Series composition of simulation-based assume-guarantee contracts for linear dynamical systems

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Abstract—We present assume-guarantee contracts for continuous-time linear dynamical systems with inputs and outputs. These contracts are used to express specifications on the dynamic behaviour of a system. Contrary to existing approaches, we use simulation to compare the dynamic behaviour of two systems. This has the advantage of being supported by efficient numerical algorithms for verification as well as being related to the rich literature on (bi)simulation based techniques for verification and control, such as those based on (discrete) abstractions. Using simulation, we define contract implementation and a notion of contract refinement. We also define a notion of series composition for contracts, which allows us to reason about the series interconnection of systems on the basis of the contracts on its components. Together, the notions of refinement and composition allow contracts to be used for modular design and analysis of interconnected systems.

I. INTRODUCTION

Contract-based design has proven to be an effective method for modular design and analysis of complex interconnected systems [1], [2], [3]. Motivated by this, we present assume-guarantee contracts for continuous-time linear dynamical systems with inputs and outputs in the spirit of [4], [5]. These contracts are defined as a pair of systems called assumptions and guarantees. The assumptions capture the available information about the dynamic behaviour of the environment in which the system is supposed to operate, while the guarantees specify the desired dynamic behaviour of the system when interconnected with a compatible environment. In contrast to [4], [5], we formalize this with the notion of simulation, which is used to compare the dynamic behaviour of two systems.

Simulation is the one-sided version of the notion of bisimulation, which is used to express (external) system equivalence. Bisimulation finds its origins in the field of computer science, where it was introduced in the context of concurrent processes [6]. In this paper, we adopt the notion of (bi)simulation for continuous-time linear dynamical system introduced in [7], see also [8], [9]. This notion is very much inspired by the work of Pappas et al. in [10], [11], [12], where the focus is on abstractions, i.e., (bi)similar systems of lower state space dimension.

Using (bi)simulation as a means of comparing system behaviour has the following advantages. First, as shown in [7], [8], [9], efficient numerical procedures for verifying (bi)simulation can be obtained using ideas from geometric control theory [13] and, in particular, the invariant subspace algorithm [14]. Second, this connection with geometric control theory allows us to use a multitude of tools in addressing problems relevant to contract-based design, such as constructing implementations and controllers for implementations. Third, the notion of (bi)simulation has been extended to more general system classes, such as hybrid and transition systems. In fact, there is a rich literature on using discrete abstractions of continuous dynamical systems for the purposes of verification and control [15], [16]. There are also alternative notions of (bi)simulation, such as approximate (bi)simulation [17], [18], and asymptotic (bi)simulation [19].

All things considered, using (bi)simulation as a means of comparing system behaviour opens the doors to the vast literature on related research.

The contributions of this paper are as follows. First, we define contracts and characterize contract implementation as a simulation of one system by another. Second, we define contract refinement, again in terms of simulation, and show that it satisfies properties which allow us to determine if a given contract expresses a stricter specification than another contract. Since simulation can be verified using efficient numerical procedures, it follows that the same procedures can be used to verify contract implementation and refinement.

Third, we define the series composition of contracts and show that it satisfies properties which allow us to reason about the series interconnection of two systems on the basis of the contracts that they implement. Together, contract refinement and the series composition of contracts have properties which enable the independent design of components within interconnected systems.

The contracts in this paper draw inspiration from the contracts introduced in [4], [5]. The main difference with [4], [5], is that here we use simulation instead of inclusion of external behaviour as a means of comparing system behaviour. As mentioned before, this enables the use of efficient computational tools that are not available for the contracts in [4], [5]. Furthermore, as noticed in the theory of concurrent processes, simulation is more powerful than behavioural inclusion for nondeterministic systems, which will be used throughout this paper.

Different types of contracts have already been used as specifications for dynamical systems. For example, parametric assume-guarantee contracts are introduced in [20] and used for control synthesis in [21], [22], while assume-guarantee contracts that can capture invariance are presented in [23] and applied in [24], [25]. Related work on contracts can also be found in [26], [27], [28]. Whereas the contracts
in this paper express specifications on the dynamics of continuous-time systems, the contracts in [20], [21], [22], [28] are defined only for discrete-time systems, and the contracts in [24], [25], [26] cannot express specifications on dynamics. In this respect, the contracts in this paper are most closely related to the contracts in [29], while [30], [31], [32] contain closely related work on compositional reasoning. A key difference with [29], however, is that there is no distinction between inputs and outputs and interconnection is defined through variable sharing.

The remainder of this paper is organized as follows. In Section II, we introduce the classes of systems considered in this paper and develop an appropriate notion of simulation. Then contracts, contract implementation and contract refinement are defined and characterized in Section III. Following this, we define and characterize the series composition of contracts in Section IV. We finish with concluding remarks in Section V.

The notation used in this paper is mostly standard. The set of nonnegative real numbers is denoted by $\mathbb{R}_{\geq 0}$. Finite-dimensional linear (sub)spaces are denoted by capital calligraphic letters. Given a linear subspace $V \subset X \times Y$, $\pi_X(V)$ denotes the projection of $V$ onto $X$, i.e.,

$$\pi_X(V) = \{ x \in X \mid \exists y \in Y \text{ s.t. } (x, y) \in V \}. \quad (1)$$

The projection $\pi_Y(V)$ is defined similarly. Given a linear map $A : X \rightarrow Y$, im$A$ and ker$A$ denote the image and kernel of $A$, respectively.

II. SYSTEM CLASSES AND SIMULATION

In this paper, we consider systems of the form

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Gd(t), \\ y(t) = Cr(t), \end{cases} \quad (2)$$

with state $x(t) \in X$, input $u(t) \in U$, output $y(t) \in Y$, and driving variable $d(t) \in D$. The driving variable $d$ can be used to model disturbance, nondeterminism, unknown inputs, or lack of knowledge about the dynamics of $\Sigma$. We treat $\Sigma$ as an open system in which the input $u$ and output $y$ are external variables that interact with the environment, whereas the state $x$ and the driving variable $d$ are internal and do not interact with the environment. As a design goal, we are interested in expressing specifications on the dynamics of the external variables of $\Sigma$. We will do this with the notion of a contract.

To define contracts and express specifications, we will make use of systems of the form

$$\Xi_i : \begin{cases} \dot{x}_i(t) = A_ix(t) + G_id_i(t), \\ w_i(t) = C_ir(t), \\ 0 = H_ix(t), \end{cases} \quad (3)$$

with state $x_i(t) \in X_i$, output $w(t) \in W_i$, and driving variable $d_i(t) \in D_i$. The main differences between $\Sigma$ and $\Xi_i$ is that $\Xi_i$ does not admit an input and includes algebraic constraints. Due to the algebraic constraints, not all initial states lead to feasible trajectories. This motivates the introduction of the consistent subspace $V_i \subset X_i$, defined as the set of initial states $x_i(0)$ for which there exists a driving variable $d_i : \mathbb{R}_{\geq 0} \rightarrow D_i$ such that the resulting state trajectory satisfies the algebraic constraint, i.e., $H_ix_i(t) = 0$ for all $t \geq 0$. It can be shown that $V_i$ is the largest subspace that satisfies

$$A_iV_i \subset V_i + \text{im} G_i \quad \text{and} \quad V_i \subset \ker H_i. \quad (4)$$

Remark 1: Including algebraic constraints in the systems $\Xi_i$ has two advantages. First, it leads to a more general class of systems, which will allow us to express more general specifications. Second, it allows for easily defining certain interconnections, which will be essential in the definitions of contract refinement and the series composition of contracts.

The theory that we will develop heavily relies on comparing the dynamics of different systems $\Xi_i$. For this, we will make use of the notion of simulation, which itself relies on the notion of simulation relation. The following definition is taken from [29], see also [8], [9].

Definition 1: A linear subspace $S \subset X_1 \times X_2$ satisfying $\pi_{X_i}(S) \subset V_i$, $i \in \{1, 2\}$, is a simulation relation of $\Xi_1$ by $\Xi_2$ if the following implication holds: for all $(x_1(0), x_2(0)) \in S$ and all $d_1 : \mathbb{R}_{\geq 0} \rightarrow D_1$ such that $x_1(t) \in V_1$ for all $t \geq 0$, there exists $d_2 : \mathbb{R}_{\geq 0} \rightarrow D_2$ such that:

1) $(x_1(t), x_2(t)) \in S$ for all $t \geq 0$;
2) $w_1(t) = w_2(t)$ for all $t \geq 0$.

Using ideas from geometric control theory [13], [14], we obtain the following equivalent characterization of a simulation relation based solely on the system matrices, see [29], [8], [9] for details.

Proposition 1: A linear subspace $S \subset X_1 \times X_2$ satisfying $\pi_{X_i}(S) \subset V_i$, $i \in \{1, 2\}$, is a simulation relation of $\Xi_1$ by $\Xi_2$ if and only if for all $(x_1, x_2) \in S$ and all $d_1 \in D_1$ such that $A_1x_1 + G_1d_1 \in V_1$, there exists $d_2 \in D_2$ such that:

1) $(A_1x_1 + G_1d_1, A_2x_2 + G_2d_2) \in S$;
2) $C_1x_1 = C_2x_2$.

Remark 2: When constructing a simulation relation $\Xi_1$, it is sometimes difficult to ensure that $\pi_{X_i}(S) \subset V_i$. For such cases, note that if $S$ satisfies the first condition in Proposition 1, then

$$A_1\pi_X(S) \subset \pi_X(S) + \text{im} G_i, \quad (5)$$

hence $\pi_{X_i}(S) \subset V_i$ if and only if $\pi_{X_i}(S) \subset \ker H_i$.

Simulation is then defined as follows.

Definition 2: A system $\Xi_1$ is simulated by $\Xi_2$, denoted as $\Xi_1 \preceq \Xi_2$, if there exists a simulation relation $S \subset X_1 \times X_2$ of $\Xi_1$ by $\Xi_2$ such that $\pi_{X_i}(S) = V_i$. A simulation relation with this property is called a full simulation relation.

If $\Xi_1 \preceq \Xi_2$, then any state trajectory of $\Xi_1$ can be matched by a state trajectory of $\Xi_2$ such that the outputs of $\Xi_1$ and $\Xi_2$ are identical. We can interpret this as $\Xi_2$ having richer dynamics than $\Xi_1$.

Remark 3: In view of (4), computing the consistent subspace of a system $\Xi_i$ amounts to computing the largest $(A_i, G_i)$-invariant subspace contained in $\ker H_i$, which can be done using the invariant subspace algorithm, see [13], [14] for details. Using the same algorithm, one can compute the largest simulation relation of $\Xi_1$ by $\Xi_2$ and thus determine whether $\Xi_1$ is simulated by $\Xi_2$, see [29, Theorem 6] and [29,
Remark 4] for details. In other words, simulation is supported by efficient numerical procedures for verification.

Remark 4: If \( \Xi_1 \preceq \Xi_2 \) and \( \Xi_2 \preceq \Xi_1 \), then \( \Xi_1 \) and \( \Xi_2 \) can be shown to be bisimilar, denoted by \( \Xi_1 \sim \Xi_2 \). Bisimilarity for systems of the form (3) is defined in [9], and a proof of this statement is given in Proposition 5.3. Loosely speaking, bisimilar systems have the same external dynamics.

Remark 5: An important property that will be used throughout this paper is that simulation is a preorder [29, Lemma 2], i.e., it is reflexive \( (\Xi_i \preceq \Xi_i) \) for all \( \Xi_i \) and transitive \( (\Xi_1 \preceq \Xi_2 \) and \( \Xi_2 \preceq \Xi_3 \) imply that \( \Xi_1 \preceq \Xi_3 \)).

III. CONTRACTS

In this section, we will define contracts, contract implementation, and a notion of contract refinement that will allow us to compare contracts. Consider a system \( \Sigma \) of the form (2). The environment \( E \) of \( \Sigma \) is a system of the form

\[
E : \begin{cases} 
\dot{x}_e = A_x x_e + G_x d_e, \\
u = C_e x_e, \\
0 = H_e x_e,
\end{cases}
\]

with \( x_e \in \mathcal{X}_e \) and \( d_e \in \mathcal{D}_e \). Here, we have omitted the time variable \( t \) for convenience. The environment \( E \) is interpreted as a system that generates inputs for \( \Sigma \), hence the interconnection of \( E \) and \( \Sigma \) is defined by

\[
E \wedge \Sigma : \begin{cases} 
\dot{x}_e = A_x x_e + G_x d_e, \\
u = C_e x_e, \\
0 = H_e x_e,
\end{cases}
\]

which we have obtained by setting the output generated by \( E \) as input to \( \Sigma \) as shown in Figure 1. We are interested in specifying the dynamic behaviour of \( E \wedge \Sigma \) only for relevant environments \( E \). This will be formalized with the notion of a contract, which will require the definition of another two systems. First, the assumptions \( A \) are a system of the form

\[
A : \begin{cases} 
\dot{x}_a = A_a x_a + G_a d_a, \\
u = C_a x_a, \\
0 = H_a x_a,
\end{cases}
\]

with \( x_a \in \mathcal{X}_a \) and \( d_a \in \mathcal{D}_a \). Assumptions have the same form as environments and they can be compared using simulation. Second, the guarantees \( \Gamma \) are a system of the form

\[
\Gamma : \begin{cases} 
\dot{x}_g = A_g x_g + G_g d_g, \\
u = C_g x_g, \\
0 = H_g x_g,
\end{cases}
\]

with \( x_g \in \mathcal{X}_g \) and \( d_g \in \mathcal{D}_g \). Guarantees have the same form as the interconnection \( E \wedge \Sigma \) and they can be compared using simulation. With assumptions and guarantees defined, we are ready to define contracts.

Definition 3: A contract \( C = (A, \Gamma) \) is a pair of assumptions and guarantees.

A contract is used as a specification in the following sense.

Definition 4: Consider a contract \( C = (A, \Gamma) \). An environment \( E \) is compatible with \( C \) if

\[
E \preceq A.
\]

A system \( \Sigma \) implements \( C \) if

\[
E \wedge \Sigma \preceq \Gamma
\]

for any environment \( E \) compatible with \( C \).

In other words, the assumptions capture the available information about the dynamics of the environments in which our system is supposed to operate, thus leading to a class of compatible environments, while the guarantees specify the desired dynamics of our system when interconnected with a compatible environment, thus leading to a class of implementations.

We can check if a given system \( \Sigma \) implements a given contract \( C = (A, \Gamma) \) without having to construct all compatible environments. To show this, we will make use of the following lemma.

Lemma 2: If \( E \preceq A \), then \( E \wedge \Sigma \preceq A \wedge \Sigma \).

Proof: Let \( S_e \) be a full simulation relation of \( E \) by \( A \). We will show that the subspace \( S \subseteq (\mathcal{X}_e \times \mathcal{X}) \) defined by

\[
S = \{(x_e, x, x_a, x) \mid (x_e, x_a) \in S_e\}
\]

is a full simulation relation of \( E \wedge \Sigma \) by \( A \wedge \Sigma \). First, note that the consistent subspace of \( E \wedge \Sigma \) is given by \( \mathcal{V}_e \times \mathcal{X} \), and the consistent subspace of \( A \wedge \Sigma \) is given by \( \mathcal{V}_e \times \mathcal{X} \), where \( \mathcal{V}_e \) and \( \mathcal{V}_a \) are the consistent subspaces of \( E \) and \( A \), respectively. Since \( \mathcal{V}_e \times \mathcal{X} \) contains \( \mathcal{V}_e \), it follows that

\[
\mathcal{V}_e \times \mathcal{X} \subseteq \mathcal{V}_e \times \mathcal{X} \quad \text{and} \quad \mathcal{V}_e \times \mathcal{X} \subseteq \mathcal{V}_e \times \mathcal{X}.
\]

Let \( (x_e, x, x_a, x) \in S \) and take \( d_e \in D_e, d \in D \) such that

\[
(A_e x_e + G_e d_e, BC_e x_e + Ax + Gd) \in \mathcal{V}_e \times \mathcal{X}.
\]

For later reference, let \( s = BC_e x_e + Ax + Gd \). As \( (x_e, x_a) \in S_e \) and \( A_e x_e + G_e d_e \in \mathcal{V}_e \), it follows from Proposition 1 that there there exists \( d_a \in D_a \) such that

\[
(A_e x_e + G_e d_e, A_a x_a + G_a d_a) \in S_e,
\]

\[
C_e x_e = C_a x_a.
\]

Then (15) implies that

\[
(A_e x_e + G_e d_e, s, A_a x_a + G_a d_a, s) \in S,
\]

while (16) implies that \( s = BC_a x_a + Ax + Gd \) and

\[
\begin{bmatrix} C_a & 0 \\ 0 & C \end{bmatrix} x_e = \begin{bmatrix} C_a & 0 \\ 0 & C \end{bmatrix} x_a.
\]

Using Proposition 1, we conclude that \( S \) is a full simulation relation of \( E \wedge \Sigma \) by \( A \wedge \Sigma \) and thus \( E \wedge \Sigma \preceq A \wedge \Sigma \).

As an almost immediate consequence of Lemma 2, we obtain the following necessary and sufficient condition for contract implementation.
Theorem 3: A system $\Sigma$ implements the contract $C = (A, \Gamma)$ if and only if

$$A \land \Sigma \preceq \Gamma. \quad (19)$$

Proof: Suppose that $\Sigma$ implements $C$. Since $A$ is an environment compatible with $C$, it follows that (19) holds. Conversely, suppose that (19) holds and let $E$ be compatible with $C$, that is, $E \preceq A$. In view of Lemma 2, we have that $E \land \Sigma \preceq A \land \Sigma$, hence $E \land \Sigma \preceq \Gamma$ because simulation is transitive. Since $E \land \Sigma \preceq \Gamma$ for any $E$ compatible with $C$, we conclude that $\Sigma$ implements $C$.

Remark 6: Clearly, two contracts define the same class of compatible environments if and only if their assumptions are bisimilar. However, two contracts can define the same class of implementations even if their guarantees are not bisimilar. For instance, $C = (A, \Gamma)$ defines the same class of implementations as $C' = (A, A \land \Gamma)$, where $A \land \Gamma$ is obtained by equating the outputs $u$ of $A$ and $\Gamma$, that is,

$$A \land \Gamma : \begin{cases}
\dot{x}_a = A_0 x_a + G_a d_g \\
u = C_a x_a \\
0 = C_a - C_g u 
\end{cases}, \quad (20)$$

Indeed, if $S$ is a full simulation relation of $A \land \Sigma$ by $\Gamma$, then

$$S' = \{(x_a, x, x_a, x_g) \mid (x_a, x, x_g) \in S\} \quad (21)$$

is a full simulation relation of $A \land \Sigma$ by $A \land \Gamma$, hence, due to Theorem 3, every implementation of $C$ is also an implementation of $C'$. Conversely, it can be shown that $A \land \Gamma$ with a full simulation relation given by

$$S = \{(x_a, x_g, x_g) \mid (x_a, x_g, x_g) \in \mathcal{V}_{A \land \Gamma}\}, \quad (22)$$

where $\mathcal{V}_{A \land \Gamma}$ is the consistent subspace of $A \land \Gamma$. Therefore, due to Theorem 3 and the transitivity of simulation, every implementation of $C'$ is also an implementation of $C$.

Remark 7: A contract $C = (A, \Gamma)$ is consistent if it can be implemented. Not every contract is consistent. To see this, note that $u$ is an input in $\Sigma$, hence any restriction on the dynamics of $u$ in $A \land \Sigma$ come from the assumptions $A$. Therefore, $A \land \Sigma \preceq \Gamma$ only if any restrictions on the dynamics of $u$ in $\Gamma$ are already present in $A$, that is, $A \preceq \Gamma$, where $\Gamma$ is obtained from $\Gamma$ by considering only $u$ as an output. Indeed, if $S$ is a full simulation relation of $A \land \Sigma$ by $\Gamma$, then it can be shown that $\pi_{X_a \times X_g}(S)$ is a full simulation relation of $A$ by $\Gamma$. Consequently, the condition $A \preceq \Gamma$ is necessary (but not sufficient) for consistency.

Next, we define the notion of refinement, which allows us to compare two contracts.

Definition 5: A contract $C_1 = (A_1, \Gamma_1)$ refines another contract $C_2 = (A_2, \Gamma_2)$, denoted as $C_1 \preceq C_2$, if

$$A_2 \preceq A_1 \quad \text{and} \quad A_2 \land \Gamma_1 \preceq \Gamma_2 \quad (23)$$

Refinement allows us to determine if a contract expresses a stricter specification than another contract. In particular, the following theorem shows that if $C_1 \preceq C_2$, then $C_1$ defines a larger class of compatible environments but a smaller class of implementations than $C_2$.

Theorem 4: If $C_1 \preceq C_2$, then the following holds:

1) any environment compatible with $C_2$ is compatible with $C_1$;
2) any implementation of $C_1$ is an implementation of $C_2$.

Proof: Let $C_1 = (A_1, \Gamma_1)$ and $C_2 = (A_2, \Gamma_2)$, and suppose that $C_1 \preceq C_2$, that is, (23) holds. Let $E$ be an environment compatible with $C_2$, that is, $E \preceq A_2$. Since $A_2 \preceq A_1$ and simulation is transitive, it follows that $E \preceq A_1$ and thus $E$ is also compatible with $C_1$. Next, suppose that $\Sigma$ implements $C_1$. Note that $A_2$ is an environment compatible with $C_1$ because $A_2 \preceq A_1$. Consequently, we must have that $A_2 \land \Sigma \preceq \Gamma_1$. As explained in Remark 6, this implies that $A_2 \land \Sigma \preceq A_2 \land \Gamma_1$, hence $A_2 \land \Sigma \preceq \Gamma_2$ because $A_2 \land \Gamma_1 \preceq \Gamma_2$ and simulation is transitive. Using Theorem 3, we conclude that $\Sigma$ also implements $C_2$.

Intuitively, Theorem 4 tells us that $C_1$ has stricter guarantees than $C_2$ that have to be met in the presence of weaker assumptions than those of $C_2$. In other words, $C_1$ expresses a stricter specification than $C_2$.

IV. SERIES COMPOSITION OF CONTRACTS

In this section, we will define the series composition of two contracts and will show that it satisfies desirable properties for modular analysis. The series composition of contracts can be used to reason about the series interconnection of systems on the basis of the contracts on its components. Loosely speaking, we want the series composition of two contracts to be implemented by the series interconnection of any of their implementations. To make this precise, we first define the series interconnection of systems of the form (2).

Definition 6: Consider systems $\Sigma_1$ and $\Sigma_2$ of the form (2). The series interconnection of $\Sigma_1$ to $\Sigma_2$, denoted as $\Sigma_1 \rightarrow \Sigma_2$, is obtained by setting the output of $\Sigma_1$ as input of $\Sigma_2$, as shown in Figure 2. In other words, the series interconnection $\Sigma_1 \rightarrow \Sigma_2$ is given by

$$\Sigma_1 \rightarrow \Sigma_2 : \begin{cases}
\dot{x}_1 = [A_1 B_{21} C_1 A_2] x_1 + [B_1 0] u \\
\dot{x}_2 = [G_1 0] x_2 \\
y = [C_2] x_1 
\end{cases}, \quad (24)$$

Therefore, given contracts $C_1$ and $C_2$ for $\Sigma_1$ and $\Sigma_2$, respectively, our goal is to define a contract $C_1 \rightarrow C_2$ which $\Sigma_1 \rightarrow \Sigma_2$ is guaranteed to implement. This will naturally
lead us to consider the series interconnection of guarantees, defined below.

**Definition 7:** Consider guarantees $\Gamma_1$ and $\Gamma_2$. The **series interconnection** of $\Gamma_1$ to $\Gamma_2$, denoted as $\Gamma_1 \rightarrow \Gamma_2$, is obtained by setting the output $y_1$ of $\Gamma_1$ equal to the output $u_2$ of $\Gamma_2$, as shown in Figure 3. In other words, the series interconnection $\Gamma_1 \rightarrow \Gamma_2$ is given by

$$
\begin{bmatrix}
\dot{x}_{g_1} \\
\dot{x}_{g_2}
\end{bmatrix} =
\begin{bmatrix}
A_{g_1} & 0 \\
0 & A_{g_2}
\end{bmatrix}
\begin{bmatrix}
x_{g_1} \\
x_{g_2}
\end{bmatrix}
+ 
\begin{bmatrix}
G_{g_1} \\
0
\end{bmatrix}
\begin{bmatrix}
d_{g_1} \\
d_{g_2}
\end{bmatrix},
$$

$$
\Gamma_1 \rightarrow \Gamma_2 : \begin{bmatrix}
u_1 \\
yy_1
\end{bmatrix} =
\begin{bmatrix}
C_{g_1} \\
0
\end{bmatrix}
\begin{bmatrix}
x_{g_1} \\
x_{g_2}
\end{bmatrix}
+ 
\begin{bmatrix}
H_{g_1} \\
0
\end{bmatrix}
\begin{bmatrix}
x_{g_1} \\
x_{g_2}
\end{bmatrix}, \quad (25)
$$

Since $\Sigma_1$ and $\Sigma_2$ are designed to work only in interconnection with environments compatible with $C_1$ and $C_2$, respectively, it is only natural to require that, for any environment $E$ compatible with $C_1 \rightarrow C_2$, the environments of $\Sigma_1$ and $\Sigma_2$ in the interconnection $E \land (\Sigma_1 \rightarrow \Sigma_2)$ are compatible with $C_1$ and $C_2$, respectively. Consequently, since the environment of $\Sigma_1$ in $E \land (\Sigma_1 \rightarrow \Sigma_2)$ is $E$ itself, it immediately follows that we must have $E \leq A_1$. On the other hand, since the environment of $\Sigma_2$ in $E \land (\Sigma_1 \rightarrow \Sigma_2)$ is $(E \land \Sigma_1)^y$, where $(E \land \Sigma_1)^y$ is obtained from $E \land \Sigma_1$ by considering only $y_1$ as an output, it follows that we must also have $(E \land \Sigma_1)^y \leq A_2$. With this in mind, consider the following definition.

**Definition 8:** Consider contracts $C_1 = (A_1, \Gamma_1)$ and $C_2 = (A_2, \Gamma_2)$. We say that $C_1$ is **series composable** to $C_2$ if

$$
(A_1 \land \Gamma_1)^y \leq A_2. \quad (26)
$$

In this case, the **series composition** of $C_1$ to $C_2$, denoted by $C_1 \rightarrow C_2$, is defined as

$$
C_1 \rightarrow C_2 = (A_1, \Gamma_1 \rightarrow \Gamma_2). \quad (27)
$$

The following theorem, whose proof can be found in the appendix of [33], shows that series composition satisfies the properties mentioned above.

**Theorem 5:** Consider contracts $C_1$ and $C_2$ such that $C_1$ is series composable to $C_2$. If $\Sigma_1$ and $\Sigma_2$ implement $C_1$ and $C_2$, respectively, and $E$ is compatible with $C_1 \rightarrow C_2$, then the following conditions hold:

1) the environment of $\Sigma_1$ in $E \land (\Sigma_1 \rightarrow \Sigma_2)$ is compatible with $C_1$;
2) the environment of $\Sigma_2$ in $E \land (\Sigma_1 \rightarrow \Sigma_2)$ is compatible with $C_2$;
3) $\Sigma_1 \rightarrow \Sigma_2$ implements $C_1 \rightarrow C_2$.

**Remark 8:** Contract refinement and the series composition of contracts have properties that enable the independent design of components within interconnected systems. As a simple example, suppose that we want to design $\Sigma_1$ and $\Sigma_2$ such that the series interconnection $\Sigma_1 \rightarrow \Sigma_2$ implements an overall contract $C$. Using the definition of the series composition, we can construct contracts $C_1$ and $C_2$ such that $C_1$ is series composable to $C_2$ and $C_1 \rightarrow C_2 \leq C$. Consequently, if $\Sigma_1$ implements $C_1$ and $\Sigma_2$ implements $C_2$, then, due to Theorem 5, we know that $\Sigma_1 \rightarrow \Sigma_2$ implements $C_1 \rightarrow C_2$, and thus, due to Theorem 4, $\Sigma_1 \rightarrow \Sigma_2$ implements $C$. This means that the designer of $\Sigma_1$ need only implement $C_1$ and need not concern themselves with the design of $\Sigma_2$ or the integration of $\Sigma_1$ into the series interconnection $\Sigma_1 \rightarrow \Sigma_2$. In other words, $\Sigma_1$ can be designed independently of $\Sigma_2$. The same is true for $\Sigma_2$, of course.

**Remark 9:** It can be shown that the series composition has the following property in relation to refinement. Suppose that $C_1$ is series composable to $C_2$ and $C_1'$ is series composable to $C_2$. If $C_1' \leq C_1$ and $C_2' \leq C_2$, then

$$
C_1' \rightarrow C_2' \leq C_1 \rightarrow C_2. \quad (28)
$$

The proof of this statement is rather long and technical, and is thus beyond the scope of this paper.

We conclude this section with a simple academic example of series composability and the series composition.

**Example 1:** Consider the contract $C = (A, \Gamma)$ with

$$
A : \begin{cases}
\dot{x}_a = d_a, \\
\dot{y} = x_a,
\end{cases} \quad \Gamma : \begin{cases}
\dot{x}_g = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} x_g + \begin{bmatrix} f & 0 \\ 0 & I \end{bmatrix} d_g,
\end{cases}
$$

$$
\begin{bmatrix}
u_a \\
y
\end{bmatrix} = \begin{bmatrix} C_a \\
0
\end{bmatrix} \begin{bmatrix}
x_{g_1} \\
x_{g_2}
\end{bmatrix} + \begin{bmatrix} H_a \\
0
\end{bmatrix} \begin{bmatrix}
x_{g_1} \\
x_{g_2}
\end{bmatrix}, \quad (29)
$$

Note that $u$ is essentially free in $A$ since the only restriction is that $\dot{u} = d_a$ for some $d_a : \mathbb{R} \rightarrow D_a$. Similarly, $u$ is essentially free in $\Gamma$, whereas $y$ is such that $\dot{y} = u$, that is, $\Gamma$ represents a single integrator. Since $u$ is essentially free in $A$, we expect that $(A \land \Gamma)^y \leq A$. Indeed, we can show that the subspace $S \subset (X_a \times X_g) \times X_a$ defined by

$$
S = \{ (x_a, x_g, x_a') | (x_a, x_g) \in V_{a \land g}, \, x_a' = \begin{bmatrix} I & 0 \end{bmatrix} x_g \}
$$

is a full simulation relation of $(A \land \Gamma)^y$ by $A$, where $A \land \Gamma$ is given in (20). To do this, let $(x_a, x_g, x_a') \in S$ and take $d_a \in D_a$ and $d_g \in D_g$ such that

$$
\begin{bmatrix} d_a \\
I \\
0
\end{bmatrix} \begin{bmatrix} x_g \\
I \\
0
\end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & I \end{bmatrix} d_g, \quad (30)
$$

Note that $d_a' = \begin{bmatrix} I & 0 \end{bmatrix} x_g$ is such that

$$
\begin{bmatrix} d_a \\
I \\
0
\end{bmatrix} \begin{bmatrix} x_g \\
I \\
0
\end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & I \end{bmatrix} d_g, d_a' \in S. \quad (31)
$$

Furthermore, since $x_a' = \begin{bmatrix} I & 0 \end{bmatrix} x_g$, it follows that

$$
\begin{bmatrix} 0 & 0 \\ 0 & I
\end{bmatrix} \begin{bmatrix} x_g \\
x_g
\end{bmatrix} = x_a', \quad (32)
$$

The proof of this statement is rather long and technical, and is thus beyond the scope of this paper.
hence $S$ is a simulation relation of $(A \land \Gamma)^y$ by $A$ due to Proposition 1. As $\pi_{Xa \times Xg} (S) = V_a \land g$, it follows that $S$ is a full simulation relation and $(A \land \Gamma)^y \preceq A$. This means that $C$ is series composable to $C$ and $C \rightarrow C = (A, \Gamma \rightarrow \Gamma)$. Note that, by Definition 7, $\Gamma \rightarrow \Gamma$ is given by

$$
\begin{align*}
\dot{x}_g &= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & I & 0
\end{bmatrix} x_g + \begin{bmatrix}
I \\
0 \\
0 \\
0
\end{bmatrix} d_g, \\
\begin{bmatrix}
u \\
y \\
0
\end{bmatrix} &= \begin{bmatrix}
I & 0 & 0 \\
0 & 0 & I \\
0 & I & -I & 0
\end{bmatrix} x_g,
\end{align*}
$$

and we have that $\bar{y} = u$, that is, $\Gamma \rightarrow \Gamma$ represents a double integrator, as expected.

V. CONCLUSION

We presented assume-guarantee contracts for linear dynamical systems with inputs and outputs. In particular, we defined contracts as a pair of linear dynamical systems called assumptions and guarantees. We defined contract implementation using the notion of simulation. We also defined and characterized notions of contract refinement and the series composition of contracts. All relevant conditions are in terms of simulation and can be verified using the efficient numerical algorithm for verifying simulation.

Future work will focus on defining different types of contract composition (e.g., feedback) in order to reason about more general system interconnections, and on addressing the problems of constructing implementations and synthesizing controllers for implementations.

REFERENCES

[1] A. Benveniste, B. Caillaud, D. Nickovic, R. Passerone, J.-B. Raclet, P. Reinkeimeier, A. Sangiovanni-Vincentelli, W. Damm, T. A. Henzinger, and K. G. Larsen, *Contracts for System Design*. Foundations and Trends in Electronic Design Automation, Now Publishers, 2018.

[2] A. Sangiovanni-Vincentelli, W. Damm, and R. Passerone, “Taming dr. Frankenstein: Contract-based design for cyber-physical systems,” *European Journal of Control*, vol. 18, no. 3, pp. 217–238, 2012.

[3] P. Nuzzo, H. Xu, N. Ozay, J. B. Finn, A. L. Sangiovanni-Vincentelli, R. M. Murray, A. Donzé, and S. A. Seshia, “A contract-based methodology for aircraft electric power system design,” *IEEE Access*, vol. 2, pp. 1–25, 2014.

[4] B. M. Shali, A. J. van der Schaft, and B. Besselink, “Behavioural contracts for linear dynamical systems: input assumptions and output guarantees,” in *Proceedings of the European Control Conference*, pp. 564–569, 2021.

[5] B. M. Shali, A. J. van der Schaft, and B. Besselink, “Behavioural assume-guarantee contracts for linear dynamical systems,” in *Proceedings of the IEEE Conference on Decision and Control*, pp. 2002–2007, 2021.

[6] R. Milner, *Communication and Concurrency*. Prentice Hall International Series in Computer Science, Prentice Hall, 1995.

[7] A. J. van der Schaft, “Equivalence of dynamical systems by bisimulation,” *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2169–2172, 2004.

[8] A. van der Schaft, “Equivalence of hybrid dynamical systems,” in *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems*, Leuven, Belgium, 2004.

[9] N. Y. Megawati and A. van der Schaft, “Bisimulation equivalence of differential-algebraic systems,” *International Journal of Control*, vol. 91, no. 1, pp. 45–56, 2016.

[10] G. Pappas, G. Lafreriere, and S. Sastry, “Hierarchically consistent control systems,” *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1144–1160, 2000.

[11] G. Pappas and S. Simic, “Consistent abstractions of affine control systems,” *IEEE Transactions on Automatic Control*, vol. 47, no. 5, pp. 745–756, 2002.

[12] G. J. Pappas, “Bisimilar linear systems,” *Automatica*, vol. 39, p. 2035–2047, Dec. 2003.

[13] G. Basile and G. Marro, *Controlled and conditioned invariants in linear system theory*. Prentice Hall, Englewood Cliffs, USA, 1992.

[14] H. L. Trentelman, A. A. Stoorvogel, and M. L. J. Hautus, *Control theory for linear systems*. London: Springer-Verlag, 2001.

[15] P. Tabuada, *Verification and Control of Hybrid Systems: A Symbolic Approach*. Springer US, 2009.

[16] C. Belta, B. Yordanov, and E. A. Gol, *Formal Methods for Discrete-Time Dynamical Systems*, vol. 89 of Studies in Systems, Decision and Control. Springer-Verlag GmbH, 2017.

[17] A. Girard and G. J. Pappas, “Approximation metrics for discrete and continuous systems,” *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 782–798, 2007.

[18] A. Girard and G. J. Pappas, “Approximate bisimulation: A bridge between computer science and control theory,” *European Journal of Control*, vol. 17, no. 5, pp. 568–578, 2011.

[19] H. Vinjamoor and A. J. van der Schaft, “Asymptotic achievability for linear time invariant state space systems,” in *Proceedings of the IEEE Conference on Decision and Control*, 2010.

[20] E. S. Kim, M. Arcak, and S. A. Seshia, “A small gain theorem for parametric assume-guarantee contracts,” in *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control*, pp. 207–216, 2017.

[21] M. Al Khutib and M. Zamani, “Controller synthesis for interconnected systems using parametric assume-guarantee contracts,” in *Proceedings of the American Control Conference*, pp. 5419–5424, 2020.

[22] Y. Chen, J. Anderson, K. Kalsi, A. D. Ames, and S. H. Low, “Safety-critical control synthesis for network systems with control barrier functions and assume-guarantee contracts,” *IEEE Transactions on Control of Network Systems*, vol. 8, no. 1, pp. 487–499, 2021.

[23] A. Saoud, A. Girard, and L. Fribourg, “Assume-guarantee contracts for continuous-time systems,” *Automatica*, vol. 134, p. 109910, 2021.

[24] D. Zonetti, A. Saoud, A. Girard, and L. Fribourg, “A symbolic approach to voltage stability and power sharing in time-varying DC microgrids,” in *Proceedings of the European Control Conference*, pp. 903–909, 2019.

[25] I. D. Loreto, A. Borri, and M. D. Di Benedetto, “An assume-guarantee approach to sampled-data quantized glucose control,” in *Proceedings of the IEEE Conference on Decision and Control*, pp. 3401–3406, 2020.

[26] S. Eqtami and A. Girard, “A quantitative approach on assume-guarantee contracts for safety of interconnected systems,” in *Proceedings of the European Control Conference*, pp. 536–541, 2019.

[27] K. Ghasemi, S. Sadraadini, and C. Bela, “Compositional synthesis via a convex parameterization of assume-guarantee contracts,” in *Proceedings of the International Conference on Hybrid Systems: Computation and Control*, pp. 1–10, 2020.

[28] M. Sharf, B. Besselink, A. Molin, Q. Zhao, and K. H. Johansson, “Assume/guarantee contracts for dynamical systems: Theory and computational tools,” in *Proceedings of the 7th IFAC Conference on Analysis and Design of Hybrid Systems*, pp. 25–30, 2021.

[29] B. Besselink, K. H. Johansson, and A. J. van der Schaft, “Contracts as specifications for dynamical systems in driving variable form,” in *Proceedings of the European Control Conference*, pp. 263–268, 2019.

[30] F. Kerber and A. van der Schaft, “Assume-guarantee reasoning for linear dynamical systems,” in *Proceedings of the European Control Conference*, pp. 5015–5020, 2009.

[31] F. Kerber and A. van der Schaft, “Compositional analysis for linear systems,” *Systems & Control Letters*, vol. 59, no. 10, pp. 645–653, 2010.

[32] F. Kerber and A. J. van der Schaft, “Decentralized control using compositional analysis techniques,” in *Proceedings of the IEEE Conference on Decision and Control and European Control Conference*, pp. 2699–2704, 2011.

[33] B. M. Shali, H. M. Heidema, A. J. van der Schaft, and B. Besselink, “Series composition of simulation-based assume-guarantee contracts for linear dynamical systems,” *arXiv e-prints*, p. arXiv:2209.01844, 2022.