CHARACTERIZING THE SPECTRA OF CARDINALITIES OF BRANCHES OF KUREPA TREES

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Abstract. We give a complete characterization of the sets of cardinals that in a suitable forcing extension can be the Kurepa spectrum, that is, the set of cardinalities of branches of Kurepa trees. This answers a question of the first named author.

1. Introduction

A tree is a Kurepa tree if it is of height \( \omega_1 \), each of its levels is countable, and it has more than \( \omega_1 \)-many cofinal (that is of order type \( \omega_1 \)) branches. In this paper we study the possible values of the branch spectrum of Kurepa trees, i.e. the set

\[
\text{Sp}_{\omega_1} = \{ \lambda : \text{there exists a Kurepa tree } T \text{ s.t. } |B(T)| = \lambda \} \subseteq [\omega_2, 2^{\omega_1}]
\]

(where \( B(T) \) stands for the set of cofinal branches of \( T \)).

The spectrum is related to the model theoretical spectrum of maximal models of \( L_{\omega_1, \omega} \)-sentences [SS17]. Also canonical topological and combinatorial structures are associated with branches of Kurepa trees possessing a remarkably wide range of nonreflecting properties [Kos05]. For higher Kurepa trees (of weakly compact height) the consistency strength of certain types of the branch spectrum was studied in [HM19].

It was first shown by Silver that the Kurepa Hypothesis (i.e. the existence of a Kurepa tree) is independent [Sil67], or see [Kun83, Ch VIII, 3.]. Moreover the non-existence of Kurepa trees is equiconsistent with the existence of an inaccessible cardinal [Kun83, Ch VII, Ex. B8.].

Questions about the possible values of the spectrum were addressed by Jin and Shelah in [JS92]. They proved (assuming an inaccessible cardinal) that consistently there are only Kurepa trees with \( \omega_3 \)-many cofinal branches while \( 2^{\omega_1} = \omega_4 \).

Building on ideas of Jin and Shelah, the first named author provided a sufficient condition for a set to be equal to \( \text{Sp}_{\omega_1} \) in a forcing extension in [Poo]. Formally, it was shown that if \( \text{GCH} \) holds, and \( 0, 1 \notin S \) is a set of ordinals such that \( S \) satisfies either

\[
\text{Case A:}
\]

\[
(i) \ 2 \in S,
\]

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the powerset of $A$

Definition 2.1. Under a sequence we mean a function defined on a set of ordinals. For sequences $s, t$ the relation $s = t \upharpoonright \text{dom}(s)$ (or equivalently $s \subseteq t$) will be also denoted by $s < t$.

**Definition 2.1.** A tree $\langle T, \prec_T \rangle$ is a partially ordered set (poset) in which for each $x \in T$ the set

$$T_{\prec x} = \{y \in T : y \prec_T x\}$$

$$\{\sup C : C \subseteq [S]^{<\omega_1} \subseteq S, \}

or

Case B:

(i) $\exists$ an inaccessible $\kappa$, 
(ii) $\{\sup C : C \subseteq [S]^{<\kappa} \subseteq S, \}

(iii) $$(\forall \alpha \in S) : (\omega \leq \text{cf}(\alpha) < \omega_2) \rightarrow (\alpha + 1 \in S),$$

then in a forcing extension we have $\{\alpha : R_{\alpha} \in \text{Sp}_{\omega_1}\} = S$ (cardinals are only collapsed in Case B, from $(\omega_1, \kappa)$). It can be easily seen that if $\text{cf}(\mu) = \omega$ and $(\text{Sp}_{\omega_1} \cap \mu)$ is cofinal in $\mu$, then there exists a Kurepa tree with $\mu$-many branches, as the union of countably many Kurepa trees is a Kurepa tree, and it is not difficult to see that the same holds if $\text{cf}(\mu) = \omega$, therefore Case A / (ii) and Case B / (ii) are in fact necessary. However, it remained a question whether the last clauses can be dropped.

In this paper as the main result we prove that assuming $\text{CH} + (2^{\omega_1} = \omega_2)$ conditions (i), (ii) (in both cases) are in fact sufficient by forcing a model of $\{\alpha : R_{\alpha} \in \text{Sp}_{\omega_1}\} = S$. Also, we can arbitrarily prescribe $2^{\omega_1}$ to be any cardinal $\lambda \geq \sup(\text{Sp}_{\omega_1})$ if in Case A the equality $\lambda^{<\omega_2} = \lambda$ holds, or in Case B $\lambda^{<\omega} = \lambda$ holds too.

Moreover, when we do not want Kurepa trees with $\omega_2$-many cofinal branches, we prove that the inaccessible is necessary by verifying that if $\omega_2$ is a successor in $L$, then there exists a Kurepa tree with only $\omega_2$-many cofinal branches in $V$. It was known that these assumptions imply that there exists a Kurepa tree even in $L[A]$ for some $A \subseteq \omega_1$ Ch VII, Ex. B8] (possibly having more than $\omega_2$-many cofinal branches in $V$). Our proof not only utilizes countable elementary submodels of initial segments of $L[A]$, but the nodes of the tree are such elementary submodels, and each cofinal branch uniquely corresponds to an initial segment of $L[A]$.

2. Preliminaries, notations

Under ordinals we always mean Neumann ordinals. For a fixed cardinal $\chi$ we will use the notation $\mathcal{H}(\chi)$ for the collection of sets of hereditary size less than $\chi$, i.e.

$$\mathcal{H}(\chi) = \{x : |\text{trcl}(x)| < \chi\},$$

where $\text{trcl}(x)$ stands for the transitive closure of $x$. In terms of forcing we will use the notations of [Kun13], e.g. $p \leq q$ means that $p$ is the stronger. If it is clear from the context and won’t make any confusion we will identify the set $x$ in the ground model with its canonical name $\dot{x}$. For a set $A$ the symbol $\mathcal{P}(A)$ denotes the powerset of $A$, and $[A]^{<\lambda}$ stands for $\{X \in \mathcal{P}(A) : |X| = \lambda\}$. For a function $s = \{(\beta, s(\beta)) : \beta \in \text{dom}(s)\}$ we will also use the following notation and refer to $s$ as

$$s_\beta : \beta \in \text{dom}(s).$$

Under a sequence we mean a function defined on a set of ordinals. For sequences $s, t$ the relation $s = t \upharpoonright \text{dom}(s)$ (or equivalently $s \subseteq t$) will be also denoted by $s < t$. 

For each $x \in T$ the set

$$T_{<x} = \{y \in T : y <_T x\}$$
is well ordered by $\prec_T$.

**Definition 2.2.** The height of $x$ in the tree $T$ is the order type of $T_{\preceq x}$

$$\text{ht}(x) = \text{otp}(T_{\preceq x}).$$

**Definition 2.3.** For each ordinal $\alpha$ the restriction of $T$ to $\alpha$ is

$$T_{\preceq \alpha} = \{ t \in T : \text{ht}(t) < \alpha \}.$$

**Definition 2.4.** The height of the tree $T$ (in symbols $\text{ht}(T)$), is the least $\beta$ such that

$$\exists t \in T : \text{ht}(t) = \beta.$$

We will need the following lemma [Kun83, Ch II. Thm. 1.6.] which we will refer to as the $\Delta$-system Lemma.

**Lemma 2.5.** Let $\kappa$ be an infinite cardinal, let $\theta > \kappa$ be regular, and satisfy $\forall \alpha < \theta (|\alpha| < \kappa |\alpha| < \theta)$. Assume that $|A| \geq \theta$, and $\forall x \in A (|x| < \kappa)$. Then there is a $D \subseteq A$, such that $|D| = \theta$, and $D$ forms a $\Delta$-system, i.e. there is a kernel set $y$ such that

$$\forall x \neq x' \in D : x \cap x' = y.$$

### 3. The Forcing

Now we can state our main theorem.

**Theorem 3.1.** Let $S_\bullet$ be a set of infinite cardinals such that $\omega, \omega_1 \notin S_\bullet$. Assume CH, and that either

**Case 1:**

(i) $\omega_2 \in S_\bullet$,

(ii) $2^{\omega_1} = \omega_2$,

(iii) $\{ \sup C : C \in [S_\bullet]^{<\omega_2} \} \subseteq S_\bullet$,

or

**Case 2:**

(i) there exists an inaccessible $\kappa$ such that $S_\bullet \cap (\omega_1, \kappa) = \emptyset$,

(ii) $\{ \sup C : C \in [S_\bullet]^{<\kappa} \} \subseteq S_\bullet$.

Then there exists a forcing extension $V^P$ such that

$$V^P \models S_\bullet = \text{Sp}_{\omega_1},$$

where $P$ only collapses cardinals in $(\omega_1, \kappa)$ in Case 2.

The key will be Lemma 3.26. After Lemma 3.29 we will put together the pieces in a short argument. Before these we need some preparation.

**Definition 3.2.** In Case 1 (i.e. $\omega_2 \in S_\bullet$) define the cardinal $\kappa$ to be $\omega_2$.

**Corollary 3.3.** No cardinal $\mu \notin (\omega_1, \kappa)$ is collapsed.

**Theorem 3.4.** Suppose that all conditions from Theorem 3.1 hold, and $\kappa$ is defined in Definition 3.2. Assume further that $\lambda$ is a cardinal which is an upper bound of $S_\bullet$ such that $\lambda^{<\kappa} = \lambda$ (thus $\text{cf}(\lambda) \geq \kappa$). Then there exists a forcing extension $V^P$ with

$$V^P \models (S_\bullet = \{ \mu : \text{there exists a Kurepa tree } T \text{ s.t. } |B(T)| = \mu \} \land (2^{\omega_1} = \lambda)).$$

**Definition 3.5.** Let $S_\bullet^+ = S_\bullet \cup \{ \kappa, \lambda \}$. 
Definition 3.6. For a cardinal $\theta \in S_\ast$ let $Q_\theta$ be the following notion of forcing. The triplet $p = (T_p, u_p, \bar{\pi}_p)$ is an element of $Q_\theta$ iff

(a) $T_p$ is a countable tree of height $\delta$ for some $\delta < \omega_1$ on the underlying set $\omega \cdot \delta$, where the $\beta$th level is $[\omega \cdot \beta, \omega \cdot (\beta+1))$, i.e. $T_{p,\leq \beta} \setminus T_{p, < \beta} = [\omega \cdot \beta, \omega \cdot (\beta+1))$ for each $\beta < \delta$.
(b) for each $t \in T_p$ and $\beta < \delta$ there exists $t' \in T_p \setminus T_{p, < \beta}$ s.t. $t <_{T_p} t'$.
(c) $u_p \in [\theta]^{<\omega}$.
(d) $\bar{\pi}_p = (\eta_{p,\alpha} : \alpha \in u_p)$, where $\eta_{p,\alpha} \subseteq T_p$ is a branch in $T_{p, < \gamma}$ for some $\gamma \in \{\beta+1 : \beta < \delta = \text{ht}(T_p)\}$ (we do it for a technical reason, we also could have stored only the maximal element instead of a chain with a maximal element).

Then $Q_\theta$ is a poset with the obvious order, i.e. $q \leq p$, if $T_q$ is an end-extension of $T_p$, formally $T_{q, < \text{ht}(T_p)} = T_p$, and for each $\alpha \in u_p$ the inclusion $\eta_{p,\alpha} \subseteq \eta_{q,\alpha}$ holds.

Let $\bar{T}_\theta, \bar{\eta}_\theta$ be the names for the generic tree and sequence, i.e. denoting the generic filter by $G_\theta$

\[
1_{Q_\theta} \models \bar{T}_\theta = \bigcup\{T_p : p \in G_\theta\} \quad \text{ and } \quad 1_{Q_\theta} \models \bar{\eta}_\theta = \left\{ \eta_{\theta,\alpha} = \bigcup\{\eta_{p,\alpha} : p \in G_\theta\} : \alpha \in \theta \right\}.
\]

Definition 3.7. For a cardinal $\theta \in S_\ast$, let $Q^+_\theta \subseteq Q_\theta$ be the following subposet.

$p \in Q^+_\theta$, iff $\text{ht}(T_p)$ is a successor, and $(\forall \alpha \in u_p) : \eta_{p,\alpha}$ is a branch through $T_p$.

Definition 3.8. If $\lambda \notin S_\ast$, then let $Q_\lambda$ be the countable supported product of $\langle\omega^2, <\rangle$-s of length $\lambda$, i.e.

$Q_\lambda = \{ p = \langle \eta_\alpha : \alpha \in u_p \rangle : (\forall \alpha \in u_p) \eta_\alpha \in \omega^2, \text{ for some } u_p \in [\lambda]^{<\omega} \}.$

Definition 3.9. If $\kappa \notin S_\ast$ (and then $\kappa > \omega^2_1$ is inaccessible), then let $Q_\kappa$ be the countable supported product of $\langle\omega^1, <\rangle$'s $\langle \gamma < \kappa \rangle$, a forcing which collapses each cardinal in $\omega^1, \kappa$:

$Q_\kappa = \{ p = \langle \eta_\alpha : \alpha \in u_p \rangle : (\forall \alpha \in u_p) \eta_\alpha \in \omega^1, \text{ for some } u_p \in [\kappa]^{<\omega} \}.$

Definition 3.10. We define the posets which we will need later.

1) For $S \subseteq S^+_\ast$ let $P_S$ be the countable supported product of $Q_{\theta}$-s ($\theta \in S$), i.e.

$P_S = \{ p \text{ is a function : } \text{dom}(p) \in [S]^{<\omega} \land (\forall \theta \in \text{dom}(p) \ p(\theta) \in Q_{\theta}) \}.$

With a slight abuse of notation for $p \in P_S$ and $\theta \in S \setminus \text{dom}(p)$ we will mean $1_{Q_{\theta}}$ under $p(\theta)$.

2) For $\theta \in S^+_\ast$, $U \subseteq \theta$ define its restriction from $\theta$ to $U$, i.e.

$Q_{\theta, U} = \{ p \in Q : \text{ u}_p \subseteq U \}.$

3) For $S \subseteq S^+_\ast$, $\bar{\sigma} = \langle U_\theta : \theta \in S \rangle \in \prod_{\theta \in S} P(\theta)$ we define $P_{S, \bar{\sigma}}$ to be $P_S$-restriction to coordinates in $U_\theta$-s, i.e.

$P_{S, \bar{\sigma}} = \{ p \in P_S : (\forall \theta \in S) \ p(\theta) \in Q_{\theta, U_\theta} \}.$

4) For $S, S' \subseteq S^+_\ast$, $\bar{\sigma} = \langle U_\theta : \theta \in S \rangle \in \prod_{\theta \in S} P(\theta), \bar{\sigma}' = \langle U'_\theta : \theta \in S \rangle \in \prod_{\theta \in S} P(\theta)$ we define

- $\bar{\sigma} + \bar{\sigma}' = \langle U_\theta \cup U'_\theta : \theta \in S \cup S' \rangle$ (where for $\theta \in S' \setminus S$ under $U_\theta$ we mean the empty set, similarly for $\theta \in S \setminus S', U'_\theta$),
prove that
\[\text{(3.3)}\]
Fix \(\alpha \subseteq y\) yields a set \(A\) if \(\alpha \subseteq y\) yields a set \(A\) for the system
\[\text{(3.4)}\]
\[P \colon \{\alpha \subseteq y \mid \alpha \subseteq y\} \to y \text{ such that } (\alpha \subseteq y) \implies \sum_{\alpha \subseteq y} \theta_{\alpha} = \left(\bigcup_{\alpha \subseteq y} \theta_{\alpha} \right) = \left(\bigcup_{\alpha \subseteq y} (W_{\alpha})_{\theta} : \theta \in S\right).
\]

5) Let \(P = \mathbb{P}_{S^+}\).
6) If \(p_{0}, p_{1}, \ldots, p_{n} \in P\) let \(\bigwedge_{i \leq n} p_{i}\) denote the greatest lower bound if exists.
7) For \(p \in P\), and \(S \subseteq S^+\), \(\bar{U} = \langle U_{\theta} : \theta \in S \rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)\) define \(p \upharpoonright \bar{U} \in \prod_{\theta \in S} \mathcal{P}(\theta)\) to be the following restriction of \(p \upharpoonright S\) in the obvious fashion

\[
\text{for each } \theta \in S : \quad (p \upharpoonright \bar{U})(\theta) = \langle T_{p\theta}(\eta_{\theta}), u_{\theta} \cap U_{\theta}, \bar{U}_{\theta} : \theta, \eta \in S \rangle.
\]

**Definition 3.11.** For \(S \subseteq S^+\) define the notion of forcing \(P^+ (\mathbb{P}_{S}, \mathbb{P}_{S \mathbb{T}}\), resp.) to be the subposet of \(P (\mathbb{P}_{S}, \mathbb{P}_{S \mathbb{T}}\), resp.) consisting of elements \(p\) for that \(p(\theta) \in \mathbb{Q}_{\theta}\) holds for each \(\theta \in S\).

**Remark 3.12.** The notion of forcing \(P^+ (\mathbb{P}_{S}, \mathbb{P}_{S \mathbb{T}}\), resp.) is a dense subposet of \(P (\mathbb{P}_{S}, \mathbb{P}_{S \mathbb{T}}\), resp.), therefore forcing with \(P^+ (\mathbb{P}_{S}, \mathbb{P}_{S \mathbb{T}}\), resp.) yields the same extensions as forcing with \(P (\mathbb{P}_{S}, \mathbb{P}_{S \mathbb{T}}\), resp.).

**Claim 3.13.** Let \(S \subseteq S^+\), \(U = \langle U_{\theta} : \theta \in S \rangle\) be fixed. Then the poset \(\mathbb{P}_{S \mathbb{T}}\) has the \(\kappa\)-cc property.

**Proof.** Suppose that \(\{p_{\alpha} : \alpha \in \kappa\} \subseteq \mathbb{P}_{S \mathbb{T}}\) is an antichain. Working in \(V^\prime\), applying the \(\Delta\)-system lemma (Lemma 3.5), for the system \(\{\text{dom}(p_{\alpha}) : \alpha \in \kappa\}\) of countable sets \(\{1\}\) from Definition 3.10, we obtain a set \(A \in [\kappa]^\kappa\), such that the \(\text{dom}(p_{\alpha})\)'s \((\alpha \in A)\) form a \(\Delta\)-system with kernel \(K \subseteq S\). Since \(K\) is obviously countable, for each \(\alpha\) we have that \(\langle T_{p_{\alpha}(\theta)} : \theta \in K \rangle\) is a countable sequence of countable trees (by \(\{[\alpha]\}\) from Definition 3.6). This means that by \(\text{CH}\) we can assume that

\[
\langle T_{p_{\alpha}(\theta)} : \theta \in K \rangle = \langle T_{p_{\beta}(\theta)} : \theta \in K \rangle \quad (\forall \alpha, \beta \in A).
\]

Now applying the \(\Delta\)-system lemma again for the system

\[U_{\alpha} = \bigcup_{\theta \in S} \left\{ \{\theta\} \times u_{p_{\alpha}(\theta)} \right\} \quad (\alpha \in \kappa)\]

yields a set \(A^\prime \in [A]^\kappa\) such that the \(U_{\alpha}\)'s \((\alpha \in A^\prime)\) form a \(\Delta\)-system with kernel \(I \subseteq \bigcup_{\theta \in S} \{\theta\} \times \theta\) (of course, in fact, \(I \subseteq \bigcup_{\theta \in K} \{\theta\} \times \theta\)). Now by (3.1) it suffices to prove that

\[
\exists \alpha \neq \beta \in A^\prime \text{ such that for each } (\theta, \delta) \in I : \quad \eta_{p_{\alpha}(\theta), \gamma} = \eta_{p_{\beta}(\theta), \gamma},
\]

for which it is enough to prove

\[
\left| \left\{ (\eta_{p_{\alpha}(\theta), \gamma} : (\theta, \delta) \in I) : \alpha \in A^\prime \right\} \right| < \kappa.
\]

Fix \(\alpha \in A^\prime\). Now for each \(\langle \theta, \gamma \rangle \in I\), if \(\theta \in S\) then \(\eta_{p_{\alpha}(\theta), \gamma} \in [\omega_1]^\omega_1\) (a branch through \(T_{p_{\alpha}(\theta)}\)).

This means that (using that \(I\) is countable)

\[
\left\{ (\eta_{p_{\alpha}(\theta), \gamma} : (\theta, \gamma) \in I, \theta \in S\) : \alpha \in A^\prime \right\} \subseteq \prod_{(\theta, \gamma) \in I, \theta \in S\} [\omega_1]^\omega_1,
\]
which latter set is of size $\omega_1$ by CH. Second, if $\theta = \lambda \in (S^+_* \setminus S_*) \cap S$, then
\[
\{ \langle \eta_{p_\alpha}(\theta),\gamma \rangle : (\theta,\gamma) \in I, \theta = \lambda \} : \alpha \in A' \} \subseteq \prod_{(\theta,\gamma) \in I, \theta = \lambda} <\omega.2.
\]
Finally we have to consider the coordinate $\theta = \kappa$ if $\kappa \in S \setminus S_*$. Then letting $\delta = \sup\{ \gamma : (\kappa,\gamma) \in I \}$ we have $\delta < \kappa$, because $I$ is countable and $\kappa$ is inaccessible. Then
\[
(3.5) \quad \{ \langle \eta_{p_\alpha}(\kappa),\gamma \rangle : (\kappa,\gamma) \in I \} \subseteq \prod_{(\kappa,\gamma) \in I} <\omega_1.\delta,
\]
and since $\kappa$ is inaccessible, this case $|\prod_{(\kappa,\gamma) \in I} <\omega_1.\delta| < \kappa$. We obtain (using $\omega_1 < \kappa$) that
\[
|\{ \langle \eta_{p_\alpha}(\theta),\gamma \rangle : (\theta,\gamma) \in I \}| \leq \omega_1 \cdot \omega_1 : \prod_{(\kappa,\gamma) \in I} <\omega_1.\delta < \kappa,
\]
therefore (3.3) holds.

Now we make the intuition behind the easy idea of first adding the trees and some branches, and then forcing over the extension precise.

**Claim 3.14.** For each $S \subseteq S^*_+$, $\mathcal{U} = \langle U_\theta : \theta \in S \rangle$ we have
\[
P_{S,\mathcal{U}} \ll P_S < P,
\]
i.e. $P_{S,\mathcal{U}}$ completely embeds into $P_S$, which completely embeds into $P$.

**Proof.** Since $P \simeq P_S \times P_{S^*_\setminus S}$, it is enough to prove that $P_{S,\mathcal{U}} \ll P_S$.

Assume that $A \subseteq P_{S,\mathcal{U}}$ is a maximal antichain in $P_{S,\mathcal{U}}$, and let $p \in P_S \setminus P_{S,\mathcal{U}}$. Then there exists $a \in A$, $a' \in P_{S,\mathcal{U}}$ such that $a' \leq a$, $a' \leq b \upharpoonright \mathcal{U}$. But then it is straightforward to check that also $a'$ and $b$ have a common lower bound. $\Box$

**Definition 3.15.** Let $S \subseteq S_*$, $\overline{S} = \langle U_\theta : \theta \in S \rangle$, $\theta_0 \in S$, $U'_0 \subseteq \theta_0 \setminus U_{\theta_0}$. Then $Q^0_{\theta_0,U'_0} = Q^0(\overline{S},\theta_0,U'_0)$ denotes the $P_{S,\mathcal{U}}$-name for a notion of forcing which adds the branches $\eta_{\theta_0,\alpha}$ ($\alpha \in U'_0$) to $\mathcal{U}$ in the following way
\[
1 \Vdash_{P_{S,\mathcal{U}}} Q^0_{\theta_0,U'_0} \sim \langle p,\eta_p \rangle : (u_p \in [U'_0]^{\leq \omega}) \land (\eta_p = \eta_{p,\alpha} : \alpha \in u_p),
\]
such that each $\eta_{p,\alpha}$ is a branch of $\mathcal{T}_{\theta_0,\leq \delta_\alpha}$, for some $\delta_\alpha \in \{ \gamma + 1 : \gamma < \omega_1 \}$.

If it is clear from the context we will use $Q^0_{\theta_0,U'_0}$, not mentioning $S$ and $\mathcal{U}$.

**Definition 3.16.** Let $S \subseteq S_*$, $\overline{S} = \langle U_\theta : \theta \in S \rangle$, $\theta_0 \in S$.

If $\theta \in S^*_+ \setminus S_*$, and $U'_0 \subseteq \theta \setminus U_{\theta_0}$, then define the $P_{S,\mathcal{U}}$-name $Q^0_{\theta,U'_0} = Q^0_{\theta,U'_0}$ to be the name for $Q^0_{\theta,U'_0}$.

**Definition 3.17.** Let $S \subseteq S^*_+$, $\overline{S} = \langle U_\theta : \theta \in S \rangle$, $\overline{U'} = \langle U'_\theta : \theta \in S \rangle \subseteq \prod_{\theta \in S} P(\theta)$, where $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$. Then $P^0_{\overline{S},\mathcal{U}} = P^0_{(\overline{S},\mathcal{U}),\overline{U'}}$ denotes the $P_{(\overline{S},\mathcal{U}),\overline{U'}}$-name for the countably supported product of $Q^0_{\theta,U'_\theta}$’s ($\theta \in S$), i.e. a notion of forcing which
adds the branches $\eta_{\theta, \alpha} (\alpha \in U'_\theta)$ to $T_\theta$ for each $\theta \in S \setminus S_\bullet$, and the sequences $\eta_{\kappa, \alpha} (\alpha \in U'_\kappa)$ if $\kappa \in S \setminus S_\bullet$, $\eta_{\lambda, \alpha} (\alpha \in U'_\lambda)$ if $\lambda \in S \setminus S_\bullet$:

\[ 1 \models_{P_{S, T}^0} \overline{\mathcal{U}} \models \left\{ p \text{ is a function } : \text{dom}(p) \in [S]^\omega \land (\forall \theta \in \text{dom}(p)) p(\theta) \in \mathcal{Q}_\theta^0 \cup U_\theta \right\} \]

Again, as in Definition 3.13 if it does not cause any confusion we only use the notation $P_{S, T}^0$ not mentioning $S$ and $T$.

The following claim is an easy observation.

**Claim 3.18.** If $G$ is a $P_{S, T}$-generic filter over $V$ (where $S \subseteq S^+$, $\overline{U} = \langle U_\theta : \theta \in S \rangle$, $\overline{U'} = \langle U'_\theta : \theta \in S \rangle \in \prod_{\theta \in S} P(\theta)$, and $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$), then with the notation from [Kun13] (by Claim 3.14) $P_{S, T}^0/G$ is generic over $V[G]$ and the evaluation of $P_{S, T}^0$ are isomorphic, i.e.

\[ V[G] \models P_{S, T}^0[G] \simeq P_{S, T}^0[G]/G. \]

Since $P_{S, T}$ completely embeds into $P_{S, T^*}$ (by Claim 3.14), [Kun13] (Lemma V.4.45.) (and [Kun13] Lemma V.4.44.) implies the following.

**Claim 3.19.** Let $S \subseteq S^+$, $\overline{U} = \langle U_\theta : \theta \in S \rangle$, $\overline{U'} = \langle U'_\theta : \theta \in S \rangle \in \prod_{\theta \in S} P(\theta)$, where $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$. Then the canonical embedding $P_{S, T}^0$ to the iteration $P_{S, T}^* (P_{S, T}^0/G)$ is a dense embedding.

Now putting together Claims 3.18 and 3.19 we have the following, meaning that instead of forcing with $P_{S, T}$ we can force with $P_{S, T}$ and then with (the evaluation of) $P_{S, T}^0$.

**Lemma 3.20.** Let $S \subseteq S^+$, $\overline{U} = \langle U_\theta : \theta \in S \rangle$, $\overline{U'} = \langle U'_\theta : \theta \in S \rangle \in \prod_{\theta \in S} P(\theta)$, where $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$. Then forcing with $P_{S, T}$ amounts to the same as forcing with $P_{S, T}$ and then with $P_{S, T}^0/G$.

**Definition 3.21.** If $S \subseteq S^+$, $\overline{U} = \langle U_\theta : \theta \in S \rangle$, $\overline{U'} = \langle U'_\theta : \theta \in S \rangle \in \prod_{\theta \in S} P(\theta)$. Now if $G$ is generic over $P = P_{S, T}$ then we define

- $G_S = G \cap P_S$,
- $G_{S, T} = G \cap P_{S, T}$,
- $G_{S, T}^0 \subseteq P_{S, T}^0[G_{S, T}] \in V[G_{S, T}]$ to be the filter given by the canonical mapping from Claims 3.18 and 3.19.

The following are basic observations. Roughly speaking, we isolate a dense $\omega_1$-closed subset of a two-step iteration similarly as in [Kun78].

**Claim 3.22.** $P^*$ (and in general each $P^*_S$) is $\omega_1$-closed, i.e. for each decreasing sequence of type $\omega$ has a lower bound. In particular if $G^* \subseteq P^*$, (or in general $G^*_{S, T} \subseteq P^*_{S, T}$) is generic over $V$, then there is no new sequence of ordinals of type $\omega$.

The last claim and Remark 3.12 obviously implies the following.
Claim 3.23. Forcing with $\mathbb{P}$ (or $\mathbb{P}_{S,\mathfrak{T}}$) doesn’t add new sequence of ordinals of type $\omega$, and for a given generic filter $G \subseteq \mathbb{P}$

$$\mathcal{H}(\omega_1)^V = \mathcal{H}(\omega_1)^{V[G]} = \mathcal{H}^{V[G_{S,\mathfrak{T}}]}.$$  

Lemma 3.24. Let $G \subseteq \mathbb{P}_{S,\mathfrak{T}}$ generic over $V$, $B \in V[G]$ where $B \subseteq \mathcal{H}(\omega_1)$. Then (in $V$) there exist $S \subseteq S$, $|S| < \kappa$ and $\mathfrak{W}' = \{W'_\gamma : \gamma \in S \} \in \prod_{\gamma \in S}[U_\gamma]^{<\kappa}$, such that $B \in V[G_{S_{\mathfrak{W}'},\mathfrak{T}}].$

Proof. Choose $p \in G$ forcing that $B \subseteq \mathcal{H}(\omega_1)$, and a nice $\mathbb{P}_{S,\mathfrak{T}}$-name for $B$, obtaining for each $x \in \mathcal{H}(\omega_1)$ an antichain $A_x \subseteq \mathbb{P}_{S,\mathfrak{T}}$ (deciding about $x \in B$). Then by $\kappa$-cc we have that each $|A_x| < \kappa$, the set $S = \bigcup_{x \in \mathcal{H}(\omega_1)} \bigcup_{a \in A_x} \text{dom}(a)$ is of size less than $\kappa$ (as $\kappa$ is either inaccessible, or $\omega_2$). Also for $\theta \in S$ the set $W^*_\theta = \bigcup_{x \in \mathcal{H}(\omega_1)} \bigcup_{a \in A_x} y_{a(\theta)}$ is smaller than $\kappa$. Now it is easy to see that $\mathfrak{W}' = \{W'_\gamma : \gamma \in S\}$ is as claimed.

Then the following immediately follows from the $\omega_1$-closedness, and $\kappa$-cc.

Claim 3.25. Forcing with $\mathbb{P}$ doesn’t collapse $\omega_1$, and cardinals at least $\kappa$. Moreover, if $G \subseteq \mathbb{P}$ is generic, then

$$V[G] \models '\kappa = \omega_2'. \tag{3.6}$$

Lemma 3.26. Let $T \in V[G_{S_{\mathfrak{W}'},\mathfrak{T}}]$ be a Kurepa tree, $S' \subseteq S$ ($S' \in V$). Then, if $b \in V[G_{S_{\mathfrak{W}'},\mathfrak{T}}]$ is a branch of $T$, then there exists a finite set $S'' \subseteq S'$, and $\mathfrak{U}' = \{U'_\theta : \theta \in S''\}$ s.t. each $U'_\theta$ is finite, and $b$ is in the model obtained by adding these finitely many $\eta_{\theta,\alpha}$’s ($\theta \in S''$, $\alpha \in U'_\theta$) to $V[G_{S_{\mathfrak{W}'},\mathfrak{T}}]$, i.e.

$$b \in V[G_{S_{\mathfrak{W}'},\mathfrak{T}}]. \tag{3.7}$$

Proof. Let $T \in V$ be a $\mathbb{P}_{S,\mathfrak{T}}$-name for $T$. Define

$$\mathbb{P}' = \mathbb{P}_{S,\mathfrak{T}} + \mathfrak{m}_{S'}. \tag{3.6}$$

Suppose that $p_* \in \mathbb{P}'$ forces that $b \in V$ is a $\mathbb{P}'$-name for a counterexample (i.e. forcing that for no such $\mathfrak{U}'$, there exists a $\mathbb{P}_{\mathfrak{U}',\mathfrak{T}}$-name $b'$ - which is of course also a $\mathbb{P}'$-name - with $b' = b$). Let $\chi$ be large enough, and let $\langle N_0, \xi \rangle \prec \langle \mathcal{H}(\chi), \xi \rangle$ be countable s.t. $p_*, b, T, S', \mathfrak{T}, \mathbb{P}_{S,\mathfrak{T}} \in N_0$.

Let $\delta_* = N_0 \cap \omega_1$. Define the countable set $N_1$ to be such that $N_0 \in N_1$, and $\langle N_1, \xi \rangle \prec \langle \mathcal{H}(\chi), \xi \rangle$. Let $X$ be set of the indices of the new branches added to $\langle \mathfrak{U}' : \theta \in S'' \rangle$ by $G_{S_{\mathfrak{W}'},\mathfrak{T}}$ that are in $V[G_{S_{\mathfrak{W}'},\mathfrak{T}}][V[G_{S_{\mathfrak{W}'},\mathfrak{T}}]]$, and belong to $N_0$, i.e.

$$X = N_0 \cap \{\langle \theta, \alpha \rangle : (\theta \in S' \wedge (\alpha \in \theta \setminus U'_\theta))\}. \tag{3.7}$$

We fix an enumeration of $X$ and define also the sequence of the first $n$ indices from this countable set, and as well for each $n$ the one-length sequence consisting only the $n$’th, that is

\begin{align*}
\mathfrak{W}_n &= \langle W_{n,\theta} : \theta \in S' \cap N_0 \rangle, \\
&\text{where } W_{n,\theta} = \{\alpha : (\theta, \alpha) = \langle \theta_j, \xi_j \rangle \text{ for some } j \leq n\}
\end{align*}  

\begin{align*}
\mathfrak{W}_n &= \langle w_{n,\theta} : \theta \in S' \cap N_0 \rangle, \\
&\text{where } w_{n,\theta} = \{\xi_n\} \text{ if } \theta = \xi_n, \ w_{n,\theta} = \emptyset \text{ otherwise.}
\end{align*}
Observe that if $p \in P \cap N_0$, then each $\theta \in \text{dom}(p)$ is an element of $N_0$ since $\text{dom}(p)$ is countable (by Definition 3.11), and similarly $T_{\text{p}(\theta)}, u_{\text{p}(\theta)} \subseteq N_0$ (by Definitions 3.6–3.9).

Working in $V$ we will construct an $N_0$-generic condition in $P'$, which will derive us to a contradiction. It is enough to prove the following claim.

**Claim 3.27.** There exists a sequence $\langle \overline{p}_n : n \in \omega \rangle \in V$, $p'_0 \in P_{S, U_x}$ and a sequence $\overline{q} = \langle q_n : n \in \omega \rangle$ such that the following holds.

\[ \begin{align*}
\text{(1)} & \; P_0 = \langle p_{0,l} : l \in \omega \rangle \text{ is such that} \\
& \quad (a) \; p_{0,0} = p_* \upharpoonright U_x, \\
& \quad (b) \; p_{0,l} \in N_0 \cap P_{S, \overline{U}_x} \text{ for each } l \in \omega, \\
& \quad (c) \; \langle p_{0,l} : l \in \omega \rangle \text{ is } \leq_\theta \text{-decreasing}, \\
& \quad (d) \; \overline{p}_0 \in N_1, \\
& \quad (e) \; \text{letting } G_0 = \{ p \in P_{S, \overline{U}_x} \cap N_0 : (\exists l) \; p \geq p_{0,l} \}, \text{ the filter } G_0 \text{ is } P_{S, \overline{U}_x} \text{-generic over } N_0.
\end{align*} \]

\[ \begin{align*}
\text{(2)} & \; p'_0 \in P_{S, \overline{U}_x} \text{ satisfies the following} \\
& \quad (a) \; p'_0 \text{ is a lower bound of } p_{0,l} \text{ for each } l \in \omega \text{ (hence forces a value to } T_{\delta,\leq \delta}) \text{ for each } \theta \in S \cap N_0, \\
& \quad (b) \; p'_0 \text{ forces a value to } T_{\delta,\leq \delta}, \text{ for each } \theta \in S \cap N_0 \text{ such that for every } \\
& \quad \delta \text{-branch } B \text{ in } T_{\delta,\leq \delta} \text{ the inclusion } B \in N_1 \text{ implies that } B \text{ has an upper bound in } T_{\delta,\leq \delta}, \\
& \quad (c) \; p'_0 \text{ forces a value to } \overline{T}_{\leq \delta}.
\end{align*} \]

\[ \begin{align*}
\text{(3)} & \; \text{for every } n > 0 \text{ the sequence } \overline{p}_n = \langle p_{n,l} : l \in \omega \rangle \text{ has the following properties.} \\
& \quad (a) \; \forall l \in \omega \; p_{n,l} \in N_0 \cap P_{S, \overline{U}_x + \overline{W}_n}, \\
& \quad (b) \; p_{n,l} \upharpoonright U_x \in G_0 \\
& \quad (c) \; \langle p_{n,l} : l \in \omega \rangle \text{ is } \leq_\theta \text{-decreasing}, \\
& \quad (d) \; \overline{p}_n \in N_1, \\
& \quad (e) \; \text{letting} \\
& \quad \quad \quad G_n = \{ p \in P_{S, \overline{U}_x + \overline{W}_n} \cap N_0 : (\exists l_0, l_1, \ldots, l_n) \; p \geq \bigwedge_{j=0}^n p_{j,l_j} \}, \\
& \quad \quad \quad \text{the filter } G_n \text{ is } P_{S, \overline{U}_x + \overline{W}_n} \text{-generic over } N_0.
\end{align*} \]

\[ \begin{align*}
\text{(4)} & \; \text{For the sequence } \overline{q} = \langle q_n : n \in \omega \rangle \\
& \quad (a) \; q_n \in N_0 \cap P_{S, \overline{U}_x + \overline{W}_n}, \text{ for each } n \in \omega, \\
& \quad (b) \; q_0 = p_*, \\
& \quad (c) \; q_n \in \omega, \text{ is } \leq_\theta \text{-decreasing,} \\
& \quad (d) \; \forall n \; q_n \upharpoonright (U_x + \overline{W}_n) \in G_n, \\
& \quad (e) \; \text{Let } \langle B_n : n \in \omega \rangle \text{ enumerate the branches of } \overline{T}_{\leq \delta} \text{ which has an upper} \\
& \quad \text{bound in } \overline{T}_{\leq \delta} \text{ (forced by } p'_0 \rangle. \text{ Then } q_{n+1} \wedge p'_0 \text{ forces that } b \neq B_n, \text{ which} \\
& \quad \text{will be guaranteed by the following requirement:} \\
& \quad \text{There exist } \delta < \delta_*, t \neq t' \in \overline{T}_{\leq \delta} \setminus \overline{T}_{\leq \delta}, \text{ such that } p'_0 \text{ forces } B_{n-s} \delta \text{ th level to be } t', \text{ and } q_{n+1} \text{ forces } t \in b, \text{ i.e.} \\
& \quad \quad \quad p'_0 \models B_n \cap (\overline{T}_{\leq \delta} \setminus \overline{T}_{\leq \delta}) = \{ t' \} \\
& \quad \quad \quad \text{ and} \\
& \quad \quad \quad q_{n+1} \models b \cap (\overline{T}_{\leq \delta} \setminus \overline{T}_{\leq \delta}) = \{ t \}.
\end{align*} \]

(Observe that the latter is a statement in $N_0$.)
Before proving Claim 3.27 we argue why this claim implies Lemma 3.26. First, the claim gives the following condition in \( P_{S, U_\ast + \text{id}_{g'}} \). For each \( n \in \omega \) let \( \eta_{\theta_n, \xi_n} \) be the branch in \( T_{\leq \theta_n, \delta} \) represented by the sequence \( \mathfrak{p}_n \), i.e.

\[
(3.10) \quad \eta_{\theta_n, \xi_n} = \bigcup \{ \eta_{p_n, i(\theta_n), \xi_n} : i \in \omega \},
\]

and note that \( \eta_{\theta_n, \xi_n} \in N_1 \) (\( n \in \omega \)) by \( \square_1^{[d]} \). Therefore by \( \square_1^{[b]} \) we can extend each \( \eta_{\theta_n, \xi_n} \) to a branch \( \eta_n' \) in \( (T_{\delta_n}(p_n))^{\leq \delta_n} + 1 \). Define the function \( p_\ast \) to be the extension of \( p'_0 \) by the \( \eta_{\theta_n, \xi_n} \)'s in the obvious way: (Note that by \( \square_1^{[d]} \) we have \( S \cap N_0 \subseteq \text{dom}(p'_0) \subseteq S \), and for each \( \theta \in S \cap N_0 \) the inclusion \( U_\ast \cap N_0 \subseteq \text{dom}(p'_0(\theta)) \subseteq U_\ast \).

Define \( p_\ast \) to be function on \( \text{dom}(p'_0) \) such that if \( \theta \notin N_0 \cap S' \), then \( p_\ast(\theta) = p'_0(\theta) \), and for \( \theta \in N_0 \cap S' \) define \( p_\ast(\theta) \) to be the following proper extension of \( p'_0(\theta) \). Let \( u_{p_\ast(\theta)} = u_{p_0(\theta)} \cup (\theta \cap N_0) \), and if \( \alpha \notin u_{p_0(\theta)} \) (when necessarily \( \alpha \notin U_\ast \)) and (by \( (3.8) \)) choose \( n > 0 \) so that

\[
(3.11) \quad \langle \theta, \alpha \rangle = (\theta_n, \xi_n), \quad \text{and let}\quad \eta_{p_\ast(\theta), \alpha} = \eta_n',
\]

Otherwise

\[
(3.12) \quad \eta_{p_\ast(\theta), \alpha} = \eta_{p_0(\theta), \alpha} \quad \text{(if } \alpha \in U_\ast \).
\]

Observe that as \( \eta_{\eta_{p_n(\theta)}}' \) was a cofinal branch in \( (T_{p_n(\eta_n)})^{\leq \delta_n} + 1 = (T_{\delta_n}(p_n))^{\leq \delta_n} + 1 \) our function \( p_\ast \) is indeed a condition in \( P_{S, U_\ast + \text{id}_{g'}} \). Moreover, the following shows that \( \forall n \in \omega \) we have \( q_n \leq q_n \). Fix \( n \in \omega \), then using \( \square_1^{[d]} \) we have \( q_n \upharpoonright (U_\ast + \mathcal{W}_n) \in G_n \), i.e. there exist \( l_0, l_1, \ldots, l_n \in \omega \), such that \( n^{l_{j=0}} p_{j, l_j} \leq q_n \upharpoonright (U_\ast + \mathcal{W}_n) \). This means that

\[
\bigwedge_{j=0}^{n} p_{j, l_j} \leq q_n \upharpoonright (U_\ast + \mathcal{W}_n) = q_n \upharpoonright (U_\ast),
\]

and for each \( 0 < j \leq n \) \( \eta_{q_n(\theta_j), \xi_j} \subseteq \eta_{p_{j, l_j}(\theta_j), \xi_j} \subseteq \eta_{p_{j, l_j}(\theta_j), \xi_j} = \eta_{p_\ast(\theta_j), \xi_j} \).

On the other hand, for \( j > n \) we have (recalling \( \mathfrak{q} = (q_n : n \in \omega) \) is \( \leq p \)-decreasing by \( \square_1^{[e]} \) that

\[
\eta_{q_n(\theta_j), \xi_j} \subseteq \eta_{q_n(\theta_j), \xi_j} \subseteq \eta_{p_{j, l_j}(\theta_j), \xi_j} = \eta_{p_\ast(\theta_j), \xi_j},
\]

to be

therefore \( p_\ast \leq q_n \), indeed.

Now assuming \( p_\ast \in G_{S, U_\ast + \text{id}_{g'}} \) will easily yield a contradiction: First recall that \( p_\ast \) (and therefore as well \( q_0 \) and \( p_\ast \)) forced that \( \check{b} \) is a branch through \( \check{T} \). Then \( \square_2^{[c]} \) implies that \( p'_0 \), thus \( p_{\ast} \) as well determines \( \check{T}_{\leq \delta_n} \), and \( p_\ast \) forces (by \( \square_2^{[c]} \)) that each element of the \( \delta_n \)-th level of \( \check{T} \) is the upper bound of \( B_i \) for some \( i \in \omega \).

This means that

\[
p_\ast \models (\exists i \in \omega) \check{b} \cap \check{T}_{\leq \delta_n} = B_i,
\]

while at the same time

\[
(q_i \wedge p'_0) \models \check{b} \neq B_i,
\]

since \( (3.9) \) holds.

This together with \( p_\ast \leq q_i, p'_0 \) gives the contradiction. Now we can turn to the proof of the claim.

**Proof.** (Claim 3.27)

For the construction of each sequence \( \mathfrak{p}_n \) and each \( q_n \) we will work in \( N_1 \). This will need a lot of preparation.

Recall that \( X \subseteq N_0 \) denoted the indices of branches added by forcing with \( P_{S, U_\ast + \text{id}_{g'}} \cap N_0 \) but missing from \( V[G_{S, U_\ast}] \) \( (3.7) \), and that for each condition \( p_\ast \),
\[ \theta \in S^*_1, \delta < \omega_1 \text{ the } \delta\text{th level of } T_{p(\theta)} \text{ is (a subset of) } [\omega \cdot \delta, \omega \cdot (\delta + 1)]. \] Define \( E \subseteq N_0 \) as follows.

\[ e \in E \iff e \in N_0, \text{ and } e = (u_e, \eta_e), \text{ where } u_e \in [X]^{\leq \omega}, \]

\[ \eta_e = \langle \eta_{e,\theta,\alpha} : (\theta, \alpha) \in u_e \rangle, \text{ such that } \eta_{e,\theta,\alpha} \subseteq \omega \cdot (\delta_{\theta,\alpha} + 1) \text{ for some } \delta_{\theta,\alpha} < \omega_1. \]

**Definition 3.28.** For each \( n, p \in P_{S, \overline{U}, \overline{V}, \overline{W}}, \) and \( e \in E, \) if each \( (\theta, \alpha) \in u_e \) we have \( \theta \in \text{dom}(p), \) and for each \( i < n \) \( \langle \eta_i, \xi_i \rangle \notin u_e \) holds then define \( p \cap e \) as

\[ \text{dom}(p \cap e) = \text{dom}(p), \]

\[ u_{(p \cap e)}(\theta) = u_p(\theta) \cup \{ \alpha : (\theta, \alpha) \in u_e \} \quad (\forall \theta \in \text{dom}(p \cap e)), \]

\[ \eta_{(p \cap e)}(\theta, \alpha) = \begin{cases} \eta_p(\theta), & \text{if } \alpha \in u_p(\theta), \\ \eta_{e,\theta,\alpha}, & \text{if } (\theta, \alpha) \in u_e, \end{cases} \]

if this is a condition in \( P \) (i.e. for each \( (\theta, \alpha) \in u_e \)

\[ \eta_{e,\theta,\alpha} \text{ is a cofinal branch of } (T_{p(\theta)})_{<\delta+1} \text{ for some } \delta \leq \text{ht}(T_{p(\theta)}), \]

otherwise \( p \cap e = \emptyset. \)

Let \( D \) denote the set of dense subsets of \( P_{S, \overline{U}, \overline{V}, \overline{W}}. \) Fix an enumeration

\[ \langle (J_i, \varepsilon_i) : i \in \omega \rangle \in N_1 \text{ of } (D \cap N_0) \times E, \]

and let \( k(D, e) \) denote the index of the pair \( (D, e) \) (i.e. \( J_{k(D, e)} = D, \varepsilon_{k(D, e)} = e \)), then we also have \( k \in N_1, \) of course.

Fix a function \( g \in N_0 \)

\[ g : P_{S, \overline{U}, \overline{V}, \overline{W}} \times D \to P_{S, \overline{U}, \overline{V}, \overline{W}}, \]

\[ (\forall p, D : \begin{align*} & 1. g(p, D) \in D, \\ & 2. g(p, D) \leq p, \end{align*} \]

(Then \( g \in N_0 \) obviously implies \( (p, D \in N_0 \Rightarrow g(p, D) \in N_0). \))

We will have to define also the auxiliary sequence \( \overline{r} = \langle r_l : l \in \omega \rangle \) with the following property:

\[ \begin{align*} & \text{1. } \overline{r} \in N_1, \\ & \text{2. for each } l \ r_l \in P_{S, \overline{U}, \overline{V}} \cap N_0, \\ & \text{3. for each } l \ p_{0,l+1} \leq r_l \leq p_{0,l}, \\ & \text{4. if there exists } p \in P_{S, \overline{U}, \overline{V}} \text{ such that } p \leq p_{0,l}, \text{ and } p \cap r_l \text{ is a condition extending } p_{0,l} \text{ in } P_{S, \overline{U}, \overline{V}, \overline{W}}, \text{ then } r_l \text{ is such that.} \end{align*} \]

Now we can construct the \( p_{0,i}'s \) and \( r_i's. \) Let \( p_{0,0} = p_* \upharpoonright \overline{U}_n. \) For obtaining the \( p_{0,i}'s \) proceed as follows. Assume we have defined \( p_{0,0}, p_{0,1}, \ldots, p_{0,l-1} \) (and as well the \( r_i's \) for \( i < l - 1 \)). Now if there exists \( p \in P_{S, \overline{U}, \overline{V}} \ p \leq p_{0,l-1}, \) s.t. \( p \cap r_{l-1} \neq \emptyset \) but a condition extending \( p_{0,l-1}, \) then let \( r_{l-1} \in N_0 \) be such a \( p \) (recall that \( r_{l-1} \in E \subseteq N_0 \) by \([3.13]\)), otherwise define \( r_{l-1} = p_{0,l} = p_{0,l-1}. \) Lastly, in the former case define \( p_{0,l} = g(r_{l-1}, D_{l-1}) \upharpoonright \overline{U}_n. \) It is clear from the construction and the definition of \( g \) that \( p_{0,l-1} \leq r_{l-1} \leq p_{0,l}, \) and \( r_{l-1}, p_{0,l} \in N_0, \) and since every object as well as the series \( \langle r_i : i \in \omega \rangle \) are elements of \( N_1, \) we obtain \( p_0, \overline{r} \in N_1, \) too.
Finally, it is straightforward to check that the filter $G_0$ generated by the $p_{0,i}$'s meets every dense subset $D \in N_0$ of $\mathbb{P}_{\mathbf{S, U}}$. Fixing such a $D$

$$D' = \{ p \in \mathbb{P}_{\mathbf{S, U}} : p \restriction \mathbf{U} \in D \}$$

is clearly a dense subset of $\mathbb{P}_{\mathbf{S, U}}$ belonging to $N_0$. This means that if $e \in E$ is the empty sequence, then there exists $i \in \omega$, such that $J_i = D'$, and $e_i = e$, therefore $p_{0,i+1} \in D$.

For $p_0'$, first consider the condition $p_0'' \in N_1$ consisting of only the generic trees given by $G_0$ (for each $\delta \in \text{dom}(p_0'') = N_0 \cap \mathbf{S}$ the tree $T_\delta'(\delta) = \cup \{ T_p(\delta) : p \in G_0 \}$ is of height $\delta$, but $u_{p_0''(\delta)} = 0$). Let $p_0'' \in \mathbb{P}_{\mathbf{S, U}}$, $p_0'' \leq p_0'$ be an extension so that for each $\delta \in \mathbf{S}' \cap N_0$ the tree $T_\delta'(\delta)$ satisfies that for each branch $B$ through $(T_\delta'(\delta))_{< \delta} = T_{p_0''}(\delta)$, if $B \in N_1$, there is an upper bound of $B$ in $T_{p_0''}(\delta)$. This can be done since $N_1$ is countable. Moreover, we choose the other part of $p_0''$ so that for each $\alpha \in N_0$, if $\alpha \subseteq \delta_0$ the chain $G_{\delta_0}(\alpha)$ (with a top element) contains the chain $\cup \{ p_{0,\alpha} : p \in G_0 \}$ which is given by $G_0$ at this coordinate. This can be done as $\cup \{ p_{0,\alpha} : p \in G_0 \} \subseteq N_1$, since $G_0, p_0 \in N_1$. Then clearly $p_0''' \leq p_{0,i}$ for each $i \in \omega$.

Finally, for the last item of $E$ first recall that $\mathbb{P}_{\mathbf{S, U}}$ is an $\omega_1$-closed dense subposet of $\mathbb{P}_{\mathbf{S, U}}$, by Remark 3.11. Then if a countable increasing sequence in $\mathbb{P}_{\mathbf{S, U}}$ (where a first element stronger than $p_0'''$) decides more and more about the $\delta_*'$th level of $\mathbf{T}$, then choosing $p_0'$ to be an upper bound will work (e.g., choose an enumeration $\langle t_i : i \in \omega \rangle$ of the $\delta_*'$th level of $\mathbf{T}$, let $\langle s_i : i \in \omega \rangle$ enumerate $T_{< \delta_*}$ in type $\omega$, and let $r_j$ decide whether the $j$'th ordered pair in the countable set $\{ s_i : i \in \omega \} \times \{ t_i : i \in \omega \}$ is in $\leq_{\mathbf{T}}$).

The next step is to construct the $\mathbf{U}_i$'s ($i > 0$) and the $q_i$'s. This will be done simultaneously by induction. The induction is carried out in $V$, but each step can be done in $N_1$, which will guarantee that each $\mathbf{U}_i \in N_1$.

It is straightforward to check that choosing $q_0 = p_*$ would satisfy our requirements, as e.g. $p_{0,0} = p_* \restriction \mathbf{U}_*$. Then fixing $n > 0$, and assuming that $\mathbf{U}_i, q_i$ are constructed for each $i < n$, first we construct $q_n$. Recall that $q_{n-1} \restriction (\mathbf{U}_* + \mathbf{W}_{n-1}) \in G_{n-1}$ (by $E$).

Recall the definition of the set $E$ (3.13), and let

$$E_{n-1} = \{ e \in E : \forall i < n \langle g_i, \xi_i \rangle \notin e \}.$$ 

Using that $p_* \in \mathbb{P}_{\mathbf{S, U} + \mathbf{W}_{n-1}}$ forced that $\dot{b}$ is not an element of $V[G_{\mathbf{S, U} + \mathbf{W}_{n-1}}]$, i.e. there is no $\mathbb{P}_{\mathbf{S, U} + \mathbf{W}_{n-1}}$-name of $\dot{b}$, we argue that

$$D = \{ p \in \mathbb{P}_{\mathbf{S, U} + \mathbf{W}_{n-1}} : \exists \epsilon, \epsilon' \in E_{n-1} (p \cap \epsilon \leq q_{n-1}, \ p \cap \epsilon' \leq q_{n-1}) \land (\exists \delta < \omega_1, \ t \neq t' \in \check{T}_{<\delta} \setminus \check{T}_{<\delta} : (p \cap e \forces t \in \dot{b}) \land (p \cap e' \forces t' \in \dot{b})) \} \setminus \mathbb{P}_{\mathbf{S, U} + \mathbf{W}_{n-1}}$$

is dense in $\mathbb{P}_{\mathbf{S, U} + \mathbf{W}_{n-1}}$ under $q_{n-1} \restriction (\mathbf{U}_* + \mathbf{W}_{n-1})$. Indeed, assume on the contrary that $q' \in \mathbb{P}_{\mathbf{S, U} + \mathbf{W}_{n-1}}$, $q' \leq q_{n-1} \restriction (\mathbf{U}_* + \mathbf{W}_{n-1})$ is such that that $D$ has no element under $q'$. Now for every $\delta < \omega_1$, consider the set

$$D_{\delta} = \{ p \in \mathbb{P}_{\mathbf{S, U} + \mathbf{W}_{n-1}} : (p \subseteq q') \land (\exists \epsilon \in E_{n-1} : (p \cap e \leq q_{n-1}) \land (\exists t_{p,\epsilon,\delta} \in \check{T}_{<\delta} \setminus \check{T}_{<\delta} : p \cap e \forces t_{p,\epsilon,\delta} \in \dot{b}))) \}.$$
which is dense under \( q' \) in \( P_{S,U}^{a} + P_{W,n}^{a} \). Now since for each \( \delta < \omega_1 \) the sets \( D \) and \( D_\delta \) are disjoint, for \( p \in D_\delta \) the witnessing \( t_{p,e,\delta} \) doesn’t depend on \( e \), therefore \( q' \land q_{n-1} \) forces that \( b \) is in \( V[G_s] \) (i.e. forces that the \( P_{S,U}^{a} + P_{W,n-1}^{a} \)-name \( \{ \langle p, t_{p,\delta} \rangle : p \in D_\delta, \delta < \omega_1 \} \) and \( b \) are equal).

Then as our set \( D \in N_0 \) is indeed dense we have that there exists a condition \( q'' \in G_{n-1} \cap D \), witnessed by \( t \neq t' \) and \( e, e' \). Finally, if \( t \in B_n \) then define \( q_n = q'' \land e' \), otherwise we can let \( q_n = q'' \land e \), which are both stronger conditions than \( q_{n-1} \) by the definition of \( D \). It is straightforward to check.

As \( q_n \) is already defined (and so are \( q_i, q'_i \) for each \( i < n \)), we turn to the definition of \( q_n \), which we will do similarly to that of \( q_0 \). Let \( p_{n,0} = q_n \upharpoonright (U_s + W_n) \), assume that \( p_{n,0}, p_{n,1}, \ldots, p_{n,t-1} \) are already chosen.

If \( \varepsilon_{t-1} \notin E_{n-1} \), then let \( p_{n,t} = p_{n,t-1} \), otherwise proceed as follows. Choose the condition \( q'' \) in \( G_{n-1} \cap D \), witnessed by \( t \neq t' \) and \( e, e' \). Finally consider the condition \( e_{n+1} \), for each \( i < n + 1 \)

\[
\begin{align*}
  1) &\quad \varepsilon_{n+1} = \varepsilon_{t-1} \text{ and } m_n = l - 1, \\
  2) &\quad \text{for each } i < n + 1
\end{align*}
\]

(3.17) \( J_{m_n} = D \land " e_i = (e_{i+1} \text{ plus } (\eta_{j,m_i} (\langle \varepsilon_i \rangle \cdot e_i) \text{, attained on } \langle \varepsilon_i, \eta_i \rangle)" \).

Provided that the \( e_j \)'s are defined for \( j > i \), and as well each \( m_j \) for \( j > i \), let \( e_i \in E \) be the element with \( e_i = u_{e_{i+1}} \cup \{ (\langle \varepsilon_i, \lambda_i \rangle, \eta_{e,\varepsilon_i} \cdot \lambda_i = \eta_{e,m_i} (\langle \varepsilon_i \rangle, \lambda_i) \} \), and let \( m_{n+1} = k(D, e_i) \). Observe that by our procedure, and by the definition of the function \( k \) (3.15) we have \( e_1 = \varepsilon_{m_0} \), and also

(3.18) \( \eta_{e_1, \varepsilon_{m_0}} = \eta_{p_{n,t-1} (\langle \varepsilon_i \rangle, \lambda_i)} \).

At some point later we will use the following fact, hence it is worth to note that for each \( i, 1 \leq i \leq n \)

(3.19) \( \pi(i, m_i) \subseteq \pi(n, l - 1), \text{ and } \pi(i, m_i) \subseteq \pi(n, l - 1) \).

Finally consider the condition \( r_{m_0} \) (from (3.16) and (3.15): if \( r_{m_0} = e_1 \) is a not a condition in \( P_{S,U} + P_{W,n} \), then let \( p_{n,l} = p_{n,t-1} \), otherwise first define the auxiliary condition

(3.20) \( r_\star = g(r_{m_0} \land e_1, D), \)

and note that in this case \( \eta_{r_{m_0} \land e_1} (\langle \varepsilon_i \rangle, \lambda_i) = \eta_{p_{n,t-1} (\langle \varepsilon_i \rangle, \lambda_i)} \) by (3.18), therefore by the properties of \( g \) we obtain

(3.21) \( \eta_{r_\star (\langle \varepsilon_i \rangle, \lambda_i)} \supseteq \eta_{p_{n,t-1} (\langle \varepsilon_i \rangle, \lambda_i)}. \)

Recall that \( p_{n,t-1} \upharpoonright U_\star \in G_0 \) by our induction hypotheses (3.13), and it can be seen from the construction of \( p_{n,t-1} \)’s that in this case \( p_{0,0} = r_\star \upharpoonright U_\star \in G_0 \). Therefore by (3.21) we have that \( (r_\star \upharpoonright U_\star + W_n) \land p_{n,t-1} \) is a condition in \( P_{S,U}^{a} + P_{W,n}^{a} \), and let

(3.21) \( p_{n,l} = (r_\star \upharpoonright U_\star + W_n) \land p_{n,t-1}. \)

Then clearly \( p_{n,l} \leq p_{n,l-1} \), and \( p_{n,l} \upharpoonright U_\star \in G_0 \). From (3.13) it only remained to check that (3.21) also hold. Since the whole construction of \( p_{n,l} \) took place in \( N_1 \) (\( k < N_1 \) and so is the enumeration \( (J_i, \varepsilon_i : i \in \omega, \varepsilon \in N_0 \), \( p_n \in N_1 \) obviously follows.

Verifying the genericity of \( G_n \) goes similarly as of \( G_0 \). Let \( D \subseteq P_{S,U}^{a} + P_{W,n}^{a}, D \in N_0 \) be a fixed dense set, and \( e' \in E \) be the empty sequence. Now, if we choose \( l \) so that \( J_{l-1} = D' = \{ p \in P_{S,U}^{a} + P_{W,n}^{a} : p \upharpoonright U_\star + W_n \in D \} \), \( \varepsilon_{l-1} = e' \), then it
follows from the construction of $p_{k,j}$’s, that of $m = m(n, l - 1)$ and $π = π(n, l - 1)$, and from (3.19) that

$$p_{i,m_i+1} = (r_i \upharpoonright U_i \cup m_i) \land p_{i,m_i} \quad \text{if } 1 \leq i \leq n,$$

and

$$p_{0,m_0+1} = g(r_{m_0} \land e_1) \upharpoonright U_*,$$

therefore

$$\bigwedge_{i \leq n} p_{i,m_i} \leq g(r_{m_0} \land e_1) \upharpoonright (U_* + W_n) \in D'.$$

□(Claim 3.27)
□(Lemma 3.26)

Lemma 3.29. Let $T \in V[G_{S,\mathcal{T}_*}]$ be a Kurepa tree, $S' \subseteq S \cap S_*$ ($S' \in V$), $G_{\mathcal{id}',\mathcal{T}_*} \subseteq \mathcal{P}_{\mathcal{id}',\mathcal{T}_*}$ be generic over $V[G_{S,\mathcal{T}_*}]$. Suppose that $b \in V[G_{S,\mathcal{T}_*}][G_{\mathcal{id}',\mathcal{T}_*}] \setminus V[G_{S,\mathcal{T}_*}]$ is a new branch of $T$, and suppose that $γ \geq κ$ is a cardinal, and for each $θ \in S'$ the inequality $|θ \setminus U^*_θ| ≥ γ$ holds. Then the filter $G_{\mathcal{id}',\mathcal{T}_*}^{\neq}$ adds at least $|γ|$-many new branches to $T$.

Proof. W.l.o.g. we can assume that $T \subseteq ω_1$, and $λ$ is a cardinal (in $V[G_{S,\mathcal{T}_*}]$).

First we will choose a system $W_0 = (W_0,θ : θ \in S') \in \prod_{θ ∈ S'} P(θ)$ with $(∀θ \in S') |W_0,θ| < κ$, and $b \in V[G_{S,\mathcal{T}_*}][G_{\mathcal{id}',\mathcal{T}_*}]$ such that $b \in V[G_{S,\mathcal{T}_*}][G_{\mathcal{id}',\mathcal{T}_*}]$, $S' \in V$ we can use Lemma 3.24 and obtain that $b \in V[G_{S,\mathcal{T}_*}][G_{\mathcal{id}',\mathcal{T}_*}] = V[G_{S,\mathcal{T}_*}].$

And because $b \in \mathcal{H}(ω_1)^V$, applying Lemma 3.24 with $S_*$ and $U = U_* + \mathcal{U}_S$, there exists $S_* \subseteq S$, $W_* \in \prod_{θ ∈ S'} P(θ)$ such that

$$b \in V[G_{S_*,\mathcal{T}_*} \cup V[G_{S,\mathcal{T}_*} + W_*] = V[G_{S,\mathcal{T}_*}][G_{\mathcal{id}',\mathcal{T}_*}],$$

where $|S_*| < κ$, and $|W^*_θ| < κ$ for each $θ \in S_*$. Then fixing $W_0 \in \prod_{θ ∈ S'} P(θ)$ so that $W_0,θ = W^*_θ \setminus U^*_θ$ if $θ \in S_*$, and $W_0,θ = 0$ for $θ \in S \setminus S_*$ has the required properties.

Now, as $|W_0,θ| < κ \leq γ$, and $γ \leq |θ \setminus U^*_θ|$ for each $θ \in S'$ we can fix for each $α < γ$ a system $W_α = (W_{α,θ} : θ \in S') ∈ \prod_{θ \in S'} P(θ)$ such that for every $θ \in S'$

(i) $W_{α,θ} \cap W_{β,θ} = 0$ for every $α < β < γ$,  
(ii) $|W_{0,θ}| = |W_{α,θ}|$ for each $α < γ$.

For each $0 < α < γ$ define the bijections

$$π_α : \bigcup_{θ \in S'} \{θ\} \times W_{0,θ} \to \bigcup_{θ \in S'} \{θ\} \times W_{α,θ}$$

where $π_α \upharpoonright \{θ\} \times W_{0,θ}$ is a bijection to $\{θ\} \times W_{α,θ}$. Then clearly each $π_α$ induces an automorphism $π_α \in V[G_{S,\mathcal{T}_*}]$ of $P_{W_0}$ and $P_{\mathcal{T}_*}$. Moreover, $π_α$ induces a natural operation $π_α^*$ from the class of $P_{W_0}$-names to the class of $P_{\mathcal{T}_*}$-names.

Now fix a $P_{W_0}$-name $b_0 \in V[G_{S,\mathcal{T}_*}]$ for our new branch $b \in V[G_{S,\mathcal{T}_*}][G_{\mathcal{id}',\mathcal{T}_*}]$, and choose an element $p_0 \in P_{W_0}$ forcing that $b_0$ is a new branch, i.e.

(3.22) $V[G_{S,\mathcal{T}_*}] \models p_0 \models b_0 \in B(T) \setminus B[V[G_{S,\mathcal{T}_*}]](T)$. 




Let \( P^\circ_\alpha = \sum_{\alpha < \gamma} \mathbb{P}_\alpha \), i.e., adding the branches \( \bigcup_{\alpha \in \gamma} W_{\alpha, \theta} \) to \( T_\theta \) for each \( \theta \in S' \), which is of course equal to the countably supported product of \( P^\circ_\alpha \)'s \((\alpha < \gamma)\), and let \( G^\circ_\alpha \) denote the generic filter \( G_{\mathbb{P}_\alpha} \cap \mathbb{P}^\circ_\alpha \).

We will show that in \( V[G_{S,T^\circ}][G^\circ_\alpha] \subseteq V[G_{S,T^\circ}][G^\circ_{\mathbb{P}_\alpha} \cap \mathbb{P}^\circ_\alpha] \) there are at least \( \gamma \)-many new branches of \( T \), i.e.

\[
|B(T) \cap \left( V[G_{S,T^\circ}][G^\circ_\alpha] \setminus V[G_{S,T^\circ}] \right) | \geq \lambda,
\]

by arguing that

\[
\otimes_1 \text{ for any } \alpha < \gamma \text{ (in } V[G_{S,T^\circ}] \text{)}
\]

\[
\hat{\pi}_\alpha(p_*) \Vdash_{\mathbb{P}_\alpha} \hat{\pi}_\alpha^*(\bar{b}_0) \notin V[G_{S,T^\circ}][G^\circ_{\mathbb{P}_\alpha} \cap \mathbb{P}^\circ_\alpha]
\]

(where \( G^\circ_{\mathbb{P}_\alpha} \) stands for \( G^\circ_{\mathbb{P}_\alpha} \cap \mathbb{P}^\circ_\alpha \)), and

\[
\otimes_2 \| \{ \alpha < \gamma : \hat{\pi}_\alpha(p_*) \in G^\circ_\alpha \} \| = \gamma.
\]

This will complete the proof of Lemma 3.29.

First we will prove \( \otimes_2 \), for which recall that we assumed that \( \gamma \) is a cardinal, and choose a system of uncountable regular cardinals \( \{ \rho_\beta : \beta < \chi < \gamma \} \), and a partition \( (I_\beta : \beta < \chi) \) of \( \gamma \) with \( \text{otp}(I_\beta) = \rho_\beta \) for each \( \beta < \chi \) (i.e. \( I_\beta \cap I_\delta = \emptyset \) for \( \beta < \delta < \rho \), and \( \bigcup_{\beta < \rho} I_\beta = \gamma \)). Then it is enough to verify

\[
(\forall \beta < \chi) \left( \{ \alpha \in I_\beta : \hat{\pi}_\alpha(p_*) \in G^\circ_\alpha \} \right) = \rho_\beta,
\]

which can be seen by a standard density argument: Fix \( \beta < \chi, \alpha \in I_\beta \), then it suffices to show that

\[
D_{\beta,\alpha} = \{ p \in P^\circ_\alpha : p \leq \hat{\pi}_\delta(p_*) \text{ for some } \delta > \alpha, \delta \in I_\beta \}
\]

is dense, which obviously holds by the regularity of the uncountable \( \rho_\beta = |I_\beta| \) (since for \( \delta \in I_\beta \) we have \( \hat{\pi}_\delta(p_*) \in P^\circ_\delta \), \( P^\circ_\delta \) is the countably supported product of \( P^\circ_\alpha \)'s \((\alpha < \gamma)\), and \( I_\beta \subseteq \gamma \)).

For \( \otimes_1 \) first consider \( P^\circ_\alpha \) as the product of \( \mathbb{P}_\alpha \sum_{\beta < \gamma, \beta \neq \alpha} W_\beta \) and \( P^\circ_\alpha \). We will need the following claim.

**Claim 3.30.** For each \( p \in P^\circ_\alpha \) \( p \leq \hat{\pi}_\delta(p_*) \) there exist \( q_0, q_1 \in P^\circ_\alpha \), \( q_0, q_1 \leq p \), and the incomparable elements \( t_0, t_1 \) of the tree \( T \) such that

\[
V[G_{S,T^\circ}][G^\circ_{\gamma \setminus \{ \alpha \}}] = q_i \Vdash \bar{t}_i \in \hat{\pi}_\alpha^*(\bar{b}_0) \text{ for each } i \in \{0,1\},
\]

where \( G^\circ_{\gamma \setminus \{ \alpha \}} = G^\circ_\alpha \cap \mathbb{P}^\circ_\alpha \sum_{\beta < \gamma, \beta \neq \alpha} W_\beta \).
We can also assume that incomparable elements $t_0, t_1$ can be in the branch $c$.

**Proof.** (Claim 3.30) First we argue that the statement holds in $V[G, S, \pi]$; i.e., for each $p \in {P}_\alpha$, $p \leq \pi_\alpha(p_*)$ there exist $q_0, q_1 \in {P}_\alpha$ $q_0, q_1 \leq p$, and the incomparable elements $t_0, t_1$ of the tree $T$ such that

\[
V[G, S, \pi] = \{ q_i \models P_\alpha \mid i \in \pi_\alpha(\check{b}_0) \} \text{ for each } i \in \{0, 1\}.
\]

Now (3.22) implies that

\[
V[G, S, \pi] = \pi_\alpha(p_*) \models P_\alpha \text{ is a counterexample, but then for the set}
\]

\[
b' = \{ t \in T : \exists q \in {P}_\alpha, q \leq p \text{ s.t. } q \models t \in \pi_\alpha(\check{b}_0) \} \in V[G, S, \pi],
\]

we have $p \models \pi_\alpha(\check{b}_0) = b'$ (since $\pi_\alpha(p_*)$ forced that $\pi_\alpha(\check{b}_0)$ is a cofinal branch in $T$), a contradiction. Finally, fixing $p \leq \pi_\alpha(p_*)$, if $q_0, q_1 \in {P}_\alpha$, $q_0, q_1 \leq p$, and the incomparable elements $t_0, t_1$ of the tree $T$ such that (3.24) holds, then

\[
V[G, S, \pi] = \{ q_i \models P_\alpha \mid i \in \pi_\alpha(\check{b}_0) \} \text{ for each } i \in \{0, 1\},
\]

since if $q_i \in H \subseteq {P}_\alpha$ is generic over $V[G, S, \pi]$, $q_i \in \pi_\alpha(\check{b}_0)|H]$, for some $i \in \{0, 1\}$), then $H$ is generic over $V[G, S, \pi]$ too, and the same holds in $V[G, S, \pi]|H]$. \hfill $\Box$

It is left to argue why Lemma 3.26 and Lemma 3.20 complete the proof of Theorem 3.11 (and Theorem 3.1). Suppose that $T \in V[G]$ is a Kurepa tree (where $G \subseteq P = {P}_*^{S_*}$ is generic), and assume on the contrary that $|B(T) \setminus B(V[G]|T)| \notin S_*$. We can also assume that $T \subseteq H(\omega_1)^V$, and by Lemma 3.24 there exists $S_* \subseteq S_*^+$, $|S_*| < \kappa$, $\bar{\omega} = (W_\theta : \theta \in S_*) \in \prod_{\theta \in S_*} [\theta]^{\kappa}$ such that $T \in V[G, S, \bar{\omega}]$. For estimating $(2^\omega)^V[G, S, \bar{\omega}]$ first a straightforward calculation yields that $\|P_{S, \bar{\omega}}\| < \kappa$: Since $|P_{S, \bar{\omega}}| = (|S_*|^{\omega_1})^\omega$ which is either $\omega \cdot \omega = \omega < \omega_2$ (if $\kappa = \omega_2$, by CH), or $\gamma^\omega < \kappa$ (for some $\gamma < \kappa$, if $\kappa$ is inaccessible). Thus recalling the definition of $Q_{\theta, W_\theta}$-s, the fact $\sum_{\theta \in S_*} |W_\theta| < \kappa$ as $\kappa$ is regular, and $\sup W_\kappa < \kappa$ (if $\kappa \in S_*$) we have the following (in both cases regardless of whether $\kappa = (\omega_2)^V$, or an inaccessible)

\[
|P_{S, \bar{\omega}}| = |P_{S, \bar{\omega}}| \cdot (\omega) \cdot \left( \sum_{\theta \in S_* \setminus \{\kappa\}} |W_\theta| \right)^\omega \cdot (|W_\kappa| \cdot \sup W_\kappa) \leq \kappa.
\]

At this point we have to discuss the two cases (i.e. whether $\kappa \in S_*$ differently, arguing that in both cases there are branches outside $V[G, S, \bar{\omega}]$.

If $\kappa = \omega_2 \in S_*$, then as

\[
V = |P_{S, \bar{\omega}}| \cdot \omega_1 \cdot |P_{S, \bar{\omega}}| = \omega_2,
\]
we have

\[ V[G_{S, \bar{\omega}}] = 2^{\omega_1} = \omega_2, \]

therefore as \( |B^{[G]}(T)| \notin S_\ast \), there are branches of \( T \) in \( V[G] \) not in \( V[G_{S, \bar{\omega}}] \).

On the other hand, if \( \kappa \notin S_\ast \) is inaccessible, then we obtain that

\[ V[G_{S, \bar{\omega}}] \models |B(T)| \leq 2^{\omega_1} < \kappa, \]

and as \( \kappa \) remains a cardinal in \( V[G] \) (by Claim \[3.23\]), and

\[ V[G] \models |B(T) \cap V[G_{S, \bar{\omega}}]| = \omega_1, \]

we conclude that this case there also must be branches of \( T \) not in \( V[G_{S, \bar{\omega}}] \) as \( T \) is a Kurepa tree in \( V[G] \).

Now let \( \bar{\mathcal{P}} \in \prod_{\theta \in S_\ast} P(\theta), R_\theta = \theta \setminus W_\theta \), then

\[ \mathbb{P} = \mathbb{P}_{S_\ast, \bar{\mathcal{P}}} = (\mathbb{P}_{S_\ast, \bar{\mathcal{P}}}, \mathcal{S}_\ast) \times (\mathbb{P}_{S_\ast, \bar{\mathcal{P}}}, \mathcal{S}_\ast) \times (\mathbb{P}_{S_\ast, \bar{\mathcal{P}}}, \mathcal{S}_\ast), \]

and there are no new sequences of type \( \omega \) in \( V[G] \) (by Claim \[3.23\]), and the second component is \( \omega_1 \)-closed, the third component has an \( \omega_1 \)-closed dense subset (which thus remain \( \omega_1 \)-closed in \( V[G_{S, \bar{\omega}}] \)) we obtain that each branch of \( T \) is added by

\[ G_{S, \bar{\omega}, \bar{\mathcal{P}}} = \mathcal{G}_{S, \bar{\mathcal{P}}}, \mathcal{S}_\ast \] (since an \( \omega_1 \)-closed forcing do not add new branches to Kurepa trees [Kun13 Lemma V.2.26]).

We only have to derive a contradiction from

\[ V[G_{S, \bar{\omega}}] \models |B(T)| \notin S_\ast. \]

Now letting \( \bar{\theta} = \bar{B}^{[G_{S, \bar{\omega}}]}(T) \notin S_\ast, S_\ast = S_\ast \cap \mathbb{P}(\theta), S_\ast = (S_\ast \cap S_\ast) \setminus S_\ast \)

by Lemma \[3.20\] we have

\[ V[G_{S, \bar{\omega}}] \models V[G_{S, \bar{\omega}}, S_\ast] = V[G_{S, \bar{\omega}}][G_{S, \bar{\omega}}]. \]

As \( \bar{\theta} \notin S_\ast, S_\ast^\dagger \), it is enough to prove that in \( V[G_{S, \bar{\omega}}, S_\ast^\dagger] \) there are less than \( \delta \)-many branches of \( T \), because if \( G_{S, \bar{\omega}} \) adds new branches, then adds

\[ \min(S_\ast^\dagger)\text{-many new branches by Lemma }3.29\text{(since each } |W_\theta^\ast| < \kappa \leq \min(S_\ast) \leq \min(S_\ast^\dagger)) \]

Now if \( \bar{\theta} = \kappa \), then \( S_\ast^\dagger = \emptyset \), we are done, so we can assume that \( \bar{\theta} > \kappa \), and

\[ \sup S_\ast^\dagger \geq \kappa. \]

As \( S_\ast < \kappa \) (in \( V \)), and our conditions (Case1 [iii]) or Case2 [ii]) states that then \( \sup(S_\ast \cap S_\ast \cap \bar{\theta}) \in S_\ast \) implying \( \sup S_\ast < \bar{\theta} \). Therefore using that

\[ W_\theta^\ast \subseteq \bar{\theta} \text{ we get } \sum_{\theta \in S_\ast} |W_\theta^\ast| = |\sup S_\ast^\dagger|^2 < \bar{\theta}. \]

Now by Lemma \[3.20\] for each branch \( b \) of \( T \) in \( V[G_{S, \bar{\omega}}, S_\ast^\dagger] = V[G_{S, \bar{\omega}}, S_\ast^\dagger] \) there exist \( \theta_0, \theta_1, \ldots, \theta_n \),

\[ U_{\theta_0}, U_{\theta_1}, \ldots, U_{\theta_n} \]

finite such that \( b \in V[G_{S, \bar{\omega}}, S_\ast^\dagger] \). Therefore, as \( |P_{S_\ast^\dagger}^\ast| = \omega_1^\ast = \omega_1 \), counting the nice \( P_{S_\ast^\dagger}^\ast \text{-names of subsets } T \) for each possible \( n \), sequence of \( \theta \)'s, and

\[ B(T) \cap (V[G_{S, \bar{\omega}}][G_{S, \bar{\omega}}]) \leq (\sup S_\ast^{\omega_1^n})^{V[G_{S, \bar{\omega}}]} \leq \sup S_\ast^{\omega_1^n}, \]

which is smaller than \( \bar{\theta} \), a contradiction.

For \( V[G] \models 2^{\omega_1} = \lambda \) we only need to show that \( 2^{\omega_1} < \lambda \). But a similar straightforward calculation yields that \( \mathbb{P} = \mathbb{P}_{S_\ast, \bar{\mathcal{P}}} \) is of cardinality \( \lambda \), and then

(using \( \kappa \)-cc and the equality \( \lambda^{<\kappa} = \lambda \)) by counting the possible nice names for subsets of \( \omega_1 \) we obtain the desired inequality.
Remark 3.31. If $S_\bullet$ also satisfies
\begin{equation}
\forall \mu \in S_\bullet : \mathrm{cf}(\mu) < \kappa \rightarrow \mu^+ \in S_\bullet,
\end{equation}
and $\text{GCH}$ holds in $V$ then $S_\bullet \setminus \{\lambda\}$ is the spectrum for the Jech-Kunen trees in $V[G]$.

(A tree $T$ of height $\omega_1$ and power $\omega_1$ is a Jech-Kunen tree if $\omega_1 < |B(T)| < 2^{\omega_1}$.)

For more on Jech-Kunen trees see also [JS93, JS92, JS94]. Note that $\text{CH}$ in the final model implies that the product of countably many Jech-Kunen trees is a Jech-Kunen tree, so is the diagonal product of $\omega_1$-many Jech Kunen trees, hence (3.25) cannot be dropped.

One can obtain similar cardinal arithmetic conditions for $\text{Sp}_\mu$ with $\mu$ large.

4. The necessity of the inaccessible cardinal

In this section we prove that if $\omega_2$ is not an element of the spectrum, then $\omega_2$ is inaccessible in $L$. The idea of using transitive collapses of elementary submodels of constructible sets as nodes of a tree goes back to Solovay’s original unpublished argument for the consistency strength of the negation of the Kurepa Hypothesis. Although the next proof is deemed to be well-known, for the sake of completeness we include the proof as there is probably no known source to cite.

Theorem 4.1. Suppose that $\omega_2^V$ is a successor in $L$. Then there exists a Kurepa tree $T$ with $\mathcal{B}^V(T) = \omega_2$.

Proof. We will use an extension of $L$, an inner model between $L$ and $V$, what serves as the motivation for the following definition of relative constructibility, which can be found in e.g. [Kan03].

Definition 4.2. For a set $A$ define $L[A] = \bigcup_{\alpha \in \text{On}} L_\alpha[A]$ by transfinite recursion as follows. $L_0[A] = \emptyset$, $L_{\alpha+1}[A] = \text{def}_A(L_\alpha[A])$, and $\alpha$ limit $L_\alpha[A] = \bigcup_{\beta < \alpha} L_\beta[A]$ (where $\text{def}_Y(X)$ are the subsets of $X$ that can be defined in the structure $(X, \in) (X \times X, Y \cap X)$ by parameters from $X$, see [Kan03] Chapter 1, §3).

The following is standard easy exercise, but for the sake of completeness we include the proof.

Claim 4.3. There exists a set $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$, $\omega_2^{L[A]} = \omega_2$.

Proof. If $\omega_2^V = (\lambda^+)^V$, where $|\lambda| = \omega_1$, then in a single subset $A$ of $\omega_1$ we can code a well-ordering of $\omega_1$ in type $\lambda$, and also for each $\alpha < \omega_1$ a well-ordering of $\omega$ in type $\alpha$ in the obvious fashion, and such that $L$ can read this coding (implying $\omega_1^{L[A]} = \omega_1$, $\omega_2^{L[A]} = \omega_2$): First let $\{X_\alpha : \alpha \leq \omega_1\} \in L$ be a set of pairwise disjoint sets of $\omega_1$ with $|X_\alpha|^L = \omega$ for each $\alpha < \omega_1$, and $|X_{\omega_1}|^L = \omega_1$, then for each $\alpha < \omega_1$ we can code the well ordering $X_\alpha$ in order type $\alpha$, and the well ordering of $\omega_2$ in type $\lambda$ in a subset $A'$ of $\bigcup_{\alpha \leq \omega_1} X_\alpha^2 \subseteq \omega_1^2$. Finally, taking the preimage of this set under a bijection $f \in L$ between $\omega_1$ and $\omega_1^2$, i.e. $A = f^{-1}(A')$ works. \hfill \Box

We have to recall a classical Lemma [Kan03] Theorem 3.3. Recall that $\mathcal{L}(R_A)$ stands for the (first-order) language of set theory extended by the unary predicate $R_A$.

Lemma 4.4. There is a sentence $\sigma \in \mathcal{L}_\in(R_A)$ such that for every transitive set $N$
\[ (N, \in, X \cap N) \models \sigma \text{ implies } N = L^\gamma[X] \text{ for some limit } \gamma. \]
In particular, if \( M \prec (L_\beta[X], \in, X \cap L_\beta[X]) \), where \( \beta \) is a limit ordinal and \( \pi \) is the collapsing isomorphism from \( M \) onto the transitive set \( \text{ran}(\pi) \), then the Mostowski collapse

\[
\text{ran}(\pi) = L_\gamma\{\pi(x) : x \in M \cap X\}
\]

for some \( \gamma \leq \beta \).

The following is immediate.

**Claim 4.5.** For each infinite ordinal \( \beta \) and \( Y \subseteq L_\beta[X] \), if \( Y \in L[X] \) and \( X \subseteq L_\beta[X] \), then \( \mu = (|\beta|^{+})^{L[X]} \) implies \( Y \in L_\mu[X] \).

(Working in \( L[X] \), if \( Y \in L_\gamma[X] \), then let \( M \prec L_\gamma[X] \) with \( \{Y\} \cup L_\beta[X] \subseteq M \), \( |M| = |L_\beta[X]| \), and apply the lemma recalling that \( \pi \upharpoonright L_\beta[X] \) is the identity.)

Now we can turn to the definition of the tree \( T \), which will be defined by its branches.

Recall that there exists a definable well-order on \( L[A] \), which is downward absolute to almost every initial segment of \( L[A] \) (to the ones indexed by limit ordinals) [Kan03, Theorem 3.3]:

**Lemma 4.6.** There exists a formula \( \varphi \in \mathcal{L}_\varepsilon(R_A) \) (i.e. in the language of set theory extended with the unary relation symbol \( A \)) which define a well-ordering on \((L[A], \in, A)\), moreover if \( \delta \) is a limit ordinal, \( x, y \in L_\delta[A] \), then

\[
(L[A], \in, A) \models \varphi(x, y) \iff (L_\delta[A], \in, A \cap L_\delta[A]) \models \varphi(x, y).
\]

From now on \( 'x <_{L[A]} y' \) abbreviates \( \varphi(x, y) \).

We will take Skolem hulls many times, thus we need to introduce the following variant of this standard notion.

**Definition 4.7.** Let \((M, \in, X, \partial), M \subseteq L[A]\) be a set model of the language \( \mathcal{L}_\varepsilon(R_A, c_\partial) \) with \( \emptyset \in M \), \( M' \subseteq M \) such that the well-ordering formula \( \varphi \in \mathcal{L}_\varepsilon(R_A) \) from Lemma 4.6 is absolute to \( M \), i.e.

\[
(\forall x, y \in M) : (L[A], \in, A) \models \varphi(x, y) \iff (M, \in, X) \models \varphi(x, y),
\]

e.g. when \( (M, \in, X) = (L_\zeta[A], \in, A \cap L_\zeta[A]) \) for some limit ordinal \( \zeta \). Then the Skolem-hull of \( M' \) in \((M, \in, X, \partial)\) (in symbols, \( \mathcal{H}(M, \in, X, \partial)(M') \)) is the closure of \( M' \) under the functions \( f^{(M, \in, X, \partial)}_\varphi \) for each formula \( \psi(v_0, \ldots, v_{n_\psi}) \in \mathcal{L}_\varepsilon(R_A, c_\partial) \) with \( n_\psi + 1 \) free variables, where the function \( f^{(M, \in, X, \partial)}_\varphi \) satisfies the following.

\[
f^{(M, \in, X, \partial)}_\varphi : M^{n_\psi} \to M
\]

is defined so that for every \( \langle x_1, x_2, \ldots, x_{n_\psi} \rangle \in M^{n_\psi} \):

- if \( \exists y ! \in M \) s.t. \( (M, \in, X, \partial) \models \psi(y, x_1, x_2, \ldots, x_{n_\psi}) \),
  then let \( f^{(M, \in, X, \partial)}_\varphi(x_1, x_2, \ldots, x_{n_\psi}) \) be the unique such \( y \),
- otherwise let \( f^{(M, \in, X, \partial)}_\varphi(x_1, x_2, \ldots, x_{n_\psi}) = \emptyset \).

Then the fact that for each formula \( \psi' \) we can define the formula saying that \( y \) is the least \( y \) (w.r.t. the well-order given by \( \varphi \)) satisfying \( \psi'(y, x_1, x_2, \ldots, x_{n_\psi}) \) together with the Tarski-Vaught criterion implies that the closure is an elementary submodel of \( M \), in symbols, \( M' \prec (M, \in, X, \partial) \).
Observe that this closure only depends on the isomorphism class of \((M, \in, X, \partial)\) by the absoluteness of the well-ordering formula \(\varphi\).

Choose \(\xi < \omega_2\) such that
\[
\text{(4.2) } \xi \text{ is the minimal ordinal } (\forall \alpha < \omega_1) \exists f_\alpha \in L_\xi[A] \text{ bijection between } \omega \text{ and } \alpha \\
\text{(which can be done due to Corollary 4.5) in fact } \xi = \omega_1, \text{ but we won’t use this equality, hence we don’t argue that).}
\]

Now we will define an operation which assigns for each \(\delta \in [\xi, \omega_2)\) the ordinal \(\delta' < \omega_2\) in the following way. We would like to choose \(\delta'\) so that in \(L_{\delta'}[A]\) it is true that for each set \(x\) there exists a surjection from \(\omega_1\) to \(x\), and for \(\delta'' \neq \delta'\) the structures \((L_{\delta''}[A], \in, A, \delta_0)\) and \((L_{\delta'}[A], \in, A, \delta_0)\) cannot be elementarily equivalent.

**Definition 4.8.** Fix \(\delta \in [\xi, \omega_2)\), and define \(\delta'\) to be the least ordinal such that
\[
a) \ \delta \in L_{\delta'}[A], \\
b) \text{ for each } x \in L_{\delta'}[A] \text{ there is a bijection } f \in L_{\delta'}[A] \text{ between } \omega_1 \text{ and } x, \\
c) \text{ taking the sentence } \sigma \text{ from Lemma 4.3 } (L_{\delta'}[A], \in, A) \models \sigma.
\]

(Using Claim 4.9 and \((L_\alpha[A] = |\alpha|L[A]) \text{ for } \alpha \geq \omega \text{ it is easy to see that we can do this closure operation, and there is such a } \delta' < \omega_2\text{.) Then we have}
\[
\text{(4.3) } (\delta' \text{ is a limit} ) \bigwedge (L_{\delta'}[A] \models \text{‘} \omega_1 \text{ is the largest cardinal}),
\]
and also the desired uniqueness by our next claim.

**Claim 4.9.** There is a statement \(\sigma' \in L\in(R_A, c_\partial)\) such that for each \(\delta \in [\xi, \omega_2)\)
\[
(L_{\delta'}[A], \in, A, \delta) \models \sigma' , \text{ moreover, for each } \delta > \omega_1 \text{ and } \delta'' > \delta
\]
\[
((L_{\delta''}[A], \in, A, \delta) \models \sigma') \Rightarrow (\delta'' = \delta').
\]

**Proof.** First define \(\sigma'' = \sigma \land (\forall y \exists f : \omega_1 \rightarrow y \text{ bijection})\) and let \(\sigma'\) be the following sentence
\[
\sigma' = \sigma'' \land (\neg (\exists X) (X \text{ is transitive}) \land (\sigma'')^X \land (\delta \in X))
\]
(where under \(\psi^X\) we always mean the formula \(\psi \in L\in(R_A, c_\partial)\) relativized to \(X\), and \(\sigma\) is from Lemma 4.3). \(\square\)

Now fix \(\delta \in [\xi, \omega_2)\), and for each ordinal \(0 < \alpha < \omega_1\) define \(M_{\delta, \alpha}\) to be the Skolem-hull
\[
M_{\delta, \alpha} = \{y \in L_{\delta'}[A], \in, A, \delta) \mid f_{\psi}^{(L_{\delta'}[A], \in, A, \delta)}(\alpha) \} \text{ (for each } \alpha < \omega_1),
\]

Also define
\[
M_{\delta, 0} = \emptyset.
\]

Then
\[
M_{\delta, \alpha} < (L_{\delta'}[A], \in, A, \delta) \text{ (for each } \alpha > 0).
\]

Observe that whenever \(M^* < (L_{\delta'}[A], \in, A, \delta)\) we have for the Skolem functions from Definition 4.7 that \(f_{\psi}^{(L_{\delta'}[A], \in, A, \delta)}(M^*) = \psi f_{\psi}^{(M^*, \in, A, \delta)}, \) hence
\[
\text{(4.7) } \forall M' \subseteq M^* < (L_{\delta'}[A], \in, A, \delta) : f_{\psi}^{(L_{\delta'}[A], \in, A, \delta)}(M') = \psi f_{\psi}^{(M^*, \in, A, \delta)}(M').
\]

Now as we defined \(\langle M_{\delta, \alpha} : \alpha < \omega_1 \rangle\) note that
\[
\text{(4.8) } (M < (L_{\delta'}[A], \in, A, \delta)) \land (|M| = \omega) \Rightarrow (M \cap \omega_1 \in \omega_1),
\]
in particular
\[
M_{\delta, \alpha} \cap \omega_1 \in \omega_1,
\]
since \((4.2)\) together with \(\xi \leq \delta < \delta'\) implies that in \(L_{\delta'}[A]\) there is an enumeration of each ordinal less than \(\omega_1\) (and \(M_{\delta,\alpha}\) is countable). This implies that
\[
(C_\delta = \{ \alpha < \omega_1 : M_{\delta,\alpha} \cap \omega_1 = \alpha \}) \text{ is a club in } \omega_1 \land (0 \in C_\delta).
\]
It is easy to see that
\[
(4.10) \quad \forall \alpha < \omega_1 : M_{\delta,\alpha} = M_{\delta,\min(C_\delta \setminus \alpha)}.
\]
For later use we verify the following statement.

**Claim 4.10.**
\[
\bigcup_{\alpha < \omega_1} M_{\delta,\alpha} = L_{\delta'}[A].
\]

**Proof.** Since the union of an increasing chain of elementary submodels is an elementary submodel, we have \(M_{\omega_1} = \bigcup_{\alpha < \omega_1} M_{\delta,\alpha} < (L_{\delta'}[A], \in, A, \delta)\). Now recall, that in \(L_{\delta'}[A]\) every set \(x\) admits a surjection from \(\omega_1\) onto \(x\), therefore \(\omega_1 \subseteq M_{\omega_1}\) implies that \(M_{\omega_1}\) is transitive. Then by Lemma \(4.4\) and \(M_{\omega_1} \models \sigma\) we have \(M_{\omega_1} = L_{\delta''}[A]\) for some \(\delta'' > \delta\). But then either \(M_{\omega_1} \subseteq L_{\delta'}[A]\), or \(M_{\omega_1} = L_{\delta'}[A]\), and because the former would contradict Claim \(4.9\) we arrive at our conclusion. \(\Box\)

For each \(\alpha \in C_\delta\) and \(\beta < \omega_1\), if \(\alpha = \max(C_\delta \cap (\beta + 1))\), then let \(N_{\delta,\beta,\alpha}\) be the range of the Mostowski-collapse \(\pi_{\delta,\alpha}\) of \((M_{\delta,\alpha}, \in)\), and let \(A_{\delta,\beta,\alpha} = \pi_{\delta,\alpha}(A), \partial_{\delta,\beta,\alpha} = \pi_{\delta,\alpha}(\delta)\):
\[
\pi_{\delta,\alpha} : M_{\delta,\alpha} \to N_{\delta,\beta,\alpha},
\]
which is of course not only an isomorphism between \((M_{\delta,\alpha}, \in)\) and \((N_{\delta,\beta,\alpha}, \in)\), but witnesses
\[
(4.12) \quad (M_{\delta,\alpha}, \in, A \cap M_{\delta,\alpha}, \delta) \simeq (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}).
\]

Now we are ready to construct the tree \(T\). For a fixed \(\delta \in [\xi, \omega_2)\), \(\alpha \in C_\delta, \beta < \omega_1\), if \(0 < \alpha = \max(C_\delta \cap (\beta + 1))\) holds then we define
\[
(4.13) \quad t_{\delta,\beta,\alpha} = (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}),
\]
\(i.e.\) the structure \((N_{\delta,\beta,\alpha}, \in)\) extended by the one-place relation for the image of \(A \in M_{\delta,\alpha}\) under the collapsing isomorphism, and the constant symbol for \(\partial_{\delta,\beta,\alpha}\).

For \(\max(C_\delta \cap (\beta + 1)) = 0\) let \(t_{\delta,\beta,0} = 0\).

Observe that given \(t = t_{\delta,\beta,\alpha}\) we can decode \(\alpha\) from \(t\), as \(\alpha\) is the first uncountable ordinal of \(t\).

**Definition 4.11.** Define
\[
T = \{(\beta, t_{\delta,\beta,\alpha}) : \delta \in [\xi, \omega_2), \beta < \omega_1, \alpha = \max(C_\delta \cap (\beta + 1))\},
\]
with the partial order \((\beta_0, t_{\delta_0,\beta_0,\alpha_0}) \leq_T (\beta_1, t_{\delta_1,\beta_1,\alpha_1})\) iff either \(\alpha_0 = 0\) (thus \(t_{\delta_0,\beta_0,\alpha_0}\) is the empty structure), or
\(\text{(i)}\) \(\beta_0 \leq \beta_1\), and
\(\text{(ii)}\) taking the Skolem-hull \(M\) of \(\alpha_0\) in
\[
t_{\delta_1,\beta_1,\alpha_1} = (N_{\delta_1,\beta_1,\alpha_1}, \in, A_{\delta_1,\beta_1,\alpha_1}, \partial_{\delta_1,\beta_1,\alpha_1})
\]
\(i.e.\) \(M = \delta^{\delta_{\xi,\beta_1,\alpha_1}}(\alpha_0)\) is isomorphic to \(t_{\delta_0,\beta_0,\alpha_0}\):
\[
(M, \in, A_{\delta_1,\beta_1,\alpha_1} \cap M, \partial_{\delta_1,\beta_1,\alpha_1}) \simeq (N_{\delta_0,\beta_0,\alpha_0}, \in, A_{\delta_0,\beta_0,\alpha_0}, \partial_{\delta_0,\beta_0,\alpha_0}),
\]
and
(iii) if $\alpha_0 < \alpha_1$, then there is no proper elementary submodel $M < (N_{\delta_1, \beta_1, \alpha_1}, \in, A_{\delta_1, \beta_1, \alpha_1}, \partial_{\delta_1, \beta_1, \alpha_1})$ with

$$\alpha_0 \cup \{\alpha_0\} \subseteq M,$$

and

$$M \cap \alpha_1 \subseteq \beta_0.$$  

Roughly speaking, in level $\beta$ we have (isomorphism types of) initial segments $M$ of models of the form $(L_{\Delta}[A], \in, A, \Delta)$ (for some $\Delta \in [\xi, \omega_2]$), such that $M \cap \omega_1 \subseteq \beta$, and $M$ is maximal w.r.t. this condition. We need to check that $T$ is a tree, its levels are countable, and that it has only $\omega_2$-many branches even in $V$.

The following claim is a standard calculation, but for the sake of completeness we include the proof.

**Claim 4.12.** Let $\delta \in [\xi, \omega_2)$ be fixed, $\beta_0 \leq \beta_1 < \omega_1$, let $\alpha_1 = \max(C_\delta \cap (\beta_1 + 1))$, $\alpha_0 = \max(C_\delta \cap (\beta_0 + 1))$. Then $(\beta_0, t_{\delta, \beta_0, \alpha_0}) \leq_T (\beta_1, t_{\delta, \beta_1, \alpha_1})$.

Moreover, the embedding $\pi_{\delta, \beta_0, \alpha_0} : N_{\delta, \beta_0, \alpha_0} \to N_{\delta, \beta_1, \alpha_1}$ is unique.

**Proof.** First observe that by (4.4) and (4.7) for $\delta \in [\xi, \omega_2)$, $\alpha_0 < \alpha_1$,

$$\tilde{\delta}^{(M_{\delta, \alpha_1}, \in, A, \delta)}(\alpha_0) = \tilde{\delta}^{(L_{\delta'}[A], \in, A, \delta)}(\alpha_0) = M_{\delta, \alpha_0},$$

therefore since $\beta_1 < \omega_1$ is such that $\alpha_1 = \max(C_\delta \cap (\beta_1 + 1))$, then applying (the restriction of) the collapsing isomorphism $\pi_{\delta, \alpha_1}$ to the left side, we obtain

$$\tilde{\delta}^{(N_{\delta, \beta_1, \alpha_1}, \in, A_{\delta, \beta_1, \alpha_1}, \partial_{\delta, \beta_1, \alpha_1})}(\alpha_0), \in \cong (M_{\delta, \alpha_0}, \in)$$

and because $\beta_0 < \beta_1$ is such that $\alpha_0 = \max(C_\delta \cap (\beta_0 + 1))$, then applying the isomorphism $\pi_{\delta, \alpha_0}$ to the right side (which fixes $\alpha_0$) we obtain

$$\tilde{\delta}^{(N_{\delta, \beta_0, \alpha_0}, \in, A_{\delta, \beta_0, \alpha_0}, \partial_{\delta, \beta_0, \alpha_0})}(\alpha_0), \in \cong (N_{\delta, \alpha_0, \alpha_0}, \in).$$

Finally, since $\pi_{\delta, \alpha_1}(A) = A_{\delta, \beta_1, \alpha_1}, \pi_{\delta, \alpha_0}(A) = A_{\delta, \beta_0, \alpha_0}, \pi_{\delta, \alpha_1}(\delta) = \partial_{\delta, \beta_1, \alpha_1}$, $\pi_{\delta, \alpha_0}(\delta) = \partial_{\delta, \beta_0, \alpha_0}$, we have

$$\tilde{\delta}^{N_{\delta, \beta_1, \alpha_1}}(\alpha_0), \in, A_{\delta, \beta_1, \alpha_1}, \partial_{\delta, \beta_1, \alpha_1},$$

is isomorphic to $(N_{\delta, \beta_0, \alpha_0}, \in, A_{\delta, \beta_0, \alpha_0}, \partial_{\delta, \beta_0, \alpha_0})$, therefore (iii) holds. The uniqueness easily follows from the facts that the embedding of $(N_{\delta, \beta_0, \alpha_0}, \in, A_{\delta, \beta_0, \alpha_0}, \partial_{\delta, \beta_0, \alpha_0})$ has to fix the ordinals less than $\alpha_0$, and elementary embeddings uniquely extend to Skolem-hulls.

For (iii) suppose that $\alpha_0 < \alpha_1$, and note that

$$\models ' \alpha_1 \text{ is the least uncountable ordinal, } \alpha_0 \text{ is countable'},$$

and for $M < (N_{\delta, \beta_1, \alpha_1}, \in, A_{\delta, \beta_1, \alpha_1}, \partial_{\delta, \beta_1, \alpha_1})$ if $\alpha_0 \cup \{\alpha_0\} \subseteq M$ then consider the corresponding submodel $M' < (M_{\delta, \alpha_1}, \in, A, \delta)$, for which $M' \supseteq M_{\delta, \alpha_1 + 1}$. But (recalling (4.8)) since $\max(C_\delta \cap (\beta_0 + 1)) = \alpha_0$ we obtain $\beta_0 \cup \{\alpha_0\} \subseteq M' \subseteq M_{\delta, \alpha_1}$, that can happen only if $\beta_0$ is smaller than the least uncountable ordinal in $N_{\delta, \beta_1, \alpha_1}$, $\alpha_1$. But then $\beta_0 \in M \cap \alpha_1$. □

The next claim will verify that $T$ is a tree of height $\omega_1$ (for the transitivity of $\leq_T$ use the claim two times).

**Claim 4.13.** For a fixed $\delta_1 \in [\xi, \omega_2)$, $\beta_0 \leq \beta_1 < \omega_1$, let $\alpha_1 = \max(C_{\delta_1} \cap (\beta_1 + 1))$, and fix arbitrary $\alpha_0 \in \omega_1$, $\delta_0 \in [\xi, \omega_2)$. Then $(\beta_0, t_{\delta_0, \beta_0, \alpha_0}) \leq_T (\beta_1, t_{\delta_1, \beta_1, \alpha_1})$ if

$$t_{\delta_0, \beta_0, \alpha_0} = t_{\delta_1, \beta_0, \max(C_{\delta_1} \cap (\beta_0 + 1))}.$$
Proof. We only have to check the 'only if' part, but first observe that Definition 1.11 clearly implies that up to isomorphism there exists only one $t$ for which $(\beta_0, t) \leq (\beta_1, t_{\delta_1, \beta_1, \alpha_1})$. Now the claim is the consequence of the fact that $t_{\delta, \beta_0, \alpha} \neq t_{\delta, \beta_0, \alpha}$, implying that they are not isomorphic as structures of the language $L_\epsilon(R_A, c_0)$: For transitive sets $N$ and $N'$ with $X, \theta \in N$, $X', \theta' \in N'$ the structures $(N, \epsilon, X, \theta)$, $(N', \epsilon, X', \theta')$ are isomorphic if and only if $N = N'$, $X = X'$ and $\theta = \theta'$ (since by the uniqueness of the Mostowski collapse we know that $(N, \epsilon) \cong (N', \epsilon)$ iff $N = N'$).

\[\begin{align*}
\text{Lemma 4.14.} & \quad \text{For each } \beta < \omega_1 \text{ the } \beta \text{th level of } T \text{ is countable.}\\
\text{Proof.} & \quad \text{By Claim 4.13 we have that the } \beta \text{th level of } T \text{ is } T_{\leq \beta} \setminus T_{< \beta} = \{ (\beta, t_{\delta, \beta, \alpha}) : \delta \in [\xi, \omega_2), \alpha = \max(C_\delta \cap (\beta + 1)) \}\.
\end{align*}\]

For a fixed $\delta \in [\xi, \omega_2)$ fix $\alpha = \max(C_\delta \cap (\beta + 1))$ too, and consider the structure $t_{\delta, \beta, \alpha} = (N_{\delta, \beta, \alpha}, \in, A_{\delta, \beta, \alpha}, \theta_{\delta, \beta, \alpha})$, where $N_{\delta, \beta, \alpha}$ is the Mostowski collapse of $(M_{\delta, \alpha}, \in)$ (by the isomorphism $\pi_{\delta, \alpha}$), and $A_{\delta, \beta, \alpha} = A \cap \alpha$. Now 4.13 states $M_{\delta, \alpha} \prec (L_\beta(\in, A))$ then (recalling $M_{\delta, \alpha} \cap \omega_1 = \alpha$, and $\pi_{\delta, \alpha} \upharpoonright \alpha = \text{id}_\alpha$) by Lemma 4.14

$$N_{\delta, \beta, \alpha} = L_\gamma[A \cap \alpha]$$

for some $\gamma = \gamma(\delta, \alpha) \in (\alpha, \omega_1)$. Now we determine an upper bound $\gamma_\alpha$ for the set $\{ (\gamma(\delta, \alpha) : \delta \in [\xi, \omega_2) \land \alpha \in C_\delta \}$. If we have such a bound for each possible $\alpha \leq \beta$, then letting $\gamma_\infty$ denote $\sup \{ \gamma_\alpha : \alpha \leq \beta \}$, we get

$$\{ (\gamma(\delta, \alpha) : \delta \in [\xi, \omega_2) \land \alpha \in C_\delta \} \subseteq \{ (L_\gamma[A \cap \alpha], \in, A \cap \alpha, \theta) : \gamma \leq \gamma_\infty, \alpha \leq \beta, \theta < \gamma \},$$

which latter set is obviously countable, this will finish the proof of the lemma. So fix $\alpha \leq \beta$ and $\delta \in [\xi, \omega_2)$ such that $\alpha \in C_\delta$. Now we have two cases depending on whether there is any (cardinal)$^{L[A \cap \alpha]}$ in $(\alpha, \omega_1)$. If $\lambda \in (\alpha, \omega_1)$ is a cardinal in the inner model $L[A \cap \alpha]$, then for each $\delta$ if $\alpha = \max(C_\delta \cap (\beta + 1))$, then the transitive set $N_{\delta, \beta, \alpha}$ cannot contain $\lambda$, as $M_{\delta, \alpha}$ sees $\omega_1$ as the largest cardinal, and $\pi_{\delta, \alpha} \upharpoonright \omega_1 = \alpha$. This case choosing $\gamma_\alpha = \lambda$ works.

On the other hand, if $|\alpha|^{+L[A \cap \alpha]} = \omega_1$, then we first prove that $\alpha \in C_\delta$ implies $|\alpha| = \omega^{L[A \cap \alpha]}$: otherwise in $M_{\delta, \alpha}$, and in $N_{\delta, \beta, \alpha}$ each ordinal less than $\alpha$ are countable, thus as well in $L[A \cap \alpha]$. Then it is easy to see that the condition

$$\lambda \text{ is the unique cardinal in } (\omega, \omega_1^{L[A \cap \alpha]}),$$

cannot hold for two different $\lambda$'s, therefore $\alpha$ can be defined in $L[A]$. But then using Claim 4.15 with $X = A \cap \alpha$ we have that for each $\zeta \in (\alpha, \omega_1)$ there is a bijection $f_\zeta \in L_{\omega_1}[A \cap \alpha]$ between $\alpha$ and $\zeta$, therefore $\alpha$ can be defined also in $L_\beta[A]$, and $M \prec (L_\beta[A], \in)$ implies $\alpha \in M$, contradicting that $M_{\delta, \alpha} \cap \omega_1 = \alpha$ (which holds by $\alpha \in C_\delta$). Then $|\alpha| = \omega^{L[A \cap \alpha]}$ and Claim 4.15 implies that there is an ordinal $\lambda < \omega_1$ such that there exists a bijection between $\alpha$ and $\omega$ in $L_\lambda[A \cap \alpha]$, implying

$$N_{\delta, \beta, \alpha} = L_\gamma(\delta, \alpha) \subseteq L_\lambda[A \cap \alpha],$$

since $\alpha$ is uncountable in $N_{\delta, \beta, \alpha}$. This case

$$\{ \gamma(\delta, \alpha) : \delta \in [\xi, \omega_2) \land \alpha \in C_\delta \} \subseteq \gamma_\alpha = \lambda,$$

which completes the proof of Lemma 4.14.
Now $T$ is obviously a Kurepa tree by the following fact and lemma.

**Fact 4.15.** The sequence $\{B_δ : δ ∈ [ξ, ω_2)\}$ lists pairwise distinct cofinal branches in $T$, where

$$B_δ = \{(β, t_δ, β, \max(C_δ ∩ (β + 1))) : β < ω_1\}.$$

**Proof.** We only need to prove that $B_δ ≠ B_γ$ if $δ ≠ γ$. But according to the second statement of Claim 4.1.2 for each $β < β' < ω_1$ there is a unique elementary embedding of $t_δ, β, \max(C_δ ∩ (β + 1))$ to $t_δ, β, \max(C_δ ∩ (β' + 1))$, therefore there is a unique direct-limit of this elementary chain, isomorphic to $∪_{α ∈ C_δ} M_δ, α$, which is $(L_δ[A], ∈, A, δ)$ by Claim 4.10.

It is only left to prove that each branch of $T$ is of the form $B_δ$ for some $δ ∈ [ξ, ω_2)$ (even in $V$). The following lemma will complete the proof of Theorem 4.1.

**Lemma 4.16.** Let $B ⊆ T$ a cofinal branch in $T$, $B ∈ V$. Then $B = B_δ$ for a unique $δ ∈ [ξ, ω_2)$.

**Proof.** Let $t_δ, β, α_δ = (N_δ, β, α_δ, ∈, A_δ, β, α_δ, θ_δ, β, α_δ)$ denote the element in $B ∩ \{T_δ, β \cap T_{< β}\}$. Working in $V$ first we define the following bonding maps: for $γ ≤ β < ω_1$ let

$$π_γ, β : N_δ, γ, α_γ → N_δ, β, α_δ$$

be the unique elementary embedding (combining Claim 4.1.3 and the second statement of Claim 4.1.2). Since elementary submodels of an elementary submodel are elementary submodels, $π_β, β, α_β$ is an elementary embedding for each $β'' ≤ β' ≤ β < ω_1$, therefore by the uniqueness

$$(∀ β'' ≤ β' ≤ β < ω_1) : \pi_β, β, α_β ∪ π_β, β, α_β' = π_β, β, α_β.$$

This elementary chain allows us to define the limit $D = (N_ω_1, E, A_ω_1, ∂_ω_1)$ of the directed system $\{t_δ, β, α_δ : β' ≤ β < ω_1\}$. Let $π_β : N_δ, β, α_δ → N_ω_1$ be the embedding, $N_β = \text{ran}(π_β)$ (hence $N_ω_1 = ∪_{β < ω_1} N_β$).

First note that $(N_ω_1, E)$ is well-founded, otherwise there would be an infinite $E$-decreasing chain in the embedded image of $N_δ, β, α_δ$ for some (in fact, every large enough) $β$, contradicting that $(N_δ, β, α_δ, ∈) ∈ E$ is well-founded. Now (by the $E$-extensionality in $N_ω_1$) we can assume that $N_ω_1$ is a Mostowski collapse, i.e. $(N_ω_1, E) = (N_ω_1, ∈)$. Then it is easy to see that if $β < ω_1$ for the elementary embedding $π_β : N_δ, β, α_δ → N_ω_1$ we have $π_β ∪ α_β = id_α$, and $π_β(α_β) = ω_1$, thus (recalling that $A_δ, β, α_δ = A(α_δ)$) we obtain $(N_ω_1, E, A_ω_1, ∂_ω_1) = (N_ω_1, ∈, A, δ_δ)$ for some $δ_δ ∈ (ω_1, ω_2)$. Now we can use Lemma 4.4 (since $(N_δ, β, α_δ, ∈, A_δ, β, α_δ) = σ)$, there exists $ζ > δ_δ$ such that

$$N_ω_1 = L_ζ[A],$$

and then

$$(N_ω_1, ∈, A, δ_δ) = (L_ζ[A], ∈, A, δ_δ).$$

Now because the formula $σ' ∈ L_ε(R_A, c_δ)$ from Claim 4.1.9 holds in $(L_δ[A], ∈, A, δ)$ (for each $δ ∈ [ξ, ω_2]$) (for our mapping $δ → σ'$ from Definition 4.8 and therefore also in $M_δ, α$’s, $N_δ, β, α_δ$’s ($δ ∈ [ξ, ω_2]$)), so it must hold in $(N_ω_1, ∈, A, δ_δ)$, which means that $δ_δ ≥ ζ$, and $ζ = δ_δ^*$, i.e.

$$(N_ω_1, ∈, A, δ_δ) = (L_δ^*[A], ∈, A, δ_δ),$$

□
Finally, we have to prove that for each $\beta < \omega_1$

$$t_{\delta, \beta, \alpha, \sigma} = (N_{\delta, \beta, \alpha, \sigma}, A_{\delta, \beta, \alpha, \sigma}, \partial_{\delta, \beta, \alpha, \sigma}) = t_{\delta, \beta, \alpha, \sigma, \text{max}(\beta + 1)}$$

by arguing (having $\beta$ fixed) that for a large enough $\gamma$

$$(\beta, t_{\delta, \beta, \alpha, \sigma, \text{max}(\beta + 1)}) \leq_T (\gamma, t_{\delta, \gamma, \alpha, \sigma})$$

Let $\alpha = \text{max}(C_0, (\beta + 1))$, $\alpha' = \text{min}(C_0 \setminus (\beta + 1))$, $\beta' = \alpha'$, and consider the models $M_{\delta, \alpha}', M_{\delta, \alpha} = (L_{\delta, \alpha}^M, \epsilon, A, \delta)$.

Choose $\gamma \geq \beta'$, $\gamma < \omega_1$ so that $N_\gamma = \pi_{\gamma'}[N_{\delta, \gamma, \alpha, \sigma}] \supseteq M_{\delta, \alpha}'. Then

$$(4.15) \quad \alpha \gamma \geq \alpha' > \beta + 1,$$

and $\alpha' \cup \{\omega_1\} \subseteq N_\gamma \supseteq (L_{\delta}^M, \epsilon, A, \delta)$ with $$(4.16) \quad \mathcal{J}(N_{\gamma}, \epsilon, A \cap N_{\gamma}, \delta) = \mathcal{J}(L_{\delta}^M, \epsilon, A, \delta).$$

Therefore in $(N_{\gamma}, \epsilon, A \cap N_{\gamma}, \delta)$ $\simeq (N_{\delta, \gamma, \alpha, \sigma}, \epsilon, A_{\delta, \gamma, \alpha, \sigma}, \partial_{\delta, \gamma, \alpha, \sigma})$ there is an elementary submodel isomorphic to $(M_{\delta, \alpha}, \epsilon, A \cap N_{\delta, \alpha}, \delta)$, which latter is isomorphic to $(N_{\delta, \beta, \alpha}, \epsilon, A \cap \delta_{\delta, \beta, \alpha})$, thus $(ii)$ from Definition 4.11 holds.

Similarly, using also $$(4.11) \quad \delta_{\gamma, \alpha, \sigma, \text{max}(\alpha + 1)} = \delta_{\delta, \alpha, \sigma, \text{max}(\alpha + 1)} = \delta_{\delta, \alpha, \sigma, \text{max}(\alpha + 1)} \supseteq \alpha' \supseteq \beta \cup \{\beta\},$$

and since the isomorphism between $(N_{\gamma}, \epsilon, A \cap N_{\gamma}, \delta)$ and $(N_{\delta, \gamma, \alpha, \sigma}, \epsilon, A_{\delta, \gamma, \alpha, \sigma}, \partial_{\delta, \gamma, \alpha, \sigma})$ fixes the ordinals less than or equal to $\alpha'$ we obtain

$$(4.15) \quad \mathcal{J}(N_{\gamma, \gamma}, \epsilon, A_{\delta, \gamma, \alpha, \sigma}, \partial_{\delta, \gamma, \alpha, \sigma}) = \mathcal{J}(L_{\delta}^M, \epsilon, A, \delta).$$

Therefore recalling $$(4.15) \quad \mathcal{J}(N_{\gamma, \gamma}, \epsilon, A_{\delta, \gamma, \alpha, \sigma}, \partial_{\delta, \gamma, \alpha, \sigma}) = \mathcal{J}(L_{\delta}^M, \epsilon, A, \delta).$$

we obtain that $$(iii) \quad (\text{of Definition 4.11})$$ holds as well.

$$\square$$

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