ORDER SPECTRUM OF THE CESÀRO OPERATOR IN BANACH LATTICE SEQUENCE SPACES

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Abstract. The discrete Cesàro operator \( C \) acts continuously in various classical Banach sequence spaces within \( C^0 \). For the coordinatewise order, many such sequence spaces \( X \) are also complex Banach lattices (e.g., \( c_0, \ell^p \) for \( 1 < p \leq \infty \), and \( \text{ces}(p) \) for \( p \in \{0\} \cup (1, \infty) \)). In such Banach lattice sequence spaces, \( C \) is always a positive operator. Hence, its order spectrum is well defined within the Banach algebra of all regular operators on \( X \). The purpose of this note is to show, for every \( X \) belonging to the above list of Banach lattice sequence spaces, that the order spectrum \( \sigma_o(C) \) of \( C \) coincides with its usual spectrum \( \sigma(C) \) when \( C \) is considered as a continuous linear operator on the Banach space \( X \).

1. Introduction

Let \( E \) be a complex Banach lattice and \( \mathcal{L}(E) \) denote the unital Banach algebra of all continuous linear operators from \( E \) into itself, equipped with the operator norm \( \| \cdot \|_{\text{op}} \). The unit is the identity operator \( I : E \to E \). Associated with each \( T \in \mathcal{L}(E) \) is its spectrum

\[ \sigma(T) := \{ \lambda \in \mathbb{C} : (\lambda I - T) \text{ is not invertible in } \mathcal{L}(E) \} \]

and its resolvent set \( \rho(T) := \mathbb{C} \setminus \sigma(T) \). An operator \( T \in \mathcal{L}(E) \) is called regular if it is a finite linear combination of positive operators. The complex vector space of all regular operators is denoted by \( \mathcal{L}_r^r(E) \); it is also a unital Banach algebra for the norm

\[ \| T \|_r := \inf \{ \| S \|_{\text{op}} : S \in \mathcal{L}(E), S \geq 0, |T(z)| \leq S(|z|) \quad \forall z \in E \}, \quad T \in \mathcal{L}_r^r(E). \]

Again \( I : E \to E \) is the unit. Moreover, \( \| T \|_{\text{op}} \leq \| T \|_r \) for \( T \in \mathcal{L}_r^r(E) \), with equality whenever \( T \geq 0 \) (i.e., if \( T \) is a positive operator). The spectrum of \( T \in \mathcal{L}_r^r(E) \), considered as an element of the Banach algebra \( \mathcal{L}_r^r(E) \), is denoted by \( \sigma_o(T) \) and is called its order spectrum. Then \( \rho_o(T) := \mathbb{C} \setminus \sigma_o(T) \) is the order resolvent of \( T \) Clearly

\[ \sigma(T) \subseteq \sigma_o(T), \quad T \in \mathcal{L}_r^r(E). \]

From the usual formula for the spectral radius, [?, Ch.1, §2, Proposition 8], it follows that the spectral radii for \( T \in \mathcal{L}_r^r(E) \) satisfy \( r(T) = r_o(T) \) whenever \( T \geq 0 \). Standard references for the above concepts and facts are [1], [2], [3], [4], for example.

It is clear from (??) that \( r(T) \leq r_o(T) \) for \( T \in \mathcal{L}_r^r(E) \). So, if \( r(T) < r_o(T) \), then (??) cannot be an equality. This is the strategy applied in [?, pp.79-80] to exhibit a regular operator for which \( \sigma(T) \nsubseteq \sigma_o(T) \). For an example of a positive operator \( T \) satisfying \( \sigma(T) \nsubseteq \sigma_o(T) \), see [?, pp.283-284]. In the contrary direction, a rich supply of classical operators \( T \) for which the equality

\[ \sigma(T) = \sigma_o(T) \]

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is satisfied arise in harmonic analysis, [?, Theorem 3.4].

The aim of this note is to contribute two further classes of operators \( T \) which satisfy (??). In Section ?? it is shown that in any Banach function space \( E \), all multiplication operators \( T \) by \( L^\infty \)-functions are regular operators and satisfy (??). This is a consequence of the fact that the algebra of such multiplication operators is maximal commutative. Let \( \mathbb{N} := \{1, 2, \ldots\} \). The remaining three sections deal with the classical Cesàro operator \( C : \mathbb{C}^N \rightarrow \mathbb{C}^N \) defined by

\[
C(x) := \left( \frac{1}{n} \sum_{k=1}^{n} x_k \right)_{n=1}^\infty \quad x = (x_n)_{n=1}^\infty \in \mathbb{C}^N,
\]

which is clearly a positive operator for the coordinatewise order in the positive cone of \( \mathbb{C}^N = \mathbb{R}^N \oplus i\mathbb{R}^N \). Section ?? establishes some general results for determining the regularity of linear operators in Banach lattice sequence spaces. These results are designed to apply to the particular operators \( (C - \lambda I)^{-1} \), where \( C \) is given in (??). In Section ?? we will consider the restriction of \( C \) to the Banach lattice sequence spaces \( c_0 \) and \( \ell^p, 1 < p \leq \infty \), and show that (??) is satisfied in all cases (with \( C \) in place of \( T \)). Section ?? is devoted to proving the same fact, but now when \( C \) acts in the discrete Cesàro spaces \( \text{ces}(p), 1 < p < \infty \), and in \( \text{ces}(0) \).

2. Multiplication operators

Let \((\Omega, \Sigma, \mu)\) be a localizable measure space (in the sense of [?, 64A]), that is, the associated measure algebra is a complete Boolean algebra and, for every measurable set \( A \in \Sigma \) with \( \mu(A) > 0 \) there exists \( B \in \Sigma \) such that \( B \subseteq A \) and \( 0 < \mu(B) < \infty \) (i.e., \( \mu \) has the finite subset property). All \( \sigma \)-finite measures are localizable, [?, 64H Proposition]. Every Banach function space \( E \) (of \( \mathbb{C} \)-valued functions) over \((\Omega, \Sigma, \mu)\) is a complex Banach lattice for the pointwise \( \mu \)-a.e. order. Given any \( \varphi \in L^\infty(\mu) \), the multiplication operator \( M_\varphi : E \rightarrow E \) defined by \( f \mapsto \varphi f \), for \( f \in E \), belongs to \( \mathcal{L}(E) \) and satisfies \( \|M_\varphi\|_{\text{op}} = \|\varphi\|_{\infty} \). Define a unital, commutative subalgebra of \( \mathcal{L}(E) \) by

\[
\mathcal{M}_E(L^\infty(\mu)) := \{ M_\varphi : \varphi \in L^\infty(\mu) \};
\]

the unit is the identity operator \( I = M_1 \) where \( 1 \) is the constant function \( 1 \) on \( \Omega \). Recall that the commutant of \( \mathcal{M}_E(L^\infty(\mu)) \) is defined by

\[
\mathcal{M}_E(L^\infty(\mu))^c := \{ A \in \mathcal{L}(E) : AM_\varphi = M_\varphi A \ \forall \varphi \in L^\infty(\mu) \} \subseteq \mathcal{L}(E).
\]

It is known that \( \mathcal{M}_E(L^\infty(\mu)) \) is a maximal commutative, unital subalgebra of \( \mathcal{L}(E) \), that is, \( \mathcal{M}_E(L^\infty(\mu)) = \mathcal{M}_E(L^\infty(\mu))^c \), [?, Proposition 2.2]. Moreover, also the bicommutant \( \mathcal{M}_E(L^\infty(\mu))^{cc} = \mathcal{M}_E(L^\infty(\mu)) \).

Proposition 2.1. Let \((\Omega, \Sigma, \mu)\) be a localizable measure space and \( E \) be a Banach function space over \((\Omega, \Sigma, \mu)\).

(i) \( \mathcal{M}_E(L^\infty(\mu)) \subseteq \mathcal{L}^*(E) \).

(ii) \( \mathcal{M}_E(L^\infty(\mu)) \) is inverse closed in \( \mathcal{L}(E) \). That is, if \( T \in \mathcal{M}_E(L^\infty(\mu)) \) is invertible in \( \mathcal{L}(E) \) (i.e., there exists \( S \in \mathcal{L}(E) \) satisfying \( ST = I = TS) \), then necessarily \( S \in \mathcal{M}_E(L^\infty(\mu)) \).

(iii) For every \( T \in \mathcal{M}_E(L^\infty(\mu)) \) we have \( \sigma_e(T) = \sigma(T) \).

Proof. (i) Let \( \varphi \in L^\infty(\mu) \). Then \( \varphi = [(\text{Re} \varphi)^+ - (\text{Re} \varphi)^-] + i[(\text{Im} \varphi)^+ - (\text{Im} \varphi)^-] \) with all four functions \( (\text{Re} \varphi)^+, (\text{Re} \varphi)^-, (\text{Im} \varphi)^+, (\text{Im} \varphi)^- \) belonging to the positive cone \( L^\infty(\mu)^+ \) of \( L^\infty(\mu) \). Since \( M_\varphi = [M_{(\text{Re} \varphi)} + M_{(\text{Re} \varphi)^-}] + i[M_{(\text{Im} \varphi)} + M_{(\text{Im} \varphi)^-}] \) is a linear combination of positive operators, it is clear that \( M_\varphi \in \mathcal{L}^*(E) \).
(ii) Since $\mathcal{M}_E(L^\infty(\mu))$ is maximal commutative in $\mathcal{L}(E)$, it follows that $\mathcal{M}_E(L^\infty(\mu))$ is inverse closed in $\mathcal{L}(E)$, [2, Ch.II, §15, Theorem 4].

(iii) In view of (??) it suffices to show that $\rho(T) \subseteq \rho_0(T)$. Suppose that $T = M_\varphi$ with $\varphi \in L^\infty(\mu)$. Fix $x \in \rho(T)$. Then $\lambda I - T = M_{(\lambda - \varphi)}$ belongs to $\mathcal{M}_E(L^\infty(\mu))$ because $(\lambda I - \varphi) \in L^\infty(\mu)$. Since $M_{(\lambda - \varphi)}$ is invertible in $\mathcal{L}(E)$, it follows from part (ii) that actually $(\lambda I - T)^{-1} \in \mathcal{M}_E(L^\infty(\mu))$ and hence, by part (i), that also $(\lambda I - T)^{-1} \in \mathcal{L}'(E)$.

\textbf{Remark 2.2.} We point out that $\|T\|_{op} = \|T\|_r$ for each $T \in \mathcal{M}_E(L^\infty(\mu))$. Indeed, let $\varphi \in L^\infty(\mu)$ satisfy $T = M_\varphi$, in which case $\|M_\varphi\|_{op} = \|\varphi\|_\infty$. Define $S := \|\varphi\|_\infty I$ and note that $S \geq 0$ with $\|S\|_{op} = \|\varphi\|_\infty$. Moreover,

$$|M_\varphi(f)| = |\varphi f| \leq \|\varphi\|_\infty |f| = S(|f|), \quad f \in E,$$

and so $\|T\|_r \leq \|S\|_{op} = \|\varphi\|_\infty = \|T\|_{op}$; see (??). The reverse inequality $\|T\|_{op} \leq \|T\|_r$ always holds.

3. The Cesàro operator in Banach sequence spaces

We begin with some preliminaries. Equipped with the topology of pointwise convergence $\mathbb{C}^\mathbb{N}$ is a locally convex Fréchet space. Let $A = (a_{nm})_{n,m=1}^{\infty}$ be any lower triangular (infinite) matrix, i.e., $a_{nm} = 0$ whenever $m > n$. Then $A$ induces the continuous linear operator $T_A : \mathbb{C}^\mathbb{N} \rightarrow \mathbb{C}^\mathbb{N}$ defined by

$$T_A(x) := \sum_{n=1}^\infty a_{nm}x_m, \quad x \in \mathbb{C}^\mathbb{N}. \quad (3.1)$$

For $x \in \mathbb{C}^\mathbb{N}$ define $|x| := (|x_n|)^{\infty}_{n=1}$. Then also $|x| \in \mathbb{C}^\mathbb{N}$. A vector subspace $X \subseteq \mathbb{C}^\mathbb{N}$ is called solid (or an ideal) if $y \in X$ whenever $x \in X$ and $y \in \mathbb{C}^\mathbb{N}$ satisfy $|y| \leq |x|$. It is always assumed that $X$ contains the vector space consisting of all elements of $\mathbb{C}^\mathbb{N}$ which have only finitely many non-zero coordinates. In addition, it is assumed that $X$ has a norm $\| \cdot \|_X$ with respect to which it is a complex Banach lattice for the coordinatewise order and such that the natural inclusion $X \subseteq \mathbb{C}^\mathbb{N}$ is continuous. Under the previous requirements $X$ is called a Banach lattice sequence space.

\textbf{Lemma 3.1.} Let $A = (a_{nm})_{n,m=1}^{\infty}$ be a lower triangular matrix with all entries non-negative real numbers and $X \subseteq \mathbb{C}^\mathbb{N}$ be a Banach lattice sequence space such that $T_A(X) \subseteq X$. Let $B = (b_{nm})_{n,m=1}^{\infty}$ be any matrix such that

$$|b_{nm}| \leq a_{nm}, \quad n, m \in \mathbb{N}. \quad (3.2)$$

Then the restricted operator $T_A : X \rightarrow X$ belongs to $\mathcal{L}(X)$. Moreover, $T_B : \mathbb{C}^\mathbb{N} \rightarrow \mathbb{C}^\mathbb{N}$ satisfies $T_B(X) \subseteq X$ and the restricted operator $T_B : X \rightarrow X$ also belongs to $\mathcal{L}(X)$. In addition, $\|T_B\|_{op} \leq \|T_A\|_{op}$.

\textbf{Proof.} Condition (??) implies that $B$ is also a lower triangular matrix. Moreover, the continuity of both $T_A : \mathbb{C}^\mathbb{N} \rightarrow \mathbb{C}^\mathbb{N}$ and of the inclusion map $X \subseteq \mathbb{C}^\mathbb{N}$ imply, via the Closed Graph Theorem in the Banach space $X$, that the restricted operator $T_A \in \mathcal{L}(X)$.

Given $x \in X$ we have for each $n \in \mathbb{N}$, via (??), that

$$(T_B(x))_n = \sum_{m=1}^\infty b_{nm}x_m \leq \sum_{m=1}^\infty |b_{nm}| \cdot |x_m| \leq \sum_{m=1}^\infty a_{nm}|x_m| = (T_A(|x|))_n.$$

Since $X$ is solid and $T_A(|x|) \in X$, these inequalities and (??) imply that $T_B(x) \in X$. Moreover, as $\| \cdot \|_X$ is a lattice norm it follows that

$$\|T_B(x)\|_X = \|(\sum_{m=1}^\infty b_{nm}x_m)_{n=1}^\infty\|_X \leq \|(\sum_{m=1}^\infty a_{nm}|x_m|)_{n=1}^\infty\|_X = \|T_A(|x|)\|_X \leq \|T_A\|_{op}\|x\|_X,$$
for each $x \in X$, where the stated series are actually finite sums. Hence, $\|T_B\|_{op} \leq \|T_A\|_{op}$ and the proof is complete. 

Since the operator $T_A$ as given in Lemma 2.2 satisfies $T_A \geq 0$, it is clearly regular.

**Corollary 3.2.** Let $A = (a_{nm})_{n,m=1}^\infty$ be a lower triangular matrix with non-negative real entries and $X \subseteq \mathbb{C}^N$ be a Banach lattice sequence space such that $T_A(X) \subseteq X$. Let $B = (b_{nm})_{n,m=1}^\infty$ be any matrix satisfying (\(2\)). Then the operator $T_B \in \mathcal{L}(X)$ is necessarily regular, that is, $T_B \in \mathcal{L}'(X)$.

**Proof.** Define the non-negative real numbers $s_{nm} := (\text{Re} \ b_{nm})^+, u_{nm} := (\text{Re} \ b_{nm})^-$, $v_{nm} := (\text{Im} \ b_{nm})^+$ and $w_{nm} := (\text{Im} \ b_{nm})^-$ for each $n, m \in \mathbb{N}$. Then $b_{nm} = (s_{nm} - u_{nm}) + i(v_{nm} - w_{nm})$ and $\{s_{nm}, u_{nm}, v_{nm}, w_{nm}\} \subseteq [0, a_{nm}]$ for $n, m \in \mathbb{N}$. Setting $S := (s_{nm})_{n,m=1}^\infty$, $V := (v_{nm})_{n,m=1}^\infty$ and $W := (w_{nm})_{n,m=1}^\infty$ it is clear from the definition (\(2\)) that each operator $T_S \geq 0, T_U \geq 0, T_V \geq 0$ and $T_W \geq 0$ (in $X$) belongs to $\mathcal{L}(X)$; see Lemma 2.2. Since $T_B = (T_S - T_V) + iT_V - iT_W$, it follows that $T_B \in \mathcal{L}'(X)$. 

Together with appropriate estimates, Corollary 2.2 will be the main ingredient required to establish (\(?\)) for $C$ (in place of $T$) when it acts in various classical Banach lattice sequence spaces $X$.

Let $\Sigma_0 := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. We recall the formula for the inverses $(C - \lambda I)^{-1} : \mathbb{C}^N \rightarrow \mathbb{C}^N$ whenever $\lambda \in \mathbb{C} \setminus \Sigma_0$, \cite[p.266]{?}. Namely, for $n \in \mathbb{N}$ the $n$-th row of the lower triangular matrix determining $(C - \lambda I)^{-1}$ has the entries

$$
\frac{1}{n \lambda^{1/k_{n+1}}(1 - \lambda^{1/k_n})}, \quad 1 \leq m < n, \quad \text{and} \quad \frac{n}{1 - \lambda} = \frac{1}{(\frac{1}{n} - \lambda)}, \quad m = n,
$$

with all other entries in row $n$ being 0. We write

$$
(C - \lambda I)^{-1} = T_{D_{\lambda}} - \frac{1}{\lambda^2} T_{E_{\lambda}},
$$

where the diagonal matrix $D_{\lambda} = (d_{nm}(\lambda))_{n,m=1}^\infty$ is given by

$$
d_{nm}(\lambda) := \left(\frac{1}{\lambda^{1/k_n}}\right) \quad \text{and} \quad d_{nm}(\lambda) := 0 \quad \text{if} \quad n \neq m.
$$

Setting $\gamma[\lambda] := \text{dist}(\lambda, \Sigma_0) > 0$ it is routine to check that

$$
|d_{nm}(\lambda)| \leq \frac{1}{\gamma[\lambda]}, \quad n \in \mathbb{N}, \quad \lambda \in \mathbb{C} \setminus \Sigma_0.
$$

Moreover, $E_{\lambda} = (e_{nm}(\lambda))_{n,m=1}^\infty$ is the lower triangular matrix given by $e_{1m}(\lambda) = 0$, for $m \in \mathbb{N}$, and for all $n \geq 2$ by

$$
e_{nm}(\lambda) := \begin{cases} 
\frac{1}{n \lambda^{1/k_n}(1 - \lambda^{1/k_n})} & \text{if} \quad 1 \leq m < n \\
0 & \text{if} \quad m \geq n.
\end{cases}
$$

**Lemma 3.3.** Let $X \subseteq \mathbb{C}^N$ be any Banach lattice sequence space. For each $\lambda \in \mathbb{C} \setminus \Sigma_0$ the diagonal operator $T_{D_{\lambda}}$, with $D_{\lambda} = (d_{nm}(\lambda))_{n,m=1}^\infty$ given by (\(2\)), is regular in $X$, that is, $T_{D_{\lambda}} \in \mathcal{L}'(X)$.

**Proof.** Fix $\lambda \not\in \Sigma_0$ and let $A := \frac{1}{\gamma[\lambda]} I$, where $I$ is the identity matrix in $\mathbb{C}^N$, in which case $T_A(X) \subseteq X$ is clear. It follows from (\(2\)) that the matrix $B := D_{\lambda}$ satisfies (\(2\)). Hence, the regularity of $T_{D_{\lambda}}$ in $X$ follows from Corollary 2.2.

**Remark 3.4.** (i) Since any Banach lattice sequence space $X \subseteq \mathbb{C}^N$ is a Banach function space over the $\sigma$-finite measure space $(\mathbb{N}, 2^\mathbb{N}, \mu)$, relative to counting measure $\mu$, and the function $n \mapsto d_{nm}(\lambda)$ on $\mathbb{N}$ belongs to $L^\infty(\mu)$ by (\(2\)), the regularity of $T_{D_{\lambda}} \in \mathcal{L}(X)$ also follows from Proposition 2.2(i).
(ii) For appropriate \( X \) and \( \lambda \notin \Sigma_0 \), it is clear from (??) and Lemma ?? that the regularity of \( (C - \lambda I)^{-1} \in \mathcal{L}(X) \) is completely determined by the matrix \( E_{\lambda} \).

The following inequalities will be needed in the sequel. For \( \alpha < 1 \) we refer to [?; Lemma 7] and for general \( \alpha \in \mathbb{R} \) to [?, Lemma 3.2(i)].

**Lemma 3.5.** Let \( \lambda \in \mathbb{C} \setminus \Sigma_0 \) and set \( \alpha := \text{Re}(\frac{1}{\lambda}) \). Then there exist positive constants \( P(\alpha) \) and \( Q(\alpha) \) such that

\[
P(\alpha) \leq \prod_{k=1}^{n} |1 - \frac{1}{k\lambda}| \leq Q(\alpha), \quad n \in \mathbb{N}.
\]

4. The Classical Spaces \( \ell^p, 1 < p \leq \infty \), and \( c_0 \)

For each \( 1 < p \leq \infty \) let \( C_p \in \mathcal{L}(\ell^p) \) denote the Cesaro operator as given by (??) when it is restricted to \( \ell^p \). As a consequence of Hardy’s inequality, [?, Theorem 326], it is known that \( \| C_p \|_{\ell^p} = p' \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \) (with \( p' := 1 \) when \( p = \infty \)). Concerning the spectrum of \( C_p \) we have

\[
\sigma(C_p) = \{ \lambda \in \mathbb{C} : |\lambda - \frac{p'}{2^p} | \leq \frac{p'}{2} \}, \quad 1 < p \leq \infty.
\]

Various proofs of (??) are known for \( 1 < p < \infty \), [?, ?, ?, ?, ?, ?]; see the discussion on p.268 of [?]. For the case \( p = \infty \) we refer to (?, Theorem 4), for example.

**Remark 4.1.** For each \( \lambda \neq 0 \) set \( \alpha := \text{Re}(\frac{1}{\lambda}) \). Then, for any \( b > 0 \) we have

\[
\alpha < \frac{1}{b} \quad \text{and only if} \quad |\lambda - \frac{b}{2} | > \frac{b}{2}.
\]

The corresponding results for \( \alpha > \frac{1}{b} \) and \( \alpha = \frac{1}{b} \) also hold.

**Proposition 4.2.** For each \( 1 < p < \infty \) the order spectrum of the positive operator \( C_p \in \mathcal{L}(\ell^p) \) satisfies

\[
\sigma_o(C_p) = \sigma(C_p).
\]

**Proof.** Via (??) it suffices to verify that \( \rho(C_p) \subseteq \rho_o(C_p) \).

With the notation of (??) and (??) it is shown on p.269 of [?], as a consequence of (??) in Lemma ?? above, that for every \( \lambda \neq 0 \) satisfying \( \alpha := \text{Re}(\frac{1}{\lambda}) < 1 \) there exists a constant \( \beta(\lambda) > 0 \) such that

\[
|e_{nm}(\lambda)| \leq \frac{\beta(\lambda)}{n^m \cdot m^n}, \quad 1 \leq m \leq n, \quad n \in \mathbb{N}.
\]

Set \( B := E_{\lambda} \) and let \( A \) be the lower triangular matrix whose entries \( a_{nm}(\lambda) \geq 0 \) are given by the right-side of (??) for each \( n \in \mathbb{N} \) and \( 1 \leq m \leq n \) (and 0 otherwise). According to (??) the matrices \( A \) and \( B \) satisfy (??). Let \( X := \ell^p \) for \( p \in (1, \infty) \) fixed. Then Corollary ?? implies that \( E_{\lambda} \) will be regular (i.e., \( T_{E_{\lambda}} \in \mathcal{L}(\ell^p) \)) whenever \( T_A(\ell^p) \subseteq \ell^p \). Note that \( T_A \in \mathcal{L}(\mathbb{C}^N) \) is given by

\[
x \mapsto \beta(\lambda) \left( \frac{1}{n^m \cdot m^n} \sum_{m=1}^{n} e_{nm}(\lambda) \right)_{n=1}^{\infty} := \beta(\lambda) G_{\lambda}(x), \quad x \in \mathbb{C}^N.
\]

So, if \( \text{Re}(\frac{1}{\lambda}) < 1 \), then (??) implies that \( T_A \in \mathcal{L}(\ell^p) \) whenever \( G_{\lambda} : \ell^p \rightarrow \ell^p \) is continuous.

Let now \( \lambda \in \rho(C_p) \), that is, \( |\lambda - \frac{p'}{2^p} | > \frac{p}{2} \). Then \( \alpha := \text{Re}(\frac{1}{\lambda}) < \frac{1}{b} \), because of Remark ??, and hence, \( (1 - \alpha)p > 1 \). Then the Proposition on p.269 of [?] yields that indeed \( G_{\lambda} \in \mathcal{L}(\ell^p) \). As noted above, this implies that \( T_{E_{\lambda}} \in \mathcal{L}(\ell^p) \). Combined with (??) and Lemma ?? it follows that \( \langle C_p - \lambda I \rangle^{-1} \in \mathcal{L}(\ell^p) \), that is, \( \lambda \in \rho_o(C_p) \). This completes the proof of (??). \qed
Recall that $\|C_\infty\|_{op} = 1$ and, from (??) for $p = \infty$, that
\begin{equation}
\sigma(C_\infty) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}
\end{equation}

**Proposition 4.3.** The order spectrum of the positive operator $C_\infty \in \mathcal{L}(\ell^\infty)$ satisfies
\begin{equation}
\sigma_o(C_\infty) = \sigma(C_\infty).
\end{equation}

**Proof.** Again by (??) it suffices to prove that $\rho(C_\infty) \subseteq \rho_o(C_\infty)$.

Fix $\lambda \in \rho(C_\infty)$. According to (??), for $b = 1$ the condition in Remark ?? is satisfied with $\alpha := \text{Re}\left(\frac{1}{\lambda}\right)$. Hence, the inequalities (??) are valid and so $A := (a_{nm}(\lambda))_{n,m=1}^\infty \geq 0$ and $B := E_\lambda$ can again be defined exactly as in the proof of Proposition ?? Then (??) is satisfied with $X := \ell^\infty$. Arguing as in the proof of Proposition ?? (via Corollary ??) it remains to verify that $T_A : \ell^\infty \to \ell^\infty$ is continuous, where $T_A$ is given by (??). To this effect, since $(1 - \alpha) > 0$ by Remark ??, it follows that
\begin{equation}
\sup_{n \in \mathbb{N}} \sum_{m=1}^\infty |a_{nm}(\lambda)| = \beta(\lambda) \sup_{n \in \mathbb{N}} \frac{1}{m^n} \sum_{m=1}^\infty \frac{1}{m^n} < \infty;
\end{equation}

this has been verified on p.778 of [7] (put $w(n) = 1$ there for all $n \in \mathbb{N}$) by considering each of the cases $\alpha < 0$, $\alpha = 0$ and $0 < \alpha < 1$ separately. But, condition (??) is known to imply that $T_A \in \mathcal{L}(\ell^\infty)$, [7, Ex.2, p.220]. The proof that $\lambda \in \rho_o(C_\infty)$ is thereby complete. \(\square\)

To conclude this section we consider the Cesàro operator $C$, as given by (??), when it is restricted to $c_0$; denote this operator by $C_0$. It is shown in [7, Theorem 3], [7], that $\|C_0\|_{op} = 1$ and
\begin{equation}
\sigma(C_0) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}.
\end{equation}

**Proposition 4.4.** The order spectrum of the positive operator $C_0 \in \mathcal{L}(c_0)$ satisfies
\begin{equation}
\sigma_o(C_0) = \sigma(C_0).
\end{equation}

**Proof.** Since (??) shows that $\sigma(C_0) = \sigma(C_\infty)$, the entire proof of Proposition ?? can be easily adapted (now for $X := c_0$ and fixed $\lambda \in \rho(C_0)$, using the same notation, up to the stage where (??) is shown to be valid. In addition to the validity of (??) it is also true that
\begin{equation}
\lim_{n \to \infty} a_{nm}(\lambda) = \frac{\beta(\lambda)}{m^n} \lim_{n \to \infty} \frac{1}{m^{1-n}} = 0, \quad m \in \mathbb{N},
\end{equation}

because $\alpha := \text{Re}\left(\frac{1}{\lambda}\right)$ satisfies $(1 - \alpha) > 0$. The two conditions (??) and (??) together are known to imply that $T_A \in \mathcal{L}(c_0)$, [7, Theorem 4.51-C]. Again via Corollary ?? and Lemma ?? we can conclude that $T_{E_\lambda} \in \mathcal{L}(c_0)$ and hence, also $(C_0 - \lambda I)^{-1}$ is regular on $c_0$. \(\square\)

5. The discrete Cesàro spaces $\text{ces}(p), 1 < p < \infty,$ and $\text{ces}(0)$

For $1 < p < \infty$ the discrete Cesàro spaces are defined by
\begin{equation}
\text{ces}(p) := \{x \in \mathbb{C}^\mathbb{N} : \|x\|_{\text{ces}(p)} := \left(\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |x_k|^p\right)^{1/p}\right) < \infty\}.
\end{equation}

In view of (??) we see that $\|x\|_{\text{ces}(p)} = \|C(|x|)\|_{lp}$ for $x \in \text{ces}(p)$. It is known that each space $\text{ces}(p), 1 < p < \infty$, is a reflexive Banach lattice sequence space for the norm $\|\cdot\|_{\text{ces}(p)}$ and the coordinatewise order. The spaces $\text{ces}(p)$ have been thoroughly treated in [7]. According to Theorem ?? of [7] the restriction of $C$ (see (??)) to $\text{ces}(p)$, denoted here by $C_{(p)}$, is continuous with $\|C_{(p)}\|_{op} = p'$ and
\begin{equation}
\sigma(C_{(p)}) = \left\{\lambda \in \mathbb{C} : |\lambda - \frac{p}{2}| \leq \frac{p'}{2}\right\}, \quad 1 < p < \infty.
\end{equation}
Proposition 5.1. For each $1 < p < \infty$ the order spectrum of the positive operator $C_{(p)} \in \mathcal{L}(\text{ces}(p))$ satisfies

\begin{equation}
\sigma_o(C_{(p)}) = \sigma(C_{(p)}).
\end{equation}

Proof. In view of (??) it suffices to verify that $\rho(C_{(p)}) \subseteq \rho_o(C_{(p)})$.

We decompose the set $\rho(C_{(p)})$ into two disjoint parts, namely the set

\begin{equation}
\rho_1 := \{ \lambda \in \mathbb{C} \setminus \{0\} : \Re\left(\frac{1}{\lambda}\right) \leq 0 \} = \{ u \in \mathbb{C} \setminus \{0\} : \Re(u) \leq 0 \}
\end{equation}

and its complement $\rho_2 := \rho(C_{(p)}) \setminus \rho_1$.

First fix $\lambda \in \rho_1$. Then $\lambda \not\in \Sigma_0$ and so we may consider $E_{\lambda} = (e_{nm}(\lambda))_{n,m=1}^{\infty}$ and $D_{\lambda} = (d_{nm}(\lambda))_{n,m=1}^{\infty}$ as specified by (??) and (??), respectively. It is shown on p.72 of [?] that

\begin{equation}
|e_{nm}(\lambda)| \leq \frac{1}{n}, \quad 1 \leq m < n, \quad n \in \mathbb{N}.
\end{equation}

Warning: In [?] the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ is used rather than $\mathbb{N} = \{1, 2, 3, \ldots\}$ which is used here and so the inequalities from [?] are slightly different when they are stated here. Back to our proof, it is clear from (??) that the matrix $A = (e_{nm})_{n,m=1}^{\infty}$ for the Cesàro operator $C$ is lower triangular with its $n$-th row, for each $n \in \mathbb{N}$, given by $e_{nm} := \frac{1}{n}$ for $1 \leq m \leq n$ and $e_{nm} := 0$ for $m > n$. Setting $B := E_{\lambda}$ it is clear from (??) that (??) is satisfied for the pair $A, B$ in the space $X := \text{ces}(p)$. Since $C_{(p)} = T_A : \text{ces}(p) \rightarrow \text{ces}(p)$ is continuous, it follows from Corollary ?? that $T_{E_{\lambda}} \in \mathcal{L}^r(\text{ces}(p))$ and hence, via Lemma ?? and (??), that also $(C_{(p)} - \lambda I)^{-1} \in \mathcal{L}^r(\text{ces}(p))$.

Consider now the set $\rho_2$. From (??) it is routine to establish that a non-zero point $z \in \mathbb{C}$ belongs to $\sigma(C_{(p)})$ if and only if $\Re\left(\frac{1}{\alpha}\right) \geq \frac{1}{p}$. From the case of equality in Remark ??, it follows that $\rho_2 = \bigcup_{0 < \alpha < 1/p} \Gamma_{\alpha}$, where

\begin{equation}
\Gamma_{\alpha} := \{ z \in \mathbb{C} \setminus \{0\} : \Re\left(\frac{1}{\alpha}\right) = \alpha \} = \{ z \in \mathbb{C} \setminus \{0\} : |z - \frac{1}{2\pi}| = \frac{1}{2\pi} \}.
\end{equation}

Fix a point $\lambda \in \rho_2$. Then there exists a unique number $\alpha \in (0, \frac{1}{p})$ such that $\lambda \in \Gamma_{\alpha}$, namely $\alpha := \Re\left(\frac{1}{\lambda}\right)$. In the notation of (??) it is shown on p.72 of [?] that

\begin{equation}
|e_{nm}(\lambda)| \leq e_{nm}\left(\frac{1}{\alpha}\right), \quad n, m \in \mathbb{N}.
\end{equation}

Note that $e_{nm}\left(\frac{1}{\alpha}\right) \geq 0$ for all $n, m \in \mathbb{N}$ follows from (??) as $0 < \alpha < \frac{1}{p}$ implies that $1 - \frac{1}{k(1/\alpha)} = (1 - \frac{2}{k}) > 0$ for $m \leq k \leq n$. Setting $\tilde{A} := E_{1/\alpha}$ and $\tilde{B} := E_{\lambda}$ it is clear from (??) that (??) is satisfied for the pair $\tilde{A}, \tilde{B}$ in place of $A, B$. Moreover, $\frac{1}{\alpha} > p$ implies that $\frac{1}{\alpha} \in \rho(C_{(p)})$, that is, $(C_{(p)} - \frac{1}{\alpha}I)^{-1} \in \mathcal{L}(\text{ces}(p))$. Since $T_{D_{1/\alpha}} \in \mathcal{L}(\text{ces}(p))$ by Lemma ?? (with $\frac{1}{\alpha}$ in place of $\lambda$), the identity $T_{E_{1/\alpha}} = \alpha^2 (T_{D_{1/\alpha}} - (C_{(p)} - \frac{1}{\alpha}I)^{-1})$ shows that $T_{\tilde{A}} \in \mathcal{L}(\text{ces}(p))$. Hence, Corollary ?? can be applied to conclude that $T_{\tilde{B}} = T_{E_{\lambda}} \in \mathcal{L}^r(\text{ces}(p))$. It then follows from (??) and Lemma 3.3 that $(C_{(p)} - \lambda I)^{-1} \in \mathcal{L}^r(\text{ces}(p))$. \hfill \Box

The remaining space to consider is $\text{ces}(0) := \{ x \in \mathbb{C}^\mathbb{N} : C(|x|) \in c_0 \}$ equipped with the norm

$$
\|x\|_{\text{ces}(0)} := \|C(|x|)\|_{c_0} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} |x_k|, \quad x \in \text{ces}(0).
$$
It is a Banach lattice sequence space for the norm \( \| \cdot \|_{\text{ces}(0)} \) and the coordinatewise order. According to [2, Theorem 6.4], the restriction of \( C \) (see (??)) to \( \text{ces}(0) \), denoted here by \( C(0) \), is continuous with \( \| C(0) \|_{\text{op}} = 1 \) and
\[
(5.7) \quad \sigma(C(0)) = \{ \lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2} \}.
\]

**Proposition 5.2.** The order spectrum of the positive operator \( C(0) \in \mathcal{L}(\text{ces}(0)) \) satisfies
\[
\sigma_0(C(0)) = \sigma(C(0)).
\]

**Proof.** As usual it suffices to show that \( \rho(C(0)) \subseteq \rho_0(C(0)) \).

Let the set \( \rho_1 \) be as in (??). For each \( \alpha \in (0, 1) \) let \( \Gamma_\alpha \) be given by (??). Then (??) ensures that we have the disjoint partition \( \rho(C(0)) = \rho_1 \cup \rho_2 \) with \( \rho_2 := \bigcup_{0 < \alpha < 1} \Gamma_\alpha \).

For any given point \( \lambda \in \rho_1 \) the estimates (??) are again valid (see [2, p.72]) and so the argument in the proof of Proposition ?? can be easily adapted (now for \( X := \text{ces}(0) \)) to again show that \( (C(0) - \lambda I)^{-1} \in \mathcal{L}^r(\text{ces}(0)) \).

Fix now \( \lambda \in \rho_2 \). Then there exists a unique \( \alpha \in (0, 1) \) such that \( \lambda \in \Gamma_\alpha \), namely \( \alpha := \text{Re}(\frac{1}{\lambda}) \). Then \( \text{Re}(1 - \frac{\lambda}{\lambda}) = (1 - \frac{\lambda}{\lambda}) \geq 0 \) for \( k \in \mathbb{N} \). Arguing as at the bottom of p.396 in [2], now with \( x \in \text{ces}(0) \) in place of \( a \in \text{ces}(2) \) there, it follows that the \( k \)-th coordinate of \( E_\lambda(x) \) is 0 and, for \( n \geq 2 \), that the \( n \)-th coordinate of \( E_\lambda(x) \) satisfies
\[
|E_\lambda(x)_n| \leq (E_{1/\alpha}(|x|))_n, \quad x \in \text{ces}(0).
\]
Substituting \( x := (\delta_{ij})_{i,j=1}^{\infty} \) into the previous estimates, for each \( r \in \mathbb{N} \), yields (??). Since \( 0 < \alpha < 1 \) implies that \( \frac{1}{\alpha} \in \rho(C(0)) \), the argument can be completed along the lines given in the proof of Proposition ?? to conclude that \( (C(0) - \lambda I)^{-1} \in \mathcal{L}^r(\text{ces}(0)) \). We again warn the reader that \( \mathbb{N} = \{0, 1, 2, \ldots \} \) is used in [2]. □

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