Global jump filters and realized volatility

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\textbf{Summary} For a semimartingale with jumps, we propose a new estimation method for integrated volatility, i.e., the quadratic variation of the continuous martingale part, based on the global jump filter proposed by Inatsugu and Yoshida \cite{InatsuguYoshida}. To decide whether each increment of the process has jumps, the global jump filter adopts the upper \(\alpha\)-quantile of the absolute increments as the threshold. This jump filter is called global since it uses all the observations to classify one increment. We give a rate of convergence and prove asymptotic mixed normality of the global realized volatility and its variant “Winsorized global volatility”. By simulation studies, we show that our estimators outperform previous realized volatility estimators that use a few adjacent increments to mitigate the effects of jumps.

\textbf{Keywords and phrases} Volatility, semimartingales with jumps, global filter, high-frequency data, order statistic, rate of convergence, asymptotic mixed normality.

\section{Introduction}

Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}\). We consider a one-dimensional semimartingale \(X = (X_t)_{t \in [0,T]}\) having a decomposition

\[ X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dw_s + J_t \quad (t \in [0,T]) \tag{1.1} \]

where \(X_0\) is an \(\mathcal{F}_0\)-measurable random variable, \(b = (b_t)_{t \in [0,T]}\) and \(\sigma = (\sigma_t)_{t \in [0,T]}\) are càdlàg \(\mathbb{F}\)-adapted processes, and \(w = (w_t)_{t \in [0,T]}\) is an \(\mathbb{F}\)-standard Wiener process. \(J = (J_t)_{t \in [0,T]}\) is

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the jump part of $X$. We will assumed that $J$ is finitely active, that is, $J_t = \sum_{s \in (0,t]} \Delta J_s$ for $\Delta J_s = J_s - J_{s-}$ and $\sum_{t \in (0,T]} 1_{\{\Delta J_t \neq 0\}} < \infty$ a.s. In this paper, we are interested in the estimation of the integrated volatility

$$\Theta = \int_0^T \sigma_t^2 dt$$  \hspace{1cm} (1.2)

based on the data $(X_{t_j})_{j=0,1,...,n}$, where $t_j = t_j^n = jT/n$.

The jump part $J$ can be endogenous or exogenous, as well as $b$ and $\sigma$, however, $J$ is a nuisance in any case. The simple realized volatility is heavily damaged when jumps exist. To avoid the effects of the jumps, various methods have been proposed so far. For example, the bipower variation (Barndorff-Nielsen and Shephard \[2\], Barndorff-Nielsen et al. \[3\]) and the minimum realized volatility (Andersen et al. \[1\]) are shown to be consistent estimators of the integrated volatility even in the presence of jumps. The idea of these methods is that, to mitigate the effect of jumps, they employ adjacent increments in constructing the estimator. Another direction to handle jumps is to introduce a threshold to detect jumps.

Parametric inference for sampled diffusion type processes was studied by Dohnal \[5\], Prakasa Rao \[16, 15\], Yoshida \[21, 22\], Kessler \[10\], Genon-Catalot and Jacod \[6\], Uchida and Yoshida \[19, 18, 20\], Ogihara and Yoshida \[14\], Kamatani and Uchida \[9\] and others. Limit theorems used to analyse the realized volatility appeared in the studies of parametric inference. If no jump part exists, then the distribution of the increment $\Delta_j X = X_{t_j} - X_{t_{j-1}}$ admits Gaussian approximation in a short time interval, and a quasi-likelihood function can be constructed with the conditional Gaussian density. When the jump part exists, the local Gaussian approximation is no longer valid. Then it is necessary to detect jumps and classify the increments to apply the local Gaussian quasi-likelihood function for estimation of the parameters in the continuous part. Threshold method was investigated by Shimizu and Yoshida \[17\] and Ogihara and Yoshida \[13\] in the context of the parametric inference for a stochastic differential equation with jumps. The idea of thresholding is rather old, going back to the studies of limit theorems for Lévy processes as latest. Mancini \[12\] used this idea in a nonparametric situation. Koike \[11\] applied the threshold method to covariance estimation for asynchronously observed semimartingales with jumps. The classical jump filters compare the size of increment with a threshold determined by a (conditionally/unconditionally deterministic) function of the length of the time interval. If an increment is so large that exceeds the threshold, it is regarded as having jumps. Otherwise, the increment is regarded as having no jump. Once classified, the increments are used to estimate the parameters in continuous and jump parts, respectively.

Though the efficiency of the traditional thresholding parametric estimators has been established theoretically, it is known that their real performance strongly depends on a choice of tuning parameters; see, e.g., Iacus and Yoshida \[7\]. Examining each individual increment without other data is not always effective in finding jumps. It sometimes overlooks relatively small jumps due to a conservative level of threshold to try to incorporate all Brownian increments. To resolve this problem, Inatsugu and Yoshida \[8\] introduced the so-called global filters that examine all increments simultaneously and regard an increment of high rank in order of absolute size as a jump. Using the information about the size of other increments helps us detect jumps more accurately than the previous methods that ignore such information. Moreover, Inatsugu and Yoshida \[8\] also removed the assumption of low intensity of small jumps, that was used in
Shimizu and Yoshida [17] and Ogihara and Yoshida [13]. This is a theoretical advantage of the global jump filters, in addition to their outperformance in practice.

In this paper, we will apply the global filtering method to nonparametric volatility estimation. Specifically, we will construct the “global realized volatility (GRV) estimator” of the integrated volatility for the semimartingale $X$ having the decomposition (1.1). Though $J$ and the jump part of $\sigma$ are assumed to be finitely active for each $n$, we permit the number of jumps to diverge as $n$ tends to infinity. We will investigate the theoretical properties of GRV and then conduct numerical simulations to study their performance compared with traditional methods, that is, the deterministic threshold estimator, the bipower variation and the minimum realized volatility.

The organization of this paper is as follows. Section 2 introduces the GRV and its variant, the winsorized GRV (WGRV). In Section 3 we introduce the local-global realized volatility (LGRV) and prove its convergence to the spot volatility. The LGRV will be used for normalizing the increments to compute the global filter. Section 4 gives the rate of convergence of the GRV and WGRV in the situation where the intensity of jumps is high. In this case, we need a high and fixed cut-off rate $\alpha$ to eliminate harmful jumps. In Section 5 we allow the cut-off rate to vary according to the sample size. This “moving threshold” method is for the situation where the intensity of jumps is moderate and small cut-off rate is applicable. Section 6 briefly discusses the situation where true volatility is constant. In this case, normalizing increments is not necessary, so the estimator gets a little simpler. Section 7 presents some simulation results to compare the real performance of the GRV, WGRV, bipower variation, and the minimum realized volatility.

Concluding, let us mention some technical aspects. The global jump filter causes theoretical difficulty. By nature, it uses all the data to classify each increment $\Delta_jX$. This completely destroys the martingale structure in the model, which makes it difficult to use orthogonality between the selected increments to validate the law of large numbers and the central limit theorem. However, it is possible to asymptotically recover the orthogonality by the glocal and global filtering lemmas presented in Sections 3 and 4. Technically, the argument here is closed within the semimartingale theory, although the global filter breaks adaptivity of the functionals, in other words, a quadratic variation with anticipative weights is treated. On the other hand, Yoshida [23] suggests a use of the Malliavin calculus to analyse robustified volatility estimators with anticipative weights.

2 Realized volatilities with a global jump filter

The global jump filter introduced by Inatsugu and Yoshida [8] uses the order statistics of the transformed increments of the observations. Suppose that an estimator $S_{n,j-1}$ of the spot volatility $\sigma(X_{t_{j-1}})^2$ (up to a common scaling factor) is given for each $j \in I_n = \{1, \ldots, n\}$. Denote $\Delta_j U = U_t - U_{t_{j-1}}$ for a process $U = (U_t)_{t \in [0,T]}$. Then the distribution of the scaled increment $S_{n,j-1}^{-1/2} \Delta_j X$ is expected to be well approximated by the standard normal distribution $N(0,1)$. Therefore, if the value

$$V_j = \left| (S_{n,j-1})^{-1/2} \Delta_j X \right|$$

(2.1)
is relatively very large among \( V_n = \{ V_k \}_{k \in I_n} \), then plausibly we can infer that the \( V_j \) involves jumps with high probability. The idea of the global jump filter is to eliminate the increment \( \Delta_j X \) from the data if the corresponding \( V_j \) is ranked within the top 100\% in \( V_n \). More precisely, let

\[
\mathcal{J}_n(\alpha) = \{ j \in I_n; V_j < V(s_n(\alpha)) \}
\]

where

\[
s_n(\alpha) = \lfloor n(1 - \alpha) \rfloor
\]

for \( \alpha \in [0, 1) \), and we denote by \( r_n(U_j) \) the rank of \( U_j \) among the variables \( \{ U_i \}_{i \in I_n} \). Let

\[
q(\alpha) = \int_{\{z \leq c(\alpha)\}} z^2 \phi(z; 0, 1) dz
\]

where \( \phi(z; 0, 1) \) is the density function of \( N(0, 1) \) and \( c(\alpha) \) defined by

\[
P[z^2 \leq c(\alpha)] = 1 - \alpha
\]

for \( \zeta \sim N(0, 1) \) and \( \alpha \in [0, 1) \). Then the global realized volatility (globally truncated realized volatility, GRV) with cut-off ratio \( \alpha \) is defined by

\[
\mathbb{V}_n(\alpha) = \sum_{j \in \mathcal{J}_n(\alpha)} q(\alpha)^{-1}[\Delta_j X]^2 K_{n,j}
\]

where \( K_{n,j} = 1_{\{|\Delta_j X| \leq n^{-1/4}\}} \). As remarked in Inatsugu and Yoshida [8], the indicator function \( K_{n,j} \) is set just for relaxing the conditions for validation. Generalization by using like \( 1_{\{|\Delta_j X| \leq B_1 n^{-\delta_1}\}} \) with constants \( B_1 > 0 \) and \( \delta_1 \in (0, 1/4] \) is straightforward, but we prefer simplicity in presentation of this article. In practice, the probability that \( K_{n,j} \) executes the task is exponentially small by the large deviation principle. However, the moments of \( \Delta J_t \) are not controllable without assumption, and we can simply avoid it by the cut-off function \( K_{n,j} \).

Winsorization is a popular technique in robust statistics. In the present context, the Winsorized global realized volatility (WGRV) is given by

\[
\mathbb{W}_n(\alpha) = \sum_{j=1}^{n} w(\alpha)^{-1}\{|\Delta_j X| \wedge (S_{n,j-1}^{1/2} V(s_n(\alpha))) \}^2 K_{n,j}
\]

where

\[
w(\alpha) = \int_{\mathbb{R}} (z^2 \wedge c(\alpha)) \phi(z; 0, 1) dz.
\]

The cut-off ratio \( \alpha \in [0, 1) \) is a tuning parameter in estimation procedures. The bigger \( \alpha \) provides the more stable estimates even in high intensity of jumps. On the other hand, the smaller \( \alpha \) gives the more precise estimates if the intensity of jumps is low. Making trade-off between stability and precision is necessary in practice. As a matter of fact, these cases require different theoretical treatments. We will consider fixed \( \alpha \) in Section 4 and shrinking \( \alpha \) in Section 5.
3 Local-global filter

An estimator $S_{n,j-1}$ for the spot volatility (up to a constant scaling) is necessary to construct a global realized volatility. Naturally, we use the data around time $t$ to estimate $\sigma_t$. Since these data are also contaminated with jumps, we need a jump filter to construct a temporally-local estimator $S_{n,j-1}$. The idea of the global jump filter with the order statistics of the data around $t$ serves to eliminate the effects of jumps, not only theoretically but also practically as demonstrated by the simulation studies of Section 7. In this section, we propose a local-global realized volatility and validate it by establishing in Section 3.2 the rate of convergence of the estimator. Since the local-global filter involves the order statistics, that destroy the martingale structure, we try to recover it by somewhat sophisticated lemmas given in Section 3.1. The minimum realized volatility (minRV) made of the temporally-local data is also a candidate of an estimator for the spot volatility. A rate of convergence of the local minRV is mentioned in Section 3.3.

3.1 Glocal filtering lemmas

For each $j \in I_n$, let

$$\tilde{j}_n = \begin{cases} 1 & (j \leq \kappa_n) \\ j - \kappa_n & (\kappa_n + 1 \leq j \leq n - \kappa_n) \\ n - 2\kappa_n & (j \geq n - \kappa_n + 1) \end{cases}$$

for $\kappa_n \in \mathbb{Z}_+$ satisfying $2\kappa_n + 1 \leq n$. Let $I_{n,j} = \{j_n, j_n + 1, \ldots, j_n + 2\kappa_n\}$. Let

$$\hat{U}_{j,k} = h^{-1/2}\sigma_{t_{j-1}}^{-1}\Delta_k X \quad \text{and} \quad W_{j} = h^{-1/2}\Delta_j w$$

for $j, k \in I_n$. Both variables $\hat{U}_{j,k}$ and $W_{j}$ depend on $n$. Let

$$\hat{R}_{j,k} = \hat{U}_{j,k} - W_k - h^{-1/2}\sigma_{t_{j-1}}^{-1}\Delta_k J$$

for $j, k \in I_n$. Denote $L^\infty = \cap_{p>1} L^p$.

Let $N = \sum_{s \in (0,\infty)} \{\Delta_J s \neq 0\}$. Let $\tilde{\sigma} = \sigma - J^\sigma$ for $J^\sigma = \sum_{s \in (0,\infty)} \Delta s$, and let $N^\sigma = \sum_{s \in (0,\infty)} \{\Delta J_s \neq 0\}$. We assume that $N^\sigma_T < \infty$ a.s. Moreover, let $\overline{N} = N + N^\sigma$. Let $\tilde{X} = X - J$. A counting process will be identified with a random measure. Let $I_{n,j} = \{t_{j-1}, t_{j-1} + 2\kappa_n\}$.

[G1] (i) For every $p > 1$, $\sup_{t \in [0,T]} \|\sigma_t\|_p < \infty$ and

$$\|\tilde{\sigma}_t - \tilde{\sigma}_s\|_p \leq C(p)|t-s|^{1/2} \quad (t, s \in [0, T])$$

for some constant $C(p)$ for every $p > 1$.

(ii) $\sup_{t \in [0,T]} \|b_t\|_p < \infty$ for every $p > 1$.

(iii) $\sigma_t \neq 0$ a.s. for every $t \in [0, T]$, an $\sup_{t \in [0,T]} \|\sigma_t^{-1}\|_p < \infty$ for every $p > 1$. 

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Lemma 3.1. Under \([G1]\),

\[
\sup_{j \in I_n} \sup_{k \in I_{n,j}} \left\| \hat{R}_{j,k} 1_{\{N^*(I_{n,j}) = 0\}} \right\|_p = O \left( \left( \frac{\kappa_n}{n} \right)^{1/2} \right) \tag{3.1}
\]
as \(n \to \infty\) for every \(p > 1\).

Proof. For \(j \in I_n\), let \(E(j) = \{N^*(I_{n,j}) = 0\}\). Then, for \(k \in I_{n,j}\),

\[
\hat{R}_{j,k} 1_{E(j)} = \left( h^{-1/2} \sigma_{I_{n-1}^j}^{-1} \Delta_k \tilde{X} - h^{-1/2} \Delta_k w \right) 1_{E(j)}
\]

\[
= h^{-1/2} \int_{I_{k-1}} \sigma_{I_{n-1}^j}^{-1} (\tilde{t} - \tilde{t}_{I_{n-1}^j}) dw_t 1_{E(j)}
\]

\[
+ h^{-1/2} \sigma_{I_{n-1}^j}^{-1} \int_{I_{k-1}} b_t dt 1_{E(j)} \tag{3.2}
\]

We obtain (3.1) by applying the Burkholder-Davis-Gundy inequality to the martingale part of (3.2) after the trivial estimate \(1_{E(j)} \leq 1\).

For \(j \in I_n\), denote by \(r_{n,j}(U_k)\) the rank of the element \(U_k\) among a collection of random variables \(\{U_k\}_{k \in I_{n,j}}\). Let

\[
0 < \eta_2 < \eta_1, \quad \pi_n = 2\kappa_n + 1, \quad a_n = \lfloor (1 - \alpha_0)\pi_n - \pi_n^{-1-\eta_2} \rfloor, \quad \hat{a}_n = \lfloor a_n - \pi_n^{-1-\eta_2} \rfloor
\]

for \(\alpha_0 \in [0, 1]\). Let

\[
L_{n,j,k} = \{ r_{n,j}(|W_k|) \leq a_n - \pi_n^{-1-\eta_2} \} \cap \{ |W|_{(j,a_n)} - |W_k| < \pi_n^{-\eta} \} \tag{3.3}
\]

where \(\{|W|_{(j,k)}\}_{k \in I_{n,j}}\) are the ordered statistics made from \(\{|W_k|\}_{k \in I_{n,j}}\). In the same way as Lemma 1 of Inatugu and Yoshida \([8]\), we obtain the following result.

Lemma 3.2. Let \(\alpha_0 \in (0, 1)\). Suppose that \(\eta_1 < 1/2\) and that \(n^{-\epsilon} \kappa_n \to \infty\) as \(n \to \infty\) for some \(\epsilon \in (0, 1)\). Then

\[
\sup_{j \in I_n} P \left[ \bigcup_{k \in I_{n,j}} L_{n,j,k} \right] = O(n^{-L})
\]
as \(n \to \infty\) for every \(L > 0\).

Define \(K_{n,j}(\alpha_0)\) by

\[
K_{n,j}(\alpha_0) = \{ k \in I_{n,j}; r_{n,j}(|\Delta_k X|) \leq (1 - \alpha_0)\pi_n \},
\]

where \(r_{n,j}(|\Delta_k X|)\) is the rank of \(|\Delta_k X|\) among \(\{|\Delta_{k'} X|\}_{k' \in I_{n,j}}\). Let

\[
\tilde{K}_{n,j}(\alpha_0) = \{ k \in I_{n,j}; r_{n,j}(|W_k|) \leq \hat{a}_n \}.
\]
Let
\[ \Omega_{n,j} = \bigcap_{k \in I_{n,j}} \left( \left\{ \hat{R}_{j,k} \big| 1_{\{N^\alpha(I_{n,j})=0\}} < 2^{-1}\pi_n^{-m} \right\} \cap L_{n,j,k}^c \right) . \]

Let
\[ L_n = \{ j \in I_n; \, \overline{N}(I_{n,j}) \neq 0 \}. \] (3.4)

**Lemma 3.3.**

(a) \( \hat{\mathcal{K}}_{n,j}(\alpha_0) \subset \mathcal{K}_{n,j}(\alpha_0) \) on \( \Omega_{n,j} \) if \( j \in L_n^c \).  

(b) \( 1_{\Omega_{n,j}} 1_{\{j \in L_n^c\}} \#(\mathcal{K}_{n,j}(\alpha_0) \setminus \hat{\mathcal{K}}_{n,j}(\alpha_0)) \leq 4 \pi_n^{1-\eta_2} \) \((j \in I_{n,j}, \, n \in \mathbb{N})\).  

**Proof.** Let \( n \in \mathbb{N} \) and suppose that \( j \in L_n^c \). We will work on \( \Omega_{n,j} \). For a pair \((k_1, k_2) \in I_{n,j}^2\), suppose that
\[ r_{n,j}(|W_{k_1}|) \leq \widehat{a}_n \quad \text{and} \quad r_{n,j}(|W_{k_2}|) \geq a_n. \] (3.5)

Then \( |\hat{U}_{j,k_1}| < |W_{k_1}| + 2^{-1}\pi_n^{-m}, \) since \( \Delta_{k_1} N = 0 \) and \( N^\alpha(I_{n,j}) = 0 \) when \( j \in L_n^c \), and then \( |\hat{R}_{j,k_1}| < 2^{-1}\pi_n^{-m} \) on \( \Omega_{n,j} \). By the first inequality of (3.5), \( r_{n,j}(|W_{k_1}|) \leq a_n - \pi_n^{1-\eta_2}, \) and hence on \( \Omega_{n,j} \subset L_{n,j,k_1}^c \), we have \(|W_{(j,a_n)} - |W_{k_1}| \geq \pi_n^{-m}\) by the definition (3.3) of \( L_{n,j,k} \). Therefore
\[ |\hat{U}_{j,k_1}| < |W_{(j,a_n)} - 2^{-1}\pi_n^{-m}|. \] (3.6)

The assumption \( j \in L_n^c \) entails \( |\hat{R}_{j,k_2}| < 2^{-1}\pi_n^{-m} \) on \( \Omega_{n,j} \), and hence \(|W_{k_2}| - 2^{-1}\pi_n^{-m} < |\hat{U}_{j,k_2}|\) due to \( \Delta_{k_2} J = 0 \). From (3.6), we have got
\[ |\hat{U}_{j,k_1}| < |\hat{U}_{j,k_2}| \] (3.7)
on \( \Omega_{n,j} \) if \( j \in L_n^c \) and if a pair \((k_1, k_2) \in I_{n,j}^2\) satisfies (3.5).

We are working on \( \Omega_{n,j} \) yet. Suppose that \( j \in L_n^c \) and \( k_1 \in \hat{\mathcal{K}}_{n,j}(\alpha_0) \). Then the inequality (3.7) holds for any \( k_2 \in I_{n,j} \) satisfying \( r_{n,j}(|W_{k_2}|) \geq a_n \). So, there are at least \([\alpha_0 \pi_n + 1](\leq \alpha_0 \pi_n + \pi_n^{1-\eta_2} + 1 \leq \pi_n - a_n + 1)\) variables \( \hat{U}_{j,k_2} \) that satisfy (3.7). Then \( r_{n,j}(|\hat{U}_{j,k_1}|) \leq (1 - \alpha_0) \pi_n, \) and hence \( k_1 \in \mathcal{K}_{n,j}(\alpha_0) \). Thus, we found 
\[ \hat{\mathcal{K}}_{n,j}(\alpha_0) \subset \mathcal{K}_{n,j}(\alpha_0) \] on \( \Omega_{n,j} \) if \( j \in L_n^c \), that is, (a).

We still work on \( \Omega_{n,j} \). Suppose that \( j \in L_n^c \) and \( k_2 \in \mathcal{K}_{n,j}(\alpha_0) \setminus \hat{\mathcal{K}}_{n,j}(\alpha_0) \). When \( r_{n,j}(|W_{k_2}|) < a_n \), since \( r_{n,j}(|W_{k_2}|) > \widehat{a}_n \) due to \( k_2 \in \hat{\mathcal{K}}_{n,j}(\alpha_0)^c \), we see
\[ 1_{\{j \in L_n^c\}} \# \{ k_2 \in \mathcal{K}_{n,j}(\alpha_0) \setminus \hat{\mathcal{K}}_{n,j}(\alpha_0); \, r_{n,j}(|W_{k_2}|) < a_n \} \leq \pi_n^{1-\eta_2} \] (3.8)
on \( \Omega_{n,j} \). When \( r_{n,j}(|W_{k_2}|) \geq a_n \), for any \( k_1 \) satisfying \( r_{n,j}(|W_{k_1}|) \leq \widehat{a}_n \), we have (3.7). Therefore
\[ \# \{ k_1 \in I_{n;j}; \, |\hat{U}_{j,k_1}| < |\hat{U}_{j,k_2}| \} \geq 1_{\{j \in L_n^c\}} 1_{\{r_{n,j}(|W_{k_2}|) \geq a_n\}} \widehat{a}_n, \]
in other words,

\[ r_{n,j}(\|\hat{U}_{j,k_2}\|) > \hat{a}_n \] (3.9)

on \( \Omega_{n,j} \) if \( j \in \mathcal{L}_n \) and \( r_{n,j}(|W_{k_2}|) \geq a_n \). Moreover, \( r_{n,j}(|\hat{U}_{j,k_2}|) \leq \|(1 - \alpha_0)\pi_n\| \) since \( k_2 \in \mathcal{K}_{n,j}(\alpha_0) \).

Combining this estimate with (3.9), we obtain

\[ 1_{\{j \in \mathcal{L}_n\}} \# \{k_2 \in \mathcal{K}_{n,j}(\alpha_0) \setminus \hat{\mathcal{K}}_{n,j}(\alpha_0); r_{n,j}(|W_{k_2}|) \geq a_n\} \leq (1 - \alpha_0)^{\pi_n} - \hat{a}_n \leq 2^{\pi_n^{1-\eta_2} + 1} \] (3.10)

From (3.8) and (3.10), we obtain (b). \( \square \)

For \( \eta_3 \in \mathbb{R}, \) \( j \in I_n \) and a sequence of random variables \((V_j)_{j \in I_n}\), let

\[ D_{n,j} = \pi_n^{\eta_3} \left| \frac{1}{\pi_n} \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} V_k - \frac{1}{\pi_n} \sum_{k \in \hat{\mathcal{K}}_{n,j}(\alpha_0)} V_k \right| \]

The following lemma follows from Lemma 3.3 immediately.

**Lemma 3.4.** (i) Let \( p \geq 1 \). Then

\[ \|D_{n,j}\|_p \leq 4 \pi_n^{\eta_3-\eta_2} \max_{k \in I_{n,j}} \|V_k|\Omega_{n,j} \cap (j \notin \mathcal{L}_n)\|_p + \pi_n^{\eta_3} \max_{k \in I_{n,j}} \|V_k|\Omega_{n,j} \cap (j \notin \mathcal{L}_n)\|_p \]

for \( j \in I_n, n \in \mathbb{N} \).

(ii) Let \( p \geq 1 \) and \( \eta_4 > 0 \). Then

\[ \|D_{n,j}\|_p \leq 4 \pi_n^{\eta_3-\eta_2} \left( \pi_n^{\eta_4} + \pi_n \max_{k \in I_{n,j}} \|V_k|\Omega_{n,j} \cap (j \notin \mathcal{L}_n)\|_p \right) \]

\[ + \pi_n^{\eta_3} \max_{k \in I_{n,j}} \|V_k|\Omega_{n,j} \cap (j \notin \mathcal{L}_n)\|_p \]

for \( j \in I_n, n \in \mathbb{N} \).

Let

\[ \hat{\mathcal{K}}_{n,j}(\alpha_0) = \{ k \in I_{n,j}; |W_k| \leq c(\alpha_0)^{1/2} \} \]

For \( \eta_3 > 0, j \in I_n \) and a sequence of random variables \((V_j)_{j \in I_n}\), let

\[ \hat{\mathcal{D}}_{n,j} = \pi_n^{\eta_3} \left| \frac{1}{\pi_n} \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} V_k - \frac{1}{\pi_n} \sum_{k \in \hat{\mathcal{K}}_{n,j}(\alpha_0)} V_k \right| \]

Let

\[ \tilde{\Omega}_{n,j} = \{ ||W|_{(j,\tilde{a}_n)} - c(\alpha_0)^{1/2} | < \tilde{C} \pi_n^{\eta_2} \} \] (3.11)

for \( j \in I_n \), where \( \tilde{C} \) is a positive constant.
Lemma 3.5. Let $\eta_3 \in \mathbb{R}$. Then

(i) For $p \geq 1$ and $j \in I_n$,
$$
\left\| \mathcal{D}_{n,j} \right\|_p \leq \pi_{n_1}^{n_2} \left\| \max_{k \in I_{n,j}} |V_k| \right\|_p \frac{1}{n} \sum_{k \in I_{n,j}} 1 \{ |W_k| < C \eta_3^{-n_2} \} \right\|_p + \pi_{n_1}^{n_2} \left\| \max_{k \in I_{n,j}} |V_k| \right\|_p $$

(ii) For $p_1 > p \geq 1$ and $j \in I_n$,
$$
\left\| \mathcal{D}_{n,j} \right\|_p \leq \pi_{n_1}^{n_2} \left\| \max_{k \in I_{n,j}} |V_k| \right\|_p \left[ \left\| |W_i| - c(\alpha_0)^{1/2} \right\| - C \eta_3^{-n_2} \right] + \pi_{n_1}^{n_2} \left\| \max_{k \in I_{n,j}} |V_k| \right\|_p \left[ \left\| |W_i| - c(\alpha_0)^{1/2} \right\| - C \eta_3^{-n_2} \right] \right\|_p \left[ \left\| |W_i| - c(\alpha_0)^{1/2} \right\| - C \eta_3^{-n_2} \right] \right\|_p 
$$

Proof. For $k \in I_{n,j}$,
$$
\mathcal{O}_{n,j} \cap \left\{ r_{n,j}(|W_k|) \leq \hat{\alpha}_n \right\} \cap \left\{ |W_k| \leq c(\alpha_0)^{1/2} \right\} = \left\{ |W_k| \leq c(\alpha_0)^{1/2} \right\} \cap \left\{ |W_k| \leq |W|_{(j, \hat{\alpha}_n)} \right\} \cap \left\{ |W_k| \leq c(\alpha_0)^{1/2} \right\} \subset \left\{ |W_k| - c(\alpha_0)^{1/2} \leq C \eta_3^{-n_2} \right\}
$$

and
$$
\mathcal{O}_{n,j} \cap \left\{ r_{n,j}(|W_k|) \leq \hat{\alpha}_n \right\} \cap \left\{ |W_k| \leq c(\alpha_0)^{1/2} \right\} = \left\{ |W_k| \leq c(\alpha_0)^{1/2} \right\} \cap \left\{ |W_k| \leq |W|_{(j, \hat{\alpha}_n)} \right\} \cap \left\{ |W_k| > c(\alpha_0)^{1/2} \right\} \subset \left\{ |W_k| - c(\alpha_0)^{1/2} \leq C \eta_3^{-n_2} \right\}.
$$

Thus we obtain (i). Property (ii) follows from (i). \hfill \Box

Lemma 3.6. If the constant $\tilde{C}$ in (4.7) is sufficiently large, then
$$
\sup_{j \in I_n} \mathbb{P}[\mathcal{O}_{n,j}] = O(n^{-L})
$$
as $n \to \infty$ for any $L > 0$.

Proof. We have
$$
\mathbb{P}[|W|_{(j, \hat{\alpha}_n)} - c(\alpha_0)^{1/2} \leq C \eta_3^{-n_2}] 
\leq \mathbb{P}
\left[
|W|_{(j, |\alpha_n - \pi_1^{-n_2} - 1|)} \leq c(\alpha_0)^{1/2} - C \eta_3^{-n_2}
\right]
\leq \mathbb{P}
\left[
\sum_{k \in I_{n,j}} 1_{A_{n,k}} \geq |\alpha_n - \pi_1^{-n_2} - 1|
\right]
\leq \mathbb{P}
\left[
\pi_1^{-n_2} \sum_{k \in I_{n,j}} \{1_{A_{n,k}} - P[A_{n,k}]\} \geq C_n
\right]
$$
(3.12)
where

\[ A_{n,k} = \{ |W_k| < c(\alpha_0)^{1/2} - \hat{C}_n^{-\eta_2} \}, \]
\[ C_n = \kappa_n^{-1/2} (a_n - \kappa_n^{1-\eta_2} - 2 - \kappa_n P[\mathbb{A}_{n,1}]). \]

By using the mean-value theorem, we obtain

\[ C_n \sim \kappa_n^{-1/2} \left[ (1 - \alpha_0)\kappa_n - 2\kappa_n^{1-\eta_2} - \kappa_n \left\{ 1 - \alpha_0 - 2\phi \left( c(\alpha_0)^{1/2}; 0, 1 \right) \hat{C}_n^{-\eta_2} \right\} \right] \]
\[ \geq \kappa_n^{-\frac{1}{2}} \eta_2 \]

as \( n \to \infty \) if we choose a sufficiently large \( \hat{C} \). Therefore, the \( L^p \)-boundedness of the random variables in (3.12) gives

\[ \sup_{j \in I_n} P \left[ |W|_{\mathbb{A}_{n,j}} - c(\alpha_0)^{1/2} < -\hat{C}_n^{-\eta_2} \right] = O(n^{-L}) \quad (3.13) \]

as \( n \to \infty \) for any \( L > 0 \). In a similar way, we know

\[ P \left[ |W|_{\mathbb{A}_{n,j}} - c(\alpha_0)^{1/2} > \hat{C}_n^{-\eta_2} \right] = O(n^{-L}) \quad (3.14) \]

as \( n \to \infty \) for any \( L > 0 \). Then we obtain the result from (3.13) and (3.14).

### 3.2 Local-global realized volatility

We introduce the local-global realized volatility (LGRV)

\[ \mathbb{L}_{n,j}(\alpha_0) = \frac{n}{\kappa_n T} \sum_{k \in K_{n,j}(\alpha_0)} q(\alpha_0)^{-1} |\Delta_k X|^2 K_{n,k}. \quad (3.15) \]

**Theorem 3.7.** Suppose that [G1] is fulfilled. For \( c_0 \in (0, 1) \) and \( B > 0 \), suppose that \( \kappa_n \sim B n^{c_0} \) as \( n \to \infty \). Then

\[ \sup_{n \in \mathbb{N}} \sup_{j \in I_n} \sup_{k \in I_{n,j}} n \gamma_* \left\| 1_{\{j \in \mathbb{L}_n \}} \left( \mathbb{L}_{n,j}(\alpha_0) - \sigma_{t_k}^2 \right) \right\|_p < \infty \quad (3.16) \]

as \( n \to \infty \) for any constant \( \gamma_* \) satisfying

\[ \gamma_* < \min \left\{ \frac{1}{2} (1 - c_0), \frac{1}{2} c_0 \right\}. \]

**Proof.** (I) We have \( \kappa_n \sim n^{c_0} \sim h^{-c_0} \) and \( n/\kappa_n \sim n^{1-c_0} \sim h^{c_0-1} \). Let

\[ D_{n,j}^* = \kappa_n^{1/2} \left\{ \frac{n}{\kappa_n} \sum_{k \in K_{n,j}(\alpha_0)} |\Delta_k X|^2 K_{n,k} - \frac{n}{\kappa_n} \sum_{k \in \hat{K}_{n,j}(\alpha_0)} |\Delta_k X|^2 K_{n,k} \right\} \]

Applied to \( V_k = n |\Delta_k X|^2 K_{n,k} 1_{\{j \in \mathbb{L}_n \}} \), Lemma 3.4 (ii) gives

\[ \left\| D_{n,j}^* 1_{\{j \in \mathbb{L}_n \}} \right\|_p \leq \Phi_{n,j}^{3.18} + \Phi_{n,j}^{3.19} \quad (3.17) \]

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for every $p > 1$, where
\[
\phi_{n,j}^{(3.18)} = 4\pi_n^{n-m} \left( \kappa_n^{n} + \pi_n \max_{k \in I_{n,j}} n|\Delta_k X|^2 1_{\{n|\Delta_k X|^2 > \kappa_n^{n} \}} 1_{\{j \in \mathcal{L}_n^c \}} \right)_{p},
\]
and
\[
\phi_{n,j}^{(3.19)} = \pi_n \left( \max_{k \in I_{n,j}} n|\Delta_k X|^2 K_{n,k} 1_{\mathcal{L}_n^c} \right)_{p}.
\]

Since there is no jump of $J$ on $\{ j \in \mathcal{L}_n^c \}$, we see
\[
\sup_{j \in I_{n,k \in l_{n,j}}} \sup_{p > 1} \left\| n|\Delta_k X|^2 1_{\{j \in \mathcal{L}_n^c \}} \right\|_p = O(1)
\]
for every $p > 1$, as a result, the $L^p$-norm on the right-hand side of (3.18) is of $O(n^{-L})$ for arbitrary $L > 0$, and hence
\[
\phi_{n,j}^{(3.18)} = O(\kappa_n^{n-n_2+n_4})
\]
as $n \to \infty$. Similarly to (3.20), we obtain
\[
\sup_{j \in I_{n}} P[\Omega_{n,j}^c] = O(n^{-L})
\]
as $n \to \infty$ for every $L > 0$, from Lemma 3.2 as well as Lemma 3.1 because $(n/\kappa_n)^{1/2} \kappa_n^{-m} \gg 1$ when $2^{-1}(c_0^{-1} - 1) > \eta_1$. Then
\[
\phi_{n,j}^{(3.19)} \leq \kappa_n^{n} n^{1/2} P[\Omega_{n,j}^c]^{1/p} = O(n^{-L})
\]
for every $L > 0$ and $p > 1$. From (3.17), (3.21) and (3.23),
\[
\left\| \mathcal{D}_{n,j}^* 1_{\{j \in \mathcal{L}_n^c \}} \right\|_p = O(\kappa_n^{n-n_2+n_4}) = O(n^{-c_0(n_2-n_3-n_4)})
\]
as $n \to \infty$ for every $p > 1$. We recall that the parameters should satisfy
\[
0 < \eta_2 < \eta_1 < \min \left\{ \frac{1}{2}, \frac{1}{2} \left( \frac{1}{c_0} - 1 \right) \right\}, \quad \eta_3 + \eta_4 < \eta_2.
\]
[ In particular, if $c_0 = 1/2$, then $0 < \eta_2 < \eta_1 < 1/2$. The positive parameters $\eta_3$ and $\eta_4$ can be sufficiently small at this stage. Remark that $c_0 \eta_2 < 1/4$ when $c_0 \leq 1/2$. ]

(II) Let
\[
\mathcal{D}_{n,j}^* = \pi_n^{\eta_3} \left\{ \frac{n}{\pi_n} \sum_{k \in \mathcal{K}_{n,j}^{\eta_3}(a_0)} |\Delta_k X|^2 K_{n,k} - \frac{n}{\pi_n} \sum_{k \in \mathcal{K}_{n,j}^{\eta_3}(a_0)} |\Delta_k X|^2 K_{n,k} \right\}.
\]

Applying Lemma 3.5 (ii) to $V_k = n|\Delta_k X|^2 K_{n,k} 1_{\{j \in \mathcal{L}_n^c \}}$, we have
\[
\left\| \mathcal{D}_{n,j}^* 1_{\{j \in \mathcal{L}_n^c \}} \right\|_p \leq \phi_{n,j}^{(3.26)} + \phi_{n,j}^{(3.27)} + \phi_{n,j}^{(3.28)}(3.25)
\]

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where

\[
\Phi_{n,j}^{3.26} = \kappa_n^{n_2} \left\| \max_{k \in I_{n,j}} n |\Delta_k X|^2 K_{n,k}1_{\{j \in \mathcal{L}_n\}} \right\|_p \left[ ||W_1| - c(\alpha_0)^{1/2}| < \tilde{C}\kappa_n^{-n_2} \right],
\]

\[
\Phi_{n,j}^{3.27} = \kappa_n^{n_2} \left\| \max_{k \in I_{n,j}} n |\Delta_k X|^2 K_{n,k}1_{\{j \in \mathcal{L}_n\}} \right\|_{pp_1(p_1 - p)^{-1}} \times \left[ \frac{1}{\kappa_n} \sum_{k \in I_{n,j}} \left( 1 \{ ||W_k| - c(\alpha_0)^{1/2}| < \tilde{C}\kappa_n^{-n_2} \} - P \left[ ||W_k| - c(\alpha_0)^{1/2}| < \tilde{C}\kappa_n^{-n_2} \right] \right) \right\|_{p_1}
\]

and

\[
\Phi_{n,j}^{3.28} = \kappa_n^{n_2} P \left[ \tilde{\sigma}_{n,j}^{1/p_1} \right] \left\| \max_{k \in I_{n,j}} n |\Delta_k X|^2 K_{n,k}1_{\{j \in \mathcal{L}_n\}} \right\|_{pp_1(p_1 - p)^{-1}}
\]

for \(j \in I_n, n \in \mathbb{N}\). Then, paying \(\kappa_n^n\) for the maximum, we have the following estimates for any \(p_1 > p \geq 1\):

\[
\sup_{j \in I_n} \Phi_{n,j}^{3.26} = O(\kappa_n^{n_3 + n_4 - n_1}) = O(n^{-c_0(\gamma_2 - \gamma_1 - 1)}),
\]

\[
\sup_{j \in I_n} \Phi_{n,j}^{3.27} = O(\kappa_n^{n_3} \times \kappa_n^{n_4} \times \kappa_n^{-(1 + \gamma_2)/2}) = O(n^{-c_0\left(1 + \frac{\gamma_2}{2} - \gamma_1 - 1\right)}),
\]

and

\[
\sup_{j \in I_n} \Phi_{n,j}^{3.28} = O(n^{-L})
\]

as \(n \to \infty\) for any \(L > 0\) for a sufficiently large \(\tilde{C}\); the estimate \(3.31\) follows from Lemma 3.6. In this way,

\[
\|\tilde{D}_{n,j}^{1_{\{j \in \mathcal{L}_n\}}}\|_p = O(n^{-c_0(\gamma_2 - \gamma_1 - 1)}) + O(n^{-c_0\left(1 + \frac{\gamma_2}{2} - \gamma_1 - 1\right)})
\]

as \(n \to \infty\) for every \(p \geq 1\).

(III) On the event \(\{j \in \mathcal{L}_n\}\), we have

\[
\sum_{k \in \tilde{K}_{n,j}(\alpha_0)} |\Delta_k X|^2 K_{n,k} = \sum_{k \in \tilde{K}_{n,j}(\alpha_0)} \left( \int_{t_{k-1}}^{t_k} \sigma_l dw_l + \int_{t_{k-1}}^{t_k} b_l dt \right)^2 K_{n,k}
\]

\[
\quad = \Phi_{n,j}^{3.34} + \Phi_{n,j}^{3.35} + \Phi_{n,j}^{3.36} + \Phi_{n,j}^{3.37} + \Phi_{n,j}^{3.38}
\]

where

\[
\Phi_{n,j}^{3.34} = \sum_{k \in I_{n,j}} \left( \sigma_{\tilde{m}}^2 \right) h W_k^2 1_{\{|W_k| < c(\alpha_0)^{1/2}\}}.
\]
\[
\phi_{n,j}^{3.35} = \sum_{k \in \mathcal{L}_{n,j}} (\sigma_{t_{\alpha}})^2 h W_k^{1/2} 1_{\{|W_k| \leq c(\alpha_0)^{1/2}\}} (K_{n,k} - 1) \\
+ 2 \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t (\bar{\sigma}_s - \bar{\sigma}_{t_{\alpha}}) dw_s \sigma_t dw_t K_{n,k} \\
+ 2 \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} \int_{t_{k-1}}^{t_k} \sigma_{t_{\alpha}} dw_s (\bar{\sigma}_t - \bar{\sigma}_{t_{\alpha}}) dw_t K_{n,k} \\
+ 2 \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} \left( \int_{t_{k-1}}^{t_k} (\bar{\sigma}_t - \bar{\sigma}_{t_{\alpha}}) dw_t \right)^2 K_{n,k},
\]

(3.35)

\[
\phi_{n,j}^{3.36} = 2 \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t b_s ds \sigma_t dw_t K_{n,k},
\]

(3.36)

\[
\phi_{n,j}^{3.37} = 2 \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t \sigma_s dw_s b_t dt K_{n,k},
\]

(3.37)

and

\[
\phi_{n,j}^{3.38} = 2 \sum_{k \in \mathcal{K}_{n,j}(\alpha_0)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t b_s ds b_t dt K_{n,k}.
\]

(3.38)

By assumption,

\[
\sup_{j \in \mathcal{L}_n} \sup_{s \in [t_{L_{n-1}} + \beta_n, t_{L_n}]} \left\| 1_{\{j \in \mathcal{L}_n\}} \left( \sigma_s^2 - \sigma_{t_{\alpha}}^2 \right) \right\|_p \\
\leq \sup_{j \in \mathcal{L}_n} \sup_{s \in [t_{L_{n-1}} + \beta_n, t_{L_n}]} \left\| \sigma_s^2 - \sigma_{t_{\alpha}}^2 \right\|_p \\
\lesssim (\kappa_n h)^{1/2} \lesssim h^{1/2} (1 - \epsilon_0) 
\]

(3.39)

for every \( p > 1 \). First, a primitive estimate gives

\[
\sup_{j \in \mathcal{L}_n} \frac{n}{\kappa_n} \| \phi_{n,j}^{3.35} 1_{\{j \in \mathcal{L}_n\}} \|_p \lesssim \frac{n}{\kappa_n} \frac{\kappa_n^{3/2}}{n^{3/2}} \lesssim h^{1/2} (1 - \epsilon_0) 
\]

(3.40)

as \( n \to \infty \); we note that the orthogonality cannot apply due to \( \mathcal{K}_{n,j}(\alpha_0) \) even after \( K_{n,k} \) is decoupled. We also have

\[
\sup_{j \in \mathcal{L}_n} \frac{n}{\kappa_n} \| \phi_{n,j}^{3.36} 1_{\{j \in \mathcal{L}_n\}} \|_p \lesssim h^{1/2}.
\]

(3.41)
For $\Phi_{n,j}$ and $\Phi_{n,j}$, by the same way, we can get
\[
\sup_{j \in I_n} \frac{n}{\kappa_n} \| \Phi_{n,j} \|_{L_p}^{3.37} 1_{\{j \in \mathcal{L}_n^c\}} \|_p \lesssim h^{1/2},
\]
and
\[
\sup_{j \in I_n} \frac{n}{\kappa_n} \| \Phi_{n,j} \|_{L_p}^{3.38} 1_{\{j \in \mathcal{L}_n^c\}} \|_p \lesssim h
\]
as $n \to \infty$. Furthermore, we have
\[
\sup_{j \in I_n} \frac{1}{\kappa_n h} \left\{ \frac{1}{\kappa_n h} \Phi_{n,j}^{3.34} - (\sigma_{t_n,1})^2 q(\alpha_0) \right\} \|_p \lesssim \sup_{j \in I_n} \left\{ \frac{1}{\kappa_n h} \Phi_{n,j}^{3.34} - \sum_{k \in I_{n,j}} (\sigma_{t_n,1})^2 q(\alpha_0) h \right\} 1_{\{j \in \mathcal{L}_n^c\}} \|_p \\
\leq \sup_{j \in I_n} \left\{ \frac{1}{\kappa_n h} \left( \frac{1}{\kappa_n h} \Phi_{n,j}^{3.34} - \sum_{k \in I_{n,j}} (\sigma_{t_n,1})^2 \right) (W_k^2 1_{\{|W_k| \leq c(\alpha_0)^{1/2}\}} - q(\alpha_0)) \right\} 1_{\{j \in \mathcal{L}_n^c\}} \|_p \\
= O(\kappa_n^{-1/2}) = O(h^{c_0/2})
\]
for every $p > 1$. Combining (3.33) and (3.39)-(3.44), we obtain
\[
\sup_{j \in I_n} \sup_{k \in I_{n,j}} \frac{n}{\kappa_n T} \left( \frac{n}{\kappa_n T} \sum_{k \in I_{n,j}} |\Delta_k X|^2 K_{n,k} - \sigma_{t_n,1}^2 q(\alpha_0) \right) \|_p \\
= O(n^{-(1-c_0)/2}) + O(n^{-c_0/2})
\]
as $n \to \infty$ for every $p > 1$.

(IV) From (3.24), (3.32) and (3.45), we obtain the estimate
\[
\sup_{j \in I_n} \sup_{k \in I_{n,j}} \frac{n}{\kappa_n T} \left( \frac{n}{\kappa_n T} \sum_{k \in I_{n,j}} |\Delta_k X|^2 K_{n,k} - \sigma_{t_n,1}^2 q(\alpha_0) \right) \|_p \\
= O(n^{-c_0(\eta_2 - \eta_3 - \eta_4)}) + \left\{ O(n^{-c_0(\eta_2 - \eta_3 - \eta_4)}) + O(n^{-c_0(\eta_2 - \eta_3 - \eta_4)}) \right\} \\
+ \kappa_n^{\eta_1} \left\{ O(n^{-(1-c_0)/2}) + O(n^{-c_0/2}) \right\} \\
= O(n^{-c_0(\eta_2 - \eta_3 - \eta_4)}) + O(n^{c_0(\eta_3 + \eta_4) - (1-c_0)/2}) =: O_n
\]
as $n \to \infty$ for every $p > 1$. Here we are assuming the parameters satisfy
\[
c_0 \in (0, 1), \quad B > 0, \quad \eta_1 \in \left( 0, \min \left\{ \frac{1}{2}(\frac{1}{c_0} - 1), \frac{1}{2} \right\} \right), \\
\eta_2 \in (0, \eta_1), \quad \eta_3 > 0, \quad \eta_4 > 0, \quad \eta_3 + \eta_4 < \eta_2.
\]
To obtain the last error bound in (3.46), we used the inequalities
\[
-c_0 \left( \frac{1 + \eta_2}{2} - \eta_3 - \eta_4 \right) < -c_0(\eta_2 - \eta_3 - \eta_4)
\]

and
\[ c_0\eta_3 - \frac{c_0}{2} < c_0\eta_3 - c_0\eta_2 < -c_0(\eta_2 - \eta_3 - \eta_4). \]

The LGRV \( L_{n,j}(\alpha_0) \) of (3.15) does not depend on \( \eta_i \) (\( i = 1, 2, 3, 4 \)) within the ranges (3.47). When \( c_0 > 1/2 \), we make
\[ \frac{1}{2} > \frac{1}{2} \left( \frac{1}{c_0} - 1 \right) > \eta_1 > \eta_2 > \eta_3 \uparrow \frac{1}{2} \left( \frac{1}{c_0} - 1 \right), \quad \eta_4 \downarrow 0 \]
to obtain \( O_n = O(1) \). When \( c_0 \leq 1/2 \), we make
\[ \frac{1}{2} > \eta_1 > \eta_2 > \eta_3 \uparrow \frac{1}{2}, \quad \eta_4 \downarrow 0 \]
to obtain \( O_n = O(1) \). Thus, the proof of Theorem 3.7 is concluded.

According to the error bound (3.16), we should in general take \( c_0 = 1/2 \), i.e., \( \kappa_n \sim Bn^{1/2} \) to obtain an optimal error estimate. However, this is not always true. If the process \( \sigma \) is (unknown) constant for example, then we do not need any spot volatility estimator to construct a global jump filter, and the convergence of the resulting estimator for \( \Theta \) becomes much faster than that in the non-constant \( \sigma \) case.

### 3.3 Local minimum RV

Estimation of spot volatilities can be done by the minimum realized volatility (minRV) method of Andersen et al. [1]. This method is localized to define the local minRV by
\[ M_{n,j} = \frac{\pi}{\pi - 2/\bar{\kappa}_nT} \sum_{k \in I_{n,j}} \{ |\Delta_kX| \land |\Delta_{k+1}X| \}^2. \]  
(3.48)

**Theorem 3.8.** Suppose that [G1] is fulfilled. For \( c_0 \in (0, 1) \) and \( B > 0 \), suppose that \( \kappa_n \sim Bn^{c_0} \) as \( n \to \infty \). Then
\[ \sup_n \sup_j \sup_{k \in I_{n,j}} n^{\gamma_{**}} \|1_{(j \in L_{n,j})} (M_{n,j} - \sigma_{k}^2)\|_p < \infty \]
as \( n \to \infty \) for any \( p > 1 \) and any constant \( \gamma_{**} \) satisfying
\[ \gamma_{**} = \min \left\{ \frac{1}{2}, \frac{1}{2}c_0, \frac{1}{2}c_0 \right\}. \]

**Proof.** Consider \( k \in I_{n,j} \) for \( j \in L_{n,j} \). Then we can decompose \( \Delta_kX \) as
\[ \Delta_kX = \sigma_{\xi_{k-1}} \Delta_k w + \int_{t_{k-1}}^{t_k} (\sigma_t - \sigma_{\xi_{k-1}}) dw_t + \int_{t_{k-1}}^{t_k} b_t dt. \]
By (3.39),

\[
\sup_{j \in I_n} \sup_{k \in I_{n,j}} \left\| \int_{t_{k-1}}^{t_k} (\sigma_t - \sigma_{t_{j_n}}) \, dw_t \mathbf{1}_{\{j \in \mathcal{L}_n^k\}} \right\|_p \\
\leq \sup_{j \in I_n} \sup_{k \in I_{n,j}} \left\| \int_{t_{k-1}}^{t_k} (\tilde{\sigma}_t - \tilde{\sigma}_{t_{j_n}}) \, dw_t \right\|_p \left\| \mathbf{1}_{\{j \in \mathcal{L}_n^k\}} \right\|_2 \\
\lesssim \sup_{j \in I_n} \sup_{k \in I_{n,j}} \sqrt{\int_{t_{k-1}}^{t_k} \left\| (\tilde{\sigma}_t - \tilde{\sigma}_{t_{j_n}})^2 \right\|_p \, dt} \\
= \sqrt{O(h \times \kappa_n h)} = O(h^{1/4})
\]

for every \( p > 1 \). Hence, we obtain

\[
|\Delta_k X|^2 = \sigma_{t_{j_n}}^2 h W_k^2 + X_k,
\]

where \( X_k \) is a random variable satisfying \( \sup_{j \in I_n} \sup_{k \in I_{n,j}} \| X_k \|_p = O(h^{1/4}) \). By using this approximation (and the equality \( a \wedge b = \frac{1}{2} (a + b - |a - b|) \) for \( a, b > 0 \)), we have

\[
|\Delta_k X|^2 \wedge |\Delta_{k+1} X|^2 = \sigma_{t_{j_n}}^2 h (W_k^2 \wedge W_{k+1}^2) + X'_k,
\]

where \( X'_k \) is a random variable satisfying \( \sup_{j \in I_n} \sup_{k \in I_{n,j}} \| X'_k \|_p = O(h^{1/4}) \). Hence, we obtain

\[
\begin{align*}
M_{n,j} - \sigma_{t_{j_n}}^2 &= \frac{\pi}{\pi - 2 \kappa_n T} \sum_{k \in I_{n,j}} \{ |\Delta_k X|^2 \wedge |\Delta_{k+1} X|^2 \} - \sigma_{t_k}^2 \\
&= \frac{1}{\kappa_n} \sum_{k \in I_{n,j}} \sigma_{t_{j_n}}^2 \left( \frac{\pi}{\pi - 2} \{ W_k^2 \wedge W_{k+1}^2 \} - 1 \right) + (\sigma_{t_{j_n}}^2 - \sigma_{t_k}^2) \\
&\quad + \frac{\pi}{\pi - 2 \kappa_n T} \sum_{k \in I_{n,j}} X'_k
\end{align*}
\]

(3.49)

The first term on the right-hand side of (3.49) is \( O(\kappa_n^{-1/2}) = O(h^{c_0/2}) \). As for the second term, (3.39) gives \( \|\sigma_{t_{j_n}}^2 - \sigma_{t_k}^2\|_p = O(h^{1/4(1-c_0)}) \). Finally, as for the third term, we can estimate as

\[
\left\| \frac{n}{\kappa_n T} \sum_{k \in I_{n,j}} X'_k \right\|_p \lesssim n \times O(h^{1/4(3-c_0)}) = O(h^{1/4(1-c_0)}).
\]

With these estimates, we obtain the desired result. \( \square \)
4 Rate of convergence of the global realized volatilities in high intensity of jumps

In this section, we present a rate of convergence of the GRV and WGRV, both defined in Section 2. When the frequency of the jumps is high, it is recommend that one should choose a value of $\alpha$ that is not extremely small in order to cover the jumps by the index set $\mathcal{F}_n(\alpha)^c$. We will assume the properties of $S_{n,j-1}$ below, that we already proved in Section 3 for the LGRV and the local minRV. Thus, GRV and WGRV with a LGRV or the local minRV are global realized volatilities.

[G2] (i) $S_{n,j-1}$ is positive a.s. and

$$\sup_{n \in \mathbb{N}} \sup_{j \in I_n} \| S_{n,j-1}^{-1} \|_p < \infty$$

for every $p > 1$.

(ii) There exist positive constants $\gamma_0$ and $c$ such that

$$\sup_{n \in \mathbb{N}} \sup_{j \in I_n} n^{\gamma_0} \left\| 1_{\{j \in \mathcal{E}_n\}} \left( \sigma_{i,j-1}^2 - c S_{n,j-1} \right) \right\|_p < \infty$$

for every $p > 1$.

In [G2], we do not assume that the value of constant $c$ is known. We note that

$$\sup_{n \in \mathbb{N}} \sup_{j \in I_n} \left\| 1_{\{j \in \mathcal{E}_n\}} S_{n,j-1} \right\|_p < \infty$$

for every $p > 1$ under [G1] and [G2]. As shown in Theorem 3.7, the LGRV in (3.15) can serve as $S_{n,j-1}$.

If $\sigma_i$ is equal to a (possibly unknown) constant, then $\gamma_0$ can be arbitrarily large since we can let $S_{n,j-1} = 1$. In other words, we do not need any pre-estimate of $\sigma_{i,j-1}^2$. So, the constant volatility case is very special and it will be discussed briefly in Section 6 separately. This section logically includes the constant volatility case (hence a less efficient way for it) but we will consider a general non-constant volatility and assume a given local estimator attains a limited rate of convergence.

Remark 4.1. When $v = 2^{-1} \inf_{t \in [0,T]} \sigma_i^2 > 0$ for a priori known constant $v$, given a local estimator $\mathbb{L}_{n,j-1}^{loc}$ of $\sigma_{i,j-1}^2$, we can use $S_{n,j-1}(v) = 1_{\mathbb{L}_{n,j}^{loc} \vee v}$ for $S_{n,j-1}$. For example, it is the case when $X$ satisfies a stochastic differential equation with jumps and its diffusion coefficient is uniformly elliptic. When $v = 0$, an appropriate modification of $\mathbb{L}_{n,j}^{loc}$ is necessary and possible. We only give an idea without going into details here. Preset a positive constant $v$. Using $S_{n,j-1}(v)$ for $S_{n,j-1}$, we obtain an estimator $\widetilde{V}_n[v]$ of $\Theta(v) = \int_0^T \sigma_i^2 \mathbb{1}_{\{\sigma_i^2 \geq v\}} dt$, and indeed, the rate of convergence $\widetilde{V}_n[v]$ is established in this paper. Then it is natural to use $\Theta[v_n]$ to estimate $\Theta = \int_0^T \sigma_i^2 dt$ with a sequence of numbers $v_n$ tending to 0 as $n \to \infty$. Consistency does not matter because the mapping $v \mapsto \Theta(v)$ is continuous and the operation $v_n \downarrow 0$ is stable. Some work is necessary to give an explicit rate of convergence since the constant of the error bound for each $v_n$ depends on $v_n$. However, the cause of the error by the truncation at level $v_n$ is the difference $\int_0^T \sigma_i^2 \mathbb{1}_{\{\sigma_i^2 < v_n\}} dt$, and it is rather easy to control for small $v_n$. 

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4.1 Rate of convergence of the GRV with a fixed \( \alpha \)

We consider the GRV given by (2.3):

\[
V_n(\alpha) = \sum_{j \in J_n(\alpha)} q(\alpha)^{-1} |\Delta_j X|^{2} K_{n,j}.
\]

Denote by \( r_n(U_j) \) the rank of \( U_j \) among the variables \( \{U_i\}_{i \in I_n} \) as before, and \( |U_{(r)}| \) denotes the \( r \)-th ordered statistic of \( \{|U_i|\}_{i \in I_n} \). Let \( 0 < \gamma_2 < \gamma_1 < \gamma_0 \), and define numbers \( a_n \) and \( \hat{a}_n \) by

\[
a_n = \lfloor (1 - \alpha)n - n^{1-\gamma_2} \rfloor \quad \text{and} \quad \hat{a}_n = \lfloor a_n - n^{1-\gamma_2} \rfloor,
\]

respectively. Define the event \( N_{n,j} \) by

\[
N_{n,j} = \{ r_n(|W_j|) \leq a_n - n^{1-\gamma_2} \} \cap \{|W|_{(a_n)} - |W_j| < n^{-\gamma_1}\}
\]

The following lemma is Lemma 2.6 of Inatsugu and Yoshida [8].

**Lemma 4.2.**

\[
P\left[ \bigcup_{j=1,\ldots,n} N_{n,j} \right] = O(n^{-L})
\]

as \( n \to \infty \) for every \( L > 0 \).

We need some notation:

\[
\tilde{J}_n(\alpha) = \{ j \in I_n; r_n(|W_j|) \leq \hat{a}_n \},
\]

\[
U_j = c^{-1/2} h^{-1/2} (S_{n,j-1})^{-1/2} \Delta_j X,
\]

\[
R_j = U_j - W_j - c^{-1/2} h^{-1/2} (S_{n,j-1})^{-1/2} \Delta_j J,
\]

as well

\[
\Omega_n = \left\{ \# \mathcal{L}_n < n^{1-\gamma_2} \right\} \bigcap \left( \bigcap_{j=1,\ldots,n} \left\{ |R_j|_{1 \{ j \in \mathcal{L}_n^c \}} < 2^{-1} n^{-\gamma_1} \right\} \cap (N_{n,j})^c \right\}.
\]

Let

\[\mathcal{L}_n = \{ j \in I_n; \Delta_j \mathcal{N} \neq 0 \}.\]  

(4.1)

The definition of \( \mathcal{L}_{n}^{(k)} \) in Inatsugu and Yoshida [8] of the extended version [arXiv:1806.10706v3] is essentially the same as \( \mathcal{L}_n \), and different from \( L_n \) defined by (3.4). The random set \( L_{n}^{(k)} \) therein corresponds to \( \mathcal{L}_n \).

We assume that the distribution of the variable \( \mathcal{N}_T \) depends on \( n \), and consider the case where \( \mathcal{N}_T \) may diverge as \( n \to \infty \). More precisely, we will assume the following situation.

\[\text{[G3]} \quad \text{There exists a constant } \xi \geq 0 \text{ such that } \| \# \mathcal{L}_n \|_p = O(n^\xi) \text{ as } n \to \infty \text{ for every } p > 1.\]

\[1\] We slightly relaxed Condition [G3] of [arXiv:2102.05307v1].
Lemma 4.3. Suppose that \([G1]\) and \([G2]\) are satisfied. Suppose that \(0 < \gamma_1 < \gamma_0 < 1/2\). Then
\[
\sup_{j \in I_n} P \left[ |R_j| 1_{j \in \mathcal{L}_n} \geq 2^{-1} n^{-\gamma_1} \right] = O(n^{-\gamma_0})
\] (4.2)
as \(n \to \infty\) for every \(L > 0\). In particular, if the conditions \(\kappa_n = O(n^{1/2})\), \(\gamma_2 < \frac{1}{2} - \xi\) and \([G3]\) are additionally satisfied, then
\[
P[\Omega_n] = O(n^{-L})
\] (4.3)
as \(n \to \infty\) for every \(L > 0\).

Proof. We have
\[
\sup_{j \in I_n} \| R_j 1_{j \in \mathcal{L}_n} \|_p = O(n^{-\gamma_0})
\]
for every \(p > 1\). The Markov inequality implies (4.2). This estimate and Lemma 4.2 give (4.3) if the Markov inequality is used with the estimate \(\| \# \mathcal{L}_n \|_p = O(n^{\xi+1/2})\) from \([G3]\). \(\square\)

Lemma 2.7 of Inatsugu and Yoshida [8] (or see an extended version arXiv:1806.10706v3) is rephrased as follows. Recall that \(\mathcal{L}_n\) is given by (3.4).

Lemma 4.4. \(\hat{J}_n(\alpha) \cap \mathcal{L}_n^c \subset J_n(\alpha)\) (4.4)
on \(\Omega_n\). In particular
\[
\# \left[ J_n(\alpha) \ominus \hat{J}_n(\alpha) \right] \leq c_* n^{1-\gamma_2} + \# \mathcal{L}_n
\] (4.5)
on \(\Omega_n\), where \(c_*\) is a positive constant. Here \(\ominus\) denotes the symmetric difference operator of sets.

For \(\gamma_3 > 0\) and random variables \((U_j)_{j=1,\ldots,n}\), let
\[
D_n = n^{\gamma_3} \left\{ \frac{1}{n} \sum_{j \in J_n(\alpha)} U_j - \frac{1}{n} \sum_{j \in \hat{J}_n(\alpha)} U_j \right\}.
\]

We refer the reader to Lemmas 2.8 and 2.9 of Inatsugu and Yoshida [8] (or see arXiv:1806.10706v3) for proof of the following two lemmas.

Lemma 4.5. (i) Let \(p_1 > 1\). Then
\[
\| D_n \|_p \leq \left( c_* n^{\gamma_3-\gamma_2} + n^{-1+\gamma_3} \| \mathcal{L}_n \|_{p_1} \right) \max_{j=1,\ldots,n} \| U_j \|_{pp_1(p_1-p)^{-1}}
+ n^{\gamma_3} \| \max_{j=1,\ldots,n} |U_j| 1_{\Omega_n} \|_p
\]
for \(p \in (1,p_1)\).
(ii) Let $\gamma_4 > 0$ and $p_1 > 1$. Then
\[
\|D_n\|_p \leq \left( c_n n^{\gamma_3} \gamma_2 + n^{-1+\gamma_3} \| \mathcal{L}_n \|_{p_1} \right)
\times \left( n^{\gamma_4} + n \max_{j=1,\ldots,n} \left\| U_j \right\|_p 1_{\{ |U_j| > n^{\gamma_4} \}} \right)
\times \left( n^{\gamma_5} \max_{j=1,\ldots,n} \left| U_j / 1_{\tilde{\Omega}_n} \right| \right)
\]
for $p \in (1, p_1)$.

Let
\[
\tilde{D}_n = n^{\gamma_3} \left\{ \frac{1}{n} \sum_{j \in \tilde{J}_n(\alpha)} U_j - \frac{1}{n} \sum_{j \in \tilde{J}_n(\alpha)} U_j \right\},
\]
for a collection of random variables $\{U_j\}_{j \in I_n}$ and
\[
\tilde{J}_n(\alpha) = \{ j \in I_n; |W_j| \leq c(\alpha)^{1/2} \}.
\]

Let
\[
\tilde{\Omega}_n = \{ |W|_{\tilde{\alpha}_n} - c(\alpha)^{1/2} < \tilde{C} n^{-\gamma_2} \},
\]
where $\tilde{C}$ is a positive constant. See Lemma 4 of Inatsugu and Yoshida [8] for a proof of the following lemma.

**Lemma 4.6.** Let $\tilde{C} > 0$ and $\gamma_3 > 0$. Then

(i) For $p \geq 1$,
\[
\|\tilde{D}_n\|_p \leq n^{\gamma_3} \max_{j=1,\ldots,n} \left| U_j \right| P \left[ |W_1| - c(\alpha)^{1/2} < \tilde{C} n^{-\gamma_2} \right]
\left[ ||W_1| - c(\alpha)^{1/2} < \tilde{C} n^{-\gamma_2} \right]
\left[ ||W_1| - c(\alpha)^{1/2} < \tilde{C} n^{-\gamma_2} \right]
\left[ ||W_1| - c(\alpha)^{1/2} < \tilde{C} n^{-\gamma_2} \right]
\]
\[
+ n^{\gamma_3} \max_{j=1,\ldots,n} \left| U_j \right| \left| U_j \right|^{pp_1(p_1-p)^{-1}}
\times \left( \frac{1}{n} \sum_{j=1}^{n} \left( 1_{\{ |W_j| - c(\alpha)^{1/2} < \tilde{C} n^{-\gamma_2} \}} - P \left[ |W_1| - c(\alpha)^{1/2} < \tilde{C} n^{-\gamma_2} \right] \right) \right)_{p_1}
\]
\[
+ n^{\gamma_3} P^{\tilde{\Omega}_n^{1/p_1}} \max_{j=1,\ldots,n} \left| U_j \right| \left| U_j \right|^{pp_1(p_1-p)^{-1}}.
\]

(ii) For $p_1 > p \geq 1$,
Lemma 4.7. If the constant $\hat{C}$ in (3.11) is sufficiently large, then

$$P[\widehat{V}_n^c] = O(n^{-L})$$

as $n \to \infty$ for any $L > 0$.

Now we shall investigate the rate of convergence of $V_n(\alpha)$ for a constant $\alpha \in (0, 1)$. We note that, under $[G1]$ and $[G3],

$$\left\| \sum_{j \in \mathcal{L}_n} |\Delta_j X|^2 K_{n,j} \right\|_p \leq n^{-1/2} \# \mathcal{L}_n \|p = O(n^{-1/2+\xi}). \quad (4.8)$$

Let

$$\hat{V}_n(\alpha) = \sum_{j \in \mathcal{J}_n(\alpha)} q(\alpha)^{-1} |\Delta_j X|^2 K_{n,j}.$$

Lemma 4.8. Suppose that $[G1]$ $[G2]$ and $[G3]$ are fulfilled. Suppose that $\xi < \frac{1}{2}$. Let $\gamma_5 < \min \{\gamma_0, \frac{1}{2} - \xi\}$ and $\kappa_n = O(n^{1/2})$. Then

$$\sup_{n \in \mathbb{N}} n^{\gamma_5} \|V_n(\alpha) - \hat{V}_n(\alpha)\|_p < \infty.$$

Proof. By (4.8), we obtain

$$\|V_n(\alpha) - \hat{V}_n(\alpha)\|_p = \left\| \sum_{j \in \mathcal{J}_n(\alpha)} q(\alpha)^{-1} |\Delta_j X|^2 1_{\{\Delta_j \mathcal{N} = 0\}} K_{n,j} \right\|_p - \sum_{j \in \mathcal{J}_n(\alpha)} q(\alpha)^{-1} |\Delta_j X|^2 1_{\{\Delta_j \mathcal{N} = 0\}} K_{n,j} \right\|_p + O(n^{-1/2+\xi}).$$

By Lemmas 4.5 and 4.3

$$n^{\gamma_3} \|V_n(\alpha) - \hat{V}_n(\alpha)\|_p \lesssim \left( c_n n^{\gamma_3 - \gamma_2 + n^{-1+\gamma_3} \# \mathcal{L}_n \|p \right) \times \left( n^{\gamma_4} + n \max_{j=1,\ldots,n} \left\| n |\Delta_j X|^2 1_{\{\Delta_j \mathcal{N} = 0\}} K_{n,j} 1_{\{n |\Delta_j X|^2 1_{\{\Delta_j \mathcal{N} = 0\}} K_{n,j} > n^{\gamma_4}\}} \|_{p^{p_1}(p_1-p)} + O(n^{-1/2+\gamma_3+\xi}) \right) + n^{\gamma_3} \max_{j=1,\ldots,n} \left( n |\Delta_j X|^2 1_{\{\Delta_j \mathcal{N} = 0\}} K_{n,j} \right) 1_{\mathcal{O}_n} \|_p + O(n^{-1/2+\gamma_3+\xi}),$$

where $1 < p < p_1$. The parameters should satisfy

$$0 < \gamma_3 < \gamma_2 < \gamma_1 < \gamma_0 < \frac{1}{2}, \quad \gamma_2 < \frac{1}{2} - \xi, \quad \gamma_4 > 0.$$

We make

$$\gamma_4 \downarrow 0, \quad \gamma_5 < \gamma_3 < \gamma_2 < \gamma_1 < \gamma_0 \min \left\{ \gamma_0, \frac{1}{2} - \xi \right\} \downarrow 0,$$
to obtain the desired exponent.

For \( \tilde{J}_n(\alpha) \) defined in (4.6), let

\[
\tilde{v}_n(\alpha) = \sum_{j \in \tilde{J}_n(\alpha)} q(\alpha)^{-1}|\Delta_j X|^2 K_{n,j}.
\]

**Lemma 4.9.** Suppose that \([G1]\) and \([G3]\) are fulfilled. Suppose that \( \xi < \frac{1}{2} \). Let \( \gamma_6 < \frac{1}{2} - \xi \).

Then

\[
\sup_{n \in \mathbb{N}} n^{\gamma_6} \| \tilde{v}_n(\alpha) - \tilde{v}_n(\alpha) \|_p < \infty.
\]

**Proof.** By (1.8), we obtain

\[
\| \tilde{v}_n(\alpha) - \tilde{v}_n(\alpha) \|_p = \left\| \sum_{j \in \tilde{J}_n(\alpha)} q(\alpha)^{-1}|\Delta_j X|^2 1_{\{\Delta_j \neq 0\}} K_{n,j} \right\|_p + O(n^{-1/2+\xi}).
\]

By Lemma 4.6, we obtain

\[
n^{\gamma_3} \| \tilde{v}_n(\alpha) - \tilde{v}_n(\alpha) \|_p \leq n^{\gamma_3} \left( \max_{j=1,\ldots,n} \left( n|\Delta_j X|^2 1_{\{\Delta_j \neq 0\}} K_{n,j} \right) \right) \left. \right|_p + O(n^{-1/2+\xi}),
\]

where \( 1 \leq p < p_1 \) and \( \gamma_4 \) is an arbitrary positive number. Lemma 4.7 was used in the above derivation. Making

\[
\gamma_4 \downarrow 0 \quad \text{and} \quad \gamma_6 < \gamma_3 \downarrow \gamma_2 \downarrow \frac{1}{2} - \xi,
\]

we conclude the proof.

**Lemma 4.10.** Suppose that \([G1]\) and \([G3]\) are satisfied. Suppose that \( \xi < 1/2 \). Then

\[
\left\| \tilde{v}_n(\alpha) - \int_0^T \sigma_i^2 dt \right\|_p = O(n^{-\frac{1}{2}+\xi})
\]

as \( n \to \infty \) for every \( p > 1 \).
Proof. Recall that $\Sigma_n$ is defined by (4.1). We have

$$
\sum_{j \in \mathcal{J}_n(\alpha)} |\Delta_j X|^2 K_{n,j} 1_{\{j \in \mathcal{E}_n\}} = \sum_{j \in \mathcal{J}_n(\alpha)} \left( \int_{t_{j-1}}^{t_j} \sigma_t dw_t + \int_{t_{j-1}}^{t_j} b_t dt \right)^2 K_{n,j} 1_{\{j \in \mathcal{E}_n\}}
= \Phi_{n,10}^{(4.10)} + \Phi_{n,11}^{(4.11)} + \Phi_{n,12}^{(4.12)}
$$

where

$$
\Phi_{n,10}^{(4.10)} = \sum_{j \in \mathcal{J}_n} \sigma_{t_{j-1}}^2 h W_j^{2,1} \{ |W_j| \leq c(\alpha)^{1/2} \},
$$

$$
\Phi_{n,11}^{(4.11)} = \sum_{j \in \mathcal{J}_n} \sigma_{t_{j-1}}^2 h W_j^{2,1} \{ |W_j| \leq c(\alpha_0)^{1/2} \} \left( K_{n,j} 1_{\{j \in \mathcal{E}_n\}} - 1 \right)
+ 2 \sum_{j \in \mathcal{J}_n(\alpha)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \left( \tilde{\sigma}_s - \tilde{\sigma}_{t_{j-1}} \right) dw_s \sigma_t dw_t K_{n,j} 1_{\{j \in \mathcal{E}_n\}}
+ 2 \sum_{j \in \mathcal{J}_n(\alpha)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \sigma_{t_{j-1}} dw_s \left( \tilde{\sigma}_t - \tilde{\sigma}_{t_{j-1}} \right) dw_t K_{n,j} 1_{\{j \in \mathcal{E}_n\}}
+ 2 \sum_{j \in \mathcal{J}_n(\alpha)} \int_{t_{j-1}}^{t_j} \tilde{\sigma}_{t_{j-1}} \left( \tilde{\sigma}_t - \tilde{\sigma}_{t_{j-1}} \right) dt K_{n,j} 1_{\{j \in \mathcal{E}_n\}}
+ \sum_{j \in \mathcal{J}_n(\alpha)} \left( \int_{t_{j-1}}^{t_j} \left( \tilde{\sigma}_t - \tilde{\sigma}_{t_{j-1}} \right) dw_t \right)^2 K_{n,j} 1_{\{j \in \mathcal{E}_n\}}
$$

and

$$
\Phi_{n,12}^{(4.12)} = 2 \sum_{j \in \mathcal{J}_n(\alpha)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} b_s dw_s \sigma_t dw_t K_{n,j} 1_{\{j \in \mathcal{E}_n\}}
+ 2 \sum_{j \in \mathcal{J}_n(\alpha)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \sigma_s dw_s b_t dt K_{n,j} 1_{\{j \in \mathcal{E}_n\}}
+ 2 \sum_{j \in \mathcal{J}_n(\alpha)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} b_s dw_s b_t dt K_{n,j} 1_{\{j \in \mathcal{E}_n\}}.
$$

(4.9)
Let $\epsilon > \xi$. For $p > 1$ and $\epsilon' > 0$,

$$\left\| \sum_{j \in I_n} \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} \left( K_{n,j} 1_{\{j \in I_n\}} - 1 \right) \right\|_p$$

$$\leq \left\| \sum_{j \in I_n} \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} \right\|_p + \left\| \sum_{j \in I_n} \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} (K_{n,j} - 1) 1_{\{j \in I_n\}} \right\|_p$$

$$\leq \left\| \sum_{j \in I_n} \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} \right\|_p + O(n^{-L})$$

$$\leq \left\| \sum_{j \in I_n} \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} \right\|_p + n^\epsilon \max_{j \in I_n} \left( \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} K_{n,j} \right)$$

$$\leq \left\| \sum_{j \in I_n} \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} \right\|_p + O(n^{-L})$$

$$\leq \sum_{j \in I_n} \left\| \sum_{j \in I_n} \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} \right\|_p + n^\epsilon \max_{j \in I_n} \left( \sigma_{t_{j-1}}^2 h W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)^{1/2}\}} K_{n,j} \right)$$

$$\lesssim h^{1/2} + \epsilon^{1-\epsilon'} \lesssim h^{1/2}$$ (4.14)

if letting $\epsilon \downarrow 1/2$ and $\epsilon' \downarrow 0$.

By the Burkholder-Davis-Gundy inequality, we have

$$\left\| \sum_{j \in I_n} \int_{t_{j-1}}^{t_j} b_s \, ds \sigma_t \, dw_t K_{n,j} 1_{\{j \in I_n\}} \right\|_p \lesssim \sum_{j \in I_n} \left\| \int_{t_{j-1}}^{t_j} b_s \, ds \sigma_t \, dw_t \right\|_p$$

$$\lesssim \sum_{j \in I_n} \left\| \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^{t} b_s \, ds \right) \, dw_t \right\|_p$$

$$\lesssim \sum_{j \in I_n} \left\| \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^{t} b_s \, ds \right) \, dw_t \right\|_p$$

$$\lesssim h^{1/2}.$$ (4.15)

From this and similar estimates, we have

$$\left\| \Phi_{\epsilon, \xi, p, \epsilon'} \right\|_p \lesssim h^{1/2}$$
as $n \to \infty$ for every $p > 1$. Moreover,

$$
\left\| \Phi_j^{(4.10)} - \sum_{j \in I_n} \sigma^2_{t_{j-1}} q(\alpha) h \right\|_p \leq \left\| h \sum_{j \in I_n} \sigma^2_{t_{j-1}} (W_j^2 1_{\{|W_j| \leq \epsilon(\alpha)\}} - q(\alpha)) \right\|_p = O(h^{1/2})
$$

(4.16)

for every $p > 1$. 

Obviously,

$$
\sup_{j \in I_n} \left\| 1_{\{j \in \mathcal{E}_n\}} (\sigma^2_{t_k} - \sigma^2_{t_{j-1}}) \right\|_p \leq \sup_{j \in I_n} \left\| \tilde{\sigma}^2_{t_k} - \tilde{\sigma}^2_{t_{j-1}} \right\|_p \lesssim h^{1/2}
$$

(4.17)

for every $p > 1$. In view of (4.17), we deduce that

$$
\left\| \sum_{j \in I_n} \sigma^2_{t_{j-1}} h - \int_0^T \sigma^2_t dt \right\|_p \\
\leq \left\| \sum_{j \in I_n} \int_{t_{j-1}}^{t_j} \tilde{\sigma}^2_t - \tilde{\sigma}^2_{t_{j-1}} dt \right\|_p + \left\| \sum_{j \in I_n} \int_{t_{j-1}}^{t_j} (\sigma^2_t - \sigma^2_{t_{j-1}}) dt 1_{\{j \in \mathcal{E}_n\}} \right\|_p \\
\leq O(h^{1/2}) + \left\| \max_{j \in I_n} \left\{ \int_{t_{j-1}}^{t_j} (\sigma^2_t + |\sigma^2_{t_{j-1}}|) dt \right\} \# \mathcal{E}_n \right\|_p
\leq O(h^{1/2}),
$$

(4.18)

following the passage from (4.13) to (4.14). 

Easily,

$$
\left\| \sum_{j \in \mathcal{E}_n(\alpha)} |\Delta_j X|^2 K_{n,j} 1_{\{j \in \mathcal{E}_n\}} \right\|_p \leq \left\| n^{-1/2} \# \mathcal{E}_n \right\|_p \lesssim n^{-\frac{1}{4} + \xi}.
$$

(4.19)

Combining (4.19), (4.9), (4.14), (4.15), (4.10) and (4.18), we obtain

$$
\left\| \check{\mathcal{V}}_n(\alpha) - \int_0^T \sigma^2_t dt \right\|_p = O(n^{-\frac{1}{2} + \xi})
$$

as $n \to \infty$ for every $p > 1$. 

\[ \square \]

**Theorem 4.11.** Suppose that $[G1]$ $[G2]$ and $[G3]$ are fulfilled. Suppose that $\xi < \frac{1}{2}$ and $\kappa_n = O(n^{1/2})$. Let $\alpha \in (0, 1)$ and $\beta_0 < \min\left\{ \gamma_0, \frac{1}{2} - \xi \right\}$. Then

$$
\left\| \mathcal{V}_n(\alpha) - \Theta \right\|_p = O(n^{-\beta_0})
$$

as $n \to \infty$ for every $p > 1$.

\[ \square \]

**Proof.** Use Lemmas 4.8, 4.9 and 4.10.
4.2 Rate of convergence of the WGRV with a fixed $\alpha$

Next, we discuss the convergence of the WGRV with a fixed $\alpha$. Recall that the WGRV is defined as

$$W_n(\alpha) = \sum_{j \in J_n} w(\alpha)^{-1} |\Delta_j X|^2 K_{n,j}.$$  

The WGRV has entirely the same rate of convergence as the GRV.

**Theorem 4.12.** Suppose that $[G1]$, $[G2]$, and $[G3]$ are fulfilled. Suppose that $\xi < \frac{1}{2}$. Let $\alpha \in (0, 1)$ and $\beta_0 < \min \{ \gamma_0, \frac{1}{2} - \xi \}$. Moreover, assume that $\kappa_n = O(n^{1/2})$. Then

$$\|W_n(\alpha) - \Theta\|_p = O(n^{-\beta_0})$$  

as $n \to \infty$ for every $p > 1$.

**Proof.** Decompose $W_n(\alpha)$ as

$$W_n(\alpha) = \sum_{j \in J_n(\alpha)} w(\alpha)^{-1} |\Delta_j X|^2 K_{n,j} + \sum_{j \in J_n(\alpha)^c} w(\alpha)^{-1} S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j}$$

$$= \frac{q(\alpha)}{w(\alpha)} \mathcal{V}_n(\alpha) + \sum_{j \in J_n(\alpha)^c} w(\alpha)^{-1} S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j}.$$  

Note that $w(\alpha) = q(\alpha) + \alpha c(\alpha)$. Hence, it suffices to show that

$$\left\| \sum_{j \in J_n(\alpha)^c} S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j} - \alpha c(\alpha) \Theta \right\|_p = O(n^{-\beta_0})$$  

as $n \to \infty$ for every $p > 1$. Decompose the left-hand side as

$$\sum_{j \in J_n(\alpha)^c} S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j} - \alpha c(\alpha) \Theta = \sum_{j \in J_n(\alpha)^c} S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j} 1_{\{j \in \mathcal{L}_n^c\}} - \alpha c(\alpha) \Theta$$

$$+ \sum_{j \in J_n(\alpha)^c} S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j} 1_{\{j \in \mathcal{L}_n \cap \mathcal{L}_n\}}$$

$$+ \sum_{j \in J_n(\alpha)^c} S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j} 1_{\{j \in \mathcal{L}_n^c \cap \mathcal{L}_n\}}$$

$$=: A_1 + A_2 + A_3.$$  

Since $S_{n,j-1} V_{s_n(\alpha)}^2 K_{n,j} \leq |\Delta_j X|^2 K_{n,j} \leq n^{-1/2}$ for $j \in J_n(\alpha)^c$, we have $\|A_2\|_p \lesssim n^{-1/2+\xi}$. As for $A_3$, note that $\#\mathcal{L}_n \lesssim n^{\xi} \times \kappa_n = O(n^{\xi+1/2})$ and that $\Delta_j X = \Delta_j \tilde{X}$ for $j \in \mathcal{L}_n^c$. Hence we have

$$\|A_3\|_p \leq \left\| \max_{j \in \mathcal{L}_n^c} |\Delta_j \tilde{X}|^2 \#(\mathcal{L}_n^c \cap \mathcal{L}_n) \right\|_p \lesssim n^{-1/2+\xi+\epsilon},$$

where $\epsilon$ is an arbitrarily small positive number.
As for $A_1$, we can set $c = 1$ in the condition $[G2]$ without loss of generality.

\[ \|A_1 - \alpha c(\alpha) \Theta\|_p \leq \left\| \left( h^{-1}V_{(s_n(\alpha))}^2 - c(\alpha) \right) h \sum_{j \in J_n(\alpha)^c} S_{n,j-1} K_{n,j} 1_{\{j \in \mathcal{L}_n^c\}} \right\|_p \]

\[ + c(\alpha) \left\| h \sum_{j \in J_n(\alpha)^c} \left( S_{n,j-1} - \sigma_{t,j-1}^2 \right) 1_{\{j \in \mathcal{L}_n^c\}} \right\|_p \]

\[ + c(\alpha) \left\| h \sum_{j \in J_n(\alpha)^c} S_{n,j-1} (1 - K_{n,j}) 1_{\{j \in \mathcal{L}_n^c\}} \right\|_p \]

\[ + c(\alpha) \left\| h \sum_{j \in J_n(\alpha)^c} \sigma_{t,j-1}^2 1_{\{j \in \mathcal{L}_n^c\}} - \alpha \Theta \right\|_p \]

\[ =: B_1 + B_2 + B_3 + B_4. \]

By condition $[G2]$, $B_2 = O(n^{-70})$. As for $B_3$, with the estimate $\|1_{\{j \in \mathcal{L}_n^c\}} (1 - K_{n,j})\|_p \leq P[\Delta_j \tilde{X}] > n^{-1/4}]^{1/p} = O(n^{-L})$ (for all $p > 1$ and $L > 0$) and the Cauchy-Schwarz inequality, we have

\[ \|B_3\|_p \leq h \sum_{j \in I_n} 1_{\{j \in \mathcal{L}_n\}} S_{n,j-1} 1_{\{j \in \mathcal{L}_n\}} (1 - K_{n,j}) 2_p = O(n^{-L}). \]

For $B_4$, we use the following decomposition:

\[ h \sum_{j \in J_n(\alpha)^c} \sigma_{t,j-1}^2 1_{\{j \in \mathcal{L}_n^c\}} - \alpha \Theta \]

\[ = \left( h \sum_{j \in I_n} \sigma_{t,j-1}^2 - \Theta \right) - h \sum_{j \in I_n} \sigma_{t,j-1}^2 1_{\{j \in \mathcal{L}_n\}} + h \sum_{j \in J_n(\alpha)} \sigma_{t,j-1}^2 1_{\{j \in \mathcal{L}_n\}} \]

\[ + \left( (1 - \alpha) \Theta - h \sum_{j \in J_n(\alpha)} \sigma_{t,j-1}^2 \right) + \left( h \sum_{j \in J_n(\alpha)} \sigma_{t,j-1}^2 - h \sum_{j \in J_n(\alpha)} \sigma_{t,j-1}^2 \right). \]

Hence, with the aid of Lemmas $4.5$ $4.6$ and the estimate $\|\mathcal{L}_n\|_p \lesssim n^{\xi + 1/2}$, we have

\[ \left\| h \sum_{j \in J_n(\alpha)^c} \sigma_{t,j-1}^2 1_{\{j \in \mathcal{L}_n^c\}} - \alpha \Theta \right\|_p \]

\[ \lesssim \left\| h \sum_{j \in I_n} \sigma_{t,j-1}^2 - \Theta \right\|_p + \left\| (1 - \alpha) \Theta - h \sum_{j \in J_n(\alpha)} \sigma_{t,j-1}^2 \right\|_p + O(n^{-\beta_0}) \quad (4.20) \]

since $\beta_0 < \frac{1}{2} - \xi$. The first term of the right-hand side of the above inequality is $O(n^{-1/2})$ by
As for the second term on the right-hand side of (4.20),
\[
\left\| (1 - \alpha)\Theta - \sum_{j \in J_n(\alpha)} \sigma^2_{j-1} \right\|_p = \left\| (1 - \alpha)\Theta - \sum_{j \in I_n} \sigma^2_{j-1} 1_{[W_j \leq c(\alpha)^{1/2}]} \right\|_p \\
\leq \left\| \sum_{j \in I_n} \sigma^2_{j-1} \left( 1_{[W_j \leq c(\alpha)^{1/2}]} - \frac{P(\{W_j \leq c(\alpha)^{1/2}\})}{P} \right) \right\|_p \\
+ (1 - \alpha) \left\| \sum_{j \in I_n} \sigma^2_{j-1} - \Theta \right\|_p \\
= O(n^{-1/2}).
\]
since Hence we have \( B_4 = O(n^{-\frac{1}{2}\xi}) \).

Finally, for \( B_1 \), it suffices to show that
\[
P(\left\{ h^{-1/2}V_{(s_n(\alpha))} - c(\alpha)^{1/2} > n^{-\beta_0} \right\} = O(n^{-L}) \tag{4.21}
\]
as \( n \to \infty \) for every \( L > 0 \) and for every \( \beta_0 < \min\{\gamma_0, \frac{1}{2} - \xi\} \). Let
\[
A_{n,j} = \left\{ h^{-1/2}V_j < c(\alpha)^{1/2} - n^{-\beta_0} \right\} \\
V_{n,j} = 1\{ |W_j| \leq c(\alpha)^{1/2} - n^{-\beta_0 + 2^{-1}n^{-\gamma}} \}
\]
and
\[
\mu_n = (1 - \alpha)n - 1 - n^{\frac{1}{2} + \xi + \xi} - nE[V_{n,j}]
\]
for \( \epsilon > 0 \). Then
\[
P(h^{-1/2}V_{(s_n(\alpha))} - c(\alpha)^{1/2} < -n^{-\beta_0}) \\
\leq P\left( \sum_{j \in I_n} 1_{A_{n,j}} \geq (1 - \alpha)n - 1 \right) \\
= P\left( \sum_{j \in I_n} 1_{A_{n,j} \cap \{ j \in \mathcal{L}_n^\epsilon \}} + \sum_{j \in I_n} 1_{A_{n,j} \cap \{ j \in \mathcal{L}_n \}} \geq (1 - \alpha)n - 1 \right) \\
\leq P\left( \sum_{j \in I_n} 1_{A_{n,j} \cap \{ j \in \mathcal{L}_n^\epsilon \}} + \# \mathcal{L}_n \geq (1 - \alpha)n - 1 \right) \\
\leq P\left( \sum_{j \in I_n} V_{n,j} \geq (1 - \alpha)n - 1 - n^{\frac{1}{2} + \xi + \epsilon} \right) + P[\# \mathcal{L}_n > n^{\frac{1}{2} + \xi + \epsilon}] + P[\Omega_n^\epsilon] \\
\leq P\left( \sum_{j \in I_n} (V_{n,j} - E[V_{n,j}]) \geq \mu_n \right) + P[\# \mathcal{L}_n > n^{\frac{1}{2} + \xi + \epsilon}] + P[\Omega_n^\epsilon].
\]
We see
\[
\mu_n \sim (1 - \alpha)n - 1 - n^{\frac{1}{2} + \xi + \epsilon} - n\{(1 - \alpha) - c^*(n^{-\beta_0} - 2^{-1}n^{-\gamma})\} \geq \frac{1}{2}c^*n^{1-\beta_0}
\]
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for large $n$, where $c^*$ is some positive constant, if we take a sufficiently small $\epsilon$ and $\gamma_1 \in (\beta_0, \gamma_0)$ thanks to $\beta_0 < \min \{\gamma_0, \frac{1}{2} - \xi\}$. Since $n^{-1/2} \mu_n \geq 2^{-1} c^* n^{1/2 - \beta_0}$, from $\beta_0 < 1/2$, we obtain

$$P \left[ n^{-1/2} \sum_{j \in I_n} (V_{n,j} - E[V_{n,j}]) \geq n^{-1/2} \mu_n \right] = O(n^{-L})$$

for every $L > 0$. Therefore,

$$P \left[ h^{-1/2} V(s_n(\alpha)) - c(\alpha)^{1/2} < -n^{-\beta_0} \right] = O(n^{-L})$$

as $n \to \infty$ for every $L > 0$. Similarly, we can obtain the estimate $P \left[ h^{-1/2} V(s_n(\alpha)) - c(\alpha)^{1/2} > n^{-\beta_0} \right] = O(n^{-L})$ to show (4.21), which concludes the proof.

5 Asymptotic mixed normality of the global realized volatilities with a moving threshold

5.1 The GRV with a moving threshold

In this section, we will consider a situation where the intensity of jumps is moderate. Then it is possible to keep the cut-off ratio of the data small, and to get a precise estimate for the integrated volatility. Let

$$\delta_0 \in (0, 1/4) \quad \text{and} \quad \delta_1 \in (0, 1/2). \quad (5.1)$$

In the context of the global jump filtering, given a collection $(\mathcal{G}_{n,j-1})_{j \in I_n}$ of nonnegative random variables, we consider the index set $\mathcal{J}_n$ given by

$$\mathcal{J}_n = \{ j \in I_n; V_j < V(s_n) \} \quad (5.2)$$

where

$$V_j = \left| (\mathcal{G}_{n,j-1})^{1/2} \Delta_j \mathbf{X} \right| \quad (5.3)$$

and

$$s_n = n - \lceil B n^{\delta_1} \rceil \quad (5.4)$$

for a positive constant $B$. Here $x^- = 1_{\{x \neq 0\}} x^{-1}$ for $x \in \mathbb{R}$.

Remark 5.1. It is natural to set a spot volatility estimator of $\sigma_{t_{j-1}}^2$ in $\mathcal{G}_{n,j-1}$ though not definitively necessary (Remark 5.2). In Section 3, we discussed some constructions of $\mathcal{G}_{n,j-1}$. In the terminology of Section 3, the cut-off rate by $\mathcal{J}_n$ is $\alpha_n = \lceil B n^{\delta_1} \rceil / n$, $\mathcal{J}_n = \mathcal{J}_n(\alpha_n)$ and $\alpha_n$ goes to 0 as $n$ tends to $\infty$. We note that the definition of $V_j$ is different from that in (2.1).
For estimation of $\Theta$ of (1.2), we consider the global realized volatility (GRV) with a moving threshold

$$G_n = \sum_{j \in J_n} q_n^{-1} |\Delta_j X|^2 H_{n,j}$$

(5.5)

where $(q_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers, and

$$H_{n,j} = 1_{|\Delta_j X| < B_0 n^{-\frac{1}{2}} - \delta_0}$$

(5.6)

for a positive constant $B_0$.

Here, $\sigma = (\sigma_t)_{t \in [0,T]}$ and $b = (b_t)_{t \in [0,T]}$ are càdlàg adapted processes. We will assume (5.1) and the following conditions.

\[ G_1 \] For every $p > 1$, $\sup_{t \in [0,T]} \|\sigma_t\|_p + \|b_t\|_p < \infty$.

\[ G_2 \] $q_n > 0 \ (n \in \mathbb{N})$ and $q_n - 1 = o(n^{-1/2})$ as $n \to \infty$.

Remark 5.2. Theoretically, we may set $G_{n,j-1} = 1$. Condition [G2] is satisfied with $q_n = 1$. Asymptotically the choice $(G_{n,j-1}, q_n) = (1, 1)$ is sufficient and valid. However, in practice, a natural choice is a pair $S_{n,j-1}$ satisfying [G2] in Section 2 and $q_n = q(\alpha_n)$, where the function $q$ is defined by (2.2) in Section 2.

For the jump part $J$ of the semimartingale $X$, we only assume

$$\Lambda_n := \# \{ j \in I_n; \Delta_j J \neq 0 \} < \infty \ \ a.s.$$ for every $n \in \mathbb{N}$, and the following estimate:

\[ G_3 \] There exists a constant $\xi \geq 0$ such that $\|\Lambda_n\|_p = O(n^\xi)$ as $n \to \infty$ for every $p > 1$.

Remark 5.3. The diverging $\Lambda_n$ models high intensity of the jump part for a fixed $n$ in practice. Mathematically, we are assuming that the process $\sigma$ is independent of $n$. This makes sense naturally in particular when the jumps are exogenous. It is sufficient for the limit theorem by using the càdlàg property of $\sigma$. On the other hand, though details are omitted, we can treat $\sigma$ depending on $n$ if uniform $L^\infty$-continuity of $\sigma$ and uniformity in [G1] are satisfied.

Define $\Gamma$ by

$$\Gamma = 2T \int_0^T \sigma_t^4 dt.$$ 

Extend $(\Omega, \mathcal{F}, P)$ so that there is a standard normal random variable $\zeta$ independent of $\mathcal{F}$ on the extension. The $\mathcal{F}$-stable convergence is denoted by $\to_{\mathcal{F}}$. We obtain asymptotic mixed normality of the global realized volatility $G_n$ with a moving threshold.

Theorem 5.4. Suppose that $[G1]$, $[G2]$ and $[G3]$ are satisfied. Suppose that $\xi < 2\delta_0$. Then

$$n^{1/2}(G_n - \Theta) \to_{\mathcal{F}} \Gamma^{1/2}\zeta$$

as $n \to \infty$.

Theorem 5.4 follows from Theorem 5.9, that is presented in a slightly more general setting.
5.2 The WGRV with a moving threshold

Suppose that a collection \((\mathcal{G}_{n,j-1})_{j \in I_n}\) of positive random variables is given. Consider constants \(\delta_0\) and \(\delta_1\) satisfying \((5.1)\), and \(V_j\) and \(s_n\) given by \((5.3)\) and \((5.4)\), respectively. We define the Winsorized global realized volatility (WGRV) with a moving threshold by

\[
W_n = \sum_{j \in I_n} q_n^{-1} \{ |\Delta_j X| \wedge \mathcal{G}_{n,j-1}^{1/2} V(s_n) \}^2 H_{n,j},
\]

where \((q_n)_{n \in \mathbb{N}}\) is a sequence of positive numbers. The error of the WGRV has the same limit as GRV \(G_n\).

**Theorem 5.5.** Suppose that \([G1^*] , [G2^*] \) and \([G3^*] \) are satisfied. Suppose that \(\xi < 2\delta_0\). Then

\[
n^{1/2} (W_n - \Theta) \to^d \Gamma^{1/2} \zeta
\]

as \(n \to \infty\) where \(\zeta\) is a standard Gaussian random variable independent of \(\mathcal{F}\).

**Proof.** Let \(\tilde{X} = X - J\). It suffices to show that \(n^{1/2} \|W_n - G_n\|_p \to 0\) as \(n \to \infty\) for every \(p > 1\). From \((5.7)\),

\[
W_n - G_n = \sum_{j \in J_n} q_n^{-1} \mathcal{G}_{n,j-1} V_{(s_n)}^2 H_{n,j}.
\]

We have

\[
\left\| \sum_{j \in J_n} \mathcal{G}_{n,j-1} V_{(s_n)}^2 H_{n,j} \right\|_p \leq \left\| \sum_{j \in J_n} |\Delta_j X|^2 1_{\{j \in \mathcal{G}_n\}} H_{n,j} \right\|_p + \left\| \sum_{j \in J_n} |\Delta_j \tilde{X}|^2 1_{\{j \in \mathcal{G}_n\}} H_{n,j} \right\|_p
\]

\[
\leq n^{-1/2+\delta_0} + n^{-1+\epsilon+\delta_1}
\]

as \(n \to \infty\) for any \(\epsilon > 0\). Since \(\delta_1 < 1/2\), \(\xi < 2\delta_0\) and \(\epsilon\) is arbitrary, we obtain the desired convergence from Theorem 5.4. \(\square\)

5.3 Stability of the realized volatility under missing

We are about establishing asymptotic mixed normality of the integrated volatility estimator having a moving threshold. We will solve this problem by showing a stability of estimation under elimination of a certain portion of the data. In other words, this is a question of stability under missing data. In what follows, we will consider the variable \(V_n\) defined by

\[
V_n = \sum_{j \in \mathcal{M}_n} q_n^{-1} |\Delta_j X|^2 H_{n,j}
\]

where \((q_n)_{n \in \mathbb{N}}\) is a sequence of positive numbers, \(H_{n,j}\) is given in \((5.6)\), and \(\mathcal{M}_n\) is an abstract random index set in \(I_n\). It is not necessary to specify \(\mathcal{M}_n\) like \(J_n\) by \((5.2)\) and \((5.4)\).

Let

\[
V_n^{\dagger} = \sum_{j \in \mathcal{M}_n} q_n^{-1} |\Delta_j \tilde{X}|^2 H_{n,j}.
\]

Recall \(\tilde{X} = X - J\).
\[ \text{[G2']} \] (i) For every \( n \in \mathbb{N} \), \( M_n \) is a random set in \( I_n \) such that \( \#(I_n \setminus M_n) \leq B_1 n^{\delta_1} \) \( (n \in \mathbb{N}) \) for some positive constant \( B_1 \).

(ii) \( q_n > 0 \) \( (n \in \mathbb{N}) \) and \( q_n - 1 = o(n^{-1/2}) \) as \( n \to \infty \).

Lemma 5.6. Suppose that [G1\(^o\)], [G2\(^o\)] and [G3\(^o\)] are satisfied. Suppose that \( \xi < 2\delta_0 \). Then
\[
 n^{1/2} \| V_n - V_n^\dagger \|_p \to 0
\]
as \( n \to \infty \) for every \( p > 1 \).

Proof. We have the estimate
\[
n^{1/2} \| V_n - V_n^\dagger \|_p \leq 2q_n^{-1} \Phi_n^{5.10} + q_n^{-1} \Phi_n^{5.11},
\]
where
\[
\Phi_n^{5.10} = n^{1/2} \left\| \sum_{j \in M_n} |\Delta_j \tilde{X} \Delta_j J| H_{n,j} \right\|_p,
\]
and
\[
\Phi_n^{5.11} = n^{1/2} \left\| \sum_{j \in M_n} |\Delta_j J|^2 H_{n,j} \right\|_p
\]
for \( p > 1 \). By using the inequality
\[
|\Delta_j J| H_{n,j} \leq \left( |\Delta_j \tilde{X}| + B_0 n^{-\frac{1}{4} - \delta_0} \right) 1_{\{\Delta_j J \neq 0\}},
\]
we obtain
\[
\Phi_n^{5.10} \leq n^{1/2} \left\| \max_{j \in I_n} \left\{ |\Delta_j \tilde{X}| \left( |\Delta_j \tilde{X}| + B_0 n^{-\frac{1}{4} - \delta_0} \right) \right\} \right\|_2 \| \Lambda_n \|_2p
\]
\[
\lesssim n^{-\frac{1}{4} - \delta_0 + \xi + \epsilon}
\]
as \( n \to \infty \) for any \( \epsilon > 0 \) and \( p > 1 \). Therefore,
\[
\Phi_n^{5.10} \to 0 \quad (5.12)
\]
for every \( p > 1 \) since \( \xi < 2\delta_0 < \frac{1}{4} + \delta_0 \). Similarly,
\[
\Phi_n^{5.11} \lesssim n^{1/2} \left\| \max_{j \in I_n} \left\{ |\Delta_j \tilde{X}|^2 + n^{-\frac{1}{2} - 2\delta_0} \right\} \right\|_2 \| \Lambda_n \|_2p
\]
\[
\lesssim n^{-2\delta_0 + \xi + \epsilon}
\]
as \( n \to \infty \) for any \( \epsilon > 0 \) and \( p > 1 \) since \( \delta_0 < 1/4 \). In particular,
\[
\Phi_n^{5.11} \to 0 \quad (5.13)
\]
as \( n \to \infty \) since \( \xi < 2\delta_0 \). Now the proof is completed with \( (5.9), (5.12) \) and \( (5.13) \).}

Define \( \tilde{V}_n \) by
\[
\tilde{V}_n = \sum_{j \in I_n} |\Delta_j \tilde{X}|^2.
\]
Lemma 5.7. Suppose that $\xi < 1/2$. Then
\[ n^{1/2} \| V^*_n - \tilde{V}_n \|_p \to 0 \]
as $n \to \infty$ for every $p > 1$.

Proof. Recall that $\delta_0 < 1/4$ and $\delta_1 < 1/2$. Define $V^*_n$ by
\[ V^*_n = \sum_{j \in M_n} q_n^{-1} |\Delta_j \tilde{X}|^2. \]
Then
\[ n^{1/2} \| V^*_n - V^\dagger_n \|_p \leq n^{1/2} \left\| \sum_{j \in M_n} |\Delta_j \tilde{X}|^2 |H_{n,j} - 1| \right\|_p + \epsilon \]
for any positive number $\epsilon > 0$. Here $L$ is an arbitrary positive number greater than $1/2$, and we used the inequality $\delta_0 < 1/4$ to get $O(n^{-L})$. Since $\xi < 1/2$, we obtain
\[ n^{1/2} \| V^*_n - V^\dagger_n \|_p = o(1) \quad \text{(5.14)} \]
as $n \to \infty$ for every $p > 1$.

From the condition $q_n - 1 = o(n^{-1/2})$ of $G2'$ (ii), obviously,
\[ n^{1/2} \| V^*_n - \tilde{V}_n \|_p \leq n^{1/2} \left\| \sum_{j \in I_n \setminus M_n} |\Delta_j \tilde{X}|^2 \right\|_p + o(1) \]
\[ \leq n^{-1/2 + \epsilon + \xi} + o(1) = o(1) \quad \text{(5.15)} \]
as $n \to \infty$ for every $p > 1$ since $\#(I_n \setminus M_n) \lesssim n^{\delta_1}$ with $\delta_1 < 1/2$ thanks to $G2'$ (i) and (5.1), and
\[ \left\| \max_{j \in I_n} |\Delta_j \tilde{X}|^2 \right\|_p = O(n^{-1+\epsilon}) \]
for any $p > 1$ and any positive number $\epsilon$. Proof ends with (5.14) and (5.15).

Lemma 5.8. Suppose that $G1^o$ is satisfied. Then
\[ n^{1/2} (\tilde{V}_n - \Theta) \to^{d_{\kappa}} \Gamma^{1/2} \zeta \]
as $n \to \infty$. 33
Proof. We have
\[ \mathring{V}_n = \sum_{j \in I_n} \left( \int_{t_{k-1}}^{t_k} \sigma_t dw_t + \int_{t_{k-1}}^{t_k} b_t dt \right)^2 \]
\[ = \Phi_{5.17} + \Phi_{5.18} + 2\Phi_{5.19} + \Phi_{5.20} \quad (5.16) \]
where
\[ \Phi_{5.17} = \sum_{j \in I_n} 2 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_i} \sigma_s dw_s \sigma_i dw_t, \quad (5.17) \]
\[ \Phi_{5.18} = \sum_{j \in I_n} \int_{t_{j-1}}^{t_j} \sigma_i^2 dt = \Theta, \quad (5.18) \]
\[ \Phi_{5.19} = \sum_{j \in I_n} \int_{t_{j-1}}^{t_j} \sigma_i dw_t \int_{t_{j-1}}^{t_j} b_i dt, \quad (5.19) \]
and
\[ \Phi_{5.20} = \sum_{j \in I_n} \left( \int_{t_{j-1}}^{t_j} b_i dt \right)^2. \quad (5.20) \]

Since \( b \) is a càdlàg process, for any \( \epsilon > 0 \), there exists a number \( \delta > 0 \) such that \( P[w'(b, \delta) \geq \epsilon] < \epsilon \). Here \( w'(b, \delta) \) is a modulus of continuity defined by
\[ w'(b, \delta) = \inf_{(s_i) \in S_\delta} \max_i \sup_{r_1, r_2 \in [s_{i-1}, s_i]} |b_{r_1} - b_{r_2}|, \]
where \( S_\delta \) is the set of sequences \((s_i)\) such that \( 0 = s_0 < s_1 < \cdots < s_v = T \) and \( \min_{i=1, \ldots, v-1} (s_i - s_{i-1}) > \delta \). Let
\[ \Phi_{5.19} = \sum_{j \in I_n} \int_{t_{j-1}}^{t_j} \sigma_i dw_t \int_{t_{j-1}}^{t_j} (b_t - b_{t_{j-1}}) dt. \quad (5.21) \]

Write
\[ E_j = \int_{t_{j-1}}^{t_j} \sigma_i dw_t, \]
\[ \mathcal{V}_j = n^{1/2} \left| \int_{t_{j-1}}^{t_j} \sigma_i dw_t \right| \left| \int_{t_{j-1}}^{t_j} (|b_t| + |b_{t_{j-1}}|) dt \right|. \]

For \( \omega \in \Omega \) such that \( w'(b(\omega), \delta) < \epsilon \), there exists a \((s_i)\) (depending on \( \omega \)) such that
\[ \max_{i} \sup_{r_1, r_2 \in [s_{i-1}, s_i]} |b_{r_1}(\omega) - b_{r_2}(\omega)| \leq \epsilon, \]
\[ \min_{i=1, \ldots, v-1} (s_i - s_{i-1}) > \delta. \]
For \( n > T/\delta \), all intervals \([t_{j-1}, t_j)\) \((j \in I_n)\) includes at most one point among \((s_i)\), therefore the number of intervals \([t_{j-1}, t_j)\) that include some one \(s_i\) is at most \(T/\delta\). The increment of \(b(\omega)\) in \([t_{j-1}, t_j)\) is less than \(\epsilon\) if \([t_{j-1}, t_j) \cap \{s_i\} = \emptyset\). Thus, we have the inequality
\[
\|n^{1/2} \Phi_n^{5.19}\|_p \leq \left\| \sum_{j \in I_n} n^{1/2} |E_j| \right\|_p \epsilon + \left\| \max_{j \in I_n} V_j \left\| \frac{T}{\delta} + \left\| \sum_{j \in I_n} \|V_j\|_2 \right\| \right\|_p \mathbb{P}[\omega'(b, \delta) \geq \epsilon]^{1/2p}
\]
for every \( p > 1 \). Therefore,
\[
\|n^{1/2} \Phi_n^{5.19}\|_p \leq C \left[ \epsilon + \left( n^{-1/2} + \sum_{j \in I_n} \|V_j 1\{V_j > n^{-1/2}\}\|_p \right) \frac{T}{\delta} + \epsilon^{1/2p} \right]
\]
\[
\leq C'(\epsilon + n^{-1/2} + \epsilon^{1/2p})
\]
for all \( n > T/\delta \), where \( C \) and \( C' \) are some constants independent of \( n \). Consequently,
\[
\lim_{n \to \infty} \|n^{1/2} \Phi_n^{5.19}\|_p = 0 \quad (5.22)
\]
for every \( p > 1 \). Moreover, for
\[
\Phi_n^{5.19} = \sum_{j \in I_n} \int_{t_{j-1}}^{t_j} \sigma_t dw_t - \int_{t_{k-1}}^{t_k} b_{t_{j-1}} dt = \sum_{j \in I_n} h b_{t_{j-1}} \int_{t_{j-1}}^{t_j} \sigma_t dw_t,
\]
we have
\[
\lim_{n \to \infty} \|n^{1/2} \Phi_n^{5.19}\|_p = 0 \quad (5.23)
\]
for every \( p > 1 \), by orthogonality. From (5.22) and (5.23),
\[
\lim_{n \to \infty} \|n^{1/2} \Phi_n^{5.19}\|_p = 0 \quad (5.24)
\]
for every \( p > 1 \).

Obviously,
\[
\|n^{1/2} \Phi_n^{5.20}\|_p = 0 \quad (5.25)
\]
for every \( p > 1 \). Now, we can show the claim of the lemma by using (5.16), (5.24) and (5.25) together with the mixture type of martingale central limit theorem applied to \(n^{1/2} \Phi_n^{5.17}\), with the aid of the càdlàg property of \(\sigma\).

**Theorem 5.9.** Suppose that \([G1']\), \([G2']\) and \([G3']\) are satisfied. Suppose that \(\xi < 2\delta_0\). Then
\[
n^{1/2} (V_n - \Theta) \Rightarrow_{d^*} \Gamma^{1/2} \xi
\]
as \( n \to \infty \).

**Proof.** Just combine Lemmas 5.6, 5.7 and 5.8. \(\square\)
6 Constant volatility

The case of constant $\sigma$ is specific and theoretical treatments can be slightly different from those of the previous sections. In this situation, we do not need to pre-estimate the local spot volatility, and hence, we can take $S_{n,j} = 1$ constantly and no approximation error is caused. $\sigma_t = \theta \tilde{\sigma}_t$ is also the case if $\tilde{\sigma}_{t-1}$ are observable. For example, the GRV with a fixed cut-off rate $\alpha$ is redefined as

$$V^0_n(\alpha) = \sum_{j \in J^0_n(\alpha)} q(\alpha)^{-1} |\Delta_j X|^2 K_{n,j},$$

where

$$J^0_n(\alpha) = \{ j \in I_n; |\Delta_j X| < |\Delta X|_{(s_n(\alpha))} \}.$$

Then we have the following theorem. Note that we do not need the condition $[G2]$, and $\gamma_0$ in $[G2](ii)$ can be arbitrarily close to $1/2$.

**Theorem 6.1.** Suppose that $[G1]$ and $[G3]$ are fulfilled. Suppose that $\xi < 1/7$. Let $\alpha \in (0, 1)$ and $\beta_0 < 1/2 - \xi$. Then

$$\|V^0_n(\alpha) - \Theta\|_p = O(n^{-\beta_0})$$

as $n \to \infty$ for every $p > 1$.

The other global-threshold estimators are discussed similarly.

7 Simulation studies

In this section, we conduct several numerical simulations to see that our global realized volatility estimators outperform those proposed in previous studies.

7.1 The case of compound Poisson jumps

Here we consider a process $X = (X_t)_{t \in [0,1]}$ satisfying the stochastic differential equation

$$dX_t = \theta X_t dt + (\sigma + \eta X_t^2)^{1/2} dw_t + dJ_t, \quad t \in [0, 1],$$

with $X_0 = 1$, where $J_t$ is the jump part of $X$. In this section, we assume that $J$ is a compound Poisson process of the form $J_t = \sum_{i=1}^{N_t} \xi_t$, where $(N_t)_t$ is a Poisson process with intensity $\lambda > 0$ and $(\xi_t)_t$ are independently and normally distributed random variable with mean $\mu$ and variance $\nu^2$. For the intensity parameter, we consider both cases where $\lambda$ is high and low. Our aim is to estimate the integrated volatility $\Theta = \int_0^1 (\sigma + \eta X^2_t)^{1/2} dt$.

By simulation, we will compare the performance of the threshold realized volatility (TRV), bipower variation (BV), minimum realized volatility (minRV), the GRV, and the WGRV, where
TRV, BV and minRV are given by

\[
\text{TRV}_n = \sum_{j=1}^{n} |\Delta_j X|^2 1_{\{1_{|\Delta_j X| \leq n^{-\rho}} \}}, \quad \rho \in (0, 1/2),
\]

\[
\text{BV}_n = \frac{\pi}{2} \sum_{j=1}^{n-1} |\Delta_j X||\Delta_{j+1} X|,
\]

\[
\minRV_n = \frac{\pi}{\pi - 2} \sum_{j=1}^{n-1} |\Delta_j X|^2 \wedge |\Delta_{j+1} X|^2,
\]

respectively. The package YUIMA (cf. [4], [7]) was used for the simulation studies below.

Note that, although TRV is based on threshold method, it is completely different from our GRV, since TRV employs a deterministic threshold and never uses information of other increments. In this sense, TRV is based on a “local” approach.

The set-up of simulation is as follows. The number of samples is \( n = 2000 \). We repeat calculating the estimators 500 times to obtain their average and quantile. The true parameters are \( \theta = 0.2 \), \( \sigma = 1 \), \( \eta = 3 \), \( \mu = 0.3 \), \( \nu = 0.2 \). Throughout this subsection, we set the cut-off ratio \( \alpha = 0.2 \) for GRV and WGRV with a local volatility estimator \( S_{n,j-1} \). That is, we trim the upper 20% of absolute increments. While it may seem that we eliminate too many observations and the estimator suffers from downside bias, GRV and WGRV estimate the integrated volatility well thanks to the adjustment coefficient by \( q(\alpha) \) and \( w(\alpha) \). In calculating the TRV, we set \( \rho = 0.45 \), \( 0.2 \), \( 0.1 \) to see the effect of the choice of this parameter on the accuracy of estimation.

Note that \( \sigma_s \) in (1.1) is not directly observable and depends on \( X_t \). Hence, we need \( S_{n,j-1} \) to normalize the increment \( \Delta_i X \) when constructing the GRV. In this simulation, we use the LGRV \( L_{n,j}(\alpha_0) \) of (3.15) with \( \alpha_0 = 0.2 \), and the local minRV \( \minRV_{n,j} \) of (3.48) for \( S_{n,j-1} \). We adopt \( \kappa_n = \lceil 10 \times n^{0.45} \rceil \) for the length of a subinterval to calculate these local volatilities. Moreover, we calculate GRV without normalization (defined in Section 6) for comparison. Note that \( \kappa_n \) depends on two tuning parameters, the choice of which can affect the precision of estimation. We argue this point in the final Section 7.3.

We use the following labels as in Table 7.1 to describe the estimators.
### Table 1: Definitions of estimators

| Label             | Method | Spot volatility | Cut-off ratio $\alpha$ | Exponent $\rho$ for truncation |
|-------------------|--------|-----------------|-------------------------|--------------------------------|
| trv[$\rho$]       | TRV    | –               | –                       | 0.45, 0.2, 0.1                 |
| bv                | BV     | –               | –                       | –                              |
| mrv               | minRV  | –               | –                       | –                              |
| grv.lgrv[$\alpha$]| GRV    | GRV             | 0.2                     | –                              |
| grv.mrv[$\alpha$]| GRV    | minRV           | 0.2                     | –                              |
| wgrv.lgrv[$\alpha$]| WGRV  | GRV             | 0.2                     | –                              |
| wgrv.mrv[$\alpha$]| WGRV  | minRV           | 0.2                     | –                              |
| grv[$\alpha$]    | GRV    | –               | 0.2, 0.1, 0.05          | –                              |
| grv.lgrv.mov      | GRV    | GRV             | depends on $n$          | –                              |
| wgrv.lgrv.mov     | WGRV  | GRV             | depends on $n$          | –                              |

#### 7.1.1 The case of high intensity: GRV with fixed cut-off ratio

First, we deal with the case of high intensity. Here we set $\lambda = 30$ so that the data includes many jumps. The example of a sample path and its increments are shown in Figure 1. Obviously, there are many large spikes in the data, suggesting the existence of jumps. Note that the volatility is non-constant here. In fact, in Panel (b) of Figure 1 the size of increments tend to increase as time passes. Hence, to estimate the volatility, we have to use estimated spot volatilities to normalize the increments.

In this example, we show the error ratios of GRV and WGRV with shrinking cut-off ratio (tuning parameters that determine the cut-off ratio $\alpha_n = \lfloor Bn^{\delta_1} \rfloor$ for these estimators are $B = 10$ and $\delta_1 = 0.45$, the same as those used in the next subsection). Theoretically, they are available in the case of moderate intensity of jumps. We show their results just for reference. We will discuss the case of moderate intensity in more detail in the next subsection.
Table 2 shows the summary of error ratios (percentage deviation of estimated values from the true value for each estimator), and Figure 2 gives their box plots. In this case, both BV (bv) and minRV (mrv) seem to suffer from upward bias due to jumps. In particular, the BV deviates from the true value considerably. On the other hand, GRV with normalization perform well with errors concentrating around zero (grv.lgrv, grv.mrv). Note that, although WGRV performs relatively well, it seems to have a small upward bias (wgrv.lgrv, wgrv.mrv). This suggests that, if there are many large jumps, using an upper quantile ($V(s_n(\alpha))$) may sometimes lead to biases rather than obtaining a robust estimate.

The three right box plots in this figure (grv[0.20], grv[0.10], grv[0.05]) are the results of GRV without normalizing increments by local-global filters, with the cut-off ratio $\alpha = 0.2, 0.1, 0.05$, respectively. We see that they seem to be less precise (especially when $\alpha$ is large extremely small) than GRV or WGRV with local volatility. This result suggests that, if we do not normalize increments by spot volatilities in the case of non-constant volatility, we end up obtaining inappropriate estimates.

Intuitively, when we ignore normalization, we tend to eliminate increments where volatility is high (because they are typically large), even if they come from the Brownian motion, while keeping relatively small jumps which we should actually remove. In addition, theoretically, the adjusting constant $q(\alpha)$ in the definition of GRV (2.3) comes from the standard normal distribution. Therefore, when the volatility is non-constant, we should normalize the increments $|\Delta_iX|$ by local volatility to make them approximately standard normally distributed.
Table 2: Summary of error ratios: $\lambda = 30$

|                  | Min. | 1st Qu. | Median | 3rd Qu. | Max. |
|------------------|------|---------|--------|---------|------|
| $\text{trv}[0.45]$ | -29.88 | -28.16  | -27.42 | -26.45  | -21.64 |
| $\text{trv}[0.20]$ | -9.40  | -3.57   | -1.95  | -0.67   | 3.54  |
| $\text{trv}[0.10]$ | 0.32   | 3.38    | 4.55   | 6.11    | 21.61 |
| $\text{bv}$      | -0.25  | 2.66    | 3.87   | 5.11    | 8.98  |
| $\text{mrv}$     | -2.61  | 0.36    | 1.43   | 2.48    | 5.75  |
| $\text{grv.1grv}[0.20]$ | -6.65  | -1.03   | -0.10  | 0.88    | 3.64  |
| $\text{grv.mrv}[0.20]$ | -6.65  | -1.33   | -0.40  | 0.64    | 3.65  |
| $\text{wgrv.1grv}[0.20]$ | -3.50  | -0.39   | 0.60   | 1.46    | 3.68  |
| $\text{wgrv.mrv}[0.20]$ | -3.53  | -0.50   | 0.44   | 1.32    | 3.72  |
| $\text{grv}[0.20]$ | -9.29  | -4.39   | -3.21  | -1.91   | 2.15  |
| $\text{grv}[0.10]$ | -10.26 | -2.84   | -1.76  | -0.76   | 2.71  |
| $\text{grv}[0.05]$ | -13.85 | -4.41   | -1.77  | -0.42   | 3.38  |
| $\text{grv.1grv.mov}$ | -9.13  | -1.18   | -0.00  | 0.95    | 3.55  |
| $\text{wgrv.1grv.mov}$ | -3.19  | -0.14   | 0.80   | 1.62    | 4.61  |
The good news is that they also perform well even in the case of extremely high intensity. We consider $\lambda = 50$ here to see their accuracy. Figure 3 shows a sample path and its increments. It is obvious there are numerous jumps and one can easily imagine that the standard realized volatility estimator can never estimate the true volatility. Table 3 and Figure 4 show error ratios of each estimator for $\lambda = 50$. It shows that GRV and WGRV with cut-off ratio $\alpha = 0.2$ perform well even in the case of high intensity of jumps.
Figure 3: Sample path of $X$ and its increments ($\lambda = 50$)
Table 3: Summary of error ratios: $\lambda = 50, \alpha = 0.2$

|                  | Min.  | 1st Qu. | Median | 3rd Qu. | Max.  |
|------------------|-------|---------|--------|---------|-------|
| trv[0.45]        | -99.52| -98.90  | -98.52 | -98.09  | -94.42|
| trv[0.20]        | -52.18| -27.61  | -19.86 | -12.98  |  2.35 |
| trv[0.10]        |  0.78 |  11.76  |  15.51 |  20.33  | 37.69 |
| bv               |  2.71 |  13.60  |  17.47 |  20.89  | 37.19 |
| mrv              | -6.36 |   3.40  |   7.72 |  11.58  | 27.20 |
| grv.lgrv[0.20]   | -38.58|   -7.51 |   -2.38|    1.82 | 18.41 |
| grv.mrv[0.20]    | -39.65|   -8.54 |   -3.51|    0.81 | 16.58 |
| wgrv.lgrv[0.20]  | -10.64|    0.87 |    3.99|    7.44 | 25.22 |
| wgrv.mrv[0.20]   | -12.60|   -0.08 |    3.18|    6.74 | 23.23 |
| grv[0.20]        | -33.92|  -15.51 |  -11.40|   -7.01 |  5.42 |
| grv[0.10]        | -56.46|  -23.57 |  -11.63|   -4.84 |  9.21 |
| grv[0.05]        | -66.14|  -40.55 |  -29.94|  -17.97 |  5.88 |
| grv.lgrv.mov     | -48.17|  -16.03 |   -6.55|   -0.14 | 16.16 |
| wgrv.lgrv.mov    | -10.31|    2.16 |    5.85|    9.61 | 31.21 |
By taking $\alpha$ larger, accuracy improves. Table 4 shows the error ratios of each estimator in the case of $\lambda = 50$, with $\alpha = \alpha_0$ ranging from 0.1 to 0.5. We can see that, for large cut-off ratio ($\alpha = \alpha_0 = 0.4, 0.5$), GRV and WGRV with spot volatilities (grv.lgrv, grv.mrv, wgrv.lgrv, wgrv.mrv) still perform well. Looking in more detail, we see that the local GRV outperforms local minRV for both GRV and WGRV. This example imply that we should take a cut-off ratio quite large in order to obtain a precise estimate.
Table 4: Error ratios [%] for the case of extremely high intensity: $\lambda = 50$

| Cut-off ratio ($\alpha = \alpha_0$) | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  |
|-------------------------------------|------|------|------|------|------|
| trv[0.45]                           | -98.40 | -98.40 | -98.40 | -98.40 | -98.40 |
| trv[0.20]                           | -20.35 | -20.35 | -20.35 | -20.35 | -20.35 |
| trv[0.10]                           | 16.19  | 16.19  | 16.19  | 16.19  | 16.19  |
| bv                                 | 17.60  | 17.60  | 17.60  | 17.60  | 17.60  |
| mrv                                | 7.71   | 7.71   | 7.71   | 7.71   | 7.71   |
| grv.lgrv[$\alpha$]                 | -18.04 | -3.49  | -0.41  | -0.96  | -1.47  |
| grv.mrv[$\alpha$]                  | -18.05 | -4.35  | -2.62  | -3.90  | -4.89  |
| wgrv.lgrv[$\alpha$]                | 10.28  | 4.05   | 2.27   | 1.35   | 0.59   |
| wgrv.mrv [$\alpha$]                | 10.87  | 3.21   | 1.06   | -0.30  | -1.40  |
| grv[0.20]                           | -11.50 | -11.50 | -11.50 | -11.50 | -11.50 |
| grv[0.10]                           | -14.60 | -14.60 | -14.60 | -14.60 | -14.60 |
| grv[0.05]                           | -28.85 | -28.85 | -28.85 | -28.85 | -28.85 |
| grv.lgrv.mov                        | -8.31  | -8.76  | -8.78  | -8.80  | -8.83  |
| wgrv.lgrv.mov                       | 4.99   | 6.26   | 6.48   | 6.63   | 6.73   |

7.1.2 The case of moderate intensity: GRV with a shrinking cut-off ratio

Next, we consider the case of low intensity. In this case, we can use shrinking cut-off rate. Recall that the shrinking cut-off rate is defined by $\alpha_n = |B n^{\delta_1}|/n$. In this simulation, we set $B = 10$ and $\delta_1 = 0.45$, so the cut-off rate is then $\alpha_n = 0.1525$.

The error ratios are shown in Table 5 Figure 5. All global-filtering estimators perform well (for GRVs with fixed cut-off ratio, we set $\alpha = 0.2$ as before). These results suggest that if there are not so many jumps in the data, it would be advisable to use as many data as possible by making the cut-off ratio small. Note that TRV still has bias, especially for $\rho = 0.45$. This implies that the accuracy of estimation is still highly vulnerable to the choice of $\rho$ for TRV, even in the case of moderate intensity of jumps.
|                | Min.  | 1st Qu. | Median | 3rd Qu. | Max.  |
|----------------|-------|---------|--------|---------|-------|
| trv[0.45]     | -29.61| -27.40  | -26.09 | -23.89  | -18.27|
| trv[0.20]     | -2.79 | -0.58   | 0.14   | 0.85    | 3.86  |
| trv[0.10]     | -2.39 | 1.16    | 2.52   | 4.25    | 13.51 |
| bv            | -2.26 | 0.31    | 1.29   | 2.24    | 6.34  |
| mr v          | -3.63 | -0.69   | 0.29   | 1.24    | 4.27  |
| grv.lgrv[0.20]| -3.62 | -1.44   | -0.55  | 0.21    | 4.79  |
| grv.mrv[0.20] | -3.59 | -1.40   | -0.55  | 0.30    | 5.00  |
| wgrv.lgrv[0.20]| -3.52| -1.11   | -0.39  | 0.44    | 4.55  |
| wgrv.mrv[0.20]| -3.53 | -1.06   | -0.40  | 0.46    | 4.62  |
| grv[0.20]     | -7.14 | -2.82   | -1.75  | -0.75   | 2.90  |
| grv[0.10]     | -5.46 | -2.15   | -1.32  | -0.46   | 3.17  |
| grv[0.05]     | -4.15 | -1.63   | -0.86  | -0.10   | 3.23  |
| grv.lgrv.mov  | -3.54 | -1.32   | -0.50  | 0.27    | 4.87  |
| wgrv.lgrv.mov | -3.15 | -0.96   | -0.34  | 0.47    | 4.17  |
Figure 5: Error ratios [%] for the case of low intensity: \( \lambda = 5 \)

For GRV and WGRV with shrinking cut-off ratio, we proved the asymptotic mixed normality. Hence, the distribution of the Studentized errors \( \Gamma^{-1/2} \sqrt{n}(V_n - \Theta) \) and \( \Gamma^{-1/2} \sqrt{n}(W_n - \Theta) \) are expected to follow the standard normally distribution.

Figure 6 shows QQ plots comparing theoretical quantiles of the standard normal distribution and the Studentized errors of GRV and WGRV estimators with shrinking threshold, BV and minRV. In this example, \texttt{wgrv.lgrv.mov} outperforms the others. It is close to the standard normal distribution. On the other hand, \texttt{grv.lgrv.mov} seems to deviate from \( N(0, 1) \). We can also see that \texttt{bv} are far from \( N(0, 1) \), implying that it is not appropriate even in the case of low intensity.
The important tuning parameter for the shrinking threshold GRV is the exponent $\delta_1$, an appropriate choice of which may strongly depend on the intensity of jumps. Recall that small $\delta_1$ means that we keep almost all the samples untrimmed. Table 6 shows average error ratios of GRV and WGRV with shrinking cut-off ratio for several values of intensity $\lambda$ and the parameter $\delta_1$. For moderate intensity ($\lambda = 5, 10$), the average ratios are not so large for small $\delta_1$. On the other hand, for high intensity ($\lambda = 30, 50$), this is not the case. Indeed, as for GRV, estimation
errors are quite large downward for small $\delta_1$. This can be interpreted that its multiplication of $q(\alpha_n)^{-1}$ for GRV is insufficient to compensate its elimination of jumps (small $\delta_1$ implies small $\alpha_n$, making $q(\alpha_n)^{-1}$ close to 1). Moreover, as for WGRV, there occur large upward biases for small $\delta_1$, since it keeps almost large increments and uses an extremely large increment for winsorization.

It is worth noting that large $\delta_1$ makes both GRV and WGRV accurate to a certain extent, even in the case of high intensity of jumps. Thus, in practice, one may use shrinking cut-off GRV and WGRV by setting the tuning parameter $\delta_1$ sufficiently close to $1/2$.

However, as Figure 7 implies, the errors are not normally distributed as theory predicts when the intensity of jumps is extremely high. We should be aware that GRV and WGRV with shrinking cut-off may suffer from some biases in the case of extremely intensive jumps. We may consider using a large fixed cut-off ratio (as discussed in the previous subsection) in such a situation.

Figure 7: QQ plot for Studentized errors: $\lambda = 50$, $\delta_1 = 0.49$
Table 6: Average error ratios [%] of GRV and WGRV with shrinking cut-off ratio

(a) GRV

| δ₀ | 5    | 10   | 30   | 50   |
|----|------|------|------|------|
| 0.10 | -0.41 | -3.07 | -23.89 | -43.57 |
| 0.20 | -0.28 | -1.56 | -18.95 | -39.12 |
| 0.30 | -0.58 | -0.63 | -11.40 | -31.14 |
| 0.40 | -1.22 | -0.63 | -2.91  | -17.39 |
| 0.45 | -1.57 | -1.09 | -0.57  | -8.76  |
| 0.49 | -1.82 | -1.50 | -0.43  | -3.01  |

(b) WGRV

| δ₀ | 5 | 10   | 30   | 50   |
|----|---|------|------|------|
| 0.10 | 2.03 | 5.88 | 60.71| 208.18 |
| 0.20 | 1.18 | 3.43 | 22.96| 66.24 |
| 0.30 | 0.43 | 1.76 | 10.27| 29.11 |
| 0.40 | -0.32| 0.60 | 4.30 | 11.13 |
| 0.45 | -0.80| 0.00 | 2.74 | 6.26  |
| 0.49 | -1.11| -0.46| 1.85 | 3.83  |

7.1.3 The case of constant volatility

Since we assumed that the volatility is location-dependent in the previous sections, the normalization by estimated spot volatilities is needed to obtain an accurate estimator. However, if the true volatility of data is constant, we may ignore normalization.

Here we set η = 0 so that the data is driven by a constant-volatility diffusion process. The intensity is λ = 30. The summary table of estimated values are shown in Table 7. Obviously, all types of GRV and WGRV outperform other estimators.

Figure 8 shows the error ratios of this case. The GRVs without normalization (grv[0.20], grv[0.10] and grv[0.05]) perform as well as those with normalization. This suggests that, if the true process can be thought as constant-volatility, we may skip normalization (calculation of spot volatilities) procedure.

However, it would be more typical that the volatility is non-constant. Thus, basically, it would be advisable to use normalization.
Table 7: Summary table of estimated values: $\lambda = 30$

|                  | Min. | 1st Qu. | Median | 3rd Qu. | Max. |
|------------------|------|---------|--------|---------|------|
| trv[0.45]        | 0.41 | 0.44    | 0.45   | 0.46    | 0.49 |
| trv[0.20]        | 0.97 | 1.11    | 1.16   | 1.22    | 1.45 |
| trv[0.10]        | 1.67 | 2.49    | 2.76   | 3.08    | 4.81 |
| bv               | 1.16 | 1.43    | 1.53   | 1.63    | 2.52 |
| mrv              | 0.92 | 1.03    | 1.07   | 1.13    | 2.52 |
| grv.lgrv[0.20]   | 0.94 | 1.01    | 1.05   | 1.07    | 1.17 |
| grv.mrv[0.20]    | 0.94 | 1.02    | 1.05   | 1.09    | 1.32 |
| wgrv.lgrv[0.20]  | 0.95 | 1.02    | 1.05   | 1.08    | 1.16 |
| wgrv.mrv[0.20]   | 0.95 | 1.02    | 1.05   | 1.08    | 1.17 |
| grv[0.20]        | 0.94 | 1.01    | 1.05   | 1.07    | 1.17 |
| grv[0.10]        | 0.95 | 1.02    | 1.06   | 1.08    | 1.16 |
| grv[0.05]        | 0.96 | 1.04    | 1.07   | 1.09    | 1.17 |
| grv.lgrv.mov     | 0.95 | 1.02    | 1.05   | 1.08    | 1.16 |
| wgrv.lgrv.mov    | 0.95 | 1.02    | 1.05   | 1.08    | 1.15 |
| True Value       | 1.00 | 1.00    | 1.00   | 1.00    | 1.00 |
7.2 The case of Neyman-Scott type clustering jumps

As the previous examples show, the minRV performs relatively well in the case of compound Poisson type jumps. However, even if the intensity of jumps is small, the minRV may suffer from an upward bias depending on the structure of jumps. In particular, if there are consecutive jumps (which is quite rare for compound Poisson processes), the minRV loses its advantage. Here we show an example of such a situation.

We consider the case that the data-generating process is given by $X = U + J$, where $U$ is the continuous part and $J$ is the jump part. Here we assume that $J$ is a marked Neyman-Scott clustering process (simply denoted by NS hereafter), instead of a compound Poisson process.

The NS process is a typical point process representing consecutive jumps. That is, there may be jumps within some consecutive intervals. This leads to upward bias of BV and minRV because the both of two adjacent increments can consist of large jumps. The NS process is constructed as follows.

(1) Set “centers” on the time interval $[0, 1]$ by a Poisson process $(N_t^0)$ with intensity $\lambda_0$. A center is defined as the point $t \in [0, 1]$ which satisfies $\Delta N_t = 1$. 

Figure 8: Error ratios [%] results for the constant volatility: $\lambda = 30$
(2) For each center \( c \in [0, 1] \), choose the number \( N_c \) of "children," assuming \( N_c \) is Poisson-distributed with mean \( \lambda_c \).

(3) For each center \( c \in [0, 1] \), generate independently and exponentially distributed random variables \( (v_i^{(c)})_{1 \leq i \leq N_c} \) with mean \( h \). Then the location of child \( i \) derived from center \( c \) is defined as \( c - v_i^{(c)} \). This defines the location of a jump.

(4) For each child \( i \), generate an independently and normally distributed random variable \( \xi_i \sim N(0, \nu_J^2) \). This determines the size and direction of a jump \( \Delta J_s \).

(5) The NS process is defined as \( J_t = \sum_{s \in [0,t]} \Delta J_s \).

We generate \( X = U + J \), where \( U \) is the Brownian semimartingale independent of \( J \), satisfying the stochastic differential equation

\[
dU_t = \theta U_t dt + (\sigma + \eta U_t^2)^{1/4} dw_t
\]

with \( U_0 = 1 \). We set \( \lambda_0 = \lambda_c = 5 \) and \( \nu_J = 0.5 \). For the continuous part \( U \), we use \( \theta = 0.2, \sigma = 1, \eta = 3 \). As before, the number \( n \) of samples is \( n = 2000 \), and the number of trials is 500.

|                  | Min.   | 1st Qu.  | Median  | 3rd Qu.  | Max.   |
|------------------|--------|----------|---------|----------|--------|
| trv[0.45]        | -97.04 | -88.67   | -82.29  | -75.74   | -59.83 |
| trv[0.20]        | -74.35 | -29.07   | -11.09  | 6.46     | 138.67 |
| trv[0.10]        | -68.02 | -14.24   | 3.17    | 24.43    | 157.66 |
| bv               | -54.60 | 5.52     | 27.19   | 69.17    | 369.71 |
| mrv              | -67.31 | -1.40    | 19.59   | 61.35    | 300.83 |
| grv.lgrv[0.20]   | -74.39 | -31.45   | -14.02  | 3.64     | 136.14 |
| grv.mrv[0.20]    | -70.53 | -26.44   | -9.19   | 8.39     | 139.11 |
| wgrv.lgrv[0.20]  | -74.49 | -31.31   | -13.35  | 4.32     | 137.83 |
| wgrv.mrv[0.20]   | -74.48 | -30.70   | -12.73  | 4.23     | 136.96 |
| grv[0.20]        | -74.89 | -33.38   | -16.67  | 0.73     | 134.64 |
| grv[0.10]        | -74.70 | -32.63   | -15.03  | 1.88     | 136.44 |
| grv[0.05]        | -74.38 | -31.68   | -14.04  | 3.40     | 136.67 |
| grv.lgrv.mov     | -74.50 | -31.63   | -13.79  | 3.84     | 138.23 |
| wgrv.lgrv.mov    | -74.32 | -30.82   | -12.89  | 4.34     | 139.78 |
Table \(8\) and Figure \(9\) show the error ratios in the case of NS jumps. Because of the possible consecutive jumps, both bipower variation and minRV have upward bias, whereas GRV and WGRV are all robust to such clustering jumps. This suggests that the GRV and WGRV perform very well for various structures of jumps.

Figure 9: Error ratios [%] for the case of Neyman-Scott clustering jumps

### 7.3 A remark on estimation of spot volatilities

Finally, we argue how estimation of spot volatilities affect the accuracy of GRV and WGRV.

We have used \(\kappa_n = [Bn^{0.45}] = 305\) for local GRV and local minRV and seen that GRV and WGRV with these spot volatilities perform highly well. However, the choice of \(\kappa_n\) may affect the accuracy of GRV and WGRV. In fact, if the true volatility varies greatly, a wide subinterval (a large \(\kappa_n\)) leads to imprecise estimation of spot volatilities and causes misdetection of jumps by using such information. Therefore, it ends up obtaining biases of GRV and WGRV.

To see this, consider the following SDE:

\[
dX_t = \theta X_t dt + (\sigma + \eta \sin^2 X_t)dw_t + dJ_t,
\]
where \( J_t = \sum_{j=1}^{N_t} \xi_j \) is the same compound Poisson process with intensity \( \lambda \) as in Section 7.1. We set \( \sigma = 1, \eta = 5, \lambda = 10, \mu = 0.3, \nu = 0.2 \). Again, the number \( n \) of samples is \( n = 2000 \), and the number of trials is 500. In this example, the volatility \((\sigma + \eta \sin^2 X_t)^2\) swings in the range \([1, 36]\). A sample path of this model is shown in Figure 10. The volatility alternates between low and high in short time intervals, so the estimation of spot volatility requires an appropriate choice of \( \kappa_n \).

![Sample path of X](image1)

![Increment of X](image2)

(a) Sample path of \( X \)  
(b) Increment of \( X \)

Figure 10: Sample path of \( X \) and its increments

Table 9 shows the summary and average error ratios of GRV and WGRV, respectively, for several values of \( c \) and \( B \) that determine the width \( \kappa_n = 2\kappa_n + 1 \) of subintervals for spot volatility estimation. This indicates that large \( B \) and \( c \) (wide subinterval) tend to give imprecise estimates. Since the volatility varies in a wide subinterval as Figure 11 shows, the estimated spot volatility is prone to deviate the true value. This leads to misdetection of jumps, and thus distorts the estimate of GRV and WGRV. For instance, an underestimated spot volatility makes normalized increments too large, so the increments are likely to be regarded as jumps and eliminated from calculation of the estimates. As a result, GRV and WGRV are underestimated. In this example, it seems that small values such as \( c = 0.1, 0.2 \) and \( B = 1, 5 \) are preferable.

This example suggests that we should choose the tuning parameters \( B \) and \( c \) carefully, especially when volatility switches between high and low states frequently. After all, the proper choice of tuning parameters, such as \( B \) and \( c \), while observing the data in detail, is needed to obtain precise estimates by GRV and WGRV.
Table 9: Average error ratios of GRV and WGRV for different $c$ and $B$ determining the width $\kappa_n$

(a) GRV with local GRV ($\text{grv\.lgrv}$)

| $B$ | $c$  | 1   | 5    | 10   | 20   |
|-----|-----|-----|------|------|------|
| 0.10| 3.41| -5.80| -9.31| -15.17|
| 0.20| -1.19| -9.31| -15.17| -24.28|
| 0.30| -5.26| -16.48| -26.09| -36.26|
| 0.40| -9.31| -27.63| -37.64| -44.13|
| 0.45| -12.36| -33.71| -42.02| -46.39|
| 0.49| -15.46| -37.92| -44.30| -47.37|

(b) GRV with local minRV ($\text{grv\.mrV}$)

| $B$ | $c$  | 1   | 5    | 10   | 20   |
|-----|-----|-----|------|------|------|
| 0.10| 6.21| -4.21| -6.92| -11.85|
| 0.20| 0.69| -6.92| -11.85| -20.91|
| 0.30| -3.85| -13.08| -22.75| -33.82|
| 0.40| -6.92| -24.44| -35.37| -43.14|
| 0.45| -9.33| -31.01| -40.38| -46.01|
| 0.49| -12.11| -35.72| -43.37| -47.22|

(c) WGRV with local GRV ($\text{wgrv\.lgrv}$)

| $B$ | $c$  | 1   | 5    | 10   | 20   |
|-----|-----|-----|------|------|------|
| 0.10| 15.27| 1.33| -3.69| -9.64|
| 0.20| 8.81| -3.69| -9.64| -17.78|
| 0.30| 1.88| -10.83| -19.33| -28.89|
| 0.40| -3.69| -20.74| -30.20| -37.17|
| 0.45| -6.97| -26.38| -34.80| -39.68|
| 0.49| -9.94| -30.50| -37.36| -40.95|

(d) WGRV with local minRV ($\text{wgrv\.mrV}$)

| $B$ | $c$  | 1   | 5    | 10   | 20   |
|-----|-----|-----|------|------|------|
| 0.10| 22.42| 3.91| -1.64| -7.54|
| 0.20| 13.26| -1.64| -7.54| -15.38|
| 0.30| 4.77| -8.71| -16.88| -26.58|
| 0.40| -1.64| -18.29| -28.05| -35.98|
| 0.45| -4.96| -23.98| -33.10| -39.12|
| 0.49| -7.78| -28.37| -36.22| -40.71|

8 Concluding remarks

In this paper, we construct the global realized volatility estimator in the nonparametric context. We proved the consistency and the asymptotic normality of GRV and WGRV, and, by numerical simulations, we show that these new approaches outperform previous studies which use increments within a single or two intervals.

Our new approach for eliminating jumps is highly versatile. For example, by normalization, it works well when the volatility of data is driven by a nonconstant-volatility process. Moreover, both GRV and WGRV are accurate enough in the case of not only compound-Poisson sporadic jumps but also Neyman-Scott consecutive jumps.
The global-filtering method could be extended to the covariance estimation even under the nonsynchronous sampling scheme. Furthermore, this approach could also be applied to construct a test statistic for jump. Also, it is valuable to apply our approach to empirical research of high-frequency time series data. These are important topics for future research.

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