Exponent Function for One Helper Source Coding Problem at Rates outside the Rate Region

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Abstract—We consider the one helper source coding problem posed and investigated by Ahlswede, Körner and Wyner. Two correlated sources are separately encoded and are sent to a destination where the decoder wishes to decode one of the two sources with an arbitrary small error probability of decoding. In this system, the error probability of decoding goes to one as the source block length \( n \) goes to infinity. This implies that we have a strong converse theorem for the one helper source coding problem. In this paper we provide the much stronger version of this strong converse theorem for the one helper source coding problem. We prove that the error probability of decoding tends to one exponentially and derive an explicit lower bound of this exponent function.

Index Terms—One helper source coding problem, strong converse theorem, exponent of correct probability of decoding

I. INTRODUCTION

We consider the one helper source coding problem posed and investigated by Ahlswede, Körner and Wyner. Two correlated sources are separately encoded and are sent to a destination where the decoder wishes to decode one of the two sources with an arbitrary small error probability of decoding. In this system, the error probability of decoding goes to one as the source block length \( n \) goes to infinity. This implies that we have a strong converse theorem for the one helper source coding problem. In this paper we provide the much stronger version of this strong converse theorem for the one helper source coding problem. We prove that the error probability of decoding tends to one exponentially and derive an explicit lower bound of this exponent function.

II. PROBLEM FORMULATION

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be finite sets and \( \{(X_t, Y_t)\}_{t=1}^\infty \) be a stationary discrete memoryless source. For each \( t = 1, 2, \ldots \), the random pair \( (X_t, Y_t) \) takes values in \( \mathcal{X} \times \mathcal{Y} \), and has a probability distribution

\[
p_{XY} = \{p_{XY}(x, y)\}_{(x,y) \in \mathcal{X} \times \mathcal{Y}}
\]

We write \( n \) independent copies of \( \{X_t\}_{t=1}^\infty \) and \( \{Y_t\}_{t=1}^\infty \), respectively as

\[
X^n = X_1, X_2, \ldots, X_n \quad \text{and} \quad Y^n = Y_1, Y_2, \ldots, Y_n.
\]

We consider a communication system depicted in Fig. 1. Data sequences \( X^n \) and \( Y^n \) are separately encoded to \( \varphi^{(1)}_1(X^n) \) and \( \varphi^{(2)}_2(Y^n) \) and those are sent to the information processing center. At the center the decoder function \( \psi^{(n)} \) observes \((\varphi^{(1)}_1(X^n), \varphi^{(2)}_2(Y^n))\) to output the estimation \( \hat{Y}^n \) of \( Y^n \).

The encoder functions \( \varphi^{(1)}_1 \) and \( \varphi^{(2)}_2 \) are defined by

\[
\varphi^{(1)}_1 : X^n \to M_1 = \{1, 2, \ldots, M_1\},
\]

\[
\varphi^{(2)}_2 : Y^n \to M_2 = \{1, 2, \ldots, M_2\},
\]

where for each \( i = 1, 2, ||\varphi^{(i)}_i|| (= M_i) \) stands for the range of cardinality of \( \varphi^{(i)}_i \). The decoder function \( \psi^{(n)} \) is defined by

\[
\psi^{(n)} : M_1 \times M_2 \to Y^n.
\]

The error probability of decoding is

\[
P_e^{(n)}(\varphi^{(1)}_1, \varphi^{(2)}_2, \psi^{(n)}) = \Pr\{\hat{Y}^n \neq Y^n\},
\]

where \( \hat{Y}^n = \psi^{(n)}((\varphi^{(1)}_1(X^n), \varphi^{(2)}_2(Y^n))) \). A rate pair \((R_1, R_2)\) is \( \varepsilon \)-achievable if for any \( \delta > 0 \), there exist a positive integer \( n_0 = n_0(\varepsilon, \delta) \) and a sequence \( \{(\varphi^{(1)}_1, \varphi^{(2)}_2, \psi^{(n)})\}_{n=n_0}^{\infty} \) such that for \( n \geq n_0 \),

\[
\frac{1}{n} \log ||\varphi^{(i)}_i|| \leq R_i + \delta \quad \text{for} \quad i = 1, 2,
\]

\[
P_e^{(n)}(\varphi^{(1)}_1, \varphi^{(2)}_2, \psi^{(n)}) \leq \varepsilon.
\]

For \( \varepsilon \in (0, 1) \), the rate region \( \mathcal{R}_{AKW}(\varepsilon|p_{XY}) \) is defined by

\[
\mathcal{R}_{AKW}(\varepsilon|p_{XY}) \triangleq \{(R_1, R_2) : (R_1, R_2) \text{ is } \varepsilon\text{-achievable for } p_{XY}\}.
\]

Furthermore, define

\[
\mathcal{R}_{AKW}(p_{XY}) \triangleq \bigcap_{\varepsilon \in (0,1)} \mathcal{R}_{AKW}(\varepsilon|p_{XY}).
\]

We can show that the two rate regions \( \mathcal{R}_{AKW}(\varepsilon|p_{XY}), \varepsilon \in (0, 1) \) and \( \mathcal{R}_{AKW}(p_{XY}) \) satisfy the following property.

Property 1:

a) The regions \( \mathcal{R}_{AKW}(\varepsilon|p_{XY}), \varepsilon \in (0, 1), \) and \( \mathcal{R}_{AKW}(p_{XY}) \) are closed convex sets of \( \mathbb{R}^2_+ \), where

\[
\mathbb{R}^2_+ \triangleq \{(R_1, R_2) : R_1 \geq 0, R_2 \geq 0\}.
\]
b) \( \mathcal{R}_{\text{AKW}}(\varepsilon|p_{XY}) \) has another form using \((n, \varepsilon)\)-rate region \( \mathcal{R}_{\text{AKW}}(n, \varepsilon|p_{XY}) \), the definition of which is as follows. We set
\[
\mathcal{R}_{\text{AKW}}(n, \varepsilon|p_{XY}) = \{(R_1, R_2) : \text{There exists } (\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}) \text{ such that } \frac{1}{n} \log ||\varphi_i^{(n)}|| \leq R_i, i = 1, 2, \]
\[
P_e^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}) \leq \varepsilon \}.
\]
Using \( \mathcal{R}_{\text{AKW}}(n, \varepsilon|p_{XY}) \), \( \mathcal{R}_{\text{AKW}}(\varepsilon|p_{XY}) \) can be expressed as
\[
\mathcal{R}_{\text{AKW}}(\varepsilon|p_{XY}) = cl \left( \bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}_{\text{AKW}}(n, \varepsilon|p_{XY}) \right).
\]

Proof of this property is given in Appendix A. It is well known that \( \mathcal{R}_{\text{AKW}}(p_{XY}) \) was determined by Ahlswede, Körner and Wyner. To describe their result we introduce an auxiliary random variable \( K \).

Theorem 2 (Ahlswede et al. [3]): For each fixed \( \varepsilon \in (0, 1) \), we have
\[
\mathcal{R}_{\text{AKW}}(\varepsilon|p_{XY}) = \mathcal{R}(p_{XY}).
\]

Gu and Effros [4] examined a speed of convergence for \( P_e^{(n)} \) to tend to 1 as \( n \to \infty \) by carefully checking the proof of Ahlswede et al. [3]. However they could not obtain a result on an explicit form of the exponent function with respect to the code length \( n \).

Our aim is to find an explicit form of the exponent function for the error probability of decoding to tend to one as \( n \to \infty \) when \((R_1, R_2) \notin \mathcal{R}_{\text{AKW}}(p_{XY}) \). To examine this quantity, we define the following quantity. Set
\[
P_{e}^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}) \triangleq 1 - P_e^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}),
\]
\[
G^{(n)}(R_1, R_2|p_{XY}) \triangleq \frac{1}{n} \log P_e^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}).
\]

By time sharing we have that
\[
G^{(n+m)}(R_1, R_2|p_{XY}) \leq nG^{(n)}(R_1, R_2|p_{XY}) + mG^{(m)}(R_1', R_2'|p_{XY}).
\]

Choosing \( R = R' \) in (4), we obtain the following subadditivity property on \( \{G^{(n)}(R_1, R_2|p_{XY}) \}_{n \geq 1} \):
\[
g^{(n+m)}(R_1, R_2|p_{XY}) \leq nG^{(n)}(R_1, R_2|p_{XY}) + mG^{(m)}(R_1, R_2|p_{XY}),
\]

from which we have that \( G^{(n)}(R_1, R_2|p_{XY}) \) exists and satisfies the following:
\[
limit_{n \to \infty} G^{(n)}(R_1, R_2|p_{XY}) = \inf_{n \geq 1} G^{(n)}(R_1, R_2|p_{XY}).
\]

The exponent function \( G(R_1, R_2|p_{XY}) \) is a convex function of \((R_1, R_2)\). In fact, from [4], we have that for any \( \alpha \in [0, 1] \)
\[
G(\alpha R_1 + \bar{\alpha} R_1', \alpha R_2 + \bar{\alpha} R_2'|p_{XY}) \leq \alpha G(R_1, R_2|p_{XY}) + \bar{\alpha} G(R_1', R_2'|p_{XY}).
\]

The region \( \mathcal{G}(p_{XY}) \) is also a closed convex set. Our main aim is to find an explicit characterization of \( \mathcal{G}(p_{XY}) \). In this paper we derive an explicit outer bound of \( \mathcal{G}(p_{XY}) \) whose section by the plane \( G = 0 \) coincides with \( \mathcal{R}_{\text{AKW}}(p_{XY}) \).

III. MAIN RESULT

In this section we state our main result. We first explain that the region \( \mathcal{R}(p_{XY}) \) has two other expressions using the supporting hyperplane. We define two sets of probability distributions on \( U \times \mathcal{X} \times \mathcal{Y} \) by
\[
\mathcal{P}_{\text{a}(p_{XY})} \triangleq \{p = p_{UXY} : |U| \leq |\mathcal{X}|, U \leftrightarrow X \leftrightarrow Y\},
\]
\[
\mathcal{Q}(p_{Y|X}) \triangleq \{q = q_{UXY} : |U| \leq |\mathcal{X}|, U \leftrightarrow X \leftrightarrow Y, q_{Y|X} = q_{Y|X}\}.
\]
For \((\alpha, \mu) \in (0, 1] \times [0, 1]\), set
\[
R^{(\mu)}(p_{XY}) \triangleq \max_{p \in \mathcal{P}_{\alpha}^{sh}(p_{XY})} \{\bar{\mu} I_p(X; U) + \mu H_p(Y|U)\},
\]
\[
\tilde{R}^{(\alpha, \mu)}(p_{XY}) \triangleq \min_{q \in \mathcal{Q}(p_{Y|X})} \left\{((1 + \mu)\bar{\alpha} D(q_X||p_X) + \alpha \bar{\mu} I_q(X; U) + \mu H_q(Y|U))\right\},
\]
\[
\mathcal{R}_{sh}(p_{XY}) \triangleq \bigcap_{\mu \in [0, 1]} \{R_1, R_2 : \bar{\mu} R_1 + \mu R_2 \geq R^{(\mu)}(p_{XY})\},
\]
\[
\tilde{\mathcal{R}}_{sh}^{(\alpha)}(p_{XY}) \triangleq \bigcap_{\alpha \in (0, 1]} \tilde{R}^{(\alpha, \mu)}(p_{XY}) \bigcap_{\mu \in [0, 1]} \{R_1, R_2 : \bar{\mu} R_1 + \mu R_2 \geq \frac{1}{\alpha} \tilde{R}^{(\alpha, \mu)}(p_{XY})\}.
\]

For \(R \subseteq \mathbb{R}^2_+\), we set
\[
\mathcal{R} - \kappa(1, 1) \triangleq \{(a - \kappa, b - \kappa) \in \mathbb{R}^2_+ : (a, b) \in \mathcal{R}\}.
\]

Then we have the following property.

**Property 2:**

a) The bound \(|U| \leq |X|\) is sufficient to describe \(R^{(\mu)}(p_{XY})\) and \(\tilde{R}^{(\alpha, \mu)}(p_{XY})\).

b) For any \(p_{XY}\) we have
\[
\mathcal{R}_{sh}(p_{XY}) = \mathcal{R}(p_{XY}).
\]

Furthermore, for any \(\alpha \in (0, \alpha_0]\), we have
\[
R^{(\mu)}(p_{XY}) - c_1 \sqrt{\frac{\alpha}{\alpha}} \log \left(\frac{c_2}{\alpha}\right) \leq \frac{1}{\alpha} \tilde{R}^{(\alpha, \mu)}(p_{XY}) \leq R^{(\mu)}(p_{XY}),
\]
where
\[
\begin{align*}
\alpha_0 &= \alpha_0(|X|, |Y|) \triangleq \left\lfloor 8 \log(|X||Y|) + 1 \right\rfloor^{-1}, \\
c_1 &= c_1(|X|, |Y|) \triangleq \frac{3}{2} \sqrt{\log(|X||Y|)^2}, \\
c_2 &= c_2(|X|, |Y|) \triangleq \frac{\log(|X||Y|)^2}{\log(|X||Y|)^2}.
\end{align*}
\]
The two inequalities of (6) implies that for each \(\alpha \in (0, \alpha_0]\),
\[
\mathcal{R}_{sh}(p_{XY}) - c_1 \sqrt{\frac{\alpha}{\alpha}} \log \left(\frac{c_2}{\alpha}\right) (1, 1) \leq \tilde{\mathcal{R}}_{sh}^{(\alpha)}(p_{XY}) \subseteq \mathcal{R}_{sh}(p_{XY}).
\]
Hence we have
\[
\tilde{\mathcal{R}}_{sh}(p_{XY}) = \mathcal{R}_{sh}(p_{XY}).
\]

Property 2 part a) is also stated as Lemma 9 in Appendix 1. Proof of this lemma is given in this appendix. Proof of Property 2 part b) is given in Appendix C. For \((\alpha, \mu) \in [0, 1]^2\), define
\[
\omega^{(\alpha, \mu)}_{q_X||p_{X},q_{XY|U}}(x, y|u)
\]
\[
\triangleq (1 + \mu)\bar{\alpha} \log \frac{q_X(x)}{p_X(x)} + \alpha \bar{\mu} \log \frac{q_{X|U}(x|u)}{q_{X|U}(y|u)} + \mu \log |X| |Y|,
\]
\[
\Omega^{(\alpha, \mu)}(q_{p_{XY}}) \triangleq \exp \left\{-\lambda \omega^{(\alpha, \mu)}_{q_X||p_{X},q_{XY|U}}(x, y|u)\right\},
\]
\[
\Omega^{(\alpha, \mu)}(p_{XY}) \triangleq \min_{q \in \mathcal{Q}(p_{Y|X})} \Omega^{(\alpha, \mu)}(q_{p_{XY}}),
\]
\[
F^{(\alpha, \mu)}(\tilde{R}_1, \mu R_2) \triangleq \frac{\Omega^{(\alpha, \mu)}(p_{XY}) - \alpha \lambda (\bar{\mu} R_1 + \mu R_2)}{1 + \lambda(1 + \mu)},
\]
\[
\Omega^{(\alpha, \mu)}(p_{XY}) \triangleq \sup_{(\alpha, \mu) \in [0, 1]^2, \lambda > 0} F^{(\alpha, \mu)}(\tilde{R}_1, \mu R_2) p_{XY}.
\]

We can show that the above functions and sets satisfy the following property.

**Property 3:**

a) The cardinality bound \(|U| \leq |X|\) in \(\mathcal{Q}(p_{Y|X})\) is sufficient to describe the quantity \(\Omega^{(\alpha, \mu)}(p_{XY})\).

b) Define a probability distribution \(q^{(\lambda)} = q^{(\lambda)}_{U|XY}\) by
\[
q^{(\lambda)}(u, x, y) \triangleq q(u, x, y) \exp \left\{-\lambda \omega^{(\alpha, \mu)}_{q_X||p_{X},q_{XY|U}}(x, y|u)\right\},
\]
\[
eq \mathbb{E}_{q}(\exp \left\{-\lambda \omega^{(\alpha, \mu)}_{q_X||p_{X},q_{XY|U}}(X, Y|U)\right\}).
\]

Then we have
\[
\frac{d}{d\lambda} \Omega^{(\alpha, \mu)}(q_{p_{XY}}) = \mathbb{E}_{q^{(\lambda)}} \left[\omega^{(\alpha, \mu)}_{q_X||p_{X},q_{XY|U}}(X, Y|U)\right],
\]
\[
\frac{d^2}{d\lambda^2} \Omega^{(\alpha, \mu)}(q_{p_{XY}}) = -\text{Var}_{q^{(\lambda)}} \left[\omega^{(\alpha, \mu)}_{q_X||p_{X},q_{XY|U}}(X, Y|U)\right].
\]

The second equality implies that \(\Omega^{(\alpha, \mu)}(q_{p_{XY}})\) is a concave function of \(\lambda > 0\).
c) Define
\[ \rho = \rho(p_{XY}) \]
\[ \Delta = \max_{q \in \mathcal{Q}(p_{XY})} \text{Var}_q \left[ \omega^{(\alpha,\mu)}_{q_{XY}}(X, Y | U) \right]. \]

Since
\[ 0 \leq \left[ \omega^{(\alpha,\mu)}_{q_{XY}}(x, y | u) \right]^2 < +\infty \]
for \((u, x, y) \in U \times X \times Y\), we have \(\rho(p_{XY}) < \infty\). Then for any \(\lambda \in (0, 1]\), we have
\[ \Omega^{(\alpha,\mu)}(q | p_{XY}) \\geq \lambda \text{Var}_q \left[ \omega^{(\alpha,\mu)}_{q_{XY}}(X, Y | U) \right] - \frac{1}{2} \rho(p_{XY}) \lambda^2. \]
Specifically, we have
\[ \Omega^{(\alpha,\mu)}(p_{XY}) \geq \lambda \tilde{R}^{(\alpha,\mu)}(p_{XY}) - \frac{1}{2} \rho(p_{XY}) \lambda^2. \]

For any \(\delta > 0\), there exists a positive number \(\nu = \nu(\delta, |X|, |Y|) \in (0, 1]\) such that for every \(\tau \in (0, \nu]\), the condition \((R_1 + \tau, R_2 + \tau) \notin \mathcal{R}(p_{XY})\) implies
\[ F(R_1, R_2 | p_{XY}) \geq \frac{\rho(p_{XY})}{2} \cdot g^2 \left( \frac{R_1 + \tau}{2\rho(p_{XY})} \right) > 0, \]
where \(g\) is the inverse function of \(\tilde{\psi}(a) \triangleq a + a^2, a > 0\).

Property 3 part a follows from Lemma 10 in Appendix E. Proof of this lemma is given in this appendix. Proofs of Property 3 parts b, c, and d are given in Appendix E. Our main result is the following:

**Theorem 3:** For any \(R_1, R_2 \geq 0\), any \(p_{XY}\), and for any \((\varphi^{(n)}_1, \varphi^{(n)}_2, \psi^{(n)})\) satisfying
\[ \frac{1}{n} \log ||\varphi^{(n)}_i|| \leq R_i, i = 1, 2, \]
we have
\[ P_c(n)(\varphi^{(n)}_1, \varphi^{(n)}_2, \psi^{(n)}) \leq 5 \exp \left\{ -nF(R_1, R_2 | p_{XY}) \right\}. \tag{9} \]

It follows from Theorem 3 and Property 3 part d) that if \((R_1, R_2)\) is outside the capacity region, then the error probability of decoding goes to one exponentially and its exponent is not below \(F(R_1, R_2 | p_{XY})\). It immediately follows from Theorem 3 that we have the following corollary.

**Corollary 1:**
\[ G(R_1, R_2 | p_{XY}) \geq F(R_1, R_2 | p_{XY}), \]
\[ G(p_{XY}) \subseteq \mathcal{G}(p_{XY}) \]
\[ = \{ R_1, R_2, G : G \geq F(R_1, R_2 | p_{XY}) \}. \]

Proof of Theorem 3 will be given in the next section. The exponent function at rates outside the rate region was derived by Oohama and Han 7 for the separate source coding problem for correlated sources 6. The techniques used by them is a method of types 8, which is not useful to prove Theorem 3. Some novel techniques based on the information spectrum method introduced by Han 9 are necessary to prove this theorem.

From Theorem 3 and Property 3 part d), we have the following corollary, which provides an explicit outer bound of \(\mathcal{R}_{AKW}(\varepsilon | p_{XY})\) with an asymptotically vanishing deviation from \(\mathcal{R}_{AKW}(p_{XY}) = \mathcal{R}(p_{XY})\). The strong converse theorem established by Ahlswede et al. 3 immediately follows from this corollary. From Theorem 3 and Property 3 part d) we have the following corollary.

**Corollary 2:** For each fixed \(\varepsilon \in (0, 1]\) and for any \(\delta > 0\), there exists a positive integer \(n_0\) with
\[ n_0 = n_0(\varepsilon, \delta, |X|, |Y|, \rho(p_{XY})) \]
such that for \(n \geq n_0\), we have
\[ \mathcal{R}_{AKW}(\varepsilon | p_{XY}) \subseteq \mathcal{R}(p_{XY}) - \kappa_n(1, 1), \]
where
\[ \kappa_n = \left\{ \sqrt{\frac{8 \rho(p_{XY})}{n} \log \left( \frac{2 \pi e n}{\rho(p_{XY})} \right)} \right\}^{\frac{1}{n}}. \]

It immediately follows from the above result that for any \(\varepsilon \in (0, 1]\), we have
\[ \mathcal{R}_{AKW}(\varepsilon | p_{XY}) = \mathcal{R}_{AKW}(p_{XY}) = \mathcal{R}(p_{XY}). \]

Proof of this corollary will be given in the next section.

**IV. PROOF OF THE MAIN RESULT**

Let \((X^n, Y^n)\) be a pair of random variables from the information source. We set \(S = \varphi^{(n)}_1(X^n)\). Joint distribution \(p_{SX^nY^n}\) of \((S, X^n, Y^n)\) is given by
\[ p_{SX^nY^n}(s, x^n, y^n) = p_{S|X^n}(s|x^n) \prod_{i=1}^{n} p_{X_iY_i}(x_i, y_i). \]

It is obvious that \(S \leftrightarrow X^n \leftrightarrow Y^n\). Then we have the following.

**Lemma 1:** For any \(\eta > 0\) and for any \((\varphi^{(n)}_1, \varphi^{(n)}_2, \psi^{(n)})\) satisfying
\[ \frac{1}{n} \log ||\varphi^{(n)}_i|| \leq R_i, i = 1, 2, \]
we have
\[ P_c(n)(\varphi^{(n)}_1, \varphi^{(n)}_2, \psi^{(n)}) \leq p_{SX^nY^n} \left\{ \begin{array}{l}
0 \geq \frac{1}{n} \log \tilde{q}_{SX^nY^n}(S, X^n, Y^n) - \eta, \\
0 \geq \frac{1}{n} \log \tilde{q}_{X^n}(X^n) - \eta, \\
0 \geq \frac{1}{n} \log p_{X^n|S}(X^n | S) - \eta, \\
R_1 \geq \frac{1}{n} \log \frac{1}{p_{Y^n|S}(Y^n | S)} - \eta \end{array} \right\} + 4e^{-n\eta}. \tag{12} \]

The probability distribution appearing in the right members of (12) have a property that we can select them arbitrary. In (10), we can choose any probability distribution \(\tilde{q}_{SX^nY^n}\) on \(S \times X^n \times Y^n\). In (11), we can choose any distribution \(\tilde{q}_{X^n}\) on \(X^n\).

Proof of this lemma is given in Appendix E. From Lemma 1 we obtain the following lemma.
Lemma 2: For any \( \eta > 0 \) and for any \((\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)})\) satisfying
\[
\frac{1}{n} \log ||\varphi_i^{(n)}|| \leq R_i, \ i = 1, 2,
\]
we have
\[
P_c^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}) \leq p_{S^X Y^n}\left\{ 0 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{q_{X_t}(X_t)}{p_{X_t}(X_t)} - \eta, \right.
\]
\[
R_i \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{p_{X_t S|X^{t-1}}(X_t|S, X^{t-1})}{p_{X_t}(X_t)} - \eta,
\]
\[
R_2 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{1}{p_{Y_t S|X^{t-1} Y^{t-1}}(Y_t|S, X^{t-1}, Y^{t-1}) - 2\eta}
\]
\[+ 4e^{-n\eta},\]
completing the proof.

Lemma 3: Suppose that for each \( t = 1, 2, \cdots, n \), the joint distribution \( p_{S^X Y^n t} \) of the random vector \( S^X Y^n t \) is a marginal distribution of \( p_{S^X Y^n} \). Then we have the following Markov chain:
\[
S^X t^{-1} \leftrightarrow X_t \leftrightarrow Y_t \tag{13}
\]
or equivalently that \( I(Y_t; S^X t^{-1}|X_t) = 0 \). Furthermore, we have the following Markov chain:
\[
Y^{t-1} \leftrightarrow S^X t^{-1} \leftrightarrow (X_t, Y_t) \tag{14}
\]
or equivalently that \( I(X_t, Y_t; Y^{t-1}|S^X t^{-1}) = 0 \). The above two Markov chains are equivalent to the following one long Markov chain:
\[
Y^{t-1} \leftrightarrow S^X t^{-1} \leftrightarrow X_t \leftrightarrow Y_t. \tag{15}
\]
Proof of this lemma is given in Appendix B. For \( t = 1, 2, \cdots, n \), let \( U_t \) be a random variable taking values in \( \mathcal{A}_1 \times \mathcal{X} \). Define \( U_t \) by \( U_t \triangleq (S^X t^{-1}). \) From Lemmas 2 and B we have the following.

Lemma 4: For any \( \eta > 0 \) and for any \((\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)})\) satisfying
\[
\frac{1}{n} \log ||\varphi_i^{(n)}|| \leq R_i, i = 1, 2,
\]
we have
\[
P_c^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}) \\
\leq p_{S^X Y^n}\left\{ 0 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{q_{X_t}(X_t)}{p_{X_t}(X_t)} - \eta, \right.
\]
\[
R_i \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{p_{X_t U_t}(X_t|U_t)}{p_{X_t}(X_t)} - \eta,
\]
\[
R_2 \geq \frac{1}{n} \sum_{t=1}^{n} \log \frac{1}{p_{Y_t U_t}(Y_t|U_t) - 2\eta}
\]
\[+ 4e^{-n\eta},\]
we have
\[
P_c^{(n)}(\phi_1^{(n)}, \phi_2^{(n)}, \psi^{(n)})
\leq 5 \exp \left\{-n \left[ 1 + \lambda(1 + \mu) \right]^{-1} \times \left[ \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}) - \alpha \lambda (\bar{\mu}R_1 + \mu R_2) \right] \right\}.
\]

**Proof:** By Lemma \[2\] for \((\alpha, \mu) \in [0, 1]^2\) and \(\lambda > 0\), we have the following chain of inequalities:

\[
P_c^{(n)}(\phi_1^{(n)}, \phi_2^{(n)}, \psi^{(n)})
\leq p_{S|X} \left\{ 0 \geq (1 + \mu)\tilde{\alpha} \left[ \frac{1}{n} \sum_{i=1}^{n} \log \frac{p_{X_i|U_i}(X_i)}{p_{X_i}(X_i)} - \eta \right],
\alpha\bar{\mu}R_1 \geq \frac{1}{n} \sum_{i=1}^{n} \log \frac{1}{p_{X_i|U_i}(X_i|U_i)} - \alpha \tilde{\mu} \eta, \right.
\]
\[
\left. \alpha\lambda R_2 \geq \frac{1}{n} \sum_{i=1}^{n} \log \frac{1}{p_{X_i|U_i}(X_i|U_i)} - 2\alpha \mu \eta \right\} + 4e^{-n\eta}
\leq p_{S|X} \left\{ (\alpha(\bar{\mu}R_1 + \mu R_2) + (1 + \mu)\eta \right.
\geq - \frac{1}{n} \sum_{i=1}^{n} \log \left[ \frac{1}{p_{X_i|U_i}(X_i|U_i)} \right]
\left. \frac{1}{p_{X_i|U_i}(X_i|U_i)} \right]\right\}
\leq \frac{1}{n} \sum_{i=1}^{n} \log \left[ \frac{1 + (1 + \mu)\tilde{\alpha}}{p_{X_i}(X_i)} \right]
\left[ \frac{1 + (1 + \mu)\tilde{\alpha}}{p_{X_i}(X_i)} \right]
\left[ \frac{1}{p_{X_i|U_i}(X_i|U_i)} \right]
\geq - \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}) + 4e^{-n\eta}
\left( \alpha \right)
\leq \exp \left\{ n \left[ \alpha \lambda (\bar{\mu}R_1 + \mu R_2) + (1 + \mu)\eta \right] - \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}) \right\} + 4e^{-n\eta}.
\]

Step (a) follows from Lemma \[5\]. We choose \(\eta\) so that
\[
- \eta = \alpha \lambda (\bar{\mu}R_1 + \mu R_2) + (1 + \mu)\eta
- \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}).
\]

Solving \[17\] with respect to \(\eta\), we have
\[
\eta = \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}) - \alpha \lambda (\bar{\mu}R_1 + \mu R_2) + (1 + \mu)\eta.
\]

For this choice of \(\eta\) and \[16\], we have
\[
P_c^{(n)}(\phi_1^{(n)}, \phi_2^{(n)}, \psi^{(n)}) \leq 5e^{-n\eta}
= 5 \exp \left\{-n \left[ 1 + \lambda(1 + \mu) \right]^{-1} \times \left[ \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}) - \alpha \lambda (\bar{\mu}R_1 + \mu R_2) \right] \right\},
\]

completing the proof.

Set
\[
\Omega^{(\alpha, \mu, \lambda)}(p_{XY})
\triangleq \inf_{n \geq 1} \min_{S \in \mathcal{M}_1} \max_{q_X \in \mathcal{Q}_X} \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}).
\]

By Proposition \[1\] we have the following corollary.

**Corollary 3:** For any \((\alpha, \mu) \in [0, 1]^2\), any \(\lambda > 0\), and any \((\phi_1^{(n)}, \phi_2^{(n)}, \psi^{(n)})\) satisfying
\[
\frac{1}{n} \log ||\phi^{(n)}|| \leq R_i, i = 1, 2,
\]
we have
\[
P_r^{(n)}(\phi_1^{(n)}, \phi_2^{(n)}, \psi^{(n)}) \leq 5 \exp \left\{-n \left[ \frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(p_{XY}) - \alpha \lambda (\bar{\mu}R_1 + \mu R_2) \right] \right\}.
\]

We shall call \(\Omega^{(\alpha, \mu, \lambda)}(p_{XY})\) the communication potential. The above inequality implies that the analysis of \(\Omega^{(\alpha, \mu, \lambda)}(p_{XY})\) leads to an establishment of a strong converse theorem for the one helper source coding problem. In the following argument we set an explicit lower bound of \(\Omega^{(\alpha, \mu, \lambda)}(p_{XY})\).

For each \(t = 1, 2, \ldots, n\), set \(u_t = (s, x^{t-1}) \in U_t\). For each \(t = 1, 2, \ldots, n\), define
\[
f^{(\alpha, \lambda)}(x_t, y_t|u_t)
\triangleq \frac{q_{X_t}^{(\alpha, \mu, \lambda)}(x_t) p_{Y_t}^{(\alpha, \mu, \lambda)}(y_t|u_t)}{p_{X_t}^{(\alpha, \mu, \lambda)}(x_t)}.\]

By definition we have
\[
\exp \left\{-\Omega^{(\alpha, \mu, \lambda)}(q_X^n, S|p_{XY}) \right\}
= \sum_{s, x^n, y^n} p_{S|X} \left\{ f^{(\alpha, \mu, \lambda)}(x_t, y_t|u_t) \right\}
\prod_{i=1}^{n} f_{q_{X_i}} \left| p_{X_i|Y_i} \right| \left( x_i, y_i | u_i \right).
\]

For each \(t = 1, 2, \ldots, n\), we define the probability distribution
\[
p^{(\alpha, \mu, \lambda)}(p_{S|X_Y}) \triangleq \left\{ p^{(\alpha, \mu, \lambda)}(s, x^t, y^t) \right\}
\end{cases}
\]
by
\[
p^{(\alpha, \mu, \lambda)}(s, x^t, y^t)
\triangleq C_t^{-1} p_{S|X_Y}(s, x^t, y^t)
\times \prod_{i=1}^{t} f_{q_{X_i}} \left| p_{X_i|Y_i} \right| \left( x_i, y_i | u_i \right),
\]
where
\[
C_t \triangleq \sum_{s, x^t, y^t} p_{S|X_Y}(s, x^t, y^t)
\times \prod_{i=1}^{t} f_{q_{X_i}} \left| p_{X_i|Y_i} \right| \left( x_i, y_i | u_i \right)
\]
are constants for normalization. For \(t = 1, 2, \ldots, n\), define
\[
\Phi_t^{(\alpha, \mu, \lambda)} \triangleq C_t C_{t-1},
\]
where we define \(C_0 = 1\). Then we have the following lemma.
Lemma 6: For each $t = 1, 2, \cdots, n$, and for any $(s, x^t, y^t) \in \mathcal{M}_1 \times \mathcal{X}^t \times \mathcal{Y}^t$, we have
\[
P_{S|X,Y}^{(\alpha,\mu,\lambda)}(s, x^t, y^t) = (\Phi_t^{(\alpha,\mu,\lambda)})^{-1} p_{S|X,Y}^{(\alpha,\mu,\lambda)}(s, x^t-1, y^t-1) \times p_{X,Y}^{(\alpha,\mu,\lambda)}(x_t, y_t|s, x^{t-1}, y^{t-1}) \times f_{\Phi_t^{(\alpha,\mu,\lambda)}}(x_t, y_t|u_t).
\]
Furthermore, we have
\[
\Phi_t^{(\alpha,\mu,\lambda)} = \sum_{s, x^t, y^t} p_{S|X,Y}^{(\alpha,\mu,\lambda)}(s, x^t-1, y^t-1) \times p_{X,Y}^{(\alpha,\mu,\lambda)}(x_t, y_t|s, x^{t-1}, y^{t-1}) \times f_{\Phi_t^{(\alpha,\mu,\lambda)}}(x_t, y_t|u_t).
\]
Proof of this lemma is given in Appendix G. Define $(\alpha,\mu,\lambda) \in \mathcal{M} = \{\Phi\}$ and choose $(\alpha,\mu,\lambda) \in \mathcal{M}_1 \times \mathcal{X}^t \times \mathcal{Y}^t$.

From Lemmas 3 and 6, we have the following lemma. Lemma 7:
\[
\Phi_t^{(\alpha,\mu,\lambda)}(u_t) = \sum_{s, x^t, y^t} p_{X,Y|U}^{(\alpha,\mu,\lambda)}(u_t)p_{X,Y|U}(x_t|u_t)p_{X,Y|U}(y_t|u_t) \times f_{\Phi_t^{(\alpha,\mu,\lambda)}}(x_t, y_t|u_t).
\]
Proof: By Lemma 5 we have
\[
p_{X,Y|U}(x_t, y_t|u_t) = p_{X,Y|U}(x_t|u_t)p_{X,Y|U}(y_t|u_t) = p_{X,Y}(x_t|u_t)p_{X,Y}(y_t|u_t) = p_{X,Y}(x_t, y_t|u_t)
\]
for $(s, x^t, y^t) \in S \times \mathcal{X}^t \times \mathcal{Y}^t$ and for $t = 1, 2, \cdots, n$. Then by Lemma 6, we have
\[
\Phi_t^{(\alpha,\mu,\lambda)}(u_t) = \sum_{s, x^t, y^t} p_{X,Y}^{(\alpha,\mu,\lambda)}(s, x^t-1, y^t-1) \times p_{X,Y}^{(\alpha,\mu,\lambda)}(x_t|s, x^{t-1}, y^{t-1}) \times f_{\Phi_t^{(\alpha,\mu,\lambda)}}(x_t, y_t|u_t).
\]

Lemma 8:
\[
\Omega_t^{(\alpha,\mu,\lambda)}(x^n|s)p_{XY} = \sum_{t=1}^n \log \Phi_t^{(\alpha,\mu,\lambda)}.
\]
Proof: From (18) we have
\[
\log \Phi_t^{(\alpha,\mu,\lambda)} = -\log C_t + \log C_{t-1}.
\]
Furthermore, by definition we have
\[
\Omega_t^{(\alpha,\mu,\lambda)}(x^n|s)p_{XY} = -\log C_n, C_0 = 1.
\]
From (22) and (23), (21) is obvious.

The following proposition is a mathematical core to prove our main result.

Proposition 2: For any $\alpha \geq 1, \mu, \lambda > 0$, we have
\[
\Omega_t^{(\alpha,\mu,\lambda)}(p_X) \leq \Omega_t^{(\alpha,\mu,\lambda)}(p_X).
\]
Proof: Set
\[
Q_n(p_{Y|X}) \triangleq \{q = q_{U|X} : |U| \leq \mathcal{M}_1 ||x^n-1|||y^{n-1}|,
\]
\[
q_{Y|X} = p_{Y|X} U \leftrightarrow X \leftrightarrow Y
\]
\[
\Omega_n^{(\alpha,\mu,\lambda)}(p_X) \triangleq \min_{q \in \mathcal{Q}_n(p_{Y|X})} \Omega_t^{(\alpha,\mu,\lambda)}(q|p_{XY}).
\]
Let $U_t$ be random variables taking values in $\mathcal{M}_1 \times \mathcal{X}^t \times \mathcal{Y}^t$. We choose $qu_t$ so that
\[
q_{X,Y|U_t}(x_t, y_t|u_t) = p_{X,Y|U_t}(x_t, y_t|u_t)
\]
and choose $p_{X,Y|U_t} (x_t, y_t|u_t)$ so that
\[
q_{X,Y|U_t}(x_t, y_t|u_t) = p_{X,Y|U_t}(x_t, y_t|u_t).
\]
Furthermore we have
\[
q_{Y|X}(x, y|t, u) \overset{(a)}{=} p_{Y|X}(y|x) = p_{Y|X}(y|x_t). \tag{26}
\]
Step (a) follows from Lemma 3. The equation (26) imply that
\[
q_t = q_{U_t|X} \in \mathcal{Q}_n(p_{Y|X}).
\]
Hence we have the following chain of inequalities:
\[
\Phi_t^{(\alpha,\mu,\lambda)}(u_t) \overset{(a)}{=} \sum_{u_t, x_t, y_t} q_{U_t}(u_t)p_{X,Y|U_t}(x_t|u_t)p_{Y|X}(y_t|x_t) \times f_{\Phi_t^{(\alpha,\mu,\lambda)}}(x_t, y_t|u_t)
\]
\[
\overset{(b)}{=} \sum_{u_t, x_t, y_t} q_{U_t}(u_t)p_{X,Y|U_t}(x_t|u_t)p_{Y|X}(y_t|x_t) \times f_{\Phi_t^{(\alpha,\mu,\lambda)}}(x_t, y_t|u_t)
\]
\[
\overset{(c)}{=} \exp \left\{ \Omega_t^{(\alpha,\mu,\lambda)}(q|p_{XY}) \right\} \equiv \exp \left\{ \Omega_t^{(\alpha,\mu,\lambda)}(p_{XY}) \right\} \tag{27}
\]
Step (a) follows from Lemma 7. Step (b) follows from (25). Step (c) follows from $q_t \in \mathcal{Q}_n(p_{Y|X})$ and the definition of $\hat{\Omega}_n^{(\alpha,\mu,\lambda)}(p_{XY})$. Step (d) follows from Lemma 10 in Appendix F. To prove this lemma we bound the cardinality $|U|$ appearing in the definition of $\hat{\Omega}_n^{(\alpha,\mu,\lambda)}(p_{XY})$ to show that the bound $|U| \leq |\mathcal{X}|$ is sufficient to describe $\hat{\Omega}_n^{(\alpha,\mu,\lambda)}(p_{XY})$. Proof
of Lemma 10 is given in Appendix B. Hence we have the following:

\[
\frac{1}{n} \Omega^{(\alpha, \mu, \lambda)}(q_{X^n}, S|p_{XY}) \leq \frac{1}{n} \sum_{i=1}^{n} \log \Phi^{(\alpha, \mu, \lambda)}_i(n, \varepsilon) \geq 0 \]

Hence we have the following:

\[
\Omega^{(\alpha, \mu, \lambda)}(p_{XY}) \geq \Omega^{(\alpha, \mu, \lambda)}(p_{XY}).
\]

Thus, Proposition 2 is proved.

**Proof of Theorem 5** We have the following:

\[
\frac{1}{n} \log \left\{ \frac{5}{P_e^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)})} \right\} \geq F(R_1, R_2|p_{XY}).
\]

Thus (3) in Theorem 3 is proved.

**Proof of Corollary 2** To prove this corollary we use the following expression of \( R_{AKW}(\varepsilon|p_{XY}) \) stated in Property 4 part d):

\[
R_{AKW}(\varepsilon|p_{XY}) = \text{cl} \left( \bigcup_{m \geq 1} \bigcap_{n \geq m} R_{AKW}(n, \varepsilon|p_{XY}) \right). \]

We assume that

\[
(R_1, R_2) \in \bigcup_{m \geq 1} \bigcap_{n \geq m} R_{AKW}(n, \varepsilon|p_{XY}).
\]

Then there exists a positive integer \( m_0 = m_0(\varepsilon) \) and some \( \{ (\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)}) \}_{n \geq m_0} \) such that for \( n \geq m_0(\varepsilon) \), we have

\[
\frac{1}{n} \log ||\varphi^{(n)}|| \leq R_{i}, i = 1, 2
\]

Then by Theorem 5 we have

\[
1 - \varepsilon \leq \frac{1}{n} \log \left( \frac{5}{P_e^{(n)}(\varphi_1^{(n)}, \varphi_2^{(n)}, \psi^{(n)})} \right) \leq 5 \exp \{-nF(R_1, R_2|p_{XY})\}
\]

for any \( n \geq m_0(\varepsilon) \). Fix any \( \delta > 0 \). We take a positive number \( \nu = \nu(\delta, |X|, |Y|) \) \( \in (0, 1] \) appearing in Property 3 part d) and set

\[
\kappa_n = \left\{ \begin{array}{ll}
2\nu(p_{XY}) \delta \left( \sqrt{\frac{2}{\nu(p_{XY})}} \log \left( \frac{5}{1-\varepsilon} \right) \right) & \text{if } n \geq n_0 \,
\end{array} \right.
\]

Since \( g \) is an inverse function of \( \vartheta \), (32) is equivalent to

\[
g \left( \frac{\kappa_n^{1+\delta}}{2\nu(p_{XY})} \right) = \sqrt{\frac{2}{\nu(p_{XY})}} \log \left( \frac{5}{1-\varepsilon} \right).
\]

We take a sufficiently large positive integer \( m_1 \) so that we have \( \kappa_n < \nu \) for \( n \geq m_1 \). Set \( n_0 = \max\{m_0, m_1\} \). We claim that for \( n \geq n_0 \), we have \( (R_1 + \kappa_n, R_2 + \kappa_n) \in \mathcal{R}(p_{XY}) \). To prove this claim we suppose that \( (R_1 + \kappa_n, R_2 + \kappa_n) \) does not belong to \( \mathcal{R}(p_{XY}) \) for some \( n^* \geq n_0 \). Since \( \mathcal{R}(p_{XY}) \) is a closed set, there exists a positive number \( \tau > \kappa_n^* \) such that

\[
\kappa_n^* < \tau \leq \nu,
\]

\[
(R_1 + \tau, R_2 + \tau) \notin \mathcal{R}(p_{XY}).
\]

Then we have the following chain of inequalities:

\[
5 \exp \{-n^*F(R_1, R_2|p_{XY})\} \leq 5 \exp \left( -n^* \frac{n^* \nu(p_{XY})}{2} \right) \leq 5 \exp \left( -n^* \nu(p_{XY}) \right) \leq 5 \exp \left( -n^* \nu(p_{XY}) \right). \]

Step (a) follows from Property 5 part d). Step (b) follows from (32). The bound (34) contradicts (31). Hence we have \( (R_1 + \kappa_n, R_2 + \kappa_n) \in \mathcal{R}(p_{XY}) \).
Fig. 2. One helper source coding system investigated by Wyner.

V. ONE HELPER PROBLEM STUDIED BY WYNER

We consider a communication system depicted in Fig. 2. Data sequences \(X^n, Y^n\), and \(Z^n\), respectively, are separately encoded to \(\varphi_1^{(n)}(X^n), \varphi_2^{(n)}(Y^n)\), and \(\varphi_3^{(n)}(Z^n)\). The encoded data \(\varphi_1^{(n)}(X^n)\) and \(\varphi_2^{(n)}(Y^n)\) are sent to the information processing center 1. The encoded data \(\varphi_1^{(n)}(X^n)\) and \(\varphi_2^{(n)}(Y^n)\) are sent to the information processing center 2. At the center 1 the decoder function \(\psi(n)\) observes \((\varphi_1^{(n)}(X^n), \varphi_2^{(n)}(Y^n))\) and \(\varphi_3^{(n)}(Z^n)\) to output the estimation \(\hat{Y}^n\) of \(Y^n\). At the center 2 the decoder function \(\phi(n)\) observes \((\varphi_1^{(n)}(X^n), \varphi_3^{(n)}(Z^n))\) to output the estimation \(\hat{Z}^n\) of \(Z^n\). The error probability of decoding is

\[
P_e(n, \varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)}, \psi(n), \phi(n)) = \Pr\{\hat{Y}^n \neq Y^n \text{ or } \hat{Z}^n \neq Z^n\},
\]

where \(\hat{Y}^n = \psi(n)(\varphi_1^{(n)}(X^n), \varphi_2^{(n)}(Y^n))\) and \(\hat{Z}^n = \phi(n)(\varphi_1^{(n)}(X^n), \varphi_3^{(n)}(Z^n))\).

A rate triple \((R_1, R_2, R_3)\) is \(\varepsilon\)-achievable if for any \(\delta > 0\), there exist a positive integer \(n_0 = n_0(\varepsilon, \delta)\) and a sequence of three encoders and two decoder functions \\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)}, \psi(n), \phi(n))\}\) such that for \(n \geq n_0\),

\[
\frac{1}{n}\|\varphi_i^{(n)}\| \leq R_i + \delta \quad \text{for } i = 1, 2, 3,
\]

\[
P_e(n, \varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)}, \psi(n), \phi(n)) \leq \varepsilon.
\]

The rate region \(\mathcal{R}(\varepsilon|p_{XYZ})\) is defined by

\[
\mathcal{R}(\varepsilon|p_{XYZ}) \triangleq \{ (R_1, R_2, R_3) : \text{there exists a rate triple } (R_1, R_2, R_3) \text{ is } \varepsilon\text{-achievable for } p_{XYZ} \}.
\]

Furthermore, define

\[
\mathcal{R}(p_{XYZ}) \triangleq \bigcap_{\varepsilon \in (0, 1)} \mathcal{R}(\varepsilon|p_{XYZ}).
\]

We can show that the two rate regions \(\mathcal{R}(\varepsilon|p_{XYZ}), \varepsilon \in (0, 1)\) and \(\mathcal{R}(p_{XYZ})\) satisfy the following property.

**Property 4:**

a) The regions \(\mathcal{R}(\varepsilon|p_{XYZ}), \varepsilon \in (0, 1)\), and \(\mathcal{R}(p_{XYZ})\) are closed convex sets of \(\mathbb{R}^3_+\).

b) We set

\[
\mathcal{R}(n, \varepsilon|p_{XYZ}) = \{(R_1, R_2, R_3) : R_1 \geq I_p(X;U), R_2 \geq H_p(Y|U), R_3 \geq H_p(Z|U)\},
\]

which is called the \((n, \varepsilon)\)-rate region. Using \(\mathcal{R}(n, \varepsilon|p_{XYZ})\), \(\mathcal{R}(p_{XYZ})\) can be expressed as

\[
\mathcal{R}(\varepsilon|p_{XYZ}) = \mathcal{R}(p_{XYZ}) = \mathcal{R}(n, \varepsilon|p_{XYZ}) = \bigcap_{m \geq 1} \bigcup_{n \geq m} \mathcal{R}(n, \varepsilon|p_{XYZ}).
\]

It is well known that \(\mathcal{R}(p_{XYZ})\) was determined by Wyner. To describe their result we introduce an auxiliary random variable \(U\) taking values in a finite set \(\mathcal{U}\). We assume that the joint distribution of \((U, X, Y)\) is

\[
p_{UXYZ}(u, x, y, z) = p_U(u)p_{X|U}(x|u)p_{YZ|X}(y, z|x).
\]

The above condition is equivalent to \(U \leftrightarrow X \leftrightarrow Y Z\). Define the set of probability distribution on \(U X Y Z\) by

\[
P(p_{XYZ}) \triangleq \{ p = p_{UXYZ} : |U| \leq |X| + 2, U \leftrightarrow X \leftrightarrow Y Z \}.
\]

Set

\[
\mathcal{R}(p) \triangleq \{(R_1, R_2, R_3) : R_1 \geq I_p(X;U), R_2 \geq H_p(Y|U), R_3 \geq H_p(Z|U)\},
\]

\[
\mathcal{R}(p_{XYZ}) \triangleq \bigcup_{p \in \mathcal{P}_{XYZ}} \mathcal{R}(p).
\]

We can show that the region \(\mathcal{R}(p_{XYZ})\) is a closed convex set of \(\mathbb{R}^3_+\). The rate region \(\mathcal{R}(p_{XYZ})\) was determined by Wyner [2]. His result is the following.

**Theorem 4 (Wyner [2]):**

\[
\mathcal{R}(p_{XYZ}) = \mathcal{R}(p_{XYZ}).
\]

On the strong converse theorem Csiszár and Körner [8] obtained the following.

**Theorem 5 (Csiszár and Körner [8]):** For each fixed \(\varepsilon \in (0, 1)\), we have

\[
\mathcal{R}(\varepsilon|p_{XYZ}) = \mathcal{R}(p_{XYZ}) = \mathcal{R}(p_{XYZ}).
\]

To examine a rate of convergence for the error probability of decoding to tend to one as \(n \to \infty\) for \((R_1, R_2, R_3) \notin \mathcal{R}( \varepsilon|p_{XYZ})\),
By time sharing we have that
\[ \sum_{(\rho,\gamma,\phi) \in \mathcal{P}} |\lambda_{\gamma} \rho \phi|^2 \leq \sum_{(\rho,\gamma,\phi) \in \mathcal{P}} \left( \frac{1}{n} \log \mathbb{P}(\lambda_{\gamma} \rho \phi) \right) \]

We define two sets whose section by the plane \( nG \) coincides with
\[ \mathcal{G}(n) = \text{argmin}_{\rho,\gamma,\phi} \left( \frac{1}{n} \log \mathbb{P}(\lambda_{\gamma} \rho \phi) \right). \]

We aim to find an explicit characterization of \( G(R_1, R_2, R_3) \).

By time sharing we have that
\[ G(R_1, R_2, R_3) \leq \frac{mG(m)(R_1, R_2, R_3) |p_{XYZ}|}{n + m}, \]
from which we have that \( G(R_1, R_2, R_3) \) exists and satisfies the following:
\[ G(R_1, R_2, R_3) = \inf_{n \geq 1} G(n)(R_1, R_2, R_3 |p_{XYZ}|). \]

The exponent function \( G(R_1, R_2, R_3 |p_{XYZ}|) \) is a convex function of \( (R_1, R_2, R_3) \). In fact, by time sharing we have that
\[ G(n) = \frac{mG(m)(R_1, R_2, R_3) |p_{XYZ}|}{n + m}, \]
from which we have that for any \( \alpha \in [0, 1] \)
\[ G(\alpha R_1 + \alpha R_2 + \alpha R_3, \alpha R_1 + \alpha R_2 + \alpha R_3 |p_{XYZ}|) \leq \alpha G(R_1, R_2, R_3 |p_{XYZ}|) + \alpha G(R_1, R_2, R_3 |p_{XYZ}|). \]

The region \( \mathcal{G}(p_{XYZ}) \) is also a closed convex set. Our main aim is to find an explicit characterization of \( \mathcal{G}(p_{XYZ}) \). In this paper we derive an explicit outer bound of \( \mathcal{G}(p_{XYZ}) \) whose section by the plane \( G = 0 \) coincides with \( \mathcal{R}_{W}(p_{XYZ}) \).

We first explain that the region \( \mathcal{R}(p_{XYZ}) \) has two other expressions using the supporting hyperplane. We define two sets of probability distributions on \( U \times X \times Y \times Z \) by
\[ \mathcal{P}(p_{XYZ}) = \{ p = p_{XYZ} : |U| \leq |X|, U \leftrightarrow X \leftrightarrow Y \times Z \}, \]
\[ \mathcal{Q}(p_{XYZ}) = \{ q = q_{XYZ} : |U| \leq |X|, p_{YZ} = q_{Z} |X, U \leftrightarrow X \leftrightarrow Y \times Z \}. \]

For \( (\alpha, \nu, \gamma) \in (0, 1) \times [0, 1]^2 \), set
\[ \mathcal{R}(\alpha, \nu, \gamma)(p_{XYZ}) \Delta = \max_{\mu \in \mathcal{P}(p_{XYZ})} \{ \mu I_{\nu}(X; U) + \nu (\gamma H_p(Y|U) + \gamma H_p(Z|U)) \}, \]
\[ \tilde{\mathcal{R}}(\alpha, \nu, \gamma)(p_{XYZ}) \Delta = \max_{\mu \in \mathcal{Q}(p_{XYZ})} \{ (1 + \mu) \tilde{\alpha} D(q_X ||p_X) + \alpha \mu I_{\nu}(X; U) + \gamma (\gamma H_p(Y|U) + \gamma H_p(Z|U)) \}, \]
\[ \mathcal{R}(p_{XYZ}) = \bigcap_{(\alpha, \nu, \gamma) \in [0, 1]^2} \{ (\alpha, \nu, \gamma) |p_{XYZ}| : |\tilde{\alpha} \gamma| \leq |\nu| \}, \]
\[ \tilde{\mathcal{R}}(\alpha, \nu, \gamma)(p_{XYZ}) = \bigcap_{(\alpha, \nu, \gamma) \in [0, 1]^2} \{ (\alpha, \nu, \gamma) |p_{XYZ}| : |\tilde{\alpha} \gamma| \leq |\nu| \}. \]

Then we have the following property.

\textbf{Property 5:}

a) The bound \( |U| \leq |X| \) is sufficient to describe \( R(\alpha, \nu, \gamma)(p_{XYZ}) \) and \( \tilde{R}(\alpha, \nu, \gamma)(p_{XYZ}) \).

b) For any \( p_{XYZ} \) we have
\[ \mathcal{R}(p_{XYZ}) = \mathcal{R}(p_{XYZ}). \]
Since the proof of Property 3 is quite similar to that of Property 2, we omit it. For \((\alpha, \mu, \gamma) \in [0,1]^3\), define
\[
\omega^{(\alpha,\mu,\gamma)}_{q_X|p_X,q_{XY}|U}(x,y,z|u) \\
= (1 + \mu)\alpha \log \frac{q_X(x)}{p_X(x)} + \alpha \left[ \log \frac{q_X|U(x|u)}{q_X(x)} \right] + \mu \left[ \gamma \log \frac{q_{XY|U}(y|u)}{q_X(x)} + \gamma \log \frac{q_{X|U}(z|u)}{q_X(x)} \right],
\]
\[
\triangleq \exp \left\{ -\omega^{(\alpha,\mu,\gamma)}_{q_X|p_X,q_{XY}|U}(x,y,z|u) \right\},
\]
\[\Omega^{(\alpha,\mu,\gamma)}(q|p_{XYZ}) \triangleq -\log E_q \left\{ \exp \left\{ -\omega^{(\alpha,\mu,\gamma)}_{q_X|p_X,q_{XY}|U}(X,Y,Z|U) \right\} \right\},
\]
\[\Omega^{(\alpha,\mu,\gamma)}(p_{XYZ}) \triangleq \min_{q \in \mathcal{Q}(p_{XYZ})} \Omega^{(\alpha,\mu,\gamma)}(q|p_{XYZ}),
\]
\[F^{(\alpha,\mu,\gamma)}(\alpha[R_1 + \mu(\gamma R_2 + \gamma R_3)]|p_{XYZ}) \triangleq \Omega^{(\alpha,\mu,\gamma)}(p_{XYZ}) - \alpha[R_1 + \mu(\gamma R_2 + \gamma R_3)] + \frac{1}{1 + \lambda(1 + \mu)},
\]
\[\mathcal{G}(p_{XYZ}) \triangleq \{(R_1, R_2, R_3, G) : G \geq F(R_1, R_2, R_3|p_{XYZ}) \}.
\]

We can show that the above functions and sets satisfy the following property.

Property 7:

a) The cardinality bound \(|\mathcal{U}| \leq |\mathcal{X}| \) in \(Q(p_{Y|X})\) is sufficient to describe the quantity \(\Omega^{(\alpha,\mu,\gamma)}(p_{XYZ})\).

b) Define a probability distribution \(q^{(\lambda)}(u,x,y,z)\) by
\[
q^{(\lambda)}(u,x,y,z) \exp \left\{ -\omega^{(\alpha,\mu,\gamma)}_{q_X|p_X,q_{XY}|U}(x,y,z|u) \right\},
\]
\[E_q \left\{ \exp \left\{ -\omega^{(\alpha,\mu,\gamma)}_{q_X|p_X,q_{XY}|U}(X,Y,Z|U) \right\} \right\}.
\]

Then we have
\[
\frac{d}{d \lambda} \Omega^{(\alpha,\mu,\gamma)}(q|p_{XYZ}) = E_q \left[ \omega^{(\alpha,\mu,\gamma)}_{q_X|p_X,q_{XY}|U}(X,Y,Z|U) \right],
\]
\[
\frac{d^2}{d \lambda^2} \Omega^{(\alpha,\mu,\gamma)}(q|p_{XYZ}) = -\text{Var}_q \left[ \omega^{(\alpha,\mu,\gamma)}_{q_X|p_X,q_{XY}|U}(X,Y,Z|U) \right].
\]

The second equality implies that \(\Omega^{(\alpha,\mu,\gamma)}(q|p_{XYZ})\) is a concave function of \(\lambda > 0\).

Proof of Theorem 6 will be given in the next section. The exponent function at rates outside the rate region was derived by Oohama and Han [7] for the separate source coding problem for correlated sources [6]. The techniques used by them is a method of types [3], which is not useful to prove Theorem 5. Some novel techniques based on the information spectrum method introduced by Han [9] are necessary to prove this theorem.
From Theorem A and Property B part d), we have the following corollary, which provides an explicit outer bound of \( R_W(\varepsilon|p_{XYZ}) \) with an asymptotically vanishing deviation from \( R_W(p_{XYZ}) = R(p_{XYZ}) \). The strong converse theorem immediately follows from this corollary.

From Theorem A and Property B part d) we have the following corollary.

**Corollary 5:** For each fixed \( \varepsilon \in (0, 1) \) and for any \( \delta > 0 \), there exists a positive integer \( n_0 \) with
\[
n_0 = n_0(\varepsilon, \delta, |X|, |Y|, |Z|, \rho(p_{XYZ})) \]
such that for \( n \geq n_0 \), we have
\[
R_W(\varepsilon|p_{XYZ}) \leq R(p_{XYZ}) - \kappa_n(1, 1, 1),
\]
where
\[
\kappa_n = \left\{ \sqrt{\frac{8p(\varepsilon|p_{XYZ})}{n}} \log \left( \frac{7}{1-\varepsilon} \right) + \frac{1}{2n} \log \left( \frac{7}{1-\varepsilon} \right) \right\}^{1/2}.
\]
It immediately follows from the above result that for any \( \varepsilon \in (0, 1) \), we have
\[
R_W(\varepsilon|p_{XYZ}) = R_W(p_{XYZ}) = R(p_{XYZ}).
\]

**APPENDIX**

A. Properties of the Rate Regions

In this appendix we prove Property B Property B part a) can easily be proved by the definitions of the rate distortion regions. We omit the proofs of this part. In the following argument we prove part b).

**Proof of Property B part b:** We set
\[
R_{AKW}(m, \varepsilon|p_{XY}) = \bigcap_{n \geq m} R_{AKW}(n, \varepsilon|p_{XY}).
\]
By the definitions of \( R_{AKW}(m, \varepsilon|p_{XY}) \) and \( R_{AKW}(\varepsilon|p_{XY}) \), we have that \( R_{AKW}(m, \varepsilon|p_{XY}) \subseteq R_{AKW}(\varepsilon|p_{XY}) \) for \( m \geq 1 \). Hence we have that
\[
\bigcup_{m \geq 1} R_{AKW}(m, \varepsilon|p_{XY}) \subseteq R_{AKW}(\varepsilon|p_{XY}). \quad (40)
\]
We now assume that \( (R_1, R_2) \in R_{AKW}(\varepsilon|p_{XY}) \). Set
\[
R_{AKW}^{(\delta)}(\varepsilon|p_{XY}) \triangleq \{ (R_1 + \delta, R_2 + \delta) : (R_1, R_2) \in R_{AKW}(\varepsilon|p_{XY}) \}
\]
Then, by the definitions of \( R_{AKW}(n, \varepsilon|p_{XY}) \) and \( R_{AKW}(\varepsilon|p_{XY}) \), we have that for any \( \delta > 0 \), there exists \( n_0(\varepsilon, \delta) \) such that for any \( n \geq n_0(\varepsilon, \delta) \),
\[
(R + \delta, R_2 + \delta) \in R_{AKW}(n, \varepsilon|p_{XY}),
\]
which implies that
\[
R_{AKW}^{(\delta)}(\varepsilon|p_{XY}) \subseteq \bigcup_{n \geq n_0(\varepsilon, \delta)} R_{AKW}(n, \varepsilon|p_{XY})
\]
\[
= R_{AKW}(n_0(\varepsilon, \delta), \varepsilon|p_{XY})
\]
\[
\subseteq \operatorname{cl} \left( \bigcup_{m \geq 1} R_{AKW}(m, \varepsilon|p_{XY}) \right). \quad (41)
\]
Here we assume that there exists a pair \( (R_1, R_2) \) belonging to \( R_{AKW}^{(\delta)}(\varepsilon|p_{XY}) \) such that
\[
(R_1, R_2) \notin \operatorname{cl} \left( \bigcup_{m \geq 1} R_{AKW}(m, \varepsilon|p_{XY}) \right). \quad (42)
\]
Since the set in the right hand side of (42) is a closed set, we have
\[
(R_1 + \delta, R_2 + \delta) \notin \operatorname{cl} \left( \bigcup_{m \geq 1} R_{AKW}(m, \varepsilon|p_{XY}) \right) \quad (43)
\]
for some small \( \delta > 0 \). On the other hand we have \( (R + \delta, R_2 + \delta) \in R_{AKW}^{(\delta)}(\varepsilon|p_{XY}) \), which contradicts (41). Thus we have
\[
\bigcup_{m \geq 1} R_{AKW}(m, \varepsilon|p_{XY})
\]
\[
\subseteq R_{AKW}(\varepsilon|p_{XY}) \subseteq \operatorname{cl} \left( \bigcup_{m \geq 1} R_{AKW}(m, \varepsilon|p_{XY}) \right). \quad (44)
\]
Note here that \( R_{AKW}(\varepsilon|p_{XY}) \) is a closed set. Then from (44), we conclude that
\[
R_{AKW}(\varepsilon|W) = \operatorname{cl} \left( \bigcup_{m \geq 1} R_{AKW}(m, \varepsilon|p_{XY}) \right) = \operatorname{cl} \left( \bigcup_{m \geq 1} \bigcap_{n \geq m} R_{AKW}(n, \varepsilon|p_{XY}) \right),
\]
completing the proof.

B. Cardinality Bound on Auxiliary Random Variables

We first prove the following lemma.

**Lemma 9:**
\[
R^{(\mu)}(p_{XY}) \triangleq \min_{p \in P(p_{XY})} \{ \tilde{\mu}I_p(X;U) + \mu H_p(Y|U) \}
\]
\[
= \min_{p \in P_{sh}(p_{XY})} \{ \tilde{\mu}I_p(X;U) + \mu H_p(Y|U) \}.
\]

**Proof:** We bound the cardinality \( |U| \) of \( U \) to show that the bound \( |U| \leq |X| \) is sufficient to describe \( R^{(\mu)}(p_{XY}) \). Observe that
\[
p_X(x) = \sum_{u \in U} p_U(u)p_{X|U}(x|u), \quad (45)
\]
\[
\tilde{\mu}I_p(X;U) + \mu H_p(Y|U) = \sum_{u \in U} p_U(u)\pi(p_{X|U}(-|u)), \quad (46)
\]
where
\[
\pi(p_{X|U}(-|u)) \triangleq \sum_{(x,y) \in X \times Y} p_{X|U}(x|u)p_Y(y|x)\log \left( \frac{p_{X|U}(x|u)}{p_X(x)} \right) \left( \sum_{\tilde{x} \notin X} p_{Y|X}(y|\tilde{x})p_{X|U}(\tilde{x}|u) \right)^{1-\mu},
\]
For each \( u \in U \), \( \pi(p_{X|U}(-|u)) \) is a continuous function of \( p_{X|U}(-|u) \). Then by the support lemma,
\[
|U| \leq |X| - 1 + 1 = |X|
\]
is sufficient to express $|X| - 1$ values of $45$ and one value of $46$.

Next we prove the following lemma.

**Lemma 10:** For each integer $n \geq 2$, we have

$$
\hat{Q}_{n}^{(\alpha, \mu, \lambda)}(p_{XY}) \triangleq \min_{q=q_{UXY}: U \rightarrow X \rightarrow Y, \ \forall x: X \rightarrow p_{X,Y}} \Omega_{n}^{(\alpha, \mu, \lambda)}(q)_{p_{XY}}
$$

$$
= \min_{q=q_{UXY}: U \rightarrow X \rightarrow Y, \ \forall x: X \rightarrow p_{X,Y}} \Omega_{n}^{(\alpha, \mu, \lambda)}(q)_{p_{XY}}
$$

Proof: We bound the cardinality $|U|$ of $U$ to show that the bound $|U| \leq |X|$ is sufficient to describe $\hat{Q}_{n}^{(\alpha, \mu, \lambda)}(p_{XY})$.

Observe that

$$
q_{X}(x) = \sum_{u \in U} q_{U}(u)q_{X \mid U}(x \mid u),
$$

$$
\exp \left\{ -\Omega_{(\alpha, \mu, \lambda)}^{(\alpha, \mu, \lambda)}(q)_{p_{XY}} \right\}
$$

$$
= \sum_{u \in U} q_{U}(u)\Pi_{q_{X \mid U}(\cdot \mid u)},
$$

where

$$
\Pi_{q_{X \mid U}(\cdot \mid u)} \overset{\Delta}{=} \sum_{(x,y) \in X \times Y} q_{X \mid U}(x \mid u)q_{Y \mid X}(y \mid x)
$$

$$
\times \exp \left\{ -\Omega_{q_{X \mid U}(\cdot \mid u)}^{(\alpha, \mu)} \right\}.
$$

For each $u \in U$, $\Pi_{q_{X \mid U}(\cdot \mid u)}$ is a continuous function of $q_{X \mid U}(\cdot \mid u)$. Then by the support lemma, $|U| \leq |X|$ is sufficient to express $|X| - 1$ values of $47$ and one value of $48$.

**C. Supporting Hyperplane Expressions of $R(p_{XY})$**

In this appendix we prove Property 2 part b). We first prepare a lemma useful to prove this property.

From the convex property of the region $R(p_{XY})$, we have the following lemma.

**Lemma 11:** Suppose that $(\hat{R}_{1}, \hat{R}_{2})$ does not belong to $R(p_{XY})$. Then there exist $\epsilon, \mu^{*} > 0$ such that for any $(R_{1}, R_{2}) \in R(p_{XY})$ we have

$$
\hat{\mu}(R_{1} - \hat{R}_{1}) + \mu^{*}(R_{2} - \hat{R}_{2}) - \epsilon \geq 0.
$$

Proof of this lemma is omitted here. Lemma 11 is equivalent to the fact that if the region $R(p_{XY})$ is a convex set, then for any point $(\hat{R}_{1}, \hat{R}_{2})$ outside the region $R(p_{XY})$, there exists a line which separates the point $(\hat{R}_{1}, \hat{R}_{2})$ from the region $R(p_{XY})$. Lemma 11 will be used to prove Property 2 part b).

**Proof of Property 2 part b):** We first recall the following definitions of $P^{*}(p_{XY})$ and $P(p_{XY})$:

$P(p_{XY}) \overset{\Delta}{=} \{ p_{UXY}: |U| \leq |X| + 1, U \leftrightarrow X \leftrightarrow Y \}$

$P_{sh}(p_{XY}) \overset{\Delta}{=} \{ p_{UXY}: |U| \leq |X|, U \leftrightarrow X \leftrightarrow Y \}$

We prove $R_{sh}(p_{XY}) \subseteq R(p_{XY})$. We assume that $(\hat{R}_{1}, \hat{R}_{2}) \notin R(p_{XY})$. Then by Lemma 11 there exist $\epsilon > 0$ and $\mu^{*} \in [0, 1]$ such that for any $(\hat{R}_{1}, \hat{R}_{2}) \in R(p_{XY})$, we have

$$
\mu^{*}\hat{R}_{1} + \mu^{*}\hat{R}_{2} \leq \hat{\mu}\hat{R}_{1} + \mu^{*}\hat{R}_{2} - \epsilon.
$$

Hence we have

$$
\hat{\mu}\hat{R}_{1} + \mu^{*}\hat{R}_{2} \leq \min_{p \in P_{sh}(p_{XY})} \{ \hat{\mu}\hat{R}_{1} + \mu^{*}\hat{R}_{2} \} - \epsilon
$$

$\leq \min_{p \in P_{sh}(p_{XY})} \{ \hat{\mu}\hat{R}_{1} + \mu^{*}\hat{R}_{2} \} - \epsilon
$$

$= R(\mu^{*})(p_{XY}) - \epsilon.
$ $(49)$

Step (a) follows from the definition of $R(p_{XY})$. The inequality $(49)$ implies that $(\hat{R}_{1}, \hat{R}_{2}) \notin R_{sh}(p_{XY})$. Thus $R_{sh}(p_{XY}) \subseteq R(p_{XY})$ is concluded. We next prove $R(p_{XY}) \subseteq R_{sh}(p_{XY})$.

We assume that $(R_{1}, R_{2}) \in R(p_{XY})$. Then there exists $q \in P(p_{XY})$ such that

$$
R_{1} \geq I_{q}(X; U), R_{2} \geq H_{q}(Y|U).
$$

Then, for each $\mu > 0$ and for $(R_{1}, R_{2}) \in R(p_{XY})$, we have the following chain of inequalities:

$$
\hat{\mu}R_{1} + \mu R_{2} \geq \hat{\mu}I_{q}(X; U) + \mu H_{q}(Y|U)
$$

$$
\geq \min_{q \in P(p_{XY})} [\hat{\mu}I_{q}(X; U) + \mu H_{q}(Y|U)]
$$

$$
= R(\mu^{*})(p_{XY}).
$$

Step (a) follows from $(49)$. Hence we have $R(p_{XY}) \subseteq R_{sh}(p_{XY})$. Next prove the second inequality of $6$ in Property 2. We have the following chain of inequalities:

$$
R(\mu)(p_{XY}) \geq \min_{q \in P_{sh}(p_{XY})} [\hat{\mu}I_{q}(X; U) + \mu H_{q}(Y|U)]
$$

$= \min_{q \in P(p_{XY})} \left\{ \frac{(1 + \mu)\hat{\alpha}}{\hat{\alpha}} D(q_{X} \mid p_{X}) + \hat{\mu}I_{q}(X; U) + \mu H_{q}(Y|U) \right\}
$$

$\geq \min_{q \in P_{sh}(p_{XY})} \left\{ \frac{(1 + \mu)\hat{\alpha}}{\hat{\alpha}} D(q_{X} \mid p_{X}) + \hat{\mu}I_{q}(X; U) + \mu H_{q}(Y|U) \right\}
$$

$= \frac{1}{\alpha} R(\alpha)(p_{XY}).
$$

Step (a) follows from that when $q \in P_{sh}(p_{XY})$, we have $D(q_{X} \mid p_{X}) = 0$. We finally prove the first inequality of $6$ in Property 2. Let

$$
q_{*, \mu} = q_{U,Y,C_* \mu} \in Q(p_{Y \mid X})
$$

be the probability distribution attaining the minimum in the definition of $R(\mu)(p_{XY})$. Let $q_{*, \mu} = q_{U,Y,C_* \mu}$ be a probability distribution with the form

$$
q_{U,Y,C_* \mu}(u, x, y) = q_{U,X,C_* \mu}(u|x)p_{X}(x)p_{Y}(y|x).
$$
By definition, we have \( \hat{q}_{\alpha, \mu} \in \mathcal{P}_{\text{sh}}(p_{XY}) \). Then we have the following chain of inequalities.

\[
\begin{align*}
\tilde{a} D(q_{X,\alpha, \mu}^* || q_{\alpha, \mu}) \\
= \tilde{a} D(q_{X,\alpha, \mu}^* || p_X) \\
\leq (1 + \mu) \tilde{a} D(q_{X,\alpha, \mu} || p_X) \\
+ a [\mu I_{q_{\alpha, \mu}}(X; U) + H_{q_{\alpha, \mu}}(Y|U)] \\
= R^{(\alpha, \mu)}(p_{XY}) \\
\leq (1 + \mu) \tilde{a} D(q_{X,\alpha, \mu} || p_X) \\
+ a [\mu I_{\hat{q}_{\alpha, \mu}}(X; U) + H_{\hat{q}_{\alpha, \mu}}(Y|U)] \\
= a [\mu I_{\hat{q}_{\alpha, \mu}}(X; U) + H_{\hat{q}_{\alpha, \mu}}(Y|U)] \\
\leq a \log(|X||Y|) \tag{51}
\end{align*}
\]

For simplicity of notation we set \( \xi \triangleq \log(|X||Y|) \). From (51) we have

\[
D(q_{\alpha, \mu}||q_{\alpha, \mu}) \leq \xi \frac{\alpha}{\alpha} \tag{52}
\]

By the Pinsker’s inequality we have

\[
\frac{1}{2} ||q_{\alpha, \mu}^* - q_{\alpha, \mu}||_1 \leq D(q_{\alpha, \mu}^* || q_{\alpha, \mu}) \leq \frac{1}{2} \tag{53}
\]

From (52) and (53), we obtain

\[
||q_{\alpha, \mu}^* - q_{\alpha, \mu}||_1 \leq \sqrt{2 \xi \alpha} \leq \frac{1}{2} \tag{54}
\]

for any \( \alpha \in (0, \alpha_0] \) with

\[
\alpha_0 = (8\xi + 1)^{-1} = [8 \log(|X||Y|)]^{-1} \tag{55}
\]

The bound (54) implies that for any \( A \subseteq \{U, X, Y, Z\} \), we have

\[
||q_{A,\alpha, \mu}^* - q_{A,\alpha, \mu}||_1 \leq \sqrt{2 \xi \alpha} \leq \frac{1}{2} \tag{56}
\]

Then for any \( \alpha \in (0, \alpha_0] \), we have the following two chains of inequalities:

\[
\begin{align*}
& - \sqrt{2 \xi \alpha} \left[ \log \left( \sqrt{\frac{2 \xi \alpha}{\alpha} \cdot \frac{1}{|U|}} \right) + \log \left( \sqrt{2 \xi \alpha} \cdot \frac{1}{|X|} \right) \right] \\
& \leq \sqrt{2 \xi \alpha} \log \left\{ \left( \frac{\alpha}{2 \xi} \right)^{\frac{1}{2}} \right\} \tag{57}
\end{align*}
\]

\[
\begin{align*}
& - \sqrt{2 \xi \alpha} \left[ \log \left( \sqrt{\frac{2 \xi \alpha}{\alpha} \cdot \frac{1}{|U|}} \right) + \log \left( \sqrt{2 \xi \alpha} \cdot \frac{1}{|Y|} \right) \right] \\
& \leq \sqrt{2 \xi \alpha} \log \left\{ \left( \frac{\alpha}{2 \xi} \right)^{\frac{1}{2}} \right\} \tag{58}
\end{align*}
\]

Steps (a) and (b) follow from (56) and LEMMA 2.7 in Section 1.2 in Csiszár and Körner [8]. Combining (57) and (58), we have

\[
\begin{align*}
& \left| \mu I_{q_{\alpha, \mu}}(X; U) + H_{q_{\alpha, \mu}}(Y|U) \right| - \left| \mu I_{\hat{q}_{\alpha, \mu}}(X; U) + H_{\hat{q}_{\alpha, \mu}}(Y|U) \right| \\
& \leq \left| \mu I_{q_{\alpha, \mu}}(X; U) - I_{\tilde{q}_{\alpha, \mu}}(X; U) \right| \\
& + \mu H_{\tilde{q}_{\alpha, \mu}}(Y|U) - H_{\hat{q}_{\alpha, \mu}}(Y|U) \\
& \leq 2 \sqrt{2 \xi \alpha} \log \left\{ \left( \frac{\alpha}{2 \xi} \right)^{\frac{1}{2}} \right\} \tag{59}
\end{align*}
\]

Step (a) follows from the cardinality bound \( |U| \leq |X| \) we have for \( q_{\alpha, \mu} \in \mathcal{Q}(p_{Y|X}) \). Then we have

\[
\begin{align*}
& 1 \tilde{a} R^{(\alpha, \mu)}(p_{XY}) \\
& = (1 + \mu) \tilde{a} D(q_{X,\alpha, \mu} || p_X) \\
& + I_{q_{\alpha, \mu}}(X; U) + H_{q_{\alpha, \mu}}(Y|U) \\
& \geq I_{\hat{q}_{\alpha, \mu}}(X; U) + H_{\hat{q}_{\alpha, \mu}}(Y|U) \tag{56}
\end{align*}
\]

\[
\begin{align*}
& \geq I_{\tilde{q}_{\alpha, \mu}}(X; U) + H_{\tilde{q}_{\alpha, \mu}}(Y|U) - c_1 \sqrt{\frac{\alpha}{\alpha}} \log \left( \frac{c_2}{\alpha} \right) \tag{60}
\end{align*}
\]

Step (b) follows from (59). Step (b) follows from that \( \hat{q}_{\alpha, \mu} \in \mathcal{Q}(p_{Y|X}) \).

\[ \blacksquare \]

D. Proof of Property 3 parts b), c), and d)

In this appendix we prove Property 3 b), c), and d). Proof of Property 3 b), c), and d): We first prove part b). For simplicity of notations, set

\[
\begin{align*}
\omega(a) & \triangleq (u, x, y), \omega(a) & \triangleq (U, X, Y), \omega(a) & \triangleq U \times X \times Y, \\
\Omega_{\alpha, \mu}(a) & \triangleq \xi(a) \tag{59}
\end{align*}
\]

Then we have

\[
\Omega_{\alpha, \mu}(a) = - \log \sum_{a \in \Delta} p_A(a) e^{-\lambda g(a)} \tag{58}
\]

We set

\[
q^{(\lambda)}(u, x, y, z) = p_A^{(\lambda)}(a). \tag{60}
\]
Then $p_A^{(l)}(\mathbf{a}) \in \mathcal{A}$ has the following form:

$$p_A^{(l)}(\mathbf{a}) = e^{\xi(\mathbf{a})} e^{-\lambda g(\mathbf{a})}.$$  

By simple computations we have

$$\xi' (\lambda) = e^{\xi(\lambda)} \left[ \sum_{\mathbf{a} \in \mathcal{A}} p_A(\mathbf{a}) g(\mathbf{a}) e^{-\lambda g(\mathbf{a})} \right],$$

$$= \sum_{\mathbf{a} \in \mathcal{A}} p_A^{(l)}(\mathbf{a}) g(\mathbf{a}).$$

$$\xi'' (\lambda) = -e^{2\xi(\lambda)} \times \left[ \sum_{\mathbf{a} \in \mathcal{A}} p_A(\mathbf{a}) p_A^{(l)}(\mathbf{b}) \begin{bmatrix} (g(\mathbf{a}) - g(\mathbf{b}))^2 \end{bmatrix}_{2} e^{-\lambda (g(\mathbf{a}) + g(\mathbf{b}))} \right],$$

$$= - \sum_{\mathbf{a} \in \mathcal{A}} p_A^{(l)}(\mathbf{a}) \left[ \sum_{\mathbf{a} \in \mathcal{A}} p_A^{(l)}(\mathbf{a}) g(\mathbf{a}) \right]^2.$$

We next prove the part c). Let $\lambda \in [0, 1]$. By the Taylor expansion of $\Omega^{(\alpha, \mu, \lambda)}(q|p_{XY})$ with respect to $\lambda$ around $\lambda = 0$, we have

$$\Omega^{(\alpha, \mu, \lambda)}(q|p_{XY}) = \xi(\lambda) = \xi(0) + \xi'(0) \lambda + \frac{1}{2} \xi''(0) (\lambda \lambda)^2$$

$$= \mathbb{E}_{\mathbf{q}} \left[ \omega^{(\alpha, \mu)}_{\mathbf{q}X \mathbf{q}Y_{XY}|U}(X, Y|U) \right] \left( 1 - \frac{1}{2} (\tau \lambda)^2 \text{Var}_{\mathbf{q}} \left[ \omega^{(\alpha, \mu)}_{\mathbf{q}X \mathbf{q}Y_{XY}|U}(X, Y|U) \right] \right)$$

for some $\tau \in [0, 1]$. Then by the definition of $\rho$, we have

$$\Omega^{(\alpha, \mu, \lambda)}(q|p_{XY}) \geq \lambda \mathbb{E}_{\mathbf{q}} \left[ \omega^{(\alpha, \mu)}_{\mathbf{q}X \mathbf{q}Y_{XY}|U}(X, Y|U) \right] - \frac{1}{2} \rho \lambda^2.$$

The second inequality is obvious from the first inequality. We finally prove the part d). By the hyperplane expression $\mathcal{R}_{sh}(p_{XY})$ of $\mathcal{R}(p_{XY})$ stated Property 2 part b) we have that when $(R_1 + \tau, R_2 + \tau) \notin \mathcal{R}(p_{XY})$, we have

$$\bar{\mu}^* (R_1 + \tau) + \mu^* (R_2 + \tau) \leq \mathcal{R}(\mu^*) (p_{XY})$$  

(61) for some $\mu^* \in [0, 1]$. Let $\alpha_0 = \alpha_0([X], [Y])$ be the quantity defined by (7) in Property 2 part b). Then from (61), we have

$$\bar{\mu}^* R_1 + \mu^* R_2 + \tau \leq \mathcal{R}(\mu^*) (p_{XY})$$

(62) for any $\tau \in (0, \alpha_0^{-1} \tau)$. For this choice of $\alpha$, we have (62) for any $\tau \in (0, \alpha_0^{-1} \tau)$ and

$$C_1 \frac{\varphi}{\alpha} \log \left( C_2 \frac{\alpha}{\varphi} \right) \times \frac{1}{\tau}$$

$$= c_1 \frac{\tau^2}{\sqrt{1 - \tau^{2+\delta}}} \log \left( C_2 \frac{1 - \tau^{2+\delta}}{\tau^{2+\delta}} \right) \rightarrow 0 \text{ as } \tau \rightarrow 0.$$  

Hence there exists a positive $\nu = \nu(\delta, [X], [Y])$ with $\nu \leq \alpha_0^{-1} \tau$ such that for $\tau \in (0, \nu) \subset (0, \alpha_0^{-1} \tau)$,

$$c_1 \cdot \frac{\alpha}{\varphi} \log \left( C_2 \frac{\alpha}{\varphi} \right) \leq \frac{\tau}{2}.$$  

(63)

The above inequality together with (62) yields that for $\tau \in (0, \nu]$,

$$\tilde{\mu}^* R_1 + \mu^* R_2 + \frac{1}{2} \tau \leq \frac{1}{\tau^{2+\delta}} \mathcal{R}(\tau^{2+\delta})(p_{XY}).$$  

(64)

Then for each positive $\tau$ with $\tau \in (0, \nu]$, we have the following chain of inequalities:

$$F(R_1, R_2)p_{XY}) \geq \sup_{\lambda > 0} F^{(\tau^{2+\delta}, \mu^*)} (\bar{\mu}^* R_1 + \mu^* R_2 | p_{XY})$$

$$= \sup_{\lambda > 0} \Omega^{(\tau^{2+\delta}, \mu^*)}(p_{XY}) - \lambda \tau^{2+\delta} (\bar{\mu}^* R_1 + \mu^* R_2)$$

$$\geq \sup_{\lambda > 0} \frac{1}{1 + 2 \lambda} \left\{ -\frac{1}{2} \rho \lambda^2 + \lambda \mathcal{R}(\tau^{2+\delta})(p_{XY}) - \lambda \tau^{2+\delta} (\bar{\mu}^* R_1 + \mu^* R_2) \right\}$$

$$\geq \sup_{\lambda > 0} \frac{1}{1 + 2 \lambda} \left\{ -\frac{1}{2} \rho \lambda^2 + \frac{1}{2} \lambda \tau^{2+\delta} \right\} \equiv \frac{\rho}{2 \bar{\rho}} \left( \frac{\tau^{2+\delta}}{2} \right).$$

Step (a) follows from Property 2 part b). Step (b) follows from (64). Step (c) follows from an elementary computation. This completes the proof of Property 3 part d).

E. Proof of Lemma 7

To prove Lemma 7 we prepare a lemma. Set

$$A_n \triangleq \left\{ (s, x^n, y^n) : \frac{1}{n} \log p_{XY}(s, x^n, y^n) \geq -\eta \right\}.$$

Furthermore, set

$$\tilde{B}_n \triangleq \left\{ x^n : \frac{1}{n} \log p_{XY}(x^n) \geq -\eta \right\},$$

$$\tilde{B}_n \triangleq \tilde{B}_n \times M_1 \times Y^n, \tilde{B}_n \triangleq \tilde{B}_n \times M_1 \times Y^n,$$

$$\tilde{C}_n \triangleq \left\{ (s, x^n) : s = \varphi_1(n)(x^n), \ p_{XY}(s|x^n) \leq M_1 e^{\eta n} \right\},$$

$$\tilde{C}_n \triangleq \tilde{C}_n \times Y^n, \tilde{C}_n \triangleq \tilde{C}_n \times Y^n,$$

$$\tilde{D}_n \triangleq \left\{ (s, x^n, y^n) : s = \varphi_1(n)(x^n), \ p_{XY}(s|y^n) \geq 1/Me^{-\eta n} \right\},$$

$$\tilde{E}_n \triangleq \left\{ (s, x^n, y^n) : s = \varphi_1(n)(x^n), \ p_{XY}(s|y^n) \geq 1/Me^{-\eta n} \right\}.$$

Then we have the following lemma.

Lemma 12:

$$p_{XY^n}(A^n) \leq e^{-\eta n}, p_{XY^n}(B^n) \leq e^{-\eta n},$$

$$p_{XY^n}(C^n) \leq e^{-\eta n}, p_{XY^n}(D^n \cap \tilde{E}_n) \leq e^{-\eta n},$$

$$p_{XY^n}(E^n) \leq e^{-\eta n}.$$
Proof: We first prove the first inequality.

\[ p_{S|X^n}(A^n_s) = \sum_{(s,x^n,y^n) \in A^n_s} p_{V_n}(s,x^n,y^n) \]

\[ \leq \sum_{(s,x^n,y^n) \in A^n_s} e^{-\eta n} q_{V_n}(s,x^n,y^n) = e^{-\eta n} q_{V_n}(A^n_s) \leq e^{-\eta n}. \]

Step (a) follows from the definition of \( A_n \). On the second inequality we have

\[ p_{S|X^n}(B^n_n) = p_{X^n}(B^n_n) = \sum_{x^n \in B^n_n} p_{X^n}(x^n) \]

\[ \leq \sum_{x^n \in B^n_n} e^{-\eta n} q_{X^n}(x^n) = e^{-\eta n} q_{X^n}(B^n_n) \leq e^{-\eta n}. \]

Step (a) follows from that the number of \( B_n \).

Proof of Lemma \([\text{I}]\) By definition we have

\[ p_{S|X^n}(A_n \cap B_n \cap C_n \cap D_n) \]

\[ = p_{S|X^n} \left\{ 1 - \frac{1}{n} \log \frac{q_{S|X^n}(S,X^n,Y^n)}{p_{S|X^n}(S,X^n,Y^n)} \right\} \geq -\eta, \]

\[ \geq -\eta, \]

\[ 0 \geq 1 - \frac{1}{n} \log \frac{p_{X^n}(Y^n)}{p_{X^n}(X^n)} - \eta, \]

\[ \frac{1}{n} \log M_1 \geq 1 - \frac{1}{n} \log \frac{p_{X^n}(Y^n)}{p_{X^n}(X^n)} - \eta, \]

\[ \frac{1}{n} \log M_2 \geq 1 - \frac{1}{n} \log \frac{p_{X^n}(Y^n)}{p_{X^n}(X^n)} - \eta \}

Then for any \((\phi_1, \phi_2, \psi(n))\) satisfying

\[ \frac{1}{n} \log ||\phi(n)|| \leq R_i, i = 1, 2, \]

we have

\[ p_{S|X^n}(A_n \cap B_n \cap C_n \cap D_n) \]

\[ \leq p_{S|X^n} \left\{ 1 - \frac{1}{n} \log \frac{q_{S|X^n}(S,X^n,Y^n)}{p_{S|X^n}(S,X^n,Y^n)} \right\} \geq -\eta, \]

\[ 0 \geq 1 - \frac{1}{n} \log \frac{p_{X^n}(Y^n)}{p_{X^n}(X^n)} - \eta, \]

\[ R_1 \geq 1 - \frac{1}{n} \log \frac{p_{X^n}(Y^n)}{p_{X^n}(X^n)} - \eta, \]

\[ R_2 \geq 1 - \frac{1}{n} \log \frac{p_{X^n}(Y^n)}{p_{X^n}(X^n)} - \eta \}

Hence, it suffices to show

\[ p_{c}(\phi_1, \phi_2, \psi(n)) \leq p_{S|X^n}(A_n \cap B_n \cap C_n \cap D_n) + e^{-\eta n} \]

to prove Lemma \([\text{I}]\) By definition we have

\[ p_{c}(\phi_1, \phi_2, \psi(n)) = p_{S|X^n}(E_n). \]

Then we have the following.

\[ p_{c}(\phi_1, \phi_2, \psi(n)) = p_{S|X^n}(E_n) \]

\[ = p_{S|X^n}(A_n \cap B_n \cap C_n \cap D_n) \]

\[ + p_{S|X^n}(A_n \cap B_n \cap C_n \cap D_n) \]

\[ \leq p_{S|X^n}(A_n \cap B_n \cap C_n \cap D_n) \]

\[ \leq p_{S|X^n}(A_n \cap B_n \cap C_n \cap D_n) + e^{-\eta n}. \]

Step (a) follows from Lemma \([\text{I2}]\) \]

F. Proof of Lemma \([\text{II}]\)

In this appendix we prove Lemma \([\text{II}]\) \]

Proof of Lemma \([\text{II}]\) We first prove the following Markov chain \([\text{I3}]\) in Lemma \([\text{II}]\)

\[ SX \leftrightarrow X_t \leftrightarrow Y_t. \]
We have the following chain of inequalities:
\[ I(Y_t;SX_{t-1}|X_t) = H(Y_t|X_t) - H(Y_t|SX_{t-1},X_t) \]
\[ \leq H(Y_t|X_t) - H(Y_t|SX^n) \]
\[ \overset{(a)}{=} H(Y_t|X_t) - H(Y_t|X^n) \]
\[ \overset{(b)}{=} H(Y_t|X_t) - H(Y_t|X_t) = 0. \]

Step (a) follows from that \( S = \varphi^{(n)}(X^n) \) is a function of \( X^n \). Step (b) follows from the memoryless property of the information source \( \{X_t,Y_t\}_t \). Next we prove the following Markov chain (14) in Lemma 3:
\[ Y^{t-1} \leftrightarrow SX^{t-1} \leftrightarrow (X_t,Y_t). \]

We have the following chain of inequalities:
\[ I(X_t,Y_t;Y^{t-1}|SX^{t-1}) \]
\[ = H(Y^{t-1}|SX^{t-1}) - H(Y^{t-1}|SX^{t-1}X_t,Y_t) \]
\[ \leq H(Y^{t-1}|X^{t-1}) - H(Y^{t-1}|X^nSY_t) \]
\[ \overset{(a)}{=} H(Y^{t-1}|X^{t-1}) - H(Y^{t-1}|X^nY_t) \]
\[ \overset{(b)}{=} H(Y^{t-1}|X^{t-1}) - H(Y^{t-1}|X^nY_t) = 0. \]

Step (a) follows from that \( S = \varphi^{(n)}(X^n) \) is a function of \( X^n \). Step (b) follows from the memoryless property of the information source \( \{X_t,Y_t\}_t \).

G. Proof of Lemma 6

In this appendix we prove Lemma 6.

Proof of Lemma 6: By the definition of \( P_{SX^nY^n}(s,x^t,y^t) \), for \( t = 1,2,\ldots,n \), we have
\[ P_{SX^nY^n}(s,x^t,y^t) = C_{t-1}^{-1}p_{SX^nY^n}(s,x^t,y^t) \]
\[ \times \prod_{i=1}^{t} f_{x_i|p_{x_i},p_{x_i,y_i}|u_i}(x_i,y_i|u_i). \]

Then we have the following chain of equalities:
\[ P_{SX^nY^n}(s,x^t,y^t) \]
\[ \overset{(a)}{=} C_{t-1}^{-1}p_{SX^nY^n}(s,x^t,y^t) \]
\[ \times \prod_{i=1}^{t} f_{x_i|p_{x_i},p_{x_i,y_i}|u_i}(x_i,y_i|u_i) \]
\[ = C_{t-1}^{-1}p_{SX^nY^n}(s,x^t,y^t-1) \]
\[ \times \prod_{i=1}^{t-1} f_{x_i|p_{x_i},p_{x_i,y_i}|u_i}(x_i,y_i|u_i) \]
\[ \times p_{X_t,Y_t|SX^{t-1}Y^{t-1}}(x_t,y_t|s,x^{t-1},y^{t-1}) \]
\[ \times f_{Y_t|p_{X_t},p_{X_t,Y_t}|u_t}(x_t,y_t|u_t) \]
\[ \overset{(b)}{=} C_{t-1}^{-1}C_{t-2}^{-1}p_{SX^nY^n}(s,x^t,y^t-1) \]
\[ \times p_{X_t,Y_t|SX^{t-1}Y^{t-1}}(x_t,y_t|s,x^{t-1},y^{t-1}) \]
\[ \times f_{Y_t|p_{X_t},p_{X_t,Y_t}|u_t}(x_t,y_t|u_t) \]
\[ = \left( \Phi^{(n)}_{t}(\alpha,\mu,\lambda) \right)^{-1} \]
\[ \times p_{SX^nY^n}(s,x^t,y^t-1) \]
\[ \times p_{X_t,Y_t|SX^{t-1}Y^{t-1}}(x_t,y_t|s,x^{t-1},y^{t-1}) \]
\[ \times f_{Y_t|p_{X_t},p_{X_t,Y_t}|u_t}(x_t,y_t|u_t). \]

Steps (a) and (b) follow from (65). From (66), we have
\[ \Phi^{(n)}_{t}(\alpha,\mu,\lambda) P_{SX^nY^n}(s,x^t,y^t) \]
\[ = p_{SX^nY^n}(s,x^t,y^t-1) \]
\[ \times p_{X_t,Y_t|SX^{t-1}Y^{t-1}}(x_t,y_t|s,x^{t-1},y^{t-1}) \]
\[ \times f_{Y_t|p_{X_t},p_{X_t,Y_t}|u_t}(x_t,y_t|u_t). \]

Taking summations of (67) and (68) with respect to \( s,x^t,y^t \), we obtain
\[ \sum_{s,x^t,y^t} \Phi^{(n)}_{t}(\alpha,\mu,\lambda) P_{SX^nY^n}(s,x^t,y^t) \]
\[ = \sum_{s,x^t,y^t} p_{SX^nY^n}(s,x^t,y^t-1) \]
\[ \times p_{X_t,Y_t|SX^{t-1}Y^{t-1}}(x_t,y_t|s,x^{t-1},y^{t-1}) \]
\[ \times f_{Y_t|p_{X_t},p_{X_t,Y_t}|u_t}(x_t,y_t|u_t), \]

completing the proof.

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