Recursive decoding of Reed-Muller codes *

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Abstract

New soft- and hard decision decoding algorithms are presented for general Reed-Muller codes \( \{m_r\} \) of length \( 2^m \) and distance \( 2^{m-r} \). We use Plotkin \((u, u + v)\) construction and decompose code \( \{m_r\} \) onto subblocks \( u \in \{m_{r-1}\} \) and \( v \in \{m_{r-1}\} \). In decoding, we first try to find a subblock \( v \) from the better protected code and then proceed with the block \( u \). The likelihoods of the received symbols are recalculated in a way similar to belief propagation. Thus, decoding is relegated to the two constituent codes. We repeat this recursion and execute decoding only at the end nodes \( \{j\} \) and \( \{j+1\} \). The overall complexity has low order of \( n \log n \). It is shown that this decoding substantially outperforms other algorithms of polynomial complexity known for RM codes. In particular, for medium and high code rates, the algorithm corrects most error patterns of weight \( d \ln d/2 \).

1 Introduction

The Reed-Muller code \( RM(r, m) \) \([7]\), which will be denoted below \( \{m_r\} \), has parameters

\[
n = 2^m, \quad k = \sum_{i=0}^{r} \binom{m}{i}, \quad d = 2^{m-r}.
\]

To construct this code, consider all polynomials \( z(x_1, \ldots, x_m) \) of degree \( r \) or less taken over \( m \) Boolean variables. Then a codeword \( z \) is an ordered set of all \( 2^m \) values that polynomial \( z \) takes on these variables. It is also well known \([7]\) that RM codes can be designed by repetitive employment of the Plotkin \((u, u + v)\) construction. Here the original block \((u, u + v) \in \{m_r\}\) is represented by two subblocks \( u \in \{m_{r-1}\} \) and \( v \in \{m_{r-1}\} \). Now we can specify general \((u, u + v)\) construction as

\[
\{m_r\} = \{m_{r-1}\} + \{m_{r-1}\}, \quad \{m_{r-1}\} = \binom{m}{r}, \quad \binom{m-1}{r-1}.
\]

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where the same codeword \( u \in \{ m^{-1} \} \) taken on both halves. In turn, we can split \( u \) and \( v \) further and obtain successively all codes \( \{ m \} \). On step \( m \), we can take the repetition code \( \{ m \} \) and full code \( \{ m \} \), while all other codes \( \{ m \} \) are obtained by recursion (1) from the previous step. Thus, any code \( \{ m \} \) can be mapped onto the \((m, r)\)-node of the Pascal triangle. Given any intermediate node \((j, s)\), we move up and left to achieve the node \((j - 1, s - 1)\); or up and right to reach the node \((j - 1, s)\). Note also that we can end our recursion on any level. In the sequel, we terminate our splitting at the nodes \((j, 1)\) (corresponding to biorthogonal codes) or at the nodes \((j, j - 1)\) (corresponding to the single parity check code). This is schematically shown in Fig. 1 for RM codes of the seventh order.

Figure 1: RM codes of length 128 on Pascal Triangle

Now let \( I^m_r \) denote the block of \( k \) information bits used to encode a vector \((u, u + v) \in \{ m \}\). It is also important that recursion (1) splits \( I^m_r \) into two subblocks \( I^{m-1}_r \) and \( I^{m-1}_{r-1} \) that correspond to vectors \( u \) and \( v \), respectively. In this way, the new information strings are split again until we arrive at the end nodes \((j, 1)\) or \((j, j - 1)\). Thus, any specific codeword can be encoded from (multiple) information strings assigned to the end nodes.

2 Background

Despite relatively bad code distance, RM codes have been considered in numerous publications thanks to efficient decoding procedures. Majority decoding developed in [2] (see also [2], [3], [8], and [10]) has complexity order at most \( nk \) and corrects all error patterns of weight below \( d/2 \). It is also known [5] that majority decoding corrects many error patterns of higher weights and can be further improved for soft-decision channels [13].

To decrease decoding complexity, recursive algorithms were also developed on both hard- ([3]) and soft-decision ([3]) channels. The algorithms provide for bounded distance decoding and have the lowest complexity order \( n \min(r, m - r) \) known for RM codes to date. Simulation results presented in [12] also showed that recursive soft-decision algorithms can increase decoding domain of bounded distance decoding. Finally, efficient permutation algorithm considered in [11] for codes \( \{ m \} \), gives a slightly higher complexity \( O(n^2m) \) while correcting most error patterns of a higher weight \( n(1-h)/2 \), where \( h \) has a vanishing order of \( (m/n)^{1/4} \) as \( m \to \infty \).

In this paper, we wish to further develop recursive algorithms for RM codes and improve their performance both on short and long lengths. Such an improvement is
especially important for short and moderate lengths on which RM codes are on par with the best codes known to date. Our goal is to use the same recursive presentation on Pascal triangle that is used above in code design. Namely, given an output $y = (y', y'')$ with halves $y'$ and $y''$, we wish to perform two steps:

1. combine $y'$ and $y''$ to find $v \in \{m-1, r-1\}$.
2. combine $(y', y'')$ and $(0, v)$ to find $u \in \{m-1, r\}$.

In turn, shorter codes $\{m-1, r-1\}$ and $\{m-1, r\}$ will be split further, while actual decoding procedures will be relegated to the end nodes. In the following sections, this procedure is discussed in more detail.

## 3 Decoding

Consider now the channel with Gaussian noise $\mathcal{N}(0, \sigma^2)$ and probability density function

$$G(y) = (1/\sqrt{2\pi}\sigma)e^{-y^2/2\sigma^2}.$$  \hfill (2)

The two symbols 0 and 1 are transmitted as +1 and −1. These two take arbitrary real values $y$ at the receiver end with probability densities $G(y + 1)$ and $G(y - 1)$, respectively. In hard decision reception, we arrive at the BSC with transition error probability $p = Q(1/\sigma)$, where

$$Q(x) = \int_x^{\infty} e^{-y^2/2}dy/\sqrt{2\pi}.$$  

In brief, we call these two channels AWGN($\sigma^2$) and BSC($p$) respectively.

Below we use the code $\{m\}$ of length $n = 2^m$. Suppose that the codeword $z$ is transmitted and $y \in \mathcal{R}^n$ is received. Given any output signal $y \in \mathcal{R}$, we can find the posterior probabilities $p \overset{\text{def}}{=} p(1|y)$ and $q \overset{\text{def}}{=} p(0|y)$. By using the Bayes’ rule we find

$$p = e^{-g/2}/(e^{g/2} + e^{-g/2}), \quad q = e^{g/2}/(e^{g/2} + e^{-g/2}).$$  \hfill (3)

Here $g$ is the *likelihood* of symbol 0:

$$g = \log(q/p) = 2y/\sigma^2.$$  \hfill (4)

Finally, we introduce the *spread* $h$ (which is the hyperbolic tangent of $g$) between the two probabilities $q$ and $p$:

$$h = q - p = (e^{g/2} - e^{-g/2})/(e^{g/2} + e^{-g/2}) = \tanh(g).$$  \hfill (5)

Given an output vector $y = (y_1, ..., y_n)$, we can find the quantities $q_j$, $p_j$, $h_j$, and $g_j$ for any position $j$. In decoding, we will use the original vector $y$, as well as the corresponding vectors $h = (h_1, ..., h_n)$ and $g = (g_1, ..., g_n)$. Note that $\tanh(g)$ is a one-to-one mapping. Therefore the three vectors are interchangeable:

$$y \leftrightarrow g \leftrightarrow h.$$  \hfill (6)

We denote such a decoding $z = \Psi_r^m(y) = \Psi_r^m(h) = \Psi_r^m(g)$, where $z \in \{m\}$ is our decoding result.
In ML decoding, we can first consider the string of hard decision outputs

\[ a_j = \begin{cases} 
0, & \text{if } g_j \geq 0, \\
1, & \text{if } g_j < 0, 
\end{cases} \quad (7) \]

and try to find the most reliable codeword

\[ z^* : \sum_{j: z_j \neq a_j} |g_j| \leq \sum_{j: z_j \neq a_j} |g_j| \]

among all codewords \( z \in \{ \mathbf{r}_m \} \). In our decoding below we also wish to minimize \( \sum_{j: z_j \neq a_j} |g_j| \). However, this will be done on the premise that \( y \) is not heavily corrupted by noise. The corresponding threshold levels will be defined for BSC(\( p \)) and AWGN(\( \sigma^2 \)) in Theorems 4 and 5, respectively.

### 3.1 Recalculating the Probabilities

Let \( z' \) and \( z'' \) denote the left- and right halves of any vector \( z \in \{ \mathbf{r}_m \} \). We also use odd positions \( j = 2s - 1 \) on the left half and their even counterparts \( 2s \) on the right one for any \( s = 1, ..., n/2 \). Similar notations are used for all other vectors, say \( y, h \) and \( g \). Given any vector \( z = (u, u + v) \) at the transmitter end, we can find

\[ v = z' + z'' \text{(mod 2)} = (z_1 + z_2, z_3 + z_4, ..., z_{n-1} + z_n). \]

By contrast, at the receiver end we know only the strings \( p = (p_1, ..., p_n) \), \( g \), and \( h \) that define the probability distribution on the transmitted symbols. Our first problem is to find the spread \( h_s \) on vectors \( v = z' + z'' \) given the original spread \( h = (h', h'') \) on vectors \( z = (z', z'') \).

**Lemma 1** (addition mod 2). Vectors \( \{z' + z''\} \text{ (mod 2)} \) have the spread

\[ h_s = h'h'' = (h_1h_2, ..., h_{n-1}h_n). \]

Given an output \( y \), we now find the probability spreads \( h \) and \( h^* \) on vectors \( z \) and \( z' + z'' \). Then our decoding \( \Psi^m_r(h) = (u, u + v) \) can first try to find \( v = \Psi^{m-1}_{r-1}(h^*) \). Once vector \( v \) is found, we wish to find the remaining block \( (u, u + v) + (0, v) = (u, u) \) in the second step of our decoding. Here we need to replace original symbols \( z_{2s} \) by symbols \( z_{2s} + v \). Correspondingly, the latter have likelihoods

\[ g_{2s} = \begin{cases} 
g_{2s}, & \text{if } v_s = 0, \\
-g_{2s}, & \text{if } v_s = 1. 
\end{cases} \]

In other words, we change the sign of \( g_{2s} \) whenever \( v_{2s} = 1 \). The result is the string \( \overline{g}(u, u) = (\overline{g'}, \overline{g''}) \), where the left half \( \overline{g'} \) is taken from the original vector \( g \) and equals \( g' \). The string \( \overline{g}(u, u) \) represents likelihoods for arbitrary vectors \( (u, u) \) with two equal parts. Then we find the likelihoods \( g_s(u) \) given the two estimates \( \overline{g'}, \overline{g''} \).

**Lemma 2** (repetition). A string of likelihoods \( (\overline{g'}, \overline{g''}) \) defined on repeated vectors \( (u, u) \) gives for vectors \( u \) the string of likelihoods

\[ g_s(u) = \overline{g'} + \overline{g''}. \]
Once the likelihoods \( g_s \) are found, we execute decoding \( \Psi_{r-1}^m(g_s) \) that finds \( u \in \{ r-1 \} \). We perform both decodings \( \Psi_{r-1}^m(h_s) \) and \( \Psi_{r-1}^m(g_s) \) in a recursive way. In this process, we only recalculate spreads \( h \) and likelihoods \( g \) while using new (shorter) codes. Our recursion moves along the edges of Pascal triangle until recalculated vectors \( h \) and \( g \) arrive at the end nodes \( (j, 1) \) and \( (j, j-1) \). At the end nodes, we perform ML decoding. Now we can describe algorithm \( \Psi_r^m \) in a general soft-decision setting.

### 3.2 Recursive decoding for codes \( \{ m \} \)

1. Receive vector \( y \in \mathcal{R}^n \). Calculate \( g = g_r^m \) and \( h = h_r^m \) according to (1) and (5). Call procedure \( \Psi_r^m \). Output decoded vector \( z_r^m \in \{ m \} \) and its information set \( I_r^m \).

2. Procedure \( \Psi_s^j \). Input \( h_s^j \) and \( g_s^j \).

   2.1. Find \( h_{s-1}^{j-1} = (h_s^j)' \cdot (h_s^j)'' \). Go to 3 if \( s = 2 \) or call \( \Psi_{s-1}^{j-1} \) otherwise.

   2.2. Find \( g_{s-1}^{j-1} = (g_s^j)' + (g_s^j)'' \). Go to 4 if \( s = j - 2 \) or call \( \Psi_{s-1}^{j-1} \) otherwise.

   2.3. Find \( I_s^j = I_{s-1}^{j-1} \cup I_{s-1}^{j-1} \) and vector \( z_s^j = (z_s^j, z_s^j - 1, z_s^j - 1) \). Return vector \( z_s^j \in \{ s \} \) and its information set \( I_s^j \).

3. Execute ML decoding \( \Psi_{j-1}^{i-1} \). Return \( z_{j-1}^i \) and \( I_{j-1}^i \).

4. Execute ML decoding \( \Psi_{j-2}^{i-1} \). Return \( z_{j-2}^i \) and \( I_{j-2}^i \).

**Qualitative analysis.** Note that increasing the noise power \( \sigma^2 \) reduces the means of the spreads \( h \) and likelihoods \( g \). In particular, it can be shown (see [13]) that for large noise \( \sigma \to \infty \), the first two moments \( Eh \) and \( Eh^2 \) of the random variable \( h = \tanh(2y/\sigma^2) \) satisfy the relation

\[
Eh \sim Eh^2 \sim \sigma^{-2}.
\]

In decoding process, we replace our original decoding \( \Psi_r^m \) by \( \Psi_{r-1}^{m-1} \). The latter operates in a less reliable setting with the lower spread \( h_{r-1}^{m-1} = (h_r^m)' \cdot (h_r^m)'' \) versus the original spread \( h_r^m \). Then the newly derived means \( E(h_{r-1}^{m-1}) \) are the componentwise products of the corresponding means \( E(h_r^m)' \) and \( E(h_r^m)'' \). According to (8), this multiplication is equivalent to replacing the original noise power \( \sigma^2 \) by the larger power \( \sigma^4 \). On the other hand, we also increase the relative distance \( d/n \) by using a better protected code \( \{ r-1 \} \) instead of the original code \( \{ m \} \). Our next decoding \( \Psi_{r-1}^{m-1} \) is performed in a better channel. Here we change the original likelihoods \( g_r^m \) for \( g_{r-1}^{m-1} = (g_r^m)' + (g_r^m)'' \). As a result, our average likelihoods are doubled. This is equivalent to the map \( \sigma^2 \Rightarrow \sigma^2/2 \). In other words, code \( \{ m \} \) operates on a channel whose noise power is reduced two times. However, we also reduce the relative distance by using a weaker code \( \{ m-1 \} \) instead of \( \{ m \} \). In the next section, we present the *quantitative* results of decoding performance.
4 Summary of results

We first consider hard decision reception on the channel BSC($p$) with transition error probability $p = (1 - h)/2$.

**Theorem 3** Recursive decoding of codes $\{m_r\}$ used on a BSC($p$), gives the output bit error probability $\alpha \leq Q(\mu)$, where

$$\mu = 2^{(m-r)/2}h^{2r-1}/\sqrt{1-h^2}, \quad p = (1 - h)/2.$$  \hspace{1cm} (9)

We then consider asymptotic capacity of recursive algorithms for long codes $\{m_r\}$ as $m \to \infty$. We consider separately low-rate codes of fixed order $r$ and those of fixed rate $R$. The latter implies that $r/m \to 0$. Let $c$ be any constant exceeding $\ln 2$.

**Theorem 4** For $m \to \infty$, recursive decoding of codes $\{m_r\}$ corrects most error patterns of weight:

$$t \leq \begin{cases} n(1 - (cm/d)^{1/2r})/2, & \text{if } r = \text{const}, \\ d(\ln d - \ln 2m)/2, & \text{if } 0 < R < 1. \end{cases}$$  \hspace{1cm} (10)

with decoding complexity order $n \log n$.

For fixed code rate $R$, the latter estimate is almost twice the bound $d \ln d/4$ known [5] for majority decoding. For low-rate codes of fixed order $r$ we correct almost $n/2$ errors. In this case, our residual term $h = 1 - 2t/n$ has vanishing order of $(cm/d)^{1/2r}$. The latter substantially reduces the former term $(m/d)^{1/2r+1}$ known [5] for majority decoding. Such a performance was known before only for $r = 2$ and was obtained in permutation decoding presented in [11]. We note that this threshold $h$ is now obtained for all orders $r$ and is achieved with a lower complexity order of $n \log n$.

Our next issue is to improve recursive decoding by using soft decision likelihoods $g$. Given any output bit error rate $\alpha < 1/2$, we compare the corresponding noise powers $\sigma_s^2$ and $\sigma_h^2$ and transition error probabilities $p_s$ and $p_h$ that sustain this probability in soft and hard decision decoding.

**Theorem 5** Given any output bit error probability $\alpha$, soft decision recursive decoding of long codes $\{m_r\}$ of fixed order $r$ increases $\pi/2$ times $\alpha$-sustainable noise power of hard decision decoding:

$$\sigma_s^2/\sigma_h^2 \to \pi/2, \quad m \to \infty.$$  \hspace{1cm} (11)

Soft decision decoding of long codes $\{m_r\}$ of fixed code rate $R$ increases $4/\pi$ times $\alpha$-sustainable transition error probability of hard decision decoding:

$$p_s/p_h \to 4/\pi, \quad m \to \infty.$$  \hspace{1cm} (12)

The above theorem shows that for long low-rate RM codes we can gain $10 \log_{10}(\pi/2) \approx 2.0$ dB over hard decision decoding for any output error rate $\alpha$. The following corollary concerns the Euclidean weights of error patterns correctable by our algorithm. We show that for fixed $r$ we exceed the bounded distance decoding weight $\sqrt{d}$ more than $2^{r/2}$ times. For fixed code rate $R$, we have a similar increase and outperform bounded distance decoding $2^{r/2}/\sqrt{m \ln 2}$ times.
Corollary 6 For $m \to \infty$, soft decision recursive decoding of codes $\{m_r\}$ corrects virtually all error patterns of Euclidean weight:

$$
\rho \leq \sqrt{n(d/2m)^{1/2}}, \quad \text{if } r = \text{const},
$$

$$
\rho \leq \sqrt{n/m \ln 2}, \quad \text{if } 0 < R < 1.
$$

5 Comparison

In Table 1 we compare the asymptotic performance of the newly developed algorithms with both the majority decoding and former recursive algorithms. This comparison is done for hard and soft decision decoding of low-rate codes of fixed order $r$ and for codes of fixed rate $R$. As above, for low-rate codes we use the residual term $h$ of our error-correcting capacity $n(1-h)/2$ for low-rate codes. Note that former recursive algorithms only provided for bounded distance capacity $d/2$. For soft decision decoding of low rate codes we use the squared Euclidean distance $\rho^2$. Again, the newly derived distance $\rho^2$ surpasses the one known for majority decoding. Finally, for medium code rates $R$ we use the threshold weight $t$, for which decoding yet corrects most error patterns. In this case, we double decoding capacity of the former algorithms as seen from Table 1.

Table 1. Comparison of decoding algorithms for $\{m_r\}$ codes.

| Decoding capacity       | Former Recursive | Majority Decoding | New Recursive |
|-------------------------|------------------|------------------|---------------|
| Hard decision, $t$ for fixed $r$ | $t = \frac{d}{2}$ | $\frac{n}{2}(1-h)$, $h = \left(\frac{m}{n}\right)^{1/2r+1}$, $h = \left(\frac{m}{n}\right)^{1/2r}$ | $\frac{n}{2}(1-h)$, $h = \left(\frac{m}{n}\right)^{1/2r}$ |
| Soft decision, $\rho^2$ for fixed $r$ | $\rho^2 = \sqrt{d}$ | $(\frac{n}{m})^{1/2r+1} \sqrt{n}$ | $(\frac{n}{m})^{1/2r} \sqrt{n}$ |
| Hard decision, $t$ for fixed $R$ | $t = \frac{d}{2}$ | $d \ln d/4$ | $d \ln d/2$ |

Below in Table 2, we present simulation results for bit error rates (BER) obtained by applying recursive and majority decoding to the code $\{\frac{9}{14}\}$ of length 512. We also exhibit the results of computer simulation presented in [12]. Here, however, block error probabilities (BLER) were used in recursive decoding.

Finally, the last row represents a refined version of recursive decoding. This improvement uses the fact that recursive decoding gives different error rates at different end nodes. In particular, the worst error rates are obtained on the leftmost node $(m - r + 1, 1)$, and the next worst results are obtained at the node $(m - r, 1)$. This asymmetrical performance can be justified by our qualitative analysis given above. We can see that the leftmost end node operates at the highest noise power $\sigma^2$.

The next important conclusion is to set the corresponding information bits as zeros. In this way we arrive at the subcodes of the original code $\{m_r\}$ obtained by eliminating only a few least protected information bits. This “expurgation” procedure gives a substantial improvement to conventional recursive algorithms as seen from Table 2. Also, such a recursion gives good block error probabilities (BLER) in contrast to most iterative algorithms developed to date. This part of the work performed jointly with K. Shabunov has been developed further and will be reported separately in more detail.
Table 2. Decoding performance for code \( \{\frac{9}{4}\} \).

| SNR (dB) | Recursive \[12\] | Majority \[13\] | Recursive (new) | BER for subcodes | BLER for subcodes |
|----------|------------------|----------------|-----------------|------------------|------------------|
| 2        | 0.9 0.5 0.2      | 0.3 0.15 0.1 | 0.2 0.03 0.002  | 0.05 0.003 \(3 \cdot 10^{-5}\) | 0.2 0.02 \(2 \cdot 10^{-4}\) |

6 Concluding remarks

It is interesting to compare the presented recursive algorithm with a few former variants considered in \[3], \[6], and \[12\]. Our algorithm is similar to these especially to the one presented in \[3\]. As a result, we achieve similar complexity of order \(n \log n\). However, there are three differences. Instead of studying bounded-distance decoding, we try to find actual decoding capacity of recursive algorithms. Secondly, in this new setting we use probabilistic tools and explicitly recalculate posterior probabilities while moving along the edges of Pascal triangle. Finally, we use a different stopping rule and terminate any branch after reaching the codes \(\{\frac{1}{1}\}\) of the first order. The algorithms considered before were terminated at codes \(\{\frac{1}{0}\}\) of order zero. By using probabilistic tools described above, one can prove that using codes of order zero gives asymptotic performance similar to that of majority decoding. Therefore our increase in decoding capacity mostly results from a different stopping rule.

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