Some families of special Lagrangian tori

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Abstract

We give a simple proof of the local version of Bryant’s result [1], stating that any 3-dimensional Riemannian manifold can be isometrically embedded as a special Lagrangian submanifold in a Calabi-Yau manifold. We then refine the theorem proving that a certain class of one-parameter families of metrics on a 3-torus can be isometrically embedded in a Calabi-Yau manifold as a one-parameter family of special Lagrangian submanifolds. Two applications of our results show how the geometry of moduli space of 3-dimesional special Lagrangian submanifolds differs considerably from the 2-dimensional one. First of all, applying our first theorem and a construction due to Calabi we show that nearby elements of the local moduli space of a special Lagrangian 3-torus can intersect themselves. Secondly, we use our examples of one-parameter families to show that the semi-flat metric on the mirror manifold proposed by Hitchin in [13] is not necessarily Ricci-flat in dimension 3.

1 Introduction

Many interesting speculations have been made about the role special Lagrangian submanifolds should play in understanding the geometry of Calabi-Yau manifolds and of Mirror Symmetry. Unfortunately the lack of examples has allowed few of these to be proved. Only recently has the number of new constructions finally begun to increase. For years, in fact, the only examples known were the ones appearing in the foundational paper by Harvey and Lawson [11], where special Lagrangian submanifolds were defined for the first time. Our paper participates in the quest for examples. We propose a

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new way to construct special Lagrangian submanifolds and one-parameter families of these and we relate them to some of the speculations which have been made about them. Let’s first recall some definitions. For us, a Calabi-Yau manifold will be a triple \((M, \Omega, \omega)\) where \(M\) is a complex \(n\)-dimensional manifold, \(\Omega\) a nowhere-vanishing holomorphic \(n\)-form on \(M\) and \(\omega\) a Kähler form related to \(\Omega\) by
\[
\omega^n = c \Omega \wedge \overline{\Omega},
\]
for some constant \(c\). By Yau’s proof of the Calabi conjecture, this triple can be constructed on any compact Kähler manifold with trivial canonical bundle. The Kähler metric \(\omega\) is Ricci-flat. An \(n\)-dimensional submanifold \(M\) is called special Lagrangian (sometimes abbreviated sLag) if it satisfies:
\[
\text{Re } \Omega|M = \text{Vol}_M,
\]
where \(\text{Vol}_M\) denotes the volume form on \(M\). Equivalently, \(M\) is special Lagrangian if and only if it satisfies the following:
\[
\begin{align*}
\text{Im } \Omega|M & = 0, \\
\omega|M & = 0.
\end{align*}
\]

In this paper we will very often refer to the work of three authors: McLean [19], Hitchin [13] and Gross [7, 8, 9]. We briefly describe here their results. Given a special Lagrangian submanifold \(M\), McLean proved that the moduli space of nearby special Lagrangian submanifolds can be identified with a smooth submanifold \(M\) of \(\Gamma(\nu(M))\), the space of sections of \(\nu(M)\), the normal bundle of \(M\). The dimension of \(M\) is \(b_1(M)\), the first Betti number of \(M\). In fact, through the map \(V \rightarrow (JV)^\flat\) (cfr. end of section for notation), which identifies a section \(V\) in \(\Gamma(\nu(M))\) with a section in \(\Omega^1(M)\), \(M\) can be viewed inside \(\Omega^1(M)\) and its tangent space at \(M\) turns out to be the vector space of harmonic one-forms on \(M\). In practice, the latter means that if we take a variation of \(M\) through special Lagrangian submanifolds with variational vector field \(V\), then \((JV)^\flat\) is a harmonic one-form. In particular, if \(M\) is a torus with non-vanishing harmonic one-forms, then McLean’s result implies that a whole open set of \(M\) around \(M\) is fibred by special Lagrangian tori. On \(M\) there is also a natural metric which is the standard \(L^2\) norm of one-forms.

In [21] the three authors conjectured, in what is now called the SYZ-conjecture, a geometric construction of Mirror Symmetry. Here, on purely physical grounds, they argued that if \(M\) is near some boundary point of its
complex moduli space then it should be possible to fibre it through special Lagrangian tori, some of which may be singular. The mirror manifold of $\overline{M}$, in the sense of Mirror Symmetry, is obtained by dualizing this fibration. Some mathematical aspects the conjecture were described and investigated by Hitchin [13] and Gross [8, 7, 9]. First Hitchin showed how $\mathcal{M}$ can be naturally identified with an open subset of $H^1(M, \mathbb{R})$ or of $H^{n-1}(M, \mathbb{R})$ and explained how the two identifications are dual to each other. According to the SYZ-conjecture, in the case $M$ is a torus, a local candidate for the mirror of $\overline{M}$ is the space

$$\mathcal{X} = \mathcal{M} \times H^1(M, \mathbb{R}/\mathbb{Z}).$$

This is a torus fibration over $\mathcal{M}$. The problem is to find, possibly in a natural way, a Calabi-Yau structure on this fibration such that the fibres are special Lagrangian tori. Using the identifications above, Hitchin explained how to construct an integrable complex structure, a Kähler form and a holomorphic n-form on $\mathcal{X}$. This metric is often called the semi-flat metric. He then showed that these forms give a Calabi-Yau structure, i.e. they are related by (1), if and only if $\mathcal{M}$ satisfies a certain condition. While this condition is known to be satisfied in the 2-dimensional case (see for example Hitchin [14]), it is one of the results of this paper that in general it is not in dimension 3.

Gross dealt with the more global aspects of the SYZ construction by treating the problem of how to include singular special Lagrangian fibres in the above picture. In fact, on the basis of the topological consequences of Mirror Symmetry, he gave a conjectural description of the singular fibres which are expected to appear and explained how to dualize them. This construction is completely understood for K3 surfaces, where special Lagrangian fibrations are just elliptic fibrations with a different complex structure.

Parallel to these speculative aspects of special Lagrangian geometry, there has been the attempt to produce examples. After the Harvey and Lawson ones, Bryant [2] and Kobayashi [15] showed how to construct special Lagrangian tori as totally real submanifolds of subvarieties of $\mathbb{C}P^n$. Lately, many examples of special Lagrangian fibrations where constructed on complete Calabi-Yau manifolds by Goldstein [4, 5, 6]. In [10] Gross used similar ideas to Goldstein’s to construct special Lagrangian fibrations on $\mathbb{C}^n/G$, where $G$ is a finite abelian subgroup of $SU(n)$. More recently Haskins [12] found more special Lagrangian cones in in $\mathbb{C}^3$. His construction was subsequently generalized by Joyce [17, 16], who also provided other examples which are not cones.
The results of this paper overlap in part with those obtained by Bryant [1]. He proved that any real-analytic, 3-dimensional Riemannian manifold $(M, g)$ with real-analytic metric $g$ can be isometrically embedded in some Calabi-Yau manifold $\mathcal{M}$. His proof used Cartan-Kähler theory, which requires the problem to be translated into one of existence of integral submanifolds of a differential ideal. Our first result (Theorem 3.1) is the local version of the same theorem, but the proof is simpler and is global in the case of the torus. We prove the following: given any pair $(U, g)$ where $U$ is some open set in $\mathbb{R}^3$ and $g$ a metric, we can isometrically embed $U$ as a special Lagrangian submanifold of some Calabi-Yau manifold $\mathcal{M}$. Our proof, as well as being simple, has other advantages. First of all we show that the complex structure of $\mathcal{M}$ around $U$ is in some sense unique and can be dealt with very concretely with a suitable choice of coordinates. Hence we prove that also the holomorphic n-form is unique, in fact it is literally the holomorphic extension of the volume form on $U$. Finally, we write the equations for the Ricci-flat Kähler metric and show that a solution always exist with three successive applications of the Cauchy-Kowalesky theorem. Using this result and the construction by Calabi of metrics on the 3-torus which admit harmonic one-forms with zeroes we show that there are special Lagrangian 3-tori which can intersect elements of the moduli space of its deformations. This did not happen in dimension 2.

The structure of the proof of our first result leads to an immediate refinement. In fact we show (Theorem 4.1) that if a one-parameter family of metrics on a 3-torus satisfies certain simple conditions, then it can always be realized as a one-parameter family of special Lagrangian tori in a Calabi-Yau manifold. The set of one-parameter families thus constructed is quite rich and provides us with many examples. Some of these also show that the condition required for Hitchin’s metric to yield a Calabi-Yau structure is in fact not satisfied. This leads to the question of how can one find such a structure.

Notations. When working in $\mathbb{C}^n$ complex coordinates are always denoted by $(z_1, \ldots, z_n)$, and real coordinates by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, where $z_k = x_k + iy_k$. Sometimes we will use $x$ (or $y$) as short for $(x_1, \ldots, x_n)$ (or $(y_1, \ldots, y_n)$). The letter $J$ is always used to denote the almost complex structure. The superscript $(V)^\flat$ stands for the element in $T^*M$ corresponding to $V$ under the identification of $TM$ and $T^*M$ induced by the metric. As usual $\ast : \Omega^k(M) \rightarrow \Omega^{n-k}(M)$ denotes the Hodge-star operator between forms. We follow the convention that given the coefficients of an invertible matrix $g_{ij}$, the terms $g^{ij}$ denote the coefficient of the inverse matrix.

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2 Complexifications

Given a real-analytic, n-dimensional manifold \( M \), a complexification of \( M \) is an n-dimensional complex manifold \( \overline{M} \) together with a real analytic embedding \( \iota : M \to \overline{M} \) such that for every \( p \in \overline{M} \) there exist holomorphic coordinates \((z_1, \ldots, z_n)\) on a neighborhood \( U \) of \( p \) such that \( q \in U \cap \iota(M) \) if and only if \( \text{Im}(z_i(q)) = 0, \ i = 1, \ldots, n \).

Example 1. Given an open set \( U \subseteq \mathbb{R}^n \), identify it with a subset of \( \mathbb{C}^n \) through the standard inclusion of \( \mathbb{R}^n \) in \( \mathbb{C}^n \) as the real part. An open neighborhood \( U_{\mathbb{C}} \) of \( U \) such that \( \text{Re}(U_{\mathbb{C}}) = U \) will be called a standard complexification of \( U \). So, \( \overline{M} \) being a complexification of \( M \) means that, locally, the pair \( (\overline{M}, M) \) is holomorphic to the pair \( (U_{\mathbb{C}}, U) \). \( \square \)

Example 2. Let \( M \) be the standard n-torus \( \mathbb{R}^n / \mathbb{Z}^n \) and \( \iota \) its obvious inclusion in \( \mathbb{C}^n / \mathbb{Z}^n \), where \( \mathbb{Z}^n \) acts through translations on the real part. Then \( (\mathbb{C}^n / \mathbb{Z}^n, \iota) \) is a complexification of \( M \). It will be referred to as a standard complexification of the n-torus. \( \square \)

Bruchat and Whitney [22] proved the following:

**Theorem 2.1 (Bruchat, Whitney)** Any paracompact, real-analytic manifold \( M \) admits a complexification. Moreover if \((\overline{M}_1, \iota_1)\) and \((\overline{M}_2, \iota_2)\) are two complexifications of \( M \), then there exist neighborhoods \( V_i \) of \( \iota_i(M) \), \( i = 1, 2 \), and a biholomorphism \( F : V_1 \to V_2 \) extending \( \iota_2 \circ \iota_1^{-1} \).

They also showed that there exists an antiholomorphic involution \( \sigma : \overline{M} \to \overline{M} \) which has \( M \) as the set of its fixed points. Identify \( \iota(M) \) with \( M \). We say that \( M \) is a totally real submanifold of a complex manifold \( \overline{M} \) if \( J(T_pM) \) is transversal to \( T_pM \), for every \( p \in M \), where \( J \) is the complex structure on \( \overline{M} \). If \( \overline{M} \) is a complexification of \( M \) then \( M \) is obviously a totally real submanifold of \( \overline{M} \). The converse is also true:

**Lemma 2.1** Let \( \iota : M \to \overline{M} \) be a real-analytic embedding of \( M \) as a totally real submanifold of the complex manifold \( \overline{M} \). Then \((\overline{M}, \iota)\) is a complexification of \( M \).
Proof. Let $p \in M$. We can assume w.l.o.g. $M = \mathbb{C}^n$, $p = 0$ and $T_p M = \{ \text{Im}(z_i) = 0, \ i = 1, \ldots, n \}$. Then there exists a neighborhood $V \subset \mathbb{C}^n$ of 0 and a real-analytic map $f : \text{Re}(V) \to \mathbb{R}^n$ such that $V \cap M = \{ x + if(x), x \in V \}$. Extend $f$ to a holomorphic function $\tilde{f} : \tilde{V} \to \mathbb{C}^n$, where $\tilde{V}$ is some neighborhood of $\text{Re}(V)$ in $\mathbb{C}^n$. Define $\tilde{F} : \tilde{V} \to \mathbb{C}^n$ by $\tilde{F}(z) = z + i\tilde{f}(z)$, then $\tilde{F}$ is a biholomorphism near 0 and $F = \tilde{F}^{-1}$ gives the complex coordinates with the required property.

In particular we have the following:

Corollary 2.1 Let $M_1$ be a Kähler manifold and $\iota_1 : M \to M_1$ a real-analytic embedding of $M$ as a Lagrangian submanifold. If $(M_2, \iota_2)$ is a complexification of $M$, then there exist neighborhoods $V_i$ of $\iota_i(M)$ and a biholomorphism $F : V_1 \to V_2$ extending $\iota_2 \circ \iota^{-1}_1$.

Proof. It follows immediately from Theorem 2.1 and Lemma 2.1 since Lagrangian submanifolds are totally real.

Notice that, since special Lagrangian submanifolds are minimal, they are also real-analytic. Hence Corollary 2.1 applies when $M$ is a special Lagrangian submanifold. In particular if $\phi : U \to M$ is a real-analytic coordinate chart, it can be extended to a holomorphic chart $\phi_C : U_C \to \overline{M}$. Also, in the case $M$ is the n-torus and $(\mathbb{C}^n / \mathbb{Z}^n, \iota)$ its standard complexification, then any special Lagrangian embedding $\tau : M \to \overline{M}$ can be extended to a holomorphic chart $F : U_C \to \overline{M}$, where $U_C$ is a sufficiently small neighborhood of $M$ in $\mathbb{C}^n / \mathbb{Z}^n$.

3 Local isometric special Lagrangian embeddings

Now let $(U, g)$ be an open neighborhood of $0 \in \mathbb{R}^3$ together with a Riemannian metric $g = (g_{ij})$. We look for isometric embeddings of $(U, g)$ as a special Lagrangian submanifold of some Calabi-Yau $M$. From the results in the previous section we may assume w.l.o.g. that $M = U_C$ for some standard complexification $U_C$. Remember that $U_C$ is a subset of $\mathbb{C}^n$, so we can use the standard complex coordinates $(z_1, \ldots, z_n)$. We will prove the following:

Theorem 3.1 On some standard complexification $U_C$ of $U$ we can find a unique holomorphic $n$-form $\Omega$ and at least one Kähler form $\omega$ satisfying the following properties:

1. $\omega^3/3! = -(i/2)^3 \Omega \wedge \overline{\Omega}$,
2. the induced metric on $U$ is $g$.

3. $\Omega|_U = \text{Vol}_U$.

The first condition is just equation (3) from the Introduction, with a choice of the constant $c$. Conditions 2 and 3 make $(U, g)$ isometrically embedded in $(U_\mathbb{C}, \Omega, \omega)$ as a special Lagrangian submanifold. In what follows we will denote by $h = (h_{ij})$ the hermitian metric associated with $\omega$. Part of the theorem is proved by the next lemma:

**Lemma 3.1** There exists a unique $\Omega$ on $U_\mathbb{C}$ satisfying conditions (1)-(3) above. In fact, in standard coordinates, $\Omega$ must be

$$\Omega = \Gamma_g(z) dz_1 \wedge dz_2 \wedge dz_3,$$

where $\Gamma_g$ denotes the holomorphic extension of $\sqrt{g} = \sqrt{\det(g_{ij})}$, the coefficient of $\text{Vol}_U$.

**Proof.** Certainly we can write

$$\Omega = f(z) dz_1 \wedge dz_2 \wedge dz_3,$$

for some holomorphic $f$. Let $f = \alpha + i\beta$, then condition (1) gives:

$$\det(h_{ij}) = \alpha^2 + \beta^2.$$

From condition (2) it follows that, along $U$, we have $h_{ij}(x, 0) = g_{ij}(x)$, giving that $\det(h_{ij})(x, 0) = g(x)$. Condition (3) implies that

$$\Omega|_U = \alpha dx_1 \wedge dx_2 \wedge dx_3 = \sqrt{g} dx_1 \wedge dx_2 \wedge dx_3.$$

Therefore we obtain that $\beta(x, 0) = 0$ and $f(x, 0) = \alpha(x, 0) = \sqrt{g}(x)$. The only holomorphic function satisfying this is precisely $\Gamma_g$. $\square$

**Proof of Theorem 3.1** We write the hermitian metric $h$ that we are looking for as $h = A + iB$, where $A = (a_{ij})$ and $B = (b_{ij})$ are real valued matrices, symmetric and antisymmetric respectively. In the basis $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})$ for $TU_\mathbb{C}$ the corresponding Kähler form can be written as a $2n \times 2n$ matrix

$$\omega = \begin{pmatrix} -B & A \\ -A & -B \end{pmatrix}.$$
In order to prove the theorem we need to solve the following "initial value"
PDE problem:

\[
\begin{cases}
\det(h) = |\Gamma_g|^2 \\
d\omega = 0 \\
A(x, 0) = g(x) \text{ and } B(x, 0) = 0 \text{ for all } x \in \mathbb{U}.
\end{cases}
\]

If we do the computations explicitly we see that (D) and (C) form the
following system of equations in the coefficients of \(\omega\):

\[
\begin{align*}
(\alpha_{22} \alpha_{33} &- \alpha_{23}^2 - \beta_{23}^2) \alpha_{11} - \beta_{13}^2 \alpha_{22} \\
- \beta_{12} \alpha_{33} - \alpha_{12}^2 \alpha_{33} - \alpha_{13}^2 \alpha_{22} + 2\alpha_{12} \alpha_{23} \alpha_{13} \\
- 2\beta_{12} \beta_{23} \alpha_{13} + 2\alpha_{12} \beta_{23} \beta_{13} + 2\beta_{12} \alpha_{23} \beta_{13} = |\Gamma_g|^2
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \beta_{ij}}{\partial y_1} &= \frac{\partial \alpha_{ij}}{\partial x_1} - \frac{\partial \alpha_{ij}}{\partial x_j} \quad (C1) \\
\frac{\partial \beta_{ij}}{\partial y_2} &= \frac{\partial \alpha_{ij}}{\partial x_2} - \frac{\partial \alpha_{ij}}{\partial x_k} \quad (C2.1) \\
\frac{\partial \alpha_{ij}}{\partial y_1} &= \frac{\partial \alpha_{ij}}{\partial x_1} + \frac{\partial \beta_{12}}{\partial x_k} \quad (C2.2) \\
\frac{\partial \alpha_{ij}}{\partial y_2} &= \frac{\partial \alpha_{ij}}{\partial x_2} + \frac{\partial \beta_{13}}{\partial x_k} \quad (C3.1) \\
\frac{\partial \alpha_{ij}}{\partial y_3} &= \frac{\partial \alpha_{ij}}{\partial x_3} + \frac{\partial \beta_{23}}{\partial x_k} \quad (C3.2) \\
\frac{\partial \beta_{23}}{\partial y_1} - \frac{\partial \beta_{13}}{\partial y_2} + \frac{\partial \beta_{12}}{\partial y_3} &= 0 \quad (C4.1) \\
\frac{\partial \beta_{23}}{\partial y_1} + \frac{\partial \beta_{13}}{\partial y_2} - \frac{\partial \beta_{12}}{\partial y_3} &= 0. \quad (C4.2)
\end{align*}
\]

Here the index \(k\) goes from 1 to 3, while \(i, j\) are such that \(i < j\).

A solution is constructed in three steps: first we find one on \(U_1^1 = \{(z_1, z_2, z_3) \in U_\mathbb{C} | y_2 = y_3 = 0\}\), then we extend it to \(U_2^2 = \{(z_1, z_2, z_3) \in U_\mathbb{C} | y_3 = 0\}\) and finally to the whole \(U_\mathbb{C}\). Notice that for the first step we
need only to look at equations (D) and (C1), which do not involve derivatives
with respect to \(y_2\) or \(y_3\). For reasons that will become apparent later we do
not assume that \(A\) is symmetric. Hence, we have four equations for twelve
unknowns (nine from \(A\) and three from \(B\)). We choose arbitrarily all \(\alpha_{ij}\)'s
on \(U_2^2\) except \(\alpha_{11}\), with the only requirements that they satisfy the initial
conditions (I), they are real-analytic and they can be coefficients of a metric
(e.g. \(\alpha_{ij} = \alpha_{ji}\)). It is now easy to see that by differentiating (D) by \(y_1\) and
substituting into it equations from (C1), (D) can be written in the form

\[
\frac{\partial \alpha_{11}}{\partial y_1} = P(x, y_1, \alpha_{11}, \beta) \frac{\partial \alpha_{11}}{\partial x_2} + Q(x, y_1, \alpha_{11}, \beta) \frac{\partial \alpha_{11}}{\partial x_3} + R(x, y_1, \alpha_{11}, \beta), \quad (D')
\]
where \( P, Q \) and \( R \) are real-analytic coefficients, which depend on the way we arbitrarily extended the other \( \alpha_{ij} \)'s. Notice that this is possible also because, with the given initial conditions, the coefficient of \( \alpha_{11} \) in (D) is different from zero near \( U \). Now equations \((D')\) and \((C1)\) are four equations in the four unknowns \( \alpha_{11}, \beta_{12}, \beta_{13}, \beta_{23} \) of the type whose solution is guaranteed to exist uniquely (at least locally) by the Cauchy-Kowalesky theorem (as stated for example in Spivak [20, Section 10.5]). The solution will also satisfy equation \((C4.1)\). In fact this is demonstrated by differentiating \((C1)\), \( i = 1, j = 2 \) by \( x_3; (C1), i = 1, j = 3 \) by \( x_2 \) and \( (C1), i = 2, j = 3 \) by \( x_1 \). From the results it follows that

\[
\frac{\partial}{\partial y_1} \left( \frac{\partial \beta_{23}}{\partial x_1} - \frac{\partial \beta_{13}}{\partial x_2} + \frac{\partial \beta_{12}}{\partial x_3} \right) = 0
\]
on \( U^1_\mathbb{C} \). This shows that since equation \((C4.1)\) holds on \( U \) it holds everywhere also on \( U^1_\mathbb{C} \).

The second step is similar. We now extend this solution to \( U^2_\mathbb{C} \) by looking at equations \((D)\) and the group \((C2)\). This time we have seven equations for twelve unknowns. We arbitrarily extend \( \alpha_{33} \) and \( \alpha_{23} = \alpha_{32} \) as before. Then, for the symmetry of \( A \), we also impose \( \alpha_{12} = \alpha_{21} \) and \( \alpha_{13} = \alpha_{31} \). Differentiating \((D)\) by \( y_2 \), again we see that we can reduce the system to one which is solvable by the Cauchy-Kowalevsky theorem, where now the evolution variable is \( y_2 \) and the initial domain is \( U^1_\mathbb{C} \). Notice that equations \((C1)\) will still hold for this extended solution. To see this, first differentiate \((C2.1)\) by \( y_1 \). Then substitute, into the result, equation \((C2.2)\), \( k = i \) differentiated by \( x_j \) and equation \((C2.2)\), \( k = j \) differentiated by \( x_i \). Thus we obtain

\[
\frac{\partial}{\partial y_2} \left( \frac{\partial \beta_{ij}}{\partial y_1} - \frac{\partial \alpha_{1j}}{\partial x_i} + \frac{\partial \alpha_{1i}}{\partial x_j} \right) = 0,
\]

which tells us that equations \((C1)\) hold for all \( y_2 \) since, by the first step, they hold for \( y_2 = 0 \). Again, the solution will satisfy also equation \((C4.1)\). This is shown by the same method as in the first step, except that we use equations \((C2.1)\) instead of \((C1)\).

The same procedure produces the third and last extension. We have ten equations for twelve unknowns. We impose \( \alpha_{23} = \alpha_{32} \) and \( \alpha_{13} = \alpha_{31} \). Notice that, because of equations \((C3.2)\), \( k = 2 \) and \((C3.3)\), \( k = 1 \), we cannot impose \( \alpha_{12} = \alpha_{21} \). So let’s treat them as separate unknowns, for the moment. As in the first and second step we find a solution to the system. Again, we must show that equations \((C1)\), \((C2.1)\) and \((C2.2)\) are
still satisfied. To prove that (C1) holds we do exactly as in step two when we proved the same thing, except that we use (C3.1) and (C3.2), in place of (C2.1) and (C2.2) respectively. We do the same to prove that (C2.1) holds, except that we use (C3.1) and (C3.3) and we differentiate with respect to $y_2$ instead of $y_1$. Notice now that from (C1), (C2.1) and (C3.1) we also obtain (C4.2). To prove that (C2.2) holds, we proceed as follows: differentiate (C3.2) by $y_2$, (C3.3) by $y_1$ and (C4.2) by $x_k$. Then, by suitably combining the results, we obtain

$$\frac{\partial}{\partial y_3} \left( \frac{\partial \alpha_{1k}}{\partial y_2} - \frac{\partial \alpha_{2k}}{\partial y_1} - \frac{\partial \beta_{12}}{\partial x_k} \right) = 0,$$

which proves (C2.2). The proof that also (C4.1) holds is just as in the previous steps. It remains to show that $\alpha_{12} = \alpha_{21}$. In fact it follows from the following:

$$\frac{\partial}{\partial y_3} (\alpha_{12} - \alpha_{21}) = \frac{\partial \alpha_{12}}{\partial y_1} + \frac{\partial \beta_{13}}{\partial x_3} - \frac{\partial \alpha_{21}}{\partial y_2} - \frac{\partial \beta_{23}}{\partial x_3} = 0,$$

where the first equality follows from subtracting (C3.2), $k = 2$ and (C3.3), $k = 1$; the second from substituting (C2.2), $k = 3$ and using the imposed symmetry of the other coefficients; the last one is just (C4.1). The proof is now complete. 

**Remark 1.** To prove his more general version of this theorem, where the open set $U$ is replaced by any manifold $M$, Bryant \[1\] had to use the fact that every 3-dimensional manifold is parallelizable. His proof then extended to higher dimensions when $M$ is assumed to be parallelizable. To prove Bryant’s theorem from our local version, one would need to understand how to glue solutions obtained from the various coordinate charts. Accomplishing this might also provide a method to prove the result without using parallelizability.

Even though this proof only works locally on a coordinate chart of the given Riemannian manifold, it is global in the important case of the torus.

**Corollary 3.1** Let $M$ be the 3-torus with any real-analytic Riemannian metric $g$, then $(M, g)$ can be isometrically embedded as a special Lagrangian submanifold of a Calabi-Yau manifold $\overline{M}$. 

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Proof. We apply Theorem 3.1 to any standard complexification $U_C$ of $M$. We view $g$ as a triply periodic metric tensor in $\mathbb{R}^3$, then we make sure that every choice involved in the three steps of the theorem is made to be triply periodic in the real part. Solutions will also be triply periodic in the real part, hence they define a Calabi-Yau structure on $U_C$. Theorem 2.1 also ensures that in this way we can describe locally all isometric special Lagrangian embeddings of $M$ in some Calabi-Yau manifold $\overline{M}$.

Given a special Lagrangian torus $M$, one of the questions which arose after the work of McLean, is whether the family of nearby special Lagrangian tori, parametrized by the moduli space $\mathcal{M}$, actually foliates a neighborhood of $M$ (cfr. Introduction). This is true in dimension two because harmonic forms of 2-tori never vanish. In dimension three instead we can construct examples where this doesn’t happen:

**Corollary 3.2** For any $k \in \mathbb{N}$, there exist Calabi-Yau manifolds with a special Lagrangian 3-torus $M$ admitting a harmonic form with $2k$ zeroes, $k$ of which of index 1 and $k$ of index $-1$. Moreover there will be elements of the moduli space of nearby special Lagrangian tori, arbitrarily close to $M$, intersecting $M$ in at least $2k$ points.

**Proof.** In [3] Calabi constructed examples of metrics on the 3-torus which admit harmonic forms with $k$ zeroes of index 1 and $k$ of index $-1$. Let $g$ be one of these metrics and $\theta$ the corresponding harmonic form with zeroes. As constructed by Calabi, $g$ is not real-analytic, but we can approximate it (in the $C^\infty$ topology) with a real-analytic one $\tilde{g}$. The $\tilde{g}$-harmonic form $\tilde{\theta}$ cohomologous to $\theta$ will also approximate $\theta$ and, by the stability of zeroes of non-zero index, $\tilde{\theta}$ will have at least the same number of zeroes if the approximation is precise enough. To the pair $(\mathcal{M}, \tilde{g})$ we can then apply Corollary 3.1 to construct the Calabi-Yau neighborhood $\overline{M}$. This proves the first claim.

McLean [19] identified the moduli space of nearby special Lagrangian tori in $\overline{M}$ with a three dimensional submanifold $\mathcal{M}$ of $\Gamma(\nu(M))$, the space of sections of the normal bundle. In fact, given $V \in \mathcal{M}$, the nearby special Lagrangian torus associated with $V$ is just $M_V = \exp_M V$. Via the identification $V \mapsto (JV)^b$, $\mathcal{M}$ may also be interpreted as a submanifold of $\Omega^1(M)$. As McLean showed, its tangent space at the zero section is the vector space of harmonic 1-forms. Now let $\xi(t)$ be a curve in $\mathcal{M}$, viewed in $\Omega^1(M)$, such that $\xi(0) = 0$ and whose tangent vector at 0 is $\theta$, the harmonic form with zeroes. Then $\lim_{t \to 0} \xi(t)/t = \tilde{\theta}$ in some $C^{k,\alpha}$ topology. Again, by the stability of zeroes of non-zero degree, this implies that, for sufficiently small $t$, $\xi(t)$
will have at least the same number of zeroes as $\tilde{\theta}$. Now if $V(t)$ is the section in $\Gamma(\nu(M))$ corresponding to $\xi(t)$, the special Lagrangian submanifold $M_{V(t)}$ will obviously intersect $T$ precisely at the zeros of $\xi(t)$. This completes the proof. \qed

4 Families of special Lagrangian tori

In the first step of Theorem 3.1, in the process of finding a solution on $U^1_C$, we were free to extend arbitrarily almost the entire matrix $A$. This matrix represents the metric induced by the horizontal slices $U_t = \{y_1 = t, y_2 = y_3 = 0\}$. So let $A_t$ be a choice of this metric for every $t$. We can, for example, ask the following question: can we choose $A_t$ so that every slice $U_t$ will also be special Lagrangian? The following theorem explains when and how this can be done:

**Theorem 4.1** Suppose that $A_t$ is a real-analytic one-parameter family of metrics on $U$. Then a Calabi-Yau metric can be constructed on $U_C$ so that each horizontal slice $U_t$ is special Lagrangian with metric $A_t$ if and only if $\det(A_t)$ does not depend on $t$ and the one form $(\frac{\partial}{\partial x_1})^\flat$ is harmonic w.r.t $A_t$ for every $t$.

**Proof.** We use the same notation as in Theorem 3.1. In particular let the initial metric $g = A_0$. In the following, $x$ will stand short for $(x_1, \ldots, x_3)$ (so, for example, $(x, t, 0, 0)$ will mean $(x_1, \ldots, x_3, t, 0, 0)$, in real coordinates for $U_C$). Imposing the special Lagrangian condition on the horizontal slices corresponds to

\[
\begin{cases}
\text{Im } \Omega_{(x,t,0,0)}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_3}) = 0 \\
B_t = 0
\end{cases}
\]

for all $t$, where $B_t$ is the value of the matrix $B$ on $U_t$. A simple computation shows that the first one of these holds if and only if:

\[
\text{Im } \Gamma_g(x,t,0,0) = 0
\]

for all $t$. Now, since $\Gamma_g$ is holomorphic, from this and from the Cauchy-Riemann equations we deduce that:

\[
\frac{\partial \Gamma_g}{\partial x_1}(x,t,0,0) = \frac{\partial \Gamma_g}{\partial y_1}(x,t,0,0) = 0,
\]
which, by the definition of $\Gamma_g$, holds if and only if
\[
\frac{\partial \sqrt{g}}{\partial x_1}(x) = 0 \quad (5)
\]
for all $x \in U$. This is only a condition on the initial data. Both conditions in (3) are satisfied if and only if equations (D) and (C1) in the previous section become
\[
\begin{align*}
\det(A_t) &= \sqrt{g}(x) \text{ for all } t, \\
\frac{\partial \alpha_{1j}}{\partial x_i} - \frac{\partial \alpha_{1i}}{\partial x_j} &= 0 \text{ on } U_C^1.
\end{align*}
\]
(6)

It is easy to see that the first equation of (6) together with (5) corresponds to the closure of $\star (\frac{\partial}{\partial x_1})^9$ while the second one to the closure of $(\frac{\partial}{\partial x_1})^9$, so that $(\frac{\partial}{\partial x_1})^9$ has to be harmonic w.r.t. to $A_t$. The first equation of (6) gives also the independence of $\det(A_t)$ on $t$. It is also easy to see that these conditions are sufficient to proceed to the construction of the Calabi-Yau metric on $U_C$ just by following the second step of Theorem 3.1.

The set of families of metrics $A_t$ satisfying the conditions in the Theorem above is quite rich. In some sense this is a problem because, for example, one can construct families with metrics degenerating quite badly. On the other hand we can also easily construct families with behaviors which we expect to observe while approaching the singular fibers described by Gross in [8]. These are are expected to appear in special Lagrangian fibrations of compact Calabi-Yau manifolds (cfr. Gross [8]), but some of them have yet to be constructed.

A fairly simple class of such families is the following:
\[
A_t(x_1, x_2, x_3) = \begin{pmatrix}
0 & 0 \\
e^{u_t(x_1)} & Q_t(x_1, x_2, x_3)
\end{pmatrix},
\]
(7)
where $u_t$ is any real-analytic function (depending only on $x_1$) and $Q_t$ is a symmetric, positive definite $2 \times 2$ matrix with real-analytic entries such that
\[
\det(Q_t) = e^{-u_t(x_1)} q(x_2, x_3),
\]
where $q$ is real-analytic and depending only on $x_2$ and $x_3$. If the functions are chosen to be periodic of period 1 in all three variables, $A_t$ defines a family of metrics on a three torus, or, if only one or two are periodic then they are metrics on a cylinder. The following is the description, in terms of Theorem 3.1, of some already known examples of one-parameter families of special Lagrangian cylinders:
Example 1. Suppose that \( \sigma : \mathbb{R}^2 \rightarrow S^5 \) is a minimal Legendrian immersion. Then it is known that the cone \( C_\sigma \) over \( \sigma(\mathbb{R}^2) \) is special Lagrangian (cfr. Haskins [12], Joyce [15]). Also, Haskins and Joyce showed that if we consider the one parameter family of curves \( \gamma_t \) in \( \mathbb{C} \) defined by \( \gamma_t = \{ z \in \mathbb{C} | \text{Im} z^3 = t, \ arg z \in (0, \pi/3) \} \) then the one parameter family of manifolds defined by \( M_t = \gamma_t \cdot \sigma(\mathbb{R}^2) \) is smooth, special Lagrangian, asymptotic to the cone \( C_\sigma \) and degenerating to the cone as \( t \rightarrow 0 \). Now parametrize \( \gamma_t \) by \( \gamma_t(x_1) = (x_1 + it)^{1/3} \) and assume, w.l.o.g., that \( \sigma \) is conformal. We can thus parametrize each \( M_t \) by the map \( F_t : \mathbb{R}^3 \rightarrow \mathbb{C}^3 \) given by

\[
F_t(x_1, x_2, x_3) = \gamma_t(x_1) \cdot \sigma(x_2, x_3).
\]

It is now easy to see that the metric \( A_t \) on \( M_t \), w.r.t. this parametrization, is

\[
A_t = \begin{pmatrix}
|\gamma_t|^2 & 0 & 0 \\
0 & |\gamma_t|^2 f & 0 \\
0 & 0 & |\gamma_t|^2 f
\end{pmatrix},
\]

where \( \dot{\gamma}_t \) is the derivative w.r.t. \( x_1 \) and \( f ds^2 \) is the conformal metric of \( \sigma \) (thus \( f \) only depends on \( x_2 \) and \( x_3 \)). It is also easy to see that \( \det A_t = f^2/9 \), in fact \( \dot{\gamma}_t^2 = \frac{1}{3 \pi^3} (\gamma_t^3) = \frac{1}{3} \). So \( A_t \) is of the type \( \mathfrak{h} \). One can also check that

\[
\frac{dF_t}{dt} = i \cdot dF_t \left( \frac{\partial}{\partial x_1} \right),
\]

i.e. that the variational vector field corresponds to the harmonic form \( \left( \frac{\partial}{\partial x_1} \right)^b \), under the identification of the normal bundle with the cotangent bundle. Of course this is also the case of the families of Theorem [4.1]. As the map \( \sigma \) we could for example use the Legendrian, conformal, harmonic maps constructed by Haskins [12] and Joyce [17].

The following two examples show how flexible this construction is. In fact we choose the family of metrics \( A_t, \ t \in [0, t_1) \), so that the tori start behaving as we would expect if the family were approaching two of the singular fibres described by Gross:

Example 2. Choose \( Q_t \), in \( \mathfrak{h} \), of the following form:

\[
Q_t = \begin{pmatrix}
1 & 0 \\
0 & e^{-u_t}
\end{pmatrix},
\]
with \( u_t \) periodic in \( x_1 \) of period 1. If the following are satisfied:

\[
\lim_{t \to t_1} u_t(1/2) = +\infty, \\
\int_0^1 e^{u_t(s)/2} ds = 1 \text{ for all } t,
\]

then these metrics describe a family of tori where the 2-cycle \( \{x_1 = 1/2\} \) collapses to a circle, while the diameter stays bounded. We expect to observe a similar behavior near a fibre of type \((2,2)\) in \( \mathcal{S} \). \( \Box \)

**Example 3.** Now assume

\[
Q_t = \begin{pmatrix} e^{v_t(x_1,x_2)} & 0 \\
0 & e^{-(u_t+v_t)} \end{pmatrix}.
\]

If \( u_t \) is as in the previous example and \( v_t \) satisfies:

\[
\lim_{t \to t_1} v_t(x_1,1/2) = +\infty \text{ for all } x_1, \\
\int_0^1 e^{v_t(x_1,s)/2} ds = 1 \text{ for all } t \text{ and } x_1,
\]

then also the 2-cycle \( \{x_2 = 1/2\} \) will collapse to a circle. This is expected to happen while approaching a fibre of type \((2,1)\). \( \Box \)

No example of special Lagrangian fibration containing a fibre of type \((2,1)\) has been constructed yet. One approach to the problem of finding one could be to try to glue this example or similar ones onto a suitable version of the singular fibre. This though seems, at the moment, a harder problem. A related question is which of these families can actually be seen in compact Calabi-Yau’s. We suspect that imposing the curvature of the ambient manifold to be bounded already provides considerable restrictions on the types of degenerations occurring in these families. In fact in Example 3, if we take \( v_t \) to depend only on \( x_2 \), one can show that the curvature of the ambient manifold blows up. For more general choices we do not know if this still happens. We hope to investigate more on these matters in the future. In the following section we use similar examples to show another instance where 3-dimensional special Lagrangian geometry differs considerably from the 2-dimensional one.

## 5 Hitchin’s metric is not always Ricci-flat

Let \( \mathcal{M} \) be the local moduli space of the deformations of a special Lagrangian \( n \)-torus \( M_0 \) inside an \( n \)-dimensional Calabi-Yau manifold \((\overline{M}, \Omega, \omega)\). For
each \( q \in \mathcal{M} \) denote by \( M_q \) the special Lagrangian submanifold corresponding to \( q \). As Hitchin [13] showed, \( \mathcal{M} \) can be naturally identified with a neighborhood of 0 in \( H^1(M_0, \mathbb{R}) \). In the same paper he also proposed the construction of a Calabi-Yau structure on the so called D-brane moduli space, i.e. on the manifold

\[
\mathcal{X} = \mathcal{M} \times H^1(M_0, \mathbb{R}/\mathbb{Z}),
\]

which according to the SYZ recipe is also a local model for the Calabi-Yau manifold mirror of \( \overline{M} \). Notice that \( \mathcal{X} \) is an \( n \)-torus fibration over \( \mathcal{M} \). Hitchin successfully showed how to construct naturally an integrable complex structure, a compatible Kähler form \( \tilde{\omega} \) and a non-vanishing holomorphic \( n \)-form \( \tilde{\Omega} \) on \( \mathcal{X} \). This metric is called semi-flat, because it induces a flat metric on the fibres. The condition required for these forms to give a Calabi-Yau structure is that they are related by the equality

\[
\tilde{\omega}^n = c \tilde{\Omega} \wedge \overline{\tilde{\Omega}}
\]

for some constant \( c \). Hitchin proved that this relation holds for the proposed forms if and only if the special Lagrangian submanifolds \( M_q \) satisfy a certain condition. One way to state this condition is the following. Fix a basis \( \Sigma_1, \ldots, \Sigma_n \) for \( H_1(M_0, \mathbb{Z}) \). If \( \mathcal{M} \) is simply connected then \( H_1(M_q, \mathbb{R}) \) can be canonically identified with \( H_1(M_0, \mathbb{Z}) \). Now, for every \( q \in \mathcal{M} \), let \( \theta_1(q), \ldots, \theta_n(q) \) be the harmonic 1-forms on \( M_q \) satisfying

\[
\int_{\Sigma_i} \theta_j = \delta_{ij}.
\]

Denote by \( \langle \theta_i(q), \theta_j(q) \rangle_{L^2} \) the usual \( L^2 \) inner product on \( \Omega^1(M_q) \) induced by the metric on \( M_q \). The condition required then is that the function

\[
\Phi : \mathcal{M} \to \mathbb{R} \quad q \mapsto \det(\langle \theta_i(q), \theta_j(q) \rangle_{L^2})
\]

is constant on \( \mathcal{M} \).

The condition does in fact always hold in the case of special Lagrangian tori in \( K3 \) surfaces, see for example Hitchin [14]. This seemed to give some hope that the same was true in higher dimensions. Unfortunately it isn’t. In this section we show that this follows from Theorem [13], which allows us to construct many counterexamples. Had this condition been true, Hitchin’s construction would have provided the first example of canonical Calabi-Yau structure on the mirror manifold. In the final remark we will also show why our counterexamples fail in dimension 2, as they should. This will highlight what goes wrong. So we have:
Corollary 5.1 There are 1-parameter families of special Lagrangian tori along which the function $\Phi$ defined in (9) is not constant.

Proof. Let $A_t$ be a family of metrics on the standard 3-torus $M = \mathbb{R}^3/\mathbb{Z}^3$ of the following type:

$$A_t = \begin{pmatrix} g_{11}(x_1, t) & 0 & 0 \\ 0 & g_{22}(x_1, t) & 0 \\ 0 & 0 & g_{33}(x_1, t) \end{pmatrix},$$

with the only condition that $\det(A_t) = g_{11}g_{22}g_{33} = 1$. Theorem 4.1 and the comments that follow show that this family can be realized as a one parameter family of special Lagrangian submanifolds of some Calabi-Yau manifold. We now show that in general the function $\Phi$ is not constant along this family. Choose as basis $\Sigma_1, \Sigma_2, \Sigma_3$ for $H_1(M, \mathbb{Z})$ the standard one. A computation shows that the forms

$$\theta_1 = \frac{g_{11}}{\int_0^1 g_{11}dx_1} dx_1,$$

$$\theta_2 = dx_2,$$

$$\theta_3 = dx_3$$

are harmonic and they satisfy (8) for every $t$. Now, since the volume form is just $dx_1 \wedge dx_2 \wedge dx_3$ and the functions given depend only on $x_1$ and $t$, we have the following:

$$|\theta_1(t)|^2_{L^2} = \frac{1}{\int_0^1 g_{11}dx_1},$$

$$|\theta_2(t)|^2_{L^2} = \int_0^1 g^{22}dx_1,$$

$$|\theta_3(t)|^2_{L^2} = \int_0^1 g^{33}dx_1,$$

$$\langle \theta_i(t), \theta_j(t) \rangle_{L^2} = 0 \text{ when } i \neq j,$$

where we also used the fact that $g^{ii} = g_{ii}^{-1}$. Now, using also the condition on the determinant of $A_t$, this implies that

$$\Phi(t) = \det(\langle \theta_i(t), \theta_j(t) \rangle_{L^2}) = \frac{\int_0^1 g^{22}dx_1 \int_0^1 g^{33}dx_1}{\int_0^1 g^{22}g^{33}dx_1},$$

which in general, for arbitrary $g^{22}$ and $g^{33}$ depending also on $t$, is not constant in $t$. \qed
Remark 1. To convince ourselves that these examples show what goes wrong in dimension 3 and certainly higher, we now demonstrate why they are not counterexamples in dimension 2, as we expect from known theory. With slight modifications, one can prove that Theorem 4.1 also holds in dimension 2. Let $A_t$ be a family of metrics on the 2-torus $M = \mathbb{R}^2/\mathbb{Z}^2$ such that $\frac{\partial}{\partial x_1}$ is harmonic and $\det(A_t) = C(x_2)$ for every $t$. Then it can be realized as a one-parameter family of special Lagrangian tori in some 2-dimensional Calabi-Yau. We now show that $\Phi$ is constant along this family.

Let $\Sigma_1, \Sigma_2$ be the standard basis for $H_1(M, \mathbb{Z})$. Then, it can be verified that

$$\theta_1 = \frac{g_{11}(\int_0^1 \sqrt{C} dx_2) dx_1 + (g_{12} \int_0^1 \sqrt{C} dx_2 - \sqrt{C} \int_0^1 g_{12} dx_2) dx_2}{\int_0^1 \sqrt{C} dx_2 \int_0^1 g_{11} dx_1},$$

$$\theta_2 = \frac{\sqrt{C}}{\int_0^1 \sqrt{C} dx_2} dx_2$$

are the harmonic 1-forms satisfying (8). Notice that $\int_0^1 \sqrt{C} dx_2$ is just a constant and in fact it represents the volume of the tori. We can thus assume, w.l.o.g., $\int_0^1 \sqrt{C} dx_2 = 1$. Also we have that $g^{11} = g_{22}/C, g^{22} = g_{11}/C$ and $g_{12} = -g_{12}/C$. Using these facts we compute the point-wise inner product:

$$|\theta_1|^2 = \frac{g_{11}(g_{11} g_{22} + (g_{12} - \sqrt{C} \int_0^1 g_{12} dx_2)^2 - 2g_{12}(g_{12} - \sqrt{C} \int_0^1 g_{12} dx_2))}{C(\int_0^1 g_{11} dx_1)^2},$$

$$= \frac{g_{11}(1 + (\int_0^1 g_{12} dx_2)^2)}{(\int_0^1 g_{11} dx_1)^2},$$

$$\langle \theta_1, \theta_2 \rangle = -\frac{g_{11} \int_0^1 g_{12} dx_2}{\int_0^1 g_{11} dx_1},$$

$$|\theta_2|^2 = g_{11}.$$

Here, to obtain the first equality we have also substituted $g_{11} g_{22} - g_{12}^2 = C$.

Now, the fact that $(\frac{\partial}{\partial x_1})$ is closed implies that $\int_0^1 g_{11} dx_1$ and $\int_0^1 g_{12} dx_2$ are constant. Thus, integrating the above functions on $M$ yields:

$$|\theta_1|_{L^2}^2 = 1 + (\int_0^1 g_{12} dx_2)^2,$$

$$\langle \theta_1, \theta_2 \rangle_{L^2} = -\int_0^1 g_{12} dx_2,$$

$$|\theta_2|_{L^2}^2 = \int_0^1 g_{11} dx_1.$$
Hence we see that:
\[
\Phi(t) = \det(\langle \theta_i, \theta_j \rangle_{L^2}) = 1,
\]
as we expected. 

References

[1] R. L. Bryant. Calibrated embeddings in the special Lagrangian and coassociative cases. e-print: math.DG/9912246 v2, January 1999.

[2] R. L. Bryant. Some examples of special lagrangian tori. Adv. Theor. Math. Phys., 3:83–90, 1999.

[3] E. Calabi. An intrinsic characterization of harmonic one–forms. In Global Analysis (Papers in Honor of K. Kodaira), pages 101–117, Tokyo, 1969. University of Tokyo Press.

[4] E. Goldstein. Calibrated fibrations. e-print: math.DG/9911093, November 1999.

[5] E. Goldstein. Calibrated fibrations on complete manifolds via torus actions. e-print: math.DG/9911093, February 2000.

[6] E. Goldstein. Special lagrangian submanifolds and algebraic complexity one torus actions. e-print: math.DG/0003220, March 2000.

[7] M. Gross. Special lagrangian fibrations I: topology. In Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997), pages 156–193, River Edge, NJ, 1998. World Sci. Publishing.

[8] M. Gross. Special Lagrangian fibrations II: geometry. In Surveys in Differential Geometry, pages 341–403, Somerville:MA, 1999. International Press.

[9] M. Gross. Topological mirror symmetry. University of Warwick Preprint, e-print: math.AG/9909015, September 1999.

[10] M. Gross. Examples of special Lagrangian fibrations. Preprint, February 2000.

[11] R. Harvey and H. B. Lawson, Jr. Calibrated geometries. Acta Math., 148:47–157, 1982.
[12] M. Haskins. Special lagrangian cones. e-print: math.dg/0005164, May 2000.

[13] N. J. Hitchin. The moduli space of special Lagrangian submanifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 25:503–515, 1998. Dedicated to Ennio De Giorgi.

[14] N. J. Hitchin. The moduli space of complex Lagrangian submanifolds. Asian J. Math., 3:77–91, 1999.

[15] D. Joyce. On counting special lagrangian homology 3-spheres. e-print: math.hep-th/9907013, July 1999.

[16] D. Joyce. Constructing special Lagrangian m–folds in C^m by evolving quadrics. e-print: math.dg/0008155, August 2000.

[17] D. Joyce. Special Lagrangian m–folds in C^m with symmetries. e-print: math.dg/0008021, August 2000.

[18] M. Kobayashi. A special Lagrangian 3–torus as a real slice. In Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997), pages 315–319, River Edge, NJ, 1998. World Sci. Publishing.

[19] C. R. McLean. Deformations of calibrated submanifolds. Comm. Anal. Geom., 6:705–747, 1998.

[20] M. Spivak. A Comprehensive Introduction to Differential Geometry, volume 5. Publish or Perish, Boston, Mass., 1975.

[21] A. Strominger, S.–T. Yau, and E. Zaslow. Mirror symmetry is T-duality. Nucl. Phys., B479:243–259, 1996.

[22] H. Whitney and F. Bruhat. Quelques propriété fondamentales des ensembles analytiques–réels. Comment. Math. Helv., 33:132–160, 1959.