Order Effects in Sequential Measurements of Non-Commuting Psychological Observables

H. Atmanspacher\textsuperscript{1,2} and H. Römer\textsuperscript{3}

1 - Institute for Frontier Areas of Psychology and Mental Health, Freiburg, Germany
2 - Collegium Helveticum, Zurich, Switzerland
3 - Physics Department, University of Freiburg, Germany

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Contents

1 Introduction\hspace{1cm} 1
  1.1 Measurement and Non-Commutativity .............................................. 1
  1.2 Non-Commutative Phenomena in Psychology ....................................... 2

2 Probabilities for Sequential Measurements ........................................... 3

3 Types of Order Effects ........................................................................... 5
  3.1 Theoretical Classification of Observed Order Effects ......................... 5
  3.2 Further Options for Future Application .............................................. 6

4 Order Effects in Hilbert Space Representations .................................... 7
  4.1 Sequential Measurements in Hilbert Space Quantum Mechanics ............ 7
  4.2 Complexity of Hilbert Space Models .................................................. 7
  4.3 Joint Measurements ........................................................................... 9

5 Summary ............................................................................................... 10
Abstract

Sequential measurements of non-commuting observables produce order effects that are well-known in quantum physics. But their conceptual basis, a significant measurement interaction, is relevant for far more general situations. We argue that non-commutativity is ubiquitous in psychology where almost every interaction with a mental system changes that system in an uncontrollable fashion. Psychological order effects for sequential measurements are therefore to be expected as a rule. In this paper we focus on the theoretical basis of such effects. We classify several families of order effects theoretically, relate them to psychological observations, and predict effects yet to be discovered empirically. We assess the complexity, related to the predictive power, of particular (Hilbert space) models of order effects and discuss possible limitations of such models.

1 Introduction

1.1 Measurement and Non-Commutativity

From a conceptual point of view, measurements are operations including interactions between a measuring system $M$, e.g. a measuring device, and a measured system $S$. Typical objects of a measurement are properties (sometimes called observables) $A, B, ...$, of system $S$ in a certain state $\rho$. Measurement results for observables $A$ and $B$ are typically quantified by real-valued numbers $a, b, ...$

A physical standard example for this terminology is a material measurement apparatus $M$ recording particular properties of a material system $S$ in state $\rho$ (with some probability $p_i$). In basic physics, observables occur in canonically conjugate pairs, such as position and momentum, exhibiting particular well-defined invariance principles and conservation laws.

In psychology we are concerned with a mental system $S$, usually of an individual subject, being in a state $\rho$. In contrast to physics, there are no canonically conjugate pairs of observables. For instance, a survey questionnaire as a measuring device $M$ can contain any questions one may want to choose as observables, to be measured in terms of answers as measurement results.

If the measurement interaction between $M$ and $S$ is weak, then the action of $M$ leaves no significant effect on $S$. In case of strong interactions, this is no longer the case, and effects of measurements due to $M$ have a non-negligible influence even on a state $\rho$ of $S$. (This brief discussion disregards that measurement affects also the state of the measuring device. This, of course, is intended and mandatory because otherwise there would be no recorded measurement result.)

In physics, the distinction of weak and strong interactions is one major criterion for delineating classical systems from quantum systems. In addition, quantum measurements – in contrast to classical measurements – are not simply registrations of pre-existing facts, but they also "establish" the fact which is registered. The key feature of this classical-quantum distinction is mathematically codified in terms of the commutativity or non-commutativity, respectively, of observables. Referring to measurements as actions of an observable on a state, $A(\rho)$ or $B(\rho)$, these actions either leave the state as it was before measurement, or they change it.
As a consequence, the successive action of two observables on a state can be commutative, $A(B(\rho)) = B(A(\rho))$, or non-commutative, $A(B(\rho)) \neq B(A(\rho))$. In physics, the measured behavior of a system is called *classical* in the former case, whereas the latter case refers to *quantum* behavior. From this perspective, quantum theory can be viewed as a general systems theory of both commuting and non-commuting observables, and classical physics is restricted to the special case of commuting observables alone (cf. Primas 1990).

Whenever the sequence of successive measurement interactions between a measuring device and a measured system makes a difference for the final result, measurements are non-commutative. Many complex systems must be expected to exhibit non-commutative properties in this sense. In psychology, where virtually every interaction of a “measuring” device with a “measured” mental state changes that state uncontrollably and where mental states are often literally established by measurements, it is highly plausible to argue that non-commutativity should be the ubiquitous rule.

### 1.2 Non-Commutative Phenomena in Psychology

Traditionally, psychology has not considered its observations as owing to effects of non-commutative observables throughout its history. But there are quite a number of psychological phenomena showing clear features of such an approach. Some of them, which have been worked out in recent years, are decision (or judgment) processes, semantic associations, bistable perception, learning, and order effects in questionnaires.

It should be noted at this point that the models developed in the mentioned areas are not all explicitly based on non-commuting observables. They feature various related concepts such as incompatibility, complementarity, entanglement, partial Boolean logic, dispersive states, quantum probability, uncertainty relations etc. However, these concepts bear close formal relationships with the property of non-commutativity and its ramifications (cf. Atmanspacher et al. 2002, 2006, Filk and Römer 2011; more technically: Rédei 1996, Rédei and Summers 2007).

A particular one among the psychological phenomena mentioned above is most illustratively addressed in terms of non-commuting operations: the phenomenon of order effects. These effects do so obviously refer to non-commutativity because they are ruled by it almost literally. For two questions (actions of observables) $A$ and $B$, their sequence makes a difference with respect to the answer (state of the system) after the second question:

$$AB \neq BA$$

Psychologists know such effects well, they are also called sequence effects or context effects. An (incomplete) list of textbooks and reviews on the topic, showing its fairly long history, is Sudman and Bradburn (1974), Schumann and Presser (1981), Schwarz and Sudman (1992), Hogarth and Einhorn (1992), Sudman, Bradburn and Schwarz (1996), Tourangeau, Rips, and Rasinski (2000). However, the proposed models for describing order effects have always been framed classically (e.g., Markov processes, Bayesian updating, etc.). Models inspired by quantum theory have been successfully applied only recently.

\footnote{For some more details and references see the review by Atmanspacher (2011), which highlights this line of thinking in Sec. 4.7. Ideas to understand consciousness in terms of actual quantum processes in the brain, also reviewed in the same article, are entirely at variance with the approach applied here.}
The first publication with corresponding indications that we know of has been due to Aerts and Aerts (1995) and essentially derived from a non-Boolean logic of propositions. With the growing success of and attention on non-classical modeling over the last decade, more work has been done along those lines. Most notable with respect to our topic in the present paper are applications to sequential decision making by Pothos and Busemeyer (2009) Busemeyer et al. (2011), Trueblood and Busemeyer (2011), and Wang and Busemeyer (2011), whose work basically utilize quantum probabilities in Hilbert space representations of mental states.

Our own formal framework for addressing non-commuting properties is somewhat more abstract, because the full-fledged formalism of (Hilbert space) quantum mechanics is generally too restrictive for modeling non-commutative situations. There are scenarios in which we cannot even require such basic concepts as a metric (linear or nonlinear) representation space for states, the formulation of dynamical laws, or a definition of probabilities. Of course, some of these may become necessary if specific examples are to be treated and empirical results are to be described. We have shown elsewhere (Atmanspacher et al. 2002, 2006, Filk and Römer 2011) how conventional quantum mechanics can be stepwise recovered if enough relevant contexts are filled into the general framework.

It is to be expected that models of psychological phenomena should be located somewhere in between the most general level and the very specific level of conventional quantum physics. For instance, psychology will arguably not have a universal quantity, such as the Planck constant, specifying the degree of non-commutativity in general. On the other hand, contact with controlled experiments will definitely make it necessary to have well-defined probabilities.

Hence, we begin this article with a brief account of probabilities for the results of sequential measurements of non-commuting observables in Section 2. Section 3 uses this framework for a theoretically framed classification of different types of order effects, some of which have been discussed in the psychological literature; others are predicted here for the first time. Section 4 addresses Hilbert space models of such order effects. It is shown that the complexity of such models, if they fit empirically obtained probabilities, is lower than the number of parameters to be fitted. In other words, the models have considerable predictive power. Finally we give an example for the limits of Hilbert space models of order effects. Section 5 summarizes the results of this article.

2 Probabilities for Sequential Measurements

Let a system be in state $\rho$ before its associated observables $A$ and $B$ are measured. Now consider subsequent measurements of first $A$ and then $B$ in the initial state $\rho$ and denote the probability of measuring the value $b_i$ of $B$ after having measured value $a_j$ of $A$ as the sequential measurement probability (SMP) $w_\rho(b_i \leftarrow a_j)$. If $A$ and $B$ do not commute, $w_\rho(b_i \leftarrow a_j)$ will be different from $w_\rho(a_j \leftarrow b_i)$ (cf. Franco 2009). For the sake of simplicity, we assume $a_j$ and $b_i$ to be discrete.

An individual measurement probability (IMP) $w_\rho(a_j)$ or $w_\rho(b_i)$ is the probability of measuring $a_j$
of \( A \) or \( b_i \) of \( B \) individually, without any prior measurement influence. The SMP is not simply the conditional probability of measuring \( b_i \) given \( a_j \) but refers to subsequent measurements of both \( A \) and \( B \) that actually have been carried out and, thus, may (and generally will) change the state \( \rho \) of the system. This will be discussed in some more detail in Sec. 4.1.

Since \( w \) is a probability (in the sense of Kolmogorov), the sum of \( w_\rho(b_i \leftarrow a_j) \) over all measurements of \( A \) and \( B \) must be normalized:

\[
\sum_{i,j} w_\rho(b_i \leftarrow a_j) = 1 \tag{1}
\]

Moreover, the sum of \( w_\rho(b_i \leftarrow a_j) \) over all measured values \( b_i \) of \( B \) is given by

\[
\sum_i w_\rho(b_i \leftarrow a_j) = w_\rho(a_j), \tag{2}
\]

whereas summing \( w_\rho(b_i \leftarrow a_j) \) over all measured values \( a_j \) of \( A \) generally yields an inequality for non-commuting \( A \) and \( B \):

\[
\sum_j w_\rho(b_i \leftarrow a_j) \neq w_\rho(b_i) \tag{3}
\]

(Of course, \( a_j \) and \( b_i \) in relations (1) to (3) must be exchanged if the sequence of measuring \( A \) and \( B \) is swapped.)

It follows that the expectation value \( \langle B \rangle_\rho \) for measuring \( B \) first (unconditioned) differs from the expectation value \( \langle B \rangle_{a_j,\rho} \) for measuring \( B \) after \( A \) (conditioned on \( a_j \))

\[
\langle B \rangle_{a_j,\rho} = \sum_i b_i w_\rho(b_i \leftarrow a_j) \neq \langle B \rangle_\rho, \tag{4}
\]

which in general also holds for any function \( f(B) \) and its expectation values. Likewise, the expectation values of the variance operators \((\Delta A)^2\) and \((\Delta B)^2\), where the operators \(\Delta A\) and \(\Delta B\) are the deviations from the expectation values of \( A \) and \( B \), differ as well:

\[
\langle (\Delta B)^2 \rangle_{a_j,\rho} = \langle (B - \langle B \rangle_{a_j,\rho})^2 \rangle_{a_j,\rho} \neq \langle (B - \langle B \rangle_\rho)^2 \rangle_\rho = \langle (\Delta B)^2 \rangle_\rho \tag{5}
\]

We can also define a joint sequential expectation value as:

\[
\langle B \leftarrow A \rangle_\rho = \sum_j a_j \langle B \rangle_{a_j,\rho} \tag{6}
\]

In general for instance in quantum mechanics), the expectation value

\[
\langle \Delta A \Delta B \rangle_\rho = \langle (A - \langle A \rangle_\rho)(B - \langle B \rangle_\rho) \rangle_\rho \tag{7}
\]

may not even be real-valued because the product operator \(\Delta A \Delta B\) is not Hermitian if \( A \) and \( B \) do not commute.

Now we define

\[
\delta_\rho(b_i) = \sum_j w_\rho(b_i \leftarrow a_j) - w_\rho(b_i) \quad \tag{8}
\]

\[
\delta_\rho(a_j) = \sum_i w_\rho(a_j \leftarrow b_i) - w_\rho(a_j) \quad \tag{9}
\]

\(^4\)Throughout this paper, we use the notation \( \langle \cdot \rangle \) for expectation values, also known as (disregarding subtle details) mean values of a distribution.
as the differences between sums of SMPs and IMPs for measuring A and B. These differences provide a convenient way to assess the effect of sequential measurements of A and B as compared to individual measurements of A or B alone. It may be mentioned that the transition probabilities for sequential measurements as introduced here represent a special case of Khrennikov's (2009) contextual probabilities.

3 Types of Order Effects

3.1 Theoretical Classification of Observed Order Effects

With the definitions in Eqs. (8,9), we can relate our probabilistic framework to consistency and contrast effects first reported by Schumann and Presser (1981) and to additive and subtractive effects found by Moore (2002). All four types have recently been investigated by Wang and Busemeyer (2011), see also Trueblood and Busemeyer (2011) and Busemeyer et al. (2011).

Additive effects (both SMPs are larger than the corresponding IMPs):

$$\delta_\rho(a_j) > 0 \quad \text{and} \quad \delta_\rho(b_i) > 0$$ (10)

Subtractive effects (both SMPs are smaller than the corresponding IMPs):

$$\delta_\rho(a_j) < 0 \quad \text{and} \quad \delta_\rho(b_i) < 0$$ (11)

Contrast effects (the difference of the two SMPs is larger than the difference of the corresponding IMPs):

$$\bigg| \sum_i w_\rho(a_j \leftarrow b_i) - \sum_j w_\rho(b_i \leftarrow a_j) \bigg| > |w_\rho(a_j) - w_\rho(b_i)|$$ (12)

Consistency effects (the difference of the two SMPs is smaller than the difference of the corresponding IMPs):

$$\bigg| \sum_i w_\rho(a_j \leftarrow b_i) - \sum_j w_\rho(b_i \leftarrow a_j) \bigg| < |w_\rho(a_j) - w_\rho(b_i)|$$ (13)

Contrast and consistency effects apply in particular to situations where $\delta_\rho(a_j)$ and $\delta_\rho(b_i)$ have different signs, $\delta_\rho(a_j)\delta_\rho(b_i) < 0$. In this case, additive or subtractive effects are not candidates anyway since there the product is always positive. Of course, all inequalities turn into equalities if A and B commute.

For a comprehensive discussion of pertinent examples of these four classes of order effects see Wang and Busemeyer (2011). They developed an approach based on quantum probabilities (cf. Sec. 4) and were able to describe empirical results of earlier surveys with excellent accuracy, far exceeding that of alternative classical models.
3.2 Further Options for Future Application

In addition to the four kinds of order effects mentioned above, there are other interesting possibilities. For instance, non-commuting observables imply a general asymmetry of SMPs

\[ w_\rho(a_j \leftarrow b_i) - w_\rho(b_i \leftarrow a_j) \neq 0, \tag{14} \]

which serves as a measure for the degree to which \( A \) and \( B \) do not commute, i.e. \( AB \neq BA \). If (and only if) an addition of observables is well-defined, this entails a non-vanishing commutator, \( AB - BA \neq 0 \).

Considering the variances according to Eq. (5), two further effects can be distinguished. If

\[ \langle (\Delta B)^2 \rangle_{a_j,\rho} < \langle (\Delta B)^2 \rangle_{\rho} , \tag{15} \]

this may be characterized as a contraction effect: the variance of measuring \( B \) decreases if \( A \) is measured first. Contraction effects indicate the degree to which \( A \) and \( B \) are interdependent (i.e. compatible). The alternative case is:

\[ \langle (\Delta B)^2 \rangle_{a_j,\rho} > \langle (\Delta B)^2 \rangle_{\rho} , \tag{16} \]

meaning that the variance of measuring \( B \) increases if \( A \) is measured first, characterizing a distraction effect. Distraction effects indicate the degree to which \( A \) and \( B \) are incompatible (i.e. non-commutative).

Such distraction effects are typical in quantum mechanics, e.g. for non-commuting observables such as position \( Q \) and momentum \( P \) of a quantum system. A measurement designed to fix \( Q \) as precisely as possible entails an increased variance of \( P \). In quantum mechanics this is expressed by Heisenberg-type uncertainty relations, where the product of the two variances is bounded from below by their commutator.

Although numerous order effects in questionnaires, surveys or polls have been found with respect to shifted expectation values, studies of uncertainty relations with respect to variances are more difficult than studies of mean shifts and have not been carried out so far. If such uncertainty relations were empirically discovered, the lower bound of the product of the variances of the distribution of sequential responses might provide an estimate for the degree to which the questions considered do not commute.

A further useful measure for testing order effects is due to correlations between measurements. Defining, as in quantum theory, \( \frac{1}{2} \langle (\Delta A \Delta B + \Delta B \Delta A) \rangle_{\rho} \) for correlations between individual measurements and \( \frac{1}{2} (\langle \Delta A \leftarrow \Delta B \rangle_{\rho} + \langle \Delta B \leftarrow \Delta A \rangle_{\rho}) \) for correlations between sequential measurements, two kinds of order effects are possible:

**Correlation enhancement:**

\[ \langle \Delta A \Delta B + \Delta B \Delta A \rangle_{\rho} < \langle \Delta A \leftarrow \Delta B \rangle_{\rho} + \langle \Delta B \leftarrow \Delta A \rangle_{\rho} \tag{17} \]

**Correlation attenuation:**

\[ \langle \Delta A \Delta B + \Delta B \Delta A \rangle_{\rho} > \langle \Delta A \leftarrow \Delta B \rangle_{\rho} + \langle \Delta B \leftarrow \Delta A \rangle_{\rho} \tag{18} \]
4 Order Effects in Hilbert Space Representations

4.1 Sequential Measurements in Hilbert Space Quantum Mechanics

In ordinary Hilbert space quantum mechanics (von Neumann 1932), both pure and mixed states can be represented by density matrices, i.e. positive normalized self-adjoint operators \( \rho \) with \( \rho^+ = \rho, \rho > 0, \text{tr} \rho = 1 \). Pure states are states of individual systems, typically represented by state vectors \( \psi \) in a Hilbert space. Every superposition of pure states represents another pure state. Pure states encode maximal information about the system. Mixed states are states of statistical ensembles of pure states with different probabilities, also called statistical states. Pure states can be represented by density operators that are idempotent, \( \rho = \rho^2 \).

In ordinary Hilbert space representations, the IMPs \( w_{\rho}(a_j) \) and \( w_{\rho}(b_i) \) are defined as

\[
\begin{align*}
    w_{\rho}(a_j) &= \text{tr}(P_{a_j} \rho) = \text{tr}(P_{a_j} \rho P_{a_j}), \\
    w_{\rho}(b_i) &= \text{tr}(P_{b_i} \rho) = \text{tr}(P_{b_i} \rho P_{b_i}),
\end{align*}
\]

(19) (20)

where \( P_{a_j} \) and \( P_{b_i} \) are the projection operators onto eigenstates of \( A \) and \( B \) with corresponding eigenvalues \( a_j \) and \( b_i \). An eigenstate of an observable \( A \) with eigenvalue \( a_j \) is a state in which a measurement of \( A \) yields \( a_j \) with probability 1.

The density matrix \( \rho_{a_j} \), i.e. the state of the system after the measurement result of \( a_j \) has been obtained, is:

\[
\rho_{a_j} = \frac{P_{a_j} \rho P_{a_j}}{\text{tr}(P_{a_j} \rho P_{a_j})}
\]

(21)

The (conditional) probability for measuring \( b_i \) after measuring \( a_j \) is:

\[
    w_{\rho}(b_i|a_j) = \text{tr} P_{b_i} \rho_{a_j} = \frac{\text{tr}(P_{b_i} P_{a_j} \rho P_{a_j} P_{b_i})}{\text{tr}(P_{a_j} \rho P_{a_j})} = w_{\rho_{a_j}}(b_i)
\]

(22)

As a consequence, the (sequential) probability of measuring first \( a_j \) and then \( b_i \) is given by:

\[
    w_{\rho}(b_i \leftarrow a_j) = w_{\rho}(a_j) w_{\rho}(b_i|a_j) = \text{tr}(P_{b_i} P_{a_j} \rho P_{a_j} P_{b_i})
\]

(23)

Likewise, we have

\[
    w_{\rho}(c_k \leftarrow b_i \leftarrow a_j) = \text{tr}(P_{c_k} P_{b_i} P_{a_j} \rho P_{a_j} P_{b_i} P_{c_k})
\]

(24)

for three successive measurements, and analogously for more.

4.2 Complexity of Hilbert Space Models

In this subsection, we investigate how the number of model parameters needed to fit experimental data depends on the dimension of the Hilbert space used to represent the states. The studies by Moore (2002) were based on binary alternatives, hence Wang and Busemeyer (2011) used a two-dimensional Hilbert space for fitting them. Furthermore, the number of parameters depends on whether pure states or mixed states are admitted. Wang and Busemeyer (2011) based their analysis on pure states. (If the states of different individuals are not assumed to be identical, it might be reasonable to employ mixed states.)
Let us first consider the situation in a two-dimensional Hilbert space. Choosing normalized eigenstates of $A$ as a basis, pure states are characterized by two real-valued parameters, because pure states are represented by complex two-dimensional vectors $\psi \neq 0$ (modulo a multiplicative complex number). By contrast, mixed states have three real-valued parameters, because $\rho$ is a self-adjoint $2 \times 2$ matrix with $\text{tr} \rho = 1$.

The relative position of a self-adjoint $2 \times 2$ matrix $B$ with respect to $A$ as a reference (corresponding to the transformation from the eigenspaces of $A$ to the eigenspaces of $B$) is determined by two real-valued parameters. Altogether, this results in four real-valued parameters for pure states, and five for mixed states.

What needs to be fitted are the measured probabilities $w_\rho(a_1), w_\rho(b_1), w_\rho(a_1 \leftarrow b_1), w_\rho(a_1 \leftarrow b_2), w_\rho(b_1 \leftarrow a_1)$, and $w_\rho(b_1 \leftarrow a_2)$—compare Eqs. (1) and (2). Hence, six empirically obtained numbers have to be fitted by four (respectively five) model parameters, which documents that two-dimensional Hilbert space models provide a non-trivial compact description of the empirical data to be fitted.

In an $n$-dimensional Hilbert space with $n > 2$ possible measurement results for each observable, the situation is analogous. For a pure (mixed, respectively) state we need $2n - 2$ ($n^2 - 1$, respectively) real-valued parameters plus $n^2 - n$ parameters for the relative positions. Together this provides $n^2 + n - 2$ parameters for pure states, and $2n^2 - n - 1$ for mixed states.

The number of probabilities to be fitted for $i, j = 1, \ldots, n$ are $2n - 2$ for the IMPs and $2n^2 - 2n$ for the SMPs (compare the 2-dimensional case), hence $2n^2 - 2$ in total. So the difference between parameters to be fitted and model parameters is $n(n - 1)$ for pure states, and $n - 1$ for mixed states. Hence, the model becomes more parsimonious with increasing $n$. As a consequence, its relative complexity (with respect to the number of parameters to be fitted) decreases with increasing $n$, and it decreases particularly fast for pure states.\footnote{The estimated numbers may be further reducible if additional knowledge is available about the state and the relative positions of $A$ and $B$. The complexity corresponding to the numbers given thus refers to the most general and least parsimonious situation within the class of Hilbert space models.} This indicates an increasing predictive power of the model with increasing $n$.

The predictive power increases dramatically if more than two observables are considered. For instance, three observables on an $n$-dimensional Hilbert space give rise to $2n^2 - 2$ (pure states) or $3n^2 - 2n - 1$ (mixed states) model parameters. The number of empirical probabilities to be fitted is $3(n - 1) + 6n(n - 1) + 6n^2(n - 1)$ in this case. For $n = 2$ this amounts to 39 probabilities to be fitted by 6 (7) model parameters for pure (mixed) states.

For empirical applications, this result suggests that scaled responses in questionnaires should be favored over binary responses to exploit the increasing parsimony of higher-dimensional Hilbert space models. In this case, however, one should keep in mind that the total number of parameters increases with Hilbert space dimension and may lead to overly high variances. In practice this can be balanced by a large number of observations.

In a finite-dimensional Hilbert space model, the state of the system determines the expectation values and variances of all observables, including the case of sequential measurements. The state $\rho$ is given by a finite number of real-valued parameters fixed by the Hilbert space dimension. By contrast, the state of a classical system is described by a multivariate distribution function which can require many parameters. Their number may be reducible by regarding only restricted families of distributions, but for order effects changes of distributions must also be taken into account.
in turn increasing the number of model parameters. Models based on stochastic processes, e.g. hidden Markov models, are typical examples of such situations.

In contrast, Hilbert space models have a tendency to be less complex (more parsimonious) because only the state and the relative positions of observables enter into the number of model parameters. This should become visible for concrete models of specific situations. Comparing Hilbert space modeling for decision making (Busemeyer et al. 2011) with classical probability models (Bayesian model selection), Shiffrin and Busemeyer (2011) found indeed that the complexity is less for Hilbert space models. It should be noted, however, that they could not explore the entire parameter space and their results have not yet been generalized beyond the one experimental situation they looked at.

4.3 Joint Measurements

In the standard formalism of quantum theory by von Neumann (1932), two observables \( A \) and \( B \) can be jointly measured only if they commute\(^6\). This commutativity corresponds to the fact that the algebra of measurements is Boolean and, thus, expresses decisions of binary alternatives. In a Hilbert space representation, measurements are projections onto subspaces of the Hilbert space, and observables are usually defined as projection-valued self-adjoint operators.

Joint measurements of observables \( A \) and \( B \) that need not commute can be described by the conjunction of their projection operators, or propositions, \( P_{a_j} \land P_{b_i} \), defined as \( P_{a_j \land b_i} \) with

\[
P_{a_j \land b_i} = P_{b_i \land a_j}
\]  

being the projections onto the intersection of the eigenspaces of \( A \) and \( B \). For questionnaires, this corresponds to the fusion of two questions into one by conjunction.

Adding a prior measurement of \( a_j \) by \( P_{a_j} \), we get

\[
P_{a_j} P_{a_j \land b_i} = P_{a_j \land b_i} P_{a_j} = P_{a_j \land b_i} = P_{b_i} P_{a_j \land b_i} = P_{a_j \land b_i} P_{b_i}
\]  

Because in Hilbert space quantum mechanics probabilities are expressed as traces of projections on states,

\[
w_\rho(a_j \land b_i \leftarrow a_j) = trP_{a_j \land b_i}P_{a_j}\rho P_{a_j} = trP_{a_j \land b_i}\rho,
\]  

we obtain the condition:

\[
w_\rho(a_j \land b_i \leftarrow a_j) = w_\rho(a_j \land b_i) = w_\rho(a_j \leftarrow a_j \land b_i)
\]  

Condition (28) is equally obtained if additional measurements are added. Consider the case of measuring \( a_j \) first, then \( b_i \), then \( a_j \land b_i \). The concatenation of these measurements in the corresponding SMP gives

\[
w_\rho(a_j \land b_i \leftarrow b_i \leftarrow a_j) = trP_{a_j \land b_i}P_{b_i}P_{a_j}\rho P_{a_j} P_{b_i} = trP_{a_j \land b_i}\rho = w_\rho(a_j \land b_i),
\]  

\(^6\)More precisely, this holds for “ideal” (von Neumann) measurements “of the first kind” (Pauli). For realistic laboratory experiments with measurements “of the second kind” positive-operator-valued measures of “unsharp” observables have been suggested by Davies, Holevo, and Ludwig in the 1970s. See Uffink (1994) for a critical discussion of “unsharp” joint measurements of non-commuting observables, and for further references.
yielding again the IMP \( w_p(a_j \land b_i) \) for combined propositions. As a result, Hilbert space quantum mechanics predicts that the SMP of a sequence of individual and joint measurements always yields the probability of the joint measurement alone. As a consequence, the propositions \( P_{a_j} \) and \( P_{a_j \land b_i} \) are always compatible so that order effects are excluded for \( P_{a_j} \) and \( P_{a_j \land b_i} \) in the projector calculus of Hilbert space quantum mechanics, and also in propositional logic.

If \( P_{a_j} \) and \( P_{a_j \land b_i} \) are empirically found to be incompatible, e.g. as in “anomalous” consistency effects or contrastive effects, this would indicate that standard Hilbert space models with self-adjoint operators are insufficient to describe the data properly. Psychological experiments with joint measurements are therefore interesting candidates for exploring potential limitations of Hilbert space modeling. It depends on the situation, which kind of extension of the model will be required or advisable in such cases.

For example, the response to a joint-measurement question \( P_{a_j \land P_{b_i}} \) does logically determine the response of each individual question \( P_{a_j} \) or \( P_{b_i} \). On the other hand, asking \( P_{a_j} \) or \( P_{b_i} \) first may lead to a response deviating from the one obtained when asking \( P_{a_j \land P_{b_i}} \) first. This is conceivable due to some “anomalous” behavior of agents in conflict with the rules of Boolean logic. In this case, sequential survey questions may produce order effects appearing as if \( P_{a_j} \) and \( P_{b_i} \) did not commute with their conjunction.

Since in a Hilbert space framework \( P_{a_j \land P_{b_i}} \) commutes with both \( P_{a_j} \) and \( P_{b_i} \), this framework is too narrow for a proper description of such order effects (with joint measurements). The challenge of such situations would be to find a non-Hilbert space representation in which both \( P_{a_j} \) and \( P_{b_i} \) in fact do not commute with their conjunction even formally. This would amount to finding a representation more general than a Hilbert space model for the particular “anomalous” behavior observed.

5 Summary

In his summary of the volume edited by Schwarz and Sudman (1992), one of the pioneers of research on sequential measurements in psychology stated (Bradburn 1992): “One of the factors that inhibited our progress in understanding order effects has been the lack of a theoretical structure within which to investigate the mechanisms by which they might occur.” The present paper presents an attempt toward such a theoretical basis.

We consider the non-commutativity of measurement operations as the formal key to order effects. This is well established in quantum theory where observables typically do not commute: The sequence in which measurement operations act upon the state of a system makes a difference for the results obtained. One reason is that a measurement operation changes the state (even it is pure) of the measured system such that a subsequent measurement operation effectively acts on another state. Related to this, a quantum measurement is not simply the registration of a pre-existing fact, but also establishes the fact that is registered.

It is highly plausible that this basic idea holds for psychological systems as well, although states, observables and their dynamics have nothing to do with quantum physics. The mathematical feature of non-commuting observables and its ramifications can be fruitfully applied to model psychological situations where order effects abound.

We proposed a classification of particular order effects, some of which were already discovered
empirically, and Hilbert space models were successfully used to analyze them: additive effects, subtractive effects, contrast effects, and consistency effects. Moreover, we predicted order effects not observed so far, which rest on uncertainty relations between variances of distributions of observables rather than on shifts of their expectation values.

We assessed the complexity of Hilbert space models for pure and mixed states, and for \( n \)-dimensional Hilbert spaces corresponding to questions (observables) with \( n \)-ary alternatives for response. It turned out that the predictive power of the model increases with increasing \( n \), and that it increases dramatically with increasing number of observables. We sketched an argument why the complexity of Hilbert space models should be generally less than that of alternative classical models.

We suggested experiments with joint measurements as interesting candidates for exploring potential limitations of Hilbert space models. If agents behave in a way conflicting with Boolean logical rules, order effects may result whose description requires non-commutative frameworks more general than standard Hilbert space models permit.

It is evident that a successful model does not entail the validity of a mechanism. However, models based on non-commuting observables stress the specific features of (i) strong measurement interactions and (ii) measurement as both establishing and registering a result. Due to these important points, which are highly plausible for observations on mental systems, non-commutative models are not only descriptively powerful but also hold the potential for explanatory surplus. In combination with their parsimony over classical models, non-commutative models offer a very attractive option for future research.

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Abstract

Sequential measurements of non-commuting observables produce order effects that are well-known in quantum physics. But their conceptual basis, a significant measurement interaction, is relevant for far more general situations. We argue that non-commutativity is ubiquitous in psychology where almost every interaction with a mental system changes that system in an uncontrollable fashion. Psychological order effects for sequential measurements are therefore to be expected as a rule. In this paper we focus on the theoretical basis of such effects. We classify several families of order effects theoretically, relate them to psychological observations, and predict effects yet to be discovered empirically. We assess the complexity, related to the predictive power, of particular (Hilbert space) models of order effects and discuss possible limitations of such models.
Order Effects in Sequential Measurements of Non-Commuting Psychological Observables

H. Atmanspacher\textsuperscript{1,2} and H. Römer\textsuperscript{3}
\textsuperscript{1} - Institute for Frontier Areas of Psychology and Mental Health, Freiburg, Germany
\textsuperscript{2} - Collegium Helveticum, Zurich, Switzerland
\textsuperscript{3} - Physics Department, University of Freiburg, Germany

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Contents

1 Introduction

1.1 Measurement and Non-Commutativity ............................................. 1
1.2 Non-Commutative Phenomena in Psychology ...................................... 2

2 Probabilities for Sequential Measurements ........................................... 3

3 Types of Order Effects

3.1 Theoretical Classification of Observed Order Effects ............................. 5
3.2 Further Options for Future Application ............................................. 6

4 Order Effects in Hilbert Space Representations ............................... 7

4.1 Sequential Measurements in Hilbert Space Quantum Mechanics .............. 7
4.2 Complexity of Hilbert Space Models ............................................ 8
4.3 Joint Measurements ..................................................................... 9

5 Summary ................................................................................. 11
1 Introduction

1.1 Measurement and Non-Commutativity

From a conceptual point of view, measurements are operations including interactions between a measuring system $M$, e.g. a measuring device, and a measured system $S$. Typical objects of a measurement are properties (sometimes called observables) $A$, $B$, ..., of system $S$ in a certain state $\rho$. Measurement results for observables $A$ and $B$ are typically quantified by real-valued numbers $a$, $b$, ...

A physical standard example for this terminology is a material measurement apparatus $M$ recording particular properties of a material system $S$ in state $\rho$ (with some probability $p_i$). In basic physics, observables occur in canonically conjugate pairs, such as position and momentum, exhibiting particular well-defined invariance principles and conservation laws.

In psychology we are concerned with a mental system $S$, usually of an individual subject, being in a state $\rho$. In contrast to physics, there are no canonically conjugate pairs of observables. For instance, a survey questionnaire as a measuring device $M$ can contain any questions one may want to choose as observables, to be measured in terms of answers as measurement results.

If the measurement interaction between $M$ and $S$ is weak, then the action of $M$ leaves no significant effect on $S$. In case of strong interactions, this is no longer the case, and effects of measurements due to $M$ have a non-negligible influence even on a state $\rho$ of $S$. (This brief discussion disregards that measurement affects also the state of the measuring device. This, of course, is intended and mandatory because otherwise there would be no recorded measurement result.)

In physics, the distinction of weak and strong interactions is one major criterion for delineating classical systems from quantum systems. In addition, quantum measurements – in contrast to classical measurements – are not simply registrations of pre-existing facts, but they also “establish” the fact which is registered. The key feature of this classical-quantum distinction is mathematically codified in terms of the commutativity or non-commutativity, respectively, of observables. Referring to measurements as actions of an observable on a state, $A(\rho)$ or $B(\rho)$, these actions either leave the state as it was before measurement, or they change it.

As a consequence, the successive action of two observables on a state can be commutative, $A(B(\rho)) = B(A(\rho))$, or non-commutative, $A(B(\rho)) \neq B(A(\rho))$. In physics, the measured behavior of a system is called classical in the former case, whereas the latter case refers to quantum behavior. From this perspective, quantum theory can be viewed as a general systems theory of both commuting and non-commuting observables, and classical physics is restricted to the special case of commuting observables alone (cf. Primas 1990).

Whenever the sequence of successive measurement interactions between a measuring device and a measured system makes a difference for the final result, measurements are non-commutative. Many complex systems must be expected to exhibit non-commutative properties in this sense. In psychology, where virtually every interaction of a “measuring” device with a “measured” mental state changes that state uncontrollably and where mental states are often literally established by measurements, it is highly plausible to argue that non-commutativity should be the ubiquitous rule.
1.2 Non-Commutative Phenomena in Psychology

Traditionally, psychology has not considered its observations as owing to effects of non-commutative observables throughout its history. But there are quite a number of psychological phenomena showing clear features of such an approach. Some of them, which have been worked out in recent years, are decision (or judgment) processes, semantic associations, bistable perception, learning, and order effects in questionnaires.

It should be noted at this point that the models developed in the mentioned areas are not all explicitly based on non-commuting observables. They feature various related concepts such as incompatibility, complementarity, entanglement, partial Boolean logic, dispersive states, quantum probability, uncertainty relations etc. However, these concepts bear close formal relationships with the property of non-commutativity and its ramifications (cf. Atmanspacher et al. 2002, 2006, Filk and Römer 2011; more technically: Rédei 1996, Rédei and Summers 2007).

A particular one among the psychological phenomena mentioned above is most illustratively addressed in terms of non-commuting operations: the phenomenon of order effects. These effects do so obviously refer to non-commutativity because they are ruled by it almost literally. For two questions (actions of observables) $A$ and $B$, their sequence makes a difference with respect to the answer (state $\rho$ of the system) after the second question,

$$AB\rho \neq BA\rho,$$

where $A$ and $B$ are operations whose actions on the state $\rho$ of the system change $\rho$. Related to this, there are no proper joint probability distributions of $A$ and $B$ in the sense of standard statistical theory, but only order-dependent probabilities to be defined in Sec. 2.

Psychologists know such effects well, they are also called sequence effects or context effects. An (incomplete) list of textbooks and reviews on the topic, showing its fairly long history, is Sudman and Bradburn (1974), Schumann and Presser (1981), Schwarz and Sudman (1992), Hogarth and Einhorn (1992), Sudman, Bradburn and Schwarz (1996), Tourangeau, Rips, and Rasinski (2000). However, the proposed models for describing order effects have always been framed classically (e.g., Markov processes, Bayesian updating, etc.). Models inspired by quantum theory have been successfully applied only recently.

The first publication with corresponding indications that we know of has been due to Aerts and Aerts (1995) and essentially derived from a non-Boolean logic of propositions. With the growing success of and attention on non-classical modeling over the last decade, more work has been done along those lines. Most notable with respect to our topic in the present paper are applications to sequential decision making by Pothis and Busemeyer (2009) Busemeyer et al. (2011), Trueblood and Busemeyer (2011), and Wang and Busemeyer (2011), whose work basically utilize quantum probabilities in Hilbert space representations of mental states.

Our own formal framework for addressing non-commuting properties is somewhat more abstract, because the full-fledged formalism of (Hilbert space) quantum mechanics is generally too restrictive for modeling non-commutative situations. There are scenarios in which we cannot even...
require such basic concepts as a metric (linear or nonlinear) representation space for states, the formulation of dynamical laws, or a definition of probabilities. Of course, some of these may become necessary if specific examples are to be treated and empirical results are to be described. We have shown elsewhere (Atmanspacher et al. 2002, 2006, Filk and Römer 2011) how conventional quantum mechanics can be stepwise recovered if enough relevant contexts are filled into the general framework.

It is to be expected that models of psychological phenomena should be located somewhere in between the most general level and the very specific level of conventional quantum physics. For instance, psychology will arguably not have a universal quantity, such as the Planck constant, specifying the degree of non-commutativity in general. On the other hand, contact with controlled experiments will definitely make it necessary to have well-defined probabilities.

Hence, we begin this article with a brief account of probabilities for the results of sequential measurements of non-commuting observables in Section 2. Section 3 uses this framework for a theoretically framed classification of different types of order effects, some of which have been discussed in the psychological literature; others are predicted here for the first time. Section 4 addresses Hilbert space models of such order effects. It is shown that the complexity of such models, if they fit empirically obtained probabilities, is lower than the number of parameters to be fitted. In other words, the models have considerable predictive power. Finally we give an example for the limits of Hilbert space models of order effects. Section 5 summarizes the results of this article.

2 Probabilities for Sequential Measurements

Let a system be in state $\rho$ before its observables $A$ and $B$ are measured. Then the individual measurement probability (IMP) $w_\rho(a_j)$ or $w_\rho(b_i)$ is the probability of measuring $a_j$ of $A$ or $b_i$ of $B$ in state $\rho$, without any prior measurement influence.

Now consider subsequent measurements of first $A$ and then $B$ in the initial state $\rho$ and denote the probability of measuring the value $b_i$ of $B$ after having measured value $a_j$ of $A$ as the sequential measurement probability (SMP) $w_\rho(b_i \leftarrow a_j)$. If $A$ and $B$ do not commute, $w_\rho(b_i \leftarrow a_j)$ will be different from $w_\rho(a_j \leftarrow b_i)$ (cf. Franco 2009). For the sake of simplicity, we assume $a_j$ and $b_i$ to be discrete.

The SMP is the fundamental quantity in the discussion to follow. It is not simply the conditional probability of measuring $b_i$ given $a_j$ but refers to subsequent measurements of both $A$ and $B$ that actually have been carried out and, thus, may (and generally will) change the state $\rho$ of the system. This will be discussed in some more detail in Sec. 4.1.

Since $w$ is a probability (in the sense of Kolmogorov), the sum of $w_\rho(b_i \leftarrow a_j)$ over all measurements of $A$ and $B$ must be normalized:

$$\sum_{i,j} w_\rho(b_i \leftarrow a_j) = 1 \quad (1)$$

3The complexity of a model is related to the number of its parameters. The model is parsimonious if this number is smaller than the number of variables to be fitted. The predictive power of a model increases with its parsimony.
Moreover, the sum of $w_\rho(b_i \leftarrow a_j)$ over all measured values $b_i$ of $B$ is given by
\begin{equation}
\sum_i w_\rho(b_i \leftarrow a_j) = w_\rho(a_j).
\end{equation}

The conditional probability
\begin{equation}
w_\rho(b_i|a_j) := \frac{1}{w_\rho(a_j)} w_\rho(b_i \leftarrow a_j)
\end{equation}
in principle contains information about both “ordinary” statistical dependence and effects of the non-commutativity of observables $A$ and $B$. In the non-commutative case, Bayes’ rule is violated:
\begin{equation}
\frac{w_\rho(b_i|a_j)}{w_\rho(a_j|b_i)} \neq \frac{w_\rho(b_i)}{w_\rho(a_j)}
\end{equation}

Concentrating on effects due to non-commuting observables alone and disregarding ordinary statistical dependence, we need to consider “pooled” probabilities $w_{A,\rho}(b_i)$, summing over all possible values $a_j$ of $A$, rather than conditional probabilities as in Eq. (3):
\begin{equation}
w_{A,\rho}(b_i) := \sum_j w_\rho(b_i \leftarrow a_j).
\end{equation}

If $w_{A,\rho}(b_i) \neq w_\rho(b_i)$, we have an unambiguous criterion for non-commuting observables. (Of course, $a_j$ and $b_i$ in relations (1) to (5) must be exchanged if the sequence of measuring $A$ and $B$ is swapped.)

It follows that the expectation value $\langle B \rangle_\rho$ for measuring $B$ alone (unconditioned) differs from the expectation value $\langle B \rangle_{a_j,\rho}$ for measuring $B$ after $A$:
\begin{equation}
\langle B \rangle_{a_j,\rho} := \sum_i b_i w_\rho(b_i|a_j) \neq \langle B \rangle_\rho = \sum_i b_i w_\rho(b_i),
\end{equation}

which in general also holds for any function $f(B)$ and its expectation values for both commuting or non-commuting observables. Again, concentrating on effects of non-commutativity we have to use pooled probabilities and get:
\begin{equation}
\langle B \rangle_{A,\rho} := \sum_i b_i w_{A,\rho}(b_i) \neq \langle B \rangle_\rho = \sum_i b_i w_\rho(b_i).
\end{equation}

Likewise, the expectation values of the variance operators $(\Delta A)^2$ and $(\Delta B)^2$, where the operators $\Delta A$ and $\Delta B$ are the deviations from the expectation values of $A$ and $B$, differ as well. In the case of both commuting or non-commuting observables, we have in general:
\begin{equation}
\langle (\Delta B)^2 \rangle_{a_j,\rho} = \langle (B - \langle B \rangle_{a_j,\rho})^2 \rangle_{a_j,\rho} \neq \langle (B - \langle B \rangle_\rho)^2 \rangle_\rho = \langle (\Delta B)^2 \rangle_\rho
\end{equation}

Concentrating on effects of non-commutativity alone, with pooled probabilities, this turns into:
\begin{equation}
\langle (\Delta B)^2 \rangle_{A,\rho} = \langle (B - \langle B \rangle_{A,\rho})^2 \rangle_{A,\rho} \neq \langle (B - \langle B \rangle_\rho)^2 \rangle_\rho = \langle (\Delta B)^2 \rangle_\rho
\end{equation}

\footnote{Throughout this paper, we use the notation $\langle \cdot \rangle$ for expectation values, also known as (disregarding subtle details) mean values of a distribution.}
Furthermore we can define a joint sequential expectation value as:

\[ \langle B \leftarrow A \rangle_\rho = \sum_{ij} a_j b_i w_\rho(b_i \leftarrow a_j) \] (10)

In general (for instance in quantum mechanics), the expectation value

\[ \langle \Delta A \Delta B \rangle_\rho = \langle (A - \langle A \rangle_\rho)(B - \langle B \rangle_\rho) \rangle_\rho \] (11)

may not even be real-valued because the product operator \( \Delta A \Delta B \) is not Hermitian if \( A \) and \( B \) do not commute. This is at variance with classical (commuting) observables as used in standard statistical theory, where the product of real-valued observables is always real-valued.

Now we define

\[ \delta_\rho(b_i) = \sum_j w_\rho(b_i \leftarrow a_j) - w_\rho(b_i) \] (12)

\[ \delta_\rho(a_j) = \sum_i w_\rho(a_j \leftarrow b_i) - w_\rho(a_j) \] (13)

as the differences between sums of SMPs and IMPs for measuring \( A \) and \( B \). These differences provide a convenient way to assess the effect of sequential measurements of \( A \) and \( B \) as compared to individual measurements of \( A \) or \( B \) alone. It may be mentioned that the transition probabilities for sequential measurements as introduced here represent a special case of Khrennikov’s (2009) contextual probabilities.

3 Types of Order Effects

3.1 Theoretical Classification of Observed Order Effects

With the definitions in Eqs. (12,13), we can relate our probabilistic framework to consistency and contrast effects first reported by Schumann and Presser (1981) and to additive and subtractive effects found by Moore (2002). All four types have recently been investigated by Wang and Busemeyer (2011), see also Trueblood and Busemeyer (2011) and Busemeyer et al. (2011).

Additive effects (both SMPs are larger than the corresponding IMPs):

\[ \delta_\rho(a_j) > 0 \quad \text{and} \quad \delta_\rho(b_i) > 0 \] (14)

Subtractive effects (both SMPs are smaller than the corresponding IMPs):

\[ \delta_\rho(a_j) < 0 \quad \text{and} \quad \delta_\rho(b_i) < 0 \] (15)

Contrast effects (the difference of the two SMPs is larger than the difference of the corresponding IMPs):

\[ \left| \sum_i w_\rho(a_j \leftarrow b_i) - \sum_j w_\rho(b_i \leftarrow a_j) \right| > \left| w_\rho(a_j) - w_\rho(b_i) \right| \] (16)
Consistency effects (the difference of the two SMPs is smaller than the difference of the corresponding IMPs):

\[
\left| \sum_i w_\rho(a_j \leftarrow b_i) - \sum_j w_\rho(b_i \leftarrow a_j) \right| < |w_\rho(a_j) - w_\rho(b_i)|
\] (17)

Contrast and consistency effects apply in particular to situations where \( \delta_\rho(a_j) \) and \( \delta_\rho(b_i) \) have different signs, \( \delta_\rho(a_j) \delta_\rho(b_i) < 0 \). In this case, additive or subtractive effects are not candidates anyway since there the product is always positive. Of course, all inequalities turn into equalities if \( A \) and \( B \) commute.

For a comprehensive discussion of pertinent examples of these four classes of order effects see Wang and Busemeyer (2011). They developed an approach based on quantum probabilities (cf. Sec. 4) and were able to describe empirical results of earlier surveys with excellent accuracy, far exceeding that of alternative classical models.

3.2 Further Options for Future Application

In addition to the four kinds of order effects mentioned above, there are other interesting possibilities. For instance, non-commuting observables imply a general asymmetry of SMPs

\[
w_\rho(a_j \leftarrow b_i) - w_\rho(b_i \leftarrow a_j) \neq 0,
\] (18)

which serves as a measure for the degree to which \( A \) and \( B \) do not commute, i.e. \( AB \neq BA \). If (and only if) an addition of observables is well-defined, this entails a non-vanishing commutator, \( AB - BA \neq 0 \).

Considering the variances according to Eq. (8), with non-pooled probabilities, two further effects can be distinguished. If

\[
\langle (\Delta B)^2 \rangle_{a_j,\rho} < \langle (\Delta B)^2 \rangle_{\rho},
\] (19)

this may be characterized as a contraction effect: the variance of measuring \( B \) decreases if \( A \) is measured first. Contraction effects indicate the degree to which \( A \) and \( B \) are interdependent (i.e. compatible). The alternative case is:

\[
\langle (\Delta B)^2 \rangle_{a_j,\rho} > \langle (\Delta B)^2 \rangle_{\rho},
\] (20)

meaning that the variance of measuring \( B \) increases if \( A \) is measured first, characterizing a distraction effect. Distraction effects indicate the degree to which \( A \) and \( B \) are incompatible (i.e. non-commutative).

For the specifically non-commutative case, we must again replace \( \langle \cdot \rangle_{a_j,\rho} \) by \( \langle \cdot \rangle_{A,\rho} \) as in Eq. (9), so that contraction is characterized by

\[
\langle (\Delta B)^2 \rangle_{A,\rho} < \langle (\Delta B)^2 \rangle_{\rho},
\] (21)

and distraction is characterized by

\[
\langle (\Delta B)^2 \rangle_{A,\rho} > \langle (\Delta B)^2 \rangle_{\rho}.
\] (22)
Distraction effects are typical in quantum mechanics, e.g. for non-commuting observables such as position \( Q \) and momentum \( P \) of a quantum system. A measurement designed to fix \( Q \) as precisely as possible entails an increased variance of \( P \). In quantum mechanics this is expressed by Heisenberg-type uncertainty relations, where the product of the two variances is bounded from below by their commutator.

Although numerous order effects in questionnaires, surveys or polls have been found with respect to shifted expectation values, studies of uncertainty relations with respect to variances are more difficult than studies of mean shifts and have not been carried out so far. If such uncertainty relations were empirically discovered, the lower bound of the product of the variances of the distribution of sequential responses might provide an estimate for the degree to which the questions considered do not commute.

A further useful measure for testing order effects is due to correlations between measurements. Defining, as in quantum theory,

\[
\langle \Delta A \Delta B + \Delta B \Delta A \rangle_\rho \quad \text{and} \quad \frac{1}{2}(\langle \Delta A \leftrightarrow \Delta B \rangle_\rho + \langle \Delta B \leftrightarrow \Delta A \rangle_\rho)
\]

for correlations between individual measurements and sequential measurements, two kinds of order effects are possible: 

**Correlation enhancement:**

\[
\langle \Delta A \Delta B + \Delta B \Delta A \rangle_\rho < \langle \Delta A \leftrightarrow \Delta B \rangle_\rho + \langle \Delta B \leftrightarrow \Delta A \rangle_\rho
\]

**Correlation attenuation:**

\[
\langle \Delta A \Delta B + \Delta B \Delta A \rangle_\rho > \langle \Delta A \leftrightarrow \Delta B \rangle_\rho + \langle \Delta B \leftrightarrow \Delta A \rangle_\rho
\]

The symmetrized product observable \( \frac{1}{2}(\langle \Delta A \Delta B + \Delta B \Delta A \rangle_\rho \), is well-defined in quantum theory, but its operationalization is not straightforward because it is not feasible by sequential measurements.

### 4 Order Effects in Hilbert Space Representations

#### 4.1 Sequential Measurements in Hilbert Space Quantum Mechanics

In ordinary Hilbert space quantum mechanics (von Neumann 1932), both pure and mixed states can be represented by density matrices, i.e. positive normalized self-adjoint operators \( \rho \) with \( \rho^\dagger = \rho \), \( \rho > 0 \), \( tr\rho = 1 \). **Pure** states are states of individual systems, typically represented by state vectors \( \psi \) in a Hilbert space. Every superposition of pure states represents another pure state. Pure states encode maximal information about the system. **Mixed** states are states of statistical ensembles of pure states with different probabilities, also called statistical states. Pure states can be represented by density operators that are idempotent, \( \rho = \rho^2 \).

In ordinary Hilbert space representations, the IMPs \( w_\rho(a_j) \) and \( w_\rho(b_i) \) are defined as

\[
\begin{align*}
    w_\rho(a_j) &= tr(P_{a_j} \rho) = tr(P_{a_j} \rho P_{a_j}), \\
    w_\rho(b_i) &= tr(P_{b_i} \rho) = tr(P_{b_i} \rho P_{b_i}),
\end{align*}
\]

where \( P_{a_j} \) and \( P_{b_i} \) are the projection operators onto eigenstates of \( A \) and \( B \) with corresponding eigenvalues \( a_j \) and \( b_i \). An eigenstate of an observable \( A \) with eigenvalue \( a_j \) is a state in which a measurement of \( A \) yields \( a_j \) with probability 1.
The density matrix $\rho_{a_j}$, i.e. the state of the system after the measurement result of $a_j$ has been obtained, is:

$$\rho_{a_j} = \frac{P_{aj}\rho P_{aj}}{tr(P_{aj}\rho P_{aj})}$$  \hspace{1cm} (27)

The (conditional) probability for measuring $b_i$ after measuring $a_j$ is:

$$w_\rho(b_i|a_j) = trP_{bj}\rho_{a_j} = \frac{tr(P_{bj}P_{aj}\rho P_{aj}P_{bj})}{tr(P_{aj}\rho P_{aj})} = w_{\rho_{a_j}}(b_i)$$  \hspace{1cm} (28)

As a consequence, the (sequential) probability of measuring first $a_j$ and then $b_i$ is given by:

$$w_\rho(b_i \leftarrow a_j) = w_\rho(a_j)w_\rho(b_i|a_j) = tr(P_{bj}P_{aj}\rho P_{aj}P_{bj})$$  \hspace{1cm} (29)

Likewise, we have

$$w_\rho(c_k \leftarrow b_i \leftarrow a_j) = tr(P_{ck}P_{bj}P_{aj}\rho P_{aj}P_{bj}P_{ck})$$  \hspace{1cm} (30)

for three successive measurements, and analogously for more.

### 4.2 Complexity of Hilbert Space Models

In this subsection, we investigate how the number of model parameters needed to fit experimental data depends on the dimension of the Hilbert space used to represent the states. The studies by Moore (2002) were based on binary alternatives, hence Wang and Busemeyer (2011) used a two-dimensional Hilbert space for fitting them. Furthermore, the number of parameters depends on whether pure states or mixed states are admitted. Wang and Busemeyer (2011) based their analysis on pure states. (If the states of different individuals are not assumed to be identical, it might be reasonable to employ mixed states.)

Let us first consider the situation in a two-dimensional Hilbert space. Choosing normalized eigenstates of $A$ as a basis, pure states are characterized by two real-valued parameters, because pure states are represented by complex two-dimensional vectors $\psi \neq 0$ (modulo a multiplicative complex number). By contrast, mixed states have three real-valued parameters, because $\rho$ is a self-adjoint $2 \times 2$ matrix with $tr\rho = 1$.

The relative position of a self-adjoint $2 \times 2$ matrix $B$ with respect to $A$ as a reference (corresponding to the transformation from the eigenspaces of $A$ to the eigenspaces of $B$) is determined by two real-valued parameters. Altogether, this results in four real-valued parameters for pure states, and five for mixed states.

What needs to be fitted are the measured probabilities $w_\rho(a_1)$, $w_\rho(b_1)$, $w_\rho(a_1 \leftarrow b_1)$, $w_\rho(a_1 \leftarrow b_2)$, $w_\rho(b_1 \leftarrow a_1)$, and $w_\rho(b_1 \leftarrow a_2)$ – compare Eqs. (1) and (2). Hence, six empirically obtained numbers have to be fitted by four (respectively five) model parameters, which documents that two-dimensional Hilbert space models provide a non-trivial compact description of the empirical data to be fitted.

In an $n$-dimensional Hilbert space with $n > 2$ possible measurement results for each observable, the situation is analogous. For a pure (mixed, respectively) state we need $2n - 2$ ($n^2 - 1$, respectively) real-valued parameters plus $n^2 - n$ parameters for the relative positions. Together this provides $n^2 + n - 2$ parameters for pure states, and $2n^2 - n - 1$ for mixed states.
The number of probabilities to be fitted for \( i, j = 1, \ldots, n \) are \( 2^n - 2 \) for the IMPs and \( 2n^2 - 2n \) for the SMPs (compare the 2-dimensional case), hence \( 2n^2 - 2 \) in total. So the difference between parameters to be fitted and model parameters is \( n(n-1) \) for pure states, and \( n-1 \) for mixed states. Hence, the model becomes more parsimonious with increasing \( n \). As a consequence, its relative complexity (with respect to the number of parameters to be fitted) decreases with increasing \( n \), and it decreases particularly fast for pure states.\(^5\) This indicates an increasing predictive power of the model with increasing \( n \).

The predictive power increases dramatically if more than two observables are considered. For instance, three observables on an \( n \)-dimensional Hilbert space give rise to \( 2n^2 - 2 \) (pure states) or \( 3n^2 - 2n - 1 \) (mixed states) model parameters. The number of empirical probabilities to be fitted is \( 3(n-1) + 6n(n-1) + 6n^2(n-1) \) in this case. For \( n = 2 \) this amounts to 39 probabilities to be fitted by 6 (7) model parameters for pure (mixed) states.

For empirical applications, this result suggests that scaled responses in questionnaires should be favored over binary responses to exploit the increasing parsimony of higher-dimensional Hilbert space models. In this case, however, one should keep in mind that the total number of parameters increases with Hilbert space dimension and may lead to overly high variances. In practice this can be balanced by a large number of observations.

In a finite-dimensional Hilbert space model, the state of the system determines the expectation values and variances of all observables, including the case of sequential measurements. The state \( \rho \) is given by a finite number of real-valued parameters fixed by the Hilbert space dimension. By contrast, the state of a classical system is described by a multivariate distribution function which can require many parameters. Their number may be reducible by regarding only restricted families of distributions, but for order effects changes of distributions must also be taken into account, in turn increasing the number of model parameters. Models based on stochastic processes, e.g. hidden Markov models, are typical examples of such situations.

In contrast, Hilbert space models have a tendency to be less complex (more parsimonious) because only the state and the relative positions of observables enter into the number of model parameters. This should become visible for concrete models of specific situations. Comparing Hilbert space modeling for decision making (Busemeyer et al. 2011) with classical probability models (Bayesian model selection), Shiffrin and Busemeyer (2011) found indeed that the complexity is less for Hilbert space models. It should be noted, however, that they could not explore the entire parameter space and their results have not yet been generalized beyond the one experimental situation they looked at.

### 4.3 Joint Measurements

In the standard formalism of quantum theory by von Neumann (1932), two observables \( A \) and \( B \) can be jointly measured (i.e. attributed definite values) without restriction only if they commute.\(^6\) This commutativity corresponds to the fact that the algebra of measurements is Boolean.
and, thus, expresses decisions of binary alternatives. In a Hilbert space representation, measurements are projections onto subspaces of the Hilbert space, and observables are usually defined as projection-valued self-adjoint operators.

Joint measurements of observables $A$ and $B$ that need not commute can be described in the following way: Let $P_{a_j}$ and $P_{b_i}$ be the propositions that a measurement of $A$ yields $a_j$ and a measurement of $B$ yields $b_i$. Then,

$$P_{a_j \land b_i} = \lim_{n \to \infty} (P_{a_j} P_{b_i})^n$$

(31)

can be shown to be the projection onto the (symmetric) intersection of the associated eigenspaces of $A$ and $B$ and corresponds to the conjunction (i.e. the fusion of two questions into one in a questionnaire)

$$P_{a_j \land b_i} = P_{b_i \land a_j}.$$  

(32)

Note that the limit in Eq. (31) does not coincide with the product $(P_{a_j} P_{b_i})$ or $(P_{b_i} P_{a_j})$ in the non-commutative case. If no common eigenstate of $(P_{a_j} P_{b_i})$ exists, we have $P_{a_j \land b_i} = 0$.

Adding a prior measurement of $a_j$ by $P_{a_j}$, we get

$$P_{a_j} P_{a_j \land b_i} = P_{a_j \land b_i} P_{a_j} = P_{a_j \land b_i} = P_{b_i} P_{a_j \land b_i} = P_{a_j \land b_i} P_{b_i}$$

(33)

Because in Hilbert space quantum mechanics probabilities are expressed as traces of projections on states,

$$w_\rho(a_j \land b_i \leftarrow a_j) = tr P_{a_j \land b_i} P_{a_j} \rho P_{a_j} = tr P_{a_j \land b_i} \rho,$$

(34)

we obtain the condition:

$$w_\rho(a_j \land b_i \leftarrow a_j) = w_\rho(a_j \land b_i) = w_\rho(a_j \leftarrow a_j \land b_i)$$

(35)

Condition (35) is equally obtained if additional measurements are added. Consider the case of measuring $a_j$ first, then $b_i$, then $a_j \land b_i$. The concatenation of these measurements in the corresponding SMP gives

$$w_\rho(a_j \land b_i \leftarrow b_i \leftarrow a_j) = tr P_{a_j \land b_i} P_{b_i} P_{a_j} \rho P_{a_j} P_{b_i} = tr P_{a_j \land b_i} \rho = w_\rho(a_j \land b_i),$$

(36)

yielding again the IMP $w_\rho(a_j \land b_i)$ for combined propositions. As a result, Hilbert space quantum mechanics predicts that the SMP of a sequence of individual and joint measurements always yields the probability of the joint measurement alone. As a consequence, the propositions $P_{a_j}$ and $P_{a_j \land b_i}$ are always compatible so that order effects are excluded for $P_{a_j}$ and $P_{a_j \land b_i}$ in the projector calculus of Hilbert space quantum mechanics, and also in propositional logic.

If $P_{a_j}$ and $P_{a_j \land b_i}$ are empirically found to be incompatible, e.g. as in “anomalous” consistency effects or contrastive effects, this would indicate that standard Hilbert space models with self-adjoint operators are insufficient to describe the data properly. Psychological experiments with joint measurements are therefore interesting candidates for exploring potential limitations of Hilbert space modeling. It depends on the situation, which kind of extension of the model will be required or advisable in such cases.

For example, the response to a joint-measurement question $P_{a_j \land b_i}$ does logically determine the response to each individual question $P_{a_j}$ or $P_{b_i}$. On the other hand, asking $P_{a_j}$ or $P_{b_i}$ first...
may lead to a response deviating from the one obtained when asking $P_{a_j} \land P_{b_i}$ first. This is conceivable if agents behave in conflict with the rules of Boolean logic. In this case, sequential survey questions may produce order effects appearing as if $P_{a_j}$ and $P_{b_i}$ did not commute with their conjunction.

Since in a Hilbert space framework $P_{a_j} \land P_{b_i}$ commutes with both $P_{a_j}$ and $P_{b_i}$, this framework is too narrow for a proper description of such order effects (with joint measurements). The challenge of such situations would be to find a non-Hilbert space representation in which both $P_{a_j}$ and $P_{b_i}$ in fact do not commute with their conjunction even formally. This would amount to finding a representation more general than a Hilbert space model for the particular “anomalous” behavior observed.

5 Summary

In his summary of the volume edited by Schwarz and Sudman (1992), one of the pioneers of research on sequential measurements in psychology stated (Bradburn 1992): “One of the factors that inhibited our progress in understanding order effects has been the lack of a theoretical structure within which to investigate the mechanisms by which they might occur.” The present paper presents an attempt toward such a theoretical basis.

We consider the non-commutativity of measurement operations as the formal key to order effects. This is well established in quantum theory where observables typically do not commute: The sequence in which measurement operations act upon the state of a system makes a difference for the results obtained. One reason is that a measurement operation changes the state (even it is pure) of the measured system such that a subsequent measurement operation effectively acts on another state. Related to this, a quantum measurement is not simply the registration of a pre-existing fact, but also establishes the fact that is registered.

It is highly plausible that this basic idea holds for psychological systems as well, although states, observables and their dynamics have nothing to do with quantum physics. The mathematical feature of non-commuting observables and its ramifications can be fruitfully applied to model psychological situations where order effects abound.

We proposed a classification of particular order effects, some of which were already discovered empirically, and Hilbert space models were successfully used to analyze them: additive effects, subtractive effects, contrast effects, and consistency effects. Moreover, we predicted order effects not observed so far, which rest on uncertainty relations between variances of distributions of observables rather than on shifts of their expectation values.

We assessed the complexity of Hilbert space models for pure and mixed states, and for $n$-dimensional Hilbert spaces corresponding to questions (observables) with $n$-ary alternatives for response. It turned out that the predictive power of the model increases with increasing $n$, and that it increases dramatically with increasing number of observables. We sketched an argument

Using the example of the conjunction fallacy (Tversky and Kahneman 1983), $P_{a_j}$ may be the response to “Linda is a bankteller”, $P_{b_i}$ the response to “Linda is a feminist”, and $P_{a_j} \land P_{b_i}$ the response to “Linda is a feminist bankteller”. It is not implausible that the state of a subject having responded with $P_{a_j} \land P_{b_i}$ differs from her state having responded with $P_{a_j}$ to a degree which creates an “anomalous” order effect defying conventional Hilbert space modeling. It remains to be explored whether or not positive-operator-valued measures as mentioned in footnote 6 would be capable of resolving such a case with a proper description.
why the complexity of Hilbert space models should be generally less than that of alternative classical models.

We suggested experiments with joint measurements as interesting candidates for exploring potential limitations of Hilbert space models. If agents behave in a way conflicting with Boolean logical rules, order effects may result whose description requires non-commutative frameworks more general than standard Hilbert space models permit.

It is evident that a successful model does not entail the validity of a mechanism. However, models based on non-commuting observables stress the specific feature of strong measurement interactions. This important point, which is highly plausible for observations on mental systems, entails that non-commutative models are not only descriptively powerful but also hold the potential for explanatory surplus. In combination with their parsimony over classical models, non-commutative models offer a very attractive option for future research.

In contrast to measurements for classical systems, registering values of observables in both quantum and cognitive systems includes manipulations of the state of the system: (1) the posed and answered measurement question brings the system into a particular state in which it has (in general) not been before the question was asked, and (2) the registration itself entails a backreaction on the system which changes its state once more, so that it differs from the state that was actually measured.

This far-reaching issue shows that a quantum theoretically inspired understanding of cognition is capable of revising plugged-in cliches of thinking in terms of classically construed concepts. The law of commutativity in elementary calculations (and the related Boolean “either–or” in logic) are special cases with their own significance. But it would be wrong to believe that their generalization is restricted to exotic particles and fields in microphysics, with no application for everyday life phenomena. The opposite is the case.

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