Fitch’s knowability axioms are incompatible with quantum theory

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How can we consistently model the knowledge of the natural world provided by physical theories? Philosophers frequently use epistemic logic to model reasoning and knowledge abstractly, and to formally study the ramifications of epistemic assumptions. One famous example is Fitch’s paradox, which begins with minimal knowledge axioms and derives the counter-intuitive result that “every agent knows every true statement.” Accounting for knowledge that arises from physical theories complicates matters further. For example, quantum mechanics allows observers to model other agents as quantum systems themselves, and to make predictions about measurements performed on each others’ memories. Moreover, complex thought experiments in which agents’ memories are modelled as quantum systems show that multi-agent reasoning chains can yield paradoxical results.

Here, we bridge the gap between quantum paradoxes and foundational problems in epistemic logic, by relating the assumptions behind the recent Frauchiger-Renner quantum thought experiment and the axioms for knowledge used in Fitch’s knowability paradox. Our results indicate that agents’ knowledge of quantum systems must violate at least one of the following assumptions: it cannot be distributive over conjunction, have a kind of internal continuity, and yield a constructive interpretation all at once. Indeed, knowledge provided by quantum mechanics apparently contradicts traditional notions of how knowledge behaves; for instance, it may not be possible to universally assign objective truth values to claims about agent knowledge. We discuss the relations of this work to results in quantum contextuality and explore possible modifications to standard epistemic logic that could make it consistent with quantum theory.

If knowledge can create problems, it is not through ignorance that we can solve them.

Isaac Asimov, Asimov’s Guide to Science

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Figure 1: **Wigner’s friend thought experiment** [1]. The subjectivity of observations in quantum theory, and the motility of the cut between quantum systems and observers, is best illustrated in Wigner’s theoretical proposal. There, an agent, Alice, holds in her lab a spin-1/2 particle $R$, initially in state $\frac{1}{\sqrt{2}}(|0\rangle_R + |1\rangle_R)$. Alice measures $R$ in a given basis $\{|0\rangle_R, |1\rangle_R\}$, and stores the outcome 0 or 1 in her memory (or notebook). This memory is itself a physical system — ultimately described by quantum theory. For simplicity, it can be modelled as another spin-1/2 system, $A$, which Alice encodes in state $|0\rangle_A$ if the observes outcome 0, and $|1\rangle_A$ if she sees 1. Therefore, from Alice’s perspective, the joint state of $R$ and her memory $A$ after the measurement is either $|0\rangle_R|0\rangle_A$ or $|1\rangle_R|1\rangle_A$. Outside her lab stands a second agent, Wigner, who models Alice’s measurement as a reversible quantum process, in accordance to standard views of quantum theory like the Copenhagen interpretation [2, 3]. While Alice observes a definite outcome, Wigner describes her as being in a superposition of having seen 0 and 1. More technically, Wigner can model Alice’s measurement as a von Neumann interaction scheme [4], through which the state or the observed system $R$ is coherently copied to the memory $A$, resulting in the final entangled state $\frac{1}{\sqrt{2}}(|00\rangle_{RA} + |11\rangle_{RA})$. Note that if Wigner applies other interpretations of quantum theory, his description of the lab’s final state may differ.

## 1 Introduction

How do abstract logical systems, used to formalize the process of rational reasoning, represent the knowledge of the natural world which is afforded to us by the physical sciences? At first sight, physical theories simply provide a description of nature, approximately valid in a given regime; agents may apply the relevant physical theory to obtain predictions, and then use the same theory to reason about the conclusions other agents would draw. Things become more subtle when the theory explicitly models the agents who use it as physical systems themselves, which undergo physical evolution. Through the analysis of settings with multiple physical agents, it becomes clear that the predictions of the theory are not strictly about nature, but about agents’ knowledge of it as well. These multi-agent scenarios also demonstrate that the process by which agents reason about the world using such a theory may not always follow the assumed rules of classical logic.

A standard example where carefully tracking how different agents simultaneously use a theory yields an unexpected outcome is the quantum thought experiment known as Wigner’s friend [5],
illustrated in Figure 1. In this setting, two agents use the same physical theory (quantum mechanics) to study the same phenomenon at different scales, but they end up reaching very different conclusions. In more recent and complex thought experiments, agents modelled by quantum theory may even reach explicit contradictions in their claims about what other agents know [6]. In this manuscript, we set out to investigate how the knowledge of the natural world provided by quantum theory affects how we may adequately model knowledge in a logical setting. For the rest of this introduction, we share some intuition about why we anticipate difficulty, before a formal analysis in the subsequent sections.

Note: The main target audience of this manuscript are philosophers and logicians. Knowledge of quantum theory is useful but not strictly necessary to follow it, as the technical quantum parts are insulated in figure captions and appendices. Rather, this paper tries to motivate why logicians should care about quantum theory beyond ordinary quantum logic, and how it impacts the manner in which we use logic to model knowledge; for modern introductions to quantum mechanics, see for example the books [7, 8]. As our results may also be of interest to physicists, we kept the sections on logic as pedagogical as possible.

1.1 Modality of knowledge in quantum theory

That our knowledge of the world is modal – insofar as we may generally only have knowledge of possible future descriptions of physical reality – is one of the fundamental properties of quantum theory. Indeed, the prominent role of probabilities in quantum theory indicates that much of the mathematical structure of the theory is meant to account for modalities – physical possibilities – which never actually occur. Such is the case with superposition, for instance. Given a photon in a superposition of horizontal and vertical polarization that is in the state $\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ which is measured in the $H/V$ basis, prior to measurement, quantum theory tells us that either measurement outcome $H$ or $V$ may occur, with equal probability. However, once the measurement takes place, only one is observed. If the measurement yields $H$, for instance, then the physical possibility of measuring $V$ never becomes a physical actuality. It has even been suggested by Clifton, Bub, and Halvorson that “quantum theory be viewed, not as first and foremost a mechanical theory of waves and particles, but as a theory about the possibilities and impossibilities of information transfer” [10].

This modality of knowledge is present in most interpretations of quantum theory at some level; indeed, there have been entire interpretations of quantum mechanics which have been built on this premise [11–14]. Modally conservative (i.e. deterministic) interpretations of quantum mechanics, such as Bohmian mechanics [15], would hold that nature is not inherently modal, and that definite predictions can be made about the dynamics of any system, but our knowledge of these definite predictions is still necessarily confined to talk of probabilities. Likewise, more modally liberal interpretations (which attribute a stronger form of reality to the modalities of quantum theory), such as the conventional many-worlds interpretation [19, 20], would contend that all modal possibilities prescribed by the theory are in fact equally real (this resembles Lewis’s notion of modal realism – see, for instance, [21]). Anti-realist interpretations (such as Bohr’s interpretation [22], operationalism [23], or QBism [24]) may remain agnostic about questions of a modal ontology, but must still

1This might not be true for some versions of many-worlds interpretations [9], where the measurement of $|\psi\rangle$ would result in a branching into two worlds, both equally physical. In this case a “physical actuality” should be understood as such for a particular branch. However, modality plays an even more prominent role in many-worlds interpretations anyway, so this does not detract from the point raised here.

2For instance, due to the sensitivity of Bohmian trajectories on initial conditions and the necessary coarse graining of realistic experimental preparations and measurements understood in the ‘typicality’ school of Bohmian mechanics [16]. See [17, 18] for detailed discussions.
account for the fact that the theory is only capable of assigning probabilities to an array of possible outcomes, and cannot determine which possible outcome will occur.

1.2 Observers are also quantum systems

Another fundamental property of quantum mechanics is that observers and observed systems are not easily distinguished from one another. Indeed, the so-called Heisenberg cut [25] separating the classical and quantum parts in a physical experiment is subjective and motile [26]. In settings with several agents who may move this cut when describing each other (as either quantum systems or classical observers), the consistency of the theory is challenged, as we will see.

Different interpretations of quantum mechanics have provided various accounts of the role of observers, but this is generally still a point of conceptual difficulty in the theory. Recently, there has been great progress in understanding how the frames of reference of different quantum agents relate to each other, in well-behaved settings (e.g. [27–30]). On the other hand, there are settings in which it may be fundamentally impossible to consistently relate the views of several agents, as shown by Frauchiger and Renner in the thought experiment to be explained in Section 3 [6]. Their result has led to a flurry of recent activity trying to make sense of multi-agent quantum mechanics how different interpretations of the theory hold up in such settings [31–35].

1.3 Quantum and epistemic logics

A third property of quantum mechanics relevant for our discussion is that it quite naturally leads to a novel propositional logic which cannot be characterized by classical logic [4]. That is, if one encodes propositions about definite measurement outcomes, which are in turn related to projective subspaces of a Hilbert space, then the resulting propositional logic does not exhibit the usual distributive properties of classical propositional logic [36, 37].

Furthermore, if quantum logical propositions pertain to outcomes of measurements, and if outcomes are experienced differently by different agents, then there is already a certain epistemic flavour to the underlying logic of quantum mechanics. Coming full circle, classical epistemic logic is generally studied as a special kind of modal logic [38–40], taking possible worlds to correspond to states of knowledge. While the modalities present in epistemic logic play a different role than the modalities present in quantum mechanics (the former pertaining to states of an agent’s knowledge, the latter generally pertaining to states of systems in the physical world), modality and quantum logic have indeed been investigated in tandem before [41, 42]. Likewise, quantum logic tools have been used to further the study of epistemic logic [43, 44]. Recently, modal logic has been deployed to study epistemological questions for quantum mechanical agents [45, 46]. It is this new thread that we pursue here.

1.4 Contribution and structure

In Section 2, we model knowledge using a fairly minimal epistemic logic system, and take three basic axioms to characterize the behaviour of knowledge within this system. From those axioms, we derive Fitch’s theorem, and the paradox that stems from it, an interesting topic in philosophical logic and epistemology in its own right. We review Fitch’s theorem explore the interpretation of knowledge through possible-worlds semantics.\(^9\)

\(^9\)While the logic constructed is classical, it is later shown that the relevant derivations also hold for an underlying quantum propositional logic.
In Section 3 we review the Frauchiger-Renner quantum thought experiment. This results in a no-go theorem for interpretations of quantum mechanics, which must violate one of four natural assumptions in order to be internally consistent.

In Section 4, we formalize the Frauchiger-Renner assumptions in the epistemic logic setting and prove that any violation of the Frauchiger-Renner assumptions leads to a violation of one or more knowledge axioms.\(^4\)

We conclude that, if any reversible physical theories which is compatible with quantum mechanics is taken as a legitimate source of knowledge, then that knowledge cannot simultaneously satisfy all three knowledge axioms stated. We summarize this result in the form of a no-go theorem for quantum mechanical knowledge. Since quantum mechanics is very well confirmed empirically, we therefore conclude that our modelling of knowledge using epistemic logic must be revised in such a way that rejects at least one of the starting knowledge axioms; under several of these choices, Fitch’s theorem becomes invalid, thereby resolving the paradox. We discuss our results and their relation to other topics in logic and quantum foundations in Section 5.

2 Fitch’s Paradox

This paper is focused on the interplay between the nature of knowledge and the role of quantum theories in providing us knowledge of the natural world. In order to make the relationship between the two precise, it is necessary that we formalize knowledge in some manner which may then be compared with certain aspects of general quantum formalisms.

In philosophical logic, the usual procedure is to represent knowledge as a special symbol in some sort of epistemic logic and then posit certain axioms for how such an abstract symbol behaves. The atomic propositional symbols of this logic are generally taken to correspond to the ‘bare’ facts about which an agent may or may not have knowledge. Epistemic logic generally presents knowledge for a given agent as a modal operator in a possible-worlds semantics. The central philosophical difficulty lies in deciding which axioms such an operator ought to obey if it represents knowledge as understood with its full epistemological complexity. While some axioms are rather uncontroversial, many ‘reasonable’ choices of knowledge axioms result in paradoxes. The discussion to follow in Section 4 shows that, in light of minimal quantum physical assumptions, certain choices of knowledge axioms are not just paradoxical but in fact provably *contradictory*. These minimal knowledge axioms are those famously associated with Fitch’s paradox. We here present an introduction to Fitch’s paradox so as to clarify how this epistemic logical system models knowledge.

2.1 Knowledge in epistemic logic

To begin, consider the following question: how does knowledge track truth? One naïve answer may be that everything which is known is true, but then the question of *which* true propositions one may have knowledge of remains open. One may suppose that, indeed, *any* true proposition is one which an agent may, in principle, know, even if they don’t know it at the moment. Conversely, however, it is unclear exactly what would be meant by a true proposition which cannot, even in principle, be known. Certainly, one cannot demonstrate that any particular statement is true but unknowable since a formal derivation of its truth would seem to require that one knows that statement. How can these conflicting ideas be reconciled?

\(^4\)Some of these derivations require a minimal reflexivity assumption on the structure of the underlying modal semantics. Two out of three of them may be derived intuitionistically.
First, to begin talking about knowledge in a logical setting, we need a formal language and semantics in place adequately equipped to handle claims of the form ‘agent \(i\) knows \(\phi\),’ and other, more complicated claims about knowledge, such as ‘agent \(i\) knows that agent \(j\) does not know \(\phi\).’ Moreover the possibility of knowledge is clearly of great importance to this discussion, so a logic capable of modeling knowledge ought to be modal. Hence, the natural setting for a discussion about the interplay between logic and knowledge is some sort of modal system equipped with a Kripke possible-worlds semantics \([47]\) and a collection of modal operators \(\{K_i\}\), where \(K_i \phi\) is understood to mean that agent \(i\) knows \(\phi\). Though epistemic modal logic is a rich area of study in its own right, we here use only a very minimally equipped Epistemic Modal Propositional Logic\(^5\) (henceforth EMPL) to discuss these ideas.

The logic EMPL is a modal propositional logic with these added \(K_i\) operators (it includes the usual possibility operator \(\Diamond\) and necessity operator \(\Box\) as well). Modal propositional logic is a system whose semantics includes a collection of so-called ‘possible worlds,’ each of which has its own truth valuations for logical formulas (thus, one speaks of truth ‘at a world’). There is then an ‘accessibility relation’ which indicates which worlds are related to each other in some way. The interpretation of accessibility depends of the context in which this logic is used. For example, for systems describing knowledge of a group of agents accessibility relation captures the settings where agents do not possess the complete information about a possible world they are in. Then they may consider other possible worlds; for instance, if the agent doesn’t know if it is raining in London, she can consider both the world where it is indeed raining in London, and the world where it isn’t. Both worlds are therefore epistemically ‘accessible’ to this agent. In a less philosophical example, one may view different arrangements of a chess board as possible worlds, and treat one world as accessible to another if it can be reached by a single legal chess move.

The details of how formulas are formed and how semantic valuations behave in EMPL are entirely standard, and provided in Appendix A. For now, the underlying propositional logic of of this system is taken to be usual classical propositional logic (axiomatized by the Hilbert-Ackermann axioms \([48]\), for instance), though there will be reason to reconsider this once quantum mechanics enters the discussion.

A note on logical notation is in order. A formula \(\phi\) is deduced from a collection of formulas \(\Sigma\) (denoted \(\Sigma \vdash \phi\)) if there is a finite sequence of formulas terminating with \(\phi\) which are all either (i) axioms, (ii) elements of \(\Sigma\), or (iii) obtained from previous formulas in the sequence by applying the permitted rules of inference. That is, every ‘line’ of the deduction is either a assumption in \(\Sigma\), an axiom, or a permitted inference. However, deductions are simply sequences of symbols. The notion of truth arises from the underlying semantics of the logic in question. If a formula \(\phi\) is semantically entailed by a collection of formulas \(\Sigma\) (i.e. truth of \(\phi\) follows from the truth of all elements of \(\Sigma\)), we instead write \(\Sigma \models \phi\).

2.2 The constructivist hypothesis

Suppose one wishes to start analyzing knowledge through the interpretation that every true assertion may, in principle, be known. This is called the constructivist hypothesis, and may be written down as a collection of axioms of the form \(\vdash \phi \rightarrow \Diamond K_i \phi\), where \(\phi\) is any well-formed formula\(^6\) \([49]\) and \(i\) is any agent (we shall henceforth simply refer to ‘formulas’ whose well-formedness is implicit unless

\(^5\)This approach to epistemic modal logic is, of course, not novel, and is based on the early ideas of Hintikka \([38]\). For a modern introduction, with notation more accessible to the reader familiar with modern Kripke semantics see \([39]\).

\(^6\)That is, a string of symbols which are put together in a grammatical way. See Appendix A for the definition of well-formedness.
otherwise stated). That is, for the constructivist, if $\phi$ is true in a particular world (with respect to some Kripke semantics), then there is an accessible world where any epistemic agent legitimately has knowledge of $\phi$.

This may appear at first glance to be a strong assumption to make. The motivation for this stance is that the constructivist often uses intuitionistic logic in their reasoning about the word [50]. Essentially, they equate truth with demonstrability; the constructivist and the intuitionist only ascribe truth to those things which can be constructively demonstrated (thus rejecting the law of the excluded middle and hence proofs by contradiction). Intuitionistic Propositional Logic (IPL), however, has a natural interpretation in the modal setting.

The modal interpretation of IPL is that possible worlds represent what may be called ‘states of deduction,’ where the set of formulas which are true at a world are precisely those which an agent has formally deduced at that world, and where a world $u$ is accessible to a world $w$ if and only if everything which has been deduced in $w$ has also been deduced in $u$, and any additional true formulas in $u$ may be deduced from the other formulas already deduced in $w$. In this way, inter-world accessibility represents the possibility of one carrying out additional deductions.

Gödel presented a formal translation taking intuitionistic formulas into modal propositional logic (MPL) formulas in the system S4 (the modal system where accessibility relations are both reflexive and transitive) [51]. The Kripke model which then represents states of deduction has the property that valuation is hereditary in the sense that, if some formula $\phi$ is true at some world $w$, then if $u$ is another world which is accessible to $w$, $\phi$ is also true at $u$. That is, the set of MPL-valid formulas in this representation must be accumulative; the intuitionist does not forget what they have already deduced; they can only ever learn more [52].

Since the Gödel translation of IPL pushes forward into the normal modal system S4, and since S4 is complete and decidable (in the sense that every formula either has a finite proof of its validity or a finite counter-model), the constructivist (who uses intuitionistic logic) could, in principle, use MPL decidability to algorithmically prove the truth or falsity of any IPL formula they are presented with in a finite amount of computing time. For the constructivist, any true proposition may be deduced in any world via decidability, and so at any given state of deduction, there is an accessible world where any other true proposition has been deduced (noting transitivity in S4). The constructivist would further contend that having a formal constructive demonstration of the truth of some formula is sufficient justification to claim that one has knowledge of that formula. Thus, these worlds as states of deduction are, for the constructivist, equivalently states of knowledge. As such, the constructivist would posit that truth of a formula strictly implies the possibility of knowing that formula.

This idea that inter-world accessibility allows one to track states of knowledge may be extended more generally so as to allow ‘knowledge’ to refer to more than mere deduction. One need not be so strict as to suppose that knowledge and deduction are equatable notions to still make use of the possible-worlds framework for studying knowledge. Thus, such a modal semantics provides a robust framework to model knowledge more generally.

It should be noted that knowledge is, of course, a time-dependent concept; generally one does not know exactly the same things later in life as they knew earlier. People learn. There is a degree of complexity which is added if we let $K_i$ depend on time, but the problem of Fitch’s paradox is a

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7The term ‘normal’ refers to a class of MPL systems, called K, T, B, S4, and S5, whose accessibility relations have certain properties (e.g. reflexivity, symmetry, and transitivity). The system S4 requires accessibility to be reflexive and transitive.

8Hintikka discusses a notion of defensible and indefensible knowledge in Section 2.6 of [38]; defensibility is analogous to the constructivist notion of possible worlds as states of knowledge accessed by deduction.
pathological feature of the $K_i$ operators irrespective of temporality [53], and so we shall here take $K_i \phi$ to mean that some epistemic agent $i$ has knowledge of $\phi$ at some time in the past, present, or future, without specification [49]. We label the constructivist hypothesis as CONST, and taking the perspective of a constructivist, we now promote CONST to a collection of axioms and further justify it in Appendix B.

2.3 Knowledge axioms

One nice feature of an epistemic modal logic is that it allows us to simply state the properties we think knowledge (i.e. the $K_i$ operators) has rather than needing to provide a full account of what knowledge is. The interesting result of Fitch is that, with very minimal such assumptions, one ends up with an apparent paradox.

We have already encountered one knowledge axiom, CONST. We now introduce another. It seems that if we know $\phi$ to be true, then is not possible to also know that we do not know $\phi$ to be true.\(^9\) This may be symbolically represented by the axiom $\neg \lozenge[(K_i \phi) \land K_i(\neg K_i \phi)]$, which we here label as KCONT (knowledge continuity). There will be another knowledge axiom which we shall assume in what is to follow, but first, we derive and discuss Fitch’s paradox using only CONST and KCONT.

2.4 Derivation of Fitch’s paradox

With this notion of knowledge in mind, drawing inspiration from Dummett’s derivation of Fitch’s paradox [49], we may now derive the intuitionistically valid version of Fitch’s paradox. Essentially, Fitch’s paradox is a result which shows that CONST and KCONT imply that every true formula is known by every agent. This seems strange, for we may wish to think that knowledge is somehow contingent, whereas logical truth is not. That is, whether or not an agent knows something is dependent on many uncontrollable factors, and thus should not be equivalent to truth. However, we have the following:

**Theorem 1 (Fitch [53])** For all MPL formulas $\phi$, and all agents $i$, if we assume CONST and KCONT,

$$\vdash (\phi \rightarrow \neg\neg K_i \phi).$$

**Proof:** For any MPL formula $\phi$, let $\psi_\phi := \phi \land \neg K_i \phi$. This formula reads ‘$\phi$ is true, but agent $i$ does not know that $\phi$ is true.’ Then we have:

1. $\psi_\phi \rightarrow \lozenge K_i \psi_\phi$ (CONST)
2. $\neg \lozenge K_i(\phi \land \neg K_i \phi) \rightarrow \neg(\phi \land \neg K_i \phi)$ (Contraposition)
3. $\neg(\phi \land \neg K_i \phi)$ Modus Ponens on 1b and 2
4. $\phi \rightarrow \neg\neg K_i \phi$ Intuitionistic Propositional Logic

\(^9\)See Chapter 2 of [38] for a thorough discussion of knowledge continuity, and Section 4.11 for a discussion of this particular schema (which Hintikka calls the epistemic version of Moore’s problem).
Thus, we see that \( \vdash (\phi \rightarrow \neg \neg K_i \phi) \) intuitionistically. If we allow classical deductions beyond those which are intuitionistically permitted, double negation elimination allows further that \( \vdash (\phi \rightarrow K_i \phi) \). This classical result is Fitch’s paradox, and it is read as ‘if \( \phi \) is true, then \( \phi \) is known by any agent \( i \).’ It appears as though if one takes this constructivist stance, then they must additionally accept that all formulas which are true are also known to be true by all epistemic agents. The intuitionistic result derived above is less problematic, but still puzzling, and it is this more minimal result of Theorem 2 which we shall label as FITCH. We shall denote the classical result \( \phi \rightarrow K_i \phi \) by FITCH*.

This appears unsettling, as naïvely some truths exist which can never be known: for instance, if someone flips a coin, and never looks at whether it landed heads or tails before returning the coin to their pocket, then there is a definite fact of the matter about the outcome of the coin flip; it is either true that it landed heads, or that it landed tails, and this fact of the matter could in principle be known. However, once the coin is returned to the pocket, it seems unreasonable to suppose that this fact of the matter may ever be known in the future, and if the coin flip was fair, then it is impossible for it to be known at any point in the past. Whence some true fact had the possibility of being known, but was never actually known. This result is perfectly consistent with the constructivist hypothesis, however, it is in direct opposition with the result of Fitch.

It seems that if one is to be a constructivist while still having a coherent theory of knowledge, they must somehow reconcile this result, either by accepting it and interpreting it in a meaningful way [49, p. 52][54], or by demonstrating that the argument by which the result was obtained is faulty [55–57]. We leave these interpretational issues of Fitch’s paradox open, and merely take it to be a theorem of a particular epistemic logical system.

2.5 Distributivity axiom

Though not strictly necessary in the derivation above, Fitch further assumes that knowledge is distributive over conjunction, that is \( K_i (\alpha \land \beta) \leftrightarrow K_i (\alpha) \land K_i (\beta) \) [53]. This means that an agent knows \( \alpha \) and \( \beta \) together if and only if they know \( \alpha \) and \( \beta \) separately: for example, I know my full name if and only if I know my first name and I know my last name.

We shall make a similar supposition in the remainder of this article as well. For our purposes here, we in fact only require one direction of this, namely that \( K_i (\alpha) \land K_i (\beta) \rightarrow K_i (\alpha \land \beta) \) which we here label as DIST. The content of DIST may seem trivially obvious; however, it is important to realize that our intuition about conjunction is prone to failure; for instance, without making use of probability theory, intuitive probabilistic reasoning can easily fall prey to the conjunction fallacy.\(^{10}\) Thus it is important to emphasize that this is an assumption being made. We take DIST to also be an axiom for knowledge.

2.6 Reflexivity assumption

In addition to these knowledge axioms, there is one more assumption which we make about the semantics of the logic in question. We shall henceforth assume that the accessibility relation \( \mathcal{R} \) is reflexive so that \( \mathcal{R} w w \) for all worlds \( w \in \mathcal{W} \). If we recall the Gödel translation of IPL into S4 modal logic, we see that in the fully constructivist reading of possible worlds as states of knowledge, accessibility is not only reflexive, but transitive as well, so this is not a bizarre requirement to

\(^{10}\)The fallacy wherein the likelihood of a conjunction of two occurrences \( A \& B \) is misjudged to be greater than the likelihood of the single occurrence of \( A \) alone [58].
Indeed, Boge [46] also argues that the modal logical representation of the Frauchiger-Renner paradox requires reflexivity, so this is not novel even in the context of modelling quantum paradoxes. We shall denote this assumption REFL and when it is necessary for the validity of some formula $\phi$, we shall write $\models_{\text{REFL}} \phi$ to make this clear.

The assumption REFL turns out to be necessary for the full generality of our claims in what is to follow. Our general result is the Frauchiger-Renner paradox from quantum theory implies that the knowledge axioms thus described yield contradictions. However, if inter-world accessibility is not reflexive, this result may be avoided. It is thus natural to now ask why we ought to think that states of knowledge obey this restriction. The answer to this is essentially that if accessibility is not reflexive, knowledge starts doing strange things; agents may be logically required to forget things or to learn specific new things.

Specifically, given an EMPL model $\langle W, R, I \rangle$, if $R$ is not reflexive, then there is some formula $\xi$ and some world $w$ for which $V_{\mathcal{F}}(\xi \rightarrow \Diamond \xi, w) = 0$. However, if there is such a formula of the form $\xi := K_i \phi$ for some formula $\phi$, then $V_{\mathcal{F}}(K_i \phi \land \neg \Diamond K_i \phi, w) = 1$, whence agent $i$ loses knowledge of $\phi$ in all future states of knowledge. Likewise, if $\xi := \neg K_i \phi$ for some $\phi$, then $V_{\mathcal{F}}(\neg K_i \land \neg \Diamond \neg K_i \phi, w)$, and so $V_{\mathcal{F}}(\neg K_i \land \Box K_i \phi, w) = 1$, whence agent $i$ necessarily acquires knowledge of $\phi$ in the next state of knowledge. Thus, if accessibility between states of knowledge is not reflexive, there can be necessary knowledge acquisition or necessary forgetfulness depending on where reflexivity fails. This is analogous to the problem of Barcan formulas in quantified modal logic [40], wherein certain features of modal semantics may necessitate existence – a very troubling prospect for metaphysicians – if not properly modified. If the $K_i$ operators have this necessitating pathology (effectively forcing knowledge or forgetfulness), they do not seem to correspond to what we want to call knowledge.

Thus, it is reasonable to suppose that REFL holds for any EMPL-model which captures knowledge.

Summarizing the important facts from this section, we can model knowledge for multiple agents using the epistemic modal logic laid out in Appendix A, wherein possible worlds encode states of knowledge. Within this system, Fitch’s result tells us the following:

$$\text{CONST} \land \text{KCONT} \rightarrow \text{FITCH}.$$  

The knowledge axioms we shall henceforth assume for the knowledge operators $K_i$ are given by:

$$\begin{align*}
\text{CONST}: & \quad \phi \rightarrow \Diamond K_i \phi \\
\text{DIST}: & \quad K_i(\alpha \land K_i(\beta)) \rightarrow K_i(\alpha \land \beta) \\
\text{KCONT}: & \quad \neg \Diamond [K_i(\phi) \land K_i(\neg K_i \phi)].
\end{align*}$$  \hspace{1cm} (1)

We shall also suppose that all relevant EMPL-models have reflexive accessibility relations, thereby satisfying REFL.

Below, we study a particular quantum mechanical thought experiment, the Frauchiger-Renner paradox [6], and show that it yields new insights into the nature of knowledge, such that at least one of the axioms in (1) fails or the accessibility relation needed to model cannot be reflexive, concluding that the derivation of Fitch’s paradox rests on an inadequate account of knowledge. More precisely, we show that if quantum mechanics gives us knowledge of the world, then knowledge must be fundamentally constrained in just the right way, possibly preventing Fitch’s paradox from getting off the ground as well, or otherwise requires irreflexive states of knowledge which pose new problems of their own.
3 Frauchiger-Renner Paradox

As we have seen in the previous section, paradoxes in philosophy may lead to more thorough understanding of how knowledge works in general, including in multi-agent scenarios. Physical paradoxes, on the other hand, give us a better understanding of just how our physical theories provide us with knowledge of the world in the first place. For example, the twin paradox in special relativity [59] highlights the importance of the concept of simultaneity in different reference frames. Therefore, paradoxes wherein the knowledge of multiple agents is an explicit part of a physical setting have the potential to be exceptionally insightful for both physics and philosophy.

An example of such a paradox is the Frauchiger-Renner paradox. Formulated within the framework of quantum theory in a multi-agent setting, this paradox presents a scenario where agents are allowed to reason about each other’s knowledge and yet, they reach contradictory descriptions of affairs [6]. The memories of agents are modeled as quantum systems. Here we provide a brief conceptually-oriented exposition of the paradox; for those who would like to read through a technical proof please refer to Appendix D.

The scenario consists of four agents: Alice, Bob, Ursula and Wigner, and two spin systems (qubits) $R$ and $S$ (Figure 2). Together, these agents perform an experiment on these systems. It can be shown that agents can make deterministic inferences about each other’s measurement outcomes at various stages of the experiment. If they are allowed to combine these inferences, they come to an apparent contradiction: whenever Ursula and Wigner both obtain a particular outcome (denoted “ok”), Wigner can reason with certainty that Alice can predict him getting a distinct outcome (denoted “fail”). The reasoning in this chain of statements bears a structural similarity to the epistemic paradox of a Liar cycle [60] in classical logic. The Liar cycle is an extension of the infamous Liar paradox, which leads to a logical inconsistency for a chain of statements of arbitrary finite length. More concretely, a Liar cycle of length $N$ has the following form:

$$\phi_1 = \text{"\phi_2 is true"}, \phi_2 = \text{"\phi_3 is true"}, \ldots, \phi_{N-1} = \text{"\phi_N is true"}, \phi_N = \text{"\phi_1 is false"}. \quad (2)$$

Each statement in the cycle refers to the truth value of the next one, the $N$th statement completing the chain by referring to the first – as in the statements produced by Alice, Bob, Ursula and Wigner. The contradiction then arises independently of the truth value one assigns to the statements in the chain.

As in any thought experiment, the result obtained here relies on the assumptions made to derive it. In the original paper [6], three assumptions are considered (Q, C and S), with a fourth implicit assumption U pointed out in [45]. Here we state them conceptually:

- **Q**: agents are allowed to use rules of quantum theory, in particular, the Born rule, to reason about other agents’ outcomes;
- **C**: agents can reason from the viewpoints of each other using the rules of quantum theory, and combine statements according to rules of classical logic;
- **S**: after performing a measurement, agents witness only single outcome;
- **U**: agents model the evolution of other agents and respective labs as a **unitary** (i.e. reversible) process.

There is a comment here to be made about what we mean by “reasoning” carried out by agents. When we speak of an agent’s “reasoning,” we mean the process by which they deduce statements
Frauchiger-Renner thought experiment [6]. The technical details of the paradox are left to Appendix D, and here we explain the paradox on the meta-level. The setting involves four agents: Alice, Bob, Ursula and Wigner, and two spin systems $R$ and $S$. The memories of the agents Alice and Bob are modeled as qubits $A$ and $B$, initialized in states $|0\rangle_A$ and $|0\rangle_B$ respectively. System $R$ is prepared in the state $\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{3}|1\rangle$. Alice measures system $R$ in basis $\{|0\rangle_R, |1\rangle_R\}$, and writes down the result in her memory ($t = 1$). If she obtains outcome $a = 0$, she prepares qubit $S$ in state $|0\rangle_S$, and otherwise in state $\frac{1}{\sqrt{2}}(|0\rangle_S + |1\rangle_S)$. Alice sends $S$ to Bob, who measures it in the basis $\{|0\rangle_S, |1\rangle_S\}$, and writes down the result into his memory ($t = 2$). Finally, Ursula and Wigner measure Alice’s and Bob’s labs respectively in the rotated bases $\{|ok\rangle_{RA}, |fail\rangle_{RA}\}$ ($t = 3$) and $\{|ok\rangle_{SB}, |fail\rangle_{SB}\}$ respectively ($t = 4$). If agents are allowed to use quantum theory and to make inferences about each other’s outcomes (here expressed in thought bubbles for each agent), they come to a contradiction.

about measurement outcomes which will occur with certainty – but what does “certainty” mean in the first place? For the purposes of the thought experiment, it suffices to take certainty as defined by Q and C: a statement is certain if it occurs with probability one according to the physical theory one is using, or according to other fellow (trustworthy) agents. We comment further on quantum mechanical knowledge in the next section.

Since the Frauchiger-Renner thought experiment yields an explicit contradiction, we must conclude that not all of these assumption — Q, U, S, and C — may be simultaneously held, and at least one must therefore be violated. Thus, the multi-agent thought experiment considered in the Frauchiger-Renner paradox provides a counter-example to the assumptions Q, U, S, and C. If we denote this counter-example by FR, the conclusion of this section, written in a compact way, is:

$$FR \implies (-Q) \lor (-U) \lor (-S) \lor (-C)$$
**Note on reversibility (unitarity).** The assumption $U$ supposes the reversibility of the theory agents are using to draw their conclusions. Not all interpretations (or modifications) of quantum mechanics satisfy this condition. For example, objective collapse theories [61] deny it by construction. They attempt to describe the quantum behavior of microscopic systems and the absence of superpositions at the macroscopic level in a unified way by modifying the reversible evolution described by Schrödinger equation.

For the sake of keeping to the original postulates of quantum theory with the unitary evolution governed by the Schrödinger equation, in this work we consider only reversible theories, and assume that $U$ is a permissible assumption. Hence, our consideration of the Frauchiger-Renner paradox calls for a rejection of one of $Q$, $S$, or $C$.

Even if one is not a realist about quantum mechanics, the Frauchiger-Renner paradox applies to how observers perceive reality and how they derive their knowledge about it. Hence, this paradox is epistemic in nature. It is therefore natural to compare it to existing results about the paradoxical nature of knowledge, such as Fitch’s result, which we do now.

### 4 From Frauchiger-Renner to Fitch

Here we investigate how the Frauchiger-Renner paradox assumptions interact with the knowledge axioms used to derive Fitch’s paradox, and demonstrate that they are fundamentally at odds with each other. This leads to a no-go theorem about quantum mechanical knowledge. To begin, it is important to understand precisely how one ought to represent the assumptions of the Frauchiger-Renner paradox in the epistemic-logical setting, but first, we must understand just how quantum mechanics is capable of providing us with knowledge in the first place.

#### 4.1 Quantum mechanical knowledge

To proceed, we must determine whether the assumptions of Frauchiger and Renner are *epistemic* or *doxastic* in nature. That is, do $Q$, $S$, and $C$ make claims about an agent’s *knowledge*, or their *belief*? Indeed, the use of the word “certain” in the original formulation of these assumptions by Frauchiger and Renner [6] may seem to allude to a notion of belief, as has been pointed out by Boge [46, pp. 1150–1151]. However, upon closer inspection, we see rather that this assumption pertains to an agent’s *knowledge* of a state of affairs, where ‘knowledge’ is understood partially in terms of the agent’s ability to apply quantum mechanical reasoning to a physical system.

It is true that $Q$, $S$, and $C$ make claims about an agent’s certainty of some class of propositions. Frauchiger and Renner assert that an agent using them would believe these particular propositions. However, there is more here than just belief; the claim is not merely that a particular class of agents *believes* these propositions, but more specifically, that they are *justified* in believing these propositions by virtue of the fact that they are using quantum mechanics to reason about the world. Thus, the Frauchiger-Renner assumptions lay claim to an agent’s justified belief. If the agent then takes quantum mechanics (in any form) to provide a *true* account of the nature of reality, these assumptions provide them with *justified true belief* (JTB). Although the idea that all knowledge takes the form of JTB is well-known to be flawed due to the existence of notable counter-examples (so-called Gettier cases [62]), the kind of JTB provided by quantum mechanics in this framework does not immediately seem to be of the pathological sort which is prone to Gettier-type criticisms. Thus, even as JTB, the Frauchiger-Renner assumptions seem to make claims about
quantum mechanical knowledge, and not just belief.

In the previous paragraph, it was mentioned that an agent must take quantum mechanics to present a ‘true’ description of reality. This may seem like a strong commitment, for it might seem as though we are requiring an agent be a realist about quantum mechanics (for instance, to follow interpretations such as Bohmian mechanics [15], Everettian many-worlds [19], or Consistent Histories [63]) and there are of course many objections to this stance. Crucially, however, we do not require that the notion of ‘truth’ here be one of correspondence which would entail such a realist reading. If an agent takes quantum mechanics to provide a true description of reality, where truth is instead understood to be a coherentist or pragmatic notion (see, for instance, [64]), they may still maintain that quantum mechanics provides them with justified true beliefs, even if their interpretation of quantum mechanics is not realist, but rather phenomenal or pragmatic [24, 65], respectively. For if they take truth to be about coherence of commitments,\(^{11}\) this corresponds to the degree to which their description the world is self-consistent, whence the neo-Kantian Copenhagen interpretation of Bohr\(^{12}\) [22] or the operationalism of Bridgman [23] may assert JTB. Likewise, if they take truth to measure to one’s ability to succeed at carrying out tasks pragmatically, then a pragmatic interpretation may still assert JTB.

This conclusion that the Frauchiger-Renner assumptions pertain to the knowledge of agents should be unsurprising. After all, they are standard formal assumptions present in essentially every interpretation of quantum mechanics. Cabello [67] notes that interpretations may generally be split into two categories; intrinsic realist interpretations which take the probabilities ascribed by quantum theory to be determined by intrinsic features of reality, and participatory realist interpretations which instead deal with the experiences of observers, rather than the system being observed. Intrinsic realist interpretations provide us with knowledge of the world because they take states to be in direct correspondence with facts of the matter; \(\psi\)-ontic variants take states to describe things in the world, and \(\psi\)-epistemic variants take states to describe our knowledge of things in the world. Participatory realist interpretations, on the other hand, are either about the knowledge an agent has of their future experiences, or the degrees of belief they have in the expected outcomes of future experiments. Cabello only lists QBism in the category of belief-oriented participatory realist interpretations, and all other categories clearly take quantum theory to be a source of knowledge. As shown above, if we take a pragmatic stance on truth, even QBism offers JTB, and thus takes quantum theory to be a source of knowledge. Therefore, there should remain no controversy that quantum theory offers agents a means for obtaining knowledge — either about the world, or otherwise about the future state of their experiences.

By establishing that the Frauchiger-Renner assumptions make claims about quantum mechanical knowledge, we are now justified in deploying an epistemic logic to model the quantum mechanical knowledge of a particular agent making use of a theory which satisfies the Frauchiger-Renner assumptions.

\(^{11}\)The coherence theory of truth asserts that truth is a measure of how well a statement ‘fits’ within a given framework and ‘coheres’ with the other elements of that framework [64]. A coherentist whose worldview is given by quantum mechanics would claim that a statement about quantum measurement outcomes in a given context is true or false based on how well that statement agrees (i.e. coheres) with the predictions of quantum mechanics, irrespective of the degree to which that statement may correspond to some fact about the world (and thus irrespective of the degree to which quantum mechanics provides a literally true description of reality, as the realist might claim).

\(^{12}\)We note that there are, in fact, two distinct Copenhagen interpretations. Howard has demonstrated that the common conception of the Copenhagen interpretation (which is often attributed to Bohr) is instead a historical artefact due to Heisenberg [66]. We refer here to the Copenhagen interpretation which may legitimately be attributed to Bohr instead.
4.2 Formalizing FR and Fitch

We now determine how the Frauchiger-Renner assumptions may be formalized in the same sort of epistemic logic as the knowledge axioms discussed in Section 2. To formalize the Frauchiger-Renner assumptions, a naïve first strategy would be to express them as constraints on admissible knowledge axioms in the same logical system (EMPL which is constructed in Appendix A) in which we first stated the knowledge axioms. However, there is an inevitable difficulty this will lead to: the underlying propositional logic of the EMPL system is classical propositional logic, while the sorts of propositions the Frauchiger-Renner assumptions pertain to (i.e. quantum measurement outcomes) obey quantum logic instead. Rather than using EMPL, we need to find an alternative system whose propositions obey quantum logic. It is still a hotly debated issue what the best way to axiomatize quantum logic might be and there have been many different approaches [68–70]; see [36] for an overview of this interesting problem.

In order to proceed, we do not construct a full epistemic modal quantum logic, as this is a hard problem. Rather, we shall continue to use the classical system EMPL, but we shall only use logical deductions which are admissible in the Birkhoff-von Neumann Hilbert space quantum logic [4]. We shall prove these deductions to be valid (according to quantum logic) using the formalism and notation developed by Svozil [37].

**Definition 2 (Quantum logic primitives)** Given a Hilbert space $\mathcal{H}$ which corresponds to the state space of the quantum system in question and a formula $\phi$:

- (correspondence) $\phi$ corresponds to a particular (projective) measurement outcome; to each formula $\phi$, there is a projective subspace $M_\phi \subseteq \mathcal{H}$;
- (truth assignment) for a system a state $|\chi\rangle \in \mathcal{H}$, $\phi$ is true if and only if $|\chi\rangle \in M_\phi$;
- (negation) the negation of a formula $\phi$ is encoded in the orthogonal complement of $M_\phi$, i.e. $M_{\neg \phi} = M_\phi^\perp$;
- (disjunctions and conjunctions) $M_{\phi \lor \psi} = M_\phi \cup M_\psi$ and $M_{\phi \land \psi} = M_\phi \cap M_\psi$;
- (conditionals) $\phi \rightarrow \psi$ is true if and only if $M_\psi \subseteq M_\phi$;
- (tautologies) if $M_\phi = \mathcal{H}$, then $\phi$ is tautologically true for any state $|\chi\rangle$; if $M_\phi = 0$ (the zero subspace) then $\phi$ is trivially false for all states $|\chi\rangle$.

In the basic quantum logic formalism defined above, we can ensure that our use of the classical logic EMPL never extends beyond where it would agree with a quantum logical variant should one be constructed. For that, it is enough to show, as we do in C, that contraposition (Lemma 9) and a particular connective identity (Lemma 10) are valid in quantum logic. This ensures that if the assumptions needed to derive Fitch’s paradox (the axioms stated in (1)) are permissible, then the Fitch’s paradox itself can be derived in the framework of Definition 2 if extended to EMPL. Let us now reformulate the Frauchiger-Renner assumptions in the language of EMPL, supposing that propositions correspond to those of the quantum logic in Definition 2.

**Definition 3 (Assumptions Q, U, S and C)** In the language of quantum logic as in Definition 2, the assumptions of Frauchiger-Renner thought experiment can be formulated as following:
• **U:** all agents apply rules of reversible evolution to describe the dynamics of the quantum systems involved in the experiment, the evolution is assumed to be applicable at all scales.

• **Q:** for any quantum system studied by agent \( i \), if the formula \( \phi \) is true of that system (according to quantum logic), then agent \( i \) knows \( \phi \), i.e. \( V_\phi(K_i \phi, w) = 1 \) for any \( \phi \) which quantum mechanics tells \( i \) is true at world \( w \).

• **S:** for all worlds \( w \) and all formulas \( \phi \), it is the case that \( V_\phi(K_i \phi \land K_i(\neg \phi), w) = 0 \).

• **C:** for all worlds \( w \) and any two agents \( i \) and \( j \) it is the case that: \( V_\phi(K_i(K_j \phi) \rightarrow K_i\phi, w) = 1 \).

We note that each inference made in the original Frauchiger-Renner thought experiment is a valid deduction in the quantum logic described in Definition 2. Using the rules governing the knowledge operators of the agents as in Definition 3, one can show that the inconsistency arises of views at the conclusion of the thought experiment is valid. The formal epistemic proof can be found in Appendix B of [45].

One may interpret the inconsistency in the Frauchiger-Renner scenario in either of two ways; (i) take it as a refutation of the Frauchiger-Renner theorem, or (ii) take it as imposing limitations on how knowledge can work in a world governed by quantum mechanics. Since the derivation appears to be legitimate, in our discussion below, we take the latter route, opting to reject at least one of the assumptions in Definition 3. We now demonstrate that, so long as \( U \) holds, a violation of any of the other assumptions (\( Q \), \( S \), or \( C \)) results in a contradiction if we suppose knowledge obeys all of the axioms in Equation 1, as well as the reflexivity assumption.

### 4.3 Violating Q

To begin, we suppose that the solution to the Frauchiger-Renner paradox is that the assumption \( Q \) is violated. In this paper, the violation of \( Q \) means that there is a proposition \( p_Q \) which follows from laws of quantum mechanics (i.e. it represents a measurement outcome which occurs with certainty) but which an agent cannot be certain of by using quantum mechanics to make inferences.

In the Theorem 11 we show that we cannot accept the axioms \( \text{CONST} \) and \( \text{KCONT} \) in (1) while also allowing \( Q \) to be violated without contradiction. In other words, to violate \( Q \) while claiming that quantum mechanics gives us knowledge about the world forces us to dispense with at least one of the starting assumptions, \( \text{CONST} \) or \( \text{KCONT} \), if we wish to maintain that the underlying logic is consistent. While we have not proven consistency for the logic in question (and indeed have not provided enough structure for the \( K \) operators to do so), it is taken to be an important value for such an epistemic logic; for other alternatives, see Section 5. Thus, either \( Q \) is not violated, or \( \text{CONST} \) and \( \text{KCONT} \) cannot both be true of knowledge, which we can express as

\[
\neg Q \iff \neg(\text{CONST} \land \text{KCONT})
\]

Accepting \( Q \) may seem necessary if we take quantum theory to be a legitimate basis for knowledge. Indeed, it seems natural that this would lead to a contradiction. We must nevertheless consider the possibility. This theorem indicates that rejecting \( Q \) obligates us to reconsider our notion of knowledge, as the knowledge axioms must be put into question.
4.4 Violating S

Here, we proceed in a similar manner as with Q, supposing in accordance with the Frauchiger-Renner paradox, that S is instead violated, and also that the premises of Fitch’s paradox are acceptable. Following Definition 3, a violation of S means there is some proposition \( p_S \), world \( w \), and agent \( i \) for which \( V_\sigma(K_i p_S \land K_i(\neg p_S), w) = 1 \). Using DIST, this further implies \( V_\sigma(K_i(p_S \land \neg p_S), w) = 1 \), and so if S is violated, there is some agent in some world who has knowledge of a contradiction. This result is formally captured in Lemma 12.

If S is violated, we must either either dispense with one of CONST and KCONT (else find a way to control contradictions by modifying EMPL to be a paraconsistent logic, see Section 5), or we must find a way to understand what it means for the accessibility relation to not be reflexive. This result is proven in Theorem 13, and can be summarized as follows:

\[ \neg S \iff \text{REFL} \neg (\text{CONST} \land \text{KCONT}). \]

4.5 Violating C

In this epistemic modal framework, C says that if one agent knows that some other agent knows some statement about the world (and assuming they can trust this other agent), then they may themselves infer knowledge of this statement. This may be formalized in terms of the scheme:

\[ K_i(K_j \phi) \rightarrow K_i \phi, \]

in all worlds, for all agents \( i \) and \( j \), and any formula \( \phi \). This scheme seems very reasonable and indeed, other epistemic logics have this as a theorem (see [38, pp. 60–61]). If C is not violated, and if we suppose the knowledge axioms CONST and KCONT as in (1) (which are enough to derive the results FITCH and FITCH*), then we may arbitrarily collapse or exchange knowledge operators for distinct agents without difficulty, noting the following (arrows indicate a modification of the formula obtained by applying a corresponding assumption above the arrow):

\[
\begin{array}{c}
   K_i \phi \\
  \downarrow \text{FITCH*}(j)
\end{array} \quad \begin{array}{c}
   K_j(K_i \phi) \\
  \downarrow \text{C}
\end{array}
\]

As such, knowledge for distinct agents may be completely decoupled. Using C and FITCH*, the knowledge of all agents is the same (there is a sort of universal, communal knowledge). If we suppose that C is violated, then there is some pair of agents \( i \) and \( j \) such that \( V_\sigma(K_i(K_j p_C) \land \neg K_i p_C, w) = 1 \) for some formula \( p_C \) and world \( w \). Using this fact, we shall be able to prove that a violation of C entails a violation of either CONST, KCONT, or DIST (assuming that accessibility is reflexive). This may be summarised as follows (Theorem 14 in the appendix):

\[ \neg C \iff \text{REFL} \neg (\text{CONST} \land \text{KCONT} \land \text{DIST}). \]

Interestingly, if DIST were to be extended to go both ways (i.e. \( K_i \alpha \land K_i \beta \leftrightarrow K_i(\alpha \land \beta) \)), then a violation of C would entail a violation of S as well; if C is violated, then some agent has knowledge of a contradiction, which, if DIST goes both ways, equivalently means that there is an instance where S is violated.
**Fitch:** \( \text{CONST} \land \text{KCONT} \implies \vdash (\phi \rightarrow \neg K_i \phi) \)  

(Section 2)

**Frauchiger-Renner:** \( \neg (Q \land U \land S \land C) \)  

(Section 3)

**This work:** \( \neg (Q \land U \land S \land C) \implies \text{REFL} \neg (\text{CONST} \land \text{KCONT} \land \text{DIST}) \)  

(Section 4)

| Assumption violated | Rejected knowledge axioms | Deduction  | REFL needed? |
|---------------------|---------------------------|------------|--------------|
| Q (quantum predictions) | CONST or KCONT | Intuitionistic | No |
| S (single outcome) | CONST or KCONT | Intuitionistic | Yes |
| C (consistency) | CONST or KCONT or DIST | Classical | Yes |
| U (large-scale quantum) | (other physical consequences) | — | — |

Table 1: A summary of the logical relations between the Frauchiger-Renner assumptions and knowledge axioms.

One way to reject the assumption C partially by not allowing it to hold for all pairs of agents \( i \) and \( j \), but only for pairs where the agent \( i \) trusts the agent \( j \) (and hence can trust her knowledge). This requires the introduction of an additional trust structure of the relations between agents; however, it does not seem to be sufficient to resolve the Frauchiger-Renner paradox for reasonably defined trust structures of the setting [45].

### 4.6 A no-go theorem

The results of the above discussion may be summarized in the following no-go theorem, which we take to be the main claim of this article. This no-go theorem essentially tells us that, from the Frauchiger-Renner paradox, we are obligated to reject at least one of the knowledge axioms we started with. This theorem is as follows (the proof is in Appendix C):

**Theorem 4** For any EMPL model \( \mathcal{M} \) which satisfies \( \text{REFL} \), it is the case that

\[
\text{FR} \land \text{CONST} \land \text{KCONT} \land \text{DIST} \models_{\text{REFL}} \bot.
\]

If we hold that the logical system ought to be consistent (at whatever cost), then it follows also that

\[
\text{FR} \models_{\text{REFL}} \neg (\text{CONST} \land \text{KCONT} \land \text{DIST}).
\]

### 5 Discussion

Our main result, Theorem 4, implies that the knowledge axioms that lead to Fitch’s result are incompatible with the quantum thought experiment of Frauchiger and Renner, unless we are willing to give up either the reflexivity of the accessibility relation of EMPL or the distributivity axiom. This means that the intuitive knowability axioms are not simultaneously acceptable in a world described by quantum mechanics. In other words, if we were to take quantum mechanics seriously as an accurate description of the world at macroscopic scales, then Fitch’s paradox would not hold in such a world. Our results are summarized in Table 1. They highlight the necessity for extending our understanding of what knowledge means for the users of physical theories, and how it may be axiomatized.
Main conclusions: letting go of classical intuitions. At a technical level, what have demonstrated is that certain features of quantum theory constrain the way in which we model knowledge, namely by requiring we reject one of four seemingly obvious assumptions (CONST, KCONT, DIST, and REFL). Out of those, the assumptions DIST and REFL seem to us to be quite minimal and philosophically reasonable. Thus, we are inclined to believe that it is either CONST or KCONT which we ought to reject. To reject CONST is to admit that there can exist unknowable truths. To reject KCONT is to admit that an agent can both know something and know that they don’t know it, similarly to rejecting the Frauchiger-Renner S assumption. On the other hand, to reject the Frauchiger-Renner C assumption is to admit that knowledge cannot always be directly passed on between agents — even perfectly rational ones.

All of these options point towards a growing perception in the quantum foundations community that knowledge — perhaps truth as well — is a relative concept which does not admit a global, unified expression. That is, Nature is such that we can only ever hope to understand relational fragments of the world from the perspectives of different agents; no complete observer-independent description is possible. This idea is not new: it can be traced back to the 1600s with Leibniz’s relational metaphysics [71], and was elegantly evoked by Pessoa in the early XX century through his heteronym Alberto Caeiro. In the natural sciences, this philosophy has taken a new life in recent relational physics research, within quantum gravity and quantum foundations in particular (see for instance [73, 74] [75, Chapter 2]). Indeed, thought experiments involving multiple quantum agents, such as the ones discussed here [1, 6, 76, 77] suggest that there might be no universal description that consistently represents distinct individual perspectives, in the traditional sense.

If we seek a logical framework capable of modelling the knowledge of quantum agents, we must dispense with some of our classical intuitions, such as universal truth assignments, consistency, or knowledge continuity between agents. Personally, we are inclined to let go of global truth assignments; research into quantum contextuality, discussed ahead, seems to favour this option. We believe that a consistent epistemic framework fully compatible with quantum theory must be able to articulate relative knowledge and subjective viewpoints, which may not always be universally shared. It is important to note that we are not calling for an abandonment of rationality; rather, we are instead suggesting that the notion of truth must be relativized in accordance with quantum theory, if it is to adequately represent the world as observed by multiple agents. While we do not make this philosophical view precise here, we note that it is related to existing forms of relativism, pluralism, and perspectivism (see for example [78]); a full account which interacts with these issues would be valuable for future philosophical investigation.

Note on the distributivity assumption. As we can see from Table 1, if it is only the knowledge distributivity assumption (DIST) that is rejected, we can manage to retain both Fitch’s theorem (as derived here) and the Frauchiger-Renner result. However, losing the option of “separating” the concatenated knowledge into individual statements poses a difficulty for classical frameworks analyzing it. After all, one of the most fundamental ways we reason about the world is by compiling multiple known facts and considering them simultaneously. It is unclear what deduction powers remain if this assumption is weakened.

13 For example, Caeiro writes “I saw that there is no Nature, that Nature does not exist — that there are hills, valleys, plains, that there are trees, flowers, grass, that there are rivers and stones, but that there isn’t a whole to which that belongs, that a real and true unity is a disease of our ideas. Nature is parts without a whole; this is perhaps the mystery they speak of.” [72]
**Paraconsistent logics.** In the above analysis, we derived contradictions between the violated Frauchiger-Renner assumptions and the knowledge axioms and concluded therefore that the knowledge axioms cannot all be simultaneously maintained. This choice was in part due to the presupposition that knowledge ought to be *consistent*; that is, any formal structure which is capable of modelling knowledge ought to be free from contradiction. This is, of course, not the only possibility. *Inconsistent logics* exist, but are generally not taken to be useful for modelling philosophical problems, because any contradiction, if not controlled, yields a so-called logical explosion wherein *any* formula may be proven (*ex contradictione quodlibet*). However, uncontrolled inconsistency is not the only possibility; it is possible to also construct formal logics which admit contradictions, but which do not allow for logical explosions. Any such logic is called *paraconsistent*. Paraconsistent logics have been studied extensively [79], and quantum logic has even been framed in a paraconsistent setting as well [80]. It would be interesting to see how such a system holds up under thought experiments such as Frauchiger-Renner — we suspect that it would still be problematic, but we do not undertake that analysis here.

**Relation to contextuality.** The restrictions that quantum mechanics imposes on knowledge have been explored before, notably by Kochen and Specker in their work on contextuality [81]. A non-contextual theory is an intuitive hidden-variable theory, in which measurements simply reveal an underlying truth — all values of all observables are determined for all systems, although these values may be hidden from the agents studying those systems, waiting to be experimentally uncovered. Kochen and Specker’s result shows that quantum theory is instead *contextual*: there is no joint probability distribution compatible with the outcomes of a set of measurements, for sufficiently complex quantum systems. In other words, the outcome of a measurement is not a preexisting fact being revealed; instead it depends on what other actions are being performed on the system (i.e. on the context). For a modern introduction to quantum contextuality, see [82].

Multi-agent paradoxes such as that of Frauchiger and Renner involve chains of statements that cannot be simultaneously true in a consistent manner, in what resembles *Liar cycles* [60]. Contextuality, on the other hand, can often be expressed in terms of the inability to consistently assign definite outcome values to a set of measurements [81, 83]. The two settings can be connected: in [84] some relations between logical paradoxes and quantum contextuality are explored; in particular, the authors point out a direct connection between contextuality and Liar cycles. It can be shown that the patterns of reasoning which are used in finding a contradiction in the Liar cycles chain of statements are similar to the reasoning we make use of in FR-type arguments, and can also be connected to the cases of the PR box [85] and Hardy’s paradox [86]. Additionally, proofs of logical pre-post selection paradoxes imply proofs of contextuality [87]; this further suggests that multi-agent paradoxes are closely linked to the notion of contextuality. Formalizing this connection is the subject of future work.

It is relevant to note that the quantum theory is not the only example of a contextual physical theory; another good example is the so-called “box world”, a special class of *general probabilistic theories* [88, 89]. In [90] it has been shown that a paradox similar to Frauchiger-Renner arises there as well. Both box world and quantum theory settings allow for different descriptions of a measurement process from the points of view of an agent performing a measurement and an agent who is observing the evolution of the agent and her measured system jointly. This shared property may be key to extend the concerns we stated in this paper to a larger set of theories.
**Application to quantum cryptography.** A direct application of this work is that one can use epistemic logic to model cryptographic scenarios and infer their security. For example, if Alice and Bob want to create a shared key $\phi$ and hide it securely from Eve, we’d want $\models K_A \phi \land K_B \phi$ and $\models \neg \Diamond (K_E \phi)$ to hold. Thus, to prove security of a protocol, we could start from the initial assumptions of their knowledge and model them over the course of the key-distribution protocol to see if the above statements held at the end. However, for fully quantum cryptography settings, in which adversarial agents may have quantum information about a shared secret, our result suggests that epistemic logics which assume the knowledge axioms described here may not be adequate. For example, if the adversary had partial access to the honest agents’ quantum computers, the effect of quantum hacking attacks may not be adequately captured by standard frameworks. In order to model the security of such settings, one needs to develop an epistemic logic system robust against the Frauchiger-Renner result — in particular, one that does without the knowability axioms in their present form, whilst taking its underlying propositional logic to be quantum logic.

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Appendix

A Syntax and semantics for epistemic logic

Throughout this article, we make use of epistemic modal propositional logic which we shall call EMPL. In Section 2, we consider its behaviour under possible axiomatic extensions. In Section 4, we likewise consider these extensions, however, we consider the case where the underlying propositional axiomatization is one of quantum – rather than classical – propositional logic instead. Here, we construct the syntax and semantics for this logic.

The syntax is given as follows. The symbols of the language $L_{EMPL}$ used are a collection of atomic propositional symbols $\{p\}$, propositional connectives $\land$, $\lor$, $\rightarrow$, negation $\neg$, a pair of dual modal operators $\Box$ (necessity), $\Diamond \equiv \neg \Box \neg$ (possibility), and a family of epistemic modal operators $\{K_i\}$ (one for each agent $i$ whose knowledge is meant to be represented by the logic — formally, $i$ just takes on a value in an indexing set). With these symbols, we can inductively construct well-formed formulas in the following manner:

**Definition 5** A string $\phi$ of symbols from $L$ is a well-formed formula exactly when:

1. $\phi \equiv p$ where $p$ is an atomic symbol.
2. $\phi \equiv \neg \alpha$ where $\alpha$ is a well-formed formula.
3. $\phi \equiv \alpha \land \beta$, or $\phi \equiv \alpha \lor \beta$, or $\phi \equiv \alpha \rightarrow \beta$, where $\alpha$ and $\beta$ are a well-formed formulas.
4. $\phi \equiv \Box \alpha$ or $\phi \equiv \Diamond \alpha$ where $\alpha$ is a well-formed formula.
5. $\phi \equiv K_i \alpha$ where $\alpha$ is a well-formed formula.

That is, given a fixed set of atomic propositional variables, these are the rules for how they may be combined to form other permitted strings of symbols which we take to be well-formed formulas (this excludes arbitrary, nonsensical strings of symbols from the language). Brackets may be applied in the usual way. Deductions in this system make use of two rules of inference; modus ponens $\{\phi \rightarrow \psi, \phi\} \vdash \psi$, and the usual modal necessitation rule:\footnote{This may seem to the reader unfamiliar with modal logic like a very strange rule, as it seems to indicate that truth implies necessity, however, this is not the case. Rather, it means that if, under a given truth valuation, the formula $\phi$ is evaluated to be true, then this will be the case in all possible worlds and so specifically, it will be true in all accessible worlds to a given world, whence it will be necessary in all worlds. The only time where truth is ‘world-relative’ is when a modal operator is involved. For a more extensive discussion of this rule of inference, see [40, p. 205].}

$$\text{(NEC)} \quad \phi \vdash \Box \phi.$$  

The classical EMPL would take as its axioms the usual Hilbert-Ackermann axioms for propositional logic together with the Kripke axiom:

$$(K) \quad \Box (\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi).$$

Following Sider [40], the modal semantics for this logic are given in the usual Kripke-style possible worlds framework, with slight modification to account for the new operators $\{K_i\}$. Specifically, we consider the structure of models of this logic.

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Following Sider [40], the modal semantics for this logic are given in the usual Kripke-style possible worlds framework, with slight modification to account for the new operators $\{K_i\}$. Specifically, we consider the structure of models of this logic.
**Definition 6** An EMPL model is an ordered triple \( \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle \). \( \mathcal{W} \) denotes a non-empty set of objects (possible worlds) and \( \mathcal{R} \) denotes a binary relation on \( \mathcal{W} \) (accessibility). \( \mathcal{I} \) denotes a two-place function taking atomic propositional symbols or well-formed formulas of the form \( \alpha \equiv K_i \phi \) (for some \( i \), and some formula \( \phi \)) and elements of \( \mathcal{W} \) into \( \{0, 1\} \). That is, \( \mathcal{I}(\alpha, w) \in \{0, 1\} \) for all \( \alpha \in \{p\} \) or \( \alpha \equiv K_i \phi \), and for all \( w \in \mathcal{W} \).

The interpretation function \( \mathcal{I} \) induces a valuation function on all EMPL formulas:

**Definition 7** Given an EMPL model \( \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle \), the associated valuation function \( V_\mathcal{I} \) takes arbitrary EMPL formula and worlds into \( \{0, 1\} \) as follows:

1. \( V_\mathcal{I}(\alpha, w) = \mathcal{I}(\alpha, w) \) for any \( \alpha \in \{p\} \) or \( \alpha \equiv K_i \phi \) at world \( w \).
2. \( V_\mathcal{I}(\neg \phi, w) = 1 \) iff \( V_\mathcal{I}(\phi, w) = 0 \).
3. \( V_\mathcal{I}(\phi \rightarrow \psi, w) = 1 \) iff either \( V_\mathcal{I}(\phi, w) = 0 \) or \( V_\mathcal{I}(\psi, w) = 1 \).
4. \( V_\mathcal{I}(\phi \land \psi, w) = 1 \) iff \( V_\mathcal{I}(\phi, w) = 0 \) and \( V_\mathcal{I}(\psi, w) = 1 \).
5. \( V_\mathcal{I}(\phi \lor \psi, w) = 1 \) iff \( V_\mathcal{I}(\phi, w) = 0 \) or \( V_\mathcal{I}(\psi, w) = 1 \).
6. \( V_\mathcal{I}(\Box \phi, w) = 1 \) iff \( V_\mathcal{I}(\phi, u) = 1 \) for all worlds \( u \) such that \( \mathcal{R}wu \).
7. \( V_\mathcal{I}(\Diamond \phi, w) = 1 \) iff \( V_\mathcal{I}(\phi, u) = 1 \) for some world \( u \) such that \( \mathcal{R}wu \).

Frequently, the world specification in \( V \) will be suppressed if it is unnecessary. The precise structure of the accessibility relation is not yet constrained (i.e. we do not require it to be symmetric, reflexive, transitive, and so on), though it will prove interesting later to consider what it means when these conditions, reflexivity in particular, fail, in the setting we are concerned with. With valuation in place, we may now define validity:

**Definition 8** An EMPL formula \( \phi \) is EMPL-valid in an EMPL model \( \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle \) and we write \( \models_{\mathcal{R}} \phi \) iff \( V_\mathcal{I}(\phi, w) = 1 \) for all \( w \in \mathcal{W} \).

Validity is dependent upon the accessibility relation \( \mathcal{R} \) because the last two conditions on \( V_\mathcal{I} \) depend upon whether or not particular worlds are accessible. Further notions of normal modal systems and semantic consequence in EMPL may readily be constructed in direct analogue with usual MPL. This much is sufficient for our discussion here. It is important, however, to realize that the system discussed throughout it not EMPL, but rather an extension of EMPL by the addition of new modal axiom schemas involving the \( K_i \) operators.

It is important to note that EMPL as thus constructed is essentially just usual modal propositional logic (MPL), with additional modal operators \( \{K_i\} \) which have not yet been included in the semantics in any serious capacity. That is, the knowledge operators are essentially inert symbols which do not yet play a role in the semantics beyond the trivial and opaque condition that if agent \( i \) knows \( \phi \), then \( K_i \phi \) is deducible. This will be filled in in Section 4 when we discuss the role of quantum mechanics as a basis for knowledge, and additionally when we begin investigating axiomatic extensions of this system.
B Constructivism and CONST

The justification for accepting CONST as a reasonable axiom lies in the fact that MPL is decidable; so long as the modal semantics in which our discussion of knowledge is situated is a normal modal system (which is thus decidable), this assumption is permissible, even if we do not directly make use of S4 as a representation of IPL. The reason for noting this is that S4 (and thus the space of ‘states of knowledge’ for the intuitionist) has a reflexive accessibility relation, but the assumption of CONST does not a priori require that inter-world accessibility be reflexive. It may seem odd to talk about states of knowledge which are not reflexive, but this will turn out to be an important feature of the discussion to follow, as we will demonstrate that taking accessibility to simply not be reflexive allows us to avoid many paradoxical results, though at other costs.

C Proofs used in the paper

Here we list proofs for lemmas and theorems from Section 4.

Lemma 9 (Contraposition) In the quantum logic defined in Definition 2, if \( \phi \rightarrow \psi \), then \( \neg \psi \rightarrow \neg \phi \).

Proof: Let \( M_\phi \) and \( M_\psi \) be the projective subspaces wherein \( \phi \) and \( \psi \) are true, respectively. Then \( \phi \rightarrow \psi \) if and only if \( M_\psi \subseteq M_\phi \). But then an element which is orthogonal to \( M_\psi \) must also be orthogonal to \( M_\phi \), and so their orthogonal compliments satisfy \( M_\phi^\perp \subseteq M_\psi^\perp \). Therefore \( M_{\neg \phi} \subseteq M_{\neg \psi} \), and so \( \neg \psi \rightarrow \neg \phi \). □

Lemma 10 (Connective Identity) In the quantum logic defined in Definition 2, It is true that \( \phi \rightarrow \psi \) if and only if it is true that \( \neg (\phi \land \neg \psi) \).

Proof: We see that:

\[
\begin{align*}
\phi \rightarrow \psi & \iff \neg \psi \rightarrow \neg \phi \\
& \iff M_\phi^\perp \subseteq M_\psi^\perp \\
& \iff M_\psi^\perp \cap M_\phi = \emptyset \\
& \iff (M_\psi^\perp \cap M_\phi)^\perp = M_\psi^{\perp \perp} \cap M_\phi^\perp = M_\psi \cap M_\phi^\perp = \mathcal{H} \\
& \iff M_{\neg (\phi \land \neg \psi)} = \mathcal{H} \\
& \iff \neg (\phi \land \neg \psi).
\end{align*}
\]

\( \text{(4)} \)

□

Theorem 11 If \( Q \) is violated, for any EMPL model \( \mathcal{M} \), we have that \( \text{CONST} \land \text{KCONT} \models_{\mathcal{M}} \bot \).
Proof: Violating Q means that there is some proposition \( p_Q \) about the world which quantum mechanics predicts is true (i.e. its corresponding measurement outcome occurs with probability 1 and so \( V_\mathcal{F}(p_Q) = 1 \)), but which agent \( i \) cannot be certain of by merely using quantum mechanics to make inferences (i.e. \( V_\mathcal{F}(K_i p_Q) = 0 \)). Thus, we have \( V_\mathcal{F}(p_Q \land \neg K_i p_Q) = 1 \).

The intuitionistically valid result FITCH, \( \vdash (\phi \rightarrow \neg \neg K_i \phi) \) (which is the weaker form of the paradox, yet still sufficient for our purposes in this section), may be re-written using the connective identity from Lemma 10 as \( \vdash \neg (\phi \land \neg K_i \phi) \) for all \( \phi \). Using the valuation function definition from Appendix A, we can see then that \( V_\mathcal{F}(\neg (p \land \neg K_i p)) = 1 \). Thus, for \( p_Q \) in particular, we have \( V_\mathcal{F}(\neg (p_Q \land \neg K_i p_Q)) = 1 \). Hence \( V_\mathcal{F}(\phi \land \neg \phi) = 1 \) where \( \phi := p_Q \land \neg K_i p_Q \), a contradiction. □

Lemma 12 If we assume CONST and KCONT, then \( \vdash \neg \Diamond K_i (\phi \land \neg \phi) \) for all \( \phi \).

Proof:

1. \( \phi \rightarrow \neg \neg K_i \phi \) (FITCH)
2. \( \neg K_i \phi \rightarrow \neg \phi \) Contraposition
3. \( \neg \Diamond K_i (\phi \land \neg K_i \phi) \rightarrow \neg \Diamond K_i (\phi \land \neg \phi) \) Modus Ponens
4. \( \neg \Diamond K_i (\phi \land \neg K_i \phi) \) (KCONT)
5. \( \neg \Diamond K_i (\phi \land \neg \phi) \) Modus Ponens

The final line reads “it is not possible for agent \( i \) to have knowledge of a contradiction.” □

Theorem 13 If \( S \) is violated, for any EMPL model \( \mathcal{M} \) which satisfies REFL, we have that \( \text{CONST} \land \text{KCONT} \models_{\mathcal{M}, \text{REFL}} \bot \).

Proof: If \( S \) is violated, then there exists some \( p_S \) and some world \( w \) for which \( V_\mathcal{F}(K_i (p_S \land \neg p_S), w) = 1 \). Recalling from Section 2 that propositions in this framework are true within a particular possible world, and that these possible worlds are connected via an accessibility relation, if the accessibility relation is reflexive, then \( \phi \vdash \Diamond \phi \) for all formulas \( \phi \). Thus, if the accessibility relation is reflexive, if \( S \) is violated, Lemma 12 implies a contradiction, because we may simultaneously obtain \( V_\mathcal{F}(\Diamond K_i (p \land \neg p), w) = 1 \) and \( V_\mathcal{F}(\neg \Diamond K_i (p \land \neg p), w) = 1 \). □

Theorem 14 If \( C \) is violated, for any EMPL model \( \mathcal{M} \) which satisfies REFL, we have that \( \text{CONST} \land \text{KCONT} \land \text{DIST} \models_{\text{REFL}} \bot \).
Proof:

1. $K_i(K_j pC) \land \neg K_i pC$ \quad \text{C violated.}
2. $\neg K_i pC \rightarrow K_j(\neg K_i pC)$ \quad \text{FITCH* applied to agent } j.
3. $K_j(\neg K_i pC) \rightarrow K_i(K_j(\neg K_i pC))$ \quad \text{FITCH* applied to agent } i.
4. $K_i(K_j pC) \land K_i(K_j(\neg K_i pC))$ \quad \text{Modus Ponens on 1 and 3.}
5. $K_i[(K_j pC) \land (K_j(\neg K_i pC))]$ \quad \text{DIST on 4 for agent } i.
6. $K_i[K_j(pC \land \neg K_i pC)]$ \quad \text{DIST on 5 for agent } j.
7. $pC \rightarrow K_i pC$ \quad \text{FITCH* for agent } i.
8. $\neg(pC \land \neg K_i pC)$ \quad \text{Connective Identity}
9. $K_i[K_j(\neg(pC \land \neg K_i pC))]$ \quad \text{FITCH* for agents } i \text{ and } j
10. $K_i[K_j((pC \land \neg K_i pC) \land \neg(pC \land \neg K_i pC))]$ \quad \text{DIST on 6 and 9.}
11. $K_i[K_j(\phi \land \neg \phi)]$ \quad \text{where } $\phi := pC \land \neg K_i pC$.

But, from Lemma 12, we see that $\vdash \neg \Diamond K_j(\phi \land \neg \phi)$. Thus, either the accessibility relation between worlds is not reflexive, or one may deduce that agent $i$ has knowledge of a contradiction, namely $K_i(\phi' \land \neg \phi')$ where $\phi' := \Diamond K_j(\phi \land \neg \phi)$, which again, by applying Lemma 12 a second time, leads either to a contradiction, or the conclusion that accessibility relation between worlds is not reflexive, thus violating the assumption that $M$ satisfies REFL. The result follows. \qed

Theorem 4 For any EMPL model $M$ which satisfies REFL, it is the case that

$$\text{FR} \land \text{CONST} \land \text{KCONT} \land \text{DIST} \models_{\text{REFL}} \bot$$

(5)

Proof: From the Frauchiger-Renner theorem described in Section 3 and derived in Appendix D, we have that

$$\text{FR} \models \neg(Q \land S \land C)$$

But from the above discussion in sections 4.3, 4.4, and 4.5, we have that

$$\neg Q \land \text{CONST} \land \text{KCONT} \models_{M} \bot$$
$$\neg S \land \text{CONST} \land \text{KCONT} \models_{\text{REFL}} \bot$$
$$\neg C \land \text{CONST} \land \text{KCONT} \land \text{DIST} \models_{\text{REFL}} \bot$$

By compositing these, we obtain

$$\neg(Q \land S \land C) \land \text{CONST} \land \text{KCONT} \land \text{DIST} \models_{\text{REFL}} \bot$$

From which it follows that
In this appendix, we present the technical derivation of the Frauchiger-Renner paradox. Here, we assume that the reader has basic knowledge of the postulates and notation of quantum theory. This derivation is taken from our previous work [45].

The experiment consists of four participants, Alice, Bob, Ursula and Wigner. Each experimenter is equipped with a quantum memory (A, B, U and W, respectively). In addition, there are two other systems, R and S, which are also modelled as qubits; the agents involved in the experiment follow the protocol outlined in Section 3:

$t = 1$. After Alice measures $R$ and records the result, the joint state of $R$ and her memory $A$ becomes entangled,

$$
\left( \sqrt{\frac{1}{3}} |0\rangle_R + \sqrt{\frac{2}{3}} |1\rangle_R \right) |0\rangle_A \rightarrow \sqrt{\frac{1}{3}} |0\rangle_R |0\rangle_A + \sqrt{\frac{2}{3}} |1\rangle_R |1\rangle_A.
$$

When Alice prepares $S$, it too becomes entangled with $R$ and $A$,

$$
\sqrt{\frac{1}{3}} |0\rangle_R |0\rangle_A |0\rangle_S + \sqrt{\frac{2}{3}} |1\rangle_R |1\rangle_A \frac{1}{\sqrt{2}}(|0\rangle_S + |1\rangle_S).
$$

$t = 2$. When Bob measures $S$ and writes the result in his memory, the global state becomes

$$
|\psi\rangle_{RASB} = \frac{1}{\sqrt{3}} (|0\rangle_R |0\rangle_A |0\rangle_S |0\rangle_B + |1\rangle_R |1\rangle_A |0\rangle_S |0\rangle_B + |1\rangle_R |1\rangle_A |1\rangle_S |1\rangle_B).
$$

Crucially, this is a Hardy state [86], and its terms can be rearranged into two convenient forms, which will be used later,

$$
|\psi\rangle_{RASB} = \frac{\sqrt{2}}{3} \frac{1}{\sqrt{2}} |0\rangle_R |0\rangle_A + |1\rangle_R |1\rangle_A |0\rangle_S |0\rangle_B + \frac{1}{\sqrt{3}} |1\rangle_R |1\rangle_A |1\rangle_S |1\rangle_B
\right)_{fail_R}.
$$

$$
= \frac{1}{\sqrt{3}} |0\rangle_R |0\rangle_A |0\rangle_S |0\rangle_B + \sqrt{\frac{2}{3}} |1\rangle_R |1\rangle_A \frac{1}{\sqrt{2}} |0\rangle_S |0\rangle_B + |1\rangle_S |1\rangle_B |1\rangle_B
\right)_{fail_S}.
$$

$t = 3$. Now Ursula and Wigner measure Alice’s and Bob’s labs in their bases listed above; the possibility of them both getting the outcome “ok” is non-zero:

$$
P[u = w = ok] = |\langle ok|_{RA}(ok|_{SB})|\psi\rangle_{RASB}|^2 = \frac{1}{12}.
$$

\[28\]
From now on, we post-select on this event. At time $t = 3$, Ursula reasons about the outcome that Bob observed at $t = 2$. Since

$$|\psi\rangle_{RASB} = \frac{\sqrt{2}}{3} |\text{fail}\rangle_R |0\rangle_A |0\rangle_S |0\rangle_B + \frac{1}{\sqrt{3}} |1\rangle_R |1\rangle_A |1\rangle_S |1\rangle_B,$$

she concludes that the only possibility with non-zero overlap with her observation of $|\text{ok}\rangle_R$ is that Bob measured $|1\rangle_S$. We can write this as “$u = \text{ok} \implies b = 1$”. She can further reason about what Bob, at time $t = 2$ thought about Alice’s outcome at time $t = 1$. Whenever Bob observes $|1\rangle_S$, he can use the same form of $|\psi\rangle_{RASB}$ to conclude that Alice must have measured $|1\rangle_R$. We can write this as “$b = 1 \implies a = 1$”. Finally, we can think about Alice’s deduction about Wigner’s outcome. Using the form

$$|\psi\rangle_{RASB} = \frac{1}{\sqrt{3}} |0\rangle_R |0\rangle_A |0\rangle_S |0\rangle_B + \frac{\sqrt{2}}{3} |1\rangle_R |1\rangle_A |\text{fail}\rangle_{SB},$$

we see that Alice reasons that, whenever she finds $R$ in state $|1\rangle_R$, then Wigner will obtain outcome “fail” when he measures Bob’s lab. That is, “$a = 1 \implies w = \text{fail}$”.

Thus, chaining together the statements (the same reasoning that allowed the reader to solve the three hats problem), we reach an apparent contradiction:

$$w = u = \text{ok} \implies b = 1 \implies a = 1 \implies w = \text{fail}.$$ 

That is, when the experiments stops with $u = w = \text{ok}$, the agents can make deterministic statements about each other’s reasoning and measurement results, concluding that Alice had predicted $w = \text{fail}$, hence arriving to a logical contradiction.
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