Algorithms and complexity for geodetic sets on planar and chordal graphs

Dibyayan Chakraborty
Indian Statistical Institute, Kolkata, India

Sandip Das
Indian Statistical Institute, Kolkata, India

Florent Foucaud
Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence, France

Harmender Gahlawat
Indian Statistical Institute, Kolkata, India

Dimitri Lajou
Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence, France

Bodhayan Roy
Indian Institute of Technology, Kharagpur

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Abstract

We study the complexity of finding the 
geodetic number
 on subclasses of planar graphs and chordal graphs. A set $S$ of vertices of a graph $G$ is a 
geodetic set
 if every vertex of $G$ lies in a shortest path between some pair of vertices of $S$. The Minimum Geodetic Set (MGS) problem is to find a geodetic set with minimum cardinality of a given graph. The problem is known to remain NP-hard on bipartite graphs, chordal graphs, planar graphs and subcubic graphs. We first study MGS on restricted classes of planar graphs: we design a linear-time algorithm for MGS on solid grids, improving on a 3-approximation algorithm by Chakraborty et al. (CALDAM, 2020) and show that it remains NP-hard even for subcubic partial grids of arbitrary girth. This unifies some results in the literature. We then turn our attention to chordal graphs, showing that MGS is fixed parameter tractable for inputs of this class when parameterized by its tree-width (which equals its clique number). This implies a polynomial-time algorithm for $k$-trees, for fixed $k$. Then, we show that MGS is NP-hard on interval graphs, thereby answering a question of Ekim et al. (LATIN, 2012). As interval graphs are very constrained, to prove the latter result we design a rather sophisticated reduction technique to work around their inherent linear structure.

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1 Introduction

A simple undirected graph $G$ has vertex set $V(G)$ and edge set $E(G)$. For two vertices $u, v \in V(G)$, let $I(u, v)$ denote the set of all vertices in $G$ that lie in some shortest path between $u$ and $v$. For a subset $S$ of vertices of a graph $G$, let $I(S) = \bigcup_{u, v \in S} I(u, v)$. We say that $T \subseteq V(G)$ is covered by $S$ if $T \subseteq I(S)$. A set of vertices $S$ is a geodetic set if $V(G)$ is covered by $S$. The geodetic number, denoted $g(G)$, is the minimum integer $k$ such that $G$ has a geodetic set of cardinality $k$. Given a graph $G$, the Minimum Geodetic Set (MGS) problem, introduced in [10], is to compute a geodetic set of $G$ with minimum cardinality. In this paper, we study the computational complexity of MGS in subclasses of planar and chordal graphs. MGS is a natural graph covering problem that falls in the class of problems dealing with the important geometric notion of convexity: see [11][21] for some
general discussion of graph convexities. The setting of MGS is quite natural, and it can be applied to facility location problems such as the optimal determination of bus routes in a public transport network [6]. See also [10] for further applications.

The algorithmic complexity of MGS has been studied actively. In 2002, Atici [2] proved that MGS is NP-hard. Later, Dourado et al. [7,8] strengthened the above result to bipartite input graphs, chordal graphs (i.e graphs with no induced cycle of order greater than 3) and chordal bipartite graphs. Recently, Bueno et al. [5] proved that MGS remains NP-hard for subcubic graphs, and Chakraborty et al. [6] proved that MGS is NP-hard for planar graphs. Kellerhals and Koana [18] studied the parameterized complexity of MGS, proving that it is unlikely to be FPT for the parameters solution size, feedback vertex set number and pathwidth, combined.

On the positive side, polynomial-time algorithms to solve MGS are known for cographs [7], split graphs [7], ptolemaic graphs [11], block cactus graphs [10], outerplanar graphs [20] and proper interval graphs [10], and the problem is FPT for parameters tree-depth and feedback edge set number [18].

A grid embedding of a graph is a set of points in 2D with integer coordinates such that each point in the set represents a vertex of the graph and, for each edge, the points corresponding to its endpoints are at Euclidean distance 1. A graph is a partial grid if it has a grid embedding. A graph is a solid grid if it has a grid embedding such that all interior faces have unit area. Chakraborty et al. [6] gave a 3-approximation algorithm for MGS on solid grids. We improve this as follows.

▶ **Theorem 1.** There is a linear-time algorithm for MGS on solid grids.

We note that researchers have designed polynomial time algorithms on solid grids [12,19,22]. Our algorithm on solid grids does not require the grid embedding to be part of the input. This is interesting since deciding whether an input graph is a solid grid is an NP-complete problem [15]. To complement Theorem 1, we prove the following.

▶ **Theorem 2.** MGS is NP-hard for subcubic partial grids of girth at least g, for any fixed integer $g \geq 4$.

We note that this result jointly strengthens three existing hardness results: for bipartite graphs [7], subcubic graphs [5] and planar graphs [6]. Moreover, partial grids are subclasses of many other important graph classes such as disk graphs, rectangle intersection graphs, etc. Hence, our result implies that MGS remains NP-hard on the aforementioned graph classes.

An interval representation of a graph $G$ is a collection of intervals on the real line such that two intervals intersect if and only if the corresponding vertices are adjacent in $G$. A graph is an interval graph if it has an interval representation. Ekim et al. [10] asked whether there is a polynomial time algorithm to solve MGS on interval graphs. We give a negative answer to their question (note that proper interval graphs are those interval graphs with no induced $K_{1,3}$).

▶ **Theorem 3.** MGS is NP-hard for interval graphs with no induced $K_{1,5}$.

This result is somewhat surprising, as most covering problems can be solved in polynomial time on interval graphs (but other distance-based problems, like Metric Dimension, are NP-complete for interval graphs [13]). Our reduction (from 3-Sat) uses a quite involved novel technique, that we hope can be used to prove similar results for other distance-related problems on interval graphs. The main challenge here is to overcome the linear structure of the graph to transmit information across the graph. To this end, we use a sophisticated
construction of many parallel \textit{tracks}, i.e. shortest paths with intervals of (mostly) the same length spanning roughly the whole graph, and such that each track is shifted with respect to the previous one. Each track represents shortest paths that will be used by solution-vertices from our variable and clause gadgets. In between the tracks, we are able to build our gadgets.

We remark that MGS admits a polynomial-time algorithm on proper interval graphs by a nontrivial dynamic programming scheme \cite{10}. Problems known to be NP-complete on interval graphs but polynomial on proper interval graphs are very rare; a single other example known to us is \textsc{Induced Subgraph Isomorphism}, which is polynomial-time solvable for connected proper interval graphs, but NP-complete for connected interval graphs \cite{17}.

To complement Theorem 3, we design an FPT algorithm for MGS on interval graphs when parameterized by its clique number \(\omega\). Observe that interval graphs are also chordal graphs, i.e. graphs without induced cycles of order greater than 3. We use dynamic programming on tree-decompositions to prove the following.

\begin{lemma} \cite{6}\end{lemma}

Any geodetic set of \(G\) contains at least one vertex from each corner path.

Moreover, any geodetic set of \(G\) contains all vertices of degree 1. We say that a vertex \(v\) of \(G\) is a corner vertex if \(v\) is an end-vertex of some corner path.

\begin{definition} We say that \(u_1, u_2, \ldots, u_k\) forms a corner sequence if for each \(1 \leq i \leq k-1\),
\begin{enumerate}
  \item there is a corner path with \(u_i\) and \(u_{i+1}\) as endpoints, and
  \item there is no corner vertex in the clockwise traversal of the boundary of \(R\) from \(u_i\) to \(u_{i+1}\).
\end{enumerate}

A corner sequence is \textit{maximal} if it is not a sub-sequence of any other corner sequence.\end{definition}

\begin{lemma} \end{lemma}

Let \(S\) be the set of all maximal corner sequences of \(G\), and let \(t\) be the number of vertices of \(G\) with degree 1. Then, \(g(G) \geq t + \sum_{S \in S} \lfloor |S|/2 \rfloor\).
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Proof. Any geodetic set of $G$ contains all vertices of degree 1 and therefore $g(G) \geq t$. Now, let $X$ be any geodetic set of $G$ and $S \subseteq X$ be an arbitrary maximal corner sequence. Assume that $u_1, u_2, \ldots, u_{|S|}$ forms the maximal sub-sequence $S$. Lemma 4 implies that for each $1 \leq j < |S|$, at least one vertex of the corner path between $u_j$ and $u_{j+1}$ must belong to $X$. Observe that two corner paths may have at most one corner vertex in common. Moreover, a corner vertex cannot be in three corner paths. Therefore, $X$ must contain at least $\left\lceil \frac{|S|}{2} \right\rceil$ vertices. Now, let $P$ be a corner path with endpoints $a, b$ and $P'$ be a corner path with endpoints $a', b'$. If $a, b$ and $a', b'$ are in different maximal corner sub-sequences, then $P$ and $P'$ have no vertex in common.

Let $\mathcal{S}$ be the set of all maximal corner sequences of $G$. For a maximal corner sequence $S = u_1, u_2, \ldots, u_k$ let $f(S)$ denote the set $\{u_2, u_4, \ldots, u_{k-1}\}$ where $k' = 0$ if $k$ is even and $k' = 1$, otherwise. Observe that $|f(S)| = \left\lfloor \frac{k}{2} \right\rfloor$. Let $V_1$ be the set of all vertices of degree 1. Now consider the sets $V_2 = \bigcup_{S \in \mathcal{S}} f(S)$ and $D = V_1 \cup V_2$. By Lemma 7, we will be done by proving that $D$ forms a geodetic set of $G$.

We shall use the following result of Ekim and Erey [9].

Theorem 8 (9). Let $F$ be a graph and $F_1, \ldots, F_k$ its biconnected components. Let $C$ be the set of cut vertices of $G$. If $X_i \subseteq V(F_i)$ is a minimum set such that $X_i \cup (V(F_i) \cap C)$ is a minimum geodetic set of $F_i$, then $\bigcup_{i=1}^k X_i$ is a minimum geodetic set of $F$.

The next observation follows from Theorem 8.

Observation 9. Let $C(G)$ be the set of cut-vertices of $G$ and let $\{H_1, H_2, \ldots, H_t\}$ be the set of biconnected components of $G$. The set $D$ is a geodetic set of $G$ if and only if $(C(G) \cap V(H_i)) \cup (D \cap V(H_i))$ is a geodetic set of $H_i$ for all $1 \leq i \leq t$.

For the remainder of this section, $C(G)$ is the set of cut vertices of $G$ and $H_1, H_2, \ldots, H_t$ are the biconnected components of $G$. Let $H$ be a biconnected component of $G$. Recall that each vertex of $H$ is a pair of integers and each edge is a line segment with unit length. Let $\mathcal{R}_H$ be the grid embedding induced by $H$ in $\mathcal{R}$. An edge $e \in E(H)$ is an interior edge if all interior points of $e$ are adjacent to only interior faces of $\mathcal{R}_H$. For a vertex $v \in V(H)$, let $P_v$ denote the maximal path such that all edges of $P_v$ are interior edges and each vertex in $P_v$ has the same $x$-coordinate as $v$. Similarly, let $P'_v$ denote the maximal path such that all edges of $P'_v$ are interior edges and each vertex in $P'_v$ has the same $y$-coordinate as $v$. A path $P$ of $H$ is a red path if (i) there exists a $v \in V(H)$ such that $P \in \{P_v, P'_v\}$ and (ii) at least one end-vertex of $P$ is a cut-vertex or a vertex of degree 4. See Figure [1] for an example. A vertex $v$ of $H$ is red if $v$ lies on some red path.

Definition 10. A subgraph $F$ of $H$ is a rectangular block if $F$ satisfies the following properties.

1. For any two vertices $(a_1, b_1), (a_2, b_2)$ of $F$, we have that any pair $(a_3, b_3)$ with $a_1 \leq a_3 \leq a_2$ and $b_1 \leq b_3 \leq b_2$ is a vertex of $F$.
2. Let $a, a'$ be the maximum and minimum $x$-coordinates of the vertices in $F$. Similarly, let $b, b'$ be the maximum and minimum $y$-coordinates of the vertices in $F$. If $x$-coordinate of a red vertex $v$ of $F$ is neither $a$ nor $a'$ then the $y$-coordinate of $v$ must equal to $b$ or $b'$.

Observe that $H$ can be decomposed into rectangular blocks such that each non-red vertex belongs to exactly one rectangular block. Let $B_1, B_2, \ldots, B_k$ be a decomposition of $H$ into rectangular blocks. Recall that $C(G)$ is the set of cut-vertices of $G$. We have the following lemma.
Figure 1 The grey solid circles in the figure represent red vertices. The highlighted region is a rectangular block.

Lemma 11. For each $1 \leq i \leq k$, there are two vertices $x_i, y_i \in (C(G) \cap V(H)) \cup (D \cap V(H))$ such that $V(B_i) \subseteq I(x_i, y_i)$.

Proof. Let $D^* = (C(G) \cap V(H)) \cup (D \cap V(H))$ and $X \in \{B_1, B_2, \ldots, B_k\}$ be an arbitrary rectangular block. A vertex $v$ of $X$ is a northern vertex if the $y$-coordinate of $v$ is maximum among all vertices of $X$. Analogously, western vertices, eastern vertices and southern vertices are defined. A vertex of $X$ is a boundary vertex if it is either northern, western, southern or an eastern vertex of $X$. Let $nw(X)$ be the vertex of $X$ which is both a northern vertex and a western vertex. Similarly, $ne(X)$ denotes the vertex which is both northern vertex and eastern vertex, $sw(X)$ denotes the vertex of $X$ which is both southern and western vertex and $se(X)$ denotes the vertex of $X$ which is both southern and eastern vertex. We shall prove the lemma by considering two cases.

First we prove the lemma assuming that all boundary vertices of $X$ are red vertices. Let $a$ (resp. $b$) denote the vertex with minimum $y$-coordinate such that $P_a$ (resp. $P_b$) contains $sw(X)$ (resp. $se(X)$). Similarly, let $c$ (resp. $d$) denote the vertex with maximum $y$-coordinate such that $P_c$ (resp. $P_d$) contains $nw(X)$ (resp. $ne(X)$). Let $d'$ (resp. $c'$) denote the vertex with minimum $x$-coordinate such that $P_{d'}$ (resp. $P'_{c'}$) contains $nw(X)$ (resp. $nw(X)$). Let $b'$ (resp. $d'$) denote the vertex with maximum $x$-coordinate such that $P'_{b'}$ (resp. $P'_{d'}$) contains $se(X)$ (resp. $se(X)$). Observe that the vertices $a', a, b, b', d, c, c'$ lie on the exterior face of the embedding.

For two vertices $i, j \in \{a', a, b, b', d', d, c, c'\}$, let $Q_{ij}$ denote the path between $i, j$ that can be obtained by traversing the exterior face of the embedding in the counter-clockwise direction starting from $i$. Observe that, if each of $Q_{a'a}$ and $Q_{d'd}$ (resp. $Q_{bb'}$ and $Q_{cc'}$) contain a vertex from $D^*$, say $f, f'$, then $\{sw(X), ne(X)\} \subseteq I(f, f')$ (resp. $\{nw(X), se(X)\} \subseteq I(f, f')$) and therefore $V(X) \subseteq I(f, f')$. Now consider the case when at least one of the paths in $\{Q_{a'a}, Q_{d'd}\}$ does not contain any vertex from $D^*$ and when at least one of the paths in $\{Q_{bb'}, Q_{cc'}\}$ does not contain any vertex from $D^*$. Due to symmetry of rotation and reflection on grids, without loss of generality we can assume that both $Q_{a'a}$ and $Q_{bb'}$ have no vertex from $D^*$. Now we prove the following claim.

Claim 12. One of the following is true. (a) The path $Q_{ab}$ contains a vertex from $D^*$ whose $x$-coordinate is at most that of $sw(X)$. (b) The path $Q_{ab}$ contains a vertex from $D^*$ whose
x-coordinate is at least that of se(X).

Assume that Case (a) does not hold. Then it must be the case that the path $P_{ab}$ contains a vertex $f$ with degree 2 whose x-coordinate is same as that of $sw(X)$. Observe that either there is a the maximal corner sequence containing $f$ or there is a cut-vertex whose x-coordinate is at least that of $se(X)$. If the later is true, then the claim is true. Otherwise, let $S$ be a maximal corner sequence containing $f$. Observe that $f$ must be the last vertex in the sequence $S$ and $|S|$ must be odd (otherwise $f \in D^*$, which contradicts the assumption that Case (a) does not hold). Hence, the vertex, say $z$, which appears just before $f$ in $S$, must belong to $D^*$. Now the x-coordinate of $z$ must be at least that of $se(X)$ (otherwise, there will be a red vertex in $X$ which is not a boundary vertex). This completes the proof of the claim.

Due to symmetry of reflection on grids, it is enough to prove the lemma only when Case (a) of Claim 12 holds. Let $z \in V(Q_{ab}) \cap D^*$ be a vertex whose x-coordinate is at most that of $sw(X)$. Observe that if $Q_{dd'}$ contains a vertex $z'$ of $D^*$, then $\{sw(X), ne(X)\} \subseteq I(z, z')$ and therefore $V(X) \subseteq I(z, z')$. Otherwise we have the following claim whose proof is similar to that of Claim 12.

**Claim 13.** One of the following is true. (a) The path $Q_{bd'}$ contains a vertex from $D^*$ whose y-coordinate is at least that of $ne(X)$. (b) The path $Q_{Vd'}$ contains a vertex from $D^*$ whose y-coordinate is at most that of $se(X)$.

Observe that if Case (a) of Claim 13 is true then there is a vertex $z' \in V(Q_{bd'}) \cap D^*$ such that $\{sw(X), ne(X)\} \subseteq I(z, z')$ and therefore $V(X) \subseteq I(z, z')$. Now assume that only Case (b) of Claim 13 is true and $z' \in V(Q_{bd'}) \cap D^*$ whose y-coordinate is at most that of $se(X)$. Observe that if $Q_{c,c'}$ has a vertex $z''$ then $\{sw(X), se(X)\} \subseteq I(z', z'')$ and therefore $V(X) \subseteq I(z', z'')$. If $Q_{dd}$ has a vertex $z''$ of $D^*$ whose x-coordinate is at most that of $nw(X)$, then $\{nw(X), se(X)\} \subseteq I(z', z'')$ and therefore $V(X) \subseteq I(z', z'')$. Otherwise, using arguments similar to that of Claim 12 we can show that the path $Q_{dd'}$ contains a vertex $z''$ from $D^*$ whose x-coordinate is at least that of $se(X)$. Observe that $\{sw(X), ne(X)\} \subseteq I(z, z'')$ and therefore $V(X) \subseteq I(z, z'')$.

Now we consider the case when there are some non-red boundary vertices of $X$. Let $v$ be a non-red vertex of $X$. Without loss of generality, we can assume that $v$ is a western vertex of $X$. Now we redefine the vertices $a, a', b, b', c, c', d, d'$ as follows. Let $a' = sw(X)$, $c = nw(X)$ and $a$ (resp. $b$) be the vertex with minimum y-coordinate such that there is a path from $a$ to $sw(X)$ (resp. from $b$ to $se(X)$) containing vertices with the same x-coordinate as that of $sw(X)$ (resp. $se(X)$). Similarly, let $c$ (resp. $d$) be the vertex with maximum y-coordinate such that there is a path from $c$ to $nw(X)$ (resp. from $d$ to $ne(X)$) containing vertices with the same x-coordinate as that of $nw(X)$ (resp. $ne(X)$). Finally, let $d'$ (resp. $b'$) be the vertex with maximum x-coordinate such that there is a path from $d'$ to $ne(X)$ (resp. from $b'$ to $se(X)$) containing vertices with the same y-coordinate as that of $ne(X)$ (resp. $se(X)$). Using similar arguments on the paths $P_{ij}$ with $i, j \in \{a', a, b, b', d, c, c', d\}$ as before, we can show that there exists vertices $f, f' \in D^*$ such that $V(X) \subseteq I(f, f')$. So we have the proof of the lemma.

**Time complexity:** All pendent vertices and corner vertices of $G$ can be obtained in $O(|V(G)|)$ time [6]. Once the corner vertices are found, the set of all maximal corner sequences and our desired geodetic set with minimum cardinality can be computed in $O(|V(G)|)$ time.
3 Algorithm for chordal graphs

We give an FPT algorithm for chordal graphs parameterized by the clique number (which is also the tree-width plus 1). We explain how to improve the complexity in the case of interval graphs after the proof of the chordal case. Our algorithm performs dynamic programming on a nice tree decomposition of the input chordal graph \[^3\].

Definition 14. A nice tree decomposition of a chordal graph \(G\) is a rooted tree \(T\) where each node \(v\) is associated to a subset \(X_v\) of \(V(G)\) called bag, and each internal node has one or two children, with the following properties.

1. The set of nodes of \(T\) containing a given vertex of \(G\) forms a nonempty connected subtree of \(T\).
2. Any two adjacent vertices of \(G\) appear in a common node of \(T\).
3. For each node \(v\) of the tree, \(G[X_v]\) is a clique.
4. Each node of \(T\) belongs to one of the following types: introduce, forget, join or leaf.
5. A join node \(v\) has two children \(v_1\) and \(v_2\) such that \(X_v = X_{v_1} = X_{v_2}\).
6. An introduce node \(v\) has one child \(v_1\) such that \(X_v \setminus \{x\} = X_{v_1}\), where \(x \in X_v\).
7. A forget node \(v\) has one son \(v_1\) such that \(X_v = X_{v_1} \setminus \{x\}\), where \(x \in X_{v_1}\).
8. A leaf node \(v\) is a leaf of \(T\) with \(X_v = \emptyset\).
9. The tree \(T\) is rooted at a leaf node \(r\) with \(X_r = \emptyset\).

In our algorithm, we traverse the nice tree decomposition in a post-order manner. At each node \(v\) of the tree we shall construct a table of size \(O \left(2^{2^{|X_v|}}\right)\) containing “partial solutions” for the graph induced by the vertices in the bags of the subtree rooted at \(v\) (let this graph be denoted as \(G_{\leq v}\)). We associate a “type” to each of these partial solutions which encodes, among other information, the effect of this partial solution to the rest of the graph and vice versa (Definition \[15\]).

To ensure that at least one of these partial solutions can be “extended” and will be part of a geodetic set with minimum cardinality of \(G\), we characterize the shortest path structure between a pair of vertices \(u, w\) where \(u \in V(G_{\leq v}), w \in V(G) \\setminus V(G_{\leq v})\) (Lemma \[16\]). We observe that the vertices in the bag \(X_v\) induce a clique cutset (clique whose removal disconnect the graph) and all shortest paths between \(u, w\) contain vertices from \(X_v\). Let \(X' \subseteq X_v\) be the vertices lying in some shortest path between \(u, w\) and have lesser distance to \(w\) than \(u\). Observe that, “pre-selecting” the vertices of \(X'\) captures the effect of \(w\) on \(G_{\leq v}\).

Hence by considering all \(2^{2^{|X_v|}}\) different collections of subsets of \(X_v\), we could capture the effect of all the vertices in \(G - G_{\leq v}\) i.e. “exterior vertices” on \(G_{\leq v}\). For different collections of subsets of \(X_v\) we have different “types” of partial solutions.

Once we have all the partial solutions for the children of a node \(v\), we show how to extend these to get the partial solutions of \(v\). It is possible that a partial solution of a node of some “type” is extended to a partial solution of its parent of a different “type”. Depending on the node under consideration, we define an exhaustive set of rules to ensure that the extended partial solutions are valid (Definitions \[20\] \[22\] \[24\] \[26\]). We prove the exhaustiveness of these rules in Lemmas \[21\] \[29\] \[29\] \[27\] and the correctness of our algorithm in Lemma \[28\].

3.1 The algorithm

We shall introduce a few definitions and notations first. Let \(G\) be a graph containing a clique \(X\) and a vertex \(y\). We say that \(y\) is close to a nonempty set \(A \subseteq X\) with respect to \(X\), if
\[ d(u, x) = d_u \text{ when } x \in A \text{ and } d(u, x) = d_u + 1 \text{ when } x \in X \setminus A. \] The set \( X \) is a clique cutset of \( G \) if \( G - S \) is disconnected.

From now on \( T \) shall denote a nice tree decomposition of \( G \). For a node \( v \in T \), let \( G_{\leq v} \) be the subgraph of \( G \) induced by the vertices present in the nodes of the subtree of \( T \) rooted at \( v \). We can define similarly \( G_{< v}, G_{\geq v} \) and \( G_{> v} \). For a node \( v \) let \( \mathcal{T}_v \) be the set of all 4-tuples \( \tau = (\tau_{\text{int}}, \tau_{\text{ext}}, \tau_{\text{cov}}, \tau_{\text{bag}}) \) where \( \tau_{\text{int}}, \tau_{\text{ext}} \) are Boolean vectors of size \( 2^{|V(G)|} \) indexed by subsets of \( V(G) \) and \( \tau_{\text{cov}}, \tau_{\text{bag}} \) are subsets of \( V(G) \). Since \(|X_v| \leq \omega(G)\), the cardinality of \( \mathcal{T}_v \) is \( 2^{O(\omega(G))} \).

For a node \( v \) and a 4-tuple \( \tau = (\tau_{\text{int}}, \tau_{\text{ext}}, \tau_{\text{cov}}, \tau_{\text{bag}}) \) let \( H^*_v \) denote the graph obtained by adding a vertex \( S \) to \( G_{\leq v} \) whenever there is a set \( S \subseteq V(G) \) with \( \tau_{\text{ext}}[S] = 1 \), and making \( S \) adjacent to each \( x \in S \). Let \( S^*_v = \{ S : S \subseteq V(G), \tau_{\text{ext}}[S] = 1 \} \) denote the newly added vertices. Observe that \( G_{\leq v} \) is an induced subgraph of \( H^*_v \) for any 4-tuple \( \tau \in \mathcal{T}_v \).

**Definition 15.** Let \( v \) be a node of \( T \). A 4-tuple \( \tau = (\tau_{\text{int}}, \tau_{\text{ext}}, \tau_{\text{cov}}, \tau_{\text{bag}}) \) of \( \mathcal{T}_v \) is a “type associated with \( v \)” if there exists a set \( D \subseteq V(H^*_v) \) such that the following hold.

1. \( S^*_v \subseteq D \) and \( \tau_{\text{bag}} = D \cap X_v \).
2. For a vertex \( w \in (V(G_{\leq v}) \setminus X_v) \cup \tau_{\text{cov}} \) there exists a pair \( w_1, w_2 \in D \) such that \( w \in I(w_1, w_2) \) and \( w_1 \in D \setminus S^*_v \).
3. For a subset \( A \subseteq X_v \), we have \( \tau_{\text{int}}[A] = 1 \) if and only if \( D \cap V(G_{\leq v}) \) contains a vertex which is closer to \( A \) with respect to \( X_v \).

Moreover, we shall say that the set \( D \setminus S^*_v \) is a “certificate” for \( (v, \tau) \).

Intuitively, for a type \( \tau \) associated with a node \( v \) and for a set \( A \subseteq X_v \) of vertices, the Boolean \( \tau_{\text{int}}[A] \) represents whether there is some vertex in the partial solution such that \( y \) is close to \( A \) with respect to \( X_v \) (“int” stands for “interior”). The Boolean \( \tau_{\text{ext}}[A] \) represents whether we need to add, at a later step of our algorithm, some vertex \( y \) which is close to \( A \) with respect to \( X_v \). Here, \( y \) is a vertex that needs to be added later to the solution, in the upper part of the tree (“ext” stands for “exterior”).

Observe that the only type associated with the root node of \( T \) is \( \tau_0 = (0, 0, 0, 0) \) where \( 0 \) denotes the vector whose all elements are 0. Now we shall characterize the types associated with different types of nodes of the tree decomposition \( T \). First we prove the following lemma which deals with how shortest paths interact with clique cutsets.

**Lemma 16.** Let \( X \) be a clique cutset of a graph \( G \) and \( u, v \) be vertices lying in two different connected components of \( G - X \). Let \( A \) and \( B \) be two subsets of \( X \) such that \( u \) (resp. \( v \)) is close to \( A \) (resp. \( B \)) with respect to \( X \). Then, a vertex \( x \in I(u, v) \cap X \) if and only if \( x \in A \cap B \) or, \( A \cap B = \emptyset \) and \( x \in A \cup B \).

**Proof.** Suppose that for all \( y \in A, d(u, y) = d_u \) and for all \( y \in B, d(v, y) = d_v \) and let \( x \in X \). Now we consider the following two cases.

1. \( A \cap B \neq \emptyset \). If \( x \in A \cap B \), then \( d(u, v) = d_u + d_v \) as any shorter path would imply the existence of a vertex \( y \in X \) such that \( d(u, y) + d(y, v) < d_u + d_v \), which would contradict the definition of \( A \) and \( B \). As \( d(u, x) + d(x, v) = d_u + d_v, x \in I(u, v) \). Conversely, if \( x \notin A \cap B \), then \( d(u, v) \leq d_u + d_v \) as \( d(u, y) + d(y, v) = d_u + d_v \). As \( d(u, x) + d(x, v) = d_u + d_v + 1, x \notin I(u, v) \).

2. \( A \cap B = \emptyset \). If \( x \in A \cup B \), then \( d(u, v) = d_u + d_v + 1 \), as any shorter path would imply the existence of a vertex \( y \in X \) such that \( d(u, y) + d(y, v) \leq d_u + d_v \), which would contradict \( A \cap B = \emptyset \). As \( d(u, x) + d(x, v) = d_u + d_v + 1, x \in I(u, v) \). Conversely, if \( x \notin A \cup B \), then \( d(u, v) \leq d_u + d_v + 1 \) as \( d(u, y) + d(y, v) = d_u + d_v + 1 \). As \( d(u, x) + d(x, v) = d_u + d_v + 2, x \notin I(u, v) \).
This completes the proof. ▶

Lemma 16 implies that to compute an optimal partial solution for a given bag \( X_v \), it is sufficient to “guess” for which subsets \( A \) of \( X_v \), there will exist (in the future solution computed that will be computed for ancestors of \( v \)) a vertex \( y \) which is close to \( A \) with respect to \( X_v \).

Suppose we have a fixed geodetic set of \( D \) of \( G \). In the following lemma we show that when \( D \) is restricted to a particular subgraph \( G_{\leq v} \), the set \( D \cap G_{\leq v} \) acts as a certificate for some \((v, \tau)\). This proves the exhaustiveness of our definition of type.

**Lemma 17.** Let \( D \) be a geodetic set of \( G \), then for each node \( v \in T \), there is a type \( \tau \) associated with \( v \) such that \( D \cap V(G_{\leq v}) \) is a certificate of \((v, \tau)\).

**Proof.** We construct \( \tau \) from \( D \) as follows. Define \( \tau^{\text{bag}} = D \cap X_v \). For each vertex \( u \in D \cap V(G_{\leq v}) \) we find the set \( Z_u \subseteq X_v \) such that \( u \) is close to \( Z_u \) with respect to \( X_v \) and put \( \tau^{\text{int}}[Z_u] = 1 \). For each \( u \in D \cap V(G_{\leq v}) \) we find the set \( Z_u \subseteq X_v \) such that \( u \) is close to \( Z_u \) with respect to \( X_v \) and put \( \tau^{\text{ext}}[Z_u] = 1 \). We put \( \tau^{\text{cov}} = X_v \setminus D \). Observe that \( D \cap V(G_{\leq v}) \) is a certificate of \((v, \tau)\).

For a node \( v \), there might be some 4-tuples in \( T \) which are not associated with \( v \). In the following lemma, establish certain restrictions that any type associated with \( v \) must follow.

**Lemma 18.** Let \( v \) be a node of \( T \) and \( \tau = (\tau^{\text{int}}, \tau^{\text{ext}}, \tau^{\text{cov}}, \tau^{\text{bag}}) \) be a type associated with \( v \). Then \( \tau \) must satisfy all of the following conditions.

(a) Whenever we have a vertex \( u \in \tau^{\text{bag}} \) we have \( \tau^{\text{int}}[\{u\}] = 1 \).
(b) \( \tau^{\text{int}}[\emptyset] = \tau^{\text{ext}}[\emptyset] = 0 \).
(c) For all \( x \in X_v \), \( \tau^{\text{int}}[A] = 1 \) and \( \tau^{\text{ext}}[B] = 1 \), if \( x \in A \cap B \) or if \( A \cap B = \emptyset \) and \( x \in A \cup B \) then \( \tau^{\text{cov}}[x] = 1 \).

From now on for a node \( v \), we will only consider the 4 tuples which satisfy the conditions of Lemma 18. We have the following observation about the leaf nodes of the nice tree decomposition.

**Observation 19.** Let \( v \) denote a leaf node. Then \( \tau_0 = (0, 0, \emptyset, \emptyset) \) is the only type associated with \( v \). Moreover, \( \tau_0 \) is the only type associated with the root node of \( T \).

Let \( v \) be an introduce node and \( u \) be its child. Let \( \tau, \tau_1 \) be types associated with \( v, u \) respectively. Below we state some rules that \( \tau \) and \( \tau_1 \) must follow so that the certificate for \((u, \tau_1)\) can be extended to a certificate for \((v, \tau)\).

**Definition 20.** Let \( v \) be an introduce node and \( \tau \) be a type associated with \( v \). Let \( u \) be the child of \( v \) and \( \tau_1 \) be a type associated with \( u \). The pair \((\tau, \tau_1)\) are compatible if the following holds.

(a) \( \tau_1^{\text{bag}} = \tau^{\text{bag}} \setminus \{x\} \).
(b) \( \tau_1^{\text{cov}} = \tau^{\text{cov}} \cap X_u \).
(c) For \( A \subseteq X_u \), \( \tau_1^{\text{ext}}[A] = 1 \) if and only if \( \tau^{\text{ext}}[A] = 1 \) or \( \tau^{\text{ext}}[A \cup \{x\}] = 1 \).
(d) \( \tau_1^{\text{ext}}[X_u] = 1 \) if and only if \( \tau^{\text{ext}}[X_u] = 1 \) and \( \tau^{\text{ext}}[\{x\}] = 1 \).
(e) If \( x \notin \tau^{\text{bag}} \) then there exist non-empty sets \( A \subseteq X_u \), \( B \subseteq X_v \setminus A \) such that \( \tau_1^{\text{int}}[A] = 1 \) and \( \tau^{\text{ext}}[B \cup \{x\}] = 1 \).
(f) \( \tau^{\text{int}}[\{x\}] = 1 \) if and only if \( x \in \tau^{\text{bag}} \).
(g) For all non empty \( A \subseteq X_u \), \( \tau^{\text{int}}[A \cup \{x\}] = 0 \),
(h) For all $A \subseteq X_v$, $\tau^{\text{int}}[A] = 1$ if and only if $\tau^{\text{int}}_1[A] = 1$,  

**Lemma 21.** Let $v$ be an introduce node, $\tau$ be a node associated with $v$, $u$ be the child of $v$ and $D$ be a minimal certificate of $(v, \tau)$. Then there exists a type $\tau_1$ associated with $u$ such that $(\tau, \tau_1)$ is a compatible pair.  

**Proof.** Let $X_v = X_u \cup \{x\}$. Define $\tau^{\text{bag}}_1 = D \cap X_u$, $\tau^{\text{cov}}_1 = \tau^{\text{cov}} \cap X_u$ and $\tau^{\text{int}}_1[A] = 1$ if and only if there exists $y \in D \setminus \{x\}$ such that $y$ is close to $A$ with respect to $X_u$. Finally define $\tau^{\text{ext}}_1$ according Conditions 22(c) and 22(d). Observe that the set $D' = D \setminus \{x\}$ satisfies all the conditions in Definition 15 for $(u, \tau_1)$ and the pair $(\tau, \tau_1)$ satisfies all conditions in Definition 20.  

Let $C_v$ denote the set of compatible pairs $(\tau, \tau_1)$ where $\tau, \tau_1$ are types associated with $v$ and $u$, respectively.  

Let $v$ be a forget node and $u$ be its child. Let $\tau, \tau_1$ be types associated with $v, u$ respectively. Below we state some rules that $\tau$ and $\tau_1$ must follow so that the certificate for $(u, \tau_1)$ can be extended to a certificate for $(v, \tau)$.  

**Definition 22.** Let $v$ be a forget node and $\tau$ be a type associated with $v$. Let $u$ be the child of $v$ and $\tau_1$ be a type associated with $u$. The pair $(\tau, \tau_1)$ are compatible if the following holds.

(a) $\tau^{\text{bag}} = \tau^{\text{bag}}_1 \setminus \{x\}$,
(b) For all $A \subseteq X_v$, $\tau^{\text{int}}_1[A] = 1$ if and only if $\tau^{\text{ext}}[A] = 1$,
(c) For all $A \subseteq X_v$, $\tau^{\text{cov}}[A \cup \{x\}] = 0$.
(d) For $A \subseteq X_v$, $\tau^{\text{int}}[A] = 1$ if and only if $\tau^{\text{int}}_1[A] = 1$ or $\tau^{\text{int}}_1[A \cup \{x\}] = 1$.
(e) $\tau^{\text{cov}}_1[A] = 1$ if and only if $\tau^{\text{cov}}_1[A \cup \{x\}] = 1$ or $\tau^{\text{cov}}_1[A \cup \{x\}] = 1$.
(f) $\tau^{\text{cov}}_1 = \tau^{\text{cov}} \cup \{x\}$.  

**Lemma 23.** Let $v$ be an forget node, $\tau$ be a node associated with $v$, $u$ be the child of $v$ and $D$ be a minimal certificate of $(v, \tau)$. Then there exists a type $\tau_1$ associated with $u$ such that $(\tau, \tau_1)$ is a compatible pair.  

**Proof.** Let $X_v = X_u \setminus \{x\}$. Define $\tau^{\text{bag}}_1 = D \cap X_u$, $\tau^{\text{cov}}_1 = \tau^{\text{cov}} \cup \{x\}$ and $\tau^{\text{int}}_1[A] = 1$ if and only if there exists $y \in D$ such that $y$ is close to $A$ with respect to $X_u$. Finally, define $\tau^{\text{ext}}_1$ according to Condition 24(b) and 24(c). Observe that the set $D$ satisfies all the conditions in Definition 15 for $(u, \tau_1)$ and the pair $(\tau, \tau_1)$ satisfies all conditions in Definition 22.  

Let $C_v$ denote the set of compatible pairs $(\tau, \tau_1)$ where $\tau, \tau_1$ are types associated with $v$ and $u$, respectively.  

Let $v$ be a join node, $u_1, u_2$ be its children. Let $\tau, \tau_1, \tau_2$ be types associated with $v, u_1, u_2$ respectively. Below we state some rules that $\tau, \tau_1, \tau_2$ must follow so that the certificate for $(u_1, \tau_1)$ and $(u_2, \tau_2)$ can be combined and extended to a certificate for $(v, \tau)$.  

**Definition 24.** Let $v$ be a join node and $\tau$ be a type associated with $v$. Let $u_1, u_2$ be the children of $v$ and $\tau_1, \tau_2$ are types associated with $u_1, u_2$ respectively. The triplet $(\tau, \tau_1, \tau_2)$ is compatible if all the following holds.

(a) $\tau^{\text{bag}} = \tau^{\text{bag}}_1 = \tau^{\text{bag}}_2$,
(b) For $i, j$ such that $\{i, j\} = \{1, 2\}$ and $A \subseteq X_v$, $\tau^{\text{int}}_1[A] = 1$ if and only if $\tau^{\text{int}}_j[A] = 1$ or $\tau^{\text{ext}}[A] = 1$.
(c) $\tau^{\text{cov}} = \tau^{\text{cov}}_1 \cup \tau^{\text{cov}}_2 \cup \text{Cov}(u_1, u_2)$ where Cov$(u_1, u_2)$ is the subset of vertices $x$ of $X_v$ such that there exists $A, B \subseteq X_v$ with $\tau^{\text{int}}_1[A] = 1$ and $\tau^{\text{int}}_2[B] = 1$ where $x \in A \cap B$ or where $x \in A \cup B$ and $A \cap B = \emptyset$.  


(d) For \(i, j\) such that \(\{i, j\} = \{1, 2\}\) and \(A \subseteq X_v, \tau_i^{\text{inf}}[A] = 1\) then \(\tau_i^{\text{inf}}[A] = 1\) and 
\(\tau_j^{\text{ext}}[A] = 1\).
(e) For \(i, j\) such that \(\{i, j\} = \{1, 2\}\) and \(A \subseteq X_v, \tau_i^{\text{inf}}[A] = 0\) and \(\tau_j^{\text{inf}}[A] = 0\) then 
\(\tau_i^{\text{ext}}[A] = 0\).

\[\begin{align*}
\text{Lemma 25.} \quad &\text{Let } v \text{ be a join node and } \tau \text{ be a type associated with } v. \text{ Let } u_1, u_2 \text{ be the children of } v. \text{ Then there are types } \tau_1, \tau_2 \text{ associated with } u_1, u_2 \text{ respectively such that } (\tau, \tau_1, \tau_2) \text{ is a compatible triplet.} \\
\text{Proof.} &\text{For each } i \in \{1, 2\} \text{ define } \tau_i^{\text{bag}} = \tau^{\text{bag}} \text{ and } \tau_i^{\text{inf}}[A] = 1 \text{ if and only if there exists } y \in D \cap G_{\leq u_i} \text{ such that } y \text{ is close to } A \text{ with respect to } X_u. \text{ Define } \tau_i^{\text{ext}} \text{ according to Conditions 24(b) and 24(b)}. \text{ Define } \tau_i^{\text{cov}} \text{ according to Condition 24(c)}. \text{ Observe that for each } i \in \{1, 2\} \text{ the sets } D \cap V(G_{\leq u_i}) \text{ satisfies all the conditions in Definition 15 for } (\tau, \tau_i) \text{ and the triplet } (\tau, \tau_1, \tau_2) \text{ is compatible.} \\
\end{align*}\]

For a join nodes \(v\) with children \(u_1, u_2\), let \(C_v\) denote the set of compatible triplets 
\((\tau, \tau_1, \tau_2)\) where \(\tau, \tau_1, \tau_2\) are types associated with \(v, u_1, u_2\), respectively. Finally we consider the root \(r\). Recall that the only type associated with \(r\) is \(\tau_0 = (0, 0, 0, 0)\).

\[\begin{align*}
\text{Definition 26.} \quad &\text{Consider the root } r, \text{ its child } u \text{ with } X_u = \{x\} \text{ and a type } \tau \text{ associated with } \\
&u. \text{ Then } (\tau_0, \tau) \text{ is a compatible pair if } x \in \tau_0^{\text{bag}} \text{ and } \tau^{\text{ext}}[A] = 0 \text{ for all } A \subseteq X_u. \\
\end{align*}\]

The proof for the following lemma is analogous to that of Lemma 24,25.

\[\begin{align*}
\text{Lemma 27.} \quad &\text{For any minimal geodetic set of } G, \text{ there is a compatible pair } (\tau_0, \tau_1) \text{ where } \\
&\tau_1 \text{ is a type associated with } u. \\
&\text{Let } C_r \text{ denote the set of compatible pairs } (\tau_0, \tau) \text{ where } \tau \text{ is a type associated with the} \\
&\text{child of } r. \\
\end{align*}\]

Now we are ready to describe our algorithm. We process the nodes of \(T\) in the post-order manner. Let \(v\) be the current node under consideration. If \(v\) is a leaf node, then define 
\[sol[v, (0, 0, 0, 0)] = \emptyset\]
Let \(v\) be an introduce node having \(u\) as child. Then for each type \(\tau \in T_v\) define 
\[sol[v, \tau] = \min_{(\tau, \tau_1) \in C_u} \left(\min_{(\tau, \tau_1) \in C_v} \left(sol[u, \tau_1] \cup \tau^{\text{bag}}\right)\right)\]
Let \(v\) be a forget node having \(u\) as child. Then for each type \(\tau \in T_v\) define 
\[sol[v, \tau] = \min_{(\tau, \tau_1) \in C_u} \left(sol[u, \tau_1]\right)\]
Let \(v\) be a join node having \(u_1, u_2\) as child. Then for each type \(\tau \in T_v\) define 
\[sol[v, \tau] = \min_{(\tau, \tau_1, \tau_2) \in C_u} \left(sol[u_1, \tau_1] \cup sol[u_2, \tau_2]\right)\]
Finally for the root \(r\) let \(u\) be its child and \(\tau_0 = (0, 0, 0, 0)\). Define 
\[sol[r, \tau_0] = \min_{(\tau_0, \tau_1) \in C_r} \left(sol[u, \tau_1]\right)\]
We shall show in the following lemma that \(sol[r, (0, 0, 0, 0)]\) gives a geodetic set of \(G\) 
with minimum cardinality. Recall the definitions of \(H^+_v\) and \(S^+_v\).
Lemma 28. For each node $v$ and type $\tau$ associated with $v$, $\text{sol} \,[v, \tau]$ is a certificate of $(v, \tau)$ with minimum cardinality.

Proof. The statement of the lemma is trivially true when $v$ is a leaf node. By induction we shall assume the lemma to be true for all nodes of the subtree rooted at $v$.

1. Assume that $v$ is an introduce node. Let $u$ be the child of $v$, $X_v = X_u \cup \{x\}$. First we show that $\text{sol} \,[v, \tau]$ is a certificate of $(v, \tau)$. Let $\tau_1$ be a type associated with $u$ such that $\text{sol} \,[v, \tau] = \text{sol} \,[u, \tau_1] \cup \tau_{bag}$ and consider the set $D = \text{sol} \,[u, \tau_1] \cup \tau_{bag} \cup X_v \cup \{x\}$. By Condition 20(a) we have that $\tau_{bag} = \tau_{bag} \setminus \{x\}$. Hence $D \cap X_v = \text{sol} \,[v, \tau] \cap X_v = (\text{sol} \,[u, \tau_1] \cap X_v \cup \{x\}) \cup (\text{sol} \,[u, \tau_1] \cap X_v \cup \{x\}) = (\text{sol} \,[u, \tau_1] \cap X_v \cup \{x\}) \cup \tau_{bag} = (\text{sol} \,[u, \tau_1] \cap X_v \cup \tau_{bag} = \tau_{bag} \cup \tau_{bag} = \tau_{bag}$. Hence $D$ satisfies Condition 13(i). Now consider any vertex $w \in (V(G_{<u}) \setminus X_u) \cup \tau_{cov}$ which is distinct from $x$. Then $w \in (V(G_{<u}) \setminus X_u) \cup \tau_{cov}$, and Condition 20(d) ensures that $\tau_{cov} = \tau_{cov} \cap X_u$ and hence $w \in (V(G_{<u}) \setminus X_u) \cup \tau_{cov}$. Now due to our induction hypothesis we have that there exists $w_1, w_2 \in \text{sol} \,[u, \tau_1] \cup S_u^0$ such that $w_1 \in I(w_1, w_2)$ and $w_1 \in \text{sol} \,[u, \tau_1]$. If $w_2 \in \text{sol} \,[u, \tau_1]$ then $w_2 \in \text{sol} \,[v, \tau]$ and therefore $w_1, w_2 \in D$. Hence Condition 13(ii) is satisfied in this case. Assume $w_2 \in S_u^0$ then there is a set $A \subseteq X_u$ such that $w_2 = A$ and $\tau_{int}[A] = 1$. If $A \neq X_u$ then due to Condition 22(c) we know that there is a set $B \supseteq A$ such that $\tau_{int}[B] = 1$. Hence there is a vertex $b \in S_u^0$ such that $b = B$ and is adjacent to all vertices of $A$. Observe that $w \in I(w_1, b)$. If $A = X_u$ then again due to Condition 20(d) and similar arguments as above we have a vertex $b' \in S_u^0$ such that $w \in I(w_1, b')$. Hence Condition 13(ii) is satisfied. Now consider the vertex $x$. If $x \in \tau_{bag}$, then $x \in D$. Now assume $x \notin \tau_{bag}$. Then due to Condition 20(b) we have sets $A \subseteq X_u$, $B \subseteq X_v \setminus A$ such that $\tau_{int}[A] = 1$ and $\tau_{int}[B \cup \{x\}] = 1$. Hence $\text{sol} \,[u, \tau_1]$ contains a vertex, say $a$, which is close to $A$ with respect to $X_u$. There also exists a vertex, say $b \in S_u^0$, such that $b = B$. By Lemma 16 $x \in I(a, b)$. Therefore Condition 13(ii) is satisfied.

Consider $A \subseteq X_u$ such that $\tau_{int}[A] = 1$. If $A = \{x\}$ then due to Condition 20(f) we have $x \in \tau_{bag}$ and therefore $x \in D$. Due to Condition 20(k) we have $A \subseteq X_u$. By Condition 20(h) $\tau_{int}[A] = 1$ and therefore $\text{sol} \,[u, \tau_1]$ contains a vertex $w$ such that $w \in V(G_{<u})$ and $w$ is closer to $A$ with respect to $X_v$. Since $v$ is an introduce node with $X_v = X_u \cup \{x\}$, $w$ must be closer to $A$ with respect to $X_v$. Hence Condition 13(iii) is satisfied. Hence, $\text{sol} \,[v, \tau]$ is a certificate of $(v, \tau)$. Now Lemma 21 implies that $\text{sol} \,[v, \tau]$ is minimum.

2. Now assume that $u$ is a forget node. Let $w$ be the child of $v$, $X_v = X_u \setminus \{x\}$. First we show that $\text{sol} \,[v, \tau]$ is a certificate of $(v, \tau)$. Let $\tau_1$ be a type associated with $u$ such that $\text{sol} \,[v, \tau] = \text{sol} \,[u, \tau_1] \cup \tau_{bag}$ and consider the set $D = \text{sol} \,[u, \tau_1] \cup \tau_{bag} \cup X_v \cup \{x\}$. By Condition 22(a) we have that $\tau_{bag} = \tau_{bag} \setminus \{x\}$. Hence $D \cap X_v = \text{sol} \,[v, \tau] \cap X_v = (\text{sol} \,[u, \tau_1] \cap X_v \cup \{x\}) \cup (\text{sol} \,[u, \tau_1] \cap X_v \cup \{x\}) = (\text{sol} \,[u, \tau_1] \cap X_v \cup \{x\}) \cup \tau_{bag} = (\text{sol} \,[u, \tau_1] \cap X_v \cup \tau_{bag} = \tau_{bag} \cup \tau_{bag} = \tau_{bag}$. Hence $D$ satisfies Condition 13(i). Now consider any vertex $w \in (V(G_{<u}) \setminus X_u) \cup \tau_{cov}$, there exist $w_1, w_2 \in \text{sol} \,[u, \tau_1] = \text{sol} \,[v, \tau]$ such that $w$ is covered by $w_1, w_2 \in H_v$. Hence Condition 13(ii) is satisfied. By Condition 22(d) for $A \subseteq X_v$ $\tau_{int}[A] = 1$ if and only if $\tau_{int}[A] = 1$ or $\tau_{int}[A \cup \{x\}] = 1$. Indeed, if some $y \in G_{<v}$ is close to $A$ or $A \cup \{x\}$ with respect to $X_u$, then it is close to $A$ with respect to $X_v$. Conversely, if there exists some $y \in G_{<v}$ close to $A$ with respect to $X_v$, then $A = B \cap X_v$ where $B$ is the set to which $y$ is close to with respect to $X_u$. The only possibilities for $B$ are $A$ and $A \cup \{x\}$. By Condition 22(o) $\tau_{int}[X_v] = 1$ if and only
if \( \tau^\text{int}_1[X_u] = 1 \), \( \tau^\text{int}_v[X_v] = 1 \) or \( \tau^\text{int}_1[x] = 1 \). Indeed, if some \( y \in G_{\leq v} \) is close to \( X_u \), \( X_v \) or \( \{x\} \) with respect to \( X_u \), then it is close to \( X_v \), with respect to \( X_v \). Conversely, if there exists some \( y \in G_{< v} \) close to \( X_v \), then \( X_v \) is included in \( A \) or \( X_u \backslash A \), where \( A \) is the set to which \( y \) is close to with respect to \( X_v \). The only possibilities for \( A \) are \( X_u \), \( X_v \) or \( \{x\} \). Hence Condition 15(iii) is satisfied. Hence, \( \text{sol} [v, \tau] \) is a certificate of \((v, \tau)\). Now Lemma 23 implies that \( \text{sol} [v, \tau] \) is minimum.

3. Assume \( v \) to be a join node. Let \( u_1, u_2 \) be the children of \( v \). Let \( \tau_1, \tau_2 \) be types associated with \( u_1, u_2 \) such that \( \text{sol} [v, \tau] = \text{sol} [u_1, \tau_1] \cup \text{sol} [u_2, \tau_2] \). Consider the set \( D = \text{sol} [v, \tau] \cup S^\tau_1 \). Due to Condition 24(a) we have that \( \tau^\text{bag}_1 = \tau^\text{bag}_2 = \tau^\text{bag} \). This implies \( \text{sol} [v, \tau^\text{bag}] \cap X_v = (\text{sol} [u, \tau_1] \cap \tau^\text{bag}_1) \cup (\text{sol} [u_2, \tau_2] \cap \tau^\text{bag}_2) = \tau^\text{bag} \).

Consider \( y \in (V(G_{\leq v}) \setminus X_v) \cup \tau^\text{cov} \). If \( y \in (V(G_{\leq v}) \setminus X_v) \cup \tau^\text{cov} \), then \( y \) is covered by a pair of vertices \( y_1 \) and \( y_2 \) in \( \text{sol} [u, \tau_1] \cup S^\tau_1 \). If \( y_1, y_2 \in \text{sol} [u_1, \tau_1] \), then \( y_1, y_2 \in D \) and we are done. Otherwise, assume without loss of generality that \( y_2 \in S^\tau_1 \setminus S^\tau_2 \). There must be a set \( A \subseteq X_u \) such that \( \tau^\text{int}_v[A] = 1 \) and \( y_2 = A \). By Condition 24(b) either \( \tau^\text{int}_v[A] = 1 \) or \( \tau^\text{int}_v[A] = 1 \). If the first case is true then there exists a \( y'_2 \in S^\tau_2 \) such that \( y \) is covered by \( y_1 \) and \( y'_2 \) in \( G_{\leq v} \). If the second case is true then there is a vertex \( y'_2 \) in \( \text{sol} [u_2, \tau_2] \) such that \( y'_2 \) is close to \( A \) with respect to \( X_u \). Due to Lemma 16 we have that \( y \) is covered by \( y_1 \) and \( y'_2 \). The case where \( y \in (V(G_{\leq v}) \setminus X_v) \cup \tau^\text{cov} \) is symmetrical. If \( y \in \text{Cov}(u_1, u_2) \), by its definition in Condition 24(c) and Lemma 22, \( y \) is covered by vertices in \( \text{sol} [v, \tau] \). Hence Condition 15(ii) is satisfied.

By Conditions 24(d) and 24(e) we have that for any \( A \subseteq X_v \), \( \tau^\text{int}_v[A] = 1 \) if and only if \( \tau^\text{int}_v[A] = 1 \) or \( \tau^\text{int}_v[A] = 1 \). Therefore by induction Condition 15(iii) is satisfied. The minimality follows from Lemma 23.

4. When \( v \) is the root node, the statement follows easily from the Definition 26 and Lemma 27.

This completes the proof.

### 3.2 The case of interval graphs

When the input graph \( G \) is an interval graph, the nice tree decomposition of \( G \) does not contain any join nodes. Moreover, the linear structure of interval graphs helps us to reduce the time complexity of the dynamic programming algorithm proposed in the previous section. Essentially, we shall show that the number of different types associated with a node \( v \) is at most \( O(2^{\omega(G)}) \). We shall use the following lemma.

**Lemma 29.** Let \( X \) be a clique cutset of an interval graph \( G \). There exists a collection \( A \) of subsets of \( X \) of size \( O(|X|) \) such that for each vertex \( v \in V(G) \), if \( v \) is close to \( A \) with respect to \( X \), then \( A \in A \).

**Proof.** If \( v \in X \), then \( A = \{v\} \). Without loss of generality, assume now that \( \min(v) < \min(X) \) (where \( \min(v) \) denotes the left endpoint of the interval associated to \( v \), and \( \min(X) \), the leftmost left endpoint of an interval of \( X \)). If \( u \in X \) such that \( d(v, u) = d \), then for every \( w \in X \) such that \( \min(w) \leq \min(u) \), \( d(v, w) \leq d \). Indeed, take a shortest path from \( v \) to \( u \) and let \( z \) be the neighbour of \( u \) in this path. Then, \( z = u \) also a neighbour of \( w \). This implies that \( v \) is close to \( A \) with respect to \( X \) which belongs to one of the following sets: \( \bigcup_{w \in X} \{w \in X | \min(w) \leq \min(u)\} \). Hence,

\[
A = \bigcup_{w \in X} \{\{w \in X | \min(w) \leq \min(u)\}, \{w \in X | \max(w) \geq \max(u)\}, \{u\}\}
\]

Observe that \( |A| \) is \( O(|X|) \).
The above lemma implies that for an interval graph, the set of 4-tuples for a node \( v \) can be chosen as a subset of \( \{0,1\}^3 \times \{0,1\}^3 \times 2^{X_v} \times 2^{Y_v} \). Hence, there are \( 2^{O(|w|)} \) types for an interval graph. This proves the statement of Theorem 4 regarding interval graphs.

4 Hardness for partial grids

We now prove Theorem 2. Let \( \mathcal{P}\mathcal{G}(3,g) \) denote the class of subcubic partial grids of girth at least \( g \). Given a graph \( G \), a subset \( S \subseteq V(G) \) is a vertex cover of \( G \) if every edge in \( E(G) \) has at least one end-vertex in \( S \). The problem MINIMUM VERTEX COVER is to compute a vertex cover of an input graph \( G \) with minimum cardinality. To prove Theorem 2, we reduce the NP-complete MINIMUM VERTEX COVER on cubic planar graphs \([14]\) to MGS on graphs in \( \mathcal{P}\mathcal{G}(3,g) \). We use the following result of Valiant \([23]\).

\[\textbf{Theorem 30} \quad (23). \text{ A planar graph } G \text{ with maximum degree 4 can be embedded in the plane using } O(|V(G)|) \text{ area in such a way that its vertices are at integer coordinates and its edges are drawn so that they are made up of line segments of the form } x = i \text{ or } y = j, \text{ for integers } i \text{ and } j.\]

Let \( R \) be an embedding of a cubic planar graph \( G \) as described in Theorem 30. Observe that one can ensure that the distance between two vertices is at least 100, and two parallel lines are at distance at least 100. (Any large constant would be sufficient). We call such an embedding a good embedding of \( G \).

A set \( S \) of vertices of a graph is an edge geodetic set if every edge lies in some shortest path between some vertices in \( S \). Note that an edge geodetic set is also a geodetic set. We need the following lemma.

\[\textbf{Lemma 31}. \text{ Let } H \text{ be a graph having a geodetic set } S \text{ which is also an edge geodetic set. If } H' \text{ denote the graph obtained by replacing each edge of } H \text{ with a path having } k \text{ edges, then } S \text{ is a geodetic set of } H'.\]

\[\textbf{Proof}. \text{ Let } w \in V(H') \text{ be a new vertex that was introduced while replacing an edge } e \text{ of } H \text{ with a path. Let } u_v, v_v \in S \text{ such that } e \text{ belongs to a shortest path } P \text{ between } u_v \text{ and } v_v. \text{ Let } P' \text{ be the path obtained by replacing each edge of } P \text{ by a path having } k \text{ edges.}\]

Let \( P' \) be a shortest path between \( u_v \) and \( v_v \) in \( H' \). Hence \( w \) belongs to a shortest path between \( u_v \) and \( v_v \) in \( H' \). Hence \( S \) is a geodetic set of \( H' \). □

\[\textbf{Overview of the reduction}. \text{ From a cubic planar graph } G \text{ with a given good embedding, first we construct a planar graph } f_1(G) \text{ having maximum degree at most 6 and girth 4. We show that } G \text{ has a vertex cover of size } k \text{ if and only if } f_1(G) \text{ has a geodetic set of size } 3|V(G)| + k. \text{ Then, we construct a graph } f_2(G) \in \mathcal{P}\mathcal{G}(3,42) \text{ such that the geodetic numbers of } f_2(G) \text{ and } f_1(G) \text{ are the same. When } g > 42, \text{ we construct a graph } f_3(G) \in \mathcal{P}\mathcal{G}(3,g) \text{ such that the geodetic numbers of } f_3(G) \text{ and } f_2(G) \text{ are the same.}\]

\[\textbf{Construction of } f_1(G). \text{ From a cubic planar graph } G \text{ with a given good embedding } R, \text{ we construct a graph } f_1(G) \text{ as follows. Each vertex } v \text{ of } G \text{ will be replaced by a vertex gadget } G_v \text{ which is shown in Figure 2. The edges outside of the vertex-gadgets will depend on } R. \text{ We assume that the edges incident with any vertex } v \text{ are labeled } e^v_i \text{ with } 0 \leq i < 3, \text{ in such a way that the labeling numbers increases counterclockwise around } v \text{ with respect to the embedding (thus the edge } vv \text{ will have two labels: } e^v_0 \text{ and } e^v_3). \text{ Consider two vertices } v \text{ and } w \text{ that are adjacent in } G, \text{ and let } e^v_i \text{ and } e^w_j \text{ be the two labels of edge } vw \text{ in } G. \text{ Add the edges } (t^v_i, t^w_j), (y^v_{i+1}, y^w_{j-1}) \text{ and } (y^v_{i-1}, y^w_{j+1}) \text{ (See Figure 2). All indices are taken modulo 3. There are no other edges in } f_1(G). \text{ Observe that } f_1(G) \text{ has maximum degree at most 6 and girth 4.}\]
Lemma 32. The graph $G$ has a vertex cover $D$ of size $k$ if and only if $f_1(G)$ has a geodetic set of size $3|V(G)| + k$.

Proof. We construct a geodetic set $S$ of $f_1(G)$ of size $3|V(G)| + k$ as follows. For each vertex $v$ in $G$, we add the three vertices $z_{i,j}^v$ $(0 \leq i < j \leq 2)$ of $G_v$ to $S$. If $v$ is in $D$, we also add vertex $c^v$ to $S$.

Let us show that $S$ is indeed a geodetic set. First, we observe that in any vertex gadget $G_v$ that is part of $f_1(G)$, the unique shortest path between two distinct vertices $z_{i,j}^v$, $z_{i',j'}^v$ has length 4 and goes through vertices $y_{i,j}^v$, $t_i^v$ and $y_{i',j'}^v$ (where $\{k\} = \{i, j\} \cap \{i', j'\}$). Thus, it only remains to show that the vertices $\{e^v, x_{i,j}^v\} (0 \leq i < j \leq 2)$ belong to some shortest path of vertices of $S$. Assume that $v$ is a vertex of $G$ in $D$. The shortest paths between $e^v$ and $z_{i,j}^v$ have length 3 and one of them goes through vertex $x_{i,j}^v$. Thus, all vertices of $G_v$ belong to some shortest path between vertices of $S$. Now, consider a vertex $w \notin D$ of $G$. Since $G$ is a cubic planar graph, all three neighbours of $w$, say, $w_1, w_2, w_3$ must lie in $D$. Let $C = \{e^{w_1}, e^{w_2}, e^{w_3}\}$ and $Z = \{z_{0,1}^w, z_{1,2}^w, z_{0,2}^w\}$. Observe that all vertices of $G_w$ lie in the set $I(C \cup Z)$. Therefore, $S$ is a geodetic set.

For the converse, assume we have a geodetic set $S'$ of $f_1(G)$ of size $3|V(G)| + k$. We will show that $G$ has a vertex cover of size $k$. First of all, observe that all the $3|V(G)|$ vertices of type $z_{i,j}^v$ are necessarily in $S'$, since they have degree 1. As observed earlier, the shortest paths between those vertices already go through all vertices of type $t_i^v$ and $y_{i,j}^v$. However, no other vertex lies on a shortest path between two such vertices: these shortest paths always go through the boundary 6-cycle of the vertex-gadgets. Let $S_0'$ be the set of the remaining $k$ vertices of $S'$. These vertices are there to cover the vertices of type $e^v$ and $x_{i,j}^v$. We construct a subset $D'$ of $V(G)$ as follows: $D'$ contains those vertices $v$ of $G$ whose vertex-gadget $G_v$ contains a vertex of $S_0'$. We claim that $D'$ is a vertex cover of $G$. Suppose by contradiction that there is an edge $vw$ of $G$ such that neither $G_v$ nor $G_w$ contains any vertex of $S_0'$. Without loss of generality assume that $e_v^w$ and $e_w^v$ are the two labels of the edge $vw$. Here also, the shortest paths between vertices of $S$ always go through the boundary 6-cycle of $G_v$ and thus, they never include vertex $x_{1,2}^v$. Let $a$ and $b$ be the neighbours of $v$ different from $w$. Observe that no shortest path between a vertex of $G_v$ and a vertex of $G_b$ contains the vertex $x_{1,2}^v$, a contradiction. Thus $S'$ is a vertex cover of $G$.  

Construction of $f_2(G)$. An edge $uv$ of $f_1(G)$ is an internal edge if both $u$ and $v$ belongs to $G_v$ for some $v \in V(G)$. The other edges of $f_1(G)$ are external edges. We construct $f_2(G)$ in three steps described below.
1. Replace each vertex of type $t_v^i$ with a new edge $T_v^i = (t_v^i t_v^{i+1})$. Replace each vertex of type $y_v^{i,j}$ with a path $Y_v^{i,j} = a_v^{i,j} y_v^{i,j} b_v^{i,j} d_v^{i,j}$. Replace each vertex of type $c_v$ with a path $C_v = a_v b_v c_v d_v$. (See Figure 3).

2. Replace each internal edge between vertices having labels $p,q$ with a new path such that the shortest path in the new graph between the vertices with label $p,q$ is 14.

3. For an edge $uv \in E(G)$, let $E_{uv}$ denote the set of three internal edges between $G_u$ and $G_v$ in $f_1(G)$. Recall that $\mathcal{R}$ is a good embedding of $G$. Let $l_{uv}$ denote the length of the edge $uv$ in $\mathcal{R}$. Replace all three external edges $p_i q_i \in E_{uv}$ $(1 \leq i \leq 3)$ with three new paths $P_i$ $(1 \leq i \leq 3)$ such that lengths of all three paths are equal and in $O(l_{uv})$.

Clearly, $f_2(G_v)$ is a partial grid for each $v \in G$ (Figure 3). It is not difficult to verify that $f_2(G)$ has maximum degree 3 and girth at least 42. Let $D$ be a vertex cover of $G$ with cardinality $k$. We construct $S$ of $f_2(G)$ of cardinality $3|V(G)| + k$ as follows. For each vertex $v$ in $G$, we add the three vertices with labels $z_v^{i,j}$ $(0 \leq i < j \leq 2)$ to $S$. If $v$ is in $D$, we also add vertex $c_v$ to $S$. From the construction of $f_2(G)$ and using arguments similar to that of Lemma 32, $G$ has a vertex cover of size $k$ if and only if $f_2(G)$ has a geodetic set of size $3|V(G)| + k$. Moreover, we can prove the following.

**Lemma 33.** The set $S$ is both a geodetic set of minimum cardinality and an edge geodetic set of minimum cardinality of $f_2(G)$.

**Completion of the proof of Theorem 2.** If $g \leq 42$, then observe that $f_2(G) \in PG(3,g)$ and from the previous discussions, we have that MGS is NP-hard for graphs in $PG(3,g)$. Otherwise, we replace every edge of $f_2(G)$ with a path of length $g$. Call this modified graph $f_3(G)$, and observe that $f_3(G) \in PG(3,g)$. By Lemma 32, $S$ is both a geodetic set of minimum cardinality and an edge geodetic set of minimum cardinality in $f_2(G)$. Now, due to Lemma 31, $S$ is a geodetic set of $f_3(G)$ of cardinality $3|V(G)| + k$. 

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**Figure 3** Construction of $f_2(G)$.
5 Hardness for interval graphs

We now prove Theorem 3. Let $F$ be an instance of 3-Sat with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$. We construct a set $D$ of intervals in polynomial time such that the geodetic number of the intersection graph of $D$ (denoted as $I(D)$) is at most $4 + 7n + 58m$ if and only if $F$ is a positive instance of 3-Sat.

The key intuition that explains why the problem is hard on interval graphs, is that considering two solution vertices $x, y$, the structure of the covered set $I(x, y)$ can be very complicated. Indeed, it can be that many vertices lying "in between" $x$ and $y$ in the interval representation, are not covered. This allows us to construct gadgets, by controlling which such vertices get covered, and which do not. Moreover, we can easily force some vertices to be part of the solution by representing them by an interval of length 0 (then, they are simplicial vertices), which is very useful to design our reduction. Nevertheless, implementing this idea turns out to be far from trivial, and to this end we need the crucial idea of tracks, which are shortest paths spanning a large part of the construction. Each track starts at a track for which $v$ is the root will almost entirely be a shortest path from the root to the rightmost end of the construction. In a way, each track "carries the effect of the root" being chosen in a solution to the rest of the graph. The tracks are shifted in a way that no shortcut can be used going from one track to another. We are then able to locally modify the tracks and place our gadgets so that the track of, say, a literal, enables the interval of that literal to cover an interval of a specific clause gadget (while the other tracks are of no use for this purpose).

5.1 Definitions

We shall use the following notations. Let $S$ be a set of intervals. For a vertex $v \in V(I(S))$, let $v = [\min(v), \max(v)]$ denote the interval corresponding to $v$ in $S$, where $\min(v)$ and $\max(v)$ refer the left boundary and right boundary of $v$, respectively. The rightmost neighbour of $v$ is the interval intersecting $v$ that has the maximum right boundary. For a set $S$ of intervals, let $\min(S) = \min\{\min(v) : v \in S\}$, $\max(S) = \max\{\max(v) : v \in S\}$. For two intervals $u, v$ we have $u < v$ if $\max(u) < \min(v)$.

Let $S$ be a set of intervals and $u, v \in S$. A shortest path between $u, v$ is a shortest path between $u, v$ in $I(S)$. The set $I(u, v)$ is the set of intervals that belongs to some shortest path between $u, v$. The geodetic set of $S$ is analogously defined. For a set of $S$ the phrase “$S$ is covered by $S'$” means that $S'$ is a geodetic set of $S$.

A point interval is an interval of the form $[a, a]$. A unit interval is an interval of the form $[a, a + 1]$. A set of interval is proper if no two intervals contain each other. A set $T = \{u_1, u_2, \ldots, u_t\}$ of intervals is a track if $\max(u_i) = \min(u_{i+1})$ for all $1 \leq i < t$. Observe that if $T$ is a track, then $I(T)$ is a path. In our construction, each track $T$ will be associated with a number of intervals called its roots. Sometimes we shall use the sentence “root $v$ of a track $T'$" to say $v \in R(T)$. For an intuition of how the tracks and roots are used, the track for which $v$ is the root will almost entirely be a shortest path from $v$ to any interval $w$ to the right of $v$ (except for some local shortcuts in the gadgets involving $w$, that can be controlled).

Definition 34. Let $T$ and $T'$ be two tracks such that $T \cup T'$ is a proper set of intervals. Then $T \prec T'$ if $\max(T) < \max(T')$.

Let $\mathcal{T}$ be a set of tracks and $T \in \mathcal{T}$. The phrase “the track just preceding $T'$" shall refer to the track $T'$ such that $T' \prec T$ and there is no $T''$ such that $T' \prec T'' \prec T$. The phrases
5.2 Overview of the reduction

There are four main stages of our reduction. We initialise it by constructing a set of intervals which we call the start gadget (denoted as $S$). The precise definitions are given in Section 5.3.

After creating the start gadget, we create the variable gadgets, which are placed consecutively, after the start gadget. For each variable $x_i$ with $1 \leq i \leq n$, we create the variable gadget $X_i$ in Section 5.5. Each variable gadget is composed of several implication gadgets. An implication gadget $IMP[\neg p \rightarrow q]$ ensures (under some extra hypotheses) that if $p$ is not chosen in a geodetic set of our constructed intervals, then $q$ must be chosen. These are used to encode the behaviour of the variables of the 3-SAT instance: there will be two possible solutions, corresponding to both truth values of $x_i$.

After creating all the variable gadgets, we create the clause gadgets, also placed consecutively, after the variable gadgets. For each clause $C_j$ with $1 \leq j \leq m$, we construct the clause gadget $C_j$ in Section 5.6. Each clause gadget is composed of a covering gadget, several implication gadgets and several AND gadgets. The covering gadget of a clause $C_i$ is denoted by $COV[i]$. For two intervals $p$ and $q$, the corresponding AND gadget is denoted by $AND[p, q]$. Together, these gadgets will ensure that all intervals of the clause gadget corresponding to the clause $C_i$ are covered by six intervals if and only if one of the intervals corresponding to the literals of $C_i$ is chosen in a geodetic set. This encodes the behaviour of the clauses of the 3-SAT instance.

After creating all the clause gadgets, we conclude our construction by creating the end gadget $E$, placed after all clause gadgets, see Section 5.7. See Figure 4 for a schematic diagram of the construction.

5.3 Construction of $S$

Let $\epsilon = \frac{1}{(n+m)^3}$ where $n$ is the number of variables and $m$ is the number of clauses. The start gadget $S$ consists of four intervals which are defined as follows: the start interval $o = [1, 1]$, $u_o = [1, 2]$, the true interval $\top = [1 + \epsilon, 1 + \epsilon]$ and $u_\top = [1 + \epsilon, 2 + \epsilon]$. Let $T_1 = \{u_o\}$ and $T_2 = \{u_\top\}$. Observe that $T_1, T_2$ are tracks and $T_1 < T_2$. 
We initialize two more sets, the set $T = \{T_1, T_2\}$ of all tracks, and the set $D = S$ of all intervals. As we proceed with the construction we shall insert more intervals in $T_1, T_2$ while maintaining that both of them are tracks. We shall also add more tracks in $T$. Let $R(T_1) = \{\emptyset\}$ and $R(T_2) = \{T\}$.

5.4 Implication gadget of a root $p$

In order to construct the variable gadgets and the clause gadgets, we need to define the implication gadget. Below we describe a generic procedure to construct implication gadgets of a root $p$ which is different from $\emptyset$ of $S$. Let $T_p \in T$ be the track such that $p \in R(T_p)$. Since $p \neq \emptyset$, $T_p$ is not the minimal element in $T$. Below we describe the three steps for constructing $IMP[\neg p \rightarrow q]$. See Figure 6 for a graphical representation of the intervals.

1. Extension of existing tracks: For each track $T \in T$, introduce three new intervals $u_T = [\max(T), \max(T) + 1]$, $v_T = [\max(T) + 1, \max(T) + 2]$ and $w_T = [\max(T) + 2, \max(T) + 3]$. Let $T_{new} = \{u_T, v_T, w_T\}$. Observe that, for two tracks $T, T' \in T$ with $T < T'$ we have $(T \cup T_{new}) < (T' \cup T'_{new})$.

2. Creation of new intervals: Let $X$ and $X'$ be the tracks that precede and follow $T_p$ in $T$, respectively. When $X'$ exists, let $\theta = \max(u_{X'})$ and $\theta' = \max(v_{X'})$. Otherwise $\theta = \max(u_{T_p}) + \epsilon$ and $\theta' = \max(v_{T_p}) + \epsilon$.

Define $q = \left[\frac{\max(u_{X}) + \max(u_{T_p})}{2}, \frac{\max(u_{T_p}) + \theta}{2}\right]$, $r_q = \left[\frac{\max(q), \max(w_{T_p}) + \theta'}{2}\right]$ and $s_q = \left[\frac{\max(r_q), \max(r_{T_p})}{2}\right]$.

3. Creation of new tracks: In this step, we shall create two new tracks. We define three more intervals as follows: $t = [\max(s_q), \max(s_q) + 1]$, $t_1 = \left[\frac{\max(q), \max(w_{T_p}) + \min(s_q)}{2}\right]$ and $t_2 = [\max(t_1), \max(t_1) + 1]$. Now let $T_1 = \{t_1, t_2\}$, $R(T_1) = \{q\}$, $T_2 = \{t\}$ and $R(T_2) = \{r_q, s_q\}$.

To complete the construction of the implication gadget, we define $IMP[\neg p \rightarrow q] = \{q, r_q, s_q\} \cup T_1 \cup T_2 \cup \bigcup_{T \in T'} \{u_T, v_T, w_T\}$. Let $D = D \cup IMP[\neg p \rightarrow q]$. For each $T \in T$, let $T = T \cup T_{new}$ and $T = T \cup \{T_1, T_2\}$. We shall use the following observations, whose proofs easily follow from the construction.

Observation 35. The leftmost neighbour of $q$ is $u_{T_p}$.
Figure 7 The covering gadget COV[i]. For drawing purposes, the proportions of the intervals were changed. Nevertheless, the obtained interval graph is unchanged.

Observation 36. We have $X < T_p < T_1 < T_2 < X'$.

5.5 Construction of variable gadgets

We construct the variable gadgets sequentially and connect each of them to the previous one ($X_1$ is connected to the start gadget $S$). Assuming that we have placed $S$, $X_1$, ..., $X_{i-1}$, we construct $X_i$ as follows. For variable $x_i$, the gadget $X_i$ consists of two implication gadgets. Let $D$ and $T$ be the set of intervals and tracks created so far. First, we construct $\text{IMP}[^\neg T \rightarrow x_i]$. Observe that the sets $D$ and $T$ have been updated after the last operation. There is an interval $x_i$ in $D$ and there is a track $T \in T$ whose root is $x_i$. Now we construct $\text{IMP}[^\neg x_i \rightarrow x_i]$. Observe that after constructing all the variable gadgets, for each literal $\ell$, there is an interval named $\ell$ in $D$.

5.6 Construction of clause gadgets

A clause gadget consists of a covering gadget, several implication gadgets and several AND gadgets. In Section 5.6.1 we describe the procedure to construct a covering gadget. In Section 5.6.2 we describe the procedure to construct an insert gadget. We need this to construct an AND gadget in Section 5.6.3. Finally, we construct the clause gadget $C_i$ corresponding to a clause $C_i$ in Section 5.6.4.

5.6.1 Construction of covering gadgets

Below we describe the three steps of constructing the covering gadget of the clause $C_i$. See Figure 7

1. Extension of existing tracks: For each track $T \in T$, introduce three new intervals $u_T^i = [\max(T), \max(T) + 1]$, $v_T^i = [\max(T) + 1, \max(T) + 2]$ and $w_T^i = [\max(T) + 2, \max(T) + 3]$. Let $T_{new} = \{u_T^i, v_T^i, w_T^i\}$. Observe that, for two tracks $T, T' \in T$ with $T < T'$ we have $(T \cup T_{new}) < (T' \cup T_{new})$.

2. Creation of new intervals: Let $T$ be the maximal track in $T$. Let $\theta = \min(u_T^i) + \epsilon$.
   We define $a_i = [\theta, \theta + a]$, $b_i = [\theta, \theta + 2a]$, $c_i = [\theta, \theta + 3a]$ and $d_i = [\theta, \theta]$. Also define $\text{cov}_i = [\max(v_T^i) + 4\epsilon, \max(v_T^i) + 7\epsilon]$, and $f_i = [\max(\text{cov}_i), \max(\text{cov}_i)]$.

3. Creation of new tracks: Now we create five more tracks as follows. Let $T_{a_i} = \{[\max(a_i) + k, \max(a_i) + k + 1] | k \in \{0, 1, 2\}\}$,
5.6.3 Construction of AND gadgets

Let \( T_p \) and \( T_q \) be two tracks of \( \mathcal{T} \) with roots \( p \) and \( q \), respectively. Without loss of generality, assume that \( T_p < T_q \). Below we describe the four steps of constructing \( \text{INS}[p, q] \). See Figure 8.

1. Extension of existing tracks: For each track \( T \in \mathcal{T} \), we introduce one new interval \( u_T = [\max(T), \max(T) + 1] \). Let \( T_{\text{new}} = \{u_T\} \). Observe that, for two tracks \( T, T' \in \mathcal{T} \) with \( T < T' \), we have \( (T \cap T_{\text{new}}) < (T' \cap T_{\text{new}}) \).

2. Creation of a new interval: Let \( X \) be the track that just follows \( T_p \) in \( \mathcal{T} \). Observe that, \( X \) always exists. Let \( \sigma(p, q) = \left[ \frac{\max(u_T) + \max(u_X)}{2}, \frac{\max(u_T) + \max(u_X)}{2} \right] \).

3. Creation of a new track: Let \( T_m = \{[\max(\sigma(p, q)), \max(\sigma(p, q)) + 1]) \} \) and \( R(T_m) = \{\sigma(p, q)\} \).

To complete the construction, we define \( \text{INS}[p, q] = \{\sigma(p, q)\} \cup \{u_T, v_T\}_{T \in \mathcal{T}} \cup T_{\text{new}} \cup T_m \). We set \( D = D \cup \text{INS}[p, q] \). For each \( T \in \mathcal{T} \), let \( T = T \cup T_{\text{new}} \) and \( \mathcal{T} = \mathcal{T} \cup \{T_m\} \). Observe that \( T_p < T_m < T_q \) in \( \mathcal{T} \).

5.6.3 Construction of AND gadgets

Let \( T_p \) and \( T_q \) be two tracks of \( \mathcal{T} \) with roots \( p \) and \( q \), respectively. Without loss of generality, assume that \( T_p < T_q \). Below we describe the construction of \( \text{AND}[p, q] \). See Figure 8.
**Observation 39.** We have $T_p < T_1 < T_m < T_2 < T_q < T_3$ in $\mathcal{T}$.

**Observation 40.** Let $X, X'$ be two tracks in $\mathcal{T} \setminus \{T_1, T_2, T_3\}$ with $X < T_p < X'$. Let $Y, Y'$ be two tracks in $\mathcal{T} \setminus \{T_1, T_2, T_3\}$ with $Y < Y'$. Let $Z, Z'$ be two tracks in $\mathcal{T} \setminus \{T_1, T_2, T_3\}$ with $Z < T_q < Z'$.
1. We have $\alpha(p, q) \subseteq v_X$ and $\alpha(p, q) \subseteq u_X$.
2. If $Y = T_p$ or $Y < T_p$ then $\gamma(p, q) \subseteq v_Y$ and if $Y' = T_q$ or $T_q < Y'$ then $\gamma(p, q) \subseteq u_{Y'}$.
3. We have $\delta(p, q) \subseteq v_Z$ and $\delta(p, q) \subseteq u_Z$.
4. The interval $\gamma(p, q)$ is contained in the only interval of $T_1$ and $\delta(p, q)$ is contained in the only interval of $T_2$.

5.6.4 Construction of $C_i$

We shall complete our construction of clause gadget $C_i$ corresponding to the clause $C_i = (e_1', e_2', e_3')$. First, we create the covering gadget $COV[i]$ and update $D, T$ as described in Section 5.6.1. Recall from the construction of $COV[i]$ that the intervals named $a_i, b_i, c_i$ exist. Also recall from the construction of variable gadgets (described in Section 5.5) that the intervals $e_1', e_2', e_3'$ and $f_1', f_2', f_3'$ exist.

Now we create, in this order, $IMP[-a_i \rightarrow a_i'], AND[a_i, e_1'], AND[a_i, f_1']$, $IMP[-b_i \rightarrow b_i'], AND[b_i, e_2'], AND[b_i, f_2']$, $IMP[-c_i \rightarrow c_i'], AND[c_i, e_3']$, and $IMP[-c_i' \rightarrow c_i'], AND[c_i', f_3']$ where $a_i', b_i'$ and $c_i'$ are three new intervals constructed in the corresponding implication gadgets. This completes the construction of $C_i$.

5.7 Construction of end gadget

For each $T \in T$, we introduce two new intervals $u_T = [\max(T), \max(T) + 1], e_T = [\max(u_T), \max(u_T)]$ and define $T = T \cup \{u_T\}, D = D \cup \{u_T, e_T\}_{T \in T}$. For each $T \in T$, let $e_T$ be the tail of $T$. The end gadget $E$ consists of all the new intervals created above.

5.8 Analysis

First, we state some observations which follow easily from the construction.

Observation 41. There are $2 + 4n + 35m$ tracks in $T$ and $4 + 6n + 52m$ point intervals in $D$. The total number of intervals in $D$ is $O((n + m)^2)$.

We define $n_{\text{point}} = 4 + 6n + 52m$. Remark that the point intervals are exactly the simplicial vertices of $D$.

Observation 42. Let $T$ be a track in $T$ such that $T = \{u_1, \ldots, u_k\}$ with $\max(u_i) = \min(u_{i+1})$ for all $i$ with $1 \leq i \leq k - 1$. For each $i$ with $1 \leq i \leq k - 1$, $u_{i+1}$ is the rightmost neighbour of $u_i$.

Now we prove the following proposition.

Proposition 43. Let $u$ and $v$ be two intervals of $D$ such that $\min(u) < \min(v)$. The path $u_0, u_1, \ldots, u_k, v$ is a shortest path from $u$ to $v$ (where $u = u_0, u_{i+1}$ is the rightmost neighbour of $u_i$ for $i \in 1 \leq i \leq k - 1$, and $u_k \notin N(v)$, while $u_k \in N(v)$).

Proof. Let $P$ be the path $u_0, u_1, \ldots, u_k, v$ and $P'$ be a shortest path from $u$ to $v$ which start by the longest common subpath with $P$: $P' = u_0, u_1, \ldots, u_i, z, \ldots, v$ with $z \notin P$. If $i = k$ then $P$ and $P'$ have the same length. Otherwise, replace $z$ in $P'$ by $u_{i+1}$ to obtain a path $P''$. It is indeed a path as $\max(z) < \max(u_{i+1})$. Moreover $P'$ and $P''$ have the same length, a contradiction with the definition of $P'$. △
Let \( P = u_1 \ldots u_k \) be a path such that \( u_{i+1} \) is the rightmost neighbour of \( u_i \) for all \( i \in 1 \leq i \leq k - 2 \) and \( u_{k+1} \) is not adjacent to \( u_k \). The path \( P \) is the shortest path between \( u_1 \) and \( u_k \) by the previous proposition. We say that such a path is a good shortest path.

An interval \( u \) is a track interval if \( u \in T \) for some \( T \in T \). From now on, \( U \) shall denote the set of track intervals. Let \( S_p \) be the set of all point intervals and recall that \( S_p \) is a subset of every geodetic set of \( D \). For an interval \( z \), let \( T(z) \) denote the track \( T \) such that \( z \in R(T(z)) \).

\[ \begin{align*}
\text{Proposition 44.} & \quad \text{If } T \text{ is a track in } T, \text{ then } T \subseteq I(o,e_T). \\
\text{Proof.} & \quad \text{Let } T = \{a_1, \ldots, a_j\} \text{ be the the interval of } T. \text{ Let } T(o) = \{b_1, \ldots, b_k\}. \text{ By Proposition 43, } o, b_1, \ldots, b_k, e_T \text{ is a shortest path from } o \text{ to } e_T \text{ where } z \text{ is the rightmost neighbour of } b_k \text{ (it belongs to the end gadget).} \text{ Remark that } a_1, \ldots, a_j, e_T \text{ is a good shortest path from } a_1 \text{ to } e_T. \text{ By induction on } i \text{ (from } i = j \text{ to } i = 1), \text{ one can see that } a_1 \text{ and } b_{k-j+i} \text{ are neighbours. Hence, the path } o, b_1, \ldots, b_{k-j+1}, a_1, \ldots, a_j, e_T \text{ is a shortest path from } o \text{ to } e_T \text{ covering } T. \\
\end{align*} \]

\[ \begin{align*}
\text{Proposition 45.} & \quad \text{For an implication gadget } IM \to, \text{ let } T \text{ be the track with root } q. \text{ Then } q \in I(p,s_q), r_q \in I(o,s_q). \\
\text{Proof.} & \quad \text{Let } T' \text{ be the track with root } p. \text{ Let } \{v_1, v_2, \ldots, v_{k+2}\} \subseteq T' \text{ such that } p \text{ and } v_1 \text{ are adjacent, } v_{i+1} \text{ is the leftmost neighbour of } v_i \text{ for } 1 \leq i \leq k + 1 \text{ and } v_k \text{ is the leftmost neighbour of } q \text{ (the existence of such an interval is guaranteed by Observation 35). By Proposition 43, } p, v_1, v_2, \ldots, v_k, e_T \text{ is a good shortest path between } p \text{ and } s_q. \text{ Hence, the path } p, v_1, v_2, \ldots, v_k, q, r_q, s_q \text{ also is a shortest path between } p \text{ and } s_q. \\
& \quad \text{Let } P = u_1, u_2, \ldots, u_k \text{ be a subpath of } T(o) \text{ such that } o, P, s_q \text{ is a good shortest path between } o \text{ and } s_q. \\
\end{align*} \]

\[ \begin{align*}
\text{Proposition 46.} & \quad \text{Consider the cover gadget } COV[i] \text{ and let } z \in \{a_i, d_i, c_i\}. \text{ Then } z \in I(d_i,e_{T(z)}), \text{ and } cov_i \in I(z,f_i). \\
\text{Proof.} & \quad \text{Let } T(z) = \{u_1, u_2, \ldots, u_{|T(z)|}\} \text{ such that } u_{j+1} \text{ is the rightmost neighbor of } u_j \text{ for all } 1 \leq j \leq k - 1. \text{ The path } d_i, z, u_1, u_2, \ldots, u_{|T(z)|}, e_{T(z)} \text{ is a shortest path between } d_i \text{ and } e_z. \text{ This proves the first part of the proposition. For the second part, consider the path } P = z, u_1, u_2, cov_i, f_i. \text{ Clearly, } P \text{ is a shortest path between } z \text{ and } f_i. \\
\end{align*} \]

\[ \begin{align*}
\text{Proposition 47.} & \quad \text{Consider an AND gadget } AND[p,q]. \text{ Let } T_1 = T(= (p,q)), T_2 = T(\gamma (p,q)) \text{ and } T_3 = \delta (p,q). \text{ Then } \\
& \quad (a) \gamma (p,q) \in I(\sigma(p,q), e_{T_2}), \\
& \quad (b) \alpha (p,q) \in I(p,e_{T_1}), \delta (p,q) \in I(q,e_{T_1}), \text{ and } \\
& \quad (c) \alpha (p,q) \in I(\beta (p,q), \gamma (p,q)) \text{ and } \delta (p,q) \in I(\gamma (p,q), e_{T_3}). \\
\text{Proof.} & \quad \text{To prove } (a) \text{ let } P = u_1, u_2, \ldots, u_j \text{ and } P' = u_{j+1}, u_2, \ldots, u_k \text{ such that } P, P' \text{ is the path induced by } T(\sigma(p,q)) \text{ where } u_i \text{ is the rightmost neighbour of } \sigma(p,q), u_{i+1} \text{ is the rightmost neighbour of } u_i \text{ for all } 1 \leq i \leq k - 1, \text{ and } u_j \text{ is the interval of } T(\sigma(p,q)) \text{ with minimal index which intersects } \gamma (p,q). \text{ Now, using Proposition 43, we can infer that the path } \sigma(p,q), P, \gamma (p,q), e_{T_2} \text{ is a good shortest path between } \sigma(p,q) \text{ and } e_{T_2}, \text{ where } z \text{ is the rightmost neighbour of } u_k. \text{ Remark that by construction, } P' \text{ and } T_2 \text{ have the same number of intervals. Hence, the path } \sigma(p,q), P, \gamma (p,q), T_2, e_{T_2} \text{ is a shortest path between } \sigma(p,q) \text{ and } e_{T_2}. \text{ The proofs of } (b) \text{ and } (c) \text{ can be done by similar arguments and are therefore omitted.} \\
\end{align*} \]
We shall show that if $F$ is satisfiable, then $D$ has a geodetic set of cardinality $4 + 7n + 58m = n_{\text{point}} + n + 6m$. Remark that the point intervals are the only simplicial vertices in $D$. Hence, they must all belong to any geodetic set of $D$. Let $\phi: \{x_1, x_2, \ldots, x_n\} \rightarrow \{1, 0\}$ be a satisfying assignment of $F$. Now, define the following sets. Let $S_1 = \{x_i: \phi(x_i) = 1\} \cup \{\overline{x_i}: \phi(\overline{x_i}) = 1\}$. Let $S_2 = \emptyset$. Now, for each clause $C_i = (\ell_1, \ell_2, \ell_3)$, do the following.

1. If $\phi(\ell_1) = 1$, then put $S_2 = S_2 \cup \{a_i, e(\ell_1, \ell_1')\}$ and if $\phi(\ell_1) = 0$ then put $S_2 = S_2 \cup \{a_i, e(\ell_1, \ell_1')\}$.
2. If $\phi(\ell_2) = 1$, then put $S_2 = S_2 \cup \{b_i, \gamma(b_i, \ell_2')\}$ and if $\phi(\ell_2) = 0$ then put $S_2 = S_2 \cup \{b_i, \gamma(b_i, \ell_2')\}$.
3. If $\phi(\ell_3) = 1$, then put $S_2 = S_2 \cup \{c_i, \gamma(c_i, \ell_3')\}$ and if $\phi(\ell_3) = 0$ then put $S_2 = S_2 \cup \{c_i, \gamma(c_i, \ell_3')\}$.

Due to Observation 11 we have that $|S_1 \cup S_2 \cup S_p| = 4 + 7n + 58m$.

**Lemma 48.** The set $S$ is a geodetic set of $D$.

**Proof.** As $S_p \subseteq S$, we know that all track vertices of $D$ are covered by Proposition 44. Moreover, every vertex of the form $r_y$ is covered by Proposition 45.

Consider any variable gadget $X_i$ corresponding to the variable $x_i$. Recall from construction, that $X_i = IMP[\neg x_i \rightarrow b_i] \cup IMP[x_i \rightarrow a_i]$. Due to Proposition 45 we have that $x_i \in I(\neg x_i, e_T(x_i))$, hence, all intervals of $IMP[\neg x_i \rightarrow b_i]$ are covered by $S$. Due to Proposition 45 either $a_i \in I(x_i, e_T(x_i))$ when $x_i \in S$ or $a_i \not\in S$ otherwise. The above arguments imply that all intervals in $X_i$ are covered by $S$.

Now, consider any clause $C_i = (\ell_1, \ell_2, \ell_3)$ and recall the construction of $\mathcal{C}_i$. Observe that there exists at least one interval $z \in \{a_i, b_i, c_i\} \cap S$. Using Proposition 46, we can infer that all intervals in $\text{COV}[z]$ are covered by $S$. Now, consider the implication gadget $IMP[\neg a_i \rightarrow a_i']$. From our definition of $S_2$, it follows that $\{a_i, a_i'\} \cap S \neq \emptyset$ and therefore using Proposition 45 we can infer that all intervals in $IMP[\neg a_i \rightarrow a_i']$ are covered by $S$. Repeating the above arguments for $IMP[\neg b_i \rightarrow b_i']$ and $IMP[\neg c_i \rightarrow c_i']$, we infer that all intervals in these implication gadgets are covered by $S$. Now, consider the AND gadget $\text{AND}[a_i, \ell_1']$. From our definition of $S_2$, it follows that either $\{a_i, \ell_1\} \subseteq S$ or $\gamma(p, q) \in S$. In both cases, we can use Proposition 47 to show that all intervals in $\text{AND}[a_i, \ell_1']$ are covered by $S$. Repeating the above arguments for all the AND gadgets in $\mathcal{C}_i$, we can show that all intervals of $\mathcal{C}_i$ are covered by $S$. This completes the proof.

Now, we shall show that if the geodetic number of $D$ is at most $4 + 7n + 58m$, then $F$ is satisfiable. Recall that $U$ is the set of all track intervals.

**Proposition 49.** There is a minimum-size geodetic set $S^*$ of $D$ such that $S^* \cap U = \emptyset$.

**Proof.** Let $S$ be a geodetic set of $D$ with minimum cardinality and, subject to this, containing minimum number of track intervals. Let $u \in S \cap U$ be the interval with $\min(S \cap U) = \min(u)$ such that $u$ belongs to a track with a root $y$. Let $A_1$ and $A_2$ be the sets of all $\gamma(p, q)$ intervals and $r_y$ intervals in $D$, respectively. Let $A_3 = \{z \in D: z \in \{a_i, b_i, c_i\}, 1 \leq i \leq m\}$ and $A = A_1 \cup A_2 \cup A_3$ and $\overline{A} = D \setminus (A \cup U \cup S_p)$. Due to Propositions 44 and 47, we will be done by proving the following claim (where $\overline{A}$ is the set of vertices that are not already covered by point intervals).

**Claim 50.** Let $v$ be any interval in $S$. We have $I(u, v) \cap \overline{A} \subseteq I(y, v) \cap \overline{A}$.
To prove the claim, define $S_1 = \{v \in S^* : \min(v) < \min(u)\}$ and $S_2 = \{v \in S^* : \min(u) < \min(v)\}$. First, assume that $v \in S_1$ and $z \in I(u, v) \cap \overline{A}$. In this case, $v$ must be a root of some track $T$ (by definition of $u$). Now, we have the following cases.

1. Assume $z = q$ for some implication gadget $IMP[\neg p \rightarrow q]$. Observe that $q$ must not be included in any interval of $T(v)$. This is only possible if $v = p$. By Proposition 45

$$q \in I(p, s_q).$$

2. Assume $z = \text{COV}[i]$ for some $1 \leq i \leq m$. Using similar arguments as above and Proposition 46, we can show that $v \in \{a_i, b_i, c_i\}$ and $q \in I(v, f_i)$.

3. Assume $z \in \{\alpha(p, q), \delta(p, q)\}$ for some AND gadget $\text{AND}[p, q]$. Using arguments as in Case 1 and Proposition 47, we can show that $v \in \{p, q\}$.

The above cases imply that when $v < u$, then $I(v, u) \subset I(v, u')$ (where $u' \in S \setminus \{u\}$). Now assume that $v \in S_2$ and $z \in (I(u, v) \cap \overline{A})$. Observe that there exists a shortest path from $u$ to $v$ that contains $z$. Consider now the good shortest path between $y$ and $u$ concatenated with the shortest path between $u$ and $v$ covering $z$. This is a shortest path between $y$ and $u$ covering $z$. The above arguments imply that $(S \setminus \{u\}) \cup \{y\}$ is also a geodetic set of $D$ with minimum cardinality of $S$. But this contradicts the minimality of $S$ (with respect to the number of track intervals).

A good geodetic set of $D$ is a geodetic set of minimum cardinality which does not contain any interval belonging to a track.

**Proposition 51.** Let $S^*$ be a good geodetic set of $D$ and $IMP[\neg p \rightarrow q]$ be an implication gadget where $p$ is the only root of $T(p)$. Then, either $p \in S^*$ or $q \in S^*$.

**Proof.** Suppose $q \notin S^*$ and let $u, v$ be two vertices in $S^*$ that covers $q$. Without loss of generality, assume that $\min(u) < \min(q)$. Let $P = u, u_1, \ldots, u_k, q$ be a good shortest path between $u$ and $q$ and $P' = q, u_{k+1}, \ldots, u_t, v$ be a good shortest path between $q$ and $v$. Note that $P, q, P'$ is a shortest path between $u$ and $v$ which covers $q$ (as otherwise, there would be a contradiction with our initial assumption). This means $u_k$ and $u_{k+1}$ do not intersect, as otherwise we could find a shorter path between $u$ and $v$. From construction it follows that $u_k$ lies in $T(p)$ and $u_{k+1}$ lies in $T(q)$ (or $u_{k+1} = r_q$). This is only possible if $u \in R(T(p))$, by definition of $P$, and thus $u = p$.

**Proposition 52.** Let $S^*$ be a good geodetic set of $D$ and $\text{AND}[p, q]$ be an AND gadget where $p$ is the only root of $T(p)$ and $q$ is the only root of $T(q)$. Then, either $\{p, q\} \subseteq S^*$, or $S^*$ contains at least one interval among $\{\alpha(p, q), \gamma(p, q), \delta(p, q)\}$.

**Proof.** Suppose that $S^* \cap \{\alpha(p, q), \gamma(p, q), \delta(p, q)\} = \emptyset$. Let $u_1, v_1, u_2, v_2$ be four vertices of $S^*$ such that $\alpha(p, q) \in I(u_1, v_1)$ and $\delta(p, q) \in I(u_2, v_2)$.

As before, decompose the shortest path $u_1$ and $v_1$ covering $\alpha(p, q)$ into a good shortest path $P$ from $u_1$ to $\alpha(p, q)$ and a good shortest path $P'$ from $\alpha(p, q)$ to $v_1$. Let $u'$ be the neighbour of $\alpha(p, q)$ in $P$ and $v'$ be the neighbour of $\alpha(p, q)$ in $P'$. Observe that $u'$ and $v'$ are not adjacent. There are only two possibilities for this to happen: either $u' \in T(p)$ (and thus $u_1 = p \in S^*$ by definition of $P$), or $u' = \delta(p, q)$ and $v_1 = \gamma(p, q)$, which cannot happen by assumption. Hence, $u_1 = p \in S^*$.

Using similar arguments we can show that $u_2 = q$.

**Proposition 53.** Let $S^*$ be a good geodetic set of $D$ and let $C_i = (\ell_1^i, \ell_2^i, \ell_3^i)$ be a clause. Then we have

\[
\begin{align*}
XX:26 & \quad \text{Algorithms and complexity for geodetic sets on planar and chordal graphs} \quad \text{Algorithms and complexity for geodetic sets on planar and chordal graphs}
\end{align*}
\]
The same holds when \( S \) is \( \alpha \)-MGS. If \( a_i \notin S^* \) then we need at least one vertex among \( \alpha (a_i, \ell^1_i) \), \( \alpha (a_i, \ell^2_i) \), \( \delta (a_i, \ell^1_i) \), \( \delta (a_i, \ell^2_i) \). The same holds when \( a_i \notin S^* \), we need at least one among \( \alpha (\ell^1_i, \ell^3_i) \), \( \alpha (\ell^1_i, \ell^2_i) \), \( \alpha (\ell^2_i, \ell^3_i) \), \( \delta (\ell^1_i, \ell^3_i) \). This implies that (a) holds. Similar arguments can be made for (b) and (c) and are therefore omitted.

** Proposition 54. ** Let \( S^* \) be a good geodetic set of \( D \) and \( C_i = (\ell^1_i, \ell^2_i, \ell^3_i) \) be a clause. If none of \( \ell^1_i, \ell^2_i, \ell^3_i \) is in \( S^* \) then \( |S^* \cap C_i| \geq 7 \).

** Proof. ** Since \( \ell^1_i \notin S^* \), due to Proposition 52 we have that at least one among \( \{ \alpha (a_i, \ell^1_i), \gamma (a_i, \ell^1_i), \delta (a_i, \ell^1_i) \} \) lies in \( S^* \). If \( a_i \notin S^* \), then \( a_i \in S^* \) and one more interval from \( \{ \alpha (\ell^1_i, \ell^3_i), \gamma (\ell^1_i, \ell^3_i), \delta (\ell^1_i, \ell^3_i) \} \) lies in \( S^* \) (Propositions 51 and 52). Now, using Proposition 53 we have the claimed statement. Now, assume that \( a_i \in S^* \). Arguing similarly as above, we can show that \( \ell^2_i \) and \( \ell^3_i \) also lie in \( S^* \). Moreover, \( S^* \) contains one interval from each of AND \( \{ a_i, \ell^1_i \} \), \( \{ b_i, \ell^2_i \} \) and \( \{ c_i, \ell^3_i \} \). Now, we will be done by showing that at least one of \( \{ a_i, b_i, c_i, \text{cov}_i \} \) must be in \( S^* \). Suppose that \( \text{cov}_i \notin S^* \) and let \( \text{cov}_i \in I(u, v) \), where \( u, v \in S^* \) and \( \min(u) < \min(v) \). Let \( P = u, u_1, \ldots, u_k, \text{cov}_i \) and \( P' = \text{cov}_i, u_{k+1}, \ldots, v \) be good shortest paths. Note that \( P, P' \) is a shortest path between \( u \) and \( v \) that covers \( \text{cov}_i \). Hence, \( u_k \) and \( u_{k+1} \) do not intersect. By construction, it follows that \( u_k \) lies in \( T(z) \), where \( z \in \{ a_i, b_i, c_i \} \). This completes the proof.

** Lemma 55. ** If there is a good geodetic set of \( D \) with cardinality \( 4 + 7n + 58m \), then \( F \) is satisfiable.

** Proof. ** Let \( S^* \) be a good geodetic set of \( D \) with cardinality \( 4 + 7n + 58m \). Recall that a variable gadget \( X_i = \text{IMP}[\neg T \rightarrow x_i] \cup \text{IMP}[\neg x_i \rightarrow \overline{x_i}] \). Due to Proposition 51, we know that at least one among \( \{ x_i, \overline{x_i} \} \) lies in \( S^* \). Let \( S_1 = S^* \cap \bigcup_{1 \leq i \leq n} X_i \) and \( S_2 = S^* \cap \left( \bigcup_{1 \leq i \leq m} \ell^1_i \right) \).

Note that \( S_1 \cup S_2 \cup S_3 \subset S^* \). We have \( |S_1| \geq n \), \( |S_2| \geq 6m \) by Proposition 53 and \( |S_3| = 4 + 6n + 52m \). Therefore, \( |S_1| = n \) as \( |S^*| \leq 4 + 7n + 58m \). This means that for each \( 1 \leq i \leq n \), exactly one of \( x_i, \overline{x_i} \) lies in \( S^* \). Based on these, we define the following truth assignment \( \phi : \{ x_1, \ldots, x_n \} \rightarrow \{ 1, 0 \} \) of \( F \). Define \( \phi(x_i) = 1 \) if \( x_i \in S^* \) and \( \phi(x_i) = 0 \), otherwise. Using Proposition 53 we can infer that for each \( 1 \leq i \leq n \), we have that \( |S^* \cap C_i| = 6 \). Due to Proposition 51 at least one of the intervals \( \ell^1_i, \ell^2_i, \ell^3_i \) lies in \( S^* \). Thus, for at least one literal \( \ell^1_i \), we have that \( \phi(\ell^1_i) = 1 \), as needed.

6 Conclusion

We gave a polynomial-time algorithm for MGS on solid grids and proved that MGS is NP-hard on partial grids. Is there a constant factor approximation algorithm for MGS on partial grids, or more generally, planar graphs? We proved that MGS is NP-hard on interval graphs and is FPT on chordal graphs when parameterized by its clique number. Are there constant factor approximation algorithms for MGS on interval graphs and chordal...
graphs? Are there FPT algorithms for MGS on interval graphs and chordal graphs when parameterized by the geodetic number? Assuming the Exponential Time Hypothesis, our reduction implies that there cannot be a $2^{o(\sqrt{n})}$ time algorithm for MGS on interval graphs of order $n$. Are there subexponential time algorithms for MGS on interval graphs or chordal graphs? (This is the case for many graph problems for geometric intersection graphs, see [4].)

We have seen that for every fixed $k$, MGS is polynomial-time for $k$-trees, and this is unlikely to be the case for partial $k$-trees by [18]. However, the case $k = 2$ (i.e. graphs of tree-width 2) remains open.

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