Use of BNNM for interference wave solutions of the gBS-like equation and comparison with PINNs

Shashank Reddy Vadyala 1*, and Sai Nethra Betgeri 1

1 Department of Computational Analysis and Modeling, Louisiana Tech University, Ruston, LA, United States

Abstract. In this work, the generalized broken soliton-like (gBS-like) equation is derived through the generalized bilinear method. The neural network model, which can fit the explicit solution with zero error, is found. The interference wave solution of the gBS-like equation is obtained by using the bilinear neural network method (BNNM) and physical informed neural networks (PINNs). Interference waves are shown well via three-dimensional plots and density plots. Compared with PINNs, the bilinear neural network method is not only more accurate but also faster.

Keywords: Bilinear neural network method · Physics-informed neural networks · Interference wave solitons · (3+1)-dimensional gBS-like equation

1 Introduction

From the atmosphere, ocean to the micro world, almost all problems can be converted into nonlinear models, so the research on nonlinear models is very important. Since the structure of neural network model has strong nonlinear characteristics, the use of neural networks to solve nonlinear problems has attracted great attention in recent years [1-7] Based on the neural network model. Physics-informed neural networks (PINNs) have been introduced by Raissi et al. [8] to find the approximate numerical solution of the nonlinear model. As a part of the neural network itself, PINNs are neural networks (NNs) that encode model equations such as partial differential equations (PDE). PDEs, fractional equations, integral-differential equations, and stochastic PDEs can all be solved using PINNs today. A NN must fit observed data while lowering a PDE residual in the multi-task learning framework that gave rise to this revolutionary technology. Instead, then attempting to determine the solution only based on data, i.e. by fitting a neural network to a set of state-value pairs, PINNs consider the underlying PDE or the physics of the problem. Early work by Owhadi which revealed the promising technique of leveraging such previous knowledge, provided the foundation for the idea of building physics-informed learning machines that make use of systematically structured prior knowledge about the solution. Raisi et al. [14, 15] accurately inferred the solution and provided uncertainty estimates for a range of physical issues by building representations of linear operator functionals using Gaussian process regression; Then, this was expanded upon in [16,17,18]. A two-part work [19, 20] that was later consolidated and published in 2019 [21] described PINNs as a new class of data-driven solvers in 2017 [21]. For the purpose of resolving nonlinear PDEs such as Schrödinger, Burgers, and Allen-Cahn equations, Raisi et al. [21] introduce and demonstrate the PINN technique. They developed physics-informed neural networks (PINNs) that can handle inverse problems in which the model parameters are inferred from observable data as well as forward problems involving estimating the solutions of governing mathematical models. The Bilinear Neural Network Method (BNNM) was introduced by Zhang et al. [9] in order to find the precise analytical solution of the nonlinear model.

In this work, the following generalized broken soliton like (gBS-like) equation is studied by using both BNNM and PINNs,
\[
\frac{9}{8} \beta \mu \left( \int v \, dx \right) \left( \int u \, dx \right)^2 + \frac{9}{4} \beta \left( \int u \, dx \right) v \mu \\
+ \frac{3}{4} \beta \left( \int u \, dx \right) \left( \int v \, dx \right) u_x + u_t + \int u_{ty} \, dx \\
+ \int u_{xz} \, dx + 3 u_x u a + 3 \alpha u^2 \left( \int u \, dx \right)
\] (1)

\[
+ \frac{3}{2} \alpha \mu \left( \int u \, dx \right)^3 + \frac{3}{2} \alpha \left( \int u \, dx \right) u_x + \frac{3}{2} \beta \mu u_y \\
+ \frac{3}{4} \beta u_y \left( \int u \, dx \right)^2 + \frac{3}{4} \beta u^2 \left( \int v \, dx \right) \\
+ \frac{3}{4} \beta \left( \int u \, dx \right)^3 v + \frac{3}{2} \beta u_x v = 0
\]

where \( v = \int u_x \, dx \). The gBS-like equation is firstly derived by us in Section 2 from a (3+1)-dimensional generalized broken soliton (gBS) equation.

\[
\int \left( u_{xt} + u_{yt} + u_{xt} \right) \, dx + \alpha u_{xxx} \\
+ \beta u_{xxy} + \gamma uu_x + \lambda uu_y + \delta u_x \int u_y \, dx = 0
\] (2)

which was firstly derived by Gai et al. [10] from a following (2+1)-dimensional breaking soliton equation

\[
u_t + \alpha u_{xxx} + \beta u_{xxy} + \gamma uu_x + \lambda uu_y + \delta u_x v = 0,
\]

\[u_y = \nu_x. \] (3)

In Section 2, BNNM will be introduced, the gBS-like equation will be derived from a (3+1)-dimensional gBS equation. In Section 3, interference wave solutions of gBS-like equation will be obtained by using BNNM and interference wave will be shown via some 3D plots and density plot. In Section 4, PINNs will be introduced and the approximate numerical solution of gBS-like equation will be obtained. The comparison of the two methods will be discussed. Section 5 will summarize this work.

2 BNNM

BNNM consists of two parts: bilinear transformation and neural networks. The algorithm flow is intuitively shown in Fig. 1. Firstly, the original equation

\[ F(u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, \ldots) = 0 \] (4)

is transformed into bilinear equation

\[ \phi(\psi_t, \psi_x, \psi_y, \psi_{tt}, \psi_{xx}, \psi_{xy}, \ldots) = 0 \] (5)

and then the neural network model is constructed as a trial function \( \psi \) to fit the bilinear equation (5). After inserting the trial function into the bilinear equation (5), collecting the coefficients in front of the independent variable \( x, y, z, t \), a nonlinear algebraic system of equations \( F(W) \) can be obtained. By solving this system, the coefficient solution of the trial function \( \psi \) can be obtained, and then the explicit solution \( u \) of the original equation (4) can be obtained.
2.1 Bilinear Transformation

Based on Hirota bilinear transformation [11] generalized bilinear operator was presented by Ma et al. [12] as follows.

\[
D_{p,x}^m D_{p,y}^n \psi \cdot \psi = \left( \frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'} \right) m \left( \frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'} \right) n
\]

\[
\psi(y, x, t) \psi(y', x', t') |_{x'=x, t'=t}
\]

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \alpha_i^p \alpha_j^p \frac{\partial^{m-i}}{\partial x^{m-i}} \frac{\partial^{i}}{\partial x_{i}} \frac{\partial^{n-j}}{\partial t^{n-j}} \frac{\partial^{j}}{\partial t_{j}}
\]

\[
\psi(y, x, t) \psi(y', x', t') |_{t'=t, x'=x}
\]

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \alpha_i^p \alpha_j^p \frac{\partial^{m+n-i-j}}{\partial x^{m-i} t^{n-j}} \psi(y, x, t)
\]

Fig.1 The algorithm flow of BNNM

The Hirota bilinear form of Eq. (2) is as,

\[
(D_x D_t + D_y D_t + D_x D_t + \alpha D_x^4 + \beta D_y^2 D_y) \psi \cdot \psi
\]

The generalized bilinear form can be obtained from Eq. (7) via Eq. (6),

\[
(D_{3,x} D_{3,t} + D_{3,y} D_{3,t} + D_{3,x} D_{3,t} + \alpha D_x^4 + \beta D_y^2 D_y) \psi \cdot \psi
\]

\[
= 2 \psi_{t,x} \psi + 2 \psi_{t,y} \psi + 2 \psi_{t,x} \psi - 2 \psi_1 (\psi_x + \psi_y + \psi_z) + 6 \alpha \psi_x^2 + 6 \beta \psi_{xy} + \psi_{xx}
\]
Through Transformation

\[ u = 2 \ln(f)_x, \quad v = 2 \ln(f)_y, \]

the \((3+1)\) gBS-like equation (1) can be derived.

### 2.2 Neural Networks Model

The multi-layer perceptron can be regarded as a mapping, such as a double-layer perceptron (Fig. 2), which can be expressed by the following mathematical expression,

\[
\begin{align*}
\psi &= F_3(\xi_3) + F_4(\xi_4) + b_5, \\
\xi_1 &= tw_{x,1} + xw_{y,1} + yw_{z,1} + zw_{x,1} + b_1, \\
\xi_2 &= tw_{x,2} + xw_{y,2} + yw_{z,2} + zw_{x,2} + b_2, \\
\xi_3 &= w_{2,3}F_2(\xi_2) + w_{1,3}F_1(\xi_1) + b_3, \\
\xi_4 &= w_{2,4}F_2(\xi_2) + w_{1,4}F_1(\xi_1) + b_4.
\end{align*}
\]

By selecting different activation functions in expression (10), the new trial function will be obtained.

![Double Layer Perception](image)

**Fig.2** Double Layer Perception

### 3 Interference wave solution of a gBK-like Equation

Making \(F_1(\bullet) = \sin(\bullet), F_2(\bullet) = \cos(\bullet), F_3(\bullet) = e(\bullet), F_4(\bullet) = (\bullet)^2, b_1 = b_2 = b_3 = b_4 = 0\) the following test function can be obtained,
$$\psi = b_5 + w_{3,\mu}e^{w_{2,3} \sin(\xi_1)+w_{1,3} \cos(\xi_2)}$$

$$+w_{4,u} \left( w_{2,4} \sin(\xi_1) + w_{1,4} \cos(\xi_2) \right)^2,$$

(11)

$$\{ \xi_1 = tw_{t,2} + xw_{x,2} + yw_{y,2} + zw_{z,2},$$

$$\xi_2 = tw_{t,1} + xw_{x,1} + yw_{y,1} + zw_{z,1} \}$$

substituting above function into Eq. (8), and collecting the coefficients in front of

t, x, y, z, \cos(\xi_1), \cos(\xi_2), \sin(\xi_1), \sin(\xi_2),$$

$$e^{w_{2,3} \sin(\xi_1)+w_{2,3} \cos(\xi_2)},$$

(12)

Where,

$$\xi_1 = tw_{t,2} + xw_{x,2} + yw_{y,2} + zw_{z,2},$$

$$\xi_2 = tw_{t,1} + xw_{x,1} + yw_{y,1} + zw_{z,1},$$

(13)

the following constraints have been got,

$$\{ \alpha = \alpha, \beta = \beta, b_5 = b_5, w_{1,3} = w_{1,3}, w_{1,4} = 0, \}$$

$$w_{2,3} = 0, w_{2,4} = w_{2,4}, w_{3,\mu} = w_{3,\mu},$$

$$w_{4,u} = w_{4,u}$$

(14)

$$w_{t,1} = w_{t,1}, w_{t,2} = w_{t,2}, w_{x,1} = -\frac{\beta w_{y,1}}{\alpha},$$

$$w_{x,2} = 0, w_{y,1} = w_{y,1}, w_{y,2} = -w_{z,2},$$

$$w_{z,1} = -w_{y,1} \frac{\alpha - \beta}{\alpha}, w_{z,2} = w_{z,2} \}$$

substituting above constraints into test function and through transformation (9), the exact analytical solution of the original function is finally obtained,

$$u = 2 \frac{\psi_{xx}}{\psi} - 2 \frac{(\psi_x)^2}{\psi^2},$$

$$\psi = w_{\theta}, \psi w_{2,3}^2 + b_5$$

$$- \left( \cos^2 \left( (-y + z)w_{x,2} + tw_{t,2} \right) \right) w_{2,4} w_{4,\psi}$$

$$w_{(1,3)} \cos \left( \frac{((-y + z)\alpha - \beta (x - z))wy_{1} + tw_{t,1} t}{\alpha} \right)$$

$$+ w_{3,\psi} e^{w_{1,3} \cos \left( \frac{((-y + z)\alpha - \beta (x - z))wy_{1} + tw_{t,1} t}{\alpha} \right)}.$$  

Letting,

$$\alpha = w_{1,3} = b_5 = w_{3,\psi} = 2,$$

$$\beta = w_{y,1} = w_{\psi}, w_{2,4} = w_{x,2} = w_{t,2} = w_{t,1} = 1,$$

(16)

we can get a unique solution,

$$u = \frac{4 \left( -\xi_1 e^{2\xi_1} + \left( \xi_2^2 + \frac{1}{2} \xi_1 - 1 \right) (\xi_2 - 3) \right) e^{2\xi_1}}{(\xi_2 - 2 e^{2\xi_1} - 3)^2},$$

(17)

Where,

$$\xi_1 = \cos \left( t - \frac{x}{2} + y - \frac{z}{2} \right),$$

(18)
\[ \xi_1 = \cos^2(t - y + z) \]

The density plot and 3D plots of above analytical solution are shown in Fig 3.

4 Comparision with Numerical Solution VIA PINNS

Due to the universality of approximate numerical solutions, approximate numerical solutions are usually used to deal with models that do not require high precision. PINNs use neural networks as the approximator of nonlinear models and do not need to select a large amount of data, which makes this method more superior than other numerical approximation methods.

The algorithm flow of PINNs is shown in Fig. 4. Through the automatic differentiation (AD) technology, the neural network model approximator is brought.

Fig.3 (Color Line) The density plot and 3D plot of Eq (17)
The flow of PNNs

Let us define $f(t, x, y, z) = 9 \frac{\partial u}{\partial t} + \frac{3}{2} \frac{\partial^2 u}{\partial x^2} + 3 \alpha u \frac{\partial u}{\partial x} + \frac{3}{2} \beta \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{3}{2} \beta \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{3}{4} \beta \frac{\partial u}{\partial x} + \frac{3}{4} \beta \frac{\partial v}{\partial x} + \frac{3}{4} \beta \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{3}{4} \beta \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{3}{2} \beta \frac{\partial u}{\partial x} \frac{\partial v}{\partial y}.$

The loss function is defined as,

$$LOSS = MSE_f + MSE_u$$

$$MSE_f = \frac{1}{N_f} \sum_{i=1}^{N_f} |f(t_f^i, x_f^i, y_f^i, z_f^i) - 0|^2$$

Where $\{t_f^i, x_f^i, y_f^i, z_f^i\}_{i=1}^{N_f}$ specify the collocations points for $f(t, x)$, and the $MSE_u$ means supervised constraint.
\[ MSE_u = \frac{1}{N_u} \sum_{i=1}^{N_u} |f(t_u^i, x_u^i, y_u^i, z_u^i) - 0|^2 \]  

Where \( \{t_u^i, x_u^i, y_u^i, z_u^i\}_{i=1}^{N_u} \) denote the boundary (and initial) training data on \( u(t, x) \). At this time, the solution of nonlinear partial differential equations is transformed into an optimization problem.

5 Conclusion

In this work, the generalized broken soliton-like (gBS-like) equation is derived through the generalized bilinear method. The neural network model, which can fit the explicit solution with zero error, is found. Interference wave solution of the gBS-like equation is obtained by using the bilinear neural network method (BNNM) and physical informed neural networks (PINNs). Interference wave are shown well via three dimensional plots and density plot. Compared with PINNs, the bilinear neural network method is not only more accurate, but also faster.

Acknowledgments:

We express our sincere thanks to Chen Zhao from Harvard University for helps on Julia Computing. This work is supported by the National Natural Science Foundation of China under grant Nos: 12061054 and 61877007; The Fundamental Research Funds for the Central Universities under No. DUT20GJ205.

Declaration:

Conflict of Interest: The authors declare that they have no conflict of interest.

Availability of data and materials: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] S. Vadyala, S. Betgeri, J. Matthews and E. Matthews, "A review of physics-based machine learning in civil engineering. Results in Engineering,", no. 3, p. 100316., 2021.
[2] S. Vadyala, B. S. N and . N. P. Betgeri, "Physics-informed neural network method for solving one-dimensional advection equation using PyTorch," Array, 13, no. 1, p. 100110, 2022.
[3] K. Zubov, Z. McCarthy, Y. Ma and P. F. Calisto, "NeuralPDE: Automating physics-informed neural networks (PINNs) with error approximations.," arXiv preprint arXiv: 2107.09443, 2021.
[4] R. Zhang and S. Bilige, "Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation.," Nonlinear Dynamics, vol. 95(4), no. 9, pp. 3041-3048, 2019.
[5] R. F. Zhang, M. C. Li, M. Albishari, F. C. Zheng and Z. Z. Lan, "Generalized lump solutions, classical lump solutions and rogue waves of the (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada-like equation.," Applied Mathematics and Computation, vol. 403, no. 6, p. 126201, 2021.
[6] R. F. Zhang, S. Bilige, J. Liu and M. Li, "Bright-dark solitons and interaction phenomenon for p-gBKP equation by using bilinear neural network method.," Physica Scripta, vol. 96(2), no. 7, p. 025224, 2020.
[7] R. F. Zhang, L. C. Ming and H. M. Yin, "Rogue wave solutions and the bright and dark solitons of the (3+1)-dimensional Jimbo–Miwa equation," Nonlinear Dynamics, vol. 103(1), no. 4, pp. 1071-1079, 2021.
[8] M. Raissi, P. Perdikaris and G. Karniadakis, "Physics-informed neural networks: A deep
learning framework for solving forward and inverse problems involving nonlinear partial differential equations," *Journal of Computational physics*, vol. 378, no. 8, pp. 686-707, 2019.

[9] R. Zhang, S. Bilige and T. Chaolu, "Fractal solitons, arbitrary function solutions, exact periodic wave and breathers for a nonlinear partial differential equation by using bilinear neural network method.,” *Journal of Systems Science and Complexity*, vol. 34(1), no. 5, pp. 122-139, 2021.

[10] L. Gai, W. X. Ma and M. Li, "Lump-type solutions, rogue wave type solutions and periodic lump-stripe interaction phenomena to a (3+1)-dimensional generalized breaking soliton equation," *Physics Letters*, vol. 384(8), p. 126178, 2020.

[11] R. Hirota, "The direct method in soliton theory," *Cambridge University Press*, no. 155., 2004.

[12] W. X. Ma, "Generalized bilinear differential equations," *Studies in Nonlinear Sciences*, vol. 2, no. 4, pp. 140-144, 2011.

[13] H. Owhadi, "Bayesian numerical homogenization," *Multiscale Modeling & Simulation*, vol. 13, no. 3, pp. 812-828., 2015.

[14] M. Raissi, P. Perdikaris and G. Karniadakis, "Inferring solutions of differential equations using noisy multi-fidelity data.,” *Journal of Computational Physics*, vol. 335, pp. 736-746, 2017.

[15] M. Raissi, P. Perdikaris and G. Karniadakis, "Machine learning of linear differential equations using Gaussian processes.,” *Journal of Computational Physics*, pp. 683-693., 2017.

[16] M. Raissi and G. Karniadakis, "Hidden physics models: Machine learning of nonlinear partial differential equations," *Journal of Computational Physics*, pp. 125-141, 2018.

[17] M. Raissi, Perdikaris, P. and Karniadakis, G. E., "Numerical Gaussian processes for time-dependent and nonlinear partial differential equations.,” *SIAM Journal on Scientific Computing*, vol. 40(1), pp. A172-A198, 2018.

[18] Vadyala, S. R., Betgeri, S. N. and Betgeri, N. P, "Physics-informed neural network method for solving one-dimensional advection equation using PyTorch," *Array 13*, p. 100110., 2022.

[19] Raissi, M., Perdikaris, P. and Karniadakis, G. E., "Physics informed deep learning (part i): Data-driven solutions of nonlinear partial differential equations.,” *arXiv preprint arXiv:1711.10561*, 2017.

[20] M. P. P. & K. Raissi, "Physics informed deep learning (part i): Data-driven solutions of nonlinear partial differential equations," *arXiv preprint arXiv:1711.10561*, 2017.

[21] Raissi, M., Perdikaris, P., & Karniadakis, G. E, "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations," *Journal of Computational physics*, vol. 378, pp. 686-707, 2019.
