamma^q) = \Xi(\Gamma^q|\beta H)$. Thus equations (27) and (29) compose a chain of recurrent relations for $\Xi(\Gamma^q|\beta H)$.

4.4 Contour models

Following [4, 5, 6] we are going to construct abstract contour models which are defined as probability distributions on collections of contours. By a contour we mean, as above, a subset of $\mathbb{Z}^{d+1}$ with a configuration on it. However, the contours constituting a collection do not necessarily agree with each other in the sense that there can be no configuration on $\mathbb{Z}^{d+1}$ that generates this collection. Given a finite volume $V \subset \mathbb{Z}^{d+1}$, denote by $\mathcal{P}(V)$ the ensemble consisting of finite collections of (not necessarily external) contours $\{\Gamma^q_1, \ldots, \Gamma^q_n\}$, with mutually distant supports that satisfy condition (28). The empty collection also belongs to $\mathcal{P}(V)$. Suppose we have a contour functional $F_q$. The contour model with the functional $F_q$ is defined as the following probability distribution on the ensemble $\mathcal{P}(V)$
\[
P_V(\{\Gamma^q_1, \ldots, \Gamma^q_n\}|F_q) = \frac{\exp \left( - \sum_i F_q(\Gamma^q_i) \right)}{\Xi(V|F_q)}. \tag{30}
\]

Here $\Xi(V|F_q)$ is a normalizing factor (called the partition function of the contour model).

Definition 22. A collection of probability distributions $P_V$ defined by (30) for all finite volumes $V$ is called a contour model.

Given a contour $\Gamma^q$, consider a set $\mathcal{G}(\Gamma^q)$ of collections of contours $g = \{\Gamma^q, \Gamma^q_1, \ldots, \Gamma^q_n\} \in \mathcal{P}(V)$ such that the only external contour of each collection is $\Gamma^q$. We define the energy of a collection $g \in \mathcal{G}(\Gamma^q)$ as
\[
E(g) = \sum_i F_q(\Gamma^q_i) + F_q(\Gamma^q).
\]

Definition 23. For a contour $\Gamma^q$ we define a virtual partition function by the formula
\[
\Xi(\Gamma^q|F_q) = \sum_{g \in \mathcal{G}(\Gamma^q)} \exp(-E(g)). \tag{32}
\]
The relation between $\Xi(V|F_q)$ and $\Xi(\Gamma^q|F_q)$ is like that one between the $(V,q)$-partition function and contour partition function of the laminated model [27]. In its turn, $\Xi(\Gamma^q|F_q)$ can be expressed through $\Xi(V|F_q)$ as

$$
\Xi(\Gamma^q|F_q) = \exp(-F_q(\Gamma^q)) \prod_m \Xi(\text{Int}_m \Gamma^q|F_q). 
$$

(33)

In [4, 5, 6], the expansion of the contour model partition function (for $\tau$-functionals) onto a ‘volume’ and ‘boundary’ terms was given. There was also given an estimate of the boundary term under the condition that $\tau$ is sufficiently large. We generalize this estimate to the case of $\pi\tau$-functionals. Let $F_q$ be $\pi\tau$-functional such that it is invariant with respect to some subgroup $\tilde{\mathbb{Z}} \subset \mathbb{Z}^{d+1}$ of finite index. We can decompose the logarithm of the contour model partition function as

$$
\log \Xi(V|F_q) = s(F_q)\|V\| + \Delta(V|F_q),
$$

(34)

$$
|\Delta(V|F_q)| < \varepsilon(\tilde{\mathbb{Z}}; \tau)\|\partial V\|,
$$

(35)

where $\varepsilon(\tilde{\mathbb{Z}}; \tau) > 0$ is such that

$$
\varepsilon(\tilde{\mathbb{Z}}; \tau) \to 0 \text{ as } \tau \to \infty.
$$

From the definition of the contour model it follows that the probability $P_V(\Gamma^q|F_q)$ for the contour $\Gamma^q$ to belong to a collection $g \in \mathcal{P}(V)$ is equal to

$$
\left| \frac{\partial}{\partial F_q(\Gamma^q)} \log \Xi(V|F_q) \right|.
$$

On the other hand,

$$
P_V(\Gamma^q|F_q) \leq \exp(-F_q(\Gamma^q)).
$$

Thus

$$
\left| \frac{\partial}{\partial F_q(\Gamma^q)} \log \Xi(V|F_q) \right| \leq \exp(-F_q(\Gamma^q)).
$$

(36)

### 4.5 Parameter contour models

Now we recall the notion of the parameter contour models [7]. Suppose we have a contour functional $F_q$ and a non-negative number $a_q$. The contour model with the functional $F_q$ and the parameter $a_q$ is defined as the following probability distribution on the ensemble $\mathcal{P}(V)$

$$
P_V(\{\Gamma_1^q, \ldots, \Gamma_n^q\}|F_q, a_q) = \frac{\exp \left( -\sum_i F_q(\Gamma_i^q) + a_q\|\bigcup_i \text{Int}\Gamma_i^q\| \right)}{\Xi(V|F_q, a_q)}. 
$$

(37)
Here \( \Xi(V|F_q, a_q) \) is a normalizing factor (the partition function of the parameter contour model). This partition function can be expressed in terms of \( \Xi(\Gamma^q_i|F_q) \) as follows

\[
\Xi(V|F_q, a_q) = \sum \prod \exp (a_q V(\Gamma^q_i)) \Xi(\Gamma^q_i|F_q).
\]  

(38)

For the \( \tilde{Z} \)-invariant functional \( F_q \) we define the boundary term \( \Delta(V|F_q, a_q) \) by

\[
\log \Xi(V|F_q, a_q) = (s(F_q) + a_q)\|V\| + \Delta(V|F_q, a_q).
\]  

(39)

We define the norm \( |F_q|_c \) by

\[
|F_q|_c = \sup_{\Gamma^q} \frac{|F_q(\Gamma^q)|}{(||\Gamma^q|| + V(\Gamma^q))c^\delta(\Gamma^q)}
\]  

(40)

where \( c \) is a constant greater than 1 and \( \delta(\Gamma^q) \) the diameter of \( \text{supp} \Gamma^q \) in the sense of metric (1).

From (36) and condition (14) it easily follows that there exists \( \gamma(\tau, c) \) such that \( \gamma(\tau, c) \to 0 \) as \( \tau \to \infty \) and for \( \tau \) sufficiently large it holds

\[
|s(F_q) - s(F_q')| < \gamma(\tau; c)|F_q - F_q'|_c
\]  

(41)

for any pair of \( \tilde{Z} \)-invariant functionals \( F_q \) and \( F_q' \).

From an obvious inequality

\[
1 \leq \Xi(V|F_q, a) \leq \exp(a\|V\|) \Xi(V|F_q)
\]  

(42)

it follows that

\[
-(a + s(F_q))\|V\| \leq \Delta(V|F_q, a) \leq \varepsilon(\tilde{Z}; \tau)\|\partial V\|.
\]  

(43)

We need to estimate the difference of the boundary terms of two different \( uv \)-functionals.

**Lemma 24.** For any pair of \( uv \)-functionals, \( F_q \) and \( F_q' \), for \( a, a' \geq 0 \), the following inequality holds:

\[
|\Delta(V|F_q, a) - \Delta(V|F_q', a')| \leq \left( \frac{1}{c - 1} + \gamma(\tau; c) \right)c^{\delta(V)} \|V\| |F_q - F_q'|_c + \|V\| |a - a'|.
\]  

(44)

**Proof** It is sufficient to consider just two cases: either \( a = a' \) or \( F_q = F_q' \). Both proofs are similar therefore we consider the case \( a = a' \).

Since

\[
|\Delta(V|F_q, a) - \Delta(V|F_q', a')| \leq | \log \Xi(V|F_q, a) - \log \Xi(V|F_q', a) | + |s(F_q) - s(F_q')| \cdot \|V\|,
\]  

(45)
and
\[ |s(F_q) - s(F'_q)| < \gamma(\tau; c)|F_q - F'_q|_c, \]
it is enough to prove the estimate
\[ |\log \Xi(V|F_q, a) - \log \Xi(V|F'_q, a)| \leq \frac{c \delta(V)}{c - 1} \|V\| |F_q - F'_q|_c. \tag{46} \]

By the definition, the probability \( P_V(\Gamma^q|F_q, a) \) for a contour \( \Gamma^q \) to belong to a collection of contours \( \{\Gamma^q_1, \ldots, \Gamma^q_n\} \in \mathcal{P}(V) \) is
\[ P_V(\Gamma^q|F_q, a) = \left| \frac{\partial}{\partial F_q(\Gamma^q)} \log \Xi(V|F_q, a) \right|. \tag{47} \]

By the Lagrange formula,
\[ |\log \Xi(V|F_q, a) - \log \Xi(V|F'_q, a)| \leq \sum_{\Gamma^q \subset V} P_V(\Gamma^q|\bar{F}_q, a) |F_q(\Gamma^q) - F'_q(\Gamma^q)|, \tag{48} \]
where \( \bar{F}_q = \theta F_q + (1 - \theta) F'_q, \ 0 \leq \theta \leq 1. \) Thus
\[ |\log \Xi(V|F_q, a) - \log \Xi(V|F'_q, a)| \leq \sum_{\Gamma^q \subset V} P_V(\Gamma^q|\bar{F}_q, a) c \delta(\Gamma^q) (\|\Gamma^q\| + V(\Gamma^q)) \cdot |F_q - F'_q|_c \]
\[ = E_V \left( \sum_{i=1}^{n} (\|\Gamma^q_i\| + V(\Gamma^q_i)) c \delta(\Gamma^q) |\bar{F}_q, a| \right) \cdot |F_q - F'_q|_c, \tag{49} \]
where \( E_V(\cdot|\bar{F}_q, a) \) is the expectation with respect to \( P_V \) and the summation \( \sum_{\Gamma^q \subset V} \) runs over all contours \( \Gamma^q \) satisfying conditions (28). Now (44) follows from the next

**Lemma 25.** For any finite volume \( V \subset \mathbb{Z}^{d+1} \) and any collection of contours \( \{\Gamma^q_1, \ldots, \Gamma^q_n\} \in \mathcal{P}(V) \) satisfying condition (28),
\[ \sum_{i=1}^{n} (\|\Gamma^q_i\| + V(\Gamma^q_i)) c \delta(\Gamma^q_i) < \frac{c \delta(V)}{c - 1} \cdot \|V\|. \tag{50} \]

**Proof** For any finite volume \( V \subset \mathbb{Z}^{d+1} \) let
\[ \varphi(V) = \frac{1}{\|V\|} \max \sum_{i=1}^{n} (\|\Gamma^q_i\| + V(\Gamma^q_i)) c \delta(\Gamma^q_i), \tag{51} \]
where the maximum is taken over all collections of contours \( \{ \Gamma_1, \ldots, \Gamma_n \} \in \mathcal{P}(V) \) (we drop the index \( q \)). Let \( \{ \Gamma_{i_1}, \ldots, \Gamma_{i_k} \} \subseteq \{ \Gamma_1, \ldots, \Gamma_n \} \) be the collection of all external contours among \( \{ \Gamma_1, \ldots, \Gamma_n \} \). Then

\[
\sum_{i=1}^{n} (\| \Gamma_i \| + V(\Gamma_i)) e^{\delta(\Gamma_i)} \leq \sum_{t=1}^{k} (\| \Gamma_{i_t} \| + V(\Gamma_{i_t})) e^{\delta(\Gamma_{i_t})} + \sum_{t=1}^{k} V(\Gamma_{i_t}) \varphi(\text{Int} \Gamma_{i_t}).
\] (52)

For any integer \( d \) denote

\[
\psi(d) = \max_{V: \delta(V) \leq d} \varphi(V).
\] (53)

Since

\[
\sum_{t=1}^{k} (\| \Gamma_{i_t} \| + V(\Gamma_{i_t})) \leq \| V \|, \quad \delta(\Gamma_{i_t}) \leq \delta(V) - 1
\] (54)

and \( \delta(\text{Int} \Gamma_{i_j}) \leq \delta(V) - 1 \), it follows from (52) that

\[
\varphi(V) \leq e^{\delta(V) - 1} + \psi(\delta(V) - 1)
\] (55)
i.e.,

\[
\psi(d) \leq e^{d - 1} + \psi(d - 1),
\] (56)

and hence

\[
\psi(d) \leq e^{d - 1} + e^{d - 2} + \cdots + 1 = \frac{e^d - 1}{c - 1}.
\] (57)

Next lemma is basic for the PS-theory of the laminated models (cf. [7]).

**Lemma 26.** For the Hamiltonian \( H(\mu) \) generated by a horizontal Hamiltonian (11) and the vertical Hamiltonian (6) for which (23) holds, there exist a unique point \( a = (a_1, \ldots, a_r) \in O_r \) and periodic \( uv \)-functionals \( F_q, 1 \leq q \leq r \), such that

\[
\Xi(\Gamma^q | \beta H) = \exp(a_q V(\Gamma^q)) \Xi(\Gamma^q | F_q),
\] (58)

\[
a_q = \beta \tilde{h}_q - s(F_q) + C,
\] (59)

where the constant \( C \) is defined from the relation \( \min_q a_q = 0 \) and does not depend on \( q \).
This lemma gives us the map \( J(\beta, \lambda) \) declared by Theorem 8.

**Proof** The Hamiltonian \( H \) is invariant with respect to the group \( \tilde{\mathbb{Z}} = \hat{\mathbb{Z}} \times l\mathbb{Z} \). Hence any solution \( F_q \) of (58) is invariant with the respect to \( \tilde{\mathbb{Z}} \).

Substituting (58) in (27) and (29) gives

\[
F_q(\Gamma^q) = \beta \Psi(\Gamma^q) - \sum_m \Delta(\text{Int}_m \Gamma^q | F_m, a_m) + \nabla(\Gamma^q | F_q),
\]

(60)

where \( \nabla(\Gamma^q | F_q) = \sum_m \Delta(\text{Int}_m \Gamma^q | F_q) \). It is supposed that \( a_q \) are defined by (59). Equations (60) and (59) form a closed system of equations for contour functionals \( F_q, 1 \leq q \leq r \).

We prove that there exists a unique solution of this system of equations in the class of \( \tilde{\mathbb{Z}} \)-invariant \( uv \)-functionals.

We solve the equations by the method of successive approximations. To this end we introduce a space of vector-functionals \( B(u,v) = \hat{F} = \{(F_1, \ldots, F_r)\} \) of \( \tilde{\mathbb{Z}} \)-invariant \( uv \)-functionals \( F_q, 1 \leq q \leq r \), with the metric

\[
|\hat{F} - \hat{F}'|_c = \max_{1 \leq q \leq r} |F_q - F_q'|_c.
\]

Then we can consider the right-hand side of (60) with \( a_m \) given by (59) as a correspondence between a collection of \( \tilde{\mathbb{Z}} \)-invariant \( uv \)-functionals \( F_q, 1 \leq q \leq r \), and a new collection of functionals. Namely, introduce the following notation

\[
T(\hat{F} | \beta \tilde{h}) = - \sum_m \Delta(\text{Int}_m \Gamma^q | F_q, a_m) + \nabla(\Gamma^q | F_q),
\]

\[
S(\hat{F} | \beta \Psi, \beta \tilde{h}) = \beta \hat{\Psi} + T(\hat{F} | \beta \tilde{h}).
\]

Then (60) becomes

\[
\hat{F} = S(\hat{F} | \beta \Psi, \beta \tilde{h}).
\]

(62)

Hence from (44), (59) and (41) we have

\[
|T(\hat{F} | \beta \tilde{h}) - T(\hat{F}' | \beta \tilde{h}')|_c \leq 2 \left( \frac{1}{c - 1} + \gamma(\tau; c) \right) |F_q - F_q'|_c
\]

\[
+ 2\beta |\tilde{h} - \tilde{h}'| + 2\gamma(\tau; c) |F_q - F_q'|_c.
\]

(63)

Let \( c = 13 \) and \( \gamma(\tau; c) < 1/12 \). Then \( T(\hat{F} | \beta \tilde{h}) \) is Lipshitz in \( \hat{F} \) with the constant 1/2 if \( \tilde{h} \) is fixed. Moreover,

\[
|T(\hat{F} | \beta \tilde{h})|_c < \infty.
\]

From (35) it follows that if \( 4(d + 1) \varepsilon(\hat{Z}, \tau) < 1 \) then \( S \) maps a collection of \( uv \)-functionals to a collection of \( uv \)-functionals. Hence the map \( S(\hat{F} | \beta \Psi, \beta \tilde{h}) \) has a unique fixed point \( \hat{F} \) for which equations (58), (59) are satisfied. \( \square \)
We remind that the Peierls condition is satisfied for the Hamiltonian $H^0$. We assume also that for a Hamiltonian $H_g = H^0 + \mu_1 H_1 + \cdots + \mu_{r-1} H_{r-1}$

the vector of parameters $\mu = (\mu_1, \ldots, \mu_{r-1})$ belongs to a neighborhood $U$ of the origin, so that inequality (23) holds true.

It follows from (63) that the fixed point $\hat{F}$ of the transformation $S(\hat{F}|\beta \hat{\Psi}, \beta \hat{h})$ continuously depends on $\beta \hat{\Psi}$ and $\beta \hat{h}$. More precisely,

$$\frac{1}{2} |\hat{F} - \hat{F}'|_c \leq \beta |\hat{\Psi} - \hat{\Psi}'|_c + 2\beta |\hat{h} - \hat{h}'|.$$  

(64)

Given $\beta > 0$, choose $l$ and $\lambda$ according to (21) and (22). Then the Hamiltonian $\beta H$ satisfies conditions of Lemma 24. Thus, each point $\mu$ in $\mathbb{R}^{r-1}$ corresponds to some point $a$ in $O_r$. We set $a = J(\beta, \lambda) \mu$. Then the coordinates $a_q$, $1 \leq q \leq r$, of $a$ are determined by the formula (59), where $\tilde{h}_q = lh_q$, $h_q = \sum e_k \mu_k$.

Conversely, given $a = \{a_q, 1 \leq q \leq r\}$, we can find $\mu = \{\mu_1, \ldots, \mu_{r-1}\}$. To this end, let us rewrite (59) as

$$\beta \tilde{h}_q = a_q + s(F_q) + \text{const},$$  

(65)

where $F_q$, $1 \leq q \leq r$, are determined by $\beta \tilde{h}_q$, $1 \leq q \leq r$. We see from (63) that the functions $F_q$, $1 \leq q \leq r$, are Lipshitz on $\beta \tilde{h}_q = \beta lh_q$, $1 \leq q \leq r$. Since $s(F_q)$ is Lipshitz on $F_q$ with a small Lipshitz constant, given $a_q$, $1 \leq q \leq r$, it is possible to find $\beta \tilde{h}_q$, $1 \leq q \leq r$, iterating (65).

Rewriting (65) as

$$h_q = \frac{1}{\beta l}(a_q + s(F_q)) + \text{const}$$

we see that the iterations do not escape from $U$ if $1/\beta l$ is small enough and $\max_q a_q$ is small enough as well. So the map $J(\beta, \lambda)$ is a homeomorphism onto the neighbourhood of the origin in $O_r$. The existence of pure thermodynamic phases corresponding to those $q$ for which $a_q = 0$ easily follows from Lemma 26. Namely, if $a_q = 0$ then equation (58) gives the equality of the partition functions of the laminated model and the contour model. Hence the distributions of the outer contours for these two models also coincide and the existence of the limit Gibbs distribution follows (see [7]).

5 (1 + 1)- laminated models

5.1 Ground states of one-dimensional models

A collection $\mathcal{A}$ of finite subsets of $\mathbb{Z}$ is called a set of patterns if for every $A \in \mathcal{A}$
1) there exists $N$ such that $A \subseteq [0, N]$,

2) $\min A = 0$.

We define a formal Hamiltonian of a one-dimensional lattice model by finite range potential functions

$$\varphi_A(s_A), \ A \in A,$$ \hspace{1cm} (66)

as

$$H(s) = \sum_{A \in A} \sum_{x \in \mathbb{Z}} \varphi_A(s_{A+x}).$$ \hspace{1cm} (67)

As above for any pair of configurations $s', s'' \in S$ which are equal almost everywhere we define the relative Hamiltonian $H(s', s'')$ \hspace{1cm} (9). We reduce the model to an equivalent one by a process we call coarse-graining. A coarse-grained model has a pair interaction between the nearest neighbour (coarse-grained) spins only.

To construct the coarse-grained model we divide $\mathbb{Z}$ into non-intersecting blocks of the size $N$ which cover $\mathbb{Z}$. That is, $\mathbb{Z} = \bigcup_{i \in \mathbb{Z}} [k_i, k_i + N)$ assuming $k_i = iN$ and $k_0 = 0$. Assign to the block $[k_i, k_i + N)$ the number $i$ and consider the set $S^{(i)} = \{s^{(i)} : [k_i, k_i + N) \to S\}$ of spin configurations on $[k_i, k_i + N)$. It is clear that $S^{(i)}$ are isomorphic for different $i$.

We define the spin space of the coarse-grained model as $\tilde{S} = S^0$. The single spin energy of the coarse-grained model is

$$\tilde{\varphi}_1(s) = \sum_{A \in A, A+x \subseteq [0,N)} \varphi_A(s_A).$$ \hspace{1cm} (68)

To define the potential $\tilde{\varphi}_2$ of two neighboring block-spins $\tilde{s}$ and $\tilde{s}'$ assume that $\tilde{s}$ is defined on $[0, N)$ and $\tilde{s}'$ is defined on $[N, 2N)$. Then

$$\tilde{\varphi}_2(\tilde{s}, \tilde{s}') = \sum_{A : A \in A, A+x \subseteq [0,2N), A+x \cap [0,N) \neq \emptyset, A+x \cap [N,2N) \neq \emptyset} \varphi_A(\text{pr}(\tilde{s} \vee \tilde{s}', A + x)), \hspace{1cm} (69)$$

where $\tilde{s} \vee \tilde{s}'$ means concatenation of $\tilde{s}$ and $\tilde{s}'$, and $\text{pr}(\tilde{s} \vee \tilde{s}', A + x)$ is the restriction of $\tilde{s} \vee \tilde{s}'$ to $A + x$.

For the coarse-grained model, the only single-site and neighboring two-sites energy is non-zero. We define $\varphi_2(\tilde{s}_1, \tilde{s}_2) = \tilde{\varphi}_2(\tilde{s}_1, \tilde{s}_2) + \tilde{\varphi}_1(\tilde{s}_1)$. Remark that generally $\varphi_2(\tilde{s}_1, \tilde{s}_2) \neq \varphi_2(\tilde{s}_2, \tilde{s}_1)$.

**Theorem 27.** Any one-dimensional model with a finite-range potential has periodic ground states.
Proof.

Let \( G = (V, E) \) be a complete oriented graph with the finite vertex set \( V \) (we will need \( V = \tilde{S} \)). Assume that every edge \( e \in E \) is supplied with an energy value \( \varphi(e) \), that is,

\[
\varphi : E \to \mathbb{R}.
\]

We define a path on \( G \) as a sequence of edges \( W = (e_1, \ldots, e_n, \ldots) \) such that the initial vertex of \( e_{i+1} \) is the final vertex of \( e_i \) for all \( i \).

A cycle \( C \) is a path \( C = (e_1, \ldots, e_n) \) such that the final vertex of \( e_n \) is the initial vertex of \( e_1 \). The specific energy of a cycle \( C \) is defined by

\[
h(C) = \frac{1}{n} \sum_{i=1}^{n} \varphi(e_i). \tag{71}
\]

Our goal is to find a cycle with minimal specific energy.

A cycle \( C = (e_1, \ldots, e_n) \) is called irreducible if all its vertices are different. Let \( \overline{C} \) be an irreducible cycle having the minimal specific energy among all irreducible cycles. The cycle \( \overline{C} \) exists, since the set of irreducible cycles is finite.

Lemma 28. The irreducible cycle having the minimal specific energy among all irreducible cycles has the minimal specific energy among all cycles.

Proof. Any cycle \( C \) with its length \( n \) can be expanded to a finite number \( k \) of irreducible cycles \( C_1, \ldots, C_k \) with lengths \( n_1, \ldots, n_k \), respectively, \( n = \sum_i n_i \). Thus

\[
h(C) = \sum \frac{n_i}{n} h(C_i). \tag{72}
\]

Since \( h(C_i) \geq h(\overline{C}) \) for any \( C_i \), we have \( h(C) \geq h(\overline{C}) \). \qed

To finish the proof of Theorem 27 we introduce the complete oriented graph with the set of vertices \( V = \tilde{S} \) as . Every configuration \( \tilde{s} : \mathbb{Z} \to \tilde{S} \) generates a path \( W = (\ldots, e_1, \ldots) \) in the graph such that \( e_1 = (\tilde{s}(0), \tilde{s}(1)), \ e_2 = (\tilde{s}(1), \tilde{s}(2)) \) etc. Any periodic configuration \( s \) generates a cycle of a finite length. Thus there exists a periodic configuration generating a cycle with minimal specific energy. \qed

Further we consider models with finite number of ground states. Making the additional coarse-graining we can count that all the ground states of our model are configurations with period 1, i.e. are constant configurations. Without loss of generality we can assume that the specific energy of the ground states is zero.

20
5.2 Peierls condition

Let $H$ be the Hamiltonian of a one-dimensional model having finite number of the ground states, and the ground states are constant configurations. Then there is a set $Q = \{q_1, \ldots, q_r\} \subseteq \tilde{S}$ such that for any $k = 1, \ldots, r$ the configuration $\tilde{s}_k(i) \equiv q_k$ is a ground state, and the specific energy $h(\tilde{s}_k) = 0$. We say that a site $i$ is regular with respect to configuration $\tilde{s}$ if $\tilde{s}(i - 1) = s(i) = s(i + 1) \in Q$. Otherwise $i$ is a boundary site.

**Proposition 3** (Peierls condition). There exists a positive constant $c$ such that for any ground state $\tilde{s}_q(i) \equiv q \in Q$ and any configuration $\tilde{s}$ that is equal to $\tilde{s}_q$ almost everywhere and has $n$ boundary sites it holds that

$$H(\tilde{s}, \tilde{s}_q) \geq cn. \quad (73)$$

**Proof.** By coarse-graining construction the ground states correspond to cycles of the form $(v, v)$, where $v \in Q$. So for any other irreducible cycle $C$ its energy $H(C)$ is strictly positive. Denote by $N(C)$ the number of boundary sites of the cycle $C$, then $H(C) > \varepsilon N(C)$ for some positive $\varepsilon$. From this inequality the Peierls condition for the Hamiltonian $H$ follows.

**Remark 29.** Return now to the $(d + 1)$-dimensional laminated model. In this section we have shown that for the case $d = 1$ any Hamiltonian with the finite number of periodic ground states satisfies the Peierls condition and so can be used as the horizontal Hamiltonian $H_g$ in the construction of the laminated model. Thus for any $(1 + 1)$-dimensional laminated model with finite number of periodic ground states we obtain the phase diagram.

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