The Logic Blog is a shared platform for
• rapidly announcing results and questions related to logic
• putting up results and their proofs for further research
• parking results for later use
• getting feedback before submission to a journal
• foster collaboration.

Each year’s blog is posted on arXiv a 2-3 months after the year has ended.

How does the Logic Blog work?

Writing and editing. The source files are in a shared dropbox. Ask André
andre@cs.auckland.ac.nz in order to gain access.

Citing. Postings can be cited. An example of a citation is:

H. Towsner, Computability of Ergodic Convergence. In André Nies (editor),
Logic Blog, 2012, Part 1, Section 1, available at http://arxiv.org/abs/1302.3686.

The logic blog, once it is on arXiv, produces citations on Google Scholar.
## Contents

### Part 1. Computability theory and randomness
1. Barmpalias and Nies: a randomness notion between ML and KL-randomness 3
2. Nies: For non-high states, strong Solovay tests are as powerful as general quantum ML tests 4
3. Franklin and Turetsky: (non)convexity of difference random degrees 5

### Part 2. Group theory and its connections to logic
4. Nies and Segal: Non-axiomatizability for classes of profinite groups 6
   4.1. Number of generators 6
   4.2. Comparison with the discrete case 7
   4.3. Finite axiomatisability implies that the set of primes that occur is finite 7
5. Nies, Perin and Segal: all f.g. profinite groups are \(\omega\)-homogeneous 8
6. Nies and Stephan: Some properties of infinitely generated nilpotent-of-class-2 groups 9
   6.1. Three properties of an elementary abelian group 9
   6.2. Nilpotent of class 2 groups 10
   6.3. FA-presentability 11

### Part 3. Computability theory and set theory
7. Greenberg and Nies: cardinal characteristics and computability 12
8. Lempp, Miller, Nies and Soskova: Analogs in computability of combinatorial cardinal characteristics 13
   8.1. Set theory. 14
   8.2. Analogous mass problems in computability 14
   8.3. The mass problems \(\mathcal{A}\) and \(\mathcal{T}\) 15
   8.4. \(\Delta^0_2\) and 1-generic sets, totally low sets, and not computing a MAD 16
   8.5. C.e. MAD sets 17
   8.6. The mass problem \(\mathcal{U}\) corresponding to the ultrafilter number, and a strict reduction \(\mathcal{T}<_{\mathcal{S}}\mathcal{U}\) 19
   8.7. Co-c.e. ultrafilter bases 20
9. Yu Liang 22
   9.1. Concerning the Joint coding theorem: a handwritten proof by Harrington 22
   9.2. A note concerning the Borelness of upper cone of hyperdegrees. 36

### Part 4. Model theory and definability
10. First-order logic, computability, and enriched structures 36
References 38
Part 1. Computability theory and randomness

1. Barmpalias and Nies: a randomness notion between ML and KL-randomness

Jason Rute [27, Section 10] introduced possible strengthenings of Kolmogorov-Loveland (KL)-randomness that are still implied by ML-randomness. In fact he worked in the general setting of computable probability spaces, where he also defined a notion of computable randomness. The following is one of his notions. It relies on the little explored notion of partial computable randomness; see [17, Ch. 7]. (A betting strategy can be undefined off the sequences it has to succeeds on, which allows for potentially indefinite waiting before making a bet.)

**Definition 1.1.** \( Z \in 2^\mathbb{N} \) is called **Rute random** if \( \Phi(Z) \) is partial computably random for each measure preserving Turing functional \( \Phi \).

A pair \( R = (\Phi, L) \) of a m.p. Turing functional and a partial computable martingale will be called a **Rute betting strategy**. We say that \( R \) is total if both \( \Phi \) and \( L \) are total functions.

To say that \( \Phi \) is measure preserving means that \( \lambda(\Phi^{-1}(C)) = \lambda(C) \) for each measurable \( C \subseteq 2^\mathbb{N} \). It suffices to require this for cylinders \( C = [\sigma]^\prec \) where \( \sigma \) is any string. Such Turing functionals are almost total: the domain is a conull \( \Pi_0^1 \) set. So \( \Phi(Z) \) is defined for any Kurtz “random” \( Z \). Its kernel \( \{(X,Y) : \Phi^X = \Phi^Y \downarrow\} \) is a \( \Pi_0^1 \) equivalence relation with all classes null. \( \Phi \) is given by a binary tree of uniformly \( \Sigma_0^1 \) sets \( U_\sigma = \Phi^{-1}([\sigma]^\prec) \) such that \( \lambda U_\sigma = 2^{-|\sigma|} \).

If \( \Phi \) is total (i.e., a tt functional) then \( \Phi \) is onto by a compactness argument. In this case the kernel is \( \Pi_0^1 \). An easy example of a m.p. total function is the shift functional \( T \) where \( T(Z) \) is \( Z \) with the first bit erased. For a more interesting example, view bit sequences as \( 2 \)-adic numbers, and consider a computable unit \( r \in \mathbb{Z}_2^* \) (anything that ends in 1). Then \( \Phi^Z = rZ \) is a m.p. total Turing functional, with inverse \( \Psi^Z = r^{-1}Z \). This example can be generalised to sequences over a \( p \)-bit alphabet, for a prime \( p \).

Each scan rule in the sense of [13, 17] induces a measure preserving tt functional on Cantor space. So Rute randomness implies KL-randomness.

The sets \( \Phi^{-1}(\{X : L(X) > 2^m\}) \) form a ML-test that succeeds on each \( Z \) on which the strategy does. So ML-randomness implies Rute randomness.

**Fact 1.2.** Given a Rute strategy \( R = (\Phi, L) \) with \( \Phi \) total but possibly \( L \) partial, there are total Rute strategies \( R_0, R_1 \) such that if \( R \) succeeds on \( Z \) then one of the \( R_i \) succeeds as well.

**Proof.** We use the similar result from [13] that given a KL-strategy \( K \) (also see [17, Ex. 7.6.25]) we can build two total KL-strategies \( K_0, K_1 \) that together cover the success set of \( K \).

In our case \( K \) is simply \( L \) (with the identity scan rule). We obtain scan rules \( K_i = (S_i, B_i) \) Let \( \Phi_i = S_i \circ \Phi \) which are m.p. total Turing functionals. Now the \( R_i \) given by \( \Phi_i, B_i \) are as needed. \( \square \)

Recall that each KL-betting strategy fails on some c.e. set (e.g [13]). This argument doesn’t quite go through. However the following is easy.
Proposition 1.3. Each Rute strategy \((\Phi, L)\) fails on a non-Kurtz random.

Proof. We may assume that \(\Phi\) is total, else it fails by not being defined. So we can also assume that \(L\) is total. Let \(R\) be a computable path along which \(L\) is bounded (say in the \(n\)-th bet, along \(R\) the martingale \(L\) goes up by at most \(2^{-n}\); this is a \(\Sigma^0_1\) event which has to happen on at least one 1-bit extension of a string). Then \(\Phi^{-1}(R)\) is a \(\Pi^0_1\) null class on which the Rute strategy fails. \(\square\)

As a consequence, if we wanted to show that Rute = ML, we’d need at least two strategies to cover all non-MLR randoms.

Question 1.4. Determine lowness for Rute randomness. Note that it is contained in Low\((\text{MLR}, \text{CR})\) which coincides with the \(K\)-trivials [15].

2. Nies: For non-high states, strong Solovay tests are as powerful as general quantum ML tests

Recall that for a non-high bit sequence \(Z\), Schnorr randomness implies ML-randomness (e.g. [17, Ch. 3]). Here we give a similar argument in the setting of randomness for infinite sequences of qubits [18].

We use the convention as in the final section of [18] that all states and projections are elementary, i.e. the relevant matrices consist of algebraic complex numbers. So it is OK to directly use a state \(\rho\) as an oracle. (In general we would have to work with elementary approximations to the density matrices \(\rho|_n\), similarly as is done in computable analysis to deal with, say, sequences of reals).

Definition 2.1 ([18], 3.14).

- A quantum Solovay test is an effective sequence \(\langle G_r \rangle_{r \in \mathbb{N}}\) of quantum \(\Sigma^0_1\) sets such that \(\sum_r \tau(G_r) < \infty\).
- We say that the test is strong if the \(G_r\) are given as projections; that is, from \(r\) we can compute \(n_r\) and a matrix of algebraic numbers in \(M_{n_r}\) describing \(G_r = p r_r\).
- For \(\delta \in (0, 1)\), we say that \(\rho\) fails the quantum Solovay test at order \(\delta\) if \(\rho(G_r) > \delta\) for infinitely many \(r\); otherwise \(\rho\) passes the qML test at order \(\delta\).
- We say that \(\rho\) is quantum Solovay-random if it passes each quantum Solovay test \(\langle G_r \rangle_{r \in \mathbb{N}}\) at each positive order, that is, \(\lim_{r} \rho(G_r) = 0\).

Tejas Bhojraj showed that q-Solovay tests are equivalent to qML tests (but the order at which the test succeeds changes). However, it is likely that strong q-Solovay tests are indeed more restricted, hence the significance of the following Resultatchen.

Proposition 2.2. Let \(\rho\) be a non-high state on the usual CAR algebra \(M_{\infty}\). If \(\rho\) fails some qML test \(\langle G_r \rangle_{r \in \mathbb{N}}\) at order \(\delta > 0\), then it fails a strong Solovay test \(\langle q_r \rangle\) at the same order. Moreover we can achieve that \(\sum_r \text{Tr}(q_r)\) is a computable real.

Proof. Given a qML-test \(\langle G_r \rangle = \langle p_r^\infty \rangle\) as defined in [18, 3.5] that \(\rho\) fails at order \(\delta\), the function \(f(r) = \mu_n \text{Tr}(\rho|_n p_n^\infty) > \delta\) is total and satisfies \(f \leq T \rho\). Since \(\rho\) is not high, there is a computable function \(h\) such that
∃r.h(r) ≥ f(r). Let now q_r = p^r_{h(r)}. This is a strong Solovay test of the required form, and ρ fails it at the same order δ. □

3. Franklin and Turetsky: (non)convexity of difference random degrees

The ML-random degrees can be divided into two sets: those which compute 0′, and those which do not compute a PA-degree. The latter correspond to the difference random degrees. The first set is known to be convex: if d_0 and d_1 are MLR degrees above 0′, then every degree d_2 with d_0 ≤ d_2 ≤ d_1 is an MLR degree. This is because every degree above 0′ contains a random. Still, it’s natural to ask whether convexity holds for the difference random degrees. Here we construct a counterexample. Our construction simply requires combining several existing results.

First we need the following:

**Proposition 3.1.** Every $\Delta^0_2$ MLR degree computes a $K$-trivial which does not have c.e. degree.

One could prove this directly, or just invoke Kučera’s result that every $\Delta^0_2$ MLR degree bounds a noncomputable c.e. degree, Hirschfeldt et al.’s [10] results showing that said degree must be $K$-trivial, and then Yates’s result that every noncomputable c.e. degree bounds a minimal degree (and then Nies [15] that $K$-triviality is downwards closed under Turing reducibility). And I guess Sacks’s density theorem to get that a minimal degree is not a c.e. degree.

Next, we need the following result of Greenberg and Turetsky (see here: https://arxiv.org/abs/1707.00258):

**Proposition 3.2.** For every $K$-trivial A there is a c.e. $K$-trivial $B \geq_T A$ such that every ML-random which computes A computes $B$.

Of course, if A does not have c.e. degree, then B will be strictly above A. Finally we’ll need this:

**Proposition 3.3.** If A does not compute B, then for almost every oracle $Z$, $A \oplus Z$ does not compute $B$.

**Proof.** Suppose otherwise. Fix an $e$ such that for positive measure of oracles $Z$, $\Phi_e(A \oplus Z) = B$. Apply Lebesgue density and then majority vote to show that A computes B, contrary to hypothesis. □

We’ll also need van Lambalgen’s theorem. We’re ready to build the counterexample.

Fix $X$ an incomplete $\Delta^0_2$ random. Fix $A \leq_T X$ a $K$-trivial which does not have c.e. degree. Fix $B >_T A$ such that every ML-random computing $A$ computes $B$. For almost every oracle $Y$, $X \oplus Y$ is random, $X \oplus Y$ does not compute $0'$, and $A \oplus Y$ does not compute $B$, so fix such a $Y$.

Our failure of convexity is provided by $Y \leq_T A \oplus Y \leq_T X \oplus Y$. Suppose $A \oplus Y$ were Turing equivalent to a random $Z$. Then $A \oplus Y \geq_T Z \geq_T A \oplus Y$, and thus $Z \geq_T A$. Then $Z \geq_T B$, by choice of $B$. So $A \oplus Y \geq_T B$, contrary to choice of $Y$. 
It’s known that convexity holds for hyperimmune-free MLR degrees. In fact, if $B$ is ML-random and of hyperimmune free degree, and $A \leq_T B$ is noncomputable, then $A$ is of ML-random Turing degree using that $A \leq_{tt} B$ and an argument of Demuth. How common are failures outside those? Does every hyperimmune MLR degree bound a failure of convexity? Is every pair of hyperimmune MLR degrees with one strictly above the other a failure of convexity?

Part 2. Group theory and its connections to logic

4. Nies and Segal: Non-axiomatizability for classes of profinite groups

Expressiveness of first-order logic for classes of groups remains an interesting topic because it straddles logic and group theory and connects these areas in novel ways. For topological groups the topic is particularly interesting, because in first-order logic (say in the language of groups) one can’t directly talk about open sets: first-order means that one can only talk about elements of the structure.

Here we show that some commonplace properties cannot be axiomatised by a single sentence.

4.1. Number of generators. The following results and proofs work with small changes also in the setting of discrete groups.

Proposition 4.1.

(a) Being $d$-generated cannot be expressed by a single first order sentence within the profinite groups.

(b) Being finitely generated cannot be expressed by a single first order sentence within the profinite groups.

Proof. Let $C_q$ denote the cyclic group of order $q$. By a result of Szmielev related to her proof that the theory of abelian groups in decidable (1950s), for each sentence $\phi$ in the language of groups the following holds: for almost all primes $q$, an abelian group $G$ satisfies $\phi$ iff $G \times C_q$ satisfies $\phi$.

Recall that $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, which is the Cartesian product of all the $\mathbb{Z}_p$ for $p$ a prime. It is 1-generated and has each finite cyclic group as a quotient (in fact it is the free profinite group of dimension 1). If $\phi$ expresses being $d$-generated, then $\hat{\mathbb{Z}}^d \models \phi$. Let $q$ be a prime as above, then also $(\hat{\mathbb{Z}})^d \times C_q \models \phi$. Yet, this group is not $d$-generated because it has $C_q^{d+1}$ as a quotient.

A similar argument shows that no sentence $\phi$ can express being finitely generated among profinite groups. Modify the argument above. Replace $(\hat{\mathbb{Z}})^d \times C_q$ by the profinite group $\hat{\mathbb{Z}} \times (C_q)^{\aleph_0}$ which is not finitely generated but would also satisfy $\phi$, for almost every prime $q$. This uses that $\phi$ can be expressed as a Boolean combination of so-called Szmielev invariant sentences, which can only contain information about finitely many primes; see Hodges [11, Thm A.2.7].

A profinite group $G$ has rank $r$ if each closed subgroup of $G$ is $r$-generated. For an abelian profinite group $G$, this is equivalent to the whole group being
r-generated, by the structure theorem that $G$ is a direct product of pro-cyclic groups and finite cyclic groups. This implies:

**Corollary 4.2.** There is no first-order sentence expressing that a profinite group $G$ has finite rank.

On the other hand, the entire theory of a profinite group $G$ contains the information whether it is (topologically) finitely generated: Jarden and Lubotzky [12] (also see [28, Thm. 4.2.3]) proved that if $G, H$ are profinite groups, $G$ is f.g. and $G \equiv H$ then $G \cong H$. Whether a profinite group is $d$-generated can also be recognized from the entire theory, because it means that each finite quotient is $d$-generated, and one can express in f.o. logic what the finite quotients are again by Jarden and Lubotzky [12].

This leaves open the question whether a uniform set of axioms can express that a profinite group is f.g.

**Question 4.3.** Is there a set of sentences $S$ in the language of group theory such that for profinite groups $G$, we have $G \models S \iff G$ is f.g.?

**4.2. Comparison with the discrete case.** Let’s compare this for a moment with the case of discrete groups where f.g. now means literally finitely generated.

The elementary theory of a group $G$ doesn’t determine whether $G$ is f.g.; by an easy elementary chain argument, for each countable group $G$ there is a countable group $H$ such that $G$ is an elementary submodel of $H$, in particular has the same theory, and $H$ is not f.g. [16, Section 4]. (This argument using elementary chains of type $\omega$ doesn’t work for profinite groups. The reason is that model theoretic constructions such as adding a constant, or union of elementary chains, don’t work in the topological setting such as for profinite structures.)

We note that the theory $T$ of f.g. groups is $\Pi^1_1$ complete by Morozov and Nies [14]. Its models could be called “pseudo-f.g.”, in analogy with the pseudofinite groups. An example of such a group that is not f.g. is the free group of infinite dimension $F_\omega$.

**4.3. Finite axiomatisability implies that the set of primes that occur is finite.** In the following “FA” is short for “finitely axiomatisable”.

We begin with an observation of T. Scanlon. Rings will be commutative with unit element. $\mathbb{Z}_p$ denotes the ring (or sometimes group) of $p$-adic integers.

**Proposition 4.4** (going back to Scanlon, 2016). Let $R$ be the profinite ring $\prod_{p \in \pi} \mathbb{Z}_p$ where $\pi$ is an infinite set of primes. Then $R$ is not FA within the class of profinite rings.

**Proof.** This follows from the Feferman-Vaught theorem (see any large text on model theory, such as [11]). For every sentence $\phi$ in the language of rings there is a finite sequence of sentences $\psi_1, \ldots, \psi_n$ in the language of rings and a formula $\theta(x_1, \ldots, x_n)$ in the language of Boolean algebras so that for any index set $I$ and any family of rings $R_i$ indexed by $I$, letting

1An fun exercise to get used to the 3-adic integers is the following. Note that $-1 = \ldots 2222222222$, and verify that $-1^2 = 1$ in $\mathbb{Z}_3$ using the multiplication algorithm from elementary school.
we have
\[ \prod_{i \in I} R_i \models \phi \text{ if and only if } \mathcal{P}(I) \models \theta(X_1, \ldots, X_n). \]

Assume for a contradiction that a sentence \( \phi \) shows that \( R \) is FA. By the pigeon hole principle, we can find two distinct primes \( r \neq q \) in \( \pi \) so that for all \( j \leq n \) we have \( Z_r | \psi_j \iff Z_q | \psi_j \). Define \( R_p := Z_p \) if \( p \neq r \) and \( R_r := Z_q \). Then \( R' := \prod R_p \models \phi \) but \( R' \not\sim R \) as, for example, \( r \) is a unit in \( R' \) but not in \( R \).

\[ \square \]

\( \text{UT}_3(R) \) denotes the usual Heisenberg group over the ring \( R \). It is well known that \( Z(\text{UT}_3(R)) \cong (R, +) \) because the centre consists of the matrices with only a nonzero entry in the upper right corner (other than the diagonal).

**Proposition 4.5.** Let \( R \) be the profinite ring \( \prod_{p \in \pi} \mathbb{Z}_p \) where \( \pi \) is a set of primes. If \( \text{UT}_3(R) \) is FA within the profinite groups then \( \pi \) is finite.

**Proof.** Note that \( \text{UT}_3(R) \) can be interpreted in \( R \) by a collection of first-order formulas that do not depend on \( R \). Hence, for each sentence \( \theta \) in the language of groups, there is a sentence \( \tilde{\theta} \) in the language of rings such that for each ring \( R \),
\[ \text{UT}_3(R) \models \theta \iff R \models \tilde{\theta}. \]

Assume for a contradiction that a sentence \( \theta \) shows that \( \text{UT}_3(R) \) is FA. within the profinite groups. Applying the above Feferman-Vaught analysis to \( \phi = \tilde{\theta} \), let \( r \neq q \) be primes as above, and let also \( R' \) defined as above. Then \( R' \models \tilde{\theta} \) and hence \( \text{UT}_3(R') \models \theta \). However, \( \text{UT}_3(R') \not\cong \text{UT}_3(R) \) because every element in the centre of \( \text{UT}_3(R') \), but not in the centre of \( \text{UT}_3(R) \), is divisible by \( r \).

\[ \square \]

A similar argument works for some other algebraic groups. For instance, for each \( n \), \( \text{GL}_n(R) \) can be interpreted in \( R \) by formulas not depending on \( R \), and its centre consists of the scalar matrices \( \alpha I_n \) where \( \alpha \) is a unit of \( R \). As mentioned, the groups of units are not isomorphic for the rings \( R' \) and \( R \) as above.

5. Nies, Perin and Segal: all f.g. profinite groups are \( \omega \)-homogeneous

Let \( G \) be a profinite group (separable, as always) and \( \overline{g} \in G^k \) where \( k \in \mathbb{N} \). We want to show that the type of \( \overline{g} \) (i.e., the first-order theory of \( (G, \overline{g}) \)) determines the orbit of \( \overline{g} \) in \( G \). In the case of countable groups such a result has been shown e.g. for the f.g. free groups [23]. The following was first noticed in discussions between Segal and Chloe Perin at an OW meeting in 2015. Her Master’s student Noam Kolodner covered it in his thesis.

**Theorem 5.1.** Let \( G, H \) be (topologically) f.g. profinite groups, \( k \in \mathbb{N} \), \( \overline{g} \in G^k \) and \( \overline{h} \in H^k \). If \( (G, \overline{g}) \cong (H, \overline{h}) \) then \( (G, \overline{g}) \cong (H, \overline{h}) \).
Here in fact it suffices to look at the $\Sigma^0_2$ formulas satisfied by $\overline{g}$ in $G$ to determine the orbit. The case $k = 0$ is due to Jarden and Lubotzky [12].

We need a lemma that is a straightforward extension of [8, Prop. 16.10.7]. Let $\text{Im}(G, \overline{g})$ denote the set of pairs $(R, \overline{r})$, $R$ a finite group, $\overline{r} \in R^k$ such that there is an epimorphism $(G, \overline{g}) \twoheadrightarrow (R, \overline{r})$, i.e. it sends $g_i$ to $r_i$ for $i < k$. (The symbol $\twoheadrightarrow$ will denote such epimorphisms.) For a profinite group $P$, $\overline{p} \in P^k$ and $N \triangleleft P$ we will write $(P, \overline{p})/N$ for the structure $(P/N, (Np_i)_{i<k})$.

**Lemma 5.2.** Suppose that $\text{Im}(H, \overline{h}) \subseteq \text{Im}(G, \overline{g})$. Then $(G, \overline{g}) \twoheadrightarrow (H, \overline{h})$.

**Proof.** Note that a f.g. profinite group $P$ has only finitely many open subgroups of each index. Let $P_n$ be the intersection of subgroups of index $\leq n$. Then $P_n$ is a characteristic subgroup of finite index.

Fix $n$. By hypothesis there is an open $N \triangleleft G$ such that $(H, \overline{h})/H_n \cong (G, \overline{g})/N$. Then $G_n \leq N$, and hence $(G, \overline{g})/G_n \twoheadrightarrow (H, \overline{h})/H_n$. Let $\Phi_n$ be the set of witnessing epimorphisms.

If $\phi \in \Phi_{n+1}$ then $\phi(G_n/G_{n+1}) \leq H_n/H_{n+1}$, so $\phi$ induces a map $\tilde{\phi} \in \Phi_n$. By Koenig’s Lemma applied to the tree where the $n$-th level is $\Phi_n$, there is an epimorphism $\psi: (G, \overline{g}) \twoheadrightarrow (H, \overline{h})$ in the inverse limit of the $\Phi_n$. \hfill $\Box$

**Proof of the Theorem.** We extend the proof in Segal [28, Thm. 4.2.3] of the Jarden-Lubotzky theorem. We show that $\text{Im}(H, \overline{h}) \subseteq \text{Im}(G, \overline{g})$. By symmetry, also $\text{Im}(G, \overline{h}) \subseteq \text{Im}(H, \overline{g})$. The lemma together with the Hopfian property of f.g. profinite groups now implies that $(G, \overline{g}) \cong (H, \overline{h})$.

Suppose that $G$ is topologically generated by $d$ elements. Recall from [22] (also [28, Ch. 4]) that a group word $w$ is called $d$-locally finite if the $d$-generated free group in the variety of groups satisfying $w$ is finite. (As an example, the word $w = [x, y]z^m$ is $d$-locally finite for each $d$ because a f.g. abelian group of exponent $m$ is finite.) A main technical result in [22] states that for such a word, there is $f = f(w,d)$ such that $w(R) = R_{w}^{f}$ for each $d$-generated finite group $R$, where $R_{w}^{f}$ is the set of products of at most $f$ many $w$ values or their inverses. Note that $R_{w}^{f}$ is definable by a $\exists \Sigma_2$ formula depending only on $f$ and $w$.

As in [28, Thm. 4.2.3] our hypothesis that $(G, \overline{g}) \equiv (H, \overline{h})$ implies that $(G, \overline{g})/G_{w}^{f} \cong (H, \overline{h})/H_{w}^{f}$. Checking the formulas reveals that only $(G, \overline{g}) \equiv \Sigma_2 (H, \overline{h})$ is needed.

Suppose $N \triangleleft H$ is open. We need to show that $(H, \overline{h})/N$ is in $\text{Im}(G, \overline{g})$. Again as in [28, Thm. 4.2.3] pick a $d$-locally finite word $w$ such that $H_{w}^{f} \leq N$. Then

$$(G, \overline{g}) \twoheadrightarrow (G, \overline{g})/G_{w}^{f} \cong (H, \overline{h})/H_{w}^{f} \twoheadrightarrow (H, \overline{h})/N,$$

as required. \hfill $\Box$

6. **Nies and Stephan: Some properties of infinitely generated nilpotent-of-class-2 groups**

6.1. **Three properties of an elementary abelian group.** Throughout fix a prime $p$. Let $G = F_{p}^{(\omega)}$ denote the elementary abelian $p$ group (i.e., vector space over the field $F_{p}$ of infinite dimension). This group has the following apparently unrelated properties.
Definition 6.1.

1. $G$ is word-automatic in the usual sense of Khoussainov and Nerode: finite automata can recognize the domain and the group operations.
2. $G$ is $\omega$-categorical: it is up to isomorphism the only countable model of its theory.
3. $G$ is pseudofinite: if a sentence $\phi$ holds in $G$ then $\phi$ also holds in a finite group.

Note that a word automatic group is called finite automata presentable in [16] in order to avoid confounding the notion with the better known notion of automatic groups due to Thurston. The group $F_p^{(\omega)}$ is $\omega$-categorical because for each $n$ there are only finitely many $n$-types. To show that $F_p^{(\omega)}$ is pseudofinite, note that its theory is axiomatized by stating that the group is infinite, has exponent $p$ and is abelian. So we may assume that a sentence $\phi$ as above is a finite conjunction of these axioms, and so clearly has a finite model.

6.2. Nilpotent of class 2 groups. Usually we will assume that $p \neq 2$. We want to study the three properties above for groups that are very close to abelian. Recall that a group $G$ is nilpotent of class 2 if each commutator $[x, y]$ is in the centre $C(G)$. In other words, $G$ is an extension of a group $N$ by a group $Q$ that is abelian, and $N$ is contained in $C(G)$ (i.e., $G$ is a central extension). It is not hard to show that such a group is entirely given by the abelian groups $N, Q$, and an alternating bilinear map $L: Q \times Q \to N$. It determines a unique central extension $G$ such that $L(Nx, Ny) = [x, y]$ (note that $[x, y]$ only depends on the cosets of $x, y$). In fact this is a special case of the well known fact that an extension of abelian group $N$ by $Q$ is given by an element of the second co-homology group $H^2(Q, N)$.

We will study our properties for three examples of infinitely generated groups of exponent $p$. We will always have $Q = F_p^{(\omega)}$, and fix a basis $\langle x_i \rangle_{i \in \mathbb{N}}$ of $Q$.

Let $L_p$ be the free nilpotent-2 exponent $p$ group of infinite rank. One has $N \cong Q$; let $y_{i,k}$ $(i < k)$ be free generators of $N$. It is given by the function $\phi(x_i, x_k) = y_{i,k}$ if $i < k$. Thus

$$L_p = \langle x_i, y_{i,k} \mid x_i^p, [x_i, x_k]y_{i,k}^{-1}, [y_{i,k}, x_r](i < k) \rangle$$

where the $y_{i,k}$ are actually redundant.

$G_p$ is the group where $Q = F_p^{(\omega)}$, $N = C_p$ (cyclic group of order $p$) and $\phi(x_i, x_k) = 1$ if $i < k$, $-1$ if $i > k$ (and of course 0 if $i = k$).

This group is extra-special in the sense of Higman and Hall. (This means that the centre is cyclic of order $p$, equals the derived subgroup, and the central quotient is an elementary abelian $p$-group.)

Felgner [7, Section 4] has proved that $G_p$ is $\omega$-categorical for $p \neq 2$. In his notation $G_p = G(p, \leq)$ where $\leq$ is the ordering of $\omega$. This is the only countably infinite extra-special group of exponent $p$.

Felgner (p. 423) also provides a recursive axiom system for the theory of $G_p$ ($p$ odd as before). One expresses that the group is of exponent $p$, that the centre is cyclic of order $p$, contains the derived subgroup, and that the derived subgroup is non-trivial (and hence equals the centre). Further one
expresses that the central quotient is infinite, using an infinite list of axioms. Note that for each odd \( k \) there is a (unique) extraspecial \( p \)-group of exponent \( p \) and order \( p^k \). (For \( k = 3 \) this is the upper triangular matrices over \( F_p \), for larger \( k \) one takes central products of these.) So, finitely many axioms can be satisfied in a finite model. Hence \( G_p \) is pseudo-finite as already noted by Felgner.

The group \( H_p \) has \( N \cong Q \), with free generators \( z_k \, (k > 0) \), and the function \( \phi(x, x_k) = z_k \) if \( i < k \):

\[
H_p = \langle x_i, z_k \mid x_i^p, [x_i, x_k]z_k^{-1}, [z_k, x_r] \mid (i < k, 0 \leq r) \rangle.
\]

In a nilpotent group, each nontrivial normal subgroup intersects the centre non-trivially. This implies that every proper quotient of \( G_p \) is abelian; in particular, \( G_p \) is not residually finite. On the other hand \( L_p \) and \( H_p \) are residually \( p \)-groups. To see this, take an element \( h \neq 1 \), where \( h \) depends on generators \( x_0, \ldots, x_{n-1} \). Then \( h \neq 1 \) in the finite quotient \( p \)-group where all the generators \( x_k \), \( k \geq n \), become trivial. This means that both groups are embedded into their pro-\( p \) completions.

**Fact 6.2.** None of the groups \( L_p, G_p, H_p \) is abelian by finite.

**Proof.** Let \( K \) be such a group and suppose \( M \) is a normal subgroup of finite index. There are \( k < r < s \) such that \( x_r x_k^{-1} \in M \) and \( x_s x_k^{-1} \in M \). Then \( [x_r x_k^{-1}, x_s x_k^{-1}] = [x_r, x_s] x_k^{-1} \), and this commutator is not \( 1 \) in \( K \). So \( M \) is not abelian. \( \square \)

### 6.3. FA-presentability

Any FA presentable structure has a decidable theory. Ershov [6] showed that a finitely generated nilpotent group has decidable first-order theory if and only if it is virtually abelian. One the other hand, Nies and Semukhin [20, Thm. 4.2] showed that an abelian group that is an extension of finite index of an FA-presentable group is in itself FA-presentable. So a finitely generated nilpotent group is FA-presentable iff it is abelian by finite. Using Mal’cev coding, Nies and Thomas [21] extended this by showing that each finitely generated subgroup of an FA-presentable group is abelian-by-finite. So the natural setting for finding interesting FA-presentable groups is among the groups that are not finitely generated.

**Fact 6.3.** \( G_p \) is FA-presentable.

**Proof.** Each element has the form \( c \cdot q \) where \( c \in N, q \in Q \). Note that \( q \) is given as a string \( \alpha \) over the alphabet of digits \( 0, \ldots, p-1 \), and \( c \) can be stored in the state. In the following \( \alpha, \beta \) denote such strings, which are thought of as extended by 0’s if necessary. Let \( \langle \alpha \rangle = \prod_i x_i^{\alpha_i} \).

It is easy to verify that, with arithmetic modulo \( p \) and component-wise addition of strings,

\[
[\alpha][\beta] = [\alpha + \beta] u^{\sum_{k>0} \alpha_k (\sum_{i<k} \beta_i)}.
\]

This is because one can calculate \( \prod_k x_k^\alpha \prod_i x_i^{\beta_i} \) by, for decreasing positive \( k \), moving \( x_k^{\alpha_k} \) to the right past the \( x_i^{\beta_i} \) for \( i = 0, \ldots, k-1 \). Each such move creates a factor \( u^{\alpha_k \beta_i} \).

A finite automaton processing \( \alpha, \beta \) on two tracks can remember the current \( \sum_{i<k} \beta_i \) in the state, and also carry out the calculation \( \alpha_k (\sum_{i<k} \beta_i) \) and store the current exponent of \( u \) in the state. \( \square \)
Fact 6.4. $H_p$ is FA-presentable.

Proof. An element $c \in N$ is of the form $\prod_i z_i^{\gamma_i}$. An element $h$ of the group $H_p$ is represented by a pair of strings $\alpha, \gamma$ such that $h = [\alpha] \prod_i z_i^{\gamma_i}$. These strings are written on two tracks, $\alpha$ on top of $\gamma$.

Note that by a similar argument as above, in $H_p$ we have $[\alpha][\beta] = [\alpha + \beta] \prod_{k>0} z_k^{\alpha_k(\sum_{i<k} \beta_i)}$.

By the argument above the FA can store the value $\alpha_k(\sum_{i<k} \beta_i)$ for increasing $k$. This shows that the group operation can be checked by finite automaton. □

Conjecture 6.5. Show that $L_p$ is not FA presentable.

If only finitely many things do not commute with the others then it is not isomorphic to the group which we constructed. So it is a further example of such a group. The reason is quite easy: If $x_1, \ldots, x_k$ do not commute with each other and $y_1, y_2, \ldots$ commute all with each other but not with $x_1, \ldots, x_k$ then every basis has two things which commute.

Fact 6.6. The pro-$p$ completion $\hat{H}_p$ of $H_p$ is Büchi presentable.

Proof. Let $\hat{Q}$ be the pro-$p$ completion of $Q$, which is $C_p^\omega$ (cartesian power) viewed as a compact group. $\hat{H}_p$ is a central extension of $\hat{Q}$ by itself where the first $\hat{Q}$ has the topological generators $z_k$, the second has the $x_i$, and we have $[x_i, x_k] = z_k$ for $i < k$, as before.

An element $h$ of the group $\hat{H}_p$ is represented by a pair of infinite strings $\alpha, \gamma$ such that $h = [\alpha] \prod_i z_i^{\gamma_i}$. So we can use the same FA as above, but working on infinite strings, to check the group operation. □

Part 3. Computability theory and set theory

7. Greenberg and Nies: cardinal characteristics and computability

Nicholas Rupprecht, in his 2010 PhD thesis supervised by Blass [25], studied the connection of cardinal characteristics of the continuum in set theory and highness properties from computability theory. He developed this connection systematically as coming from a single source. Somewhat unfortunately, the thesis is somewhat technically written and the notation is hard to access, which may be a reason for the fact that no immediate reaction by either of the two communities to this unexpected connection resulted.

The cardinal characteristics given by null sets correspond to highness properties defined in terms of algorithmic randomness. For instance, $\text{non}(\mathcal{N})$, the least size of a non-null set of reals, corresponds to the strength of an oracle that computes a Schnorr test containing all recursive bit sequences. Rupprecht published a single paper mainly on the computability theoretic impact of that notion [26]. Rupprecht called this property “Schnorr covering”, which is confusing because it does not correspond to the characteristic
cov(\mathcal{N}) (the least size of a class of null sets with union \mathbb{R}); later on, Brendle et al. suggested the term “weakly Schnorr engulfing” [4]. We note that Rupprecht had attended the NSF FRG-funded school on computability and randomness at Gainesville in 2008, where Nies gave a tutorial on randomness; this may have prompted him to make the connection between measure theoretic cardinal invariants and randomness. He started a career as an investor shortly after his defence. According to LinkedIn his last job was as the Global Director of Operations at Virtu Financial. Blass mentioned he didn’t work there any longer. Apparently he has disappeared from view.

A quote from Brendle et al. [4]: the analogy occurred implicitly in the work of Terwijn and Zambella [31], who showed that being low for Schnorr tests is equivalent to being computably traceable. (These are lowness properties, saying the oracle is close to being computable; we obtain highness properties by taking complements.) This is the computability theoretic analog of a result by Bartoszyński [2] that the cofinality of the null sets (how many null sets does one need to cover all null sets?) equals the domination number for traceability. (Curiously, Terwijn and Zambella alluded to some connections with set theory in their work. However, it was not Bartoszyński’s work, but rather work on rapid filters by Raisonnier [24].) Their proof bears striking similarity to Bartoszyński’s; for instance, both proofs use measure-theoretic calculations involving independence. See also the book references [1, 2.3.9] and [17, 8.3.3]. (End quote.)

8. LEMPP, MILLER, NIES AND SOSKOVA: ANALOGS IN COMPUTABILITY OF COMBINATORIAL CARDINAL CHARACTERISTICS

Brief summary. Our basic objects are infinite sets of natural numbers. In set theory, the MAD number is the least cardinality of a maximal almost disjoint class of sets of natural numbers. The ultrafilter number is the least size of an ultrafilter base.

We study computability theoretic analogs of these cardinals. In our approach, all the basic objects will be infinite computable sets. A class of such basic objects is encoded as the set of “columns” of a set R, which allows us to study the Turing complexity of the class.

We show that each non-low oracle computes a MAD class R, give a finitary construction of a c.e. MAD set (compatible with permitting), and on the other hand show that a 1-generic below the halting problem does not compute a MAD class. We also provide some initial results on ultrafilter bases in this setting.

The original impetus for this work resulted from some interactions of Nies with Brendle, Greenberg and others at the August meeting on set theory of the reals at Casa Matemática Oaxaca in early August, which was organised by Hrusak, Brendle, and Steprans. The computability results reported below are due to Lempp, Miller, Nies and Soskova (in prep.), based on joint work that happened during Nies’ visit in Madison end of August, 2 weeks after the end of the CMO meeting. Downey’s visit in Madison in September further advanced it.
8.1. **Set theory.** Let \( b \) be the unbounding number (the least size of a class of functions on \( \omega \) that is not dominated by a single function).

Several cardinal characteristics are based on cardinals of subclasses of \([\omega]^\omega/\ast\), i.e. the infinite subsets of \( \omega \) under almost equality. One of them is called the almost disjointness number, denoted \( a \). This is the minimal size of a maximal almost disjoint (MAD) family of subsets of \( \omega \). The following is well known.

**Fact 8.1.** \( b \leq a \).

*Proof.* Suppose that the infinite cardinal \( \kappa \) is less than \( b \). Let \( \mathcal{V} \) be an almost disjoint family of size \( \kappa \). We show that \( \mathcal{V} \) is not MAD. Let \( \langle r_n \rangle, n \in \omega \), be a sequence of pairwise distinct elements of \( \mathcal{V} \). Given \( y \in \mathcal{V} \) let

\[
   f_y(n) = \max(y \cap r_n)
\]

with the convention that \( \max \emptyset = 0 \). Let \( g \) be an increasing function such that \( g(n) \in r_n \) for each \( n \) and \( \forall y \in \mathcal{V} \forall \omega n [g(n) > f_y(n)] \). It is clear that the family \( \mathcal{V} \cup \{\text{range}(g)\} \) is almost disjoint. \( \square \)

Other cardinal characteristics are based on properties of subclasses of the infinite sets modulo almost equality:

- the ultrafilter number \( u \) (the least size of a set with upward closure a free ultrafilter on \( \omega \)),
- the tower number \( t \) (the minimum size of a linearly ordered subset that can’t be extended by putting a new element below all given elements),
- the independence number \( i \)

and several others. See e.g. [3] or the talk by Diana Montoya at the CIRM meeting on descriptive set theory in June 2018. This talk contained a Hasse diagram of these cardinals with ZFC inequalities, reproduced here with permission:

![Hasse Diagram](image)

Note that \( r \) and \( s \) are the unreaping and splitting numbers, respectively. They are given by relations, and their analogs in computability have been studied [4]. Furthermore, \( e \) is the escaping number due to Brendle and Shelah. Its analog has recently been considered in computability theory by Ivan Ongay Valverde and Paul Tveite, two PhD students from Madison. Finally, \( h \) is the distributivity number. For another diagram see Soukup [30].

8.2. **Analogous mass problems in computability.** For a set \( F \),

\( F^{[n]} \) denotes the column \( \{x : \langle x, n \rangle \in F\} \).
We will also denote this by $F_n$ if no confusion is possible.

A systematic way to translate these characteristics into highness properties of oracles is as follows. We consider subclasses of $\text{Rec}^* \setminus \{0\}$, identified with classes of infinite recursive sets in the context of almost inclusion. Such a subclass $C$ is encoded by a set $F$ such that

$$C = C_F = \{F^n; \ n \in \mathbb{N}\}.$$

We view these classes as mass problems. We compare them via Muchnik reducibility $\leq_W$ and the stronger, uniform Medvedev reducibility $\leq_S$.

### 8.3. The mass problems $\mathcal{A}$ and $\mathcal{T}$.

We say that (the class $C_F$ described by) $F \subseteq \mathbb{N}$ is a **almost disjoint, AD in brief**, if each $F_n$ is infinite, and $F_n \cap F_k$ is finite for each $n \neq k$.

**Definition 8.2.** The mass problem $\mathcal{A}$ corresponding to the almost disjointness number $a$ is the class of sets $F$ such that $C_F$ is maximal almost disjoint (MAD) at the recursive sets. Namely, $C_F$ is AD, and for each infinite recursive set $R$ there is $n$ such that $R \cap F_n$ is infinite.

We need to get this out of the way:

**Fact 8.3.** No MAD set $F$ is computable.

**Proof.** Suppose $F$ is AD and computable. Let $r_{-1} = 0$, and $r_n$ be the least number $r > r_{n-1}$ such that $r \in F_n - \bigcup_{i < n} F_i$. Then the computable set $R = \{r_0, r_1, \ldots\}$ shows that $F$ is not MAD. \qed

**Definition 8.4.** We say that $G \subseteq \mathbb{N}$ is a **tower** (or $C_G$ is a tower) if for each $n$ we have $G_{n+1} \subseteq^* G_n$ and $G_n - G_{n+1}$ is infinite.

By a function associated to $G$ we mean an increasing function $p$ satisfying the conditions $p(n) \in \bigcap_{i \leq n} G_i$.

**Definition 8.5.** The mass problem $\mathcal{T}$ corresponding to the tower number $t$ is the class of sets $G$ such that $C_G$ is a tower that is maximal in the recursive sets. Namely, for each infinite recursive set $R$ there is $n$ such that $R \cap G_n$ is infinite.

**Fact 8.6.** No tower is c.e.

Otherwise there is a computable associated function $p$. The range of $p$ would extend the tower.

**Fact 8.7.** $\mathcal{A} \leq_S \mathcal{T} \leq_S \mathcal{A}$.

**Proof.** To check that $\mathcal{A} \leq_S \mathcal{T}$, given a set $G$ let $\text{Diff}(G)$ be the set $D$ such that $D_n = G_n - G_{n+1}$. Clearly the operator $\text{Diff}$ can be seen as a Turing functional. If $G$ is a maximal tower then $D = \text{Diff}(G)$ is MAD. For, if $R$ is infinite recursive then $R \cap G_n$ is infinite for some $n$, and hence $R \cap D_i$ is infinite for some $i < n$.

For $\mathcal{T} \leq_S \mathcal{A}$, given a set $F$ let $G = \text{Cp}(F)$ be the set such that

$$x \in G_n \iff \forall i < n [x \notin F_n].$$

Again Cp is a Turing functional. If $F$ is AD then $G$ is a tower, and if $F$ is MAD then $G$ is a maximal tower. \qed

Recall that a characteristic index for a set $M$ is an $e$ such that $\chi_M = \phi_e$. 

Proposition 8.8. Suppose $F \in A$. While each $F_n$ is computable, $\emptyset'$ is not able to compute, from input $n$, a characteristic index for $F_n$.

Proof. Assume the contrary. Then there is a computable function $f$ such that $\phi_{\lim s f(n,s)}$ is the characteristic function of $F_n$.

Let $\hat{F}$ be defined as follows. Given $n, x$ compute the least $s > x$ such that $\phi_{f(n,s),s}(x) \downarrow$. If the value is not 0 put $x$ into $\hat{F}_n$.

Clearly $\hat{F}$ is computable and $F_n =^* \hat{F}_n$ for each $n$. So $\hat{F}$ is MAD, contrary to Fact 8.3. □

This research started at the CMO by looking at an analog of $b \leq a$. This isn’t quite the right approach as we will see. At least, it is misleading.

The analog of $b$ is the property of an oracle $C$ to be high, namely $\emptyset'' \leq_T C'$; for detail on this see [26, 4]. (To see this as a mass problem, take the functions eventually dominating all computable functions, instead.) We want to show that an analog of the relationship $b \leq a$ holds: every high oracle computes a set in $A$. The inversion that occurs here is part of the generally accepted setup of analogy between the two areas of logic. One can show that each high oracle $C$ computes a set in $A$. The proof is a straightforward application of a dominating function computed by $C$.

Lempp, Miller, Nies and Soskova (in prep.) have strengthened the result: each non-low oracle computes a set in $T$, and hence in $A$. The result is uniform in the sense of mass-problems. Let NonLow denote the class of oracles $X$ such that $X' \not\leq_T \emptyset'$.

Theorem 8.9. $T \leq_S$ NonLow.

Proof. In the following $x, y, z$ denote binary strings; we identify $x$ with the number $1x$ via the binary expansion. Define a Turing functional $\Phi$ for the Medvedev reduction, as follows. $\Phi^Z = G$, where for each $n$

$$G_n = \{x : |x| = s \geq n \land Z'_s |n = x |n\}.$$  

It is clear that for each $n$ we have $G_{n+1} \subseteq^* G_n$ and $G_n - G_{n+1}$ is infinite. Also, for each $n$, for large $s$ the string $Z'_s |n$ settles, so $G_n$ is computable.

Suppose now that $R$ is an infinite set such that $R \subseteq^* G_n$ for each $n$. Then $Z(k) = \lim_{x \in G, |x| > k} x(k)$, and hence $Z' \leq_T R'$. So if $Z \in$ NonLow then $R$ cannot be computable, and hence $\Phi^Z \in T$. □

8.4. $\Delta^0_2$ and 1-generic sets, totally low sets, and not computing a MAD. We provide a type of sets which doesn’t compute a MAD set. Joe Miller and Mariya Soskova showed (during the weekend after Andre had left) that if $L$ is $\Delta^0_2$ and 1-generic, then $L$ does not compute a MAD family. Downey’s visit in Madison before the Lempp ’60 conference brought another interesting lowness property into the game: totally low oracles. This property, derived from Joe’s and Mariya’s proof, is in between 1-generic $\Delta^0_2$, and not computing a MAD. No separation is known at present.

Note that for a $\Delta^0_2$ oracle $L$, $\emptyset'$ can compute from $e$ an index for $\Phi^L_e$, provided that it’s computable. To be totally low means that $\emptyset'$ suffices.

Definition 8.10. We call an oracle $L$ totally low if $\emptyset'$ can compute from $e$ an index for $\Phi^L_e$, whenever $\Phi^L_e$ is computable. In other words, there is a functional $\Psi$ a such that
\( \Phi_e^L \) computable \( \Rightarrow \) \( \Psi(\emptyset'; e) \) is an index for it.

No assumption is made on the convergence of \( \Psi(\emptyset'; e) \) in case \( \Phi_e^L \) is not a computable function. A total function is computable iff its graph is. A moment’s thought using this fact shows that we may restrict ourselves to \( e \) such that \( \Phi_e^L \) is \( \{0,1\} \)-valued. Clearly, being totally low is closed downward under \( \leq_T \).

By Prop 8.8, a totally low oracle \( D \) does not compute a MAD set. In particular by Theorem 8.9, \( D \) is low (as the name suggests)

**Theorem 8.11.** Suppose \( L \) is \( \Delta^0_2 \) and 1-generic. Then \( L \) is totally low.

**Proof.** Suppose \( F = \Phi_e^L \) and \( F \) is a computable set. Let

\[
S_e = \{ \sigma: (\exists \tau_1 \succeq \sigma)(\exists \tau_2 \succeq \sigma)(\exists p)(\Phi^\tau_e(p) \neq \Phi_e^\tau(p)) \}.
\]

Suppose that \( S_e \) is dense along \( L \). We claim that the set

\[
C_e = \{ \tau: (\exists p)(\Phi^\tau_e(p) \neq F(p)) \}
\]

is also dense along \( L \): i.e. for every \( k \) there is some \( \tau \succeq L \upharpoonright k \) such that \( \tau \in C_e \). Indeed, let \( \sigma \succeq L \upharpoonright k \) be a member of \( S_e \) and let \( \tau_1 \) and \( \tau_2 \) and \( p \) witness that. Let \( \tau_i \) where \( i = 1 \) or \( 2 \) be such that \( \Phi^\tau_e(p) \neq F(p) \). Then \( \tau_i \succeq L \upharpoonright k \) is in \( C_e \). The set \( C_e \) is c.e. and hence \( L \) meets \( C_e \), contradicting our assumption that \( F = \Phi_e^L \).

It follows that \( S_e \) is not dense along \( L \). In other words, there is some least \( k_e \) such that there is no splitting of \( \Phi_e \) above \( L \upharpoonright k_e \). On input \( e \) the oracle \( \emptyset' \) can compute \( k_e \) and \( L \upharpoonright k_e \). This allows \( \emptyset' \) to find an index for \( F \), given by the following procedure. To compute \( F(p) \), find the least \( \tau \succeq L \upharpoonright k_e \) such that \( \Phi^\tau_e(p) \downarrow \) (in \( |\tau| \) many steps). Such a \( \tau \) exists because \( \Phi^\tau_e(p) \downarrow \). By our choice of \( k_e \) it follows that \( \Phi^\tau_e(p) = \Phi_e^\tau(p) = F(m) \).

By EX one denotes the class of computable functions \( f \) so that an index can be learned in the limit: there is a machine \( M \) taking values \( f(0), f(1), \ldots \) such that \( \lim_m M(f(0), \ldots, f(s)) \) exists and is an index for \( M \). To relativize this to oracle \( X \), allow \( M \) access to \( X \) (but the functions are still computable). Recall that \( A \) is low for EX learning if \( \text{EX}^A = \text{EX} \). Slaman and Solovoy [29] showed that

\( A \) is low for EX-learning \( \iff A \) is \( \Delta^0_2 \) and 1-generic.

We summarize the known implications of lowness notions.

\( \Delta^0_2 \) and 1-generic \( \Rightarrow \) Totally low \( \Rightarrow \) computes no MAD \( \Rightarrow \) low.

The last arrow doesn’t reverse by what follows; the others might.

8.5. C.e. MAD sets. It may come as a surprise that a set in \( A \) can be c.e., and in fact can be built by a priority construction with finitary requirements, akin to Post’s construction of a simple set. Such a construction is compatible with permitting. The result to follow is part of the joint work mentioned above.

**Theorem 8.12.** For each incomputable c.e. set \( A \), there is a MAD c.e. set \( R \leq_T A \).
Proof. Let \( \langle V_e \rangle_{e \in \mathbb{N}} \) be a u.c.e. sequence of sets such that \( V_{2e} = W_e \) and \( V_{2e+1} = \mathbb{N} \) for each \( e \). We will build an auxiliary c.e. set \( S \leq_T A \) and let the c.e. set \( R \leq_T A \) be defined by \( R^{[e]} = S^{[2e]} \cup S^{[2e+1]} \). The role of the \( V_{2e+1} \) is to make the sets \( S^{[2e+1]} \), and hence the \( R^{[e]} \) infinite. The construction also ensures that \( S \), and hence \( R \), is AD, and that \( \bigcup_n S^{[n]} \) is cofinite.

We will write \( S_e \) for \( S^{[e]} \). We provide a stage-by-stage construction to meet the requirements \( P_n, n = (e, k) \) given by

\[
P_n : V_e - \bigcup_{i < n} S_i \text{ infinite } \Rightarrow |S_e \cap V_e| \geq k.
\]

At stage \( s \) we say that \( P_n \) is satisfied if \( |S_{e,s} \cap V_{e,s}| \geq k \).

Construction.

Stage \( s > 0 \). For each \( n < s \) such that \( P_n \) is not satisfied, \( n = (e, k) \), if there is \( x \in V_{e,s} - \bigcup_{i < n} S_{i,s} \) such that \( x > \max(S_{e,s-1},) \), \( x \geq 2n \) and \( A_s |x \neq A_{s-1} |x \), then put \( (x, e) \) into \( S \) (i.e., put \( x \) into \( S_e \)).

Verification. Each \( S_e \) is computable, being enumerated in an increasing fashion.

Each \( P_n \) is active at most once. This ensures that \( \bigcup_e S_e \) is cofinite: for each \( N \), if \( x < 2N \) enters this union then this is due to the action of a requirement \( P_n \) with \( n \leq N \), so there are at most \( N \) many such \( x \).

To see that a requirement \( P_n, n = (e, k) \), is met, suppose that its hypothesis holds. Then there are potentially infinitely many candidates \( x \) that can go into \( S_e \). Since \( A \) is incomputable, one of them will be permitted.

Now, by the choice of \( V_{2e+1} \), each \( S_{2e+1} \), and hence each \( R_e \) is infinite. By construction, for \( e < m \) we have \( |S_e \cap S_m| \leq m \). So the family described by \( S \) and therefore also the one described by \( R \) is almost disjoint.

To show \( R \) is MAD, it suffices to verify that if \( V_e \) is infinite then \( V_e \cap R_p \) is infinite for some \( p \). If all the \( P_{e,k} \) are satisfied during the construction then we let \( p = e \). Otherwise let \( k \) be least such that \( P_n \) is never satisfied where \( n = (e, k) \). Then its hypothesis fails, so \( V_e \subseteq^* \bigcup_{i < n} S_i \).

By the argument for the reduction \( T \leq_S A \) in Fact 8.7, we obtain

**Corollary 8.13.** For each incomputable c.e. set \( A \), there is a co-c.e. set \( G \leq_T A \) such that \( G \) describes a maximal tower, i.e. \( G \in T \).

Namely, \( G = \text{Cp}(A) \) i.e.,

\[
x \in G_n \iff \forall i < n [x \notin A_n].
\]

**Fact 8.14.** From a co-c.e. tower \( G \) one can uniformly compute an infinite co-c.e. set \( B \) such that \( B \subseteq^* G_n \) for each \( n \). Moreover, the index for \( B \) is also obtained uniformly.

**Proof.** We may assume that \( G_n \supseteq G_{n+1} \) for each \( n \). Let \( \gamma_{n,s} \) be the \( n \)-th element of \( G_{n,s} \). Clearly \( \gamma_{n,s} \) is monotonic in \( s \) and strictly monotonic in \( n \). Let \( \gamma_n = \lim_s \gamma_{n,s} \). The set \( B = \{ \gamma_n : n \in \omega \} \) is as required.

Since a totally low set computes no MAD set, we obtain:

**Corollary 8.15.** No incomputable, c.e. set \( L \) is totally low.
Downey and Nies in October (during Nies’ Wellington visit where Nies gave a seminar talk about this stuff) have given a direct proof of this. Assume for contradiction that \( L \) is incomputable, c.e., and totally low. Define Turing functionals \( \Gamma_e \), uniformly in \( e \). By the recursion theorem, we know in advance a computable function \( p \) such \( \Gamma_e^X(n) = \Phi_{p(e)}(n) \) for each \( X, n \).

We meet for each \( e \) the requirement

\[ P_e: \Gamma_e^L \text{ is computable but } \Phi_{\emptyset'}(p(e)) \text{ is not an index for it.} \]

Write \( \alpha(e, s) \) for \( \Phi_{\emptyset'}(p(e))[s] \).

**Construction.** Assign an increasing sequence of followers \( x_e \in \omega^e \) to \( P_e \).

Initially we set \( \Gamma_e^L(x_e) = 0 \) with use \( x_e \). Once \( \phi_{\alpha(e, s)}(x_e) = 0 \), we call \( x_e \) certified for \( e \). If \( L \) at a certain stage \( t \) permits a certified \( x_e \) that is larger than any follower that is fulfilled, we change the value \( \Gamma_e^L(x_e) \) to 1, and call \( x_e \) fulfilled.

**Verification.** Since followers are enumerated in an increasing fashion, \( \Gamma_e^L \) is computable. There is a final value \( \lim s \alpha(e, s) \), which hence is an index for \( \Gamma_e^L \). After \( \alpha(e, s) \) has stabilized to a value \( v \), infinitely many followers get certified. So \( L \) permits at a stage \( t \) a certified follower for \( P_e \). This follower gets fulfilled at stage \( t \), which causes the index \( v \) to become incorrect.

**Contradiction.**

8.6. The mass problem \( U \) corresponding to the ultrafilter number, and a strict reduction \( T <_S U \).

**Definition 8.16.** Let \( U \) be the class of sets \( F \) such that \( F \) is tower as in Def. 8.4, and \( \mathcal{C}_F \) is an ultrafilter base within the recursive sets: for each recursive set \( R \) there is \( n \) such that \( F_n \subseteq^* R \) or \( F_n \subseteq^* \bar{R} \).

Since \( U \subseteq T \) we trivially have \( T \leq_S U \) via the identity reduction. Since our properties of (codes for) classes are arithmetical, each of the mass problems introduced contains an element computable from \( \emptyset^{(n)} \) for sufficiently large \( n \). For instance, \( \emptyset'' \) computes an oracle in \( U \). To see this, take any \( r \)-cohesive set \( C \). By definition of \( r \)-cohesiveness, the recursive sets \( R \) such that \( C \subseteq^* R \) is cofinite form an ultrafilter. If e.g. \( \overline{C} \) is \( r \)-maximal, we can write this ultrafilter as \( \mathcal{C}_F \) for some \( F \leq_T \emptyset'' \).

For the next fact we follow the upcoming work mentioned above due to Lempp, Miller, Nies and Soskova. In the cardinal setting we have \( t \leq u \), which suggests that each maximal tower should compute an ultrafilter base. But this is not true by the following.

**Proposition 8.17.** No ultrafilter base \( R \) is computably dominated.

**Proof.** Let \( p(n) \) be an associated function as in Def. 8.4, i.e. an increasing function \( p \) such that \( p(n) \in \bigcap_{i \leq n} R_i \). Then \( p \leq_T R \). Assume that there is a computable function \( f \geq p \). The conditions \( n_0 = 1 \) and \( n_{k+1} = f(n_k) \) define a computable sequence. So the set

\[ E = \bigcup_i [n_{2i}, n_{2i+1}) \]

is computable. Clearly \( R_n \not\subseteq^* E \) and \( R_n \not\subseteq^* \overline{E} \) for each \( n \). So \( R \) is not an ultrafilter base. 

\[ \square \]
On the other hand, each non-low set computes a set in $\mathcal{A} \equiv_\mathcal{T}$ as shown above, and of course a computably dominated oracle can be non-low.

8.7. Co-c.e. ultrafilter bases. Recall that no tower, and hence no ultrafilter base, is c.e.

**Fact 8.18.** Every co-c.e. ultrafilter base $G$ is high.

**Proof.** Let $B$ be a co-c.e. set as obtained in Fact 8.14. Clearly $B \subseteq^* R$ or $B \subseteq^* \overline{R}$ for each computable $R$. Hence $B$ is $r$-maximal, and therefore high. Since $B \leq_T G$, $G$ is also high. □

Let us say for the moment that a tower $F$ is a ultrafilter base for c.e. sets if for each computably enumerable set $R$ we have $F_n \subseteq^* R$ or $F_n \subseteq^* \overline{R}$. By Fact 8.14 there is no co-c.e. ultrafilter base for c.e. sets. To build a co-c.e. ultrafilter base we need to make use of the fact that we are given c.e. index for a computable set and also one for its complement.

**Theorem 8.19.** There is a co-c.e. ultrafilter base $F$.

**Proof.** We build the co-c.e. tower $F$ by providing uniformly co-c.e. sets $F_e$, $e \in \mathbb{N}$ that form a descending sequence: $F_e \supset F_{e+1}$. If we remove $x$ from $F_e$ at a stage $s$ we also remove it from all $F_i$ for $i > e$, without further mention.

Let $\langle (V_{e,0}, V_{e,1}) \rangle$ be an effective listing of the pairs of disjoint c.e. sets. The construction will ensure that the following requirements are met.

$M_e: F_e \setminus F_{e+1}$ is infinite.

$P_e: V_{e,0} \cup V_{e,1} = \mathbb{N} \Rightarrow F_{e+1} \subseteq^* V_{e,0} \vee F_{e+1} \subseteq^* V_{e,1}$.

This suffices to establish that $F$ is an ultrafilter base.

The tree of strategies is $T = \{0, 1, 2\}^{<\omega}$. Each string $\alpha \in T$ of length $e$ is associated with $M_e$ and also $P_e$. We write $\alpha: M_e$ and $\alpha: P_e$ to indicate that we view $\alpha$ as a strategy of the respective type.

**Streaming.** For each string $\alpha \in T$, $|\alpha| = e$, at each stage of the construction we have a set $S_\alpha$, thought of as a stream of numbers used by $\alpha$. Each time $\alpha$ is initialized, $S_\alpha$ is made empty and its content removed from $F_{e+1}$. Also, $S_\alpha$ is enlarged only at stages at which $\alpha$ appears to be on the true path. We will verify the following properties.

1. $S_\emptyset = \omega$;
2. if $\alpha$ is not the empty node then $S_\alpha$ is a subset of $S_{\alpha^-}$ (where $\alpha^-$ is the immediate predecessor of $\alpha$);
3. At every stage $S_\alpha \cap S_\beta = \emptyset$ for incomparable strings $\gamma$ and $\beta$;
4. at the time a number $x$ first enters $S_\alpha$, $x$ is in $F_{e+1}$; and
5. if $\alpha$ is along the true path of the construction then $S_\alpha$ is an infinite computable set.

Thus $S_\alpha$ can be thought of as a set d. c. e. uniformly in $\alpha$; $S_\alpha$ is finite if $\alpha$ is to the left of the true path of the construction; an infinite computable set if $S_\alpha$ is along the true path; and empty if $\alpha$ is to the right of the true path.

**Construction.**

Stage 0. Let $\delta_0$ be the empty string. Let $F_e = \mathbb{N}$ for each $e$. 
Stage $s+1$. Let $S_{0,s} = [0,s)$.

Substage $e < s$. We suppose that $\alpha = \delta_{s+1} \upharpoonright e$ has been defined.

Strategy $\alpha$: $M_e$ removes every other element of $S_\alpha$ from $F_{e+1}$. Let $S'_\alpha$ denote the set of remaining numbers. More precisely, if we have $S_\alpha = \{r_0 < \ldots < r_k\}$, then $S'_\alpha$ contains the numbers of the form $r_{2i}$ while the numbers of the form $r_{2i+1}$ are removed from $F_{e+1}$.

Strategy $\alpha$: $P_e$ picks the first applicable case below.

1. Each reserved number of $\alpha$ has been processed.
   - If there is a number $x$ from $S'_\alpha$ greater than $\alpha$’s last reserved number (if any) and greater than $s_0$, pick $x$ least and reserve it. Initialize $\alpha \uparrow 2$. Let $\alpha \uparrow 2$ be eligible to act next.
   - If (1) doesn’t apply then $\alpha$ has a reserved, unprocessed number $x$.

2. $[0,x] \subseteq V_{e,0} \cup V_{e,1}$ and $x \in V_{e,0}$.
   - Let $s_0$ be greatest stage $< s$ at which $\alpha$ was initialized. Add $x$ to $S_a$ and remove from $F_{e+1}$ all numbers in the interval $(s_0,x)$ which are not in $S_a$. Declare that $\alpha$ has processed $x$. Let $\alpha \uparrow 0$ be eligible to act next.

3. $[0,x] \subseteq V_{e,0} \cup V_{e,1}$ and $x \in V_{e,1}$.
   - Let $s_0$ be greatest stage $< s$ at which $\alpha$ was initialized or $\alpha \uparrow 0$ was eligible to act. Add $x$ to $S_a$ and remove from $F_{e+1}$ all numbers in the interval $(s_0,x)$ which are not in $S_a$. Declare that $\alpha$ has processed $x$. Let $\alpha \uparrow 1$ be eligible to act next.

4. Otherwise.
   - Let $s_0$ be greatest stage $< s$ at which $\alpha$ was initialized or $\alpha \uparrow 2$ was eligible to act. Let $S_{a2} = S'_\alpha \cap (s_0,s)$. Let $\alpha \uparrow 2$ be eligible to act next.

We define $\delta_{s+1}(e) = i$ where $\alpha \uparrow i$, $0 \leq i \leq 2$, has been declared eligible to act next.

Verification. By construction and our convention above, $F_e \supseteq F_{e+1}$ for each $e$, and $F$ is co-c.e.

Let $g \in 2^\omega$ denote the true path, namely the leftmost path in $\{0,1,2\}^\omega$ such that $\forall e \exists s \exists e \uparrow \delta_s$. In the following, given $e$ let $\alpha = g \upharpoonright e$, and let $s_\alpha$ be the largest stage $s$ such that $\alpha$ is initialized at stage $s$. We verify a number of claims.

Claim 8.20. The “streaming” properties (1)-(5) hold.

(1,2) hold by construction.

(3) Assume this fails for incomparable $\gamma, \beta$, so $x \in S_\gamma \cap S_\beta$ at stage $s$. We may as well assume that $\gamma = \alpha \uparrow i$ and $\beta = \alpha \uparrow k$ where $i < k$. By construction $k \leq 1$ is not possible, so $k = 2$. Since $x \in S_\alpha \uparrow i$ adn $i \leq 1$, $x$ was reserved by $\alpha$ at some stage $t < s$. So $x$ can never go into $S_\alpha \uparrow 2$ by the initialization of $\alpha \uparrow 2$ when $x$ was reserved.

(4) is true by construction.

(5) holds by definition of the true path, and because $S_\alpha$ is enumerated in increasing fashion at stages $\geq s_\alpha$.

Claim 8.21. (i) $F_e \subseteq S_\alpha$. (ii) $S_\alpha \subseteq F_e$. 
The claim is verified by induction on $e$. It holds for $e = 0$ because $F_0 = S_0 = \mathbb{N}$. Suppose the claim is true for $e$. To verify it for $e + 1$, let $\gamma = g \upharpoonright_{e+1}$, and let $s_\gamma$ be the largest stage $s$ such that $\gamma$ is initialized at stage $s$.

(i) Suppose $x \in F_{e+1}$. Then $x \in F_e$, so inductively $x \in S_\alpha$ for almost all such $x$. By construction, any element $x$ that isn’t promoted to $S_\gamma$ is also removed from $F_{e+1}$, unless $x$ is the last element $\alpha$ reserves. However, in that case necessarily $\gamma = \alpha \hat{2}$, so this leads to at most one new element in $F_{e+1} \setminus S_\gamma$.

(ii) Suppose $x \in S_\gamma$. Then $x \in S_\alpha$, so inductively $x \in F_e$ for almost all such $x$. At stage $s \geq s_\gamma$ an element $x$ of $S_\alpha$ can not be removed from $F_{e+1}$ by a strategy $\beta >_L \alpha$ because $S_\beta \cap S_\alpha = \emptyset$ because of (3) verified above, and since $\beta$ can only remove elements from $S_\beta$. So $x$ can only be removed by a strategy tied to $\alpha$.

If $\alpha : M_e$ removes $x$ from $F_{e+1}$, then $x \not\in S_\alpha$, contradiction. So by construction the only way $x$ can be removed from $F_{e+1}$ is by $\alpha : P_e$, which for a sufficiently large $x$ means that $x$ does not get promoted to $S_\gamma$ either.

Claim 8.22. $M_e$ is met, namely, $F_e \setminus F_{e+1}$ is infinite.

To see this, recall $\alpha = g \upharpoonright_e$. By the foregoing claim, the action of $\alpha : M_e$ removes infinitely many elements of $S_\alpha \subseteq F_e$ from $F_{e+1}$.

Claim 8.23. $P_e$ is met.

Suppose the hypothesis of $P_e$ holds. Then every number $\alpha$ reserves eventually gets processed. So either $g(e) = 0$, in which case $F_{e+1} \subseteq^* V_{e,0}$ by Claim 8.21, or $g(e) = 1$, in which case $F_{e+1} \subseteq^* V_{e,1}$ by Claim 8.21.

Added Jan. 2020: Lempp, Miller, Nies and Soskova (in prep.) have shown that both $\mathcal{U}$ and the analog of the independence number are Medvedev equivalent to the mass problem of dominating functions. The paper will be posted.

9. **Yu Liang**

9.1. **Concerning the Joint coding theorem: a handwritten proof by Harrington.** The following theorem has appeared in [5].

**Theorem 9.1.** Given two uncountable $\Sigma^1_1$-sets $A_0, A_1$, for any real $z$, there are reals $x_0 \in A_0$ and $x_1 \in A_1$ so that $x_0 \oplus x_1 \equiv_h z \oplus \mathcal{O}$.

On 11/July/2019, Steve Simpson sent us a manuscript by Harrington dated on 9/1975 in which Theorem 9.1 was already proved. Harrington’s proof is model theoretic and certainly deserves to be studied in detail. The manuscript follows.
The hyperdegrees of reals in products of uncountable $\Sigma_1^1$ sets

Leo Harrington

Theorem Let $A_0, A_1$ be two uncountable $\Sigma_1^1$ sets of reals (real = subset of $\omega$). For all reals $b \geq_{\text{hyp}} \emptyset$ there exist reals $a_0, a_1$ s.t. $a_0 \in A_0, a_1 \in A_1$ and $b \equiv_{\text{hyp}} \langle a_0, a_1 \rangle$.

Proof

Let $\omega_1 = 1^{st}$ non-recursive ordinal.

Let $B$ be the smallest collection of sentences s.t.

"$j \in x$" and "$j \notin x$" are in $B$ for each $j \in \omega$;

if $X \in B$ then $X \in L_{\omega_1}$ then $WX$ and $M X$ are both in $B$.

For a a real $b$ for $\varphi \in B$, $a \equiv_{\text{hyp}} \varphi$ iff $\varphi$ is true when $a$ is substituted for $x$ in $\varphi$.

By making $A_0 \cap A_1$ smaller if necessary we get that $a \in A_\varepsilon$ ($\varepsilon = 0$ or 1) $\Rightarrow$

$a$ is not hyp $b \circ \neq_{\text{hyp}} a$. 

9/75
Thus we can find (by the Specter-Gandy Theorem) \( T_0, T_1 \leq \text{O3} \) both \( \Sigma_1 \) over \( \mathbb{L} \) s.t.

for all real \( a \), \( a \in A_\varepsilon \) (\( \varepsilon = 0 \) or 1) 
iff: \( a \perp T_\varepsilon \) and \( \mathbb{L} A_\varepsilon \) is admissible.
Also \( T_\varepsilon \vdash \neg \text{I is not hyp} \).

View integers \( e \) as indexing \( \Sigma_1 \) over \( \mathbb{L} \) 
subset \( \mathcal{U} (e) \), of \( \mathbb{B} \).

Wma that this indexing is s.t.
for all \( \langle \nu_i \rangle \) wu, a \( \Sigma_1 \) over \( \mathbb{L} \) 
enumeration of \( \Sigma_1 \) over \( \mathbb{L} \) subset of 
\( \mathbb{B} \), there is \( e \in w \) s.t.
\( \forall i \exists \mathcal{U} (\langle e, i \rangle) = \nu_i \).

also wma that!

given \( e_1, \ldots, e_r \) we can effectively find 
\( e_1 \wedge \ldots \wedge e_r \) s.t. 
\( \mathcal{U} (e_1 \wedge \ldots \wedge e_r) = \mathcal{U} (e_1) \cup \ldots \cup \mathcal{U} (e_r) \).

Let \( C \) be a \( \Sigma_1 \) formula s.t. for \( e \in w \),
\( C(e) \equiv \exists X \exists W \exists P (w \text{ witnesses that } X \subseteq \mathcal{U}(e) \) 
and \( P \) is a proof of a contradiction from \( X \).
Let $f : \mathcal{A} \rightarrow \mathcal{A}_1$ be the word onto func. For $\varepsilon = 0$ or 1, let $\bar{\varepsilon} = 1 - \varepsilon$.

For each $\eta \in \mathcal{A}_w^w$, each $x \in \mathcal{A}_1$ and each $\varepsilon$, we will (attempt to) define an integer $d(\eta, x, \varepsilon)$ as follows:

If $\eta = \emptyset$ then let $d(\eta, x, \varepsilon) = \text{def. \, \varepsilon(\emptyset)}$ (where $\varepsilon(\emptyset)$ is some previously chosen index for $\emptyset$).

If $\eta = \tau \in \mathcal{A}_w$ and if $\varepsilon \neq \varepsilon(\tau)$ has been defined, then:

View $d(\tau, x, \varepsilon)$ coding a pair $<\varepsilon, e>$. Let $\Pi(\eta, x) = 1^* \text{ new } s.t.$

Let $\Pi(\eta, x) = 1^* \text{ new } s.t.$

If there is no such $n$, let $\Pi(\eta, x) = \infty$.

[Def. For $\varepsilon \in 2^w$ and for $e \in \mathcal{A}_w$, let $e \circ \varepsilon$ be an index for $\{ t(e) \mid t(e) = \varepsilon \} \cup \{ \neg j \in \mathcal{A}^w \mid \tau(j) = 1 \}$. (i.e., $\Pi(e \circ \varepsilon) = 1^*$.)]
we define a real \( c(t, \alpha) \) by:

\[
c(t, \alpha)(k) =
\begin{cases}
0 & \text{if } L_2 \not\in C(d(t, \alpha, \varepsilon) \circ (\overline{c(t, \alpha)}(k) \circ 0)) \\
1 & \text{o.w.}
\end{cases}
\]

Since \( L_2 \not\in C(d(t, \alpha, \varepsilon)) \), we have,

\[
\forall k \quad L_2 \not\in C(d(t, \alpha, \varepsilon) \circ (\overline{c(t, \alpha)}(k) \circ 0)).
\]

We call \( \overline{c(t, \alpha)}(k) \) a branching pt. of \( c(t, \alpha) \) if:

\[
L_2 \not\in C(d(t, \alpha, \varepsilon) \circ (\overline{c(t, \alpha)}(k) \circ 0))
\]

and \( L_2 \not\in C(d(t, \alpha, \varepsilon) \circ (\overline{c(t, \alpha)}(k) \circ 1)) \).

For \( i \in \mathbb{N} \), let \( t_i(t, \alpha) \) = the \( i^{th} \) branching pt. of \( c(t, \alpha) \) (if it exists).

Let \( \beta(t, \alpha) = 1^{st} \) ord \( \leq \alpha \) s.t. \( \forall i < \beta(t, \alpha) \).

\[
L_{\beta(t, \alpha)} = C(d(t, \alpha, \varepsilon) \times \langle \varepsilon, n_7 \rangle).
\]

Let \( p(t, \alpha) = 1^{st} p \) in \( \mathcal{O} \) s.t. \( f(p) = \beta(t, \alpha) \).

let \( q(t, \alpha) = \langle p(t, \alpha), m \rangle \).
If $\beta(T, x) \neq x$, or if $g(T, x, 2) \in \omega$ is not defined, then leave $d(T, x, 0) \& d(T, x, 1)$ undefined.

Otherwise, let $d(T, x, 2) = d(T, x, 3) \setminus \langle c, \varepsilon(T, x) \rangle$, and let $d(T, x, 3) = d(T, x, 3) \setminus d(T, x, 2).

(Notation: For $z \leq \omega_1$, let $U^x(z) = \{ z \in B \mid z \cup x \subseteq U(z) \}.$)

**Facts**

1. $d(T, x, 2)$ is defined for all $z \notin \varepsilon.$
2. If $d(T, x, 2)$ is defined, then:
   
   $\varepsilon(T, x, 2)$ is defined to $\varepsilon(T, x, 2) = d(T, x, 2) \setminus d(T, x, 3).$

And: $L_z \not\subseteq C(d(T, x, 2)).$

3. If $\eta$ is incomparable, and if $T \in d(T, x, 2) \cup d(T, x, 3)$ are defined, then:

   Either $U^x(d(T, x, 0)) \cup U^x(d(T, x, 0))$
   
   Or $U^x(d(T, x, 1)) \cup U^x(d(T, x, 1))$

   is inconsistent.

4. If $x \leq \omega_1$ is a limit ordinal, and
if \( \theta' \) is defined, then:

2. For all large enough \( \lambda' < \lambda \), \( \theta' \) is defined and 
\[
d(\lambda', \lambda, \varepsilon) = d(\lambda', \lambda, \varepsilon),
\]

Furthermore, for all \( K \in \omega \)

6. For all large enough \( \lambda' < \lambda \)

\[
C(\lambda', \lambda')(K) = C(\lambda, \lambda')(K), \quad \text{and}
\]

\[
C(\lambda', \lambda')(K) \text{ is a branching pt. of } C(\lambda', \lambda'),
\]

iff it \( \theta' \) is a branching pt. of \( C(\lambda, \lambda') \).

Proof of 4: 1 is obvious for \( \tau = \emptyset \).

Assume 2 is true for \( \tau \). We will then prove 6 for \( \tau \), and then prove 2 for \( \varepsilon = \tau \). Assume \( \ell(\tau) \text{ codo}(\varepsilon, \varepsilon) \)

6: For all \( n = 0, 1 \), \( \forall \mu \in 2^{<\omega} \) s.t. \( \ell(\mu) \leq n \)

if \( L_\lambda = C(\theta, \lambda, \varepsilon) \circ (\mu \downarrow n) \), then

\[
L_\lambda = C(\theta, \lambda, \varepsilon) \circ (\mu \downarrow n).
\]

But there are only finitely many such \( \mu, \downarrow n \), and for all (enough large \( \lambda' < \lambda \))

\[
d(\lambda', \lambda, \varepsilon) = d(\lambda', \lambda, \varepsilon) \quad (\text{by 4} \text{ for } \tau),
\]

thus
for all large enough \( \alpha < \lambda \), \( \theta \in 0 \cup \omega \) s.t. \( \theta(\mu) \leq \lambda \),
\( L_{\lambda} \in C(d(\tau, x, \bar{e}) \circ (m \circ \nu)) \) if \( \theta \in 0 \cup \omega \) s.t. \( \theta(\mu) \leq \lambda \).
This implies (4) (5).

(3) for \( \tau = \tau^m \):
assume both \( d(\tau, x, \bar{e}) \) and \( d(\tau, \lambda, \bar{e}) \) are defined.
Thus \( \beta(\tau, x) < \lambda \).
So for all large enough \( \alpha < \lambda \), \( \theta(x) < \lambda \),
\( L_{\lambda} \in C(d(\tau, x, \bar{e}) < e, \tilde{\alpha}(\tau, x)) \).
Since \( L_{\lambda} \in C(d(\tau, x, \bar{e}) < e, \tilde{\alpha}(\tau, x)) \),
we have \( \hat{\alpha}(\tau, x) \geq \beta(\tau, x) \).
Thus \( \tilde{\alpha}(\tau, x) = \hat{\alpha}(\tau, x) + e(\tau, x) = \bar{\beta}(\tau, x) \).
So \( \tilde{q}(\tau, x) = q(\tau, x) \).
Thus by (4) (5) for \( \tau \),
for all large enough \( \alpha < \lambda \),
\( \tilde{q}(\theta, x)(\tau, x) = \tilde{q}(\theta, x)(\tau, \lambda) \).
Thus $d(\eta, x, \varepsilon) \# d(\eta, x, \bar{\varepsilon})$ are defined and equal to $d(\eta, x, \varepsilon) + d(\eta, x, \bar{\varepsilon})$. \( \square \)

5 Let $\lambda \leq \omega_1$ be a limit ordinal, and let $\alpha_0, \alpha_1$ be reals neither of which is in $L^{\lambda+1}$, and let $s \in \omega$.

Assume: for unboundedly many $\eta < \lambda$ there is $\eta \in \omega$ s.t. $E(\eta) = s$ and s.t. $\forall \varepsilon \in d(\eta, x, \varepsilon)$ is defined and $\alpha_\varepsilon \in \bigcup^\lambda d(\eta, x, \varepsilon)$.

Then: there is $\eta \in \omega$ s.t. $E(\eta) = s$ and s.t. $\forall \varepsilon \in d(\eta, x, \varepsilon)$ is defined and $\alpha_\varepsilon \in \bigcup^\lambda d(\eta, x, \varepsilon)$.

**Proof of 5:** By induction on $s$. If $s = 0$ this is trivial. So assume $s = 0 + 1$.

By assumption we have $B \subseteq \lambda$ s.t. $B$ is unbounded s.t. $\forall \xi \in B$ there is $\eta = \eta(x)$ s.t. $E(\eta) = s$ and s.t. $\forall \varepsilon \in d(\eta, x, \varepsilon)$ is defined and $\alpha_\varepsilon \in \bigcup^\lambda d(\eta, x, \varepsilon)$.

By 3 $\eta(x)$ is unique.

Since 5 is true for 0, we have $\forall \varepsilon \in d(\eta, x, \varepsilon)$

...
s.t. \( \ell(t) = 0 \) \& \( \forall \epsilon d(\ell(t), \lambda, \epsilon) \) is defined \& \( a_\epsilon = U^1(d(\ell(t), \lambda, \epsilon)) \).

By \((\circ)\) \( \tau \) is unique.

By \((\circ \circ) \& (\circ)\), \( \forall \lambda \in \mathcal{B}, \tau \in \mathcal{C}(\lambda), \quad \mathcal{C}(\tau)(\lambda) \) and \( \forall \epsilon d(\ell(t), \lambda, \epsilon) = d(\ell(t), \lambda, \epsilon) \) def

Let \( \sigma \) code \( \langle \epsilon, e \rangle \).

Since \( \mathcal{C}(\tau, \lambda) \in \mathcal{L}_{\lambda+1} \), we have that \( \mathcal{C}(\tau, \lambda) \neq a_\epsilon \). So there is a \( K \) s.t.

\( \mathcal{C}(\tau, \lambda)(K) \neq a_\epsilon \). By \((\circ \circ)\) \( \forall \lambda \in \mathcal{B} \)

\( \mathcal{C}(\tau, \lambda)(K) = \mathcal{C}(\tau, \lambda)(K) \).

Since \( a_\epsilon = U^1(d(\ell(t), \lambda, \epsilon)) \), \( \forall \lambda \in \mathcal{B} \), we have that \( a_\epsilon \leq t_\mathcal{C}(\tau, \lambda)(\epsilon), \forall \lambda \in \mathcal{B}, \) and thus \( q(\mathcal{C}(\tau, \lambda))(\epsilon) \leq K, \forall \lambda \in \mathcal{B} \).

Thus \( \forall \lambda \) that there is a \( q \) s.t. \( \forall \lambda \in \mathcal{B} \)

\( q(\mathcal{C}(\tau, \lambda))(\epsilon) = q_1 \) and \( \{ \) since \( t_\mathcal{C}(\tau, \lambda)(\epsilon) \leq \mathcal{C}(\tau, \lambda)(K) \) by \((\circ \circ)\) \( \forall \lambda \) that \( t_\mathcal{C}(\tau, \lambda)(\epsilon) = t_\mathcal{C}(\tau, \lambda)(\epsilon) \).

\( q = \langle p, m \rangle \). So \( \forall \lambda \in \mathcal{B} \) \( p(\tau, \lambda) = p_1 \) and \( \forall \epsilon \eta(\lambda) = \tau^m \). Let \( \beta = f(p) \). Thus,
\[ \beta = \beta(\tau, x), \; \forall \alpha \in B. \]

Let \( n = 1 + n \text{ th} \text{ u.s.} \)

\[ \lambda \not\in \mathcal{C}(d(e) \times \langle e, n \rangle); \]

and let \( n = \infty \) if there is no such \( n \).

By definition of \( \beta(\tau, x) \), \( \forall \alpha \in B, \)

\[ \lambda \not\in \mathcal{C}(d(e) \times \langle e, \infty \rangle). \text{ Thus} \]

\[ \lambda \not\in \mathcal{C}(d(e) \times \langle e, \infty \rangle), \text{ so } \beta(\tau, \lambda) = \beta, \]

and \( \beta(\tau, \lambda) = \beta \).

Let \( \eta = \tau \times \infty. \)

Thus \( q(\eta, \lambda) = q \).

Thus \( t \eta q(\eta, \lambda), (\tau, \lambda) \) is defined,

and \( \beta(\tau, \lambda) < \lambda \). So \( d(\eta, \lambda, \tau) \) and \( d(\eta, \alpha, \tau) \)

are both defined and are equal to \( d(\eta, \lambda, \tau) \), \( \forall \alpha \in B. \)

Since \( \forall \nu = 0 \circ \tau \), \( \forall \alpha \in \mathcal{U} \times (d(\eta, \lambda, \nu)) \)

for all \( \alpha \in B, \) we have

\[ \alpha \in \mathcal{U} \times (d(\eta, \lambda, \nu)). \quad \square \]
Now:
let \( b : w \rightarrow w \).
Let \( U_e = U \cup \mathcal{D}(\mathcal{E}(s), w_1, e) \).

It is easy to see that \( U_e \) is a complete consistent extension of \( Te \).

Let \( a_e \overset{\text{def}}{=} \{ s \mid \langle e, x \rangle \in U_e \} \).

So \( a_e \subseteq U_e \). By the construction we have:

given \( e \), if \( \forall i \ a_e \not\in U(\langle e, i \rangle) \) then
\( \exists \beta < w_1 \text{ s.t. } \forall i \ a_e \not\in U(\langle e, i \rangle) \).

Thus \( \mathcal{L}_2 \exists a_e \) is admissible \( \forall e \) so \( a_e \in A_e \).

Since \( a_e \) is constructed, using \( b \) as a guide, we have
\( a_e \leq \text{hyp} \langle b, \emptyset \rangle \).

Since \( b \) is coded into the construction of \( a_0, a_1 \), we have
\( b \leq \text{hyp} \langle a_0, a_1, \emptyset \rangle \).

Thus,
it suffice to show that $\mathfrak{A} \leq \text{hyp} \langle \alpha_0, \alpha_1 \rangle$.

By (4) (applied to $\lambda = \omega_1$), we can find $x(0) < x(1) < \ldots < x(\eta)$ s.t.

$<x(\eta)>_{\mathfrak{A}}$ is $\Delta_1$ over $L_{\mathfrak{A}}[<\alpha_0, \alpha_1>^3]$, and s.t. $\forall \xi < \eta$ s.t. $\xi(\eta) = s \in 

s.t. \exists \theta (d(\eta, x(\xi)), \varepsilon) \text{ is defined to } \alpha_\varepsilon \in \mathcal{U} \lambda(\theta(\varsigma, x(\xi), 1)).$

Let $\lambda = \sup x(\xi)$. If $\lambda = \omega_1$ then $L_{\mathfrak{A}}[<\alpha_0, \alpha_1>]$ is not admissible and $\mathfrak{A} \leq \text{hyp} \langle \alpha_0, \alpha_1 \rangle$.

If $\lambda < \omega_1$, then since $\alpha_\varepsilon \in A_\varepsilon$, $\alpha_\varepsilon \in L_{\lambda+1}$. Thus by (5) and (3) there is a real $b'$ s.t. $\forall \theta \in d(b', x), \varepsilon)$ is defined to $\alpha_\varepsilon \in \mathcal{U} \lambda(\theta(b', x), 1, \varepsilon)$.

But then, for all $\varepsilon$, if $\alpha_\varepsilon \notin \mathcal{U} \lambda(\varepsilon, i)$ for all $i$, then $\exists \beta < \lambda$ s.t. $\alpha_\varepsilon \notin \mathcal{U} \lambda(b', \varepsilon)$ for all $i$. 
The $L[A]_c$ is admissible, which is absurd. □

This result was obtained in 1974, a few days after Martin & Friedman had independently shown:

If $A$ is an uncountable $\Delta^1_1$ set of reals, and if $b \equiv_{\text{hyp}} A$, then there is $a \in A$ s.t. $a \equiv_{\text{hyp}} b$.

Our proof here relies on Friedman's proof of the above. (And Friedman's proof in turn relied on our earlier proof of the above for the case $b = \omega$, which in turn was motivated by Friedman's even earlier proof of the above for the case when $A$ contains no hyp reals.)
9.2. A note concerning the Borelness of upper cone of hyperdegrees.

**Proposition 9.2.** The following statement is independent from ZFC: for any real $x$, the set $A_x = \{ y \mid y \geq_h x \}$ is Borel.

**Proof.** Clearly if $V = L$, then the set $A_x = \{ y \mid y \geq_h x \}$ is Borel for any real $x$.

Now suppose that $\omega_1 = (\omega_1)^L$ and there is an $L$-random real $r$. We prove that $A_r$ is not Borel. Otherwise, there is a countable ordinal $\alpha$ so that
\[
\forall z (z \in A_r \implies r \in L_\alpha[z]).
\]

Let $y \in L$ be a real so that $\omega_1^y > \alpha$. By Martin’s theorem relative to $y$, there is a $\Delta^1_1(y)$-Sacks generic real $z$ be a so that $z \oplus y \geq_h r$. Then $r$ is $L_\alpha[y \oplus z]$-random and so $r \notin L_\alpha[y \oplus z]$ but $r \leq_h y \oplus z$, a contradiction. \(\Box\)

**Note:** Actually it can be proved that $A_x$ is Borel if and only if $x \in L$. The idea is as follows: Suppose that $\alpha$ is a countable ordinal and $x \notin L$. Let $\beta > \alpha$ be a countable admissible ordinal. Then there is a real $y$ so that $\omega_1^y = \beta$ but $x \not<_h y$. Then one can prove that there is a real $z \geq_T y$ so that $z \geq_h y$ but $x \notin L_\beta[z]$.

---

**Part 4. Model theory and definability**

10. **First-order logic, computability, and enriched structures**

A structure in mathematical logic is a non-empty domain (a set) with some relations and functions defined on it. Examples are graphs, and groups. First-order logic is made for dealing with them: the relations and functions are described by atomics formulas, the quantifiers range over the set. When the domain is countably infinite, the notions of computability also work well with such structures: via some coding, one can see the domain elements as inputs to machines, and one requires that the relations and functions be computable.

Many kinds of structures that mathematicians study are not of this kind. Often they study “enriched structures”: a non-empty domain, some relations and functions on it, and something else, some extra structure. Examples are:

1. metric structures: there is a distance function producing a topology, the relations are closed, the functions continuous.
2. Hilbert spaces, with a C-valued inner product.
3. $C^*$-algebras, which are Banach $*$-algebras satisfying $\|xx^*\| = \|x\|^2$.
4. tracial von Neumann algebras $N$, such as the hyperfinite $II_1$-factor $R$.
   
   The extra structure is a trace, a positive functional sending $1_N$ to 1, corresponding to the (normalised) trace of matrices
5. Lie groups: groups on a manifold.
6. Profinite groups: a group on a compact, totally disconnected space with group operations continuous.

And these are often the most relevant for applications elsewhere: e.g. Hilbert spaces and operator algebras are used in quantum physics. Lie groups also play a major role in modern physics.
Remark 10.1. The last two examples suggest that the proposed view may be that of a logician. The extra structure can also be viewed as the primary one, with the algebraic structure sitting on top of it. For instance, Lie groups are group objects in the category of manifolds (which has Cartesian products); these group objects form a new category where the morphisms are the Lie group homomorphisms. Ditto for profinite groups, which are group objects on compact totally disconnected topological spaces.

How do we extend the methods of first-order logic, and of computability, to such structures? There are two pathways.

A. The first pathway, not followed here, is to also extend the language, or to adapt the notion of computability to the new setting. For the language, this has been done in many of the cases above. E.g., continuous logic has been introduced to deal with metric structures, and also with tracial von Neumann algebras. Relations are now real valued, and the distance function plays the role that formerly equality had played. For topological groups, one can use a two-sorted language: one sort for the elements, and another for the open sets, say.

For computability, there are Blum-Shub-Smale (BSS) machines that directly work on reals, as an example.

B. The second pathway is to retain the bare tools of first-order logic and computability developed for the naked structures. The approach to the enriched structures will then necessarily be indirect. We still express things about, or compute with, the elements of the domain with their basic relationships; the expressivity is enhanced only because the semantics is different. For instance, one can study the expressivity of first-order logic in profinite groups. If the theory determines the group within a class of groups, this is called quasi-axiomatisable (QA) relative to the class. Jarden and Lubotzky [12] (also see [28, Thm. 4.2.3]) showed that each topologically f.g. profinite group is QA within the profinite groups. Nies, Segal and Tent [19] show that for many profinite groups there is in fact a single sentence determining it up to topological isomorphism within the class of profinite groups. This property is called finitely axiomatisable (FA) relative to the class. Examples of FA groups include $SL_n(\mathbb{Z}_p)$ for any odd prime $p$ not dividing $n$.

In the computability setting, the problem is that many of the structures are now uncountable, e.g. Polish metric spaces. We have to reduce these enriched structures to a countable core, which can be done by choosing a dense countable subset $D$. In this way we can define computable metric spaces: each element is given a the limit of a certain fast converging Cauchy sequence of elements of $D$. For profinite groups the most natural way to extend computability to the whole structure is by asking that the group be an effective inverse limit of finite groups with onto projections. This is e.g. the case for the additive group of $p$-adics, $\mathbb{Z}_p = \text{proj lim}_n C_{p^n}$. One also gets an ultrametric out of this, relative to which the group operations are computable.

In the case of the von Neumann hyperfinite $II_1$ factor $\mathcal{R}$ it is easy to determine the canonical computable structure as a metric space. $\mathcal{R}$ can be introduced as the completion of the $\ast$-algebra $\bigcup_n M_{2^n} (\mathbb{C})$ with respect to norm given by the inner product $\langle A, B \rangle = \tau_n (B^* A)$, where $\tau_n$ is the
normalized trace given by \( \tau_n(C) = 2^{-n} \sum_i C_{ii} \); see e.g. the Goldbring 2013 notes [9].

Consider a class \( \mathcal{C} \) of enriched structures. To structures in \( \mathcal{C} \) can be algebraically isomorphic, and fully isomorphic, i.e. also the extra structure is preserved. How much stronger is the second of these notions? For Polish groups (topological groups where the topology is Polish), any Baire measurable isomorphism is continuous. So, if the Axiom of Determinacy (AD) holds (which contradicts the axiom of choice), then there is no difference. So, to build profinite groups that are only algebraically isomorphic one needs to use the axiom of choice. For instance let \( A \) be the product of all cyclic groups of order \( p^n \), \( n \in \mathbb{N} - \{0\} \). Then \( A \) is algebraically isomorphic to \( A \times \mathbb{Z}_p \) by Cor 3.2 in Kiehlmann (J. Group Theory 16 (2013), 141-157), which uses AC. They are not topologically isomorphic because the torsion subgroup is dense in \( A \) but not in \( A \times \mathbb{Z}_p \). As alternative to Kiehlmann’s proof one can use Ulm invariants for abelian \( p \) groups. One can show the dual fact in abelian torsion groups, that \( \bigoplus_n C_{p^n} \cong \bigoplus_n C_{p^n} + C_{p^\infty} \). To make Ulm invariants work one again needs AC (in the guise of the well ordering theorem).

If an enriched structure \( M \) is quasi-axiomatizable within \( \mathcal{C} \) then each algebraic isomorphism with another structure in \( \mathcal{C} \) is automatically a full isomorphism. This can be used to find examples of structures \( M \) that are not QA in their class. E.g. Kiehlmann’s group \( A \) is not QA in the profinite groups (assuming AC).

References

[1] T. Bartoszyński and H. Judah. Set Theory. On the structure of the real line. A K Peters, Wellesley, MA, 1995. 546 pages.
[2] Tomek Bartoszyński. Additivity of measure implies additivity of category. Trans. Amer. Math. Soc., 281(1):209–213, 1984.
[3] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In Matthew Foreman and Akihiro Kanamori, editors, Handbook of Set Theory, volume 1, pages 395–489. Springer, Dordrecht Heidelberg London New York, 2010.
[4] J. Brendle, A. Brooke-Taylor, Keng Meng Ng, and A. Nies. An analogy between cardinal characteristics and highness properties of oracles. In Proceedings of the 13th Asian Logic Conference: Guangzhou, China, pages 1–28. World Scientific, 2013. http://arxiv.org/abs/1404.2839.
[5] C. T. Chong and Liang Yu. Randomness in the higher setting. J. Symb. Log., 80(4):1131–1148, 2015.
[6] Y. Ershov. Elementary group theories. In Doklady Akademii Nauk, volume 203, pages 1240–1243. Russian Academy of Sciences, 1972.
[7] U. Felgner. On \( \aleph_0 \)-categorical extra-special \( p \)-groups. Logique et Analyse, pages 407–428, 1975.
[8] M. Fried and M. Jarden. Field arithmetic, volume 11. Springer Science & Business Media, 2006.
[9] Isaac Goldbring. A gentle introduction to von neumann algebras for model theorists, 2013.
[10] D. Hirschfeldt, A. Nies, and F. Stephan. Using random sets as oracles. J. Lond. Math. Soc. (2), 75(3):610–622, 2007.
[11] W. Hodges. Model Theory. Encyclopedia of Mathematics. Cambridge University Press, Cambridge, 1993.
[12] M. Jarden and A. Lubotzky. Elementary equivalence of profinite groups. Bulletin of the London Mathematical Society, 40(5):887–896, 2008.
[13] W. Merkle, J. Miller, A. Nies, J. Reimann, and F. Stephan. Kolmogorov-Loveland randomness and stochasticity. *Ann. Pure Appl. Logic*, 138(1-3):183–210, 2006.

[14] A. Morozov and A. Nies. Finitely generated groups and first-order logic. *J. Lond. Math. Soc.*, 71(2):545–562, 2005.

[15] A. Nies. Lowness properties and randomness. *Adv. in Math.*, 197:274–305, 2005.

[16] A. Nies. Describing groups. *Bull. Symbolic Logic*, 13(3):305–339, 2007.

[17] A. Nies. *Computability and Randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009. 444 pages. Paperback version 2011.

[18] A. Nies and V. Scholz. Martin-Löf random quantum states. *Journal of Mathematical Physics*, 60(9):092201, 2019. available at doi.org/10.1063/1.5094660.

[19] A. Nies, D. Segal, and K. Tent. Finite axiomatizability for profinite groups i: group theory. http://arxiv.org/abs/1907.02262, 29 pages, 2019.

[20] A. Nies and P. Semukhin. Finite automata presentable abelian groups. *Annals of Pure and Applied Logic*, 161(3):458–467, 2009.

[21] A. Nies and R. Thomas. Fa-presentable groups and rings. *Journal of Algebra*, 320(2):569–585, 2008.

[22] N. Nikolov and D. Segal. On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Ann. of Math. (2)*, 165(1):171–238, 2007.

[23] C. Perin and R. Sklinos. Homogeneity in the free group. *Duke Mathematical Journal*, 161(13):2635–2668, 2012.

[24] Jean Raisonnier. A mathematical proof of S. Shelah’s theorem on the measure problem and related results. *Israel J. Math.*, 48:48–56, 1984.

[25] N. Rupprecht. *Effective correspondents to cardinal characteristics in Cichoń’s diagram*. PhD thesis, University of Michigan, 2010.

[26] N. Rupprecht. Relativized Schnorr tests with universal behavior. *Arch. Math. Logic*, 49(5):555–570, 2010.

[27] J. Rute. Computable randomness and betting for computable probability spaces. *Mathematical Logic Quarterly*, 62(4-5):335–366, 2016.

[28] D. Segal. *Words: notes on verbal width in groups*, volume 361. Cambridge University Press, 2009.

[29] T. Slaman and R. Solovay. When oracles do not help. In *Proceedings of the fourth annual workshop on Computational learning theory*, pages 379–383. Morgan Kaufmann Publishers Inc., 1991.

[30] D. Soukup. Two infinite quantities and their surprising relationship. *arXiv preprint arXiv:1803.04331*, 2018.

[31] S. Terwijn and D. Zambella. Algorithmic randomness and lowness. *J. Symbolic Logic*, 66:1199–1205, 2001.