Linear Operators and Operator Functions
Associated with Spectral Boundary Value Problems

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Abstract. The paper develops a theory of spectral boundary value problems from the perspective of general theory of linear operators in Hilbert spaces. An abstract form of spectral boundary value problem with generalized boundary conditions is suggested and results on its solvability complemented by representations of weak and strong solutions are obtained. Existence of a closed linear operator defined by a given boundary condition and description of its domain are studied in detail. These questions are addressed on the basis of Krein’s resolvent formula derived from the explicit representations of solutions also obtained here. Usual resolvent identities for two operators associated with two different boundary conditions are written in terms of the so called M-function. Abstract considerations are complemented by illustrative examples taken from the theory of partial differential operators. Other applications to boundary value problems of analysis and mathematical physics are outlined.

Keywords: Spectral boundary value problem, singular perturbations, M-function, Krein’s resolvent formula, linear operators, open systems theory

1 Introduction

Close relationships between studies of boundary value problems and the linear operator theory are well known to specialists in both disciplines. As an example, one can only mention numerous attempts to translate properties of solutions to boundary value problems into the operator-theoretic language that culminated in the development of an important branch of contemporary mathematics, the interpolation theory of linear operators and scales of Banach and Hilbert spaces [10], [45], [49]. Another achievement of the abstract operator theory in relation to boundary problems arising in applications is the extension theory of symmetric operators. With its origin in quantum mechanics and operating in the setting of Hilbert spaces, the extension theory suggests a convenient model of boundary value problems rooted in Hilbert space operator theory [2]. Although the abstract approach often turns out to be too generic and therefore additional considerations are required to complete the study, the extension theory of symmetric operators continues to be an important and widely used tool in the studies
of boundary value problems in abstract settings. During past decades it was substantially enhanced and enriched by various applications to the operator theory itself, to the classical and functional analysis, and to the mathematical physics. The long list of publications \[7, 9, 11, 14, 15, 17, 18, 21, 22, 23, 24, 25, 26, 29, 30, 31, 35, 37, 38, 42, 43, 48, 52, 53, 54, 55, 64, 65, 66, 67, 68, 85\] reflects only a small portion of the sheer amount of ongoing studies in the field of extensions theory of symmetric operators.

The present paper offers an operator-theoretic treatment of boundary value problems. The main topic under discussion is the existence of Hilbert space operators corresponding to abstract linear boundary value problems defined by suitably generalized boundary conditions. As is well known, many applications of the partial differential equations theory entail problem statements characterized by certain types of formally written boundary conditions. In the case of second order partial differential equations on bounded or unbounded domains such conditions are usually rendered in terms of linear combinations of boundary values of solutions and traces of their derivatives evaluated on the domain boundary. Typical examples include the Laplacian in a domain of Euclidian space with Dirichlet, Neuman, or Robin boundary conditions. Depending on the nature of these conditions, the resulting problem may or may not give rise to a closed linear operator in a Hilbert or Banach space. If such an associated operator exists, then the study is effectively reduced to the analysis of its properties. The paper describes a wide class of boundary conditions that determine a closed linear operator in a Hilbert space and studies its spectral characteristics in the general setting. In a sense, the goal pursued here is opposite to the treatment by G. Grubb \[35\] where the existence of boundary conditions corresponding to a given closed realization of an elliptic operator is investigated.

An essential part of present research is the formulation of generic linear boundary value problems in the language of Hilbert space operator theory. Within this framework, boundary conditions are defined by two parameters, two closed linear operators acting in the “boundary space.” Reiteration of the material developed earlier in \[71, 73\] is followed by a more detailed inquiry into the properties of solutions, which in turn leads to Krein’s resolvent formula and usual resolvent identities for closed operators acting on the “main space” corresponding to various boundary conditions. Spectral properties of these operators are described in terms of the so called M-function. \[1\] Several examples offered throughout the text illustrate the main ideas.

Ongoing study of M-functions, also known as m-functions, Q-functions, Weyl-Titchmarsh functions, Steklov-Poincaré operator, Dirichlet-to-Neumann maps, transfer functions, etc. forms a significant part of the contemporary boundary value problems studies. The M-functions theory originates in the concept of m-function for singular Sturm-Liouville differential equations \[83\]. Since then

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\[1\] Values of all M-functions under consideration are closed linear operators acting in Hilbert spaces, with one exception of the example given in Section 7 where M-function is a matrix function. Sometimes the term “M-operators” is used in order to stress the operator theoretic nature of M-function \[8\].
the notion of M-functions has been generalized to other settings and followed by deep results on M-function properties and applications. We only mention a few relevant papers concerning topics in scattering theory [63], [83], Schrödinger and Sturm-Liouville operators theory [8], inverse problems [39], [80], the spectral asymptotic [27], [75], extensions of symmetric operators and adjoint pairs [14], [15], [17], [21], [22], [23], [24], [25], [29], [30], [31], [32], [33], [34], [35], [36], numerous studies on partial differential operators including operators in non-smooth domains [7], [28], [29], [30], [31], [32], [33], the numerical spectral analysis [16], [56], singular perturbations [64], [65], [66], and the linear systems theory [71], [73]. In the present paper’s context M-functions are realized as operator-functions with values in the set of closed linear operators acting in the “boundary space,” a Hilbert space associated with the “boundary.”

Operator theoretic parts of the present work have some overlaps with the extensions theory of linear symmetric operators and relations in Hilbert and Krein spaces based on the notion of so called boundary triplets [18], [42] and their generalizations. This approach relies on on properties of the abstract Green’s formula that involves a linear symmetric operator (a linear symmetric relation in general case) and two linear boundary maps into the “boundary space.” The theory of boundary triplets is one of the most generic treatements of general boundary relations available today. The interested reader is referred to the original papers [21], [22], [23], [24], [25], [26], [52], [53], [54], [55], [56] where further references can be found. Another successful approach to the extension theory of symmetric operators was elaborated by A. Posilicano in works [64], [65], [66], [67], [68]. It is rooted in a close relationship between singular perturbations of elliptic differential operators and the extensions theory [4], [5]. In comparison with these studies, the present study follows the line of reasoning found in [71], [73]. The ideas expounded below are inspired by the Birman-Krein-Vishik method of extensions of positive operators in Hilbert space [11], [44], [85] (see also [35], [37] and [8]), the Weyl decomposition [55], the open systems theory [61], and the theory of linear systems with boundary control [77]. As a result, the framework in this paper is not centered around symmetric operators and does not involve any notions specific to the extensions theory. It is built from the first principles concerning linear operators and their domains, as well as properties of linear sets in Hilbert spaces. The linear systems theory conveniently provides an adequate language to communicate the underpinnings of this approach. With some risk of oversimplification, the last part of Section 2 explains these ideas in more depth by connecting them to the objects of systems theory.

The paper’s treatment of boundary value problems from the abstract point of view opens a possibility to consider classical and non-classical applications from a uniform perspective. As an example, it turns out that obtained results offer a straightforward interpretation of boundary value problems when the “boundary” does not exist a priori and has to be constructed artificially. This type of problems has been well studied in the literature and is usually referred to as singular perturbations of differential operators characterized by perturbations supported on the sets of Lebesque measure zero (often called the null sets).
well known quantum mechanical model of point interactions \cite{4, 5} and the study of more general Schrödinger operators with potentials supported by null sets \cite{3} are typical examples. This fact underlines close connections of the present material to the papers \cite{64, 65, 66, 67, 68} devoted to the study of extensions of symmetric operators and singular perturbations. In the field of linear systems theory singular perturbations represent the procedure of “channels opening” connecting an initially closed systems to its environment \cite{51}. From this point of view, the M-function is naturally identified with the transfer function of the resulting open system interacting with its environment by means of these channels. Operator theoretic treatment also illuminates ideas behind the so-called “Dirichlet decoupling” \cite{20} also known as “Glazman’s splitting procedure” \cite{34} and establishes connections to the analog of Weyl-Titchmarsh function of multidimensional Schrödinger operator \cite{8, 73}. It appears relevant to other problems of mathematical physics, e. g. the exterior complex scaling in the theory of resonances \cite{76} and the R-matrix method well known in nuclear physics \cite{47}. Some of these applications are discussed in Section 7 and in the last part of Section 2 where relevant bibliographical references can be found; these ideas are the topics of further research.

The approach to spectral boundary value problems adopted in the paper has certain limitations. One of them is the assumption of selfadjointness and bounded invertibility of the “main” operator (denoted $A_0$ throughout) acting in the Hilbert space $H$. The requirement of bounded invertibility of $A_0$ can be weakened to the condition $\text{Ker}(A_0 + cI) = \{0\}$ for some $c \in \mathbb{R}$, but the selfadjointness is essential. Nevertheless, the schema can be extended to the case $A_0 \neq A_0^*$, but only at the expense of introducing the so called dual pairs \cite{53, 54, 55} (see also \cite{14, 15, 17}), which makes the study much more involved. Another limitation is the preference to work with linear operators, rather than with linear relations (multivalued operators) which appears to be the recent trend in the literature, see especially \cite{22, 23}. The language of single valued operators stands more in line with the classical approach of operator theory and is preferred here. One more requirement is that the operator $A_0$ must be unbounded so that the range of $A_0^{-1}$ is dense in $H$. Fortunately, all these restrictions do not impede the study of the main question addressed in the paper, that is, the description of operators corresponding to boundary value problems defined in terms of boundary conditions.

Let us now briefly overview the paper’s structure. Section 2 offers an accessible introduction into the setting of boundary problems and M-functions. It serves as a guideline for the topics discussed later and provides a concise exposition of the operator theoretic framework of spectral boundary value problems independent of the symmetric operators theory. The main example is the well known spectral problem for Dirichlet Laplacian in a smooth domain in $\mathbb{R}^n$, $n \geq 3$. An adequate language for study of this operator is the language of Green’s functions, integral equations and layer potentials. When necessary, relevant results are freely borrowed from the standard references \cite{1, 57, 58}. By this example all essential ingredients of the following exposition are explicitly formulated.
and finally compiled in a short catalog. Relationships to the extension theory of symmetric operators, Krein’s resolvent formula, and resolvent identities are also discussed. Since the linear systems theory plays an important role for the approach employed in this work, a brief explanation of the principal ideas of this theory is provided for reader’s convenience. At the end of section other cases of partial differential operators that can be treated in a similar fashion are mentioned.

Section 3 develops the machinery required for the purposes of the paper. The main objective here is to formulate notions useful for the study of spectral boundary value problems and associated M-functions given in terms of their basic underlying objects. Such objects are two Hilbert spaces and three closed linear operators satisfying certain compatibility conditions. The solvability theorem is proven and ensuing definitions of weak and strong solutions are discussed. The section concludes with alternative descriptions of M-functions and some comments regarding their properties.

Spectral boundary value problems with general boundary conditions are investigated in Section 4. After the problem statement the solvability theorem is proven and expressions for the solutions corresponding to various boundary conditions are obtained. The last part of the section explores a general definition of M-functions associated with two different boundary conditions.

Section 5 is the main contribution of the paper. We discuss the existence of closed linear operators corresponding to spectral boundary value problems and subsequent study of their properties. Formal expressions for the resolvents and parameters of “boundary conditions” are derived from the general representation of solutions obtained in the previous section. These expressions are rigorously justified alongside with the study of spectral properties of respective operators and detailed descriptions of their domains. Relations to the extension theory of symmetric operators are also explained. The section closes with a brief digression into the original Birman-Krein-Vishik theory [11], [44], [85] and remarks on its connections to the present study.

Section 6 offers a sketch of scattering theory for operators associated with boundary value problems. Simultaneously, by virtue of the paper’s approach, all arguments of this section remain valid for singular perturbations of partial differential operators by “potentials” concentrated on null sets. Form the operator-theoretic point of view, the primary interest here is the link between the boundary value problems theory and the functional model of nonselfadjoint operators established by means of Cayley transform applied to the M-function. It is shown that the ideas of papers [60], [61] devoted to the functional model based approach to the scattering theory are easily adopted and are fully applicable for the comprehensive development of scattering theory of linear boundary value problems, selfadjoint and nonselfadjoint alike. Section 6 can be seen as a groundwork for the future study in this direction.

The last section is an illustration of the boundary value problem technique discussed in the paper in application to singular perturbations of multidimensional differential operators. A simple example of the quantum mechanical model
for a finite number of point interactions in $L^2(\mathbb{R}^3)$ (see [4], [5]) is studied. A familiar interpretation in the form of Schrödinger operator with $\delta$-potentials is given and additional comments regarding singular perturbations concentrated on the null sets are supplied. Reported results are by no means new; most of them can be easily found in the relevant literature cited in the text. The objective of this section is to demonstrate how the abstract schema presented in earlier chapters can be put to practice for the study of particular cases of multidimensional differential operators.

The concept of this paper took shape during my visit to Cardiff University in April 2008. I would like to express my sincere gratitude to Prof. Marco Marletta and Prof. Malcolm Brown for the invitation, hospitality, and inspiring discussions. I am also grateful to Prof. Serguei Naboko for his interest to the work, multiple discussions concerning its style, and continuous encouragement. Last but not least, I am indebted to an anonymous referee for his/her thorough examination of the manuscript and the list of relevant research papers published after the present work was completed, see [87], [88], [89], [90], [91], [92], [93], [94], [95], [96], [97], [98]. The comments that I was able to act on definitely improved the text; as for the others, I am hopeful to be able to address them in future publications.

This work is dedicated to the memory of Boris Pavlov (1936 – 2016), the role model of my academic career.

Notation Symbols $\mathbb{R}$, $\mathbb{C}$, $\mathrm{Im}(z)$ stand for the real axis, the complex plane, and the imaginary part of a complex number $z \in \mathbb{C}$, respectively. The upper and lower half planes are the open sets $\mathbb{C}_\pm := \{z \in \mathbb{C} \mid \pm \mathrm{Im}(z) > 0\}$. If $A$ is a linear operator on a separable Hilbert space $H$, the domain, range and null set of $A$ are denoted $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathrm{Ker}(A)$, respectively. For two separable Hilbert spaces $H_1$ and $H_2$ the notation $A : H_1 \to H_2$ is used for a bounded linear operator $A$ defined everywhere in $H_1$ with the range in the space $H_2$. The symbol $\rho(A)$ is used for the resolvent set of $A$. For a Hilbert space $H$ the term subspace always denotes a closed linear set in $H$. The closure of operators and sets is denoted by the horizontal bar over the corresponding symbol. All Hilbert spaces are assumed separable.

2 Boundary Value Problems by Example

In this introductory section we recall the classical example of the boundary value problem and M-function associated with the Dirichlet Laplacian in a simply connected bounded domain with smooth boundary in the Euclidian space. The purpose of this exposition is twofold. First, it reminds the reader of the concept of M-functions, and secondly it brings together facts that serve as a foundation for the general approach developed further. The italic typeface is used to highlight those observations which are essential for the investigations of the present paper. Results cited below hold true under much weaker assumptions, e. g. for elliptic differential operators on non-smooth domains including Lipschitz subdomains.
of Riemannian manifolds, see [11, 58, 59] and references therein. For further details the reader is referred to many expositions of the boundary integral equations method in application to boundary value problems for elliptic equations and systems, see [11, 57, 58] for relevant references.

**Dirichlet problem** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded simply connected domain with $C^{1,1}$-boundary $\Gamma$. The Laplace differential expression $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ defined on smooth functions in $\Omega$ generates the Dirichlet Laplacian $\Delta_D$ in $L^2(\Omega)$. The domain of $A_0 := -\Delta_D$ consists of functions from the Sobolev class $H^2(\Omega)$ with null traces on $\Gamma$. The operator $A_0$ is selfadjoint and boundedly invertible in $L^2(\Omega)$.

**Harmonic functions and operator of harmonic continuation** Let $\gamma_0$ be the trace operator that maps continuous functions $u$ defined in the closure $\Omega$ of $\Omega$ into their traces on the boundary, $\gamma_0 : u \mapsto u|_\Gamma$. It follows from the definition of $-\Delta_D$ that $\gamma_0 A_0^{-1} = 0$. For $\varphi \in C(\Gamma)$ denote $h_\varphi$ the solution of the Dirichlet problem in $\Omega$:

$$\Delta u = 0, \quad \gamma_0 u = \varphi, \quad \text{where} \quad \varphi \in C(\Gamma)$$

The operator $\Pi : \varphi \mapsto h_\varphi$ is bounded as a mapping from $L^2(\Gamma)$ into $L^2(\Omega)$ and $\text{Ker}(\Pi) = \{0\}$, see [58]. It is readily seen that $\Pi$ is the classical operator of harmonic continuation from the boundary $\Gamma$ into the domain $\Omega$ uniquely extended to the bounded linear map defined on the space $L^2(\Gamma)$. The equality $\gamma_0 \Pi \varphi = \varphi$ continues to hold for $\varphi \in L^2(\Gamma)$ and moreover $\Delta h_\varphi = 0$ for $h_\varphi = \Pi \varphi$ in the sense of distributions [58]. Observe that the (unbounded and not closed) operator $A : A_0^{-1} f + \Pi \varphi \mapsto f$, $f \in L^2(\Omega)$, $\varphi \in L^2(\Gamma)$ is well defined since the domain of operator $A_0$ and the set $\mathcal{R}(\Pi)$ do not have nontrivial common elements, otherwise $A_0$ would not be boundedly invertible:

$$\exists A_0^{-1} \Rightarrow \mathcal{D}(A_0) \cap \mathcal{R}(\Pi) = \{0\}$$

The same argument shows that $\mathcal{D}(A_0)$ does not contain any nontrivial functions from $H^2(\Omega)$ satisfying the homogenous equation $(-\Delta - zI)h = 0$ under the assumption $z \in \rho(A_0)$. Obviously, $A_0$ is a restriction of $A$ to $\mathcal{D}(A_0)$. Notice also that $A$ does not coincide with the “maximal operator” defined as an adjoint to the map $u \mapsto -\Delta u$, where $u \in L^2(\Omega)$ belongs to the class $C_0^\infty$ of infinitely differentiable functions vanishing in the vicinity of boundary $\Gamma$.

**Adjoint of the harmonic continuation operator** Let $G(x,y)$ be the Green’s function of $A_0 = -\Delta_D$, so that $(A_0^{-1} f)(x) = \int_\Omega G(x,y) f(y) \, dx$ for $f \in L^2(\Omega)$, see [58]. The kernel $G(\cdot, \cdot)$ is symmetric and real-valued: $G(x,y) = G(y,x)$ and $G(x,y) = G(x,y)$. Denote by $d\sigma$ the normalized Lebesgue surface measure on $\Gamma$. Then the operator $\Pi$ can be expressed as an integral operator with Poisson kernel

$$\Pi : \varphi \mapsto -\int_\Gamma \varphi(y) \frac{\partial}{\partial y} G(x,y) \, d\sigma_y$$
where \( \frac{\partial}{\partial \nu} \) is the derivative along the outside pointing normal at the boundary \( \Gamma \). For a smooth function \( f \) in \( \Omega \)

\[
(\Pi \varphi, f) = -\int_{\Gamma} \left( \int_{\Gamma} \varphi(y) \frac{\partial}{\partial \nu_y} G(x, y) \, d\sigma_y \right) f(x) \, dx
\]

and due to Fubini’s theorem and properties of \( G(\cdot, \cdot) \),

\[
(\Pi \varphi, f) = -\int_{\Gamma} \varphi(y) \frac{\partial}{\partial \nu_y} \left( \int_{\Omega} f(x, \cdot) G(x, y) \, dx \right) \, d\sigma_y = -\left\langle \varphi, \frac{\partial}{\partial \nu} \left( \int_{\Omega} G(x, \cdot) f(x) \, dx \right) \right\rangle
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2(\Gamma) \). Since \( G(x, y) = G(y, x) \) is the integral kernel of \( A_0^{-1} \), we obtain the representation for \( \Pi^* \), the adjoint of \( \Pi \),

\[
\Pi^* = \gamma_1 A_0^{-1}
\]

where \( \gamma_1 : u \mapsto -\gamma_0 \frac{\partial u}{\partial \nu} = -\frac{\partial u}{\partial \nu} \big|_{\Gamma} \). We will use the symbol \( \partial_\nu \) for the map \( u \mapsto \frac{\partial u}{\partial \nu} \big|_{\Gamma} \), so that \( \gamma_1 = -\partial_\nu \).

**The spectral problem** The spectral Dirichlet boundary value problem for the differential expression \( \Delta = \sum_i \frac{\partial^2}{\partial x_i^2} \) in \( \Omega \) is defined by the system of equations for \( u \in \mathcal{D}(A) := \mathcal{D}(A_0) \oplus \mathcal{R}(\Pi) \)

\[
\begin{cases}
(A - zI)u = 0, \\
\gamma_0 u = \varphi
\end{cases} \tag{2.1}
\]

where \( A : u \mapsto -\Delta u, \varphi \in L^2(\Gamma) \), and the number \( z \in \mathbb{C} \) plays the role of spectral parameter. For \( z \in \rho(A_0) \) the distributional solution \( u^z_\varphi \) can be obtained from the harmonic function \( \Pi \varphi \) by the formula \( u^z_\varphi = (I - zA_0^{-1})^{-1} \Pi \varphi \). Indeed, since \( (I - zA_0^{-1})^{-1} = I + z(A_0 - zI)^{-1} \) and \( A\Pi \varphi = 0 \) in the distributional sense, we have

\[
(A - zI)u^z_\varphi = (A - zI) \left( \Pi \varphi + z(A_0 - zI)^{-1} \Pi \varphi \right) = -z\Pi \varphi + z\Pi \varphi = 0
\]

due to the identity \( (A - zI)(A_0 - zI)^{-1} = I \). Therefore the vector \( u^z_\varphi \) is a solution to the equation \( (A - zI)u = 0 \). Further, \( \gamma_0 u^z_\varphi = \gamma_0 \Pi \varphi = \varphi \). Hence the vector \( u^z_\varphi = (I - zA_0^{-1}) \Pi \varphi \) is a solution to the spectral problem \([2.1]\) for \( \varphi \in L^2(\Gamma) \) and \( z \in \rho(A_0) \).

**Solution Operator and DN-Map** For the spectral problem \([2.1]\) with \( \varphi \in L^2(\Gamma) \) and \( z \in \rho(A_0) \) introduce the solution operator

\[
S_z : \varphi \mapsto (I - zA_0^{-1})^{-1} \Pi \varphi \tag{2.2}
\]
Operator $S_z$ is bounded as a mapping from $L^2(\Gamma)$ into $L^2(\Omega)$. For $\varphi \in C^2(\Gamma)$ the inclusion $S_z \varphi \in H^2(\Omega)$ holds and therefore the expression $\gamma_1 S_z \varphi$ is well defined. The operator function $M(z)$ defined by

$$M(z) : \varphi \mapsto \gamma_1 S_z \varphi, \quad \varphi \in C^2(\Omega)$$

(2.3)

is analytic in $z \in \rho(A_0)$. It is called the Dirichlet-to-Neumann map (DN-map) or, more generally, the M-function of $A = -\Delta$ in the domain $\Omega$. By construction, $-\partial_\nu u = M(z) (u|_\Gamma)$ for $u \in \text{Ker}(A - zI)$ as long as the function $\gamma_0 u = u|_\Gamma$ is sufficiently smooth on $\Gamma$. In fact, it can be shown that values of so defined $M(z)$, $z \in \rho(A_0)$ are closed operators acting in $L^2(\Gamma)$ with the domain $H^1(\Gamma)$, see [81] and references therein.

The representation $S_z = (I - z A_0^{-1})^{-1} \Pi$ and equality $\Pi^* = \gamma_1 A_0^{-1}$ imply

$$(S_z)^* = \gamma_1 A_0^{-1} (I - z A_0^{-1})^{-1} = \gamma_1 (A_0 - z I)^{-1}$$

(2.4)

Therefore $S_z = [\gamma_1 (A_0 - z I)^{-1}]^*$ and the M-function $M(z)$ can be rewritten:

$$M(z) = \gamma_1 [\gamma_1 (A_0 - z I)^{-1}]^*$$

In particular, $M(0) = \gamma_1 (\gamma_1 A_0^{-1})^* = \gamma_1 \Pi$. It can be shown (see [81]) that the operator $M(0) = \gamma_1 \Pi$ defined on the domain $\mathcal{D}(M(0)) = H^1(\Gamma)$ is selfadjoint in $L^2(\Gamma)$. Operator $M(0)$ turns out to be a rather important object; it is convenient to use a special notation for it:

$$A = M(0) = \gamma_1 \Pi, \quad \mathcal{D}(A) = H^1(\Gamma)$$

Robin Boundary Conditions Let $\beta \in L^\infty(\Gamma)$ be a bounded function defined almost everywhere on the boundary $\Gamma$. In what follows we also denote $\beta$ the bounded operator of multiplication $\varphi \mapsto \beta \varphi$, $\varphi \in L^2(\Gamma)$ acting in the space $L^2(\Gamma)$. Consider the boundary value problem

$$
\begin{cases}
(A - zI)u = 0, \\
-\partial_\nu u + \beta u|_\Gamma = \varphi
\end{cases}
$$

(2.5)

with $\varphi \in L^2(\Gamma)$. In particular, for $\beta = 0$ we recover the classical Neumann problem for the Laplacian in $\Omega$. For nontrivial $\beta$ the system (2.5) is called the boundary problem of third type, or Robin problem. Assume $z \in \rho(A_0)$ and let $u^\varphi_\zeta$ be a smooth solution to the first equation, that is $(A - zI)u^\varphi_\zeta = 0$. Because $\gamma_1 u^\varphi_\zeta = M(z) \gamma_0 u^\varphi_\zeta$, the second equation for the trace $\psi := \gamma_0 u^\varphi_\zeta$ becomes $(\beta + M(z)) \psi = \varphi$. Suppose the map $(\beta + M(z))$ is boundedly invertible as an operator in $L^2(\Gamma)$. Then the boundary equation for $\psi$ can be solved explicitly: $\psi = (\beta + M(z))^{-1} \varphi$. In turn, the solution $u^\varphi_\zeta$ is recovered from its trace $\psi = \gamma_0 u^\varphi_\zeta$ by the mapping $S_z$:

$$u^\varphi_\zeta = (I - z A_0^{-1})^{-1} \Pi \gamma_0 u^\varphi_\zeta = (I - z A_0^{-1})^{-1} \Pi (\beta + M(z))^{-1} \varphi,$$

(2.6)
where \( z \in \rho(A_0) \) is such that \((\beta + M(z))^{-1}\) exists. Observe that application of \( \gamma_1 \) to both sides of this equality yields the expression for the map \( \varphi \mapsto \gamma_1 u^\varphi \), which by analogy with the DN-map can be called the Robin-to-Neumann map:

\[
M_{RN}(z) = M(z)(\beta + M(z))^{-1}
\]

Similarly, application of \( \gamma_0 \) yields an expression for the Robin-to-Dirichlet map:

\[
M_{RD}(z) = (\beta + M(z))^{-1}
\] (2.7)

**Krein’s resolvent formula and Hilbert resolvent identity** Equations (2.5) give rise to another boundary problem, namely the problem for an unknown function \( u \) in \( \Omega \) satisfying

\[
\begin{aligned}
(A - zI)u &= f, \\
\gamma_1 u + \beta \gamma_0 u &= 0
\end{aligned}
\] (2.8)

with \( f \in L^2(\Omega) \), where \( \gamma_1 u = -\partial_\nu u = -\frac{\partial u}{\partial \nu}|_\Gamma \) and \( \gamma_0 u = u|_\Gamma \). It is customary to look for a solution to (2.8) in the form

\[
u^f = (A_0 - zI)^{-1}f + S_z \psi = (A_0 - zI)^{-1}f + (I - zA_0^{-1})^{-1}\Pi \psi \] (2.9)

with \( z \in \rho(A_0) \) and some \( \psi \in L^2(\Gamma) \) to be determined. Since \((A - zI)(A_0 - zI)^{-1}f = f \) and \((A - zI)S_z \psi = 0\), the first equation (2.8) is satisfied by (2.9) automatically; therefore we only need to find \( \psi \in L^2(\Gamma) \) such that (2.9) obeys the boundary condition in (2.8). Applying \( \gamma_0 \) and \( \gamma_1 \) to (2.9) we obtain

\[
\begin{aligned}
\gamma_0 u^f &= \gamma_0 S_z \psi = \psi, \\
\gamma_1 u^f &= \gamma_1 (A_0 - zI)^{-1}f + \gamma_1 S_z \psi = \Pi^*(I - zA_0^{-1})^{-1}f + M(z)\psi
\end{aligned}
\]

Now the relation \( \Pi^* = \gamma_1 A_0^{-1} \), properties of solution operator \( S_z \) and the definition of \( M(z) \), lead to the following equation for the unknown function \( \psi \)

\[
0 = (\gamma_1 + \beta \gamma_0) u^f = \Pi^*(I - zA_0^{-1})^{-1}f + (\beta + M(z))\psi
\]

Again, assuming \( z \in \rho(A_0) \) is such that \((\beta + M(z))\) is boundedly invertible, the formula for \( \psi \) follows:

\[
\psi = -(\beta + M(z))^{-1}\Pi^*(I - zA_0^{-1})^{-1}f
\]

Substitution into (2.9) yields the result

\[
u^f = (A_0 - zI)^{-1}f - (I - zA_0^{-1})^{-1}\Pi(\beta + M(z))^{-1}\Pi^*(I - zA_0^{-1})^{-1}f \] (2.10)

This expression certainly requires some justification as the second summand need not be smooth and thereby the normal derivative \(-\partial_\nu u^f\) that appears in the boundary condition may be undefined for some \( f \in L^2(\Omega) \). But let us defer discussion of this difficulty to the main body of the paper and turn instead to the operator-theoretic interpretation of the equations (2.8) and their solution (2.10).
The system (2.8) represents a problem of finding a vector \( u \) from the domain of operator \( A_\beta \) defined as a restriction of \( A \) to the set of functions \( u \in L^2(\Omega) \) satisfying the boundary condition \( (\gamma_1 + \beta\gamma_0)u = 0 \) in some yet undefined sense. It is clear that \( A_\beta \) also can be treated as an extension of the so-called minimal operator defined as \( A = -\Delta \) restricted to the set \( C_0^\infty(\Omega) \) of infinitely differentiable functions in \( \Omega \) that vanish in some neighborhood of \( \Gamma \) along with all their partial derivatives. Assuming for the sake of argument that each vector \( u \in D(A_\beta) \) satisfies the condition \( (\gamma_1 + \beta\gamma_0)u = 0 \) literally, that is the expression \( (\gamma_1 + \beta\gamma_0)u \) makes sense for each \( u \in D(A_\beta) \), the problem (2.8) with \( f \in L^2(\Omega) \) is the familiar resolvent equation \( (A_\beta - zI)u = f \) for the operator \( A_\beta \). Therefore the solution (2.10) for \( z \in \rho(A_\beta) \) coincides with \( (A_\beta - zI)^{-1}f \). We see that the resolvents of \( A_0 \) and \( A_\beta \) for \( z \in \rho(A_0) \cap \rho(A_\beta) \) are related by the following identity commonly known as Krein’s resolvent formula

\[
(A_\beta - zI)^{-1} = (A_0 - zI)^{-1} - (I - zA_0^{-1})^{-1} \Pi(\beta + M(z))^{-1} \Pi^*(I - zA_0^{-1})^{-1}
\]  

(2.11)

Notice that the right hand side of (2.11), depends on \( (\beta + M(z))^{-1} \) which is exactly the M-function (2.7). Under assumption of bounded invertibility of \( \beta + M(0) \) in \( L^2(\Gamma) \) we have

\[
A_\beta^{-1} = A_0^{-1} - \Pi(\beta + M(0))^{-1} \Pi^*
\]  

(2.12)

This expression shows in particular that while the difference of \( A_\beta \) and \( A_0 \) is only defined a priori on the set of smooth functions \( u \) vanishing on the boundary \( \Gamma \) along with their first derivatives where \((A_\beta - A_0)u = 0\), the difference of their inverses \( A_\beta^{-1} - A_0^{-1} \) is a nontrivial bounded operator in \( L^2(\Omega) \). As a consequence, if \( \beta = \beta^* \), then the operator \( A_\beta \) is selfadjoint as an inverse of a sum of two bounded selfadjoint operators. Moreover, the formula (2.12) can be successfully employed for the investigation into spectral properties of \( A_\beta \), as it reduces the boundary problem setting to the well-developed case of perturbation theory for bounded operators (cf. [35]).

Krein’s formula (2.11) implies another useful identity relating resolvents of \( A_0 \) and \( A_\beta \) to each other. According to the definition of solution operator \( S_\gamma \) the identity \( \gamma_0(I - zA_0^{-1})^{-1}\Pi = I \) holds for any \( z \in \rho(A_0) \). Hence, application of \( \gamma_0 \) to both sides of (2.11) leads to

\[
\gamma_0(A_\beta - zI)^{-1} = (\beta + M(z))^{-1} \Pi^*(I - zA_0^{-1})^{-1}
\]  

(2.13)

Krein’s formula can now be rewritten in the form

\[
(A_\beta - zI)^{-1} - (A_0 - zI)^{-1} = -(I - zA_0^{-1})^{-1} \Pi \gamma_0(A_\beta - zI)^{-1}
\]

By substituting the adjoint of \( S_\gamma = (I - zA_0^{-1})^{-1}\Pi \) from (2.4) we obtain the following variant of Hilbert resolvent identity for \( A_0 \) and \( A_\beta \) (cf. [30], [31])

\[
(A_0 - zI)^{-1} - (A_\beta - zI)^{-1} = [\gamma_1(A_0 - zI)^{-1}]^* \gamma_0(A_\beta - zI)^{-1}, \quad z \in \rho(A_0) \cap \rho(A_\beta)
\]  

(2.14)

Finally, notice that all considerations above are valid at least formally if the symbol \( \beta \) in the condition (2.8) represents a linear bounded operator acting on the Hilbert space \( L^2(\Gamma) \).
Observations of this section lay down a foundation for the study of boundary value problems and M-functions presented in the paper. For further convenience, this preliminary discussion concludes by summing up properties of operators $A_0$ and $\Pi$ and their relationships to the boundary maps $\gamma_0$, $\gamma_1$ that are relevant for our study.

- Operator $A_0^{-1}$ is bounded, selfadjoint, and $\ker(A_0^{-1}) = \{0\}$
- Operator $\Pi$ is bounded and $\ker(\Pi) = \{0\}$
- The intersection $\mathcal{D}(A_0) \cap \mathcal{R}(\Pi) = \mathcal{R}(A_0^{-1}) \cap \mathcal{R}(\Pi)$ is trivial
- The left inverse of $\Pi$ is the trace operator $\gamma_0$ restricted to $\mathcal{R}(\Pi)$, that is $\gamma_0 \Pi \varphi = \varphi$ for $\varphi \in L^2(\Gamma)$.
- The set $\mathcal{D}(A_0) = \mathcal{R}(A_0^{-1})$ is included into the null space of $\gamma_0$, so that $\gamma_0 A_0^{-1} = 0$
- The adjoint operator of $\Pi$ is expressed in terms of $\gamma_1$ and $A_0$ as $\Pi^* = \gamma_1 A_0^{-1}$
- Operator $\Lambda = \gamma_1 \Pi$ is selfadjoint (and unbounded) in $L^2(\Gamma)$.

Further, the spectral boundary value problem $(A - zI)u = 0$, $\gamma_0 u = \varphi$, where $A$ is an extension of $A_0$ to the set $\mathcal{D}(A_0) + \mathcal{R}(\Pi)$ defined as $Ah = 0$ for $h \in \mathcal{R}(\Pi)$, gives rise to the solution operator $S_z$ and to the M-function $M(z)$, $z \in \rho(A_0)$.

- The solution operator has the form $S_z = (I - zA_0^{-1})^{-1} \Pi$, $z \in \rho(A_0)$
- The M-function is formally defined by the equality $M(z) = \gamma_1 S_z$, $z \in \rho(A_0)$

Finally, the boundary condition associated with the expression $\gamma_1 + \beta \gamma_0$ where $\beta$ is a linear operator in $L^2(\Gamma)$ defines the Robin boundary value problem and the corresponding linear operator $A_\beta$.

- The resolvents of $A_\beta$ of $A_0$ are related by Krein’s formula (2.11) expressed in terms of M-function (2.7)
- Hilbert resolvent identity (2.14) holds.

The linear systems theory perspective As stated in Introduction, ideas underlying the operator theoretic framework employed for the paper’s purpose are partially inspired by the approach to boundary value problems found in the linear systems theory. These ideas are best illustrated by considering the following variant of problem (2.1)

\[
\begin{aligned}
Au &= f, \\
\gamma_0 u &= \varphi
\end{aligned}
\]  

(2.15)

where all participating objects are as in (2.1) and the vector $f$ is an arbitrary function from $L^2(\Omega)$. From the point of view of linear systems theory, equations (2.15) describe a linear system with the state space $H = L^2(\Omega)$, the input-output space $E = L^2(\Gamma)$ and the main operator $A$. Solutions to (2.15) are called “internal states” of the system and vectors $\varphi \in L^2(\Gamma)$ are interpreted as the system’s input. The system’s output is defined by the operator $\gamma_1$ that maps internal states of the system to elements of the input-output space $E$.

When the input in (2.15) is absent ($\varphi = 0$), the corresponding internal state is obviously $u^f = A_0^{-1} f$. This situation corresponds to the closed system, that is,
the system that is isolated from the external influences modeled by inputs \( \varphi \in E \).

The closed system still has a nontrivial output given by \( \gamma_1 : u^{f,\varphi} \mapsto \gamma_1 A_0^{-1} f = \Pi^* f \). Introduction of the non-zero input \( \varphi \in E \) in (2.19) is a way to open the system to external influences. As can be easily verified, the procedure of system opening results in an additional term in the expression for the state vectors, \( u^{f,\varphi} = A_0^{-1} f + \Pi \varphi \). The output of the system defined by the operator \( \gamma_1 \) results in the mapping from internal states to outputs in the form \( \gamma_1 : u^{f,\varphi} \mapsto \gamma_1 A_0^{-1} f + \gamma_1 \Pi \varphi \). At this point we need to take into consideration the unboundedness of the trace operator \( \gamma_1 \) and only choose inputs resulting in the outputs that belong to \( E = L^2(\Gamma) \). All such inputs (admissible inputs) therefore are functions \( \varphi \in L^2(\Gamma) \) for which the harmonic continuations \( \Pi \varphi \) into the domain \( \Omega \) possess normal derivatives with traces on \( \Gamma \) from the space \( L^2(\Gamma) \). With an appropriate choice of inputs \( \varphi \in L^2(\Gamma) \), the system’s output is determined by the map \( \gamma_1 : u^{f,\varphi} \mapsto \Pi^* f + A \varphi \), where we employed notation \( A = \gamma_1 \Pi \) introduced earlier and used the equality \( \gamma_1 A_0^{-1} = \Pi^* \).

The restriction of admissible inputs to a smaller set in this example is dictated by the choice of output operator \( \gamma_1 \) that cannot be defined on all attainable internal states \( \{ u^{f,\varphi} \mid f \in H, \varphi \in E \} \) of the system. Such a restriction however does not create any inconvenience. Quite the opposite, this feature can be perceived as an advantage of the approach, because it allows for the definition of inputs according to the particular problem at hand\(^2\) Note also that an alternative approach consists of suitable alterations of outputs that do not change essential properties of the system under investigation, and at the same time widen the set of admissible inputs (see [15], [17] in this regard for an example of “regularization procedure” applied to the system’s output).

The internal states of the linear system described by equations (2.15) are therefore represented as the sum of two components, \( u^{f,\varphi} = A_0^{-1} f + \Pi \varphi \). The first term is always a function from the domain of Dirichlet Laplacian, and the second term needs not be smooth and belong to the domain of \( A = -\Delta \) at all. It is a function from the range of the operator of harmonic continuation from the boundary, \( \Pi : L^2(\Gamma) \rightarrow L^2(\Omega) \). These two components are linearly independent in the sense of equivalence \( \{ u^{f,\varphi} = 0 \} \iff \{ A_0^{-1} f = 0, \Pi \varphi = 0 \} \). In other words, the internal states of the system are vectors from the direct sum \( D(A_0) + R(\Pi) \). In the language of linear systems theory the second summand is associated with the set of controls imposed on the system. The equality \( \text{Ker}(\Pi) = \{ 0 \} \) means that this set stands in a one-to-one correspondence with the set of all inputs. Operator \( \Pi \) that maps inputs into controls is often called the control operator.

For the "spectral" case of linear system described by equations (2.1) the system’s input are again vectors \( \varphi \in L^2(\Gamma) \) and the internal state is determined by the solution operator \( \varphi \mapsto S_{z\varphi} \) for \( z \in \rho(A_0) \), see (2.2). Following the systems

\(^2\)This situation is common in practical applications of the systems theory where the set of inputs is always subject to the real world limitations. For instance, it is clear that only smooth functions from \( L^2(\Gamma) \) can be realized in practice as the system’s inputs. Therefore the ability to choose input vectors freely conforms to the standard assumptions of systems theory.
theory language, if the output is defined by means of operator $\gamma_1$ as $\gamma_1 S_z \varphi$, then the M-function (2.3) is nothing but the transfer function of this system that maps the input $\varphi$ into the output $\gamma_1 S_z \varphi$ (for suitable $\varphi \in E$). The resolvent identity and formula $\Pi^* = \gamma_1 A_0^{-1}$ allow to rewrite (2.3) as

$$\nu(z) = A + z \Pi^* (I - z A_0^{-1})^{-1} \Pi, \quad z \in \rho(A_0)$$

(2.16)

This representation has important consequences.

First, the function (2.16) is expressed in terms of three linear operators, $A_0^{-1}$, $\Pi$, and $A$, playing very specific and well defined roles in the description of linear system corresponding to (2.1). Namely, many applications of the systems theory interpret the spectral parameter $z \in \mathbb{C}$ in (2.1) as the frequency of oscillations taking place inside $\Omega$. Typical and well known examples are classical acoustic and electromagnetic waves existing in the domain $\Omega$. The operator $A = M(0)$ then has the meaning of system’s response at zero frequency, and can be interpreted as the operator of static reaction. Since its independence on the spectral parameter it maps inputs directly to the outputs without applying any $z$-dependent (therefore, frequency dependent) transformations. In the systems theory terms it maps inputs to the outputs without applying any $z$-dependent transformations. In the systems theory terms the operator $A$ is usually called the feedthrough operator. Consequently, with a given input $\varphi \in E$ the second term in (2.16) describes oscillations of the system around its “static reaction” $A \varphi$. Notice that for $z \in \rho(A_0)$ the second term is a bounded operator in $E$. Also of interest is the observation that the feedthrough operator $A$ is independent of operators $A_0$ and $\Pi$ describing oscillations, and therefore can be chosen to suit specific requirements of the given application.

Secondly, as described above, the operator of harmonic continuation $\Pi : L^2(\Gamma) \to L^2(\Omega)$ translates inputs into controls. Its adjoint $\Pi^*$ is called the observation operator because according to (2.16) it maps internal states $S_z \varphi = (I - z A_0^{-1})^{-1} \Pi \varphi$ into the system’s output, thereby making internal states available to the external observer. The equality $\Pi^* = \gamma_1 A_0^{-1}$ is crucial for the representation (2.16) of M-operator initially defined as $\nu(z) = \gamma_1 S_z$. For the model example of the Laplacian discussed above the identity $\Pi^* = \gamma_1 A_0^{-1}$ is a consequence of Fubini’s theorem and properties of Green’s function. To ensure validity of the representation (2.16) within the general framework, the definition of abstract counterpart of $\gamma_1$ given below explicitly involves operator $\Pi^*$, see Definition 3.3.

Finally, from the theoretical point of view the system is considered a “black box,” with the transfer function being the only source of information about its internals available to the observer. It follows that the linear system defined by equations (2.1) or (2.15) with the internal states-outputs map $\gamma_1$ is completely described by the operators $A_0^{-1}$, $\Pi$, and $A$ participating in the representation (2.16) of its transfer function. In other words, the study of (2.1) from the systems theory perspective is equivalent to the study of the set $\{A_0^{-1}, \Pi, A\}$.

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3See publications [15], [17] as an example, where the authors modify the system’s output by subtracting the static reaction, thereby working with the system with the output defined as $(\gamma_1 - A \gamma_0) u^x$ and consequently with the null feedthrough operator. Here $\varphi \in E$ is the input and $u^x \in H$ is the corresponding internal state.
The transition from the system defined in terms of \( \{ A, \gamma_0, \gamma_1 \} \) to the system defined by \( \{ A^{-1}_0, \Pi, A \} \) is known in the systems theory as reciprocal transform, see [19], [77] and [73]. These two systems share the state and input-output spaces, their transfer functions coincide, but their defining operators are different. One advantage of the reciprocal transform is that it translates operators \( \{ A, \gamma_0, \gamma_1 \} \) that are often difficult to describe in practical applications into the set of well defined and closed operators \( \{ A^{-1}_0, \Pi, \Lambda \} \), two of which are bounded. For instance, the Laplacian \( A = -\Delta \) in the domain \( \Omega \) from the model example above, when defined in its “natural domain,” that is, the Sobolev space \( H^2(\Omega) \), is not a closed operator in \( L^2(\Omega) \). At the same time the mappings \( \gamma_0 \) and \( \gamma_1 \) are well defined on \( H^2(\Omega) \), although they are not closed on their domains either. The procedure of operator theoretic closure of \( A = -\Delta \) in the space \( L^2(\Omega) \) results in the operator \( \overline{A} = \text{clos}(-\Delta) \) with the domain that contains elements from \( L^2(\Omega) \setminus H^2(\Omega) \). Because the null set of a closed operator is always closed, \( \mathcal{D}(\overline{A}) \) contains at least the \( L^2 \)-closure of all harmonic functions continuous in \( \Omega \). This set includes functions that need not to possess boundary values on \( \Gamma \), so that the boundary mappings \( \gamma_0, \gamma_1 \) can not be defined on all elements from \( \mathcal{D}(\overline{A}) \). Therefore the choice of suitable domain for \( A \) and subsequent expressions for boundary operators are not always obvious (except for the simplest cases, involving a boundary space of finite dimensionality as one example). In contrast, operators of the reciprocal system \( \{ A^{-1}_0, \Pi, \Lambda \} \) are all well defined and always closed. They are the solution operator of the Dirichlet problem in the domain \( \Omega \), the operator of harmonic continuation from the boundary \( \Gamma \) into \( \Omega \), and the classical Dirichlet-to-Neumann map for the Laplacian in \( \Omega \), respectively.

References to the reciprocal transform also help to clarify the relationship between the paper’s framework and the mentioned earlier approach based on the notion of boundary triples. The starting point for the latter is the set \( \{ \overline{A}, \gamma_0, \gamma_1 \} \) (where \( \overline{A} \) is the operator-theoretic closure of \( A \)) that gives rise to an abstract Green’s formula, as opposed to the discussion below carried out on the basis of operators \( \{ A^{-1}_0, \Pi, A \} \) that define the “reciprocal” system. In order to circumvent the described above difficulties with the operator domains the earlier versions of boundary triples approach [24], [25], [26] severely limited its applicability by requesting the operator \( A \) to be closed, \( \gamma_0, \gamma_1 \) to be bounded in the graph norm of \( A \), and the ranges of \( \gamma_0, \gamma_1 \) to coincide with the boundary space \( E \). The last assumption is the most restrictive, as it automatically excludes from consideration unbounded M-functions. These limitations were removed only recently, see papers [9], [22], opening further possibilities of non trivial applications to the partial differential operators. In contrast, the approach based on the set \( \{ A^{-1}_0, \Pi, A \} \) offers a framework free of these restrictions. It not only allows one to work with closed and bounded operators, but also gives an option to selectively choose inputs from the boundary space \( E \), thus eliminating the concern of a suitable domain definition for boundary mappings and removing the assumption of closedness (and even closability) of \( A \). It is also worth mentioning that when operators \( \{ \overline{A}, \gamma_0, \gamma_1 \} \) form a ”boundary triplet,” all three of them are mutually interdependent. Their domains must be suitably chosen and
their definitions must fit together in order for the Green’s formula to hold. In the “reciprocal” approach, only two operators, \( A_0 \) and \( \Pi \), are interdependent (the intersection of their ranges must be trivial), whereas the operator \( \Lambda \) (both its action and its domain) can be selected arbitrarily. Qualitatively speaking, one may say that the boundary triples method goes “from the inside to the outside” relating elements of the state space \( H \) to elements in the boundary space \( E \) by means of operators \( \gamma_0 \) and \( \gamma_1 \), whereas the approach adopted in this paper goes in the opposite direction by introducing the control operator \( \Pi \) that maps elements from the boundary space into elements of the state space. Operator \( \Lambda \), then, as a feedthrough operator acting on the boundary space, is an arbitrary parameter that does not have to be closed and even closable.

Applications

Translation of classic boundary value problems and their solution procedures to the operator theoretic language suggests applicability of the obtained results in various settings. As one example, it seems rather natural to consider a more general type of boundary conditions (2.8) written as 

\[(a\gamma_1 + \beta\gamma_0)u = 0\]

with some linear operators \( a, \beta \) acting on \( L^2(\Omega) \) (or even bounded operator valued functions \( a(z), \beta(z) \) of the spectral parameter \( z \in \mathbb{C} \)). If \( \beta = \chi_E \) is the characteristic function of a non-empty measurable set \( E \subset \Gamma \) of positive Lebesgue surface measure on \( \Gamma \) and \( \alpha = 1 - \chi_E \), then the boundary condition above takes the form \((1 - \chi_E)\partial_n u + \chi_E u\Gamma = 0\). It describes the so-called mixed boundary value problem (Zaremba’s problem) with the Dirichlet boundary condition on \( E \) and the Neumann condition on \( \Gamma \setminus E \) (cf. [58]).

The abstract operator theoretic technique elaborated in the paper can be successfully applied to the study of boundary value problems of classic and modern complex analysis. In particular, it is possible to reformulate within the abstract framework classic problems of Poincaré, Hilbert, and Riemann for harmonic and analytic functions in bounded simply connected and sufficiently smooth domains of the complex plane, see [74]. The generic boundary conditions in the form \((a\gamma_1 + \beta\gamma_0)u = 0\) appear rather naturally in these cases.

One more example is based on the earlier study [73] and is discussed here at some length. Using the above notation, it concerns the transmission type boundary condition imposed on solutions to the equation \((-\Delta - \zeta I)u = 0\) inside and outside of \( \Omega \). It is convenient to rewrite this equation as \((A - zI)u = 0\) with \( A = -\Delta + I \) and \( z = \zeta + 1 \) for reasons that will be clarified shortly. Denote \( u^\pm \) its solutions in the domains \( \Omega^\pm \) where \( \Omega^- = \Omega \) and \( \Omega^+ = \mathbb{R}^n \setminus \bar{\Omega} \). Then the boundary condition \((\partial_n u^-)|_\Gamma - (\partial_n u^+)|_\Gamma = \varphi \) with \( \varphi \in L^2(\Gamma) \) defines a variant of transmission problem. Here \((\partial_n u^\pm)|_\Gamma\) are boundary values on \( \Gamma \) of the normal derivatives of functions \( u^\pm \) in the direction of outer normal to the domain \( \Omega \).

The solution to this problem is given by the single layer potential

\[ (S_z \varphi)(x) := \int_{\Gamma} G(x,y,z)\varphi(y)d\sigma_y, \quad x \in \mathbb{R}^n \]

where \( G(\cdot, \cdot, z) \) is the standard Green’s function of the differential operator \((-\Delta + I - zI)\) and \( d\sigma_y \) is the Euclidian surface measure on \( \Gamma \). In order to include
this problem into the paper’s framework, define operators \( \gamma_0 \) and \( \gamma_1 \) acting on linear combinations of smooth functions \( v^\pm \in L^2(\Omega^\pm) \) with the property \( v^-|_\Gamma = v^+|_\Gamma = v|_\Gamma \in C(\Gamma) \) as maps

\[
\gamma_0 : v \mapsto (\partial_\nu v^-)|_\Gamma - (\partial_\nu v^+)|_\Gamma, \quad \gamma_1 : v \mapsto v|_\Gamma
\]

where we put \( v := v^+ + v^- \in L^2(\mathbb{R}^n) \). Properties of single layer potentials are such that boundary values on \( \Gamma \) of the function \( \mathcal{S}_\pm \varphi \) taken from \( \Omega^+ \) and \( \Omega^- \) coincide almost everywhere. Moreover, the difference of boundary values of normal derivatives of \( \mathcal{S}_\pm \varphi \) from inside and outside of \( \Omega \) are equal to \( \varphi \) almost everywhere. In other words, \( \gamma_0 \mathcal{S}_\pm \varphi \) and \( \gamma_1 \mathcal{S}_\pm \varphi \) are well defined and \( \gamma_0 \mathcal{S}_\pm \varphi = \varphi \). Now it is only a matter of interpretation to treat this transmission problem as a spectral problem in the form \((2.1)\). The solution operator \( S_\varphi \) obviously coincides with \( \varphi \mapsto \mathcal{S}_\varphi \) and the choice of operator \( \gamma_1 \) made above leads to the \( M \)-function being the single layer potential restricted to \( \Gamma \), that is, \( M(\varphi) = \mathcal{S}_\varphi|_\Gamma \). Corresponding expressions for \( \Pi \) and \( M(0) = \gamma_1 \Pi \) easily follow from their definitions. More precisely, since \( \Pi = \mathcal{S}_\varphi|_{\Gamma = 0} \) we have \( \Pi \varphi = \mathcal{S}_\varphi \) and \( M(0)\varphi = \mathcal{S}_\varphi \).

The expression for \( A_0 = A|_{\ker(\gamma_0)} \) deserves further discussion. Since \( A \) is initially defined on the domain of all functions \( v \in L^2(\mathbb{R}^n) \), smooth in \( \Omega^\pm \) and continuous in \( \mathbb{R}^n \), the condition \( \gamma_0 v = 0 \) makes \( A_0 \) equal to \( -\Delta + I \) defined on the domain of standard Laplacian \( -\Delta \) in \( L^2(\mathbb{R}^n) \) (after the conventional operator closure procedure). This fact follows from the embedding theorems for Sobolev classes \( H^2 \), according to which the function \( v = v^- + v^+ \), where \( v^\pm \in H^2(\Omega^\pm) \) belongs to \( H^2(\mathbb{R}^n) \) if \( v^-|_\Gamma = v^+|_\Gamma \) and \( (\partial_\nu v^-)|_\Gamma = (\partial_\nu v^+)|_\Gamma \) almost everywhere on \( \Gamma \). Also note that the addition of identity operator \( I \) to the Laplacian \( -\Delta \) ensures bounded invertibility of \( A_0 \). Operator defined by \((2.3)\) with \( \beta = 0 \) (that is, \( \gamma_1 u = 0 \)) is the orthogonal sum of two Dirichlet Laplacians acting in \( L^2(\Omega^-) \oplus L^2(\Omega^+) \). A more general transmission problem corresponding to the boundary condition \( \alpha(v)|_\Gamma + \beta((\partial_\nu v^-)|_\Gamma - (\partial_\nu v^+)|_\Gamma) = \varphi \) with \( \varphi \in L^2(\Gamma) \) and bounded operators \( \alpha, \beta \) acting in \( E = L^2(\Gamma) \) is a particular case of problems investigated in the present paper.

It is also clear that the setting of transmission problem can be interpreted as a case of singular perturbations of quantum mechanics \[3, 4, 5\], where the “free” Laplacian defined initially in all space \( \mathbb{R}^n \) is perturbed by the “potential” supported by the surface \( \Gamma \). Various boundary conditions in the form \( (\alpha \gamma_0 + \beta \gamma_1)u = 0 \) with \( \gamma_0, \gamma_1 \) as above and suitable choice of linear operators \( \alpha, \beta \) acting in \( L^2(\Gamma) \) reflect the variety of possible “parameterizations” available in this model. Another illustration of the point of view based on the theory of singular perturbations is given in the last section.

Naturally, the same considerations are applicable to more generic elliptic differential operators in place of the Laplacian, as long as the single layer potential constructed by the Green’s function of such operators possesses the same boundary properties as the conventional “acoustic” potential \( \mathcal{S}_\pm \), see \[1, 57, 58\]. In particular, the Schrödinger operator \(-\Delta + q(x)\) in \( L^2(\mathbb{R}^n) \) with sufficiently regular real valued function \( q(x) \) satisfies this condition. It is a remarkable fact that
when \( n = 3, q \in L^\infty(\mathbb{R}^3) \) and \( \Omega = \{ x \in \mathbb{R}^3 \mid |x| < 1 \} \) the M-function defined by the theory elaborated in the paper coincides with the Weyl-Titchmarsh function of the three-dimensional Schrödinger operator obtained in [8] by the multidimensional analogue of the classical nesting procedure of the Sturm-Liouville theory [83] (see [73] for the proof). Thus the single layer potential constructed by the Green’s function of Schrödinger operator with the density supported by the unit sphere in \( \mathbb{R}^3 \) is a direct multidimensional equivalent of the celebrated Weyl-Titchmarsh \( m \)-function.

A similarly developed theory for double layer potentials results in another type of transmission boundary conditions; the M-function in this case coincides with the (unbounded) hypersingular integral operator acting in \( L^2(\Gamma) \). The “unperturbed” operator \( A_0 \) then is the “free” Laplacian acting in \( L^2(\mathbb{R}^n) \), whereas the operator defined by the condition \( \gamma_1 u = 0 \) is the orthogonal sum of two Neumann Laplacians acting in \( L^2(\Omega^-) \oplus L^2(\Omega^+) \). The interested reader is referred to the publication [73] for proofs and further details.

3 Spectral Boundary Value Problem and its M-function

This section is concerned with a framework used in the study of spectral boundary value problems conducted in Sections 5 and 6. A substantial part of the material covered here is an exposition of certain facts that can be found in the literature. For the most general perspective, the reader is referred to the works [21], [22], [23] and references therein carried out in a very generic setting of abstract boundary relations. In fact, principal results communicated here can be derived from the exhaustive treatment of [22] as a particular case. Remark 3.6 at the end of section outlines a possible approach for such a derivation and also clarifies existing relationships between [22] and the setting of present paper. The main goal of this section is to give a concise account of all relevant facts in the form convenient for the present study alongside with adequate proofs. Topics covered include the definition of spectral boundary value problem complemented by a discussion of properties of its solutions and the definition of corresponding M-function. An abstract analogue of the operator \( \gamma_1 \) from Section 2 leading to the Green’s formula and to the concept of weak solutions is elaborated in some depth. The study is conducted under the following assumption.

Let \( H, E \) be two separable Hilbert spaces, \( A_0 \) be a linear operator in \( H \) defined on the dense domain \( \mathcal{D}(A_0) \) in \( H \) and let \( \Pi : E \to H \) be a bounded linear mapping.

**Assumption 1** Suppose the following:

- Operator \( A_0 \) is selfadjoint and boundedly invertible in \( H \).
- Mapping \( \Pi \) possesses the left inverse \( \tilde{I}_0 \) defined on \( \mathcal{R}(\Pi) \) by \( \tilde{I}_0 : \Pi \varphi \mapsto \varphi, \varphi \in E \).
- The intersection of \( \mathcal{D}(A_0) \) and \( \mathcal{R}(\Pi) \) is trivial, \( \mathcal{D}(A_0) \cap \mathcal{R}(\Pi) = \{0\} \).
Remark 3.1 As shown in [22], conditions of Assumption 1 can be substantially weakened. In particular, boundary mappings $\Gamma_0$ and $\Gamma_1$ in the context of [22] are multivalued operators (linear relations) defined on the graph of operator $A$ that need not be single-valued, nor have a dense domain (compare with the definitions of $\Gamma_0$, $\Gamma_1$, and $A$ in our case below). In addition, bounded invertibility of $A_0$ is not required for validity of a number of statements found in this section.

Under Assumption 1 neither of sets $D(A_0)$ and $R(\Pi)$ coincides with the whole space $H$. In follows that $A_0$ is necessarily unbounded. Furthermore, existence of the left inverse of $\Pi$ implies $\ker(\Pi) = \{0\}$. The condition $D(A_0) \cap R(\Pi) = \{0\}$ is essential. It guarantees existence of (unbounded) projections from the direct sum $D(A_0) + R(\Pi)$ into the each component parallel to another. In turn, it ensures correctness of definitions of operators $A$ and $\Gamma_0$ in the next paragraph. Finally, note that for a non-invertible selfadjoint operator $A_0$ with a real regular point $c \in \rho(A_0)$ the invertibility condition can be easily satisfied by considering the operator $A_0 - cI$ in place of $A_0$.

Introduce two linear operators $A$ and $\Gamma_0$ on the domain $D(A) = D(\Gamma_0) \subset H$ by

$$D(A) := D(A_0) + R(\Pi) = \{A_0^{-1}f + \Pi\varphi \mid f \in H, \varphi \in E\} \quad (3.1)$$

$$A : A_0^{-1}f + \Pi\varphi \mapsto f, \quad \Gamma_0 : A_0^{-1}f + \Pi\varphi \mapsto \varphi, \quad f \in H, \varphi \in E \quad (3.2)$$

Operators $A$ and $\Gamma_0$ are extensions of $A_0$ and $\tilde{\Gamma}_0$ to $D(A)$ defined to be the null mapping on the complementary subsets $R(\Pi)$ and $D(A_0)$, respectively. Observe that $\ker(A) = R(\Pi)$ and $\ker(\Gamma_0) = R(A_0^{-1}) (= D(A_0))$ since $\ker(A|_{D(A_0)})$ and $\ker(\Gamma_0|_{R(\Pi)})$ are trivial by construction.

Definition 3.1 Spectral boundary problem associated with the pair $A_0$, $\tilde{\Gamma}_0$ satisfying Assumption 1 consists of the system of linear equations for an unknown element $u \in D(A)$

$$\begin{cases} (A - zI)u = f \\ \Gamma_0 u = \varphi \end{cases} \quad f \in H, \varphi \in E \quad (3.3)$$

where $z \in \mathbb{C}$ is the spectral parameter.

Theorem 3.1 For $z \in \rho(A_0)$ and any $f \in H$, $\varphi \in E$ there exists a unique solution $u_{z}^{f,\varphi}$ to the problem (3.3) given by the formula

$$u_{z}^{f,\varphi} = (A_0 - zI)^{-1}f + (I - zA_0^{-1})^{-1}\Pi\varphi \quad (3.4)$$

Moreover, if for some $f \in H$ and $\varphi \in E$ the vector defined by the right hand side of (3.4) is null, then $f = 0$ and $\varphi = 0$.

Proof. We will show that the first term in (3.4) is a solution to the system (3.3) with $\varphi = 0$, $f \neq 0$ and the second one solves the system (3.3) for $f = 0$, $\varphi \neq 0$. 

To that end let us verify first that \((I - zA_0^{-1})^{-1}II\varphi\) belongs to \(\text{Ker}(A - zI)\). We have

\[
(A - zI)(I - zA_0^{-1})^{-1}II\varphi = (A - zI)(I + z(A_0 - zI)^{-1})II\varphi = (A - zI + zI)II\varphi = AII\varphi = 0
\]

since \(A_0 \subset A\) and \(\text{Ker}(A) = \mathcal{R}(II)\). Therefore

\[
(A - zI)u_z^{I\varphi} = (A - zI)(A_0 - zI)^{-1}f = f
\]

For the second equation (3.3) and \(u_z^{I\varphi}\) as in (3.4),

\[
\Gamma_0 u_z^{I\varphi} = \Gamma_0 [(I - zA_0^{-1})^{-1}II\varphi] = \Gamma_0 [(I + z(A_0 - zI)^{-1})II\varphi] = \Gamma_0 II\varphi = \varphi
\]

because \(\text{Ker}(\Gamma_0) = \mathcal{D}(A_0) = \mathcal{R}((A_0 - zI)^{-1})\). Both equations (3.3) are therefore satisfied.

Uniqueness of the solution (3.4) is a direct consequence of assumption \(z \in \rho(A_0)\). For \(z = 0\) the implication \(u_z^{I\varphi} = 0 \Rightarrow f = 0, \varphi = 0\) trivially holds due to uniqueness of the decomposition \(u_z^{I\varphi} = A_0^{-1}f + II\varphi\) into the sum of two terms from disjoint sets and equalities \(\text{Ker}(A_0^{-1}) = \{0\}, \text{Ker}(II) = \{0\}\). For \(z \in \rho(A_0)\) with the help of identity \((I - zA_0^{-1})^{-1} = I + z(A_0 - zI)^{-1}\) the representation (3.4) can be rewritten as

\[
u_z^{I\varphi} = (A_0 - zI)^{-1}(f + zII\varphi) + II\varphi
\]

The first summand here belongs to \(\mathcal{D}(A_0)\) and the second to \(\mathcal{R}(II)\). Since the intersection of these two sets is trivial, the equality \(u_z^{I\varphi} = 0\) implies \(II\varphi = 0\) and thus \(\varphi = 0\). Then \((A_0 - zI)^{-1}f = 0\) and therefore \(f = 0\).

**Definition 3.2** Assuming \(z \in \rho(A_0)\) denote \(R_z = (A_0 - zI)^{-1}\) the resolvent of \(A_0\) and introduce the solution operator \(S_z : E \rightarrow E\)

\[
S_z : \varphi \mapsto (I - zA_0^{-1})^{-1}II\varphi = (I + zR_z)II\varphi, \quad \varphi \in E, z \in \rho(A_0)
\]

**Remark 3.2** An alternative name for the solution operator commonly accepted in the theory of linear symmetric operators and relations is \(\gamma\)-field, see [21], [22], [23] and references therein. The present paper follows the terminology inherited from the theory of boundary value problems [37] in order to stress out the role mapping \(S_z\) plays in the considerations below.

**Remark 3.3** Important properties of the solution operator follow from its definition and the resolvent identity (see [22], Proposition 4.11 for the general case). Suppose \(z \in \rho(A_0)\). Then \(\Gamma_0 S_z = I\) and \(\mathcal{R}(S_z) = \text{Ker}(A - zI)\). Moreover,

\[
S_z - S_{\zeta} = (z - \zeta)R_zS_{\zeta}, \quad z, \zeta \in \rho(A_0)
\]
Proof. The first claim follows from Theorem 3.1. The same theorem shows that the range of \( S_z \) is included into \( \text{Ker}(A - zI) \). To show that \( \mathcal{R}(S_z) = \text{Ker}(A - zI) \) assume \( u = A_0^{-1}f + \Pi \varphi \) with \( f \in H, \varphi \in E \) is such that \( u \in \text{Ker}(A - zI) \). Then

\[
0 = (A - zI)u = (A - zI)(A_0^{-1}f + \Pi \varphi) = (I - zA_0^{-1})f - z\Pi \varphi
\]

so that \( f = z(I - zA_0^{-1})^{-1}\Pi \varphi \). Substitution into \( u = A_0^{-1}f + \Pi \varphi \) gives

\[
u = A_0^{-1}f + \Pi \varphi = [zA_0^{-1}(I - zA_0^{-1})^{-1} + I] \Pi \varphi = (I - zA_0^{-1})^{-1}\Pi \varphi = S_z \varphi
\]

The last statement is easily verified by the direct calculation based on the resolvent identity

\[
(I - zA_0^{-1})^{-1} - (I - \zeta A_0^{-1})^{-1} = z(A_0 - zI)^{-1} - \zeta(A_0 - \zeta I)^{-1}
\]

\[
= (A_0 - zI)^{-1}(zI - \zeta(I - zA_0^{-1})(I - \zeta A_0^{-1})^{-1})
\]

\[
= (z - \zeta)(A_0 - zI)^{-1}(I - \zeta A_0^{-1})^{-1}
\]

Multiplication by \( \Pi \) from the right concludes the proof. \( \Box \)

Now an analogue of the “second boundary operator” \( \gamma_1 \) described in Section 1 can be introduced.

**Definition 3.3** Let \( A \) be a linear operator in \( E \) with the domain \( \mathcal{D}(A) \subset E \). Define the linear mapping \( \Gamma_1 \) on the subset \( \mathcal{D} := \mathcal{D}(A) + \Pi \mathcal{D}(A) \) by

\[
\Gamma_1 : A_0^{-1}f + \Pi \varphi \mapsto \Pi^*f + A\varphi, \quad f \in H, \varphi \in \mathcal{D}(A)
\] (3.6)

Note that according to this definition \( A = \Gamma_1 \Pi \) and \( \Pi = (\Gamma_1 A_0^{-1})^* \). In particular, for the solution operator \( S_z = (I - zA_0^{-1})^{-1}\Pi = A_0(A_0 - zI)^{-1}\Pi \) we obtain

\[
(S_z)^* = \Gamma_1(A_0 - zI)^{-1} = \Gamma_1 R_z, \quad z \in \rho(A_0)
\] (3.7)

**Assumption 2** Operator \( A = \Gamma_1 \Pi \) is selfadjoint (and thereby densely defined).

**Remark 3.4** In the sequel it is always assumed that the set \( \{A_0^{-1}, \Pi, A\} \) satisfies both Assumptions 1 and 2.

**Theorem 3.2** (Green’s Formula)

\[
(Au, v)_H - (u, A(v))_H = (\Gamma_1 u, \Gamma_1 v)_E - (\Gamma_0 u, \Gamma_1 v)_E, \quad u, v \in \mathcal{D}
\]

Proof. Let \( u = A_0^{-1}f + \Pi \varphi, \ v = A_0^{-1}g + \Pi \psi \) with \( f, g \in H, \varphi, \psi \in \mathcal{D}(A) \). We have \( Au = f, \ Av = g \), and due to selfadjointness of \( A_0^{-1} \) and \( A \),

\[
(Au, v)_H - (u, Av)_H = (f, A_0^{-1}g + \Pi \psi) - (A_0^{-1}f + \Pi \varphi, g) = (f, \Pi \psi) - (\Pi \varphi, g) = (\Pi^* f, \psi) - (\varphi, \Pi^* g) = (\Pi^* f + A\varphi, \psi) - (\varphi, \Pi^* g + A\psi) = (\Gamma_1 u, \Gamma_1 v) - (\Gamma_0 u, \Gamma_1 v)
\]

since both Assumptions 1 and 2 are valid. \( \Box \)
Introduction of the second boundary operator $\Gamma_1$ and Theorem 3.2 lead to the concept of weak solutions to the problem (3.3) defined as solutions to a certain “variational” problem.

**Definition 3.4** The weak solution of the problem (3.3) is an element $w_z^{f,\varphi} \in H$ satisfying

$$
(w_z^{f,\varphi}, (A_0 - \bar{z}I)v) = (f, v) + (\varphi, \Gamma_1 v) \quad \text{for any } v \in \mathcal{D}(A_0) \quad (3.8)
$$

Let us verify that this definition is consistent with the solvability statement of Theorem 3.1. In other words, we need to show that for $z \in \rho(A_0)$ the vector $u_z^{f,\varphi}$ from (3.4) solves the variational problem (3.8). Indeed, for $u_z^{f,\varphi} = R_z f + S_z \varphi$ and any $v \in \mathcal{D}(A_0)$ we have

$$
(u_z^{f,\varphi}, (A_0 - \bar{z}I)v) = (R_z f, (A_0 - \bar{z}I)v) + (S_z \varphi, (A_0 - \bar{z}I)v)
$$

$$
= (f, v) + (\varphi, (S_z)^* (A_0 - \bar{z}I)v) = (f, v) + (\varphi, \Gamma_1 v)
$$

according to (3.7) and the claim is proved.

**Remark 3.5** The notion of weak solution suggests that the applicability of representation (3.4) is wider than that described in Theorem 3.1. Firstly, rewrite the right hand side of (3.8) as

$$
(f, v)_H + (\varphi, \Gamma_1 v)_E = (A_0^{-1} f, A_0 v) + (\varphi, \Pi^* A_0 v) = (A_0^{-1} f + \Pi \varphi, A_0 v) \quad (3.9)
$$

Recall now that $\mathcal{R}(A_0) = H$. Therefore the concept of weak solutions can be extended to the case when $f$ and $\varphi$ are chosen from spaces wider than $H$ and $E$ as long as the sum $A_0^{-1} f + \Pi \varphi$ belongs to $H$. As an illustration consider a simple example when $f$ and $\varphi$ are such that both summands on the left side of (3.9) are finite. Let $H_- \supset H$ and $E_- \supset E$ be Hilbert spaces obtained by completion of $H$ and $E$ with respect to norms $\|f\|_- = \|A_0^{-1} f\|_H$ and $\|\varphi\|_- = \|\Pi \varphi\|_H$, where $f \in H$, $\varphi \in E$, correspondingly. Since both $\ker(A_0^{-1})$ and $\ker(\Pi)$ are trivial, these norms are non-degenerate. For each $v \in \mathcal{D}(A_0)$ the usual estimates hold

$$
\| (f, v) \| \leq \| A_0^{-1} f \| \cdot \| A_0 v \| = \| f \|_- \cdot \| A_0 v \|
$$

$$
| (\varphi, \Gamma_1 v) | = | (\varphi, \Gamma_1 A_0^{-1} A_0 v) | = | (\Pi \varphi, A_0 v) | \leq \| \Pi \varphi \| \cdot \| A_0 v \| = \| \varphi \|_- \cdot \| A_0 v \|
$$

Thus the right hand side of (3.9) is finite for any $v \in \mathcal{D}(A_0)$ so that $A_0^{-1} f + \Pi \varphi \in H$ as long as $f \in H_-$ and $\varphi \in E_-$. It follows that the vector $u_z^{f,\varphi} = R_z f + S_z \varphi$ defined for $z \in \rho(A_0)$ by the formula (3.4) is the weak solution of (3.3) with $f \in H_-$, $\varphi \in E_-$. Introduce the notion of M-function (M-operator) as follows.

**Definition 3.5** Operator-valued function $M(z)$ defined on the domain $\mathcal{D}(A)$ for $z \in \rho(A_0)$ by the formula

$$
M(z) \varphi = \Gamma_1 S_z \varphi = \Gamma_1 (I - z A_0^{-1})^{-1} \Pi \varphi
$$

is called the M-function of the problem (3.3).
Theorem 3.3 1. The representation is valid
\[ M(z) = A + z\Pi^*(I - zA_0^{-1})^{-1}II, \quad z \in \rho(A_0) \quad (3.10) \]
2. For each \( \varphi \in D(A) \) the vector function \( M(z)\varphi, z \in \rho(A_0) \) with values in \( E \) is analytic for.
3. For \( z, \zeta \in \rho(A_0) \) the operator \( M(z) - M(\zeta) \) is bounded and
\[ M(z) - M(\zeta) = (z - \zeta)(S_z)^*S_\zeta \]
In particular, \( \text{Im} \ M(z) = (\text{Im} z)(S_z)^*S_z \) and \( (M(z))^* = M(\bar{z}) \) where \( \text{Im} M(\cdot) \) denotes the imaginary part of operator \( M(\cdot) \).
4. For \( u_z \in \text{Ker}(A - zI) \cap D = \text{Ker}(A - zI) \cap \{ D(A_0) + \Pi D(A) \} \) the formula holds
\[ M(z)I_1u_z = I_1u_z \quad (3.11) \]
Proof. (1) The claim follows from the identities \( \Lambda = I_1II, \Pi^* = I_1A_0^{-1} \), the elementary computation
\[ (I - zA_0^{-1})^{-1} = I + z(A_0 - zI)^{-1} = I + zA_0^{-1}(I - zA_0^{-1})^{-1}, \quad z \in \rho(A_0) \]
and the definition \( M(z) = I_1(I - zA_0^{-1})^{-1}III \).
(2) As the term \( z\Pi^*(I - zA_0^{-1})^{-1}II \) is a bounded analytic operator-function of \( z \in \rho(A_0) \) the statement is a consequence of the representation obtained in (1).
(3) We have
\[
M(z) - M(\zeta) = \Pi^* [z(I - zA_0^{-1})^{-1} - \zeta(I - \zeta A_0^{-1})^{-1}] II
\]
\[
= \Pi^* (I - zA_0^{-1})^{-1} [z(I - \zeta A_0^{-1})^{-1} - \zeta(I - zA_0^{-1})^{-1}] (I - \zeta A_0^{-1})^{-1}II
\]
\[
= (z - \zeta)\Pi^* (I - zA_0^{-1})^{-1}(I - \zeta A_0^{-1})^{-1}II = (z - \zeta)(S_z)^*S_\zeta.
\]
The equality \( (M(z))^* = M(\bar{z}) \) is valid due to selfadjointness of \( A \).
(4) Any vector \( u_z \in \text{Ker}(A - zI) \) is uniquely represented in the form \( u_z = S_zI_1u_z \). In the case \( u_z \in D \) either side belongs to \( D(I_1) \). Therefore, \( I_1u_z = I_1S_zI_1u_z = M(z)I_1u_z \).

Remark 3.6 Results of [22] suggest an alternative approach to build the framework described in this section. As an illustration of this possibility, and in order to explain relationships between [22] and the present paper, let us derive the representation \( (3.10) \) for Weyl function \( M(z) \) within the scope of [22]. The key component here is the Example 6.6 of [22]. Using notations of this Example, substitution of \( D = A_0^{-1}, B = II, \) and \( E = -A \) yields the following form of boundary relation \( \Gamma : H \oplus H \to E \oplus E \)
\[ \Gamma = \left\{ \left( \frac{f}{A_0^{-1}f + II\varphi}, \left(-A\varphi - \Pi^*f\right) \right) \right\}, \quad f \in H, \varphi \in D(A) \]
Formula (3.6) of [22] splits \( \Gamma \) into two boundary mappings, \( \hat{\Gamma}_0 \) and \( \hat{\Gamma}_1 \)
\[ \hat{\Gamma}_0 = \left\{ \left( \frac{f}{A_0^{-1}f + II\varphi}, \left(\varphi \right) \right) \right\}, \quad \hat{\Gamma}_1 = \left\{ \left( \frac{f}{A_0^{-1}f + II\varphi}, \left(-A\varphi - \Pi^*f\right) \right) \right\} \]
where \( f \in H, \varphi \in \mathcal{D}(A) \). Note that the mapping \( \hat{\Gamma}_0 \) can be extended to the subset \( \{ f, A_0^{-1} f + \Pi \varphi \} \) with \( f \in H, \varphi \in E \). Comparison to expressions (3.2) and (3.6) for operators \( \hat{\Gamma}_0 \) and \( \hat{\Gamma}_1 \) clarifies relationships between \( \{ \hat{\Gamma}_0, \hat{\Gamma}_1 \} \) and \( \{ I_0, I_1 \} \). More precisely, for \( f \in H, \varphi \in \mathcal{D}(A) \)

\[
\hat{\Gamma}_0 = \begin{cases} 
\begin{pmatrix} f \\ A_0^{-1} f + \Pi \varphi \end{pmatrix}, & I_0 \begin{pmatrix} A_0^{-1} f + \Pi \varphi \\ 0 \end{pmatrix} \\
\end{cases}, \\
\hat{\Gamma}_1 = \begin{cases} 
\begin{pmatrix} f \\ A_0^{-1} f + \Pi \varphi \end{pmatrix}, & -I_1 \begin{pmatrix} A_0^{-1} f + \Pi \varphi \\ 0 \end{pmatrix} \\
\end{cases}
\]

Weyl family \( \hat{M}(z) \) corresponding to \( \Gamma \) is the relation

\[
\hat{M}(\lambda) = \{ \hat{\varphi} \in E \oplus E \mid \{ \hat{\lambda}, \hat{\varphi} \} \in \Gamma \text{ for some } \hat{\lambda} = \{ f, \lambda f \} \in H \oplus H \}
\]

(see Definition 3.3 of [22]). For any element \( \hat{\lambda} = \{ f, \lambda f \} \in H \oplus H \) the condition \( \{ \hat{f}_\lambda, \hat{\varphi} \} \in \Gamma \) implies \( \hat{\lambda} \in \mathcal{D}(\Gamma) \), which leads to the equation \( \lambda f = A_0^{-1} f + \Pi \varphi \) for vectors \( f \) and \( \varphi \). It follows that \( f = (\lambda I - A_0^{-1})^{-1} \Pi \varphi \) at least for \( \text{Im}(\lambda) \neq 0 \). If this equality holds, then the relation \( \Gamma \) takes the form

\[
\Gamma = \left\{ \begin{pmatrix} f \\ \lambda f \end{pmatrix}, \begin{pmatrix} \varphi \\ -A \varphi - \Pi^* (\lambda I - A_0^{-1})^{-1} \Pi \varphi \end{pmatrix} \right\}, \quad f \in H, \varphi \in \mathcal{D}(A)
\]

and therefore the Weyl family is the relation defined for \( \varphi \in \mathcal{D}(A) \) as

\[
\hat{M}(\lambda) = \{ \varphi, -(A + \Pi^* (\lambda I - A_0^{-1})^{-1} \Pi) \varphi \}
\]

Additionally, decomposition of \( \Gamma \) into two boundary mappings \( \hat{\Gamma}_0 \) and \( \hat{\Gamma}_1 \) yields

\[
\hat{\Gamma}_0 = \begin{cases} 
\begin{pmatrix} f \\ \lambda f \end{pmatrix}, & \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \\
\end{cases}, \\
\hat{\Gamma}_1 = \begin{cases} 
\begin{pmatrix} f \\ \lambda f \end{pmatrix}, & \begin{pmatrix} 0 \\ -(A + \Pi^* (\lambda I - A_0^{-1})^{-1} \Pi) \varphi \end{pmatrix} \\
\end{cases}
\]

and therefore \( \hat{\Gamma}_0 \hat{\lambda} = \hat{M}(\lambda) \hat{\Gamma}_1 \hat{\lambda} \) for any \( \hat{\lambda} = \{ f, \lambda f \} \in \mathcal{D}(\Gamma) \) (cf. (3.7) of [22] and (5.11) above).

Finally, the relation \( \hat{M}(\lambda) \) is the graph of a linear operator in \( E \) (also denoted \( \hat{M}(\lambda) \)) with the domain \( \mathcal{D}(\lambda) \) and \( M(z) = -\hat{M}(1/z) \), \( \text{Im}(z) \neq 0 \) where \( M(z) \) is the M-function (5.10) of boundary value problem (3.3).

4 Boundary Conditions

This section explores other types of boundary value problems for the operator \( A \) and boundary mappings \( I_0, I_1 \) introduced in Section 3. The problems under consideration are defined in terms of certain linear “boundary conditions.” More precisely, given two linear operators \( \beta_0, \beta_1 \) acting in the space \( E \) we are formally looking for solutions to the equation \( (A - zI)u = f \) satisfying condition \( (\beta_0 I_0 + \beta_1 I_1)u = \varphi \) where \( f \in H, \varphi \in E, \) and \( z \in \mathbb{C} \). The exact meaning of this
problem statement and the solvability theorem are the main results of this section. Definitions and some properties of associated \( M \)-functions are also briefly reviewed.

Everywhere below \( \beta_0, \beta_1 \) are two linear operators in \( E \) such that \( \beta_0 \) is defined on the domain \( \mathcal{D}(\beta_0) \supset \mathcal{D}(A) \) and \( \beta_1 \) is defined everywhere on \( E \) and bounded. Consider the following spectral boundary value problem for \( w \in H \) associated with the set \( \{ A_0^{-1}, \Pi, A \} \) and the pair \( (\beta_0, \beta_1) \)

\[
\begin{align*}
(A - zI)w &= f, \\
(\beta_0 I_0 + \beta_1 I_1)w &= \varphi
\end{align*}
\]

(4.1)

where \( z \in \mathbb{C} \) plays the role of a spectral parameter.

The first goal in the study of (4.1) is clarification of the equality \( (\beta_0 I_0 + \beta_1 I_1)w = \varphi \). Having this objective in mind, observe that the sum \( \beta_0 I_0 + \beta_1 I_1 \) is defined at least on \( S_z \mathcal{D}(A) \) for \( z \in \rho(A_0) \) and

\[
(\beta_0 I_0 + \beta_1 I_1)S_z\varphi = (\beta_0 + \beta_1 M(z)) \varphi, \quad \varphi \in \mathcal{D}(A)
\]

(4.2)

according to the properties of \( S_z \) and definition of \( M(z) \). Rewrite the right hand side using the representation \( M(z) = A + z\Pi^*(I - zA_0^{-1})^{-1}\Pi \) in the form

\[
(\beta_0 I_0 + \beta_1 I_1)S_z\varphi = (\beta_0 + \beta_1 A)\varphi + z\Pi^*(I - zA_0^{-1})^{-1}\Pi \varphi, \quad \varphi \in \mathcal{D}(A)
\]

(4.3)

The second term on the right is bounded for \( z \in \rho(A_0) \), thus the mapping properties of the sum \( \beta_0 I_0 + \beta_1 I_1 \) as an operator from \( H \) into \( E \) are fully determined by the map \( \beta_0 + \beta_1 A \). The following closability condition is assumed to be always satisfied.

**Assumption 3** The operator \( \beta_0 + \beta_1 A \) defined on \( \mathcal{D}(A) \) is closable in \( E \). Let \( \mathscr{B} = \beta_0 + \beta_1 A \) be its closure.

**Remark 4.1** It follows from (4.2) and (4.3) that under this assumption all operators \( \beta_0 + \beta_1 M(z) \) are also closable for \( z \in \rho(A_0) \) and the domain of their closures coincides with \( \mathcal{D}(\mathscr{B}) \). Equality (4.3) therefore can be extended to the set \( \varphi \in \mathcal{D}(\mathscr{B}) \). However, the operator sum \( \beta_0 I_0 + \beta_1 I_1 \) needs not be closed on the linear set \( \{ S_z\varphi \mid \varphi \in \mathcal{D}(\mathscr{B}) \} \) and in general cannot be treated as a sum of two separate operators, \( \beta_0 I_0 \) and \( \beta_1 I_1 \).

**Definition 4.1** Let \( \mathcal{H}_{\mathscr{B}} \) be the linear set of elements

\[
\mathcal{H}_{\mathscr{B}} = \{ A_0^{-1}f + \Pi \varphi \mid f \in H, \varphi \in \mathcal{D}(\mathscr{B}) \}
\]

Notice that since \( \mathcal{D}(A) \subseteq \mathcal{D}(\mathscr{B}) \subseteq E \), the inclusions \( \mathcal{D} \subseteq \mathcal{H}_{\mathscr{B}} \subseteq \mathcal{D}(A) \) hold, where \( \mathcal{D} = \{ A_0^{-1}f + \Pi \varphi \mid f \in H, \varphi \in \mathcal{D}(A) \} \), as defined in Section 3.

The set \( \mathcal{H}_{\mathscr{B}} \) can be turned into a (closed) Hilbert space by introducing a certain non-degenerate metric. Then the map \( \beta_0 I_0 + \beta_1 I_1 \) is bounded as an operator from \( \mathcal{H}_{\mathscr{B}} \) into \( E \). More precise result is given by the following Lemma.
Lemma 4.1 The set $\mathcal{H}$ is a Hilbert space with the norm
$$
\|u\|_{\mathcal{H}} = (\|f\|^2_{\mathcal{H}} + \|\varphi\|^2_{\mathcal{E}} + \|\mathcal{B}\varphi\|^2)^{1/2}.
$$
The operator $\beta_0 I_0 + \beta_1 I_1 : \mathcal{H} \to E$ is bounded.

Proof. The proof is based on the density of $\mathcal{D}(A)$ in the domain $\mathcal{D}(B)$ equipped with the graph norm of operator $\mathcal{B}$, which in turn implies density of $\mathcal{D}$ in $\mathcal{H}$ in the norm $\|\cdot\|_{\mathcal{B}}$.

Let $\{u_n\}_{n=1}^{\infty} \subset \mathcal{D}$ be a Cauchy sequence in the norm of $\mathcal{H}$, that is $\|u_n - u_m\|_{\mathcal{H}} \to 0$ as $n, m \to \infty$. Each vector $u_n$ is represented as the sum $u_n = A_0^{-1} f_n + \Pi \varphi_n$ with uniquely defined $f_n \in H, \varphi_n \in \mathcal{D}(A)$. We have
$$
\|u_n - u_m\|^2_{\mathcal{H}} = \|f_n - f_m\|^2 + \|\varphi_n - \varphi_m\|^2 + \|\mathcal{B}(\varphi_n - \varphi_m)\|^2 \to 0
$$
as $n, m \to \infty$.

The first summand here tends to zero, and therefore $f_n \to f_0 \in H$ for some $f_0 \in H$ as $n \to \infty$. The sum of second and third terms is the norm of $\varphi_n - \varphi_m$ in the graph norm of $\mathcal{B}$. Because operator $\mathcal{B}$ defined on $\mathcal{D}(A)$ is closeable, there exists a vector $\varphi_0 \in \mathcal{D}(\mathcal{B})$ such that $\varphi_n \to \varphi_0$ as $n \to \infty$. The limit of the sequence $\{u_n\}_{n=1}^{\infty}$ therefore is represented in the form $A_0^{-1} f_0 + \Pi \varphi_0$ where $f_0 \in H$ and $\varphi_0 \in \mathcal{D}(\mathcal{B})$. Hence $\mathcal{H}$ is closed in the norm $\|\cdot\|_{\mathcal{B}}$.

The second statement follows directly from the norm estimate for elements of $\mathcal{D}$. When $f \in H$ and $\varphi \in \mathcal{D}(A)$, the sum $u = A_0^{-1} f + \Pi \varphi$ belongs to the set $\mathcal{D}(I_0) \cap \mathcal{D}(I_1)$ and
$$
(\beta_0 I_0 + \beta_1 I_1)u = \beta_1 I_1 A_0^{-1} f + (\beta_0 I_0 + \beta_1 I_1) \Pi \varphi
$$
$$
= \beta_1 \Pi^* f + (\beta_0 + \beta_1 A) \varphi = \beta_1 \Pi^* f + \mathcal{B} \varphi.
$$

Because operator $\beta_1 \Pi^*$ is bounded, the following estimates hold
$$
\| (\beta_0 I_0 + \beta_1 I_1) u \| \leq C \| u \|_{\mathcal{H}}, \quad u = A_0^{-1} f + \Pi \varphi, \quad f \in H, \varphi \in \mathcal{D}(A).
$$

The set $\{A_0^{-1} f + \Pi \varphi \mid f \in H, \varphi \in \mathcal{D}(A)\}$ is dense in $\mathcal{H}$; hence the operator $\beta_0 I_0 + \beta_1 I_1$ is bounded as a mapping from $\mathcal{H}$ into $E$.

Remark 4.2 The symbol $\beta_0 I_0 + \beta_1 I_1$ will be used for the extension of operator of Lemma 4.1 to the space $\mathcal{H}$, although two terms in the sum $(\beta_0 I_0 + \beta_1 I_1) u$ need not exist separately for an arbitrary $u \in \mathcal{H}$.

Taking Lemma 4.1 into consideration, we shall look for solutions of the problem (4.1) that belong to $\mathcal{H}$.

Theorem 4.1 Suppose $z \in \rho(A_0)$ is such that the closed operator $\beta_0 + \beta_1 M(z)$ defined on $\mathcal{D}(\mathcal{B})$ is boundedly invertible in the space $E$. Then the problem (4.1) is uniquely solvable and the solution $w_{z, \varphi}^f \in \mathcal{H}$ is given by the formula
$$
w_{z, \varphi}^f = (A_0 - z I)^{-1} f_{z} + (I - z A_0^{-1})^{-1} \Pi \Psi_{z, \varphi}^f
$$
where $\Psi_{z, \varphi}^f$ is a vector from $\mathcal{D}(\mathcal{B})$
$$
\Psi_{z, \varphi}^f = (\beta_0 + \beta_1 M(z))^{-1} (\varphi - \beta_1 \Pi^* (I - z A_0^{-1})^{-1} f)
$$

(4.5)
Remark 4.3 According to this Theorem the problem (4.1) is reduced to the problem (3.3) with \( \varphi \) replaced by the vector \( \Psi_z^{\l,\varphi} \) defined in (4.2). This observation makes the concept of weak solutions applicable to the problem (4.1), see Definition 3.4 and Remark 3.5.

The proof of Theorem 4.1 is given at the end of this section.

In order to discuss the notion of M-operators associated with the boundary value problem (4.1) define the corresponding M-operators as follows. The solution \( w_z^{\varphi} := w_z^{\l,\varphi} \) is obtained in the closed form by putting \( f = 0 \) in (4.1) and (4.2):

\[
w_z^{\varphi} = (I - zA_0^{-1})^{-1} \Pi (\beta_0 + \beta_1 M(z))^{-1} \varphi = S_z (\beta_0 + \beta_1 M(z))^{-1} \varphi
\]

Vector \( w_z^{\varphi} \) belongs to the domain of \( I_0 \) for any \( \varphi \in E \) and

\[
I_0 w_z^{\varphi} = (\beta_0 + \beta_1 M(z))^{-1} \varphi
\]

Hence the operator \( (\beta_0 + \beta_1 M(z))^{-1} \) could be termed “\( (\beta_0 \beta_1) \)-to-(I, 0) map.” The notation “\((I, 0)\)” reflects equalities \( \beta_0 = I, \beta_1 = 0 \) that correspond to the condition \( I_0 w = 0 \) in (4.1). At the same time the inclusion \( w_z^{\varphi} \in D(I) \) needs not be valid for an arbitrary \( \varphi \in E \). However, if there exists a set of \( \varphi \in E \) such that vectors \( (\beta_0 + \beta_1 M(z))^{-1} \varphi \) lie in \( D(A) \), similar arguments lead to the definition of \( (\beta_0 \beta_1) \)-to-(0, I) map \( M(z)(\beta_0 + \beta_1 M(z))^{-1} \) that may be unbounded and even non-densely defined as an operator in \( E \).

This argumentation is easily extendable to the definition of M-operators as \( (\beta_0 \beta_1) \)-to-(\( \alpha_0 \alpha_1 \))-maps, where \( \alpha_0, \alpha_1 \) is another pair of “boundary operators” from the boundary condition \( (\alpha_0 I_0 + \alpha_1 I_1)u = \psi \). Such a map is formally given by the “linear-fractional transformation with operator coefficients” \( (\alpha_0 + \alpha_1 M(z))(\beta_0 + \beta_1 M(z))^{-1} \). The precise meaning of this formula needs to be clarified in each particular case at hand. Operator transformations of this kind (with \( z \)-dependent coefficients) are typical in the systems theory where M-functions are realized as transfer functions of linear systems, see [73, 77]. For the cases when \( \text{Ker}(\beta_0 + \beta_1 M(z)) \neq \{0\} \) relevant results are given by the boundary triplets approach in [21, 23, 25] in terms of linear boundary relations in Hilbert and Krein spaces.

The section concludes with the proof of Theorem 4.1.

Proof. As clarified in Remark 4.1 and Remark 4.3, operators \( \beta_0 + \beta_1 M(z) \) are closed on \( D(\mathcal{B}) \) simultaneously for all \( z \in \rho(A_0) \) and in accordance with Theorem 3.1 the vector \( w_z^{\l,\varphi} \) from (4.1), (4.2) is a solution to the system (3.3) with \( \varphi \) replaced by \( \Psi_z^{\l,\varphi} \). In particular, Theorem 3.1 implies that \( I_0 w_z^{\l,\varphi} = \Psi_z^{\l,\varphi} \) and the solution \( w_z^{\l,\varphi} \in \text{Ker}(A - zI) \) is unique.

Assume the vector \( \Psi_z^{\l,\varphi} \) defined by (4.5) belongs to \( D(A) \) so that \( w_z^{\l,\varphi} \in D(I) \). Then

\[
I_1 w_z^{\l,\varphi} = I_1 (A_0 - zI)^{-1} f + I_1 (I - zA_0^{-1})^{-1} \Pi \Psi_z^{\l,\varphi} = \Pi^* (I - zA_0^{-1})^{-1} f + M(z)\Psi_z^{\l,\varphi}
\]
Therefore 
\[
(\beta_0 I_0 + \beta_1 I_1)w^f_\varphi = (\beta_0 + \beta_1 M(z))\Psi^f_\varphi + \beta_1 \Pi^* (I - z A_0^{-1})^{-1} f \\
= \varphi - \beta_1 \Pi^* (I - z A_0^{-1})^{-1} f + \beta_1 \Pi^*(I - z A_0^{-1})^{-1} f = \varphi
\]

Hence both equations (4.1) are satisfied if \(\Psi^f_\varphi \in \mathcal{D}(A)\).

In the general case when \(\Psi^f_\varphi \in \mathcal{D}(\mathcal{B})\) the vector \(w^f_\varphi\) from (4.3) belongs to \(\mathcal{K}\) and therefore the expression \((\beta_0 I_0 + \beta_1 I_1)w^f_\varphi\) is well defined in accordance with Lemma 4.1. We need only show that it is equal to \(\varphi\), as required by the second equation in (4.1). Consider the sequence \(v_n \in \mathcal{D}(A), n = 0, 1, \ldots\) such that \(\Psi_n \rightarrow \Psi^f_\varphi\) in the graph norm of operator \(\mathcal{B}\). Then vectors \(w_n \in \mathcal{D}\) defined by (4.3) with \(\Psi^f_\varphi\) replaced by \(\Psi_n\) converge to \(w^f_\varphi\) in the metric of \(\mathcal{K}\) as \(n \rightarrow \infty\). Due to the boundedness of expression \((\beta_0 I_0 + \beta_1 I_1)\) as an operator from \(\mathcal{K}\) to \(E\),

\[
\lim_{n \rightarrow \infty} (\beta_0 I_0 + \beta_1 I_1)w_n = (\beta_0 I_0 + \beta_1 I_1)w^f_\varphi \tag{4.6}
\]

From the other side,

\[
(\beta_0 I_0 + \beta_1 I_1)w_n = (\beta_0 I_0 + \beta_1 I_1)((A_0 - z I)^{-1} f + (I - z A_0^{-1})^{-1} f) \\
= \beta_0 \Psi_n + \beta_1 I_1(A_0 - z I)^{-1} f + \beta_1 M(z) \Psi_n
\]

Since \(\beta_0 \Psi_n + \beta_1 M(z) \Psi_n \rightarrow (\beta_0 + \beta_1 M(z))\Psi^f_\varphi\) as \(n \rightarrow \infty\), we see that

\[
\lim_{n \rightarrow \infty} (\beta_0 I_0 + \beta_1 I_1)w_n = (\beta_0 + \beta_1 M(z))\Psi^f_\varphi + \beta_1 I_1(A_0 - z I)^{-1} f
\]

Direct substitution of \(\Psi^f_\varphi\) from (4.4) yields \((\beta_0 I_0 + \beta_1 I_1)w_n \rightarrow \varphi\) as \(n \rightarrow \infty\). In accordance with (4.6), the equality \((\beta_0 I_0 + \beta_1 I_1)w^f_\varphi = \varphi\) follows.

5 Linear Operators of Boundary Value Problems

Let \(A_{00}\) be the minimal operator defined as a restriction of \(A\) to the set of elements \(u \in \mathcal{D}\) satisfying conditions \(I_0 u = I_1 u = 0\). This section is concerned with extensions of \(A_{00}\) to operators corresponding to “boundary conditions” of the form \((\beta_0 I_0 + \beta_1 I_1)u = 0\). These operators are first defined via their resolvents given by a version of Krein’s resolvent formula [44], [48]. More conventional definitions via boundary conditions are provided in terms of extensions of \(A_{00}\). The groundwork for the study is laid down in Theorem 4.1.

**Definition 5.1** Let \(A_{00}\) be the restriction of \(A\) to the linear set

\[
\mathcal{D}(A_{00}) = \text{Ker}(I_0) \cap \text{Ker}(I_1) = \mathcal{D}(A_0) \cap \text{Ker}(I_1),
\]

that is, \(A_{00} = A|_{\mathcal{D}(A_{00})}\). We call \(A_{00}\) the minimal operator.

The next characterization of \(\mathcal{D}(A_{00})\) is more universal since it does not involve the map \(I_1\). Recall that \(\text{Ker}(A) = \mathcal{R}(\Pi)\) by definition of \(A\).
Remark 5.1 The domain \( \mathcal{D}(A_{00}) \) is described as follows
\[
\mathcal{D}(A_{00}) = \{ u \in \mathcal{D}(A_0) \mid A_0 u \perp \mathcal{R}(\Pi) \} = A_0^{-1} (\mathcal{R}(\Pi)^\perp)
\]
where \( \mathcal{R}(\Pi)^\perp \) is the orthogonal complement to the range of \( \Pi \). The range of \( A_{00} \) is closed in \( H \) and coincides with the subspace \( \mathcal{R}(\Pi)^\perp = H \ominus \text{Ker}(A) \).

Proof. Indeed, if \( u \in \mathcal{D}(A_0) \) then \( u = A_0^{-1} f \) with some \( f \in H \). The condition \( \Gamma \_1 u = 0 \) means that \( \Gamma \_1 A_0^{-1} f = 0 \), or \( f \in \text{Ker}(\Pi^*) \) (since \( \Gamma \_1 A_0^{-1} = \Pi^* \)), which is equivalent to \( f \perp \mathcal{R}(\Pi) \). The second statement holds because \( A_{00} A_0^{-1} (\mathcal{R}(\Pi)^\perp = \mathcal{R}(\Pi)^\perp = H \ominus \text{Ker}(A) \). □

Remark 5.2 The equality \( \mathcal{D}(A_{00}) = A_0^{-1} (\mathcal{R}(\Pi)^\perp \) shows in particular that the operator \( A_{00} \) does not depend on any given choice of \( \Lambda \). Moreover, \( A_{00} \) is symmetric but need not be densely defined. The operator \( A_0 \) is a selfadjoint extension of \( A_{00} \) contained in \( A \).

Relations (4.3) and (4.4) offer a rather natural way to define the resolvent of an operator associated with the “boundary condition” \((\beta_0 \Gamma_0 + \beta_1 \Gamma_1) u = 0 \). By putting \( \varphi = 0 \) and inserting (4.3) into (4.4) a suitable candidate for the role of resolvent is obtained:
\[
\mathcal{R}_{\beta_0, \beta_1}(z) = (A_0 - zI)^{-1} - (I - zA_0^{-1})^{-1} \Pi (\beta_0 + \beta_1 M(z))^{-1} (I - zA_0^{-1})^{-1} \quad (5.1)
\]

As will be shown, the operator function \( (5.1) \) is indeed the resolvent of some closed linear operator \( A_{\beta_0, \beta_1} \) in \( H \) whose domain \( \mathcal{D}(A_{\beta_0, \beta_1}) \) coincides with the set \( \text{Ker}(\beta_0 \Gamma_0 + \beta_1 \Gamma_1) \). Assuming the conditions of Theorem 4.1 are satisfied, denote
\[
Q_{\beta_0, \beta_1}(z) = - (\beta_0 + \beta_1 M(z))^{-1} \beta_1
\]
The operator-function \( Q_{\beta_0, \beta_1}(z) \) is analytic and bounded as long as \( z \in \rho(A_0) \) satisfies conditions of Theorem 4.1 The expression (5.1) for \( \mathcal{R}_{\beta_0, \beta_1}(z) \) takes the form
\[
\mathcal{R}_{\beta_0, \beta_1}(z) = R_z + S_z Q_{\beta_0, \beta_1}(z) S_z^* \quad (5.2)
\]
where \( R_z = (A_0 - zI)^{-1} \) is the resolvent of \( A_0 \) and \( S_z = (I - zA_0^{-1})^{-1} \Pi \) is the solution operator. For simplicity, the indices in \( Q_{\beta_0, \beta_1} \) will be omitted and the notation \( Q(z) \) will be used for \( Q_{\beta_0, \beta_1}(z) \) when it does not lead to confusion. An important analytical property of \( Q(z) \) is formulated in the next Lemma.

Lemma 5.1 For \( z, \zeta \in \rho(A_0) \) satisfying assumptions of Theorem 4.1 the equality holds
\[
Q(z) - Q(\zeta) = (z - \zeta) Q(z) S_z^* S_\zeta Q(\zeta)
\]
Proof. By virtue of formula (3) from Theorem 3.3 we have for \( \varphi \in \mathcal{D}(A) \)
\[
(z - \zeta) \beta_1 S_z^* S_\zeta \varphi = \beta_1 [M(z) - M(\zeta)] \varphi = (\beta_0 + \beta_1 M(z)) \varphi - (\beta_0 + \beta_1 M(\zeta)) \varphi = (\beta_0 + \beta_1 M(z)) \left[ (\beta_0 + \beta_1 M(\zeta))^{-1} - (\beta_0 + \beta_1 M(z))^{-1} \right] (\beta_0 + \beta_1 M(\zeta)) \varphi
\]
Therefore

$$(z - \zeta)Q(z)S_\zeta^* S_\zeta Q(\zeta) = \left[ (\beta_0 + \beta_1 M(\zeta))^{-1} - (\beta_0 + \beta_1 M(z))^{-1} \right] \beta_1$$

$$= Q(z) - Q(\zeta)$$
as stated.

The main Theorem of this section reads as follows.

**Theorem 5.1** Assume $z \in \rho(A_0)$ is such that the closed operator $\bar{\beta_0} + \beta_1 M(z)$ defined on $\mathcal{D}(\mathcal{B})$ is boundedly invertible in the space $E$. Then the operator function $\mathcal{R}_{\beta_0, \beta_1}(z)$ defined by (5.1) is the resolvent of a closed densely defined operator $A_{\beta_0, \beta_1}$ in $\mathcal{H}$. For $A_{\beta_0, \beta_1}$ the inclusions are valid

$$A_{\beta_0} \subset A_{\beta_0, \beta_1} \subset A,$$

The domain of $A_{\beta_0, \beta_1}$ satisfies

$$\mathcal{D}(A_{\beta_0, \beta_1}) = \{ u \in \mathcal{H} : (\beta_0 \Im \zeta + \beta_1 \Im \zeta) u = 0 \} = \text{Ker}(\beta_0 \Im \zeta + \beta_1 \Im \zeta)$$

In addition,

$$\Im \zeta(A_{\beta_0, \beta_1} - zI)^{-1} = Q(z)\Im \zeta(A_0 - zI)^{-1}$$

and the resolvent identity holds:

$$(A_{\beta_0, \beta_1} - zI)^{-1} - (A_0 - zI)^{-1} = \left[ \Im \zeta(A_0 - zI)^{-1} \right] \Im \zeta(A_{\beta_0, \beta_1} - zI)^{-1}$$

**Proof.** Operator function $\mathcal{R}(z) = \mathcal{R}_{\beta_0, \beta_1}(z)$ is bounded and analytic for suitable $z \in \mathbb{C}$. To show that $\mathcal{R}(z)$ is a resolvent, we need to check three conditions [10]. They are: 1) Ker($\mathcal{R}(z)$) = \{0\}, 2) $\mathcal{R}(\mathcal{R}(z))$ is dense in $\mathcal{H}$, and 3) the function $\mathcal{R}(z)$ satisfies the first resolvent equation

$$\mathcal{R}(z) - \mathcal{R}(\zeta) = (z - \zeta)\mathcal{R}(z)\mathcal{R}(\zeta)$$

The equality Ker($\mathcal{R}(z)$) = \{0\} follows directly from the last statement of Theorem 3.1. The same argument applied to $\mathcal{R}(z)^*$ in conjunction with boundedness of $Q(z)$ and equality Ker($\mathcal{R}(z)^*$) = $\mathcal{H} \ominus \mathcal{R}(\mathcal{R}(z))$ shows that the range of $\mathcal{R}(z)$ is dense in $\mathcal{H}$.

We shall verify the resolvent identity for $\mathcal{R}(\cdot)$ written in simplified notation (5.2).

$$\mathcal{R}(z)\mathcal{R}(\zeta) = (R_z + S_z Q(z)S_z^*) \times (R_\zeta + S_\zeta Q(\zeta)S_\zeta^*)$$

$$= R_z R_\zeta + R_z S_z Q(\zeta)S_\zeta^* + S_z Q(z)S_z^* R_\zeta + S_z Q(z)S_z^* S_\zeta Q(\zeta)S_\zeta^*$$

Multiplying by $(z - \zeta)$ and noticing that $R_z - R_\zeta = (z - \zeta)R_z R_\zeta$ due to the resolvent identity for $A_0$, the identity (5.7) is rewritten as

$$S_z Q(z)S_z^* - S_\zeta Q(\zeta)S_\zeta^* = (z - \zeta) \left[ R_z S_\zeta Q(\zeta)S_\zeta^* + S_z Q(z)S_z^* R_\zeta \right]$$

$$+ (z - \zeta) S_z Q(z)S_z^* S_\zeta Q(\zeta)S_\zeta^*$$
By virtue of (5.1), its adjoint, and Lemma 5.1, the right hand side of this equality is
\[(S_z - S_2)Q(\zeta)S^*_2 + S_2 Q(z)(S^*_2 - S^*_1) + S_z (Q(z) - Q(\zeta))S^*_1\]
which coincides with the left hand side. The existence of a closed densely defined operator \(A_{\beta_0, \beta_1}\) with the resolvent \((A_{\beta_0, \beta_1} - zI)^{-1}\) defined by (5.1) thereby is proven.

Turning to the proof of (5.3), notice that in accordance with (5.2) the range of \((A_{\beta_0, \beta_1} - zI)^{-1}\) is contained in \(\mathcal{D}(A)\) and since \(S_z f \in \text{Ker}(A - zI)\) for \(f \in H\),
\[(A - zI)(A_{\beta_0, \beta_1} - zI)^{-1} f = (A - zI)(R_z + S_z Q(z)S^*_1) f = (A - zI)R_z f = f\]
Hence \(A_g = A_{\beta_0, \beta_1} g\) for \(g \in \mathcal{D}(A_{\beta_0, \beta_1})\), which means \(A_{\beta_0, \beta_1} \subset A\).

To prove the inclusion \(\mathcal{D}(A_{\beta_0, \beta_1}) \subset \text{Ker}(\beta_0 I_0 + \beta_1 I_1)\) in (5.1), note that, as follows from (4.5) with \(\varphi = 0\), the vector \(w^*_z = (A_{\beta_0, \beta_1} - zI)^{-1} f\) is represented as
\[(A_{\beta_0, \beta_1} - zI)^{-1} f = R_z f + S_z \Psi^*_z\]
for each \(f \in H\), where \(\Psi^*_z = Q(z)S^*_z f \in \mathcal{D}(\mathcal{B})\). Therefore \(w^*_z \in \mathcal{H}_{\beta_0, \beta_1}\) and \((\beta_0 I_0 + \beta_1 I_1)w^*_z = 0\) by Theorem (4.1) with \(\varphi = 0\).
Hence \(\mathcal{D}(A_{\beta_0, \beta_1})\) is included into \(\text{Ker}(\beta_0 I_0 + \beta_1 I_1)\).

In order to prove the inverse inclusion first consider \(u \in \mathcal{D}\) in the form \(u = R_z f + S_z \varphi \in \text{Ker}(\beta_0 I_0 + \beta_1 I_1)\) with \(f \in H\) and \(\varphi \in \mathcal{D}(A)\). Then \(u \in \mathcal{D}(I_0) \cap \mathcal{D}(I_1)\) and the operator sum \(\beta_0 I_0 + \beta_1 I_1\) can be calculated for the element \(u\) termwise, i.e. \((\beta_0 I_0 + \beta_1 I_1)u = \beta_0 I_0 u + \beta_1 I_1 u,\)
\[(\beta_0 I_0 + \beta_1 I_1)u = (\beta_0 I_0 + \beta_1 I_1)(R_z f + S_z \varphi)\]
\[= \beta_0 I_0 S_z \varphi + \beta_1 I_1 (A_0 - zI)^{-1} f + \beta_1 I_1 S_z \varphi\]
\[= \beta_1 I_1^*(I - z A_0^{-1})^{-1} f + (\beta_0 + \beta_1 M(z)) \varphi\]

If \(u \in \text{Ker}(\beta_0 I_0 + \beta_1 I_1)\), the left hand side of (5.8) equals zero, so that
\[\varphi = -(\beta_0 + \beta_1 M(z))^{-1} \beta_1 I_1^*(I - z A_0^{-1})^{-1} f = Q_{\beta_0, \beta_1}(z)S^*_z f\]
by virtue of invertibility of \(\beta_0 + \beta_1 M(z)\). Thus, vector \(u = R_z f + S_z \varphi\) due to (5.2) is
\[u = (R_z + S_z Q_{\beta_0, \beta_1}(z)S^*_z) f = \mathcal{B}_{\beta_0, \beta_1}(z) f\]
Since \(\mathcal{B}_{\beta_0, \beta_1}(z)\) is the resolvent of \(A_{\beta_0, \beta_1}\), we have \(u \in \mathcal{D}(A_{\beta_0, \beta_1})\).

Consider now the general case of element \(v = R_z f + S_z \varphi \in \mathcal{H}_{\beta_0, \beta_1}, f \in H,\)
\(\varphi \in \mathcal{D}(\mathcal{B})\) so that \(v \notin \mathcal{D}\) and the operator sum \(\beta_0 I_0 + \beta_1 I_1\) calculated on \(v\) cannot be computed termwise. Since the set \(\mathcal{D}\) is dense in the Hilbert space \(\mathcal{H}_{\beta_0, \beta_1}\), see Definition (4.1) and Lemma (4.1) there exists a sequence \(v_n = R_z f_n + S_z \varphi_n\)
with \(f_n \in H\) and \(\varphi_n \in \mathcal{D}(A)\) converging to \(v\) in \(\mathcal{H}_{\beta_0, \beta_1}\) as \(n \to \infty\). It means in particular that \(\varphi_n \to \varphi\) and \((\beta_0 + \beta_1 M(z)) \varphi_n \to (\beta_0 + \beta_1 M(z)) \varphi\) for \(n \to \infty\).
Because \(\beta_0 I_0 + \beta_1 I_1\) is bounded as an operator from \(\mathcal{H}_{\beta_0, \beta_1}\) to \(E\) by virtue of Lemma (4.1) we have
\[(\beta_0 I_0 + \beta_1 I_1) v_n \to (\beta_0 I_0 + \beta_1 I_1) v, \quad n \to \infty\]
Expression for \((\beta_0 I_0 + \beta_1 I_1) v_n\) follows from (5.8)
\[(\beta_0 I_0 + \beta_1 I_1) v_n = \beta_1 I_1^*(I - z A_0^{-1})^{-1} f_n + (\beta_0 + \beta_1 M(z)) \varphi_n\]
Because of the boundedness of $\beta_1P^*(I-zA_0^{-1})^{-1}$ and closability of $\beta_0+\beta_1M(z)$ this leads to
\[
(\beta_0I_0+\beta_1I_1)v_n \to \beta_1P^*(I-zA_0^{-1})^{-1}f + (\beta_0+\beta_1M(z))\varphi, \quad n \to \infty
\]

Comparison with (5.9) gives
\[
(\beta_0I_0+\beta_1I_1)v = \beta_1P^*(I-zA_0^{-1})^{-1}f + (\beta_0+\beta_1M(z))\varphi
\]

Therefore if $v \in \text{Ker}(\beta_0I_0+\beta_1I_1)$, then under conditions of Theorem
\[
\varphi = -(\beta_0+\beta_1M(z))^{-1}\beta_1P^*(I-zA_0^{-1})^{-1}f = Q_{\beta_0,\beta_1}(z)S_\gamma^*f
\]

Hence, $v = (R_zf+S_zQ_{\beta_0,\beta_1}(z)S_\gamma^*f) = R_{\beta_0,\beta_1}(z)f$ and $v \in D(A_{\beta_0,\beta_1})$.

To prove that $A_{\beta_0} \subset A_{\beta_0,\beta_1}$ in (5.3) we need to show that any vector $u$ from $D(A_{\beta_0})$ belongs to $D(A_{\beta_0,\beta_1})$, in other words, can be represented in the form $u = (A_{\beta_0} - zI)^{-1}u$ with some $f \in H$. Suppose $u \in D(A_{\beta_0})$ and let us choose $f = (A_0 - zI)u$. Then $f = (A_0 - zI)u$ because $A_0 \subset A_0$ and
\[
(A_{\beta_0,\beta_1} - zI)^{-1}f = (A_{\beta_0,\beta_1} - zI)^{-1}(A_0 - zI)u
\]
\[
= (R_z + S_zQ(z)S_\gamma^*)(A_0 - zI)u = u + S_zQ(z)S_\gamma^*(A_0 - zI)u = u
\]

The last equality holds due to identities
\[
S_\gamma^*(A_0 - zI)u = P^*(I-zA_0^{-1})^{-1}(A_0 - zI)u = \Gamma_1u, \quad u \in \mathcal{D}
\]
and $\Gamma_1u = 0$ for $u \in D(A_{\beta_0})$. All claims (5.3) and (5.4) are proven.

Finally, in the notation above the formula (5.5) is equivalent to the already established relation $I_0w^*_\varphi = \Psi$ the resolvent identity (5.0) is obtained from (5.1) by (5.5) and equality $\Gamma_1(A_0 - zI)^{-1} = P^*(I-zA_0^{-1})^{-1}$.

\[\square\]

**Remark 5.3** Equalities (5.1) and (5.0) are correspondingly Krein’s formula and Hilbert resolvent identity for $A_0$ and $A_{\beta_0,\beta_1}$.

**Remark 5.4** Let $\tilde{\beta}_0$ and $\tilde{\beta}_1$ be two linear operators with the same properties as $\beta_0$ and $\beta_1$ in Theorem (5.1). A natural question arises as to whether the boundary conditions $(\tilde{\beta}_0I_0+\tilde{\beta}_1I_1)u=0$ discussed in Theorem. One obvious answer is that when $\tilde{\beta}_0 = C\beta_0$ and $\tilde{\beta}_1 = C\beta_1$ with some operator $C$ such that $\text{Ker}(C) = \{0\}$ then the equality $A_{\tilde{\beta}_0,\tilde{\beta}_1} = A_{\beta_0,\beta_1}$ holds because the null sets $\text{Ker}(\tilde{\beta}_0I_0+\tilde{\beta}_1I_1)$ and $\text{Ker}(\beta_0I_0+\beta_1I_1)$ are equal. Necessary and sufficient condition follows from the formula (5.1). Namely, the identity $\Pi Q_{\beta_0,\beta_1}(z)\Pi^* = \Pi Q_{\tilde{\beta}_0,\tilde{\beta}_1}(z)\Pi^*$ for $z$ in a (non-empty) domain of the complex plane is equivalent to the identity of resolvents of $A_{\beta_0,\beta_1}$ and $A_{\tilde{\beta}_0,\tilde{\beta}_1}$, thus to the equality $A_{\beta_0,\beta_1} = A_{\tilde{\beta}_0,\tilde{\beta}_1}$.

**Corollary 5.1** Assume the operator $\mathcal{B} = \beta_0 + \beta_1A$ is boundedly invertible in $E$. Then $A_{\beta_0,\beta_1}$ is boundedly invertible in $H$,
\[
A_{\beta_0,\beta_1}^{-1} = A_0^{-1} - \Pi(\beta_0 + \beta_1A)^{-1}\beta_1\Pi^* = A_0^{-1} + \Pi Q(0)\Pi^*,
\]
and \( Q(z) \) has the representation

\[
Q(z) = Q(0) + zQ(0)\Pi^*(I - zA_{\beta_0,\beta_1}^{-1})^{-1}I\Pi Q(0)
\]

at least in a small neighborhood of \( z = 0 \).

**Proof.** Noting that \( Q(0) = -(\beta_0 + \beta_1 A)^{-1} \beta_1 \) is bounded, invertibility of \( A_{\beta_0,\beta_1} \) and the formula for \( A_{\beta_0,\beta_1}^{-1} \) follow directly from (5.1) or (5.2). Existence of \( Q(z) = -(\beta_0 + \beta_1 M(z))^{-1} \beta_1 \) for small \( |z| \) results from analyticity and invertibility of \( \beta_0 + \beta_1 M(z) \) at \( z = 0 \). Lemma 5.1 with \( \zeta = 0 \) yields

\[
Q(z) = Q(0) + zQ(z)S_z^*S_0Q(0)
\]  \hfill (5.10)

Observe now that \( Q(z)S_z^* = Q(z)I_1(I_0 - zI)^{-1} \), thus according to (5.5),

\[
Q(z)S_z^* = \Gamma_0(A_{\beta_0,\beta_1} - zI)^{-1} = \Gamma_0A_{\beta_0,\beta_1}^{-1}(I - zA_{\beta_0,\beta_1}^{-1})^{-1}
\]

Formula (5.5) for \( z = 0 \) gives \( \Gamma_0A_{\beta_0,\beta_1}^{-1} = Q(0)I_1A_0^{-1} = Q(0)\Pi^* \) so that

\[
Q(z)S_z^* = Q(0)\Pi^*(I - zA_{\beta_0,\beta_1}^{-1})^{-1}
\]

In combination with \( S_0Q(0) = I\Pi Q(0) \) the expression (5.10) yields the required representation for \( Q(z) \). \( \square \)

**Corollary 5.2** Assume conditions of Corollary 5.1 are satisfied, operators \( \beta_0, \beta_1 \) and \( \Lambda \) are bounded, and \( \beta_{0*} \) is selfadjoint. Then \( A_{\beta_0,\beta_1} \) is selfadjoint.

**Proof.** Since \( A_{\beta_0,\beta_1}^{-1} - (A_{\beta_0,\beta_1}^{-1})^* = \Pi \left[ (\beta_0 + \beta_1 A)^{-1} \beta_1 - \beta_{0*}^* (\beta_0^* + \Lambda \beta_1^*)^{-1} \right] \Pi^* \)

\[
= \Pi(\beta_0 + \beta_1 A)^{-1} [\beta_1(\beta_0^* + \Lambda \beta_1^*) - (\beta_0 + \beta_1 A)\beta_1^*] (\beta_0^* + \Lambda \beta_1^*)^{-1} \Pi^* = 0
\]

under assumption \( \beta_1 \beta_0^* = \beta_0 \beta_1^* \), the operator \( A_{\beta_0,\beta_1} \) is an (unbounded) inverse of the bounded selfadjoint operator. \( \square \)

A special case of operator \( A_{\beta_0,\beta_1} \) in Theorem 5.1 with \( \beta_0 = 0, \beta_1 = I \) is of particular interest. It can be seen as an abstract analogue of the Laplacian with Neumann boundary condition from Section 2. Note that in this case \( Q(z) = -(\Lambda(M(z))^{-1}) \) and \( Q(0) = -\Lambda^{-1} \).

**Corollary 5.3** Suppose \( \Lambda \) is boundedly invertible. Then operator \( A_1 \) defined as a restriction of \( A \) to the set \( D(A_1) = \{u \in \mathcal{D} | \Gamma_1u = 0\} \) is selfadjoint and boundedly invertible. For \( z \in \rho(A_0) \cap \rho(A_1) \)

\[
(A_1 - zI)^{-1} = (A_0 - zI)^{-1} - (I - zA_0^{-1})^{-1}I\Pi(M(z))^{-1}I\Pi^*(I - zA_0^{-1})^{-1}
\]

where \( (M(z))^{-1} = \Lambda^{-1} - z\Lambda^{-1} \Pi^*(I - zA_1^{-1})^{-1}I\Pi\Lambda^{-1}, \quad z \in \rho(A_1). \) \hfill (5.11)

Moreover, for \( z \in \rho(A_0) \cap \rho(A_1) \)

\[
(A_1 - zI)^{-1} = (A_0 - zI)^{-1} - (I - zA_1^{-1})^{-1}(I - zA_1^{-1})^{-1}\pi M(z)\pi^*(I - zA_1^{-1})^{-1}, \quad (5.12)
\]

where \( \pi = (I_0A_1^{-1})^* \) is bounded with \( \mathcal{R}(\pi^*) \subset D(A). \)

In particular, \( A_1^{-1} = A_0^{-1} - \Pi\Lambda^{-1}\Pi^* = A_0^{-1} - \pi A_1 \) where both \( \Pi\Lambda^{-1}\Pi^* \) and \( \pi A_1 \) are bounded operators.
Theorem 5.2
ions are discussed in [15], [17].

M-functions [24], [25] report some general results obtained within the boundary triplet framework [23] and spectral characteristics of nonselfadjoint operators and their M-functions [24], [25] can be analytically continued from a neighborhood of the origin \( z = 0 \) to all \( z \in \rho(A_1) \). The alternative representation (5.12) is obtained from (5.11) with the help of equalities (5.5) and (5.6). Boundedness of the operator function \( \pi M(z) \pi^* \), \( z \in \rho(A_0) \), or equivalently of the operator \( \pi A \pi^* \), is ensured by the calculations

\[
\pi^* = \Gamma_0 A_1^{-1} = \Gamma_0(A_0^{-1} - \Pi \Lambda^{-1} \Pi^*) = -\Gamma_0 \Pi \Lambda^{-1} \Pi^* = -A^{-1} \Pi^*,
\]

so that \( A \pi^* = -\Pi^* \). This equality also follows from (5.5) with \( z = 0 \).

There exists a close relationship between analytical properties of the operator-function \( Q_{\beta_0, \beta_1}(z) \) and spectral characteristics of \( A_{\beta_0, \beta_1} \). For example, papers [24], [25] report some general results obtained within the boundary triplet framework when \( \beta_1 = I \), \( \beta_0 \) is closed, \( \mathcal{R}(I_0) = \mathcal{R}(I_1) = E \), and therefore \( M(z) \) is bounded. The next theorem concerning the point spectrum of \( A_{\beta_0, \beta_1} \) renders similar results in the paper’s setting. Much more complicated relationships between spectral properties of nonselfadjoint operators and their M-functions are discussed in [15], [17].

Theorem 5.2
Assume the operator \( \mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 A \) is boundedly invertible. Then for any \( z \in \rho(A_0) \) the mapping \( \varphi \mapsto S_z \varphi \) establishes an one-to-one correspondence between \( \{ \varphi \in \mathcal{D}(\mathcal{B}) \mid (\mathcal{B}_0 + \mathcal{B}_1 M(z)) \varphi = 0 \} \) and \( \text{Ker}(A_{\beta_0, \beta_1} - z I) \). In particular, \( \text{Ker}(\mathcal{B}_0 + \mathcal{B}_1 M(z)) = \{ 0 \} \) is equivalent to \( \text{Ker}(A_{\beta_0, \beta_1} - z I) = \{ 0 \} \) for \( z \in \rho(A_0) \).

Proof. We start with the observation that under the Theorem’s assumptions the operator \( Q(0) = -\mathcal{B}^{-1} \beta_1 \) is bounded. Hence, according to Corollary 5.1, \( A_{\beta_0, \beta_1} \) is boundedly invertible and \( A_{\beta_0, \beta_1}^{-1} = A_0^{-1} + \Pi Q(0) \Pi^* \).

Assume that \( (\mathcal{B}_0 + \mathcal{B}_1 M(z)) \varphi = 0 \) for some \( z \in \rho(A_0) \) and \( \varphi \in \mathcal{D}(\mathcal{B}) \). Let \( u = S_z \varphi \) be the corresponding solution to the equation \( (A - z I) u = 0 \) satisfying condition \( I_0 u = \varphi \). Then

\[
0 = (\mathcal{B}_0 + \mathcal{B}_1 M(z)) \varphi = (\mathcal{B}_0 + \mathcal{B}_1 A) \varphi + z \beta_1 \Pi^*(I - z A_0^{-1})^{-1} \Pi \varphi
\]

and therefore \( \varphi \) can be expressed in terms of \( u = (I - z A_0^{-1})^{-1} \Pi \varphi \) as follows

\[
\varphi = -z(\mathcal{B}_0 + \mathcal{B}_1 A)^{-1} \beta_1 \Pi^*(I - z A_0^{-1})^{-1} \Pi \varphi = -z \mathcal{B}^{-1} \beta_1 \Pi^* S_z \varphi = z Q(0) \Pi^* u
\]

Owing to identity \( (I - z A_0^{-1})^{-1} = I + z A_0^{-1}(I - z A_0^{-1})^{-1} \) we obtain

\[
u = (I - z A_0^{-1})^{-1} \Pi \varphi = \Pi \varphi + z A_0^{-1}(I - z A_0^{-1})^{-1} \Pi \varphi = \Pi \varphi + z A_0^{-1} S_z \varphi = z Q(0) \Pi^* u + z A_0^{-1} u = z(A_0^{-1} + \Pi Q(0) \Pi^*) u = z A_{\beta_0, \beta_1}^{-1} u
\]

It means inclusion \( u \in \mathcal{D}(A_{\beta_0, \beta_1}) \). It follows that \( (A_{\beta_0, \beta_1} - z I) u = (A - z I) u = 0 \) since \( A_{\beta_0, \beta_1} \subset A \).
Suppose now that $u \in \text{Ker}(A_{\beta_0, \beta_1} - zI)$ and denote $\varphi = \Gamma_0 u \in E$. Then $u$ has the form $u = (I-zA_0^{-1})^{-1} zI \varphi$ because $A_{\beta_0, \beta_1} \subset A$ and therefore $u \in \text{Ker}(A-zI)$. We need to show that $\varphi$ belongs to the domain of $\mathcal{B} = \beta_0 + \beta_1 A$ and $\mathcal{B} \varphi = -z \beta_1 \Pi^* u$. The equality $(A_{\beta_0, \beta_1} - zI) u = 0$ implies $(I-zA_0^{-1}) u = 0$. Hence $u = zA_0^{-1} \varphi = z (A_0^{-1} + \Pi Q(0) \Pi^*) u$. Application of $\Gamma_0$ to both sides yields $\varphi = \Gamma_0 u = zQ(0) \Pi^* u$. Recall now that $Q(0) = -(\beta_0 + \beta_1 A)^{-1} \beta_1 = -\mathcal{B}^{-1} \beta_1$ and the required identity $\mathcal{B} \varphi = -z \beta_1 \Pi^* u$ follows. \hfill $\square$

The rest of this section is devoted to the special case of operators $A_{\beta_0, \beta_1}$ inspired by the Birman-Krein-Vishik theory of extensions of positive symmetric operators \cite{11, 44, 85}. Only a simplified version of this theory is considered assuming that the extension parameter (operator $B$ below) is densely defined and boundedly invertible in the space $\mathcal{R}(\Pi)$. For the general case of the Birman-Krein-Vishik theory the reader is referred, apart from the original publications cited above, to the work \cite{35} for the exhaustive treatment and to the paper \cite{6} for an overview.

Denote $\mathcal{H} := \overline{\mathcal{R}(\Pi)} = \overline{\text{Ker}(A)}$. Recall that according to Remark \ref{5.1} the orthogonal complement of $\mathcal{H}$ is the subspace $\mathcal{H}^\perp = H \ominus \text{Ker}(A) = \mathcal{R}(A_0)$. Let $B$ be a closed densely defined operator in $\mathcal{H}$ such that $\mathcal{D}(B) \supset \Pi \mathcal{D}(A)$. Consider the restriction $L_B$ of $A$ to the set

$$\mathcal{D}(L_B) = \{A_0^{-1}(f_\perp + Bh) + h \mid f_\perp \in \mathcal{H}^\perp, h \in \Pi \mathcal{D}(A)\}$$

Since $L_B \subset A$ by definition, we have

$$L_B : A_0^{-1}(f_\perp + Bh) + h \mapsto f_\perp + Bh, \quad f_\perp \in \mathcal{H}^\perp, h \in \Pi \mathcal{D}(A) \quad (5.13)$$

Clearly $A_0 \subset L_B$ because $\mathcal{D}(A_0) = A_0^{-1} \mathcal{H}^\perp \subset \mathcal{D}(L_B)$. We would like to show that $L_B$ is closed and $L_B = A_{\beta_0, \beta_1}$ for some $\beta_0$, $\beta_1$. To simplify the matter, additional conditions of the boundedness and invertibility of $\Pi^* B \Pi$ are imposed in the following Theorem.

**Theorem 5.3** Suppose the set $\Pi \mathcal{D}(A)$ is dense in $\mathcal{H}$ and the operator $\Pi^* B \Pi$ is bounded and boundedly invertible in $E$. Then the inverse $L_B^{-1}$ exists and

$$L_B^{-1} = A_0^{-1} + \Pi (\Pi^* B \Pi)^{-1} \Pi^* \quad (5.14)$$

Moreover, $L_B = A_{\beta_0, \beta_1}$ with $\beta_1 = -I_E$ and $\beta_0 = A + \Pi^* B \Pi$. In particular, if the function

$$M_B(z) = A_B + z \Pi^* (I-zA_0^{-1})^{-1} \Pi, \quad \text{with} \quad A_B = -\Pi^* B \Pi$$

is boundedly invertible for some $z \in \rho(A_0)$, then $z \in \rho(L_B)$ and

$$(L_B - zI)^{-1} = (A_0 - zI)^{-1} - (I-zA_0^{-1})^{-1} \Pi M_B^{-1}(z) \Pi^* (I-zA_0^{-1})^{-1}$$

**Proof.** Formula \ref{5.14} is verified by direct computations.

Assuming \( u = A_0^{-1}(f_\perp + B\Pi \varphi) + \Pi \varphi \) with \( f_\perp \in \mathcal{H}_\perp = \ker(\Pi^*) \) and \( \varphi \in \mathcal{D}(\Lambda) \) we have

\[
(A_0^{-1} + \Pi(\Pi^* B\Pi)^{-1} \Pi^*) L_B u = (A_0^{-1} + \Pi(\Pi^* B\Pi)^{-1} \Pi^*)(f_\perp + B\Pi \varphi) \\
= A_0^{-1}(f_\perp + B\Pi \varphi) + \Pi(\Pi^* B\Pi)^{-1} \Pi^* B\Pi \varphi = A_0^{-1}(f_\perp + B\Pi \varphi) + \Pi \varphi = u
\]

From the other side, consider \( f \in H \) in the form \( f = f_\perp + B\Pi \varphi \) with \( f_\perp \in \mathcal{H}_\perp \), \( \varphi \in \mathcal{D}(\Lambda) \). By assumptions the set of such vectors \( f \) is dense in the space \( H \). Analogously, for the right hand side of (5.14)

\[
(A_0^{-1} + \Pi(\Pi^* B\Pi)^{-1} \Pi^*) f = (A_0^{-1} + \Pi(\Pi^* B\Pi)^{-1} \Pi^*)(f_\perp + B\Pi \varphi) = u
\]

where \( u = A_0^{-1}(f_\perp + B\Pi \varphi) + \Pi \varphi \). Application of \( L_B \) defined in (5.13) to both sides gives the desired result (here \( h = \Pi \varphi \)):

\[
L_B (A_0^{-1} + \Pi(\Pi^* B\Pi)^{-1} \Pi^*) f = L_B (A_0^{-1}(f_\perp + B\Pi \varphi) + \Pi \varphi) \\
= f_\perp + B\Pi \varphi = f
\]

The formula \( L_B^{-1} = A_0^{-1} + \Pi(\Pi^* B\Pi)^{-1} \Pi^* \) now follows from the usual density arguments.

Consider operator-function \( Q(z) = Q_{\beta_0, \beta_1}(z) \) with \( \beta_0 = \Lambda + \Pi^* B\Pi \) and \( \beta_1 = -I \).

\[
Q(z) = -(\beta_0 + \beta_1 M(z))^{-1} \beta_1 = (\Lambda + \Pi^* B\Pi - M(z))^{-1} = -(M_B(z))^{-1}
\]

Since \( Q(0) = -(M_B(0))^{-1} = -A_B^{-1} = (\Pi^* B\Pi)^{-1} \) is bounded by assumption, operators \( Q_{\beta_0, \beta_1}(z) \) exist and are bounded at least for small \( |z| \). According to Theorem 5.1 and representation (5.1) (see Corollary 5.1) the inverse \( (A_{\beta_0, \beta_1})^{-1} \) is bounded and

\[
(A_{\beta_0, \beta_1})^{-1} = A_0^{-1} + \Pi Q(0) \Pi^* = A_0^{-1} - \Pi A_B^{-1} \Pi^* = A_0^{-1} + \Pi(\Pi^* B\Pi)^{-1} \Pi^*
\]

which coincides with \( L_B^{-1} \). The last assertion again follows from Theorem 5.1 \( \square \)

Simple corollaries of Theorem 5.3 and definition of \( L_B \) are given below. Their consequences will not be pursued here, see [71], [73] for further details.

**Remark 5.5** Statements of Theorem 5.3 can be used to describe dependence of \( M \)-operator on the particular choice of \( \Lambda \) in Definition 5.3 of boundary operator \( \Gamma_1 \). Obviously, if \( B = B^* \) and the operator \( \Gamma_1 \) is defined with \( \Lambda \) replaced by \( \Lambda_B \), i. e. as \( \Gamma_1 : A_0^{-1} f + \Pi \varphi \mapsto \Pi^* f + \Lambda_B \varphi \) for \( f \in H \), \( \varphi \in \mathcal{D}(\Lambda) \), then all results remain valid with \( L_B \) playing the role of operator \( A_1 \) with \( M_B(z) \) being the \( M \)-function.

**Remark 5.6** The equality \( \Lambda = \Lambda_B \) is only possible if

\[
\Gamma_1(I + A_0^{-1} B) \Pi \varphi = 0, \quad \varphi \in \mathcal{D}(\Lambda)
\]
by virtue of representations \( \Lambda = \Gamma_1 \Pi \) and \( A_B = -\Pi^* B \Pi = -\Gamma_1 A_0^{-1} B \Pi \). It does not follow that the solution to this equation is \( B = -A_0 \). In fact, this equality contradicts the assumption \( \mathcal{D}(B) \supset \Pi \mathcal{D}(A) \) about operator \( B \) since \( \mathcal{D}(A_0) \cap \mathcal{R}(\Pi) = \{0\} \). When \( B \) is such that \( \Lambda = \Lambda_B \) then \( L_B \) and \( A_1 \) coincide due to Corollary 5.5 (or the equalities \( \beta_1 = -I, \beta_0 = 0 \) that follow from Theorem 5.3).

**Remark 5.7** The operator \( A_K \) corresponding to \( B = 0 \) is an analogue of Krein’s extension of \( A_{00} \) characterized by the boundary condition \( (\Gamma_1 - A_0)u = 0 \) see [7], [15], [20], [36]. Note that the semiboundedness of \( A_{00} \) is not required for this definition of Krein’s extension (cf. [38]). The formal equality \( B = \infty \) corresponds to the operator \( L_B = A_0 \).

6 Cayley Transform of M-function. Applications to the scattering theory

This section outlines basic results on the scattering theory for operators corresponding to boundary conditions studied in the previous section. In order to investigate selfadjoint and nonselfadjoint cases simultaneously the schema based on the functional model for linear operators suggested by S. Naboko in papers [60], [61] is proposed. The central ingredient of this schema is a dissipative operator whose B. Sz.-Nagy and C. Foias functional model [62] serves as the model space for all operators under consideration. Papers [60], [61] are devoted to the case of additive perturbations of a selfadjoint operator and offer an explicit form for the “model” dissipative operator used in the study. The crucial element of the approach is availability of a certain factorization of the perturbation which allows the subsequent model construction. These assumptions regarding perturbations render the schema of [60], [61] not applicable to linear operators associated with boundary value problems because they can not be represented as additive perturbations of one another. The obvious way to circumvent this difficulty, at least in case of elliptic boundary value problems, is to investigate inverse operators instead of the “direct” ones [12]. Since the inverses are bounded, their differences are well defined bounded operators, and the method of [60], [61] is fully applicable provided the “model” dissipative operator is suitably chosen. This chapter suggests such an operator based on considerations involving the Cayley transform of M-function. In addition, the required factorizations turn out to be direct consequences of formulas of previous section, see especially Corollary 5.1. Necessary connections to the theory of dissipative operators and functional models are established by the relationship between the M-function and the so-called characteristic function of a “minimal” symmetric operator discussed in papers [24], [32], [33], [79] in other contexts. The exposition in this section is carried out in the spirit of nonselfadjoint operator theory and concludes with a brief sketch illustrating the proposed approach in Remark 6.3.

It is convenient to begin with the following observation. Since values of \( M(z), z \in \mathbb{C}_+ \) are (possibly unbounded) operators with positive imaginary part, operators \( M(z) + iI \) are boundedly invertible for \( z \in \mathbb{C}_+ \). Moreover, a
short argument shows that the Cayley transform of $M(z)$ defined as $\Theta(z) = (M(z) - iI)(M(z) + iI)^{-1}$ is analytic and contractive for $z \in \mathbb{C}_+$. It turns out that $\Theta(z)$ for $z \in \mathbb{C}_+$ is the characteristic function of some dissipative operator $L$ in the sense of A. Štraus [78]. This fact was first observed in [42], [43], [79] for the characteristic function of Cayley transform of $A_{00}$ extended by the null map on $[\mathcal{R}(A_{00} + iI)]^1$ to the partial isometry defined everywhere in $H$, and then reformulated in the setting of boundary triplets (under assumptions $\mathcal{R}(I_0) = \mathcal{R}(I_1) = E$ and $M(z)$ bounded) for the characteristic function of respective dissipative operator in [24]. The following Theorem renders these results in the form convenient for the discussion of nonselfadjoint scattering theory at the end of this section. Note that boundedness of $M(z)$ below is not required.

**Theorem 6.1** Operator $L$ defined by the boundary condition $(I_1 - iI_0)u = 0$ according to the Theorem [78] with $\beta_0 = -iI$, $\beta_1 = I$ is dissipative and boundedly invertible. The inverse of $L$ is the operator $T = A_0^{-1} - \Pi(A - iI)^{-1}\Pi^*$. For $z \in \mathbb{C}_+$ the characteristic function of $L$ is given by the formula:

$$\Theta(z) = (A - iI)(A + iI)^{-1} + 2iz(A + iI)^{-1}\Pi^*(I - zT^*)^{-1}\Pi(A + iI)^{-1}$$

For $z \in \mathbb{C}_+$ this function coincides with the Cayley transform of $M(z)$,

$$\Theta(z) = (M(z) - iI)(M(z) + iI)^{-1}, \ z \in \mathbb{C}_+$$

Before turning to the proof, let us recall the definition of characteristic function of $L$ according to [78]. This definition is equivalent to the definition given by M. Livšic [50] and independently by B. Sz.-Nagy and C. Foias [62] and has been proven more convenient in practical applications.

Following [78], introduce a sesquilinear form $\Psi(\cdot, \cdot)$ defined on the domain $\mathcal{D}(L) \times \mathcal{D}(L)$:

$$\Psi(u, v) = \frac{1}{2i}[\langle Lu, v \rangle_H - \langle u, Lv \rangle_H], \ u, v \in \mathcal{D}(L)$$

and a linear set $\mathcal{G}(L) = \{v \in \mathcal{D}(L) \mid \Psi(u, v) = 0, \forall v \in \mathcal{D}(L)\}$. Define the linear space $\mathcal{L}(L)$ as a closure of the quotient space $\mathcal{D}(L)/\mathcal{G}(L)$ endowed with the inner product $[\xi, \eta]_\mathcal{L} = \Psi(f, g)$, where $\xi, \eta \in \mathcal{L}(L)$ and $u \in \xi$, $v \in \eta$. Obviously, $\mathcal{L}(L) = \{0\}$ if $L$ is symmetric. A *boundary space* for the operator $L$ is any linear space $\mathcal{L}$ which is isomorphic to $\mathcal{L}(L)$. A *boundary operator* for the operator $L$ is the linear map $\Gamma$ with the domain $\mathcal{D}(L)$ and the range in the boundary space $\mathcal{L}$ such that

$$[\Gamma u, \Gamma v]_\mathcal{L} = \Psi(u, v), \ u, v \in \mathcal{D}(L)$$

Let $\mathcal{L}'$ with the inner product $[\cdot, \cdot]'$ be a boundary space for $-L^*$ with the boundary operator $\Gamma'$ mapping $\mathcal{D}(L^*)$ onto $\mathcal{L}'$. A *characteristic function* of the operator $L$ is an operator-valued function $\theta$ defined on the set $\rho(L^*)$ whose values $\theta(z)$ map $\mathcal{L}$ into $\mathcal{L}'$ according to the equality

$$\theta(z)\Gamma u = \Gamma'(L^* - zI)^{-1}(L - zI)u, \ u \in \mathcal{D}(L).$$
Since the right hand side of this formula is analytic with regard to \( z \in \rho(L^*) \), the function \( \theta \) is analytic on \( \rho(L^*) \).

This construction needs to be applied to the operator \( L \) from Theorem 6.1 defined by the boundary condition \((\Gamma_1 - iI_0)u = 0\). To that end, notice that \( \beta_0 = -iI \) and \( \beta_1 = I \), and therefore \( \mathcal{B} = \beta_0 + \beta_1 A = -iI + A \) is boundedly invertible as \( A = A^* \). The operator function \( Q(z) = -(\beta_0 + \beta_1 M(z))^{-1} \beta_1 \) has the representation \( Q(z) = -(M(z) - iI)^{-1} \) and is bounded for \( z \in \mathbb{C}_- \). In accordance with Theorem 5.1 and Corollary 5.1 the inverse \( L^{-1} \) exists and

\[
L^{-1} = A_0^{-1} + \Pi Q(0) \Pi^* = A_0^{-1} - \Pi(A - iI)^{-1} \Pi^*
\]

Denote \( T = L^{-1} \) and compute the imaginary part of \( T \) defined as \( \text{Im}(T) = (T - T^*)/2i \). We have

\[
T - T^* = L^{-1} - (L^{-1})^* = -\Pi(A - iI)^{-1} \Pi^* + \Pi(A + iI)^{-1} \Pi^*
\]

\[
= \Pi(A + iI)^{-1} [(A - iI) - (A + iI)] (A - iI)^{-1} \Pi^*
\]

\[
= -2i\Pi(A + iI)^{-1}(A - iI)^{-1} \Pi^*
\]

Therefore

\[
\text{Im}(T) = \frac{T - T^*}{2i} = -\Pi(A + iI)^{-1}(A - iI)^{-1} \Pi^*
\]

which shows that \( T^* \) is dissipative:

\[
\text{Im}(T^*) = \Pi(A + iI)^{-1}(A - iI)^{-1} \Pi^* \geq 0 \tag{6.1}
\]

The proof of Theorem 6.1 is based on direct computations that closely follow the schema of A. Štraus [78].

**Proof.** Suppose \( u, v \in \mathcal{D}(L) \) and denote \( Lu = f, Lv = g \). Then \( f = Tu, g = Tv \) where \( T = L^{-1} \) and for the form \( \Psi(\cdot, \cdot) \) we have

\[
\Psi(u, v) = \frac{1}{2i}[(Lu, v) - (u, Lv)] = \frac{1}{2i}[(f, Tg) - (Tf, g)]
\]

\[
= \left( \frac{T^* - T}{2i} f, g \right) = \text{Im}(T^*) f, g = ((A - iI)^{-1} \Pi^* f, (A - iI)^{-1} \Pi^* g)
\]

\[
= ((A - iI)^{-1} \Pi^* Lu, (A - iI)^{-1} \Pi^* Lv)
\]

Thus, the boundary space \( \mathfrak{L} \) for \( L \) can be chosen as a closure of \( \mathcal{R}((A - iI)^{-1} \Pi^*) \) with the boundary operator \( \Gamma = (A - iI)^{-1} \Pi^* L \):

\[
\mathfrak{L} = \overline{\mathcal{R}((A - iI)^{-1} \Pi^* L)}, \quad \Gamma : u \mapsto (A - iI)^{-1} \Pi^* Lu, \quad u \in \mathcal{D}(L)
\]

Note that the metric in \( \mathfrak{L} \) is positive definite, and \( \mathfrak{L} \) is in fact a Hilbert space. Analogous computations for \( -(L^*)^* \) justify the following choice of boundary space \( \mathfrak{L}' \) and boundary operator \( \Gamma' \):

\[
\mathfrak{L}' = \overline{\mathcal{R}((A + iI)^{-1} \Pi^* L^*)}, \quad \Gamma' : v \mapsto (A + iI)^{-1} \Pi^* L^* v, \quad v \in \mathcal{D}(L^*)
\]
Here $\mathfrak{L}'$ is a Hilbert space.

In order to calculate the characteristic function $\Theta(z)$ of operator $L$ corresponding to this choice of boundary spaces and operators, set again $u = Tf$ with $f \in H$ so that $f = Lu$. For $z \in \rho(L^*)$ we have

$$
\Gamma'(L^* - zI)^{-1}(L - zI)u = (A + iI)^{-1}L^*(L^* - zI)^{-1}(L - zI)L^{-1}f
$$

$$
= (A + iI)^{-1}L^*(I - zT^*)^{-1}(I - zT)f
$$

$$
= (A + iI)^{-1}L^*[I + 2iz(I - zT^*)^{-1}(\text{Im}(T^*))] f
$$

$$
= (A + iI)^{-1}L^*[I + 2iz(I - zT^*)^{-1}P(A + iI)^{-1}(A - iI)^{-1}Pi^*f]
$$

$$
= [(A - iI)(A + iI)^{-1} + 2iz(A + iI)^{-1}P(A - iI)^{-1}Pi^*] \times (A - iI)^{-1}Pi^* f
$$

Since $(A - iI)^{-1}Pi^* f = (A - iI)^{-1}Pi^* Lu = \Gamma u$, this formula shows that the characteristic function of $L$ coincides with the expression in brackets, that is, the function $\Theta(z)$ from the Theorem statement.

For the verification of identity $\Theta = (M - iI)(M + iI)^{-1}$ write down the adjoint of function $\Theta$

$$
[\Theta(\bar{z})]^* = (A + iI)(A - iI)^{-1} - 2iz(A - iI)^{-1}Pi^*(I - zT)^{-1}Pi(A - iI)^{-1}, \quad z \in \mathbb{C}_-
$$

By virtue of equality $Q(0) = -(A - iI)^{-1}$ and Corollary 5.1

$$
z(A - iI)^{-1}Pi^*(I - zT)^{-1}Pi(A - iI)^{-1}
$$

$$
= zQ(0)Pi^*(I - zL)^{-1}PiQ(0) = Q(z) - Q(0) = -(M(z) - iI)^{-1} + (A - iI)^{-1}
$$

Therefore

$$
[\Theta(\bar{z})]^* = (A + iI)(A - iI)^{-1} + 2i(M(z) - iI)^{-1} - 2i(A - iI)^{-1}
$$

$$
= I + 2i(M(z) - iI)^{-1} = (M(z) + iI)(M(z) - iI)^{-1}
$$

By passing to the adjoint operators and noticing that $[M(\bar{z})]^* = M(z)$ the claimed identity follows. \Box

**Remark 6.1** The characteristic function of a linear operator is not determined uniquely [13], [62], [78]. Namely, consider two isometries $\tau : \mathfrak{L} \to \mathfrak{L}$ and $\tau' : \mathfrak{L}' \to \mathfrak{L}'$ of the boundary spaces $\mathfrak{L}$, $\mathfrak{L}'$ of operator $L$ to another pair of spaces $\mathfrak{L}$, $\mathfrak{L}'$. It is easy to see that the characteristic function of $L$ corresponding to the pair $\mathfrak{L}$, $\mathfrak{L}'$ is the function $\theta(z) = \tau' \circ \theta(z) \circ \tau^*$. In application to the characteristic function $\Theta(z)$ above observe that the operator $U = (A - iI)(A + iI)^{-1}$ is an unitary in $H$. Therefore, both functions $U^* \Theta(z)$ and $\Theta(z) U^*$, $z \in \mathbb{C}_+$

$$
U^* \Theta(z) = I + 2iz(A - iI)^{-1}Pi^*(I - zT^*)^{-1}Pi(A + iI)^{-1}
$$

$$
\Theta(z) U^* = I + 2iz(A + iI)^{-1}Pi^*(I - zT^*)^{-1}Pi(A - iI)^{-1}
$$

are characteristic functions of $L$, although corresponding to alternative choices of boundary spaces and operators.
Remark 6.2 Direct calculations according to the schema outlined above yield the following expression for the characteristic function \( \vartheta(z) \) of dissipative operator \( T^* = (L^*)^{-1} \):

\[
\vartheta(\zeta) = I + 2i(\Lambda + iI)^{-1}\Pi^*(T - \zeta I)^{-1}\Pi(\Lambda - iI)^{-1}, \quad \zeta \in \mathbb{C}_+
\]

By virtue of [6.1] this characteristic function is given in its “standard” form, which is consistent with the expression for characteristic function \( W(z) = I + 2iK^*(A^* - zI)^{-1}K \) of a bounded dissipative operator \( A = R + iQ \) with \( R = R^* \), \( Q = Q^* \geq 0 \) and \( Q = KK^* \) (or the corresponding operator node) that can be found in the literature [13], [62]. A close relationship between \( \vartheta(\zeta) \) and \( \Theta(z) \) is clarified by the substitution \( \zeta \to z = 1/\zeta \)

\[
\vartheta(1/z) = I - 2iz(A + iI)^{-1}\Pi^*(I - zT)^{-1}\Pi(A - iI)^{-1}, \quad z \in \mathbb{C}_-
\]

Comparison with the expression for the adjoint of \( [\Theta(z)U] \) leads to the identity

\[
\vartheta(1/z) = U [\Theta(z)]^*, \quad z \in \mathbb{C}_-
\]

where \( U = (\Lambda - iI)(\Lambda + iI)^{-1} \) is an unitary.

Remark 6.3 Dissipative operator \( T^* = (L^{-1})^* = A_0^{-1} - \Pi(\Lambda + iI)^{-1}\Pi^* \) can be employed for the development of scattering theory of (in general, nonselfadjoint) operators \( L^\ast \) defined by boundary conditions \( (I_1 + \varepsilon I_0)u = 0 \) with \( \varepsilon : E \to E \). Assume \( \Lambda + \varepsilon \) is boundedly invertible. Then the inverse \( T_\varepsilon = (L^\ast)^{-1} \) exists and \( T_\varepsilon = A_0^{-1} - \Pi(\Lambda + \varepsilon)^{-1}\Pi^* \) by Corollary 5.1. The functional model construction for additive perturbations [60] is fully applicable to \( A_0^{-1}, T^*, T_\varepsilon \), which makes possible development of the scattering theory for \( A_0^{-1} \) and \( T_\varepsilon \). Application of the invariance principle for the function \( t \to (1/t), t \in \mathbb{R}, t \neq 0 \) yields existence and completeness results for the local wave operators for the pairs \( (A_0, L^\ast) \), and \( (L^\ast, A_0) \). The interested reader is referred to the works [60], [61], [62], [72] for further details on the functional model of nonselfadjoint operators and its applications to the scattering theory.

7 Singular Perturbations

The schema developed in preceding sections is essentially axiomatic. The only condition imposed on the set \( \{A_0^{-1}, \Pi, \Lambda\} \) is the validity of two Assumptions from Section 6 whereas nothing specific is requested of the “boundary”. Due to this fact, our approach is applicable in situations not readily covered by the traditional boundary problems technique. For instance, it makes possible a construction of “boundary value problem” when no boundary is given a priori. Introduction of an artificial boundary is a certain form of perturbation that is not “regular” in the traditional sense. Such “singular” perturbations are typical in the open systems theory where they are identified with the open channels connecting the system with its environment [51]. From this point of view, the selfadjoint operator \( A_0 \) acting in the “inner space” \( H \) describes the “ unperturbed
system” coupled with the “external space” \( E \) by means of the “channel” operator \( \Pi : E \to H \). The “coupling” takes place at the “boundary”. More details on connections to the open systems theory can be found in [73].

This section offers an illustration of these ideas by means of an elementary example considered previously within the framework of boundary triples in [70]. We study the physical model of a quantum particle in the potential field of finite number of singular interactions modeled by Dirac’s \( \delta \)-functions. The free particle is described by the Hamiltonian operator which in this case is the “free” Laplacian acting in \( L^2(\mathbb{R}^3) \), and the point interactions define “perturbations” of the unperturbed system (see [3], [4], [5] and references therein). Within the paper’s context, the points where the interactions are situated form the “boundary” of the “boundary value problem.”

Let \( H := L^2(\mathbb{R}^3) \). Denote \( A_0 \) the selfadjoint boundedly invertible operator \( I - \Delta \) in \( H \) with domain \( D(A_0) := H^2(\mathbb{R}^3) \). The fundamental solution to the equation \( ((I - \Delta) - zI)u = 0 \), \( z \in \mathbb{C} \setminus [1, \infty) \) is the square summable function \( \varphi_{z}(x) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|x|^2}{4z}\right) \). Fix a finite set of distinct points \( x_j \in \mathbb{R}^3 \), \( j = 1, 2, \ldots, n \) and introduce \( n \) functions \( \varphi_j(x, z) := \varphi_{z}(x - x_j) \). Formally, each \( \varphi_j(x, z) \) is the solution to the partial differential equation \( ((I - \Delta) - zI)u = \delta(x - x_j) \). Any function \( \varphi_j(x, z) \) is infinitely differentiable in any domain that does not contain \( x_j \). Because of the singularity at \( x \to x_j \) functions \( \varphi_j(x, z) \) are not in \( D(A_0) \). However, for any \( z, \zeta \in \mathbb{C} \setminus [1, \infty) \) the difference \( \varphi_j(x, z) - \varphi_j(x, \zeta) \) lies in \( D(A_0) \). In the following the abridged notation \( \varphi_j \) for \( \varphi_j(x, 0) \) will be used. Notice that \( \varphi_j \) are linearly independent as elements of \( H = L^2(\mathbb{R}^3) \).

Choose the space \( E \) to be the \( n \)-dimensional Euclidian \( E = \mathbb{C}^n \) with the orthonormal basis \( \{e_j\}_1^n \) and define the operator \( \Pi : E \to H \) on \( \{e_j\}_1^n \) by \( \Pi : e_j \mapsto \varphi_j \). It follows that \( \Pi : a \mapsto \sum a_j \varphi_j \) where \( a = \sum a_j e_j \) is an element of \( E \). Since \( \mathcal{R}(\Pi) \cap D(A_0) = \{0\} \) and the inverse to \( \Pi \) is the mapping \( \sum a_j \varphi_j \mapsto \{a_j\}_1^n \). Assumption \( \Pi \) holds. Therefore we can introduce the operator \( A \) on domain \( \mathcal{D}(A) := D(A_0) + \mathcal{H} \), where \( \mathcal{H} := \mathcal{R}(\Pi) = \bigvee \varphi_j \). According to the Section \( \Pi \) \( A : A_0^{-1}f + \sum a_j \varphi_j \mapsto f \), \( f \in H \). The equality \( \text{Ker}(A) = \mathcal{H} \) can be understood literally, because \( (I - \Delta)\varphi_j = \delta(x - x_j) \) and the right hand side is supported on the set of zero Lebesgue measure in \( \mathbb{R}^3 \). Further, the boundary operator \( I_0 \) defined on \( D(I_0) = D(A) \) acts according to the rule \( I_0 : f_0 + \sum a_j \varphi_j \mapsto \{a_j\}_1^n \), where \( f_0 \in D(A_0) \) and \( \{a_j\}_1^n \in E \). Due to identity \( I_0 \varphi_j = c_j \) we have \( \text{Ker}(I_0) = D(A_0) \). The requirements \( I_0 \Pi = I_E \) and \( \Pi I_0 \varphi_j = \varphi_j \) therefore are met.

The operator \( S_z \) maps \( a \in E \) into a unique solution \( u_z \) of the equation \( (A - zI)u = 0 \) satisfying condition \( I_0u = a \). It is not difficult to see that \( S_z \) has the form

\[
S_z : \{a_j\}_1^n \mapsto u_z = \sum_j a_j \varphi_j(x, z), \quad z \in \mathbb{C} \pm
\]
Indeed, the fact $G_j(x, z) \in \text{Ker}(A - zI)$ was discussed above, and the boundary condition is verified by direct computations. For $a = \sum_j a_j e_j$ we have

$$I_0 S_e a = \sum_j a_j I_0 G_j(x, z) = \sum_j a_j I_0 G_j + \sum_j a_j I_0 (G_j(x, z) - G_j) = \sum_j a_j e_j = a$$

because $I_0 G_j = I$ and the difference $G_j(x, z) - G_j$ belongs to $\mathcal{D}(A_0)$, therefore to $\text{Ker}(I_0)$.

To calculate the adjoint $\Pi^* : H \to E$ and choose the operator $A$ in the representation $\Pi_1 = \Pi^* A + A_0 I_0$ appropriately suppose $a = \sum_j a_j e_j$ and $f \in H$. Then $(\Pi a, f) = \sum_j a_j (G_j, f) = \langle a, \sum (f, G_j)e_j \rangle$, hence $\Pi^*$ is defined as $\Pi^* : f \mapsto \sum (f, G_j)e_j$. If $f = A_0 f_0$ with some $f_0 \in \mathcal{D}(A_0)$, then $\Pi^* A_0 f_0 = \Pi^* A_0 f_0 = \sum (A_0 f_0, G_j)e_j$. Summands here are easy to compute. It follows from the properties of fundamental solutions $G_j$ that $(A_0 f_0, G_j) = f_0(x_j)$, therefore $\Pi_1|_{\mathcal{D}(A_0)} = \Pi^* A_0 : f_0 \mapsto \sum f_0(x_j)e_j$ for $f_0 \in \mathcal{D}(A_0)$.

The operator $A$ describing $\Pi_1$ restricted to the set $\mathcal{R}(\Pi)$ can be chosen arbitrarily as long as it is selfadjoint. For example, it could be taken as the identity $A = I_E$ or the null operator $A : a \mapsto 0$, $a \in E$. However, it is convenient to define the action of $\Pi_1$ on $\mathcal{R}(\Pi)$ consistently with its action on $\mathcal{D}(A_0)$. Since $\Pi_1|_{\mathcal{D}(A_0)}$ evaluates functions $f_0 \in \mathcal{D}(A_0)$ at the points $\{x_j\}_1^n$, and then builds a corresponding vector $\{f_0(x_j)\}_1^n$ in $E = \mathbb{C}^n$, we would like $\Pi_1|_{\mathcal{R}(\Pi)}$ to act similarly. Functions $G_j(x)$ are easily evaluated at $x_s$ for $s \neq j$, but $G_j$ is not defined at $x = x_j$; thus it is not possible to define $\Pi_1$ on $\mathcal{R}(\Pi) = \bigvee G_j$ to be the evaluation operator. To circumvent this problem recall that in the neighborhood of $x_j$ the function $G_j(x - x_j)$ has the following asymptotic expansion

$$G_j(x - x_j) = \frac{1}{4\pi} \exp \left( i \sqrt{z - 1} |x - x_j| \right)$$

$$\sim \frac{1}{4\pi} \left( \frac{1}{|x - x_j|} + i \sqrt{z - 1} + O(\frac{1}{|x - x_j|}) \right)$$

Define the action $\Pi_1$ on the vector $G_j(x - x_j)$ as

$$\Pi_1 : G_j(x - x_j) \mapsto \frac{i \sqrt{z - 1}}{4\pi} e_j + \sum_{s \neq j} G_s(x_j - x_s)e_s$$

where $\frac{i \sqrt{z - 1}}{4\pi}$ is the coefficient in the asymptotic expansion above corresponding to $|x - x_j|$ to the power 0. In particular, for $z = 0$

$$\Pi_1 : G_j \mapsto - \frac{1}{4\pi} e_j + \sum_{s \neq j} G_s|_{x = x_s} e_s$$

where $G_j|_{x = x_s} = G_j(x_s, 0) = \mathcal{G}_0(x_j - x_s)$, $s \neq j$. Thus for $a = \{a_j\}_1^n \in E$

$$\Pi_1 : \Pi a = \sum_j a_j G_j \mapsto \left\{ - a_j \frac{1}{4\pi} + \sum_{s \neq j} a_s G_s|_{x = x_s} \right\}_{j=1}^n$$
The next step is the calculation of M-operator of $A$. Quite analogously to the computation of $\Gamma \Pi$ above we have for $a = \{a_j\}_1^n = \sum_j a_je_j \in E$

$$\Gamma_1 : \sum_j a_j \mathcal{G}_j(x,z) \mapsto \left\{ \frac{i\sqrt{z - 1}}{4\pi} + \sum_{s \neq j} a_s \mathcal{G}_s(x_j,z) \right\} = a$$

Since $S_z a = \sum_j a_j \mathcal{G}_j(x,z)$, this formula yields for $M(z) a = \Gamma_1 S_z a$

$$M(z) a = \Gamma_1 \left( \sum_j a_j \mathcal{G}_j(x,z) \right) = \frac{1}{4\pi} \left\{ \frac{i\sqrt{z - 1}}{4\pi} + \sum_{s \neq j} a_s \exp \left( \frac{i\sqrt{z - 1}|x_j - x_s|}{|x_j - x_s|} \right) \right\} = a$$

Therefore the operator-function $M(z)$ is the $n \times n$-matrix function with elements

$$M_{js}(z) = \frac{1}{4\pi} \left\{ \begin{array}{ll} \frac{i\sqrt{z - 1}}{4\pi}, & j = s \\ \exp \left( \frac{i\sqrt{z - 1}|x_j - x_s|}{|x_j - x_s|} \right), & j \neq s \end{array} \right\}$$

By the change of variable $z \mapsto z + 1$ the matrix $M(z + 1)$ can be interpreted as the M-function of the Laplacian $-\Delta = A - I$ in $L_2(\mathbb{R}^3)$ perturbed by a set of point interactions $\{\delta(x - x_j)\}_1^n$. To elaborate more on this statement consider extensions of symmetric operator $A_{00}$ defined as $-\Delta + I$ on the domain

$$\mathcal{D}(A_{00}) = \{ u \in \mathcal{D}(A_0) \mid \Gamma_1 u = 0 \} = \{ u \in H^2(\mathbb{R}^3) \mid u(x_s) = 0, s = 1, 2, \ldots n \}$$

Suppose the operator $A^\beta$ is defined as a restriction of $A$ to domain $\mathcal{D}(A^\beta) = \{ u \in \mathcal{D}(A) \mid (\beta_0 \Gamma_0 + \beta_1 \Gamma_1) u = 0 \}$ where $\beta_0, \beta_1$ are arbitrary $n \times n$-matrices. The resolvent of $A^\beta$ is described in Theorem [5.1]. In particular, assuming that $\beta_0 + \beta_1 A$ where $A = M(0)$ is boundedly invertible, the inverse of $A^\beta$ as given by Corollary [5.1] is

$$(A^\beta)^{-1} = A^{-1} - \Pi(\beta_0 + \beta_1 A)^{-1}\beta_1 \Pi^*$$

Consider sesquilinear forms of both sides of this identity on a pair of vectors $f, g \in H$. Since $\mathcal{R}(A_0) = H$, vectors $f$ and $g$ can be represented as $f = A_0 u$, $g = A_0 v$ with some $u, v \in \mathcal{D}(A_0)$. Then the form on the right is

$$(A_0^{-1} f, g) - (\Pi(\beta_0 + \beta_1 A)^{-1}\beta_1 \Pi^* f, g) = (A_0 u, v) - ((\beta_0 + \beta_1 A)^{-1}\beta_1 \Gamma_1 u, \Gamma_1 v)$$

due to equalities $\Pi^* f = \Gamma_1 A_0^{-1} A_0 u = \Gamma_1 u$ and $\Pi^* g = \Gamma_1 v$. Notice that vectors $\Gamma_1 u$ and $\Gamma_1 v$ are known explicitly, namely $\Gamma_1 u = \{ u(x_j) \}_1^n$ and $\Gamma_1 v = \{ v(x_j) \}_1^n$. In order to clarify meaning of the form $((A^\beta)^{-1} f, g)$ of the operator on the left hand side of (7.1) we need to recall some basic concepts from the theory of scales of Hilbert spaces [10]. Introduce the rigging $H^+ \subset H \subset H^-$ of $H$ constructed by the positive boundedly invertible operator $A_0 = -\Delta + I$. The positive space $H^+$...
consists of elements from $\mathcal{D}(A_0)$ and is equipped with the norm $\|u\|_0 = \|A_0 u\|_H$, $u \in \mathcal{D}(A_0)$. It follows that $A_0$ acts as an isometry from $H^+$ onto $H$. The dual space $H^-$ is identified with the Hilbert space of all antilinear functionals over elements from $H^+$ with respect to the inner product in $H$. In the usual way, the product $(f, g)_H$ of two vectors $f, g \in H$ is naturally extended to the duality relation between $f \in H^-$ and $g \in H^+$. This construction allows one to consider a continuation $A_0^+$ of $A_0$ from the domain $\mathcal{D}(A_0)$ to the whole of $H$. The map $A_0^+$ is defined on $H$ by the formula $(A_0^+ f, v) = (f, A_0 v)$, $f \in H$, $v \in H^+$ and its range coincides with $H^-$. The sesquilinear form of $(A^\beta)^{-1}$ on the left hand side of (7.1) calculated on the pair $A_0 u, A_0 v$ now can be written as

$$((A^\beta)^{-1} A_0 u, A_0 v) = (A_0^+ (A^\beta)^{-1} A_0 u, v), \quad u, v \in H^+$$

Thus the operator $\\mathcal{A}^\beta := A_0^+ (A^\beta)^{-1} A_0$ acts from $H^+$ into $H^-$ and its sesquilinear form is

$$(\\mathcal{A}^\beta u, v) = (u, v) + (-\Delta u, v) + \sum_{j,k} \alpha_{jk} u(x_k) \bar{v}(x_j), \quad u, v \in H^2(\mathbb{R}^3) \quad (7.2)$$

where $\alpha_{jk}$ are the matrix elements of the operator $-(\beta_0 + \beta_1 A)^{-1} \beta_1$ in the basis $\{e_j\}^n$.

Formula (7.2) relates ideas of this section to the conventional theory of point interactions. It is easily seen that the mapping $L^\beta = A_0^+ (A^\beta)^{-1} A_0$ is formally represented as $-\Delta + I + \alpha (\cdot, \delta) \delta$ where $\delta = \{\delta(x-x_j)\}^n$ and $\alpha$ is the matrix $\alpha = \|\alpha_{jk}\|$. Non-diagonal elements of $\alpha$ describe pairwise interactions between points $\{x_j\}$ themselves (the so called “non-local model” [40]), whereas the standard case of $n$ mutually independent point interactions is recovered from (7.2) when the matrix $\alpha$ is diagonal. Under assumption $\beta_0 \beta_1^* = \beta_1 \beta_0^*$ the operator $A^\beta$ is selfadjoint according to Corollary 5.2. Finally, Theorem 5.2 reduces the question of point spectrum of $A^\beta$ to the study of $\det(\beta_0 + \beta_1 M(z))$, where $M(z)$ is the M-function discussed above. The point spectrum in the case $\beta_1 = I$ and the matrix $\beta_0$ diagonal was investigated in the work [82].

Notice in conclusion that considerations of this section suggest a consistent way to construct singular perturbations of differential operators by “potentials” supported by sets of Lebesgue measure zero in $\mathbb{R}^n$, cf. [3].

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