New Decay Results for a Partially Dissipative Viscoelastic Timoshenko System with Infinite Memory

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Abstract

In this paper, we consider the following dissipative viscoelastic with memory-type Timoshenko system

\[
\begin{aligned}
\rho_1 \phi_{tt} - \kappa (\phi_x + \psi)_x + \kappa \int_0^\infty g(s)(\phi_x + \psi)_x(t-s) \, ds &= 0 & \text{in } (0,L) \times \mathbb{R}^+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + \kappa (\phi_x + \psi) - \kappa \int_0^\infty g(s)(\phi_x + \psi)(t-s) \, ds &= 0 & \text{in } (0,L) \times \mathbb{R}^+, \\
\phi(0,t) &= \phi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, & t \geq 0, \\
\phi_x(x,-t) &= \phi_0(x,t), & \phi_t(x,0) = \phi_1(x), & x \in (0,L), \\
\psi(x,-t) &= \psi_0(x,t), & \psi_t(x,0) = \psi_1(x), & x \in (0,L),
\end{aligned}
\]

with Dirichlet boundary conditions, where \( g \) is a positive non-increasing function satisfying, for some nonnegative functions \( \xi \) and \( H \),

\[
g'(t) \leq -\xi(t)H(g(t)), \quad \forall \ t \geq 0.
\]

Under appropriate conditions on \( \xi \) and \( H \), we establish some new decay results for the case of equal-speeds of propagation that generalize and improve many earlier results in the literature.

Keywords: Timoshenko system, Infinite memory, General decay, Convex functions, equal wave speeds.

AMS Classification: 35B37 · 35L55 · 74D05 · 93D15 · 93D2.

1 Introduction.

We consider the following Timoshenko system with a viscoelastic dissipation mechanism coupled on the shear force:

\[
\begin{aligned}
\rho_1 \phi_{tt} - \kappa (\phi_x + \psi)_x + \kappa \int_0^\infty g(s)(\phi_x + \psi)_x(t-s) \, ds &= 0 & \text{in } (0,L) \times \mathbb{R}^+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + \kappa (\phi_x + \psi) - \kappa \int_0^\infty g(s)(\phi_x + \psi)(t-s) \, ds &= 0 & \text{in } (0,L) \times \mathbb{R}^+, \\
\phi(0,t) &= \phi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, & t \geq 0, \\
\phi_x(x,-t) &= \phi_0(x,t), & \phi_t(x,0) = \phi_1(x), & x \in (0,L), \\
\psi(x,-t) &= \psi_0(x,t), & \psi_t(x,0) = \psi_1(x), & x \in (0,L),
\end{aligned}
\]

where \((x,t) \in (0,L) \times (0,\infty)\), \( \rho_1, \rho_2, b \) and \( \kappa \) are positive constants, \( L > 0 \) is the length of the beam and \( \mathbb{R}^+ = (0,\infty) \). Corresponding to the unknown variables \( \phi \) and \( \psi \), and \( \phi_0, \)

\( \phi_1, \psi_0 \) and \( \psi_1 \) as an initial data and \( g \) is the relaxation function which is also known as memory kernel satisfying some conditions to be specified in the next section. The Timoshenko model was first derived in 1921 by Timoshenko \([1, 2]\) to describe the dynamics of a beam by taking the transverse shear strain into consideration.

Giorgi et al. \([3]\) considered the following semilinear hyperbolic equation with linear memory in a bounded domain \( \Omega \subset \mathbb{R}^3 \)

\[
    u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1.1}
\]

with \( K(0), K(+\infty) > 0 \) and \( K' \leq 0 \) and proved the existence of global attractors for the solutions. Conti and Pata \([4]\) considered the following semilinear hyperbolic equation:

\[
    u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1.2}
\]

where the memory kernel is a convex decreasing smooth function such that \( K(0) > K(+\infty) > 0 \) and \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) is a nonlinear term of at most cubic growth satisfying some conditions. They proved the existence of a regular global attractor. In \([5]\), Appleby et al. studied the linear integro-differential equation

\[
    u_{tt} + Au(t) + \int_{-\infty}^{t} K(t-s)Au(s)ds = 0 \quad \text{for} \quad t > 0, \tag{1.3}
\]

and established an exponential decay result for strong solutions in a Hilbert space. Pata \([6]\) discussed the decay properties of the semigroup generated by the following equation:

\[
    u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)ds = 0 \quad \text{for} \quad t > 0, \tag{1.4}
\]

where \( A \) is a strictly positive self-adjoint linear operator and \( \alpha > 0, \beta \geq 0 \) and the memory kernel \( \mu \) is a decreasing function satisfying specific conditions. Subsequently, they established necessary as well as the sufficient conditions for the exponential stability. In \([7]\), Guesmia considered

\[
    u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)ds = 0 \quad \text{for} \quad t > 0, \tag{1.5}
\]

and introduced a new ingenuous approach for proving a more general decay result based on the properties of convex functions and the use of the generalized Young inequality. He used a larger class of infinite history kernels satisfies the following condition

\[
    \int_0^{+\infty} \frac{g(s)}{H^{-1}(-g'(s))}ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{H^{-1}(-g'(s))} < +\infty, \tag{1.6}
\]

such that

\[
    H(0) = H'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} H'(t) = +\infty, \tag{1.7}
\]

where \( H : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing strictly convex function. Using this approach, Guesmia and Messaoudi \([8]\) later looked into

\[
    u_{tt} - \Delta u + \int_0^t g_1(t-s)\text{div}(a_1(x)\nabla u(s))ds + \int_0^{+\infty} g_2(s)\text{div}(a_2(x)\nabla u(t-s))ds = 0,
\]
in a bounded domain and under suitable conditions on \(a_1\) and \(a_2\) and for a wide class of relaxation functions \(g_1\) and \(g_2\) that are not necessarily decaying polynomially or exponentially and established a general decay result from which the usual exponential and polynomial decay rates are only special cases. Recently, Al-Mahdi [9] consider the following viscoelastic plate problem with a velocity-dependent material density and a logarithmic nonlinearity:

\[
|u_t|^\sigma u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^{+\infty} g(s)\Delta^2 u(t-s)ds = ku \ln |u| \quad \text{in } \Omega \times (0, \infty),
\]

where \(\Omega\) is a bounded domain of \(\mathbb{R}^2\), with a smooth boundary \(\partial \Omega\). He established an explicit and general decay rate results with imposing a minimal condition on the relaxation function, that is,

\[
g'(t) \leq -\xi(t)H(g(t)),
\]

where the two functions \(\xi\) and \(H\) satisfy some conditions. Very recently, Al-Mahdi [10] considered the following plate problem:

\[
u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \int_0^{+\infty} g(s)\Delta^2 u(t-s)ds = 0,
\]

and proved that the stability of this problem holds for which the relaxation function \(g\) satisfies the condition (1.9).

For Timoshenko systems with infinite memory, Rivera et al. [11] considered vibrating systems of Timoshenko type with past history acting only in one equation. They showed that the dissipation given by the history term is strong enough to produce exponential stability if and only if the equations have the same wave speeds. In the case that the wave speeds of the equations are different, they showed that the solution decays polynomially to zero if the corresponding system does not decay exponentially as time goes to infinity, with rates that can be improved depending on the regularity of the initial data. Guesmia et al. [12] have adopted the method introduced in [7] with some necessary modifications to establish a general decay of the solution for a vibrating system of Timoshenko type in a one-dimensional bounded domain with an infinite history acting in the equation of the rotation angle. Guesmia and Messaoudi [13] discussed a Timoshenko system in the presence of an infinite memory, where the relaxation function satisfies \(g'(t) \leq -\xi(t)G(g(t)), \forall t \geq 0\) and established some general decay results for the equal and nonequal speed propagation cases. Recently, Guesmia [14] adapted the approach of [15] to two models of wave equations with infinite memory and proved, under the condition \(g'(t) \leq -\xi(t)G(g(t)), \forall t \geq 0\), where \(\xi\) is satisfying \(\int_0^{+\infty} \xi(s)ds = +\infty\) decay rate of solutions and the growth of \(g\) at infinity. Al-Mahdi [9, 10] also adapted the approach of [15] to some viscoelastic plate equations with relaxation functions satisfy the condition \(g'(t) \leq -\xi(t)G(g(t)), \forall t \geq 0\). The results of [14] and [9, 10] improved and generalized the ones of [10, 7, 18, 19] and [20] by getting a better decay rate and deleted some assumptions on the boundedness of initial data.

The rest of this paper is organized as follows. In section 2, we present some assumptions and material needed for our work. Some technical lemmas are presented and proved in section 3. Finally, we state and prove our main decay results and provide some examples in section 4.
2 Preliminaries

In this section, we introduce some notation and assumption, present some useful lemmas and state the existence theorem. Let us start by introducing the following standard functional spaces:

\[
L^2 := L^2(0, L), \quad ||u||^2 = \int_0^L |u(x)|^2 dx,
\]

\[
H^1 := H^1(0, L), \quad ||u||_{H^1} = ||u||^2 + ||u'||^2,
\]

\[
L^2_s := L^2_s(0, L) = \left\{ u \in L^2(0, L); \frac{1}{t} \int_0^L u(x) dx = 0 \right\},
\]

\[
H^1_0 := H^1_0(0, L) = \left\{ u \in H^1(0, L); u(0) = u(L) = 0 \right\},
\]

\[
H^1_s := H^1_s(0, L) = \left\{ u \in H^1(0, L); \frac{1}{t} \int_0^L u(x) dx = 0 \right\}.
\]

Due to Poincaré’s inequality, we can also consider the equivalent norms in \( H^1_0 \) and \( H^1_s \),

\[
||u||_{H^1_0} = ||u_x||_2 \quad \text{and} \quad ||u||_{H^1_s} = ||u_x||_2,
\]

respectively. In this work, we will always denote by \( c_p > 0 \) the Poincaré constant.

**Assumptions:** We assume that the relaxation function \( g \) satisfy the following hypotheses.

(A1) \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) is a non-increasing differentiable function such that

\[
g(0) > 0 \quad \text{and} \quad \ell := 1 - \int_0^\infty g(s) ds > 0. \tag{2.1}
\]

and

\[
\int_0^\infty g(s) ds \geq \max \left\{ \frac{1}{32}, \frac{64\rho_1 L^2}{31} \right\}. \tag{2.2}
\]

(A2) There exist a non-increasing differentiable function \( \xi : \mathbb{R}^+ \to \mathbb{R}^+ \) and a \( C^1 \)-function \( H : [0, +\infty) \to [0, +\infty) \) which is either linear or strictly increasing and strictly convex \( C^2 \)-function on \((0, r]\) for some \( r > 0 \) with \( H(0) = H'(0) = 0, \lim_{s \to +\infty} H'(s) = +\infty, \ s \mapsto s H'(s) \) and \( s \mapsto s (H')^{-1} (s) \) are convex on \((0, r]\). Moreover, there exists a positive non-increasing differentiable function \( \xi \) such that

\[
g'(t) \leq -\xi(t) H(g(t)), \quad \forall \ t \geq 0. \tag{2.3}
\]

**Remark 2.1.** The condition (2.2) means that the area under the graph of \( g \) is bounded below. It does not contradict the relation (2.1) since

\[
C_0 := \max \left\{ \frac{1}{32}, \frac{64\rho_1 L^2}{31} \right\} < 1.
\]

**Remark 2.2.** From (2.2) we infer that there exists a time \( t_0 > 0 \) large enough such that

\[
1 - h(t) = \int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds := g_0 > C_0 \quad \forall t \geq t_0. \tag{2.4}
\]

**Remark 2.3.** If \( H \) is a strictly increasing, strictly convex \( C^2 \) function over \((0, r]\) and satisfying \( H(0) = H'(0) = 0 \), then it has an extension \( \overline{H} \), that is also strictly increasing.
and strictly convex $C^2$ over $(0, \infty)$. For example, if $H(r) = a, H'(r) = b, H''(r) = c$, and for $t > \varepsilon$, $H(t)$ can be defined by
\[
H(t) = \frac{c}{2}t^2 + (b - cr)t + \left(a + \frac{c}{2}r^2 - br\right).
\]
(2.5)

For simplicity, in the rest of this paper, we use $H$ instead of $H$.

**Remark 2.4.** Since $H$ is strictly convex on $(0, r]$ and $H(0) = 0$, then
\[
H(\theta z) \leq \theta H(z), \quad 0 \leq \theta \leq 1, \quad z \in (0, r].
\]
(2.6)

**Theorem 2.5.** Under the Assumptions (A1)-(A2) and taking $(\phi_0, \phi_1, \psi_0, \psi_1) \in H^1_0 \times L^2 \times H^1_0 \times L^2_0$, there exists a unique weak solution $(\phi, \psi)$ of problem (P) in the class
\[
(\phi, \psi) \in C(\mathbb{R}^+; H^1_0 \times H^1_0) \cap C^1(\mathbb{R}^+; L^2 \times L^2).
\]
Furthermore, if $(\phi_0, \phi_1, \psi_0, \psi_1) \in (H^2 \cap H^1_0) \times H^1_0 \times (H^2 \cap H^1_0) \times H^1_0$, then there exists a unique strong solution $(\phi, \psi)$ of problem (P) in the class
\[
(\phi, \psi) \in C(\mathbb{R}^+; (H^2 \cap H^1_0) \times (H^2 \cap H^1_0)) \cap C^1(\mathbb{R}^+; H^1_0 \times H^1_0).
\]

**Proof.** The proof of this theorem can be achieved by using the pattern of the Faedo-Galerkin method (see Lions book [23]) as applied to wave equations with memory.

**Lemma 2.6.** If $(\phi, \psi) \in L^2(0, T; H^1_0 \times H^1_0), T > 0$, then
\[
p(\cdot, t) := \phi(\cdot, t) + \psi(\cdot, t) \in H^1_0(0, L).
\]
(2.7)

**Proof.** The proof follows from direct computations.

We are going to see that problem (P) is dissipative with only one damping mechanism given by the convolution term involving the shear force component. Indeed, under the above notation and given a weak solution $(\phi, \psi)$ of problem (P), we define the corresponding energy functional
\[
E(t) = E(\phi(t), \psi(t), \phi_t(t), \psi_t(t)), \quad t \geq 0, \quad \text{by}
\]
\[
E(t) := \frac{\rho_1}{2} ||\phi_t(t)||^2 + \frac{\rho_2}{2} ||\psi_t(t)||^2 + \frac{b}{2} ||\psi_x(t)||^2 + \frac{\kappa}{2} ||p_x(t)||^2 + \frac{\kappa}{2} (g \circ p_x)(t),
\]
(2.8)
where $\gamma := 1 - \int_0^\infty g(s)ds$ and $p(x, t) := \int_0^x u(y, t)dy$ and
\[
(g \circ u)(t) := \int_0^\infty g(s)||u(t) - u(t - s)||^2 ds.
\]
(2.9)

**Lemma 2.7.** The energy $E(t)$ satisfies the following identity:
\[
\frac{d}{dt} E(t) = -\frac{\kappa}{2} (g' \circ p_x)(t) \leq 0, \quad t > 0.
\]
(2.10)

**Proof.** Taking the multipliers $\phi_t$ and $\psi_t$ in the first two equation of (P), respectively, a straightforward computation leads to (2.10).
As in [24], we set, for any $0 < \alpha < 1$,

$$C_\alpha := \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)} \, ds \quad \text{and} \quad \mu(t) := \alpha g(t) - g'(t). \quad (2.11)$$

**Remark 2.8.** Using the fact that $\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$ and recalling the Lebesgue dominated convergence theorem, we can easily deduce that

$$\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} \, ds \to 0 \text{ as } \alpha \to 0. \quad (2.12)$$

**Lemma 2.9.** ([24]). Assume that assumption (A1) holds. Then for any $v \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(0, L))$, we have

$$\int_0^L \left( \int_0^\infty g(s)(v(t) - v(t-s)) \, ds \right)^2 \, dx \leq C_\alpha (\mu \circ v)(t), \quad \forall \, t \geq 0. \quad (2.13)$$

### 3 Technical lemmas

In this section, we state and proof some lemmas needed to establish our main results.

**Lemma 3.1.** There exist a positive constant $M_0$ such that

$$\int_t^{+\infty} g(s) ||p_x(t) - p_x(t-s)||^2_2 \, ds \leq M_0 h_0(t), \quad (3.1)$$

where $h_0(t) = \int_0^{+\infty} g(t+s) (1 + ||p_{x0x}(s)||^2) \, ds$.

**Proof.** The proof of (3.1) is identical to the one in [11] and [10]. Indeed, we have

$$\int_t^{+\infty} g(s) ||p_x(t) - p_x(t-s)||^2_2 \, ds \leq 2 ||p_x(t)||^2 \int_t^{+\infty} g(s) \, ds$$

$$+ 2 \int_t^{+\infty} g(s) ||p_x(t-s)||^2_2 \, ds$$

$$\leq 2 \sup_{s \geq 0} ||p_x(s)||^2 \int_0^{+\infty} g(t+s) \, ds + 2 \int_0^{+\infty} g(t+s)||p_{x0x}(s)||^2_2 \, ds$$

$$\leq \frac{4E(s)}{\kappa \gamma} \int_0^{+\infty} g(t+s) \, ds + 2 \int_0^{+\infty} g(t+s)||p_{x0x}(s)||^2_2 \, ds$$

$$\leq \frac{4E(0)}{\kappa \gamma} \int_0^{+\infty} g(t+s) \, ds + 2 \int_0^{+\infty} g(t+s)||p_{x0x}(s)||^2_2 \, ds$$

$$\leq M_0 \int_0^{+\infty} g(t+s) (1 + ||p_{x0x}(s)||^2) \, ds.$$  

where $M_0 = \max \{2, \frac{4E(0)}{\kappa \gamma}\}$. \qed

**Lemma 3.2.** Under assumption (A1) and (A2), the functionals $\chi_1(t), \chi_2(t)$ and $\chi_3(t)$ defined by

$$\chi_1(t) := \int_0^L (\rho_1 \phi \phi_t + \rho_2 \psi \psi_t) \, dx \quad (3.3)$$
\[\chi_2(t) := -\rho_1 \int_0^L \phi_t(t) \int_0^\infty g(s)(p(t) - p(t - s))dsdx, \quad (3.4)\]
\[\chi_3(t) := -\rho_2 \int_0^L \psi_t \left( h(t)p_x(t) + \int_0^\infty g(s)(p_x(t) - p_x(t - s))ds \right) dx - \rho_2 \int_0^L \phi_t \psi_x dx. \quad (3.5)\]
satisfies, along with the solution of (P), the estimates
\[\chi_1'(t) \leq \rho_1 \left| \phi_t(t) \right|^2_2 + \rho_2 \left| \psi(t) \right|^2_2 \leq \frac{\kappa}{2} \phi_t(t) \left| p_x(t) \right|^2_2 + \frac{\kappa}{2} C_\alpha(\mu \circ p_x)(t). \quad (3.6)\]
\[\chi_2'(t) \leq -\frac{\rho_1}{\rho_2} \frac{\phi_t(t)}{\phi_t(t)} \left| \phi_t(t) \right|^2_2 + \frac{\kappa(1 - \rho_0)\rho_1L^2}{\rho_2} \left| p_x(t) \right|^2_2 + \frac{\rho_1L^2}{\rho_2} \left| \psi(t) \right|^2_2 + \rho_2 \int_0^L \left( \int_0^\infty g'(s)(p_x(t) - p_x(t - s))ds \right)^2 dx + \kappa \left( 1 + \frac{\rho_2}{4(1 - \rho_0)\rho_1L^2} \right) C_\alpha(\mu \circ p_x)(t). \quad (3.7)\]
for all \( t \geq t_0 \), where \( c_\rho > 0 \) is the Poincaré constant.
\[\chi_3'(t) \leq -\frac{\rho_2}{2} \left| \psi(t) \right|^2_2 + 2\kappa \left| h(t) \right|^2_2 \left| p_x(t) \right|^2_2 + 2\kappa C_\alpha(\mu \circ p_x)(t) \]
\[+ \rho_2 g(0)(0) \left| p_x(t) \right|^2_2 + \rho_2 \int_0^L \left( \int_0^\infty g'(s)(p_x(t) - p_x(t - s))ds \right)^2 dx. \quad (3.8)\]

**Proof.** Taking the derivative of \( \chi_1(t) \) defined in (3.3), Using equations of (P), and performing some integration by parts, we arrive at
\[\chi_1'(t) = \rho_1 \left| \phi_t(t) \right|^2_2 + \rho_2 \left| \psi(t) \right|^2_2 + \rho_1 \int_0^L \phi \phi_t dx + \rho_2 \int_0^L \psi \psi_t dx \]
\[= \rho_1 \left| \phi_t(t) \right|^2_2 + \rho_2 \left| \psi(t) \right|^2_2 + \rho_1 \left| \phi_t(t) \right|^2_2 + \rho_2 \left| \psi(t) \right|^2_2 \]
\[+ \kappa \int_0^L p_x(t) \int_0^\infty g(s)(p_x(t) - p_x(t - s))dsdx. \]
Applying Cauchy-Schwarz’s, Young’s inequalities and (2.13) we obtain
\[\chi_1'(t) \leq \rho_1 \left| \phi_t(t) \right|^2_2 + \rho_2 \left| \psi(t) \right|^2_2 - b \left| \psi(t) \right|^2_2 - \kappa \left( h(t) - \frac{\ell}{2} \right) \left| p_x(t) \right|^2_2 + \frac{\kappa}{2} C_\alpha(\mu \circ p_x)(t). \]
Note that \( h(t) - \frac{\ell}{2} \geq \frac{h(t)}{2} \) for all \( t > 0 \), which proved (3.6).

Deriving the functional \( \chi_2(t) \) set in (3.4) and using the first equation of (P) we get
\[\chi_2'(t) = -\rho_1 \int_0^L \phi_t \phi_t \left[ \int_0^\infty g(s)(p(t) - p(t - s))ds \right] dx \]
\[+ \rho_1 \int_0^L \phi_t \left[ \int_0^\infty g(s)(p(t) - p(t - s))ds \right] dx \]
\[= -\rho_1 \int_0^L \phi_t \left[ \int_0^\infty g(s)(p(t) - p(t - s))ds \right] dx \]
\[+ \rho_1 \int_0^L \phi_t \left[ \int_0^\infty g(s)(p(t) - p(t - s))ds \right] dx \]
\[+ \kappa \int_0^L \left( \int_0^\infty g(s)(p_x(t) - p_x(t - s))ds \right)^2 dx \]
\[+ \kappa h(t) \int_0^L p_x(t) \int_0^\infty g(s)(p_x(t) - p_x(t - s))dsdx. \quad (3.9)\]
Applying again Cauchy-Schwarz’s and Young’s inequalities one has
\[\left| \int_0^L \phi_t \int_0^\infty \left( g'(s)(p(t) - p(t - s))ds \right) dx \right| \]
\[
\frac{g_0}{4}||\phi_t(t)||_2^2 + \frac{1}{g_0} \int_0^L \left( \int_0^\infty g'(s)(p(t) - p(t-s))ds \right)^2 dx,
\]
(3.10)
\[
\left| \int_0^L \phi_t \tilde{\psi}_t(t) dx \right| \leq \frac{1}{2} ||\phi_t(t)||_2^2 + \frac{1}{2} ||\tilde{\psi}_t(t)||_2^2,
\]
(3.11)
\[
\left| \int_0^L p_x \int_0^\infty g(s)(p_x(t) - p_x(t-s))ds dx \right| \\
\leq (1 - g_0) \frac{\rho_1 L^2}{\rho_2} ||p_x(t)||_2^2 + \frac{\rho_2}{4(1 - g_0)\rho_1 L^2} C_\alpha (\mu \circ p_x)(t).
\]
(3.12)

Replacing (3.10) and (3.12) in (3.9) we obtain
\[
\chi_2'(t) \leq -\frac{\rho_2}{2} \left( 1 - h(t) - \frac{g_0}{\rho_2} \right) ||\phi_t(t)||_2^2 + \kappa (1 - g_0) \frac{\rho_1 L^2}{\rho_2} h(t) ||p_x(t)||_2^2 \\
+ \frac{\rho_2}{2} (1 - h(t)) ||\tilde{\psi}_t(t)||_2^2 + \kappa \left( 1 + \frac{\rho_2}{4(1 - g_0)\rho_1 L^2} \right) C_\alpha (\mu \circ p_x)(t)
\]
(3.13)
\[
\chi_3'(t) = -\rho_2 ||\psi_t(t)||_2^2 + \kappa [h(t)]^2 ||p_x(t)||_2^2 - \rho_2 \int_0^L \phi_{tt} \psi_t dx \\
+ 2\kappa h(t) \int_0^L p_x(t) \int_0^\infty g(s)(p_x(t) - p_x(t-s))ds dx \\
+ \kappa \int_0^L \left( \int_0^\infty g(s)(p_x(t) - p_x(t-s))ds \right)^2 dx \\
+ b \int_0^L \psi_{xx} \left( gp_x + \int_0^\infty g'(s)(p_x(t) - p_x(t-s))ds \right) dx
\]
(3.14)
\[
\chi_3'(t) = -\rho_2 ||\psi_t(t)||_2^2 + \kappa [h(t)]^2 ||p_x(t)||_2^2 + \left( \frac{b\kappa}{\kappa} - \rho_2 \right) \int_0^L \phi_{tt} \psi_t dx \\
+ 2\kappa h(t) \int_0^L p_x(t) \int_0^\infty g(s)(p_x(t) - p_x(t-s))ds dx \\
+ \kappa \int_0^L \left( \int_0^\infty g(s)(p_x(t) - p_x(t-s))ds \right)^2 dx \\
+ \rho_2 \int_0^L \psi_t \left( gp_x + \int_0^\infty g'(s)(p_x(t) - p_x(t-s))ds \right) dx
\]
(3.15)

Using over again Cauchy-Schwarz’s, Young’s inequalities and (2.13), and recalling that \( g(t) \leq g(0) \) for all \( t > 0 \), we deduce
\[
\left| \int_0^L \psi_t \left( gp_x - \int_0^\infty g'(s)(p_x(t) - p_x(t-s))ds \right) dx \right| \\
\leq \frac{1}{2} ||\psi_t(t)||_2^2 + g(0) ||p_x(t)||_2^2 + \int_0^L \left( \int_0^\infty g'(s)(p_x(t) - p_x(t-s))ds \right)^2 dx,
\]
(3.16)
Thus, regarding the equal wave speeds assumption ($\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$), we have

$$h(t) \int_0^L p_x \int_0^\infty g(s)(p_x(t) - p_x(t-s))ds dx \leq \frac{|h(t)|^2}{2}||p_x(t)||^2 + \frac{1}{2}C_\alpha(\mu \circ p_x)(t). \quad (3.17)$$

Replacing (3.16) and (3.17) in (3.15) and recalling the condition $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ we have

$$\chi'(t) \leq -\frac{\kappa}{2}||\psi_t(t)||^2 + 2\kappa|h(t)|^2||p_x(t)||^2 + 2\kappa C_\alpha(\mu \circ p_x)(t) + \rho_2 g(0)g(t)||p_x(t)||^2 + \rho_2 \int_0^L \left( \int_0^\infty g'(s)(p_x(t) - p_x(t-s))ds \right)^2 dx.$$  

Thus, regarding the equal wave speeds assumption ($\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$), we conclude from (3.18) that (3.8) satisfied.

**Lemma 3.3.** Let $N, \eta_1, \eta_2 > 0$ be constants (to be determined later) and

$$L(t) := NE(t) + \eta_1 I_1(t) + \eta_2 I_2(t) + I_3(t), \quad (3.19)$$

satisfy, for some constant $C > 0$, the estimate

$$L'(t) \leq -CE(t) + \frac{1}{4}(g \circ p_x)(t) - 4(1-\ell)||p_x(t)||^2, \quad \forall t \geq t_0, \quad (3.20)$$

**Proof.** Taking the derivative of $\chi(t)$ and using the estimates (3.6), (3.7), and (3.8), we have, for all $t \geq t_0$,

$$\chi'(t) \leq -\rho_1 \left( \eta_2 \frac{g_0}{2} - \eta_1 \right) ||\phi_t(t)||^2 - \eta_1 b ||\psi_x(t)||^2 - \left( \frac{\eta_2}{2} - \eta_1 \rho_2 - \eta_2 \frac{\rho_2 L^2}{2} \right) ||\psi_t(t)||^2$$

$$-\kappa \left( \frac{\eta_2}{2} - \eta_2(1-g_0) \frac{\rho_2 L^2}{\rho_2} - 2h(t) \right) ||\phi_t(t)||^2 - \left( \frac{\kappa}{2} - C_1 \right) ||\psi_x(t)||^2$$

$$- \left( \frac{\kappa}{4} - \rho_2 g(0)g(t) \right) ||\psi_t(t)||^2 + \left( \frac{\eta_2}{g_0} + \rho_2 \right) \int_0^L \left( \int_0^\infty g'(s)(p_x(t) - p_x(t-s))ds \right)^2 dx,$$

where $C_1 = \kappa \left[ \frac{\eta_2}{2} + \eta_2 \left( 1 + \frac{\rho_2}{4(1-g_0)\rho_1 L^2} \right) + 2 \right]$.

Now using (A1), Remark 2.1 and Remark 2.2 we notice that

$$g_0 > C_0 \geq \frac{64\rho_1 L^2}{64\rho_1 L^2 + \rho_2},$$

then

$$\frac{32(1-g_0)}{g_0} < \frac{\rho_2}{2\rho_1 L^2}.$$ 

So, it is possible to select $\eta_2$ such that

$$\frac{32(1-g_0)}{g_0} < \eta_2 < \frac{\rho_2}{2\rho_1 L^2}. \quad (3.22)$$

Similarly, using Remark’s 2.1 and 2.2 one can show that

$$8(1-g_0) < \frac{1}{4} \min\{\eta_2 g_0, 1\}.$$
which implies that we can select $\eta_1$ such that

$$8(1 - g_0) < \eta_1 < \frac{1}{4} \min\{\eta_2 g_0, 1\}$$  \hspace{1cm} (3.23)

From the choices in (3.22) and (3.23) we observe that

- $\eta_2 g_0 - \eta_1 > 0$,
- $\eta_2 g_0 - \frac{\eta_1 L^2}{2} > 0$,
- $\eta_2 (1 - g_0) - \frac{\eta_1 L^2}{2} - 2h(t) > \frac{3}{2} (1 - g_0) > 0$ \hspace{0.5cm} for all $t \geq t_0$.

In this case, combining (3.22) and (3.21) we arrive at

$$\chi'(t) \leq -CE(t) - \left( \frac{N}{2} - C_1 C_\alpha \right) (\mu \circ p_x)(t) - \left( \frac{N_k}{4} - \rho_2 g(0) g(t) \right) \|p_x(t)\|_2^2,$$  \hspace{1cm} (3.24)

Now, using remark 2.8 there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8C_1}.$$  \hspace{1cm} (3.25)

Next, we choose $N$ large enough so that

$$\frac{N_k}{4} - \rho_2 g(0) g(t) > 4(1 - \ell) \hspace{0.5cm} \text{and} \hspace{0.5cm} \alpha = \frac{1}{2N} < \alpha_0.$$  

Therefore, (3.24) reduces to (3.20).

**Lemma 3.4.** The functional

$$I_4(t) = \int_0^t h(t-s) \|p_x(t)\|_2^2 ds dx,$$

satisfies, along the solution of (14), the estimate

$$I_4'(t) \leq -\frac{1}{2} (g \circ p_x)(t) + 3(1 - \ell) \|p_x(t)\|_2^2 dx + \frac{1}{2} \int_t^{+\infty} g(s) \|p_x(t)\|_2^2 ds.$$  \hspace{1cm} (3.26)

where $h(t) = \int_t^{+\infty} g(s) ds$.

**Proof.** As in [25], we have the following

$$h'(t) = -g(t), \hspace{0.5cm} \int_0^t g(t-s) ds = \int_0^t g(s) ds = \int_0^\infty g(s) ds - \int_t^\infty g(s) ds = h(0) - h(t).$$  \hspace{1cm} (3.27)
Now, direct differentiation of $I_4$, we have

$$I'_4(t) = h(0)||p_x(t)||_2^2 + \int_0^t h'(t-s)||p_x(s)||_2^2 ds$$

$$= h(0)||p_x(t)||_2^2 - \int_0^t g(t-s)||p_x(s)||_2^2 ds$$

$$= h(0)||p_x(t)||_2^2 - \int_0^t g(t-s)||p_x(s) - p_x(t) + p_x(t)||_2^2 ds$$

$$= h(0)||p_x(t)||_2^2 - \int_0^t g(t-s)||p_x(s) - p_x(t)||_2^2 ds - 2\int_0^t g(t-s)||p_x(s) - p_x(t)||_2||p_x(t)||_2 ds$$

$$= h(t)||p_x(t)||_2^2 - \int_0^t g(t-s)||p_x(t)||_2^2 ds$$

$$+ 2(1 - \ell)||p_x(t)||_2^2 + \int_0^t \frac{g(s)ds}{2(1 - \ell)} \int_0^t g(t-s)||p_x(t) - p_x(s)||_2^2 ds$$

$$\leq 3(1 - \ell)||p_x(t)||_2^2 - \int_0^t g(t-s)||p_x(t) - p_x(s)||_2^2 ds + \frac{1}{2} \int_0^t g(t-s)||p_x(t) - p_x(s)||_2^2 ds$$

(3.28)

$$\leq 3(1 - \ell)||p_x(t)||_2^2 - \frac{1}{2} \int_0^t g(t-s)||p_x(t) - p_x(s)||_2^2 ds$$

$$\leq 3(1 - \ell)||p_x(t)||_2^2 - \frac{1}{2} \int_0^\infty g(t-s)||p_x(t) - p_x(s)||_2^2 ds + \frac{1}{2} \int_0^\infty g(t-s)||p_x(t) - p_x(s)||_2^2 ds$$

$$\leq 3(1 - \ell)||p_x(t)||_2^2 - \frac{1}{2} (g \circ p_x)(t) + \frac{1}{2} \int_0^\infty g(t-s)||p_x(t) - p_x(s)||_2^2 ds.$$

(3.29)

Then (3.20) is established.

Lemma 3.5. Assume that

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}.$$  (3.30)

Then, the energy functional satisfies, for all $t \in \mathbb{R}^+$, the following estimate

$$\int_0^t E(s) ds < \hat{m}f(t),$$  (3.31)

where $f(t) = 1 + \int_0^t h_0(s) ds$ and $h_0$ is defined in Lemma 3.2.

Proof. let $F(t) = L(t) + I_4(t)$, then, we obtain, for all $t \in \mathbb{R}_+$,

$$L'(t) \leq -mE(t) + \frac{1}{2} \int_t^\infty g(s)||p_x||_2^2 ds,$$
where \( m \) is some positive constant. Therefore, using (3.1), we obtain
\[
m \int_0^t E(s) ds \leq F(0) - F(t) + \frac{M_0}{2} \int_0^t \int_0^{+\infty} g(\tau + s) (1 + |(p_x)_{0x}(s)|^2) d\tau ds
\]
\[
\leq F(0) + \frac{M_0}{2} \int_0^t h_0(s) ds.
\]
Hence, we get
\[
\int_0^t E(s) ds \leq F(0) + \frac{M_0}{2} \int_0^t h_0(s) ds \leq \tilde{m} \left( 1 + \int_0^t h_0(s) ds \right),
\]
where \( \tilde{m} = \max \{ \frac{F(0)}{m} , \frac{M_0}{2m} \} \).

**Corollary 3.1.** There exists \( 0 < q_0 < 1 \) such that, for all \( t \geq 0 \), we have the following estimate:
\[
\int_0^t g(s) ||p_x(t) - p_x(t-s)||_2^2 ds \leq \frac{1}{q(t)} H^{-1} \left( \frac{q(t)\mu(t)}{\xi(t)} \right)
\]
where \( H \) is defined earlier in Remark 2.1.
\[
\mu(t) := -\int_0^t g'(s) ||p_x(t) - p_x(t-s)||_2^2 ds \leq -cE'(t)
\]
and
\[
q(t) := \frac{q_0}{f(t)}.
\]

**Proof.** As in [14], using (2.8) and (3.31), we have
\[
\int_0^t ||p_x(t) - p_x(t-s)||_2^2 ds
\]
\[
\leq 2 \int_0^L \int_0^t \left( |p_x(t)|^2 + |p_x(t-s)|^2 \right) ds dx
\]
\[
\leq \frac{4}{\kappa \gamma} \int_0^t (E(t) + E(t-s)) ds dx
\]
\[
\leq \frac{8}{\kappa \gamma} \int_0^t E(s) ds dx \leq \frac{8}{\kappa \gamma} \tilde{m} f(t), \quad \forall \ t \in \mathbb{R}_+.
\]
Thanks to (3.31), we have for all \( t \geq 0 \) and for \( 0 < q_0 < \min \{ 1, \frac{\kappa \gamma}{8\tilde{m}} \} \), \( q(t) < 1 \) and
\[
q(t) \int_0^t ||p_x(t) - p_x(t-s)||_2^2 ds < 1.
\]
So, the proof of (3.34) can be archived as the one given in [15].

### 4 A decay result for equal speeds of wave propagation

In this section, we state and prove a new general decay result in the case of equal speeds of wave propagation (3.30). As in [10], we introduce the following functions:
\[
G_1(t) := \int_0^1 \frac{1}{sH'(s)} ds,
\]
\begin{align*}
G_2(t) &= tH'(t), \quad G_3(t) = t(H')^{-1}(t), \quad G_4(t) = G_3^*(t). \tag{4.2}
\end{align*}

Further, we introduce the class \( S \) of functions \( \chi : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying for fixed \( c_1, c_2 > 0 \) (should be selected carefully in (4.17)):

\[ \chi \in C^1(\mathbb{R}_+), \quad \chi \leq 1, \quad \chi' \leq 0, \tag{4.3} \]

and

\[ c_2G_4 \left[ \frac{c}{d}q(t)h_0(t) \right] \leq c_1 \left( G_2 \left( \frac{G_5(t)}{\chi(t)} \right) - G_2(0) \right), \tag{4.4} \]

where \( d > 0, c \) is a generic positive constant which may change from line to line, \( h_0 \) and \( q \) are defined in Lemma 3.1 and Corollary 3.1 and

\[ G_5(t) = G_1^{-1} \left( c_{1} \int_{0}^{t} \xi(s)ds \right), \tag{4.5} \]

**Theorem 4.1.** Let \((\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H^2_0(0, L) \times L^2(0, L). \) Assume that the hypotheses (A.1) and (A.2) hold and

\[ \frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{4.6} \]

then there exists a strictly positive constant \( C \) such that the solution of (4.1) satisfies, for all \( t \geq 0, \)

\[ E(t) \leq \frac{CG_5(t)}{\chi(t)q(t)}. \tag{4.7} \]

**Remark 4.2.** [10] According to the properties of \( H \) introduced in (A2), \( G_2 \) is convex increasing and defines a bijection from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \), \( G_1 \) is decreasing defines a bijection from \( (0, 1] \) to \( \mathbb{R}_+ \), and \( G_3 \) and \( G_4 \) are convex and increasing functions on \([0, r]\). Then the set \( S \) is not empty because it contains \( \chi(s) = \varepsilon G_5(s) \) for any \( 0 < \varepsilon \leq 1 \) small enough. Indeed, (4.3) is satisfied (since (4.1) and (4.5)). On the other hand, we have \( q(t)h_0(t) \) is nonincreasing, \( 0 < G_5 \leq 1, \) and \( H' \) and \( G_4 \) are increasing, then (4.4) is satisfied if

\[ c_2G_4 \left[ \frac{c}{d}q_0h_0(0) \right] \leq c_{1} \left( H'(\frac{1}{\varepsilon}) - H'(1) \right) \]

which holds, for \( 0 < \varepsilon \leq 1 \) small enough, since \( \lim_{t \to +\infty} H'(t) = +\infty. \) But with the choice \( \chi = \varepsilon G_5, \) (4.7) (below) does not lead to any stability estimate. The idea is to choose \( \chi \) satisfy (4.3) and (4.4) such that (4.7) gives the best possible decay rate for \( E. \)

**Remark 4.3.** [25] The stability estimate (4.7) holds for any \( \chi \) satisfying (4.3) and (4.4). But (4.7) does not lead in general to the asymptotic stability \( \lim_{t \to +\infty} E(t) = 0 \) (like in case of the choice \( \chi \) = \( \varepsilon G_5 \)) indicated in Remark 4.1, where (4.7) becomes just an upper bound estimate for \( E. \) The idea is to choose \( \chi \) satisfying (4.3) and (4.4) such that (4.7) gives the best possible decay rate for \( E. \) This choice can be done by taking \( \chi \) satisfying (4.3) and (4.4) such that the decay rate of the function in the right hand side of (4.4) has the closet decay rate to the one of the function in the left hand side of (4.4). So such choice of \( \chi \) can be seen from each specific considered functions \( g \) and \( \psi_{0\pi} \) (see the particular example considered below).
Proof. Now we start proof our main Theorem 4.1.
We combine (3.1), (3.20), (3.34), to have, for some \( m > 0 \) and for any \( t \geq 0 \), we have
\[
L'(t) \leq -mE(t) + \frac{c}{q(t)}H^{-1}\left(\frac{q(t]\mu(t)}{\xi(t)}\right) + ch_0(t). \tag{4.8}
\]
Without loss of generality, one can assume that \( E(0) > 0 \). For \( \varepsilon_0 < r \), let the functional \( \mathcal{F} \) defined by
\[
\mathcal{F}(t) := H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)L(t),
\]
which satisfies \( \mathcal{F} \sim E \). By noting that \( H'' \geq 0 \), \( q' \leq 0 \) and \( E' \leq 0 \), we get
\[
\mathcal{F}'(t) = \varepsilon_0\frac{(qE)'(t)}{E(0)}H''\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)L(t) + H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)L'(t)
\leq -mE(t)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{c}{q(t)}H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)H^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) + ch_0(t)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right). \tag{4.9}
\]
Let \( H^* \) be the convex conjugate of \( H \) in the sense of Young (see [20]), then
\[
H^*(s) = s(H')^{-1}(s) - H\left[(H')^{-1}(s)\right], \quad \text{if } s \in (0, H'(r)] \tag{4.10}
\]
and satisfies the following generalized Young inequality
\[
AB \leq H^*(A) + H(B), \quad \text{if } A \in (0, H'(r)], \quad B \in (0, r]. \tag{4.11}
\]
So, with \( A = H'\left(\varepsilon_0\frac{E(t)\mu(t)}{E(0)}\right) \) and \( B = H^{-1}\left(\frac{\mu(t)q(t)}{\xi(t)}\right) \) and using (4.10)-(4.11), we arrive at
\[
\mathcal{F}'(t) \leq -mE(t)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{c}{q(t)}H^*\left(H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right) + c\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + c\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + c\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) \tag{4.12}
\]
So, multiplying (4.12) by \( \xi(t) \) and using (3.35) and the fact that \( \varepsilon_0\frac{E(t)q(t)}{E(0)} < r \) gives
\[
\xi(t)\mathcal{F}'(t) \leq -m\xi(t)E(t)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + c\xi(t)\varepsilon_0E(t)E(0)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + cm(t)q(t) + c\xi(t)h_0(t)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) \leq -\varepsilon_0\frac{mE(0)}{\xi_0} - c\xi(t)E(t)E(0)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) - cE'(t) + c\xi(t)h_0(t)H'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right). \tag{4.13}
\]
Consequently, recalling the definition of $G_2$ and choosing $\varepsilon_0$ small enough so that $k = \left(\frac{mE(0)}{\varepsilon_0} - c\right) > 0$, we obtain, for all $t \in \mathbb{R}_+$,

\[
\mathcal{F}_1(t) \leq -k' \xi(t) \left(\frac{E(t)}{E(0)}\right) H' \left(\frac{\varepsilon_0 E(t)q(t)}{E(0)}\right) + c' \xi(t) h_0(t) H' \left(\frac{\varepsilon_0 E(t)q(t)}{E(0)}\right) \leq -k' \frac{\xi(t)}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + c' \xi(t) h_0(t) H' \left(\frac{\varepsilon_0 E(t)q(t)}{E(0)}\right),
\]

where $\mathcal{F}_1 = \xi \mathcal{F} + cE \sim E$ and satisfies for some $\alpha_1, \alpha_2 > 0$.

\[
\alpha_1 \mathcal{F}_1(t) \leq E(t) \leq \alpha_2 \mathcal{F}_1(t).
\]

Since $G'_2(t) = H'(t) + tH''(t)$, then, using the strict convexity of $H$ on $(0, r]$, we find that $G''_2(t), G_2(t) > 0$ on $(0, r]$. Let $d > 0$, use the general Young inequality (4.11) on the last term in (4.13) with $A = H' \left(\frac{E(0)\varepsilon_0}{E(0)}\right)$ and $B = [\frac{\xi}{\varepsilon_0} h_0(t)]$, to get

\[
ch_0(t) H' \left(\frac{\varepsilon_0 E(t)q(t)}{E(0)}\right) = \frac{d}{q(t)} \left[\frac{c}{d} q(t)h_0(t)\right] \left(\frac{\varepsilon_0 E(t)q(t)}{E(0)}\right) \leq \frac{d}{q(t)} G_3 \left(\frac{\varepsilon_0 E(t)q(t)}{E(0)}\right) \left(\frac{\varepsilon_0 E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_4 \left[\frac{c}{d} q(t)h_0(t)\right]
\]

\[
\leq \frac{d}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_4 \left[\frac{c}{d} q(t)h_0(t)\right] = \frac{d}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_4 \left[\frac{c}{d} q(t)h_0(t)\right].
\]

Now, combining (4.13) and (4.15) and choosing $d$ small enough so that $k_1 = (k - d) > 0$, we arrive at

\[
\mathcal{F}_1(t) \leq -k' \frac{\xi(t)}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_4 \left[\frac{c}{d} q(t)h_0(t)\right]
\]

\[
\leq -k_1 \frac{\xi(t)}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_4 \left[\frac{c}{d} q(t)h_0(t)\right].
\]

Using the equivalent property in (4.14) and since $G_2$ is increasing, we have, for some $d_0 = \frac{a_1}{E(0)} > 0$,

\[
G_2 \left(\frac{E(t)q(t)}{E(0)}\right) \geq G_2 \left(d_0 \mathcal{F}_1(t)q(t)\right).
\]

Letting $\mathcal{F}_2(t) := d_0 \mathcal{F}_1(t)q(t)$ and recalling that $q' \leq 0$, then we obtain for some constant $c_1 = d_0 k_1 > 0$ and $c_2 = d_0 d > 0$,

\[
\mathcal{F}_2(t) \leq -c_1 \xi(t) G_2(\mathcal{F}_2(t)) + c_2 \xi(t) G_4 \left[\frac{c}{d} q(t)h_0(t)\right].
\]

Since $d_0 q(t)$ is non-increasing, then use of the equivalent property $\mathcal{F}_1 \sim E$ implies that there exists $d_0 > 0$ such that $\mathcal{F}_2(t) \geq d_0 E(t)q(t)$. Let $t \in \mathbb{R}_+$ and $\chi(t)$ satisfy (4.13) and (4.14). If

\[
b_0 q(t) E(t) \leq \frac{2 G_5(t)}{\chi(t)},
\]

then...
then, we have
\[ E(t) \leq \frac{2}{b_0} \frac{G_5(t)}{\chi(t) q(t)}. \]  
(4.19)

If
\[ b_0 q(t) E(t) > \frac{2 G_5(t)}{\chi(t)}, \]  
(4.20)

then, for any \( 0 \leq s \leq t \), we get
\[ b_0 q(s) E(s) > \frac{2 G_5(t)}{\chi(t)}. \]  
(4.21)

since, \( q(t) E(t) \) is nonincreasing function. Therefore, we have for any \( 0 \leq s \leq t \),
\[ F_2(s) > \frac{2 G_5(t)}{\chi(t)}. \]  
(4.22)

Using \([4.22]\), the definition of \( G_2 \), the fact that \( G_2 \) is convex and \( 0 < \chi \leq 1 \), we have, for any \( 0 \leq s \leq t \) and \( 0 < \epsilon_1 \leq 1 \),
\[ G_2 \left( \epsilon_1 \chi(s) F_2(s) - \epsilon_1 G_5(s) \right) \leq \epsilon_1 \chi(s) G_2 \left( F_2(s) - \frac{G_5(s)}{\chi(s)} \right) \]
\[ \leq \epsilon_1 \chi(s) F_2(s) H' \left( \frac{F_2(s)}{\chi(s)} \right) - \epsilon_1 \chi(s) G_5(s) H' \left( \frac{F_2(s)}{\chi(s)} \right) \]
\[ \leq \epsilon_1 \chi(s) F_2(s) H' \left( \frac{F_2(s)}{\chi(s)} \right) - \epsilon_1 \chi(s) G_5(s) H' \left( \frac{F_2(s)}{\chi(s)} \right). \]  
(4.23)

Now, we let
\[ F_3(t) = \epsilon_1 \chi(t) F_2(t) - \epsilon_1 G_5(t), \]  
(4.24)

where \( \epsilon_1 \) is small enough so that \( F_3(0) \leq 1 \). Then \([4.23]\) becomes, for any \( 0 \leq s \leq t \),
\[ G_2 \left( F_3(s) \right) \leq \epsilon_1 \chi(s) G_2 \left( F_2(s) \right) - \epsilon_1 \chi(s) G_2 \left( \frac{G_5(s)}{\chi(s)} \right). \]  
(4.25)

Further, we have
\[ F'_3(t) = \epsilon_1 \chi'(t) F_2(t) + \epsilon_1 \chi(t) F'_2(t) - \epsilon_1 G'_5(t). \]  
(4.26)

Since \( \chi' \leq 0 \) and using \([4.17]\), then for any \( 0 \leq s \leq t \), \( 0 < \epsilon_1 \leq 1 \), we obtain
\[ F'_3(t) \leq \epsilon_1 \chi(t) F'_2(t) - \epsilon_1 G'_5(t) \]
\[ \leq -c_1 \epsilon_1 \xi(t) \chi(t) G_2 (F_2(t)) + c_2 \epsilon_1 \xi(t) \chi(t) G_4 \left[ \frac{c}{d} q(t) h_0(t) \right] - \epsilon_1 G'_5(t). \]  
(4.27)

Then, using \([4.18]\) and \([4.25]\), we get
\[ F'_3(t) \leq -c_1 \epsilon_1 \xi(t) G_2 (F_3(t)) + c_2 \epsilon_1 \xi(t) \chi(t) G_4 \left[ \frac{c}{d} q(t) h_0(t) \right] \]
\[ - c_1 \epsilon_1 \xi(t) \chi(t) G_2 \left( \frac{G_5(t)}{\chi(t)} \right) - \epsilon_1 G'_5(t). \]  
(4.28)

From the definition of \( G_1 \) and \( G_5 \), we have
\[ G_1 (G_5(s)) = c_1 \int_0^s \xi(\tau) d\tau, \]
hence,

\[ G'_5(s) = -c_1 \xi(s) G_2(G_5(s)). \]  

(4.29)

Now, we have

\[
\begin{align*}
&c_2 \epsilon_1 \xi(t) \chi(t) G_4 \left[ \frac{c}{d} q(t) h_0(t) \right] - c_1 \epsilon_1 \xi(t) \chi(t) G_2 \left( \frac{G_5(t)}{\chi(t)} \right) - \epsilon_1 G'_5(t) \\
&= c_2 \epsilon_1 \xi(t) \chi(t) G_4 \left[ \frac{c}{d} q(t) h_0(t) \right] - c_1 \epsilon_1 \xi(t) \chi(t) G_2 \left( \frac{G_5(t)}{\chi(t)} \right) + c_1 \epsilon_1 \xi(t) G_2(G_5(t)) \\
&= \epsilon_1 \xi(t) \chi(t) \left( c_2 G_4 \left[ \frac{c}{d} q(t) h_0(t) \right] - c_1 G_2 \left( \frac{G_5(t)}{\chi(t)} \right) + c_1 G_2(G_5(t)) \right).
\end{align*}
\]

(4.30)

Then, according to (4.4), we get

\[
\epsilon_1 \xi(t) \chi(t) \left( c_2 G_4 \left[ \frac{c}{d} q(t) h_0(t) \right] - c_1 G_2 \left( \frac{G_5(t)}{\chi(t)} \right) - c_1 G_2(G_5(t)) \right) \leq 0
\]

Then (4.28) gives

\[ F'_3(t) \leq -c_1 \xi(t) G_2(F_3(t)). \]  

(4.31)

Thus from (4.31) and the definition of \( G_1 \) and \( G_2 \) in (4.1) and (4.2), we obtain

\[
\left( G_1(F_3(t)) \right)' \geq c_1 \xi(t).
\]  

(4.32)

Integrating (4.32) over \([0, t]\), we get

\[
G_1(F_3(t)) \geq c_1 \int_0^t \xi(s) ds + G_1(F_3(0)).
\]  

(4.33)

Since \( G_1 \) is decreasing, \( F_3(0) \leq 1 \) and \( G_1(1) = 0 \), then

\[
F_3(t) \leq G_1^{-1} \left( c_1 \int_0^t \xi(s) ds \right) = G_5(t).
\]  

(4.34)

Recalling that \( F_3(t) = \epsilon_1 \chi(t) F_2(t) - \epsilon_1 G_5(t) \), we have

\[
F_2(t) \leq \frac{(1 + \epsilon_1) G_5(t)}{\epsilon_1} \chi(t).
\]  

(4.35)

Similarly, recall that \( F_2(t) := d_0 F_1(t) q(t) \), then

\[
F_1(t) \leq \frac{(1 + \epsilon_1) G_5(t)}{d_0 \epsilon_1} \chi(t) q(t).
\]  

(4.36)

Since \( F_1 \sim E \), then for some \( b > 0 \), we have \( E(t) \leq b F_1 \); which gives

\[
E(t) \leq \frac{b(1 + \epsilon_1) G_5(t)}{d_0 \epsilon_1} \chi(t) q(t).
\]  

(4.37)

From (4.19) and (4.37), we obtain the following estimate

\[
E(t) \leq c_3 \frac{G_5(t)}{\chi(t) q(t)},
\]  

(4.38)

where \( c_3 = \max \left\{ \frac{2}{b_0}, \frac{b(1 + \epsilon_1)}{d_0 \epsilon_1} \right\} \). This complete the proof of Theorem 4.1.  

\[ \square \]
Example 4.1 [10], [14]: Let \( g(t) = \frac{a}{(1+t)^r} \), where \( \nu > 1 \) and \( 0 < a < \nu - 1 \) so that (A1) is satisfied. In this case \( \xi(t) = \nu a \frac{1}{t^\nu} \) and \( G(t) = t^{\frac{-1}{\nu}} \). Then, there exist positive constants \( a_i (i = 0, \ldots, 3) \) depending only on \( a, \nu \) such that

\[
\begin{align*}
G_1(t) &= a_0 t^{\frac{-1}{\nu^2}}, \\
G_2(t) &= a_1 t^{\frac{1}{\nu^1}}, \\
G_3(t) &= a_2 (t^{\frac{1}{\nu^2}} - 1), \\
G_4(t) &= (a_4 t + 1)^{-\nu}, \\
G_5(t) &= a_3 t^{\nu+1}.
\end{align*}
\]

We will discuss two cases:

Case 1: if \( m_0 (1 + t)^r \leq 1 + ||p_{x0x}||^2 \leq m_1 (1 + t)^r \) \hspace{1cm} (4.40)

where \( 0 < r < \nu - 1 \) and \( m_0, m_1 > 0 \), then we have, for some positive constants \( a_i (i = 5, \ldots, 8) \) depending only on \( a, \nu, m_0, m_1, r \), the following:

\[
a_5 (1 + t)^{-\nu+1+r} \leq h_0(t) \leq a_6 (1 + t)^{-\nu+1+r},
\]

(4.41)

\[
\begin{align*}
\frac{q_0}{q(t)} &\geq a_7 \left\{ \begin{array}{ll}
1 + \ln(1 + t), & \nu - r = 2; \\
2, & \nu - r > 2; \\
(1 + t)^{-\nu+r+2}, & 1 < \nu - r < 2.
\end{array} \right. \\
\frac{q_0}{q(t)} &\leq a_8 \left\{ \begin{array}{ll}
1 + \ln(1 + t), & \nu - r = 2; \\
2, & \nu - r > 2; \\
(1 + t)^{-\nu+r+2}, & 1 < \nu - r < 2.
\end{array} \right.
\end{align*}
\]

(4.42) \hspace{1cm} (4.43)

We notice that condition (4.4) is satisfied if

\[
(t + 1)^\nu q(t) h_0(t) \chi(t) \leq a_9 \left( 1 - \left( \chi \right)^{\frac{1}{\nu}} \right)^{\frac{-1}{\nu} t^{\frac{1}{\nu}}},
\]

(4.44)

where \( a_9 > 0 \) depending on \( a, \nu, c_1 \) and \( c_2 \). Choosing \( \chi(t) \) as the following

\[
\chi(t) = \lambda \left\{ \begin{array}{ll}
(1 + t)^{p-r}, & p = r + 1, \nu - r \geq 2; \\
(1 + t)^{-p}, & p = \nu - 1, \nu - r < 2.
\end{array} \right.
\]

(4.45)

so that (4.3) is valid. Moreover, using (4.41) and (4.42), we see that (4.4) is satisfied if \( 0 < \lambda \leq 1 \) is small enough, and then (4.3) is satisfied. Hence (4.7) and (4.43) imply that, for any \( t \in \mathbb{R}_+ \)

\[
E(t) \leq a_{10} \left\{ \begin{array}{ll}
1 + \ln(1 + t), (1 + t)^{-(\nu-r-1)}, & \nu - r = 2; \\
(1 + t)^{-(\nu-r-1)}, & \nu - r > 2; \\
(1 + t)^{-(\nu-r-1)}, & 1 < \nu - r < 2.
\end{array} \right.
\]

(4.46)

Thus, the estimate (4.46) gives \( \lim_{t \to +\infty} E(t) = 0 \).

Case 2: if \( m_0 \leq 1 + ||p_{x0x}||^2 \leq m_1 \). That is \( r = 0 \) in (4.46).

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References

[1] Timoshenko, S.P., 1921. LXVI. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 41(245), pp.744-746.

[2] Timoshenko, S.P., 1922. X. On the transverse vibrations of bars of uniform cross-section. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 43(253), pp.125-131.

[3] Giorgi, C., Rivera, J.E.M. and Pata, V., 2001. Global attractors for a semilinear hyperbolic equation in viscoelasticity. Journal of Mathematical Analysis and Applications, 260(1), pp.83-99.

[4] Conti, M. and Pata, V., 2005. Weakly dissipative semilinear equations of viscoelasticity. Communications on Pure & Applied Analysis, 4(4), p.705.

[5] Appleby, J.A., Fabrizio, M., Lazzari, B. and Reynolds, D.W., 2006. On exponential asymptotic stability in linear viscoelasticity. Mathematical Models and Methods in Applied Sciences, 16(10), pp.1677-1694.

[6] Pata, V., 2009. Stability and exponential stability in linear viscoelasticity. Milan journal of mathematics, 77(1), pp.333-360.

[7] Guesmia, A., 2011. Asymptotic stability of abstract dissipative systems with infinite memory. Journal of mathematical analysis and applications, 382(2), pp.748-760.

[8] Guesmia, A. and Messaoudi, S.A., 2012. A general decay result for a viscoelastic equation in the presence of past and finite history memories. Nonlinear Analysis: Real World Applications, 13(1), pp.476-485.

[9] Al-Mahdi, A.M., 2020. Stability result of a viscoelastic plate equation with past history and a logarithmic nonlinearity. Boundary Value Problems, 2020(1), pp.1-20.

[10] Al-Mahdi, A.M., 2020. General stability result for a viscoelastic plate equation with past history and general kernel. Journal of Mathematical Analysis and Applications, 490(1), p.124216.

[11] Fernández Sare, H.D. and Munoz Rivera, J.E., 2008. Stability of Timoshenko systems with past history. J. Math. Anal. Appl, 339(1), pp.482-502.

[12] Guesmia, A., Messaoudi, S.A. and Soufyane, A., 2012. Stabilization of a linear Timoshenko system with infinite history and applications to the Timoshenko-heat systems. Electronic Journal of Differential Equations, (193).

[13] Guesmia, A. and Messaoudi, S.A., 2014. A general stability result in a Timoshenko system with infinite memory: a new approach. Mathematical Methods in the Applied Sciences, 3(37), pp.384-392.

[14] Guesmia, A., 2020. New general decay rates of solutions for two viscoelastic wave equations with infinite memory. Mathematical Modelling and Analysis, 25(3), pp.351-373.
[15] Mustafa, M.I., 2018. Optimal decay rates for the viscoelastic wave equation. Mathematical Methods in the Applied Sciences, 41(1), pp.192-204.

[16] Guesmia, A. and Tatar, N.E., 2015. Some well-posedness and stability results for abstract hyperbolic equations with infinite memory and distributed time delay. Communications on Pure and Applied Analysis, 14(2), pp.457-491.

[17] Guesmia, A. and Messaoudi, S.A., 2014. A new approach to the stability of an abstract system in the presence of infinite history. Journal of Mathematical Analysis and Applications, 416(1), pp.212-228.

[18] Youkana, A., 2018. Stability of an abstract system with infinite history. arXiv preprint arXiv:1805.07964.

[19] Al-Gharabli, M.M., 2019. New general decay results for a viscoelastic plate equation with a logarithmic nonlinearity. Boundary Value Problems, 2019(1), pp.1-21.

[20] Al-Mahdi, A.M. and Al-Gharabli, M.M., 2019. New general decay results in an infinite memory viscoelastic problem with nonlinear damping. Boundary Value Problems, 2019(1), pp.1-15.

[21] Alves, M.O., Gomes Tavares, E.H., Jorge Silva, M.A. and Rodrigues, J.H., 2019. On modeling and uniform stability of a partially dissipative viscoelastic Timoshenko system. SIAM Journal on Mathematical Analysis, 51(6), pp.4520-4543.

[22] Guesmia, A., 2015. Asymptotic behavior for coupled abstract evolution equations with one infinite memory. Applicable Analysis, 94(1), pp.184-217.

[23] Lions, J.L., 1969. Quelques méthodes de résolution de problèmes aux limites non linéaires.

[24] Jin, K.P., Liang, J. and Xiao, T.J., 2014. Coupled second order evolution equations with fading memory: Optimal energy decay rate. Journal of Differential Equations, 257(5), pp.1501-1528.

[25] Al-Mahdi, A.M., Al-Gharabli, M.M., Guesmia, A. and Messaoudi, S.A., 2021. New decay results for a viscoelastic-type Timoshenko system with infinite memory. Zeitschrift für angewandte Mathematik und Physik, 72(1), pp.1-24.

[26] Arnol’d, V.I., 2013. Mathematical methods of classical mechanics (Vol. 60). Springer Science & Business Media.