Weyl geometry approach to describe planetary systems

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Abstract

In the present work we show that planetary mean distances can be calculated through considering the Weyl geometry. We interpret the Weyl gauge field as a vector field associated with the hypercharge of the particles and apply the gauge concept of the Weyl geometry. The results obtained are shown to agree with the observed orbits of all the planets and of the asteroid belt in the solar system, with some empty states.

1 Introduction

In recent years, some quantum mechanical approaches have been presented to calculate the planetary orbits in solar system [1, 2, 3]. Most of these theoretical approaches are based on self-similarity concept which emphasizes the presence of quantization at different scales. These methods implies the existence of a re-scaled Plank constant or fine structure constant as $\hbar^* \approx 10^{42}$ js whose physical origin is still undetermined. In this work we proceed to consider the Weyl geometry instead of Riemannian and show that Weyl geometry introduces automatically quantization conditions for the planetary orbits. Furthermore, we present a different interpretation of the Weyl gauge field, in that we associate it with the spin-one hyperphoton coupled to hypercharge, rather than the charge. Using this interpretation, we can give a physical meaning to the re-scaled fine structure constant (or Plank constant) mentioned in the other approaches. All results will be obtained with just one input parameter, namely a fundamental radius around 0.04 AU which is predicted by some authors and several planets have been recently discovered orbiting at this radius in extra solar systems. In this paper we first briefly review the structure of Weyl geometry and present a new interpretation of the Weyl gauge field. Then we calculate the spherical solution of gravitational field equation in Weyl-Dirac theory in section two. The planetary mean distances are calculated through applying the gauge concept of the Weyl geometry in the last section.

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2 Weyl Geometry

After Einstein put forth his general theory of relativity, which provided a geometrical description of gravitation, Weyl proposed a more general theory that also included a geometrical description of electromagnetism. In the case of general relativity one has a Riemannian geometry with a metric tensor $g_{\mu\nu}$. If a vector undergoes a parallel displacement in this geometry, its direction may change, but not its length. In Weyl geometry, there is a given vector $k_{\mu}$ which, together with $g_{\mu\nu}$, characterizes the geometry. For any given vector $\xi^\mu$ undergoing parallel displacement in this geometry, not only the direction but also the length $\xi$ may change and this change depends on $k^\mu$ according to the relation

$$d\xi = \xi_k dx^\mu \quad \text{or} \quad \xi = \xi_0 e^{\int k_\mu dx^\mu},$$

(1)

where $\xi_0$ is the length of original vector before displacement. The change in the length of $\xi$ in going from one point to another depends on the path followed, i.e., length is not integrable. This mathematical theory of displacement provides a great flexibility in the choice of the standard of length. Thus, one can introduce an arbitrary standard of length, or gauge, at each point

$$ds' = e^\lambda ds$$

(2)

here $\lambda$ is an arbitrary function of the coordinates. To preserve equation (1) under this gauge transformation, $k_\mu$ transforms according to

$$k_\mu \rightarrow k'_\mu = k_\mu + \partial_\mu \lambda.$$  

(3)

Straightforward generalization of Einstein-Hilbert action to Weyl geometry leads to a higher order theory [4]. Dirac introduced a new action called Weyl-Dirac action, by including a new scalar field avoiding the presence of higher order terms in action. The Weyl-Dirac action is given by [5]

$$S[\phi, k_{\mu}] = \frac{1}{2} \int d^4 x \sqrt{-g} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + R \phi^2 + \alpha \phi_{\mu} \phi^\mu + \right.$$  

$$\left. (\alpha - 6) \phi^2 k_{\mu} k^\mu + 2(\alpha - 6) \phi k_{\mu} \phi_{\mu} + \frac{\lambda}{2} \phi^4 + 6(\phi^2 k_{\mu})_{\mu} \right\}$$

(4)

where $F_{\mu\nu} = k_{\nu,\mu} - k_{\mu,\nu}$. This action is invariant under the gauge transformations:

$$g_{\mu\nu} \rightarrow e^{2\lambda} g_{\mu\nu}$$

$$\phi \rightarrow e^{-\lambda} \phi$$

$$k_{\mu} \rightarrow k_{\mu} + \partial_\mu \lambda.$$  

(5)

Varying the action (4) with respect to $g_{\mu\nu}$, $k_{\mu}$ and $\phi$ yields

$$G_{\mu\nu} = \phi^{-2}[T_{\mu\nu} + \tau_{\mu\nu} + t_{\mu\nu}]$$

(6)

$$-(F^{\nu\mu})_{\mu} + (\alpha - 6)(\phi^2 k^\mu + \phi \phi^\mu) = 0$$

(7)

$$\alpha \phi_{\mu}^\mu - R \phi - (\alpha - 6)[\phi k_{\mu} k^\mu - \phi k^\mu_{\mu}] - \lambda \phi^3 = 0$$

(8)

here

$$T_{\mu\nu}[\phi] = (-2 + \alpha \phi_{\alpha} \phi^\alpha + (2 - \alpha) \phi_{\mu} \phi^\mu - 2 g_{\mu\nu} \phi(\phi^\alpha)_{\alpha} + 2 \phi^\mu_{\nu} + \frac{\lambda}{4} g_{\mu\nu} \phi^4$$

(9)

$$\tau_{\mu\nu} = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} - F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta}$$

(10)
and

\[ t_{\mu \nu} = (\alpha - 6) [\phi^2 (-k_\mu k_\nu + \frac{1}{2} g_{\mu \nu} k^\alpha k_\alpha) - \phi (k_\mu \phi_\nu + k_\nu \phi_\mu - k_\alpha \phi_\alpha g_{\mu \nu})]. \]  

(11)

Equation (3) is a familiar transformation that one has in the Maxwell theory for the electromagnetic vector potential, and it may lead to the identification of \( k_\mu \) as the electromagnetic vector field. But, it was shown that the usual spin \( 1/2 \) fermions (such as quarks or electrons) and the vector particles (such as Yang-Mills field or photons) cannot couple in minimal form to the Weyl gauge field \( k_\mu \) \([6]\). Also, on the basis of the Ehlers-Pirani-Schild axioms, \( F_{\mu \nu} \) must be viewed as the formal description of a phenomenon which has not been observed in nature \([7]\). In this paper we present a different interpretation of the Weyl gauge field. We assume that the vector field \( k^\mu \) is associated with the spine-one hyperphoton coupled to hypercharge, rather than the charge (hypercharge is defined as the sum of the baryon number and the strangenesses). Since the discovery that \( K_0^0 \) as the long lived component of the \( K^0 \) decays into two \( \pi \) mesons (a CP violating interaction), it was pointed out that the observed effect can be interpreted by supposing a new long-range interaction between \( K \) meson and our galaxy \([8]\). It was postulated that this interaction can be mediated by a vector field analogous to the electromagnetic field but coupled to hypercharge rather than the charge. This interaction produces a potential energy equal in magnitude and opposite in sign for particles and antiparticles \([8]\). The configuration for this vector field analogous with the electrodynamic is

\[ A_0 = gH/r, \quad A_i = 0 \]  

(12)

Here \( g \) is the hypercharge coupling constant and \( H \) is the number of nuclei in the central mass. Therefore, we assume that the Weyl gauge field \( k_\mu \) is the vector field analogous whit this large range potential.

### 3 Spherical solution

In this section we attempt to obtain spherical solution of the field equation (6). We assume the factor \( (\alpha - 6) \) to be small \([9]\). So, we neglect the terms including this factor. Let us take the line element as

\[ ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2. \]  

(13)

where \( \nu \) and \( \lambda \) are functions of \( t \) and \( r \). The absence of a generalized Birkhoff theorem in Weyl space allows the existence of the dynamical solutions in the spherical symmetric model under consideration \([10]\). Solving the equation (7) leads to

\[ k_{0r} = \frac{\gamma(t) e^{(\nu + \lambda)}}{r^2} \]  

(14)

where \( \gamma(t) \) is an arbitrary function of time arising from integration. The field equation (6) together with (14) gives

\[ e^{-\lambda} \left( -\frac{\lambda r}{r^2} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{1}{\phi} e^{-\nu} \left( \phi_{,t} \lambda_{,t} + \frac{3\phi^2}{\phi^2} \right) - \frac{\gamma^2}{r^4 \phi^2} \]  

(15)

\[ e^{-\lambda} \left( \frac{\nu r}{r^2} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{1}{\phi} e^{-\nu} \left( \phi_{tt} - \frac{\phi^2}{\phi^2} - \phi_{,t} \nu_{,t} \right) - \frac{\gamma^2}{r^4 \phi^2} \]  

(16)
Here the subscripts denote partial derivatives. In view of the forms of the left sides of equations (15,16), we assume that

$$e^\nu = f(t)e^{-\lambda}$$

Making use of (17) and (18), one finds that equations (15) and (16) become identical provided \(f(t)\) satisfies

$$f_t/f = 2\phi_{tt}/\phi_t - 4\phi_t/\phi$$

this gives

$$f = \left(\phi_t/\alpha^2\phi^4\right)^2$$

After some manipulation we obtain the generalization of the result derived by Rosen [9], as

$$e^\nu = (\phi^2/\alpha^2\phi^4)e^{-\lambda}$$

with

$$e^{-\lambda} = \frac{1}{2} - \frac{m_0}{\phi^r} + \frac{b^2}{\phi^2r^2} + \Delta$$

$$\Delta = \left[\left(\frac{1}{2} - \frac{m_0}{\phi^r} + \frac{b^2}{\phi^2r^2}\right)^2 + \alpha^2\phi^2r^2\right]^{\frac{1}{2}}$$

Here we assume that \(\gamma(t) = b\phi^{-1}\) in which \(b\) is a constant such as \(m_0\). One can interpret \(m_0\) and \(b\) as the mass and hypercharge of the central mass respectively. The above solution is valid for any gauge function \(\phi(t)\). We choose the Einstein gauge \((ds^2 \rightarrow d\bar{s}^2 = \phi^2 ds^2, \phi \rightarrow \bar{\phi} = 1)\), thus one gets

$$ds^2 = -e^{-\lambda}dT^2 + e^{2\alpha T}(e^\lambda dr^2 + r^2d\Omega^2)$$

where

$$dT = \frac{\phi_t}{\alpha \phi}dt = \frac{d\phi}{\alpha \phi}$$

so that, with a suitable choice of the origin of \(T\) in the natural units \((c = \hbar = 1)\), we have

$$\phi = e^{\alpha T}$$

From the coefficient of \(T_{\mu\nu}\) in equation (6) one can postulate that the gravitational coupling \(G\) behaves as

$$G = G_0 \phi^{-2}$$

where \(G_0\) is the gravitational constant in the present epoch. Comparing this equation with equation (25), shows

$$G \approx G_0 \left(1 - 2T/T_0\right)$$

where \(T_0\) is the present age of the universe and \(T\) is the time elapsed from the present epoch. One can see that the spherical solutions of the Weyl-Dirac theory leads to the time varying gravitational constant in accordance with [11].
4 Orbit of planets

We proceed now to show that planetary mean distances can be calculated in a Weyl spacetime. We investigate the motion of a planet in the gravitational field of the sun and apply to it the gauge concept of the Weyl geometry. This method was also used for the motion of an electron in the electromagnetic field of a proton in a hydrogen atom [12]. These considerations provides for obtaining the Bohr radii of quantum theory. Thus, the Weyl geometry introduces automatically quantization conditions for the orbits in the hydrogen atom. We now apply this model to the solar system. In this scale one can make the reasonable assumption that there is no pure electric charge, but instead, the hypercharge will be dominant. We investigate the effects of the hypercharge of the sun on the motion of the planets through considering the gauge field $k_\mu$ as the hypercharge vector field induced by the long range potential. So, the non-vanishing component of $k_\mu$ is $k_0 = gA_0 = g^2H/r$. Since we interpret $b$ as the hypercharge of the central mass in the equation (22) one can compute this constant in terms of the hypercharge of the sun as $(b = g^2H)$ which according to Bell’s estimation is approximately $b \approx 10^9$ [8]. Now, we first calculate the gravitational potential of the sun by using the spherical solution of the Weyl-Dirac gravitational equation. Let $g_{00}$ be the component of the metric (13) in the conventional units in the present epoch, then we get from (22)

$$g_{00} = \frac{1}{2} - \frac{G_0 M}{r} + \left(\frac{g^2 H}{r}\right)^2 + \Delta$$

(28)

where

$$\Delta = \left[\left(\frac{1}{2} - \frac{G_0 M}{r} + \left(\frac{g^2 H}{r}\right)^2 + \left(\frac{r}{T_0}\right)^2\right)^{1/2}\right]^2.$$

(29)

and $M$ is mass of the sun. It is obvious that the effect of the term proportional to $1/r^2$ in the large scale is ignorable compared to the term proportional to $1/r$. Thus, by expanding the above equation to the first order we obtain

$$g_{00} = 1 - \frac{2G_0 M}{r} + \frac{2G_0 M r}{T_0^2} + \frac{r^2}{T_0^2} - 8\left(\frac{G_0 M}{T_0}\right)^2.$$  
(30)

One can see that the existence of the linear terms in the potential are due to the variable gravitational coupling, in accordance with [11]. Now, we consider the motion of a planet in the gravitational potential of the sun, then for a given orbit we have

$$\frac{G_0 M}{r^2} + \frac{G_0 M}{T_0^2} + \frac{r}{T_0^2} = \frac{v^2}{r}.$$  
(31)

We assume that the orbits of the planets are integrable in the Weyl geometry. It means that the frequency of spectral lines evidently does not depend on the history of the radiating matter.

$$\exp\left(\int_0^\tau k_0 d\tau\right) = 1$$  
(32)

here $\tau$ is the period of motion of a planet around it’s orbit. This relation seems like a quantization condition. By using equations (31) and (32) the possible radius $r$ for the orbits around which the length scale is preserved in the solar scale are given by

$$r = \frac{n^2 G_0 M}{g^4 H^2}.$$  
(33)
which corresponds to the derived expression for the radii of orbits by Agnese and Festa [1], but there is a difference. There is a constant parameter in the result derived in [1] which is called re-scaled Planck constant [3] or re-scaled fine structure constant [1] and there is no physical origin accounting for them. Here the term $g^2 H$ plays the role of the re-scaled fine structure constant considered in [1]. As one can see in the present model the Weyl geometry can provide a physical meaning for it.

Now, we assume a fundamental radius $r_1 = 0.04 AU$ for $n = 1$, as was predicted by Nottale [2] and also by Agnese [13]. Several extra solar planets recently discovered lie at this distance from their star [14]. Using this assumption, we can estimate the hypercharge coupling constant $g$, as $g^2 = 10^{-53}$ which is roughly related to the Bell’s estimation [8]. By this consideration we can obtain a sequence of values that assures very well the observed values of the orbital radii in our solar system. For $n$ running from 3 to 6, we obtain orbital radii of Mercury, Venus, Earth and Mars; for $n = 8$ we have the asteroid Ceres, and for $n = 11, 15, 21, 26$ and 30, the orbital radii of Jupiter, Saturn, Uranus, Neptune and Pluto follow, in agreement with the observed values. The problem here, as we can see, is that we fall into a lot of empty orbital positions which are predicted by the formula $r = n^2 r_1$, but not occupied by any observed body, particularly for large values of $n$.

5 Conclusion

In this paper we have presented a new model for obtaining the orbits of the planets and the asteroid belts. Our approach is in the direction of many recent works that have been done in geometrization of quantum processes. In this manner we have shown, using Weyl-Dirac approach to gravity that one can describe solar scale quantization. A major ingredient of this model is the Weyl gauge field which can be interpreted as a gauge field describing the hypercharge of particles. This is the dominant field at large scales and would provide a physical meaning to the re-scaled Planck constant in solar and also in cosmic scales.

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