NONINVERTIBILITY, SEMISUPERMANIFOLDS AND CATEGORIES REGULARIZATION*

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Abstract

The categories with noninvertible morphisms are studied analogously to the semisupermanifolds with noninvertible transition functions. The concepts of regular \( n \)-cycles, obstruction and the regularization procedure are introduced and investigated. It is shown that the regularization of a category with noninvertible morphisms and obstruction form a 2-category. The generalization of functors, Yang-Baxter equation, (co-) algebras, (co-) modules and some related structures to the regular case is given.

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1 Introduction

In the supermanifold noninvertible generalization approach [1,4,3] we study here the obstructed cocycle conditions in the category theory framework and extend them to such structures as categories, functors, (co-) algebras, (co-) modules etc. This approach is connected with the higher regularity concept [4] and reconsidering the role of identities [5]. The introduced category regularization together with obstruction form a 2-category. Similar abstract structure generalizations were considered in topological QFT [8,7], for n-categories [5,10], near-group categories [11,12] (with noninvertible elements) and weak Hopf algebras [13,14] in which the counit does not satisfy \( \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \) or satisfy first order (in our classification) regularity conditions [13,14]. We first show how to deal with noninvertibility in the supermanifold theory [17,18] and then apply this approach to more general structures.

2 Supermanifolds and semisupermanifolds

In the supermanifold theory [17,18,19] the phenomenon of noninvertibility obviously arises from odd nilpotent elements and zero divisors of Grassmann algebras (also in the infinite dimensional case [20]). Despite the invertibility question is quite natural, the answer is not so simple and in some cases can be nontrivial, e.g. in some superalgebras one can introduce invertible analog of an odd symbol [21], or construct elements without number part which are not nilpotent even topologically [22]. Several guesses concerning inner noninvertibility inherent in the supermanifold theory were made before, e.g. “...there may be no inverse projection at all” [23], “…a general SRS needs not have a body” [24], or “…a body may not even exist in the most extreme examples” [25]. It were also considered pure odd supermanifolds [26,27] which give an important counterexample to the Coleman-Mandula theorem “...and provides us with a new, missed so far, version of the Poincaré supergroup” [28], exotic supermanifolds with nilpotent even coordinates [29] and supergravity with noninvertible vierbein [30]. Some problems with odd directions and therefore connected with noninvertibility in either event are described in [31,32], and a perspective list of supermanifold problems was stated by D. Leites in [33].

\(^0\)number part.
The patch definition of a supermanifold $\mathcal{M}_0$ in most cases differs from the patch definition of an ordinary manifold $\mathcal{M}$ by “super-” terminology only and is well-known $[36]$. Let $\bigcup \{U_\alpha, \varphi_\alpha\}$ is an atlas of a supermanifold $\mathcal{M}_0$, then its gluing transition functions $\Phi_{\alpha\beta} = \varphi_\alpha \circ \varphi^{-1}_\beta$ satisfy the cocycle conditions

$$\Phi_{\alpha\beta}^{-1} = \Phi_{\beta\alpha}, \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} = 1_{\alpha\alpha}$$ (1)

on overlaps $U_\alpha \cap U_\beta$ and on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ respectively, where $1_{\alpha\alpha} \overset{\text{def}}{=} id (U_\alpha)$. To obtain a patch definition of an object analogous to supermanifold we try to weaken demand of invertibility of coordinate maps $\varphi_\alpha$. Consider a generalized superspace $\mathcal{M}$ covered by open sets $U_\alpha$ as $\mathcal{M} = \bigcup_\alpha U_\alpha$.

We assume here that the maps $\varphi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^{n|m}$ are not all homeomorphisms, i.e. among them there are noninvertible maps $\mathcal{M}$.

**Definition 1.** A semisupermanifold is a noninvertibly generalized superspace $\mathcal{M}$ represented as a semiatlas $\mathcal{M} = \bigcup_\alpha \{U_\alpha, \varphi_\alpha\}$ with invertible and noninvertible coordinate maps $\varphi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^{n|m}$.

We do not concretize here the details, how the invertibility appears here, but instead we will describe it by some general relations between semitransition functions and other objects. We The noninvertibly extended gluing semitransition functions of a semisupermanifold are defined by the equations

$$\Phi_{\alpha\beta} \circ \varphi_\beta = \varphi_\alpha, \quad \Phi_{\beta\alpha} \circ \varphi_\alpha = \varphi_\beta$$ (2)

instead of $\Phi_{\alpha\beta} = \varphi_\alpha \circ \varphi^{-1}_\beta$, which obviously extends the class of functions to noninvertible ones. Then we assume that instead of (1) the semitransition functions $\Phi_{\alpha\beta}$ of a semisupermanifold $\mathcal{M}$ satisfy the following relations

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}$$ (3)

on $U_\alpha \cap U_\beta$ overlaps (invertibility is extended to regularity) and

$$\begin{align*}
\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} &= \Phi_{\alpha\beta}, \\
\Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} &= \Phi_{\beta\gamma}, \\
\Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} &= \Phi_{\gamma\alpha}
\end{align*}$$ (4, 5, 6)

$^1$Under $\mathbb{R}^{n|m}$ we imply some its noninvertible generalization $\mathcal{M}$.
on triple overlaps \( U_\alpha \cap U_\beta \cap U_\gamma \) and

\[
\begin{align*}
\Phi_{\alpha \beta} \circ \Phi_{\beta \gamma} \circ \Phi_{\gamma \rho} \circ \Phi_{\rho \alpha} \circ \Phi_{\alpha \beta} &= \Phi_{\alpha \beta}, \\
\Phi_{\beta \gamma} \circ \Phi_{\gamma \rho} \circ \Phi_{\alpha \beta} \circ \Phi_{\beta \gamma} &= \Phi_{\beta \gamma}, \\
\Phi_{\gamma \rho} \circ \Phi_{\rho \alpha} \circ \Phi_{\alpha \beta} \circ \Phi_{\beta \gamma} &= \Phi_{\gamma \rho}, \\
\Phi_{\rho \alpha} \circ \Phi_{\alpha \beta} \circ \Phi_{\beta \gamma} \circ \Phi_{\gamma \rho} \circ \Phi_{\rho \alpha} &= \Phi_{\rho \alpha}
\end{align*}
\]

(7) (8) (9) (10)

on \( U_\alpha \cap U_\beta \cap U_\gamma \cap U_\rho \). We can write similar cycle relations to infinity and call them tower relations which satisfy identically in the standard invertible case \([36]\).

**Remark 1.** In any actions with noninvertible functions \( \Phi_{\alpha \beta} \) we are not allowed to cancel by them, because the semigroup of \( \Phi_{\alpha \beta} \)'s is a semigroup without cancellation, and we are forced to exploit the corresponding semigroup methods \([37, 38]\).

**Conjecture 2.** The functions \( \Phi_{\alpha \beta} \) satisfying the relations (3)–(10) can be viewed as some noninvertible generalization of the transition functions as cocycles in the corresponding Čech cohomology of coverings \([37, 40]\).

### 3 Obstructedness and additional orientation on semisupermanifolds

The semisupermanifolds defined above belong to a class of so called obstructed semisupermanifolds \([1, 3]\) in the following sense. Let us rewrite relations (11) as the infinite series

\[
\begin{align*}
n = 1 : \ & \Phi_{\alpha \alpha} = 1_{\alpha \alpha}, \\
n = 2 : \ & \Phi_{\alpha \beta} \circ \Phi_{\beta \alpha} = 1_{\alpha \alpha}, \\
n = 3 : \ & \Phi_{\alpha \beta} \circ \Phi_{\beta \gamma} \circ \Phi_{\gamma \alpha} = 1_{\alpha \alpha}, \\
n = 4 : \ & \Phi_{\alpha \beta} \circ \Phi_{\beta \gamma} \circ \Phi_{\gamma \delta} \circ \Phi_{\delta \alpha} = 1_{\alpha \alpha}
\end{align*}
\]

(11) (12) (13) (14)

\[
\cdots \cdots
\]
Definition 3. A semisupermanifold is called obstructed, if some of the cocycle conditions (11)–(14) are broken.

It can happen that starting from some \( n = n_m \) all higher cocycle conditions hold valid.

Definition 4. Obstructedness degree of a semisupermanifold is a maximal \( n_m \) for which the cocycle conditions (11)–(14) are broken. If all of them hold valid, then \( n_m \overset{\text{def}}{=} 0 \).

Obviously, that ordinary manifolds [35] (with invertible transition functions) have vanishing obstructedness, and the obstructedness degree for them is equal to zero, i.e. \( n_m = 0 \).

Remark 2. The obstructed semisupermanifolds may have nonvanishing ordinary obstruction which can be calculated extending the standard methods [17] to the noninvertible case.

Therefore, using the obstructedness degree \( n_m \), we have possibility to classify semisupermanifolds properly. Moreover, the pure soul supernumbers do not contain unity. Obviously that obstructed semisupermanifolds cannot have identity semitransition functions.

The orientation of ordinary manifolds is determined by the Jacobian sign of transition functions \( \Phi_{\alpha\beta} \) written in terms of local coordinates on \( U_\alpha \cap U_\beta \) overlaps [14, 35]. Since this sign belong to \( \mathbb{Z}_2 \), there exist two orientations on \( U_\alpha \). Two overlapping charts are consistently oriented (or orientation preserving) if \( \Phi_{\alpha\beta} \) has positive Jacobian, and a manifold is orientable if it can be covered by such charts, thus there are two kinds of manifolds: orientable and nonorientable [35]. In supersymmetric case the role of Jacobian plays Berezinian [17] which has a “sign” belonging to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), and so there are four orientations on \( U_\alpha \) and five corresponding kinds of supermanifold orientability [41, 42].

Definition 5. In case a nonvanishing Berezinian of \( \Phi_{\alpha\beta} \) is nilpotent (and so has no definite sign in the previous sense) there exists additional nilpotent orientation on \( U_\alpha \) of a semisupermanifold.

A degree of nilpotency of Berezinian allows us to classify semisupermanifolds having nilpotent orientability (see e.g. [35, 44]).
4 Higher regularity and obstruction

The above constructions have the general importance for any set of non-invertible mappings. The extension of $n = 2$ cocycle given by (3) can be viewed as some analogy with regular [45] or pseudoinverse [46] elements in semigroups or generalized inverses in matrix theory [47], category theory [48] and theory of generalized inverses of morphisms [49]. The relations (4)–(10) and with other $n$ can be considered as noninvertible analogue of regularity for higher cocycles. Therefore, by analogy with (3)–(10) it is natural to formulate the general

**Definition 6.** An noninvertible mapping $\Phi_{\alpha\beta}$ is $n$-regular, if it satisfies on overlaps $U_\alpha \cap U_\beta \cap \ldots \cap U_\rho$, to the following conditions

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \ldots \circ \Phi_{\rho\alpha} = \Phi_{\alpha\beta} + \text{perm.}$$  \hspace{1cm} (15)

The formula (3) describes 3-regular mappings, the relations (4)–(8) correspond to 4-regular ones, and (7)–(10) give 5-regular mappings. Obviously that 3-regularity coincides with the ordinary regularity.

Let us consider a series of the selfmaps $e_{\alpha\alpha}^{(n)} : U_\alpha \to U_\alpha$ of a semisupermanifold defined as

$$e_{\alpha\alpha}^{(1)} = \Phi_{\alpha\alpha},$$  \hspace{1cm} (16)

$$e_{\alpha\alpha}^{(2)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha},$$  \hspace{1cm} (17)

$$e_{\alpha\alpha}^{(3)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha},$$  \hspace{1cm} (18)

$$e_{\alpha\alpha}^{(4)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\delta} \circ \Phi_{\delta\alpha}$$  \hspace{1cm} (19)

We will call $e_{\alpha\alpha}^{(n)}$’s tower identities (or obstruction of $U_\alpha$). From (11)–(14) it follows that for ordinary supermanifolds obstruction coincide with the usual identity map

$$e_{\alpha\alpha}^{(n), \text{ordinary}} = 1_{\alpha\alpha}.$$  \hspace{1cm} (20)
So the obstructedness degree can be treated as a maximal $n = n_m$ for which tower identities differ from the identity, i.e. (20) is broken. The obstruction gives the numerical measure of distinction of a semisupermanifold from an ordinary supermanifold. When morphisms are noninvertible (a semisupermanifold has a nonvanishing obstructedness), we cannot “return to the same point”, because in general $e^{(n)}_{\alpha\alpha} \neq 1_{\alpha\alpha},$ and we have to consider “nonclosed” diagrams due to the fact that the relation $e^{(n)}_{\alpha\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}$ is noncancellative now (see Remark 1).

Summarizing the above statements we propose the following intuitively consistent changing of the standard diagram technique as applied to noninvertible morphisms. In every case we get a new arrow which corresponds to the additional multiplier, and so for $n = 2$ we obtain

\[
\begin{array}{ccc}
\text{Invertible morphisms} & \Phi_{\alpha\beta} & \Phi_{\beta\alpha} \\
\text{Noninvertible morphisms} & \Phi_{\alpha\beta} & \Phi_{\beta\alpha}
\end{array}
\]

which describes the transition from (12) to (3) and presents the ordinary regularity condition for morphisms [48, 49]. The most intriguing semicommutative diagram is the triangle one

\[
\begin{array}{ccc}
\text{Invertible morphisms} & \Phi_{\alpha\beta} & \Phi_{\beta\gamma} \\
\text{Noninvertible morphisms} & \Phi_{\alpha\beta} & \Phi_{\beta\gamma} + \text{perm.}
\end{array}
\]

which generalizes the cocycle condition (1).

The higher $n$-regular semicommutative diagrams can be considered in the framework of generalized categories [4, 12, 50] in the following way.
5 Categories and 2-categories

There is an algebraic approach to the formalism considered in previous sections based on the category theory [5, 4]. A category $\mathcal{C}$ contains a collection $\mathcal{C}_0$ of objects and a collection $\text{hom}(\mathcal{C})$ of arrows (morphisms) (see e.g. [51]). The collection $\text{hom}(\mathcal{C})$ is the union of mutually disjoint sets $\text{hom}_C(X, Y)$ of arrows $X \xrightarrow{f} Y$ from $X$ to $Y$ defined for every pair of objects $X, Y \in \mathcal{C}$. It may happen that for a pair $X, Y \in \mathcal{C}$ the set $\text{hom}_C(X, Y)$ is empty. The associative composition of morphisms is also defined. By an equivalence in $\mathcal{C}$ we mean a class of morphisms $\text{hom}'(\mathcal{C}) = \bigcup_{X, Y \in C} \text{hom}'_C(X, Y)$ where $\text{hom}'_C(X, Y)$ is a subset of $\text{hom}_C(X, Y)$. Two objects $X, Y$ of the category $\mathcal{C}$ is equivalent if and only if there is a morphism $X \xrightarrow{s} Y$ in $\text{hom}'_C(X, Y)$ such that

$$s^{-1} \circ s = id_X, \quad s \circ s^{-1} = id_Y$$

(21)

Let $X = (X_1, \ldots, X_n)$ be a sequence of objects of $\mathcal{C}$. Our category can contains a class of noninvertible morphisms [48, 4]. A (strict) 2-category $\mathcal{C}$ consists of a collection $\mathcal{C}_0$ of objects as 0-cells and two collections of morphisms: $\mathcal{C}_1$ and $\mathcal{C}_2$ called 1-cells and 2-cells, respectively [52]. For every pair of objects $X, Y \in \mathcal{C}_0$ there is a category $\mathcal{C}(X, Y)$ whose objects are 1-cell $f : X \rightarrow Y$ in $\mathcal{C}_1$ and whose morphisms are 2-cells. For a pair of 1-cells $f, g \in \mathcal{C}_1$ there is a 2-cell $s : f \rightarrow g$ in $\mathcal{C}_2$. For every three objects $X, Y, Z \in \mathcal{C}_0$ there is a bifunctor

$$c : \{\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)\}$$

(22)

which is called a composition of 1-cells. There is an identity 1-cell $id_X \in \mathcal{C}(X, X)$ which acts trivially on $\mathcal{C}(X, Y)$ or $\mathcal{C}(Y, X)$. There is also 2-cell $id_{id_X}$ which acts trivially on 2-cells.

Let $\mathcal{C}$ be a category with equivalence. Then one can see that collection of all equivalence classes of objects of $\mathcal{C}$ forms a 2-category $\mathcal{C}(\mathcal{C})$. These classes are 0-cells of $\mathcal{C}(\mathcal{C})$, 1-cells are classes of morphisms of $\mathcal{C}$, and 2-cells are maps between these classes. Observe that 1-cells of $\mathcal{C}(\mathcal{C})$ can be represented by morphisms of the underlying category $\mathcal{C}$, but such representation is not unique. One equivalence class can be represented by several equivalent morphisms. One can define 2-morphisms on equivalence classes, and $\mathcal{C}(\mathcal{C})$ becomes a 2-category. If the category $\mathcal{C}$ is equipped with certain additional structures, then one can transform them into $\mathcal{C}(\mathcal{C})$. If for instance
\(\mathcal{C}\) is monoidal category with product \(\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}\), then \(\mathcal{C}(\mathcal{C})\) becomes the so-called semistrict monoidal 2-category. This means that the product \(\otimes\) (under some natural conditions) is defined for all cells of the 2-category \(\mathcal{C}(\mathcal{C})\). In the case of braided categories one can obtain the semistrict braided monoidal category \([52]\). Algebras, coalgebras, modules and comodules can be also included in this procedure. We apply such method to regularize categories with noninvertible morphisms and obstruction \([5, 4]\).

6 Categories and regularization

Let \(\mathcal{C}\) be a category with invertible and noninvertible morphisms \([\text{[5]}\) and equivalence. The equivalence in \(\mathcal{C}\) is here defined as the class of invertible morphisms in the category \(\mathcal{C}\).

**Definition 7.** A sequence of morphisms

\[
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1
\]

such that there is an (endo-)morphism \(e^{(n)}_{X_1}: X_1 \rightarrow X_1\) defined uniquely by the following equation

\[
e^{(n)}_{X_1} := f_n \circ \cdots \circ f_2 \circ f_1
\]

and subjects to the relation \(f_1 \circ f_n \circ \cdots f_2 \circ f_1 = f_1\) is said to be a regular \(n\)-cycle on \(\mathcal{C}\) and it is denoted by \(f = (f_1, \ldots, f_n)\).

The (endo-)morphisms \(e^{(n)}_{X_i}: X_i \rightarrow X_i\) corresponding for \(i = 2, \ldots, n\) are defined by a suitable cyclic permutation of above sequence.

**Definition 8.** The morphism \(e^{(n)}_X\) is said to be an obstruction of \(X\). The mapping \(e^{(n)}: X \in \mathcal{C}_0 \rightarrow e^{(n)}_X \in \text{hom}(X, X)\) is called a regular \(n\)-cycle obstruction structure on \(\mathcal{C}\).

If

\[
X_1 \xrightarrow{g_1} X'_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} X'_n \xrightarrow{g_n} X_1
\]

is an another \(n\)-tuple of morphisms such that \(e^{(n)}_{X_1}: g_n \circ \cdots \circ g_2 \circ g_1\), then we assume that \(X'_i\) is equivalent to \(X_i\), for \(i = 2, \ldots, n\).
**Definition 9.** A map \( s : f \Rightarrow g \) which sends the object \( X_i \) into equivalent object \( X'_i \) and morphism \( f_i \) into \( g_i \) is said to be obstruction \( n \)-cycle equivalence.

We have the diagram

\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & \cdots \xrightarrow{f_{n-1}} & X_n \\
\downarrow & & & \downarrow \\
X_1 & \xrightarrow{s} & X_1 \\
\downarrow & & & \downarrow \\
X'_2 & \xrightarrow{g_2} & \cdots \xrightarrow{g_{n-1}} & X'_n
\end{array}
\]

(25)

**Lemma 10.** There is a one to one correspondence between equivalence classes of regular \( n \)-cycles and regular \( n \)-cycle obstruction structures.

If \( f = (f_1, \ldots, f_n) \) is a class of regular \( n \)-cycles, then there is the corresponding regular \( n \)-cycle obstruction structure \( e : X \in \mathcal{C}_0 \to e_X \in \text{hom}(X, X) \) such that the relation (24) holds true. Let \( e^{(n)} : X \in \mathcal{C}_0 \to e_X^{(n)} \in \text{hom}(X, X) \) be a regular \( n \)-cycle obstruction in \( \mathcal{C} \).

**Definition 11.** A morphism \( \alpha : X \to Y \) of the category \( \mathcal{C} \) such that

\[
\alpha \circ e_X^{(n)} = e_Y^{(n)} \circ \alpha
\]

is said to be a regular \( n \)-cycle obstruction morphism from \( X \) to \( Y \).

It follows from (24) that the morphism \( \alpha \) is in fact a sequence of morphism \( \alpha := (\alpha_1, \ldots, \alpha_n) \) such that the diagram

\[
\begin{array}{cccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & Y_1
\end{array}
\]

(27)

is commutative.

**Definition 12.** A collection of all equivalence classes of objects \( \mathcal{C}_0 \) with obstruction structures \( e^{(n)} : X \in \mathcal{C}_0 \to e_X^{(n)} \in \text{hom}(X, X) \) is denoted by \( \text{Reg}_n(\mathcal{C}) \) and called an obstruction \( n \)-cycle regularization of \( \mathcal{C} \). The class of all regular \( n \)-cycle morphisms from \( X \) to \( Y \) is denoted by \( \text{Reg}_n(\mathcal{C})(X, Y) \).
Corollary 13. It follows from the Lemma 10 that the map $s : \alpha \rightarrow \beta$ which sends an arbitrary regular $n$-cycle morphisms $\alpha \in \text{Reg}_n(C)(X, X')$ into a regular $n$-cycle morphisms $\beta \in \text{Reg}_n(C)(X, X')$ is a regular obstruction $n$-cycle equivalence.

One can define 2-morphisms and an associative composition of 2-morphisms such that $\text{Reg}_n(C)(X, Y)$ becomes a category for every two objects $X, Y \in C_0$. If $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow Z$ are two $n$-cycle morphisms, then the composition $\beta \circ \alpha : X \rightarrow Z$ is also a $n$-cycle morphism. In this way we obtain the composition as bifunctors

$$c^{\text{Reg}_n} := \{\text{Reg}_n(C)(X, Y) \times \text{Reg}_n(C)(Y, Z) \rightarrow \text{Reg}_n(C)(X, Z)\} \quad (28)$$

We summarize our considerations in the following lemma:

Lemma 14. The class $\text{Reg}_n(C)$ forms a (strict) 2-category whose 0-cells are equivalence classes of objects of $C$ with obstructions, whose 1-cells are regular $n$-cycle obstruction morphisms, and whose 2-cells are regular obstruction $n$-cycle 2-morphisms.

7 Regularization of monoidal categories functors and Yang-Baxter equation

Let $C = C(I, \otimes)$ be a monoidal category, where $I$ is the unit object and $\otimes : C \times C \rightarrow C$ is the monoidal product [53, 54]. If the following relation

$$e^{(n)}_X \otimes e^{(n)}_Y = e^{(n)}_{X \otimes Y}. \quad (29)$$

holds true, then we have

Proposition 15. The monoidal product of two regular $n$-cycles $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with obstruction $e^{(n)}_{X_1}$, and $e^{(n)}_{Y_1}$, respectively, is the regular $n$-cycle

$$X_1 \otimes Y_1, \otimes \cdots \otimes X_n \otimes Y_n$$

with the obstruction $e^{(n)}_{X \otimes Y}$.
One can see that in this case \( \text{Reg}_n(C) \) is the so-called semistrict monoidal category \([52]\).

Let \( C \) and \( D \) be two monoidal categories and let \( \text{Reg}_n(C), \text{Reg}_n(D) \) be their regularization 2-categories. We can introduce the notion of regular 2-functors, pseudonatural transformations and modifications. All definitions do not changed, but the preservation of the identity can be replaced by the requirement of preservation of obstruction morphisms \( e_X^{(n)} \) and the invertibility is replaced by regularity. If, for instance, there is a regular 2-functor \( \mathcal{F} : \text{Reg}_n(C) \rightarrow \text{Reg}_n(C) \), then in addition to the standard definition \([51]\) we have the following relation

\[
\mathcal{F}(e_X) = e_{\mathcal{F}(X)}.
\]

(30)

In the same manner we can “regularize” pseudo-natural transformations and modifications \([50]\). Let \( \text{Reg}_n(C) \) be a semistrict monoidal 2-category. A pseudo-natural transformations \( B = \{B_{X,X'} : X \otimes X' \rightarrow X' \otimes X\} \) and two regular modifications \( B_{X \otimes Y,Z}, B_{X,Y \otimes Z} \) such that

\[
B_{X \otimes Y,Z}
\]

\[
\begin{array}{c}
X \otimes Y \otimes Z \\
B_{X,Y} \otimes e_Z \downarrow \\
Y \otimes X \otimes Z
\end{array}
\]

\[
\mathcal{F}(e_Y) \otimes B_{X,Z}
\]

and

\[
B_{X,Y \otimes Z}
\]

\[
\begin{array}{c}
X \otimes Y \otimes Z \\
e_X \otimes B_{Y,Z} \downarrow \\
X \otimes Z \otimes Y
\end{array}
\]

\[
\mathcal{F}(e_Y) \otimes B_{X,Z}
\]

and

\[
B_{X,X'} \circ e_{X \otimes X'} = e_{X' \otimes X} \circ B_{X,X'},
\]

(33)

are said to be a regular \( n \)-cycle braiding. Obviously, these operations must satisfying all conditions of \([52]\) with two changes indicated at the beginning of this section. Then the 2-category \( \text{Reg}_n(C) \) is called a semistrict regular \( n \)-cycle braided monoidal category. This allows us to obtain here the following regular \( n \)-cycle Yang–Baxter equation \([5, 4]\)

\[
B^{(1)}_{Y,Z,X} \circ B^{(2)}_{Y,X,Z} \circ B^{(1)}_{X,Y,Z} = B^{(2)}_{Z,X,Y} \circ B^{(1)}_{X,Z,Y} \circ B^{(2)}_{X,Y,Z},
\]

(34)
where the notation
\[ B^{(1)}_{X,Y,Z} = B_{X,Y} \otimes e_Z, \quad B^{(2)}_{X,Y,Z} = e_X \otimes B_{Y,Z} \]
has been used and the obstruction \( e_X \) is exploited instead of the identity \( Id_X \).
Solutions of the regular \( n \)-cycle Yang–Baxter equation (34) can be found by application of the endomorphism semigroup methods used in [55, 16].

8 Regularization of algebras, coalgebras, modules and comodules

Let \( (\mathcal{C}) \) be a monoidal category and \( \text{Reg}_n(\mathcal{C}) \) be its regularization. It is known that an associative algebra in the category \( \mathcal{C} \) is an object \( \mathcal{A} \) of this category such that there is an associative multiplication \( m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) which is also a morphism of this category. If the multiplication is in addition a regular \( n \)-cycle morphism, then the algebra \( \mathcal{A} \) is said to be a regular \( n \)-cycle algebra. This means that we have the relation
\[ m \circ (e_{\mathcal{A}} \otimes e_{\mathcal{A}}) = e_{\mathcal{A}} \circ m. \]  

(35)

Obviously such multiplication not need to be unique. Denote by \( \text{Reg}_n(\mathcal{C})(\mathcal{A} \otimes \mathcal{A}) \) a class of all such multiplications. We can see that a regular \( n \)-cycle 2-morphisms \( s : m \Rightarrow n \) which send the multiplication \( m \) into a new one \( n \) should be an algebra homomorphism. One can define regular \( n \)-cycle coalgebra or bialgebra in a similar way. A comultiplication \( \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \) can be regularized according to the relation
\[ \Delta \circ e_{\mathcal{A}} = (e_{\mathcal{A}} \otimes e_{\mathcal{A}}) \circ \Delta. \]  

(36)

In this case we obtain a class \( \text{Reg}_n(\mathcal{C})(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}) \) of comultiplications.

Let \( \mathcal{A} \mathcal{C} \) be a category of all left \( \mathcal{A} \)-modules, where \( \mathcal{A} \) is a bialgebra. For the regularization \( \text{Reg}_n(\mathcal{A} \mathcal{C}) \) of the \( \mathcal{A} \)-module action \( \rho_M : \mathcal{A} \otimes M \to M \) we use the following formula
\[ \rho_M \circ (e_{\mathcal{A}} \otimes e_M) = e_M \circ \rho_M, \]  

(37)

where \( \rho_M : \mathcal{A} \otimes M \to M \) is the left module action of \( \mathcal{A} \) on \( M \). The class of all such module actions is denoted by \( \text{Reg}_n(\mathcal{A} \mathcal{C})(\mathcal{A} \otimes \mathcal{M}, \mathcal{M}) \). The
monoidal operation in this category is given as the following tensor product of $A$-modules

$$
\rho_{M \otimes N} := (id_M \otimes \tau \otimes id_N) \circ (\rho_M \otimes \rho_N) \circ (\Delta \otimes id_{M \otimes N}),
$$

(38)

where $\tau : A \otimes M \rightarrow M \otimes A$ is the twist, i. e. $\tau(a \otimes m) := m \otimes a$ for every $a \in A, m \in M$.

**Lemma 16.** For the tensor product of module actions we have the following formula

$$
\rho_{M \otimes N} \circ (e_A \otimes e_{M \otimes N}) = e_{M \otimes N} \circ \rho_{M \otimes N}.
$$

(39)

This lemma means that the tensor product of two module actions satisfy our regularity condition if and only if these two actions also satisfy the regularity condition (37).

Observe that there is also a category $\mathcal{C}^A$ of right $A$-comodules, where $A$ is an algebra. We can regularize this category in the following way. For the coaction we have

$$
\rho \circ e_A = (e_M \otimes e_A) \circ \rho_M,
$$

(40)

and

$$
\rho_{M \otimes N} := (id_M \otimes m_A) \circ (id_M \otimes \tau \otimes id_N) \circ (\rho_M \otimes \rho_N),
$$

(41)

where $\tau : M \otimes N \rightarrow N \otimes M$ is the twist, $m_A : A \otimes A \rightarrow A$ is the multiplication in $A$.

**Conclusions**

Thus noninvertible extension of many abstract structures can be done in common general way: by introduction of the obstructions (or $n$-cycles) $e$ which are analogs of units of the invertible case. In search of possible analogies we observe that “ln $e$” can play the role of first “fundamental group” for “space” of categories and vanishes for invertible morphisms, while its difference from “zero” can be treated as nontrivial “noninvertible topology” of such “space”. We also note that “nil-” extension of supermanifolds – semisupermanifolds $\mathcal{S}$, $\mathcal{S}^*$ – can be compared with the “meta-” extension of supermanifolds –
metamanifolds \[57, 58\] – to find their complimentarity or additivity and possibly for further generalizations simultaneously in both ways.

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