DIMENSION INVARIANTS FOR GROUPS SATISFYING PROPERTIES (M) AND (NM)

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Abstract. We prove that the Bredon cohomological dimension and the virtual cohomological dimension coincide for groups that admit a cocompact model for $EG$ and satisfy properties (M) and (NM). Among the examples of groups satisfying these hypothesis are cocompact and arithmetic Fuchsian groups, one relator groups, the Hilbert modular group and 3-manifold groups.

1. Introduction

For a group $G$ consider the following properties:

(M) Every non-trivial finite subgroup of $G$ is contained in a unique maximal finite subgroup of $G$.

(NM) If $M$ is a non-trivial maximal finite subgroup of $G$ then $N_G(M) = M$, where $N_G(M)$ denotes the normalizer of $M$ in $G$.

In this paper $\mathcal{F}$ will always denote the family of finite subgroups of $G$. A model for the classifying space $EG$ (usually denoted $EG$), is a $G$-CW-complex $X$ such that every isotropy group is finite and the fixed point set $X^H$ is contractible for every finite subgroup $H$ of $G$. Equivalently, $X$ is a model for $EG$ if for every $G$-CW-complex $Y$ with finite isotropy groups, there exists a map, unique up to $G$-homotopy, $Y \to X$. In particular any two models for $EG$ are $G$-homotopy equivalent. We say $G$ is of type $\mathcal{F}$ if $G$ admits a cocompact model for $EG$.

The orbit category $\mathcal{O}_\mathcal{F}G$ of $G$ with respect to $\mathcal{F}$ is the category of homogeneous $G$-sets $G/F$, where $F \in \mathcal{F}$, and morphisms are given by $G$-maps. A Bredon module is a contravariant functor from $\mathcal{O}_\mathcal{F}G$ to the category of abelian groups, and a morphism of Bredon modules is a natural transformation. The category of Bredon modules, for $G$ and $\mathcal{F}$ choosen, is abelian with enough projectives. The Bredon cohomological dimension (or proper cohomological dimension) $\text{cd}(G)$ of $G$ is the length of the shortest projective resolution of the constant module $\mathbb{Z}_\mathcal{F}$. If $G$ is torsion free then $\text{cd}(G)$ is the classical cohomological dimension $cd(G)$ of $G$.

On the other hand, provided that $G$ is a virtually torsion free group, the virtual cohomological dimension $\text{vcd}(G)$ of $G$ is, by definition, the cohomological dimension of a finite index torsion free group. By a well-known theorem due to Serre $\text{vcd}(G)$ does not depend on the finite index subgroup of $G$ that we choose, hence $\text{vcd}(G)$ is a well-defined invariant of $G$.

For every group $G$ we have the following inequality $\text{vcd}(G) \leq \text{cd}(G)$. This inequality can be strict due to examples constructed in [LN03, LP17, DS17].

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factors [Lac19], and groups acting chamber transitively on a Euclidean building [DMP16].

In this note we prove \( \text{vcd}(G) = \text{cd}(G) \) for groups of type \( F \) that satisfy properties (M) and (NM). This implies, due to the main theorem of [LM00], the existence of a cocompact model for \( E_G \) of dimension max\( \{3, \text{vcd}(G)\} \). Among the examples of groups that satisfy these properties we have: cocompact Fuchsian groups and arithmetic Fuchsian groups, one relator groups, the Hilbert modular group, and 3-manifold groups. It is worth noticing that the class of groups for which our main theorem applies is closed under taking free products, this is a particular case of Theorem 3.1.

The proof is very short and relies on [ADMPS17, Corollary 3.4] (Theorem 2.1 below), which reduces the proof of the main theorem (Theorem 2.5) to computing the dimension of certain fixed point sets of a cocompact \( X \) model for \( E_G \). For groups satisfying properties (M) and (NM) we prove that \( X \) can be chosen in such a way that the relevant fixed point sets are one-point spaces (Lemma 2.4), hence they have dimension 0.

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2. Preliminaries and main theorem

We have the following criterion that we will use to prove \( \text{vcd}(G) = \text{cd}(G) \) under the hypothesis of our main theorem.

If \( G \) acts on a space \( X \), and \( A \) is an element of \( G \), we denote \( X^A \) the set consisting of all elements of \( X \) fixed by \( A \). In a similar way we define \( X^H \) for a subgroup \( H \) of \( G \).

**Theorem 2.1.** [ADMPS17 Corollary 3.4] Let \( G \) be a virtually torsion-free group of type \( F \) with a cocompact model \( X \) for \( E_G \). Let \( K \) be the kernel of the \( G \)-action on \( X \). If \( \dim(X^A) < \text{vcd}(G) \) for every finite order element \( A \) of \( G \setminus K \), then \( \text{vcd}(G) = \text{cd}(G) \).

**Remark 2.2.** Note that if \( G \) is as in Theorem 2.1 then \( \text{vcd}(G) \) is finite. Let \( H \) be a finite index torsion-free subgroup of \( G \). Then a cocompact model \( X \) for \( E_G \) is, by restriction, a cocompact model for \( E_H = E_H \). Therefore \( \text{vcd}(G) = \text{cd}(H) = \dim(X/H) < \infty \).

Our next lemma characterizes properties (M) and (NM) in terms of the existence of a model for \( E_G \) with small fix point sets. Recall that \( F \) is the family of finite subgroups of \( G \).

**Lemma 2.3.** Let \( G \) be a group. Then the following to conditions are equivalent

1. There exists a model \( X \) for \( E_G \) with the property that \( X^H \) consists of exactly one point for every non-trivial finite subgroup \( H \) of \( G \).
2. Properties (M) and (NM) are true for \( G \).

**Proof.** Assume \( X \) is a model for \( E_G \) such that \( X^H = \{x_H\} \) for every non-trivial finite subgroup \( H \) of \( G \). Let \( K \) be a non-trivial finite subgroup of \( G \). Then \( K \) is contained in the stabilizer \( S \) of \( x_H \) for a unique \( H \in F \). Note that \( S \) is a maximal finite subgroup of \( G \). In fact, if \( S \) is contained in \( F \in F \), then \( F \) must fix a unique point \( y \) of \( X \) since \( X^F \) consists of a point. In particular, \( S \) fixes \( y \). Hence by uniqueness \( F \) and \( S \) fix the same point of \( X \). Therefore \( F \leq S \). This proves that the stabilizer of any \( x_H \) is a finite maximal subgroup of \( G \), for all \( H \in F \). Hence \( G \)
satisfies (M). The normalizer $N_G(S)$ acts on $X^H$, hence $N_G(S) \leq S$. Therefore $G$ satisfies (NM).

Assume that $G$ satisfies (M) and (NM). Let $Y$ by any model for $EG$. Let $I$ be the set of finite maximal subgroups of $G$. Note that $G$ acts on $I$ by conjugation. Moreover, the stabilizer of $M \in I$ is $N_G(M) = M$, since $G$ satisfies (NM). Then, the join $X = Y * I$ is a model for $EG$, where the $G$–action on $X$ is the diagonal action. In fact, $X$ is contractible since it can be seen as the union of copies cones of $Y$ glued all together by their common base. Let $H \in \mathcal{F}$, then $X^H$ consists of the conic point represented by the unique finite maximal subgroup that contains $H$.

Our next lemma tells us that we can collapse down to a point the fixed point sets of any cocompact model for $EG$.

**Lemma 2.4.** Let $G$ be a group of type $\Sigma$ that satisfies properties (M) and (NM). Then $G$ admits a cocompact model $X$ for $EG$ such that $X^H$ consists of exactly one point for every non-trivial finite subgroup $H$ of $G$.

**Proof.** Let $Y$ be any cocompact model for $EG$. Given a point $y \in Y$, we denote by $Gy$ the $G$-orbit of $y$. Denote by $Y_{sing}$ the subspace of $Y$ consisting of points with non-trivial isotropy. Note that $Y_{sing}$ is a $G$-CW-subcomplex of $Y$.

We claim that there exist a finite number of points $y_1, \ldots, y_m$ of $Y$ such that the disjoint union $Gy_1 \sqcup \cdots \sqcup Gy_m$ is a $G$-deformation retract of $Y_{sing}$.

By Lemma 2.3 there is a model $Z$ for $EG$ such that $Z^H$ consists of exactly one point for every non-trivial finite subgroup $H$ of $G$. On the other hand we have unique (up to $G$-homotopy) $G$-maps $f: Y \to Z$ and $g: Z \to Y$ such that $f \circ g$ and $g \circ f$ are $G$-homotopic to the corresponding identity functions. These functions induce $G$-maps $f': Y_{sing} \to Z_{sing}$ and $g': Z_{sing} \to Y_{sing}$, and also, by restriction $f' \circ g'$ and $g' \circ f'$ are $G$-homotopic to the corresponding identity functions. By construction of $Z$, $Z_{sing}$ is of the form $\bigsqcup_{M \in \mathcal{M}} Gz_M$, where $\mathcal{M}$ is the set of representatives of conjugacy classes of maximal finite subgroups of $G$ and the isotropy of $z_M$ is $M$.

Since $G$ admits a cocompact model for $EG$, by [Lie00] Theorem 4.2 $G$ has a finite number of conjugacy classes of finite subgroups. Therefore $Z_{sing} = Gz_1 \sqcup \cdots \sqcup Gz_m$ for certain points $z_1, \ldots, z_m$ of $Z$. Define $y_i = f'(z_i)$ for $i = 1, \ldots, m$. We can conclude that $Gy_1 \sqcup \cdots \sqcup Gy_m$ is a $G$-deformation retract of $Y_{sing}$. Moreover, if $r: Y_{sing} \to Gy_1 \sqcup \cdots \sqcup Gy_m$ is the mentioned retraction, then $r^{-1}(gy_i)$ is contractible and consists of all points $x$ of $Y_{sing}$ such that $G_x \leq G_{gy_i} = gy_i, g^{-1}$. Hence the setwise stabilizer of $r^{-1}(gy_i)$ is $N_G(G_{gy_i}) = G_{gy_i}$.

Define $X$ to be the $G$-CW-complex defined by $Z/\sim$, where $\sim$ is the relation generated by $x \sim y$ if and only if $r(x) = r(y)$. Hence $X$ is $G$-homotopically equivalent to $Z$. Therefore $X$ is a model for $EG$. Clearly $X$ is cocompact and by construction $X^H$ consists of exactly one point if $H$ is a non-trivial finite subgroup of $X$.

Now we are ready to prove our main theorem.

**Theorem 2.5.** Let $G$ be a virtually torsion-free group of type $\Sigma$ that satisfies properties (M) and (NM). Then vcd($G$) = cd($G$).

**Proof.** If $G$ is finite then there is nothing to prove. From now on we assume $G$ is infinite.

By Lemma 2.4 there exists a cocompact model $X$ for $EG$ satisfying that $X^H$ consists of exactly one point for every non-trivial finite subgroup $H$ of $G$. 

On the other hand, since $G$ is infinite and virtually torsion free, we conclude that $vcd(G) > 0$. Hence we have $\dim(X^F) = 0 < vcd(G)$ for every non-trivial finite subgroup $F$ of $G$. Therefore by Theorem 2.4 we have $vcd(G) = cd(G)$. □

3. Examples

Next, we will describe some examples of groups satisfying the hypothesis of Theorem 2.5.

3.1. Groups in the literature.

1. Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ such that $F$ is finite and the conjugation action of $F$ on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$, and $G$ is of type $\mathbb{F}$. Properties (M) and (NM) for this groups are stablished in [DL03]. We do not know if in general these groups are of type $\mathbb{F}$.

2. Cocompact Fuchsian groups and arithmetic Fuchsian groups. Let $G$ be a Fuchsian group, i.e. $G$ acts properly discontinuously and by orientation-preserving isometries on the hyperbolic plane $\mathbb{H}$. A subgroup of $G$ is finite and non-trivial if and only if it fixes a unique point in $\mathbb{H}$. Also any element of infinite order does not fix any point of $\mathbb{H}$ because the action is proper. This implies that $\mathbb{H}$ is a model for $EG$ such that the point set $\mathbb{H}$ consists of one point for every non-trivial finite subgroup $H$ of $G$. Thus by Lemma 2.3 we have that $G$ satisfies properties (M) and (NM). If additionally $G$ acts cocompactly on $\mathbb{H}$, then clearly satisfies the hypothesis of Theorem 2.5. If $G$ is an arithmetic Fuchsian group, then the Borel-Serre bordification of $\mathbb{H}$ is a cocompact model for $EG$ with $X^H$ is a one-point space for $H$ finite non-trivial. For more information about Fuchsian groups see [Fre90].

3. One relator groups admiting a cocompact model for $EG$. Properties (M) and (NM) are verified in [DL03].

4. The Hilbert modular group. A totally real number field $K$ is an algebraic extension of $\mathbb{Q}$ such that all its embeddings $\sigma_i : K \to \mathbb{C}$ have image contained in $\mathbb{R}$. Let $k$ denote a totally real number field of degree $n$ and $O_k$ its ring of integers. The Hilbert modular group is by definition $PSL_2(O_k)$. If $K = \mathbb{Q}$ we recover the classical modular group $PSL_2(\mathbb{Z})$. Properties (M) and (NM) are verified for the Hilbert modular group in [BSSn16, Lemma 4.3]. Since the Hilbert modular group is a lattice in $PSL_2(\mathbb{R}) \times \cdots \times PSL_2(\mathbb{R})$, then it is an arithmetic group acting diagonally in the symmetric space $\mathbb{H} \times \cdots \times \mathbb{H}$. Hence the Borel-Serre bordification again provides a cocompact model for $EG$. See [Fre90] for more information of the Hilbert modular group.

3.2. Groups acting on trees and properties (M) and (NM). Let us quickly recall the notation of graph of groups from [Ser03]. A graph of groups $\mathcal{Y}$ consists of a graph $Y$ (in the sense of Serre), one group $Y_y$ for every edge $y$ of $Y$, one group $Y_P$ for each vertex $P$ of $Y$, and injective homomorphism $Y_e \to Y_P$ of $P$ is a vertex of the edge $e$. Recall that associated to $Y$ we have the fundamental group $\pi_1(\mathcal{Y})$ and the Bass-Serre tree $T$, in such a way that $\pi_1(\mathcal{Y})$ acts on $T$ by simplicial automorphism and the quotient graph is isomorphic to $Y$. Denote by $f : T \to Y$ the quotient projection.

We will be able to construct more examples using the following theorem.

**Theorem 3.1.** Let $\mathcal{Y}$ be a graph of groups in the sense of Serre with compact underlying graph. Assume that the vertex groups are of type $\mathbb{F}$ and satisfy properties (M) and (NM), and assume that the edge groups are torsion free. Then the fundamental group $\pi_1(\mathcal{Y})$ of $Y$ is of type $\mathbb{F}$ and satisfies properties (M) and (NM).
Proof. Let $T$ be the Bass-Serre tree of $Y$. Denote $G = \pi_1(Y)$. Choose cocompact models $X_y$ and $X_P$ for $EY_y$ and $EY_P$ respectively. Then we can construct a model $X$ for $EG$ as follows. Replace each vertex $v$ of $T$ by the corresponding $X_{f(v)}$, and each edge $e$ of $T$ by $X_{p(e)} × [0,1]$. Next if $v$ is a vertex of $e$ glue one one of the $X_{p(e)} × 0$ to $X_v$ using map induced by the homomorphism $X_{f(v)} → X_{p(e)}$ (and $X_{p(e)} × 1 → X_{p(v)}$ where $v'$ is the other vertex of $e$). Hence $X$ inherits a $G$-action and we can easily verify that it is a model for $EG$ (compare with [JLSU] Proposition 4.8]). Moreover, the orbit space $X/G$ can be constructed using a similar construction using instead $Y$, $X_P/Y_P$ and $X_y/Y_y$. Therefore, since $Y$ is compact and each $X_P/Y_P$ and $X_y/Y_y$ are compact, we have that $X/G$ is also compact. This proves that $G$ admits a cocompact model for $EG$.

Assume now that each $X_y$ and $X_P$ are models satisfying the conclusion of Lemma 2.3. Let $H$ be a non-trivial finite subgroup of $G$. Then $H$ cannot fix any edge of $T$, because every edge group of $Y$ is torsion free. But, since $H$ is finite, has to fix one vertex of $T$. Hence $H$ fixes a unique vertex $v$ of $T$. Hence $H$ acts on $X_{p(v)}$, so $H$ fixes a unique point of $X_{p(v)}$. Therefore every non-trivial finite subgroup of $G$ fixes a unique point of $X$, and by Lemma 2.3 we conclude that $G$ satisfies properties (M) and (NM). □

Our final example are 3-manifold groups. For more information about 3-manifold groups, JSJ-decomposition, and the geometrization theorem see [Mor05].

3.3. 3-manifold groups. Let $M$ be a closed, orientable, connected 3-manifold with fundamental group $G$. We claim that $G$ is of type $F$ and satisfies properties (M) and (NM). The prime decomposition $M = N_1 \# \cdots \# N_m$ induces a splitting of $G = G_1 * \cdots * G_m$. By Theorem 3.1 it is enough to prove that each $G_i$ is of type $\overline{\mathbb{F}}$ and satisfies properties (M) and (NM).

From now on assume $M$ is prime. Using the Perelman-Thurston geometrization theorem we can chop off $M$ along tori to obtain pieces that are either hyperbolic or Seifert fibered. This is the so-called JSJ-decomposition. More explicitly, we can find a collection of tori (possibly empty) $T_1, \ldots, T_r$ embedded in $M$ such that (abusing of notation) $M - \bigcup_i T_i$ is a disjoint union of manifolds (with boundary if the collection of tori is not empty) such that each piece is either hyperbolic or Seifert fibered. Hence $G$ is the fundamental group of a graph of groups $Y$ with vertex groups the fundamental groups of Seifert fibered manifolds or hyperbolic manifolds, and edge group isomorphic to $\mathbb{Z}^2$. Again, by Theorem 3.1 it is enough to prove that all vertex groups in $Y$ are of type $\overline{\mathbb{F}}$ and satisfy properties (M) and (NM). If the collection of tori is empty, then $M$ itself is either hyperbolic or Seifert fibered. If $M$ is hyperbolic, then $G$ is torsion free since $M$ is aspherical, thus $G$ satisfies properties (M) and (NM). Additionally the universal cover of $M$ is a cocompact model for $EG$. If $M$ is Seifert fibered, then $M$ is aspherical unless is covered by the three sphere $S^3$ or by $S^2 \times \mathbb{R}$. In the $S^3$ case $G$ is finite, while in the $S^2 \times \mathbb{R}$ case $G$ is either isomorphic to $\mathbb{Z}$ or to the infinite dihedral subgroup $D_{\infty}$. In both cases $G$ satisfies properties (M) and (NM) and is of type $\overline{\mathbb{F}}$. Finally, we have to deal with the case of a non-trivial JSJ-decomposition. In this case we can verify case by case that every hyperbolic and Seifert fibered manifold in the JSJ-decomposition is an aspherical manifolds, and therefore their fundamental groups are torsion free. Hence all vertex and edge groups of $Y$ are torsion free, thus $G = \pi_1(Y)$ is torsion free. Also, we have that $M$ is a cocompact model for $G$.

We can conclude that the fundamental group of every prime manifold is of type $\overline{\mathbb{F}}$ and satisfies properties (M) and (NM). Therefore every 3-manifold group is of type $\overline{\mathbb{F}}$ and satisfies properties (M) and (NM).
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