A simple but accurate explicit finite difference method for the advection-diffusion equation

Febi Sanjaya\(^1\) and Sudi Mungkasi\(^2\)

\(^1\) Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University, Mricon, Tromol Pos 29, Yogyakarta 55002, INDONESIA
\(^2\) Department of Mathematics, Faculty of Science and Technology, Sanata Dharma University, Mricon, Tromol Pos 29, Yogyakarta 55002, INDONESIA

E-mail: febi@usd.ac.id, sudi@usd.ac.id

Abstract. We consider the advection-diffusion equation in one dimension. The equation is solved using an explicit finite difference method. As the method is explicit, it is simple to implement. We investigate the performance of the method. We obtain that the method is indeed accurate.

1. Introduction
The advection-diffusion equation is a model that can be used for simulation of the spreading of pollutant. It governs the process of advection and diffusion concurrently. A large number of authors (to mention some of them, see [1-15]) have attempted to solve and use this equation in their simulations.

In this paper, we investigate the performance of a finite difference method due to Karahan [4] for solving the advection-diffusion equation of pollutant transports. The finite difference approach is appropriate because the equation is a partial differential equation having the parabolic type. We use the MATLAB software in the implementation of the finite difference method. The iterations are started from a given initial condition using an arbitrary space width and time step.

This paper is organised simply as follows. Sections 2-5 contains the mathematical model, numerical method, numerical results, and conclusion, respectively.

2. Advection-diffusion equation
The mathematical model describing the pollutant transport is the one-dimensional advection-diffusion equation [4]

\[
\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2}, \quad 0 < x < L, \quad 0 < t \leq T
\]  

(1)

with the initial condition

\[ c(x, 0) = f(x), \quad 0 \leq x < L \]  

(2)

and boundary conditions

\[ c(0, t) = g(t), \quad 0 < t \leq T \]  

(3)

\[ c(L, t) = h(t), \quad 0 < t \leq T \]  

(4)
in a finite space domain of \( x \) and a finite time domain of \( t \). The variable \( c = c(x,t) \) denotes the concentration of a pollutant. In addition, \( f, g \) and \( h \) are known functions; \( u, x, D \) respectively represent the velocity, the direction, and the dispersion coefficient. Here \( u \) and \( D \) are assumed to be positive constants. As advection-diffusion equation contains advection and diffusion phenomena, the solution should reveal these two phenomena.

3. Finite difference method

We use the finite difference method due to Karahan [4] to solve the problem. Note that the equation is a partial differential equation of the parabolic type, so finite difference methods should be able to solve the problem. This is because finite difference methods are based on the direct-discretisation of differential equations. Solutions of parabolic partial differential equations are always continuous even when the initial condition is discontinuous.

We discretise the problem by choosing a stepsize \( \Delta x \) as the spatial step in \( x \) and stepsize \( \Delta t \) as the temporal step in \( t \). We also approximate the concentration \( c \) on a grid of point in the \( xt \)-plane. Moreover,

\[
x_i = i\Delta x, \  i = 1, 2, 3, ..., M, \ \Delta x = \frac{L}{M}
\]

\[
t_n = n\Delta t, \ t = 1, 2, 3, ..., N, \ \Delta t = \frac{T}{N}
\]

Furthermore, we take the following discretisations

\[
\frac{\partial c}{\partial t} = \frac{c(i,n+1) - c(i,n)}{\Delta t}
\]

\[
u \frac{\partial c}{\partial x} = u \left( \frac{1}{2\Delta x} \right) \left[ c(i,n+1) - c(i-1,n+1) + c(i+1,n) - c(i,n) \right]
\]

\[
D \frac{\partial^2 c}{\partial x^2} = D \left[ \frac{\theta}{(\Delta x)^2} \right] \left[ c(i-1,n+1) - c(i,n+1) + c(i+1,n) - c(i,n) \right] + D \left[ \frac{1-\theta}{(\Delta x)^2} \right] \left[ c(i-1,n) - 2c(i,n) + c(i+1,n) \right]
\]

where \( \theta \) is the weighting factor. Here the time integration (7) is forward difference, so it has the first order of accuracy.

Substituting the equations above into equation (1), we have

\[
\frac{c(i,n+1) - c(i,n)}{\Delta t} + u \left( \frac{1}{2\Delta x} \right) \left[ c(i,n+1) - c(i-1,n+1) + c(i+1,n) - c(i,n) \right] = D \left[ \frac{\theta}{(\Delta x)^2} \right] \left[ c(i-1,n) - 2c(i,n) + c(i+1,n) \right]
\]

for \( 1 \leq i \leq M \) and \( 1 \leq n \leq N \). Equation (10) is equivalent to

\[
\left[ 1 + \frac{Cr}{2} + \theta \frac{Cr}{Pe} \right] c(i,n+1) - \left[ \frac{Cr}{2} + \theta \frac{Cr}{Pe} \right] c(i-1,n+1) = \left[ \frac{Cr}{Pe} - \theta \frac{Cr}{Pe} \right] c(i-1,n)
\]

\[
+ \left[ 1 + \frac{Cr}{2} - 2 \frac{Cr}{Pe} + \theta \frac{Cr}{Pe} \right] c(i,n) + \left[ \frac{Cr}{Pe} - \frac{Cr}{2} \right] c(i+1,n)
\]

where the Courant number \( Cr = u\Delta t / \Delta x \), and the Peclet number \( Pe = u\Delta x / D \). The boundary condition, equation (3), makes the equation is explicit, so we obtain
\[ c(i,n+1) = \left[ \frac{Cr}{Pe} - \theta \left( \frac{Cr}{Pe} \right) \right] c(i-1,n) + \left[ 1 + \left( \frac{Cr}{2} \right) - 2 \left( \frac{Cr}{Pe} \right) + \theta \left( \frac{Cr}{Pe} \right) \right] c(i,n) + \left[ \frac{Cr}{Pe} - \frac{Cr}{2} \right] \]

Equation (12) is the explicit finite difference method due to Karahan [4] for solving the advection-diffusion equation (1).

4. Results and discussion

In this section, we present our numerical results and discuss them. We assume that all quantities are in SI units with the MKS system.

As the numerical test, we consider the following settings. The initial condition is

\[ c(x,0) = \exp \left( -\frac{(x-1)^2}{D} \right) \].

This initial condition is illustrated in Figure 1. Parameters being used are \( D = 0.005 \) and \( u = 0.8 \). The spatial step is \( \Delta x = 0.025 \), and the temporal step is \( \Delta t = 0.0125 \). We consider the spatial domain \( 0 \leq x \leq 9 \). We seek the solution at time \( t = 2.5 \) and \( t = 5 \).

![Figure 1. Initial condition (t = 0) of an advection-diffusion problem.](image)

The analytical solution for this problem is [4]

\[ c(x,t) = \frac{1}{\sqrt{4t+1}} \exp \left( -\frac{(x-1-ut)^2}{D (4t+1)} \right) \].

Note that this problem is chosen, because the analytical solution has been found (see Sankaranarayanan et al. [11]). In general, the analytical solution to the advection-diffusion equation is not available. Therefore, we do need numerical methods to solve the advection-diffusion equation.

Numerical results show that the method is simple to implement, yet gives accurate solutions. This is shown in Figure 2 as representatives of our numerical results. This figure shows the solution at time \( t = 2.5 \) and \( t = 5 \), respectively. Notice that our solutions reveal advection and diffusion phenomena at the same time. That is, as time evolves the solution translates to the right (advection phenomenon) and it diffuses from the centre of the concentration position (diffusion phenomenon).
Figure 2. Numerical and analytical solutions using $\Delta x = 0.025$ and $\Delta t = 0.0125$ at time $t = 2.5$ (left figure) and $t = 5.0$ (right figure) respectively. We observe that the numerical solution is very accurate, as the error is quite small with respect to the analytical solution.

Figure 3. Numerical and analytical solutions using $\Delta x = 0.05$ and $\Delta t = 0.0125$ at time $t = 2.5$ (left figure) and $t = 5.0$ (right figure) respectively. We observe that artificial oscillations occur if the spatial step is taken too large.

We should not take the spatial step $\Delta x$ too large, as it may not be able to cover the accuracy of our numerical discretisation. Taking the spatial step $\Delta x$ too large may lead to unphysical solutions. That is, artificial oscillations may occur in the numerical solutions if the spatial step $\Delta x$ is too large. This is shown in Figure 3. This figure shows the solution at time $t = 2.5$ and $t = 5.0$, respectively using the same numerical settings as before, except the spatial step here is $\Delta x = 0.05$.

5. Conclusion
We have investigated the performance of an explicit finite difference method for solving the advection-diffusion equation. As we use the forward difference for the time derivative, numerical solutions should have the first order of accuracy at best. The general analytical solution to the advection-diffusion equation is not available, so we need the numerical method to solve the advection-diffusion equation for the general case. The proposed method can be extended for solving the advection-diffusion equation with a source term.
Acknowledgment
This research was financially supported by Universitas Sanata Dharma (Sanata Dharma University) and Direktorat Riset dan Pengabdian kepada Masyarakat of Direktorat Jenderal Penguatan Riset dan Pengembangan of Kementerian Riset, Teknologi, dan Pendidikan Tinggi of Republik Indonesia in the financial year 2017 with Research Contract Number 051/HB-LIT/IV/2017 dated 14 April 2017 (under DIPA-042.06-0.1.401516/2017 funds). Both authors are very grateful for the financial support.

References
[1] Benito M, Charpentier C and Kojouharov H V 2013 An unconditionally positivity preserving scheme for advection-diffusion reaction equations Mathematical and Computer Modelling 57 2177
[2] Company R, Ponsoda E, Romero J V and Roselló M D 2009 A second order numerical method for solving advection-diffusion models Mathematical and Computer Modelling 50 806
[3] Karahan H 2007 A third-order upwind scheme for the advection-diffusion equation using spreadsheets Advances in Engineering Software 38 688
[4] Karahan H 2007 Unconditional stable explicit finite difference technique for the advection-diffusion equation using spreadsheets Advances in Engineering Software 38 80
[5] Mohebbi A and Dehghan M 2010 High-order compact solution of the one-dimensional heat and advection-diffusion equations Applied Mathematical Modelling 34 3071
[6] Mojtabi A and Deville M O 2015 One-dimensional linear advection-diffusion equation: Analytical and finite element solutions Computers & Fluids 107 189
[7] Nazir T, Abbas M, Ismail A I M, Majid A A and Rashid A 2016 The numerical solution of advection-diffusion problems using new cubic trigonometric B-splines approach Applied Mathematical Modelling 40 4586
[8] Nishikawa H 2014 First, second, and third order finite-volume schemes for advection-diffusion Journal of Computational Physics 273 287
[9] Prieto F U, Muñoz J J B, Corvinos L G 2011 Application of the generalized finite difference method to solve the advection-diffusion equation Journal of Computational and Applied Mathematics 235 1849
[10] Ricchiuto M, Villeldeiu N, Abgrall R, and Deconinck H 2008 On uniformly high-order accurate residual distribution schemes for advection-diffusion Journal of Computational and Applied Mathematics 215 547
[11] Sankaranarayanan S, Shankar N J and Cheong H F 1998 Three-dimensional finite difference model for transport of conservative pollutants Ocean Engineering 25 425
[12] Saul'Yev V K 1964 Integration of Equations of Parabolic Type by the Method of Nets (Oxford: Pergamon Press)
[13] Savovic S and Djordjevich A 2012 Finite difference solution of the one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media International Journal of Heat and Mass Transfer 55 4291
[14] Sousa E 2009 Finite difference approximations for a fractional advection diffusion problem Journal of Computational Physics 228 4038
[15] Witek M L, Teixeira J and Flatau P J 2008 On stable and explicit numerical methods for the advection-diffusion equation Mathematics and Computers in Simulation 79 561