LOCAL COHOMOLOGY MODULES WITH INFINITE DIMENSIONAL SOCLES

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Abstract. In this paper we prove the following generalization of a result of Hartshorne: Let $T$ be a commutative Noetherian local ring of dimension at least two, $R = T[x_1, \ldots, x_n]$, and $I = (x_1, \ldots, x_n)$. Let $f$ be a homogeneous element of $R$ such that the coefficients of $f$ form a system of parameters for $T$. Then the socle of $H^0_I(R/fR)$ is infinite dimensional.

1. Introduction

The third of Huneke’s four problems in local cohomology \cite{Hu} is to determine when $H^i_I(M)$ is Artinian for a given ideal $I$ of a commutative Noetherian local ring $R$ and finitely generated $R$-module $M$. An $R$-module $N$ is Artinian if and only $\text{Supp}_R N \subseteq \{m\}$ and $\text{Hom}_R(R/m, N)$ is finitely generated, where $m$ is the maximal ideal of $R$. Thus, Huneke’s problem may be separated into two subproblems:

- When is $\text{Supp}_R H^i_I(M) \subseteq \{m\}$?
- When is $\text{Hom}_R(R/m, H^i_I(M))$ finitely generated?

This article is concerned with the second question. For an $R$-module $N$, one may identify $\text{Hom}_R(R/m, N)$ with the submodule $\{x \in N \mid mx = 0\}$, which is an $R/m$-vector space called the socle of $N$ (denoted $\text{soc}_R N$). It is known that if $R$ is an unramified regular local ring then the local cohomology modules $H^i_I(R)$ have finite dimensional socles for all $i \geq 0$ and all ideals $I$ of $R$ (\cite{HS}, \cite{L1}, \cite{L2}).

The first example of a local cohomology module with an infinite dimensional socle was given in 1970 by Hartshorne \cite{Ha}: Let $k$ be a field, $R = k[[u, v]][x, y]$, $P = (u, v, x, y)R$, $I = (x, y)R$, and $f = ux + vy$. Then $\text{soc}_{R_P} H^2_I(R_P/fR_P)$ is infinite dimensional. Of course, since $I$ and $f$ are homogeneous, this is equivalent to saying that $\text{Hom}_R(R/P, H^2_I(R/fR))$ (the “socle” of $H^2_I(R/fR)$) is infinite dimensional. Hartshorne proved this by exhibiting an infinite set of linearly independent elements in the “socle” of $H^2_I(R)$.

In the last 30 years there have been few results in the literature which explain or generalize Hartshorne’s example. For affine semigroup rings, a remarkable result proved by Helm and Miller \cite{HM} gives necessary and sufficient conditions
(on the semigroup) for the ring to possess a local cohomology module (of a finitely generated module) having infinite dimensional socle. Beyond that work, however, little has been done.

In this paper we prove the following:

**Theorem 1.1.** Let \((T, m)\) be a Noetherian local of dimension at least two. Let \(R = T[x_1, \ldots, x_n]\) be a polynomial ring in \(n\) variables over \(T\), \(I = (x_1, \ldots, x_n)\), and \(f \in R\) a homogeneous polynomial whose coefficients form a system of parameters for \(T\). Then the \(*\)socle of \(H^n_I(R/fR)\) is infinite dimensional.

Hartshorne’s example is obtained by letting \(T = k[[u, v]]\), \(n = 2\), and \(f = ux + vy\) (homogeneous of degree 1). Note, however, that we do not require the coefficient ring to be regular, or even Cohen-Macaulay. As a further illustration, consider the following:

**Example 1.2.** Let \(R = k[[u^4, u^3v, uv^3, v^4]][x, y, z], I = (x, y, z)R\), and \(f = u^4x^2 + v^8yz\). Then the \(*\)socle of \(H^3_I(R/fR)\) is infinite dimensional.

Part of the proof of Theorem 1.1 was inspired by the recent work of Katzman [Ka] where information on the graded pieces of \(H^n_I(R/fR)\) is obtained by examining matrices of a particular form. We apply this technique in the proof of Lemma 2.8.

Throughout all rings are assumed to be commutative with identity. The reader should consult [Mat] or [BH] for any unexplained terms or notation and [BS] for the basic properties of local cohomology.

## 2. The Main Result

Let \(R = \oplus R_\ell\) be a Noetherian ring graded by the nonnegative integers. Assume \(R_0\) is local and let \(P\) be the homogeneous maximal ideal of \(R\). Given a finitely generated graded \(R\)-module \(M\) we define the \(*\)socle of \(M\) by

\[
* \text{soc}_R M = \{ x \in M \mid Px = 0 \} \\
\cong \text{Hom}_R(R/P, M).
\]

Clearly, \(* \text{soc}_R M \cong \text{soc}_{R_P} M_P\). An interesting special case of Huneke’s third problem is the following:

**Question 2.1.** Let \(n := \mu_R(R_+/PR_+)\), the minimal number of generators of \(R_+\). When is \(* \text{soc} H^n_{R_+}(R)\) finitely generated?

For \(i \in \mathbb{N}\) it is well known that \(H^i_{R_+}(R)\) is a graded \(R\)-module, each graded piece \(H^i_{R_+}(R)_\ell\) is a finitely generated \(R_0\)-module, and \(H^i_{R_+}(R)_\ell = 0\) for all sufficiently large integers \(\ell\) ([BS, 15.1.5]). If we know \textit{a priori} that \(H^n_{R_+}(R)_\ell\) has finite length for all \(\ell\) (e.g., if Supp\(R H^n_{R_+}(R) \subseteq \{ P \}\)), then Question 2.1 is equivalent to:

**Question 2.2.** When is \(\text{Hom}_R(R/R_+, H^n_{R_+}(R))\) finitely generated?
We give a partial answer to these questions for hypersurfaces. For the remainder of this section we adopt the following notation: Let \((T, m)\) be a local ring of dimension \(d\) and \(R = T[x_1, \ldots, x_n]\) a polynomial ring in \(n\) variables over \(T\). We endow \(R\) with an \(\mathbb{N}\)-grading by setting \(\deg T = 0\) and \(\deg x_i = 1\) for all \(i\). Let \(I = R_+ = (x_1, \ldots, x_n)R\) and \(P = m + I\) the homogeneous maximal ideal of \(R\). Let \(f \in R\) be a homogeneous element of degree \(p\) and \(C_f\) the ideal of \(T\) generated by the nonzero coefficients of \(f\).

Our main result is the following:

**Theorem 2.3.** Assume \(d \geq 2\) and the (nonzero) coefficients of \(f\) form a system of parameters for \(T\). Then \(\ast \text{soc}_R H^n_I(R/fR)\) is not finitely generated.

The proof of this theorem will be given in a series of lemmas below. Before proceeding with the proof we make a couple of remarks:

**Remark 2.4.**
(a) If \(d \leq 1\) in Theorem 2.3 then \(\ast \text{soc}_R H^n_I(R/fR)\) is finitely generated. This follows from [DM, Corollary 2] since \(\dim R/I = \dim T \leq 1\).

(b) The hypothesis that the nonzero coefficients of \(f\) form a system of parameters for \(T\) is stronger than our proof requires. One only needs that \(C_f\) be \(m\)-primary and that there exists a dimension 2 ideal containing all but two of the coefficients of \(f\). (See the proof of Lemma 2.8.)

The following lemma identifies the support of \(H^n_I(R/fR)\) for a homogeneous element \(f \in R\). This lemma also follows from a much more general result recently proved by Katzman and Sharp [KS, Theorem 1.5].

**Lemma 2.5.** Let \(f \in R\) be a homogeneous element. Then
\[
\text{Supp}_R H^n_I(R/fR) = \{Q \in \text{Spec } R \mid Q \supseteq I + C_f\}.
\]

**Proof:** It is enough to prove that \(H^n_I(R/fR) = 0\) if and only if \(C_f = T\). As \(H^n_I(R/fR)_k\) is a finitely generated \(T\)-module for all \(k\), we have by Nakayama that \(H^n_I(R/fR) = 0\) if and only if \(H^n_I(R/fR) \otimes_T T/m = 0\). Now
\[
H^n_I(R/fR) \otimes_T T/m \cong H^n_I(R/fR) \otimes_T T/m \\
\cong H^n_N(S/fS)
\]
where \(S = (T/m)[x_1, \ldots, x_n]\) is a polynomial ring in \(n\) variables over a field and \(N = (x_1, \ldots, x_n)S\). As \(\dim S = n\), we see that \(H^n_N(S/fS) = 0\) if and only if the image of \(f\) modulo \(m\) is nonzero. Hence, \(H^n_I(R/fR) = 0\) if and only if at least one coefficient of \(f\) is a unit, i.e., \(C_f = T\).

We are mainly interested in the case the coefficients of \(f\) generate an \(m\)-primary ideal:

**Corollary 2.6.** Let \(f \in R\) be homogeneous and suppose \(C_f\) is \(m\)-primary. Then
\[
\text{Supp}_R H^n_I(R/fR) = \{P\}.
\]

Our next lemma is the key technical result in the proof of Theorem 2.3.
Lemma 2.7. Suppose \( u, v \in T \) such that \( \text{ht}(u, v)T = 2 \). For each integer \( n \geq 1 \) let \( M_n \) be the cokernel of \( \phi_n : T^{n+1} \to T^n \) where \( \phi_n \) is represented by the matrix

\[
A_n = \begin{pmatrix}
u & v & 0 & \cdots & 0 & 0 \\
u & v & 0 & \cdots & 0 & 0 \\
0 & 0 & u & v & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & u & v
\end{pmatrix}_{n \times (n+1)}
\]

Let \( J = \cap_{n \geq 1} \text{ann}_T M_n \). Then \( \dim T/J = \dim T \).

**Proof:** Let \( \hat{T} \) denote the \( m \)-adic completion of \( T \). Then \( \text{ht}(u, v)\hat{T} = 2, \text{ann}_T M_n = \text{ann}_T(M_n \otimes_T \hat{T}) \cap T \), and \( \dim T/(I \cap T) \geq \dim \hat{T}/I \) for all ideals \( I \) of \( \hat{T} \). Thus, we may assume \( T \) is complete. Now let \( p \) be a prime ideal of \( T \) such that \( \dim T/p = \dim T \). Since \( T \) is catenary, \( \text{ht}(u, v)T/p = 2 \). Assume the lemma is true for complete domains. Then \( \cap_{n \geq 1} \text{ann}_T/M_n \otimes_T T/p) = p/p \). Hence

\[
J = \cap_{n \geq 1} \text{ann}_T M_n \\
\subseteq \cap_{n \geq 1} \text{ann}_T(M_n \otimes_T T/p) \\
= p,
\]

which implies that \( \dim T/J \geq \dim T/p = \dim T \). Thus, it suffices to prove the lemma for complete domains.

As \( T \) is complete, the integral closure \( S \) of \( T \) is a finite \( R \)-module. Since \( \text{ht}(u, v)S = 2 \) ([Ma, Theorem 15.6]) and \( S \) is normal, \( \{u, v\} \) is a regular sequence on \( S \). It is easily seen that \( I_n(A_n) \), the ideal of \( n \times n \) minors of \( A_n \), is \( (u, v)^nT \). By the main result of [BE] we obtain \( \text{ann}_S(M_n \otimes_T S) = (u, v)^nS \). Hence \( \text{ann}_T M_n \subseteq (u, v)^nS \cap T \). As \( S \) is a finite \( T \)-module there exists an integer \( k \) such that \( \text{ann}_T M_n \subseteq (u, v)^n-kT \) for all \( n \geq k \). Therefore, \( \cap_{n \geq 1} \text{ann}_T M_n = (0) \), which completes the proof.

Lemma 2.8. Assume \( d \geq 2 \) and let \( f \in R \) be a homogeneous element of degree \( p \) such that the coefficients of \( f \) form a system of parameters for \( T \). Then \( \dim T/\text{ann}_T \overline{H}_T^n(R/fR) \geq 2 \).

**Proof:** Let \( c_1, \ldots, c_d \) be the nonzero coefficients of \( f \). Let \( T' = T/(c_3, \ldots, c_d)T \) and \( R' = T'[x_1, \ldots, x_n] \cong R/(c_3, \ldots, c_d)R \cong R \otimes_T T' \). Since

\[
\dim T/\text{ann}_T \overline{H}_T^n(R/fR) \geq \dim T/\text{ann}_T(\overline{H}^n_T(R/fR) \otimes_T T') \\
= \dim T'/\text{ann}_T \overline{H}^n_{T'}(R'/fR'),
\]

we may assume that \( \dim T = 2 \) and \( f \) has exactly two nonzero terms.

For any \( w \in R \) there is a surjective map \( H^r_T(R/wfR) \to H^r_T(R/fR) \). Hence, \( \text{ann}_T \overline{H}_T^n(R/wfR) \subseteq \text{ann}_T \overline{H}_T^n(R/fR) \). Thus, we may assume that the terms of \( f \) have no (nonunit) common factor. Without loss of generality, we may write \( R = T[x_1, \ldots, x_k, y_1, \ldots, y_r] \) and \( f = u x_1^{d_{1}} \cdots x_k^{d_{k}} + v y_1^{e_{1}} \cdots y_r^{e_{r}} = u x_1^{d} + v y_{1}^{e} \), where \( \{u, v\} \) is a system of parameters for \( T \). As \( f \) is homogeneous, \( p = \sum_{i} d_{i} = \sum_{i} e_{i} \).
Applying the right exact functor $H^n_I(\cdot)$ to $R(-p) \xrightarrow{f} R \to R/fR \to 0$ we obtain the exact sequence

$$H^n_I(R)_{-\ell} \xrightarrow{f} H^n_I(R)_{-\ell} \to H^n_I(R/fR)_{-\ell} \to 0$$

for each $\ell \in \mathbb{Z}$. For each $\ell$, $H^n_I(R)_{-\ell}$ is a free $T$-module with basis

$$\{x^{-\alpha}y^{-\beta} | \sum_{i,j} \alpha_i + \beta_j = \ell, \alpha_i > 0, \beta_j > 0 \forall i, j\}$$

(e.g., [BS, Example 12.4.1]). Let $q$ be an arbitrary positive integer and let $\ell(q) = qp + k + r$. Define $L_{-\ell(q)}$ to be the free $T$-summand of $H^n_I(R)_{-\ell(q)}$ spanned by the set

$$\{x^{-sd-1}y^{-te-1} | s + t = q, s, t \geq 0\}.$$ 

Then the cokernel of $\delta_q : L_{-\ell(q+1)} \xrightarrow{f} L_{-\ell(q)}$ is a direct summand (as a $T$-module) of $H^n_I(R/fR)_{-\ell(q)}$. For a given $q$ we order the basis elements for $L_{-\ell(q)}$ as follows:

$$x^{-sd-1}y^{-te-1} > x^{-s'd-1}y^{-t'e-1}$$

if and only if $s > s'$. With respect to these ordered bases, the matrix representing $\delta_q$ is

$$\begin{pmatrix}
  u & v & 0 & 0 & \cdots & 0 & 0 \\
  0 & u & v & 0 & \cdots & 0 & 0 \\
  0 & 0 & u & v & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & u & v
\end{pmatrix}_{(q+1) \times (q+2)}.$$

By Lemma 2.7, if $J = \bigcap_{q \geq 1} \text{ann}_T \text{coker} \delta_q$ then $\dim T/J = \dim T = 2$. As $\text{coker} \delta_q$ is a direct $T$-summand of $H^n_I(R/fR)$, we have $\text{ann}_T H^n_I(R/fR) \subseteq J$. This completes the proof. \qed

**Lemma 2.9.** Under the assumptions of Lemma 2.8, $\text{Hom}_R(R/I, H^n_I(R/fR))$ is not finitely generated as an $R$-module. Consequently, $\text{Hom}_R(R/I, H^n_I(R/fR))_k \neq 0$ for infinitely many $k$.

**Proof:** Suppose $\text{Hom}_R(R/I, H^n_I(R/fR))$ is finitely generated. By Lemma 3.5 of [MV] we have that $I + \text{ann}_R H^n_I(R/fR)$ is $P$-primary. (One should note that the hypothesis in [MV, Lemma 3.5] that the ring be complete is not necessary.) This implies that $\text{ann}_R H^n_I(R/fR) \cap T = \text{ann}_T H^n_I(R/fR)$ is $m$-primary, contradicting Lemma 2.8. \qed

We now give the proof of our main result:

**Proof of Theorem 2.3:** By Corollary 2.9, $\text{Supp}_R H^n_I(R/fR) = \{P\}$. Thus, $\text{Hom}_R(R/I, H^n_I(R/fR))_k$ has finite length as a $T$-module for all $k$ and is nonzero for infinitely many $k$ by Lemma 2.9. Consequently,

$$\text{Hom}_R(R/P, H^n_I(R/fR))_k = \text{Hom}_T(T/m, \text{Hom}_R(R/I, H^n_I(R/fR))_k)$$
is nonzero for infinitely many \( k \). Hence

\[
\text{soc}_R(H^n_I(R/fR)) = \text{Hom}_R(R/P, H^n_I(R/fR))
\]

is not finitely generated. \( \square \)

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