SB-LABELINGS, DISTRIBUTIVITY, AND BRUHAT ORDER ON SORTABLE ELEMENTS

HENRI MÜHLE

ABSTRACT. In this article, we investigate the set of $\gamma$-sortable elements, associated with a Coxeter group $W$ and a Coxeter element $\gamma \in W$, under Bruhat order, and we denote this poset by $B_\gamma$. We prove that this poset belongs to the class of SB-lattices recently introduced by Hersh and Mészáros. We use an observation of Armstrong, namely that $B_\gamma$ is a join-distributive lattice, to generalize the previous result, and to show that all join-distributive lattices are SB-lattices. Subsequently, we investigate for which finite Coxeter groups $W$ and which Coxeter elements $\gamma \in W$ the lattice $B_\gamma$ is distributive. It turns out that this is the case for the “coincidental” Coxeter groups, namely the groups $A_n$, $B_n$, $H_3$, and $I_2(k)$. We conclude this article with a conjectural characterization of the Coxeter elements $\gamma$ of the said groups for which $B_\gamma$ is distributive in terms of forbidden orientations of the Coxeter diagram.

1. Introduction

Recently, Hersh and Mészáros introduced a new class of lattices, so-called SB-lattices, see [11]. They showed that these lattices admit a certain edge-labeling, which implies that the order complex of every open interval of this lattice is homotopy equivalent to a sphere or a ball (hence the name). Equivalently, the Möbius function of such a lattice takes values only in $\{-1, 0, 1\}$. In the same paper they showed that every distributive lattice admits an SB-labeling, and they showed that the same is true for the weak order on a Coxeter group and for the Tamari lattice. Inspired by their results, we investigate the set of $\gamma$-sortable elements on a Coxeter group, denoted by $C_\gamma$, as defined by Reading and Speyer, see [17, 19]. If we equip $C_\gamma$ with the Bruhat order, then we obtain an infinite lattice, denoted by $B_\gamma$. As a first result, we show that $B_\gamma$ admits a very natural SB-labeling. Moreover, it follows from Armstrong’s work on $\omega$-sorting orders on Coxeter groups, see [2], that $B_\gamma$ is a join-distributive lattice, i.e. it can be realized as the lattice of feasible sets of an antimatroid, see [7]. This connection allows us to show that every join-distributive lattice admits an SB-labeling. This result generalizes Hersh’s and Mészáros’ result on distributive lattices. To summarize, the first main result of this paper is the following.

2010 Mathematics Subject Classification. 20F55 (primary), and 06D75, 06A07 (secondary).
Key words and phrases. SB-Labeling, Möbius function, Distributive lattice, Join-distributive lattice, Antimatroids, Bruhat order, Sortable Elements, Coxeter groups.
This work was funded by the FWF Research Grant No. Z130-N13.
Theorem 1.1. Every join-distributive lattice is an SB-lattice. In particular, the Bruhat order on $\gamma$-sortable elements is an SB-lattice for every Coxeter group $W$ and every Coxeter element $\gamma \in W$.

Join-distributivity is a generalization of distributivity, and while working with the poset $B_\gamma$ we observed that for some Coxeter groups and for some Coxeter elements the lattice $B_\gamma$ is in fact distributive. This led us to the question whether we can characterize the (finite) Coxeter groups $W$ and the Coxeter elements $\gamma \in W$ for which $B_\gamma$ is distributive. We approach this problem by looking for forbidden orientations of the Coxeter diagram of $W$, and we prove the following result.

Theorem 1.2. Let $W$ be a finite Coxeter group. There exists a Coxeter element $\gamma \in W$ such that $B_\gamma$ is distributive if and only if $W$ is of type $A_n, B_n, H_3$ or $I_2(k)$.

The finite Coxeter groups appearing in Theorem 1.2 are sometimes called the "coincidental types", since these groups enjoy a list of properties that distinguishes them from the other finite (complex) reflection groups, see [10, Theorems 8.5 and 10.2], [12, Theorem 14], [18], and [23, Remark 3.1.26]. Theorem 1.2 adds another property to this list. We conclude this article with a conjectural characterization of the Coxeter elements $\gamma \in W$ for which $B_\gamma$ is distributive.

2. Preliminaries

In this section, we recall the basic concepts needed in this article. For more background on SB-labelings, we refer to [11]. For any undefined notation and additional information on Coxeter groups and sortable elements, we refer to [5] and [19], respectively.

2.1. SB-Labelings. Let $\mathcal{P} = (P, \leq)$ be a (possibly infinite) poset. An element $p \in P$ is covered by another element $q \in P$ (denoted by $p < q$) if $p < q$ and there exists no $z \in P$ with $p < z < q$. Accordingly, $q$ covers $p$ and the elements $p$ and $q$ form a cover relation or an edge in $\mathcal{P}$. The set $\mathcal{E}(\mathcal{P}) = \{(p, q) \mid p < q\}$ is the Hasse diagram of $\mathcal{P}$.

For $p \leq q$ we call a set of the form $[p, q] = \{z \in P \mid p \leq z \leq q\}$ a (closed) interval of $\mathcal{P}$. A saturated chain in an interval $[p, q]$ is a sequence $(p, z_1, z_2, \ldots, z_{k-1}, q)$, where $z_i \in P$ for $i \in \{1, 2, \ldots, k-1\}$ and $p \lessdot z_1 \lessdot z_2 \lessdot \cdots \lessdot z_{k-1} \lessdot q$.

A poset $\mathcal{P}$ is a lattice if any two elements in $\mathcal{P}$ have a least upper and a greatest lower bound, denoted by $\vee$ and $\wedge$, respectively. A lattice is locally finite if every interval is finite.

An edge-labeling of $\mathcal{P}$ is a map $\lambda : \mathcal{E}(\mathcal{P}) \to \Lambda$, where $\Lambda$ is some set of labels. An SB-labeling of a lattice $\mathcal{P}$ is an edge-labeling $\lambda$ of $\mathcal{P}$ that satisfies the following properties for every $p, p_1, p_2 \in P$ with $p \lessdot p_1, p_2$:

(i) $\lambda(p, p_1) \neq \lambda(p, p_2)$;
(ii) each saturated chain in the interval $[p, p_1 \vee p_2]$ uses both labels $\lambda(p, p_1)$ and $\lambda(p, p_2)$ a positive number of times; and
(iii) none of the saturated chains in the interval $[p, p_1 \vee p_2]$ uses any other label besides $\lambda(p, p_1)$ and $\lambda(p, p_2)$.

A locally finite lattice with a least element that admits an SB-labeling is called an SB-lattice.
Remark 2.1. In fact, the original definition of an SB-labeling given in [11, Definition 3.2] was phrased a bit differently, but it was shown in [11, Theorem 3.5] that the above definition is equivalent to the original definition.

SB-lattices enjoy the following nice property.

Theorem 2.2 ([11, Theorem 3.8]). The Möbius function of an SB-lattice takes values only in \{-1, 0, 1\}.

2.2. Coxeter Groups. A Coxeter group is a group \( W \) admitting a presentation
\[
W = \langle s_1, s_2, \ldots, s_n \mid (s_is_j)^{m_{ij}} = e, \text{ for } i, j \in \{1, 2, \ldots, n\} \rangle,
\]
where \( e \in W \) is the identity and the numbers \( m_{ij} \) are either positive integers or the formal symbol \( \infty \) for all \( i, j \in \{1, 2, \ldots, n\} \) such that \( m_{ij} \geq 2 \) if \( i \neq j \), and \( m_{ii} = 1 \). (We use the convention that \( \infty \) is formally larger than every integer.) The elements in \( S = \{s_1, s_2, \ldots, s_n\} \) are the Coxeter generators of \( W \), and \( n \) is the rank of \( W \). A subgroup of \( W \) that is generated by a subset \( J \subseteq S \) is a Coxeter group in its own right and is called a standard parabolic subgroup of \( W \). A Coxeter group is called irreducible if it is not isomorphic to a direct product of Coxeter groups of smaller rank. The finite irreducible Coxeter groups were completely classified by Coxeter in [6]. This classification is best visualized using so-called Coxeter diagrams. The Coxeter diagram of \( W \), denoted by \( \Gamma(W) \), is a labeled graph whose vertices are the Coxeter generators of \( W \), and two generators \( s_i \) and \( s_j \) are connected by an edge if and only if \( m_{ij} \geq 3 \). In addition, the edge between \( s_i \) and \( s_j \) is labeled by \( m_{ij} \) if \( m_{ij} \geq 4 \). It is not hard to see that \( W \) is irreducible if and only if its Coxeter diagram is connected. Figure 1 shows the Coxeter diagrams of the finite irreducible Coxeter groups.

Since \( S \) is a generating set of \( W \), we can write every \( w \in W \) as a product of Coxeter generators. The least number of generators needed to form \( w \), \( \ell(w) \), is called the Coxeter length of \( w \), and will be written as \( \ell_S(w) \). We say that a word \( w = s_{i_1}s_{i_2}\cdots s_{i_k} \) is reduced if \( \ell_S(w) = k \).

2.3. Sortable Elements. Let \( W \) be a Coxeter group of rank \( n \). An element \( \gamma \in W \) is called a Coxeter element of \( W \) if \( \gamma = s_{\pi(1)}s_{\pi(2)}\cdots s_{\pi(n)} \) for some permutation \( \pi \) of \( \{1, 2, \ldots, n\} \). Without loss of generality, we can restrict our attention to the Coxeter element \( \gamma = s_1s_2\cdots s_n \). Consider the half-infinite word
\[
\gamma^\infty = s_1s_2\cdots s_n|s_1s_2\cdots s_n|s_1\cdots.
\]
The vertical bars have no influence on the structure of the word, but shall serve for a better readability. Clearly, for every \( w \in W \), every reduced word for \( w \) can be written as a subword of \( \gamma^\infty \). We call the lexicographically first subword of \( \gamma^\infty \) that is a reduced word for \( w \), the \( \gamma \)-sorting word of \( w \), and we denote it by \( \gamma(w) \).

We can write
\[
\gamma(w) = s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}}|s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}}|s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}}|\cdots |s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}}|s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}}|s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}}|s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}}|s_{\delta_{i_1}}s_{\delta_{i_2}}\cdots s_{\delta_{i_l}},
\]
for \( l \in \mathbb{N} \) and \( \delta_{i,j} \in \{0, 1\} \) for \( i \in \{1, 2, \ldots, l\} \) and \( j \in \{1, 2, \ldots, n\} \). The \( i \)-th block of \( w \) is the set \( b_i = \{s_j \mid \delta_{i,j} = 1\} \). We say that \( w \) is \( \gamma \)-sortable if \( b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l \).
and we write $C_\gamma$ for the set of $\gamma$-sortable elements of $W$. Further, define the set of filled positions of $w$ as

$$\alpha_\gamma(w) = \{(i-1)n + j \mid \delta_{ij} = 1\}.$$ 

We notice that $\alpha_\gamma$ depends on the choice of reduced word for $\gamma$, while $C_\gamma$ does not.

**Remark 2.3.** The concept of $\gamma$-sortability was introduced by Reading in [17] as a generalization of stack-sortability, and was used to define the family of Cambrian lattices associated with a Coxeter group [15,19]. The number of $\gamma$-sortable elements of $W$ is the $W$-Catalan number, defined in [4, Section 5.2], and Reading used the $\gamma$-sortable elements to provide a bridge between the noncrossing partitions of $W$ and the clusters of $W$, see [16].

The concept of $\gamma$-sortability has been further extended by Armstrong in [2], where he defined $\omega$-sortability for an arbitrary, not necessarily reduced word $\omega$ in the Coxeter generators of $W$, and if $\omega = \gamma^\infty$, then one obtains precisely the $\gamma$-sortable elements.
3. The Bruhat Order on Sortable Elements

Instead of considering the usual order on $\gamma$-sortable elements we equip $C_\gamma$ with the Bruhat order to be defined next.

**Definition 3.1.** For $u, v \in W$ we write $u \leq_B v$ if and only if there exists a reduced word $v = a_1a_2 \cdots a_l$ and indices $1 \leq i_1 < i_2 < \cdots < i_k \leq l$ such that $u = a_1a_{i_2} \cdots a_{i_k}$. The partial order $\leq_B$ is called the **Bruhat order** on $W$.

Clearly the identity $e$ is the least element with respect to $\leq_B$. Moreover, the poset $(W, \leq_B)$ is graded by $\ell_S$, but it is in general not a lattice. Thus we restrict our attention to the subposet $B_\gamma = (C_\gamma, \leq_B)$, and in what follows, we index poset-theoretic notions that refer to the Bruhat order on $\gamma$-sortable elements by "B", i.e. an interval in the poset $B_\gamma$ will be denoted by $[u, v]_B$, and likewise for joins and meets. Recall that a lattice is finitary if every principal order ideal is finite. In particular, finitary lattices are locally finite.

**Theorem 3.2.** The poset $B_\gamma$ is a finitary lattice for every Coxeter group $W$ and every Coxeter element $\gamma \in W$.

**Proof.** First of all, let $w \in C_\gamma$ with $\ell_S(w) = k$. The interval $[e, w]_B$ is certainly finite, since $w$ has finite length. Moreover, using the terminology from above, it is easy to see that $w \leq_B w'$ if and only if $\alpha_\gamma(w) \subseteq \alpha_\gamma(w')$.

Let $u, u' \in C_\gamma$. The word $\bar{u}$ defined by $\alpha_\gamma(\bar{u}) = \alpha_\gamma(u) \cup \alpha_\gamma(u')$ is again $\gamma$-sortable. In particular, $\bar{u}$ is the least upper bound for both $u$ and $u'$. Hence the interval $[e, \bar{u}]_B$ is finite, and analogously to before we see that any two elements in this interval possess a join. Hence it is a classical lattice-theoretic result that $[e, \bar{u}]_B$ is a lattice. It follows immediately that the meet of $u$ and $u'$ exists as well, and the proof is complete. \hfill $\square$

Theorem 3.2 was already mentioned in [2, Section 6]. It should be remarked that in general $B_\gamma$ is an infinite lattice with no greatest element, which implies in particular that $B_\gamma$ is no complete lattice.

**Theorem 3.3.** For every Coxeter group $W$ and every Coxeter element $\gamma \in W$ the lattice $B_\gamma$ admits an SB-labeling.

**Proof.** Let $u, u_1, u_2 \in C_\gamma$ such that $u \leq_B u_1, u_2$. Since $B_\gamma$ is graded by $\ell_S$ it follows that there are integers $i_1$ and $i_2$ such that $
_1(u_1) = \n_1(u) \cup \{i_1\}$ and $
_1(u_2) = \n_1(u) \cup \{i_2\}$. In particular, the join $\bar{u} = u_1 \lor_B u_2$ is uniquely defined by the property $\n(u) = \n(u) \cup \{i_1, i_2\}$, and in particular satisfies $u_1, u_2 \leq_B \bar{u}$. This implies also that the interval $[u, \bar{u}]_B$ consists only of the four elements $u, u_1, u_2, \bar{u}$. Define

$$
\eta : \mathbb{N} \rightarrow S, \quad i \mapsto \begin{cases} s_n, & \text{if } i \equiv 0 \pmod{n}, \\ s_{i \text{ mod } n}, & \text{otherwise}. \end{cases}
$$

Hence the labeling

$$
b_\gamma : \mathcal{E}(B_\gamma) \rightarrow S, \quad (u, v) \mapsto \eta(\n_\gamma(v) \setminus \n_\gamma(u))
$$

is an SB-labeling of $B_\gamma$. \hfill $\square$

---

1 Usually, the set of $\gamma$-sortable elements of a Coxeter group is equipped with the weak order, and the resulting poset is the so-called $\gamma$-Cambrian semilattice. See [19] for more information.
Corollary 3.4. For every Coxeter group $W$ and every Coxeter element $\gamma \in W$, the Möbius function of $B_\gamma$ takes values only in $\{-1, 0, 1\}$.

Remark 3.5. The reasoning in the proof of Theorem 3.3 can be carried over almost verbatim to show that each of Armstrong’s $\omega$-sorting orders admits an SB-labeling.

Example 3.6. Figure 2 shows two Bruhat lattices labeled by the SB-labeling defined in (1). The lattice in Figure 2(a) is associated with the Coxeter group $A_3$ and the Coxeter element given by the oriented Coxeter diagram $s_1 \overset{1}{\rightarrow} s_2 \overset{2}{\rightarrow} s_3$. Figure 2(b) shows the first seven ranks of the lattice associated with the affine Coxeter group $\tilde{C}_2$ subject to the Coxeter element given by the oriented Coxeter diagram $s_1 \overset{1}{\rightarrow} s_2 \overset{2}{\rightarrow} s_3 \overset{3}{\rightarrow} s_1$. (See Section 4 for an explanation of the connection between Coxeter elements and orientations of the Coxeter diagram.)
3.1. **Join-Distributive Lattices.** Armstrong remarked in [1] that $B_3$ is a join-distributive lattice. A lattice $\mathcal{P} = (P, \leq)$ is **join-distributive** if it is both meet-semidistributive\(^2\) and upper semimodular\(^3\). See [7] for more information on join-distributive lattices. Armstrong’s remark together with Theorem 3.3 led us to the question whether all join-distributive lattices admit an SB-labeling.

Before we answer that question, we recall some more theory. An **antimatroid** is a pair $(M, \mathcal{F})$, where $M$ is a set and $\mathcal{F} \subseteq \wp(M)$ is a family of subsets of $M$ that satisfies the following properties:

1. $\emptyset \in \mathcal{F}$;
2. If $X, Y \in \mathcal{F}$ with $Y \subseteq X$, then there exists some $x \in X \setminus Y$ such that $X \cup \{x\} \in \mathcal{F}$.

The elements of $\mathcal{F}$ are called the **feasible sets** of $(M, \mathcal{F})$. We have the following result.

**Theorem 3.7** ([7, Theorem 3.3]). A lattice $\mathcal{P}$ is join-distributive if and only if there exists an antimatroid $(M, \mathcal{F})$ such that $\mathcal{P} \cong (\mathcal{F}, \subseteq)$.

In view of this correspondence we can now conclude the following result.

**Theorem 3.8.** Every join-distributive lattice admits an SB-labeling.

*Proof.* Let $\mathcal{P}$ be a join-distributive lattice. In view of Theorem 3.7, we can view $\mathcal{P}$ as a lattice of feasible sets of some antimatroid $(M, \mathcal{F})$, and thus every edge in $\mathcal{P}$ is determined by a pair $X, Y \in \mathcal{F}$ with $Y \setminus X = \{x\}$. This induces an edge-labeling of $\mathcal{P}$, which we will denote by $\lambda_\mathcal{F}$. Since $\mathcal{P}$ is upper-semimodular, it follows that for any $p, p_1, p_2 \in \mathcal{P}$ with $p \ll p_1, p_2$, where we write $\bar{p} = p_1 \lor p_2$, we have $p_1, p_2 \ll \bar{p}$. Since $\mathcal{P}$ is meet-semidistributive, it follows that the interval $[p, \bar{p}]$ consists only of the four elements $p, p_1, p_2, \bar{p}$. Thus we have $\lambda_\mathcal{F}(p, p_1) = \lambda_\mathcal{F}(p_2, \bar{p})$ and $\lambda_\mathcal{F}(p, p_2) = \lambda_\mathcal{F}(p_1, \bar{p})$, as well as $\lambda_\mathcal{F}(p, p_1) \neq \lambda_\mathcal{F}(p, p_2)$. Hence $\lambda$ is an SB-labeling of $\mathcal{P}$.

**Corollary 3.9.** The Möbius function of a join-semidistributive lattice takes values only in $\{-1, 0, 1\}$.

*Proof.* This follows from Theorems 2.2 and 3.8.

**Proof of Theorem 1.1.** This follows from Theorems 3.3 and 3.8.

**Remark 3.10.** Join-distributivity can be seen as a generalization of distributivity, see (2) and (3) below. In that sense, Theorem 3.8 generalizes [11, Theorem 5.1], which stated that every distributive lattice is an SB-lattice.

**Remark 3.11.** All join-distributive lattices are by definition meet-semidistributive. Lattices that satisfy the meet-distributive law and the corresponding dual law are called **semidistributive**. Obviously, every distributive lattice is also semidistributive, but semidistributive lattices need no longer be graded. It is known that the Möbius function of a semidistributive lattice takes values only in $\{-1, 0, 1\}$, [8], and it would be interesting whether such lattices are always SB-lattices.

---

\(^2\)A lattice is **meet-semidistributive** if for every three elements $p, q, r$ with $p \land q = p \land r$ we have $p \land q = p \land (q \lor r)$.

\(^3\)A lattice is **upper semimodular** if for every two elements $p, q$ with $p \land q \ll p, q$, we have $p, q \ll p \lor q$. 
Remark 3.12. Another generalization of distributivity to ungraded lattices, so-called *trimness*, was introduced by Thomas in [22]. It is the statement of [22, Theorem 7] that the Möbius function of a trim lattice also takes values only in \{-1, 0, 1\}, and again it would be interesting to know whether trim lattices are always SB-lattices.

An important example of lattices that belong to both previously mentioned classes of lattices are Reading’s Cambrian semilattices, see [19, Theorem 8.1] and [14, Theorem 1.1(iii)]. The Cambrian semilattices generalize the Tamari lattice to all Coxeter groups. Theorem 5.5 in [11] states that the Tamari lattices are SB-lattices. We could produce SB-labelings for some small Cambrian semilattices, but we could not find a uniform definition of such a labeling. We nevertheless pose the following conjecture.

**Conjecture 3.13.** Let \( W \) be a Coxeter group, and let \( \gamma \in W \) be a Coxeter element. The \( \gamma \)-Cambrian semilattice, i.e. the set \( C_\gamma \) equipped with the weak order on \( W \), is an SB-lattice.

### 4. Distributivity of the Bruhat Order on Sortable Elements

Recall that a lattice \( P = (P, \leq) \) is *distributive* if it satisfies one of the two following, equivalent, properties for all \( p, q, r \in P \):

\[
\begin{align*}
(2) & \quad p \land (q \lor r) = (p \land q) \lor (p \land r) \\
(3) & \quad p \lor (q \land r) = (p \lor q) \land (p \lor r)
\end{align*}
\]

Armstrong remarked in [2] that for a certain Coxeter element of the Coxeter group \( A_n \) the lattice \( B_\gamma \) coincides with the lattice of order ideals of the root poset of \( A_n \). (For any undefined terminology, we refer once more to [5].) Hence this particular lattice is distributive. However, Armstrong remarked that this “phenomenon, unfortunately, does not persist for all types”. In this section we partially answer the question for which finite Coxeter groups and which Coxeter elements the lattice \( B_\gamma \) is distributive.

**Lemma 4.1.** Let \( W \) be a Coxeter group, and \( \gamma \in W \) a Coxeter element. If \( W' \) is a standard parabolic subgroup of \( W \) and \( \gamma' \in W' \) denotes the restriction of \( \gamma \) to \( W' \), then \( B_{\gamma'} \) is an order ideal of \( B_{\gamma} \).

**Proof.** Since \( W' \) is a subgroup of \( W \), every element \( w' \in W' \) lies in \( W \) as well. Since \( W' \) is a standard parabolic subgroup of \( W \), we conclude that there exists some \( J \subseteq S \) such that \( W' \) is generated by \( J \). Since \( \gamma' \) is the restriction of \( \gamma \) to \( W' \), we conclude that \( \gamma' \) is the subword of \( \gamma \) that is obtained by deleting the letters not in \( J \). Hence if \( w' \in W' \) is \( \gamma' \)-sortable, then it is also \( \gamma \)-sortable. It follows immediately if \( w \in W \) and \( w' \in W' \) satisfy \( w \leq_B w' \), then we have \( w \in W' \).

(Otherwise, the \( \gamma \)-sorting word of \( w \) contains a letter not in \( J \), which then implies \( a_{\gamma}(w) \not\subseteq a_{\gamma'}(w') \). This, however, contradicts \( w \leq_B w' \)). \( \square \)

**Remark 4.2.** In particular, if \( W \) is finite, then each standard parabolic subgroup of \( W \) induces an interval of \( B_{\gamma} \).
Let $W$ be a Coxeter group, let $\gamma \in W$ be a Coxeter element, and let $\Gamma_\gamma(W)$ be the Coxeter diagram of $W$ with the orientation induced by $\gamma$. If $\Gamma_\gamma(W)$ contains one of the following induced subgraphs, then $B_\gamma$ is not distributive:

(i) $s_i \overset{a}{\longrightarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3}$ for $i_1, i_2, i_3 \in \{1, 2, \ldots, n\}$, and $a, b \geq 3$,

(ii) $s_i \overset{a}{\longrightarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3}$ for $i_1, i_2, i_3 \in \{1, 2, \ldots, n\}$, and $a \geq 4$,

(iii) $s_i \overset{a}{\longrightarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3}$ for $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$,

(iv) $s_i \overset{a}{\longrightarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3}$ for $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$,

(v) $s_i \overset{a}{\longrightarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3} \overset{c}{\longrightarrow} s_{i_4}$ for $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$, and $a \geq 4$,

(vi) $s_i \overset{a}{\longrightarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3} \overset{c}{\longrightarrow} s_{i_4}$ for $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$, and $a \geq 5$, or

(vii) $s_i \overset{a}{\longrightarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3} \overset{c}{\longrightarrow} s_{i_4}$ for $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$, and $a \geq 5$.

Proof. Suppose that $\Gamma_\gamma(W)$ contains an induced subgraph of form (i). Then, in particular, we have the $\gamma$-sortable elements $x = s_{i_2}s_{i_1}|s_{i_2}y = s_{i_2}s_{i_1}s_{i_3}$, and $z = s_{i_2}s_{i_3}|s_{i_2}$. We have

$$x \land_B (y \lor_B z) = x \land_B s_{i_2}s_{i_1}s_{i_3}|s_{i_2}$$

$$= s_{i_2}s_{i_1}|s_{i_2},$$

and

$$(x \land_B y) \lor_B (x \land_B z) = s_{i_2}s_{i_1} \lor_B s_{i_2}$$

$$= s_{i_2}s_{i_1},$$

which contradicts (2).

If $\Gamma_\gamma(W)$ contains an induced subgraph of form (ii), then consider the elements $x = s_{i_2}s_{i_1}|s_{i_2}y = s_{i_2}s_{i_1}s_{i_3}$, and $z = s_{i_2}s_{i_3}|s_{i_2}$. We have

$$x \land_B (y \lor_B z) = x \land_B s_{i_2}s_{i_1}s_{i_3}|s_{i_2}s_{i_2}$$

$$= s_{i_2}s_{i_1}|s_{i_2},$$

and

$$(x \land_B y) \lor_B (x \land_B z) = s_{i_2}s_{i_1} \lor_B s_{i_2}$$

$$= s_{i_2}s_{i_1},$$

which contradicts (2).

If $\Gamma_\gamma(W)$ contains an induced subgraph of the form (iii), then consider the elements $x = s_{i_4}s_{i_2}s_{i_3}|s_{i_2}y = s_{i_4}s_{i_2}s_{i_3}$, and $z = s_{i_4}s_{i_3}|s_{i_2}$. We have

$$x \land_B (y \lor_B z) = x \land_B s_{i_4}s_{i_2}s_{i_3}|s_{i_1}s_{i_2}s_{i_2}$$

$$= s_{i_4}s_{i_3}|s_{i_2},$$

and

$$(x \land_B y) \lor_B (x \land_B z) = s_{i_4}s_{i_2}s_{i_3} \lor_B s_{i_1}s_{i_2}s_{i_2}$$

$$= s_{i_4}s_{i_3}|s_{i_2},$$

which contradicts (2).
(x \land_B y) \lor_B (x \land_B z) = s_1 s_2 s_{i_3} \lor_B s_1 s_2 s_{i_2} \\
= s_1 s_2 s_{i_3},

which contradicts (2).

If \( \Gamma_\gamma(W) \) contains an induced subgraph of the form (iv), then consider the elements \( x = s_1 s_2 s_{i_3} s_{i_2} | s_1 s_2 s_{i_3} s_{i_4} | s_1 s_2 s_{i_3} s_{i_5} | s_1 s_2 s_{i_3} s_{i_6}, \) \( y = s_1 s_2 s_{i_3} s_{i_7} | s_1 s_2 s_{i_3} s_{i_8} | s_1 s_2 s_{i_3} s_{i_9}, \) and \( z = s_1 s_2 s_{i_3} s_{i_10} | s_1 s_2 s_{i_3} s_{i_11} | s_1 s_2 s_{i_3} s_{i_12}. \) We have

\[
(x \land_B (y \lor_B z)) = x \land_B s_1 s_2 s_{i_3} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_5} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \\
= s_1 s_2 s_{i_3} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_5} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \quad \text{and}
\]

\[
(x \land_B (y \lor_B z)) = s_1 s_2 s_{i_3} s_{i_10} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_11} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \quad \text{and}
\]

which contradicts (2).

If \( \Gamma_\gamma(W) \) contains an induced subgraph of the form (v), then consider the elements \( x = s_1 s_2 s_{i_3} s_{i_4} | s_1 s_2 s_{i_3} s_{i_5} | s_1 s_2 s_{i_3} s_{i_6} | s_1 s_2 s_{i_3} s_{i_7} | s_1 s_2 s_{i_3} s_{i_8} | s_1 s_2 s_{i_3} s_{i_9} \), \( y = s_1 s_2 s_{i_3} s_{i_10} s_{i_4} s_{i_7} s_{i_4} s_{i_5} s_{i_7}, \) \( z = s_1 s_2 s_{i_3} s_{i_11} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9}. \) We have

\[
(x \land_B (y \lor_B z)) = x \land_B s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} | s_1 s_2 s_{i_3} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \\
= s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \quad \text{and}
\]

\[
(x \land_B (y \lor_B z)) = s_1 s_2 s_{i_3} s_{i_10} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \\
= s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \quad \text{and}
\]

which contradicts (2).

If \( \Gamma_\gamma(W) \) contains an induced subgraph of the form (vi), then consider the elements \( x = s_1 s_2 s_{i_3} s_{i_4} | s_1 s_2 s_{i_3} s_{i_5} | s_1 s_2 s_{i_3} s_{i_6} | s_1 s_2 s_{i_3} s_{i_7} | s_1 s_2 s_{i_3} s_{i_8} | s_1 s_2 s_{i_3} s_{i_9} \), \( y = s_1 s_2 s_{i_3} s_{i_10} s_{i_4} s_{i_7} s_{i_4} s_{i_5} s_{i_7}, \) and \( z = s_1 s_2 s_{i_3} s_{i_11} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \). We have

\[
(x \land_B (y \lor_B z)) = x \land_B s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} | s_1 s_2 s_{i_3} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \\
= s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \quad \text{and}
\]

\[
(x \land_B (y \lor_B z)) = s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \\
= s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \quad \text{and}
\]

which contradicts (2).

If \( \Gamma_\gamma(W) \) contains an induced subgraph of the form (vii), then consider the elements \( x = s_1 s_2 s_{i_3} s_{i_4} | s_1 s_2 s_{i_3} s_{i_5} | s_1 s_2 s_{i_3} s_{i_6} | s_1 s_2 s_{i_3} s_{i_7} | s_1 s_2 s_{i_3} s_{i_8} | s_1 s_2 s_{i_3} s_{i_9} \), \( y = s_1 s_2 s_{i_3} s_{i_10} s_{i_4} s_{i_7} s_{i_4} s_{i_5} s_{i_7}, \) \( z = s_1 s_2 s_{i_3} s_{i_11} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \). We have

\[
(x \land_B (y \lor_B z)) = x \land_B s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} | s_1 s_2 s_{i_3} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \\
= s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_5} s_{i_7} s_{i_4} s_{i_6} | s_1 s_2 s_{i_3} s_{i_2} \quad \text{and}
\]

\[
(x \land_B (y \lor_B z)) = s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \\
= s_1 s_2 s_{i_3} s_{i_4} s_{i_5} s_{i_7} s_{i_4} s_{i_5} s_{i_9} \quad \text{and}
\]

which contradicts (2).

We obtain the following corollary immediately.
Corollary 4.4. If \( W = D_n \), for \( n \geq 4 \), \( W = E_n \), for \( n \in \{6, 7, 8\} \), \( W = F_4 \), or \( W = H_4 \), and \( \gamma \in W \) is a Coxeter element, then \( B_\gamma \) is not distributive.

Proof. First consider \( W = D_4 \). The eight orientations of \( \Gamma(D_4) \) are shown below.

\[
\begin{array}{cccc}
  s_1 & s_2 & s_3 & s_4 \\
  s_1 & s_2 & s_3 & s_4 \\
  s_1 & s_2 & s_3 & s_4 \\
  s_1 & s_2 & s_3 & s_4 \\
\end{array}
\]

The first four orientations in the first row correspond to case (i) in Proposition 4.3, the first three orientations in the second row correspond to case (iii) in Proposition 4.3, and the fourth orientation in the second row corresponds to case (iv) in Proposition 4.3. Hence \( B_\gamma \) cannot be distributive for Coxeter elements inducing these orientations.

If \( W = D_n \), for \( n > 4 \), or \( W = E_n \), for \( n \in \{6, 7, 8\} \), then we conclude from Figure 1 that \( W \) has a standard parabolic subgroup isomorphic to \( D_4 \). In view of Lemma 4.1 and Remark 4.2, we conclude that \( B_\gamma \) contains a non-distributive interval, and hence cannot be distributive itself.

Now let \( W = F_4 \). The eight orientations of \( \Gamma(F_4) \) are shown below.

\[
\begin{array}{cccc}
  s_1 & s_2 & 4 & s_3 & s_4 \\
  s_1 & s_2 & 4 & s_3 & s_4 \\
  s_1 & s_2 & 4 & s_3 & s_4 \\
  s_1 & s_2 & 4 & s_3 & s_4 \\
\end{array}
\]

The first four orientations correspond to case (i) in Proposition 4.3, the last two orientations in the second row correspond to case (ii) in Proposition 4.3, and the two orientations in the third row correspond to case (v) in Proposition 4.3. Hence \( B_\gamma \) cannot be distributive for Coxeter elements inducing these orientations.

Now let \( W = H_4 \). The eight orientations of \( \Gamma(H_4) \) are shown below.

\[
\begin{array}{cccc}
  s_1 & s_2 & 5 & s_3 & s_4 \\
  s_1 & s_2 & 5 & s_3 & s_4 \\
  s_1 & s_2 & 5 & s_3 & s_4 \\
  s_1 & s_2 & 5 & s_3 & s_4 \\
\end{array}
\]

The first four orientations correspond to case (i) in Proposition 4.3, and the last two orientations in the second row correspond to case (ii) in Proposition 4.3. The first orientation in the third row corresponds to case (vi) in Proposition 4.3, and the second orientation in the third row corresponds to case (vii) in Proposition 4.3. \( \square \)

Proposition 4.5. If \( W = A_n \) for \( n \geq 1 \), \( W = B_n \) for \( n \geq 2 \), \( W = H_3 \) or \( W = I_2(k) \) for \( k \geq 5 \), then there exists a Coxeter element \( \gamma \in W \) such that \( B_\gamma \) is distributive.
Proof. Let $W = A_n$ and let $\gamma$ be the Coxeter element that induces the linear orientation $s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_n$. It follows from the bijection in [3] that $B_\gamma$ is isomorphic to the lattice of classical Dyck paths under dominance order. This lattice is known to be distributive, see for instance [9, Corollary 2.2]. The same bijection implies also that $B_\gamma$ is isomorphic to the lattice of order ideals of the root poset of $A_n$.

Let $W = B_n$ and let $\gamma$ be the Coxeter element that induces the linear orientation $s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_n-1 \rightarrow s_n$. It follows from the bijection in [21, Section 3] that $B_\gamma$ is isomorphic to the lattice of type-B Dyck paths under dominance order. This lattice is known to be distributive, see [13, Theorem 2.9]. The same bijection implies also that $B_\gamma$ is isomorphic to the lattice of order ideals of the root poset of $B_n$.

Let $W = I_2(k)$ for $k \geq 5$, and denote by $s_1$ and $s_2$ the Coxeter generators of $W$. We have $B_{s_1s_2} \cong B_{s_2s_1}$, and this lattice is trivially distributive. It is also isomorphic to the lattice of order ideals of the “root poset” of $I_2(k)$ defined by Armstrong in [1, Figure 5.15].

Let $W = H_3$ and let $\gamma$ be the Coxeter element that induces the linear orientation $s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_3$ . We can easily check by computer that the resulting lattice $B_\gamma$ is distributive. However, in this case, $B_\gamma$ is not isomorphic to the lattice of order ideals of the “root poset” of $H_3$ defined by Armstrong in [1, Figure 5.15]. \qed

Proof of Theorem 1.2. This follows immediately from Proposition 4.5. \qed

We conclude this section with the following conjecture.

Conjecture 4.6. For finite Coxeter groups, the list in Proposition 4.3 is exhaustive, i.e. if $W$ is a finite Coxeter group, $\gamma \in W$ is a Coxeter element and the orientation $\Gamma_\gamma(W)$ of the Coxeter diagram of $W$ induced by $\gamma$ does not contain one of the induced subgraphs listed in Proposition 4.3, then $B_\gamma$ is distributive.

Remark 4.7. The claim of Conjecture 4.6 for $W = H_3$ can be verified by computer. For $W = B_3$, the only orientation other than the one in Proposition 4.5 that is conjectured to yield a distributive lattice $B_\gamma$ is $s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_n-1 \rightarrow s_n$. For $W = A_n$ there are several more options.

References

[1] Drew Armstrong, Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups, Memoirs of the American Mathematical Society 202 (2009).

[2] ________, The Sorting Order on a Coxeter Group, Journal of Combinatorial Theory (Series A) 116 (2009), 1285–1305.

[3] Jason Bandlow and Kendra Killpatrick, An Area-to-Inv Bijection Between Dyck Paths and 312-Avoiding Permutations, The Electronic Journal of Combinatorics 8 (2001).

[4] David Bessis, The Dual Braid Monoid, Annales Scientifiques de l’Ecole Normale Superieure 36 (2003), 647–683.

[5] Anders Björner and Francesco Brenti, Combinatorics of Coxeter Groups, Springer, New York, 2005.

[6] Harold S. M. Coxeter, The Complete Enumeration of Finite Groups of the Form $R^2 = (R_iR_j)^{k_{ij}} = 1$, Journal of the London Mathematical Society 10 (1935), 21–25.

[7] Paul. H. Edelman, Meet-Distributive Lattices and the Anti-Exchange Closure, Algebra Universalis 10 (1980), 290–299.
[8] Jonathan D. Farley, 2013. Personal Communication.
[9] Luca Ferrari and Renzo Pinzani, *Lattices of Lattice Paths*, Journal of Statistical Planning and Inference 135 (2005), 77–92.
[10] Sergey Fomin and Nathan Reading, *Generalized Cluster Complexes and Coxeter Combinatorics*, International Mathematics Research Notices 44 (2005), 2709–2757.
[11] Patricia Hersh and Karola Mészáros, *SB-Labelings and Posets with each Interval Homotopy Equivalent to a Sphere or a Ball* (2014), available at arXiv:1407.5311.
[12] Alex R. Miller, *Foulkes Characters for Complex Reflection Groups* (2013), available at http://www-users.math.umn.edu/~mill1966/ARMillerFoulkesPaper.pdf.
[13] Henri Mühle, *A Heyting Algebra on Dyck Paths of Type A and B* (2013), available at arXiv:1312.0551.
[14] , *Structural Properties of the Cambrian Semilattices – Consequences of Semidistributivity* (2013), available at arXiv:1312.4449.
[15] Nathan Reading, *Cambrian Lattices*, Advances in Mathematics 205 (2006), 313–353.
[16] , *Clusters, Coxeter-Sortable Elements and Noncrossing Partitions*, Transactions of the American Mathematical Society 359 (2007), 5931–5958.
[17] , *Sortable Elements and Cambrian Lattices*, Algebra Universalis 56 (2007), 411–437.
[18] , *Chains in the Noncrossing Partition Lattice*, SIAM Journal on Discrete Mathematics 22 (2008), 875–886.
[19] Nathan Reading and David E. Speyer, *Sortable Elements in Infinite Coxeter Groups*, Transactions of the American Mathematical Society 363 (2011), 699–761.
[20] Jian-yi Shi, *The Enumeration of Coxeter Elements*, Journal of Algebraic Combinatorics 6 (1997), 161–171.
[21] Christian Stump, *More Bijective Catalan Combinatorics on Permutations and Signed Permutations* (2008), available at arXiv:0808.2822.
[22] Hugh Thomas, *An Analogue of Distributivity for Ungraded Lattices*, Order 23 (2006), 249–269.
[23] Nathan Williams, *Cataland*, Dissertation, University of Minnesota, 2013.

Fak. für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
E-mail address: henri.muehle@univie.ac.at