Stable triples, equivariant bundles and dimensional reduction

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October 29, 2018

1 Introduction

The Hitchin-Kobayashi correspondence between stable bundles and solutions to the Hermitian-Einstein equations allows one to apply analytic methods to the study of stable bundles. One such analytic technique, which has not yet been much exploited, is that of dimensional reduction. This is a useful tool for studying certain special solutions to partial differential equations; in particular it is useful for studying solutions which are invariant under the action of some symmetry group. When applied to the Hermitian-Einstein equations, it thus provides a way of looking at holomorphic bundle structures which are both stable and invariant under some group action on the bundle, i.e. of looking at equivariant stable bundles.

The main idea in dimensional reduction is the following: Suppose we have a partial differential equation defined on a space which has a symmetry, i.e. which supports some group action. Then by integrating over the group orbits, any solution which is invariant under the group action becomes an object defined in a space of lower dimension than that of the original setting for the general solutions. This lower dimensional space is the orbit space of the group action, and in that space the special solutions to the original equations can be re-interpreted as ordinary solutions to some new set of equations.

In particular, suppose that we start with a 4-manifold, $M$, a Lie group $G$ which acts on it, and a complex vector bundle $E$ to which this action lifts. In such a situation, there can be equivariant solutions to the Hermitian-Einstein equations, and dimensional reduction can be applied. The orbit space $E/G$ will be a new bundle over $M/G$, the orbit space of the group action on the original 4-manifold. The equivariant solutions to the original Hermitian-Einstein equations will be solutions to a new set of equations on the bundle $E/G \longrightarrow M/G$. For example, the equations introduced by Hitchin in [H], namely the Anti-Self-Dual equations on a Riemann surface, can be viewed as arising in this way.

*Supported in part by a NSF-NATO Postdoctoral Fellowship and NSF grant DMS 93–03545
A special case of the above situation occurs when the 4-manifold $M$ is a complex surface, and the orbit space $M/G$ also admits a complex structure. Solutions to the Hermitian-Einstein equations then correspond to stable holomorphic structures on the bundle $E$. In such a situation, dimensional reduction acquires an extra, holomorphic interpretation. It results in information about the equivariant stable bundles on $M$ being encoded in a holomorphic interpretation for the dimensionally reduced equations on $M/G$.

These sort of ideas are developed in [GP3], where they are applied to certain $SU(2)$-equivariant bundles over $X \times P^1$. Here $X$ is a closed Riemann surface and the $SU(2)$-action is trivial on $X$ and the standard one on $P^1$. In this case, the equivariant holomorphic bundles over $X \times P^1$ correspond to holomorphic pairs (i.e. bundles plus prescribed global sections) over $X$. The dimensional reduction of the Hermitian-Einstein equations gives the vortex equations, and the stable equivariant bundles on $X \times P^1$ correspond (by dimensional reduction) to $\tau$-stable holomorphic pairs on $X$, with $\tau$-stability as defined in [B2] and [GP3].

However not all the $SU(2)$-equivariant holomorphic bundles over $X \times P^1$ correspond to holomorphic pairs on $X$. In fact those that do form a rather restricted subset of the set of all such equivariant bundles. A very natural relaxation of this restriction leads to a class of equivariant bundles on $X \times P^1$ which still corresponds to data on the (lower dimensional) space $X$, but not necessarily to holomorphic pairs. What such bundles correspond to is a pair of bundles plus a holomorphic homomorphism between them. We call such data a holomorphic triple.

In this paper we undertake a detailed investigation of holomorphic triples over the closed Riemann surface $X$. In particular, we define, in Section 3, a notion of stability for such objects. We explore the relationship between the stability of a triple and the stability of the corresponding equivariant bundle over $X \times P^1$. An important feature of the definition is that, like in the case of holomorphic pairs, it involves a real parameter. This can be traced back to the fact that the definition of stability for a bundle over $X \times P^1$ depends on the polarization (choice of Kähler metric) on $X \times P^1$. We discuss the nature of this parameter, and its influence on the properties of the stable triples. We show for example that

*In all cases, with one exception, the parameter in the definition of triples stability lies in a bounded interval. The interval is partitioned by a finite set of non-generic values.*

Our main result is given in Section 4. Loosely speaking, it is that the stable triples over $X$ can be considered the dimensional reduction of the stable equivariant bundles over $X \times P^1$. In other words,

*A holomorphic triple over $X$ is stable if and only if the corresponding $SU(2)$-equivariant extension over $X \times P^1$ is stable.*

In [GP3] dimensional reduction is applied to the Hermitian-Einstein equation on equivariant bundles over $X \times P^1$. The result is that on bundles corresponding to triples over $X$, the equivariant solutions correspond to solutions to a pair of Coupled Vortex
Equations on the two bundles in the triple. By combining this result, our dimensional reduction result for stable bundles, and the Hitchin-Kobayashi correspondence, we can thus show

There is a Hitchin-Kobayashi correspondence between stability of a triple and existence of solutions to the Coupled Vortex Equations.

This is discussed in Section 5. In Section 6 we discuss the moduli spaces of stable triples. By identifying these as fixed point sets of an $SU(2)$-action on the moduli spaces of stable bundles over $X \times P^1$, we obtain results such as

For fixed value of the stability parameter, the moduli space of stable triples is a quasi-projective variety. For generic values of the parameter, and provided the ranks and degrees of the two bundles satisfy a certain coprimality condition, the moduli space is projective.

In Section 2 we have collected together the basic definitions and background material that we will need.

## 2 Background and Preliminaries

### 2.1 Basic Definitions

Let $X$ be a compact Riemann surface.

**Definition 2.1** A holomorphic triple on $X$ is a triple $(E_1, E_2, \Phi)$ consisting of two holomorphic vector bundles $E_1$ and $E_2$ on $X$ together with a homomorphism $\Phi: E_2 \to E_1$, i.e. an element $\Phi \in H^0(\text{Hom}(E_2, E_1))$.

In this paper we will develop a theory of holomorphic triples as objects in their own right. We will also show how they arise from $SU(2)$-equivariant holomorphic vector bundles over $X \times P^1$.

Let $SU(2)$ act on $X \times P^1$ trivially on $X$ and in the standard way on $P^1$, that is we regard $P^1$ as the homogeneous space $SU(2)/U(1)$. Let $F$ be a $C^\infty$ complex vector bundle over $X \times P^1$.

**Definition 2.2** The bundle $F$ is said to be $SU(2)$-equivariant if there is an action of $SU(2)$ on $F$ covering the action on $X \times P^1$. Similarly a holomorphic vector bundle $F$ is $SU(2)$-equivariant if it is $SU(2)$-equivariant as a $C^\infty$ bundle and in addition the action of $SU(2)$ on $F$ is holomorphic.

### 2.2 Smooth and Holomorphic Equivariant bundles

Our main objective in this section will be the study of $SU(2)$-equivariant holomorphic vector bundles on $X \times P^1$, however, before addressing this, we shall analyse the much
easier problem of classifying the SU(2)-equivariant $C^\infty$ ones. Let $p$ and $q$ be the projections from $X \times P^1$ to the first and second factors respectively.

**Proposition 2.3** Every SU(2)-equivariant $C^\infty$ vector bundle $F$ over $X \times P^1$ can be equivariantly decomposed, uniquely up to isomorphism, as

$$F = \bigoplus_i p^* E_i \otimes q^* H^{\otimes n_i},$$

where $E_i$ is a $C^\infty$ vector bundle over $X$, $H$ is the $C^\infty$ line bundle over $P^1$ with Chern class 1, and $n_i \in \mathbb{Z}$ are all different.

**Proof.** See [GP3, Proposition 3.1].

We shall describe now SU(2)-invariant holomorphic structures on a fixed SU(2)-equivariant $C^\infty$ vector bundle over $X \times P^1$. We shall restrict ourselves, however, to the case which is relevant in connection to holomorphic triples. Let $E_1$ and $E_2$ be $C^\infty$ vector bundles on $X$ and let $H$ be as in Proposition 2.3. Consider the SU(2)-equivariant $C^\infty$ vector bundle

$$F = p^* E_1 \oplus p^* E_2 \otimes q^* H^{\otimes 2}, \quad (1)$$

Note that the total space of $p^* E_1$ is $E_1 \times P^1$, and the action of SU(2) that we are considering is trivial on $E_1$ and the standard one on $P^1$; similarly for $p^* E_2$. On the other hand, recall that the SU(2)-equivariant line bundle $H^{\otimes 2}$ over $P^1 \cong SU(2)/U(1)$ corresponds to the one dimensional representations of $U(1)$ given by $e^{i\alpha}$, i.e. $H^{\otimes 2} = SU(2) \times_{U(1)} C$, where $(g, v) \sim (g', v')$ if there is an $e^{i\alpha} \in U(1)$ such that $g' = e^{-i\alpha} g$ and $v' = e^{i2\alpha} v$.

The action of SU(2) on $SU(2) \times C$, given by

$$\gamma.(g, v) = (\gamma g, v) \quad \text{for} \quad \gamma \in SU(2) \quad \text{and} \quad (g, v) \in SU(2) \times C,$$

descends to an action on $H^{\otimes 2}$.

In order to avoid the introduction of more notation we shall denote a $C^\infty$ vector bundle and the same bundle endowed with a holomorphic structure by the same symbol. The distinction will be made explicit unless it is obvious from the context.

### 2.3 Dimensional Reduction of Bundles

**Proposition 2.4** Every SU(2)-equivariant holomorphic vector bundle $F$ with underlying SU(2)-equivariant $C^\infty$ structure given by (1) is in one-to-one correspondence with a holomorphic extension of the form

$$0 \rightarrow p^* E_1 \rightarrow F \rightarrow p^* E_2 \otimes q^* \mathcal{O}(2) \rightarrow 0, \quad (2)$$

where $E_1$ and $E_2$ are the bundles over $X$ defining (1) equipped with holomorphic structures.

Moreover, every such extension is defined by an element $\Phi \in \text{Hom}(E_2, E_1)$, and is thus in one-to-one correspondence with the holomorphic triple $(E_1, E_2, \Phi)$ on $X$. 


Proof. We shall give here only a brief sketch of the proof (see [GP3, Proposition 3.9] for details). Let $E_1$ and $E_2$ be two holomorphic vector bundles over $X$. The extensions over $X \times P^1$ of the form (2) are parametrized by
\[ H^1(X \times P^1, p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2)); \]
but this is isomorphic to
\[ H^0(X, E_1 \otimes E_2^*) \otimes H^1(P^1, \mathcal{O}(-2)) \cong H^0(X, E_1 \otimes E_2^*), \]
by means of the K"unneth formula, the fact that $H^0(P^1, \mathcal{O}(-2)) = 0$ and
\[ H^1(P^1, \mathcal{O}(-2)) \cong H^0(P^1, \mathcal{O})^* \cong \mathbb{C}. \]
Therefore after fixing an element in $H^1(P^1, \mathcal{O}(-2))$, the homomorphism $\Phi$ can be identified with the extension class defining $F$.

Certainly, since the action of $SU(2)$ on the extension class is trivial (note that this action is induced from the action on $E_1 \otimes E_2^*$, which is trivial), then the bundle $F$ defined by the triple $(E_1, E_2, \Phi)$ is an $SU(2)$-equivariant holomorphic vector bundle. One can show with a little bit more work that in fact every $SU(2)$-equivariant holomorphic structure on the $SU(2)$-equivariant $C^\infty$ bundle (1) defines an extension of the form (2).

What this Proposition says is that holomorphic triples over $X$ can be regarded as a “dimensional reduction” of certain $SU(2)$-equivariant holomorphic vector bundles over $X \times P^1$.

2.4 Subtriples

Definition 2.5 A triple $T' = (E'_1, E'_2, \Phi')$ is a subtriple of $T = (E_1, E_2, \Phi)$ if

1. $E'_i$ is a coherent subsheaf of $E_i$, for $i = 1, 2$
2. $\Phi' = \Phi|_{E'_2}$, i.e. $\Phi'$ is the restriction of $\Phi$.

In other words we have the commutative diagram
\[
\begin{array}{ccc}
E_2 & \xrightarrow{\Phi} & E_1 \\
\uparrow & & \uparrow \\
E'_2 & \xrightarrow{\Phi'} & E'_1.
\end{array}
\]

If $E'_1 = E'_2 = 0$, the subtriple is called the trivial subtriple.

Remark. When studying stability criteria, it will suffice, as usual, to consider saturated subsheaves, that is subsheaves whose quotient sheaves are torsion free. On a Riemann surface these are precisely subbundles.
With this definition, subobjects of the triple \((E_1, E_2, \Phi)\) are related to subsheaves of the corresponding \(SU(2)\)-equivariant bundle \(F \to X \times P^1\) in an appropriate way. First note that the correspondence between triples on \(X\) and bundles on \(X \times P^1\) can be extended more generally to arbitrary coherent sheaves. Namely, if \(S_1\) and \(S_2\) are two coherent sheaves on \(X\) and \(\Psi \in \text{Hom}(S_2, S_1)\) the triple \((S_1, S_2, \Psi)\) defines a coherent sheaf \(U\) over \(X \times P^1\). This sheaf is given, as for bundles, as an extension

\[
0 \to p^* S_1 \to U \to p^* S_2 \otimes \mathcal{O}(2) \to 0.
\]

The proof is very much as for the case of bundles, once we have fixed \(S_1\) and \(S_2\), the extensions as \((\mathbb{R})\) are parametrized by

\[
\text{Ext}^1_{X \times P^1}(p^* S_2 \otimes q^* \mathcal{O}(2), p^* S_1).
\]

But, by the K"unneth formula for the Ext groups, this group is isomorphic to

\[
\text{Hom}_X(S_2, S_1) \otimes \text{Ext}^1_{P^1}(\mathcal{O}, \mathcal{O}(2)) \oplus \text{Ext}^1_X(S_2, S_1) \otimes \text{Hom}_{P^1}(\mathcal{O}(2), \mathcal{O}).
\]

This reduces to \(\text{Hom}_X(S_2, S_1)\), since \(\text{Hom}_{P^1}(\mathcal{O}(2), \mathcal{O}) \cong H^0(P^1, \mathcal{O}(-2)) = 0\) and

\[
\text{Ext}^1_{P^1}(\mathcal{O}, \mathcal{O}(2)) \cong H^1(P^1, \mathcal{O}(-2)) \cong C.
\]

**Lemma 2.6** Let \(F \to X \times P^1\) be the bundle associated to a triple \((E_1, E_2, \Phi)\). Then every \(SU(2)\)-invariant coherent subsheaf \(F' \subset F\) is an extension of the form

\[
0 \to p^* E'_1 \to F' \to p^* E'_2 \otimes q^* \mathcal{O}(2) \to 0,
\]

with \(E'_1 \subset E_1\) and \(E'_2 \subset E_2\) coherent subsheaves, making the following diagram commutative

\[
\begin{array}{cccccc}
0 & \to & p^* E'_1 & \to & F' & \to & p^* E'_2 \otimes q^* \mathcal{O}(2) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}
\]

Thus \(F'\) corresponds to a triple \((E'_1, E'_2, \Phi')\), for \(\Phi' \in \text{Hom}(E'_2, E'_1)\).

**Proof.** Let \(f : F' \to p^* E_2 \otimes q^* \mathcal{O}(2)\) be the composition of the injection \(F' \to F\) with the surjective map \(F \to p^* E_2 \otimes q^* \mathcal{O}(2)\). Consider the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & p^* E_1 & \to & F & \to & p^* E_2 \otimes q^* \mathcal{O}(2) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \to & \text{Ker } f & \to & F' & \to & \text{Im } f & \to & 0.
\end{array}
\]

The \(SU(2)\)-invariance of \(F'\) implies that of \(\text{Ker } f\) and \(\text{Im } f\). It suffices therefore to show that if \(E\) is a holomorphic vector bundle over \(X\) and if \(p^* E\) is the pull-back to \(X \times P^1\), then every \(SU(2)\)-invariant subsheaf of \(p^* E\) is isomorphic to a sheaf of the form \(p^* E'\) for \(E'\) a subsheaf of \(E\). Indeed, the action of \(SU(2)\) on \(p^* E\) can be extended to an action of \(SL(2, C)\). Let \(F' \subset p^* E\) be a \(SL(2, C)\)-invariant coherent
subsheaf. Consider the action of a subgroup \( C^* \subseteq SL(2, C) \) on \( X \times C \subseteq X \times P^1 \) and let \( A = H^0(X, E) \) be the space of global sections. Clearly \( H^0(X \times C, F') \subseteq A[t] \), that is, \( H^0(X \times C, F') = \bigoplus_{k=0}^N B_k \), where an element of \( B_k \) is of the form \( st^k \) for \( s \in A \). The action of \( \alpha \in C^* \) is given by

\[
\alpha(st^k) = s\alpha^kt^k.
\]

By choosing another subgroup \( C^* \subseteq SL(2, C) \), the \( SL(2, C) \)-invariance of \( F' \) implies that \( H^0(X \times C, F') = B_0 \) and hence \( F' = p^*E' \) for \( E' \subseteq E \) a coherent subsheaf.

We shall show in the next lemma that the triple associated to \( F' \subseteq F \) is in fact a subtriple of \((E_1, E_2, \Phi)\), and conversely, every subtriple of \((E_1, E_2, \Phi)\) defines a unique \( SU(2) \)-invariant coherent subsheaf of \( F \).

**Lemma 2.7** Let \( E'_1 \subseteq E_1 \) and \( E'_2 \subseteq E_2 \) be coherent subsheaves and let \( \Phi' \subseteq Hom(E'_2, E'_1) \). Let \( F' \) be the coherent sheaf over \( X \times P^1 \) defined by the triple \((E'_1, E'_2, \Phi')\). Then \( F' \) is a subsheaf of \( F \) making the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & p^*E_1 & \rightarrow & F & \rightarrow & p^*E_2 \otimes q^*\mathcal{O}(2) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & p^*E'_1 & \rightarrow & F' & \rightarrow & p^*E'_2 \otimes q^*\mathcal{O}(2) & \rightarrow & 0,
\end{array}
\]

commutative if and only if \((E'_1, E'_2, \Phi')\) is a subtriple of \((E_1, E_2, \Phi)\).

**Proof.** Consider the diagram

\[
\text{Hom}(E'_2, E'_1) \xrightarrow{i} \text{Hom}(E'_2, E_1) \xleftarrow{j} \text{Hom}(E_2, E_1).
\]

To say that \((E'_1, E'_2, \Phi')\) is a subtriple of \((E_1, E_2, \Phi)\) is equivalent to saying that

\[
i(\Phi') = j(\Phi).
\]

Under the isomorphisms

\[
\text{Hom}(E'_2, E'_1) \cong \text{Ext}^1(p^*E'_1, p^*E'_2 \otimes q^*\mathcal{O}(2))
\]

\[
\text{Hom}(E'_2, E_1) \cong \text{Ext}^1(p^*E_1, p^*E'_2 \otimes q^*\mathcal{O}(2))
\]

\[
\text{Hom}(E_2, E_1) \cong \text{Ext}^1(p^*E_1, p^*E_2 \otimes q^*\mathcal{O}(2)),
\]

\(i(\Phi')\) defines an extension \( \tilde{F}^{(i)} \) which makes the following diagram commutative

\[
\begin{array}{ccccccccc}
0 & \rightarrow & p^*E_1 & \rightarrow & \tilde{F}^{(i)} & \rightarrow & p^*E'_2 \otimes q^*\mathcal{O}(2) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & p^*E'_1 & \rightarrow & F' & \rightarrow & p^*E'_2 \otimes q^*\mathcal{O}(2) & \rightarrow & 0.
\end{array}
\]

In particular \( F' \) is a subsheaf of \( \tilde{F}^{(i)} \). On the other hand \( j(\Phi) \) defines an extension \( \tilde{F}^{(j)} \) which fits in the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & p^*E_1 & \rightarrow & F & \rightarrow & p^*E_2 \otimes q^*\mathcal{O}(2) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & p^*E'_1 & \rightarrow & \tilde{F}^{(j)} & \rightarrow & p^*E'_2 \otimes q^*\mathcal{O}(2) & \rightarrow & 0,
\end{array}
\]

and in particular \( \tilde{F}^{(j)} \) is a subsheaf of \( F \). Since \( i(\Phi') = j(\Phi) \), \( \tilde{F}^{(i)} \cong \tilde{F}^{(j)} \) and we can compose the above two diagrams to obtain the desired result.
2.5 Simple Triples

Definition 2.8 Let

\[ H^0(E_1, E_2, \Phi) = \{(u, v) \in H^0(\text{End } E_1) \oplus H^0(\text{End } E_2) \mid u\Phi = \Phi v\}. \] (7)

We say a holomorphic triple \((E_1, E_2, \Phi)\) is simple if \(H^0(E_1, E_2, \Phi) \simeq C\), i.e. if the only elements in \(H^0(E_1, E_2, \Phi)\) are of the form \(\lambda(I_1, I_2)\) where \(\lambda\) is a constant and \((I_1, I_2)\) denote the identity maps on \(E_1\) and \(E_2\).

This definition too is dictated by the correspondence between triples on \(X\) and equivariant holomorphic extensions over \(X \times P^1\):

Proposition 2.9 The triple \((E_1, E_2, \Phi)\) is simple if and only if the \(SU(2)\)-equivariant bundle \(F\) associated to \((E_1, E_2, \Phi)\) is \(SU(2)\)-equivariantly simple.

Proof. The definition of simplicity for an equivariant bundle is the obvious generalization of that for an ordinary holomorphic bundle. Namely, an equivariant bundle is said to be equivariantly simple if it has no other invariant endomorphisms than the constant multiples of the identity. The proof of the Proposition follows from the following lemma.

Lemma 2.10 Let \(T = (E_1, E_2, \Phi)\) and \(T' = (E'_1, E'_2, \Phi')\) be two holomorphic triples over \(X\) and \(F\) and \(F'\) be the corresponding \(SU(2)\)-equivariant holomorphic vector bundles over \(X \times P^1\). Then, every \(SU(2)\)-equivariant homomorphism \(g : F \rightarrow F'\) induces homomorphisms \(u : E_1 \rightarrow E'_1\) and \(v : E_2 \rightarrow E'_2\) such that

\[ u\Phi = \Phi'v. \] (8)

Conversely, given morphisms \(u\) and \(v\) satisfying (3) there exists a unique morphism \(g : F \rightarrow F'\) inducing \(u\) and \(v\).

Proof. The map \(g\) can be decomposed as

\[ g = \begin{pmatrix} g_1 & f_1 \\ f_2 & g_2 \end{pmatrix}, \]

where \(g_1 : p^*E_1 \rightarrow p^*E'_1, g_2 : p^*E_2 \otimes q^*O(2) \rightarrow p^*E'_2 \otimes q^*O(2), f_1 : p^*E_2 \otimes q^*O(2) \rightarrow p^*E'_1\) and \(f_2 : p^*E_1 \rightarrow p^*E'_2 \otimes q^*O(2).\) By invariance it is very easy to see (cf. [GP3, Proposition 3.9]) that \(g_1 = p^*u, g_2 = p^*v,\) and \(f_1 = 0 = f_2.\) Equation (3) follows from the commutativity of the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & p^*E_1 & \rightarrow & F & \rightarrow & p^*E_2 \otimes q^*O(2) & \rightarrow & 0 \\
p^*u\downarrow & & g\downarrow & & p^*v\downarrow & \\
0 & \rightarrow & p^*E'_1 & \rightarrow & F' & \rightarrow & p^*E'_2 \otimes q^*O(2) & \rightarrow & 0.
\end{array}
\]
From this lemma it follows that every $SU(2)$-invariant endomorphism of $F$ induces endomorphisms $u : E_1 \to E_1$ and $v : E_2 \to E_2$, satisfying $u \Phi = \Phi v$. And conversely, given endomorphisms $u$ and $v$ satisfying $u \Phi = \Phi v$ there exists a unique endomorphism of $F$ inducing $u$ and $v$.

The definition of simplicity for a triple given above is also motivated by a deformation theory description of the “tangent space” to the space of triples. We will say more about this in Section 6.

2.6 Reducible Triples

A related, but inequivalent, notion to simplicity is that of irreducibility. We make the following definitions.

**Definition 2.11** We say the triple $T = (E_1, E_2, \Phi)$ is reducible if there are direct sum decompositions $E_1 = \bigoplus_{i=1}^{n} E_{1i}$, $E_2 = \bigoplus_{i=1}^{n} E_{2i}$, and $\Phi = \bigoplus_{i=1}^{n} \Phi_i$, such that $\Phi_i \in \text{Hom}(E_{2i}, E_{1i})$. We adopt the convention that if $E_{2i} = 0$ or $E_{1i} = 0$ for some $i$, then $\Phi_i$ is the zero map. With $T_i = (E_{1i}, E_{2i}, \Phi_i)$, we write $T = \bigoplus_{i=1}^{n} T_i$. Thus $T$ is reducible if it has a decomposition as a direct sum of subtriples.

If $T$ is not reducible, we say $T$ is irreducible.

**Proposition 2.12** If a triple $T = (E_1, E_2, \Phi)$ is simple, then it is irreducible.

**Proof.** Suppose $T$ is reducible, with $T = \bigoplus_{i=1}^{n} T_i$. Then we can define $(u, v) \in H^0(E_1, E_2, \Phi)$ by $u = \bigoplus_{i=1}^{n} \lambda_i I_{1i}$, $v = \bigoplus_{i=1}^{n} \lambda_i I_{2i}$, where for each $i$, $\lambda_i \in \mathbb{C}$ and $I_{1i}(I_{2i})$ is the identity map on $E_{1i}(E_{2i})$. Clearly $T$ is not simple.

**Proposition 2.13** A holomorphic triple $T = (E_1, E_2, \Phi)$ over $X$ is irreducible if and only if the corresponding $SU(2)$-equivariant extension $F \to X \times P^1$ is equivariantly irreducible, i.e. cannot be decomposed as a sum of $SU(2)$-equivariant extensions of the form (4).

**Proof.** This follows directly from the relation between subtriples of $T$ and $SU(2)$-equivariant subbundles of $F$ (cf. Lemmas 2.6 and 2.7).

2.7 Equations for special Metrics

Given a holomorphic vector bundle over a compact Riemann surface there is a natural condition for a Hermitian metric on it: that of being projectively flat. By choosing a metric on $X$ one can rewrite this condition in a way which turns out to be the right generalization for higher dimensional manifolds: the Hermitian–Einstein condition.
Since we shall use this notion on $X$ as well as on $X \times P^1$, we shall define it on a compact Kähler manifold of arbitrary dimension $(M, \omega)$.

Let $E$ be a holomorphic vector bundle over $M$ and $h$ be a Hermitian metric on $E$. Recall that there is on $E$ a unique connection compatible with both the metric and the holomorphic structure—the so-called metric connection. Let $F_h$ be its curvature and $\Lambda F_h$ be the contraction of $F_h$ with the Kähler form $\omega$. $\Lambda F_h$ is hence a smooth section of $\text{End} E$. The metric $h$ is said to be Hermitian–Einstein with respect to $\omega$ if

$$\sqrt{-1} \Lambda F_h = \lambda I_E,$$

where $I_E \in \Omega^0(\text{End} E)$ is the identity and $\lambda$ is a constant which is determined by integrating the trace of (9).

Using that the degree of $E$, defined as

$$\text{deg } E = \frac{1}{(m-1)!} \int_M c_1(E) \wedge \omega^{m-1},$$

where $m$ is the dimension of $M$ and $c_1(E)$ is the first Chern class of $E$, given via Chern–Weil theory by

$$\text{deg } E = \frac{i}{2\pi} \int_M \text{Tr}(\Lambda F_h) \frac{\omega^m}{m!},$$

we obtain

$$\lambda = \frac{2\pi}{\text{Vol } M} \frac{\text{deg } E}{\text{rank } E}.$$

Coming back to our compact Riemann surface $X$, let us choose a metric on $X$ with Kähler form $\omega_X$ and volume normalized to one. Given a holomorphic triple $(E_1, E_2, \Phi)$ on $X$ it was shown in [GP3] that there are natural equations for metrics on the bundles $E_1$ and $E_2$. These equations, formally similar to the Hermitian–Einstein equations, involve in a natural way the endomorphism $\Phi$. If $E_1$ and $E_2$ are endowed with Hermitian metrics one can form smooth sections of $\text{End } E_1$ and $\text{End } E_2$ respectively by taking the compositions $\Phi \Phi^*$ and $\Phi^* \Phi$. Here $\Phi^*$ is the adjoint of $\Phi$ with respect to the metrics of $E_1$ and $E_2$. The equations for the metrics $h_1$ and $h_2$ on $E_1$ and $E_2$, respectively, are given by

$$\sqrt{-1} \Lambda F_{h_1} + \Phi \Phi^* = 2\pi \tau I_{E_1},$$
$$\sqrt{-1} \Lambda F_{h_2} - \Phi^* \Phi = 2\pi \tau' I_{E_2},$$

where $\tau$ and $\tau'$ are real parameters.

We first observe that, in order to solve (10), the parameters $\tau$ and $\tau'$ must be related. Indeed, by adding the trace of the two equations in (10), and since $\text{Tr}(\Phi \Phi^*) = \text{Tr}(\Phi^* \Phi)$, we get

$$\sqrt{-1} \text{Tr}(\Lambda F_{h_1}) + \sqrt{-1} \text{Tr}(\Lambda F_{h_2}) = 2\pi r_1 \tau + 2\pi r_2 \tau',$$

where $r_1$ and $r_2$ are the ranks of $E_1$ and $E_2$ respectively. By integrating this equation and, since

$$\text{deg } E_1 = \frac{\sqrt{-1}}{2\pi} \int_X \text{Tr}(\Lambda F_{h_1}) \omega \quad \text{and} \quad \text{deg } E_2 = \frac{\sqrt{-1}}{2\pi} \int_X \text{Tr}(\Lambda F_{h_2}) \omega,$$
we obtain
\[ r_1 \tau + r_2 \tau' = \deg E_1 + \deg E_2. \tag{11} \]

There is therefore just one independent parameter, that we choose to be \( \tau \). These equations are called the \textit{coupled \( \tau \)-vortex equations} by analogy with the vortex equations on a single bundle studied in \([B1, B2, GP1, GP2]\).

2.8 Dimensional Reduction of Equations

The coupled vortex equations have similar interpretations to the Hermitian–Einstein equation and the vortex equations on a single bundle. They can be interpreted both as the equations satisfied by the minima of a certain gauge-theoretical functional—a generalized Yang–Mills–Higgs-type functional—as well as moment map equations in the sense of symplectic geometry (see \([GP3\text{, Section 2}]\) for details). In fact, the relation between the coupled vortex equations and the Hermitian–Einstein equation that we shall exploit here is of a more intimate nature. Namely, the coupled vortex equations are a dimensional reduction of the Hermitian–Einstein equation under the action of \( SU(2) \) on \( X \times P^1 \). Of course, in order to talk about the Hermitian–Einstein equation on \( X \times P^1 \) one needs to choose a Kähler metric. We shall consider the one-parameter family of \( SU(2) \)-invariant Kähler metrics with Kähler form
\[
\omega_\sigma = \frac{\sigma}{2} p^* \omega_X \oplus \omega_{P^1},
\]
where \( \omega_{P^1} \) is the Fubini-Study Kähler form normalized to volume one, and \( \sigma \in \mathbb{R}^+ \).

Proposition 2.14 Let \( T = (E_1, E_2, \Phi) \) be a holomorphic triple and \( F \) be the \( SU(2) \)-equivariant holomorphic bundle over \( X \times P^1 \) associated to \( T \), that is given as an extension
\[
0 \longrightarrow p^* E_1 \longrightarrow F \longrightarrow p^* E_2 \otimes q^* \mathcal{O}(2) \longrightarrow 0. \tag{12}
\]
Suppose that \( \tau \) and \( \tau' \) are related by \([L1]\) and let
\[
\sigma = \frac{(r_1 + r_2) \tau - (\deg E_1 + \deg E_2)}{r_2}. \tag{13}
\]
Then \( E_1 \) and \( E_2 \) admit metrics satisfying the coupled \( \tau \)-vortex equations if and only if \( F \) admits an \( SU(2) \)-invariant Hermitian–Einstein metric with respect to \( \omega_\sigma \).

Proof. We shall give here just a sketch of the proof (see \([GP3, Proposition 3.11]\) for details). First one has the following result, which is a special case of the general characterization of an \( SU(2) \)-invariant Hermitian metric on an \( SU(2) \)-equivariant vector bundle over \( X \times P^1 \).

Lemma 2.15 Let \( h \) be an \( SU(2) \)-invariant Hermitian metric on the bundle \( F \longrightarrow X \times P^1 \) associated to the triple \( (E_1, E_2, \Phi) \). Then \( h \) is of the form
\[
h = p^* h_1 \oplus p^* h_2 \otimes q^* h'_2, \tag{14}
\]
where $h_1$ and $h_2$ are metrics on $E_1$ and $E_2$, respectively, and $h_2'$ is an $SU(2)$-invariant metric on $O(2)$. Conversely, given metrics $h_1$, $h_2$ and $h_2'$ as above, (14) defines an $SU(2)$-invariant metric on $F$.

Let $F_1$ and $F_2$ be the curvatures of the metric connections of $p^*h_1$ and $p^*h_2 \otimes q^*h_2'$ respectively. Then

\[
F_1 = p^*F_{h_1}, \\
F_2 = p^*F_{h_2} \otimes 1 + I_{E_2} \otimes q^*F_{h_2'}.
\]

The curvature of the metric connection corresponding to $h$ is given by

\[
F_h = \begin{pmatrix} F_1 - \beta \wedge \beta^* & D' \beta \\ -D'' \beta^* & F_2 - \beta^* \wedge \beta \end{pmatrix},
\]

where $\beta \in \Omega^{0,1}(X \times P^1, p^*(E_1 \otimes E_2^*) \otimes q^*O(-2))$ is a representative of the extension class in $H^1(X \times P^1, p^*(E_1 \otimes E_2^*) \otimes q^*O(-2))$ defining (12), and

\[
D : \Omega^1(X \times P^1, p^*(E_1 \otimes E_2^*) \otimes q^*O(-2)) \rightarrow \Omega^2(X \times P^1, p^*(E_1 \otimes E_2^*) \otimes q^*O(-2))
\]

is built from the metric connections of $p^*h_1$ and $p^*h_2 \otimes q^*h_2'$.

As explained in Proposition 2.4, $\beta = p^*\Phi \otimes q^*\alpha$, where $\alpha \in \Omega^{0,1}(P^1, O(-2))$ is the unique $SU(2)$-invariant representative of the element in $H^1(P^1, O(-2))$, which has to be fixed in order to associate the extension (12) to $(E_1, E_2, \Phi)$. One can choose the constant in $H^1(P^1, O(-2)) \cong C$ such that $\alpha \wedge \alpha^* = \frac{1}{\sigma} \omega_{P^1}$.

Let $\Lambda_{\sigma}$ be the contraction with the Kähler form $\omega_{\sigma}$. A straightforward computation shows that if $\sigma$ is related to $\tau$ by (13), then $h$ is Hermitian–Einstein with respect to $\omega_{\sigma}$. That is

\[
\sqrt{-1} \Lambda_{\sigma} F_h = \lambda I_F,
\]

if and only if $h_1$ and $h_2$ satisfy the coupled $\tau$-vortex equations.

We have assumed that if the relation between $\sigma$ and $\tau$ is given by (13), then $\sigma > 0$. However, we will show in Section 3 that this can actually be derived from the coupled vortex equations.

**Remark.** The choice of the Kähler metric on $X \times P^1$ that we have made differs from the one made in [GP3]. There the parameter $\sigma$ is multiplying the metric on $P^1$, i.e. $\omega_{\sigma} = p^*\omega_X \oplus \sigma q^*\omega_{P^1}$. This, and the fact that the volume of $X$ was not normalized to one, explains why the relation between $\tau$ and $\sigma$ given there is the inverse of (13).

### 2.9 Invariant Stability and the Hitchin-Kobayashi Correspondence

It is very well-known that the existence of a Hermitian–Einstein metric on a holomorphic vector bundle is governed by the algebraic-geometric condition of stability. Recall
that a holomorphic vector bundle $E$ over a compact Kähler manifold $(M, \omega)$ is said to be stable if
\[ \mu(E') < \mu(E) \]
for every non-trivial coherent subsheaf $E' \subset E$. Where
\[ \mu(E') = \frac{\deg E'}{\text{rank } E'} \]
is the slope of $E'$.

The precise relation between the Hermitian–Einstein condition and stability is given by the so-called Hitchin–Kobayashi correspondence, proved by Donaldson [D1, D2] in the algebraic case and by Uhlenbeck and Yau [U-Y] for an arbitrary compact Kähler manifold (see also [Ko, L, A-B, N-S]):

**Theorem 2.16** Let $E$ be a holomorphic vector bundle over a compact Kähler manifold $(M, \omega)$. Then $E$ admits a Hermitian–Einstein metric if and only if $E$ is polystable, that is a direct sum of stable bundles of the same slope.

From this theorem and Proposition 2.14 we conclude that the existence of solutions to the coupled vortex equations must be dictated by the stability of the bundle $F \to X \times P^1$ associated to the triple $(E_1, E_2, \Phi)$. In fact, since the Hermitian–Einstein metric on $F$ is $SU(2)$-invariant, the condition that $F$ has to satisfy is a slightly weaker condition than stability, namely that of invariant stability. Let $(M, \omega)$ be a compact Kähler manifold and $G$ be a compact Lie group acting on $M$ by isometric biholomorphisms. Let $E$ be a $G$-equivariant holomorphic vector bundle over $M$. We say that $E$ is $G$-invariantly stable if
\[ \mu(E') < \mu(E) \]
for every $G$-invariant non-trivial coherent subsheaf $E' \subset E$.

The basic relation between $G$-invariant stability and ordinary stability is given by the following theorem (cf. [GP2, Theorem 4]).

**Theorem 2.17** Let $E$ be a $G$-invariant holomorphic vector bundle as above. Then $E$ is $G$-invariantly stable if and only if $E$ is $G$-indecomposable and is of the form
\[ E = \bigoplus_{i=1}^{n} E_i \]
where $E_i$ is a stable bundle, which is the transformed of $E_1$ by an element of $G$.

As a corollary of Theorems 2.16 and 2.17 one obtains a $G$-invariant version of the Hitchin-Kobayashi correspondence (cf. [GP2, Theorems 4 and 5]):
Theorem 2.18 Let $E$ be a $G$-equivariant holomorphic vector bundle over a compact Kähler manifold $(M, \omega)$. Then $E$ admits a $G$-invariant Hermitian–Einstein metric if and only if $E$ is $G$-invariantly polystable, that is a direct sum of $G$-invariantly stable bundles of the same slope.

From Proposition 2.14 and Theorem 2.18 we obtain the following existence theorem.

Theorem 2.19 Let $T = (E_1, E_2, \Phi)$ be a holomorphic triple over a compact Riemann surface $X$ equipped with a metric. Let $F \to X \times P^1$ be the bundle associated to $T$ as above. Let $\sigma$ and $\tau$ be real parameters related by (13). Then $E_1$ and $E_2$ admit metrics satisfying the coupled $\tau$-vortex equations if and only if $F$ is a $SU(2)$-invariantly polystable bundle with respect to the Kähler form $\omega_\sigma$ defined above.

3 Definition and Properties of Stability for Triples

The existence theorem 2.19 gives conditions on the extension $F \to X \times P^1$ for existence of solutions to the coupled vortex equations on $(E_1, E_2, \Phi)$. We would like to express these conditions entirely in terms of the data on $(E_1, E_2, \Phi)$. Indeed this is one of our primary objectives in this paper. To achieve this, we will need an appropriate notion of stability for a triple. In this section we define such a concept for holomorphic triples, and discuss some properties that follow from the definition.

Keeping our earlier notation, we let $E_1$ and $E_2$ be holomorphic vector bundles over a Riemann surface $X$. We denote their ranks by $r_1$ and $r_2$ respectively, and their degrees by $d_1$ and $d_2$. We let $\Phi : E_2 \to E_1$ be a holomorphic bundle homomorphism, i.e. $\Phi \in H^0(\text{Hom}(E_2, E_1))$.

Our definition of stability for the triple $T = (E_1, E_2, \Phi)$ has two equivalent formulations. The first has some advantages when considering the relation between stability and the coupled vortex equations, while the second has the virtue that it is in the style of the definition of parabolic stability, and thus looks more familiar. Both definitions involve a real parameter, with the result that there is a 1-parameter family of stability criteria for triples. This is the same phenomenon as is observed in the case of holomorphic pairs. All our results can be of course be stated in terms of either definition, and for the sake of completeness we will give both versions.

Definition 3.1 Let $T' = (E'_1, E'_2, \Phi)$ be a nontrivial subtriple of $(E_1, E_2, \Phi)$, with rank $E'_1 = r'_1$ and rank $E'_2 = r'_2$. For any real $\tau$ define

$$\theta_\tau(T') = (\mu(E'_1 \oplus E'_2) - \tau) - \frac{r'_1 r_2 + r'_2 r_1}{r'_2 + r'_1} (\mu(E_1 \oplus E_2) - \tau). \tag{15}$$

The triple $T = (E_1, E_2, \Phi)$ is called $\tau$-stable if $\theta_\tau(T') < 0$. 

for all nontrivial subtriples \( T' = (E'_1, E'_2, \Phi) \). The triple is called \( \tau \)-semistable if for all subtriples
\[
\theta_\tau(T') \leq 0.
\]

**Definition 3.2** With \( \sigma \) a real number, define the \( \sigma \)-degree and \( \sigma \)-slope of a subtriple \( T' = (E'_1, E'_2, \Phi) \) by
\[
\deg_{\sigma}(T') = \deg(E'_1 \oplus E'_2) + r'_2 \sigma,
\]
and
\[
\mu_{\sigma}(T') = \frac{\deg_{\sigma}(T')}{r'_1 + r'_2}.
\]

The triple \( T = (E_1, E_2, \Phi) \) is called \( \sigma \)-stable if for all nontrivial subtriples \( T' = (E'_1, E'_2, \Phi) \) we have
\[
\mu_{\sigma}(T') < \mu_{\sigma}(T).
\]

A straightforward computation shows the equivalence of these two definitions.

**Proposition 3.3** Fix \( \tau \) and \( \sigma \) such that
\[
\sigma = \frac{r_1 + r_2}{r_2} (\tau - \mu(T)) ,
\]
or equivalently
\[
\tau = \mu_{\sigma}(T).
\]
Then for any subtriple \( T' = (E'_1, E'_2, \Phi) \), the following are equivalent:

1. \( \theta_\tau(T') < 0 \),
2. \( \mu_{\sigma}(T') < \mu_{\sigma}(T) \).

That is, the triple is \( \tau \)-stable if and only if it is \( \sigma \)-stable. A similar result holds with “<” replaced by “=”.

**Remark.** There are two special cases where the notion of stability for a triple is especially simple, namely when \( \Phi = 0 \), and when \( E_2 \) is a line bundle.

**Lemma 3.4** Suppose that \( \Phi = 0 \). The degenerate holomorphic triple \( (E_1, E_2, 0) \) is \( \tau \)-semistable if and only if \( \tau = \mu(E_1) \) and both bundles are semistable. Such triple cannot be \( \tau \)-stable.

**Proof.** Subtriples of \( T = (E_1, E_2, 0) \) are all of the form \( T' = (E'_1, E'_2, 0) \), with \( E'_1 \) and \( E'_2 \) being any holomorphic subbundles of \( E_1 \) and \( E_2 \) respectively. Applying the condition \( \theta_\tau(T') \leq 0 \) to subtriples of the form \( T' = (E'_1, 0, 0) \) gives
\[
\mu(E'_1) \leq \tau,
\]

(16)
while applying the condition to subtriples of the form \( T' = (E'_1, E_2, 0) \) gives stability

\[
\mu(E_1/E'_1) \geq \tau.
\]  

(17)

These two inequalities imply

\[
\mu(E_1) \leq \tau \leq \mu(E_1).
\]  

(18)

That is, \( \tau = \mu(E_1) \), and hence \( E_1 \) is a semistable bundle. Similarly, by considering the subtriples \((0, E'_2, 0)\) and \((E_1, E'_2, 0)\), we see that \( E_2 \) is also semistable. Notice that the inequalities in (16) and (17) cannot be made strict without leading to a contradiction in (18).

**Corollary 3.5** The map \( \Phi \) cannot be identically zero in a \( \tau \)-stable triple.

**Lemma 3.6** In the case where \( E_2 = L \) is a line bundle, i.e. \( r_2 = 1 \), the above definition is equivalent to the notion of \( \tau \)-stability defined in [GP3]. It thus corresponds to the \((\tau - \deg L)\)-stability for the holomorphic pair \((E_1 \otimes L^*, \Phi)\).

**Proof.** In this case there are only two types of subtriple possible, corresponding to \( r'_2 = 0 \) or \( r'_2 = 1 \). In the first case the subtriples are of the form \((E'_1, 0, 0)\), where \( E'_1 \) is an arbitrary holomorphic subbundle of \( E_1 \). The condition \( \theta_r(T') < 0 \) then reduces to

\[
\mu(E'_1) < \tau.
\]

In the second case, the subtriples are of the form \((E'_1, E_2, \Phi)\) where \( E'_1 \) is a holomorphic subbundle such that \( \Phi(E_2) \subset E'_1 \). For such subtriples the condition \( \theta_r(T') < 0 \) is equivalent to

\[
(r'_1 + 1)\mu(E'_1 \oplus E_2) - (r_1 + 1)\mu(E_1 \oplus E_2) - (r'_1 - r_1)\tau < 0,
\]

i.e.

\[
\mu(E_1/E'_1) > \tau.
\]

Definition 3.1 can thus be considered a natural extension of the \( \tau \)-stability for pairs defined in [B2]. For the more general triples which we are considering here however, the number of different possibilities for subtriples is too large to reformulate the definition of \( \tau \)-stability in the style of [GP3] or [B2], i.e. in terms of separate slope conditions on the various families of subtriples. The \( \tau \)-stability of a triple does however imply the following conditions on subtriples:

**Proposition 3.7** Let \((E_1, E_2, \Phi)\) be a \( \tau \)-stable triple. Let \( \tau' \) be related to \( \tau \) by

\[
r_1\tau + r_2\tau' = \deg E_1 + \deg E_2.
\]  

(19)

Then
(1) \( \mu(E'_1) < \tau \) for all holomorphic subbundles \( E'_1 \subset E_1 \),
(2) \( \mu(E'_2) < \tau' \) for all holomorphic subbundles \( E'_2 \subset E_2 \) such that \( E'_2 \subset \text{Ker}(\Phi) \),
(3) \( \mu(E''_2) > \tau' \) for all holomorphic quotients of \( E_2 \),
(4) \( \mu(E''_1) > \tau \) for all holomorphic quotients of \( E_1 \) such that \( \pi \circ \Phi(E_2) = 0 \),
where \( \pi : E_1 \rightarrow E''_1 \) denotes projection onto the quotient.

Proof. These are immediate consequences of the stability condition, i.e.
\[
\theta_{\tau}(E'_1, E'_2, \Phi') < 0,
\]
applied to the following special subtriples

1. \((E'_1, 0, \Phi)\),
2. \((0, E'_2, \Phi)\),
3. \((E_1, E'_2, \Phi), \) with \( E''_2 = E_2 / E'_2 \),
4. \((E'_1, E_2, \Phi), \) with \( E''_1 = E_1 / E'_1 \).

Notice that (19) can be expressed as
\[
\tau' = \mu(E_1 \oplus E_2) - \frac{r_1}{r_1 + r_2} \sigma.
\]

An equivalent formulation of Proposition 3.7 is thus

Proposition 3.8 Let \( T = (E_1, E_2, \Phi) \) be a \( \sigma \)-stable triple. Then

1. \( \mu(E'_1) < \mu(T) + \frac{r_2}{r_1 + r_2} \sigma \) for all holomorphic subbundles \( E'_1 \subset E_1 \),
2. \( \mu(E'_2) < \mu(T) - \frac{r_1}{r_1 + r_2} \sigma \) for all holomorphic subbundles \( E'_2 \subset E_2 \) such that \( E'_2 \subset \text{Ker}(\Phi) \),
3. \( \mu(E''_2) > \mu(T) - \frac{r_1}{r_1 + r_2} \sigma \) for all holomorphic quotients, \( E''_2 \), of \( E_2 \),
4. \( \mu(E''_1) > \mu(T) + \frac{r_2}{r_1 + r_2} \sigma \) for all holomorphic quotients, \( E''_1 \), of \( E_1 \) such that \( \pi \circ \Phi(E_2) = 0 \), where \( \pi : E_1 \rightarrow E''_1 \) denotes projection onto the quotient.

3.1 Stable implies simple

An important consequence of stability for holomorphic bundles is that the only automorphisms of a stable bundle are the constant multiples of the identity, i.e. stable bundles are simple. We now show that this remains true in the case of holomorphic triples, where the definition of simplicity is that given in Definition 2.8. The key result is the following Proposition.

Proposition 3.9 Let \((E_1, E_2, \Phi)\) be a \( \tau \)-stable holomorphic triple. Let \((u, v)\) be in \( H^0(E_1, E_2, \Phi) \). Either \((u, v)\) is trivial, or both \( u \) and \( v \) are isomorphisms.
Proof. Suppose that \( u \) and \( v \) are both neither trivial nor isomorphisms. Consider the triples \( K = (\text{Ker } u, \text{Ker } v, \Phi) \) and \( I = (\text{Im } u, \text{Im } v, \Phi) \), where Ker and Im denotes the kernels and images of the maps. Since \( u\Phi = \Phi v \), these are both proper subtriples of \((E_1, E_2, \Phi)\), and thus the \( \tau \)-stability condition gives

\[
\theta_\tau(K) < 0, \tag{20}
\]

and

\[
\theta_\tau(I) < 0. \tag{21}
\]

We also have the exact sequences

\[
0 \longrightarrow \text{Ker } u \longrightarrow E_1 \longrightarrow \text{Im } u \longrightarrow 0,
\]

and

\[
0 \longrightarrow \text{Ker } v \longrightarrow E_2 \longrightarrow \text{Im } v \longrightarrow 0.
\]

Let \( \sigma_u \) and \( \rho_u \) denote the ranks of \( \text{Ker } u \) and \( \text{Im } u \), and similarly for \( \sigma_v \) and \( \rho_v \). Then from the exact sequences we get

\[
(\sigma_u + \sigma_v)\mu(\text{Ker } u \oplus \text{Ker } v) + (\rho_u + \rho_v)\mu(\text{Im } u \oplus \text{Im } v) = (r_1 + r_2)\mu(E_1 \oplus E_2). \tag{22}
\]

But by definition of \( \theta_\tau(K) \),

\[
(\sigma_u + \sigma_v)\mu(\text{Ker } u \oplus \text{Ker } v) = (\sigma_u + \sigma_v)\theta_\tau(K) + \frac{\sigma_u}{r_2}(r_1 + r_2)\mu(E_1 \oplus E_2) + (\sigma_u + \sigma_v) - \frac{\sigma_u}{r_2}(r_1 + r_2))\tau,
\]

with a similar expression for \((\rho_u + \rho_v)\mu(\text{Im } u \oplus \text{Im } v)\). Also, \( \sigma_u + \rho_u = r_1 \), and \( \sigma_v + \rho_v = r_2 \). Hence from (22) we obtain

\[
r_1\theta_\tau(K) + r_2\theta_\tau(I) = 0.
\]

This is incompatible with (20) and (21).

**Corollary 3.10** If \((E_1, E_2, \Phi)\) is \( \tau \)-stable, then it is simple.

**Proof.** Let \((u, v)\) be a nontrivial element in \( H^0(E_1, E_2, \Phi) \). By the above Proposition, both \( u \) and \( v \) are isomorphisms. Fix a point \( p \) on the base of the bundles, and let \( \lambda \) be an eigenvalue of \( v : E_2|_p \longrightarrow E_2|_p \), i.e. of \( v \) acting on the fibre over \( p \).

Now define

\[
\hat{u} = u - \lambda I_1,
\]

\[
\hat{v} = v - \lambda I_2.
\]

Clearly \((\hat{u}, \hat{v})\) is in \( H^0(E_1, E_2, \Phi) \), but since \( \hat{u} \) is not an isomorphism, it follows from Proposition 3.9 that both are identically zero, i.e.

\[
(u, v) = \lambda(I_1, I_2).
\]

18
We see, in particular, that stable triples are necessarily irreducible. For reducible triples, we can however define a notion of polystability. This will be useful when we consider the relation between stability and the coupled vortex equations.

**Definition 3.11** Let $T = (E_1, E_2, \Phi)$ be a reducible triple, with $T = \bigoplus_{i=1}^n T_i$. Suppose that in each summand $T_i = (E_{1i}, E_{2i}, \Phi_i)$, the map $\Phi_i$ is non-trivial unless $E_{1i} = 0$ or $E_{2i} = 0$. Fix value of $\tau$, and let $\tau'$ be related to $\tau$ as in (19). We say that $T$ is $\tau$-polystable if for each summand $T_i$

1. if $\Phi_i \neq 0$, then $T_i$ is $\tau$-stable,
2. if $E_{1i} = 0$, then $E_{2i}$ is a stable bundle of slope $\tau'$,
3. if $E_{2i} = 0$, then $E_{1i}$ is a stable bundle of slope $\tau$.

**3.2 Duality for triples**

Associated to a triple $T = (E_1, E_2, \Phi)$ there is always a dual triple $T^\ast = (E_2^\ast, E_1^\ast, \Phi^\ast)$, where $\Phi^\ast$ is the transpose of $\Phi$, i.e. the image of $\Phi$ via the canonical isomorphism

$$\text{Hom}(E_2, E_1) \cong \text{Hom}(E_1^\ast, E_2^\ast).$$

It is reasonable that the stability of $T$ should be related to that of $T^\ast$. More precisely.

**Proposition 3.12** $T = (E_1, E_2, \Phi)$ is $\tau$-stable if and only if $T^\ast = (E_2^\ast, E_1^\ast, \Phi^\ast)$ is $(-\tau')$-stable, where $\tau'$ is related to $\tau$ by (19). Equivalently, $T$ is $\sigma$-stable if and only if $T^\ast$ is $\sigma$-stable.

**Proof.** Let $T' = (E_1', E_2', \Phi')$ be a subtriple of $T$. This defines a quotient triple $T'' = (E_1'', E_2'', \Phi'')$, where $E_1'' = E_1/E_1'$, $E_2'' = E_2/E_2'$, and $\Phi''$ is the morphism induced by $\Phi$. $T'''' = (E_2'', E_1'', \Phi''')$ is the desired subtriple of $T^\ast$. Since one has the isomorphism $T \cong T''''$ we can conclude that there is a one-to-one correspondence between subtriples of $T$ and subtriples of $T^\ast$. It is not difficult to verify that $\theta_\tau(T') < 0$ is equivalent to $\theta_{-\tau'}(T''') < 0$. The equivalence of the $\sigma$-stability for $T$ and $T^\ast$ now follows from the fact that if $\tau = \mu_\sigma(T)$, then

$$-\tau' = -\mu(E_1 \oplus E_2) - \frac{r_1}{r_1 + r_2}\sigma = -\mu_\sigma(T^\ast).$$

**3.3 Constraints on the parameters**

**Proposition 3.13** Let $(E_1, E_2, \Phi)$ be a $\tau$-stable triple, and let $\tau'$ be as above. Then
(1) $\tau > \mu(E_1)$,
(2) $\tau' < \mu(E_2)$, and
(3) $\tau - \tau' > 0$,

Equivalently, if $(E_1, E_2, \Phi)$ is $\sigma$-stable, then

(1) $\sigma > \mu(E_1) - \mu(E_2)$,
(2) $\sigma > 0$.

**Proof.** The first two statement follows from cases (1) and (3) in Proposition 3.7 with $E'_1 = E_1$, and $E''_2 = E_2$ respectively.

To prove the third statement, let $K$ be the subbundle of $E_2$ generated by the kernel of $\Phi$, and let $I$ be the subbundle of $E_1$ generated by the image of $\Phi$. Since the triple is assumed to be $\tau$-stable, $\Phi$, and therefore $I$, is non-trivial. By (1) in Proposition 3.7 we thus have

$$\mu(I) < \tau.$$  \hspace{1cm} (23)

But we also have $0 \rightarrow K \rightarrow E_2 \rightarrow I \rightarrow 0$, i.e. $I$ is a quotient of $E_2$. It thus follows from (3) in Proposition 3.7 that

$$\mu(I) > \tau'.$$  \hspace{1cm} (24)

The bounds on $\sigma$ can be obtained from those on $\tau$ by substituting $\tau = \mu_\sigma(T)$, and using the fact that

$$\sigma = \tau - \tau'$$

if $\tau'$ is as above.

Part (1) of this proposition gives the lower bound on the allowed range for $\tau$. In almost all cases the rank and degree of $E_1$ and $E_2$ also impose an upper bound on $\tau$. In fact

**Proposition 3.14** Let $(E_1, E_2, \Phi)$ be a triple with $r_1 \neq r_2$. If the triple is $\tau$-stable then

$$\tau < \mu(E_1) + \frac{r_2}{|r_1 - r_2|}(\mu(E_1) - \mu(E_2))$$  \hspace{1cm} (25)

Equivalently, if the triple is $\sigma$-stable, then

$$\sigma < (1 + \frac{r_1 + r_2}{|r_1 - r_2|})(\mu(E_1) - \mu(E_2)).$$  \hspace{1cm} (26)

**Proof.** Let $K = \text{Ker} \Phi$ and $I = \text{Im} \Phi$. Consider the subtriples

$$T_1 = (0, K, \Phi) \quad \text{and} \quad T_2 = (I, E_2, \Phi).$$

Since $r_1 \neq r_2$, $\Phi$ cannot be an isomorphism and at least one of these must be a proper subtriple. Let $r'_2 = \text{rank} K$, $r''_2 = \text{rank} I$, $d'_2 = \text{deg} K$ and $d''_2 = \text{deg} I$. 

20
A straightforward computation shows that
\[ \theta_r(T_1) < 0 \iff d'_1 - r'_2(d_1 + d_2) + r_1 r'_2 \tau < 0 \]  
\[ \theta_r(T_2) < 0 \iff d''_2 - d_1 + (r_1 - r''_2) \tau < 0. \]  

Adding (27) to \( r \) times (28), and noting that \( d_2 = d'_2 + d''_2 \) and \( r_2 = r'_2 + r''_2 \), we get that
\[ r_2(d_2 - d_1) - r'_2(d_1 + d_2) + (r_2(r_1 - r_2) + r'_2(r_1 + r_2)) \tau < 0. \]  
On the other hand combining (3) in Proposition 3.13 and (19) we obtain
\[ d_1 + d_2 - (r_1 + r_2) \tau < 0. \]  
Adding (29) to \( r'_2 \) times (30) we get
\[ (r_1 - r_2) \tau < d_1 - d_2. \]  
If now \( r_1 > r_2 \), then we get
\[ \tau < \frac{d_1 - d_2}{r_1 - r_2}, \]  
or equivalently
\[ \tau < \mu(E_1) + \frac{r_2}{r_1 - r_2} (\mu(E_1) - \mu(E_2)). \]  

To obtain the bound in the case \( r_1 < r_2 \), note that by Proposition 3.12 the \( \tau \)-stability of \((E_1, E_2, \Phi)\) is equivalent to the \((-\tau')\)-stability of the dual triple \((E^*_2, E^*_1, \Phi^*)\), where \( \tau' \) is given by
\[ r_1 \tau + r_2 \tau' = d_1 + d_2. \]  
Hence we can apply (31) to \((E^*_2, E^*_1, \Phi^*)\) to get that
\[ -\tau' < \frac{d_1 - d_2}{r_2 - r_1}, \]  
which together with (32) leads to
\[ \tau < \frac{(2r_2 - r_1)d_1 - r_1 d_2}{(r_2 - r_1)r_1}, \]  
i.e.
\[ \tau < \mu(E_1) + \frac{r_2}{r_2 - r_1} (\mu(E_1) - \mu(E_2)). \]  

Combining the lower and upper bounds on \( \tau \) (or \( \sigma \)) we can deduce

**Corollary 3.15** If rank \( E_1 \) and rank \( E_2 \) are unequal, then a triple \((E_1, E_2, \Phi)\) cannot be stable unless \( \mu(E_2) < \mu(E_1) \).
Furthermore, by the proof of Proposition 3.14 we get the following corollary.

**Corollary 3.16** Let $(E_1, E_2, \Phi)$ be $\tau$-stable and suppose that $r = s$. If $\Phi$ is not an isomorphism, then $d_1 > d_2$. In particular, in any $\tau$-stable triple $(E_1, E_2, \Phi)$, the bundle map $\Phi$ is an isomorphism if and only if $r_1 = r_2$ and $d_1 = d_2$.

*Proof.* The fact that $d_1 > d_2$ follows from the inequality

$$(r_1 - r_2) \tau < d_1 - d_2,$$

which applies if $\Phi$ is not an isomorphism. In particular, if $\Phi$ is not an isomorphism then $d_1 \neq d_2$. Conversely, if $\Phi$ is an isomorphism, then clearly $r_1 = r_2$ and $d_1 = d_2$.

It is when $\Phi$ is an isomorphism that the range for $\tau$ can fail to be bounded. For example

**Proposition 3.17** Suppose that $E_1$ and $E_2$ are both stable bundles of rank $r$ and degree $d$, and that $\Phi : E_2 \rightarrow E_1$ is non trivial. Then for any $\tau > \mu(E_1)$ the holomorphic triple $(E_1, E_2, \Phi)$ is $\tau$-stable.

*Proof.* Let $\mu(T) = \mu(E_1 \oplus E_2)$, and for a subtriple $T' = (E_1', E_2', \Phi)$ set $\mu(T') = \mu(E_1' \oplus E_2')$. Since $E_1$ and $E_2$ are stable and of equal slope, we have $\mu(T') < \mu(T)$ for all subtripes. Thus

$$\theta_{\tau}(T') \leq (\mu(T) - \tau) \frac{r_1' - r_2'}{r_1' + r_2'}.$$

Since $\Phi$ is a nontrivial map between stable bundles of the same rank and degree, it must be a multiple of the identity (cf. [O-S-S]). In particular $\Phi$ is injective and hence $r_1' - r_2' \geq 0$. Thus $\theta_{\tau}(T') \leq 0$. In fact, $\theta_{\tau}(T') < 0$ unless $r_1' = r_2'$. But in that case, we can write

$$\theta_{\tau}(T') = r_1' (\mu(E_1') - \mu(E_1)) + r_1' (\mu(E_2') - \mu(E_2)),$$

which is strictly negative.

### 3.4 Special values for $\tau$ and $\tau$-semistability

In principle $\tau$ is a continuously varying real parameter. The stability properties of a given triple do not likewise vary continuously, but can change only at certain rational values of $\tau$. This is the same phenomenon as appears in the case of stable pairs. In both cases it is due to the fact that, except for $\tau$ itself, all numerical quantities in the definition of stability are rational numbers with bounded denominators. In the case of holomorphic pairs, this has the additional consequence that for the generic choice of $\tau$ there is no distinction between stability and semistability. This is in contrast to the case of pure bundles, where the notions of stability and semistability coincide only when the rank and degree of the bundle are coprime. The next proposition shows that for a holomorphic triple both the value of $\tau$ and the greatest common divisor of the rank and degree, are relevant.
**Proposition 3.18** Let $T = (E_1, E_2, \Phi)$ be a $\tau$-semistable triple, and let $T' = (E'_1, E'_2, \Phi')$ be a subtriple such that $\theta_r(T') = 0$. Then either
\begin{equation}
r_1 r'_2 = r_2 r'_1 \text{ and } \mu(E'_1 \oplus E'_2) = \mu(E_1 \oplus E_2),
\end{equation}
or
\begin{equation}
\frac{r_2(r'_1 + r'_2)\mu(T') - r'_2(r_1 + r_2)\mu(T)}{r_2 r'_1 - r_1 r'_2} = \tau.
\end{equation}

In particular, if $r_1 + r_2$ and $d_1 + d_2$ are coprime, and $\tau$ is a not rational number with denominator of magnitude less than $r_1 r_2$, then all $\tau$-semistable triples are $\tau$-stable.

**Proof.** From the definition of $\theta_r$, we see that $\theta_r(T') = 0$ is equivalent to
\[ (\mu(E'_1 \oplus E'_2) - \frac{r'_1 r_1 + r'_2 r_2}{r_1 r'_1 + r_2 r'_2} \mu(E_1 \oplus E_2)) = \frac{\tau}{r_1 r_2}. \]
If $r'_1 r_2 - r_1 r'_2 \neq 0$ we get (??), and if $r'_1 r_2 - r_1 r'_2 = 0$ then
\[ \frac{r'_2 r_1 + r_2}{r_2 r'_1 + r'_2} = 1 \]
and we get (33).

Next we compare the stability conditions for a triple and for the two bundles in the triple.

**Proposition 3.19** Let $(E_1, E_2, \Phi)$ be a non-degenerate holomorphic triple. There is an $\epsilon > 0$, which depends only on the degrees and ranks of $E_1$ and $E_2$, and such that for $\mu(E_1) < \tau < \mu(E_1) + \epsilon$ the following is true:

1. If $(E_1, E_2, \Phi)$ is a $\tau$-stable triple, then both $E_1$ and $E_2$ are semistable bundles.
2. Conversely, if $E_1$ and $E_2$ are stable bundles, then $(E_1, E_2, \Phi)$ will be a $\tau$-stable triple for any choice of $\Phi \in H^0(\text{Hom}(E_2, E_1))$.

**Proof.** For all subbundles $E'_1 \subset E_1$ the slope $\mu(E'_1)$ is a rational number with denominator less than $r_1$. Clearly, if we pick $\epsilon$ small enough then the interval $(\mu(E_1), \mu(E_1) + \epsilon)$ contains no rational numbers with denominator less than $r_1$. The condition $\mu(E'_1) < \tau$ is thus equivalent to the condition $\mu(E'_1) \leq \mu(E_1)$, i.e. to the semistability of $E_1$.

Furthermore, as noted above, if $\tau < \mu(E_1) + \epsilon$ then $\tau' > \mu(E_2) - \frac{\epsilon}{r_2}$. Hence if $\frac{\epsilon}{r_2}$ is small enough, then the condition $\mu(E_2/E'_2) > \tau'$ for all subbundles $E'_2 \subset E_2$ becomes equivalent to the condition that $\mu(E_2/E'_2) \geq \mu(E_2)$.

Conversely, suppose $\tau = \mu(E_1) + \delta$ for some $\delta > 0$, and that $\Phi$ is any section of $H^0(\text{Hom}(E_2, E_1))$. Then for any subtriple $(E'_1, E'_2, \Phi')$ we get
\[ (r'_1 + r'_2)\theta_r(E'_1, E'_2, \Phi') = r'_1(\mu(E'_1) - \mu(E_1)) + r'_2(\mu(E'_2) - \mu(E_2)) + (r_2 r'_1 - r_1 r'_2)\delta, \]
where $r'_1 = \text{rank } E'_1$ and $r'_2 = \text{rank } E'_2$. If $E_1$ and $E_2$ are stable, and $\delta$ is small enough, then it follows from this that $\theta_r(E'_1, E'_2, \Phi') < 0$ for all subtriples.
4 Main theorem

In this section we shall show how the stability of a holomorphic triple over $X$ relates to the stability of the associated (SU(2)-equivariant) bundle over $X \times P^1$. As in Section 2, let $F \to X \times P^1$ be the extension associated to the triple $(E_1, E_2, \Phi)$, i.e. let $F$ be

$$0 \to p^*E_1 \to F \to p^*E_2 \otimes q^*\mathcal{O}(2) \to 0,$$

(35)

where $p$ and $q$ are the projections from $X \times P^1$ to $X$ and $P^1$ respectively, and $\mathcal{O}(2)$ is the line bundle of degree 2 over $P^1$. To relate the $\tau$-stability of $(E_1, E_2, \Phi)$ to the stability of $F$ we need to consider some Kähler polarization on $X \times P^1$. The parameter $\tau$ will be encoded in this polarization. Let us choose a metric on $X$ with Kähler form $\omega_X$, with volume normalized to one. The metric we shall consider on $X \times P^1$ will be, as in Section 2, the product of a the metric on $X$ with a coefficient depending on a parameter $\sigma > 0$, and the Fubini–Study metric on $P^1$ with volume also normalized to one. The Kähler form corresponding to this metric depending on the parameter $\sigma$ is

$$\omega_\sigma = \frac{\sigma}{2} p^*\omega_X \oplus q^*\omega_{P^1}.$$  

(36)

We can now state the main result of this section.

**Theorem 4.1** Let $(E_1, E_2, \Phi)$ be a holomorphic triple over a compact Riemann surface $X$. Let $F$ be the holomorphic bundle over $X \times P^1$ defined by $(E_1, E_2, \Phi)$ as in Proposition 2.4, and let

$$\sigma(\tau) = \frac{(r_1 + r_2)\tau - (\deg E_1 + \deg E_2)}{r_2}.$$  

(37)

Suppose that in $(E_1, E_2, \Phi)$ the two bundles $E_1$ and $E_2$ are not isomorphic. Then $(E_1, E_2, \Phi)$ is $\tau$-stable (equivalently $\sigma$-stable) if and only if $F$ is stable with respect to $\omega_\sigma$.

In the case that $E_1 \cong E_2 \cong E$, the triple $(E, E, \Phi)$ is $\tau$-stable (equivalently $\sigma$-stable) if and only if $F$ decomposes as a direct sum

$$F = p^*E \otimes q^*\mathcal{O}(1) \oplus p^*E \otimes q^*\mathcal{O}(1),$$

and $p^*E \otimes q^*\mathcal{O}(1)$ is stable with respect to $\omega_\sigma$.

**Proof.** As mentioned in §3, if $E_2$ is a line bundle the $\tau$-stability of $(E_1, E_2, \Phi)$ is equivalent to the $\tau$-stability of the pair $(E_1 \otimes E_2^*, \Phi)$ in the sense of Bradlow [B2]. In this case Theorem 4.1 has been proved in [GP3, Theorem 4.6]. The main ideas of that proof extend to the general case in a rather straightforward manner.

Recall that the bundle $F$ associated to $(E_1, E_2, \Phi)$ comes equipped with a holomorphic action of SU(2). It makes sense therefore to talk about the SU(2)-invariant stability of $F$. As explained in §3, this is like ordinary stability, but the slope condition has to be satisfied only for SU(2)-invariant subsheaves of $F$. In order to prove the theorem we shall prove first the following slightly weaker result.
Proposition 4.2 Let \((E_1, E_2, \Phi)\) be a holomorphic triple over a compact Riemann surface \(X\). Let \(F\) be the holomorphic bundle over \(X \times \mathbb{P}^1\) defined by \((E_1, E_2, \Phi)\) and let \(\sigma\) and \(\tau\) be related by (37). Then \((E_1, E_2, \Phi)\) is \(\tau\)-stable (equivalently \(\sigma\)-stable) if and only if \(F\) is \(SU(2)\)-invariantly stable with respect to \(\omega_{\sigma}\).

Proof. We saw in §2 (Lemmas 2.6 and 2.7) that there is a one-to-one correspondence between subtriples \(T' = (E'_1, E'_2, \Phi')\) of \(T\) and \(SU(2)\)-invariant coherent subsheaves \(F' \subset F\). Moreover, the subsheaf \(F'\) defined by \(T'\) is an extension of the form

\[
0 \to p^*E'_1 \to F' \to p^*E'_2 \otimes q^*\mathcal{O}(2) \to 0.
\]

(38)

In terms of the parameter \(\tau'\) as defined in (19), the relation between \(\sigma\) and \(\tau\), given by (37), can be rewritten as

\[
\sigma = \tau - \tau'.
\]

If \((E_1, E_2, \Phi)\) is \(\tau\)-stable (equivalently, \(\sigma\)-stable) it follows from (3) in Proposition 3.13 that \(\sigma\) defined by (37) is positive. The slope of \(F'\) with respect to \(\omega_{\sigma}\) is defined as

\[
\mu_{\sigma}(F') = \frac{\text{deg}_{\sigma} F'}{\text{rank} F'},
\]

where \(\text{deg}_{\sigma} F'\) is the degree of \(F'\), which is given by

\[
\text{deg}_{\sigma} F = \frac{1}{2} \int_{X \times \mathbb{P}^1} c_1(F) \wedge \omega_{\sigma}.
\]

The proposition follows now from the following lemma.

Lemma 4.3 Let \(T'\) be a subtriple of \(T\) and \(F'\) the corresponding \(SU(2)\)-invariant subsheaf of \(F\). Let \(\sigma\) be as in Proposition 4.2. The following are equivalent

1. \(\mu_{\sigma}(F') < \mu_{\sigma}(F)\)
2. \(\theta_{\sigma}(T') < 0\)
3. \(\mu_{\sigma}(T') < \mu_{\sigma}(T)\).

Proof. The equivalence between (2) and (3) corresponds, of course, to the two equivalent definitions of stability for \(T\) (cf. Proposition 3.3).

From (33) and (38) we obtain that

\[
\mu_{\sigma}(F) = \frac{\deg E_1 + \deg E_2 + \sigma r_2}{r_1 + r_2},
\]

and

\[
\mu_{\sigma}(F') = \frac{\deg E'_1 + \deg E'_2 + \sigma r'_2}{r'_1 + r'_2},
\]

where \(r'_1 = \text{rank} E'_1\) and \(r'_2 = \text{rank} E'_2\). From Definition 3.1 we immediately obtain the equivalence between (1) and (3).
Remark. As usual, in order for \( F \) to be \( SU(2) \)-invariantly stable it is enough to check condition (1) of Lemma 4.3 only for saturated \( SU(2) \)-invariant subsheaves, that is \( SU(2) \)-invariant subsheaves \( F' \) such that the quotient \( F/F' \) is torsion-free. Such a subsheaf \( F' \) is a subbundle outside of a set of codimension greater or equal than 2. Hence by \( SU(2) \)-invariance one concludes that \( F' \) must be actually a subbundle of \( F \) over the whole \( X \times P^1 \). It is easy to see that the saturation of \( F' \) implies that of \( p^*E_1' \) and \( p^*E_2' \otimes q^*\mathcal{O}(2) \) in (38), and hence \( E_1' \subset E_1 \) and \( E_2' \subset E_2 \) are in fact subbundles. In other words, the one-to-one correspondence between \( SU(2) \)-invariant subsheaves of \( F \) and subtriples of \( T \) established in Lemma 2.7 sends saturated subsheaves into saturated subtriples.

To prove the theorem, we first observe that if \( F \) is \( SU(2) \)-invariantly stable then, from Theorem 2.17, it decomposes as a direct sum

\[
F = F_1 \oplus F_2 \oplus ... \oplus F_k
\]  

(39)

of stable bundles, where \( F_i \) is the transformed by an element of \( SU(2) \) of a fixed subbundle \( F_1 \) of \( F \).

For the remaining parts of the theorem, the proof splits into two cases, corresponding to whether \( E_1 \) and \( E_2 \) are isomorphic (as holomorphic bundles) or not. We treat the non-isomorphic case first. Notice that in this case, the map \( \Phi \) certainly cannot be an isomorphism. Clearly if \( F \) is stable it is in particular \( SU(2) \)-invariantly stable and hence by the previous Proposition, the corresponding triple will be \( \tau \)-stable. Suppose now that \( (E_1, E_2, \Phi) \) is \( \tau \)-stable, and that \( \Phi \) is not an isomorphism. Our strategy to prove the stability of \( F \) will be to prove that \( F \) is simple, that is \( H^0(\text{End} F) \cong \mathcal{C} \), and hence there must be just one summand in the decomposition of \( F \) given by (39).

To compute \( H^0(\text{End} F) \cong H^0(F \otimes F^*) \) let us tensor (33) with \( F^* \). We obtain the short exact sequence

\[
0 \longrightarrow p^*E_1 \otimes F^* \longrightarrow F \otimes F^* \longrightarrow p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^* \longrightarrow 0,
\]

and the corresponding sequence in cohomology

\[
0 \longrightarrow H^0(p^*E_1 \otimes F^*) \longrightarrow H^0(F \otimes F^*) \longrightarrow H^0(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \longrightarrow .
\]

(40)

We first compute \( H^0(p^*E_1 \otimes F^*) \). Dualizing (35), tensoring with \( p^*E_1 \), and using that \( H^0(p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2)) = 0 \), we have the sequence in cohomology

\[
0 \longrightarrow H^0(p^*E_1 \otimes F^*) \longrightarrow H^0(p^*(E_1 \otimes E_2^*)) \xrightarrow{g} H^1(p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2)).
\]

(41)

By the Künneeth formula

\[
H^0(p^*(E_1 \otimes E_1^*)) \cong H^0(E_1 \otimes E_1^*) \quad \text{and} \quad H^1(p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2)) \cong H^0(E_1 \otimes E_2^*).
\]

Now, thanks to these isomorphisms, \( g \) can be interpreted as the map \( H^0(E_1 \otimes E_1^*) \to H^0(E_1 \otimes E_2^*) \) defined by \( \Phi \), i.e. \( g(u) = u\Phi \).
Now, from the $\tau$-stability of $(E_1, E_2, \Phi)$ one has from Corollary 3.10 that $(E_1, E_2, \Phi)$ is simple. Thus $\text{Ker } g \cong 0$ and from the exactness of (41) one obtains
\[ H^0(p^*E_1 \otimes F^*) = 0. \] (42)

To compute $H^0(p^*E_2 \otimes q^*O(2) \otimes F^*)$, we dualize (35) and tensor it with $p^*E_2 \otimes q^*O(2)$, to get the sequence
\[ 0 \to H^0(p^*(E_2 \otimes E_2^*)) \to H^0(p^*E_2 \otimes q^*O(2) \otimes F^*) \to H^0(p^*(E_1^* \otimes E_2) \otimes q^*O(2)). \] (43)

\textbf{Lemma 4.4} Let $(E_1, E_2, \Phi)$ be $\tau$-stable and suppose that $\Phi$ is not an isomorphism, then $H^0(E_1^* \otimes E_2) = 0$.

\textit{Proof.} Suppose that there is a non-zero homomorphism $\Psi : E_1 \to E_2$. Let $u = \Phi\Psi \in H^0(\text{End } E_1)$ and $v = \Psi\Phi \in H^0(\text{End } E_2)$. Then $u\Phi = \Phi v$ and, since $(E_1, E_2, \Phi)$ is simple, we have that
\[ u = \lambda I_{E_1} \quad \text{and} \quad v = \lambda I_{E_2}, \quad \text{for } \lambda \in C. \]

If $\lambda \neq 0$, we easily see that $\Phi$ is an isomorphism, contradicting the assumption of the Lemma. Thus $\lambda = 0$ and then
\[ \text{Im } \Psi \subseteq \text{Ker } \Phi \quad \text{and} \quad \text{Im } \Phi \subseteq \text{Ker } \Psi. \]

We can therefore consider the subtriples of $(E_1, E_2, \Phi)$
\[ T_1 = (K, E_2, \Phi) \quad \text{and} \quad T_2 = (0, I, \Phi), \]
where $K = \text{Ker } \Psi$ and $I = \text{Im } \Psi$. Let $r'_1 = \text{rank } K$, $r''_1 = \text{rank } I$, $d'_1 = \text{deg } K$ and $d''_1 = \text{deg } I$. Applying the $\tau$-stability condition to $T_1$ and $T_2$ we get the inequalities
\[ r_2 d''_1 - r'_1 (d_1 + d_2) + r_1 r''_1 \tau < 0 \]
\[ d'_1 - d_1 + (r_1 - r'_1) \tau < 0. \]

From this and using that from
\[ 0 \to K \to E_1 \to I \to 0, \]
br'_1 = r'_1 + r''_1 and $d_1 = d'_1 + d''_1$, we obtain that
\[ \tau < \mu(E_1 \oplus E_2), \]
which is equivalent to $\sigma(\tau) < 0$, contradicting the $\tau$-stability of $(E_1, E_2, \Phi)$.

From the Künneth formula and Lemma 4.4 we get that
\[ H^0(p^*(E_1^* \otimes E_2) \otimes q^*O(2)) \cong H^0(E_1^* \otimes E_2) \otimes H^0(O(2)) \cong 0, \]
and from (43)

\[ H^0(p^*E_2 \otimes q^*O(2) \otimes F^*) \cong H^0(E_2 \otimes E_2^*). \]

From this and (42) the first three terms in (40) reduce to

\[ 0 \longrightarrow H^0(F \otimes F^*) \overset{i}{\longrightarrow} H^0(E_2 \otimes E_2^*). \]

Since \( F \) is \( SU(2) \)-invariantly stable then it is \( SU(2) \)-invariantly simple, i.e. the only \( SU(2) \)-invariant endomorphisms are multiples of the identity. Let \( \Psi \in H^0(F \otimes F^*) \) be a non \( SU(2) \)-invariant endomorphism of \( F \), i.e. \( \Psi^g \neq \Psi \) for some \( g \in SU(2) \). Since \( i \) must be compatible with the action of \( SU(2) \), we get

\[ i(\Psi^g) = (i(\Psi))^g. \]

On the other hand, since the action of \( SU(2) \) on \( H^0(E_2 \otimes E_2^*) \) is trivial

\[ (i(\Psi))^g = i(\Psi), \]

hence \( i(\Psi) = i(\Psi^g) \) contradicting the injectivity of \( i \). Thus \( H^0(F \otimes F^*) \cong C^* \), which concludes the proof of our theorem for the case where \( E_1 \) and \( E_2 \) are not isomorphic.

Now suppose that \( E_1 \cong E_2 \). We first prove

**Lemma 4.5** Suppose \( E_1 \cong E_2 \). Then for any \( \tau > \mu(E_1) \), the triple \((E_1, E_2, \Phi)\) is \( \tau \)-stable if and only if \( \Phi \) is an isomorphism and \( E_1 \) is stable.

**Proof.** Suppose that the triple is \( \tau \)-stable. Then (by Corollary 3.8) \( \Phi \) is an isomorphism. Now consider the subtriples of the form \( T' = (\Phi(E_2'), E_2', \Phi) \). These have \( r_1' = r_2' \) and, since \( \Phi \) is an isomorphism, \( \mu(T') = \mu(E_2') \). Hence \( \theta_r(T') = \mu(E_2') - \mu(E_2) \), and thus the \( \tau \)-stability of the triple implies the stability of \( E_2 \). Conversely, if \( E_2 \cong E_1 \) and both are stable, then all non trivial \( \Phi \) are isomorphisms. It now follows as a special case of Proposition 3.6 that the triple \((E_1, E_2, \Phi)\) is \( \tau \)-stable.

Suppose that \( E_1 \cong E_2 \cong E \) and that the bundle \( F \) associated to \((E, E, \Phi)\) is of the form

\[ F = p^*E \otimes q^*O(1) \oplus p^*E \otimes q^*O(1). \]

If we assume now that \( p^*E \otimes q^*O(1) \) is stable, then \( E \) is also stable and hence \( H^0(E \otimes E^*) \cong C \). Thus there is only one non-trivial extension class (corresponding to \( \Phi = \lambda I \)).

We must now examine this (unique) non-trivial extension

\[ 0 \longrightarrow p^*E \longrightarrow F \longrightarrow p^*E \otimes q^*O(\varepsilon) \longrightarrow 0. \]

This is of course nothing else but the pull-back to \( X \times P^1 \) of the non-trivial extension on \( P^1 \)

\[ 0 \longrightarrow O \longrightarrow O(1) \oplus O(1) \longrightarrow O(2) \longrightarrow 0, \]

28
tensored with \( p^* E \). Thus the action of \( SU(2) \) permutes the two summands in (14) and from Theorem 2.17 we conclude that \( F \) is an \( SU(2) \)-invariantly stable bundle. The \( \tau \)-stability of \((E_1, E_2, \Phi)\) follows now from Proposition 4.2.

Conversely, suppose that \((E_1, E_2, \Phi)\) is \( \tau \)-stable, then from Lemma 4.5 we obtain that \( E_1 \cong E_2 \cong E \) is stable and \( \Phi \) is hence a non-trivial multiple of the identity. From the above discussion we conclude that

\[
F = p^* E \otimes q^* O(1) \oplus p^* E \otimes q^* O(1).
\]

On the other hand, from Proposition 4.2, we argue as before that \( F \) is certainly invariantly stable. Also (14), (11) and (12) show that we have an exact sequence

\[
0 \rightarrow H^0(F \otimes F^*) \rightarrow H^0(p^* E \otimes q^* O(\varepsilon) \otimes F^*). 
\]

Using that \( H^0(E \otimes E^*) \cong C \), the exact sequence (13) becomes,

\[
0 \rightarrow C \rightarrow H^0(p^* E \otimes q^* O(\varepsilon) \otimes F^*) \rightarrow C^3.
\]

From these two exact sequences, we see that

\[
1 \leq h^0(F \otimes F^*) \leq h^0(p^* E \otimes q^* O(\varepsilon) \otimes F^*) \leq \Delta .
\] (45)

But since \( F \) is invariantly stable it is given by the direct sum (39). In that case,

\[
H^0(F \otimes F^*) \cong GL(k, C),
\]

where \( k \) is the number of stable summands in \( F \). It follows that since \( h^0(F \otimes F^*) \neq 1 \), then \( h^0(F \otimes F^*) = k^2 - 1 \) for some integer \( k \). The only possibility consistent with the constraint (13) is thus \( h^0(F \otimes F^*) = 3 \), i.e. \( k = 2 \). Hence the bundle \( p^* E \otimes q^* O(1) \) in the decomposition of \( F \) is stable, which finishes the proof of the Theorem.

Notice that the conclusion of Proposition 4.2 extends straightforwardly to cover polystable objects. We thus get the following corollary, which will be useful in the next section.

**Theorem 4.6** Let \((E_1, E_2, \Phi)\) be a holomorphic triple and \( F \) be the corresponding holomorphic bundle over \( X \times P^1 \). Let \( \tau \) and \( \sigma \) be related as above. Then \((E_1, E_2, \Phi)\) is \( \tau \)-polystable if and only if \( F \) is \( SU(2) \)-invariantly polystable with respect to \( \omega_\sigma \).

## 5 Relation to vortex equations

In this section we relate the \( \tau \)-stability of a triple to the existence of solutions to the coupled \( \tau \)-vortex equations. Using the idea of dimensional reduction this can be viewed as simply a special case of the Hitchin-Kobayashi correspondence between stable
bundles and Hermitian-Einstein metrics. Indeed, specializing to the case of SU(2)-equivariant bundles over $X \times P^1$ as in §2 (i.e. to the case of the bundles associated with triples over $X$), we have already assembled seen the results we need.

From [GP3] (cf. also Theorem 2.18) we know that the SU(2)-equivariant bundle $F$ admits a Hermitian-Einstein metric with respect to $\omega_\sigma$ if and only if $F$ is polystable with respect to $\omega_\sigma$. By Theorem 4.6 we know that $F$ is SU(2) invariantly polystable with respect to $\omega_\sigma$ if and only if the corresponding triple $(E_1, E_2, \Phi)$ is $\tau$-polystable with $\tau$ and $\sigma$ related by (37). Finally, by Proposition 2.14 there is a bijective correspondence between the SU(2)-equivariant Hermitian-Einstein metrics on $F$ and the solutions to the coupled vortex equations (10) on $(E_1, E_2, \Phi)$.

Putting all this together, we see that we have filled in three sides of the following “commutative diagram” for a holomorphic triple over $X$ and the corresponding SU(2)-equivariant bundle over $X \times P^1$ (with Kähler form $\omega_\sigma$)

| SPECIAL METRICS | Hitchin-Kobayashi correspondence | HOLOMORPHIC INTERPRETATION |
|-----------------|---------------------------------|-----------------------------|
| On $X \times P^1$ SU(2)-invariant Solutions to Hermitian-Einstein Equations | Th. 2.18 SU(2)-invariantly Polystable Extensions |
| dim. red. | Prop. 2.14 | Th. 4.6 |
| On $X$ | Solutions to Coupled $\tau$-Vortex Equations | $\tau$-stable holomorphic Triples |

By tracing around the three sides of this diagram which already have arrows filled in, we can fill in the arrows on the fourth side and prove

**Theorem 5.1** Let $T = (E_1, E_2, \Phi)$ be a holomorphic triple. Then the following are equivalent.

1. The bundles support Hermitian metrics $h_1, h_2$ such that the coupled $\tau$-vortex equations are satisfied, i.e. such that
   \begin{align*}
   \sqrt{-1} \Lambda F_{h_1} + \Phi^* = 2\pi \tau I_{E_1}, \\
   \sqrt{-1} \Lambda F_{h_2} - \Phi^* \Phi = 2\pi \tau' I_{E_2},
   \end{align*}
   with
   \begin{equation}
   r_1 \tau + r_2 \tau' = \deg E_1 + \deg E_2,
   \end{equation}

2. The triple is $\tau$-polystable.

Note that the statement and conclusion of this theorem make no mention of $X \times P^1$ or the SU(2)-equivariant bundle $F$. One might thus expect a more direct
proof that does not use dimensional reduction. We will not attempt to prove both
directions of the biconditional in the theorem in this way, but the one direction is quite
easily seen. That is, one can show how the $\tau$-stability condition can be derived directly
as a consequence of the coupled vortex equations. We do this as fo llows.

Let $T' = (E'_1, E'_2, \Phi)$ be a holomorphic saturated sub-triple of $T$, and let

$$E_1 = E'_1 \oplus E''_1,$$

and

$$E_2 = E'_2 \oplus E''_2$$

be smooth orthogonal splittings of $E_1$ and $E_2$. With respect to these splittings, we get
a block diagonal decomposition of $\sqrt{-1}\Lambda F_{h_1}$, $\sqrt{-1}\Lambda F_{h_2}$ as

$$\sqrt{-1}\Lambda F_{h_i} = \begin{pmatrix} \sqrt{-1}\Lambda F'_i + \Pi_i & * \\ * & \sqrt{-1}\Lambda F''_i - \Pi_i \end{pmatrix}, \quad (49)$$

where $\Pi_i$ is a positive definite endomorphism coming from the second fundamental
form for the inclusion of $E'_i$ in $E_i$. We also get a decomposition of $\Phi$ as

$$\Phi = \begin{pmatrix} \Phi' & \Theta \\ 0 & \Phi'' \end{pmatrix}. \quad (50)$$

The coupled vortex equations thus split into equations on the summands of $E_1$ and $E_2$
to yield

$$\sqrt{-1}\Lambda F'_1 + \Pi_1 + \Phi'\Phi'^* + \Theta\Theta^* = 2\pi\tau'I'_1; \quad (51)$$

$$\sqrt{-1}\Lambda F''_1 - \Pi_1 + \Phi''\Phi''* = 2\pi\tau'I''_1; \quad (52)$$

$$\sqrt{-1}\Lambda F'_2 + \Pi_2 - (\Phi')\Phi' = 2\pi\tau'I'_2; \quad (53)$$

$$\sqrt{-1}\Lambda F''_2 - \Pi_2 - (\Phi'')\Phi'' - \Theta^*\Theta = 2\pi\tau'I''_2. \quad (54)$$

We now integrate the trace of these equations over the base manifold $X$. We use
the notation

$$d'_i = \deg E'_i = \frac{\sqrt{-1}}{2\pi} \int_X \Tr(\Lambda F'_i),$$

$$d''_i = \deg E''_i = \frac{\sqrt{-1}}{2\pi} \int_X \Tr(\Lambda F''_i),$$

$$||\Pi_i||^2 = \frac{1}{2\pi} \int_X \Tr(\Pi_i),$$

$$||\Theta||^2 = \frac{1}{2\pi} \int_X \Tr(\Theta\Theta^*),$$

From equations (46) and (48) we get

$$d''_1 + d''_2 - (||\Pi_1||^2 + ||\Pi_2||^2) - ||\Theta||^2 = \tau(r_1 - r'_1) + \tau'(r_2 - r'_2), \quad (55)$$

31
and from \((\text{47})\) and \((\text{51})\) we get
\[
d_1' + d_2' + (||\Pi_1||^2 + ||\Pi_2||^2) + ||\Theta||^2 = \tau r_1' + \tau' r_2'. \tag{56}
\]
Using the fact that \(d_i = d_i' + d_i''\), these equations can be combined to give
\[
\text{deg}(E_1' \oplus E_2') + 2(||\Pi_1||^2 + ||\Pi_2||^2 + ||\Theta||^2) = r_1' \tau + r_2' \tau' \tag{57}
\]
It now follows by the positivity of the terms \(||\Pi_i||^2\) and \(||\Theta||^2\), that
\[
\mu(T') \leq \frac{r_1' \tau + r_2' \tau'}{r_1' + r_2'}. \tag{58}
\]
But from equation \((\text{48})\) we have that
\[
\frac{r_1 \tau + r_2 \tau'}{r_1 + r_2} = \mu(T),
\]
and by using this to solve for \(\tau'\), one sees that \((\text{58})\) is equivalent to the condition \(\theta_r(T') \leq 0\). Furthermore, it follows from \((\text{57})\) that in order to have \(\theta_r(T') = 0\), one needs
\[
||\Pi_1||^2 = ||\Pi_2||^2 = ||\Theta||^2 = 0.
\]
This means that the bundles split holomorphically as \(E_i = E_i' \oplus E_i''\) and \(\Phi = \Phi' \oplus \Phi''\), i.e. the triple \(T = (E_1, E_2, \Phi)\) splits as \(T = (E_1', E_2', \Phi') \oplus (E_1'', E_2'', \Phi'')\). Each summand separately supports a solution to the coupled \(\tau\)-vortex equations. It is possible that in this splitting \(\Phi'\) or \(\Phi''\) is trivial. The bundles in the degenerate subtriple then each supports solutions to the Hermitian-Einstein equations. We have thus proven the following proposition:

**Proposition 5.2** Let \(T = (E_1, E_2, \Phi)\) be a holomorphic triple in which the bundles support Hermitian metrics \(h_1, h_2\) such that the coupled \(\tau\)-vortex equations are satisfied. Then

1. \(T\) splits as a direct sum of triples \((E_{1i}, E_{2i}, \Phi_i)\), i.e. \(E_1 = \bigoplus E_{1i}, E_2 = \bigoplus E_{2i}\), and \(\Phi = \bigoplus \Phi_i\),
2. each summand \((E_{1i}, E_{2i}, \Phi_i)\) is either a \(\tau\)-stable triple, or \(\Phi_i = 0\) and both bundles are stable. In the latter case, the slope of \(E_{1i}\) is \(\tau\) and the slope of \(E_{2i}\) is \(\tau'\).

That is \(T\) is \(\tau\)-polystable.

### 6 Moduli spaces
6.1 Moduli spaces of stable triples

Recall that two triples \( T = (E_1, E_2, \Phi) \) and \( T' = (E'_1, E'_2, \Phi') \) are isomorphic if there exist isomorphisms \( u : E_1 \rightarrow E'_1 \) and \( v : E_2 \rightarrow E'_2 \) making the following diagram commutative

\[
\begin{array}{ccc}
E_2 & \xrightarrow{\Phi} & E_1 \\
v\downarrow & & u\downarrow \\
E'_2 & \xrightarrow{\Phi'} & E'_1
\end{array}
\]

After fixing the topological invariants of our bundles, that is the ranks \( r_1 \) and \( r_2 \) and the first Chern classes \( d_1 \) and \( d_2 \), let \( \mathcal{M} \) be the set of equivalence classes of holomorphic triples and \( \mathcal{M}_\tau \subset \mathcal{M} \) be the subset of equivalence classes of \( \tau \)-stable triples. Our goal in this section is to show that \( \mathcal{M}_\tau \) has the structure of an algebraic variety, more precisely:

**Theorem 6.1** Let \( X \) be a compact Riemann surface of genus \( g \) and let us fix ranks \( r_1 \) and \( r_2 \) and degrees \( d_1 \) and \( d_2 \). The moduli space of \( \tau \)-stable triples \( \mathcal{M}_\tau \) is a complex analytic space with a natural Kähler structure outside of the singularities. Its dimension at a smooth point is

\[
1 + r_2d_1 - r_1d_2 + (r_1^2 + r_2^2 - r_1r_2)(g - 1).
\]

The moduli space, \( \mathcal{M}_\tau \) is non-empty if and only if \( \tau \) is inside the interval

\[
(\mu(E_1), \mu_{\text{MAX}})
\]

where

\[
\mu_{\text{MAX}} = \mu(E_1) + \frac{r_2}{|r_1 - r_2|}(\mu(E_1) - \mu(E_2))
\]

if \( r_1 \neq r_2 \), and \( \mu_{\text{MAX}} = \infty \) if \( r_1 = r_2 \).

Moreover \( \mathcal{M}_\tau \) is in general a quasi-projective variety. It is in fact projective if \( r_1 + r_2 \) and \( d_1 + d_2 \) are coprime and \( \tau \) is generic.

**Proof.** There are several approaches one can take to prove this theorem. One can use standard Kuranishi deformation methods as done in \([BD1, BD2]\) for the construction of the moduli spaces of stable pairs. Alternatively one can use geometric invariant theory methods to give an algebraic geometric construction of our moduli spaces, generalizing the construction of the moduli space of stable pairs given in \([B3, I]\). We will leave these two direct methods for a future occasion and instead will exploit the relation between \( \tau \)-stable triples and equivariant bundles over \( X \times \mathbb{P}^1 \). This method, which is used in \([GP3]\) to construct the moduli spaces of triples when \( E_2 \) is a line bundle, leads also to an alternative construction of the moduli spaces of stable pairs. Apart from smoothness considerations, which we shall discuss later, the arguments of the proof are the same that those in \([GP3]\).
Let $\sigma$ be related to $\tau$ by (37) and let $\omega_{\sigma}$ be the Kähler form on $X \times P^1$ defined by (36). Let $M_{\sigma}$ be the moduli space of stable bundles with respect to $\omega_{\sigma}$ whose underlying smooth bundle is defined by (1). Let us exclude for the moment the case $r_1 = r_2$ and $d_1 = d_2$. Let $F \rightarrow X \times P^1$ be the bundle associated to $(E_1, E_2, \Phi)$ as in Proposition 2.4. Theorem 4.1 says that the correspondence $(E_1, E_2, \Phi) \mapsto F$ defines a map

$\mathcal{M}_\tau \rightarrow \mathcal{M}_\sigma$.

The action of $SU(2)$ on $X \times P^1$ defined in Section 2 induces an action on $M_{\sigma}$ and, since the bundle $F$ associated to $(E_1, E_2, \Phi)$ is $SU(2)$-equivariant the image of the above map is contained in $\mathcal{M}_{SU(2)}$—the set of fixed points of $\mathcal{M}_\sigma$ under the $SU(2)$ action. As proved in [GP3, Proposition 5.3] the set $\mathcal{M}_{SU(2)}$ can be described as a disjoint union of a finite number of sets

$\mathcal{M}_{SU(2)} = \bigcup_{i \in I} \mathcal{M}_i$.

The index $I$ ranges over the set of equivalence classes of different smooth $SU(2)$-equivariant structures on the smooth bundle $F$ defined by (1). Of course the way of writing $F$ in (1) already exhibits a particular $SU(2)$-equivariant structure, but in principle the bundle $F$ might admit different ones. The set $\mathcal{M}_i$ corresponds to the set of equivalence classes in $\mathcal{M}_\sigma$ admitting a representative which is $SU(2)$-equivariant for the smooth equivariant structure defined by $i \in I$. An equivariant smooth structure defines an action on the space of smooth automorphisms of the bundle $F$ and, as shown in [GP3, Theorem 5.6] the sets $\mathcal{M}_i$ can be described as the set of equivalence classes of $SU(2)$-equivariant holomorphic structures on the underlying smooth $SU(2)$-equivariant bundle defined by $i$, modulo $SU(2)$-equivariant isomorphisms.

Let $i_0$ be the $C^\infty$ $SU(2)$-equivariant structure on $F$ defined by (1). As shown in Proposition 2.4 there is a one-to-one correspondence

$\{\text{holomorphic triples}\} \leftrightarrow \{i_0\text{-equivariant holomorphic vector bundles}\}$.

(62)

On the other hand by Lemma 2.10 the equivariant homomorphisms between two equivariant holomorphic bundles $F$ and $F'$ corresponding to triples $T$ and $T'$, respectively, are in one-to-one correspondence with the morphisms between $T$ and $T'$. In fact the correspondence (62) descends to the quotient and thus from Theorem 4.1 we can identify $\mathfrak{M}_\tau$ with $\mathcal{M}_{i_0}$. The properties of $\mathfrak{M}_\tau$ follow now from standard facts about the more familiar moduli spaces of stable bundles $\mathcal{M}_\sigma$ [D-K, G, M, Ko], and more particularly of the fixed-point sets $\mathcal{M}_i$ (see [GP3, Theorem 5.6] for details). Namely,

**Theorem 6.2** $\mathcal{M}_i$ is a complex analytic variety. A point $[F] \in \mathcal{M}_i$ is non-singular if it is non-singular as a point of $\mathcal{M}_\sigma$. The tangent space at such a point can be identified with the $SU(2)$-invariant part of $H^1(X \times P^1, \text{End } F)$. $\mathcal{M}_i$ has a natural Kähler structure induced from that of $\mathcal{M}_\sigma$. Moreover if $\sigma$ is a rational number then $\mathcal{M}_i$ is a quasi-projective variety.
From this theorem and the identification of $\mathcal{M}_r$ with $\mathcal{M}_g^{\mu}$ we deduce that $\mathcal{M}_r$ is a complex analytic variety with a Kähler metric outside the singularities. To compute the dimension of the tangent space at a smooth point $[T]$ it suffices to compute the dimension of the $SU(2)$-invariant part of $H^1(X \times P^1, \text{End } F)$. This can be done in a similar way to that of [GP3, Theorem 5.13] to obtain that
\[
\dim \mathcal{M}_r = 1 + \chi(E_1 \otimes E_2^*) - \chi(\text{End } E_1) - \chi(\text{End } E_2),
\]
which by Riemann-Roch yields (59).

We consider now the case $r_1 = r_2 = r$ and $d_1 = d_2 = d$. In this case by Lemma 3.18 we can identify the moduli space $\mathcal{M}_r$ with the moduli space of stable bundles of rank $r$ and degree $d$ on $X$. The theorem follows now from well-known results about this moduli space [A-B, N-S].

The fact that $\mathcal{M}_r$ is empty outside the interval (60) if $r_1 \neq r_2$ and outside $(\mu(E_1), \infty)$ if $r_1 = r_2$ follows from Proposition 3.14. As explained in Proposition 3.18 the non-generic values divide this intervals in subintervals in such a way that the stability properties of a given triple do not change for two values of $\tau$ in the same subinterval. Therefore we can always choose $\tau$ (and hence $\sigma$) to be rational, which by Theorem 6.2 gives that $\mathcal{M}_r$ is quasi-projective.

To show the compactness of $\mathcal{M}_r$ when $r_1 + r_2$ and $d_1 + d_2$ are coprime and $\tau$ is generic (we are also assuming that $r_1 \neq r_2$ or $d_1 \neq d_2$) we consider a sequence of points in $\mathcal{M}_r^{\mu}$—the Uhlenbeck compactification of $\mathcal{M}_r$. Using $SU(2)$-invariance one can see that the limit has to correspond to a polystable element, but by Proposition 3.18 this has to be actually stable, that is the limit must be in $\mathcal{M}_r$ and hence in $\mathcal{M}_r^{\mu}$ since this is closed. The compactness when $r_1 = r_2$ and $d_1 = d_2$ follows from the compactness of the moduli space of stable bundles of rank $r$ and degree $d$ when $r$ and $d$ are coprime. The compactness of $\mathcal{M}_r$ can also be obtained (as it is done for pairs in [B-D]) from the fact that it can be identified with the moduli space of solutions to the coupled vortex equations and these are moment map equations as we shall explain later.

It was shown in [GP3, Theorem 5.13] that when $E_2$ is a line bundle our moduli spaces are smooth for every value of $\tau$. This does not seem to be the case when $E_2$ is of arbitrary rank. However we can show the following

**Proposition 6.3** Let $T = (E_1, E_2, \Phi)$ be a holomorphic triple such that $\Phi$ is either injective or surjective, then $[T]$ is a smooth point of $\mathcal{M}_r$.

**Proof.** Let
\[
0 \rightarrow p^*E_1 \rightarrow F \rightarrow p^*E_2 \otimes q^*\mathcal{O}(2) \rightarrow 0,
\]
be the extension over $X \times P^1$ corresponding to $T$. To prove the smoothness of $\mathcal{M}_r$ at the point $[(E_1, E_2, \Phi)]$ it suffices to show that $H^2(X \times P^1, \text{End } F) = 0$. Tensoring (63) with $F^*$ the last terms in the corresponding long exact sequence are
\[
H^2(p^*E_1 \otimes F^*) \rightarrow H^2(F \otimes F^*) \rightarrow H^2(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \rightarrow 0.
\]
By Serre duality
\[
H^2(p^*E_1 \otimes F^*) \cong H^0(p^*(E_1^* \otimes K) \otimes F^*),
\]
\[
H^2(p^*E_2 \otimes q^*\mathcal{O}(2) \otimes F^*) \cong H^0(p^*(E_2^* \otimes K) \otimes q^*\mathcal{O}(-4) \otimes F^*),
\]
where $K$ is the canonical line bundle of $X$.

It is easy to see that $H^0(p^*(E_2^* \otimes K) \otimes q^*\mathcal{O}(-4) \otimes F^*) \cong 0$. To analyse $H^0(p^*(E_2^* \otimes K) \otimes q^*\mathcal{O}(-2))$, where $(E_1, E_2, d)$ is the Seshadri compactification of the moduli space of stable bundles of rank $r, d$ over $X$, that is, the space of $S$-equivalence classes of semistable bundles.

Assume now that $\Phi$ is injective, if we prove that $f$ is also injective, by the exactness of $(\mathcal{E})$ we would be done. Suppose that $\text{Ker } f \neq 0$. This means that there exists a non-zero map $\Psi : E_1 \rightarrow E_2 \otimes K$, and since $\Psi \circ \Phi = 0$, $\text{Im } \Psi$ is a non-trivial subsheaf contained in $\text{Ker } \Phi$ contradicting the injectivity.

To prove smoothness when $\Phi$ is surjective, we consider the dual triple $T^* = (E_2^*, E_1^*, \Phi^*)$. $\Phi^*$ is now injective and the result follows from the fact that $T = (E_1, E_2, \Phi)$ is a smooth point if and only if $T^* = (E_2^*, E_1^*, \Phi^*)$ is a smooth point.

### 6.2 Abel–Jacobi maps

As shown in Proposition 3.19 there is a range for the parameter $\tau$ such that the $\tau$-stability of a triple $(E_1, E_2, \Phi)$ implies the semistability of $E_1$ and $E_2$. Let $\mathcal{M}_0$ be the moduli space of $\tau$-stable triples for $\tau$ in such a range. Let $N(r, d)$ be the Seshadri compactification of the moduli space of stable bundles of rank $r$ and degree $d$ over $X$, that is, the space of $S$-equivalence classes of semistable bundles.

There are natural “Abel–Jacobi” maps $\pi_1$ and $\pi_2$

\[
\mathcal{M}_0 \xrightarrow{\pi_2} N(r_2, d_2) \xrightarrow{\pi_1} N(r_1, d_1)
\]
defined as

$\pi_1([(E_1, E_2, \Phi)]) = [E_1]$ and $\pi_2([(E_1, E_2, \Phi)]) = [E_2]$.

We know also from Proposition 3.19 that if both $E_1$ and $E_2$ are stable then the intersection of the fibres $\pi_1^{-1}([E_1])$ and $\pi_2^{-1}([E_2])$ can be identified with $P(H^0(E_1 \otimes E_2))$. In general, though, this intersection for non-stable points is hard to describe.
If \( \mu(E_1 \otimes E_2^*) > 2g - 2 \), that is if

\[
r_2d_1 - r_1d_2 > r_1r_2(2g - 2),
\]

where \( g \) is the genus of \( X \), then \( H^1(E_1 \otimes E_2^*) = 0 \) for \( E_1 \) and \( E_2 \) stable and the projection from \( \mathcal{M}_0 \) to \( N(r_1, d_1) \times N(r_2, d_2) \) is a fibration on the stable part.

Recall that if \( (r_1, d_1) = 1 \) and \( (r_2, d_2) = 1 \) then stability and semistability coincide and there exist universal bundles \( E_1 \rightarrow X \times N(r_1, d_1) \) and \( E_2 \rightarrow X \times N(r_2, d_2) \).

Let us denote by \( p_1, p_2 \) and \( \pi \) the projections from \( X \times N(r_1, d_1) \times N(r_2, d_2) \) to \( X \times N(r_1, d_1), X \times N(r_2, d_2), \) and \( N(r_1, d_1) \times N(r_2, d_2) \) respectively. It is clear that \( \mathcal{M}_0 \) can be identified with

\[
P(\pi_*(p_1^*E_1 \otimes p_2^*E_2^*)).
\]

But in the non-coprime situation we have no universal bundles \( E_1 \) and \( E_2 \) available and the analogue of (66) has to be constructed as a moduli space in its own right.

As explained in Theorem 6.1 the moduli space of \( \tau \)-stable triples is non-empty if and only \( \tau \) is in the interval \( I = (\mu(E_1), \mu_{MAX}) \). We saw in §3.4 that the stability properties of a given triple can change only at certain rational values of \( \tau \) (the critical values) which divide \( I \) in a finite number of subintervals. The moduli spaces for values of \( \tau \) in the same open subinterval are then isomorphic, and they might change only when crossing one of the critical values. We expect that, as in the case of stable pairs [B-D-W, T], the moduli spaces for consecutive intervals must be related by some sort of flip-type birational transformation. This, as well as the construction of a “master” space for triples (cf. [B-D-W]) containing the moduli space of triples for all possible values of \( \tau \), will be dealt with in a future paper.

### 6.3 Vortices

Thanks to our existence theorem the moduli space of stable triples can be interpreted as the moduli space of solutions to the coupled vortex equations. To understand the meaning of this statement one needs to regard the vortex equations as equations for unitary connections instead of equations for metrics. This point of view corresponds to the fact that fixing a holomorphic structure and varying the metric on a vector bundle is equivalent to fixing the metric and varying the holomorphic structure—or the corresponding connection. Recall that the space of unitary connections on a smooth Hermitian vector bundle can be identified with the space of \( \partial \)-operators which in turn corresponds with the space of holomorphic structures on our bundle.

Let \( E_1 \) and \( E_2 \) be smooth vector bundles over \( X \) and \( h_1 \) and \( h_2 \) be Hermitian metrics on \( E_1 \) and \( E_2 \) respectively. Let \( A_1 \) (resp. \( A_2 \)) be the space of unitary connections on \( (E_1, h_1) \) (resp. \( (E_2, h_2) \)). Let \( (A_1, A_2, \Phi) \in A_1 \times A_2 \times \Omega^0(\text{Hom}(E_2, E_1)) \). The vortex
equations can be regarded as the equations for \((A_1, A_2, \Phi)\)
\[
\begin{align*}
\overline{\partial}_{A_1*A_2}\Phi &= 0 \\
\sqrt{-1}\Lambda F_{A_1} + \Phi\Phi^* &= 2\pi\tau I_{E_1} \\
\sqrt{-1}\Lambda F_{A_2} - \Phi^*\Phi &= 2\pi\tau'I_{E_2}
\end{align*}
\]
(67)

The connections \(A_1\) and \(A_2\) induce holomorphic structures on \(E_1\) and \(E_2\) and the first equation in (67) simply says that \(\Phi\) must be holomorphic.

Let \(G_1\) and \(G_2\) be the gauge groups of unitary transformations of \((E_1, h_1)\) and \((E_2, h_2)\) respectively. \(G_1 \times G_2\) acts on \(A_1 \times A_2 \times \Omega^0(\text{Hom}(E_2, E_1))\) by the rule
\[
(g_1, g_2). (A_1, A_2, \Phi) = (g_1 A_1 g_1^{-1}, g_2 A_2 g_2^{-1}, g_1 \Phi g_2^{-1}).
\]
The action of \(G_1 \times G_2\) preserves the equations and the moduli space of \textit{coupled \(\tau\)-vortices} is defined as the space of all solutions to (67) modulo this action.

The moduli space of \(\tau\)-vortices can be obtained as a symplectic reduction (see [GP3, Section 2.2]) in a similar way to the moduli space of Hermitian–Einstein connections: \(A_1 \times A_2 \times \Omega^0(\text{Hom}(E_2, E_1))\) admits a Kähler structure which is preserved by the action of \(G_1 \times G_2\). Associated to this action there is a moment map given precisely by
\[
(A_1, A_2, \Phi) \rightarrow (\Lambda F_{A_1} - \sqrt{-1}\Phi\Phi^* + 2\sqrt{-1}\pi\tau, \Lambda F_{A_2} + \sqrt{-1}\Phi^*\Phi + 2\sqrt{-1}\pi\tau').
\]
(68)

Let \(\mu\) be this moment map restricted to the subvariety
\[
\mathcal{N} = \{(A_1, A_2, \Phi) \in A_1 \times A_2 \times \Omega^0(\text{Hom}(E_2, E_1)) \mid \overline{\partial}_{A_1*A_2}\Phi = 0\}.
\]
The moduli space of \(\tau\)-vortices is then nothing else but the symplectic quotient
\[
\mu^{-1}(0, 0)/G_1 \times G_2,
\]
and Theorem 5.1 can be reformulated by saying that there is a one-to-one correspondence
\[
\mu^{-1}(0, 0)/G_1 \times G_2 \leftrightarrow \mathcal{M}_\tau.
\]

### 7 Some generalizations

1. Although for simplicity we have worked on a Riemann surface, most of our results extend in a straightforward manner to a compact complex manifold of arbitrary dimension. Of course, as in ordinary stability, one needs to choose a Kähler metric in order to define the degree of a coherent sheaf and hence the slopes involved in the definition of \(\tau\)-stability for a triple.

2. One of our main goals in this paper has been to show that our stability condition for a triple corresponds to the existence of solutions to the coupled vortex equations. This is the main reason for defining our stability criterium only for vector bundles.
One can more generally define \( \tau \)-stability for a triple consisting of two (torsion free) coherent sheaves and a morphism between them. The main results of Sections 3 and 4 go through in this more general situation.

3. A. King \([K]\) has been able to characterize all \( SU(2) \)-equivariant holomorphic vector bundles on \( X \times P^1 \). Generalizing the results in Section 2, he has shown that these bundles are in one-to-one correspondence with \( (2n - 1) \)-tuples consisting of \( n \) holomorphic vector bundles \( E_1, \ldots, E_n \) over \( X \) and a chain of morphisms

\[
E_n \xrightarrow{\phi_{n-1}} E_{n-1} \rightarrow \ldots \rightarrow E_2 \xrightarrow{\phi_1} E_1.
\]

He has defined a stability condition for such a \( (2n - 1) \)-tuple which involves \( (n - 1) \) parameters and that specializes to our stability condition for a triple when \( n = 2 \). In fact he considers this notion for more general diagrams than the one above. Presumably this stability condition governs, as for triples, the existence of Hermitian metrics on the bundles \( E_i \) satisfying some generalized vortex equations naturally associated to the \( (2n - 1) \)-tuple.

4. Our results have also been extended in a different direction \([B-GP]\) to parabolic triples, that is to triples in which the bundles are endowed with parabolic structures. The Higgs field can be either a parabolic morphism or a meromorphic morphism with simple poles at the parabolic points and whose residues respect the parabolic structure in some precise sense. In both cases one can prove a Hitchin-Kobayashi correspondence, although the metrics involved now have singularities at the parabolic points.

Acknowledgements. The authors would like to thank Alastair King and Jun Li for helpful conversations and the following institutions for their hospitality during the course of this project: The Mathematics Institute of the University of Warwick, England; I.H.E.S., France; the Mathematics Department of UC Berkeley, USA; and C.I.M.A.T., Mexico.

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