OPTIMIZATION OF RELATIVE ARBITRAGE

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Abstract. In stochastic portfolio theory, a relative arbitrage is an equity portfolio which outperforms a benchmark portfolio over a specified horizon. When the market is diverse and sufficiently volatile, and the benchmark is the market or a buy-and-hold portfolio, functionally generated portfolios provide a systematic way of constructing relative arbitrages. In this paper we show that if the market portfolio is replaced by the equal or entropy weighted portfolio among many others, no relative arbitrages can be constructed using functionally generated portfolios. We also introduce and study a shaped-constrained optimization problem for functionally generated portfolios in the spirit of maximum likelihood estimation of a log-concave density.

1. Introduction

A major aim of stochastic portfolio theory (see [Fer02] and [FK09] for an introduction) is to uncover relative arbitrage opportunities under realistic assumptions on the behavior of equity markets. Consider an equity market with \( n \) stocks. The market weight \( \mu_i(t) \) of stock \( i \) at time \( t \) is the market capitalization of stock \( i \) divided by the total capitalization of the market. The vector \( \mu(t) = (\mu_1(t), ..., \mu_n(t)) \) of market weights takes value in the open unit simplex \( \Delta^{(n)} \) in \( \mathbb{R}^n \) defined by

\[
\Delta^{(n)} = \left\{ p = (p_1, ..., p_n) : p_i > 0, \sum_{i=1}^n p_i = 1 \right\}.
\]

Loosely speaking, the market is said to be diverse if the market weights do not approach the corners or the boundary of the simplex, and it is sufficiently volatile if the market weights fluctuate continuously. As shown in [FKK05, FK05, BF08, PW13], in a diverse and sufficiently volatile market it is possible to construct relative arbitrages with respect to the market portfolio whose portfolio weights are \((\mu_1(t), ..., \mu_n(t))\).

In fact, it is possible to construct relative arbitrages whose portfolio weights are deterministic functions of the current market weights (so that forecasts of expected returns and the covariance matrix are not required). These portfolios, first introduced by Fernholz [Fer99], are said to be functionally generated. This is in accordance with the observation by many academics and practitioners (see for example [FGH98, DGU09, BNPS12]) that simple portfolio rules such as the equal and diversity weighted portfolios often beat the market over long periods. Indeed,
in [PW14] we showed that a relative arbitrage portfolio (more precisely a *pseudo-arbitrage*, see below) depending only on the market weights must be functionally generated.

There are two important questions that are not fully addressed by the existing theory. First, what happens if we replace the market portfolio by another benchmark? In [Str12] the concept of functionally generated portfolio and the key ‘master equation’ (see Lemma 3.3 below) are extended to arbitrary benchmark portfolios, but little is known about the existence of relative arbitrage under general conditions such as diversity and sufficient volatility. For example, can we beat the equal-weighted portfolio by a functionally generated portfolio in a diverse and sufficiently volatile market, in the same way a functionally generated portfolio beats the market portfolio? More generally, does there exist an infinite hierarchy of relative arbitrages?

Second, is there a sound and applicable theory of optimization for relative arbitrages and functionally generated portfolios? Such a theory is clearly of great interest and this problem was raised already in Fernholz’s monograph [Fer02 Problems 3.1.7-8]. To the best of our knowledge, limited progress has been made to optimization of functionally generated portfolios (see [PW13 Section 5] for an attempt in the two asset case). On the theoretical side, if the market model is known it is sometimes possible to characterize the highest return relative to the market or a given trading strategy that can be achieved using nonanticipative investment rules over a given time horizon. See [FK10] for the case of Markovian markets, [FK11] for a more general setting which allows uncertainty regarding the drift and diffusion coefficients, and [Ruf11] which expresses optimal relative arbitrages with respect to Markovian trading strategies as delta hedges. For optimization of functionally generated portfolios, a major difficulty is that the class of functionally generated portfolios is a function space and the optimization has to be nonparametric. Ideally, given historical data or a stochastic model of the dynamics of the market weights, we want to be able to pick an optimal functionally generated portfolio subject to appropriate constraints.

The present paper attempts to give answers to both questions. In this paper we interpret relative arbitrage by what we call *pseudo-arbitrage* in [PW14]. This is a model-free concept and the precise definition will be stated in Section 2. Let \( \tau \) and \( \pi \) be portfolios, and suppose we invest $1 in each portfolio at the beginning. We denote by \( V(t) \) the ratio of the value of portfolio \( \tau \) to that of \( \pi \) at time \( t \). Informally, we say that \( \tau \) is a pseudo-arbitrage relative to \( \pi \) if this ratio is bounded below by a positive constant whenever the market weights are confined to a fixed region, and for certain paths of market weights in this region the ratio tends to infinity as \( t \to \infty \). That is, the downside risk relative to \( \pi \) is uniformly bounded below regardless of the market movements in that region, and there is a possibility of unbounded gain. Under appropriate conditions on diversity and market volatility, a pseudo-arbitrage is a relative arbitrage.

Regarding the ‘hierarchy of relative arbitrages’, given a portfolio \( \pi \), we give a simple, easy-to-test sufficient condition to guarantee that there is no functionally generated portfolio which is a pseudo-arbitrage relative to \( \pi \). Let \( D_i \) be the partial derivative in the direction of \( x_i \) in \( \mathbb{R}^n \). Let \( \Phi \) be a \( C^2 \) (twice continuously differentiable) positive concave function defined on an open neighborhood of \( \Delta^{(n)} \) in \( \mathbb{R}^n \). Using Fernholz’s original formulation, we say that \( \pi \) is *generated* by \( \Phi \) if the
portfolio weights \((\pi_1, ..., \pi_n)\) are given by
\[
\pi_i = \mu_i \left(D_i \log \Phi(\mu) + 1 - \sum_{j=1}^{n} \mu_j D_j \log \Phi(\mu)\right), \quad i = 1, ..., n.
\]
We say that \(\Phi\) is a measure of diversity if in addition \(\Phi\) is symmetric (invariant under permutations of the coordinates). Let \(e(1) = (1, 0, ..., 0)\) be a vertex of \(\Delta^{(n)}\) and \(\bar{\pi} = (\frac{1}{n}, ..., \frac{1}{n})\) be the barycenter of \(\Delta^{(n)}\).

**Theorem 1.1.** Suppose \(\pi\) is generated by a measure of diversity \(\Phi\). If
\[
\int_0^1 \frac{1}{\Phi(te(1) + (1-t)\bar{\pi})^2} dt = \infty,
\]
there is no functionally generated pseudo-arbitrage relative to \(\pi\).

This sufficient condition is satisfied by the equal and entropy weighted portfolios among many others. For the market portfolio the generating function is a positive constant and so the integral in (1.2) converges.

Regarding optimization of functionally generated portfolios, we formulate a shape-constrained optimization problem in the spirit of maximum likelihood estimation of log-concave density in nonparametric statistics \[DR09, CSS10, CS10, KM10, SW10\]. Following \[PW14\], we associate to each functionally generated portfolio a discrete energy functional \(T(\cdot | \cdot)\) (see Definition 3.1). Intuitively, \(T(q | p)\) measures the potential profit from volatility as the market weight jumps from \(p\) to \(q\) in \(\Delta^{(n)}\). Let \(\mathbb{P}\) be an intensity measure over the jumps \((p, q)\) which can be defined in terms of data or a given model (examples will be given in Section 5). We maximize
\[
\int T(q | p) d\mathbb{P}
\]
over all functionally generated portfolios (with or without constraints). This optimization problem is shape-constrained because the generating function of a functionally generated portfolio is concave. We prove that the optimization problem is well-posed and is in a suitable sense consistent when interpreted as a statistical estimation problem. In this paper we implement this optimization for the case of two assets (analogous to univariate density estimation) and a general algorithm will be the topic of future research. We illustrate a typical application in portfolio management with a case study.

The paper is organized as follows. In Section 2 we set up the notations and recall the definitions of pseudo-arbitrage and functionally generated portfolio. In Section 3 we use the framework of \[PW14\] to characterize pseudo-arbitrages with respect to a benchmark portfolio which is functionally generated. Using a relative concavity lemma given in \[CDO07\], we prove in Section 4 that there are no functionally generated pseudo-arbitrages relative to portfolios including the equal and entropy weighted portfolios. Optimization of functionally generated portfolios is studied in Section 5 and empirical examples are given in Section 6. Several proofs of a more technical nature are gathered in Appendix A.
2. Pseudo-arbitrage and functionally generated portfolio

2.1. Portfolio and pseudo-arbitrage. We work under the set-up of Pal and Wong [PW14 Section 1.1] which we briefly recall here. Let $n \geq 2$ be the number of stocks or assets in the market. We endow the open unit simplex $\Delta^{(n)}$ with the Euclidean metric. The open ball in $\Delta^{(n)}$ centered at $p$ with radius $\delta$ is denoted by $B(p, \delta) \subset \Delta^{(n)}$. A tangent vector of $\Delta^{(n)}$ is a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{n} v_i = 0$. We denote the vector space of tangent vectors of $\Delta^{(n)}$ by $T\Delta^{(n)}$. For $i = 1, \ldots, n$, we let $e(i) = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vertex of $\Delta^{(n)}$ in the $i$-th direction. If $a$ and $b$ are vectors in $\mathbb{R}^n$, we let $\langle a, b \rangle$ be the Euclidean inner product. The Euclidean norm is denoted by $\| \cdot \|$. If $b$ has nonzero entries, we denote by $a/b$ the vector of the componentwise ratios $a_i/b_i$.

Let $X_i(t) > 0$ be the market capitalization of stock $i$ at time $t$. The total capitalization of the market is then $X_1(t) + \cdots + X_n(t)$. The market weight of stock $i$ is defined by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \quad i = 1, \ldots, n.$$ 

The vector $\mu(t) = (\mu_1(t), \ldots, \mu_n(t))$ takes values in $\Delta^{(n)}$ and represents the relative sizes of the firms. As the stock prices change the market weights will fluctuate accordingly.

As in [PW13, PW14], the stock market is modeled as a path $\{\mu(t)\}_{t \geq 0}$ taking values in $\Delta^{(n)}$, and time can be either discrete ($t = 0, 1, 2, \ldots$) or continuous. In discrete time $\{\mu(t)\}_{t \geq 0}$ is any deterministic sequence in $\Delta^{(n)}$. In continuous time $\{\mu(t)\}_{t \geq 0}$ is a typical path of any continuous semimartingale with state space $\Delta^{(n)}$ defined on a given filtered probability space. We stress that our framework does not depend on any specific stochastic model for the market weights.

We consider a price taker in this market who cares about the value of his or her portfolio relative to that of the entire market. We restrict ourselves to portfolios which are deterministic functions of the current market weights. Short sales are not allowed and we assume there is no transaction cost.

**Definition 2.1 (Portfolio and relative value process).** A portfolio is a Borel measurable map $\pi : \Delta^{(n)} \to \Delta^{(n)}$. The market portfolio is the identity map $p \mapsto p$. Given a portfolio $\pi$, its relative value process $\{V_\pi(t)\}_{t \geq 0}$ is defined by $V_\pi(0) = 1$ and

$$\frac{V_\pi(t + 1)}{V_\pi(t)} = 1 + \langle \pi(\mu(t)), \mu(t + 1) - \mu(t) \rangle,$$

in discrete time, and

$$\frac{dV_\pi(t)}{V_\pi(t)} = \sum_{i=1}^{n} \pi_i(\mu(t)) \frac{d\mu_i(t)}{\mu_i(t)},$$

in continuous time.

The weight ratio of the portfolio at $p \in \Delta^{(n)}$ is the vector $\frac{\pi(p)}{p} = \left(\frac{\pi_1(p)}{p_1}, \ldots, \frac{\pi_n(p)}{p_n}\right)$.

The relative value $V_\pi(t)$ can be interpreted as the ratio of the growth of $\$1$ invested in the portfolio to that of $\$1$ invested in the market portfolio. If $V_\pi(t + 1) > V_\pi(t)$, the portfolio outperforms the market portfolio over the period $[t, t + 1]$. As mentioned in [PW14], it is helpful to think of the weight ratio $p \mapsto \frac{\pi(p)}{p}$ as a vector field on $\Delta^{(n)}$. From the first equation of (2.1), the portfolio outperforms the market
if the inner product between the displacement $\mu(t+1) - \mu(t)$ of market weights and the weight ratio is positive. This means on average the portfolio puts more weight on the assets which perform well relative to the rest of the market.

Given two portfolios $\pi$ and $\tau$, we want to study whether one beats the other. We adopt the following slightly extended definition from [PW14, Section 1.1].

**Definition 2.2** (Pseudo-arbitrage). Let $\pi$ and $\tau$ be portfolios. We say that $\tau$ is a pseudo-arbitrage relative to $\pi$ (written $\tau \triangleright \pi$) if the following holds:

(i) For any compact subset $K$ of $\Delta^n$, there exists a constant $C = C(K, \pi, \tau) \geq 0$ such that for any path $\{\mu(t)\} \subset K$, we have

$$\log \frac{V_\tau(t)}{V_\pi(t)} \geq -C + A(t), \quad t \geq 0,$$

where $A(t)$ satisfies $A(0) = 0$ and is non-decreasing.

(ii) For some compact subset $K$ of $\Delta^n$, there are paths $\{\mu(t)\} \subset K$ along which the process $\{A(t)\}$ in (2.2) satisfies $A(t) \to \infty$. (In continuous time, this means there exists a continuous semimartingale $\{\mu(t)\}$ with values in $K$ such that $A(t) \to \infty$ with positive probability.)

We write $\tau \trianglerightequal \pi$ if only (i) holds.

The requirement $\mu(t) \in K$ is a diversity assumption on the market, and the statement $A(t) \to \infty$ refers to the presence of sufficient volatility as will become clear below.

2.2. **Functionally generated portfolios.** Functionally generated portfolios were first introduced in a general form by Fernholz in [Fer99]. We will follow the treatment in [PW14, Section 2] which is intrinsic and emphasizes the relationship with convex analysis. Throughout the paper we will rely heavily on results from convex analysis and a standard reference is [Roc97].

**Definition 2.3** (Functionally generated portfolios). Let $\pi$ be a portfolio and $\Phi : \Delta^n \to (0, \infty)$ be a concave function. We say that $\pi$ is generated by $\Phi$ if the inequality

$$1 + \left\langle \frac{\pi(p)}{p}, q - p \right\rangle \geq \frac{\Phi(q)}{\Phi(p)}$$

holds for all $p, q \in \Delta^n$. We call $\Phi$ the generating function of $\pi$. We denote by $FG$ the collection of all functionally generated portfolios $(\pi, \Phi)$ where $\pi$ is generated by the concave function $\Phi$.

Let $\Phi$ be a concave function on $\Delta^n$ and $p \in \Delta^n$. The superdifferential of $\Phi$ at $p$ is the set $\partial \Phi(p)$ defined by

$$\partial \Phi(p) = \{\xi \in T\Delta^n : \Phi(p) + \langle \xi, q - p \rangle \geq \Phi(q) \quad \forall q \in \Delta^n\}.$$

Note that in [PW14] we allowed $\xi$ to be any vector in $\mathbb{R}^n$ satisfying the inequality in definition of $\partial \Phi(p)$. Also, if $\Phi$ is concave and positive, it is easy to show that $\log \Phi$ is also a concave function, and

$$\partial \log \Phi(p) = \frac{1}{\Phi(p)} \partial \Phi(p) = \left\{ \frac{1}{\Phi(p)} \xi : \xi \in \partial \Phi(p) \right\}.$$
To see this, note that by [Roc97, Theorem 23.2], we have
\[ \partial \Phi(p) = \{ \xi \in T\Delta^{(n)} : \Phi'(p; v) \leq \langle \xi, v \rangle \ \forall v \in T\Delta^{(n)} \}, \]
where \( \Phi'(p; v) \) is the directional derivative at \( p \) in the direction of \( v \). Then divide both sides of the inequality by \( \Phi(p) \) and use the chain rule for directional derivative:
\[
(\log \Phi)'(p; v) = \frac{1}{\Phi(p)} \Phi'(p; v).
\]

**Lemma 2.4.** Let \( \Phi \) be a positive concave function on \( \Delta^{(n)} \).

(i) Suppose the portfolio \( \pi \) is generated by \( \Phi \). Then for \( p \in \Delta^{(n)} \), the tangent vector \( v = (v_1, \ldots, v_n) \) defined by
\[
(2.6) \quad v_i = \frac{\pi_i(p)}{p_i} - \frac{1}{n} \sum_{j=1}^{n} \frac{\pi_j(p)}{p_j}, \quad i = 1, \ldots, n,
\]
belongs to \( \partial \log \Phi(p) \).

(ii) Conversely, if \( v \in \partial \log \Phi(p) \), then the vector \( \pi = (\pi_1, \ldots, \pi_n) \) defined by
\[
(2.7) \quad \frac{\pi_i}{p_i} = v_i + 1 - \sum_{j=1}^{n} p_j v_j, \quad i = 1, \ldots, n,
\]
is an element of \( \Delta^{(n)} \). In particular, any measurable selection of \( \partial \log \Phi \) (a Borel measurable map \( \xi : \Delta^{(n)} \to T\Delta^{(n)} \) such that \( \xi(p) \in \partial \log \Phi(p) \) for all \( p \in \Delta^{(n)} \)) defines via \( 2.7 \) a portfolio generated by \( \Phi \). (By [RW98, Theorem 14.56], there is always a measurable selection of \( \partial \log \Phi \).)

Moreover, the operations \( \pi \mapsto v \) and \( v \mapsto \pi \) defined by \( 2.6 \) and \( 2.7 \) are inverses of each other.

**Proof.** (i) Let \( p \in \Delta^{(n)} \). By \( 2.3 \), we have
\[
1 + \left\langle \frac{\pi(p)}{p}, q - p \right\rangle \geq \frac{\Phi(q)}{\Phi(p)}
\]
for all \( q \in \Delta^{(n)} \). Note that \( \frac{\pi(p)}{p} \) is not a tangent vector of \( \Delta^{(n)} \). The normalization \( 2.6 \) ‘projects’ \( \frac{\pi(p)}{p} \) to \( v \) which is a tangent vector. Since \( \frac{\pi(p)}{p} - v \) is perpendicular to \( T\Delta^{(n)} \), the inner product does not change if \( \frac{\pi(p)}{p} \) is replaced by \( v \). Hence by \( 2.5 \) we have \( v \in \partial \log \Phi(p) \).

(ii) It is easy to verify that \( \sum_{i=1}^{n} \pi_i = 1 \). To see that \( \pi_i \geq 0 \) for each \( i \), let \( q - p = t(e(i) - p) \) for \( 0 < t < 1 \) in \( 2.4 \), and we have
\[
-\Phi(p) \leq \Phi(p + t(e(i) - p)) - \Phi(p) \quad \text{(since } \Phi(q) > 0) \leq \langle \Phi(p) v, t(e(i) - p) \rangle
\]
\[
= t \Phi(p) \left( v_i + \sum_{j=1}^{n} p_j v_j \right).
\]

Letting \( t \uparrow 1 \) and dividing both sides by \( \Phi(p) \), we get the desired inequality \( \pi_i \geq 0 \).

That \( \pi \mapsto v \) and \( v \mapsto \pi \) are inverses of each other can be verified by a direct computation. \( \square \)
If $\pi$ is generated by $\Phi$, the weight ratio vector field $\frac{\pi}{p}$ is conservative on $\Delta^{(n)}$ and its potential function is given by the logarithm of the generating function $\Phi$. Here is a precise statement (see the proof of [PW14, Theorem 7] for details). Let $\pi$ be a portfolio. If $\gamma : [0, 1] \to \Delta^{(n)}$ is a piecewise linear path in $\Delta^{(n)}$, consider the line integral of the weight ratio $\frac{\pi}{p}$ along $\gamma$:

$$I_\pi(\gamma) := \int_0^1 \frac{\pi}{p} dp = \int_0^1 \sum_{i=1}^n \frac{\pi_i(\gamma(t))}{p_i(\gamma(t))} \gamma_i'(t) dt.$$

If $\pi$ is functionally generated, the weight ratio $\frac{\pi}{p}$ is conservative in the sense that this line integral is zero whenever $\gamma$ is closed, i.e., $\gamma(0) = \gamma(1)$. Moreover, for any $p, q \in \Delta^{(n)}$ we have

$$\log \Phi(q) - \log \Phi(p) = I_\pi(\gamma),$$

where $\gamma$ is any piecewise linear path from $p$ to $q$. In classical terminology, $\log \Phi$ is then the potential function of the weight ratio vector field. Fernholz’s decomposition (see Lemma 3.3 below) shows that the log relative value $\log V_\pi(t)$ can be decomposed as the increment of $\log \Phi(\mu(t))$ plus a non-decreasing process related to market volatility.

With these definitions, the main result of [PW14] can be summarized as follow.

**Theorem 2.5** (Pseudo-arbitrages relative to the market portfolio). [PW14, Theorem 7, Theorem 12] A portfolio $\pi$ is a pseudo-arbitrage relative to the market portfolio $\mu$ if and only if $\pi$ is generated by a concave function $\Phi : \Delta^{(n)} \to (0, \infty)$ which is not affine. Moreover, these portfolios correspond to solutions of an optimal transport problem.

In Section 4 we focus on functionally generated portfolios whose generating functions are $C^2$ on $\Delta^{(n)}$. In this case the portfolio weights are given by explicit formulas. Recall that $\Phi'(p; v)$ is the directional derivative.

**Lemma 2.6.** [PW14, Lemma 3] If $\pi$ is a portfolio generated by a continuously differentiable concave function $\Phi : \Delta^{(n)} \to (0, \infty)$, then

$$\frac{\pi_i(p)}{p_i} = 1 + (\log \Phi)'(p; e(i) - p)$$

for all $p \in \Delta^{(n)}$ and $i = 1, ..., n$.

It can be verified that (2.10) and Fernholz’s formulation (1.1) are equivalent.

**Definition 2.7.**

(i) We denote by $FG^2$ the collection of functionally generated portfolios whose generating functions are $C^2$ and concave. An element of $FG^2$ is denoted by either $\pi$, $\Phi$ or $(\pi, \Phi)$ where $\pi$ is generated by $\Phi$.

(ii) A positive $C^2$ concave function $\Phi$ on $\Delta^{(n)}$ is called a measure of diversity if it is symmetric, i.e.,

$$\Phi(p_1, ..., p_n) = \Phi(p_{\sigma(1)}, ..., p_{\sigma(n)})$$

for all $p \in \Delta^{(n)}$ and any permutation $\sigma$ of $\{1, ..., n\}$.
Table 1. Examples of functionally generated portfolios

| Name           | Portfolio weights | Generating function |
|----------------|-------------------|---------------------|
| Market         | \( \pi_i(p) = p_i \) | \( \Phi(p) = 1 \) |
| Diversity-weighted \((0 < r < 1)\) | \( \pi_i(p) = \frac{p_i^r}{\sum_{j=1}^n p_j^r} \) | \( \Phi(p) = \left( \sum_{j=1}^n p_j^r \right)^{\frac{1}{r}} \) |
| Equal-weighted | \( \pi_i(p) = \frac{1}{n} \) | \( \Phi(p) = (p_1 p_2 \cdots p_n)^{\frac{1}{n}} \) |
| Entropy-weighted | \( \pi_i(p) = \frac{-p_i \log p_i}{\sum_{j=1}^n -p_j \log p_j} \) | \( \Phi(p) = \sum_{j=1}^n -p_j \log p_j \) |

The notation in Definition 2.7(i) is justified since the generating function is unique up to a positive multiplicative constant by [PW14, Lemma 2]. On the other hand, a non-smooth concave function \( \Phi \) generates multiple portfolios but they differ only on the set where \( \Phi \) is not differentiable (i.e., the superdifferential \( \partial \log \Phi(p) \) has more than one element), and this set has measure zero by [Roc97, Theorem 25.5]. Measures of diversity were introduced by Fernholz in [Fer99, Section 4]. Some examples will be given in Section 3 and more can be found in [Fer02, Section 3.4].

2.3. Maximal portfolio. The notion of pseudo-arbitrage can be regarded as a kind of partial ordering among portfolios. It is natural to study the maximal elements.

**Definition 2.8** (Maximal portfolio). Let \( S \) be a family of portfolios and \( \pi \in S \). We say that \( \pi \) is maximal in \( S \) on \( \Delta^{(n)} \) if there is no pseudo-arbitrage with respect to \( \pi \) belonging to \( S \).

A maximal portfolio is one which is impossible to beat assuming only diversity and sufficient volatility, in the sense of Definition 2.2, by any other portfolio in the class. Note that a maximal portfolio strategy may not exist and may not be unique in the given class. In Section 4 we will study the maximal portfolios for \( S = \mathcal{FG}^2 \).

3. Benchmarking a functionally generated portfolio

Fix a portfolio \( \pi \) generated by a concave function \( \Phi : \Delta^{(n)} \to (0, \infty) \) and call it the benchmark portfolio. Some examples we have in mind are given in Table 1. All of these portfolios are generated by measures of diversity.

As mentioned in the introduction, it can be proved that many functionally generated portfolios (including the three nontrivial examples above) outperform the market with probability one over sufficiently long periods under the assumptions of diversity and sufficient volatility. As these hypotheses appear to hold empirically, many functionally generated portfolios outperform the market over long periods, see [Fer02, Chapter 6] for several case studies using data of the US stock market. Since these portfolios contain no proprietary modeling, behave reasonably well and are easily replicable, they also serve as alternative benchmarks as discussed in the practitioner papers [FGH98, HCKL11]. It is natural to ask whether we can construct relative or pseudo-arbitrages with respect to these portfolios.

Our first objective is to give a characterization of pseudo-arbitrages with respect to the given benchmark portfolio. First we recall the notion of discrete energy functional introduced in [PW14] which is crucial for our development.
Figure 1. Hypothetical performance of a functionally generated portfolio. If the market weight $\mu(t)$ stays within a subset $K \subset \Delta^{(n)}$, the relative value process will stay within the dashed curves which are vertical translations of the drift process $\Theta(t)$. The width of the ‘sausage’ is given by the oscillation of $\log \Phi$ on $K$ defined by $\osc_K(\log \Phi) = \sup_{p,q \in K} |\log \Phi(q) - \log \Phi(p)|$.

**Definition 3.1 (Discrete energy functional).** Let $\pi$ be a portfolio generated by a concave function $\Phi : \Delta^{(n)} \rightarrow (0, \infty)$. The discrete energy functional of the pair $(\pi, \Phi)$ is the function $T : \Delta^{(n)} \times \Delta^{(n)} \rightarrow [0, \infty)$ defined by

$$T(q | p) = \log \left(1 + \frac{\pi(p)}{p} \cdot q - p\right) - \log \frac{\Phi(q)}{\Phi(p)}, \quad p,q \in \Delta^{(n)}.$$ 

It can be shown that $T(q | p) \geq 0$ and $T(q | p) = 0$ if and only if $\Phi$ is affine on the line segment between $p$ and $q$. In discrete time, the cumulated energy $\sum_t T(\mu(t+1) | \mu(t))$ measures the potential profit from market volatility. Applying Taylor approximation to (3.1) when $q \approx p$, we get the following quadratic form which features in the continuous time framework.

**Definition 3.2 (Drift quadratic form).** Suppose $(\pi, \Phi) \in \mathcal{FG}^2$. Its drift quadratic form $H_\pi \equiv H_\Phi$ is defined by

$$H_\pi(p)(v,v) = H_\Phi(p)(v,v) := \frac{-1}{2\Phi(p)} \Hess \Phi(p)(v,v),$$

where $p \in \Delta^{(n)}$ and $v \in T\Delta^{(n)}$. Here $\Hess \Phi$ is the Hessian of $\Phi$ regarded as a quadratic form. By definition, it is given by

$$\Hess \Phi(p)(v,v) = \left. \frac{d^2}{dt^2} \Phi(p + tv) \right|_{t=0}.$$ 

To motivate these functionals we state a decomposition formula of the relative value of a functionally generated portfolio which can be motivated by the vector field interpretation discussed in Section 2.2.

**Lemma 3.3 (Fernholz’s decomposition).** [Fer99, Theorem 3.1] [PW14, Lemma 6]

If $\pi$ is generated by a concave function $\Phi$, the relative value process $V_\pi$ has the
following decomposition in discrete time:

\[
\log V_\tau(t) = \log \frac{\Phi(\mu(t))}{\Phi(\mu(0))} + A(t),
\]

where \( A(t) = \sum_{k=0}^{t-1} T(\mu(k+1) | \mu(k)) \) is the cumulative energy and is non-decreasing.

If \((\pi, \Phi) \in \mathcal{F}\mathcal{G}^2\), the same formula holds in continuous time if \( A(t) \) is replaced by the stochastic integral

\[
\Theta(t) = \int_0^t H_\pi(\mu(t))(d\mu(t), d\mu(t))
\]

using the notation of [PW14, (27)]. The process \( \Theta \) is also non-decreasing. We call \( A(t) \) or \( \Theta(t) \) the drift process of the portfolio.

The key idea of the decomposition is that over any period \([t_0, t_1]\) where \( \log \Phi(\mu(t_1)) \) and \( \log \Phi(\mu(t_0)) \) are approximately equal, the portfolio will outperform the market by an amount equal to the cumulated energy over \([t_0, t_1]\), see Figure 1 for an illustration. For this reason, the drift process \( \Theta(t) \) can be thought of as the cumulative amount of market volatility captured by the portfolio. Empirical studies (see for example [FK09, Figure 11.2]) show that \( \Theta \) increases roughly at a linear rate depending on the portfolio and market volatility. Thus, as long as the fluctuation of \( \log \Phi(\mu(t)) \) is bounded, the drift process will dominate in the long run and the portfolio will outperform the market. The assumption on diversity is imposed precisely to bound \( \log \Phi(\mu(t)) \). For (say) the entropy-weighted portfolio, \( \log \Phi(\mu(t)) \) is bounded as long as \( \max_{1 \leq i \leq n} \mu_i(t) \leq 1 - \delta \) for some \( \delta > 0 \) (this is the definition of diversity stated in [Fer99] and [FK09]). For other portfolios such as the equal-weighted portfolio, this condition is not enough and we require that \( \mu(t) \) stays within a compact subset of \( \Delta(n) \). Fernholz’s decomposition is implemented in the \textit{R} package \textit{RelValAnalysis} (available on CRAN) written by the author.

In [PW14] pseudo-arbitrages with respect to the market portfolio are characterized in terms of a property called \textit{multiplicative cyclical monotonicity}, which is a variant of cyclical monotonicity in convex analysis (see [Roc97, Section 24]). Intuitively, this property requires that the portfolio outperforms the market whenever the market weight returns to an earlier position. It is natural to extend the definition as follow.

**Definition 3.4** (Relative multiplicative cyclical monotonicity - RMCM). Let \( \pi \) and \( \tau \) be portfolios. We say that \( \tau \) satisfies multiplicative cyclical monotonicity relative to \( \pi \) if over any discrete loop \( \mu(0), \mu(1), ..., \mu(m), \mu(m+1) = \mu(0) \) in \( \Delta(n) \), we have

\[
V_\tau(m+1) \geq V_\tau(m+1).
\]

The following result parallels [PW14, Theorem 3.1].

**Theorem 3.5.** Let \( \pi \) be a portfolio generated by a concave function \( \Phi : \Delta(n) \to (0, \infty) \), and let \( \tau \) be a portfolio.

(i) \( \tau \) satisfies MCM relative to \( \pi \) if and only if there exists a concave function \( \Psi : \Delta(n) \to (0, \infty) \) such that \( \tau \) is generated by \( \Psi \), and

\[
1 + \left\langle \frac{\tau(p)}{p}, q - p \right\rangle \geq \frac{\Psi(q)/\Psi(p)}{\Phi(q)/\Phi(p)},
\]

(ii) \( \tau \) satisfies RMCM relative to \( \pi \) if and only if there exists a concave function \( \Psi : \Delta(n) \to (0, \infty) \) such that \( \tau \) is generated by \( \Psi \), and

\[
1 + \left\langle \frac{\tau(p)}{p}, q - p \right\rangle \geq \frac{\Psi(q)/\Psi(p)}{\Phi(q)/\Phi(p)},
\]

for all \( p, q \in \Delta(n) \).

Theorem 3.5 parallels the result in [Fer99, Section 24] for the entropy-weighted portfolio.
for any \( p, q \in \Delta(n) \). Equivalently, the discrete energy \( T_\tau(q \mid p) \) of \((\tau, \Psi)\) dominates \( T_\pi(q \mid p) \) of \((\pi, \Phi)\) in the sense that

\[
T_\tau(q \mid p) \geq T_\pi(q \mid p)
\]

for all \( p, q \in \Delta(n) \). In this case, we have in discrete time the decomposition

\[
\log \frac{V_\tau(t)}{V_\tau(t)} = \log \frac{\Psi(\mu(t))}{\Psi(\mu(0))} + A(t),
\]

where \( A(t) \) is the difference between the cumulated discrete energies of \((\tau, \Psi)\) and \((\pi, \Phi)\) and is non-decreasing. An analogous formula holds in continuous time if \((\pi, \Phi), (\tau, \Psi) \in \mathcal{FG}^2\).

(ii) In discrete time, \( \tau \) is a pseudo-arbitrage with respect to \( \pi \) (i.e., \( \tau \gg \pi \)) if and only if \( \tau \) satisfies MCM relative to \( \pi \) and the difference \( T_\tau(q \mid p) - T_\pi(q \mid p) \) is not identically zero in \( p, q \in \Delta(n) \).

(iii) Suppose \((\pi, \Phi), (\tau, \Psi) \in \mathcal{FG}^2\). If \( H_\tau \geq H_\pi \) in the sense that

\[
H_\tau(p)(v, v) \geq H_\pi(p)(v, v)
\]

for all \( p \in \Delta(n) \) and tangent vectors \( v \) (i.e., the difference \( H_\tau - H_\pi \) is non-negative definite), then \( \pi \leq \tau \) in continuous time.

Proof. The proof of (i) is similar to that of [PWT14] Theorem 3.1 and we only highlight the differences. Suppose \( \tau \) satisfies MCM relative to \( \pi \). It is clear that \( \tau \) itself satisfies MCM [PWT14] Definition 4], so \( \tau \) is functionally generated and has a generating function \( \Psi \).

Let \( p, q \in \Delta(n) \) with \( p \neq q \). Let \( \{q = \mu(1), ..., \mu(m), \mu(m + 1) = p\} \) be a partition of the line segment \([q, p]\). Then if \( \mu(0) = p \), \( \{\mu(k)\}_{k=0}^{m+1} \) is a loop which starts at \( p \), jumps to \( q \) and then returns to \( p \) along the partition. Then the RMCM inequality [3.4] implies

\[
\left(1 + \frac{\tau(p)}{\mu}, q - p\right) \prod_{k=1}^{m} \left(1 + \frac{\tau(\mu(k))}{\mu(k)}, \mu(k) + 1 - \mu(k)\right)
\geq \left(1 + \frac{\pi(p)}{\mu}, q - p\right) \prod_{k=1}^{m} \left(1 + \frac{\pi(\mu(k))}{\mu(k)}, \mu(k) + 1 - \mu(k)\right).
\]

Taking log on both sides, we have

\[
\log \left(1 + \frac{\tau(p)}{\mu}, q - p\right) + \sum_{k=1}^{m} \log \left(1 + \frac{\tau(\mu(k))}{\mu(k)}, \mu(k) + 1 - \mu(k)\right)
\geq \log \left(1 + \frac{\pi(p)}{\mu}, q - p\right) + \sum_{k=1}^{m} \log \left(1 + \frac{\pi(\mu(k))}{\mu(k)}, \mu(k) + 1 - \mu(k)\right).
\]

By the fundamental theorem of calculus for concave function and Taylor approximation, we can choose a sequence of partitions with mesh size going to zero, along which

\[
\sum_{k=1}^{m} \log \left(1 + \frac{\pi(\mu(k))}{\mu(k)}, \mu(k) + 1 - \mu(k)\right) \to \int_{\gamma} \frac{\pi}{\mu} d\mu = \log \frac{\Phi(p)}{\Phi(q)},
\]

\[
\sum_{k=1}^{m} \log \left(1 + \frac{\tau(\mu(k))}{\mu(k)}, \mu(k) + 1 - \mu(k)\right) \to \int_{\gamma} \frac{\tau}{\mu} d\mu = \log \frac{\Psi(p)}{\Psi(q)}.
\]
where $\gamma$ is the line segment from $q$ to $p$. Taking the corresponding limit in (3.7), we obtain the desired inequality (3.5). The converse is straightforward and can be proved in the same way as in [PW14]. The statements (ii) and (iii) follow directly from the definition and Lemma 3.3. □

The following lemma shows that for portfolios in $FG^2$, if $\tau \succeq \pi$ holds in discrete time then the same is true in continuous time. Thus, to check that a portfolio in $FG^2$ is maximal in $FG^2$, it suffices to show that its drift quadratic form cannot be strictly dominated in the sense of (3.6).

Lemma 3.6. Let $(\pi, \Phi), (\tau, \Psi) \in FG^2$, and let $T_\pi$ and $T_\tau$ be their discrete energy functionals respectively. If $T_\pi (q \mid p) \geq T_\tau (q \mid p)$ for all $p, q \in \Delta^{(n)}$, then $H_\pi \succeq H_\tau$.

Proof. The lemma follows immediately from the Taylor approximation

$$T_\pi (p + tv \mid p) = \frac{-1}{2\Phi(p)} \text{Hess } \Phi(p)(tv, tv) + o(t^2),$$

where $p \in \Delta^{(n)}$, $v$ is a tangent vector, and $t \in \mathbb{R}$ is small. □

Example 3.7 (Diversity-weighted portfolio). For $0 < r < 1$, the diversity-weighted portfolio $\pi$ introduced at the beginning of this section is generated by the function

$$\Phi(p) = \left( \sum_{j=1}^{n} p_j^r \right)^{\frac{1}{r}} .$$

It is easy to show that $\Phi$ is bounded below by $n^{\frac{1}{r} - 1}$. Let $\tau$ be the portfolio generated by $\Psi := \Phi - n^{\frac{1}{r} - 1}$. Then $\tau$ is a pseudo-arbitrage relative to $\pi$. To see this (say in continuous time), we simply note that

$$\frac{-1}{2\Phi(p)} \text{Hess } \Psi(p)(v, v) = \frac{-1}{2\Phi(p)} \text{Hess } \Phi(p)(v, v) \geq \frac{-1}{2\Phi(p)} \text{Hess } \Phi(p)(v, v)$$

for all $p \in \Delta^{(n)}$ and $v \in T\Delta^{(n)}$. In general, it is easy to see that for a portfolio $(\pi, \Phi)$ to be maximal in $FG^2$ on $\Delta^{(n)}$, it is necessary that the continuous extension of $\Phi$ to the closure $\overline{\Delta^{(n)}}$ (which exists by [Roc97, Theorem 10.3]) vanishes at all the vertices $e(1), ..., e(n)$. (This is because subtracting an affine function from $\Phi$ does not affect the Hessian.) However this condition is not sufficient.

4. Relative concavity and maximal portfolios

4.1. Two asset case. In this section we study the maximal portfolios in $FG^2$ on $\Delta^{(n)}$. Before we state the main result, to illustrate the ideas involved we first give a proof of the maximality of the equal-weighted portfolio on $\Delta^{(n)}$ for $n = 2$.

Proposition 4.1. For $n = 2$, the equal-weighted portfolio $\pi \equiv \left( \frac{1}{2}, \frac{1}{2} \right)$ generated by the geometric mean $\Phi(p) = \sqrt{p_1p_2}$ is maximal in $FG^2$ on $\Delta^{(2)}$.

Proof. Define $u(x) = \Phi(x, 1 - x) = \sqrt{x(1-x)}$, $x \in (0, 1)$. Suppose $(\tau, \Psi) \in FG^2$ is a pseudo-arbitrage relative to $(\pi, \Phi)$, and let $v(x) = \Psi(x, 1 - x)$. Then $u$ and $v$ are positive $C^2$ concave functions on $(0, 1)$, and from (3.2) we have

$$-v''(x) \geq -u''(x) \frac{u(x)}{v(x)} = \frac{1}{4(x(1-x))^2}, \quad x \in (0, 1).$$

We claim that $v$ also generates the equal-weighted portfolio.
We will use a transformation which amounts to a change of numéraire using
\( y = \log \frac{x}{1-x} \). See the binary tree model in [PW13, Section 4] for the motivation of
this transformation and related results. Define a function \( \tau_1 : (0, 1) \to [0, 1] \) by
\[
\tau_1(x) = x + x(1-x) \frac{v'(x)}{v(x)} = x \left[ 1 + (1-x)(\log v)'(x) \right].
\]
By Lemma 2.6, this is the portfolio weight of the first stock generated by \( v \) and \( \tau_1 \) takes value in \([0, 1]\). Let \( y = \log \frac{x}{1-x} \), so \( x = \frac{e^y}{1+e^y} \). Define \( q : \mathbb{R} \to [0, 1] \) by
\[
q(y) = \tau_1(x) = \frac{e^y}{1+e^y} + \frac{e^y}{(1+e^y)^2} \frac{v'(x)}{v(x)}, \quad x = \frac{e^y}{1+e^y}, \quad y \in \mathbb{R}.
\]
For the equal-weighted portfolio the corresponding function is identically \( \frac{1}{2} \). It
follows from a straightforward computation that
\[
q(y)(1-q(y)) - q'(y) = -\frac{e^{2y}}{(1+e^y)^2} \frac{v''(x)}{v(x)}.
\]
Now (4.1) can be rewritten in the form
(4.3) \[ q(y)(1-q(y)) - q'(y) \geq \frac{1}{4}, \quad y \in \mathbb{R}. \]
The proof is then completed by the following lemma.

**Lemma 4.2.** Suppose \( q : \mathbb{R} \to [0, 1] \) is differentiable and \( q(1-q) - q' \geq 1/4 \) on \( \mathbb{R} \). Then \( q \equiv 1/2 \).

**Proof.** Since \( 0 \leq q(y) \leq 1 \), we have
\[
q' \leq q(1-q) - \frac{1}{4} \leq \frac{1}{4} - \frac{1}{4} = 0,
\]
and so \( q \) is non-increasing. If \( q(y_0) = q_0 < \frac{1}{2} \) for some \( y_0 \), then on \( y \in [y_0, \infty] \), \( q \)
must satisfy the differential inequality
\[
q'(y) \leq q_0(1-q_0) - \frac{1}{4} < 0,
\]
which contradicts the fact that \( q(y) \geq 0 \). Similarly, if \( q(y_0) = q_0 > \frac{1}{2} \) for some \( y_0 \),
the same inequality is satisfied on \((-\infty, y_0]\), again a contradiction. Thus we get
\( q(y) \equiv \frac{1}{2} \) for all \( y \in \mathbb{R} \).

The main idea of the proof of Proposition 4.1 is that for a portfolio to be a
pseudo-arbitrage relative to the equal-weighted portfolio \( \pi \), it must be more aggres-
sive than \( \pi \) everywhere on the simplex. This means buying more and more
the underperforming stock at a sufficiently fast rate satisfying (4.3), but this is
impossible to continue up to the boundary of the simplex. While there is a multi-
dimensional analogue of the inequality (4.3) (see [PW14, Theorem 9]), we are unable
to extend this proof to the general case since the market and portfolio weights can
move in many directions. Instead, we will work with portfolio generating functions
and use the simple but powerful tools of convex analysis.
4.2. Main result. Recall that for $i = 1, \ldots, n$, $e(i) = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the vertex of $\Delta^{(n)}$ in the $i$-th direction, and $\bar{e} = (\frac{1}{n}, \ldots, \frac{1}{n})$ is the barycenter of $\Delta^{(n)}$.

**Theorem 4.3.** Let $\pi \in \mathcal{FG}^2$ be generated by a measure of diversity $\Phi$. If

$$
\left(4.4\right)
\int_0^1 \frac{1}{\Phi((1-t)\bar{e} + te(1))^2} dt = \infty,
$$

then $\pi$ is maximal in $\mathcal{FG}^2$ on $\Delta^{(n)}$. For $n = 2$, this condition is also necessary for $\pi$ to be maximal in $\mathcal{FG}^2$ on $\Delta^{(2)}$.

Before we turn to the proof, we show that condition (4.4) is sufficient to capture many important examples.

**Corollary 4.4.** The following portfolios are maximal in $\mathcal{FG}^2$ on $\Delta^{(n)}$.

(i) The equal-weighted portfolio $\pi \equiv (\frac{1}{n}, \ldots, \frac{1}{n})$ generated by the geometric mean $\Phi(p) = (p_1 \cdots p_n)^\frac{1}{n}$.

(ii) The entropy-weighted portfolio $\pi_i = -(p_i \log p_i) / \Phi(p)$ generated by the Shannon entropy $\Phi(p) = -\sum_{j=1}^n p_j \log p_j$.

**Proof.** For the equal-weighted portfolio, we have

$$
\Phi((1-t)\bar{e} + te(1)) = \left(\frac{1}{n} + \left(1 - \frac{1}{n}\right)t\right)^\frac{1}{n} \left(\frac{1-t}{n}\right)^\frac{n-1}{n}.
$$

Note that the first coordinate of $(1-t)\bar{e} + te(1)$ is bounded away from zero for $t \in [0, 1)$. Since $\frac{2(n-1)}{n} \geq 1$, it follows that

$$
\int_0^1 \frac{1}{\Phi((1-t)\bar{e} + te(1))^2} dt \geq n \int_0^1 \frac{1}{(1-t)^{2(n-1)/n}} dt = \infty.
$$

Hence the equal-weighted portfolio is maximal in $\mathcal{FG}^2$ on $\Delta^{(n)}$.

Next we analyze the entropy-weighted portfolio. For notational convenience we consider instead

$$
t\bar{e} + (1-t)e(1) = \left(1 - \frac{n-1}{n} t, \frac{t}{n}, \ldots, \frac{t}{n}\right),
$$

so that the singularity is at $t = 0$. Now

$$
\Phi(t\bar{e} + (1-t)e(1)) = -\left(1 - \frac{n-1}{n} t\right) \log \left(1 - \frac{n-1}{n} t\right) - (n-1) \frac{t}{n} \log \frac{t}{n}.
$$

By l'Hôpital’s rule, for any $a, b > 0$ we have

$$
\lim_{t \to 0} \frac{-at \log at}{-(1-bt) \log(1-bt)} = \infty.
$$

So when $t > 0$ is small, we have the estimate

$$
\Phi(t\bar{e} + (1-t)e(1)) \leq C \left(- \frac{t}{n} \log \frac{t}{n}\right)
$$

where $C > 0$ is a constant. Since

$$
\int_0^1 \frac{1}{\left(\frac{t}{n} \log \frac{t}{n}\right)^2} dt = \infty,
$$

we see that the entropy-weighted portfolio is maximal in $\mathcal{FG}^2$ on $\Delta^{(n)}$ as well. □
The most important tool we need for the proof of Theorem 4.3 is the following ingenious observation taken from [CDO07] and [CDOS09, Lemma 2] (it is called the relative convexity lemma in these references). It can be proved by direct differentiation.

**Lemma 4.5** (Relative concavity lemma). [CDO07] Let $-\infty < a < b \leq \infty$ and $c, C : [a, b) \to \mathbb{R}$ be continuous. Suppose $u, v : [a, b) \to (0, \infty)$ are $C^2$ and satisfy the differential equations

$$u''(x) + c(x)u(x) = 0, \quad x \in [a, b),$$

$$v''(x) + C(x)v(x) = 0, \quad x \in [a, b).$$

Define $F : [a, b) \to [0, \infty)$ by

$$F(x) = \int_a^x \frac{1}{u(t)^2}dt, \quad x \in [a, b).$$

Let $G$ be the inverse of $F$ defined on $[0, \ell)$, where $\ell = \lim_{x \to b} F(x)$. Then the function $w(y) := v(G(y))u(G(y))^{\lambda} \in [0, \ell)$ satisfies the differential equation

$$w''(y) = -(C(x) - c(x))c(x)w(y), \quad 0 \leq y < \ell, \quad x = G(y).$$

In particular, if $C(x) \geq c(x)$ on $[a, b)$, then $w$ is concave on $[0, \ell)$.

We also need some convex analytic properties of functionally generated portfolios.

**Lemma 4.6.** Suppose $\pi^{(1)}, \pi^{(2)} \in \mathcal{F}\mathcal{G}$ are generated by $\Phi^{(1)}$ and $\Phi^{(2)}$ respectively, and $\lambda \in [0, 1]$. Then the portfolio

$$\pi(p) := \lambda \pi^{(1)}(p) + (1 - \lambda)\pi^{(2)}(p), \quad p \in \Delta^n,$$

belongs to $\mathcal{F}\mathcal{G}$. Indeed, $\pi$ is generated by the geometric mean $\Phi := (\Phi^{(1)})^\lambda (\Phi^{(2)})^{1-\lambda}$ of the two generating functions.

**Proof.** For $C^2$ generating functions this result is stated in [Fer02, Page 50]. The same is true in the general case where the generating functions are not necessarily smooth. To prove this, we need to check that $\pi = \lambda \pi^{(1)} + (1 - \lambda)\pi^{(2)}$ satisfies the defining inequality (2.3). Using (2.3) for $\pi^{(1)}$ and $\pi^{(2)}$, we see that

$$1 + \left< \frac{\pi(p)}{p}, q - p \right> \geq \lambda \frac{\Phi^{(1)}(q)}{\Phi^{(1)}(p)} + (1 - \lambda) \frac{\Phi^{(2)}(q)}{\Phi^{(2)}(p)}.$$

Using the elementary AM-GM inequality

$$\lambda x + (1 - \lambda)y \geq x^\lambda y^{1-\lambda}$$

which holds for $x, y > 0$ and $\lambda \in [0, 1]$, we can bound (4.5) below by

$$\frac{(\Phi^{(1)}(q))^\lambda (\Phi^{(2)}(q))^{1-\lambda}}{(\Phi^{(1)}(p))^\lambda (\Phi^{(2)}(q))^{1-\lambda}} = \frac{\Phi(q)}{\Phi(p)}.$$
Lemma 4.7. The discrete energy functional and the drift quadratic form are concave in the portfolio weights in the following sense. Let \((\pi^{(1)}, \Phi^{(1)}), (\pi^{(2)}, \Phi^{(2)}) \in \mathcal{F}\). For \(\lambda \in [0, 1]\), let \(\pi = \lambda \pi^{(1)} + (1 - \lambda) \pi^{(2)}\) and let \(\Phi = (\Phi^{(1)})^\lambda (\Phi^{(2)})^{1-\lambda}\) be the generating function of \(\pi\). Let \(T, T^{(1)}\) and \(T^{(2)}\) be the discrete energy functionals of \((\pi, \Phi), (\pi^{(1)}, \Phi^{(1)})\) and \((\pi^{(2)}, \Phi^{(2)})\) respectively. Then

\[
T(q \mid p) \geq \lambda T^{(1)}(q \mid p) + (1 - \lambda) T^{(2)}(q \mid p), \quad p, q \in \Delta^{(n)}.
\]

If \(\Phi^{(1)}\) and \(\Phi^{(2)}\) are \(C^2\), then \(H_{\pi} \geq \lambda H_{\pi}^{(1)} + (1 - \lambda) H_{\pi}^{(2)}\) in the sense that

\[
H_{\pi}(p)(v, v) \geq \lambda H_{\pi}^{(1)}(p)(v, v) + (1 - \lambda) H_{\pi}^{(2)}(p)(v, v)
\]

for all \(p \in \Delta^{(n)}\) and \(v \in T\Delta^{(n)}\).

Proof. To prove (4.6) we write \(T(q \mid p)\) for a functionally generated portfolio \((\pi, \Phi)\) in the form

\[
T(q \mid p) = \log \left(1 + \frac{\pi(p)}{\pi(q)} - 1\right) - I_{\pi}(\gamma),
\]

where (as in (2.8)) \(I_{\pi}(\gamma) = \int_\gamma \frac{\pi}{p} dp\) is the line integral of the weight ratio along the line segment from \(p\) to \(q\). Since the line integral is linear in \(\pi\) and the logarithm is concave, we see that \(T(q \mid p)\) is concave in \(\pi\). The statement for the drift quadratic form follows from the Taylor approximation (3.8).

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. Suppose \(\Psi \in \mathcal{F}^2\) generates a portfolio \(\tau\) which is a pseudo-arbitrage relative to the portfolio \(\pi\) generated by \(\Phi\). We will prove that \(\Psi\) is a constant positive multiple of \(\Phi\), so \(\Psi\) generates \(\pi\) as well. By scaling, we may assume that \(\Psi(e) = \Phi(e)\). We divide the proof into the following steps.

Step 1 (Symmetrization). Let \(S_n\) be the set of permutations of \(\{1, \ldots, n\}\). For \(\sigma \in S_n\), define \(\Psi_\sigma\) by relabelling the coordinates, i.e.,

\[
\Psi_\sigma(p) = \Psi(p_{\sigma(1)}, \ldots, p_{\sigma(n)}).
\]

Since \(\tau \succeq \pi\), we have \(H_{\Psi_\sigma} \geq H_{\Phi_\sigma}\) for all \(\sigma \in S_n\). But \(\Phi\) is a measure of diversity, so \(\Phi_\sigma = \Phi\) by symmetry and we have \(H_{\Psi_\sigma} \geq H_{\Phi}\) for all \(\sigma \in S_n\). Let

\[
\bar{\Psi} = \prod_{\sigma \in S_n} (\Psi_\sigma)^{\frac{1}{n!}}
\]

be the symmetrization of \(\Psi\). By Lemma 4.6, it generates the symmetrized portfolio

\[
\bar{\tau}(p) = \frac{1}{n!} \sum_{\sigma \in S_n} \tau(p_{\sigma(1)}, \ldots, p_{\sigma(n)}), \quad p \in \Delta^{(n)}.
\]

By Lemma 4.7, we have

\[
H_{\bar{\Psi}} \geq \frac{1}{n!} \sum_{\sigma \in S_n} H_{\Psi_\sigma} \geq H_{\Phi}.
\]

It follows that \(\bar{\tau} \succeq \pi\). Clearly \(\bar{\Psi}\) is a measure of diversity and by symmetry it achieves its maximum at \(\tau\).
Step 2 ($\tilde{\Psi} \leq \Phi$). We claim that $\tilde{\Psi} \leq \Phi$ on $\Delta^{(n)}$. Let $p \in \Delta^{(n)}$ be fixed, and consider the one-dimensional concave functions

\begin{equation}
\begin{aligned}
u(t) &= \Phi((1-t)p + tp) \\
u(t) &= \tilde{\Psi}((1-t)p + tp)
\end{aligned}
\end{equation}

(4.9)
defined on $[0,1]$. We have $u(0) = v(0)$ and $u'(0) = v'(0) = 0$ since both $\Phi$ and $\tilde{\Psi}$ achieve their maximums at $p$. Since $H_{\tilde{\Psi}} \geq H_\Phi$, we have

$$\frac{-\nu''(t)}{\nu(t)} \geq \frac{-\nu''(t)}{u(t)}, \quad t \in [0,1].$$

By the relative concavity lemma,

\begin{equation}
w(y) = \frac{v(G(y))}{u(G(y))}
\end{equation}

(4.10)
is a positive concave function on $[0,\ell]$, where $\ell = \int_0^1 \frac{1}{u(t)} dt$, with $w(0) = 1$ and $w'(0) = 0$ (by the quotient rule). This implies that $w$ is non-increasing and thus $w(1) = \tilde{\Psi}(p)/\Phi(p) \leq 1$.

Step 3 ($\tilde{\Psi} \equiv \Phi$). Let $Z = \{p \in \Delta^{(n)} : \tilde{\Psi}(p) = \Phi(p)\}$ and we claim that $Z = \Delta^{(n)}$. Here we follow an idea in the proof of \cite[Theorem 3]{CDOS09}. Define $u$ and $v$ on $[0,1]$ by (4.9) with $p$ replaced by $e(1)$. Then the function $w$ defined as in (4.10) is positive and concave on $[0,\ell]$ since the integral in (4.4) (which defines $\ell$) diverges. Again $w$ satisfies $w(0) = 1$ and $w'(0) = 0$. But since $w$ is defined on an infinite interval, if $w(y) < 0$ for some $y$, then $w$ must hit zero as $w'$ is non-increasing by concavity. This contradicts the positivity of $w$, and so $w$ is identically one on $[0,\infty)$. It follows that $\tilde{\Psi} = \Phi$ on the line segment $[\pi,e(1))$. By symmetry, $Z$ contains the segments $[\pi,e(i))$ for all $i$.

Next we show that the set $Z$ is convex. Let $p, q \in Z$. Again we consider the pair of functions

\begin{equation}
\begin{aligned}
u(t) &= \Phi((1-t)p + tp) \\
u(t) &= \tilde{\Psi}((1-t)p + tp)
\end{aligned}
\end{equation}

(4.11)
on $[0,1]$. Let $\tilde{w}(t) = \frac{\nu(t)}{\nu'(t)}$, $t \in [0,1]$. By the relative concavity lemma again, we know that $\tilde{w}$ is a concave function after a reparameterization. But $\tilde{w}(t) \leq 1$ by Step 2 and $\tilde{w}$ equals one at the endpoints. By concavity, $\tilde{w}$ is identically one on $[0,1]$. Hence if $Z$ contains $p$ and $q$, it also contains the line segment $[p,q]$. Now $Z$ is a convex set which contains $[\pi,e(i))$ for all $i$. It is easy to see that $Z$ is then the simplex $\Delta^{(n)}$, and hence $\tilde{\Psi}$ equals $\Phi$ identically.

Step 4 (Desymmetrization). We have shown that $\tilde{\Psi} \equiv \Phi$, and so $H_{\tilde{\Psi}} = H_\Phi$. By (4.8), we have

$$H_\Phi = H_{\tilde{\Psi}} \geq \frac{1}{n!} \sum_{\sigma \in S_n} H_{\Psi^{\sigma}} \geq H_\Phi.$$  

Since each $H_{\Psi^{\sigma}} \geq H_\Phi$ (as quadratic forms), we see that $H_{\Psi^{\sigma}} = H_\Phi$ for all $\sigma$. In particular, taking $\sigma$ to be the identity, we have $H_{\Psi} = H_\Phi$. We claim that $\Psi$ equals $\Phi$ identically. (Recall that we assume $\Psi(\pi) = \Phi(\pi)$.)
Fix $i \in \{1, ..., n\}$ and consider
\[ u(t) = \Phi((1 - t)e + te(i)) \]
\[ v(t) = \Psi((1 - t)e + te(i)) \]
for $t \in [0, 1)$. By the argument in Step 3, if \( (\frac{v}{u})'(0) \leq 0 \), the integral condition (4.4) implies that \( \frac{v}{u} \) is identically one, and so \( (\frac{v}{u})'(0) \leq 0 \) implies \( (\frac{v}{u})'(0) = 0 \).

For $\sigma \in S_n$ let
\[ v_\sigma(t) = \Psi((1 - t)e + te(\sigma(i))). \]
Since $\tilde{\Psi} = \Phi$, we have
\[ \prod_{\sigma \in S_n} \left( \frac{v_\sigma(t)}{u(t)} \right)^{\frac{1}{n}} = 1. \]
Taking logarithm on both sides and differentiating, we see that the average of the derivatives \( (\frac{v}{u})'(0) \) over $i$ is 0 (recall that $\Phi$ is symmetric). Since all derivatives are non-negative by the above argument, in fact they are all 0, and so $\Psi = \Phi$ on $[e, e(\sigma(i))]$ for all $i$.

Since the vectors $e(i) - e$ span the plane parallel to $\Delta^{(n)}$, the graphs of $\Psi$ and $\Phi$ have the same tangent plane at $e$. Since $\Phi$ achieves its maximum at $e$, we see that $\Psi$ achieves its maximum at $e$ as well. Now we may apply Steps 2 and 3 to conclude that $\Psi$ equals $\Phi$ identically on $\Delta^{(n)}$.

Finally, we note that [CDOS09, Theorem 3] shows that the integral condition (4.4) is also necessary for $\pi$ to be maximal in $\mathcal{FG}^2$ when $n = 2$. Let $u(x) = \Phi(x, 1 - x)$. The idea is that if the integral converges, we can solve the initial value problem
\[ \frac{d^2}{dx^2}v(x) + \left( \frac{d}{dx} \left( \frac{u''(x)}{u(x)} \right) + s(x) \right) v(x) = 0, \quad x \in (0, 1), \]
\[ v \left( \frac{1}{2} \right) = u \left( \frac{1}{2} \right), \quad v' \left( \frac{1}{2} \right) = u' \left( \frac{1}{2} \right) = 0, \]
for some appropriately chosen function $s(x)$ such that $s(x) \geq 0$, $s(x) \neq 0$ and $s$ is symmetric about $\frac{1}{2}$. Sturm’s comparison theorem implies that the solution $v(x)$ is positive (and concave) on $(0, 1)$. So $\Psi(p) = v(p_1)$ is a portfolio generating function, and the corresponding portfolio is a pseudo-arbitrage relative to $\pi$.

\[ \square \]

**Problem 4.8.** Characterize all maximal portfolios in $\mathcal{FG}^2$ on $\Delta^{(n)}$.

5. Optimization of functionally generated portfolios

5.1. A shape-constrained optimization problem. In this section we work under the discrete time framework which is more convenient for applications. Let $(\pi, \Phi) \in \mathcal{FG}$ be a functionally generated portfolio. Recall that
\[ \log V_\pi(t) = \log \Phi(\mu(t)) + A(t), \]
where
\[ A(t) = \sum_{k=0}^{t-1} \mathbf{T}(\mu(k + 1) | \mu(k)) \]
is the drift process.
If we have a stochastic model for the market weight process \( \{ \mu(t) \} \), a natural optimization problem is to maximize the expected growth rate of the drift process over some horizon. To this end, suppose we are given an intensity measure \( P \) of the increments \( (\mu(t), \mu(t+1)) \) modeled as a Borel probability measure on \( \Delta^{(n)} \times \Delta^{(n)} \). We assume that \( P \) is either discrete (with countably many masses) or continuous, i.e., \( P \) has a density relative to the measure \( \nu := m \otimes m \) on \( \Delta^{(n)} \times \Delta^{(n)} \), where \( m \) is the surface measure of \( \Delta^{(n)} \) in \( \mathbb{R}^n \) (which should be thought of as the Lebesgue measure on \( \Delta^{(n)} \)). For technical reasons, we will assume that \( P \) is supported on \( K \times K \) for some compact subset \( K \) of \( \Delta^{(n)} \times \Delta^{(n)} \).

**Example 5.1.** Suppose \( \{(\mu(t-1), \mu(t))\} \) is an ergodic Markov chain on \( \Delta^{(n)} \times \Delta^{(n)} \). We can take \( P \) to be the stationary distribution of \( (\mu(t-1), \mu(t)) \). It is easy to see that the optimal portfolio in (5.1) maximizes the asymptotic growth rate \( \lim_{t \to \infty} \frac{1}{t} A(t) \) of the drift process. This portfolio can be regarded as the growth optimal portfolio among the functionally generated portfolios.

**Example 5.2.** We model \( \{\mu(t)\} \) as a stochastic process. Let \( K \) be a compact subset of \( \Delta^{(n)} \) containing \( \mu(0) \). Let \( \tau \) be the first exit time of \( K \), i.e.,

\[
\tau = \inf\{t \geq 0 : \mu(t) \notin K\}.
\]

Consider the Borel measure \( G \) on \( K \times K \) defined by

\[
G(A) := E \left[ \sum_{t=1}^{\tau-1} \mathbf{1}_{\{ (\mu(t-1), \mu(t)) \in A \}} \right], \quad A \subset K \times K \text{ measurable}.
\]

If the process \( \{(\mu(t-1), \mu(t))\} \) is Markovian, \( G \) is the Green kernel of the process killed at time \( \tau \). Suppose \( G(K \times K) = E(\tau - 1) < \infty \), i.e., the exit time has finite expectation. Then

\[
P(\cdot) := \frac{1}{G(K \times K)} G(\cdot)
\]

is a probability measure on \( K \times K \). This measure will be used in the empirical example in Section 6.

Given the intensity measure \( P \), we consider the optimization problem

\[
\max_{(\pi, \Phi) \in \mathcal{F}G} \int T(q | p) \, dP.
\]

This is a shape-constrained optimization problem because the generating function \( \Phi \) is required to be concave. We will first study some theoretical properties of this abstract (unconstrained) optimization problem, and then focus on a discrete special case where numerical solutions are possible and further constraints are imposed. In contrast to classical portfolio selection theory where the portfolio weights are optimized period by period, in (5.1) we optimize the portfolio weights over a region simultaneously.

Throughout the development it is helpful to keep in mind the analogy between (5.1) and the maximum likelihood estimation of a log-concave density. In that context, we are given a random sample \( X_1, \ldots, X_N \) from a log-concave density \( f_0 \) on \( \mathbb{R}^d \) (i.e., \( \log f_0 \) is concave). The log-concave MLE \( \hat{f} \) is the solution of

\[
\max_{f} \sum_{j=1}^{N} \log f(X_j),
\]
where \( f \) ranges over all log-concave densities on \( \mathbb{R}^d \). It can be shown that the MLE exists almost surely (when \( N \geq d + 1 \) and the support of \( f_0 \) has full dimension) and is unique; see [CSS10] for precise statements of these results. We remark that (5.1) is more complicated than (5.2) because the portfolio weights correspond to selections of the superdifferential \( \partial \log \Phi \), whereas (5.2) involves only the values of the density.

5.2. Theoretical properties. It is easy to check that (5.1) is a convex optimization problem since the discrete energy functional is concave in the portfolio weights (Lemma 4.7). First we show that (5.1) has an optimal solution and study in what sense the solution is unique.

If the measure \( \mathbb{P} \) is continuous, it can be decomposed in the form
\[
\mathbb{P} = g(p)h(q \mid p)\,dm(q)\,dp(p),
\]
where \( g(p) \) represents the density of the starting point of the increment and \( h(q \mid p) \) is the conditional density given that the starting point is \( p \).

Definition 5.3 (Support condition). Suppose \( \mathbb{P} \) is a continuous probability measure on \( \Delta^{(n)} \times \Delta^{(n)} \) given by (5.3). We say that \( \mathbb{P} \) satisfies the support condition if for \( m \)-almost all \( p \) for which \( g(p) > 0 \) the support of \( h(\cdot \mid p) \) has full dimension, i.e., the affine hull of the support of \( h(\cdot \mid p) \) contains \( \Delta^{(n)} \).

The support condition says that the directions of possible jumps span \( T\Delta^{(n)} \). We have the following result which is analogous to [CSS10, Theorem 1].

Theorem 5.4. Consider the optimization problem (5.1) where \( \mathbb{P} \) is a discrete or continuous Borel probability measure on \( \Delta^{(n)} \times \Delta^{(n)} \) supported on \( K \times K \) with \( K \subset \Delta^{(n)} \) compact.

(i) The problem has an optimal solution.
(ii) If \( \pi^{(1)} \) and \( \pi^{(2)} \) are optimal solutions, then
\[
\left\langle \frac{\pi^{(1)}(p)}{p} , q - p \right\rangle = \left\langle \frac{\pi^{(2)}(p)}{p} , q - p \right\rangle
\]
for \( \mathbb{P} \)-almost all \( (p,q) \). In particular, if \( d\mathbb{P} = g(p)h(q \mid p)\,dm(q)\,dp(p) \) is continuous and satisfies the support condition, then \( \pi^{(1)} = \pi^{(2)} \) \( m \)-almost everywhere on \( \{ p : g(p) > 0 \} \).

The proofs of Theorem 5.4 and Theorem 5.5 below are given in Appendix A.

Suppose \( \mathbb{P} \) is continuous and \( \{ \mathbb{P}_N \}_{N \geq 1} \) is a sequence of probability measures converging weakly to \( \mathbb{P} \). By definition, this means that
\[
\lim_{N \to \infty} \int f\,d\mathbb{P}_N = \int f\,d\mathbb{P}
\]
for all bounded continuous functions on \( \Delta^{(n)} \times \Delta^{(n)} \). For example, one may sample i.i.d. observations \( \{(p(j),q(j))\}_{j=1}^N \) from \( \mathbb{P} \) and take \( \mathbb{P}_N \) to be the empirical measure \( \frac{1}{N} \sum_{j=1}^N \delta_{(p(j),q(j))} \), where \( \delta_{(p(j),q(j))} \) is the point mass at \( (p(j),q(j)) \). From the perspective of statistical inference, the optimal portfolio \( (\hat{\pi}^{(N)}, \hat{\Phi}^{(N)}) \) for \( \mathbb{P}_N \) can be regarded as a point estimate of the optimal portfolio \( (\pi, \Phi) \) for \( \mathbb{P} \). The following result states that the estimator is consistent. See [CSS10, Theorem 4] for an analogous statement in the context of log-concave density estimation.
| Constraint | Interpretation |
|-----------|----------------|
| $a_i \leq \pi_i(p) \leq b_i$ | Box constraints on portfolio weights |
| $m_i \leq \frac{\pi_i(p)}{p_i} \leq M_i$ | Box constraints on weight ratios |
| $(\pi(p) - p)' \Sigma(\pi(p) - p) < \varepsilon$ | Constraint on tracking error given a covariance matrix |

Table 2. Examples of additional constraints imposed for $p \in \{p(1), ..., p(N)\}$. The parameters may be given functions of $p$.

**Theorem 5.5.** Let $(\pi, \Phi)$ be the optimal portfolio in problem (5.1) for $\mathbb{P}$, where $d\mathbb{P} = g(p)h(q | p) dm(q)dm(p)$ is continuous, supported on $K \times K$ with $K \subset \Delta^{(n)}$ compact, and satisfies the support condition. Let $\{\mathbb{P}_N\}$ be a sequence of discrete or continuous probability measures on $K \times K$ such that $\mathbb{P}_N \rightarrow \mathbb{P}$ weakly, and suppose $(\hat{\pi}^{(N)}, \hat{\Phi}^{(N)})$ is optimal for the measure $\mathbb{P}_N$, $N \geq 1$. Then $\hat{\pi}^{(N)} \rightarrow \pi$ m-almost everywhere on $\{p : g(p) > 0\}$.

5.3. **Discrete special case.** Without further constraints, the optimal portfolio weights of (5.1) may be highly irregular. Now we restrict to the special case where

\[
\mathbb{P} = \frac{1}{N} \sum_{j=1}^{N} \delta_{(p(j),q(j))}
\]

is a discrete measure and $(p(j), q(j)) \in \Delta^{(n)} \times \Delta^{(n)}$ for $j = 1, ..., N$. This presents no great loss of generality because in practice the market weights have finite precision and we can choose the pairs $(p(j), q(j))$ to take values on a grid approximating $\Delta^{(n)} \times \Delta^{(n)}$. Moreover, from Theorem 5.5 we expect that when $N$ is large the optimal solution approximates that of the continuous counterpart. Consider the modified optimization problem

\[
\text{maximize} \quad \int T(q | p) \, d\mathbb{P} \\
\text{subject to} \quad (\pi(p(1)), ..., \pi(p(N))) \in C,
\]

where $C$ is a given closed convex subset of $\Delta^{(n)}_N$. Some examples of $C$ are given in Table 2 where each constraint is a cylinder set of the form $\{\pi(p(j)) \in C_j\}$ with $C_j$ a closed convex set of $\Delta^{(n)}$. ‘Global’ constraints on the weights can be imposed, see Section 6 for an example. It can be verified easily that the proof of Theorem 5.4 goes through without changes with these constraints, so (5.6) has an optimal solution. Moreover, if $\pi^{(1)}$ and $\pi^{(2)}$ are optimal solutions, then

\[
\left\langle \frac{\pi^{(1)}(p(j))}{p(j)}, q(j) - p(j) \right\rangle = \left\langle \frac{\pi^{(2)}(p(j))}{p(j)}, q(j) - p(j) \right\rangle, \quad j = 1, ..., N.
\]

For maximum likelihood estimation of a log-concave density, it is shown in [CSS10] that the logarithm of the MLE $\hat{f}$ is polyhedral, i.e., log $\hat{f}$ is the pointwise minimum of several affine functions (see [Roc97, Section 19]). In particular, there exists a triangulation of the data points over which log $\hat{f}$ is piecewise affine. We show that
Figure 2. The figure on the left shows the growth of $1 for each asset, and the one on the right shows the time series of the market weight $\mu_1(t)$ of US. The vertical dotted line divides the data set into the training and testing periods respectively.

an analogous statement holds for (5.6). Let $D = \{p(j), q(j) : j = 1, ..., N\}$ be the set of data points.

**Theorem 5.6.** Let $(\pi, \Phi)$ be an optimal portfolio for the problem (5.6) where $\mathbb{P} = \frac{1}{N} \sum_{j=1}^{N} \delta_{(p(j), q(j))}$. Let $\overline{\Phi} : \Delta^{(n)} \to (0, \infty)$ be the smallest positive concave function on $\Delta^{(n)}$ such that $\overline{\Phi}(x) \geq \Phi(x)$ for all $x \in D$. Then $\overline{\Phi}$ is a polyhedral positive concave function on $\Delta^{(n)}$ satisfying $\overline{\Phi} \leq \Phi$ and $\overline{\Phi}(p(j)) = \Phi(p(j))$, $\overline{\Phi}(q(j)) = \Phi(q(j))$ for all $j$. Moreover, $\overline{\Phi}$ generates a portfolio $\pi$ such that $\pi(p(j)) = \pi(p(j))$ for all $j$. In particular, $(\pi, \overline{\Phi})$ is also optimal for the problem (5.6).

**Proof.** It is a standard result in convex analysis that $\overline{\Phi}$ such defined is finitely generated (see [Roc97, Section 19]). By [Roc97, Corollary 19.1.2], $\overline{\Phi}$ is a polyhedral concave function. By definition of $\overline{\Phi}$ and concavity of $\Phi$, we have $\overline{\Phi}(p(j)) = \Phi(p(j))$ and $\overline{\Phi}(q(j)) = \Phi(q(j))$ for all $j$. This implies that $\partial \log \overline{\Phi}(p(j)) \subset \partial \log \Phi(p(j))$ for all $j$, and so $\overline{\Phi}$ generates a portfolio $\pi$ which agrees with $\pi$ on $\{p(1), ..., p(N)\}$. It follows that (using obvious notations)

$$T(q(j)|p(j)) = T(q(j)|p(j))$$

for all $j$, and hence $(\pi, \overline{\Phi})$ is optimal for (5.6).

Theorem 5.6 reduces the shape-constrained problem (5.6) to a finite-dimensional problem. In the next section we present an elementary implementation for the case $n = 2$ (analogous to univariate density estimation) and illustrate its application in portfolio management with a case study.

6. **Empirical examples**

6.1. **A case study.** In global portfolio management, an important topic is the determination of the aggregate portfolio weights for countries. In this example
we consider two countries: US and China. We represent them by the S&P US BMI index (asset 1) and the S&P China BMI index (asset 2) respectively. The ‘market’ consists of these two assets. We collect monthly data from January 2001 to June 2014 using Bloomberg. The benchmark portfolio is taken to be the buy-and-hold portfolio starting with weights (0.5, 0.5) at January 2001. Here the initial market weights (0.5, 0.5) are chosen arbitrarily. The data from January 2001 to December 2010 will be used as the training data to optimize the portfolio which will be backtested in the subsequent period. The market weights at January 2011 are (0.1819, 0.8191). The data is plotted in Figure 2.

Let $K \subset \Delta^{(2)}$ be the compact set defined by

$$K = \{p = (p_1, p_2) \in \Delta^{(2)} : 0.1 \leq p_1 \leq 0.3\}. \tag{6.1}$$

Our objective here is to optimize a functionally generated portfolio to be held as long as the market weights stay within $K$. If the market weight of US approach these boundary points (regarded as a regime change), a new portfolio will be chosen, so 0.1 and 0.3 can be thought of as the **trigger points**.

6.2. **The intensity measure and constraints.** Suppose $t = 0$ corresponds to January 2011. We model $\{\mu(t)\}_{t \geq 0}$ as a discrete-time stochastic process (time is monthly) where $\mu(0)$ is constant. We take $\mathbb{P}$ to be the measure in Example 5.2 where $\tau$ is the first exit time of $K$ given in (6.1).

If a stochastic model is given, we may approximate $\mathbb{P}$ by simulating paths of $\{\mu(t)\}$ killed upon exiting $K$. The resulting discrete measure $\mathbb{P}_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{(\mu(j), q(j))}$ is then taken as the underlying measure of the optimization problem (5.6).

Since our main concern is the implementation of the optimization problem (5.6), sophisticated modeling of $\{\mu(t)\}$ will not be attempted and we will use a simple method to simulate paths of $\{\mu(t)\}$. Namely, starting at $\mu(0) = (0.1819, 0.8191)$, we simulate paths of $\{\mu(t)\}_{t=0}^{\tau-1}$ by bootstrapping the past returns of the two assets and computing the corresponding market weight series. However, in view of the possible recovery of US and slowdown of China, we recentered the past returns...
0.10 0.15 0.20 0.25 0.30
0.10 0.15 0.20 0.25 0.30

Figure 4. The portfolio weight and the generating function of the optimized portfolio.

so that they both have mean zero over the training period. (Essentially, only the difference in returns matter for the evolution of market weights.) We simulated 50 such paths and obtained $N = 3115$ pairs $(p(j), q(j))$ in $K \times K$. A density estimate of $P_N$ (in terms of the market weight of US) is plotted in Figure 3. To reduce the number of variables, the market weights are rounded to 3 decimal places, so the market weights of US take values in the set $D = \{0.100, 0.101, \ldots, 0.299, 0.300\}$.

We also need to specify the constraints for \{\pi(p_1) := \pi(p_1, 1 - p_1) : p_1 \in D\}. (This notation should cause no confusion since the market weight of China is determined by that of US.) First, we require that $\pi_1(p_1)$ is non-decreasing in $p_1$, i.e.,

$$\pi_1(0.100) \leq \pi_1(0.101) \leq \cdots \leq \pi_1(0.300).$$

This imposes a shape constraint on the portfolio weights which guarantees that the portfolio weights always move in the direction of market movement. Moreover, to control the concentration of the portfolio we require that the weight ratio of US satisfies $0.5 \leq \pi_1(p_1) p_1 \leq 2$ for $p_1 \in D$ (since there are only two assets, this implies a weight ratio bound for China). These constraints determine the convex set $C$ in the optimization problem (5.6) we are about to solve.

6.3. Optimization procedure. By Theorem 5.6, it suffices to optimize over generating functions which are piecewise linear over the data points. First we introduce some simplifying notations. Write the set of grid points as $D = \{x_1 < x_2 < \cdots < x_m\}$ and let $x_0 = 0$, $x_{m+1} = 1$ be the endpoints of the interval. Let the decision variables be

$$z_j := \pi(x_j, 1 - x_j), \quad j = 1, \ldots, m,$$
$$\varphi_j := \Phi(x_j, 1 - x_j), \quad j = 0, \ldots, m + 1.$$

By scaling, we may assume $\varphi_1 = 1$. The constraints on \{\varphi_j\} are

$$\varphi_j \geq 0, \quad j = 0, \ldots, m + 1, \quad \varphi_1 = 1, \quad \text{(non-negativity)}$$

$$s_0 \geq s_1 \geq \cdots \geq s_m, \quad s_j := \frac{\varphi_{j+1} - \varphi_j}{x_{j+1} - x_j}, \quad \text{(concavity)}$$
We require that \( \pi \) is generated by \( \Phi \). By (4.2) and Lemma 2.4 it can be seen that \( z_j \) satisfies the inequality

\[
(6.4) \quad x_j + x_j (1 - x_j) \frac{\delta_j}{\varphi_j} \leq z_j \leq x_j + x_j (1 - x_j) \frac{\delta_{j-1}}{\varphi_j}, \quad j = 1, \ldots, m. \quad ((\pi, \Phi) \in \mathcal{FG})
\]

We require that \( z_j \) is non-decreasing in \( j \):

\[
(6.5) \quad z_1 \leq z_2 \leq \cdots \leq z_m. \quad \text{(monotonicity)}
\]

Finally, we require that the weight ratios are bounded between 0.5 and 2:

\[
(6.6) \quad 0.5 \leq \frac{z_j}{x_j} \leq 2, \quad j = 1, \ldots, m. \quad \text{(weight ratios)}
\]

With the constraints (6.2)-(6.6) we maximize

\[
\int T(q \mid p) \, d\mathbb{P} = \frac{1}{N} \sum_{j=1}^{N} T(q(j) \mid p(j))
\]

over \( \{z_j\} \) and \( \{\varphi_j\} \). This is a standard non-linear, but smooth, constrained optimization problem (convexity is lost because \( \Phi \) is now piecewise linear). We implement this optimization problem using the \texttt{fmincon} function in MATLAB. The optimal portfolio weights together with the generating function are plotted in Figure 4. It turns out that the optimal portfolio is close to constant-weighted (with weights \((0.2331, 0.7669)\)). Note that the constraint on the weight ratio limits the deviation of \( \pi_1(p_1) \) from the market weight \( p_1 \). If the weight ratio constraint was not imposed (while the monotonicity constraint was kept), the optimal portfolio would be the equal-weighted portfolio \( \pi \equiv (0.5, 0.5) \), and the reason can be seen from the proof of Lemma 4.2.
6.4. Backtesting the portfolio. Finally, we compute the performance of the optimized portfolio over the testing period January 2011 to June 2014. The result (plotted using the function `FernholzDecomp` of the `RelValAnalysis` package) is shown in Figure 5. Over the testing period, the portfolio beats the market by nearly 2% in log scale and its performance has been steady. From the decomposition, about half of the outperformance is attributed to the increase of the generating function (note that the market weight of US becomes closer to 0.2331 where the generating function attains its maximum), and the rest comes from the cumulated energy. We remark that market volatility decreased after the financial crisis, so the drift process has only moderate growth rate. That the optimal portfolio is close to constant-weighted may not be very interesting, but this is a consequence of the data and our choice of constraints and is by no means obvious. Our optimization framework allows many other possibilities especially when there are multiple assets. Other useful constraints and efficient algorithms are natural subjects of further research.

Appendix A.

In this appendix we present the proofs of Theorem 5.4 and Theorem 5.5. First we will state and prove some lemmas from convex analysis.

Lemma A.1. Let \( p_0 \in \Delta^{(n)} \) be fixed and let \( C_0 \) be the collection of positive concave functions \( \Phi \) on \( \Delta^{(n)} \) satisfying \( \Phi(p_0) = 1 \). Then any sequence in \( C_0 \) has a subsequence which converges locally uniformly on \( \Delta^{(n)} \) to a function in \( C_0 \).

Proof. By [Roc97, Theorem 10.9], it suffices to prove that \( C_0 \) has a uniform upper bound (the lower bound is immediate since functions in \( C_0 \) are non-negative). We first derive an upper bound in the one-dimensional case. Let \( f \) be a non-negative concave function on the real interval \([a, b]\). Let \( x_0 \in (a, b) \) and suppose \( f(x_0) = 1 \). Let \( x \in [a, x_0] \) and write \( x_0 = \lambda x + (1 - \lambda)b \) for some \( \lambda \in [0, 1] \). By concavity, \[
1 = f(x_0) \geq \lambda f(x) + (1 - \lambda)f(b) \geq \lambda f(x).
\]

Thus \[
f(x) \leq \frac{1}{\lambda} = \frac{b - x}{b - x_0} \leq \frac{b - a}{b - x_0}, \quad x \in [a, x_0].
\]
The case \( x \in [x_0, b] \) can be handled similarly, and we get

(A.1) \[
f(x) \leq \frac{b - a}{\min \{|x_0 - a|, |x_0 - b|\}}, \quad x \in [a, b].
\]

Now let \( \Phi \in C_0 \). Applying (A.1) to the restrictions of \( \Phi \) to line segments in \( \Delta^{(n)} \) containing \( p_0 \), we get

\[
\Phi(p) \leq \frac{\text{diam}(\Delta^{(n)})}{\text{dist}(p_0, \partial\Delta^{(n)})}, \quad p \in \Delta^{(n)}.
\]

where \( \text{diam}(\Delta^{(n)}) \) is the diameter of \( \Delta^{(n)} \) and \( \text{dist}(p_0, \partial\Delta^{(n)}) \) is the distance from \( p_0 \) to the boundary of \( \Delta^{(n)} \). This completes the proof of the lemma. \( \square \)

Lemma A.2. Let \((\pi, \Phi), (\pi^{(k)}, \Phi^{(k)}) \in \mathcal{FG}, k \geq 1 \). Suppose \( \Phi^{(k)} \) converges locally uniformly on \( \Delta^{(n)} \) to \( \Phi \). Let \( p \in \Delta^{(n)} \) be a point at which \( \Phi \) is differentiable. Then given \( \varepsilon > 0 \), there exists \( \delta > 0 \) and a positive integer \( k_0 \) such that \( \|\pi^{(k)}(q) - \pi(p)\| < \varepsilon \) whenever \( k \geq k_0 \) and \( q \in B(p, \delta) \). In particular, \( \pi^{(k)} \) converges \( m \)-almost everywhere to \( \pi \) as \( k \to \infty \).
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Proof. It is clear that \( \log \Phi^{(k)} \) also converges locally uniformly to \( \log \Phi \). We will use a well-known convergence result for the superdifferentials of concave functions, see [HUL96, Theorem 6.2.7]. Indeed, the proof of [HUL96, Theorem 6.2.7] implies a slightly stronger statement than the theorem. Namely, for any \( p \in \Delta^{(n)} \) and any \( \varepsilon > 0 \), there exists a positive integer \( k_0 \) and \( \delta > 0 \) such that

\[
\partial \log \Phi^{(k)}(q) \subset \partial \log \Phi(p) + B(0, \varepsilon), \quad k \geq k_0, \quad q \in B(p, \delta),
\]

\[
\partial \log \Phi(q) \subset \partial \log \Phi(p) + B(0, \varepsilon), \quad q \in B(p, \delta).
\]

(A.2)

Suppose \( \Phi \) is differentiable at \( p \). Then \( \partial \log \Phi(p) \) is a singleton. By Lemma 2.4 there are measurable selections \( \xi^{(k)} \) and \( \xi \) of \( \partial \log \Phi^{(k)} \) and \( \partial \log \Phi \) respectively such that

\[
\pi_i^{(k)}(q) = q_i \left( \xi_i^{(k)}(q) + 1 - \sum_{j=1}^{n} p_j \xi_j^{(k)}(q) \right),
\]

\[
\pi_i(q) = q_i \left( \xi_i(q) + 1 - \sum_{j=1}^{n} p_j \xi_j(q) \right),
\]

for all \( q \in \Delta^{(n)}, i = 1, \ldots, n, \) and \( k \geq 1 \).

For each \( i = 1, \ldots, n \), consider the map \( G_i \) defined by

\[(q, \xi) \in \Delta^{(n)} \times T \Delta^{(n)} \mapsto q_i \left( \xi_i + 1 - \sum_{j=1}^{n} q_j \xi_j \right).\]

The map \( G = (G_1, \ldots, G_n) \) is clearly jointly continuous. We have \( \pi(q) = G(q, \xi(q)) \) and \( \pi^{(k)}(q) = G(q, \xi^{(k)}(q)) \).

By (A.2), for any \( \varepsilon > 0 \), there exists \( k_0 \) and \( \delta > 0 \) such that

\[(\forall k \geq k_0, q \in B(p, \delta)) \quad \| \xi^{(k)}(q) - \xi(p) \| < \varepsilon, \quad \| \xi(q) - \xi(p) \| < \varepsilon\]

for all \( k \geq k_0 \) and \( q \in B(p, \delta) \). The claim (A.2) follows from (A.3) and the joint continuity of \( G \) at \( (q, \xi(q)) \) (the last statement follows since a finite concave function \( \log \Phi \) on \( \Delta^{(n)} \) is differentiable m-almost everywhere [Roc97, Theorem 25.5]).

Proof of Theorem 5.4. (i) The existence of an optimal solution will be proved by a compactness argument. Suppose \( (\pi^{(k)}, \Phi^{(k)}) \) is a maximizing sequence for (5.1). By scaling, we may assume \( \Phi^{(k)}(p_0) = 1 \) where \( p_0 \in \Delta^{(n)} \) is fixed. By Lemma A.1, we may replace it by a subsequence such that \( \Phi^{(k)} \) converges locally uniformly on \( \Delta^{(n)} \) to a positive concave function \( \Phi \) on \( \Delta^{(n)} \). By Lemma 2.4, \( \Phi \) generates a portfolio \( \pi \).

Case 1. \( \mathbb{P} \) is continuous. By Lemma A.2, \( \pi^{(k)} \) converges m-almost everywhere to \( \pi \). Let \( T^{(k)} \) and \( T \) be the discrete energy functionals of \( (\pi^{(k)}, \Phi^{(k)}) \) and \( (\pi, \Phi) \) respectively. Recall that \( \mathbb{P} \) is supported on \( K \times K \) where \( K \subset \Delta^{(n)} \) is compact. Note that for \( x \in \Delta^{(n)} \) and \( p, q \in K \) we have

\[
1 + \left< \frac{x}{p}, q - p \right> = \sum_{i=1}^{n} x_i \frac{q_i - p_i}{p_i} \leq \sum_{i=1}^{n} \frac{x_i}{p_i} \leq \frac{1}{\min_{p \in K, 1 \leq i \leq n} p_i}.
\]

Also \( \Phi^{(k)} \to \Phi \) uniformly on \( K \). Hence the family of discrete energy functionals \( \{T, T^{(1)}, T^{(2)}, \ldots\} \) is uniformly bounded on \( K \times K \). By Lebesgue’s dominated
convergence theorem, we have
\[ \lim_{k \to \infty} \int T^{(k)}(q \mid p) \, dP = \int T(q \mid p) \, dP. \]
Thus \((\pi, \Phi)\) is optimal.

**Case 2.** If \(\mathbb{P}\) is discrete and has masses at \((p(j), q(j))\). Since \(\overline{\Delta^{(n)}}\) is compact, by a diagonal argument we can extract a further subsequence (still denoted by \(\{(\pi(k), \Phi(k))\}\)) such that \(\lim_{k \to \infty} \pi(k)(p(j))\) exists for each \(j\). Now we can redefine \(\pi\) on \(\{p(1), p(2), ...\}\) such that \(\pi(p(j)) = \lim_{k \to \infty} \pi(k)(p(j))\) for each \(j\). Since we only modify \(\pi\) at countably many points, \(\pi\) is still Borel measurable. Now we may apply Lebesgue’s dominated convergence theorem and conclude that \((\pi, \Phi)\) is optimal.

(ii) Suppose \((\pi^{(1)}, \Phi^{(1)})\) and \((\pi^{(2)}, \Phi^{(2)})\) are optimal solutions. Define \(\pi = \frac{1}{2} \pi^{(1)} + \frac{1}{2} \pi^{(2)}\) which is generated by the geometric mean \(\Phi = \sqrt{\Phi^{(1)} \Phi^{(2)}}\) (Lemma 4.6). Also let \(T, T^{(1)}\) and \(T^{(2)}\) be the discrete energy functionals of \((\pi, \Phi), (\pi^{(1)}, \Phi^{(2)})\) and \((\pi^{(2)}, \Phi^{(2)})\) respectively. By concavity of the discrete energy functional (Lemma 4.7), we have
\[ (A.5) \quad \int T(q \mid p) \, d\mathbb{P} \geq \frac{1}{2} \left( \int T^{(1)}(q \mid p) \, d\mathbb{P} + \int T^{(2)}(q \mid p) \, d\mathbb{P} \right). \]
Hence \((\pi, \Phi)\) is also optimal. It follows from \((A.5)\) and the strict concavity of the logarithm that
\[ \left\langle \frac{\pi^{(1)}(p)}{p}, q - p \right\rangle = \left\langle \frac{\pi^{(2)}(p)}{p}, q - p \right\rangle \]
for \(\mathbb{P}\)-almost all \((p, q)\).

If \(d\mathbb{P} = g(p)h(q \mid p) \, dm(q)dm(p)\) is continuous and satisfies the support condition, then for \(m\)-almost all \(p\) for which \(g(p) > 0\), we have
\[ \left\langle \frac{\pi^{(1)}(p)}{p}, v \right\rangle = \left\langle \frac{\pi^{(2)}(p)}{p}, v \right\rangle \]
for all tangent vectors \(v\). This and the fact that \(\pi^{(1)}(p), \pi^{(2)}(p) \in \overline{\Delta^{(n)}}\) imply that \(\pi^{(1)}(p) = \pi^{(2)}(p)\) \(m\)-almost everywhere on \(\{p : g(p) > 0\}\).

**Proof of Theorem 5.5.** By scaling, we may assume that \(\tilde{\Phi}^{(N)}(p_0) = \Phi(p_0) = 1\) for all \(N \geq 1\). By Lemma A.1, any subsequence of \(\{\tilde{\Phi}^{(N)}\}\) has a further subsequence which converges locally uniformly to a positive concave function \(\tilde{\Phi}\) on \(\Delta^{(n)}\). Replacing \(\tilde{\Phi}^{(N)}\) by such a convergent subsequence, we may assume that \(\tilde{\Phi}^{(N)} \to \tilde{\Phi}\) locally uniformly on \(\Delta^{(n)}\). Let \(\tilde{\pi}\) be any portfolio generated by \(\tilde{\Phi}\) (which exists by Lemma 2.4). We claim that \((\pi, \Phi)\) is optimal and hence \(\tilde{\pi} = \pi\) \(m\)-almost everywhere on \(\{p : g(p) > 0\}\).

Let \(\tilde{T}^{(N)}, \tilde{T}\) and \(T\) be the discrete energy functionals of \((\tilde{\pi}^{(N)}, \tilde{\Phi}^{(N)}), (\tilde{\pi}, \tilde{\Phi})\) and \((\pi, \Phi)\) respectively. By the optimality of \((\pi^{(N)}, \Phi^{(N)})\) for the measure \(\mathbb{P}_N\), we have
\[ (A.6) \quad \int \tilde{T}^{(N)}(q \mid p) \, d\mathbb{P}_N \geq \int T(q \mid p) \, d\mathbb{P}_N, \quad N \geq 1. \]

We would like to let \(N \to \infty\) in \((A.6)\). The discrete energy functional \(T(q \mid p)\) is clearly continuous on \(K \times K\) (note that \(K\) is compact). By the definition of weak
convergence, we have
\[ \lim_{N \to \infty} \int T(q \mid p) \, d\mathbb{P}_N = \int T(q \mid p) \, d\mathbb{P}. \]
Suppose we can prove that
\[ (A.7) \lim_{N \to \infty} \int \hat{T}^{(N)}(q \mid p) \, d\mathbb{P}_N = \int \hat{T}(q \mid p) \, d\mathbb{P}. \]
Then letting \( N \to \infty \) in \( (A.6) \), we have
\[ \int \hat{T}(q \mid p) \, d\mathbb{P} \geq \int T(q \mid p) \, d\mathbb{P}, \]
so \((\hat{T}, \hat{\Phi})\) is optimal for the measure \( \mathbb{P} \). Since \( \mathbb{P} \) satisfies the support condition by assumption, by Theorem 5.4(ii) \( \hat{T} \) and \( \pi \) are equal \( m \)-almost everywhere on \( \{ p : g(p) > 0 \} \).

Thus we only need to prove \((A.7)\). Here the technicality lies in the fact that both the integrands and the measures change with \( N \), so standard integral convergence theorems do not apply.

The main idea is to use the local uniform convergence property in Lemma \( A.2 \) and approximate the integrals in \((A.7)\) by Riemann sums. Let \( \varepsilon > 0 \) be given. We will construct two partitions \( \{ A_k \}_{k=0}^{k_0} \), \( \{ B_\ell \}_{\ell=1}^{\ell_0} \) of \( K \), points \( p_k \in A_k, q_\ell \in B_\ell \) and a positive integer \( N_0 \) with the following properties:

(i) \( A_k \times B_\ell \) is a \( \mathbb{P} \)-continuity set, i.e., \( \mathbb{P}(\partial(A_k \times B_\ell)) = 0 \). Thus, by the Portmanteau theorem (see \[ B109 \]), we have
\[ \lim_{N \to \infty} \mathbb{P}_N(A_k \times B_\ell) = \mathbb{P}(A_k \times B_\ell). \]
So for \( N \geq N_0 \) where \( N_0 \) is sufficiently large, we have
\[ |\mathbb{P}_N(A_k \times B_\ell) - \mathbb{P}(A_k \times B_\ell)| < \frac{\varepsilon}{k_0 \ell_0} \]
for all \( k, \ell \).
(ii) \( \mathbb{P}(A_0 \times K) < \varepsilon \) and \( \mathbb{P}_N(A_0 \times K) < \varepsilon \) for \( N \geq N_0 \).
(iii) For \( N \geq N_0, p \in A_k, q \in B_\ell \), \( 1 \leq k \leq k_0 \) and \( 1 \leq \ell \leq \ell_0 \), we have
\[ \left| \hat{T}^{(N)}(q \mid p) - \hat{T}(q_\ell \mid p_k) \right| < \varepsilon, \left| \hat{T}(q \mid p) - \hat{T}(q_\ell \mid p_k) \right| < \varepsilon. \]
(iv) \( |\log \hat{\Phi}^{(N)}(p) - \log \hat{\Phi}(p)| < \varepsilon \) for \( k \in K \) and \( N \geq N_0 \). (This is immediate since \( \hat{\Phi}^{(N)} \) converges uniformly to \( \hat{\Phi} \) on \( K \) and \( \hat{\Phi} \) is positive on \( K \).)

Suppose these objects have been constructed. Then for \( N \geq N_0 \) we can approximate the integrals as follows. By (ii) and (iii), we have
\[ (A.8) \left| \int \hat{T}(q \mid p) \, d\mathbb{P} - \sum_{\ell=1}^{\ell_0} \sum_{k=1}^{k_0} \hat{T}(q_\ell \mid p_k) \mathbb{P}(A_k \times B_\ell) \right| \]
\[ \leq \left| \int_{A_0 \times K} \hat{T}(q \mid p) \, d\mathbb{P} \right| + \sum_{\ell=1}^{\ell_0} \sum_{k=1}^{k_0} \int_{A_k \times B_\ell} \left| \hat{T}(q \mid p) - \hat{T}(q_\ell \mid p_k) \right| \, d\mathbb{P} \]
\[ \leq \varepsilon \max_{p,q \in K} \left| \hat{T}(q \mid p) \right| + \varepsilon. \]
Similarly, we have
\[
\left| \int \hat{T}^{(N)}(q \mid p) d\mathbb{P}_N - \sum_{\ell=1}^{k_0} \sum_{k=1}^{\ell_0} \hat{T}(q_\ell \mid p_k) \mathbb{P}_N(A_k \times B_\ell) \right| \\
\leq \varepsilon \max_{p,q \in K} \hat{T}^{(N)}(q \mid p) + \varepsilon.
\]

By (A.4) and uniform convergence of \( \{ \hat{\Phi}^{(N)} \} \) on \( K \), we can bound \( \max_{p,q \in K} \hat{T}^{(N)}(q \mid p) \) and \( \max_{p,q \in K} \hat{T}^{(N)}(q \mid p) \) by a constant \( C \). Using (i) and (iii), we get
\[
\left| \sum_{k,\ell} \hat{T}(q_\ell \mid p_k) \mathbb{P}_N(A_k \times B_\ell) - \sum_{k,\ell} \hat{T}(q_\ell \mid p_k) \mathbb{P}(A_k \times B_\ell) \right| \\
\leq \sum_{k,\ell} \left| \hat{T}(q_\ell \mid p_k) \mathbb{P}_N(A_k \times B_\ell) - \mathbb{P}(A_k \times B_\ell) \right| \\
\leq k_0 \ell_0 C \varepsilon = C \varepsilon.
\]

Combining (A.8), (A.9) and (A.10), we have the estimate
\[
\left| \int \hat{T}^{(N)}(q \mid p) d\mathbb{P}_N - \int \hat{T}(q \mid p) d\mathbb{P} \right| \leq (3C + 2)\varepsilon, \quad N \geq N_0,
\]
and so (A.7) holds.

Thus it remains to construct the sets \( \{ A_k \}, \{ B_\ell \} \), the points \( p_k, q_\ell \) and \( N_0 \) satisfying (i)-(iv). Before we begin, we note the fact that the boundary of any convex subset of \( \Delta^{(n)} \) has \( m \)-measure zero [Lan86, Theorem 1]. Let \( \varepsilon > 0 \) be given. By [Roc97, Theorem 10.6], the family \( \{ \hat{\Phi}, \hat{\Phi}^{(1)}, \hat{\Phi}^{(2)}, \ldots \} \) is uniformly Lipschitz on \( K \). Also, it is not difficult to verify that there exists a constant \( L > 0 \) so that
\[
\left| \log \left( 1 + \left\langle \frac{x}{p}, q - p \right\rangle \right) - \log \left( 1 + \left\langle \frac{x}{p}, q' - p' \right\rangle \right) \right| \leq L (\|p - p'\| + \|q - q'\|)
\]
for all \( x \in \Delta^{(n)} \) and \( p, p', q, q' \in K \). It follows that the family of discrete energy functionals \( \{ \hat{T}, \hat{T}^{(1)}, \hat{T}^{(2)}, \ldots \} \) is uniformly Lipschitz on \( K \times K \). Thus there exists \( \delta_0 > 0 \) such that if \( p, p', q, q' \in \Delta^{(n)} \), then
\[
\hat{T}^{(N)}(q' \mid p') - \hat{T}(q' \mid p) < \varepsilon \quad \text{and} \quad \hat{T}(q' \mid p') - \hat{T}(q' \mid p) < \varepsilon
\]
whenever \( \|q - q'\| < \delta_0, \|p - p'\| < \delta_0 \).

Let \( D \) be the set of points in \( K \) at which \( \hat{\Phi} \) is differentiable. Then \( K \setminus D \) has \( m \)-measure zero by [Roc97, Theorem 25.5]. Let \( \varepsilon' > 0 \) be arbitrary. By Lemma A.2 for each \( p \in D \) there exists \( 0 < \delta(p) \leq \delta_0 \) and a positive integer \( N_0(p) \) such that \( \|\hat{\pi}^{(N)}(q) - \hat{\pi}(p)\| < \varepsilon' \) for all \( N \geq N_0(p) \) and \( q \in B(p, \delta(p)) \).

Since \( K \) is compact, it is separable, and so is \( D \) as a subset of \( K \). The collection \( \{ B(p, \delta(p)) \}_{p \in D} \) forms an open cover of \( D \) and hence there exists a countable sub-cover. By the continuity of measure, for any \( \eta > 0 \) there exists \( p_1, \ldots, p_{j_0} \in D \) such that
\[
m(A_0) < \eta, \quad A_0 := K \setminus \bigcup_{j=1}^{j_0} B(p_j, \delta(p_j)),
\]
where \( \delta(p_j) = \delta(p_j) \).
Since \( \partial A_0 \subset \partial K \cup \bigcup_j \partial B(p_j, \delta(p_j)) \), \( \partial(A_0 \times K) \) has \( m \)-measure zero and hence \( A_0 \times K \) is a \( \mathbb{P} \)-continuity set. Since \( \mathbb{P} \) is absolutely continuous, choosing \( \eta > 0 \) sufficiently small we have
\[
\mathbb{P}(A_0 \times K) < \varepsilon,
\]
and by weak convergence we have \( \mathbb{P}_N(A_0 \times K) < \varepsilon \) for \( N \) sufficiently large, so (ii) holds. Let \( A_1 = B(p_1, \delta(p_1)) \cap K \) and define \( A_k = \{ p_k \} \cup (B(p_k, \delta(p_k)) \cap K) \setminus (A_1 \cup \cdots \cup A_{k-1}) \), \( j = 2, \ldots, k_0 \). If \( N \geq \max_{1 \leq k \leq k_0} N_0(p_k) \), we have
\[
(A.12) \quad \| \hat{\pi}^N(p) - \hat{\pi}(p_k) \| < \varepsilon', \quad p \in A_k, \quad k = 1, \ldots, k_0.
\]
Next choose \( q_1, \ldots, q_{k_0} \in K \) such that \( K \subset \bigcup_{j=1}^{k_0} B(q_j, \delta_0) \). Define \( B_1 = B(q_1, \delta_0) \cap K \) and \( B_j = \{ q_j \} \cup (B(q_j, \delta_0) \cap K) \setminus (B_1 \cup \cdots \cup B_{j-1}) \), \( j = 2, \ldots, k_0 \). Again it is clear that \( \partial(A_k \times B_j) \) has \( m \)-measure zero and is a \( \mathbb{P} \)-continuity set. So (i) holds for \( N \) sufficiently large. Finally, if we choose \( \varepsilon' > 0 \) small enough in (A.12), we have
\[
\left| \hat{T}^N(q | p) - \hat{T}(q | p_k) \right| < \frac{\varepsilon}{2}, \quad p \in B(p_k, \delta_0), \quad q \in \Delta^{(n)}
\]
for \( N \) sufficiently large. This and (A.11) imply (iii) and the proof of Theorem 5.5 is complete. \( \square \)

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