Compatible nonlocal Poisson brackets of hydrodynamic type, and integrable hierarchies related to them

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1 Introduction. Basic definitions

In the present work, the integrable bi-Hamiltonian hierarchies related to compatible nonlocal Poisson brackets of hydrodynamic type are effectively constructed. For achieving this aim, first of all, the problem on the canonical form of a special type for compatible nonlocal Poisson brackets of hydrodynamic type is solved. The compatible pairs of nonlocal Poisson brackets of hydrodynamic type have a more simple description in special coordinates in which the metrics corresponding to these brackets are diagonal (see [1], [2]), but for an effective construction of the hierarchies we need a different approach developed in this paper. For compatible local Poisson brackets of hydrodynamic type (the Dubrovin–Novikov brackets [3]), the corresponding integrable bi-Hamiltonian hierarchies were constructed by the present author in the papers [4], [5], and for compatible nonlocal Mokhov–Ferapontov brackets [6] generated by metrics of constant Riemannian curvature, the bi-Hamiltonian hierarchies were constructed in [7].

1.1 Local Poisson brackets of hydrodynamic type

An arbitrary local homogeneous first-order Poisson bracket, that is, a Poisson bracket of the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x)) \delta_x(x - y) + b^{ij}_k(u(x)) u_k^x \delta(x - y),$$

where $u^1, ..., u^N$ are local coordinates on a certain given smooth $N$-dimensional manifold $M$, is called a local Poisson bracket of hydrodynamic type or Dubrovin–Novikov bracket [3]. Here $u^i(x), 1 \leq i \leq N$, are functions (fields) of single independent variable $x$, the coefficients $g^{ij}(u)$ and $b^{ij}_k(u)$ of bracket (1.1) are smooth functions of local coordinates.

In other words, for arbitrary functionals $I[u]$ and $J[u]$ on the space of fields $u^i(x), 1 \leq i \leq N$, a bracket of the form

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x)) u^k_x \right) \frac{\delta J}{\delta u^j(x)} dx$$

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is defined and it is required that this bracket is a Poisson bracket, that is, it is skew-symmetric:
\[
\{I, J\} = -\{J, I\},
\]
and satisfies the Jacobi identity
\[
\{\{I, J\}, K\} + \{\{J, K\}, I\} + \{\{K, I\}, J\} = 0
\]
for arbitrary functionals \(I[u], J[u]\), and \(K[u]\).

A local bracket \((1.2)\) is called nondegenerate if \(\det(g_{ij}(u)) \neq 0\). For the general nondegenerate brackets \((1.2)\), Dubrovin and Novikov proved the following important theorem.

**Theorem 1.1 (Dubrovin, Novikov [3])** If \(\det(g_{ij}(u)) \neq 0\), then bracket \((1.2)\) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. \(g_{ij}(u)\) is an arbitrary flat pseudo-Riemannian contravariant metric (a metric of zero Riemannian curvature),
2. \(b_{ij}^k(u) = -g^{is}(u)\Gamma_{sk}^j(u)\), where \(\Gamma_{sk}^j(u)\) is the Riemannian connection generated by the contravariant metric \(g_{ij}(u)\) (the Levi–Civita connection).

Consequently, for any local nondegenerate Poisson bracket of hydrodynamic type, there always exist local coordinates \(v^1, \ldots, v^N\) (flat coordinates of the metric \(g_{ij}(u)\)) in which all the coefficients of the bracket are constant:
\[
\tilde{g}^{ij}(v) = \eta^{ij} = \text{const}, \quad \tilde{\Gamma}_{jk}^i(v) = 0, \quad \tilde{b}_{ij}^k(v) = 0,
\]
that is, the bracket has the form
\[
\{I, J\} = \int \frac{\delta I}{\delta v^i(x)} \eta^{ij} \frac{d}{dx} \frac{\delta J}{\delta v^j(x)} dx,
\]
where \((\eta^{ij})\) is a nondegenerate symmetric constant matrix:
\[
\eta^{ij} = \eta^{ji}, \quad \eta^{ij} = \text{const}, \quad \det(\eta^{ij}) \neq 0.
\]

The local Poisson brackets of hydrodynamic type \((1.1)\) were introduced and studied by Dubrovin and Novikov in [3]. In this paper, they proposed a general local Hamiltonian approach (this approach corresponds to the local brackets of form \((1.1)\)) to the so-called homogeneous systems of hydrodynamic type, that is, to evolutionary quasilinear systems of first-order partial differential equations
\[
u^i_t = V^i_j(u) u^j_x.
\]

This Hamiltonian approach was motivated by the study of the equations of Euler hydrodynamics and the Whitham averaging equations describing the evolution of slowly modulated multiphase solutions of partial differential equations (see [8]). In [9], [10] Tsarev constructed the theory of integrating the class of diagonalizable Hamiltonian (and also semi-Hamiltonian) homogeneous systems of hydrodynamic type.
1.2 Nonlocal Poisson brackets of hydrodynamic type

Nonlocal Poisson brackets of hydrodynamic type (the Mokhov–Ferapontov brackets) were introduced and studied in the work of the present author and Ferapontov \[6\] (see also \[11\]–\[13\]). They have the following form:

\[
\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b_k^{ij}(u(x)) u^k_x + Ku^j_x \left( \frac{d}{dx} \right)^{-1} \frac{d}{dx} \right) \frac{\delta J}{\delta u^j(x)} dx,
\]

(1.7)

where \(K\) is an arbitrary constant.

A bracket of form (1.7) is called nondegenerate if \(\det(g^{ij}(u)) \neq 0\).

**Theorem 1.2** ([6]) If \(\det(g^{ij}(u)) \neq 0\), then bracket (1.7) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. \(g^{ij}(u)\) is an arbitrary pseudo-Riemannian contravariant metric of constant Riemannian curvature \(K\),
2. \(b_k^{ij}(u) = -g^{is}(u) \Gamma^j_{sk}(u)\), where \(\Gamma^j_{sk}(u)\) is the Riemannian connection generated by the contravariant metric \(g^{ij}(u)\) (the Levi–Civita connection).

In \[6\] Ferapontov introduced and studied more general nonlocal Poisson brackets of hydrodynamic type (the Ferapontov brackets), namely, the Poisson brackets of the form

\[
\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b_k^{ij}(u(x)) u^k_x + \sum_{\alpha=1}^L \varepsilon_\alpha (w^\alpha)_k^i(u(x)) u^k_x \left( \frac{d}{dx} \right)^{-1} (w^\alpha)_j^k(u(x)) u^k_x \right) \frac{\delta J}{\delta u^j(x)} dx, \quad \det(g^{ij}(u)) \neq 0.
\]

(1.8)

**Theorem 1.3** ([11]) Bracket (1.8) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. \(b_k^{ij}(u) = -g^{is}(u) \Gamma^j_{sk}(u)\), where \(\Gamma^j_{sk}(u)\) is the Riemannian connection generated by the contravariant metric \(g^{ij}(u)\) (the Levi–Civita connection),
2. the metric \(g^{ij}(u)\) and the set of the affinors \((w^\alpha)_i^j(u)\) satisfies relations

\[
g_{ik}(u)(w^\alpha)_k^i(u) = g_{jk}(u)(w^\alpha)_i^k(u), \quad \alpha = 1, \ldots, L,
\]

(1.9)

\[
\nabla_k(w^\alpha)_i^j(u) = \nabla_j(w^\alpha)_k^i(u), \quad \alpha = 1, \ldots, L,
\]

(1.10)

\[
R^j_{ik}(u) = \sum_{\alpha=1}^L \varepsilon_\alpha \left( (w^\alpha)_j^i(u)(w^\alpha)_k^l(u) - (w^\alpha)_j^k(u)(w^\alpha)_i^l(u) \right).
\]

(1.11)

In addition, the family of the affinors \(w^\alpha(u)\) is commutative: \([w^\alpha, w^\beta] = 0\).
Let us write out all the relations on the coefficients of the nonlocal Poisson bracket (1.8) in a convenient form for further repeated use.

**Lemma 1.1** Bracket (1.8) is a Poisson bracket if and only if its coefficients satisfy the relations

\[ g^{ij} = g^{ji}, \]  
\[ \frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k, \]  
\[ g^{is}b^{jk}_s = g^{js}b^{ik}_s, \]  
\[ g^{is}(w^\alpha)_s^j = g^{js}(w^\alpha)_i^j, \]  
\[ (w^\alpha)_s^j(w^\beta)_r^s = (w^\beta)_s^j(w^\alpha)_r^s, \]  
\[ g^{is}g^{jr}\frac{\partial(w^\alpha)_s^l}{\partial u^r} - g^{jr}b^{sk}_s(w^\alpha)_r^k = g^{is}g^{ir}\frac{\partial(w^\alpha)_s^k}{\partial u^r} - g^{ir}b^{sj}_s(w^\alpha)_r^s, \]  
\[ g^{is}\left( \frac{\partial b^{jk}_s}{\partial u^r} - \frac{\partial b^{jk}_r}{\partial u^s} \right) + b^{jk}_sb^{sj}_r - b^{sj}_sb^{jk}_r = \sum_{\alpha=1}^L \xi_\alpha g^{is}\left( (w^\alpha)_s^j(w^\alpha)_r^k - (w^\alpha)_s^j(w^\alpha)_r^k \right). \]  

1.3 Compatible Poisson brackets

In [14] Magri proposed a bi-Hamiltonian approach to the integration of nonlinear systems. This approach demonstrated that integrability is closely related to the bi-Hamiltonian property, that is, the property of a system to have two compatible Hamiltonian representations.

**Definition 1.1** (Magri [14]) Two Poisson brackets \{·, ·\}_1 and \{·, ·\}_2 are called compatible if an arbitrary linear combination of these Poisson brackets

\[ \{·, ·\} = \lambda_1 \{·, ·\}_1 + \lambda_2 \{·, ·\}_2, \]  

where \(\lambda_1\) and \(\lambda_2\) are arbitrary constants, is also a Poisson bracket. In this case, we shall also say that the brackets \{·, ·\}_1 and \{·, ·\}_2 form a pencil of Poisson brackets.

As was shown by Magri in [14], compatible Poisson brackets generate integrable hierarchies of systems of differential equations. In particular, for a system, the bi-Hamiltonian property generates recurrent relations for the conservation laws of this system.

Here the integrable hierarchies related to compatible nonlocal Poisson brackets of hydrodynamic type are constructed.
2 Pencil of nonlocal Poisson brackets of hydrodynamic type

Let us describe all nonlocal Poisson brackets (1.8) compatible with the constant nondegenerate Poisson bracket of hydrodynamic type

\[ \{ I, J \}_2 = \int \frac{\delta I}{\delta u^i(x)} \frac{d}{dx} \frac{\delta J}{\delta u^j(x)} dx, \]  

where \((\eta^{ij})\) is an arbitrary nondegenerate symmetric constant matrix: \(\text{det}(\eta^{ij}) \neq 0\), \(\eta^{ij} = \eta^{ji}\), \(\eta^{ij} = \text{const}\), that is, let us classify all the following pencils of nonlocal Poisson brackets:

\[ \{ I, J \}_\lambda = \{ I, J \}_1 + \lambda \{ I, J \}_2, \]  

where \(\{ I, J \}_1\) is a Poisson bracket of form (1.8).

Lemma 2.1 The Poisson brackets (2.1) and (1.8) are compatible if and only if the following relations are satisfied:

\[ \eta^{is} b^{jk}_s = \eta^{js} b^{ik}_s, \]  

\[ \eta^{is} (w^\alpha)^j_s = \eta^{js} (w^\alpha)^i_s, \]  

\[ \frac{\partial (w^\alpha)^i_j}{\partial u^k} = \frac{\partial (w^\alpha)^j_i}{\partial u^k}, \]  

\[ \frac{\partial b^{jk}_r}{\partial u^s} - \frac{\partial b^{jk}_s}{\partial u^r} = \sum_{\alpha=1}^L \varepsilon_\alpha \left( (w^\alpha)^j_s (w^\alpha)^k_r - (w^\alpha)^j_r (w^\alpha)^k_s \right). \]

It is important to note that the relation

\[ \eta^{ir} b^{lk}_s (w^\alpha)^r_s = \eta^{ir} b^{lk}_s (w^\alpha)^r_s, \]  

derived from (1.17) as one of the compatibility conditions for Poisson brackets (2.1) and (1.8) follows from relations (2.3)–(2.5) for every Poisson bracket (1.8). Actually, from (2.4) we get

\[ \eta^{ir} b^{lk}_s (w^\alpha)^r_s = \eta^{ir} b^{lk}_s (w^\alpha)^r_s, \]  

and from relation (2.3) we also have

\[ \eta^{is} b^{jk}_s (w^\alpha)^j_r = \eta^{is} b^{jk}_s (w^\alpha)^j_r. \]  

Consequently relation (2.7) is reduced to the relation

\[ \eta^{ijs} b^{jk}_s (w^\alpha)^j_r = \eta^{ijs} b^{jk}_s (w^\alpha)^j_r, \]  

that is,

\[ b^{jk}_s (w^\alpha)^j_r = b^{jk}_s (w^\alpha)^j_r. \]
Let us prove that relation (2.11) is satisfied for any Poisson bracket (1.8) for which relation (2.5) is valid. In fact, in this case relation (1.17) for the Poisson bracket (1.8) takes the form
\[ g^{jr} b^{ik}_{a} (w^{\alpha})^s_r = g^{ir} b^{jk}_{a} (w^{\alpha})^s_r. \] (2.12)
From relation (1.15) we get
\[ g^{jr} b^{ik}_{a} (w^{\alpha})^s_r = g^{sr} b^{ik}_{a} (w^{\alpha})^j_r, \] (2.13)
and from relation (1.14) we have
\[ g^{sr} b^{ik}_{a} (w^{\alpha})^j_r = g^{si} b^{rk}_{a} (w^{\alpha})^j_r, \] (2.14)
that is, relation (2.12) is reduced to the relation
\[ g^{si} b^{rk}_{a} (w^{\alpha})^j_r = g^{ir} b^{jk}_{a} (w^{\alpha})^s_r, \] (2.15)
which is equivalent to relation (2.11).

3 Canonical form of compatible pairs of brackets

Theorem 3.1 An arbitrary nonlocal Poisson bracket \( \{I, J\}_1 \) of form (2.1) if and only if it has the form
\[ \{I, J\}_1 = \int \delta I \frac{\delta J}{\delta u^i(x)} \left[ \eta^{is} \frac{\partial F^j}{\partial u^s} + \eta^{js} \frac{\partial F^i}{\partial u^s} - \eta^{ik} \sum_{\alpha=1}^{L} \varepsilon_{\alpha} \frac{\partial \psi^{\alpha}_i}{\partial u^s} \frac{\partial \psi^{\alpha}_j}{\partial u^r} \right] \frac{d}{dx} + \eta^{is} \eta^{jr} \sum_{\alpha=1}^{L} \varepsilon_{\alpha} \frac{\partial^2 \psi^{\alpha}_i}{\partial u^s \partial u^k} \frac{\partial \psi^{\alpha}_j}{\partial u^r} \frac{u^k_x}{u^p} \right] u^k_x + \eta^{ip} \eta^{jr} \sum_{\alpha=1}^{L} \varepsilon_{\alpha} \frac{\partial^2 \psi^{\alpha}_i}{\partial u^p \partial u^k} \frac{d}{dx} \left( \frac{1}{\frac{\partial \psi^{\alpha}_i}{\partial u^j}} \frac{\partial^2 \psi^{\alpha}_j}{\partial u^p \partial u^k} \right) \frac{\delta J}{\delta u^j(x)} dx, \] (3.1)
where \( F^i(u), 1 \leq i \leq N, \) and \( \psi^{\alpha}(u), 1 \leq \alpha \leq L, \) are smooth functions defined in a certain domain of local coordinates.

It follows immediately from relation (2.3) that there locally exist functions \( (\varphi^{\alpha})^i(u), 1 \leq i \leq N, 1 \leq \alpha \leq L, \) such that
\[ (w^{\alpha})^i_j = \frac{\partial (\varphi^{\alpha})^i}{\partial u^j}. \] (3.2)

Then relation (2.6) takes the form
\[ \frac{\partial b^{jk}_{a}}{\partial u^s} - \frac{\partial b^{jk}_{a}}{\partial u^r} = \sum_{\alpha=1}^{L} \varepsilon_{\alpha} \left( \frac{\partial (\varphi^{\alpha})^j}{\partial u^s} \frac{\partial \varphi^{\alpha}_k}{\partial u^r} - \frac{\partial (\varphi^{\alpha})^j}{\partial u^r} \frac{\partial \varphi^{\alpha}_k}{\partial u^s} \right). \] (3.3)
Let us introduce the function
\[ A_{ij}^k(u) = b_{ij}^k(u) - \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i \frac{\partial(\varphi^\alpha)^j}{\partial u^k}. \] (3.4)

Then using (3.3) we get
\[
\begin{align*}
\frac{\partial A_{ij}^k}{\partial u^l} &= \frac{\partial b_{ij}^k}{\partial u^l} - \sum_{\alpha=1}^{L} \varepsilon_{\alpha} \frac{\partial(\varphi^\alpha)^i}{\partial u^l} \frac{\partial(\varphi^\alpha)^j}{\partial u^k} - \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i \frac{\partial^2(\varphi^\alpha)^j}{\partial u^k \partial u^l} = \\
\frac{\partial b_{ij}^k}{\partial u^l} - \sum_{\alpha=1}^{L} \varepsilon_{\alpha} \frac{\partial(\varphi^\alpha)^i}{\partial u^k} \frac{\partial(\varphi^\alpha)^j}{\partial u^l} - \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i \frac{\partial^2(\varphi^\alpha)^j}{\partial u^k \partial u^l} &= \frac{\partial A_{ij}^k}{\partial u^l}. \end{align*}
\] (3.5)

Consequently there exist functions \( P_{ij}^k(u) \), \( 1 \leq i, j \leq N \), such that
\[ A_{ij}^k(u) = \frac{\partial P_{ij}^k}{\partial u^k}. \] (3.6)

Thus,
\[ b_{ij}^k(u) = \frac{\partial P_{ij}^k}{\partial u^k} + \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i \frac{\partial(\varphi^\alpha)^j}{\partial u^k}. \] (3.7)

From relation (1.13) for the Poisson bracket \( \{I, J\}_1 \) we get
\[
\begin{align*}
\frac{\partial g_{ij}^k}{\partial u^k} &= b_{ij}^k + b_{ji}^k = \frac{\partial P_{ij}^k}{\partial u^k} + \frac{\partial P_{ji}^k}{\partial u^k} + \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i \frac{\partial(\varphi^\alpha)^j}{\partial u^k} + \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^j \frac{\partial(\varphi^\alpha)^i}{\partial u^k}, \end{align*}
\] (3.8)

that is, using (1.12) we have
\[ g_{ij}^k = P_{ij}^k + P_{ji}^k + \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i (\varphi^\alpha)^j + c_{ij}^k + c_{ji}^k, \] (3.9)

where \((c_{ij}^k)\) is an arbitrary constant matrix, \( c_{ij} = \text{const} \). Defining the function \( R_{ij}^k(u) \) by the formula
\[ R_{ij}^k = P_{ij}^k + c_{ij}^k, \] (3.10)

we get the proof of the Liouville property for the Poisson bracket \( \{I, J\}_1 \) in the considered local coordinates:
\[ g_{ij}^k = R_{ij}^k + R_{ji}^k + \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i (\varphi^\alpha)^j, \] (3.11)

\[ b_{ij}^k = \frac{\partial R_{ij}^k}{\partial u^k} + \sum_{\alpha=1}^{L} \varepsilon_{\alpha}(\varphi^\alpha)^i \frac{\partial(\varphi^\alpha)^j}{\partial u^k}. \] (3.12)
In the next section, see about the Liouville property more in detail.

It follows from relation (2.4) that
\[
\eta^s \frac{\partial (\varphi^\alpha)^j}{\partial u^s} = \eta^j \frac{\partial (\varphi^\alpha)^i}{\partial u^s},
\]
that is,
\[
\frac{\partial (\eta^j (\varphi^\alpha)^j)}{\partial u^p} = \frac{\partial (\eta^j (\varphi^\alpha)^j)}{\partial u^i}.
\]
Thus there exist functions \(\psi^\alpha(u), 1 \leq \alpha \leq L\), such that
\[
\eta_{rs} (\varphi^\alpha)^s = \partial \psi^\alpha \partial u^r \quad (3.15)
\]
or
\[
(\varphi^\alpha)^s = \eta_{sr} \partial \psi^\alpha \partial u^r. \quad (3.16)
\]

It follows from relation \((2.3)\) that
\[
\eta^s \left( \frac{\partial R^{jk}}{\partial u^s} + \eta^k \eta^{kp} \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial^2 \psi^\alpha}{\partial u^r \partial u^s} \right) = \eta^j \left( \frac{\partial R^{ik}}{\partial u^s} + \eta^i \eta^{kp} \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial^2 \psi^\alpha}{\partial u^r \partial u^s} \right),
\]
or
\[
\frac{\partial (\eta^j R^{jk})}{\partial u^s} + \eta^k \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial^2 \psi^\alpha}{\partial u^r \partial u^s} = \frac{\partial (\eta^j R^{ik})}{\partial u^s} + \eta^i \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial^2 \psi^\alpha}{\partial u^r \partial u^s}. \quad (3.17)
\]

Thus,
\[
\frac{\partial (\eta^j R^{jk})}{\partial u^s} + \eta^k \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial^2 \psi^\alpha}{\partial u^r \partial u^s} = \frac{\partial (\eta^j R^{ik})}{\partial u^s} + \eta^k \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial^2 \psi^\alpha}{\partial u^r \partial u^s}.
\]

Consequently there exist functions \(F^j(u)\) such that
\[
\eta^j R^{jk} + \eta^k \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial \psi^\alpha}{\partial u^r} = \frac{\partial F^k}{\partial u^r}.
\]

that is, we get
\[
R^{ij} = \eta^s \frac{\partial F^j}{\partial u^s} - \eta^s \eta^{jk} \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^s} \frac{\partial \psi^\alpha}{\partial u^k},
\]
\[
g^{ij} = \eta^s \frac{\partial F^j}{\partial u^s} + \eta^s \frac{\partial F^i}{\partial u^s} - \eta^s \eta^{jk} \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^\alpha}{\partial u^s} \frac{\partial \psi^\alpha}{\partial u^k}.
\]
Here it is easy to check that for the Poisson bracket (3.1) all the relations of compatibility (2.3)–(2.6) are satisfied.

4 Integrable equations for canonical compatible pair of brackets

**Theorem 4.1**
Nonlocal bracket (3.1) is a Poisson bracket if and only if the following relations are satisfied:

\[
\begin{align*}
\frac{\partial^2 Q_1}{\partial u^i \partial u^s} \eta^{s \rho} \frac{\partial^2 Q_2}{\partial u^\rho \partial u^j} &= \frac{\partial^2 Q_2}{\partial u^i \partial u^s} \eta^{s \rho} \frac{\partial^2 Q_1}{\partial u^\rho \partial u^j}, \\
g^{ij}_{can} \eta^{s \rho} \frac{\partial^2 Q}{\partial u^s \partial u^r} &= g^{js}_{can} \eta^{s \rho} \frac{\partial^2 Q}{\partial u^s \partial u^r},
\end{align*}
\]

(4.1), (4.2)

where \( g^{ij}_{can}(u) \) is the metric of bracket (3.1):

\[
g^{ij}_{can}(u) = \eta^{is} \frac{\partial F^j}{\partial u^s} + \eta^{is} \frac{\partial F^j}{\partial u^s} - \eta^{is} \eta^{jk} \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial \psi^{\alpha \ast}}{\partial u^s} \frac{\partial \psi^{\alpha \ast}}{\partial u^k},
\]

(4.3)

the functions \( Q(u) \), \( Q_1(u) \), and \( Q_2(u) \) are arbitrary from the functions \( F^i(u) \), \( 1 \leq i \leq N \), and \( \psi^\alpha(u) \), \( 1 \leq \alpha \leq L \).

The nonlinear system (4.1), (4.2) is integrable by the method of inverse scattering problem. The procedure of the integration for the system (4.1), (4.2) will be published in our next paper.

5 Liouville and special Liouville coordinates

Local coordinates \( u = (u^1, ..., u^N) \) are called Liouville for an arbitrary Poisson bracket \( \{I, J\} \) if the functions (the fields) \( u^i(x) \) are densities of integrals in involution with respect to this bracket, that is,

\[
\{U^i, U^j\} = 0, \quad 1 \leq i, j \leq N,
\]

(5.1)

where \( U^i = \int u^i(x) dx \), \( 1 \leq i \leq N \). In this case the Poisson bracket is also called Liouville in these coordinates. Liouville coordinates naturally arise and play an essential role in the Dubrovin–Novikov procedure of averaging of Hamiltonian equations [3]. Physical coordinates derived by averaging of densities of participating in the Dubrovin–Novikov procedure \( N \) involutive local integrals of an initial Hamiltonian system are always Liouville for corresponding averaged bracket. This property was a motivation for the definition of Liouville coordinates.
coordinates for local Poisson brackets of hydrodynamic type in [3]. For general nonlocal Poisson brackets of hydrodynamic type (1.8), Liouville coordinates were introduced in [15].

A nonlocal Poisson bracket of hydrodynamic type (1.8) is Liouville in the local coordinates $u = (u^1, \ldots, u^N)$ if and only if there exist functions $(\varphi^\alpha)^i(u), 1 \leq i \leq L$, and a matrix function $\Phi^{ij}(u)$ such that the bracket has the following form (see [15], where name only this characteristic is taken for the definition of the Liouville property of bracket (1.8)):

$$\{I, J\}_1 = \int \frac{\delta I}{\delta u^i(x)} \left( \Phi^{ij}(u) + \Phi^{ji}(u) - \sum_{\alpha=1}^L \varepsilon_\alpha (\varphi^\alpha)^i(\varphi^\alpha)^j \right) \frac{du^k}{dx} +$$

$$\left[ \frac{\partial \Phi^{ij}}{\partial u^k} - \sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial (\varphi^\alpha)^j}{\partial u^k} \varphi^\alpha_i \right] u^k_x +$$

$$\sum_{\alpha=1}^L \varepsilon_\alpha \frac{\partial (\varphi^\alpha)^i}{\partial u^k} u^k_x \left( \frac{d}{dx} \frac{1}{\partial u^s} \partial u^s_x \right) \frac{\delta J}{\delta u^j(x)} dx.$$ (5.2)

From theorem 3.1, it follows

**Theorem 5.1** Flat coordinates of an arbitrary nondegenerate local Poisson bracket of hydrodynamic type $\{I, J\}_2$ are always Liouville for any nonlocal Poisson bracket $\{I, J\}_1$ (1.8) compatible with $\{I, J\}_2$. Moreover, in addition the corresponding Liouville functions $\Phi^{ij}(u)$ and the functions $(\varphi^\alpha)^i(u)$ always have the special form

$$\Phi^{ij}(u) = \eta^{is} \frac{\partial F^j}{\partial u^s}, \quad (\varphi^\alpha)^i(u) = \eta^{is} \frac{\partial \psi^\alpha}{\partial u^s}.$$ (5.3)

Local coordinates $u = (u^1, \ldots, u^N)$ are called special Liouville coordinates [6], [7] for an arbitrary Poisson bracket $\{I, J\}$ if there exists a nonzero constant symmetric matrix $(\eta_{ij})$ such that the functions (the fields) $u^i(x), 1 \leq i \leq N$, and $\eta_{ij} u^i(x) u^j(x)$ are densities of integrals in involution with respect to this bracket, that is,

$$\{U^i, U^j\} = 0, \quad 1 \leq i, j \leq N + 1,$$ (5.4)

where $U^i = \int u^i(x) dx, 1 \leq i \leq N, U^{N+1} = \int \eta_{ij} u^i(x) u^j(x) dx$. In this case the Poisson bracket is also called special Liouville in these coordinates. The special Liouville coordinates were introduced in [6], [7]. The most important case is the case of nondegenerate matrix $\eta_{ij}$.

**Theorem 5.2** An arbitrary Poisson bracket of form (1.8) is special Liouville in local coordinates $u = (u^1, \ldots, u^N)$ if and only if it is Liouville with a special Liouville function $\Phi^{ij}(u)$ and functions $(\varphi^\alpha)^i(u)$ of the special form such that

$$\eta_{ks} \Phi^{ij}(u) = \frac{\partial F^j}{\partial u^k}, \quad \eta_{ks} (\varphi^\alpha)^i(u) = \frac{\partial \psi^\alpha}{\partial u^k}.$$ (5.5)
In this case, for a nondegenerate matrix \((\eta_{ij})\), we get exactly our bracket \((3.1)\) from the canonical compatible pair.

Thus our problem on compatible nonlocal Poisson brackets of hydrodynamic type is equivalent to the problem of classification of the special Liouville coordinates for nonlocal Poisson brackets of hydrodynamic type.

**Theorem 5.3** An arbitrary nonlocal Poisson bracket of hydrodynamic type of form \((1.8)\) is compatible with the constant Poisson bracket \((2.1)\) if and only if the functions \(u^i(x), 1 \leq i \leq N, \) and \(\eta_{ij}u^j(x)u^i(x), \eta^{is}\eta_{sj} = \delta_j^i,\) are densities of integrals in involution with respect to the Poisson bracket \((2.1)\).

Note that \(u^i(x), 1 \leq i \leq N, \) are the densities of the annihilators of the bracket \((2.1),\) and \(\frac{1}{2}\eta_{ij}u^i(x)u^j(x)\) is the density of the momentum of bracket \((2.1).\)

**Theorem 5.4** An arbitrary nonlocal Poisson bracket of hydrodynamic type of form \((1.8)\) is compatible with an arbitrary nondegenerate local Poisson bracket of hydrodynamic type \((1.2)\) if and only if \(N\) annihilators and the momentum of bracket \((1.2)\) are integrals in involution with respect to the Poisson bracket \((1.8)\).

### 6 Integrable bi-Hamiltonian hierarchies

Consider a pair of compatible Hamiltonian operators of hydrodynamic type \(P_1^{ij}\) and \(P_2^{ij}\), one of which, let us assume \(P_2^{ij}\), is local, and another is an arbitrary nonlocal operator of form \((1.8)\). Apparently, one of the nonlocal Hamiltonian operators of hydrodynamic type can always be reduced to the canonical constant form by series of reciprocal transformations if it is nondegenerate (an analog of the classical Darboux theorem in symplectic geometry), so that the considered case is, in fact, general. If the local Hamiltonian operator \(P_2^{ij}\) is nondegenerate, then it follows from theorem \((3.1)\) that, by local change of coordinates, the pair of compatible Hamiltonian operators \(P_1^{ij}\) and \(P_2^{ij}\) can be reduced to the following canonical form:

\[
P_2^{ij}[u(x)] = \eta^{ij} \frac{d}{dx},
\]

\[
P_1^{ij} = \left(\eta^{is} \frac{\partial F^j}{\partial u^s} + \eta^{is} \frac{\partial F^j}{\partial u^s} - \eta^{is} \eta^{jk} \sum_{\alpha=1}^{L} \xi_{\alpha} \frac{\partial \psi^\alpha}{\partial u^s} \frac{\partial \psi^\alpha}{\partial u^k} \right) \frac{d}{dx} + \]

\[
\sum_{\alpha=1}^{L} \xi_{\alpha} \frac{\partial^2 \psi^\alpha}{\partial u^s \partial u^p} \left( \eta^{is} \eta^{jp} \eta^{qr} \sum_{\alpha=1}^{L} \xi_{\alpha} \frac{\partial^2 \psi^\alpha}{\partial u^s \partial u^k} \frac{\partial \psi^\alpha}{\partial u^r} \frac{\partial \psi^\alpha}{\partial u^q} \right) \frac{d}{dx} u^k + \]

\[
\eta^{hp} \frac{\partial^2 \psi^\alpha}{\partial u^p \partial u^k} \frac{d}{dx} \left( \frac{d}{dx} \right)^{-1} \frac{\partial^2 \psi^\alpha}{\partial u^r \partial u^s} u^s_x.
\]
where \((\eta^{ij})\) is an arbitrary nondegenerate constant symmetric matrix: \(\det(\eta^{ij}) \neq 0\), \(\eta^{ij} = \text{const}\), \(\eta^{ij} = \eta^{ji}\); \(F^i(u), 1 \leq i \leq N\), and \(\psi^\alpha(u), 1 \leq \alpha \leq L\), are smooth functions defined in a certain domain of local coordinates such that operator (6.2) is Hamiltonian, that is, the functions \(F^i(u)\) and \(\psi^\alpha(u)\) satisfy the integrable equations (4.1), (4.2) (see theorem 4.1 above).

**Remark 6.1** It is obvious that here we can always consider that \(\eta^{ij} = \varepsilon^i \delta^{ij}\), \(\varepsilon^i = 1\) for \(i \leq p\), \(\varepsilon^i = -1\) for \(i > p\), where \(p\) is the positive index of inertia of the metric, \(0 \leq p \leq N\), and, in addition, it is necessary to classify the Hamiltonian operators (6.2) with respect to the action of the group of motions for the corresponding \(N\)-dimensional pseudo-Euclidean space \(\mathbb{R}^N\), but for our purposes it is sufficient (and more convenient) to use the indicated above representation for canonical compatible pair (“conventionally canonical” representation).

Consider the recursion operator generated by the canonical compatible Hamiltonian operators (6.1), (6.2):

\[
R^i_l[v(x)] = \left[ P_1[v(x)] (P_2[v(x)])^{-1} \right]^i_l = \left( \eta^{is} \frac{\partial F^j}{\partial u^s} + \eta^{is} \frac{\partial F^i}{\partial u^s} - \eta^{is} \frac{\partial^2 F^j}{\partial u^s \partial u^k} \eta^{jk} L \sum_{\alpha=1}^L \varepsilon^s_{\alpha} \frac{\partial^2 \psi^\alpha}{\partial u^s \partial u^k} \frac{\partial \psi^\alpha}{\partial u^p} \right) u^k_x + \eta^p \eta^{s_1} \frac{\partial^2 \psi^s}{\partial u^p \partial u^k} u^k_x \left( \frac{d}{dx} \right)^{-1} u^s_x \eta^{s_1} \left( \frac{d}{dx} \right)^{-1} \quad (6.3)
\]

(what about recursion operators generated by pairs of compatible Hamiltonian operators, see [18]–[23].

Let us apply the derived recursion operator (6.3) to the system of translations with respect to \(x\), that is, the system of hydrodynamic type

\[
u_i^t = u^i_x, \quad (6.4)
\]

which is, obviously, Hamiltonian with the Hamiltonian operator (6.1):

\[
u_i^t = u^i_x \equiv P_2^{ij} \delta H \frac{\delta u^j}{\delta u^i(x)}; \quad H = \frac{1}{2} \int \eta_{ij} u^j(x) u^i(x) dx. \quad (6.5)
\]

Any system from hierarchy

\[
u^i_n = (R^n)_j^i u^j_x, \quad n \in \mathbb{Z}, \quad (6.6)
\]

is a multi-Hamiltonian integrable system.

In particular, any system of the form

\[
u^i_t = R^i_j u^j_x, \quad (6.7)
\]
that is, the system of hydrodynamic type

\[
\begin{align*}
\eta^{is} \frac{\partial F^j}{\partial u^s} + \eta^{js} \frac{\partial F^i}{\partial u^s} - \eta^{is} \eta^{jk} \sum_{\alpha=1}^{L} \varepsilon_\alpha \left( \frac{\partial \psi^\alpha}{\partial u^s} \frac{\partial \psi^\alpha}{\partial u^k} \right) 
\end{align*}
\]

\[
\begin{align*}
\eta^{ip} \eta^{jr} L \sum_{\alpha=1}^{L} \varepsilon_\alpha \left( \frac{\partial^2 \psi^\alpha}{\partial u^p \partial u^k} \right) u^k_x \equiv
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} \int \eta_t u^j(x) u^i(x) dx,
\end{align*}
\]

\[
\begin{align*}
\int \left( \eta_{jk} F^k(u(x)) u^j(x) - \frac{1}{2} \sum_{\alpha=1}^{L} \varepsilon_\alpha (\psi^\alpha(u))^2 \right) dx.
\end{align*}
\]

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