Wigner Rotations, Bell States, and Lorentz Invariance of Entanglement and von Neumann Entropy

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We compute, for massive particles, the explicit Wigner rotations of one-particle states for arbitrary Lorentz transformations; and the explicit Hermitian generators of the infinite-dimensional unitary representation. For a pair of spin 1/2 particles, Einstein-Podolsky-Rosen-Bell entangled states and their behaviour under the Lorentz group are analysed in the context of quantum field theory. Group theoretical considerations suggest a convenient definition of the Bell states which is slightly different from the conventional assignment. The behaviour of Bell states under arbitrary Lorentz transformations can then be described succinctly. Reduced density matrices applicable to systems of identical particles are defined through Yang’s prescription. The von Neumann entropy of each of the reduced density matrix is Lorentz invariant; and its relevance as a measure of entanglement is discussed, and illustrated with an explicit example. A regularization of the entropy in terms of generalized zeta functions is also suggested.

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I. INTRODUCTION AND OVERVIEW

It can be argued that, aside from theories with infinite number of particle types such as string theory, quantum field theory is the only way to reconcile the principles of quantum mechanics with those of special relativity. In quantum field theory, vanishing correlations for space-like separated operators are ensured; whereas most efforts in quantum computation have so far relied upon non-relativistic quantum mechanics which is not fully compatible with Lorentz invariance and the causal structure of space-time. Recently however, several groups (see, for instance, Refs. [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]) have focused their investigations on relativistic effects in quantum information science. The issues include Lorentz invariance of entanglement, the behaviour of Einstein-Podolsky-Rosen-Bell states in different inertial frames, and possible modifications to the degree of Bell inequality violations for moving observers. These relativistic effects may alter the efficiency of eavesdropper detection in quantum cryptography and compromise the security of quantum protocols. It is also expected that future applications in quantum teleportation, entanglement-enhanced communication, high-precision quantum clock synchronization based on shared entanglement, and quantum-enhanced positioning will also require relativistic treatments of quantum systems; and in particular, the careful analysis of the properties of entangled particles under Lorentz transformations and the construction of meaningful measures of entanglement. In this article, we study the behaviour of Bell states under Lorentz transformations and consider von Neumann entropy as a Lorentz invariant characterization of entanglement.

We compute, for massive particles, the explicit Wigner rotations of one-particle states; and the explicit generators of the unitary representation of the Lorentz group. Unitary representations of the Poincare group acting on physical states are founded upon Wigner’s seminal work; but the explicit expressions of Wigner rotations for massive particles have been computed, with some difficulty by direct matrix multiplication, only for rotations and boosts considered separately. Our derivation, carried out in Section II, is somewhat simpler, and it permits the explicit general result for arbitrary infinitesimal Lorentz transformations to be stated as in Eq. (2.7). Moreover, with the infinitesimal Wigner angle at hand, the explicit infinite-dimensional Hermitian generators of the unitary representation can be worked out, as in Section III. Wigner rotation for a finite general Lorentz transformation is slightly more complicated, but the explicit form is also listed in Appendix A.

As basic entangled states, Bell states figure prominently in the literature on quantum information science and in quantum computational schemes. Their behaviour under Lorentz transformations is therefore of interest and...
important. In this article we focus on Bell states of two spin $\frac{1}{2}$ massive fermions. Under Lorentz transformations, each one-particle state of the entangled pair undergoes an $SU(2)$ Wigner rotation. Since $[SU(2) \times SU(2)]/Z_2$ is isomorphic to $SO(4)$, which contains 3-dimensional rotations $SO(3)$ as a sub-group; in group theoretical terms the entangled system transforms as $2 \otimes 2 = 4 = 1 \oplus 3$, in which the final step denotes the behaviour under $SO(3)$. Thus by forming a 4-vector of $SO(4)$ out of the rotational singlet and triplet Bell states, the complete, and explicit behaviour given the Wigner angle, of these Bell states under Lorentz transformations can be succinctly stated, as in Eq.(4.4).

We therefore advocate for Bell states the convention in Eq.(4.2). Details of the setup are presented in Section IV. In Section V, we introduce reduced density matrices, defined through Yang’s prescription[21], for systems of identical particles; and analyze their behaviour under Lorentz transformations. Without taking Lorentz symmetry into account, quantum correlations and entanglement of identical fermions have also been studied in Refs.[22, 23, 24, 25]. Here we are able to show that the von Neumann entropy of each of the constructed reduced density matrix is Lorentz invariant; and we illustrate the usefulness of the von Neumann entropy as a Lorentz-invariant measure of entanglement with a worked example comparing unentangled and Bell states. In the ensuing subsection, we present a relation between generalized zeta functions and von Neumann entropy which may be useful for the regularization of the entropy of infinite-dimensional density matrices. Further comments and conclusions are presented in the final section.

As far as it is convenient to do so, we shall follow the conventions and normalizations in Weinberg’s tome[1], with $\eta^{\mu\nu} = diag(-1, +1, +1, +1)$. Space-time Lorentz indices are denoted by Greek letters, spatial indices by Latin letters; and summation over repeated discrete index is assumed. Our computations shall concentrate, for convenience, only on the homogeneous Lorentz group since translations can be incorporated rather readily[1] once the behaviour of fields under the homogeneous group has been worked out.

II. WIGNER ROTATIONS

In quantum field theory, one-particle states $|p, s\rangle$ are classified by eigenvalues of the Casimir invariants of the Poincaré group. Any value of the momentum, $p^\mu$, can be reached by Lorentz transformation $L(p)$ on a standard $k^\mu$ for which $p^\mu = L^\mu_0(p)k^\mu$; and the states can be defined as

$$|p, s\rangle = \sqrt{\frac{k^0}{p^0}} U(L(p)) |k, s\rangle,$$

with $P^\mu|p, s\rangle = p^\mu|p, s\rangle$. It follows[see, for instance, Ref.[1]] that the effect of an arbitrary Lorentz transformation $\Lambda$ unitarily implemented as $U(\Lambda)$ on one-particle states is

$$|p, s\rangle' = U(\Lambda)|p, s\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{s'}(W(\Lambda, p)) |\Lambda p, s'\rangle,$$

and

$$W(\Lambda, p) = L^{-1}(\Lambda p)A L(p)$$

is a Wigner transformation which leaves $k^\mu$ invariant[19], and $D(W)$ represents its action on the state (summation convention over the repeated index $s'$ is assumed). The explicit form of $L(p)$ is dependent on the class of the four-momenta. For massive particles, $p^2p_\mu = -m^2 < 0$, and a convenient choice for the standard vector is $k^\mu = (m, 0)$. It is then obvious that the set of Wigner transformations leaving $k^\mu$ unchanged is just the rotation group $SO(3)$. Furthermore, $L(p)$ can then be taken as the pure Lorentz boost

$$L^0_0(p) = \cosh \chi$$
$$L^0_i(p) = L^i_0(p) = \hat{p}_i \sinh \chi$$
$$L^i_j(p) = \delta^i_j + (\cosh \chi - 1)\hat{p}^i\hat{p}_j,$$

with $\tanh \chi = \frac{|p|}{\sqrt{|p|^2 + m^2}}$. In this parametrization, $L(p) = \exp(-i \chi \hat{p} \cdot \hat{K})$ where $K^i = M^{0i}$ is the boost generator[1].

We may rely upon the analytic nature of Lie groups for the computation of the Wigner angle, and the corresponding infinitesimal Wigner rotation can be evaluated by Taylor expansion. Details of the rest of the computations are delegated to Appendix A. The end result is that the infinitesimal Wigner rotation of a massive particle is

$$W(\Lambda, p) = I + \frac{i}{2} \left[ \omega_{ij} - \frac{1}{p^0 + m}(p_i\omega_{j0} - p_j\omega_{i0}) \right] M^{ij}$$

$$= I + i \theta_W \cdot J,$$
with the infinitesimal Wigner angle denoted as

$$\theta_W = \theta - \frac{p \times \tau}{p^0 + m} \equiv \theta + \phi_1. \quad (2.9)$$

This agrees with the results of Ref. [20] when rotations and boosts are considered separately. It should be noted that the generators of the Wigner rotations are indeed $J_i = \frac{1}{2} \epsilon_{ijk} M^{jk}$, but the complete Wigner angle receives contributions from both the boost and rotation parameters, $\tau^i = \omega^i_0$ and $\theta_i = \frac{1}{2} \epsilon_{ijk} \omega^k$ respectively, of $\Lambda(\omega)$. In the absence of boosts, Wigner rotations are degenerate with ordinary rotations i.e. $\theta_W = \theta$. Although arbitrary finite Lorentz transformations of $\Lambda(\omega) = \exp(\frac{1}{2} \omega_{\alpha\beta} M^{\alpha\beta})$ can be evaluated as $\lim_{N \to \infty} [I + \frac{1}{N} \omega_{\alpha\beta} M^{\alpha\beta}]^N$, it is not simple to express the corresponding finite Wigner rotations in closed form via products of infinitesimal rotations, essentially because Wigner rotations are also functions of the momenta. We may however consider a general Lorentz transformation relating two frames as a product of a pure boost in an arbitrary direction, $L(\alpha) = \exp(-\alpha \cdot K)$, followed by an arbitrary rotation $R(\psi)$. Using the multiplication rule for Wigner transformations, (Eq. (4.7) in Appendix A), the closed form Wigner rotation can be expressed as in Eqs. (A16)-(A17). Note also that for the special case of $U(\Lambda) = U(L(p))$ acting on $|k, s\rangle$ the Wigner transformation is $W(L(p), k) = [L^{-1}(L(p)k)]L(p)L(k) = L^{-1}(p)L(p)I = I$, which consistently produces no rotation in spin space, as Eq. (2.1) demands.

### III. Explicit Generators of the Unitary Representation

One-particle states are defined through the action of creation operator $a^\dagger(p, s)$ on the vacuum as $|p, s\rangle = a^\dagger(p, s)|0\rangle$. It follows from Eq. (2.2) that under a Lorentz transformation annihilation and creation operators in quantum field theory behave as

$$a^\dagger(p, s) = U(\Lambda) a^\dagger(p, s) U^\dagger(\Lambda) \quad (3.1)$$

$$= \int d^3p \left[ U^* (p, s', s) (\Lambda) \right] a^\dagger(p', s') \quad (3.2)$$

$$= \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{s's} (W(\Lambda, p)) a^\dagger(p, s') \quad (3.3)$$

with $p^\prime_\Lambda \equiv \Lambda^\prime_\mu p^\mu$; and by taking the adjoint,

$$a(p, s) = U(\Lambda) a(p, s) U^\dagger(\Lambda) \quad (3.4)$$

$$= \int d^3p \left[ U(p, s)(p', s') (\Lambda) \right] a(p', s') \quad (3.5)$$

$$= \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{s's'} (W^{-1}(\Lambda, p)) a(p, s'). \quad (3.6)$$

We may hence deduce from the last equation that

$$U(p, s)(p', s') (\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{s's'} (W^{-1}(\Lambda, p)) \delta(p' - p_\Lambda), \quad (3.7)$$

and verify by direct calculation, using $D^\dagger(W) = [D(W)]^{-1} = D(W^{-1})$, that the transformation $U(\Lambda)$ is indeed unitary i.e.

$$\int d^3p'' U(p, s)(p''', s''')(\Lambda) U^\dagger(p', s')(p'', s') (\Lambda) = \delta(p - p') \delta_{ss'}. \quad (3.8)$$

But with the formula of the infinitesimal Wigner rotation of Eq. (2.7) at hand, it is possible to proceed even further, to obtain the explicit generators. By considering an infinitesimal transformation with $U(\Lambda) = I + \frac{1}{2} \omega_{\mu\nu} M^{\mu\nu}$ we may
express Eq. 3.1 as

\[ U(\Lambda) a(p, s) U(\Lambda)^\dagger = a(p, s) + \frac{i}{2} \omega_{\mu\nu} [M^{\mu\nu}, a(p, s)] \]  

(3.9)

\[ = (1 + \omega_0 \frac{p^\mu}{2p^0}) \left( I - i [\sigma^i + \omega_0 k^i J^i p^\mu + \frac{p^\mu}{2i p^0}] \right) a(p, s) \]  

(3.10)

\[ = (1 - i \omega_0 \left( \frac{\epsilon_{ijk} J^j p^k}{p^0 + m} + V_{n} \frac{\partial}{\partial p^j} + \frac{p^j}{2ip^0} \right)) \frac{\omega_{ijk}}{2} (J^k + \epsilon^{klm} p^l \frac{\partial}{\partial p^m}) a(p, s) \]  

(3.11)

\[ = a(p, s) - \frac{i}{2} \omega_{\mu\nu} \left( \int d^3 p \tilde{M}^{\mu\nu}_{ss'} a(p, s') \right) \]  

(3.12)

In terms of creation and annihilation operators, the explicit infinite-dimensional Hermitian generators of unitary transformations \( U(\Lambda) = \exp(\frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}) \) for the non-compact Lorentz group are

\[ \epsilon_{ijk} M_{jk} = \int d^3 p a^\dagger(p, s) \left( J^i + \epsilon_{ijk} p^j \frac{\partial}{\partial p^k} \right) a(p, s'), \]  

(3.13)

\[ M^{\mu i} = \int d^3 p a^\dagger(p, s) \left( \frac{\epsilon_{ijk} J^j p^k}{p^0 + m} + V_{n} \frac{\partial}{\partial p^j} + \frac{p^i}{2ip^0} \right) a(p, s'). \]  

(3.14)

In general, we may also introduce the particle species label \( n_i \) for the creation \( a^\dagger(p, s, n_i) \) and annihilation \( a(p, s, n_i) \) operators; and the expression of the generators will then include summing over all \( n_i \).

It can be verified that the explicit generators

\[ \mathbf{J} = \mathbf{J} + \frac{\partial}{\partial \mathbf{p}} \]  

(3.15)

\[ \mathbf{K} = \left( \mathbf{J} \times \frac{\mathbf{p}}{p^0 + m} + p^0 \frac{\partial}{\partial \mathbf{p}} + \frac{\mathbf{p}}{2ip^0} \right), \]  

(3.16)

do satisfy the commutation relations of the Lie algebra of the Lorentz group:

\[ [J^i, J^j] = i \epsilon^{ijk} J^k, \quad [J^i, K^j] = i \epsilon^{ijk} K^k, \quad [K^i, K^j] = -i \epsilon^{ijk} J^k. \]  

(3.17)

Likewise \( \tilde{M}^{\mu\nu} \) of Eqs. (3.13) and (3.14) obey the similar commutation relations. The expression of the boost generator of Ref. 1 (following Ref. 21) and Ref. 20 which differs from ours in not having the final term of Eq. 3.17 can be rendered Hermitian provided the measure \( d^3p/p^0 \) is adopted instead of Weinberg’s.

IV. BELL STATES AND LORENTZ TRANSFORMATIONS

In this section we specialize to the case with \( \mathbf{J} = \sigma^i/2 \) for spin \( \frac{1}{2} \) particles. For each massive spin \( \frac{1}{2} \) particle, the Lie group of all Wigner rotations is \( SU(2) \). A two-particle state, and in particular an entangled Bell pair, should transform according to the \( SU(2) \times SU(2) \) representation. As we shall see, the isomorphism between \( [SU(2) \times SU(2)]/Z_2 \) and \( SO(4) \) permits us to describe the behaviour of the four basis Bell states of spin space for fixed momenta under Lorentz transformations succinctly. Since \( SO(3) \) is a subgroup of \( SO(4) \), in group theoretical terms if we denote the spin \( \frac{1}{2} \) doublet as the \( 2 \) of \( SU(2) \), then a 2-particle state behaves as

\[ 2 \otimes 2 = 4 = 1 + 3 \]  

(3.18)

where the final step denotes its behaviour under \( SO(3) \). In other words, we may express the two-particle state in terms of a four-vector (the \( 4 \) of \( SO(4) \) which transforms as singlet and triplet states under \( SO(3) \). Indeed it is known the four Bell states are expressible as a singlet and a triplet under ordinary 3-dimensional \( SO(3) \) rotations, and as we shall show, they undergo \( SO(4) \) Wigner rotations among themselves under Lorentz transformations.

The quadruplet \((\mu = 0, 1, 2, 3)\) of Bell states can be conveniently defined in the following manner:

\[ |B^\mu(p_1, p_2)\rangle = \frac{1}{\sqrt{2}} (\bar{a}^\mu \sigma^2)_{ss'} a^\dagger(p_1, s; n_1) a^\dagger(p_2, s'; n_2) |0\rangle \]  

(3.19)
with $\tilde{\sigma}^0 = i\mathbb{I}_2$ and $\tilde{\sigma}^i(i = 1, 2, 3)$ being the Pauli matrices $\sigma^i$. In discussing an entangled pair of identical as well as distinguishable particles, the additional species labels, $n_{1,2}$, can be introduced for generality. Let us proceed to show that the states defined above are indeed Bell states. The spin indices $s, s'$ are summed over $\pm \frac{1}{2}$ (which we shall denote as $\pm$ for simplicity). Focusing on the spin part of the states, (and ignoring for the moment normalization factors and species labels), it can be checked that the quadruplet is simply

\[ |B^0\rangle \propto |+, 1; -1, 2\rangle - |-, 1; +1, 2\rangle \]
\[ |B^1\rangle \propto i(|+, 1; +1, 2\rangle - |-, 1; -1, 2\rangle) \]
\[ |B^2\rangle \propto |+, 1; +1, 2\rangle + |-, 1; -1, 2\rangle \]
\[ |B^3\rangle \propto -i(|+, 1; +1, 2\rangle + |-, 1; -1, 2\rangle). \]

For ease of comparison we have also simplified the momenta indices $p_{1,2}$ to 1, 2 respectively. Therefore, apart from multiplicative constants, these are, respectively, the familiar “singlet” ($|B^0\rangle$) and “triplet” ($|B^{1,2,3}\rangle$) Bell states encountered in non-relativistic quantum information science. The conventional assignment of the four Bell states is related to the present one by $|\beta_{11}\rangle = |B^0\rangle, |\beta_{10}\rangle = -i|B^1\rangle, |\beta_{00}\rangle = |B^2\rangle, |\beta_{01}\rangle = i|B^3\rangle$. But we would like to advocate the convention in Eq.(4.2) because it is the particular combination of ($\tilde{\sigma}^u\sigma^v$)$_{ss'}a^i(p_1, s; n_1)a^i(p_2, s'; n_2)|0\rangle$ which provides us with complete and concise description of the behaviour of Bell states under arbitrary Lorentz transformations. The upshot is that under generic $\Lambda$, these Bell states undergo a rotation among themselves, and transform as

\[ |B^\mu(p_1, p_2)\rangle' = U(\Lambda)|B^\mu(p_1, p_2)\rangle \]
\[ \equiv \sqrt{(\Lambda p_1)^0(\Lambda p_2)^0 \over p_1^0 p_2^0} \quad R_{\nu} |\Lambda(p_1, p_2)\rangle |B^\nu(p_1, p_2)\rangle, \]

for which $R_{\nu} |\Lambda(p_1, p_2)\rangle \in SO(4)$ is explicitly listed in Appendix B.

**A. Bell states, and the isomorphism between $[SU(2) \times SU(2)]/Z_2$ and $SO(4)$**

We shall briefly recap the group isomorphism to establish the notations, and to explain our line of reasoning. Let us label elements of the two distinct $SU(2)$ groups by $U_{1,2}(\Lambda)$. For our purposes, and following the analysis of Wigner rotations performed earlier, we should explicitly use

\[ U_1(\Lambda) = \exp(i{\sigma \over 2} \cdot \theta_W(p_1)) \]
\[ U_2(\Lambda) = \exp(i{\sigma \over 2} \cdot \theta_W(p_2)), \]

for which $\theta_W(p_{1,2})$ are the Wigner angles for particles with momenta $p_{1,2}$. To set up the group isomorphism, we may consider

\[ X \equiv x_\mu \tilde{\sigma}^\mu = \left( \begin{array}{ccc} ix_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & ix_0 - x_3 \end{array} \right) = \left( \begin{array}{ccc} v & w \\ w^* & -v^* \end{array} \right). \]

Given $U_{1,2} \in SU(2)$, it can be verified that $X' \equiv U_1 X U_2^{-1}$ is also of the form

\[ X' = \left( \begin{array}{ccc} v' & w' \\ w'^* & -v'^* \end{array} \right). \]

It follows that we may write $X' = x'_\mu \tilde{\sigma}^\mu$. Since $X = -x_\mu x_\mu = \det X = -x'_\mu x'_\mu$, this implies for each $x_\mu$ there is an $R_{\mu} \in SO(4)$ such that $x'_\mu = R_{\mu} x_\mu$. Hence we infer

\[ X' = U_1(x_\mu \tilde{\sigma}^\mu)U_2^{-1} = R_{\mu} x_\mu \tilde{\sigma}^\mu, \]

yielding the identity

\[ U_1 \tilde{\sigma}^\mu U_2^{-1} = R_{\mu} \tilde{\sigma}^\nu. \]

Returning to one-particle states, the Lorentz transformation for each party of the entangled pair is (disregarding, for the moment, normalization factors which are not relevant to the discussion below)

\[ |p_{1, s, n_1}\rangle \rightarrow U_1(\Lambda)_{s's}|p_{1\Lambda, s', n_1}\rangle \]
\[ |p_{2, s, n_2}\rangle \rightarrow U_2(\Lambda)_{s'n}|p_{2\Lambda, s', n_2}\rangle. \]
The crucial observation is that the quadruplet \((\mu = 0, 1, 2, 3)\) of states defined by \((\hat{\sigma}^\mu \sigma^2)_{ss'}|p_1, s, n_1; p_2, s', n_2\rangle\) transforms as

\[
(\hat{\sigma}^\mu \sigma^2)_{ss'}|p_1, s, n_1; p_2, s', n_2\rangle \rightarrow [U_1(\hat{\sigma}^\mu \sigma^2)U_2^\dagger]_{ss'}|p_{1\Lambda}, s, n_1; p_{2\Lambda}, s', n_2\rangle = R^\mu_{ss'}(\hat{\sigma}^\nu \sigma^2)_{ss'}|p_{1\Lambda}, s, n_1; p_{2\Lambda}, s', n_2\rangle.
\]

In arriving at the last result we have used \(\sigma^2U_2^{-1}U_2^{-1}\) which follows from \(\sigma^2\sigma^2 = -\sigma^2\); as well as the identity of Eq. (4.13) in the final step. As a consequence, Eq. (4.14) is therefore valid. Furthermore Eq. (4.13) also yields the relation

\[
R^\mu_{ss'}(\Lambda) = \frac{1}{2} \eta_{\alpha\beta} Tr[U_1(\Lambda)\hat{\sigma}^\nu U_2^{-1}(\Lambda)\hat{\sigma}^\alpha] \tag{4.12}
\]

In passing we mention two special cases: For Lorentz transformations which are pure rotations, \((X = -Y = \theta)\) as \(\phi(p_{1,2}) = 0\); the explicit form of the \(SO(4)\) matrix in Appendix B then yields

\[
R^\mu_{ss'} = \begin{pmatrix}
1 & 0 \\
0 & \cos \theta \delta_{ij} + \epsilon_{ijk} (\sin \theta) \hat{\sigma}^k + (1 - \cos \theta) \hat{\sigma}^i \hat{\sigma}^j
\end{pmatrix},
\]

which means that, as expected, the three Bell states \(|B^{1,2,3}(p_1, p_2)\rangle\) form an \(SO(3)\) rotation triplet while \(|B^0(p_1, p_2)\rangle\) is a singlet. For an equal-mass entangled pair in the center-of-momentum (COM) frame \((p_1 + p_2 = 0)\) and pure boost in the perpendicular direction \((\mathbf{r} \cdot p_{1,2} = 0)\), Eqs. (A9)-(A13) of Appendix A reveal that the resultant Wigner rotations are related by \(\phi(\mathbf{r}, p_1) = -\phi(\mathbf{r}, p_2) \equiv \phi\). Thus with \((X = Y = \phi)\) the result is

\[
R^\mu_{ss'} = \begin{pmatrix}
\cos \phi & (\sin \phi) \hat{\sigma}^i \\
-(\sin \phi) \hat{\sigma}^i & \delta_{ij} + (\cos \phi - 1) \hat{\sigma}^j \hat{\sigma}^i
\end{pmatrix}.
\]

In general, rotational singlet and triplet states do not belong to invariant subspaces when boost transformations are also included. Special cases of the behaviour of Bell states for massive particles under Lorentz transformations have also been calculated in Refs. [2, 6, 9]. As we have shown the precise and complete behaviour under arbitrary Lorentz transformations can be succinctly stated, as in Eq. (4.12).

### B. Two-particle states as superpositions of Bell states

In the previous sections we discussed Bell states with infinitely sharp momenta, but it is possible to generalize the discussion to generic superpositions of Bell states

\[
|\Psi\rangle = \int d^3p_1 \int d^3p_2 C_\mu(p_1, n_1; p_2, n_2) |B^\mu(p_1, n_1; p_2, n_2)\rangle. \tag{4.13}
\]

Moreover, it is actually possible to think of any two-particle state, entangled or otherwise, of the form

\[
|\Psi\rangle = \int d^3p_1 \int d^3p_2 f(p_1, s_1, n_1; p_2, s_2, n_2) a^\dagger(p_1, s_1, n_1) a^\dagger(p_2, s_2, n_2) |0\rangle
\]

in terms of Bell states. Clearly a relation between the coefficients given by

\[
f(p_1, s_1, n_1; p_2, s_2, n_2) = C_\mu(p_1, n_1; p_2, n_2)(\hat{\sigma}^\mu \sigma^2)_{s_1s_2}\tag{4.14}
\]

works. The relation is invertible as

\[
C_\mu(p_1, n_1; p_2, n_2) = \frac{1}{2} \eta_{\mu\nu} f(p_1, s_1, n_1; p_2, s_2, n_2)(\sigma^2 \hat{\sigma}^\nu)_{s_1s_2} \tag{4.15}
\]

Thus \(C_\mu(p_1, n_1; p_2, n_2)\) can be written down given \(f(p_1, s_1, n_1; p_2, s_2, n_2)\), and vice versa. Note however that in quantum field theory all states transform unitarily \((|\Psi\rangle = U(\Lambda)|\Psi\rangle)\) under Lorentz transformations, no matter how complicated the superposition is. The coefficients \(f(p_1, s_1, n_1; p_2, s_2, n_2)\) and \(C_\mu(p_1, n_1; p_2, n_2)\) are not operator-valued, and commute with \(U(\Lambda)\).
V. REDUCED DENSITY MATRICES, IDENTICAL PARTICLES, AND LORENTZ-INVARIA NCE OF VON NEUMANN ENTROPY

We next introduce reduced density matrices and their properties. While it is possible to generalize, we shall choose to concentrate on systems with identical spin $\frac{1}{2}$ massive fermions e.g. electrons.

Given an $N$-particle system of identical particles with density matrix $\rho$ (which is not restricted to a pure state density matrix, but may also correspond to a mixed configuration $\text{Tr} \rho^2 \neq \text{Tr} \rho$), the $m$-particle $(m < N)$ reduced density matrices can be defined as

$$\rho_m \equiv \frac{1}{(m!)^2} |i_1i_2...i_m \rangle \langle j_1j_2...j_m| \text{Tr} \{ a_{i_1} a_{i_2}...a_{i_m} \rho a_{j_{m-1}...j_1}^{\dagger} \} (j_1j_2...j_m|.$$  \hfill (5.1)

This is equivalent to Yang’s definition\textsuperscript{[21]}. Note that $\rho_n$ is an $m$-particle operator; and we have simplified all the quantum numbers of the creation operator $a_{i_k}^{\dagger}$ to the label $i_k$. It can then be worked out that Eq. (5.1) implies

$$\frac{\langle i_1...i_m| \rho_m |j_1...j_m \rangle}{m!(\text{Tr} \rho_m)} = \frac{\langle i_1...i_m k_{m+1}...k_n | \rho_n |j_1...j_m k_{m+1}...k_n \rangle}{n!(\text{Tr} \rho_n)}, \quad \forall m < n, \quad 1 < n \leq N,$$ \hfill (5.2)

with

$$\frac{\rho_n}{\text{Tr} \rho_n} = \frac{\rho}{\text{Tr} \rho}.$$ \hfill (5.3)

Thus these reduced density matrices are defined by the partial traces of higher particle number density operators.

It is worth emphasizing in quantum field theory Lorentz transformations are implemented \textit{unitarily} on \textit{physical states}. Under any Lorentz transformation $\Lambda$, all creation and annihilation operators transform as $U(\Lambda) a_i^{\dagger} U(\Lambda)^{\dagger}$ and $U(\Lambda) a_i U(\Lambda)^{\dagger}$. It follows that (we also assume a Lorentz invariant vacuum $U(\Lambda)|0\rangle = |0\rangle$) all states obtained through the action of creation and annihilation operators on the vacuum must transform unitarily as $|\Psi\rangle' = U(\Lambda)|\Psi\rangle$ and $\langle \Psi\rangle' = \langle \Psi| U(\Lambda)^{\dagger}$. As a consequence, under Lorentz transformations all reduced density matrices defined above also transform unitarily as $\rho' = U(\Lambda) \rho U(\Lambda)^{\dagger}$.

The von Neumann entropy of a density matrix is

$$S = -\text{Tr} (\rho \ln \rho) = - \sum_{n=1}^{N} \lambda_n \ln \lambda_n.$$ \hfill (5.4)

with $\lambda_n$ being the eigenvalues of $\rho$ (we assume normalization of $\text{Tr} \rho = 1$ has been carried out in the definition of the von Neumann entropy). Since $\rho$ is Hermitian it can be diagonalized, and we may write $\rho = V |\text{diag}(\lambda_1, ..., \lambda_N)\rangle V^{\dagger}$. It follows that the eigenvalues are \textit{invariant} under unitary transformations (and, in particular, Lorentz transformations) since $\rho' = U(\Lambda) \rho U(\Lambda)^{\dagger} = U(\Lambda) V |\text{diag}(\lambda_1, ..., \lambda_N)\rangle V^{\dagger} U(\Lambda)^{\dagger}$ obviously has the same eigenvalues as $\rho$. Thus the von Neumann entropy is \textit{Lorentz invariant}. Moreover the von Neumann entropy of all reduced density matrices defined by $S_m = -\text{Tr} (\rho_m \ln \rho_m)$ with $(m = 1, ..., N)$ are also invariant for the same reasons. A physical system defined by $\rho$ can thus be parametrized by a set of Lorentz invariant measures $\{S_1, ..., S_m, ..., S_N = S\}$. We shall proceed to show that $S_m$ can be useful measures of entanglement shortly.

A. Reduced density matrix, Bell states, and Lorentz-invariant entanglement: a worked example

Consider a system of two identical fermions, and for the first part of the illustration let us follow the discussion of Ref.\textsuperscript{[27]}. A two-particle state of identical fermions may be written as

$$|\Psi\rangle = C_{ij} a_i^{\dagger} a_j^{\dagger} |0\rangle \quad i, j = 1, 2, ..., N.$$ \hfill (5.5)

$C_{ij}$ is anti-symmetric and can be set into block diagonal form\textsuperscript{[21]} via

$$U C U^{\dagger} = \bigoplus_{i=1}^{N_f} \left( \begin{array}{cc} 0 & c_i \\ -c_i & 0 \end{array} \right) \quad \text{for } N \text{ odd}$$ \hfill (5.6)

with $N_f \equiv (N/2)$ for even $N$, and for odd $N$, $N_f \equiv (N - 1)/2$; and $U$ is a unitary matrix. Considering a redefinition with $a_i^{\dagger} = U c_i^{\dagger} a_i^{\dagger}$, we may also rewrite

$$|\Psi\rangle = 2 \sum_{i=1}^{N_f} c_i a_i^{\dagger} a_{i+1}^{\dagger} |0\rangle.$$ \hfill (5.7)
This is the analog of “Schmidt decomposition” for identical fermion systems. The total system has density matrix
\[ \rho = |\Psi\rangle\langle\Psi| \] and entropy \( S = 0 \). Following the prescription for density matrix reduction discussed earlier, the one-particle reduced density matrix is then
\[
\langle i|\rho_1|j \rangle = \text{Tr} \{ a_i \rho a_j^\dagger \} = 4 (CC^d)_{ij}.
\]
(5.8)

In terms of \( (UCU^T) \) combination of Eq. (5.10), we note that \( \rho_1 = 4U^\dagger [(UCU^T)(UCU^T)^\dagger](U) \); thus
\[
U \rho_1 U^\dagger = 4 \bigg\{ \bigg[ \sum_{i=1}^{N_f} |c_i|^2 \bigg] \bigg\} \bigg[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bigg] \bigg[ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \bigg].
\]
(5.9)

The normalization condition is
\[
\langle \Psi|\Psi \rangle = 4 \sum_{i=1}^{N_f} |c_i|^2 = 1 \quad \Rightarrow \text{Tr} \rho_1 = 8 \sum_{i=1}^{N_f} |c_i|^2 = 2;
\]
and the von Neumann entropy of the reduced density matrix is therefore
\[
S_1 = -\text{Tr} \left( \frac{\rho_1}{\text{Tr} \rho_1} \ln \frac{\rho_1}{\text{Tr} \rho_1} \right) = -4 \sum_{i=1}^{N_f} |c_i|^2 \ln(2|c_i|^2);
\]
(5.10)

which is bounded by \( \ln 2 \leq S_1 \leq \ln(2N_f) \). The upper limit is obtained by maximizing \( S_1 \) subject to the constraint \( 0 \leq S_1 \leq \ln(2N_f) \); while the lower bound occurs when there is only a single non-vanishing anti-symmetric block in Eq. (5.6).

It may appear disconcerting that this lower bound for fermions is \( \ln 2 \), rather than zero, whereas the analogous result for a system of two bosons is \( 0 \leq S_1 \leq \ln 2 \) (5.10); while the lower bound occurs when there is only a single non-vanishing anti-symmetric block in Eq. (5.6) [23, 25].

Consider instead the Bell state
\[
|\Psi \rangle = |B^0(p_1, p_2)\rangle
\]
(5.16)

yielding the non-vanishing part of the C-matrix as
\[
C \equiv \begin{pmatrix} (p_1, +) & (p_2, -) \\ (p_2, +) & (p_1, -) \end{pmatrix}.
\]
Following earlier derivations, we obtain the normalized one-particle reduced density matrix as
\[
\rho_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix};
\]
(5.15)

giving \( S_1 = -2(\frac{1}{2}) \ln \frac{1}{2} = \ln 2 \), which is the minimum von Neumann entropy of the reduced 1-particle density matrix for two identical fermions. We chose to compute \( C \) explicitly to illustrate the consistency of the approach, but it is not necessary to go through this step. Given \( \rho \), computation of \( \rho_1 \) can also be done directly through Eq. (5.12).

Consider instead the Bell state
\[
|\Psi \rangle = |B^0(p_1, p_2)\rangle
\]
(5.16)

now yielding the non-vanishing part of the C-matrix as
for taming and isolating divergences\cite{29, 30}. Here we briefly touch upon its relation to the von Neumann entropy.

Since its generalized zeta function as a Lorentz invariant any of the reduced density matrices in quantum field theory will be entanglement entropy of value $2 \ln 2$. Moreover, as explained, for any physical system the von Neumann entropy of any of the reduced density matrices in quantum field theory will be Lorentz invariant.

\[ C = \begin{pmatrix} (p_1, +) & (p_2, -) & (p_2, +) & (p_1, -) \\ 0 & \frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{2}} \end{pmatrix} , \]

for which the normalized reduced density matrix is

\[ \rho_1 = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} . \]

The corresponding entropy is now $S_1 = -4(\frac{1}{4} \ln \frac{1}{4}) = 2 \ln 2$ instead. This is a von Neumann entanglement entropy (of the reduced density matrix) which is $\ln 2$ units higher than the lowest value thus justifying that the Bell state $|B_0(p_1, p_2)\rangle$ is an entangled state. Similarly all the Bell states $|B_\mu(p_1, p_2)\rangle$ discussed previously have von Neumann entanglement entropy of value $2 \ln 2$. Moreover, as explained, for any physical system the von Neumann entropy of any of the reduced density matrices in quantum field theory will be Lorentz invariant.

B. Generalized zeta functions and von Neumann Entropy

Zeta function regularizations have been employed in quantum field theory as gauge and Lorentz-invariant methods for taming and isolating divergences\cite{29, 30}. Here we briefly touch upon its relation to the von Neumann entropy.

The generalized zeta function of an operator can be defined to be

\[ \zeta_\hat{O}(s) \equiv \sum_n \frac{1}{\lambda_n^s} \]

where $\lambda_n$ are eigenvalues of the operator $\hat{O}$. There is an interesting relation between generalized zeta function and von Neumann entropy. Recall that a density matrix $\rho$ (or, for this matter, $\rho_m$, a reduced density matrix) has von Neumann entropy

\[ S = -\text{Tr}(\rho \ln \rho) = -\sum_n \lambda_n \ln \lambda_n \]

with $\lambda_n$ being the eigenvalues of $\rho$. Note that in the sum, zero eigenvalues do not pose a problem (by L’Hospital rule $\lambda_n \ln \lambda_n$ has no contribution for zero eigenvalues). However in quantum field theory, without regularization, the entropy can suffer from divergences. For instance, $\rho = e^{-\beta H}/\text{Tr}(e^{-\beta H})$ leads to the thermodynamic relation $S = \beta \langle H \rangle$, and the expectation value is in general divergent in quantum field theory.

If we adopt $\hat{O} = \rho$, then $\zeta_\rho(s) = \sum_n \frac{1}{\lambda_n^s} = \sum_n \exp(-s \ln \lambda_n)$. We note that

\[ \left. \frac{d\zeta_\rho}{ds} \right|_{s=-1} = -\lim_{s=-1} \sum_n \exp(-s \ln \lambda_n) \ln \lambda_n \]

\[ = -\sum_n \lambda_n \ln \lambda_n . \]

Thus it is possible to define $S(s) \equiv \frac{d\zeta_\rho}{ds}(s)$, and analytically continue from values for which it is defined to $s = -1$. However, depending on the form of the operator $\rho$, it may still be that $s = -1$ is a pole of $\zeta_\rho(s)$.

An alternative then is to note the sum for $S$ in Eq.(5.21) does not include zero eigenvalues. Therefore we may substitute $\rho$ with its sub-matrix $\rho'$ which contains no zero eigenvalues. Its inverse, $\rho'^{-1}$, exists; and we may construct its generalized zeta function as

\[ \zeta_{\rho'^{-1}}(s) = \sum_n (\lambda_n)^s . \]

Since $\rho$ is a density matrix, its eigenvalues satisfy $0 \leq \lambda_n \leq 1 \ \forall n$; so for positive $s$, the sum is bounded above by the
dimension of $\rho$, and therefore converges for any finite-dimensional $\rho$. Furthermore

$$S(s)\Big|_{s=1} \equiv -\frac{d\zeta_{\rho^\prime-1}(s)}{ds}\Big|_{s=1} = -\lim_{s\to 1} \sum_n \exp(s \ln \lambda_n) \ln \lambda_n$$

(5.25)

and

$$= -\sum_n \lambda_n \ln \lambda_n,$$

(5.26)

for infinite-dimensional $\rho$ we may also define the von Neumann entropy $S$ by analytic continuation of the zeta function $\zeta_{\rho^\prime-1}(s)$ and $S(s)$ to $s = 1$. A further generalization is

$$S \equiv \frac{1}{\alpha} \frac{d\zeta_{\rho^\prime\alpha}(s)}{ds}\Big|_{s=-1/\alpha},$$

(5.27)

with $\rho^\prime$ substituting for $\rho$ when $\alpha < 0$.

VI. FURTHER COMMENTS AND CONCLUSIONS

Our work links the Wigner rotations of spins to the behaviour of Bell States under arbitrary Lorentz transformations. We discussed reduced density matrices for identical particle systems, established the Lorentz invariance of their von Neumann entropies, and suggested an invariant regularization through generalized zeta function. In addition, we worked out the explicit expressions of the infinite-dimensional Hermitian generators in the momentum representation. We hope that the results presented here will help to place Relativistic Quantum Information Science on the surer foundations of quantum field theory which is fully compatible with Lorentz symmetry and causality. It is worth emphasizing that we considered the full reduced density matrix with all the degrees of freedom, including the momentum with the reduction prescription of Section V when we go from $N$ particles to $m < N$ particles. This is different from the reduced density matrix used, for instance in Ref. [3], in which “reduction” to the spin degree of freedom is applied even to a single particle. The von Neumann entropy is invariant in our case because Lorentz symmetry is unitarily implemented in quantum field theory; each creation operator transforms unitarily under the Lorentz group, hence the prescribed reduction of density matrices in Section V will result in Lorentz-invariant von Neumann entropy for whatever resultant reduced density matrix we have. In contraposition, Lorentz symmetry is not implemented unitarily in non-relativistic treatments of “wavefunctions”.

It is known through earlier efforts by others that for a system of two particles in a total pure state, the von Neumann entropy of the reduced density matrix is a good measure of the entanglement. We demonstrated that such a characterization is in fact Lorentz invariant. Although a system of $n$ particles can be similarly parametrized by the Lorentz-invariant von Neumann entropy each of the reduced density matrices, the characterization of entanglement in terms of these $n$ numbers, and especially in the case when the total system is a mixed configuration rather than a pure state, is still not completely clear and merits further studies.

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APPENDIX A: COMPUTATION OF THE COMPLETE WIGNER ANGLE

We shall first calculate the infinitesimal Wigner angle by Taylor expansion as

$$W(\Lambda, p) = W(\Lambda, p) \left|_{\omega=0} + \frac{\omega_{\alpha\beta}}{2} \frac{dW}{d\omega_{\alpha\beta}} \right|_{\omega=0} + \cdots$$  \hspace{1cm} (A1)

$$= I + \frac{\omega_{\alpha\beta}}{2} \left[ dL^{-1}(\Lambda p) \right]_0 L(p) + \frac{\omega_{\alpha\beta}}{2} L^{-1}(p) \frac{dL(\omega)}{d\omega_{\alpha\beta}}_0 L(p) + \cdots$$  \hspace{1cm} (A2)

$$= 1 + \frac{\omega_{\alpha\beta}}{2} \left[ L^{-1}(\Lambda p) \right]_0 \frac{dL(\Lambda p)}{d\omega_{\alpha\beta}} \equiv \frac{\omega_{\alpha\beta}}{2} L^{-1}(p) M^{\alpha\beta} L(p).$$  \hspace{1cm} (A3)

to first order in $\omega$; with $\Lambda(\omega) = I + \frac{\omega}{2} \omega_{\mu\nu} M^{\mu\nu}$ and $M^{\mu\nu} = \frac{\partial}{\partial p^\nu} + \frac{\omega}{2} (\omega_{\alpha\beta} M^{\alpha\beta})_{\mu}^{\rho} p^{\nu} = \delta^\mu_{\nu} + \omega^\mu_{\nu}$.

Straightforward and careful calculations using the explicit matrix elements,

$$(K^i)_{ab} = i(\delta^i_a \delta_{0\nu} + \delta_{0a} \delta_i^\nu)$$  \hspace{1cm} (A4)

$$(J^i)_{ab} = -\epsilon_{iab},$$  \hspace{1cm} (A5)

doing, Wigner transformations satisfy the multiplication rule

$$W(\Lambda_2 \Lambda_1, p) = W(\Lambda_2, \Lambda_1 p) \cdot W(\Lambda_1, p).$$  \hspace{1cm} (A7)

To compute the Wigner rotations for general Lorentz transformations with 6 independent parameters, we may consider the decomposition $\Lambda = R(\psi) \cdot L(\alpha)$. The multiplication rule leads to

$$W(\Lambda, p) = W(R(\psi), L(\alpha) p) \cdot W(L(\alpha), p).$$  \hspace{1cm} (A8)

Regardless of the momenta, Wigner angles are degenerate with ordinary rotation angles when the boost parameters are zero; thus the first factor is just $\exp(\psi \cdot J)$, while the second factor is the Wigner rotation for an arbitrary pure boost. Exploiting the homomorphism between $SL(2, C)$ and $SO(3, 1)$, the Wigner angle of this remaining factor has been successfully computed by Halpern[31] by direct multiplication of $2 \times 2$ $SL(2, C)$ matrices representing the Lorentz transformations in Eq.(2.4-2.6). We may express the result of Halpern as

$$W(L(\tau), p) \equiv \exp(i\phi(\tau) \cdot J),$$  \hspace{1cm} (A9)

with

$$\cos \phi = \frac{[cosh \tau + cosh \chi + sinh \tau sinh(\hat{\tau} \cdot \hat{p}) + (cosh \tau - 1)(cosh \chi - 1)(\hat{\tau} \cdot \hat{p})^2]}{[1 + cosh \tau cosh \chi + sinh \tau sinh(\hat{\tau} \cdot \hat{p})]}$$  \hspace{1cm} (A10)

$$= \frac{[m cosh \tau + p^0 + sinh(\hat{\tau} \cdot \hat{p}) + (cosh \tau - 1)(p^0 - m)(\hat{\tau} \cdot \hat{p})^2]}{[m + p^0 cosh \tau + sinh(\tau(\hat{\tau} \cdot \hat{p})]}},$$  \hspace{1cm} (A11)

$$sinh \chi = \frac{[sinh \tau sinh \chi + (cosh \tau - 1)(cosh \chi - 1)(\hat{\tau} \cdot \hat{p})]}{[1 + cosh \tau cosh \chi + sinh \tau sinh(\hat{\tau} \cdot \hat{p})]}(\hat{\tau} \times \hat{p})$$  \hspace{1cm} (A12)

$$= \frac{[|p| sinh \tau + (p^0 - m)(cosh \tau - 1)(\hat{\tau} \cdot \hat{p})]}{[m + p^0 cosh \tau + sinh(\tau(\hat{\tau} \cdot \hat{p})]}(\hat{\tau} \times \hat{p});$$  \hspace{1cm} (A13)

and the rapidity $\chi$ is related to $p$ by $sinh \chi = \frac{|p|}{m}, cosh \chi = \frac{p^0}{m}$. It is easy to confirm the infinitesimal limit is indeed

$$\phi(\tau) \to \phi_1 = \frac{p \times \tau}{p^0 + m}.$$  \hspace{1cm} (A14)
The complete expression of the Wigner rotation of (A12) is therefore

\[ W(\Lambda, p) = \exp(i\theta_W \cdot J) = \exp(i\psi \cdot J) \cdot \exp(i\phi(\alpha) \cdot J); \quad (A15) \]

which yields the explicit relations

\[ \cos(\frac{\theta_W}{2}) = (\cos \frac{\psi}{2})(\cos \frac{\phi}{2}) - (\sin \frac{\psi}{2})(\sin \frac{\phi}{2})(\psi \cdot \phi), \]

\[ \sin(\frac{\theta_W}{2}) = (\cos \frac{\psi}{2})\hat{\phi} + (\sin \frac{\psi}{2})(\cos \frac{\phi}{2})\hat{\psi} + (\sin \frac{\psi}{2})(\sin \frac{\phi}{2})(\hat{\phi} \times \hat{\psi}). \]

\[ \text{APPENDIX B: EXPLICIT FORM OF THE ROTATION MATRIX } R_{\mu}^{\nu}(\Lambda, p_1, p_2) \]

The rotation matrix among the four Bell states has been shown to be

\[ R_{\mu}^{\nu} = \frac{1}{2}\eta_{\nu\alpha}\mathrm{Tr}[U_1(\Lambda)\tilde{\sigma}_{\mu}U_2^{\dagger}(\Lambda)\tilde{\sigma}_{\alpha}] \]

\[ = \frac{1}{2}\eta_{\nu\alpha}\mathrm{Tr}[\exp(i\frac{X}{2} \cdot \tilde{\sigma}_{\mu}) \exp(i\frac{Y}{2} \cdot \tilde{\sigma}_{\alpha})]; \quad (B2) \]

with the definitions

\[ X \equiv \theta_W(p_1), \]
\[ Y \equiv -\theta_W(p_2). \]

Straightforward computations yield the explicit matrix elements

\[ R_{0}^{0} = (\cos \frac{X}{2})(\cos \frac{Y}{2}) - \hat{X} \cdot \hat{Y}(\sin \frac{X}{2})(\sin \frac{Y}{2}) \]
\[ R_{i}^{0} = -R_{0}^{i} = -(\cos \frac{X}{2})(\sin \frac{Y}{2})\hat{Y} - (\sin \frac{X}{2})(\cos \frac{Y}{2})\hat{X} + (\sin \frac{X}{2})(\sin \frac{Y}{2})\epsilon_{ijk}\hat{X}_{j}\hat{Y}_{k} \]
\[ R_{ij} = (\cos \frac{X}{2})(\cos \frac{Y}{2})\delta_{ij} - (\cos \frac{X}{2})(\sin \frac{Y}{2})\epsilon_{ijm}\hat{Y}_{m} + (\sin \frac{X}{2})(\cos \frac{Y}{2})\epsilon_{ijm}\hat{X}_{m} + (\sin \frac{X}{2})(\sin \frac{Y}{2})[\hat{X} \cdot \hat{Y}\delta_{ij} - \hat{X}_{i}\hat{Y}_{j} - \hat{X}_{j}\hat{Y}_{i}]. \]

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