On Hamiltonian formulations of the Schrödinger system

László Á. Gergely
Astronomical Observatory and Department of Experimental Physics,
University of Szeged, Dóm tér 9, Szeged, H-6720 Hungary

We review and compare different variational formulations for the Schrödinger field. Some of them rely on the addition of a conveniently chosen total time derivative to the hermitic Lagrangian. Alternatively, the Dirac-Bergmann algorithm yields the Schrödinger equation first as a consistency condition in the full phase space, second as canonical equation in the reduced phase space. The two methods lead to the same (reduced) Hamiltonian. As a third possibility, the Faddeev-Jackiw method is shown to be a shortcut of the Dirac method. By implementing the quantization scheme for systems with second class constraints, inconsistencies of previous treatments are eliminated.

I. INTRODUCTION

Outstanding equation of modern physics, the Schrödinger equation has multiple and deep connections with integral principles. Historically, Schrödinger obtained his equation guided by the beautiful analogy between the Fermat principle and the principle of least action [1]. Moreover, motivated by a remark of Dirac [2], Feynman has derived the Schrödinger equation from the Huygens principle, realizing the first step towards his path integral approach [3], [4]. (For recent recent reviews see [5], [6].) The Schrödinger field is equally a popular choice to illustrate how second quantization proceeds.

At a closer inspection, however, the power and beauty of the variational approach is obstructed by an aesthetical bug. The reason: the abundance of dynamical variables, some of them being redundant. The Lagrangian yielding the nonrelativistic Schrödinger equation is linear in the time derivatives of the fields $\Psi$ and its complex conjugate $\Psi^\ast$. Thus the Legendre transformation does not lead to an unambiguous Hamiltonian. Various Hamiltonians, all having Schrödinger’s equation as canonical equation can be found in textbooks. They depend either on two pairs of canonical variables [7], or - as a result of adding a total time derivative to the hermitic Lagrangian - just on a single (complex or real) canonical pair [8]- [10]. In the next section we briefly review these approaches, pointing out both the weakness and ingenuity in sweeping away the problem.

There are two equivalent methods to circumvent this difficulty. In section 3 we treat the Schrödinger field as a constrained system, applying the Dirac-Bergmann algorithm [11]- [13]. As the system has second class constraints, the dynamics involves Dirac brackets. We present two alternatives to the existing derivations of the Schrödinger equation. First, the consistency requirement yields the Schrödinger equation in the form of a weak equation on the full phase space. Second, by a suitable canonical transformation we introduce new canonical coordinates, containing the constraints. The Dirac bracket of the full phase space becomes the Poisson bracket of the reduced phase space and the Schrödinger equation is a canonical equation. As a bonus we recover the Hamiltonian obtained by addition of a properly chosen time-derivative. At the end of Section 3 we present an alternative discussion in terms of real variables: the real and imaginary parts of the complex field $\Psi$.

In Section 4 we apply the Faddeev-Jackiw scheme developed for Lagrangians, which are first order in "velocities" [14], [15]. We verify that the fundamental brackets of the Faddeev-Jackiw approach coincide with the Dirac brackets. Finally in the fifth section we follow the canonical quantization scheme [11]- [13], [16], giving an operator representation of the Dirac bracket algebra of the canonical variables. This is equivalent with the quantization of the Faddeev-Jackiw fundamental bracket. Our approach avoids the interpretational inconsistencies of some standard treatments [7], [8], already pointed out by Tassie [17] and is a viable alternative to the existing reduced phase space quantization schemes [9], [10]. By imposing the second class constraints as operator identities, second quantization proceeds smoothly.

II. STANDARD VARIATIONAL PROCEDURES. A REVIEW

A. Two pairs of complex variables

The action for the Schrödinger field cf. Henley and Thirring [7] is:

$$S[\psi, \psi^\ast] = \int dt \int d\mathbf{r} \mathcal{L}$$

(2.1)
\[ L = \frac{i\hbar}{2} \left( \psi^* \dot{\psi} - \psi \dot{\psi}^* \right) - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi - V(r,t) \psi^* \psi . \]

This Lagrangian density \( L \) is hermitian\(^1\). Variation of (2.1) with respect to \( \psi^* \) and \( \psi \) gives the Schrödinger equation

\[ i\hbar \dot{\psi} + \frac{\hbar^2}{2m} \Delta \psi - V \psi = 0 , \]  

(2.2)

and its complex conjugate. The canonical momenta of the complex conjugate variables are complex conjugates too:

\[ \pi = \frac{i\hbar}{2} \psi^* , \quad \pi^* = -\frac{i\hbar}{2} \dot{\psi} . \]

(2.3)

According to Henley and Thirring, the Hamiltonian density \( \mathcal{H} \), when written in terms of the phase space variables \( (\psi, \psi^*, \pi, \pi^*) \), has to be:

\[ S[\psi, \pi, \psi^*, \pi^*] = \int dt \int d\mathbf{r} \left( \pi \dot{\psi} + \pi^* \dot{\psi}^* - \mathcal{H} \right) \]

\[ \mathcal{H} = \frac{i\hbar}{2m} \left( \nabla^* \nabla \pi - \nabla \psi \nabla \pi^* \right) - \frac{i}{\hbar} V(r,t)(\psi \pi - \psi^* \pi^*) . \]

(2.4)

Variations with respect to \( \psi, \psi^* \) give the Schrödinger equation and its complex conjugate. The same equations emerge as a result of the relations (2.3) and the variations with respect to \( \pi, \pi^* \). The Schrödinger equation as such a frequent outcome indicates that there are redundant variables in the formalism.

According to Henley and Thirring, the Hamiltonian density \( \mathcal{H} \), when written in terms of the phase space variables \( (\psi, \psi^*, \pi, \pi^*) \), has to be:

\[ S[\psi, \pi, \psi^*, \pi^*] = \int dt \int d\mathbf{r} \left( \pi \dot{\psi} + \pi^* \dot{\psi}^* - \mathcal{H} \right) \]

\[ \mathcal{H} = \frac{i\hbar}{2m} \left( \nabla^* \nabla \pi - \nabla \psi \nabla \pi^* \right) - \frac{i}{\hbar} V(r,t)(\psi \pi - \psi^* \pi^*) . \]

(2.4)

Variations with respect to \( \psi, \psi^* \) give the Schrödinger equation and its complex conjugate. The same equations emerge as a result of the relations (2.3) and the variations with respect to \( \pi, \pi^* \). The Schrödinger equation as such a frequent outcome indicates that there are redundant variables in the formalism.

Furthermore, the requirement of hermiticity is not even a necessary one. Schiff [8] derives the Schrödinger equation from a non-hermitic Lagrangian, found from (2.1) by adding the total time derivative \( i\hbar(\psi \psi^*) / 2 \):

\[ S[\psi, \psi^*] = \int dt \int d\mathbf{r} \ L_S \]

\[ L_S = i\hbar \dot{\psi}^* \dot{\psi} - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi - V(r,t) \psi^* \psi . \]

(2.5)

Variation of the action (2.5) with respect to \( \psi \) and \( \psi^* \) gives the Schrödinger equation (2.2) and its complex conjugate. However the canonical momenta are not complex conjugate any more:

\[ \pi_S := \frac{\delta L_S}{\delta \dot{\psi}} = i\hbar \psi^* , \quad \pi^*_S := \frac{\delta L_S}{\delta \dot{\psi}^*} = 0 . \]

(2.6)

The first of these equations is used to eliminate \( \psi^* \) from the action, which becomes:

\[ \text{B. One pair of complex variables} \]

Furthermore, the requirement of hermiticity is not even a necessary one. Schiff [8] derives the Schrödinger equation from a non-hermitic Lagrangian, found from (2.1) by adding the total time derivative \( i\hbar(\psi \psi^*) / 2 \):

\[ S[\psi, \psi^*] = \int dt \int d\mathbf{r} \ L_S \]

\[ L_S = i\hbar \dot{\psi}^* \dot{\psi} - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi - V(r,t) \psi^* \psi . \]

(2.5)

Variation of the action (2.5) with respect to \( \psi \) and \( \psi^* \) gives the Schrödinger equation (2.2) and its complex conjugate. However the canonical momenta are not complex conjugate any more:

\[ \pi_S := \frac{\delta L_S}{\delta \dot{\psi}} = i\hbar \psi^* , \quad \pi^*_S := \frac{\delta L_S}{\delta \dot{\psi}^*} = 0 . \]

(2.6)

The first of these equations is used to eliminate \( \psi^* \) from the action, which becomes:
\[ S[\psi, \pi_S] = \int dt \int d\mathbf{r} (\pi_S \dot{\psi} - \mathcal{H}_S) \]
\[ \mathcal{H}_S = -\frac{i\hbar}{2m} \nabla \psi \nabla \pi_S - \frac{i}{\hbar} V(\mathbf{r}, t) \psi \pi_S. \] 

Thus no starred field shows up as canonical variable. The canonical equation obtained by varying \( \psi \) is the Schrödinger equation. The other canonical equation from variation of \( \pi_S \), together with the first relation (2.6) gives the complex conjugate Schrödinger equation. Some of the superfluous variables were removed by the addition of a total time derivative, achieving a partial reduction to the true degrees of freedom. This Hamiltonian description is the simplest one in terms of complex fields.

C. One pair of real variables

A more efficient way to look on the variational problem for quantum mechanics is described by Kuchař [9] and alternatively by Cohen-Tannoudji, Dupont-Roc and Grynberg [10]. These approaches rely on the decomposition of the field \( \psi \) in real and imaginary parts:

\[ q = \frac{1}{\sqrt{2}} (\psi + \psi^\ast), \quad p = -\frac{i\hbar}{\sqrt{2}} (\psi - \psi^\ast). \] 

In terms of the real fields \( q \) and \( p \) the action (2.1) becomes:

\[ S[q, p] = \int dt \int d\mathbf{r} \mathcal{L} \]
\[ \mathcal{L} = \frac{1}{2} (\dot{q} - \dot{p}) - \frac{\hbar^2}{4m} \left[ (\nabla q)^2 + \left( \frac{\nabla p}{\hbar} \right)^2 \right] - \frac{V}{2} \left( q^2 + \frac{p^2}{\hbar^2} \right). \] 

By adding the total time derivative \((pq)/2\) to the Lagrangian density, the action takes an already Hamiltonian form. The field \( p \) turns to be the conjugate momentum to \( q \):

\[ S[q, p] = \int dt \int d\mathbf{r} \mathcal{L} \]
\[ \mathcal{L} = \frac{\hbar^2}{4m} \left[ (\nabla q)^2 + \left( \frac{\nabla p}{\hbar} \right)^2 \right] + \frac{V}{2} \left( q^2 + \frac{p^2}{\hbar^2} \right). \] 

The two canonical equations obtained by variations with respect to \( q \) and \( p \) are:

\[ \dot{q} = -\frac{1}{2m} \Delta p + \frac{V}{\hbar^2} p, \quad \dot{p} = \frac{\hbar^2}{2m} \Delta q - V q. \] 

Up to global factors, the first equation of (2.11) is the real part, while the second the imaginary part of the Schrödinger equation (2.2).

In this latest approach a full reduction of the phase space to the true degrees of freedom was achieved, as the Schrödinger equation emerges only once in the Hamiltonian formalism. In terms of real fields this is the simplest description, which again relies on the addition of a properly chosen total time derivative to the Lagrangian density. We will see in the next section that the addition of specific total time derivative terms to the Lagrangian (2.1) is not compulsory. The standard Dirac-Bergmann algorithm leads directly either to the Hamiltonian density (2.7), (in a description in terms of complex fields) or the Hamiltonian density (2.10) (if real fields are introduced).

III. THE DIRAC-BERGMANN ALGORITHM

Constrained systems are characterized by the singularity of the inertia matrix, whose elements are given by the second derivatives of the Lagrangian with respect to the generalized velocities. This property always holds if a Lagrangian density is linear in the time derivatives of fields [13], as is the case for (2.1). Then the Dirac-Bergmann algorithm takes the role of the Legendre transformation.
A. Complex fields

The momenta (2.3) provide two primary Hamiltonian constraints:

\[ \phi_1 := \pi - \frac{i\hbar}{2} \psi^* = 0, \quad \phi_2 := \pi^* + \frac{i\hbar}{2} \psi = 0. \]  

(3.1)

Time evolution of an arbitrary phase space function \( f \) is generated through the Poisson bracket by the primary Hamiltonian density \( \mathcal{H}_P \) rather than the canonical Hamiltonian density \( \mathcal{H}_C \):

\[ \dot{f} \approx \{ f, H_P \}, \quad H_P = \int d\mathbf{r} \mathcal{H}_P \]

\[ \mathcal{H}_P := \mathcal{H}_C + \dot{\psi}_1 + \dot{\psi}^* \phi_2 \]

\[ \mathcal{H}_C := \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi + V(\mathbf{r}, t) \psi^* \psi. \]  

(3.2)

Here the velocities \( \dot{\psi}, \dot{\psi}^* \) are unknown functions of the phase space variables. The symbol \( \approx \) denotes weak equality, holding only on the hypersurface determined by the constraints (3.1).

Consistency requires the time derivatives of the primary constraints \( \phi_1, \phi_2 \) to vanish:

\[ 0 \approx \dot{\phi}_1 \approx -i\hbar \psi^* + \frac{\hbar^2}{2m} \Delta \psi^* - V \psi^* \]

\[ 0 \approx \dot{\phi}_2 \approx i\hbar \dot{\psi} + \frac{\hbar^2}{2m} \Delta \psi - V \psi^*. \]  

(3.3)

These are the Schrödinger equation and its complex conjugate. They emerge as weak equalities; a price one has to pay for working on the complete phase space \((\psi, \pi, \psi^*, \pi^*)\). No secondary constraint appears in the theory since Eqs. (3.3) are relations determining the unknown functions \( \dot{\psi} \) and \( \dot{\psi}^* \).

The Poisson bracket of the two constraints shows that they are of second class:

\[ \{ \phi_1, \phi_2 \} := \{ \phi_1(\mathbf{r}, t), \phi_2(\mathbf{r}', t) \} = -i\hbar \delta(\mathbf{r} - \mathbf{r}') . \]  

(3.4)

Time evolution can be given in the alternative form:

\[ \dot{f} = \{ f, H_C \}_D, \quad H_C = \int d\mathbf{r} \mathcal{H}_C \]  

(3.5)

in terms of the Dirac bracket [11]:

\[ \{ f, g \}_D := \{ f, g \} - \sum_{i,j=1,2} \{ f, \phi_i \} \{ \{ \phi_k, \phi_l \} \}^{-1} \{ \phi_j, g \} . \]  

(3.6)

Here \( \{ \{ \phi_k, \phi_l \} \}^{-1} \) denotes the inverse of the matrix with elements given by the Poisson brackets of the constraints. Straightforward computation using (3.4) gives the following expression for the Dirac bracket:

\[ \{ f, g \}_D := \frac{1}{2} \{ f, g \} - \frac{i}{\hbar} \int d\mathbf{r} \left( \frac{\delta f}{\delta \psi} \frac{\delta g}{\delta \psi^*} - \frac{\delta f}{\delta \psi^*} \frac{\delta g}{\delta \psi} \right) + \frac{i\hbar}{4} \int d\mathbf{r} \left( \frac{\delta f}{\delta \pi} \frac{\delta g}{\delta \pi^*} - \frac{\delta f}{\delta \pi^*} \frac{\delta g}{\delta \pi} \right) \]  

(3.7)

From here it is immediate to write the Dirac brackets of the canonical data:

\[ \{ \psi(\mathbf{r}), \psi^*(\mathbf{r}') \}_D = -\frac{i\hbar}{\hbar} \delta(\mathbf{r} - \mathbf{r}'), \quad \{ \psi(\mathbf{r}), \psi(\mathbf{r}') \}_D = \{ \psi^*(\mathbf{r}), \psi^*(\mathbf{r}') \}_D = 0, \]  

(3.8)

\[ \{ \pi(\mathbf{r}), \pi^*(\mathbf{r}') \}_D = \frac{i\hbar}{4} \delta(\mathbf{r} - \mathbf{r}'), \quad \{ \pi(\mathbf{r}), \pi(\mathbf{r}') \}_D = \{ \pi^*(\mathbf{r}), \pi^*(\mathbf{r}') \}_D = 0, \]  

(3.9)

\[ \{ \psi(\mathbf{r}), \pi^*(\mathbf{r}') \}_D = \{ \psi^*(\mathbf{r}), \pi(\mathbf{r}') \}_D = 0, \]  

(3.10)
the reduced phase space is immediate:

\[ \{ \psi(r), \pi(r') \}_D = \frac{1}{2} \{ \psi(r), \pi(r') \} = \frac{1}{2} \delta(r - r') \quad (3.11) \]

\[ \{ \psi^*(r), \pi^*(r') \}_D = \frac{1}{2} \{ \psi^*(r), \pi^*(r') \} = \frac{1}{2} \delta(r - r') \quad (3.12) \]

These Dirac brackets do not contain phase-space functions, thus no operator ordering difficulties will occur during quantization.

Reduction. Dirac brackets of second class constraints with arbitrary functions vanish. Thus the constraints can be solved prior to calculating the Dirac brackets, by reducing the phase space to the physical degrees of freedom. As each second class constraint reduces the dimension of the phase space by one, a basis in the reduced phase space is provided by a single pair of complex canonical data. A suitable canonical transformation turns the constraints into the other pair of canonical data:

\[
\begin{pmatrix}
\psi \\
\pi
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\psi_1 = \psi/2 + i\pi^*/\hbar \\
\pi_1 = \pi + i\hbar\psi^*/2
\end{pmatrix} \quad \psi_2 = -i\phi_2/\hbar \quad \pi_2 = \phi_1
\quad (3.13)
\]

A straightforward check shows that the Dirac bracket (3.7) written in terms of the new coordinates becomes the Poisson bracket on the reduced phase space, coordinatized by \(\psi_1, \pi_1\):

\[
\{ f, g \}_D = \int dx \left( \frac{\delta f}{\delta \psi_1} \frac{\delta g}{\delta \pi_1} - \frac{\delta f}{\delta \pi_1} \frac{\delta g}{\delta \psi_1} \right)
\quad (3.14)
\]

Now it is immediate to verify the generic property that the Dirac brackets of the constraints \(\psi_2, \pi_2\) with arbitrary functions vanish.

The Hamiltonian density on the reduced phase space is the one introduced by Schiff (2.7), with \(\psi_1, \pi_1\) in place of \(\psi, \pi\). The canonical equations are the Schrödinger equation and its complex conjugate.

B. Real fields

Our starting point in this section is the action (2.9) written in terms of the real fields \(q, p\). As in the previous section we have presented in detail the method, here we merely list the results. From the definition of the momenta \(P_{q,p}\) canonically conjugate to the \(q, p\) variables we find the constraints which are second class:

\[
\Phi_1 := P_q - \frac{p}{2} \quad \Phi_2 := P_p + \frac{q}{2}
\quad (3.15)
\]

The consistency requirements \(\Phi_{1,2} \approx 0\) are the real and imaginary parts of the Schrödinger equation, Eqs. (2.11).

Reduction. The constraints (3.15) already form a canonical pair of variables, thus a canonical transformation to the reduced phase space is immediate:

\[
\begin{pmatrix}
q \\
P_q
\end{pmatrix} \rightarrow
\begin{pmatrix}
Q_1 = q/2 - P_p \\
Q_2 = \Phi_2
\end{pmatrix} \quad (3.16)
\]

Again the Dirac bracket on the full phase space \((Q_1, P_1, q, p)\) becomes the Poisson bracket on the reduced phase space \((Q_1, P_1)\). The Hamiltonian density on this reduced phase space is \(\mathcal{H}_K\) given in (2.10), with \(Q_1, P_1\) in place of \(q, p\). So the canonical equations on the reduced phase space are the real and imaginary parts of the Schrödinger equation.

IV. THE FADDEEV-JACKIW APPROACH

Developed as a Hamiltonian formulation of dynamical systems with Lagrangians linear in velocities, the Faddeev-Jackiw method provides undoubtfully the shortest path toward a fundamental bracket in the phase space.

The kinetic part of the Lagrangian (2.1) with \(2 \times \infty\) basic variables \(\xi^i(r) = (\psi(r'), \psi^*(r'))\) determines the symplectic 2-form with the inverse

\[
\omega^{ij}(r', r'') = \frac{1}{i\hbar} \begin{pmatrix}
0 & \delta(r' - r'') \\
-\delta(r' - r'') & 0
\end{pmatrix}
\quad (4.1)
\]
in terms of which the time evolution of the fundamental variables is

\[ \dot{\xi}^i(r) = \omega^i_{\cdot\cdot}(r, r') \frac{\delta}{\delta \xi^i(r')} H_C . \] (4.2)

This is nothing but the Schrödinger equation and its complex conjugate. Eq. (4.2) represents a Hamiltonian evolution if the fundamental bracket obeys

\[ \{\xi^i(r), \xi^j(r')\}_F = \omega^i_{\cdot\cdot}(r, r') , \] (4.3)

which is nothing but a shorthand notation for Eqs. (3.8).

V. SECOND QUANTIZATION

The standard procedure for canonical quantization of systems with second class constraints is to turn the Dirac brackets into commutators cf. the scheme:

\[ \{f, g\}_D = l \rightarrow [\hat{f}, \hat{g}] = i\hbar \hat{l} . \] (5.1)

Here \( f, g \) and \( l \) are phase-space functions, \( \hat{f}, \hat{g} \) and \( \hat{l} \) are operators. The complex conjugate \( f^* \) of a function \( f \) becomes the adjoint operator \( \hat{f}^\dagger \). When we apply these prescriptions to the Dirac brackets (3.8)-(3.12), we get:

\[ [\hat{\psi}(r), \hat{\psi}^\dagger(r')] = \delta(r - r') , \quad [\hat{\psi}(r), \hat{\psi}(r')] = [\hat{\psi}^\dagger(r), \hat{\psi}^\dagger(r')] = 0 , \] (5.2)

\[ [\hat{\pi}(r), \hat{\pi}^\dagger(r')] = -\frac{\hbar^2}{4} \delta(r - r') , \quad [\hat{\pi}(r), \hat{\pi}(r')] = \{\hat{\pi}^\dagger(r), \hat{\pi}^\dagger(r')\} = 0 , \] (5.3)

\[ [\hat{\psi}(r), \hat{\pi}(r')] = [\hat{\psi}^\dagger(r), \hat{\pi}(r')] = \frac{i\hbar}{2} \delta(r - r') , \quad [\hat{\psi}(r), \hat{\pi}(r')] = [\hat{\psi}^\dagger(r), \hat{\pi}(r')] = 0 . \] (5.4)

The second class constraints (3.1) of the theory become operator identities:

\[ \hat{\pi} = \frac{i\hbar}{2} \hat{\psi}^\dagger , \quad \hat{\pi}^\dagger = -\frac{i\hbar}{2} \hat{\psi} . \] (5.5)

By inserting Eqs. (5.5) in Eqs. (5.3) and (5.4) we recover again Eqs. (5.2).

The commutators (5.2) between \( \hat{\psi} \) and \( \hat{\psi}^\dagger \) represent the starting point in the second quantization, as indicated by Henley and Thirring [7] and described in detail by Schiff [8]. While Eqs. (5.2) have emerged in a natural way from the Dirac bracket quantization, they had to be imposed ”by hand” in the previous approaches, a feature already criticized by Tassie [17]. Schiff arrives to Eqs. (5.2) by imposing the canonical commutation rules:

\[ [\hat{\psi}, \hat{\pi}_S] = [\hat{\psi}^\dagger, \hat{\pi}^\dagger_S] = i\hbar \delta(r - r') , \] (5.6)

however his treatment also requires \( \hat{\pi}^\dagger_S = 0 \) as can be seen from Eq. (2.6), which is in obvious contradiction with the second commutator (5.6). Meanwhile, the treatment of Henley and Thirring starts by postulating Eqs. (5.4), in other words with the surprising statement that what was canonically conjugate in the classical theory in not any more canonically conjugate in the quantum theory (see the extra factor of 1/2). They impose the commutators (5.4) motivated by the analogy with the variational problem of the harmonic oscillator, written in complex coordinates. Tassie proposes a solution to these conceptual problems. By working on the momentum space, he essentially eliminates the imaginary part of \( \psi \) and gives a description in terms of real fields without encountering the above-mentioned inconvenience.

No problems appear in the approaches employing real fields. This is because any description in terms of real fields [9,10,17] essentially means that we have eliminated the redundant variables, thus we are quantizing on the reduced phase-space.

No difficulties appear in the Dirac bracket quantization either. The Dirac bracket (3.11) of the variables \( \psi \) and \( \pi \), canonically conjugate at the classical level, is one-half their Poisson bracket, thus no reason to wonder why the
corresponding commutator contains the factor of $1/2$. Consistent canonical quantization of the complex Schrödinger field requires the Dirac bracket.

Stated in other way, if we start from the canonical chart (3.13), compute the Dirac bracket cf. Eq. (3.14) and apply the prescription (5.1) for the variables spanning the reduced phase space, we find:

$$\{\psi_1(r), \pi_1(r')\}_D = \delta(r - r') \quad \rightarrow \quad [\hat{\psi}_1(r), \hat{\pi}_1(r')] = i\hbar\delta(r - r') .$$

But modulo the constraints this is consistent with Eq. (5.4). Now, in contrast with the treatment of Schiff, we can impose $\hat{\psi}_2 = \hat{\pi}_2 = 0$, because in the framework of the constrained systems these canonical variables are second class constraints and they should not be turned into canonically conjugate operators [11].

VI. CONCLUDING REMARKS

We have reviewed how the Schrödinger equation can be found from various Hamiltonians, representing different stages of reduction. Hopefully our treatment shed light on the many Hamiltonian formulations of the Schrödinger system and their multiple interconnections. We have seen how the construction of these Hamiltonians requires some artwork, like the addition of appropriately chosen time derivative terms to the Lagrangian.

Alternatively the Schrödinger field appears as a computationally simple example for constrained systems. By employing the characteristic toolchest, we have found the Hamiltonian and the Schrödinger equation via the Dirac-Bergmann algorithm and the consistency requirement, respectively. We have achieved the reduction to the physical degrees of freedom by suitable canonical transformations. The canonical equation in the reduced phase space is again the Schrödinger equation.

We have shown how second quantization of the complex Schrödinger field by turning the Dirac bracket (or the equivalent fundamental bracket in the Faddeev-Jackiw approach) into commutators on the one side avoids interpretational difficulties, on the other side leads to the same quantum theory, which emerges from quantization on the reduced phase space.

The Schrödinger equation completes the list of famous equations of modern physics, like Maxwell and Einstein equations, not covered by the "usual" variational treatments. The exceptions turn out to be rather generic. However, there is a major difference: the Schrödinger constraints are second class as opposed to the first class constraints of electrodynamics and general relativity. Following the equivalent Faddeev-Jackiw approach, there are no constraints at all. This latter approach with uncontested simplicity yields the correct fundamental bracket for the Schrödinger system, but the role of the previously found Hamiltonians is revealed only by the Dirac method.

VII. ACKNOWLEDGMENTS

The author is grateful to Karel Kuchař for his constructive criticism on an early version of this paper, to Mihály Benedict for bringing into his attention Cohen-Tannoudji’s approach and to János Polonyi for his encouragement to pursue the topic. This work has been completed under the support of the Zoltán Magyary Fellowship.

[1] E. Schrödinger, Ann. Phys. 79, 361-376 (1926).
[2] P. A. M. Dirac, Phys. Zeits. Sowjetunion 3, 64-72 (1933).
[3] R. P. Feynman, Science 153, 699-706 (1966).
[4] R. P. Feynman, Rev. Mod. Phys. 20, 367-387 (1948).
[5] C. Tzanakis, Eur. J. Phys. 19 69-75 (1998).
[6] D. Derbes, Am. J. Phys. 64, 881-884 (1996).
[7] E. M. Henley, W. Thirring, Elementary Quantum Field Theory (McGraw-Hill Book Company, 1962), pp. 32-33.
[8] L. I. Schiff, Quantum Mechanics (McGraw-Hill Book Company, 1988), pp. 498-504.
[9] K. V. Kuchař, Selected Topics in Quantum Mechanics (Lecture notes, Univ. of Utah, Spring Quarter 1978)
[10] C. Cohen-Tannoudji, J. Dupont-Roc, G.Grynberg, Photons and Atoms - Introduction to Quantum Electrodynamics (John Wiley & Sons, 1989), pp. 154-168.
[11] P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, NJ, 1964), pp. 5-43.
[12] E. C. G. Sudarshan, N. Mukunda, *Classical Dynamics: A Modern Perspective* (John Wiley and Sons, 1974), pp. 78-137.
[13] K. Sundermayer, *Constrained Dynamics* (Springer Verlag, 1982), pp. 38-109.
[14] L. Faddeev, R. Jackiw, Phys. Rev. Lett. **60**, 1692-1694 (1988).
[15] R. Jackiw, in *Constraint Theory and Quantization Methods*, eds. F. Colomo, L. Lusanna and G. Marmo (World Scientific 1994), pp. 163-175.
[16] P. A. M. Dirac, Can. J. Math. **2**, 129-148 (1950).
[17] L. J. Tassie, Am. J. Phys. **32**, 609-611 (1964).