Instability results for an elliptic equation on compact Riemannian manifolds with non-negative Ricci curvature

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Abstract

We prove nonexistence of nonconstant local minimizers for a class of functionals, which typically appears in the scalar two-phase field model, over a smooth $N$–dimensional Riemannian manifold without boundary with non-negative Ricci curvature. Conversely for a class of surfaces possessing a simple closed geodesic along which the Gauss curvature is negative we prove existence of nonconstant local minimizers for the same class of functionals.

Key words. Riemannian manifold, Ricci curvature, local minimizer, Gamma-convergence, reaction-diffusion equations.

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1 Introduction

Let $\mathcal{M}$ be a smooth $N$–dimensional compact Riemannian manifold without boundary and consider the functional $\mathcal{E} : H^1(\mathcal{M}) \to \mathbb{R}$ given by

$$\mathcal{E}(u) = \int_{\mathcal{M}} \left\{ \frac{|\nabla u|^2}{2} - F(u) \right\} d\mu$$

where $F$ is a $C^2$ real function and $H^1(\mathcal{M})$ the usual Sobolev space.

In this work we are interested in the question of how locally minimizing functions of $\mathcal{E}$ are related to the geometry of $\mathcal{M}$.

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We will say that \( u_0 \in C^\infty(M) \) is a local minimizer of \( \mathcal{E} \) if \( \exists \delta > 0 \) such that
\[
\mathcal{E}(u_0) \leq \mathcal{E}(u) \quad \text{whenever} \quad \|u - u_0\|_{H^1(M)} \leq \delta.
\]

In case the first inequality is strict, i.e., \( \mathcal{E}(u_0) < \mathcal{E}(u) \), \( u_0 \) is said to be a local isolated minimizer. Our main results are stated in the following theorems.

**Theorem 1.** Suppose that the Ricci curvature of \( M \) is non-negative. Then any local minimizer of \( \mathcal{E} \) is a constant function.

An interesting condition that shows up in the computations of Theorem 1 provides some insight on the structure of \( M \). For any \( u \in H^1(M) \) we denote by \( \mathcal{E}''(u) \) the second variation of \( \mathcal{E} \) at \( u \).

**Theorem 2.** Keep the hypothesis of Theorem 1. Let \( u \) be a non-constant critical point of \( \mathcal{E} \) and set \( v = |\nabla u| \). If
\[
(\mathcal{E}''(u)v, v) = 0
\]
then there exist a complete riemannian submanifold \( N \subset M \), a real geodesic line subbundle \( \mathcal{I} \subset TM \) and an isometric regular covering map \( \varphi : \mathbb{R} \times N \rightarrow M \). Denoting by \( K \) the group of covering transformations of \( \varphi \), then \( K \) is made of isometries, and \( M \) is isometric to the quotient \( (\mathbb{R} \times N)/K \). If \( \mathcal{I} \) is orientable then \( K \) is generated by a nontrivial (affine) translation of \( \mathbb{R} \) with some isometry of \( N \). Otherwise \( K \) is generated by two involutions of \( \mathbb{R} \times N \).

Regarding Theorem 1 we show how to construct non-constant local minimizers on some non-convex surfaces. To that purpose we introduce a small positive parameter \( \varepsilon \) in the functional thus writing
\[
(3) \quad \mathcal{E}_\varepsilon(u) = \int_M \left\{ \varepsilon \frac{|\nabla u|^2}{2} - \varepsilon^{-1}F(u) \right\} d\mu
\]
and take for \( F \) a suitable nonnegative double-well potential which vanishes only at \( \alpha \) and \( \beta \) (\( \alpha < \beta \)). As usual \( \chi_A \) will stand for the characteristic function of a set \( A \).

**Theorem 3.** Let \( M \) be a surface diffeomorphic to \( S^2 \). Assume that there exists a simple closed geodesic \( \gamma_0 \subset M \) so that the Gauss curvature \( K \) of \( M \) is negative along \( \gamma_0 \). Then for \( \varepsilon \) small enough there is a non-constant family \( \{u_\varepsilon\}_{\varepsilon > 0} \) of local minimizers of \( \mathcal{E}_\varepsilon \). Moreover it holds that \( u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u_0 \) in \( L^1(M) \) where \( u_0 = \alpha \chi_{M_\alpha} + \beta \chi_{M_\beta} \) and \( M = M_\alpha \cup \gamma_0 \cup M_\beta \) is the partition of \( M \) determined by \( \gamma_0 \).
The association of local minimizers of $\mathcal{E}$ to the geometry of the domain goes back to 1978 when the authors in [4] and [9] considered the evolution problem

\[
\begin{cases}
    u_t = \Delta u + f(u) & \text{in } \mathbb{R}^+ \times \Omega \\
    \partial_{\nu} u = 0 & \text{on } \mathbb{R}^+ \times \partial \Omega
\end{cases}
\]

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $f \in C^2(\Omega)$ and $\partial_{\nu} u$ stands for the exterior normal derivative.

They showed that if $\Omega$ is convex then any non-constant solution to (4) is unstable in the Lyapunov sense. In this case it amounts to saying that any local minimizer of the corresponding energy functional is a constant function.

Still for bounded convex domains with homogeneous zero Neumann boundary condition, the same kind of result was obtained for systems of reaction-diffusion equations [13] and [17], Ginzburg-Landau equation [16], reaction-diffusion systems with skew-gradient structure [14], geometric parabolic equation [15] and in the context of permanent currents for the full bi-dimensional Ginzburg-Landau functional in [16], among others. In all of these works the proofs make use in a strong way of the homogeneous Neumann boundary condition on a convex domain.

When $\mathcal{M}$ is a general Riemannian boundaryless manifold the Euler-Lagrange equation for $\mathcal{E}$ yields the stationary solutions of the reaction-diffusion equation

\[
\begin{cases}
    u_t = \Delta u + f(u) & \text{in } \mathbb{R} \times \mathcal{M}
\end{cases}
\]

The only result of this type regarding (5) over surfaces was considered in [11] where it was shown that if $\mathcal{M} \subset \mathbb{R}^3$ is a convex surface of revolution then the only stable solutions are the constant ones. Actually the prove consists of showing that $\mathcal{I}$, with $F' = f$, has no nonconstant local minimizer.

In this particular case writing the planar curve that generates the surface in appropriate coordinates reduces the domain to an interval thus making the underlying analysis much easier than the general case considered here.

In case $\mathcal{M}$ is a bounded domain in $\mathbb{R}^N$ typically $\mathcal{E}_\epsilon$ models the phase separation phenomenon in the context of van der Waals-Cahn-Hilliard theory whereby $u$ represents the density of a two-phase fluid and is also associated to the motion of phase boundaries (interfaces) by mean curvature (see [19], for instance).

Equation (5) has been studied in the context of pattern formation, i.e., existence of nonconstant stable (in the sense of Lyapunov) stationary solu-
tion. It may model bio-chemical processes over cell surfaces or propagation of calcium waves over the surface of a fertilized egg, for instance.

In particular Theorem 1 implies that (5) has no pattern as long as \( M \) has non-negative Ricci curvature. On the other hand Theorem 3 gives an example of \( M \) for which (5), after a suitable scaling, develops patterns.

Setting \( f = F' \) then clearly critical points of \( E \) satisfy the semi-linear elliptic equation

\[
\Delta u + f(u) = 0 \quad \text{on} \quad M.
\]

A smooth solution \( u \) of the above equation is said to be weakly stable if the quadratic form

\[
E(\varphi) = \int_M \left\{ \frac{|\nabla \varphi|^2}{2} - f'(u)\varphi^2 \right\} d\mu \geq 0
\]

in \( H^1(M) \). Otherwise \( u \) is called weakly unstable. Then it follows immediately from the proof of Theorem 1 that any nonconstant solution to the above equation is weakly unstable as long as \( M \) has non-negative Ricci curvature.

The paper is organized as follows. In Section 2 in addition to recalling some notation of Riemmanian Geometry we prove some preliminary results, Section 3 is devoted to the proofs of Theorem 1 and Theorem 2 and Section 4 to the proof of Theorem 3.

2 Geometric Background

Throughout this section \( M \) will denote an \( N \)-dimension (\( N \geq 2 \)) riemannian manifold without boundary, and \( TM, T^*M \) its tangent and cotangent bundles, respectively. We shall deal with the tensor bundles \( T^r_s(M) = (TM)^\otimes r \otimes (T^*M)^\otimes s \), for non-negative integers \( r \) and \( s \). For an integer \( k \geq 0 \) let \( A^k T^*M \) be the alternate k-bundle of \( T^*M \). Notice that \( T^0_0(M) = A^0 T^*M \) is the trivial bundle \( M \times \mathbb{R} \).

Given any real vector bundle \( F \) over \( M \) we denote by \( G(F) \) the set of its smooth sections and by \( G^k(F) = G(A^k T^*M \otimes F) \) the smooth sections of \( k \)-forms of \( M \) with coefficients in \( F \).

The contraction is a natural coupling \( c : T^1_1(M) \rightarrow T^0_0(M) \) given by \( c(v \otimes \omega) = \omega(v) \), where \( v \otimes \omega \) is a decomposable tensor of \( TM \otimes T^*M \). The contraction extends to \( c : T^r_s(M) \rightarrow T^{r-1}_{s-1}(M) \) for any \( r, s \geq 1 \), by putting \( c(v_1 \otimes \cdots \otimes v_r \otimes \omega_s \otimes \cdots \otimes \omega_1) = \omega_1(v_1) v_2 \otimes \cdots \otimes v_r \otimes \omega_s \otimes \cdots \otimes \omega_2 \). Indeed, when \( r = s = 1 \) the contraction is just the trace operator on linear homomorphisms \( TM \rightarrow TM \).
Let \( \nabla : G(T^1_0(\mathcal{M})) \to G^1(T^1_0(\mathcal{M})) \) be the Levi-Civita connection on \( \mathcal{M} \). It is well known that \( \nabla \) can be extended in a unique way to an operator \( \overline{\nabla} : G(T^s_0(\mathcal{M})) \to G^1(T^s_0(\mathcal{M})) \) such that Leibnitz rule is preserved and commutes with the contraction \([\mathcal{L}]\). We abuse notation and write \( \overline{\nabla} = \nabla \) whenever \( r,s \) are not both zero. When \( f \in G(T^0_0(\mathcal{M})) \) is just a smooth function we preserve the usual notation \( \nabla f = (df)^* \in G(T^1_0(\mathcal{M})) \). It then follows

\[
\nabla(T \otimes W) = \nabla T \otimes W + T \otimes \nabla W \\
\forall T \in G(T^s_0(\mathcal{M})) \text{ and } \forall W \in G(T^p_0(\mathcal{M})) , \text{ and }
\]

\[
\nabla(c(T) = c(\nabla T), \\
\text{for a contraction } c : T^r_s(\mathcal{M}) \to T^{r-1}_{s-1}(\mathcal{M}).
\]

Notice that we identify

\[
(TM)^{\otimes r} \otimes (T^s\mathcal{M})^{\otimes s} \otimes (TM)^{\otimes p} \otimes (T^s\mathcal{M})^{\otimes q} \cong \\
(TM)^{\otimes r} \otimes (TM)^{\otimes p} \otimes (T^s\mathcal{M})^{\otimes s} \otimes (T^s\mathcal{M})^{\otimes q}
\]

and similarly, by sticking the 1-form component of a section of \( \mathcal{A}^1 T^r\mathcal{M} \otimes (T^s_0(\mathcal{M})) \) on the left of the covariant part we have \( \mathcal{A}^1 T^r\mathcal{M} \otimes T^s_0(\mathcal{M}) \cong T^r_{s+1}(\mathcal{M}) \). These identifications are necessary for \([\mathcal{L}]\) and \([\mathcal{D}]\) to make sense. They also allow us to define the composition \( \nabla(\nabla T) \) for any \( T \in G(T^s_0(\mathcal{M})) \).

Some combinations of \( \otimes \) and \( c(\cdot) \) deserve special notation. For tensors \( T \in G(T^1_s(\mathcal{M})) \) and \( W \in G(T^1_0(\mathcal{M})) \) we write \( TW = c(W \otimes T) \). When \( s = 1 \) and \( q = 1 \), \( TW \) is the composition of the endomorphisms \( T \) with \( W \), and if \( q = 0 \) \( TW \) is the image of the vector \( W \) under \( T \). In particular, if \( s \geq 2 \) and \( W_1, W_2 \) are vector fields we set \( T(W_1, W_2) = [TW_2]W_1 \).

Let \( F \in T^3_2(\mathcal{M}) \) be the Riemann tensor of \( \mathcal{M} \). The tensor \( F \) can be seen as a two form with values in the endomorphism bundle of \( TM \) or \( F \in G^2(TM \otimes T^s\mathcal{M}) \). For any vector fields \( X, Y, Z \) and \( W \) locally defined we have

\[
F(X,Y,Z,W) \overset{\text{def}}{=} \langle [FZ](Y, X), W \rangle = \\
= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, W \rangle .
\]

The proof of the next lemma is straightforward and will be omitted.

**Lemma 4.** Let \( V \in G(T^0_0(\mathcal{M})) \). Then the skew-symmetric component with respect to the cotangent factors of \( \nabla(\nabla V) \) is \( FV \). This is equivalent to

\[
[\nabla(\nabla V)](X,Y) - [\nabla(\nabla V)](Y,X) = [FV](Y, X)
\]

for any vectors \( X, Y \).
We define the Ricci tensor of $M$ as $\text{Ric}(V,W) = -c([FW]V)$, for any $V, W$ vector fields. Observe that if $\{s_i | i = 1, \ldots, n\}$ is any local orthonormal basis of $T_M$ then $\text{Ric}(V,W) = \sum_{i=1}^{n} F(s_i,V,W,s_i)$.

**Definition.** A non-negative Ricci manifold $M$ is one that satisfies $\text{Ric}(V,V) \geq 0$ for any $V \in T_M$.

The following lemma will be useful in our approach.

**Lemma 5.** Let $V$ and $W$ be vector fields over $U \subset M$ open. Then

$$c([\nabla(\nabla V)]W - \nabla W(\nabla V)) = \text{Ric}(V,W) .$$

**Proof.** We choose an orthonormal basis $\{s_1, s_2, \ldots, s_n\}$ locally defined and compute

$$c([\nabla(\nabla V)]W - \nabla W(\nabla V)) =$$

$$= \sum_i \langle [\nabla_{s_i}(\nabla V)]W - [\nabla W(\nabla V)]s_i, s_i \rangle =$$

$$= \sum_i (\nabla_{s_i}[(\nabla V)W] - (\nabla V)\nabla_{s_i}W - \nabla_W[(\nabla V)s_i] + (\nabla V)\nabla_{W}s_i, s_i) =$$

$$= \sum_i \langle \nabla_{s_i} \nabla W V - \nabla_W \nabla_{s_i} V - \nabla_{[s_i, W]}V, s_i \rangle =$$

$$= \sum_i F(s_i, W, V, s_i) = \text{Ric}(V,W) .$$

Let $\mathcal{M}$ be a Riemann surface and $\gamma_0 \subset \mathcal{M}$ be a simple closed geodesic. We assume local orientability of $\mathcal{M}$ in a neighborhood of $\gamma_0$, i.e., there exists a smooth unitary orthogonal vector field $\eta$ defined on $\gamma_0$. Standard arguments (see [3]) allow $\eta$ to be extended to a geodesic vector field on a vicinity $V$ of $\gamma_0$. Let $\varphi(t) = \varphi(t, p)$ be the flow of $\eta$. Restricting $V$ if necessary, one can choose $\delta > 0$ so that the map $\varphi : [-\delta, \delta] \times \gamma_0 \rightarrow V$ is a diffeomorphism. In all computations it is implicitly assumed that $\gamma_0$ is arcwise parametrized, so that $\gamma'_0$ is a well defined unitary vector field along $\gamma_0$.

Let $t$ and $x$ be the coordinate functions of the inverse map $\varphi^{-1} : V \rightarrow [-\delta, \delta] \times \gamma_0$, $\varphi^{-1}(p) = (t(p), x(p))$. For any $\sigma : [0, 1] \rightarrow V$ a smooth curve we denote by $\overline{\sigma}$ its projection over $\gamma_0$,

$$\overline{\sigma}(s) = x \circ \sigma(s) , \quad 0 \leq s \leq 1 .$$
Notice that we abuse language and denote by $\sigma$ either a curve or its trace, according to the context. Similarly, $|\sigma|$ denotes the length of the curve, but for a two dimensional region $U \subset \mathcal{M}$, $|U|$ denotes its area.

The contents of the next lemma are well known to geometers, and can be found in the literature. Nevertheless we choose to state and proof the precise statements we need for the sake of completeness.

**Lemma 6.** Suppose that the gaussian curvature $K$ is strictly negative on $\mathcal{V}$. We have:

(a) Let $p_0, p_1 \in \mathcal{V}$ and $\sigma$ be any smooth simple curve joining $p_0$ and $p_1$. Then

\[ |\sigma| \geq |t(p_1) - t(p_0)| \quad \text{and equality holds if and only if } \sigma \text{ reparametrizes the geodesic segment } t \mapsto \varphi_t(p) \text{ between } p_0 \text{ and } p_1. \]

(b) Let $J \subset \gamma_0$ be an interval or $J = \gamma_0$. Let $0 < \delta_0 \leq \delta$ and $U$ be any of the sets $\varphi([0, \delta_0] \times J)$ or $\varphi([-\delta_0, 0] \times J)$. Then $|U| > \delta_0 |J|$.

**Proof.** Let $W_p = (d\varphi_t)_x \cdot \gamma_0(x)$ for any $p = \varphi(t, x) \in \mathcal{V}$. Then $W$ is a smooth vector field on $\mathcal{V}$. Using the symmetry of the Levi-Civita connection together with $|\eta| \equiv 1$ one gets

\[
\frac{d}{dt} \langle \eta, W \rangle = \eta \langle \eta, W \rangle = \langle \nabla_\eta \eta, W \rangle + \langle \eta, \nabla_\eta W \rangle = \\
= \langle \eta, \nabla W \eta + [\eta, W] \rangle = \\
= \frac{1}{2} W \langle \eta, \eta \rangle + \langle \eta, [\eta, W] \rangle = 0,
\]

and therefore $\langle \eta, W \rangle$ is constant along the flow of $\eta$. Over $\gamma_0$ we know $W = \gamma_0'$, from what we obtain $\langle \eta, W \rangle = 0$ on $\mathcal{V}$. The field $W$ is nowhere singular in $\mathcal{V}$, and we set the orientation of $\mathcal{V}$ as given by the orthogonal basis $\{\eta, W\}$.

Let $x(s)$ and $t(s)$ be the local coordinate functions of a given $\sigma(s)$, so that

$\sigma(s) = \varphi(t(s), x(s))$, for $0 \leq s \leq 1$. Let $p_0 = \sigma(0)$ and $p_1 = \sigma(1)$. Notice that $x(s)$ belongs to the trace of $\gamma_0$ and its derivative is a multiple of $\gamma_0'$, but we abuse language and set $x'(s) = \langle \frac{dx}{ds}, \gamma_0' \rangle_{x(s)}$. Then $\sigma'(s) = t'(s)\eta + x'(s)W$. Since $\sigma$ has no self-intersections it follows

\[
|\sigma| = \int_0^1 |\sigma'(s)| \, ds = \int_0^1 \sqrt{(t')^2 + (x')^2 |W|^2} \, ds \\
\geq \int_0^1 |t'| \, ds \geq |t(p_1) - t(p_0)|.
\]
Equality in (15) occurs if and only if \( x' \equiv 0 \) and \( t' \) does not change sign. This implies \( x(s) = x_0 \in \gamma_0 \) is constant, hence \( \sigma(s) = \varphi_{t(s)}(x_0) \) is just a parametrization of an arc of geodesic, what proves part (a1). A computation similar to (17) yields

\[
|\sigma| = \int_0^1 \sqrt{(t')^2 + (x')^2} |W|^2 \, ds \geq \int_0^1 |x'||W| \, ds.
\]

We show that \( |W_p| \geq 1 \) with equality only when \( p \in \gamma_0 \). It suffices showing that the function \( t \mapsto |W_{(t,x)}|^2 \) is convex in \([-\delta, \delta]\), with a strict minimum attained in \( t = 0 \). Indeed,

\[
\frac{d}{dt} \bigg|_{t=0} |W|^2 = 2 \langle \nabla_{\eta} W, W \rangle_{t=0} = 2 \langle \nabla_{W_\eta} W \rangle_{t=0} = 0,
\]

since \( \nabla_{\eta} W = 0 \) along \( \gamma_0 \). The second derivative gives

\[
\frac{d^2}{dt^2} |W|^2 = 2 (\langle \nabla_{\eta} \nabla_{\eta} W, W \rangle + \langle \nabla_{W_\eta} W, \nabla_{\eta} W \rangle) = 2 (\langle \nabla_{\eta} W \eta - \nabla_{W} \nabla_{\eta} \eta - \nabla_{[\eta,W]} \eta, W \rangle + |\nabla_{\eta} W|^2) = 2(-K + |\nabla_{\eta} W|^2) |W|^2,
\]

so it is strictly positive for any \( t \), under the hypothesis \( K < 0 \). Back to (18) we have

\[
|\sigma| \geq \int_0^1 |x'||W| \, ds \geq \int_0^1 |x'| \, ds = |\bar{\eta}|,
\]

with equality \( |\sigma| = |\bar{\eta}| \) if and only if \( t(s) \equiv 0 \), or \( \sigma = \bar{\eta} \) is an arc of the geodesic \( \gamma_0 \). This proves (a2).

Now assume \( U = \varphi([0, \delta_0] \times J) \). We consider an orthonormal basis of 1-forms \( \{\omega_1, \omega_2\} \) dual to \( \{\eta, \frac{W}{|W|}\} \). The area element is \( \omega_1 \wedge \omega_2 \). Let \( J \subset \gamma_0 \) be arclength parametrized by the interval \( [s_0, s_1] \subset \mathbb{R} \) so that \( \{\eta, J'(s)\} \) preserves the orientation of \( V \) over \( \gamma_0 \). Using the local chart \( \varphi \) to write the integral on the plane and applying Fubini Theorem the area of \( U \) is computed as

\[
|U| = \iint_U \omega_1 \wedge \omega_2 = \int_{[0, \delta_0] \times [s_0, s_1]} \varphi^*(\omega_1 \wedge \omega_2) = \int_{s_0}^{s_1} \int_0^{\delta_0} |W| \, dt \, ds > \int_{s_0}^{s_1} \int_0^{\delta_0} \, dt \, ds = \delta_0 |J|,
\]

thus proving the theorem. \( \square \)
3 Nonexistence of nonconstant minimizers

This section is devoted to the proofs of Theorems 1 and 2, which in turn will be applications of the identities established in the next two lemmas.

Recall that the Riemannian metric of $\mathcal{M}$ induces metrics in any tensor product $T^r_s(\mathcal{M})$, as well as in their spaces of sections. If $T, W \in T^1_1(\mathcal{M})$ then their inner-product (fiberwise) is computed as $\langle T, W \rangle = c(c(T \otimes W^*))$, being $W^*$ the (metric) transpose of the endomorphism $W : T\mathcal{M} \to T\mathcal{M}$.

If $V$ is a $C^1$ vector field on $\mathcal{M}$ we set $\text{div}(V) = c(\nabla V)$. The hessian of a $C^2$ function $u$ on $\mathcal{M}$ is $H_u = \nabla(\nabla u)$. The Laplacean of $u$ is then $\Delta u = c(H_u) = \text{div}(\nabla u)$.

The Riemannian measure on $\mathcal{M}$ will be denoted by $d\mu$. By a component of a topological space we always mean a connected component.

Lemma 7. Let $V$ be a $C^2$ vector field on $\mathcal{M}$ and $u$ a $C^3$ function on $\mathcal{M}$. Then

\begin{equation}
\Delta(Vu) - V(\Delta u) = \text{div}((\nabla V)^*\nabla u) + \langle H_u, \nabla V \rangle + \text{Ric}(\nabla u, V).
\end{equation}

Proof. We first notice that

\begin{equation}
\nabla(Vu) = [d(Vu)]^* = (\nabla V)^*\nabla u + H_u V.
\end{equation}

Then,

\begin{align}
\Delta(Vu) - V(\Delta u) &= c(\nabla(\nabla V)^*\nabla u + H_u V) - c(\nabla V H_u) \\
&= c(\nabla((\nabla V)^*\nabla u)) + c((\nabla H_u)V + H_u \nabla V - \nabla V H_u)
\end{align}

\begin{equation}
= \text{div}((\nabla V)^*\nabla u) + c((\nabla H_u)V - \nabla V H_u) + c(H_u \nabla V).
\end{equation}

Applying Lemma 5 to the second summand of term (27) and observing that $c(H_u \nabla V) = \langle H_u, \nabla V \rangle$ we arrive at

\begin{equation}
\Delta(Vu) - V(\Delta u) = \text{div}((\nabla V)^*\nabla u) + \text{Ric}(\nabla u, V) + \langle H_u, \nabla V \rangle,
\end{equation}

and the proof is complete. \hfill \Box

Remark 1. Lemma 7 is central in the next constructions of this section. Indeed, it somehow appears in [11], where its full geometric significance is shadowed by the high symmetry of that case. The main idea there, which holds in general, is a commutation relation between the Laplacean operator and a particular directional derivative, namely, the normalized gradient of $u$. 

Let \( u \) be a non-constant critical point of \( E \) with \( F' = f \). Then

\[
\frac{d}{dt} E(u + tv)|_{t=0} = -\int_{\mathcal{M}} (\Delta u + f(u)) v \, d\mu = 0, \forall v \in H^1(\mathcal{M}).
\]

The linearization of the operator \( \Delta + f(\cdot) \) at \( u \) yields an operator \( \mathcal{L} : H^1(\mathcal{M}) \rightarrow H^{-1}(\mathcal{M}) \) defined by

\[
\mathcal{L}(u)v = \Delta v + i(f'(u) v),
\]

where \( i : H^1(\mathcal{M}) \rightarrow H^{-1}(\mathcal{M}) \) is the Sobolev inclusion \( H^1 \subset H^{-1} \). Let \((\cdot, \cdot) : H^{-1} \times H^1 \rightarrow \mathbb{R}\) be the canonical pairing of a vector space and its dual. Then

\[
\frac{d^2}{dt^2} E(u + tv)|_{t=0} = (E''(u)v, v) = -(\mathcal{L}(u)v, v).
\]

For the next lemma we temporarily drop any hypothesis about Ricci curvature. It will be immediate that for Ricci non-negative manifolds the quadratic form associated to \( \mathcal{L} \) is not sign definite. Define

\[
U = \{ \nabla u \neq 0 \} \subset \mathcal{M}.
\]

Let \( V \) be the unitary vector field \( V = \frac{\nabla u}{|\nabla u|} \) over \( U \).

**Lemma 8.** Let \( v = |\nabla u| \). Then

\[
(\mathcal{L}(u)v, v) = \int_{\mathcal{M}} |\nabla u|^2 (|\nabla V|^2 + \text{Ric}(V, V)) \, d\mu.
\]

**Proof.** The function \( u \) is of class \( C^3 \), hence \( V \) is \( C^2 \). In the open set \( U \) we have \( V(\Delta u + f(u)) = 0 \), thus

\[
\Delta(Vu) + f'(u)(Vu) = \Delta(Vu) - V(\Delta u).
\]

Applying Lemma 4 directly to the righthand side of (16) we get

\[
\Delta(Vu) + f'(u)(Vu) = \text{div}((\nabla V)^4 \nabla u) + \langle H_u, \nabla V \rangle + \text{Ric}(\nabla u, V).
\]

The covariant derivative of \( V \) is given by

\[
\nabla V = \frac{1}{|\nabla u|} H_u - \nabla u \otimes \frac{(H_u \nabla u)^*}{|\nabla u|^3}.
\]

A computation shows that \( \nabla V \) is orthogonal to the tensor \( \nabla u \otimes \frac{(H_u \nabla u)^*}{|\nabla u|^3} \). Recalling that \( v = |\nabla u| = Vu \) we obtain

\[
\langle H_u, \nabla V \rangle = |\nabla u| \left\langle \frac{1}{|\nabla u|} H_u - \nabla u \otimes \frac{(H_u \nabla u)^*}{|\nabla u|^3}, \nabla V \right\rangle = v |\nabla V|^2.
\]
Let \( W \) be any vector in the tangent space over a point of \( U \). Since \( V \) is unitary we have

\[
\langle (\nabla V)^* V, W \rangle = \langle V, \nabla W V \rangle = \frac{1}{2} W |V|^2 = 0 .
\]

Thus \( \text{div}((\nabla V)^* \nabla u) = \text{div}(v(\nabla V)^* V) \) vanishes identically. With the help of (34) equation (32) turns into

\[
\Delta v + f'(u)v = v |\nabla V|^2 + v \text{Ric}(V, V) .
\]

Notice that \( v \) vanishes in \( M - U \). Looking at the left-hand side of the above identity as a distribution it becomes clear that its support is contained in \( U \). Therefore, applying it on \( v \in H^1(M) \) one obtains

\[
(\mathcal{L}(u)v, v) = \int_M |\nabla u|^2 (|\nabla|^2 + \text{Ric}(V, V)) d\mu ,
\]

which proves the Lemma.

**Remark 2.** Let \( p \in M \) be a non-critical point of \( u \). The level set \( S = \{ x \mid u(x) = u(p) \} \) is a regular hypersurface near \( p \). It can be seen that \( \nabla V = A + (\nabla V)^* \otimes V^* \), where \( A : TS \to TS \) is the shape operator respect to \( V \) of the second fundamental form of the inclusion \( S \subset M \). By setting \( c = |\nabla V| \) the squared norm of \( \nabla V \) becomes

\[
|\nabla V|^2 = |A|^2 + c^2 .
\]

Therefore \( |\nabla V|^2 \) is the sum of the square of the principal curvatures of \( S \) plus the square of the curvature of the flow of \( \nabla u \).

**Remark 3.** In the unidimensional case \( M = S^1 \) a direct proof of instability can be given. Endow \( S^1 \) with a metric so that \( |S^1| = l \). Functions on \( S^1 \) are identified with functions on \([0, l]\) satisfying certain boundary conditions. In this case the Euler-Lagrange equation for \( E \) is

\[
\begin{cases}
u''(t) + f(u(t)) = 0, & 0 < t < l \\
u(0) = u(l) \\nu'(0) = u'(l)
\end{cases}
\]

It’s linearization becomes \( \mathcal{L}(u)v = v'' + f'(u)v \). Assume by contradiction that \( u \) is a non-constant local minimizer of \( E \). Then \( (\mathcal{L}(u)v, v) \leq 0 \), and due to Lemma \( \S \) we get \( \mathcal{L}(u)v = 0 \). Hence \( v = |u'| \) is an eigenfunction associated to the zero eigenvalue.
A direct computation shows that $u'$ is also an eigenfunction of the zero eigenvalue of $L(u)$. Then $w = u' + |u'|$ is an eigenfunction and since $w$ vanishes in an open interval the Unique Continuation Theorem gives us $w \equiv 0$. Hence $u' \equiv 0$, what goes against the hypothesis. This shows that the first eigenvalue of $L(u)$ is positive and there are no non-constant local minimizers of $E$.

In view of Lemma 8 the proof of Theorem 1 is now immediate if we strengthen the hypothesis to $\text{Ric} > 0$ on $\mathcal{M}$. Indeed, one can show that $\text{Ric} > 0$ on some open set of $\mathcal{M}$ suffices for the positivity of $(L(u)v,v)$, by using the Unique Continuation Theorem together with the contradiction assumption that the first eigenvalue of $L(u)$ is zero.

We will rather give a unified proof for the case $\text{Ric} \geq 0$. This requires a few more lemmas dealing with the more delicate case $\nabla V = 0$ and $\text{Ric} = 0$ on $U$. It will follow after a series of steps rich on tricky details. The main ingredients are the level sets of $u$ and the behaviour of the geodesics of $\mathcal{M}$ respect to the critical points of $u$.

The remaining results of this section do not demand that $u$ be bounded or belong to any particular Sobolev Space. We will skip for a while any functional analytic concerns, and assume that $\mathcal{M}$ is an arbitrary complete, not necessarily compact, Riemann manifold, and $u$ is a classical solution to equation (6). The compactness of $\mathcal{M}$ will be implicitly invoked back only in the proofs of Theorems 1 and 2.

For the next six Lemmas and Corollaries we thus assume

\begin{equation}
|\nabla V| = 0 \quad \text{in } U,
\end{equation}

unless otherwise stated. In particular we obtain that $V$ is a parallel vector field over $U$. From equation (33) we also get

\begin{equation}
H_u = V \otimes (H_u V)^* = \Delta u V \otimes V^*.
\end{equation}

For any $p \in \mathcal{M}$ define $N_p$ as the component of the level set $\{x \in \mathcal{M} \mid u(x) = u(p)\}$ that contains $p$.

**Lemma 9.** If $p \in U$ then $N_p \subset U$. Further, $N_p$ is a complete geodesic riemannian submanifold of codimension 1 of $\mathcal{M}$ and $|\nabla u| > 0$ is constant on $N_p$.

**Proof.** Let $U_p$ be a component of $U$ and $C_p$ a component of $U_p \cap N_p$ so that $p \in C_p$. Clearly $C_p$ is a codimension 1 submanifold of $\mathcal{M}$. If $X,Y \in T(C_p) \subset TU$ we have

\begin{equation}
\langle \nabla X Y, V \rangle = X \langle Y, V \rangle - \langle Y, \nabla X V \rangle = 0
\end{equation}
for $V$ is parallel and $X, Y$ are orthogonal to $V$. This shows that $C_p$ is geodesic.

Letting $q \in C_p$ and $X \in T_q(C_p)$, we have $\nabla_X \nabla u = H_u(X) = 0$. Therefore $\nabla u$ is parallel and $|\nabla u| \neq 0$ is constant along $C_p$. If $\overline{q}$ is an adherent point of $C_p$ then $\nabla u(\overline{q})$ is non-zero so that $\overline{q} \in U_p$. This shows that $C_p$ is closed in $\mathcal{M}$, and since $U_p$ is open, $C_p$ is also open as a topological subspace of $\mathcal{N}_p$. Therefore by the conexity we have $C_p = \mathcal{N}_p \subset U_p$.

The geodesic completeness of $\mathcal{N}_p$ follows from the Theorem of Rinow and Hopf [3] and the fact that $\mathcal{M}$ is complete.

**Lemma 10.** Let $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ be an arclength parametrized geodesic, and $h(t) = u(\gamma(t))$ for all $t \in \mathbb{R}$. Assume that $h$ is non-constant, and let $(a, b)$ be a component of $\gamma^{-1}(U)$. Then

(a) $h$ is strictly monotone in $(a, b)$.

(b) Assume $a \in \mathbb{R}$, and let $p = \gamma(a)$. Then $p$ is a critical point of $u$ and $H_u(p) \neq 0$.

(c) Under the same hypothesis as (b) let $r = b - a \in \mathbb{R} \cup \{+\infty\}$. Then $(a - r, a)$ is also a component of $\gamma^{-1}(U)$. Further, $h(t)$ is symmetric respect to $t = a$, i.e., $h(a - s) = h(a + s)$ for all $s \in \mathbb{R}$.

(d) Under the same hypothesis as (c), assume also $b \in \mathbb{R}$. Then $h$ is periodic of period $2r$.

**Proof.** For all $t \in \mathbb{R}$ we have $h'(t) = \langle \nabla u, \gamma'(t) \rangle$. This justifies the existence of the interval $(a, b)$, since $h$ is non-constant. For all $t \in (a, b)$ we can write $h'(t) = |\nabla u| \langle V, \gamma'(t) \rangle$. Both of $V$ and $\gamma'$ are parallel along $\gamma$, hence $\langle V, \gamma'(t) \rangle = k$ is a constant in $(a, b)$. We must have $k \neq 0$, otherwise the geodesic $\gamma$ would be entirely contained in $\mathcal{N}_{\gamma(a_0)}$, for any $t_0 \in (a, b)$, and $h$ would be constant. Hence $k$ and $|\nabla u|$ are non-zero in $(a, b)$ and part (a) is proved.

We compute the second derivative of $h$ for any $t \in (a, b)$,

$$h''(t) = \frac{d}{dt} \langle \nabla u, \gamma'(t) \rangle = \langle H_u(\gamma'(t)), \gamma'(t) \rangle = \Delta u(\gamma(t)) k^2,$$

in view of equation (41). Then $h(t)$ is a solution to the 2nd order equation

$$h'' + k^2 f(h) = 0$$

on $(a, b)$. If $a \in \mathbb{R}$, $h$ satisfies the initial condition $h(a) = u(p)$, $h'(a) = 0$. By uniqueness of the Initial Value Problem the constant function $t \mapsto u(p)$ is not a solution of that problem, and therefore $u(p)$ is not a root of $f$. Hence,
Due to \( h''(a) \neq 0 \) there is a small left open neighborhood of \( a \) where \( h'(t) \neq 0 \), and hence \( \gamma(t) \in U \) for \( t < 0 \) small. Therefore there is a component of \( \gamma^{-1}(U) \) of the form \( (c, a) \), for some \( c \in (-\infty, a) \). Let \( J = (0, \min\{r, a-c\}) \).

We define \( h_-(s) = h(a - s) \) and \( h_+(s) = h(a + s) \) for all \( s \in \mathbb{R} \). Then \( h_-(0) = h_+(0) = h(a) \), \( h'_-(0) = h'_+(0) = 0 \). Further, for \( s \in J \) there are suitable constants \( k_-, k_+ \) that play the role of \( k \) on (44):

\[
\begin{align*}
    h''_+ + k_+^2 f(h_+) &= 0, \\
    h''_- + k_-^2 f(h_-) &= 0.
\end{align*}
\]

Again uniqueness for this problem will give us \( h_- \equiv h_+ \) as long as we show that \( k_-^2 = k_+^2 \).

Let \( V_-(s) = V(\gamma(a - s)) \) and \( V_+(s) = V(\gamma(a + s)) \) for all \( s > 0 \) small. Both of \( V_- \) and \( V_+ \) can be continuously extended by parallel transport along \( \gamma \) to vectors \( \tilde{V}_- \) and \( \tilde{V}_+ \), respectively, on \( T_pM \). We claim that the (unitary) vectors \( \tilde{V}_- \) and \( \tilde{V}_+ \) are colinear. The (symmetric polynomials on the) eigenvalues of the continuous symmetric tensor \( H_u \) are continuous. The special form of \( H_u \) on \( U \), given by equation (41), implies that for all small \( s > 0 \), \( H_u(\gamma(a \pm s)) \) has a zero eigenvalue of multiplicity at least \( N - 1 \), which is inherited by \( H_u(p) \). The remaining eigenvalue of \( H_u(p), \Delta_u(p) \), has to be non-zero (after part (b)) and simple. This is an open condition, and the eigenspace associated to this eigenvalue varies continuously, close to \( p \). It is generated by \( V \) on \( U \), therefore, we have \( \tilde{V}_- = \pm \tilde{V}_+ \). Since

\[
\begin{align*}
    k_- &= \lim_{s \to 0^+} \langle V_-(s), \gamma'(a - s) \rangle = \langle \tilde{V}_-, \gamma'(a) \rangle, \\
    k_+ &= \lim_{s \to 0^+} \langle V_+(s), \gamma'(a + s) \rangle = \langle \tilde{V}_+, \gamma'(a) \rangle,
\end{align*}
\]

we get \( |k_-| = |k_+| \), hence \( h_-(s) = h_+(s) \) for \( s \in J \). Critical points of \( h_- \) and \( h_+ \) happen together in this range and correspond to intersections of \( \gamma(t) \) with the border of \( U \). Therefore \( 0 < s \mapsto \gamma(a - s) \) cannot leave \( U \) before \( s = r \), and since the argument is symmetric, we conclude that \( a - c = r \) and \( \gamma^{-1}(U) \) contains \( (a - r, a) \) as a component, which proves part (c).

Part (d) is now immediate. Clearly the symmetry of \( h(t) \) holds respect to any critical point of \( h \). If \( r = b - a \) is finite then we get \( h(a + r + s) = h(a + r - s) = h(a - r + s) \) for any \( 0 < s < r \). In particular, an inductive argument shows that \( \{a + mr \mid m \in \mathbb{Z}\} \) are all critical points of \( h(t) \). The period of \( h \) is \( 2r \) since it intercalates increasing with decreasing intervals between consecutive critical points. \( \Box \)
Remark 4. From part (c) of the Lemma we have \( h'(a + s) = -h'(a - s) \), and picking \( s > 0 \) small we obtain

\[
(46) \quad h'(a + s) = k_+ |\nabla u|_{\gamma(a+s)} = -k_- |\nabla u|_{\gamma(a-s)} = -h'(a - s) .
\]

Therefore, \( k_- = -k_+ \) and \( \tilde{V}_- = -\tilde{V}_+ \).

As a consequence of Lemma 10 we get \( H_u(p) \neq 0 \) and \( \Delta u(p) \neq 0 \) for any critical point \( p \) of \( u \), since there is a point \( q \in M \) with \( u(q) \neq u(p) \) and a geodesic \( \gamma(t) \) joining \( p \) to \( q \). Further, the set of critical points of \( u \) is \( \partial U = M - U \).

We are now ready to give the

Proof of Theorem 1. By Lemma 8 along with the condition \( \text{Ric} \geq 0 \) we deduce that \( (L(u)v, v) \geq 0 \). We will show that this inequality is strict, so \( u \) cannot be a local minimum of \( E \). The case where \( \nabla V \neq 0 \) is straightforward from the Lemma, so we assume in the sequel that \( \nabla V \equiv 0 \) on \( U \).

Suppose by contradiction that the first eigenvalue of \( L(u) \) is non-positive. Then \( (L(u)v, v) = 0 \) and \( v \) must be an eigenfunction of \( L(u) \) associated to the zero eigenvalue. Since \( f'(u)v \) is continuous, standard elliptic regularity applied to

\[
(47) \quad \Delta v + f'(u) v = 0 \quad \text{on} \quad M
\]

gives us \( v \in C^2(M) \). Computing the gradient of \( v \) in \( U \) we obtain

\[
(48) \quad \nabla v = \nabla |\nabla u| = H_u(V) = \Delta u V .
\]

Let \( p \) be a critical point of \( u \) and \( \gamma(t) \) be a geodesic satisfying the hypotheses on Lemma 10 so that \( \gamma(0) = p \). Following the notation in the proof of the Lemma we have, by part (b), that \( \Delta u(p) \neq 0 \). On the other hand, Remark 4 gives us

\[
(49) \quad \lim_{t \to 0^+} V_{\gamma(t)} = -\lim_{t \to 0^-} V_{\gamma(t)} \neq 0 .
\]

This shows that \( \nabla v \) is not even continuous at \( p \), what contradicts the \( C^2 \) regularity of \( v \). The only remedy is granting that the first eigenvalue of \( L(u) \) is positive, which finishes the proof of the Theorem.

Notice that \( V \) defines a line subbundle of \( T_M|_U \) that can be extended over \( \partial U \) by taking the only simple eigenspace of \( H_u \) (associated to the non-zero eigenvalue) near critical points. This justifies the next
Corollary 11. There exists a geodesic line bundle $\mathcal{I} \subset TM$ so that $\mathcal{I}|_U$ is spanned by $V$.

Choose a point $p_0 \in U$ and let $U_0$ be its correspondent component of $U$. Denote $N_0 = N_{p_0}$. We would like to extend the field $V|_{U_0}$ to the whole of $\mathcal{M}$ by means of the bundle $\mathcal{I}$. The flow of such extension would, then, be generated by isometries, and routine arguments would give us a covering map $\varphi : \mathbb{R} \times N_0 \to \mathcal{M}$, from which one would quickly derive the results of Theorem 2. This case has already been researched in greater generality, for instance, in [2].

Here is where the orientability of $\mathcal{I}$ comes in. Clearly, such an extension of $V|_{U_0}$ is possible if and only if $\mathcal{I}$ is orientable (as a real vector bundle). Both of orientable and non-orientable cases can happen to $\mathcal{I}$, leading to two different constructions for $\mathcal{M}$. In order to keep generality and short the proofs, we give a definition of $\varphi$ independent of $\mathcal{I}$.

For any $p \in N_0$ let $t \mapsto \varphi_t(p)$ be the geodesic defined by $\varphi_0(p) = p$ and $\varphi_0'(p) = V_p$. Then $\varphi : \mathbb{R} \times N_0 \to \mathcal{M}$ is smooth.

Lemma 12. There is an open interval $(a, b)$ so that $\varphi : (a, b) \times N_0 \to U_0$ is an isometry.

Proof. Let $(a, b) \ni 0$ be the maximal interval for which $\varphi_t(p_0)$ belongs to $U_0$. If $q \in N_0$ is any other point we see that $u(\varphi_t(p_0)) = u(\varphi_t(q))$ for $t \in \mathbb{R}$, since both functions satisfy the same differential equation (44) with same initial conditions. Due to Lemma 10 it follows that $(a, b)$ keeps the maximality property above stated, for any $q \in N_0$.

Since $V$ is parallel and equals $\varphi'_t(p)$ on $p$, it holds $\varphi'_t(p) = V_{\varphi_t(p)}$ for all $t \in (a, b)$. Therefore $t \mapsto \varphi_t$ are integral curves of $V|_{U_0}$. Two such curves do not intersect, and because $u(\varphi_t(p))$ is monotone the curve $\varphi_t(q)$ cannot be a reparametrization of $\varphi_t(p)$, for any $(s, q) \in (a, b) \times N_0$ with $q \neq p$. This concludes injectivity of $\varphi : (a, b) \times N_0 \to U_0$. Notice that $\varphi$ is the flow of $V$ restricted to $N_0$, hence it is an isometry with its image. The set $\varphi((a, b) \times N_0)$ is open.

Now we show that the image of $\varphi$ is closed in $U_0$. Let $q \in U_0$ be an adherent point of $\varphi((a, b) \times N_0)$, and $\sigma : [0, 1] \to U_0$ be a smooth curve with $\sigma(0) = p_0$, $\sigma(1) = q$. Let $I = \sigma^{-1}(U_0)$, $I$ is open in $[0, 1]$ and non-empty. Using that $\varphi$ is a local isometric coordinate chart one see that $I$ is closed, hence $I = [0, 1]$ and $q$ belongs to the image of $\varphi$. The image of $\varphi$ is then open and closed in $U_0$, and by conexity, we have $\varphi((a, b) \times N_0) = U_0$. \hfill $\Box$

Following the notation of Lemmas 10 and 12 we consider the case $b \in \mathbb{R}$. Then $\varphi_b(N_0) \subset \partial U_0$. Let $\hat{p} = \varphi_b(p_0)$. We have $\varphi_b(N_0) = N_{\hat{p}}$, since $\varphi_t$
preserves level sets of \( u \). Surprisingly, \( N_\hat{b} \) may not be isometric to \( N_0 \). This question relates to whether the curve \( t \mapsto \varphi_t(p) \) does leave \( U_0 \) when it crosses the border at \( t = b \).

Let \( U_1 \) be the component of \( U \) that contains \( \varphi_t(N_0) \) for all \( b < t < 2b - a \).

**Lemma 13.** \( N_\hat{b} \) is a geodesic complete submanifold of \( \mathcal{M} \). The map \( \varphi_b : N_0 \rightarrow N_\hat{b} \) is a local isometry. It is a bijection if and only if \( \mathcal{I}|_{U_0 \cup N_\hat{b}} \) is orientable, and it holds \( U_1 \neq U_0 \). Otherwise \( \varphi_b \) is a two-fold covering map onto \( N_\hat{b} \) and \( U_1 = U_0 \).

**Proof.** Let \( p \in N_0 \) and \( \mathcal{V} \ni \varphi_b(p) \) be a simply connected open neighborhood of \( \varphi_b(p) \). There is a local trivialization of \( \mathcal{I}|_{\mathcal{V}} \) by means of a unitary parallel vector field \( \hat{V} \), so that \( \hat{V}_{\varphi_b} = \varphi_b(p) \). By continuity, \( \varphi_b'(q) = \hat{V}_{\varphi_b(q)} \) for any \( q \in \varphi_b^{-1}(\mathcal{V}) \). Again, uniqueness of the parallel trasport along a curve subject to the same initial conditions gives us \( \hat{V}_{\varphi_{b+s}}(q) = \varphi_b'(q) \) for all \( s \) small enough. Restricting \( \mathcal{V} \) if necessary we see that \( \varphi \) is the flow of a unitary killing field defined on the open set \( \varphi((a, b + \varepsilon) \times \varphi_b^{-1}(\mathcal{V})) \cup \mathcal{V} \), for some \( \varepsilon > 0 \) small. Hence \( \varphi_b \) is a local isometry of \( \mathcal{N}_0 \) onto \( N_\hat{b} \). From that it also follows that \( N_\hat{b} \) is geodesic and complete.

Now assume \( \varphi_b \) is injective. Then \( (t, q) \in (a, b) \times N_0 \mapsto \varphi_t(q) \) is a well defined trivialization of \( \mathcal{I}|_{U_0 \cup N_\hat{b}} \), so it is orientable. If \( \varphi_t(p) \) belongs to \( U_0 \) for some \( t \in (b, 2b - a) \) then there is \( s \in (a, b) \) and \( q \in N_0 \) with \( \varphi_s(q) = \varphi_t(p) \). Both geodesics have velocities on the bundle \( \mathcal{I} \), so they must be opposite since \( u(\varphi_t(p)) \) is decreasing on \( t \). Therefore \( \varphi_t(p) \) is a backward reparametrization of \( \varphi_s(q) \) and we get \( \varphi_b(p) = \varphi_b(q) \), contradicting injectivity. Hence there must be \( U_0 \neq U_1 \).

On the other hand, if there are distinct points \( p, q \in N_0 \) with \( \varphi_b(p) = \varphi_b(q) \) one clearly has \( \varphi_b'(p) = -\varphi_b'(q) \), since both velocities lie in the same fiber of \( \mathcal{I} \) and cannot be equal. Therefore no orientation of \( \mathcal{I}|_{U_0} \) can be extended to a larger set on \( \mathcal{M} \) containing \( N_\hat{b} \), i.e., \( \mathcal{I}|_{U_0 \cup N_\hat{b}} \) is non-orientable. In this case it holds \( \varphi_{2b}(p) = q \), hence \( \varphi_{2b}(N_0) = N_\hat{b} \), what indicates that \( U_0 = U_1 \). Restricting \( \varphi_b \) to suitable vicinities \( \mathcal{V}_p, \mathcal{V}_q \) of \( p \) and \( q \), respectively, we may write \( \varphi_{2b}|_{\mathcal{V}_p} = (\varphi_b|_{\mathcal{V}_q})^{-1} \circ \varphi_b|_{\mathcal{V}_p} \), what shows that \( \varphi_{2b} \) is locally an isometry without fixed points and \( \varphi_{2b}^2 = Id_{N_0} \). This finishes the proof that \( \varphi_b : N_0 \rightarrow N_\hat{b} \) is a two-fold covering map.

Recall that an involution of a riemannian manifold is an isometry \( I \) such that \( I^2 = id \).

**Lemma 14.** \( \varphi : \mathbb{R} \times N_0 \rightarrow \mathcal{M} \) is a regular isometric covering map. Denote by \( K = Aut(\mathbb{R} \times N_0, \varphi) \) the group of covering transformations of \( \varphi \). Then, if
$I$ is orientable, $K$ is either trivial or cyclic generated by the metric product of a translation of $\mathbb{R}$ with an isometry of $N_0$. If $I$ is not orientable $K$ is generated by at most two involutions of $\mathbb{R} \times N_0$.

**Proof.** If $u$ has no critical points then $U_0 = U = \mathcal{M}$ and $\varphi$ is the (regular) trivial covering map, $I$ is orientable and $K = \{Id\}$. Otherwise $\partial U_0 \neq \emptyset$ and we assume $b$ on Lemma $12$ is finite.

Following Lemma $13$ we let $N_0 = \varphi_b(N_0)$ be a component of the border of $U_0$. If there is another component $U_1$ of $U$ that cobounds $U_0$ through $N_0$ then we can choose $p_1 \in U_1$ with $u(p_1) = u(p_0)$ and let $N_1 = N_0$. Let $\psi : (a,b) \times N_1 \to U_1$ be the map analogous to $\varphi$. It can be seen from the proof of Lemma $13$ that $(a,b) \times N_1 \to U_1$ be the map analogous to $\varphi$. It can be seen from the proof of Lemma $13$ that $\varphi_t(p) \in I_{\varphi_t}(p)$ for all $t \in \mathbb{R}, p \in N_0$. Then $\psi_b(N_1) = \varphi_b(N_0) = N_0$. It is clear that $\varphi_{2b}(N_0) = N_1$ and $\varphi_{b+s}(p) = \psi_{b+s}(\varphi_{2b}(p))$ for all $s \in \mathbb{R}, p \in N_0$. Therefore $\varphi$ is an isometry from $(a,2b-a) \times N_0$ onto $U_0 \cup N_0 \cup U_1$.

On the other hand, if $U_0$ self-bounds at $N_0$ as described by Lemma $13$ the function $\psi$ above defined equals $\varphi$, and $N_1 = N_0$. Hence $\varphi : (a,2b-a) \times N_0 \to U_0 \cup N_0 \cup U_1$ is a two-fold isometric covering map.

If $a = -\infty$ we are done. Otherwise there is another component $N_0$ of $\partial U_1, N_0 \neq N_0$. The above constructions can be repeated, extending the isometric covering property of $\varphi$ to the interval $(a,3b-2a)$. This can also be performed backwards on $t$, starting on $t = a$. An inductive argument gives us that $\varphi : \mathbb{R} \times N_0 \to \mathcal{M}$ is a covering map, and a local isometry.

If $\varphi$ is injective we have again the trivial covering, and $K = \{Id\}$. In this case one clearly has $I$ orientable. We assume in the remaining of this proof that $\varphi$ is not injective.

Suppose first that $I$ is orientable. Let $\varphi_{t_1}(p_1) = \varphi_{t_2}(p_2)$ for some $(t_1,p_1), (t_2,p_2) \in \mathbb{R} \times N_0$ distinct. Then $\varphi_{t_1}(p_1) = \varphi_{t_2}(p_2)$, so $\varphi_t(p_1)$ is an orientation preserving reparametrization of $\varphi(p_2)$. There is $\tau > 0$ with $\varphi_\tau(N_0) = N_0$, and $\tau$ can be taken the smallest positive number with such property. Then $\varphi_\tau$ is an isometry of $N_0$.

Consider the automorphism of the covering space $\mathbb{R} \times N_0$ given by $g_\tau(t,p) = (t-\tau, \varphi_\tau(p))$. A quick computation shows that the subgroup generated by $g_\tau$ acts transitively on the preimage $\varphi^{-1}(q)$ for all $q \in \mathcal{M}$. Since $K$ is completely defined by some subgroup of the permutations of $\varphi^{-1}(q)$ it becomes $K = \{g^n | n \in \mathbb{Z}\}$, and the covering map is regular.

Now consider $I$ not orientable. Reasoning similarly to the previous case we can find $C \neq 0$ so that $\varphi_C : N_0 \to N_0, \dot{p} = \varphi_C(p_0)$, is a two-fold covering, and $\varphi_{2C} : N_0 \to N_0$ is an involution. We can pick $C$ so that $|C| > 0$ is minimum. Then $g_C(t,p) = (2C-t, \varphi_{2C}(p))$ is an involution of $\mathbb{R} \times N_0$ and
a covering transformation. If \( \varphi \) is a two-fold covering then the orbits of \( \{ \text{Id}, g_C \} \) acting on \( \mathbb{R} \times \mathcal{N}_0 \) are all the preimages of points of \( \mathcal{M} \). Hence \( \varphi \) is regular and \( K = \{ \text{Id}, g_C \} \).

If \( \varphi \) is not a two-fold covering let \((t_2, p_2), (t_1, p_1)\) and \( g_C(t_1, p_1) \) be three distinct points in the preimage of a fixed point \( q \in \mathcal{M} \). The velocities of the geodesics \( s \mapsto \varphi_s(p_1) \) and \( s \mapsto \varphi_{2C-s}(\varphi_{2C}(p_1)) \) are opposite over \( q \), and we can assume, without loss of generality, that \( \varphi'_{t_2}(p_2) = \varphi'_{t_1}(p_1) \). Again there is \( \tau > 0 \) minimum such that \( \varphi_\tau(\mathcal{N}_0) = \mathcal{N}_0 \) and \( \varphi'_\tau(p) = V_{\varphi_\tau}(p) \) for any \( p \in \mathcal{N}_0 \). Define \( g_\tau \) as in the \( I \) orientable case.

Now let \((t, p)\) be any point in \( \varphi^{-1}(q) \ni (t_1, p_1) \). If \( \varphi'_t(p) = \varphi'_{t_1}(p_1) \) then there is an integer \( n \) such that \((t, p) = g^n_\tau \circ g_C(t_1, p_1) \). Otherwise \((t, p) = g^n_\tau \circ g_C(t_1, p_1) \). This shows that the action of \( K \) is transitive on the preimages and the covering map is regular. Further \( K \) is generated by \( \{ g_\tau, g_C \} \). A careful check traveling forth and back on the geodesics \( t \mapsto \varphi_t(p) \) reveals that \( \varphi_\tau \circ \varphi_{2C} \circ \varphi_\tau \circ \varphi_{2C} = \text{Id}_{\mathcal{N}_0} \). Defining \( D = C - \frac{t}{2} \) and \( g_D(t, p) = (2D - t, \varphi_{2D}(p)) \) we see that \( g_D = g_\tau \circ g_C \) is an involution of \( \mathbb{R} \times \mathcal{N}_0 \) and \( \{ g_C, g_D \} \) generates \( K \). This finishes the proof of the Lemma.

\[ \square \]

**Proof of Theorem 2** Let \( u \) be a non-constant critical point of \( \mathcal{E} \) with \( (\mathcal{L}(u)v, v) = - (\mathcal{E}'(u)v, v) = 0 \). Clearly the manifold \( \mathcal{N} \) in the Theorem stands for \( \mathcal{N}_0 \).

The proof then follows from the sequence of the Lemmas and Corollaries numbering from \([9]\) through \([14]\). The assertion \( \mathcal{M} \simeq (\mathbb{R} \times \mathcal{N}) / K \) is a standard fact in Topology \([10]\) and the metric is induced from \( \mathbb{R} \times \mathcal{N} \) through the local isometry \( \varphi \).

\[ \square \]

### 4 Existence of nonconstant minimizers

This section is devoted to show that if \( \mathcal{M} \) fails to have non-negative Ricci curvature then Theorem 1 may not hold. This will be accomplished by showing that there are non-convex surfaces for which \( \mathcal{E}_\varepsilon \) has non-constant local minimizers, for \( \varepsilon \) small enough.

The procedure we follow consists of finding the limit of the energies \( \mathcal{E}_\varepsilon \) in the sense of \( \Gamma \)-convergence and then using a result of De Giorgi which roughly states that close (in some specified topology) to an isolated minimizer of the \( \Gamma \)-limit problem there is a minimizer of the original one.

Throughout this section, \( \mathcal{M} \) will denote a surface diffeomorphic to \( S^2 \). For the reader’s convenience we give the definition of \( \Gamma \)-convergence which is going to be used.
A family \( \{ \Lambda_\varepsilon \}_{0 < \varepsilon \leq \varepsilon_0} \) of real-extended functionals defined in \( L^1(M) \) is said to \( \Gamma \)-converge in \( L^1(M) \), as \( \varepsilon \to 0 \), to a functional \( \Lambda_0 : L^1(M) \to \mathbb{R} \cup \{ \infty \} \), if:

- For each \( v \in L^1(M) \) and for any family \( \{ v_\varepsilon \} \) in \( L^1(M) \) such that \( v_\varepsilon \to v \) in \( L^1(M) \), as \( \varepsilon \to 0 \), it holds that \( \Lambda_0(v) \leq \lim \inf_{\varepsilon \to 0} \Lambda_\varepsilon(v_\varepsilon) \).
- For each \( v \in L^1(M) \) there is a family \( \{ w_\varepsilon \} \) in \( L^1(M) \) such that \( w_\varepsilon \to v \) in \( L^1(M) \), as \( \varepsilon \to 0 \) and \( \Lambda_0(v) \geq \lim \sup_{\varepsilon \to 0} \Lambda_\varepsilon(w_\varepsilon) \).

Convergence in this sense will be denoted by \( \Gamma^- \lim_{\varepsilon \to 0^+} \Lambda_\varepsilon = \Lambda_0 \). The definitions and results we need about functions of bounded variation defined on \( M \) are provided below.

We set

\[
G(M) \overset{\text{def}}{=} \{ g \mid g \text{ is a } C^1 \text{ section of } TM, \ |g(x)| \leq 1, \ \forall x \in M \}
\]

and let \( \mathcal{H}^N \) denote the usual \( N \)-dimensional Hausdorff measure.

Given \( u : M \to \mathbb{R} \) we define

\[
|Du|(M) \overset{\text{def}}{=} \sup_{g \in G(M)} \int_M u \text{ div}(g) \, d\mathcal{H}^2.
\]

A real function \( u \in L^1(M) \) has bounded variation in \( M \) if \( |Du|(M) < \infty \). See [5] when \( M \) is a bounded domain in \( \mathbb{R}^N \). The set

\[
BV(M) \overset{\text{def}}{=} \{ u : M \to \mathbb{R}; \ u \in L^1(M) \text{ and } |Du|(M) < \infty \}
\]

is a Banach space with the norm \( \|u\|_{BV} = \|u\|_{L^1} + |Du|(M) \).

Letting \( \chi_A \) denoting the characteristic function of a set \( A \subseteq M \) we have

\[
|D\chi_A|(M) = \sup_{g \in G(M)} \int_A \text{ div}(g) \, d\mathcal{H}^2.
\]

The perimeter of a set \( A \subseteq M \) is defined by \( \text{Per}_M(A) := |D\chi_A|(M) \). If the border of \( A \) in \( M \) is at least \( C^2 \) then \( |D\chi_A|(M) = \mathcal{H}^1(\partial A \cap M) \).

Throughout this section we assume that the potential \( F \) in (3) satisfies:

- \( F : \mathbb{R} \to \mathbb{R} \) is \( C^2 \)
- \( F \geq 0 \) and \( F(t) = 0 \) if and only if \( t \in \{ \alpha, \beta \}, \ \alpha < \beta \).
- \( \exists t_0 > 0, \ c_1 > 0, \ c_2 > 0, \ k > 2 \) such that \( c_1 t^k \leq F(t) \leq c_2 t^k \), for \( |t| \geq t_0 \).
For convenience we denote the space of functions of bounded variation in $\mathcal{M}$ taking only two values, $\alpha$ and $\beta$, by $BV(\mathcal{M}, \{\alpha, \beta\})$.

The computation of the $\Gamma$–limit of $\mathcal{E}_\varepsilon$ when $\mathcal{M}$ is a bounded domain in $\mathbb{R}^N$ is standard by now. However no such result is available in the literature when $\mathcal{M}$ is a surface. Nevertheless the proof found in [1] can be adapted to our case in a natural manner thus yielding

**Theorem 15.** Let $\mathcal{E}_\varepsilon : L^1(\mathcal{M}) \to \mathbb{R}$ be defined by

\[
\mathcal{E}_\varepsilon(u) = \begin{cases} 
\int_\mathcal{M} \left[\frac{\varepsilon |\nabla u|^2}{2} - \varepsilon^{-1} F(u)\right] d\mathcal{H}^2 & \text{if } u \in H^1(\mathcal{M}) \\
\infty & \text{if } u \in L^1(\mathcal{M}) \setminus H^1(\mathcal{M})
\end{cases}
\]

Then $\Gamma^\varepsilon \lim_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon = \mathcal{E}_0$ where

\[
\mathcal{E}_0(u) = \begin{cases} 
\lambda |D\chi_{\{u=\alpha\}}|(\mathcal{M}) & \text{if } u \in BV(\mathcal{M}, \{\alpha, \beta\}) \\
\infty & \text{otherwise}
\end{cases}
\]

and

\[
\lambda = \int_0^1 \sqrt{F(s)} \, ds .
\]

We say that $v_0 \in L^1(\mathcal{M})$ is an $L^1$-local minimizer of the functional $\Lambda_0 : L^1(\mathcal{M}) \to \mathbb{R} \cup \{\infty\}$ if there is $r > 0$ such that

\[
\Lambda_0(v_0) \leq \Lambda_0(v) \quad \text{whenever } 0 < \|v - v_0\|_{L^1(\mathcal{M})} < r .
\]

Moreover if $\Lambda_0(v_0) < \Lambda_0(v)$ for $0 < \|v - v_0\|_{L^1(\mathcal{M})} < r$, then $v_0$ is called an isolated $L^1$-local minimiser of $\Lambda_0$.

The next result, which we use in order to find a family of minimizers for $\mathcal{I}$, is due to De Giorgi and can be found in its abstract form in [15]. A proof, with the hypotheses on $F$ given above, can be found in [8], since the replacement of Lebesgue measure with Haussdorf measure does not affect the arguments used.

**Theorem 16.** Suppose that a sequence of real-extended functionals $\{\Lambda_\varepsilon\}$ and $\Lambda_0$ satisfy

(i) $\Gamma^\varepsilon \lim_{\varepsilon \to 0^+} \Lambda_\varepsilon = \Lambda_0$

(ii) Any sequence $\{v_\varepsilon\}_{\varepsilon > 0}$ such that $\Lambda_\varepsilon(v_\varepsilon) \leq C < \infty$ for all $\varepsilon > 0$, is compact in $L^1(\mathcal{M})$. 

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(iii) There exists an isolated $L^1$-local minimizer $v_0$ of $\Lambda_0$.

Then $\exists \varepsilon_0 > 0$ and a family $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ such that

- $v_\varepsilon$ is an $L^1$-local minimiser of $\Lambda_\varepsilon$, and
- $\|v_\varepsilon - v_0\|_{L^1(M)} \to 0$, as $\varepsilon \to 0$.

The growth condition on $F$ is required in order to have the hypothesis on compactness (ii) satisfied. We also take, without loss of generality, $\lambda = 1$ on equation (55).

For any $u \in BV(M,\{\alpha,\beta\})$ we denote by $\gamma$ its boundary curve, i.e., $\gamma = \partial\{p \in M \mid u(p) = \alpha\}$. Similarly, for any such $\gamma$ there are exactly two distinct functions in $BV(M,\{\alpha,\beta\})$ with $\gamma$ as boundary curve. It holds $E_0(u) = |\gamma|$. Given $r > 0$ there exists $\tilde{u} \in BV(M,\{\alpha,\beta\})$ so that $\tilde{\gamma}$ is the disjoint union of a finite number of smooth closed curves satisfying

- $\|u - \tilde{u}\|_{BV} < r$;
- $|\gamma| \geq |\tilde{\gamma}|$.

We set

$$BV_s(M,\{\alpha,\beta\}) = \{u \in BV(M,\{\alpha,\beta\}) \mid \gamma \subset M$$

is a smooth 1-dimensional submanifold}.

Now we assume that a simple closed geodesic $\gamma_0$ is separable, i.e., $M - \{\gamma_0\}$ has two components. Let $u_0 \in BV_s(M,\{\alpha,\beta\})$ be the function associated to $\gamma_0$ so that $u_0 = \alpha \chi_{M_\alpha} + \beta \chi_{M_\beta}$ with $M_i = \{p \in M \mid u_0(p) = i\}$ ($i = \alpha,\beta$).

**Theorem 17.** Under the hypotheses and notation of Theorem 3 it holds that $u_0$ is an $L^1(M)$-local isolated minimizer of $E_0$.

**Proof.** Let $V$ be the neighborhood constructed in preparation for Lemma 8. We choose $0 < \delta_0 < \delta$ and define $V_0 = \varphi([-\delta_0,\delta_0] \times \gamma_0)$. We claim that any $r > 0$ with

$$r < |\beta - \alpha| \delta_0 \min \left\{ \delta - \delta_0, \frac{|\gamma_0|}{2} \right\}$$

will verify $E_0(u) > E_0(u_0)$ whenever $u \in BV(M,\{\alpha,\beta\})$ and $0 < \|u - u_0\|_{L^1} < r$. 

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The discussion prior to the theorem allows us to restrict our attention to competing functions \( u \in BV_s(M, \{\alpha, \beta\}) \). Let \( \gamma \) be the boundary curve of a given \( u \). A differential topology argument (see [6]) allows us to consider \( \gamma \) in generic position with \( \partial V_0 \) and \( \partial V \), or equivalently, \( \gamma \) is transversal to the boundaries of \( V_0 \) and \( V \). In particular, each connected component of \( \gamma \cap V_0 \) is diffeomorphic to either \( S^1 \subset \text{int} V_0 \) or \([0, 1] \subset V_0 \) and endpoints contained in \( \partial V_0 \). We define

\[
D = \{ \sigma \mid \sigma \text{ is a component of } \gamma \cap V_0 \},
\]

\[
I = \bigcup_{\sigma \in D} \sigma \subset \gamma_0 .
\]

**Lemma 18.** Let \( u \in BV_s(M, \{\alpha, \beta\}) \) with \( \|u - u_0\| < r \). Then \(|I| > \max \left\{ |\gamma_0| - (\delta - \delta_0), \frac{\|\gamma_0\|}{2} \right\} \).

**Proof.** For each \( \sigma \in D, \bar{\sigma} \) is a closed segment of \( \gamma_0 \). Hence,

\[
J \overset{\text{def}}{=} \gamma_0 - I = \bigcup_{i=1}^{m} J_i ,
\]

where each \( J_i \) is an open interval of \( \gamma_0 \), and the \( J_i \)'s are pairwise disjoint. The construction leading to \( J \) clearly yields

\[
\gamma \cap \varphi([-\delta_0, \delta_0] \times J_i) = \emptyset \quad \text{for } 1 \leq i \leq m .
\]

Therefore, \( u \) is constant in \( \varphi([-\delta_0, \delta_0] \times J_i) \). Since \( u_0 \) switches its value over \( J_i \) we conclude that \( |u - u_0| = |\beta - \alpha| \) in one of the regions \( \varphi([-\delta_0, 0] \times J_i) \) or \( \varphi([0, \delta_0] \times J_i) \). Applying Lemma 3 part (b) we derive

\[
\|u - u_0\|_{L^1(\varphi([-\delta_0, \delta_0] \times J_i))} > |\beta - \alpha| \delta_0 |J_i| .
\]

Thus

\[
r > \|u - u_0\|_{L^1} > \sum_{i=1}^{m} |\beta - \alpha| \delta_0 |J_i| = |\beta - \alpha| \delta_0 (|\gamma_0| - |I|)
\]

\[
\Rightarrow |I| > |\gamma_0| - \frac{r}{|\beta - \alpha| \delta_0} .
\]

Together with (57) the above inequality readily implies the Lemma. \( \square \)
We set a little more notation: for any \( \sigma \in D \) let \( \rho = \rho(\sigma) \) be the component of \( \gamma \) that contains \( \sigma \) as an arc. We are led to three cases:

(i) If there is some \( \rho(\sigma) \not\subset V \) then there is an arc \( \tilde{\sigma} \subset \rho \) joining a point of \( \partial V_0 \) to a point of \( \partial V \). Lemma 6 (part (a)) gives us \( |\tilde{\sigma}| \geq \delta - \delta_0 \) and then

\[
|\gamma| \geq |\tilde{\sigma}| + \sum_{\sigma \in D} |\sigma| \geq \delta - \delta_0 + |I|
\]

in view of Lemma 18.

(ii) If there is some \( \rho(\sigma) \subset V \) that is freely homotopic to \( \gamma_0 \) within \( V \) then the intersection number of \( \rho \) with any geodesic ray \( t \mapsto \varphi_t(x) \) is \( \pm 1 \). Denoting by \( \bar{\rho} \) the projection of \( \rho \) over \( \gamma_0 \) we get \( \bar{\rho} = \gamma_0 \). Hence, Lemma 6 part (a2) gives us \( |\gamma| \geq |\rho| \geq |\gamma_0| \). The strictness \( |\gamma| > |\gamma_0| \) comes from \( \|u - u_0\|_{L^1} > 0 \), since there must be another component \( \rho' \neq \rho \) of \( \gamma \) or \( \rho \) is not equal to \( \gamma_0 \).

(iii) Assume that neither (i) nor (ii) occurs. If for some \( \sigma \in D \) we have \( \bar{\rho} = \gamma_0 \) we conclude similarly to case (ii) above, hence \( |\gamma| > |\gamma_0| \). Otherwise, let \( p \) and \( q \) be points of \( \rho \) so that their projections over \( \gamma_0 \) are the end points of the segment \( \bar{\rho} \subset \gamma_0 \). Let \( \sigma_1 \) and \( \sigma_2 \) be the two distinct arcs of \( \rho \) joining \( p \) and \( q \) (\( \sigma_i \subset V \), \( i = 1, 2 \)), with projections respectively \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \). Since the intersection number of \( \rho \) with the ray \( t \mapsto \varphi_t(x) \) is 0 we have \( \bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\rho} \). Hence \( |\rho| = |\sigma_1| + |\sigma_2| > 2|\sigma_1| \). Fixing \( \rho \) we see that any \( \sigma \in D \) that is an arc of \( \rho \) satisfies \( \bar{\sigma} \subset \bar{\sigma}_1 \). Then,

\[
|\gamma| \geq |\bar{\sigma}_1| < \frac{1}{2} |\rho| ,
\]

from which we derive

\[
|\gamma| = \sum_{\rho \text{ a component of } \gamma} |\rho| > 2|I| > |\gamma_0| .
\]

Therefore \( E_0(u) = |\gamma| > |\gamma_0| = E_0(u_0) \) if \( 0 < \|u - u_0\|_{L^1} < r \) and the theorem is proved.

\textbf{Proof of Theorem 2} As mentioned before, Theorem 2 is just an application of Theorem 16 for \( \Lambda_\varepsilon = E_\varepsilon \), whose hypotheses we now verify. Indeed (i) is nothing but Theorem 15 and (ii) may be found in [12], for instance. Although the proof of (ii) in [12] is rendered for \( M \) a bounded domain in \( \mathbb{R}^N \) the proof holds equally well in our case.

As for (iii) it has been verified in Theorem 17 above.
The following result seems to be known, though we have not been able to find it in the literature. It is a consequence of the procedure used in this section along with Theorem[1]

**Lemma 19.** Let $M$ be a compact Riemann surface with no boundary and having nonnegative Gaussian curvature. Then $M$ has no closed nonintersecting isolated minimizing geodesic.

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