Effective $H^\infty$ interpolation constrained by Hardy and Bergman weighted norms

Rachid Zarouf

Abstract

Given a finite set $\sigma$ of the unit disc $\mathbb{D}$ and a holomorphic function $f$ in $\mathbb{D}$ which belongs to a class $X$ we are looking for a function $g$ in another class $Y$ which minimizes the norm $\|g\|_Y$ among all functions $g$ such that $g|_{\sigma} = f|_{\sigma}$. Generally speaking, the interpolation constant considered is $c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_{\sigma} = f|_{\sigma} \}$. When $Y = H^\infty$, our interpolation problem includes those of Nevanlinna-Pick (1916), Carathéodory-Schur (1908). Moreover, Carleson’s free interpolation (1958) has also an interpretation in terms of our constant $c(\sigma, X, H^\infty)$.

If $X$ is a Hilbert space belonging to the scale of Hardy and Bergman weighted spaces, we show that $c(\sigma, X, H^\infty) \leq a_{\varphi_X} \left(1 - \frac{1-r}{n}\right)$ where $n = \# \sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space $X$. The upper bound is sharp over sets $\sigma$ with given $n$ and $r$.

If $X$ is a general Hardy-Sobolev space or a general weighted Bergman space (not necessarily of Hilbert type), we also found upper and lower bounds for $c(\sigma, X, H^\infty)$ (sometimes for special sets $\sigma$) but with some gaps between these bounds.

This constrained interpolation is motivated by some applications in matrix analysis and in operator theory.

INTRODUCTION

1. Statement and historical context of the problem. Let $\text{Hol}(\mathbb{D})$ be the space of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We consider here the following problem: given two Banach spaces $X$ and $Y$ of holomorphic functions on the unit disc $\mathbb{D}$, $X, Y \subset \text{Hol}(\mathbb{D})$, and a finite set $\sigma \subset \mathbb{D}$, what is the best possible interpolation by functions of the space $Y$ for the traces $f|_{\sigma}$ of functions of the space $X$, in the worst case? The case $X \subset Y$ is of no interests, and so one can suppose that either $Y \subset X$ or $X, Y$ are incomparable. Here and everywhere below, $H^\infty$ stands for the space (algebra) of bounded holomorphic functions in the unit disc $\mathbb{D}$ endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$.

Precisely, our problem is to compute or estimate the following interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_{\sigma} = f|_{\sigma} \}.$$

If $r \in [0, 1)$ and $n \geq 1$, we also define

$$C_{n, r}(X, Y) = \sup \{ c(\sigma, X, Y) : \# \sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma \}.$$
Let us first explain why the classical interpolation problems, those of Nevanlinna-Pick and Carathéodory-Schur (see [N2] p.231 for these two problems) on the one hand, and Carleson’s free interpolation problem (1958) (see [N1] p.158) on the other hand, are of the nature of our interpolation problem.

(i) The Carleson interpolation problem

We suppose here that the sequence $\sigma$ can be infinite. Let $l^\infty(\sigma)$ be the space of complex sequences $(a_\lambda)_{\lambda \in \sigma}$ of support $\sigma$ endowed with the norm $\|a\|_{l^\infty(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$. Carleson’s free interpolation is to compare the norms $\|a\|_{l^\infty(\sigma)}$ and

$$\inf \{ \|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma \}. $$

In other words, it deals with the interpolation constant defined as

$$c(\sigma, l^\infty(\sigma), H^\infty) = \sup_{a \in l^\infty(\sigma), \|a\|_{l^\infty(\sigma)} \leq 1} \inf \{ \|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma \} :$$

looking at our interpolation constant $c(\sigma, X, H^\infty)$, we have replaced the holomorphic functions space $X$ by the space of sequences $l^\infty(\sigma)$.

Now on the contrary to Carleson’s free interpolation, the Nevanlinna-Pick and Carathéodory-Schur interpolation problems are “individual”, in the sense that they look simply to compute the norms $\|f\|_{H^\infty(\sigma)}$ or $\|f\|_{H^\infty/\lambda_i^H^\infty}$ for a given $f$. We first recall those two classical problems (see (ii) and (iii) below) and then we explain in (iv) in which way they are included into our interpolation problem.

(ii) The Nevanlinna-Pick interpolation problem

Given $\Lambda = (\lambda_1, ..., \lambda_n)$ in $\mathbb{D}^n$ and $W = (w_1, ..., w_n) \in \mathbb{C}^n$, we are looking for

$$C(\Lambda, W) = \inf \{ \|f\|_\infty : f(\lambda_i) = w_i, i = 1..n \}. $$

The classical answer of Pick is the following :

$$C(\Lambda, W) = \inf \left\{ c > 0 : \left( \frac{c^2 - \sum_{i,j} w_i w_j}{1 - \sum_{i,j} \lambda_i \lambda_j} \right)_{1 \leq i, j \leq n} \gg 0 \right\}, $$

where for any $n \times n$ matrix $M$, $M \gg 0$ means that $M$ is positive definite.

(iii) The Carathéodory-Schur interpolation problem

Given $A = (a_0, ..., a_n) \in \mathbb{C}^{n+1}$, we are looking for

$$C(A) = \inf \{ \|f\|_\infty : f(z) = a_0 + a_1 z + ... + a_n z^n + ... \}. $$

The classical answer of Schur is the following :

$$C(A) = \| (T_\varphi)_n \|,$$

where $T_\varphi$ is the Toeplitz operator associated with a symbol $\varphi$, $(T_\varphi)_n$ is the compression of $T_\varphi$ on $P_n$, the space of analytic polynomials of degree less or equal than $n$, and $\varphi$ is the polynomial $\sum_{k=0}^n a_k z^k$. 
Notice that the Carathéodory-Schur interpolation theorem can be seen as a particular case of the famous commutant lifting theorem of Sarason and Sz-Nagy-Foias (1968) see [N2] p.230, Theorem 3.1.11.

(iv) How are the Nevanlinna-Pick and the Caratheodory-Schur interpolation problems included into ours?

From a modern point of view, the two interpolation problems (ii) and (iii) are included in the following mixed problem: given

- \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{D} \), the finite Blaschke product \( B_\sigma = \prod_{j} b_{\lambda_j} \), where \( b_{\lambda} = \frac{\lambda - z}{1 - \lambda z} \), \( \lambda \in \mathbb{D} \),
- \( f \in Hol(\mathbb{D}) \),

we would like to compute or estimate

\[ \|f\|_{H^\infty/BH^\infty} = \inf \{ \|g\|_{\infty} : f - g \in B_\sigma Hol(\mathbb{D}) \} . \]

Now if we want to compare \( \|f\|_{H^\infty/BH^\infty} \) with the size of \( f \) (measured in a certain Banach space \( X \)), then looking at our interpolation constant \( c(\sigma, X, Y) \) we have

\[ c(\sigma, X, Y) = \sup_{f \in X, \|f\|_{X} \leq 1} \|f\|_{Y/BY} . \]

As a consequence, the classical Nevanlinna-Pick problem corresponds to the case \( Y = H^\infty \), and the one of Carathéodory-Schur to the case where \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0 \) and \( Y = H^\infty \).

Remarks. a. Looking at this problem, say, in the form of computing/estimating the interpolation constant \( c(\sigma, X, Y) \) which is nothing but the norm of the embedding operator \( (X|_{\sigma}, \|\cdot\|_{X|_{\sigma}}) \to (Y|_{\sigma}, \|\cdot\|_{Y|_{\sigma}}) \), one can think, of course, on passing (after) to the limit - in the case of an infinite sequence \( \{\lambda_j\} \) and its finite sections \( \{\lambda_j\}_{j=1}^n \), in order to obtain a Carleson type interpolation theorem \( X|_{\sigma} = Y|_{\sigma} \). But not necessarily. In particular, even the classical Nevanlinna-Pick theorem (giving a necessary and sufficient condition on a function \( a \) for the existence of \( f \in H^\infty \) such that \( \|f\|_{\infty} \leq 1 \) and \( f(\lambda) = a_\lambda, \lambda \in \sigma \)), does not lead immediately to Carleson’s criterion for \( H^\infty|_{\sigma} = l^\infty(\sigma) \). (In fact, a direct deduction of Carleson’s theorem from Pick’s result was done by P. Koosis [K] in 1999 only). Similarly, the problem stated for \( c(\sigma, X, Y) \) is of interest in its own. It is a kind of “effective interpolation” because we are looking for sharp estimations or a computation of \( c(\sigma, X, Y) \) for a variety of norms \( \|\cdot\|_X, \|\cdot\|_Y \).

b. An interesting case occurs when \( X \) is larger than \( Y \), and the sense of the issue lies in comparing \( \|\cdot\|_X \) and \( \|\cdot\|_Y \) when \( Y \) interpolates \( X \) on the set \( \sigma \). For example, we can wonder what happens when \( X = H^p \), the classical Hardy spaces of the disc or \( X = L^p_a \), the Bergman spaces, etc..., and when \( Y = H^\infty \), but also \( Y = W \) the Wiener algebra (of absolutely converging Fourier series) or \( Y = B^0_{\infty,1} \), a Besov algebra (an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition).
2. Motivations.

We also can add some more motivations to our problem:

a. One of the most interesting cases is \( Y = H^\infty \). As we have seen it in the paragraph 1 of the Introduction (above), in this case the quantity \( c(\sigma, X, H^\infty) \) has a meaning of an intermediate interpolation between the Carleson one (when \( \| f \|_{X|_\sigma} \asymp \sup_{1 \leq i \leq n} | f(\lambda_i) | \)) and the individual Nevanlinna-Pick interpolation (no conditions on \( f \)).

b. There is a straight link between the constant \( c(\sigma, X, Y) \) and numerical analysis.

b.1 For example, in matrix analysis, it is of interest to bound the norm of an \( H^\infty \)-calculus
\[ \| f(A) \| \leq c \| f \|_\infty, \quad \forall f \in H^\infty, \] for an arbitrary Banach space \( n \)-dimensional contraction \( A \) with a given spectrum \( \sigma(A) \subset \sigma \). The best possible constant is \( c = c(\sigma, H^\infty, W) \), so that

\[ c(\sigma, H^\infty, W) = \sup_{\| f \|_\infty \leq 1} \sup \{ \| f(A) \| : A : (\mathbb{C}^n, |.|) \to (\mathbb{C}^n, |.|), \| A \| \leq 1, \sigma(A) \subset \sigma \}, \]

where \( W = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \sum_{k \geq 0} | \hat{f}(k) | < \infty \right\} \) stands for the Wiener algebra, and the interior sup is taken over all contractions on \( n \)-dimensional Banach spaces \( (\mathbb{C}^n, |.|) \).

b.2 In the same spirit, our constant \( c(\sigma, X, H^\infty) \) is straightly linked with the well known Von-Neumann’s inequality for contractions on Hilbert spaces, which asserts that if \( A \) is a contraction on a Hilbert space and \( f \in H^\infty \), then the operator \( f(A) \) satisfies the inequality

\[ \| f(A) \| \leq \| f \|_\infty. \]

Using this inequality we get the following interpretation of our interpolation constant \( c(\sigma, X, H^\infty) \) : it is the best possible constant \( c \) such that \( \| f(A) \| \leq c \| f \|_X, \quad \forall f \in X, \)

\[ c(\sigma, X, H^\infty) = \sup_{\| f \|_X \leq 1} \sup \{ \| f(A) \| : A : (\mathbb{C}^n, |.|_2) \to (\mathbb{C}^n, |.|_2), \| A \| \leq 1, \sigma(A) \subset \sigma \}, \]

where the interior sup is taken over all contractions \( A \) on \( n \)-dimensional Hilbert spaces \( (\mathbb{C}^n, |.|_2) \), with a given spectrum \( \sigma(A) \subset \sigma \).

b.3 As in b.1 and b.2 we can choose \( Y = B^0_{\infty,1} \), a Besov algebra instead of \( Y = H^\infty \) or \( Y = W \). Our interpolation constant \( c(\sigma, X, B^0_{\infty,1}) \) is defined in the same way and presents an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition.

For b.1, b.2 and b.3, an interesting case occurs for \( f \) such that \( f |_{\sigma} = \frac{1}{z} |_{\sigma} \) (estimation of condition numbers and the norm of inverses of \( n \times n \) matrices) or \( f |_{\sigma} = \frac{1}{\lambda - z} |_{\sigma} \) (for estimation of the norm of the resolvent of an \( n \times n \) matrix), see for instance [Z4].
c. Let us notice that the following question (the special case \(X = H^2, Y = H^\infty\)) was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]): given a set \(\sigma \subset \mathbb{D}\), how to estimate \(c(\sigma, H^2, H^\infty)\) in terms of \(n = \text{card}(\sigma)\) and \(\max_{\lambda \in \sigma} |\lambda| = r\) only? Here and everywhere below, \(H^2\) is the standard Hardy space of the disc,

\[
H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|_{H^2} = \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{1/2} < \infty \right\}.
\]

The issue of estimating/computing \(c(\sigma, H^2, H^\infty)\) has been treated in [Z2], in which we also considered some others partial cases of the problem: more precisely, we obtained sharp estimations for the constant \(C_{n,r}(X, Y)\) (see Theorem A below) for the cases \(X = H^p, L^2_a,\) and \(Y = H^\infty,\) where \(H^p (1 \leq p \leq \infty)\) stand for the standard Hardy spaces of the unit disc (see [N2] for the definition) and \(L^2_a\) stands for the Bergman space of all holomorphic functions \(f\) such that

\[
\int_{\mathbb{D}} |f(\omega)|^2 \, dA < \infty,
\]

where \(dA\) stands for the area measure. Here is the main result of [Z2]:

**Theorem A.** Let \(n \geq 1, r \in [0, 1), p \in [1, +\infty)\) and \(\lambda, |\lambda| \leq r.\) Then,

\[
1 \leq \frac{n}{32} \left( \frac{1}{1 - |\lambda|} \right) \leq c(\sigma_{n,\lambda}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left( \frac{n}{1 - r} \right)^{\frac{1}{p}},
\]

\[
(2) \quad \frac{1}{32} \frac{n}{1 - |\lambda|} \leq c(\sigma_{n,\lambda}, L^2_a, H^\infty) \leq C_{n,r}(L^2_a, H^\infty) \leq \sqrt{\frac{n}{1 - r}},
\]

where \(\sigma_{n,\lambda} = \{\lambda, \ldots, \lambda\}\) is the one-point set of multiplicity \(n\) corresponding to \(\lambda,\) \(A_p\) is a constant depending only on \(p\) and the left-hand side inequality from (1) is valid only for \(p \in 2\mathbb{Z}_+.\) For \(p = 2,\) we have \(A_2 = \sqrt{2}\).

3. **Main results.** Theorems B, C, D&E below in this paragraph, were already announced in the note [Z1]. In this paper, we extend the above results to the case where \(X\) is a weighted space \(X = l^p_a(w_k),\)

\[
l^p_a(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\},
\]

with a weight \(w\) satisfying \(w_k > 0\) for every \(k \geq 0\) and \(\lim_k (1/w_k^{1/k}) = 1.\) The latter condition implies that \(l^p_a(w_k)\) is continuously embedded into the space of holomorphic functions \(Hol(\mathbb{D})\) on the unit disc \(\mathbb{D}\) (and not on a larger disc, i.e. \(l^p_a(w_k)\) does not contained in \(Hol(r\mathbb{D})\) for every \(r > 1)\).

First, we study the special case \(p = 2\) and \(w_k = (k + 1)^\alpha, \alpha \leq 0.\) Then \(X = l^2_a((k + 1)^\alpha),\) are the spaces of all \(f(z) = \sum_{k \geq 0} \hat{f}(k)z^k\) satisfying

\[
\sum_{k \geq 0} |\hat{f}(k)|^2 (k + 1)^{2\alpha} < \infty.
\]
Notice that \( H^2 = l_a^2(1) \) and \( L_a^2 = l_a^2 \left( \frac{1}{(k+1)^{\frac{1}{2}}} \right) \). An equivalent description of this scale of spaces is:

\[
X = L_a^2 \left( (1 - |z|^2)^{\beta} \, dA \right), \quad \beta = -2\alpha - 1 > -1,
\]

the weighted Bergman spaces of holomorphic functions such that

\[
\int_D |f(z)|^2 \left( 1 - |z|^2 \right)^{\beta} \, dA < \infty.
\]

We shorten the notation and write

\[
L_a^2 (\beta) = L_a^2 \left( (1 - |z|^2)^{\beta} \, dA \right),
\]

for every \( \beta > -1 \). For the case \( \beta = 0 \), we have \( L_a^2 (0) = L_a^2 \).

Our principal result is the following.

**Theorem B.** Let \( n \geq 1 \), \( r \in [0, 1) \), \( \alpha \in (-\infty, 0] \) and \( \lambda \in r\overline{D} \). Then,

\[
a \left( \frac{n}{1 - |\lambda|} \right)^{\frac{1-2\alpha}{2}} \leq c \left( \sigma_n, \lambda, l_a^2 ((k + 1)^{\alpha}) , H^\infty \right) \leq C_{n,r} \left( l_a^2 ((k + 1)^{\alpha}) , H^\infty \right) \leq A \left( \frac{n}{1 - r} \right)^{\frac{1-2\alpha}{2}}.
\]

Furthermore if \( \beta \in (-1, +\infty) \), then

\[
a' \left( \frac{n}{1 - |\lambda|} \right)^{\frac{\beta+2}{2}} \leq c \left( \sigma_n, \lambda, L_a^2 (\beta) , H^\infty \right) \leq C_{n,r} \left( L_a^2 (\beta) , H^\infty \right) \leq A' \left( \frac{n}{1 - r} \right)^{\frac{\beta+2}{2}},
\]

where \( A = A(\alpha) \) and \( a = a(\alpha) \) are constants depending only on \( \alpha \), \( A' = A'(\beta) \) and \( a' = a'(\beta) \) are constants depending only on \( \beta \), and both of the two left-hand side inequalities are valid only for \( \alpha \) and \( \beta \) satisfying \( 1 - 2\alpha \in \mathbb{N} \) and \( \frac{\beta+1}{2} \in \mathbb{N} \).

Right-hand side inequalities from Theorem B are proved in Section 1 whereas the left-hand side ones are shown in Section 7.

**Remark.** On the size of the constants \( a, A, a' \) and \( A' \) of Theorem B : if \( N = \lfloor 1 - 2\alpha \rfloor \) stands for the integer part of \( 1 - 2\alpha \), then \( a \) and \( A \) are such that \( a \asymp \frac{1}{2^{\lfloor N/2 \rfloor}!} \) and \( A \asymp N^{2N} \). In the same way, if \( N' = \lfloor 2 + \beta \rfloor \) stands for the integer part of \( 2 + \beta \), then \( a' \) and \( A' \) are such that \( a' \asymp \frac{1}{2^{\lfloor N'/2 \rfloor}!} \) and \( A' \asymp N'^{2N'} \). (The notation \( x \asymp y \) means that there exist numerical constants \( c_1, c_2 > 0 \) such that \( c_1 y \leq x \leq c_2 y \).

In Sections 2, 3, 4 and 5, we deal with an upper estimate for \( C_{n,r} (X, H^\infty) \) in the scale of spaces \( X = l_p^a ((k + 1)^{\alpha}) \), \( \alpha \leq 0 \), \( 1 \leq p \leq +\infty \). We start giving a result for \( 1 \leq p \leq 2 \).

**Theorem C.** Let \( r \in [0, 1] \), \( n \geq 1 \), \( 1 \leq p \leq 2 \), and \( \alpha \leq 0 \). We have

\[
B \left( \frac{1}{1 - r} \right)^{\frac{1-2\alpha}{2}} \leq C_{n,r} \left( l_p^a ((k + 1)^{\alpha}) , H^\infty \right) \leq A \left( \frac{n}{1 - r} \right)^{\frac{1-2\alpha}{2}},
\]
where $A = A(\alpha, p)$ is a constant depending only on $\alpha$ and $p$ and $B = B(p)$ is a constant depending only on $p$.

It is very likely that the bounds of Theorem C are not sharp. The sharp one should be probably \((\frac{n}{1-r})^{1-a-\frac{1}{p}}\). In the same way, for $2 \leq p \leq \infty$, we give the following theorem, in which we feel again that the upper bound \((\frac{n}{1-r})^{\frac{3}{2}-a-\frac{1}{p}}\) is not sharp. The sharp one probably should be the lower bound \((\frac{n}{1-r})^{1-a-\frac{1}{p}}\).

**Theorem D.** Let $r \in [0, 1[, n \geq 1, 2 \leq p \leq \infty$, and $\alpha \leq 0$. We have

$$B \left(\frac{1}{1-r}\right)^{1-a-\frac{1}{p}} \leq C_{n,r} (L^p_a ((k+1)^n), H^{\infty}) \leq A \left(\frac{n}{1-r}\right)^{\frac{3}{2}-a-\frac{1}{p}},$$

where $A = A(\alpha, p)$ is a constant depending only on $\alpha$ and $p$ and $B = B(p)$ is a constant depending only on $p$.

In Section 6, we suppose that $X$ is equal to $L^p_a(\beta)$, $\beta > -1$, $1 \leq p \leq 2$. Our goal in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

$$c(\sigma_n, \lambda, X, H^{\infty}) = \sup \{ \|f\|_{H^{\infty}/b^{n}_{\lambda}H^{\infty}} : f \in X, \|f\|_{X} \leq 1 \},$$

where $\|f\|_{H^{\infty}/b^{n}_{\lambda}H^{\infty}} = \inf \{ \|f + b^{n}_{\lambda}g\|_{H^{\infty}} : g \in X \}, \lambda \in \mathbb{D}$ and $\sigma_n, \lambda$ is (again) the one-point set of multiplicity $n : \sigma_n, \lambda = \{\lambda, ..., \lambda\}$. The corresponding interpolation problem is : given $f \in X$, to minimize $\|h\|_{\infty}$ such that $h^{(j)} (\lambda) = f^{(j)} (\lambda), 0 \leq j < n$.

For this partial case, we have the following generalization of the estimate from Theorem B.

**Theorem E.** Let $\lambda \in \mathbb{D}$, $\beta \in (-1, +\infty)$ and $p \in [1, 2]$. Then,

$$c(\sigma_n, \lambda, L^p_a (\beta), H^{\infty}) \leq A' \left(\frac{n}{1-|\lambda|}\right)^{\frac{2}{p}+\frac{2}{p}},$$

where $A' = A'(\beta, p)$ is a constant depending only on $\beta$ and $p$.

4. **Technical tools ("about the proofs"**). In order to find an upper bound for $c(\sigma, X, H^{\infty})$, we first use a linear interpolation:

$$f \mapsto \sum_{k=1}^{n} \langle f, e_k \rangle e_k,$$

where $\langle .., \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k)\overline{g(k)}$, and $(e_k)_{k=1}^{n}$ is the explicitly known Malmquist basis of the space $K_B = H^2\Theta BH^2$, $B = \Pi_{i=1}^{n} b_{\lambda_i}$ being the corresponding Blaschke product, $b_{\lambda} = \frac{\lambda-\overline{\lambda}}{\lambda-\overline{\lambda}}$ (see [N1] p. 117 or Definition 1.1 below). Next, we use the complex interpolation between Banach spaces, (see [Tr] Theorems 1.9.3-(a) p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^{n} \langle f, e_k \rangle e_k\|_{\infty}$ (in terms of $\|f\|_{X}$), the most important is a Bernstein-type inequality (used by induction) $\|f\|_{p} \leq c_p \|B\|_{\infty} \|f\|_{p}$ for a (rational)
function $f$ in the star-invariant subspace $H^p \cap BzH^p$ generated by a (finite) Blaschke product $B$, (K. Dyakonov [Dya1],[Dya2]). For $p = 2$, it is given in [Z2] an alternative proof of the needed Bernstein-type estimate.

The lower bound problem (for $C_{n,r}(X, H^\infty)$) is treated by using the “worst” interpolation $n$–tuple $\sigma = \sigma_{n,\lambda} = \{\lambda, ..., \lambda\}$, a one-point set of multiplicity $n$ (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to $\lambda$. We notice that spaces $X = l^p_\alpha((k + 1)^\alpha)$ satisfy the condition $X \circ b_\lambda \subset X$ when $p = 2$, whereas this is not the case for $p \neq 2$ and this makes the problem of upper/lower bound harder.

Before starting Section 1 and studying upper estimates for $c(\sigma, X, H^\infty)$, we give the following lemma which is going to be useful throughout this paper, in particular in view of applying interpolation between Banach spaces. Its proof is obvious.

Lemma 0. Let $X$ be a Banach space of holomorphic functions in the unit disc $D$ and $\sigma = \{\lambda_1, \lambda_2, ..., \lambda_n\} \subset D$ a finite subset of the disc. We define the Blaschke product $B_\sigma = \prod_{i=1}^{n} b_{\lambda_i}$ where $b_{\lambda_i} = \frac{\lambda_i - z}{1 - \lambda_i z}$ is an elementary Blaschke factor for $\lambda_i \in D$.

**Definition 1.1.** Malmquist family. For $k \in [1, n]$, we set $f_k = \frac{1}{1 - \lambda_k z}$, and define the family $(e_k)_{k=1}^{n}$, (which is known as Malmquist basis, see [N1] p.117), byand

The paper is organized as follows. In Section 1 we prove the upper bounds from Theorem B. Sections 2&3 are devoted to the proof of Theorem C, and Sections 4&5 to the proof of Theorem D. Theorem E is proved in Section 6. Finally, in Section 7 we prove the lower bounds from Theorem B.

**1. AN UPPER BOUND FOR** $c(\sigma, l^2_\alpha((k + 1)^\alpha), H^\infty)$

In this Section, we prove the right-hand side inequalities from Theorem B. That is to say that we give an upper bound for $C_{n,r}(X, H^\infty)$ where $X = l^2_\alpha((k + 1)^\alpha)$, $\alpha \leq 0$. We recall also that $H^2 = l^2_\alpha(1)$ and $L^2_\alpha = l^2_\alpha(\frac{1}{(k + 1)^2})$. As it is mentionned in the paragraph 4 of the introduction, the main technical tool used in the proof of the upper estimate for

$C_{n,r}(l^2_\alpha((k + 1)^\alpha), H^\infty)$,

is a Bernstein-type inequality applied to a rational function.

In Definitions 1.1, 1.2, 1.3 and in Remark 1.4 below, $\sigma = \{\lambda_1, ..., \lambda_n\}$ is a sequence in the unit disc $D$, $B_\sigma = \prod_{i=1}^{n} b_{\lambda_i}$ is the finite Blaschke product corresponding to $\sigma$, where $b_{\lambda} = \frac{\lambda - z}{1 - \lambda z}$ is an elementary Blaschke factor for $\lambda \in D$.

**Definition 1.1.** Malmquist family. For $k \in [1, n]$, we set $f_k = \frac{1}{1 - \lambda_k z}$, and define the family $(e_k)_{k=1}^{n}$, (which is known as Malmquist basis, see [N1] p.117), byand
(1.1) \( e_1 = \frac{f_1}{\|f_1\|_2} \) and \( e_k = (\Pi_{j=1}^{k-1}b_{\lambda_j}) \frac{f_k}{\|f_k\|_2} \), for \( k \in [2, n] \), where \( \|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2} \).

**Definition 1.2.** The model space \( K_{B_\sigma} \). We define \( K_{B_\sigma} \) to be the \( n \)-dimensional space:

(1.2) \( K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \Theta B_\sigma H^2 \).

**Definition 1.3.** The orthogonal projection \( P_{B_\sigma} \) on \( K_{B_\sigma} \). We define \( P_{B_\sigma} \) to be the orthogonal projection of \( H^2 \) on its \( n \)-dimensional subspace \( K_{B_\sigma} \).

**Remark 1.4.** The Malmquist family \( (e_k)_{k=1}^n \) corresponding to \( \sigma \) is an orthonormal basis of \( K_{B_\sigma} \). In particular,

(1.4) \( P_{B_\sigma} = \sum_{k=1}^n (., e_k)_{H^2} e_k \),

where \( (., .)_{H^2} \) means the scalar product on \( H^2 \).

### 1.1. Bernstein-type inequalities for rational functions

Bernstein-type inequalities for rational functions were the subject of a number of papers and monographs (see, for instance, [L], [BoEr], [DeLo], [B]). Perhaps, the stronger and closer to the one we need here (Lemma 1.1.1) of all known results are due to K. Dyakonov [Dya1] & [Dya2]. First, we recall the following lemma (proved in Proposition 4.1 of [Z2]).

**Lemma 1.1.1.** Let \( B = \Pi_{j=1}^{n}b_{\lambda_j} \), be a finite Blaschke product (of order \( n \)), \( r = \max_j |\lambda_j| \), and \( f \in K_B \). Then,

\[
\| f' \|_{H^2} \leq \frac{5}{2} \frac{n}{1 - r} \| f \|_{H^2}.
\]

Lemma 1.1.1 is in fact a partial case \((p = 2)\) of the following K. Dyakonov’s result [Dya1] (which is, in turn, a generalization of M. Levin’s inequality [L] corresponding to the case \( p = \infty \)) : it is proved in [Dya1] that the norm \( \| D \|_{K_B^p \to H^p} \) of the differentiation operator \( Df = f' \) on the star-invariant subspace of the Hardy space \( H^p \), \( K_B^p := H^p \cap BzH^p \), (where the bar denotes complex conjugation) satisfies the following inequalities

\[
\frac{c_2'}{p} \| B' \|_\infty \leq \| D \|_{K_B^p \to H^p} \leq \frac{c_2}{p} \| B' \|_\infty;
\]

for every \( p, 1 \leq p \leq \infty \) where \( c_p \) and \( c_2' \) are positives constants depending on \( p \) only, \( B \) is a finite Blaschke product and \( \| . \|_\infty \) means the norm in \( L^\infty (\mathbb{T}) \). For the partial case considered in Lemma 1.1.1 above, the constant \( \frac{5}{2} \) is slightly better. More precisely, it is proved in [Dya1] that \( c_2' = \frac{1}{36c} \), \( c_2 = \frac{26\sqrt{3}}{27} \) and \( c = 2\sqrt{3\pi} \) (as one can check easily (\( c \) is not precised in [Dya1])). It implies the inequality of Lemma 1.1 with a constant about \( \frac{13}{2} \) instead of \( \frac{5}{2} \).

The sharpness of the inequality from Lemma 1.1.1 is discussed in [Z3]. Here we use it by induction in order to get the following corollary.
Corollary 1.1.2. Let $B = \prod_{j=1}^{n} b_{\lambda_j}$ be a finite Blaschke product (of order $n$), $r = \max_j |\lambda_j|$, and $f \in K_B$. Then,

$$\|f^{(k)}\|_{H^2} \leq k! \left( \frac{r}{2} \right)^k \left( \frac{n}{1 - r} \right)^k \|f\|_{H^2},$$

for every $k = 0, 1, \ldots$

Indeed, since $z^{k-1} f^{(k-1)} \in K_B$, we obtain applying Lemma 1.1.1 for $B^k$ instead of $B$,

$$\|z^{k-1} f^{(k)} + (k - 1) z^{k-2} f^{(k-1)}\|_{H^2} \leq \frac{5}{2} \frac{kn}{1 - r} \|z^{k-1} f^{(k-1)}\|_{H^2} = \frac{5}{2} \frac{kn}{1 - r} \|f^{(k-1)}\|_{H^2},$$

since $|z^{k-1}| = 1$, $\forall z \in \mathbb{T}$. In particular,

$$\|z^{k-1} f^{(k)}\|_{H^2} - \|(k - 1) z^{k-2} f^{(k-1)}\|_{H^2} \leq \frac{5}{2} \frac{kn}{1 - r} \|f^{(k-1)}\|_{H^2},$$

which gives, since $|z^{k-1}| = |z^{k-2}| = 1$, $\forall z \in \mathbb{T}$,

$$\|f^{(k)}\|_{H^2} \leq \frac{5}{2} \frac{kn}{1 - r} \|f^{(k-1)}\|_{H^2} + (k - 1) \|f^{(k-1)}\|_{H^2} \leq \left( \frac{5}{2} \frac{n}{1 - r} + 1 \right) k \|f^{(k-1)}\|_{H^2} = \frac{5}{2} \frac{kn}{1 - r} \left( 1 + \frac{2}{5} \right) \|f^{(k-1)}\|_{H^2} \leq \frac{7}{2} \frac{kn}{1 - r} \|f^{(k-1)}\|_{H^2}.$$

By induction,

$$\|f^{(k)}\|_{H^2} \leq k! \left( \frac{7}{2} \frac{n}{1 - r} \right)^k \|f\|_{H^2}.$$

\[\Box\]

1.2. The proof of Theorem B (the upper bound only)

The first consequence of Corollary 1.1.2 in the following one.

Corollary 1.2.1. Let $N \geq 0$ be an integer. Then,

$$C_{n,r} \left( l_a^2 \left( (k+1)^{-N} \right), H^\infty \right) \leq A \left( \frac{n}{1 - r} \right)^{\frac{2N+1}{2}},$$

for all $r \in [0, 1]$, $n \geq 1$, where $A = A(N)$ is a constant depending on $N$ (of order $N^{2N}$ from the proof below).

Indeed, let $H = l_a^2 \left( (k+1)^{-N} \right)$ and $B = B_\sigma$ the finite Blaschke product corresponding to $\sigma$. Let $\tilde{P}_B$ be the orthogonal projection of $H$ onto $K_B = K_B(H^2)$. Then $\tilde{P}_{B|H^2} = P_B$, where $P_B$ is defined in 4.3. We notice that $\tilde{P}_B : H \rightarrow H$ is a bounded operator and the adjoint $\tilde{P}_B^* : H^* \rightarrow H^*$ of $\tilde{P}_B$ relatively to the Cauchy pairing $\langle \cdot, \cdot \rangle$ satisfies $\tilde{P}_B^* \varphi = \tilde{P}_B \varphi = P_B \varphi$, $\forall \varphi \in H^* \subset H^2$, where $H^* = l_a^2 \left( (k+1)^N \right)$ is the dual of $H$ with respect to this pairing. If $f \in H$, then $|\tilde{P}_B f(\zeta)| = |\langle \tilde{P}_B f, k_\zeta \rangle| = |\langle f, \tilde{P}_B^* k_\zeta \rangle|$, with $k_\zeta = (1 - \bar{\zeta}z)^{-1} \in H^2$ and

$$|\tilde{P}_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left( \|P_B k_\zeta\|_{H^2} + \|\tilde{P}_B k_\zeta\|_{H^2} \right).$$
where

\[ K_N = \max \left\{ N^N, \sup_{k \geq N} \frac{(k + 1)^N}{k(k - 1) \ldots (k - N + 1)} \right\} = \max \left\{ N^N, \frac{(N + 1)^N}{N!} \right\} = \left\{ \begin{array}{ll} N^N, & \text{if } N \geq 3 \\ \frac{(N+1)^N}{N!}, & \text{if } N = 1, 2 \end{array} \right.. \]

(Indeed, the sequence \( \frac{(k+1)^N}{k(k-1) \ldots (k-N+1)} \) is decreasing since \( (1+x)^{-N} \geq 1-Nx \) for all \( x \in [0, 1] \), and \( \left[ N^N > \frac{(N+1)^N}{N!} \right] \Leftrightarrow N \geq 3 \). Since \( P_B k_\zeta \in K_B \), Corollary 1.1.2 implies

\[
|\widetilde{P}_B f(\zeta)| \leq \|f\|_H K_N \left( \|P_B k_\zeta\|_{H^2} + N! \left( \frac{7}{2} \frac{n}{1-r} \right)^N \|P_B k_\zeta\|_{H^2} \right) \leq A(N) \left( \frac{n}{1-r} \right)^{N+\frac{1}{2}} \|f\|_H,
\]

where \( A(N) = \sqrt{2} K_N \left( 1 + N! \left( \frac{7}{2} \right)^N \right) \), since \( \|P_B k_\zeta\|_2 \leq \frac{\sqrt{2}}{\sqrt{1-r}} \). \( \square \)

**Proof of Theorem B (the right-hand side inequality only).** Applying Lemma 0 with \( X = l^2_a ((k+1)\alpha) \), we get

\[
\|T\|_{l^2_a ((k+1)\alpha) \rightarrow H^\infty / B_\sigma H^\infty} = c \left( \sigma, l^2_a ((k+1)\alpha), H^\infty \right),
\]

where \( T \) and \( B_\sigma \) are defined in Lemma 0. Moreover, there exists an integer \( N \) such that \( N - 1 \leq -\alpha \leq N \). In particular, there exists \( 0 \leq \theta \leq 1 \) such that \( -\alpha = (1-\theta)(N-1) + \theta N \). And since (as in the proof of Theorem C of [Z2], we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

\[
\left[ \left( \frac{1}{(k+1)^{N-1}} \right), l^2_a (1) \right]_{\theta, 2} = l^2_a \left( \left( \frac{1}{(k+1)^{N-1}} \right)^{2N+1 \theta}, \left( \frac{1}{(k+1)^N} \right)^{2N+1 \theta} \right) = l^2_a \left( \frac{1}{(k+1)^{(1-\theta)(N-1)+\theta N}} \right) = l^2_A ((k+1)^{\alpha}),
\]

this gives, using Corollary 1.2.1 and [Tr] Theorem 1.9.3-(a) p.59,

\[
\|T\|_{l^2_a ((k+1)\alpha) \rightarrow H^\infty / B_\sigma H^\infty} \leq \left( c \left( \sigma, l^2_a \left( \frac{1}{(k+1)^{N-1}} \right), H^\infty \right) \right)^{1-\theta} \left( c \left( \sigma, l^2_a \left( \frac{1}{(k+1)^{N}} \right), H^\infty \right) \right)^{\theta} \leq \left( A(N-1) \left( \frac{n}{1-r} \right)^{2N+1 \theta}, \left( A(N) \left( \frac{n}{1-r} \right)^{2N+1 \theta} \right)^{\theta} = A(N-1)^{1-\theta} A(N)^{\theta} \left( \frac{n}{1-r} \right)^{\frac{(2N-1)(1-\theta)}{2} + \frac{(2N+1)\theta}{2}}.
\]

It remains to use \( \theta = 1 - \alpha - N \) and set \( A(\alpha) = A(N-1)^{1-\theta} A(N)^{\theta} \). \( \square \)
2. AN UPPER BOUND FOR $c \left( \sigma, l^1_a (w_k), H^\infty \right)$

The aim of this Section is to prove the right-hand side inequality of Theorem C for the partial case $p = 1$, in which the upper bound $\left( \frac{n}{1-r} \right)^{N+\frac{1}{2}}$ is not as sharp as in Section 1. We suspect $\left( \frac{n}{1-r} \right)^{-\alpha}$ is the sharp bound for the quantity $C_{n,r} \left( l^1_a ((k+1)^\alpha), H^\infty \right)$. First, we recall the following lemma already proved in [Z2] (see Lemma 1.1.5 of [Z2]).

**Lemma 2.0.** Let $\sigma = \{ \lambda_1, \ldots, \lambda_n \}$ be a sequence in the unit disc $\mathbb{D}$ and $(e_k)_{k=1}^n$ the Malmquist family (see 4.1) corresponding to $\sigma$. The map $\Phi : \text{Hol}(\mathbb{D}) \to Y \subset \text{Hol}(\mathbb{D})$ defined by

$$
\Phi : f \mapsto \sum_{k=1}^n \left( \sum_{j \geq 0} \hat{f}(j)e_k(j) \right) e_k,
$$

is well defined (if $h \in \text{Hol}(\mathbb{D})$ and $k \in \mathbb{N}$, $\hat{h}(k)$ stands for the $k^{th}$ Taylor coefficient of $h$) and has the following properties:

(a) $\Phi_{|H^2} = P_{B_\sigma}$,

(b) $\Phi$ is continuous on $\text{Hol}(\mathbb{D})$ for the uniform convergence on compact sets of $\mathbb{D}$,

(c) Let $\Psi = \text{Id}_{|X} - \Phi_{|X}$, then $\text{Im}(\Psi) \subset B_\sigma X$,

where $B = B_\sigma$ is the finite Blaschke product corresponding to $\sigma$ and $P_{B_\sigma}$ is defined in 1.3.

Now we prove the following partial case of Theorem C.

**Lemma 2.1.** Let $N \geq 0$ be an integer. Then,

$$
C_{n,r} \left( l^1_a ((k+1)^{-N}), H^\infty \right) \leq A_1 \left( \frac{n}{1-r} \right)^{N+\frac{1}{2}},
$$

for all $r \in [0, 1]$, $n \geq 1$, where $A_1 = A_1(N)$ is a constant depending only on $N$ (of order $N^{2N}$ from the proof below).

Indeed, the proof is exactly the same as in Corollary 1.2.1: if $\sigma$ is a sequence of $\mathbb{D}$ with $\# \sigma \leq n$, and $f \in l^1_a \left( \frac{1}{(k+1)^N} \right) = X$, then

$$
\Phi(f)(\zeta) = \sum_{k=1}^n \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1}^n e_k(\zeta)e_k \right\rangle,
$$

where $\langle ., . \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k)\overline{g(k)}$. In particular,

$$
|\Phi(f)(\zeta)| = |\langle f, P_B k_\zeta \rangle|,
$$

where $B = B_\sigma$ is the finite Blaschke product corresponding to $\sigma$, $P_B$ is defined in 1.3 and $k_\zeta = (1 - \zeta)^{-1}$. Denoting $X^*$ the dual of $X$ with respect to this pairing, $X^* = l^\infty_a ((k+1)^N)$, we get,

$$
|\Phi(f)(\zeta)| \leq \|f\|_X \|P_B^* k_\zeta\|_{X^*} = \|f\|_X \|P_B k_\zeta\|_{X^*} \leq \left(\sup_{0 \leq k \leq N-1} \left| \overrightarrow{P_B^* k_\zeta}(k) \right|, \sup_{k \geq N} \left( \overrightarrow{P_B^* k_\zeta}^{(N)}(k - N) \right) \right) \leq \|f\|_X K_N \max \left\{ \|P_B k_\zeta\|_{H^2}, \left\| P_B k_\zeta(\zeta)^{(N)} \right\|_{H^2} \right\},
$$
where $K_N$ is defined in the the proof of Corollary 1.2.1. Since $P_Bk_\zeta \in K_B$, Corollary 1.1.2 implies

$$|\Phi(f)\zeta| \leq \|f\|_X K_N \left(\|P_Bk_\zeta\|_{H^2} + N! \left(\frac{7}{2}\right)^N \left(\frac{n}{1-r}\right)^N \|P_Bk_\zeta\|_{H^2}\right) \leq K_N \|f\|_X \|P_Bk_\zeta\|_2 \left(1 + N! \left(\frac{7}{2}\right)^N \left(\frac{n}{1-r}\right)^N\right),$$

which completes the proof setting $A_1(N) = 2\sqrt{2N! \left(\frac{7}{2}\right)^N K_N}$.

\[\square\]

**Proof of Theorem C for $p = 1$ only (the right-hand inequality only).** This is the same reasoning as in Theorem B. Applying Lemma 0 with $X = l_1^1 ((k+1)^\alpha)$, we get

$$\|T\|_{l_1^1 ((k+1)^\alpha) \rightarrow H^\infty / B_\sigma H^\infty} = c(\sigma, l_1^1 ((k+1)^\alpha), H^\infty),$$

where $T$ and $B_\sigma$ are defined in Lemma 0. It remains to use both Theorem B, Theorem C for the special case $p = 1$ (already proved in Section 2), and (again) [Tr] Theorem 1.9.3-(a) p.59 to complete the proof.

\[\square\]

### 3. An upper bound for $c(\sigma, l_p^p ((w_k), H^\infty), 1 \leq p \leq 2$

The aim of this section is to generalize the result of Section 2 to the case $p \in [1, 2]$. In other words we prove Theorem C, in which again, the upper bound $\left(\frac{n}{1-r}\right)^{\frac{1-2\alpha}{2}}$ is not sharp as sharp as in Section 1. We suppose that the sharp upper (and lower) bound here should be of the order of $\left(\frac{n}{1-r}\right)^{1-\alpha-\frac{\alpha}{p}}$.

**Proof of Theorem C.** We first prove the right hand side inequality. The scheme of the proof is completely the same as in Theorem B, but we simply use interpolation between $l^1$ and $l^2$ (the classical Riesz-Thorin theorem). Applying Lemma 0 with $X = l_p^p ((k+1)^\alpha)$, we get

$$\|T\|_{l_p^p ((k+1)^\alpha) \rightarrow H^\infty / B_\sigma H^\infty} = c(\sigma, l_p^p ((k+1)^\alpha), H^\infty),$$

where $T$ and $B_\sigma$ are defined in Lemma 0. It remains to use both Theorem B, Theorem C for the special case $p = 1$ (already proved in Section 2), and (again) [Tr] Theorem 1.9.3-(a) p.59 to complete the proof of the right hand side inequality.

Now, we prove the left hand side one. First, it is clear that

$$C_{n,r} (l_p^p ((k+1)^\alpha), H^\infty) \geq \|\varphi_r\|_{l_p^p ((k+1)^\alpha)} = \left(\sum_{k \geq 0} (k+1)^{\alpha-1} (r^p)^k \right)^\frac{1}{p},$$

where $\varphi_r$ is the evaluation functional

$$\varphi_r(f) = f(r), \ f \in X,$$
and \( p' \) is the conjugate of \( p : \frac{1}{p} + \frac{1}{p'} = 1 \). Now, since
\[
\sum_{k \geq 1} k^s x^k \sim \int_1^\infty t^s x' dt \sim \Gamma(s + 1)(1 - x)^{-s - 1}, \quad \text{as } x \to 1,
\]
for all \( s > -1 \), where \( \Gamma \) stands for the usual Gamma function, \( \Gamma(z) = \int_0^{+\infty} e^{-s} s^{z-1} ds \), we get
\[
\sum_{k \geq 0} (k + 1)^{(\alpha - 1)p'} \left( r^{p'} \right)^k \sim \Gamma \left( (\alpha - 1)p' + 1 \right) \left( 1 - r^{p'} \right)^{-(\alpha - 1)p' - 1}, \quad \text{as } r \to 1.
\]
But,
\[
\left( 1 - r^{p'} \right)^{-(\alpha - 1)p' - 1} = \left( \frac{r - 1}{r^{p'} - 1} \right)^{(\alpha - 1)p' + 1} \sim \left( \frac{1}{p' - 1} \right)^{(\alpha - 1)p' + 1}, \quad \text{as } r \to 1.
\]
As a result,
\[
\sum_{k \geq 0} (k + 1)^{(\alpha - 1)p'} \left( r^{p'} \right)^k \sim \frac{\Gamma \left( (\alpha - 1)p' + 1 \right)}{p'^{(\alpha - 1)p' + 1}} \left( \frac{1}{1 - r} \right)^{(\alpha - 1)p' + 1}, \quad \text{as } r \to 1,
\]
and
\[
\left( \sum_{k \geq 0} (k + 1)^{(\alpha - 1)p'} \left( r^{p'} \right)^k \right)^{\frac{1}{p'}} \sim \left( \frac{1}{p'} \right)^{\frac{1}{p'} + (\alpha - 1)} \left( \Gamma \left( (\alpha - 1)p' + 1 \right) \right)^{\frac{1}{p'}} (1 - r)^{-(\alpha - 1) - \frac{1}{p'}}, \quad \text{as } r \to 1.
\]
This completes the proof since \( \frac{1}{p'} = 1 - \frac{1}{p'} \). □

4. AN UPPER BOUND FOR \( c \left( \sigma, l^\infty_a \left( w_k \right), H^\infty \right) \)

In this Section, we prove the right-hand side inequality from Theorem D for \( p = \infty \) only, in which -again- the upper bound \( \left( \frac{n}{r} \right)^{\frac{3}{2} - 2\alpha} \) is not as sharp as in Section 1. We can suppose here that the constant \( \left( \frac{n}{r} \right)^{1 - \alpha} \) is the sharp bound for the quantity \( C_{n,r} \left( l^\infty_a \left( (k + 1)^\alpha \right), H^\infty \right) \). First we prove the following partial case of Theorem D.

**Corollary 4.1** Let \( N \geq 0 \) be an integer. Then,
\[
C_{n,r} \left( l^\infty_a \left( (k + 1)^{-N} \right), H^\infty \right) \leq A_\infty \left( \frac{n}{1 - r} \right)^{N + \frac{3}{2}},
\]
for all \( r \in [0, 1[, \, n \geq 1 \), where \( A_\infty = A_\infty(N) \) is a constant depending on \( N \) (of order \( N^{2N} \) from the proof below).

**Proof.** We use literally the same method as in Corollary 1.2.1 and Lemma 2.1. Indeed, if \( f \in l^\infty_a \left( \frac{1}{(k + 1)^\alpha} \right) = X \), then
\[
|\Phi(f)(\zeta)| = |\langle f, P_B k_\zeta \rangle|,
\]
where \( \Phi \) is defined in Lemma 2.0, \( B = B_\sigma \) is the finite Blaschke product corresponding to \( \sigma \), \( P_B \) is defined in 1.3, \( k_\zeta = (1 - \zeta z)^{-1} \) and \( \langle ., . \rangle \) means the Cauchy pairing. Denoting \( X^* \) the dual of \( X \)
with respect to this pairing, \(X^* = l^1_\alpha ((k + 1)^N)\), we get

\[
|\Phi(f)(\zeta)| \leq \|f\|_X \|P_B k\zeta\|_X^* \leq \|f\|_H K_N \left(\|P_B k\zeta\|_W + \|P_B k\zeta\|^{(N)}_W\right),
\]

where \(W = \left\{f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_W = \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}\) stands for the Wiener algebra, and \(K_N\) is defined in Corollary 1.3. Now, applying Hardy’s inequality (see \([N2]\) p.370, 8.7.4 (c)),

\[
|\Phi(f)(\zeta)| \leq \|f\|_X K_N \left(\pi \left\|P_B k\zeta\right\|_{H^1} + \left\|P_B k\zeta\right\|(0) + \pi \left\|P_B k\zeta\right\|^{(N+1)}_{H^1} + \left\|P_B k\zeta\right\|(N)(0)\right) \leq
\]

\[
\leq \|f\|_X K_N \left(\pi \left\|P_B k\zeta\right\|_{H^2} + \left\|P_B k\zeta\right\|(0) + \pi \left\|P_B k\zeta\right\|^{(N+1)}_{H^2} + \left\|P_B k\zeta\right\|(N)(0)\right),
\]

which gives using both Lemma 1.1 and Corollary 1.2,

\[
|\Phi(f)(\zeta)| \leq \frac{\alpha}{1 - r} \left(\frac{5}{2}\right) \|P_B k\zeta\|_{H^2} + \|P_B k\zeta\|(0) + (N + 1)! \left(\frac{7}{2}\right)^{N+1} \|P_B k\zeta\|_{H^2} + \left\|P_B k\zeta\right\|(N)(0) \leq \frac{\alpha}{1 - r} \left(\frac{5}{2}\right) \|P_B k\zeta\|_{H^2} + \|P_B k\zeta\|_{H^2} + N! \|P_B k\zeta\|_{H^2}.
\]

This completes the proof since \(\|P_B k\zeta\|_{H^2} \leq (\frac{2m}{1 - r})^{\frac{1}{2}}\). \(\square\)

Proof of Theorem D for \(p = \infty\) (the right-hand side inequality only). This is the same application of interpolation between Banach spaces, as before (Theorem 1.0&2.0) excepted that this time we apply Lemma 0 with \(X = l^\infty_\alpha ((k + 1)^\alpha)\) to get

\[
\|T\|_{l^\infty_\alpha ((k+1)^\alpha)} \rightarrow H^{\infty} / B_\alpha H^{\infty} = c(\sigma, l^\infty_\alpha ((k + 1)^\alpha), H^{\infty}),
\]

where \(T\) and \(B_\sigma\) are defined in Lemma 0.

Applying Lemma 4.1 and using (again) \([Tr]\) Theorem 1.9.3-(a) p.59, we can complete the proof. \(\square\)

5. AN UPPER BOUND FOR \(c(\sigma, l^p_\alpha (w_k), H^{\infty}), 2 \leq p \leq \infty\)

The aim of this section is to prove Theorem D. As before, the upper bound \((\frac{n}{1 - r})^{\frac{1}{2} - \frac{1}{p}}\) is not as sharp as in Section 1. We can suppose here the constant \((\frac{n}{1 - r})^{1 - \frac{1}{p}}\) should be a sharp upper (and lower) bound for the quantity \(C_{a, r} (l^p_\alpha ((k + 1)^\alpha), H^{\infty}), 2 \leq p \leq +\infty\).

Proof of Theorem D. We first prove the right hand side inequality. The proof repeats the scheme from previous theorems and from Theorem C in particular. We have already seen (in Theorem C) that
\[ \|T\|_{L^p_a((k+1)^{\alpha}) \to H^\infty / B_\sigma H^\infty} = c(\sigma, I_p^a((k+1)^{\alpha}), H^\infty), \]

where \( T \) and \( B_\sigma \) are defined in Lemma 0. Now, using both Theorem B, Theorem D for the particular case \( p = \infty \) (already proved in Section 4), and [Tr] Theorem 1.9.3-(a) p.59, we complete the proof. The proof of the left hand side inequality is exactly the same as in Theorem C. \( \square \)

6. CARATHÉODORY-SCHUR INTERPOLATION IN WEIGHTED BERGMAN SPACES

We suppose that \( X = L^p_a((1-|z|^2)^{\beta} \, dA) \), \( \beta > -1 \) and \( 1 \leq p \leq 2 \). Our aim in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

\[ c(\sigma_{n,\lambda}, X, H^\infty) = \sup \{ \|f\|_{H^\infty / b^{n}_\lambda H^\infty} : f \in X, \|f\| \leq 1 \}, \]

where \( \|f\|_{H^\infty / b^{n}_\lambda H^\infty} = \inf \{ \|f + b^{n}_\lambda g\| \, : \, g \in X \} \), and \( \sigma_{n,\lambda} = \{\lambda, \lambda, \ldots, \lambda\} \), \( \lambda \in \mathbb{D} \). The corresponding interpolation problem is: given \( f \in X \), to minimize \( \|h\|_{\infty} \) such that \( h^{(j)}(\lambda) = f^{(j)}(\lambda) \), \( 0 \leq j < n \).

For this partial case, we prove Theorem E which is a generalization of the estimate from Theorem B.

We first need a simple equivalent to \( I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^{\beta} \, dr, \beta > -1 \).

**Lemma 6.1.** Let \( k \geq 0 \), \( \beta > -1 \) and \( I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^{\beta} \, dr \). Then,

\[ I_k(\beta) \sim \frac{1}{2} \frac{\Gamma(\beta + 1)}{\beta^{k+1}}, \]

for \( k \to \infty \), where \( \Gamma \) stands for the usual Gamma function, \( \Gamma(z) = \int_0^{+\infty} e^{-s}s^{z-1} \, ds \).

**Proof.** Let \( a = \frac{1}{\sqrt{k+1}}, b = \max(1, a^{\beta}) \). Since \( 1 - e^{-u} \sim u \) as \( u \to 0 \), we have

\[
I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^{\beta} \, dr = \int_0^\infty e^{-(2k+1)t}(1-e^{-2t})^{\beta}e^{-t} \, dt = \\
= \int_0^a e^{-2(k+1)t}(1-e^{-2t})^{\beta} \, dt + \int_a^\infty e^{-2(k+1)t}(1-e^{-2t})^{\beta} \, dt = \\
= \int_0^a e^{-2(k+1)t}(1-e^{-2t})^{\beta} \, dt + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
= (1+o(1)) \int_0^a e^{-2(k+1)t}(2t)^{\beta} \, dt + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
= (1+o(1)) \int_0^{(2k+1)a} e^{-s}\left(\frac{s}{k+1}\right)^{\beta} \, ds + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
= \frac{1}{2} \frac{1}{(k+1)^{\beta+1}}(1+o(1)) \int_0^{2(k+1)a} e^{-s}s^{\beta} \, ds + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) =
\]
\[
\frac{1}{2} \Gamma(\beta + 1) (1 + o(1)) + O\left(\frac{b}{k + 1} e^{-2\alpha(k+1)}\right) = \\
= \frac{1}{2} \Gamma(\beta + 1) (1 + o(1)) \sim \frac{1}{2} \Gamma(\beta + 1),
\]
which completes the proof. \(\square\)

**Proof of Theorem E.** **Step 1.** We start to prove the Theorem for \(p = 1\).

Let \(f \in X = L^1_a \left( (1 - |z|^2)^\beta \, dA \right) \) such that \(\|f\|_X \leq 1\). Since \(X \circ b_\lambda = X\), we have
\(f \circ b_\lambda = \sum_{k \geq 0} a_k z^k \in X\). Let \(p_n = \sum_{k=0}^{n-1} a_k z^k\) and \(g = p_n \circ b_\lambda\). Then, \(f \circ b_\lambda - p_n \in z^n X\) and \(f - p_n \circ b_\lambda \in (z^n X) \circ b_\lambda = b_\lambda^n X\). Now, \(p_n \circ b_\lambda = \sum_{k=0}^{n-1} a_k b_\lambda^k\) and

\[
\|p_n \circ b_\lambda\|_\infty = \|p_n\|_\infty \leq A_n \|f \circ b_\lambda\|_X,
\]
where \(A_n = \|\sum_{k \geq 0} a_k z^k \mapsto \sum_{k=0}^{n-1} a_k z^k\|_{X \to H^\infty}\). Now,
\[
\|f \circ b_\lambda\|_X \leq \int_D |f (b_\lambda(z)) (1 - |z|^2)^\beta \, dA = \int_D |f (w) (1 - |b_\lambda(w)|^2)^\beta |b_\lambda^\prime (w)|^2 \, dA = \\
\leq 2^\beta \int_D |f (w)| \left( \frac{(1 - |\lambda|^2)}{|1 - \lambda w|^2} \right) \left( \frac{(1 - |\lambda|^2)}{|1 - \lambda w|^2} \right) \, dA = \\
= \int_D |f (w)| (1 - |w|^2)^\beta \left( \frac{(1 - |\lambda|^2)}{|1 - \lambda w|^2} \right)^{2 + \beta} \, dA \leq \\
\leq \sup_{w \in D} \left( \frac{(1 - |\lambda|^2)}{|1 - \lambda w|^2} \right)^{2 + \beta} \int_D |f (w)| (1 - |w|^2)^\beta \, dA \leq \left( \frac{(1 - |\lambda|^2)}{(1 - |\lambda|)^2} \right)^{2 + \beta} \|f\|_X,
\]
which gives,
\[
\|f \circ b_\lambda\|_X \leq \left( \frac{1 + |\lambda|}{1 - |\lambda|} \right)^{2 + \beta} \|f\|_X.
\]

We now give an estimation for \(A_n\). Let \(g(z) = \sum_{k \geq 0} \hat{g}(k) z^k \in X\), then
\[
\left\| \sum_{k=0}^{n-1} \hat{g}(k) z^k \right\|_\infty \leq \sum_{k=0}^{n-1} |\hat{g}(k)|.
\]
Now, noticing that
\[
\int_D g(w) |w|^\beta (1 - |w|^2)^\beta \, dA = \int_0^1 \int_0^{2\pi} f(re^{it}) r^k e^{-ikt} (1 - r^2)^\beta r \, dt \, dr = \\
= \int_0^1 (1 - r^2)^\beta r^{k+1} \int_0^{2\pi} f(re^{it}) e^{-ikt} \, dt \, dr = \int_0^1 \hat{g}_r(k) r^{k+1} (1 - r^2)^\beta \, dr,
\]
where \( g_r(z) = g(rz) \), \( \tilde{g}_r(k) = r^k \tilde{g}(k) \). Setting \( I_k(\beta) = \int_0^1 r^{2k+1}(1 - r^2)^\beta dr \), we get
\[
\tilde{g}(k) = \frac{1}{I_k(\beta)} \int_D g(w) \overline{w}^k (1 - |w|^2)^\beta dA.
\]
Then,
\[
|\tilde{g}(k)| = \frac{1}{I_k(\beta)} \left| \int_D g(w) \overline{w}^k (1 - |w|^2)^\beta dA \right| \leq \frac{1}{I_k(\beta)} \|g\|_X,
\]
which gives
\[
\left\| \sum_{k=0}^{n-1} \tilde{g}(k) z^k \right\|_\infty \leq \left( \sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \right) \|g\|_X.
\]
Now using Lemma 6.1,
\[
\sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \sim_{n \to \infty} \frac{2}{\Gamma(\beta + 1)} \sum_{k=0}^{n-1} k^{\beta + 1} \sim \frac{2c_\beta}{\Gamma(\beta + 1)} n^{\beta + 2},
\]
where \( c_\beta \) is a constant depending on \( \beta \) only. This gives
\[
\left\| \sum_{k=0}^{n-1} \tilde{g}(k) z^k \right\|_\infty \leq C_\beta n^{\beta + 2} \|g\|_X,
\]
where \( C_\beta \) is also a constant depending on \( \beta \) only. Finally, we conclude that \( A_n \leq C_\beta n^{\beta + 2} \), and as a result,
\[
\|p_n \circ b_\lambda\|_\infty \leq C_\beta n^{\beta + 2} \left( \frac{1 + |\lambda|}{1 - |\lambda|} \right)^{2+\beta} \|f\|_X,
\]
which proves the Theorem for \( p = 1 \).

**Step 2.** This step of the proof repetes the scheme from Theorems C&D. Let \( T : L_\alpha^p \left( (1 - |z|^2)^\beta \ dA \right) \rightarrow H^\infty / b_\lambda^p H^\infty \) be the restriction map defined by
\[
Tf = \left\{ g \in H^\infty : f - g \in b_\lambda^p L_\alpha^p \left( (1 - |z|^2)^\beta \ dA \right) \right\},
\]
for every \( f \). Then applying Lemma 0,
\[
\|T\|_{L_\alpha^p \left( (1 - |z|^2)^\beta \ dA \right) \rightarrow H^\infty / b_\lambda^p H^\infty} = c \left( \sigma, L_\alpha^p \left( (1 - |z|^2)^\beta \ dA \right), H^\infty \right).
\]
Now, let \( \gamma > \beta \) and \( P_\gamma : L^p \left( (1 - |z|^2)^\beta \ dA \right) \rightarrow L_\alpha^p \left( (1 - |z|^2)^\beta \ dA \right) \) be the Bergman projection, (see [H], p.6), defined by
\[
P_\gamma f = (\gamma + 1) \int_D \frac{(1 - |w|^2)^\gamma}{(1 - \overline{z}w)^{2+\gamma}} f(w) dA(w),
\]
for every $f$. $P_\gamma$ is a bounded projection from $L^p \left( (1 - |z|^2)^\beta \, dA \right)$ onto $L^p_a \left( (1 - |z|^2)^\beta \, dA \right)$ (see [H], Theorem 1.10 p.12), (since $1 \leq p \leq 2$). Moreover, since $L^p_a \left( (1 - |z|^2)^\beta \, dA \right) \subset L^p \left( (1 - |z|^2)^\gamma \, dA \right)$, we have $P_\gamma f = f$ for all $f \in L^p \left( (1 - |z|^2)^\beta \, dA \right)$, (see [H], Corollary 1.5 p.6). As a result,

$$\|TP_\gamma\|_{L^p \left( (1 - |z|^2)^\beta \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty}} \leq \|TP_\gamma\|_{L^p \left( (1 - |z|^2)^\gamma \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty}} ;$$

for all $1 \leq p \leq 2$. We set

$$c_i(\beta) = \|P_\gamma\|_{L^i \left( (1 - |z|^2)^\beta \, dA \right) \to L^\lambda_a \left( (1 - |z|^2)^\beta \, dA \right)} ,$$

for $i = 1, 2$. Then,

$$\|TP_\gamma\|_{L^i \left( (1 - |z|^2)^\beta \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty}} \leq \|T\|_{L^i \left( (1 - |z|^2)^\beta \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty} \|P_\gamma\|_{L^i \left( (1 - |z|^2)^\beta \, dA \right) \to L^\lambda_a \left( (1 - |z|^2)^\beta \, dA \right)} =$$

$$= c \left( \sigma, L^1_a \left( (1 - |z|^2)^\beta \, dA \right), H^{\infty} \right) c_1(\beta) \leq \leq A'(\beta, 1) \left( \frac{n}{1 - |\lambda|} \right)^{\beta + 2} c_1(\beta) ,$$

using Step 1. In the same way,

$$\|TP_\gamma\|_{L^2 \left( (1 - |z|^2)^\beta \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty}} \leq \|T\|_{L^2_a \left( (1 - |z|^2)^\beta \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty} \|P_\gamma\|_{L^2 \left( (1 - |z|^2)^\beta \, dA \right) \to L^\lambda_a \left( (1 - |z|^2)^\beta \, dA \right)} =$$

$$= \left( \sigma, L^2_a \left( (1 - |z|^2)^\beta \, dA \right), H^{\infty} \right) c_2(\beta) ,$$

Now, we recall that

$$L^2_a \left( (1 - |z|^2)^\beta \, dA \right) = L^2 \left( (k + 1)^{-\frac{\beta + 1}{2}} \right) , \beta > -1 .$$

As a consequence,

$$\|T\|_{L^p \left( (1 - |z|^2)^\beta \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty}} = c \left( \sigma, L^2 \left( (k + 1)^{-\frac{\beta + 1}{2}} \right), H^{\infty} \right) ,$$

and, applying Theorem B,

$$\|TP_\gamma\|_{L^2 \left( (1 - |z|^2)^\beta \, dA \right) \to H^{\infty} / b^p_\infty H^{\infty}} \leq c_2(\beta) A' \left( \frac{\beta + 1}{2} , 2 \right) \left( \frac{n}{1 - |\lambda|} \right)^{\frac{\beta + 1 + 1}{2}} =$$

$$= c_2(\beta) A' \left( \frac{\beta + 1}{2} , 2 \right) \left( \frac{n}{1 - |\lambda|} \right)^{\frac{\beta + 2}{2}} .$$
We finish the reasoning applying Riesz-Thorin Theorem, (see [Tr] for example), to the operator \(TP\). If \(p \in [1, 2]\), there exists \(0 \leq \theta \leq 1\) such that

\[
\frac{1}{p} = (1 - \theta) \frac{1}{1} + \theta \frac{1}{2} = 1 - \frac{\theta}{2},
\]

and then,

\[
\left[ L_a^1 \left( (1 - |z|^2)^\beta \, dA \right), L_a^2 \left( (1 - |z|^2)^\beta \, dA \right) \right] = L_a^p \left( (1 - |z|^2)^\beta \, dA \right),
\]

and

\[
\|TP\|_{L^p\left( (1 - |z|^2)^\theta \, dA \right) \to H^\infty / b_n^\lambda H^\infty} \leq \left( \|TP\|_{L^1\left( (1 - |z|^2)^\beta \, dA \right) \to H^\infty / b_n^\lambda H^\infty} \right)^{1-\theta} \left( \|TP\|_{L^2\left( (1 - |z|^2)^\theta \, dA \right) \to H^\infty / b_n^\lambda H^\infty} \right)^{\theta} \leq \left( c_1(\beta) A'(\beta, 1) \left( \frac{n}{1 - |\lambda|} \right)^{\beta+2} \right)^{1-\theta} \left( c_2(\beta) A' \left( \frac{\beta + 1}{2}, 2 \right) \left( \frac{n}{1 - |\lambda|} \right)^{\frac{\beta+2}{2}} \right)^{\theta} = \left( c_1(\beta) A'(\beta, 1) \right)^{1-\theta} \left( c_2(\beta) A' \left( \frac{\beta + 1}{2}, 2 \right) \right)^{\theta} \left( \frac{n}{1 - |\lambda|} \right)^{(\beta+2)(1-\theta)+\theta \frac{\beta+2}{2}}.
\]

Now, since \(\theta = 2(1 - \frac{1}{p})\), \((\beta + 2)(1 - \theta) + \theta \frac{\beta+2}{2} = \beta - (1 - \frac{1}{p})\beta + 2 - 2 + \frac{2}{p} = \frac{\beta+2}{p}\), and

\[
\|T\|_{L^p\left( (1 - |z|^2)^\beta \, dA \right) \to H^\infty / b_n^\lambda H^\infty} \leq \|TP\|_{L^p\left( (1 - |z|^2)^\beta \, dA \right) \to H^\infty / b_n^\lambda H^\infty} ;
\]

we complete the proof. □

7. A LOWER BOUND FOR \(C_{n,r}(l_a^2(w_k), H^\infty)\)

Here, we consider the weighted spaces \(H = l_a^2(w_k)\) of polynomial growth and the problem of lower estimates for the one point special case \(\sigma_{n,\lambda} = \{\lambda, \lambda, ..., \lambda\}\), \((n\text{ times})\ \lambda \in \mathbb{D}\). Recall the definition of the interpolation constant

\[
c(\sigma_{n,\lambda}, H, H^\infty) = \sup \left\{ \|f\|_{H^\infty / b_n^\lambda H^\infty} : f \in H, \|f\|_H \leq 1 \right\},
\]

where \(\|f\|_{H^\infty / b_n^\lambda H^\infty} = \inf \{\|f + b_n^\lambda g\|_\infty : g \in H\}\). In particular, our aim is to prove the sharpness of the upper estimate for the quantity
In particular, for the Hardy space $H^2$, endowed with the norm $\|f\|_{H^2} = \sum_{k \geq 0} \|\hat{f}(k)\|^2 w_k^p < \infty$, for the Bergman space $L^2$, equivalent to $(\text{reproducing kernel of } l^p)$ study the case $(n)=0$, so that $K_{\lambda}(w) = (1-\lambda z)^{-1}$, where $\lambda, z \in \mathbb{D}$, corresponding Hilbert spaces $H^p$ satisfy $\sum_{k \geq 0} \hat{h}(k)\bar{g}(k)w_k^p \geq 0$. Since one has $f(\lambda) = \sum_{k \geq 0} \hat{f}(k)\lambda^k \frac{1}{w_k} w_k^p$ ($\lambda \in \mathbb{D}$), it follows that

$$k_{\lambda}(z) = \sum_{k \geq 0} \frac{\lambda^k z^k}{w_k^p}, \quad z \in \mathbb{D}.$$ 

In particular, for the Hardy space $H^2 = l^2(1)$, we get the Szegö kernel $k_{\lambda}(z) = (1-\lambda z)^{-1}$, for the Bergman space $L^2 = l^2(1)$ - the Bergman kernel $k_{\lambda}(z) = (1-\lambda z)^{-2}$.

Conversely, following the Aronszajn theory of RKHS (see, for example [A] or [N2] p.317), given a positive definite function $\langle \lambda, z \rangle \longrightarrow k(\lambda, z)$ on $\mathbb{D} \times \mathbb{D}$ (i.e. such that $\sum_{i,j} \bar{a}_i a_j k(\lambda_i, \lambda_j) > 0$ for all finite subsets $(\lambda_i) \subset \mathbb{D}$ and all non-zero families of complex numbers $(a_i)$) one can define the corresponding Hilbert spaces $H(k)$ as the completion of finite linear combinations $\sum_i \bar{a}_i k(\lambda_i, \cdot)$ endowed with the norm

$$\left\| \sum_i \bar{a}_i k(\lambda_i, \cdot) \right\|^2 = \sum_{i,j} \bar{a}_i a_j k(\lambda_i, \lambda_j).$$

When $k$ is holomorphic with respect to the second variable and anti-holomorphic with respect to the first one, we obtain a RKHS of holomorphic functions $H(k)$ embedded into $\text{Hol}(\mathbb{D})$.

For functions $k$ of the form $k(\lambda, z) = K(\lambda z)$, where $K \in \text{Hol}(\mathbb{D})$, the positive definiteness is equivalent to $\hat{K}(j) > 0$ for every $j \geq 0$, where $\hat{K}(j)$ stands for Taylor coefficients, and in this case we have $H(k) = l^2(w_j)$, where $w_j = 1/\sqrt{\hat{K}(j)}$, $j \geq 0$. In particular, for $K(w) = (1-w)^{-\beta}$,
\( k_\lambda(z) = (1 - \bar{\lambda}z)^{-\beta}, \beta > 0, \) we have \( \hat{K}(j) = (\frac{\beta+j-1}{\beta-1}) \) (binomial coefficients), and hence \( w_j = \left( \frac{1}{\beta-1} \right)^{j} \). Indeed, deriving \( \frac{1}{1-z} \), we get by induction

\[
(1-z)^{-\beta} = \frac{1}{(\beta-1)!} \sum_{j \geq 0} (j+\beta-1)...(j+1)z^j = \sum_{j \geq 0} (\beta+j-1)z^j.
\]

Clearly, \( w_j \simeq 1/j^{\frac{\beta}{2}} \), where \( a_j \simeq b_j \) means that there exist constants \( c_1 > 0, c_2 > 0 \) such that \( c_1a_j \leq b_j \leq c_2a_j \) for every \( j \). Therefore, \( H(k) = l^2_a \left( \frac{1}{(k+1)^{\frac{\beta}{2}} \!} \right) \) (a topological identity: the spaces are the same and the norms are equivalent).

We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given the reproducing kernel \( k \) of \( H^2 \) and \( \varphi \in \{z^N : N \in \mathbb{N} \setminus \{0\}\} \), the function \( \varphi \circ k \) is also positive definit and the corresponding Hilbert space is

\[
H_{\varphi} = \varphi(H^2) = l^2_a \left( \frac{1}{(k+1)^{\frac{\beta}{2}} \!} \right).
\]

It satisfies the following property: for every \( f \in H^2, \varphi \circ f \in \varphi(H^2) \) and \( \|\varphi \circ f\|_{\varphi(H^2)}^2 \leq \varphi(\|f\|_{H^2})^2 \) (see [N2] p.320). In particular, if \( \varphi \) is a polynomial of degree \( N \) and \( k \) is the Szegö kernel then \( \varphi \circ k_\lambda(z) = \sum_{j \geq 0} c_j \lambda^j z^j \) with \( c_k \simeq (k+1)^{N-1} \), and hence

\[
H_{\varphi} = l^2_a \left( \frac{1}{(k+1)^{N-\frac{1}{2}}} \right),
\]

(a topological identity: the spaces are the same and the norms are equivalent). In particular

\[
H_{z^N} = l^2_a \left( \frac{1}{(k+1)^{\frac{N-1}{2}}} \right),
\]

for all \( N \in \mathbb{N} \setminus \{0\} \). The link between spaces of type \( l^2_a \left( \frac{1}{(k+1)^{\frac{\beta}{2}} \!} \right) \) and of type \( \varphi(H^2) = H_{\varphi} \) being established, we give the proof of the left-hand side from Theorem F.

**Proof of Theorem B (the left-hand side inequalities only).**

0) Setting \( N = 1 - 2\alpha \in \mathbb{N} \setminus \{0\}, \alpha = \frac{1-N}{2} \) and

\[
l^2_a ((k+1)^{\alpha}) = l^2_a \left( \frac{1}{(k+1)^{\frac{N-1}{2}}} \right).
\]

1) We set

\[
Q_n = \sum_{k=0}^{n-1} b_k \lambda^k \frac{(1 - |\lambda|^2)^{1/2}}{1 - \lambda z}, \quad H_n = \varphi \circ Q_n, \quad \Psi = bH_n.
\]
Then \( \|Q_n\|_2^2 = n \), and hence by the Aronszajn-deBranges inequality, see [N2] p.320, point (k) of Exercise 6.5.2, with \( \varphi(z) = z^N \) and \( K(\lambda, z) = k_\lambda(z) = \frac{1}{1-\lambda z} \), and noticing that \( H(\varphi \circ K) = H_\varphi \),

\[
\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi \left( \|Q_n\|_2^2 \right) = b^2 \varphi(n).
\]

Let \( b > 0 \) such that \( b^2 \varphi(n) = 1 \).

2) Since the spaces \( H_\varphi \) and \( H^\infty \) are rotation invariant, we have \( c(\sigma_{n, \lambda}, H_\varphi, H^\infty) = c(\sigma_{n, \mu}, H_\varphi, H^\infty) \) for every \( \lambda, \mu \) with \( |\lambda| = |\mu| = r \). Let \( \lambda = -r \). To get a lower estimate for \( \|\Psi\|_{H_\varphi/b^2 H_\varphi} \) consider \( G \) such that \( \Psi - G \in b H_{\varphi} \text{Hol}(D) \), i.e. such that \( b H_{\varphi} \circ b - G \circ b \in z^n H_{\varphi} \text{Hol}(D) \).

3) First, we show that

\[
\psi =: \Psi \circ b_\lambda = b H_{\varphi} \circ b_\lambda
\]
is a polynomial (of degree \( nN \)) with positive coefficients. Note that

\[
Q_n \circ b_\lambda = \sum_{k=0}^{n-1} z^k \left( \frac{(1-|\lambda|^2)^{1/2}}{1-\lambda b(z)} \right) = (1-|\lambda|^2)^{-1/2} \left( 1 + (1-\lambda) \sum_{k=1}^{n-1} z^k - \lambda z^n \right) = (1-r^2)^{-1/2} \left( 1 + (1+r) \sum_{k=1}^{n-1} z^k + rz^n \right) =: (1-r^2)^{-1/2} \psi_1.
\]

Hence, \( \psi = \Psi \circ b_\lambda = b H_{\varphi} \circ b_\lambda = b \varphi \circ (1-r^2)^{-1/2} \psi_1 \) and

\[
\varphi \circ \psi_1 = \sum_{k=0}^N a_k \psi^k_1(z).
\]

(In fact, we can simply assume that \( \varphi \circ \psi_1 = \psi^N_1(z) \) since \( H_\varphi = b^2 \left( \frac{1}{(k+1)^{\alpha}} \right) = H_\varphi^N \). Now, it is clear that \( \psi \) is a polynomial of degree \( Nn \) such that

\[
\psi(1) = \sum_{j=0}^{Nn} \psi(j) = b \varphi ((1-r^2)^{-1/2}(1+r)n) = b \varphi \left( \sqrt{\frac{1+r}{1-r}} n \right) > 0.
\]
4) Next, we show that there exists a constant \( c = c(\varphi) > 0 \) (for example, \( c = \alpha/2^{2N}(N - 1)! \), \( \alpha \) is a numerical constant) such that
\[
\sum_{j=0}^{m} \hat{\psi}(j) = c \sum_{j=0}^{Nn} \hat{\psi}(j) = c\psi(1),
\]
where \( m \geq 1 \) is such that \( 2m = n \) if \( n \) is even and \( 2m - 1 = n \) if \( n \) is odd.

Indeed, setting
\[
S_n = \sum_{j=0}^{n} z^j,
\]
we have
\[
\sum_{j=0}^{m} \psi_1^j = \sum_{j=0}^{m} \left( \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + rz^n \right)^k \right) \geq \sum_{j=0}^{m} \left( S_{n-1}^k \right).
\]
Next, we obtain
\[
\sum_{j=0}^{m} \left( S_{n-1}^k \right) = \sum_{j=0}^{m} \left( \frac{1}{1 - z} \right)^k = \sum_{j=0}^{m} \left( \frac{1}{1 - z} \right)^k = \sum_{j=0}^{m} \left( \frac{1}{1 - z} \right)^k \geq \alpha \frac{m^k}{(k - 1)!},
\]
where \( \alpha > 0 \) is a numerical constant. Finally,
\[
\sum_{j=0}^{m} \psi_1^j \geq \alpha \frac{m^k}{(k - 1)!} \geq \alpha \frac{(n/2)^k}{(k - 1)!} = \frac{\alpha}{2^k(k - 1)!} \cdot \frac{(1 + r)^n}{(1 + r)^k(k - 1)!} \cdot (\psi_1(1))^k \geq \frac{\alpha}{2^N(1 + r)^N(N - 1)!} \cdot (\psi_1(1))^k.
\]
Summing up these inequalities in \( \sum_{j=0}^{m} \psi = b \sum_{j=0}^{m} (\varphi \circ \psi_1) = b \sum_{k=0}^{N} a_k (1 - r^2)^{-k/2} \sum_{j=0}^{m} \psi_1^j \) (or simply taking \( k = N \), if we already supposed \( \varphi = z^N \)), we obtain the result claimed.

5) Now, using point 4) and the preceding Fejer kernel argument and denoting \( F_n = \Phi_m + z^m \Phi_m \), where \( \Phi_k \) stands for the \( k \)-th Fejer kernel, we get
\[
\|\Psi\|_{H^\infty / \psi_n H^\infty} = \|\psi\|_{H^\infty / \psi_n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty \geq \frac{1}{2} \sum_{j=0}^m \hat{\psi}(j) \geq \frac{c}{2} \psi(1) = \frac{c}{2} \psi\left(\sqrt{\frac{1+r}{1-r}}\right) = \frac{c}{2} \varphi\left(\sqrt{\frac{1+r}{1-r}}\right) \varphi(n)^{1/2} \geq \frac{a(\varphi)}{\left(\frac{n}{1-r}\right)^{m}}.
\]

(assuming that \(\varphi = z^N\))

6) In order to conclude, it remains to use (7.1).

\[\square\]

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CMI-LATP, UMR 6632, Université de Provence, 39, rue F.-Joliot-Curie, 13453 Marseille cedex 13, France

E-mail address : rzarouf@cmi.univ-mrs.fr