Distance Computations in the Hybrid Network Model via Oracle Simulations

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--- Abstract ---

The Hybrid network model was introduced in [Augustine et al., SODA ’20] for laying down a theoretical foundation for networks which combine two possible modes of communication: One mode allows high-bandwidth communication with neighboring nodes, and the other allows low-bandwidth communication over few long-range connections at a time. This fundamentally abstracts networks such as hybrid data centers, and class-based software-defined networks.

Our technical contribution is a density-aware approach that allows us to simulate a set of oracles for an overlay skeleton graph over a Hybrid network.

As applications of our oracle simulations, with additional machinery that we provide, we derive fast algorithms for fundamental distance-related tasks. One of our core contributions is an algorithm in the Hybrid model for computing exact weighted shortest paths from $\tilde{O}(n^{1/3})$ sources which completes in $\tilde{O}(n^{1/3})$ rounds w.h.p. This improves, in both the runtime and the number of sources, upon the algorithm of [Kuhn and Schneider, PODC ’20], which computes shortest paths from a single source in $\tilde{O}(n^{2/5})$ rounds w.h.p.

We additionally show a 2-approximation for weighted diameter and a $(1 + \epsilon)$-approximation for unweighted diameter, both in $\tilde{O}(n^{1/3})$ rounds w.h.p., which is comparable to the $\tilde{\Omega}(n^{1/3})$ lower bound of [Kuhn and Schneider, PODC ’20] for a $(2 - \epsilon)$-approximation for weighted diameter and an exact unweighted diameter. We also provide fast distance approximations from multiple sources and fast approximations for eccentricities.

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1 Introduction

The Hybrid model of computation was recently introduced by Augustine et al. [7, SODA’20], for abstracting networks which can utilize both high-bandwidth local communication links, as well as very few low-bandwidth global communication channels. This model abstracts fundamental systems, such as a combination of device-to-device communication with cellular networks (e.g. 5G) [5], wired data centers with wireless links (hybrid DCNs) [14,22,25,38], and Class-Based Hybrid software-defined networks (SDNs) [37].

The pioneering works of [7,17,26] provide fast algorithms for various distance-related tasks in the Hybrid model. At the heart of many of these algorithms lies a framework for using skeleton overlay graphs for computation and approximation of distances, as well as for fast communication.

In this paper, we show how to efficiently simulate oracles over skeleton graphs in the Hybrid model. Using additional machinery that we provide, the implications of our simulations are faster algorithms for distance computations in the Hybrid model. Our oracle models could also be of independent interest, presenting a generic approach which can potentially be applied elsewhere.

1.1 Our Contributions

The Hybrid model, which we consider in this paper, abstracts a synchronous network of nodes, where in each round, every node can send and receive many messages of $O(\log n)$ bits to/from each of its neighbors (over local edges) and an additional $O(\log n)$ messages, in total, to/from any other nodes in the network (over global edges). The high bandwidth permissible over the local edges is aligned with previous research in the Hybrid model as well as with the extensively studied LOCAL distributed model.

The main idea which our simulations of oracles hinge upon is exploiting an inherent asymmetry in the Hybrid model which we observe. This asymmetry allows nodes with dense neighborhoods to effectively receive significantly more information. To see this, note that every node can use the global edges of the Hybrid model to send and receive a limited number of messages every round. However, since in the next round a node can communicate with its neighborhood in the graph using local edges and share with them the information which it received, this implies that a node is able to learn much more information from the entire graph if it is in a more dense neighborhood. Thus, density-aware algorithms are inherently useful in the Hybrid model.

To capture what an oracle can do, we introduce the Oracle and Tiered Oracles models. Roughly speaking, in the Oracle model there is a node $\ell$, the oracle, which can receive $\deg(v)$ messages from each node $v$, within a single round. In particular, this implies that the oracle can learn the entire communication graph.

We cannot afford to directly simulate the Oracle model as it requires too much communication in the Hybrid model. Instead, we simulate the Oracle model over a skeleton graph. Roughly speaking, given an input graph $G$, a skeleton graph is a subset of the nodes of $G$, connected by virtual edges which represent paths in $G$. Skeleton graphs are a common tool for distance computations in various models [7,26,30,34,36], and it has been shown in [7,26] that some distances on the skeleton graph can be efficiently extended to distances on the entire graph in the Hybrid model.

As a warm-up, we show that a single round of the Oracle model over certain skeleton
graphs can be simulated in $\tilde{O}(n^{1/3})$ rounds, w.h.p.\footnote{As common, w.h.p. indicates a probability that is at least $1 - n^{-c}$, for some constant $c > 1$.} in the Hybrid model. Combining this with a simple, constant-round algorithm for exact weighted single source shortest paths (SSSP) which we show in the Oracle model, gives the following.

**Theorem 1 (Exact SSSP).** Given a weighted graph $G = (V, E)$, there is an algorithm in the Hybrid model that computes an exact weighted SSSP in $\tilde{O}(n^{1/3})$ rounds w.h.p.

This result should be compared with the previous state-of-the-art algorithms for exact weighted SSSP in $\tilde{O}(n^{2/3})$ rounds \cite{26}, and in $\tilde{O}(\sqrt{\text{SPD}})$ rounds \cite{7}, where \text{SPD} is the length of the shortest path diameter. Further, it improves upon the $\tilde{O}(n^{1/3}/\epsilon^3)$ round algorithm for a $(1 + \epsilon)$-approximation of weighted SSSP \cite{7}, in both the runtime and in being exact. We stress that this is a warm up, and later on we extend this result to shortest path distances from $O(n^{1/3})$ sources, instead of a single source, in the same round complexity of $\tilde{O}(n^{1/3})$.

It is well known that one can approximate the diameter using a solution to SSSP, and so as a byproduct we get the following result.

**Corollary 2 (2-Approx. Weighted Diameter).** There is an algorithm in the Hybrid model that computes a $2$-approximation of weighted diameter in $\tilde{O}(n^{1/3})$ rounds w.h.p.

Notably, $\tilde{O}(n^{1/3})$ rounds are necessary for a $(2 - \epsilon)$-approximation for weighted diameter \cite{26}. Our algorithm in Corollary 2 thus raises the interesting open question of whether one can go below this complexity for a $2$-approximation.

While efficiently simulating an oracle is powerful, it still does not exploit the full capacity of the Hybrid model. This observation brings us to enhance the Oracle model and introduce the Tiered Oracles model, which consists of multiple oracles with varying abilities. In a nutshell, in the Tiered Oracles model, in each round every node $v$ can send (the same) $\deg(v)$ messages to all nodes $u$ with $\deg(u) \geq \deg(v)/2$. This basically means that nodes are bucketed according to degrees and each node is an oracle for all nodes in buckets below it. One can notice that the node with the highest degree in the graph is equivalent to the oracle in the Oracle model, but here, the other nodes in the graph also have some partial oracle capabilities.

We show how to simulate the Tiered Oracles model over skeleton graphs in the Hybrid model within $O(n^{1/3})$ rounds. Subsequently, we present an algorithm which solves all pairs shortest paths (APSP) using one round of the Tiered Oracles model and $O(\log n)$ rounds of the Congested Clique model\footnote{The Congested Clique is a synchronous distributed model where every two nodes in the graph can exchange messages of $O(\log n)$ bits in every round.}. We then utilize our Tiered Oracles model simulation, along with a previously known simulation of the Congested Clique model from \cite{26}, to simulate the APSP algorithm over skeleton graphs in the Hybrid model. Our efficient computation of APSP over a skeleton graph in the Hybrid model then leads to computing multi-source shortest parts from random sources in the Hybrid model.

Shortest paths from random sources is a crucial stepping stone for our later results. We show that computing shortest path distances from random sources to the entire graph, allows us to subsequently obtain fast algorithms for other distance problems.

**Theorem 3 (Exact Shortest Paths, Sources Sampled i.i.d.).** Given a graph $G = (V, E)$, $0 < x < 1$, and a set of nodes $M$ sampled independently with probability $n^{2x-1}$, there is an algorithm in the Hybrid model that ensures that every $v \in V$ knows the exact, weighted distance from itself to every node in $M$ within $\tilde{O}(n^{1-x} + n^{2x-1})$ rounds w.h.p.
We complement Theorem 3 with a lower bound, following the lines of [26], for approximating distances from many random sources, to any reasonable approximation factor, which tightly matches the upper bound when $x = 2/3$.

**Theorem 4 (Lower Bound Exact Shortest Paths, Sources Sampled i.i.d.).** Let $p = \Omega(\log n/n)$ and $\alpha < \sqrt{n/p} \cdot \log(n)/2$. Any $\alpha$-approximate unweighted algorithm from random sources sampled independently with probability $p$ in the Hybrid network model takes $\Omega(\sqrt{p \cdot n / \log n})$ rounds w.h.p.

We leverage our near-optimal (tight up to polylogarithmic factors) algorithm for shortest paths from a set $M$ of $\tilde{O}(n^{2/3})$ random sources in order to obtain exact weighted shortest paths from any given set $U$ of $O(n^{1/3})$ sources. We achieve this by adapting the behavior of the given fixed source nodes to the density of their neighborhoods, as follows. A source node $s \in U$ in a sparse neighborhood broadcasts the distances to all the random source nodes from $M$ it sees in its neighborhood. A source node $s \in U$ in a dense neighborhood takes control of one of the random sources in $M$ in its neighborhood and uses it as a proxy in order to communicate enough information to all the other nodes in the graph so that they could determine their distances distance from $s$. We remark that this proxy approach is a key insight which we later encapsulate as a general tool in the Hybrid model and may potentially be of independent interest. Our approach gives the following.

**Theorem 5 (Exact $n^{1/3}$ Sources Shortest Paths).** Given a weighted graph $G = (V, E)$, and a set of sources $U$, such that $|U| = O(n^{1/3})$, there exists an algorithm, at the end of which each $v \in V$ knows its distance from every $s \in U$, which runs in $O(n^{1/3})$ rounds w.h.p.

Theorem 5 raises an interesting open question of whether the complexity of SSSP in the Hybrid model is below that of computing shortest paths from $\tilde{O}(n^{1/3})$ sources.

We also exploit our aforementioned solution for computing APSP on the skeleton graph to obtain approximate distances from a larger set of given sources, as follows.

**Theorem 6 (Approximate Multiple Source Shortest Paths).** Given a graph $G = (V, E)$, a set of sources $U$, where $|U| = \tilde{O}(n^y)$ for some constant $0 < y < 1$, and a value $0 < \epsilon$, there is an algorithm in the Hybrid model which ensures that every node $v \in V$ knows an approximation to its distance from every $s \in U$, where the approximation factor is $(1 + \epsilon)$ if $G$ is unweighted and $3$ if $G$ is weighted. The complexity of the algorithm is $\tilde{O}(n^{1/3}/\epsilon + n^{y/2})$ rounds, w.h.p.

This result improves both in round complexity and approximation factors upon the previous results in [26]. The reason for this is that we compute APSP over skeleton graphs using the efficient, exact algorithm from the Tiered Oracles oracle model, while [26] simulate the slower, approximate algorithms of [11,12] in the Congested Clique model. Particularly, this result is tight up to polylogarithmic factors for $y \geq 2/3$ due to a lower bound of [26].

We can also approximate unweighted eccentricities by a combination of computing shortest path distances from $n^{2/3}$ random sources and performing local explorations using the local edges of the model. For approximating weighted eccentricities, this is insufficient, and here our approach is to additionally broadcast required information from each random source node regarding its $\tilde{O}(n^{1/3})$-hop neighborhood in the graph. We obtain the following.

**Theorem 7 (Approx. Eccentricities).** Given a graph $G = (V, E)$, there is an algorithm in the Hybrid model that computes a $(1 + \epsilon)$-approximation of unweighted and $3$-approximation of weighted eccentricities in $\tilde{O}(n^{1/3}/\epsilon)$ rounds, w.h.p.

Finally, the unweighted eccentricities approximation directly implies a $(1 + \epsilon)$ approximation for unweighted diameter. This should be compared with the lower bound of $\tilde{O}(n^{1/3})$ rounds for exact unweighted diameter due to [26].
Corollary 8 ((1 + ε)-Approx. Unweighted Diameter). Let $G = (V, E)$ be an unweighted graph, and let $\epsilon > 0$. There exists an algorithm in the Hybrid model which computes a $(1 + \epsilon)$-approximation of the diameter in $\tilde{O}(n^{1/3}/\epsilon)$ rounds, w.h.p.

We refer the reader to Table 1 for a visual summary of our end results with comparison to related work.

Roadmap: The remainder of the current section is dedicated towards surveying related work. In Section 2, we provide all formal definitions. Next, in Section 3 we formally define the Oracle and Tiered Oracles models, and show how to simulate them over skeleton graphs in the Hybrid model. Finally, Section 4 gives our algorithms for distance problems in the oracle models and, using our simulations, also in the Hybrid model. Some additional results are deferred to the appendix. We show the approximation for shortest path from $\mathbb{R}^x$ given sources in Appendix C.3. In Appendices C.4 and C.5 we provide eccentricities and diameter approximations, respectively. We wrap up with our lower bound for shortest paths from sources sampled i.i.d. in Appendix D.

| Problem   | Variant               | Approximation | This work         | Previous works                      |
|-----------|-----------------------|---------------|-------------------|-------------------------------------|
| SSSP      | weighted              | exact         | $\tilde{O}(n^{1/3})$ | $\tilde{O}(n^{2/3})$, $O(\sqrt{SPD})$ [7] |
|           | weighted              | $1 + \epsilon$ | $O(n^{1/3}/\epsilon)$ | $O(n^{2/3} \cdot \epsilon^{-\beta})$ [7]$^7$ |
| $n^x$-RSSP| unweighted            | $\tilde{O}(n^{1-x/2})$ | $\tilde{O}(n^{1-x})$ | $\tilde{O}(n^{1-x} + n^{2x-1})$ [26] |
|           | weighted              | exact         | $\tilde{O}(n^{1/3})$ | $O(n^{1/3}/\epsilon)$ [26] |
| $n^{1/3}$-SSP | unweighted       | $1 + \epsilon$ | $\tilde{O}(n^{1/3})$ | $\tilde{O}(n^{1/3}/\epsilon)$ [26] |
|           | weighted              | $3 + \epsilon$ | $\tilde{O}(n^{1/3})$ | $\tilde{O}(n^{1/3}/\epsilon)$ [26] |
| $n^x$-SSP | unweighted            | $\tilde{O}(n^{1-x/2})$ | $\tilde{O}(n^{1/3}/\epsilon + n^{x/2})$ | $\tilde{O}(n^{1/3}/\epsilon + n^{x/2})$ [26] |
|           | weighted              | $1 + \epsilon$ | $\tilde{O}(n^{1/3}/\epsilon + n^{x/2})$ | $\tilde{O}(n^{1/3}/\epsilon + n^{x/2})$ [26] |
|           | weighted              | $3 + \epsilon$ | $\tilde{O}(n^{1/3} + n^{x/2})$ | $\tilde{O}(n^{1/3}/\epsilon + n^{x/2})$ [26] |
|           | weighted              | $7 + \epsilon$ | $\tilde{O}(n^{1/3} + n^{x/2})$ | $\tilde{O}(n^{1/3}/\epsilon + n^{x/2})$ [26] |
| eccentricities | unweighted       | $1 + \epsilon$ | $O(n^{1/3}/\epsilon)$ | $O(n^{1/3}/\epsilon)$ [26] |
|           | weighted              | $3$           | $O(n^{1/3}/\epsilon)$ | $O(n^{1/3}/\epsilon)$ [26] |
| diameter  | unweighted            | exact         | $\tilde{O}(n^{1/3}/\epsilon)$ | $\tilde{O}(n^{1/3}/\epsilon)$ [26] |
|           | weighted              | $1 + \epsilon$ | $\tilde{O}(n^{1/3}/\epsilon)$ | $\tilde{O}(n^{1/3}/\epsilon)$ [26] |
|           | unweighted            | $3/2 + \epsilon$ | $\tilde{O}(n^{1/3}/\epsilon)$ | $\tilde{O}(n^{1/3}/\epsilon)$ [26] |
|           | weighted              | $2 - \epsilon$ | $\tilde{O}(n^{1/3})$ | $\tilde{O}(n^{1/3})$ [26] |
|           | weighted              | $2 + \epsilon$ | $\tilde{O}(n^{1/3})$ | $\tilde{O}(n^{1/3})$ [26] |
|           | weighted              | $2 \cdot (1/\epsilon)^{O(1/\epsilon)}$ | $\tilde{O}(n^{1/3}/\epsilon)$ | $\tilde{O}(n^{1/3}/\epsilon)$ [26] |

Table 1 Comparison of our results. SPD is the length of the shortest path diameter. The results for $n^x$-RSSP, and weighted diameter approximation upper bounds from previous works are implicit in [7, 26]. Our upper bound for $n^{2/3}$-RSSP is tight up to poly-logarithmic factors due to our lower bound. Our approximations for $n^{x}$-SSP are also tight up to poly-logarithmic factors for $x \geq 2/3$, due to [26].
1.2 Related Work

Hybrid Models. The Hybrid network model was studied in [7,17,26]. In [17], distance results are obtained in one of the harsher variants of the model, where the local edges are restricted to have log n bandwidth. However, these apply only to extremely sparse graphs of at most \( n + O(n^{1/3}) \) edges and cactus graphs. In [21], a slightly different model of hybrid nature is studied.

Augustine et al. [6] proposed the Node-Capacitated Clique model, which is similar to the Congested Clique model, but each node has log n bandwidth. This model is also a special case of the generalised Hybrid model [7] without local edges. This allows one to use the results from the Node-Capacitated Clique model in the Hybrid model without modifications.

Distributed Distance Computations. Distance related problems have been extensively studied in many distributed models. For example, in the CONGEST model, there is a long line of research on APSP [2–4,9,16,29,31,33] which culminated in tight, up to polylogarithmic factors, \( \tilde{O}(n) \) round exact weighted APSP randomized algorithm of Bernstein and Nanongkai [9] and a \( \tilde{O}(n^{4/3}) \) round deterministic algorithm of Agarwal and Ramachandran [3]. [31,33] develop an \( \tilde{O}(n) \) round algorithm, optimal up to polylogarithmic factors, for unweighted APSP. The study of approximate SSSP algorithms was the focus of many recent paper [8,23,35] and lately Becker et al. [8] showed the solution which is close to the lower bound of Das Sarma et al. [35]. In case of exact SSSP, after recent works [16,18,20], there still is a gap between upper and lower bounds. The diameter and eccentricities problems are studied in the CONGEST model in [1,19,33].

In the Congested Clique model, k-SSP, APSP and diameter are extensively studied in [12,13,27,32] and approximate versions of the k-SSP and APSP problem are solved in polylogarithmic [10] and even polyloglogarithmic [15] time. In the more restricted Broadcast Congested Clique model, in which each message a node sends in a round is the same for all recipients, distance computations are researched by [8,24].

2 Preliminaries

We provide here some definitions and claims that are critical for reading the main part of the paper. Appendix A contains additional definitions and basic claims. We use the following variant of the Hybrid model, introduced in [7].

Definition 9 (Hybrid Model). In the Hybrid model, a synchronous network of \( n \) nodes with identifiers in [\( n \)], is given by a graph \( G = (V,E) \). In each round, every node can send and receive \( \lambda \) messages of \( O(\log n) \) bits to/from each of its neighbors (over local edges) and an additional \( \gamma \) messages in total to/from any other nodes in the network (over global edges). If in some round more than \( \gamma \) messages are sent via global edges to/from a node, only \( \gamma \) messages selected adversarially are delivered.

We follow the previous work of [7,26] and consider \( \lambda = \infty, \gamma = O(\log n) \). Notice that the Hybrid model can also capture the classic LOCAL model, with \( \lambda = \infty, \gamma = 0 \), the classic CONGEST model, with \( \lambda = O(1), \gamma = 0 \), the Congested Clique model, with \( \lambda = O(1), \gamma = 0 \) and \( G \) being a clique, and the Node-Capacitated Clique model [6], with \( \lambda = 0, \gamma = O(\log(n)) \).

\(^3\) The LOCAL and CONGEST models are synchronous distributed models where every two neighbors in the graph can exchange messages of unlimited size or of \( O(\log n) \) bits, respectively, in each round.
Many of our results hold for weighted graphs $G = (V, E, w)$. We assume an edge weight is given by a function $w : E \rightarrow \{1, 2, \ldots, W\}$ for a $W$ which is polynomial in $n$. When we send an edge as part of a message in any algorithm, we assume it is sent along with its weight.

### 2.1 Notations and Problem Definitions

We use the following definitions related to graphs. Given a graph $G = (V, E)$ and a pair of nodes $u, v \in V$, we denote by $h-hop(u, v)$ the hop distance between $u$ and $v$, by $N_{h}^{G}(v)$ the $h$-hop neighborhood of $v$, by $d_{G}^{h}(u, v)$ the weight of the lightest path between $u$ and $v$ of at most $h$-hops, and if there is no path of at most $h$-hops then $d_{G}^{h}(u, v) = \infty$. In the special case of $h = 1$, we denote by $N_{G}(v)$ the neighbors of $v$ and in the special case of $h = \infty$, we denote by $d_{G}(u, v)$ the weight of the lightest path between $u$ and $v$. We also denote by $\text{deg}_{G}(v)$ the degree of $v$ in $G$. Whenever it is clear from the context we drop the subscript of $G$ and just write $N$, $N^{h}$, $d$, $d^{h}$ or $\text{deg}(v)$.

We define the following problems in the Hybrid model.

- **Definition 10 (k-Source Shortest Paths (k-SSP)).** Given a graph $G = (V, E)$, and a set $S \subseteq V$ of $k$ sources. Every $u \in V$ is required to learn the distance $d_{G}(u, s)$ for each $s \in S$. The case where $k = 1$, is called the single source shortest paths problem (SSSP).

- **Definition 11 (n$^2$-Random Sources Shortest Paths (n$^2$-RSSP)).** Given a graph $G = (V, E)$, and a set $M \subseteq V$ of sources, such that each $v \in V$ is sampled independently with probability $n^{-1}$ to be in $M$. Every $u \in V$ is required to learn the distance $d_{G}(u, s)$ for each $s \in M$.

In the approximate versions of these problems, each $u \in V$ is required to learn an $(\alpha, \beta)$-approximate distance $\tilde{d}(u, v)$ which satisfies $\tilde{d}(u, v) \leq d(u, v) + \beta$, and in case $\beta = 0$, $\tilde{d}(u, v)$ is called an $\alpha$-approximate distance.

- **Definition 12 (Eccentricity and diameter).** Given a graph $G = (V, E)$ and node $v \in V$, the eccentricity of $v$ is the farthest shortest path distance from $v$, i.e., $\text{ecc}(v) = \max_{u \in V} d(v, u)$ and the diameter $D = \max_{v \in V} \{\text{ecc}(v)\}$ is the maximum eccentricity. An $\alpha$-approximation of all eccentricities is a function $\text{ecc}(v)$ which satisfies $\text{ecc}(v)/\alpha \leq \text{ecc}(v) \leq \text{ecc}(v)$ for all nodes $v$. An $\alpha$-approximation of the diameter is a value $\tilde{D}$ which satisfies $\tilde{D}/\alpha \leq \tilde{D} \leq D$.

### 2.2 Skeleton Graphs

In a nutshell, given a graph $G = (V, E)$, a skeleton graph $S_{x} = (M, E_{S})$, for some constant $0 < x < 1$, is generated by letting every node in $V$ independently join $M$ with probability $n^{-x}$. Two nodes in $M$ have an edge in $E_{S}$ if there exists a path between them in $G$ of at most $h = \tilde{O}(n^{1-x})$ hops. This graph w.h.p. satisfies many useful properties in terms of distance computation, which for simplicity of presentation we add to its definition, provided below. A crucial property is that for any two nodes, if the shortest path between them in $G$ has more than $h$ hops, then there exists a shortest path between them in $G$ on which every roughly $h$ nodes there is a node from $M$ (all such skeleton properties hold w.h.p.).

- **Definition 13 (Skeleton Graph, Combined Definition of [7,26]).** Given a graph $G = (V, E)$ and a value $0 < x < 1$, a graph $S_{x} = (M, E_{S})$ is called a skeleton graph in $G$, if all of the following hold.
  1. Each $v \in V$ included to $M$ independently with probability $n^{-x}$.
  2. $\{v, u\} \in E_{S}$ if and only if there is a path of at most $h = \tilde{O}(n^{1-x})$ edges between $v, u$ in $G$.
  3. Every node $v \in M$ knows all its incident edges in $E_{S}$.
4. $S_x$ is connected.
5. For any two nodes $v, v' \in M$, $d_S(v, v') = d_G(v, v')$.
6. For any two nodes $u, v \in V$ with $\text{hop}(u, v) \geq h$, there is at least one shortest path $P$ from $u$ to $v$ in $G$, such that any sub-path $Q$ of $P$ with at least $h$ nodes contains a node $w \in M$.
7. $|M| = \tilde{O}(nx)$.

The following claim summarizes what is proven in [7] regarding the construction of a skeleton graph from a set of random marked nodes, w.h.p.

\begin{itemize}
  \item \textbf{Claim 14 (Skeleton from Random Nodes).} Given a graph $G = (V, E)$, a value $0 < x < 1$, and a set of nodes $M$ marked independently with probability $nx^{-1}$, there is an algorithm which constructs a skeleton graph $S_x = (M, E_S)$ in $\tilde{O}(n^{1-x})$ rounds w.h.p. If also given a single node $s \in V$, it is possible to construct $S_x = (M \cup \{s\}, E_S)$ without damaging the properties of $S_x$.
\end{itemize}

We extract the following basic claim, used in the proof of [7] Theorem 2.7 for a $(1 + \epsilon)$-approximation for SSSP, and slightly extend it to use for multiple source problem and arbitrary approximation factors. It states that given a skeleton graph and a set of sources, if every skeleton node knows any approximation to its distance from every source, then it is possible to efficiently reach a state where every node in the graph knows the approximation for its own distance from any of the sources. We give the proof for the sake of self-containment in Appendix A. The idea is that each node locally explores its $\tilde{O}(n^{1-x})$ neighborhood and identifies for each source the best skeleton node in its neighborhood to go through.

\begin{itemize}
  \item \textbf{Claim 15 (Extend Distances).} [7, Theorem 2.7] Let $G = (V, E)$, let $S_x = (M, E_S)$ be a skeleton graph, and let $V' \subseteq V$ be the set of source nodes. If for each source node $s \in V'$, each skeleton node $v \in M$ knows the $(\alpha, \beta)$-approximate distance $d(v, s)$ such that $d(v, s) \leq \tilde{d}(v, s) \leq \alpha d(v, s) + \beta$, then each node $u \in V$ can compute for all source nodes $s \in V'$, a value $\tilde{d}(u, s)$ such that $d(u, s) \leq \tilde{d}(u, s) \leq \alpha d(u, s) + \beta$ in $\tilde{O}(n^{1-x})$ rounds.
\end{itemize}

3 Oracles in the Hybrid model

This section is split into three parts. Initially, as preliminaries, we show simulations of the LOCAL and Congested Clique models in the Hybrid model, citing [26] for the Congested Clique simulation. Then, we devote a section to each of the two new oracle models in order to introduce them and present their simulations in the Hybrid model.
3.1 Model Simulation Preliminaries

We will use simulations of the LOCAL and Congested Clique models, as follows.

► **Lemma 16** (LOCAL Simulation). Given a graph $G = (V, E)$, and a skeleton graph $S_x = (M, E_S)$, it is possible to simulate one round of the LOCAL model over $S_x$ within $\tilde{O}(n^{1-x})$ rounds in $G$ in the Hybrid model. That is, within $\tilde{O}(n^{1-x})$ rounds in $G$ in the Hybrid model, any two adjacent nodes in $S_x$ can communicate any amount of data between each other.

The proof follows trivially due to the definition of the Hybrid model and Property [2] in the definition of a skeleton graph $S_x$, since in $S_x$ two skeleton nodes are connected if they are within $\Theta(n^{1-x})$ hops in the original graph $G$. Thus, one round of the LOCAL model over $S_x$ is obtained in the Hybrid network in $\tilde{O}(n^{1-x})$ rounds, by having neighboring skeleton nodes communicate through the local edges.

► **Lemma 17** (Congested Clique Simulation). [26, Corollary 4.1.] Given a graph $G = (V, E)$, and a skeleton graph $S_x = (M, E_S)$, for some constant $0 < x < 1$, it is possible to simulate one round of the Congested Clique model over $S_x$ in $\tilde{O}(n^{2x-1} + n^{2x})$ rounds of the Hybrid model on $G$, w.h.p. That is, within $\tilde{O}(n^{2x-1} + n^{2x})$ rounds of the Hybrid model on $G$, w.h.p., every node $v \in M$ can, for each node $u \in M$, each send a unique $O(\log n)$ bit message to $u$.

3.2 Simulating the Oracle Model

Here, we define the Oracle model and then show how to efficiently simulate it over a skeleton graph in the Hybrid model.

► **Definition 18** (Oracle Model). In the Oracle model over a network $G$, there exists one oracle node $\ell$, which in every round can send to and receive from every node $v$ a number of $O(\log n)$-bit messages that is equal to the degree of $v$ in $G$.

► **Theorem 19** (Oracle Simulation). Given a graph $G = (V, E)$, for every constant $0 < x < 1$, there is an algorithm which simulates one round of the Oracle model, on a skeleton graph $S_x = (M, E_S)$, in $\tilde{O}(n^{1-x} + n^{2x-1})$ rounds of the Hybrid model on $G$, w.h.p.

**Proof.** We prove the claim by showing how to simulate a round of the Oracle model in $O(1)$ rounds of the Congested Clique model and 1 round of the LOCAL model. Then, invoking the simulations of Lemmas [16] and [17] gives the desired round complexity in the Hybrid model.

We show how to send messages to the oracle, and the receiving part is symmetric. The pseudocode is given by Algorithm [1]. First, each $v \in M$ broadcasts its degree in $S_x$ using one round of the Congested Clique model (Line [1]) and selects as an oracle $\ell$ the node with largest degree in $S_x$, breaking ties by identifier (Line [2]). Then, the identifiers of the neighbors of $\ell$ are broadcast using one round of the Congested Clique model (Line [3]). The actual messages are sent to these neighbors instead of to $\ell$ itself (Line [4]) and $\ell$ learns all these messages in 1 round of the LOCAL model in Line [5].

Clearly, all the nodes select the same oracle $\ell$ (Line [2]). Due to the definition of the Oracle model, each node $v \in M$ has $\deg_{S_x}(v)$ messages to send, and since $\deg_{S_x}(\ell) \geq \deg_{S_x}(v)$, there are enough neighbors of $\ell$ to receive one message from $v$ per neighbor, which is why Line [4] can work.

\[\blacktriangleleft\]
3.3 Simulating the Tiered Oracles Model

We further enhance our Oracle model and define the Tiered Oracles model, where, roughly speaking, all nodes, in parallel, can learn all the edges adjacent to nodes with degrees in lower degree buckets. To simulate the stronger Tiered Oracles model over a skeleton graph in the Hybrid model, we need additional insights. Here, we use the fact that when we scatter messages independently at random, denser neighborhoods are more likely to receive a given message than sparse neighborhoods. In other words, while for simulating the Oracle model, we used the LOCAL round only to concentrate information in a single node \( \ell \), here we exploit the information that each node can gather from its neighborhood.

**Definition 20 (Tiered Oracles Model).** In the Tiered Oracles model over a network \( G \), in every round, suppose each node \( v \) has a set of \( O(\log n) \)-bit messages \( M_v \) of size \( |M_v| = \deg(v) \), then each node \( u \) can receive all messages in \( M_v \) for every \( v \) such that \( \deg(u) \geq \deg(v)/2 \).

To simulate the Tiered Oracles model, we first prove the following model-independent tool, the proof of which is deferred to Appendix B.

**Lemma 21 (Degree-Based Simulation Tool).** Consider a graph \( G = (V,E) \), and a routing algorithm \( \text{ALG}_c \), for some value \( c \), in some model, which receives from every node \( v \) any set of \( c \) messages \( S_v \), and delivers each message in \( S_v \) to a node chosen uniformly and independently at random over \( V \). Let there be for each node \( v \in V \) a message set \( M_v \), such that \( |M_v| \leq c \). Then, it is possible within \( O(\log n) \) invocations of \( \text{ALG}_c \), followed by a single round of the LOCAL model over \( G \), to ensure that for every pair of nodes \( u,v \in V \), if \( \deg(u) \geq |M_v|/c \), then \( u \) knows all of \( M_v \), w.h.p.

**Theorem 22 (Tiered Oracles Simulation).** Given a graph \( G = (V,E) \), for every constant \( 0 < x < 1 \), there is an algorithm which simulates one round of the Tiered Oracles model, on a skeleton graph \( S_v = (M,E_S) \), in \( \tilde{O}(n^{1-x} + n^{2x-1}) \) rounds of the Hybrid model on \( G \), w.h.p.

**Proof.** We prove the claim by showing that we can simulate the Tiered Oracles model in the Congested Clique and LOCAL models, and then by invoking Lemmas 16 and 17 the round complexity in the Hybrid model is obtained. For each \( v \in V \), let \( M_v \) be the set of messages, of size \( |M_v| = \deg(v) \), which \( v \) desires to broadcast. We apply Lemma 21 with \( ALG_{2|M_v|/c} \) (described below) which for each \( v \in S_v \) scatters a set of \( c = 2 |M| \) messages to nodes sampled independently and uniformly from \( M \). Lemma 21 implies that there is an algorithm, which invokes \( ALG_{2|M|/c} \) \( \tilde{O}(1) \) times and uses one additional round of the LOCAL model, and ensures that for each \( u,v \in M \) such that \( \deg_S(u) \geq \frac{\deg(v)|M|}{2|M|} = \frac{1}{2} \deg_S(v) \), node \( u \) learns \( M_v \) w.h.p. This satisfies the definition of the Tiered Oracles model.

The implementation of \( ALG_{2|M|/c} \) in the Congested Clique model is straightforward. As stated in Lemma 21, \( ALG_{2|M|/c} \) needs to route from each node \( v \in M \) a set of \( S_v \) messages, where \( |S_v| = 2 |M| \). Thus, \( ALG_{2|M|/c} \) works as follows. Each node \( v \in M \) for each message in \( m \in S_v \) samples uniformly and independently the receiver \( u \in M \) for \( m \) and then uses...
the well known routing theorem of Lenzen [28, Theorem 3.7] (which allows delivering $O(n)$ messages to and from each node within $O(1)$ rounds of the Congested Clique model) to deliver the messages in $\tilde{O}(1)$ rounds of the Congested Clique model w.h.p.

In total, we used $\tilde{O}(1)$ rounds of the Congested Clique model and one round of the LOCAL model on $S_x$.

4 Shortest Paths Algorithms

4.1 Warm-Up: Exact SSSP

As a warm-up, we show how to compute exact SSSP in the Oracle model, and then we simulate this on a skeleton graph in the Hybrid model in order to get exact SSSP in the Hybrid model within $\tilde{O}(n^{1/3})$ rounds. We note that later, in Section 4.3, we obtain this complexity for exact distances from a much larger set, of $O(n^{1/3})$ sources.

\begin{lemma}[Exact SSSP in the Oracle Model]
There is a deterministic algorithm in the Oracle model that given a weighted graph $G = (V,E)$ and source $s \in V$ solves exact SSSP in $O(1)$ rounds.
\end{lemma}

\begin{proof}
Let $s \in V$ be the source node.

We solve the problem in two communication rounds. On the first round, oracle $\ell$ learns all of $E$ by receiving from each node $v$ its adjacent edges. Afterwards, oracle $\ell$, given all the edges in the graph $G$, locally computes the distance from $s$ to every other node. On the second round, oracle $\ell$ sends for each $v \in V$ the value $d(s,v)$. It is clear that the algorithm computes SSSP from $s \in V$, and that it takes two rounds in the Oracle model.
\end{proof}

\begin{theorem}[Exact SSSP]
Given a weighted graph $G = (V,E)$, there is an algorithm in the Hybrid model that computes an exact weighted SSSP in $\tilde{O}(n^{1/3})$ rounds w.h.p.
\end{theorem}

\begin{proof}
Let $s$ be the source node, and let $x = 2/3$. We start by constructing a skeleton graph $S_x = (M,E_S)$, by sampling nodes with probability $n^{-1/3}$ and using Claim 14 (Skeleton from Random Nodes). Then, we simulate the algorithm given in Lemma 23 in the Oracle model, which computes the distance $d_S(s,v)$ from $s$ to each node $v \in M$. By Property 5 of the skeleton graph, for every $v \in M$, it holds that $d_S(s,v) = d_G(s,v)$. To extend this and compute the distance from $s$ for each node $v \in V$, we apply Claim 15 (Extend Distances).

Constructing the skeleton graph takes $O(h) = \tilde{O}(n^{1/3})$ rounds w.h.p., by Claim 14 (Skeleton from Random Nodes). Simulating the algorithm from Lemma 23 completes in $O(n^{1/3})$ rounds w.h.p. by Theorem 19 (Oracle Simulation). Applying Claim 15 (Extend Distances) takes $O(n^{1/3})$ rounds. Therefore, overall, the execution of the algorithm completes in $O(n^{1/3})$ rounds w.h.p.
\end{proof}

4.2 Exact $n^k$-RSSP

Recall that in Definition 11 ($n^k$-Random Sources Shortest Path ($n^k$-RSSP)) we are given set of roughly $n^k$ sources sampled independently with probability $n^{k-1}$, and we need for each node to compute its distance to each source. We do so by constructing a skeleton graph $S_x$ from the random sources. We show that using one round of the Tiered Oracles model, and $O(\log n)$ rounds of the Congested Clique model, one can solve APSP over $S_x$. To do so, we
split the nodes of the graph into \([\log n]\) tiers by degree and compute APSP by proceeding
tier after tier and computing distances from current tier to all the tiers below.

**Lemma 24** (APSP in Congested Clique with Tiered Oracles). There is a deterministic
algorithm which, given a weighted graph \(G = (V,E)\), solves exact APSP on \(G\) using \(O(\log |V|)\)
rounds of the Congested Clique model and one round of the Tiered Oracles model.

**Proof Sketch (full proof appears in Appendix C.1).** The pseudocode for the algorithm appears in Algorithm 2. We partition the nodes \(V\) by their degrees into \([\log |V|]\) tiers, \(T_j = \{ v \in V : 2^j \leq \deg(v) < 2^{j+1} \}\) for \(0 \leq j < \log |V|\). Denote by \(T_{\geq i} = \bigcup_{k \geq i} T_k\) the nodes in all tiers \(k \geq i\) and by \(T_{< i} = \bigcup_{k < i} T_k\) the nodes in all tiers \(k < i\). Similarly, define \(T_{\geq i}\) and \(T_{< i}\). Denote by \(d_{\leq i}(u,v)\) the weight of the shortest path between \(u\) and \(v\) that uses
only edges adjacent to at least one node in \(T_{< i}\).

**Algorithm 2** Exact-APSP: Computes exact APSP using the Congested Clique and
Tiered Oracles models

1. Tiered Oracles model: each \(v \in T_i\) broadcasts to \(u \in T_{\geq i}\) its incident edges
2. for \(i = \lceil \log |V| \rceil - 1\) downto 0 do
3. For each node \(u \in T_{< i}\), each node \(v \in T_i\) computes
   \(\tilde{d}(v,u) \leftarrow \min \{ d_{\leq i}(v,u), \min_{w \in T_{\geq i}} \{ \tilde{d}(v,w) + d_{\leq i}(w,u) \} \}\)
4. Congested Clique model: \(v \in T_i\) sends to \(u \in T_{< i}\), the value \(\tilde{d}(v,u)\)

The outline of our algorithm is as follows. We start by having each node \(v \in T_i\) broadcast
its incident edges to all the nodes in tiers greater than or equal to its own, that is, to all
\(u \in T_{\geq i}\), using one round of the Tiered Oracles model (Line 1). Afterwards, in the loop
in Line 2 we compute the solution tier by tier, starting from the topmost tier, which contains
nodes knowing all the edges in the graph. While processing the \(i\)-th tier, every node \(v \in T_i\)
already knows its distance to every node in \(T_{\geq i}\), and so computes its distances to every node
\(u \in T_{< i}\). A shortest path between such \(v\) and \(u\) can either pass through edges which are
all known to \(v\), or be broken into a subpath from \(v\) to some node \(w \in T_{\geq j}\) and then a path
from \(w\) to \(u\) which is known to \(v\). Thus, we compute the distance from \(v \in T_i\) to the nodes
\(T_{< i}\) (Line 3). On Line 4 node \(v \in T_i\), which knows for each node \(u \in T_{< i}\) the distance to \(u\),
sends it to \(u\).

For each \(u,v \in V\), Algorithm 2 outputs a value \(\tilde{d}(u,v)\). We show that it is the correct
distance in \(G\), that is \(\tilde{d}(u,v) = d(u,v)\). Notice that for each pair \(u,v \in V\), the value \(\tilde{d}(u,v)\)
is set only once in the algorithm, and so once we show that it is set to the correct value
\(d(u,v)\), we do not worry that this will later change.

One round of the Tiered Oracles model suffices for ensuring that for each tier, \(T_i\), every
node \(v \in T_i\) knows all the edges incident to all the nodes \(u \in T_{< i}\). Let \(v \in T_i\), and \(u \in T_j\)
such that \(i \geq j\), observe that it holds that \(\deg(v) \geq 2^i \geq 2^j = \frac{1}{2} \deg(u)\), and therefore
after Line 1 node \(v\) knows the edges incident to \(u\). Thus, each node \(v \in T_i\) knows enough
information to compute the function \(d_{\leq i}\), which is the distance function in \(G\) limited to edges
incident to nodes in \(T_{< i}\).

By induction, which appears in the full proof in Appendix C.1 on tier index \(i\), we show
that after iteration \(i\) of the loop in Line 2, all the nodes in \(V\) know the exact distances to all
nodes in tiers \(T_{\geq i}\).

Lines 1 and 4 each take a single round of the Tiered Oracles model and the Congested Clique
model, respectively, and thus the execution of the entire algorithm takes \(O(\log |V|)\) rounds
of the Congested Clique model and one round of the Tiered Oracles model.
By simulating the algorithm given in Lemma 24 (APSP in Congested Clique with Tiered Oracles) using Theorem 22 (Tiered Oracles Simulation) and Lemma 17 (Congested Clique Simulation) we get exact APSP over the skeleton graph, as follows.

**Corollary 25 (Exact APSP on Skeleton Graph).** For any constant $0 < x < 1$, there is an algorithm in the Hybrid model that computes an exact weighted APSP on a skeleton graph $S_x = (M, E_S)$, in $\tilde{O}(n^{1-x} + n^{2x-1})$ rounds w.h.p.

Finally, we extend the result to $n^x$-RSSP on $G$, by having each node in the graph learn the information stored in the skeletons in its $O(n^{1-x})$ neighborhood.

**Theorem 3 (Exact Shortest Paths, Sources Sampled i.i.d.).** Given a graph $G = (V, E)$, $0 < x < 1$, and a set of nodes $M$ sampled independently with probability $n^{x-1}$, there is an algorithm in the Hybrid model that ensures that every $v \in V$ knows the exact, weighted distance from itself to every node in $M$ within $\tilde{O}(n^{1-x} + n^{2x-1})$ rounds w.h.p.

**Proof.** Given $M$, we use [Claim 14 (Skeleton from Random Nodes)] to build a skeleton graph $S_x = (M, E_S)$ in $\tilde{O}(n^{1-x})$ rounds w.h.p. Then, we compute exact APSP on the skeleton graph using [Corollary 25 (Exact APSP on Skeleton Graph)] in $\tilde{O}(n^{1-x} + n^{2x-1})$ rounds w.h.p. By Property 3 of the skeleton graph, for each $v, u \in M$ it holds that $d_S(v, u) = d(v, u)$, where $d_S(v, u)$ is the distance in the skeleton graph. So, we apply [Claim 15 (Extend Distances)] with $\alpha = 1$ and set of sources $V' = M$, to compute an exact weighted shortest paths distances, from $M$ to all of $V$, in additional $\tilde{O}(n^{1-x})$ rounds w.h.p.

Instantiating Theorem 3 with $x = 2/3$ gives $n^{2/3}$-RSSP in $\tilde{O}(n^{1/3})$ rounds w.h.p., which is tight due to our lower bound given in [Theorem 4 (Lower Bound Exact Shortest Paths, Sources Sampled i.i.d.)]. We extensively use our $n^{2/3}$-RSSP algorithm for our following results.

### 4.3 Exact $n^{1/3}$-SSP

We now present an improvement over the warm-up exact SSSP algorithm which we showed previously, by providing an algorithm for exact shortest paths from a given set of $n^{1/3}$ nodes ($n^{1/3}$-SSP) in $\tilde{O}(n^{1/3})$ rounds. To do so, we create a skeleton graph and use our algorithm for $n^{2/3}$-RSSP algorithm to compute exact distances from the skeleton nodes to the entire graph. Then, we adapt the behavior of the source nodes depending on the number of skeleton nodes in their neighborhood (which is proportional to the density of the neighborhoods). That is, nodes in sparse neighborhoods can broadcast the distances from themselves to all the skeleton nodes which they see surrounding them, while a node in dense neighborhoods can take over a skeleton node surrounding it and use it as a proxy to communicate efficiently with the other skeleton nodes in the graph. We formalize this in this section, as well as refer to [Lemma 29 (Reassign Skeletons)] which appears in Appendix C.2 and is a generic tool which performs this action of taking over skeleton nodes as proxies. The full proof of our following theorem appears in Appendix C.2 and here we provide its main ideas.

**Theorem 5 (Exact $n^{1/3}$ Sources Shortest Paths).** Given a weighted graph $G = (V, E)$, and a set of sources $U$, such that $|U| = O(n^{1/3})$, there exists an algorithm, at the end of which each $v \in V$ knows its distance from every $s \in U$, which runs in $\tilde{O}(n^{1/3})$ rounds w.h.p.
Proof Sketch. In order to solve exact $n^{1/3}$-SSP from some set of sources $U$ in $\tilde{O}(n^{1/3})$ rounds, we select a random set $M$ of $\tilde{O}(n^{2/3})$ skeleton nodes and use our $n^{2/3}$-RSSP algorithm from Theorem 3 to let every node in the graph know its distance to $M$.

Next, each node $u \in U$ observes its $\tilde{O}(n^{1/3})$-hop neighborhood. If it sees at most $\tilde{O}(n^{1/3})$ skeleton nodes, then it tells all the graph its distance from those skeleton nodes. To do so nodes participate in a broadcast protocol Claim 28, which w.h.p. broadcasts $\tilde{O}(n^{1/3} \cdot |U|) = \tilde{O}(n^{2/3})$ messages, where each node initially holds $\tilde{O}(n^{1/3})$ messages, in $\tilde{O}(n^{1/3})$ rounds.

We are left with the nodes in $U$ which have $\Omega(n^{1/3})$ skeleton nodes in their neighborhood. We show in Lemma 29 (Reassign Skeletons) that since $|U| = \tilde{O}(n^{1/3})$, each such source $u$ which has many skeletons in its neighborhood, can claim at least one skeleton in its neighborhood to help it. We simulate one round of the Congested Clique where $v$ tells all skeleton nodes in $M$ their distance from $u$ (which is known to $u$ since we computed sources from $M$ to the entire graph).

Finally, every node in $M$ knows its distance to every node in $U$, and so all the nodes in the graph look at the skeletons nodes in their neighborhood and deduce their own distances to the nodes of $U$.

Remark: We show the approximation for shortest path from $n^2$ given sources in Appendix C.3. In Appendices C.4 and C.5 we provide eccentricities and diameter approximations, respectively. We wrap up with our lower bound for shortest paths from sources sampled i.i.d. in Appendix D.

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A Preliminaries – Missing proofs

We slightly extend the claim used in the proof of [7, Theorem 2.7] – a basic claim regarding usage of skeleton graphs for purposes of distance computations in the Hybrid model. We include a proof for completeness.

Claim 15 (Extend Distances). [7, Theorem 2.7] Let $G = (V,E)$, let $S_x = (M,E_S)$ be a skeleton graph, and let $V' \subseteq V$ be the set of source nodes. If for each source node $s \in V'$, each skeleton node $v \in M$ knows the $(\alpha,\beta)$-approximate distance $\tilde{d}(v,s)$ such that $d(v,s) \leq \tilde{d}(v,s) \leq \alpha d(v,s) + \beta$, then each node $u \in V$ can compute for all source nodes $s \in V'$, a value $\hat{d}(u,s)$ such that $d(u,s) \leq \hat{d}(u,s) \leq \alpha d(u,s) + \beta$ in $\tilde{O}(n^{1-x})$ rounds.

Proof. First, each node $u \in V$ learns $\tilde{d}(v,s)$ for each source node $s \in V'$ and skeleton node $v \in M$ in its $h = \Theta(n^{1-x})$-hop neighborhood. Then, node $u \in V$ approximates its distance to each source $s \in V'$ using Equation (1), as follows.

$$\hat{d}(u,s) = \min \left\{ d^h(u,s), \min_{v \in M \cap N^h_G(s)} \left\{ d^h(u,v) + \tilde{d}(v,s) \right\} \right\} \quad (1)$$

For each node $u \in V$, skeleton node $v \in M \cap N^h_G$, and source $s \in V'$, $d^h(u,v) \geq d(u,v)$, it holds that $\tilde{d}(v,s) \geq d(v,s)$ and $d^h(u,v) \geq d(u,v)$, by triangle inequality $d(u,v) + d(v,s) \geq d(u,s)$, thus $\hat{d}(u,s) \geq d(u,s)$. To show that $\hat{d}(v,s) \leq \alpha \cdot d(v,s) + 2\beta$, we consider two cases. If there is a shortest path of at most $h$ hops between $s$ and $v$, then its length appears as $d^h(u,v)$ in the Equation (1). Otherwise, by the Property 6 of the definition of the skeleton graph, there is a shortest path from $u$ to $s$ on which there is a node $v' \in M$ in one of its $h$ first notes. This means that $d(u,v') + d(v',s) = d(u,s)$ and also $d^h(u,v') = d(u,v')$ (as each sub-path of shortest path is a shortest path) and so for this $v = v'$ the corresponding element $d^h(u,v') + \tilde{d}(v',s)$ appears as an option in Equation (1). This implies that $\hat{d}(u,s) \leq d^h(u,v') + \tilde{d}(v',s) \leq d(u,v') + \alpha \cdot d(v',s) + \beta = \alpha \cdot d(u,v') + d(v',s) + \beta \leq \alpha \cdot d(u,s) + \beta$. Thus, in both cases $d(u,s) \leq \alpha \cdot d(u,s) + \beta$, and therefore $\hat{d}$ is an $(\alpha,\beta)$-approximation for the shortest path.

Learning the information in the $h$-hop neighborhood requires $h = \tilde{O}(n^{1-x})$ rounds and the rest is done locally, so overall the algorithm runs in $\tilde{O}(n^{1-x})$ rounds of the Hybrid model. ▷
B Oracles in the Hybrid Model – Missing proof

We show the following tool which is model oblivious and used to prove Tiered Oracles simulation.

Lemma 21 (Degree-Based Simulation Tool). Consider a graph $G = (V, E)$, and a routing algorithm $ALG_c$, for some value $c$, in some model, which receives from every node $v$ any set of $c$ messages $S_v$, and delivers each message in $S_v$ to a node chosen uniformly and independently at random over $V$. Let there be for each node $v \in V$ a message set $M_v$, such that $|M_v| \leq c$. Then, it is possible within $O(\log n)$ invocations of $ALG_c$, followed by a single round of the LOCAL model over $G$, to ensure that for every pair of nodes $u, v \in V$, if $\deg(u) \geq |M_v|n/c$, then $u$ knows all of $M_v$, w.h.p.

Proof. We invoke algorithm $ALG_c$ for $O(\log n)$ times, where for each node $v$, we use $S_v$ as $\lceil c/|M_v| \rceil$ copies of $M_v$. Then, we execute one round of the LOCAL model over $G$, where every node learns all the messages that its neighbors received throughout the invocations of $ALG_c$.

We now prove that for every pair of nodes $u, v \in V$, if $\deg(u) \geq |M_v|n/c$, then $u$ knows all of $M_v$, w.h.p. Let $u, v \in V$, where $\deg(u) \geq |M_v|n/c$. We strive to show that after the above invocations of $ALG_c$, the nodes in $N(u)$ collectively know $M_u$ w.h.p. For a given message $m \in M_v$, since $|c/|M_v|| \geq c/(2|M_v|)$, on expectation, the nodes $N(u)$ receive $c|N(u)|/(2|M_v||n) \geq 1/2$ copies of $m$ within every invocation of $ALG_c$, and therefore $O(\log n)$ copies, in expectation, over all the invocations of $ALG_c$. Thus, since for each message, sent by $ALG_c$, the receiver is selected independently, due to a Chernoff concentration bound, and by applying the union bound over all the messages, the claim of the statement holds w.h.p.

C Shortest Path Algorithms – Extended

A useful communication routine which we use in this section is Claim 26 (Aggregate And Broadcast). It was proved for a weaker Node-Capacitated Clique model [6] Theorem 2.2], and thus directly holds in the Hybrid model.

Claim 26 (Aggregate And Broadcast). Let $S$ be some ground set and $f$ be a globally known function, which maps some multiset $S' \subseteq S$ to a value $f(S') \in S$. Assume that there exists a globally known function $g$, such that for any multiset $S'$ and any partition $S_1, \cdots, S_k$ of $S'$ holds $f(S) = g(f(S_1), \ldots, f(S_k))$. Assume also that each node $u$ has an input value $val(u) \in S$. Then there is an algorithm in the Hybrid model, which runs for $O(\log n)$ rounds, after which each node $u$ knows $f(v_1, \ldots, v_n)$.

For example, we use Claim 26 to sum numbers that each node has and make the result globally known.

C.1 Exact $n^2$-RSSP – Missing Proof

Lemma 24 (APSP in Congested Clique with Tiered Oracles). There is a deterministic algorithm which, given a weighted graph $G = (V, E)$, solves exact APSP on $G$ using $O(\log |V|)$ rounds of the Congested Clique model and one round of the Tiered Oracles model.

Almost entire proof already appears in Section 1.2.
Proof. The pseudocode for the algorithm appears in Algorithm 2. We partition the nodes \( V \) by their degrees into \( \lceil \log |V| \rceil \) tiers, \( T_j = \{ v \in V : 2^j \leq \deg(v) < 2^{j+1} \} \) for \( 0 \leq j < \lceil \log |V| \rceil \). Denote by \( T_{\leq i} = \bigcup_{\substack{0 \leq j \leq i}} T_j \) the nodes in all tiers \( k \leq i \) and by \( T_{<i} = \bigcup_{\substack{0 \leq j < i}} T_j \) the nodes in all tiers \( k \leq i \). Similarly, define \( T_{\leq i} \) and \( T_{<i} \). Denote by \( d_{\leq i}(u, v) \) the weight of the shortest path between \( u \) and \( v \) that uses only edges adjacent to at least one node in \( T_{\leq i} \).

The outline of our algorithm is as follows. We start by having each node \( v \in T_i \) broadcast its incident edges to all the nodes in tiers greater than or equal to its own, that is, to all \( u \in T_{>i} \), using one round of the Tiered Oracles model (Line 1). Afterwards, in the loop in Line 2 we compute the solution tier by tier, starting from the topmost tier, which contains nodes knowing all the edges in the graph. While processing the \( i \)-th tier, every node \( v \in T_i \) already knows its distance to every node in \( T_{>i} \), and so computes its distances to every node \( u \in T_{\leq i} \). A shortest path between such \( v \) and \( u \) can either pass through edges which are all known to \( v \), or be broken into a subpath from \( v \) to some node \( w \in T_{>i} \), and then a path from \( w \) to \( u \) which is known to \( v \). Thus, we compute the distance from \( v \) to the nodes \( T_{\leq i} \) (Line 3). On Line 3 node \( v \in T_i \), which knows for each node \( u \in T_{\leq i} \) the distance to \( u \), sends it to \( u \).

For each \( u, v \in V \), Algorithm 2 outputs a value \( \tilde{d}(u, v) \). We show that it is the correct distance in \( G \), that is \( \tilde{d}(u, v) = d(u, v) \). Notice that for each pair \( u, v \in V \), the value \( \tilde{d}(u, v) \) is set only once in the algorithm, and so once we show that it is set to the correct value \( d(u, v) \), we do not worry that this will later change.

One round of the Tiered Oracles model suffices for ensuring that for each tier, \( T_i \), every node \( v \in T_i \) knows all the edges incident to all the nodes \( u \in T_{\leq i} \). Let \( v \in T_i \), and \( u \in T_j \) such that \( i \geq j \), observe that it holds that \( \deg(v) \geq 2^i \geq \frac{1}{4} 2^j = \frac{1}{4} \deg(u) \), and therefore after Line 1 node \( v \) knows the edges incident to \( u \). Thus, each node \( v \in T_i \) knows enough information to compute the function \( d_{\leq i} \), which is the distance function in \( G \) limited to edges incident to nodes in \( T_{\leq i} \).

By induction on tier index \( i \), we show that after iteration \( i \) of the loop in Line 2 all the nodes in \( V \) know the exact distances to all nodes in tiers \( T_{\geq i} \).

Base case: In iteration \( i = \lfloor \log |V| \rfloor - 1 \), node \( v \in T_{\lfloor \log |V| \rfloor - 1} \) (if exists) in the topmost tier knows about all the edges in \( E \) since it knows about all edges incident to nodes \( T_{\leq \lfloor \log |V| \rfloor - 1} = V \). Thus, \( v \) can compute the solution to the entire APSP on \( G \), since \( d_{\leq \lfloor \log |V| \rfloor - 1} = d \). Since the set
\[
\{ d(v, w) + d_{\leq \lfloor \log |V| \rfloor - 1}(w, u) \}_{w \in T_{\geq \lfloor \log |V| \rfloor - 1}}
\]
is empty, we get that \( \tilde{d}(v, u) = d_{\leq \lfloor \log |V| \rfloor - 1}(v, u) = d(v, u) \). That is, node \( v \in T_{\lfloor \log |V| \rfloor - 1} \) computes for each other node \( u \in V \) its weighted distance \( d(v, u) \) and sends it to \( u \) on Line 2.

Induction Step: In iteration \( i < \lfloor \log |V| \rfloor - 1 \), consider \( v \in T_i \) and \( u \in T_{\leq i} \), and let \( P \) be a shortest path between them. Recall that node \( v \) can locally compute \( d_{\leq i} \), and thus knows the value \( d_{\leq i}(v, u) \) and for each \( w \in T_{>i} \), it knows the value \( d_{\leq i}(w, u) \). Further, for each \( w \in T_{>i} \), the value \( \tilde{d}(v, w) = d(v, w) \) is known to \( v \) from one of the previous iterations of the loop in Line 2 by the induction assumption. All values in the set \( \{ \tilde{d}(v, w) + d_{\leq i}(w, u) \}_{w \in T_{>i}} \cup \{ d_{\leq i}(v, u) \} \) are either infinite or correspond to some (not necessary simple) path from \( v \) to \( u \), thus \( \tilde{d}(v, u) \geq d(v, u) \). To show that \( \tilde{d}(v, u) \leq d(v, u) \), we consider two cases. If \( P \) does not contain nodes from \( T_{>i} \), then \( d_{\leq i}(v, u) = d(v, u) \) is the length of \( P \). Otherwise, let \( w' \in T_{>i} \) be the last node on \( P \) (closest to \( u \)) which belongs to \( T_{>i} \). By the induction hypothesis, \( v \) knows \( \tilde{d}(v, w') = d(v, w') \). Moreover, the subpath from \( w' \) to \( u \) only contains edges with at least one endpoint incident to node in \( T_{\leq i} \), thus \( d_{\leq i}(w', u) = d(w', u) \). For this node \( w' \) the value \( \{ \tilde{d}(v, w') + d_{\leq i}(w', u) \} \) belongs
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to \{ \hat{d}(v, w) + d_{\leq i}(w, u) \}_{w \in T_3}^i . Thus, in both cases the computed \( \hat{d}(v, u) \) is at most the weighted length of \( P \). Hence, \( \hat{d}(v, u) = d(v, u) \). On Line 4 node \( v \) informs \( u \) about the correct \( d(v, u) \), which completes the induction proof.

Lines 2 and 3 each take a single round of the Tiered Oracles model and the Congested Clique model, respectively, and thus the execution of the entire algorithm takes \( O(\log |V|) \) rounds of the Congested Clique model and one round of the Tiered Oracles model.

C.2 Exact \( n^{1/3} \)-SSP – Missing Proof

The first tool we need for our exact \( n^{1/3} \)-SSP algorithm is a token dissemination algorithm, given in \([7, \text{Theorem 2.1}]\).

Definition 27 (Token Dissemination Problem). The problem of making \( k \) distinct tokens globally known, where each token is initially known to one node, and each node initially knows at most \( \ell \) tokens is called the \((k, \ell)\)-Token Dissemination (TD) problem.

Claim 28 (Token Dissemination). \([26, \text{Theorem 2.2}]\) There is an algorithm that solves \((k, \ell)\)-TD in the Hybrid model in \( \tilde{O}(\sqrt{k} + \ell) \) rounds, w.h.p.

A second tool that we need is the following fundamental algorithm, which allows assigning skeletons to help other skeletons. That is, given a set of nodes \( A \) where each node in \( A \) sees many skeleton nodes in its neighborhood, it is possible to assign skeleton nodes to service the nodes of \( A \). We use this to increase sending and receiving capacity of the nodes of \( A \). This is a key tool which we use in the proof of Theorem 5 (Exact \( n^{1/3} \) Sources Shortest Paths) and we believe it may be useful for additional tasks.

Lemma 29 (Reassign Skeletons). Given graph \( G = (V, E) \), a skeleton graph \( S_x = (M, E_S) \), a value \( k \) which is known to all the nodes, and nodes \( A \subseteq V \) such that each \( u \in A \) has at least \( \tilde{O}(k \cdot |A|) \) nodes \( M_u \subseteq M \) in its \( \tilde{O}(n^{1-x}) \) neighborhood, there is an algorithm that assigns \( K_u \subseteq M_u \) nodes to \( u \), where \( |K_u| = \tilde{O}(k) \), such that each node in \( M \) is assigned to at most \( \tilde{O}(1) \) nodes in \( A \). With respect to the set \( A \), it is only required that every node in \( G \) must know whether or not it itself is in \( A \) – that is, the entire contents of \( A \) do not have to be globally known. The algorithm runs in \( \tilde{O}(n^{1-x}) \) rounds in the Hybrid model, w.h.p.

Proof. The pseudocode is provided by Algorithm 3

Algorithm 3 Reassign-Skeletons\((A, k)\)

1. Compute \(|A|\) by running Aggregate-And-Broadcast
2. Skeleton node \( v \in M \) learns its \( \tilde{O}(n^{1-x}) \)-hop neighborhood
3. Skeleton node \( v \in M \) samples each \( u \in A \cap N^S_G(n^{1-x})(v) \) with probability \( \frac{1}{|A|} \)
4. Skeleton node \( v \in M \) informs each sampled node \( u \) about \( v \in K_u \)

First, each node \( w \) learns the size of the set \( A \) by invoking Claim 26 (Aggregate And Broadcast) with value 1 if \( w \in A \) and 0 otherwise, and the summation function (Line 1). Then, each skeleton node \( v \in M \), learns its \( \tilde{O}(n^{1-x}) \)-hop neighborhood (Line 2), and in particular it learns the nodes \( A_v = A \cap N^S_G(n^{1-x})(v) \). Then, \( v \) samples each \( u \in A_v \) independently with probability \( \frac{1}{|A_v|} \) (Line 3). Afterwards, \( v \) informs each node \( u \) it sampled on the previous stage that \( v \in K_u \) (Line 4).
For every $v \in M$, since $|A_v| \leq |A|$, and since $v$ samples nodes from there $A_v$ independently with probability $\frac{1}{|A|}$, by Chernoff Bounds each $v$ assigns itself to at most $\tilde{O}(1)$ nodes $u \in A_v$ w.h.p. Hence, by a union bound over all skeleton nodes, each skeleton node is assigned to $\tilde{O}(1)$ nodes w.h.p.

For every $u \in A$, since it is sampled by at least $\tilde{O}(k \cdot |A|)$ skeleton nodes independently with probability $\frac{1}{|A|}$, by Chernoff Bounds it is sampled by $|K_u| = \Omega(1)$ skeleton nodes w.h.p. Thus, by union bound over all skeleton nodes, each $u \in A$ has $|K_u| = \Omega(1)$ assigned nodes w.h.p.

By Claim 26 (Aggregate And Broadcast) Line 1 takes $\tilde{O}(1)$ rounds w.h.p., and Lines 2 and 4 take $\tilde{O}(n^{1-x})$ rounds, and thus the entire execution completes in $\tilde{O}(n^{1-x})$ rounds w.h.p.

Now we apply Claim 28 (Token Dissemination) or Lemma 29 (Reassign Skeletons) depending on density of each source’s neighborhood and show how to compute exact $n^{1/3}$-SSP in $\tilde{O}(n^{1/3})$ rounds. For sources in “sparse” neighborhoods, in which there is a small number of skeleton nodes, we use Claim 28 (Token Dissemination) to inform all nodes about their distances to those skeletons. For source $v$ with “dense” neighborhood, in which there are many skeleton nodes, we use Lemma 29 (Reassign Skeletons) to get at least one skeleton node $u$ which participates in the round of the Congested Clique model on behalf of that source and sends each other skeleton node $v'$ the distance $d(v, v')$.

▶ Theorem 5 (Exact $n^{1/3}$ Sources Shortest Paths). Given a weighted graph $G = (V, E)$, and a set of sources $U$, such that $|U| = O(n^{1/3})$, there exists an algorithm, at the end of which each $v \in V$ knows its distance from every $s \in U$, which runs in $\tilde{O}(n^{1/3})$ rounds w.h.p.

Proof. The pseudocode for the algorithm appears in Algorithm 4.

\begin{algorithm}[h]
\caption{Exact-$n^{1/3}$-SSP: Computes an exact weighted $n^{1/3}$-SSP. Routine for node $u \in V$}
1 Join $M$ independently with probability $n^{-1/3}$
2 Compute $n^{2/3}$-RSSP from $M$
3 Construct skeleton graph $S_{2/3} = (M, E_S)$
4 Learn $h = \tilde{O}(n^{1/3})$-hop neighborhood
5 if $u \in U$ then
6   if $|N^h(u) \cap M| = \tilde{O}(n^{1/3})$ then
7     Participate in Token-Dissemination with a token $\langle u, v', d(v', u) \rangle$ for each $v' \in M \cap N^h(u)$
8   else
9     $K_u \leftarrow$ Reassign-Skeletons ($\{ u \mid N^h(u) \cap M = \tilde{O}(n^{1/3}) \}$, $\tilde{O}(1)$)
10    Send each $v \in K_u$ the values $d(v, v')$ for each $v' \in M$
11 if $u \in M$ then
12   In the Congested Clique model: for each $v' \in M$ and each $v \in U$ such that $u \in K_v$
13     send $d(v, v')$ to $v'$
14   For each $s \in U$, compute $\tilde{d}(u, s)$ by Equation (2) and output it
14 Apply Claim [15], which given distances from skeleton to sources $\tilde{d}$: $M \times U \mapsto \mathbb{N}$
15 extends it to distances from each nodes to sources $d$: $V \times U \mapsto \mathbb{N}$
\end{algorithm}
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Without loss of generality the set of nodes $U$ is globally known (it can be disseminated in $\tilde{O}(n^{1/3})$ rounds w.h.p. using Claim 28 (Token Dissemination)). We build $M \subseteq V$ by marking nodes independently with probability $n^{-1/3}$ (Line 1). Then we run the algorithm from Theorem 3 (Exact Shortest Paths, Sources Sampled i.i.d.) with $x = 2/3$ to obtain w.h.p. $n^{2/3}$-RSSP from the set of nodes $M$ (Line 2), such that w.h.p. every $u \in V$ knows its distance to every node in $M$. Afterwards, we apply Claim 14 (Skeleton from Random Nodes) to construct a skeleton graph $S_{2/3} = (M, E_2)$ w.h.p. Then, each source learns the information in its $h$-hop neighborhood (Line 4), for $h \in \tilde{O}(n^{1/3})$. In particular, it counts the skeleton nodes in its $h$-hop neighborhood.

If a source finds that the number of skeleton nodes in its $h$-hop neighborhood is $\tilde{O}(n^{1/3})$, then it participates in a token dissemination protocol (Claim 28 (Token Dissemination)) and w.h.p. informs all the graph about its distance to these skeleton nodes (Line 7) w.h.p. There node $v$ by Property 6 w.h.p. there exists a shortest path between $v$ to the set over which we take the minimum and thus $\tilde{d}(v, s)$ that some (not necessary simple) path, thus, since the graph has non-negative weights, it holds

Each element in $\{d^h(u, s)\} \cup \{d(u, v') + d^h(v', s)\}_{v' \in M}$ is either infinite, or corresponds to some (not necessary simple) path, thus, since the graph has non-negative weights, it holds that $\tilde{d}(u, s) \geq d(u, s)$. We show that $d(u, s) = \tilde{d}(u, s)$ by considering two cases. If shortest path between $u$ and $s$ has less than $h$ hops, then $d(u, s) = d^h(u, s)$ appears as an argument to the set over which we take the minimum and thus $\tilde{d}(u, s) \leq d^h(u, s) = d(u, s)$. Otherwise, by Property 6 w.h.p. there exists a shortest path between $u$ and $s$ such that there is a node $v'' \in M$ in one of its $h$ last (closest to $s$) nodes.

4 Note that difference from Equation (1). There node $u$ does not know precise distance to skeleton, but only $h$-limited distance, while here it knows the precise distance after Line 2. Similarly with the $d^h(v', s)$ term.
Thus, the corresponding element \(d(u,v') + d^0(v',s) = d(u,s)\) appears in \(\{d(u,v') + d^0(v',s)\}_{v' \in M}\) for \(v' = v''\). Therefore, in both cases \(\bar{d}(u,s) \leq d(u,s)\).

We continue the proof of Theorem 5. By Lemma 30 each node in \(M\) knows the distance to each node in \(U\), thus by Claim 15 (Extend Distances) with \(\alpha = 1, \beta = 0\) there is an algorithm to compute shortest paths distance from \(U\).

By Theorem 3 (Exact Shortest Paths, Sources Sampled i.i.d.) with \(x = \frac{2}{3}\), Line 2 completes in \(\tilde{O}(n^{1/3})\) rounds w.h.p. For Line 3 by Claim 14 (Skeleton from Random Nodes) the round complexity, w.h.p., is \(\tilde{O}(n^{1/3})\) as well. Line 4 completes in \(\tilde{O}(n^{1/3})\) rounds. Since there are at most \(\ell = \tilde{O}(n^{1/3})\) tokens per source and \(k = \Omega(n^{x/3})\) tokens overall, Line 7 takes \(\tilde{O}(n^{1/3})\) rounds w.h.p. by Claim 28 (Token Dissemination). By Lemma 29 (Reassign Skeletons) Line 9 takes \(\tilde{O}(n^{1/3})\) rounds w.h.p. All skeleton nodes are assigned to some helpers in their \(\tilde{O}(n^{1/3})\)-hop neighborhood by Line 9 so Line 14 takes \(\tilde{O}(n^{1/3})\) rounds. Since each skeleton selects \(\tilde{O}(1)\) sources w.h.p. in Line 9 by Lemma 29 (Reassign Skeletons) Line 12 simulates \(\tilde{O}(1)\) rounds of the Congested Clique model and takes \(\tilde{O}(n^{1/3})\) rounds by Lemma 17 (Congested Clique Simulation) w.h.p. Finally by Claim 15 (Extend Distances) Line 14 for \(x = \frac{2}{3}\) takes \(\tilde{O}(n^{1/3})\) rounds as well. Thus, the overall execution of the algorithm takes \(\tilde{O}(n^{1/3})\) rounds.

### C.3 Approximating \(n^{x/3}\)-SSP from a Given Set of Sources

We previously showed how to approximate distances to a set of independently, uniformly, randomly chosen sources, and here we leverage this to the case of a given set of sources, which is the case we care more about. We show an algorithm for approximating distances from a given set of sources, which is tight, matching the lower bound of [26], when the number of sources is at least \(\Omega(n^{2/3})\): given \(O(n^x)\) sources, our algorithm takes \(\tilde{O}(n^{1/3} + n^{x/2})\) rounds for a \((1 + \epsilon)\)-approximation in unweighted graphs, and a 3-approximation in weighted graphs.

The proof of the following, appears as a part of [26] Theorem 4.2, which shows how to obtain distance approximation algorithms in the Hybrid model from distance approximation algorithms in the Congested Clique model. A similar approach is also used in the proof of [7] Theorem 2.3. We find this tool useful in general in order to obtain, given some algorithm which approximates distances on a skeleton graph, an algorithm for approximating distances from a set of nodes to the entire graph, and hence we extract it here from the proof of [26] Theorem 4.2.

\[\triangleright \text{Claim 31 (}n^x\text{-SSP from APSP on Skeleton Graph).} \quad [26] \text{Theorem 4.2] Consider a graph } \quad G = (V,E), \text{ its skeleton graph } \quad S_x = (M,E_x), \text{ for some constant } \quad 0 < x < 1 \text{ and a set of sources } \quad U, \text{ where } |U| = \tilde{O}(n^y), \text{ for some constant } \quad 0 < y < 1. \text{ Let } A_x \text{ be a } T\text{-round algorithm in the Hybrid model, such that given a skeleton graph } S_x, \text{ A_x ensures that for every pair } \quad v, v' \in M, \text{ both } v \text{ and } v' \text{ know an } (\alpha, \beta)\text{-approximation for the distance between them in } G. \quad \text{Then, for any value } \eta \geq 1, \text{ there is an algorithm } B \text{ for the Hybrid model which takes } \tilde{O}(T + n^{\frac{x}{2}} + \eta \cdot n^{1-x}) \text{ rounds over } G, \text{ and ensures that for every } v \in V, s \in U, \text{ node } v \text{ knows an approximation for the distance from } v \text{ to } s \text{ in } G, \text{ where the approximation factor is } (2\alpha + 1, \beta) \text{ if } G \text{ is weighted, and } \left(\alpha + \frac{2}{\eta}, \beta\right) \text{ if } G \text{ is unweighted.}\]

Proof. We follow the proof of [26] Theorem 2.2].

We apply \(A_x\) on \(S\), and denote by \(d^S\) the computed \((\alpha, \beta)\)-approximation for the distances amongst the set of nodes \(M\).
Afterwards, nodes in $V$ learn their $(\eta \cdot h)$-hop neighborhoods, where $h = \tilde{O}(n^{1-x})$. As in [26, Theorem 4.2], each source $s \in U$ selects its representative — the closest node $r_s \in M$ found in its $h$-hop neighborhood — and participates in a token dissemination protocol with the token $(r_s, d^b(r_s, s))$. Each node $u \in V$, for each source $s \in U$, outputs $\tilde{d}(u, s) = \min \{ d^b(u, s), \min_{v' \in M \cap N_2^d(u)} \{ d^b(u, v') + \tilde{d}(v', r_s) \} \}$.

Each node $u \in V$ knows $\tilde{d}(v', r_s)$ since it is an output of the algorithm $A_x$ for nodes $v' \in M \cap N_2^d(u)$ and $u \in V$ learns this information from each $v'$ in its $h$-hop neighborhood. Node $u$ also knows $d^b(r_s, s)$ w.h.p., due to Claim 28 (Token Dissemination).

There are at most $O(n^9)$ source nodes w.h.p. due to Property 7, so the token dissemination protocol with $k = \tilde{O}(n^9)$ and $\ell = 1$ takes $\tilde{O}(\sqrt{n^9}) = \tilde{O}(n^{\tilde{2}})$ rounds w.h.p., by Claim 28 (Token Dissemination). Overall, the number of rounds is $\tilde{O}(T + n^{\tilde{2}} + \eta \cdot n^{1-x})$, as needed.

To show the approximation, first notice that the algorithm does not underestimate the distances, since each value in the set over which the minimum is taken is either infinite or corresponds to the approximate weight of a not necessarily simple path. This also implies that in case shortest path between $u$ and $s$ has less than $\eta h$ hops, it holds that $\tilde{d}(u, s) = d^b(u, s) = d(u, v)$, so we assume without loss of generality that shortest path between $u$ and $v$ at least $h$ hops and also $d(u, s) \geq \text{hop}(u, s) > \eta \cdot h$.

Take two nodes $u, s \in V$, $s \in U$, by Property 3 there is a shortest path between $u$ and $s$ such that there is a node $u' \in M$ in the first $h$ nodes (from the $u$-side) and a node $v'' \in M$ in the last $h$ nodes (from the $v$-side). Let $d' = \arg \min_{v' \in M} \{ d^b(u, v') + \tilde{d}(v', r_s) \} + d^b(r_s, s)$.

We upper bound $\tilde{d}$ for the weighted case:
\[
\tilde{d}(u, s) \leq (d^b(u, v') + \tilde{d}(v', r_s)) + d^b(r_s, s) \\
\leq (d(u, v') + d^d(v', r_s)) + d^b(r_s, s) \\
\leq (d(u, u') + \tilde{d}(v', s)) + \alpha \cdot d(u', v'') + \beta \\
\leq d(u, s) + \alpha \cdot (d(u', s) + d(s, r_s)) + \beta \\
\leq d(u, s) + \alpha \cdot (d(u, s) + d(v, v')) + \beta \\
\leq (2\alpha + 1) \cdot d(u, s) + \beta,
\]
where the first transition is due to the computation of $\tilde{d}(u, s)$, the second is implied by the choice of $v'$ and the representative $r_s$, the third follows from the definition of an $(\alpha, \beta)$-approximation of the distances and Property 3 of the skeleton graph, the forth is due to the non-negative weights and the triangle inequality, the fifth is due to the non-negative weights and the choice of the representative $r_s$, and the last one is due to the non-negative weights. Also, we may use the fact that $d(u, v) \geq \eta \cdot h$ and obtain a purely multiplicative approximation factor of $2\alpha + 1 + \frac{\beta}{\eta \cdot \pi}$.

However, for the unweighted case we can upper bound $\tilde{d}$ slightly better:
\[
\tilde{d}(u, s) \leq (d(u, v') + \tilde{d}(v', r_s)) + d^b(r_s, s) \\
\leq (d(u, u') + \tilde{d}(v', v')) + d^b(r_s, s) \\
= 2 \cdot h + \alpha \cdot d(u', v') + \beta \\
< 2 \cdot \frac{d(u, v)}{\eta} + \alpha \cdot d(u, v) + \beta,
\]
where the first transition is due to the computation of $\tilde{d}(u, s)$, the second is implied by the choice of $v'$, the third follows from $u'$ and $r_s$ being in $h$-hop neighborhood of $u$ and $s$, respectively, and the definition of a $(\alpha, \beta)$-approximation, and the forth is due to the assumption that $d(u, s) > \eta \cdot h$. Again, using the fact that $d(u, v) \geq \eta \cdot h$, we obtain a purely multiplicative approximation factor of $\alpha + \frac{2}{\eta} + \frac{\beta}{\eta \cdot \pi}$.
\[\]
We use the claim above to compute our tight approximation of the $n^{9/3}$-SSP.

**Theorem 6 (Approximate Multiple Source Shortest Paths).** Given a graph $G = (V, E)$, a set of sources $U$, where $|U| = \Theta(n^y)$ for some constant $0 < y < 1$, and a value $0 < \epsilon < 1$, there is an algorithm in the Hybrid model which ensures that every node $v \in V$ knows an approximation to its distance from every $s \in U$, where the approximation factor is $(1 + \epsilon)$ if $G$ is unweighted and $3$ if $G$ is weighted. The complexity of the algorithm is $\tilde{O}(n^{1/3}/\epsilon + n^{y/2})$ rounds, w.h.p.

**Proof.** We use $x = 2/3$ and let $A_{2/3}$ be the algorithm from [Corollary 25 (Exact APSP on Skeleton Graph)] on a skeleton graph and not a random set $M$, yet it is possible to convert $M$ to a skeleton graph using [Claim 14 (Skeleton from Random Nodes)] in $\tilde{O}(n^{1/3})$ rounds w.h.p. Further, [Corollary 25 (Exact APSP on Skeleton Graph)] computes the distances between the nodes of $M$ over a skeleton graph and not over $G$, as required of $A_{2/3}$ in [Claim 31 (n$^\alpha$-SSP from APSP on Skeleton Graph)] yet, this is equivalent due to Property 5 in the definition of a skeleton graph. Using $A_{2/3}$, the rest of the proof follows directly from [Claim 31 (n$^\alpha$-SSP from APSP on Skeleton Graph)] with $\alpha = 1, \beta = 0, \eta = 2/3, x = 2/3$, $T = \tilde{O}(n^{1/3})$ in $\tilde{O}(n^{1/3} + n^{y/2} + n^{y/3}/\epsilon) = \tilde{O}(n^{1/3}/\epsilon + n^{y/2})$ rounds leading to the approximation factor of $(2 \cdot 1 + 1 = 3, 0) = (3, 0)$ for weighted graphs and $(1 + 2/(2/\epsilon), 0) = (1 + \epsilon, 0)$.

Notice, that for $y \geq 2/3$, the complexity is $\tilde{O}(n^{2/3})$, which is tight due to the lower bound of [26, Theorem 1.5].

### C.4 Approximations of Eccentricities

**Lemma 32 ((1 + \epsilon)-Approx. Unweighted Eccentricities).** Let $G = (V, E)$ be an unweighted graph, and let $\epsilon > 0$. Then there exists an algorithm in the Hybrid model which, for each $v$, computes a $(1 + \epsilon)$-approximation of unweighted $ecc(v)$ in $\tilde{O}(n^{1/3}/\epsilon)$ rounds w.h.p.

**Proof.** We run the algorithm from [Theorem 3 (Exact Shortest Paths, Sources Sampled i.i.d.)] with $x = 2/3$ to obtain $n^{2/3}$-RSSP from a set of nodes $M$ selected independently randomly with probability $n^{-1/3}$ from $V$, such that every $v \in V$ knows its distance to every node in $M$. This takes $\tilde{O}(n^{1/3})$ rounds. Then, each node $v$ learns its $(1 + \frac{1}{x})$-hop neighborhood, where $h = \tilde{O}(n^{1/3})$, in $\tilde{O}(n^{1/3}/\epsilon)$ rounds. The approximate eccentricity $\tilde{ecc}(v)$ of node $v$ is the maximum between the distance to the farthest node in $M$ from $v$, and distance to the farthest node in the $(1 + \frac{1}{x})$-hop neighborhood of $v$.

We now prove the approximation factors. Notice, that there exists a node $w \in V$ such that $\tilde{ecc}(v) = d(v, w) \leq ecc(v)$, so we do not overestimate the eccentricity. If $ecc(v) \leq (1 + \frac{1}{x}) h$, the farthest node is in the explored neighborhood and the returned $\tilde{ecc}(v)$ is exact in this case. Let $u$ be farthest node from $v$. By Claim 14 and Property 6

there exists a node $w \in M$ on some shortest path between $v$ and $u$ within hop-distance at most $h$ from $u$ w.h.p. By definition, $\tilde{ecc}(v) \geq d(v, w) \geq ecc(v) - h \geq ecc(v) - \frac{ecc(v)}{1+\frac{1}{x}} = (1 - \frac{1}{1+\frac{1}{x}}) ecc(v) = (1 - \frac{x}{1+\frac{1}{x}}) ecc(v) = (\frac{1+x}{1+\frac{1}{x}} - 1) ecc(v) = \frac{1}{1+\frac{1}{x}} ecc(v)$. ▶

**Lemma 33 (3-Approx. Weighted Eccentricities).** Let $G = (V, E)$ be a weighted graph. There is an algorithm in the Hybrid model that computes a 3-approximation of weighted eccentricities in $\tilde{O}(n^{1/3})$ rounds, w.h.p.

**Proof.** We run the algorithm from [Theorem 3 (Exact Shortest Paths, Sources Sampled i.i.d.)] with $x = 2/3$ to obtain $n^{2/3}$-RSSP from a set of nodes $M$ selected independently randomly


with probability $n^{-1/3}$ from $V$, such that every $u \in V$ knows its distance to every node in $M$. Then, each $v \in M$ learns its $h$-hop neighborhood, where $h = \Theta(n^{1/3})$, and we use token dissemination, Claim 28 (Token Dissemination) to ensure that all the nodes in the graph know $ecc_h(v) = \max_{w \in N^h_G(v)} \{d(v, w)\}$ for each $v \in M$. The approximate eccentricity $\tilde{ecc}(u)$ of node $u \in V$, is the value of Equation 3.

$$\tilde{ecc}(u) = \frac{1}{3} \max_{v \in M} \{d(u, v) + ecc_h(v)\} \quad (3)$$

To show the approximation factor for some $u \in V$, let $v' = \arg \max_{v \in M} \{d(u, v) + ecc_h(v)\}$ and $v'' \in N^h_G(v)$ be the farthest node from $v'$ in its $h$-hop neighborhood, that is $ecc_h(v') = d(v', v'')$. Further, let $p \in V$ be the farthest node from $u \in V$, in other words, $ecc(u) = d(u, p)$, and $p' \in M$ be some marked node in the $h$-hop neighborhood of $p$, which exists w.h.p. by Chernoff bound. Finally, let $p'' \in V$ be the farthest node from $p'$ in its $h$-hop neighborhood, that is, $ecc(p') = d(p', p'')$. Notice that $\tilde{ecc}(u) = d(u, v') + d(v', v'') \geq d(u, p') + d(p', p'') > d(u, p') + d(p', v'') \geq \frac{3}{4} d(u, p) = \frac{3}{4} ecc(u)$, where the first inequality is due to the choice of $v'$, the second inequality follows from the choice of $p''$ and the third is implied by triangle inequality. Also, we do not overestimate the eccentricity, since $\tilde{ecc}(u) = d(u, v') + d(v', v'') \leq d(u, v') + d(v', v'') \leq ecc(u)$, where the first inequality is due to triangle inequality and the second is due to the graph being undirected and due to the definition of eccentricity.

By Theorem 3 (Exact Shortest Paths, Sources Sampled i.i.d) the round complexity of the $n^{2/3}$-RSSP is $\tilde{O}(n^{1/3})$ w.h.p. and token dissemination with a total of $k = |M| = \tilde{O}(n^{2/3})$ tokens w.h.p. (by Chernoff bound) and $\ell = 1$ tokens per node takes $\tilde{O}(n^{1/3})$ rounds w.h.p. due to Claim 28 (Token Dissemination).

Combining Lemma 32 ((1 + $\epsilon$)-Approx. Unweighted Eccentricities) and Lemma 33 (3-Approx. Weighted Eccentricities) gives Theorem 7 (Approx. Eccentricities).

### C.5 Diameter Approximations

**Corollary 8 ((1 + $\epsilon$)-Approx. Unweighted Diameter).** Let $G = (V, E)$ be an unweighted graph, and let $\epsilon > 0$. There exists an algorithm in the Hybrid model which computes a $(1 + \epsilon)$-approximation of the diameter in $\tilde{O}(n^{1/3} / \epsilon)$ rounds, w.h.p.

**Proof.** To achieve this result we first use the algorithm from Lemma 32 ((1 + $\epsilon$)-Approx Unweighted Eccentricities) to compute a $(1 + \epsilon)$-approximate eccentricities, for each $v \in V$ a value $\tilde{ecc}(v)$, in $\tilde{O}(n^{1/3} / \epsilon)$ rounds, and then use Claim 26 (Aggregate And Broadcast) to compute the maximum between the approximate eccentricities $ecc(v)$ in an additional number of $\tilde{O}(1)$ rounds. Notice that the maximum of $(1 + \epsilon)$-approximate eccentricities is indeed a good approximation of the diameter, as

$$D = \max \{ ecc(v) \} \geq \max \left\{ \frac{ecc(v)}{1 + \epsilon} \right\} = \frac{\max \{ ecc(v) \}}{1 + \epsilon} = \frac{D}{1 + \epsilon}.$$

On the other hand, since the approximate eccentricities $\tilde{ecc}(v)$ do not overestimate the real eccentricities, their maximum does not overestimate the true diameter. Thus, $\hat{D} = \max \{ \tilde{ecc}(v) \} \leq \max \{ ecc(v) \} = D$, which completes the proof.
Our oracle-based techniques allow us to solve weighted SSSP fast. After we do it, the following well-known simple reduction allows us to compute 2 approximate weighted diameter.

\[\alpha\leq 2.\]  

\[\begin{align*}
\alpha & \leq 2.
\end{align*}\]

**Claim 34 (Diameter From SSSP).** Given a graph \(G = (V, E)\), a value \(\alpha > 0\) and an algorithm which computes an \(\alpha\) approximation of weighted SSSP in \(T\) rounds of the Hybrid model, there is an algorithm which computes a 2\(\alpha\)-approximation of the weighted diameter in \(T + \tilde{O}(1)\) rounds of the Hybrid model.

**Proof.** We compute SSSP from an arbitrary node \(s\). Then, each node \(v\) proposes its candidate for the diameter \(\frac{1}{\alpha}\tilde{d}(v, s)\), which it computes after SSSP. The approximation is the maximum of all the proposals \(\tilde{D} = \max_{v \in V} \frac{1}{\alpha}\tilde{d}(v, s)\), and we computed using Claim 26 (Aggregate And Broadcast).

Since for each \(v \in V\), it holds that \(\tilde{d}(v, s) \leq \alpha d(v, s)\) is a shortest path between \(v\) and \(s\), \(D \geq \tilde{D} = \max_{v \in V} \frac{1}{\alpha}d(v, s)\). Let \(u\) and \(v\) be the endpoints of some path whose length is the diameter (i.e., \(\tilde{D} = d(u, v)\)), by the triangle inequality, \(d(u, s) + d(s, v) \geq d(u, v)\). Thus, at least one of the terms is greater than \(\frac{d(u, v)}{2}\), assume without loss of generality that \(d(u, s) \geq \frac{d(u, v)}{2}\). Since \(\tilde{d}(u, s) \geq d(u, s)\) and the proposal of \(u\) is at least \(\frac{1}{\alpha}\tilde{d}(u, s)\), it holds that \(\tilde{D} \geq d(u, s) \geq \frac{d(u, v)}{2} = \frac{D}{2\alpha}\).

The round complexity for computing the SSSP approximation is \(T\), and the round complexity for the aggregation operation is \(\tilde{O}(1)\) by Claim 26 (Aggregate And Broadcast).

**Corollary 2 (2-Approx. Weighted Diameter).** There is an algorithm in the Hybrid model that computes a 2-approximation of weighted diameter in \(\tilde{O}(n^{1/3})\) rounds w.h.p.

**Proof.** The claim follows immediately from Theorem 1 (Exact SSSP) and Claim 34 (Diameter From SSSP) with \(\alpha = 1\).

**D. A Lower Bound for \(n^2\)-RSSP**

We use the following Claim 35 (Lower Bound Random Variable) to obtain our lower bound.

\[\begin{align*}
\alpha & \leq 2.\]  

**Claim 35 (Lower Bound Random Variable).** [7, Lemma 4.12] Let \(G = (V, E)\) be an \(n\)-node graph that consists of a subgraph \(G' = (V', E')\) and a path of length \(L\) (edges) from some node \(a \in V \setminus V'\) to \(b \in V'\) and that except for node \(b\) is vertex-disjoint from \(V'\). Assume further that the nodes in \(V'\) are collectively given the state of some random variable \(X\) and that node \(a\) needs to learn the state of \(X\). Every randomized algorithm that solves this problem in the Hybrid network model requires \(\Omega\left(\min\left\{ L, \frac{H(X)}{E} \frac{1}{\log n} \right\} \right)\) rounds, where \(H(X)\) denotes the Shannon entropy of \(X\).

**Theorem 4 (Lower Bound Exact Shortest Paths, Sources Sampled i.i.d.).** Let \(p = \Omega (\log n / n)\) and \(\alpha < \sqrt{n/p} \cdot \log(n)/2\). Any \(\alpha\)-approximate unweighted algorithm from random sources sampled independently with probability \(p\) in the Hybrid network model takes \(\Omega \left(\sqrt{n/p} / \log n \right)\) rounds w.h.p.

**Proof.** We use the same base construction as in [7, Theorem 2.5] and in [26, Theorem 1.5] with only slight modifications. However, for the sake of self-containment, we recall here the entire construction.

We construct an unweighted graph \(G = (V, E)\) which contains a path, with endpoints at nodes \(a\) and \(c\), and two fooling sets of nodes \(S_b, S_c\), both of size \(y = \left\lfloor \frac{1}{\alpha} \right\rfloor\). Every node in \(S_c\) has an edge to \(c\). At hop-distance \(L = \left\lfloor \sqrt{\frac{n}{\log n}} \right\rfloor \geq 1\) from \(a\) there is a node \(b\). Each node in \(S_b\)
has an edge to $b$. The segment of path between $b$ and $c$ contains all other nodes, thus the hop distance between $b$ and $c$ is $z = n - 2y - L - 1$, and the hop distance between $a$ and $c$ is $x = z + L = n - 2y - 1$. See Figure 2.}

We follow the scheme of [7], Theorem 2.5 and reserve $2y$ identifiers from the set $[2y]$ for nodes in $S = S_b \cup S_c$ and assign additional $n - 2y$ identifiers to other nodes $V \setminus S$ in a globally known manner. The nodes from $S = S_b \cup S_c$ are assigned identifiers randomly from the reserved set $[2y]$, as follows. Each identifier from $[y]$ (the first half of identifiers) is assigned with probability $1/2$ to the next currently unlabeled node in $S_b$ and with probability $1/2$ to the next currently unlabeled node in $S_c$. The other $y$ reserved identifiers are arbitrarily assigned to nodes left unlabeled in $S$.

Let $S'$ be the set of nodes which are sampled to be sources. Consider the set $S'' = [y] \cap S'$ of sources with identifiers in $[y]$. For $1 \leq i \leq y$ we define $X_i$ to be an indicator of whether identifier $i$ is one of the sampled source identifiers. Notice that in expectation there are $E \left( \sum_{i=1}^{y} X_{i}\right) = p \cdot y = p \left( \frac{\log n}{n} \right)$ sampled nodes with identifiers in $[y]$. Moreover, for a large enough $n$, since $p = \Omega \left( \frac{\log n}{n} \right)$, by Chernoff bounds, there exists a constant $\beta > 0$ such that w.h.p. $\sum_{i=1}^{y} X_i \geq \beta p \cdot n$, which means that there are at least $\beta \cdot p \cdot n = \Omega (p \cdot n)$ source nodes in $S''$. These may be in $S_b$ or $S_c$.

By definition, an $\alpha$-approximation of distances from sources $S'$ is a function $\text{hop}: (V, S') \rightarrow \mathbb{Z}$ which satisfies for every $v \in V, u \in S'$: $\text{hop}(v, u) \leq \text{hop}(v, u) \leq \alpha \text{hop}(v, u)$, where $\text{hop}(u, v)$ denotes the hop distance between $u$ and $v$. In particular, the node $a \in V$ should compute the values $\text{hop}(a, \cdot)$, which satisfy for every $u \in S''$: $\text{hop}(a, u) \leq \text{hop}(a, u) \leq \alpha \text{hop}(a, u)$. If $u \in S_c$ then $\text{hop}(a, u) = x + 1$, otherwise, if $u \in S_b$ then $\text{hop}(a, u) = L + 1$. Thus, if $a$ does not know whether $u \in S_b$ or $u \in S_c$, it has to be on the safe side and return $\delta (a, u) = x + 1 \geq L + 1$.

The approximation factor for $u \in S_b$ is

$$
\alpha = \frac{x + 1}{L + 1} = \frac{n - 2y}{L + 1} = \frac{n - 2 \left\lfloor \frac{n}{2} \right\rfloor}{\log n} \geq \frac{1}{2} \sqrt{\frac{n}{p} \log n}.
$$

Given $S'$, let $Y \in \{b, c\}^{\lvert S''\rvert}$ be a random variable indicating for each sampled node $S''$ whether it belongs to $S_b$ or $S_c$. As in the proof of [7], Theorem 2.5, to get any better approximation, node $a$ has to learn for each node $s \in S''$ whether it is in $S_b$ or $S_c$. For this $a$ has to learn the random variable $Y$ whose entropy is w.h.p.

$$
H (Y) = \sum_{q \in \{b, c\}^{\lvert S''\rvert}} 2^{-q} \log (2^q) = \Omega (\lvert S''\rvert) = \Omega (p \cdot n).
$$

By Claim 35 \textit{(Lower Bound Random Variable)} $\Omega \left( \sqrt{\frac{n}{\log n}} \right)$ rounds are required for $a$ to learn the variable $Y$ w.h.p.