ON THE BIRATIONALITY OF COMPLETE INTERSECTIONS ASSOCIATED TO NEF-PARTITIONS

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Abstract. We prove that generic complete intersections associated to double mirror nef-partitions are all birational. This result answers a question asked by Batyrev and Nill in [6].

1. Introduction

Mirror symmetry was first discovered in string theory as a duality between families of 3-dimensional Calabi-Yau manifolds. It drew much attention of physicists and mathematicians for more than twenty years. Mirror symmetry predicts symmetric properties between mirror pairs. The celebrated homological mirror symmetry conjecture of Kontsevich [16] says: if $X, Y$ is a mirror pair, then the derived category on coherent sheaves of $X$ is equivalent to the Fukaya category of $Y$.

Batyrev [2] used $\Delta$-regular hypersurfaces in toric varieties associated to reflexive polytopes as a way of constructing a large family of mirror pairs. In this case, the mirror pair consists of the family of $\Delta$-regular hypersurfaces associated to a reflexive polytope and the family of $\Delta$-regular hypersurfaces associated to its dual polytope. Borisov [7] generalized this by considering nef-partitions of reflexive polytopes. A nef-partition of a reflexive polytope corresponds to a decomposition of the boundary divisor into nef divisors. In this case, the mirror pairs are constructed as the family of complete intersections associated to a nef-partition and the family of complete intersections associated to its dual nef-partition. These complete intersections are Calabi-Yau varieties, and their string-theoretic Hodge numbers behave as predicted by mirror symmetry [4].

Compared to hypersurfaces, complete intersections associated to nef-partitions are more complicated. In particular, they may exhibit non-trivial double mirror phenomenon, i.e. two families of Calabi-Yau varieties have the same mirror. If this is the case, the homological mirror symmetry conjecture implies that the derived category of coherent sheaves on the two complete intersections $X, \bar{X}$ are equivalent. Batyrev
and Nill [6] asked whether toric double mirrors are birational. We give an affirmative answer to this question in Theorem 4.6:

**Theorem** Let $X, \tilde{X}$ be toric double mirrors and $D$ be the determinantal variety, if they are all irreducible with $\dim D = \dim X$, then $X, \tilde{X}$ are birational.

The outline of the proof is as follows: first, we need to decompose the lattice according to nef-partitions. In the process, combinatorial properties of nef-partitions are used in an essential way, and due to the possible non-saturatedness, we have to work in some auxiliary lattices. Then, we construct a determinantal variety by the results of lattice decomposition. Finally, we show that the generic complete intersections are both birational to this determinantal variety, and hence birational to each other. In the proof, we use the fact that a generic complete intersection has a compactification which is a Calabi-Yau variety with canonical, Gorenstein singularities. This is based on $\Delta$-regularity. We leave its proof and somewhat lengthy discussions of the properties of $\Delta$-regularity to the appendix.

We describe briefly the content of each section:

In Section 2, we fix the notations used throughout the paper. We give necessary background on reflexive Gorenstein cones, nef-partitions, and their relations. In the end of this section, we prove Proposition 2.5 which generalizes the relation between nef-partitions and reflexive Gorenstein cones. This will be used to reformulate Batyrev and Nill’s original question in the language of Gorenstein cones. We also give a constructive proof of its converse in Proposition 2.6. In section 3, we reformulate the question of Batyrev and Nill using reflexive Gorenstein cones. We also discuss the motivation of this question and give an example which motivates our proof. In section 4, we give a proof for the main result Theorem 4.6. We also discuss the necessity of its assumptions. In section 5, we present some open questions related to the subject. In the appendix, we give the definition of $\Delta$-regularity and discuss its properties. We show that the singularities of $\Delta$-regular intersections are inherited from the ambient toric variety. In particular, the complete intersections considered in the paper are Calabi-Yau varieties with canonical, Gorenstein singularities. This fact is used in the proof of the main theorem.

**Acknowledgements.** I would like to thank my advisor Prof. Lev Borisov for his patient guidance and constant encouragement. Also, I would like to thank Howard Nuer for useful discussions related to the subject.
2. Background

2.1. Gorenstein cones and Nef-partitions. We fix the following notations throughout the paper. Let $M \cong \mathbb{Z}^d$ be a lattice of rank $d$, and $N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ be its dual lattice with pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. Let $M_\mathbb{R} := M \otimes \mathbb{Z} \mathbb{R}$, and $N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}$ be the $\mathbb{R}$-linear extensions. The pairing between $M, N$ can be extended to $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$.

Let $M = \mathbb{Z}^s \oplus M$ be the lattice extended from $M$, and $N = \mathbb{Z}^s \oplus N$ be its dual lattice with pairing:

$$\langle a_1, \cdots, a_s, m \rangle \times \langle b_1, \cdots, b_s, n \rangle \mapsto \sum_{i=1}^s a_i b_i + \langle m, n \rangle,$$

where the integer $s$ should be obvious from the context.

The purpose of introducing notations $M, N$ will become clear in a moment: if a nef-partition lives in $M$ (or $N$), then the corresponding reflexive Gorenstein cone will live in $M$ (or $N$). Sometimes we also use lattice $M_1$ and its dual lattice $N_1$. The convention is as follows: we always use $M$ (or $N$) to denote the lattice where polytopes live, if the cones come from nef-partitions, we use $M$ (or $N$) to denote the lattice where they live. However, when talking about general cones which do not come from nef-partitions, we use $M_1$ (or $N_1$) to denote the lattice where they live. We also introduce the auxiliary lattices $M', N'$ in Section 4.

Let $S \subset M_\mathbb{R}$ be a set, we use Conv$(S)$ to denote its convex hull.

If $\Delta \subset M_\mathbb{R}$ is a lattice polytope (i.e. the convex hull of finite lattice points), then $\Delta^\vee := \{ y \in N_\mathbb{R} \mid \langle x, y \rangle \geq -1, \forall x \in \Delta \}$ is its dual polytope. We use Vert$(\Delta)$ to denote the set of vertices of a lattice polytope $\Delta$, and $l(\Delta)$ to denote the set of its lattice points, i.e. $l(\Delta) = \Delta \cap M$.

**Definition 2.1.** Let $\Delta$ be a lattice polytope with the origin $0 \in \Delta$ as an interior point. If the dual polytope $\Delta^\vee$ is also a lattice polytope, then $\Delta$ is called reflexive polytope.

**Definition 2.2.** A $d$-dimensional rational polyhedral cone $K \subset (M_1)_\mathbb{R}$ is called a Gorenstein cone, if it is generated by lattice points which are contained in an affine hyperplane $\{ x \in (M_1)_\mathbb{R} \mid \langle x, n \rangle = 1 \}$ for some $n \in N_1$.

This $n$ is uniquely determined if dim $K = \text{rank } M_1$, and this is the only case considered in the paper. We denote this unique element by
deg^∨, and call it the degree element. By definition, deg^∨ must live in
K^∨ \cap N_1, where K^∨ := \{y \in (N_1)_{\mathbb{R}} \mid \langle x, y \rangle \geq 0, \forall x \in K\} is the dual cone of K.

In general, K is a Gorenstein cone does not imply K^∨ is a Gorenstein cone. However, if this is the case, we arrive at the notion of reflexive Gorenstein cone.

Definition 2.3. A Gorenstein cone K is called reflexive Gorenstein cone if K^∨ is also a Gorenstein cone. Let deg \in K, deg^∨ \in K^∨ be the degree elements in K, K^∨ respectively, then \langle deg, deg^∨ \rangle is called the index of this pair of dual reflexive Gorenstein cones.

We will see in a moment how reflexive Gorenstein cones relate to nef-partitions. Before doing this we should briefly recall the notion of nef-partition. In the projective toric variety defined by a reflexive polytope, a nef-partition is equivalent to a decomposition of the boundary divisor into a summation of nef divisors. On the other hand, there exists a purely combinatorial definition of nef-partition without invoking toric variety constructions. For simplicity, we use this combinatorial definition here. The readers can find its equivalent form and its motivation in Borisov’s original paper [7].

Definition 2.4. If the Minkowski sum of s lattice polytopes \sum_{i=1}^{s} \Delta_i is a reflexive polytope, and the origin 0 \in \Delta_i (0 may not be an interior point) for each i, then \{\Delta_i \mid i = 1, \ldots, s\} is called a length s nef-partition of the convex hull Conv(\bigcup_{i=1}^{s} \Delta_i).

Nef-partitions arise in pairs [7]: if we fixed a nef-partition \{\Delta_i \mid i = 1, \ldots, s\} with \Delta_i \subset M_\mathbb{R}, then there exists a dual nef-partition \{\nabla_i \mid i = 1, \ldots, s\} with \nabla_i \subset N_\mathbb{R}. The relations between them are

\[(\sum_{i=1}^{s} \Delta_i)^\vee = \text{Conv}(\bigcup_{i=1}^{s} \nabla_i)\]

\[(\sum_{i=1}^{s} \nabla_i)^\vee = \text{Conv}(\bigcup_{i=1}^{s} \Delta_i)\].

Furthermore, they satisfy the property

\[\min \langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij},\]

and \forall w_j \in \text{Vert}(\nabla_j) - \{0\}, the minimum value can be achieved, that is

\[\min_{x \in \Delta_i} \langle x, w_j \rangle = -\delta_{ij}.\]
2.2. Relationship between nef-partitions and reflexive Gorenstein cones. From a nef-partition, one can construct a reflexive Gorenstein cone [5]. On the other hand, from a reflexive Gorenstein cone associated to a nef-partition, if we have a decomposition of the degree element \( \deg^\vee \), we can construct another nef-partition. Now we will give a precise statement of the above relations, which appeared in a slightly different form in [6]. In fact, we will prove a general result.

Let \( K, K^\vee \) be full dimensional reflexive Gorenstein cones in \( (M_1)_\mathbb{R}, (N_1)_\mathbb{R} \), with degree elements \( \deg, \deg^\vee \) in \( K, K^\vee \) respectively. Suppose the index is \( \langle \deg, \deg^\vee \rangle = s \) and 

\[
\deg^\vee = \sum_{i=1}^{s} e_i,
\]

with \( e_i \in N_1 \cap K^\vee, e_i \neq 0 \).

Let 

\[
S = \{ x \in K \mid \langle x, \deg^\vee \rangle = 1 \}
\]

\[
S_i = \{ x \in K \mid \langle x, e_i \rangle = 1, \langle x, e_j \rangle = 0, j \neq i \}
\]

\[
T = \{ y \in K^\vee \mid \langle \deg, y \rangle = 1 \}.
\]

Because \( K \) is a Gorenstein cone, any vertex \( v \) of \( S \) is a lattice point. Thus \( \langle v, e_i \rangle \) are nonnegative integers which add up to 1. Hence, there exists precisely one \( e_i \) such that \( \langle v, e_i \rangle = 1 \). On the other hand, for any \( e_j \), because \( e_j \neq 0 \) and \( K \) is a full dimensional cone, there exists at least one vertex \( w \) of \( S \) such that \( \langle w, e_j \rangle = 1 \). Using these facts, one can show that \( \{ e_1, \ldots, e_s \} \) must be part of a \( \mathbb{Z} \)-basis of \( N_1 \).

Let 

\[
\text{Ann}(e_1, \ldots, e_s) := \{ m \in M_1 \mid \langle m, e_i \rangle = 0, \forall i, 1 \leq i \leq s \}
\]

be a sublattice of \( M_1 \) (we also use \( \text{Ann}(e) \) for simplicity if no confusion arises), and 

\[
\text{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\} := \sum_{i=1}^{s} \mathbb{Z}e_i
\]

be a sublattice of \( N_1 \). From the fact that \( \{ e_1, \ldots, e_s \} \) is part of a \( \mathbb{Z} \)-basis, it follows that the pairing between \( M \) and \( N \) induces a pairing 

\[
\text{Ann}(e_1, \ldots, e_s) \times (N_1 / \text{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\}) \to \mathbb{Z}.
\]

Under this pairing, \( \text{Ann}(e_1, \ldots, e_s) \) and \( N_1 / \text{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\} \) can be identified as dual lattices.
Proposition 2.5. Under the above notations, the lattice polytope
\[ \sum_{i=1}^{s} S_i - \deg \subset \operatorname{Ann}(e_1, \ldots, e_s)_\mathbb{R} \]

is a reflexive polytope.

Proof. We will show that the dual polytope of \( \sum_{i=1}^{s} S_i - \deg \) is exactly \( \bigcap_{i=1}^{s} S_i - \deg \subset (N_1 / \operatorname{Span}_\mathbb{Z}\{e_1, \ldots, e_s\})_\mathbb{R} \), where \( \bigcap_{i=1}^{s} S_i - \deg \) is the image of \( T \) under the projection \( (N_1)_\mathbb{R} \to (N_1 / \operatorname{Span}_\mathbb{Z}\{e_1, \ldots, e_s\})_\mathbb{R} \).

First, we show that \( \bigcap_{i=1}^{s} S_i - \deg \) has 0 as an interior point. Because \( \deg^\vee \) is in the interior of \( K^\vee \), \( \frac{1}{s} \deg^\vee \) is also in the interior of \( K^\vee \), and thus in the interior of \( T \). This property is kept under the projection map \( T \to \bigcap_{i=1}^{s} S_i - \deg \). The image of \( \frac{1}{s} \deg^\vee \) is 0 in \( (N_1 / \operatorname{Span}_\mathbb{Z}\{e_1, \ldots, e_s\})_\mathbb{R} \), and thus 0 is in the interior of \( \bigcap_{i=1}^{s} S_i - \deg \).

Second, we show \( 0 \in \bigcap_{i=1}^{s} S_i - \deg \) is an interior point. Let \( \deg = \sum_{i \in I} \lambda_i v_i \), where \( \lambda_i \in \mathbb{R} \) and \( v_i \in \operatorname{Vert}(S) \) be vertices of \( S \). We have
\[ 1 = \langle \deg, e_1 \rangle = \langle \sum_{i \in I} \lambda_i v_i, e_1 \rangle = \sum_{i \in I} \lambda_i, \]
where \( I_1 = \{ i \in I \mid \langle v_i, e_1 \rangle = 1 \} \). This implies \( \sum_{i \in I_1} \lambda_i v_i \in S_1 \). By continuing this procedure, one can show \( \deg \in \sum_{i=1}^{s} S_i \). If \( w \in \operatorname{Vert}(T) \) is a vertex of \( T \), then \( \langle \sum_{i=1}^{s} S_i, w \rangle \) cannot always be zero. Indeed otherwise, all the \( S_i \) would be contained in a facet of \( K \), which is impossible. Thus, for any \( w \in \operatorname{Vert}(T) \), there exists \( v \in \sum_{i=1}^{s} S_i \) such that \( \langle v - \deg, w \rangle \geq 0 \). If \( \sum_{i=1}^{s} S_i - \deg \) did not have 0 as an interior point, then \( \mathbb{R}_{\geq 0} (\sum_{i=1}^{s} S_i - \deg) \neq (M_1)_\mathbb{R} \). We have already showed that \( \bigcap_{i=1}^{s} S_i - \deg \) had 0 as an interior point, so \( \mathbb{R}_{\geq 0} \bigcap_{i=1}^{s} S_i - \deg \neq (M_1)_\mathbb{R} \). In particular, there exists a vertex \( \bar{w} \) of \( \bigcap_{i=1}^{s} S_i - \deg \), and thus a vertex \( w \in T \), such that \( \langle \sum_{i=1}^{s} S_i - \deg, w \rangle < 0 \), a contradiction.

Next, we show that \( \sum_{i=1}^{s} S_i - \deg \subset \operatorname{Ann}(e_1, \ldots, e_s)_\mathbb{R} \) is a reflexive polytope with dual \( \bigcap_{i=1}^{s} S_i - \deg \subset (N_1 / \operatorname{Span}_\mathbb{Z}\{e_1, \ldots, e_s\})_\mathbb{R} \). Because
\[ \min(\sum_{i=1}^{s} S_i - \deg, T) = \min(\sum_{i=1}^{s} S_i - \deg, T) \]
\[ = \min(\sum_{i=1}^{s} S_i, T) - \langle \deg, T \rangle \geq 0 - 1 = -1, \]
we have \( \bigcap_{i=1}^{s} S_i - \deg \subset (\sum_{i=1}^{s} S_i - \deg)^\vee \).

We only need to show the other inclusion \( \bigcap_{i=1}^{s} S_i - \deg \supset (\sum_{i=1}^{s} S_i - \deg)^\vee \). Let \( y \in (\sum_{i=1}^{s} S_i - \deg)^\vee \) such that there exists \( x \in \sum_{i=1}^{s} S_i - \deg \) with
\( \langle x, y \rangle = -1 \) (this \( y \) corresponding to some boundary point of the dual polytope of \( \sum_{i=1}^{s} S_i - \deg \)). We will show for this \( y, y \in \overline{T} \). Then it follows for arbitrary \( y \in (\sum_{i=1}^{s} S_i - \deg)^\vee, y \in \overline{T} \).

Let \( \theta_i = \min_{x \in S_i} \langle x, y \rangle \) and set \( y' = y - \sum_{i=1}^{s} \theta_i e_i \). We claim \( y' \in K^\vee \).

Indeed, \( K = \sum_{i=1}^{s} t_i S_i \) with \( t_i \geq 0 \), and we have

\[
\min \langle K, y' \rangle = \min \left( \sum_{i=1}^{s} t_i \langle S_i, y \rangle - \sum_{i=1}^{s} \langle S_i, \theta_i e_i \rangle \right) = \sum_{i=1}^{s} (t_i (\min \langle S_i, y \rangle - \theta_i)) \geq 0.
\]

Finally, we will show \( y' \in T \) and this will imply \( y \in \overline{T} \). By the assumption on \( y \), we have \( \min (\sum_{i=1}^{s} S_i - \deg, y) \geq -1 \), and there exists \( x \in \sum_{i=1}^{s} S_i - \deg, \) such that \( \langle x, y \rangle = -1 \). Let \( x = \sum_{i=1}^{s} x_i - \deg \) with \( x_i \in S_i \), then we must have \( \langle x_i, y \rangle = \theta_i \). Indeed, otherwise there exists \( k \) such that \( \langle x_k, y \rangle > \theta_k \), and all the others satisfy \( \langle x_i, y \rangle \geq \theta_i \).

Thus

\[
-1 = \min (\sum_{i=1}^{s} S_i - \deg, y) = \left( \sum_{i=1}^{s} \min \langle S_i, y \rangle \right) - \langle \deg, y \rangle
\]

and this implies \( y' \in T \).

The converse is proved in [6] Theorem 2.6. We will give a direct proof by constructing the dual cone \( K^\vee \) explicitly.

**Proposition 2.6.** Let \( \Delta_1, \ldots, \Delta_s \subset M_\mathbb{R} \) be lattice polytopes such that the Minkowski sum \( \sum_{i=1}^{s} \Delta_i \) has dimension \( \dim(M_\mathbb{R}) \) and \( \sum_{i=1}^{s} \Delta_i - m \) is a reflexive polytope for some \( m \in M \). Let \( \overline{M} = \mathbb{Z}^s \oplus M \), then the associated cone in \( \overline{M}_\mathbb{R} \)

\[
K = \{(a_1, \ldots, a_s; \sum_{i=1}^{s} a_i \Delta_i) \mid a_i \geq 0\}
\]

is a reflexive Gorenstein cone of index \( \langle \deg, \deg \rangle = s \).
if necessary, we have reflexive polytopes. Let \( v \) have used the 1 correspondence between vertices and facets in dual. Particularly, it is on some facet \( \Delta_i \), and because we have used the 1 – 1 correspondence between vertices and facets in dual reflexive polytopes. Let \( v = \sum_{i=1}^{s} v_i - m \), with \( v_i \) a vertex of \( \Delta_i \), and \( t = \sum_{j} \lambda_j t_j, \lambda_j \geq 0, \sum_j \lambda_j = 1 \), with \( t_j \) vertices of \( F_v \). Then because

\[
-1 + \langle m, t \rangle = \min \langle \sum_{i=1}^{s} \Delta_i, t \rangle = \sum_{i=1}^{s} \min \langle \Delta_i, t \rangle = \sum_{i=1}^{s} \langle v_i, t \rangle,
\]

we have

\[
- \min \langle \Delta_i, t \rangle = -\langle v_i, t \rangle = -\sum_j \lambda_j \langle v_i, t_j \rangle = -\sum_j \lambda_j \min \langle \Delta_i, t_j \rangle.
\]

The last equation uses the fact that \( t_j \in F_v \), and because

\[
-1 = \min \langle \sum_{i=1}^{s} \Delta_i - m, t_j \rangle = \left( \min \sum_{i=1}^{s} \langle \Delta_i, t_j \rangle \right) - \langle m - t_j \rangle
\]

\[
= \langle \sum_{i=1}^{s} v_i - m, t_j \rangle = \left( \sum_{i=1}^{s} \langle v_i, t_j \rangle \right) - \langle m, t_j \rangle,
\]
we must have $\langle \Delta_i, t_j \rangle = \langle v_i, t_j \rangle$. Putting everything together, we have

$$(-\min \langle \Delta_1, t \rangle, \ldots, -\min \langle \Delta_s, t \rangle, t) = \sum_j \lambda_i (-\min \langle \Delta_1, t_j \rangle, \ldots, -\min \langle \Delta_1, t_j \rangle; t_j).$$

This proves the claim $K^\vee \subseteq C$.

In order to show $K^\vee$ is also a Gorenstein cone, let $\deg = (1,1,\ldots,1; m)$. By using the property $\min(\sum s_{i=1}^s \Delta_i - m, v_j) = -1$, it is straightforward to show that for the vertex $v_j$ of $\nabla$,

$$\langle \deg, (m_{1j}, m_{2j}, \ldots, m_{rj}; v_j) \rangle = 1$$

and

$$\langle \deg, (0,\ldots,1,\ldots,0;0) \rangle = 1.$$

Thus we finish the argument $K$ is a reflexive Gorenstein cone. Because $\deg^\vee = (1,1,\ldots,1;0)$, the index is $\langle \deg, \deg^\vee \rangle = s$.

The above theorem can be applied to the case of nef-partitions, where $\sum s_{i=1}^s \Delta_i$ itself is a reflexive polytope with dual polytope $(\sum s_{i=1}^s \Delta_i)^\vee = \text{Conv}(\cup s_{i=1}^s \nabla_i)$. Because $0 \in \nabla_i$, and $\min(\Delta_i, \nabla_j) = -\delta_{ij}$, we can write the reflexive Gorenstein cones associated to this pair of nef-partitions in a symmetric way

$$K = \{(a_1, \ldots, a_s; \sum s_{i=1}^s a_i \Delta_i) \subset (\overline{M})_\mathbb{R} \mid a_i \geq 0\}$$

$$K^\vee = \{(b_1, \ldots, b_s; \sum s_{i=1}^s b_i \nabla_i) \subset (\overline{N})_\mathbb{R} \mid b_i \geq 0\}.$$ 

This result can also be proved directly as in [5].

Now we start off with a nef-partition $\{\Delta_i \mid 1 \leq i \leq s\}$, and let $K$ be the reflexive Gorenstein cone associated to it as above with the degree element $\deg^\vee \in K^\vee$. If $\deg^\vee = \sum s_{i=1}^s \tilde{e}_i$, with $\tilde{e}_i \neq 0, \tilde{e}_i \in K^\vee \cap \overline{N}$, then we define $\tilde{S}_i$ as in Proposition 2.5. In this case, $\left(\sum s_{i=1}^s \tilde{S}_i - \deg^\vee\right)$ is a reflexive polytope in $\text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)$. Without lost of generality, we can assume

$$\tilde{e}_i = (0,\ldots,1,\ldots,0; p_i) \in \mathbb{Z}^s \oplus N.$$

We claim that there exists a lattice isomorphism

$$\phi : \text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s) \to M$$
defined by restricting to the projection \( p : \mathbb{Z}^s \oplus M \to M \). In fact, if \( \phi(x) = 0 \), then \( x = (a_1, \ldots, a_s; 0) \), but \( x \in \text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s) \) implies that \( \forall \ i, a_i = 0 \), thus \( \phi \) is injective. The surjectivity comes from the fact that for \( m \in M \), if we let \( a_i = -\langle m, p_i \rangle \), then \( (a_1, \ldots, a_s; m) \in \text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s) \) maps to \( m \) under \( \phi \).

Under this isomorphism, we can identify \( \text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s) \) with \( M \). Let \( \tilde{\Delta}_i = p(\tilde{S}_i) \), one can show that

\[
\text{Conv}(\cup_{i=1}^s \tilde{\Delta}_i) = \text{Conv}(\cup_{i=1}^s \Delta_i).
\]

Moreover, since \( \phi(\text{deg}) = 0 \), and by Proposition 2.5, \( (\sum_{i=1}^s \tilde{S}_i - \text{deg}) \) is a reflexive polytope in \( \text{Ann}(e_1, \ldots, e_s)_{\mathbb{R}} \). Hence, \( (\sum_{i=1}^s \Delta_i) \) is a reflexive polytope in \( M \). Because \( (0, \ldots, 1, \ldots, 0; 0) \in \tilde{S}_i \), we have \( 0 \in \tilde{\Delta}_i \), and this implies \( \{\tilde{\Delta}_i \mid 1 \leq i \leq s\} \) is another nef-partition of \( \text{Conv}(\cup_{i=1}^s \Delta_i) \) (see Definition 2.1).

One cannot exhaust \textit{all} the nef-partitions of length \( s \) of \( \text{Conv}(\cup_{i=1}^s \Delta_i) \) using the above process (i.e. first construct reflexive Gorenstein cone \( K, K^\vee \), then decompose \( \text{deg}^\vee = \sum_{i=1}^s \tilde{e}_i \), and finally construct \( \tilde{\Delta}_i \)). However, the above process will give exactly the combinatorial data for toric double mirrors (details see Theorem 3.3).

Let \( X(\Sigma) \) be the toric variety defined by the fan \( \Sigma := \{0\} \cup \{\mathbb{R}_{\geq 0}\theta \mid \theta \subset \text{Conv}(\cup_{i=1}^s \Delta_i) \} \) is a face}, and

\[
\mathcal{L}_i = \sum_{\rho \in \text{Vert}(\Delta_i) \setminus \{0\}} D_{\rho}, \quad \tilde{\mathcal{L}}_i = \sum_{\rho \in \text{Vert}(\tilde{\Delta}_i) \setminus \{0\}} D_{\rho}
\]

be the nef divisors corresponding to \( \{\Delta_i\} \), \( \{\tilde{\Delta}_i\} \) respectively, where \( D_{\rho} \) is the torus invariant divisor associated to the primitive element \( \rho \). The following result gives a characterization of the nef-partition obtained from reflexive Gorenstein cones as above.

**Proposition 2.7.** The nef-partition \( \{\tilde{\Delta}_i \mid 1 \leq i \leq s\} \) of \( \text{Conv}(\cup_{i=1}^s \Delta_i) \) is obtained from the reflexive Gorenstein cone if and only if the corresponding divisors \( \{\tilde{\mathcal{L}}_i \mid 1 \leq i \leq s\} \) and \( \{\mathcal{L}_i \mid 1 \leq i \leq s\} \) are pairwise linearly equivalent.

**Proof.** Suppose \( \text{deg}^\vee = \sum_{i=1}^s \tilde{e}_i = \sum_{i=1}^s e_i \). Without lost of generality, we can assume \( \tilde{e}_i - e_i = p_i \in N \). Then one can check that \( \tilde{\mathcal{L}}_i - \mathcal{L}_i \) is exactly the principle divisor \( (X^{p_i}) \) on \( X(\Sigma) \).

On the other hand, suppose \( \tilde{\mathcal{L}}_i, \mathcal{L}_i \) are linearly equivalent divisors for each \( i \), then there exists \( p_i \in N \) such that \( \tilde{\mathcal{L}}_i - \mathcal{L}_i = (X^{p_i}) \), one can check that \( \tilde{e}_i = e_i + (0; p_i) \) satisfies the requirement.
We will prove the birationality for the $\Delta$-regular complete intersections associated to nef-partitions which are obtained from above.

3. The main question

3.1. The main question and its motivation. After establishing the relation between reflexive Gorenstein cones and nef-partitions, we are ready to state the question asked in [6] more explicitly.

Let us repeat the construction in the last part of Section 2 in order to extract the main ingredients. Let $\Delta \subset M$ be a reflexive polytope, $\text{Conv}(\bigcup_{i=1}^{s} \Delta_i) = \Delta$, and $\{\Delta_i \mid i = 1, \ldots, s\}$ be a nef-partition of $\Delta$. Let $\overline{M} = \mathbb{Z}^s \oplus M$, $\overline{N} = \mathbb{Z}^s \oplus N$, and $K \subset \overline{M}_\mathbb{R}$ be the reflexive Gorenstein cone associated to this nef-partition. The dual cone of $K$ is $K^\vee \subset \overline{N}_\mathbb{R}$ and $\text{deg}^\vee \in K^\vee$ is the degree element. Then $\text{deg}^\vee = \sum_{i=1}^{s} e_i$ with $e_i = (0, \ldots, 1, \ldots, 0; 0)$ gives back the original nef-partition $\{\Delta_i\}$. If there exists another decomposition $\text{deg}^\vee = \sum_{i=1}^{s} \tilde{e}_i$ with $\tilde{e}_i \neq 0, \tilde{e}_i \in K^\vee \cap \overline{N}$, then we can associate $\{\tilde{\Delta}_i\}$ which gives another nef-partition of $\Delta$.

Whenever one has a polytope, there is a family of Laurent polynomials associated to it. Let $l(\Delta_i)$ be the set of lattice points in $\Delta_i$, then the family of Laurent polynomials associated to $\Delta_i$ is

$$f_i = \sum_{v \in l(\Delta_i)} c_v X^v \in \mathbb{C}[M],$$

where $c_v$ is a complex coefficient only depends on the vertex $v$. Here we abuse notations, using $v$ to represent the lattice point as well as its coordinate in $M$. For example, if $v = (a_1, \ldots, a_n) \in M$, then $X^v = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{C}[M]$. In the same fashion, $\tilde{\Delta}_j$ produces a family of Laurent polynomials

$$\tilde{f}_j = \sum_{v \in l(\tilde{\Delta}_j)} c_v X^v \in \mathbb{C}[M].$$

Remark 3.1. We should emphasize that for the same vertex $v, v \neq 0$, the coefficient $c_v$ is the same in all Laurent polynomials. However, the coefficient of the origin, $c_0$ (i.e. the constant term) might be different in different Laurent polynomials. We abuse notations to avoid writing $c_{0,j}$ in place of $c_0$. 

We can take the zero locus of all $f_i$ in $(\mathbb{C}^*)^d = \text{Spec}(\mathbb{C}[M])$, and denote this variety as $X_{(\Delta_i)}$. To be precise $X_{(\Delta_i)} \subset (\mathbb{C}^*)^d$ is defined by:

$$X_{(\Delta_i)} : f_1 = f_2 = \cdots = f_s = 0,$$

and similarly, $X_{(\tilde{\Delta}_i)} \subset (\mathbb{C}^*)^d$ is defined by

$$X_{(\tilde{\Delta}_i)} : \tilde{f}_1 = \tilde{f}_2 = \cdots = \tilde{f}_s = 0.$$

**Remark 3.2.** From toric variety point of view, this construction can be stated as follows. Let $X := X(\Sigma(\nabla))$ be the projective toric variety associated to the polytope $\sum_{i=1}^s \Delta_i$, $T \subset X$ be the big torus. Let $\mathcal{L}_i$ be the line bundle associated to the dual nef-partition $\{\nabla_i \mid 1 \leq i \leq s\}$. Generic global sections in $H^0(X, \mathcal{L}_i)$ can be identified with Laurent polynomials with Newton polytopes $\Delta_i$. In particular, for $1 \leq i \leq s$, $f_i = \sum_{v \in l(\Delta_i)} c_v X^v \in H^0(X, \mathcal{L}_i)$. Let $(f_i)_0$ be the zero locus of $f_i$, then

$$X_{(\Delta_i)} = T \cap (f_1)_0 \cap \cdots \cap (f_s)_0.$$

We will return to this point of view in the appendix.

The following question was asked by Batyrev and Nill in [6] Question 5.2:

**(Nef-partition version)**

Are the mirror Calabi-Yau complete intersections $X_{(\Delta_i)}$ and $X_{(\tilde{\Delta}_i)}$ birational to each other?

We can reformulate this question in terms of reflexive Gorenstein cones.

Let $\tilde{S} = \{x \in K \mid \langle x, \deg^\vee \rangle = 1\}$, $\tilde{S}_i = \{v \in K \mid \langle v, \deg^\vee \rangle = \langle v, \tilde{e}_i \rangle = 1\}$. Because $\deg^\vee = \sum_{i=1}^s \tilde{e}_i$, for each lattice point $v$ in $\tilde{S}$, that is $v \in l(\tilde{S})$, there exists a unique $i$, such that $\langle v, \tilde{e}_i \rangle = 1$. We have a disjoint union $l(\tilde{S}) = \coprod_{i=1}^s l(\tilde{S}_i)$. One can define a Laurent polynomial in $\mathbb{C}[\tilde{\mathcal{M}}]$ by setting:

$$\tilde{g}_i = \sum_{v \in l(\tilde{S}_i)} c_v X^v.$$

For any lattice point $w_i$ such that $\langle w_i, \tilde{e}_i \rangle = 1, \langle w_i, \tilde{e}_j \rangle = 0, i \neq j$, $X^{-w_i} \cdot \tilde{g}_i$ is a Laurent polynomial in $\mathbb{C}[[\text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)]]$. We can similarly define an intersection $X_{(\tilde{e}_i)} \subset (\mathbb{C}^*)^d = \text{Spec}(\mathbb{C}[[\text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)]]$ by

$$X_{(\tilde{e}_i)} : X^{-w_1} \cdot \tilde{g}_1 = X^{-w_2} \cdot \tilde{g}_2 = \cdots = X^{-w_s} \cdot \tilde{g}_s = 0.$$

This intersection does not depend on the choice of $w_i$, because any other choice will differ by a factor $X^w, w \in \mathbb{C}[[\text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)]]$ and this will not affect the zero loci defined in $(\mathbb{C}^*)^d$. 
Similarly, we can construct \( S_i \) and \( g_i \) associated to the decomposition \( \deg^\vee = \sum_{i=1}^r e_i \), and an intersection \( X_{(e_i)} \subset (\mathbb{C}^*)^d = \text{Spec}(\mathbb{C}[\text{Ann}(e_1, \ldots, e_s)]) \) by
\[
X_{(e_i)} : X^{-w'_1} \cdot g_1 = X^{-w'_2} \cdot g_2 = \cdots = X^{-w'_s} \cdot g_s = 0.
\]
We can compare the equations defined by these intersections with the equations defined the intersections above by nef-partitions. Because the lattice isomorphism
\[
\phi : \text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_r) \to M
\]
sends \( \tilde{S}_i - \deg \) to \( \tilde{\Delta}_i \), we can identify \( \tilde{g}_i \in \mathbb{C}[\text{Ann}(\tilde{e}_1, \ldots, \tilde{e}_r)] \) with \( f_i \in \mathbb{C}[M] \) up to a factor \( X^{e_i}, v_i \in \mathbb{C}[M] \). Hence, \( X_{(\tilde{e}_i)} \) and \( X_{(\Delta_i)} \) are isomorphic varieties. The same thing is true for \( X_{(e_i)} \) and \( X_{(\Delta_i)} \) as well.

The importance of the above construction is explained in the following theorem.

**Theorem 3.3.** The complete intersections \( X_{(\tilde{e}_i)} \) and \( X_{(e_i)} \) are toric double mirror in the sense that they both mirror to the same family.

We abuse the notations: \( X_{(\tilde{e}_i)} \) here means the family parameterized by the coefficients \( c_v \), and the same for \( X_{(e_i)} \).

**Proof.** By the toric mirror construction in [2] [3], the mirror of \( X_{(\tilde{e}_i)} \) is a family of generic complete intersections defined by divisors \( \{ \tilde{L}_i \} | 1 \leq i \leq s \} \) in the toric variety \( X(\Sigma) \) (see the notations above Proposition 2.7). Likewise, the mirror of \( X_{(e_i)} \) is a family of generic complete intersections defined by divisors \( \{ L_i \} | 1 \leq i \leq s \} \) in \( X(\Sigma) \). By Proposition 2.7, \( \{ \tilde{L}_i \} | 1 \leq i \leq s \}\), \( \{ L_i \} | 1 \leq i \leq s \} \) consist of pairwise linearly equivalent divisors and hence they defined the same family of complete intersections which is the mirror of both \( X_{(\tilde{e}_i)} \) and \( X_{(e_i)} \).

\( \square \)

Viewing the original question from this perspective, we can ask:

**(Reflexive Gorenstein cone version)**

Are the toric double mirror \( X_{(e_i)}, X_{(\tilde{e}_i)} \) birational?

We give an affirmative answer to this question in Theorem 4.6 under some technical assumptions.

**3.2. Example.** In this section, we will illustrate the basic idea of the proof by an explicit example.
Let \( \{u_1, \ldots, u_{15}\} \) be a basis of \( \mathbb{Z}^{15} \), and we consider a sublattice \( M \subset \mathbb{Z}^{15} \) which is defined by

\[
M := \{ \sum_{i=1}^{15} l_i u_i \in \mathbb{Z}^{15} \mid \sum_{i=1}^{5} l_i = \sum_{i=6}^{10} l_i = \sum_{i=11}^{15} l_i \}. \]

The rank of \( M \) is 13, it contains a cone \( K = \mathbb{Z}_0^{15} \cap M \) which is defined by nonnegativity of all \( l_i \). The 125 generators of rays of \( K \) are given by \( u_i + u_j + u_k \) with \( 5j - 4 \leq i, j \leq 5j \), and let \( c_{ijk} \in \mathbb{C} \) denote coefficients. Suppose \( \{v_1, \ldots, v_{15}\} \) is the dual basis of \( \{u_1, \ldots, u_{15}\} \), then the dual lattice \( M^\vee \) is the quotient of \( \mathbb{Z}^{15} \):

\[
M^\vee = \mathbb{Z}^{15} / \text{Span}_\mathbb{Z} \{ \sum_{i=1}^{5} v_i - \sum_{i=6}^{10} v_i, \sum_{i=1}^{5} v_i - \sum_{i=11}^{15} v_i \}. \]

The dual cone \( K^\vee \) is the image of \( \mathbb{Z}_0^{15} \) in \( M^\vee \), and its rays are generated by \( v_i \), \( 1 \leq i \leq 15 \). The degree elements \( \deg^\vee \) are given by \( \sum_{i=1}^{15} u_i \) and \( \sum_{i=1}^{5} v_i \) respectively.

There are three different ways of decomposing \( \deg^\vee \) as a summation of lattice points in \( K^\vee \):

\[
\deg^\vee = \sum_{i=1}^{5} v_i, \quad \deg^\vee = \sum_{i=6}^{10} v_i, \quad \deg^\vee = \sum_{i=11}^{15} v_i. \]

This gives three different complete intersections in \( \mathbb{P}^4 \times \mathbb{P}^4 \).

For \( \deg^\vee = \sum_{i=1}^{5} v_i \), the equations of this decomposition can be expressed as

\[
\sum_{1 \leq j, k \leq 5} c_{1jk} x_1 y_j z_k = 0 \\
\sum_{1 \leq j, k \leq 5} c_{2jk} x_2 y_j z_k = 0 \\
\vdots \\
\sum_{1 \leq j, k \leq 5} c_{5jk} x_5 y_j z_k = 0.
\]

Here \([x_1, \ldots, x_5]\) are homogenous coordinates of \( \mathbb{P}^4 \), and similarly for \( y_j, z_k \).
As explained before, we can multiply each equation a factor in order to make it well defined in \(M \cap \text{Ann}(v_1, \ldots, v_5)\). Hence, let
\[
f_i(y, z) = x_i^{-1} \sum_{1 \leq j, k \leq 5} c_{ijk} \ x_i y_j z_k = \sum_{1 \leq j, k \leq 5} c_{ijk} \ y_j z_k = 0, \quad 1 \leq i \leq 5.
\]
This can be viewed as five bidegree \((1, 1)\) equations in \(\mathbb{P}^4 \times \mathbb{P}^4\). Similarly, for \(\deg^\vee = \sum_{i=6}^{10} v_i\) and \(\deg^\vee = \sum_{i=11}^{15} v_i\) we have defining equations:
\[
g_j(x, z) = \sum_{1 \leq i, k \leq 5} c_{ijk} \ x_i z_k = 0, \quad 1 \leq j \leq 5,
\]
\[
h_k(x, y) = \sum_{1 \leq i, j \leq 5} c_{ijk} \ x_i y_j = 0, \quad 1 \leq k \leq 5.
\]
Our question thus becomes whether these three complete intersections are birational for generic choice of \(c_{ijk}\).

Let \(X_1\) be the variety defined by \(f_i = 0, 1 \leq i \leq 5\). Let \(A_1(z)\) be \(5 \times 5\) matrix
\[
A_1(z) = \left( \sum_{k=1}^{5} c_{ijk} z_k \right)_{ij}, \quad 1 \leq i, j \leq 5,
\]
then \(f_i = 0, 1 \leq i \leq 5\) can be written as a matrix equation
\[
A_1(z) \begin{pmatrix} y_1 \\ \vdots \\ y_5 \end{pmatrix} = 0.
\]
Notice that \(([y_1, \ldots, y_5], [z_1, \ldots, z_5]) \in \mathbb{P}^4 \times \mathbb{P}^4\) satisfy \(f_i = 0, 1 \leq i \leq 5\) if and only if \(\det(A_1(z)) = 0\) in \(\mathbb{P}^4\). Let \(D_1\) denote the variety defined by \(\det(A_1(z)) = 0\). For generic coefficients, one can show \(X_1\) and \(D_1\) are birational.

Similarly, the variety \(X_2\) defined by \(g_j = 0, 1 \leq j \leq 5\) can be written as
\[
(x_1, \cdots, x_5) A_2(z) = 0
\]
where
\[
A_2(z) = \left( \sum_{k=1}^{5} c_{ijk} z_k \right)_{ij}, \quad 1 \leq i, j \leq 5.
\]
Let \(D_2\) be the variety defined by \(\det(A_2(z)) = 0\). The same argument as above shows that \(X_2\) is birational to \(D_2\). On the other hand, \(D_1\) and \(D_2\) are the same varieties, and hence \(X_1, X_2\) are birational. We notice that despite drastically different defining equations, the three complete intersections are all birational.
This example suggests us to look at the determinantal variety defined by a “common” matrix of different nef-partitions. However, it is not very clear how to construct this “common” matrix at present stage. Besides that, there are following more pressing issues: (1) the dimension of $\text{Span}_\mathbb{R}\{\tilde{e}_1 - e_1, \ldots, \tilde{e}_s - e_s\}$ might be smaller than $s - 1$ which leads to considering the intersection of several determinantal varieties; (2) “non-saturatedness” might occur, which forces us to work in auxiliary lattices; (3) in order to show the birationality, we have to take into account of the singularities of the complete intersection. This leads us to consider $\Delta$-regular intersections.

4. THE MAIN THEOREM

4.1. Results on the decomposition of lattices. Let $\Delta$ be a reflexive polytope, $\{\Delta_i \mid 1 \leq i \leq s\}$ be a nef-partition of $\Delta$, and $\{\nabla_i \mid 1 \leq i \leq s\}$ be its dual nef-partition. In the following, we assume $\dim \Delta = \dim M_\mathbb{R}$. Because

$$\Delta \subset \sum_{i=1}^{s} \Delta_i,$$

we have $\dim(\sum_{i=1}^{s} \Delta_i) = \dim M_\mathbb{R}$. We use $\text{Span}_\mathbb{R}\{p_1, \ldots, p_s\}$ to denote the vector space spanned by $p_i \in N_\mathbb{R}, 1 \leq i \leq s$. The following lemma is crucial for our argument.

**Lemma 4.1.** Let $p_i \in \nabla_i$, if $\sum_{i=1}^{s} p_i = 0$, and

$$\dim(\text{Span}_\mathbb{R}\{p_1, \ldots, p_s\}) = s - r,$$

then there exists disjoint sets $I_k, 1 \leq k \leq r$, such that $\bigcap_{k=1}^{r} I_k = \{1, \ldots, s\}$. Moreover, for each $k$, we have $\sum_{i \in I_k} p_i = 0$.

**Proof.** Suppose $l$ is the maximum number such that there exists $l$ nonempty disjoint sets $I_j, 1 \leq j \leq l$ satisfying

$$I_1 \prod \cdots \prod I_l = \{1, \ldots, s\},$$

and $\forall j, \sum_{i \in I_j} p_i = 0$.

Because these $l$ equations are linearly independent, we have

$$s - r = \dim(\text{Span}_\mathbb{R}\{p_1, \ldots, p_s\}) \leq s - l,$$

and hence $l \leq r$. All we need to show is $l = r$.

Otherwise, suppose $l < r$, then there must exist at least one equation

$$\sum_{1 \leq i \leq s} a_i p_i = 0,$$

which is not a linear combination of $\sum_{i \in I_j} p_i = 0$. Hence, there must exist an index $j$, such that for $i \in I_j, a_i$ are not identically the same. Suppose $a_m$ is a minimal element in $\{a_i \mid i \in I_j\}$. 

After reindexing the set, we can assume $j = 1$ and $m = 1$. Let $C$ be a sufficiently large number, then

$$0 = \sum_{1 \leq i \leq s} a_i p_i - a_1 \sum_{i \in I_1} p_i + C \cdot \sum_{i \in I_2 \cup \ldots \cup I_l} p_i = \sum_{2 \leq i \leq s} b_i p_i$$

satisfies $b_i > 0$ when $i \in I_2 \cup \ldots \cup I_l$, and $b_i \geq 0$ when $i \in I_1$. Moreover, there exists at least one element $t \in I_1$ such that $b_t > 0$ (because $a_i$ are not identically the same for $i \in I_1$). Let $S = \{i \mid b_i \neq 0\}$ be the index set corresponding to nonzero coefficients.

Set $P = \sum_{i \in S} p_i = \sum_{i \in S} (1 - cb_i) p_i$ with $c$ sufficiently big such that $\forall i, (1 - cb_i) < 0$. When $k \notin S$, we have

$$\langle \Delta_k, \sum_{i \in S} p_i \rangle \geq 0$$

$$\langle \Delta_k, \sum_{i \in S} (1 - cb_i) p_i \rangle \leq 0.$$

Hence $\langle \Delta_k, P \rangle = 0$ for $k \notin S$.

In the following, we will show $P = 0$. Otherwise, there exists $v \in M_\mathbb{R}$ such that $\langle v, P \rangle > 0$. Because $M_\mathbb{R} = \sum_{i=1}^s \mathbb{R}_{\geq 0} \Delta_i$, we can chose $v = \sum_{1 \leq i \leq s} v_i$ with $v_i \in \Delta_i$. Then we have

$$\langle v, P \rangle = \langle \sum_{i \in S} v_i + \sum_{i \notin S} v_i, P \rangle = \sum_{i \in S} \langle v_i, - \sum_{j \notin S} p_j \rangle + \sum_{i \notin S} \langle v_i, P \rangle.$$

We use the assumption $\sum_{j=1}^s p_j = 0$, and thus $P = -\sum_{j \notin S} p_j$ in the second equation. However, $\sum_{i \in S} \langle v_i, - \sum_{j \notin S} p_j \rangle \leq 0$, and $\sum_{i \notin S} \langle v_i, P \rangle = 0$ because $\langle \Delta_k, P \rangle = 0$ for $k \notin S$. This contradiction implies $P = \sum_{i \in S} p_i = 0$.

Because $I_1 \cap S \neq \emptyset$ and $I_1 \not\subseteq S$, the index set $I_1' := I_1 \cap S$ must satisfy $\emptyset \not\subsetneq I_1' \subsetneq I_1$. Since $I_2 \cup \ldots \cup I_l \subset S$, we have

$$\sum_{j \in I_1'} p_j = P - \sum_{i \in I_2 \cup \ldots \cup I_l} p_i = 0.$$

But this implies

$$\sum_{j \in I_1'} p_j = \sum_{j \in I_1 \setminus I_1'} p_j = 0$$

which gives a further decomposition of $I_1$. This is a contradiction to the maximality of $l$.

\[\square\]

**Remark 4.2.** Under the notation of lemma, we observe that for each $k$, $\dim(\text{Span}_\mathbb{R}\{p_i \mid i \in I_k\}) = \#(I_k) - 1$. 
Let $\overline{M} = \mathbb{Z}^s \oplus M$, and $K \subset \overline{M}_{\mathbb{R}}$ be the reflexive Gorenstein cone associated to a nef-partition $\{\Delta_1, \ldots, \Delta_s\}$ in $M_{\mathbb{R}}$ as it is in Proposition 2.6. This nef-partition corresponds to $\deg^\vee = \sum_{i=1}^s e_i \in K^\vee$, where $e_i = (0, \ldots, 1, \ldots, 0; 0)$. If $\deg^\vee = \sum_{i=1}^s \tilde{e}_i$ with $\tilde{e}_i \neq 0, \tilde{e}_i \in N \cap K^\vee$, then we can assume without lost of generality that

$$\tilde{e}_i = (0, \ldots, 1, \ldots, 0; p_i), \quad p_i \in N \cap \nabla_i.$$  

Because $\dim(\text{Span}_\mathbb{R}\{e_1, \ldots, e_s, \tilde{e}_1, \ldots, \tilde{e}_s\}) = s + \dim(\text{Span}_\mathbb{R}\{p_1, \ldots, p_s\})$. If $\dim(\text{Span}_\mathbb{R}\{p_1, \ldots, p_s\}) = s - r$, by Lemma 4.1 there exists disjoint index sets $I_k, 1 \leq k \leq r$, such that $\bigsqcup_{k=1}^r I_k = \{1, \ldots, s\}$. For each $k$, we have $\sum_{i \in I_k} p_i = 0$, with $\dim(\text{Span}_\mathbb{R}\{p_i \mid i \in I_k\}) = \#(I_k) - 1$.

Let $n_k = \#(I_k)$ from now on, and let

$$\text{Ann}(e) := \text{Ann}(e_1, \ldots, e_s) = \{m \in \overline{M} \mid \langle m, e_i \rangle = 0, \forall 1 \leq i \leq s\}.$$  

If rank $M = d$, then $\text{Ann}(e)$ is a sublattice of $\overline{M}$ with rank $d$, and

$$\text{Ann}(e, \tilde{e}) := \text{Ann}(e_1, \ldots, e_s, \tilde{e}_1, \ldots, \tilde{e}_s)$$

$$= \{m \in \overline{M} \mid \langle m, e_i \rangle = \langle m, \tilde{e}_i \rangle = 0, \forall 1 \leq i \leq s\}$$

a sublattice of $\overline{M}$ with rank $d + r - s$.

For our convenience, we use $\{(k1), (k2), \ldots, (kn_k)\}$ as the index set of $I_k$, and reindex the corresponding elements. For example

$$\sum_{i \in I_k} p_i = 0$$

becomes

$$\sum_{i=1}^{n_k} p_{ki} = 0$$

under the new index.

Because $\dim(\text{Span}_\mathbb{R}\{p_{k1}, \ldots, p_{kn_k}\}) = n_k - 1$, we can choose

$$\{p_{12}, \ldots, p_{1n_1}, \ldots, p_{r2}, \ldots, p_{rn_r}\}$$

as a $\mathbb{R}$-linearly independent set. Due to the fact that $N$ sublattice $\sum_{k=1}^r \sum_{i=1}^{n_k} \mathbb{Z}p_{ki}$ may not be saturated (i.e. $N/ (\sum_{k=1}^r \sum_{i=1}^{n_k} \mathbb{Z}p_{ki})$ may have torsion), we need to work in an auxiliary sublattice where it is saturated. To be precise, let $\xi_1, \ldots, \xi_{d+r-s} \in N$ such that $\xi_1, \ldots, \xi_{d+r-s}$ generate the non-torsion part of $N/ (\sum_{k=1}^r \sum_{i=1}^{n_k} \mathbb{Z}p_{ki})$. Let

$$N' = \text{Span}_\mathbb{Z}\{p_{12}, \ldots, p_{1n_1}, \ldots, p_{r2}, \ldots, p_{rn_r}, \xi_1, \ldots, \xi_{d+r-s}\},$$
then $N' \subset N$ is a sublattice with finite index $[N : N']$. In particular, 
\( \{p_{12}, \ldots, p_{1n_1}, \ldots, p_{r2}, \ldots, p_{rn_r} \} \) is part of a $\mathbb{Z}$-basis of $N'$.

Let $M' \supset M$ be the dual lattice of $N'$, and $\overline{N'} := \mathbb{Z}^s \oplus N' \subset \overline{N}$ with the dual lattice $\overline{M'} := \mathbb{Z}^s \oplus M' \supset \overline{M}$. Because $e_i, \tilde{e}_j$ are also in $\overline{N'}$, we can similarly define

\[
\text{Ann}(e)' = \{m' \in \overline{M'} \mid \langle m', e_i \rangle = 0 \ \forall \ i \} \subset \overline{M'},
\]
and

\[
\text{Ann}(e, \tilde{e})' = \{m' \in \overline{M'} \mid \langle m', e_i \rangle = \langle m', \tilde{e}_j \rangle = 0 \ \forall \ i, j \} \subset \overline{M'}.
\]

Moreover, one can directly verify that $\text{Ann}(e, \tilde{e})' \subset \overline{M'}$ is the same lattice as $\text{Ann}(e, \tilde{e}) \subset \overline{M}$, and hence we also use $\text{Ann}(e, \tilde{e})'$ to denote the sublattice $\text{Ann}(e, \tilde{e})'$ in the sequence.

**Lemma 4.3.** The lattice $\text{Ann}(e)' \subset \overline{M'}$ can be decomposed as follows:

\[
\text{Ann}(e)' = \text{Ann}(e, \tilde{e}) \\
\quad \oplus \mathbb{Z}[w_{12}] \oplus \cdots \oplus \mathbb{Z}[w_{1n_1}] \\
\quad \oplus \cdots \\
\quad \oplus \mathbb{Z}[w_{r2}] \oplus \cdots \oplus \mathbb{Z}[w_{rn_r}].
\]

Where $w_{ki} \in \overline{M'}$ satisfies the following requirements (where by our indexing, $w_{ki}$ starts from $w_{k2}$):

1. $\langle w_{ki}, \tilde{e}_{k1} \rangle = -1$, $\langle w_{ki}, \tilde{e}_{ki} \rangle = 1$ for $i \geq 2$.
2. $\langle w_{ki}, \tilde{e}_{ij} \rangle = 0$ for all $\tilde{e}_{ij} \neq \tilde{e}_{k1}, \tilde{e}_{ki}$.
3. $\langle w_{ki}, e_{ij} \rangle = 0$ for all $e_{ij}$.

**Proof.** First, if we already have $w_{ki}$ satisfying the given properties, then by definition, we have

\[
\text{Ann}(e, \tilde{e}) \oplus \mathbb{Z}[w_{12}] \oplus \cdots \oplus \mathbb{Z}[w_{1n_1}] \oplus \cdots \oplus \mathbb{Z}[w_{r2}] \oplus \cdots \oplus \mathbb{Z}[w_{rn_r}] \subset \text{Ann}(e)'
\]
as a sublattice. On the other hand, $\forall \ m \in \text{Ann}(e)'$, we set

\[
m - \sum_k \sum_{i \geq 2} \langle m, \tilde{e}_{ki} \rangle w_{ki},
\]

then by definition, one can check

\[
m - \sum_k \sum_{i \geq 2} \langle m, \tilde{e}_{ki} \rangle w_{ki}
\]

\[
\in \text{Ann}(e)' \cap \text{Ann}(\tilde{e}_{12}, \cdots, \tilde{e}_{1n_1}, \cdots, \tilde{e}_{r2}, \cdots, \tilde{e}_{rn_r})'.
\]
Using the fact that $\forall k, \sum_{t \in I_k} e_t = \sum_{t \in I_k} \tilde{e}_t$, we have
$$m - \sum_k \sum_{k_i \geq 2} \langle m, \tilde{e}_{k_i} \rangle w_{k_i} \in \text{Ann}(e, \tilde{e}).$$

Thus, we only need to show the existence of $w_{k_i}$. Let lattice map
$$\theta : M' \to \mathbb{Z}^{s-r}$$
be defined by
$$m \mapsto (\langle m, p_{12} \rangle, \ldots, \langle m, p_{1n_1} \rangle, \ldots, \langle m, p_{r2} \rangle, \ldots, \langle m, p_{rn_r} \rangle).$$
We claim that $\theta$ is a surjective lattice map. By construction, $$\{p_{12}, \ldots, p_{1n_1}, \ldots, p_{r2}, \ldots, p_{rn_r}\}$$ forms part of $\mathbb{Z}$-basis of $N'$. It follows that $\theta$ is surjective.

We can choose $m$ such that $\langle m, p_{ij} \rangle = 0 \forall j \geq 2$ except $\langle m, p_{ki} \rangle = 1$, and set
$$w_{k_i} = (0, 0, \ldots, 0; m) \in \overline{M'},$$
then $w_{k_i}$ satisfies the required properties.

Now let
$$L' = \text{Span}_\mathbb{Z}\{w_{12}, \ldots, w_{1n_1}, \ldots, w_{r2}, \ldots, w_{rn_r}\} \subset \overline{M'}$$
and $L := L' \cap \overline{M}$, then we have
$$\text{Ann}(e) = \text{Ann}(e, \tilde{e}) \oplus L.$$ 

Because of the above decomposition of lattice, we have a corresponding decomposition of toric varieties:
$$\text{Spec}(\mathbb{C}[\text{Ann}(e)']) = \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})]) \times \text{Spec}(\mathbb{C}[L']),$$
and
$$\text{Spec}(\mathbb{C}[\text{Ann}(e)]) = \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})]) \times \text{Spec}(\mathbb{C}[L]).$$

For any closed point in $\text{Spec}(\mathbb{C}[\text{Ann}(e)'])$ with coordinate $x'$, we can write $x' = (y', \omega')$ with $y' \in \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})])$, $\omega' \in \text{Spec}(\mathbb{C}[L'])$ respectively. Similarly, we can write the closed point $x = (y, \omega) \in \text{Spec}(\mathbb{C}[\text{Ann}(e)])$ with $y \in \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})])$, $\omega \in \text{Spec}(\mathbb{C}[L])$.

Because $\text{Ann}(e) \subset \text{Ann}(e')$ is a finite index sublattice, we have a finite morphism $\rho : \text{Spec}(\mathbb{C}[\text{Ann}(e)']) \to \text{Spec}(\mathbb{C}[\text{Ann}(e)])$. Suppose the image of $(y', \omega')$ is $(y, \omega)$, then $y' = y$ as the same closed point in $\text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})])$. This means $\rho$ is the identity map when restricting to $\text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})]) \to \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})])$. This simple observation is crucial in the subsequent argument.
Although, $X^{-w'_i} \cdot g_i$ are elements in $\mathbb{C}[\text{Ann}(e)]$ (see Section 3.1), they still make sense in $\mathbb{C}[\text{Ann}(e)]$, and we can define the variety:

$$X'_{(e_i)} : X^{-w'_1} \cdot g_1 = X^{-w'_2} \cdot g_2 = \cdots = X^{-w'_s} \cdot g_s = 0$$

in $\text{Spec}(\mathbb{C}[\text{Ann}(e)]')$. Then $\rho$ induces a morphism from $X'_{(e_i)}$ to $X_{(e_i)}$ which we still denote by $\rho$. Generically, $X'_{(e_i)} \to X_{(e_i)}$ is a finite morphism of degree $[\text{Ann}(e): \text{Ann}(e)]$.

4.2. Construction of the determinantal variety. The main ingredient in the proof of Theorem 4.6 is a determinantal variety $D$ which serves as a bridge to connect two complete intersections. We will show that $X_{(e_i)}$ and $X_{(\tilde{e}_i)}$ are both birational to this determinantal variety. However, due to the possible non-saturatedness of the sublattice $\sum_{k=1}^n \sum_{i=1}^n \mathbb{Z}e_{ki} \subset N$, we have to first construct the morphism $\pi'$ from $X'_{(e_i)}$ to $D$. Then we use the fact that $\rho : X'_{(e_i)} \to X_{(e_i)}$ have the same coordinates in $\text{Spec} \mathbb{C}[\text{Ann}(e, \tilde{e})]$ parts to construct a morphism $\pi$ from $X_{(e_i)}$ to $D$. Everything are the same for the $X'_{(\tilde{e}_i)}$, $X_{(\tilde{e}_i)}$ side.

This is illustrated in the following picture:

Now, let

$$S_{i,j} = \{ v \in K | \langle v, \text{deg}^\vee \rangle = 1, \langle v, e_i \rangle = \langle v, \tilde{e}_j \rangle = 1 \}$$

be a polytope, and

$$g_{i,j} = \sum_{v \in (S_{i,j})} c_v X^v$$

be the Laurent polynomial associated to $S_{i,j}$ with coefficients $c_v \in \mathbb{C}$. Let $u_{ki} \in \overline{M}$ satisfy:

1. $\langle u_{ki}, e_{ki} \rangle = \langle u_{ki}, \tilde{e}_{k1} \rangle = 1$
2. $\langle u_{ki}, e_{lj} \rangle = 0$ for all $e_{lj} \neq e_{ki}$
3. $\langle u_{ki}, \tilde{e}_{lj} \rangle = 0$ for all $\tilde{e}_{lj} \neq \tilde{e}_{k1}$.

We point out that unlike those $w_{ki}$ constructed before, $u_{ki}$ starts from $u_{k1}$ for each $k$. The existence of $u_{ki}$ follows from the similarly reason as in Lemma 4.3, and we do not repeat it here.
Next, we proceed to the construction of the determinantal variety $D$.

Let $A_k(y)$ be the $n_k \times n_k$ matrix with entries in $\mathbb{C}[M']$,

\[
A_k(y) = \begin{pmatrix}
X^{-u_{k1}g_{k1,1}} & X^{-u_{k1}w_{k1,k1}}g_{k1,k1} & \cdots & X^{-u_{k1}w_{knk}g_{knk,k1}} \\
X^{-u_{k2}g_{k2,1}} & X^{-u_{k2}w_{k2,k1}}g_{k2,k1} & \cdots & X^{-u_{k2}w_{knk}g_{knk,k2}} \\
\vdots & \vdots & \ddots & \vdots \\
X^{-u_{knk}g_{knk,k1}} & X^{-u_{knk}w_{knk,k2}}g_{knk,k2} & \cdots & X^{-u_{knk}w_{knk}g_{knk,knk}} 
\end{pmatrix}
\]

Notice that the first column is not constructed identically as the rest. The reason for writing the matrix $A_k(y)$ as a function of $y$ is that every entry of this matrix is in $\mathbb{C}[\text{Ann}(e, \hat{e})]$, as one can verify. Thus, according to the above decomposition $\text{Spec}(\mathbb{C}[\text{Ann}(e)']) = \text{Spec}(\mathbb{C}[\text{Ann}(e, \hat{e})]) \times \text{Spec}(\mathbb{C}[L'])$, we use $y$ to represent the corresponding coordinates in $\text{Spec}(\mathbb{C}[\text{Ann}(e, \hat{e})])$.

Next, we define $n_k \times 1$ matrix

\[
w_k = (1, X^{w_{k2}}, \ldots, X^{w_{knk}})^t,
\]

where $t$ means the transpose of a matrix. And also define the $1 \times n_k$ matrix

\[
u_k = (X^{u_{k1}}, X^{u_{k2}}, \ldots, X^{u_{knk}}).
\]

We claim that the condition

\[A_k(y) \cdot w_k = 0\]

is exactly the same as

\[
\begin{pmatrix}
X^{-u_{k1}g_{k1}} \\
\vdots \\
X^{-u_{knk}g_{knk}}
\end{pmatrix} = 0.
\]

Indeed, recall (Section 3) by definition, we have

\[g_{ki} = \sum_{v \in (S_{ki})} c_v X^v\]

where

\[S_{ki} = \{v \in K \mid \langle v, \deg v \rangle = 1, \langle v, e_{ki} \rangle = 1 \}\]

(Notice: this is not the same as $S_{k,i}$ defined before).

Because of the relation $\sum_{i \in I_k} e_i^* = \sum_{i \in I_k} \hat{e}_i$, for any $v \in l(S_{ki})$, $\langle v, \sum_{i \in I_k} e_i \rangle = 1$ implies $\langle v, \sum_{i \in I_k} \hat{e}_i \rangle = 1$, thus there exists $kj$, such
that \( v \in l(\tilde{S}_{kj}) \), where \( \tilde{S}_{kj} = \{ v \in K \mid \langle v, \deg v' \rangle = 1, \langle v, \tilde{e}_{kj} \rangle = 1 \} \). This means \( v \in l(S_{ki,kj}) \), and in particular, we have a disjoint union
\[
l(S_{ki}) = \bigcup_{kj \in I_k} l(S_{ki,kj}).
\]
Hence, \( g_{ki} = \sum_{kj \in I_k} g_{ki,kj} \), and this justifies the claim.

On the other hand, \( u_k \cdot A_k(y) = 0 \)
is exactly the same as
\[
(X^{-w_{k1}} \tilde{g}_{k1}, \ldots, X^{-w_{knk}} \tilde{g}_{knk}) = 0,
\]
where \( \tilde{g}_{kj} = \sum_{v \in l(\tilde{S}_{ki})} c_v X^v = \sum_{ki \in I_k} g_{ki,kj} \) because of the disjoint union
\[
l(\tilde{S}_{ki}) = \bigcup_{ki \in I_k} l(\tilde{S}_{ki,kj}).
\]

Let
\[
D_k := \{ \text{det} A_k(y) = 0 \}
\]
in \( \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})]) \). Let
\[
D = \bigcap_{k=1}^r D_k
\]
with its reduced induced subscheme structure. This \( D \) is the variety which will serve as a bridge to prove the birationality of two complete intersections.

**Remark 4.4.** We have \( \text{det} A_k(y) \not\equiv 0 \) for generic coefficients because we always have
\[
(0, \ldots, 1, \ldots, 0; 0) \in S_{ii}.
\]
Thus by the definition of determinant, these elements will always give a nonzero summand in \( \text{det} A_k(y) \), so \( D_k \) is a hypersurface in \( \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})]) \), and \( \dim D_k = d + r - s - 1 \), with \( d = \text{rank} M \).

We state without proof the following lemma.

**Lemma 4.5.** Let \( f : X \to Y \) be a dominant morphism of varieties over \( \mathbb{C} \). Suppose \( [K(X) : K(Y)] = n \). Then there exists a dense open subset \( U \) of \( Y \) such that \( f^{-1}(y) \) consists of \( n \) (distinct) points for all \( y \in U \).

In particular, if \( f \) is a dominant, injective morphism, then \( [K(X) : K(Y)] = 1 \), so \( X, Y \) are birational.
4.3. **Proof of the main theorem.** Let $M, N$ be rank $d$ lattices, and $K, K^\vee$ be reflexive Gorenstein cones associated to a length $s$ nef-partition. Let $\deg e_i = \sum_{i=1}^{s} e_i$, where $e_i, \tilde{e}_i \in K^\vee \cap N$, $e_i, \tilde{e}_i \neq 0$ as before. Again, without loss of generality, we assume $\forall \ 1 \leq i \leq s$,

\[ e_i = (0, \ldots, 1, \ldots, 0; 0), \quad \tilde{e}_i = (0, \ldots, 1, \ldots, 0; p_i) \in N. \]

By Lemma 4.1, we have a decomposition of $\{p_1, \ldots, p_s\}$ into subsets $I_k = \{p_{k1}, \ldots, p_{kn_k}\}$, $1 \leq k \leq r$. We define the intersections $X_{(e_i)}$, $X_{(\tilde{e}_i)}$ as in Section 3. Using these notations, we have the following birationality result:

**Theorem 4.6.** For generic coefficients, if $X_{(e_i)}$, $X_{(\tilde{e}_i)}$, $D$ are irreducible with $\dim D = \dim X_{(e_i)}$, then the complete intersections $X_{(e_i)}$ and $X_{(\tilde{e}_i)}$ are birational.

**Proof.** When $s = 1$, then $X_{(e_i)} = X_{(\tilde{e}_i)}$, so nothing needs to be proved. Now we assume $s \geq 2$.

By the discussion after Lemma 4.3, we have

\[ \text{Ann}(e) = \text{Ann}(e, \tilde{e}) \oplus L \subset M. \]

For any $x \in X_{(e_i)}$, we can write $x = (y, \omega) \in \text{Spec}(\mathbb{C}[\text{Ann}(e)])$ with $y \in \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})])$, $\omega \in \text{Spec}(\mathbb{C}[L])$ respectively. We claim that there exists a morphism $\pi$:

\[ \pi : X_{(e_i)} \to D \]

defined by $x \mapsto y$.

Indeed, we define a subvariety $X'_k$ of $\text{Spec}(\mathbb{C}[\text{Ann}(e)^r])$ by

\[ X'_k : X^{-w'_k1} \cdot X^{-w'_k2} \cdot \cdots = X^{-w'_kn_k} = 0 \]

where

\[ g_{ki} = \sum_{v \in (S_{ki})} c_v X^v \]

and $w'_ki$ satisfying $\langle w'_ki, e_{ki} \rangle = 1$, $\langle w'_ki, e_{kj} \rangle = 0 \ \forall \ k \neq j$ (certainly, $w'_ki$ exist and $X'_k$ does not depend on the choice of $w'_ki$). We have

\[ X'_{(e_i)} = \bigcap_{k=1}^{r} X'_k. \]

By Lemma 4.3, we have lattice decomposition in $M'$

\[ \text{Ann}(e)^r = \text{Ann}(e, \tilde{e}) \oplus \mathbb{Z}[w_{12}] \oplus \cdots \oplus \mathbb{Z}[w_{1n_1}] \oplus \cdots \oplus \mathbb{Z}[w_{r2}] \oplus \cdots \oplus \mathbb{Z}[w_{rn_r}]. \]
By the construction of $A_k(y)$, the following matrix equation

$$
\begin{pmatrix}
A_1(y) \\
A_2(y) \\
\vdots \\
A_r(y)
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_r
\end{pmatrix} = 0
$$

gives the variety $X'_{(e_i)}$, where $w_k = (1, X^{w_{k1}}, \ldots, X^{w_{kn}})^t$.

Hence, when $(y, \omega') \in X'_{(e_i)}$ with $y \in \text{Spec}(\mathbb{C}[\text{Ann}(e, \bar{e})])$ and $\omega' \in \text{Spec}(\mathbb{C}[L'])$, we have $\forall k, (y, \omega') \in X'_k$, and thus satisfies $A_k(y)w_k = 0$. Because $w_k \neq 0$, we must have $\det(A_k(y)) = 0$. Hence $\forall k, y \in D_k$ and thus $y \in D = \cap_{k=1}^r D_k$. This shows that the natural projection $\text{Spec}(\mathbb{C}[\text{Ann}(e)])' \to \text{Spec}(\mathbb{C}[\text{Ann}(e, \bar{e})])$ maps $X'_{(e_i)}$ to $D$. We denote this morphism by $\pi'$.

We have morphism

$$X_{(e_i)} \hookrightarrow \text{Spec}(\mathbb{C}[\text{Ann}(e)]) \to \text{Spec}(\mathbb{C}[\text{Ann}(e, \bar{e})])$$

where the second morphism is the natural projection. Because of the morphism $X'_{(e_i)} \xrightarrow{\rho} X_{(e_i)} \to D$ and the surjectivity of $\rho$, we know that $X_{(e_i)} \to \text{Spec}(\mathbb{C}[\text{Ann}(e, \bar{e})])$ maps to $D$, and denote this morphism by $\pi$.

Next, we show that $\pi$ is generically injective, that is, $\pi$ is injective on a nonempty open subset of $X_{(e_i)}$. Roughly speaking, the idea of the proof is that Calabi-Yau variety cannot be uniruled. We show that if $\pi$ is not generically injective, then $X'_{(e_i)}$ is a uniruled variety. However, we can construct a compactification $\overline{X}_{(e_i)}$ of $X_{(e_i)}$ which is a projective, Calabi-Yau variety with canonical, Gorenstein singularities. Put these facts together, we get a contradiction. The details are given in the follows:

If $\pi$ is not generically injective, then $\pi'$ is not generically injective either. In that case, We show that $X'_{(e_i)}$ is a uniruled variety (in fact ruled variety), and hence $X_{(e_i)}$ is also uniruled by definition.

By a theorem of Chevalley ([10]), there exists a nonempty open set $V \subset \pi'(X'_{(e_i)})$ such that over $V$, the fibres have the same dimension $h$. Let $y \in V$, and $(X'_{(e_i)})_y$ be the fibre over $y$. We claim that there exists a birational morphism

$$\theta_y : (X'_{(e_i)})_y \to \mathbb{P}^h.$$

In fact, let

$$\Omega_y = \{\omega \in \text{Spec}(\mathbb{C}[L']) \mid (y, \omega) \in (X'_{(e_i)})_y\},$$
and
\[ W_y = \{ \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i \mid \omega_i \in \Omega_y, \lambda_i \in \mathbb{C} \} \]
be the affine space of \( \mathbb{C}^{s-r} \) generated by \( \Omega_y \), where \( I \) is some finite index set. We claim that \( \Omega_y \subset W_y \) is a dense open subvariety. To see this, whenever
\[ \sum_{i \in I, \#(I) < \infty} \lambda_i \neq 0 \text{ and } \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i \in \text{Spec}(\mathbb{C}[L']) \]
by the matrix equation
\[
\begin{pmatrix}
A_1(y) \\
A_2(y) \\
\vdots \\
A_r(y)
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_r
\end{pmatrix} = 0,
\]
we have
\[ (y, \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i) \in (X_{(e_i)})_y. \]

However, the closed points in \( W_y \) satisfying
\[ \sum_{i \in I, \#(I) < \infty} \lambda_i \neq 0 \text{ and } \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i \in (\mathbb{C}^*)^{s-r} \]
is an open variety, and this justifies the claim. Now, \( \dim \Omega_y = h \) implies \( \dim W_y = h \) and the natural morphism
\[ \theta_y : (X'_{(e_i)})_y \hookrightarrow W_y \hookrightarrow \mathbb{P}^h \]
is a birational morphism. Moreover, if \( \pi \) is not generically injective, then \( h \geq 1 \) (one might need to pass to some small open subvariety of \( V \) in order to make the fibre contains distinct closed points). Suppose this is the case, then we can construct a birational morphism
\[ \pi^{-1}(V) \rightarrow V \times \mathbb{P}^h \]
\[ (y, \omega) \mapsto (y, \theta_y(\omega)). \]
This shows that \( X'_{(e_i)} \) is a ruled variety. Because of the surjective morphism \( \rho : X'_{(e_i)} \rightarrow X_{(e_i)} \), \( X_{(e_i)} \) is a uniruled variety.

In the appendix (cf. Remark 6.4, Proposition 6.10), we construct a compactification \( \overline{X_{(e_i)}} \) of \( X_{(e_i)} \), such that \( \overline{X_{(e_i)}} \) is a projective, Calabi-Yau variety with canonical, Gorenstein singularities. Let \( \overline{X_{(e_i)}} \) be a desingularization of \( \overline{X_{(e_i)}} \). It is also a uniruled variety.
a Calabi-Yau variety with canonical singularity. The canonical divisor $K(e_i)$ of $\tilde{X}(e_i)$ is
\[ K(e_i) = \sum c_j E_j, \quad c_j \geq 0, \]
where $E_j$ are the exceptional divisors. Hence, $H^0(\tilde{X}(e_i), \mathcal{O}(K(e_i))) \neq 0$.

However, because $\tilde{X}(e_i)$ is a smooth, proper uniruled variety over $\mathbb{C}$, $H^0(\tilde{X}(e_i), \mathcal{O}(K(e_i))) = 0$ \([14] \text{ IV Corollary 1.11}\). This is a contradiction, and hence $\pi$ is generically injective.

Let $U \subset X(e_i)$ be an open set where $\pi|_U$ is injective. Because for generic coefficients, $X(e_i)$ is smooth of dimension $d - s$ (Proposition 6.3), $\pi(U)$ is a constructible subset of $D$ with dimension $d - s$. This is the same dimension as $D$ by assumption. Thus $\pi$ is dominant as well. By Lemma 4.5, $X(e_i)$ is birational to $D$.

Because the construction is quite symmetric in natural, $X(\tilde{e}_i)$ is birational to $D$ can be shown in the same vein. We just give a sketch below:

First, by the proof of Lemma 4.3, one has a decomposition of lattice $\text{Ann}(\tilde{e})' = \text{Ann}(e, \tilde{e})$
\[ = \mathbb{Z}[u_{12} - u_{11}] \oplus \cdots \oplus \mathbb{Z}[u_{1n - 1} - u_{11}] \oplus \cdots \oplus \mathbb{Z}[u_{r2} - u_{r1}] \oplus \cdots \oplus \mathbb{Z}[u_{rn - 1}]. \]

where $u_{ki}$ is defined in Section 4. We can view $u_{ki} - u_{k1}$ as $w_{ki}$ when $i \geq 2$ because it satisfies the required relation in Lemma 4.3 (with $e_{ki}, \tilde{e}_{ki}$ switched), and this is enough for the existence of the decomposition. Correspondingly, we have a decomposition of the torus:
\[ \text{Spec}(\mathbb{C}[\text{Ann}(\tilde{e})']) = \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})]) \times (\mathbb{C}^*)^{s-r}. \]

We can similarly define $X'_(\tilde{e}_i) \to D$ and surjective $X'_(\tilde{e}_i) \to X(e_i)$. This gives $X(\tilde{e}_i) \to D$ which factoring through $X'_(\tilde{e}_i) \to D$.

For the same reason as before, $X(\tilde{e}_i) \to D$ is a birational morphism.

Hence $X(e_i)$ and $X(\tilde{e}_i)$ are both birational to $D$, and this completes the proof.

\[ \square \]

Remark 4.7. It is necessary for our argument to require $D$ to be irreducible. When examine the case $s = 2$, with $S_{2,1} = \emptyset$, we see that $D$ is a union of zero loci of $g_{1,1}, g_{2,2}$, where $g_{k,i} = \sum_{v \in l(S_{i,j})} c_v X^v, i = 1, 2$.

By the proof of the theorem, we see $X(e_i)$ is birational to the zero locus
of \( g_{2,2} \), but \( X(\tilde{e}_i) \) is birational to the zero locus of \( g_{1,1} \). A priori, one cannot expect that the two loci are birational.

There is a result due to Batyrev and Borisov ([3] Theorem 3.3) which asserts that \( X(e_i) \) is irreducible if the nef-partition is 2-independent. This means there exists no integer \( n > 0 \) and subset of nef-partition \( \{\Delta_k_1, \ldots, \Delta_k_n\} \subset \{\Delta_1, \ldots, \Delta_s\} \), such that \( \text{dim}(\Delta_k_1 + \cdots + \Delta_k_n) \leq n \).

**Remark 4.8.** It is reasonable to require that \( \text{dim} D = \text{dim} X(e_i) = d - s \). Indeed \( D = \cap_{i=1}^r D_i \) is a variety in \( \text{Spec}(\mathbb{C}[\text{Ann}(e, \tilde{e})]) \cong (\mathbb{C}^*)^{d-(s-r)} \) defined by the intersection of \( r \) hypersurfaces. Thus \( D \) is expected to have dimension \( d - s \) for generic choice of coefficients.

5. **Open Questions: \( D \)-equivalence and \( K \)-equivalence**

Let \( D^b(Coh(X)) \) be the derived category of bounded complexes of coherent sheaves on \( X \). For smooth varieties \( X, Y \), if \( D^b(Coh(X)) \) is equivalent to \( D^b(Coh(Y)) \) as derived categories, then \( X, Y \) are called \( D \)-equivalent.

Let \( K_X, K_Y \) be canonical divisors of \( X \) and \( Y \) respectively. If there exists a birational correspondence

\[
X \xleftarrow{\pi_X} Z \xrightarrow{\pi_Y} Y
\]

such that \( \pi_X^*K_X \cong \pi_Y^*K_Y \), then \( X, Y \) are called \( K \)-equivalent. There is a surprising relation between \( D \)-equivalence and \( K \)-equivalence [11]. A theorem of Kawamata [12] says: if \( X, Y \) are projective smooth varieties of general type over an algebraically closed field, then \( X, Y \) are \( D \)-equivalent implies they are \( K \)-equivalent. We have the following conjecture of Kawamata [12]

**Conjecture 5.1.** If \( X, Y \) are smooth projective varieties, then \( X, Y \) are \( K \)-equivalent implies they are \( D \)-equivalent.

This conjecture has been settled for smooth Calabi-Yau threefolds [8] and toroidal varieties [13].

Back to the case considered in this paper. We have proved that \( X(e_i), X(\tilde{e}_i) \) are birational Calabi-Yau varieties, and their compactifications \( \overline{X(e_i)}, \overline{X(\tilde{e}_i)} \) are automatically \( K \)-equivalent. According to the conjecture, we expect to have \( D \)-equivalence \( D^b(Coh(\overline{X(e_i)})) \cong D^b(Coh(\overline{X(\tilde{e}_i)})) \).

**Conjecture 5.2** ([6] Conjecture 5.3). There exists an equivalence (of Fourier-Mukai type) between the derived category of coherent sheaves on the two Calabi-Yau complete intersections \( \overline{X(e_i)} \) and \( \overline{X(\tilde{e}_i)} \).
One might consider sheaves on smooth DM-stacks associated to $X_{(e_i)}$, $X_{(\tilde{e}_i)}$ because of the possible singularities. Moreover, when we consider the homological mirror symmetry conjecture, it is plausible to have such $D$-equivalence.

6. Appendix: $\Delta$-regularity, singularities and Calabi-Yau varieties

Roughly speaking, $\Delta$-regularity is a condition on the smoothness of stratifications with correct dimension. In this appendix, we generalize the concept of $\Delta$-regularity [1] [2] of a hypersurface to an intersection of several hypersurfaces in toric varieties. We will show that for generic coefficients (meaning for a nonempty open set of the parameter space of coefficients), the complete intersections defined by a nef-partition are $\Delta$-regular, and thus form a large family of intersections associated to a nef-partition. Under the $\Delta$-regular assumption, the singularities of the complete intersection are inherited from the ambient toric variety. Using these results, we will show that a $\Delta$-regular complete intersection associated to a nef-partition is a Calabi-Yau variety with canonical, Gorenstein singularities. This fact is used in the proof of Theorem 4.6 by showing that the morphism $\pi$ is generically injective.

6.1. $\Delta$-regularity. Let $\Sigma \subset N_\mathbb{R}$ be a fan, and $X(\Sigma)$ be the toric variety defined by $\Sigma$. If $\sigma \in \Sigma$ is a cone, let $T_\sigma$ be the torus corresponding to $\sigma$. Then we have the following stratification:

$$X(\Sigma) = \bigcup_{\sigma \in \Sigma} T_\sigma.$$  

Definition 6.1. Let $V_i, 1 \leq i \leq s$, be hypersurfaces of $X(\Sigma)$, and let $V = \cap_i V_i$ be the intersection. Then $V$ is called $\Delta$-regular if and only if $\forall \sigma \in \Sigma$, $T_\sigma \cap V$ is either empty or a smooth variety of codimension $s$ in $T_\sigma$.

Remark 6.2. The $\Delta$-regular condition is equivalent to the linear independence of the cotangent spaces at a common intersection point. This takes care both of smoothness and of codimension.

We use the name $\Delta$-regularity following Batyrev [1] [2], where $\Delta$ is a polytope, and the regularity is about a hypersurface defined by a Laurent polynomial with Newton polytope inside $\Delta$.

One can consider the family of $\Delta$-regular complete intersections associated to a nef-partition. In fact, let $\Delta \subset M_\mathbb{R}$ be a reflexive polytope with nef-partition $\{\Delta_i \mid 1 \leq i \leq s\}$, in particular, we have
Conv(∪_{i=1}^s \Delta_i) = \Delta. Let \{\nabla_i \mid 1 \leq i \leq s\} be the dual nef-partition, then

\nabla = \text{Conv}\left(\bigcup_{i=1}^s \nabla_i\right) = \left(\sum_{i=1}^s \Delta_i\right)^\vee.

Let

\Sigma(\nabla) = \{0\} \cup \{R \geq 0 \mid \theta \text{ is a face of } \nabla\}

be a fan, and \(X(\Sigma(\nabla))\) be the toric variety defined by fan \(\Sigma(\nabla)\). One can show that \(X(\Sigma(\nabla))\) is the same as the projective toric variety associated to the polytope \(\sum_{i=1}^s \Delta_i\).

By the construction of a nef-partition, we have a nef torus invariant (Cartier) divisor \(L_i\):

\[L_i = \sum_{\rho \in \text{Vert}(\nabla_i) \setminus \{0\}} D_\rho\]

where \(D_\rho\) is the torus invariant divisor associated to the primitive element \(\rho\).

One can identify the global sections of \(L_i\) with Laurent polynomials associated to \(\Delta_i\) [9]:

\[H^0(X(\Sigma(\nabla)), L_i) \cong \{\sum_{v \in \ell(\Delta_i)} c_v X^v \mid c_v \in \mathbb{C}\}.

Let \(g_i = \sum_{v \in \ell(\Delta_i)} c_v X^v\) and \(V_i = (g_i)_0\) be the zero locus of \(g_i\) on \(X(\Sigma(\nabla))\). Then

\[\{V = \bigcap_{i=1}^s V_i \mid V_i = (g_i)_0, g_i \in H^0(X(\Sigma(\nabla)), L_i)\}\]

is a family of complete intersections in \(X(\Sigma(\nabla))\) parameterized by the coefficients of \(g_i, 1 \leq i \leq s\). The following result shows that a large part of them are \(\Delta\)-regular.

**Proposition 6.3.** For generic coefficients \(c_v \in \mathbb{C}\) of \(g_i = \sum_{v \in \ell(\Delta_i)} c_v X^v, 1 \leq i \leq s\), the complete intersection \(V = \bigcap_{i=1}^s V_i\) is \(\Delta\)-regular.

**Proof.** Using the same notation as before. Because nefness and basepoint freeness are equivalent on toric varieties, the linear system \(|L_i|\) is basepoint free.

Next, we generalize Bertini’s theorem ([10] III Corollary 10.9 and Remark 10.9.2) to show that for generic coefficients, either \(T_\sigma \cap V\) is empty or smooth of codimension \(s\), where \(\sigma \in \Sigma\). Because \(T_\sigma\) is a
locally closed subvariety of $X(\Sigma)$, the restriction $\tilde{L}_i$ of $L_i$ is also base-point free. If the dimension of the linear system $|\tilde{L}_i|$ is $n_i$, then together they define a morphism

$$f : T_\sigma \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}.$$ 

Let $\mathbb{P} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, and we consider it as a homogeneous space under the action of $G := \text{PGL}(n_1) \times \cdots \times \text{PGL}(n_s)$. Let $H_i \to \mathbb{P}^{n_i}$ be the inclusion of a hyperplane $H_i \cong \mathbb{P}^{n_i-1}$, and

$$g : H_1 \times \cdots \times H_s \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$$

be the product of these inclusions.

Next, we set $H := H_1 \times \cdots \times H_s$, and for $\tau \in G$, let $H^\tau$ be $H$ with the morphism $\tau \circ g$ to $\mathbb{P}$. We can apply Kleiman’s theorem ([10] III Theorem 10.8) to $g$ and conclude that there exists a nonempty open set $W \subset G$, such that $\forall \tau \in W, T_\sigma \times_\mathbb{P} H^\tau$ is nonsingular and either empty or of codimension $s$. However, one can show that $f^{-1}(H^\tau)$ is exactly the intersection $T_\sigma \cap V$ defined by the linear systems $\tilde{L}_i$, $1 \leq i \leq s$. This completes the proof.

\[\square\]

Remark 6.4. If the nef-partition $\{\Delta_i \mid 1 \leq i \leq s\}$ comes from $\deg^s = \sum_{i=1}^s e_i$ as in Section 3, then $V \cap (\mathbb{C}^*)^d = X(e_i)$. In other words, $V$ is a projective compactification of $X(e_i)$, and we denote it by $X(e_i)$ in Theorem 4.6.

Remark 6.5. It worth while to point out that not only the complete linear system $|L_\Delta|$ is basepoint free, but also the linear system $\{\sum_{v \in \text{Vert}(\Delta_i)} c_v X^v \mid c_v \in \mathbb{C}\}$ is basepoint free, where $\text{Vert}(\Delta_i)$ denotes the set of vertices of $\Delta_i$.

6.2. Singularities of the $\Delta$-regular variety. In this section, we will show that the singularities of a $\Delta$-regular variety is inherited from the singularities of the ambient toric variety.

Toric Gorenstein, canonical and terminal singularities are characterized by the combinatoric properties of cones [18] (See also [1]):

Proposition 6.6. Let $n_1, \ldots, n_r \in \mathbb{N}$ be primitive integral generators of all 1-dimensional faces of a cone $\sigma \subset \mathbb{N}_\mathbb{R}$.

(1) $U_\sigma$ has Gorenstein singularity if and only if $n_1, \ldots, n_r$ are contained in an affine hyperplane

$$H_\sigma := \{y \in \mathbb{R}^n \mid \langle k_\sigma, y \rangle = 1\},$$

for some $k_\sigma \in M$. 

Let $X(\Sigma)$ be the toric variety defined by a fan $\Sigma$. If $X(\Sigma)$ has Gorenstein, canonical (resp. terminal) singularities, then a $\Delta-$regular complete intersection in $X(\Sigma)$ also has Gorenstein, canonical (resp. terminal) singularities.

Proof. For $\sigma \in \Sigma$, let $T_\sigma$ be the torus corresponding to $\sigma$. Let $U_{\sigma,N}$ be the toric variety associated to the cone $\sigma$ in the lattice $N$, and $N(\sigma)$ be the lattice $N \cap \mathbb{R} \sigma$. Then we have

$$U_{\sigma,N} \cong U_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{d-l},$$

with rank $N = d$, rank $N(\sigma) = l$.

Under this identification, $T_\sigma \cong p_\sigma \times (\mathbb{C}^*)^{d-l}$, where $p_\sigma \in U_{\sigma,N(\sigma)}$ is the unique torus invariant point with coordinate $(0, \ldots, 0)$. Let $f_1, \ldots, f_s$ be the restriction of $\Delta$-regular Laurent polynomials on $U_{\sigma,N}$. This should be understood as follows: since $U_{\sigma,N(\sigma)} = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M(\sigma)])$, $(\mathbb{C}^*)^{d-l} = \text{Spec}(\mathbb{C}[x_1^\pm, \ldots, t_{n-l}^\pm])$, $f_1, \ldots, f_s$ should be viewed as elements in $\mathbb{C}[x_1, \ldots, x_l; t_1^\pm, \ldots, t_{n-l}^\pm]$. By the $\Delta$-regular assumption, if $V_{f_i}$ denotes the zero locus of $f_i$, for any $(0, \ldots, 0; a_1, \ldots, a_{n-l}) \in T_\sigma \cap V_{f_1} \cap \cdots \cap V_{f_s}$, the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial t_j}(0, \ldots, 0; a_1, \ldots, a_{n-l})\right)_{ij}, \quad 1 \leq i \leq s, 1 \leq j \leq n - l$$

has rank $s = \text{dim}(T_\sigma) - \text{dim}(T_\sigma \cap V_{f_1} \cap \cdots \cap V_{f_s})$. By continuity, in an analytic neighborhood of $(0, \ldots, 0; a_1, \ldots, a_{n-l}) \in \mathbb{C}^l \times (\mathbb{C}^*)^{n-l}$, the matrix

$$\left(\frac{\partial f_i}{\partial t_j}(x_1, \ldots, x_l; a_1, \ldots, a_{n-l})\right)_{ij}, \quad 1 \leq i \leq s, 1 \leq j \leq n - l$$

has rank $s$. Without lost of generality, we can assume the $s \times s$ minor with $1 \leq i \leq s, l + 1 \leq j \leq l + s$ is nonvanishing. Thus, we can apply the implicit function theorem to $f_1, \ldots, f_s$. It shows that there are $s$ analytic functions $u_1, \ldots, u_s$ defined on an open neighborhood of $(0, \ldots, 0; a_{s+1}, \ldots, a_{n-l}) \in \mathbb{C}^l \times (\mathbb{C}^*)^{n-l-s}$ such that for points satisfying $f_1 = \cdots = f_s = 0$ on $\mathbb{C}^l \times (\mathbb{C}^*)^{n-l}$, we have

$$f_i = f_i(x_1, \ldots, x_l; u_1, \ldots, u_s, t_{l-s+1}, \ldots, t_{n-l}), \quad 1 \leq i \leq s.$$
Thus, when we restrict to a neighborhood of

\[(0, \ldots, 0; a_1, \ldots, a_{n-l}) \in U_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{n-l} \subset \mathbb{C}^l \times (\mathbb{C}^*)^{n-l},\]

it is locally, analytically isomorphic to a product of a neighborhood of 

\[p_\sigma = (0, \ldots, 0) \text{ in } U_{\sigma,N(\sigma)},\]

with a neighborhood of \((a_{s+1}, \ldots, a_{n-l})\) in \((\mathbb{C}^*)^{n-l-s+1}\). Moreover, Gorenstein singularity is a locally analytic property. This is because the completion of the local ring of a variety is the same as the completion of the local ring of the analytic space associated to that variety, and a local ring is Gorenstein if and only if its completion is Gorenstein. Likewise, canonical and terminal singularities are both local analytic property ([17] Proposition 4-4-4). Hence, we have proved the claim.

\[\square\]

The same argument also shows that a \(\Delta\)-regular complete intersection is normal, because \(X(\Sigma)\) is normal, and normality is preserved under analytic isomorphism.

**Corollary 6.8.** The \(\Delta\)-regular complete intersection \(V\) of \(X(\Sigma(\nabla))\) in Proposition 6.3 has Gorenstein canonical singularities.

**Proof.** Because

\[\Sigma(\nabla) = \{0\} \cup \{\mathbb{R}_{\geq 0}\theta \mid \theta \text{ is a face of } \nabla\},\]

and \(\nabla\) is a reflexive polytope, Proposition 6.6 implies that \(X(\Sigma(\nabla))\) has Gorenstein canonical singularities. Then the result follows by Proposition 6.7.

\[\square\]

In the last part of this section, we apply the adjunction formula to a \(\Delta\)-regular complete intersection of a nef-partition to show that it is a Calabi-Yau variety. As Proposition 6.3, we assume \(V\) to be a \(\Delta\)-regular intersection associated to a nef-partition. First recall following proposition about the adjunction formula on a Cohen-Macaulay scheme ([15] Proposition 5.73).

**Proposition 6.9.** Let \(P\) be a projective Cohen-Macaulay scheme of pure dimension \(n\) over a field \(k\), and \(D \subset P\) an effective Cartier divisor. Then \(\omega_D \cong \omega_P(D) \otimes O_D\). Here \(\omega_D, \omega_P\) are dualizing sheaves of \(D, P\) respectively.

Applying this result and combining with Theorem 6.7, we have the following proposition.

**Proposition 6.10.** If \(V\) is a complete intersection of generic elements of \(|\mathcal{L}_i|, 1 \leq i \leq s\), then \(V\) is a Calabi-Yau variety with canonical, Gorenstein singularities.
Proof. By Corollary 6.8, $V$ has canonical, Gorenstein singularities.

Let $V_i \in |L_i|$ be the generic element which is a Weil divisor and associates to the effective Cartier divisor $L_i$. By definition (see Proposition 6.3), we have $V = \bigcap_i^s V_i$.

A Gorenstein ring is naturally Cohen-Macaulay, so $X := X(\sum(\nabla))$ is a Cohen-Macaulay scheme, and we can apply Proposition 6.9 to get

$$\omega_{V_1} \cong \omega_X(V_1) \otimes \mathcal{O}_{V_1}$$
$$\omega_{V_1 \cap V_2} \cong \omega_X(V_1 + V_2) \otimes \mathcal{O}_{V_1 \cap V_2}$$
$$\vdots$$
$$\omega_V \cong \omega_X(V_1 + V_2 + \cdots + V_s) \otimes \mathcal{O}_V.$$

Because of the nef-partition, we have

$$-K_X \cong \sum_{i=1}^s V_s.$$

We have

$$\omega_X(V_1 + V_2 + \cdots + V_s) \cong \mathcal{O}_X(-K_X + V_1 + V_2 + \cdots + V_s) \cong \mathcal{O}_X,$$

hence $\omega_V \cong \mathcal{O}_V$. On a normal variety, the dualizing sheaf is equivalent to the canonical sheaf ([15]Proposition 5.77). Using the fact that $V$ is a normal variety, we have $K_V = 0$. This shows that $V$ is a Calabi-Yau variety.

In summary, we have proved that for generic coefficients, the complete intersection associated to a nef-partition is a $\Delta$-regular Calabi-Yau variety with canonical, Gorenstein singularities.

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