Empirical Variance Minimization with Applications in Variance Reduction and Optimal Control

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We study the problem of empirical minimization for variance-type functionals over functional classes. Sharp non-asymptotic bounds for the excess variance are derived under mild conditions. In particular, it is shown that under some restrictions imposed on the functional class fast convergence rates can be achieved including the optimal non-parametric rates for expressive classes in the non-Donsker regime under some additional assumptions. Our main applications include variance reduction and optimal control.

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1. Introduction

Empirical Risk Minimization (ERM) plays a central role in statistics and machine learning nowadays. The goal of learning is usually to find a model which delivers good generalization performance over an underlying distribution of the data. Let \( P_{X,Y} \) be a joint distribution of the vector \((X, Y) \in \mathbb{R}^d \times \mathbb{R}\). Given a loss function \( \ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), one aims at minimizing the risk \( R(h) = \mathbb{E}[\ell(h(X), Y)] \) over a class \( \mathcal{H} \) of functions (predictors) \( h : \mathbb{R}^d \to \mathbb{R} \). Since the distribution \( P_{X,Y} \) is usually unknown, one usually replaces \( P_{X,Y} \) by its empirical counterpart and considers

\[
h_n \in \arg \min_{h \in \mathcal{H}} R_n(h),
\]

where

\[
R_n(h) := \frac{1}{n} \sum_{i=1}^{n} \ell(h(X_i), Y_i),
\]

and \( \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \) is a sample from \( P_{X,Y} \) called “data” or “training set”. ERM covers many popular methods and is widely used in practice. For example, if \( \mathcal{H} = \{h : h(x) = \theta^\top x \text{ for } \theta \in \mathbb{R}^p\} \) and \( \ell(y, p) = (y - p)^2 \), then ERM becomes the well-known linear least squares estimator. The celebrated maximum likelihood principle can also be cast as an instance of ERM where the loss function is taken to be the negative log-likelihood function. In turn, ERM itself can be seen as a special case of a more
general empirical minimization problem of the form
\[ h \in \arg \min_{h \in \mathcal{H}} U_n((h(X_1), Y_1), \ldots, (h(X_n), Y_n)), \]  
(1)
where \( U_n : \mathbb{R}^{2n} \to \mathbb{R} \) is a function which estimates \( \mathbb{E}[U_n((h(X_1), Y_1), \ldots, (h(X_n), Y_n))] \). Having the minimal variance among all unbiased estimates of the mean, see [Hoeffding, 1948], \( U \)-statistics naturally appear in this context. Following, for instance, [Joly and Lugosi, 2016], they are defined as
\[ U_n = \left( \frac{n}{m} \right)^{-1} \sum_{1 \leq i_1 < \ldots < i_m \leq n} W((h(X_{i_1}), Y_{i_1}), \ldots, (h(X_{i_m}), Y_{i_m})), \] 
(2)
for a fixed positive integer \( m \leq n \) and a symmetric measurable function \( W : \mathbb{R}^{2n} \to \mathbb{R} \) satisfying \( \mathbb{E}[W((h(X_1), Y_1), \ldots, (h(X_m), Y_m))] < \infty \). The summation in (2) is taken over all \( m \)-element subsets of the set \( \{1, \ldots, n\} \). For the special case \( m = 2 \), and for an appropriate choice of \( W \), we recover the empirical variance
\[ U_n = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (Y_i - h(X_i) - Y_j + h(X_j))^2, \]
with \( \mathbb{E}[U_n] = \text{Var}[Y - h(X)] \). In fact, \( U \)-statistics generalize common notions of unbiased estimation such as the sample mean, the unbiased sample variance, and the third \( k \)-statistic which estimates the third cumulant. After the seminal papers of Halmos [1946] and Hoeffding [1948], it became clear that many meaningful statistics were \( U \)-statistics. We refer to the monograph [Korolyuk and Borovskich, 2013] for a complete exposition of the theory. Currently, \( U \)-statistics play an important role in statistics and probability since they appear in many problems such as clustering, ranking, and learning on graphs. Recently one witnessed a growing interest for concentration properties of \( U \)-statistics in the context of empirical minimization problems of the form (1). We refer to [Clémençon, Lugosi, and Vayatis, 2008], [Clémençon, 2011], [Clémençon, 2014], [Joly and Lugosi, 2016], [Maurer and Pontil, 2019], [Laforgue, Clémençon, and Bertail, 2019], and references therein.

In this paper we address the problem of empirical minimization for variance-type functionals, also referred to as Empirical Variance Minimization (EVM). This problem appears in several important applications including variance reduction and optimal control which we discuss in detail in Section 3. We formulate the problem as follows.

Let \( \mathcal{X} \subseteq \mathbb{R}^d \) and \( P \) be a measure on \( \mathcal{X} \). Furthermore, let \( \mathcal{H} \) be a class of functions \( h : \mathcal{X} \to \mathbb{R} \) such that \( Ph := \mathbb{E}_P[h(X)] = \mathcal{E} \) for all \( h \in \mathcal{H} \) and some constant \( \mathcal{E} \). The last assumption is a natural convention since the variance as well as the empirical variance are translation invariant. The analysis for a general case follows without changes. In this paper we study non-asymptotic properties of the empirical minimizer
\[ h_n \in \arg \min_{h \in \mathcal{H}} V_n(h) \]
with
\[ V_n(h) := \frac{1}{n-1} \sum_{k=1}^{n} (h(X_k) - P_n h)^2 = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (h(X_i) - h(X_j))^2, \] 
(3)
being the empirical variance based on an i.i.d. sample \( X_1, \ldots, X_n \) from \( P \) and \( P_n h \) — the empirical mean with respect to the sample \( X_1, \ldots, X_n \). Our goal is to investigate the magnitude of the excess...
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variance

\[ V(h_n) - V(h^*), \]

where

\[ h^* \in \arg \min_{h \in \mathcal{H}} V(h), \]

and \( V(h) := \text{Var}_P [h(X)] \) is the true variance of \( h(X) \) under \( P \). For simplicity, we assume that minimizers \( h_n \) and \( h^* \) exist as all the arguments can easily be adapted by considering approximate minimizers. We show, for instance, that under suitable conditions on the class \( \mathcal{H} \), the excess variance can be of order up to \( O(n^{-1}) \) with high probability. Finally, we consider applications of these results to problems of variance reduction and optimal control. The paper is organized as follows. In Section 2 we present our main results for the EVM problem including a bound on the variance excess under various complexity assumptions on the class \( \mathcal{H} \). Section 3 contains some applications including variance reduction and optimal control. The proofs of the main results are collected in Section 4. Some auxiliary results can be found in Appendix A.

Notation. We use the standard notation for \( L^r(P) \)-norms, \( 1 \leq r \leq \infty \), and denote the set of all functions \( h \) with \( \| h \|_{L^r(P)} < \infty \) by \( L^r(P) \). We write \( \mathcal{N}(\mathcal{H}, \| \cdot \|_{L^r(P)}, \varepsilon) \) for the \( \varepsilon \)-covering number of \( \mathcal{H} \subset L^r(P) \), that is, the minimal number of balls of radius \( \varepsilon > 0 \) w.r.t. distance \( \| \cdot \|_{L^r(P)} \) needed to cover \( \mathcal{H} \). The natural logarithm of the covering number is called metric entropy of \( \mathcal{H} \). Further, given two functions \( g_1, g_2 \in L^r(P) \) such that \( g_1 \leq g_2 \), with probability one, the bracket \( [g_1, g_2] \) is the set of all functions \( h \in L^r(P) \) satisfying \( g_1 \leq h \leq g_2 \), with probability one. The size of the bracket \( [g_1, g_2] \) is defined as \( \| g_1 - g_2 \|_{L^r(P)} \). The \( \varepsilon \)-bracketing number \( \mathcal{N}_{\| \cdot \|_{L^r(P)}, \varepsilon} \) of the set \( \mathcal{H} \subset L^r(P) \) is the minimal number of brackets of size less or equal to \( \varepsilon > 0 \) necessary to cover \( \mathcal{H} \). The natural logarithm of the bracketing number is called the bracketing entropy of \( \mathcal{H} \). Finally, for any estimator \( h_n \) based on the sample \( X_1, \ldots, X_n \) the variance \( V(h_n) \) is assumed to be taken conditionally on this sample and is a \( (X_1, \ldots, X_n) \)-dependent random variable, that is, for a new \( X \sim P \), independent of \( (X_1, \ldots, X_n) \),

\[ V(h_n) := \text{Var}_P [h_n(X)|X_1, \ldots, X_n]. \]

Comparison with previous results and techniques. The analysis of empirical minimization of \( U \)-statistics is explored in several papers. Under the margin assumption, the seminal paper by Clémençon, Lugosi, and Vayatis [2008] provides fast rates (up to \( O(n^{-1}) \)) for \( U \)-statistics in the context of ranking problems. An important difference with our bounds is that their results take the form of non-exact oracle inequalities (see [Clémençon et al., 2008, Theorem 5]), namely, bounds (in our context) on \( V(h_n) - 2V(h^*) \). In contrast, our results present exact oracle inequalities, these are the bounds on the excess variance \( V(h_n) - V(h^*) \). Non-exact oracle inequalities are known to be easier to establish, compared to more informative bounds on \( V(h_n) - V(h^*) \), and fast rates can be obtained even in the cases where exact oracle inequalities only admit slow rates of convergence [Lecué and Mendelson, 2012]. In particular, non-exact oracle inequalities controlling \( V(h_n) - 2V(h^*) \) only require the boundedness of the loss and exploit neither the convexity of the class nor the margin assumption. In the context of empirical variance minimization, our results provide the first analysis of \( U \)-statistics that implies the exact oracle inequality with the optimal rate of convergence. More recent results [Clémençon, Colin, and Bellet, 2016, Theorem 12] require that the optimal rule belongs to the class in order to provide an exact oracle inequality. In order to simplify the proofs only finite classes are considered in [Clémençon, 2014] and [Joly and Lugosi, 2016]. In contrast, we are considering a larger spectrum of classes: from small parametric classes to expressive classes in the non-Donsker regime.
There is also a recent line of research [Maurer and Pontil, 2019] providing bounds for general non-linear statistics satisfying certain weak interaction assumptions. However, these techniques applied to the special case of $U$-statistics seem to only recover slow rates of convergence.

Finally, the problem of Empirical Variance Minimization was previously announced in [Belomestny, Iosipoi, and Zhivotovskiy, 2018]. The main difference with this work lies in the fact that the proposed estimators $\tilde{h}_n = \arg\min_{h \in \tilde{H}} V_n(h)$ were computed by minimization over a finite approximation ($\varepsilon$-net) $\tilde{H}$ of $H$, which simplifies the analysis, in the spirit of [Devroye et al., 1996] or [van de Geer, 2000].

It has been established that $U$-statistics of order two have concentration properties (for example, the Bernstein inequality) analogous to those of sums of independent random variables [Hoeffding, 1963]. Our contribution builds upon the observation that a similar phenomenon happens in the localized analysis of empirical minimization of $U$-statistics over expressive classes of functions: the classical rates of parametric and non-parametric regression are also achievable when minimizing certain $U$-statistics. In particular, our analysis avoids referring to the order two Rademacher chaos as in the analysis by Clémençon et al. [2008] and leads to the first sharp oracle inequalities with fast rates of convergence.

On the technical side, this paper presents a self-contained version of the localized analysis of Koltchinskii [2006] adapted to a more challenging case of $U$-statistics. We remark that the original approach of Koltchinskii has close similarities to the localized analysis of Massart [2007] and of Bartlett, Bousquet, and Mendelson [2005]. Our analysis further extends these techniques beyond the standard i.i.d. setting.

2. Main results

As we mentioned above, without the loss of generality we consider the class of functions $H$ such that $Ph = \mathcal{E}$ for all $h \in H$ and some constant $\mathcal{E}$. We work under the following assumptions on the class $H$.

(A1) There exists $b > 0$ such that $\sup_{h \in H} \|h\|_{L^{\infty}(P)} \leq b$.

(A2) The class $H$ is star-shaped around $h^*$, that is, for any $h \in H$ and any $t \in [0,1]$, the function $th + (1-t)h^*$ belongs to $H$.

Assumptions (A1) and (A2) are fairly standard in the analysis based on the empirical process theory (see, for instance, the monographs [Koltchinskii, 2011] and [Wainwright, 2019]). Note that Assumption (A2) is clearly satisfied if $H$ is convex.

Our first result addresses the case where the class $H$ has either polynomial covering number in $L^2(P_n)$-norm or polynomial bracketing number in $L^2(P)$-norm. These classes are sometimes referred to as parametric or VC-type classes (see, for example, [Yang and Barron, 1999], [Rakhlin, Sridharan, and Tsybakov, 2017]). We remark that the results expressed in terms of bracketing numbers is of importance in our application Section 3.

**Theorem 2.1.** Suppose that Assumptions (A1) and (A2) hold. Suppose also that, for all $0 < u \leq b$, either

$$\mathcal{N}(H, \| \cdot \|_{L^2(P_n)}, u) \leq \left( \frac{C}{u} \right)^{\alpha} \text{ a.s., or } \mathcal{N}(H, \| \cdot \|_{L^2(P)}, u) \leq \left( \frac{C}{u} \right)^{\alpha}$$

(4)
for some constants \(c > b\) and \(\alpha > 0\). Then, for all \(n \geq 1\) and all \(t > 0\), with probability at least \(1 - 4e^{-t}\),

\[
V(h_n) - V(h^*) \leq A \max \left\{ \frac{\log n}{n}, \frac{t}{n} \right\},
\]

where \(A > 0\) does not depend on \(n, t\) and is explicitly given in the proof up to a multiplicative constant.

In the above result, the restrictions on the complexity of class \(\mathcal{H}\) applies to a fairly large set of situations. However, it excludes classes \(\mathcal{H}\) of non-parametric nature such as Hölder or Sobolev balls (see, for example, Chapters 2.6 and 2.7 in [van der Vaart and Wellner, 1996]). The following result addresses this case and applies to the situation where the covering number or bracketing number of \(\mathcal{H}\) is exponential in \(1/u\).

**Theorem 2.2.** Suppose that Assumptions (A1) and (A2) hold. Suppose also that, for all \(0 < u \leq 2b\), either

\[
\log \mathcal{N}(\mathcal{H}, \| \cdot \|_{L^2(P)}, u) \leq \left( \frac{c}{u} \right)^\alpha \text{ a.s., or } \log \mathcal{N}(\mathcal{H}, \| \cdot \|_{L^2(P)}, u) \leq \left( \frac{c}{u} \right)^\alpha
\]

for some constants \(c > 0\) and \(\alpha > 0\). Then, for all \(n \geq 1\) and all \(t > 0\), with probability at least \(1 - 4e^{-t}\),

\[
V(h_n) - V(h^*) \leq A \max \left\{ \vartheta_n \frac{t}{n} \right\}, \quad \vartheta_n = \begin{cases} n^{-\frac{2}{2+\alpha}}, & \text{if } \alpha < 2, \\ (\log n)n^{-1/2}, & \text{if } \alpha = 2, \\ n^{-1/\alpha}, & \text{if } \alpha > 2, \end{cases}
\]

where \(A > 0\) does not depend on \(n, t\) and is explicitly given in the proof up to a multiplicative constant.

The rates displayed in Theorem 2.1 and Theorem 2.2 are typical for ERM over classes with polynomial and exponential covering numbers respectively. Up to our knowledge, these results are the first fast rates (that is, of order \(n^{1/2}\), exponential and polynomial respectively. Up to our knowledge, these results are the first fast rates (that is, of order \(o(n^{-1/2})\), at least in Theorem 2.1 and in Theorem 2.2 for \(\alpha < 2\)) for empirical variance minimizers. As already mentioned, fast rates in empirical minimization of U-statistics have been first obtained by Clémençon et al. [2008] in the context of ranking problems but only in the sense of non-exact oracle inequalities.

As was mentioned in the Introduction, the proof of Theorem 2.1 and Theorem 2.2 is based on the localized analysis of Koltchinskii [2006]. We show that the excess risk \(\delta_n := V(h_n) - V(h^*)\) satisfies the so-called fixed point equation \(\delta_n \leq \phi_n(\delta_n)\) for some function \(\phi_n\) depending on \(n\) and on certain “geometric” and “complexity” properties of the function class \(\mathcal{H}\). By solving the inequality \(\delta \leq \phi_n(\delta)\) for \(\delta > 0\), one can construct a high-probability bound on \(\delta_n\) (see Lemma 4.1 which contains a simplification of arguments from [Koltchinskii, 2006]; it can be of independent interest). The function \(\phi_n\) for general classes \(\mathcal{H}\) is given in Lemma 4.2. Based on complexity assumptions on \(\mathcal{H}\), we upper bound the excess risk \(\delta_n\) in Section 4.2 and Section 4.3 for Theorem 2.1 and Theorem 2.2 correspondingly. Note that the empirical risk \(V_n(h)\) is not a sum of independent random variables \(h(X_1), \ldots, h(X_n)\) contrary to classical problems in empirical processes. Instead, \(V_n(h)\) is a particular case of U-statistics. This fact complicates the application of the general approach from [Koltchinskii, 2006].

The rates \(\vartheta_n\) of Theorem 2.2 are well-known in the non-parametric regression literature. In particular, the rate \(n^{-2/(2+\alpha)}\) is minimax optimal under various mild assumptions, see [Barron, Birgé, and Massart, 1999]. The situation \(\alpha > 2\), also known as the non-Donsker regime [van der Vaart and Wellner, 1996], corresponds to very expressive classes and is more subtle. It was first shown by Birgé
There is an absolute constant $c_0$ [Belomestny et al., 2018]. Before we proceed, we formulate the following assumption.

Assumption (A3) There is an absolute constant $c_1 > 0$ and a constant $c_2 > 0$, which may depend on the parameters of the problem, such that for any $h \in \mathcal{H}$,

$$V(h) - V(h^*) \leq c_1 \|h - h^*\|^2_{L^2(P)} + c_2 n^{-\frac{2}{2+\alpha}}. $$

This assumption is rather restrictive, though it is always satisfied if the class $\mathcal{H}$ contains a function with small variance. Indeed, using $(a + b)^2 \leq 2a^2 + 2b^2$ and $Ph = Ph^*$, we have

$$V(h) - V(h^*) = \|h - Ph\|^2_{L^2(P)} - \|h^* - Ph^*\|^2_{L^2(P)} \leq 2\|h - h^*\|^2_{L^2(P)} + V(h^*). \quad (6)$$

Hence, provided that $V(h^*) \leq c_2 n^{-\frac{2}{2+\alpha}}$ for a fixed constant $c_2 > 0$, we see that (A3) holds with $c_1 = 2$. Assuming that the variance $V(h^*)$ is decreasing as the sample size $n$ grows is quite natural in some situations. Indeed, due to the non-asymptotic nature of our results, one may choose $\mathcal{H}$ according to the value of the sample size. When the sample size $n$ is getting larger, it is reasonable to consider a more expressive class $\mathcal{H}$. As a result, the minimum variance in the class can decrease. We are now ready to state our last excess variance bound which can be seen as a strengthening of Theorem 2.2 under additional assumptions.

**Theorem 2.3.** Suppose that Assumptions (A1), (A2), (A3) hold. Suppose also that, for all $0 < u \leq b$,

$$\log \mathcal{N}(\mathcal{H}, \ell_{L^2(P)}, u) \leq \left(\frac{c}{u}\right)^{\alpha}$$

for some constants $c > 0$ and $\alpha > 0$. Then there is an estimator $\hat{h}_n$ such that, for all $n \geq 1$ and all $t > 0$, with probability at least $1 - 4e^{-t}$,

$$V(\hat{h}_n) - V(h^*) \leq A \max \left\{ n^{-\frac{2}{2+\alpha}}, \frac{t}{n} \right\},$$

where $A > 0$ does not depend on $n, t$, but depends on $c_2$ in Assumption (A3) and is explicitly given in the proof.

**Remark.** Observe that in Theorem 2.3 the assumption $\log \mathcal{N}(\mathcal{H}, \ell_{L^2(P)}, u) \leq (c/u)^{\alpha}$ is replaced by the weaker assumption $\log \mathcal{N}(\mathcal{H}, \ell_{L^2(P)}, u) \leq (c/u)^{\alpha}$.

**Remark.** It follows from the proof of Theorem 2.3 that in the special case where $V(h^*) = 0$, that is, $\mathcal{H}$ contains a constant function, Assumption (A2) is not required and Assumption (A3) is automatically satisfied due to (6).
Let us now present the estimator of Theorem 2.3. We fix
\[ \varepsilon = b^2 + 2 \alpha n^{-1} + \alpha, \tag{7} \]
and consider the set \( \mathcal{H}_\varepsilon \subseteq \mathcal{H} \) to be a minimal \( \varepsilon \)-net of \( \mathcal{H} \) with respect to the \( L^2(P) \) distance. Define
\[ \tilde{h}_n = \arg \min_{h \in \mathcal{H}_\varepsilon} V_n(h). \tag{8} \]
The estimators of the form (8) are analyzed in statistical literature and are referred to as the skeleton or sieve estimators. The known results imply the bounds for procedures used in density estimation, non-parametric regression and classification (see [Wong and Shen, 1995], [Devroye, Györfi, and Lugosi, 1996], and [van de Geer, 2000]). Observe that the construction of the set \( \mathcal{H}_\varepsilon \) depends on the \( L^2(P) \) structure of \( \mathcal{H} \) and therefore, requires some prior knowledge on \( P \) or an access to additional unlabelled data points which allows to estimate the \( L^2(P) \) distances between functions in \( \mathcal{H} \).

### 3. Applications

In this section we provide several applications of our general results.

#### 3.1. Variance reduction

Suppose that we wish to compute \( Pf \), where \( f : \mathcal{X} \to \mathbb{R} \) is a function from \( L^2(P) \). A natural estimate for \( Pf \) is the Monte Carlo estimate \( P_n f \). This estimate is unbiased, that is, \( \mathbb{E}[P_n f] = Pf \), and has variance \( \text{Var}[P_n f] = V(f)/n \). From the point of view of applications, an important problem is to improve accuracy of the Monte Carlo estimate \( P_n f \), which in turn means reducing its variance. This can be achieved by increasing the sample size \( n \) but in some cases this solution may not be practical. Another approach to reduce the variance is to decrease the value \( V(f) \) by considering a new Monte Carlo estimate \( P_n f \), which in turn means reducing its variance. Such techniques are called variance reduction methods, see [Robert and Casella, 1999], [Rubinstein and Kroese, 2016], and [Glasserman, 2013] for an introduction to this field. The method of control variates is one of the few generic variance reduction methods. The idea behind this method is to choose a class \( \mathcal{G} \subset L^2(P) \) of functions such that any \( g \in \mathcal{G} \) satisfies \( Pg = 0 \) (such functions are called control variates), and to consider a new Monte Carlo estimate
\[ P_n (f - g^*) = \frac{1}{n} \sum_{k=1}^{n} (f(X_k) - g^*(X_k)), \]
where
\[ g^* \in \arg \min_{g \in \mathcal{G}} V(f - g). \]
If the class \( \mathcal{G} \) of control variates is properly chosen, this approach may significantly reduce the variance, that is, \( V(f - g^*) \ll V(f) \). Since the true variance \( V(f - g) \) for all \( g \in \mathcal{G} \) is not assumed to be known and its computation is a more difficult problem than the initial one, a natural approach to pick a control variate is to minimize the empirical variance
\[ g_n \in \arg \min_{g \in \mathcal{G}} V_n(f - g). \tag{9} \]
Consequently, since the true variance is replaced by its empirical counterpart, the variance of \( f - g_n \) can be decomposed as

\[
V(f - g_n) = V(f - g_n) - \inf_{g \in \mathcal{G}} V(f - g) + \inf_{g \in \mathcal{G}} V(f - g) .
\]

So there is a natural trade-off between the stochastic and approximation errors. Namely, as the set \( \mathcal{G} \) increases the approximation error decreases while the estimation error typically increases. In the particular case where the function \( g^\ast = f - Pf \) belongs to \( \mathcal{G} \) the approximation error is equal to zero, hence the variance of \( f - g_n \) includes only the stochastic error.

In order to bound the stochastic error, Theorem 2.1, Theorem 2.2, or Theorem 2.3 may be used with \( \mathcal{H} = \{ f - g : g \in \mathcal{G} \} \). These results show that, under some technical assumptions, the stochastic error is of order \( O(n^{-1}) \) when \( \mathcal{H} \) has a polynomial covering number as in (4) and ranges from \( O(n^{-2/2+\alpha}) \) for \( \alpha < 2 \) to \( O(n^{-1/\alpha}) \) for \( \alpha > 2 \) when \( \mathcal{H} \) has an exponential covering number as in (5) (note that covering numbers of \( \mathcal{H} \) and \( \mathcal{G} \) are the same). Similar results were previously announced in [Belomestny, Iosipoi, and Zhivotovskiy, 2018] but only for the sieve estimator \( \tilde{g}_n \) of the form (8) under more restrictive assumptions. Their results also imply that the stochastic error is of order up to \( O(n^{-1}) \). In [Belomestny, Iosipoi, Moulines, Naumov, and Samsonov, 2020a,b] the authors study a more sophisticated setting where \( \tilde{g}_n \) is estimated based on dependent samples generated by MCMC algorithms. In this case the stochastic error for the sieve estimator is bounded by \( O(n^{-1/2}) \).

The effect of control variates on the Monte Carlo estimate \( Pf \) can be illustrated by constructing confidence intervals before and after the variance reduction. Namely, using a sample of length \( N \) and the central limit theorem, the following asymptotic confidence interval for \( Pf \) holds

\[
P_{Nf} \pm q \sqrt{\frac{V(f)}{N}},
\]

where \( q \) is a quantile of the standard normal distribution. If we now apply the EVM variance reduction approach with a class \( \mathcal{G} \) having polynomial covering number and construct \( g_n \) using a new independent sample of the length \( n \), then the confidence interval becomes

\[
P_{N(f - g_n)} \pm q \sqrt{\frac{V(f - g_n)}{N}} = P_{Nf} \pm q \sqrt{\frac{\inf_{g \in \mathcal{G}} V(f - g)}{N}} + O \left( \sqrt{\frac{1}{nN}} \right),
\]

which can be significantly tighter than (10) if \( n \) is large. Loosely speaking, this approach may reduce the length of the asymptotic confidence interval from order \( O(N^{-1/2}) \) to \( O(n^{-1/2}N^{-1/2}) \), provided that the class \( \mathcal{G} \) is chosen so that \( \inf_{g \in \mathcal{G}} V(f - g) \) is small enough (this actually means that the class of control variates \( \mathcal{G} \) possesses nice approximation properties). Of course, it comes at a price — this procedure requires computational resources to choose a good control variate (that is, to solve the optimization problem (9)).

Furthermore, let us examine separately the case of Stein control variates (see [Assaraf and Caffarel, 1999], [Mira et al., 2013], and [Oates et al., 2017]), where the class \( \mathcal{G} \) is constructed by substitution of various smooth functions \( \phi : \mathcal{X} \to \mathbb{R}^d \) into

\[
g_\phi = \langle \phi, \nabla \log \pi \rangle + \text{div}(\phi),
\]

where \( \pi \) is the density of the distribution \( P \), \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^d \), and \( \text{div}(\phi) \) stands for the divergence of \( \phi \). Under rather mild conditions on \( \pi \) and \( \phi \), integration by parts implies that \( P g_\phi = 0 \) (see [Mira et al., 2013, Propositions 1 and 2]). In what follows, \( W^{s,p}(\lambda) \) denotes
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the Sobolev space with respect to the Lebesgue measure $\lambda$, that is, $W^{s,p}(\lambda) = \{ u \in L^p(\lambda) : D^\alpha u \in L^p(\lambda) \text{ for all } |\alpha| \leq s \}$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index with $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $D^\alpha$ is the differential operator of the form $D^\alpha = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_d}_{x_d}$ (here partial derivatives are understood in the generalized sense of distributions). We write $W^{s,p}_R(\lambda)$ for the class of functions $h \in W^{s,p}(\lambda)$ supported on a closed ball $B_R = \{ x \in X : |x| \leq R \}$. The Sobolev norm for $u \in W^{s,p}(\lambda)$ or $u \in W^{s,p}_R(\lambda)$ is defined as $\| u \|_{W^{s,p}(\lambda)} = \sum_{|\alpha| \leq s} \| D^\alpha u \|_{L^p(\lambda)}$. By a slight abuse of notation, we continue to write $\phi \in W^{s,p}_R(\lambda)$ for functions $\phi : X \to \mathbb{R}^d$ meaning that any coordinate of $\phi$ belongs to $W^{s,p}(\lambda)$. Given a class $\Phi$ of smooth functions $\phi : X \to \mathbb{R}^d$, define the corresponding minimizers

$$\phi_n \in \arg \min_{\phi \in \Phi} V_n(f - g_\phi) \quad \text{and} \quad \phi^* \in \arg \min_{\phi \in \Phi} V(f - g_\phi).$$

**Proposition 3.1.** Fix some $R > 0$ and let $\Phi \subset W^{s,2}_R(\lambda)$ be a class of $s$-smooth functions $\phi : X \to \mathbb{R}^d$ with $s - d/2 > 0$. Let also $\pi \in W^{1,2}(\lambda)$ be a probability density function satisfying $1/l \leq \pi(x) \leq L$ for some $L > 0$ and all $x \in B_R$. If $\mathcal{H} = \{ f - g_\phi : \phi \in \Phi \}$ satisfies (A1) and (A2), then for all $n \geq 1$ and $t > 0$, with probability at least $1 - 4e^{-t}$,

$$V(f - g_{\phi_n}) - V(f - g_{\phi^*}) \leq A \max \left\{ \vartheta_n, \frac{b_t^2}{n} \right\}, \quad \vartheta_n = \begin{cases} \frac{n^{-s/d^2}}{n}, & d^2/s < 2, \\ \frac{\log n}{n^{1/2}}, & d^2/s = 2, \\ \frac{n^{s/d^2}}{n}, & d^2/s > 2, \end{cases}$$

where $A > 0$ is a constant not depending on $n$.

**Remark.** It is worth mentioning that in Proposition 3.1 we consider the case where $\Phi$ has large complexity. If, for example, covering number of $\Phi$ is polynomial, the rates may be improved by applying Theorem 2.1 instead of Theorem 2.2 (assumption $\Phi \subset W^{s,2}_R(\lambda)$ corresponds to exponential covering number). Furthermore, if one has an access to a minimal $\varepsilon$-net of $\Phi$, Theorem 2.3 can be applied. In this case $\hat{h}_n$ will not depend on any properties of $f$ other than its values observed on the sample.

### 3.2. Optimal control

We consider the optimal control of a discrete-time Markov process with a finite time horizon $T$. On a filtered measurable probability space $(\Omega, \mathcal{F})$ with $\mathcal{F} = (\mathcal{F}_r)_{r=0,1,\ldots,T}$, $T \in \mathbb{Z}_+$, we define an adapted control process $\mathbf{a} : \Omega \times \{0, \ldots, T-1\} \to A$, control for short, where $(A, \mathcal{B})$ is a measurable space. We assume a given set of admissible controls which is denoted by $A$. Given a control $\mathbf{a} = (a_0, a_1, \ldots, a_{T-1}) \in A$, we consider a controlled Markov process $X$ valued in some measurable space $(S, \mathcal{S})$ and defined on a probability space $(\Omega, \mathcal{F}, P^A)$ with $X_0 = x_0$ a.s. and transition kernel of the type

$$P^A_r(x, dy) = P^A(X_{r+1} \in dy \mid X_r = x), \quad 0 \leq r < T.$$ 

So, it is assumed that the distribution of $X_{r+1}$ conditional on $\mathcal{F}_r$ is governed by a (one-step) transition kernel $P^A_r(X_r, dy)$ which is in turn controlled by $a_r$. In this way to each admissible control function, one associates a random variable and a measure. One can therefore also associate to the admissible control $\mathbf{a}$ the mean $E^A$ of the alternative random variable with respect to the alternative measure $P^A$. This procedure defines a continuous function from the set of admissible control functions to the real numbers. We refer to this function as the reward of the underlying optimal control problem. The desired
unknown value that we wish to estimate is equal to the global maximum of the reward with respect to all admissible controls. In this way we may consider the general optimal control problem of the form:

\[
Y^*_0 = \sup_{a \in A} \mathbb{E}^a \left[ \sum_{r=0}^{T-1} f_r(X_r, a_r) + g_T(X_T) \right] \tag{11}
\]

for given functions \( f_r, r = 0, \ldots, T-1, \) and \( g_T \). Introduce the process

\[
Y^*_r = \sup_{a \in A_r} \mathbb{E}^a \left[ \sum_{s=r}^{T-1} f_s(X_s, a_s) + g_T(X_T) \bigg| F_r \right], \quad 0 \leq r \leq T, \tag{12}
\]

with \( A_r \) being the set of all admissible controls \( a : \Omega \times \{r, \ldots, T-1\} \rightarrow A \). Then there exists a vector \( h^* = (h^*_0, \ldots, h^*_T) \) of measurable functions on \( S \), such that \( Y^*_r = h^*_r(X_r) \) and \( h^* \) satisfies the dynamic programming principle:

\[
h^*_r(x) = (\mathcal{L}h^*)_r(x), \quad 0 \leq r < T, \quad h^*_T(x) = g_T(x),
\]

where \( \mathcal{L} \) is a Bellman-type operator defined by

\[
(\mathcal{L}h)_r(x) = \sup_{a \in A} \left[ f_r(x, a) + P^a h_{r+1}(x) \right]
\]

and

\[
P^a h_{r+1}(y) := \int P^a(x, dy) h_{r+1}(y).
\]

We now assume that there exists a reference measure \( P^* \) equivalent to \( P^a \), such that

\[
P^a(x, dy) = \varphi(x, y, a) P^*(x, dy), \quad a \in A,
\]

with \( P^*(x, dy) := P^*(X_{r+1} \in dy \mid X_r = x) \) and the function \( \varphi(x, y, a) \) satisfying \( \varphi \geq 0 \) and

\[
\int P^*(x, dy) \varphi(x, y, a) = 1.
\]

Denote by \( C_b(S) \) the set of continuous bounded functions on \( S \). As shown in [Rogers, 2007] the value of the problem (11) in a dual form can be expressed as an infimum over a family of martingales \( M_j = (\mathcal{L}h)_j(X_j) - h_j(X_j), j = 1, \ldots, T, h \in C_b(S) \) of an expectation, under measure \( P^* \), of a pathwise supremum adjusted by a weighted sum of \( M_j \).

**Theorem (Rogers).** Let \( Y^*_r \) be a solution of the optimal control problem (12), then the following representation holds

\[
Y^*_0 = \inf_{h \in C_b^{T+1}(S)} \mathbb{E}^* \left[ h_0(X_0) + \sum_{j=0}^{T-1} W_j \left( (\mathcal{L}h)_j(X_j) - h_j(X_j) \right) \right], \tag{13}
\]

where

\[
W_j = \sup_{a \in A} \left[ \prod_{l=0}^{j-1} \varphi(X_l, X_{l+1}, a_l) \right]
\]

and \( \mathbb{E}^* \) stands for expectation under \( P^* \).
As shown in [Belomestny and Schoenmakers, 2018] the optimization problem (13) can be alternatively formulated as a problem of variance minimization

\[ Y_0^* = \min_{h \in C_{b}^{T+1}(S)} \text{Var} \left[ h_0(X_0) + \sum_{j=0}^{T-1} W_j \left( (\mathcal{L}h)_j(X_j) - h_j(X_j) \right) \right], \tag{14} \]

provided the set \( A \) of control values is finite. To solve this problem, one can use Monte Carlo approach and cast the problem (14) into the empirical variance minimization

\[ h_n \in \arg\inf_{h \in \mathcal{H}} \left\{ \frac{1}{n(n-1)} \sum_{1 \leq k < l \leq n} \left( \phi_h(X_0^{(k)}, \ldots, X_T^{(k)}) - \phi_h(X_0^{(l)}, \ldots, X_T^{(l)}) \right)^2 \right\}, \]

where \( \mathcal{D}_n = \{(X_0^{(k)}, \ldots, X_T^{(k)}), k = 1, \ldots, n\} \) is a set of \( n \) independent trajectories of the chain \((X_j)_{j \geq 0}\), \( \mathcal{H} \subset C_{b}^{T+1}(S) \), and

\[ \phi_h(X_0, \ldots, X_T) = h_0(X_0) + \sum_{j=0}^{T-1} W_j \left( (\mathcal{L}h)_j(X_j) - h_j(X_j) \right). \]

Theorem 2.1 implies that, for all \( n \geq 1 \) and all \( t > 0 \), with probability at least \( 1 - 4e^{-t} \),

\[ \text{Var}[\phi_h_n(X_0, \ldots, X_T) | \mathcal{D}_n] - \inf_{h \in \mathcal{H}} \text{Var}[\phi_h(X_0, \ldots, X_T)] \leq A \max \left\{ \frac{\log n}{n}, \frac{t}{n} \right\} \]

for some \( A > 0 \) not depending on \( n, t \), provided that the class of functions \( \Phi = \{\phi_h : h \in \mathcal{H}\} \) satisfies the assumptions (A1), (A2) and

\[ \log \mathcal{N}(\Phi, \parallel \cdot \parallel_{L_2(P^*_\times \ldots \times P^*_s)}, u) \leq \left( \frac{c}{u} \right)^2. \]

Note that the latter condition is, for example, fulfilled if \( \Phi \) is a ball in Sobolev space \( W^{s,2}(S^{T+1}) \) for \( s \) large enough. This condition in turn can be translated into a smoothness assumption imposed on the class \( \mathcal{H} \) and the transition densities \( \varphi(x, y, a), a \in A \), see the proof of Proposition 3.1.

4. Proofs

4.1. Outline of the proofs of Theorem 2.1 and Theorem 2.2

Below, we present an outline of the proof of Theorem 2.1 and Theorem 2.2 based on technical results proved in Appendix A. The proof consists in an adaptation of the general strategy for bounding empirical risk minimizers devised in [Koltchinskii, 2006]. The main difference being that we deal with the quadratic functionals \( V(h) \) and \( V_n(h) \) instead of \( P h \) and \( P_n h \) respectively. Essentially, we prove that the strategy extends nicely to our setting using Hoeffding decomposition for \( U\)-Statistics. Let

\[ \delta_n := V(h_n) - V(h^*). \]
Notice that by the definition of $h_n$, we get
\[
\delta_n \leq (V(h_n) - V(h^*)) - (V_n(h_n) - V_n(h^*))
\]
\[
= P((h_n - \mathcal{E})^2 - (h^* - \mathcal{E})^2) - (V_n(h_n) - V_n(h^*))
\]
where we recall that $\mathcal{E} = Ph$ for all $h \in \mathcal{H}$. Observe that, for all $h \in \mathcal{H}$, $V_n(h)$ may be rewritten as
\[
V_n(h) = \frac{1}{2n(n-1)} \sum_{i \neq j} (h(X_i) - h(X_j))^2
\]
\[
= P_n(h - \mathcal{E})^2 - \frac{1}{n(n-1)} \sum_{i \neq j} (h(X_i) - \mathcal{E})(h(X_j) - \mathcal{E}).
\]
This expression is a simple instance of the well-known Hoeffding decomposition for $U$-statistics (due to Hoeffding [1948] and Hoeffding [1961]). Combining this representation with the above inequality yields
\[
\delta_n \leq T_n(h_n) + W_n(h_n),
\]
where
\[
T_n(h) := (P - P_n)((h - \mathcal{E})^2 - (h^* - \mathcal{E})^2),
\]
and
\[
W_n(h) = w_n(h) - w_n(h^*) \quad \text{with} \quad w_n(h) = \frac{1}{n(n-1)} \sum_{i \neq j} (h(X_i) - \mathcal{E})(h(X_j) - \mathcal{E}).
\]
As a result, we arrive at the central observation that the excess risk $\delta_n$ satisfies
\[
\delta_n \leq \phi_n(\delta_n),
\]
where, for any $\delta > 0$,
\[
\phi_n(\delta) := \sup_{h \in \mathcal{H}(\delta)} \{T_n(h) + W_n(h)\} \quad \text{and} \quad \mathcal{H}(\delta) = \{h \in \mathcal{H} : V(h) - V(h^*) \leq \delta\}. \quad (15)
\]

The main technical result we will invoke, in order to bound $\delta_n$, is the following lemma. This result follows from a combination of arguments presented in Theorem 4.1, Corollary 4.1, and Theorem 4.3 in [Koltchinskii, 2011]. We simplify these arguments and, for the sake of completeness, provide the proof in Appendix A.4.

**Lemma 4.1.** Let $\{\phi(\delta) : \delta \geq 0\}$ be non-negative random variables (indexed by all deterministic $\delta \geq 0$) such that, almost surely, $\phi(\delta) \leq \phi(\delta')$ if $\delta \leq \delta'$. Let $\{\beta(\delta, t) : \delta \geq 0, t \geq 0\}$, be (deterministic) real numbers such that
\[
P(\phi(\delta) \geq \beta(\delta, t)) \leq e^{-t}. \quad (16)
\]
Finally, let $\hat{\delta}$ be a nonnegative random variable, a priori upper bounded by a constant $\delta > 0$, and such that, almost surely,
\[
\hat{\delta} \leq \phi(\delta).
\]
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Then defining, for all $t \geq 0$,

$$\beta(t) := \inf \left\{ \tau > 0 : \sup_{\delta \geq \tau} \frac{\beta(\delta, b\delta^2 \tau)}{\delta} \leq \frac{1}{2} \right\}, \quad \text{(17)}$$

we obtain, for all $t \geq 0$,

$$P(\hat{\delta} \geq \beta(t)) \leq 2e^{-t}.$$  

The result of Lemma 4.1 is a version of the fixed point argument which is widely applied in the statistical literature to prove the rates of convergence faster than $O(n^{-1/2})$. For a deeper discussion on this argument we refer the reader to the papers [Birgé and Massart, 1993], [Barron et al., 1999], [Bartlett et al., 2005] and the monographs [van de Geer, 2000], [van der Vaart and Wellner, 1996], [Massart, 2007], [Wainwright, 2019].

According to Lemma 4.1, it remains to bound $\phi_n(\delta)$ with high probability for any fixed $\delta > 0$. The next lemma reduces the problem to that of bounding the expected suprema of empirical processes. Below, for every $h \in \mathcal{H}$, we denote

$$\ell(h) := (h - \mathcal{E})^2 - \left( h^* - \mathcal{E} \right)^2.$$ \quad \text{(18)}

Lemma 4.2. Suppose that Assumptions (A1) and (A2) hold. Then, for every $\delta > 0$ and any $t > 0$,  

$$P\left( \phi_n(\delta) \geq \beta_n(\delta, t) \right) \leq 2e^{-t},$$  

where,

$$\beta_n(\delta, t) := A \left( E \sup_{h \in \mathcal{H}(\delta)} (P - P_n) \ell(h) + \left( E \sup_{h \in \mathcal{H}(\delta)} \left| (P - P_n) h \right| \right)^2 + b\sqrt{\frac{t\delta}{n}} + \frac{b^2(1 + t)}{n} \right).$$

$\ell(h)$ is defined by (18), and $A > 0$ is a universal constant.

Proof. Let us write

$$\phi_n(\delta) \leq \phi_n^{(1)}(\delta) + \phi_n^{(2)}(\delta),$$

where

$$\phi_n^{(1)}(\delta) := \sup_{h \in \mathcal{H}(\delta)} T_n(h)$$ and $$\phi_n^{(2)}(\delta) := \sup_{h \in \mathcal{H}(\delta)} W_n(h).$$

Bound on $\phi_n^{(1)}(\delta)$. Under (A1), we have $|\ell(h)| \leq 4b^2$ for all $h \in \mathcal{H}$. It then follows from a version of Talagan's inequality due to Bousquet (see Lemma A.1 in Appendix A.1) that, with probability at least $1 - e^{-t}$,

$$\phi_n^{(1)}(\delta) \leq E\phi_n^{(1)}(\delta) + \sqrt{\frac{2t}{n} \sigma^2(\delta) + 16b^2 E\phi_n^{(1)}(\delta)} + \frac{8b^2t}{3n},$$

where

$$\sigma^2(\delta) = \sup_{h \in \mathcal{H}(\delta)} P(\ell(h))^2.$$
Using basic inequalities $\sqrt{u + v} \leq \sqrt{u} + \sqrt{v}$ and $2\sqrt{uv} \leq u + v$ for positive numbers $u$ and $v$, we further deduce that, with probability at least $1 - e^{-t}$,
\[
\phi_n^{(1)}(\delta) \leq 2E\phi_n^{(1)}(\delta) + \sigma(\delta) \sqrt{\frac{t}{n} + \frac{32b^2t}{3n}}. \tag{19}
\]

Now let us upper bound the term $\sigma^2(\delta)$. Observe that under (A1), for every $h \in \mathcal{H}$,
\[
|h(h)| = |(h - h^*)(h + h^* - 2\mathcal{E})| \leq 4|h - h^*|,
\]
so that
\[
P(\ell(h))^2 \leq 16b^2P(h - h^*)^2.
\]
Then, using the simple identity $(h - h^*)^2 = 2\ell(h) - 4\ell((h + h^*)/2)$, we deduce that
\[
P(h - h^*)^2 = 2P\ell(h) - 4P\ell\left(\frac{h + h^*}{2}\right) \leq 2P\ell(h), \tag{20}
\]
where the inequality follows by definition of $h^*$ and the fact that, since $\mathcal{H}$ is star-shaped around $h^*$ according to (A2), the function $(h + h^*)/2$ belongs to $\mathcal{H}$. Hence we have proven that, for every $h \in \mathcal{H}$,
\[
P(\ell(h))^2 \leq 32b^2P(h).
\]
This inequality is an instance of the Bernstein assumption [Bartlett and Mendelson, 2006]. By the definition (15) of $\mathcal{H}(\delta)$, it follows that
\[
\sigma^2(\delta) \leq 32b^2\delta. \tag{21}
\]
Combining (19) and (21), we deduce that
\[
\phi_n^{(1)}(\delta) \leq 2E \sup_{h \in \mathcal{H}(\delta)} (P - P_n)\ell(h) + 8b \sqrt{\frac{t\delta}{n} + \frac{32b^2t}{3n}}. \tag{22}
\]

**Bound on $\phi_n^{(2)}(\delta)$.** Notice that, for all $h \in \mathcal{H}$,
\[
w_n(h) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (h(X_i) - \mathcal{E})(h(X_j) - \mathcal{E}) - \frac{1}{n(n-1)} \sum_{i=1}^{n} (h(X_i) - \mathcal{E})^2
\]
\[
= \frac{n}{n-1} (P_n(h - \mathcal{E}))^2 - \frac{1}{n-1} P_n(h - \mathcal{E})^2
\]
\[
= \frac{n}{n-1} ((P - P_n)h)^2 - \frac{1}{n-1} P_n(h - \mathcal{E})^2.
\]
Omitting negative terms and using (A1) we get, for all $h \in \mathcal{H}$ and all $n \geq 2$,
\[
W_n(h) \leq \frac{n}{n-1} ((P - P_n)h)^2 + \frac{1}{n-1} P_n(h^* - \mathcal{E})^2
\]
\[
\leq 2((P - P_n)h)^2 + \frac{8b^2}{n}.
\]
As a result, for any \( \delta > 0 \), we have

\[
\phi_n^{(2)}(\delta) \leq 2 \left( \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \right)^2 + \frac{8b^2}{n}
\]

\[
\leq 4 \left( \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| - \mathbb{E} \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \right)^2 + 4 \left( \mathbb{E} \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \right)^2 + \frac{8b^2}{n}.
\]

A classical application of the bounded differences inequality [Boucheron et al., 2013, Theorem 6.2] implies that, with probability at least \( 1 - e^{-t} \),

\[
\left( \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| - \mathbb{E} \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \right)^2 \leq \frac{2b^2 t}{n}.
\]

Hence, we deduce from the above that, for every \( \delta > 0 \) and every \( t > 0 \),

\[
\phi_n^{(2)}(\delta) \leq 4 \left( \mathbb{E} \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \right)^2 + \frac{8b^2(1 + t)}{n},
\]

with probability at least \( 1 - e^{-t} \). Combining (22) and (23) we obtain the desired conclusion.

The rest of the proofs of Theorem 2.1 and Theorem 2.2 is given in Section 4.2 and Section 4.3 correspondingly. There, depending on the assumption on the entropy of the class \( \mathcal{H} \), we explicitly compute \( \beta_n(\delta, t) \) defined in Lemma 4.2 and substitute it in Lemma 4.1 to obtain the final bound on the excess risk \( \delta_n \).

### 4.2. End of the proof of Theorem 2.1

Throughout the proof, let \( A > 0 \) be a universal constant whose value may change from line to line. It follows from Lemma 4.2 that, for every \( \delta \geq 0 \) and every \( t > 0 \),

\[
P(\phi_n(\delta) \geq \beta_n(\delta, t)) \leq 2e^{-t},
\]

where

\[
\beta_n(\delta, t) := A \left( \mathbb{E} \sup_{h \in \mathcal{H}(\delta)} (P - P_n)\ell(h) + \left( \mathbb{E} \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \right)^2 + b \sqrt{\frac{t \delta}{n} + \frac{b^2(1 + t)}{n}} \right).
\]

Let us now bound the expected suprema of empirical processes given in \( \beta_n(\delta, t) \) under the assumptions of Theorem 2.1 on either the covering or bracketing numbers. Under (A1), we have \(|\ell(h)| \leq 4b^2\) for any \( h \in \mathcal{H} \). Moreover, for any \( h_1, h_2 \in \mathcal{H} \),

\[
|\ell(h_1) - \ell(h_2)| \leq |(h_1 - h_2)(h_1 + h_2 - 2\mathcal{E})| \leq 4b|h_1 - h_2|,
\]

and hence

\[
\mathcal{N}(L(\delta), \| \cdot \|_{L^2(P_n), u}) \leq \mathcal{N}(\mathcal{H}(\delta), \| \cdot \|_{L^2(P_n), \frac{u}{4b}}).
\]
The same result holds also for the bracketing numbers. From (21) we also see that
\[
\sup_{h \in H(\delta)} P(\ell(h))^2 \leq \sqrt{32b^2}.
\]
Combining these facts and taking \( \sigma = \sqrt{32b^2} \) in either Lemma A.3 for covering numbers or Lemma A.6 for the bracketing numbers (both results are presented in Appendix A), we get
\[
E \sup_{h \in H(\delta)} \left| \left( P - P_n \right) h \right| \leq A \left( \sqrt{\frac{32\alpha b^2}{n} \log \left( \frac{4c}{\sqrt{32\delta}} \right)} + \frac{\alpha b}{n} \log \left( \frac{4c}{\sqrt{32\delta}} \right) \right).
\]

Now let us turn to the second summand in \( \beta_n(\delta, t) \). Letting \( \bar{H}(\delta) := H(\delta) \cup (-H(\delta)) \), we obtain
\[
\mathcal{N}(\bar{H}(\delta), \| \cdot \|_{L^2(P_n), u}) \leq 2\mathcal{N}(H(\delta), \| \cdot \|_{L^2(P_n), u})
\]
and
\[
E \sup_{h \in \bar{H}(\delta)} \left| (P - P_n) h \right| = E \sup_{h \in H(\delta)} \left| (P - P_n) h \right|.
\]
Using either Lemma A.3 for the covering number assumption or Lemma A.6 for the bracketing number assumption, we get with \( \sigma = b \) that
\[
E \sup_{h \in H(\delta)} \left| (P - P_n) h \right| \leq A \left( \sqrt{\frac{\alpha b^2}{n} \log \left( \frac{c}{b} \right)} + \frac{\alpha b}{n} \log \left( \frac{c}{b} \right) \right).
\]
Hence, whenever \( n \geq \alpha \log(c/b) \),
\[
\left( E \sup_{h \in H(\delta)} \left| (P - P_n) h \right| \right)^2 \leq A \frac{\alpha b^2}{n} \log \left( \frac{c}{b} \right).
\]
Combining these we conclude
\[
\beta_n(\delta, t) \leq A \left( \sqrt{\frac{\alpha b^2 \delta}{n} \log \left( \frac{c^2 A}{\delta} \right)} + \frac{\alpha b^2}{n} \log \left( \frac{c^2 A}{\delta} \right) + b\sqrt{\frac{\delta}{n}} + \frac{b^2 (1 + t)}{n} \right).
\]
Lemma 4.1 now implies that
\[
P(\delta_n \geq \beta_n(t)) \leq 4e^{-t} \quad \text{with} \quad \beta_n(t) := \inf \left\{ \tau > 0 : \sup_{\delta \geq \tau} \frac{\beta_n \left( \delta, \frac{\delta t}{\tau} \right)}{\delta} \leq \frac{1}{2} \right\}.
\]
It remains only to compute an upper bound for \( \beta_n(t) \). To that aim, it may be easily checked that, for any \( \tau > 0 \),
\[
\sup_{\delta \geq \tau} \frac{\beta_n \left( \delta, \frac{\delta t}{\tau} \right)}{\delta} = A \left( \sqrt{\frac{\alpha b^2}{\tau n} \log \left( \frac{c^2 A}{\tau} \right)} + \frac{\alpha b^2}{\tau n} \log \left( \frac{c^2 A}{\tau} \right) + b\sqrt{\frac{t}{\tau n}} + \frac{b^2 (1 + t)}{\tau n} \right).
\]
Next, notice that for any non-increasing functions \(y_1, \ldots, y_k\), we have

\[
\inf\{\tau > 0 : y_1(\tau) + \cdots + y_k(\tau) \leq 1\} \leq \max_{1 \leq i \leq k} \inf\{\tau > 0 : y_i(\tau) \leq \frac{1}{k}\}.
\]

As a result, we deduce that

\[
\beta_n(t) \leq \max_{1 \leq i \leq 4} \beta_n^{(i)}(t),
\]

where

\[
\beta_n^{(1)}(t) = \inf\left\{\tau > 0 : A \sqrt{\frac{\alpha b^2}{\tau n}} \log\left(\frac{c^2 A}{\tau}\right) \leq \frac{1}{8}\right\},
\]

\[
\beta_n^{(2)}(t) = \inf\left\{\tau > 0 : A \frac{b^2}{\tau n} \log\left(\frac{c^2 A}{\tau}\right) \leq \frac{1}{8}\right\},
\]

\[
\beta_n^{(3)}(t) = \inf\left\{\tau > 0 : Ab \sqrt{\frac{t}{\tau n}} \leq \frac{1}{8}\right\},
\]

\[
\beta_n^{(4)}(t) = \inf\left\{\tau > 0 : A \frac{b^2(1 + t)}{\tau n} \leq \frac{1}{8}\right\}.
\]

To upper bound \(\beta_n^{(1)}(t)\) and \(\beta_n^{(2)}(t)\), we use the fact that for all \(u > 0\) and \(v > 0\),

\[
\inf\left\{\tau > 0 : \frac{u}{n} \log\frac{v}{\tau} \leq 1\right\} \leq \frac{u \log n}{n},
\]

as soon as \(\log n \geq v/u\) (this fact can be checked by a substitution of the given bound). For \(\beta_n^{(3)}(t)\) and \(\beta_n^{(4)}(t)\), we compute the infimums directly. We finally obtain

\[
\beta_n^{(1)}(t) \leq \frac{64 A^2 \alpha b^2 \log n}{n}, \quad \beta_n^{(2)}(t) \leq \frac{8 A c b^2 \log n}{n}, \quad \beta_n^{(3)}(t) = \frac{64 A^2 b^2 t}{n}, \quad \beta_n^{(4)}(t) = \frac{8 A b^2 (1 + t)}{n},
\]

which finishes the proof for large enough \(n\), that is, when \(\log n \geq A c^2/(\alpha b^2)\). For smaller \(n\) the desired formula obviously holds for large \(A > 0\).

### 4.3. End of the proof of Theorem 2.2

As before, throughout the proof, let \(A > 0\) be a universal constant whose value may change from line to line. It follows from Lemma 4.2 that, for every \(\delta \geq 0\) and every \(t > 0\),

\[
P\left(\phi_n(\delta) \geq \beta_n(\delta, t)\right) \leq 2e^{-t},
\]

where, for a universal constant \(A > 0\),

\[
\beta_n(\delta, t) := A \left(\mathbb{E}_{h \in \mathcal{H}(\delta)} (P - P_n) h(h) + \left(\mathbb{E}_{h \in \mathcal{H}(\delta)} |(P - P_n) h|\right)^2 + b \sqrt{\frac{t \delta}{n} + \frac{b^2 (1 + t)}{n}}\right).
\]
Let us now bound the expected suprema of empirical processes given in $\beta_n(\delta, t)$ under the assumptions of Theorem 2.2 on either the metric or bracketing entropy. Consider the following cases of values $\alpha$.

**Case** $\alpha < 2$. Here the proof repeats in many ways the proof of Theorem 2.1, so some details are omitted. Taking $\sigma = \sqrt{32b^2\delta}$ in either Lemma A.4 for the metric entropy assumption or Lemma A.7 for the bracketing entropy assumption, we get

$$
E \sup_{h \in \mathcal{H}(\delta)} (P - P_n) \ell(h) \leq A \left( \sqrt{\frac{b^2\delta}{(2 - \alpha)^2 n}} \left( \frac{c}{\sqrt{\delta}} \right)^\alpha + \frac{b}{(2 - \alpha)^2 n} \left( \frac{c}{\sqrt{\delta}} \right)^\alpha \right).
$$

Again depending on the entropy assumption we use, either Lemma A.4 or Lemma A.7 with $\sigma = b$ yields

$$
E \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \leq A \left( \sqrt{\frac{b^2}{(2 - \alpha)^2 n}} \left( \frac{c}{b} \right)^\alpha + \frac{b}{(2 - \alpha)^2 n} \left( \frac{c}{b} \right)^\alpha \right).
$$

Hence, whenever $n \geq (2 - \alpha)^{-2}(c/b)^\alpha$, we have

$$
E \sup_{h \in \mathcal{H}(\delta)} |(P - P_n)h| \leq A \frac{b^2}{(2 - \alpha)^2 n} \left( \frac{c}{b} \right)^\alpha.
$$

Combining these bounds we conclude

$$
\beta_n(\delta, t) \leq A \left( \sqrt{\frac{b^2\delta}{(2 - \alpha)^2 n}} \left( \frac{c}{\sqrt{\delta}} \right)^\alpha + \frac{b^2}{(2 - \alpha)^2 n} \left( \frac{c}{\sqrt{\delta}} \right)^\alpha + b\sqrt{\frac{t\delta}{n}} + \frac{b^2(1 + t)}{n} \right).
$$

Lemma 4.1 now implies that

$$
P(\delta_n \geq \beta_n(t)) \leq 4e^{-t} \quad \text{with} \quad \beta_n(t) := \inf \left\{ \tau > 0 : \sup_{\delta \geq \tau} \frac{\beta_n(\delta, \tau)}{\delta} \leq \frac{1}{2} \right\}.
$$

It remains only to compute an upper bound for $\beta_n(t)$. To that aim, it may be easily checked that, for any $\tau > 0$,

$$
\sup_{\delta \geq \tau} \frac{\beta_n(\delta, \tau)}{\delta} = A \left( \sqrt{\frac{b^2}{(2 - \alpha)^2 \tau n}} \left( \frac{c}{\sqrt{\tau}} \right)^\alpha + \frac{b^2}{(2 - \alpha)^2 \tau n} \left( \frac{c}{\sqrt{\tau}} \right)^\alpha + b\sqrt{\frac{t\tau}{\tau n}} + \frac{b^2(1 + t)}{\tau n} \right).
$$

As a result, we deduce that

$$
\beta_n(t) \leq \max_{1 \leq i \leq 4} \beta_n^{(i)}(t),
$$

where

$$
\beta_n^{(1)}(t) = \inf \left\{ \tau > 0 : A \sqrt{\frac{b^2}{(2 - \alpha)^2 \tau n}} \left( \frac{c}{\sqrt{\tau}} \right)^\alpha \leq \frac{1}{8} \right\} = \left( \frac{64A^2b^2c^\alpha}{(2 - \alpha)^2 n} \right)^{\frac{1}{\alpha + 2}}.
$$
\[ \beta_n(t) = \inf \left\{ \tau > 0 : \frac{b^2}{(2 - \alpha)^2 \tau n} \left( \frac{c}{\sqrt{\tau}} \right)^\alpha \leq 1 \right\} = \left( \frac{8Ab^2 c^\alpha}{(2 - \alpha)^2 n} \right)^{\frac{2}{\tau + \alpha}}, \]

\[ \beta_n(t) = \inf \left\{ \tau > 0 : A b \sqrt{\frac{t}{\tau n}} \leq 1 \right\} = 64A b^2 t, \]

\[ \beta_n(t) = \inf \left\{ \tau > 0 : A b^2 \left( 1 + \frac{t}{\tau n} \right) \leq 1 \right\} = 8Ab^2 \left( 1 + \frac{t}{n} \right). \]

This completes the proof for $\alpha < 2$. 

**Case** $\alpha = 2$. Repeated application of either Lemma A.4 for the metric entropy assumption or Lemma A.7 for the bracketing entropy assumption allows us to write, whenever $n \geq (b \log n/c)^2$, that

\[ \beta_n(t) \leq A \left( \frac{c b \log n}{\sqrt{n}} + \frac{b}{n} + b \sqrt{\frac{t \delta}{n}} + \frac{b^2 (1 + t)}{n} \right). \]

Lemma 4.1 now implies that

\[ P(\delta_n \geq \beta_n(t)) \leq 4e^{-t}, \]

where

\[ \beta_n(t) \leq A \max \left\{ \frac{c b \log n}{\sqrt{n}}, \frac{b}{n}, \frac{b^2 (1 + t)}{n} \right\}. \]

This finishes the proof for $\alpha = 2$. 

**Case** $\alpha > 2$. Similarly, repeated application of either Lemma A.4 for the metric entropy assumption or Lemma A.7 for the bracketing entropy assumption enables us to write, whenever $n \geq \max\{b^{\alpha}, c^{\alpha/2} b^{-\alpha - 1} (\alpha - 2)^{-\alpha}\}$, that

\[ \beta_n(t) \leq A \left( \frac{b}{n^{1/\alpha}} + \frac{c^{\alpha/2} b}{(\alpha - 2) n^{1/\alpha}} + b \sqrt{\frac{t \delta}{n}} + \frac{b^2 (1 + t)}{n} \right). \]

Lemma 4.1 now implies that

\[ P(\delta_n \geq \beta_n(t)) \leq 4e^{-t}, \]

where

\[ \beta_n(t) \leq A \max \left\{ \frac{b}{n^{1/\alpha}}, \frac{c^{\alpha/2} b}{(\alpha - 2) n^{1/\alpha}}, \frac{b^2 t}{n}, \frac{b^2 (1 + t)}{n} \right\}. \]

This completes the proof of Theorem 2.2.
4.4. Proof of Theorem 2.3

The main technical result we will invoke is the Bernstein-type inequality for \( U \)-statistics (see, for example, inequality A.1 in [Clémençon et al., 2008]). By definition, we can write for any \( h \in \mathcal{H} \)

\[
V_n(h) - V_n(h^*) = \frac{1}{n(n-1)} \sum_{1 \leq i, j \leq n} \frac{(h(X_i) - h(X_j))^2 - (h^*(X_i) - h^*(X_j))^2}{2}.
\]

Using (A1), the kernel of this \( U \)-statistic satisfies for any \( 1 \leq i, j \leq n \)

\[
\frac{|(h(X_i) - h(X_j))^2 - (h^*(X_i) - h^*(X_j))^2|}{2} \leq 2b^2.
\]

Now the Bernstein inequality immediately implies that, for all \( \delta \geq 0 \),

\[
P\left(\left|V_n(h) - V_n(h^*) - (V(h) - V(h^*))\right| \geq \delta\right) \leq 2 \exp\left(- \frac{n\delta^2}{\sigma^2 + 4b^2\delta/3}\right),
\]

(24)

where \( \sigma^2 = \text{Var}\left[(h(X_i) - h(X_j))^2 - (h^*(X_i) - h^*(X_j))^2\right] \). A simple computation gives

\[
\sigma^2 \leq E\left[ (h(X_i) - h(X_j))^2 - (h^*(X_i) - h^*(X_j))^2 \right]^2 \leq 16b^2 E\left[ (h(X_i) - h(X_j) - h^*(X_i) + h^*(X_j))^2 \right] \leq 32b^2 P(h - h^*)^2.
\]

Using the fact that \( P(h - h^*)^2 \leq 2(V(h) - V(h^*)) \), see (20) in the proof of Lemma 4.2, we get

\[
\sigma^2 \leq 32b^2 P(h - h^*)^2 \leq 64b^2 (V(h) - V(h^*)).
\]

The main idea of the proof is to consider concentration of \( V_n(h) - V_n(h^*) \) not around the mean \( V(h) - V(h^*) \), but around \( c(V(h) - V(h^*)) \) for some \( c > 1 \). This idea is often used to obtain fast rates, see, for example, [Bartlett et al., 2005] and [Koltchinskii, 2006]. Combining the last bound for \( \sigma^2 \) with (24) we obtain for all \( \delta > 0 \),

\[
P\left(V_n(h) - V_n(h^*) - 2(V(h) - V(h^*)) \geq \delta\right)
\]

\[
\leq P\left(V_n(h) + V_n(h^*) - (V(h) - V(h^*)) \geq \delta + V(h) - V(h^*)\right)
\]

\[
\leq 2 \exp\left(- \frac{n(\delta + V(h) - V(h^*))^2}{64b^2 (V(h) - V(h^*)) + 4b^2(\delta + V(h) - V(h^*)/3)}\right)
\]

\[
\leq 2 \exp\left(- \frac{n(\delta + V(h) - V(h^*))}{(64 + 4/3)b^2}\right)
\]

\[
\leq 2 \exp\left(- \frac{n\delta}{66b^2}\right),
\]

(25)
where the last inequality follows from the fact that $V(h) - V(h^*) \geq 0$. Similarly,

$$P\left( V(h) - V(h^*) - 2\left( V_n(h) - V_n(h^*) \right) \geq \delta \right) \leq P\left( \frac{V(h) + V(h^*) - (V_n(h) - V_n(h^*))}{2} \geq \frac{\delta + V(h) - V(h^*)}{2} \right) \leq 2 \exp \left( -\frac{n\delta}{260b^2} \right). \quad (26)$$

Having disposed of this preliminary step, we can now return to the proof. Fix any $\varepsilon > 0$ and let $\mathcal{H}_\varepsilon$ be a minimal $\varepsilon$-net of $\mathcal{H}$ with respect to $L_2(P)$. Choose any $\hat{h}_n \in \arg \min_{h \in \mathcal{H}_\varepsilon} V_n(h)$ and any $h^*_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\left\| h^*_\varepsilon - h^* \right\|_{L_2(P)} \leq \varepsilon$. We have the following decomposition

$$V(\hat{h}_n) - V(h^*) \leq V(\hat{h}_n) - V(h^*) - 2(V_n(\hat{h}_n) - V_n(h^*)) = V(\hat{h}_n) - V(h^*) - 2(V_n(\hat{h}_n) - V_n(h^*)) + 2(V_n(h^*_\varepsilon) - V_n(h^*)) \leq \sup_{h \in \mathcal{H}_\varepsilon} \{ V(h) - V(h^*) - 2(V_n(h) - V_n(h^*)) \}. \quad (27)$$

Let us bound all the terms in (27) separately. First, we get by (26) and the union bound,

$$P\left( \sup_{h \in \mathcal{H}_\varepsilon} \{ V(h) - V(h^*) - 2(V_n(h) - V_n(h^*)) \} \geq \delta \right) \leq 2\mathcal{N}(\mathcal{H}, \| \cdot \|_{L_2(P)}, \varepsilon) \exp \left( -\frac{n\delta}{260b^2} \right).$$

Choosing $\delta = 260b^2n^{-1}(\mathcal{N}(\mathcal{H}, \| \cdot \|_{L_2(P)}, \varepsilon) + t)$ and using the assumption on the covering number of $\mathcal{H}$, we obtain that, with probability at least $1 - 2e^{-t}$,

$$\sup_{h \in \mathcal{H}_\varepsilon} \{ V(h) - V(h^*) - 2(V_n(h) - V_n(h^*)) \} \leq \frac{260b^2}{n} \left( \frac{c}{\varepsilon} \right)^\alpha + \frac{260b^2t}{n}.$$

Further, for the second term in (27) we have

$$2(V_n(h^*_\varepsilon) - V_n(h^*)) = 2(V_n(h^*_\varepsilon) - V_n(h^*)) - 4(V(h^*_\varepsilon) - V(h^*)) + 4(V(h^*_\varepsilon) - V(h^*)).$$

It follows from (25) that, with probability at least $1 - 2e^{-t}$,

$$2(V_n(h^*_\varepsilon) - V_n(h^*)) - 4(V(h^*_\varepsilon) - V(h^*)) \leq \frac{132b^2t}{n}.$$ 

Assumption (A3) implies that

$$V(h^*_\varepsilon) - V(h^*) \leq c_1\varepsilon^2 + c_2n^{-\frac{2}{2+\alpha}}.$$

Combining all the bounds, we conclude that, with probability at least $1 - 4e^{-t}$,

$$V(\hat{h}_n) - V(h^*) \leq \frac{260b^2}{n} \left( \frac{c}{\varepsilon} \right)^\alpha + \frac{392b^2t}{n} + 4c_1\varepsilon^2 + 4c_2n^{-\frac{2}{2+\alpha}}.$$

The proof is finished by taking $\varepsilon = b^2/(2+\alpha)n^{-1/(2+\alpha)}$. 

\textit{Empirical Variance Minimization}
4.5. Proof of Proposition 3.1

With notation $\mathcal{H} = \{f - g_\phi, \phi \in \Phi\}$ and $\mathcal{G} = \{g_\phi, \phi \in \Phi\}$, it obviously holds that

$\mathcal{N}_1[\mathcal{H}, \| \cdot \|_{L^2(P)}, u] = \mathcal{N}_1[\mathcal{G}, \| \cdot \|_{L^2(P)}, u]$.

For any two functions $\phi_1, \phi_2 \in \Phi$, we have

$$\|g_{\phi_1} - g_{\phi_2}\|_{L^2(P)} \leq \left\| \langle \phi_1 - \phi_2, \nabla \log \pi \rangle \right\|_{L^2(P)} + \|\text{div}(\phi_1 - \phi_2)\|_{L^2(P)}.$$ 

Let us denote the coordinates of $\phi_k : \mathcal{X} \to \mathbb{R}^d$ by $\phi^i_k, i = 1, \ldots, d, k = 1, 2$. By our assumptions and the Cauchy-Schwarz inequality, the first term can be bounded as

$$\left\| \langle \phi_1 - \phi_2, \nabla \log \pi \rangle \right\|_{L^2(P)} \leq \sqrt{L} \left( \int_{B_R} |\phi_1(x) - \phi_2(x)|^2 |\nabla \pi(x)|^2 \, dx \right)^{1/2} \leq M \sqrt{L} \sum_{i=1}^d \left( \int_{B_R} |\phi^i_1(x) - \phi^i_2(x)|^2 \, dx \right)^{1/2},$$

provided that $\|\pi\|_{W^{1,2}(\lambda)} \leq M$ for some $M > 0$. Further, using the Minkowski inequality,

$$\|\text{div}(\phi_1 - \phi_2)\|_{L^2(P)} = \left( \int_{B_R} \left( \sum_{i=1}^d \frac{\partial}{\partial x_i} (\phi^i_1(x) - \phi^i_2(x)) \right)^2 \pi(x) \, dx \right)^{1/2} \leq \sqrt{L} \sum_{i=1}^d \left( \int_{B_R} \left( \frac{\partial}{\partial x_i} (\phi^i_1(x) - \phi^i_2(x)) \right)^2 \, dx \right)^{1/2}.$$ 

Combining these bounds, we conclude

$$\|g_{\phi_1} - g_{\phi_2}\|_{L^2(P)} \leq \sqrt{L}(M + 1) \sum_{i=1}^d \|\phi^i_1 - \phi^i_2\|_{W^{1,2}(\lambda)}.$$ 

By the arithmetic properties of the bracketing entropy, for all $u > 0$,

$$\mathcal{N}_1[\mathcal{G}, \| \cdot \|_{L^2(P)}, u] \leq \prod_{i=1}^d \mathcal{N}_1[\Phi^i, \| \cdot \|_{W^{1,2}(\lambda)}, \frac{u}{\sqrt{L}(M + 1)d}],$$

where $\Phi^i$ denotes the class generated by $i$-th coordinate of $\phi \in \Phi$, that is, $\Phi^i = \{\phi^i, \phi \in \Phi\}$. Using the classic results for the bracketing entropy of Sobolev spaces defined on $B_R$ (see, for example, [Nickl and Pötscher, 2007, Corollary 4 and Section 3.3.4]), we get

$$\mathcal{N}_1[\mathcal{G}, \| \cdot \|_{L^2(P)}, u] \leq \prod_{i=1}^d c \left( \frac{\sqrt{L}(M + 1)d}{u} \right)^{d/s} \leq \left( \frac{c^{s/d} \sqrt{L}(M + 1)d}{u} \right)^{d^2/s}$$

for some $c > 0$ not depending on $u$. The result now follows from Theorem 2.2.
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Appendix A: Supplementary Material

In this section we have compiled some standard facts on empirical processes. Namely, we start with Bousquet’s concentration inequality given in Appendix A.1. Then we provide bounds on the expectation of supremum of empirical process based on either covering number or bracketing number assumptions in Appendix A.2 and Appendix A.3 respectively. Throughout the section, $\mathcal{H}$ stands for a class of functions $h : \mathcal{X} \to \mathbb{R}$.

A.1. Bousquet’s concentration inequality

We start with the well-known concentration result, known as Bousquet’s form of Talagrand’s inequality for empirical processes. It involves a notion of variance of the empirical process

$$\sigma_{\mathcal{H}} := \sup_{h \in \mathcal{H}} \sqrt{P h^2},$$

which plays a crucial role in many modern proof techniques involving the local behaviour of the supremum of empirical process. The proof of the following lemma can be found in [Bousquet, 2002] or [Giné and Nickl, 2016].

Lemma A.1. Suppose that all functions in $\mathcal{H}$ are $[a, b]$-valued, for some $a < b$. Then, for all $n \geq 1$ and all $t > 0$,

$$\sup_{h \in \mathcal{H}} (P - P_n)h \leq \mathbb{E} \sup_{h \in \mathcal{H}} (P - P_n)h + \sqrt{\frac{2t}{n} \left( \sigma^2_{\mathcal{H}} + 2(b - a) \mathbb{E} \sup_{h \in \mathcal{H}} (P - P_n)h \right)} + \frac{(b - a)t}{3n},$$

with probability larger than $1 - e^{-t}$.

A.2. Bounds on expected supremum of empirical process using metric entropy

In this section we derive explicit bounds for the expected supremum of empirical process under specific assumptions on the metric entropy of $\mathcal{H}$. We let $\zeta_1, \ldots, \zeta_n$ be a sequence of i.i.d. random signs, that is, $P(\zeta_i = -1) = P(\zeta_i = 1) = 1/2$, independent from the sample $X_1, \ldots, X_n$. We denote

$$R_n(h) := \frac{1}{n} \sum_{i=1}^n \zeta_i h(X_i),$$

the Rademacher process indexed by $\mathcal{H}$. The next result is a generalized version of Dudley’s entropy bound; its proof can be found, for instance, in [Srebro et al., 2010, Lemma A.3].

Lemma A.2. Let $X_1, \ldots, X_n$ be a sample associated to the empirical distribution $P_n$ and suppose that $\mathcal{H} \subset L^2(P_n)$. Then, for all $\varepsilon > 0$,

$$\mathbb{E} \left[ \sup_{h \in \mathcal{H}} R_n(h) \left| X_1, \ldots, X_n \right. \right] \leq 4 \varepsilon + \frac{12}{\sqrt{n}} \int_{\varepsilon}^{\sigma_n} \sqrt{\log \mathcal{N}(\mathcal{H}, \|\cdot\|_{L^2(P_n)}, u)} \, du,$$

where $\sigma_n := \sup_{h \in \mathcal{H}} \sqrt{P_n h^2}$. 

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The next two lemmas apply Lemma A.2 to derive the upper bounds on the expected supremum of the empirical process under the polynomial and exponential covering number assumptions. They consist in a simple version of Theorem 3.12 in [Koltchinskii, 2011] and can be also found in, for example, [Giné and Koltchinskii, 2006]. We start with the case of a class $\mathcal{H}$ with polynomial covering number.

**Lemma A.3.** Let $\mathcal{H}$ be a class of $[-b, b]$-valued functions for some $b > 0$. Suppose that, for all $0 < u \leq b$,
\[
\mathcal{N}(\mathcal{H}, \| \cdot \|_{L^2(P_n)}, u) \leq \left( \frac{c}{u} \right)^\alpha \text{ almost surely},
\]
for some constants $c > b$ and $\alpha > 0$. Then, for any $\sigma \in [\sigma_{\mathcal{H}}, b]$ and all $n \geq 1$,
\[
E \sup_{h \in \mathcal{H}} (P - P_n)h \leq A \left( \sqrt{\alpha \sigma_n^2} \log \left( \frac{c}{\sigma} \right) + \frac{\alpha b}{n} \log \left( \frac{c}{\sigma} \right) \right),
\]
where $A > 0$ is an absolute constant computed explicitly in the proof.

**Proof.** To simplify the notation throughout the proof, we denote
\[
R := E \sup_{h \in \mathcal{H}} R_n(h).
\]
From the symmetrization principle, we get
\[
E \sup_{h \in \mathcal{H}} (P - P_n)h \leq 2R.
\]
Now it remains to bound $R$. Taking $\varepsilon = 0$ in Lemma A.2, yields
\[
R \leq \frac{12}{\sqrt{n}} E \left[ \int_0^{\sigma_n} \sqrt{\log \left( \frac{c}{u} \right)} \, du \right],
\]
where we have used the assumption on the covering numbers of $\mathcal{H}$ and the fact that, almost surely, $\sigma_n \leq b$. Since the function
\[
\varepsilon \mapsto \int_0^\varepsilon \sqrt{\log \left( \frac{c}{u} \right)} \, du,
\]
is concave on $(0, c]$, we deduce from Jensen’s inequality that
\[
R \leq \frac{12}{\sqrt{n}} \int_0^{\sigma_n} \sqrt{\log \left( \frac{c}{u} \right)} \, du.
\]
Now let us bound the value of $E[\sigma_n]$. Using once again Jensen’s inequality, we obtain
\[
E[\sigma_n] = E \left[ \sup_{h \in \mathcal{H}} \sqrt{P_n h^2} \right] \leq \sqrt{E \sup_{h \in \mathcal{H}} P_n h^2}.
\]
The symmetrization and contraction principles imply
\[
E[\sigma_n] \leq \sqrt{\sigma^2 + E \sup_{h \in \mathcal{H}} (P_n - P)h^2}
\]
\[
\sqrt{\sigma^2 + 2E \sup_{h \in \mathcal{H}} R_n(h^2)} \\
\leq \sqrt{\sigma^2 + 4bR} \\
\leq \sigma + 2\sqrt{bR}.
\]

Since \(\sigma_n \leq b\) almost surely, we deduce that
\[
E[\sigma_n] \leq B := \min \left\{ b, \sigma + 2\sqrt{bR} \right\}.
\]

Consequently, after a change of variable we get
\[
R \leq 12 \sqrt{\frac{\alpha}{n}} \int_0^B \sqrt{\log \left( \frac{c}{u} \right)} \, du = 12c \sqrt{\frac{\alpha}{n}} \int_{c/B}^{+\infty} u^{-2} \sqrt{\log(u)} \, du. \tag{28}
\]

Observe that, by assumptions, \(c/B \geq e\). Using integration by parts, it follows that
\[
\sup_{t \geq e} \frac{\int_t^{+\infty} u^{-2} \sqrt{\log(u)} \, du}{t^{-1} \sqrt{\log(t)}} \leq 1 + \sup_{t \geq e} \frac{1}{2} \int_t^{+\infty} u^{-2} (\log(u))^{-3/2} \, du \leq \frac{3}{2}.
\]

Therefore, we have
\[
\int_{c/B}^{+\infty} u^{-2} \sqrt{\log(\alpha u)} \, du \leq \frac{3}{2} \frac{B}{c} \sqrt{\log \left( \frac{c}{B} \right)} \leq \frac{3}{2} \frac{B}{c} \sqrt{\log \left( \frac{c}{\sigma} \right)}, \tag{29}
\]

where, in the last inequality, we have used that \(B \geq \sigma\). Now combining (28) and (29), we deduce that
\[
R \leq 18(\sigma + 2\sqrt{bR}) \sqrt{\frac{\alpha}{n} \log \left( \frac{c}{\sigma} \right)}.
\]

Since \(R \leq x + y\sqrt{R}\) implies \(R \leq 2x + 2y^2\) for any \(x, y > 0\), we obtain
\[
R \leq 36 \sqrt{\frac{\alpha \sigma^2}{n} \log \left( \frac{c}{\sigma} \right) + 2592 \frac{\alpha b}{n} \log \left( \frac{c}{\sigma} \right)},
\]

which completes the proof. \(\square\)

Next we study the case of classes \(\mathcal{H}\) for which the covering number grows exponentially. The important observation here is that the square root of the entropy might not be integrable around zero anymore.

**Lemma A.4.** Let \(\mathcal{H}\) be a class of \([-b, b]\)-valued functions for some \(b > 0\). Suppose that, for all \(0 < u \leq b\),
\[
\log N(\mathcal{H}, \| \cdot \|_{L^2(P_n)}, u) \leq \left( \frac{c}{u} \right)^\alpha,
\]

for some constants \(c > 0\) and \(\alpha > 0\). Then for some absolute constant \(A > 0\) and all \(n \geq 1\) the following holds.
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1. If $\alpha < 2$, then, for any $\sigma \in [\sigma_H, b]$,
   \[
   E \sup_{h \in H} (P - P_n) h \leq A \left( \frac{\sigma}{(2 - \alpha) \sqrt{n}} \left( \frac{c}{\sigma} \right)^{\alpha/2} + \frac{b}{(2 - \alpha)^2 n} \left( \frac{c}{\sigma} \right)^{\alpha} \right).
   \]

2. If $\alpha = 2$, then
   \[
   E \sup_{h \in H} (P - P_n) h \leq A \left( \frac{c \log n}{\sqrt{n}} + \frac{b}{n} \right).
   \]

3. If $\alpha > 2$, then
   \[
   E \sup_{h \in H} (P - P_n) h \leq \frac{Ab}{n^{1/\alpha}} \left( 1 + \frac{1}{(\alpha - 2)} \left( \frac{c}{b} \right)^{\alpha/2} \right).
   \]

**Proof.** From the symmetrization principle, we get
   \[
   E \sup_{h \in H} (P - P_n) h \leq 2R,
   \]
where $R := E \sup_{h \in H} R_n(h)$ for brevity. Now it remains to bound $R$. Consider the following cases of values $\alpha$.

**Case of $\alpha < 2$.** Here the square root of the entropy is still integrable around zero. Using Lemma A.2 with $\varepsilon = 0$, we get
   \[
   R \leq \frac{12}{\sqrt{n}} E \left[ \int_0^{\sigma_n} \left( \frac{c}{u} \right)^{\alpha/2} du \right].
   \]
As in the proof of Lemma A.3, using Jensen’s inequality together with the symmetrization and contraction principles, we obtain
   \[
   R \leq \frac{12}{\sqrt{n}} E \left[ \int_0^{B} \left( \frac{c}{u} \right)^{\alpha/2} du \right],
   \]
where $B = \min\{b, \sigma + 2\sqrt{bR}\}$. Integrating and using the fact that $B \geq \sigma$, yields
   \[
   R \leq \frac{24 c^{\alpha/2}}{(2 - \alpha) \sqrt{n}} B^{1-\alpha/2} \leq \frac{24 c^{\alpha/2}}{(2 - \alpha) \sqrt{n}} \sigma^{1-\alpha/2} \leq \frac{24 c^{\alpha/2}}{(2 - \alpha) \sqrt{n}} \sigma^{-\alpha/2} (\sigma + 2\sqrt{bR}).
   \]
Observing that $R \leq x + y\sqrt{R}$ implies $R \leq 2x + 2y^2$ for any $x, y > 0$, we obtain
   \[
   R \leq \frac{48 \sigma}{(2 - \alpha) \sqrt{n}} \left( \frac{c}{\sigma} \right)^{\alpha/2} + \frac{4608 b}{(2 - \alpha)^2 n} \left( \frac{c}{\sigma} \right)^{\alpha}.
   \]

**Case of $\alpha = 2$.** Since the square root of the entropy is not anymore integrable around zero, we apply Lemma A.2 with $\varepsilon = \sigma_n/n$ and obtain
   \[
   R \leq \frac{4}{n} E[\sigma_n] + \frac{12 c}{\sqrt{n}} E \left[ \int_{\sigma_n/n}^{\sigma_n} \frac{du}{u} \right] \leq \frac{4b}{n} + \frac{12c \log n}{\sqrt{n}}.
   \]
where in the last inequality we have used the fact that, almost surely, $\sigma_n \leq b$.

**Case of $\alpha > 2$.** It follows from Lemma A.2 that, for all $\varepsilon > 0$,

$$R \leq \mathbb{E} \left[ 4\varepsilon + \frac{12}{\sqrt{n}} \int_{\varepsilon}^{+\infty} \sqrt{\log \mathcal{N} \left( \mathcal{H}, \| \cdot \|_{L^2(P_n)}, u \right)} \, du \right].$$

Taking $\varepsilon = b/n^{1/\alpha}$ gives us

$$R \leq \frac{4b}{n^{1/\alpha}} + \frac{12}{\sqrt{n}} \mathbb{E} \left[ \int_{b/n^{1/\alpha}}^{+\infty} \left( \frac{c}{u} \right)^{\alpha/2} du \right] \leq \frac{4b}{n^{1/\alpha}} + \frac{24b}{(\alpha - 2)n^{1/\alpha}} \left( \frac{c}{b} \right)^{\alpha/2}.$$  

This finishes the proof. \(\square\)

### A.3. Bounds on the expected supremum of empirical process using the bracketing entropy

In this section we derive explicit bounds for the expected supremum of empirical process under specific assumptions on the bracketing entropy of a class $\mathcal{H}$. We recall that the variance $\sigma_{\mathcal{H}}$ of the empirical process is defined as

$$\sigma_{\mathcal{H}} := \sup_{h \in \mathcal{H}} \sqrt{Ph^2}.$$  

The following result is an analogue of Dudley’s entropy bound, see Lemma A.2, but for the bracketing entropy. This result follows from Lemma 7 in [Han et al., 2019] with the only difference that we present this result for $[-b,b]$-valued functions, not for $[-1,1]$-valued functions. However, a simple rescaling argument $\mathcal{H} \mapsto \mathcal{H}/b$ proves the claim.

**Lemma A.5.** Let $\mathcal{H}$ be a class of $[-b,b]$-valued functions for some $b > 0$. Then for any $\sigma > \sigma_{\mathcal{H}}$, any $\varepsilon \in [0,\sigma/3]$, and any $n \geq 1$, it holds that

$$\mathbb{E} \sup_{h \in \mathcal{H}} \left| (P - P_n)h \right| \leq A \left( \varepsilon + \frac{1}{\sqrt{n}} \int_{\varepsilon}^{2\sigma} \sqrt{\log \mathcal{N}_{[-1]} \left( \mathcal{H}, \| \cdot \|_{L^2(P)}, u \right)} \, du + \frac{b}{n} \log \mathcal{N}_{[-1]} \left( \mathcal{H}, \| \cdot \|_{L^2(P)}, \sigma \right) \right),$$

where $A > 0$ is an absolute constant. In particular, for $\sigma = 2b$, we have

$$\mathbb{E} \sup_{h \in \mathcal{H}} \left| (P - P_n)h \right| \leq A \left( \varepsilon + \frac{1}{\sqrt{n}} \int_{\varepsilon}^{2b} \sqrt{\log \mathcal{N}_{[-1]} \left( \mathcal{H}, \| \cdot \|_{L^2(P)}, u \right)} \, du + \frac{b}{n} \right).$$

The next two lemmas apply Lemma A.5 to derive explicit upper bounds on the expected supremum of empirical process under specific assumptions on the bracketing metric entropy of $\mathcal{H}$. The first case we study is whenever the class $\mathcal{H}$ has polynomial complexity.

**Lemma A.6.** Let $\mathcal{H}$ be a class of $[-b,b]$-valued functions for some $b > 0$. Suppose that, for all $0 < u \leq b$,

$$\mathcal{N}_{[-1]} \left( \mathcal{H}, \| \cdot \|_{L^2(P)}, u \right) \leq \left( \frac{c}{u} \right)^{\alpha},$$

where $\alpha > 2$.
for some constants $c > b$ and $\alpha > 0$. Then, for any $\sigma \in [\sigma_H, b]$ and all $n \geq 1$,

$$
E \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \sqrt{\frac{\alpha \sigma^2}{n} \log \left( \frac{c}{\sigma} \right)} + \frac{\alpha b}{n} \log \left( \frac{c}{\sigma} \right) \right),
$$

where $A > 0$ is an absolute constant.

**Proof.** Fix any $\sigma \in [\sigma_H, b]$. It follows from Lemma A.5 with $\varepsilon = 0$ that

$$
E \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \frac{1}{\sqrt{n}} \int_0^\sigma \sqrt{\log N [H, \| \cdot \|_{L^2(P)}, u]} \, du + \frac{b}{n} \log N [H, \| \cdot \|_{L^2(P), \sigma}] \right),
$$

(30)

where $A > 0$ is an absolute constant. Using the assumption on the bracketing number of $\mathcal{H}$, we get

$$
\int_0^\sigma \sqrt{\log N [H, \| \cdot \|_{L^2(P)}, u]} \, du \leq \sqrt{\alpha} \int_0^\sigma \sqrt{\log \left( \frac{c}{u} \right)} \, du = \sqrt{\alpha} \int_{c/\sigma}^{+\infty} u^{-2} \sqrt{\log (u)} \, du.
$$

Observe that, by assumptions, $c/\sigma \geq e$. Now, integration by parts yields

$$
\sup_{t \geq e} \int_t^{+\infty} u^{-2} \sqrt{\log (u)} \, du \leq 1 + \sup_{t \geq e} \frac{1}{2} \int_t^{+\infty} u^{-2} \left( \log (u) \right)^{-3/2} \, du \leq \frac{3}{2}.
$$

Therefore, we have

$$
\int_0^\sigma \sqrt{\log N [H, \| \cdot \|_{L^2(P)}, u]} \, du \leq \sqrt{\frac{9}{2} \alpha \sigma^2 \log \left( \frac{c}{\sigma} \right)}.
$$

Substituting this into (30), for some new absolute constant $A > 0$, we obtain

$$
E \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \sqrt{\frac{\alpha \sigma^2}{n} \log \left( \frac{c}{\sigma} \right)} + \frac{\alpha b}{n} \log \left( \frac{c}{\sigma} \right) \right),
$$

which completes the proof.

We turn to the case where the complexity of $\mathcal{H}$ is exponential. Similar to the metric entropy case, the square root of the bracketing entropy might not be integrable around zero anymore.

**Lemma A.7.** Let $\mathcal{H}$ be a class of $[-b, b]$-valued functions for some $b > 0$. Suppose that, for all $0 < u \leq 2b$,

$$
\log N [H, \| \cdot \|_{L^2(P)}, u] \leq \left( \frac{c}{u} \right)^\alpha,
$$

(31)

for some constants $c > 0$ and $\alpha > 0$. Then for some absolute constant $A > 0$ and all $n \geq 1$ the following holds.

1. If $\alpha < 2$, then, for any $\sigma \in [\sigma_H, b]$,

$$
E \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \frac{\sigma}{(2 - \alpha) \sqrt{n}} \left( \frac{c}{\sigma} \right)^{\alpha/2} + \frac{b}{n} \left( \frac{c}{\sigma} \right)^\alpha \right).
$$
2. If $\alpha = 2$, then
\[
\mathbb{E} \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \frac{c \log n}{\sqrt{n}} + \frac{b}{n} \right).
\]

3. If $\alpha > 2$, then
\[
\mathbb{E} \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq \frac{Ab}{n^{1/\alpha}} \left( 1 + \frac{1}{(\alpha - 2)} \left( \frac{c}{b} \right)^{\alpha/2} \right).
\]

**Proof.** Throughout the proof, let $A > 0$ be a universal constant whose value may change from line to line. Consider the following cases.

**Case of $\alpha < 2$.** Fix any $\sigma \in [\sigma_{\mathcal{H}}, b]$. It follows from Lemma A.5 with $\varepsilon = 0$ that
\[
\mathbb{E} \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \frac{1}{\sqrt{n}} \int_0^{\sigma} \sqrt{\log \mathcal{N}[\mathcal{H}, \| \cdot \|_{L^2(P)}, u]} \, du + \frac{b}{n} \log \mathcal{N}[\mathcal{H}, \| \cdot \|_{L^2(P)}, \sigma] \right).
\]

Using the assumption on the bracketing number of $\mathcal{H}$, we get
\[
\int_0^{\sigma} \sqrt{\log \mathcal{N}[\mathcal{H}, \| \cdot \|_{L^2(P)}]} \, du \leq \int_0^{\sigma} \left( \frac{c}{u} \right)^{\alpha/2} \, du = \frac{2 \alpha^{\alpha/2} \sigma^{1-\alpha/2}}{2 - \alpha}.
\]

Therefore, we conclude
\[
\mathbb{E} \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \frac{\sigma}{(2 - \alpha)\sqrt{n}} \left( \frac{c}{\sigma} \right)^{\alpha/2} + \frac{b}{n} \left( \frac{c}{\sigma} \right)^{\alpha} \right).
\]

**Case of $\alpha = 2$.** Since the square root of the bracketing entropy is not anymore integrable around zero, we apply Lemma A.5 with $\sigma = 2b$ and $\varepsilon = b/n$ and obtain for $n \geq 3/2$
\[
\mathbb{E} \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \frac{b}{n} + \frac{1}{\sqrt{n}} \int_{b/n}^{2b} \frac{c}{u} \, du + \frac{b}{n} \right) \leq A \left( \frac{b}{n} + \frac{c \log n}{\sqrt{n}} \right).
\]

It is clear that the same bound holds for $n = 1$.

**Case of $\alpha > 2$.** Using Lemma A.5 with $\sigma = 2b$ and $\varepsilon = b/n^{1/\alpha}$, we obtain for $n \geq 3/2$
\[
\mathbb{E} \sup_{h \in \mathcal{H}} |(P - P_n)h| \leq A \left( \frac{b}{n^{1/\alpha}} + \frac{1}{\sqrt{n}} \int_{b/n^{1/\alpha}}^{+\infty} \left( \frac{c}{u} \right)^{\alpha/2} \, du + \frac{b}{n} \right) \leq A \left( \frac{b}{n^{1/\alpha}} + \frac{2b}{(\alpha - 2)n^{1/\alpha}} \left( \frac{c}{b} \right)^{\alpha/2} \right).
\]

The same bound obviously holds for $n = 1$. This finishes the proof of the lemma. \qed

**A.4. Proof of Lemma 4.1**

As it was mentioned, this result follows from a combination of arguments presented in [Koltchinskii, 2011]. We devide the proof into two steps.
Step 1. Let \( \delta_j, j \geq 0 \), be a strictly decreasing sequence of positive numbers with \( \delta_0 = \bar{\delta} \) and let \( t_j, j \geq 0 \), be an arbitrary sequence of positive numbers. For all \( \delta \geq 0 \), denote

\[
\bar{\beta}(\delta) = \sum_{j=0}^{+\infty} \beta(\delta_j, t_j) 1 \{ \delta_{j+1} < \delta \leq \delta_j \},
\]

and set

\[
\delta^* = \sup \{ \delta \geq 0 : \delta \leq \bar{\beta}(\delta) \}.
\]

The goal of this first step is to prove that, for all \( \delta \geq \delta^* \),

\[
P(\hat{\delta} \geq \delta) \leq \sum_{j: \delta_j \geq \delta} e^{-t_j}.
\]

Fix any \( \delta > \delta^* \). For all \( j \geq 0 \), define \( E_j = \{ \phi(\delta_j) < \bar{\beta}(\delta_j) \} \) and set

\[
E = \bigcap_{j: \delta_j \geq \delta} E_j.
\]

Combining property (16) with the definition of \( \bar{\beta} \) (in particular that \( \bar{\beta}(\delta_j) = \beta(\delta_j, t_j) \) by construction) yields

\[
P(E) \geq 1 - \sum_{j: \delta_j \geq \delta} e^{-t_j}.
\]

On the event \( E \) and for all \( \delta' \geq \delta \), we have \( \phi(\delta') \leq \bar{\beta}(\delta') \), by the monotonicity of \( \phi \) and by the definition of \( \bar{\beta} \). Thus, on the event \( \{ \hat{\delta} \geq \delta \} \cap E \) we obtain

\[
\hat{\delta} \leq \phi(\hat{\delta}) \leq \bar{\beta}(\hat{\delta}),
\]

which implies that \( \delta \leq \hat{\delta} \leq \delta^* \). Since this contradicts \( \delta > \delta^* \), we deduce that \( \{ \hat{\delta} \geq \delta \} \subset E^c \) and hence

\[
P(\hat{\delta} \geq \delta) < \sum_{j: \delta_j \geq \delta} e^{-t_j}.
\]

By continuity, this also holds for \( \delta = \delta^* \).

Step 2. Fix \( t > 0 \) and set \( \tau = \beta(t) \) with \( \beta(t) \) defined in (17) (there is no loss of generality in assuming that \( \beta(t) \) is finite). Set for \( j \geq 0 \),

\[
\delta_j = \frac{\bar{\delta}}{2^j} \quad \text{and} \quad t_j = \frac{t\bar{\delta}}{2^j}.
\]

Now for this specific choice of \( \delta_j \) and \( t_j \) observe that, for any \( \delta \geq \tau \),

\[
\frac{\bar{\beta}(\delta)}{\delta} = \sum_{j=0}^{+\infty} \frac{\bar{\delta}}{2^j} \left( \frac{\bar{\delta}}{2^j} \right)^{-1} \beta \left( \frac{\bar{\delta}}{2^j}, \frac{t\bar{\delta}}{2^j} \right) 1 \left\{ \frac{\bar{\delta}}{2^{j+1}} < \delta \leq \frac{\bar{\delta}}{2^j} \right\}
\]
\[
\leq 2 \sum_{j=0}^{+\infty} \left( \frac{\delta}{2^j} \right)^{-1} \beta \left( \frac{\delta}{2^j}, \frac{t\delta}{\tau 2^j} \right) 1 \left\{ \frac{\delta}{2^j+1} < \delta \leq \frac{\delta}{2^j} \right\}
\]
\[
\leq 2 \sup_{\delta \geq \tau} \frac{\beta(\delta, t\delta/\tau)}{\delta}
\leq 1,
\]

where the last inequality follows from the definition of \( \beta(t) \). Hence
\[
\delta \geq \delta^* = \sup \left\{ \delta \geq 0 : \delta \leq \bar{\beta}(\delta) \right\}.
\]

Then, according to the first step above,
\[
P(\hat{\delta} \geq \delta) \leq \sum_{j: \frac{\delta}{2^j} \geq \delta} e^{-t\frac{\delta}{2^j}}. \tag{32}
\]

The sum on the right-hand side of (32) may be bounded as follows. Let
\[
j^* = \max \left\{ j \geq 0 : \frac{\delta}{2^j} \geq \tau \right\}.
\]

Then
\[
\sum_{j: \frac{\delta}{2^j} \geq \delta} e^{-t\frac{\delta}{2^j}} = \sum_{j=0}^{j^*} e^{-t\frac{\delta}{2^j}} \leq \sum_{j=0}^{j^*} e^{-t2(j^*-j)} \leq \sum_{j=0}^{+\infty} e^{-t2^j},
\]

and
\[
\sum_{j=0}^{+\infty} e^{-t2^j} \leq e^{-t} + \sum_{j=1}^{+\infty} (2^j - 2^{j-1}) e^{-t2^j} \leq e^{-t} + \int_{1}^{+\infty} e^{-tx} \, dx = e^{-t}.
\]

Finally, we have proved that, for all \( t \geq 0 \) and for all \( \delta > \tau = \beta(t) \) we have
\[
P(\hat{\delta} \geq \delta) \leq 2e^{-t}.
\]

By continuity, this result also holds for \( \delta = \beta(t) \).