A Short Remark on Analogical Reasoning *

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Abstract
We discuss the problem of defining a logic for analogical reasoning, and sketch a solution in the style of the semantics for Counterfactual Conditionals, Preferential Structures, etc.

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1 Introduction

We consider here (largely verbatim, only punctually slightly modified) excerpts from [SEP13], see also [SEP19c], to set the stage for Section 2 (page 4).

1.1 Section 2.2, p. 5 of [SEP13]

Definition 1.1
An analogical argument has the following form:
1. $S$ is similar to $T$ in certain (known) respects.
2. $S$ has some further feature $Q$.
3. Therefore, $T$ also has the feature $Q$, or some feature $Q^*$ similar to $Q$.

(1) and (2) are premises. (3) is the conclusion of the argument. The argument form is inductive; the conclusion is not guaranteed to follow from the premises.

$S$ and $T$ are referred to as the source domain and target domain, respectively. A domain is a set of objects, properties, relations and functions, together with a set of accepted statements about those objects, properties, relations and functions. More formally, a domain consists of a set of objects and an interpreted set of statements about them. The statements need not belong to a first-order language, but to keep things simple, any formalizations employed here will be first-order. We use unstarred symbols ($a, P, R, f$) to refer to items in the source domain and starred symbols ($a^*, P^*, R^*, f^*$) to refer to corresponding items in the target domain.

Definition 1.2
Formally, an analogy between $S$ and $T$ is a one-to-one mapping between objects, properties, relations and functions in $S$ and those in $T$.

1.2 Section 2.2, pp. 6-7 of [SEP13]

In an earlier discussion of analogy, Keynes, in [Key21], introduced some terminology that is also helpful.

(1) Positive analogy.

Let $P$ stand for a list of accepted propositions $P_1, \ldots, P_n$ about the source domain $S$. Suppose that the corresponding propositions $P^*_1, \ldots, P^*_n$, abbreviated as $P^*$, are all accepted as holding for the target domain $T$, so that $P$ and $P^*$ represent accepted (or known) similarities. Then we refer to $P$ as the positive analogy.

(2) Negative analogy.

Let $A$ stand for a list of propositions $A_1, \ldots, A_r$ accepted as holding in $S$, and $B^*$ for a list $B_1^*, \ldots, B_s^*$ of propositions holding in $T$. Suppose that the analogous propositions $A^*_1 = A_1^*, \ldots, A_r^*$ fail to hold in $T$, and similarly the propositions $B = B_1, \ldots, B_s$ fail to hold in $S$, so that $A, \neg A^*$ and $\neg B, B^*$ represent accepted (or known) differences. Then we refer to $A$ and $B$ as the negative analogy.

(3) Neutral analogy.

The neutral analogy consists of accepted propositions about $S$ for which it is not known whether an analogue holds in $T$.

(4) Hypothetical analogy.

The hypothetical analogy is simply the proposition $Q$ in the neutral analogy that is the focus of our attention.

These concepts allow us to provide a characterization for an individual analogical argument that is somewhat richer than the original one.
**Definition 1.3**

(Augmented representation)

Correspondence between SOURCE (S) and TARGET (T)

1. Positive analogy:
   \[ P \Leftrightarrow P^* \]

2. Negative analogy:
   \[ A \Leftrightarrow \neg A^* \]
   and
   \[ \neg B \Leftrightarrow B^* \]

3. Plausible inference:
   \[ Q \Leftrightarrow Q^* \]

An analogical argument may thus be summarized: It is plausible that \( Q^* \) holds in the target because of certain known (or accepted) similarities with the source domain, despite certain known (or accepted) differences.

**1.3 Section 2.4 of [SEP13]**

Scepticism:

Of course, it is difficult to show that no successful analogical inference rule will ever be proposed. But consider the following candidate, formulated using the concepts of the schema in Definition 1.3 (page 3) and taking us only a short step beyond that basic characterization.

**Definition 1.4**

Suppose \( S \) and \( T \) are the source and target domains. Suppose \( P_1, \ldots, P_n \) (with \( n = 1 \)) represents the positive analogy, \( A_1, \ldots, A_r \) and \( \neg B_1, \ldots, \neg B_s \) represent the (possibly vacuous) negative analogy, and \( Q \) represents the hypothetical analogy. In the absence of reasons for thinking otherwise, infer that \( Q^* \) holds in the target domain with degree of support \( p > 0 \), where \( p \) is an increasing function of \( n \) and a decreasing function of \( r \) and \( s \).

(Definition 1.4 (page 3) is modeled on the straight rule for enumerative induction and inspired by Mill’s view of analogical inference, as described in [SEP13] above. We use the generic phrase “degree of support” in place of probability, since other factors besides the analogical argument may influence our probability assignment for \( Q^* \).)

It is pretty clear that the schema in Definition 1.4 (page 3) is a non-starter. The main problem is that the rule justifies too much.

So, how do we choose the “right one”?

**1.4 A Side Remark**

The author was surprised to find a precursor to his concept of homogenousness in the work of J. M. Keynes, [Key21], quoted in Section 4.3 of [SEP13].
2 The Idea

We now describe the idea, and compare it to other ideas in philosophical and AI related logics. But first, we formalize above ideas into a definition.

**Definition 2.1**

Let $L$ be an alphabet.

1. Let $L_\alpha \subseteq L$, and $\alpha : L_\alpha \to L$ an injective function, preserving the type of symbol, e.g.,
   - if $x \in L_\alpha$ stands for an object of the universe, then so will $\alpha(x)$
   - if $X \in L_\alpha$ stands for a subset of the universe, then so will $\alpha(X)$
   - if $P(.) \in L_\alpha$ stands for an unary predicate of the universe, then so will $\alpha(P(.))$
   - etc., also for higher symbols, like $f : \mathcal{P}(U) \to \mathcal{P}(U)$, $U$ the universe.

2. Let $F_\alpha$, a subset of the formulas formed with symbols from $L_\alpha$.
   For $\phi \in F_\alpha$, let $\alpha(\phi)$ be the obvious formula constructed from $\phi$ with the function $\alpha$.

3. We now look at the truth values of $\phi$ and $\alpha(\phi)$, $v(\phi)$ and $v(\alpha(\phi))$. In particular, there may be $\phi$ s.t. $v(\phi)$ is known, $v(\alpha(\phi))$ not, and we extrapolate that $v(\phi) = v(\alpha(\phi))$, this is then the analogical reasoning based on $\alpha$.

   More precisely:

   1. There may be $\phi$ s.t. $v(\phi)$ is not known, $v(\alpha(\phi))$ is known or not, such $\phi$ do not interest us here.
      Assume in the following that $v(\phi)$ is known.

   2. $v(\phi)$ and $v(\alpha(\phi))$ are known, and $v(\phi) = v(\alpha(\phi))$. The set of such $\phi$ is the positive support of $\alpha$, $\alpha^+.$

   3. $v(\phi)$ and $v(\alpha(\phi))$ are known, and $v(\phi) \neq v(\alpha(\phi))$. The set of such $\phi$ is the negative support of $\alpha$, $\alpha^-.$

   4. $v(\phi)$ is known, $v(\alpha(\phi))$ is not known. The set of such $\phi$ is denoted $\alpha^\ast.$

   The “effect” of $\alpha$ is to conjecture, by analogy, that $v(\phi) = v(\alpha(\phi))$ for such $\phi.$

Intuitively, $\alpha^+$ strengthens the case of $\alpha$, $\alpha^-$ weakens it - but these need not be the only criteria, see also [SEP13] and [SEP19c].

Let $\mathcal{A}$ be a set of functions $\alpha$ as defined in Definition 2.1 (page 3).

We may close $\mathcal{A}$ under combinations, as illustrated in the following Example 2.1 (page 4), or not.

**Example 2.1**

Consider $\alpha, \alpha'$.

Let $x, x', P, Q \in L_\alpha = L_{\alpha'}$, $Q(x), Q(x') \in \alpha^, \alpha'^\ast$.

1. $\alpha$ works well for $x$, but not for $x'$: $P(x) = \alpha(P)(x)$, $P(x') \neq \alpha(P)(x')$, so $P(x) \in \alpha^+$, $P(x') \in \alpha^-$,
2. $\alpha'$ works well for $x'$, but not for $x$: $P(x') = \alpha'(P)(x')$, $P(x) \neq \alpha'(P)(x)$, so $P(x) \in \alpha'^-$, $P(x') \in \alpha'^+$.

Let further $\alpha(Q)(x) \neq \alpha'(Q)(x)$ and $\alpha(Q)(x') \neq \alpha'(Q)(x')$.

What shall we do, should we chose one, $\alpha$ or $\alpha'$, for guessing, or combine $\alpha$ and $\alpha'$ to $\alpha''$, choosing $\alpha'' = \alpha$ for expressions with $x$, and $\alpha'' = \alpha'$ for expressions with $x'$, more precisely $\alpha''(Q)(x) := \alpha(Q)(x)$, and $\alpha''(Q)(x') := \alpha'(Q)(x')$?

The idea is now to push the choice of suitable $\alpha \in \mathcal{A}$ into a relation $\prec$, expressing quality of the analogy. E.g., in Example 2.1 (page 3), $\alpha'' \prec \alpha$ and $\alpha'' \prec \alpha'$ - for historical reasons, smaller elements will be “better”.

Usually, this “best” relation will be partial only, and there will be many “best” $f$. Thus, it seems natural to conclude the properties which hold in ALL best $f$. 
Definition 2.2
Let $A$ be a set of functions as described in Definition 2.1 (page 4), and $\prec$ a relation on $A$ (expressing “better” analogy wrt. the problem at hand).

We then write $A \models \prec \phi$ iff $\phi$ holds in all $\prec$-best $f \in A$.
(This is a sketch only, details have to be filled in according to the situation considered.)

2.1 Discussion
This sounds like cheating: we changed the level of abstraction, and packed the question of “good” analogies into the $\prec$-relation.
But when we look at the Stalnaker-Lewis semantics of counterfactual conditionals, see [Sta68], [Lew73], the preferential semantics for non-monotonic reasoning and deontic logic, see e.g. [Han69], [KLM90], [Sch04], [Sch18], the distance semantics for theory revision, see e.g. [LMS01], [Sch04], this is a well used “trick” we need not be ashamed of.

In above examples, the comparison was between possible worlds, here it is between usually more complicated structures (functions), but this is no fundamental difference.

But even if we think that there is a element of cheating in our idea, we win something: properties which hold in ALL preferential structures, and which may be stronger for stronger relations $\prec$, see Fact 2.1 (page 5) below.

2.2 Problems and solutions
(1) In the case of infinitely many f’s we might have a definability problem, as the resulting best guess might not be definable any more - as in the case of preferential structures, see e.g. [Sch04].

(2) Abstract treatment of representation problems for abovementioned logics work with arbitrary sets, so we have a well studied machinery for representation results for various types of relations of “better” analogies - see e.g. [LMS01], [Sch04], [Sch18].

To give the reader an idea of such representation results, we mention some, slightly simplified.

Definition 2.3
(1) Let again $\prec$ be the relation, and $\mu(X) := \{x \in X : \neg \exists x' \in X. x' \prec x\}$,
(2) $\prec$ is called smooth iff for all $x \in X$, either $x \in \mu(X)$ or there is $x' \in \mu(X), x' \prec x$,
(3) $\prec$ is called ranked iff for all $x, y, z$, if neither $x \prec y$ nor $y \prec x$, then if $z \prec x$, then $z \prec y$, too, and, analogously, if $x \prec z$, then $y \prec z$, too.

We then have e.g.

Fact 2.1
(2.1) General and transitive relations are characterised by
$(\mu \subseteq)\mu(X) \subseteq X$
and
$(\mu PR) X \subseteq Y \rightarrow \mu(Y) \cap X \subseteq \mu(X)$

(2.2) Smooth and transitive smooth relations are characterised by $(\mu \subseteq), (\mu PR)$, and the additional property
$(\mu CUM)\mu(X) \subseteq Y \subseteq X \rightarrow \mu(X) = \mu(Y)$

(2.3) Ranked relations are characterised by $(\mu \subseteq), (\mu PR)$, and the additional property
$(\mu =) X \subseteq Y, \mu(Y) \cap X \neq \emptyset \rightarrow \mu(X) = \mu(Y) \cap X$.

For more explanation and details, see e.g. [Sch18], in particular Table 1.6 there.
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