Cycle Embedding in Generalized Recursive Circulant Graphs

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SUMMARY Generalized recursive circulant graphs (GRCGs for short) are a generalization of recursive circulant graphs and provide a new type of topology for interconnection networks. A graph of $n$ vertices is said to be $s$-pancyclic for some $3 \leq s \leq n$ if it contains cycles of every length $t$ for $s \leq t \leq n$. The pancyclicity of recursive circulant graphs was investigated by Araki and Shibata (Inf. Process. Lett. vol.81, no.4, pp.187–190, 2002). In this paper, we are concerned with the $s$-pancyclicity of GRCGs.

key words: interconnection networks, generalized recursive circulant graphs, recursive circulant graphs, cycle embedding, pancyclicity

1. Introduction

Interconnection networks are usually modeled as undirected simple graphs $G = (V, E)$, where the vertex set $V(= V(G))$ and the edge set $E(= E(G))$ represent the set of processors and the set of communication channels between processors, respectively. For graph embedding, it has many practical applications such as allocating concurrent processes to processors in networks, transplanting or simulating parallel algorithms developed for one network to a different one, and so forth. In particular, cycle embedding is an extension of the theoretical research on Hamiltonicity in graph theory. A ring structure is often used as an interconnection architecture for local area network and as a control and data flow structure in parallel and distributed networks. Many efficient algorithms with low communication cost have been developed based on the ring structure. Accordingly, cycle embedding is an important measurement in evaluating the efficiency of interconnection networks (see [13], [28], [29] and quotes therein).

Adding chords to a ring is to enhance its fault-tolerance capability. A circulant graph $C(n; c_1, c_2, \ldots, c_k)$ is a ring of $n$ ($\geq 3$) vertices in which vertices $u$ and $v$ are adjacent if and only if $u \equiv v \pm c_i \pmod{n}$ where $u, v \in \{0, 1, \ldots, n-1\}$ and $1 \leq c_i < c_{i+1} \leq \lfloor n/2 \rfloor$ for $1 \leq i \leq k − 1$ (see [7] and [8, pp. 73–75]). For example, two circulant graphs $C(24; 1, 3, 12)$ and $C(24; 1, 4, 8)$ are shown in Figs. 1(a) and 1(b), respectively. In this figure, vertices are labeled by serial numbers within a circle. Since the two graphs belong to a class of generalized recursive circulant graphs (defined later in Sect. 2), there is another labeling of vertices to simplify the structure representation. Circulant graphs, which are vertex-symmetric, form a subclass of Cayley graphs [1], [4].

A subclass of circulant graphs with recursive struc-
ture was proposed by Park and Chwa [22] in the earlier stage. A circulant graph is called a recursive circulant graph (RCG for short), denoted by \( R(c dm, d) \), if \( n = c \cdot d^m \) with positive integers \( c < d \) such that \( c_i = d^{i-1} \) for \( i = 1, 2, \ldots, k \) and \( k = \lceil \log_d n \rceil \). Note that if \( c > 1 \), then \( k = m + 1 \); otherwise, \( k = m \). An RCG can also be constructed by \( d \) disjoint copies of \( R(c dm-1, d) \) recursively, this is the reason why the name of these graphs contains the word “recursive”. In \( R(c dm, d) \), since \( c_1 = d^0 = 1 \), all edges of \((u, u + 1 \pmod{n})\) form a Hamiltonian cycle and we call this Hamiltonian cycle the basic cycle. As a famous network topology, properties and algorithms on RCGs have been widely studied, such as Hamiltonian decomposition [5], [20], super-connectivity [27], fault-tolerant Hamiltonicity [10], [21], [26], independent spanning trees [30], [31], disjoint path covers [16], [17], and routine and broadcasting schemes [12], [15], [23], [24]. Moreover, embedding schemes on RCGs are of particular interest for many researches, e.g., path and cycle embeddings [2], [3], [21], tree embeddings [14], [18], and hypercube and mesh embeddings [23].

Although the class of RCGs can provide a good flexibility on designing network systems, due to the restriction of \( 0 < c < d \) we can find that \( C(12; 1, 4) \) is an RCG (i.e., it is isomorphic with \( R(3 \cdot 4^1, 4) \)), while \( C(12; 1, 3) \) is not. To remove this restriction, Tang et al. [25] proposed a more generalized definition of RCGs, called generalized recursive circulant graphs (defined later in Sect. 2), which can also be constructed recursively.

To study cycle embedding, we need the following graph terminology. A graph of \( n \) vertices is said to be s-pancyclic for some \( 3 \leq s \leq n \) if it contains cycles of every length \( t \) for \( s \leq t \leq n \). If \( s = 3 \), then s-pancyclic is the so-called pancyclic [3], [21], [29]. A graph is called bipancyclic if it has cycles of every even length.

2. Generalized Recursive Circulant Graphs

A \( k \)-dimensional generalised recursive circulant graph (k-GRCG for short) is denoted by \( GR(h_k, h_{k-1}, \ldots, h_1) \), where \( h_i \geq 2 \) is the size of the \( i \)th dimension for \( 1 \leq i \leq k \). Every vertex \( x \) in the graph is labeled by a \( k \)-tuple \((x_k, x_{k-1}, \ldots, x_1)\) with \( 0 \leq x_i \leq h_i - 1 \) for \( 1 \leq i \leq k \), which is a mixed radix number representation, such that it is adjacent to vertices \((x_k, x_{k-1} + 1, \ldots, x_1)\) and \((x_k, x_{k-1} - 1, \ldots, x_1)\), where the addition and subtraction in each dimension are with the carry and borrow. That is, for \( x = (x_k, x_{k-1}, \ldots, x_1 + 1, \ldots, x_1) \), if \( x_1 + 1 = h_1 \), then a carry occurs at dimension \( i \) and \( x \) is indeed \((x_k, x_{k-1}, \ldots, x_1 + 1, 0, x_{1-1}, \ldots, x_1)\). Similarly, for \( x = (x_k, x_{k-1}, \ldots, x_1 - 1, \ldots, x_1) \), if \( x_1 = 0 \), then \( x_1 - 1 \) will borrow 1 from \( x_1+1 \), and the resulting label will be \((x_k, x_{k-1}, \ldots, x_1 + 1, h_1 - 1, x_{1-1}, \ldots, x_1)\). Furthermore, when the carry and borrow operations occur at dimension \( k \), it is manipulated as that there is an invisible dimension \( k + 1 \) in which no carry and borrow will occur. This means that vertices \((0, x_{k-1}, \ldots, x_1, x_{k-1})\) and \((h_k - 1, x_{k-1}, \ldots, x_1, x_{k-1})\) are adjacent.

![Fig. 2](image-url) The embedding of an 8 \( \times \) 3 mesh on GR(2, 4, 3).

Clearly, \( GR(h_k, h_{k-1}, \ldots, h_1) \) contains \( \Pi_i h_i \) vertices, and in general, it is \( 2k \)-regular except for \( 2k - 1 \)-regular when \( h_k = 2 \) (since \( x_k + 1 \equiv x_k - 1 \pmod{h_k} \) in this case). In [25], Tang et al. gave a mapping from \( GR(h_k, h_{k-1}, \ldots, h_1) \) to \( C(n; c_1, c_2, \ldots, c_k) \) which is described as follows.

**Theorem 1 ([25]):** A circulant graph \( C(n; c_1, c_2, \ldots, c_k) \) is \( GR(h_k, h_{k-1}, \ldots, h_1) \) if and only if \( c_i = n/\Pi_{j=1}^k h_j \) for \( i = 1, 2, \ldots, k \).

By Theorem 1, we can find that \( C(24; 1, 3, 12) \) and \( C(24; 1, 4, 8) \) shown in Figs. 1 (a) and 1 (b), respectively, are \( GR(2, 3, 4) \) and \( GR(3, 2, 4) \).

The following lemma provides a simple rule to determine whether a \( k \)-GRCG is bipartite or not.

**Lemma 2 ([25]):** \( GR(h_k, h_{k-1}, \ldots, h_1) \) is a bipartite graph if and only if \( h_1 \) is even and \( h_i \) is odd for \( i = 1, 2, \ldots, k \).

A mesh network is defined as the Cartesian product \( P_r \times P_s \) of undirected paths \( P_r \) and \( P_s \), denoted by \( M_{r,s} \). Lemma 3 gives a condition that a mesh can be embedded in a \( k \)-GRCG. For example, we consider mesh embedding in \( GR(2, 4, 3) \). According to this lemma, all subgraphs of mesh \( M_{2 \times 12} \) or \( M_{8 \times 3} \) can be embedded in \( GR(2, 4, 3) \). See Fig. 2 as an example.

**Lemma 3 ([25]):** For a mesh \( M_{r,s} \) and a \( k \)-GRCG \( GR(h_k, h_{k-1}, \ldots, h_1) \), if there exists an integer \( j \) with \( 1 \leq j \leq k \) such that \( 2 \leq r \leq \Pi_{i=1}^j h_i \) and \( 2 \leq s \leq \Pi_{i=j+1}^k h_i \), then \( M_{r,s} \) can be embedded in \( GR(h_k, h_{k-1}, \ldots, h_1) \).

For more results related to \( k \)-GRCGs, we refer to [9], [11].

3. The Pancyclicity of \( k \)-GRCGs

In [3], Araki and Shibata addressed the pancyclicity on RCGs. In this section, we are concerned with the pancyclicity of \( k \)-GRCGs. Since the class of RCGs is a subclass of \( k \)-GRCGs, our results also hold for RCGs. We first examine the bipancyclicity of \( k \)-GRCGs. Clearly, a 1-GRCG is a cycle and is not pancyclic except \( GR(3) \). Thus, we consider hereafter only \( k \)-GRCGs with \( k \geq 2 \). For simplicity,
hereafter GR\((h_k, h_{k-1}, \ldots, h_1)\) is abbreviated as GR when the context is clear.

By Lemma 3, we know that a mesh \(M_{r,s}\) with \(r, s \geq 2\) and \(r \cdot s = \prod_{i=1}^{k} h_i\) can be embedded in a \(k\)-GRGC. It has been proved that a mesh \(M_{r,s}\) is bipancyclic if both \(r, s \geq 2\) (see [19, Lemma 2.1]). Since, by definition, the size of each dimension in a \(k\)-GRGC is at least 2, the following result directly holds.

**Lemma 4:** For \(k \geq 2\), any \(k\)-GRGC with \(n \geq 4\) vertices is bipancyclic.

In the following, we shall investigate the existence conditions of odd cycles in \(k\)-GRGCs. For ease of description, we define some terms for using later. Hereafter, we assume a \(k\)-GRGC is at least 2, the following result directly holds.

**Lemma 6:**

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**Lemma 7:** If GR\((h_k, h_{k-1}, \ldots, h_1)\) has \(h_k = \delta\), then it contains odd cycles of every length in the range from \(\delta\) to \(n\).

**Proof.** By Lemma 3, a mesh \(M_{r,s}\) with \(r = h_k\) and \(s = \prod_{i=1}^{k} h_i\) can be embedded in GR. By definition, vertices in the first column of \(M_{r,s}\) form an \(r\)-cycle. Let \(C_r = (0, 0, \ldots, 0) \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k\) be the \(r\)-cycle in the first column, and let \(C_4\) be the cycle \((0, 0, \ldots, 0)\) \(\times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k\) in \(M_{r,s}\). Then, \(C_r \circ C_4\) results in a \(C_r \circ C_4\) (see Fig. 3(a)). Again, \(C_r \circ C_4\) for the above \(C_r \circ C_4\) and \(C_4\) = \((0, 0, \ldots, 1) \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k\) will result in a \(C_r \circ C_4\).

By using the similar technique repeatedly on the first two rows of \(M_{r,s}\), we can build odd cycles with lengths in the range from \((\delta + 6)\) to \((\delta + 12)\) (see Fig. 3(b)). Then, by using the above technique on every two rows of \(M_{r,s}\), every odd cycle in the range from \((\delta + 2)\) to \((\delta + 12)\) can be built (see Fig. 3(c)). To include the vertices in the last row into the above \((\delta + 2)\)-cycle, we can construct disjoint \(4\)-cycles in the last two rows one by one from the second column of \(M_{r,s}\). Then use the combining operator “\(\circ\)” to combine the newly created cycle and a \(4\)-cycle for obtaining a larger odd cycle (see Figs. 3(d) and 3(e) for \(n\) is even and odd, respectively). This establishes the lemma.

**Lemma 8:** If GR\((h_k, h_{k-1}, \ldots, h_1)\) has \(h_1 = \delta - 1\), then it contains odd cycles of every length in the range from \(\delta\) to \(n\).

**Proof.** By Lemma 3 again, a mesh \(M_{r,s}\) with \(r = \prod_{i=2}^{k} h_i\) and \(s = h_1\) can be embedded in GR. By Lemma 6, \((0, 0, \ldots, 0) \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k \times \delta h_k\) is a \(\delta\)-cycle (see Fig. 4(a)). To construct cycles of lengths in the range from \(\delta + 2\) to \(\delta + 2(r - 1)\), we can use the vertices in the second and third columns of \(M_{r,s}\) to build adjacent \(C_4\)’s one by one from the first row to the last row of \(M_{r,s}\). Note that the bottom edge of a \(4\)-cycle will be the top edge of the successive \(4\)-cycle. Then use the combining operator “\(\circ\)” to combine
the newly created cycle and a \( C_4 \) for getting a larger odd cycle (see Figs. 4(b) and 4(c)). By using a similar technique on every two columns, i.e., columns 4 and 5, 6 and 7, etc., we can construct odd cycles with lengths in the range from \( \delta + 2r \) to \( \delta + (s - 2)(r - 1) \) (see Fig. 4(d)). In the above construction, we can find that only two vertices in the first column and one vertex in the last column, namely vertices \((0, 0, \ldots , 0), (0, 0, \ldots , 1, 0)\), and \((0, 0, \ldots , h_1)\), are used in constructing aforementioned odd cycles. By combining the previous constructed \( C_{\delta + (s - 2)(r - 1)} \) and the \( C_4 = (0, 0, \ldots , 1, 0) \), an odd cycle of length \( \delta + (s - 2)(r - 1) + 2 \) is built (see Fig. 4(e)). Similarly, by shifting down the above \( C_4 \) one row, another \( C_4 \) can be constructed, and, then combining with the newly created cycle, a larger odd cycle is built. By repeating the above procedure until the vertex in the lower left corner, namely vertex \((h_{k-1} - 1, h_{k-1} - 1, \ldots , h_2 - 1, 0)\), is included, we have constructed all odd cycles for every length in the range from \( \delta \) to \( r \cdot s - 1 \) (see Fig. 4(f)). This completes the proof. \( \square \)

**Lemma 9:** If \( \text{GR}(h_k, h_{k-1}, \ldots, h_1) \) is not a bipartite graph, then it contains odd cycles of every length in the range from \( \delta \) to \( n \).

**Proof.** Since GR is not bipartite, by Lemma 2, either \( h_k \) is odd or \( h_i \) is even for some \( i \in \{1, 2, \ldots , k - 1\} \). By Lemmas 7 and 8, if \( h_k = \delta \) or \( h_1 = \delta - 1 \), then this lemma holds. It remains to consider the case that \( h_i = \delta - 1 \) for some \( 1 < i < k \). Let \( \text{GR}(h_k, h_{k-1}, \ldots, h_1) \) (GR for short) be the \( j \)th copy of \( \text{GR}(h_k, h_{k-1}, \ldots, h_1) \), and, by definition, there are \( \prod_{i=1}^{k} h_i \) such copies. By Lemma 3, there exists a mesh \( M(= M_{s\times r}) \) with \( r = \prod_{i=1}^{k} h_i \) and \( s = \prod_{i=1}^{k-1} h_i \) such that \( \text{GR}_j \) is embedded in the \( j \)th column of \( M \). Note that the vertices in each column of \( M \) are arranged according to their corresponding ordering in the largest odd cycle, say \( C_{\ell} \), in \( \text{GR}_1 \). We use \( M(= M_{s\times r}) \) to denote the submesh of \( M \) which is the mesh without containing the first column of \( M \) (i.e., \( s' = s - 1 \)). Also, we denote \( M(p, q) \) the vertex at the \( p \)th row and the \( q \)th column of \( M \). By Lemma 8, there exist odd cycles of every length in the range from \( \delta \) to \( r \) in \( \text{GR}_1 \). All we have to prove is that there exist odd cycles of every length in the range from \( \ell + 2 \) to \( n \). Since \( r \) is an even number, \( C_{\ell} = C_{\ell-1} \). We consider the following cases.

**Case 1.** \( s > 2 \). In this case, \( M' \) is still a mesh having more than one column. Since \( M' \) is biparticyclic, it contains every cycle of even length in the range from 4 to \( r \cdot s' \). It is obvious that we can construct every even-length cycle in \( M' \) such that there is an edge in the first column of \( M' \). Let \( C_{x} \) be such an even cycle with an edge in the first column of \( M' \) which connects vertices \( M'(y, 1) \) and \( M'(y + 1, 1) \), namely \( M(y, 2) \) and \( M(y + 1, 2) \), respectively. Let \( C_{x} = (M(y, 1), M(y, 2), M(y + 1, 2), M(y + 1, 1), M(y, 1)) \). Since the cycle \( C_{x} \) in \( \text{GR}_1 \) has an edge between \( y \) and \( y + 1 \) for some \( y \), we can find that \( (C_{x} \circ C_{2}) \circ C_{x} \) is an odd cycle of length \( \ell + x \). Note that \( C_{x} \circ C_{4} \) is an odd cycle of length \( \ell + 2 \). Therefore, in this case, there exists every odd cycle with length in the range from \( \delta \) to \( n - 1 \).

**Case 2.** \( s = 2 \). In this case, there is only one column in \( M' \). However, there exists every even-length cycle, say \( C_{x} \) for \( 4 \leq x \leq r \), in \( \text{GR}_2 \) passing through the edge between vertices \( M(1, 2) \) and \( M(2, 2) \). Let \( C_{x} = (M(1, 1), M(1, 2), M(2, 2), M(2, 1), M(1, 1)) \). Thus, \( C_{x} \circ C_{4} \) is an odd cycle of length \( \ell + 2 \) and \( (C_{x} \circ C_{4}) \circ C_{1} \) is an odd cycle of length \( \ell + x \). This establishes the lemma. \( \square \)

**Lemma 10:** If \( \text{GR}(h_k, h_{k-1}, \ldots, h_1) \) is not a bipartite graph, then no odd cycle has length smaller than \( \delta \).

**Proof.** If \( k = 1 \), then there is exactly one cycle \( C_{h_1} \) in \( \text{GR} \), and this lemma holds directly. Therefore, we consider the case where \( k \geq 2 \) in the following. Suppose to the contrary that there is an odd cycle \( C_{\ell} \) in \( \text{GR} \) with \( \ell < \delta \) and no smaller odd cycle (including \( C_{\ell} \)) is totally contained in some \( \text{GR}(h_k, h_{k-1}, \ldots, h_2) \) for \( 1 \leq i \leq h_1 \); otherwise, we can consider the subgraph \( \text{GR}_i \). Assume that \( C_{\ell} = x_{j_1 j_2 \ldots j_l} \) where \( j_i \) is a jump for \( i = 1, 2, \ldots, l \). By Proposition 5, we can swap jumps in \( j_1, j_2, \ldots, j_l \) such that all jumps \( 1^* \) are in the beginning, then follow by all jumps \( 1^- \), finally all other jumps appear without changing their order. For simplicity, we assume that the resulting jump sequence is \( 1^{*+}1^{-+}j_1 j_2 \ldots j_l j_1^* \) for some nonnegative integers \( t_1 \) and \( t_2 \) and \( j_1 \notin \{1^*, 1^-\} \) for \( t_1 + t_2 + 1 \leq i \leq \ell \). Clearly,

\[
\begin{align*}
\begin{array}{c}
1^{*+}1^{-+}j_1j_2\ldots j_lj_1^* \rightarrow x.
\end{array}
\end{align*}
\]

In the following, we only consider the case where \( t_1 \geq t_2 \). The case where \( t_1 \leq t_2 \) can be handled similarly. By the possible values of \( t_2 \), there are two cases to be considered.

**Case 1.** \( t_2 = 0 \). In this case, \( t_1 \) cannot be equal to 0 either; for otherwise, \( C_{\ell} \) will be totally contained in some \( \text{GR}_i \) which is a contradiction. Therefore, \( t_1 \) must be equal to \( h_1 \). If \( h_1 \) is even, then \( x_{1^{*+}1^{-+}j_1j_2\ldots j_lj_1^*} \) will be an odd cycle of length \( h_1 + 1 \leq \ell \). However, by the definition of \( C_\delta \) and the above inequality, \( \delta \leq h_1 + 1 \leq \ell \). This contradicts the assumption that \( \ell < \delta \). Therefore, \( h_1 \) must be an odd number. However, in this case, by replacing \( 1^{*+} \) with \( 2^- \) in the jump sequence \( 1^{*+}j_1j_2\ldots j_lj_1^* \), this results in a path \( P = x_{2^-j_1j_2\ldots j_lj_1^*} \times \) which is totally contained in some \( \text{GR}_i \). Note that the length of \( P \) is equal to \( \ell - t_1 + 1 \) which is less than \( \ell \) and is an odd number. If every vertex in \( P \) appears at most once, then \( P \)
is actually an odd cycle which contradicts the assumption. If some vertex in $P$ appears more than once, then a smaller odd cycle can be found in $P$. This is also a contradiction. Therefore, this case is impossible.

**Case 2.** $t_2 \neq 0$. Obviously, if $t_2 \neq 0$, then $x$ taking jumps $1^{t_2}2^{t_2}$ will reach itself. This means that $x$ taking the remaining jump sequence $1^{t_1}2^{t_2+1}, \ldots, j_\ell$ still can reach itself, namely

$$x = 1^{t_1}2^{t_2+1}j_\ell x.$$

Note that $\ell - 2t_2$ is still an odd number. Then, by a similar argument as Case 1, this case is also impossible. This completes the proof. \hfill \Box

By combining Lemma 4 and Lemmas 7–10, we summarize our results as the following theorem.

**Theorem 11:** $GR(h_k, h_{h-1}, \ldots, h_1)$ is $\delta$-pancyclic if $\delta$ exists, where $\delta$ is the minimum odd number in the set \{$(h_k, h_{h-1} + 1, h_{h-2} + 1, \ldots, h_1 + 1)$\}.

**Corollary 12:** A generalized recursive circulant graph $GR(h_k, h_{h-1}, \ldots, h_1)$ is pancyclic if and only if $\delta = 3$.

4. **Conclusion Remarks**

In this paper, we study the cycle embedding in a class of circulant graphs, which is a generalization of the class of RCGs. At first, we prove that a $k$-GRCG with two or more dimensions must be bipancyclic. Further, if we can determine the smallest odd $s$-cycle in a $k$-GRCG, then the graph also contains all odd cycles with length greater than $s$.

With a structure similar to multidimensional torus networks [6], the topology of the $k$-GRCGs provides an alternative for designing the parallel computer. For a future research, many network properties, as well as combinatorial problems can be studied on $k$-GRCGs.

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