ON DYNAMIC CONTACT PROBLEM WITH GENERALIZED COULOMB FRICTION, NORMAL COMPLIANCE AND DAMAGE

LESZEK GASIŃSKI

Pedagogical University of Krakow, Department of Mathematics
ul. Podchorążych 2, 30-084 Kraków, Poland

PIOTR KALITA

Jagiellonian University, Faculty of Mathematics and Computer Science
ul. Łojasiewicza 6, 30-348 Kraków, Poland

Dedicated to Professor Meir Shillor on the occasion of his 70th birthday

Abstract. We formulate a dynamic problem which governs the displacement of a viscoelastic body which, on one hand, can come into frictional contact with a penetrable foundation, and, on the other hand, may undergo material damage. We formulate and prove the theorem on the existence and uniqueness of the weak solution to the formulated problem.

1. Introduction. This paper is a follow up of [15], [14], and [12]. We study an evolution of the displacement of a Kelvin–Voigt viscoelastic body which can come into contact with a foundation. We formulate an initial and boundary value problem which governs the unknown displacement \( u : \Omega \times [0, T] \to \mathbb{R}^d \), where \( d = 2, 3 \), \( \Omega \subset \mathbb{R}^d \) is the reference domain and \( T > 0 \) is the time during which the process takes place.

Additionally, we assume that the material from which the body is made, can be damaged, and the damage evolution is governed by the variational inequality, according to the model introduced by Frémond [10, 11]. According to this model, the damage is represented by a function \( \beta : \Omega \times [0, T] \to [0, 1] \), where the value 0 means that the body is fully damaged, and the value 1 means that it is not damaged at all. Such approach has been recently extensively studied for springs (see [2, 6]), beams (see [1, 18]) and for three-dimensional deformable solids (see [4, 12, 14, 19, 20, 25, 27]).

The contact conditions assumed here are the same as in [12]. The foundation is penetrable and the reaction force density depends on the penetration depth.
through so called normal compliance law. Moreover we consider here the generalized Coulomb friction law, where the friction force density depends on the slip rate through the multivalued relation given by the Clarke subdifferential of a certain locally Lipschitz and possibly nonconvex functional $j$. The friction law considered here is on one hand a generalization of Coulomb law of dry friction considered in [15], given by

$$\dot{\nu}(t) = 0 \Rightarrow \|\sigma(t)\|_{\mathbb{R}^d} \leq k|\sigma(\nu)|,$$

(1.1)

$$\dot{\nu}(t) \neq 0 \Rightarrow -\sigma(t) = k|\sigma(\nu)| \frac{\dot{\nu}(t)}{\|\dot{\nu}(t)\|_{\mathbb{R}^d}},$$

(1.2)

and on the other hand a generalization of a Tresca-type law considered in [14]

$$-\sigma(t) \in F(\dot{\nu}(t)).$$

(1.3)

Our friction law, considered already in [12], has the form

$$-\sigma(t) \in |\sigma(\nu)| F(\dot{\nu}(t)).$$

(1.4)

Such friction law is a generalization of both (1.1)–(1.2) and (1.3). First, similar as (1.1)–(1.2) it allows us to reflect the fact that the friction threshold, at which the passage from stick to slip phenomenon occurs, depends on the normal stress, in particular when there is no contact, i.e. when the stress is equal to zero, in contrast to (1.3), there is no friction in the model. Second, like (1.3), we can take nonmonotone $F$ and such choice leads to the nonmonotone dependence of the friction on the tangential velocity, which reflects the fact that kinetic friction is less than static friction, i.e. after the slip appears, there occurs the drop in friction value. For more information about the multivalued contact conditions and subdifferentials see the monographs [7, 8, 21, 22, 24, 26].

The novelty of this article with respect to [12] lies in the fact that while in [12] the model was assumed to be quasistatic, here we assume that it is dynamic, i.e., instead of the equilibrium equation

$$-\text{Div } \sigma(t) = f_0(t),$$

studied in [12], we consider the momentum equation

$$\ddot{u}(t) - \text{Div } \sigma(t) = f_0(t).$$

As it turns out, this, more general, model enables to remove the unnecessary restrictions of [12]: here we do not need neither presence on nontrivial clamped section of the domain boundary, nor the smallness assumption on the problem data (cf. assumption $H_0(f)$ in [12]).

The main result of this paper, Theorem 1, states the existence and uniqueness of the weak solution. Similar as in [14] and [12], the main ingredient of the proof is based on the introduction of auxiliary problems which are constructed such that the two unknowns, $u$ and $\beta$, are decoupled from each other, and application of fixed point argument of Banach type. However, we are able to simplify the proofs with respect to [12] and [14]: it turns out that in place of two fixed point arguments present in both above papers, it is sufficient to use only one fixed point argument.

The rest of the paper is structured as follows. The model is described in detail in Section 2. Its weak formulation as well as needed assumptions on the problem data are listed in Section 3. The main result of the paper, Theorem 1, which states the existence and uniqueness of the solution is proved in Section 4. Finally, in
Appendix, some mathematical notions and tools used throughout the article are recalled.

2. The model and assumptions. We study the evolution of the displacement of a viscoelastic body occupying a reference domain \( \Omega \subseteq \mathbb{R}^d \) (in applications \( d = 2, 3 \)) with a \( C^{0,1} \) boundary \( \partial \Omega \). The boundary of \( \Omega \) denoted by \( \partial \Omega \) is divided into three mutually disjoint and relatively open sets \( \Gamma_D, \Gamma_C, \) and \( \Gamma_N \) such that \( \Gamma_D \cup \Gamma_N \cup \Gamma_C = \partial \Omega \) and each of them may be possibly an empty set. The body is clamped on \( \Gamma_D \) and surface tractions of density \( f_N \) act on \( \Gamma_N \). The part of surface that can be in contact with the foundation is \( \Gamma_C \) and the gap given by \( g \) is measured along the outward normal. We assume that the volume forces of density \( f \) act in \( \Omega \). We denote by \( [0,T] \) the time interval of interest, with \( T > 0 \), and use the notation \( Q = \Omega \times (0,T) \).

On the contact boundary \( \Gamma_C \) we assume the normal compliance condition, where the normal stress depends on the normal displacement through the function \( p \) and, in tangential direction, a subdifferential Coulomb friction condition. The process is assumed to be dynamic and we use the Kelvin–Voigt viscoelastic constitutive law with damage effect. The function \( \beta : Q \to \mathbb{R} \) which attains values between 0 and 1 is used to describe the damage, where value 0 means that the material is fully destroyed and value 1 means that it is not destroyed at all. The damage function \( \beta \) is governed by a parabolic differential inclusion and satisfies a homogeneous Neumann boundary condition.

The model is given as follows.

**Problem P:** Find a displacement field \( u : Q \to \mathbb{R}^d \), a stress field \( \sigma : Q \to \mathbb{S}^d \) and a damage function \( \beta : Q \to \mathbb{R} \), such that for all \( t \in (0,T) \) we have

\[
\begin{align*}
\sigma(t) &= \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{G}(\varepsilon(u(t)), \beta(t)) \quad \text{in} \; \Omega, \\
\dot{u}(t) - \text{Div} \; \sigma(t) - f_0(t) &= 0 \quad \text{in} \; \Omega, \\
\dot{\beta}(t) - \kappa \Delta \beta(t) + \partial \psi_{[0,1]}(\beta(t)) &\ni \phi(t, \varepsilon(u(t)), \beta(t)) \quad \text{in} \; \Omega, \\
\frac{\partial \beta(t)}{\partial \nu} &= 0 \quad \text{on} \; \partial \Omega, \\
u(t) &= 0 \quad \text{on} \; \Gamma_D, \\
\sigma(t) \nu &= f_N(t) \quad \text{on} \; \Gamma_N, \\
- \sigma_{\nu}(t) &= p(u_{\nu}(t) - g) \quad \text{on} \; \Gamma_C, \\
- \sigma_{\tau}(t) &\in [\sigma_{\nu}(t)] \partial \nu(u_{\tau}(t)) \quad \text{on} \; \Gamma_C, \\
u(0) &= u_0, \quad \dot{u}(0) = u_1, \quad \beta(0) = \beta_0 \quad \text{in} \; \Omega.
\end{align*}
\]

Here, \( \mathbb{S}^d \) denotes the space of second-order symmetric \( d \times d \) matrices, \( \nu \) represents the unit outward normal on \( \partial \Omega \), \( \partial \beta/\partial \nu \) is the normal derivative on \( \partial \Omega \), \( \sigma_{\nu} \) and \( \sigma_{\tau} \) stand for the normal and tangential components of \( \sigma \nu \) on \( \Gamma_C \), respectively, and \( u_\nu \) and \( u_\tau \) are the normal and tangential components of displacement \( u \) and velocity \( \dot{u} \) on \( \Gamma_C \), respectively. The linearized strain tensor is given as the symmetric part of the displacement gradient \( \varepsilon(u) = \frac{1}{2} (\nabla u + \nabla u^T) \), and the divergence operator is given by

\[
\text{Div} \; \sigma = \left( \sum_{i=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} \right)_{i=1,...,d}.
\]
By $\partial \psi_{[0,1]}$ we denote the convex subdifferential of $\psi$, the indicator function of $[0,1]$, and by $\partial j$ we mean the Clarke subdifferential of $j$.

The viscoelastic constitutive law (2.1) depends on the velocity through the nonlinear viscosity operator $A: \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ which satisfies the following assumptions

$H(A)(a)$ $A(\cdot, \varepsilon)$ is measurable on $\Omega$ for all $\varepsilon \in \mathbb{S}^d$;

$H(A)(b)$ $A(x, \cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $x \in \Omega$;

$H(A)(c)$ there exist $a_0 \in L^2(\Omega)$, $a_0 \geq 0$ and $a_1 > 0$ such that

$$\|A(x, \varepsilon)\|_{\mathbb{S}^d} \leq a_0(x) + a_1 ||\varepsilon||_{\mathbb{S}^d}$$

for every $\varepsilon \in \mathbb{S}^d$ and a.e. $x \in \Omega$;

$H(A)(d)$ there exists $m_A > 0$ such that

$$(A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A ||\varepsilon_1 - \varepsilon_2||_{\mathbb{S}^d}^2$$

for every $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$;

$H(A)(e)$ $A(x, 0) = 0$ for a.e. $x \in \Omega$.

Dependence of constitutive law (2.1) on the displacement holds through the nonlinear elasticity operator $G: \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$, which also includes effect of material damage. It satisfies the assumptions

$H(G)(a)$ $G(\cdot, \varepsilon, \beta)$ is measurable on $\Omega$ for all $\varepsilon \in \mathbb{S}^d$, $\beta \in \mathbb{R}$;

$H(G)(b)$ there exists $L_G > 0$ such that

$$\|G(x, \varepsilon_1, \beta_1) - G(x, \varepsilon_2, \beta_2)\|_{\mathbb{S}^d} \leq L_G (||\varepsilon_1 - \varepsilon_2||_{\mathbb{S}^d} + |\beta_1 - \beta_2|)$$

for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, $\beta_1, \beta_2 \in \mathbb{R}$ and a.e. $x \in \Omega$;

$H(G)(c)$ $G(\cdot, 0, 0) \in L^2(\Omega, \mathbb{S}^d)$.

We note that as the main focus of the article is put on the conditions on $\Gamma_C$, the assumptions on $A$ and $G$ are rather strong and standard. Since we assume that the process is dynamic, we use the equation (2.2) to describe the evolution of the mechanical state of the body. One of the novelties of the present paper with respect to [12] is that we consider the extra term $\bar{u}(t)$ in (2.2). The damage process is described here exactly as in [12, 14], namely, the evolution of the damage function $\beta$ is described by the parabolic inclusion (2.3) with the damage source function $\phi: Q \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ and boundary condition (2.4). We make the following assumptions on the function $\phi$.

$H(\phi)(a)$ $\phi(\cdot, \varepsilon, \beta)$ is measurable on $Q$ for all $\varepsilon \in \mathbb{S}^d$, $\beta \in \mathbb{R}$;

$H(\phi)(b)$ there exists $L_\phi > 0$ such that

$$|\phi(x, t, \varepsilon_1, \beta_1) - \phi(x, t, \varepsilon_2, \beta_2)| \leq L_\phi (||\varepsilon_1 - \varepsilon_2||_{\mathbb{S}^d} + |\beta_1 - \beta_2|)$$

for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, $\beta_1, \beta_2 \in \mathbb{R}$ and a.e. $(x, t) \in Q$;

$H(\phi)(c)$ $\phi(x, \cdot, \cdot, \beta)$ is continuous on $[0, T]$ for all $\varepsilon \in \mathbb{S}^d$, $\beta \in \mathbb{R}$ and a.e. $x \in \Omega$;

$H(\phi)(d)$ $|\phi(x, t, 0, 0)| \leq \bar{\phi}(x)$ for all $t \in [0, T]$ and almost all $x \in \Omega$ where $\bar{\phi} \in L^2(\Omega)$.

Equations (2.5) and (2.6) represent the displacement and traction boundary conditions, respectively. Contact conditions here are the same as in [12]: the normal compliance condition (2.7), and the generalized Coulomb friction law (2.8). The normal compliance function $p: \Gamma_C \times \mathbb{R} \to [0, \infty)$ present in (2.7) is assumed to satisfy the following assumptions.

$H(p)(a)$ $p(\cdot, r)$ is measurable on $\Gamma_C$ for all $r \in \mathbb{R}$;

$H(p)(b)$ there exists $L_v > 0$ such that

$$|p(x, r_1) - p(x, r_2)| \leq L_v |r_1 - r_2|$$

for every $r_1, r_2 \in \mathbb{R}$ and a.e. $x \in \Gamma_C$;

$H(p)(c)$ $p(\cdot, r) = 0$ for $r \leq 0$ on $\Gamma_C$;

$H(p)(d)$ $p(\cdot, r) \leq c_p$ for all $r > 0$ on $\Gamma_C$ with a constant $c_p > 0$. 


The friction potential \( j: \Gamma_C \times \mathbb{R}^d \to \mathbb{R} \) present in (2.8) satisfies.

\( H(j)(a) \) \( j(\cdot, \xi) \) is measurable on \( \Gamma_C \) for all \( \xi \in \mathbb{R}^d \);

\( H(j)(b) \) \( j(x, \cdot) \) is Lipschitz on \( \mathbb{R}^d \) with a constant \( c_r > 0 \) for a.e. \( x \in \Gamma_C \);

\( H(j)(c) \) the one sided Lipschitz constant given by

\[
L(x, \partial j) = \inf_{\xi_1, \xi_2 \in \mathbb{R}^d, \xi_1 \neq \xi_2} \frac{(\eta_2 - \eta_1) \cdot (\xi_2 - \xi_1)}{\|\xi_2 - \xi_1\|^2_{\mathbb{R}^d}}
\]

satisfies \( L(x, \partial j) \geq -L_r \) on \( \Gamma_C \), for certain \( L_r \geq 0 \).

We remark that by \( \partial j(x, \xi) \) we mean the Clarke subdifferential of \( j \) taken with respect to the second variable \( \xi \). Note that \( H(j)(b) \) implies that

\[
\max_{\eta \in \partial j(x, \xi)} \|\eta\|_{\mathbb{R}^d} \leq c_r \text{ for all } \xi \in \mathbb{R}^d \text{ and for a.e. } x \in \Gamma_C.
\] (2.10)

Often, for the sake of the ease of notation we do not write explicitly the dependence of various quantities on the variable \( x \), i.e. in place of \( j(x, \xi) \) we write only \( j(\xi) \).

**Remark 1. (a)** Every convex function \( j: \mathbb{R}^d \to \mathbb{R} \) satisfies assumption \( H(j)(c) \) with \( L_r = 0 \).

**Remark 1. (b)** The condition \( H(j)(c) \) is equivalent to the fact that the functional \( j(x, \cdot) + L_r \frac{\|\cdot\|^2_{\mathbb{R}^d}}{2} \) is convex, or, equivalently, the multivalued map of \( \xi \to \partial j(x, \xi) + L_r \xi \) is monotone.

**Remark 1. (c)** In contrast to the quasistatic case (see Gasiński-Kalita [12]) no smallness assumption on the problem constants: \( c_p, L_r \) and \( m_A \) is needed.

Finally, the initial conditions for displacement, velocity, and damage are given by (2.9).

### 3. Notation and weak formulation

By \( V \) we denote the closed subspace of \( H^1(\Omega)^d \), given by

\[
V = \left\{ v \in H^1(\Omega)^d : v = 0 \text{ a.e. on } \Gamma_D \right\},
\]

where \( \Gamma_D \subset \partial \Omega \) is possibly empty. Since \( \partial \Omega \) is Lipschitz, the Korn inequality holds (see, e.g., [3])

\[
\|v\|^2_{H^1(\Omega)^d} + \|\varepsilon(v)\|^2_{L^2(\Omega;\mathbb{S}^d)} \geq c\|v\|^2_{H^1(\Omega)^d} \quad \text{for all } v \in V,
\]

(3.1)

where, here and below, \( c \) represents a positive constant, which may change from line to line and may depend on the data. We define the seminorm and norm on \( V \) by

\[
|v|_V = \|\varepsilon(v)\|_{L^2(\Omega;\mathbb{S}^d)}, \quad \|v\|^2_V = |v|^2_V + \|v\|^2_{L^2(\Omega)^d} \quad \text{for all } u \in V.
\]

(3.2)

It follows that \( \|\cdot\|_{H^1(\Omega)^d} \) and \( \|\cdot\|_V \) are equivalent norms on \( V \). Whenever we refer to measurability or measure on subsets of \( \partial \Omega \) we mean the \( d-1 \) dimensional Hausdorff measure which we denote by \( \text{meas}_{d-1} \). Moreover, for a measurable set \( \Gamma \subset \partial \Omega \) we denote by \( \gamma_{\Gamma}: V \to L^2(\Gamma)^d \) the trace operator, by \( \|\gamma_{\Gamma}\| \) its norm in \( L^2(\Gamma)^d \), and by \( \gamma_{\Gamma}^*: L^2(\Gamma)^d \to V^* \) the adjoint operator to \( \gamma_{\Gamma} \).

The indices \( i \) and \( j \) always run between 1 and \( d \) and the summation convention over repeated indices is used. In \( \mathbb{R}^d \) by \( \mathbf{u} \cdot \mathbf{v} = u_i v_i \), we denote the inner product and by \( \|v\|_{\mathbb{R}^d} = \sqrt{v \cdot v} \) the euclidean norm. On \( \mathbb{S}^d \) we use the inner product \( \sigma \cdot \tau = \sigma_{ij} \tau_{ij} \) and the associated norm \( \|\tau\|_{\mathbb{S}^d} = \sqrt{\tau \cdot \tau} \).
For an element \( v \in H^1(\Omega)^d \) we denote by \( v \) its trace on \( \partial \Omega \) and by \( v_\nu = v \cdot \nu \) and \( v_\tau = v - v_\nu \nu \) its normal and tangential components on the boundary.

The following Green formula holds for smooth functions \( \sigma : \Omega \to \mathbb{S}^d \) and \( v : \Omega \to \mathbb{R}^d \)

\[
(\sigma, \varepsilon(v))_{L^2(\Omega; \mathbb{S}^d)} + (\text{Div} \sigma, v)_{L^2(\Omega)^d} = \int_{\partial \Omega} \sigma \nu \cdot v \, dS. \tag{3.3}
\]

Finally, for a real Hilbert space \((X, \| \cdot \|_X)\) we use the standard notation for Bochner–Lebesgue space \( L^2(0, T; X) \), Bochner–Sobolev space \( H^1(0, T; X) \) and space of vector-valued continuous functions \( C([0, T]; X) \).

We now derive the variational formulation of Problem P. We consider the function \( f : (0, T) \to V^* \), given by

\[
\langle f(t), v \rangle_{V^* \times V} = (f_0(t), v)_{L^2(\Omega)^d} + (f_N(t), \gamma_{\Gamma_N} v)_{L^2(\Gamma_N)^d} \text{ for all } v \in V \text{ a.e. } t \in [0, T],
\]

and the set of admissible damage functions

\( \mathcal{X} = \{ \zeta \in H^1(\Omega) : 0 \leq \zeta \leq 1 \text{ a.e. in } \Omega \} \).

Assume that \((u, \sigma, \beta)\) are sufficiently smooth functions that solve \((2.1)-(2.9)\), \( v \in V, \zeta \in \mathcal{X} \) and \( t \in [0, T] \). First we use the equation of motion \((2.2)\) and the Green formula \((3.3)\) to obtain

\[
\langle \bar{u}(t), v \rangle_{V^* \times V} + (\sigma(t), \varepsilon(v))_{L^2(\Omega; \mathbb{S}^d)} = (f_0(t), v)_{L^2(\Omega)^d} + \int_{\partial \Omega} \sigma(t) \nu \cdot v \, dS. \tag{3.5}
\]

Taking into account the boundary conditions \((2.5)-(2.8)\), in a standard way, we obtain

\[
\int_{\partial \Omega} \sigma(t) \nu \cdot v \, dS = \int_{\Gamma_N} f_N(t) \cdot v \, dS
\]

\[
- \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu \, dS - \int_{\Gamma_C} p(u_\nu(t) - g)\xi(t) \cdot \nu \, dS, \tag{3.6}
\]

for \( \xi(t) \in S^2_{\partial j(\bar{u}^e(t))} \) (for a multifunction \( F : \Gamma_C \to 2^{\mathbb{R}^d} \) we use the symbol \( S^2_{\partial j(\cdot)} \) to denote all of its selections of class \( L^2(\Gamma_C)^d \)). Hence, using \((3.6)\) and \((3.4)\) in \((3.5)\), we have

\[
\langle \bar{u}(t), v \rangle_{V^* \times V} + (\sigma(t), \varepsilon(v))_{L^2(\Omega; \mathbb{S}^d)}
\]

\[
+ \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu \, dS + \int_{\Gamma_C} p(u_\nu(t) - g)\xi(t) \cdot \nu \, dS = \langle f(t), v \rangle_{V^* \times V}.
\]

Next, using the definition of the subdifferential of the indicator function \( \psi_{[0,1]} \) and integration by parts, we see that

\[
0 \geq (\phi(t, \varepsilon(u(t)), \beta(t)) - \beta(t), \zeta - \beta(t))_{L^2(\Omega)^d} - \kappa(\nabla \beta(t), \nabla \zeta - \nabla \beta(t))_{L^2(\Omega)^d}.
\]

Collecting these relations and inequalities leads to the following weak formulation of Problem P.

**Problem P**: Find \( \beta \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), u \in L^2(0, T; V) \) with \( u \in L^2(0, T; \mathbb{R}^d) \) and \( \bar{u} \in L^2(0, T; V^*) \), \( \sigma \in L^2(0, T; L^2(\Omega; \mathbb{S}^d)) \), and \( \xi \in \mathcal{X} \),
$L^2(0,T;L^2(\Gamma_C)^d)$ such that
\[
\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{G}(\varepsilon(u(t)), \beta(t)), \tag{3.7}
\]
\[
\langle \ddot{u}(t), v \rangle_{V^* \times V} + (\sigma(t), \varepsilon(v))_{L^2(\Omega;\mathbb{S}^d)} + \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu \, dS \\
+ \int_{\Gamma_C} p(u_\nu(t) - g)\xi(t) \cdot v_\nu \, dS = \langle f(t), v \rangle_{V^* \times V}
\text{ for all } v \in V \text{ and a.e. } t \in (0,T), \tag{3.8}
\]
\[
\xi(t) \in S^2_{\partial_j(\dot{u}_\nu(t))} \text{ for a.e. } t \in (0,T); \tag{3.9}
\]
\[
(\dot{\beta}(t), \zeta - \beta(t))_{L^2(\Omega)} + \kappa(\nabla \beta(t), \nabla \zeta - \nabla \beta(t))_{L^2(\Omega)^d} \\
\geq (\phi(t, \varepsilon(u(t)), \beta(t)), \zeta - \beta(t))_{L^2(\Omega)}
\text{ for all } \zeta \in \mathcal{K} \text{ and a.e. } t \in (0,T), \tag{3.10}
\]
\[
u(t) = u_0, \quad \dot{u}(0) = u_1, \quad \beta(0) = \beta_0. \tag{3.12}
\]

4. Existence and uniqueness results. The main existence result of this paper is the following.

**Theorem 1.** In addition to the hypotheses stated in Section 2, assume that $u_0 \in V$, $u_1 \in L^2(\Omega)^d$, $\beta_0 \in \mathcal{K}$, $f_0 \in L^2(0,T;L^2(\Omega)^d)$, $J_{N} \in L^2(0,T;L^2(\Gamma_N)^d)$, $g \in L^\infty(\Gamma_C)$, $g \geq 0$. Then Problem $\mathcal{P}_V$ admits a unique solution $(u, \sigma, \xi, \beta)$.

The proof of the above theorem is based on the fixed point theorem of Banach type (see Theorem 3). The proof also simplifies the method used in Gasiński-Ochal-Shillor [14] and Gasiński-Kalita [12] for quasistatic problems, namely, in contract to the above papers, it turns out that only one step of the fixed point procedure is sufficient in the proof. For this reason we formulate two auxiliary problems (for displacement and damage separately) and we prove the existence and uniqueness of their solutions. First, let us assume that the elastic part of the stress $\eta \in C([0,T];L^2(\Omega;\mathbb{S}^d))$, the friction term $z \in C([0,T];L^2(\Gamma_C))$ and the damage source function $\theta \in C([0,T];L^2(\Omega))$ are given, and consider the following two auxiliary problems, one for the displacement and another one, for the damage function.

**Problem $\mathcal{P}_{\eta}^\xi$:** Find $\sigma_{\eta} \in L^2(0,T;L^2(\Omega;\mathbb{S}^d))$ and $u_{\eta} \in L^2(0,T;V)$ with $\dot{u}_{\eta} \in L^2(0,T;V^*)$ such that
\[
\sigma_{\eta}(t) = \mathcal{A}(\varepsilon(\dot{u}_{\eta}(t))) + \eta(t), \tag{4.1}
\]
\[
\langle \ddot{u}_{\eta}(t), v \rangle_{V^* \times V} + (\sigma_{\eta}(t), \varepsilon(v))_{L^2(\Omega;\mathbb{S}^d)} + \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu \, dS \\
+ \int_{\Gamma_C} p(u_\nu(t) - g)\xi(t) \cdot v_\nu \, dS = \langle f(t), v \rangle_{V^* \times V}
\text{ for all } v \in V \text{ and a.e. } t \in (0,T), \tag{4.2}
\]
\[
\xi(t) \in S^2_{\partial_j(\dot{u}_\nu(t))} \text{ for a.e. } t \in (0,T), \tag{4.3}
\]
\[
u_{\eta}(0) = u_0, \quad \dot{u}_{\eta}(0) = u_1. \tag{4.4}
\]
Proposition 1. Under the hypotheses of Theorem 1, for every \( \eta \in C([0,T]; L^2(\Omega; S^d)) \) and \( z \in C([0,T]; L^2(\Gamma_C)) \), Problem \( P_{z\eta} \) admits a unique solution.

Proof. The existence of a solution to Problem \( P_{z\eta} \) follows from the abstract existence result of Theorem 4. For this purpose let \( U := L^2(\Gamma_C)^d \), \( H := L^2(\Omega)^d \), \( A: V \to V^* \) be given by (4.9), \( F: (0,T) \times U \to 2^{U^*} \) be defined by
\[
F(t,y) = \{ \kappa \in U : \kappa = p(z(t))\xi \text{ for some } \xi \in S^2_{\partial j}(y) \} \quad \text{for all } (t,y) \in (0,T) \times U.
\]
Let \( \iota: V \to L^2(\Gamma_C)^d \) be defined by \( \iota(v) = v_\tau \) on \( \Gamma_C \). This is a compact operator and its Nemytskii operator \( \iota: M^{2,2}(0,T; V; V^*) \to L^2(0,T; L^2(\Gamma_C)^d) \) is compact too (see the discussion in Section 8 of [16]). Now Problem \( P_{z\eta} \) can be equivalently formulated as:

Problem \( P_{z\eta} \): Find \( \eta \in C([0,T]; L^2(\Omega; S^d)) \) and \( z \in C([0,T]; L^2(\Gamma_C)) \) such that
\[
\dot{z}(t) + A(z(t)) + \iota^* F(t, \iota z(t)) \ni \bar{f}(t) \quad \text{for a.e. } t \in (0,T),
\]
\[
z(0) = u_1.
\]

To apply Theorem 4, we need to check hypotheses \( H(A), H(F), H_0 \) and \( H(U) \) (see the Appendix). It suffices to check \( H(A) \) and \( H(F) \). Verification of these hypotheses is technical, however, for the exposition completeness, we provide it here.

By hypothesis \( H(A)(c) \) we get that \( A \) is bounded and

\[
\|A(u)\|_{V^*} = \sup_{\|v\|_V = 1} \langle A(u), v \rangle_{V^* \times V} = \sup_{\|v\|_V = 1} \langle A(\varepsilon(u)), \varepsilon(v) \rangle_{L^2(\Omega; \mathbb{R}^d)} \\
\leq \sup_{\|v\|_V = 1} (\|a_0\|_{L^2(\Omega)} + a_1 \|u\|_V) \|v\|_V = \|a_0\|_{L^2(\Omega)} + a_1 \|u\|_V \quad \text{for} \quad u \in V.
\]

Next, by hypotheses \( H(A)(d) \) and \( (e) \), we get that the operator \( A \) is coercive in the sense that

\[
\langle A(u), u \rangle_{V^* \times V} \geq m_A \|u\|^2_V = m_A (\|u\|^2_V - \|u\|^2_H) \quad \text{for} \quad u \in V. \tag{4.15}
\]

Moreover, the operator \( A \) is monotone (by hypothesis \( H(A)(c) \)), and, finally, \( A \) is continuous (by a standard argument which uses \( H(A)(b), H(A)(c) \), and the Lebesgue dominated convergence theorem), i.e., if the sequence \( \{u_n\}_{n \geq 1} \subseteq V \) is such that \( u_n \to u \) in \( V \), then \( A(u_n) \to A(u) \) in \( V^* \). Since the operator \( A \) is bounded, monotone and hemicontinuous, it is also pseudomonotone (see [21, Theorem 3.69(i)]). So, hypotheses \( H(A) \) are satisfied.

Next, we check hypotheses \( H(F) \). For a fixed \( y \in U \), the set \( S^2_{\partial j}(y) \) is convex (as the values of the Clarke subdifferential are convex) and bounded, see (2.10). Moreover, from Theorem 5.6.39 of [8] it follows that this set is nonempty. For any \((t, y) \in (0, T) \times U \), the set \( F(t, y) \) is obviously nonempty and convex. We need to show that it is also closed. Let \( \{\kappa_n\}_{n \geq 1} \subseteq F(t, y) \) be a sequence such that \( \kappa_n \to \kappa \) in \( L^2(\Gamma_C)^d \). Then, for each \( n \geq 1 \), we can find \( \xi_n \in S^2_{\partial j}(y) \) such that \( \kappa_n = p(z(t))\xi_n \).

As the set \( S^2_{\partial j}(y) \) is bounded, passing to a subsequence if necessary, we may assume that \( \xi_n \to \xi \) weakly in \( U \) for some \( \xi \in U \). As \( p(z(t)) \in L^\infty(\Gamma_C) \) it holds

\[
(p(z(t))\xi_n, w)_V \to (p(z(t))\xi, w)_V \quad \text{for all} \quad w \in U,
\]

so \( \kappa = p(z(t))\xi \). We must prove that \( \xi \in S^2_{\partial j}(y) \), for which it suffices to show that \( \xi(x) \in \partial j(x, y(x)) \) for a.e. \( x \in \Gamma_C \). Using Proposition 4 and the bound (2.10) it follows that

\[
\xi(x) \in \text{conv} \left( \limsup_{n \to \infty} \{\xi_n(x)\} \right) \quad \text{for a.e.} \quad x \in \Gamma_C,
\]

where \( \limsup_{n \to \infty} \) is the Kuratowski-Painlevé upper limit of sets in \( \mathbb{R}^d \). Furthermore

\[
\text{conv} \left( \limsup_{n \to \infty} \{\xi_n(x)\} \right) \subset \text{conv} \left( \limsup_{n \to \infty} \partial j(x, y(x)) \right) = \partial j(x, y(x)),
\]

for a.e. \( x \in \Gamma_C \) and the assertion follows. The proof of closedness, and of \( H(F)(ii) \), is complete.

Fix \( y \in U \). For \( t \in (0, T) \), let us define \( N(t) = F(t, y) \). We need to check that the multifunction \( t \mapsto N(t) \) is measurable. For this purpose, as it has weakly compact, convex values, it is enough to show that for any \( h \in U \), the corresponding support function

\[
(0, T) \ni t \mapsto \sigma(h, N(t)) = \sup_{\kappa \in N(t)} \int_{\Gamma_C} h(x) \cdot \kappa(x) \, dx
\]
is measurable (see for example Theorem 3.6 in [21]). Because the set $S^2_{\partial j}(y)$ is separable and metrizable for the weak topology (see for example Theorem 3.6.17 in [8]), we can choose a weakly dense sequence $\{\xi_n\}_{n \geq 1} \subseteq S^2_{\partial j}(y)$. Then we have

$$
\sigma(h, N(t)) = \sup_{\kappa \in N(t)} \int_{\Gamma_C} h \cdot \kappa \, dS
$$

$$
= \sup_{\xi \in S^2_{\partial j}(y)} \int_{\Gamma_C} h \cdot (p(z(t))\xi) \, dS
$$

$$
= \sup_{n \geq 1} \int_{\Gamma_C} h \cdot (p(z(t))\xi_n) \, dS.
$$

But for each $n \geq 1$, the function $t \mapsto \int_{\Gamma_C} h(x) \cdot \xi_n(x) p(x, z(x, t)) \, dS$ is measurable. Thus, the function $t \mapsto \sigma(h, N(t))$ is measurable. This proves the measurability of $F(\cdot, y)$, so hypothesis $H(F)(i)$ holds.

Fix $t \in (0, T)$. We need to show that the multifunction $F(t, \cdot)$ is $(s-U) \times (w-U)$-upper semicontinuous. For this purpose, first observe that the graph of $F(t, \cdot): U \rightarrow 2^U$ is sequentially $(s-U) \times (w-U)$-closed. Indeed, let $\{y_n\}_{n \geq 1} \subseteq U$ be a sequence such that $y_n \rightarrow y$ in $U$, let $\kappa_n \in F(t, y_n)$ for $n \geq 1$ and assume that $\kappa_n \rightarrow \kappa$ weakly in $U$. From the definition of $F$, we see that for each $n \geq 1$ there exists $\xi_n \in S^2_{\partial j}(y_n)$ such that $\kappa_n = p(z(t))\xi_n$. Since, by (2.10), we have $\|\xi_n\|_{L^\infty(\Gamma_C)} \leq c_t$, it follows that $\{\xi_n\}$ is bounded also in $U$ and, for a subsequence, still denoted by the same notation,

$$
\xi_n \rightarrow \xi \text{ weakly in } U
$$

for certain $\xi \in U$. As $p(z(t)) \in L^\infty(\Gamma_C)$, we have $(\kappa_n, w)_U = (p(z(t))\xi_n, w)_U \rightarrow (p(z(t))\xi, w)_U = (\kappa, w)_U$. We must show that $\xi \in S^2_{\partial j}(y)$. To this end it suffices to prove that $\xi(x) \in \partial j(x, y(x))$ for a.e. $x \in \Gamma_C$. The bound (2.10) as well as Proposition 4 imply that

$$
\xi(x) \in \overline{\text{conv}}(\limsup_{n \rightarrow \infty} \{\xi_n(x)\}) \subset \overline{\text{conv}}(\limsup_{n \rightarrow \infty} \partial j(x, y_n(x)))
$$

for a.e. $x \in \Gamma_C$. Since, by Proposition 3, the multifunction $\partial j(x, \cdot)$ has closed graph, the pointwise convergence $y_n(x) \rightarrow y(x)$ of a subsequence implies that

$$
\overline{\text{conv}}(\limsup_{n \rightarrow \infty} \partial j(x, y_n(x))) \subset \overline{\text{conv}} \partial j(x, y(x)) = \partial j(x, y(x)) \text{ for a.e. } x \in \Gamma_C.
$$

This proves the $(s-U) \times (w-U)$-closedness of $\text{Gr} F(t, \cdot)$. Now, let $\{(y_n, \kappa_n)\}_n$ be a net in $\text{Gr} F(t, \cdot)$ such that $y_n \rightarrow y$ strongly in $U$. From this net we can choose a convergent (countable) subsequence $\{y_n\}$ such that $y_n \rightarrow y$ strongly in $U$. From the bound on $\partial j$, cf. (2.10), we see that the corresponding sequence $\{\kappa_n\}_{n \geq 1}$ is bounded, thus passing to a next subsequence if necessary, we may assume that $\kappa_n \rightarrow \kappa$ weakly in $U$ for some $\kappa \in U$. But we have shown that $\text{Gr} F(t, \cdot)$ is sequentially $(s-U) \times (w-U)$-closed. Thus $\kappa \in F(t, y)$. So, the net $\{\kappa_n\}_n$ has a $w-U$ cluster point in $F(t, y)$. Now using Proposition 5 we see that $F(t, \cdot)$ is upper semicontinuous in appropriate topologies. This proves hypothesis $H(F)(iii)$.

Finally, for $(t, y) \in (0, T) \times U$, we have

$$
\inf_{\kappa \in F(t, y)} (\kappa, y)_U = \inf_{\kappa \in F(t, y)} \int_{\Gamma_C} \kappa \cdot y \, dS
$$

$$
= \inf_{\xi \in S^2_{\partial j}(y)} \int_{\Gamma_C} p(z(t))\xi \cdot y \, dS
$$

...
\[ \geq - \int_{\Gamma_C} c_p c_\tau \| y \|_{\mathbb{R}^d} dS \geq -\varepsilon \| y \|_{L_1}^2 - c(\varepsilon) \]

(see hypothesis \( H(p)(d) \) and the bound (2.10)), for some constant \( c(\varepsilon) > 0 \), where \( \varepsilon > 0 \) is chosen small enough as it is needed in hypothesis \( H(F)(iv) \). Also, for \( \kappa \in F(t, y) \), we have

\[ \| \kappa \|_{L_1}^2 \leq \int_{\Gamma_C} |p(z(t))\xi|_{\mathbb{R}^d}^2 dS \leq c_p^2 c_\tau^2 (\text{meas}_{d-1}(\Gamma_C))^2, \]

where \( \xi \in S_{\partial\Omega}^2(y) \) is such that \( \kappa = p(z(t))\xi \). Thus all requirements of hypothesis \( H(F)(iv) \) are satisfied.

Now we can use Theorem 4 to conclude that Problem \( P_{z\eta} \) admits a solution \( w_{z\eta} \).

To show the uniqueness of the solution of Problem \( P_{z\eta} \), let \( w_{z\eta}^1 \) and \( w_{z\eta}^2 \) be two such solutions, with the corresponding selections \( \xi^1 \in S_{\partial\Omega}^2(w_{z\eta}^1) \), \( \xi^2 \in S_{\partial\Omega}^2(w_{z\eta}^2) \). Subtracting equation (4.10) for \( w_{z\eta}^1 \) and \( w_{z\eta}^2 \), we obtain

\[ \langle w_{z\eta}^1(t) - w_{z\eta}^2(t), v \rangle_{\mathcal{V}^* \mathcal{V}} + (A(\varepsilon(w_{z\eta}^1(t))) - A(\varepsilon(w_{z\eta}^2(t))), \varepsilon(v))_{L^2(\Omega; \mathbb{R}^d)} + \int_{\Gamma_C} p(z(t)) (\xi(t) - \xi(t)) \cdot v, dS = 0 \quad \text{for all } v \in \mathcal{V} \text{ and a.e. } t \in (0, T). \]

Taking as a test function \( v = w_{z\eta}^1(t) - w_{z\eta}^2(t) \) and using hypothesis \( H(A)(d) \), we get

\[ \frac{1}{2} \frac{d}{dt} \| w_{z\eta}^1(t) - w_{z\eta}^2(t) \|^2_{L_1} + \int_{\Omega} m_A \| \varepsilon(w_{z\eta}^1(t) - w_{z\eta}^2(t)) \|^2 dx + \int_{\Gamma_C} p(z(t)) (\xi(t) - \xi(t)) \cdot (w_{z\eta}^1(t) - w_{z\eta}^2(t)) dS \leq 0 \quad \text{for a.e. } t \in (0, T). \]

Next, using hypotheses \( H(p)(d) \) and \( H(j)(c) \), we have

\[ \frac{d}{dt} \| w_{z\eta}^1(t) - w_{z\eta}^2(t) \|^2_{L_1} + 2m_A \| w_{z\eta}^1(t) - w_{z\eta}^2(t) \|^2_{\mathcal{V}} - 2L c_p \| \gamma(w_{z\eta}^1(t) - w_{z\eta}^2(t)) \|^2_{(\Gamma_C; \mathbb{R}^d)} \leq 0 \quad \text{for a.e. } t \in (0, T). \]

We use the following trace inequality, which is a consequence of the fact that the trace operator is defined on \( H^{1-\delta}(\Omega)^d \), where \( \delta \in (0, 1/2) \), the embedding \( V \subset H^{1-\delta}(\Omega)^d \) is compact, and of the Ehrlich lemma (cf., e.g., Lemma 7.6 in [23])

\[ \| \gamma v \|^2_{(\Gamma_C; \mathbb{R}^d)} \leq \varepsilon |v|^2_{\mathcal{V}} + C(\varepsilon) \| v \|^2_{L_1} \quad \text{for } v \in \mathcal{V}, \quad (4.16) \]

valid for any \( \varepsilon > 0 \) and with \( C(\varepsilon) \) independent of \( v \), choosing \( \varepsilon = \frac{m_A}{L c_p} \), we obtain

\[ \frac{d}{dt} \| w_{z\eta}^1(t) - w_{z\eta}^2(t) \|^2_{L_1} \leq c \| w_{z\eta}^1(t) - w_{z\eta}^2(t) \|^2_{L_1} \quad \text{for a.e. } t \in (0, T), \]

with a constant \( c > 0 \). Since \( w_{z\eta}^1(0) = w_{z\eta}^2(0) = u_1 \), using the Gronwall inequality, we conclude that \( w_{z\eta}^1(t) = w_{z\eta}^2(t) \) for all \( t \in (0, T) \). Thus the solution of Problem \( P_{z\eta} \) is unique. \( \square \)

Now we recall the result on existence and uniqueness of solution to Problem \( P_\theta \) (with the fixed damage source function \( \theta \)).

**Proposition 2.** Under the hypotheses of Theorem 1, for every \( \theta \in C([0, T]; L^2(\Omega)) \) and \( \beta_0 \in \mathcal{K} \), Problem \( P_\theta \) admits a unique solution \( \beta_0 \).

**Proof.** The proof follows from standard results for parabolic variational inequalities (see e.g., Barbu [3, p. 124]). \( \square \)
It remains to prove Theorem 1.

Proof of Theorem 1. Let us denote
\[ \mathcal{Y} = C([0, T]; L^2(\Gamma_C) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega)) \]
and let us consider the operator \( \Lambda: \mathcal{Y} \to \mathcal{Y} \), given by
\[ \Lambda(z, \eta, \theta) = (\pi, \eta, \theta), \]
where \( \pi, \eta, \theta \) are defined as follows. For fixed \( (z, \eta, \theta) \in \mathcal{Y} \), let \( w_{z\eta} \) be the unique solution of Problem \( \mathbf{P}_{z\eta} \) and let \( \beta_0 \) be the unique solution of Problem \( \mathbf{P}_{\theta} \). Let
\[ u_{z\eta}(t) = u_0 + \int_0^t w_{z\eta}(s) \, ds \quad \text{for all} \quad t \in [0, T] \]
and
\[ \pi(t) = (u_{z\eta})_{\nu}(t) - g \quad \text{for all} \quad t \in [0, T]. \]
Then \( \pi \in C([0, T]; L^2(\Gamma_C)) \). Also, let
\[ \mathcal{H}(t) = \mathcal{G}(\mathbf{e}(u_{z\eta}(t)), \beta_0(t)) \quad \text{and} \quad \mathcal{H}(t) = \phi(t, \mathbf{e}(u_{z\eta}(t)), \beta_0(t)). \]
Using hypotheses \( H(\mathcal{G})(b) \) and \( (c) \) we have that \( \pi \in C([0, T]; L^2(\Omega; \mathbb{R}^d)) \) and from hypotheses \( H(\phi)(b) \) and \( (d) \) we have that \( \theta \in C([0, T]; L^2(\Omega)) \). Thus the operator \( \Lambda \) is well defined.

We show that operator \( \Lambda \) has a unique fixed point \( (z^*, \eta^*, \theta^*) \in \mathcal{Y}. \) To this end let \( (z_1, \eta_1, \theta_1), (z_2, \eta_2, \theta_2) \in \mathcal{Y}. \) We denote \( u_i = u_{z_i\eta_i}, w_i = u_{z_i\eta_i}, \beta_i = \beta_{\theta_i}, \)
\( \xi_i = \xi_{z_i\eta_i} \) for \( i = 1, 2 \) (where \( \xi_1, \xi_2 \) are the corresponding selections of \( S^2_{\beta_0((u_1)_{\nu}(t))} \) and \( S^2_{\beta_0((u_2)_{\nu}(t))} \), respectively as in Problem \( \mathbf{P}_{z\eta} \)). Using hypotheses \( H(\mathcal{G})(b) \) and \( H(\phi)(b) \) we deduce that for all \( t \in [0, T], \)
\[
\|\Lambda(z_1, \eta_1, \theta_1)(t) - \Lambda(z_2, \eta_2, \theta_2)(t)\|_{\mathcal{Y}} \\
= \|\mathcal{H}(u_1(t)) - \mathcal{H}(u_2(t))\|_{L^2(\Gamma_C)} + \|\mathcal{G}(\mathbf{e}(u_1(t)), \beta_1(t)) - \mathcal{G}(\mathbf{e}(u_2(t)), \beta_2(t))\|_{L^2(\Omega; \mathbb{R}^d)} \\
+ \|\phi(t, \mathbf{e}(u_1(t)), \beta_1(t)) - \phi(t, \mathbf{e}(u_2(t)), \beta_2(t))\|_{L^2(\Omega)} \\
\leq (\|\gamma\| + L_G + L_\phi)(|u_1(t) - u_2(t)|_V + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Omega)}). \tag{4.17}
\]
Since \( u(0) = u_0 = u_0, \) using (4.8), we get
\[
|u_1(t) - u_2(t)|_V \leq \int_0^t |w_1(s) - w_2(s)|_V \, ds \quad \text{for all} \quad t \in [0, T]. \tag{4.18}
\]
For a.e. \( s \in (0, t), \) subtracting (4.10) for \( w_1(s) \) and \( w_2(s), \) and taking the test function \( v = w_1(s) - w_2(s), \) we get
\[
(u_1(s) - u_2(s), w_1(s) - w_2(s))_{V^* \times V} \\
+ (A(\mathbf{e}(w_1(s))) - A(\mathbf{e}(w_2(s))), \mathbf{e}(w_1(s)) - \mathbf{e}(w_2(s)))_{L^2(\Omega; \mathbb{R}^d)} \\
= \int_{\Gamma_C} (p(z_2(s)) - p(z_1(s))) ((u_1)_{\nu}(s) - (u_2)_{\nu}(s)) \, dS \\
+ \int_{\Gamma_C} (p(z_2(s))\xi_2(s) - p(z_1(s))\xi_2(s)) \\
+ p(z_1(s))\xi_2(s) - p(z_1(s))\xi_1(s) \cdot ((u_1)_{\nu}(s) - (u_2)_{\nu}(s)) \, dS \\
+ (\eta_2 - \eta_1, \mathbf{e}(w_1(s)) - \mathbf{e}(w_2(s)))_{L^2(\Omega; \mathbb{R}^d)},
\]
so, by $H(A)(d)$ we get

$$
\frac{1}{2} \frac{d}{dt} \| w_1(s) - w_2(s) \|_{L^2(\Omega)^d}^2 + m_A \| w_1(s) - w_2(s) \|_{V}^2
\leq \int_{\Gamma_c} |p(z_1(s)) - p(z_2(s))| |(w_1)_\nu(s) - (w_2)_\nu(s)| dS \\
+ \int_{\Gamma_c} |p(z_1(s)) - p(z_2(s))| |(w_1)_{\tau}(s) - (w_2)_{\tau}(s)|_{H^1(\Gamma_c)} dS \\
+ \int_{\Gamma_c} p(z_1(s))(\xi_2(s) - \xi_1(s)) \cdot ((w_1)_{\tau}(s) - (w_2)_{\tau}(s)) dS \\
+ \| \eta_1(s) - \eta_2(s) \|_{L^2(\Gamma_c)^d} \| w_1(s) - w_2(s) \|_V.
$$

Using hypotheses, $H(p)(b)(d)$ and $H(j)(d)$ and the bound (2.10) we get

$$
\frac{1}{2} \frac{d}{dt} \| w_1(s) - w_2(s) \|_{L^2(\Omega)^d}^2 + m_A \| w_1(s) - w_2(s) \|_{V}^2
\leq L_p \| z_1(s) - z_2(s) \|_{L^2(\Gamma_c)} \| (w_1)_\nu(s) - (w_2)_\nu(s) \|_{L^2(\Gamma_c)} \\
+ L_{\nu} \| z_1(s) - z_2(s) \|_{L^2(\Gamma_c)} \| (w_1)_{\tau}(s) - (w_2)_{\tau}(s) \|_{H^1(\Gamma_c)} \\
+ L_{\tau} \| (w_1)_{\tau}(s) - (w_2)_{\tau}(s) \|_{L^2(\Gamma_c)}^2 \\
+ \| \eta_1(s) - \eta_2(s) \|_{L^2(\Omega)^d} \| w_1(s) - w_2(s) \|_V.
$$

By the Young inequality, and the trace inequality (4.16), for a fixed $\varepsilon > 0$ we can choose $c(\varepsilon) > 0$ such that

$$
\frac{1}{2} \frac{d}{dt} \| w_1(s) - w_2(s) \|_{L^2(\Omega)^d}^2 + m_A \| w_1(s) - w_2(s) \|_{V}^2
\leq \varepsilon \| w_1(s) - w_2(s) \|_{V}^2 + c(\varepsilon) \| w_1(s) - w_2(s) \|_{L^2(\Omega)^d}^2 \\
+ c(\varepsilon) \| \eta_1(s) - \eta_2(s) \|_{L^2(\Omega)^d}^2 + \| z_1(s) - z_2(s) \|_{L^2(\Gamma_c)}^2.
$$

Choosing $\varepsilon = \frac{m_A}{2}$, we get

$$
\frac{d}{dt} \| w_1(s) - w_2(s) \|_{L^2(\Omega)^d}^2 + m_A \| w_1(s) - w_2(s) \|_{V}^2
\leq c \| w_1(s) - w_2(s) \|_{L^2(\Omega)^d}^2 \\
+ c \| \eta_1(s) - \eta_2(s) \|_{L^2(\Omega)^d}^2 + \| z_1(s) - z_2(s) \|_{L^2(\Gamma_c)}^2) \quad \text{for a.e. } s \in (0, T).
$$

(4.19)

Using the Gronwall inequality, we get

$$
\| w_1(t) - w_2(t) \|_{L^2(\Omega)^d}^2
\leq c \int_0^t \left( \| \eta_1(s) - \eta_2(s) \|_{L^2(\Omega)^d}^2 + \| z_1(s) - z_2(s) \|_{L^2(\Gamma_c)}^2 \right) ds,
$$

(4.20)

for all $s \in [0, T]$. Integrating (4.19) over $(0, t)$ and using (4.20), we get

$$
\int_0^t \| w_1(s) - w_2(s) \|_{V}^2 ds
\leq c \int_0^t \left( \| \eta_1(s) - \eta_2(s) \|_{L^2(\Omega)^d}^2 + \| z_1(s) - z_2(s) \|_{L^2(\Gamma_c)}^2 \right) ds.
$$

(4.21)
From the above estimate and (4.18) we obtain
\[
\left| u_1(t) - u_2(t) \right|^2 \leq c \int_0^t \left( \| \eta_1(s) - \eta_2(s) \|^2_{L^2(\Omega;\mathbb{C}^d)} + \| z_1(s) - z_2(s) \|^2_{L^2(\Gamma_C)} \right) ds
\]
for all \( t \in [0,T] \). (4.22)

Writing the inequality (4.5) for \( \beta_1(s) \) with \( \zeta = \beta_2(s) \), then for \( \beta_2(s) \) with \( \zeta = \beta_1(s) \) and adding the resulting inequalities, we obtain
\[
(\beta_1(s) - \beta_2(s), \beta_1(s) - \beta_2(s))_{L^2(\Omega)} + \kappa(\nabla \beta_1(s) - \nabla \beta_2(s), \nabla \beta_1(s) - \nabla \beta_2(s))_{L^2(\Omega;\mathbb{R}^d)}
\leq (\theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s))_{L^2(\Omega)} \quad \text{for a.e.} \quad s \in (0,T).
\]

Integrating this inequality over \((0,t)\) for \( t \in [0,T] \) and using integration by parts, we get
\[
\frac{1}{2} \| \beta_1(t) - \beta_2(t) \|^2_{L^2(\Omega)} - \frac{1}{2} \| \beta_1(0) - \beta_2(0) \|^2_{L^2(\Omega)} + \kappa \int_0^t \| \nabla \beta_1(s) - \nabla \beta_2(s) \|^2_{L^2(\Omega;\mathbb{R}^d)} ds
\leq \int_0^t \| \theta_1(s) - \theta_2(s) \|_{L^2(\Omega)} \| \beta_1(s) - \beta_2(s) \|_{L^2(\Omega)} ds \quad \text{for all} \quad t \in [0,T].
\]

Since \( \beta_1(0) = \beta_2(0) = \beta_0 \), we get
\[
\| \beta_1(t) - \beta_2(t) \|^2_{L^2(\Omega)} \leq \int_0^t \| \theta_1(s) - \theta_2(s) \|^2_{L^2(\Omega)} ds + \int_0^t \| \beta_1(s) - \beta_2(s) \|^2_{L^2(\Omega;\mathbb{R}^d)} ds,
\]
for all \( t \in (0,T) \). Using the Gronwall inequality yields
\[
\| \beta_1(t) - \beta_2(t) \|^2_{L^2(\Omega)} \leq c \int_0^t \| \theta_1(s) - \theta_2(s) \|^2_{L^2(\Omega)} ds \quad \text{for all} \quad t \in [0,T]. \quad (4.23)
\]

Applying (4.22) and (4.23) in (4.17), we obtain
\[
\| \Lambda(z_1, \eta_1, \theta_1)(t) - \Lambda(z_2, \eta_2, \theta_2)(t) \|^2_{Y}
\leq c(\| u_1(t) - u_2(t) \|^2_{Y} + \| \beta_1(t) - \beta_2(t) \|^2_{L^2(\Omega)})
\leq c \int_0^t \left( \| z_1(s) - z_2(s) \|^2_{L^2(\Gamma_C)} + \| \eta_1(s) - \eta_2(s) \|^2_{L^2(\Omega;\mathbb{C}^d)} + \| \theta_1(s) - \theta_2(s) \|^2_{L^2(\Omega;\mathbb{R}^d)} \right) ds
\leq c \int_0^t \| (z_1, \eta_1, \theta_1)(s) - (z_2, \eta_2, \theta_2)(s) \|^2_Y ds.
\]

It follows from Theorem 3 that \( \Lambda \) has a unique fixed point \((z^*, \eta^*, \theta^*)\).

We now establish the existence of a solution to Problem \( P_V \). Let \((u_{z^*,\eta^*}, \sigma_{z^*,\eta^*}, \xi_{z^*,\eta^*})\) be the solution of Problem \( P_{z^*} \) for \( z = z^* \) and \( \eta = \eta^* \) (Proposition 1) and let \( \beta_{\theta} \) be the solution of Problem \( P_{\theta} \) for \( \theta = \theta^* \) (Proposition 2). From the definition of \( \Lambda \) we have that
\[
z^* = (u_{z^*,\eta^*}, \sigma_{z^*,\eta^*}, \xi_{z^*,\eta^*}) \quad \text{and} \quad \theta^* = \phi(\cdot, \epsilon(u_{\eta^*}, \beta_{\theta})),
\]
therefore,
\[
(u_{z^*,\eta^*}, \sigma_{z^*,\eta^*}, \xi_{z^*,\eta^*}, \beta_{\theta^*}) \quad \text{is a solution of Problem} \quad P_V,
\]
where \( \xi = \in L^2(0,T;L^2(\Gamma_C)) \) and \( \xi(t) \in \mathbb{S}^2_{(u_{z^*,\eta^*}, \tau(t))} \) for a.e. \( t \in (0,T) \). The uniqueness of the solution for problem \( P_V \) follows, exactly as in Theorem 5.1 in [14] from the uniqueness of fixed point of \( \Lambda \), the fact that if \((u, \sigma, \xi, \beta)\) is a solution of Problem \( P_V \), then \((z, \eta, \theta) = (u_{\eta} - g, G(\epsilon(u), \beta), \phi(\cdot, \epsilon(u), \beta))\) is a fixed point of \( \Lambda \) and the uniqueness of solutions to Problems \( P_{z\eta} \) (Proposition 1) and \( P_{\theta} \) (Proposition 2). The theorem is proved. \( \square \)
Appendix. Here we provide several definitions, notations, and results needed in the article. If $X$ is a reflexive Banach space, we denote by $X^*$ its topological dual and $\langle \cdot, \cdot \rangle_{X \times X}$ denotes the duality pairing between $X$ and $X^*$. A mapping $A: X \to X^*$ is called bounded if $A$ maps bounded sets of $X$ into bounded sets of $X^*$. It is called monotone if $\langle Au - Az, u - z \rangle_{X \times X} \geq 0$ for all $u, z \in X$. It is pseudomonotone if $u_n \to u$ weakly in $X$ and $\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle_{X \times X} \leq 0$ imply

$$\langle Au, u - v \rangle_{X \times X} \leq \liminf_{n \to \infty} \langle Au_n, u_n - v \rangle_{X \times X} \quad \text{for all} \quad v \in X.$$  

We recall some basic tools from convex analysis and nonsmooth analysis, cf., e.g., [7].

Let $\varphi : \mathbb{R} \to (-\infty, \infty]$ be a convex and lower semicontinuous function which is not identically $+\infty$. Then $\text{dom } \varphi = \{ x \in \mathbb{R} : \varphi(x) \neq +\infty \}$. The mapping $\partial \varphi : \mathbb{R} \to 2^\mathbb{R}$ defined by $\partial \varphi(x) = \emptyset$ for $x \notin \text{dom } \varphi$ and

$$\partial \varphi(x) = \{ z \in \mathbb{R} : z(v - x) \leq \varphi(v) - \varphi(x) \text{ for all } v \in \mathbb{R} \} \quad \text{otherwise}$$

is called the subdifferential of $\varphi$.

Next, we recall the Clarke subdifferential. This notion is valid for potentials which are not necessarily convex, however, they must be locally Lipschitz and hence they must attain only finite values. While the notion of the Clarke subdifferential is valid for any Banach space, we restrict to the case of $\mathbb{R}^d$, as the bigger generality is not needed in the article.

**Definition 2.** Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of $\varphi$ at $x \in \mathbb{R}^d$ in the direction $v \in \mathbb{R}^d$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \to x, t \to 0} \frac{\varphi(y + tv) - \varphi(y)}{t}$$

and the generalized gradient (subdifferential) of $\varphi$ at $x$, denoted by $\partial \varphi(x)$, is a subset of $\mathbb{R}^d$ given by

$$\partial \varphi(x) = \{ \zeta \in \mathbb{R}^d : \varphi^0(x; v) \geq \zeta \cdot v \text{ for all } v \in \mathbb{R}^d \}.$$

**Proposition 3.** If $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz function, then for every $x \in \mathbb{R}^d$ the generalized gradient $\partial \varphi(x)$ is a nonempty, convex, and compact subset of $\mathbb{R}^d$, and the graph of the generalized gradient $\partial \varphi$ is a closed set in $\mathbb{R}^d \times \mathbb{R}^d$.

The proof of the next fixed point theorem, which is used in the article, is similar to that presented in Migórski-Ochal-Sofonea [21, pp. 107–108].

**Theorem 3.** If $X$ is a Banach space and $\Lambda : C([0, T]; X) \to C([0, T]; X)$ is an operator for which there exists $c > 0$ such that

$$\| (Au)(t) - (Av)(t) \|_X^2 \leq c \int_0^t \| u(s) - v(s) \|_X^2 ds \text{ for all } u, v \in C([0, T]; X), t \in [0, T],$$

then $\Lambda$ has a unique fixed point in $C([0, T]; X)$.

We use the notion of Kuratowski–Painlevé upper limit of a sequence of sets $A_n \subset \mathbb{R}^d$, given by

$$\limsup_{n \to \infty} A_n = \{ x \in \mathbb{R}^d : x = \lim_{k \to \infty} x_{n_k}, x_{n_k} \in A_{n_k}, 0 < n_1 < n_2 < \ldots < n_k < \ldots \}.$$
We recall the following result on pointwise behavior of weakly convergence sequences (see Proposition 3.16 in [21])

**Proposition 4.** Let \((\Gamma, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, and let \(d \geq 1\) be a natural number. Let \(u_n \to u\) weakly in \(L^2(\Gamma)^d\) and \(u_n(x) \in G(x)\) for \(\mu\)-a.e. \(x \in \Gamma\) and all \(n \in \mathbb{N}\), where \(G(x)\) is a bounded set for \(\mu\)-a.e. \(x \in \Gamma\), then

\[
    u(x) \in \text{conv}\left(\limsup_{n \to \infty} \{u_n(x)\}\right) \quad \text{for } \mu\text{-a.e. } x \in \Gamma,
\]

where \(\limsup_{n \to \infty}\) stands for the Kuratowski–Painlevé upper limit of sets.

The next result concerns the upper semicontinuity of multifunctions (see Proposition 4.1.11 in [8])

**Proposition 5.** Let \(X\) and \(Y\) be Hausdorff topological vector spaces. A multifunction \(F: X \to 2^Y \setminus \emptyset\) is compact valued and upper semicontinuous if and only if for every net \(\{(x_\alpha, y_\alpha)\}_\alpha \subseteq \text{Gr} F\) with \(x_\alpha \to x\) in \(X\), the net \(\{y_\alpha\}_\alpha\) has a cluster point in \(F(x)\).

Next, following the paper of Kalita [17], we present the existence result for an abstract parabolic inclusion, which was used in the proof of Proposition 1. Let \(V \subset H \subset V^*\) be an evolution triple of spaces where all the embeddings are assumed to be continuous, dense and compact. The space \(V\) is assumed to be a separable and reflexive Banach space while the space \(H\) is a Hilbert space. Let \(U\) be any reflexive Banach space and let \(\iota\) be a linear, continuous and compact mapping \(\iota: V \to U\). By \(\iota^*: U^* \to V^*\) we denote the mapping adjoint to \(\iota\) defined as \(\langle \iota^* u, v \rangle_{V^* \times Y} = \langle u, \iota v \rangle_{U^* \times U}\). We write \(V = L^2(0,T;V), \ H = L^2(0,T;H), \ V^* = L^2(0,T;V^*)\) and \(U = L^2(0,T;U)\). The Nemytskii mappings for \(\iota\) and \(\iota^*\) will be denoted by the same symbols. Let \(u: [0,T] \to X\), where \(X\) is a Banach space. The 2-variation seminorm is defined as

\[
    \|u\|_{BV^2(0,T;X)}^2 = \sup_{\pi \in F} \sum_{(a,b) \in \pi} \|u(b) - u(a)\|^2_X,
\]

where the supremum is taken over all all partitions of \([0,T]\) into disjoint open time subintervals. We denote by \(BV^2(0,T;X)\) the set of all functions for which the 2-variation seminorm is finite. For Banach spaces \(X,Y\) such that \(X \subset Y\) we define the following Banach space (see [16])

\[
    M^{2,2}(I;X,Y) = L^2(I;X) \cap BV^2(I;Y),
\]

We consider the following abstract parabolic inclusion:

**Problem P\(A\) :** Find \(w \in V\) with \(\dot{w} \in V^*\) such that

\[
    \dot{w}(t) + A(w(t)) + \iota^* F(t,\iota w(t)) \ni f(t) \quad \text{for } a.e. \ t \in (0,T), \quad (4.24)
\]

\[
    w(0) = w_0, \quad (4.25)
\]

The problem data \(A, F, f\) and \(w_0\) are assumed to satisfy the following conditions:

**H(A) :** \(A: V \to V^*\) is the mapping such that

(i) \(A\) is pseudomonotone,

(ii) \(A\) is coercive, i.e., \(\langle A(v), v \rangle \geq \alpha \|v\|^2_V - \beta \|v\|^2_H\) for all \(v \in V\) with \(\alpha > 0\) and \(\beta \geq 0\),

(iii) \(A\) satisfies the growth condition \(\|A(v)\|_{V^*} \leq a(\|v\|_H)(1 + \|v\|_V)\) for all \(v \in V\) with \(a: [0,\infty) \to [0,\infty)\) nondecreasing.
Theorem 4. \( H(F) : F : (0, T) \times U \to 2^{U^*} \) is the multifunction such that
(i) \( F(\cdot, u) \) is measurable for all \( u \in U \),
(ii) the set \( F(t, u) \) is nonempty, closed and convex in \( U^* \) for all \( u \in U \) and a.e. \( t \in (0, T) \),
(iii) the mapping \( F(t, \cdot) \) is upper semicontinuous from the strong topology of \( U \) into weak topology of \( U^* \) for a.e. \( t \in (0, T) \),
(iv) for all \( u \in U \) we have \( \inf_{\xi F(t, u)} \langle \xi, u \rangle_{U^*, U} \geq g(t) - \lambda \|u\|_{L^2}^2 \), where \( 0 < \lambda < \frac{\alpha}{\|\xi\|_{L^2(V; U)}} \) and \( g \in L^1(0, T) \), and \( F \) satisfies the growth condition
\[ \|\xi\|_{U^*} \leq d_1 + d_2 \|u\|_U \] for all \( \xi \in F(t, u) \), all \( u \in U \) and a.e. \( t \in (0, T) \) with \( d_1 \geq 0 \) and \( d_2 \geq 0 \).

\( H_0 : f \in V^*, w_0 \in H \).

\( H(U) : \) the mapping \( \iota \) is linear, continuous and compact, and the Nemytskii mapping \( \iota : M^{2,2}(0, T; V; V^*) \to U \) is also compact.

Theorem 4. Under assumptions \( H(A), H(F), H(U) \) and \( H_0 \), Problem \( P_A \) admits a solution \( w \).

For the proof of the above theorem we refer to Kalita [17, Theorem 1].

REFERENCES

[1] K. T. Andrews and M. Shillor, Thermomechanical behaviour of a damageable beam in contact with two stops, Appl. Anal., 85 (2006), 845–865.
[2] K. T. Andrews, S. Anderson, R. S. R. Menike, M. Shillor, R. Swaminathan and J. Yuzwalk, Vibrations of a damageable string, Fluids and waves, Contemp. Math., Amer. Math. Soc., Providence, RI, 440 (2007), 1–13.
[3] V. Barbu, Optimal Control of Variational Inequalities, Research Notes in Mathematics, 100. Pitman, Boston, MA, 1984.
[4] E. Bonetti and G. Schimperna, Local existence for Frémond’s model of damage in elastic materials, Continuum Mech. Therm., 16 (2004), 319–335.
[5] P. G. Ciarlet, On Korn’s inequality, Chin. Ann. Math. Ser. B, 31 (2010), 607–618.
[6] J. C. Chipman, A. Roux, M. Shillor and M. Sofonea, A damageable spring, Machine Dyn. Res., 35 (2011), 82–96.
[7] F. H. Clarke, Optimization and Nonsmooth Analysis, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1983.
[8] Z. Denkowski, S. Migórski and N. S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic/Plenum Publishers, Boston, 2003.
[9] Z. Denkowski, S. Migórski and N. S. Papageorgiou, An Introduction to Nonlinear Analysis: Applications, Kluwer Academic Publishers, Boston, MA, 2003.
[10] M. Frémond, Non-Smooth Thermomechanics, Springer-Verlag, Berlin, 2002.
[11] M. Frémond, K. L. Kuttler and M. Shillor, Existence and uniqueness of solutions for a one-dimensional damage model, J. Math. Anal. Appl., 229 (1999), 271–294.
[12] L. Gasiński and P. Kalita, On quasi-static contact problem with generalized Coulomb friction, normal compliance and damage, Euro. J. Appl. Math., 27 (2016), 625–646.
[13] L. Gasiński and A. Ochal, Dynamic thermoviscoelastic problem with friction and damage, Nonlinear Anal. Real World Appl., 21 (2015), 63–75.
[14] L. Gasiński, A. Ochal and M. Shillor, Variational-hemivariational approach to a quasistatic viscoelastic problem with normal compliance, friction and material damage, Z. Anal. Anwend., 34 (2015), 251–275.
[15] W. M. Han, M. Shillor and M. Sofonea, Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage, J. Comput. Appl. Math., 137 (2001), 377–398.
[16] P. Kalita, Convergence of Rothe scheme for hemivariational inequalities of parabolic type, Int. J. Numer. Anal. Mod., 10 (2013), 445–465.
[17] P. Kalita, Semidiscrete variable time-step \( \theta \)-scheme for nonmonotone evolution inclusion, arXiv:1402.3721v1.
[18] K. L. Kuttler, J. Purcell and M. Shillor, Analysis and simulations of a contact problem for a nonlinear dynamic beam with a crack, Q. J. Mech. Appl. Math., 65 (2012), 1–25.

[19] K. L. Kuttler and M. Shillor, Quasistatic evolution of damage in an elastic body, Nonlinear Anal. Real World Appl., 7 (2006), 674–699.

[20] Y. X. Li and Z. H. Liu, A quasistatic contact problem for viscoelastic materials with friction and damage, Nonlinear Anal., 73 (2010), 2221–2229.

[21] S. Migórski, A. Ochal and M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics, 26. Springer, New York, 2013.

[22] Z. Naniewicz and P. D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Monographs and Textbooks in Pure and Applied Mathematics, 188. Marcel Dekker, Inc., New York, 1995.

[23] T. Roubíček, Nonlinear Partial Differential Equations with Applications, Second edition, International Series of Numerical Mathematics, 153. Birkhäuser/Springer Basel AG, Basel, 2013.

[24] M. Shillor, M. Sofonea and J. J. Telega, Models and Analysis of Quasistatic Contact, Lecture Notes in Physics, 655. Springer, Berlin, 2004.

[25] M. Sofonea, C. Avramescu and A. Matei, A fixed point result with applications in the study of viscoplastic frictionless contact problems, Commun. Pure Appl. Ana., 7 (2008), 645–658.

[26] M. Sofonea, W. M. Han and M. Shillor, Analysis and Approximations of Contact Problems with Adhesion or Damage, Pure and Applied Mathematics, 276. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[27] P. Szafraniec, Dynamic nonsmooth frictional contact problems with damage in thermoviscoelasticity, Math. Mech. Solids, 21 (2016), 525–538.

Received for publication October 2019.

E-mail address: leszek.gasinski@up.krakow.pl
E-mail address: piotr.kalita@ii.uj.edu.pl