ASYMPTOTIC ANALYSIS OF THE SUBSTRATE EFFECT FOR AN ARBITRARY INDOCTOR

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Summary

A quasistatic unilateral frictionless contact problem for a rigid axisymmetric indenter pressed into a homogeneous, linearly elastic and transversely isotropic elastic layer bonded to a homogeneous, linearly elastic and transversely isotropic half-space is considered. Using the general solution to the governing integral equation of the axisymmetric contact problem for an isotropic elastic half-space, we derive exact equations for the contact force and the contact radius, which are then approximated under the assumption that the contact radius is sufficiently small compared to the thickness of the elastic layer. An asymptotic analysis of the resulting non-linear algebraic problem corresponding to the fourth-order asymptotic model is performed in the case when the contact radius becomes vanishingly small compared with the layer thickness. A special case of the indentation problem for a blunt punch of power-law profile is studied in detail. Approximate force-displacement relations are obtained in explicit form, which is most suited for development of indentation tests.

1. Introduction

In recent years, development of minimally invasive measurement techniques for determining the biomechanical properties of biological and artificial replacement tissues has become a problem of paramount importance in medicine. The mechanical properties of biomaterials can be evaluated by means of simple indentation techniques (1), which have been proved useful for both identification of mechanical properties of a soft biological tissue, like articular cartilage (2), and assessing its viability during arthroscopy (3). Indentation tests are performed with various indenter geometries, and among the most frequently used types of indenters are axisymmetric (in particular, cylindrical, spherical and conical) indenters.

Development of an indentation test for the articular cartilage layer requires solving the corresponding contact problem for an elastic layer (see Fig. 1a). The widely used mathematical model for indentation tests of articular cartilage developed by Hayes et al. (2) is based on the analytical solution (4) of the frictionless contact problem for an elastic layer bonded to a rigid base. This model takes the thickness effect into account, but it neglects the substrate effect. The viscoelastic effect was
taken into account in [5], where one can also find an example output concerning spherical indentation of articular cartilage. In order to take into account the influence of the substrate deformation, we consider the frictionless contact problem for an elastic layer bonded to an elastic half-space (see Fig. 1b).

In the present study, we consider the quasistatic unilateral contact problem for a rigid axisymmetric indenter pressed into a homogeneous, linearly elastic and transversely isotropic elastic layer bonded to a homogeneous, linearly elastic and transversely isotropic half-space. We focus on deriving asymptotic approximations for the force-displacement relation (that is the relation between the contact force $P$ and the indenter displacement $w$), which constitutes the basis for developing depth-sensing indentation tests. In the isotropic case, the substrate effect in the flat-ended and spherical indentation was studied recently in [6] based on the asymptotic solutions obtained in [7, 8]. The main novelty of the present paper consists in considering the case of an arbitrary axisymmetric indenter.

As a special case, we study the indentation problem for a blunt punch of power-law profile. The main difficulty for an analytical analysis of the axisymmetric unilateral contact problem is associated with the fact that the radius $a$ of the circular contact area $\omega$ is not known in advance and must be determined as a part of the solution. Apparently the first asymptotic analysis of the axisymmetric contact problem for an arbitrary axisymmetric indenter dates back to England, whose work [9] was published in 1962. As a special case, England [9] considered the indentation problem for a hemispherically ended punch and obtained asymptotic expansions in terms of powers of a small parameter $\mu = R/h$, where $R$ is the curvature radius of the hemispherical end, $h$ is the thickness of the elastic layer.

Our asymptotic analysis is based on a small parameter $\epsilon = a/h$ defined as the ratio of the contact radius to the layer thickness. It should be noted that in contrast to the method originally developed by England [9], we construct asymptotic expansions in terms of an unknown quantity in the same way as in the asymptotic method of large $\lambda$ developed by Vorovich et al. [7]. However, our approach differs from previous studies [7, 10] in two ways. First, we do not construct an asymptotic expansion for the contact pressure, but for the aim of deriving the force-displacement relation, following [11, 12], we derive exact equations for the contact force and the contact radius and operate with integral characteristics of the contact pressure. Second, we perform asymptotic analysis of the resulting
non-linear algebraic problem obtaining the approximate force-displacement relation in explicit form, which is most suited for development of indentation tests.

2. Indentation problem formulation

We consider the unilateral frictionless contact problem for an elastic layer bonded to an elastic half-space, which is made of a different material. Let the composite elastic medium occupying the half-space \( x_3 > 0 \) be indented by an arbitrary frictionless axisymmetric indenter. Assuming that the \( x_3 \)-axis coincides with the axis of symmetry of the indenter, we will describe its surface by the equation

\[
x_3 = \Phi(x_1, x_2),
\]

where \( \Phi(x_1, x_2) \) is the shape function whose dependence on the Cartesian coordinates \( x_1 \) and \( x_2 \) comes through a polar radius \( r = \sqrt{x_1^2 + x_2^2} \). Without loss of generality, we may assume that \( \Phi(0, 0) = 0 \).

Under the assumption that the indenter shape function \( \Phi(x_1, x_2) \) is convex and the indenter is loaded with a normal force, \( P \), which is directed along the positive \( x_3 \)-axis, contact between the indenter and the surface of the elastic medium will be established over a circular region, \( \omega \), say, of radius \( a \). For a blunt indenter, the contact area \( \omega \) is variable and depends on the contact force \( P \).

Further, let \( G_3(x_1, x_2, 0) \) be the surface influence function for the elastic medium that gives the vertical surface displacement of the elastic layer under a unit point force applied at the origin of coordinates and directed along \( x_3 \)-axis. Then, the unilateral contact problem under consideration, which involves an a priori unknown contact area \( \omega \), reduces to that of solving the following integral equation:

\[
\int_\omega G_3(x_1 - y_1, x_2 - y_2, 0) p(y) \, dy = w - \Phi(x_1, x_2).
\]

(2.2)

Here, \( p(x_1, x_2) \) is the contact pressure, \( w \) is the indenter displacement.

It should be emphasized that (2.2) implicitly contains an a priori unknown radius \( a \) of the contact area \( \omega \), which is to be determined as a part of solution from the following conditions:

\[
p(y) > 0 \quad \text{for} \quad 0 \leq |y| < a, \quad p(y) = 0 \quad \text{for} \quad |y| = a.
\]

(2.3)

Here, \( |y| = \sqrt{y_1^2 + y_2^2} \) is a polar radius. The integral Equation (2.2) with the positiveness condition (2.3) constitute the so-called unilateral contact problem for a semi-infinite linearly elastic body. This problem can be reformulated in terms of variational inequalities (13, 14).

By the equilibrium equation, the external contact force \( P \) is related to the contact pressure density \( p(x_1, x_2) \) as follows:

\[
\int_\omega p(y) \, dy = P.
\]

(2.4)

Due to the axisymmetry of the elastic medium, the surface influence function can be represented as

\[
G_3(y_1, y_2, 0) = \frac{1}{2\pi\theta} \left( \frac{1}{|y|} - \frac{1}{h^2} G\left( \frac{|y|}{h} \right) \right),
\]

(2.5)

where \( h \) is the layer thickness, \( \theta \) is an elastic constant related to the first layer, \( G(t) \) is a dimensionless function.
It can be shown (7) that the following integral representation holds:

$$G(t) = \int_{0}^{\infty} \left[ 1 - L(u) \right] J_0(ut) \, du. \quad (2.6)$$

Here, $J_0(x)$ is the Bessel function of the first kind, the kernel function $L(u)$ depends both on the material properties of the elastic layer and the elastic half-space.

According to (7), there is a neighbourhood of zero on which the function (2.6) is expanded into an absolutely convergent power series

$$G(t) = \sum_{m=0}^{\infty} a_m t^{2m} \quad (2.7)$$

with the coefficients

$$a_m = \frac{(-1)^m}{[2m]!!} \int_{0}^{\infty} \left[ 1 - L(u) \right] u^{2m} \, du. \quad (2.8)$$

Thus, the indentation problem is to find the force-displacement relationship, $P(w)$. In the present study, we employ an asymptotic approach to derive approximate analytical formulas for $P(w)$ in the small-contact limit that is under the assumption that the contact radius $a$ is small compared with the layer thickness $h$.

3. Equation for the radius of the contact area

In the axisymmetric case, (2.2) can be reduced to a simpler form (see, for example, (7, 10))

$$\frac{2}{\pi} \int_{0}^{a} K \left( \frac{2\sqrt{\rho r}}{\rho + r} \right) p(\rho) \rho \, d\rho = \theta \left( w - \Phi(r) \right) + \frac{1}{h} \int_{0}^{a} F(\frac{\rho}{h}, \frac{r}{h}) p(\rho) \rho \, d\rho, \quad (3.1)$$

where $K(x)$ is the complete elliptic integral of the first kind, $F(\sigma, \tau)$ is a dimensionless function given by

$$F(\sigma, \tau) = \int_{0}^{\infty} \left[ 1 - L(u) \right] J_0(\sigma u) J_0(\tau u) \, du. \quad (3.2)$$

In the neighbourhood of the origin, the function $F(\sigma, \tau)$ has the following absolutely convergent expansion (10):

$$F(\sigma, \tau) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} \sigma^{2i} \tau^{2j}, \quad (3.3)$$

$$b_{ij} = \frac{(m!)^2}{(i!)^2 (j!)^2} a_m \quad (m = i + j),$$

where $a_m$ are given by (2.3).
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Now, introducing an auxiliary function $v(r)$ by the formula

$$
\frac{1}{2\pi} v(r) = \theta (w - \Phi(r)) + \frac{1}{h} \int_0^a \mathcal{F} \left( \frac{\rho}{h}, \frac{r}{h} \right) p(\rho) \rho \, d\rho
$$

(3.4)

and making use of the general solution of the axisymmetric contact problem for an elastic isotropic half-space [15–17], we represent the solution to (3.1) in the form

$$
p(r) = \frac{V(a)}{\pi \sqrt{a^2 - r^2}} - \frac{1}{\pi} \int_r^a \frac{V'(s) \, ds}{\sqrt{s^2 - r^2}},
$$

(3.5)

$$
\pi V(r) = v(0) + \int_0^r \frac{v'(s) \, ds}{\sqrt{r^2 - s^2}}.
$$

(3.6)

From the boundary condition (2.3) of vanishing contact pressure at the edge contour of the contact area, it necessarily follows that

$$
V(a) = 0.
$$

(3.7)

Hence, in view of (3.7), formula (3.5) reduces to

$$
p(r) = -\frac{1}{\pi} \int_r^a \frac{V'(s) \, ds}{\sqrt{s^2 - r^2}}.
$$

(3.8)

Now, according to (3.4) and the assumption $\Phi(0) = 0$, we will have

$$
\frac{1}{2\pi} v(0) = \theta w + \frac{1}{h} \int_0^a \mathcal{F} \left( \frac{\rho}{h}, 0 \right) p(\rho) \rho \, d\rho,
$$

(3.9)

$$
\frac{1}{2\pi} v'(r) = -\theta \Phi'(r) + \frac{1}{h^2} \int_0^a \frac{\partial \mathcal{F}}{\partial r} \left( \frac{\rho}{h}, \frac{r}{h} \right) p(\rho) \rho \, d\rho.
$$

(3.10)

The substitution of the expression (3.2) into (3.9), (3.10) and then into (3.6) yields the following formulas (11):

$$
V(r) = 2\theta w - 2\theta r \int_0^r \frac{\Phi'(s) \, ds}{\sqrt{s^2 - r^2}} + \frac{2}{h} \int_0^a S_0 \left( \frac{\rho}{h}, \frac{r}{h} \right) p(\rho) \rho \, d\rho,
$$

(3.11)

$$
S_0(\sigma, \tau) = \int_0^\infty \left[ 1 - L(u) \right] J_0(\sigma u) \cos \tau u \, du.
$$

(3.12)

Thus, (3.7) for determining the contact radius $a$ takes the form

$$
w = a \int_0^a \frac{\Phi'(\rho) \, d\rho}{\sqrt{\rho^2 - a^2}} - \frac{1}{\theta h} \int_0^a S_0 \left( \frac{\rho}{h}, \frac{a}{h} \right) p(\rho) \rho \, d\rho.
$$

(3.13)

It is clear that if an approximation for the contact pressure $p(r)$ is known, the second term of the right-hand side of (3.13) will give a correction to the corresponding equation of the Galin–Sneddon theory of axisymmetric elastic contact [15–19].
4. Equation for the contact force
Substituting (3.8) into the equilibrium equation (2.4), we will have

\[ P = 2 \int_{0}^{a} \mathcal{V}(s) \, ds. \]

(4.1)

Now, the substitution of (3.11) for \( \mathcal{V}(s) \) into (4.1) yields

\[ P = 4 \theta wa - 4 \theta \int_{0}^{a} \Phi'(\rho) \sqrt{a^2 - \rho^2} \, d\rho + 4 \int_{0}^{a} S_2 \left( \frac{\rho}{h}, \frac{a}{h} \right) p(\rho) \rho \, d\rho, \]

(4.2)

where

\[ S_2(\sigma, \alpha) = \int_{0}^{\infty} \left[ 1 - L(u) \right] J_0(\sigma u) \sin \alpha u \, du. \]

(4.3)

Finally, taking into account (3.13), we eliminate the variable \( w \) in (4.2) as follows:

\[ P = 4 \theta \int_{0}^{a} \Phi'(\rho) \sqrt{a^2 - \rho^2} \, d\rho + 4 \int_{0}^{a} \tilde{S}_2 \left( \frac{\rho}{h}, \frac{a}{h} \right) p(\rho) \rho \, d\rho, \]

(4.4)

Here we introduced the notation

\[ \tilde{S}_2(\sigma, \alpha) = \int_{0}^{\infty} \left[ 1 - L(u) \right] J_0(\sigma u) \left( \frac{\sin \alpha u}{u} - \alpha \cos \alpha u \right) \, du. \]

(4.5)

Again, if we have an approximation for the contact pressure \( p(r) \), then the second term of the right-hand side of (4.4) will produce a correction to the corresponding equation of the Galin–Sneddon theory of axisymmetric elastic contact [18, 19].

5. The fourth-order asymptotic model
According to (3.12) and (4.5), the following series expansions hold:

\[ S_0(\sigma, \alpha) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}^{(0)} \sigma^{2i} \alpha^{2j}, \]

(5.1)

\[ \tilde{S}_2(\sigma, \alpha) = \alpha \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \tilde{b}_{ij}^{(2)} \sigma^{2i} \alpha^{2j}, \]

(5.2)

\[ b_{ij}^{(0)} = \frac{2i(2i)!}{(i!)^2(2j)!} a_m, \quad \tilde{b}_{ij}^{(2)} = -\frac{2^{i+j}j(2j+1)!}{(i!)^2(2j+1)!} a_m \quad (m = i + j). \]

Now, keeping terms of the infinite series (5.1) and (5.2) that contain only the coefficients \( a_0 \) and \( a_1 \), we get

\[ S_0(\sigma, \alpha) \simeq a_0 + a_1(\sigma^2 + 2\alpha^2), \quad \tilde{S}_2(\sigma, \alpha) \simeq -\frac{4}{3} a_1 \alpha^3. \]

(5.3)
The substitution of the asymptotic expressions (5.3) into (3.13) and (4.4) results in the following approximate relations:

\[
\begin{align*}
    w & \simeq a \int_0^a \frac{\Phi'(\rho)}{\sqrt{a^2 - \rho^2}} d\rho - \frac{1}{2\pi \theta h} \left\{ P \left( a_0 + 2a_1 \frac{a^2}{h^2} \right) - P_2 \frac{a_1}{h^2} \right\}, \\
    P & \approx 4\theta \int_0^a \frac{\Phi'(\rho)^2}{\sqrt{a^2 - \rho^2}} d\rho - \frac{8a_1 a^3}{3\pi h^3} P.
\end{align*}
\]

(5.4)

(5.5)

Here, \( P \) and \( P_2 \) are, respectively, the contact force and the polar moment of inertia of the contact pressure,

\[
\begin{align*}
    P & = 2\pi \int_0^a p(\rho) \rho d\rho, \\
    P_2 & = 2\pi \int_0^a \rho^2 p(\rho) \rho d\rho.
\end{align*}
\]

(5.6)

Substituting the expression (3.8) for the contact pressure into (5.6), we obtain

\[
P_2 = 4 \int_0^a V(s)s^2 ds.
\]

(5.7)

Now, the substitution of (3.11) for \( V(s) \) into (5.7) yields

\[
P_2 \simeq 8\theta \left( \frac{a_3}{3} w - \int_0^a \Phi(\rho) \frac{2\rho^2 - a^2}{\sqrt{a^2 - \rho^2}} \rho d\rho \right) + \frac{8a_3^3}{3\pi} \int_0^a S_4 \left( \frac{\rho}{h}, \frac{a}{h} \right) p(\rho) \rho d\rho.
\]

(5.8)

Here we introduced the notation

\[
S_4(\sigma, \alpha) = \int_0^\infty [1 - L(u)] J_0(\sigma u) \left\{ \sin \alpha u \frac{\sin \alpha u}{\alpha u} - \frac{2}{(\alpha u)^2} \left( \sin \alpha u - \cos \alpha u \right) \right\} du.
\]

(5.9)

In view of the fact that the quantity \( P_2 \) enters (5.4) with the coefficient \( a_1 \), (5.8) may be reduced to the following one being in the same range of accuracy with (5.4) and (5.5):

\[
P_2 \simeq 8\theta \left( \frac{a_3}{3} w - \int_0^a \Phi(\rho) \frac{2\rho^2 - a^2}{\sqrt{a^2 - \rho^2}} \rho d\rho \right) + \frac{4a_3^3}{3\pi h} a_0 P.
\]

(5.10)

Note that in writing down (5.10), it is assumed that

\[
S_4(\sigma, \alpha) \simeq \frac{1}{3} a_0.
\]

(5.11)

Thus, the three relations (5.4) (5.5) and (5.10) form the resulting algebraic problem with respect to the four variables \( w, P, P_2 \) and \( a \).
Eliminating the variable $P_2$ from (5.4) by the substitution (5.7), we obtain
\[ w \left( 1 + \frac{4a_1 a^3}{3\pi h^3} \right) \approx 4a_1 \int_0^a \frac{d\rho}{\sqrt{a^2 - \rho^2}} + 4a_1 \int_0^a \frac{\Phi'(\rho) \rho^2 d\rho}{\sqrt{a^2 - \rho^2}} \rho d\rho - \frac{P}{2\pi h} \left( a + 2a_1 a^3 + \frac{4a_1 a_3}{3\pi h^3} \right). \] (5.12)

On the other hand, (5.5) yields
\[ \frac{P}{a} \left( 1 + \frac{8a_1 a^3}{3\pi h^3} \right) \approx 4\int_0^a \frac{\Phi'(\rho) \rho^2 d\rho}{\sqrt{a^2 - \rho^2}}. \] (5.13)

With the same asymptotic accuracy as that of (5.4), (5.5) and (5.10), we obtain from (5.13) the following approximate relation:
\[ P \approx \left( 1 - \frac{8a_1 a^3}{3\pi h^3} \right) 4\int_0^a \frac{\Phi'(\rho) \rho^2 d\rho}{\sqrt{a^2 - \rho^2}}. \] (5.14)

Now, substituting the expression (5.14) into (5.12), we arrive after some asymptotic simplifications at the following expansion:
\[ w \approx \left( 1 - \frac{4a_1 a^3}{3\pi h^3} \right) 4a_1 \int_0^a \frac{d\rho}{\sqrt{a^2 - \rho^2}} + 4a_1 \int_0^a \frac{\Phi'(\rho) \rho^2 d\rho}{\sqrt{a^2 - \rho^2}} \rho d\rho - \frac{2}{\pi h} \left( a_0 + 2a_1 a^3 - \frac{8a_1 a_3}{3\pi h^3} \right) \int_0^a \frac{\Phi'(\rho) \rho^2 d\rho}{\sqrt{a^2 - \rho^2}}. \] (5.15)

Thus, (5.14) and (5.15) define the force displacement relation in a parametric way.

We note that the fourth-order asymptotics of the axisymmetric unilateral contact problem for an elastic layer was previously derived by Alexandrov and Pozharskii (10) in a somewhat different way (see (10), Ch. 1.3, Eqs. (25) and (26))). It can be shown that the two solutions are asymptotically equivalent up to terms of order $O(\varepsilon^5)$ as $\varepsilon \to 0$, where $\varepsilon = a/h$.

6. The case of a blunt indenter

Let us consider indentation of the elastic medium by a blunt indenter with the shape function
\[ \Phi(r) = Ar^\lambda, \] (6.1)

where $\lambda > 1$ is an arbitrary real number, $A$ is a constant having the dimension $[L^{1-\lambda}]$ with $L$ being the dimension of length. For an elastic half-space, analytical relations between the contact force, contact radius and depth of indentation were given by Galin in 1946 (18). It was shown relatively recently by Borodich et al. (20) that such a geometrical description is related to real indenters.
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Taking into account the integral
\[ \int_0^1 \frac{t^{\lambda+1} dt}{\sqrt{1-t^2}} = \frac{2^{\lambda-2}\lambda}{(\lambda+1)} \Gamma\left(\frac{\lambda}{2}\right)^2 \Gamma\left(\lambda^{-1}\right), \]
we reduce (5.14) and (5.15) as follows:

\[ P \simeq \theta A F_1(\lambda) a^{\lambda+1} \left(1 - \varepsilon^{3/8} a_1^{3/\pi}\right), \quad \text{for} \quad \lambda \gg 1, \quad (6.2) \]

\[ w \simeq A F_2(\lambda) a^{\lambda} \left\{1 - \varepsilon - \frac{2\lambda a_0}{\pi (\lambda + 1)} - \varepsilon^{3/8} \frac{8\lambda (2\lambda + 5) a_1}{3\pi (\lambda + 3) (\lambda + 1)} + \varepsilon^{4/16} \frac{16\lambda a_0 a_1}{3\pi^2 (\lambda + 1)}\right\}, \quad \text{for} \quad \lambda \gg 1, \quad (6.3) \]

Here we introduced the notation

\[ F_1(\lambda) = \frac{\lambda^{2\lambda}}{(\lambda + 1)^2} \Gamma\left(\frac{\lambda}{2}\right)^2 \Gamma\left(\lambda^{-1}\right), \quad (6.4) \]

\[ F_2(\lambda) = \lambda^{2\lambda-2} \Gamma\left(\frac{\lambda}{2}\right)^2 \Gamma\left(\lambda^{-1}\right). \quad (6.5) \]

Following (21) and using the asymptotic method based on Lagrange's formula for solving algebraic equations (22), we invert (6.3) to find the contact radius as a function of the indenter displacement in following form

\[ a/h \simeq \sigma \left(1 + B_1 \sigma + B_2 \sigma^2 + B_3 \sigma^3 + B_4 \sigma^4\right) \quad \text{for} \quad \lambda \gg 1, \quad (6.6) \]

with the coefficients given by

\[ B_1 = \frac{2a_0}{\pi (\lambda + 1)}, \quad B_2 = \frac{(\lambda + 3)}{2 (\lambda + 1)^2} \left(\frac{2a_0}{\pi}\right)^2, \]

\[ B_3 = \frac{(\lambda + 4) (\lambda + 2)}{3 (\lambda + 1)^3} \left(\frac{2a_0}{\pi}\right)^3 + \frac{(2\lambda + 5)}{(\lambda + 1) (\lambda + 3)} \frac{8a_1}{3\pi}, \]

\[ B_4 = \frac{(2\lambda + 5) (\lambda + 5) (3\lambda + 5) (\lambda + 3)}{24 (\lambda + 1)^4 (\lambda + 3)} \left(\frac{2a_0}{\pi}\right)^4 + \frac{(\lambda^2 + 11\lambda + 22)}{(\lambda + 1)^2 (\lambda + 3)^2} \frac{16a_0 a_1}{3\pi^2}. \]

Here we introduced the notation

\[ \sigma = \frac{1}{h} \left(\frac{w}{AF_2(\lambda)}\right)^{1/\lambda}. \quad (6.7) \]

Now, substituting the expansion (6.6) into (6.2), we obtain

\[ P \simeq \theta A^{-\frac{1}{2}} F_3(\lambda) w^{\frac{1}{\lambda+1}} \left(1 + C_1 \sigma + C_2 \sigma^2 + C_3 \sigma^3 + C_4 \sigma^4\right), \quad (6.8) \]
The obtained results are in agreement with those in (6.1). Note that (6.6) and (6.8) generalize the second-order asymptotic model developed in (23).

6.1 Example: Paraboloid indenter

In the case $\lambda = 2$ and $A = 1/(2R)$, (6.1) takes the form

$$\Phi(r) = \frac{r^2}{2R},$$

(6.11)

where $R$ is the radius of curvature of the indenter's surface at its vertex.

According to (6.4), (6.5), (6.7) and (6.9), we will have

$$F_1(2) = \frac{16}{3}, \quad F_2(2) = 2, \quad F_3(2) = \frac{4\sqrt{2}}{3}, \quad \sigma = \frac{\sqrt{2}}{h},$$

$$B_1 = \frac{2a_0}{3\pi}, \quad B_2 = \frac{10a_0}{9\pi}, \quad B_3 = \frac{64a_0^3}{27\pi^3} + \frac{8a_1}{5\pi}, \quad B_4 = \frac{154a_0^4}{27\pi^4} + \frac{256a_0a_1}{45\pi^2},$$

$$C_1 = \frac{2a_0}{\pi}, \quad C_2 = \frac{14a_0^2}{3\pi}, \quad C_3 = \frac{320a_0^3}{27\pi^3} + \frac{32a_1}{15\pi}, \quad C_4 = \frac{286a_0^4}{9\pi^4} + \frac{64a_0a_1}{5\pi^2}.$$

The obtained results are in agreement with those in (23).

6.2 Example: Cone indenter

For a cone indenter, we have $\lambda = 1$ and $A = \tan \gamma$, whereas (6.1) takes the form

$$\Phi(r) = r \tan \gamma,$$

where $\gamma$ is the angle between the contact surface and the side surface of the cone.

According to (6.4), (6.5), (6.7) and (6.9), now we have

$$F_1(1) = \pi, \quad F_2(1) = \frac{\pi}{2}, \quad F_3(1) = \frac{4}{\pi}, \quad \sigma = \frac{2w \cot \gamma}{\pi h},$$

$$B_1 = \frac{a_0}{\pi}, \quad B_2 = \frac{2a_0^2}{\pi}, \quad B_3 = \frac{5a_0^3}{\pi^3} + \frac{7a_1}{3\pi}, \quad B_4 = \frac{14a_0^4}{\pi^4} + \frac{34a_0a_1}{3\pi^2},$$

$$C_1 = \frac{2a_0}{\pi}, \quad C_2 = \frac{5a_0^2}{\pi^2}, \quad C_3 = \frac{14a_0^3}{\pi^3} + \frac{2a_1}{\pi}, \quad C_4 = \frac{42a_0^4}{\pi^4} + \frac{14a_0a_1}{\pi^2}.$$
Substituting the above coefficients into (6.6) and (6.8), we arrive at the following approximate relations:

\[ a \simeq \frac{2 \cot \gamma}{\pi} w \left\{ 1 + \frac{2a_0}{\pi^2} \cot \gamma \frac{w}{h} + \frac{8a_0^2}{\pi^4} \cot^2 \gamma \frac{w^2}{h^2} \right. \]
\[ \left. + \left( \frac{40a_0^3}{\pi^6} + \frac{56a_0}{3\pi^4} \right) \cot^3 \gamma \frac{w^3}{h^3} + \left( \frac{224a_0^4}{\pi^8} + \frac{544a_0a_1}{3\pi^6} \right) \cot^4 \gamma \frac{w^4}{h^4} \right\}. \] (6.12)

\[ P \simeq \frac{4\theta \cot \gamma}{\pi} w^2 \left\{ 1 + \frac{4a_0}{\pi^2} \cot \gamma \frac{w}{h} + \frac{20a_0^2}{\pi^4} \cot^2 \gamma \frac{w^2}{h^2} \right. \]
\[ \left. + \left( \frac{112a_0^3}{\pi^6} + \frac{16a_1}{\pi^4} \right) \cot^3 \gamma \frac{w^3}{h^3} + \left( \frac{672a_0^4}{\pi^8} + \frac{224a_0a_1}{\pi^6} \right) \cot^4 \gamma \frac{w^4}{h^4} \right\}. \] (6.13)

Equations (6.12) and (6.13) generalize the second-order asymptotic model developed in (23). (We note the following misprints in (23): missing factors of \( \cot \gamma \) and \( \cot^2 \gamma \) in (31) and an extra factor of \( \vartheta \) in (32).)

7. The case of a hemispherically ended indenter

For a hemispherical indenter, we will have

\[ \Phi(r) = R - \sqrt{R^2 - r^2}, \] (7.1)

where \( R \) is the curvature radius of the hemispherical end.

In what follows, we make use of the identities

\[ \int_0^a \Phi' (\rho) \frac{d\rho}{\sqrt{a^2 - \rho^2}} = \frac{1}{2} \mathcal{L}(\alpha), \quad \int_0^a \frac{\Phi' (\rho) \rho^2 d\rho}{\sqrt{a^2 - \rho^2}} = R^2 \left\{ -\frac{\alpha}{2} + \frac{1}{4} (1 + \alpha^2) \mathcal{L}(\alpha) \right\}. \]

\[ \int_0^a \Phi (\rho) \frac{(2\rho^2 - a^2)}{\sqrt{a^2 - \rho^2}} \rho d\rho = R^4 \left\{ \frac{\alpha}{4} + \frac{a^3}{12} - \frac{1}{8} \left( 1 - \alpha^4 \right) \mathcal{L}(\alpha) \right\}. \]

Here we introduced the notation

\[ \alpha = \frac{a}{R} \quad \text{and} \quad \mathcal{L}(\alpha) = \ln \left( \frac{1 + \alpha}{1 - \alpha} \right). \] (7.2)

Thus, (5.14) and (5.15) can be represented as follows:

\[ P \simeq \theta R^2 \left( 1 + \alpha^2 \right) \mathcal{L}(\alpha) - 2\alpha \left( 1 - \mu^3 \alpha^3 \frac{8a_1}{3\pi} \right). \] (7.3)

\[ \frac{w}{R} \simeq \frac{\alpha}{2} \mathcal{L}(\alpha) - \mu \frac{a_0}{2\pi} \left( 1 + \alpha^2 \right) \mathcal{L}(\alpha) - 2\alpha + \mu^3 \frac{a_1}{6\pi} \left( 2\alpha(7\alpha^2 + 3) - (7\alpha^4 + 6\alpha^2 + 3) \mathcal{L}(\alpha) \right) \]
\[ + \mu^4 \frac{4a_0a_1}{3\pi^2} \alpha^3 \left( 1 + \alpha^2 \right) \mathcal{L}(\alpha) - 2\alpha. \] (7.4)
Here we introduced the notation
\[ \mu = \frac{R}{h}. \quad (7.5) \]

Now, following England \(9\), we construct the approximate solution for the contact radius in the form of a power expansion with respect to the small parameter \(\mu\) as
\[ a/R \simeq a_0 + \mu a_1 + \mu^2 a_2 + \mu^3 a_3 + \mu^4 a_4. \quad (7.6) \]
The coefficients in (7.6) are determined according to (7.3). By a perturbation method, we get
\[ a_1 = 0, \quad a_2 = 0, \quad a_4 = 0, \quad (7.7) \]
\[ a_3 = \frac{4a_1 a_0^2 (1 - a_0^2)}{3\pi} \left[ \frac{1 + a_0^2}{L(a_0)} - 2a_0 \right] \quad (1 - \alpha_0^2) L(\alpha_0), \quad (7.8) \]
while \(a_0\) is a unique root of the equation
\[ \frac{P}{\theta R^2} = \left( 1 + a_0^2 \right) \ln \left( \frac{1 + a_0}{1 - a_0} \right) - 2a_0. \quad (7.9) \]

In view of (7.7), it is readily seen that the expansion (7.6) simplifies to
\[ a/R = a_0 + \mu^3 a_3 + O(\mu^5). \quad (7.10) \]
The substitution of (7.10) into (7.4) yields the following result after expanding up to the fourth order in \(\mu\):
\[ \frac{w}{R} = \frac{a_0}{2} L(a_0) - \frac{a_0}{2\pi} \left( 1 + a_0^2 \right) L(a_0) - 2a_0 - \mu^3 \frac{a_1}{6\pi} \left( 3a_0^4 + 2a_0^2 + 3 \right) L(a_0) - 6a_0 \left( 1 + a_0^2 \right) + O(\mu^5). \quad (7.11) \]
Formulas (7.6)–(7.11) are in complete agreement with the corresponding formulas from \(9\), provided the following relations hold:
\[ \left( \frac{k_1}{1 + k_1} - \frac{k_2}{1 + k_2} \right) \frac{1}{y_1 - y_2} = \frac{1}{\theta}, \quad K_0 = -\frac{2a_0}{\pi}, \quad K_1 = \frac{-4a_1}{\pi}. \]
Here, \(k_1, k_2, y_1, y_2\) are elastic constants used in \(9\), \(K_0\) and \(K_1\) are the corresponding asymptotic constants.

Finally, observe that in the case of a paraboloid indenter (according to Section 5.1 and 5.14) and 5.15) the following relation hold:
\[ P \simeq \theta \frac{8}{3R} a^3 \left( 1 - \varepsilon^3 \frac{8a_1}{3\pi} \right), \quad w \simeq \frac{a^2}{R} \left( 1 - \varepsilon^3 \frac{4a_0}{5\pi} - \varepsilon^3 \frac{16a_1}{5\pi} + \varepsilon^4 \frac{32a_0a_1}{9\pi^2} \right). \]
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On the other hand, (7.3) and (7.4) can be rewritten in the form

\[ P \simeq \frac{8}{3R} a^3 \Xi_0(\alpha) \left( 1 - \varepsilon^3 \frac{8a_1}{3\pi} \right) , \]
\[ w \simeq \frac{a^2}{R} \Upsilon_0(\alpha) \left\{ 1 - \varepsilon \frac{4a_0}{3\pi} \Upsilon_1(\alpha) - \varepsilon^3 \frac{16a_1}{5\pi} \Upsilon_3(\alpha) + \varepsilon^4 \frac{32a_0a_1}{9\pi^2} \Upsilon_4(\alpha) \right\} . \]

Here we introduced the notation

\[ \Xi_0(\alpha) = \frac{3}{8a^3} \left( 1 + a^2 \right) \mathcal{L}(\alpha) - 2\alpha , \quad \Upsilon_0(\alpha) = \frac{1}{2\alpha} \mathcal{L}(\alpha) , \quad \Upsilon_1(\alpha) = \Upsilon_4(\alpha) = \frac{\Xi_0(\alpha)}{\Upsilon_0(\alpha)} , \]
\[ \Upsilon_3(\alpha) = -\frac{1}{16} \frac{2\alpha (7a^2 + 3) - (7a^4 + 6a^2 + 3) \mathcal{L}(\alpha)}{a^4 \mathcal{L}(\alpha)} . \]

The variation of the factors \( \Xi_0(\alpha) \) and \( \Upsilon_0(\alpha) \) with \( \alpha \) is shown in Fig. 2. The substrate effect slightly increases the effect of deviation from sphericity, because the coefficients \( \Upsilon_0(n), \) \( n = 1, 3, 4, \) are greater than unity. However, as shown in Fig. 2b, the error in replacing these coefficients with 1 is quite negligible since the ratio \( \alpha = a/R \) is usually small.

8. Indentation scaling factor for a blunt indenter

Let us rewrite (6.8) in the form

\[ P = \theta A^{-\frac{1}{2}} F_3(\lambda) w^{\frac{1}{2}} f_3(\sigma) , \]  
(8.1)

introducing the indentation scaling factor \( f_3(\sigma) \). According to (6.8), we have

\[ f_3(\sigma) \simeq 1 + C_1 \sigma + C_2 \sigma^2 + C_3 \sigma^3 + C_4 \sigma^4 , \]
(8.2)

where the coefficients \( C_n \) are given by (6.10).
Observe that the factor \( f_\lambda(\sigma) \) depends on the dimensionless indentation parameter \( \sigma \) defined by (6.7). In view of (6.3) and (6.7), we have

\[
\sigma \simeq \varepsilon \left\{ 1 - \varepsilon \left( \frac{a_0}{\pi} \right) \right. \\
\left. - \varepsilon^3 \left( \frac{(\lambda - 1) (2\lambda - 1) 8a_1}{6 (\lambda + 1)^3} \pi \right) + \frac{8a_1 (2\lambda + 5)}{3\pi (\lambda + 3) (\lambda + 1)} \right\}
\]

\[ \vdash + \varepsilon^4 \left( \frac{(\lambda - 1) (2\lambda - 1) (3\lambda - 1)}{24 (\lambda + 1)^4} \right) \]

Substituting the value \( \lambda = 2 \) into (8.3), we express the indentation factor in terms of the relative contact radius by introducing a new notation

\[ \kappa_\lambda(\varepsilon) = f_\lambda(\sigma(\varepsilon)). \]

Thus, from (8.3) and (8.4), it follows that

\[ \kappa_\lambda(\varepsilon) \simeq 1 + \varepsilon \left( \frac{a_0}{\pi} \right) + \varepsilon^2 \frac{2a_0}{2 (\lambda + 1)} \left( \frac{2a_0}{\pi} \right)^2 \\
+ \varepsilon^3 \left( \frac{(2\lambda + 1) (3\lambda + 1)}{6 (\lambda + 1)^3} \right) \left( \frac{2a_0}{\pi} \right)^3 + \frac{8a_1 (\lambda - 2)}{3\pi (\lambda + 3)} \] 

\[ + \varepsilon^4 \left( \frac{(2\lambda + 1) (3\lambda + 1) (4\lambda + 1)}{24 (\lambda + 1)^4} \right) \left( \frac{2a_0}{\pi} \right)^4 + \frac{16a_0a_1 (2\lambda^2 + 4\lambda - 1)}{3\pi^2 (\lambda + 1) (\lambda + 3)} \].

Substituting the value \( \lambda = 2 \) into (8.5), we readily get

\[ \kappa_2(\varepsilon) \simeq 1 + \varepsilon \left( \frac{a_0}{\pi} \right) + \varepsilon^2 \frac{10a_0^2}{3\pi^2} + \varepsilon^3 \left( \frac{140a_0^3}{27\pi^3} + \frac{32a_1}{15\pi} \right) + \varepsilon^4 \left( \frac{70a_0^4}{9\pi^4} + \frac{16a_0a_1}{3\pi^2} \right). \]

In order to check the asymptotic formula (8.5), we pass to the limit as \( \lambda \rightarrow +\infty \), obtaining the following result:

\[ \kappa_\infty(\varepsilon) \simeq 1 + \varepsilon \left( \frac{a_0}{\pi} \right) + \varepsilon^2 \left( \frac{2a_0}{\pi} \right)^2 + \varepsilon^3 \left( \frac{2a_0}{\pi} \right)^3 + \frac{8a_1}{3\pi} \varepsilon^4 \left( \frac{2a_0}{\pi} \right)^4 + \frac{32a_0a_1}{3\pi^2} \].

The case \( \lambda = +\infty \) corresponds to a cylindrical indenter. The asymptotic expansion (8.7) is in complete agreement with the fourth-order asymptotics derived previously in (6.24).

9. Displacement-force relationship for a blunt indenter

According to the notation (6.7), the force-displacement relation (6.8) can be rewritten as

\[ P \simeq \theta \mathcal{F}_1(\lambda) h^{\lambda^2 + 1} \left( 1 + C_1 \sigma + C_2 \sigma^2 + C_3 \sigma^3 + C_4 \sigma^4 \right), \]

\[
\text{Displacement-force relationship for a blunt indenter}
\]

\[
\text{According to the notation (6.7), the force-displacement relation (6.8) can be rewritten as}
\]

\[
P \simeq \theta \mathcal{F}_1(\lambda) h^{\lambda^2 + 1} \left( 1 + C_1 \sigma + C_2 \sigma^2 + C_3 \sigma^3 + C_4 \sigma^4 \right),
\]
where the factor $F_1(\lambda)$ is given by (6.4). Inverting the relation (9.1), we obtain
\[ \varpi \simeq \tilde{P} \left( 1 + D_1 \tilde{P} + D_2 \tilde{P}^2 + D_3 \tilde{P}^3 + D_4 \tilde{P}^4 \right). \] (9.2)

where we introduced the notation
\[ \tilde{P} = \left( \frac{\theta A F_1(\lambda)}{\lambda + 1} \right)^{-1} h^{-1} P^{-1}. \] (9.3)

Now, in view of (6.7), from (9.2) it follows that
\[ w \simeq A \frac{1}{\lambda + 1} \left( \frac{P}{\theta F_3(\lambda)} \right)^{1/\pi} \left( 1 - \frac{2\lambda a_0}{\pi(\lambda + 1)} \tilde{P} - \frac{8\lambda(\lambda + 2)a_1}{3\pi(\lambda + 1)(\lambda + 3)} \tilde{P}^3 \right). \] (9.4)

Formula (9.4) represents the sought-for relation between the indenter displacement $w$ and the contact force $P$.

Substituting now the expansion (9.2) into (6.6), we derive the corresponding relation between the contact radius $a$ and the contact force $P$ in the simple form
\[ a \frac{h}{\tilde{h}} \simeq \tilde{P} \left( 1 + \frac{8a_1}{3\pi(\lambda + 1)} \right). \] (9.5)

Observe that (9.5) can be directly obtained from (6.2).

10. Incremental indentation stiffness for a blunt indenter

During the depth-sensing indentation, the incremental indentation stiffness can be evaluated according to the relation
\[ \frac{dP}{dw} = \frac{dP/\partial a}{dw/\partial a}. \] (10.1)

Substituting (6.2) and (6.3) into (10.1), we arrive at the approximate relation
\[ \frac{dP}{dw} \simeq 4\theta a \left[ 1 - \frac{2a_0}{\pi} \varepsilon - \frac{8a_1}{3\pi} \left( \frac{\lambda + 4}{\lambda + 1} \right)^3 \varepsilon^3 + \frac{16a_0 a_1}{3\pi^2} \left( \frac{\lambda + 4}{\lambda + 1} \right)^4 \varepsilon^4 \right]. \] (10.2)

Note that in deriving (10.2), the following relation was taken into account (see (6.4) and (6.5)):
\[ \frac{(\lambda + 1) F_1(\lambda)}{\lambda F_2(\lambda)} = 4. \]

Now, expanding the right-hand side of (10.2) into a power series in $\varepsilon$, we obtain
\[ \frac{dP}{dw} \simeq 4\theta a \left\{ 1 + \varepsilon \frac{2a_0}{\pi} + \varepsilon^2 \left( \frac{2a_0}{\pi} \right)^2 + \varepsilon^3 \left( \frac{8a_1}{3\pi} \right)^3 + \varepsilon^4 \left( \frac{8a_1}{3\pi} \right)^4 \right\}. \] (10.3)

It is interesting to observe that the coefficients of the asymptotic expansion (10.3) do not depend on the parameter $\lambda$, which describes the indenter shape.
On the other hand, the incremental indentation stiffness for a blunt indenter can be evaluated by differentiating the force-displacement relationship (9.1) with respect to the indenter displacement. Thus, by taking into account the formula

\[
\frac{d\sigma}{dw} = \frac{\sigma^{1-\lambda}}{\lambda F_2(\lambda)Ah},
\]

we get the relation

\[
\frac{dP}{dw} \approx \theta \left(\frac{\lambda+1}{\lambda}\right) F_3(\lambda)F_2(\lambda)^{\frac{1}{2}} h\sigma \left\{1 + \sigma C_1 \frac{(\lambda + 2)}{\lambda + 1} + \sigma^2 C_2 \frac{(\lambda + 3)}{\lambda + 1} \right. \\
+ \sigma^3 C_3 \frac{(\lambda + 4)}{\lambda + 1} + \sigma^4 C_4 \frac{(\lambda + 5)}{\lambda + 1} \right\},
\]

(10.4)

where the coefficients \(C_n\) are given by (6.10).

Now, substituting the expression (8.3) for \(\sigma\) into (10.4) and expanding the result in terms of powers of the parameter \(\varepsilon\), we again arrive at (10.3) by noting that

\[
\left(\frac{\lambda+1}{\lambda}\right) F_3(\lambda)F_2(\lambda)^{\frac{1}{2}} = 4.
\]

Finally, by comparing (10.3) and (8.7), it is readily seen that (10.3) can be rewritten as

\[
\frac{dP}{dw} = 4\theta a\kappa_\infty(\varepsilon),
\]

(10.5)

Here, \(\kappa_\infty(\varepsilon)\) is the indentation scaling factor for a cylindrical indenter.

Observe that (10.5) can be regarded as a generalization of Barber’s theorem (25) for the incremental indentation stiffness established in the case of an isotropic elastic half-space.

Note also that in context of indentation testing, (10.5) should be rewritten as

\[
\frac{dP}{dw} = 4\theta \sqrt{A/\pi} \kappa_\infty(\varepsilon),
\]

(10.6)

where \(A\) is the current area of contact.

In contrast to the Bulychev–Alekhin–Shorshorov (BASH) relation established in (26–28) in the case of an isotropic elastic half-space, (10.6) contains the extra factor \(\kappa_\infty(\varepsilon)\), which accounts for both the thickness and substrate effects. Note that in the case of isotropy, we have \(\theta = E^{(1)}/[2(1 - (\nu^{(1)})^2)]\), where \(E^{(1)}\) and \(\nu^{(1)}\) are Young’s modulus and Poisson’s ratio of the layer material.

11. Discussion

In the present study, various special cases of the indenter geometry are examined and agreement is obtained with previously published particular results. In particular, a complete second-order asymptotic analysis has been carried out with a complete set of analytical results obtained. In the
case of a blunt indenter, the force-displacement relationship has been obtained in the following form (see (6.8)):

\[
P \simeq \theta A^{-\frac{1}{2}} F_3(\lambda)w^{-\frac{1}{2}} \left( 1 + \sum_{n=1}^{4} \frac{C_n}{h^n} \left( \frac{w}{AF_2(\lambda)} \right)^n \right).
\]

The inverse relation is given by (9.4). The incremental indentation stiffness is approximated by the expansion (10.4), while the indentation scaling factor is given by (8.5).

The obtained asymptotic solutions depend on the three parameters \( \theta, a_0 \) and \( a_1 \), which are defined in terms of the Green’s function via (2.5) and (2.7) and so relate to the material properties and geometry of the layer and the half-space. As the matter of fact, the present asymptotic analysis is based on the approximation

\[
G_3(y_1, y_2, 0) \simeq \frac{1}{2\pi \theta} \left( \frac{1}{|y|} - \frac{1}{h} \left( a_0 + a_1 \frac{|y|^2}{h^2} \right) \right).
\]

From here it is seen that the dimensionless coefficients \( a_0 \) and \( a_1 \) do not depend on the layer thickness (because the problem for the Green’s function does not contain some other parameter with the dimension of length) that is they do not depend on the geometry of the layer and the half-space. The elastic constant \( \theta \) is solely related to the first elastic layer, while the asymptotic constants \( a_0 \) and \( a_1 \) depend on Poisson’s ratios of both layers as well as on the ratios of their elastic moduli.

Based on the obtained analytical results, it is possible to develop an identification procedure that allows to make some deductions about the elastic moduli of the layer and the half-space. However, solving the inverse problem requires the development of new methods and this is beyond of the scope of this article. We note that the identification of Young’s modulus of an isotropic elastic material by means of the usual frictionless depth-sensing indentation test requires some assumption about the value of Poisson’s ratio, because the elastic constant \( \theta \) combines them both. In the special case when the material parameters are known, while the layer thickness is unknown, the latter can be determined by one-parameter fitting.

12. Conclusion

An asymptotic analysis of the substrate effect has been performed for deriving approximate solutions of the quasistatic indentation problem for an arbitrary frictionless indenter. Explicit asymptotic formulas are derived in the framework of the fourth-order asymptotic model, which has been previously shown to be effective for describing the thickness effect for a spherical indenter on an elastic layer. The results obtained can be applied for development of indentation tests.

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APPENDIX A

Kernel function for a transversely isotropic layer bonded to a transversely isotropic half-space

The constitutive relationship for a transversely isotropic material referred to the Cartesian coordinates \((x_1, x_2, x_3)\) with the Ox1x2 plane coinciding with the plane of elastic symmetry can be written in the matrix form as follows (29):

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{13} \\
\sigma_{23} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\
A_{12} & A_{11} & A_{13} & 0 & 0 & 0 \\
A_{13} & A_{13} & A_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 2A_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 2A_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & A_{11} - A_{12}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{13} \\
\varepsilon_{23} \\
\varepsilon_{12}
\end{pmatrix}.
\]

For a transversely isotropic material, only five independent elastic constants are needed to describe its deformational behaviour. The elastic moduli \(A_{11}, A_{12}, A_{13}, A_{33}\) and \(A_{44}\) can be expressed in terms of the engineering elastic constants as follows (30):

\[
A_{11} = \frac{E(1 - \nu)}{(1 + \nu)(1 - \nu)^2} v^2 + \frac{E}{1 - \nu^2}, \quad A_{12} = \frac{E(v^2 + \nu)}{(1 + \nu)(1 - \nu^2)} v^2 + \frac{E}{1 - \nu^2},
\]

\[
A_{13} = \frac{E\nu^2}{1 - 2\nu}, \quad A_{33} = \frac{E'(1 - \nu)}{(1 - 2\nu) v^2}, \quad A_{44} = G'.
\]
According to (1), the functions $M(x_1, y_1)$ and $N(x_1, y_1)$ are given by

$$
M(x_1, y_1) = -\left( g_1^{(1)} a_{12} - g_2^{(2)} a_{21} \right)x_1 y_1 - \left( g_1^{(1)} a_{11} x_1^2 + g_2^{(2)} a_{22} x_1^2 \right) - (a_{12} a_{21} - a_{11} a_{22}) x_1^2 y_1^2,
$$

$$
N(x_1, y_1) = 1 + 2 \left( g_1^{(1)} a_{12} - g_2^{(2)} a_{21} \right)x_1 y_1 + \left( g_1^{(1)} + g_2^{(2)} \right) \left( a_{11} x_1^2 - a_{22} y_1^2 \right) + (a_{12} a_{21} - a_{11} a_{22}) x_1^2 y_1^2.
$$

According to (3), Eqs. (51)–(54)), the coefficients $a_{11}, a_{12}, a_{21}$ and $a_{22}$ are defined as

$$
a_{11} = 1 + \frac{2y_2^{(1)}}{Z} \left( -\frac{H}{2} \left[ y_2^{(2)} - \left( m_1^{(2)} \right)^2 y_2^{(2)} \right] + gH \left[ 2 \left( y_1^{(2)} - m_1^{(2)} y_2^{(2)} \right) \right] \right),
$$

$$
a_{12} = \frac{2y_2^{(1)}}{Z} \left( \frac{g^2 m_1^{(1)} - gH \left[ g_1^{(2)} - \left( m_1^{(2)} \right)^2 g_2^{(2)} \right] \left( m_1^{(1)} + 1 \right) + H^2 \left[ g_1^{(2)} - \left( m_1^{(2)} \right)^2 g_2^{(2)} \right] \right) \right),
$$

$$
a_{21} = \frac{2y_2^{(1)}}{Z} \left( \frac{g^2 m_1^{(1)} - gH \left[ g_1^{(2)} - \left( m_1^{(2)} \right)^2 g_2^{(2)} \right] \left( m_1^{(1)} + 1 \right) + H^2 \left[ g_1^{(2)} - \left( m_1^{(2)} \right)^2 g_2^{(2)} \right] \right) \right),
$$

$$
a_{22} = 1 + \frac{2y_2^{(1)}}{Z} \left( \left[ H^2 \left[ y_2^{(2)} - \left( m_1^{(2)} \right)^2 y_2^{(2)} \right] - gH \left[ 2m_1^{(1)} \left( y_1^{(2)} - m_1^{(2)} y_2^{(2)} \right) \right] \right.ight.
$$

$$
\left. - \left( m_1^{(1)} - 1 \right) \left( m_1^{(2)} - 1 \right) y_1^{(1)} \right] \frac{1}{y_1^{(2)} - y_2^{(2)}} + g^2 \left( m_1^{(1)} \right)^2 \right).
$$

Here the following notation is used (see (3), Eqs. (48), (59) and (60)):

$$
Z = H^2 \left[ \left( y_1^{(1)} - y_2^{(1)} \right) \left( g_1^{(2)} - \left( m_1^{(2)} \right)^2 g_2^{(2)} \right) \right]
$$

$$
- \frac{gH}{y_1^{(2)} - y_2^{(2)}} \left[ \left( m_1^{(1)} - 1 \right) \left( m_1^{(2)} - 1 \right) \left( y_1^{(1)} y_2^{(1)} + y_1^{(2)} y_2^{(2)} \right) \right]
$$

$$
+ 2 \left( \frac{g^2 m_1^{(1)} - gH \left[ g_1^{(2)} - \left( m_1^{(2)} \right)^2 g_2^{(2)} \right] \left( m_1^{(1)} + 1 \right) + H^2 \left[ g_1^{(2)} - \left( m_1^{(2)} \right)^2 g_2^{(2)} \right] \right)}{y_1^{(2)} - y_2^{(2)}} \left( y_1^{(1)} - m_1^{(1)} y_1^{(2)} \right) \right] + g^2 \left[ y_1^{(1)} - \left( m_1^{(2)} \right)^2 \right].
$$
ASYMPTOTIC ANALYSIS OF THE SUBSTRATE EFFECT

\( g(1) = \gamma(1)_1 - \gamma(1)_2 \), \( g(2) = \gamma(2)_1 - \gamma(2)_2 \),
\( H = \frac{H(2)(m_1^{(1)} - 1)}{H(1)(m_2^{(2)} - 1)} \), \( g = \frac{\gamma(1)_1 - \gamma(1)_2}{\gamma(2)_1 - \gamma(2)_2} \).

We note two misprints in (31), Eq. (59)) for \( Z \): one closing bracket is missing and the product \( gH \) is misprinted as \( Hg \).

The layer material is characterized by the elastic constants \( H(1), m_1^{(1)}, \gamma(1)_1, \gamma(1)_2 \) and so on, while the material of the elastic half-space is characterized by the elastic constants \( H(2), m_1^{(2)}, \gamma(2)_1, \gamma(2)_2 \) and so on. So, we have
\[
H(n) = \frac{(\gamma(n)_1 + \gamma(n)_2)A(n)_{11}}{2\pi (A(n)_{11}A(n)_{33} - (A(n)_{13})^2)}
\]
and
\[
m_1^{(n)} = \frac{A(n)_{11}(\gamma(n)_1)^2}{A(n)_{11} + A(n)_{44}}, \quad m_2^{(n)} = \frac{A(n)_{11}(\gamma(n)_2)^2 - A(n)_{44}}{A(n)_{13} + A(n)_{44}}.
\]
Finally, the dimensionless parameters \( \gamma(n)_1, \gamma(n)_2 \) are defined as the roots of the equation
\[
\gamma^4 A_{11}^{(n)} A_{44}^{(n)} - \gamma^2 \left(A_{11}^{(n)} A_{33}^{(n)} - A_{13}^{(n)} \left(A_{13}^{(n)} + 2A_{44}^{(n)}\right)\right) + A_{33}^{(n)} A_{44}^{(n)} = 0. \tag{A.1}
\]
In the case of isotropy, we have
\[
L(u) = 1 + Q(u), \quad Q(u) = -2e^{-2u} \frac{d_1 + d_2 e^{-2u}}{1 + d_1 e^{-2u} + d_2 e^{-4u}}, \quad \theta = \frac{E(1)}{2(1 - (\nu(1))^2)}.
\]
where \( E(1) \) and \( \nu(1) \) are Young’s modulus and Poisson’s ratio of the layer material, while the coefficients \( d_1, d_2 \) and \( d_3 \) are given by (31) Eqs. (105)–(108)). We note that in this particular case, Fabrikant’s solution obtained in (31) is in agreement with Burmister’s solution (33).