Sobolev spaces on locally compact abelian groups

Przemysław Górka\textsuperscript{1} and Enrique G. Reyes\textsuperscript{2}

\textsuperscript{1}Instituto de Matemática y Física, Universidad de Talca
Casilla 747, Talca, Chile, and
Department of Mathematics and Information Sciences,
Warsaw University of Technology,
Pl. Politechniki 1, 00-661 Warsaw, Poland.

\textsuperscript{2}Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile
Casilla 307 Correo 2, Santiago, Chile.

May 2, 2014

Abstract

Motivated by a class of nonlinear equations of interest for string theory, we introduce Sobolev spaces on arbitrary locally compact abelian groups and we examine some of their properties. Specifically, we focus on analogs of the Sobolev embedding and Rellich-Kondrachov compactness theorems. As an application, we prove the existence of continuous solutions to a generalized euclidean bosonic string equation posed on an arbitrary compact abelian group.

Keywords: Abstract harmonic analysis; Sobolev spaces; embedding theorems; locally compact groups; nonlinear pseudodifferential equations.

AMS Classification: 43A15; 43A25; 46E35.

1 Introduction

In this work we introduce Sobolev spaces on arbitrary locally compact abelian groups and we examine analogs to the Sobolev embedding and Rellich-Kondrachov compactness theorems. Sobolev spaces are well understood on (domains of) $\mathbb{R}^n$, see \cite{1}, compact and complete Riemannian manifolds $\textsuperscript{2}$\cite{17}, and metric measure spaces (the so-called Hajlasz-Sobolev spaces, see \cite{15} \cite{16}, and Newtonian spaces \cite{24}). There are also some works on Sobolev spaces in the $p$-adic context, see \cite{22} and references therein, and in special cases of locally compact groups such as the Heisenberg group \cite{3}.

Besides their intrinsic interest, we are interested in Sobolev spaces in this general context because we wish to consider some nonlinear equations appearing in physical theories \cite{4} \cite{10} \cite{11} \cite{28} \cite{30} in settings beyond Riemannian manifolds, and also because we wish to use them...
as a tool for a better understanding of pseudodifferential operators defined on locally compact abelian groups, motivated by the papers [22] and [14], and also by the recent treatise [23] in which the authors study in detail pseudo-differential operators on compact Lie groups. As a first application of our Sobolev-type theorems, we investigate the existence of continuous solutions to the \textit{generalized euclidean bosonic string equation}

$$\Delta e^{-c\Delta} \phi = U(x, \phi) , \quad c > 0 ,$$

introduced in [5] (in a Lorentzian context) and recently considered in [10, 12]. We stress that studying such an equation in the general setting of topological groups is not simply a technical exercise. Equation (1) is obtained formally in [5] as the Euler-Lagrange equation of the "nonlocal Lagrangian"

$$L(\phi) = \phi \Delta e^{-c\Delta} \phi - U(x, \phi) , \quad c > 0 ,$$

and this Lagrangian is but an approximation to the highly sophisticated bosonic string action considered in [30] which contains an infinite number of fields and yields —via a formal application of the variational principle— an infinite number of equations for infinitely many variables, see for instance [21]. Topological groups appear therefore as a natural testing ground for gathering a better understanding of (1) and (2). For instance, we can consider Equation (1) for functions \(\phi\) "depending on an infinite number of variables", if we pose it on an infinite product of spheres.

We start with some standard notation from harmonic analysis [18, 19]. Let us fix a locally compact abelian group \(G\). We denote by \(\mu_G\) the unique Haar measure of \(G\). We also consider the dual group of the group \(G\) (that is, the locally compact abelian group of all continuous group homomorphisms from \(G\) to the circle group \(T\)), and we denote it by \(G^\wedge\). \(L^p\) spaces over \(G\) are defined as usual,

$$L^p_{\mu_G}(G) = \left\{ f : G \to \mathbb{C} : \int_G |f(x)|^p d\mu_G(x) < \infty \right\} ,$$

and the Fourier transform on \(G\) is defined as follows: if \(f \in L^1_{\mu_G}(G)\), then it Fourier transform is the function \(\hat{f} : G^\wedge \to \mathbb{C}\) given by

$$\hat{f}(\xi) = \int_G \xi(x)f(x)d\mu_G(x) .$$

Next, we denote by \(\Gamma\) the following set

$$\Gamma = \left\{ \gamma : G^\wedge \to [0, \infty) : \exists c_\gamma \forall \alpha, \beta \in G^\wedge \gamma(\alpha \beta) \leq c_\gamma \left[ \gamma(\alpha) + \gamma(\beta) \right] \right\} ,$$

and we are in position to introduce Sobolev spaces:

**Definition 1.** Let us fix a map \(\gamma \in \Gamma\) and a nonnegative real number \(s\). We shall say that \(f \in L^2_{\mu_G}(G)\) belongs to \(H^s_\gamma(G)\) if the following integral is finite:

$$\int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 d\mu_{G^\wedge}(\xi) .$$
Moreover, for \( f \in H^s_\gamma(G) \) its norm \( \| f \|_{H^s_\gamma(G)} \) is defined as follows:

\[
\| f \|_{H^s_\gamma(G)} = \left( \int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 d\mu_{G^\wedge}(\xi) \right)^{\frac{1}{2}}.
\]  

(4)

Remark: We note that by taking appropriate functions \( \gamma \) we obtain the classical Sobolev spaces on \( \mathbb{T}^n \) and \( \mathbb{R}^n \), see [9] and [27, Chp. 4]. The use of the Fourier transform and the duality theory of locally compact abelian groups is crucial in the present general context, since we do not have differential calculus at our disposal. A particular instance of Definition 1 appears in the paper [9] by H. G. Feichtinger and T. Werther. The function \( \gamma \) used in that article is called by their authors a \textit{weakly subadditive weight}. We also note that in \( p \)-adic analysis Sobolev spaces are defined in a way analogous to our Definition 1: if we take \( \gamma(\xi) = \| \xi \|_p \), where \( \| . \|_p \) is a \( p \)-adic norm on \( \mathbb{Q}_p^n \simeq \mathbb{Q}_p^{n^\wedge} \), then (3) and (4) allow us to recover the \( p \)-adic Sobolev spaces considered in [22].

Remark: It is important to take \( \gamma \in \Gamma \) in order to prove that —as it happens in standard contexts, see [9, 27]— our spaces \( H^s_\gamma(G) \) are Banach algebras under some assumptions on \( s \). We show this fact in Theorem 2 below. The other theorems of Sections 2 and 3 of this paper hold true without this assumption on \( \gamma \).

Remark: Our spaces \( H^s_\gamma(G) \) are contained in the \( A_{\omega,\omega}^w(G) \) spaces introduced by H.G. Feichtinger and A.T. Gürkanli in [8] in the following sense: if (notation as in [8]) \( w \in L^2_{\mu_G}(G) \) and we take \( \omega = (1 + \gamma^2)^{s/2} \), then, \( H^s_\gamma(G) \hookrightarrow A_{\omega,\omega}^w(G) \).

2 Continuous embedding theorems

Embedding properties of Sobolev spaces are essential for proving existence and regularity of solutions to partial differential equations [27] and for the analysis of pseudo-differential operators, see for instance [22]. Thus, we begin by proving a Sobolev embedding type theorem in our general setting.

Let us start with two elementary observations. First, we show in Proposition 1 that our spaces \( H^s_\gamma(G) \) are included in \( L^2_{\mu_G}(G) \). Then, we prove in Proposition 2 that in fact we have a “scale” of spaces:

**Proposition 1.** If \( G \) is a locally compact abelian group then,

\[
H^s_\gamma(G) \hookrightarrow L^2_{\mu_G}(G) .
\]

Moreover, for each \( f \in H^s_\gamma(G) \) the following inequality holds:

\[
\| f \|_{L^2_{\mu_G}(G)} \leq \| f \|_{H^s_\gamma(G)} .
\]

**Proof.** By Pontriagin duality and a basic inequality we get

\[
\| f \|_{L^2_{\mu_G}(G)} = \| \hat{f} \|_{L^2(G^\wedge)} = \left( \int_{G^\wedge} |\hat{f}(\xi)|^2 d\mu_{G^\wedge}(\xi) \right)^{\frac{1}{2}} \leq \left( \int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 d\mu_{G^\wedge}(\xi) \right)^{\frac{1}{2}} = \| f \|_{H^s_\gamma(G)} .
\]

\( \square \)
Proposition 2. If $s > \sigma$, then $H^s_\gamma(G) \hookrightarrow H^\sigma_\gamma(G)$. Moreover, the inequality

$$\|f\|_{H^\sigma_\gamma(G)} \leq \|f\|_{H^s_\gamma(G)}$$

holds.

Proof. The proof follows from an elementary inequality. \qed

The classical Sobolev embedding theorem, see for instance [1], reads in our context as follows:

Theorem 1. If $\frac{1}{(1+\gamma(\cdot)^2)^s} \in L^1(G^\wedge)$, then

$$H^s_\gamma(G) \hookrightarrow C(G),$$

in which $C(G)$ denotes the space of continuous complex-valued functions on $G$. Moreover, there exists a constant $C(\gamma, s)$ such that for each $f \in H^s_\gamma(G)$, the following inequality holds:

$$\|f\|_{C(G)} \leq C(\gamma, s)\|f\|_{H^s_\gamma(G)}.$$

Proof. Using the formula for the inverse Fourier transform,

$$f(x) = \int_{G^\wedge} \hat{f}(\xi)\xi(x) \, d\mu_{G^\wedge}(\xi),$$

we get

$$|f(x)| = \left| \int_{G^\wedge} \hat{f}(\xi)\xi(x) \, d\mu_{G^\wedge}(\xi) \right| \leq \int_{G^\wedge} \left| \hat{f}(\xi) \right| \, d\mu_{G^\wedge}(\xi) \leq \left( \int_{G^\wedge} (1 + \gamma(\xi)^2)^s \left| \hat{f}(\xi) \right|^2 \, d\mu_{G^\wedge}(\xi) \right)^{\frac{1}{2}} \left\| \frac{1}{(1+\gamma(\cdot)^2)^s} \right\|_{L^1(G^\wedge)}^{\frac{1}{2}}.$$

Finally, since $\hat{f} \in L^1(G^\wedge)$ we get that $f \in C(G)$ (see [19], Theorem 31.5). \qed

The following theorem tells us that, under a technical assumption involving the exponent $s$ and our function $\gamma$, the space $H^s_\gamma(G)$ is a Banach algebra. It is well-known that such a property is important for instance, for the study of existence of solutions to partial differential equations. A recent example appears in our paper [13].

Theorem 2. If $\frac{1}{(1+\gamma(\cdot)^2)^s} \in L^1(G^\wedge)$, then $H^s_\gamma(G)$ is a Banach algebra. Moreover, there exists a constant $D(\gamma, s)$ such that for each $f, g \in H^s_\gamma(G)$, the following inequality holds

$$\|fg\|_{H^s_\gamma(G)} \leq D(\gamma, s)\|f\|_{H^s_\gamma(G)}\|g\|_{H^s_\gamma(G)}.$$

Proof. First of all let us notice that for each $\xi, \eta \in G^\wedge$ the following inequality holds

$$(1 + \gamma(\xi)^2) \leq (2 + 2c_\gamma^2)(2 + \gamma(\xi\eta^{-1})^2 + \gamma(\eta)^2).$$
Hence, we obtain
\[
(1 + \gamma(\xi)^2)^{\frac{s}{2}} \hat{f} g(\xi) = \int_{G^\infty} (1 + \gamma(\xi)^2)^{\frac{s}{2}} \hat{f}(\xi \eta^{-1}) \hat{g}(\eta) \mu_{G^\infty}(\eta) \leq
\]
\[
2^{\frac{s}{2}} (1 + c_\gamma^2)^{\frac{s}{2}} \int_{G^\infty} (1 + \gamma(\xi \eta^{-1})^2 + 1 + \gamma(\eta)^2)^{\frac{s}{2}} |\hat{f}(\xi \eta^{-1})\hat{g}(\eta)| \mu_{G^\infty}(\eta) \leq
\]
\[
2^{s-1} (1 + c_\gamma^2)^{\frac{s}{2}} \int_{G^\infty} (1 + \gamma(\xi \eta^{-1})^2)^{\frac{s}{2}} |\hat{f}(\xi \eta^{-1})\hat{g}(\eta)| \mu_{G^\infty}(\eta) +
\]
\[
2^{s-1} (1 + c_\gamma^2)^{\frac{s}{2}} \int_{G^\infty} (1 + \gamma(\eta)^2)^{\frac{s}{2}} |\hat{f}(\xi \eta^{-1})\hat{g}(\eta)| \mu_{G^\infty}(\eta) =
\]
\[
2^{s-1} (1 + c_\gamma^2)^{\frac{s}{2}} \left((1 + \gamma(\xi)^2)^{\frac{s}{2}} |\hat{f}| \hat{g} + |\hat{f} \ast (\hat{g}(1 + \gamma(\xi)^2)^{\frac{s}{2}})|\right).
\]
Thus
\[
\|fg\|_{L^2(G^\infty)} = \|(1 + \gamma(\xi)^2)^{\frac{s}{2}} \hat{f}g(\xi)\|_{L^2(G^\infty)} \leq
\]
\[
2^{2s-1} (1 + c_\gamma^2)^s \left(\|(1 + \gamma(\xi)^2)^{\frac{s}{2}} \hat{f} \hat{g}\|_{L^2(G^\infty)}^2 + \|\hat{f} \ast (\hat{g}(1 + \gamma(\xi)^2)^{\frac{s}{2}})\|_{L^2(G^\infty)}^2\right).
\]
Next, by Young inequality \(\|u \ast v\|_{L^2(G^\infty)} \leq c_g \|u\|_{L^2(G^\infty)} \|v\|_{L^1(G^\infty)}\), we obtain
\[
\|fg\|_{H^s(G^\infty)}^2 \leq 2^{2s-1} (1 + c_\gamma^2)^s c_g^2 \left(\|f\|_{H^s(G)}^2 \|g\|_{L^1(G^\infty)}^2 + \|\hat{f}\|_{L^1(G^\infty)}^2 \|g\|_{H^s(G)}^2\right).
\]
Finally, from the proof of the previous theorem we can finish the proof. \(\square\)

Now we prove a second embedding result. While Theorem 1 tells us that functions in \(H^s_\alpha(G)\) are continuous, Theorem 3 tells us that they possess “higher integrability properties”:

**Theorem 3.** If \(\alpha > s\) and \(\frac{1}{(1 + \gamma(\xi)^2)^{1+s}} \in L^\alpha(G^\infty)\), then
\[
H^s_\alpha(G) \hookrightarrow L^{\alpha^*}(G),
\]
where \(\alpha^* = \frac{2\alpha s}{\alpha + s}\). Moreover, there exists a constant \(D(\gamma, s)\) such that for each \(f \in H^s_\gamma(G)\), the following inequality holds
\[
\|f\|_{L^{\alpha^*}(G)} \leq D(\gamma, s)\|f\|_{H^s(G)}.
\]

**Proof.** By a standard corollary of Hausdorff-Young inequality (see [19]) we have
\[
\|f\|_{L^{\alpha^*}(G)} \leq \|\hat{f}\|_{L^p(G^\infty)},
\]
where \(p\) is the conjugate of \(\alpha^*\), i.e. \(p = \frac{2\alpha s}{\alpha + s}\). Next, using Hölder inequality with exponents \(\frac{2}{p}\) and \(\frac{2}{2-p}\) we get
\[
\|\hat{f}\|_{L^p(G^\infty)} = \left(\int_{G^\infty} \left|\hat{f}(\xi)\right|^p \left(\frac{1 + \gamma(\xi)^2)^{\frac{2}{p}}}{(1 + \gamma(\xi)^2)^{\frac{2}{p}}} \mu_{G^\infty}(\xi)\right)^{\frac{1}{p}} \leq
\]
\[
\leq \|f\|_{H^s(G)} \left(\int_{G^\infty} \frac{1}{(1 + \gamma(\xi)^2)^{\frac{2}{2-p}}} \mu_{G^\infty}(\xi)\right)^{\frac{2}{2-p}}.
\]
Since $\frac{sp}{2-p} = \alpha$, we get
\[ \|f\|_{L^{\alpha^*}(G)} \leq \left\| \frac{1}{(1 + \gamma(\cdot)^2)} \right\|_{L^{\alpha(G^\wedge)}}^{\frac{1}{2}} \|f\|_{H^s(\gamma)(G)} . \]

3 Compact embedding theorems

In this section we prove a Rellich-Kondrachov type theorem. As is well-known, this theorem plays a crucial role in proving compactness of operators and in fixed point arguments. Now, the standard Rellich-Kondrachov theorem [1, 17] is valid only on spaces with finite measure such as compact Riemannian manifolds. It is then natural to assume that in our case the condition $\mu_G(G) < \infty$ holds, or equivalently (see [18]), that the locally compact abelian group $G$ is actually compact. We stress that even with this restriction our results go beyond the standard case: besides infinite products of basic examples of compact abelian groups, other interesting instances of compact groups are the dyadic group (see for instance [26]) and the compact group appearing in the recent preprint [25].

In the theorem below we use the following convention: $g(h) \to 0$ as $h \to e$ means that for all $\epsilon > 0$, there exists an open set $U_\epsilon$ with $e \in U_\epsilon$ such that for all $h \in U_\epsilon$ we have $|g(h)| \leq \epsilon$. Also, the notation $A \hookrightarrow B$ means that the space $A$ is compactly embedded into $B$.

Remark: It is known that in the case of $\mathbb{R}^n$, the Kolmogorov-Riesz-Weil theorem (see [20] and [29]) can be used to prove the Rellich-Kondrachov theorem. Similar compactness results exist for locally compact abelian groups, see [7], which presumably would yield another approach to the problem of compact embeddings. We present a direct proof.

Theorem 4. Let $\frac{1}{(1 + \gamma(\cdot)^2)} \in L^\alpha(G^\wedge)$ for some $\alpha > s$ and assume that
\[ \frac{|\xi(h) - 1|}{(1 + \gamma(\xi)^2)^s} \to 0 \quad \text{uniformly with respect to } \xi \in G^\wedge . \] (5)
If $G$ is compact, then for all $p < \alpha^*$,
\[ H^s(\gamma)(G) \hookrightarrow \hookrightarrow L^p_{\mu_G}(G) . \]

Before proving Theorem 4 we note that if $G = \mathbb{R}^n$ or $\mathbb{T}^n$, Condition (5) is satisfied. Indeed, if $G = \mathbb{R}^n$, then $G^\wedge = \mathbb{R}^n$ and a straightforward calculation yields
\[ \frac{|\xi(h) - 1|}{(1 + \gamma(\xi)^2)^s} = \frac{|h|}{(1 + |\xi|^2)^s} \leq |h| . \]
If $G = \mathbb{T}^n$, then $G^\wedge = \mathbb{Z}^n$ and we can show as before that
\[ \frac{|n(h) - 1|}{(1 + \gamma(n)^2)^s} = \frac{|nh - 1|}{(1 + |n|^2)^s} \leq |h| . \]

Proof. Let us start with the following lemma.
Lemma 1. Let \( f \in H^*_\gamma(G) \) and assume that \( \frac{|\xi(h)-1|}{(1+\gamma(\xi))^s} \rightarrow 0 \) uniformly with respect to \( \xi \in G^\wedge \).
Then, for each \( h \in G \)
\[
\int_G |f(h) - f(x)|^2 d\mu_G(x) \leq C(h) \| f \|_{H^*_\gamma(G)}^2 ,
\]
where \( C(h) \rightarrow 0 \).

Proof. By Pontriagin duality we have
\[
\int_G |f(h) - f(x)|^2 d\mu_G(x) = \int_{G^\wedge} |\hat{f}(\xi) - \hat{f}(\xi)|^2 d\mu_G^\wedge(\xi) .
\]
Since the measure \( \mu_G \) is invariant, we obtain
\[
\hat{f}(h)(\xi) = \int_G \bar{\xi}(x)f(hx)d\mu_G(x) = \int_G \bar{\xi}(yh^{-1})f(y)d\mu_G(y) = \int_G \bar{\xi}(y)\bar{\xi}(h^{-1})f(y)d\mu_G(y) = \xi(h)\hat{f}(\xi) .
\]
Hence, we get
\[
\int_G |f(h) - f(x)|^2 d\mu_G(x) = \int_{G^\wedge} |\hat{f}(\xi)|^2 |\xi(h) - 1|^2 d\mu_G^\wedge(\xi) = \int_{G^\wedge} |\hat{f}(\xi)|^2 (1 + \gamma(\xi)^2)^s \frac{|\xi(h) - 1|^2}{(1 + \gamma(\xi)^2)^s} d\mu_G^\wedge(\xi) \leq C(h) \| f \|_{H^*_\gamma(G)}^2 ,
\]
where \( C(h) = \| \frac{|\xi(h)-1|^2}{(1+\gamma(\xi)^2)^s} \|_{L^\infty(G^\wedge)} \rightarrow 0 . \)

We continue the proof of Theorem 3. Let \( I \) be the set of all symmetric unit-neighborhoods, partially ordered by the inverse inclusion. Then, using Urysohn lemma we can construct the so-called Dirac net \((\phi_U)_{U \in I}\) in \( C_c(G) \) (see [6]). Each function \( \phi_U \) is nonnegative, satisfies \( \int_G \phi_U(x)d\mu_G(x) = 1 \) and the support of \( \phi_U \) shrinks. We are in position to formulate the next lemma:

Lemma 2. Let \((\phi_U)_{U \in I}\) be a Dirac net and \( f \in H^*_\gamma(G) \). Then
\[
\int_G |f * \phi_U(x) - f(x)|^2 d\mu_G(x) \leq \| f \|_{H^*_\gamma(G)}^2 \sup_{y \in U} C(y) .
\]

Proof.
\[
|f * \phi_U(x) - f(x)|^2 = \left| \int_G \phi_U(y)f(y^{-1}x)d\mu_G(y) - f(x) \right|^2 = \left( \int_G \phi_U(y)(f(y^{-1}x) - f(x))d\mu_G(y) \right)^2 \leq \int_G \phi_U(y)|f(y^{-1}x) - f(x)|^2 d\mu_G(y) = \int_U \phi_U(y)|f(y^{-1}x) - f(x)|^2 d\mu_G(y) .
\]
Hence, by Fubini theorem and the invariance of the measure we get
\[
\int_G |f * \phi_U(x) - f(x)|^2 d\mu_G(x) \leq \int_G \int_U \phi_U(y) |f(y^{-1}x) - f(x)|^2 d\mu_G(y) d\mu_G(x)
\]
\[
= \int_U \phi_U(y) \int_G |f(y^{-1}x) - f(x)|^2 d\mu_G(x) d\mu_G(y)
\]
\[
= \int_U \phi_U(y) \int_G |f(z) - f(yz)|^2 d\mu_G(z) d\mu_G(y).
\]

By the previous lemma
\[
\int_G |f * \phi_U(x) - f(x)|^2 d\mu_G(x) \leq \int_U \phi_U(y) \|f\|_{H^2_\gamma(G)}^2 C(y) d\mu_G(y) \leq \int_U \phi_U(y) d\mu_G(y) \|f\|_{H^2_\gamma(G)}^2 \sup_{y \in U} C(y) = \|f\|_{H^2_\gamma(G)}^2 \sup_{y \in U} C(y).
\]

This finishes the proof of the lemma.

Now we can finish the proof of the theorem. Let us take any sequence \( f_n \) bounded in the space \( H^q_\gamma(G) \), then by Theorem 3 the sequence is bounded in \( L^\alpha_{\mu_G}(G) \). Hence, there exists a subsequence \( f_{n_k} \) of \( f_n \) such that
\[
f_{n_k} \rightarrow f \quad \text{in} \quad L^\alpha_{\mu_G}(G).
\]

We claim that \( f_{n_k} \rightarrow f \) in \( L^q_{\mu_G}(G) \), where \( q < \alpha^\gamma \): for every \( f \in L^2_{\mu_G}(G) \) we denote by \( f_U \) the function \( f_U = f * \phi_U \). Also, for simplicity, we write \( f_n \) instead of \( f_{n_k} \). By lemma 2 we get
\[
\sup_n \int_G |f_{n(U)}(x) - f_n(x)|^2 d\mu_G(x) \leq \sup_n \|f_n\|_{H^2_\gamma(G)}^2 \sup_{y \in U} C(y) \leq C \sup_{y \in U} C(y).
\]

Moreover, we can show that \( \|f_U - f\|_{L^2_{\mu_G}(G)} \rightarrow 0 \) in the sense that for each \( \epsilon > 0 \) there exists \( U_\epsilon \) such that for each \( U \in \mathcal{I}, U \subset U_\epsilon \) the inequality holds \( \|f_U - f\|_{L^2_{\mu_G}(G)} \leq \epsilon \). Next, by Minkowski inequality we have
\[
\|f_n - f\|_{L^2_{\mu_G}(G)} \leq \|f_n - f_{n(U)}\|_{L^2_{\mu_G}(G)} + \|f_{n(U)} - f_U\|_{L^2_{\mu_G}(G)} + \|f_U - f\|_{L^2_{\mu_G}(G)}.
\]

Now we fix \( \epsilon > 0 \). There exists \( U_\epsilon \in I \) such that for each \( U \in \mathcal{I}, U \subset U_\epsilon \) the following inequality holds
\[
\|f_n - f\|_{L^2_{\mu_G}(G)} \leq \frac{2}{3} \epsilon + \|f_{n(U)} - f_U\|_{L^2_{\mu_G}(G)}.
\]

Thus, in order to show that \( \|f_n - f\|_{L^2_{\mu_G}(G)} \rightarrow 0 \), it is enough to check the limit
\[
\|f_{n(U_\epsilon)} - f_U\|_{L^2_{\mu_G}(G)} \rightarrow 0.
\]

In fact, since \( f_n \rightarrow f \) in \( L^\alpha_{\mu_G}(G) \), we have
\[
f_{n(U_\epsilon)}(x) = \int_G \phi_{U_\epsilon}(xy^{-1}) f_n(y) d\mu_G(y) \rightarrow \int_G \phi_{U_\epsilon}(xy^{-1}) f(y) d\mu_G(y) = f_U(x).
\]
Moreover, since $G$ is abelian we get
\[ |f_{n(U_\epsilon)} - f(U_\epsilon)|^2 = \left| \int_G (f_n(y) - f(y))\phi_{U_\epsilon}(y^{-1}x) d\mu_G(y) \right|^2 \leq \int_G |f_n(y) - f(y)| \phi_{U_\epsilon}(y^{-1}x) d\mu_G(y), \]
and finally, since we are assuming that $G$ is of finite measure, we can apply the Lebesgue theorem and obtain
\[ \| f_{n(U_\epsilon)} - f(U_\epsilon) \|_{L^2_{\mu_G}(G)} \to 0 \text{ as } n \to \infty. \]
So, we have obtained that $f_{n_k} \to f$ in $L^2_{\mu_G}(G)$. Finally, since the sequence is bounded in $L^{\alpha^*}_{\mu_G}(G)$ we can apply Vitali convergence theorem and we obtain that $f_{n_k} \to f$ in $L^p_{\mu_G}(G)$, where $p < \alpha^*$.

4 An application: the generalized euclidean bosonic string

We recall that the generalized euclidean bosonic string equation \[ \Delta e^{-c\Delta} \phi = U(x, \phi), \quad c > 0. \] (6)
Classically, Equation (6) is known as an equation with an infinite number of derivatives (see [4] and references therein). Such equations have been considered in the mathematical literature since the 1930’s, but only recently physicists have found reasons to study nonlinear equations such as (6), see for instance [4, 5, 21, 28, 30]. The existence of very serious proposals claiming that $p$-adic and non-commutative mathematics are relevant to physics (see for instance [28, 30]), makes it natural, even necessary, to consider equations of physical relevance in contexts other than Euclidean space or (pseudo-)Riemannian manifolds, as stated in Section 1.

Suppose for a moment that we are working on Euclidean space and that $f$ is the real function $f(s) = s \exp(-cs)$ so that, formally, the left hand side of (6) is $f(\Delta)$. We expand $f$ as a power series, $f(s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} s^n$. Then, formally, we should have
\[ f(\Delta)u = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \Delta^n u. \]
Applying Fourier transform we obtain (we set $\hat{f} = \mathcal{F}(f)$ for clarity)
\[ \mathcal{F}(f(\Delta)u)(\xi) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mathcal{F}(\Delta^n u) \]
\[ = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (-|\xi|^2)^n \mathcal{F}(u) = f(-|\xi|^2) \mathcal{F}(u)(\xi), \]
so that, naturally, we may interpret $f(\Delta)u$ in a way that reminds us of the classical definitions of pseudo-differential operators, see for instance [23], as
\[ f(\Delta)u = \mathcal{F}^{-1} \left( f(-|\xi|^2) \mathcal{F}(u)(\xi) \right) = -\mathcal{F}^{-1} \left( |\xi|^2 e^{c|\xi|^2} \mathcal{F}(u)(\xi) \right). \] (7)
Motivated by these remarks, we make two definitions. First, we write down the correct domain for the operator $L_c = \Delta e^{-c\Delta} - Id$. Then, we define the action of $L_c$:

**Definition 2.** The space $\mathcal{H}^{c,\infty}(G)$, $c > 0$, is given by

$$\mathcal{H}^{c,\infty}(G) = \left\{ f \in L^2_{\mu_G}(G) : \int_{G^\wedge} \left( 1 + \gamma(\xi)^2 e^{c\gamma(\xi)^2} \right)^2 |\hat{f}(\xi)|^2 \, d\mu_{G^\wedge}(\xi) < \infty \right\}.$$  

**Definition 3.** The operator $L_c = \Delta e^{-c\Delta} - Id$ is defined as

$$L_c u = -F^{-1} \left( \mathcal{F}(u) + \gamma(\xi)^2 e^{c\gamma(\xi)^2} \mathcal{F}(u) \right),$$

for any $u \in \mathcal{H}^{c,\infty}(G)$.

We note that $L_c$ is an isometry from $\mathcal{H}^{c,\infty}(G)$ into $L^2_{\mu_G}(G)$; we also remark that analogous definitions of pseudo-differential operators appear in $p$-adic analysis, see for instance [22].

We state two important technical observations on the structure of the space $\mathcal{H}^{c,\infty}(G)$:

**Lemma 3.** 1. For each non-negative $s \in \mathbb{R}$ the embedding $\mathcal{H}^{c,\infty}(G) \hookrightarrow H^s_{\gamma}(G)$ holds. In other words, $\|f\|_{H^s_{\gamma}(G)} \leq C(s) \|f\|_{\mathcal{H}^{c,\infty}(G)}$ for some constant $C(s) > 0$.

2. Assume that $\frac{1}{(1 + \gamma(\xi)^2)^2} \in L^1(G^\wedge)$. Then, the embedding $\mathcal{H}^{c,\infty}(G) \hookrightarrow C(G)$ holds.

**Proof.** The first claim follows immediately from the elementary properties of the map $x \mapsto e^x$. The second claim is a consequence of the first one combined with our Sobolev embedding result, Theorem 1.

Now we show that the linear problem $L_c u = g$, $g \in L^2_{\mu_G}(G)$, can be solved completely using our set-up:

**Theorem 5.** For each $c > 0$ and $g \in L^2_{\mu_G}(G)$, there exists a unique solution $u_g \in \mathcal{H}^{c,\infty}(G)$ to the linear problem

$$L_c u = g.$$  

Moreover, the equation

$$\|u_g\|_{\mathcal{H}^{c,\infty}(G)} = \|g\|_{L^2_{\mu_G}(G)}$$

holds.

**Proof.** It is easy to see that $u_g$ given by

$$u_g = -F^{-1} \left( \frac{\mathcal{F}(g)}{1 + \gamma(\xi)^2 e^{c\gamma(\xi)^2}} \right),$$

is an element of $\mathcal{H}^{c,\infty}(G)$ which solves Equation (9). Now, applying Fourier transform we get

$$\left( 1 + \gamma(\xi)^2 e^{c\gamma(\xi)^2} \right) \mathcal{F}(u_g) = \mathcal{F}(g),$$

and so the Plancherel Theorem implies that $\|u_g\|_{\mathcal{H}^{c,\infty}(G)} = \|g\|_{L^2_{\mu_G}(G)}$. Equation (10) tells us that the operator $L_c$ has trivial kernel. Uniqueness then follows immediately.
We are ready to show that the generalized bosonic string equation (6) admits continuous solutions:

**Theorem 6.** Assume that $G$ is a compact abelian group, and that $\frac{1}{(1+\gamma)^{2}} \in L^{\delta}(G^{\wedge})$, where $\delta > 1$. Let $U : G \times \mathbb{R} \rightarrow \mathbb{R}$ be a function which is differentiable with respect to its second argument, and suppose that there exist constants $\alpha > 1$, $\beta \in [0, \alpha - 1]$, $C > 0$, and functions $h \in L^{2}_{\mu_{G}}(G)$ and $f \in L^{2\alpha}_{\mu_{G}}(G)$, such that the following two inequalities hold:

$$\|u(x, y) - y\| \leq C(\|h(x)\| + |y|^{\alpha}), \quad \left| \frac{\partial}{\partial y}(U(x, y) - y) \right| \leq C \left( |f(x)| + |y|^{\beta} \right). \quad (11)$$

If $\|h\|_{L^{2\alpha}_{\mu_{G}}(G)}$ is suitably small and $\frac{|\xi(h)| - 1}{\alpha(\xi(h))^{\frac{\alpha}{\alpha - 1}} \rightarrow 0}$ uniformly with respect to $\xi \in G^{\wedge}$, then there exists a solution $\phi \in H^{\infty}(\mathbb{R}, \mathbb{R})$ to the nonlinear problem

$$\Delta e^{\Delta} \phi - U(x, \phi) = 0. \quad (12)$$

**Proof.** Let us set $V(\cdot, u) = U(\cdot, u) - u$. Then, the nonlinear equation (12) is formally equivalent to $L(\cdot, u) = V(\cdot, u)$, and it is easy to see that the function $V$ belongs to $L^{2\alpha}_{\mu_{G}}(G)$. We define the set

$$Y_{\epsilon} = \left\{ u \in L^{2\alpha}_{\mu_{G}}(G) : \|u\|_{L^{2\alpha}_{\mu_{G}}(G)} \leq \epsilon \right\}$$

for $\epsilon > 0$. It is easy to see that $Y_{\epsilon}$ is a bounded, closed, convex and nonempty subset of the Banach space $L^{2\alpha}_{\mu_{G}}(G)$. We define a map $\mathcal{G}$ as follows:

$$\mathcal{G} : Y_{\epsilon} \rightarrow L^{2\alpha}_{\mu_{G}}(G), \quad \mathcal{G}(u) = \tilde{u},$$

where $\tilde{u}$ is the unique solution to the non-homogeneous linear problem

$$L_{c}\tilde{u} = V(\cdot, u).$$

Lemma 3 and Theorem 3 imply that $\mathcal{G}$ is well defined. We show that there exists $\epsilon > 0$ such that $\mathcal{G} : Y_{\epsilon} \rightarrow Y_{\epsilon}$. Indeed, let us take $u \in Y_{\epsilon}$, then we get, using (11),

$$\|\mathcal{G}(u)\|_{H^{\infty}(G)}^{2} = \|V(\cdot, u)\|_{L^{2\alpha}_{\mu_{G}}(G)}^{2} \leq C^{2} \int_{G} |h(x)| + |u(x)|^{\alpha} d\mu_{G}(x)$$

$$\leq 2C^{2} \int_{G} |h(x)|^{2} + |u(x)|^{2\alpha} d\mu_{G}(x) \leq 2C^{2} \left( \|h\|_{L^{2\alpha}_{\mu_{G}}(G)}^{2} + \|u\|_{L^{2\alpha}_{\mu_{G}}(G)}^{2\alpha} \right). \quad (13)$$

Now, let us fix $s \in (\delta - \frac{\delta}{\alpha}, \delta)$. Using again Lemma 3 and Theorem 3, we have $H^{s}(G) \hookrightarrow H^{\delta}(G) \hookrightarrow L^{2\alpha}_{\mu_{G}}(G)$.

Hence, since $u \in Y_{\epsilon}$, the inequality (13) there exists a constant $D$ such that

$$\|\mathcal{G}(u)\|_{L^{2\alpha}_{\mu_{G}}(G)} \leq D \left( \|h\|_{L^{2\alpha}_{\mu_{G}}(G)} + \epsilon^{2\alpha} \right).$$

Since we are assuming that $\|h\|_{L^{2\alpha}_{\mu_{G}}(G)}$ is suitably small and $\alpha > 1$, we can find $\epsilon$ such that $\|\mathcal{G}(u)\|_{L^{2\alpha}_{\mu_{G}}(G)} \leq \epsilon$. This implies that $\mathcal{G} : Y_{\epsilon} \rightarrow Y_{\epsilon}$.

Now we apply a fixed point argument. We skip the details, as similar proofs appear in (12).

First, we note that Theorem 4 implies that $H^{s}(G) \hookrightarrow L^{2\alpha}_{\mu_{G}}(G)$, and therefore the map $\mathcal{G}$ is compact. Second, a standard reasoning using the Mean Value Theorem and our assumptions on the derivative of $V$, implies that the map $\mathcal{F}$ is continuous. Application of Schauder’s fixed point theorem finishes the proof. \qed
Acknowledgements

We are most grateful to Professor H. G. Feichtinger for his comments on a first version of this paper, and for directing us to references [7] [8] [9]. P. Górka is partially supported by FONDECYT grant #3100019; E.G. Reyes is partially supported by FONDECYT grants #1070191 and #1111042.

References

[1] R. A. Adams, Sobolev Spaces Academic Press, New York, 1975.

[2] T. Aubin, Some nonlinear problems in Riemannian geometry. Springer-Verlag, Berlin, 1998.

[3] H. Bahouri, C. Fermanian-Kammerer and I. Gallagher, Analyse de l’espace des phases et calcul pseudo-differential sur le groupe de Heisenberg. C. R. Math. Acad. Sci. Paris 347 (2009), no. 17-18, 1021–1024.

[4] N. Barnaby and N. Kamran, Dynamics with infinitely many derivatives: the initial value problem. J. High Energy Physics 2008 no. 02, Paper 008, 40 pp.

[5] G. Calcagni, M. Montobbio and G. Nardelli, Localization of nonlocal theories. Physics Letters B 662 (2008), 285-289.

[6] A. Deitmer, S. Echterhoff, Principles of harmonic analysis, Springer 2009.

[7] H.G. Feichtinger, Compactness in translation invariant Banach spaces of distributions and compact multipliers. J. Math. Anal. Appl. 102 (1984), 289–327.

[8] H.G. Feichtinger and A. T. Gürkanli, On a family of weighted convolution algebras. Internat. J. Math. & Math. Sci. 13 (1990), 517–526.

[9] H. G. Feichtinger and T. Werther, Robustness of regular sampling in Sobolev algebras. In: Sampling, Wavelets and Tomography, John Benedetto and Ahmed I. Zayed (Eds.), Birkhuser (2004), p.83–113.

[10] P. Górka, H. Prado and E.G. Reyes, Nonlinear equations with infinitely many derivatives. Complex Analysis and Operator Theory 5 (2011), 313–323.

[11] P. Górka, H. Prado and E.G. Reyes, Functional calculus via Laplace transform and equations with infinitely many derivatives. Journal of Mathematical Physics 51, 103512 (2010).

[12] P. Górka, H. Prado and E.G. Reyes, Linear and nonlinear equations with infinitely many derivatives. Submitted, May 2011.

[13] P. Górka and E.G. Reyes, The modified Camassa-Holm equation. International Mathematics Research Notices, September 2010, doi: 10.1093/imrn/rnq163.
[14] K. Gröchenig and T. Strohmer, Pseudodifferential operators on locally compact abelian groups and Sjstrand’s symbol class. *J. Reine Angew. Math.* 613 (2007), 121–146.

[15] P. Hajłasz, Sobolev spaces on an arbitrary metric space. *Potential Anal.* 5 (1996), no. 4, 403–415.

[16] P. Hajłasz and P. Koskela, Sobolev met Poincaré. *Mem. Amer. Math. Soc.* 145 (2000), no. 688.

[17] E. Hebey, *Sobolev spaces on Riemannian manifolds.* Lecture Notes in Mathematics, 1635. Springer-Verlag, Berlin, 1996. x+116 pp.

[18] E. Hewitt, K. A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory. Groups representations.* Die Grundlehren der mathematischen Wissenschaften, Band 115 Springer-Verlag, New York-Berlin 1963.

[19] E. Hewitt, K. A. Ross, *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Band 152 Springer-Verlag, New York-Berlin, 1970.

[20] A.N. Kolmogorov. Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel, *Nachr. Ges. Wiss. Göttingen* 9 (1931), 60–63.

[21] L. Rastelli, Open string fields and D-branes. *Fortschr. Phys.* 52 (2004), 302–337.

[22] J.J. Rodríguez-Vega and W. A. Zúñiga-Galindo, Elliptic pseudodifferential equations and Sobolev spaces over p-adic fields. *Pacific Journal of Mathematics* Vol. 246 (2010), No. 2, 407-420.

[23] M. Ruzhansky and V. Turunen, *Pseudo-Differential Operators and Symmetries. Back- ground Analysis and Advanced Topics.* Birkhäuser, Basel - Boston - Berlin, 2010.

[24] N. Shanmugalingam, Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana* 16 (2000), no. 2, 243–279.

[25] D.V. Tausk, A locally compact non divisible abelian group whose character group is torsion free and divisible. Preprint. [http://arxiv.org/abs/1002.4164v2](http://arxiv.org/abs/1002.4164v2)

[26] J. Tateoka, On the characterization of Hardy-Besov spaces on the dyadic group and its applications. *Studia Mathematica* 110 (1994), 127–148.

[27] M. E. Taylor, *Partial Differential Equations Vol. I.* Springer-Verlag (1996).

[28] V.S. Vladimirov, The equation of the $p$-adic open string for the scalar tachyon field. *Izvestiya: Mathematics* 69 (2005), 487–512.

[29] A. Weil. *L’intégration dans les groupes topologiques et ses applications.* Hermann et Cie., Paris, 1940.

[30] E. Witten, Noncommutative geometry and string field theory. *Nuclear Physics B* 268 (1986), 253–294.