Attenuation and shock waves in linear hereditary viscoelastic media. Strick-Mainardi and Jeffreys-Lomnitz-Strick creep compliances.

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Abstract

Dispersion, attenuation and wavefronts in a class of linear viscoelastic media proposed by Strick and Mainardi in 1982 and a related class of models due to Lomnitz, Jeffreys and Strick are studied by a new method due to the Author. Unlike the previously studied explicit models of relaxation modulus or creep compliance, these two classes support propagation of discontinuities. Due to an extension made by Strick either of these two classes of models comprise both viscoelastic solids and fluids.

1 Introduction.

In most explicit analytic models of viscoelastic media the attenuation as a function of frequency tends to infinity according to a power law. As a consequence in such viscoelastic media initial discontinuities and discontinuous source signals are immediately smoothed out. In those media in which additionally disturbances are bounded in space by a wavefront propagating at a finite speed the wavefield must decay to zero with all its derivatives at the wavefront. Consequently the peak of a pulse stays behind the wavefront and is preceded by a flat pedestal (Strick 1970). The pedestal widens with the propagation and the delay of the signal with respect to the wavefront increases in time (Hanyga and Seredyńska 2002, 1999a). The delay of seismic signals with respect to the wavefronts has to be taken into account in seismic inversion (Strick 1971, Hanyga and Seredyńska 1999b).

Viscoelastic models with a power law behavior in the high frequency limit are common in materials science (e.g. in polymer and rubber theory) and in the
theory of ultrasound in biotissues [Szabo and Wu (2000); Szabo (2004)]. Biot’s theory of poroelasticity ([Biot 1956a, 1962, 1956b,c]) leads to similar results ([Hanyga and Carcione 2000, Hanyga and Lu 2005a, Lu and Hanyga 2005a]).

In this paper we shall present a class of creep compliances proposed by seismologists Lomnitz, Jeffreys, Mainardi and Strick ([Lomnitz 1962a, Jeffreys 1967, Strick and Mainardi 1982, Strick 1984]). These creep compliances were originally considered in connection with the constant Q hypothesis. They however deserve attention because of another peculiarity: in the viscoelastic media defined by these creep compliances the attenuation function is bounded and therefore such media support propagation of discontinuities at the wavefront. Green’s function for such media can be locally decomposed into a discontinuity wave and a continuous remainder

$$G(t, x) = a(x) \theta(t - F(x)) + G_1(t, x)$$

If a pulse $$f'(t) \delta(x)$$ is sent from a point source then the wavefield

$$u(t, x) = f'(t) * G(t, x) = a(x) f(t - F(x)) + f'(t) * G_1(t, x)$$

where

$$\varphi_1(t) * \varphi_2(t) := \int_0^t \varphi_1(s) \varphi_1(t - s) \, ds$$

denotes the Volterra convolution with respect to time. Equation (2) shows that the pulse travels with the speed of the wavefront. This is an assumption commonly made in seismic inversion. It is clear that a careful analysis of viscoelastic models of wave propagation is overdue.

Wavefronts in Jeffreys media ($$\alpha > 0$$) were previously studied numerically by Buchen (1974), who summed ray expansions and compared the wavefronts for various pulse shapes and material parameters. Our objective is to put wavefront discontinuities and the attenuation functions in the same perspective. Low-frequency attenuation has often been studied by experimental methods in materials science, bio-tissues and in seismology. Wavefront singularities provide additional information on attenuation in the high-frequency range. Wavefront singularities are also relevant for a correct definition of travel time ([Hanyga and Seredyńska 1999a]). However, since attenuation is often considered independently of wavefront singularities and pulse propagation, viscoelastic models for these two kinds of phenomena are often inconsistent.

It was shown in [Hanyga and Seredyńska (2012)] and [Hanyga (2013)] that the propagation speed $$c(\omega)$$ and the attenuation function $$A(\omega)$$ in a viscoelastic medium with a creep compliance which is a Bernstein function can be expressed in terms of a Radon measure (essentially a locally finite measure) called the attenuation spectrum. Only the low-frequency behavior of the propagation speed and the attenuation function is available to experiments ([Hanyga 2013, Nasholm and Holm 2011]). In [Hanyga (2014)] it was shown that the high-frequency asymptotics of the attenuation function determines the regularity of viscoelastic Green’s functions. In [Hanyga and Seredyńska (2012)] and
Hanyga (2013) a causal function \( g(t) \) was defined such that \( \mathcal{A}(\omega) = \text{Re}[\tilde{p} \tilde{g}(p)] \).

In Hanyga (2014a) asymptotic estimates and upper bounds of the Green’s functions near the wavefront have been expressed in terms of the function \( g(t) \).

We shall apply this analytic toolbox to the analysis of attenuation, dispersion and discontinuity waves in two classes of viscoelastic models: Strick-Mainardi models and the Jeffreys-Lomnitz-Strick models. The attenuation function and the function \( g(t) \) can be explicitly calculated for Strick-Mainardi models. Both classes comprise viscoelastic solids (\( \alpha < 0 \)) and viscoelastic fluids (\( \alpha \geq 0 \)).

2 Mathematical preliminaries.

We shall consider the Initial-Value Problem (IVP)

\[
\rho u_{tt} = \nabla \cdot [G(t) \ast \nabla u] + \delta(x) \delta(t), \quad t \geq 0, \quad x \in \mathbb{R}
\]

\[ u(0,x) = 0; \quad u_t(0,x) = 0 \tag{5} \]

for the particle velocity \( u \) in a hereditary viscoelastic medium. It is assumed that the relaxation modulus \( G(t) \) (defined for \( t > 0 \)) is completely monotonic (CM), i.e. it has derivatives \( D^n G \) of arbitrary order and these derivatives satisfy the inequalities

\[
(-1)^n D^n G(t) \geq 0 \quad \text{on } \mathbb{R} \quad \text{for } n = 0, 1, 2, \ldots
\]

It is also assumed that \( G \) is locally integrable, or, equivalently

\[
\int_0^1 G(s) \, ds < \infty
\]

We shall use the abbreviation LICM for locally integrable completely monotonic functions. It follows [Hanyga and Seredyńska 2007] that the creep compliance \( J(t) \) (\( t \geq 0 \)), related to the relaxation modulus by the equation

\[
\int_0^t G(s) J(t-s) \, ds = t \quad \text{for } t \geq 0
\]

is a Bernstein function (BF), i.e. it is non-negative, differentiable and its derivative \( J' \) is LICM [Schilling et al. 2010]. Conversely, for a given BF \( J \) equation (6) has a unique solution \( G \) and the solution \( G \) is LICM [Hanyga and Seredyńska 2007]. We also recall that \( 0 \leq J_0 := J(0) < \infty \) and \( J_0 = 0 \) if and only if \( \lim_{t \to 0^+} G(t) = \infty \).

The solution of the IVP (4,5) is given by the formula

\[
u(t,x) = \frac{1}{4\pi i} \int_{-\infty+\epsilon}^{\infty+\epsilon} \frac{\kappa(p)}{2p^2} e^{pt-k(p)|x|} \, dp
\tag{7}
\]

where

\[
\kappa(p) := \rho^{-1/2} p \left[p \tilde{J}(p) \right]^{1/2} \tag{8}
\]
and $\varepsilon > 0$.

In [Hanyga and Seredyńska (2012) and Hanyga (2013)] it was showed that $\kappa(p)$ is a complete Bernstein function (CBF) ([Schilling et al., 2010; Jacob, 2001], i.e.

$$\kappa(p) = p^2 \tilde{F}(p),$$

where $F$ is a Bernstein function. Furthermore $\kappa(0) = 0$. Consequently $\kappa$ has an integral representation of the following form

$$\kappa(p) = p/c_0 + p \int_{[0,\infty]} \frac{\nu(dr)}{p + r} \quad (9)$$

where $\nu$ is a positive Radon measure satisfying the inequality

$$\int_{[0,\infty]} \frac{\nu(dr)}{1 + r} < \infty \quad (10)$$

([Schilling et al., 2010] and $c_0$ is a constant satisfying the inequalities $0 < c_0 \leq \infty$, defined by the formula

$$1/c_0 := \lim_{p \to \infty} \kappa(p)/p \quad (11)$$

Note that

$$1/c_0 = \rho^{1/2} \lim_{p \to \infty} \left[ p \tilde{J}(p) \right]^{1/2} = [\rho J_0]^{1/2} \quad (12)$$

The dimension of $\kappa(p)$ and $\nu(dr)$ is $1/L$. We shall assume that $J_0 > 0$ and $c_0 < \infty$. This excludes some viscoelastic models used in seismology in connection with the constant $Q$ hypothesis (e.g. [Kjartansson (1979)]) and in materials science in connection with the power law attenuation (e.g. [Kelly et al., 2008]).

If $J_0 > 0$ then the constant $c_0$ defines the wavefronts $|x| = c_0 t$ such that $u(t,x) = 0$ for $t > |x|/c_0$, otherwise $c_0 = \infty$ and the solution $u(t,x)$ does not vanish anywhere in the space-time.

The attenuation function $\text{Re} \kappa(-i\omega)$ and the dispersion function $-\text{Im} \kappa(-i\omega)$ of the medium can be expressed in terms of the Radon measure $\nu$, hence the Radon measure $\nu$ is called the dispersion-attenuation measure in [Hanyga (2013)]

$$\mathcal{A}(\omega) = \omega^2 \int_{[0,\infty]} \frac{\nu(dr)}{r^2 + \omega^2} \quad (13)$$

$$\mathcal{D}(\omega) = \omega \int_{[0,\infty]} \frac{r \nu(dr)}{r^2 + \omega^2} \quad (14)$$

The attenuation function $\mathcal{A}(\omega)$ is non-decreasing and therefore it tends to a finite limit $\mathcal{A}_\infty := \lim_{\omega \to \infty} \mathcal{A}(\omega)$ if it is bounded. If $\nu$ has finite mass $N := \nu([0,\infty[) < \infty$ then $\lim_{\omega \to \infty} \mathcal{A}(\omega) = N$ by the Lebesgue Dominated Convergence Theorem. In particular $N < \infty$ if the support of $\nu$ is bounded.
Conversely, if \( A(\omega) \) is bounded, then, by the Fatou lemma (Rudin 1976, Theorem 11.31) and equation (13) \( \int_0^\infty \nu(dr) \leq \lim_{\omega \to \infty} A(\omega) \) and the attenuation-dispersion spectral measure \( \nu \) has finite mass. By the preceding argument \( A_\infty = N \). We have thus proved that \( A_\infty = N \) and both numbers can be finite or infinite.

The following theorem can be used to check whether the attenuation measure \( \nu \) has finite total mass.

**Theorem 2.1**

\[
\lim_{p \to \infty} \left[ p \left( \frac{\kappa(p)}{p} - \frac{1}{c_0} \right) \right] = \int_{[0, \infty]} \nu(dr) \tag{15}
\]

where the right-hand side can be infinite.

**Proof.** The left-hand side of equation (15) equals \( \int_{[0, \infty]} (p/(r + p)) \nu(dr) \). The theorem follows by the Monotone Convergence Theorem (Rudin, 1976, Sec. 11.28).

In terms of the creep compliance

\[
\lim_{p \to \infty} \left\{ p \left( p \tilde{J}(p) \right)^{1/2} - (\rho J_0)^{1/2} \right\} = \int_{[0, \infty]} \nu(dr) \tag{16}
\]

The Radon measure \( \nu \) can be calculated using equation (9). If \( \nu(dr) = h(r) dr \), then

\[
h(r) = \frac{1}{\pi} \text{Im} \left[ \frac{\kappa(p)}{p} \big|_{p=r} \exp(-\pi i) \right] \tag{17}
\]

(Hanyga and Seredyńska 2012; Hanyga 2013), or, using equation (8),

\[
h(r) = \frac{\rho^{1/2}}{\pi} \text{Im} \left\{ \left[ \rho \tilde{J}(p) \right]^{1/2} \right\} \tag{18}
\]

Recall that every LICM function \( \varphi \) has the integral representation

\[
\varphi(t) = a + \int_{[0, \infty]} e^{-rt} \nu(dr) \tag{19}
\]

where \( \nu \) is a positive Radon measure satisfying the inequality (Gripenberg et al. 1990). Define the function function \( g \) by the formula

\[
g(t) = \int_{[0, \infty]} e^{-rt} \nu(dr) \tag{20}
\]

where the Radon measure \( \nu \) is defined by equation (9). We then have an important formula

\[
\kappa(p) = \frac{p}{c_0} + p \tilde{g}(p) \tag{21}
\]

The function \( g \) is LICM and \( \lim_{t \to \infty} g(t) = 0 \). The dimension of \( g(t) \) is 1/L. The function \( g(t) \) assumes a finite value at 0 if \( \nu \) has a finite mass.
any function $\kappa$ given by equation (21), where $g$ is a LICM function, is a CBF on account of equation (19) and equation (9). Furthermore, it is proved in [HANYGA (2014a)] that

$$g(0+) = \rho c_0 J'(0+)/2.$$  \hfill (22)

or, equivalently,

$$g(0+)) = J'(0+)/2J_0 c_0$$ \hfill (23)

Furthermore

$$g(t) \leq \rho c_0 J'(t)/2$$ \hfill (24)

If the attenuation function is bounded then $g(0+) = \int_{0,\infty} \nu(dr) = A_\infty < \infty$.

Green’s function $G$ can be approximated by an explicit function $H(t,x)$

$$G(t,x) = 1/2 \rho H(t - |x|/c_0, |x|) [1 + O(t - |x|/c_0)]$$ \hfill (25)

where $H(\cdot, r)$ is a non-negative non-decreasing function defined by the equation

$$e^{-p \tilde g(p)} r/p = \int_0^\infty e^{-p t} H(t, r) dt$$ \hfill (26)

It is then proved in [HANYGA (2014a)] that

$$H(t, r) \sim_{t \to 0} e^{-g(t)r}$$ \hfill (27)

In view of equation (25) this implies that

$$G(t, x) \sim_{t \to |x|/c_0 + 0} 1/2 \rho e^{-g(0+) r}$$ \hfill (28)

If $g(0+) < \infty$ then it is also true that

$$\lim_{t \to |x|/c_0 - 0} G(t, x) = 1/2 \rho e^{-g(0+) r}$$ \hfill (29)

while $\lim_{t \to |x|/c_0 - 0} G(t, x) = 0$. Hence in this case the wavefront carries a jump discontinuity $\exp(-g(0+) r)/(2\rho)$.

3 The Strick-Mainardi creep compliance.

Consider the following function

$$F_\alpha(\Omega, p) := \frac{1}{\alpha} \left[ \left( 1 + \frac{\Omega}{p} \right)^\alpha - 1 \right], \quad -1 < \alpha < 1, \quad \alpha \neq 0$$ \hfill (30)

and its limit for $\alpha \to 0$:

$$F_0(\Omega, p) = \ln \left( 1 + \frac{\Omega}{p} \right)$$ \hfill (31)

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We suppose that $F_\alpha$ is a Laplace transform and try to find its original $K_\alpha$:

$$K_\alpha(t, \Omega) = \frac{1}{2\pi i} \int_G e^{pt} \left[ \left( 1 + \frac{\Omega}{p} \right)^\alpha - 1 \right] dp$$

It follows from the asymptotic estimate $(1 + \Omega/p)^\alpha - 1 \sim \alpha \Omega/p$ that the integrand tends to zero uniformly for $p \to \infty$ in the left complex half-plane $\text{Re } p \leq 0$. The integrand on the right-hand side does not have any singularities outside the cut along the negative real semi-axis. There is no contribution of the small circle of radius $\varepsilon$ centered at the origin. By Jordan’s lemma the Bromwich contour can be replaced by the Hankel loop encircling the negative semi-axis in the positive direction. Setting $p = re^{i\pi}$ for the part of the contour running above the cut yields the following expression:

$$K_\alpha(t, \Omega) = -\frac{1}{\alpha \pi} \int_0^\infty e^{-rt} \text{Im} \left( 1 + e^{-i\pi \frac{\Omega}{r}} \right)^\alpha dr$$

(32)

where the limits from the upper/lower half of the complex $p$-plane are identified by the phases $\arg(p) = \pm \pi$. On $[\Omega, \infty[$ the function $(1 + e^{-i\pi \Omega/r}) = 1 - \Omega/r$ is non-negative and the integrand of (32) vanishes. On $[0, \Omega[$ however $1 + e^{-i\pi \Omega/r} = (\Omega/r - 1) e^{-i\pi}$ and thus the integrand of the right-hand side of equation (32) does not vanish. Thus

$$K_\alpha(t, \Omega) = \frac{\sin(\alpha \pi)}{\alpha \pi} \int_0^\Omega e^{-r't} \int_0^\Omega e^{-r t} r^{-\alpha} (\Omega - r)^\alpha dr = \frac{\Omega}{\alpha \sin(\alpha \pi)} \int_0^1 e^{-\Omega t y} y^{-\alpha} (1 - y)^\alpha dy$$

Hence

$$\int_0^t K_\alpha(t, \Omega) dt = \frac{\sin(\alpha \pi)}{\alpha \pi} \int_0^1 (1 - e^{-\Omega t y}) y^{-\alpha-1} (1 - y)^\alpha dy$$

(33)

The integral on the right-hand side of equation (33) converges if $-1 < \alpha < 1$, and represents a Bernstein function. Comparison with the integral representation of the confluent hypergeometric function (Abramowitz and Stegun (1970) Sec. 13.2.1) and the relation $\text{1F1}( -\alpha, 1; 0 ) = 1$ shows that

$$\frac{\sin(\alpha \pi)}{\alpha \pi} \int_0^1 (1 - e^{-\Omega t y}) y^{-\alpha-1} (1 - y)^\alpha dy = \left[ 1 \text{F1}( -\alpha, 1; -\Omega t ) - 1 \right] / \alpha$$

(34)

provided $-1 < \alpha < 0$. Note also that this expression vanishes at 0. Consequently if $J_0, M_0 \geq 0$ and $-1 < \alpha < 0$ then

$$J^{(\alpha, \Omega)}(t) := J_0 + \frac{M_0}{\alpha} \left[ 1 \text{F1}( -\alpha, 1; -\Omega t ) - 1 \right], \quad t > 0$$

(35)

is a creep compliance. This creep compliance was introduced by E. Strick and F. Mainardi (Strick, 1982; Strick and Mainardi, 1982). Note that
\[
\lim_{t \to 0^+} J^{(\alpha, \Omega)}(t) = \lim_{p \to \infty} \left[ p \tilde{J}^{(\alpha, \Omega)}(p) \right] = J_1, \quad \text{where} \quad J_1 := J_0 - M_0/\alpha, \quad \text{is finite and} \quad J_{\infty} \geq J_0.
\]

The Laplace transform of the creep compliance is given by the formula

\[
p \tilde{J}^{(\alpha, \Omega)}(p) = J_0 + \frac{M_0}{\alpha} \left[ \left( 1 + \frac{\Omega}{p} \right)^{\alpha} - 1 \right]
\]

and the retardation spectral density can be calculated from equation (34):

\[
H^{(\alpha, \Omega)}(r) = \frac{\sin(\alpha \pi)}{\alpha \pi} M_0 r^{-\alpha - 1} (1 - r/\Omega)^{\alpha} \theta(1 - r/\Omega)
\]

The case of \( \alpha = 0 \) will be treated in a similar way.

\[
K_0(t, \Omega) = \frac{1}{2\pi i} \int_{\mathcal{B}} e^{pt} \ln \left( 1 + \frac{\Omega}{p} \right) dp = -\frac{1}{\pi} \int_0^\infty e^{-rt} \text{Im} \ln \left( 1 + \frac{\Omega}{r} e^{-i\pi} \right) dr
\]

The logarithm in the integrand is real for \( r > \Omega \). On \([0, \Omega]\) however it has the imaginary part \(-\pi\). Hence

\[
K_0(t, \Omega) = \int_0^\Omega e^{-rt} dr = \frac{1}{t} (1 - e^{-\Omega t})
\]

The indefinite integral of \( K_0(t, \Omega) \)

\[
\int_0^t K_0(s, \Omega) ds = \int_0^\Omega \frac{1}{y} (1 - e^{-yt}) dy
\]

is thus a Bernstein function. It is recognized as the modified exponential integral \( \text{Ein}(\Omega t) \) (Abramowitz and Stegun, 1970). We can now define Becker’s creep compliance (Becker, 1925)

\[
J^{(0, \Omega)}(t) = J_0 + M_0 \text{Ein}(\Omega t), \quad t \geq 0
\]

where \( J_0 \) is a non-negative constant. Applying the limit \( \alpha \to 0 \) in (36) yields

\[
p \tilde{J}^{(0, \Omega)}(p) = J_0 + M_0 \ln \left( 1 + \frac{\Omega}{p} \right)
\]

and the retardation spectral density is \( H^{(0, \Omega)}(r) = M_0 \theta(1 - r/\Omega)/r \).

We have thus proved that the left-hand side of equation (34) is defined for \(-1 < \alpha < 1\) and is obviously an analytic function of \( \alpha \). The confluent hypergeometric function is however an analytic function of the first argument. Equation (34) therefore holds for \(-1 < \alpha < 1\) by analytic continuation, with the value at \( \alpha = 0 \) given by \( \text{Ein}(a t) \).

The creep compliances \( J^{(\alpha, \Omega)}(t) \) are shown in Figure 1.

The asymptotic behavior of Strick-Mainardi creep compliance follows from the formulae

\[
\text{iF}_1(a, 1; -z) \sim z^{-a}/\Gamma(1 - a)
\]

\[
\text{Ein}(z) \sim \ln(z) + \gamma + e^{-z}/z
\]
Figure 1: Strick-Mainardi creep compliance $J^{(\alpha,1)}$ for $J_0 = 4.1 \times 10^{-11} \text{ Pa}^{-1}$, $M_0 = 16 \times 10^{-11} \text{ Pa}^{-1}/(\pi \ast 50)$ and for $\alpha = -0.5$ (dot-dashed line), $\alpha = 0$ (solid line) and $\alpha = 0.5$ (dashed line).

\[ J^{(\alpha,\Omega)}(t) \sim_{\infty} \begin{cases} J_0 + M_0 (\Omega t)^{\alpha} / \alpha & \alpha > 0 \\ J_0 + M_0 [1 - (\Omega t)^{\alpha}] / |\alpha| & \alpha < 0 \\ J_0 + M_0 \ln(\Omega t) & \alpha = 0 \end{cases} \]  

Note that the creep compliance for $\alpha < 0$ is bounded and in the remaining cases it is unbounded. Hence for $\alpha \geq 0$ the low-frequency limit of creep compliance $J_\infty = \infty$ and therefore $G_\infty = 0$. Consequently the medium is a viscoelastic solid if $\alpha < 0$ and a viscoelastic fluid if $\alpha \geq 0$.

For $t$ small we can use the Taylor expansions of the confluent hypergeometric function and the modified exponential integral:

\[ _1F_1(a,; z) \sim_{0} 1 + a z/b \]  

\[ \text{Ein}(z) \sim_{0} z \]

4 Attenuation and dispersion in the Strick-Mainardi media.

We shall now consider the attenuation and dispersion in materials characterized by the Strick-Mainardi creep compliance $J^{(\alpha,\Omega)}(t)$, where $-1 < \alpha < 1$, $\alpha \neq 0$ and $\Omega > 0$:

\[ \hat{J}^{(\alpha,\Omega)}(p) = J_1 + M_1 \left(1 + \frac{\Omega}{p}\right)^{\alpha} \]
where $M_1 := M_0/\alpha$ and $J_1 := J_0 - M_1$. The wavenumber function $\kappa(p) = \rho^{1/2} p \left[ \overline{p J(\alpha, \Omega)}(p) \right]^{1/2} = p/c_0 + \beta(p)$, where $1/c_0 = (\rho J_0)^{1/2}$ if $J_0 > 0$.

The density of the attenuation-dispersion measure $\nu$ will be calculated from the formula (18):

$$h(r) = \frac{\rho^{1/2}}{\pi} \text{Im} Z^{1/2}$$

where $Z := J_1 + M_1 (1 + \Omega/r \exp(-i\alpha)/\alpha)$. Note that

$$\left(1 + \frac{\Omega}{r \exp(-i\alpha)} \right)^\alpha = \begin{cases} (\Omega/r - 1)\alpha e^{i\alpha}, & r < \Omega \\ (1 - \Omega/r)^\alpha, & r > \Omega \end{cases}$$

and $\text{Im} Z^{1/2} = -\frac{1}{\sqrt{2}} \sqrt{X^2 - Y^2 - X}$, where $X := \text{Re} Z$ and $Y := \text{Im} Z$. It follows that $h(r) = 0$ for $r > \Omega$ and

$$h(r) = \frac{\rho^{1/2}}{\sqrt{2\pi}} \sqrt{X(r)^2 + Y(r)^2 - X(r)}, \quad r < \Omega$$

where $X(r) := J_1 + M_1 \cos(\alpha \pi) (\Omega/r - 1)^\alpha$ and $Y(r) := M_1 \sin(\alpha \pi) (\Omega/r - 1)^\alpha$ for $0 \leq r \leq \Omega$.

The case of $\alpha = 0$ requires some calculi. We note that $h(r)$ is given by equation (45) with $Y(r) = \pi M_0$ and $X(r) = J_0 + M_0 \ln(|\Omega/r - 1|)$, both for $r \leq \Omega$. Hence $h(r)$ vanishes for $r > \Omega$ and

$$h(r) \sim_0 \frac{1}{2\pi c_0} \sqrt{\frac{J_0^2 + \pi^2 M_0^2}{J_0 M_0}} \ln^{-1/2} \left(\frac{\Omega}{r} - 1\right), \quad r \leq \Omega$$

The attenuation and dispersion can now be determined by substituting (45) in the equations

$$A(\omega) = \omega^2 \int_{0, \infty} h(r) \frac{r h(r)}{\omega^2 + r^2} \, dr$$

$$D(\omega) = \omega \int_{0, \infty} \frac{r h(r)}{\omega^2 + r^2}$$

[HANYGA, 2013]. The attenuation-dispersion spectrum (the support of the function $h(r)$) of the materials with the Strick-Mainardi creep compliance is bounded. This implies that the attenuation function tends to a finite value at infinite frequency. In particular, if $J_1 = 0$ then

$$h(r) = \frac{\sqrt{\rho |M_1|}}{\pi} |\sin(\alpha \pi/2)| (\Omega/r - 1)^{\alpha/2}, \quad 0 \leq r \leq \Omega$$

and

$$\int_0^\Omega (\Omega/r - 1)^{\alpha/2} \, dr = \Omega \int_0^\infty (1 + y)^{-2} y^{\alpha/2} \, dy = \frac{\Omega \alpha \pi/2}{\sin(\alpha \pi/2)}$$
Figure 2: Attenuation and phase speed in a medium with the Strick-Mainardi creep compliance with $c_0 = 2851 \text{ m/s}$, $J_0 = 4.1 \times 10^{-11} \text{ Pa}^{-1}$, $M_0 = 0.026$ corresponding to $Q = 50$. Solid curves: $\alpha = 0.3$, dashed curves: $\alpha = -0.3$.

hence

$$\lim_{\omega \to \infty} A(\omega) = \int_0^\Omega h(r) \, dr = -|\alpha| \Omega \sqrt{\frac{\rho M_1}{2}}$$  \hspace{1cm} \text{(49)}$$

Attenuation in acoustics is usually expressed in $\text{db/m}$, $A_{\text{dbm}}(\omega) := \log_{10} \left(e^{-A(\omega)}\right)$, where $A(\omega)$ is expressed in $\text{m}^{-1}$, in terms of the quality factor $Q(\omega) = \omega/[2c(\omega) A(\omega)]$.

Figure 2 shows that the bounded and unbounded Strick-Mainardi creep compliances yield very similar dispersion and attenuation.

5 Wavefronts in Strick-Mainardi models.

Strick-Mainardi creep models have a bounded creep rate, i.e. $J^{(\alpha, \Omega)}(0) = M_0 \Omega$ is finite and the jump of Green’s function at the wavefront assumes the special form $\exp(-M_0 \Omega r/(2c_0 J_0))$. The values of the Young modulus $1/J_0$ and wavefront velocity $c_0$ or density $\rho$ are known for many materials and we can only speculate about the creep parameter $M_0$ and the creep time scale $2\pi/\Omega$. The ratio $M_0/J_0$ controls the rate of gradual creep to instantaneous elastic strain following application of a unit stress. For $\alpha < 0$ this parameter controls the saturation creep $J_\infty = \lim_{t \to \infty} J^{(\alpha, \Omega)}(t) = 1/G_\infty$, where $G_\infty$ is the equilibrium elastic modulus. We recall that $J^{(\alpha, \Omega)}(t)$ tends to infinity as $t \to \infty$ if $\alpha \geq 0$. For a fixed $M_0/J_0$ ratio the logarithmic decay of the wavefront jump is controlled by the wavefront attenuation length scale $2\pi c_0/\Omega$.

The sign of $\alpha$ determines the long-time asymptotics of the function $g$ and the rate of growth of Green’s function away from the wavefront. Let $0 < \alpha < 1$. The asymptotics of $h(r)$ for $r \to 0$ can be easily calculated:

$$h(r) \sim 0 \sqrt{\frac{\rho M_0 \Omega^2}{\alpha}} \frac{\sin(\alpha \pi/2)}{\pi} r^{-\alpha/2}$$
Thus $h(r)$ is regularly varying at 0 and

$$g(t) \sim \Gamma(1 - \alpha/2) \sqrt{\frac{\rho M_0 \Omega^\alpha}{\alpha} \sin(\alpha \pi/2)} t^{\alpha/2 - 1}$$

(50)

by the Karamata Abelian Theorem. Thus $g(t)$ decreases slower than $t^{-1}$ in this case.

If $-1 < \alpha < 0$, then $(\Omega/r - 1)^\alpha \to 0$ as $r \to 0$ and therefore $\lim_{r \to 0} h(r) = 0$. Consequently close to the origin the function $g(t)$ decreases faster than $t^{-1}$.

The case of $\alpha = 0$ has to be considered separately. Equation (46) shows that $h$ is slowly varying at 0. Denote the right-hand side of (46) by $l(\Omega/r)$. It is a function of dimension $T/L$. We then have

$$g(t) \sim l(\Omega t)/t = C/\left[ t \ln^{1/2}(\Omega t) \right]$$

(51)

where $C$ is a constant of dimension $T/L$. Note that $g$ decreases faster than $1/t$.

The function $g$ can be calculated in closed form in the case of $J_1 = 0$:

$$g(t) = \frac{\alpha \sqrt{\rho |M_1| \Omega}}{2} \sin(\alpha \pi/2) \frac{t^{1-\alpha}}{F_1(1-\alpha/2, 2; -\Omega t)} \quad t \geq 0$$

(52)

Asymptotic estimates of Green’s function in a neighborhood of the wavefront for $\Omega = M_0 = 1$ are plotted in Figure 3. Exaggerated values of material parameters have been chosen for illustrative purposes. For metals Young’s modulus is of order of a few hundreds of GPa. In this case we should assume $J_0 \sim 10^{-11} \text{Pa}^{-1}$ and the function $g$ is of order of $10^{-6} \text{m}^{-1}$. With these parameter values the wavefront is hardly distinguishable from the simple step function. Bio-tissues such as liver have however much lower Young’s modulus of order of hundreds Pa. In this case Green’s function exhibits significant variation behind the wavefront.

6 Jeffreys-Lomnitz creep compliance, attenuation and wavefronts.

The Lomnitz logarithmic law was suggested in the context of the constant $Q$ hypothesis. The Jeffreys-Lomnitz-Strick creep compliance is defined by the equation

$$J_{\alpha, \Omega}(t) = J_0 + \begin{cases} J_0 + M_0 \alpha t^{\alpha - 1}, & \alpha \neq 0 \\ J_0 + M_0 \ln(1 + \Omega t), & \alpha = 0 \end{cases}$$

(53)

for $\alpha \leq 1$, $J_0, M_0, \Omega \geq 0$. The logarithmic law ($\alpha = 0$) is due to Lomnitz (1957, 1962), the extension to $\alpha > 0$ was made by Jeffreys (1967) and the extension to negative values of $\alpha$ is due to Strick and Mainardi (1982). Strick and Mainardi also compared the Jeffreys-Lomnitz-Strick law with Becker’s creep compliance, focusing however on the values of $Q$ predicted by these theories.
(a) Evolution of the wavefront profile for $\alpha = 0.5$, $\Omega = M_0 = 1$, $c_0 = 1\text{km/s}$.

(b) Evolution of the wavefront profile for $\alpha = -0.5$, $\Omega = M_0 = 1$, $c_0 = 1\text{km/s}$.

(c) Dependence of the wavefront signal on $\alpha$ at $r = 5\text{km}$.

Figure 3: Green’s function of Strick’s creep compliance model near the wavefront.
More recently, the Jeffreys-Lomnitz-Strick law and the associated material response functions were examined by Mainardi and Spada (2012b).

The retardation spectral density $H_{\alpha,\Omega}(r)$ of the Jeffreys-Lomnitz-Strick media can be calculated using the identity (Mainardi and Spada, 2012)

$$
\frac{1}{\Gamma(1-\beta)} \int_0^\infty e^{-rt} e^{-r^\beta} \, dr = (1+t)^{\beta-1}
$$

for $\beta < 1$. This identity is easily proved by substituting $s = (1+t)\, r$. It follows that

$$
J_{\alpha,\Omega}(t) = J_0 + \frac{M_0}{\Omega \Gamma(-\alpha)} \int_0^\infty (1-e^{-rt}) \, e^{-r/\Omega} \, (r/\Omega)^{-\alpha} \, dr
$$

for $\alpha \neq 0$. For $\alpha = 0$ we note that equation (54) follows from the identity

$$
F(x) := \int_0^\infty \left(1-e^{-xy}\right) e^{-y^\alpha} \, dy = \ln(1+x)
$$

Indeed, $F'(x) = \int_0^\infty e^{-xy^\alpha} \, dy = 1/(1+x)$. Hence the Jeffreys-Lomnitz-Strick retardation spectral density is given by the formula

$$
H_{\alpha,\Omega}(r) = \frac{M_0}{\Omega \Gamma(-\alpha)} e^{-r/\Omega} \, (r/\Omega)^{-\alpha}
$$

Note that $\int_0^\infty H_{\alpha,\Omega}(r) \, dr < \infty$.

The attenuation spectral density is more difficult to calculate. The Laplace transform of the Jeffreys-Lomnitz-Strick creep compliance can be expressed in terms of the exponential integral (Abramowitz and Stegun, 1970, Chap. 5)

$$
E_{\alpha}(q) := \int_1^\infty e^{-q \, r} \, r^{-\alpha} \, dr
$$

(do not confuse this notation with the Mittag-Leffler function) by the formula

$$
\widetilde{J}_{\alpha,\Omega}(p) = \left\{ J_0 + M_0 \left( p e^{p/\Omega} E_{\alpha}(p/\Omega) / \Omega - 1 \right) / p \right\} / p, \quad \alpha \neq 0
$$

For $\alpha = 0$ note that

$$
\int_0^\infty e^{-pt} \ln(1+\Omega \, t) \, dt = \frac{e^{p/\Omega}}{\Omega} \int_1^\infty e^{-py/\Omega} \ln(y) \, dy = \frac{e^{p/\Omega}}{p} \int_1^\infty e^{-py}/y \, dy
$$

Hence

$$
\widetilde{J}_{\alpha,\Omega}(p) = \left[ J_0 + M_0 \, e^{p/\Omega} \, E_1(p/\Omega) \right] / p
$$

The exponential integral has a branching cut along the entire negative axis. We thus do not expect the attenuation spectrum to be bounded, but we shall show that the attenuation measure has finite total mass and therefore the attenuation function is bounded. This implies that $g(0+) < \infty$ and shock wave discontinuities propagate at the wavefronts. Consequently Jeffreys-Lomnitz-Strick media support shock waves.
The asymptotic formula

\[ E_{-\alpha}(z) \sim \infty \left( e^{-z}/z \right) \left[ 1 + \alpha/z + O \left( z^{-2} \right) \right] \]

(Abramowitz and Stegun [1970], Sec. 5.1.51) implies that

\[ \lim_{p \to \infty} \{ p \left[ (\rho p \mathcal{J}_{\alpha,\Omega}(p))^{1/2} - (\rho J_0)^{1/2} \right] \} = N \]

where \( N = M_0/2c_0 J_0 \) < \( \infty \) for both \( \alpha \neq 0 \) and \( \alpha = 0 \). Hence, by Theorem 2.1 the attenuation measure \( \nu \) has finite total mass. It follows that \( \lim_{\omega \to \infty} A(\omega) = \int_{0,\infty} \nu(\omega) \, d\omega \), hence the attenuation function is bounded.

In the case at hand the attenuation function and the phase speed can be calculated using the equations \( A(\omega) = \omega \Im \left[ -i\omega \mathcal{J}_{\alpha,\Omega}(-i\omega) \right]^{1/2} \), \( D(\omega) = \omega \Re \left[ -i\omega \mathcal{J}_{\alpha,\Omega}(-i\omega) \right]^{1/2} \), and \( 1/c(\omega) = 1/c_0 + D(\omega)/\omega \), where \( c_0 = (\rho J_0)^{-1/2} \).

The results for selected parameters are shown in Figure 4.

7 A viscoelastic medium exhibiting the pedestal effect.

The Jeffreys creep compliance with \( 0 < \alpha < 1 \) can be recast in the form

\[ J_{\alpha,\tau}(t) = J_0 + J_2 \left( \tau + t \right)^\alpha \]

(58)

where \( \tau = 1/\Omega \) and \( J_2 = M_0 \Omega^\alpha/\alpha \). \( J_{\alpha,0} \) is a special case of the Andrade creep compliance

\[ J_{\alpha}(t) = J_0 + J_1 t + J_2 t^\alpha \]

(59)

where \( J_0, J_1, J_2 \geq 0 \) and \( 0 < \alpha \leq 1 \). The second term is known as linear creep; it dominates at long observation times. Pure linear creep is equivalent to Newtonian viscosity. The third term dominates for shorter observation times and
is attributed to dislocation motion. Andrade creep was originally observed in metals (Da Andrade 1910, 1912; Cottrell 1996; Nabarro 1997; Miguel et al. 2002) with $\alpha = 1/3$, but it was subsequently found in other materials, including rocks (Lockner 1993; Murrell and Chakravarty 1973; Gribb and Cooper 1998). $J_\alpha$ is clearly a Bernstein function.

Even though the Andrade creep compliance has been obtained by a limiting process from the Jeffreys creep compliance it is radically different from the latter because $J'_\alpha(t)$ tends to infinity for $t \to 0^+$. It is shown in (Hanyga 2014a) that viscoelastic media with this property do not support discontinuity waves.

Note that $\tilde{p}J_\alpha(p) = J_0 + J_1/p + J_2/p^{-\alpha}$ and $\kappa(p) = p \left[ 1 + J_1/(J_0 p) + J_2/J_0 p^{-\alpha} \right]^{1/2} / c_0$, where $c_0$ is given by equation (12). Hence

$$\lim_{p \to \infty} \left[ \kappa(p) - p/c_0 \right] = \frac{1}{2c_0} \lim_{p \to \infty} \left[ \frac{J_1}{J_0} + \frac{J_2}{J_0} p^{1-\alpha} \right] = \infty$$

Hence the attenuation measure $\nu$ has infinite total mass and the attenuation spectrum is the entire positive semi-axis.

The attenuation function can be explicitly calculated

$$A(\omega) = \Re \kappa(-i\omega) = \frac{\omega}{c_0} \Im \left[ 1 + \frac{J_1}{-i\omega J_0} + \frac{J_2}{J_0} (-i\omega)^{-\alpha} \right] = \frac{\omega}{c_0 \sqrt{2}} \left\{ \sqrt{1 + J_1^2/(J_0^2 \omega^2)} + J_2^2/J_0 \omega^{-2\alpha} + J_2/J_0 \left[ \sin(\pi\alpha/2) + \cos(\pi\alpha/2) \right] \omega^{-\alpha} - 1 - J_2/J_0 \cos(\pi\alpha/2) \omega^{-\alpha} \right\}$$

In the high-frequency range

$$A(\omega) \sim \infty \sin^{1/2}(\pi\alpha/2) \left( \frac{J_2}{J_0} \right)^{1/2} \frac{\omega^{\gamma}}{\sqrt{2}}$$

where $\gamma := 1 - \alpha/2$ satisfies the inequalities $1/2 < \gamma < 1$. It follows from the theory developed by Hanyga (2014a) that Green’s functions for the Andrade viscoelastic media are infinitely smooth at the wavefronts. This in turn implies that acoustic pulses follow the wavefront with a delay and are preceded by a pedestal in Strick’s terminology (Strick 1970). The effective travel time of a seismic signal is thus greater than the wavefront travel time, which is directly linked to the wavefront speed. In seismic inversion the effective travel time is determined (Hanyga and Seredynska 1999a). If it is believed that the attenuation function increases at a rate higher than logarithmic, as is the case in Andrade viscoelastic media, then the standard methods of seismic inversion misposition the scatterers. This error was pointed out by Hanyga and Seredynska (1999a) and in a different context by Strick (1971).

The Andrade model was conceived as a fit to creep data rather than wave propagation. It is nevertheless an instructive example of the possibility of an entirely different wave propagation pattern, with important consequences for the identification of travel times and location of scatterers in seismic applications.
Attenuation and dispersion in Andrade viscoelastic media was recently studied by semi-numerical methods by [Ben Jazia et al. (2013)]. The discrete approximation of the Andrade creep compliance applied in this paper does not reflect the unboundeness of the attenuation spectrum and of the attenuation function. Consequently it does not account for the wavefront smoothness.

8 Concluding remarks.

Strick-Mainardi and Jeffreys-Lomnitz-Strick viscoelastic models provide the only known examples of a closed form creep compliance consistent with the propagation of shock waves. The former models are characterized by bounded attenuation and retardation spectra while the latter have integrable attenuation and retardation spectral densities. The Strick-Mainardi retardation and attenuation-dispersion spectral measures are given by elementary functions and the function $g$ is easy to analyze. The Jeffreys-Lomnitz-Strick models are not amenable to such a detailed analysis but numerical analysis shows that they are qualitatively fairly similar. The similarity is due to the fact that both classes of models defined in terms of the power function, which is invariant with respect to the Carson-Laplace transform up to a numerical factor. In the context of the $Q$ factor such striking similarities were discovered in [Strick and Mainardi 1982].

Short-time creep and its singularity at 0 affects Green’s functions at the wavefront. The exact time dependence of the wave field at the wavefront is however represented by the function $g(t)$, which is indirectly related to the creep rate function. The wavefront behavior of Green’s functions provides a constraint on the creep rate at short times.

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