Tait colorings, and an instanton homology for webs and foams

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Tait colorings, and an instanton homology for webs and foams

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Abstract. We use SO(3) gauge theory to define a functor from a category of unoriented webs and foams to the category of finite-dimensional vector spaces over the field of two elements. We prove a non-vanishing theorem for this SO(3) instanton homology of webs, using Gabai’s sutured manifold theory. It is hoped that the non-vanishing theorem may support a program to provide a new proof of the four-color theorem.

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1 Introduction

1.1 Statement of results

By a web in $\mathbb{R}^3$ we shall mean an embedded trivalent graph. More specifically, a web will be a compact subset $K \subset \mathbb{R}^3$ with a finite set $V$ of distinguished points, the vertices, such that $K \setminus V$ is a smooth 1-dimensional submanifold of $\mathbb{R}^3$, and such that each vertex has a neighborhood in which $K$ is diffeomorphic to three distinct, coplanar rays in $\mathbb{R}^3$. We shall refer to the components of $K \setminus V$ as the edges. Note that our definition allows an edge to be a (possibly knotted) circle. The vertex set may be empty.

An edge $e$ of $K$ is an embedded bridge if there is a smoothly embedded 2-sphere in $\mathbb{R}^3$ which meets $e$ transversely in a single point and is otherwise disjoint from $K$. We use the term “embedded bridge” to distinguish this from the more general notion of a bridge, which is an edge whose removal increases the number of connected components of $K$.

A Tait coloring of $K$ is a function from the edges of $K$ to a 3-element set of “colors” $\{1, 2, 3\}$ such that edges of three different colors are incident at each vertex. Tait colorings which differ only by a permutation of the colors will still be regarded as distinct; so if $K$ is a single circle, for example, then it has three Tait colorings.

In this paper, we will show how to associate a finite-dimensional $\mathbb{F}$-vector space $J^\#(K)$ to any web $K$ in $\mathbb{R}^3$, using a variant of the instanton homology for knots from [5] and [15], but with gauge group $SO(3)$ replacing the group $SU(2)$ as it appeared in [15]. (Here $\mathbb{F}$ is the field of 2 elements.) We will establish the following non-vanishing property for $J^\#$.

**Theorem 1.1.** For a web $K \subset \mathbb{R}^3$, the vector space $J^\#(K)$ is zero if and only if $K$ has an embedded bridge.

Based on the evidence of small examples, and some general properties of $J^\#$, we also make the following conjecture:

**Conjecture 1.2.** If the web $K$ lies in the plane, so $K \subset \mathbb{R}^2 \subset \mathbb{R}^3$, then the dimension of $J^\#(K)$ is equal to the number of Tait colorings of $K$.

The conjecture is true for planar webs which are bipartite. Slightly more generally, we know that a minimal counterexample cannot contain any squares, triangles, bigons or circles. If the conjecture is true in general, then, together with the preceding theorem, it would establish that every bridgeless, planar trivalent...
graph admits a Tait coloring. The existence of Tait colorings for bridgeless trivalent graphs in the plane is equivalent to the four-color theorem (see [22]), so the above conjecture would provide an alternative proof of the theorem of Appel and Haken [1]: every planar graph admits a four-coloring.

1.2 Functoriality

The terminology of graphs as ‘webs’ goes back to Kuperberg [16] and was used by Khovanov in [7], where Khovanov’s $\mathfrak{s}_3$ homology, $F(K)$, was defined for oriented, bipartite webs $K$. The results of [7] have motivated a lot of the constructions in this paper, but our webs are more general, being unoriented and without the bipartite restriction. At least for planar webs, one can repeat Khovanov’s combinatorial definitions without the bipartite condition, but only by passing to $\mathbb{F}$ coefficients. (We will take this up briefly in section 8.3.) In this way one can sketch a possible combinatorial counterpart to $J^\#(K)$, though it is not clear to the authors how to calculate it, or even that this combinatorial version is finite-dimensional in general.

A key property of $J^\#(K)$ is its functoriality for certain singular cobordisms between webs. Following [7], these singular cobordisms will be called foams, though the foams that are appropriate here are more general than those of [7], in line with our more general webs (see also [17]). A (closed) foam $\Sigma$ in $\mathbb{R}^4$ will be a compact 2-dimensional subcomplex decorated with “dots”. The subcomplex is required to have one of the following models at each point $x \in \Sigma$:

(a) a smoothly embedded 2-manifold in a neighborhood of $x$;

(b) a neighborhood modeled on $\mathbb{R} \times K_3$ where $K_3 \subset \mathbb{R}^3$ is the union of three distinct rays meeting at the origin; or

(c) a cone in $\mathbb{R}^3 \subset \mathbb{R}^4$ whose vertex is $x$ and whose base is the union of 4 distinct points in $S^2$ joined pairwise by 6 non-intersecting great-circle arcs, each of length at most $\pi$.

Points of the third type will be called tetrahedral points, because a neighborhood has the combinatorics of a cone on the 1-skeleton of a tetrahedron. (Singular points of this type appear in the foams of [17].) The points of the second type form a union of arcs called the seams. The seams and tetrahedral points together form an embedded graph in $\mathbb{R}^4$ whose vertices have valence 4. The complement of the tetrahedral points and seams is a smoothly embedded 2-manifold whose
components are the *facets* of the foam. Each facet is decorated with a number of *
dots*: a possibly empty collection of marked points. No orientability is required of the facets of $\Sigma$.

Given webs $K$ and $K'$ in $\mathbb{R}^3$, we can also consider *foams with boundary*, $\Sigma \subset [a, b] \times \mathbb{R}^3$, modeled on $[a, a + \epsilon) \times K$ at one end and $(b - \epsilon, b] \times K'$ at the other. The seams and tetrahedral points comprise a graph with interior vertices of valence 4 and vertices of valence 1 at the boundary. are either circles or arcs whose endpoints lie on the boundary. We will refer to such a $\Sigma$ as a (foam)-cobordism from $K$ to $K'$. A cobordism gives rise to linear map,

$$J^\#(\Sigma) : J^\#(K) \to J^\#(K').$$

Cobordisms can be composed in the obvious way, and composite cobordisms give rise to composite maps, so we can regard $J^\#$ as a functor from a suitable category of webs and foams to the category of vector spaces over $\mathbb{F}$.

### 1.3 Remarks about the proof of Theorem 1.1

Theorem 1.1 belongs to the same family as other non-vanishing theorems for instanton homology proved by the authors in [13] and [15]. As with the earlier results, the proof rests on the Gabai’s existence theorem for sutured manifold hierarchies, for taut sutured manifolds [6].

Given a web $K \subset \mathbb{R}^3$, let us remove a small open ball around each of the vertices, to obtain a manifold with boundary, each boundary component being sphere with 3 marked points. Now identify these boundary components in pairs (the number of vertices in a web is always even) to obtain a link $K^+$ in the connected sum of $\mathbb{R}^3$ with a number of copies of $S^1 \times S^2$. Add a point at infinity to replace $\mathbb{R}^3$ by $S^3$. Let $M$ be the complement of a tubular neighborhood of $K^+$. Form a sutured manifold $(M, \gamma)$ by putting two meridional sutures on each of the torus boundary components of $M$. By a sequence of applications of an excision principle, we will show that the non-vanishing of $J^\#(K)$ is implied by the non-vanishing of the *sutured instanton homology*, $SHI(M, \gamma)$, as defined in [13]. Furthermore, $SHI(M, \gamma)$ will be non-zero provided that $(M, \gamma)$ is a *taut* sutured manifold in the sense of [6]. Finally, by an elementary topological argument, we show that the tautness of $(M, \gamma)$ is equivalent to the conditions that $K \subset \mathbb{R}^3$ is not split (i.e. admits no separating embedded 2-sphere) and has no embedded bridge. This establishes Theorem 1.1 for the case of non-split webs, which is sufficient because of a multiplicative property of $J^\#$ for split unions.
1.4 Outline of the paper

Section 2 deals with the gauge theory that is used in the definition of \( J^\# \). Section 3 defines the functor \( J^\# \) and establishes its basic properties. We define \( J^\# \) more generally than in this introduction, considering webs in arbitrary 3-manifolds. In section 6, we establish properties of \( J^\# \) that are sufficient to calculate \( J^\#(K) \) at least when \( K \) is a bipartite planar graph, as well as some other simple cases. The arguments in section 6 depend on applications of an excision principle which is discussed in section 4. The proof of Theorem 1.1 is given in section 7. Some further questions are discussed in section 8.

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2 Preliminaries

2.1 Orbifolds

We will consider 3- and 4-dimensional orbifolds with the following local models. In dimension 3, we require that every singular point of the orbifold has a neighborhood which is modeled either on \( \mathbb{R} \times (\mathbb{R}^2/\{\pm1\}) \) or on \( \mathbb{R}^3/V_4 \), where \( V_4 \) is a standard Klein 4-group in \( SO(3) \). In the first of these cases, the singular set is locally a smooth 1-manifold, and in the second case the singular set is three half-lines meeting at a single vertex. In dimension 4, we allow local models which are either products of one of the above two 3-dimensional models with \( \mathbb{R} \), or one additional case, the quotient of \( \mathbb{R}^4 \) by the group \( V_8 \subset SO(4) \) consisting of all diagonal matrices with entries \( \pm1 \) and determinant 1.

Every point in such an orbifold has a neighborhood \( U \) which is the codomain of an orbifold chart,

\[
\phi : \tilde{U} \to U.
\]

The map \( \phi \) is a quotient map for an action of a group \( H \) which is either trivial, \( F \), \( V_4 \) or \( V_8 \). If \( x \) is a point in \( U \) and \( \tilde{x} \) a preimage point, we write \( H_x \) for the stabilizer of \( \tilde{x} \).

We introduce a term for this restricted class of orbifolds:

Definition 2.1. A bifold will mean a 3- or 4-dimensional orbifold whose local models belong to one of the particular types described above.
Our bifolds will be equipped with an orbifold Riemannian metric, which we can regard as a smooth metric \( \tilde{g} \) on the complement of the singular set with the property that the pull-back of \( \tilde{g} \) via any orbifold chart \( \phi \) extends to a smooth metric on the domain \( \tilde{U} \) of \( \phi \). In the 3-dimensional case, the singular set is an embedded web and the total space of the bifold is topologically a manifold. The Riemannian metric will have cone-angle \( \pi \) along each edge of the graph. In the 4-dimensional case, the singular set is an embedded 2-dimensional complex \( \Sigma \) with the same combinatorial structure as a foam, and cone-angle \( \pi \) along the facets of \( \Sigma \).

### 2.2 Bifolds from embedded webs and foams

Given a smooth Riemannian 3-manifold \( Y \) containing a web \( K \), smoothly embedded in the sense described in the introduction, we can construct a Riemannian bifold \( \tilde{Y} \) with a copy of \( K \) as its singular set by modifying the Riemannian metric in a neighborhood of \( K \) to introduce the required cone angles. To do this, we first adjust \( K \) so that, at each vertex the incident edges arrive with tangent directions which are coplanar and at equal angles \( 2\pi/3 \). This can be done in a standard way, because the space of triples of distinct points in \( S^2 \) deformation-retracts to the subspace of planar equilateral triangles. Once this is done, the neighborhoods of the vertices are standard and we can use the smooth Riemannian metric \( g \) on \( Y \) to introduce coordinates in which to modify the Riemannian metric, making it isometric to \( \mathbb{R}^3/V_4 \) in a neighborhood of the singular set near the vertex. Along each edge, we continue the modification, using exponential normal coordinates to modify the metric and introduce a cone-angle \( \pi \).

In dimension 4, we can similarly start with a smooth Riemannian 4-manifold \((X,g)\) and an embedded foam \( \Sigma \), and modify the metric near \( \Sigma \) to produce a bifold \((\tilde{X},\tilde{g})\). Again, the first step is to modify \( \Sigma \) near the tetrahedral points and seams so that, at each point \( x \) on a seam, the tangent planes of the three incident branches of \( \Sigma \) lie in a single 3-dimensional subspace of \( T_x\mathbb{R}^4 \) and are separated by angles \( 2\pi/3 \). The structure of \( \Sigma \) is then locally standard, and we can use normal coordinates as before.

We will often pass freely from a pair \((Y,K)\) or \((X,\Sigma)\), consisting of a Riemannian 3- or 4-manifold with an embedded web or foam, to the corresponding Riemannian bifold \( \tilde{Y} \) or \( \tilde{X} \).
2.3 Orbifold connections

Let $\hat{Y}$ be a closed, connected 3-dimensional bifold, and $K \subset \hat{Y}$ the singular part (which we may regard as a web). A $C^\infty$ orbifold $SO(3)$-connection over $\hat{Y}$ means an oriented $\mathbb{R}^3$-vector bundle $E$ over $\hat{Y} \setminus K$ with an $SO(3)$ connection $A$ having the property that the pull-back of $(E, A)$ via any orbifold chart $\phi : \hat{U} \to U$ extends to a smooth pair $(\hat{E}, A)$ on $\hat{U}$. (It may be that the bundle $E$ cannot be extended to all of $\hat{Y}$ as a topological vector bundle.)

If $U$ is the codomain of an orbifold chart around $x$ and $\hat{x}$ is a preimage in $\hat{U}$, then the stabilizer $H_x$ acts on the fiber $\hat{E}_{\hat{x}}$. We will require that this action is non-trivial at all points where $H_x$ has order 2, i.e. at all points on the edges of the web $K$. We introduce a name for orbifold connections of this type:

**Definition 2.2.** A bifold connection on a bifold $\hat{Y}$ will mean an $SO(3)$ connection for which all the order-2 stabilizer groups $H_x$ act non-trivially on the corresponding fibers $\hat{E}_{\hat{x}}$.

This determines the action of $H_x$ on $\hat{E}_{\hat{x}} \cong \mathbb{R}^3$ also at the vertices of $K$, up to conjugacy: we have a standard action of $V_4$ on the fiber. If $\gamma$ is a standard meridional loop of diameter $\epsilon$ about an edge of $K$, then the holonomy of $A$ will approach an element of order 2 in $SO(3)$ as $\epsilon$ goes to zero.

Given an orbifold connection $(E, A)$, we can use the connection $A$ and the Levi-Civita connection of the Riemannian metric $\hat{g}$ to define Sobolev norms on the spaces of sections of $E$ and its associated vector bundles. We fix a sufficiently large Sobolev exponent $l$ ($l \geq 3$ suffices) and we consider orbifold $SO(3)$-connections $A$ of class $L^2_l$. By this we mean that there exists a $C^\infty$ bifold connection $(E, A_0)$ so that $A$ can be written as $A_0 + a$, where $a$ belongs to the Sobolev space

$$L^2_{l} (\hat{Y} \setminus K, \Lambda^1 Y \otimes E).$$

Under these circumstances, the Sobolev norms $L^2_{l,A}$ and $L^2_{l,A_0}$ are equivalent for $j \leq l + 1$.

If we have two $SO(3)$ bifold connections of class $L^2_l$, say $(E, A)$ and $(E', A')$, then an isomorphism between them is a bundle map $\tau : E \to E'$ over $\hat{Y} \setminus K$ of class $L^2_{l+1}$ such that $\tau^*(A') = A$. The group $\Gamma_{E,A}$ of automorphisms of $(E, A)$ can be identified as usual with the group of parallel sections of the associated bundle with fiber $SO(3)$. It is isomorphic to either the trivial group, the group of order 2, the group $V_4$, the group $O(2)$ or all of $SO(3)$. If $K$ has at least one vertex, then $\Gamma_{E,A}$ is no larger than $V_4$. 
We write $\mathcal{B}_l(\tilde{Y})$ for the space of all isomorphism classes of bifold connections of class $L^2_l$. By the usual application of a Coulomb slice, the neighborhood of an isomorphism class $[E, A]$ in $\mathcal{B}_l(\tilde{Y})$ can be given the structure $S/\Gamma_{E,A}$, where $S$ is a neighborhood of 0 in a Banach space and the group acts linearly. In particular, if $K$ has at least one vertex, then $\mathcal{B}_l(\tilde{Y})$ is a Banach orbifold.

All the content of this subsection carries over without essential change to the case of a 4-dimensional bifold $\tilde{X}$, where we also have a space $\mathcal{B}_l(\tilde{X})$ of isomorphism classes of $SO(3)$ bifold connections of class $L^2_l$. We still require that the order-2 stabilizers $H_x$ act non-trivially on the fibers, and this condition determines the action of the order-8 stabilizers $V_8$ at the tetrahedral points, up to conjugacy.

In the 4-dimensional case (and sometimes also in the 3-dimensional case, if $Y \setminus K$ has non trivial homology) the space $\mathcal{B}_l(\tilde{X})$ or $\mathcal{B}_l(\tilde{Y})$ will have than one component, because the $SO(3)$ bifold connections belong to different topological types.

### 2.4 Marked connections

Because $SO(3)$ bifold connections may have non-trivial automorphism groups, we introduce marked connections. By marking data $\mu$ on a 3-dimensional bifold $\tilde{Y}$ we mean a pair $(U_\mu, E_\mu)$, where:

- $U_\mu \subset \tilde{Y}$ is any subset; and
- $E_\mu \to U_\mu \setminus K$ is an $SO(3)$ bundle (where $K$ denotes the singular set).

An $SO(3)$ connection marked by $\mu$ will mean a triple $(E, A, \sigma)$, where

- $(E, A)$ is a bifold $SO(3)$ connection as before; and
- $\sigma : E_\mu \to E|_{U_\mu \setminus K}$ is an isomorphism of $SO(3)$ bundles.

An isomorphism between marked connections $(E, A, \sigma)$ and $(E', A', \sigma')$ is a bundle map $\tau : E' \to E$ such that

- $\tau$ respects the connections, so $\tau^*(A) = A'$; and
- the automorphism $\sigma^{-1} \tau \sigma' : E_\mu \to E_\mu$

lifts to the “determinant-1 gauge group”. That is, when we regard $\sigma^{-1} \tau \sigma'$ as a section of the bundle $SO(3)_{E_\mu}$ with fiber $SO(3)$ (associated to $E_\mu$ by the
adjoint action), we require that this section lift to a section of the bundle $SU(2)_{E_\mu}$ associated to the adjoint action of $SO(3)$ on $SU(2) = Spin(3)$.

We will say that the marking data $\mu$ is strong if the automorphism group of every $\mu$-marked bifold connection on $\tilde{Y}$ is trivial.

**Lemma 2.3.** The marking data $\mu$ on a 3-dimensional bifold $\tilde{Y}$ is strong if either of the following hold.

(a) The set $U_\mu$ contains a neighborhood of a vertex of $K$.

(b) The set $U_\mu$ contains a 3-ball $B$ which meets $K$ in a Hopf link $H$ contained in the interior of $B$, and $w_2(E_\mu)|_B$ is Poincaré dual to an arc joining the two components of $H$.

**Proof.** From a $\mu$-marked bundle $(E, A, \tau)$, we obtain by pull-back a connection $A_\mu$ in the bundle $E_\mu$. The automorphisms of $(E, A, \tau)$ are a subgroup of the group of $A_\mu$-parallel sections of the associated bundle $SU(2)_\mu$ with fiber $SU(2)$.

Pick a point $y$ in $U_\mu \setminus K$ and consider the holonomy group of $A_\mu$ at this point, as a subgroup of $SO(3)$ (the automorphisms of the fiber of $E_\mu$ at $p$). In case (a), the closure of the holonomy group contains $V_4$. We can find a subset of $U_\mu$ with trivial second homology over which the closure of the holonomy group still contains $V_4$. Over this subset, we can lift $E_\mu$ to an $SU(2)$ bundle and lift $A_\mu$ to an $SU(2)$ connection $\hat{A}_\mu$. The holonomy group of $\hat{A}_\mu$ contains lifts of the generators of $V_4$, which implies that its commutant in $SU(2)$ is trivial.

In case (b), pick one component of the Hopf link and a point $q$ on this component. Using loops based at $p$ running around small meridional loops near $q$, we see that the closure of the holonomy group contains an element $h_q$ of order 2 in $SO(3)$. As $q$ varies, this element of order 2 cannot be constant, for otherwise $w_2$ would be zero. So the holonomy group contains two distinct involutions $h_q$ and $h_q'$. We can now lift to $SU(2)$ as in the previous case to see that the commutant in $SU(2)$ is trivial. \qed

We write $\mathcal{B}_l(\tilde{Y}; \mu)$ for the space of isomorphism classes of $\mu$-marked bifold connections of class $L^2$ on the bifold $\tilde{Y}$. If the marking data is strong, then $\mathcal{B}_l(\tilde{Y}; \mu)$ is a Banach manifold modeled locally on the Coulomb slices. There is a map that forgets the marking,

$$\mathcal{B}_l(\tilde{Y}; \mu) \to \mathcal{B}_l(\tilde{Y}).$$

We write $\mathcal{B}_l(\tilde{Y}; \mu)$ for the space of isomorphism classes of $\mu$-marked bifold connections of class $L^2$ on the bifold $\tilde{Y}$. If the marking data is strong, then $\mathcal{B}_l(\tilde{Y}; \mu)$ is a Banach manifold modeled locally on the Coulomb slices. There is a map that forgets the marking,
Its image consists of a union of some connected components of $B_l(\check{\mathcal{Y}})$, namely the components comprised of isomorphism classes of connections $(E,A)$ for which the restriction of $E$ to $U_\mu \setminus K$ is isomorphic to $E_\mu$.

Slightly more generally, we can consider the case that we have two different marking data, $\mu$ and $\mu'$ with $U_\mu \subset U_{\mu'}$ and $E_\mu = E_{\mu'}|_{U_\mu \setminus K}$. In this case, there is a forgetful map

$$r : B_l(\check{\mathcal{Y}}; \mu') \to B_l(\check{\mathcal{Y}}; \mu).$$

(1)

**Lemma 2.4.** The image of the map (1) consists of a union of connected components of $B_l(\check{\mathcal{Y}}; \mu)$. Over these components, the map $r$ is a covering map with covering group an elementary abelian 2-group, namely the group which is the kernel of the restriction map

$$H^1(U_\mu \setminus K; \mathbb{F}) \to H^1(U_{\mu'} \setminus K; \mathbb{F}).$$

**Proof.** The image consists of isomorphism classes of triples $(E,A,\tau)$ for which the map $\tau : E_\mu \to E$ can be extended to some $\tau' : E_{\mu'} \to E$ over $U_\mu \setminus K$. This set is open and closed in $B_l(\check{\mathcal{Y}}; \mu)$, so it is a union of components of this Banach manifold.

If $(E,A,\tau')$ and $(E,A,\tau'')$ are two elements of a fiber of $r$, then the difference of $\tau'$ and $\tau''$ is an automorphism $\sigma : E_{\mu'} \to E_{\mu'}$ whose restriction to $E_\mu$ lifts to determinant 1. The fiber consists of such automorphisms $\sigma$ modulo those that lift to determinant 1 on the whole of $E_{\mu'}$. These form a group isomorphic to the kernel of the restriction map

$$H^1(U_\mu \setminus K; \mathbb{F}) \to H^1(U_{\mu'} \setminus K; \mathbb{F}).$$

Over the components that form its image, the map $r$ is a quotient map for this elementary abelian 2-group. $\square$

All of the above definitions can be formulated in the 4-dimensional case. So for a compact, connected 4-dimensional bifold $\check{X}$ we can talk about marking data $\mu$ in the same way. We have a space $B_l(\check{X}; \mu)$ parametrizing isomorphism classes of $\mu$-marked $SO(3)$ connections, and this space is a Banach manifold if $\mu$ is strong.

### 2.5 ASD connections and the index formula

Let $\check{X}$ be a closed, oriented Riemannian bifold of dimension 4, and let $(X, \Sigma)$ be the associated pair. We can, as usual, consider the anti-self-duality condition,

$$F_A^+ = 0$$
in the bifold setting. We write

\[ M(\hat{X}) \subset \mathcal{B}_l(\hat{X}) \]

for the moduli space of anti-self-dual bifold connections. It is independent of the choice of Sobolev exponent \( l \geq 3 \). We can also introduce marking data \( \mu \), and consider the moduli space

\[ M(\hat{X}; \mu) \subset \mathcal{B}_l(\hat{X}; \mu). \]

We write \( \kappa(E,A) \) for the orbifold version of the characteristic class \(-\frac{1}{4}p_1(E)\), which we can compute as the Chern-Weil integral,

\[
\kappa(E,A) = \frac{1}{32\pi^2} \int_{\hat{X}} \text{tr}(F_A \wedge F_A). \tag{2}
\]

Our normalization means that \( \kappa \) coincides with \( c_2(\hat{E})[X] \) if \( E \) lifts to an SU(2) bundle \( \hat{E} \) and \( \Sigma \) is absent. In general \( \kappa \) may be non-integral. We refer to \( \kappa \) as the (topological) action.

The formal dimension of the moduli space in the neighborhood of \([E,A]\) is given by the index of the linearized equations with gauge fixing, which we write as

\[
d(E,A) = \text{index}(-d_A^* \oplus d_A^*). \tag{3}
\]

This definition does not require \( A \) to be anti-self-dual and defines a function which is constant on the components of \( \mathcal{B}_l(\hat{X}) \). If the marking is strong, then for a generic metric \( \hat{g} \) on \( \hat{X} \), the marked moduli space \( M(\hat{X}; \mu) \) is smooth and of dimension \( d(E,A) \) in the neighborhood of any anti-self-dual connection \((E,A)\), unless the connection is flat.

We shall give a formula for the formal dimension \( d(E,A) \) in terms of \( \kappa \) and the topology of \((X,\Sigma)\). To do so, we must digress to say more about foams \( \Sigma \).

We shall define a self-intersection number \( \Sigma \cdot \Sigma \) which coincides with the usual self-intersection number of a (not necessarily orientable) surface in \( X \) if there are no seams. We can regard \( \Sigma \) as the image of an immersion of a surface with corners, \( i: \Sigma^+ \rightarrow X \), which is injective except at \( \partial \Sigma^+ \), which is mapped to the seams as a 3-fold covering. The corners of \( \Sigma^+ \) are mapped to the tetrahedral points of \( \Sigma \). The tangent spaces to the three branches of \( \Sigma \) span a 3-dimensional subspace \( V_q \) at each point \( q \) on the seams and vertices. This determines a 3-dimensional subbundle

\[
V \subset i^*(TX)|_{\partial \Sigma^+}.
\]
Since $V$ contains the directions tangent to $i(\Sigma^+)$, it determines a 1-dimensional subbundle $W$ of the normal bundle $N \to \Sigma^+$ to the immersion:

$$W \subset N|_{\partial \Sigma^+}.$$ 

Although the 2-plane bundle $N \to \Sigma^+$ is not necessarily orientable, it has a well-defined “square”, $N^{[2]}$. (Topologically, this is the bundle obtained by identifying $n$ with $-n$ everywhere.) The orientation bundles of both $N$ and $N^{[2]}$ are canonically identified with the orientation bundle of $\Sigma^+$, using the orientation of $X$. The subbundle $W$ determines a section $w$ of $N^{[2]}|_{\partial \Sigma^+}$, and there is a relative Euler number,

$$e(N^{[2]}, w)[\Sigma^+, \partial \Sigma^+]$$ \hspace{1cm} (4) 

obtained from the pairing in (co)homology with coefficients in the orientation bundle.

**Definition 2.5.** We define $\Sigma \cdot \Sigma$ to be half of the relative Euler number (4).

If $V$ is orientable, then so is $W$. In this case, we may choose a trivialization of $W$ and obtain a section of $N$ along $\partial \Sigma^+$. This removes the need to pass to the square of $N$ and also shows that $\Sigma \cdot \Sigma$ is an integer. If $V$ is non-orientable, then $\Sigma \cdot \Sigma$ may be a half-integer.

With this definition out of the way, we can state the index theorem for the formal dimension $d(E,A)$.

**Proposition 2.6.** The index $d(E,A)$ is given by

$$d(E,A) = 8\kappa - 3(1 - b^1(X) + b^+(X)) + \chi(\Sigma) + \frac{1}{2} \Sigma \cdot \Sigma - \frac{1}{2} |\tau|,$$ \hspace{1cm} (5) 

where $\chi(\Sigma)$ is the ordinary Euler number of the 2-dimensional complex underlying the foam $\Sigma$, the integer $|\tau|$ is the number of tetrahedral points of $\Sigma$, and $\Sigma \cdot \Sigma$ is the self-intersection number from Definition 2.5.

**Proof.** In the case that $\Sigma$ has no seams, the result coincides with the index formula in [15]. (If $\Sigma$ is also orientable then the formula appears in [12].) So the result is known in this case.

Consider next the case that $\Sigma$ has no tetrahedral points. The seams of $\Sigma$ form circles, and the neighborhood of each circle has one of three possible types: the three branches of the foam are permuted by the monodromy around the circle, and permutation may be trivial, an involution of two of the branches, or a cyclic
permutation of the three. So (in the absence of tetrahedral points) an excision
argument shows that it is sufficient to verify the index formula for just three
examples, one containing seams of each of three types. A standard model for a
neighborhood of a seam \(s \cong S^1\) is a foam in \(S^1 \times B^3\). By doubling this standard
model, we obtain a foam in \(X = S^1 \times S^3\) with two seams of the same type. If
the branches are permuted non-trivially by the monodromy, we can now pass
to a 2-fold or 3-fold cyclic cover of \(X\), and so reduce to the case of seams with
trivial monodromy. (Both sides of the formula in the Proposition are multiplica-
tive under finite covers.) When the monodromy permutation of the branches is
trivial, the pair \((X, \Sigma)\) that we have described carries a circle action along the \(S^1\)
factor and there is a flat bundle \((E, A)\) with holonomy group \(V_4\), also acted on by
\(S^1\). The circle action on the bundle means that the index \(d(E, A)\) is zero, as is the
right-hand side. This completes the proof in the absence of tetrahedral points.

Consider finally the case that \(\Sigma\) has tetrahedral points. By taking two copies
if \((X, \Sigma)\) if necessary, we may assume that the number of tetrahedral points is
even. By an excision argument, it is then enough to verify the formula in the
case of a standard bifold \(\hat{X}\) with two tetrahedral points, namely the quotient of
\(S^4\) by the action of

\[
V_8 \subset SO(4) \subset SO(5).
\]

In this model case, there is again a flat bifold connection with holonomy \(V_4\),
obtained as a global quotient of the trivial bundle on \(S^4\). For this example we
have \(d(E, A) = 0\), as one can see by looking at corresponding operator on the
cover \(S^4\). The pair \((X, \Sigma)\) is topologically a 4-sphere containing a foam which is
the suspension of the 1-skeleton of a tetrahedron. The Euler number \(\chi(\Sigma)\) is 4,
and the terms \(b_1(X), b^+(X)\) and \(\Sigma \cdot \Sigma\) in (5) are zero. So the formula is verified in
this case:

\[
0 = -3 + 4 - \frac{1}{2} \times 2.
\]

This completes the proof. \(\square\)

From the formula, one can see that \(\kappa(E, A)\) belongs to \((1/32)\mathbb{Z}\), because \(2(\Sigma \cdot \Sigma)\)
is an integer.

### 2.6 Bubbling

Uhlenbeck’s theorem [23] applies in the orbifold setting. So if \([E_i, A_i] \in M(\hat{X})\) is a
sequence of anti-self-dual bifold connections on the closed oriented bifold \(\hat{X}\) and
if their actions $\kappa(E_i, A_i)$ are bounded, then there exists a subsequence $\{i'\} \subset \{i\}$, an anti-self-dual connection $(E, A)$, a finite set of points $Z \subset \tilde{X}$ and isomorphisms

$$
\tau_{i'} : (E, A)|_{\tilde{X}\setminus(\Sigma \cup Z)} \to (E_{i'}, A_{i'})|_{\tilde{X}\setminus(\Sigma \cup Z)}
$$

such that the connections $(E, \tau^*(A_{i'}))$ converge on compact subsets of $\tilde{X}\setminus(K \cup Z)$ to $(E, A)$ in the $C^\infty$ topology. Furthermore,

$$
\kappa(E, A) \leq \lim \inf \kappa(E_{i'}, A_{i'}),
$$

and if equality holds then $Z$ can be taken to be empty and the convergence is strong (i.e. the convergence is in the topology of $M(\tilde{X})$).

When the inequality is strict, the difference is accounted for by bubbling at the points $Z$. The difference is therefore equal to the sum of the actions of some finite-action solutions on bifold quotients of $\mathbb{R}^4$ by either the trivial group, the group of order 2, the group $V_4$, or the group $V_8$. Any solution on such a quotient of $\mathbb{R}^4$ pulls back to a solution on $\mathbb{R}^4$ with integer action; so on the bifold, the action of any bubble lies in $(1/8)\mathbb{Z}$. If there are no tetrahedral points, then the action lies in $(1/4)\mathbb{Z}$. Thus we have:

**Lemma 2.7.** In the Uhlenbeck limit, either the action inequality (6) is an equality and the convergence is strong, or the difference is at least $1/8$. If the difference is exactly $1/8$, then the set of bubble-points $Z$ consists of a single point which is a tetrahedral point of $\Sigma$.

If there are no tetrahedral points, and the energy-loss is non-zero, then the difference is at least $1/4$. If the difference is exactly $1/4$, then the set of bubble-points $Z$ consists of a single point which lies on a seam of $\Sigma$.

Suppose that $\Sigma$ has no tetrahedral points. From the dimension formula, Proposition 2.6, we see that when bubbling occurs, the dimension of the moduli space drops by at least 2. Let us write $M_d(\tilde{X}; \mu)$ for the moduli space of $\mu$-marked instantons $(E, A, \tau)$ whose action $\kappa$ is such that the formal dimension at $[E, A, \tau]$ is $d$. Suppose that the marking data $\mu$ is strong, that $M_0(\tilde{X}; \mu)$ is regular, and that all moduli space of negative formal dimension are empty. Uhlenbeck’s theorem and the Lemma tell us that $M_0(\tilde{X}; \mu)$ is a finite set and that $M_d(\tilde{X}; \mu)$ has a compactification,

$$
\tilde{M}_2 = M_2 \cup s \times M_0,
$$

where $s$ is the union of the seams of $\Sigma$. 
Proposition 2.8. In the above situation, when $\Sigma$ has no tetrahedral point, for each $q \in s$ and $\alpha \in M_0$, an open neighborhood of $(q, \alpha)$ in $\bar{M}_2$ is homeomorphic to

$$N(q) \times T$$

where $N(q) \subset s$ is a neighborhood of $q$ in the seam $s$, and $T$ is a cone on 4 points (i.e. the union of 4 half-open intervals $[0,1)$ with their endpoints 0 identified). In particular, if $M_2$ is regular, then $\bar{M}_2$ is homeomorphic to an identification space of a compact 2-manifold with boundary, $S$, by an identification which maps $\partial S$ to $s$ by a 4-sheeted covering.

Proof. This proposition is an orbifold adaptation of the more familiar non-orbifold version [3] which describes the neighborhood of the stratum $X \times M_0$ in the Uhlenbeck compactification of $M_8$. In the non-orbifold version, the local model near $(x, [A])$ is $N(x) \times \mathrm{Cone}(\mathrm{SO}(3))$, where $N(x)$ is a 4-dimensional neighborhood of $x \in X$ and $\mathrm{SO}(3)$ arises as the gluing parameter. The above proposition is similar, except that the cone on $\mathrm{SO}(3)$ has been replaced by $T$, which is a cone on the Klein 4-group $V_4$.

Let $\mathbb{R}^4$ be Euclidean 4-space and let $\mathbb{R}^4$ be its bifold quotient by $V_4$ acting on the last three coordinates. Let $s \subset \mathbb{R}^4$ be the seam, i.e. the line $\mathbb{R} \times 0$ fixed by $V_4$. The standard 1-instanton moduli space on $\mathbb{R}^4$ is the 5-dimensional space with center and scale coordinates:

$$M_5(\mathbb{R}^4) = \mathbb{R}^4 \times \mathbb{R}^+.$$ 

The moduli space of solutions with $\kappa = 1/4$ on $\mathbb{R}^4$ is the set of fixed points of the $V_4$ action on $M_5$. This space is 2-dimensional, with a center coordinate constrained to lie on $s$ and scale coordinate as before:

$$M_2(\mathbb{R}^4) = s \times \mathbb{R}^+.$$ 

On $\mathbb{R}^4$, we may pass to the 8-dimensional framed moduli space $\tilde{M}_8(\mathbb{R}^4)$ parametrizing isomorphism classes of instantons $A$ together with an identification of the limiting flat connection on the sphere at infinity $S^3_\infty$ with the trivial connection in $\mathbb{R}^3 \times S^3_\infty$. This is the product,

$$\tilde{M}_8(\mathbb{R}^4) = \mathbb{R}^4 \times \mathbb{R}^+ \times \mathrm{SO}(3).$$ 

In the bifold case, the framing data is an identification of the flat bifold connection on $S^3_\infty/V_4$ with a standard bifold connection whose holonomy group is $V_4$. The
choice of identification is now the commutant of $V_4$ in $SO(3)$, namely $V_4$ itself. So the framed moduli space is

$$\tilde{M}_2(\mathbb{R}^4) = s \times \mathbb{R}^+ \times V_4.$$  

With this moduli space understood, the usual proof from the non-orbifold case carries over without change. □

When tetrahedral points are present, bubbling is a codimension-1 phenomenon, meaning that even 1-dimensional moduli spaces may be non-compact. We have the following counterpart of the previous proposition. Consider a 1-dimensional moduli space $M_1$ of bifold connections on $(X, \Sigma)$. Its Uhlenbeck compactification is the space

$$\tilde{M}_1 = M_1 \cup \tau \times M_0,$$

where $\tau$ is the finite set of tetrahedral points of $\Sigma$.

**Proposition 2.9.** In the above situation, for each tetrahedral point $q$ and $\alpha \in M_0$, an open neighborhood of $(q, \alpha)$ in $\tilde{M}_1$ is homeomorphic to $T$, i.e. again the union of 4 half-open intervals $[0, 1)$ with their endpoints 0 identified. In particular, if $M_1$ is regular, then $\tilde{M}_1$ is homeomorphic to an identification space of a compact 1-manifold with boundary, by an identification which identifies the boundary points in sets of four.

**Proof.** The proof is much the same as the proof of the previous proposition. □

### 2.7 The Chern-Simons functional

Let $\hat{Y}$ be a closed, oriented, 3-dimensional bifold and let $\mu$ be strong marking data on $\hat{Y}$. The tangent space to $B_l(\hat{Y}; \mu)$ at a point represented by a connection $(E, A)$ is isomorphic to the kernel of $d_A^*$ acting on $L^2(\hat{Y}; \Lambda^1 \hat{Y} \otimes E)$. As usual, there is a locally-defined smooth function, the Chern-Simons function, on $B_l(\hat{Y}; \mu)$ whose formal $L^2$ gradient is $* F_A$. On the universal cover of each component of $B_l(\hat{Y}; \mu)$, the Chern-Simons function is single-valued and well-defined up to the addition of a constant.

If we have a closed loop $z$ in $B_l(\hat{Y}; \mu)$, then we have a 1-parameter family of marked connections,

$$\zeta(t) = (E(t), A(t), \tau(t))$$

parametrized by $[0, 1]$ together with an isomorphism $\sigma$ from $\zeta(0)$ to $\zeta(1)$. The isomorphism $\sigma$ is determined uniquely by the data, because the marking is strong,
which means that $\zeta(0)$ has no automorphisms. Identifying the two ends, we have a connection $(E_z, A_z)$ over the bifold $S^1 \times \hat{Y}$.

This 4-dimensional connection has an index $d(E_z, A_z)$ and an action $\kappa(E_z, A_z)$. We can interpret both as usual, in terms of the Chern-Simons function. Up to a constant factor, $\kappa(E_z, A_z)$ is equal to minus the change in CS along the path $\zeta(t)$:

$$\kappa(E_z, A_z) = \frac{1}{32\pi^2} (\text{CS}(\zeta(0)) - \text{CS}(\zeta(1))).$$

The index $d(E_z, A_z)$ is equal to the spectral flow of a family of elliptic operators related to the formal Hessian of CS, along the path $\zeta$: that is,

$$d(E_z, A_z) = \text{sf}_\zeta(D_A)$$

where $D_A$ is the “extended Hessian” operator on $\hat{Y}$,

$$D_A = \begin{bmatrix} 0 & -d_A^* \\ -d_A & *d_A \end{bmatrix}$$

acting on $L_2^2$ sections of $(\Lambda^0 \oplus \Lambda^1) \otimes E$.

An important consequence of the index formula, Proposition 2.6, is the proportionality of these of two quantities. For any path $\zeta$ corresponding to a closed loop $z$ in $B_l(\hat{Y}; \mu)$, we have

$$\text{CS}(\zeta(0)) - \text{CS}(\zeta(1)) = 4\pi^2 \text{sf}_\zeta(D_A).$$

(7)

Since the spectral flow is always an integer, we also see that CS defines a single-valued function with values in the circle $\mathbb{R}/(4\pi^2\mathbb{Z})$.

2.8 Representation varieties

The critical points of $\text{CS} : B_l(\hat{Y}; \mu) \to \mathbb{R}/(4\pi^2\mathbb{Z})$ are the isomorphism classes of $\mu$-marked connections $(E, A, \tau)$ for which $A$ is flat. We will refer to the critical point set as the representation variety and write

$$\mathcal{R}(\hat{Y}; \mu) \subset B_l(\hat{Y}; \mu).$$

We may also write this as $\mathcal{R}(Y, K; \mu)$. In the absence of the marking, this would coincide with a space of homomorphisms from the orbifold fundamental group $\pi_1(\hat{Y}, y_0)$ to $\text{SO}(3)$, modulo the action of $\text{SO}(3)$ by conjugation. In terms of the pair $(Y, K)$, we are looking at conjugacy classes of homomorphisms

$$\rho : \pi_1(Y \setminus K, y_0) \to \text{SO}(3)$$
satisfying the constraint that, for each edge $e$ and any representative $m_e$ for the conjugacy class of the meridian of $e$, the element $\rho(m_e)$ has order 2 in $SO(3)$.

We can adapt this description to incorporate the marking data $\mu = (U_\mu, E_\mu)$ as follows. Let $w \subset U_\mu$ be a closed 1-dimensional submanifold dual to $w_2(E_\mu)$. One $U_\mu \setminus w$, we can lift $E_\mu$ to an $SU(2)$ bundle $\tilde{E}_\mu$, and we fix an isomorphism $\rho : \text{ad}(\tilde{E}_\mu) \to E_\mu|_{U_\mu \setminus w}$. Via $\rho$, any flat $SO(3)$ connection $A_\mu$ on $E_\mu$ gives rise to a flat $SU(2)$ connection $\tilde{A}_\mu$ on $\tilde{E}_\mu$ with the property that, for each component $v \subset w$, the holonomy the meridian $m_v$ of $v$ is $-1 \in SU(2)$. If we pick a basepoint $y_0$ in $U_\mu \setminus (K \cup w)$, then we have the following description. The representation variety $\mathcal{R}(Y; \mu)$ corresponds to $SO(3)$-conjugacy classes of pairs $(\rho, \tilde{\rho}_\mu)$, where

(a) $\rho : \pi_1(Y \setminus K, y_0) \to SO(3)$ is a homomorphism with the property that $\rho(m_e)$ has order 2, for every edge $e$ of $K$;

(b) $\tilde{\rho}_\mu : \pi_1(U_\mu \setminus (K \cup w), y_0) \to SU(2)$ is a homomorphism with the property that $\tilde{\rho}_\mu(m_v)$ is $-1$, for every component $v$ of $w$;

(c) the diagram formed from $\rho$, $\tilde{\rho}_\mu$, the adjoint homomorphism $SU(2) \to SO(3)$ and the inclusion $U_\mu \setminus (K \cup w) \to Y \setminus K$ commutes.

Note that these conditions imply that $\tilde{\rho}_\mu(m_e)$ has order 4 in $SU(2)$, for every meridian of $K$ contained in $U_\mu$. We give two basic examples.

**Example: the theta graph.** Take $Y = S^3$ and $K = \Theta$ an unknotted theta graph: two vertices joined by three coplanar arcs. Take $U_\mu$ to be a ball containing $\Theta$ and take $E_\mu$ to be the trivial $SO(3)$ bundle. Lemma 2.3 tells us that $\mu$ is strong. Using the above description, we see that $\mathcal{R}(S^3, \Theta; \mu)$ is the space of conjugacy classes of homomorphism $\tilde{\rho} : \pi_1(B^3 \setminus \Theta) \to SU(2)$ mapping meridians to elements of order 4. Up to conjugacy, there is exactly one such homomorphism. Its image is the quaternion group of order 8. So $\mathcal{R}(S^3, \Theta; \mu)$ consists of a single point.

**Example: a Hopf link.** Take $Y = S^3$ again and take $K = H$, a Hopf link. Take $U_\mu$ to be a ball containing $H$ and take $w_2(E_\mu)$ to be dual to an arc $w$ joining the two components of $H$. This example is similar to the previous one, and was exploited in [15]. The representation variety $\mathcal{R}(S^3, H; \mu)$ is again a single point,
corresponding to a unique homomorphism with image the quaternion group in $SU(2)$.

As an application of these basic examples, we have the following observation.

**Lemma 2.10.** Let $K \subset Y$ be an embedded web. Let $B$ be a ball in $Y$ disjoint from $K$, and let $K^\#$ be the disjoint union of $K$ with either a theta graph $\Theta \subset B$ or a Hopf link $H \subset B$. Let $\mu$ be marking data for $(Y, K^\#)$, with $U_\mu = B$. In the Hopf link case, take $w_3(E_\mu)$ to be dual to an arc joining the two components. Then the representation variety

$$R(Y, K^\#; \mu)$$

can be identified with the space of homomorphisms

$$\rho : \pi_1(Y \setminus K, y_0) \to SO(3)$$

which map meridians of $K$ to elements of order 2.

Note that the space described in the conclusion of the lemma is not the quotient by the action of $SO(3)$ by conjugation, but simply the space of homomorphisms. We introduce abbreviated terminology for the versions of these representation varieties that we use most:

**Definition 2.11.** For web $K \subset \mathbb{R}^3$, we denote by $R^\#(K)$ the space of homomorphisms

$$R^\#(K) = \{ \rho : \pi_1(\mathbb{R}^3 \setminus K) \to SO(3) | \rho(m_e) \text{ has order 2 for all edges } e \},$$

a space that we can identify with $R(S^3, K^\#; \mu)$ by the above lemma. We write $R(K)$ for the quotient by the action of conjugation,

$$R(K) = R^\#(K) / SO(3).$$

**2.9 Examples of representation varieties**

We examine the representation varieties $R^\#(K)$ and $R(K)$ for some webs $K$ in $\mathbb{R}^3$. First, we make some simple observations. As in Section 2.3, as long as $K$ is non-empty, a flat $SO(3)$ bifold connection $(E, A)$ has automorphism group $\Gamma$ described by one of the following cases, according to the image of the corresponding representation $\rho$ of the fundamental group:

(a) the image of $\rho$ is a 2-element group and the automorphism group $\Gamma$ is $O(2)$;
(b) the image of $\rho$ is $V_4$ and the $\Gamma$ is also $V_4$;

(c) the image of $\rho$ is contained in $O(2)$ and strictly contains $V_4$, so that the automorphism group $\Gamma(E,A)$ is $\mathbb{Z}/2$;

(d) the image of $\rho$ is not contained in a conjugate of $O(2)$ and the automorphism group $\Gamma$ is trivial.

The first case arises only if $K$ has no vertices. We refer to the remaining three cases as $V_4$ connections, fully $O(2)$ connections, and fully irreducible connections respectively. By an $O(2)$ connection, we mean either a fully $O(2)$ connection or a $V_4$ connection.

For a conjugacy class of representations $\rho$ in $\mathcal{R}(K)$, the preimage in $\mathcal{R}^\sharp(K)$ is a copy of $SO(3)/\Gamma$, where $\Gamma$ is described by the above cases. The quotient $SO(3)/O(2)$ is $\mathbb{RP}^2$, the quotient $SO(3)/V_4$ is the flag manifold of real, unoriented flags in $\mathbb{R}^3$, while $SO(3)/(\mathbb{Z}/2)$ is a lens space $L(4,1)$.

The following is a simple observation. (In the language of graph theory, it is the statement that Tait colorings of a cubic graph are the same as nowhere-zero 4-flows.)

**Lemma 2.12.** If $K \subset \mathbb{R}^3$ is any web, then the $V_4$ connections in $\mathcal{R}(K)$ correspond bijectively to the set of all Tait colorings of $K$ modulo permutations of the three colors.

*Proof.* Since $V_4$ is abelian, a homomorphism from $\pi_1(\mathbb{R}^3 \setminus K)$ to $V_4$ which maps each meridian to an element of order 2 is just the same as an edge-coloring of $K$ by the set of three non-trivial elements of $V_4$. \[\square\]

So the $V_4$ connections contribute one copy of the flag manifold $SO(3)/V_4$ to $\mathcal{R}^\sharp(K)$ for each permutation-class of Tait coloring of $K$.

**Example: the unknot.** Let $K$ be an unknotted circle in $S^3$. The fundamental group of the complement is $\mathbb{Z}$, and $\mathcal{R}(K)$ contains a single point corresponding to the representation with image $\mathbb{Z}/2$. The stabilizer is $O(2)$ and $\mathcal{R}^\sharp(K)$ is homeomorphic to $\mathbb{RP}^2$.

**The theta graph.** For the theta graph $K$, the representation variety $\mathcal{R}(K)$ consists of a single $V_4$ representation, corresponding to the unique class of Tait colorings for $K$. 

The tetrahedron graph. Let $K$ be the 1-skeleton of a standard simplex in $\mathbb{R}^3$, viewed as a web. All representations in $\mathcal{R}(K)$ are again $V_4$ connections, and there is only one of these, because there is only way to Tait-color this graph, up to permutation of the colors. So $\mathcal{R}^\sharp(K)$ is a copy of the flag manifold again.

The Hopf link with a bar. Let $K$ be formed by taking the two circles of the standard Hopf link and joining the two components by an unknotted arc. The complement deformation-retracts to a punctured torus in such a way that the two generators for the torus are representatives for the meridians of the Hopf link, and the puncture is a meridian for the extra arc. If $a$, $b$ and $c$ are these classes, then a presentation of $\pi_1$ is $[a,b] = c$. For any $\rho \in \mathcal{R}^\sharp(K)$, the $\rho(a)$ and $\rho(b)$ are rotations about axes in $\mathbb{R}^3$ whose angle is $\pi/4$, because the commutator needs to be of order 2. The stabilizer is $\mathbb{Z}/2$ and $\mathcal{R}^\sharp(K)$ is therefore a copy of $L(4,1)$.

The dodecahedron graph. Let $K$ be the 1-skeleton of a regular dodecahedron in $\mathbb{R}^3$. The representation variety $\mathcal{R}^\sharp(K)$ is described in [11]. We summarize the results here. The dodecahedron has 60 different Tait colorings which form 10 orbits under the permutation group of the colors. These contribute 10 $V_4$-connections to $\mathcal{R}(K)$ and thence 10 copies of the flag manifold to $\mathcal{R}^\sharp(K)$. There also exist exactly two fully irreducible representations in $\mathcal{R}(K)$. These make a contribution of 2 copies of $SO(3)$ to $\mathcal{R}^\sharp(K)$. The representation variety $\mathcal{R}^\sharp(K)$ consists therefore of 10 copies of the flag manifold and two copies of $SO(3)$.

3 Construction of the functor $J$

3.1 Discussion

In this section we will define the instanton Floer homology $J(\hat{Y}; \mu)$ for any closed, oriented bifold $\hat{Y}$ with strong marking data $\mu$. The approach is, by now, very standard. We start with the circle-valued Chern-Simons function on $\mathcal{B}_l(\hat{Y}; \mu)$ and add a suitable real-valued function $f$ as a perturbation. The perturbation is chosen so as to make $CS + f$ formally Morse-Smale. We then construct the Morse complex $(C,d)$ for this function, with $\mathbb{F}$ coefficients, and we define $J(\hat{Y}; \mu)$ to be the homology of the complex. Nearly all the steps are already laid out elsewhere in the literature: the closest model is the exposition in [15], which we will follow closely. That paper in turn draws on the more complete expositions in [14] and [8], to which we eventually refer for details of the proofs.
However, there is one important issue that arises here that is new. The codimension-2 bubbling phenomenon detailed in Lemma 2.7 and Proposition 2.8 means that there are extra considerations in the proof that $d^2 = 0$ in the Morse complex.

### 3.2 Holonomy perturbations

The necessary material on holonomy perturbations can be drawn directly from [15], with slight modifications to deal with the marking data. Fix $\tilde{Y}$ with marking data $\mu = (U_\mu, E_\mu)$. Choose a lift of $U_\mu$ to a $U(2)$ bundle $\tilde{E}_\mu$, and fix a connection $\theta$ on its determinant. A $\mu$-marked connection $(E, A, \tau)$ on $\tilde{Y}$ determines a $U(2)$ connection $\tilde{A}_\mu$ on $E_\mu$, with determinant $\theta$. Let $q$ be a smooth loop in $Y \setminus K$ based at $y$. If we identify the fiber $E_y$ with $\mathbb{R}^3$, then the holonomy of $A$ around the loop becomes an element of $SO(3)$. If $y$ belongs to $U_\mu$, let us identify the fiber of $\tilde{E}_\mu$ with $C^2$. If the loop is entirely contained in $U_\mu$, then using $\tau$ we can interpret the holonomy as an element of $U(2)$.

As in [15], we now consider a collection $q = (q_1, \ldots, q_k)$ where each $q_i$ is an immersion

$$q_i : S^1 \times D^2 \to Y \setminus K.$$  

We suppose that all the $q_i$ agree on $p \times D^2$, where $p$ is a basepoint. Let us suppose also that the image of $q_i$ is contained in $U_\mu$ for $1 \leq i \leq j$. Choose a trivialization of $\tilde{E}_\mu$ on the image of $p \times D^2$. Then for each $x \in D^2$, the holonomy of $A$ around the $k$ loops determined by $x$ gives a well-defined element of

$$U(2)^j \times SO(3)^{k-j}. \quad (8)$$

Fix any conjugation-invariant function $h : U(2)^j \times SO(3)^{k-j} \to \mathbb{R}$

Denoting by $\text{Hol}_x$ the holonomy element in 8, we obtain for each $x$ a well-defined function $H_x = h \circ \text{Hol}_x : \mathcal{B}_t(\tilde{Y} ; \mu) \to \mathbb{R}$. We then integrate this function over $D^2$ with respect to a compactly supported bump-form $\nu$ with integral 1, to obtain a function

$$f_q : \mathcal{B}_t(\tilde{Y} ; \mu) \to \mathbb{R}$$

$$f_q(E, A, \tau) = \int_{D^2} H_x(A) \nu.$$

We refer to such functions as cylinder functions, as in [15].
Next, following [15], we introduce a suitable Banach space $\mathcal{P}$ of real sequences $\pi = \{\pi_i\}$, and a suitable infinite collection of cylinder functions $\{f_i\}$ so that for each $\pi \in \mathcal{P}$, the sum

$$f_\pi = \sum_i \pi_i f_i$$

is convergent and defines a smooth function on $\mathcal{B}_l(\tilde{Y}; \mu)$. Furthermore, we arrange that the formal $L^2$ gradient of $f_\pi$ defines a smooth vector field on the Banach manifold $\mathcal{B}_l(\tilde{Y}; \mu)$, which we denote by $V_\pi$. With suitable choices for $\mathcal{P}$ and the $f_i$, we arrange that the analytic conditions of [14, Proposition 3.7] hold. In order to have a large enough space of perturbations, we form the countable collection $f_i$ by taking, for every $k$, a countable $C^\infty$-dense set in the space of $k$-tuples of immersions $(q_1, \ldots, q_k)$; and for each of these $k$-tuples of immersions, we take a $C^\infty$-dense collection of conjugation-invariant functions $h$. Cylinder functions separate points and tangent vectors in $\mathcal{B}_l(\tilde{Y}; \mu)$, because the SO(3) holonomies already do so, up to finite ambiguity, and the extra data from the loops in $U_\mu$ is enough to resolve the remaining ambiguity. So we can achieve the transversality that we need. We state these consequences now, referring to [14, 8] for proofs.

Let $\mathcal{C}_\pi \subset \mathcal{B}_l(\tilde{Y}; \mu)$ be the set of critical points of $CS + f_\pi$. The space $\mathcal{C}_0$ coincides with the representation variety $\mathcal{R}(\tilde{Y}; \mu)$. For any $\pi$, the space $\mathcal{C}_\pi$ is compact, and for a residual set of perturbations $\pi$ all the critical points are non-degenerate, in the sense that the Hessian operator has no kernel. In this case, $\mathcal{C}_\pi$ is a finite set.

Now fix such a perturbation $\pi_0$ with the property that $\mathcal{C}_\pi$ is non-degenerate. For any pair of critical points $\alpha$, $\beta$ in $\mathcal{C}_\pi$, we can then form the moduli space $M(\alpha, \beta)$ of formal gradient-flow lines of $CS + f_\pi$ from $\alpha$ to $\beta$. This can be interpreted as a moduli space of solutions to a the perturbed anti-self-duality equations on the 4-dimensional bifold $\mathbb{R} \times \tilde{Y}$. Note that the (non-compact) foam $\mathbb{R} \times K$ in $\mathbb{R} \times Y$ has no tetrahedral points. Each component of $M(\alpha, \beta)$ has a formal dimension, given by the spectral flow of the Hessian. We write $M_d(\alpha, \beta)$ for the component of formal dimension $d$. The group $\mathbb{R}$ acts by translations, and we write

$$M'_d(\alpha, \beta) = M_d(\alpha, \beta)/\mathbb{R}.$$ 

The relation (7) implies a bound on the action $\kappa$ depending only on $d$, across moduli spaces $M_{d'}(\alpha, \beta)$ for all $\alpha$ and $\beta$ and all $d' \leq d$. This is the “monotone” condition of [14].

We can now find a perturbation

$$\pi = \pi_0 + \pi_1$$
such that $f_{\pi}$ vanishes in a neighborhood of the critical point set $\mathbb{C}_{\pi_0} = \mathbb{C}_{\pi}$, and such that the moduli spaces $M(\alpha, \beta)$ are regular for all $\alpha$ and $\beta$.

We shall call a perturbation $\pi$ a good perturbation if the critical set $\mathbb{C}_{\pi}$ is non-degenerate and all moduli spaces $M(\alpha, \beta)$ are regular. We shall call $\pi$ $d$-good if it satisfies the weaker condition that the moduli spaces $M_d(\alpha, \beta)$ are regular for all $d' \leq d$.

Suppose $\pi$ is 2-good. Then the moduli space $M'_1(\alpha, \beta)$ is a finite set of points. Write

$$n_{\alpha, \beta} = \#M'_1(\alpha, \beta) \mod 2.$$  

Let $C$ be the $\mathbb{F}$ vector space with basis $\mathbb{C}_{\pi}$ and let $d : C \to C$ be the linear map whose matrix entry from $\alpha$ to $\beta$ is $n_{\alpha, \beta}$.

### 3.3 Proof that $d^2 = 0 \mod 2$

We wish to show that $(C, d)$ is a complex whenever $\pi$ is good. Let us first summarize the usual argument, in order to isolate where a new issue arises. To show that $d^2$ is zero is to show, for all $\alpha$ and $\beta$ in $\mathbb{C}_{\pi}$,

$$\sum_{\gamma} n_{\alpha, \gamma} n_{\gamma, \beta} = 0 \quad (\text{mod } 2).$$

One considers the 1-dimensional moduli space $M'_2(\alpha, \beta) = M_2(\alpha, \beta)/\mathbb{R}$, of trajectories from $\alpha$ to $\beta$, and one proves that it has a compactification obtained by adding broken trajectories. The broken trajectories correspond to elements of $M'_1(\alpha, \gamma) \times M'_1(\gamma, \beta)$ for some critical point $\gamma$, and the number of broken trajectories is the sum above. Finally, one must show that the compactification of $M'_2(\alpha, \beta)$ has the structure of a manifold with boundary, and uses the fact that the number of boundary points of a compact 1-manifold is even.

The extra issue in the bifold case is that 2-dimensional moduli spaces such as $M_2(\alpha, \beta)$ may have non-compactness due to bubbling, just as in the case of closed manifolds, as described in Lemma 2.7. There are no tetrahedral points, so the codimension-2 bubbles can only arise on the seams, which in the cylindrical case means that lines $\mathbb{R} \times V$, where $V \subset \bar{Y}$ is the set of vertices of the web $K$. After dividing by translations, we have the following description of a compactification of $M'_2(\alpha, \beta)$:

$$\tilde{M}'_2(\alpha, \beta) = M'_2(\alpha, \beta) \cup \left( \bigcup_{\gamma} M'_1(\alpha, \gamma) \times M'_1(\gamma, \beta) \right) \cup (V \times M'_0(\alpha, \beta)).$$
Note that $M'_0(\alpha, \beta)$ is empty unless $\alpha = \beta$, because it consists only of constant trajectories. So the matrix entry of $d^2$ from $\alpha$ to $\beta$ can be shown to be zero for $\alpha \neq \beta$ as usual. For $\alpha = \beta$, we have

\[
\tilde{M}'_2(\alpha, \alpha) = M'_2(\alpha, \alpha) \cup \left( \bigcup_{\gamma} M'_1(\alpha, \gamma) \times M'_1(\gamma, \alpha) \right) \cup V.
\]

We would like to be able to apply Proposition 2.8 to the cylinder $\mathbb{R} \times \tilde{Y}$, to conclude that a neighborhood of each vertex $v \in V$ in $\tilde{M}'_2(\alpha, \alpha)$ is homeomorphic to 4 intervals joined at one endpoint. This would tell us that 4 ends of the 1-manifold $M'_2(\alpha, \alpha)$ are incident at each point of $V$ in the compactification. Since 4 is even (and indeed, since the number of vertices is also even), it would follow that $d^2 = 0$.

However, Proposition 2.8 requires gluing theory which has not been carried out in the situation where holonomy perturbations are present at the point where the bubble occurs. For a holonomy perturbation $f_\pi$, the $\tilde{Y}$-support of $f_\pi$ will mean the closure in $\tilde{Y}$ of the union of the images of all the immersions $q : (S^1 \times D^2) \to \tilde{Y} \setminus K$ used in the cylinder functions for $f_\pi$. If $f_\pi$ is a finite sum of cylinder functions, then the $\tilde{Y}$-support is a compact subset of $\tilde{Y} \subset K$ and is therefore disjoint from some open neighborhoods of the vertex set $V$. In this case, the gluing theory for bubbles at $V$ is standard, and we can apply Proposition 2.8 to conclude that $d^2 = 0$.

It may not be the case that there is a good perturbation which is a finite of cylinder functions. However, the proof that $d^2 = 0$ only involves moduli spaces with $d \leq 2$. So the following Proposition is the key.

**Proposition 3.1.** There exists a $2$-good perturbation $f_\pi$ which is a finite sum of cylinder functions.

**Proof.** We can approximate any $\pi$ by a finite sum, so the issue is openness of the required regularity conditions. Recall that we construct good perturbations $\pi$ as $\pi_0 + \pi_1$, where $\pi_0$ is chosen first to make $\mathcal{C}_{\pi_0}$ non-degenerate. The non-degeneracy of the compact critical set $\mathcal{C}_{\pi_0}$ is an open condition, so we certainly arrange that $f_{\pi_0}$ is a finite sum by truncating $\pi_0$ after finitely many terms.

With $\pi_0$ fixed, choose a good $\pi = \pi_0 + \pi_1$ as before, with $f_{\pi} = f_{\pi_0}$ in a neighborhood of $\mathcal{C}_{\pi}$. Choose an approximating sequence

\[
\pi^m \to \pi
\]
in \( \mathcal{P} \), with the property that each \( f^m \) is a finite sum and \( f^m = f_{0} \) on the same neighborhood of \( \mathcal{C} \). From the fact that \( \pi \) is 2-good, we wish to conclude that \( \pi^m \) is 2-good for sufficiently large \( m \).

Let \( M(\alpha, \beta)_m \) denote the moduli spaces for the perturbation \( \pi_m \). For \( d \leq 1 \), there is a straightforward induction argument to show that \( \pi^m \) is \( d \)-good for \( m \) sufficiently large. This starts with the observation that, for \( d \) sufficiently negative, the moduli spaces \( M_d(\alpha, \beta)_m \) are empty because they have negative action \( \kappa \), as a consequence of equation (7). If \( d_0 \leq 0 \) is the first dimension for which any the moduli spaces \( M'_d(\alpha, \beta)_m \) is non-empty, then these moduli spaces are compact, and their regularity is therefore an open condition. It follows that these moduli spaces are regular (i.e. empty if \( d_0 < 0 \)) for \( m \) sufficiently large.

Compactness also holds for \( M'_1(\alpha, \beta) \) once the lower moduli spaces are regular. So we may assume that all the \( \pi^m \) are 1-regular. We must prove that they are then 2-regular for large \( m \).

Suppose the contrary. Passing to a subsequence, we may assume that there is a critical point \( \alpha \) and for each \( m \) a perturbed anti-self-dual solution \( A_m \) representing a point of \( M_2(\alpha, \alpha)_m \) which is not regular. This means that the cokernel of the linearized operator is not surjective at \( A_m \), and we can choose a non-zero solution

\[
\omega_m \in L^2_t(\mathbb{R} \times \tilde{Y}; \Lambda^+ \otimes E)
\]

of the formal adjoint equation (i.e. the operator \( d^*_{A_m} \) plus a perturbation determined by the holonomy of \( A_m \)).

In seeking a contradiction, the interesting cases are when \( A_m \) is either converging to a broken trajectory or is approaching a point in the Uhlenbeck compactification with a bubble at a point on a seam. The case of a broken trajectory is standard: if a sequence \( A_m \) converges to a broken trajectory and the components of the broken trajectory are regular, then \( A_m \) is regular for large \( m \). This is a standard consequence of the proof of the gluing theorem for broken trajectories.

So let us suppose that \( A_m \) converges to a point \((p,A)\) in the Uhlenbeck compactification, where \( p \) is a point on a seam \( \mathbb{R} \times \{v\} \) and \( A \in M_0(\alpha, \alpha) \) is the constant solution. Let us apply a conformal change to \( \mathbb{R} \times \tilde{Y} \), as indicated in Figure 1, so that the new manifold \( \tilde{X}_m \) contains a cylinder on \( S^3/V_4 \) of length \( T_m + 2S \). The length \( S \) will be fixed and large. We can choose the lengths \( T_m \) and the centers of the conformal transformations so that the \( A_m \) converge in the sense of broken trajectories as \( T_m \to \infty \). In the limit, we obtain two pieces. We obtain an orbifold instanton on \( \mathcal{B} = B^4/V_4 \) equipped with a cylindrical end (or equivalently \( S^4/V_4 \)), with action \( \kappa = 1/4 \); and we obtain the conformal transformation of the
constant solution $A$ on $(\mathbb{R} \times \check{Y}) \setminus \{p\}$.

On $\check{X}_m$, the equations satisfied by $A_m$ are of the form $F_{A_m}^+ = W_m$, where $W_m$ is obtained from the holonomy perturbation. In the original metric on $\mathbb{R} \times \check{Y}$, these perturbing terms satisfy a uniform $L^\infty$ bound. So in the conformally equivalent metric of $\check{X}_m$, there is a bound of the form

$$\|W_m\|_{L^\infty(Z_m)} \leq c_1 e^{-c_2 S}$$

on the cylinder $Z_m$ (see Figure 1). For any $\epsilon$, we may arrange, by increasing $S$, that the connections $A_m$ are all $\epsilon$-close to the constant flat $V_4$-connection in $L^p_1$ norm on $Z_m$, for any $p$.

Consider now the cokernel elements $\omega_m$. Scale these so that their $L^2$ norm is 1 on $\check{X}_m$. Since the $A_m$ are converging locally in $L^p_1$ for all $p$, we have a uniform $C^0$ bound on $\omega_m$ in the metric of $\check{X}_m$, and (after passing to a subsequence) convergence to a limit $\omega$. We may assume that the $\omega_m$ have uniform exponential decay on the two ends of $\mathbb{R} \times \check{Y}$, for otherwise we are in the standard case where the bubble plays no role. Because $M_0(\alpha, \alpha)$ is regular, the limit $\omega$ must be zero on $\check{X}_m \setminus \check{B}^4$, and the uniform exponential decay on the two ends means that

$$\int_{\check{X}_m \setminus \check{B}^4} |\omega_m|^2 \to 0.$$
Because all orbifold instantons on $\mathbb{R}^4/V^4$ are also regular, we have
\[
\int_{\tilde{B}} |\omega_m|^2 \to 0
\]
also. It follows that
\[
\int_{Z_m} |\omega_m|^2 \to 1.
\]
However, $\omega_m$ is converging to zero on the two boundary components of the cylinder $Z_m = [0, T_m] \times S^3/V_4$, and satisfies an equation which is schematically of the shape
\[
d^* \omega_m + A_m \cdot \omega_m = Q_m,
\]
where $Q_m$ is the contribution coming from the non-local perturbation (and is therefore dependent on the value of $\omega$ on parts of $\tilde{X}_m$ that are not in $Z_m$). The $L^p_1$ convergence of the terms $A_m$ mean that the corresponding multiplication operators that appear on the left can be taken to be uniformly small in operator norm, as operators either from $L^2$ to $L^2$, or as operators on weighted Sobolev spaces, $L^2_{1,\delta} \to L^2_{\delta}$, weighted by $e^{\delta t}$. The terms $Q_m$ are also uniformly bounded in weighted $L^2$ norms. So standard arguments (see [8] for example) establish that the $L^2$ norm of $\omega_m$ on length-1 cylinders $[t, t+1] \times (S^3/V_4)$ is uniformly controlled by a function that decays exponentially towards the middle of the cylinder. It follows that the integral of $|\omega_m|^2$ on $Z_m$ is also going to zero, which is a contradiction. \qed

The Proposition shows that we can construct 2-good perturbations for which $d^2 = 0$. It follows that, in fact, $(C, d)$ is a complex for all 2-good perturbations, and therefore for all good perturbations. This is because, up to isomorphism, the complex $C$ and boundary map $d$ are stable under small changes in the perturbation, by the compactness of $\mathcal{C}_\pi$ and $M'_\alpha(\alpha, \beta)$. So we can always approximate a given good perturbation by one for which the proof that $d^2 = 0$ applies, without actually changing $d$. Thus, we have:

**Proposition 3.2.** The map $d : C \to C$ satisfies $d^2 = 0$, for all good perturbations $\pi$.

We can now define the instanton Floer homology:

**Definition 3.3.** For any closed, oriented bifold $Y$ with strong marking data $\mu$, we define $J(\tilde{Y}; \mu)$ to be the homology of the complex $(C, d)$ constructed above, for any choice of orbifold Riemannian metric $\tilde{g}$ and any good perturbation $\pi$. When $\tilde{Y}$ is obtained from a 3-manifold $Y$ with an embedded web $K$, we may write $J(Y, K; \mu)$. 
3.4 Functoriality of $J$

In the above definition, the group $J(\tilde{Y}; \mu)$ depends on a choice of metric and perturbation. As usual with Floer theories, the fact that $J(\tilde{Y}; \mu)$ is a topological invariant is a formal consequence of the more general functorial property of instanton homology.

Let $\tilde{Y}_1$ and $\tilde{Y}_0$ be two 3-dimensional bifolds with orbifold metrics $\tilde{g}_1$ and $\tilde{g}_0$. Let $\tilde{X}$ be an oriented cobordism from $\tilde{Y}_1$ to $\tilde{Y}_0$. Equip $\tilde{X}$ with an orbifold metric that is a product in a collar of each end. Let $\mu_1$ and $\mu_0$ be marking data for the two ends, and let $\nu$ be marking data for $\tilde{X}$ that restricts to $\mu_1$ and $\mu_0$ at the boundary. Let $\pi_1$ and $\pi_0$ be good perturbations on $\tilde{Y}_1$ and $\tilde{Y}_0$. If we attach cylindrical ends to $\tilde{X}$ and choose auxiliary perturbations in the neighborhoods of the two boundary components [15, 14, 8], then we have moduli spaces of solutions to the perturbed anti-self-duality equations on the bifold:

$$M(\tilde{X}, \nu; \alpha, \beta).$$

(9)

Here $\alpha$ and $\beta$ are critical points for the perturbed Chern-Simons functional, i.e. generators for the chain complexes $(C_1, d_1)$ and $(C_0, d_0)$ associated with $(\tilde{Y}_1, \mu_1)$ and $(\tilde{Y}_0, \mu_0)$. Thus we can use $(\tilde{X}; \nu)$ to define a linear map

$$(C_1, d_1) \rightarrow (C_0, d_0)$$

between the respective complexes, by counting solutions to the perturbed equations in zero-dimensional moduli spaces. The usual proof that the linear map defined in this way is a chain map involves moduli spaces only of dimension 0 and 1. In particular, the codimension-2 bubbling phenomenon of Proposition 2.8 does not enter into the argument (unlike the proof that $d^2 = 0$). However, if the foam $\Sigma$ has tetrahedral points, then the codimension-1 bubbling phenomenon of Proposition 2.9 comes into play. That proposition tells us that the number of ends of each 1-dimensional moduli space that are accounted for by bubbling in the Uhlenbeck compactification is a multiple of 4, and in particular even. So the usual proof carries over as long as we are using coefficients $\mathbb{F}$.

In this way, $(\tilde{X}; \nu)$ defines a map on homology,

$$J(\tilde{X}; \nu) : J(\tilde{Y}_1; \mu_1) \rightarrow J(\tilde{Y}_0; \mu_0).$$

Furthermore, a chain-homotopy argument shows that the map is independent of the choice of metric on the interior of $\tilde{X}$ and is independent also of the auxiliary perturbations on the cobordism.
Consider now the composition of two cobordisms. Suppose we have two cobordisms \((\tilde{X}; \nu)\) and \((\tilde{X}''; \nu'')\), where the first is a cobordism to \((\tilde{Y}; \mu)\) and the second is a cobordism from \((\tilde{Y}; \mu)\). This means that the restrictions of \(E_{\nu}\) and \(E_{\nu''}\) to the common subset \(E_\mu \subset \tilde{Y}\) are the same bundle (not just isomorphic), so together they define a bundle \(E_{\nu} \cup E_{\nu''}\). We can therefore join the cobordisms canonically to make a composite cobordism with marking data,

\[(X; \nu) = (\tilde{X} \cup \tilde{X}'', \nu' \cup \nu'').\]

In the above situation it is not always the case that 

\[J(\tilde{X}; \nu) = J(\tilde{X}''; \nu'') \circ J(\tilde{X}'', \nu').\]  

To see the reason for this, consider the problem of joining together marked connections. Suppose we have marked connection \((E', A', \sigma')\) and \((E'', A'', \sigma'')\) on \((\tilde{X}; \nu')\) and \((\tilde{X}''; \nu'')\) respectively. By restriction, we obtain two \(\mu\)-marked connections on \((\tilde{Y}; \mu)\), say \((F', B', \tau')\) and \((F'', B'', \tau'')\). Let us suppose these are isomorphic: they define the same point in \(\mathcal{B}(\tilde{Y}; \mu)\). To avoid the issue of differentiability, let us also suppose that \((E', A', \sigma')\) coincides with the pull-back of \((F', B', \tau')\) in a collar of \(\tilde{Y}\), and similarly with \(E''\). Because the marking is strong, there is a unique isomorphism, which is a bundle isomorphism

\[\phi : F' \to F''\]

such that \(\phi^*(B'') = B'\) and such that the map

\[\psi = (\tau'')^{-1}\phi \tau' : E_\mu \to E_\mu\]

lifts to determinant 1.

These conditions on \(\phi\) are not enough to allow us to construct a \(\nu\)-marked connection \((E, A, \sigma)\) on the union \(\tilde{X}\). Certainly we can use \(\phi\) to glue \(E'\) to \(E''\) along the common boundary, and the union of the connections \(A'\) and \(A''\) will give us a connection \(A\). So we do have an unmarked \(SO(3)\) connection \((E, A)\) on \(\tilde{X}\). However, \((E, A)\) does not have a \(\nu\)-marking. Instead of having a bundle isomorphism

\[\sigma : E_\nu \to E_{\mid U_\nu \setminus \Sigma}\]

we have a pair of bundle isomorphisms \(\tilde{\sigma}'\) and \(\tilde{\sigma}''\) (essentially the maps \(\sigma'\) and \(\sigma''\)) defined over \(U_\nu\) and \(U_{\nu''}\). To form \(\sigma\), we would want \(\tilde{\sigma}'\) and \(\tilde{\sigma}''\) to be equal on their common domain \(E_\mu\). Instead, we are only given that \((\tilde{\sigma}'')^{-1}\sigma'|_{E_\mu}\) lifts to determinant 1.
For any $SO(3)$ bundle $F$, let $\delta(F)$ denote the group of bundle automorphisms of $F$ modulo those that lift to determinant 1. From the above discussion, we see that we need the following diagram to be a fiber product:

\[
\begin{array}{ccc}
\delta(E_{\nu'}) & \rightarrow & \delta(E_{\nu}) \\
\downarrow & & \downarrow \\
\delta(E_{\nu''}) & \rightarrow & \delta(E_{\mu})
\end{array}
\]

Since $\delta(F)$ is isomorphic to $H^1$ of the base with $\mathbb{F}$ coefficients, this is equivalent to the exactness of the sequence

\[
0 \rightarrow H^1(W; \mathbb{F}) \rightarrow H^1(W'; \mathbb{F}) \oplus H^1(W''; \mathbb{F}) \rightarrow H^1(W' \cap W''; \mathbb{F}) \rightarrow 0
\]  

(11)

where $W = U_{\nu} \setminus \Sigma \subset \tilde{X}$ and so on. To summarize this, we have:

**Proposition 3.4.** The composition law (10) holds provided that the sequence (11) is exact.

Using the Mayer-Vietoris sequence, we see that one way to ensure exactness of (11) is to require that the maps

\[
H^i(W'; \mathbb{F}) \rightarrow H^i(W' \cap W''; \mathbb{F})
\]

are surjective for $i = 0$ and 1. So we make the following definition.

**Definition 3.5.** If $(\tilde{X}, \nu)$ is a marked cobordism from $(\tilde{Y}_1, \mu_1)$ to $(\tilde{Y}_0, \mu_0)$, we say that the marking data $\nu$ is right-proper if the map

\[
H^i(U_{\nu} \setminus \Sigma; \mathbb{F}) \rightarrow H^i(U_{\mu_0} \setminus K_0; \mathbb{F})
\]

is surjective for $i = 0$ and 1. We define left-proper similarly.

Thus the composition law (10) always holds if the marking data $\nu'$ on $(\tilde{X}', \nu')$ is right-proper, or if the marking data $\nu''$ on $(\tilde{X}'', \nu'')$ is left-proper. Furthermore, if we compose two cobordisms with right-proper marking, then the composite marking is right-proper also (and similarly with left-proper).
Passing to the language of smooth 3-manifolds $Y$ with embedded webs $K$, and smooth 4-manifolds $X$ with embedded foams $\Sigma$, we have the following description. According to the definition, a foam comes with marked points (dots) on the faces, but we consider first the case that there are no dots. There is a category in which an object is a quadruple $(Y, K, \mu, a)$, where $Y$ is a closed, oriented, connected 3-manifold, $K$ is an embedded web, $\mu$ is strong marking data, and $a$ is auxiliary data consisting of a choice of orbifold metric $\bar{g}$ on a corresponding orbifold $\bar{Y}$ and a choice of good perturbation $\pi \in \mathcal{P}$. A morphism in this category from $(Y_1, K_1, \mu_1, a_1)$ to $(Y_0, K_0, \mu_0, a_0)$ is an isomorphism class of compact, oriented, 4-manifold-with-boundary, $X$, containing a foam $\Sigma$ and right-proper marking data $\nu$, together with orientation-preserving identifications $\iota$ of $(\partial X, \partial \Sigma, \partial \nu)$ with $(-Y_1, K_1, \mu_1) \cup (Y_0, K_0, \mu_0)$.

The construction $J$ defines a functor from this category to vector spaces over $\mathbb{F}$.

If only the auxiliary data $a_i$ differ, then there is a canonical product morphism from $(Y, K, \mu, a_1)$ to $(Y, K, \mu, a_0)$, which is an isomorphism. So we can drop the auxiliary data $a$ and take an object in our category to be a triple $(Y, K, \mu)$ and a morphism to be (still) an isomorphism class of triples $(X, \Sigma, \nu)$.

**Proposition 3.6.** Let $C_0$ be the category in which an object is a triple $(Y, K, \mu)$, consisting a closed, connected, oriented 3-manifold $Y$, an embedded web $K$, and strong marking data $\mu$, and in which the morphisms are isomorphism classes of foam cobordisms $(X, \Sigma, \nu)$ with right-proper marking data $\nu$. (Here again $X$ is an oriented 4-manifold with boundary and $\Sigma$ is an embedded foam, without dots.) Then $J$ defines a functor from $C_0$ to vector spaces over $\mathbb{F}$.

We use the terminology $C_0$ rather than $C$ because we reserve $C$ for the category in which the foams are allowed to have dots. We turn to this matter next.

### 3.5 Foams with dots

Until now, the foam $\Sigma$ has been dotless. We now explain how to extend the definition of the linear maps defined by cobordisms to the case of foams with dots. For now, we work in the general setting of cobordisms between 3-dimensional bifolds equipped with strong marking data.

So consider a 4-manifold $X$ with a foam $\Sigma \subset X$ and strong marking data $\nu$. Let $\tilde{X}$ be a corresponding 4-dimensional Riemannian bifold. Let $\delta \in \Sigma$ be a point
lying on a face (a “dot”). Associated with \( \delta \) is a real line bundle over the space of connections,
\[
L_\delta \to \mathcal{B}_i(\tilde{X}; \nu),
\]
defined as follows. Let \( B_\delta \subset X \) be a ball neighborhood of \( \delta \), and consider marking data \( \mu \) given by the trivial bundle over \( U_\mu = B_\delta \setminus \Sigma \). There is a double covering,
\[
\mathcal{B}_i(\tilde{X}; \nu, \mu) \to \mathcal{B}_i(\tilde{X}; \nu)
\]
where the left-hand side is the space of isomorphism classes of bifold connections \((E, A)\) equipped with two markings: a \( \nu \)-marking \( \tau \), and a \( \mu \)-marking \( \sigma \). The covering map is the map that forgets \( \sigma \):
\[
(E, A, \tau, \sigma) \mapsto (E, A, \tau).
\]
The fact that this is a double cover is essentially the same point as in Lemma 2.4. We take \( L_\delta \) to be the real line bundle associated to this double cover.

The line bundle \( L_\delta \) can be regarded as being pulled back from the space of connections over a suitable open subset of \( X \). Specifically, suppose \( X_\delta \subset X \) is a connected open set such that the two restriction maps
\[
H^1(U_\nu; F) \to H^1(X_\delta \cap U_\nu; F)
\]
and
\[
H^1(U_\mu; F) \to H^1(X_\delta \cap U_\mu; F)
\]
are injective. On \( X_\delta \) we have marking data \( \nu_\delta \) and \( \mu_\delta \) by restriction; and the line bundle \( L_\delta \) can be regarded as pulled back via the restriction map
\[
\mathcal{B}_i(\tilde{X}; \nu) \to \mathcal{B}_i(\tilde{X}_\delta; \nu_\delta).
\]
We will choose \( X_\delta \) so that it is disjoint from the foam and the boundary of \( X \). (Think of a regular neighborhood of a collection of loops in \( X \).)

In this way, given a collection of points \( \delta_i \) \((i = 1, \ldots, n)\), we obtain a collection of real line bundles \( L_\delta \), which we may regard as pulled back from spaces of marked connections over disjoint open sets \( X_\delta \subset X \). Let \( s_i \) be a section of \( L_\delta \) that is also pulled back from \( X_\delta \), and let
\[
V(\delta_i) \subset \mathcal{B}_i(\tilde{X}; \nu)
\]
be its zero-set. If $\tilde{X}$ is a bifold cobordism to which we attach cylindrical ends, we can then consider the moduli spaces of solutions to the perturbed anti-self-duality equations (9) cut down by the $V(\delta_i)$:

$$M(\tilde{X}, v; \alpha, \beta) \cap V(\delta_1) \cap \cdots \cap V(\delta_n).$$

The sections $s_{\delta_i}$ can be chosen so that all such intersections are transverse. The disjointness of the $X_{\delta_i}$ ensures the necessary compactness properties when bubbles occur and these moduli spaces define chain maps in the usual way. The resulting maps on homology are the required maps defined by the cobordism with dots $\delta_1, \ldots, \delta_n$ on the foam. We have:

**Proposition 3.7.** The above construction extends $J$ from the category $C_0$ of dotless foams to the category $C$ of foams with dots, in which an object is a triple $(Y, K, \mu)$, consisting a closed, connected, oriented 3-manifold $Y$, an embedded web $K$, and strong marking data $\mu$, and in which the morphisms are isomorphism classes of foam cobordisms $(X, \Sigma, v)$ with right-proper marking data $v$.

As a special case of the construction, given a web $K \subset Y$ we can define a collection of operators on $J(Y, K, \mu)$, one for each edge of $K$:

**Definition 3.8.** For each edge $e$ of a web $K$, we write

$$u_e : J(Y, K, \mu) \to J(Y, K, \mu)$$

for the operator corresponding to cylindrical cobordism with a single dot on the face corresponding to the edge $e$.

The operators are $u_e$ are not independent. Consider a morphism $(X, \Sigma, v)$ in the foam category $C$, let $s$ be a seam in $\Sigma$, and let $\delta_1, \delta_2$ and $\delta_3$ be points on the three facets of $\Sigma$ that are locally incident along $s$. Let $\Sigma(\delta_i)$ be the foam obtained from $\Sigma$ be placing one additional dot at $\delta_i$. See Figure 2. Then we have relation (which is also part of the set-up of the $sl_3$ homology of [7]):

**Proposition 3.9.** The linear maps $J(\delta_i) = J(X, \Sigma(\delta_i), v)$ satisfy the relation

$$J(\delta_1) + J(\delta_2) + J(\delta_3) = 0.$$ 

In particular (and equivalently), if $e_1, e_2$ and $e_3$ are the edges incident at a vertex of $K$, then the three operators $u_{e_1}$ from Definition 3.8 satisfy $u_{e_1} + u_{e_2} + u_{e_3} = 0$. 
Figure 2: The three foams $\Sigma(\delta_i) \ (i = 1, 2, 3)$ that appear in Proposition 3.9.

Proof. The corresponding real line bundles $L_i$ satisfy

$$L_1 \otimes L_2 \otimes L_3 = \mathbb{R}$$

so the homology classes dual to the zero-sets $V_i$ of sections of these line bundles add up to zero. Furthermore, this is already true in the space of connections on a connected open set $Z \subset \hat{X}$ that includes the domain of the marking data $\nu$ together with a neighborhood of a 2-sphere meeting $\Sigma$ in the three points $\delta_i$ (i.e. a 2-sphere which is a link of the seam). We can use $Z$ in place of $X_{\delta_i}$ in defining the maps corresponding to the foams with dots, and the result follows. □

There is an alternative way to incorporate dots into our definitions. The authors do not know in general whether this definition is equivalent to our standard definition (the one above). Given a foam with dots, $\Sigma \subset X$, we can construct a new foam $\Sigma' \subset X$ as follows. We choose standard 4-balls $B_i$ centered on each dot $\delta_i$ and meeting $\Sigma$ in standard 2-disks $D_i$. We define $\Sigma'$ to be the union of $\Sigma$ with collection of genus-1 surfaces with boundary, $T_i$. Each $T_i$ is contained in the corresponding ball $B_i$, where it is isotopic to a standardly-embedded surface in $B^3$, with $\partial T_i$ meeting $D_j$ in a circle so as to form a new circular seam in $\Sigma'$. See Figure 3. This construction is functorial, from the category in which the morphisms are foams possibly with dots, to the category of dotless foams. We extend the definition of $J$ to foams with dots by composing with this functor.

We have now defined two different ways to extend the functoriality of $J$, from foams without dots to foams with dots. Since it is not clear that the definitions are the same, we take the first as standard. It turns out that the two definitions are the same in the more restricted setting of the functor $J^\#$ defined in the next subsection.
3.6 Defining $J^\#$

The objects in the category $C$ on which $f$ is a functor are required to be equipped with strong marking data, in order to avoid reducibles. Following [15], we introduce a variant of the construction which applies to arbitrary webs in 3-manifolds. Suppose we are given a closed, connected, oriented 3-manifold $Y$ with a framed basepoint $y_0$. Given a web $K \subset Y$ disjoint from the basepoint, we can form the union $K \cup H$, where the Hopf link $H$ is contained in a standard ball centered on $y_0$, disjoint from $K$. The framing at $y_0$ is used to put the Hopf link $H$ in a standard position. Let $\mu$ be the strong marking data with $U_\mu$ a ball containing $H$ and $E_\mu$ a bundle with $w_2$ dual to an arc joining the two components of $H$ (as in Lemma 2.10). We consider $K^\# = K \cup H$ as a web in $Y$, with strong marking data $\mu$, and so we can define

$$J^\#(Y, K) = J(Y, K^\#, \mu) \quad (12)$$

Given a foam cobordism $(X, \Sigma)$ from $(Y_1, K_1)$ to $(Y_0, K_0)$ and a framed arc in $X$ joining the framed basepoints in $Y_1$ and $Y_2$, then we can insert $[0,1] \times H$ as a foam in $X$ along the arc in a standard way, and take standard strong marking data $\nu = [0,1] \times \mu$ in $X$. In this way, $(X, \Sigma)$ gives rise to a homomorphism,

$$J^\#(X, \Sigma) : J^\#(Y_1, K_1) \to J^\#(Y_0, K_0).$$

(We omit the basepoints and arcs from our notation, but this is not meant to imply that the map $J^\#(X, \Sigma)$ is independent of the choice of arc.)

**Definition 3.10.** We define a category $C^\#$ whose objects are pairs $(Y, K, y)$ consisting of a closed, oriented connected 3-manifold $Y$ containing a web $K$ and framed basepoint $y$. The morphisms are isomorphism classes of triples $(X, \Sigma, \gamma)$,
where $X$ is a connected oriented 4-manifold, $\Sigma$ is a foam, and $\gamma$ is a framed arc joining the basepoints. Thus $J^\sharp$ is a functor,

$$J^\sharp : C^\# \to \text{(Vector spaces over } F\text{)}.$$ 

by composing “Sharp” with $J$.

Sometimes we will regard $J^\#$ as defining a functor from the category of webs in $\mathbb{R}^3$ and isotopy classes of foams in $[0, 1] \times \mathbb{R}^3$, by compactifying $\mathbb{R}^3$ and putting the framed basepoint at infinity. We will then just write $J^\#(K)$ etc. for a web $K$ in $\mathbb{R}^3$.

There is a variant of this definition that we can consider. We can define

$$I^\# : C^\# \to \text{(Vector spaces over } F\text{)}$$

in much the same way, except that instead of using the marking $\mu$ with $U_\mu = B^3$ as above, we instead use the marking $\mu'$ with $U_{\mu'}$ the whole of the three-manifold $Y$. We still take $E_{\mu'}$ to be a bundle with $w_2$ dual to the same arc. Thus,

$$I^\#(Y, K) = J(Y, K \cup H; \mu').$$  \hfill (13)

In the case that $K$ has no vertices, this is precisely the knot invariant $I^\#(K)$ defined in [15], except that we are now using $F$ coefficients. Over $F$ at least, our definition extends that of [15] by allowing embedded webs rather than just knots and links.

There is another simple abbreviation to our notation that will be convenient. It is a straightforward fact that $J^\#(S^3, \emptyset)$ is $F$ (see below), so a cobordism from $(S^3, \emptyset)$ to $(Y, K)$ in the category $C^\#$ determines a vector in $J^\#(Y, K)$. This allows us to adopt the following two conventions:

**Definition 3.11.** Let $X$ be compact, oriented 4-manifold with boundary the connected 3-manifold $Y$, and let $\Sigma$ be a foam in $X$ with boundary a web $K \subset Y$. Let a basepoint $y \in Y$ be given. Then we write $J^\#(X, \Sigma)$ for the element of $J^\#(Y, K)$ corresponding the cobordism from $(S^3, \emptyset)$ to $(Y, K)$ obtained from $(X, \Sigma)$ by removing a ball from $X$ and connecting a point on its boundary to $y$ by an arc.

Similarly, given closed, connected manifold $X$ containing a closed foam $\Sigma$, we will write $J^\#(X, \Sigma)$ for the scalar in $F$ corresponding to the morphism from $(S^3, \emptyset)$ to itself, obtained from $(X, \Sigma)$ by removing two balls disjoint from $\Sigma$ and joining them by an arc. We refer to $J^\#(X, \Sigma)$ in this context as the $J^\#$-evaluation of the closed foam.
We can also use the language of bifolds directly, rather than manifolds with embedded webs. We will consider 3-dimensional bifolds $\tilde{Y}$ with a framed basepoint $y_0$ (in the smooth part of $\tilde{Y}$, not an orbifold point). For such a $\tilde{Y}$, we write the corresponding invariant as $J^\#(\tilde{Y})$. Given a bifold cobordism $\tilde{X}$ from $\tilde{Y}_1$ to $\tilde{Y}_0$ and a framed arc joining basepoints in these, we obtain a homomorphism

$$J^\#(\tilde{X}) : J^\#(\tilde{Y}_1) \to J^\#(\tilde{Y}_0).$$

We will also allow the conventions of Definition 3.11 in this context, and so talk about $J^\#(\tilde{X})$ for an orbifold with connected boundary, or with no boundary.

### 3.7 Simplest calculations

We have the following three simple examples of calculation of $J^\#$ for webs $K$ in $\mathbb{R}^3$.

**Proposition 3.12.** We have the following special cases.

(a) If $K$ is the empty web in $\mathbb{R}^3$, then $J^\#(K) = \mathbb{F}$.

(b) If the web $K \subset \mathbb{R}^3$ has an embedded bridge, then $J^\#(K) = 0$.

**Proof.** In both cases, we start by applying Lemma 2.10, which tells us that the unperturbed set of critical points for the Chern-Simons functional for $(S^3, K \cup H, \mu)$ can be identified with

$$\mathcal{R}(K) = \{ \rho : \pi_1(\mathbb{R}^3 \setminus K) \to SO(3) | \rho(m_e) \text{ has order 2 for all edges } e \}.$$

In the first case of the proposition, the representation variety is a single point (the trivial representation of the trivial group), and this point is non-degenerate as a critical point of Chern-Simons. So $J^\#$ has rank 1.

In the case of an embedded bridge $e$, the meridian $m_e$ is null-homotopic, so there is no $\rho$ such that $\rho(m_e)$ has order 2. The representation variety is therefore empty, and $J^\#(K) = 0$. 

### 4 Excision

#### 4.1 Floer’s excision argument

Floer’s excision theorem [2, 13] can be applied in the setting of the $SO(3)$ instanton homology $J$. We spell out statements of the basic result for cutting and
gluing along tori (which is the original version), as well as cutting an gluing along 2-spheres with three orbifold points.

Let \((\tilde{Y}, \mu)\) be a 3-dimensional bifold containing a web with strong marking data \(\mu\). We will temporarily allow the case that \(\tilde{Y}\) is not connected, in which case the marking data \(\mu\) is strong only if its restriction to each component of \(\tilde{Y}\) is strong. Let \(T_1, T_2\) be disjoint, oriented, embedded 2-tori in \(U_\mu\). (In particular, they lie in the smooth part of \(Y\).) We require the additional property that the restrictions, \(E_i\), of the marking bundle \(E_\mu\) to the tori \(T_i\) have non-zero \(w_2\). Pick an identification between the tori, \(T_1 \rightarrow T_2\), and lift it to an isomorphism between the bundles. By cutting along \(T_1, T_2\) and regluing (using the preferred identifications in an orientation-preserving way), we obtain a new object \((\tilde{Y}', \mu')\). In \(\tilde{Y}'\), we also end up with a pair of embedded tori \(T'_1\) and \(T'_2\).

We will be concerned with the case that \(T_1\) and \(T_2\) lie in distinct components \(\tilde{Y}_1, \tilde{Y}_2\) of \(\tilde{Y}\). In this case, cutting along \(T_i\) cuts \(\tilde{Y}_i\) into pieces \(\tilde{Y}_i^+\) and \(\tilde{Y}_i^-\). The new bifold \(\tilde{Y}'\) has components \(\tilde{Y}'_i = \tilde{Y}_i^+ \cup \tilde{Y}_i^-\) etc.

**Theorem 4.1 (Floer’s excision).** In the above setting, when \(T_1, T_2\) are separating tori in distinct components \(\tilde{Y}_1, Y_2\), we have

\[
J(\tilde{Y}, \mu) = J(\tilde{Y}', \mu').
\]

**Proof.** The proof is a straightforward adaption of the original argument of Floer [2]. The case of separating tori was given in [15].

There is a variant of the excision argument, in which the tori \(T_1, T_2\) are replaced by spheres \(S_1, S_2\) each with three orbifold points. That is, we ask that \(S_1, S_2\) are 2-dimensional sub-orbifolds of \(Y\), meeting the web \(K\) of orbifold points in three points each. We still require that \(S_i \setminus K\) lies in \(U_\mu\), and require the same separation hypotheses. By cutting and gluing, we form \((Y', \mu')\) just as in the torus case. The same result holds:

**Theorem 4.2.** In the above setting, when \(S_1, S_2\) are separating 3-pointed spheres in distinct components \(\tilde{Y}_1, Y_2\), we have

\[
J(\tilde{Y}, \mu) = J(\tilde{Y}', \mu').
\]

**Proof.** This adaptation of the excision theorem is contained in [21], in a version for any number orbifold points on the spheres. In the case of 3 points, the argument from the torus case needs no serious adaptation: the essential point is the existence of a unique \(SO(3)\) connection on the 2-dimensional orbifold.
Remark. In the non-separating case, some extra care is needed to obtain correct statements, because of the double-covers that correspond to possibly different strong marking data.

4.2 Applications of excision

If $\tilde{Y}_1$ and $\tilde{Y}_2$ are bifolds with framed basepoints (at non-orbifold points), then there is a preferred connected sum $\tilde{Y} = \tilde{Y}_1 \# \tilde{Y}_2$ obtained by summing at the basepoints. We can give the new bifold a preferred framed basepoint, on the 2-sphere where the sum is made. Given bifold cobordisms with framed arcs joining the basepoints, say

$$\tilde{X}_i : \tilde{Y}_i \to \tilde{Y}_i', \quad (i = 1, 2),$$

then we can form a cobordism

$$\tilde{X} : \tilde{Y} \to \tilde{Y}$$

by summing along the embedded arcs.

**Proposition 4.3.** In the above situation, we have an isomorphism

$$j^\#(\tilde{Y}) = j^\#(\tilde{Y}_1) \otimes j^\#(Y_2).$$

Furthermore, this isomorphism is natural for the maps corresponding to cobordisms $\tilde{X}_i$ as above, so that

$$j^\#(\tilde{X}) = j^\#(\tilde{X}_1) \otimes j^\#(\tilde{X}_2).$$

The same applies with $I^\#$ in place of $J^\#$.

**Corollary 4.4.** Suppose $K = K_1 \cup K_2$ is a split web in $\mathbb{R}^3$, meaning that there is an embedded 2-sphere $S$ which separates $K_1$ from $K_2$. Then there is an isomorphism,

$$j^\#(K) = j^\#(K_1) \otimes j^\#(K_2).$$

Moreover, if $\Sigma$ is a split cobordism, meaning that $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma$ is disjoint from $[0, 1] \times S$, then

$$j^\#(\Sigma) = j^\#(\Sigma_1) \otimes j^\#(\Sigma_2).$$

The same applies with $I^\#$ in place of $J^\#$. 
These excision results will be often applied through the following corollaries, which we state first in the language of bifold. Let $\hat{Q}$ be a closed, oriented, connected 3-dimensional bifold. Let $\hat{P}_i$ be 4-dimensional bifolds with boundary $\hat{Q}$, for $i = 1, \ldots, n$. In the notation of Definition 3.11, these have invariants

$$ J^\#(\hat{P}_i) \in J^\#(\hat{Q}). $$

Suppose that we also have a collection of cobordisms

$$ \hat{X}_i : \hat{Y} \to \hat{Y}', \quad i = 1, \ldots, n, $$

such that $\hat{X}_i$ contains in its interior an embedded copy of $\hat{P}_i$. Suppose the complements $\hat{X}_i \setminus \text{int}(\hat{P}_i)$ are all identified, in a way that restricts to the identity on their common boundary components $\hat{Y}, \hat{Y}'$ and $\hat{Q}$. We then have

**Corollary 4.5.** If the elements of $J^\#(\hat{Q})$ defined by the bifolds $\hat{P}_i$ satisfy a relation

$$ \sum_{i=1}^n J^\#(\hat{P}_i) = 0, $$

then it follows also that

$$ \sum_{i=1}^n J^\#(\hat{X}_i) = 0 $$

as maps from $J^\#(\hat{Y})$ to $J^\#(\hat{Y}')$. The same applies with $I^\#$ in place of $J^\#$.

We next restate the last corollary as it applies to the case of webs. Let $W$ be a web in a connected 3-manifold $Q$. For $i = 1, \ldots, n$, let $P_i$ be a connected, oriented 4-manifold with boundary $Q$, and let $V_i \subset P_i$ be a foam with boundary $W$. These determine elements

$$ J^\#(P_i, V_i) \in J^\#(Q, W). $$

Suppose we also have a collection of foam cobordisms

$$ (X_i, \Sigma_i) : (Y, K) \to (Y', K'). $$

Suppose that each $(X_i, \Sigma_i)$ contains an interior copy of $(P_i, V_i)$, and that the complements are all identical. We then have the following restatement:

**Corollary 4.6.** If $\sum J^\#(P_i, V_i) = 0$, then $\sum J^\#(X_i, \Sigma_i) = 0$, as maps from $J^\#(Y, K) \to J^\#(Y', K')$. The same holds with $I^\#$ in place of $J^\#$. 
5 Calculations

5.1 The unknot and the sphere with dots

**Proposition 5.1.** For the unknotted circle, the homology $J^\#(K)$ has rank 3. Equipped with the operator $u_e : J^\#(K) \to J^\#(K)$ (see Definition 3.8), it is isomorphic to $\mathbb{F}[u]/u^3$ as a module over the polynomial algebra.

*Proof.* The representation variety $\mathcal{R}^\#(K)$ is a copy of $\mathbb{R}P^2$ (as in section 2.9). We can choose a small, good perturbation $\pi$ so that $f_\pi$ restricts to a standard Morse function on $\mathbb{R}P^2$, leading to three critical points $\alpha_0$, $\alpha_1$ and $\alpha_2$. For the special case of the unknot the Floer homology is $\mathbb{Z}/4$ graded, and the only 1-dimensional moduli spaces are therefore $M_1(\alpha_2, \alpha_1)$ and $M_1(\alpha_1, \alpha_0)$. These moduli spaces approximate the Morse trajectories on $\mathbb{R}P^2$, so $J^\#(K) = H_*(\mathbb{R}P^2; \mathbb{F})$, which has rank 3.

For the module structure, a dot $\delta$ on the knot, the double-cover which corresponds to the line bundle $L_\delta$ is the non-trivial double cover of $\mathbb{R}P^2$. The calculation only involves Morse trajectories on $\mathbb{R}P^2$, so the result is the same as for a calculation of the cap product with the one-dimensional cohomology class acting on the homology of $\mathbb{R}P^2$. □

As a simple application of Corollary 4.6, we have the following result.

**Proposition 5.2.** Let $K \subset Y$ be a web, let $e$ be an edge of $K$, and $u_e$ the corresponding operator on $J^\#(Y,K)$. (See Definition 3.8.) Then $u^3_e = 0$.

*Proof.* We apply Corollary 4.6, to the case that $M = B^3$, the web $W$ is a single arc in $B^3$, and $(N,V)$ is a disk with 3 dots. We see that it is sufficient to check that a disk with three dots defines the zero element in $J^\#(K)$ when $K$ is the unknot. This in turn follows from Proposition 5.1. □

The calculation of $J^\#$ with its module structure for the unknot $K$ is closely related to the evaluation of the closed foam $S(k)$, consisting of an unknotted 2-sphere $S$ in $\mathbb{R}^4$ or $S^4$, with $k$ dots. Before continuing, it will be useful to make some general remarks about closed foams.

A closed foam $\Sigma$ in $S^4$ evaluates to 0 or 1,

$$J^\#(\Sigma) \in \mathbb{F}$$

by the convention of Definition 3.11. Because the definition involves counting solutions in 0-dimensional moduli spaces, we can read off the action $\kappa$ of
the relevant solutions from the dimension formula (5). Taking account of the codimension-1 constraints corresponding to the dots on \( \Sigma \), and writing \( k \) for the number of dots, we must have

\[
8\kappa + \chi(\Sigma) + \frac{1}{2} \Sigma \cdot \Sigma - \frac{1}{2} |\tau| - k = 0.
\]

This gives us some necessary conditions for a non-zero evaluation. Since \( \kappa \) is non-negative, we must have

\[
k \geq \chi(\Sigma) + \frac{1}{2} \Sigma \cdot \Sigma - \frac{1}{2} |\tau|.
\]

(14)

We evaluate \( J^\#(\Sigma) \) for a sphere with dots.

**Proposition 5.3.** Let \( S(k) \) denote the unknotted 2-sphere with \( k \) dots, as a foam in \( \mathbb{R}^4 \) or \( S^4 \). Then \( J^\#(S(k)) = 1 \) if \( k = 2 \) and is 0 otherwise.

**Proof.** The fact that \( J^\#(S(k)) = 0 \) for \( k \geq 3 \) follows from Proposition 5.2. The inequality (14) requires \( k \geq 2 \) in this case. So \( S(2) \) is the only case where the evaluation can be non-zero. Furthermore, for the case \( k = 2 \), the action \( \kappa \) for the relevant moduli spaces is zero: we are looking for flat connections. As in the case of the unknot considered earlier, the representation variety of flat connections is \( \mathbb{R}P^2 \), and the real line bundle over \( \mathbb{R}P^2 \) corresponding to each dot is the non-trivial one. So \( J^\#(S(2)) \) is obtained by evaluating the square of the basic 1-dimensional class on \( \mathbb{R}P^2 \), and the answer is 1 as claimed. \( \Box \)

If \( D(k) \) is a 2-disk in \( B^4 \) with boundary a standard circle in \( S^3 \), regarded as a foam with \( k \) dots, then a formal corollary of the above proposition is:

**Corollary 5.4.** The elements \( J^\#(D(0)), J^\#(D(1)) \) and \( J^\#(D(2)) \) are a basis for \( J^\#(K) \), where \( K \subset S^3 \) is a standard circle (the unknot).

Using Corollary 4.4, we obtain from this a simple criterion for testing whether an element of the Floer homology of an unlink is zero:

**Lemma 5.5.** Let \( (S^3, K_n) \) be an \( n \)-component unlink in the 3-sphere. Let \( (X, \Sigma_i) \) be webs with boundary \( (S^3, K_n) \). Then, in order that a relation

\[
\sum_i J^\#(X, \Sigma_i) = 0
\]
holds in \( J^\#(S^3, K_n) \), it is necessary and sufficient to test, for all \( n \)-tuples \((k_1, \ldots, k_n) \in \{0, 1, 2\}^n \), that the relation

\[
\sum_i J^\#(\bar{X}, \bar{\Sigma}_i) = 0,
\]

holds in \( \mathbb{F} \), where \( \bar{X} \) is obtained from \( X \) by adding a ball, and \( \bar{\Sigma}_i \) is obtained from \( \Sigma_i \) by adding a standard disk with \( k_m \) dots, \( D(k_m) \subset B^4 \), to the \( m \)'th component of \( K_n \).

5.2 The theta foam and the theta web

A theta foam is a closed foam formed from three disks in \( \mathbb{R}^3 \) meeting along a common circle, the seam. We write \( \Theta \) for a typical theta-foam in \( \mathbb{R}^4 \), and \( \Theta(k_1, k_2, k_3) \) for the same foam with the addition of \( k_i \) dots on the \( i \)'th disk.

Proposition 5.6. For the theta foam with dots, \( \Theta(k_1, k_2, k_3) \), we have

\[
\Theta(k_1, k_2, k_3) = 1
\]

if \((k_1, k_2, k_3) = (0, 1, 2) \) or some permutation thereof. In all other cases the evaluation is zero.

Proof. In the case of the theta foam in \( \mathbb{R}^4 \), any solution of the equation has action satisfying \( 4\kappa \in \mathbb{Z} \). One can verify this by considering a bifold connection on \((S^4, \Theta)\): any such gives rise to an ordinary \( SO(3) \) action of the \( V_4 \)-cover \( S^4 \), where the action must be an integer. The relation in Proposition 3.9 tells us that

\[
J^\#(\Theta(k_1, k_2, k_3)) = J^\#(\Theta(k_1 - 1, k_2 + 1, k_3)) + J^\#(\Theta(k_1 - 1, k_2, k_3 + 1))
\]

as long as \( k_1 \neq 0 \). So the evaluation for \( \Theta(k_1, k_2, k_3) \) will be completely determined once we know the evaluations for the cases with \( k_1 = 0 \). So we examine \( \Theta(0, k_2, k_3) \). For a non-zero evaluation, we require \( k_i \leq 2 \), because of Proposition 5.2. The inequality (14) requires \( k_2 + k_3 \geq 3 \) in this case. Furthermore, \( k_2 + k_3 \) must be odd, because \( 4\kappa \) is an integer. These constraints mean we need only look at \( \Theta(0, 1, 2) \), and here the moduli space we are concerned with is a moduli space of flat connections. The representation variety in this example is the flag manifold \( F = SO(3)/V_4 \), and the three real line bundles corresponding to dots on the three sheets are the three tautological line bundles \( L_1, L_2, L_3 \) on \( F \). The evaluation of \( \Theta(0, 1, 2) \) is therefore equal to

\[
\langle w_1(L_2)w_1(L_3)^2, [F] \rangle,
\]

which is 1. \( \square \)
Let $K$ be a theta web. Corresponding to the three edges, there are three operators $u_i : J^b(K) \to J^b(K)$.

**Proposition 5.7.** For the theta web $K$, the instanton homology $J^b(K)$ can be identified with the ordinary $\mathbb{F}$ homology of the flag manifold $F = SO(3)/V_4$ in such a way that the operators $u_i$ correspond to the operation of cap product with the classes $w_i(L_i)$, where $L_1$, $L_2$, $L_3$ are the tautological line bundle on the flag manifold. Concretely, this means that the dimension is 6 and that the instanton homology is a cyclic module over the algebra generated by the $u_i$ which satisfy the relations

$$u_1 + u_2 + u_3 = 0$$
$$u_1u_2 + u_2u_3 + u_3u_1 = 0$$
$$u_1u_2u_3 = 0.$$

**Proof.** The representation variety for the theta web is the flag manifold, so the rank of the instanton homology is at most 6. On the other hand, we can see that the rank is at least 6 as follows. Consider a theta-foam cut into two pieces by a hyperplane, so as to have cobordisms $\Theta_-$ from $\emptyset$ to $K$ and $\Theta_+$ from $K$ to $\emptyset$. Putting $k_i$ dots on the $i$'th facet of $\Theta_\pm$ and applying $J^b$, we obtain vectors $x_-(k_1, k_2, k_3)$ in $J^b(K)$ and covectors $x_+(k_1, k_2, k_3)$. The pairings between these can be evaluated using the knowledge of the closed theta foam (Proposition 5.6). If we restrict to the elements $x_+(0, k_1, k_2)$ with $k_1 \leq 1$ and $k_2 \leq 2$, then the resulting matrix of pairings is non-singular. So the corresponding 6 elements $x_-(0, k_1, k_2)$ in $J^b(K)$ are independent. The relations satisfied by the operators $u_i$ can be read off similarly. \qed

As a corollary of the above proposition and excision, we have:

**Proposition 5.8.** Let $K$ be any web in a 3-manifold $Y$, and let $u_1$, $u_2$, $u_3$ be the operators corresponding to three edges of $K$ that are incident at a common vertex. Then the operators satisfy the same relations,

$$u_1 + u_2 + u_3 = 0$$
$$u_1u_2 + u_2u_3 + u_3u_1 = 0$$
$$u_1u_2u_3 = 0.$$

(15)

In particular, any monomial $u_1^a u_2^b u_3^c$ of total degree 4 or more is zero.
5.3 The tetrahedron web and its suspension

Let $K$ be the tetrahedron web: the graph drawn in Figure 4, with the edges labeled as shown. Let $T \subset B^4$ be the cone on $K$, regarded as a foam with one tetrahedral point. Let $E_i$ and $F_i$ be the facets of $T$ corresponding to the edges $e_i$ and $f_i$ of $K$. Let $T(k_1, k_2, k_3)$ denote the foam $T$ with the addition of $k_i$ dots on the facet $E_i$. Let $S \subset S^4$ be the double of $T$, the suspension of $K$ with two tetrahedral points. Let $S(k_1, k_2, k_3)$ be defined similarly.

We can evaluate the closed foams $S(k_1, k_2, k_3)$ just as we did for the theta foam, and the answers are the same, as the next proposition states.

**Proposition 5.9.** For the foam $S(k_1, k_2, k_3)$, we have

$$S(k_1, k_2, k_3) = 1$$

if $(k_1, k_2, k_3) = (0, 1, 2)$ or some permutation thereof. In all other cases the evaluation is zero.

**Proof.** In the case that $k = k_1 + k_2 + k_3$ is less than 3, we obtain 0. In the case that $k = 3$, we obtain the evaluation of the corresponding ordinary cohomology class on the flag manifold, as in the case of the theta foam. When $k \geq 4$, the evaluation is zero, by an application of excision to reduce to the case of the theta foam. □

**Proposition 5.10.** The instanton homology $J^\#(K)$ has dimension 6. It is generated by the elements $J^\#(T(0, k_2, k_3))$ with $0 \leq k_2 \leq 1$ and $0 \leq k_3 \leq 2$. The operators
$u_1, u_2, u_3$ corresponding to the edges $e_1, e_2, e_3$ satisfy the relations of the cohomology of the flag manifold (15) and $J^\#(K)$ is a cyclic module over the corresponding polynomial algebra, with generator $J^\#(T)$.

**Proof.** The representation variety of $K^\#$ is again the flag manifold, so the dimension of $J^\#(K)$ is at most 6. On the other hand, we can compute the pairings of the classes $J^\#(T(k_1, k_2, k_3))$ with all their doubles, using our knowledge of $S(k_1, k_2, k_3)$, and from this we see that the six elements $J^\#(0, k_2, k_3)$ for $k_2$ and $k_3$ in the given range are independent. So the rank is exactly 6. □

To complete our description of $J^\#(K)$, we must describe the operators corresponding to the other three edges, $f_1, f_2, f_3$.

**Proposition 5.11.** The operators $v_1, v_2, v_3$ on $J^\#(K)$ corresponding to the edges $f_1, f_2, f_3$ are equal to $u_1, u_2, u_3$ respectively.

**Proof.** We will show that $v_1 = u_1$. Let $S_1$ denote the foam $S$ equipped with a single dot on the facet corresponding to $f_1$. Let $S_1(k_1, k_2, k_3)$ denote the same foam with addition dots on the facets corresponding to the edges $e_i$. It will suffice to show that

$$S_1(0, k_2, k_3) = S(1, k_2, k_3)$$

for all values of $k_2$ and $k_3$. As usual if $1 + k_2 + k_3 \leq 2$, then the evaluation is zero, while if $1 + k_2 + k_3 = 3$ we are evaluating an ordinary cohomology class on the flag manifold. Since the line bundle corresponding to a dot on $f_1$ is the same as the line bundle corresponding to a dot on $e_1$, equality does hold when $1 + k_2 + k_3 = 3$. The only case still in doubt is when $1 + k_2 + k_3 = 4$, and the only potentially non-zero evaluations here are $S_1(0, 1, 2)$ or $S_1(0, 2, 1)$ (which are certainly equal, by symmetry). We must show that $S_1(0, 1, 2) = 0$.

Suppose instead that $S_1(0, 1, 2) = 1$. With this complete information about $S_1(0, k_1, k_2)$ we can evaluate the matrix elements of $v_1$ on our standard basis, and we find that $v_1 + u_1 = 1$ as operators on $J^\#(K)$. But $u_1$ and $v_1$ are both nilpotent because $u_1^3 = v_1^3 = 0$, so this is impossible. □

### 5.4 Upper bounds for some prisms

Let $L_n$ be the planar trivalent graph formed from two concentric $n$-gons with edges joining each vertex of the outer $n$-gon to the corresponding vertex of the inner $n$-gon. We call $L_n$ the $n$-sided prism. Later we will be able to determine $J^\#(L_n)$ entirely; but for now we can obtain upper bounds on the rank of $L_n$ for
small $n$. These upper bounds will be used later, in section 6, after which we will have the tools to show that these upper bounds are exact values.

**Lemma 5.12.** The dimension of $\mathcal{J}^\#(L_n)$ is at most 12 for $n = 2$, at most 6 for $n = 3$, and at most 24 for $n = 4$.

**Proof.** For $n = 2$ and $n = 3$, the representation varieties $\mathcal{R}^\#(L_n)$ consists of 2 (respectively, 1) copy of the flag manifold $SO(3)/V_4$. These are the orbits of the flat connections with image $V_4$, of which there are two distinct classes in the case of $L_2$. In both cases, the Chern-Simons function is Morse-Bott, so an upper bound for the ranks of $\mathcal{J}^\#(L_n)$ are the dimension of the ordinary homology of the representation variety, which are 12 and 6 respectively.

For the case $n = 4$, the representation variety is not Morse-Bott. The un-based representation variety $\mathcal{R}(L_4)$ consists of three arcs with one endpoint in common. The common point and the three other endpoints of the arc correspond to $V_4$ representations in $\mathcal{R}(L_4)$. The interiors of the three arcs parametrize flat connections with image contained in $O(2)$ but not in $V_4$. Using an explicit holonomy perturbation, one can perturb the Chern-Simons function by a small function $f$ which has critical points at the common endpoint and the three other endpoints of the arc, but whose restriction to the interiors of the three arcs has no critical points. After adding this perturbation, the critical set in $\mathcal{R}^\#(L_4)$ is Morse-Bott and comprises four copies of $SO(3)/V_4$. We omit the details for this explicit construction, because there is a more robust argument based on the exact triangle satisfied by $\mathcal{J}^\#$ which will be treated in a subsequent paper [9]. □

6 Relations

Sections 6.1–6.4 establish properties of $\mathcal{J}^\#$ that aid calculation. With the exception of the triangle relation (section 6.3), these properties (and some of their proofs) are motivated by [7], where similar properties of Khovanov’s $\mathfrak{sl}_3$ homology are either proved or (in the case of the neck-cutting relation) taken as axioms.

6.1 The neck-cutting relation

The following proposition describes what happens when a foam is surgered along an embedded disk. The statement of the proposition is illustrated in Figure 5. See also [7].
Figure 5: The neck-cutting relation, cf. [7]. The disk $D$ is not part of the original foam.

**Proposition 6.1 (Neck-cutting).** Let $\langle X, \Sigma \rangle$ be a cobordism defining a morphism in $C^\sharp$. Suppose that $X$ contains an embedded disk $D$ whose boundary lies in the interior of a facet of $\Sigma$, which it meets transversely. Suppose that the trivialization of the normal bundle to $D$ at the boundary which $\Sigma$ determines extends to a trivialization over the disk. Let $\Sigma'$ be the foam obtained by surgering $\Sigma$ along $D$, replacing the annular neighborhood of $\partial D$ in $\Sigma$ with two parallel copies of $D$. Let $\Sigma'(k_1, k_2)$ be obtained from $\Sigma'$ by adding $k_i$ dots to the $i$'th copy of $D$. Then

$$J^\sharp(X, \Sigma) = J^\sharp(X, \Sigma'(0, 2)) + J^\sharp(X, \Sigma'(1, 1)) + J^\sharp(X, \Sigma'(2, 0)).$$

**Proof.** Using the excision principle, Corollary 4.6, we see that it is enough to test this in the local case that $X$ is a 4-ball, $\Sigma$ is a standard annulus, $\Sigma'$ is a pair of disks, and we regard $\langle X, \Sigma \rangle$ etc. as cobordisms from $\emptyset$ to the unlink of 2 components. Thus the relation to be proved is a relation in $J^\sharp(K_2)$, where $K_2 \subset S^3$ is the 2-component unlink. To prove this relation in $J^\sharp(K_2)$, we use Lemma 5.5. This reduces the question to the $J^\sharp$-evaluation of some closed foams in the 4-sphere: the foams all consist of spheres $S$ with a certain number of dots, and the relation to be proved is

$$J^\sharp(S(k_1 + k_2)) = J^\sharp(S(k_1))J^\sharp(S(k_2 + 2)) + J^\sharp(S(k_1 + 1))J^\sharp(S(k_2 + 1)) + J^\sharp(S(k_1 + 2))J^\sharp(S(k_2)).$$

This relation follows directly from Proposition 5.3 $\square$

Using neck-cutting, we can evaluate closed surfaces, as long as they are standardly embedded.

**Proposition 6.2.** Let $\Sigma_g \subset \mathbb{R}^4$ be a genus-$g$ surface embedded as the boundary of a standard handlebody in $\mathbb{R}^3$. Then $J^\sharp(\Sigma_1) = 1$ and $J^\sharp(\Sigma_g) = 0$ for $g \geq 2$. If $g \geq 1$ and $\Sigma_g$ has one or more dots, then its evaluation is zero.
Proof. Use the neck-cutting relation to reduce to the case of a sphere with dots.

As in [7], one can use the evaluation of spheres and theta foams together with the neck-cutting relation to obtain other relations. In particular:

**Proposition 6.3 (Bubble-bursting).** Let $D$ be an embedded disk in the interior of a facet of a foam $\Sigma \subset X$. Let $\gamma$ be the boundary of $D$, and let $\Sigma'$ be the foam $\Sigma' = \Sigma \cup D'$, where $D'$ is a second disk meeting $\Sigma$ along the circle $\gamma$, so that $D \cup D'$ bounds a 3-ball. Let $\Sigma'(k)$ denote $\Sigma'$ with $k$ dots on $D'$. Let $\Sigma(k)$ denote $\Sigma$ with $k$ dots on $D$. Then we have

$$j^\#(\Sigma'(k)) = j^\#(\Sigma(k - 1))$$

for $k = 1$ or 2, and $j^\#(\Sigma'(k)) = 0$ otherwise.

Proof. Let $\gamma_1$ be a circle parallel to $\gamma$ in $\Sigma \setminus D$. Apply the neck-cutting relation to the surgery of $\Sigma'$ along a disk with boundary $\gamma'$. The surgered foam is the union of a foam isotopic to $\Sigma$ with theta foam.

Instead of adding a disk $D'$ to $\Sigma$, we can add a standard genus-1 surface $T$ with boundary $\gamma$. The resulting foam $\Sigma \cup T$ was considered earlier in section 3.5. Let $\Sigma(k)$ again denote $\Sigma$ with $k$ dots in the disk. Then we have

**Proposition 6.4.** In the above situation, $j^\#(\Sigma \cup T) = j^\#(\Sigma(1))$.

Proof. By two surgeries, the foam $\Sigma \cup T$ becomes a split union of $\Sigma$, a theta foam and a standard torus. Apply the neck-cutting relation and the known evaluations for theta foams and tori.

The proposition above justifies the alternative description of how to incorporate dots on foams, from section 3.5.

### 6.2 The bigon relation

Let $K \subset Y$ be a web containing a bigon: a pair of edges spanning standard disk in $Y$, and let $K'$ be obtained from $K$ by collapsing the bigon to a single edge. (See Figure 6.)

**Proposition 6.5.** The dimension of $j^\#(K)$ is twice the dimension of $j^\#(K')$.
Proof. The proof mirrors Khovanov’s argument in [7]. There four standard morphisms,

\[ A, B : K' \to K \]
\[ C, D : K \to K' \]

as shown in Figure 6, and these give rise to maps

\[ a, b : J^\#(K') \to J^\#(K) \]
\[ c, d : J^\#(K) \to J^\#(K'). \]

The composite foams \( CA \) and \( DB \) are foams to which the bubble-bursting relation applies, from which we see that \( ca = 1 \) and \( db = 1 \). Similarly, \( cb = da = 0 \). From this it follows that \( a \oplus b \) maps \( J^\#(K') \oplus J^\#(K') \) injectively into \( J^\#(K) \) and that

\[ ac + bd : J^\#(K) \to J^\#(K) \]

is a projection on the image of \( a \oplus b \). If we can show that

\[ ac + bd = 1, \]

then we will be done. By the excision principle in the form of Corollary 4.6, this is equivalent to checking a relation

\[ \alpha + \beta = \epsilon \]
Figure 7: The foams $\alpha$, $\beta$ and $\epsilon$.

Figure 8: An isotopy of foams showing that $ac(\epsilon) = \alpha$.

in $\mathcal{J}(L_2)$, where $\alpha$, $\beta$ and $\epsilon$ are the dotted foams show in Figure 7.

In the special case that $K'$ is the theta web and $K$ is a prism $L_2$, we know from Lemma 5.12 that $\mathcal{J}(K)$ has dimension at most 12, which is twice the dimension of $\mathcal{J}(K')$. So equality must hold, and in this special case we have $ac + bd = 1$. Applying this to the element $\epsilon \in \mathcal{J}(K) = \mathcal{J}(L_2)$, we obtain

$$ac(\epsilon) + bd(\epsilon) = \epsilon.$$  

By simple isotopies, we see that $ac(\epsilon) = \alpha$ (see Figure 8) and $bd(\epsilon) = \beta$, so the result follows. \hfill \qed

6.3 The triangle relation

Let $K$ be a web containing three edges which form a triangle lying in a standard 2-disk in $Y$. and $v$ a vertex of $K'$. Let $K'$ be obtained from $K$ by collapsing the triangle to a single vertex, following the standard planar model shown in Figure 9. There are standard morphisms given by foams with one tetrahedral point each,

$$A : K' \rightarrow K$$

$$C : K \rightarrow K',$$
Figure 9: The triangle move.

Figure 10: The foams $C$, $A$ and their composite $CA$, as a foam cobordism from $K'$ to $K'$.

and there are resulting linear maps

$$a : J^\#(K') \to J^\#(K)$$
$$c : J^\#(K) \to J^\#(K')$$

**Proposition 6.6.** The instanton homologies $J^\#(K')$ and $J^\#(K)$ are isomorphic, and the maps $a$ and $c$ are mutually inverse isomorphisms.

**Proof.** The proof follows the same plan as in the case of the bigon relation. To show that composite foam $CA$ gives the identity map on $J^\#(K')$, we apply the excision principle, Corollary 4.6, to the part of the composite shown on the left-hand side in Figure 10. The left- and right-hand sides define elements in $J^\#$ of the
theta graph $K_\theta$, which we must show are equal. We can demonstrate equality by checking that we have the same evaluation of closed foams when we pair both sides with the known basis of $J^\#(K_\theta)$ coming from the foams $\Theta_-(0,k_2,k_3)$. The required evaluations become (respectively) $J^\#(S(0,k_2,k_3))$, where $S$ is the cone on the tetrahedron web, and $J^\#(\Theta(0,k_2,k_3))$, where $\Theta$ is the closed theta foam. These evaluations are equal by Proposition 5.9 and Proposition 5.6. This verifies that $ca = 1$.

To show that $ac = 1$, we must show that

$$\alpha = \epsilon$$

where $\alpha$ and $\epsilon$ are the elements in the instanton homology $J^\#(L_3)$ corresponding to the foams in Figure 11. From the relation $ca = 1$, it already follows that $AC$ is a projection onto the injective image of $J^\#(K)$ in $J^\#(K')$. In the special case that $K'$ is the tetrahedron graph and $K$ is the prism $L_3$, we already know that $\dim J^\#(K) \leq \dim J^\#(K')$ from Lemma 5.12, so in this special case we know that $ac = 1$ as endomorphisms of $J^\#(L_3)$. Applying this to the element $\epsilon \in J^\#(L_3)$, we obtain

$$ac(\epsilon) = \epsilon.$$  

By an isotopy, we see that $ac(\epsilon) = \alpha$, and the proof is complete. $\Box$

We can reinterpret the central calculation in the above proof. Let $(X, \Sigma)$ be a foam cobordism inducing a map $J^\#(X, \Sigma) : J^\#(Y_1, K_1) \to J^\#(Y_0, K_0)$. Suppose $x_1, x_2$ are two distinct tetrahedral points in $\Sigma$ connected by a seam-edge $\gamma$. A regular neighborhood of the arc $y$ is a 4-ball meeting $\Sigma$ in a standard foam $W \subset B^4$ whose boundary is the prism web $L_3$, the same foam that appears in the left-hand side of Figure 11. Let $W' \subset B^4$ be the foam with the same boundary shown in the right-hand side of the figure, and let $\Sigma'$ be obtained from $\Sigma$ by replacing $W$ with $W'$. We refer to the process of passing from $\Sigma$ to $\Sigma'$ as canceling tetrahedral points.
Figure 12: The webs $K$, $K'$ and $K''$ involved in the square relation: $f^\#(K) = f^\#(K') \oplus f^\#(K'')$.

**Proposition 6.7 (Canceling tetrahedral points).** In the above situation, we have $f^\#(X, \Sigma) = f^\#(X, \Sigma')$.

**Proof.** By excision this reduces to the same problem as before, namely proving that $\alpha = \epsilon$ (equation (16)) in $f^\#(S^3, L_3)$. □

### 6.4 The square relation

Like the bigon and triangle relations above, our statement and proof of the square relation follows [7], with the same sorts of adaptations as in the earlier cases. Let $K \subset Y$ be a web containing a square: four edges connecting four vertices in a standard disk. Let $K'$ and $K''$ be obtained from $K$ as shown in Figure 12.

**Proposition 6.8.** If the web $K$ contains a square and $K'$, $K''$ are as shown in Figure 12, then

$$f^\#(K) = f^\#(K') \oplus f^\#(K'').$$

**Proof.** There are standard morphisms

$$
\begin{align*}
    K' & \xrightarrow{A} K \xleftarrow{C} K'' \\
    K & \xrightarrow{D} K''
\end{align*}
$$

A model for the foam $A$ is shown in Figure 13, and the others are similar. Let $a, b, c, d$ be the corresponding linear maps on $f^\#(K')$ etc. By an application of neck-cutting and bubble-bursting, one sees that $ca = 1$, and similarly $db = 1$. By an application of bubble-bursting, one sees that $da = 0$, and similarly $cb = 0$. It follows $ac + bd$ is a projection onto the injective image of $f^\#(K') \oplus f^\#(K'')$, and to complete the proof we must show

$$ac + bd = 1.$$
In the case that $K$ is a cube, we know that $J^\sharp(K)$ has dimension at most 24 by Lemma 5.12. The webs $K'$ and $K''$ in that case are the same, and can both be reduced to the theta web by the collapsing of a bigon; so $J^\sharp(K')$ and $J^\sharp(K'')$ both have rank 12. So we learn that $ac + bd = 1$ in the case that $K$ is a cube. The general case is now proved by the same strategy as in the bigon and triangle relations. □

6.5 Simple graphs

To the bigon, triangle and square relations above, we can add the relations that hold for a ‘0-gon’ and a ‘1-gon’. A 1-gon is an edge-loop which bounds a disk whose interior is disjoint from $K$. If $K$ has a 1-gon, then it has an embedded bridge, and $J^\sharp(K) = 0$. A 0-gon is a vertexless circle bounding a disk whose interior is disjoint from $K$. If $K'$ is obtained from $K$ by deleting the circle, then $\dim J^\sharp(K) = 3 \dim J^\sharp(K')$, by the multiplicative property, Corollary 4.4, and Proposition 5.1. If $\tau(K)$ denotes the number of Tait colorings of $K$, then the relations that $\dim J^\sharp$ satisfies as a consequence of these relations for $n$-gons, $0 \leq n \leq 4$, are satisfied also by $\tau(K)$ (as is well known and easy to verify). It follows that if the square, triangle and bigon moves can be used to reduce a web $K \subset \mathbb{R}^3$ to webs that are either unlinks or contain embedded bridges, then $\dim J^\sharp(K) = \tau(K)$. Conjecture 1.2 is the statement that $\dim J^\sharp(K) = \tau(K)$, and we see that a minimal counterexample cannot have any $n$-gons with $n \leq 4$. 
7 Proof of non-vanishing, Theorem 1.1

7.1 Passing from \( J^\# \) to \( I^\# \)

From a finite chain complex \((C,d)\) over \( \mathbb{F} \) with a basis \( \mathcal{C} \), we can form a graph \( \Gamma \) with vertices \( \mathcal{C} \) and with edges corresponding to non-zero matrix entries of \( d \). Let us say that a chain complex \((\tilde{C},\tilde{d})\) is a double-cover of \((C,d)\) if there is a basis \( \mathcal{C} \) for \( C \) and basis \( \tilde{\mathcal{C}} \) for \( \tilde{C} \) such that the corresponding graph \( \tilde{\Gamma} \) is a double-cover of \( \Gamma \).

Lemma 7.1. If the homology of \((\tilde{C},\tilde{d})\) is non-zero, then so is the homology of \((C,d)\).

Proof. This is a corollary of the standard mod 2 Gysin sequence for the double cover, i.e. the long exact sequence corresponding to the short exact sequence of chain complexes

\[
0 \to C \to \tilde{C} \to C \to 0.
\]

On the bases \( \mathcal{C} \) and \( \tilde{\mathcal{C}} \), the first map sends a basis element to the sum of its two preimages, and the second map is the projection. \( \square \)

If we have a Morse function \( f \) on a compact Riemannian manifold \( B \), satisfying the Morse-Smale condition, and if we pull it back to a double-cover \( \tilde{B} \), then the pull-back will still be Morse-Smale. Furthermore, the associated Morse complex \((\tilde{C},\tilde{d})\) will be a double-cover of \((C,d)\) in the above sense. The same applies equally to a Morse complex of the perturbed Chern-Simons functional on \( \mathcal{B}_l(\tilde{Y};\mu) \). This gives us the following result.

Lemma 7.2. Let \( \tilde{Y} \) be a bifold with singular set \( K \), and suppose we have two different strong marking data, \( \mu \) and \( \mu' \), with \( U_\mu \subset U_{\mu'} \) and \( E_\mu = E_{\mu'}|_{U_\mu \setminus K} \). Suppose \( J(\tilde{Y};\mu) \) is non-zero. Then \( J(\tilde{Y};\mu') \) is also non-zero.

Proof. According to Lemma 2.4, the map \( \mathcal{B}_l(\tilde{Y};\mu') \to \mathcal{B}_l(\tilde{Y};\mu) \) is an iterated double-covering of some union of components of the latter. It follows that the corresponding chain complex \((C',d')\) is an iterated double-cover of some direct summand \((C_0,d_0)\) of the complex \((C,d)\). The previous lemma tells us that \( H_*(C_0,d_0) \) is non-zero. Therefore \( H_*(C,d) \) is non-zero too. \( \square \)

Corollary 7.3. If \( K \) is a web in \( \mathbb{R}^3 \) and \( I^\#(K) \) is non-zero, then \( J^\#(K) \) is also non-zero.

Proof. This follows from the definition of \( I^\# \) and \( J^\# \) (Section 3.6): they are defined using the same bifold, but \( J^\# \) is defined using a larger marking set. \( \square \)
Using also the multiplicative property for split webs, Corollary 4.4, we learn that, in order to prove the non-vanishing theorem, Theorem 1.1, it will suffice to prove the following variant:

**Variant 7.4.** If $K \subset \mathbb{R}^3$ is a non-split web with no embedded bridge, then the $\mathbb{F}$ vector space $I^\sharp(K)$ is non-zero.

At this point in the argument, we can dispense with $J^\sharp$.

### 7.2 Removing vertices

As mentioned in Section 3.6, the construction $I^\sharp$ is an extension to webs of the invariant of links (also called $I^\sharp(K)$) which was defined in [15]. We now adopt some of the notation and concepts from [15] to continue the argument.

Let $(Y, K)$ be a 3-manifold with an embedded web, and let $\mu$ be marking data in which the subset $U_\mu$ is all of $Y$. In this case, let us represent $w_2(E_\mu)$ as a the dual class to a closed 1-dimensional submanifold $w \subset Y \setminus K$, consisting of a collection of circles and arcs joining points on edges of $K$. Since $w$ determines the marking, we may write

$$I^w(Y, K)$$

for the invariant that we have previously called $J(Y, K; \mu)$. This is the notation used in [15], though we are still using $\mathbb{F}$ coefficients. The group is defined only if $(Y, K, w)$ satisfies a condition which is a counterpart of the strong marking condition: a sufficient condition for this is that there is a non-integral surface $S$ in $Y$: i.e. a closed orientable surface which either

(a) has odd intersection with $K$, or

(b) is disjoint from $K$ and has odd intersection with $w$.

See [15, Definition 3.1]. The first condition holds, in particular, if $K$ has a vertex. In this notation, if $H \subset S^3$ is a Hopf link near infinity in $S^3$ and $u$ is an arc joining its two components, then for a web $K \subset \mathbb{R}^3$, we have

$$I^\sharp(K) = I^u(S^3, K \cup H).$$

(17)

There is another notation used in [15, Section 5.2] that we also need to adopt here. On $Y$, the obstruction to lifting an $SO(3)$ gauge transformation $u$ to the determinant-1 gauge group is an element $\varnothing(u)$ in $H^1(Y; \mathbb{F})$. So the usual configuration space of connections (the quotient by the determinant-1 gauge group) is
acted on by $H^1(Y; \mathbb{F})$. Given a choice of subgroup $\phi \subset H^1(Y; \mathbb{F})$, we can form the quotient of the configuration space by $\phi$ and compute the Morse theory of the perturbed Chern-Simons functional on this quotient. The resulting group

$$I^w(Y, K)^\phi$$

is defined whenever $\phi$ acts freely on the set of critical points. A sufficient condition of this is that there be a non-integral surface $S$ such that the restriction of $\phi$ to $S$ is zero. (See [15] once more.) In our applications, this condition will always be easy to verify, because there will be a 2-sphere meeting $K$ in three points, on which $\phi$ will be zero.

We return to a general $(Y, K)$, with the marking data now represented by $w$ as above. Let $v_1, v_2$ be two vertices of $K$, and let $B_1, B_2$ be standard ball neighborhoods of these, each meeting $K$ in three arcs joined at the vertex, disjoint from $w$. Let $Y^+$ be the oriented 3-manifold obtained by removing the balls $B_i$ and identifying the two $S^2$ boundary components. (Topologically this is $Y\#(S^1 \times S^2)$.) We make the identification so that the three points on $\partial B_1$ where $K$ meets $\partial (Y \setminus B_1)$ are identified (in any order) with the corresponding points on $\partial B_2$. In this way we obtain a new pair $(Y^+ K^+)$, though not uniquely. Because it is disjoint from the balls, we may regard $w$ as a submanifold also of $Y^+$.

**Proposition 7.5.** In the above situation, if $I^w(Y^+, K^+)$ is non-zero, then so too is $I^w(Y, K)$.

**Proof.** This is an application of excision. Let $S_1$ and $S_2$ be the spheres with three marked points, $\partial B_1$ and $\partial B_2$. If we cut $(Y, K)$ along $S_1$ and $S_2$ and re-glue, then we obtain a disconnected manifold with two components:

$$(Y', K') = (Y^+, K^+) \cup (S^3, \Theta).$$

Here $\Theta$ is a standard Theta graph in $S^3$. Let $S \subset Y^+$ be the new 2-sphere where the two surfaces are identified. Combing the argument of [15, Theorem 5.6] with Theorem 4.2, there is an excision isomorphism,

$$I^w(Y, K) \cong I^w(Y', K')^\psi = I^w(Y^+, K^+)^\psi \otimes I(S^3, \Theta),$$

where the subgroup $\psi \subset H^1(Y'; \mathbb{F})$ is the 2-element subgroup generated by the class dual to $S$.

Since the $I(S^3, \Theta) = \mathbb{F}$, we see that non-vanishing of $I^w(Y, K)$ is equivalent to non-vanishing of $I^w(Y^+, K^+)^\psi$. By the double-cover argument of Section 7.1, this is implied by the non-vanishing of $I^w(Y^+, K^+)$. □
If $K \subset S^3$ is a web, let $2n$ be the number of vertices (which is always even). We may pair these up arbitrarily, and apply the above topological construction $n$ times to obtain a new pair $(Y^+, K^+)$ with no vertices at all. The manifold $Y^+$ will be a connected sum of $n$ copies of $S^1 \times S^2$. We call this process excising all the vertices.

### 7.3 A taut sutured manifold

Let us turn again to the situation of Proposition 7.4. In the notation (17), we are looking at $I^w(S^3, K \cup H)$. Excising all the vertices and applying Proposition 7.5, we see that it suffices to prove the following variant:

**Variant 7.6.** Suppose $K \subset S^3$ is a non-split web without an embedded bridge and let $H$ and $u$ be as before. Let $(Y^+, K^+)$ be obtained from $(S^3, K)$ by excising all the vertices. Then $I^w(Y^+, K^+ \cup H)$ is non-zero.

Our next step deals with the Hopf link $H$. We state a lemma which involves summing a Hopf link onto one of the components of a link $K$.

**Lemma 7.7.** Let $K \subset Y$ be a web and $w \subset Y \setminus K$ a union of arcs representing a choice of $w_2$. Let $H$ and $u$ be a Hopf link and arc as usual, contained in a ball disjoint from $K$. Let $m$ be a meridian of some edge $e$ of $K$, i.e. the boundary of a disk transverse to $e$, and let $v$ be an arc joining $m$ to $e$ in the disk. Then (with coefficients still $F$), we have

$$\dim I^{w+u}(Y, K \cup H) = 2 \dim I^{w+u}(Y, K \cup m).$$

**Proof.** We shall compare both the left- and right-hand sides of the equation to the dimension of the instanton homology when both $m$ and $H$ are included,

$$I^{w+u+v}(Y, K \cup m \cup H).$$

The skein exact sequence of [15] gives an exact triangle

$$\cdots \to I^{w+u}(Y, K \cup H) \to I^{w+u+v}(Y, K \cup m \cup H) \to I^{w+u}(Y, K \cup H) \xrightarrow{\partial} \cdots.$$

In the special case that $K$ is the unknot in $S^3$ and $w$ is empty, we know that $I^{w+u}(Y, K \cup H)$ has dimension 2, and $I^{w+u+v}(Y, K \cup m \cup H)$ has dimension 4; so in this case the connecting homomorphism

$$I^{w+u}(Y, K \cup H) \xrightarrow{\partial} I^{w+u}(Y, K \cup H)$$
is zero (with \( \mathbb{F} \) coefficients). By an application of Corollary 4.6 it follows that \( \partial = 0 \) for all \( K \) and \( w \), and hence

\[
\dim I^{w+u+u}(Y, K \cup m \cup H) = 2 \dim I^{w+u}(Y, K \cup H).
\]

On the other hand we also have

\[
\dim I^{w+u+u}(Y, K \cup m \cup H) = 4 \dim I^{w+u}(Y, K \cup m). \tag{18}
\]

To see this we can take the disjoint union of \( Y \) with a 3-sphere containing two Hopf links,

\[
(Y, K \cup m) \amalg (S^3, H_1 \cup H_2),
\]

and apply the excision principle, cutting and along tori to see that the instanton Floer homology of this union is the same as that of

\[
(Y, K \cup m \cup H) \amalg (S^3, H).
\]

The formula (18) then follows from the fact that

\[
\dim I^{u_1+\cdots+u_r}(S^3, H_1 \cup \cdots \cup H_r) = 4 \dim I^u(S^3, H).
\]

\(\square\)

In the situation described in Variant 7.6, let \( r \) be the number of components of the link \( K^+ \). (This number will depend on how we chose to make the identifications when we excised all the vertices.) To prove non-vanishing of \( I^u(Y^+, K^+ \cup H) \), it will suffice to prove non-vanishing of

\[
I^{u_1+\cdots+u_r}(Y^+, K^+ \cup H_1 \cup \cdots \cup H_r)
\]

because of the tensor product rule for \( \ast \) of disjoint unions. By the Lemma above, the rank of this group is \( 2^r \) times the rank of the group

\[
I^u(Y^+, K^+ \cup m)
\]

where \( m = m_1 \cup \cdots \cup m_r \) is a collection of meridians, one for each component of \( K^+ \), and \( v = v_1 + \cdots + v_r \) is a collection of standard arcs. We arrive at the following variant which is sufficient to prove Theorem 1.1.
**Variant 7.8.** Let the web $K \subset S^3$ be non-split and without an embedded bridge, let $(Y^+, K^+)$ be obtained by excising all the vertices, let $m$ be a collection of meridians of $K^+$, one for each edge, and let $v$ be a collection of standard arcs joining each meridian to the corresponding component. Then

$$I^v(Y^+, K^+ \cup m)$$

is non-zero.

If $K^+$ has only one component, then $I^v(Y^+, K^+ \cup m)$ is exactly the group called $I^v(Y^+, K^+)$ in [15], though over $\mathbb{F}$ here. It is in fact possible to choose the identifications when excising the vertices so that $r = 1$, but we will continue without making any restriction on $r$.

We now use excision, to remove all the components of $K^+$ and $m$, in the same way that was done for the case $r = 1$ in [15]. (See in particular [15, Proposition 5.7] for essentially the same argument that we present now.) For each component $K^+_i \subset K^+$ and corresponding meridian $m_i$, let $T'_i$ and $T''_i$ be boundaries of disjoint tubular neighborhoods of $K^+_i$ and $m_i$ respectively. Let $\tilde{Y}^+$ be the 3-manifold obtained from $Y^+$ by removing all the tubular neighborhoods and gluing $T'_i$ to $T''_i$ for all $i$. The gluing is done in such a way that a longitude of $K^+_i$ on $T'_i$ is glued to a meridian of $m_i$ on $T''_i$, and vice versa. In $\tilde{Y}^+$, let $\tilde{T}_i$ be the torus obtained by the identification of $T'_i$ and $T''_i$. We arrange the arc $v_i$, for each $i$, so that each hits both $T'_i$ and $T''_i$ in one point, so that $\tilde{Y}^+$ contains closed curves $\tilde{v}_i$ transverse to each $\tilde{T}_i$. By excision argument from the proof of [15, Proposition 5.7], we have an isomorphism

$$I^v(Y^+, K^+ \cup m) \cong I^\tilde{v}(\tilde{Y}^+)\tilde{\phi}.$$  

(Here $\tilde{\phi}$, of course, is the sum of the $\tilde{v}_i$. Note also that there is no link or web $K$ in $\tilde{Y}^+$.) Applying the double-cover argument again, we see that we will be done if we can show that $I^\tilde{v}(\tilde{Y}^+)$ is non-zero.

At this point, we appeal to the universal coefficient theorem, which tells us that a finite complex over $\mathbb{Z}$ has non-trivial homology over $\mathbb{F}$ if it has non-zero homology over $\mathbb{Q}$. So we will aim to prove that $I^\tilde{v}(\tilde{Y}^+; \mathbb{Q})$ is non-zero. As in Section 5 of [15], this last group is essentially the same as the sutured Floer homology of an associated sutured manifold. To state it precisely, let $M$ be complement in $Y^+$ of a tubular neighborhood of $K^+$. Let $(M, \gamma)$ be the sutured manifold obtained by putting two meridional sutures on each boundary component. Then we have

$$\dim I^\tilde{v}(\tilde{Y}^+; \mathbb{Q}) = 2 \dim SHI(M, \gamma),$$
where \( SHI \) is the sutured instanton Floer homology from [13]. Our proof will be complete if we can show that \( SHI(M, \gamma) \) is non-zero. According to [13, Proposition 7.12], the sutured instanton homology is non-zero provided that \((M, \gamma)\) is \textit{taut} in the sense of [6]. (The proof of that Proposition rests on the existence of sutured manifold hierarchies for taut sutured manifolds, proved in [6].) So the following variant will suffice:

\textbf{Variant 7.9.} Let the web \( K \subset S^3 \) be non-split and without an embedded bridge, let \((Y^+, K^+)\) be obtained by excising all the vertices (so \( Y^+ \) is a connected sum of \( n \) copies of \( S^1 \times S^2 \)), and let \((M, \gamma)\) be the sutured link complement. Then \((M, \gamma)\) is taut.

\section*{7.4 Proof of tautness}

We will now show that \((M, \gamma)\) is taut. From the definition, this means establishing:

(a) every 2-sphere in \( M \) bounds a ball; and

(b) no meridional curve on \( \partial M \) bounds a disk.

(The meridional curves are the curves parallel to the sutures.)

By its construction, \( M \) contains \( n \) pairs of pants \( P_1, \ldots, P_n \), obtained from the standard 2-spheres in \( \#(S^1 \times S^2) \) by removing the neighborhood of \( K^+ \). Write \( P \) for their union. If we cut \( M \) along \( P \), we obtain a 3-manifold diffeomorphic to the complement of a regular neighborhood of \( K \subset S^3 \). The hypothesis that \( K \) is non-split can therefore be restated as the condition that every 2-sphere in \( M \setminus P \) bounds a ball in \( M \setminus P \). Similarly, the hypothesis that \( K \) has no embedded bridge can be stated as the condition that no meridional curve on \( (\partial M \setminus P) \) bounds a disk in \( M \setminus P \).

So, for a proof by contradiction, it will suffice to establish the following:

\textbf{Lemma 7.10.} If there is a sphere \( S \subset M \) contradicting (a), or a disk \( D \subset M \) contradicting (b), then there is another sphere \( S' \) or disk \( D' \) (not necessarily “respectively”) contradicting (a) or (b) with the additional property that it is disjoint from \( P \).

\textit{Proof.} Suppose first that \( D \) is a disk contradicting (b). We may assume that the boundary of \( D \) is disjoint from \( P \) and that \( D \) meets \( P \) transversely in a collection of disjoint simple closed curves. Assume there is no disk \( D' \) with meridional boundary disjoint from \( P \). Then the number of curves in \( D \cap P \) is always non-zero, and we can choose \( D \) to minimize this number.
Any curve on $P$ either bounds a disk or is parallel to a boundary component of $P$. Given any collection of curves $C$ on $P$, there is always at least one curve among them that either bounds a disk $\delta$ in $P$ disjoint from the other components of $C$, or cobounds an annulus $\alpha$ which is disjoint from the other curves in $C$. Applying this statement to $C = D \cap P$, we let $\gamma$ be a component of $D \cap P$ that either bounds a disk $\delta$, or cobounds an annulus $\alpha$, disjoint from the other curves of intersection. Suppose $\gamma$ bounds a disk $\delta$. The curve $\gamma$ also lies on the embedded disk $D$. Let $A \subset D$ be the annulus bounded by $\partial D$ and $\gamma$. The interior of $A$ may intersect $P$, but will do so in a collection of curves that is smaller than $D \cap P$. From the union of $A$ and a parallel push-off of the disk $\delta$ (pushed off to be disjoint from $P$, on the same side locally as $A$), we form a new disk $D^*$ with the same boundary as $D$ but with fewer curves of intersection with $P$. This contradicts the choice of $D$. In the case that $\gamma$ cobounds an annulus $\alpha \subset P$, and the interior of $\alpha$ is disjoint from $D$, we define $A$ as before and consider the disk formed as the union of $\alpha$ and $D \setminus A$. Pushing $\alpha$ off $P$ as before, we obtain a disk $D^*$ with meridional boundary having fewer curves of intersection with $P$ than $D$ did: the same contradiction.

Now suppose there is a sphere $S \subset M$ contradicting (a). Suppose that every sphere $S'$ in $M \setminus P$ bounds a ball in $M \setminus P$. Then $S$ must intersect $P$. Arrange that $S \cap P$ is transverse and let $C$ be this collection of curves. If a component $\gamma \subset C$ is isotopic in $P$ to a boundary curve, then let $\alpha \subset P$ be the annulus bounded by $\gamma$ and the boundary of $P$. The curve $\gamma$ cuts $S$ into two disks. The union of one of these disks with $\alpha$ is an immersed disk in $M$ with meridional boundary. By Dehn’s lemma, there is also an embedded disk $D$ in $M$, with the same boundary. This takes us back to case (b), which we have already dealt with.

There remains the case that every component of $C = S \cap P$ bounds a disk in $P$. Let us choose $S$ so that the number of components of $C$ is as small as possible. Among the components of $C$, there is a curve $\gamma$ that bounds a disk $\delta \subset P$ which is disjoint from the other curves in $C$. If we surger $S$ along $\delta$, we obtain two spheres $S_1$ and $S_2$. Let $C_1$ and $C_2$ be their curves of intersecton with $P$. We have $C_1 \cup C_2 = C \setminus \gamma$. So, by our minimality hypothesis on $S$, it follows that both $S_1$ and $S_2$ bound balls, $B_1$ and $B_2$ in $M$. The union of the balls, together with a two-sided collar of $\delta$, is an immersed ball $B$ in $M$ with boundary $S$. (The ball is only immersed because $B_1$ and $B_2$ may not be disjoint.) An embedded sphere which bounds an immersed ball also bounds an embedded ball. So $S$ bounds a ball in $M$, contrary to hypothesis. □

The proof of this lemma completes the proof of Theorem 1.1. It is clear from the proof that a more general result holds, which we state here:
Theorem 7.11. Let $Y$ be a closed, oriented 3-manifold and $K \subset Y$ an embedded web. Suppose that the complement $K \setminus Y$ is irreducible (every 2-sphere bounds a ball), and that $K$ has no embedded bridge (no meridian of $K$ bounds a disk in the complement). Then $J^{\#}(Y,K)$ is non-zero.

Proof. From $K \subset Y$, we form $(Y^+,K^+)$ as before by excising all vertices. So $Y^+$ is the connected sum of $Y$ with $n$ copies of $S^1 \times S^2$ and $K^+$ is an $r$-component link in $Y^+$. Let $(M,\gamma)$ be the sutured link complement, with 2 meridional sutures on each component of the $r$ components of the boundary. The same sequence of variants applies, and we left to show again that $(M,\gamma)$ is taut. The proof of Lemma 7.10 still applies, without change. □

8 Tait-colorings, $O(2)$ representations and other topics

8.1 $O(2)$ connections and Tait colorings of spatial and planar graphs

Let $(Y,K)$ be a connected 3-manifold containing a web $K$ with at least one vertex. As discussed in Section 2.9, an $SO(3)$ bifold connection $(E,A)$ has automorphism group $\Gamma(E,A)$ which is either $V_4$, $\mathbb{Z}/2$ or trivial, according as the corresponding connection is a $V_4$-connection, a fully $O(2)$-connection, or a fully irreducible connection respectively. We write $\mathcal{R}(Y,K)$ for the space of flat bifold connections.

The following lemma is a straightforward observation. (See also Lemma 2.12.)

Lemma 8.1. If $K$ is a planar graph and $\mathcal{R}(K)$ contains an $O(2)$ connection, then $\mathcal{R}(K)$ also admits a $V_4$ connection, and $K$ therefore admits a Tait coloring.

Proof. The are two different conjugacy classes of elements of order 2 in $O(2)$. There is the central element, $i$, and the conjugacy class of “reflections” that comprise the non-identity component of $O(2)$. If $K \subset \mathbb{R}^3$ is a (not necessarily planar) web and $\rho$ a fully $O(2)$ connection in $\mathcal{R}(K)$, then each edge of $K$ is labeled by an element that is either a reflection or central. Furthermore, two reflection edges meet at each trivalent vertex. The reflection edges therefore form a 2-factor for $K$ (by definition, a set of edges with two edges incident at each vertex). The edges in the 2-factor form a collection of closed curves.

If the web is planar, then we can give a standard meridian generator for each edge, by connecting a standard loop to a point above the plane (as one does for the Wirtinger presentation of a knot group). For each edge $e$, we therefore have a well-defined element $\rho_e$ in $O(2)$. Along the loops formed by the edges of the 2-factor, the elements $\rho_{e_m}$, $\rho_{e_{m+1}}$ corresponding to consecutive edges differ by
multiplication by \( i \). There are therefore an even number of edges in each loop in the 2-factor. There is therefore a Tait coloring of \( K \), in which the edges in the 2-factor are labeled alternately \( j \) and \( k \), and the remaining edges are labeled \( i \). \( \square \)

From these lemmas, we obtain an elementary reformulation of the four-color theorem as the statement that, for any bridgeless planar web \( K \), the representation variety \( \mathcal{R}(K) \) contains an \( O(2) \) connection.

For non-planar webs in \( \mathbb{R}^3 \), the existence of an \( O(2) \) connection does not imply the existence of a Tait coloring. Nor is it true that, for every web \( K \) in \( \mathbb{R}^3 \) without a spatial bridge, the representation variety \( \mathcal{R}(K) \) contains an \( O(2) \) representation. (An example with no \( O(2) \) representation can be constructed by starting with a non-split 2-component \( L \) whose two components bound disjoint Seifert surfaces, and taking \( K \) to be the union of \( L \) with an extra edge joining the two components.)

### 8.2 Foams with non-zero evaluation, and \( O(2) \) connections

The remarks about \( O(2) \) representations for webs in the previous subsection are motivated in part by the following positive result about \( O(2) \) bifold connections for foams in 4-manifolds.

**Proposition 8.2.** Let \( \Sigma \) be a closed foam in a closed 4-manifold \( X \) (possibly with dots). If \( J^b(X, \Sigma) \) is non-zero, then for every Riemannian metric on the corresponding bifold \( \tilde{X} \), there exists an anti-self-dual \( O(2) \) bifold connection.

Furthermore, if \( X \) admits an orientation-reversing diffeomorphism that fixes \( \Sigma \) pointwise, then there exists a flat \( O(2) \) bifold connection.

**Proof.** First of all, we can immediately reduce to the case of dotless foams by applying Proposition 6.4.

Let \( Z \) be the product \( S^1 \times S^3 \) containing the foam \( S^1 \times H \), where \( H \) is the Hopf link. Let \( \tilde{Z} \) be the corresponding bifold. The non-vanishing of \( J^b(X, \Sigma) \) implies the existence, for all metrics, of an anti-self-dual \( SO(3) \) bifold connection on the connected sum \( \tilde{X} \# \tilde{Z} \). (The connected sum \( X \# Z \) is the manifold we obtain when we regard \( X \) as a cobordism from \( S^3 \) to \( S^3 \) by removing two balls, and then glue the two boundary components together.) Furthermore, these anti-self-dual connections lie in 0-dimensional moduli spaces.

By the usual connected sum argument, either \( \tilde{X} \) or \( \tilde{Z} \) carries a solution lying in a negative-dimensional moduli space. The \( \kappa = 0 \) moduli space on \( \tilde{Z} \) has formal dimension 0, so in fact \( \tilde{X} \) must carry an anti-self-dual connection in a
negative-dimensional moduli spaces. Furthermore, this must remain true for any
Riemannian metric on \( \tilde{X} \) and any small 4-dimensional holonomy perturbation on
\( \tilde{X} \). The following lemma therefore completes the proof of the first assertion in
the proposition.

**Lemma 8.3.** *If a moduli space on \( \tilde{X} \) has negative formal dimension and consists
only of fully irreducible connections, then a small holonomy perturbation can be
chosen so as to make the moduli space empty.*

*Proof.* We adapt the argument in [4] to our case of \( SO(3) \) connections. The map
\( \psi : SO(3) \to so(3) \) given by \( A \mapsto A - A^T \) is equivariant for the adjoint action of
\( SO(3) \), and has the property that if \( H \subset SO(3) \) is any subgroup that is not con-
tained in a conjugate of \( O(2) \) then \( \psi(H) \) spans \( so(3) \). Using \( \psi \) to construct pertur-
bations as in [4, Section 2(b)], one shows the existence of holonomy perturbation
such that the moduli space is regular at all fully irreducible solutions. \( \square \)

To prove the last statement in Proposition 8.2, we note that any \( O(2) \) connec-
tion determines a flat real line bundle on \( \xi \ X \setminus \Sigma \) that has non-trivial holonomy
on the links of some 2-dimensional surface \( S \subset \Sigma \subset X \) comprised of a some sub-
set of the facets. The curvature of an anti-self-dual \( O(2) \) bifield connection can
be interpreted as an anti-self-dual harmonic 2-form, \( f \in H^-(\xi) \), with values in
\( \xi \) and lying in certain lattice in \( H^2(\xi) \). An adaptation of the usual argument for
reducible connections in instanton moduli spaces shows that, for generic met-
ric, any such orbifold 2-form is zero unless \( \dim H^+(\xi) = 0 \) and \( \dim H^-((\xi) \neq 0 \).
However, if \( X \) has an orientation-reversing diffeomorphism fixing \( \Sigma \), then these
two vector spaces have the same dimension. \( \square \)

**Corollary 8.4.** *Let \( \Sigma \) be a closed foam in \( \mathbb{R}^3 \subset \mathbb{R}^4 \). If \( \mathcal{J}_\bigwedge(\Sigma) \) is non-zero, then the
foam \( \Sigma \) admits a “Tait coloring”: a coloring of the facets of \( \Sigma \) by three colors so that
all three colors appear on the facets incident to each seam.*

*Proof.* By the previous proposition, there is a flat \( O(2) \) bifield connection on
(\( \mathbb{R}^4, \Sigma \)). We shall imitate the argument from the 3-dimensional case, Lemma 8.1,
to see that the existence of a flat \( O(2) \) connection implies the existence of flat \( V_4 \)
connection when \( \Sigma \) is contained in \( \mathbb{R}^3 \).

Whether or not \( \Sigma \) is contained in \( \mathbb{R}^3 \), given an \( O(2) \) bifield connection and
corresponding homomorphism \( \rho \) from \( \pi_1(X \setminus \Sigma) \), a representative of the conju-
gacy class corresponding to the link of each facet of \( \Sigma \) in \( X \) is mapped by \( \rho \) either
to the central element \( i \) in \( O(2) \) or to a reflection. The reflection facets constitute
what we can call a “2-factor” for the foam: a subset of the facets such that, along
each seam, two of the three facets which meet locally there belong to this subset. The facets belonging to the 2-factor form a collection of closed surfaces in $S \subset X$.

When $\Sigma$ is contained in $\mathbb{R}^3$, then there is a standard representative in $\pi_1(X \setminus \Sigma)$ for the link of each facet, so $\rho$ gives a map from facets to elements of order 2 in $O(2)$, much as in the case of a planar web. So for each facet $f$, we have an element $\rho_f$. For facets in the 2-factor, $\rho_f$ is a reflection, and if two such facets $f_1$ and $f_2$ are separated by a seam then $\rho_{f_1} = i\rho_{f_2}$. On each component of the closed surface $S$, the facets are therefore labeled by only 2 distinct reflections, in a checkerboard fashion. We can therefore give a Tait coloring of the facets by coloring the rotation facets with $i$ and the reflection facets alternately by $j$ and $k$. □

Corollary 8.5. If $\Sigma$ is a closed foam in $\mathbb{R}^3$ with $J^4(\Sigma) \neq 0$, then the seams of $\Sigma$ form a bipartite 4-valent graph.

Proof. The proof of the previous corollary shows that the seam $\Gamma$ is a 4-valent graph lying on a closed surface $S \subset \mathbb{R}^3$, and that the facets into which $S$ is divided admit a coloring with two colors called $j$ and $k$. For each connected component $S_m$ of $S$, the corresponding subgraph of $\Gamma$ is formed as the transverse intersection of a collection of circles $\gamma_{m,l}$ arising as boundary components of the facets labeled by the central element $i \in O(2)$ (i.e. the facets not contained in $S$). The surface $S_m$ separates $\mathbb{R}^3$ into two components, and we can therefore partition the circles $\gamma_{m,l}$ into two types, according to the component in which the corresponding facet of $\Sigma \setminus S$ lies. Circles $\gamma$ and $\gamma'$ belonging to the same type do not intersect. We can now distinguish two different types of intersection points, as follows. After choosing an orientation of $S_m$, we can choose an oriented chart at each intersection point in $S_m$ which maps the circle $\gamma$ of the first type to the $x$ axis and the circle of the second type to the $y$ axis. The first quadrant is then colored by either $j$ or $k$. In this way we partition the vertices of $\Gamma$ into two sets, in a way that exhibits $\Gamma$ as bipartite. □

Corollary 8.6. Let $\Sigma \subset X$ be a closed foam in a closed 4-manifold $X$ with $J^4(X,\Sigma) \neq 0$. Let $K \subset Y$ be the transverse intersection of $\Sigma$ with a closed 3-manifold $Y \subset X$. Then $\mathcal{R}(Y,K)$ contains a flat $O(2)$ connection.

Furthermore, if $K$ is planar, meaning that it is contained in a standard 2-disk in $Y$. Then $K$ admits a Tait coloring.

Proof. We know from Proposition 8.2 that, for every metric on the bifold $\tilde{X}$, there exists an anti-self-dual $O(2)$ connection on $\tilde{X}$. Furthermore, the action of this connection has an upper bound that is independent of the metric. If we choose
a sequence of metrics on $\tilde{X}$ that contain cylinders on $\tilde{Y}$ of increasing length, then standard applications of the compactness theorems will lead to a flat $O(2)$ connection on $\tilde{Y}$. If $K$ is planar, then the $O(2)$ connection leads to a Tait coloring as before.

Because of the previous corollaries, an affirmative answer to the following question would provide a proof of the four-color theorem.

**Question 8.7.** If $K \subset \mathbb{R}^3$ is bridgeless and planar, does it arise as $\mathbb{R}^3 \cap \Sigma$ for some foam $\Sigma \subset \mathbb{R}^4$ with $J^\delta(\Sigma) \not= 0$?

In the situation considered by the question above, we can regard the evaluation of $J^\delta(\Sigma)$ as a pairing of an element of $J^\delta(K)$ with an element in the dual space $J^\delta(-K)$, where $-K$ is the mirror image. Because we know that $J^\delta(K)$ is non-trivial, an affirmative answer to the next question would also suffice.

**Question 8.8.** Let $K \subset S^3$ be a planar web. Is it always true that $J^\delta(K)$ is generated by the elements $J^\delta(\Sigma)$ for foams $\Sigma \subset (\mathbb{R}^4)^+$ with boundary $K$?

**Remarks.** If $K \subset \mathbb{R}^3$ is spatial web that is not planar, it may be that there are no $O(2)$ representations in $\mathcal{R}(K)$, as noted previously. For such a $K$, there cannot be foam $\Sigma$ with $J^\delta(\Sigma) \not= 0$ such $\Sigma \cap \mathbb{R}^3 = K$. So one cannot expect an affirmative answer to either of the two questions above if one drops the planar hypothesis for $K$.

Whether it is reasonable to conjecture an affirmative answer to this question for planar graphs is not at all clear. For example, in the case of the dodecahedral graph $K$, an affirmative answer would probably require that the elements $J^\delta(\Sigma)$ corresponding to foams $\Sigma$ with boundary $K$ span a space of dimension at least 60 (since this is the number of Tait colorings). Calculations by the authors failed to show that the span had dimension any larger than 58.

### 8.3 A combinatorial counterpart

Motivated by the partial information that we have about the evaluation of closed foams, we can try to define a purely combinatorial counterpart to $J^\delta$ for planar webs $K$, by closely imitating the construction of Khovanov’s $\mathfrak{sl}_3$ homology from [7]. We shall describe this combinatorial version here, though we do run into a question of well-definedness.

Let us consider closed *pre-foams*, by which we mean abstract 2-dimensional cell complexes, with finitely many cells, with a local structure modeled on a foam,
in the sense defined in this paper. Unlike the pre-foams in [7], no orientations are required, and we allow pre-foams to have tetrahedral points. We allow our pre-foams to have decoration by dots.

We can try and define an evaluation, \( J^\flat(S) \in \mathbb{F} \), for such prefoams \( S \) by applying the following rules.

(a) The seams of the pre-foam form a 4-valent graph. If this graph is not bipartite, then we define \( J^\flat(S) = 0 \).

(b) If the seams form a bipartite graph, we can pair up the tetrahedral points and cancel them in pairs (Figure 11) to obtain a new pre-foam \( S' \) with no tetrahedral points. We declare that \( J^\flat(S) = J^\flat(S') \), and so reduce to the case that there are no tetrahedral points.

(c) When there are no tetrahedral points, the seams form a union of circles where three facets meet locally. If the monodromy of the three sheets is a non-trivial permutation around any of these circles, then we define \( J^\flat(S) = 0 \). In this way we reduce to the case that the model for the neighborhood of each component of the seams is a product \( S^1 \times Y \), where \( Y \) is a standard \( Y \)-shape graph.

(d) For each component of the seam having a neighborhood of the form \( S^1 \times Y \), we apply (abstract) neck-cutting (i.e. surgery) on the three circles parallel to the seam in each of the three neighboring facets, as in [7]. In this way we reduce to the case that \( S \) is a disjoint union of theta-foams and closed surfaces.

(e) The theta-foams are evaluated using the rule for \( J^\# \) (Proposition 5.6) to leave only closed surfaces. The evaluation \( J^\flat(S) \) for a sphere with two dots is 1, for a torus with no dots is 1, and for everything else is zero (including non-orientable components).

The issue of well-definedness arises here because of the choice of how to pair up the tetrahedral points, but the following two conjectures both seem plausible.

**Conjecture 8.9.** The above rules lead to a unique, well-defined evaluation \( J^\flat(S) \in \mathbb{F} \) for any pre-foam \( S \).

**Conjecture 8.10.** For a closed foam \( \Sigma \) in \( \mathbb{R}^3 \) with underlying pre-foam \( S \), we have \( J^\flat(\Sigma) = J^\flat(S) \).
The difficulty with establishing the second conjecture arises from the fact that the abstract surgeries required to do neck-cutting on a pre-foam $S$ may not always be achieved by embedded surgeries when the pre-foam is embedded as a foam in $\mathbb{R}^3$.

Assuming the first of the two conjectures, one can associate an $\mathbb{F}$ vector space $J^b(K)$ to a planar web $K$ by taking as generators the set of all foams $\Sigma \subset (\mathbb{R}^3)^-$ with boundary $K$ and relations

$$\sum [\Sigma_i] = 0$$

whenever the collection of foams $\Sigma_i$ satisfies

$$\sum J^b(\Sigma_i \cup T) = 0$$

for all foams $T \subset (\mathbb{R}^3)^+$ with $\partial T = -K$. This definition imitates [7]; and by extending the arguments of that paper one can show that $J^b(K)$ satisfies the bigon, triangle and square relations. In particular (still contingent on Conjecture 8.9), $J^b(K)$ is isomorphic to $J^\flat(K)$ at least for simple planar graphs in the sense of section 6.5.

If both of the above conjectures are true, then $J^b(K)$ is a subquotient of $J^\flat(K)$: it is the subspace of $J^\flat(K)$ generated by the foams with boundary $K$, divided by the annihilator of the dual space generated by the foams with boundary $-K$.

Remark. Assuming Conjecture 8.9, the authors examined $J^\flat(K)$ in the case of the dodecahedral graph [10], and showed that its dimension in this case is at least 58. See also the remarks at the end of the previous subsection.

### 8.4 Gradings

Although it plays no real role in the earlier parts of this paper, we consider now whether there is a natural $\mathbb{Z}/d$ grading on the homology group $J^\flat(K)$, or more generally on $J(Y, K; \mu)$, for a web $K \subset Y$ with strong marking data $\mu$. To formulate the question precisely, we write $\tilde{Y}$ as usual for the corresponding bifold, and $B_l(\tilde{Y}; \mu)$ for the space of marked bifold connections. For any closed loop $\zeta$ in $B_l(\tilde{Y}; \mu)$ there is a corresponding spectral flow $sf_\zeta$; and if $sf_\zeta \in d\mathbb{Z}$ for all loops $\zeta$, then we can regard $J(Y, K; \mu)$ as $\mathbb{Z}/d$ graded, in the usual way. With this understood, we have the following result.

**Proposition 8.11.** If $K \subset Y$ has no vertices, or if the abstract graph underlying $K$ is bipartite, then $J(Y, K; \mu)$ may be given a $\mathbb{Z}/2$ grading. If the graph is not bipartite,
then in general there may exist closed loops with spectral flow 1, and there is no non-trivial \( \mathbb{Z}/d \) grading for any \( d \).

Proof. The spectral flow \( \text{sf}_\zeta \) is equal to the index \( d(E,A) \) for the corresponding 4-dimensional bifold connection on \( S^1 \times \tilde{Y} \). By the formula 2.6, this in turn is equal to \( 8\kappa \), where \( \kappa \) is the action of \( (E,A) \).

If \( K \) has no vertices, then the symbol of the linearized equations with gauge fixing on the bifold \( S^1 \times Y \) is homotopic to the symbol of a complex-linear operator, for we can choose an almost-complex structure on \( S^1 \times Y \) such that the submanifold \( S^1 \times K \) is almost complex: the appropriate complex-linear operator is then the operator \( \bar{\partial} + \text{bar} \bar{\partial}^* \) coupled to the complexification of \( E \) on the complex orbifold. It follows that the real index \( d(E,A) \) is even in this case.

In the general case, let \( e \) be an edge of \( K \), let \( m_e \) be a meridional circle linking \( e \), and consider the torus \( S^1 \times m_e \) in \( S^1 \times (Y \setminus K) \). We can evaluate \( w_2(E) \) on \( T_e \), and we call the result \( \eta(e) \):

\[
\eta(e) = \langle w_2(E), [S^1 \times m_e] \rangle \\
\in \{0,1\}.
\]

If edges \( e_1, e_2, e_3 \) meet at a vertex \( v \), then the sum of the meridians \( [m_{e_1}] + [m_{e_2}] + [m_{e_3}] \) is zero in homology, so

\[
\eta(e_1) + \eta(e_2) + \eta(e_3) = 0.
\]

It follows that, at each vertex \( v \), the number edges which are locally incident at \( v \) and have \( \eta(e) = 1 \) is either 0 or 2. Let us say that a vertex \( v \) of \( K \) has Type I if all incident edges have \( \eta(e) = 0 \), and Type II if two of the incident edges have \( \eta(e) = 1 \).

Lemma 8.12. If \( N \) is the number of vertices of \( K \) with Type II, then the action \( \kappa \) is equal to \( N/8 \) modulo \((1/4)\mathbb{Z}\).

Proof of the lemma. Let \( v \) be any vertex of \( K \), let \( B_v \) be a ball neighborhood of \( v \), and consider the restriction of the bifold connection subset \( S^1 \times B_v \) in \( S^1 \times \tilde{Y} \). Using radial parallel dilation in the \( B_v \) directions, we can alter \( (E,A) \) so that the bifold connection is flat on \( S^1 \times B_v \). The holonomy group of \( (E,A) \) on the slice \( B_v \) is the Klein 4-group \( V_4 = 1, a, b, c \). On \( S^1 \times B_v \), the flat connection is determined by the holonomy around four loops: first the three loops which are the meridians of the three edges in \( B_v \), and fourth the loops \( S^1 \times p \) for some point \( p \) in \( B_v \setminus K \). After re-ordering the edges, the first three holonomies are \( a, b \) and \( c \), while the
holonomy around $S^1 \times p$ is either 1, $a$, $b$ or $c$. If the holonomy around $S^1 \times p$ is 1, then $v$ has Type I, otherwise it has Type II.

We see from this description that if $v_1$ and $v_2$ are vertices of the same type (either both Type I or both Type II), then, after modifying the bifold connection a little, we can find an identification of the neighborhoods

$$\tau: S^1 \times B_{v_1} \to S^2 \times B_{v_2}$$

such that the bifold connections $(E,A)$ and $\tau^*(E,A)$ are isomorphic on $S^1 \times B_{v_1}$. We can then surger $(Y,K)$ by removing the two balls and identifying the boundaries using $\tau$, and we obtain a new bifold connection $(E',A')$ for the new pair $(Y',K')$. The new bifold connection has the same action as the original. If the number of Type II vertices is even, then the number of Type I vertices is even also (because any web has an even number of vertices in all); so we can then pair up vertices of the same type, and apply the above surgery repeatedly to arrive at the case where there are no vertices. When there are no vertices, we have already seen that the index $d$ is even, which means that $\kappa$ belongs to $(1/4)\mathbb{Z}$.

To complete the proof of the lemma, it is sufficient to exhibit a single example of a bifold connection $(E,A)$ on some $S^1 \times \tilde{Y}$ whose action is $1/8$. So consider the standard 1-instanton, as an anti-self-dual $SO(3)$-connection on $\mathbb{R} \times S^3$, invariant the action of $SO(4)$ on $S^3$. Let $\Gamma \subset SO(4)$ be the group of order 8 generated by the reflections in the coordinate 2-planes of $\mathbb{R}^4$. The standard 1-instanton descends to an anti-self-dual bifold connection $(E,A)$ on $\mathbb{R} \times (S^3/\Gamma)$. The bifold $\tilde{Y} = S^3/\Gamma$ corresponds to the pair $(S^3,K)$, where the web $K$ is the 1-skeleton of a tetrahedron. If we modify the connection to be flat near $\pm \infty$, we can glue the two ends to obtain a bifold connection on $S^1 \times \tilde{Y}$ with action $1/8$. \qed

We return to the proof of the proposition. Since two edges are incident at each vertex of Type II in $K$, the edges $e$ with $\eta(e) = 1$ form cycles. If the graph is bipartite, then every cycle contains an even number of vertices, and it follows that there are an even number of vertices of Type II. By the lemma, the action is therefore in $(1/4)\mathbb{Z}$ in the bipartite case, and so the index $d(E,A)$ is even. By contrast, in the non-bipartite case, the example of the tetrahedron web in the proof of the lemma above shows that the action may be $1/8$, and the spectral flow $sf_\zeta$ around a closed loop $\zeta$ may therefore be 1. \qed

### 8.5 The relation between $I^\#$ and $J^\#$

For a web $K$ in $\mathbb{R}^3$, we have already considered the relation between $I^\#(K)$ and $J^\#(K)$ in a very simple form in Corollary 7.3. The essential point in the proof of
that corollary was the mod-2 Gysin sequence relating the homology of a double-cover to the homology of the base, as a mapping-cone. In the proof of Corollary 7.3, passing from $J^\#(K)$ to $I^\#(K)$ involves taking iterated double-covers of the spaces of marked bifold connections, so the complex that computes $I^\#(K)$ can be constructed as an iterated mapping cone. In this section we will describe the complex a little more fully, as a “cube”.

Consider first the situation of a finite simplicial complex $S$ with a covering $f : \tilde{S} \to S$ which is the quotient map for an action of the group $G = (\mathbb{Z}/2)^n$. We use multiplicative notation for the group. We write $\mathbb{F}[G]$ for the group ring of $G$ over the field $\mathbb{F}$ of 2 elements. With coefficients $\mathbb{F}$, we can describe the homology of $\tilde{S}$ almost tautologically as the homology of the base $S$ with coefficients in a local system $\square$ whose stalk is the vector space $\mathbb{F}[G]$ at each point. Following [18] we can construct a spectral sequence abutting to this homology group as follows.

The ring $\mathbb{F}[G]$ contains an ideal $I$, the augmentation ideal

$$I = \left\{ \sum_{g \in G} \lambda_g g \mid \sum_{g \in G} \lambda_g = 0 \right\}.$$ 

The powers of the augmentation ideal define a filtration $\{I^m\}$ of the group ring. We write $\text{Gr}$ for the associated graded:

$$\text{Gr} = \bigoplus_m \text{Gr}_m = \bigoplus_m I^m/I^{m+1}.$$ 

The dimension of the vector space $\text{Gr}_m$ is $\binom{n}{m}$.

For any $g \in G$, the element $1 + g$ belongs to $I$, so multiplication by $1 + g$ lowers the filtration level. So multiplication by $g$ induces the identity operator on the associated graded $\text{Gr}$. The filtration of $I$ of $\mathbb{F}[G]$ gives rise to a filtration of the local system $\Gamma$ on $S$, so there is also an associated graded object $\text{Gr}_*(\Gamma)$. The local system $\text{Gr}_*(\Gamma)$ on $S$ is trivial, because the monodromy elements $g \in G$ act trivially on $\text{Gr}$.

Consider now the cohomology of $S$ with coefficients in $\Gamma$. (We choose cohomology instead of homology for a while, for the sake of the exposition below.) The cochain complex $C^*(S; \Gamma)$ has a filtration by subcomplexes $C^*(S; I^m)$ and there is an associated spectral sequence. The $E_1$ page of the spectral sequence consists of the homology groups

$$H^*(S; \text{Gr}_*(\Gamma));$$
and since the local system $\text{Gr}_*(\Gamma)$ is trivial, this is simply

$$H^*(S; \mathbb{F}) \otimes \text{Gr}_* \cong H^*(S; \mathbb{F}) \otimes \mathbb{F}^{2n}.$$  

The individual summands here are

$$H_*(S; \mathbb{F}) \otimes \text{Gr}_m \cong H_*(S; \mathbb{F}) \otimes \mathbb{F}^{(m)}.$$  

We therefore have:

**Proposition 8.13 ([18]).** There is a spectral sequence whose $E_1$ page is $H^*(S; \mathbb{F}) \otimes \text{Gr}$ and which abuts to $H^*(\hat{S}; \mathbb{F})$.

After selecting a generating set $x_1, \ldots, x_n$ for $G$, we can unravel the picture a little further. For any subset $A \subset \{1, \ldots, n\}$, let $G_A \leq G$ be the subgroup generated by the elements $x_i$ with $i \in A$, and let

$$\xi_A = \sum_{g \in G_A} g \in \mathbb{F}[G].$$

As $A$ runs through all subsets, the elements $\{\xi_A\}_A$ form a basis for $\mathbb{F}[G]$. In this basis, the multiplication by $x_i$ has the form

$$x_i \xi_A = \begin{cases} 
\xi_A, & i \in A, \\
\xi_A + \xi_{A \cup \{i\}}, & i \notin A.
\end{cases}$$

For a general element of $G$ of the form $x_I = x_{i_1} x_{i_2} \ldots x_{i_k}$ (the product of $k$ distinct generators), we similarly have

$$x_I \xi_A = \sum_{A \subset A' \subset A \cup I} \xi_{A'}.$$  \hspace{1cm} (19)

In terms of the basis $\xi_A$, we can describe the augmentation ideal and its powers by

$$\mathcal{F}^m = \text{span}\{ \xi_A \mid |A| \geq m \}.$$  

The formula (19) shows that, when $|A| = m$,

$$x_I \xi_A = \xi_A \pmod{\mathcal{F}^{m+1}}.$$  \hspace{1cm} (20)

Consider again the local system $\Gamma$ on $S$, and regard $\Gamma$ as having trivialized stalks: so at each vertex of $S$ the stalk is $\mathbb{F}[G]$, and to each 1-simplex $\sigma$ is assigned multiplication by $x_I$ for some $I = I(\sigma)$. From the filtration $\{I^m\}$ of $\mathbb{F}[G]$, ...
we obtain a filtration of the cochain complex $C^*(S;\Gamma)$, as discussed earlier. If we write $d_\Gamma$ for the coboundary operator, we can compare $d_\Gamma$ to the ordinary coboundary operator $d$ with trivial constant coefficients $\mathbb{F}[G]$. We use the basis $\xi_A$ to write

$$C^*(S;\Gamma) = \bigoplus_A C^*(S;\mathbb{F})$$

so that

$$d_\Gamma = \sum_{B \supseteq A} d_{AB},$$

where $d_{AB}$ is the component from the summand indexed by $A$ to the summand indexed by $B$. For each 1-simplex $e$ and each $i \in \{1, \ldots, n\}$ let

$$u_i(e) = \begin{cases} 1, & i \in I(e) \\ 0, & i \notin I(e). \end{cases}$$

These 1-cochains $u_i$ on $S$ are cocycles representing the cohomology classes in $H^1(S;\mathbb{F})$ which are the dual basis to the elements $x_i \in G \cong H_1(S;\mathbb{F})$. For $I = \{i_1, \ldots, i_k\}$, define a cochain $U_I$ by

$$U_I(e) = u_{i_1}(e)u_{i_2}(e) \cdots u_{i_k}(e).$$

(This could equivalently be written as

$$U_I = u_{i_1} \sim_1 u_{i_2} \sim_1 \cdots \sim_1 u_{i_k}$$

where $\sim_1$ is the reduced product of [20].) From (19), we obtain the formulae

$$d_{AA} = d,$$

and for $B \supseteq A$,

$$d_{AB}\alpha = U_{B\setminus A} \sim \alpha.$$

If we think of the subsets $A$ as indexing the vertices of a cube, then we have a complex computing $H^*(\tilde{S};\mathbb{F})$ which consists of a copy of $C^*(S;\mathbb{F})$ at each vertex. In the spectral sequence corresponding to the filtration $\mathcal{F}^m$, the $E_0$ page is a direct sum of copies of the complex $(C^*(S),d)$, one at each vertex of the cube. The $E_1$ page has a copy of $H^*(S;\mathbb{F})$ at each vertex of the cube and the differential on $E_1$ is the sum of maps corresponding to edges of the cube: each such map is multiplication by a class $[u_i]$ in $H^1(S;\mathbb{F})$. 
To describe essentially the same situation in using Morse homology in the context of $J^\sharp$ and $l^\sharp$, let $K \subset \mathbb{R}^3$ be a web, let $K^\sharp = K \cup H$ be the web in $S^3$ obtained by adding a Hopf link $H$ near infinity. Recall that the groups $J^\sharp(K)$ and $l^\sharp(K)$ are defined using two different marking data $\mu$ and $\mu'$ for bifold connections on $(S^3, K \cup H)$. If $\mathcal{B}$ and $\mathcal{B}'$ denote the corresponding spaces of bifold connections, then there is a map (see Lemma 2.4)

$$f : \mathcal{B}' \to \mathcal{B}$$

which is a covering of a connected component of $\mathcal{B}$. The covering group is the group

$$G = H^1(\mathbb{R}^3 \setminus K; \mathbb{F}),$$

and the image of $f$ consists of bifold connections $(E, A)$ such that $w_2(E)$ is zero on $\mathbb{R}^3 \setminus K$ (regarded as a subset of $S^3 \setminus K^\sharp$). Because we wish to write the group $G$ multiplicatively and $H^1(\mathbb{R}^3 \setminus K; \mathbb{F})$ additively, we write $x^h \in G$ for the element corresponding to $h \in H^1(\mathbb{R}^3 \setminus K; \mathbb{F})$. We also write $x^h$ for the corresponding linear operator on $\mathbb{F}[G]$, given by translation. The marking data $\mu$ ensures in any case that $w_2(E)$ is zero on the sphere at infinity in $\mathbb{R}^3$; and if $K$ is connected, this means that $f$ is surjective. For connected webs, we are therefore in the situation that $\mathcal{B}$ is the quotient of $\mathcal{B}'$ by the action of $G$.

At this point, for connected webs $K$, we can regard $l^\sharp(K)$ as the Floer homology of the perturbed Chern-Simons functional on $\mathcal{B}$ with coefficients in a local system with fiber $\mathbb{F}[G]$. Using the filtration of $\mathbb{F}[G]$ by the powers $I^m$ of the augmentation ideal as above, we obtain a filtration of the complex for $l^\sharp(K)$, and hence a spectral sequence:

**Proposition 8.14.** For a connected web in $\mathbb{R}^3$, there is a spectral sequence whose $E_1$ page is the vector space

$$E_1 = J^\sharp(K) \otimes \text{Gr},$$

graded by the grading of $\text{Gr}$, and which abuts to $l^\sharp(K)$.

After choosing a basis for $H^1(\mathbb{R}^3 \setminus K; \mathbb{F})$ (i.e. an $n$-element generating set for $G$), we can describe the $E_1$ page as a direct sum of $2^n$ copies of $J^\sharp(K)$, indexed by the vertices $A$ of a cube of dimension $n$.

In the Morse theory picture that underlies $l^\sharp$, we can describe the total differential on the $E_0$ page. Let $(C^\sharp(K), d)$ be the instanton Morse complex which computes $J^\sharp(K)$, whose basis is the set of critical points of the perturbed Chern-Simons functional on the corresponding space of bifold connections $\mathcal{B}^\sharp$. Fix a
basis for $H^1(\mathbb{R}^3 \setminus K; \mathbb{F})$ (or equivalently a dual basis $\{\gamma_i\}$ in $H_1(\mathbb{R}^3 \setminus K; \mathbb{F})$), so that the $E_0$ page of the spectral sequence can be written as a direct sum of $2^n$ copies of the complex $C^\#(K)$ indexed by the vertices $A$ of the cube:

$$E_0 = \sum_{A \subset \{1, \ldots, n\}} C^\#(K)_A.$$ 

For each basis element $\gamma_i$, there is a corresponding chain map

$$u_i : C^\#(K) \to C^\#(K).$$

Its matrix entries can be defined by first choosing a suitable codimension-1 submanifold $V_i \subset B^3$ representing the corresponding element of $H^1(B^3)$ and then defining the matrix entry from $\alpha$ to $\beta$ to be count of index-1 instanton trajectories from $\alpha$ to $\beta$ whose intersection with $V_i$ is odd. More generally, for each subset $I \subset \{1, \ldots, n\}$, there is a map (not a chain map in general),

$$U_I : C^\#(K) \to C^\#(K)$$

obtained by counting trajectories having odd intersection with $V_i$ for every $i \in I$. (Thus $U_I = d$ in the case that $I$ is empty.) The complex that computes $I^\#(K)$ is then $(E_0, D)$ where $D$ has components as follows. For every pair of subsets $A \subset B$, there is a component

$$D^{A,B} : C^\#(K)_B \to C^\#(K)_A$$

given $D^{A,B} = U_B\setminus A$.

8.6 Replacing $SO(3)$ by $SU(3)$

One can construct a variant of $J^\#(K)$ using the group $SU(3)$ in place of $SO(3)$. Given a web $K$ in a 3-manifold $Y$, form the disjoint union $K \cup H$, where $H$ is a Hopf link contained in a ball $B \subset Y$. Consider $(Y, K \cup H)$ as defining a bifold, just as in the $SO(3)$ case. Fix a $U(3)$ bifold bundle with a reduction to $SU(3)$ outside the ball $B$, whose determinant is non-trivial on the peripheral torus of the components of $H$. (This non-triviality of the bundle plays the same role as the strong marking data in the $SO(3)$ case.) The projectively flat connections with fixed determinant on such a bifold bundle correspond to representations

$$\rho : \pi_1(Y \setminus K, y_0) \to SU(3)$$
satisfying the constraint that, for each edge $e$ of $K$ and any representative $m_e$ for the conjugacy class of the meridian of $e$, the element $\rho(m_e)$ has order 2 in $SU(3)$. (Compare Definition 2.11.) The $SO(3)$ representation variety $R^\#(Y,K)$ sits inside the larger $SU(3)$ representation variety as the fixed-point set of complex conjugation acting on the matrix entries.

One can go on to construct the corresponding instanton homology groups, which we temporarily denote by $L^\#(K)$. Unlike the $SO(3)$ case, this can now be done with integer coefficients. For webs in $\mathbb{R}^3$, with $F$ coefficients, it turns out that the dimension $L^\#(K;F)$ is equal to the number of Tait colorings of $K$. In particular, it is independent of the spatial embedding of the web. A proof in the $F$ case can be given using the ideas of [9]. The same result holds with rational coefficients, but with a different proof. One can imagine that a Smith-theory argument [19] might establish an inequality

$$\dim L^\#(K;F) \geq \dim J^\#(K)$$

for planar webs $K$, in which case the four-color theorem would follow from Theorem 1.1. However, there is not a version of Smith theory that applies to these variants of Floer homology, and one must use the hypothesis of planarity.

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