RANDOM PERTURBATIONS OF STOCHASTIC CHAINS WITH
UNBOUNDED VARIABLE LENGTH MEMORY

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ABSTRACT. We consider binary infinite order stochastic chains perturbed by a random noise. This means that at each time step, the value assumed by the chain can be randomly and independently flipped with a small fixed probability. We show that the transition probabilities of the perturbed chain are uniformly close to the corresponding transition probabilities of the original chain. As a consequence, in the case of stochastic chains with unbounded but otherwise finite variable length memory, we show that it is possible to recover the context tree of the original chain, using a suitable version of the algorithm Context, provided that the noise is small enough.

1. Introduction

The original motivation of this paper is the following question. Is it possible to recover the context tree of a variable length Markov chain from a noisy sample of the chain. We recall that in a variable length Markov chain the conditional probability of the next symbol, given the past depends on a variable portion of the past whose length depends on the past itself. This class of models were first introduced by Rissanen (1983) who called them finite memory sources or tree machines. They recently became popular in the statistics literature under the name of variable length Markov chains (VLMC) (Bühlmann and Wyner; 1999).

The notion of variable memory model can be naturally extended to a non markovian situation where the contexts are still finite, but their lengths are no longer bounded (see for
example Ferrari and Wyner (2003), Csiszár and Talata (2006) and Duarte et al. (2006)). This leads us to consider not only randomly perturbed unbounded variable length memory models, but more generally randomly perturbed infinite order stochastic chains.

We will consider binary chains of infinite order in which at each time step the value assumed by the chain can be randomly and independently flipped with a small fixed probability. Even if the original chain is markovian, the perturbed chain is in general a chain of infinite order (we refer the reader to Fernández et al. (2001) for a self contained introduction to chains of infinite order). We show that the transition probabilities of the perturbed chain are uniformly close to the corresponding transition probabilities of the original chain. More precisely, we prove that the difference between the conditional probabilities of the next symbol given a finite past of any fixed length is uniformly bounded above by the probability of flipping, multiplied by a fixed constant. This is the content of our first theorem.

Using this result we are able to solve our original problem of recovering the context tree of a chain with unbounded variable length from a noisy sample. To make this point clear, we must explain the notion of context. In his original 1983 paper, Rissanen used the word context to designate the minimal suffix of the string of past symbols which is enough to define the probability of the next symbol. Rissanen also observed that this notion is interesting only if the set of all contexts satisfies the suffix property, which means that no context is a proper suffix of another context. This property allows to represent the set of all contexts as the set of leaves of a rooted labeled tree, henceforth called the context tree of the chain. With this representation the process is described by the tree of all contexts and an associated family of probability measures on $A$, indexed by the leaves of the tree. Given a context, its associated probability measure gives the probability of the next symbol for any past having this context as a suffix.

Rissanen (1983) not only introduced the class of variable memory models but he also introduced the algorithm Context to estimate both the context tree and the associated family of probability transition. The way the algorithm Context works can be summarized as follows. Given a sample produced by a chain with variable memory, we start with a maximal tree of candidate contexts for the sample. The branches of this first tree are then pruned until we obtain a minimal tree of contexts well adapted to the sample.
From Rissanen (1983) to Galves et al. (2006), passing by Ron et al. (1996) and Bühlmann and Wyner (1999), several variants of the algorithm Context have been presented in the literature. In all the variants the decision to prune a branch is taken by considering a gain function. A branch is pruned if the gain function assumes a value smaller than a given threshold. The estimated context tree is the smallest tree satisfying this condition. The estimated family of probability transitions is the one associated to the minimal tree of contexts.

Rissanen (1983) proved the weak consistency of the algorithm Context when the tree of contexts is finite. Bühlmann and Wyner (1999) proved the weak consistency of the algorithm also in the finite case without assuming a prior known bound on the maximal length of the memory but instead using a bound allowed to grow with the size of the sample. In both papers the gain function is defined using the log likelihood ratio test to compare two candidate trees and the main ingredient of the consistency proofs was the chi-square approximation to the log likelihood ratio test for Markov chains of fixed order.

The unbounded case was considered by Ferrari and Wyner (2003), Duarte et al. (2006), Csiszár and Talata (2006) and Leonardi (2007). The first two papers essentially extend to the unbounded case the original chi-square approach introduced by Rissanen. Instead of the chi-square, the last two papers use penalized likelihood algorithms, related to the Bayesian Information Criterion (BIC), to estimate the context tree. We refer the reader to Csiszár and Talata (2006) for a nice description of other approaches and results in this field, including the context tree maximizing algorithm by Willems et al. (1995).

In the present paper we use a variant of the algorithm Context introduced in Galves et al. (2006) for finite trees and extended to unbounded trees in Galves and Leonardi (2007). In this variant, the decision of pruning a branch is taken by considering the difference between the estimated conditional probabilities of the original branch and the pruned one, using a suitable threshold. Using exponential inequalities for the estimated transition probabilities associated to the candidate contexts, these papers not only show the consistency of this variant of the algorithm Context, but also provide an exponential upper bound for the rate of convergence.

This version of the algorithm Context does not distinguish transition probabilities which are closer than the threshold level used in the pruning decision. Our first theorem assures that this is what happens between the conditional probabilities of the original variable memory
chain and the perturbed one, if the probability of random flipping is small enough. Hence it is natural to expect that with this version of the algorithm Context, one should be able to retrieve the original context tree out from the noisy sample. This is actually the case, as we prove in the second theorem.

The paper is organized as follows. In section 2 we give the definitions and state the main results. Section 3 and 4 are devoted to the proof of Theorem 1 and 2, respectively.

2. Definitions and results

Let $A$ denote the binary alphabet $\{0, 1\}$ with size $|A| = 2$. Given two integers $m \leq n$, we will denote by $w_{m}^{n}$ the sequence $(w_{m}, \ldots, w_{n})$ of symbols in $A$. The length of the sequence $w_{m}^{n}$ is denoted by $\ell(w_{m}^{n})$ and is defined by $\ell(w_{m}^{n}) = n - m + 1$. Any sequence $w_{m}^{n}$ with $m > n$ represents the empty string and is denoted by $\lambda$. The length of the empty string is $\ell(\lambda) = 0$.

Given two sequences $w$ and $v$, we will denote by $vw$ the sequence of length $\ell(v) + \ell(w)$ obtained by concatenating the two strings. In particular, $\lambda w = w \lambda = w$. The concatenation of sequences is also extended to the case in which $v$ denotes a semi-infinite sequence, that is $v = v_{-\infty}^{-1}$.

We say that the sequence $s$ is a suffix of the sequence $w$ if there exists a sequence $u$, with $\ell(u) \geq 1$, such that $w = us$. In this case we write $s \prec w$. When $s \prec w$ or $s = w$ we write $s \preceq w$. Given a sequence $w$ we denote by $\text{suf}(w)$ the largest suffix of $w$.

In the sequel $A^{j}$ will denote the set of all sequences of length $j$ over $A$ and $A^{*}$ represents the set of all finite sequences, that is

$$A^{*} = \bigcup_{j=1}^{\infty} A^{j}.$$ 

**Definition 2.1.** A countable subset $T$ of $A^{*}$ is a tree if no sequence $s \in T$ is a suffix of another sequence $w \in T$. This property is called the *suffix property*.

We define the height of the tree $T$ as

$$\ell(T) = \sup\{\ell(w) : w \in T\}.$$ 

In the case $\ell(T) < +\infty$ it follows that $T$ has finite cardinality. In this case we say that $T$ is bounded and we will denote by $|T|$ the number of sequences in $T$. On the other hand, if
ℓ(T) = +∞ then T has a countable number of sequences. In this case we say that the tree T is unbounded.

Given a tree T and an integer K we will denote by T|_K the tree T truncated to level K, that is

\[ T|_K = \{ w \in T : \ell(w) \leq K \} \cup \{ w : \ell(w) = K \text{ and } w \prec u, \text{ for some } u \in T \}. \]

We will say that a tree is irreducible if no sequence can be replaced by a suffix without violating the suffix property. This notion was introduced in Csiszár and Talata (2006) and generalizes the concept of complete tree.

**Definition 2.2.** A probabilistic context tree over A is an ordered pair \((T, p)\) such that

1. \(T\) is an irreducible tree;
2. \(p = \{ p(\cdot | w) ; w \in T \}\) is a family of transition probabilities over \(A\).

Consider a stationary stochastic chain \(\{X_t : t \in \mathbb{Z}\}\) over \(A\). Given a sequence \(w \in A^j\) we denote by

\[ p(w) = \mathbb{P}(X_1^j = w) \]

the stationary probability of the cylinder defined by the sequence \(w\). If \(p(w) > 0\) we write

\[ p(a|w) = \mathbb{P}(X_0 = a | X_{-1} = w). \]

**Definition 2.3.** A sequence \(w \in A^j\) is a context for the process \(\{X_t : t \in \mathbb{Z}\}\) if \(p(w) > 0\) and for any semi-infinite sequence \(x_{-\infty}^{-1}\) such that \(w\) is a suffix of \(x_{-\infty}^{-1}\) we have that

\[ \mathbb{P}(X_0 = a | X_{-\infty}^{-1} = x_{-\infty}^{-1}) = p(a|w), \text{ for all } a \in A, \quad (2.4) \]

and no suffix of \(w\) satisfies this equation.

**Definition 2.5.** We say that the process \(\{X_t : t \in \mathbb{Z}\}\) is compatible with the probabilistic context tree \((T, \bar{p})\) if the following conditions are satisfied

1. \(w \in T\) if and only if \(w\) is a context for the process \(\{X_t : t \in \mathbb{Z}\}\).
2. For any \(w \in T\) and any \(a \in A\), \(\bar{p}(a|w) = \mathbb{P}(X_0 = a | X_{-1} = w)\).

In the unbounded case, the compactness of \(A^\mathbb{Z}\) assures that there is at least one stationary stochastic chain compatible with a probabilistic context tree. The uniqueness requires further conditions, such as the ones presented in Fernández and Galves (2002).
Definition 2.6. A probabilistic context tree \((T, p)\) is of type B if it satisfies the following conditions

1. **Non-nullness**, that is
   \[
   \alpha := \inf_{w \in T, a \in A} p(a|w) > 0;
   \]

2. **Log-continuity**, that is
   \[
   \beta_k \to 0 \text{ when } k \to \infty,
   \]

   where the sequence \(\{\beta_k\}_{k \in \mathbb{N}}\) is defined by
   \[
   \beta_k := \sup\{|1 - \frac{p(a|w)}{p(a|v)}|: a \in A, v, w \in T \text{ with } w \overset{k}{=} v\}.
   \]

   Here, \(w \overset{k}{=} v\) means that there exists a sequence \(u\), with \(\ell(u) = k\) such that \(u \prec w\) and \(u \prec v\). The sequence \(\{\beta_k\}_{k \in \mathbb{N}}\) is called the continuity rate.

For a probabilistic context tree of type B with summable continuity rate, the maximal coupling argument used in Fernández and Galves (2002) implies the uniqueness of the law of the chain consistent with it. Then, we will assume here that the continuity rate is summable, that is

\[
\beta := \sum_{k \in \mathbb{N}} \beta_k < +\infty. \tag{2.7}
\]

This condition immediately implies that

\[
1 \leq \beta^* := \prod_{k=0}^{+\infty} (1 + \beta_k) < +\infty.
\]

Given an integer \(k \geq 1\) we define

\[
D_k = \min_{w \in T, \ell(w) \leq k} \max_{a \in A} \{|p(a|w) - p(a|\text{suf}(w))|\}. \tag{2.8}
\]

In this paper we are interested on the effect of a Bernoulli noise flipping independent from the successive symbols produced by the chain. Namely, let \(\{\xi_t: t \in \mathbb{Z}\}\) be an i.i.d. sequence of random variables taking values in the alphabet \(A\), independent of \(\{X_t: t \in \mathbb{Z}\}\), with

\[
\mathbb{P}(\xi_t = 0) = 1 - \epsilon,
\]

where \(\epsilon\) is a fixed noise parameter in \([0, 1]\). For \(a\) and \(b\) in \(A\), we define

\[
a \oplus b = a + b \pmod{2},
\]
and $\bar{a} = 1 \oplus a$. We now define the stochastically perturbed chain $\{Z_t: t \in \mathbb{Z}\}$ by

$$Z_t = X_t \oplus \xi_t.$$

The process $\{Z_t: t \in \mathbb{Z}\}$ is an example of a hidden Markov model. In the case $\epsilon = 1/2$, $\{Z_t: t \in \mathbb{Z}\}$ is an i.i.d. uniform Bernoulli. However, in general it is not a chain of finite order.

We will use the shorthand notation

$$q(w^1) = \mathbb{P}(Z^1 = w^1)$$

and

$$q(a|w^{-1}_j) = \mathbb{P}(Z^0 = a \mid Z^{-1}_j = w^{-1}_j)$$

to denote the probabilities corresponding to the process $\{Z_t: t \in \mathbb{Z}\}$. We also define

$$q_k = \min\{q(w): \ell(w) \leq k \text{ and } q(w) > 0\}. \tag{2.9}$$

We can now state our first theorem.

**Theorem 1.** Assume the chain $\{X_t: t \in \mathbb{Z}\}$ has summable continuity rate. Then, for any $\epsilon \in [0, 1]$ and for any $j \geq 0$

$$\sup_{w^0_j} |\mathbb{P}(Z^0 = w_0 \mid Z^{-1}_j = w^{-1}_j) - \mathbb{P}(X^0 = w_0 \mid X^{-1}_j = w^{-1}_j)| \leq (1 + \frac{4\beta^*}{\alpha}) \epsilon.$$

To state the second theorem we first need to present the version of the Algorithm Context introduced in Galves et al. (2006) and Galves and Leonardi (2007).

In what follows we will assume that $z_1, z_2, \ldots, z_n$ is a sample of the observed chain $\{Z_t: t \in \mathbb{Z}\}$ and that the underlying chain $\{X_t: t \in \mathbb{Z}\}$ is compatible with the probabilistic context tree $(T, p)$.

For any finite string $w$ with $\ell(w) \leq n$, we denote by $N_n(w)$ the number of occurrences of the string in the sample; that is

$$N_n(w) = \sum_{t=0}^{n-\ell(w)} 1\{z_{t+\ell(w)} = w\}. \tag{2.10}$$

For any element $a \in A$ and any finite sequence $w \in A^*$, the empirical transition probability $\hat{q}_n(a|w)$ is defined by

$$\hat{q}_n(a|w) = \frac{N_n(aw) + 1}{N_n(w \cdot) + |A|}. \tag{2.11}$$
where
\[ N_n(w) = \sum_{b \in A} N_n(wb). \]

This definition of \( \hat{q}_n(a|w) \) is convenient because it is asymptotically equivalent to \( \frac{N_n(wa)}{N_n(w)} \) and it avoids an extra definition in the case \( N_n(w) = 0 \).

The variant of Rissanen’s algorithm Context we will use is defined as follows. First of all, let us define for any finite string \( w \in A^* \):
\[ \Delta_n(w) = \max_{a \in A} |\hat{q}_n(a|w) - \hat{q}_n(a|\text{suf}(w))|. \]

The \( \Delta_n(w) \) operator computes a distance between the empirical transition probabilities associated to the sequence \( w \) and the one associated to the sequence \( \text{suf}(w) \).

**Definition 2.12.** Given \( \delta > 0 \) and \( d < n \), the tree estimated with the algorithm Context is
\[ \hat{T}_{\delta,d} = \{ w \in A_1^d : \Delta_n(w) > \delta \text{ and } \Delta_n(uw) \leq \delta \ \forall u \in A_1^{d-\ell(u)} \}, \]
where \( A_r^1 \) denotes the set of all sequences of length at most \( r \). In the case \( \ell(w) = d \) we have \( A_1^{d-\ell(w)} = \emptyset \).

It is easy to see that \( \hat{T}_{\delta,d} \) is a tree. Moreover, the way we defined \( \hat{q}_n(\cdot|\cdot) \) in (2.11) associates a probability distribution to each sequence in \( \hat{T}_{\delta,d} \).

We may now state our second theorem.

**Theorem 2.** Let \( K \) be an integer and let \( z_1, z_2, \ldots, z_n \) be a sample of the perturbed chain \( \{Z_t : t \in \mathbb{Z}\} \). Then, for any \( d \) satisfying
\[ d > \max_{w \in T_K} \min \{ \ell(v) : v \in T, v \succeq w \}, \]
for any \( \delta \) such that \( 2(1 + \frac{433^*}{\alpha})\epsilon < \delta < D_d - 2(1 + \frac{433^*}{\alpha})\epsilon \) and for any \( n \) such that
\[ n > \frac{4(|A| + 1)}{\min(\delta, D_d - \delta) - 2\epsilon(1 + \frac{433^*}{\alpha})} + d \]
we have
\[ \mathbb{P}(\hat{T}_{\delta,d}|_K \neq T|_K) \leq 4 \epsilon \left( |A| + 1 \right) |A|^{d+1} \exp\left[ -(n - d) \frac{\min(\delta, D_d - \delta) - 2\epsilon(1 + \frac{433^*}{\alpha})}{256\epsilon(1 + \beta)|A|^2(d + 1)} \right]^2. \]

As a consequence we obtain the following strong consistency result.
Corollary 1. For any integer $K$ and for almost all infinite sample $z_1, z_2 \ldots$ there exists a $\tilde{n}$ such that, for any $n \geq \tilde{n}$ we have

$$T^n_{\delta, d}|_K = T|_K,$$

where $d$ is given by (2.13) and $\delta$ is such that $2(1 + \frac{4\beta^*}{\alpha}) \epsilon < \delta < D_d - 2(1 + \frac{4\beta^*}{\alpha}) \epsilon$.

3. Proof of Theorem 1

We start by proving three preparatory lemmas.

Lemma 3.1. For any $k > j \geq 0$ and any $\epsilon \in [0, 1]$ we have

$$\sup_{w_0^0, a, b} |\mathbb{P}(X_0 = w_0 \mid X_j^{-1} = w_j^{-1}, X_{j-1} = a, Z_{j-1} = b, Z_k^{-j-2} = w_k^{-j-2}) - p(w_0 \mid w_0^{-\infty})| \leq \beta_j.$$

Proof. We observe that for $j \geq 0$ it follows from the independence of the flipping sequence that

$$\mathbb{P}(X_0 = w_0 \mid X_j^{-1} = w_j^{-1}, X_{j-1} = a, Z_{j-1} = b, Z_k^{-j-2} = w_k^{-j-2})$$

$$= \sum_{u_k^{-j-2}} \mathbb{P}(X_k^{-j-1} = u_k^{-j-2} a, w_k^{-j}) \mathbb{P}(Z_k^{-j-1} = w_k^{-j-2} b \mid X_k^{-j-1} = u_k^{-j-2} a).$$

It is easy to see using conditioning on the infinite past that

$$\inf_{v_j^{-j-1}} \mathbb{P}(X_0 = w_0 \mid X_j^{-1} = w_j^{-1}, X_{-\infty} = v_{-\infty}^{-j-1})$$

$$\leq \mathbb{P}(X_0 = w_0 \mid X_k^{-1} = u_k^{-j-2} a, w_k^{-j})$$

$$\leq \sup_{v_j^{-j-1}} \mathbb{P}(X_0 = w_0 \mid X_j^{-1} = w_j^{-1}, X_{-\infty} = v_{-\infty}^{-j-1}).$$

Then, using continuity we have

$$p(w_0 \mid w_0^{-\infty}) - \beta_j \leq \mathbb{P}(X_0 = w_0 \mid X_k^{-1} = u_k^{-j-2} a, w_k^{-j}) \leq p(w_0 \mid w_0^{-\infty}) + \beta_j$$

and the assertion of the Lemma follows immediately. \(\square\)

Lemma 3.2. For any $\epsilon \in [0, 1]$ and for any $k \geq 0$ we have

$$\inf_{w_0^{-k}} \mathbb{P}(Z_0 = w_0 \mid Z_k^{-1} = w_k^{-1}) \geq \alpha.$$
and
\[ \inf_{w_{-k}^0} \mathbb{P}(X_0 = w_0 \mid Z_{-1}^{-1} = w_{-1}^{-1}) \geq \alpha. \]

Moreover, for any \(0 \leq j \leq k\) we have
\[ \inf_{w_{-k}^{-1}} \mathbb{P}(X_{-j-1} = w_{-j-1}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-j}^{-2} = w_{-j}^{-2}) \geq \frac{\alpha}{\beta^*}. \]

**Proof.** We first observe that
\[ \mathbb{P}(Z_0 = w_0 \mid Z_{-1}^{-1} = w_{-1}^{-1}) = (1 - \epsilon) \mathbb{P}(X_0 = w_0 \mid Z_{-1}^{-1} = w_{-1}^{-1}) + \epsilon \mathbb{P}(X_0 = \bar{w}_0 \mid Z_{-1}^{-1} = w_{-1}^{-1}). \]

It is therefore enough to prove the second assertion. From the independence of the flipping procedure we have
\[
\begin{align*}
\mathbb{P}(X_0 = w_0 \mid Z_{-1}^{-1} = w_{-1}^{-1}) &= \\
&= \lim_{t \to \infty} \frac{(1 - \epsilon)^k \sum_{w_{-l}^{-1}} p(w_0 \mid u_{-l}^{-1}w_{-l}^{-1}) \mathbb{P}(X_{-l}^{-1} = u_{-l}^{-1} \mid X_{-l}^{-1} = w_{-l}^{-1})(\epsilon/(1 - \epsilon))^{\sum_{j=1}^{l-1} u_j w_j}}{(1 - \epsilon)^k \sum_{w_{-l}^{-1}} \mathbb{P}(X_{-l}^{-1} = u_{-l}^{-1} \mid X_{-l}^{-1} = w_{-l}^{-1})(\epsilon/(1 - \epsilon))^{\sum_{j=1}^{l-1} u_j w_j}} \\
&\geq \alpha.
\end{align*}
\]

For the last assertion we first observe that
\[
\frac{\mathbb{P}(X_{-j-1} = w_{-j-1}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-j}^{-2} = w_{-j}^{-2})}{\mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1}, X_{-j}^{-2} = w_{-j}^{-2})} = \\
\frac{\sum_{x_{-j}^{-2}} \mathbb{P}(Z_{-j}^{-2} = w_{-j}^{-2} \mid X_{-j}^{-1} = x_{-j}^{-2}) \mathbb{P}(X_{-j-1}^{-1} = w_{-j-1}^{-1}, X_{-j}^{-1} = x_{-j}^{-2})}{\sum_{x_{-j}^{-2}} \mathbb{P}(Z_{-j}^{-2} = w_{-j}^{-2} \mid X_{-j}^{-1} = x_{-j}^{-2}) \mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1}, X_{-j}^{-2} = x_{-j}^{-2})}.
\]

Moreover,
\[
\frac{\mathbb{P}(X_{-j-1}^{-1} = w_{-j-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2})}{\mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1}, X_{-j}^{-2} = x_{-j}^{-2})} = \\
\frac{\prod_{l=1}^{j+1} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2}) \prod_{l=j+2}^{k} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2}) \prod_{l=j+2}^{k} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2})}{\prod_{l=1}^{j} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2}) \prod_{l=j+2}^{k} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2})}
\]
\[\geq \mathbb{P}(X_{-j-1} = w_{-j-1}^{-1} \mid X_{-j}^{-2} = x_{-j}^{-2}) \prod_{l=1}^{j} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2}) \prod_{l=j+2}^{k} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2}) \prod_{l=j+2}^{k} \mathbb{P}(X_{-l} = w_{-l} \mid X_{-l-1}^{-1} = w_{-l-1}^{-1}, X_{-j}^{-2} = x_{-j}^{-2})
\]
and using non-nullness and log-continuity this is bounded below by
\[
\alpha \prod_{l=1}^{j} \frac{1}{1 + \beta_{j-l}} \geq \frac{\alpha}{\beta^*}.
\]
This finishes the proof of the Lemma. \hfill \Box

Lemma 3.3. For any \( k > j \geq 0 \) and any \( \epsilon \in [0,1] \)

\[
\sup_{\omega_{-k}} \mathbb{P}(X_{-j-1} = \bar{w}_{-j-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1}) \leq \frac{\beta^*}{\alpha} \epsilon.
\]

Proof. We have

\[
\mathbb{P}(X_{-j-1} = \bar{w}_{-j-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1}) = \frac{\mathbb{P}(X_{-j-1} = \bar{w}_{-j-1}, Z_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1})}{\mathbb{P}(Z_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1})}.
\]

It follows from Lemma 3.2 that

\[
\mathbb{P}(Z_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1}) = (1 - \epsilon) \mathbb{P}(X_{-j-1} = w_{-j-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1}) + \epsilon \mathbb{P}(X_{-j-1} = \bar{w}_{-j-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1}) \geq \frac{\alpha}{\beta^*}.
\]

This concludes the proof of Lemma 3.3. \hfill \Box

Proof of Theorem 1. We first observe that

\[
\mathbb{P}(Z_0 = w_0 \mid Z_{-k}^{-1} = w_{-k}^{-1}) = (1 - \epsilon) \mathbb{P}(X_0 = w_0 \mid Z_{-k}^{-1} = w_{-k}^{-1}) + \epsilon \mathbb{P}(X_0 = \bar{w}_0 \mid Z_{-k}^{-1} = w_{-k}^{-1}).
\]

Therefore,

\[
|\mathbb{P}(Z_0 = w_0 \mid Z_{-k}^{-1} = w_{-k}^{-1}) - \mathbb{P}(X_0 = w_0 \mid Z_{-k}^{-1} = w_{-k}^{-1})| \leq \epsilon
\]

and if \( k = 0 \) the Theorem is proved. We will now assume \( k \geq 1 \) and we write

\[
\mathbb{P}(X_0 = w_0 \mid Z_{-k}^{-1} = w_{-k}^{-1}) - \mathbb{P}(X_0 = w_0 \mid X_{-k}^{-1} = w_{-k}^{-1}) = \sum_{j=0}^{k-1} \mathbb{P}(X_0 = w_0 \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1}) - \mathbb{P}(X_0 = w_0 \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-k}^{-1} = w_{-k}^{-1}).
\]
We will bound each term in the sum separately. We can write

\[ \mathbb{P}(X_0 = w_0 \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-j-1} = w_{-k}^{-j-2}) - \mathbb{P}(X_0 = w_0 \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-k}^{-j-2} = w_{-k}^{-j-2}) \]

\[ = \sum_{b \in \{0, 1\}} \left[ \mathbb{P}(X_0 = w_0 \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-j-1} = b, Z_{-k}^{-j-1} = w_{-k}^{-j-1}) \right. \]

\[ \left. - \mathbb{P}(X_0 = w_0 \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-k}^{-j-2} = w_{-k}^{-j-2}) \right] \]

\[ \times \mathbb{P}(X_{-j-1} = b \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-j-1} = w_{-k}^{-j-1}). \]

The above sum is a sum of two terms, one with \( b = \bar{w}_{-j-1} \), the other one with \( b = w_{-j-1} \).

We will bound above these two terms separately. For the first term we have the bound

\[ |\mathbb{P}(X_0 = w_0 \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-j-1} = \bar{w}_{-j-1}, Z_{-k}^{-j-1} = w_{-k}^{-j-1}) \]

\[ - \mathbb{P}(X_0 = w_0 \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-k}^{-j-2} = w_{-k}^{-j-2}) | \]

\[ \times \mathbb{P}(X_{-j-1} = \bar{w}_{-j-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-j-1} = w_{-k}^{-j-1}) \]

\[ \leq \sum_{a \in \{0, 1\}} |\mathbb{P}(X_0 = w_0 \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-j-1} = w_{-j-1}, Z_{-k}^{-j-1} = w_{-k}^{-j-1}) \]

\[ - \mathbb{P}(X_0 = w_0 \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-j-1} = a, Z_{-k}^{-j-2} = w_{-k}^{-j-2}) | \]

\[ \times \mathbb{P}(Z_{-j-1} = a \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-k}^{-j-2} = w_{-k}^{-j-2}) \]

\[ \times \mathbb{P}(X_{-j-1} = w_{-j-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Z_{-k}^{-j-1} = w_{-k}^{-j-1}). \]

Using the fact that the term in the sum with \( a = w_{-j-1} \) vanishes this is bounded above by

\[ |\mathbb{P}(X_0 = w_0 \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-j-1} = w_{-j-1}, Z_{-k}^{-j-1} = w_{-k}^{-j-1}) \]

\[ - \mathbb{P}(X_0 = w_0 \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-j-1} = \bar{w}_{-j-1}, Z_{-k}^{-j-2} = w_{-k}^{-j-2}) | \]

\[ \times \mathbb{P}(Z_{-j-1} = \bar{w}_{-j-1} \mid X_{-j-1}^{-1} = w_{-j-1}^{-1}, Z_{-k}^{-j-2} = w_{-k}^{-j-2}) \]

\[ \leq 2 \beta_j \epsilon \]
from Lemma 3.1. Putting all the above bounds together we get
\[
\left| \mathbb{P}(Z_0 = w_0 \mid Z_{-k}^{-1} = w_{-k}^{-1}) - \mathbb{P}(X_0 = w_0 \mid X_{-k}^{-1} = w_{-k}^{-1}) \right| \leq \epsilon + \frac{2\beta\beta^*}{\alpha} \epsilon + 2\beta \epsilon
\]
and the Theorem follows. \qed

4. Proof of Theorem 2

We start by proving four new auxiliary Lemmas.

Lemma 4.1. For any \( i \geq 1 \), any \( k > i \), any \( j \geq 1 \) and any finite sequence \( w_1^j \), the following inequality holds
\[
\sup_{x_1^i, \theta_1^i \in A^i} \mathbb{P}(Z_k^{k+j-1} = w_1^j \mid X_1^i = x_1^i, \xi_1^i = \theta_1^i) - q(w_1^j) \leq j \beta_{k-i-1}.
\]

Proof. Observe that for any \( x_1^i, \theta_1^i \in A^i \)
\[
\mathbb{P}(Z_k^{k+j-1} = w_1^j \mid X_1^i = x_1^i, \xi_1^i = \theta_1^i) - q(w_1^j)
\]
\[
= \sum_{x_k^{k+j-1} \in A^j} \mathbb{P}(X_k^{k+j-1} = x_k^{k+j-1}, Z_k^{k+j-1} = w_1^j \mid X_1^i = x_1^i, \xi_1^i = \theta_1^i) - q(w_1^j)
\]
\[
= \sum_{x_k^{k+j-1} \in A^j} \mathbb{P}(Z_k^{k+j-1} = w_1^j \mid X_k^{k+j-1} = x_k^{k+j-1}) \mathbb{P}(X_k^{k+j-1} = x_k^{k+j-1} \mid X_1^i = x_1^i, \xi_1^i = \theta_1^i)
\]
\[
- q(w_1^j)
\]
by the independence of the flipping procedure. The last term can be bounded above by
\[
= \sum_{x_k^{k+j-1} \in A^j} \mathbb{P}(Z_k^{k+j-1} = w_1^j \mid X_k^{k+j-1} = x_k^{k+j-1}) \mathbb{P}(X_k^{k+j-1} = x_k^{k+j-1} \mid X_1^i = x_1^i)
\]
\[
- \mathbb{P}(X_k^{k+j-1} = x_k^{k+j-1} \mid x_k^{k+j-1})
\]
Then, we can use Lemma 3.6 in Galves and Leonardi (2007) to bound above the last sum with
\[
\sum_{x_k^{k+j-1} \in A^j} j \beta_{k-i-1} \mathbb{P}(X_k^{k+j-1} = x_k^{k+j-1})
\]
We conclude the proof of Lemma 4.1. \qed
Lemma 4.2. For any finite sequence $w$ and any $t > 0$ we have
\[
\mathbb{P}(|N_n(w) - (n - \ell(w) + 1)q(w)| > t) \leq e^{\frac{t^2}{2}} \exp \left[ \frac{-t^2}{4e(1 + \beta)(\ell(w)(n - \ell(w)+1))} \right].
\]
Moreover, for any $a \in A$ and any $n > \frac{|A|+1}{tq(w)} + \ell(w)$ we have
\[
\mathbb{P}(|\hat{g}_n(a|w) - q(a|w)| > t) \leq \left( |A| + 1 \right) e^{\frac{t}{2}} \exp \left[ - (n - \ell(w)+1) \frac{|t - \frac{|A|+1}{(n - \ell(w)+1)q(w)}|^2[q(w) + \frac{|A|}{n\ell(w)+1}]^2}{16e(1 + \beta)|A|^2(\ell(w)+1)} \right].
\]

Proof. Observe that for any finite sequence $w^j_1 \in A^j$
\[
N_n(w^j_1) = \sum_{t=0}^{n-j} \prod_{i=1}^{j} \left( 1_{\{X_{t+i}=w_i\}}1_{\{\xi_{t+i}=0\}} + 1_{\{X_{t+i}=w_i\}}1_{\{\xi_{t+i}=1\}} \right).
\]
Define the process $\{U_t : t \in \mathbb{Z}\}$ by
\[
U_t = \prod_{i=1}^{j} \left( 1_{\{X_{t+i-1}=w_i\}}1_{\{\xi_{t+i-1}=0\}} + 1_{\{X_{t+i-1}=w_i\}}1_{\{\xi_{t+i-1}=1\}} \right) - q(w^j_1)
\]
and denote by $\mathcal{M}_i$ the $\sigma$-algebra generated by $U_1, \ldots, U_i$. Applying Proposition 4 in Dedecker and Doukhan (2003) we obtain that, for any $r \geq 2$
\[
||N_n(w^j_1) - (n - j + 1)q(w^j_1)||_r 
\leq \left( 2r \sum_{i=1}^{n-j+1} \max_{i \leq \ell \leq n-j+1} ||U_i \ell \sum_{k=i}^{\ell} \mathbb{E}(U_k|\mathcal{M}_t)||_r/2 \right)^{\frac{1}{2}}
\leq \left( 2r \sum_{i=1}^{n-j+1} ||U_i||_r^{\frac{1}{2}} \sum_{k=i}^{n-j+1} ||\mathbb{E}(U_k|\mathcal{M}_t)||_\infty \right)^{\frac{1}{2}}
\leq \left( 2r \sum_{i=1}^{n-j+1} \left( \sum_{k=i}^{n-j+1} \sup_{\sigma'_1 \in A^i} ||\mathbb{E}(U_k|U^j_1 = \sigma'_1)||_r \right)^{\frac{1}{2}}
\leq \left( 2r \sum_{i=1}^{n-j+1} \left( \sum_{k=i}^{n-j+1} \sup_{x'_1, \theta'_1 \in A^i} \mathbb{E}(U_k|X^j_1 = x'_1, \xi^j_1 = \theta'_1) \right)^{\frac{1}{2}}
\leq \left( 2r \sum_{i=1}^{n-j+1} \left( \sum_{k=i}^{n-j+1} \sup_{x'_1, \theta'_1 \in A^i} \mathbb{P}(Z^{k+j-1}_k = w^j_1|X^j_1 = x'_1, \xi^j_1 = \theta'_1 - q(w^j_1)) \right)^{\frac{1}{2}}
\right)^{\frac{1}{2}}
\right).
\]
Using Lemma 4.1 we can bound above the last expression by
\[
|2r(1 + \beta)\ell(w)(n - j + 1)|^{\frac{1}{2}}.
\]
Then, as in Galves and Leonardi (2007) we obtain
\[
\mathbb{P}(|N_n(w) - (n - \ell(w) + 1)q(w)| > t) \leq e^{\frac{1}{2}} \exp\left[\frac{-t^2}{4e(1 + \beta)\ell(w)(n - \ell(w) + 1)}\right]
\]
and
\[
\mathbb{P}(|\hat{q}_n(a|w) - q(a|w)| > t) \leq (|A| + 1) e^{\frac{1}{2}} \exp\left[\frac{1}{16e(1 + \beta)|A|^2(\ell(w) + 1)}\right].
\]
This concludes the proof of Lemma 4.2 

\[\square\]

**Lemma 4.3.** For any \(\delta > 2(1 + \frac{4\beta^*}{\alpha})\epsilon\), for any
\[
n > \frac{2(|A| + 1)}{(\frac{\delta}{2} - \epsilon (1 + \frac{4\beta^*}{\alpha}))q_d + d}
\]
and for any \(w \in T\), \(uw \in \hat{T}^d\) we have that
\[
\mathbb{P}(\Delta_n(uw) > \delta) \leq 2|A| (|A| + 1) e^{\frac{1}{2}} \exp\left[-(n - d) \frac{\left(\frac{\delta}{2} - \epsilon (1 + \frac{4\beta^*}{\alpha})\right)^2q_d^2}{32e(1 + \beta)|A|^2(d + 1)}\right].
\]

**Proof.** Recall that
\[
\Delta_n(uw) = \max_{a \in A} |\hat{q}_n(a|uw) - \hat{q}_n(a|\text{suf}(uw))|.
\]
Note that the fact \(w \in T\) implies that for any finite sequence \(u\) and any symbol \(a \in A\) we have \(p(a|uw) = p(a|\text{suf}(uw))\). Hence,
\[
|\hat{q}_n(a|uw) - \hat{q}_n(a|\text{suf}(uw))| \leq |\hat{q}_n(a|uw) - q(a|uw)| + |q(a|uw) - p(a|uw)|
\]
\[
+ |q(a|\text{suf}(uw)) - p(a|\text{suf}(uw))| + |\hat{p}_n(a|\text{suf}(uw)) - q(a|\text{suf}(uw))|.
\]
Then, using Theorem 1 we have that
\[
\mathbb{P}(\Delta_n(uw) > \delta) \leq \sum_{a \in A} \left[\mathbb{P}(|\hat{q}_n(a|uw) - q(a|uw)| > \frac{\delta}{2} - \epsilon (1 + \frac{4\beta^*}{\alpha})) \right]
\]
\[
+ \mathbb{P}(|\hat{q}_n(a|\text{suf}(uw)) - q(a|\text{suf}(uw))| > \frac{\delta}{2} - \epsilon (1 + \frac{4\beta^*}{\alpha})) \right].
\]
Now, for
\[
n > \frac{2(|A| + 1)}{(\frac{\delta}{2} - \epsilon (1 + \frac{4\beta^*}{\alpha}))q_d + d}
\]
we can bound above the right hand side of the expression above using Lemma 4.2 by

\[
2 |A| (|A| + 1) e^{\frac{1}{2}} \exp \left[ -(n-d) \frac{\frac{1}{2} - \epsilon (1 + \frac{4\beta^*}{\alpha})^2 q_d^2}{32 \epsilon (1 + \beta) |A|^2 (d+1)} \right].
\]

\[ \square \]

**Lemma 4.4.** For any \( d \) satisfying (2.13), for any \( \delta < D_d - 2\epsilon (1 + \frac{4\beta^*}{\alpha}) \), for any

\[
n > \frac{4(|A| + 1)}{(D_d - 2\epsilon (1 + \frac{4\beta^*}{\alpha}) - \delta) q_d} + d
\]

and for any \( w \in \hat{T}_n^d \) with \( \ell(w) < K \) we have that

\[
\mathbb{P}( \bigcap_{uw \in T^d} \{ \Delta_n(uw) \leq \delta \}) \leq 2(|A| + 1) e^{\frac{1}{2}} \exp \left[ -(n-d) \frac{[D_d - 2(1 + \frac{4\beta^*}{\alpha}) \epsilon - \delta] q_d^2}{256 \epsilon (1 + \beta) |A|^2 (d+1)} \right].
\]

**Proof.** As \( d \) satisfies (2.13) we have that there exists a \( \bar{u} \bar{w} \in T^d \) such that \( \bar{u} \bar{w} \in T \). Then

\[
\mathbb{P}( \bigcap_{uw \in T^d} \{ \Delta_n(uw) \leq \delta \}) \leq \mathbb{P}(\Delta_n(\bar{u} \bar{w}) \leq \delta).
\]

Observe that for any \( a \in A \),

\[
\begin{align*}
|q_n(a|\text{suf}(\bar{u} \bar{w})) - \hat{q}_n(a|\bar{u} \bar{w})| & \geq |p(a|\text{suf}(\bar{u} \bar{w})) - p(a|\bar{u} \bar{w})| - |q_n(a|\text{suf}(\bar{u} \bar{w})) - q(a|\text{suf}(\bar{u} \bar{w}))| - \\
& \quad |\hat{q}_n(a|\bar{u} \bar{w}) - q(a|\bar{u} \bar{w})| - |q(a|\text{suf}(\bar{u} \bar{w})) - p(a|\text{suf}(\bar{u} \bar{w}))| - \\
& \quad |q(a|\bar{u} \bar{w}) - p(a|\bar{u} \bar{w})|.
\end{align*}
\]

Hence, we have that for any \( a \in A \)

\[
\Delta_n(\bar{u} \bar{w}) \geq D_d - 2\epsilon (1 + \frac{4\beta^*}{\alpha}) - |q_n(a|\text{suf}(\bar{u} \bar{w})) - q(a|\text{suf}(\bar{u} \bar{w}))| - |\hat{q}_n(a|\bar{u} \bar{w}) - q(a|\bar{u} \bar{w})|.
\]

Therefore,

\[
\mathbb{P}(\Delta_n(\bar{u} \bar{w}) \leq \delta) \leq \mathbb{P}( \bigcap_{a \in A} \{ |q_n(a|\text{suf}(\bar{u} \bar{w})) - q(a|\text{suf}(\bar{u} \bar{w}))| \geq \frac{D_d - 2\epsilon (1 + \frac{4\beta^*}{\alpha}) - \delta}{2} \})
\]

\[
+ \mathbb{P}( \bigcap_{a \in A} \{ |\hat{q}_n(a|\bar{u} \bar{w}) - q(a|\bar{u} \bar{w})| \geq \frac{D_d - 2\epsilon (1 + \frac{4\beta^*}{\alpha}) - \delta}{2} \}).
\]

As \( \delta < D_d - 2\epsilon (1 + \frac{4\beta^*}{\alpha}) \) and

\[
n > \frac{4(|A| + 1)}{(D_d - 2\epsilon (1 + \frac{4\beta^*}{\alpha}) - \delta) q_d} + d
\]
we can use Lemma 4.2 to bound above the right hand side of the inequality above by
\[ 2(|A|+1)e^\frac{1}{4}\exp[-(n-d)\frac{[D_d-2(1+\frac{4\beta\beta^*}{\alpha})\epsilon-\delta]q_d}{256e(1+\beta)|A|^2(d+1)}]. \]

This concludes the proof of Lemma 4.4 \(\square\)

Now we proceed with the proof of our main result.

**Proof of Theorem 2.** Define
\[ O_{n,\delta}^{K,d} = \bigcup_{w \in T} \bigcup_{\ell(w)<K} \{ \Delta_n(uw) > \delta \}, \]
and
\[ U_{n,\delta}^{K,d} = \bigcup_{w \in \hat{T}_n^{\delta,d}} \bigcap_{\ell(w)<K} \{ \Delta_n(uw) \leq \delta \}. \]

Then, if \( d < n \) we have that
\[ \{ \hat{T}_n^{\delta,d} | K \neq T | K \} = O_{n,\delta}^{K,d} \cup U_{n,\delta}^{K,d}. \]

Using the definition of \( O_{n,\delta}^{K,d} \) and \( U_{n,\delta}^{K,d} \) we have that
\[ \mathbb{P}(\hat{T}_n^{\delta,d} | K \neq T | K) \leq \sum_{w \in T} \sum_{\ell(w)<K} \mathbb{P}(\Delta_n(uw) > \delta) + \sum_{w \in \hat{T}_n^{\delta,d}} \mathbb{P}(\bigcap_{\ell(w)<K} \Delta_n(uw) \leq \delta). \]

Applying Lemma 4.3 and Lemma 4.4 we obtain, for
\[ n > \frac{4(|A|+1)}{\min(\delta,D_d-\delta) - 2\epsilon(1+\frac{4\beta\beta^*}{\alpha})} + d, \]
the inequality
\[ \mathbb{P}(\hat{T}_n^{\delta,d} | K \neq T | K) \leq 4e^\frac{1}{4}(\frac{1}{4})(|A|+1)|A|^{d+1}\exp[-(n-d)\frac{\min(\delta,D_d-\delta) - 2\epsilon(1+\frac{4\beta\beta^*}{\alpha})^2q_d}{256e(1+\beta)|A|^2(d+1)}]. \]

We conclude the proof of Theorem 2. \(\square\)

**Proof of Corollary 1.** It follows from Theorem 2, using the first Borel-Cantelli Lemma and the fact that the bounds for the error estimation of the context tree are summable in \( n \) for a fixed \( d \) satisfying (2.13). \(\square\)
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