Research on Solution Space of Bipartite Graph Vertex-Cover by Maximum Matchings

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Abstract. Some rigorous results and statistics of the solution space of Vertex-Covers on bipartite graphs are given in this paper. Based on the König’s theorem, an exact solution space expression algorithm is proposed and statistical analysis of the nodes’ states is provided. The statistical results fit well with the algorithmic results until the emergence of the unfrozen core, which makes the fluctuation of statistical quantities and causes the replica symmetric breaking in the solutions. Besides, the entropy of bipartite Vertex-Cover solutions is calculated with the clustering entropy using a cycle simplification technique for the unfrozen core. Furthermore, as generalization of bipartite graphs, bipartite core graph is proposed, the solution space of which can also be easily determined; and based on these results, how to generate a König–Egerváry subgraph is studied by a growth process of adding edges. The investigation of solution space of bipartite graph Vertex-Cover provides intensive understanding and some insights on the solution space complexity, and will produce benefit for finding maximal König–Egerváry subgraphs, solving general graph Vertex-Cover and recognizing the intrinsic hardness of NP-complete problems.

Keywords: disordered systems (theory), classical phase transitions (theory), cavity and replica method, phase diagrams (theory)
1. Introduction

Minimum vertex cover (Vertex-Cover) problem, as one of Karp’s 21 NP-complete problems and a classical graph theoretical computational problem [1], has a central status in the research of computational complexity and attracts the interests of many mathematicians, physicists and computer scientists, which also has a large number of applications such as immunization strategies in networks [2], the prevention of denial-of-service attacks [3] and monitoring of internet traffic [4]. Till now, it is known that Vertex-Cover problem is NP-complete even in cubic graphs [5] and planar graphs [6]. But for bipartite graphs, König’s theorem proves a fact that its minimum coverage is equal to its maximum matchings [7]. Besides, for graphs without leaf-removal core, a simple algorithm can help to find a minimal vertex-cover in polynomial time [8].

In statistical mechanics, there are a lot of results obtained on the research of Vertex-Cover problem. In [9, 10], theoretical analysis of Vertex-Cover problem on random graphs is provided to investigate how to solve it efficiently. In [11], a significant structure named long-range frustration, is proposed to explain the strong correlations among the nodes in Vertex-Cover, and in [12] and [13], the replica symmetric breaking (RSB) analysis on the minimum coverage and its entropy of random graphs is discussed. Recently, another relationship between the nodes, mutual-determination, is investigated to detect the solution space of Vertex-Cover [14], which can express the solution space exactly when the graph has no leaf-removal core. And, a solution number counting algorithm is given based on the solution space expression of Vertex-Cover [15]. In these research, most results can be proved right only when the graph is of simple structure/topology, such as trees and graphs with no leaf-removal core, which has to ensure the validity of replica symmetry assumption. Compared with the Vertex-Cover problem, the satisfiability problem especially k-SAT problem is known as another famous NP-complete problem [16], but k-XORSAT problem can be solved in polynomial time which undergoes both the replica symmetry and one step replica symmetric breaking phases (1-RSB) [17]. Can we find a similar easily-solving subset of Vertex-Cover like that k-XORSAT in k-SAT?

Based on the König’s theorem [7] and the existing reduced solution graph analysis [15], the complete solution space of Vertex-Cover on bipartite graphs will be studied in the paper, and a polynomial-time algorithm to obtain the exact solution space expression will be provided. By the results, statistical analysis of the nodes’ states in the solution space is done, which will lose its effectiveness when a structure named unfrozen core emerges. The existence of the unfrozen core accords with the replica symmetric breaking, as some nodes can have very large influence on others of the graph, which implicates the long-range correlation among nodes. Besides, the entropy and the clustering entropy of bipartite graph Vertex-Cover are investigated. As some generalizations, some easily-solving Vertex-Cover instances are constructed and how to generate a König–Egerváry subgraph is discussed.
2. Vertex-Cover solution space of bipartite graphs

As is known to all, the classical Vertex-Cover problem is one of the six basic NP-complete problems [6]. For arbitrary graphs, till now there is no algorithm in polynomial time to find the minimal vertex-covers, and to detect its solution space is even harder. In this section, we will focus on minimal Vertex-Cover on bipartite graphs which belongs to the $P$ class in computational complexity, and achieve some rigorous results on the solution space and its statistics.

2.1. Rigorous solution space expression of Vertex-Cover on bipartite graphs

In [14], the reduced solution graph is defined to describe the solution space of Vertex-Cover: double edges are used to identify the mutual-determinations, in which two ends of the edge are unfrozen and can determine the states of each other; red or black nodes are used to identify the positive or negative backbones, which should be always uncovered or covered in the solution space of Vertex-Cover; single edges retained from the original graph keep the relationship among the nodes by Vertex-Cover. By the results in [14], the reduced solution graph can exactly express the whole solution space of Vertex-Cover when there is no leaf-removal core [8] on the given graph. And, in the following, this result can be generalized to bipartite graphs.

**Theorem:** The solution space of Vertex-Cover for bipartite graphs with $n$ nodes can be exactly represented by the reduced solution graph in $O(n^2)$ steps, i.e., the states (unfrozen or backbone) of any node can be easily determined for bipartite graphs.

**proof:** Our inference is based on a very important theorem in graph theory, the König’s theorem [7]: In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex-cover. And, the maximum matching can be solved by an algorithm within polynomial time for general graphs [18], especially for the bipartite graph it can be solved in $O(n^2)$ steps by the Hungarian algorithm [19].

For a bipartite graph $G$, we can first get a maximum matching by the Hungarian algorithm, e.g., $(i_1, j_1), (i_2, j_2), \cdots, (i_m, j_m)$. By the König’s theorem, if the maximum matching has $m$ independent edges (edges without common ends), there should be $m$ covered nodes in the minimal vertex-covers, which means that each independent edge must have one and only one covered node. Then, there should be one and only one covered node in each such pair of nodes $(i_k, j_k), k = 1, \cdots, m$, which suggests that the relation in each pair of matched nodes should be mutual-determinations (double edges) in the solution space of Vertex-Cover; and no other nodes except $i_1, j_1, i_2, j_2, \cdots, i_m, j_m$ can be covered in the minimal vertex-covers, which suggests that all the other nodes except those matched have to be always uncovered (positive backbones) in the solution space of Vertex-Cover.

For the obtained graph, if the positive backbones have neighbor nodes on the original graph $G$, these neighbors should have their states to be always covered (negative backbones); if any new produced negative backbone, say $i_k$, has one mutual-
Figure 1. An instance for the process of obtaining the reduced solution graph of Vertex-Cover for bipartite graphs.

determination $(i_k,j_k)$ neighbor node $j_k$, $j_k$ should be always uncovered (positive backbones) by the mutual-determination relation. This rule is followed by the requirements of Vertex-Cover and should be performed until no such conditions exist, which costs at most $O(n)$ steps.

**Bipartite Graph Solution Space Expression Algorithm**

**INPUT:** Bipartite graph $G$

**OUTPUT:** Reduced solution graph $\mathcal{S}(G)$ of $G$

**Freezing-Influence** (Graph $G$, Matching $\mathcal{M}$)

begin

$\mathcal{S}(G) =$ make all matching edges double edges

while (unmatched nodes exist)

$\mathcal{S}(G) =$ make all unmatched nodes uncovered backbones

$\mathcal{S}(G) =$ make uncovered backbones’ neighbors covered backbone

while (any double edge $(i,j)$ has node $i$ covered backbone)

$\mathcal{S}(G) =$ make $j$ uncovered backbone

$\mathcal{S}(G) =$ make $j$’s neighbors covered backbone

return($\mathcal{S}(G)$)

end

**main**

begin

$\mathcal{M} =$maximum matching of $G$

$\mathcal{S}(G) =$Freezing-Influence($G$, $\mathcal{M}$)

end

Then, the final obtained graph is the reduced solution graph of $G$. As the whole process is done according to the Kőnig’s theorem and the requirements of Vertex-Cover,
the reduced solution graph is an exact representation of the solution space of Vertex-Cover and the total time consumption is at most $O(n^2)$. □

According to the proof of the above theorem, a Bipartite Graph Solution Space Expression Algorithm is provided to obtain the reduced solution graph of a given bipartite graph $G$, and an instance for this process is shown in Fig.1.

2.2. Statistical analysis of bipartite Vertex-Cover nodes’ states

Denote $G = (X_1, X_2, E)$ as a bipartite graph with two independent node sets $X_1$ and $X_2$, and the edge set $E$ only involves edges connecting $X_1$ and $X_2$. Define $n_1, n_2$ as the sizes of $X_1$ and $X_2$, $m$ as the size of $E$, $c_1 = \frac{m}{n_1}$, $c_2 = \frac{m}{n_2}$ as the average degrees in $X_1$ and $X_2$, $n = n_1 + n_2$ as the total nodes number, and the average degree for the whole graph is $c = 2c_1c_2 / (c_1 + c_2)$. Then, the probability $p$ for the appearance of each edge satisfies $p = \frac{m}{n_1 \times n_2} = \frac{c_1 n_2}{n_1}$, and the degree distributions of $X_1$ and $X_2$ are separately $p_1(k) = C^n_k p^k (1 - p)^{n-k} = \frac{c_1}{e^{c_1} k!}$ and $p_2(k) = \frac{c_2}{e^{c_2} k!}$ when $n_1, n_2$ are sufficiently large [20].

Let $Q$ be the size of giant connected component of the bipartite graph, $Q_1$ the node ratio in $X_1$ and $Q_2$ for $X_2$. Thus, one node in $X_1$ belongs to the giant connected component, if and only if at least one of its neighbors in $X_2$ is in the giant connected component. So, we have

$$Q_1 = 1 - \sum_k p_1(k) (1 - Q_2)^k = 1 - e^{-c_1 Q_2}, \quad (1)$$

$$Q_2 = 1 - \sum_k p_2(k) (1 - Q_1)^k = 1 - e^{-c_2 Q_1}. \quad (2)$$

Thus, the size of giant component for the whole graph $G$ is $Q = \frac{Q_1 c_2 + Q_2 c_1}{c_1 + c_2}$.

Next, we will calculate the coverage ratio of a random bipartite graph. Taking the analysis in [21], define $\pi_1$ as the probability of an edge entering a node in $X_1$ with coverage requirement, i.e., it has not been covered, and $\pi_2$ is the same but with entering node in $X_2$. Thus, an edge entering a node has coverage requirement only when the other end of the edge has no coverage requirement from the rest edges, and we have

$$\pi_1 = \sum_{k_1=0}^{\infty} q_1(k_1)(1 - \pi_2)^{k_1}, \quad (3)$$

$$\pi_2 = \sum_{k_2=0}^{\infty} q_2(k_2)(1 - \pi_1)^{k_2}, \quad (4)$$

where $q_i(k) = (k + 1)p_i(k + 1)/c_i = p_i(k), \ i = 1, 2$ are the probabilities of an edge entering a node with degree $k + 1$ in $X_i$. Then, the above equations can be simplified by

$$\pi_1 = e^{-c_1 \pi_2}, \quad \pi_2 = e^{-c_2 \pi_1}. \quad (5)$$
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Using the analysis above, a node \( i \) in \( X_1 \) should be uncovered only in the following two cases: all its neighbors have coverage requirements from their edges; all its neighbors except one (say \( j \)) have coverage requirements from their edges, and at this time nodes \( i \) and \( j \) can either be covered. The analysis is the same for nodes in \( X_2 \), so we have the coverage ratios in \( X_1 \) and \( X_2 \) as

\[
x_1 = 1 - \sum_{k=0}^{\infty} p_1(k)(1-\pi_2)^k - \sum_{k=1}^{\infty} p_1(k)(1-\pi_2)^{k-1}\frac{k\pi_2}{2} = 1 - e^{-c_1\pi_2} - \frac{c_1\pi_2}{2} e^{-c_1\pi_2},
\]

(6)

\[
x_2 = 1 - \sum_{k=0}^{\infty} p_2(k)(1-\pi_1)^k - \sum_{k=1}^{\infty} p_2(k)(1-\pi_1)^{k-1}\frac{k\pi_1}{2} = 1 - e^{-c_2\pi_1} - \frac{c_2\pi_1}{2} e^{-c_2\pi_1}.
\]

(7)

Thus, the coverage ratios of the whole graph \( G \) is \( x = \frac{x_1 + x_2}{c_1 + c_2} \).

Using similar analysis in \([11, 14]\), we derive the ratios of backbones and unfrozen nodes in the following. Let \( q_1^+, q_1^-, q_0^+, q_2^+, q_2^- \) be the ratios of uncovered backbones, covered backbones and unfrozen nodes in \( X_1 \) and \( X_2 \) separately, and evidently \( q_1^+ + q_1^- + q_0^+ = 1, q_2^+ + q_2^- + q_0^0 = 1 \). Then, one new added node should be an uncovered backbone only when all its neighbors are unfrozen or covered backbones; one new added node should be an unfrozen node only when there is exactly one uncovered backbone in its neighbors. Thus, we have

\[
q_1^+ = \sum_{k=0}^{\infty} p_1(k)(1-q_2^+)^k = e^{-c_1 q_2^+},
\]

\[
q_2^+ = \sum_{k=0}^{\infty} p_2(k)(1-q_1^+)^k = e^{-c_2 q_1^+},
\]

(8)

\[
q_1^0 = \sum_{k=1}^{\infty} p_1(k)k q_2^+(1-q_2^+)^{k-1} = c_1 q_2^+ e^{-c_1 q_2^+},
\]

\[
q_2^0 = \sum_{k=1}^{\infty} p_2(k)k q_1^+(1-q_1^+)^{k-1} = c_2 q_1^+ e^{-c_1 q_1^+}.
\]

(9)

And, the ratios of the uncovered backbones and unfrozen nodes for the whole graph \( G \) are separately \( q^+ = \frac{q_1^+ + q_2^+}{c_1 + c_2} \) and \( q^0 = \frac{q_1^0 + q_2^0}{c_1 + c_2} \). Then, the coverage ratio of the whole graph \( G \) is \( x = 1 - q^+ - q^0/2 \), and it is easy to see that the results of coverage by equations (6-7) and equations (8-9) are actually the same. By the shape of equations (5) and (8), the probability of an edge entering a node with coverage requirement is the same as the probability of a node being positive backbone.

For Vertex-Cover of random graphs as we know, when there is a leaf-removal core, as the existence of long-range correlations \([8, 11, 22]\), we cannot obtain the exact results of covered and uncovered backbones, and the key difficulty is how to deal with the correlations among the neighbors of a node. Fortunately, the correlation among the
neighbors is easy to deal with on the bipartite graphs: if a new added node \( i \) has no neighbors of positive backbones, \( i \) being uncovered requires that all its neighbors change to be covered, and it will cause no confliction among these neighbors. Assuming two unfrozen neighbors \( j_1, j_2 \) of \( i \) cannot be covered simultaneously, i.e., \( j_1 \) being covered causes an influence propagation to affect the state of \( j_2 \) and requires \( j_2 \) being uncovered, by the organization of the reduced solution graph, the number of nodes in the influence chain between \( j_1 \) and \( j_2 \) should be even, e.g., the intermediate nodes \( k_1, k_2, \ldots, k_{2n} \), and we can find that nodes \( i, j_1, k_1, k_2, \ldots, k_{2n}, j_2 \) form a cycle having \( 2n + 3 \) nodes, which conflicts with the organization of bipartite graph. Thus, we can always have that the unfrozen neighbors of a node can be covered simultaneously for bipartite graphs. Therefore, the long-range correlations of Vertex-Cover on bipartite graphs are easygoing and produce no trouble for determining nodes’ states by a node-adding process [11, 14]. The local evolution of the unfrozen node and negative backbone can be obtained by the same analysis.

In Fig.2 the results of minimal coverage, unfrozen nodes and negative backbones on random bipartite graphs are provided, which fit very well with the analysis of equations in this subsection. These statistical quantities evolve quite like those on random graphs [11], but the minimal coverage becomes almost unchangeable after some average degrees: for \( c_1 : c_2 = 4 : 1 \), the minimal coverage almost stays to be \( x = 0.2 \) when \( c >\sim 3.3 \); for \( c_1 : c_2 = 4 : 2 \), the minimal coverage almost stays to be \( x = \frac{1}{3} \) when \( c >\sim 5.5 \); for

Figure 2. The results of minimal coverage, unfrozen nodes and negative backbones for different ratios of \( n_1 : n_2 = 4:1, 4:2, 4:3 \) with different average degrees. The theoretical results are shown by lines with different colors and the experimental results are by data with red symbols. The experimental results fit very well with those by equations (6-9). All the experimental results are obtained by 1000 instances with size \( n_1 + n_2 = 2000 \).
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The results of minimal coverage, unfrozen nodes and negative backbones for \( n_1 : n_2 = 1 \) with different average degrees, shown separately by blue, green and black lines/symbols. The experimental results fit well with those by equations (7-11) when \( c < e \), but diverge when the average degree gets larger. All the experimental results are obtained by 1000 instances with size \( n_1 + n_2 = 2000 \).

When \( c_1 : c_2 = 4 : 3 \), the minimal coverage almost stays to be \( x = \frac{3}{7} \) when \( c \sim 8.5 \). By the organization of bipartite graphs, it suggests that after some large average degrees, the minimal coverage, i.e., number of maximum matchings, should be fulfilled by the nodes in the independent set with smaller size; on the reduced solution graph, after some average degrees, almost all the nodes in the smaller independent set become negative backbones, and almost all the nodes in the other part become positive backbones.

2.3. Fluctuation and unfrozen core of bipartite Vertex-Cover solution space

In the above subsection, a statistical analysis for bipartite Vertex-Cover is provided with unequal sizes of two independent sets. When \( c_1 : c_2 = 1 : 1 \), the coverage obtained by these equations also fits very well with the experiments, but it is surprising that the results of equations (8-9) deviate from the experimental ones on calculating positive backbones and unfrozen nodes when the average degree \( c > e \), which are shown in Fig.3. In this subsection, a new concept, unfrozen core, will be introduced to explain this phenomenon.

Unfrozen core, an organization on the reduced solution graph, is the leaf-removal core for the unfrozen nodes. In the unfrozen core, some unfrozen nodes couple together and have close connections, which suggests that there are strong correlations among them. When a new node is added with connection to the unfrozen core, it being uncovered will lead to a large number of nodes in the unfrozen core to be frozen. Thus, the unfrozen core is very vulnerable under the local evolution of the node states.
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Ave.\( \text{Deg} \)  \( c = 0.5 \)  \( c = 1 \)  \( c = 1.5 \)  \( c = 2 \)  \( c = 2.5 \)  \( c = 3 \)  \( c = 3.5 \)
\( \rho(\text{big ratio}) \)  1  1  1  1  1  0.919  0.930

Ave.\( \text{Deg} \)  \( c = 4 \)  \( c = 4.5 \)  \( c = 5 \)  \( c = 5.5 \)  \( c = 6 \)  \( c = 6.5 \)  \( c = 7 \)
\( \rho(\text{big ratio}) \)  0.914  0.877  0.860  0.796  0.702  0.636  0.458

Ave.\( \text{Deg} \)  \( c = 7.5 \)  \( c = 8 \)  \( c = 8.5 \)  \( c = 9 \)  \( c = 9.5 \)  \( c = 10 \)  \( c = 10.5 \)
\( \rho(\text{big ratio}) \)  0.380  0.231  0.164  0.095  0.068  0.030  0.021

Ave.\( \text{Deg} \)  \( c = 11 \)  \( c = 11.5 \)  \( c = 12 \)  \( c = 12.5 \)  \( c = 13 \)  \( c = 13.5 \)  \( c = 14 \)
\( \rho(\text{big ratio}) \)  0.019  0.005  0.003  0.001  0.001  0.000  0.000

Table 1. The sizes \( \rho \) of big ratio class of positive backbones for different average degrees. All the experimental results are obtained by 1000 instances with size \( n_1 + n_2 = 2000 \).

In Fig.4, an interesting fluctuation phenomenon emerges: when the average degree is small (\( c = 2 \)), the positive backbone ratios (the blue cycles) fluctuate around the average value (the red line) and the average is a typical value; when the average degree gets larger (\( c = 6 \)), the positive backbone ratios split into two different classes, in which some are around 0.5 (big ratio) and the others are around 0 (small ratio); when \( c = 10 \), this phenomenon gets clearer and the class of big ratio greatly reduces; when \( c = 14 \), the big ratio class disappears. In Table.I, the sizes \( \rho \) of the big ratio class are provided to show its evolution.

In Fig.5, the experimental results of sizes for giant component, leaf-removal core and unfrozen core on random bipartite graphs with \( n_1 : n_2 = 1 : 1 \) are provided, in
which the emergence of leaf-removal core and unfrozen core is simultaneously. By the analysis in [14, 15], the emergence of the unfrozen core means the existence of long-range correlations.

For any specific instance, the local evolution of the node states can be exact, which derives from the theorem of the reduced solution graph for bipartite graphs. Why the statistical results for positive backbones/unfrozen nodes in equations (8-9) lose their effect? For statistics in equations (8-9), there is a basic assumption that adding a new node cause little influence on the macroscopic statistical results. However, when the unfrozen core exists and a new node is added, if the new node is connected to the unfrozen core, there is a large probability that many nodes' states will be changed to be frozen in the core, which is shown in Fig. 4 by the existence of big ratio class of positive backbones. Indeed, by the results in Fig. 4 and Fig. 5 when $c > e$, some instances have unfrozen core, but the others have no unfrozen core as many nodes in the core are forced to be frozen by some few nodes and the core is broken. So, one single node in microscopic scale, the influence of which should be neglected in statistics, may have profound influence on the statistical results in macroscopic scale when the unfrozen core exists. Thus, the unfrozen core can survive after the node-adding process for some instances, and for other instances, many of the unfrozen cores get damaged or even disappear, which are determined by the specific details of the organizations of bipartite graphs. Therefore, the statistical results by equations (8-9) for the node states are unstable and not exact when the unfrozen core exists after $c > e$. Similar as the analysis in [17], the existence of the unfrozen core is quite like the leaf-removal core in $k$-XORSAT, under which the replica symmetric breaking works. The long-range
correlation among nodes in the unfrozen core can also be reflected in the core entropy calculated in next section.

3. Counting the solution number of bipartite Vertex-Cover

By the above analysis, we can see that there is no odd-cycle-breaking \([14]\) in obtaining the reduced solution graph of bipartite Vertex-Cover, and reduced solution graph for any bipartite graph can be exactly achieved. To see the organization complexity of the solution space of Vertex-Cover on bipartite graphs, the entropy \(h_s = \frac{\log S_n}{n}\) of whole solution space and core entropy \(h_c = \frac{\log S_c}{n}\) \([17]\) are studied in this section, in which \(S_n, S_c\) are separately the total number of minimal vertex-covers and just the number of minimal vertex-covers in the unfrozen core.

For an obtained reduced solution graph (neglecting the backbones), to know the core entropy, the leaf-removal unfrozen core should be achieved first, which is shown in Fig.5 with instances in Fig.6 (a-c). However, by the results in Fig.5 not all nodes in the leaf-removal core belong to the unfrozen core, the unfrozen core excludes the backbones and performs as the intrinsic complexity, and the nodes in the unfrozen core have strong correlations and long-range influence \([15]\). By the knowledge of Vertex-Cover on random graph cases \([9, 10, 11]\), solution space (ground states in the terminology of statistical mechanics) undergoes the replica symmetric breaking phenomenon when the leaf-removal core exists at the average degree \(c > e\), and it should have a replica symmetry phase when there is no leaf-removal core. So, for the strong correlations among the nodes in the unfrozen core, we can recognize that the clustering entropy \([16]\) is caused by the minimal vertex-covers in the unfrozen core, i.e., each proper assignment in the unfrozen core leads a cluster of solutions on the whole reduced solution graph, and the core entropy is indeed the clustering entropy.

To calculate the core entropy \(h_c\), we indeed calculate the number of minimal vertex-covers in the unfrozen core of the reduced solution graph. Here, a technique named cycle simplification can be performed: when there is some cycle with alternatively double and single edges in the unfrozen core, the nodes on the cycle mutually determine each other, and they can be simplified by two nodes. In Fig.6 (a-b), for the six nodes in the unfrozen core, it is easy to verify that the yellow nodes \(i_1, i_2, i_3\) must take the same value in the solution space and so do the green nodes \(j_1, j_2, j_3\), and in Fig.6 (c) one yellow node \(i\) and one green node \(j\) are used to substitute these two classes of nodes, in which nodes outside the unfrozen core connect \(i\) when it originally connects \(i_k, k = 1, 2, 3\) or \(j\) when it originally connects \(j_k, k = 1, 2, 3\). It is evident that the cycle simplification does not change the solution number but leads to new reduced solution graph with smaller size. Then, for the simplified core, the Vertex-Cover Solution Number Counting Algorithm \([15]\) can be used to calculate the clustering entropy \(h_c\). Then, a result for the unfrozen core of the reduced solution graph after cycle simplification will be given:

**Proposition:** After the cycle simplification operations for \(\mathcal{S}(G)\), the new reduced solution graph \(\mathcal{S}'(G)\) has no unfrozen core, i.e., all the unfrozen nodes can be removed
after the leaf-removal steps.

**proof**: Here, we use proof by contradiction and assume that there can be leaf-removal unfrozen core $U_c(G)$ after all possible cycle simplification operations. In the following, we will prove that there must be cycles satisfying the requirement of cycle simplification in $U_c(G)$. For the unfrozen core $U_c(G)$ of $\mathcal{S}''(G)$, as the connectivity of all the nodes in it, a spanning tree $T$ containing all the double edges of $U_c(G)$ can be easily obtained. For there is no leaf in $U_c(G)$, any additional single link of $U_c(G) - T$ must produce cycles.

For a pendant $l_1$ of one leaf as the end of one branch of $T$, there must be some edge $e_1$ in $U_c(G) - T$ with end $l_1$, which connects some other node in $T$ and form a cycle with $l_1$, and then we consider $T \cup e_1$. When the other end (except $l_1$) of edge $e_1$ is one of its ancestors in its branch, by the bipartite property, the resulted cycle must be
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Experimental Results of the Unfrozen Core Entropy of Bipartite Vertex-Cover

Instances with $c=3$

Instances with $c=3.5$

Instances with $c=4$

Instances with $c=4.5$

Experimental Results of Bipartite Vertex-Cover Entropy

Figure 7. The experimental results of entropy and core entropy of bipartite Vertex-Cover for random bipartite graphs of $n_1 : n_2 = 1$ with different average degrees. All the experimental results are obtained by 1000 instances with size $n_1 + n_2 = 2000$.

For the entropy $h_s = \log_2 \frac{S_n}{n}$, the number of solutions does not change after the cycle simplifications, the simplified unfrozen core with the removed leaves should be consider together, and the calculation of $h_s$ is mainly based on the Vertex-Cover Solution Number Counting Algorithm [15], which is a complete algorithm when the reduced solution graph is exact and performs well when there is no unfrozen core. In Fig.7, the entropy of random bipartite Vertex-Cover with equal independent sets is provided, which evolves similarly as that of random graphs [13]; the core entropy for 1000 instances with average degrees $c = 3, 3.5, 4, 4.5$ are also given and most instances have very low entropy (very few solutions). As the average degree increases, the unfrozen core increases as that in
Fig. but the solutions in the unfrozen core are very few, which also reveals the strong correlations among nodes in the unfrozen core.

4. Easy-solving Vertex-Cover instances and König – Egerváry subgraph

4.1. Vertex-Cover solution space of bipartite core graph

By the results in [8] and [11], we know that the minimal vertex-covers can be easily obtained when there is no leaf-removal core for a graph. And, by the analysis of the above sections, the solution space of the bipartite graph can be easily obtained by the maximum matchings. Thus, if a graph has no leaf-removal core or the leaf-removal core is a bipartite graph, we name it as bipartite core graph, and its solution space expression/reduced solution graph can be exactly achieved using the Mutual-determination and Backbone Evolution Algorithm [14] and the Bipartite Graph Solution Space Expression Algorithm.

Indeed, these two algorithms can be combined simply using the idea of maximum matching for the bipartite core graph: each leaf (a petiole with one pendant point connecting it) can be viewed as one matching, so the maximum matching number (except the core) is the same as the number of leaves; for the bipartite leaf-removal core, the maximum matchings are easily to be obtained. By the minimal coverage, the covered nodes should be in these two kinds of matchings, and nodes that are not contained in the matchings must be uncovered backbones. Thus, for a bipartite core graph $G$, we can first find its maximum matchings, mark them as double edges and unmatched nodes uncovered backbones; then consider the freezing influence propagation caused by the uncovered backbones, and fix the states of some nodes like that in the Bipartite Graph Solution Space Expression Algorithm; at last, check whether the odd cycle breaking operation [14] should be performed, and do the freezing influence operation [14] for the new fixed backbones. Finally, the obtained compatible graph is the reduced solution graph $\mathcal{S}(G)$, and the algorithm is given as follows.

As the leaves of a bipartite core graph may not be bipartite, e.g., odd cycles can exist (that is why odd-cycle-breaking operations are needed), the bipartite core graph involves not only the bipartite graph and it belongs to the König – Egerváry graph. On the contrary, a König – Egerváry graph cannot always be a bipartite core graph, and the structures or characteristics of the König – Egerváry graph are not very clear till now [24]. However, if it is known that $G$ is a König – Egerváry graph, its solution space can be determined similarly as the Bipartite Core Graph Solution Space Expression Algorithm, as all the covered nodes must be involved in the maximum matchings.
Bipartite Core Graph Solution Space Expression Algorithm

INPUT: Bipartite core graph $G$
OUTPUT: Reduced solution graph $\mathcal{S}(G)$ of $G$

begin
    Find all the leaves by the leaf-removal algorithm
    $\mathcal{M}_1$: assign one matching in each leaf
    $\mathcal{M}_2$: maximum matching of bipartite leaf-removal core of $G$
    $\mathcal{M} = \mathcal{M}_1 \bigcup \mathcal{M}_2$
    $\mathcal{S}(G) = \text{Freezing-Influence}(G, \mathcal{M})$
    while (Odd-Cycle-Breaking needed)
        $\mathcal{S}(G) = \text{Odd-Cycle-Breaking}(\mathcal{S}(G))$
        $\mathcal{S}(G) = \text{Freezing-Influence}(\mathcal{S}(G))$
end

4.2. Generate König–Egerváry subgraph by bipartite Vertex-Cover solution space

By the results in [7, 24], König–Egerváry graph has its minimum coverage the same as its maximum matching number, and finding the maximum König–Egerváry subgraph of a given graph $G(V, E)$ is an NP-complete problem. König–Egerváry subgraph provides a witness of the minimum coverage having the maximum matchings’ number as a lower bound. However, to find such a maximal König–Egerváry subgraph is not always easy as it is NP-complete. Here, we aim to analyze the König–Egerváry subgraph using an edge-adding process, in order to see the organization of König–Egerváry graph in the viewpoint of reduced solution space. In the following, one possible way based on our above analysis is provided to obtain a König–Egerváry subgraph.

We will start the process by a bipartite subgraph $B_G$, which is easy to be obtained from $G$. Based on $B_G$ (a König–Egerváry subgraph), we will show how to enlarge it to keep the König–Egerváry property by adding new edges and using the solution space expression of Vertex-Cover. Certainly, any new edge between the two independent sets $X_1$ and $X_2$ for a bipartite graph will keep the König–Egerváry property, as it still leads to a bipartite subgraph. Thus, the following steps are provided as a Growth Process for König–Egerváry Subgraph.

**Step 1** Start with the graph $G(V, \emptyset)$, find a group of maximum matchings of $G$ and add all the matching edges to $G(V, \emptyset)$.

**Step 2** Divide the vertices into two different sets $X_1$ and $X_2$ by which the two ends of any matching are in different sets, add all the edges between different sets, and a bipartite subgraph $G_B$ is obtained.

**Step 3** Using the method of Bipartite Graph Solution Space Expression Algorithm, the solution space $\mathcal{S}(G_B)$ of the obtained bipartite graph $G_B$ can be achieved.

Till now, all the edges between $X_1$ and $X_2$ have been added, and then edges in the same sets will be considered. By the following steps, it is ensured that each obtained
Figure 8. The generating process of a König – Egerváry subgraph for a given graph $G$. Subgraph (a) is the original graph, and (b) is its bipartite subgraph expressed by the reduced solution graph. Subgraphs (c-f) reveal the growth process of the König – Egerváry subgraphs, which are for the cases in Steps (4-7) correspondingly.

Subgraph is a König – Egerváry one and denoted by $G_{KE}$. Evidently, before these steps, $G_{KE} = G_B$. When a new edge $e \in G - G_{KE}$ is considered to be added to $G_{KE}$, the following steps work:

**Step 4** If one of its two ends is a covered backbone in $\mathcal{S}(G_{KE})$, the edge can be added directly without changing the solution space of $G_{KE}$.

**Step 5** If both of its two ends are uncovered backbones in $\mathcal{S}(G_{KE})$, the edge cannot be added into $G_{KE}$ and should be discarded, as it will cause the energy increase.

**Step 6** If one of its two ends is an uncovered backbone with the other unfrozen in $\mathcal{S}(G_{KE})$, by changing the unfrozen end to covered backbone and doing a Freezing-Influence operation on the reduced solution graph of $G_{KE}$, the edge can be added to $G_{KE}$.

**Step 7** If both of its two ends are unfrozen in $\mathcal{S}(G_{KE})$, the edge can be added to $G_{KE}$ and an Odd-Cycle-Breaking operation should be perform if necessary.

**Step 8** The above Steps (4-7) are repeated until no new edge in $G - G_{KE}$ can be added, and the final resulted $G_{KE}$ is obtained.

In Fig 8 an example is provided to show the process of getting a König – Egerváry subgraph, which indeed provides a maximal König – Egerváry subgraph of $G$. By the steps (1-3) and repeating steps (4-7), the obtained subgraph is a König – Egerváry
one, and its coverage equals to the maximum matching number. However, the above process cannot always ensure a maximal König–Egerváry subgraph of a general graph $G$. Indeed, inaccuracy arises from the sequence of adding these edges and the Step 7. In Step 7, as there may be many choices to do the Odd-Cycle-Breaking operation \cite{14}, it may lead to a contraction of the solution space of $G_{KE}$, and make wrong decisions on discarding more edges in the following steps. Here, the inaccuracy is a complicated topic for the NP-completeness of maximal König–Egerváry subgraph, which will not be studied in this paper. Certainly, we can record the solution space for different Odd-Cycle-Breaking choices by more than one reduced solution graphs, but when facing many cases of Step 7, the space complexity increases exponentially, which leads to the NP-completeness of determining maximal König–Egerváry subgraphs. Generally speaking, at least we can make the following conclusion:

**Proposition**: The Growth Process for König–Egerváry Subgraph keeps the system energy unchanged and the König–Egerváry property for each step, and results in a König–Egerváry subgraph.

5. Conclusion and Discussion

The solution space description for Vertex-Cover of bipartite graphs with some further analysis is investigated in this paper. The exact solution space expression is provided by the Bipartite Graph Solution Space Expression Algorithm for bipartite graphs and by the Bipartite Core Graph Solution Space Expression Algorithm for bipartite core graphs. The states of the nodes on the bipartite graphs, e.g., backbones and unfrozen nodes, are determined by the algorithms, and experiments are given to support the analysis of their statistics. Specially, as the emergence of the unfrozen core, the statistical results become not typical, which implicates the existence of long-range influence of some nodes and the replica symmetric breaking phenomenon. The entropy and clustering entropy are calculated separately, and some technique named cycle simplification is proposed to simplify the unfrozen core. Besides, as extended research, some analysis on the König–Egerváry (sub)graph is given to detect the scope of applications for the proposed algorithms.

Though the Vertex-Cover problem on bipartite graph is a $P$ problem in computational complexity, clear understanding of its solution space and nodes’ correlations can help us a lot to recognize the core difficulty of solving Vertex-Cover on general graphs. Indeed, bipartite graphs can be imbedded as subgraphs to view general graphs and the modes of nodes’ evolution in it can be used for reference to detect the solution organization mechanism on general graphs. Especially, as the proposed Growth Process for König–Egerváry Subgraph, the study on bipartite graph can also provide us a schematic view of the obstacles of achieving a maximal König–Egerváry subgraph, which is another NP-complete problem.
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