Energy-balance for the incompressible Euler equations with stochastic forcing

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Abstract

We establish energy-balance for weak solutions of the stochastically forced incompressible Euler equations, enjoying Hölder regularity $C^{\alpha}$, $\alpha > 1/3$. It is well known as the Onsager’s conjecture for the deterministic incompressible Euler equations, which describes the energy conservation of weak solutions having Hölder regularity $C^{\alpha}$, $\alpha > 1/3$. Additionally, we obtain energy-balance for the inhomogeneous incompressible Euler system driven by cylindrical Wiener process.

Keywords: Incompressible Euler equations, Onsager’s conjecture, Energy-balance, Stochastic forcing, cylindrical Wiener process, Itô’s formula, Commutator estimate, Besov space.

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1. Introduction

In this paper we prove Energy-balance of the following two incompressible Euler equations driven by cylindrical Wiener process -

(Eq.:1) Incompressible Euler equations with stochastic forcing:-
\begin{align*}
\frac{d}{dt}v + [\text{div}_x(v \otimes v) + \nabla_x p] dt &= G(v) dW_t, \\
\text{div}_x v &= 0. 
\end{align*}

(Eq.:2) Inhomogeneous incompressible Euler equations with stochastic forcing:-
\begin{equation}
\frac{d}{dt} \rho + \text{div}_x (\rho v) dt = 0,
\end{equation}
\[ d(\rho \mathbf{v}) + \text{div}_x (\rho \mathbf{v} \otimes \mathbf{v}) \, dt + \nabla_x p \, dt = \tilde{G}(\rho, \rho \mathbf{v}) \, dW_t, \]  \hspace{1cm} (1.4)
\[ \text{div}_x \mathbf{v} = 0. \]  \hspace{1cm} (1.5)

In above equations vector \( \mathbf{v}(t, x) \) denotes the velocity of the fluid particle which occupies the point \( x \) at time \( t \), \( \rho(t, x) \) denotes the hydrodynamic pressure and \( \rho(t, x) > 0 \) is the scalar density of a fluid. Both the equations are driven by a cylindrical Wiener process \( \{W_t\}_{t \geq 0} \) in a separable Hilbert space \( \mathcal{U} \), defined on some filtered probability space \( (\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathbb{P}) \) with a complete, right-continuous filtration. We work with periodic boundary conditions with period box \( \mathbb{T}^n \equiv [0, 1]^n \) for \( n = 2, 3 \). The diffusion coefficients \( \mathbb{G} \) and \( \tilde{\mathbb{G}} \) take values in \( L_2(\mathcal{U}; L^2(\mathbb{T}^n)) \) which is the space of Hilbert-Schmidt operators.

In case of deterministic incompressible Euler equation, the famous Onsager’s conjecture \cite{22} describes energy conservation \cite{10} for weak solutions which are Hölder regular \( C^\alpha \), for \( \alpha > 1/3 \). Another direction of Onsager’s conjecture i.e. when weak solutions are \( C^\alpha, \alpha < 1/3 \), then energy may not be conserved, was shown in \cite{8, 20}. Let us mention some other existing literature about Onsager’s conjecture in the deterministic setup - e.g. \cite{3, 12, 13, 15}. Onsager’s conjecture for general system of conservation laws are presented in \cite{1, 2}. Energy conservation for deterministic inhomogeneous incompressible Euler equations has been proved in \cite{14}. Existence and uniqueness results in various situations for stochastically forced incompressible Euler equations have been studied in \cite{4, 6, 7, 9, 18}.

The goal of this article is to investigate stochastic counterpart of energy-balance of the incompressible Euler equation, whose weak solutions are \( C^\alpha \) Hölder regular, for \( \alpha > 1/3 \). Last part of the paper is devoted to energy-balance for the inhomogeneous incompressible Euler equation driven by cylindrical Wiener process. We prove in more general setting by considering solutions in Besov space. Note that, due to the presence of a noise term in the systems, we obtain energy balance equations corresponding to both the systems \( \text{(Eq. 1)} \) and \( \text{(Eq. 2)} \).

The paper is organized as follows. We describe Besov space, cylindrical Wiener process, diffusion coefficients and required lemmas in section 2. Energy-balance for the stochastically forced incompressible Euler equation is presented in section 3 and at the end in section 4 we prove energy balance for the inhomogeneous incompressible Euler equation driven by cylindrical Wiener process.

2. Preliminaries

In this present article, we consider that weak solutions possess Besov regularity, defined as follows: let \( \alpha \in (0, 1), 1 \leq q < \infty \). Then for an open bounded set \( \Pi \subset \overline{\Pi} \subset \mathbb{T}^n \), Besov space \( B^{\alpha, \infty}_q(\Pi, \mathbb{R}^n) \) is a subset of \( L^q(\Pi, \mathbb{R}^n) \) such that \( f \in B^{\alpha, \infty}_q(\Pi, \mathbb{R}^n) \) if the following seminorm is finite,

\[ |f|_{B^{\alpha, \infty}_q(\Pi, \mathbb{R}^n)} := \sup_{0 \neq \zeta \in \mathbb{R}^n, \zeta + \Pi \subset \mathbb{T}^n} \frac{\|f(\cdot + \zeta) - f(\cdot)\|_{L^q(\Pi, \mathbb{R}^n)}}{|\zeta|^\alpha} < \infty. \]

We note that \( C^\alpha(\Pi) \subset B^{\alpha, \infty}_q(\Pi) \) for \( 1 \leq q < \infty \).

The cylindrical Wiener process \( \{W_t\}_{t \geq 0} \) considered here are defined over a separable Hilbert space \( \mathcal{U} \) with respect to a filtered probability space \( (\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathbb{P}) \) where \( \{\mathfrak{F}_t\}_{t \geq 0} \) is a complete and right-continuous filtration. \( W_t \) can be the formally written as

\[ W_t := \sum_{j=1}^{\infty} e_j W_t^j, \]  \hspace{1cm} (2.1)

where, \( \{e_j\}_{j \geq 1} \) is an orthonormal basis of \( \mathcal{U} \) and \( \{W_t^j\}_{j \geq 1} \) is a sequence of mutually independent \( \mathbb{R} \)-valued Brownian motions with respect to \( (\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathbb{P}) \). We describe the diffusion coefficients \( \mathbb{G} \) and \( \tilde{\mathbb{G}} \) respectively.
Let \( v \in L^2(\mathbb{T}^n) \), then we define the noise coefficient \( \mathcal{G}(v) : \mathcal{U} \rightarrow L^2(\mathbb{T}^n; \mathbb{R}^n) \) by
\[
\mathcal{G}(v) e_j := \mathcal{G}_j(\cdot, v(\cdot)).
\]
The coefficients \( \mathcal{G}_j = \mathcal{G}_j(x, v) : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) are \( C^1 \)-functions that satisfy uniformly in \( x \in \mathbb{T}^n \)
\[
\mathcal{G}_j(\cdot, 0) = 0, \\
|\nabla_v \mathcal{G}_j| \leq g_j, \quad [\text{where sequence } \{g_j\} \subset (0, \infty)] \\
\sum_{j \geq 1} \tilde{g}_j^2 < \infty.
\]

If \( \mathcal{G} \) satisfies (2.2) and \( v \) are \( \{\tilde{\mathcal{F}}_t\} \)-progressively measurable \( L^2(\mathbb{T}^n) \)-valued process such that
\[
v \in L^2(\Omega \times [0, T]; L^2(\mathbb{T}^n; \mathbb{R}^n)),
\]
then the following integral is well-defined
\[
\int_0^t \mathcal{G}(v) \, dW_s = \sum_{j \geq 1} \int_0^t \mathcal{G}_j(\cdot, v) \, dW^j_s.
\]

Let \( \rho \in L^2(\mathbb{T}^n) \), \( \rho \geq 0 \) and \( m := (\rho v) \in L^2(\mathbb{T}^n) \), then we define the diffusion coefficient \( \mathcal{G}(\rho, m) : \mathcal{U} \rightarrow L^2(\mathbb{T}^n; \mathbb{R}^n) \) by
\[
\mathcal{G}(\rho, m) e_j := \mathcal{G}_j(\cdot, \rho(\cdot), m(\cdot)),
\]
where coefficients \( \mathcal{G}_j = \mathcal{G}_j(x, \rho, m) : \mathbb{T}^n \times [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) are \( C^1 \)-functions and there exists a non-negative real numbers sequence \( \{\tilde{g}_j\} \) such that the following holds uniformly in \( x \in \mathbb{T}^n \),
\[
\mathcal{G}_j(\cdot, 0, 0) = 0, \quad |\partial_\rho \mathcal{G}_j| + |\nabla_m \mathcal{G}_j| \leq \tilde{g}_j \quad \text{and} \quad \sum_{j \geq 1} \tilde{g}_j^2 < \infty.
\]

Note that, if \( \mathcal{G} \) satisfies (2.3) and \( \rho, (\rho v) \) are \( \{\tilde{\mathcal{F}}_t\} \)-progressively measurable \( L^2(\mathbb{T}^n) \)-valued process such that
\[
\rho \in L^2(\Omega \times [0, T]; L^2(\mathbb{T}^n)), \quad (\rho v) \in L^2(\Omega \times [0, T]; L^2(\mathbb{T}^n; \mathbb{R}^n)),
\]
then the following is a well-defined \( \{\tilde{\mathcal{F}}_t\} \)-martingale in \( L^2(\mathbb{T}^n; \mathbb{R}^n) \),
\[
\int_0^t \mathcal{G}(\rho, \rho v) \, dW_s := \sum_{j \geq 1} \int_0^t \mathcal{G}_j(\cdot, \rho, \rho v) \, dW^j_s.
\]

The infinite sum in (2.1) does not converge in probabilistic sense as a random variable in \( \mathcal{U} \).
But, we can construct an auxiliary space \( \mathcal{U}_0 \supset \mathcal{U} \), where the sum converges. Define the space
\[
\mathcal{U}_0 := \left\{ u = \sum_{j \geq 1} u_j e_j; \sum_{j \geq 1} \frac{u_j^2}{j^2} < \infty \right\},
\]
with the following norm
\[
\|u\|_{\mathcal{U}_0}^2 := \sum_{j \geq 1} \frac{u_j^2}{j^2}.
\]
Note that the embedding \( \mathcal{U} \hookrightarrow \mathcal{U}_0 \) is Hilbert-Schmidt. The trajectories of \( \{W_t\} \subset C([0, T]; \mathcal{U}_0) \), \( \mathbb{P} \)-a.s. In this article, the separable Hilbert space is chosen to be \( \mathcal{U} = L^2(\mathbb{T}^n) \).

For detailed discussions about cylindrical Wiener process one can look [5, chapter 2.3], [16, chapter 2], [11], [23].

Throughout the article we assume \( \nu^\varepsilon := \nu * \eta_\epsilon \), where \( \{\eta_\epsilon\}_{\epsilon > 0} \) denotes the standard mollifiers sequence in space variable.
We state the following three lemmas, which we use to prove Energy-balance theorems.
Lemma 2.1 ([17]). Let \( q \geq 2 \). Let \( U_t \) be a process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) such that
\[
U_t \in L^\infty(0, T; L^2(\mathbb{T}^n)), \quad \text{with} \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|U_t\|_{L^2}^{2\lambda} \right] < \infty \text{ for } 1 \leq \lambda < \infty.
\]

Let \( V_t \) be progressively measurable with \( V_t \in L^2(\Omega; L^2(0, T; L_2(\mathcal{U}; L^{2\lambda}\mathbb{T}^n))) \) with
\[
\mathbb{E} \left( \sum_{k \geq 1} \int_0^T \|V_t(e_k)\|_{L^{2\lambda}\mathbb{T}^n}^2 dt \right)^\lambda < \infty \quad \text{for } 1 \leq \lambda < \infty.
\]

Then the following holds, up to a subsequence, \( \mathbb{P} \)-a.s. as \( \epsilon \to 0 \)
\[
\int_0^T \left( \int_{\mathbb{T}^n} U_t^\epsilon V_t^\epsilon dx \right) dW_t \to \int_0^T \left( \int_{\mathbb{T}^n} U_t V_t dx \right) dW_t.
\]

The lemma 2.1 can be proved by applying Burkholder-Davis-Gundy inequality [5, 16]. Here we omit the proof and it can be found in the appendix of [17].

Lemma 2.2. Let \( Z_t, Y_t \) be processes on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) such that
\[
Z_t \in L^\infty(0, T; L^p(\mathbb{T}^n, \mathbb{R}^N)), \quad \text{with} \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|Z_t\|_{L^p}^{2\lambda} \right] < \infty \text{ for } 1 \leq \lambda < \infty,
\]
\[
Y_t \in L^\infty(0, T; L^q(\mathbb{T}^n, \mathbb{R}^N)), \quad \text{with} \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|Y_t\|_{L^q}^{2\lambda} \right] < \infty \text{ for } 1 \leq \lambda < \infty,
\]
such that \( 1 \leq p, q \leq \infty \) satisfying \( p^{-1} + q^{-1} = 1 \). Then, for a subsequence, we get the following \( \mathbb{P} \)-a.s. as \( \epsilon \to 0 \)
\[
\int_0^T \int_{\mathbb{T}^n} Z_t^\epsilon \cdot Y_t dxdt \to \int_0^T \int_{\mathbb{T}^n} Z_t \cdot Y_t dxdt.
\]

The proof of Lemma 2.2 is standard and it follows from properties of \( L^p \) functions.

The following commutator estimate lemma is useful for proving energy balance equation.

Lemma 2.3 (Commutator estimate, [10, 14, 19]). Let \( K, M \in \mathbb{N} \) and \( \mathcal{O}, \mathcal{O}_1 \subset \mathbb{R}^d \) be two open sets which are bounded and \( \mathcal{O} \subset \mathcal{O}_1 \). Let \( f \in B_{3,\infty}^\alpha(\mathcal{O}, \mathbb{R}^M) \) and \( \phi \in B_{3,\infty}^\beta(\mathcal{O}_1, \mathbb{R}^M) \) for \( \alpha, \beta \in (0, 1) \). Let \( \mathcal{H} : \mathcal{U} \to \mathbb{R}^{K \times M} \) be a \( C^2 \) function where \( \mathcal{U} \) is a bounded open convex set such that it contains the closure of range of \( f \). Then we get
\[
\| (\mathcal{H}(f^\epsilon) - \mathcal{H}(f))^\epsilon \|_{L^1(\mathcal{O})} \leq C_0 \| f \|_{B_{3,\infty}^\alpha(\mathcal{O}_1)}^2 \| \phi \|_{B_{3,\infty}^\beta(\mathcal{O})}^{2\alpha + \beta - 1},
\]
where \( C_0 \) depends on \( \sup \{ \| \nabla^2 u \mathcal{H} \| ; u \in \mathcal{U} \} \) and domain \( \mathcal{O} \).

The proof of Lemma 2.3 is omitted here. We refer [14, 19] for a complete proof of it.

3. Energy-balance for the stochastically forced incompressible Euler equation

This section deals with energy-balance for the incompressible Euler equation driven by cylindrical Wiener process.

Definition 3.1 (pathwise weak solutions of incompressible Euler equations with stochastic forcing). Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be a complete filtered probability space with right-continuous filtration,
\( \{ W_t \}_{t \geq 0} \) be a cylindrical Wiener process with respect to filtration \( \{ \mathfrak{F}_t \}_{0 \leq t \leq T} \) and \( \mathbb{G} \) satisfy condition (2.2). We say that \((v, p, \tau)\) is a pathwise weak solution of (1.1)-(1.2), if the following holds:

(i) \( \tau > 0 \) is an a.s. strictly positive \( \{ \mathfrak{F}_t \} \)-stopping time.

(ii) For each \( \varphi \in C_c^\infty (\mathbb{T}^n, \mathbb{R}^n) \), \( t \mapsto \int_{\mathbb{T}^n} v \cdot \varphi \, dx \in C([0, \tau)) \) and the process \( t \mapsto \int_{\mathbb{T}^n} v \cdot \varphi \, dx \), is progressively measurable with respect to filtration \( \{ \mathfrak{F}_t \} \), such that \( \mathbb{P} \)-a.s. \( v \in C \left([0, \tau); L^2(\mathbb{T}^n, \mathbb{R}^n)\right) \cap L^\infty ([0, \tau) \times \mathbb{T}^n; \mathbb{R}^n) \).

(iii) For each \( \varphi \in C_c^\infty (\mathbb{T}^n) \), \( t \mapsto \int_{\mathbb{T}^n} \phi \varphi \, dx \) is progressively measurable with respect to filtration \( \{ \mathfrak{F}_t \} \), such that \( \mathbb{P} \)-a.s. \( p \in L^2 ([0, \tau) \times \mathbb{T}^n, \mathbb{R}) \).

(iv) The following equation holds \( \mathbb{P} \)-a.s., for \( 0 \leq t < \tau \)

\[
\int_{\mathbb{T}^n} v(t, \cdot) \cdot \nabla_x \varphi \, dx = 0.
\]

(v) For all \( \varphi \in C_c^\infty (\mathbb{T}^n, \mathbb{R}^n) \) the following equation holds \( \mathbb{P} \)-a.s., for \( 0 \leq t < \tau \)

\[
\int_{\mathbb{T}^n} v(t, \cdot) \cdot \varphi \, dx = \int_{\mathbb{T}^n} v(0, \cdot) \cdot \varphi \, dx + \int_0^t \int_{\mathbb{T}^n} [(v \otimes v) : \nabla_x \varphi + p(x) \nabla_x \varphi] \, dx \, ds + \int_0^t \left( \int_{\mathbb{T}^n} \mathbb{G}(v) \cdot \varphi \, dx \right) \, dW_s.
\]

**Theorem 3.2** (Energy-balance for the stochastically forced incompressible Euler equations). Let \((v, p, \tau)\) be a pathwise weak solution of (1.1)-(1.2), as in definition 3.1 and \( v \in L^3 ([0, \tau); B_3^{\alpha \infty} (\mathbb{T}^n, \mathbb{R}^n)) \).

Then, for \( \alpha > \frac{1}{3} \), the following Energy-balance equation holds, \( \mathbb{P} \)-a.s., for each \( t \in [0, \tau) \)

\[
\int_{\mathbb{T}^n} |v(t, x)|^2 \, dx = \int_{\mathbb{T}^n} |v(0, x)|^2 \, dx + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^n} |\mathbb{G}(v(s, x))(e_k)|^2 \, dx \, ds \tag{3.1}
\]

\[ + \int_0^t \int_{\mathbb{T}^n} 2 \, v(s, x) \cdot \mathbb{G}(v(s, x)) \, dx \, dW_s.
\]

**Proof of Theorem 3.2.** After mollifying (1.1)-(1.2) in space, we obtain

\[
dv^t + [\nabla_x (v^t \otimes v^t) + \nabla_x p^t] \, dt = \nabla_x [(v^t \otimes v^t) - (v \otimes v)^t] \, dt + \mathbb{G}(v)^t \, dW_t,
\]

\[
\nabla_x v^t = 0. \tag{3.3}
\]

Strong solution of (3.2), for \( 0 \leq t < \tau \), \( \mathbb{P} \)-a.s.

\[
v^t(t, \cdot) = v^t(0, \cdot) + \int_0^t \left( - [\nabla_x (v^t \otimes v^t) + \nabla_x p^t] + \nabla_x [(v^t \otimes v^t) - (v \otimes v)^t] \right) \, ds + \int_0^t \mathbb{G}(v)^t \, dW_s.
\]

Writing Itô’s formula for \( r \mapsto \| r \|_{L^2(\mathbb{T}^n)}^2 \), [16, Theorem 2.10], for \( 0 \leq t < \tau \), \( \mathbb{P} \) a.s., we obtain

\[
\| v^t(t, x) \|_{L^2(\mathbb{T}^n)}^2 = \| v^t(0, x) \|_{L^2(\mathbb{T}^n)}^2 - \int_0^t \langle 2 \, v^t(s, x), [\nabla_x (v^t \otimes v^t) + \nabla_x p^t] (s, x) \rangle_{L^2(\mathbb{T}^n)} \, ds
\]

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+ \int_0^t \langle 2 \mathbf{v}^f(s, x), \text{div}_x[(\mathbf{v}^f \otimes \mathbf{v}^f) - (\mathbf{v} \otimes \mathbf{v})^f] (s, x) \rangle \, ds
+ \sum_{k \geq 1} \int_0^t \| \mathbb{G}(\mathbf{v}(s, x))^f(e_k) \|_{L^2(\mathbb{T}^d)}^2 \, ds + \int_0^t \langle 2 \mathbf{v}^f(s, x), \mathbb{G}(\mathbf{v}(s, x))^f \, dW_s \rangle_{L^2(\mathbb{T}^d)}.

Now applying stochastic Fubini [16, Theorem 2.8], we get
\begin{align*}
\int_{\mathbb{T}^d} |\mathbf{v}^f(t, x)|^2 \, dx &= \int_{\mathbb{T}^d} |\mathbf{v}^f(0, x)|^2 \, dx - \int_0^t \int_{\mathbb{T}^d} 2 \mathbf{v}^f(s, x) \cdot [\text{div}_x(\mathbf{v}^f \otimes \mathbf{v}^f) + \nabla_x p^f] (s, x) \, dx \, ds \\
&+ \int_0^t \int_{\mathbb{T}^d} 2 \mathbf{v}^f(s, x) \cdot \text{div}_x [(\mathbf{v}^f \otimes \mathbf{v}^f) - (\mathbf{v} \otimes \mathbf{v})^f] (s, x) \, dx \, ds \\
&+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^d} |\mathbb{G}(\mathbf{v}(s, x))^f(e_k)|^2 \, dx \, ds + \int_0^t \int_{\mathbb{T}^d} 2 \mathbf{v}^f(s, x) \cdot \mathbb{G}(\mathbf{v}(s, x))^f \, dx \, dW_s.
\end{align*}

As $\epsilon \to 0$, by lemma 2.2 we have $\mathbb{P}$-a.s.,
\begin{align*}
\int_{\mathbb{T}^d} |\mathbf{v}^f(t, x)|^2 \, dx &\to \int_{\mathbb{T}^d} |\mathbf{v}(t, x)|^2 \, dx, \\
\int_{\mathbb{T}^d} |\mathbf{v}^f(0, x)|^2 \, dx &\to \int_{\mathbb{T}^d} |\mathbf{v}(0, x)|^2 \, dx.
\end{align*}

Then, consider the integrand of the 2nd term of r.h.s. of (3.4) and apply integration by parts together with (3.3), to obtain
\begin{align*}
\int_{\mathbb{T}^d} 2 \mathbf{v}^f(s, x) \cdot [\text{div}_x(\mathbf{v}^f \otimes \mathbf{v}^f) + \nabla_x p^f] (s, x) \, dx \\
= - \int_{\mathbb{T}^d} 2 \nabla_x \mathbf{v}^f(s, x) : (\mathbf{v}^f \otimes \mathbf{v}^f)(s, x) \, dx - \int_{\mathbb{T}^d} 2 \text{div}_x \mathbf{v}^f(s, x) p^f(s, x) \, dx \\
= - \int_{\mathbb{T}^d} 2 \mathbf{v}^f(s, x) \cdot \nabla_x \frac{1}{2} |\mathbf{v}^f(s, x)|^2 \, dx - 0 \\
= \int_{\mathbb{T}^d} \text{div}_x \mathbf{v}^f(s, x) |\mathbf{v}^f(s, x)|^2 \, dx \\
= 0.
\end{align*}

By Commutator estimate lemma 2.3, as $\epsilon \to 0$, 3rd term of r.h.s. of (3.4)
\begin{equation}
\int_0^t \int_{\mathbb{T}^d} 2 \mathbf{v}^f(s, x) \cdot \text{div}_x [(\mathbf{v}^f \otimes \mathbf{v}^f) - (\mathbf{v} \otimes \mathbf{v})^f] (s, x) \, dx \, ds \to 0, \mathbb{P} - \text{a.s.}
\end{equation}

Again, by lemma 2.2, as $\epsilon \to 0$, 4th term of the r.h.s. of (3.4)
\begin{equation}
\sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^d} |\mathbb{G}(\mathbf{v}(s, x))^f(e_k)|^2 \, dx \, ds \to \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^d} |\mathbb{G}(\mathbf{v}(s, x))(e_k)|^2 \, dx \, ds, \mathbb{P} - \text{a.s.}
\end{equation}

At the end, as $\epsilon \to 0$, by lemma 2.1, 5th term of r.h.s. of (3.4)
\begin{equation}
\int_0^t \int_{\mathbb{T}^d} 2 \mathbf{v}^f(s, x) \cdot \mathbb{G}(\mathbf{v}(s, x))^f \, dx \, dW_s \to \int_0^t \int_{\mathbb{T}^d} 2 \mathbf{v}(s, x) \cdot \mathbb{G}(\mathbf{v}(s, x)) \, dx \, dW_s,
\end{equation}

converges in mean square, hence in probability. Therefore, combining (3.5)-(3.9) in (3.4), as $\epsilon \to 0$, for a subsequence, we obtain energy-balance equation (3.1), in $\mathbb{P}$-a.s. sense, for each $t \in [0, \tau)$. This concludes the proof. \hfill \square
4. Energy-balance for the stochastically forced inhomogeneous incompressible Euler equation

In this section, we prove energy-balance equation for the inhomogeneous incompressible Euler equation driven by the cylindrical Wiener process.

**Definition 4.1** (pathwise weak solutions of inhomogeneous incompressible Euler equations with stochastic forcing). Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\) be a complete filtered probability space with right-continuous filtration, \(\{W_t\}_{t \geq 0}\) be a cylindrical Wiener process with respect to the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) and \(\mathbb{G}\) satisfy condition (2.3). \((\rho, \mathbf{v}, p, \tau)\) is called a pathwise weak solution of (1.3)-(1.5), if the following holds:

(i) \(\tau > 0\) is an a.s. strictly positive \(\{\mathcal{F}_t\}\)-stopping time.

(ii) The density \(\rho\) is progressively measurable with respect to the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\). There exists \(\bar{r} > 0\) such that the following holds for \(\mathbb{P}\)-a.s.

\[
\rho \geq \bar{r} \quad \text{and} \quad \rho \in C \left(\{0, \tau\}; L^2(\mathbb{T}^n)\right) \cap L^\infty \left(\{0, \tau\} \times \mathbb{T}^n\right).
\]

(iii) For each \(\Phi \in C^\infty_c(\mathbb{T}^n, \mathbb{R}^n)\), \(t \mapsto \int_{\mathbb{T}^n} (\rho \mathbf{v}) \cdot \Phi \, dx \in C([0, \tau])\) and the process \(t \mapsto \int_{\mathbb{T}^n} (\rho \mathbf{v}) \cdot \Phi \, dx\), is progressively measurable with respect to the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\), such that for \(0 \leq t < \tau\), \(\mathbb{P}\)-a.s.

\[
(\rho \mathbf{v}) \in C \left(\{0, \tau\}; L^2(\mathbb{T}^n; \mathbb{R}^n)\right) \cap L^\infty \left(\{0, \tau\} \times \mathbb{T}^n; \mathbb{R}^n\right).
\]

(iv) The pressure \(p\) is progressively measurable with respect to the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) and following holds \(\mathbb{P}\)-a.s.

\[
p \in C \left(\{0, \tau\}; L^2(\mathbb{T}^n)\right) \cap L^\infty \left(\{0, \tau\} \times \mathbb{T}^n\right).
\]

(v) For all \(\phi \in C^\infty_c(\mathbb{T}^n, \mathbb{R}^n)\), the map \(t \mapsto \int_{\mathbb{T}^n} \rho \phi \, dx\), is progressively measurable with respect to the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\), such that the following equation holds \(\mathbb{P}\)-a.s., for \(0 \leq t < \tau\),

\[
\int_{\mathbb{T}^n} \rho(t, \cdot) \phi \, dx = \int_{\mathbb{T}^n} \rho(0, \cdot) \phi \, dx + \int_0^t \int_{\mathbb{T}^n} \rho \mathbf{v}(s, \cdot) \cdot \nabla_x \phi \, dx \, ds.
\]

(vi) The following equation holds \(\mathbb{P}\)-a.s., \(\forall \Phi \in C^\infty_c(\mathbb{T}^n, \mathbb{R}^n)\) for \(0 \leq t < \tau\),

\[
\int_{\mathbb{T}^n} \rho \mathbf{v}(t, \cdot) \cdot \Phi \, dx = \int_{\mathbb{T}^n} \rho \mathbf{v}(0, \cdot) \cdot \Phi \, dx
\]

\[
+ \int_0^t \int_{\mathbb{T}^n} [(\rho \mathbf{v} \otimes \mathbf{v}) : \nabla_x \Phi + p \, \text{div}_x \Phi] \, dx \, ds + \int_0^t \left(\int_{\mathbb{T}^n} \mathbb{G}(\rho, \rho \mathbf{v}) \cdot \Phi \, dx\right) \, dW_s.
\]

(vii) The following equation holds \(\mathbb{P}\)-a.s., \(\forall \psi \in C^\infty_c(\mathbb{T}^n)\) for \(0 \leq t < \tau\),

\[
\int_{\mathbb{T}^n} \mathbf{v}(t, \cdot) \cdot \nabla_x \psi \, dx = 0.
\]

**Theorem 4.2** (Energy-balance for the stochastically forced inhomogeneous incompressible Euler equation). Let \((\rho, \mathbf{v}, p, \tau)\) be a pathwise weak solution of (1.3)-(1.5), as in definition 4.1 and \((\rho, \mathbf{v}, p) \in L^3 ([0, \tau); B^3_{2, \infty}(\mathbb{T}^n, (0, \infty) \times \mathbb{R}^n \times \mathbb{R}))\).
Then, for $\alpha > \frac{1}{3}$, the following Energy-balance equation holds, $\mathbb{P}$- a.s., for all $\theta \in C^\infty_c(0, \infty)$ and $\Psi \in C^\infty_c(\mathbb{T}^n)$, for each $t \in [0, \tau)$

\[
\int_0^t \int_{\mathbb{T}^n} \partial_t \theta \frac{1}{2} \rho |v|^2 \Psi \, dz \, ds + \int_0^t \int_{\mathbb{T}^n} \left( v \left( \frac{1}{2} \rho |v|^2 + p \right) \right) \cdot \nabla \Psi \, dz \, ds
\]

\[
+ \int_0^t \int_{\mathbb{T}^n} \sum_{k \geq 1} \frac{1}{2\rho} |\tilde{G}(\rho, \rho v)(e_k)|^2 \Psi \, dz \, ds + \int_0^t \int_{\mathbb{T}^n} v \cdot \tilde{G}(\rho, \rho v) \Psi \, dz \, dW_s = 0.
\]

\[\text{(4.1)}\]

**Proof of Theorem 4.2.** Mollifying the system of equations (1.3)-(1.5), we obtain

\[
d\rho^\epsilon + \text{div}_x (\rho v)^\epsilon \, dt = 0, \quad (4.2)
\]

\[
d(\rho v)^\epsilon + \text{div}_x (\rho v \otimes v)^\epsilon \, dt + \nabla_x p^\epsilon \, dt = \tilde{G}(\rho, \rho v)^\epsilon \, dW_t, \quad (4.3)
\]

\[
\text{div}_x v^\epsilon = 0. \quad (4.4)
\]

We observe that $\rho^\epsilon \geq \bar{\rho}$ for $\epsilon > 0$. Let $m := \rho v$. Note that, now onwards in the proof we work with $(\rho, m)$ variable instead of $(\rho, v)$, because (4.3) is driven by Wiener process, hence we avoid time mollification and work with only space mollified version of the system. Now, from (4.2)

\[
d\rho^\epsilon = - \text{div}_x m^\epsilon \, dt.
\]

Applying Itô’s formula, for the function $r \mapsto \frac{1}{2r}$,

\[
d \left( \frac{1}{2\rho^\epsilon} \right) = \frac{1}{2(\rho^\epsilon)^2} \text{div}_x m^\epsilon \, dt. \quad (4.5)
\]

Again, from (4.3)

\[
d m^\epsilon
\]

\[
= - \text{div}_x \left( \frac{m \otimes m}{\rho} \right)^\epsilon \, dt - \nabla_x p^\epsilon \, dt + \tilde{G}(\rho, m)^\epsilon \, dW_t
\]

\[
= \left[ - \text{div}_x \left( \frac{m^\epsilon \otimes m^\epsilon}{\rho^\epsilon} \right) - \nabla_x p^\epsilon + \mathcal{R}^\epsilon \right] \, dt + \tilde{G}(\rho, m)^\epsilon \, dW_t,
\]

where $\mathcal{R}^\epsilon$ is defined as,

\[
\mathcal{R}^\epsilon := \text{div}_x \left( \frac{m^\epsilon \otimes m^\epsilon}{\rho^\epsilon} \right) - \text{div}_x \left( \frac{m \otimes m}{\rho} \right)^\epsilon.
\]

Applying Itô’s formula [21], for the function $r \mapsto |r|^2$,

\[
d \left( |m^\epsilon|^2 \right)
\]

\[
= \left( 2m^\epsilon \cdot \left[ - \text{div}_x \left( \frac{m^\epsilon \otimes m^\epsilon}{\rho^\epsilon} \right) - \nabla_x p^\epsilon + \mathcal{R}^\epsilon \right] + \sum_{k \geq 1} \left| \tilde{G}(\rho, m)^\epsilon (e_k) \right|^2 \right) \, dt
\]

\[
+ 2m^\epsilon \cdot \tilde{G}(\rho, m)^\epsilon \, dW_t. \quad (4.6)
\]
By Itô’s product rule [5, Proposition 2.4.2., chapter 2], between (4.5) and (4.6)
\[
\begin{align*}
&\quad \left( \left\langle \frac{\mathbf{m}^e}{\rho^e} \right\rangle \right)^2 \\
&= \left( \left\langle \frac{\mathbf{m}^e}{\rho^e} \cdot \left[ -\text{div}_x \left( \frac{\mathbf{m}^e \otimes \mathbf{m}^e}{\rho^e} \right) - \nabla_x p^e + \mathcal{R}^e \right] \right\rangle + \sum_{k \geq 1} \frac{1}{2\rho^e} \left\langle \mathcal{G}(\rho, \mathbf{m})' (\epsilon_k) \right\rangle^2 \right) dt \\
&\quad + \frac{\mathbf{m}^e}{\rho^e} \cdot \ddot{\mathcal{G}}(\rho, \mathbf{m})^e dW_t + \frac{\left\langle \mathbf{m}^e \right\rangle^2}{2(\rho^e)^2} \text{div}_x \mathbf{m}^e dt \\
&= - (\text{div}_x \mathbf{m}^e) \frac{\left\langle \mathbf{m}^e \right\rangle^2}{2(\rho^e)^2} dt - \mathbf{m}^e \cdot \nabla_x \left( \frac{\left\langle \mathbf{m}^e \right\rangle^2}{\rho^e} \right) dt - \frac{\mathbf{m}^e}{\rho^e} \cdot \nabla_x p^e dt + \mathcal{R}^e \cdot \frac{\mathbf{m}^e}{\rho^e} dt \\
&\quad + \sum_{k \geq 1} \frac{1}{2\rho^e} \left\langle \mathcal{G}(\rho, \mathbf{m})' (\epsilon_k) \right\rangle^2 dt + \frac{\mathbf{m}^e}{\rho^e} \cdot \ddot{\mathcal{G}}(\rho, \mathbf{m})^e dW_t \\
&= - (\text{div}_x \mathbf{m}^e) \frac{\left\langle \mathbf{m}^e \right\rangle^2}{2(\rho^e)^2} dt - \mathbf{m}^e \cdot \nabla_x p^e dt + \mathcal{R}^e \cdot \frac{\mathbf{m}^e}{\rho^e} dt \\
&\quad + \sum_{k \geq 1} \frac{1}{2\rho^e} \left\langle \mathcal{G}(\rho, \mathbf{m})' (\epsilon_k) \right\rangle^2 dt + \frac{\mathbf{m}^e}{\rho^e} \cdot \ddot{\mathcal{G}}(\rho, \mathbf{m})^e dW_t.
\end{align*}
\] (4.7)

Fix a \( \theta \in C^\infty_c(0, \infty) \) and a \( \Psi \in C^\infty_c(\mathbb{T}^n) \). Then integrating both sides of (4.7) over time and space, against \( \theta \Psi \) and applying stochastic Fubini [11, Theorem 4.33], we obtain
\[
\begin{align*}
&\quad - \int_0^t \partial_s \theta \int_{\mathbb{T}^n} \left( \frac{\left\langle \mathbf{m}^e \right\rangle^2}{2\rho^e} \right) \Psi dx ds \\
&= - \int_0^t \int_{\mathbb{T}^n} \text{div}_x \left[ \mathbf{m}^e \left( \frac{\left\langle \mathbf{m}^e \right\rangle^2}{2(\rho^e)^2} \right) \right] \Psi dx ds \\
&\quad - \int_{\mathbb{T}^n} \theta \int_0^t \frac{\mathbf{m}^e}{\rho^e} \cdot \nabla_x p^e \Psi dx ds + \int_{\mathbb{T}^n} \theta \int_0^t \mathcal{R}^e \cdot \frac{\mathbf{m}^e}{\rho^e} \Psi dx ds \\
&\quad + \int_{\mathbb{T}^n} \int_0^t \sum_{k \geq 1} \frac{1}{2\rho^e} \left\langle \mathcal{G}(\rho, \mathbf{m})' (\epsilon_k) \right\rangle^2 \Psi dx ds + \int_0^t \theta \int_{\mathbb{T}^n} \frac{\mathbf{m}^e}{\rho^e} \cdot \ddot{\mathcal{G}}(\rho, \mathbf{m})^e \Psi dx dW_s.
\end{align*}
\] (4.8)

Note that,
\[
\text{div}_x \left( \frac{\mathbf{m}^e}{\rho^e} p^e \right) = \frac{\mathbf{m}^e}{\rho^e} \cdot \nabla_x p^e + \text{div}_x \left( \frac{\mathbf{m}^e}{\rho^e} \right) p^e.
\] (4.9)

Therefore from (4.9), by applying integration by parts, 2nd term of r.h.s. of (4.8) becomes
\[
\begin{align*}
&\quad \int_{\mathbb{T}^n} \theta \int_0^t \frac{\mathbf{m}^e}{\rho^e} \cdot \nabla_x p^e \Psi dx ds \\
&= \int_{\mathbb{T}^n} \theta \int_0^t \text{div}_x \left( \frac{\mathbf{m}^e}{\rho^e} \right) \Psi dx ds - \int_0^t \theta \int_{\mathbb{T}^n} \text{div}_x \left( \frac{\mathbf{m}^e}{\rho^e} \right) p^e \Psi dx ds \\
&= - \int_0^t \theta \int_{\mathbb{T}^n} \left( \frac{\mathbf{m}^e}{\rho^e} \right) \cdot \nabla_x \Psi dx ds - \int_0^t \theta \int_{\mathbb{T}^n} \left[ \text{div}_x \left( \frac{\mathbf{m}^e}{\rho^e} \right) \right] \Psi dx ds.
\end{align*}
\] (4.10)
Hence, plugging (4.10) in (4.8), we get
\[
- \int_0^t \partial_t \theta \int_{\mathcal{T}^n} \left( \frac{m^2}{2 \rho^2} \right) \Psi \, dx \, ds \\
= \int_0^t \theta \int_{\mathcal{T}^n} \left[ m^\epsilon \left( \frac{m^2}{2 \rho^2} \right) \right] \cdot \nabla_x \Psi \, dx \, ds + \int_0^t \theta \int_{\mathcal{T}^n} \left( \frac{m^\epsilon \rho^2}{2} \right) \cdot \nabla_x \Psi \, dx \, ds \\
+ \int_0^t \theta \int_{\mathcal{T}^n} \left[ \text{div}_x \left( \frac{m^\epsilon}{\rho^2} \right) - \text{div}_x \left( \frac{m^\epsilon}{\rho} \right) \right] p^\epsilon \Psi \, dx \, ds + \int_0^t \theta \int_{\mathcal{T}^n} \mathcal{R}^\epsilon \cdot \frac{m^\epsilon}{\rho^2} \Psi \, dx \, ds \\
+ \int_0^t \theta \int_{\mathcal{T}^n} \frac{1}{2 \rho^2} |\mathcal{G}(\rho, m)^\epsilon(e_k)|^2 \Psi \, dx \, ds + \int_0^t \theta \int_{\mathcal{T}^n} \frac{m^\epsilon}{\rho^2} \cdot \mathcal{G}(\rho, m)^\epsilon \Psi \, dx \, dW_s 
\]
(4.11)

Now, as \( \epsilon \to 0 \), by lemma 2.2, l.h.s. of (4.11):
\[
- \int_0^t \partial_t \theta \int_{\mathcal{T}^n} \left( \frac{|m|^2}{2 \rho^2} \right) \Psi \, dx \, ds \to - \int_0^t \partial_t \theta \int_{\mathcal{T}^n} \left( \frac{|m|^2}{2 \rho} \right) \Psi \, dx \, ds, \ \mathbb{P} \text{-a.s.} \tag{4.12}
\]

Then, as \( \epsilon \to 0 \), by lemma 2.2, 1st term on the r.h.s. of (4.11):
\[
\int_0^t \theta \int_{\mathcal{T}^n} \left[ m^\epsilon \left( \frac{|m|^2}{2 \rho^2} + p^\epsilon \right) \right] \cdot \nabla_x \Psi \, dx \, ds \to \int_0^t \theta \int_{\mathcal{T}^n} \left[ \frac{m}{\rho} \left( \frac{|m|^2}{2 \rho} + p \right) \right] \cdot \nabla_x \Psi \, dx \, ds, \ \mathbb{P} \text{-a.s.} \tag{4.13}
\]

Next, as \( \epsilon \to 0 \), by Commutator estimate lemma 2.3, 2nd term on the r.h.s. of (4.11):
\[
\int_0^t \theta \int_{\mathcal{T}^n} \left[ \text{div}_x \left( \frac{m^\epsilon}{\rho^2} \right) - \text{div}_x \left( \frac{m^\epsilon}{\rho} \right) \right] p^\epsilon \Psi \, dx \, ds \to 0, \ \mathbb{P} \text{-a.s.} \tag{4.14}
\]

again, as \( \epsilon \to 0 \), by Commutator estimate lemma 2.3, 3rd term on the r.h.s. of (4.11):
\[
\int_0^t \theta \int_{\mathcal{T}^n} \mathcal{R}^\epsilon \cdot \frac{m^\epsilon}{\rho^2} \Psi \, dx \, ds \to 0, \ \mathbb{P} \text{-a.s.} \tag{4.15}
\]

Then, as \( \epsilon \to 0 \), by lemma 2.2, 4th term on the r.h.s. of (4.11):
\[
\int_0^t \theta \int_{\mathcal{T}^n} \frac{1}{2 \rho^2} |\mathcal{G}(\rho, m)^\epsilon(e_k)|^2 \Psi \, dx \, ds \to \int_0^t \theta \int_{\mathcal{T}^n} \frac{1}{2 \rho^2} |\mathcal{G}(\rho, m)(e_k)|^2 \Psi \, dx \, ds, \tag{4.16}
\]
\( \mathbb{P} \text{-a.s.} \]

At the end, as \( \epsilon \to 0 \), by lemma 2.1, 5th term on the r.h.s. of (4.11):
\[
\int_0^t \theta \int_{\mathcal{T}^n} \frac{m^\epsilon}{\rho^2} \cdot \mathcal{G}(\rho, m)^\epsilon \Psi \, dx \, dW_s \to \int_0^t \theta \int_{\mathcal{T}^n} \frac{m}{\rho} \cdot \mathcal{G}(\rho, m) \Psi \, dx \, dW_s, \tag{4.17}
\]

converges in mean square, hence in probability. Therefore, as \( \epsilon \to 0 \), combining (4.12) - (4.17) in (4.11) and replacing \( m \) with \( \rho \mathbf{v} \), for a subsequence, we obtain energy-balance equation (4.1), in \( \mathbb{P} \text{-a.s.} \) sense, for each \( t \in [0, \tau] \). This concludes the proof.
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