On the smoothness of the critical sets of the cylinder at spatial infinity in vacuum spacetimes

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April 16, 2018

Abstract

We analyze the appearance of logarithmic terms at the critical sets of Friedrich’s cylinder representation of spatial infinity. It is shown that if the radiation field vanishes at all orders at the critical sets no logarithmic terms are produced in the formal expansions. Conversely, it is proved that, under the additional hypothesis that the spacetime has constant (ADM) mass aspect and vanishing dual (ADM) mass aspect, this condition is also necessary for a spacetime to admit a smooth representation at the critical sets.

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*Preprint UWThPh-2018-14
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1 Introduction

1.1 Asymptotically flat spacetimes

The notion of asymptotic flatness is a delicate issue in general relativity due to the absence of non-dynamical background fields. Penrose [47, 48] provided an elegant geometric approach to resolve this issue, cf. [4, 34]. A spacetime is regarded as asymptotically flat if it admits a smooth conformal compactification at infinity. By this it is meant that, after some appropriate conformal rescaling, one can attach a conformal boundary $I$ to the spacetime through which the rescaled metric admits a smooth extension. The picture behind this notion is that such an extension is possible whenever the gravitational field admits an asymptotically Minkowski-like fall-off behavior.

In this paper we are interested in the vacuum case $\tilde{R}_{\mu\nu} = 0$. Then $I$ is a null hypersurface. Imposing the “natural” topological restriction $I \cong \mathbb{R} \times S^2$ one can show that the Weyl tensor needs to vanish on a smooth $I$ (cf. [34, 48]). Even more, the smoothness of $I$ is related to specific peeling properties of the Weyl tensor, cf. [32] for an overview. This raised the question whether the Einstein equation are compatible with Penrose’s notion of asymptotic flatness in the sense that they admit a sufficiently large class of solutions for which the Weyl tensor shows this peeling behavior. Meanwhile, Klainerman and Nicolò have shown that appropriate, sufficiently small asymptotically Euclidean Cauchy data generate vacuum spacetimes where the Weyl tensor does have the peeling properties [40], leaving the question open, whether a smooth $I$ is generated as well.

Besides, Penrose’s approach provided a tool to construct asymptotically flat vacuum spacetimes. Instead of studying the long-term behavior of the gravitational field by limiting processes one can start from the outset in a conformally rescaled spacetime and work on bounded domains, which is also very convenient from a PDE point of view. The relevant substitute to Einstein’s field equations are Friedrich’s conformal field equations (CFE) [19, 20] which are equivalent to Einstein’s if the conformal factor is non-zero, but which remain regular where it vanishes. In particular, this conformal approach permits the formulation of an asymptotic initial value problem where data are prescribed either on a portion of $I^-$ and an incoming null hypersurface [39], or on $I^-$ as the future light-cone of a regular point $i^-$ representing past timelike infinity [14, 31]. While this shows that there is a large class of asymptotically flat vacuum spacetimes one does
not gain any insights how ‘generic’ these solutions are, as the smoothness of \( I \) is built in from the outset.

One therefore needs to study initial value problems where data are prescribed on ordinary (i.e. non-asymptotic) hypersurfaces. To avoid difficulties at spatial infinity one is led to study, as a first step, a hyperboloidal Cauchy problem, where data are prescribed on a spacelike hypersurface which intersects \( I \) in a spherical cross-section. Supposing that the relevant initial data for the CFE admit smooth extensions through \( I \), Friedrich proved local well-posedness \[21\]. For small data one can use Cauchy stability of the underlying symmetric hyperbolic system contained in the CFE to show that the solution is complete and even admits a smooth future timelike infinity \( i^+ \) \[23\]. However, it turns out that generic solutions to the vacuum constraint equations do not admit a smooth but a polyhomogeneous expansion at \( I^- \), and that certain mild regularity conditions need to be imposed on the asymptotic behavior of the freely prescribable ‘seed’ data to end up with a solution to the constraints which is smooth at \( I \) \[2, 3\].

The same phenomenon can be observed for a characteristic Cauchy problem with data on either a future light-cone or two transversally intersecting null hypersurfaces. Assuming that the data for the CFE are smoothly extendable through the cross-section, where the initial surface intersects \( I \), local well-posedness holds as the spacetime admits a piece of a smooth \( I \) \[7\]. Again, one finds that generically solutions of the characteristic constraint equations constructed from smooth “seed” data develop logarithmic terms at \( I \), while certain mild regularity conditions ensure that this does not happen \[15, 45\].

While these types of initial value problems permit the construction of large classes of non-generic asymptotically flat vacuum spacetimes, they also show that this is only possible if the leading order terms of the seed data are subject to certain regularity conditions.

To fully analyze this issue, one needs to construct asymptotically flat spacetimes from an ordinary Cauchy problem. In fact, this has been done via gluing techniques. Friedrich’s results on the hyperboloidal Cauchy problem can be used to construct asymptotically flat, and in fact asymptotically simple (where, in addition, a certain completeness condition is imposed), vacuum spacetimes from Cauchy data which are glued to stationary data near spatial infinity \[11\].

All these constructions have in common that they circumvent issues arising at spatial infinity by either ignoring this part of the spacetime completely, or choosing the spacetime to be stationary near \( i^0 \). However, in order to gain a full understanding of the obstructions coming along with Penrose’s notion of asymptotic flatness and whether this definition is broad enough to include all cases of physical interest, an understanding of the behavior of the gravitational field at spatial infinity is essential.

The results described above might be viewed as an indication that a polyhomogeneous \( I \) where the asymptotic expansion of the gravitational field is allowed to have logarithmic terms is somewhat more natural and generic. However, to give a fully satisfactory answer, spatial infinity, where Cauchy surfaces with asymptotically Euclidean data “touch” infinity, needs to be taken into account. In a recent breakthrough result Hintz and Vasy \[38\] were able to construct spacetimes with a polyhomogeneous \( I \) from asymptotically Euclidean Cauchy data sets. They use a gauge where log terms are inevitably produced unless the ADM mass vanishes, so a priori their result does not give hints concerning the smoothness of \( I \) (in an appropriate gauge).

Ideally one would like to mimic the results on the hyperboloidal Cauchy problem and construct asymptotically Euclidean Cauchy data which admit smooth extensions through spatial infinity and use standard result on symmetric hyperbolic systems. However, it is well-known that in the “classical” representation of spatial infinity as a point \( i^0 \), this point cannot be regular unless the ADM mass vanishes. This is somehow intuitive as spatial infinity is compressed to a single point. It led Friedrich to introduce a blow-up of this point to a cylinder \( I \) \[26\]. While the metric is singular there, the frame field is not, and it becomes possible to construct non-trivial
asymptotically Euclidean Cauchy data sets such that all the fields which appear in the CFE admit smooth expansions through the 2-sphere \( I^0 \) where the Cauchy surface intersects \( I \) [26, 54]. Nevertheless, some difficulties remain (cf. [27, 56] for an overview). The hyperbolicity of the CFE breaks down at the critical sets \( I^\pm \) where the cylinder “touches” \( \mathcal{I}^\pm \). Related to this is the following property: The CFE provide inner equations on the cylinder so that in principle all fields (including all radial derivatives) can be determined on the cylinder from the (asymptotic part of the) Cauchy data. It turns out, though, that, in general, logarithms arise at the critical sets. While this seems similar to e.g. the hyperboloidal Cauchy problem, the problem is in fact much more severe: In the latter case, once one makes sure that no logarithmic terms arise when solving the constraints equations, i.e. that the restriction of the relevant fields to the initial surface admits smooth extension through \( \mathcal{I} \), no logarithmic terms arise in higher order transverse derivatives (in an appropriate gauge). At the critical sets this is no longer true. In principle, logarithmic terms can arise at any order, while all lower orders admit smooth expansions at \( I^- \).

A priori, non-smoothness of the critical sets is irrelevant as the critical sets do not belong to \( \mathcal{I} \). However, one expects that these log terms spread over to \( \mathcal{I} \) and therefore produce a \( \mathcal{I} \) which is not smooth but merely polyhomogeneous. In the spin-2 case this can explicitly be shown [28], and there is no reason to expect that the situation is better in the non-linear case. For this reason one would like to understand the meaning of logarithmic terms at the critical sets, and characterize initial data which admit smooth extensions through \( I^\pm \).

Friedrich [26] (in the time-symmetric case) and Valiente Kroon [54] (in the general case) analyzed this issue starting from an ordinary Cauchy problem. They derived a couple of necessary conditions on the Cauchy data to get rid of the log terms. One can also establish sufficient conditions for the non-appearance of logarithmic terms at the critical sets, and at \( \mathcal{I}^- \): Dain [16] has shown, cf. [17], that an asymptotically Euclidean spacetime which is stationary near spatial infinity admits a smooth \( \mathcal{I} \) (at least near spacelike infinity). Friedrich [30] showed that Cauchy data which are static also admit smooth critical sets, a result which has been generalized by Aceña and Valiente Kroon [1] to the stationary case. One may regard this as another indication that there is a relation between the appearance of log terms at the critical sets and at \( \mathcal{I} \). Their results imply that the inner equations on the cylinder do not produce logarithmic terms if the data are merely asymptotically stationary.

This raises the question whether asymptotic stationarity (staticity in the time-symmetric case) is also necessary for the non-appearance of log terms. That the notion of asymptotic staticity plays a distinguished role in view of the appearance of log terms at the critical sets was first observed in [26]. Some evidence that this might be true is provided by the result [55] that staticity is necessary for time-symmetric, conformally flat data.

The main purpose of this paper is to analyze these kind of issues from \( \mathcal{I}^- \). More precisely, we consider an asymptotic characteristic initial value problem, where data are prescribed on \( \mathcal{I}^- \) (the data on the incoming null hypersurface will be largely irrelevant), and analyze the appearance of log terms approaching the critical set \( I^- \) from \( \mathcal{I}^- \). Assuming that the data generate a spacetime with a smooth cylinder we will then analyze the appearance of log terms when approaching \( I^- \) from \( I \), as well, where we assume that the data for the transport equations on \( I \) are induced by the limit of the corresponding fields on \( \mathcal{I}^- \) to \( I^- \).

A main advantage of this approach as compared to the ordinary Cauchy problem is that the critical set arises as a future boundary of the initial surface, whence it is easier to control the fields there. Another advantage is that the no-logs conditions one obtains from \( \mathcal{I}^- \) turn out to be somewhat easier to handle. Finally, the no-logs conditions depend crucially on the radiation field, which essentially provides the freely prescribable data near \( I^- \) (together with certain “integration functions” prescribed at \( I^- \), cf. Appendix A.2).

Omitting some technical details our main result can be stated as follows:
Theorem 1.1  (i) Assume that a smooth vacuum spacetime with smooth $\mathcal{I}^-$, $I$ and $I^-$ has been given and assume further that it has constant (ADM) mass aspect and vanishing dual (ADM) mass aspect,\(^1\) then the radiation field vanishes at $I^-$ at any order.

(ii) Conversely, the restriction of all the fields appearing in the CFE to both $\mathcal{I}^-$ and $I$, and all derivatives thereof, admit smooth extensions through $I^-$ if the radiation field vanishes there at any order.

Remark 1.2 This raises the question, to be analyzed elsewhere, whether an asymptotic characteristic Cauchy problem with data on $\mathcal{I}^-$ and some incoming null hypersurface generates a vacuum spacetime which admits a smooth cylinder with smooth critical sets if the prescribed radiation field vanishes at $I^-$ at any order.

1.2 Overview

In Section 2 we recall the conformal field equations as well as the conformal Gauss gauge, which provides a natural, geometric gauge in the conformal setting we will use. In particular we explain how a conformal Gauss gauge is constructed from $\mathcal{I}^-$. We further describe the additional gauge data which one may specify on $\mathcal{I}^-$. Finally we provide the evolution equations in this gauge, and, as we will analyze the constraint equation using adapted null coordinates, we give the relation between frame components and coordinate components on $\mathcal{I}^-$. In Section 3 we derive conditions on the gauge data at $\mathcal{I}^-$ in order to end up with a spacetime which admits a finite representation of spatial infinity. Starting from the Minkowski spacetime as an explicit example we then introduce the cylinder representation of spatial infinity. We also extract another gauge freedom. The section is closed by defining a what we call “(weakly) asymptotically Minkowski-like conformal Gauss gauge”, were a certain asymptotic behavior of the gauge data at $\mathcal{I}^-$ is imposed when approaching $I^-$ which turns out to be very convenient for the subsequent analysis.

Section 4 is devoted to an analysis of the behavior of all the unknowns which appear in the CFE and all transverse derivatives thereof when approaching the critical set $I^-$ from $\mathcal{I}^-$. In the zeroth order this can and will be done explicitly. For higher order transverse derivatives we will merely derive the structure of the equations and of the no-logs condition, as the gauge will be still quite general at this stage.

In Section 5 we provide a corresponding analysis of the behavior of the fields and their radial derivatives when approaching the critical set $I^-$ from $\mathcal{I}^-$. In the zeroth order one recovers the Minkowskian values. We will also study the first-order radial derivatives explicitly and derive the structure of the equations and of the no-logs condition for radial derivatives of higher orders. The no-logs conditions adopt here a somewhat more difficult form, as they are PDEs rather than ODEs. However, expanding the fields in spherical harmonics they can be transformed into hypergeometric ODEs which are analyzed.

In Section 6 we show that transforming a spacetime which admits a smooth representation of $\mathcal{I}^- \cup I^- \cup I$ from one weakly asymptotically Minkowski-like conformal Gauss gauge into another one is accompanied by a smooth (coordinate and conformal) transformation unless the gauge data are badly chosen in such a way that the congruence of conformal geodesics produces conjugate point directly on $I^-$. From this we conclude that if a spacetime is smooth near $I^-$ in one weakly asymptotically Minkowski-like conformal Gauss gauge then the same is true in any other one.

This allows us to restrict attention in Section 7 to a more restricted gauge which we call asymptotically Minkowski-like conformal Gauss gauge at each order (which is also introduced

\(^1\)The dual mass aspect introduced later might be regarded as a generalized NUT-like parameter.
in Section 3). In this particular gauge we then show that for a radiation field which vanishes asymptotically at \( I^- \) at any order no logarithmic terms are produced when approaching \( I^- \) from both \( \mathcal{I}^- \) and \( I \). This can be done because in this gauge one can show that simply no terms arise at the critical orders which produce logarithmic terms. On \( \mathcal{I}^- \) it can be shown that the fields are basically polynomials of a sufficiently low degree, supplemented by terms which decay arbitrarily fast, while on \( I \) all fields decay sufficiently fast at \( I^- \). It is illuminating to bring the Kerr metric into such a gauge (at least up to some order).

In Section 8 we study the massless spin-2 equation. On the one hand this provides a toy model for the full non-linear case where the whole analysis concerning the appearance of log terms can be done very explicitly. In particular one finds if-and-only-if conditions for the appearance of logarithmic terms at the critical sets. On the other hand this allows us to compare the no-logs conditions of the spin-2 case with the general case, which gives some insights what the sources of additional or more restrictive no-logs conditions are in the general case (or even on a curved background).

In Section 9 we restrict attention to data with constant (ADM) mass aspect and vanishing (ADM) dual mass aspect (by which we mean the limit of the Bondi mass and dual mass aspect to \( I^- \)). We will explain why this simplifies the analysis considerably. We then explicitly show that the no-logs conditions are indeed more restrictive as compared to the spin-2 case (in fact this is already known from the ordinary Cauchy problem [53]). In a next step we show that logarithmic terms are inevitably produced, unless the radiation field vanishes at any order at \( I^- \), which does not need to be the case in the spin-2 case. This requires some rather lengthy computations as also next-to-leading order terms need to be taken into account.

In Appendix A we review the constraint equations in adapted null coordinates for the asymptotic characteristic initial value problem. We also provide a slight variation of the standard approach which allows to shift the freedom to prescribe certain data on the intersection sphere of the two null hypersurfaces to \( I^- \), so that as many data as possible are directly prescribed on \( \mathcal{I}^- \) and its future boundary \( I^- \).

1.3 Notation

Here let us given an overview over some frequently used notation:

1. \( \eta = \text{diag}(-1,1,1,1) \)
2. \( (\cdot)_{\text{tf}} \) denotes the trace-free part
3. \( g = \Theta^2 \tilde{g} \), where \( \tilde{g} \) is the physical metric, the inverse metric is denoted by \( g^\flat \)
4. \( \mathcal{I}^\pm \) denotes future and past null infinity, \( i^0 \) spatial infinity, in particular if represented as a point, \( I \) denotes the cylinder representation of spatial infinity, \( I^\pm \) the critical sets where \( \mathcal{I}^\pm \) and \( I \) “touch”
5. Coordinate spacetime indices are denoted by \( \mu, \nu, \sigma, \ldots \), spatial coordinate indices are denoted by \( \alpha, \beta, \gamma, \ldots \), and angular coordinate indices by \( \hat{A}, \hat{B}, \hat{C}, \ldots \)
6. Frame spacetime indices are denoted by \( i, j, k, \ldots \), spatial frame indices are denoted by \( a, b, c, \ldots \), and angular coordinate indices by \( A, B, C, \ldots \)
7. Objects associated to the Weyl connection are decorated with \( ^\hat{\cdot} \)
8. Objects associated to \( g = g_{\hat{A}\hat{B}}dx^A dx^B |_{\mathcal{I}^-} \) are denoted by \( \hat{\nabla}, \hat{\Gamma} \) etc.
9. The Levi-Civita covariant derivative associated to the standard metric $s_{AB}dx^Adx^B$ on $S^2$ is denoted by $\mathcal{D}$, the Christoffel symbols by $\tilde{\Gamma}$ and the volume form by $\epsilon_{AB}$.

10. $\nu_r = g_{rr}|_{\mathcal{I}^-}$, $\nu_A = g_{rA}|_{\mathcal{I}^-}$

11. The Hodge decomposition of a 1-form $f_A$ on $S^2 := (S^2, s_{AB})$ is written as $f_A = \mathcal{D}_A f + \epsilon_A^B \mathcal{D}_B \tilde{\Theta}$, that of a symmetric trace-free tensor $\mathfrak{t}_{AB}$ as $\mathfrak{t}_{AB} = (\mathcal{D}_A \mathcal{D}_B \tilde{\Theta})_{\mathcal{I}^+} + \epsilon_{(A}^C \mathcal{D}_B) \mathcal{D}_C \tilde{\Theta}$

12. $f^{(m)} = \frac{1}{m!} \partial^m_r f |_{\mathcal{I}^-}$ denotes expansion coefficients in the radial coordinate $r$ ($\Theta^{(n)}$ and $b_i^{(n)}$ are exceptions, they denote the expansion coefficients in $1 + \tau$)

13. $f^{(m,n)} = \frac{1}{m! n!} \partial^m_r \partial^n_r f |_{\mathcal{I}^-}$

14. the independent components of the rescaled Weyl tensor $W_{ijkl}$ we will use are: $U_{AB} = W_{0101} \eta_{AB} + W_{01AB}$, $V^\pm_{AB} = (W_{1A}B)_{\mathcal{I}^+} \pm W_{0(A|B)1}$ and $W^\pm_A = W_{010A} \pm W_{011A}$.

15. $M$ denotes the (ADM) mass aspect, and $N$ the dual (ADM) mass aspect, we also use the notation $\mathcal{M}_A = \mathcal{D}_AM + \epsilon_A^B \mathcal{D}_BN$ and $\mathcal{M}_A = \mathcal{D}_AM - \epsilon_A^B \mathcal{D}_BN$ (more precisely, the limit of the Bondi mass and dual mass aspect on $\mathcal{I}^-$ to $I^-$).

2 General conformal field equations and conformal Gauss gauge

2.1 General conformal field equations

A breakthrough on the way to gain a better understanding of the compatibility of Penrose’s notion of asymptotic simplicity and Einstein’s field equations was obtained by Friedrich [19, 20]. He derived a set of equations, the so-called conformal field equations (CFE), which substitute Einstein’s vacuum field equations in Penrose’s conformally rescaled spacetimes. They are equivalent to the vacuum equations in regions where the conformal factor does not vanish, and remain regular at points where it vanishes. This result offered the possibility to study the evolution of initial data sets in the conformally rescaled spacetime from the outset. In particular, it permitted the formulation of an asymptotic initial value problem where an appropriate set of data is prescribed “at infinity”, i.e. at $\mathcal{I}^-$.

Beside the usual gauge freedom arising from the freedom to choose coordinates, frame field etc., the CFE contain an additional gauge freedom which arises from the artificially introduced conformal factor $\Theta$ which relates the physical spacetime with its conformally rescaled counterpart. For an analysis of the gravitational field near spacelike infinity, though, it turned out that the introduction of additional gauge degrees of freedom, which even more exploits the conformal structure, can simplify the analysis considerably. They are obtained when replacing the Levi-Civita connection by some appropriately chosen Weyl connection. This way one is led to the so-called general conformal field equations (GCFE), introduced by Friedrich in [25], cf. [26, 27, 30]. In the following we will recall these equations and sum up some of the results.

Let $(\mathcal{M}, \tilde{g})$ be a smooth Lorentzian manifold, and denote by $g = \Theta^2 \tilde{g}$ a conformally rescaled metric. We denote by $\tilde{\nabla}$ and $\nabla$ the Levi-Civita connection of $\tilde{g}$ and $g$, respectively.

Let $\tilde{f}$ be a smooth 1-form on $\mathcal{M}$. There exists a unique torsion-free connection $\tilde{\nabla}$, the so-called Weyl connection, which satisfies

$$\tilde{\nabla}_\sigma \tilde{g}_{\mu\nu} = -2\tilde{f}_\sigma \tilde{g}_{\mu\nu}. \quad (2.1)$$
Then
\[ \hat{\nabla} = \nabla + S(\tilde{f}), \quad \text{where} \quad S(\tilde{f})_{\mu \nu} := 2\delta_{(\mu} \tilde{f}_{\nu)} - \tilde{g}_{\mu\nu} \tilde{g}^{\rho\sigma} \tilde{f}_\rho, \quad (2.2) \]
or, equivalently,
\[ \hat{\nabla} = \nabla + S(f), \quad \text{where} \quad f = \tilde{f} - \Theta^{-1} d\Theta. \quad (2.3) \]
We observe that \( S(\tilde{f}) \) depends merely on the conformal class of \( \tilde{g} \).

Let \( e_k \) be a frame field satisfying \( g(e_i, e_j) = \eta_{ij} \equiv \text{diag}(-1,1,1,1) \). We define the connection coefficients of \( \hat{\nabla} \) in this frame field by
\[ \hat{\nabla}_i e_j = \hat{\Gamma}_i^k j e_k. \quad (2.4) \]

Note that
\[ \hat{\Gamma}_i^k j = \Gamma_i^k j + S(f)_{i \ j}, \quad \text{and} \quad f_i = \frac{1}{4} \hat{\Gamma}_i^k k. \quad (2.5) \]

A Weyl connection respects the conformal class in the sense that for any \( C^1 \)-curve \( \gamma : (-\varepsilon, \varepsilon) \to \mathcal{M} \) and any frame field \( e_k \) which is parallely transported along \( \gamma \) w.r.t. \( \hat{\nabla} \), there exists a function \( \Omega_\tau > 0 \) along \( \gamma(\tau) \) such that \( \tilde{g}(e_i, e_j)(\gamma(\tau)) = (\Omega_\tau)^2 g(e_i, e_j)|_{\gamma(0)} \).

Finally, set
\[ b := \Theta \tilde{f} = \Theta f + d\Theta, \quad (2.6) \]
and denote by
\[ \tilde{W}^\mu_{\nu\sigma\rho} = \Theta^{-1} \hat{\mathcal{C}}^\mu_{\nu\sigma\rho}, \quad (2.7) \]
\[ \hat{L}_{\mu\nu} = \frac{1}{2} \hat{R}_{(\mu\nu)} - \frac{1}{4} \hat{R}_{[\mu\nu]} - \frac{1}{12} \hat{R} g_{\mu\nu}, \quad (2.8) \]
the rescaled Weyl tensor and Schouten tensor of \( \hat{\nabla} \), respectively. We note that
\[ \hat{L}_{\mu\nu} = L_{\mu\nu} - \nabla_\mu f_\nu + \frac{1}{2} S(f)_{\mu \sigma \nu} f_\sigma, \quad (2.9) \]
while the rescaled Weyl tensor does not depend on the Weyl connection,
\[ \tilde{W}^\mu_{\nu\sigma\rho} = W^\mu_{\nu\sigma\rho}. \quad (2.10) \]

Let now \((\mathcal{M}, \tilde{g})\) be a solution to Einstein's vacuum field equations
\[ \hat{R}_{\mu\nu} = \lambda \tilde{g}_{\mu\nu}. \quad (2.11) \]
Then the tuple
\[ \tilde{f} := (e^\mu_k, \hat{\Gamma}_i^k j, \hat{L}_{ij}, W^i_{\ jkl}), \quad (2.12) \]
where \( e^\mu_k := (dx^\mu, e_k) \), satisfies the general conformal field equations (GCFE) \([25]\)
\[ [e_p, e_q] = 2\hat{\Gamma}_i^k [p q] e_i, \quad (2.13) \]
\[ e_p(\hat{\Gamma}_i^k j) = \hat{\Gamma}_i^k [p j] + \hat{\Gamma}_i^k [q] - \hat{\Gamma}_i^k [p b_i q] - \delta_i^p \hat{L}_{[jq]} - \eta_{ij} b_i q] - \delta_i^p \hat{L}_{[jq]} - \eta_{ij} b_i q] + \Theta W^i_{jpq}, \quad (2.14) \]
\[ 2\hat{\Gamma}_k [L_{ij}] = b_i W^i_{jkl}, \quad (2.15) \]
\[ \hat{\nabla}_{i} W_{jkl} = \frac{1}{4} \hat{\Gamma}_i ^{p} W_{jkl}. \quad (2.16) \]
The fields \( \Theta \) and \( b \) reflect the conformal gauge freedom.
2.2 Conformal geodesics and conformal Gauss gauge

2.2.1 Definition and properties of conformal geodesics

A conformal geodesic for \((\mathcal{M}, \bar{g})\) (cf. e.g. [29, 33]) is a curve \(x(\tau)\) in \(\mathcal{M}\) for which a 1-form \(\tilde{f} = \tilde{f}(\tau)\) exists along \(x(\tau)\) such that the pair \((x, \tilde{f})\) solves the conformal geodesics equations

\[
\begin{align*}
(\nabla_x \tilde{x})^\mu + S(\tilde{f})_{\lambda}^{\mu, \rho} \tilde{x}^\rho &= 0, \\
(\nabla_x \tilde{f})_\nu - \frac{1}{2} \tilde{f}_\mu S(\tilde{f})_{\lambda}^{\mu, \nu} \tilde{x}^\lambda &= \tilde{L}_{\lambda \nu} \tilde{x}^\lambda.
\end{align*}
\]

Given data \(x_* \in \mathcal{M}, \dot{x}_* \in T_{x_*} \mathcal{M}\) and \(\tilde{f}_* \in T^* \mathcal{M}\) there exists a unique solution \(x(\tau), \tilde{f}(\tau)\) to (2.17)-(2.18) near \(x_*\) satisfying, for given \(\tau_* \in \mathbb{R}\),

\[
x(\tau_*) = x_*, \quad \dot{x}(\tau_*) = \dot{x}_*, \quad \tilde{f}(\tau_*) = \tilde{f}_*.
\]

Conformal geodesics are curves which are associated with the conformal structure in a similar way as geodesics are associated with the metric. They are conformally invariant in the following sense: Let \(b\) be a smooth 1-form on \(\mathcal{M}\). Then \((x(\tau), \tilde{f}(\tau))\) solves (2.17)-(2.18) if and only if \((x(\tau), \tilde{f}(\tau) - b(\tau))\) solves (2.17)-(2.18) with \(\nabla\) and \(\tilde{L}\) replaced by \(\nabla = \nabla + S(b)\) and \(\tilde{L}\), respectively. The conformal geodesic \(x(\tau)\) and its parameter \(\tau\) are independent of the Weyl connection in the conformal class w.r.t. which (2.17)-(2.18) are written (in particular, they do not depend on the metric in the conformal class chosen to write the conformal geodesics equations).

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It follows from (2.17) that the sign of \(\tilde{g}(\dot{x}, \dot{x})\) is preserved along any conformal geodesic,

\[
\nabla_x \tilde{g}(\dot{x}, \dot{x}) = -2(\tilde{f}, \dot{x})\tilde{g}(\dot{x}, \dot{x}),
\]

i.e. conformal geodesics preserve their causal character.

**Lemma 2.1 ([29])** Consider a conformal geodesic \(x(\tau)\). Changes of its initial data \(\dot{x}_*\) and \(f_*\) for the conformal geodesics equations which locally preserve the point set spread out by the curve \(x(\tau)\) (i.e. which change only its parameterization) are given by

\[
\begin{align*}
\dot{x}_* &\mapsto \varphi_* \dot{x}_*, \quad \varphi_* \in \mathbb{R} \setminus \{0\}, \\
f_* &\mapsto f_* + \psi_*(\dot{x}_*, \cdot), \quad \psi_* \in \mathbb{R}.
\end{align*}
\]

Consider now a congruence \(\{x(\tau, \rho), f(\tau, \rho)\}\) of conformal geodesics, set \(x' := \partial x/\partial \rho\), and denote by \(F := \nabla_x f\) the deviation 1-form. The conformal Jacobi equation reads [29]

\[
\nabla_x \nabla_x x' = \text{Ric}(\dot{x}, x') \dot{x} - S(F)(\dot{x}, \cdot, \cdot) - 2S(f)(\dot{x}, \cdot, \nabla_x F).
\]

It is convenient to introduce conformal Gauss coordinates, a geometrically defined coordinate system, where the time-like coordinate lines are generated by timelike conformal geodesics. As metric geodesics, conformal geodesics may develop caustics. Even worse, they may, in addition, become tangent to each other. However, because of (2.23) one may hope that curvature induced tendencies to develop caustics may be counteracted by the 1-form \(f\). This is a reason why one expects a gauge which is based on conformal geodesics to provide a more convenient setting to cover large domains of spacetime than the classical Gauss gauge does. In fact, Friedrich showed that the Schwarzschild-Kruskal spacetime permits a global coordinate system based on conformal geodesics [29]. Also in the case of a hyperboloidal initial value problem with data sufficiently close to Minkowskian hyperboloidal data it can be shown that conformal Gauss coordinates exist globally [43].
2.2.2 Conformal Gauss gauge

We construct a conformal Gauss gauge adapted to a congruence of conformal geodesics, which employs the fact that a congruence of conformal geodesics distinguishes the Weyl connection associated to the 1-form $f$. This will be done from an initial surface $S$ (spacelike or null) which intersects the congruence transversally and meets each of the curves exactly once. This way we mimic the construction in [25, 26]. Here, though, it provides the starting point to construct such a gauge for the special case where $S$ is identified with past null infinity (for non-negative cosmological constant).

Consider a smooth congruence of conformal geodesics which covers an open set $U$ of $\tilde{M}$ (we do not need to impose the vacuum equations at this stage), and assume that the associated 1-form $\tilde{f}$ defines a smooth tensor field on $U$. As above, we denote by $\hat{\nabla}$ the Weyl connection which satisfies $\hat{\nabla}_\xi x = 0$, $\hat{\nabla}_\xi f = 0$. Then (2.17)-(2.18) adopt the simple form

\[
\hat{\nabla}_\xi x = 0, \quad \hat{L}(\xi, \cdot) = 0.
\] (2.24)

We would like to construct a gauge where $\hat{\nabla}$ preserves the conformal structure, i.e. where

\[
\hat{\nabla}_\xi g = 0.
\] (2.25)

It follows from (2.1) that

\[
(2.25) \iff \hat{\nabla}_\xi \Theta = \Theta(\xi, \tilde{f}),
\] (2.26)

i.e. such a conformal gauge can always be realized by an appropriate choice of the conformal factor $\Theta$.

Let us start with the physical spacetime. Let $\tilde{S} \subset \tilde{M}$ be a hypersurface (for definiteness take a spacelike or characteristic one). We require the fields $x$, $\tilde{f}$ and $\Theta$ to satisfy (2.17), (2.18) and (2.25). This conformal gauge needs to be complemented by the “gauge data”

\[
x|_{\tilde{S}}, \quad \tilde{f}|_{\tilde{S}}, \quad \Theta|_{\tilde{S}} > 0,
\] (2.27)

with $x|_{\tilde{S}}$ being transversal to $\tilde{S} \subset \tilde{M}$. In this article we will always assume $x|_{\tilde{S}}$ to be timelike, $\tilde{g}(x, x)|_{\tilde{S}} < 0$. It follows from (2.26) that instead of (2.27) one may prescribe

\[
x|_{\tilde{S}}, \quad \tilde{f}|_{\tilde{S}}, \quad \Theta|_{\tilde{S}} > 0, \quad \nabla_\xi \Theta|_{\tilde{S}},
\] (2.28)

where $\tilde{g}(x, x)|_{\tilde{S}} < 0$. Here $\tilde{f}|_{\tilde{S}}$ denotes the pull back of $\tilde{f}$ on $\tilde{S}$.

Furthermore, we choose the frame field on $\tilde{S}$ in such a way that

\[
\Theta^2 g(e_i, e_j)|_{\tilde{S}} = \eta_{ij} \iff g(e_i, e_j)|_{\tilde{S}} = \eta_{ij}.
\] (2.29)

The frame field is then parallely propagated w.r.t $\hat{\nabla}$ along the conformal geodesics, whence, by (2.25)

\[
g(e_i, e_j) = \eta_{ij} \quad \text{on } U.
\] (2.30)

Let us now pass to a conformally rescaled space-time and replace $\tilde{f}$ by $f = \tilde{f} - d \log \Theta$. Expressed in terms of the Levi-Civita connection of $g$ and the 1-form $f$ the conformal geodesics equations read

\[
\nabla_\xi x = -S(f)(x, \cdot),
\] (2.31)

\[
\nabla_\xi f = \frac{1}{2} S(f)(x, f, \cdot) + L(x, \cdot),
\] (2.32)
while (2.26) becomes

\[ \langle \dot{x}, f \rangle = 0. \] (2.33)

These equations contain the conformal factor only implicitly. However, from the above considerations it is clear how such a gauge can be constructed starting from the physical, non-rescaled spacetime \((\mathcal{M}, \tilde{g})\) and the 1-form \(\tilde{f}\). The free gauge data on the (spacelike or null) hypersurface \(\tilde{S} \subset \mathcal{M}\) for a conformal gauge adapted to a congruence of timelike conformal geodesics are

\[ \dot{x}|_S, \ f|_S, \ \Theta|_S > 0, \ \nabla_x \Theta|_S, \ \text{with} \ \ g(\dot{x}, \dot{x})|_S < 0. \] (2.34)

### 2.2.3 Construction of the conformal Gauss gauge from null infinity

Let us analyze the construction of such a conformal gauge more detailed in the case where the initial surface belongs to (past) null infinity, \(S \subset \mathcal{I}^-\). More specifically, we consider a smooth \(\lambda \geq 0\)-vacuum spacetime \((\mathcal{M}, \tilde{g})\) which admits a conformal representation \((\mathcal{M}, g)\) and a smooth \(\mathcal{I}^-\) à la Penrose [47, 48] (so that \(\mathcal{I}^-\) is either a spacelike or a null hypersurface). In the conformal picture this corresponds to a metric \(g\) and a conformal factor \(\Theta\), which satisfies \(\Theta|_{\mathcal{I}^-} = 0\) and \(d\Theta|_{\mathcal{I}^-} \neq 0\), such that \((g, \Theta)\) solves the conformal field equations with non-negative cosmological constant \(\lambda\).

Given initial data \(\dot{x}|_{\mathcal{I}^-}\) and \(f|_{\mathcal{I}^-}\) on \(\mathcal{I}^-\), we solve the conformal geodesic equations (2.31)-(2.32) by standard results on ODEs. In particular, this singles out the Weyl connection \(\nabla = \nabla + S(f)\). In addition, though, we want to choose \(\Theta\) in such a way that \(\langle \dot{x}, f \rangle = 0\) holds in the region \(U\) covered by the congruence of conformal geodesics, which will, in general, not be the case. The “wrong” conformal factor needs to be rescaled by some positive function \(\phi\). The gauge condition (2.33) adopts the form

\[ \nabla_x \phi = \phi \langle \dot{x}, f \rangle, \] (2.35)

and, once this equation has been solved, \(g, \Theta\) and \(f\) need to be replaced by

\[ g^\text{new} = \phi^2 g, \ \ \Theta^\text{new} = \phi \Theta, \ \ f^\text{new} = f - d \log \phi. \] (2.36)

When solving the ODE (2.35) for \(\phi\) there remains the freedom to prescribe \(\phi|_{\mathcal{I}^-}\). We further observe that the freedom to prescribe \(\langle \dot{x}, f \rangle|_{\mathcal{I}^-}\) can be identified with the freedom to prescribe \(\nabla_x \phi|_{\mathcal{I}^-}\), while there remains the freedom to prescribe the pull back \(f^\text{new}|_{\mathcal{I}^-}\) of \(f^\text{new}\) on \(\mathcal{I}^-\).

We want to identify these gauge degrees of freedom with certain gauge data for the initial value problem. Since it is the case we are mainly interested in we will focus on the \(\lambda = 0\)-case. In that case \(\mathcal{I}^-\) represents a null hypersurface in \((\mathcal{M}, g)\). In contrast to the case of a positive cosmological constant the Weyl tensor does not need to vanish on \(\mathcal{I}^-\). Here, we assume that \(\mathcal{I}^-\) has the “natural” topology \([34, 37]\)

\[ \mathcal{I}^- \cong \mathbb{R} \times S^2. \] (2.37)

This topology will be crucial for some of the computations below,\(^2\) and in that case it is well-known [34] that the Weyl tensor vanishes on \(\mathcal{I}^-\).

The initial data for the GCFE need to satisfy certain constraint equations on \(\mathcal{I}^-\). Now, the frame we are dealing with will be adapted to an ordinary spacelike Cauchy problem and to the cylinder at spacelike infinity rather than to \(\mathcal{I}^-\). For this reason it is convenient to choose coordinates which are adapted to \(\mathcal{I}^-\). Solutions to the constraint equation are then constructed in these coordinates. The frame coefficients needed for the GCFE will be computed afterwards.

\(^2\)e.g. we will employ that, given a smooth 1-form \(\nu_A\) such that its Hodge decomposition scalars do not contain \(\ell = 1\)-spherical harmonics, the PDE \(\partial_B \nu_A = \nu_B\) admits a unique symmetric and trace-free solution \(\nu:\) on the round sphere, and that there are no non-trivial harmonic 1-forms on the round sphere.
Assuming \( \lambda = 0 \), let us introduce adapted null coordinates \((\tau, r, x^A)\) on \( \mathcal{I}^- \cong \mathbb{R} \times S^2 \). They are defined in such a way that \( \mathcal{I}^- = \{ \tau = -1 \} \), \( r \) parameterizes the null geodesic generators of \( \mathcal{I}^- \), and the \( x^A \)'s are local coordinates on the \( \Sigma_r := \{ \tau = -1, r = \text{const.} \} \cong S^2 \)-level sets (cf. [8] for more details). Because \( \mathcal{I}^- \) is a null hypersurface \( g(\partial_\tau, \partial_\tau)|_{\mathcal{I}^-} = 0 \). Since \( n = \text{grad}(\tau) \) is another null vector normal and tangent to \( \mathcal{I}^- \), \( n \) and \( \partial_\tau \) have to be proportional, which implies that \( g(\partial_\tau, \partial_A)|_{\mathcal{I}^-} = 0 \), so that the metric adopts the form

\[
g|_{\mathcal{I}^-} = g_{\tau\tau}d\tau^2 + 2\nu_\tau d\tau dr + 2\nu_A d\tau dx^A + g_{AB}dx^Adx^B. \tag{2.38}
\]

Here an henceforth we use \( ^\prime \) to denote angular coordinate indices. The remaining metric coefficients are determined by the constraint equations and how the coordinates are extended off \( \mathcal{I}^- \).

At each \( p \in \Sigma_r \) we denote by \( \ell^\pm \) the future-directed null vectors orthogonal to \( \Sigma_r \) and normalized in such a way that \( g(\ell^+, \ell^-) = -2 \). In adapted null coordinates they read

\[
\ell^+ = \partial_\tau, \quad \ell^- = -2\nu^\tau \partial_\tau - g^{rr}\partial_r - 2g^{A\bar{A}}\partial_A,
\]

where \( \nu^\tau := \nu^{-1}_\tau \). We denote by \( \theta^\pm \) the divergences of the null hypersurfaces emanating from \( \Sigma_r \) tangentially to \( \ell^\pm \). For the computation of \( \theta^\pm \) it does not matter how \( \ell^\pm \) are extended off \( \mathcal{I}^- \), so we may use (2.39) for all values of \( \tau \). A somewhat lengthy calculation making extensively use of the formulae in [8, Appendix A] reveals that

\[
\theta^+(r, x^A) \equiv [g^{\mu\nu} + (\ell^+)\mu(\ell^-)\nu]\nabla_\mu \ell^+_\nu |_{\Sigma_r} = \frac{1}{2}g^{AB}\partial_\tau g_{AB},
\]

\[
\theta^-(r, x^A) \equiv [g^{\mu\nu} + (\ell^+)\mu(\ell^-)\nu]\nabla_\mu \ell^-_\nu |_{\Sigma_r} = 2\nu^\tau \nabla^A \nu_A - \theta^+ g^{rr} - \nu^\tau g^{AB}\partial_\tau g_{AB},
\]

where \( \nabla \) denotes the Levi-Civita connection associated to the one-parameter family \( r \mapsto \tilde{\phi} = g_{AB}|_{\mathcal{I}^-}dx^Adx^B \) on \( S^2 \).

Let us consider the behavior of \( \theta^\pm \) under conformal rescaling \( \Theta \mapsto \phi \Theta \),

\[
\theta^+_{\text{new}} = \theta^+ + 2\partial_\tau \log \phi,
\]

\[
\theta^-_{\text{new}} = \phi^{-2}\left( \theta^- - 4\nu^\tau \partial_\tau \log \phi - 4g^{A\bar{A}}\partial_A \log \phi - 2g^{rr}\partial_r \log \phi \right).
\]

We conclude that for a timelike congruence of conformal geodesics the freedom to prescribe \( \phi |_{\mathcal{I}^-} \) and \( \nabla_\phi |_{\mathcal{I}^-} \) can be employed to prescribe the divergences \( \theta^\pm \) on \( \mathcal{I}^- \). However, we observe that (2.42) does not fully determine \( \phi \) since it leaves the gauge freedom \( \phi \mapsto \alpha(x^A)\phi \). For this reason it is more convenient to employ this gauge freedom to prescribe \( \nabla_\phi \Theta |_{\mathcal{I}^-} \), which transforms as

\[
\nabla_\phi \Theta_{\text{new}} |_{\mathcal{I}^-} = \phi \nabla_\phi \Theta.
\]

Let us merely remark that for \( \lambda > 0 \) a corresponding analysis of the behavior of the Ricci scalar \( R^{(3)} \) of the induced metric and the mean curvature \( K \) on \( \mathcal{I}^- \) under conformal rescalings shows that these two functions can be identified as gauge degrees of freedom.

---

3 One may think that, instead of \( \theta^- \), it should be possible to exploit the gauge freedom \( \nabla_\phi \phi |_{\mathcal{I}^-} \) to prescribe the function \( \nabla_\phi \nabla_\phi \Theta |_{\mathcal{I}^-} \). However, this does not work as will become clear later. The reason for this basically involves a transverse derivative of \( \tilde{x} \) which is only determined by the conformal geodesics equations. Instead, \( \nabla_\phi \nabla_\phi \Theta |_{\mathcal{I}^-} \) can be identified with a certain gauge freedom to choose coordinates on the initial surface (cf. (2.115)).
By way of summary, a gauge based on a congruence of timelike conformal geodesics, which requires the equations (2.31)-(2.33) to be fulfilled, comes along with the additional gauge freedom to prescribe

\[
\dot{x}\big|_{\mathcal{I}^-}, \quad f_{\mathcal{I}^-}, \quad \hat{\nabla}_2 \Theta\big|_{\mathcal{I}^-} > 0, \quad \theta^- \quad \text{for} \quad \lambda = 0, \quad (2.45)
\]

\[
\dot{x}\big|_{\mathcal{I}^-}, \quad f_{\mathcal{I}^-}, \quad R^{(3)}, \quad K \quad \text{for} \quad \lambda > 0, \quad (2.46)
\]

in either cases we choose \(g(\dot{x}, \dot{x})\big|_{\mathcal{I}^-} < 0\).

We introduce the initial frame field as follows: Let \(e_0\), be a future-directed timelike vector field on \(\mathcal{I}^-\), and denote by \(e_{\alpha}, \alpha = 1, 2, 3\), spacelike vectors which complement \(e_0\), to an orthonormal frame \(g_0(e_{\alpha}, e_{\beta}) = \eta_{\alpha\beta}\). For \(\lambda = 0\), to obtain \((e_\alpha)\) we consider any 2-sphere which is transversally intersected by the null geodesic generators of \(\mathcal{I}^-\). We choose two spacelike orthonormal frame vectors \(e_A, A = 2, 3\) tangent to that sphere which we complement by another spacelike vector \(e_1\) and a future-directed timelike vector \(e_0\) to an orthonormal frame on this 2-sphere. Parallel transport along the null geodesic generators of \(\mathcal{I}^-\) yields an orthonormal frame on \(\mathcal{I}^-\).

The frame field is then parallelly propagated w.r.t. \(\hat{\nabla}\) along the congruence of conformal geodesics, i.e. it is required to satisfy the transport equation

\[
\hat{\nabla}_2 e_k = 0, \quad (2.47)
\]

starting from the initial frame field \(e_{\alpha}^*\) on \(\mathcal{I}^-\) (i.e. it is Fermi-propagated w.r.t. \(\nabla\)). Then, by (2.25),

\[
g(e_i, e_j) = \eta_{ij}. \quad (2.48)
\]

Finally, a coordinate system is obtained as follows: We choose \(x^0 = \tau\). Moreover, let \((x^\alpha)\), \(\alpha = 1, 2, 3\), be local coordinates on \(\mathcal{I}^-\) (for \(\lambda = 0\) we will take adapted null coordinates \((r, x^A)\)). The coordinates \((x^\alpha)\) are then dragged along the conformal geodesics.

For given “conformal gauge data” (2.45) and (2.46), respectively, at least locally a conformal gauge which satisfies (2.24), (2.25) and (2.47) can be constructed: Through each point \(x_* \in \mathcal{I}^-\) there exists a unique solution

\[
\tau \mapsto (x(\tau), f(\tau), \Theta(\tau), e_k(\tau))
\]

which yields a smooth orthonormal frame field \(e_k\), conformal factor \(\Theta\), and a coordinate system. The parameter \(\tau\) along the conformal geodesics is chosen such that \(\mathcal{I}^- = \{\tau = -1\}\). The so-obtained conformal geodesics define in some neighborhood of \(\mathcal{I}^-\) a smooth caustic-free congruence.

Coordinates, frame field, and conformal factor constructed this way are said to form a conformal Gauss gauge (cf. [25, 26]). There still remains some gauge freedom which arises from the freedom to choose coordinates \(x^\alpha\) and frame field \(e_\alpha\) on \(\mathcal{I}^-\). This freedom will be addressed below. In a conformal Gauss gauge the following relations are fulfilled,

\[
\dot{x} = e_0 = \partial_\tau, \quad g(e_i, e_j) = \eta_{ij}, \quad \hat{L}_{0k} = 0, \quad \hat{\Gamma}^k_{0j} = 0. \quad (2.49)
\]

2.2.4 Gauge freedom associated to \(\nabla_2 \Theta\big|_{\mathcal{I}^-} \) and \(\theta^-\)

Geometrically the freedom to prescribe \(\dot{x}\big|_{\mathcal{I}^-} := \dot{x}_*\) and \(f_{\mathcal{I}^-} := f_*\) clearly corresponds to the choice of a congruence of conformal geodesics. As the remaining conformal gauge data we have identified \(\nabla_2 \Theta\big|_{\mathcal{I}^-} := \Theta^{(1)}\) and \(\theta^-\) (which are supplemented by the gauge freedom to choose frame and coordinates).

Recall Lemma 2.1. The transformation \(x_* \mapsto \alpha_* x_*\) and \(f_* \mapsto f_* + \psi_* g(\dot{x}, \cdot)\) corresponds only to a change of the parameterization of the conformal geodesics, i.e. these transformations...
locally preserve the point set spread out by each conformal geodesic in the congruence. The first transformation will in general violate \( g(\dot{x}, \dot{x}) = -1 \) and thus require a conformal rescaling of \( g \) as well, \( g \mapsto \alpha^{-2} g \). Applying both transformations yields \( \Theta^{(1)} \mapsto \alpha \Theta^{(1)} \) and we observe that \( \Theta^{(1)} \) can be any positive prescribed function.

The second transformation will in general lead to a violation of the gauge condition \( \langle \dot{x}, f \rangle = 0 \) and therefore also requires a conformal transformation \( g \mapsto \phi^2 g \) with

\[
\nabla_{\dot{x}} \phi = \phi \langle \dot{x}, f \rangle + \psi \phi g(\dot{x}, \dot{x}) .
\]

(2.50)

Since \( \phi \ast g(\dot{x}, \dot{x}) \) is non-zero in our setting, the initial datum \( \phi \ast \) can be adjusted in such a way that \( \nabla_{\dot{x}} \phi |_{I^-} \) and thus \( \theta^- \) becomes any prescribed function.

The freedom to prescribe \( \Theta^{(1)} \) and \( \theta^- \) is in this sense related to the freedom to choose a parameterization of the conformal geodesics.

2.2.5 Some crucial relations

The GCFE have been formulated in terms of the gauge fields \( \Theta \) and \( b \equiv \Theta f + d\Theta \). These are determined by the gauge conditions (2.31)-(2.33) which, expressed in terms of \( \Theta \) and \( b \), read

\[
\nabla_{\dot{x}} \dot{x} = 0 , \quad \tilde{L}(\dot{x}, \cdot) = 0 , \quad \nabla_{\dot{x}} \Theta = \langle \dot{x}, b \rangle .
\]

(2.51)

In principle these equations need to be employed to supplement the GCFE to a closed system. However, a very remarkable result by Friedrich [25] shows that this is not necessary. The fields \( \Theta \) and \( b \) can be explicitly determined in the conformal Gauss gauge, the latter one in terms of its frame components \( b_k = e^\mu_k b_\mu \) of a parallely propagated w.r.t. \( \nabla \) frame \( e_k \). The corresponding expressions can then be simply inserted into the GCFE.

**Lemma 2.2** ([25]) In the conformal Gauss gauge the following relations hold:

(i) \( \nabla_{\dot{x}} \nabla_{\dot{x}} \nabla_{\dot{x}} \Theta = 0 \), and

(ii) \( \nabla_{\dot{x}} \nabla_{\dot{x}} b_k = 0 \), where \( b_k \equiv \langle b, e_k \rangle \).

**Remark 2.3** For \( \lambda = 0 \) the initial data for these ODEs in terms of the “gauge data” (2.45) at \( \mathcal{J}^- \) are computed in Section 2.5.3 below.

**Remark 2.4** The conformal Gauss gauge is distinguished by the fact that it is geometric and deeply intertwined with the conformal structure. It has the decisive property to supply explicit knowledge about the fields \( \Theta \) and \( b_k \). These fields can be computed explicitly along any conformal geodesic of the congruence at hand. In particular, one gains an a priori knowledge of the location of \( \mathcal{J}^- \) (supposing that the solution extends that far).

**Proof:** In terms of the Levi-Civita connection \( \nabla \) (2.47) and (2.51) read

\[
\Theta \nabla_{\dot{x}} \dot{x} = -g^2(b - d\Theta, \cdot),
\]

(2.52)

\[
\Theta \nabla_{\dot{x}} (b - d\Theta) = \langle \dot{x}, d)(b - d\Theta) - \frac{1}{2} g^2(b - d\Theta, b - d\Theta)g(\dot{x}, \cdot) + \Theta^2 L(\dot{x}, \cdot),
\]

(2.53)

\[
\nabla_{\dot{x}} \Theta = \langle \dot{x}, b \rangle ,
\]

(2.54)

\[
\Theta \nabla_{\dot{x}} e_k = -(b - d\Theta, e_k)\dot{x} + g(\dot{x}, e_k)g^2(b - d\Theta, \cdot),
\]

(2.55)
where $g^\delta$ denotes the inverse metric. We contract (2.53) with $\dot x$ and $b$, respectively. Using also (2.52) we deduce
\begin{equation}
\Theta(g^\delta(d\Theta,d\Theta) - g^\delta(b,b)) + \Theta^2 L(\dot x,\dot x), \tag{2.56}
\end{equation}
\begin{equation}
\Theta g^\delta(b,\nabla_\dot x(b - d\Theta)) = \frac{\langle \dot x, b \rangle}{2}
\left(g^\delta(b,b) - g^\delta(d\Theta,d\Theta)\right) + \Theta^2 g^\delta(\cdot,\cdot) L(\dot x,\cdot). \tag{2.57}
\end{equation}
On the other hand, it follows from (2.52) and (2.54) that
\begin{equation}
\Theta \nabla_\dot x \nabla_\dot x \Theta = \Theta \nabla_\dot x \langle \dot x, b \rangle = \Theta \langle \dot x, \nabla_\dot x b \rangle \tag{2.58}
\end{equation}
\begin{equation}
= -g^\delta(b,b) + g^\delta(d\Theta,b) + \Theta \langle \dot x, \nabla_\dot x b \rangle. \tag{2.59}
\end{equation}
Combined, that yields
\begin{equation}
\Theta^2 L(\dot x,\dot x) = \frac{1}{2}g^\delta(b,b) - g^\delta(d\Theta,b) + \frac{1}{2}g^\delta(d\Theta,d\Theta). \tag{2.60}
\end{equation}
The CFE (A.3) and (A.5) in Appendix A.2, cf. [27]) imply
\begin{equation}
\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + \frac{1}{2} \Theta^{-1} \left( \nabla_\alpha \Theta \nabla_\alpha \Theta + \frac{\lambda}{3} \right) g_{\mu\nu}. \tag{2.61}
\end{equation}
Contracting this twice with $\dot x$ and using (2.52) and (2.60) yields
\begin{equation}
\nabla_\dot x \nabla_\dot x \Theta = g(\nabla_\dot x \dot x, d\Theta) - \Theta L(\dot x,\dot x) - \frac{1}{2} \Theta^{-1} \left(g^\delta(d\Theta,d\Theta) + \frac{\lambda}{3}\right) = -\frac{1}{2} \Theta^{-1} \left(g^\delta(b,b) + \frac{\lambda}{3}\right), \tag{2.62}
\end{equation}
while contraction with $\dot x$ and $b$ leads to
\begin{equation}
g^\delta(b,\nabla_\dot x d\Theta) = -\Theta g^\delta(\cdot,\cdot) L(\dot x,\cdot) + \frac{1}{2} \Theta^{-1} \left(g^\delta(d\Theta,d\Theta) + \frac{\lambda}{3}\right) \langle \dot x, b \rangle. \tag{2.63}
\end{equation}
We insert the latter equation into (2.57),
\begin{equation}
\nabla_\dot x g^\delta(\cdot,\cdot) = \Theta^{-1} \langle \dot x, b \rangle \left(g^\delta(b,b) + \frac{\lambda}{3}\right). \tag{2.64}
\end{equation}
Taking now the derivative of (2.62) along the conformal geodesics and using (2.54) and (2.64), we obtain (i).

Next, we insert (2.61), contracted with $\dot x$, into (2.53),
\begin{equation}
\Theta \nabla_\dot x b = \left(g^\delta(b,d\Theta) - \frac{1}{2}g^\delta(b,b) + \frac{\lambda}{6}\right) g(\dot x,\cdot) + \langle \dot x, b \rangle (b - d\Theta). \tag{2.65}
\end{equation}
From (2.65) and (2.55) we deduce
\begin{equation}
\nabla_\dot x \langle b, e_k \rangle = \frac{1}{2} \Theta^{-1} \left(g^\delta(b,b) + \frac{\lambda}{3}\right) g(\dot x, e_k). \tag{2.66}
\end{equation}
Differentiating this one more time along the conformal geodesics and using (2.54) and (2.64), we find (ii).
2.3 Evolution equations for Schouten tensor, connection and frame coefficients

In the conformal Gauss gauge the GCFE split into evolution and constraint equations. The constraint equations will be analyzed in Section 4.1 for $\lambda = 0$ and in adapted null coordinates (and then rewritten in terms of frames). Here we want to derive a somewhat more explicit form of the evolution equations. Let us start with Schouten tensor, connection coefficients, and frame field. Recall (2.49), so we do not need equations for $\tilde{L}_{a0}$, $\tilde{\Gamma}^i_{0j}$ and $e^\mu_0$. The GCFE (2.13)-(2.15) imply the following system of evolution equations for the remaining components (cf. e.g. [27]),

$$\begin{align*}
\partial_\tau \tilde{L}_{a0} &= b_i W^i_{0ja0} - \tilde{\Gamma}^b_{a0} \tilde{L}_{bj}, \\
\partial_\tau \tilde{L}_{ab} &= b_i W^i_{b0a} - \tilde{\Gamma}^c_{a0} \tilde{L}_{cb} , \\
\partial_\tau \tilde{\Gamma}^i_{a0} &= -\tilde{\Gamma}^b_{i} \tilde{\Gamma}^j_{a0} + 2\delta_{i0} \tilde{L}_{aj} - \eta_{ij} \tilde{L}_a^i + \Theta W^i_{j0a} , \\
\partial_\tau e^\mu_a &= -\tilde{\Gamma}^b_{a0} \delta^\mu_b - \tilde{\Gamma}^b_{0c} e^\mu_b.
\end{align*}$$

The Levi-Civita connection satisfies $\Gamma^i_{(jk)} = 0$, equivalently, $\tilde{\Gamma}^i_{(jk)} = \eta_{jk} f^i$. If follows that the Weyl connection has the following (anti-)symmetric properties, we will make extensively use of,

$$\tilde{\Gamma}^1_{a0} = \tilde{\Gamma}^0_{a1}, \quad \tilde{\Gamma}^0_{a0} = \tilde{\Gamma}^1_{a1} = \frac{1}{2} \tilde{\Gamma}^A_{aA}, \quad \eta_{AB} \tilde{\Gamma}^B_{a1} = -\tilde{\Gamma}^1_{aA}, \quad \eta_{AB} \tilde{\Gamma}^A_{a0} = \tilde{\Gamma}^0_{aA}. \quad (2.70)$$

As the relevant independent components of the Weyl connection one may regard

$$\tilde{\Gamma}^0_{a1}, \quad \tilde{\Gamma}^1_{a0}, \quad \tilde{\Gamma}^{[BC]}_{a}. \quad (2.71)$$

Using the algebraic symmetries of the Weyl tensor (cf. the next section) one ends up with a system of evolution equations for Schouten tensor, connection and frame coefficients,

$$\begin{align*}
\partial_\tau \tilde{L}_{a0} &= b_i W^i_{00a} - \tilde{\Gamma}^b_{a0} \tilde{L}_{bj}, \\
\partial_\tau \tilde{L}_{ab} &= b_i W^i_{b0a} - \tilde{\Gamma}^c_{a0} \tilde{L}_{cb} , \\
\partial_\tau \tilde{\Gamma}^0_{a0} &= -\tilde{\Gamma}^b_{c} \tilde{\Gamma}^c_{a0} + \tilde{L}_{ab} - \Theta W^i_{0ab}, \\
\partial_\tau \tilde{\Gamma}^1_{a0} &= -\tilde{\Gamma}^b_{c} \tilde{\Gamma}^c_{a0} + \delta^1_{b} \tilde{L}_{a0} - \Theta W^i_{0ab1}, \\
\partial_\tau \tilde{\Gamma}^A_{B0} &= -\tilde{\Gamma}^B_{C} \tilde{\Gamma}^C_{A0} + \delta^A_{B} \tilde{L}_{AB0} - \Theta W^i_{A0B1}, \\
\partial_\tau e^\mu_a &= -\tilde{\Gamma}^b_{a0} \delta^\mu_b - \tilde{\Gamma}^b_{0c} e^\mu_b. \quad (2.72-2.78)
\end{align*}$$

2.4 Bianchi equation

As 10 independent components of the rescaled Weyl tensor in an orthonormal frame one can identify ("tf" denotes the trace-free part w.r.t. to the $(AB)$-angular"-components),

$$W_{0101}, \quad W_{011A}, \quad W_{010A}, \quad W_{01AB}, \quad (W_{1A1B})_{tf}, \quad (W_{0(AB)}1)_{tf}. \quad (2.79)$$

The remaining components are related to these ones in the following way:

$$\begin{align*}
W_{0(AB)1} &= -\frac{1}{2} W_{01AB}, \quad \eta^{AB} W_{0AB1} = 0, \quad (2.80) \\
W_{0ABC} &= -2 W_{011C} \eta_{B}^{A} , \quad W_{1ABC} = -2 W_{010C} \eta_{B}^{A} , \\
\eta^{AB} W_{0AB} &= -W_{0101}, \quad (W_{0AB})_{tf} = (W_{1A1B})_{tf}, \quad (2.81-2.83)
\end{align*}$$
where we have employed all the algebraic symmetries of the Weyl tensor. It is convenient to make the following definitions:

\[
V_{AB}^\pm := (W_{LAIB})_{t} \pm W_{0(AB)1},
\]

(2.84)

\[
W_{A}^\pm := W_{010A} \pm W_{011A},
\]

(2.85)

\[
U_{AB} := W_{01AB} + \eta_{AB}W_{0101},
\]

(2.86)

which capture all independent components. We will use \(U_{AB}\) (instead of \(W_{0101}\) and \(W_{01AB}\)) only occasionally.

Let us consider the Bianchi equation (2.16). The independent components are provided by the \(j = a\) components. Expressed in terms of the connection coefficients these components read

\[
\partial_\tau W_{0a0b} = e^\mu c e^\mu d W_{c0a0b} - \hat{\Gamma}_{c0}^a W_{0a0b} + \hat{\Gamma}_{c0}^d W_{d0a0b} - \hat{\Gamma}_{c0}^a W_{c0a0b} - \hat{\Gamma}_{c0}^d W_{d0a0b},
\]

(2.87)

\[
- 2\hat{\Gamma}_{c0}^a W_{0a0b} - \hat{\Gamma}_{c0}^d W_{d0a0b} + \hat{\Gamma}_{c0}^d W_{c0a0b},
\]

and

\[
\partial_\tau W_{0abc} = e^\mu d \partial_\mu W_{d0abc} - \hat{\Gamma}_{d0}^a W_{0abc} + \hat{\Gamma}_{d0}^c W_{c0abc} - \hat{\Gamma}_{d0}^a W_{d0abc} - \hat{\Gamma}_{d0}^c W_{d0abc},
\]

(2.88)

We will need all of them for our analysis of the critical sets where spatial infinity touches null infinity. For this we rewrite them in terms of (2.84)-(2.86). It is convenient to define the operator \(\hat{\nabla}\) as follows,

\[
\hat{\nabla}_{A}^\mu B := e^\mu A \partial_\mu v_B - \hat{\Gamma}_{A}^C B v_C.
\]

(2.89)

and similarly for tensors of higher valence. From a lengthy calculation we obtain,

\[
\partial_\tau W_{0101} = \left(-\frac{1}{2} \hat{\nabla}_{A} + 2\hat{\Gamma}_{A0}^0 + \frac{1}{2} \hat{\Gamma}_{A0}^0 \right)W_{A}^+ + \left(\frac{1}{2} \hat{\nabla}_{A} - 2\hat{\Gamma}_{A0}^0 + \frac{1}{2} \hat{\Gamma}_{A0}^0 \right)W_{A}^- + \frac{3}{2} \hat{\Gamma}_{A0}^0 W_{0101} - \frac{3}{2} \hat{\Gamma}_{A0}^0 W_{0101} + \frac{1}{2} \left(\hat{\Gamma}_{AB}^1 \hat{\nabla}_{A} - \hat{\Gamma}_{AB}^0 \right) v_{AB} + \frac{1}{2} \left(\hat{\Gamma}_{AB}^1 \hat{\nabla}_{A} - \hat{\Gamma}_{AB}^0 \right) v_{AB},
\]

(2.90)

\[
\partial_\tau W_{01AB} = \left(\hat{\nabla}_{A} - 2\hat{\Gamma}_{A|0}^0 - \hat{\Gamma}_{A|0}^0 \right)W_{B}^+ + \left(\hat{\nabla}_{A} - 2\hat{\Gamma}_{A|0}^0 - \hat{\Gamma}_{A|0}^0 \right)W_{B}^- + \frac{3}{2} \hat{\Gamma}_{A|0}^0 W_{0101} - \frac{3}{2} \hat{\Gamma}_{A|0}^0 W_{0101} + \frac{1}{2} \left(\hat{\Gamma}_{AC}^1 \hat{\nabla}_{A} - \hat{\Gamma}_{AC}^0 \right) v_{AC} + \frac{1}{2} \left(\hat{\Gamma}_{AC}^1 \hat{\nabla}_{A} - \hat{\Gamma}_{AC}^0 \right) v_{AC},
\]

(2.91)

\[
\partial_\tau W_{A}^- = \left(\hat{\nabla}_{B} - 3\hat{\Gamma}_{B0}^0 + 2\hat{\Gamma}_{B0}^0 \right) W_{AB}^+ + \frac{1}{2} \hat{\nabla}_{B} U_{BA} - \frac{3}{2} \hat{\Gamma}_{B0}^0 U_{BA} + \frac{1}{2} \left(\hat{\Gamma}_{B1}^1 \hat{\nabla}_{B} - \hat{\Gamma}_{B0}^0 \right) W_{A}^- + \frac{1}{2} \left(\hat{\Gamma}_{B1}^1 \hat{\nabla}_{B} - \hat{\Gamma}_{B0}^0 \right) W_{A}^-.
\]

(2.92)
\begin{align}
\partial_r W_A^+ &= (-\tilde{\nabla} B + 3\tilde{\nabla} B_0 + 2\tilde{\nabla} B_1) V_{A\theta}^- - \frac{1}{2} \nabla B U_{A\theta} + \frac{3}{2} \tilde{\nabla} B_0 U_{A\theta}
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - 3\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta}) \nabla B^\theta + \frac{1}{2} (3\tilde{\nabla} B_0^\theta + \hat{\Gamma}_B^1) W_A^+ \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - 3\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta}) W_B^+ - \frac{1}{2} (\tilde{\nabla} B_1 + \hat{\Gamma}_B^0) W_A^- ,
\end{align} 

(2.93)

\begin{align}
(e^\tau + 1) \partial_r V_{A\theta}^+ &= - e^{\alpha_1} \partial_\alpha V_{A\theta}^+ + (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} - \hat{\Gamma}_C^1 |_{A \theta}) V_{A\theta}^- \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} + \hat{\Gamma}_C^0 |_{A \theta} - \hat{\Gamma}_C^1 |_{A \theta}) V_{A\theta}^- \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} + \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- ,
\end{align} 

(2.94)

\begin{align}
(e^\tau - 1) \partial_r V_{A\theta}^- &= - e^{\alpha_1} \partial_\alpha V_{A\theta}^- + (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} + \hat{\Gamma}_C^0 |_{A \theta} - \hat{\Gamma}_C^1 |_{A \theta}) V_{A\theta}^- \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) V_{A\theta}^- \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- ,
\end{align} 

(2.95)

and,

\begin{align}
(\partial_r - e^{\alpha_1} \partial_\alpha) W_{0101} &= (\tilde{\nabla} A^0 - A^0 u_0 + A^0 - \hat{\Gamma}_A^1 |_{A \theta} - \hat{\Gamma}_A^0) W_A^+ \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} - \hat{\Gamma}_C^1 |_{A \theta}) W_A^- \\
&- (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} + \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) W_A^- \\
&- 3 \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + \frac{1}{2} \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - \frac{1}{2} \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} W_{0101} ,
\end{align} 

(2.96)

\begin{align}
(\partial_r - e^{\alpha_1} \partial_\alpha) W_{\alpha \beta \gamma \delta} &= (\tilde{\nabla} B - 6\tilde{\nabla} B_0 + 4\tilde{\nabla} B_1 - \hat{\Gamma}_B^0 |_{A \theta}) \nabla B^\theta + \frac{3}{2} (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - \hat{\Gamma}_d^0 |_{A \theta}) U_{A\theta} \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} - \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} + \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- \\
&- (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- \\
&- 6 \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + \frac{1}{2} \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - \frac{1}{2} \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} W_{0101} ,
\end{align} 

(2.97)

\begin{align}
(\partial_r - e^{\alpha_1} \partial_\alpha) W_A^+ &= (\tilde{\nabla} B - 6\tilde{\nabla} B_0 + 4\tilde{\nabla} B_1 - \hat{\Gamma}_B^0 |_{A \theta}) \nabla B^\theta + \frac{3}{2} (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - \hat{\Gamma}_d^0 |_{A \theta}) U_{A\theta} \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} - \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- \\
&+ (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} + \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- \\
&- (\tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + 2\tilde{\nabla} \hat{\Gamma}_d^1 |_{A \theta} - \hat{\Gamma}_C^0 |_{A \theta} + \hat{\Gamma}_C^1 |_{A \theta}) W_{A\theta}^- \\
&- 6 \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} + \frac{1}{2} \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} - \frac{1}{2} \tilde{\nabla} \hat{\Gamma}_d^0 |_{A \theta} W_{0101} ,
\end{align} 

(2.98)

At times we will call (2.90)-(2.95) “evolution equations” and (2.96)-(2.99), whose evaluation on \( \mathcal{A} \) does not contain transverse derivatives, “constraint equations”. However, in this work we do not attempt to solve the evolution problem or show preservation of the constraints under evolution (for an ordinary Cauchy problem this has been done in [25]). We therefore do not care here whether this is the “appropriate” split of the Bianchi equation.
2.5 Frame field and coordinates at $\mathcal{I}^-$ for $\lambda = 0$

2.5.1 Adapted null coordinates

We assume henceforth that the cosmological constant vanishes,

$$\lambda = 0. \quad (2.100)$$

We introduce adapted null coordinates $(\tau, r, x^A)$ on $\mathcal{I}^- \cong \mathbb{R} \times S^2$ (cf. their definition prior to (2.38)). In conformal Gaussian coordinates we require, in addition, $g_{\tau\tau} = g(\dot{x}, \dot{x}) = -1$. It is well-known that the shear tensor vanishes on $\mathcal{I}^-$. This implies that $g_{AB} = \Omega^2(r, x^C)\delta_{AB}$ for some $r$-independent Riemannian metric on $S^2$. Since any smooth Riemannian metric on $S^2$ is conformal to the standard metric $\delta_{AB}dx^A dx^B$ we may simply assume by redefining $\Omega$ that $h_{AB} = s_{AB}$. By way of summary, the line element takes the following form on $\mathcal{I}^-,$

$$g|_{\mathcal{I}^-} = -d\tau^2 + 2\nu^A(r, x^C)d\tau dr + 2\nu^A(r, x^C)dr dx^A + \Omega^2(r, x^C)s_{AB}(x^C)dx^A dx^B, \quad (2.101)$$

which is a regular Lorentzian metric supposing that $\nu, \Omega \neq 0$. Adapted null coordinates are used as “initial coordinates” which are then dragged along the congruence of conformal geodesics. The following relation holds between $\Omega$ and the divergence $\theta^+$ of the null geodesic generators of $\mathcal{I}^-$ [8],

$$\theta^+ = 20\lambda \log \Omega. \quad (2.102)$$

We have already mentioned that there is still a gauge freedom left, namely to reparameterize the null geodesic generators of $\mathcal{I}^-$. This gauge freedom, $r \mapsto \tilde{r} = r(r, x^A)$, can be employed to prescribe the function $\kappa$ [8], given by

$$\nabla_r \ell = \kappa \ell. \quad (2.103)$$

It measures the deviation of the coordinate $r$ to be an affine parameter. This does not completely fix the $r$-coordinate. The remaining gauge freedom will be considered below. There also remains the gauge freedom to choose coordinates $(x^A)$ on $\{ \tau = -1, r = \text{const.} \} \cong S^2$, whose specific choice will be irrelevant for us.

A list of all the relevant Christoffel symbols, or rather their restriction to $\mathcal{I}^-$, is provided in Appendix A.1, (A.22)-(A.34) (recall that in conformal Gauss coordinates we, in addition, require $g_{\tau\tau} = -1$). We remark that (A.26) and (A.28) may be regarded as definitions of $\kappa$ and $\xi_A$, while the trace-free part of (A.32) may be regarded as the definition of $\Xi_{\dot{A}\dot{B}}$. Equivalently, they can be defined as

$$\kappa = \nu^A \partial_r \nu_A \frac{1}{2} \nu^B \partial_r g_{rr}|_{\mathcal{I}^-}, \quad (2.104)$$

$$\xi_A = -\nu^A (\partial_A \nu^r - \partial_r \nu_A)|_{\mathcal{I}^-} + \partial_r \nu_A - \theta^+ \nu_A, \quad (2.105)$$

$$\Xi_{\dot{A}\dot{B}} = \nu^C (\partial_C g_{\dot{A}\dot{B}})|_{\mathcal{I}^-} - 2 \nabla_{(\dot{A}} \nu_{\dot{B})}\ell, \quad (2.106)$$

where $\nu^A := (\nu_r)^{-1}$. The indices of $\xi_A$ and $\Xi_{\dot{A}\dot{B}}$ will be raised and lowered with $g_{\dot{A}\dot{B}}|_{\mathcal{I}^-} = \Omega^2 s_{\dot{A}\dot{B}}$.

2.5.2 Frame field

We need to choose an initial frame field $\{e_i\}$ on $\mathcal{I}^-$ which satisfies

$$g(e_i, e_j) = \eta_{ij} \quad \text{and} \quad e_0 = \partial_r. \quad (2.107)$$
A frame field which fulfills these requirements is provided by

\[ e_{0*} = \partial_\tau , \quad e_{1*} = \partial_\tau + \nu^\tau \partial_r , \quad e_{A*} = \Omega^{-1} \dot{e}^A_A (\partial_A - \nu^A \nu_A \partial_r) , \]

where \( (\dot{e}_A) \), \( A = 2, 3 \), denotes an orthonormal frame field on the round sphere \( S^2 := (S^2, s_{AB}) \). All other frame fields which satisfy (2.107) arise from this one as

\[ \dot{e}_{0*} = e_{0*} , \quad \dot{e}_{a*} = M(r, x^A) \cdot e_{a*} , \quad M(r, x^A) \in O(3) . \]

We consider a conformal Gauss gauge based on adapted null coordinates and a frame field given by (2.108)-(2.110).

### 2.5.3 Initial data for \( \Theta \) and \( b \)

According to Lemma 2.2 the conformal factor \( \Theta \) and the 1-form \( b \) are globally of the form

\[ \Theta = \Theta^{(1)} (1 + \tau) + \Theta^{(2)} (1 + \tau)^2 , \]

\[ b_i = b_i^{(0)} + b_i^{(1)} (1 + \tau) . \]

We want relate the values of the integration functions \( \Theta^{(n)} = \Theta^{(n)}(r, x^A) \) and \( b_i^{(n)} = b_i^{(n)}(r, x^A) , \)

\( n = 0, 1 \), in terms of the gauge data (2.45). First of all, we observe that \( \Theta^{(1)} = \nabla_x \Theta|_{\mathcal{F}^{-}} > 0 \) can be directly identified as a conformal gauge freedom.

Let us express \( \Theta^{(2)} = \frac{1}{2} (\nabla_x \nabla_x \Theta)|_{\mathcal{F}^{-}} \) in terms of data on \( \mathcal{F}^{-} \). For this, we contract equation (2.61) twice with \( \dot{x} \) as well as with \( \dot{x} \) and \( \ell \). Eliminating the second term on the right-hand side yields

\[ \Theta^{(2)} = - \frac{1}{2} \nu^\tau \left( \nabla_x \Gamma_{\tau r} - \nu^\tau \Gamma_{\tau r} \right) \Theta^{(1)} .\]

Below (cf. (2.145)) we will show that \( \Gamma_{\tau r} \big|_{\mathcal{F}^-} = - f^r = - \nu^\tau f_r \). Using also (A.27), it follows that

\[ \Theta^{(2)} = - \frac{1}{2} \left( \nabla_x + \kappa + \langle \ell, f \rangle \right) \left( \frac{\Theta^{(1)}}{g(\dot{x}_+, \ell)} \right) .\]

The freedom to prescribe \( \kappa \) can be replaced by the freedom to prescribe \( \Theta^{(2)} \), whence one may regard \( \Theta^{(2)} \) as a (coordinate) gauge freedom.

Let us consider the 1-form \( b \equiv \Theta f + d\Theta \). It follows straightforwardly from (2.108)-(2.110) that

\[ b_i^{(0)} = \Theta^{(1)} , \quad b_i^{(0)} = \Theta^{(1)} , \quad b_A^{(0)} = 0 .\]

To obtain the first-order expansion coefficients we employ (2.78),

\[ b_i^{(1)} = 2 \Theta^{(2)} , \]

\[ b_i^{(1)} = - \Gamma_i^1 0 \Theta^{(1)} + 2 \Theta^{(2)} + \nu^r \partial_r \Theta^{(1)} , \]

\[ b_A^{(1)} = - \Gamma_A^1 0 \Theta^{(1)} + e^r_A \partial_r \Theta^{(1)} + e^A_A \partial_A \Theta^{(1)} .\]

Using the formulas (2.163) and (2.168) derived below,

\[ \Gamma_i^1 0 \big|_{\mathcal{F}^-} = - (\partial_r + \kappa + f_r) \nu^r , \]

\[ \Gamma_A^1 0 \big|_{\mathcal{F}^-} = \frac{1}{2} \xi_A + \nu^r \partial_A \nu_r + \nu_A \left( \partial_r - \frac{1}{2} \theta^r + \kappa \right) \nu^r ,\]
the constraint equations (cf. (A.10) and (A.14)),
\[
(\partial_t - \frac{1}{2} \theta_1^+ + \kappa)(\nu^T \Theta_1) = 0, \\
\xi_A - 2 \nabla_A \log(\nu^T \Theta_1) = 0,
\]

as well as (2.115), we end up with the following expressions for b,
\[
b_0^{(1)} = 2\Theta^{(2)}, \quad b_1^{(1)} = 0, \quad b_A^{(1)} = 0.
\]

This shows that the frame components of b are fully determined by \(\Theta^{(1)}\) and \(\Theta^{(2)}\). They do not depend on \(f\). In particular, the gauge data \(f_{\mathcal{A}}\) cannot be identified with certain components of b.

### 2.5.4 Gauge data at \(\mathcal{I}^-\)

Let us analyze the freedom to choose the initial direction \(\hat{x}_*\) of the conformal geodesics somewhat more detailed. For this let \(v\) be an arbitrary timelike vector field on \(\mathcal{J}^-\), i.e. with \(v^\mu v_\mu < 0\).

We introduce arbitrary adapted null coordinates \((\tau, r, x^A)\), \(v|_{\mathcal{J}^-} = (\nabla^\tau, \nabla^r, \nabla^A)\). We want to transform into new adapted null coordinates where \(\hat{v}|_{\mathcal{J}^-} = (1, 0, 0, 0)\). For this, we introduce new coordinates \((\hat{\tau}, \hat{r}, \hat{x}^\hat{A})\) by
\[
1 + \hat{\tau} = (1 + \tau)/\nabla^\tau, \quad \hat{r} = r - (1 + \tau)\nabla^r/\nabla^\tau, \quad \hat{x}^\hat{A} = x^A - (1 + \tau)\nabla^A/\nabla^\tau.
\]

Note that the \(r\)- and \(x^\hat{A}\)-coordinates remain unchanged on \(\mathcal{J}^- = \{\tau = -1\}\) under such a transformation. We find that
\[
\hat{\nu}^\tau|_{\mathcal{J}^-} = \frac{\partial \hat{\tau}}{\partial x^\mu} \hat{\nu}^\mu = 1, \\
\hat{\nu}^r|_{\mathcal{J}^-} = \frac{\partial \hat{r}}{\partial x^\mu} \hat{\nu}^\mu = 0, \\
\hat{\nu}^A|_{\mathcal{J}^-} = \frac{\partial \hat{x}^\hat{A}}{\partial x^\mu} \hat{\nu}^\mu = 0.
\]

Under this coordinate transformation we have
\[
g_{\tau\tau}|_{\mathcal{J}^-} = \frac{\partial \hat{\tau}}{\partial x^\mu} \frac{\partial \hat{\tau}}{\partial x^\nu} \hat{g}_{\mu\nu} = \frac{\hat{g}_{\tau\tau}}{\nabla^\tau}, \\
g_{\tau A}|_{\mathcal{J}^-} = \frac{\partial \hat{\tau}}{\partial x^\mu} \frac{\partial \hat{x}^\hat{A}}{\partial x^\nu} \hat{g}_{\mu\nu} = \frac{\hat{g}_{\tau A}}{\nabla^\tau} - \frac{\hat{g}_{\hat{A} B}}{\nabla^r}, \\
g_{\tau r}|_{\mathcal{J}^-} = \frac{\partial \hat{\tau}}{\partial x^\mu} \frac{\partial \hat{r}}{\partial x^\nu} \hat{g}_{\mu\nu} = \frac{\hat{g}_{\tau r}}{\nabla^\tau} - 2 \frac{\hat{g}_{\tau \hat{A} \hat{B}}}{(\nabla^\tau)^2} + \frac{\hat{g}_{\hat{A} B} \nabla^\hat{A} \nabla^B}{(\nabla^\tau)^2}.
\]

We conclude that the gauge freedom to prescribe \(\hat{x}_*\) can be identified with the freedom to prescribe, in a fixed adapted null coordinate system, the metric coefficients \(g_{\tau\mu}\) on \(\mathcal{J}^-\),
\[
g_{\tau\tau}|_{\mathcal{J}^-}, \quad \nu_\tau, \quad \nu_\hat{A}.
\]

In the conformal Gauss gauge the vector \(\hat{x}\) is normalized to 1, whence \(g_{\tau\tau}|_{\mathcal{J}^-} = -1\).

In other words, the gauge freedom to choose the initial direction of the conformal geodesics is chosen in such a way that the metric components \(\nu_\tau\) and \(\nu_\hat{A}\) take certain prescribed values in
the associated conformal Gauss coordinates. Instead of \( \hat{x} \), they may therefore be regarded as
gauge degrees of freedom.

Let us also take a look at the 1-form \( f \). From (2.145) below we deduce (recall that \( f_0 = (f, \hat{x}) = 0 \))

\[
\Gamma^\mu_{\tau\tau}|_x = -f^\mu \quad \iff \quad \begin{cases} f_\tau|_x = -\nu_\tau \Gamma^\tau_{\tau\tau} = \frac{1}{2} \partial_\tau g_{\tau\tau} - \partial_x g_{\tau\tau}, \\
\partial_\tau g_{\tau\tau}|_x = \nu_\tau \Gamma^\tau_{\tau\tau} = \frac{1}{2} \partial_\tau g_{\tau\tau} - \partial_x g_{\tau\tau}, \\
\partial_\tau g_{\tau\tau}|_x = \nu_\tau \Gamma^\tau_{\tau\tau} = \frac{1}{2} \partial_\tau g_{\tau\tau} - \partial_x g_{\tau\tau}.
\end{cases}
\]

(2.133)

The freedom to prescribe \( f_\alpha, \alpha = 1, 2, 3, \) on \( \mathcal{J}^- \) therefore corresponds to the freedom to prescribe \( \partial_\tau g_{\tau\alpha}|_x \) cannot be considered as a gauge function (or rather it
could if the normalization condition on \( \hat{x} \) is dropped). However, in this work we prefer to regard
\( f_x \) as gauge functions.

Accordingly, as gauge data to realize a conformal Gauss gauge from null infinity one can identify

\[ \nu_\tau, \ \nu_A, \ \Gamma_x, \ \kappa, \ \theta^-, \ \Theta^{(1)}. \]

(2.134)

As indicated above, the gauge freedom \( r \mapsto r' = r'(r, x^A) \) is not completely exhausted by
these gauge data. The remaining freedom can be used to prescribe certain functions at spatial
infinity, by which we mean the future boundary of \( \mathcal{J}^- \). We will analyze this for the cylinder
representation in Section 3.4.

### 2.6 Realization of conformal Gauss coordinates

Consider a solution \((\mathcal{M}, \tilde{g}, \tilde{\Theta})\) of the CFE with \( \lambda = 0 \) which admits a smooth \( \mathcal{J}^- \). Moreover, choose any functions

\[ \Theta^{(1)}(r, x^A) > 0, \ \kappa(r, x^A), \ \theta^-(r, x^A), \ \nu_\tau(r, x^A) > 0, \ \nu_A, \ \Gamma_x, \ \Theta^{(1)}. \]

(2.135)

We will describe how conformal Gauss coordinates with this choice of gauge data on \( \mathcal{J}^- \) can be realized.

For this choose an adapted null coordinate system \((\tilde{r}, \tilde{\tau}, \tilde{x}^A)\) (in particular \( \mathcal{J}^- = \{ \tilde{r} = 1 \})
and extend it off \( \mathcal{J}^- \) in any way. One would like to start with a conformal transformation which
realizes \( \Theta^{(1)} \) (and \( \theta^- \)) followed by a coordinate transformation which realizes the remaining gauge
data. Then a solution to the conformal geodesics equations would determine the coordinate
transformation off \( \mathcal{J}^- \). However, there is a problem: \( \Theta^{(1)} \) is given w.r.t. the new \( r \)-coordinate.
The relation between the new and the old \( r \)-coordinate is determined by \( \kappa \) and \( \tilde{\kappa} \), which are not
invariant under conformal transformations. The transformations to \( \Theta^{(1)} \) and \( \kappa \) therefore need to
be accomplished simultaneously. We further note that \( \Theta^{(1)} \) is not invariant under rescaling of \( \tau \),
so also the transformation to \( \nu_\tau \) needs to be taken into account.

We therefore consider a coordinate transformation of the form

\[ \tilde{r} \mapsto r = r(\tilde{r}, x^A), \ \tilde{\tau} \mapsto \tilde{\tau} = \psi(\tilde{r}, x^A) \tilde{\Theta}, \ \ 1 + \tilde{r} \mapsto 1 + \tau = h(\tilde{r}, x^A)(1 + \tilde{\tau}). \]

(2.136)

Taking the behavior of connection coefficients under conformal and coordinate transformations
into account, we find that the function \( r \) is given by (we suppress dependence on the angular
coordinates),

\[
\kappa(r(\tilde{r})) = \frac{\partial \tilde{x}^\alpha}{\partial r} \frac{\partial \tilde{x}^\beta}{\partial r} \frac{\partial r}{\partial x^\gamma} \left( \tilde{\Gamma}_{\alpha\beta} - \psi^{-1}(2\delta_\gamma(\nu_\beta)\psi - g_{\alpha\beta}g^{\gamma\lambda}\partial_\lambda \psi) + \frac{\partial r}{\partial \tilde{x}^\alpha} \frac{\partial^2 \tilde{x}^\alpha}{\partial r^2} \right) + \frac{\partial \tilde{r}}{\partial r} [\kappa(\tilde{r}) - 2\partial_\tilde{r} \log \psi(\tilde{r})] + \frac{\partial \tilde{r}}{\partial r} \frac{\partial^2 \tilde{r}}{\partial r^2},
\]

(2.137)
or, equivalently,

\[ \frac{\partial^2 \tau}{\partial r^2} = \frac{\partial}{\partial r}[\tilde{\kappa}(\tau)] - 2\partial_r \log \psi(\tau)] - \left( \frac{\partial \tau}{\partial r} \right)^2 \kappa(\tau(\tau)) . \]  

The function \( \psi \) is given by

\[ \psi(\tau) = h(\tau) \frac{\Theta^{(1)}(r(\tau))}{\Theta^{(1)}(\tau)} , \quad \text{where} \quad h(\tau) = \frac{\partial \tau}{\partial r} \nu'(r(\tilde{\tau})) . \]  

We construct the gauge from some cut \( \{ \tilde{\tau} = \text{const.} \} \) of \( \mathcal{I}^- \). The ODE (2.138) is of the following form

\[ \frac{\partial^2 \tau}{\partial r^2} = F(r, \partial_r \tau, \tilde{\tau}) , \]  

where \( F \) is some smooth function, and this equation can at least locally be solved. The freedom to choose the initial data will be specified later (and not at a cut of \( \mathcal{I}^- \) but at the critical set \( I^- \) of spatial infinity). Choosing \( \psi \) and \( h \) as above the desired values for \( \Theta^{(1)}, \kappa \) and \( \nu_\tau \) are realized.

Next, a coordinate transformation of the form

\[ x^\alpha \mapsto x^\alpha + f^\alpha(x^\beta)(1 + \tau) \]  

realizes \( g_{\tau\tau} |_{\mathcal{I}^-} = -1 \) and the prescribed value for \( \nu_A \). A conformal transformation

\[ \Theta \mapsto [1 + \phi(r, x^A)(1 + \tau)]\Theta , \]  

with an appropriately chosen \( \phi \) transforms to the right value for \( \theta^- \), cf. (2.43). Note that \( \Theta^{(1)}, \kappa \) and \( \nu_\tau \) remain invariant under (2.141)-(2.142). Then we solve the conformal geodesics equations with initial data

\[ \dot{x} |_{\mathcal{I}^-} = \partial_\tau , \quad f_{\tau} |_{\mathcal{I}^-} = 0 , \quad f_{\tau} |_{\mathcal{I}^-} = f_{r+} , \quad f_{\Lambda} |_{\mathcal{I}^-} = f_{\Lambda+} . \]  

That yields a vector field \( \dot{x} \) and a 1-form \( f \) on \( \mathcal{M} \) (at least in some neighborhood of \( \mathcal{I}^- \)). The gauge condition \( (\dot{x}, f) = 0 \) is realized by another conformal transformation \( \Theta \mapsto \Psi \Theta \). There is no freedom to choose the initial datum \( \Psi |_{\mathcal{I}^-} \) which needs to be 1 in order to preserve the gauge functions we have already realized. Finally, a coordinate transformation is necessary to transform \( \dot{x} \) to \( \partial_\tau \). Since \( \dot{x} |_{\mathcal{I}^-} = \partial_\tau \) it is of the form \( x^\mu \mapsto x^\mu + O(1 + \tau)^2 \) and therefore does not affect the gauge functions we have realized in the previous steps.

### 2.7 Connection coefficients

We want to compute the connection coefficients of the Weyl connection w.r.t. the frame field \( (e_i) \) on \( \mathcal{I}^- \) in terms of the connection coefficients associated with the adapted null coordinates (2.101). We have

\[ \hat{\Gamma}^k_{ij} = e^\nu_i \nabla_\nu e^\mu_j = e^\nu_i (\partial_\nu e^\mu_j + \Gamma^\mu_{\nu\sigma} e^\sigma_j) + 2e^\nu_i(f_j - \eta_{ij} f^\mu) . \]  

Recall that the frame field \( (e_i) \) has been constructed such that \( \hat{\Gamma}_0^k = 0 \), so that the \( i = 0 \)-components of (2.144) yield with (2.108)-(2.110)

\[ \Gamma^\mu_{\tau\tau} = -f^\mu , \]  

(this relation holds globally), and

\[ \partial_\tau e^\mu_1 |_{\mathcal{I}^-} = -f_1 e^\mu_0 - \Gamma^\mu_{\tau\tau} - \nu^\nu \Gamma^\nu_{\tau\tau} , \]  

\[ \partial_\tau e^\mu_A |_{\mathcal{I}^-} = -f_A e^\mu_0 - e^A_{\Lambda}(\Gamma^\mu_{\tau\lambda} - \Gamma^\nu_{\tau\nu} \nu_A) . \]
We set \( \hat{e}^A_A := e^A_A|_{\mathcal{X}} = \Omega^{-1}e^A_A \). Then \((\hat{e}^A_A)\) is an orthonormal frame for \( g_{AB}|_{\mathcal{X}} \).

For \( i = A \) we obtain from (2.144) (set \( \nu_A := \hat{e}^A_A \) and use (A.22), (A.23) and (A.25))

\[
\begin{align*}
\hat{\Gamma}^k_A \theta^k_A|_{\mathcal{X}} &= \hat{e}^A_A(\Gamma^A_B - \nu^B \nu_A \Gamma_{\tau B}^A + \delta^B \nu_A), \\
\hat{\Gamma}^k_A \theta^k_B|_{\mathcal{X}} &= \hat{e}^A_A(\nu^B \Gamma^A_B - \nu^B \nu_A \hat{e}^A_B - \nu^B \nu_A \nu^B \nu_A), \\
\hat{\Gamma}^k_A \theta^k_C|_{\mathcal{X}} &= \hat{e}^A_A(\nu^B \nu_A \nu^B \nu_A - \nu^B \nu_A \nu^B \nu_A).
\end{align*}
\]

We deduce the following relations, where we denote by \((\hat{e}^A_A)\) the co-frame of \((\hat{e}^A_A)\), and by \( \Gamma^C_B \) the connection coefficients of \((\hat{e}^A_A)\),

\[
\begin{align*}
\hat{\Gamma}_A^k \theta^k_A|_{\mathcal{X}} &= \hat{e}^A_A(\Gamma^A_B - \nu^B \nu_A \Gamma_{\tau B}^A), \\
\hat{\Gamma}_A^k \theta^k_B|_{\mathcal{X}} &= \hat{e}^A_A(\nu^B \Gamma^A_B - \nu^B \nu_A \nu^B \nu_A), \\
\hat{\Gamma}_A^k \theta^k_C|_{\mathcal{X}} &= \hat{e}^A_A(\nu^B \nu_A \nu^B \nu_A - \nu^B \nu_A \nu^B \nu_A).
\end{align*}
\]

For \( i = 1 \) we obtain from (2.144), using (2.146)-(2.147)

\[
\begin{align*}
\hat{\Gamma}_1^k \theta^k_A|_{\mathcal{X}} &= \Gamma^A_B - \nu^B \Gamma^A_B, \\
\hat{\Gamma}_1^k \theta^k_B|_{\mathcal{X}} &= \nu^B \nu_A \nu^B \nu_A - \nu^B \nu_A \nu^B \nu_A + 2 \epsilon^B \nu_A \nu^B \nu_A + 2 \epsilon^B \nu_A \nu^B \nu_A, \\
\hat{\Gamma}_1^k \theta^k_C|_{\mathcal{X}} &= \nu^B \nu_A \nu^B \nu_A - \nu^B \nu_A \nu^B \nu_A + 2 \epsilon^B \nu_A \nu^B \nu_A + 2 \epsilon^B \nu_A \nu^B \nu_A,
\end{align*}
\]

whence

\[
\begin{align*}
\hat{\Gamma}_1^k \theta^k_A|_{\mathcal{X}} &= \Gamma^A_B - \nu^B \Gamma^A_B, \\
\hat{\Gamma}_1^k \theta^k_B|_{\mathcal{X}} &= \nu^B \nu_A \nu^B \nu_A - \nu^B \nu_A \nu^B \nu_A, \\
\hat{\Gamma}_1^k \theta^k_C|_{\mathcal{X}} &= \nu^B \nu_A \nu^B \nu_A - \nu^B \nu_A \nu^B \nu_A.
\end{align*}
\]

Finally, we insert the expressions (A.22)-(A.34) to end up with the following list for the relevant components of the connection coefficients

\[
\begin{align*}
\hat{\Gamma}_A^k \theta^k_A|_{\mathcal{X}} &= \nu^B \nu_A \nu^B \nu_A, \\
\hat{\Gamma}_A^k \theta^k_B|_{\mathcal{X}} &= \nu^B \nu_A \nu^B \nu_A, \\
\hat{\Gamma}_A^k \theta^k_C|_{\mathcal{X}} &= \nu^B \nu_A \nu^B \nu_A.
\end{align*}
\]

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Here we have set \( \xi_A = \hat{\xi}_A \), and \( \Xi_{AB} = \hat{\xi}_A \hat{\xi}_B \). \( \hat{\nabla}_A = \hat{\xi}_A \nabla_A \) refers to the Levi-Civita covariant derivative associated to the family \( r \mapsto \hat{g} = g_{AB}dx^A dx^B \) of Riemannian metrics.

For the remaining connection coefficients we find with (2.145)

\[
\begin{align}
\hat{\Gamma}_{11|g}^{1} &= f_1, \\
\hat{\Gamma}_{10|g}^{1} &= -(\partial_r + \kappa)\nu^r - f_1, \\
\hat{\Gamma}_{1|g}^{A} &= \nu^r (\partial_r + \kappa - \nu^r \partial_r \nu_r) \nu^A + \frac{1}{2} \xi^A - f^A, \\
\hat{\Gamma}_{1|g}^{A} &= \hat{\Gamma}^{A} G, \\
\hat{\Gamma}_{1|g}^{A} &= f_1 \delta^A B.
\end{align}
\]

### 2.8 Schouten tensor

We compute the Schouten tensor associated to the Weyl connection. First of all we express its frame components in terms of coordinate components of the adapted null coordinate system (2.101),

\[
\begin{align}
\tilde{L}_{11|g}^{r} &= \tilde{L}_{11}^r - (\nu^r)^2 \nu^A \nu_B \tilde{L}_{rrA}, \\
\tilde{L}_{1A|g}^{r} &= \nu^r \hat{\xi}_A \tilde{L}_{rrA} - (\nu^r)^2 \nu_A \tilde{L}_{rr}, \\
\tilde{L}_{A1|g}^{r} &= \nu^r (\nu_r \hat{\xi}_A \tilde{L}_{rr} - \nu^B \nu_A \tilde{L}_{rr} + (\nu^r)^2 \nu_A \nu_B \tilde{L}_{rr} + \nu^r \nu^B \tilde{L}_{rrB}), \\
\tilde{L}_{AB|g}^{r} &= \nu^r (\nu_r \hat{\xi}_A \tilde{L}_{rr} - \nu^B \nu_A \tilde{L}_{rr} + (\nu^r)^2 \nu_A \nu_B \tilde{L}_{rr} + \nu^r \nu^B \tilde{L}_{rrB}), \\
\tilde{L}_{10|g}^{r} &= \nu^r \hat{\xi}_A \tilde{L}_{rrA} - (\nu^r)^2 \tilde{L}_{rr}, \\
\tilde{L}_{A0|g}^{r} &= \nu^r \hat{\xi}_A \tilde{L}_{rrA} - (\nu^r)^2 \nu^A \tilde{L}_{rr}.
\end{align}
\]

Moreover, by (2.9) we have

\[
\begin{align}
\tilde{L}_{rr|g}^{r} &= L_{rr} - (\partial_r + \kappa - f_r) f_r, \\
\tilde{L}_{rA|g}^{r} &= L_{rA} - (\partial_r - \frac{1}{2} \theta^+ \phi + f_A) \phi r = \frac{1}{2} \xi_A, \\
\tilde{L}_{A|g}^{r} &= L_{rA} - (\phi + f_A) \phi r = f_A (\hat{\xi}_A - \frac{1}{2} \xi_A), \\
\tilde{L}_{rA|g}^{r} &= \nu^r \hat{\xi}_A \tilde{L}_{rrA} - (\nu^r)^2 \nu^A \tilde{L}_{rr} + \nu^r \nu^B \tilde{L}_{rrB}, \\
\tilde{L}_{A0|g}^{r} &= \nu^r \hat{\xi}_A \tilde{L}_{rrA} - (\nu^r)^2 \nu^A \tilde{L}_{rr} + \nu^r \nu^B \tilde{L}_{rrB}.
\end{align}
\]

The relevant components of the Schouten tensor associated to the Levi-Civita connection are given in Appendix A.1 (the \( L^{rr} \)-component is not needed), (A.11), (A.16), (A.17), (A.19), (A.36),
We consider a conformally rescaled vacuum spacetime

\[ L_{rr} |_{\mathcal{F}_-} = -\frac{1}{2} \left( \partial_r + \frac{1}{2} \theta^+ - \kappa \right) \theta^+ , \quad (2.178) \]

\[ L_{rA} |_{\mathcal{F}_-} = -\frac{1}{2} \left( \nabla_A + \frac{1}{2} \xi_A \right) \theta^+ , \quad (2.179) \]

\[ L_{r} |_{\mathcal{F}_-} = \frac{1}{4} \left( \partial_r + \kappa \right) \theta^+ + \frac{1}{4} \left( \nabla_A - \frac{1}{2} \xi_A \right) \xi^A - \frac{1}{4} \theta^+ - \frac{1}{4} R , \quad (2.180) \]

\[ L_{\bar{A}B} |_{\mathcal{F}_-} = -\frac{1}{2} \left( \partial_r - \frac{1}{2} \theta^+ + \kappa \right) \xi_{\bar{A}B} + \frac{1}{2} \left( \nabla_{(\bar{A})} \xi_{\bar{B}} \right) |_{\mathcal{F}_-} - \frac{1}{4} \left( \xi_{\bar{A}B} \right) |_{\mathcal{F}_-} + \frac{1}{4} \left( R + \frac{1}{2} \theta^+ \theta^- \right) g_{\bar{A}B} . \quad (2.181) \]

\[ L_{A} |_{\mathcal{F}_-} = \frac{1}{2} \left( \nabla^A - \frac{1}{2} \xi^A \right) (\xi_{\bar{A}B} + \frac{1}{2} \theta^- g_{\bar{A}B}) - \frac{1}{4} \theta^+ \left( \nabla_A + \frac{1}{2} \xi_A \right) \theta^+ . \quad (2.182) \]

This allows us to compute \( \hat{L}_{ij} \) in terms of the coordinate components of the Schouten tensor in adapted null coordinates as computed from the constraint equations given in Appendix A.1. This will be done explicitly in Section 7.2 for a specific choice of the gauge data (2.134).

### 3 Cylinder representation of spatial infinity

So far, we have described the construction of a gauge based on conformal geodesics starting from \( \mathcal{F}_- \) which does not care about any representation of spatial infinity. In fact, depending on the choice of the conformal gauge data at \( \mathcal{F}_- \), \( \nu_\tau, \nu_\lambda, f_{\mathcal{F}_-}, \kappa, \theta^+, \Theta^1 \) (cf. (A.10) in Appendix A.1) the conformal Gauss gauge leads to different representations of spatial infinity such as the “classical” point representation or Friedrich’s cylinder representation. The behavior of the fields near the critical sets of a cylinder representing spatial infinity, tough, is what we are interested in.

#### 3.1 Spatial infinity

We consider a conformally rescaled vacuum spacetime \( (\mathcal{M}, g, \Theta) \) which admits a smooth \( \mathcal{F}_- \), and we introduce adapted null coordinates at \( \mathcal{F}_- \). For \( (\mathcal{M}, g, \Theta) \) to admit a (finite) representation of spatial infinity, \( d\Theta \) needs to vanish along each null geodesic generator of \( \mathcal{F}_- \) for some (finite) value of \( r \), i.e. for each \( x^A \) the function \( \partial_r \Theta |_{\mathcal{F}_-} \) needs to have a zero for some (finite) value \( r = r_1(x^A) \). We are interested in the possible behavior of the functions \( \partial_r \Theta |_{\mathcal{F}_-}, \nu_\tau, \theta^+ \), and \( \kappa \) near \( i^0 \). The constraint equations on \( \mathcal{F}_- \) imply that the function \( \partial_r \Theta |_{\mathcal{F}_-} \) satisfies the ODE

\[ \left( \partial_r - \frac{1}{2} \theta^+ + \kappa - \nu^\tau \partial_r \nu^\tau \right) \partial_r \Theta |_{\mathcal{F}_-} = 0 . \quad (3.1) \]

This equation can be integrated,

\[ \partial_r \Theta |_{\mathcal{F}_-}(r, x^A) = e^{\int_{r_0}^{r} \left( \frac{1}{2} \theta^+ - \kappa + \nu^\tau \partial_r \nu^\tau \right) r^2 d\tau \} \partial_r \Theta |_{\mathcal{F}_-}(r_0, x^A) . \quad (3.2) \]

for some initial value \( \partial_r \Theta |_{\mathcal{F}_-}(r_0, x^A) \). The solution will vanish at \( r_1 \) if and only if

\[ \int_{r_0}^{r_1} \left( \frac{1}{2} \theta^+ - \kappa + \nu^\tau \partial_r \nu^\tau \right) r^2 d\tau = -\infty . \quad (3.3) \]

We deduce that whenever a vacuum spacetime admits a piece of a smooth \( \mathcal{F}_- \) as well as some representation of spatial infinity \( i^0 \), then, along any null geodesic generator of \( \mathcal{F}_- \) at least one of the following scenarios happens in adapted null coordinates on \( \mathcal{F}_- \):

\[ \int_{r_0}^{r_1} \left( \frac{1}{2} \theta^+ - \kappa + \nu^\tau \partial_r \nu^\tau \right) r^2 d\tau = -\infty . \quad \]
\[ \lim_{r \to i_0} \nu_r = 0, \]
\[ \int_0^\infty \kappa = \infty, \]
\[ \int_0^\infty \theta^+ = -\infty. \]

Remark 3.1 The divergence of \( \theta^+ \) along each null geodesic generator of \( \mathcal{I}^- \) indicates the presence of a conjugate point. One therefore should expect that a gauge where (iii) is realized at \( r_1 < \infty \) leads to the usual representation of \( i^0 \) as a point. This point is known to be singular for non-vanishing ADM mass. In any gauge where (i) or (ii) are realized the inverse metric or the derivative \( \partial_r g_{rr} \big|_{\mathcal{I}^-} = 2(\partial_\kappa - \kappa)\nu_r \) diverge. A certain singular behavior at spatial infinity therefore seems to be unavoidable regardless of the gauge condition.

Let us compute how the conditions (i)-(iii) behave under reparameterizations \( r \mapsto r' = r'(r, x^A) \),

\[ \lim_{r' \to i_0} \nu_r' = \lim_{r \to i_0} \left( \frac{\partial r}{\partial r'} \nu_r \right), \]
\[ \int_0^{i_0'} \kappa' \, dr' = \int_0^\infty \left( \frac{\partial r}{\partial r'} \kappa(r(r')) - \partial_{\nu_r'} \log \left| \frac{\partial r'}{\partial r} \right| \right) \, dr' = \int_0^\infty \kappa \, dr - \lim_{r \to i_0} \log \left| \frac{\partial r'}{\partial r} \right| + \text{const.}, \]
\[ \int_0^{i_0'} \theta^+ \, dr' = \int_0^\infty \frac{\partial r}{\partial r'} \theta^+(r(r')) \, dr' = \int_0^\infty \theta^+ \, dr. \]

While (iii) is invariant, (ii) is invariant at least as long as \( \lim_{r \to i_0} |\nu_{\nu_r'}| \neq \infty \). However, these considerations suggest to combine (i) and (ii) into one condition

\[ \int_0^{i_0} \nu^r \partial_r g_{rr} \big|_{\mathcal{I}^-} \equiv 2 \int_0^\infty (\nu^r \partial_r \nu_r - \kappa) = -\infty \iff \lim_{r \to i_0} \log |\nu^r| + \int_0^\infty \kappa = \infty \]

Indeed, under the transformation \( r \mapsto r' = r'(r, x^A) \) this behaves as

\[ \lim_{r' \to i_0} \log |\nu^r'| + \int_0^{i_0'} \kappa' \, dr' = \lim_{r \to i_0} \log |\nu^r| + \int_0^\infty \kappa \, dr + \text{const.}, \]

so that (3.7) is invariant under reparameterizations of \( r \).

We have proved:

**Lemma 3.2** Assume that a vacuum spacetime admits a piece of a smooth \( \mathcal{I}^- \) as well as some representation of spatial infinity \( i^0 \). Consider any adapted null coordinate system at \( \mathcal{I}^- \) which admits a finite coordinate representation of \( i^0 \). Then along each null geodesic generator of \( \mathcal{I}^- \) at least one of the following scenarios happens:

(i) \( \int_0^{i_0} \nu^r \partial_r g_{rr} \big|_{\mathcal{I}^-} = -\infty \) (equivalently (3.7)), or

(ii) \( \int_0^{i_0} \theta^+ = -\infty \) (which indicates that \( i^0 \) is a conjugate point).

These conditions are invariant under reparameterizations of \( r \).

Remark 3.3 A similar analysis can be applied to timelike infinity.

Next, we present and discuss two explicit gauge choices for the Minkowski spacetime where the different scenarios (i)-(ii) are realized and yield qualitatively different representations on spatial infinity. We will see that (ii) corresponds to the classical point representation of spatial infinity while (i) yields a representation as a cylinder.
3.2 Example: Minkowski spacetime

3.2.1 Point representation of spatial infinity

Via a conformal rescaling and suitable coordinate transformations (compare [48]) the Minkowski metric \( \tilde{\eta} = -(dT)^2 + (dR)^2 + R^2 s_{\hat{A}\hat{B}} dx^{\hat{A}} dx^{\hat{B}} \) can be brought into the form

\[
\eta = \Theta^2 \tilde{\eta} = -d\tau^2 - 2d\tau dr + \sin^2(r) s_{\hat{A}\hat{B}} dx^{\hat{A}} dx^{\hat{B}},
\]  

with

\[
\Theta = 4 \sin \frac{1 + \tau}{2} \sin \left( r + \frac{1 + \tau}{2} \right),
\]

and with \( s_{\hat{A}\hat{B}} dx^{\hat{A}} dx^{\hat{B}} \) being the standard metric on \( S^2 \). This is realized as follows: First of all one introduces the retarded time \( U \),

\[
U := T - R,
\]

so that the Minkowski metric becomes

\[
\tilde{\eta} = -dU^2 - 2dU dR + R^2 s_{\hat{A}\hat{B}} dx^{\hat{A}} dx^{\hat{B}}.
\]

We then apply the coordinate transformation

\[
R \mapsto r := \arccot(2U) - \arccot(2(U + 2R)),
\]

\[
U \mapsto \tau := 2\arccot(2(U + 2R)) - 1.
\]

The inverse transformation reads

\[
\tau \mapsto U = \frac{1}{2} \cot \left( r + \frac{1 + \tau}{2} \right), \quad r \mapsto R = \frac{\sin r}{\Theta},
\]

and we have

\[
dU = -\frac{4 \sin^2 \frac{1 + \tau}{2} dr}{\Theta^2} - \frac{8 \sin^2 \frac{1 + \tau}{2} dr}{\Theta^2},
\]

\[
dR = -\frac{2 \sin^2 \left( r + \frac{1 + \tau}{2} \right) - \sin^2 \frac{1 + \tau}{2} dr}{\Theta^2} + \frac{4 \sin^2 \frac{1 + \tau}{2} dr}{\Theta^2}.
\]

In the conformally rescaled spacetime, past timelike infinity \( i^- \) can be identified with the point \( (\tau = -1, r = 0) \), past null infinity \( \mathcal{J}^- \) corresponds to the set \( \{ \tau = -1, \ r \in (0, \pi) \} \) and spacelike infinity \( i^0 \) is given by the point \( (\tau = -1, r = \pi) \).

We find that

\[
\theta^+ = 2 \cot r, \quad \theta^- = -2 \cot r, \quad \kappa = 0,
\]

\[
\partial_\tau \Theta |_{\mathcal{J}^-} = 2 \sin r, \quad \nu_\tau = -1, \quad \nu_\lambda = 0, \quad \partial_\tau g_{\tau\tau} |_{\mathcal{J}^-} = 0.
\]

which clearly belongs to case (ii) of Lemma 3.2 (the integrand in (i) vanishes). The null geodesics emanating from past timelike infinity \( i^- \) meet again at \( i^0 \), as indicated by the divergence of the expansion \( \theta^+ \).
3.2.2 Cylinder representation of spatial infinity and conformal Gauss coordinates

In fact, we are more interested in case (i) of Lemma 3.2. Again, as an example let us study the Minkowski spacetime for which we want to find a conformal representation which admits a cylinder representation of spatial infinity and which we aim to express in conformal Gauss coordinates (cf. [52]).

Consider the Minkowski spacetime in standard Cartesian coordinates \((y^\mu)\),

\[
\tilde{\eta} = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2 .
\]  

(3.20)

In the domain \(\{y_\mu y^\mu > 0\}\) we introduce new coordinates \((x^\mu)\) via

\[
x^\mu := -\frac{y^\mu}{y^\nu y_\nu} \iff y^\mu = -\frac{x^\mu}{x^\nu x_\nu}.
\]  

(3.21)

This coordinate patch excludes causal future and past of the origin, whence there will be no representation of timelike infinity.

The Minkowski line element becomes

\[
\tilde{\eta} = \frac{1}{(x^\mu x_\mu)^2} \left( - (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right).
\]  

(3.22)

Let now \(r\) denote the standard radial coordinate associated with the spatial coordinates \(x^\alpha\), \(\alpha = 1,2,3\), and set \(\tau := x^0/r\). Replacing \((x^\mu)\) by polar coordinates \((r,x^3)\), \(\tilde{\eta}\) takes the form

\[
\tilde{\eta} = \frac{1}{r^2(1-\tau^2)} \left( - dr^2 - 2 \frac{\tau}{r} d\tau dr - \frac{\tau^2-1}{r^2} dr^2 + s_{A\Bar{B}} d\xi^A d\xi^\Bar{B} \right).
\]  

(3.23)

We choose the conformal factor,

\[
\Theta := r(1-\tau^2),
\]  

(3.24)

which yields the following conformal representation of Minkowski spacetime,

\[
\eta = \Theta^2 \tilde{\eta} = -dr^2 - 2 \frac{\tau}{r} d\tau dr + \frac{1-\tau^2}{r^2} dr^2 + s_{A\Bar{B}} d\xi^A d\xi^\Bar{B}, \quad |\tau| < 1, \quad r > 0.
\]  

(3.25)

Future and past null infinity can be identified with \(\mathcal{J}^\pm = \{\tau = \pm 1, r > 0\}\). The set \(\{r = 0\}\) represents spacelike infinity. By introducing \(\hat{\tau} := -\log r\) as a new coordinate one shows that the set \(\{\hat{\tau} = 0\}\), where the metric coefficients in (3.25) become singular, has cylinder topology \([-1,1] \times \mathbb{S}^2\). We denote the 2-spheres where the cylinder touches \(\mathcal{J}\) by \(I^\pm := \{\tau = \pm 1, r = 0\}\), while the “proper part” of spacelike infinity is denoted \(I := \{|\tau| < 1, r = 0\}\). \(I^\pm\) are called critical sets. We have

\[
L_{\tau\tau} = \frac{1}{2}, \quad L_{\tau r} = \frac{\tau}{2r}, \quad L_{\tau A} = 0,
\]  

(3.26)

and one checks that

\[
\hat{x} = \partial_{\tau}, \quad f_r = 0, \quad f_r = r^{-1}, \quad f_A = 0
\]  

(3.27)

solves the conformal geodesics equations (2.17)-(2.18), so that (3.25) provides a conformal representation of (a subset of) Minkowski spacetime in conformal Gauss coordinates.

The coordinate transformation which relates (3.20) and (3.25) is given by

\[
y^0 = \frac{-\tau}{r(1-\tau^2)}, \quad y^1 = \frac{-\sin \theta \cos \phi}{r(1-\tau^2)}, \quad y^2 = \frac{-\sin \theta \sin \phi}{r(1-\tau^2)}, \quad y^3 = \frac{-\cos \theta}{r(1-\tau^2)}.
\]  

(3.28)
The inverse transformation takes the form
\[
\begin{align*}
\tau &= \frac{y^0}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}, \\
\theta &= \arccos\left(\frac{y^3}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}\right), \\
\phi &= \arcsin\left(\frac{y^2}{\sqrt{(y^1)^2 + (y^2)^2}}\right).
\end{align*}
\tag{3.29}
\]
These conformal Gauss coordinates correspond to the following gauge data,
\[
\nu_r = \frac{1}{r}, \quad \nu_A = 0, \quad f_1|_{\mathcal{J}^+} = 1, \quad f_A|_{\mathcal{J}^+} = 0, \quad \kappa = -\frac{2}{\nu}, \quad \theta^- = 0, \quad \Theta^{(1)} = 2r. \tag{3.31}
\]
Moreover, we anticipate (this will be relevant in view of Section 3.4 below, where \(v^{(2)}_A\) is defined),
\[
g_{\hat{A}\hat{B}}|_{\mathcal{I}^-} = s_{\hat{A}\hat{B}}, \quad \mathcal{Q}^A v^{(2)}_A = 0, \tag{3.32}
\]
where \(\mathcal{Q}\) denotes the Levi-Civita connection of the standard metric on \(S^2\). It follows from \(A.10\) that in this gauge \(\mathcal{J}^-\) has vanishing divergence, \(\theta^+ = 0\). That means that this gauge cannot admit a (finite) representation of a (regular) past timelike infinity \(\mathcal{I}^-\) as a tip of a cone. Instead, \(\mathcal{I}^-\) is shifted to infinity in these coordinates (we observe this directly when applying the coordinate transformation (3.21)). We also note that case (ii) of Lemma 3.2 is violated while (i) is fulfilled (here we have \(\partial_\tau g_{rr},|_{\mathcal{J}^-} = 2/\nu^2\) whence \(\nu^2 \partial_\tau g_{rr} = 2/\nu\)).

### 3.3 Cylinder representation and a priori restrictions on the gauge functions

We want to derive restrictions on the asymptotic behavior of the gauge functions appearing in the conformal Gauss gauge scheme at \(\mathcal{I}^-\), necessary to obtain a spacetime which admits a smooth cylinder representation of spatial infinity. As already mentioned before, to obtain some representation of spatial infinity the differential of the conformal factor \(\Theta\) needs to become zero somewhere along the null geodesic generators of \(\mathcal{J}^-\). We will choose the \(r\)-coordinates in such a way that spatial infinity is located at \(r = 0\). It follows that the gauge function \(\Theta^{(1)}\) needs to satisfy \(\Theta^{(1)} = o(1)\). We are interested in the construction of smooth spacetimes, which admit a smooth extension through null infinity, spatial infinity, which we want to represent as a cylinder, and therefore also through its critical sets. This leads to more restrictions than those obtained Section 3.1. First of all we need to require
\[
\Theta^{(1)} = \mathcal{O}(r). \tag{3.33}
\]
Here the symbol \(\mathcal{O}(r)\) is defined as follows: We say that a function \(f = \mathcal{O}(r^n), n \geq 0\,\), if it is a smooth function of \(r\) and \(x^A\), and if it Taylor expansion at \(r = 0\) starts with a term of \(n\)th-order. We say that \(f = \mathcal{O}(r^{-n})\) if \(r^n f = \mathcal{O}(1)\).

Let us now focus on the specifics of the cylinder representation. It is obtained by imposing a specific behavior on the gauge functions \(2.134\) near spatial infinity. A characteristic feature of the cylinder representation is that the Riemannian metric \(g_{\hat{A}\hat{B}}|_{\mathcal{J}^-}\) does not degenerate at spatial infinity (as compared to e.g. the point representation of spatial infinity). It follows from \(2.101\) and \(2.102\) that \(g_{\hat{A}\hat{B}}|_{\mathcal{J}^-}\) satisfies
\[
g_{\hat{A}\hat{B}}|_{\mathcal{J}^-} = e^{\int^r \theta^+ d\nu} s_{\hat{A}\hat{B}}. \tag{3.34}
\]
We thus need to require
\[-\infty < \int_{I^-} \theta^+ \, dr < \infty, \tag{3.35}\]
i.e.
\[\theta^+ = \mathcal{O}(1). \tag{3.36}\]
In a conformal Gauss gauge the conformal factor satisfies globally \[\Theta = \Theta^{(1)}(1+\tau) + \Theta^{(2)}(1+\tau)^2.\]
To end up with a spacetime where \(\mathcal{I}^+ = \{\tau = +1\}\) the relation
\[\Theta^{(2)} = -\frac{1}{2} \Theta^{(1)} \tag{3.37}\]
needs to be satisfied.

“Natural” requirements on \(\Theta\) at \(I\) are, as on \(\mathcal{I}\), \(\Theta|_{\mathcal{I}} = 0\) and \(d\Theta|_{\mathcal{I}} \neq 0\). The gauge function \(\Theta^{(1)}\) therefore needs to satisfy the following condition,
\[\Theta^{(1)} = \Theta^{(1,1)}(r^4) + \mathcal{O}(r^2), \quad \Theta^{(1,1)} > 0. \tag{3.38}\]

It is clear that in our smooth setting we need to impose
\[f_{\mu} |_{\mathcal{I}^-} = \mathcal{O}(1). \tag{3.39}\]
Moreover, we require the frame coefficients (which appear as unknowns in the GCFE) to be regular at spatial infinity. This will be the case if (cf. \((2.108)-(2.110))
\[\nu^\tau = \mathcal{O}(1), \quad \nu_\lambda = \mathcal{O}(1). \tag{3.40}\]
We apply Lemma 3.2 to deduce that necessarily
\[\int_{I^-} \kappa d\tilde{r} = \infty. \tag{3.41}\]
It follows from \((2.168)\) that
\[(\partial_\tau + \kappa)\nu^\tau = \mathcal{O}(1) \quad \overset{(3.40)}{\Leftrightarrow} \quad \kappa \nu^\tau = \mathcal{O}(1), \tag{3.42}\]
which is only possible if
\[\nu^\tau = \mathcal{O}(r). \tag{3.43}\]
Because of this behavior, the frame vectors \(e_0\) and \(e_1\) \((2.108)-(2.109)\) become linearly depend at spatial infinity which implies that \(I\) is a total characteristic. This also implies that \(\nu_\tau\) diverges at spatial infinity. Let us impose the condition that this divergence is as weak as possible,
\[\nu^\tau = \nu^{(1)}(r) + \mathcal{O}(r^2), \quad \nu^{(1)}(r) \neq 0, \tag{3.44}\]
equivalently,
\[d\sqrt{-\det g} |_{\mathcal{I}^-} \neq 0. \tag{3.45}\]
Taking \((3.1)\) into account it follows that \(\kappa\) cannot diverge faster than \(r^{-1}\) and that
\[\kappa = -\frac{2}{r} + \mathcal{O}(1). \tag{3.46}\]
In particular, any affine parameter along the null geodesics generating \( \mathcal{I}^- \) diverges when approaching \( \mathcal{I}^- \),

\[
\kappa_{\text{aff}} = 0 \iff r_{\text{aff}}(r,x^A) = \left( \frac{\partial r_{\text{aff}}}{\partial r} \right)\bigg|_{r=r_0} \int_{r_0}^r f_a^{\wedge} \kappa_{\text{aff}} \, \text{d}r + r_{\text{aff}}\big|_{r=r_0}. \tag{3.47}
\]

From the trace of (2.164) we deduce that

\[
\theta^- = \mathcal{O}(r). \tag{3.48}
\]

Moreover, (2.115) together with (3.37) gives

\[
(f \mid \mathcal{I}^-) = \nu_\tau f_1
\]

\[
f_1\big|_{\mathcal{I}^-} = 1 - (\Theta^{(1)})^{-1}(\partial_\tau + \kappa)(\nu_\tau \Theta^{(1)}), \tag{3.49}
\]

i.e. to make sure that \( \mathcal{I}^+ = \{ \tau = +1 \} \) the freedom to choose \( f_1\big|_{\mathcal{I}^-} \) is lost.

Because the conformal factor \( \Theta \) vanishes on \( I \), the equations (2.72)-(2.78) for connection coefficients, Schouten tensor and frame field decouple, on the cylinder, from those for the rescaled Weyl tensor (cf. [27]),

\[
\frac{\nabla_\tau \hat{\mathcal{L}}_{ab}}{I} = -\hat{\Gamma}_a^b_0 \hat{\mathcal{L}}_{b0}, \tag{3.50}
\]

\[
\frac{\nabla_\tau \hat{\mathcal{L}}_{ab}}{I} = -\hat{\Gamma}_a^c_0 \hat{\mathcal{L}}_{cb}, \tag{3.51}
\]

\[
\frac{\nabla_\tau \hat{\mathcal{L}}_{0b}}{I} = -\hat{\Gamma}_a^0_0 \hat{\mathcal{L}}_{ab}, \tag{3.52}
\]

\[
\frac{\nabla_\tau \hat{\mathcal{L}}_{a1}}{I} = -\hat{\Gamma}_c^1_0 \hat{\mathcal{L}}_{a0} + \delta^1_0 \hat{\mathcal{L}}_{a0}, \tag{3.53}
\]

\[
\frac{\nabla_\tau \hat{\mathcal{L}}_{01}}{I} = -\hat{\Gamma}_a^1_0 \hat{\mathcal{L}}_{a0} + \delta^1_0 \hat{\mathcal{L}}_{a0}, \tag{3.54}
\]

\[
\frac{\nabla_\tau \hat{\mathcal{L}}_{00}}{I} = -\hat{\Gamma}_a^0_0 \hat{\mathcal{L}}_{a0} + \delta^0_0 \hat{\mathcal{L}}_{a0}. \tag{3.55}
\]

The divergence of \( \nu_\tau \equiv (\nu^-)^{-1} \) does not matter as the frame field remains regular and the metric itself does not appear as an unknown in the GCFE.

Finally, it follows from (2.164) that necessarily

\[
\Xi_{AB} = \Xi_{AB}^{(1)} r + \mathcal{O}(r^2). \tag{3.57}
\]

**Lemma 3.4** *The gauge data need to satisfy the following a priori restrictions to obtain a smooth representation of spatial infinity as a cylinder \( I = \{ r = 0, |\tau| < 1 \} \) and a smooth representation of null infinity \( \mathcal{I}^\pm = \{ \tau = \pm 1, r > 0 \} \), which, in addition, satisfies (3.45),

\[
\nu_\tau = \nu_\tau^{(1)} r^{-1} + \mathcal{O}(1), \quad \nu_A = \mathcal{O}(1) \tag{3.58}
\]

\[
f_a = \mathcal{O}(1), \quad \kappa = \frac{2}{r} + \mathcal{O}(1), \quad \theta^- = \mathcal{O}(r), \tag{3.59}
\]

\[
\Theta^{(1)} = \Theta^{(1,1)}(x^A)r + \mathcal{O}(r^2), \tag{3.60}
\]

where \( \nu_\tau^{(1)} \neq 0 \) and \( \Theta^{(1,1)} > 0 \). Moreover, the data \( \Xi_{AB} \) need to be of the form (3.57), and the gauge function \( f_1\big|_{\mathcal{I}^-} \) needs to fulfill (3.49).*

**Remark 3.5** In the next step these expansions are inserted into the constraint equations computed in Appendix A.1. In turns out that further restrictions need to be imposed to make sure that the restriction of the rescaled Weyl tensor is bounded at \( \mathcal{I}^- \) and does not produce logarithmic terms there.

However, in view of an analysis of the constraint equations on the cylinder it is very convenient if the gauge functions approach the “Minkowskian values” (3.31). In particular, this makes sure that the system (3.50)-(3.56) can be solved explicitly. The analysis of the no-logs condition will therefore be carried out only for gauge functions of a form as in Definition 3.6 & 3.8 below.
3.4 Yet another gauge freedom

Before we proceed it is important to note that there is still some gauge freedom left. We have already mentioned that the gauge function $\kappa$ does not fully determine the $r$-coordinate. When transforming to a prescribed $\kappa$ via the transformation $r \mapsto \tilde{r} = \tilde{r}(r,x^A)$ one solves a second-order ODE, so there remains the freedom to choose the integration functions. The precise role of these integration functions depends on the asymptotic behavior of $\kappa$ near spatial infinity. In a conformal Gauss gauge which satisfies (3.80) below we can work out what this gauge freedom corresponds to. For this let us assume that all the other gauge data have already been transformed to their desired values.

We consider a transformation as in (2.136) which leads us to the ODE (2.138)

\[
\frac{\partial^2 r}{\partial \tilde{r}^2} = \frac{\partial r}{\partial \tilde{r}} \left[ \kappa(\tilde{r}) - 2 \partial_{\tilde{r}} \log \psi(\tilde{r}) \right] - \left( \frac{\partial r}{\partial \tilde{r}} \right)^2 \kappa(\tilde{r}), \tag{3.61}
\]

with

\[
\psi(\tilde{r}) = \frac{\partial \tilde{r}}{\partial r} \nu^r(r(\tilde{r})) \Theta^{(1)}(r(\tilde{r})) \Theta^{(1)}(\tilde{r}). \tag{3.62}
\]

Here, we want to solve this equation from $I^-$. In a conformal Gauss gauge which satisfies (3.80) below it has the form

\[
\frac{\partial^2 r}{\partial \tilde{r}^2} = \left( \frac{\partial r}{\partial \tilde{r}} \right)^2 \left( \frac{2}{r} + \mathcal{D}(r) \right) - \frac{\partial r}{\partial \tilde{r}} \left( \frac{2}{r} + \mathcal{D}(\tilde{r}) \right). \tag{3.63}
\]

Set $u := \partial \kappa \log(\tilde{r}/r)$ and $v := r/\tilde{r}$. Then this singular ODE becomes a regular first-order system,

\[
\begin{align*}
\partial_{\tilde{r}} u &= -u^2 - v^2 (1-u^2)^2 \mathcal{D}((v\tilde{r})^0) + (1-u^2) \mathcal{D}(\tilde{r}^0), \tag{3.64} \\
\partial_{\tilde{r}} v &= -uv. \tag{3.65}
\end{align*}
\]

The solution is of the form

\[
r = f_{p,q}(\tilde{r}, \tilde{x}^A) = \mathcal{D}(\tilde{r}), \quad \text{where} \quad \partial_{\tilde{r}} f_{p,q} |_{I^-} = p(x^A) > 0 \quad \text{and} \quad \partial_{\tilde{r}}^2 f_{p,q} |_{I^-} = q(x^A) \tag{3.66}
\]

are the initial data. Note that this transformation does not change the location of $I^- = \{r = 0\}$. (In the special case where $\kappa = -2/r$ the solution can be determined explicitly, $f_{p,q} = 2p^2\tilde{r}/(2p - q\tilde{r})$.)

One then proceeds as described in Section 2.6, where coordinate and conformal transformations are chosen in such a way that the other gauge data remain invariant. Under these transformations

\[
\bar{g}_{\bar{A}\bar{B}}(\tilde{r}, x^\bar{C})|_{\tilde{x}^-} \mapsto g_{\bar{A}\bar{B}}(r, x^\bar{C})|_{x^-} = (\psi(\tilde{r}(r), x^\bar{C}))^2 g_{\bar{A}\bar{B}}(\tilde{r}(r), x^\bar{C}), \tag{3.67}
\]

We have

\[
\psi |_{I^-} = p(x^A), \tag{3.68}
\]

whence

\[
g_{\bar{A}\bar{B}} |_{I^-} = (p(x^A))^2 \bar{g}_{\bar{A}\bar{B}}. \tag{3.69}
\]

The gauge freedom coming along with $p(x^A)$ can therefore be employed to conformally rescale $g_{\bar{A}\bar{B}}$ in any convenient manner. Let us consider the behavior of $\Xi_{\bar{A}\bar{B}}$ under the conformal and
coordinate transformations of Section 2.6. Since we want to leave \( g_{\hat{A}\hat{B}} \) invariant we set \( p(\hat{x}^\hat{A}) = 1 \). Then

\[
\Xi_{\hat{A}\hat{B}} = -2 \left( \frac{\partial \hat{x}^\alpha}{\partial x^A} \frac{\partial \hat{x}^\beta}{\partial x^B} \frac{\partial r}{\partial x^\gamma} \hat{\kappa}_{\alpha\beta} + \frac{\partial r}{\partial x^\alpha} \frac{\partial^2 \hat{x}^\beta}{\partial x^A \partial x^B} \right)_{\alpha\beta} 
\]

\[
= -2 \left( \frac{\partial r}{\partial x^A} \frac{\partial r}{\partial x^B} \frac{\partial r}{\partial x^\alpha} \hat{\kappa}_{\alpha\beta} + 2 \frac{\partial \hat{r}}{\partial x^A} \frac{\partial \hat{r}}{\partial x^B} r_B + 2 \frac{\partial \hat{r}}{\partial x^A} \frac{\partial \hat{r}}{\partial x^B} r_B \hat{\kappa}_{\alpha\beta} + \frac{\partial r}{\partial x^A} \frac{\partial r}{\partial x^B} \hat{\kappa}_{\alpha\beta} + \frac{\partial r}{\partial x^A} \frac{\partial^2 \hat{r}}{\partial x^A \partial x^B} \right)_{\alpha\beta} 
\]

\[
= \frac{\partial r}{\partial r} \left( -2(\hat{\kappa} - \hat{\theta}^+ \right) \nabla_A \hat{\kappa} \nabla_B \hat{r} + 2 \hat{\kappa}_{(A} \nabla_B \hat{r} + \Xi_{AB} - 2 \nabla_A \nabla_B \hat{r}^2 \right)_{\alpha\beta} + \frac{\partial r}{\partial x^A} \frac{\partial r}{\partial x^B} \right)_{\alpha\beta} 
\]

\[
= (1 + q r)(\Xi_{AB} - r^2 \hat{\kappa}_{(A} \nabla_B q + r^2 \nabla_A \nabla_B q)_{\alpha\beta} + \frac{\partial r}{\partial x^A} \frac{\partial r}{\partial x^B} \right)_{\alpha\beta}. 
\]  

(3.70)

In a conformal Gauss gauge which satisfies (3.80) below we have \( \xi_A = \Theta(\hat{r}) \) (cf. (A.14)).

In Section 4.1 we will see that boundedness of the rescaled Weyl tensor \( I^- \) requires the data \( \Xi_{AB} \) to be of the form \( \Xi_{AB} = \Xi^{(2)}_{AB} r^2 + \Sigma(r^3) \). It follows from (3.70) that the leading order term transforms as

\[
\Xi^{(2)}_{AB} \mapsto \Xi^{(2)}_{AB} + (\nabla_A \nabla_B q)_{\alpha\beta}. \quad (3.71)
\]

It is convenient to set

\[
v_A := \nabla_B \Xi^{(2)}_{AB}. \quad (3.72)
\]

It follows from the Hodge-decomposition theorem (cf. e.g. [58]) that on a closed Riemannian manifold (\( \Sigma, h \)) a (smooth) 1-form \( \omega \) admits the decomposition

\[
\omega_A = \nabla_A \omega + f_A B \nabla_B \omega + \lambda_A \quad \text{with} \quad \Delta_h \lambda_A = 0, \quad (3.73)
\]

where \( f_{AB} \) denotes the volume form associated with \( h \). If (\( \Sigma, h \)) is compact and has non-negative Ricci curvature which is positive at one point it follows from Bochner’s theorem (cf. e.g. [49]) that all harmonic 1-forms are identically zero. In that case any (smooth) vector field admits a decomposition of the form

\[
\omega_A = \nabla_A \omega + f_A B \nabla_B \omega. \quad (3.74)
\]

In particular on a Riemannian 2-sphere all vector fields can be decomposed this way, whence the expansion coefficients of \( v_A \) can be written as

\[
v_A^{(n)} = \nabla_A \omega^{(n)} + f_A B \nabla_B \omega^{(n)}. \quad (3.75)
\]

Consider a gauge where \( g_{\hat{A}\hat{B}}|_{\Omega^-} \) is the standard metric \( s_{AB} \) on \( S^2 \). We observe that in that case \( \Xi^{(2)} \) transforms as

\[
\Xi^{(2)} \mapsto \Xi^{(2)} + \frac{1}{2}(\Delta_s + 2) q. \quad (3.76)
\]

Recall that \( v_A^{(2)} \) arises as the divergence of a symmetric trace-free tensor, \( v_A^{(2)} = g_{AB} \Xi^{(2)}_{AB} \). As a consequence of York splitting (cf. e.g. [9]), the fact that there are no non-trivial TT-tensors on \( S^2 \), and Hodge decomposition [6], on \( S^2 \), any symmetric trace-free tensor \( \tau \) admits a decomposition of the form

\[
\tau_{AB} = (\varphi_A \varphi_B)_{\alpha\beta} + (\varphi_A \varphi_B)_{\alpha\beta} + \epsilon_{(A} C \varphi_B) \varphi_{C\alpha\beta}, \quad (3.77)
\]

for appropriately chosen 1-forms \( \varphi_A \) and functions \( \tau \) and \( \tilde{\tau} \). Its divergence reads

\[
\varphi_{AB} \tau_{AB} = \frac{1}{2} \varphi_A (\Delta_s + 2) q + \frac{1}{2} \epsilon_{AB} C \varphi_{B}(\Delta_s + 2) \tilde{\tau}. \quad (3.78)
\]
It follows that $v^{(2)}$ and $\mathcal{v}^{(2)}$ cannot have $\ell = 0, 1$-spherical harmonics in their harmonic decomposition. We will make frequently use of the Hodge decompositions described here.

By way of summary, assuming a conformal Gauss gauge which satisfies (3.80) below (and $g_{\tilde{A}\tilde{B}}|_{I^-} = s_{\tilde{A}\tilde{B}}$) the remaining gauge freedom can be employed to prescribe the divergence of $\nu^{(2)}_A$ by solving a Laplace-like equation. Since $v^{(2)}$ does not contain $\ell = 1$-spherical harmonics, the kernel of the operator in (3.76) does not provide any obstructions. One then may proceed as described in Section 2.6 to transform the remaining gauge functions to their desired form.

### 3.4.1 Dual mass aspect

It is convenient so set

$$N := \frac{1}{8} \Delta_s v^{(2)} \iff N = -\frac{1}{8} \epsilon^{AB} \mathcal{G}_{AB} v^{(2)}.$$ \hspace{1cm} (3.79)

Later on we shall see (cf. (4.57)-(4.58)) that the function $N$ can be identified with the leading order term of a certain rescaled Weyl tensor component at $I^-$, and this component is dual to the one which involves the (ADM) mass aspect $M$, by which we mean the limit of the Bondi mass aspect at $I^-$. In the case of e.g. the Taub-NUT spacetime, cf. [35], this component is constant and can be identified with the NUT-parameter (note that this spacetime is not asymptotically flat, whence $N$ can be constant and non-zero, which it cannot be in our setting). In this sense $N$ may be regarded as a generalized NUT-like or twist parameter.

In [50], cf. [5], a so-called ‘dual Bondi 4-momentum’ has been introduced, leading in particular to the notion of a ‘dual Bondi mass’, or ‘magnetic Bondi mass’. As the Bondi mass it is defined as the integral of a ‘dual Bondi mass aspect’ over cuts of $\mathcal{J}$. Since the function $N$ arises as a limit thereof at $I^-$, we will call it dual (ADM) mass aspect.

It follows immediately from (3.79) that the dual mass, i.e. the integral of $N$ over $I^\sim \simeq S^2$ vanishes. This is in accordance with the results in [5], that the dual mass has to vanish in a spacetime with a regular $I^-$ with topology $\mathbb{R} \times S^2$.

### 3.5 Asymptotically Minkowski-like conformal Gauss gauge

In Section 3.3 we have derived some a priori restrictions on the gauge functions in order to end up with a smooth cylinder representation of spatial infinity. However, it is useful and convenient to impose some weak additional restrictions on the asymptotic behavior of the gauge functions at $I^-$. The equations for Weyl connection, Schouten tensor etc. derived in Section 2.3 & 2.4 involve terms which are quadratic in the unknowns. This implies that the structure of the equations for the nth-order radial derivatives on the cylinder depends crucially on terms of 0th-order (in particular of connection and frame coefficients).

In the case of a smooth critical set $I^-$, the integration functions for the transport equations on the cylinder are determined at $I^-$ by the limit of the corresponding fields on $\mathcal{J}^-$. The initial data for the 0th-order equations, (3.50)-(3.56), which are non-linear, are determined by the asymptotic behavior of the gauge functions on $\mathcal{J}^-$. In order to study the transport equations for radial derivatives of order $m \geq 1$, a simple, explicit form for the terms of 0th-order on $I$ is beneficial. We will therefore fix the leading order term of the asymptotic expansion of the gauge functions (the condition below on the next-to-leading order term for $\nu_\tau$ corresponds to a restriction on the leading-order of the divergence $\theta^\tau$). Guided by the representation (3.25) of the Minkowski spacetime we will restrict attention henceforth to gauge functions of the following form, for which, indeed, (3.50)-(3.56) can be solved explicitly, which will be accomplished in Section 5.1.
Definition 3.6 We call a conformal Gauss gauge “weakly asymptotically Minkowski-like” if the
gauge functions are of the following form,
\[
\nu_r = \frac{1}{r} + \Theta^{(1,2)} + O(r), \quad \nu_A = O(r), \quad \Theta^{(1)} = 2r + \Theta^{(1,2)} r^2 + O(r^3), \quad \kappa = -\frac{2}{r} + O(r),
\]
\[
\theta^- = O(r^3), \quad g_{\hat{A} \hat{B}}|_{\tau^-} = s_{\hat{A} \hat{B}}, \quad f_r|_{\tau^-} = \frac{1}{r} + O(1), \quad f_{\hat{A}}|_{\tau^-} = O(r).
\]

Remark 3.7 In Section 3.3 we have assumed that \( \mathcal{I}^+ \) is located at \( \{ \tau = \pm 1 \} \), so that (3.37) holds, in order to motivate (3.38). Since we are mainly interested in the behavior of the fields near \( I^- \) we do not include (3.37) in this definition so that all gauge functions are independent.

It turns out that connection and frame coefficients on \( I \) do not depend on the physical, non-gauge data, while their 1st-order radial derivatives (and the restriction to \( I \) of the rescaled Weyl tensor) depend on the radiation field (and the angular momentum). Since we know that (3.25) provides a smooth representation of Minkowski, it therefore seems reasonable to expect that (3.80)-(3.81) do not impose restrictions on the non-gauge data to produce a spacetime which admits a smooth critical set \( I^- \).

For later reference, we also add the following

Definition 3.8 We call a conformal Gauss gauge “asymptotically Minkowski-like at each order” if the gauge functions are of the following form,
\[
\nu_r = \frac{1}{r} + O(r^\infty), \quad \nu_A = O(r^\infty), \quad \Theta^{(1)} = 2r + O(r^\infty), \quad \kappa = -\frac{2}{r} + O(r^\infty),
\]
\[
\theta^- = O(r^\infty), \quad g_{\hat{A} \hat{B}}|_{\tau^-} = s_{\hat{A} \hat{B}}, \quad \mathcal{G}_A^{(2)} = 0, \quad f_r|_{\tau^-} = \frac{1}{r} + O(r^\infty), \quad f_{\hat{A}}|_{\tau^-} = O(r^\infty),
\]
i.e. if the gauge functions have the same expansions at \( I^- \) as in (3.31)-(3.32).

We will use this gauge in Section 7 to establish sufficient conditions for the non-appearance of logarithmic terms at the critical sets.

4 Appearance of log terms: Approaching \( I^- \) from \( \mathcal{I}^- \)

Our goal is as follows: We assume we have been given asymptotic initial data, which will be the radiation field on \( \mathcal{I}^- \) supplemented by certain “integration functions” at \( I^- \) such as the (ADM) mass aspect, cf. Appendix A.2. Then we solve the characteristic constraint equations to determine all the relevant data for the evolution equations, and analyze the appearance of logarithmic terms at \( I^- \).

A related problem for an ordinary (i.e. non-asymptotic) characteristic initial value problem with one initial surface going all the way to null infinity has been analyzed in [7, 15, 45]. There it turns out that, in an appropriate gauge, if the constraint equations do not produce logarithmic terms, the solution will be smooth, in particular higher-order transverse derivatives will not pick up log terms when approaching null infinity.

When approaching spatial infinity the situation turns out to be completely different, as logarithmic terms can appear in transverse derivatives of arbitrary high orders with all lower orders being smooth. We thus need to take higher order transverse derivatives into account as well, and analyze their behavior when approaching \( I^- \), which makes the problem significantly harder to deal with. We will do this by determining expansions of all the relevant fields on \( \mathcal{I}^- \) (and transverse derivatives thereof) when approaching \( I^- \). Later on, we will study the same issue when approaching \( I^- \) from the cylinder \( I \).
4.1 Solution of the asymptotic constraint equations

We assume a weakly asymptotically Minkowski-like conformal Gauss gauge (3.80)-(3.81). The constraint equations in adapted null coordinates are listed in Appendix A.1. Recall that the data \( \Xi_{AB} \) need to be of the form (3.57). Those constraint equations (A.9)-(A.19) which do not involve the radiation field can be straightforwardly solved,

\[
\Sigma = 2r^2 + \Sigma^{(4)}r^4 + \Sigma^{(5)}r^5 + \mathcal{O}(r^6),
\]
\[
\theta^+ = \theta^{(1)} + \theta^{(2)}r^2 + \mathcal{O}(r^3), \quad \text{where} \quad \theta^{(1)} = 2(\kappa^{(1)} + \Sigma^{(4)}),
\]
\[
g_{\bar{A}\bar{B}}|_{r^-} = \left(1 + \frac{1}{2} \theta^{(1)}r^2 + \frac{1}{3} \theta^{(2)}r^3\right) s_{\bar{A}\bar{B}} + \mathcal{O}(r^4),
\]
\[
R = 2 - \frac{1}{2} \left(\Delta_s + 2\right) \theta^{(1)}r^2 - \frac{1}{3} \left(\Delta_s + 2\right) \theta^{(2)}r^3 + \mathcal{O}(r^4),
\]
\[
L_{rr}|_{r^-} = - \frac{3}{2} \theta^{(1)} + \mathcal{O}(r),
\]
\[
\xi_{\bar{A}} = \hat{\nabla}_{\bar{A}} \Sigma^{(4)}r^2 + \hat{\nabla}_{\bar{A}} \Sigma^{(5)}r^3 + \mathcal{O}(r^4),
\]
\[
L_{r\bar{A}}|_{r^-} = - \frac{1}{2} \hat{\nabla}_{\bar{A}} \theta^{(1)}r - \frac{1}{2} \hat{\nabla}_{\bar{A}} \theta^{(2)}r^2 + \mathcal{O}(r^3),
\]
\[
g^{\bar{A}\bar{B}} L_{\bar{A}\bar{B}}|_{r^-} = 1 - \frac{1}{4} \left(\Delta_s + 2\right) \theta^{(1)}r^2 + \mathcal{O}(r^3),
\]
\[
L_{r\bar{r}}|_{r^-} = - \frac{1}{4} \left(\theta^{(3)} + \Delta_s \Sigma^{(4)} + \frac{1}{2} \left(\Delta_s - 4\right) \theta^{(1)}\right)r^2 + \frac{1}{3} \left(2\theta^{(4)} - 4\theta^{(2)} + \Delta_s \Sigma^{(5)} - 3g^{rr(3)} \theta^{(1)} + \frac{1}{3} \left(\Delta_s + 2\right) \theta^{(2)}\right) r^3 + \mathcal{O}(r^4).}
\]

Here \((\cdot)^{(n)}\) denotes the \(n\)th-order expansion coefficient at \(r = 0\). The values for \(\Sigma^{(4)}, \Sigma^{(5)}\) and \(\theta^{(2)}\) are determined by \(\nu_r, \Theta^{(1)}\) and \(\kappa\); the precise relation is irrelevant here. Integration of (A.36) and (A.39) yields

\[
(L_{\bar{A}\bar{B}})_{rr}|_{r^-} = \frac{1}{2}\mathcal{L}_{\bar{A}\bar{B}} + \mathcal{O}(r^2),
\]
\[
L_{\bar{A}}^{\bar{r}}|_{r^-} = \frac{1}{2} \nu^{(1)}_{\bar{A}} r + \mathcal{O}(r^2).
\]

Then we employ (A.40)-(A.41) to obtain

\[
W_{r\bar{A}r\bar{B}}|_{r^-} = \mathcal{O}(r^{-1}),
\]
\[
W_{r\bar{A}r}|_{r^-} = - \frac{1}{4r^2} \nu^{(1)}_{\bar{A}} + \mathcal{O}(1),
\]

which implies that the frame component (recall (2.108)-(2.110))

\[
W_{010A} - W_{011A}|_{r^-} = \nu^{r}e^{\bar{A}}_{\bar{A}} W_{r\bar{A}r}^r - (\nu^r)^2 \nu_{\bar{B}}e^{\bar{A}}_{\bar{A}} W_{r\bar{A}r\bar{B}} = -\frac{1}{4r^2} \nu^{(1)}_{\bar{A}} + \mathcal{O}(r)
\]

is unbounded at \(I^-\) whenever \(v^{(1)}_{\bar{A}} \neq 0\). We deduce the regularity condition

\[
v^{(1)}_{\bar{A}} = 0 \iff \mathcal{L}_{\bar{A}}^{(1)} = 0.
\]

\[\text{\footnotesize\textsuperscript{4}}\text{Alternatively, one could analyze the constraints directly in a conformal Gauss gauge and the associated frame. Since the constraint equations in adapted null coordinates have been derived in [44], the coordinates are adapted to the geometry of \(\mathcal{J}^-\), and since we also have identified the remaining gauge degrees of freedom using coordinates, it seems convenient to start with them and determine the behavior in the conformal Gauss gauge afterwards.}\]
In the analysis the radiation field $W_{\alpha rB}$ plays a distinguished role; in turn out the the expressions below take the most compact form when expressed in term of $W_{\alpha rB}$ rather than $\Xi_{\alpha B}$, which, tough, does not comprise the integration functions $\Xi_{\alpha B}^{(1)}$ and $\Xi_{\alpha B}^{(2)}$. It is convenient to make the following definitions,

$$w_A := \nabla^B W_{rA rB}|_{r = -}, \quad w_A^{(n)} := \mathcal{D}^B W_{rA rB}^{(n)}|_{r = -}.$$  \hspace{1cm} (4.16)

Recall that $v_A^{(n)} = \mathcal{D}_A v_A^{(n)} + \epsilon_A^B \mathcal{D}_B v_B^{(n)}$, Definition 3.79, and that $\Xi$ may be regarded as a gauge function. From (A.36), (A.39)-(A.42) we obtain ($\epsilon_{\alpha B}$ denotes the volume form of the round sphere)

$$(L_{\alpha B})_{tt}|_{r = -} = -\frac{1}{2} \bigl(2\Xi_{\alpha B} - \frac{1}{2} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B}\bigr)_{tt} r^2$$ \hspace{1cm} (4.17)

$L_A^{|A, B}|_{r = -} = -\frac{1}{2} \Xi_{\alpha B} + \frac{1}{2} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} - \frac{1}{4} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} \mathcal{D}_B \Xi_{\alpha B} - \frac{1}{4} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} \mathcal{D}_B \Xi_{\alpha B}$

$W_{rA rB}|_{r = -} = -\frac{1}{2} \Xi_{\alpha B} + \frac{1}{2} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} - \frac{1}{4} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} \mathcal{D}_B \Xi_{\alpha B} + \mathcal{D}_B \Xi_{\alpha B} + \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B}$

$w_A = -\frac{1}{2} \Xi_{\alpha B} + \frac{1}{2} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} - \frac{1}{4} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} \mathcal{D}_B \Xi_{\alpha B} + \mathcal{D}_B \Xi_{\alpha B} + \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B}$

The ODE (A.43) for $W_{rA rB}$

$$(\partial_r + \frac{3}{2} \theta^{(1)} + \mathcal{D}(r^2)) W_{rA rB}|_{r = -} = \frac{1}{2} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} - \frac{1}{4} \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B} \mathcal{D}_B \Xi_{\alpha B} + \mathcal{D}_B \Xi_{\alpha B} + \mathcal{D}_A \mathcal{D}_B \Xi_{\alpha B}$$

does not produce log-terms. Its solution is of the form

$W_{rA rB}|_{r = -} = 2M + \frac{1}{2} \Xi_{\alpha B} w_A^{(1)} + \frac{1}{2} \Xi_{\alpha B} w_A^{(0)} + \frac{1}{2} \Xi_{\alpha B} w_A^{(0)} + \frac{1}{2} \Xi_{\alpha B} w_A^{(0)} - \frac{3}{4} \mathcal{D}_B \Xi_{\alpha B} \mathcal{D}_B \Xi_{\alpha B}$

As explained in Appendix A.2.3, the integration function $M$ – as the ones which appear below – may be regarded as part of the freely prescribable initial data. The function $M$ can be identified with the ADM mass aspect, or rather the limit of the Bondi mass aspect at $r^{-}$. There are results [41] which show that for a certain class of data it is this limit, whence we will call it (ADM) mass aspect.

We consider the ODE (A.44) for $W_{A r A r B}$

$$(\partial_r - \frac{2}{r} + \mathcal{D}(r^2)) W_{A r A r B}|_{r = -} = \mathcal{D}_A M + \epsilon_A^B \mathcal{D}_B N + \frac{1}{2} \mathcal{D}_A \mathcal{D}_B w_A^{(0)} - \frac{1}{4} \mathcal{D}_B \Xi_{\alpha B} w_A^{(1)} - \frac{1}{4} \mathcal{D}_B \Xi_{\alpha B} w_A^{(1)} - \frac{1}{4} \mathcal{D}_B \Xi_{\alpha B} w_A^{(1)} + \mathcal{D}(r^2).$$ \hspace{1cm} (4.25)
The term of order \( r \) on the right-hand side produces log-terms. We therefore need to impose the "no-logs-condition"

\[
(\Delta_s - 1)w_A^{(-1)} - 2\mathcal{D}_A \mathcal{D}^B w_B^{(-1)} = 0. \tag{4.26}
\]

Again, we use Hodge decomposition,

\[
w_A^{(n)} = \mathcal{D}_A \mathcal{D}^B w_B^{(n)} + \epsilon_A B \mathcal{D}^B \overline{w}^{(n)}. \tag{4.27}
\]

By taking the divergence and curl of (4.26) it follows that

\[
\Delta_s \Delta_s w_A^{(-1)} = 0 \implies \Delta_s w_A^{(-1)} = \text{const.} \tag{4.28}
\]

\[
\Delta_s \Delta_s \overline{w}^{(-1)} = 0 \implies \Delta_s \overline{w}^{(-1)} = \text{const.} \tag{4.29}
\]

According to Gauss’ theorem, on \( S^2 \), a solution \( w_A^{(-1)} \) and \( \overline{w}^{(-1)} \), respectively, exists if and only if the corresponding constant in the equation vanishes. In that case \( w_A^{(-1)} \) and \( \overline{w}^{(-1)} \) need to be constant and the no-logs condition (4.26) becomes

\[
w_A^{(-1)} = 0 \iff W_A^{(-1)} = 0. \tag{4.30}
\]

At this stage it seems remarkable that all gauge functions, which in principle provide contributions to this order, cancel out. In particular they cannot be employed to fulfill the no-logs conditions, at least not at this order (of course, in principle it is conceivable that the gauge functions of this order can be used to get rid of log terms which appear in higher orders). We will return to this observation later on and in particular in Section 6.

Assuming that (4.30) holds, the ODE for \( W_A^{r} \) takes the form

\[
\left( \partial_r - 2/r + (\kappa^{(1)} + \frac{1}{2} \theta^{(1)}) r + \mathcal{O}(r^2) \right) W_A^{r} \big|_{r^*} = 0.
\]

whence, for some integration function \( \hat{L}_A \),

\[
W_A^{r} \big|_{r^*} = - \mathcal{M}_A r + \hat{L}_A r^2 + \left( (\kappa^{(1)} + \frac{1}{4} \theta^{(1)}) \mathcal{M}_A - \frac{1}{12} (\Delta_s - 3) w_A^{(0)} + \frac{1}{6} \mathcal{D}_A \mathcal{D}^B w_B^{(0)} - \frac{3}{2} \mathcal{D}_A \Sigma (4) M - \frac{3}{2} N \mathcal{D}^B \Sigma (4) \epsilon_{A\hat{B}} \right) r^3 + \mathcal{O}(r^4).
\]

Finally, we consider the ODE (4.47) which determines \( (W_A^{r})_{tt} \),

\[
\left( \partial_r - 4/r + (2\kappa^{(1)} - \frac{1}{2} \theta^{(1)}) r + \mathcal{O}(r^2) \right) (W_A^{r})_{tt} \big|_{r^*} = 0
\]

\[
= - \left( \mathcal{D}_A \mathcal{M}_B \right)_{tt} r^2 + \left( \mathcal{D}_A \hat{L}_B \right)_{tt} + \frac{3}{2} M \mathcal{E} (2)_{AB} + \frac{3}{2} N \mathcal{E} (2)_{AB} \mathcal{E} (2)_{CD} \right) r^2
\]

\[
+ \left( (\kappa^{(1)} - \frac{1}{4} \theta^{(1)}) \mathcal{D}_A \mathcal{M}_B - \frac{1}{12} \mathcal{D}_A (\Delta_s - 1) w_B^{(0)} + \frac{1}{6} \mathcal{D}_A \mathcal{D}^B \mathcal{D}^C w_C^{(0)} \right)_{tt} r^3 + \mathcal{O}(r^4). \tag{4.31}
\]
The solution contains logarithmic term unless the following no-logs-condition holds,
\[
\mathcal{D}(\Delta_s - 1)w^{(0)}_B = 2\mathcal{D}_A \mathcal{D}_B \mathcal{D}^C w^{(0)}_C.
\] (4.32)

Then, the solution takes the form (for some integration function \(\epsilon^{(2,0)}_{AB}\))
\[
(W^{(r)}_{AB})_{t|\mathcal{I}^-} = \frac{1}{2}(\mathcal{D}(\mathcal{M}^{\mathcal{B}}_B)_{|\mathcal{I}^-}) r^2 - \left( (\mathcal{D}(\dot{\mathcal{L}}_{\mathcal{B}})_{|\mathcal{I}^-}) + \frac{3}{2}M^{(2)}_{AB} + \frac{3}{2}N^{(2)}_{(A} \epsilon^{(2)}_{B)C} \right) r^3 + \epsilon^{(2,0)}_{AB} r^4 + \mathcal{O}(r^5).
\] (4.33)

To analyze (4.32) we decompose, as above, \(w^{(0)}_A\) as \(w^{(0)}_A = \mathcal{D}_A \mathcal{w}^{(0)} + \epsilon_A^B \mathcal{D}_B \mathcal{w}^{(0)}\) and apply \(\mathcal{D}_A \mathcal{D}_B\),
\[
\Delta_s \Delta_s (\Delta_s + 2) w^{(0)} = 0.
\]

It follows that \(w^{(0)}_A\) needs to be a linear combination of \(\ell = 0, 1\)-spherical harmonics. Next we apply \(\epsilon^{AC} \mathcal{D}_C \mathcal{D}_B\) to (4.32) to obtain an identical equation for \(\mathcal{w}^{(0)}\), \(\Delta_s \Delta_s (\Delta_s + 2) \mathcal{w}^{(0)} = 0\). Consequently, \(\mathcal{w}^{(0)}\) can be represented by \(\ell = 0, 1\)-spherical harmonics as well. Equivalently, \(w^{(0)}_A\) is a conformal Killing 1-form on \(\mathcal{S}^2\).

However, recall that \(w^{(0)}_A\) was defined as the divergence of a symmetric trace-free tensor, \(w^{(0)}_A = \mathcal{D}_B w^{(0)}_{AB}\), whence, on \(\mathcal{S}^2\), \(w^{(0)}\) and \(\mathcal{w}^{(0)}\) are not allowed to have \(\ell = 1\)-spherical harmonics in their decomposition, cf. (3.78). The no-logs condition (4.32) thus requires that also this expansion coefficient of the radiation field needs to vanish, again regardless of the choice of the gauge functions,
\[
W^{(0)}_{r|\mathcal{A}B} = 0.
\] (4.34)

Altogether, assuming that the no-logs conditions (4.30) and (4.34) as well as the boundedness-condition (4.15) hold, the restriction of the rescaled Weyl tensor to \(\mathcal{I}^-\) extends smoothly across \(I^-\) and admits an expansion of the form,
\[
W_{r|\mathcal{A}B} = 0 \mathcal{D}(r),
\]
\[
W_{r|\mathcal{A}B} = 0 \mathcal{D}(r^2),
\]
\[
W_{\mathcal{A}B} = -2N \epsilon_{\mathcal{A}B} + \frac{1}{2} N \theta^{(1)} \epsilon_{\mathcal{A}B} r^2 + \mathcal{D}(r^3),
\]
\[
W_{r|\mathcal{A}B} = -2M - \frac{3}{2} M \theta^{(1)} r^2 + \mathcal{O}(r^3),
\]
\[
W_{r|\mathcal{A}B} = -2M - \frac{3}{2} M \theta^{(1)} r^2 + \mathcal{O}(r^3),
\]
\[
(W_{r|\mathcal{A}B})_{t|\mathcal{I}^-} = \frac{1}{2}(\mathcal{D}(\mathcal{M}^{\mathcal{B}}_B)_{|\mathcal{I}^-}) r^2 - \left( (\mathcal{D}(\dot{\mathcal{L}}_{\mathcal{B}})_{|\mathcal{I}^-}) + \frac{3}{2}M^{(2)}_{AB} + \frac{3}{2}N^{(2)}_{(A} \epsilon^{(2)}_{B)C} \right) r^3 + \epsilon^{(2,0)}_{AB} r^4 + \mathcal{O}(r^5).
\] (4.40)

We have proved the following

**Proposition 4.1** Consider asymptotic initial data \((W^{(1)}_{r|\mathcal{A}B}, \mathcal{M}, N, \tilde{\mathcal{L}}_A, \epsilon^{(2)}_{AB})\) for the GCFE on \(\mathcal{I}^-\) in a weakly asymptotically Minkowski-like conformal Gauss gauge (3.80)-(3.81). Then the solution of the vacuum constraint equations for the GCFE on \(\mathcal{I}^-\) admits a smooth expansion.

---

5 To have a well-posed initial value problem the data on \(\mathcal{I}^-\) need to be supplemented by appropriate data on e.g. an incoming null hypersurface, cf. Appendix A.2.
through $I^-$ if and only if the boundedness condition (4.15) holds, i.e. $\Xi^{(1)}_A\mapsto 0$, and the data $W_{\hat{A}\hat{B}}$ admit an expansion of the form $W_{\hat{A}\hat{B}} \equiv \mathcal{O}(r)$ at $I^-$, i.e. if and only if the two leading order terms of the radiation field vanish.\(^6\) In that case the expansion of the rescaled Weyl tensor takes the form (4.35)-(4.40). In the frame (2.108)-(2.110) its expansion is given by (4.57)-(4.62) below.

### 4.1.1 Frame coefficients

So far we have solved the asymptotic constraint equations on $\mathcal{I}^-$ in adapted null coordinates. Now we want to compute the corresponding frame coefficients associated to the frame (2.108)-(2.110) in our current weakly asymptotically Minkowski-like conformal Gauss gauge, where

\[ e_1^\tau|_{\mathcal{I}^-} = 1, \quad e_1^\nu|_{\mathcal{I}^-} = \nu^r = r + \mathcal{O}(r^2), \quad e_1^{A}|_{\mathcal{I}^-} = 0, \quad (4.41) \]

\[ e_1^A|_{\mathcal{I}^-} = 0, \quad e_1^A|_{\mathcal{I}^-} = \mathcal{O}(r^2), \quad e_1^A|_{\mathcal{I}^-} = \hat{e}^A_A + \mathcal{O}(r^2). \quad (4.42) \]

It follows from (2.167)-(2.171) that

\[ \hat{\Gamma}_1^1|_{\mathcal{I}^-} = 1 + f^{(1)}_1 r + \mathcal{O}(r^2), \quad (4.43) \]

\[ \hat{\Gamma}_1^0|_{\mathcal{I}^-} = -f^{(1)}_1 r + \mathcal{O}(r^2), \quad (4.44) \]

\[ \hat{\Gamma}_1^A|_{\mathcal{I}^-} = -f^{A(1)} r + \mathcal{O}(r^2), \quad (4.45) \]

\[ \hat{\Gamma}_1^A|_{\mathcal{I}^-} = \hat{\Gamma}_1^A_0, \quad (4.46) \]

\[ (\hat{\Gamma}_1^A_B)_n|_{\mathcal{I}^-} = 0, \quad (4.47) \]

while (2.162)-(2.166) give (with $\nu^{(n)}_A := \hat{e}^A_A \nu^{(n)}_A$),

\[ \hat{\Gamma}_1^A|_{\mathcal{I}^-} = f^{(1)}_A r + \mathcal{O}(r^2), \quad (4.48) \]

\[ \hat{\Gamma}_1^0|_{\mathcal{I}^-} = \left(\mathcal{O}(r^2) - \nu^{(1)}_A\right)r + \mathcal{O}(r^2), \quad (4.49) \]

\[ \hat{\Gamma}_1^B|_{\mathcal{I}^-} = \frac{1}{2} f^{(2)}_A r + \mathcal{O}(r^2), \quad (4.50) \]

\[ \hat{\Gamma}_1^B|_{\mathcal{I}^-} = \hat{\Gamma}_1^B_0 + \left(1 + f^{(1)}_1 r + \mathcal{O}(r^2)\right)\delta^B_A, \quad (4.51) \]

\[ \hat{\Gamma}_1^B|_{\mathcal{I}^-} = \hat{\Gamma}_1^B_0 + \left(2\delta^C_A f^{(1)}_B - \eta_{AB} f^{(1)}_C\right)r + \mathcal{O}(r^2). \quad (4.52) \]

From Section 2.8 we obtain

\[ \hat{L}_{1j}|_{\mathcal{I}^-} = \mathcal{O}(r^2), \quad (4.53) \]

\[ \hat{L}_{11}|_{\mathcal{I}^-} = \left(\frac{1}{2} f^{(2)}_A + f^{(1)}_A - f^{(0)}_B + \frac{1}{2} d_{AB} g^{(3)rr} \right)r + \mathcal{O}(r^2), \quad (4.54) \]

\[ \hat{L}_{AB}|_{\mathcal{I}^-} = \left(- f^{(1)}_B - f^{(0)}_B + \frac{1}{2} \eta_{AB} g^{(3)rr} \right)r + \mathcal{O}(r^2), \quad (4.55) \]

\[ \hat{L}_{AB}|_{\mathcal{I}^-} = \frac{1}{2} \left( f^{(2)}_A - \frac{1}{2} \eta_{AB} g^{(3)rr} \right)r + \mathcal{O}(r^2). \quad (4.56) \]

\(^6\) An $r^{-2}$-term would yield an unbounded frame component.
For the rescaled Weyl tensor we find, redefining the integration functions $\hat{L}_A$ and $\hat{\epsilon}^{(2,0)}_{AB}$ (now denoted without $\hat{}$),

$$W_{0101}|_{\mathcal{F}^-} = 2M + \mathcal{O}(r^2),$$

$$W_{01AB}|_{\mathcal{F}^-} = 2\lambda_{AB} + \mathcal{O}(r^2),$$

$$W^-_{A}|_{\mathcal{F}^-} = \mathcal{O}(r^2),$$

$$W^+_{A}|_{\mathcal{F}^-} = -2\lambda_{RA} + 2L_{AR} + \mathcal{O}(r^2),$$

$$V_{AB}|_{\mathcal{F}^-} = \mathcal{O}(r^2),$$

$$V^-_{AB}|_{\mathcal{F}^-} = (\mathcal{D}_{(A}\lambda_{B)})_{tt} + \left(-2(\mathcal{D}_{(A\lambda B)})_{tt} - 3M\xi^{(2)}_{AB} - 3N\xi^{(2)}_{(A}\lambda_{B)C}
$$

$$- 2(\lambda_{(A}\mathcal{D}_{B)})_{tt} + 2(\lambda^{(1)}_{(A}\mathcal{D}_{B)})_{tt}\right)_{tt} + \mathcal{O}(r^2).$$

(4.62)

4.2 Higher-order derivatives: Structure of the equations and no-logs

In the previous section we derived conditions which make sure that the restriction to $\mathcal{F}^-$ of the fields appearing in the GCFE admit smooth extensions through $I^-$. Here we devote attention to the issue under which conditions the statement remains true for transverse derivatives of these fields as well. For this let us assume that all the fields $\mathbf{f} = (e^{n}, \hat{\Gamma}_{ijk}, \hat{L}_{ij}, W_{ij;k})$ have transverse derivatives $\partial^n f|_{\mathcal{F}^-}$ which admit smooth extensions across $I^-$ for $0 \leq k \leq n - 1$. We aim to find conditions which guarantee that this also holds true for $\partial^n f|_{\mathcal{F}^-}, n \geq 1$.

Recall the evolution equations (2.72)-(2.78). We apply $\partial^n$, which yields algebraic equations for $(\partial^n e^{n}, \partial^n \hat{\Gamma}_{i}^{j}, \partial^n \hat{L}_{ij})|_{\mathcal{F}^-}$,

$$\partial^n e^n|_{\mathcal{F}^-} = \mathcal{O}(1), \quad \partial^n \hat{\Gamma}_{a}^{i}|_{\mathcal{F}^-} = \mathcal{O}(1), \quad \partial^n \hat{L}_{ai}|_{\mathcal{F}^-} = \mathcal{O}(1),$$

whence the restrictions to $\mathcal{F}^-$ of the $n$th-order $\tau$-derivatives of frame field, connection coefficients and Schouten tensor are smooth at $I^-$, supposing that this is the case for all derivatives of $\mathbf{f}$ up to and including order $n - 1$.

Let us consider the evolution equations (2.90)-(2.94) for the rescaled Weyl tensor. Again we apply $\partial^n$ and take the restriction to $\mathcal{F}^-$. We obtain a set of equations which determines all independent components (except $\partial^n V_{AB}|_{\mathcal{F}^-}$) algebraically in terms of $(\partial^n e^n, \partial^n \hat{\Gamma}_{a}^{i}, \partial^n \hat{L}_{ij})|_{\mathcal{F}^-}$ and lower-order derivatives, which are already known at this stage. In particular these components are smooth at $I^-$, as well,

$$\partial^n U_{AB}|_{\mathcal{F}^-} = \mathcal{O}(1), \quad \partial^n W_{A}^{+}|_{\mathcal{F}^-} = \mathcal{O}(1), \quad \partial^n V_{AB}|_{\mathcal{F}^-} = \mathcal{O}(1).$$

(4.64)

The missing components $\partial^n V^-_{AB}|_{\mathcal{F}^-}$ are determined by (2.95). We apply $\partial^n$ and take its restriction to $\mathcal{F}^-$. In this case it is not an algebraic equation but an ODE for $\partial^n V^-_{AB}|_{\mathcal{F}^-}$ along the null geodesic generators of $\mathcal{F}^-$ (the 0th-order recovers the constraint (A.47) for $(W_{AB})_{tt}$ in frame components),

$$(\nu^{*} \partial_{\nu} + n \partial_{\nu} e^{n}) \partial^n V^-_{AB}|_{\mathcal{F}^-} = (\hat{\Gamma}_{0}^{0} + 2\hat{\Gamma}_{1}^{1} + \hat{\Gamma}_{C}^{C})_{tt} + \mathcal{O}(1),$$

$$- \left(\hat{\Gamma}_{C}^{0} + \hat{\Gamma}_{C}^{1} - 2\hat{\Gamma}_{A}^{C} \right)_{tt} + \mathcal{O}(1),$$

(4.65)

where $\mathcal{O}(1)$ only involves terms such as (4.64) which are in principle known at this stage, and known to be smooth at $I^-$. Using (2.162)-(2.171) and taking into account that by (2.78) we have

$$\partial_{\nu} e^{n}|_{\mathcal{F}^-} = (\partial_{\nu} + \kappa)\nu^{*},$$

(4.66)
this can be written as
\[
\left(\nu^2 \partial_\nu + \frac{1}{2} \theta^2 \nu^2 + (n + 2)(\partial_\nu + \kappa) \nu^2 \right) \partial_n^\nu V^-_{AB} |_{\mathcal{I}^-} = \mathcal{O}(1),
\]
(4.67)
or,
\[
r^{n+3} \left( \partial_\nu + \mathcal{O}(1) \right) (r^{-n-2} \partial_\nu^\nu V^-_{AB}) |_{\mathcal{I}^-} = \mathcal{O}(1),
\]
(4.68)
Equation (4.68) suggests that in general one should expect the appearance of logarithmic terms. Under the premise that everything is smooth up to and including the \((n-1)\)st-order, \(n \geq 1\), logarithmic terms in the expansions in \(r\) of the \(n\)th-order transverse derivatives can appear at most in the expansion of \(\partial_n^\nu V^-_{AB}|_{\mathcal{I}^-}\). To check whether this is indeed the case, one needs to compute the expansions in \(r\) of all the other fields up to and including the order \(n + 2\): An \(r^{n+2}\)-contribution on the right-hand side of (4.68) produces log terms. The observation that log terms can in principle appear at arbitrary high orders makes the analysis cumbersome. In the following we will analyze the mechanism how logarithmic terms arise via (4.68) in more detail. In Section 7 we will provide some more explicit calculations in an asymptotically Minkowski-like conformal Gauss gauge at each order, where the asymptotic behavior of the gauge functions is fixed.

**Proposition 4.2** Consider asymptotic initial data \((W_{r^A B}|_{\mathcal{I}^-}, \Xi_{AB}^{(1)}, M, N, L_A, (r^{(n+2,n)}_{AB})_{n \geq 0})\)\(^7\) for the GCFE on \(\mathcal{I}^-\) in a weakly asymptotically Minkowski-like conformal Gauss gauge (3.80)-(3.81). Then the restrictions to \(\mathcal{I}^-\) of all the fields \((\partial_\nu^\nu \nu^\nu, \partial_\nu^\nu \gamma^\nu, \partial_\nu^\nu L_{ij}, \partial_\nu^\nu W_{ijk}^\nu)\), \(n \in \mathbb{N}\), admit smooth extensions through \(I^-\) if and only if this \(\Xi_{AB}^{(1)} = 0\), the no-logs conditions (4.30) and (4.34) are fulfilled by the radiation field, or, equivalently, \(W_{r^A B}|_{\mathcal{I}^-} = \mathcal{O}(r)\), and (4.68) does not produce log terms \(\forall n \geq 1\).

### 4.3 No-logs condition for \(V^-_{AB}\)

The no-logs condition (4.34) for \(V^-_{AB}|_{\mathcal{I}^-}\) arises as Laplace-like equation on the expansion coefficient \(W^{(0)}_{r^A B}|_{\mathcal{I}^-}\) (equivalently \(\Xi_{AB}^{(k)}\)) of the radiation field. This leads to the question whether also in higher orders the no-logs condition for \(\partial_n^\nu V^-_{AB}|_{\mathcal{I}^-}\) can be read as a Laplace equation for \(W^{(n)}_{r^A B}|_{\mathcal{I}^-}\), or, alternatively, \(\Xi_{AB}^{(n+4)}\). To get some insights, set
\[
f^{(m,n)} := \frac{1}{m! n!} r^m \partial_\nu^n f |_{I^-}.
\]
(4.69)
Moreover, we write
\[
f = O_{\Xi}(n)
\]
(4.70)
if the function \(f\) is smooth at \(I^-\) and depends only on \(\Xi_{AB}^{(k)}\) with \(k \leq n\) and possibly the gauge functions and the integration functions \(M, N, L_A\) and \(c_{AB}^{(k+2,k)}\) but not on \(\Xi_{AB}^{(k)}\) with \(k \geq n + 1\). In this section it is convenient to express everything in terms of \(\Xi_{AB}\) rather than \(W_{r^A B}\).

From the constraint equations derived in Appendix A.1 we deduce that only the following

---

\(^7\)The \(c_{AB}^{(n+2,n)}\)'s are integration functions on \(I^-\) which arise from the \(\partial_n^\nu V^-_{AB}|_{\mathcal{I}^-}\)-equation, cf. Appendix A.2.3.
coordinate components depend on $\Xi_{\tilde{A}\tilde{B}}$

\[
(L_{\tilde{A}\tilde{B}})_{(m,0)}^{(m,0)} = -\frac{m - 1}{2} \Xi_{\tilde{A}\tilde{B}}^{(m+1)} + O_{\Xi}(m),
\]

\[
L_{\tilde{A}}^{r(m,0)} = \frac{1}{2} \Xi_{\tilde{A}}^{(m)} + O_{\Xi}(m - 1),
\]

\[
W_{r\tilde{A}\tilde{B}}^{(m,0)} = -\frac{(m + 2)(m + 3)}{4} \Xi_{\tilde{A}\tilde{B}}^{(m+4)} + O_{\Xi}(m + 3),
\]

\[
W_{r\tilde{A}\tilde{r}}^{(m,0)} = \frac{m + 1}{4} v_{\tilde{A}}^{(m+3)} + O_{\Xi}(m + 2),
\]

\[
W_{\tilde{A}\tilde{B}r}^{(m,0)} = \frac{1}{2} \Xi_{\tilde{A}\tilde{B}}^{(m+2)} + O_{\Xi}(m + 1),
\]

\[
W_{r\tilde{r}}^{(m,0)} = -\frac{1}{4} \Xi_{\tilde{A}}^{(m+2)} + O_{\Xi}(m + 1),
\]

\[
W_{A}^{r} r_{A}^{(m,0)} = \frac{1}{8(m - 2)} \left(\Delta_{\tilde{s}} - (m - 1)(m - 2) - 1)v_{\tilde{A}}^{(m+1)} - 2\Xi_{\tilde{A}}^{(m+1)} + O_{\Xi}(m),
\]

\[
(W_{A}^{r} r_{B})_{(m,0)}^{(m,0)} = \frac{1}{8(m - 3)(m - 4)} \left(\Xi_{\tilde{A}}^{(m+1)} + O_{\Xi}(m + 1) + \Xi_{\tilde{B}}^{(m+3)} + O_{\Xi}(m + 1)
\]

\[
= \frac{(m - 2)(m - 1)}{16} \Xi_{\tilde{A}\tilde{B}}^{(m)} + O_{\Xi}(m - 1).
\]

Terms with vanishing denominator, such as the first one on the right-hand side in (4.78) for $m = 3, 4$, are defined to be zero. For the frame components we then obtain (cf. the formulas in Section 2.7 & 2.8),

\[
\hat{L}_{ij}^{(m,0)} = O_{\Xi}(m + 1),
\]

\[
\hat{e}_{i j}^{(m,0)} = O_{\Xi}(m + 1),
\]

\[
e_{i}^{\mu}^{(m,0)} = O_{\Xi}(m + 1),
\]

\[
V_{i A}^{(m,0)} = -\frac{m(m + 1)}{8} \Xi_{A}^{(m+2)} + O_{\Xi}(m + 1),
\]

\[
V_{A}^{(m,0)} = \frac{m}{4} v_{A}^{(m+2)} + O_{\Xi}(m + 1),
\]

\[
W_{010}^{(m,0)} = -\frac{1}{4} \Xi_{A}^{(m+2)} + O_{\Xi}(m + 1),
\]

\[
W_{01AB}^{(m,0)} = -\frac{1}{2} \Xi_{A}^{(m+2)} + O_{\Xi}(m + 1),
\]

\[
W_{i A}^{r} r_{A}^{(m,0)} = \frac{1}{4(m - 1)} \left(\Delta_{\tilde{s}} - 1\right) v_{A}^{(m+2)} - \frac{1}{2(m - 1)} \Xi_{A}^{(m+2)} + O_{\Xi}(m + 1),
\]

\[
V_{i A}^{r} r_{A}^{(m,0)} = \frac{1}{8(m - 1)(m - 2)} \left(\Xi_{A}^{(m+1)} + O_{\Xi}(m + 1) + \Xi_{B}^{(m+3)} + O_{\Xi}(m + 1)
\]

From the evolution equations (2.72)-(2.78) we deduce by induction over $n$, and assuming that the solution is smooth up to and including the order $n - 1$,

\[
e_{i}^{\mu}^{(m,n)} = O_{\Xi}(m + 1),
\]

\[
\hat{L}_{ij}^{(m,n)} = O_{\Xi}(m + 1),
\]

\[
\hat{L}_{ij}^{(m,n)} = O_{\Xi}(m + 1).
\]

Similarly, taking the $(n - 1)$st-order $\tau$-derivatives of (2.90)-(2.94), we deduce that $W_{ijkl}^{(m,n)} = O_{\Xi}(m + 2)$, except possibly for $V_{ijkl}^{(m,n)}$, which so far does not even need to exist if logarithmic
terms appear. We want to work out the dependence on $\Xi^{(m+2)}_{AB}$ explicitly. For this the following observation is important: It follows from (5.6)-(5.9) below that in our current weakly asymptotically Minkowski gauge the Schouten tensor as well as frame and connection coefficients are $\tau$-independent on $I^-$, except for $e^{-1}|_{\tau} = -\tau$. In particular a regular $I^-$ requires the following relations on $\mathcal{F}$ for $p \leq n - 1$,

$$\partial^p \xi_{\tau}^j |_{\mathcal{F}} = O(r), \quad \text{for } p \geq 1,$$
$$\partial^p e_{\tau}^\mu |_{\mathcal{F}} = O(r), \quad \text{for } p \geq 2,$$
$$\partial^p \partial_\tau e_{\tau}^\mu |_{\mathcal{F}} = -\delta_\mu^0 + O(r),$$
$$\partial^p \partial_\tau e_{\tau}^\mu A |_{\mathcal{F}} = O(r),$$

whence most terms in the Bianchi equation do not contribute to a $\Xi^{(m+2)}_{AB}$-term. We evaluate the $(n - 1)$st-order $\tau$-derivative of (2.90)-(2.91) and (2.94), and the $n$th-order $\tau$-derivative of (2.98) for $m, n \geq 1$,

$$nW_{0101}^{(m,n)} = -\frac{1}{2} \mathcal{D}^A W_{A}^{(m,n-1)} + \frac{1}{2} \mathcal{D}^A W_{A}^{(m,n-1)} + O_\Xi(m + 1),$$
$$nW_{01AB}^{(m,n)} = \mathcal{D}_{[A} W_{B]}^{+,(m,n-1)} + O_\Xi(m + 1),$$
$$nV_{AB}^{+,(m,n)} = \frac{1}{2} (n + m - 1) V_{AB}^{+,(m,n-1)} + \frac{1}{2} (\mathcal{D}_A W_{B}) - (m,n-1) + O_\Xi(m + 1),$$

$$(n - m + 2) W_{A}^{-(m,n)} = 2 \mathcal{D}_B V_{AB}^{+(m,n)} + O_\Xi(m + 1).$$

We further evaluate the $n$th-order $\tau$-derivative of (5.39) and the $(n - 1)$st-order $\tau$-derivative of (5.41) below (which arise as linear combinations of (2.90)-(2.99))

$$(n - m + 2) V_{AB}^{-(m,n)} = - (\mathcal{D}_{[A} W_{B]}^{-(m,n)})_{\tau} + O_\Xi(m + 1),$$
$$nW_{A}^{+(m,n)} = - \mathcal{D}_B V_{AB}^{-,(m,n-1)} + \frac{1}{2} (n - m - 2) W_{A}^{+(m,n-1)} + O_\Xi(m + 1).$$

These equations can be decoupled to provide recursive formulas for various components of the transverse derivatives of the rescaled Weyl tensor at $I^-$,

$$(n - m + 1) W_{A}^{-(m,n)} = \frac{1}{2n} (\Delta_\tau + (m - n)(m - n + 1) - 1) W_{A}^{-(m,n-1)} + O_\Xi(m + 1),$$
$$(n - m - 2) \mathcal{D}_B V_{AB}^{+(m,n)} = \frac{1}{2n} (\Delta_\tau + (m - n)(m - n + 1) - 1) \mathcal{D}_B V_{AB}^{+(m,n-1)} + O_\Xi(m + 1),$$
$$m - n - 1) W_{A}^{+(m,n)} = - \frac{1}{2n} (\Delta_\tau + (m - n)(m - n + 1) - 1) W_{A}^{+(m,n-1)} + O_\Xi(m + 1),$$
$$m - n - 2) \mathcal{D}_B V_{AB}^{-(m,n)} = - \frac{1}{2n} (\Delta_\tau + (m - n)(m - n + 1) - 1) \mathcal{D}_B V_{AB}^{-(m,n-1)} + O_\Xi(m + 1).$$

The no-logs condition for $\partial^\mu V_{AB}^{+}$ takes the form

$$\mathcal{D}_{[A} W_{B]}^{+(n+2,n)} = O_\Xi(n + 3),$$

its divergence reads

$$\Delta_\tau + 1) W_{A}^{+(n+2,n)} = O_\Xi(n + 3).$$
In the special case where \( m = n + 2 \) we express \( W_A^{(n+2,n)} \) in terms of the initial data on \( \mathcal{I}^- \). Using (4.101) we obtain
\[
W_A^{(n+2,n)} = - \frac{1}{2n} (\Delta + 5) W_A^{(n+2,n-1)} + O_\Xi(n + 3)
\]
\[
= \frac{1}{8n(n-1)} (\Delta + 5)(\Delta + 11) W_A^{(n+2,n-2)} + O_\Xi(n + 3)
\]
\[
= \ldots
\]
\[
= \frac{(-1)^n}{2n(n-1)!} \prod_{\ell=2}^{n+1} (\Delta + \ell(\ell + 1) - 1) W_A^{(n+2,0)} + O_\Xi(n + 3)
\]
The no-logs condition (4.104) therefore adopts the form
\[
\prod_{\ell=1}^{n+1} (\Delta + \ell(\ell + 1) - 1) ((\Delta - 1)v_A^{(n+4)} - 2\partial_A \partial_B v_B^{(n+4)}) = O_\Xi(n + 3).
\]
Curl and divergence read
\[
\prod_{\ell=0}^{n+1} (\Delta + \ell(\ell + 1)) \partial_A v_A^{(n+4)} = O_\Xi(n + 3), \tag{4.105}
\]
\[
\prod_{\ell=0}^{n+1} (\Delta + \ell(\ell + 1)) (\epsilon^{AB} \partial_B v_B^{(n+4)}) = O_\Xi(n + 3). \tag{4.106}
\]
We may regard the no-logs condition on \( \partial_A^2 V_{AB} \) as a condition on the \( (n+4) \)th-order expansion coefficient of \( \Xi_{AB} \) (equivalently, on the \( n \)th-order expansion coefficient of the radiation field \( W_{A,B} \)). In general, though, this Laplace-like equation does not need to admit a solution: By construction from a symmetric trace-free tensor, the right-hand sides do not contain \( \ell = 0 \), 1-spherical harmonics in their decomposition (cf. (3.77)-(3.78)). Nonetheless, they may contain spherical harmonics with \( 2 \leq \ell \leq n + 1 \) which straightaway suppress the existence of a solution.

If a solution exists, i.e. if and only if no such spherical harmonics arise, there is the freedom to choose the spherical harmonics in the harmonics decomposition of the Hodge decomposition functions \( \Xi^{(n+4)} \) and \( \Xi^{(n+4)} \) of \( \Xi^{(n+4)} \) up to and including the order \( \ell = n + 1 \).

In Section 7 we will show that for a more restricted class of gauge functions a radiation field which vanishes at any order at \( I^- \) satisfies the no-logs conditions (4.105)-(4.106) for any \( n \). However, it is not clear to us whether a radiation field with a non-trivial expansion at \( I^- \) exists which fulfills the no-logs conditions, i.e. where its asymptotic expansion is adjusted in such a way that the right-hand sides of (4.105)-(4.106) do not contain spherical harmonics with \( 2 \leq \ell \leq n + 1 \). One might expect this to be very restrictive, and, if possible at all, might impose restriction not only on \( \Xi_{AB} \) but also on the integration functions such as the mass aspect \( M \) etc. We will discuss this more detailed in Section 9, where it is shown that the radiation does need to have a trivial expansion at least for constant \( M \) and vanishing \( N \).

So far we have not analyzed the impact of the gauge functions, which one might think could be employed to get rid of the disturbing spherical harmonics. However, when computing the 0th- 1st-order transverse derivatives we have seen that the gauge function drop out, and in Section 6 we will show that logarithmic terms cannot be eliminated by appropriately adjusted gauge functions.
5 Appearance of log terms: Approaching $I^-$ from $I$

In the previous sections we have analyzed the appearance of logarithmic terms when approaching the critical set $I^-$ from $\mathcal{I}^-$. The aim of this section is to carry out a corresponding analysis when approaching $I^-$ from the cylinder $I$.

Our goal is as follows: We assume that we have been given, in a weakly asymptotically Minkowski-like conformal Gauss gauge, a smooth solution of the GCFE which admits a smooth $\mathcal{I}^-$ and a smooth spatial infinity $I$. We have already seen above that, in general, solutions cannot be expected to be smooth at the critical set $I^-$ where $I$ and $\mathcal{I}^-$ intersect, due to the appearance of logarithmic terms. We therefore aim to extract conditions on the initial data which are compatible with smoothness at $I^-$ of all the relevant fields (and radial derivatives thereof) when approaching $I^-$ from $I$, under the assumption that the initial data for the transport equations on $I$ are induced by the limit of the corresponding fields on $\mathcal{I}^-$ to $I^-$. 

5.1 Solution of the inner equations on $I$ for connection coefficients, Schouten tensor and frame field

We want to solve the transport equations (3.50)-(3.56) for connection coefficients, Schouten tensor and frame field on the cylinder $I$ (in our setting the cylinder “touches” $\mathcal{I}^-$ at $I^- = \{ \tau = -1, r = 0 \}$). By assumption, the initial data for the transport equations are determined by taking the limit of the corresponding fields on $\mathcal{I}^-$ to $I^-$. It follows from Section 4.1.1 that

\[
\begin{align*}
    e^r_1 |_{I^-} &= 1, & e^r_1 |_{I^-} &= 0, & e^A_1 |_{I^-} &= 0, \\
    e^r_A |_{I^-} &= 0, & e^r_A |_{I^-} &= 0, & e^A_A |_{I^-} &= \hat{e}^A_A, \\
    \hat{\Gamma}^1_{1j} |_{I^-} &= \delta^i_j, & \hat{\Gamma}^a_{a0} |_{I^-} &= 0, & \hat{\Gamma}^A_{A1} |_{I^-} &= 0, & \hat{\Gamma}^B_{A1} |_{I^-} &= \delta^B_A, & \hat{\Gamma}^C_{AB} |_{I^-} &= \hat{\Gamma}^C_{AB}, \\
    \hat{L}_{ij} |_{I^-} &= 0.
\end{align*}
\]

(5.1) (5.2) (5.3) (5.4)

Although the equations (3.50)-(3.56) are not linear, they can be solved explicitly due to the fact that the initial data are almost trivial: We observe that (3.51) and (3.52), are decoupled from the other ones. Since the initial data for these equations vanish we conclude $\hat{\Gamma}^a_{a0}$ and $\hat{L}_{ab}$ vanish on $I$. It then follows from (3.50) that $\hat{L}_{a0}$ vanishes as well. The remaining equations,

\[
\begin{align*}
    \partial_\tau \hat{\Gamma}^1_{a} |_{I} &= 0, & \partial_\tau \hat{\Gamma}^A_{A} |_{I} &= 0, & \partial_\tau \hat{\Gamma}^B_{C} |_{I} &= 0, & \partial_\tau \hat{\Gamma}^a_{a} |_{I} &= -\hat{\Gamma}^0_{a0} \delta^0_a,
\end{align*}
\]

(5.5)

can then be straightforwardly integrated. Altogether, we obtain the following solution

\[
\begin{align*}
    e^r_1 |_{I} &= -\tau, & e^r_1 |_{I} &= 0, & e^A_1 |_{I} &= 0, \\
    e^r_A |_{I} &= 0, & e^r_A |_{I} &= 0, & e^A_A |_{I} &= \hat{e}^A_A, \\
    \hat{\Gamma}^1_{1j} |_{I} &= \delta^i_j, & \hat{\Gamma}^a_{a0} |_{I} &= 0, & \hat{\Gamma}^A_{A1} |_{I} &= 0, & \hat{\Gamma}^B_{A1} |_{I} &= \delta^B_A, & \hat{\Gamma}^C_{AB} |_{I} &= \hat{\Gamma}^C_{AB}, \\
    \hat{L}_{ij} |_{I} &= 0.
\end{align*}
\]

(5.6) (5.7) (5.8) (5.9)

Recall that in a conformal Gauss gauge the relations $\hat{\Gamma}^0_{ij} = 0$, $\hat{L}_{0i} = 0$ and $e^{\mu} = \delta^{\mu}_{\tau}$ hold globally.

5.2 Solution of the Bianchi equation on $I$

Next, let us analyze the Bianchi equation for the rescaled Weyl tensor on the cylinder $I$ at spatial infinity. Although this is not necessary for our purposes, let us analyze the full system.
Evaluation of (2.90)-(2.99) on \( I \) using (5.6)-(5.9) yields equations which can be written as

\[
(1 - \tau^2)\partial_\tau W_+^A |_{I} = -W_+^A - 2\mathcal{D}B V_{AB}, \\
(1 + \tau)\partial_\tau W_+^A |_{I} = W_-^A + 2\mathcal{D}B V_{AB}, \\
(1 - \tau^2)\partial_\tau W_{0101} |_{I} = -\frac{1}{2}(1 + \tau)\mathcal{D}A W_+^A + \frac{1}{2}(1 - \tau)\mathcal{D}A W_-^A, \\
(1 - \tau^2)\partial_\tau W_{01AB} |_{I} = (1 + \tau)\mathcal{D}[A W_+^B] + (1 - \tau)\mathcal{D}[A W_-^B], \\
\partial_\tau[(1 - \tau)^2V_+^A] |_{I} = -(1 + \tau)(\mathcal{D}[A W_+^B])_H, \\
\partial_\tau[(1 + \tau)^2V_-^A] |_{I} = -(1 + \tau)(\mathcal{D}[A W_+^B])_H,
\]

and

\[
(1 + \tau)\mathcal{D}A W_+^A |_{I} = -(1 - \tau)\mathcal{D}A W_-^A, \\
(1 + \tau)\mathcal{D}[A W_+^B] |_{I} = -(1 - \tau)\mathcal{D}[A W_-^B], \\
\tau W_+^A |_{I} = \frac{1}{2}(1 - \tau)\mathcal{D}B U_{AB} - (1 + \tau)\mathcal{D}B V_{AB}, \\
\tau W_-^A |_{I} = -\frac{1}{2}(1 + \tau)\mathcal{D}B U_{BA} + (1 - \tau)\mathcal{D}B V_{AB}.
\]

The subsystem (5.10)-(5.15) provides transport equations for all independent components of the rescaled Weyl tensor on \( I \). The subsystem (5.16)-(5.19) can be regarded as the constraint part of the Bianchi system on \( I \). A straightforward computation shows that the constraint equations are preserved under the evolution of (5.10)-(5.15), and therefore merely need to be satisfied initially at \( I^- \).

We want to decouple the evolution equations. Differentiation of the equations for \( W_+^A \) by \( \tau \) yields with (5.12)-(5.13) and (5.16)-(5.19)

\[
(((1 - \tau^2)\partial_\tau^2 - \Delta_\tau \pm 2(1 + \tau)\partial_\tau + 1)W_+^A |_{I} = 0.
\]

Let us also take into account that the constraints merely need to be satisfied at \( I^- \) where they read

\[
\mathcal{D}A W_+^A |_{I^-} = 0, \quad \mathcal{D}[A W_+^B] |_{I^-} = 0, \quad W_-^A |_{I^-} = -2\mathcal{D}B V_{AB}, \quad W_+^A |_{I^-} = -\mathcal{D}B U_{AB}.
\]

On \( I^- \cong S^2 \) this can only be satisfied if \( W_-^A \) and \( V_{AB}^+ \) vanish there altogether. We conclude that the system (5.10)-(5.19) is equivalent to the following one,

\[
(((1 - \tau^2)\partial_\tau^2 - \Delta_\tau + 2(1 - \tau)\partial_\tau + 1)W_+^A |_{I} = 0, \\
(((1 - \tau^2)\partial_\tau^2 - \Delta_\tau - 2(1 + \tau)\partial_\tau + 1)W_+^A |_{I} = 0,
\]

\[
(1 - \tau^2)\partial_\tau W_{0101} |_{I} = -\frac{1}{2}(1 + \tau)\mathcal{D}A W_+^A + \frac{1}{2}(1 - \tau)\mathcal{D}A W_-^A, \\
(1 - \tau^2)\partial_\tau W_{01AB} |_{I} = (1 + \tau)\mathcal{D}[A W_+^B] + (1 - \tau)\mathcal{D}[A W_-^B], \\
\partial_\tau[(1 - \tau)^2V_+^A] |_{I} = -(1 + \tau)(\mathcal{D}[A W_+^B])_H, \\
\partial_\tau[(1 + \tau)^2V_-^A] |_{I} = -(1 + \tau)(\mathcal{D}[A W_+^B])_H.
\]
and
\[ W^+_{[\text{I}^-]} = 0, \quad \partial_\tau W^+_{[\text{I}^-]} = \frac{1}{2} \mathcal{D}^B U_{BA} = \mathfrak{M}_A, \]  
(5.28)
\[ W^-_{[\text{I}^-]} = -\mathcal{D}^B U_{AB} = -2\mathfrak{A}_A, \quad \lim_{\tau \to -1} [(1 + \tau) W^+_{[\text{I}^-]} = 0, \]  
(5.29)
\[ V^+_{AB}[\text{I}^-] = 0, \]  
(5.30)
with
\[ \mathfrak{A}_A \equiv \mathcal{D}_A M + \epsilon_A^B \mathcal{D}_B N, \quad \mathfrak{M}_A \equiv \mathcal{D}_A M - \epsilon_A^B \mathcal{D}_B N. \]  
(5.31)
The data at \( \text{I}^- \) can be computed from (4.57)-(4.62) by continuity at \( \text{I}^- \). The additional conditions in (5.28)-(5.29) are needed since we have replaced the first-order equations for \( W^\pm_A \) by second-order ones. The analysis in Section 5.6 below shows that these are indeed the “right”, i.e. freely prescribable, data. It is further shown there that the solutions are regular at \( \text{I}^- \). They admit the following expansions,
\[ W^-_{[\text{I}^-]} = \mathfrak{M}_A M(1 + \tau) + \frac{1}{2}(\Delta_s + 1)\mathfrak{M}_A(1 + \tau)^2 + \mathcal{O}(1 + \tau)^3, \]  
(5.32)
\[ W^+_{[\text{I}^-]} = -2\mathfrak{A}_A - \frac{1}{2}(\Delta_s - 1)\mathfrak{A}_A(1 + \tau) + \mathcal{O}(1 + \tau)^2. \]  
(5.33)
We observe that once \( W^\pm_{[\text{I}^-]} \) are known, the remaining evolution equations are merely ODEs (some of them of Fuchsian type) which can be straightforwardly integrated. No logarithmic terms arise when integrating (5.24)-(5.27). We obtain the following expansions,
\[ W_{0101}[\text{I}^-] = M + \frac{1}{2} \Delta_s M(1 + \tau) + \mathcal{O}(1 + \tau)^2, \]  
(5.34)
\[ W_{01AB}[\text{I}^-] = N\epsilon_{AB} + \frac{1}{2} \Delta_s N\epsilon_{AB}(1 + \tau) + \mathcal{O}(1 + \tau)^2, \]  
(5.35)
\[ V^+_{AB}[\text{I}^-] = \frac{1}{8} (\mathcal{D}(\mathfrak{M}_B))_{tr}(1 + \tau)^2 + \mathcal{O}(1 + \tau)^3, \]  
(5.36)
\[ V^-_{AB}[\text{I}^-] = \frac{1}{2} (\mathcal{D}(\mathfrak{M}_B))_{tr} + \mathcal{O}(1 + \tau). \]  
(5.37)
The expansions are compatible at \( \text{I}^- \) with the corresponding ones computed on \( \mathcal{I}^- \). In general, a solution \( V^-_{AB} \) to (5.27) will be unbounded at \( \text{I}^- \) whence there is no freedom to choose initial data if one requires the solution to be bounded.

5.3 Rewriting the Bianchi equation
For the analysis on the cylinder it turns out that it is convenient to use a different subsystem of the Bianchi equation as compared to our analysis on \( \mathcal{I}^- \) to evolve the independent components of the radial derivatives of the rescaled Weyl tensor (in this paper we do not care whether the subsystem used to determine higher order derivatives forms a symmetric hyperbolic system in spacetime). The following system is obtained by taking appropriate linear combinations of
\[(2.90)-(2.99),
\]
\[
(e^{\mu_1}_1 \vartheta_{\mu} + \vartheta_{\mu}) V_{AB}^{+} = \left(\hat{\Gamma}^{0}_{1,0} + 2\hat{\Gamma}^{1}_{1,0} - \hat{\Gamma}^{0}_{1,0} - \hat{\Gamma}^{1}_{1,0} V_{AB}^{+} - \left(2\hat{\Gamma}^{0}_{1,0} - \hat{\Gamma}^{0}_{1,1} \right) V_{AB}^{-}\right) + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} + \hat{\Gamma}^{1}_{1,0} V_{AB}^{-}\right) + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} - \hat{\Gamma}^{0}_{1,1} \right) V_{AB}^{-},
\]
\[(5.38)\]
\[
(e^{\mu_1}_1 \vartheta_{\mu} - \vartheta_{\mu}) V_{AB}^{-} = - e^{\alpha}_1 \vartheta_{\alpha} V_{AB}^{-} - \left(\hat{\Gamma}^{0}_{1,0} + 2\hat{\Gamma}^{1}_{1,0} + \hat{\Gamma}^{0}_{1,1} \right) V_{AB}^{-} + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} - \hat{\Gamma}^{0}_{1,1} \right) V_{AB}^{-},
\]
\[(5.39)\]
\[
(\vartheta_{\mu} + e^{\mu_1}_1 \vartheta_{\mu}) W_{A}^{+} = - \left(2\hat{\Gamma}^{0}_{1,0} - 4\hat{\Gamma}^{0}_{1,0} - 2\hat{\Gamma}^{1}_{1,0} - \hat{\Gamma}^{1}_{1,0} \right) W_{A}^{+} + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} + \hat{\Gamma}^{1}_{1,0} \right) W_{A}^{+},
\]
\[(5.40)\]
\[
(\vartheta_{\mu} + e^{\mu_1}_1 \vartheta_{\mu}) W_{0101} = (1+\tau) \left(-\frac{1}{2} \hat{\Gamma}^{A}_{0} + \frac{1}{2} \hat{\Gamma}^{A}_{0} \right) W_{A}^{+} + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} - \hat{\Gamma}^{1}_{1,0} \right) W_{A}^{+} + \frac{3}{2} \left(\hat{\Gamma}^{A}_{0} + \hat{\Gamma}^{A}_{0} \right) W_{A}^{+},
\]
\[(5.41)\]
\[
(\vartheta_{\mu} + e^{\mu_1}_1 \vartheta_{\mu}) W_{01AB} = (1+\tau) \left(-\frac{1}{2} \hat{\Gamma}^{A}_{0} + \frac{1}{2} \hat{\Gamma}^{A}_{0} \right) W_{B}^{+} + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} - \hat{\Gamma}^{1}_{1,0} \right) W_{B}^{+} + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} + \hat{\Gamma}^{0}_{1,0} \right) W_{B}^{+},
\]
\[(5.42)\]
\[
(\vartheta_{\mu} + e^{\mu_1}_1 \vartheta_{\mu}) W_{01AB} = (1+\tau) \left(-\frac{1}{2} \hat{\Gamma}^{A}_{0} + \frac{1}{2} \hat{\Gamma}^{A}_{0} \right) W_{B}^{+} + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} - \hat{\Gamma}^{1}_{1,0} \right) W_{B}^{+} + \frac{3}{2} \left(\hat{\Gamma}^{0}_{1,0} + \hat{\Gamma}^{0}_{1,0} \right) W_{B}^{+},
\]
\[(5.43)\]
Taking radial derivatives of (5.38)-(5.43) and evaluating them on I gives the desired equations. Note that the equations for $\partial_{\mu}^{n} W_{0101}$ and $\partial_{\mu}^{n} W_{01AB}$ are algebraic (cf. (5.6)), supposing that all lower order derivatives are known and supposing that $n \geq 1$. For $n = 0$ (5.42)-(5.43) need to be replaced by e.g. (2.90)-(2.91), and this case has already been treated in the previous section.

5.4 First-order radial derivatives

Let us consider the case $n = 1$ for the first-order radial derivatives on I explicitly. For this, we differentiate the evolution equations (2.72)-(2.78) and (5.38)-(5.43) by $r$. Taking their restrictions to the cylinder and using the results of Section 5.1 & 5.2 we obtain transport equations on I for $(e^{\mu_1}_{1}, \tilde{\Gamma}_{1,0}^{A}, \tilde{\Gamma}_{1,1}^{A}, W_{ijkl})$. 

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First of all note that with regard to (2.72)-(2.78)
\[ \partial_r \Theta|_I = 1 - \tau^2, \quad b_0|_I = 0, \quad (5.44) \]
\[ \partial_r \theta|_I = -2\tau, \quad \partial_r b_1|_I = 2, \quad \partial_r b_A|_I = 0. \quad (5.45) \]

For the Schouten tensor we obtain the following set of equations,
\[ \partial_r \partial_r \hat{L}_{10}|_I = -4M + \mathcal{O}(1 + \tau), \quad (5.46) \]
\[ \partial_r \partial_r \hat{L}_{11}|_I = -4M + \mathcal{O}(1 + \tau), \quad (5.47) \]
\[ \partial_r \partial_r \hat{L}_{1A}|_I = \mathcal{O}(1 + \tau), \quad (5.48) \]
\[ \partial_r \partial_r (\hat{L}_{A0} - \hat{L}_{A1})|_I = \mathcal{O}(1 + \tau), \quad (5.49) \]
\[ \partial_r \partial_r (\hat{L}_{A0} + \hat{L}_{A1})|_I = 4\mathcal{M}_A + \mathcal{O}(1 + \tau), \quad (5.50) \]
\[ \partial_r \partial_r \hat{L}_{AB}|_I = 2M\eta_{AB} + 2N\epsilon_{AB} + \mathcal{O}(1 + \tau). \quad (5.51) \]
Integration yields (the integration functions are determined by (4.53)-(4.56))
\[ \partial_r \hat{L}_{10}|_I = -4M(1 + \tau) + \mathcal{O}(1 + \tau)^2, \quad (5.52) \]
\[ \partial_r \hat{L}_{11}|_I = -4M(1 + \tau) + \mathcal{O}(1 + \tau)^2, \quad (5.53) \]
\[ \partial_r \hat{L}_{1A}|_I = \mathcal{O}(1 + \tau)^2, \quad (5.54) \]
\[ \partial_r (\hat{L}_{A0} - \hat{L}_{A1})|_I = \mathcal{P}_A\nu^{(0)}_r + \mathcal{P}_A f^{(1)}_A - f^{(1)}_A + \mathcal{O}(1 + \tau)^2, \quad (5.55) \]
\[ \partial_r (\hat{L}_{A0} + \hat{L}_{A1})|_I = v^{(2)}_A + f^{(1)}_A - \mathcal{P}_A f^{(1)}_A + \mathcal{P}_A \nu^{(0)}_A + 4\mathcal{M}_A(1 + \tau) + \mathcal{O}(1 + \tau)^2, \quad (5.56) \]
\[ \partial_r \hat{L}_{AB}|_I = -\mathcal{P}_A f^{(1)}_B - \frac{1}{2} \mathcal{S}^{(2)}_{AB} - f^{(1)}_A \eta_{AB} + 2(M\eta_{AB} + N\epsilon_{AB})(1 + \tau) + \mathcal{O}(1 + \tau)^2. \quad (5.57) \]

For the connection coefficients we end up with the following equations
\[ \partial_r \partial_r \hat{\Gamma}^0_{11}|_I = \mathcal{O}(1 + \tau), \quad (5.58) \]
\[ \partial_r \partial_r \hat{\Gamma}^0_{1A}|_I = \mathcal{O}(1 + \tau), \quad (5.59) \]
\[ \partial_r \partial_r \hat{\Gamma}^0_{A1}|_I = \partial_r \hat{L}_{A1}|_I + \mathcal{O}(1 + \tau), \quad (5.60) \]
\[ \partial_r \partial_r \hat{\Gamma}^0_{AB}|_I = \partial_r \hat{L}_{AB}|_I + \mathcal{O}(1 + \tau), \quad (5.61) \]
\[ \partial_r \partial_r \hat{\Gamma}^1_{11}|_I = -\partial_r \hat{\Gamma}^1_{01}|_I - \mathcal{O}(1 + \tau), \quad (5.62) \]
\[ \partial_r \partial_r \hat{\Gamma}^1_{1A}|_I = \partial_r \hat{\Gamma}^1_{0A}|_I - \mathcal{O}(1 + \tau), \quad (5.63) \]
\[ \partial_r \partial_r \hat{\Gamma}^1_{A1}|_I = -\partial_r \hat{\Gamma}^1_{A0}|_I - \partial_r \hat{L}_{A0}|_I - \mathcal{O}(1 + \tau), \quad (5.64) \]
\[ \partial_r \partial_r \hat{\Gamma}^1_{AB}|_I = \partial_r \hat{\Gamma}^1_{AB}|_I + \mathcal{O}(1 + \tau), \quad (5.65) \]
\[ \partial_r \partial_r \hat{\Gamma}^2_{1B}|_I = -\partial_r \hat{\Gamma}^2_{1B}|_I - \mathcal{O}(1 + \tau), \quad (5.66) \]
\[ \partial_r \partial_r \hat{\Gamma}^2_{A|_I} = -\hat{\Gamma}^2_{B|_I} \partial_r \hat{\Gamma}^2_{A|_I} + \partial_r \hat{L}_{A0}|_I + \mathcal{O}(1 + \tau). \quad (5.67) \]
The solutions have the expansions (the integration functions follow from (4.43)-(4.52))
\[
\begin{align*}
\partial_r \hat{\Gamma}^0_{11}|_I &= -f^{(1)}_A + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^0_{A1}|_I &= -f^{(1)}_A + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^{0}_{A1}|_I &= \mathcal{D}_A \nu^{(0)}_A - \nu^{(1)}_A + \left(\frac{1}{2} \hat{f}^{(2)}_A + f^{(1)}_A - \mathcal{D}_A f^{(1)}_A\right)(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^1_{A1}|_I &= \frac{1}{2} \hat{e}^{(2)}_{AB} - \mathcal{O}(f^{(1)}_B + \frac{1}{2} \hat{e}^{(2)}_{AB} + f^{(1)}_A \eta_{AB})(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^{1}_{A1}|_I &= f^{(1)}_A(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^1_{1A}|_I &= f^{(1)}_A - f^{(1)}_A(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^{1}_{1A}|_I &= f^{(1)}_A(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^{1}_{B1}|_I &= \frac{1}{2} \hat{e}^{(2)}_{A1} + \nu^{(1)}_A(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^{1}_{B1}|_I &= -\frac{1}{2} \hat{e}^{(2)}_{AB} - f^{(1)}_A \eta_{AB} + \frac{1}{2} \hat{e}^{(2)}_{AB}(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^{1}_{C1}|_I &= \mathcal{D}_A \nu^{(0)}_A f^{(1)}_C + \frac{1}{2} \hat{e}^{(2)}_{C} (1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \hat{\Gamma}^{1}_{C1}|_I &= 2 \delta^{B}(A f^{(1)}_B - \eta_{AC} f^{(1)}_B) - \frac{1}{2} \hat{e}^{(2)}_{B} \nu^{(0)}_A + \mathcal{D}_A f^{(1)}_C (1 + \tau).
\end{align*}
\]  

In fact, we have
\[
\partial_r \partial_r (\hat{\Gamma}^0_{11} + \hat{\Gamma}^1_{1A})|_I = -f^{(1)}_A + \mathcal{O}(1 + \tau)^2, 
\]
whence
\[
\partial_r (\hat{\Gamma}^0_{11} + \hat{\Gamma}^1_{1A})|_I = -f^{(1)}_A(1 + \tau) + \mathcal{O}(1 + \tau)^3,
\]
which will be relevant below. The equations for the frame coefficients read
\[
\begin{align*}
\partial_r \partial_r e^r|_I &= -f^{(1)}(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \partial_r e^A|_I &= 0, \\
\partial_r \partial_r e^r|_I &= f^{(1)}(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \partial_r e^A|_I &= \left(\nu^{(1)}_A - \mathcal{D}_A \nu^{(0)}_A - f^{(1)}_A\right)(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r \partial_r e^r|_I &= 0, \\
\partial_r \partial_r e^A|_I &= -\frac{1}{2} \hat{e}^{(2)}_{B} e^A_B + \mathcal{O}(1 + \tau),
\end{align*}
\]
from which we obtain the solutions (with data induced by (4.41)-(4.42))
\[
\begin{align*}
\partial_r e^r|_I &= -f^{(1)}(1 + \tau)^2 + \mathcal{O}(1 + \tau)^3, \\
\partial_r e^r|_I &= 1, \\
\partial_r e^A|_I &= f^{(1)}(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r e^r|_I &= \left(\nu^{(1)}_A - \mathcal{D}_A \nu^{(0)}_A - f^{(1)}_A\right)(1 + \tau) + \mathcal{O}(1 + \tau)^2, \\
\partial_r e^r|_I &= 0, \\
\partial_r e^A|_I &= -\frac{1}{2} \hat{e}^{(2)}_{B} e^A_B(1 + \tau) + \mathcal{O}(1 + \tau)^2.
\end{align*}
\]
From the equations (5.38)-(5.41) for the rescaled Weyl tensor we obtain

\[
\partial_r [(1 - \tau) \partial_r V_{A B}^+] I = \left(3(M \mathcal{D}(A f_{B}^{(1)}) + N \mathcal{D} f_{(A C} C_{B)})^{\tau} - 5(f_{(A}^{(1)} \mathcal{M} B)^{\tau})^{(1 + \tau)} \right) + (\mathcal{D} f_{(A}^{(1)} \mathcal{M} B)^{\tau})^{(1 + \tau)}^2, \tag{5.92}
\]

\[
\partial_r [(1 + \tau) \partial_r V_{A B}^-] I = - (\mathcal{D}(A \partial_r W_{B}^{+})^{\tau} + \mathcal{D}(1), \tag{5.93}
\]

\[
(1 - \tau) \partial_r \partial_r W_{A}^{+} I = - 2 \partial_r W_{A}^{+} - 2 \mathcal{D}^{B} \partial_r W_{A B}^{+} + \mathcal{D}(1), \tag{5.94}
\]

\[
(1 + \tau) \partial_r \partial_r W_{A}^{-} I = 2 \partial_r W_{A}^{-} + 2 \mathcal{D}^{B} \partial_r W_{A B}^{-} + 3 f^{B}(M \eta_{A B} - N \epsilon_{A B})(1 + \tau) + 2 \left(\mathcal{D} \partial_r W_{B}^{+} + 3 \mathcal{M}^{B} \mathcal{D}_{A f_{B}^{(1)}} + \mathcal{M}^{A} \mathcal{D}_{B f_{B}^{(1)}} \right)(1 + \tau)^2 + \mathcal{D}(1 + \tau)^2. \tag{5.95}
\]

Taking the divergence of the first two equations and inserting the result into the latter two yields decoupled equations for \( \partial_r W_{A}^{+} I \),

\[
[(1 - \tau)^2 \partial_r^2 - 2 \partial_r - (\Delta_A - 1)] \partial_r W_{A}^{-} I = 6 f^{(1)}(M \eta_{A B} - N \epsilon_{A B}) + 3(\Delta_A - 1) \left(\mathcal{D} f_{(A}^{(1)} \mathcal{M} B)^{\tau} \right)(1 + \tau) + \mathcal{D}(1 + \tau)^2, \tag{5.96}
\]

\[
[(1 - \tau^2) \partial_r^2 + 2 \partial_r - (\Delta_A - 1)] \partial_r W_{A}^{+} I = \mathcal{D}(1). \tag{5.97}
\]

The analysis in Section 5.6 shows that the solutions are regular at \( I^- \) for any data \( \partial_r W_{A}^{-} I^- \), \( \partial_r^2 \partial_r W_{A}^{-} I^- \) and \( \partial_r W_{A}^{+} I^- \) (the second integration function for \( W_{A}^{+} I \) comes along with a log term and therefore needs to vanish). We have (cf. (4.59)-(4.60))

\[
\partial_r W_{A}^{-} I^- = 0, \quad \partial_r W_{A}^{+} I^- = 2 L_A, \tag{5.98}
\]

the datum \( \partial_r^2 \partial_r W_{A}^{-} I^- \) is irrelevant for our purposes. Moreover, the solutions admit an expansion of the form

\[
\partial_r W_{A}^{-} I = - 3 f^{(1)}(M \eta_{A B} - N \epsilon_{A B})(1 + \tau) + \mathcal{D}(1 + \tau)^2, \tag{5.99}
\]

\[
\partial_r W_{A}^{+} I = 2 L_A + \mathcal{D}(1 + \tau). \tag{5.100}
\]

From this we compute \( \partial_r V_{A B}^{\pm} I \). The integration function for \( \partial_r V_{A B}^{+} I \) is determined by continuity from (4.61), while that for \( \partial_r V_{A B}^{-} I \) needs to vanish in order to get a bounded solution.

\[
\partial_r V_{A B}^{+} I = \mathcal{D}(1 + \tau)^2, \tag{5.101}
\]

\[
\partial_r V_{A B}^{-} I = \left(- 2 \mathcal{D}(A L_B) - 3 \Xi^{(2)}(A \mathcal{M} B C + N \epsilon_{B C}) + 2 \nu^{(1)}(A \mathcal{M} B) - 2 \mathcal{M}^{(A} \mathcal{D}_{B) \nu^{(1)}} \right) I + \mathcal{D}(1 + \tau). \tag{5.102}
\]

Finally, the radial derivative of (5.42)-(5.43) yields,

\[
\partial_r U_{A B} I = \mathcal{D}(1 + \tau). \tag{5.103}
\]

Again, one checks that all values at \( I^- \) are in accordance with (4.57)-(4.62). We conclude:

**Lemma 5.1** Under the same hypotheses as in Proposition 4.1, no additional restrictions need to be imposed on the data, to get regular expansions of \((e_{i}^{\mu}, \Gamma_{ij}^{k}, L_{ij}, W_{ijkl}) I \) and their first order radial derivatives at \( I^- \).
5.5 Higher-order derivatives: Structure of the equations and no-logs conditions

As in Section 4.2 we want to derive the overall structure of the transport equations on the cylinder for radial derivatives of any order of the fields involved, in particular concerning the appearance of logarithmic terms. For this we assume that appropriate data have been prescribed on \( \mathcal{I}^- \) and some incoming null hypersurface which generate a smooth solution \( f = (e^\nu, \Gamma_i^j\kappa, \hat{L}_{ij}, W_{ijkl}) \) to the GCFE in a weakly asymptotically Minkowski-like conformal Gauss gauge which admits a smooth \( \mathcal{I}^- \) and a smooth cylinder \( I^- \). We further assume that all transverse derivatives \( \partial^\nu f|_{\mathcal{I}^-} \) are smooth at \( I^- \), and induce there the data for the transport equations on \( I^- \). Our goal is to analyze the smoothness of \( \partial^\nu f|_{\mathcal{I}^-} \) at \( I^- \).

Previously we have shown that no additional restrictions apart from those in Proposition 4.1 are needed for the smoothness of \( f|_{I^-} \) and \( \partial f|_{I^-} \). Let us assume now that \( \partial^\nu f|_{I^-} \) is smooth at \( I^- \) for \( 0 \leq k \leq n-1 \).

We consider the evolution equations (2.72)-(2.78), and apply \( \partial^\nu \). With (5.6)-(5.9) that yields ODEs for \((\partial^\nu e^\nu, \partial^\nu \hat{\Gamma}_i^j\kappa, \partial^\nu \hat{L}_{ij})|_{I^-}\),

\[
\begin{align*}
\partial_\tau \partial^\nu \hat{L}_{a0}|_{I^-} &= \mathcal{O}(1), \\
\partial_\tau \partial^\nu \hat{L}_{ab}|_{I^-} &= \mathcal{O}(1), \\
\partial_\tau \partial^\nu \hat{\Gamma}_a^0|_{I^-} &= \mathcal{O}(1), \\
\partial_\tau \partial^\nu \hat{\Gamma}_a^1|_{I^-} &= -\hat{\Gamma}_c^1 \partial^\nu \hat{\Gamma}_a^c|_{I^-} + \delta^1_a \partial^\nu \hat{\Gamma}_{a0}|_{I^-} + \mathcal{O}(1) = \mathcal{O}(1), \\
\partial_\tau \partial^\nu \hat{\Gamma}_a^A|_{I^-} &= -\hat{\Gamma}_c^A \partial^\nu \hat{\Gamma}_a^c|_{I^-} + \delta^A_a \partial^\nu \hat{\Gamma}_{a0}|_{I^-} + \mathcal{O}(1) = \mathcal{O}(1), \\
\partial_\tau \partial^\nu \hat{\Gamma}_A^B|_{I^-} &= -\hat{\Gamma}_D^B \partial^\nu \hat{\Gamma}_A^D|_{I^-} + \delta^B_A \partial^\nu \hat{\Gamma}_{A0}|_{I^-} + \mathcal{O}(1) = \mathcal{O}(1),
\end{align*}
\]

The ODEs can be straightforwardly integrated with initial data computed from the data on \( \mathcal{I}^- \), or rather their limit to \( I^- \). It follows that the restrictions to \( I^- \) of the \( n \)-th-order \( \tau \)-derivatives of frame field, connection coefficients and Schouten tensor are smooth at \( I^- \), supposing that this is the case for all derivatives of \( f \) up to and including order \( n-1 \),

\[
\begin{align*}
\partial^\nu \hat{L}_{a0}|_{I^-} &= \mathcal{O}(1), \\
\partial^\nu \hat{\Gamma}_a^i|_{I^-} &= \mathcal{O}(1), \\
\partial^\nu e^\nu|_{I^-} &= \mathcal{O}(1).
\end{align*}
\]

We also apply \( \partial^\nu \) to the equations (5.38)-(5.43) for the rescaled Weyl tensor, and take their restriction to \( I^- \),

\[
\begin{align*}
\partial_\tau [1 - (1 - \tau)^{2-n} \partial^\nu V^+_{AB}]|_{I^-} &= (1 - \tau)^{1-n}(\partial_\tau \partial^\nu W^+_{AB})|_{I^-} + \mathcal{O}(1), \\
\partial_\tau [(1 + \tau)^{2-n} \partial^\nu V^-_{AB}]|_{I^-} &= (1 + \tau)^{1-n}(\partial_\tau \partial^\nu W^-_{AB})|_{I^-} + \mathcal{O}(1 + \tau)^{1-n}, \\
[(1 + \tau)\partial_\tau - (n+1)]\partial^\nu W^-_{AB}|_{I^-} &= 2\partial^\nu \partial^\nu V^+_{AB} + \mathcal{O}(1), \\
[(1 - \tau)\partial_\tau + (n+1)]\partial^\nu W^+_{AB}|_{I^-} &= -2\partial^\nu \partial^\nu V^-_{AB} + \mathcal{O}(1), \\
n\partial^\nu W^-_{0101}|_{I^-} &= -\frac{1}{2}(1 + \tau)\partial^\nu \partial^\nu W^+_{A} - \frac{1}{2}(1 - \tau)\partial^\nu \partial^\nu W^+_{A} + \mathcal{O}(1), \\
n\partial^\nu W^-_{01AB}|_{I^-} &= (1 + \tau)(\partial_\tau \partial^\nu W^+_{AB}) - (1 - \tau)(\partial_\tau \partial^\nu W^-_{AB}) + \mathcal{O}(1).
\end{align*}
\]
We take the divergence of (5.114) and (5.115), and insert them into (5.116) and (5.117), respectively, to get decoupled equations

\[
(1 - \tau^2)\partial^2_t + 2((n-1)\tau - \Delta_s + n^2 - n - 1)\partial_t - (\Delta_s + n^2 - n - 1)\partial^n_\tau W^-_\ell = \mathcal{O}(1), \tag{5.120}
\]

\[
(1 - \tau^2)\partial^2_t + 2((n-1)\tau + 1)\partial_t - (\Delta_s + n^2 - n - 1)\partial^n_\tau W^+_\ell = \mathcal{O}(1). \tag{5.121}
\]

The regularity of solutions to this equation at \( I^- \) is discussed in Section 5.6. For the time being, let us assume that the data are such that the solutions are smooth at \( I^- \). Then (5.114) can be integrated for initial data induced by \( V^+_{AB}[\mathcal{I}] \). The solution \( \partial^n_\tau V^+_{AB}[\mathcal{I}] \) will be smooth at \( I^- \). The equations (5.118)-(5.119) determine \( \partial^n_\tau W^+_{0101}[\mathcal{I}] \) and \( \partial^n_\tau W^+_{01AB}[\mathcal{I}] \) algebraically and the components will be smooth at \( I^- \), as well.

It remains to compute \( \partial^n_\tau V^-_{AB}[\mathcal{I}] \). We observe that, in contrast to \( n = 0, 1 \), for \( n \geq 2 \) the solution to (5.115),

\[
\partial_t[(1 + \tau)^{-2 + n} \partial^n_\tau V^-_{AB}[\mathcal{I}] |_{I^-} = \mathcal{O}(1 + \tau)^{1-n} \tag{5.122}
\]

will be bounded at \( I^- \) for any choice of the initial data, which are given by the integration functions \( \partial^n_\tau V^-_{AB}[\mathcal{I}^-] = c_{AB}^{n(n-2)}, \ n \geq 2 \), which can be regarded as part of the freely prescribable data, cf. Appendix A.2.3.

We further observe that, for \( n \geq 2 \), the solution will develop logarithmic terms at \( I^- \) unless the right-hand side does not have a term of order \((1 + \tau)^{-1}\) in its expansion at \( I^- \). This is another no-logs condition which needs to be imposed.

Comparing this with (4.68) we observe that (5.122) is very similar to the corresponding one \( \mathcal{I}^- \) (cf. Section 5.7),

\[
(\partial_t + \mathcal{O}(1))(r^{-n+2}\partial^n_\tau V^-_{AB}[\mathcal{J}] = \mathcal{O}(r^{-n-3}).
\]

In both cases \( \partial^n_\tau V^-_{AB}[\mathcal{I}] \) diverges at \( I^- \) for some \( k \) if logarithmic terms appear.

### 5.6 Analysis of the singular wave equation on \( I \)

We want to analyze (5.120)-(5.121), as well as (5.22)-(5.23) and (5.96)-(5.97). To deal with scalar equations we take curl and divergence. Let

\[
\phi_n^{\pm} \in \{ \mathcal{D}^A \partial^n_\tau W^\pm_A, \mathcal{A}_{AB} \partial^n_\tau W^\pm_B \}, \tag{5.123}
\]

then we are led to study the following linear PDE of Fuchsian type

\[
(1 - \tau^2)\partial^2_t + 2((n-1)\tau \pm 1)\partial_t - [\Delta_s + n(n-1)]\phi^{\pm} = g_n^{\pm}. \tag{5.124}
\]

on \([-1, 1] \times S^2\) for a given smooth source \( g_n^{\pm} \), and with \( s = d\theta^2 + \sin^2\theta d\varphi^2 \). Eventually we are interested in smooth solutions which allow a decomposition into spherical harmonics. Since, by construction, \( \phi_n^{\pm} \) and \( W_n^{\pm} \) are divergence or curl of a 1-form, their harmonic decompositions will not contain \( \ell = 0\)-spherical harmonics,

\[
\phi_n^{\pm}(\tau, \theta, \varphi) = \sum_{\ell=1}^\infty \sum_{m=-\ell}^{+\ell} \phi_{n\ell m}^{\pm}(\tau)Y_{\ell m}(\theta, \varphi), \tag{5.125}
\]

\[
g_n^{\pm}(\tau, \theta, \varphi) = \sum_{\ell=1}^\infty \sum_{m=-\ell}^{+\ell} g_{n\ell m}^{\pm}(\tau)Y_{\ell m}(\theta, \varphi), \tag{5.126}
\]

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\[
\Delta_s Y_{\ell m} = -\ell(\ell + 1)Y_{\ell m}.
\]  
(5.127)

That yields ODEs for the expansion coefficients,
\[
\left( (1 - \tau^2)\partial_\tau^2 + 2[(n - 1)\tau \pm 1]\partial_\tau + (\ell + n)(\ell - n + 1) \right) \phi^\pm_{\ell n m}(\tau) = q^\pm_{\ell n m}(\tau).
\]  
(5.128)

Set
\[
z := \frac{1 + \tau}{2}, \quad a_{n \ell} := -(\ell + n), \quad b_{n \ell} := \ell - n + 1, \quad c^\pm_n := n - 1 \mp 1,
\]
then (5.124) becomes,
\[
\left( z(1 - z)\partial_z^2 - (c^\pm_n + (a_{n \ell} + b_{n \ell} + 1)z)\partial_z - a_{n \ell}b_{n \ell}\right) \phi^\pm_{\ell n m}(z) = q^\pm_{\ell n m}(z),
\]  
(5.129)

which is a hypergeometric equation with source term which can be solved using e.g. Frobenius method, cf. e.g. [18] and the references given there. Such an equation already appears in [26], cf. also [52], where, among other things, the general solutions to its homogeneous counterpart is constructed in terms of generalized Jacobi polynomials. The coefficients there differ slightly from (5.128), because the equations are expressed in terms of different components (equations for divergence and curl of \( \partial^I \partial_\alpha V_{AB} \mid_I \) would yield identical equations). In particular it becomes clear from the discussion there, that non-smoothness of the solutions is actually due to the appearance of logarithmic terms, which will not be immediate from our subsequent discussion.

In the following let us first focus on the case \( n \geq 2 \). We are interested in the behavior at \( \tau = -1 \), i.e. at \( z = 0 \). By assumption, the source term is smooth in \( z \) and therefore admits an expansion of the form
\[
q^\pm_{\ell n m} \sim \sum_{k=0}^{\infty} q^\pm_{k\ell n m} z^k.
\]

Any smooth solution \( \phi^\pm_{\ell n m} \) admits an expansion at \( z = 0 \) of the form,
\[
\phi^\pm_{\ell n m} \sim \sum_{k=0}^{\infty} \phi^\pm_{k\ell n m} z^k.
\]

We plug it in
\[
(k + 1)(k - c^\pm_n)\phi^\pm_{(k+1)\ell n m} - \left( (k(k - 1) + k(a_{n \ell} + b_{n \ell} + 1) + a_{n \ell}b_{n \ell}\right) \phi^\pm_{k\ell n m} = q^\pm_{k\ell n m}.
\]  
(5.130)

If the solution is smooth at \( I^- \) this system needs to admit a solution. The solution is determined by regarding the system as a hierarchy of equations for \( \{\phi^\pm_{k\ell n m}\}_{k \in \mathbb{N}} \). This imposes the restriction,
\[
(\ell(\ell + 1) - 1) \phi^\pm_{c^\pm_n\ell n m} = q^\pm_{c^\pm_n\ell n m}.
\]  
(5.131)

Supposing that (5.131) holds, the solution to (5.124) will be of the form \( \phi^\pm_n = \Omega(1) \).

Let us analyze this condition in detail. We begin with \( \phi^- \). Note that \( c^-_n = n \), and that (5.131) can be written as
\[
\Delta_s \partial_\phi^\pm \phi^- = -\partial_\phi^\pm q^-.
\]  
(5.132)

The first factor in (5.131) is nonzero for all \( \ell \geq 1 \). For fixed \( n \), \( \ell \) and \( m \), \( \phi^-_{n\ell n m} \) is determined from (5.130) by solving a hierarchical system for \( \phi^\pm_{k\ell n m} \), \( 1 \leq k \leq n \), in terms of the initial data.
\(\phi_{0n\ell m}\) and the source \(q_{kn\ell m}\), \(k \leq n - 1\). Then the coefficients \(\phi_{kn\ell m}\), \(k \geq n + 1\), are determined in terms of the data \(\phi_{(n+1)\ell m}\) and the source \(q_{kn\ell m}\), \(k \geq n + 1\). The “integration functions”

\[
\phi_{0n\ell m}, \quad \phi_{(n+1)\ell m},
\]

are determined from the data on \(\mathcal{J}^-\), i.e. from the solution of the \(W_A^-\)-constraint and its \((n+1)\)st-order transverse derivative.

Next we consider condition (5.131) for \(\phi^+\). Note that \(c^+_n = n - 2\), and that (5.131) can be written as

\[
(\Delta_s + 2)\partial_I^{-2}\phi_n^{+} = -\partial_I^{-2}q_{n}^{+}.
\]

For \(n \geq 2\) the first factor in (5.131) is non-zero for \(\ell \geq 2\). In that case, again, \(\phi_{(n-2)\ell m}^{+}\) is determined from (5.130) by solving a hierarchical system for \(\phi_{kn\ell m}\), \(1 \leq k \leq n - 2\), in terms of the initial data \(\phi_{0n\ell m}^{+}\) and the source \(q_{kn\ell m}^{+}\), \(k \leq n - 3\). Then the coefficients \(\phi_{kn\ell m}^{+}\), \(k \geq n - 1\), are determined in terms of the data \(\phi_{(n-1)\ell m}^{+}\) and the source \(q_{kn\ell m}^{+}\), \(k \geq n - 1\). The “integration functions”

\[
\phi_{0n\ell m}^{+}, \quad \phi_{(n-1)\ell m}^{+},
\]

are determined by \(W_A^{+}|_{\mathcal{J}^-}\) and its \((n - 1)\)st-order transverse derivative. For \(n \geq 2\) and \(\ell = 1\) (5.131) becomes a condition on the source term,

\[
q_{(n-2)\ell m}^{+} = 0.
\]

Some consequences of the above considerations are provided by the following lemma:

**Lemma 5.2** Take \(n \geq 2\).

(i) Assume that the initial data at \(I^-\) satisfy

\[
\phi_{n}^{\pm}|_{I^-} = \sum_{\ell=1}^{n-1} \sum_{m=-\ell}^{+\ell} \phi_{0n\ell m}^{\pm} Y_{\ell m}(\theta, \phi),
\]

and that the source term satisfies

\[
q_{n}^{\pm}|_{I^-} = \sum_{\ell=1}^{n-1} \sum_{m=-\ell}^{+\ell} q_{0n\ell m}^{\pm}(\tau) Y_{\ell m}(\theta, \phi) + O(1 + \tau)^{n+1}.
\]

Then the solution is smooth at \(I^-\) if \(q_{0n\ell m}^{\pm}(\tau) = \mathcal{P}^{n-\ell-2}\), where \(\mathcal{P}^k\) denotes a polynomial in \((1 + \tau)\) of degree \(k\).

(ii) Assume that \(\phi_{n}^{\pm}|_{I^-} \neq 0\), \(q_{n}^{\pm}|_{I^-} = O(1 + \tau) c_{n}^{\pm}\) with \(\partial_{\tau}^{\pm} q_{n}|_{I^-} \neq 0\). Then the solution cannot be smooth at \(I^-\).

(iii) Assume that \(\langle \phi_{n}^{\pm}|_{I^-}, Y_{\ell m}, m_\star \rangle \neq 0\) for some \(\ell_\star \geq n\) and some \(-\ell_\star \leq m_\star \leq \ell_\star\), and \(\langle q_{n}^{\pm}|_{I^-}, Y_{\ell m}, m_\star \rangle = O(1 + \tau)^{n+1}\). Then the solution cannot be smooth at \(I^-\).

(iv) Assume that \(\langle \partial_{\tau}^{\pm} q_{n}^{\pm}|_{I^-}, Y_{1 m}, m_\star \rangle \neq 0\) for \(m_\star \in \{-1, 0, 1\}\). Then the solution cannot be smooth at \(I^-\).
Proof: (i): We need to check whether (5.131) holds. The data and the source term has then been chosen in such a way, that \( \phi_{k,n}^{+} = 0 \) and \( q_{kn}^{\pm} = 0 \) for \( \ell \geq n \) and \( k \leq c^+_n \), in particular (5.131) holds for \( \ell \geq n \) and \( \ell = 0 \). To deal with \( 1 \leq \ell \leq n - 1 \), we consider (5.130) from which we obtain

\[
\phi_{(k+1)ntm}^{\pm} = \frac{\ell}{(k+1)(k+c^{-}_n)} \phi_{kn}^{\pm} + \frac{1}{(k+1)(k-c^{-}_n)} q_{kn}^{\pm}.
\]  

(5.137)

The zeros of the numerator

\[
k(k-1) + k(a_{nt} + b_{nt} + 1) + a_{nt} b_{nt} = k(k-1) - 2k(n-1) - (\ell + n)(\ell + n - 1)
\]

are given by \( k = n + \ell \) and \( k = n - \ell - 1 \). We are only interested in those zeros where \( k \) is an integer in the interval \([0, n]\) for \( \phi^- \) and where \( k \) is an integer in the interval \([0, n - 2]\) for \( \phi^+ \). Recall that \( \ell \geq 1 \). Zeros of \( k \) in the desired range appear if and only if \( \ell \) satisfies \( 1 \leq \ell \leq n - 1 \),

\[
k_* = n - \ell - 1.
\]

In particular,

\[
\phi_{(k+1)ntm}^{\pm} = \frac{1}{(k_* + 1)(k_* - c^{-}_n)} q_{k_*ntm}^{\pm},
\]

and \( \phi_{kn}^{\pm} \) with \( k_* + 1 \leq k \leq c^+_n \) merely depends on the source \( q_n \) but not on the second initial datum. Now if the source has been chosen in such a way that \( q_{kn}^{\pm} = 0 \) for \( k_* \leq k \leq c^+_n \), it follows that \( \phi_{kn}^{\pm} = 0 \) for \( k_* + 1 \leq k \leq c^+_n \), and (5.131) is fulfilled (for \( \phi^* \) and \( \ell = 1 \), where \( k_* + 1 > c^+_n \) we have (5.136), and (5.131) holds as well).

(ii): This is straightforward.

(iii): By assumption there exist \( \ell_* \geq n \) and \( m_* \) such that \( \phi_{0nt, m_*}^{\pm} \neq 0 \). Taking also the assumption on the source term into account we deduce from (5.130)

\[
\phi_{(k+1)nt, m_*}^{\pm} = \frac{k(k-1) + k(a_{nt*} + b_{nt*} + 1) + a_{nt*} b_{nt*}}{(k+1)(k-c^{-}_n)} \phi_{kn, m_*}^{\pm},
\]

and the considerations above show that the numerator does not have integer zeros in \([0, c^+_n]\). It follows that \( \phi_{kn, m_*}^{\pm} \neq 0 \) for \( 0 \leq k \leq c^+_n \) and the smoothness condition (5.131) is violated.

(iv): That is (5.136).

Let us now consider the remaining cases where \( n = 0, 1 \). For \( n = 0 \) there is no source term, cf. (5.22)-(5.23), and (5.130) becomes

\[
(k + 1)(k + 2) \phi_{(k+1)0t}^{\pm} - \left(k(k-1) + k(\ell+1)\right) \phi_{k0t}^{\pm} = 0, \quad k \geq -1,
\]

(5.139)

\[
(k + k) \phi_{(k+1)0t}^{\pm} - \left(k(k-1) + k(\ell+1)\right) \phi_{k0t}^{\pm} = 0, \quad k \geq 0.
\]

(5.140)

We observe that \( \phi_{-10t}^{\pm} \) is an integration function which produces a divergence term at \( \Gamma^- \). Also in the second case one integration function is lost by the smoothness requirement as can be seen by evaluating (5.140) for \( k = 0 \).

In both cases, to get a smooth solution at \( \tau = -1 \) there is only one freely prescribable datum, while the other one needs to vanish,

\[
\phi_{-10t}^{\pm} = 0, \quad \phi_{00t}^{+}, \quad \phi_{00t}^{-} = 0, \quad \phi_{00t}^{-}.
\]

(5.141)
Note that the data (5.28)-(5.29) are indeed of this form.

For \( n = 1 \) the hypergeometric equation is of the form, cf.(5.96)-(5.97),

\[
\left( (1 - \tau^2) \partial_x^2 + 2 \partial_x + \ell(\ell + 1) \right) \phi^+_{1\ell m}(\tau) = \mathcal{O}(1),
\]
\[
\left( (1 - \tau^2) \partial_x^2 - 2 \partial_x + \ell(\ell + 1) \right) \phi^-_{1\ell m}(\tau) = \mathcal{O}(1 + \tau)^2.
\]

In this case we are led to the system

\[(k + 1)(k \pm 1) \phi^\pm_{(k+1)\ell m} - \left(k(k - 1) - \ell(\ell + 1)\right) \phi^\pm_{k\ell m} = q^\pm_{k\ell m}, \quad (5.144)\]

In the “−”-case the free data are \( \phi^-_{01\ell m} \) and \( \phi^-_{21\ell m} \) and the solution is smooth at \( \tau = -1 \) since the source is of order \( (1 + \tau)^2 \). In the “+”-case there is only one free datum (the second one is not visible as it comes along with a logarithmic term and therefore needs to vanish), namely \( \phi^+_{01\ell m} \), which generates a smooth solution.

### 5.7 Comparison: Approaching \( I^- \) from \( \mathcal{J}^- \) and \( I \)

We have analyzed the appearance of logarithmic terms when approaching the critical set \( I^- \) from both \( \mathcal{J}^- \) and \( I \). Let us assume that data have been constructed such that all relevant fields and all of their transverse derivatives remain smooth at \( I^- \) when taking their limit from \( \mathcal{J}^- \). Then the question arises whether this already implies the existence of a smooth cylinder \( I \) and a smooth critical set \( I^- \). We do not attempt here to solve the evolution problem. On the level of constraint/transport equations, though, this leads to the question whether the no-logs conditions on \( I^- \) viewed from \( I \) do impose additional restrictions on the data if \( I^- \) is known to be smooth when approached from \( \mathcal{J}^- \).

When analyzing the appearance of log terms at \( I^- \) from \( \mathcal{J}^- \) and from the cylinder \( I \) it was convenient to use different subsystems of the Bianchi equations whence we have obtained “more” no-logs conditions when approaching \( I^- \) from \( I \). However, we could have derived an analog to the singular wave equation we just discussed on \( \mathcal{J}^- \) as well. And, we will see this more explicitly in the case of the spin-2 equation discussed in Section 8, due to the constraint propagation, one should expect the no-logs conditions for the \( V_{AB} \) and the \( W^A \)-equation to be equivalent, though this does not follow from the considerations we made here.

We have shown that the no-logs condition arising from the \( V_{AB} \)-equation evaluated on \( \mathcal{J}^- \) and on \( I \) adopt a very similar form, (4.68) and (5.122). Both equations are obtained by differentiating (2.95) by \( r \) and \( \tau \), and in both cases the appearance of a log term becomes evident by the divergence of \( \partial_x^\mu \partial_x^\nu V_{AB} \) at \( I^- \) for some \( \nu \), which one may also regard as an indication that the no-logs condition on \( \mathcal{J}^- \) implies that on \( I \). In Section 9 below we will show that for \( M = \text{const.} \) and \( N = 0 \) the radiation field necessarily needs to vanish asymptotically at \( I^- \) at any order to have a regular \( I^- \) viewed from \( \mathcal{J}^- \), and we will see in Section 7.6 that in that case the expansions coming from \( I \) do not produce log-terms either.

To get a satisfactory answer to this question one needs to solve the evolution problem through the critical set \( I^- \). However, by the above considerations one might be led to the expectation that the no-logs condition (4.68) characterizes data which generate a spacetime with a smooth cylinder \( I \) and a smooth critical set \( I^- \).
6 Gauge independence of the no-logs conditions

In all the previous considerations we have assumed that the gauge functions at $\mathcal{I}^-$ (or rather their expansions at $\mathcal{I}^-$) have been given, and tried to construct data which do not produce logarithmic terms by deriving “no-logs conditions”.

A priori one might expect that the gauge functions such as $f_n$, $\nu_\mathcal{A}$, $\nu_\tau$ etc. appear in the right-hand sides of (4.105)-(4.106). In that case the appearance of logarithmic terms would depend on the gauge. Conversely, one should get rid of many logarithmic terms by an appropriately adjusted gauge. However, if one computes in which way the expansion coefficients $f^{(n+2)}_n$, $\nu^{(n+2)}_\mathcal{A}$, $\nu^{(n+2)}_\tau$ etc. enter the no-logs conditions (4.105)-(4.106) for $\partial^k V_{AB} |_{\mathcal{I}^-}$, one finds that they cancel out.\footnote{We have seen this explicitly for the lower orders in Section 4.1, cf. also Section 5.4. As the computations are not very illuminating we forgo to present the general case.}

Unfortunately, on the level of formal expansions, there seems to be no chance to get insights in which way and whether at all they enter the no-logs conditions for $\partial^k V_{AB} |_{\mathcal{I}^-}$ for $k \geq n + 1$. To analyze the gauge-dependence of the appearance of logarithmic terms at the critical sets of spatial infinity we will therefore consider the behavior of smooth solutions to the GCFE under coordinate transformation which are associated with changes in the gauge data at $\mathcal{I}^-$.

Let us assume we have been given a smooth solution $\left(e^{\psi_i}, \hat{\Gamma}^i_{jk}, \hat{L}_{ij}, W_{ijkl}\right)$ to the GCFE in a weakly asymptotically Minkowski-like conformal Gauss gauge (in fact some steps rely on (3.80)-(3.81)) with gauge functions

$$\nu_\tau, \quad \nu_\mathcal{A}, \quad \Theta^{(3)}_\mathcal{A}, \quad \kappa, \quad \theta^\gamma, \quad f_\mathcal{A}, \quad f_\tau, \quad f_\mu, \quad \Theta, \quad \nu_\gamma, \quad \psi,$$  \hspace{1cm} (6.1)

which admits a smooth representation of $\mathcal{I}^- \cup \mathcal{I}^- \cup I$. Let us consider another weakly asymptotically Minkowski-like conformal Gauss gauge, given by the gauge functions

$$\nu^{\text{new}}_\tau, \quad \nu^{\text{new}}_\mathcal{A}, \quad \Theta^{(1)}^{\text{new}}_\mathcal{A}, \quad \kappa^{\text{new}}, \quad \theta^{\text{new}}, \quad f^{\text{new}}_\mathcal{A}, \quad f^{\text{new}}_\tau.$$  \hspace{1cm} (6.2)

To transform into the new gauge, we first apply a combination of a conformal and a coordinate transformation of the form (cf. (2.136))

$$r \mapsto r^{(1)}_{\text{new}}(r, x^A), \quad 1 + \tau \mapsto 1 + \tau^{(1)}_{\text{new}} = h(r, x^A)(1 + \tau), \quad \Theta \mapsto \Theta^{(1)}_{\text{new}} = \psi(r, x^A)\Theta.$$  \hspace{1cm} (6.3)

The function $r^{(1)}_{\text{new}}$ is given by (2.138),

$$\frac{\partial^2 r^{(1)}_{\text{new}}}{\partial r^2} = \frac{\partial r^{(1)}_{\text{new}}}{\partial r} \left[ \kappa(r) - 2 \partial_\tau \log \psi^{(1)}(r) \right] - \left( \frac{\partial r^{(1)}_{\text{new}}}{\partial r} \right)^2 \kappa^{(1)}_{\text{new}}(r^{(1)}_{\text{new}}(r)), \quad (6.3)$$

with, cf. (2.139),

$$\psi(r) = h(r) \frac{\Theta^{(1)}(r^{(1)}_{\text{new}}(r))}{\Theta^{(1)}(r)}, \quad h(r) = \frac{\partial r}{\partial r^{(1)}_{\text{new}}} \frac{\nu^{\text{new}}_\gamma(r^{(1)}_{\text{new}}(r))}{\nu^{\text{new}}_\gamma(r)}.$$  \hspace{1cm} (6.4)

In a weakly asymptotically Minkowski-like conformal Gauss gauge this equation is of the form (cf. (3.63))

$$\frac{\partial^2 r^{(1)}_{\text{new}}}{\partial r^2} = \left( \frac{\partial r^{(1)}_{\text{new}}}{\partial r} \right)^2 \frac{2}{r^{(1)}_{\text{new}}(r)} + 2 \partial r^{(1)}_{\text{new}}(r) \frac{2}{r^{(1)}_{\text{new}}(r)} + \Theta(r^{(1)}_{\text{new}}(r)),$$  \hspace{1cm} (6.5)
With \( u := \partial_{\tau} \log(r/r_{\text{new}}(1)) \) and \( v := r_{\text{new}}(1)/r \) this singular ODE becomes a regular first-order system (cf. (3.64)-(3.65)),

\[
\begin{align*}
\partial_{\tau} u &= -u^2 - v^2 (1 - ur)^2 \Omega^2 ((vr)^0) + (1 - ur) \Omega(r^0), \\
\partial_{\tau} v &= -uv. 
\end{align*}
\] (6.6) (6.7)

In fact to get this regular system it is crucial that \( \Theta^{(1)}, \kappa \) and \( \nu_\tau \) have an asymptotic behavior at \( I^- \) as required by the weakly asymptotically Minkowski-like gauge condition. The solution is of the form (with \( p(x^A) > 0 \))

\[
r_{\text{new}}(1)(r, x^A) = p(x^A)r + \Omega(r^2), \quad r_{\text{new}}(1)(r, x^A) = \frac{1}{p(x^A)}r_{\text{new}}(1) + \Omega(r_{\text{new}}^1)^2.
\]

The function \( p \) is determined as described in Section 3.4 (as the second datum \( q \) which we do not need to consider here explicitly). In particular \( \psi(\tau_{\text{new}}(1), x^A) \) and \( h(\tau_{\text{new}}(1), x^A) \) will be smooth,

\[
\psi(r_{\text{new}}(1), x^A) = p(x^A) + \Omega(r_{\text{new}}^1), \\
h(r_{\text{new}}(1), x^A) = 1 + \Omega(r_{\text{new}}^1).
\] (6.8) (6.9)

We then consider the coordinate transformation (2.141) (for this let us denote the just obtained \( r_{\text{new}}, \tau_{\text{new}} \) and \( \Theta_{\text{new}} \) by \( r, \tau \) and \( \Theta \)),

\[
\begin{align*}
\tau &\mapsto \tau_{\text{new}} := \tau + r_{\text{new}}(2)(r, x^A)(1 + \tau), \\
x^A &\mapsto x_{\text{new}}^A := x^A + h^A(r, r^B)(1 + \tau).
\end{align*}
\]

The functions \( r_{\text{new}}(2) \) and \( h^A \) are chosen in such a way that \( \nu^A_{\text{new}} \) is realized and \( g_{\tau \tau}, x^- = -1 \),

\[
\begin{align*}
h^A &= g^A_B (\nu^B - \nu^B_{\text{new}}) = \Omega(r), \\
r_{\text{new}}(2) &= -\nu_{\text{new}}^\tau (h^A_{\text{new}} - \frac{1}{2} h^A_B g_{AB}) = \Omega(r^3).
\end{align*}
\]

A conformal transformation (2.142) yields the desired value \( \theta_{\text{new}}^- \); (cf. (2.43)),

\[
\Theta \mapsto [1 + \phi(r, x^A)(1 + \tau)] \Theta,
\] (6.10)

where

\[
\phi(r, x^A) = \frac{1}{4} \nu_{\tau} (\theta^+ - \theta_{\text{new}}^-) = \Omega(r^2),
\] (6.11)

because of (3.81). We deduce that the combination of conformal and coordinate transformations which realize (6.2) is smooth at \( I^- \).

Before we proceed, let us direct attention to some consequences of our smoothness assumption on \((e^{\alpha}_\iota, \tilde{\Gamma}^{\iota}_{\jmath \kappa}, \tilde{L}_{ij}, W_{ijkl})\). It implies that also the fields \((e^{\alpha}_\iota, \Gamma^{\iota}_{\jmath \kappa}, L_{ij}, W_{ijkl})\) are smooth. In the next step the conformal geodesic equations need to be solved with initial data \( (\tilde{x}^\alpha)|_{x^-} = (1, 0, 0, 0) \) and \( (f_\mu)|_{x^-} = (0, f^r_{\text{new}}, f^A_{\text{new}}) \). We analyze the conformal geodesic equations in a frame, as the frame components are regular at \( I^- \). The initial data then read

\[
\begin{align*}
(\tilde{x}^\alpha)|_{x^-} &= (1, 0, 0, 0), \\
(f_\mu)|_{x^-} &= (0, \nu^r f^r_{\text{new}}, \nu^A f^A_{\text{new}}) = \Omega(r^0).
\end{align*}
\]
In frame components the conformal geodesic equations read
\[ \dot{x}^i e^\mu_i \partial_\mu \dot{x}^j + \Gamma^j_{\mu \nu} \dot{x}^\nu \dot{x}^j = -2\dot{x}^j f_j \dot{x}^i + \dot{x}^j \dot{x}^i f^j, \]
\[ \dot{x}^i e^\mu_i \partial_\mu f_i - \Gamma^j_{k \mu} \dot{x}^k f_i = \dot{x}^j f_j f_i - \frac{1}{2} f_j f^j \dot{x}_i + \dot{x}^i L_{ij}. \]
This is a regular symmetric hyperbolic system which gives a smooth solution \((\dot{x}^i, f_i)\) in some neighborhood of the initial surface, including some neighborhood of \(I^-\). In particular \(\dot{x}^\mu = e^\mu_i x^i\) is smooth. Next, we apply a coordinate transformation \(x^\mu \mapsto \tilde{x}^\mu\) which transforms \(\dot{x}^\mu\) to \(\partial_\tau\).

\[ 1 = \frac{\partial \tilde{x}^\tau}{\partial x^\mu} \tilde{x}^\mu, \quad 0 = \frac{\partial \tilde{x}^\tau}{\partial x^\mu} \tilde{x}^\mu, \quad 0 = \frac{\partial \tilde{x}^A}{\partial x^\mu} \tilde{x}^\mu. \]

As initial data we take \(\tilde{x}^\mu|_{\mathcal{I}^-} = x^\mu\). Note that, near \(\mathcal{I}^-\) the transformation is of the form \(x^\mu \mapsto x^\mu + \mathcal{O}(1 + \tau)^2\), so that the initial gauge conditions realized above are preserved.

It is instructive to evaluate the conformal geodesics equations on \(I\). For this note that (5.6)-(5.9) hold, and one checks that \((\dot{x}^\mu)|_I = (1, 0, 0, 0)\) and \((f_i)|_I = (0, 1, 0, 0)\). The coordinate transformation (6.12) therefore reduces to the identity on \(I\), whence the leading-order behavior of all fields is unaffected at \(I\). Of course this is to be expected as the fields acquire their “weakly asymptotically Minkowski-like conformal Gauss gauge” values there.

The final gauge is obtained by another conformal transformation \(g \mapsto \Psi^2 g, \Theta \mapsto \Psi \Theta\), which is determined by the equation
\[ \nabla_\xi \Psi = \Psi \langle \dot{x}, f \rangle \iff \partial_\xi \Psi = \Psi f_\xi, \]
with initial data \(\Psi|_{\mathcal{I}^-} = 1\). Since \(f_\xi|_{\mathcal{I}^-} = 0\) we have, near \(\mathcal{I}^-\), \(\Psi = 1 + \mathcal{O}(1 + \tau)^2\). Near \(I\) we have \(\Psi = 1 + \mathcal{O}(\tau)\). Note that \(g(\dot{x}, \dot{x})|_{\mathcal{I}^-} = -1\), so by (2.20) \(\dot{x}\) is globally normalized to \(-1\).

The conformal Gauss coordinates underlying the weakly asymptotically Minkowski-like conformal Gauss gauge as determined by the gauge data (6.2) is obtained from the original one by a combination of a conformal transformation and a coordinate transformation both of which are smooth near \(I^-\). The transformed fields \((e^\mu_i, \tilde{\Gamma}^i_{jk}, \tilde{L}_{ij}, W_{ijkl})\) which appear in the GCFE and which are determined by \(g\) as well as \(\Theta\) and \(f\) are therefore smooth as well. This is as one should expect since, by choice of the gauge data (6.1), the congruence of conformal geodesics on which this gauge is based does not have conjugate point near \(\mathcal{I}^- \cup I^-\).

**Lemma 6.1** Consider a solution \((e^\mu_i, \tilde{\Gamma}^i_{jk}, \tilde{L}_{ij}, W_{ijkl})\) of the GCFE in a weakly asymptotically Minkowski-like conformal Gauss gauge which is smooth at \(\mathcal{I}^-, I\) and \(I^-\). Then the validity of all the no-logs conditions obtained in Section 4 & 5 are preserved under gauge transformations which transform into any other weakly asymptotically Minkowski-like conformal Gauss gauge.

### 7 Asympt. Minkowski-like conformal Gauss gauge

#### 7.1 Solution of the constraint equations

In Section 4.1 we have studied the constraint equations listed in Appendix A.1 in a weakly asymptotically Minkowski-like gauge. For a further analysis concerning the appearance of logarithmic terms at \(I^-\) it is convenient, and by Lemma 6.1 without restriction, to assume an asymptotically Minkowski-like conformal Gauss gauge at each order (cf. Definition 3.8), where the asymptotic expansions of the gauge functions are completely fixed. Then the computations are much simpler. As “physical” initial data on \(\mathcal{I}^-\) we regard \(\Xi_{AB}^{(1)}\) rather than the radiation field \(W_{rAB}\).

By (7.5) below they are – apart from the integration functions \(\Xi_{AB}^{(1)}\) and \(\Xi_{AB}^{(2)}\) – in one-to-one correspondence, and also their expansion coefficients are, cf. (7.23).
From the constraint equations derived in Appendix A.1 we obtain

\[ g_{AB}|_{\mathcal{J}} = s_{AB} + \mathfrak{D}(r^{\infty}), \quad \theta^+ = \mathfrak{D}(r^{\infty}), \quad \xi_A = \mathfrak{D}(r^{\infty}), \quad (7.1) \]

\[ L_{rr}|_{\mathcal{J}} = \mathfrak{D}(r^{\infty}), \quad L_{rA}|_{\mathcal{J}} = \mathfrak{D}(r^{\infty}), \quad (7.2) \]

\[ (L_{AB})_{tt}|_{\mathcal{J}} = -\frac{1}{2} \left( \partial_r - \frac{2}{r} \right) \varepsilon_{AB} + \mathfrak{D}(r^{\infty}), \quad s^{AB} L_{\mathcal{J}} = 1 + \mathfrak{D}(r^{\infty}), \quad (7.3) \]

\[ L_{r}|_{\mathcal{J}} = -\frac{1}{2} + \mathfrak{D}(r^{\infty}), \quad L_{A}|_{\mathcal{J}} = \frac{1}{2} \varepsilon_A + \mathfrak{D}(r^{\infty}), \quad (7.4) \]

\[ W_{rA\bar{B}}|_{\mathcal{J}} = -\frac{1}{4r^2} \partial_r \left( \partial_r - \frac{2}{r} \right) \varepsilon_{AB} + \mathfrak{D}(r^{\infty}), \quad (7.5) \]

\[ W_{rA}\bar{r}|_{\mathcal{J}} = -\frac{1}{4r^2} \left( \partial_r - \frac{2}{r} \right) \varepsilon_A + \mathfrak{D}(r^{\infty}), \quad (7.6) \]

\[ W_{A\bar{B}r}|_{\mathcal{J}} = -\frac{1}{2r^2} \left( \partial_{[\mathcal{A}]} \varepsilon_{\mathcal{B}]} + \frac{1}{2} \varepsilon_{\mathcal{A}C} \partial_r \varepsilon_{\mathcal{B}[C} \right) + \mathfrak{D}(r^{\infty}), \quad (7.7) \]

\[ (\partial_r + \mathfrak{D}(r^{\infty})) W_{rA}\bar{r}|_{\mathcal{J}} = -\frac{1}{4r^2} \partial_r \left( \partial_r - \frac{2}{r} \right) \bar{\partial}_{A} + \frac{1}{2} \mathfrak{D}_{\mathcal{A}B} W_{r,A\bar{B}} + \mathfrak{D}(r^{\infty}), \quad (7.8) \]

\[ \left( \partial_r - \frac{2}{r} + \mathfrak{D}(r^{\infty}) \right) W_{A}\bar{r}^r|_{\mathcal{J}} = -\frac{1}{8} \left( \partial_r - \frac{2}{r} \right) \left( \partial_r - \frac{2}{r} \right) \varepsilon_A - \frac{1}{2} \mathfrak{D}_{\mathcal{A}B} W_{A\bar{B}r} \right. \]

\[ + \frac{1}{2} \varepsilon_{\mathcal{A}} W_{\mathcal{A}}\mathcal{B}^r - \mathfrak{D}_{\mathcal{A}} W_{r\bar{B}r} + \mathfrak{D}(r^{\infty}), \quad (7.9) \]

and

\[ \left( \partial_r - \frac{4}{r} + \mathfrak{D}(r^{\infty}) \right) \left( (W_{A\bar{B}})_{tt} - \frac{r^4}{4} W_{rA\bar{B}} \right)|_{\mathcal{J}} = \left( \partial_{(A} \mathfrak{D}_{B)} r - \frac{r^2}{2} \partial_{(A} W_{B)r} \right)_{tt} \]

\[ + \frac{3}{4} \mathfrak{D}_{\mathcal{A}B} W_{A\bar{B}r} - \frac{3}{4} \mathfrak{D}_{(A} \mathfrak{D}_{B)\mathcal{C}r} + \mathfrak{D}(r^{\infty}). \quad (7.10) \]

Let us compute the relevant frame components. Note that in an asymptotically Minkowski-like conformal Gauss gauge at each order, on \( \mathcal{J} \),

\[ e_0|_{\mathcal{J}} = \partial_\tau, \quad e_1|_{\mathcal{J}} = \partial_r + r \partial_\tau + \mathfrak{D}(r^{\infty}), \quad e_A|_{\mathcal{J}} = \mathfrak{D}_{\mathcal{A}} \partial_\tau + \mathfrak{D}(r^{\infty}). \quad (7.11) \]

Using the formulas derived in Section 2.7 we find for the connection coefficients

\[ \tilde{\Gamma}_{A}^{1i}|_{\mathcal{J}} = \frac{\delta}{i} + \mathfrak{D}(r^{\infty}), \quad \tilde{\Gamma}_{A}^{10}|_{\mathcal{J}} = \mathfrak{D}(r^{\infty}), \quad \tilde{\Gamma}_{A}^{00}|_{\mathcal{J}} = \mathfrak{D}(r^{\infty}), \quad (7.12) \]

\[ \tilde{\Gamma}_{A B}^{00}|_{\mathcal{J}} = -\frac{1}{2} \varepsilon_{AB} + \mathfrak{D}(r^{\infty}), \quad (7.13) \]

\[ \tilde{\Gamma}_{A B}^{11}|_{\mathcal{J}} = -\frac{1}{2} \varepsilon_{AB} + 2B + \mathfrak{D}(r^{\infty}), \quad (7.14) \]

\[ \tilde{\Gamma}_{A C}^{C B}|_{\mathcal{J}} = -\partial_{A} B_{C} + \mathfrak{D}(r^{\infty}). \quad (7.15) \]

For the components of the Schouten tensor the results of Section 2.8 yield

\[ L_{A A}|_{\mathcal{J}} = \mathfrak{D}(r^{\infty}), \quad (7.16) \]

\[ L_{A 1}|_{\mathcal{J}} = \frac{1}{2r^2} \varepsilon_{A} + \mathfrak{D}(r^{\infty}), \quad (7.17) \]

\[ L_{A B}|_{\mathcal{J}} = -\frac{1}{2} \left( \partial_r - \frac{4}{r} \right) \varepsilon_{AB} + \mathfrak{D}(r^{\infty}), \quad (7.18) \]

\[ L_{A 0}|_{\mathcal{J}} = \frac{1}{2r^2} \varepsilon_{A} + \mathfrak{D}(r^{\infty}). \quad (7.19) \]
For the solutions of the constraint equations for the rescaled Weyl tensor to be smooth at $I^-$ \( \Xi_{AB} \) necessarily needs to admit an expansion of the form (3.57)

$$
\Xi_{AB} \sim \sum_{m=1}^{\infty} \Xi_{AB}^{(m)} r^m,
$$

(7.20)

where the \( \Xi_{AB}^{(m)} \)'s denote trace-free tensors on the round 2-sphere. Recall that

$$
v_A \equiv \nabla_B \Xi_A^B, \quad v_A^{(m)} \equiv \mathcal{D}_B \Xi_A^{(m)B}.
$$

(7.21)

We assume that all smoothness conditions in Proposition 4.1 are satisfied, i.e.

$$
\Xi_{AB}^{(1)} = \Xi_{AB}^{(3)} = \Xi_{AB}^{(4)} = 0.
$$

(7.22)

Then the restriction of the rescaled Weyl tensor to \( \mathscr{S}^- \) extends smoothly across \( I^- \). We determine its expansion coefficients (terms with vanishing denominator are defined to be zero),

$$
W_{rABr}^{(m)} = - \frac{(m+3)(m+2)}{4} \Xi_{AB}^{(m+4)},
$$

(7.23)

$$
W_{rA}^{(m)} = \frac{m+1}{4} \Xi_{AB}^{(m+3)},
$$

(7.24)

$$
W_{ABr}^{(m)} = \frac{1}{2} \mathcal{D}_{[A} \Xi_{B]}^{(m+2)} + \frac{1}{4} \sum_k k \Xi^{(m-k+3)}_{AB \dot{C}} \cdot \Xi_{\dot{C}C}^{(k)},
$$

(7.25)

$$
W_{A}^{(m)} r = 2M \delta A 0 - \frac{1}{4} \mathcal{D}_A \Xi_{AB}^{(m+2)} - \frac{1}{8} \sum_k (k+1)(k+2) \Xi^{(m-1-k)}_{A \dot{B}} \Xi_{\dot{B}C}^{(k+4)} - \Xi_{AB}^{(k+2)}.
$$

(7.26)

$$
W_{A}^{(m)} r = 2M \delta A 0 - \frac{1}{4} \mathcal{D}_A \Xi_{AB}^{(m+2)} - \frac{1}{8} \sum_k (k+1)(k+2) \Xi^{(m-1-k)}_{A \dot{B}} \Xi_{\dot{B}C}^{(k+4)} - \Xi_{AB}^{(k+2)}.
$$

(7.27)

Recall that \( M, L_A, \) and \( c_A^{(2,0)} \) arise as integration functions. For the frame components we have

$$
V_{AB}^{(x) -} = \frac{\epsilon^2}{2} \epsilon_A^B B W_{rABr},
$$

(7.29)

$$
V_{AB}^{(x) -} = - \frac{\epsilon_A^B B}{\epsilon^2} \left( \frac{r^2}{r^2} W_{rABr} - \frac{2}{r^2} (W_{rABr})_{(x)} \right),
$$

(7.30)

$$
W_{A}^{(x) -} = \epsilon_A^B W_{rA Br} + 2r^{-1} \epsilon_A^B W_{A Br},
$$

(7.31)

$$
W_{A}^{(x) -} = \epsilon_A^B W_{rA Br},
$$

(7.32)

$$
W_{010}^{(x) -} = W_{rA Br},
$$

(7.33)

$$
W_{01A}^{(x) -} = - \epsilon_A^B B W_{ABr},
$$

(7.34)

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whence (again, terms with vanishing denominator are defined to be zero)

\[ V_{AB}^{\pm(m)} = - \frac{m(m+1)}{8} \epsilon_{AB}^{(m+2)}, \]  
\[ W_A^{-\prime (m)} = \frac{m}{4} \epsilon_{A}^{(m+2)}, \]  
\[ W_{0101}^{(m)} = 2M \delta^m_0 - \frac{1}{4} \epsilon^{A}_{A} \epsilon^{(m+2)}_A - \frac{1}{8m} \sum_k (k+3)(k+2) \epsilon^{(m-1-k)AB} \epsilon^{(k+4)}_{AB}, \]  
\[ W_{01AB}^{(m)} = - \frac{1}{2} \epsilon[A^{(m+2)}_A] - \frac{1}{4} \sum_k k \epsilon^{(m-k+3)}_A \epsilon^{(k)}_{AB}, \]  
\[ W_A^{\prime (m)} = 2 \delta^m_1 L_A + \frac{1}{m-1} \left( \epsilon^{B}_{A} \epsilon^{(m)}_{AB} - \frac{1}{2} \sum_k k \epsilon^{(m-k+1)}_A \epsilon^{(k+2)}_{B} \right), \]  
\[ V_{AB}^{-\prime (m)} = 2 \delta^m_2 c_{AB}^{(2,0)} + \frac{1}{m-2} \left( \epsilon_{A}^{(m)} \epsilon_{B}^{(m)} + \frac{3}{2} \sum_k \epsilon^{(m-k+1)}_A \epsilon^{(k)}_{B} \right). \]

In particular, we find for the leading orders (recall that \( N \equiv -\frac{1}{8} \epsilon^{AB} \epsilon_{AB}^{(2)} \))

\[ W_{0101}|_{\sigma} = 2M - \frac{r^3}{4} \epsilon^{A}_{A} + \Omega(r^4), \]  
\[ W_A^{-\prime}|_{\sigma} = \frac{3}{4} r^3 \epsilon^{A}_{A} + \Omega(r^4), \]  
\[ W_A^{\prime}|_{\sigma} = -2M A + 2L_A r + \frac{r^3}{8} \left( (\Delta - 1) \epsilon^{A}_{A} - 2 \epsilon^{B}_{A} \epsilon^{(5)}_{B} \right) + \Omega(r^4), \]  
\[ W_{01AB}|_{\sigma} = 2N \epsilon_{AB} - \frac{3}{2} \epsilon^{A}_{A} + \Omega(r^4), \]  
\[ V_{AB}^{-\prime}|_{\sigma} = (\epsilon_{A}^{0,1} \epsilon^{(2)}_{A}) + \left( 2(\epsilon_{A}^{0,1} \epsilon^{(5)}_{A}) + 3M \epsilon^{(2)}_{AB} + N \epsilon_{A}^{C \epsilon^{(2)}_{B}} \right) r \]  
\[ + c_{AB}^{(2,0)} \frac{3}{8} \left( (\Delta - 4) \epsilon^{A}_{A} \epsilon^{C}_{B} - 2 \epsilon^{B}_{A} \epsilon^{C}_{A} \right) + \Omega(r^4). \]

### 7.2 First-order transverse derivatives on \( J^- \)

To get a better idea what is going on let us consider the 1st-order derivatives (i.e. the \( n = 1 \) case) explicitly, as well. In particular we want to determine the source terms in (4.105)-(4.106). Evaluation of (2.72)-(2.78) on \( J^- \) by using all the expressions we have derived in the previous section gives the following relations for connection and frame coefficients, and Schouten tensor
For the relevant transverse derivatives of the frame components of the rescaled Weyl tensor we obtain from (2.90)-(2.94) (observe that $\Xi_A^C \Xi_{BC} = \frac{1}{2} \Box^2 \eta_{AB}$)

\[
\partial_r (\hat{L}_{10} - \hat{L}_{11})|_{\sigma^*} = O(r^\infty), \hspace{1cm} (7.47)
\]
\[
\partial_r (\hat{L}_{10} + \hat{L}_{11})|_{\sigma^*} = -4rW_{0101} + O(r^\infty), \hspace{1cm} (7.48)
\]
\[
\partial_r (\hat{L}_{00} - \hat{L}_{A1})|_{\sigma^*} = O(r^\infty), \hspace{1cm} (7.49)
\]
\[
\partial_r (\hat{L}_{00} + \hat{L}_{A1})|_{\sigma^*} = -2rW^{+} - 2rW^{-} - \frac{1}{2r^2}\Xi_A^B v_B + O(r^\infty), \hspace{1cm} (7.50)
\]
\[
\partial_r \hat{L}_{1A}|_{\sigma^*} = -2rW^{-} + O(r^\infty), \hspace{1cm} (7.51)
\]
\[
\partial_r \hat{L}_{AB}|_{\sigma^*} = -2rV^{+} + rU_{AB} + \frac{1}{4r}\Xi_A^C \left( \partial_t - \frac{1}{r} \right) \Xi_{BC} + O(r^\infty), \hspace{1cm} (7.52)
\]
\[
\partial_r \hat{\Gamma}^A_{1k}|_{\sigma^*} = O(r^\infty), \hspace{1cm} (7.53)
\]
\[
\partial_r (\hat{\Gamma}_A^0 - \hat{\Gamma}_{A1})|_{\sigma^*} = O(r^\infty), \hspace{1cm} (7.54)
\]
\[
\partial_r (\hat{\Gamma}_A^0 + \hat{\Gamma}_{A1})|_{\sigma^*} = \frac{1}{r} v_A + O(r^\infty), \hspace{1cm} (7.55)
\]
\[
\partial_r (\hat{\Gamma}_A^0 B - \hat{\Gamma}_{A1} B)|_{\sigma^*} = -\frac{1}{2r^2} |\Xi|^2 \delta_{AB} - \frac{1}{2} \partial_t \Xi_{AB} + O(r^\infty), \hspace{1cm} (7.56)
\]
\[
\partial_r (\hat{\Gamma}_A^0 + \hat{\Gamma}_B^1)|_{\sigma^*} = -\frac{1}{2} \left( \partial_t - \frac{2}{r} \right) \Xi_{AB} + O(r^\infty), \hspace{1cm} (7.57)
\]
\[
\partial_r \hat{\Gamma}_A^B C\big|_{\sigma^*} = \frac{1}{2r} \delta^B_C v_A - \frac{1}{2r} \hat{\Gamma}^D_B C \Xi_A^D + O(r^\infty), \hspace{1cm} (7.58)
\]
\[
\partial_r \mu^\mu|_{\sigma^*} = -\delta^\mu_0 + O(r^\infty), \hspace{1cm} (7.59)
\]
\[
\partial_r \mu^\mu_A|_{\sigma^*} = -\frac{1}{2r} \Xi_A^B \tau^\mu_B + O(r^\infty), \hspace{1cm} (7.60)
\]

where we have used that in an asymptotically Minkowski-like conformal Gauss gauge (cf. Section 2.5.3)

\[
\Theta = r(1 - r^2) + O(r^\infty), \hspace{1cm} b_0 = -2r \tau + O(r^\infty), \hspace{1cm} b_1 = 2r + O(r^\infty), \hspace{1cm} b_A = O(r^\infty). \hspace{1cm} (7.61)
\]
Finally, evaluation of the \( \tau \)-derivative of (2.95) on \( \mathcal{I}^– \) yields
\[
\begin{align*}
r^4 \partial_\tau (r^{-3} \partial_\tau V_{AB}) |_{\mathcal{I}^–} = & \left( \mathcal{D}_A \partial_\tau W^+_B - \frac{1}{2r} (\xi (A^C \mathcal{D}_C + 4v(A)W^+_B) - 3/4 \partial_\tau \xi (A^C U_{B}) + 3/2r \xi (A^C \partial_\tau U_{B}) \right) \bigg|_{\mathcal{I}^–} + \mathcal{D}(\tau^\infty) \\
= & \left( - \mathcal{D}_A \mathcal{D}^C V_{B}^+ - 1/2 \mathcal{D}_A \mathcal{D}^C U_{B}^+ - 3/4 \partial_\tau \xi (A^C U_{B}) - \mathcal{D}_A W^+_B - 5/4r \xi (A^C \mathcal{D}_C W^+_{B}) \\
+ & 3/4r \xi (A^C \mathcal{D}_B) W^+_C + 2/r v(A W^+_B) + 1/r W_{C} \mathcal{D}^B/A \xi (A^C) - 3/4r \xi (A^C \mathcal{D}_C W^+_{B}) - 7/4r \xi (A^C \mathcal{D}_B) W^+_{C} \right) \bigg|_{\mathcal{I}^–} \\
+ & \left( \frac{3}{4r} \xi_{A B} \mathcal{D}^C W^+_C - \frac{3}{4r} \xi_{A B} \mathcal{D}^C W^+_{C} + 3/4r^2 \xi^2 V^+_A - 3/2r^2 \xi_{A B} \mathcal{D}^C V^+_C + \mathcal{D}(\tau^\infty). \right) \tag{7.67}
\end{align*}
\]

We obtain a smooth solution \( V_{AB}^- \) whenever the term of order \( r^3 \) on the right-hand side of (7.67) vanishes. Taking the expansions computed in Section 7.1 into account we find that this will be the case whenever
\[
\begin{align*}
0 = & \left( \mathcal{D}_A \mathcal{D}^C V_{B}^{(3)} + 1/2 \mathcal{D}_A \mathcal{D}^C U_{B}^{(3)} + 3/2r \mathcal{D}^{(2)} U_{B}^{(2)} + \mathcal{D}_A W^{(3)} + 2v^{(2)} W^+_B \right) \\
+ & 2\xi^{(2)} (A^C \mathcal{D}_C W_{B}^+) + \mathcal{D}(\xi^{(2)} C W_{B}^+) - \frac{5}{2} \mathcal{D}^{(2)} B^C W_{B}^+ \bigg|_{\mathcal{I}^–} \\
= & \left( \frac{1}{16} \mathcal{D}_A \Delta A \mathcal{D}s v_{B}^{(5)} - \frac{1}{8} \mathcal{D}_A \mathcal{D}_B \Delta s \mathcal{D}^{C} v_{C}^{(5)} + \frac{3}{4} \mathcal{D}_A \mathcal{D}_B \mathcal{D}^{C} v_{C}^{(5)} - \frac{5}{16} \mathcal{D}_A v_{B}^{(5)} + \frac{1}{4} \mathcal{D}_A \Delta A \mathcal{D}s v_{B}^{(5)} \right) \bigg|_{\mathcal{I}^–}. \tag{7.68}
\end{align*}
\]

We compute the divergence,
\[
\left( \Delta A \Delta A + 5 \Delta A - 5 \right) v_{A}^{(5)} = 2 \left( \Delta A \Delta A + 8 \Delta A \Delta A + 12 \Delta A \right) \mathcal{D}_A v_{B}^{(5)} = 0 .
\]

Taking divergence and curl yields equations for divergence and curl of \( v_{A}^{(5)} \).
\[
\begin{align*}
(\Delta A + 6)(\Delta A + 2) \Delta A v_{A}^{(5)} = & 0 , \tag{7.69} \\
(\Delta A + 6)(\Delta A + 2) \Delta A (\epsilon^{AB} \mathcal{D}_A v_{B}^{(5)}) = & 0 . \tag{7.70}
\end{align*}
\]

Comparison with (4.105)-(4.106) shows that the source terms vanish for \( n = 1 \). For the no-logs condition to be fulfilled \( v_{A}^{(5)} \) needs to admit a Hodge decomposition \( v_{A}^{(5)} = \mathcal{D}_A \mathcal{D}_B v_{B}^{(5)} + \epsilon^{A B} \mathcal{D}_B v_{B}^{(5)} \), where \( \omega^{(5)} \) and \( \pi^{(5)} \) are linear combinations of \( \ell = 0, 1 \)-spherical harmonics. However, since \( v_{A} \) arises as a divergence of a trace-free, symmetric tensor, \( \ell = 0, 1 \)-spherical harmonics cannot arise. We observe that (7.69)-(7.70) is equivalent to (7.68). Altogether, the no-logs condition (7.68) holds if and only if
\[
\Xi_{AB}^{(5)} = (\mathcal{D}_A \mathcal{D}_B \Xi^{(5)}) \big|_{\mathcal{I}^–} + \epsilon^{A C} \mathcal{D}_B \mathcal{D}^{C} \Xi^{(5)} \text{ for some } \ell = 2 \text{ spherical harmonics } \omega^{(5)} \text{ and } \pi^{(5)} . \tag{7.71}
\]

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7.3 Second-order transverse derivatives on $\mathcal{J}^-$

In anticipation of later computations, let us also compute the second-order transverse derivatives of frame and connection coefficients, which we obtain by differentiating (2.72)-(2.78) by $\tau$,

\[
\begin{align*}
\partial^2_{\tau} e^\mu_{\nu}|_{\mathcal{J}^-} &= \mathcal{D}(r^\infty), \\
\partial^2_{\tau} e_{\nu A}|_{\mathcal{J}^-} &= -\frac{1}{r} v_A + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} e_{\nu A}|_{\mathcal{J}^-} &= -\frac{1}{2} v_A + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} e_{\nu A}|_{\mathcal{J}^-} &= -\frac{1}{2} \left( \partial_{\tau} - \frac{1}{r} \right) \Xi_A B e^A_B + \frac{1}{4\tau^2} |\Xi|^2 e^A_A + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} \Gamma^0_{1 A}|_{\mathcal{J}^-} &= -4 r W_{0101} + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} \Gamma^1_{1 A}|_{\mathcal{J}^-} &= -2 r W_{0101} + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} (\hat{\Gamma}^0_{1 A} - \hat{\Gamma}^1_{1 A})|_{\mathcal{J}^-} &= -2 (W^+_A + W^-_A) r + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} (\hat{\Gamma}^0_{1 A} + \hat{\Gamma}^1_{1 A})|_{\mathcal{J}^-} &= -4 W_A r + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} (\hat{\Gamma}^1_{1 A} B)|_{\mathcal{J}^-} &= 2 r W^A B_0 + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} \hat{\Gamma}^0_{1 A}|_{\mathcal{J}^-} &= -\frac{1}{2r^2} \Xi_A B v_B - 2r (W^+_A + W^-_A) r + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} \hat{\Gamma}^1_{1 A}|_{\mathcal{J}^-} &= -\frac{1}{2r^2} \Xi_A B v_B - r (W^+_A + W^-_A) r + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} (\hat{\Gamma}^0_{1 A} B - \hat{\Gamma}^1_{1 A} B)|_{\mathcal{J}^-} &= \frac{1}{4r^3} |\Xi|^2 \Xi_{AB} - \frac{1}{4r^2} |\Xi|^2 \eta_{AB} + \frac{1}{r} \Xi (\xi^C \partial_{\tau} \Xi_{BC}) - \frac{1}{2} \left( \partial_\tau - \frac{1}{r} \right) \Xi_{AB} - 2 V_A^+ r - 2 V^+_{AB} r + 2 r W_{0101} \eta_{AB} + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} (\hat{\Gamma}^0_{1 A} B + \hat{\Gamma}^1_{1 A} B)|_{\mathcal{J}^-} &= \frac{1}{2r^2} |\Xi|^2 \eta_{AB} + \frac{1}{2r} \Xi_A C \partial_{\tau} \Xi^C B - \frac{1}{2} \left( \partial_\tau - \frac{1}{r} \right) \Xi_{AB} - 4 V^+_{AB} r + 2 r U_{AB} + \mathcal{D}(r^\infty), \\
\partial^2_{\tau} \hat{\Gamma}^1_{1 A} B|_{\mathcal{J}^-} &= - \frac{1}{2r} \delta^B_C v_A + \frac{1}{4r^2} \hat{\Gamma}^1_{1 A} B C |\Xi|^2 + \frac{1}{2} \hat{\Gamma}^1_{1 A} B C \left( \partial_\tau - \frac{1}{r} \right) \Xi^D + \frac{1}{2r^2} \delta^B_C \Xi_A D - r (W^+_A + W^-_A) \delta^B_C - 2r (W^+_A - W^-_A) \eta_{BC} + \mathcal{D}(r^\infty). 
\end{align*}
\]

7.4 A sufficient condition for the non-appearance of logs: Approaching $I^-$ from $\mathcal{J}^-$

We have seen in Section 4.3 that whether the no-logs conditions (4.105)-(4.106), regarded as equations on the expansion coefficients of the radiation field at $I^-$, can be fulfilled or not, depends on the harmonic decomposition of the source terms in (4.105)-(4.106). The source terms in turn are determined from lower-order expansion coefficients (and certain integration functions such as the mass aspect etc.). Here, we want to construct data for which these lower order terms can be controlled, and in order to be able to do that, besides imposing the asymptotically Minkowski-like conformal Gauss gauge at each order, we assume that the radiation field vanishes at each order at $I^-$, equivalently,

$$
\Xi_{AB} = \Xi_{AB}^{(2)} r^2 + \mathcal{D}(r^\infty).
$$

The idea is that for these class of data no terms of order $n+2$ arise on the right-hand side of (4.68) and produce logarithmic terms, i.e. the source terms in (4.105)-(4.106) vanish (as we have already shown above for $n = 1$).
The asymptotic expansions of frame and connection coefficients as well as the Schouten tensor at $I^−$ (including their 1st-order transverse derivatives) follow straightforwardly from (7.11)-(7.19) and (7.47)-(7.60) by inserting (7.86). For convenience let us give the expansion of the rescaled Weyl tensor (cf. (7.35)-(7.40)),

\[
W_{0101}|_{I^−} = 2M + \mathcal{O}(r^∞),
\]
\[
W_{01AB}|_{I^−} = 2N\epsilon_{AB} + \mathcal{O}(r^∞),
\]
\[
W_{−A}|_{I^−} = \mathcal{O}(r^∞),
\]
\[
W_{+A}|_{I^−} = -2\mathcal{M}_A + 2\mathcal{L}_Ar + \mathcal{O}(r^∞),
\]
\[
V_{−AB}|_{I^−} = \mathcal{O}(r^∞),
\]
\[
V_{+AB}|_{I^−} = (\mathcal{P}_{(A}\mathcal{M}_B)|_{I^−} = \mathcal{P}_{(A}\mathcal{M}_B)|_{I^−} = \mathcal{O}(r^∞),
\]
\[
\partial_τW_{0101}|_{I^−} = \Delta_s M - \mathcal{G}^A L_A r + \mathcal{O}(r^∞),
\]
\[
\partial_τW_{01AB}|_{I^−} = \Delta_s N\epsilon_{AB} + 2\mathcal{G}(A L_B)r + \mathcal{O}(r^∞),
\]
\[
\partial_τW_{−A}|_{I^−} = \mathcal{M}_A + \mathcal{O}(r^∞),
\]
\[
\partial_τW_{+A}|_{I^−} = -\frac{1}{2}(\Delta_s - 1)\mathcal{M}_A + (\mathcal{G}^B H_{AB} - 2\mathcal{L}_A)r - 2\mathcal{G}^B c^{(2)}_{AB}r^2 + \mathcal{O}(r^∞),
\]
\[
\partial_τV_{−AB}|_{I^−} = \mathcal{O}(r^∞),
\]
\[
\partial_τV_{AB}|_{I^−} = (V_{−01}|_{I^−} + V_{01}|_{I^−}) + V_{−11}|_{I^−} + V_{02}|_{I^−}r + V_{−12}|_{I^−}r^2 + V_{03}|_{I^−}r^3 + \mathcal{O}(r^∞),
\]

where the precise form of the coefficients is irrelevant for our purposes.
Lemma 7.1 Assume that (7.86) holds. Then, in an asymptotically Minkowski-like conformal Gauss gauge at each order the following relations hold for all $k \geq 1$ (cf. Lemma 7.6),

\[
\begin{align*}
\partial^k_r \tilde{\Gamma}_{11|,\sigma} &= \mathcal{O}(r^{k-1}) + \mathcal{O}(r^\infty), \\
\partial^k_r \tilde{\Gamma}_{00|,\sigma} &= \mathcal{O}(r^{k-1}) + \mathcal{O}(r^\infty), \\
\partial^k_r \tilde{\Gamma}_{A1|,\sigma} &= \mathcal{O}(r^k) + \mathcal{O}(r^\infty), \\
\partial^k_r \tilde{\Gamma}_{AA|,\sigma} &= \mathcal{O}(r^k) + \mathcal{O}(r^\infty), \\
\partial^k_r (\bar{\Gamma}^A_{0A} + \bar{\Gamma}^A_{1A})|_{\sigma} &= \mathcal{O}(r^{k-2}) + \mathcal{O}(r^\infty), \\
\partial^k_r (\bar{\Gamma}^A_{0A} - \bar{\Gamma}^A_{1A})|_{\sigma} &= \mathcal{O}(r^k) + \mathcal{O}(r^\infty), \\
\partial^k_r (\bar{\Gamma}^A_{0B} + \bar{\Gamma}^A_{1B})|_{\sigma} &= \mathcal{O}(r^{k-1}) + \mathcal{O}(r^\infty), \\
\partial^k_r (\bar{\Gamma}^A_{0B} - \bar{\Gamma}^A_{1B})|_{\sigma} &= \mathcal{O}(r^{k+1}) + \mathcal{O}(r^\infty), \\
\partial^k_r \tilde{\Lambda}_{AB} |_{\sigma} &= \mathcal{O}(r^{k+1}) + \mathcal{O}(r^\infty), \\
\partial^k_r e^A |_{\sigma} &= \mathcal{O}(r^{k-1}) + \mathcal{O}(r^\infty), \\
\partial^k_r e^A |_{\sigma} &= \mathcal{O}(r^k) + \mathcal{O}(r^\infty), \\
\partial^k_r e^A |_{\sigma} &= \mathcal{O}(r^{k-1}) + \mathcal{O}(r^\infty), \\
\partial^k_r U_{AB} |_{\sigma} &= \mathcal{O}(r^k) + \mathcal{O}(r^\infty), \\
\partial^k_r W_{AB}^A |_{\sigma} &= \mathcal{O}(r^{k+1}) + \mathcal{O}(r^\infty), \\
\partial^k_r V_{AB}^A |_{\sigma} &= \mathcal{O}(r^{k+2}) + \mathcal{O}(r^\infty),
\end{align*}
\]

where $\mathcal{O}(r^k)$ denotes a polynomial in $r$ of degree $\leq k$ (the zero-polynomial if $k$ is negative).

Proof: This is proved by induction. The case $k = 1$ follows from the above considerations. So let us assume that the assertion holds for $1 \leq k \leq n - 1$ We then apply $\partial^k_r$ to the evolution equations (2.72)-(2.78), (2.90)-(2.94). Taking the induction hypothesis as well as the 0th-order expansions into account, we straightforwardly deduce that the assertion of the lemma holds for $k = n$ for all components, excluding for the time being $\partial^0_r (\bar{\Gamma}^A_{0B} + \bar{\Gamma}^A_{1B})$, $\partial^0_r (\bar{\Gamma}^A_{0A} - \bar{\Gamma}^A_{1A})$ and $\partial^0_r \tilde{\Lambda}_{AB}$.

Let us consider the behavior of these connection components somewhat more detailed. From (2.72)-(2.78) we have

\[
\partial^0_r (\bar{\Gamma}^A_{0B} + \bar{\Gamma}^A_{1B})|_{\sigma} = \partial^0_r \tilde{\Gamma}_{0B} + \partial^0_r \tilde{\Lambda}_{AB} + \mathcal{O}(r^\infty).
\]

We need to show that the terms of order $n$ in $\partial^0_r \tilde{\Gamma}_{0B}$ and $\partial^0_r \tilde{\Lambda}_{AB}$ cancel each other. We have

\[
\begin{align*}
\partial^n_r \tilde{\Lambda}_{AB} |_{\sigma} &= -(n - 1)r \partial^{n-2} V_{AB} - \partial^{n-1}_r (\bar{\Gamma}^C_{0A} \tilde{\Lambda}_{CB}) + \mathcal{O}(r^\infty), \\
\partial^n_r \tilde{\Gamma}_{0B} |_{\sigma} &= - \partial^n_r (\bar{\Gamma}^C_{0B} \tilde{\Gamma}_{A0}) - (n - 1)r \partial^{n-2} V_{AB} + \mathcal{O}(r^\infty),
\end{align*}
\]
whence
\[ \partial_\tau^n (\hat{L}_{AB} + \hat{\Gamma}_A^0 B)|_{\mathcal{I^+}} = - \partial_\tau^{n-1}[(\hat{\Gamma}_A^C C_0 (\hat{L}_{CB} + \hat{\Gamma}_C^0 B)] + \mathcal{P}^n + \mathcal{O}(r^\infty). \]

By induction we conclude (one easily checks by using (7.47)-(7.60) that this is satisfied for \( n = 1 \),
\[ \partial_\tau^n (\hat{L}_{AB} + \hat{\Gamma}_A^0 B) = \mathcal{P}^n + \mathcal{O}(r^\infty), \]
whence it follows readily that
\[ \partial_\tau^n (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B) = \mathcal{P}^{n-1} + \mathcal{O}(r^\infty). \]

Similarly, we have
\[
\begin{align*}
\partial_\tau^n (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B)|_{\mathcal{I^+}} &= \partial_\tau^{n-1} (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B) + \mathcal{P}^{n-2} + \mathcal{O}(r^\infty), \\
\partial_\tau^n (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B)|_{\mathcal{I^+}} &= \partial_\tau^{n-1} (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B) - \frac{r}{2} \partial_\tau^{n-1}[(1 - r^2) W_1] + \mathcal{P}^{n-1} + \mathcal{O}(r^\infty), \\
\partial_\tau^n (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B)|_{\mathcal{I^+}} &= \partial_\tau^{n-1} (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B) + r \partial_\tau^{n-1}[(1 + \tau) W_1] + \mathcal{P}^{n-1} + \mathcal{O}(r^\infty),
\end{align*}
\]
whence
\[
\begin{align*}
\partial_\tau^n (\hat{L}_{AB} + \hat{\Gamma}_A^0 B)|_{\mathcal{I^+}} &= - \partial_\tau^{n-1}[(\hat{L}_{BA} + \hat{\Gamma}_B^0 A) \hat{\Gamma}_A^1 B] + \frac{r}{2} \partial_\tau^{n-1}[(1 + \tau) W_1] + \mathcal{P}^{n-1} + \mathcal{O}(r^\infty) \\
&= \mathcal{P}^{n-1} + \mathcal{O}(r^\infty),
\end{align*}
\]
and
\[ \partial_\tau^n (\hat{\Gamma}_A^0 B + \hat{\Gamma}_A^1 B)|_{\mathcal{I^+}} = \mathcal{P}^{n-2} + \mathcal{O}(r^\infty). \]

Finally, we consider the equation (2.95) for \( V_{AB}^- \), to which we apply \( \partial_\tau^n \),
\[ r^{n+3} \partial_\tau^n (r^{n-2} \partial_\tau^n V^-_{AB})|_{\mathcal{I^+}} = \mathcal{P}^{n+1} + \mathcal{O}(r^\infty), \]
and we observe that no logarithmic terms arise,
\[ \partial_\tau^n V^-_{AB}|_{\mathcal{I^+}} = \mathcal{P}^{n+1} + c^{n+2,n}_{AB} r^n + \mathcal{O}(r^\infty), \]
and the lemma is proved. \( \square \)

Let us return to Proposition 4.2. A sufficient condition which makes sure that (4.68) holds is given by the following result. It is a corollary of the previous lemma, which shows that the source term in (4.68) is of the form \( \mathcal{P}^{n+1} + \mathcal{O}(r^\infty) \):

**Corollary 7.2** Assume that the radiation field vanishes at any order at \( \mathcal{I}^- \), (7.86). Then the restrictions to \( \mathcal{F}^- \) of all the fields appearing in the GCFE including their transverse derivatives of all orders admit smooth extensions through \( \mathcal{I}^- \) in an asymptotically Minkowski-like conformal gauged at each order (cf. Corollary 7.8).

**Remark 7.3** If \( \partial_\tau^n \Xi_{AB}|_{\mathcal{I}^-} = 0 \) only for \( 3 \leq k \leq n \), then the restrictions to \( \mathcal{F}^- \) of all the fields appearing in the GCFE including their transverse derivatives up to and including the order \( n - 4 \) admit smooth extensions through \( \mathcal{I}^- \) in an asymptotically Minkowski-like conformal gauged at each order.

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Let us assume that $\Xi_{AB}^{(4)}$ vanishes for $3 \leq k \leq n - 1$, $n \geq 5$ (we already know that $\Xi_{AB}^{(3)}$ and $\Xi_{AB}^{(4)}$ necessarily need to vanish). Then, by the previous considerations, the no-logs conditions for $\partial^4_r^{-4} V_{AB}|_{\mathcal{J}^-}$, (4.105)-(4.106), take the form

$$
\prod_{\ell=0}^{n-3} (\Delta_\nu + \ell(\ell + 1)) \mathcal{D}_A v_{A}^{(n)} = 0, \tag{7.120}
$$

$$
\prod_{\ell=0}^{n-3} (\Delta_\nu + \ell(\ell + 1)) (\varepsilon^{AB} \mathcal{D}_A v_{B}^{(n)}) = 0. \tag{7.121}
$$

Unfortunately, since the operator appearing on the left-hand side has a non-trivial kernel, we cannot conclude at this stage that the converse to the corollary is also true, i.e. that (7.86) is necessary for the non-appearance of logarithmic terms. The two functions $\xi^{(n)}$ and $\eta^{(n)}$ appearing in the Hodge decomposition of $v_{A}^{(n)} = \mathcal{D}_A (\xi^{(n)}) + \varepsilon_{AC} \mathcal{D}_B (\eta^{(n)})$ are allowed to have spherical harmonics with $2 \leq \ell \leq n - 3$. That yields a weakened converse to the above corollary.

**Corollary 7.4** Consider smooth initial data of the form $\Xi_{AB} = O(r^2)$. Then the expansion coefficients admit a Hodge decomposition of the form $\Xi_{AB}^{(n)} = (\mathcal{D}_A \mathcal{D}_B \xi^{(n)})_{\text{II}} + \varepsilon_{AC} \mathcal{D}_B (\eta^{(n)})$. Assume that $\xi^{(n)}$ and $\eta^{(n)}$ do not contain spherical harmonics with $2 \leq \ell \leq n - 3$ for $n \geq 3$. Then, in an asymptotically Minkowski-like conformal Gauss gauge, the restrictions to $\mathcal{J}^-$ of all the fields appearing in the GCFF including their transverse derivatives of all orders admit smooth extensions through $\mathcal{I}^-$ if and only if (7.86) holds.

**Remark 7.5** In Section 8 we will see that for the massless spin-2 equation a vanishing radiation field is not necessary for the non-appearance of logarithmic terms. The results in Section 9 show that non-linear effects, or rather a non-vanishing mass, do impose additional restrictions on the radiation field: Elements in the kernel of the operator in (4.105)-(4.106) cause a violation of the no-logs condition in higher orders.

### 7.5 Transport equations on $I$

Here we want to analyze the appearance of logarithmic terms when approaching $I^-$ from $I$ for all radial derivatives of the relevant fields. The 9th order has been computed in Section 5, (5.6)-(5.9) and (5.32)-(5.37) (we assume that the no-logs conditions (7.22) are satisfied). The 1st-order radial derivatives have been computed in Section 5.4. Let us sum up the results in our current asymptotically Minkowski-like conformal Gauss gauge at each order

$$
\partial_\nu \hat{L}_{10}|_I = -4M(1 + \tau) + O(1 + \tau)^2, \tag{7.122}
$$

$$
\partial_\nu \hat{L}_{11}|_I = -4M(1 + \tau) + O(1 + \tau)^2, \tag{7.123}
$$

$$
\partial_\nu \hat{L}_{1A}|_I = O(1 + \tau)^2, \tag{7.124}
$$

$$
\partial_\nu (\hat{L}_{00} - \hat{L}_{A1})|_I = O(1 + \tau)^2, \tag{7.125}
$$

$$
\partial_\nu (\hat{L}_{00} + \hat{L}_{A1})|_I = v_{A}^{(2)} + 4\mathcal{M}A(1 + \tau) + O(1 + \tau)^2, \tag{7.126}
$$

$$
\partial_\nu \hat{L}_{AB}|_I = -\frac{1}{2} \Xi_{AB}^{(2)} + 2\left(M\eta_{AB} + N\epsilon_{AB}\right)(1 + \tau) + O(1 + \tau)^2, \tag{7.127}
$$

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\[\partial_r \hat{\Gamma}^i_{\ j}|_{I} = \mathcal{O}(1 + \tau)^2, \tag{7.128}\]
\[\partial_r (\hat{\Gamma}^0_{\ A} + \hat{\Gamma}^1_{\ A})|_{I} = \mathcal{O}(1 + \tau)^3, \tag{7.129}\]
\[\partial_r \hat{\Gamma}^0_{\ A}|_{I} = \frac{1}{2} \hat{\nu}^0_A (1 + \tau) + \mathcal{O}(1 + \tau)^2, \tag{7.130}\]
\[\partial_r \hat{\Gamma}^1_{\ A}|_{I} = \frac{1}{2} \hat{\nu}^1_A (1 + \tau) + \mathcal{O}(1 + \tau)^2, \tag{7.131}\]
\[\partial_r (\hat{\Gamma}^0_{\ B} + \hat{\Gamma}^1_{\ B})|_{I} = \mathcal{O}(1 + \tau)^2, \tag{7.132}\]
\[\partial_r (\hat{\Gamma}^0_{\ B} - \hat{\Gamma}^1_{\ B})|_{I} = \Xi^{(2)}_{AB} - \Xi^{(2)}_{AB}(1 + \tau) + \mathcal{O}(1 + \tau)^2, \tag{7.133}\]
\[\partial_r \hat{e}^A_{\ C}|_{I} = \frac{1}{2} \left( \hat{\nu}^0_A \delta^B_C - \hat{\nu}^0_B \delta^C_A \right) (1 + \tau) + \mathcal{O}(1 + \tau)^2, \tag{7.134}\]
\[\partial_r \hat{e}^0_{\ 1}|_{I} = \mathcal{O}(1 + \tau)^3, \tag{7.135}\]
\[\partial_r \hat{e}^1_{\ 1}|_{I} = 1, \tag{7.136}\]
\[\partial_r \hat{e}^0_{\ A}|_{I} = \mathcal{O}(1 + \tau)^2, \tag{7.137}\]
\[\partial_r \hat{e}^1_{\ A}|_{I} = \mathcal{O}(1 + \tau)^2, \tag{7.138}\]
\[\partial_r \hat{e}^0_{\ A}|_{I} = 0, \tag{7.139}\]
\[\partial_r \hat{e}^1_{\ A}|_{I} = -\frac{1}{2} \Xi^{(2)}_{AB} \hat{e}^A_{\ B} (1 + \tau) + \mathcal{O}(1 + \tau)^2, \tag{7.140}\]
\[\partial_r \hat{W}^-_{\ A}|_{I} = \mathcal{O}(1 + \tau)^2, \tag{7.141}\]
\[\partial_r \hat{W}^+_{\ A}|_{I} = 2L_A + \mathcal{O}(1 + \tau), \tag{7.142}\]
\[\partial_r \hat{V}^+_{\ AB}|_{I} = \mathcal{O}(1 + \tau)^3, \tag{7.143}\]
\[\partial_r \hat{V}^-_{\ AB}|_{I} = -2 (\hat{\mathcal{D}}_{(A} L_{B)})_{|I} - 3 \Xi^{(2)}_{(A} C (M_{(B)C} + N_{(B)C}) + \mathcal{O}(1 + \tau), \tag{7.144}\]
\[\partial_r \hat{U}_{AB}|_{I} = \mathcal{O}(1 + \tau). \tag{7.145}\]

Note that, as compared to (5.101), \(\partial_r \hat{V}^+_{\ AB}\) has a better decay in an asymptotically Minkowski-like conformal Gauss gauge at each order. This follows straightforwardly from (5.92).

We consider radial derivatives of higher order. In our setting the initial data for the transport equations are determined by the data on \(\mathcal{F}^+\), which we assume to be smooth at any order at \(I^-\), cf. (7.11)-(7.19) and (7.35)-(7.40). The missing data for the singular wave equations (5.120)-(5.121) are provided by \(\partial_r \hat{W}^-_{\ A}|_{I^-}\) and \(\partial_r \hat{W}^+_{\ A}|_{I^-}\), which are also determined from \(\mathcal{F}^+\) but will be irrelevant for our purposes.

### 7.6 A sufficient condition for the non-appearance of logs: Approaching \(I^-\) from \(I\)

As in Section 7.4 let us assume that the radiation field vanishes at any order at \(I^-\), i.e.
\[\Xi_{AB} = \Xi^{(2)}_{AB} r^2 + \mathcal{O}(r^{\infty}). \tag{7.146}\]

In that case the following data, relevant to solve the GCFE on \(I\), are induced on \(I^-\) for \(p \geq 2,\)
\[\partial_r \hat{W}^-_{\ A}|_{I^-} = 0, \quad \partial_r \hat{W}^+_{\ A}|_{I^-} = 0, \quad \partial_r \hat{U}_{AB}|_{I^-} = 0, \quad \partial_r \hat{V}^+_{\ AB}|_{I^-} = 0, \quad \partial_r \hat{V}^-_{\ AB}|_{I^-} = \frac{1}{p!} \mathcal{C}^{(p,p-2)}_{AB}. \tag{7.147}\]
In Section 7.4 we have shown that data on \( \mathcal{F}^- \) which satisfy (7.146) do not produce logarithmic terms when approaching \( I^- \) from \( \mathcal{F}^- \). Here we aim to show that the same class of data also satisfies all no-logs conditions at \( I^- \) when coming from the cylinder \( I^- \). More specifically, let us assume that no logs arise up to and including the order \( n - 1 \) at \( I^- \). We want to show that the singular wave equations (5.120)-(5.121) and the \( \partial^k_{\tau} V_{AB} \)-equation (5.115) do not produce logarithmic terms.

**Lemma 7.6** Assume that (7.146) holds. Then for \( k \geq 1 \) the radial derivatives have the following fall-off behavior at \( I^- \) in an asymptotically Minkowski-like conformal Gauss gauge at each order (cf. Lemma 7.1):

\[
\begin{align*}
\partial^k_{\tau} & \hat{L}_{11} |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & (\hat{L}_{A0} + \hat{L}_{A1}) |_{I^-} = \mathcal{O}(1 + \tau)^{k-1}, \\
\partial^k_{\tau} & (\hat{L}_{A0} - \hat{L}_{A1}) |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & \hat{L}_{AB} |_{I^-} = \mathcal{O}(1 + \tau)^{k-1}, \\
\partial^k_{\tau} & \hat{\Gamma}_{10}^0 |_{I^-} = \mathcal{O}(1 + \tau)^{k+1}, \\
\partial^k_{\tau} & \hat{\Gamma}_{A0}^0 |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & \hat{\Gamma}_{11}^0 |_{I^-} = \mathcal{O}(1 + \tau)^{k+1}, \\
\partial^k_{\tau} & (\hat{\Gamma}_{10}^0 + \hat{\Gamma}_{11}^0) |_{I^-} = \mathcal{O}(1 + \tau)^{k+2}, \\
\partial^k_{\tau} & (\hat{\Gamma}_{10}^0 - \hat{\Gamma}_{11}^0) |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & \hat{\Gamma}_{A0}^0 |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & \hat{\Gamma}_{A1}^0 |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & \hat{\Gamma}_{A0}^1 |_{I^-} = \mathcal{O}(1 + \tau)^{k-1}, \\
\partial^k_{\tau} & \hat{\Gamma}_{A1}^1 |_{I^-} = \mathcal{O}(1 + \tau)^{k+1}, \\
\partial^k_{\tau} & \hat{\Gamma}_{A0}^{\mu} |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & \hat{\epsilon}_{1}^0 |_{I^-} = \mathcal{O}(1 + \tau)^{k+2}, \\
\partial^k_{\tau} & \hat{\epsilon}_{A}^0 |_{I^-} = \mathcal{O}(1 + \tau)^{k+1}, \\
\partial^k_{\tau} & \hat{\epsilon}_{A}^{\mu} |_{I^-} = \mathcal{O}(1 + \tau)^k, \\
\partial^k_{\tau} & \hat{e}_{0}^1 |_{I^-} = \mathcal{O}(1 + \tau)^{k+1}, \\
\partial^k_{\tau} & \hat{e}_{1}^1 |_{I^-} = \delta^k + \mathcal{O}(1 + \tau)^{k+1}, \\
\partial^k_{\tau} & \hat{V}_{AB}^0 |_{I^-} = \mathcal{O}(1 + \tau)^{k+2}, \\
\partial^k_{\tau} & \hat{W}_{A}^0 |_{I^-} = \mathcal{O}(1 + \tau)^{k+1}, \\
\partial^k_{\tau} & \hat{U}_{AB}^0 |_{I^-} = \mathcal{O}(1 + \tau)^k.
\end{align*}
\]

**Remark 7.7** While in Lemma 7.1 we had polynomials of a sufficiently small degree which ensured that terms of the critical, logarithms producing order in the \( \partial^k_{\tau} V_{AB} \)-equation do not appear, here the components show a sufficiently fast decay.

**Proof:** As in the proof of Lemma 7.1 we use an induction argument. The above considerations show that the lemma is true for \( k = 1 \). So let us assume that it is true for \( 1 \leq k \leq n - 1 \). We want to show that it is also true for \( k = n \geq 2 \).
From (2.72)-(2.78), (5.6)-(5.9) and (5.32)-(5.37) we deduce that

\[ \partial_r \partial^r \hat{L}_0\big|_I = \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r \hat{L}_1\big|_I = \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r \hat{L}_A|_I = \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r (\hat{L}_A + \hat{L}_A)|_I = \mathcal{O}(1 + \tau)^{n-2}, \]
\[ \partial_r \partial^r (\hat{L}_A - \hat{L}_A)|_I = \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r \hat{L}_{AB}|_I = n(1 + \tau)\partial_r^{-1}V_{AB} - \partial_r^r(\hat{\Gamma}^C_A \hat{C}_{CB}) + \mathcal{O}(1 + \tau)^{n-1} = \mathcal{O}(1 + \tau)^{n-2}, \]

(the intermediate step in the last line will be relevant below). For initial data (7.147) we obtain the asserted decay for the Schouten tensor.

Moreover,

\[ \partial_r \partial^r \hat{\Gamma}_0^0\big|_I = - \partial_r^r \hat{\Gamma}_0^0 + \mathcal{O}(1 + \tau)^n, \]
\[ \partial_r \partial^r \hat{\Gamma}_A^0\big|_I = - \partial_r^r \hat{\Gamma}_A^0 + \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r \hat{\Gamma}_B^0\big|_I = \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r (\hat{\Gamma}_A^0 + \hat{\Gamma}_A^0)|_I = \partial_r^r (\hat{\Gamma}_A^0 + \hat{\Gamma}_A^0) + \mathcal{O}(1 + \tau)^{n+1}, \]
\[ \partial_r \partial^r (\hat{\Gamma}_A^0 - \hat{\Gamma}_A^0)|_I = - \partial_r^r \hat{\Gamma}_A^0 + \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r \hat{\Gamma}_A^B\big|_I = \mathcal{O}(1 + \tau)^n, \]
\[ \partial_r \partial^r (\hat{\Gamma}_A^0 - \hat{\Gamma}_A^0)|_I = \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^r (\hat{\Gamma}_A^0 - \hat{\Gamma}_A^0)|_I = \mathcal{O}(1 + \tau)^{n-2}, \]
\[ \partial_r \partial^r (\hat{\Gamma}_A^B - \hat{\Gamma}_A^B)|_I = \mathcal{O}(1 + \tau)^n. \]

Similar to the proof of Lemma 7.1 one needs some auxiliary equations (which also follow from (2.72)-(2.78))

\[ \partial_r \partial^r (\hat{L}_A + \hat{\Gamma}_A^B)|_I = - \partial_r^r [\hat{\Gamma}^C_A \hat{C}_{CB} + \hat{\Gamma}^C_A \hat{C}_{CB}] + \mathcal{O}(1 + \tau)^{n-1}, \]

which we use to show, by induction,

\[ \partial_r^r (\hat{L}_A + \hat{\Gamma}_A^B)|_I = \mathcal{O}(1 + \tau)^n. \]

Note that it follows from (7.122)-(7.145) that this is true for \( n = 1 \).

Furthermore,

\[ \partial_r \partial^r (\hat{L}_A + \hat{\Gamma}_A^B) = - \partial_r^r [\hat{\Gamma}^C_A \hat{C}_{CB} + \hat{\Gamma}^C_A \hat{C}_{CB}] + \frac{n}{2} (1 + \tau)^2 \partial_r^{n-1} W_A^n + \mathcal{O}(1 + \tau)^n \]

which we use to show, again by induction (it follows from (7.122)-(7.145) that this holds for \( n = 1 \))

\[ \partial_r^r (\hat{L}_A + \hat{\Gamma}_A^B)|_I = \mathcal{O}(1 + \tau)^{n+1}. \]
For initial data (7.147) we then end up with desired decay for the connection coefficients by integrating all the above ODEs.

For the frame coefficients we find

\[ \partial_r \partial^\tau e^\tau_1|_I = \mathcal{O}(1 + \tau)^{n+1}, \]
\[ \partial_r \partial^\tau e^\hat{A}_1|_I = \mathcal{O}(1 + \tau)^n, \]
\[ \partial_r \partial^\tau e^\tau_1|_I = \mathcal{O}(1 + \tau)^n, \]
\[ \partial_r \partial^\tau e^\tau_A|_I = \mathcal{O}(1 + \tau)^n, \]
\[ \partial_r \partial^\tau e^\hat{A}_A|_I = \mathcal{O}(1 + \tau)^{n-1}, \]
\[ \partial_r \partial^\tau e^\tau_A|_I = \mathcal{O}(1 + \tau)^{n-1}. \]

For initial data (7.147) at \( I^- \) that yields the desired result.

It remains to consider the radial derivatives of the rescaled Weyl tensor. Evaluation of the \( n \)-th order radial derivative of (5.38)-(5.43) yields

\[ \partial_r [(1 - \tau)2^{-n} \partial^n V^+_{AB}]|_I = \frac{1}{(1 - \tau)^{n-1}}[\mathcal{P}(A)\partial^n W^+_{AB}]_{ht} + \mathcal{O}(1 + \tau)^{n+1}, \]
\[ \partial_r [(1 + \tau)2^{-n} \partial^n V^-_{AB}]|_I = -\frac{1}{(1 + \tau)^{n-1}}[\mathcal{P}(A)\partial^n W^-_{AB}]_{ht} + \mathcal{O}(1), \]
\[ (1 + \tau)\partial_r \partial^n \mathcal{D}_{AB} W^+_{A} = 2\mathcal{P} B \partial^n \mathcal{D}_{AB} W^+_{A} + (n + 1)\partial^n \mathcal{D}_{AB} W^-_{A} + \mathcal{O}(1 + \tau)^{n+2}, \]
\[ (1 - \tau)\partial_r \partial^n \mathcal{D}_{AB} W^+_{A} = -2\mathcal{P} B \partial^n \mathcal{D}_{AB} W^-_{A} - (n + 1)\partial^n \mathcal{D}_{AB} W^+_{A} + \mathcal{O}(1 + \tau)^{n-2}, \]
\[ n\partial^n \mathcal{D}_{A} W^+_{10} = \frac{1}{2}(1 + \tau)\partial^n \mathcal{D}_{A} W^+_{10} - \frac{1}{2}(1 - \tau)\partial^n \mathcal{D}_{A} W^-_{10} + \mathcal{O}(1 + \tau)^{n}, \]
\[ n\partial^n \mathcal{D}_{A} W^+_{0AB} = (1 + \tau)\partial^n \mathcal{D}_{A} W^+_{0AB} - (1 - \tau)\partial^n \mathcal{D}_{A} W^-_{0AB} + \mathcal{O}(1 + \tau)^n. \]

As in Section 5.5 we derive decoupled equations for \( \partial^n V^+_{AB} \) from this system,

\[ (1 - \tau^2)\partial^2 \partial^n \mathcal{D}_{A} W^+_{A} = -\mathcal{D}_s + (n - 1)n - 1)\partial^n \mathcal{D}_{A} W^+_{A} - 2(1 - (n - 1)\tau)]\partial^n \mathcal{D}_{A} W^+_{A} + \mathcal{O}(1 + \tau)^{n+1}, \]
\[ (1 - \tau^2)\partial^2 \partial^n \mathcal{D}_{A} W^+_{A} = (\mathcal{D}_s + (n - 1)n - 1)\partial^n \mathcal{D}_{A} W^+_{A} - 2[1 + (n - 1)\tau)]\partial^n \mathcal{D}_{A} W^+_{A} + \mathcal{O}(1 + \tau)^{n-1}. \]

It follows from (7.148) and Lemma 5.2 (i) that the solutions are smooth,

\[ \partial^n \mathcal{D}_{A} W^+_{A} = \mathcal{O}(1 + \tau)^{n+1}, \]
\[ \partial^n \mathcal{D}_{A} W^+_{A} = \mathcal{O}(1 + \tau)^{n-1}. \]

Using (7.148) one obtains the desired decay for the remaining components of the Weyl tensor by solving the above transport equations (in particular the \( \partial^n V^+_{AB} \)-equation does not produce log-terms), which completes the proof. \( \square \)

**Corollary 7.8** Assume that (7.86) holds, i.e. that the radiation field vanishes at any order at \( I^- \), in a spacetime which admits a smooth \( I \), and where the data for the transport equations on \( I \) at \( I^- \) are induced by the limits of the corresponding data on \( \mathcal{F}^- \). Then the restrictions to \( I \) of all the fields appearing in the GCFE including their radial derivatives of all orders admit smooth extensions through \( I^- \) in an asymptotically Minkowski-like conformal gauss gauge at each order (cf. Corollary 7.2).
7.7 Conformal Gauss coordinates at $\mathcal{I}^-$

We determine the expansion near $\mathcal{I}^-$ of a line element of a vacuum spacetime which satisfies

$$\Xi_{AB} = \mathcal{O}(r^\infty)$$  \hspace{1cm} (7.149)

in conformal Gaussian coordinates at $\mathcal{I}^-$ in an asymptotically Minkowski-like conformal Gauss gauge at each order. In particular this will be useful to determine Kerr data on $\mathcal{I}^-$ for this gauge.

The restriction of the metric to $\mathcal{I}^-$ follows immediately from the gauge data (3.80).

$$g|_{\mathcal{I}^-} = -d\tau^2 + \frac{2}{r}d\tau dr + s_{AB}dx^A x^B + \mathcal{O}(r^\infty).$$  \hspace{1cm} (7.150)

Higher-order derivatives are obtained from expansions of the frame coefficients. The first-order terms have been computed in (7.59)-(7.60) For the second- and third-order derivatives a computation which uses (7.149) and (2.72)-(2.78) reveals

$$\partial^2 g_{\mu\nu}|_{\mathcal{I}^-} = \mathcal{O}(r^\infty),$$  \hspace{1cm} (7.151)

$$\partial^3 e^\tau|_{\mathcal{I}^-} = 12rM + \mathcal{O}(r^\infty),$$  \hspace{1cm} (7.152)

$$\partial^3 e^\tau|_{\mathcal{I}^-} = 8r^2M + \mathcal{O}(r^\infty),$$  \hspace{1cm} (7.153)

$$\partial^3 e^A|_{\mathcal{I}^-} = 6r^2L^A + \mathcal{O}(r^\infty),$$  \hspace{1cm} (7.154)

$$\partial^3 e^A|_{\mathcal{I}^-} = 6r^2L_A + \mathcal{O}(r^\infty),$$  \hspace{1cm} (7.155)

$$\partial^3 e^A|_{\mathcal{I}^-} = 4r^3L_A + \mathcal{O}(r^\infty),$$  \hspace{1cm} (7.156)

$$\partial^3 g_{A|\mathcal{I}^-} = -2r^2\tilde{e}^B_{AB} + O_r^2(C\Xi_{(A}L_{C)}) + 2r^2\tilde{e}^B_{AB}L_A - 4rM\tilde{e}^A_{BA} + \mathcal{O}(r^\infty).$$  \hspace{1cm} (7.157)

We then obtain

$$g = -d\tau^2 + \frac{2}{r}d\tau dr + s_{AB}dx^A x^B + (1 + \tau)\left(\frac{2}{r^2}d\tau^2 - \frac{2}{r}d\tau dr\right)$$

$$- \frac{1}{r^2}(1 + \tau)^2dr^2 + \frac{2}{3}(1 + \tau)^3\left[2Mdr + r^2L_A d\tau dx^A - 6r^{-1}Mdr^2 + 6L_A d\tau dx^A + \left(2rMs_{AB} + r^2(C\Xi_{(A}L_{B)})\right)dx^A x^B\right] + \mathcal{O}(1 + \tau)^3 + \mathcal{O}(r^\infty).$$  \hspace{1cm} (7.158)

7.7.1 Example: Kerr spacetime

It is quite illuminating to calculate which data on $\mathcal{I}^-$ are needed in our gauge to generate a spacetime which belongs to the Kerr family. For this purpose let us compute the Kerr metric in conformal Gauss coordinates in an asymptotically Minkowski-like conformal Gauss gauge at each order, or rather its asymptotic expansion at $\mathcal{I}^-$. In Kerr-Schild Cartesian coordinates the Kerr line elements reads (cf. e.g. [57]),

$$\tilde{g} = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2 + \frac{2mR^3}{R^4 + a^2(y^r)^2}\ell \otimes \ell$$  \hspace{1cm} (7.159)

where

$$\ell = dy^0 + \frac{R y^3 + ay^2}{a^2 + R^2}dy^1 + \frac{R y^2 - ay^1}{a^2 + R^2}dy^2 + \frac{y^3}{R}dy^3.$$  \hspace{1cm} (7.160)
The function $R$ is given by

\[(y^1)^2 + (y^2)^2 + (y^3)^2 = R^2 + a^2 \left( 1 - \frac{(y^3)^2}{R^2} \right). \tag{7.161}\]

Observe that

\[\eta^\phi(\ell, \ell) = 0. \tag{7.162}\]

We apply the same coordinate transformation (3.28) as for Minkowski and choose the same conformal factor (3.24). That yields

\[g = -dr^2 - 2\frac{\tau}{r}drd\tau + \frac{1 - \tau^2}{r^2}d\theta^2 + d\phi^2 + \frac{2mR^3\Theta^4}{R^4\Theta^2 + a^2 \cos^2 \theta} \ell \otimes \ell, \tag{7.163}\]

where

\[R = -\frac{1}{\sqrt{2}} \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2 - a^2 + \sqrt{((y^1)^2 + (y^2)^2 + (y^3)^2 - a^2)^2 + 4a^2(y^3)^2}}\]

whence

\[R\Theta = -1 + 2a^2r^2 \sin^2 \theta(1 + \tau)^2 + O(1 + \tau)^4. \tag{7.164}\]

We have

\[\ell_\tau = \frac{(a^2 + R^2)(2\tau - R\Theta(1 + \tau^2)) - 2a^2\tau \sin^2 \theta}{R(a^2 + R^2)\Theta^2} = -\left( \frac{1}{4\tau^2} + ra^2 \sin^2 \theta \right) + O(1 + \tau), \tag{7.165}\]

\[\ell_r = \frac{-(a^2 + R^2)(1 + \tau^2) + a^2 \sin \theta}{R(a^2 + R^2)\Theta^2} = \frac{1}{2r^2} + O(1 + \tau), \tag{7.166}\]

\[\ell_\theta = -\frac{a^2 \cos \theta \sin \theta}{a^2 + R^2} = O(1 + \tau), \tag{7.167}\]

\[\ell_\phi = -\frac{a \sin^2 \theta}{(a^2 + R^2)\Theta^2} = -a \sin^2 \theta + O(1 + \tau)^2. \tag{7.168}\]

For the prefactor of $\ell \otimes \ell$ in (7.163) we find the expansion

\[\frac{2mR^3\Theta^4}{R^4\Theta^2 + a^2 \cos^2 \theta} = -16mr^3(1 + \tau)^3 + O(1 + \tau)^4. \tag{7.169}\]

Altogether the Kerr metric adopts the form

\[g = -dr^2 - \frac{2}{r}d\tau d\tau + d\theta^2 + \sin^2 \theta d\phi^2 + (1 + \tau)\left( \frac{2}{r^2}d\tau^2 - \frac{2}{r}d\tau dr \right) - \frac{1}{r^2}(1 + \tau)^2 dr^2 + 4mr(1 + \tau)^3 \left[ -\frac{r^2}{4} \left( \frac{1}{r} + 4ra^2 \sin^2 \theta \right) d\tau^2 - r^{-2}d\tau dr - 4a^2 r^2 \sin^4 \theta d\phi^2 + \left( \frac{1}{r} + 4ra^2 \sin^2 \theta \right) dr d\tau \right] + 2ar^2 \sin^2 \theta \left( \frac{1}{r} + 4ra^2 \sin^2 \theta \right) d\tau d\phi + 4a \sin^2 \theta d\tau d\phi + O(1 + \tau)^4. \tag{7.170}\]
We need to make sure that this is the right gauge. Comparison with (7.158) shows that this is the case only up to and including terms of order \((1 + \tau)^2\). Straightforward transformation of \(\tau\), \(r\) and \(\phi\)

\[
\tau \mapsto \tau + 2mr\left(\frac{1}{3} + 2a^2r^2\sin^2\theta\right)(1 + \tau)^4, \quad \text{(7.171)}
\]

\[
r \mapsto r + mr\left(\frac{3}{8} + 2a^2r^2\sin^2\theta - 2a^4r^4\sin^4\theta\right)(1 + \tau)^4, \quad \text{(7.172)}
\]

\[
\phi \mapsto \phi - 2mar\left(\frac{1}{3} + 2a^2r^2\sin^2\theta\right)(1 + \tau)^4, \quad \text{(7.173)}
\]

accompanied by a conformal transformation \(\Theta \mapsto \Omega \Theta\) with

\[
\Omega = 1 + 2mr\left(\frac{1}{3} + 2a^2r^2\sin^2\theta\right)(1 + \tau)^3 \quad \text{(7.174)}
\]

brings the line element into the desired form

\[
g = -d\tau^2 + 2\frac{r}{r^2}\frac{dr}{d\tau}rdr + d\theta^2 + \sin^2\theta d\phi^2 + (1 + \tau)^3\left(\frac{2}{r^2}dr^2 - \frac{2}{r}\frac{dr}{d\tau}rdright) - \frac{1}{r^2}(1 + \tau)^2dr^2
\]

\[
+ 4mr(1 + \tau)^3\left[\frac{1}{3r}rdr - r^{-2}dr^2 + \frac{1}{3}(d\theta^2 + \sin^2\theta d\phi^2) - \frac{2}{3}ar\sin^2\theta d\tau d\phi + 4a\sin^2\theta d\tau d\phi
\]

\[
+ 2a^2r^2\sin^2\theta\left(d\theta^2 - \sin^2\theta d\phi^2\right)\right] + O(1 + \tau)^4. \quad \text{(7.175)}
\]

Comparison with (7.158) shows that

\[
M = m, \quad L_\theta = 0, \quad L_\phi = -4ma\sin^2\theta, \quad \text{(7.176)}
\]

\[
c_{\theta \theta} = -12ma^2\sin^2\theta, \quad c_{\phi \phi} = 12ma^2\sin^4\theta, \quad c_{\theta \phi} = 0. \quad \text{(7.177)}
\]

Note that \(L_A\) is a conformal Killing 1-form, and that (we have not attempted to compute the higher-order integration functions \(c^{(p+2, p)}_{\hat{A}\hat{B}}\))

\[
c^{(2, 0)}_{\hat{A}\hat{B}} = \frac{3}{2m}(L_A \otimes L_B)_{\mu \nu}. \quad \text{(7.178)}
\]

8 Toy model: Massless Spin-2 equation

We have seen so far that when computing all the fields and their transverse and radial derivatives at \(I^-\) one generically should expect logarithmic terms even if the seed data, i.e. the radiation field, is smooth at \(I^-\). These logarithmic terms can arise at arbitrary high order. Although we have established a sufficient condition which ensures that no logarithmic terms arise in the formal expansions it seems much harder to establish necessary-and-sufficient conditions. The main issue is that the appearance of log terms does not only depend on the leading-order terms, which are the only ones which are controllable without too much effort.

The purpose of this section is to consider a similar problem which is much simpler to deal with, namely we consider the massless spin-2 equation (cf. \([28, 52]\))

\[
\hat{\nabla}_i W^i_{jkl} = \frac{1}{4}P_p P^p_{i j} W^i_{jkl} \quad \text{(8.1)}
\]

on a flat background, as which we take the Minkowski metric in the form (3.25)

\[
\eta = -d\tau^2 - 2\frac{\tau}{r}\frac{dr}{d\tau}rdr + \frac{1 - \tau^2}{r^2}dr^2 + d\theta^2 + \sin^2\theta d\phi^2. \quad \text{(8.2)}
\]
In the gauge (3.31) we have
\[ e_0 = \partial_\tau, \quad e_1 = -\tau \partial_\tau + r \partial_r, \quad e_A = \hat{e}_A, \quad f_1 = 1, \quad f_A = 0. \] (8.3)

We give a list of the non-vanishing Christoffel symbols
\[ \Gamma^{\tau}_{\tau r} = \tau, \quad \Gamma^{r}_{\tau r} = \frac{\tau^2}{r}, \quad \Gamma^{r}_{rr} = -\frac{\tau}{r^2}(1-\tau^2), \quad \Gamma^{\tau}_{rr} = -\tau, \]
\[ \Gamma^{\gamma}_{rr} = -\tau, \quad \Gamma^{\gamma}_{rr} = -\frac{1+\tau^2}{r}, \quad \Gamma^{\gamma}_{AB} = \hat{\Gamma}^{\gamma}_{AB}. \]

For the Weyl connection coefficients we then find
\[ \hat{\Gamma}^{1}_{00} = 1, \quad \hat{\Gamma}^{1}_{10} = 0, \quad \hat{\Gamma}^{1}_{01} = 0, \quad \hat{\Gamma}^{1}_{11} = 0, \quad \hat{\Gamma}^{1}_{AA} = 0, \quad \hat{\Gamma}^{1}_{BB} = 0, \quad \hat{\Gamma}^{1}_{AA} = 0, \quad \hat{\Gamma}^{1}_{BB} = 0. \] (8.4)

\[ \hat{\Gamma}^{1}_{01} = 0, \quad \hat{\Gamma}^{1}_{10} = 0, \quad \hat{\Gamma}^{1}_{01} = 0, \quad \hat{\Gamma}^{1}_{11} = 0, \quad \hat{\Gamma}^{1}_{AA} = 0, \quad \hat{\Gamma}^{1}_{BB} = 0. \] (8.5)

The spin-2 equation then takes the form (cf. (2.90)-(2.99))
\[ \mathcal{W}_0 := \partial_\tau W_{0101} + \frac{1}{2} \mathcal{D}^A W^+_A - \frac{1}{2} \mathcal{D}^A W^-_A = 0, \] (8.6)
\[ \mathcal{W}_{AB} := \partial_\tau W_{01AB} - \mathcal{D}[A W^+_B] - \mathcal{D}[A W^-_B] = 0, \] (8.7)
\[ \mathcal{W}^-_A := \partial_\tau W^-_A - \mathcal{D}^B V^+_B - \frac{1}{2} \mathcal{D}^B U_A = 0, \] (8.8)
\[ \mathcal{W}^+_A := \partial_\tau W^+_A + \mathcal{D}^B V^-_B - \frac{1}{2} \mathcal{D}^B U_A = 0, \] (8.9)
\[ \mathcal{W}^+_{AB} := (1 - \tau) \partial_\tau V^+_{AB} + (r \partial_r - 2) V^+_{AB} - \mathcal{D}(A W^+_{B}) = 0, \] (8.10)
\[ \mathcal{W}^-_{AB} := (1 + \tau) \partial_\tau V^-_{AB} - (r \partial_r - 2) V^-_{AB} + \mathcal{D}(A W^-_{B}) = 0, \] (8.11)

and,
\[ (1 + \tau) \partial_\tau W_{0101} = r \partial_r W_{0101} + \mathcal{D}^A W^-_A, \] (8.12)
\[ (1 + \tau) \partial_\tau W_{01AB} = r \partial_r W_{01AB} + 2 \mathcal{D}(A W_B), \] (8.13)
\[ (1 + \tau) \partial_\tau W^-_A = (r \partial_r - 1) W^-_A + 2 \mathcal{D}^B V^+_B, \] (8.14)
\[ (1 + \tau) \partial_\tau W^+_A = (r \partial_r - 1) W^+_A - 2 \mathcal{D}^B U_A. \] (8.15)

Using the evolution equations we rewrite the latter ones as the following set of constraint equations,
\[ \mathcal{C} := r \partial_r W_{0101} + \frac{1}{2} (1 + \tau) \mathcal{D}^A W^+_A + \frac{1}{2} (1 - \tau) \mathcal{D}^A W^-_A = 0, \] (8.16)
\[ \mathcal{C}_{AB} := r \partial_r W_{01AB} - (1 + \tau) \mathcal{D}[A W^+_B] + (1 - \tau) \mathcal{D}[A W^-_B] = 0, \] (8.17)
\[ \mathcal{C}^- := (r \partial_r - \tau) W^-_A + (1 - \tau) \mathcal{D}^B V^+_B - \frac{1}{2} (1 + \tau) \mathcal{D}^B U_A = 0, \] (8.18)
\[ \mathcal{C}^+ := (r \partial_r + \tau) W^+_A + (1 + \tau) \mathcal{D}^B V^-_B - \frac{1}{2} (1 - \tau) \mathcal{D}^B U_A = 0, \] (8.19)
which one easily checks to be preserved under evolution (8.6)-(8.11). Here, though, we will use a different set of evolution equations.
8.1 Rewriting the equations

We want to decouple the evolution equations. For this we take the divergence of (8.10) and (8.11) and eliminate $V_{AB}^+$ via (8.14) and a linear combination of (8.9) and (8.15),

$$ (1 - \tau)\partial_r W_A^- - (r\partial_r - 2) W_A^+ - 2\mathcal{B}^B_{AB} = 0, $$

(8.20)

respectively. That yields decoupled equations for $W_A^\pm$ which we supplement by the remaining evolution equations for $V_{AB}^\pm$ and $U_{AB}$,

$$(1 - \tau^2)\partial_\tau^2 W_A^- = 2\left(1 - \tau(r\partial_r - 1)\right)\partial_r W_A^- + \left(\Delta_s + (r\partial_r - 2)(r\partial_r + 1) + 1\right)W_A^-, $$

(8.21)

$$(1 - \tau^2)\partial_\tau^2 W_A^+ = 2\left(1 - \tau(r\partial_r - 1)\right)\partial_r W_A^+ + \left(\Delta_s + (r\partial_r - 2)(r\partial_r + 1) + 1\right)W_A^+, $$

(8.22)

$$(1 - \tau)\partial_r V_{AB} = - (r\partial_r - 2)V_{AB}^+ + (\mathcal{D}(A W_B^-))_\tau, $$

(8.23)

$$(1 + \tau)\partial_r V_{AB} = (r\partial_r - 2)V_{AB}^+ - (\mathcal{D}(A W_B^+))_\tau $$

(8.24)

$$ \partial_\tau W_{0101} = - \frac{1}{2} \mathcal{A}^A W_A^+ + \frac{1}{2} \mathcal{A}^A W_A^-, $$

(8.25)

$$ \partial_r W_{01AB} = \mathcal{D}(A W_B^+) + \mathcal{D}(A W_B^-). $$

(8.26)

By derivation these equation follow from the spin-2 equation. To obtain conditions which ensure that they are also sufficient we find that they imply the following set of equations (note that (8.6)-(8.7) and (8.10)-(8.11) are trivially satisfied),

$$ \left(1 - \tau\right)\partial_r + r\partial_r - 2\right)\left(1 + \tau\right)\mathcal{M}_A^- - \mathcal{C}_A^- - 2\mathcal{B}^B_{AB} = 0, $$

(8.27)

$$ \left(1 - \tau\right)\partial_r + r\partial_r - 2\right)\left(1 + \tau\right)\mathcal{M}_A^+ + \mathcal{C}_A^+ - 2\mathcal{B}^B_{AB} = 0, $$

(8.28)

$$ \partial_r \mathcal{C}_A - \frac{1}{2} \left(1 + \tau\right)\mathcal{A}^A \mathcal{M}_A^- - \frac{1}{2} \left(1 - \tau\right)\mathcal{A}^A \mathcal{M}_A^+ - \frac{1}{2} \mathcal{A}^A \mathcal{C}_A^- + \frac{1}{2} \mathcal{A}^A \mathcal{C}_A^+ = 0, $$

(8.29)

$$ \partial_r \mathcal{C}_A + \left(1 + \tau\right)\mathcal{D}(A \mathcal{M}_B^+) - \left(1 - \tau\right)\mathcal{D}(A \mathcal{M}_B^-) - \mathcal{D}(A \mathcal{C}_B^-) - \mathcal{D}(A \mathcal{C}_B^+) = 0, $$

(8.30)

$$ (\partial_r - 1)\mathcal{C}_A^- - (r\partial_r - \tau)\mathcal{M}_A^- - \frac{1}{2} \mathcal{A} \mathcal{C} + \frac{1}{2} \mathcal{B} \mathcal{C}_AB = 0, $$

(8.31)

$$ (\partial_r + 1)\mathcal{C}_A^+ - (r\partial_r + \tau)\mathcal{M}_A^+ + \frac{1}{2} \mathcal{A} \mathcal{C} + \frac{1}{2} \mathcal{B} \mathcal{C}_AB = 0. $$

(8.32)

One needs to characterize data which ensure that the trivial solution to (8.27)-(8.32) is the only one. Here, however, we are interested in the appearance of logarithmic terms: Once we know that, for a given solution of the spin-2 equation, the $W_A^\pm$-components are smooth at $\Gamma^-$, it follows immediately from (8.18), (8.20), (8.25)-(8.26) that the other components need to be smooth there as well. So our focus will be on an analysis of (8.21)-(8.22) near $\Gamma^-$. 

8.2 Appearance of logarithmic terms

We consider (8.21)-(8.22). Expanding $W_A^\pm$ in terms of $r$ one obtains (5.120)-(5.121). The crucial difference is that in this linearized case there is no source term: The no-logs condition is a condition on the $n$th-order expansion coefficient of the radiation field, and independent of all expansion coefficients of different orders. The no-logs condition at a given order is thus completely independent of lower order terms. As a corollary of Lemma 5.2 we obtain the following
**Proposition 8.1** Let \( W_{ijkl} \) be a smooth solution of the massless spin-2 equation on the Minkowski background (8.2) which is smooth at \( \mathcal{I}^- \). The data at \( \mathcal{I}^- \) are a tracefree, symmetric, tensor \( V_{AB} \) which admits a Hodge decomposition of the form

\[
V_{AB} = (D_A D_B V_f + \epsilon(A^C D_B) \partial_C V),
\]

where

\[
V_f = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \sum_{\ell=0}^{n-m} (-1)^{n-m-\ell} \sum_{\mu=0}^{\ell} \sum_{\nu=0}^{\ell} (-1)^{\ell+\mu+\nu} \frac{\ell!}{\mu! \nu!} \partial_{\ell-\mu-\nu} V^{(n)}_{\ell \mu \nu} Y_{\ell \mu \nu}(\theta, \phi).
\]

The solution \( W_{ijkl} \) satisfies all no-logs conditions when approaching \( \mathcal{I}^- \) from \( \mathcal{I}^+ \) if and only if the data are of the following form

\[
V_f \sim \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \sum_{\ell=0}^{n-m} (-1)^{n-m-\ell} \sum_{\mu=0}^{\ell} \sum_{\nu=0}^{\ell} (-1)^{\ell+\mu+\nu} \frac{\ell!}{\mu! \nu!} \partial_{\ell-\mu-\nu} V^{(n)}_{\ell \mu \nu} Y_{\ell \mu \nu}(\theta, \phi),
\]

and

\[
\nabla V \sim \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \sum_{\ell=0}^{n-m} (-1)^{n-m-\ell} \sum_{\mu=0}^{\ell} \sum_{\nu=0}^{\ell} (-1)^{\ell+\mu+\nu} \frac{\ell!}{\mu! \nu!} \partial_{\ell-\mu-\nu} \nabla V^{(n)}_{\ell \mu \nu} Y_{\ell \mu \nu}(\theta, \phi).
\]

In that case also all no-logs conditions are fulfilled when approaching \( \mathcal{I}^- \) from \( \mathcal{I}^+ \), supposing that \( \mathcal{I}^- \) is smooth and that the data for the transport equations on \( \mathcal{I}^- \) are induced by the limits of the corresponding fields on \( \mathcal{I}^- \).

**Remark 8.2** A corresponding analysis which analyzes the appearance of logarithmic terms starting from an ordinary Cauchy problem for the spin-2 equation has been carried out in [52], cf. [26, 28]. It is shown there that no logs arise if and only if all symmetrized trace-free derivatives of the linearized Cotton tensor of the induced metric on the initial surface vanish at spatial infinity, to which (8.33)-(8.34) is the analog at \( \mathcal{I}^- \).

**9 Constant (ADM) mass aspect and vanishing dual mass aspect**

Let us compare Proposition 8.1 with the full non-linear case: In an asymptotically Minkowski-like conformal Gauss gauge at each order which admits a smooth \( \mathcal{I}^- \) data of the form (8.33)-(8.34) provide the maximal part of the radiation field which one can freely prescribe. Spherical harmonics with \( \ell \geq n \) which may appear in the harmonic decomposition of \( \nabla V^{(n)} \) and \( \nabla V^{(n)} \) are determined by the no-logs conditions (4.105)-(4.106).

While we have shown that a radiation field which has a trivial expansion at \( \mathcal{I}^- \) does not produce log-terms at any order, it is not clear at all how necessary-and-sufficient conditions look like, even on the level of formal expansions we are interested in. This is due to the fact that the no-logs conditions are not decoupled equations for the expansion coefficients \( V_f^{(n)} \) and \( \nabla V^{(n)} \) as they are in the spin-2 case considered above.

If the \( n \)th-order expansion coefficient of the radiation field is the first non-trivial one, no log-terms are produced if and only if its Hodge-decomposition scalars have only \( 0 \leq k \leq n-1 \) spherical harmonics in their decomposition. However, when passing to the \( (n+1) \)st-order, the \( n \)th-order expansion coefficient appears in the source, and the source is not allowed to have \( 0 \leq k \leq n \)-spherical harmonics. One should therefore expect additional restrictions arising from this as compared to the spin-2 case, where this cannot happen, unless there are some magic cancellations or Laplacian-like operators which project out all the problematic terms. In fact, it is shown in [53], where smoothness is analyzed from an ordinary Cauchy problem, that the “linear spin-2 condition”, i.e. the condition on the Cotton tensor mentioned in Remark 8.2, is not sufficient to exclude logarithmic terms. One therefore must expect non-trivial additional restrictions as compared to the spin-2 case.

To make computations feasible we restrict attention henceforth to a more restricted class of initial data where

\[
M = \text{const.} \neq 0 \quad \text{and} \quad N = 0.
\]
we deduce let us apply, now with

\[ W^\pm_A|_I = 0, \quad V^\pm_{AB}|_I = 0, \quad W_{001}|_I = 2M, \quad W_{01AB}|_I = 0. \]  

(9.2)

We will analyze this problem approaching \( I^- \) from \( \mathcal{J}^- \), which is more natural in our setting. Of course one could do a similar analysis from \( I^+ \). However, it turns out that the equations which arise on \( \mathcal{J}^- \) are somewhat more manageable. For instance on \( I^- \) a decomposition in spherical harmonics comes in at a very early stage when solving the Bianchi equation, while on \( \mathcal{J}^- \) it needs to be taken into account only when the actual no-logs condition is derived.

### 9.1 Second-order transverse derivatives on \( \mathcal{J}^- \)

In order to get some insights concerning the expected additional “non-linear” restrictions let us consider the second-order transverse derivatives on \( \mathcal{J}^- \) first, before we analyze the general case. We want to search for additional restrictions on \( \Xi_{AB}^{(5)} \), in addition to (7.71), which ensure that no log-terms arise in the expansions of the second-order transverse derivatives. Since we already know how the contribution from \( \Xi_{AB}^{(6)} \) looks like, and we are only interested in the source, i.e. the right-hand side of (4.3), we also assume \( \Xi_{AB}^{(6)} = 0 \) for the computation.

The second-order transverse derivative of (2.95) reads (using (7.12)-(7.15) and (7.47)-(7.60)),

\[
\begin{align*}
\rho^5 \partial_r (r^{-4} \partial^2 V_{AB}^-)|_{\mathcal{J}^-} &= - \partial^2 r \partial_\mu r^2 \partial_\mu V_{AB}^- + (\partial_r (\partial^2 W_{AB}^+)_{\mathcal{J}^-}) \\
&\quad + (\partial^2 W_{AB}^+ - \partial^2 \mathcal{G}^+_{AB} W_{AB}^+ + \partial^2 \mathcal{G}^+_{AB} W_{AB}^+) \\
&\quad + \partial^2 (\hat{\Gamma}^0_{10} + 2 \hat{\Gamma}^1_{10} + \hat{\Gamma}^0_{0} - \hat{\Gamma}^1_{0} - \hat{\Gamma}^1_{1} + V_{AB}^-) \\
&\quad + \left[ - \partial^2 (\hat{\Gamma}^0_{C0} - \hat{\Gamma}^1_{C0} - 2 \hat{\Gamma}^1_{C0}) + \frac{3}{2} \partial^2 (\hat{\Gamma}^0_{C0} - \hat{\Gamma}^1_{C0} + \hat{\Gamma}^1_{C0} + 2 \hat{\Gamma}^1_{C0}) W_{AB}^+ \right]_{\mathcal{J}^-} + \mathcal{O}(\rho^5) .
\end{align*}
\]

The first-order transverse derivative of (2.93) gives with (7.62)-(7.66) (alternatively, one could compute \( (\partial^2 W_{AB}^+)^{(4)} \) from (5.41))

\[
(\partial^2 W_{AB}^+)^{(4)} = (-\mathcal{G}^B \partial_r V_{AB}^- - \frac{1}{2} \mathcal{G}^B \partial_r W_{0101} - \frac{1}{2} \mathcal{G}^B \partial_r W_{01AB} - \partial_r W_{AB}^+)^{(4)} + \frac{3}{2} M \mathcal{G}_{AB}^{(5)}
\]

\[
= \frac{3}{4} M (\Delta_+ + 15) v_{AB}^{(5)} ,
\]

where we used that by (7.67)

\[
(\partial_r V_{AB}^-)^{(4)} = - \frac{3}{2} M \left( (\mathcal{G}^B_{AB})_{\mathcal{J}^-} + 5 \Xi_{AB}^{(5)} \right) .
\]

---

9 In fact similar restrictions should be expected if one linearizes around e.g. the Schwarzschild metric, i.e. the additional restrictions seem mainly be due to a (dual) mass rather than non-linearities.
Using (7.72)-(7.85) to determine the second-order transverse derivatives of connection and frame coefficients, a calculation reveals that

\[
(\partial_r (r^{-4}\partial^2 V_B^{-k} ))^{-1} = (\mathcal{R}_A (\partial^2 W^+ )^{(4)} )_{\mathcal{T}} - \frac{15}{4} M \left( (\Delta_s - 4)\mathcal{R}_A v_B^{(5)} - 2\mathcal{R}_A \mathcal{R}_B \mathcal{R}_C v_C^{(5)} \right)_{\mathcal{T}} + 15 \Xi_{AB}^{(5)} M.
\]

We deduce that for this order to be smooth at \( I^- \) the following smoothness condition needs to be satisfied (in addition to the requirement on \( \Xi_{AB}^{(5)} \) to arise from a linear combination of \( 2 \leq \ell \leq 3 \) spherical harmonics),

\[
(\Delta_s - 8)(\mathcal{R}_A v_B^{(5)})_{\mathcal{T}} - \frac{5}{2} \mathcal{R}_A \mathcal{R}_B \mathcal{R}_C v_C^{(5)}_{\mathcal{T}} - 5 \Xi_{AB}^{(5)} = 0. \quad (9.3)
\]

We compute the divergence

\[
(\Delta_s - 5)(\Delta_s + 1)v_A^{(5)} - \frac{5}{2} \mathcal{R}_A (\Delta_s + 2) \mathcal{R}_B v_B^{(5)} - 10 v_A^{(5)} = 0. \quad (9.4)
\]

Divergence and curl of this equation read

\[
(3\Delta_s \Delta_s + 14\Delta_s + 36)\mathcal{R}_A v_A^{(5)} = 0 \implies \mathcal{R}_A v_A^{(5)} = 0, \quad (9.5)
\]

\[
(\Delta_s \Delta_s - 2\Delta_s - 18)\epsilon^{AB} \mathcal{R}_{[A} v_{B]}^{(5)} = 0 \implies \epsilon^{AB} \mathcal{R}_{[A} v_{B]}^{(5)} = 0, \quad (9.6)
\]

and we deduce the smoothness condition

\[
\Xi_{AB}^{(5)} = 0. \quad (9.7)
\]

While a non-trivial \( \Xi_{AB}^{(5)} = (\mathcal{R}_A \mathcal{R}_B \Xi^{(5)} )_{\mathcal{T}} + \epsilon A \mathcal{R}_B \mathcal{R}_C \Xi^{(5)} \) does not produce logarithmic terms in the expansion of \( \partial_r V_{AB} |_{\mathcal{T}} \) as long as \( \Xi^{(5)} \) and \( \Xi^{(5)} \) are linear combinations of \( \ell = 2 \)-spherical harmonics, it does produce log terms in the next order, namely for \( \partial^2 V_{AB} |_{\mathcal{T}} \), at least supposing that the mass aspect \( M \) is constant and non-zero, and that the dual mass aspect \( N \) vanishes.

### 9.2 A necessary condition for the non-appearance of log terms

We want to generalize the above computation to any order. For this we will extend the computations of Section 4.3 to determine the \( \Xi_{AB}^{(m_0 + 2)} \)-contribution to the no-logs condition (4.105)-(4.106),

\[
\prod_{\ell=0}^{m_0} (\Delta_s + \ell(\ell + 1)) v^{(m_0 + 3)} = O(\ell^{m_0 + 2}), \quad v^{(m_0 + 3)} \in \{ \mathcal{R}_A v_A^{(m_0 + 3)}, \epsilon^{AB} \mathcal{R}_{[A} v_{B]}^{(m_0 + 3)} \}. \quad (9.8)
\]

For this we consider a scenario where the \( \Xi_{AB}^{(k)} \)'s vanishes for \( 3 \leq k \leq m_0 + 1 \). In an asymptotically Minkowski-like conformal Gauss gauge at each order we then compute the \( \Xi_{AB}^{(m_0 + 2)} \)-contribution, which, somewhat surprisingly, can be done explicitly.

More precisely, let us assume that (9.1) holds and consider initial data of the form

\[
\Xi_{AB} = \Xi_{AB}^{m_0 + 2} + \Xi_{AB}^{m_0 + 3} v^{(m_0 + 3)} + \mathcal{D} v^{(m_0 + 4)}, \quad m_0 \geq 3. \quad (9.9)
\]
It follows from the results in Section 7.4 that in order for $I^-$ to be smooth the two functions appearing in the Hodge decomposition of $\Xi_{AB}^{(m_0+2)} = (\mathcal{D}_A \mathcal{D}_B \Xi_{AB}^{(m_0+2)})_{I^+} + \epsilon(A_C \mathcal{D}_B \Xi_{AB}^{(m_0+2)})$ need to be linear combinations of $1 \leq \ell \leq m_0 - 1$-spherical harmonics,

$$\Xi_{AB}^{(m_0+2)} = \sum_{\ell=1}^{m_0-1} \sum_{m=-\ell}^{\ell} \Xi_{\ell m}^{(m_0+2)} Y_{\ell m}(\theta, \phi), \quad \Xi^{(m_0+2)} = \sum_{\ell=1}^{m_0-1} \sum_{-\ell}^{\ell} \Xi_{\ell m}^{(m_0+2)} Y_{\ell m}(\theta, \phi).$$

(9.10)

If this is the case, no log terms arise up to and including the $(m_0-2)$-nd order transverse derivatives of the rescaled Weyl tensor. Our goal is to compute the $(m_0-1)$-st order transverse derivatives. We will see that a non-trivial $\Xi_{AB}^{(m_0+2)}$ contribution does produce non-trivial $2 \leq \ell \leq m_0 - 1$-spherical harmonics in the source term of the no-logs condition of order $m_0 + 1$. We will thus be able to deduce that in a smooth setting the radiation field necessarily needs to vanish at all orders at $I^-$.  

9.2.1 First-order radial derivatives on $I$

Recall the expressions (5.6)-(5.9) of the 0th-order radial derivatives on $I$,

$$e^7_1|_I = -\tau, \quad e^r_1|_I = 0, \quad e^A_1|_I = 0, \quad e^r_A|_I = 0, \quad e^A_A|_I = e^A_A,$$

$$\tilde{\Gamma}^i_{1j}|_I = \delta^i_j, \quad \tilde{\Gamma}^b_{a0}|_I = 0, \quad \tilde{\Gamma}^1_A|_I = 0, \quad \tilde{\Gamma}^B_A|_I = \delta^B_A, \quad \tilde{\Gamma}^C_B|_I = \tilde{\Gamma}^C_B,$$

$$\tilde{L}_{ij}|_I = 0.$$

(9.11)

(9.12)

(9.13)

and that in our current setting we have

$$W^+_A|_I = 0, \quad V^+_A|_I = 0, \quad W_{0101}|_I = 2M, \quad W_{01AB}|_I = 0.$$  

(9.14)

From (2.72)-(2.78) we compute the first-order radial derivatives on $I$ for Schouten tensor, connection and frame coefficients (recall (5.6)-(5.9), $\Theta = r(1 - \tau^2), b_0 = -2r\tau, b_1 = 2r, b_A = 0$). For trivial data as computed from (7.16)-(7.19) we find for the Schouten tensor

$$\partial_r \tilde{L}_{10}|_I = -4M(1 + \tau),$$

$$\partial_r \tilde{L}_{A0}|_I = 0,$$

$$\partial_r \tilde{L}_{11}|_I = -2M(1 - \tau^2),$$

$$\partial_r \tilde{L}_{1A}|_I = 0,$$

$$\partial_r \tilde{L}_{A1}|_I = 0,$$

$$\partial_r \tilde{L}_{AB}|_I = M(1 - \tau^2)\eta_{AB}.$$

(9.15)

(9.16)

(9.17)

(9.18)

(9.19)

(9.20)
With trivial data as induced by (7.12)-(7.15) we end up with the following expressions for the connection coefficients,

\[ \partial_r \hat{\Gamma}_{1}^{0} |_{I} = -4M \left( (1 + \tau)^2 - \frac{1}{3} (1 + \tau)^3 \right), \]
\[ \partial_r \hat{\Gamma}_{1}^{A} |_{I} = 0, \]
\[ \partial_r \hat{\Gamma}_{A}^{0} |_{I} = 0, \]
\[ \partial_r \hat{\Gamma}_{A}^{B} |_{I} = 2M \left( (1 + \tau)^2 - \frac{1}{3} (1 + \tau)^3 \right) \eta_{AB}, \]
\[ \partial_r \hat{\Gamma}_{1}^{1} |_{I} = -2M \left( (1 + \tau)^2 - \frac{2}{3} (1 + \tau)^3 + \frac{1}{6} (1 + \tau)^4 \right), \]
\[ \partial_r \hat{\Gamma}_{1}^{A} |_{I} = 0, \]
\[ \partial_r \hat{\Gamma}_{A}^{A} |_{I} = 0, \]
\[ \partial_r \hat{\Gamma}_{A}^{B} |_{I} = \frac{2}{3} M \left( (1 + \tau)^3 - \frac{1}{4} (1 + \tau)^4 \right) \hat{\Gamma}_{A}^{B} C. \]

Finally, we have

\[ \partial_r e^{r} |_{I} = 2M \left( (1 + \tau)^3 - \frac{5}{6} (1 + \tau)^4 + \frac{1}{6} (1 + \tau)^5 \right), \]
\[ \partial_r e^{A} |_{I} = 1, \]
\[ \partial_r e^{A} |_{I} = 0, \]
\[ \partial_r e^{A} |_{I} = 0, \]
\[ \partial_r e^{A} |_{I} = 0, \]
\[ \partial_r e^{A} |_{I} = -\frac{2}{3} M \left( (1 + \tau)^3 - \frac{1}{4} (1 + \tau)^4 \right) e^{A}. \]

We will also need some second-order radial derivatives,

\[ \partial_r^2 e^{r} |_{I} = \frac{8}{3} M \left( (1 + \tau)^3 - \frac{1}{4} (1 + \tau)^4 \right), \]
\[ \partial_r^2 e^{A} |_{I} = 0. \]

These results will be crucial for the computations on \( \mathcal{I}^- \) because it provides information concerning the decay of connection and frame coefficients, in particular we e.g. find that

\[ \partial_r^2 \Gamma_{ijk} |_{\mathcal{I}^-} = O(r^2) \quad \text{for all } n \geq 5. \]

Because of this only a bounded number of terms will contribute to the critical logarithmic terms producing order in the Bianchi equation for transverse derivatives of any order.

9.2.2 Some expansion coefficients at \( I^- \)

In analogy to the proof of Lemma 7.1, replacing \( r^\infty \) there by \( m_0 + m_1 \), where \( m_1 \) depends on the first-order contribution, cf. (7.47)-(7.67), one shows that for \( 1 \leq k \leq m_0 + 1 \) (\( 1 \leq k \leq m_0 \) for
\[ \partial_v^k V_{AB}^\pm |_{x-} \] since the order \( m_0 + 1 \) may have log terms
\[
\begin{align*}
\partial_v^k \hat{\Gamma}^1_{11} |_{x-} &= p^{k-1} + O(r^{m_0+1}), \\
\partial_v^k \hat{\Gamma}^0_{11} |_{x-} &= p^{k-1} + O(r^{m_0+1}), \\
\partial_v^k \hat{\Gamma}^A_{A1} |_{x-} &= p^k + O(r^{m_0+1}), \\
\partial_v^k \hat{\Gamma}^0_A |_{x-} &= p^k + O(r^{m_0+1}), \\
\partial_v^k (\hat{\Gamma}^0_{1A} + \hat{\Gamma}^1_{1A}) |_{x-} &= p^{k-1} + O(r^{m_0+1}), \\
\partial_v^k (\hat{\Gamma}^0_{1A} - \hat{\Gamma}^1_{1A}) |_{x-} &= p^k + O(r^{m_0+1}), \\
\partial_v^k (\hat{\Gamma}^A_{A1} B - \hat{\Gamma}^1_{1A} B) |_{x-} &= p^{k-1} + O(r^{m_0+1}), \\
\partial_v^k (\hat{\Gamma}^A_{A1} B - \hat{\Gamma}^1_{1A} B) |_{x-} &= p^{k+1} + O(r^{m_0+1}), \\
\partial_v^k \hat{\Gamma}^A_{A1} B |_{x-} &= p^{k-1} + O(r^{m_0+1}), \\
\partial_v^k \hat{\Gamma}^A_{A1} C |_{x-} &= p^k + O(r^{m_0+1}), \\
\partial_v^k \hat{L}_{11} |_{x-} &= p^k + O(r^{m_0+1}), \\
\partial_v^k (\hat{L}_{A0} + \hat{L}_{A1}) |_{x-} &= p^{k+1} + O(r^{m_0+1}), \\
\partial_v^k (\hat{L}_{A0} - \hat{L}_{A1}) |_{x-} &= p^k + O(r^{m_0+1}), \\
\partial_v^k e^r |_{x-} &= - \delta r + p^{k-2} + O(r^{m_0+1}), \\
\partial_v^k e^r A |_{x-} &= p^{k-1} + O(r^{m_0+1}), \\
\partial_v^k e^r A |_{x-} &= p^{k-1} + O(r^{m_0+1}), \\
\partial_v^k e^r |_{x-} &= p^{k-1} + O(r^{m_0+2}), \\
\partial_v^k e^r A |_{x-} &= p^k + O(r^{m_0+2}), \\
\partial_v^k U_{AB} |_{x-} &= p^k + O(r^{m_0}), \\
\partial_v^k W^\pm |_{x-} &= p^{k+1} + O(r^{m_0}), \\
\partial_v^k V_{AB}^\pm |_{x-} &= p^{k+2} + O(r^{m_0}).
\end{align*}
\]

\textbf{Remark 9.1} To obtain the error term for \( \partial_v^k e^r A |_{x-} \) one uses that \( \partial_v^k \Gamma_{A0} |_{x-} = O(r) \) and \( \partial_v^k e^r A |_{x-} = O(r) \) for all \( n \) by (9.11)-(9.13).

Recall the notation introduced in (4.69). We further introduce the notation
\[
V^\pm_A := \mathcal{G}^B V^\pm_{AB}.
\]

As a consequence of Lemma 7.1 and (4.93)-(4.102) we have

\textbf{Lemma 9.2} Let \( n \geq 1 \), then
\[
\begin{align*}
V^+_A (m_0, n) &= \frac{1}{2} (n - m_0 - 1) W^-_A (m_0, n), \quad n \leq m_0, \\
V^-_A (m_0, n) &= - \frac{1}{2(n - m_0 + 2)} (\Delta + 1) W^+_A (m_0, n), \quad n + 3 \leq m_0.
\end{align*}
\]

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and
\[
W_{0101}^{(m_0,n)} = -\frac{1}{2} \partial_A W_A^{+(m_0,n-1)} + \frac{1}{2} \partial_A W_A^{-(m_0,n-1)}, \quad n + 1 \leq m_0,
\]
\[
W_{01AB}^{(m_0,n)} = \partial_A W_B^{+(m_0,n-1)} + \partial_A W_B^{-(m_0,n-1)}, \quad n + 1 \leq m_0,
\]
\[
(m_0 - n + 1)W_A^{-(m_0,n)} = -\frac{1}{2n}\left( \Delta_s + (m_0 - n)(m_0 - n + 1) - 1 \right)W_A^{-(m_0,n-1)}, \quad n \leq m_0,
\]
\[
(m_0 - n + 2)V_A^{+(m_0,n)} = -\frac{1}{2n}\left( \Delta_s + (m_0 - n)(m_0 - n + 1) - 1 \right)V_A^{+(m_0,n-1)}, \quad n - 1 \leq m_0,
\]
\[
(m_0 - n + 1)V_A^{+(m_0,n)} = -\frac{1}{2n}\left( \Delta_s + (m_0 - n)(m_0 - n + 1) - 1 \right)V_A^{+(m_0,n-1)}, \quad n + 2 \leq m_0,
\]
\[
(m_0 - n - 2)V_A^{-(m_0,n)} = -\frac{1}{2n}\left( \Delta_s + (m_0 - n)(m_0 - n + 1) - 1 \right)V_A^{-(m_0,n-1)}, \quad n + 3 \leq m_0.
\]

**Remark 9.3** Given $m_0$, for transverse derivatives of order $n$, with $n$ in a certain range, we have the same recursion relations at $I^-$ as for the spin-2 equation.

**Proof:** For initial data $\Xi_{AB}$ which vanish asymptotically up to an including the order $m_0 + 1$ the error terms in (4.93)-(4.102) are the same as for a radiation field which vanishes asymptotically at any order (in fact if $\Xi^{(m_0+2)}$ is the first non-trivial term this influences the expansion in $r$ of transverse derivatives of any order only for $m \geq m_0 + m_1$ as in (9.39)-(9.61)). It then follows from Lemma 7.1 that these error terms are polynomials as in (7.117)-(7.119). In particular if $m_0$ is sufficiently large as compared to the number of transverse derivatives (as in the formulation of the lemma), the polynomials are in the kernel and their contribution vanishes.

Applying this formula recursively, the corresponding expansion coefficients can be expressed in terms of the initial data given at $\mathcal{I}^-$:

**Corollary 9.4** The following relations hold at $I^-$,
\[
W_A^{-(m_0,m_0-k)} = \frac{(-1)^{m_0-k} k!}{2^{m_0-k}(m_0-k)!m_0} \prod_{\ell=k}^{m_0-1} \left( \Delta_s + \ell(\ell + 1) - 1 \right)W_A^{-(m_0,0)}, \quad k \geq 0,
\]
\[
V_A^{+(m_0,m_0-k)} = \frac{(-1)^{m_0-k} (k+1)!}{2^{m_0-k}(m_0-k)!(m_0+1)!} \prod_{\ell=k}^{m_0-1} \left( \Delta_s + \ell(\ell + 1) - 1 \right)V_A^{+(m_0,0)}, \quad k \geq -1,
\]
\[
W_A^{+(m_0,m_0-k)} = \frac{(-1)^{m_0-k} (k-2)!}{2^{m_0-k}(m_0-k)!(m_0-2)!} \prod_{\ell=k}^{m_0-1} \left( \Delta_s + \ell(\ell + 1) - 1 \right)W_A^{+(m_0,0)}, \quad k \geq 2,
\]
\[
V_A^{-(m_0,m_0-k)} = -\frac{(-1)^{m_0-k} (k-3)!}{2^{m_0-k}(m_0-k)!(m_0-3)!} \prod_{\ell=k}^{m_0-1} \left( \Delta_s + \ell(\ell + 1) - 1 \right)V_A^{-(m_0,0)}, \quad k \geq 3.
\]

Recall that by (7.35)-(7.40) for $m_0 \geq 3$
\[
V_A^{+(m_0,0)} = -\frac{m_0(m_0+1)}{8} v_A^{(m_0+2)},
\]
\[
W_A^{-(m_0,0)} = -\frac{m_0}{4} v_A^{(m_0+2)},
\]
\[
W_A^{+(m_0,0)} = -\frac{1}{4(m_0-1)} \left( (\Delta_s - 1) v_A^{(m_0+2)} - 2 \partial_A \partial_B v_B^{(m_0+2)} \right),
\]
\[
V_A^{-(m_0,0)} = \frac{1}{8(m_0-1)(m_0-2)} \left( (\Delta_s + 1) v_A^{(m_0+2)} - 2 \partial_A \partial_B v_B^{(m_0+2)} \right),
\]
so that the above expansion coefficients of the rescaled Weyl tensor can directly be expressed in terms of the asymptotic initial data $\Xi_{AB}$.

9.2.3 Higher-order transverse derivatives on $\mathcal{I}^-$

We apply $\partial_\tau$ to (2.95) and employ the formulas derived in Section 9.2.1 & 9.2.2 which ensure that only very few terms contribute to the critical order $m_0 + 1$. For $n \leq m_0$ we find

$$(r \partial_\tau - n - 2)\partial_\tau^n V_{AB}|_{\mathcal{I}^-} = - \left( \begin{array}{c} n \\ 3 \end{array} \right) \partial_\tau^2 e_1^\tau \partial_\tau^{n-2} V_{AB} - \left( \begin{array}{c} n \\ 4 \end{array} \right) \partial_\tau^4 e_1^\tau \partial_\tau^{n-3} V_{AB} - \left( \begin{array}{c} n \\ 5 \end{array} \right) \partial_\tau^5 e_1^\tau \partial_\tau^{n-4} V_{AB}$$

$$- \left( \begin{array}{c} n \\ 3 \end{array} \right) \partial_\tau^3 e_1^\tau \partial_\tau \partial_\tau^{n-3} V_{AB} - \left( \begin{array}{c} n \\ 4 \end{array} \right) \partial_\tau^4 e_1^\tau \partial_\tau \partial_\tau^{n-4} V_{AB}$$

$$+ \left( \begin{array}{c} n \\ 2 \end{array} \right) \partial_\tau^2 \left( 3 \tilde{\Gamma}_1 \tilde{\Gamma}_1 + 2 \tilde{\Gamma}_1 \tilde{\Gamma}_1 + \frac{1}{2} \tilde{\Gamma}_C C + \frac{1}{2} \tilde{\Gamma}_C C \right) \partial_\tau^{n-2} V_{AB}$$

$$+ \left( \begin{array}{c} n \\ 3 \end{array} \right) \partial_\tau^3 \left( 3 \tilde{\Gamma}_1 \tilde{\Gamma}_1 + 2 \tilde{\Gamma}_1 \tilde{\Gamma}_1 + \frac{1}{2} \tilde{\Gamma}_C C + \frac{1}{2} \tilde{\Gamma}_C C \right) \partial_\tau^{n-3} V_{AB}$$

$$+ \left( \begin{array}{c} n \\ 4 \end{array} \right) \partial_\tau^4 \left( 3 \tilde{\Gamma}_1 \tilde{\Gamma}_1 + 2 \tilde{\Gamma}_1 \tilde{\Gamma}_1 + \frac{1}{2} \tilde{\Gamma}_C C + \frac{1}{2} \tilde{\Gamma}_C C \right) \partial_\tau^{n-4} V_{AB}$$

$$+ \left( \begin{array}{c} n \\ 3 \end{array} \right) \left( \partial_\tau^2 e_1^A \partial_\tau \partial_\tau^{n-3} W_{AB} - \partial_\tau \partial_\tau^{n-3} W_{AB} \right)_{\mathcal{I}^-}$$

$$+ \left( \begin{array}{c} n \\ 4 \end{array} \right) \left( \partial_\tau^4 e_1^A \partial_\tau \partial_\tau^{n-4} W_{AB} - \partial_\tau \partial_\tau^{n-4} W_{AB} \right)_{\mathcal{I}^-}$$

$$(\mathcal{D}_A \partial_\tau^{n+1} W^+_{AB})_{\mathcal{I}^-} + 3 M \partial_\tau (\hat{\Gamma}_A - \hat{\Gamma}_B)_{\mathcal{I}^-} + \mathcal{P}^{n+1} + \mathcal{D}(r^{m_0+2})$$

$$= - 12 M \left[ \left( \begin{array}{c} n \\ 3 \end{array} \right) + 2 \left( \begin{array}{c} n \\ 2 \end{array} \right) \right] \partial_\tau^{n-2} V_{AB} + 8 M \left[ \left( \begin{array}{c} n \\ 4 \end{array} \right) - (m_0 - 5) \left( \begin{array}{c} n \\ 3 \end{array} \right) \right] \partial_\tau^{n-3} V_{AB}$$

$$- 4 M \left[ \left( \begin{array}{c} n \\ 5 \end{array} \right) - (2m_0 - 7) \left( \begin{array}{c} n \\ 4 \end{array} \right) \right] \partial_\tau^{n-4} V_{AB}$$

$$- 4 M \left[ \left( \begin{array}{c} n \\ 3 \end{array} \right) \left( \mathcal{D}_A \partial_\tau^{n-3} W^+_{AB} \right)_{\mathcal{I}^-} + 4 M \left( \begin{array}{c} n \\ 4 \end{array} \right) \left( \mathcal{D}_A \partial_\tau^{n-4} W^+_{AB} \right)_{\mathcal{I}^-} \right. $$

$$+ \left( \mathcal{D}_A \partial_\tau^{n+1} W^+_{AB} \right)_{\mathcal{I}^-} + 3 M \partial_\tau (\hat{\Gamma}_A - \hat{\Gamma}_B)_{\mathcal{I}^-} + \mathcal{P}^{n+1} + \mathcal{D}(r^{m_0+2}) \right.$$.}

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A similar computation when \( \partial_{n}^{\nu} \) is applied to (5.41) yields for \( n \leq m_{0} \),

\[
2\partial_{n+1}^{\nu+1}W_{A}^{+}|_{x-} = - \left( \frac{n}{4} \right) \partial_{n}^{3}e_{r}^{+1}\partial_{n}^{-2}W_{A}^{+} - \left( \frac{n}{5} \right) \partial_{n}^{5}e_{r}^{+1}\partial_{n}^{-3}W_{A}^{+} - \left( \frac{n}{2} \right) \partial_{n}^{2}\left( 3\tilde{\Gamma}_{1,0}^{0} - 2\tilde{\Gamma}_{B}^{B} - 2\tilde{\Gamma}_{B}^{B} + \tilde{\Gamma}_{1,0}^{1} \right) \partial_{n}^{-2}W_{A}^{+} + \left( \frac{n}{3} \right) \partial_{n}^{3}\left( 3\tilde{\Gamma}_{1,0}^{0} - 2\tilde{\Gamma}_{B}^{B} - 2\tilde{\Gamma}_{B}^{B} + \tilde{\Gamma}_{1,0}^{1} \right) \partial_{n}^{-3}W_{A}^{+} + \left( \frac{n}{4} \right) \partial_{n}^{2}\left( 3\tilde{\Gamma}_{1,0}^{0} - 2\tilde{\Gamma}_{B}^{B} - 2\tilde{\Gamma}_{B}^{B} + \tilde{\Gamma}_{1,0}^{1} \right) \partial_{n}^{-4}W_{A}^{+} - 2\partial_{n}^{2}\partial_{n+1}^{\nu}V_{A}^{+} - (r\partial_{n} - n + 1)\partial_{n}^{2}W_{A}^{+} - 3M\partial_{n}^{2}(\tilde{\Gamma}_{1,0}^{0} - \tilde{\Gamma}_{1,0}^{1}) + \mathcal{P}_{n+1} + \mathcal{O}(r^{m_{0}+2})
\]

\[
= -12Mr \left[ \left( \frac{n}{3} \right) + 3 \left( \frac{n}{2} \right) \right] \partial_{n}^{-2}W_{A}^{+} + 8Mr \left[ 5 \left( \frac{n}{4} \right) - (m_{0} - 8) \left( \frac{n}{3} \right) \right] \partial_{n}^{-3}W_{A}^{+} + 8Mr \left[ 5 \left( \frac{n}{5} \right) - (m_{0} - 5) \left( \frac{n}{4} \right) \right] \partial_{n}^{-4}W_{A}^{+} - 2\partial_{n}^{2}\partial_{n+1}^{\nu}V_{A}^{+} - (r\partial_{n} - n + 1)\partial_{n}^{2}W_{A}^{+} - 3M\partial_{n}^{2}(\tilde{\Gamma}_{1,0}^{0} - \tilde{\Gamma}_{1,0}^{1}) + \mathcal{P}_{n+1} + \mathcal{O}(r^{m_{0}+2})
\]

For \( n \leq m_{0} - 1 \) we combine both equations to obtain the expansion coefficient in \( r \) of order \( m_{0} + 1 \).

\[
(n + 1)(m_{0} - n - 1)W_{A}^{+(m_{0}+1,n+1)} + \frac{1}{2} \left( \Delta_{s} + (m_{0} - n)(m_{0} - n + 1) - 1 \right)W_{A}^{+(m_{0}+1,n)}
\]

\[
= 2(n + 4)MV_{A}^{-(m_{0},n-2)} - \frac{1}{3}(7n - 6m_{0} + 7)MV_{A}^{-(m_{0},n-3)} + \frac{2}{3}M(n - m_{0})V_{A}^{-(m_{0},n-4)} - M(m_{0} - n - 1)(n + 7)W_{A}^{+(m_{0},n-2)} + \frac{1}{6}M(m_{0} - n - 1)(5n - 4m_{0} + 17)W_{A}^{+(m_{0},n-3)} + \frac{1}{12}M(\Delta_{s} + 1)W_{A}^{+(m_{0},n-4)} - 3M\partial_{n}B(\tilde{\Gamma}_{(A,B)}^{0,1} - \tilde{\Gamma}_{(A,B)}^{1,0})_{tt}^{(m_{0}+1,n)} - \frac{3}{2}M(m_{0} - n - 1)(\tilde{\Gamma}_{1,0}^{0} - \tilde{\Gamma}_{1,0}^{1})^{(m_{0}+1,n)}
\]

\[
= -M(m_{0} - n - 1)(n + 7)W_{A}^{+(m_{0},n-2)} - M\frac{n + 4}{n - m_{0}}(\Delta_{s} + 1)W_{A}^{+(m_{0},n-2)} - \frac{1}{6}M(m_{0} - n - 1)(5n - 4m_{0} + 17)W_{A}^{+(m_{0},n-3)} + \frac{1}{6}M\frac{9n - 8m_{0} + 5}{n - m_{0} - 1}(\Delta_{s} + 1)W_{A}^{+(m_{0},n-3)} + \frac{1}{6}M(m_{0} - n - 1)\frac{5n - 5m_{0} - 2}{n - m_{0} - 2}(\Delta_{s} + 1)W_{A}^{+(m_{0},n-4)} - 3M\partial_{n}B(\tilde{\Gamma}_{(A,B)}^{0,1} - \tilde{\Gamma}_{(A,B)}^{1,0})_{tt}^{(m_{0}+1,n)} - \frac{3}{2}M(m_{0} - n - 1)(\tilde{\Gamma}_{1,0}^{0} - \tilde{\Gamma}_{1,0}^{1})^{(m_{0}+1,n)}
\]
where we have employed Lemma 9.2. This formula holds for \( 0 \leq n \leq m_0 - 1 \) if one defines derivatives of negative order to be zero.

We still need to find expressions for \((\tilde{\Gamma}^0_{(A \ B)})_{hl} - \tilde{\Gamma}^1_{(A \ B)})_{hl}\) and \(\dot{\tilde{\Gamma}}^0_{A \ A} - \dot{\tilde{\Gamma}}^1_{A \ A}\) in terms of \(W^+_{\tau}\).

From the \((n-1)\)-st order transverse derivatives of \((2.72)-(2.78)\) we find for \(n \geq 1\) (recall \((7.61)\)),

\[
\partial^n_t \tilde{L}_{1A} \big|_{\sigma^*} = -\partial^n_t (\tilde{\Gamma}^0_{(A \ 0 \ B)_{hl}})_{hl} - \partial^n_t (\tilde{\Gamma}^1_{(A \ B \ 0)_{hl}})_{hl} - 2r \partial^n_t W^-_{A} + (n-1)r \partial^n_t (W^+_{A} + W^-_{A}) + \mathcal{O}(r^\infty),
\]

\[
\partial^n_t \tilde{\Gamma}^1_{(A \ 0 \ B)}_{hl} \big|_{\sigma^*} = -\partial^n_t (\tilde{\Gamma}^0_{(A \ 0 \ B)_{hl}})_{hl} - \partial^n_t (\tilde{\Gamma}^1_{(A \ B \ 0)_{hl}})_{hl} + (n-1)r \partial^n_t (W^+_{A} + W^-_{A}) + \mathcal{O}(r^\infty),
\]

as well as

\[
(\partial^n_t \tilde{L}_{(AB)})_{hl} \big|_{\sigma^*} = -2r \partial^n_t W^-_{A} + (n-1)r \partial^n_t (W^+_{A} + W^-_{A}) + \mathcal{O}(r^\infty),
\]

\[
(\partial^n_t \tilde{\Gamma}^0_{(A \ B)})_{hl} \big|_{\sigma^*} = -2r \partial^n_t (W^+_{A} + W^-_{A}) - \partial^n_t (\tilde{\Gamma}^1_{(A \ B \ 0)_{hl}})_{hl} - (n-1)r \partial^n_t (W^+_{A} + W^-_{A}) + \mathcal{O}(r^\infty),
\]

\[
(\partial^n_t \tilde{\Gamma}^1_{(A \ B)})_{hl} \big|_{\sigma^*} = -2r \partial^n_t (W^+_{A} + W^-_{A}) - \partial^n_t (\tilde{\Gamma}^1_{(A \ B \ 0)_{hl}})_{hl} + \mathcal{O}(r^\infty).\]

From this we deduce, for \(2 \leq n \leq m_0 + 2\),

\[
\partial^n_t (\tilde{L}_{1A} - \tilde{\Gamma}^1_{(A \ 0 \ B)})_{hl} \big|_{\sigma^*} = -2r \partial^n_t W^-_{A} + 2(n-1)r \partial^n_t (W^+_{A} + W^-_{A}) - \frac{r}{2} (n+1)(n-2) \partial^n_t (W^+_{A} + W^-_{A}) + \mathcal{O}(r^{m_0+2}),
\]

and

\[
(\partial^n_t (\tilde{L}_{(AB)} - \tilde{\Gamma}^0_{(A \ B)})_{hl} \big|_{\sigma^*} = -2r \partial^n_t (W^+_{A} + W^-_{A}) + 2(n-1)r \partial^n_t (W^+_{A} + W^-_{A}) + \mathcal{O}(r^{m_0+2}).
\]
We then determine, for $3 \leq n \leq m_0 + 2$ (one checks that this actually holds for $n \geq 1$),
\[
\partial^n_r (\widehat{f}_1^0 A - \widehat{f}_1^1 A),_r - 2(n-1)r\partial^n_r W_A^+ + r(n-1)(n-2)\partial^n_r W_A^- + \mathcal{O}(r^{\infty})
\]
and, for $3 \leq n \leq m_0 + 2$ (this formula is wrong for $n \in \{1, 2\}$),
\[
(\partial^n_r (\widehat{f}_A^0 B) - \widehat{f}_A^1 B),_r - 2(n-1)r\partial^n_r V_{AB}^+ + r(n-1)(n-2)\partial^n_r V_{AB}^- + \mathcal{O}(r^{\infty})
\]
That yields
\[
(\widehat{f}_1^0 A - \widehat{f}_1^1 A)^{(m_0+1,n)} = \frac{1}{2(n-1)(n-2)} W_A^+(m_0,n-4), \quad 3 \leq n \leq m_0 + 2.
\]
This formula holds for $n \in \{1, 2\}$ if one defines terms with $W_A^+(m_0,n)$, $n < 0$, to vanish. Using Lemma 9.2 we also obtain
\[
(\partial^n_r (\widehat{f}_A^0 B) - \widehat{f}_A^1 B),_r - 2(n-1)r\partial^n_r V_{AB}^+ + r(n-1)(n-2)\partial^n_r V_{AB}^- + \mathcal{O}(r^{m_0+2})
\]
Moreover (cf. Section 7.2 & 7.3),
\[
\mathcal{D}^B(\tilde{\Gamma}(A,B) - \tilde{\Gamma}(A_1,B_1))_{\ell}^{(m_0+1,2)} = \frac{(m_0 + 1)(m_0 + 2)}{8} e_A^{(m_0+2)} - \frac{1}{8(m_0 - 1)(m_0 - 2)}(\Delta_s + 1)(\Delta_s - 1) e_A^{(m_0+2)} - 2 \mathcal{D}^B e_B^{(m_0+2)}.
\]

Altogether we end up with the following recursion relation for \(3 \leq n \leq m_0 - 1\),
\[
(n + 1)(m_0 - n - 1)W_A^{+(m_0+1,n+1)} + \frac{1}{2}(\Delta_s + (m_0 - n)(m_0 - n + 1) - 1)W_A^{+(m_0,n)}
\]
\[
= - \frac{M}{n} \left[ (m_0 - n - 1)(n^2 + 7n - 3) + \frac{n^2 + 4n + 3}{n - m_0}(\Delta_s + 1) \right] W_A^{+(m_0,n-2)} + \frac{M}{6}(m_0 - n - 1)(5n - 4m_0 + 17 - \frac{9(n + 1)}{n(n - 1)})W_A^{+(m_0,n-3)}
\]
\[
+ \frac{M}{6}(m_0 - n - 1)(9n - 8m_0 + 5 + \frac{9(n + 1)}{n(n - 1)})(\Delta_s + 1)W_A^{+(m_0,n-3)} + \frac{M}{12(n - m_0 - 2)(5n - 5m_0 - 2 + \frac{9}{n - m_0 - 2})}(\Delta_s + 1)W_A^{+(m_0,n-4)}
\]
\[
- \frac{M}{n(n - 1)}W_A^{-(m_0,n-2)} - \frac{5}{n(n - 2)}W_A^{-(m_0,n-3)} + \frac{3}{(n - 1)(n - 2)}W_A^{-(m_0,n-4)}.
\]

Using Lemma 9.2 this can be written for \(4 \leq n \leq m_0 - 1\),
\[
(n + 1)(m_0 - n - 1)W_A^{+(m_0+1,n+1)} + \frac{1}{2}(\Delta_s + (m_0 - n)(m_0 - n + 1) - 1)W_A^{+(m_0,n)}
\]
\[
= - \frac{M}{6} \left[ \frac{1}{2(n - 3)(m_0 - n + 2)} \left[ 3 \left( (m_0 - n - 1)(n^2 + 7n - 3) + \frac{n^2 + 4n + 3}{n - m_0}(\Delta_s + 1) \right) \right.ight.
\]
\[
\left. \left. \times (\Delta_s + (m_0 - n + 2)(m_0 - n + 3) - 1) \right] (m_0 - n - 1)(5n - 4m_0 + 17 - \frac{9(n + 1)}{n(n - 1)}) - \frac{1}{m_0 - n + 1}(9n - 8m_0 + 5 + \frac{9(n + 1)}{n(n - 1)})(\Delta_s + 1) \right]
\]
\[
\times \left( \Delta_s + (m_0 - n + 3)(m_0 - n + 4) - 1 \right)
\]
\[
- \frac{M}{2(n - 1)(n - 2)}(5n - 5m_0 - 2 + \frac{9}{n - m_0 - 2})(\Delta_s + 1) \right] \right] \times W_A^{+(m_0,n-4)}
\]
\[
- \frac{3M}{n - 2} \left[ \frac{1}{n(n - 3)(m_0 - n + 4)} \left[ 1 \left( \Delta_s + (m_0 - n + 2)(m_0 - n + 3) - 1 \right) + \frac{5}{2} \right] \times (\Delta_s + (m_0 - n + 3)(m_0 - n + 4) - 1) + \frac{3}{2(n - 1)} \right] W_A^{-(m_0,n-4)}.
\]

For \(n \in \{0, 1, 2, 3\}\) one has (we assume \(m_0 \geq 4\) which is fine as the case \(m_0 = 3\) is covered by...
Section 9.1)

\[4(m_0 - 4)W_A^{+(m_0+1,4)} + \frac{1}{2} \left( \Delta_s + (m_0 - 3)(m_0 - 2) - 1 \right) W_A^{+(m_0+1,3)}\]

\[-\frac{M}{6} (23m_0^2 - 39m_0 - 212)W_A^{+(m_0,0)} + \frac{M(11m_0^2 - 227m_0 + 486)}{6(m_0 - 2)(m_0 - 3)} (\Delta_s + 1)W_A^{+(m_0,0)}\]

\[-4M \frac{1}{(m_0 - 2)(m_0 - 3)} (\Delta_s + 1)(\Delta_s + 1)W_A^{+(m_0,0)}\]

\[+ M \left((m_0 + 4)W_A^{-(m_0,0)} + \frac{1}{m_0} (\Delta_s - 1)W_A^{-(m_0,0)} \right),\]  

\[9.62\]

\[3(m_0 - 3)W_A^{+(m_0+1,3)} + \frac{1}{2} \left( \Delta_s + (m_0 - 2)(m_0 - 1) - 1 \right) W_A^{+(m_0+1,2)}\]

\[= - \frac{15}{2} M(m_0 - 3)W_A^{+(m_0,0)} + \frac{15}{2m_0 - 2} (\Delta_s + 1)W_A^{+(m_0,1)} - \frac{3(3m_0 + 1)}{m_0} W_A^{-(m_0,0)},\]  

\[9.63\]

\[2(m_0 - 2)W_A^{+(m_0+1,2)} + \frac{1}{2} \left( \Delta_s + (m_0 - 1)m_0 - 1 \right) W_A^{+(m_0+1,1)} = \frac{6}{m_0} M(m_0 + 2)W_A^{-(m_0,0)},\]  

\[9.64\]

\[(m_0 - 1)W_A^{+(m_0+1,1)} + \frac{1}{2} \left( \Delta_s + m_0(m_0 + 1) - 1 \right) W_A^{+(m_0+1,0)} = - \frac{12}{m_0} MW_A^{-(m_0,0)},\]  

\[9.65\]

where we have used (7.72)-(7.85), (7.35)-(7.40), and (7.62)-(7.66),

\[W_A^{+(m_0,0)} = \frac{m_0}{4} \nu_A^{(m_0+2)},\]

\[V_A^{+(m_0,0)} = - \frac{m_0 + 1}{2} W_A^{-(m_0,0)},\]

\[W_A^{+(m_0,0)} = \frac{1}{m_0 - 1} \partial^B \nu_A^{(m_0,0)},\]

\[V_A^{-(m_0,0)} = \frac{1}{2(m_0 - 2)} (\Delta_s + 1)W_A^{+(m_0,0)},\]

\[W_A^{-(m_0,1)} = - \frac{m_0 - 1}{2} W_A^{-(m_0,0)} - \frac{1}{2m_0} (\Delta_s - 1)W_A^{-(m_0,0)},\]

\[W_A^{+(m_0,1)} = - \frac{m_0 + 1}{2} W_A^{+(m_0,0)} - \frac{1}{2(m_0 - 2)} (\Delta_s + 1)W_A^{+(m_0,0)}.\]

9.2.4 Recursion relation

We observe that the recursion relation has the following structure for \(4 \leq n \leq m_0 - 1,\)

\[(n + 1)(m_0 - n - 1)W_A^{+(m_0+1,n+1)} + \frac{1}{2} \left( \Delta_s + (m_0 - n)(m_0 - n + 1) - 1 \right) W_A^{+(m_0+1,n)}\]

\[= a_{m_0,n} W_A^{+(m_0,n-4)} + b_{m_0,n} W_A^{-(m_0,n-4)},\]

where \(a_{m_0,n}\) and \(b_{m_0,n}\) are operators, more precisely they are polynomials in the Laplacian \(\Delta_s.\)

In fact, now it is convenient to set

\[W_A^{-(m_0,n)} := W_A^{-(m_0,0)}\]

for \(n \leq -1\) \hspace{1cm} (9.66)

as then the formula remains true for \(n \in \{0, 1, 2, 3\}\) with appropriately chosen \(a_{m_0,n}\) and \(b_{m_0,n}\)

which can be read off from (9.62)-(9.65).
The no-logs-condition for $\partial^{m_0-1}V_A^-|_\ell$ can be written as (cf. (4.104))

\[(\Delta_s + 1)W^{+(m_0+1,m_0-1)}_A - 2a_{m_0,m_0-1}W^{+(m_0,m_0-5)}_A - 2b_{m_0,m_0-1}W^{-(m_0,m_0-5)}_A = 0.\] (9.67)

By recursion one shows that

\[W^{+(m_0+1,m_0-1)}_A = 1 \left( \frac{1}{8(m_0-1)(m_0-2)} (\Delta + 5)(\Delta + 11)W^{+(m_0+1,m_0-3)}_A \right) - \frac{1}{4(m_0-1)(m_0-2)} (\Delta + 5) \left[ a_{m_0,m_0-3}W^{+(m_0,m_0-7)}_A + b_{m_0,m_0-3}W^{-(m_0,m_0-7)}_A \right] + \frac{1}{m_0-1} \left[ a_{m_0,m_0-2}W^{+(m_0,m_0-6)}_A + b_{m_0,m_0-2}W^{-(m_0,m_0-6)}_A \right] = \ldots \]

\[= \frac{(-1)^{m_0-1}}{2^{m_0-1}(m_0-1)!} \prod_{\ell=2}^{m_0} (\Delta_s + \ell(\ell + 1) - 1) W^{+(m_0+1,0)}_A + \sum_{k=0}^{m_0-2} \frac{(-1)^k(m_0 - k - 2)!}{2k(m_0 - 1)!(k + 1)!} a_{m_0,m_0-2-k} \prod_{\ell_1=2}^{k+1} (\Delta_s + \ell_1(\ell_1 + 1) - 1) W^{+(m_0,m_0-k-6)}_A + \sum_{k=0}^{m_0-2} \frac{(-1)^k(m_0 - k - 2)!}{2k(m_0 - 1)!(k + 1)!} b_{m_0,m_0-2-k} \prod_{\ell_1=2}^{k+1} (\Delta_s + \ell_1(\ell_1 + 1) - 1) W^{-(m_0,m_0-k-6)}_A.\]

We use Corollary 9.4 to conclude that the no-logs condition (9.67) becomes (recall (9.66)),

\[0 = \frac{(-1)^{m_0-1}}{2^{m_0-1}(m_0-1)!} \prod_{\ell=1}^{m_0} (\Delta_s + \ell(\ell + 1) - 1) W^{+(m_0+1,0)}_A + \sum_{k=1}^{m_0-6} \frac{(-1)^k(m_0 - k - 2)!}{m_0(l(m_0 - 1))(k + 1)!} a_{m_0,m_0-2-k} \prod_{\ell_1=1}^{k+1} (\Delta_s + \ell_1(\ell_1 + 1) - 1) W^{+(m_0,0)}_A + \frac{1}{2^{m_0-6}} \sum_{k=1}^{m_0-6} \frac{(-1)^k(m_0 - k - 2)!}{m_0^2(m_0 - 1)!} b_{m_0,m_0-2-k} \prod_{\ell_1=1}^{k+1} (\Delta_s + \ell_1(\ell_1 + 1) - 1) W^{-(m_0,0)}_A + \sum_{k=m_0-5}^{m_0-2} \frac{(-1)^k(m_0 - k - 2)!}{2k(m_0 - 1)!} a_{m_0,m_0-2-k} \prod_{\ell_1=1}^{k+1} (\Delta_s + \ell_1(\ell_1 + 1) - 1) W^{+(m_0,0)}_A + \sum_{k=m_0-5}^{m_0-2} \frac{(-1)^k(m_0 - k - 2)!}{2k(m_0 - 1)!} b_{m_0,m_0-2-k} \prod_{\ell_1=1}^{k+1} (\Delta_s + \ell_1(\ell_1 + 1) - 1) W^{-(m_0,0)}_A.\] (9.68)
Because of (9.10) the Hodge decomposition scalars of \( W^{\pm(m_0,0)}_A \) only involve spherical harmonics up to and including \( \ell = m_0 - 1 \). With (7.35)–(7.40) we deduce that a necessary condition for the non-appearance of logarithmic terms at this order is

\[
\Xi^{(m_0+3)} = \sum_{\ell=1}^{m_0} \sum_{m=-\ell}^{+\ell} \Xi^{(m_0+3)}_{\ell m}(\theta, \phi), \quad \Xi^{(m_0+3)} = \sum_{\ell=1}^{m_0} \sum_{m=-\ell}^{+\ell} \Xi^{(m_0+3)}_{\ell m}(\theta, \phi), \tag{9.69}
\]

and then the term in the first line vanishes.

We further observe that for each \( 1 \leq \ell \leq m_0 - 1 \) in the harmonic decomposition of \( W^{\pm(m_0,0)}_A \) there are terms in the above sum, for which the Laplacian does not project out their contribution.

To deduce that the no-log condition is actually violated by a non-trivial \( \Xi^{(m_0+2)}_{AB} \), we also need to make sure that the coefficients are non-zero. Since the initial data (9.9) enter this condition linearly, and \( a_{m,n} \) and \( b_{m,n} \) only involve the Laplacian and \( x^A \)-independent coefficients, we may assume w.l.o.g. initial data of the following form

\[
\Xi^{(m_0+2)}_{AB} = \Xi^{(m_0+2)}_{AB} r^{m_0+2} + \Sigma (r^{m_0+3}), \quad m_0 \geq 3, \tag{9.70}
\]

where \( \Xi^{(m_0+2)}_{AB} = (\mathcal{D}_A \mathcal{D}_B \Xi^{(m_0+2)})_{\ell m} + \epsilon_{AC} \mathcal{D}_C \Xi^{(m_0+2)}_{\ell m} \) with

\[
\Xi^{(m_0+2)}_{\ell m} = \sum_{m=-\ell}^{+\ell} \Xi^{(m_0+2)}_{\ell m}(\theta, \phi), \quad \Xi^{(m_0+2)}_{\ell m} = \sum_{m=-\ell}^{+\ell} \Xi^{(m_0+2)}_{\ell m}(\theta, \phi), \quad 2 \leq \ell \leq m_0 - 1.
\]

Let us first analyze the case where \( 1 \leq \ell \leq m_0 - 4 \). Then the last two lines in (9.68) vanish. We further observe that there are at most 4 terms in (9.68) which come along with \( W^{+(m_0,0)}_A \) and \( W^{-}(m_0,0) \), respectively, which are not projected out by the Laplacians. The no-logs condition thus becomes

\[
0 = \sum_{k=\ell-5}^{\ell-2} \frac{(m_0 - k - 2)!(k + 4)!}{(m_0 - 1)!(m_0 - 2)!(k + 1)!(m_0 - k - 6)!} a_{m_0,m_0-2-k} \times \prod_{\ell_1=1}^{k+1} \left( \Delta_s + \ell_1(\ell_1 + 1) - 1 \right) m_0^{-1} \prod_{\ell_2=\ell+6}^{m_0-1} \left( \Delta_s + \ell_2(\ell_2 + 1) - 1 \right) W^{+(m_0,0)}_A \]

\[
+ \sum_{k=\ell-5}^{\ell-2} \frac{(m_0 - k - 2)!(k + 6)!}{m_0!(m_0 - 1)!(k + 1)!(m_0 - k - 6)!} b_{m_0,m_0-2-k} \times \prod_{\ell_1=1}^{k+1} \left( \Delta_s + \ell_1(\ell_1 + 1) - 1 \right) m_0^{-1} \prod_{\ell_2=\ell+6}^{m_0-1} \left( \Delta_s + \ell_2(\ell_2 + 1) - 1 \right) W^{-}(m_0,0).
\]

To ensure that \( k \) is in \([ -1, m_0 - 6 ]\) as required by (9.68) it is convenient to set

\[
a_{m_0,n} = b_{m_0,n} = 0 \quad \text{for} \quad n \leq 3 \quad \text{and} \quad n \geq m_0. \tag{9.71}
\]

It seems important to emphasize that this is only for the evaluation of the above term, the coefficients do not vanish when the contributions from the last two lines in (9.68) are determined below.

We take divergence and curl of this equation. Then we replace the Laplacian \( \Delta_s \) in the resulting formula by the corresponding eigenvalue (be aware that it also appears in the coefficients...
$a_{m,n,k}$ and $b_{m,n,k}$. Assuming that $\Xi^{(m_0+2)}_\ell \neq 0$ and $\Xi^{(m_0+2)}_\ell \neq 0$, respectively, the no-logs condition reads

$$0 = \sum_{k=\ell-5}^{\ell-2} \frac{(k+4)!}{(k+1)!} \prod_{\ell_1=1}^{k+1} \left( \ell_1(\ell_1+1) - \hat{\ell}(\hat{\ell}+1) \right) \prod_{\ell_2=\ell+6}^{m_0-1} \left( \ell_2(\ell_2+1) - \hat{\ell}(\hat{\ell}+1) \right) \times \left( \hat{\ell}(\hat{\ell}+1)a_{m_0,m_0-2-k} \pm (k+5)(k+6)b_{m_0,m_0-2-k} \right),$$

(9.72)

where we used that (cf. Corollary 9.4)

$$\mathcal{D}^A \mathcal{W}^{-,(m_0,0)}_A = \frac{m_0}{4} \mathcal{D}^A \mathcal{V}^{(m_0+2)}_A = \frac{m_0}{8} \left( \hat{\ell}(\hat{\ell}+1) - 2 \right) \hat{\ell}(\hat{\ell}+1) \Xi^{(m_0+2)}_\ell,$$

$$\mathcal{D}^A \mathcal{W}^{+(m_0,0)}_A = -\frac{1}{4(m_0-1)} \Delta \mathcal{D}^A \mathcal{V}^{(m_0+2)}_A = \frac{1}{8(m_0-1)} \left( \hat{\ell}(\hat{\ell}+1) - 2 \right) \hat{\ell}(\hat{\ell}+1)^2 \Xi^{(m_0+2)}_\ell,$$

$$\epsilon^{AB} \mathcal{D}^A \mathcal{V}^{(m_0,0)}_{B^0} = \frac{m_0}{4} \epsilon^{AB} \mathcal{D}^A \mathcal{V}^{(m_0+2)}_{B^0} = \frac{m_0}{8} \left( \hat{\ell}(\hat{\ell}+1) - 2 \right) \hat{\ell}(\hat{\ell}+1) \Xi^{(m_0+2)}_\ell,$$

$$\epsilon^{AB} \mathcal{D}^A \mathcal{V}^{+(m_0,0)}_{B^0} = -\frac{1}{4(m_0-1)} \Delta \epsilon^{AB} \mathcal{D}^A \mathcal{V}^{(m_0+2)}_{B^0} = -\frac{1}{8(m_0-1)} \left( \hat{\ell}(\hat{\ell}+1) - 2 \right) \hat{\ell}(\hat{\ell}+1)^2 \Xi^{(m_0+2)}_\ell.$$

The “+”-sign appears for the divergence, the “−”-sign for the curl. For $4 \leq n \leq m_0 - 1$ the coefficients in (9.72) are given by

$$\hat{a}_{m_0,n} = -M \left( \frac{(m_0 - n + 3)(m_0 - n + 4) - \hat{\ell}(\hat{\ell}+1)}{4n(n-2)(n-3)(m_0 - n + 1)(m_0 - n + 2)} \right) \left( \frac{\hat{\ell}(\hat{\ell}+1)}{m_n - n} \right),$$

$$\quad \quad \times \left( \frac{(m_0 - n - 1)(n^2 + 7n - 3) + n^2 + 4n + 3}{m_n - n} \right) \left( \frac{5n - 4m_0 + 17 - 9(n+1)}{n(n-1)} \right),$$

$$- M(m_0 - n - 1) \left( \frac{(m_0 - n + 3)(m_0 - n + 4) - \hat{\ell}(\hat{\ell}+1)}{12(n-3)(m_0 - n + 2)} \right) \left( \frac{5n - 4m_0 + 17 - 9(n+1)}{n(n-1)} \right),$$

$$+ M \left( \frac{(m_0 - n + 3)(m_0 - n + 4) - \hat{\ell}(\hat{\ell}+1)}{12(n-3)(m_0 - n + 1)(m_0 - n + 2)} \right) \left( \frac{9n - 8m_0 + 5 + 9(n+1)}{n(n-1)} \right),$$

$$+ M \left( \frac{\hat{\ell}(\hat{\ell}+1) - 2}{12(m_0 - n + 2)} \right) \left( \frac{5m_0 - 5n + 2}{(n-1)(n-2)} \right),$$

$$+ M \left( \frac{\hat{\ell}(\hat{\ell}+1) - 2}{12(m_0 - n + 2)} \right) \left( \frac{5m_0 - 5n + 2}{(n-1)(n-2)} \right),$$

$$\hat{b}_{m_0,n} = -3M \left( \frac{(m_0 - n + 3)(m_0 - n + 4) - \hat{\ell}(\hat{\ell}+1)}{n(n-1)(n-2)(n-3)(m_0 - n + 3)(m_0 - n + 4)} \right) \left( \frac{\hat{\ell}(\hat{\ell}+1)}{2n(n-2)(n-3)(m_0 - n + 4)} \right),$$

$$- 15M \left( \frac{(m_0 - n + 3)(m_0 - n + 4) - \hat{\ell}(\hat{\ell}+1)}{2n(n-2)(n-3)(m_0 - n + 4)} \right) \left( \frac{9M}{2(n-1)(n-2)} \right).$$

One checks that (9.72) is equivalent to the following equation,

$$0 = (\hat{\ell} - 2)(\hat{\ell} - 3)(\hat{\ell} + 1)(\hat{\ell} + 3)(m_0 - \hat{\ell} + 3)(m_0 - \hat{\ell} + 2)(m_0 - \hat{\ell} + 1) \left( a_{m_0,m_0-\hat{\ell}+3}^{(m_0+2)} + b_{m_0,m_0-\hat{\ell}+3}^{(m_0+2)} \right)$$

$$- 3(\hat{\ell} - 1)(\hat{\ell} - 2)(\hat{\ell} + 3)(m_0 - \hat{\ell} + 3)(m_0 - \hat{\ell} + 1)(m_0 - \hat{\ell} + 2)(m_0 - \hat{\ell} + 1) \left( \hat{a}_{m_0,m_0-\hat{\ell}+2}^{(m_0+2)} + \hat{b}_{m_0,m_0-\hat{\ell}+2}^{(m_0+2)} \right)$$

$$+ 3(\hat{\ell} - 1)(\hat{\ell} - 2)(m_0 - \hat{\ell} + 3)(m_0 - \hat{\ell} + 1)(m_0 - \hat{\ell} + 2)(m_0 - \hat{\ell} + 1) \left( \hat{a}_{m_0,m_0-\hat{\ell}+1}^{(m_0+2)} + \hat{b}_{m_0,m_0-\hat{\ell}+1}^{(m_0+2)} \right)$$

$$- (\hat{\ell} - 2)\hat{a}_{m_0,m_0-\hat{\ell}} \pm (\hat{\ell} + 1)\hat{b}_{m_0,m_0-\hat{\ell}}.$$
Taking into account that the relevant range of \( \hat{\ell} \) is \( 2 \leq \hat{\ell} \leq m_0 - 4 \) we observe that (9.71) is actually not needed as the corresponding coefficients vanish anyway. This equation can be determined explicitly. A Mathematica computation shows that there is no contribution by the \( b_{m_0,n}^\ell \)-terms,\(^\text{10}\) and that the right-hand side is given by the surprisingly simple expression

\[
P^\ell_{m_0} = \frac{M}{2} \left[ 5 \left( \ell^2 + \ell - 6 \right) m_0^3 - \left( 132 - 24\ell - 55\ell^2 + 3\ell^3 + 4\ell^4 \right) m_0^2 \right. \\
- \left. \left( 186 - 61\ell - 193\ell^2 + 77\ell^3 + 139\ell^4 + 52\ell^5 \right) m_0 - 84 + 54\ell + 193\ell^2 - 87\ell^3 - 257\ell^4 - 147\ell^5 - 32\ell^6 \right],
\]

valid for \( 1 \leq \hat{\ell} \leq m_0 - 4 \).

For \( m_0 - 3 \leq \hat{\ell} \leq m_0 - 1 \) we find with (9.70) that divergence and curl of the no-logs condition (9.68) become

\[
0 = 360 \frac{(m_0 - 4)!}{(m_0 - 7)!} \left[ (m_0 - 1)m_0 - \hat{\ell}(\hat{\ell} + 1) \right] \left( m_0 - 2 \right) (m_0 - 1) - \hat{\ell}\hat{\ell} + 1 \right) \\
\times \left( \hat{\ell}(\hat{\ell} + 1)a_{m_0,6}^\ell \pm (m_0 - 3)(m_0 - 2)b_{m_0,6}^\ell \right) \\
+ 120 \frac{(m_0 - 3)!}{(m_0 - 6)!} \prod_{\ell_1 = \max(1,m_0 - 6)}^{m_0 - 6} \left( \ell_1 \left( \ell_1 + 1 \right) - \hat{\ell}(\hat{\ell} + 1) \right) \left( m_0 - 1 \right)m_0 - \hat{\ell}\hat{\ell} + 1 \right) \\
\times \left( \hat{\ell}(\hat{\ell} + 1)a_{m_0,5}^\ell \pm (m_0 - 2)(m_0 - 1)b_{m_0,5}^\ell \right) \\
+ 24 \frac{(m_0 - 2)!}{(m_0 - 5)!} \prod_{\ell_1 = \max(1,m_0 - 5)}^{m_0 - 5} \left( \ell_1 \left( \ell_1 + 1 \right) - \hat{\ell}(\hat{\ell} + 1) \right) \left( \hat{\ell}(\hat{\ell} + 1)a_{m_0,4}^\ell \pm (m_0 - 1)m_0b_{m_0,4}^\ell \right) \\
- 3(m_0 - 2)(m_0 - 3) \prod_{\ell_1 = \max(1,m_0 - 4)}^{m_0 - 4} \left( \ell_1 \left( \ell_1 + 1 \right) - \hat{\ell}(\hat{\ell} + 1) \right) \left( \hat{\ell}(\hat{\ell} + 1)a_{m_0,3}^\ell \pm m_0(m_0 - 1)b_{m_0,3}^\ell \right) \\
+ \frac{1}{2} \frac{(m_0 - 2)!}{(m_0 - 6)!} \prod_{\ell_1 = \max(1,m_0 - 6)}^{m_0 - 4} \left( \ell_1 \left( \ell_1 + 1 \right) - \hat{\ell}(\hat{\ell} + 1) \right) \left( \hat{\ell}(\hat{\ell} + 1)a_{m_0,2}^\ell \pm m_0(m_0 - 1)b_{m_0,2}^\ell \right) \\
- \frac{1}{8} \prod_{\ell_1 = \max(1,m_0 - 6)}^{m_0 - 2} \left( \ell_1 \left( \ell_1 + 1 \right) - \hat{\ell}(\hat{\ell} + 1) \right) \left( \hat{\ell}(\hat{\ell} + 1)a_{m_0,1}^\ell \pm m_0(m_0 - 1)b_{m_0,1}^\ell \right). \tag{9.74}
\]

To ensure that \( k \) is in is in the right range as required by (9.68) we set (cf. (9.71)),

\[
a_{m_0,n} = b_{m_0,n} = 0 \quad \text{for} \quad n \geq m_0. \tag{9.75}
\]

We have already considered the case \( m_0 = 3 \) in Section 9.1. To avoid a tedious case distinction it is convenient to check first that the above condition is violated for \( m_0 = 4, 5, 6, 7 \) and \( m_0 - 3 \leq \hat{\ell} \leq m_0 - 1 \), which is just a matter of computation. We may then assume \( m_0 \geq 8 \), for which we

\(^{10}\) Because of this the no-logs conditions for divergence and curl take an identical form. In Section 9.1 this was not the case. The reason for that is that for the derivation in this section we have used (9.10), while we have not used in Section 9.1 that \( \mathfrak{S}^{(3)} \) and \( \mathfrak{S}^{(5)} \) are \( \ell = 2 \)-spherical harmonics.
obtain from (9.74)

\[
0 = -20(m_0 - 5)(m_0 - 6)(m_0 - 2)(2m_0 - 3)\left( a_{m_0,6}^{m_0 - 3} + b_{m_0,6}^{m_0 - 3} \right) \\
+ 20(m_0 - 4)(m_0 - 5)(2m_0 - 3) \left( (m_0 - 3)a_{m_0,5}^{m_0 - 3} \pm (m_0 - 1)b_{m_0,5}^{m_0 - 3} \right) \\
- 4(m_0 - 4)(2m_0 - 7) \left( (m_0 - 2)(m_0 - 3)a_{m_0,4}^{m_0 - 3} \pm (m_0 - 1)m_0b_{m_0,4}^{m_0 - 3} \right) \\
- (m_0 - 3)(2m_0 - 7) \left( (m_0 - 2)(m_0 - 3)a_{m_0,3}^{m_0 - 3} \pm m_0(m_0 - 1)b_{m_0,3}^{m_0 - 3} \right), \tag{9.76}
\]

\[
0 = -20(m_0 - 1)(m_0 - 4)(m_0 - 5) \left( a_{m_0,5}^{m_0 - 2} \pm b_{m_0,5}^{m_0 - 2} \right) \\
+ 12(m_0 - 3)(m_0 - 4) \left( (m_0 - 2)a_{m_0,4}^{m_0 - 2} \pm m_0b_{m_0,4}^{m_0 - 2} \right) \\
+ 3(m_0 - 3)(2m_0 - 5) \left( (m_0 - 2)a_{m_0,3}^{m_0 - 3} \pm m_0b_{m_0,3}^{m_0 - 3} \right) \\
+ (m_0 - 2)(2m_0 - 5) \left( (m_0 - 2)a_{m_0,2}^{m_0 - 3} \pm m_0b_{m_0,2}^{m_0 - 3} \right), \tag{9.77}
\]

\[
0 = 8(m_0 - 3)(m_0 - 4) \left( a_{m_0,4}^{m_0 - 1} \pm b_{m_0,4}^{m_0 - 1} \right) + 6(m_0 - 2)(m_0 - 3) \left( a_{m_0,3}^{m_0 - 1} \pm b_{m_0,3}^{m_0 - 1} \right) \\
+ 2(2m_0 - 3)(m_0 - 2) \left( a_{m_0,2}^{m_0 - 1} \pm b_{m_0,2}^{m_0 - 1} \right) \\
+ (2m_0 - 3)(m_0 - 1) \left( a_{m_0,1}^{m_0 - 1} \pm b_{m_0,1}^{m_0 - 1} \right), \tag{9.78}
\]

where, in addition to the above expressions for \( a_{m_0,n}^{\ell} \) and \( b_{m_0,n}^{\ell} \) with \( n \geq 4 \), we obtain from (9.62)-(9.65),

\[
\begin{align*}
 a_{m_0,3}^{\ell} &= \frac{M}{6} \left( 23m_0^5 - 39m_0 - 212 \right) + \frac{M(11m_0^2 - 227m_0 + 486)}{6(m_0 - 2)(m_0 - 3)} \left( -\ell(\ell + 1) + 2 \right) \\
&\quad - \frac{4M}{(m_0 - 2)(m_0 - 3)} \left( -\ell(\ell + 1) + 2 \right)^2, \\
 b_{m_0,3}^{\ell} &= M(m_0 + 4) - \frac{M}{m_0} \ell(\ell + 1), \\
 a_{m_0,2}^{\ell} &= -\frac{15}{2} M(m_0 - 3) + M \frac{15}{2(m_0 - 2)} \left( -\ell(\ell + 1) + 2 \right), \\
 b_{m_0,2}^{\ell} &= -\frac{M}{m_0} \frac{3(3m_0 + 1)}{m_0}, \\
 a_{m_0,1}^{\ell} &= 0, \\
 b_{m_0,1}^{\ell} &= M \frac{6(m_0 + 2)}{m_0}.
\end{align*}
\]

The right-hand sides of (9.76)-(9.78) yield the following polynomials, where, again, the \( b \)-terms drop out so that we get the same polynomials for divergence and curl,

\[
\begin{align*}
 p_{m_0}^{m_0 - 1} &= \frac{M}{3} \left( 22m_0^6 - 275m_0^5 + 1345m_0^4 - 3358m_0^3 + 4777m_0^2 - 3657m_0 + 1146 \right), \tag{9.79}
\end{align*}
\]

\[
\begin{align*}
 p_{m_0}^{m_0 - 2} &= \frac{M}{3} \left( 22m_0^5 - 147m_0^4 + 334m_0^3 - 354m_0^2 + 343m_0 - 90 \right), \tag{9.80}
\end{align*}
\]

\[
\begin{align*}
 p_{m_0}^{m_0 - 3} &= -\frac{2}{3} M \left( 11m_0^4 - 26m_0^3 + 4m_0^2 - m_0 + 48 \right). \tag{9.81}
\end{align*}
\]

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9.2.5 Analysis of the no-logs condition

We want to show that the no-logs condition (9.68) is violated for data of the form (9.70) with a non-trivial $\Xi^{(m+2)}$-term, i.e. that such data inevitably produce logarithmic terms. This will be the case if and only if the polynomials $P_m$ given by (9.73), (9.79)-(9.81) do not have integer roots $\hat{m}$ in the interval $[2, m_0 - 1]$ for any integer $m_0 \geq 4$ (the case $m_0 = 3$ was treated in Section 9.1).

We start with (9.79)-(9.81). For $m_0 \geq 17$ we have

$$22m_0^6 > 275m_0^4 + 1345m_0^3 + 3358m_0^2 + 4777m_0 + 3657m_0 + 1146, \quad (9.82)$$

$$22m_0^5 > 147m_0^4 + 334m_0^3 + 354m_0^2 + 343m_0 + 90, \quad (9.83)$$

$$11m_0^4 > 26m_0^3 + 4m_0^2 + 60m_0 + 48, \quad (9.84)$$

so that the polynomials cannot have any roots. A straightforward computation shows that they do not have integer roots in the interval $[4, 16]$.

For the analysis of (9.73) it is convenient to treat $\hat{m}$ as a parameter and read $P_{\hat{m}}$ as a polynomial in $m_0$. First of all for $\hat{m} = 2$ we have

$$P_{m_0}^2 = 6M\left(4m_0^2 - 343m_0 - 897\right),$$

which has no integer roots. It remains to consider the cases where $\hat{m} \geq 3$. One easily checks that $P_{m_0}$ is negative at $m_0 = 0$, goes to plus infinity as $m_0 \to \infty$ and has one stationary point for $m_0 > 0$. It follows that $P_{m_0}$ has exactly one root for $m_0 > 0$. We need to ensure that this cannot be an integer. As a polynomial of degree 3 the root can be computed explicitly,

$$\hat{m}_0(\hat{m}) = \frac{4\hat{m}^3 + 3\hat{m}^2 - 55\hat{m}^2 - 24\hat{m} + 132}{15(\hat{m}^2 + \hat{m} - 6)} + \frac{1}{15(\hat{m}^2 + \hat{m} - 6)} \left\{ \left( a_{\hat{m}} + i \sqrt{b_{\hat{m}}^2 - a_{\hat{m}}^2} \right)^{1/3} + \left( a_{\hat{m}} - i \sqrt{b_{\hat{m}}^2 - a_{\hat{m}}^2} \right)^{1/3} \right\} \quad (9.85)$$

$$= \frac{4\hat{m}^3 + 3\hat{m}^2 - 55\hat{m}^2 - 24\hat{m} + 132}{15(\hat{m}^2 + \hat{m} - 6)} + \frac{2\sqrt{b_{\hat{m}}^2}}{15(\hat{m}^2 + \hat{m} - 6)} \cos \left( \frac{1}{3} \sqrt{b_{\hat{m}}^2 / a_{\hat{m}}^2 - 1} \right) \quad (9.86)$$

$$= \frac{16\hat{m}^2 + 256\hat{m}^3 - 3545}{20} + q(\hat{m})\hat{m}^{-1}, \quad (9.87)$$

with

$$a_{\hat{m}} := 64\hat{m}^2 + 4824\hat{m}^3 + 30768\hat{m}^4 + 9180\hat{m}^5 - 284037\hat{m}^6 - 406854\hat{m}^7 + 585521\hat{m}^8 + 1228797\hat{m}^9 + 291384\hat{m}^{10} - 293463\hat{m}^{11} - 191484\hat{m}^{12} - 8748\hat{m}^{13} + 6048\hat{m}^{14},$$

$$b_{\hat{m}} := 16\hat{m}^6 + 804\hat{m}^7 + 2734\hat{m}^8 - 1662\hat{m}^9 - 12113\hat{m}^{10} - 7308\hat{m}^{11} + 5301\hat{m}^{12} + 1944\hat{m}^{13} + 684.$$

We want to derive an estimate for $q(\hat{m})$. Note that $a_{\hat{m}}, b_{\hat{m}}$ and $b_{\hat{m}}^2/a_{\hat{m}}^2 - 1$ are positive in the range of interest, so $m_0(\hat{m})$ is a real function. We set $x := 1/\hat{m}$, then

$$q(\hat{m}(x)) = -\frac{32 + 804x - 9935x^2 - 15147x^3 + 63282x^4 - 8\sqrt{b_{\hat{m}}^2} \cos \left( \frac{1}{3} \sqrt{b_{\hat{m}}^2 / a_{\hat{m}}^2 - 1} \right)}{60x^3(1 + x - 6x^2)},$$

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with
\[ a_x := 64 + 4824x + 30768x^2 + 9180x^3 - 284037x^4 - 406854x^5 + 585521x^6 + 1228797x^7 + 291384x^8 - 293463x^9 - 191484x^{10} - 8748x^{11} + 6048x^{12}, \]
\[ b_x := 16 + 804x + 2734x^2 - 1662x^3 - 12113x^4 - 7308x^5 + 5301x^6 + 1944x^7 + 684x^8. \]

A Taylor expansion yields for \( x < 10^{-3} \)
\[ \sqrt{b_x} = 4 + \frac{201}{2}x - \frac{29465}{32}x^2 + q^{(1)}(x)x^3, \quad |q^{(1)}| < 2 \times 10^4, \]
\[ \frac{1}{60(1 + x - 6x^2)} = \frac{1}{60} - \frac{1}{60}x + \frac{7}{60}x^2 + q^{(2)}(x)x^3, \quad |q^{(2)}| < 2. \]

We further find for \( x < 10^{-6} \) (set \( c_x := b_x/a_x^2 - 1 \))
\[ |\partial_x^3 c_x| < 6.8 \times 10^5, \quad \left| \frac{(\partial_x c_x)^3}{c_x} \right| < 20, \quad \left| \frac{(\partial_x c_x)^3}{c_x^2} - 2\frac{\partial_x c_x}{c_x} \partial_x^2 c_x \right| < 9.1 \times 10^5, \]
which yields for \( x < 10^{-6} \)
\[ \left| \partial_x^3 \cos \left( \sqrt{\frac{c_x}{3}} \right) \right| = \frac{1}{24} \left[ \cos \left( \sqrt{\frac{c_x}{3}} \right) - \frac{3}{\sqrt{c_x}} \sin \left( \sqrt{\frac{c_x}{3}} \right) \left( \frac{(\partial_x c_x)^3}{c_x^2} - 2\frac{\partial_x c_x}{c_x} \partial_x^2 c_x \right) \right. \]
\[ + \left. \frac{1}{9\sqrt{c_x}} \sin \left( \sqrt{\frac{c_x}{3}} \right) \left( \frac{(\partial_x c_x)^3}{c_x} - 36\partial_x^3 c_x \right) \right| < 10^6. \]

From this we obtain
\[ \cos \left( \frac{1}{3} \sqrt{\frac{b_x}{a_x^2} - 1} \right) = 1 - \frac{10275}{128}x^2 + q^{(3)}(x)x^3, \quad |q^{(3)}| < 1.7 \times 10^5, \]
and finally, for \( x < 10^{-6} \)
\[ |q(\hat{\ell}(x))| < 1.2 \times 10^5. \]

It follows that, for \( \hat{\ell} > 10^6 \)
\[ \left| \frac{m_0(\hat{\ell})}{m_0(\ell)} - 16\hat{\ell}^2 + 256\hat{\ell} - 3545 \right| < 1.2 \times 10^5 \times \hat{\ell}^{-1}. \]

The numerator is always an odd number, so the fraction differs from an integer by at least 1/20, i.e. the polynomial cannot have integer roots if the right-hand side is smaller than 1/20, i.e. for \( \hat{\ell} > 2.4 \times 10^6 \). It remains to be checked whether there are integer roots for \( 3 \leq \hat{\ell} \leq 2.4 \times 10^6 \). Note that given \( \hat{\ell} \) there is only one root and this is given by (9.87). A mathematica computation shows that there are no integer roots in this range of \( \hat{\ell} \). Here a remark is in order: Due to the appearance of roots there might arise a problem to recognize \( m_0(\hat{\ell}) \) as an integer due to numerical errors. We therefore rounded \( m_0(\hat{\ell}) \) to the nearest integers and plugged it in into the polynomial to check whether it is a root.

By way of summary, at least in the setting of constant (ADM) mass aspect \( M \) and vanishing dual (ADM) mass aspect \( N \) we conclude that while \( 2 \leq \ell \leq m_0 - 1 \)-spherical harmonics in \( \Xi^{m_0 + 2} \) do not produce logarithmic terms at order \( m_0 - 2 \), they do produce logarithmic terms in the next order. Taking Lemma 6.1 into account we end up with the following result:
Theorem 9.5 Consider a solution \((e^\mu_i, \hat{\Gamma}_{ij}^k, \hat{L}_{ij}, W_{ijkl})\) of the GCFE with constant mass aspect \(M\) and vanishing dual mass aspect \(N\) which is smooth at \(I^- \cup I^- \cup I^\infty\) in some weakly asymptotically Minkowski-like conformal Gauss gauge. Then the expansion of the radiation field vanishes at \(I^-\) at any order.

Remark 9.6 If the solution is only \(C^3\) the above computation shows that \(\Xi^{(m+2)}_{AB} = 0\) for \(m \leq m_0\).

Remark 9.7 We expect this result to remain true for arbitrary \(M\) and \(N\).

Acknowledgments

The author wishes to thank Helmut Friedrich and Juan A. Valiente Kroon for useful discussions and comments on the manuscript. The author is also thankful to the Max Planck Institute for Gravitational Physics in Golm, Germany, for hospitality, where part of the work on this paper has been done. Financial support by the Austrian Science Fund (FWF) P 28495-N27 is gratefully acknowledged, as well.

A Asymptotic initial value problem

A.1 Characteristic constraint equations on \(\scri^-\)

There are different versions of the CFE using different sets of unknowns. Here let us focus on the metric conformal field equations (MCFE) [27], where, besides rescaled Weyl tensor \(W_{\mu\nu\sigma\rho}\), Schouten tensor \(L_{\mu\nu}\) and conformal factor \(\Theta\), the metric \(g_{\mu\nu}\) and a certain scalar \(s\) are regarded as unknowns,

\[
\begin{align*}
\nabla_\rho W_{\mu\nu\sigma\rho} &= 0, \\
\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} &= \nabla_\rho \Theta W_{\nu\mu\rho}, \\
\nabla_\mu \nabla_\nu \Theta &= -\Theta L_{\mu\nu} + s g_{\mu\nu}, \\
\nabla_\mu s &= -L_{\mu\nu} \nabla_\nu \Theta, \\
2\Theta s - \nabla_\rho \Theta \nabla^\rho \Theta &= \lambda/3, \\
R_{\mu\nu\sigma}^\kappa[g] &= \Theta W_{\mu\nu\sigma}^\kappa + 2 \left( g_{\sigma[\mu} L_{\nu]\kappa} - \delta_{\mu[\kappa} L_{\nu]\sigma] \right).
\end{align*}
\]

In the following we assume that the cosmological constant vanishes,

\[\lambda = 0.\]

We will recall the constraint equations induced by the MCFE in a generalized wave-map gauge in adapted null coordinates \([8, 22] (\tau, r, x^A)\) on \(\scri^-\), cf. Section 2.2.3. It seems worth to emphasize that we do not assume the existence of a regular point \(i^-\) representing past timelike infinity. We assume that the null geodesics generating \(\scri^-\) emanate from \(O = \{\tau = -1, r = r_0\}\) which could be a point which represents a (possibly regular) \(i^-\), but which also could be a topological 2-sphere. Spatial infinity (at least its “intersection” with \(\scri^-\)), which also could be a point or a 2-sphere, is located at \(i^0 = \{\tau = -1, r = r_1\}\).

In comparison with [44] we present here a slightly modified system, which permits gauges where the scalar \(s := \frac{1}{4} \nabla_\rho \Theta + \frac{A}{4} R \Theta\) vanishes on \(\scri^-\), as crucial in view of a cylinder representation of spatial infinity (in fact the function \(s\) is not needed in this scheme). To do that it is
convenient to regard
\[ \Sigma := \nabla^r \Theta|_{\mathscr{I}^-} \neq 0, \quad \kappa, \quad \text{and the gauge source functions } W^\mu \] (A.7)
as the “gauge data” (rather than \( \kappa, s|_{\mathscr{I}^-}, W^\mu \)). These data are supplemented by “non-gauge”
data. On \( \mathscr{I}^- \), we take (note that this differs slightly from the data used in [44]), and that further
data need to be prescribed on an incoming null hypersurface as described in Appendix A.2
\[ \Xi_{AB} := -2(\Gamma^r_{AB})_{tt} = \nu^r (\partial_r g_{AB})_{tt} - 2\nu^r (\nabla (A\nu_B))_{tt}. \] (A.8)

It is related to the radiation field via (A.40).

The constraint equations on \( \mathscr{I}^- \) form a hierarchical system of ODEs and algebraic equations
(cf. [44], but note that in [44] a regular vertex has been assumed whence some equations take a
slightly different form here),
\[ \sigma_{AB} = 0, \quad \tau = \frac{2}{\Sigma}(\partial_r + \kappa)\Sigma, \] (A.9)
\[ L_{rr}|_{\mathscr{I}^-} = -\frac{1}{2}\left(\partial_r + \frac{1}{2}\theta^r - \kappa\right)\theta^r, \] (A.10)
\[ \left(\partial_r + \frac{1}{2}\theta^r + \kappa\right)\nu^r = -\frac{1}{2}W^r, \] (A.11)
\[ \partial_r \Theta|_{\mathscr{I}^-} = \nu_r \Sigma, \] (A.12)
\[ \xi_A = 2\nabla_A \log |\Sigma|, \] (A.13)
\[ \left(\partial_r + \frac{1}{2}\theta^r + \kappa\right)g^rA|_{\mathscr{I}^-} = \frac{1}{2}\left(\xi^A - W^A + g^{BC} F^A_{BC}\right), \] (A.14)
\[ L_{rA}|_{\mathscr{I}^-} = -\frac{1}{2}\left(\nabla^A + \frac{1}{2}\xi_A\right)\theta^r, \] (A.15)
\[ g^{A\dot{B}}L_{A\dot{B}}|_{\mathscr{I}^-} = \frac{1}{4}\theta^r \theta^r + \frac{1}{2} R, \] (A.16)
\[ \left(\partial_r + \frac{1}{2}\theta^r + \kappa\right)g^{rr}|_{\mathscr{I}^-} = \frac{1}{2}\theta^r - W^r, \] (A.17)
\[ 4L_{r^r}|_{\mathscr{I}^-} = (\partial_r + \kappa)\theta^r + \left(\nabla^A - \frac{1}{2}\xi_A\right)\xi^A - g^{rr}\left(\partial_r + \frac{1}{2}\theta^r - \kappa\right)\theta^r - R. \] (A.18)

For completeness let us also provide the constraint for the function \( s \),
\[ s|_{\mathscr{I}^-} = \frac{1}{2}\theta^r \Sigma. \] (A.20)

Here \( R \) denotes the curvature scalar associated to the Riemannian family \( r \mapsto g = g_{\dot{A}\dot{B}} dx^\dot{A} dx^\dot{B}|_{\mathscr{I}^-} \). Moreover, \( \sigma_{\dot{A}\dot{B}} \) denotes the shear, while the divergences \( \theta^+ \) and \( \theta^- \) are defined in (2.40)-(2.41). \( \kappa \) and \( \xi_A \) may be regarded here as auxiliary quantities.

In fact, it is more convenient to regard the metric coefficients \( g^{r\mu}|_{\mathscr{I}^-} \) as gauge functions
which determine \( W^\mu \) on \( \mathscr{I}^- \) [13]. Off \( \mathscr{I}^- \) we use a conformal Gauss gauge, whence the gauge
source functions are basically irrelevant for our purposes.

In a wave-map gauge one usually regards the curvature scalar \( R \) as a gauge function. Its
restriction to \( \mathscr{I}^- \) is related to \( \theta^- \) (which we regard here as gauge function) as follows,
\[ R|_{\mathscr{I}^-} = 3\left(\partial_r + \frac{1}{2}\theta^r + \kappa\right)\theta^- + 3\left(\nabla^A - \frac{1}{2}\xi_A\right)\xi^A. \]

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One therefore needs to make sure that also the integration function which arises when solving this equation can be regarded as a conformal gauge freedom in order to make sure that $\theta^-$ can, indeed, be treated as a gauge function. However, this is precisely what we have accomplished in Section 2.2.3.

We observe that the constraint imply the following useful relations

$$\left( \partial_\tau + \frac{1}{2} \theta^+ - \kappa \right) \Sigma = 0 , \quad \partial_\tau \xi_A = \nabla_A (\theta^+ - 2\kappa) . \quad (A.21)$$

Note further that when prescribing $g^{\tau \nu}|_{\mathcal{J}^-}$ and $\theta^-$ instead of $W^\mu|_{\mathcal{J}^-}$ and $R|_{\mathcal{J}^-}$ the whole system becomes a system of algebraic equations (in that case there remains the gauge freedom to extend $W^\mu|_{\mathcal{J}^-}$ and $R|_{\mathcal{J}^-}$, as computed algebraically from the constraints, off $\mathcal{J}^-$). The components considered so far do not involve the “physical” data $\Xi_{\alpha \beta}$ and are purely determined by the gauge.

Before we continue let us provide a list of the Christoffel symbols, or rather their restriction

$$\Gamma_{rr}^\tau|_{\mathcal{J}^-} = 0 = \Gamma_{rA}^|_{\mathcal{J}^-} = \Gamma_{rr}^C|_{\mathcal{J}^-} , \quad (A.22)$$

$$\Gamma_{rA}^C|_{\mathcal{J}^-} = -\frac{1}{2} \theta^+ \delta_A^C , \quad (A.23)$$

$$\Gamma_{AB}^\tau|_{\mathcal{J}^-} = -\frac{1}{2} \theta^+ \nu^C g_{AB} , \quad (A.24)$$

$$\Gamma_{AC}^B|_{\mathcal{J}^-} = \frac{C}{AB} + \frac{1}{2} \theta^+ \nu^C g_{AB} , \quad (A.25)$$

$$\Gamma_{\tau r}|_{\mathcal{J}^-} = \kappa , \quad (A.26)$$

$$\Gamma_{\tau r}|_{\mathcal{J}^-} = \nu^C \partial_\tau \nu^C - \kappa , \quad (A.27)$$

$$\Gamma_{\tau A}|_{\mathcal{J}^-} = -\frac{1}{2} \xi_A , \quad (A.28)$$

$$\Gamma_{\tau A}|_{\mathcal{J}^-} = \frac{1}{2} \xi_A + \nu^C \nabla_A \nu^C - \frac{1}{2} \theta^+ \nu^A , \quad (A.29)$$

$$\Gamma_{\tau r}|_{\mathcal{J}^-} = \frac{1}{2} \nu^C \xi_C + \left( \partial_\tau + \frac{1}{2} \theta^+ + \kappa - \nu^A \partial_\tau \nu^A \right) \nu^C , \quad (A.30)$$

$$\Gamma_{\tau r}|_{\mathcal{J}^-} = -\frac{1}{2} \nu^C \xi_A - \frac{1}{2} \nu^C (\partial_\tau + 2\kappa) g^{\tau r} , \quad (A.31)$$

$$\Gamma_{AB}|_{\mathcal{J}^-} = -\frac{1}{2} \Xi_{\alpha \beta} + \frac{1}{4} (\theta^ - + \theta^+ g^{\tau r}) g_{AB} , \quad (A.32)$$

$$\Gamma_{AC}^B|_{\mathcal{J}^-} = \frac{1}{2} \nu^C \xi_A + \left( \nabla_A - \frac{1}{2} \xi_A + \frac{1}{2} \nu^C \nu^C - \nu^A \nabla_A \nu^C \right) \nu^C - \frac{1}{4} (\theta^- + \theta^+ g^{\tau r}) \nu_C \delta_A^C , \quad (A.33)$$

$$\Gamma_{\tau A}|_{\mathcal{J}^-} = -\frac{1}{2} \nu^C (\nabla_A - \xi_A) g^{\tau r} - \frac{1}{2} \nu^C \Xi_{AB} + \frac{1}{4} (\theta^- + \theta^+ g^{\tau r}) \nu_A \xi_B \quad (A.34)$$

A somewhat lengthy calculation, which makes heavily use of these formulas for the Christoffel symbols and the constraint equations reveals that (this computation as the ones below have not
been carried out in [44] for a general wave-map gauge),\(^{11}\)

\[
(L_{A}B)_{ht} |_{\mathcal{S}^{-}} = \frac{1}{2} (\partial_{r} \Gamma_{AB}^{\mu} - \partial_{A} \Gamma_{B}^{\mu} + \Gamma_{AB}^{\mu} \Gamma_{\mu}^{\nu} - \Gamma_{A}^{\mu} \Gamma_{B}^{\nu})_{ht} \\
= - \frac{1}{2} \left( \partial_{r} - \frac{1}{2} \theta^{+} + \kappa \right) \Xi_{AB} + \frac{1}{2} \left( \nabla_{(A} \xi_{B)} \right)_{ht} - \frac{1}{4} \left( \xi_{A} \xi_{B} \right)_{ht},
\]

\[
L_{A}^{r} |_{\mathcal{S}^{-}} = \nu^{r} L_{rA} + g^{rr} L_{rA} + g^{r} B L_{AB} \\
= - \nu^{r} g_{AB} g^{BC} R_{rBC}^{D} + \nu_{A}^{r} \nu^{2} (2 L_{r} \theta - 2 g^{rr} L_{rr} - g^{r} B L_{rA}) + g^{rr} L_{rA} \\
= \frac{1}{2} \left( \nabla^{B} - \frac{1}{2} \xi^{B} \right) \left( \Xi_{AB} + \frac{1}{2} \theta^{2} \right) - \frac{1}{4} g^{rr} \left( \nabla_{A} + \frac{1}{2} \xi_{A} \right) \theta^{+}.
\]

Let us compute the independent components of the rescaled Weyl tensor. First of all, we have

\[
W_{rA} B |_{\mathcal{S}^{-}} = - \frac{1}{2 \Sigma} \left( \partial_{r} \left( \left( \partial_{r} - \theta^{+} + \kappa \right) \Xi_{AB} - \left( \nabla_{(A} \xi_{B)} \right)_{ht} + \frac{1}{2} \left( \xi_{A} \xi_{B} \right)_{ht} \right) \right) \left( \nabla_{A} + \frac{1}{2} \xi_{A} \right) \theta^{+}.
\]

In [44] we have used certain components of \( \nabla_{\mu} W_{\nu \sigma \rho} = 0 \) to determine \( W_{rA} B \) and \( W_{rA} A \) on \( \mathcal{S}^{-} \) by integrating ODEs along the null geodesic generators of \( \mathcal{S}^{-} \). However, it is more convenient to employ appropriate components of \( 2 \nabla_{\mu} L_{\nu} \sigma \rho = \nabla_{\mu} \Theta \nabla_{\nu} \sigma \rho \), which yields algebraic equations from the outset.

\[
W_{rA}^{\nu} |_{\mathcal{S}^{-}} = \frac{1}{\Sigma} \left( \partial_{r} \left( \left( \partial_{r} - \theta^{+} + \kappa \right) \Xi_{AB} - \left( \nabla_{(A} \xi_{B)} \right)_{ht} + \frac{1}{2} \left( \xi_{A} \xi_{B} \right)_{ht} \right) \right) \left( \nabla_{A} + \frac{1}{2} \xi_{A} \right) \theta^{+}.
\]

We shall use the constraints on \( \mathcal{S}^{-} \) which yields ODEs for \( W_{rA}^{\nu} |_{\mathcal{S}^{-}} \) and \( W_{rA} |_{\mathcal{S}^{-}} \),

\[
\left( \partial_{r} + \frac{3}{2} \theta^{+} \right) W_{rA}^{r} |_{\mathcal{S}^{-}} = - \left( \nabla^{A} + \frac{1}{2} \xi^{A} \right) W_{rA}^{r} + \frac{1}{4} \Xi_{AB} W_{rA} B,
\]

\[
\left( \partial_{r} + \frac{3}{2} \theta^{+} + \kappa \right) W_{A}^{r} |_{\mathcal{S}^{-}} = \frac{1}{4} g^{rr} \left( \nabla^{B} + \frac{1}{2} \xi^{B} \right) W_{rA}^{r} + \frac{1}{2} \left( \nabla_{A} + \frac{3}{2} \xi_{A} \right) W_{A}^{r} - \Xi_{A} B W_{rA}^{r} + \frac{1}{2} \left( \nabla_{B} + \frac{3}{2} \xi_{B} \right) W_{AB}^{r}.
\]

From the CFE \( \nabla_{r} L_{rr} - \nabla_{A} L_{rr} - g^{rr} \nabla_{r} L_{A} r = \Sigma W_{r}^{r} r \) and using the Bianchi identity as well as one more time, the above formulas for the Christoffel symbols, we obtain

\[
2 \left( \partial_{r} + \frac{1}{2} \theta^{+} + 2 \kappa \right) L_{rr} |_{\mathcal{S}^{-}} = \left( \partial_{r} + \partial_{r} + \frac{1}{4} \theta^{+} + 2 \kappa \right) g^{rr} \left( \nabla^{A} + \frac{5}{2} \xi^{A} \right) L_{A} r + \frac{1}{4} \Xi_{AB} L_{AB} \\
+ \frac{1}{2} \left( \left( \nabla^{A} + \frac{5}{2} \xi^{A} \right) g^{rr} \right) L_{rA} - \left( \nabla^{A} + \frac{5}{2} \xi^{A} \right) L_{A} r + \frac{1}{2} \Xi_{AB} L_{AB} \\
- \frac{1}{8} \left( \partial_{r} + \frac{1}{2} \theta^{+} + 2 \kappa \right) \left( R + \frac{1}{2} \theta^{+} \right) + \frac{1}{6} \nabla_{r} R + \Sigma W_{r}^{r} r.
\]

\(^{11}\)It seems worth to stress that an analog to the to a large extent gauge-independent field \( \Xi_{AB} \) can be defined for the ordinary characteristic Cauchy problem as well. For this one sets on a characteristic initial surface \( \Sigma \),

\[
\Xi_{AB} := - 2 (\Gamma_{AB})_{ht} - \theta^{+} \sigma_{AB} \chi.
\]

One then checks that it satisfies the equation,

\[
\left( \partial_{r} - \frac{3}{2} \theta^{+} + \kappa \right) \Xi_{AB} - \left( \nabla_{(A} \xi_{B)} \right)_{ht} + \frac{1}{2} \left( \xi_{A} \xi_{B} \right)_{ht} - \frac{1}{2} \theta^{+} \sigma_{AB} = - 2 (L_{AB})_{ht},
\]

cf. (A.77). Note that in the vacuum case the right-hand side is determined by the Einstein equations.

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Finally, let us derive an equation for \((W^r_{\ A\ B\ r\ r})_t\), somewhat more explicitly as compared to [44]. From the algebraic symmetries of the Weyl tensor it follows that

\[
W^r_{\ (\mathcal{A}\mathcal{B})r} |_{\mathcal{S}} = g^{\mathcal{A}\mathcal{B}} W^{\mathcal{A}\mathcal{B}} |_{\mathcal{S}} = -\frac{1}{2} g^{\mathcal{A}\mathcal{B}} W_{\mathcal{A}\mathcal{B}r} - \frac{1}{4} g^{\mathcal{A}\mathcal{B}} g^{\mathcal{C}\mathcal{D}} g^{\mathcal{E}\mathcal{F}} W_{\mathcal{C}\mathcal{D}\mathcal{E}\mathcal{F}r},
\]

\[
W^r_{\ \mathcal{A}\mathcal{B}\mathcal{C}} |_{\mathcal{S}} = g^{\mathcal{A}\mathcal{B}} W^{\mathcal{A}\mathcal{B}\mathcal{C}} = (W^r_{\ \mathcal{A}\mathcal{B}r} - g^{\mathcal{A}\mathcal{B}} W^r_{\ r\ r\ r}) g^{\mathcal{A}\mathcal{C}} + f_C g^{\mathcal{A}\mathcal{B}},
\]

where the specific form of \(f_C\) is irrelevant. The Bianchi identity and the algebraic symmetries of the rescaled Weyl tensor imply

\[
(\nabla_r W^r_{\ A\ B\ r\ r})_t |_{\mathcal{S}} = (\nu^r \nabla_r W^r_{\ (\mathcal{A}\mathcal{B})r} + \nabla_C W^1_{\ (\mathcal{A}\mathcal{B})r})_t
\]

\[
= (\nu^r \nabla_r W^r_{\ (\mathcal{A}\mathcal{B})r} - \frac{1}{2} g^{\mathcal{A}\mathcal{B}} g^{\mathcal{C}\mathcal{D}} \nabla_C W_{\mathcal{A}\mathcal{B}r} + \nabla_{(\mathcal{A} W^r_{\ B\ r})} - g^{\mathcal{A}\mathcal{B}} \nabla_{(\mathcal{A} W^r_{\ B\ r})})_t,
\]

as well as

\[
(\nabla_r W^r_{\ (\mathcal{A}\mathcal{B})r})_t |_{\mathcal{S}} = -(\nabla_r W^r_{\ (\mathcal{A}\mathcal{B})r} - \nabla_C W^C_{\ (\mathcal{A}\mathcal{B})r})_t
\]

\[
= (\nabla^a W^a_{\ (\mathcal{A}\mathcal{B})r} - \nu_{(\mathcal{A} W^r_{\ B})r})_t
\]

\[
= \left(\frac{1}{2} \nu^r (g^{rr}) \nabla_r W_{\mathcal{B}r} - \nu^r \nabla_r W^r_{\ A\ B} + 2 \nu_{(\mathcal{A} W^r_{\ B}r)} - g^{rr} \nu_{(\mathcal{A} W^r_{\ B}r)} + \nabla_C W^C_{\ (\mathcal{A}\mathcal{B})r} - \nu_{(\mathcal{A} W^r_{\ B}r)} \right)_t
\]

\[
= \nu^r \left(\frac{1}{2} g^{rr} \nabla_r + g^{rr} \nabla_C\right) W_{\mathcal{A}\mathcal{B}r} - \nabla_r W^r_{\ A\ B} + \nabla_{(\mathcal{A} W^r_{\ B}r)}.
\]

Altogether that yields

\[
(\nabla_r W^r_{\ A\ B\ r\ r})_t = \frac{1}{4} (g^{rr})^2 \nabla_r W_{\mathcal{B}r} - \frac{1}{2} g^{rr} (\nabla_{(\mathcal{A} W^r_{\ B}r)})_t + (\nabla_{(\mathcal{A} W^r_{\ B}r)})_t,
\]

equivalently,

\[
\left(\partial_t - \frac{1}{2} g^{rr} + 2 \kappa\right) \left( (W^r_{\ A\ B\ r\ r})_t - \frac{1}{4} (g^{rr})^2 W_{\mathcal{A}\mathcal{B}r} \right)
\]

\[
= \left[ \left( \nabla_{(\mathcal{A} W^r_{\ B}r)} - \frac{3}{4} \nabla_{(\mathcal{A} W^r_{\ B}r)} \right) \right]_t + \frac{3}{4} \tilde{\nabla}_{\mathcal{A}\mathcal{B}} W^r_{\ A\ B\ r} - \frac{3}{4} \tilde{\nabla}_{\mathcal{A}} W^r_{\ B\ r}.
\]

A remark concerning the integration functions is in order which arises when integrating those constraints which are ODEs rather than algebraic equations. We are interested in an analysis of the constraints near spatial infinity, and an asymptotic expansion of the solutions. The integration functions bring in a global aspect which encodes information of the data on the whole of null infinity and not just its asymptotic part near spatial infinity (such as e.g. the (ADM) mass aspect). In this context let us mention two possibilities to set up an asymptotic characteristic initial value problem:

(i) The first one [39] is to start with two characteristic surfaces intersecting a spherical cross section \(S\), and with one of these surfaces representing \(\mathcal{J}^-\). In that case the initial data for the constraint ODEs are determined at \(S\). Some of the data will be determined by continuity requirements at \(S\) whereas other can be prescribed freely (cf. Appendix A.2).

(ii) Alternatively [14, 31] one may prescribe data on \(\mathcal{J}^-\), regarded as a future light-cone emanating from some point \(i^-\) which represents past timelike infinity. Assuming this point to be regular in the spacetime to be constructed, the initial data are determined by regularity conditions there [10].
A.2 Asymptotic initial value problem with prescribed (ADM) mass and dual mass aspect

The conformal field equations (CFE) \cite{19, 20} permit the formulation of an asymptotic Cauchy problem where some of the data are prescribed on (a piece of) null infinity $\mathcal{I}$. The simplest situation arises when considering two null hypersurfaces which intersect transversally in a smooth spherical cross section $S$, one of them representing an incoming null hypersurface and the other one null infinity. In \cite{39} Kánnár has proved local well-posedness for the CFE in some future-neighbourhood of $S$. For this he employs that the CFE contain a symmetric hyperbolic system of evolution equations and a system of constraint equations, preserved under evolution. Solutions to the constraints are constructed from suitably chosen freely prescribable “seed data”, and well-posedness for the evolution equations follows from Rendall’s result \cite{51}, which guarantees existence in some neighborhood to the future of the intersection sphere $S$. This result has been improved recently in \cite{7}, where it is shown that a solution exists in fact in some neighborhood to the future of the whole initial surface (or rather of that part where the constraints admit a solution as there might be obstructions due to the non-linear Raychaudhuri equation).

The purpose of this appendix is to split the required data on the initial surface into “gauge data”, whose description is just a matter of choice, and the remaining “physical data”. For this it is convenient to somewhat reformulate the asymptotic Cauchy problem where the freedom to prescribe data on the incoming null hypersurface is, to some extent, shifted to the freedom to prescribe certain global quantities such as the mass aspect on the critical set $I^-$. Our aim is to set up a scheme where as many data as possible can be freely prescribed on $\mathcal{I}^-$ and its future boundary $I^-$. Such a scheme turns out to be very convenient for the analysis of the appearance of logarithmic terms at the critical sets of spatial infinity. In doing so we will choose a wave-map gauge which admits a representation of spatial infinity as a cylinder à la Friedrich.

A.2.1 Gauge freedom

Consider two null hypersurfaces $\mathcal{N}$ and $\Sigma$ with transverse intersection along a smooth submanifold $S \cong S^2$. We introduce adapted null coordinates $(\tau, r, x^A)$ (cf. \cite{8}) so that $\Sigma$ coincides with the set $\{\tau = -1\}$, while $\mathcal{N}$ is given by $\{r = r_N > 0\}$ and the intersection sphere $S$ corresponds to the set $\{\tau = -1, r = r_N\}$. The conformal factor is to be chosen in such a way that $\Sigma$ can be identified with $\mathcal{I}^-$ and its future boundary $I^-$ in the emerging vacuum spacetime.

The conformal gauge freedom hidden in the MCFE (A.1)-(A.6) arises from the freedom to choose the conformal factor $\Theta$. It can be exploited in such a way that e.g.

$$R = R^*, \quad \theta^*_N = 0, \quad \theta^*_\Sigma = 0, \quad g_{AB}|_S = s_{AB}. \quad \text{(A.48)}$$

Such a gauge can be realized as follows: Assume we have been given a spacetime $(\mathcal{M}, g)$ and a conformal factor $\Theta$. We apply a conformal rescaling $\Theta \rightarrow \Theta \Theta$ and $g \rightarrow g^2 g$ with $\Theta > 0$. To realize the condition $R = R^*$ the function $\Theta$ needs to satisfy a wave equation. This leaves the freedom to prescribe $\phi$ on $\mathcal{N} \cup \Sigma$. On $\mathcal{N}$ we have $\theta^*_N \rightarrow \theta^*_N + 2\partial_\tau \log \phi|_\mathcal{N}$, which becomes zero if the restriction of $\phi$ to $\mathcal{N}$ satisfies an appropriate ODE along each of the null geodesic generators on $\mathcal{N}$. Since any Riemannian metric on the 2-sphere is conformal to the standard metric, the initial data for $\phi|_\mathcal{N}$ at $S$ can be employed to arrange that $g_{AB}|_S = s_{AB}$. Finally, $\theta^*_\Sigma \rightarrow \theta^*_\Sigma + 2\partial_\tau \log \phi|_\Sigma$, whence $\theta^*_\Sigma = 0$ is realized by a function $\phi|_\Sigma$ which satisfies an ODE. The initial data follow from $\phi|_\mathcal{N}$ and continuity at $S$. (Note that the solutions of both ODEs $\phi|_\mathcal{N}$ and $\phi|_\Sigma$ will be positive since $\phi|_S > 0$, which in turn implies that $\phi$ will be positive at least sufficiently close to $\mathcal{N} \cup \Sigma$.)

Next, we want exploit the coordinate gauge freedom in such a way that

$$\kappa_N = 0, \quad \kappa_\Sigma = -\frac{2}{r}, \quad \partial_\tau \Theta|_S = 2, \quad g_{\tau\tau}|_S = g^*_{\tau\tau}. \quad \text{(A.49)}$$
For this, we consider coordinate transformations of the form \( \tau \mapsto \tilde{\tau} = \tilde{\tau}(\tau, x^\hat{A}) \) and \( r \mapsto \tilde{r} = \tilde{r}(r, x^\hat{A}) \). First of all we observe that (A.48) remains invariant. The gauge conditions \( \kappa_N = 0 \) and \( \kappa_\Sigma = -\frac{1}{2} \) are arranged by solving second-order ODEs for \( \tilde{\tau} \) and \( \tilde{r} \) along the null geodesic generators of \( N \) and \( \Sigma \), respectively. This still leaves the freedom to apply transformations of the form \( \tau \mapsto p^{(1)} \tau + p^{(2)} \) on \( N \) and \( r \mapsto q^{(1)}r + q^{(2)} \) on \( \Sigma \) with \( p^{(a)} \) and \( q^{(a)} \) some functions on \( S^2 \). We have imposed the conditions that \( \mathscr{J}^+ = \{ \tau = -1 \}, \ S = \{ \tau = -1, r = r_N \} \) and \( I^- = \{ \tau = -1, r = 0 \} \). This requires \( p^{(2)} = p^{(1)} - 1 \) and \( q^{(1)} = 1 + q^{(2)}r_N \). Applying both transformations we find that

\[
\partial_\tau \Theta|_S \mapsto \frac{1}{p^{(1)}} \partial_\tau \Theta, \quad g_{\tau\tau}|_S \mapsto \frac{1}{p^{(1)}(1 + q^{(2)}r_N)} g_{\tau\tau}, \tag{A.50}
\]

which clearly can be employed to realize (A.49). The remaining gauge freedom will be fixed below. Let us choose

\[
r_N = 1. \tag{A.51}
\]

In addition to (A.48) and (A.49) there remains the freedom to prescribe the gauge source functions \( W^\mu \) (cf. [8, 22, 24]) which capture the freedom to choose coordinates off the initial surface. The gauge source functions (or rather their restrictions to \( N \cup \Sigma \)) can be chosen in such a way that [44, 46]

\[
g^{\tau\tau}|_N = r_N = 1, \quad g^{\tau\tau}|_\Sigma = r, \quad g^{\tau\hat{A}}|_N = g^{\tau\hat{A}}|_\Sigma = 0, \quad g^{\tau\hat{A}}|_\Sigma = \chi(r). \tag{A.52}
\]

The function \( \chi(r) \) is a smooth, non-increasing cut-off function which is one on \([0, 1/3]\) and zero on \([2/3, 1]\). The reason for the cut-off is that \( \partial_\tau \) should be a null vector on \( \Sigma \) close to \( S \), so that it provides a parameterization of the null geodesic generators of \( N \), while we want it to be timelike close to \( I^- \) to get there conformal Gauss coordinates based on a congruence of timelike conformal geodesics.

Finally we choose

\[
R^\tau|_N = 0, \quad R^\tau|_\Sigma = 0, \quad \partial_\tau R^\tau|_\Sigma = 0. \tag{A.53}
\]

The gauge source functions and the curvature scalar are then extended to smooth spacetime functions, e.g. in such a way that one obtains conformal Gauss coordinates near spatial infinity. In which way they are chosen off the initial surface will be irrelevant for the following considerations. Once this has been done, the gauge is fixed, apart from the freedom to choose coordinates \( (x^\hat{A}) \) on \( S^2 \) (which will be irrelevant for us).

### A.2.2 Constraint equations on an incoming null hypersurface

In the gauge described in Section A.2.1 the constraint equations on \( \mathscr{J}^- \cong \Sigma \) have been derived in Appendix A.1. On \( \Sigma \) it is convenient to regard \( \Xi_{A\hat{B}}^\Sigma \) equivalently the radiation field \( W_{rA\hat{B}}|\Sigma \) supplemented by \( \Xi_{A\hat{B}}^\Sigma |_S \) and \( \partial_\tau \Xi_{A\hat{B}}^\Sigma |_S \), as the free “physical” initial data. In the “standard” approach the initial data for the ODEs for \( W_{r^\tau r^\tau}, W_{A^\tau r^\tau}, W_{A\hat{B} r^\tau} \) and \( L^{\tau\tau} \) cannot be specified freely, but follow from the data given at \( N \) and the continuity requirement at \( S \). Here, we want to present an approach where this procedure is reserved: We prescribe initial data for the ODEs at \( I^- \), i.e. at \( r = 0 \). Then we solve the ODEs and determine the data they induce at \( S \), i.e. at \( r = r_N = 1 \) and choose the data on \( N \) in such a way that all the field are continuous at \( S \) so that the results in [7, 39, 51] apply. For this it becomes necessary to discuss the constraint equations on \( \mathcal{N}^- \) as well.

In the gauge constructed above we have

\[
R|_{\mathcal{N}^-} = 0, \quad \theta^\tau|_{\mathcal{N}^-} = \kappa_{\mathcal{N}^-} = 0, \quad g^{\tau\tau}|_{\mathcal{N}^-} = 1, \quad g^{\tau\hat{A}}|_{\mathcal{N}^-} = g^{\tau\hat{A}}|_{\Sigma} = 0, \quad g_{A\hat{B}}|_S = s_{A\hat{B}}. \tag{A.54}
\]
As “physical” initial data we regard as e.g. in \([8, 51]\) the

\[ \text{conformal class of } g_{\hat{A}\hat{B}}|_\mathcal{N}dx^\hat{A}dx^\hat{B}, \]  

which is a smooth 1-parameter family of Riemannian metrics, defined at least in some neighborhood of \(S\), i.e. on \(\mathcal{N} \cong [0, \varepsilon) \times S^2\). Denote by \(\gamma_{\hat{A}\hat{B}}dx^\hat{A}dx^\hat{B}\) a representative of that conformal class. The conformal factor \(\Omega > 0\) relating \(\gamma_{\hat{A}\hat{B}}\) and \(g_{\hat{A}\hat{B}}\). \(g_{\hat{A}\hat{B}} = \Omega^2 \gamma_{\hat{A}\hat{B}}\) needs to be chosen in such a way that \(\theta^\mathcal{N} = 0\), or, equivalently,

\[ \partial_\tau \log \Omega = -\frac{1}{4} \gamma_{\hat{A}\hat{B}} \partial_\tau \gamma_{\hat{A}\hat{B}}. \]  

For a given initial datum \(\Omega|_S > 0\), which is computed from the gauge condition \(\Omega^2 \gamma_{\hat{A}\hat{B}}|_S = s_{\hat{A}\hat{B}}\), this equation determines a positive function \(\Omega\) and thus a Riemannian family \(g_{\hat{A}\hat{B}}|_\mathcal{N}\).

For smooth seed data \(\gamma_{\hat{A}\hat{B}}, g_{\hat{A}\hat{B}}|_\mathcal{N}\) admits an expansion at the intersection sphere \(S \cong S^2\) of the form

\[ g_{\hat{A}\hat{B}}|_\mathcal{N} \sim s_{\hat{A}\hat{B}} + \sum_{n=1}^{\infty} h_{\hat{A}\hat{B}}^{(n)}(1 + \tau)^n. \]  

In fact this expansion will be the only relevant part of the data with regard to the problem we are interested in. As “non-gauge”-part of the asymptotic expansion (A.57) one may regard the trace-free part of the \(h^{(n)}\)'s: Indeed, instead of \(\gamma_{\hat{A}\hat{B}}\) we may prescribe a set of \(s\)-tracefree tensors \((h_{\hat{A}\hat{B}}^{(n)})_{\mathcal{H}^t}, n \in \mathbb{N}, \mathcal{H}^t \subset S\). The gauge condition \(\theta^\mathcal{N} = 0\) then determines all the traces \(s_{\hat{A}\hat{B}} h_{\hat{A}\hat{B}}^{(n)}\) by solving a hierarchical system of algebraic equations. This determines the expansion (A.57) which then can be extended in any way to a \(\theta^\mathcal{N} = 0\)-family of Riemannian metrics on \(\mathcal{N}\).

Continuity of \(\partial_\tau g_{\hat{A}\hat{B}}\) at \(S\) requires

\[ \mathcal{X}_{\hat{A}\hat{B}}|_S = (\partial_\tau g_{\hat{A}\hat{B}})_\mathcal{H}^t|_S = (h_{\hat{A}\hat{B}}^{(1)})_{\mathcal{H}^t}. \]  

The shear \(\sigma^\mathcal{N}\) of \(\mathcal{N}\) depends only on the conformal class of \(\gamma_{\hat{A}\hat{B}}\), cf. [8]. Its expansion at \(S\) reads

\[ \sigma^\mathcal{N} = \frac{1}{2}(g^{\hat{B}\hat{C}} \partial_\tau g_{\hat{A}\hat{C}})|_\mathcal{N} = \frac{1}{2} (h_{\hat{A}\hat{B}}^{(1)})_{\mathcal{H}^t} + (h_{\hat{A}\hat{B}}^{(2)})_{\mathcal{H}^t}(1 + \tau) + O(1 + \tau)^2. \]  

Angular indices which refer to fields defined on \(\mathcal{N}\) are raised and lowered with \(g_{\hat{A}\hat{B}}|_\mathcal{N}\) while those of their expansion coefficients at \(S\) are raised and lowered with \(s_{\hat{A}\hat{B}}\).

Let us determine all the remaining fields on \(\mathcal{N}\) which are needed as initial data for the symmetric hyperbolic system of evolution equations implied by the MCFE. From the definition of the Schouten tensor in terms of \(g\) one finds that

\[ L_{\tau\tau}|_\mathcal{N} = \frac{1}{2} R_{\tau\tau}[g] = -\frac{1}{2} |\sigma^\mathcal{N}|^2, \]  

where \(|\sigma|^2 := \sigma^\hat{A}\hat{B} \sigma_{\hat{A}\hat{B}}\). Here (and in what follows) we make extensively use of the expressions for the Christoffel symbols in adapted null coordinates computed in [8, Appendix A].

Next, we evaluate the \((\tau\tau)\)-component of (A.3),

\[ \partial^2_{\tau\tau} \Theta|_\mathcal{N} = -\Theta L_{\tau\tau}, \quad \text{with} \quad \Theta|_S = 0, \quad \partial_\tau \Theta|_S = 2. \]  

The first initial datum makes sure that \(S\) correspond to a cross-section of \(\mathcal{J}^-\) while the second datum is our gauge condition (A.49). In particular this yields the expansion

\[ \Theta = 2(1 + \tau) + \frac{1}{24} |h_{\mathcal{H}^t}^{(1)}|^2 (1 + \tau)^3 + O(1 + \tau)^4. \]
The \((\tau A)\)-component of (A.3) together with the definition of the Schouten tensor yields an expression for \(\xi^N_A \equiv -\frac{1}{2} \Gamma^N_{\tau A}\big|_N\) and \(L_{\tau A}\big|_N\):

\[
(\partial_\tau - \frac{2}{1+\tau} + O(1+\tau))\xi^N_A = 2\nabla_B \sigma^{NB}_A + 4 \frac{\partial_r \sigma^N_A}{\Theta} - 4 \sigma^{NB}_A \frac{\partial_r \Theta}{\Theta},
\]

(A.63)

\[
L_{\tau A}\big|_N = \frac{1}{2} \nabla_B \sigma^{NB}_A - \frac{1}{4} \partial_r \xi^N_A.
\]

(A.64)

The Levi-Civita connection associated to \(g_{AB}\big|_N\) is denoted by \(\nabla\). The ODE for \(\xi^N_A\) takes the asymptotic form

\[
(\partial_\tau - \frac{2}{1+\tau} + O(1+\tau))\xi^N_A = \mathcal{D}_B (h^{(1) B}_A)_{tt} + 2(1+\tau)\mathcal{D}_B (h^{(2) B}_A)_{tri} + O(1+\tau)^2,
\]

(A.65)

where \(\mathcal{D}\) denotes the Levi-Civita connection of \(s_{AB}\). This is a Fuchsian ODE and there remains a gauge freedom to prescribe

\[
\varsigma_A := \partial^2 r \xi^N_A |_S.
\]

(A.66)

This corresponds to the freedom to prescribe the torsion 1-form on the intersection surface of two null hypersurfaces intersecting transversally in the physical spacetime \((\mathcal{M}, \tilde{g})\) (cf. e.g. [13]).

In general, the asymptotic expansion of the solution of the \(\xi^N_A\)-equation will involve logarithmic terms. The solution will be smooth at \(S\) if and only if a no-logs-condition holds,

\[
\nabla_b (h^{(2) B}_A)_{tt} = 0 \iff (h^{(2) B}_A)_{tt} = 0.
\]

(A.67)

This recovers the no-logs-condition derived in [15] expressed in the conformally rescaled spacetime and in our current gauge. We assume that this condition is satisfied. Then

\[
\xi^N_A = -\mathcal{D}_B (h^{(1) B}_A)_{tt} (1+\tau) + \frac{1}{2} \varsigma_A (1+\tau)^2 + O(1+\tau)^3.
\]

(A.68)

The results in [7, 15, 45] then tell us that this already implies that there exists a smooth extension through \(\mathcal{F}^-\).

Taking the trace of the \((AB)\)-component of (A.3) and combining it with (A.5) and the \(\tau\)-component of (A.4), the definition of the Schouten tensor and the gauge condition \(R|_N = 0\), we obtain the following system \((R^N\text{ denotes the curvature scalar associated to }g_{AB}\big|_N)\):

\[
(\partial_\tau - \frac{\partial_r \Theta}{\Theta} - \frac{\partial^2 r \Theta}{\partial_r \Theta}) \partial_r \theta_N = - \left( \partial_\tau - \frac{\partial^2 r \Theta}{\partial_r \Theta} \right) \left( R^N + \nabla^A \xi^N_A \frac{1}{2} |\xi^N|^2 - \frac{2 \partial_\tau \theta}{\Theta} \right)
\]

\[
+ \frac{2 \partial^2 r \Theta}{\Theta^2 \partial_r \Theta} \left( \partial_r \Delta \Theta - \nabla_A \Theta \nabla^A \Theta \right) - \frac{4 L_{\tau A} \nabla^A \Theta}{\Theta},
\]

(A.69)

\[
s|_N = \frac{1}{4} \left( R^N + \nabla^A \xi^N_A \frac{1}{2} |\xi^N|^2 + \partial_r \theta_N \right) + \frac{\Delta \Theta}{2} - \frac{\theta_N}{4} \partial_r \Theta,
\]

(A.70)

\[
\partial_r \theta|_N = \frac{1}{\partial_r \Theta} \left[ \Theta s - \frac{1}{2} \nabla_A \Theta \nabla^A \Theta \right],
\]

(A.71)

\[
g^{\hat{A} \hat{B}} L_{\hat{A} \hat{B}}|_N = \frac{1}{2} R^N + \frac{1}{2} \nabla^A \xi^N_A \frac{1}{4} |\xi^N|^2 + \frac{1}{2} \partial_r \theta_N,
\]

(A.72)

\[
L_{\tau \tau}|_N = - \frac{1}{2} g^{\hat{A} \hat{B}} L_{\hat{A} \hat{B}}.
\]

(A.73)

Near \(S\) the ODE for \(\theta^{-N}\) takes the form (note that \(R^N + \nabla^A \xi^N_A = 2 + O(1+\tau)^2\))

\[
(\partial_\tau - \frac{1}{1+\tau} + O(1+\tau)) \partial_r \theta_N = O(1+\tau),
\]

(A.74)
and the boundary conditions are $\theta_N|_s = -2\theta^e_S = 0$ and $\zeta := \partial^2_{\tau\tau} \theta_N|_s$, whence

$$
\theta_N = \frac{1}{2} \zeta (1 + \tau)^2 + O(1 + \tau)^3.
$$

Moreover,

$$
g^{AB} L_{AB}|_N = 1 + \frac{1}{2} \zeta (1 + \tau) + O(1 + \tau)^2.
$$

The tracefree part of the $(\hat{A}\hat{B})$-component of (A.3) combined with the definition of the Schouten tensor in terms of $g$ provides the following equations

$$
\left( \partial_{\tau} - \frac{\partial_{\tau} \Theta}{\Theta} \right) \Xi^N_{AB} = \left( \nabla^N_{(A} \xi^N_{B)} - \frac{1}{2} \Sigma^N_{ AB} + 2 \Theta^{-1} \nabla^A \nabla^B \Theta \right)_{\tau\tau} + \sigma^N_{ AB} \left( \frac{\theta_N}{2} + 2 \frac{\partial_{\tau} \Theta}{\Theta} \right),
$$

$$
(L_{AB}^N)_{\tau\tau} = \left( \frac{1}{2} \nabla^N_{(A} \xi^N_{B)} - \frac{1}{4} \Sigma^N_{ AB} + \frac{1}{4} \theta_N \sigma^N_{ AB} - \frac{1}{2} \Theta \Xi^N_{AB} \right)_{\tau\tau},
$$

where

$$
\Xi^N_{ AB} := -2(\Gamma^N_{AB})_{\tau\tau} - g^{\tau\tau} \sigma^N_{AB}|_\Sigma.
$$

Near $S$, the ODE for $\Xi^N_{AB}$ is of the form

$$
\left( \partial_{\tau} - \frac{1}{1 + \tau} + O(1) \right) \Xi^N_{AB} = O(1 + \tau),
$$

and the data $\partial_{\tau} \Xi^N_{AB}|_S$ are determined by the data $\Xi^N|_S$ given on $\Sigma$,

$$
\partial_{\tau} \Xi^N_{AB}|_S = \partial_{\tau} \Xi^N_{AB}|_S = \Sigma^{(1)}_{AB}.
$$

From the definition of the Weyl tensor we find

$$
\Theta W_{\tau\tau\tau\tau AB}|_N = R_{\tau\tau AB} - g_{AB} L_{\tau\tau\tau} = -g_{\hat{B}\hat{C}} \partial_{\tau} \sigma^{\hat{C}\hat{C}}_{\hat{A}} = -(\hat{\theta}^{(2)}_{AB})_{\tau\tau} + O(1 + \tau) = O(1 + \tau),
$$

as follows from the no-logs condition (A.67).

Using the algebraic symmetries of the Weyl tensor we extract from (A.2) the following set of equations,

$$
\partial_{\tau} \Theta W_{\tau\tau\tau\tau AB}|_N = \partial_{\tau} L_{\tau AB} - \left( \nabla^A \xi^N_{AB} + \frac{1}{2} \xi^N_{AB} \right) L_{\tau\tau\tau} + \sigma^N_{ AB} L_{\tau\tau\tau} + \nabla^B \Theta W_{\tau\tau AB},
$$

$$
\partial_{\tau} \Theta W_{\tau\tau\tau\tau AB} = 2 \left( \nabla^N_{(A} \xi^N_{B)} - \frac{1}{2} \Sigma^N_{ AB} \right) L_{\tau\tau\tau} - 2 \sigma^N_{ AB} (L_{(B)\tau\tau})_{\tau\tau} + 2 \nabla^A \Theta W_{\tau\tau AB},
$$

$$
\partial_{\tau} L_{\tau AB}|_N = \sigma^N_{ AB} (L_{\tau\tau\tau})_{\tau\tau} - 2 \partial_{\tau} L_{\tau\tau\tau} - \left( \nabla^A - \frac{1}{2} \xi^N_{AB} \right) L_{\tau AB} + \frac{1}{2} \theta^{-N} L_{\tau\tau\tau} + \nabla^A \Theta W_{\tau\tau\tau AB},
$$

$$
\partial_{\tau} L_{\tau AB}|_N = \nabla^N_{(A} \Theta W_{\tau\tau\tau\tau} - \frac{1}{2} \xi^N_{AB} \Theta W_{\tau\tau\tau\tau} + \frac{1}{4} \nabla^B \Theta W_{\tau\tau\tau AB} + \nabla^N \Theta L_{\tau\tau\tau},
$$

$$
\partial_{\tau} \Theta W_{\tau\tau\tau\tau AB} = \nabla^N L_{\tau\tau\tau} - \nabla^B (L_{AB})_{\tau\tau\tau} + \sigma^N_{ AB} L_{\tau\tau\tau} + \frac{1}{2} \sigma^N_{ AB} L_{\tau\tau\tau} + \frac{1}{4} \theta^{-N} L_{\tau\tau\tau}
$$

The initial data for (A.85) follow from the data on $\Sigma$ by continuity, $L_{\tau AB}|_S = 0$. 

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The \((\tau \tau \tau)\)-component of (A.2) together with the contracted second Bianchi identity provides an ODE for \(L_{\tau \tau |N}\):

\[
2\partial_{\tau} L_{\tau |N} = \partial_{\tau} W_{\tau \tau \tau \tau} + \nabla^A \Theta W_{\tau \tau \tau A} + \theta^{-N} L_{\tau \tau} - \left( \nabla^A - \frac{3}{2} \ell^A \right) L_{\tau \tau}^A + \frac{1}{2} \Xi_{AB} (L_{A B})_{\tau \tau} + \frac{1}{6} \partial_\tau R.
\]

(A.87)

Again, the data at the intersection sphere follow from those on \(\Sigma\), \(L_{\tau \tau |S} = 0\).

The remaining components for the rescaled Weyl tensor follow from the Bianchi equation and algebraic symmetries of the Weyl tensor,

\[
\partial_{\tau} (W_{A B})_{\tau \tau} \bigg|_N = \frac{3}{4} \xi^N (W_{A B})_{C} + \frac{3}{4} \Xi_{A B} W_{\tau \tau \tau r} + \left( (\nabla^A - \frac{7}{2} \ell^A) W_{B \tau \tau r} \right)_{\tau \tau}
\]

(A.88)

where the initial data are determined by the radiation field at \(S\).

From all these equations one may determine smooth expansions of all the relevant fields on \(\mathcal{N}\) near \(S\) (assuming, as a matter of course, that the no-logs condition (A.67) holds).

We obtain the following

**Proposition A.1**

1. Consider two smooth null hypersurfaces \(\mathcal{N}\) and \(\Sigma\) in a 3+1-dimensional manifold with transverse intersection along a smooth submanifold \(S \cong S^2\) in adapted null coordinates (so that \(\mathcal{N} = \{ r = 1 \}, \Sigma = \{ r = -1 \}\) and \(S = \{ r = -1, r = 1 \}\)). Given an initial data set which consists of

   (i) a smooth family \(\tau \mapsto \gamma_{A B}(\tau)\) of Riemannian metrics on \(\mathcal{N}\),

   (ii) a smooth family \(r \mapsto W_{A B}(r)\) of symmetric, s-trace-free tensor fields on \(\Sigma\) representing the radiation field,

   (iii) a function \(\zeta\), a 1-form \(\varsigma_A\) and two symmetric trace-free tensors \(\Sigma^{(n)}_{A B}\), \(n = 0, 1\) on \(S\),

   and assume that the no-logs condition (A.67) is satisfied at \(S\). Then there exists a unique smooth (continuous at \(S\)) solution \((\Theta, s, g_{\mu \nu}, L_{\mu \nu}, W_{\mu \nu r})\) to the characteristic constraint equations induced by the MCFE on \(\mathcal{N} \cup \Sigma\) in the gauge described in Section A.2.1 such that

   (a) \(\gamma_{A B} = [g_{A B}]|_\mathcal{N}\),

   (b) \(W_{A B} = [W_{A B}]|_\Sigma\),

   (c) \(\varsigma = 4 W_{\tau \tau \tau |S}, \varsigma_A = 8 W_{\tau \tau \tau A |S}, \Sigma^{(0)}_{A B} = \Xi_{A B}|_S\) and \(\Sigma^{(1)}_{A B} = \partial_r \Xi_{A B}|_S\),

   (d) \(\Sigma = \mathcal{J}^-.\)

   One may regard the set \((\gamma_{A B}, W_{A B}, \varsigma, \varsigma_A, \Sigma^{(0)}_{A B}, \Sigma^{(1)}_{A B})\) as “physical” seed data for the evolution equations.\(^{12}\)

2. Given seed data \((\gamma_{A B}, W_{A B}, \varsigma, \Sigma^{(0)}_{A B}, \Sigma^{(1)}_{A B})\) it follows from the results in [7, 39, 51] that an \((up to conformal diffeomorphisms)\) unique solution to the MCFE exists in some neighborhood to the future of \(\mathcal{J}^-\) to the future of \(\mathcal{N}\), supposing that the Raychaudhuri equation does not produce conjugate points. Since our main interest lies in the behavior near \(\mathcal{J}^-\) and the critical set \(I^-\), this is irrelevant for our purposes.

\(^{12}\) As trace-free symmetric tensors on \(S^2\), \(\Sigma^{(0)}_{A B}\) and \(\Sigma^{(1)}_{A B}\) are both determined (via Hodge decomposition) in terms of 2 functions. Here we use a gauge where \(r_{\mathcal{N}} = 1\). In fact this gauge freedom can alternatively be employed to prescribe one of these functions. This freedom will be relevant when it is shifted to \(I^-\) in the next section, and it is shown in Section 3.4 that this is possible.
A.2.3 An alternative initial data set

In view of an analysis of the behavior of the CFE at spatial infinity it is convenient to pre-

scribe as many data as possible at the critical set $I^-$ \( \equiv S^2 \). Instead of \( (r \rightarrow r_B^A(\tau), r \rightarrow W_A^B(r), \xi, \zeta, \Sigma_A^{(0)}, \Sigma_A^{(1)}, \zeta_A^{(0)}, \zeta_A^{(1)}) \) let us therefore consider the following initial data set

(i) a smooth family \( r \mapsto W_A^B(r) \) of symmetric, \( s \)-tracefree tensor fields on \( \mathcal{R}^- \) representing the radiation field,

(ii) two functions \( \hat{M} = \frac{1}{r} W_A^B(r) \big|_{\mathcal{R}^-} \) (which can be identified as the (ADM) mass aspect, cf. Section 4.1) and \( N := -\frac{1}{r} \hat{A} B \hat{C} \hat{C} \partial_I^2 \hat{C} \mathcal{R}^- \big|_{\mathcal{R}^-} \) (which can be identified as the dual mass aspect, i.e. a NUT-like parameter, cf. Section 3.4.1).

(iii) a 1-form \( L_A = 2 \partial_I^2 W_A^B \big|_{\mathcal{R}^-} \) on \( \mathcal{R}^- \) (which is related to the angular momentum), and

(iv) a set \( \{ c_{A B}^{(n+2,n)} \} \) of symmetric, \( s \)-tracefree tensors on \( \mathcal{R}^- \), where \( c_{A B}^{(n+2,n)} \) corresponds to the \((n + 2)\)th-order expansion coefficient of \( \partial_I^2 (W_A^B)\big|_{\mathcal{R}^-} \), \( n \geq 0 \).

\( M \) and \( L_A \) provide the initial data for the ODEs (A.43)-(A.44) for \( W_A^B \big|_{\mathcal{R}^-} \) and \( W_A^B \big|_{\mathcal{R}^-} \) (note that the latter one is of Fuchsian type at \( I^- \)). They substitute the data \( \xi \) and \( \zeta_A \) at \( S \), to which they are, via the constraints, in one-to-one correspondence. Similarly, the freedom to prescribe \( \Sigma_A^{(0)} = \Xi \mathcal{R}^- \big|_{\mathcal{R}^-} \) and \( \Sigma_A^{(1)} = \zeta \mathcal{R}^- \big|_{\mathcal{R}^-} \) can be shifted by the second-order ODE (A.40) to the freedom to prescribe \( \partial_I \Xi \mathcal{R}^- \big|_{\mathcal{R}^-} \) and \( \partial_I^2 \Xi \mathcal{R}^- \big|_{\mathcal{R}^-} \). In the previous section we have chosen a gauge where \( r_N \rightarrow 1 \). This gauge freedom, which arises from a freedom to rescale \( r \) can be used to prescribe \( \mathcal{R}^- \mathcal{R}^- \partial_I^2 \Xi \mathcal{R}^- \big|_{\mathcal{R}^-} \) instead (cf. Section 3.4), so that, via Hodge decomposition, the function \( N \) is left as “physical” part of the data. The second datum \( \partial_I \Xi \mathcal{R}^- \big|_{\mathcal{R}^-} \) needs to vanish if one requires the rescaled Weyl tensor to be bounded at \( I^- \) (cf. (4.15)), whence we do not consider it here (it is tacitly assumed to be trivial).

Let us derive equations for \( \partial_I^2 (W_A^B) \big|_{\mathcal{R}^-} \). For this set \( \nabla_{\tau}^{(n)} := \nabla_{\tau} \ldots \nabla_{\tau} \). Suppose that the fields \( \partial_I^{k+1} \Theta, \partial_I^k s, \partial_I^{k+1} g_{\mu \nu}, \partial_I^k L_{\mu \nu}, \partial_I^k W_{\mu \nu \rho \sigma} \big|_{\mathcal{R}^-} \), \( k \leq n - 1 \), have been computed from appropriate smooth seed data. We employ the MCFE to compute \( \partial_I^{n+1} \Theta, \partial_I^n s, \partial_I^{n+1} g_{\mu \nu}, \partial_I^n L_{\mu \nu}, \partial_I^n W_{\mu \nu \rho \sigma} \) on \( \mathcal{R}^- \). It turns out that the equations are algebraic, except the ones for \( \partial_I L_{\tau \tau} \), and for certain components of the metric and the rescaled Weyl tensor. For \( n \geq 1 \) we have

\[
\nabla_{\tau}^{(n+1)} \Theta \big|_{\mathcal{R}^-} = - \nabla_{\tau}^{(n+1)} (\Theta L_{\tau \tau}) + g_{\tau \tau} \nabla_{\tau}^{(n-1)} s, \tag{A.89}
\]

\[
\nabla_{\tau}^{(n)} s \big|_{\mathcal{R}^-} = - \nabla_{\tau}^{(n-1)} (L_{\tau \tau} \nabla_{\tau} \Theta), \tag{A.90}
\]

\[
\nabla_{\tau}^{(n)} L_{\alpha \beta} \big|_{\mathcal{R}^-} = - \nabla_{\tau}^{(n-1)} \nabla_{\tau} L_{\alpha \beta} - \nabla_{\tau}^{(n-1)} (\nabla_{\tau} \Theta W_{\tau \alpha \beta}), \tag{A.91}
\]

\[
g_{\tau \tau} \nabla_{\tau}^{(n)} W_{\mu \nu \sigma \tau} \big|_{\mathcal{R}^-} = - g_{\tau \tau} \nabla_{\tau}^{(n-1)} \nabla_{\tau} W_{\mu \nu \sigma \tau} - g_{\tau \tau} \nabla_{\tau}^{(n-1)} \nabla_{\tau} W_{\mu \nu \sigma \tau} - g_{\tau \tau} \nabla_{\tau}^{(n-1)} \nabla_{\tau} W_{\mu \nu \sigma \tau} - g_{\tau \tau} \nabla_{\tau}^{(n-1)} \nabla_{\tau} W_{\mu \nu \sigma \tau}, \tag{A.92}
\]

\[
2g_{\tau \tau} \nabla_{\tau}^{(n)} \nabla_{\tau} L_{\tau \tau} \big|_{\mathcal{R}^-} = \frac{1}{6} \nabla_{\tau}^{(n+1)} R - g_{\tau \tau} \nabla_{\tau}^{(n)} \nabla_{\tau} L_{\tau \tau} - g_{\tau \tau} \nabla_{\tau}^{(n)} \nabla_{\tau} L_{\tau \tau} + \nabla_{\tau}^{(n)} (\nabla_{\tau} W_{\tau \tau} \rho), \tag{A.93}
\]

\[
\nabla_{\tau}^{(n)} R_{\mu \nu} \big|_{\mathcal{R}^-} = -2 \nabla_{\tau}^{(n)} L_{\mu \nu} + \frac{1}{6} \nabla_{\tau}^{(n)} R_{\mu \nu}, \tag{A.94}
\]

where \( R_{\mu \nu}^{(H)} \) denotes the wave-map gauge reduced Ricci tensor [8]. Indeed, we observe that (A.89)-(A.92) provide algebraic equations for \( \partial_I^{n+1} \Theta, \partial_I^n s, \partial_I^{n+1} g_{\mu \nu}, \partial_I^n L_{\mu \nu}, \partial_I^n W_{\mu \nu \rho \sigma} \big|_{\mathcal{R}^-} \) in terms of the
known fields \((\partial^k_{\nu} + 1) \Theta, \partial^k_{\mu}, \partial^k_{\rho} g_{\mu\nu}, \partial^k_{\mu} L_{\nu\rho}, \partial^k_{\mu} W_{\mu\nu\rho\sigma})\), \(k \leq n - 1\), while (A.93) provides an ODE for \(\partial^k_{\nu} L_{\nu\rho\sigma}\) with initial data determined by \(L_{\tau\tau}\) on \(N\), (A.60). Then (A.94) provides ODEs for \(\partial^k_{\nu} g_{\mu\nu}\). The initial data at \(S\) are determined by \(g_{\mu\nu}|_{S}\).

Note that, due to the divergence of \(g_{\tau\tau}|_{S}\), these ODEs are of Fuchsian type at \(I^-\), and note further that the solutions to e.g. (A.92) might be unbounded at \(I^-\). For our current analysis, though, this does not cause any problems.

Finally, the second Bianchi identity and the algebraic symmetries of the rescaled Weyl tensor yield (cf. [44])

\[
0 = \nabla^{(n)}_{\tau}(\nabla_{\rho} W_{0(\bar{A},\bar{B})}^{\rho})|_{S^-} = \frac{1}{2} g^{\tau r} \nabla^{(n)}_{\tau}(W_{r(\bar{A},\bar{B})})_{hf} - g^{\tau r} \nabla^{(n)}_{\tau}(W_{r(\bar{A},\bar{B})})_{hf} + \frac{1}{2} g^{\tau r} (g^{\tau r})^2 \nabla^{(n)}_{\tau}(W_{r(\bar{A},\bar{B})})_{hf} + g^{\tau r} \nabla^{(n)}_{\tau}(\nabla_{\rho} W_{\rho(\bar{A},\bar{B})})_{hf} + \frac{1}{2} g^{\tau r} \nabla^{(n)}_{\tau}(\nabla_{\rho} W_{\rho(\bar{A},\bar{B})})_{hf},
\]

Equation (A.97) is of the form (recall that \(\sigma^{\tau^+}_{AB} = 0, g^{\tau r}|_{S^-} = r, g^{\tau r}|_{S^-} = r^2\) and \(\kappa = -2/r, r > 1/3\),

\[
(\partial_{\tau} - \frac{n + 2}{r}) \partial^\tau_{\tau}(W_{r(\bar{A},\bar{B})})_{hf}|_{S^-} = \text{known smooth function}.
\]

This equation is also valid for \(n = 0\). In the usual approach the initial data for these ODEs follow from (A.81) and the continuity requirement at \(S\) (the right-hand side of (A.81) divided by \(\Theta\) is regular at \(S\)). What actually matters from the data given on \(N\) is thus the expansion of \(W_{r(\bar{A},\bar{B})}|_{N}\) at \(S\), and this is determined by the functions \((h_{AB}^{(k)})_{hf}, k \geq 3\).

Here we want to prescribe data at \(I^- = \{r = 0\}\). The data which can be specified for \(\partial^\tau_{\tau}(W_{r(\bar{A},\bar{B})})_{hf}|_{S^-}, n \geq 0\), correspond to its \((n + 2)nd\-order expansion coefficient \(c_{AB}^{(n+2,n)}\) at \(I^-\).

We then compute all the \((h_{AB}^{(k)})_{hf} s, k \geq 3,\) at \(S\) by solving the hierarchical system above, and, using Borel summation (cf. e.g. [12]), extend them to data \(\gamma_{AB}\) on \(N\). Our analysis at \(I^-\) in this work does not depend on this extension. Note that \((h_{AB}^{(1)})_{hf}\) is determined by (A.40) while \((h_{AB}^{(2)})_{hf}\) follows from the no-logs condition. A solution to (A.97) will generally be polyhomogeneous at \(I^-\). If this already happened for some \(k < n\) the right-hand side might be polyhomogeneous at \(I^-\) as well. For our current discussion non-smoothness at \(I^-\) is irrelevant.

**Proposition A.3** The data \((W_{AB}^{(r)}, M, N, L_A, c_{AB}^{(n+2,n)})\) for the asymptotic characteristic initial value problem determine a unique (up to gauge) solution of the MCFE supposing that an extension of the data \(\gamma_{AB}|_{N}\) has been given, whose Taylor expansion at \(S\) is determined by \((W_{AB}^{(r)}, M, N, L_A, c_{AB}^{(n+2,n)})\). All solutions with bounded rescaled Weyl tensor \(W_{ijkl}|_{S^-}\) at \(I^-\) can be generated by such data.
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