1 Introduction

In this paper, we consider nonlinear PDEs of the form
\[ \partial_t x^\alpha(z, t) = f^\alpha(z, t, x(z, t), \partial_x x(z, t), \partial_x^2 x(z, t), u(t)), \quad \alpha = 1, \ldots, n_x \]  
(1)
on a 1-dimensional spatial domain \( \Omega = (0, 1) \subset \mathbb{R} \) with a single input \( u(t) \), boundary conditions
\[ g^\lambda(t, x(0, t), \partial_x x(0, t)) = 0, \quad \lambda = 1, \ldots, n_A \]
\[ h^\mu(t, x(1, t), \partial_x x(1, t)) = 0, \quad \mu = 1, \ldots, n_B, \]
(2)
and an output function
\[ y(t) = c(t, x(z_0, t), \partial_x x(z_0, t)) \]
(3)
defined at some point \( z_0 \in \bar{\Omega} \). By \( \bar{\Omega} = [0, 1] \) we denote the closure of \( \Omega \). The functions \( f^\alpha, g^\lambda, h^\mu, \) and \( c \) are smooth, and we assume that solutions of the PDEs (1) with the boundary conditions (2) exist and are uniquely determined by the initial condition \( x(z, 0) \) and the input function \( u(t) \) (well-posedness, see e.g. [1]).

A pair of initial conditions of such a system is said to be indistinguishable, if for every admissible trajectory \( u(t) \) of the input the system generates for both initial conditions the same trajectory \( y(t) \) of the output. The system is said to be observable, if (locally) there exists no pair of indistinguishable initial conditions (see e.g. [2]). In this contribution, we want to use the existence of certain symmetry groups for proving that a system is not observable.

In [3], symmetry groups have already been used to deal with a special case of the observability problem with a fixed choice of the input trajectory \( u(t) \). Furthermore, in [4] they have also been used to study the accessibility of a system.

2 Geometric Framework

The mathematical framework for the calculation of symmetry groups is differential geometry. For a differential-geometric representation of the system, we present conditions for the existence of special symmetry groups that do not change the trajectories of the input and the output. If such a symmetry group exists, the system cannot be observable.

3 Symmetry Groups and Non-Observability

Roughly speaking, a symmetry group of a system of PDEs is a (local) group of transformations acting on the manifold \( \mathcal{E} \), with the property that it transforms solutions of the system into other solutions, see [5]. A transformation group on \( \mathcal{E} \) is generated by a smooth vector field \( v \) on \( \mathcal{E} \), and the conditions, which a transformation group must satisfy to qualify as a symmetry group of the system of PDEs, can be formulated in terms of this vector field. These conditions involve the prolongations \( j^1(v) \) and \( j^2(v) \) of \( v \) to the first and the second jet manifold. In the following theorem, we have supplemented the conditions that can
be found in [5] by additional conditions that ensure that the symmetry group also respects the boundary conditions, and that it does not change the trajectories \( u(t) \) and \( y(t) \) of the input and the output. If such a special symmetry group exists, the system (1) – (3) cannot be observable.

**Theorem 3.1** Consider the system (1) with boundary conditions (2) and output (3). If there exists a smooth vector field \( v = v^\alpha(z,t,x,u)\partial_{x^\alpha} \) on \( \mathcal{E} \) with \( \partial_{x^\alpha} v^\alpha|_{t=0} = 0 \), \( \alpha = 1, \ldots, n_x \) and \( v^\alpha|_{t=0} \neq 0 \) that satisfies the conditions

\[
L^2_j(v) (x^\alpha - f^\alpha(z,t,x,x_z,x_{zz},u)) = 0
\]

on the submanifold \( S^2 \subset J^2(\mathcal{E}) \), the conditions

\[
L^j_1(v) g^\lambda(t,x,x_z) \big|_{z=0} = 0
\]

\[
L^j_1(v) h^\mu(t,x,x_z) \big|_{z=1} = 0
\]

on the submanifolds \( S^1_A \subset B_A \) and \( S^1_B \subset B_B \), and the condition

\[
L^j_1(v) c(t,x,x_z) \big|_{z=0} = 0,
\]

then the system is not observable.

The conditions (4) and (5) guarantee that the vector field \( j^2(v) \) generates a symmetry group of the system with boundary conditions. Geometrically, condition (4) means that the vector field \( j^2(v) \) is tangent to the submanifold \( S^2 \subset J^2(\mathcal{E}) \) determined by the system equations, and condition (5) means that the vector fields \( j^1(v) \big|_{z=0} \) and \( j^1(v) \big|_{z=1} \) are tangent to the submanifolds \( S^1_A \subset B_A \) and \( S^1_B \subset B_B \) determined by the boundary conditions. Since the vector field \( v \) has no component in \( \partial_{x^\alpha} \)-direction and because of condition (6), the symmetry group transforms every solution into other solutions without changing the trajectories of input and output. The initial condition of a transformed solution is indistinguishable from the initial condition of the original solution (in the sense of Section 1). Thus, for every initial condition the symmetry group allows to construct a set of indistinguishable initial conditions, and we can conclude that the system is certainly not observable.

It is worth mentioning that the obtained conditions (4), (5), and (6) are linear PDEs in the unknown coefficients \( v^\alpha \) of the vector field \( v \), even though the original system is nonlinear.

### 4 Linear Systems

For linear systems, because of the superposition principle it is sufficient if we consider in Theorem 3.1 only vector fields of the form \( v = v^\alpha(z,t)\partial_{x^\alpha} \), where the coefficients do not depend on \( x \) and \( u \). If we apply the conditions of Theorem 3.1 to the linear time-invariant version

\[
\partial_t x^\alpha(z,t) = A^\alpha_0(z) x^\alpha(z,t) + A^\alpha_\beta(z) \partial_x x^\beta(z,t) + A^\alpha_{zz,\beta}(z) \partial_{x_z} x^\beta(z,t) + B^\alpha(z) u(t), \quad \alpha = 1, \ldots, n_x
\]

\[
G^\alpha_0 x^\beta(0,t) + G^\alpha_\beta \partial_x x^\beta(0,t) = 0, \quad \lambda = 1, \ldots, n_A
\]

\[
H^\alpha_\beta x^\beta(1,t) + H^\alpha_{zz,\beta} \partial_{x_z} x^\beta(1,t) = 0, \quad \mu = 1, \ldots, n_B
\]

\[
y(t) = C_\beta x^\beta(z_0,t) + C_{z,\beta} \partial_{x_z} x^\beta(z_0,t)
\]

of the system (1) – (3), it turns out that the coefficients \( v^\alpha(z,t) \) of the vector field \( v \) must be solutions of the homogenous part of the PDEs (7) that satisfy the boundary conditions (8) and produce an output (9) which is identically zero. Since it is well-known that a linear system allows such solutions if and only if it is not observable (see e.g. [6]), in the linear case the conditions of Theorem 3.1 become necessary and sufficient for non-observability. Furthermore, the symmetry group which is generated by \( v \) transforms solutions exactly in such a way that the difference of the initial conditions is an element of the non-observable subspace.

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