Dimensionally reduced gravity theories are asymptotically safe

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Abstract

4D Einstein gravity coupled to scalars and abelian gauge fields in its 2-Killing vector reduction is shown to be quasi-renormalizable to all loop orders at the expense of introducing infinitely many essential couplings. The latter can be combined into one or two functions of the ‘area radius’ associated with the two Killing vectors. The renormalization flow of these couplings is governed by beta functionals expressible in closed form in terms of the (one coupling) beta function of a symmetric space sigma-model. Generically the matter coupled systems are asymptotically safe, that is the flow possesses a non-trivial UV stable fixed point at which the trace anomaly vanishes. The main exception is a minimal coupling of 4D Einstein gravity to massless free scalars, in which case the scalars decouple from gravity at the fixed point.

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1. Introduction: asymptotic safety and dimensional reduction

Roughly speaking asymptotic safety is a property of a non-renormalizable quantum field theory replacing asymptotic freedom for a renormalizable one \([1]\). Unsurprisingly not many examples are known so far. Weinberg’s original idea was that 4D quantum Einstein gravity might have this property in a suitable (non-perturbative) formulation, a scenario for which only recently non-trivial evidence has been reported \([2, 3, 4, 5]\). Nevertheless it is unlikely that a definite conclusion can be reached on the issue in the near future. In view of this it seems worthwhile to look for simpler systems where a definite conclusion can be reached. For dimensionally reduced gravity theories this turns out to be the case.

Generalizing the results for pure gravity in \([6, 7]\) we consider here the 2-Killing vector reduction of 4D Einstein gravity coupled to abelian gauge fields and scalars in a way as they arise from \(D \geq 4\) dimensional (super-) gravity theories \([12]\). Coupling gravity to scalars and/or gauge fields is known to destroy even one loop renormalizability \([8, 9]\) so that the status of the asymptotic safety scenario is of particular interest; see \([10]\) for some indicative results. We consider the 2-Killing vector reduction of the large class of 4D matter coupled systems studied in \([12]\); see also \([13, 15]\) for their higher dimensional origin. For our purposes this class of matter couplings is natural because in their 2 Killing vector reduction all but two of the 4D gravitational + 4D matter degrees of freedom can be arranged to parameterize a non-compact symmetric space \(G/H\). The reduced action becomes that of 2D gravity non-minimally coupled to a \(G/H\) sigma-model, where the coupling is via the “area radius” \(\rho\) of the two Killing vectors. This and the residual conformal factor \(\sigma\) drastically change the classical and quantum dynamics as compared to the same noncompact \(G/H\) sigma-model without coupling to gravity \([23]\). The qualitative differences are summarized in the table below.

| \(G/H\) sigma-model | dim. red. gravity with \(G/H\) |
|----------------------|-----------------------------|
| renormalizable       | non-renormalizable          |
| one essential coupling| \(\infty\) essential couplings |
| \(\lambda\)          | function \(h(\cdot)\)       |
| flow is formally infrared free | flow is asymptotically safe |
| trivial fixed point  | non-trivial fixed point    |
| \(\lambda = 0\)      | \(h^{\beta}(\cdot)\)       |
| formally IR stable   | UV stable                  |
| trace anomalous      | trace anomaly vanishes     |
The result is surprising for several reasons. First because the stability properties of the renormalization flow are reversed, for which there is no obvious 2D reason. Second, a non-trivial fixed point exists in all cases, where gravity remains self-interacting and coupled to matter. The trace anomaly then vanishes and the quantum constraints (stemming from the residual 2D diffeomorphism invariance) can in principle be imposed. Third, although the structure of the symmetric space \( G/H \) depends on the signature of the Killing vectors and the details of the reduction procedure, the generalized beta function governing the functional \( h \)-flow is independent of it. These features are not built into the renormalization procedure as is highlighted by the fact that when a collection of 4D free massless fields is minimally coupled to 4D Einstein gravity its 2-Killing vector reduction only has a trivial fixed point where the scalars decouple from gravity. The latter is a special case of ‘augmented’ scalar matter which we also consider.

The article is organized as follows. After introducing the class of 4D gravity theories considered along with their 2-Killing vector reduction, the renormalization of the basic Lagrangian is performed in section 3 to all loop orders and the generalized beta function for the essential couplings is derived. The existence of an UV stable fixed point for their renormalization flow is established in section 4 and is extended to additional scalar matter in section 5. In all cases the vanishing of the beta function(s) is subsequently found to entail the vanishing of the trace anomaly modulo an improvement term, and vice versa. Some technical material is relegated to appendices.

2. Einstein gravity coupled to abelian gauge fields and scalars

Here we describe the class of 4D gravity theories considered and outline the reduction procedure. We largely follow the treatment in \[12, 13\]. The higher dimensional origin of their 3D reductions is systematically explored in \[14, 15\].

We consider 4D Einstein gravity coupled to \( k \) abelian gauge fields and \( \tilde{n} \) scalars through the following action

\[
S_4 = \int d^4x \sqrt{-g} \left[ -R(g) + \frac{1}{2} (T^\alpha_J, J_\alpha)_g - \frac{g}{4} F^T_{\alpha\beta} (\mu F^{\alpha\beta} - \nu^* F^{\alpha\beta}) \right]. \tag{2.1}
\]

Here \( g_{\alpha\beta}, 1 \leq \alpha, \beta \leq 4 \), is the spacetime metric with eigenvalues \((+, -, -, -)\), \( R(g) \) is its scalar curvature and indices are raised with \( g^{\alpha\beta} \). There are \( k \) real abelian vector fields
arranged in a column \( B_\alpha = (B_\alpha^I) \), \( I = 1, \ldots, k \), with field strength \( F_{\alpha\beta} = \partial_\alpha B_\beta - \partial_\beta B_\alpha \) and dual field strength \( *F^{\alpha\beta} = \frac{1}{2\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \). The scalars \( \phi^i \), \( i = 1, \ldots, n \), parameterize a non-compact riemannian symmetric space \( \mathcal{G}/\mathcal{H} \) with metric \( \overline{m}_{ij}(\phi) \). Its \( \text{dim} \mathcal{G} \) Killing vectors give rise to a Lie algebra valued Noether current \( J_\alpha \). In terms of them the sigma-model Lagrangian for the scalars can be written as \( \langle J^i, J_\alpha \rangle_{\bar{g}} \), where \( \langle \cdot, \cdot \rangle_{\bar{g}} \) is an invariant scalar product on the Lie algebra \( \bar{g} \). Finally the coupling matrices \( \mu = \mu(\phi) \) and \( \nu = \nu(\phi) \) are symmetric \( k \times k \) matrices that depend on the scalars; the constant \( q > 0 \) has been extracted for normalization purposes. The vector fields are supposed to contribute positively to the energy density which requires that \( \mu \) is a positive definite matrix. As such it has a unique positive square root \( \mu^{1/2} \) to be used later. The coupling matrices \( \mu \) and \( \nu \) are now chosen in a way that renders the field equations derived from \( S_4 \) – though in general not the action itself – \( \mathcal{G} \)-invariant.

The field equation for the gauge fields \( \nabla_\alpha (\mu F^{\alpha\beta} - \nu *F^{\alpha\beta}) = 0 \) can be interpreted as the Bianchi identity for a field strength \( G_{\alpha\beta} = \partial_\alpha C_\beta - \partial_\beta C_\alpha \) derived from dual potentials \( C_\alpha \). For later convenience one chooses \( \eta (\mu F_{\alpha\beta} - \nu *F_{\alpha\beta}) \) with some constant orthogonal matrix \( \eta \). In view of \( **F = -F \) they satisfy the linear relation

\[
\begin{pmatrix} F \\ G \end{pmatrix} = \Upsilon \mathcal{V}_c \mathcal{V}_c^T \begin{pmatrix} *F \\ *G \end{pmatrix} \text{ with } \mathcal{V}_c = \begin{pmatrix} \mu^{1/2} & \nu \mu^{-1/2} \\ 0 & \eta \mu^{-1/2} \end{pmatrix}, \ \Upsilon = \begin{pmatrix} 0 & \eta^T \\ -\eta & 0 \end{pmatrix}, \ (2.2)
\]

where the subscript \( c \) is mnemonic for ‘coupling’. If one now assumes that the column \( \begin{pmatrix} F \\ G \end{pmatrix} \) transforms linearly under a faithful \( 2k \) dimensional real matrix representation \( c \) of \( \mathcal{G} \), i.e. \( \begin{pmatrix} F \\ G \end{pmatrix} \mapsto c(\bar{g}^{-1})^T \begin{pmatrix} F \\ G \end{pmatrix} \), \( \bar{g} \in \mathcal{G} \), one finds that \( (2.2) \) transforms covariantly if \( \mathcal{V}_c \mapsto c(\bar{g}) \mathcal{V}_c \mathcal{h}_c \), with an orthogonal matrix \( \mathcal{h}_c \) and \( c(\bar{g}^{-1})^T = \Upsilon c(\bar{g}) \Upsilon^{-1} \). Comparing this with the transformation law of the \( \mathcal{G} \)-valued coset representatives \( \mathcal{V}_c \) in appendix A one sees that these conditions are satisfied if \( c(\mathcal{V}_c) = \mathcal{V}_c \) \( (\ast) \) and \( c(\bar{\tau}(\bar{g})) = c(\bar{g}^{-1})^T \), \( \bar{g} \in \mathcal{G} \), where \( \bar{\tau} \) is the involution whose set of fixed points defines \( \mathcal{H} \). Clearly this restricts the allowed cosets \( \mathcal{G}/\mathcal{H} \). For the admissible ones Eq. \( (\ast) \) then determines the couplings \( \mu(\phi), \nu(\phi) \) as functions of the scalars. Since \( c \) is faithful the determination is unique for a given choice of section \( \mathcal{V}_c \). Since \( \mathcal{V}_c \mathcal{V}_c^T = c(\mathcal{V}_c \bar{\tau}(\mathcal{V}_c^{-1})) \) the result does not depend on the choice of section, i.e. \( \tilde{\mathcal{V}}_c = \mathcal{V}_c h \) for some \( H \)-valued function \( h \) determines the same \( \mu(\phi) \) and \( \nu(\phi) \).

Under the above conditions the 1-Killing vector reduction of \( S_4 \) can be brought into a form which – after a partial dualization – is manifestly \( \mathcal{G} \)-invariant (while in the 4D theory \( \mathcal{G} \) was only an on-shell symmetry). By definition the fields entering the action are assumed to have vanishing Lie derivatives \( \mathcal{L}_K g_{\alpha\beta} = \mathcal{L}_K B_\alpha = \mathcal{L}_K \phi = 0 \), where \( K^\alpha \)
is the Killing vector and for the vector fields a gauge has been chosen. The Killing vector can be spacelike \((\epsilon_1 = +1)\) or timelike \((\epsilon_1 = -1)\). As indicated we distinguish both cases by a sign and write \(K^\alpha K_\alpha = -\epsilon_1 \Delta\), with \(\Delta > 0\). Any tensor can then decomposed into components parallel and orthogonal to \(K^\alpha\); the respective projectors are \(-\epsilon_1 \Delta^{-1} K_\alpha K^\beta\) and \(\delta_\alpha^\beta + \epsilon_1 \Delta^{-1} K_\alpha K^\beta\). In particular the components of the metric orthogonal and parallel to \(K^\alpha\) are \(g_{\alpha\beta} + \epsilon_1 \Delta^{-1} K_\alpha K_\beta =: \epsilon_1 \Delta \gamma_{\alpha\beta}\) and \(K_\alpha =: -\epsilon_1 \Delta k_\alpha\), where we extracted \(\Delta\) for later convenience. To solve the Killing equations one chooses adapted coordinates in which \(K = K^\alpha \partial_\alpha\) acts by translations; ordering coordinates according to the assumed \((+,-,-,-)\) eigenvalues of the metric we take \(K = \partial_3\) for \(\epsilon_1 = 1\) and \(K = \partial_0\) for \(\epsilon_1 = -1\). The fields then only depend on the remaining coordinates which we rename \(x^\hat{\alpha}, \hat{\alpha} = 0, 1, 2\). In these coordinates we write \(\epsilon_1 \Delta \gamma_{\hat{\alpha}\hat{\beta}}\) for the metric components orthogonal to \(K^\alpha\) and \(-\epsilon_1 \Delta k_\hat{\alpha}\) for those parallel to it. Note that \(\gamma_{\hat{\alpha}\hat{\beta}}\) has eigenvalues \((+,-,-)\) for \(\epsilon_1 = +1\) and \((+,+,-)\) for \(\epsilon_1 = -1\). For the vector fields one then has \(B_{\hat{\alpha}} = k_{\hat{\alpha}} B + B_{\hat{\alpha}}^\perp\), with \(B = B_\alpha K^\alpha\) and \(B_{\hat{\alpha}}^\perp K^\alpha = 0\).

Entering with this decomposition into the action \(S_4\) yields a reduced action \(S_3^{\text{direct}}\), where a 3D Einstein-Hilbert term for the metric \(\gamma\) criticality”, see e.g. [19]) the field equations derived from \(S_3^{\text{direct}}\) coincide with the reduction of the original field equations. In particular the ones for \(\epsilon_1 = 1\) and \(\epsilon_1 = -1\) can be interpreted as Bianchi identities implying the existence of scalar potentials \(\psi\) and \(C = (C^I), I = 1, \ldots, k\), where the latter are just the parallel components of the 4D dual vector potentials, \(C = C_\alpha K^\alpha\). The defining relations are

\[
\partial^\hat{\alpha} k^\beta - \partial^\hat{\beta} k^\hat{\alpha} = \frac{1}{\sqrt{\gamma}} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \frac{1}{\Delta^2} \omega_{\hat{\gamma}} \quad \text{with} \quad \omega_{\hat{\alpha}} = \partial_{\hat{\alpha}} \psi + \frac{q}{2} [C^T \eta \partial_{\hat{\alpha}} B - B^T \eta^T \partial_{\hat{\alpha}} C],
\]

\[
\partial^\hat{\alpha} (B_{\hat{\alpha}}^\perp)^\hat{\beta} - \partial^\hat{\beta} (B_{\hat{\alpha}}^\perp)^\hat{\alpha} + (\partial^\hat{\alpha} k^\hat{\beta} - \partial^\hat{\beta} k^\hat{\alpha}) B = - \frac{\epsilon_1}{\sqrt{\gamma}} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \frac{1}{\Delta^2} \mu^{-1} [\nu \partial_{\hat{\gamma}} B - \eta^T \partial_{\hat{\alpha}} C]. \quad (2.3)
\]

Performing a Legendre transformation to the scalar variables one obtains [12]

\[
S_3 = \int d^3 x \sqrt{\gamma} \left[ R^{(3)}(\gamma) - \frac{1}{2} \gamma^{\hat{\alpha}\hat{\beta}} m_{ij}(\varphi) \partial_{\hat{\alpha}} \varphi^i \partial_{\hat{\beta}} \varphi^j \right], \quad \text{with} \quad (2.4)
\]

\[
m_{ij}(\varphi) \partial_{\hat{\alpha}} \varphi^i \partial_{\hat{\beta}} \varphi^j = \frac{1}{\Delta^2} (\partial_{\hat{\alpha}} \Delta \partial_{\hat{\beta}} + \omega_{\hat{\alpha}} \omega_{\hat{\beta}}) + \langle \vec{J}_{\hat{\alpha}}, \vec{J}_{\hat{\beta}} \rangle_k + \epsilon_1 \frac{q}{\Delta} \partial_{\hat{\alpha}} A^T \nabla \varphi \nabla^T \partial_{\hat{\beta}} A.
\]

Here the scalars \(B, C\) have been arranged into a 2k dimensional column \(A = (B)^T\) and since \(\omega_{\hat{\alpha}} = \partial_{\hat{\alpha}} \psi - \frac{q}{2} A^T \gamma \partial_{\hat{\alpha}} A\) the action \((2.3)\) is manifestly \(G\)-invariant. Moreover we anticipated that the 3D ‘matter’ combines into a nonlinear sigma-model. The target space has dimension \(2 + \bar{n} + 2k\), we take \(\varphi^T = (\Delta, \psi, \varphi^T, A^T)\) as field coordinates, and
with the normalization \( \langle J_\alpha, J_\beta \rangle_{\bar{g}} = \bar{m}_{ij}(\varphi) \partial_\alpha \varphi^i \partial_\beta \varphi^j \) the metric comes out as

\[
m(\varphi) = \begin{pmatrix}
\frac{1}{\Delta^2} & 0 & 0 & 0 \\
0 & \frac{1}{\Delta^2} & \bar{m}(\varphi) & -\frac{q}{\Delta^2} A^T \Upsilon \\
0 & \frac{q}{\Delta^2} \Upsilon A & \epsilon_1 \frac{q}{2} V_c V_c^T - \frac{q^2}{4 \Delta^2} \Upsilon A \otimes A^T \Upsilon & 0 \\
\end{pmatrix}.
\]

It is riemannian for \( \epsilon_1 = +1 \) and pseudo-riemannian for \( \epsilon_1 = -1 \), with \( 2k \) negative eigenvalues. Note also that due to the dualization the part of the coset space parameterized by the gravitational potentials \((\Delta, \psi)\) always has eigenvalues \((+++)\).

We briefly digress on the isometries of (2.5). By virtue of the \( G \) invariance of the action \( m \) has \( \text{dim} G \) Killing vectors of which \( \bar{n} = \text{dim} G/\bar{H} \) are algebraically independent. The residual gauge transformations \( A \mapsto A + a, \psi \mapsto \psi - \frac{q}{2} A^T \Upsilon a \), with a constant \( 2k \) column \( a \) give rise to \( 2k \) Killing vectors. Finally constant translations in \( \psi \) and scale transformations \((\Delta, \psi, \varphi^T, A^T) \mapsto (\lambda \Delta, \lambda \psi, \varphi^T, \lambda^{1/2} A^T), \lambda > 0 \), are obvious symmetries of the action. The associated Killing vectors \( e, h \) of \( m \) generate a Borel subalgebra of \( sl_2 \), i.e. \([h, e] = -2e\). In contrast the last \( sl_2 \) generator \( f \) is only a Killing vector of \( m \) under certain conditions on \( \overline{G}/\overline{H} \). If these are satisfied a remarkable ‘symmetry enhancement’ takes place in that \( m \) is the metric of a much larger symmetric space \( G/H \), where \( G \) is a non-compact real form of a simple Lie group with \( \text{dim} G = \text{dim} \overline{G} + 4k + \text{dim} SL(2) \). The point is that if \( f \) exists as Killing vector its commutator with the gauge transformations is nontrivial and yields \( 2k \) additional symmetries (generalized “Harrison transformations”). Since \( m \) always has \( \text{dim} \overline{G} + 2k + 2 \) Killing vectors the additional \( 1 + 2k \) then match the dimension of \( G \). For the number of dependent Killing vectors, i.e. the dimension of the putative maximal subgroup \( H \subset G \) one expects \( \text{dim} H = \text{dim} \overline{H} + 1 + 2k \). Indeed under the conditions stated the symmetric space \( G/H \) exists and is uniquely determined by \( \overline{G}/\overline{H} \) and the sign \( \epsilon_1 \). See [12] for a complete list. Evidently the gauge fields are crucial for the symmetry enhancement. among the systems in [12] only pure gravity has \( k = 0 \).

We proceed by performing the reduction with respect to the second Killing vector. Clearly if the first Killing vector was timelike it can only be spacelike \( (\epsilon_2 = +1) \) while if the first Killing vector was spacelike the second can be either spacelike \( (\epsilon_2 = +1) \) or timelike \( (\epsilon_2 = -1) \). Again one chooses adapted coordinates in which the second Killing vector acts by translations. The fields in \( S_3 \) will then only depend on the remaining two non-Killing coordinates which we denote by \( x^\mu, \mu = 0, 1 \). The decomposition of the 3D metric \( \gamma_{\alpha\beta} \) can be performed as before. Anticipating that the off-diagonal components

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turn out to be non-dynamical we write $-\epsilon_1 \epsilon_2 \rho^2$ for the component contracted with the Killing vector and $\gamma_{\mu\nu}$ for the block orthogonal to it, which is diffeomorphic to $\text{diag}(\epsilon_2 e^\sigma, -\epsilon_1 e^\sigma)$. One readily checks that the sign pattern matches the $(+, -\epsilon_1, -\epsilon_1)$ eigenvalues of the 3D metric. Entering with this decomposition into $S_2$ yields the 2D action $S_2 = \int d^2x L_{h=\rho}$, which we shall take as the starting point to develop the quantum theory. We write the Lagrangian as a special case of a family of Lagrangians $L_h$ labeled by a real function $h$ of one variable and two parameters $a, b \in \mathbb{R}$, $b \neq 0$, because this is what is needed in the quantum theory. The generalized Lagrangian reads

$$L_h(\varphi, \rho, \sigma) = \frac{1}{2\lambda} h(\rho) \sqrt{\gamma^{\mu\nu}} \left[ m_{ij}(\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j + a \rho^{-2} \partial_\mu \rho \partial_\nu \rho \right] + \frac{1}{2\lambda} f(\rho) \sqrt{\gamma R(2)}(\gamma). \quad (2.6)$$

Here $\lambda$ is Newton’s constant per unit volume of the internal space. The 2D metric $\gamma_{\mu\nu}$ is diffeomorphic to $\eta_{\mu\nu} e^\sigma$, where $\eta_{\mu\nu}$ is constant with eigenvalues $(\epsilon_2, -\epsilon_1)$. Its scalar curvature $R(2)(\gamma)$ is normalized such that $R(2)(e^\sigma \eta) = -e^{-\sigma} \partial^2 \sigma$. The target space metric is that of Eq. (2.5); note that its signature only depends on $\epsilon_1$, i.e. on the spacelike/timelike character of the first Killing vector. In the quantum theory the function $h$ parameterizes an infinite set of essential couplings (e.g. via the expansion coefficients with respect to some basis) while classically $h(\rho) = \rho$. The function $f$ is given in terms of $h$ by

$$f(\rho) = 2b \int^\rho \frac{du}{u} h(u), \quad (2.7)$$

for some non-zero constant $b$. Likewise $a$ is a real constant; the values $b = -1, a = 0$ for $h(\rho) = \rho$ are the ones that come out of the reduction procedure.

Since $\rho$ will play a pivotal role in the following it is worth pointing out its geometrical meaning as the ‘area radius’ associated with the two Killing vectors. Let $\kappa_1, \kappa_2$ and $\kappa_{12}$ denote the norm of the two Killing vectors and their inner product, respectively, with respect to the 4D metric $g_{\alpha\beta}$. By definition $-\epsilon_1 \kappa_1^2 = \Delta \geq 0$ and $-\epsilon_2 \kappa_2^2 \geq 0$. Spelling out the net parameterization of $g_{\alpha\beta}$ induced by the above procedure (but without dualizing the $k_\alpha$ via (2.3)) one finds

$$\rho^2 = \epsilon_1 \epsilon_2 (\kappa_1^2 \kappa_2^2 - \kappa_{12}^2) \geq 0, \quad g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho = \epsilon_1 \Delta e^{-\sigma} \partial^\mu \rho \partial_\mu \rho. \quad (2.8)$$

If one of the Killing vectors has compact orbits of length $2\pi$, the area swept out per unit length of the other is $2\pi \rho$. Whence the term “area radius” [20] which we retain also in the general case. The gradient of $\rho$ is spacelike with respect to the 4D metric if one of the Killing vectors is timelike and indefinite if both are spacelike.
The action (2.6) is manifestly invariant under 2D diffeomorphisms. The associated hamiltonian and 1D diffeomorphism constraints are \( H_0 = T_{00} \) and \( H_1 = T_{01} \), where \( T_{\mu\nu} \) is the energy momentum tensor of the (flat space) action (2.6) in conformal gauge \( \gamma_{\mu\nu} = e^\sigma \eta_{\mu\nu} \). As usual light cone components \( T_{\pm\pm}(\rho) = H_0 \pm H_1 \) are convenient

\[
\lambda T_{\pm\pm} = h(\rho) [m_{ij}(\varphi) \partial_\pm \varphi^i \partial_\pm \varphi^j + a \rho^{-2} (\partial_\pm \rho)^2] + \partial_\pm \sigma \partial_\pm f - \partial_\pm^2 f .
\]

They constitute a pair of first class constraints and generate two commuting copies of a centerless Virasoro-Witt algebra with respect to the Poisson structure induced by (2.6). Importantly the latter coincides with the symplectic structure induced from the higher dimensional theory.

Even in conformal gauge the classical action (2.6) ‘remembers’ its gravitational origin. Recall that in a diffeomorphism invariant theory the Lagrangian can always be written as a total divergence on-shell. For the action (2.6) in conformal gauge there exist two currents \( C_\mu, D_\mu \) obeying

\[
\partial^\mu C_\mu = \rho \partial_\rho \ln h \cdot L_h , \quad C_\mu = b \frac{\rho}{2\lambda} h(\rho) \partial_\mu (2\sigma + \frac{a}{b} \ln \rho) ,
\]

\[
\partial^\mu D_\mu = - \ln \rho \cdot \rho \partial_\rho \ln h \cdot L_h , \quad D_\mu = b \frac{\rho}{\lambda} h(\rho) (\sigma \partial_\mu \ln \rho - \ln \rho \partial_\mu \sigma) .
\]

In particular there are (exactly) two choices for \( h(\rho) \) for which the Lagrangian is a total divergence on-shell: \( h(\rho) \sim \rho^p, p \neq 0 \), and \( h(\rho) \sim \ln \rho \). The first one corresponds to the outcome of the classical reduction procedure (and, as will become clear below, it is also necessary and sufficient for one-loop renormalizability).

In summary, starting from 4D gravity coupled to vectors and scalars according to (2.1) we performed a two step dimensional reduction procedure. In the first step only one Killing vector is used and in the resulting 3D theory all vectors are replaced by dual scalar potentials. The original and the induced scalars combine under certain conditions to parameterize a symmetric space of the form \( G/H \), where \( G \) is a non-compact real form of a simple Lie group and \( H \) is a maximal subgroup (whose properties depend on the signature of the Killing vector). The resulting 3D action \( S_3 \) couples 3D gravity minimally to a \( G/H \) sigma-model. Finally the reduction with respect to the second Killing vector is performed. This leaves the coset \( G/H \) unaffected while the 3D metric gives rise to the scalars \( \rho \) and \( \sigma \). The classical action \( S_3 \) derives from the Lagrangian (2.6) with \( h(\rho) = \rho \) and \( b = -1, a = 0 \). The generalized form (2.6) anticipates the structure of the renormalized Lagrangian.
3. Universal dressing of the beta function

The goal in the following is to construct a perturbative quantum theory based on the Lagrangian (2.6). We fix the conformal gauge for $\gamma_{\mu\nu}$ and aim at a Dirac quantization. Further the renormalization strategy of [7] will be adopted, in particular we use the covariant background field expansion, dimensional regularization, and minimal subtraction to determine the counter terms. For the cancellation of the counter terms nonlinear field renormalizations are allowed, nevertheless one finds that the system cannot be renormalized with finitely many essential couplings beyond one loop. Hence an infinite set of essential couplings is required, identifying the systems as non-renormalizable ones. The systems remain manageable because these couplings can be combined into (the expansion coefficients of) one function $h$ of $\rho$ which enters the renormalized Lagrangian in the way anticipated in (2.6). The fact that $h$ indeed qualifies as an “essential” set of couplings can be seen from the identities (2.10). Beyond one loop one is forced to take $h(\rho)$ different from $\rho^p$ upon variation of the renormalization scale. Then $\partial^\mu C_\mu \not\sim L_h$ and the expansion coefficients of $h$ with respect to some basis are essential couplings. Though we shall only be interested in the systems (2.6) arising by dimensional reduction, the results of section 3.1 and 3.2 hold for any symmetric space $G/H$ with $G$ simple (i.e. not only for the specific non-compact ones in (2.5)) as well as for direct products thereof with identical factors. The non-compactness of $G/H$ will be crucial in section 4.

3.1 Renormalization of the Lagrangian to all loop orders

In order to determine the counter terms we interpret (2.6) as the action of a riemannian sigma-model in the sense of Friedan [27]. Introducing two coordinates $\phi^{n+1} := \rho, \phi^{n+2} := \sigma$ in addition to $\phi^i = \varphi^i, i = 1, \ldots, n$, one can regard (2.6) as a sigma-model with a fiducial non-homogeneous target manifold of dimension $n + 2$. In conformal gauge $\gamma_{\mu\nu} = e^\sigma \eta_{\mu\nu}$ and after integration by parts one has $L_h(\varphi, \rho, \sigma) = \frac{1}{2\pi} g_{ij}(\varphi) \partial \varphi^i \partial \varphi^j$, with

$$g_{ij} = h(\rho) \begin{pmatrix} (m_{ij})_{1 \leq i,j \leq n} & 0 & 0 \\ 0 & a/\rho^2 & b/\rho \\ b/\rho & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.1)$$

Here $m_{ij}$ is the metric (2.5); as remarked above the results of this and the following section remain valid if it is replaced by the metric of an arbitrary symmetric space $G/H$. 

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with $G$ simple. In addition to the Killing vectors associated with $m$ the metric (3.1) possesses two conformal Killing vectors $t^+ = \rho \partial_\rho - \frac{\sigma}{\rho} \partial_\sigma$ and $d = -\rho \ln \rho \partial_\rho + (\sigma + \frac{\sigma}{\rho} \ln \rho) \partial_\sigma$, which together with $t^- = \partial_\sigma$ generate the isometries of $\mathbb{R}^{1,1}$. The currents associated with $t^+$ and $d$ are those in Eq. (2.10).

In dimensional regularization ($\int d^2 x \rightarrow \int d^d x$) the $l$-loop counter terms contain poles of order $\nu \leq l$ in $(2 - d)$. We denote the coefficient of the $\nu$-th order pole by $T^{(\nu,l)}_{ij}(g)$. In principle the higher order pole terms are determined recursively by the residues $T^{(1,l)}_{ij}(g)$ of the first order poles. Taking the consistency of the cancellations for granted one can focus on the residues of the first order poles, which we shall do throughout. In appendix B we show that to all loop orders they have the following structure:

$$T^{(1,l)}_{ij}(g) = \frac{1}{h(\rho)^{l-1}} \begin{pmatrix} \zeta_l (m_{ij})_{1 \leq i,j \leq n} & 0 \\ 0 & n \rho^2 S_l(\rho) \end{pmatrix}, \quad \forall l \geq 1. \quad (3.2)$$

The $\zeta_l$ are constants defined through the curvature scalars of $m_{ij}$. The $S_l(\rho)$ are differential polynomials in $h$ invariant under constant rescalings of $h$ and normalized to vanish for constant $h$. The first three are:

$$S_1(h) = -\frac{1}{2}(\rho \partial_\rho)^2 \ln h + \frac{1}{4}(\rho \partial_\rho \ln h)^2, \quad S_2(h) = 0,$$

$$S_3(h) = -\frac{\zeta_2}{4}(\rho \partial_\rho)^2 \ln h + \frac{\zeta_2}{12}(\rho \partial_\rho \ln h)^2. \quad (3.3)$$

The counter terms (3.2) ought to be absorbed by nonlinear field renormalizations

$$\phi^B_\mu = \phi^j + \frac{1}{2 - d} \Xi^j(\phi, \lambda) + \ldots, \quad \text{with} \quad \Xi^j = \sum_{l \geq 1} \left( \frac{\lambda}{2 \pi} \right)^l \phi^j_l(\phi), \quad (3.4)$$

and a renormalization of the function $h$

$$h_0(\rho) = \mu^{d-2} h(\rho, \lambda) \left[ 1 + \frac{1}{2 - d} H(\rho, \lambda) + \ldots \right], \quad H(\rho, \lambda) = \sum_{l \geq 1} \left( \frac{\lambda}{2 \pi} \right)^l H_l(\rho), \quad (3.5)$$

where $\mu$ is the renormalization scale. Note that on both sides of (3.5) the argument is the renormalized field. The renormalized $h$ function is allowed to depend on $\lambda$; specifically we assume it to have the form

$$h(\rho, \lambda) = \rho^p + \frac{\lambda}{2 \pi} h_1(\rho) + \left( \frac{\lambda}{2 \pi} \right)^2 h_2(\rho) + \ldots, \quad (3.6)$$
where the first term ensures standard renormalizability at the 1-loop level – and is determined by this requirement up the power $p \neq 0$. The power has no intrinsic significance; one could have chosen a parameterization of the 4D spacetime metric $g_{\alpha\beta}$ such that $h(\rho) = \rho^{p}$ in (2.4) was the outcome of the classical reduction procedure. In particular the sectors $p > 0$ and $p < 0$ are equivalent and we assume $p > 0$ throughout.

Combining (3.1), (3.4), (3.5) and (3.2) one finds that the first order poles cancel in the renormalized Lagrangian iff the following “finiteness condition” holds:

$$L_{\Xi} g_{ij} + H(\rho, \lambda) g_{ij} = \lambda T^{(1)}_{ij}(g/\lambda) ,$$

where $T^{(1)}(g) = \sum_{l \geq 1} \left( \frac{1}{2\pi} \right)^{l} T^{(1,l)}(g)$ and $L_{\Xi} g$ is a Lie derivative. The $\rho$-dependence of $H$ marks the deviation from conventional renormalizability. Guided by the structure of (3.1) and (3.2) we search for a solution with $\Xi^{j} = (0, \ldots, 0, \Xi^{\rho}(\rho, \lambda), \Xi^{\sigma}(\rho, \lambda))$, where here and later on we also use $\rho = n + 1, \sigma = n + 2$ for the index labeling. The Lie derivative term with this $\Xi^{j}$ is

$$L_{\Xi} g_{ij} = \begin{pmatrix} \Xi^{\rho}(\rho) \partial_{\rho} h \left( m_{ij} \right)_{1\leq i,j \leq n} & 0 \\ 0 & L_{\Xi} g_{\rho\rho} & L_{\Xi} g_{\rho\sigma} \\ L_{\Xi} g_{\rho\rho} & 0 \end{pmatrix},$$

$$L_{\Xi} g_{\rho\rho} = \frac{a}{\rho^{2}} \left[ \partial_{\rho} h \Xi^{\rho} + 2h \rho \partial_{\rho} \left( \frac{\Xi^{\rho}}{\rho} \right) \right] + 2b \frac{h}{\rho} \partial_{\rho} \Xi^{\sigma} , \quad L_{\Xi} g_{\rho\sigma} = b \partial_{\rho} \left( \frac{h\Xi^{\rho}}{\rho} \right).$$

The finiteness condition (3.7) then is equivalent to a simple system of differential equations whose solution is

$$H(\rho/\lambda) = -\frac{1}{h(\rho, \lambda)} \rho \partial_{\rho} \left[ h(\rho, \lambda) \Xi^{\rho}(\rho/\lambda) \right] ,$$

$$\Xi^{\rho}(\rho/\lambda) = -\rho \int^{\rho} \frac{du}{u} B_{\lambda} \left( \frac{\lambda}{h(u, \lambda)} \right) ,$$

$$\Xi^{\sigma}(\rho/\lambda) = -\frac{a}{2b} \Xi^{\rho}(\rho/\lambda) + \frac{1}{2b} \int^{\rho} \frac{du}{u} S(u, \lambda) .$$

Here we set

$$B_{\lambda}(\lambda) := \sum_{l \geq 1} c_{l} \left( \frac{\lambda}{2\pi} \right)^{l} , \quad S(\rho, \lambda) := n \sum_{l \geq 1} \left( \frac{\lambda}{2\pi} \right)^{l} h^{-l} S_{l}(\rho) ,$$

10
and slightly adjusted the notation to stress the functional dependence on \( h/\lambda \). Possibly \( \lambda \)-dependent integration constants have been absorbed into the lower integration boundaries of the integrals. Throughout these solutions should be read as shorthands for their series expansions in \( \lambda \) with \( h \) of the form (3.6). For example

\[
\Xi^p(\rho, \lambda) = \frac{\lambda}{2\pi} \zeta_1 \rho^{-p+1} + \left( \frac{\lambda}{2\pi} \right)^2 \rho \int_0^\infty \frac{du}{u^2} \left[ \zeta_2 - \zeta_1 h_1(u) \right] + O(\lambda^3). \tag{3.11}
\]

For the derivation of (3.7) and (3.9) we fixed a coordinate system in which the target space metric takes the form (3.1). Under a change of parameterization \( \phi^j \to \hat{\phi}^j(\hat{\phi}) \) the finiteness condition (3.7) should transform covariantly, and indeed it does. The constituents transform as

\[
\hat{g}_{ij}(\hat{\phi}) = \frac{\partial \phi^k}{\partial \hat{\phi}^i} \frac{\partial \phi^m}{\partial \hat{\phi}^j} g_{km}(\phi), \quad T^{(1)}_{ij}(\hat{\phi}) = \frac{\partial \phi^k}{\partial \hat{\phi}^i} \frac{\partial \phi^m}{\partial \hat{\phi}^j} T^{(1)}_{km}(\phi),
\]

\[
\hat{\Xi}^i(\hat{\phi}) = \frac{\partial \hat{\phi}^j}{\partial \phi^k} \Xi^k(\phi), \quad (L \hat{g})_{ij}(\hat{\phi}) = \frac{\partial \hat{\phi}^j}{\partial \phi^k} \frac{\partial \hat{\phi}^m}{\partial \phi^l} (L g)_{km}(\phi). \tag{3.12}
\]

The covariance of the counter terms as a function of the full field is nontrivial \([28, 24]\) and is one of the main advantages of the covariant background field expansion. The relations (3.12) can be used to convert the solutions (3.9) of the finiteness condition into any desired coordinate system on the target space. The coordinates \( \sigma \) and \( \rho \) used in (3.1) are adapted to the Killing vector \( t_- \) and the conformal Killing vectors \( t_+, d \).

### 3.2 Universal dressing of the beta function

So far the renormalization has been performed at some fixed normalization scale \( \mu \). Changing the scale gives rise to renormalization flow equations of which the one for the essential couplings \( h \) is of primary interest. We first present the result and then outline the derivation. Denoting the ‘running’ coupling function by \( \hat{h}(\cdot, \mu) \) the flow equation reads

\[
\mu \frac{d}{d\mu} \hat{h} = \lambda \beta_h(\hat{h}/\lambda). \tag{3.13}
\]

The associated beta functional is given by

\[
\lambda \beta_h(h/\lambda) = -\rho \partial \rho \left[ h \int_0^\infty \frac{du}{u} \frac{h(u)}{\lambda} \beta_h \left( \frac{\lambda}{h(u)} \right) \right]. \tag{3.14}
\]
Here $\beta_\lambda(\lambda) = \sum_{l \geq 1} \zeta_l \frac{1}{2\pi} l^2$ is the conventional beta function of the $G/H$ symmetric space sigma-model without coupling to gravity, defined e.g. in the minimal subtraction scheme. Thus $\beta_h(h)$ can be viewed as a “gravitationally dressed” version of $\beta_\lambda(\lambda)$, akin to the phenomenon in [36, 35]. Remarkably the “dressing” is universal, i.e. independent of the symmetric space considered, and can be given in closed form to all loop orders.

The flow equation (3.13) of course has to be supplemented by a boundary condition. The condition $h(\rho, \lambda)/\rho^d \to 1$ for $\rho \to \infty$ entails that $\beta_h(h)$ vanishes for $\rho \to \infty$. This asymptotics is therefore preserved by the flow. Moreover is can be seen to ensure that the flow is entirely driven by the counter terms, as it should. We therefore adopt this asymptotic boundary condition throughout.

The derivation of (3.13), (3.14) is straightforward once $H$ is known explicitly as a functional of $h$ via (3.9). Starting from (3.5) one determines how the renormalized $h(\cdot)$ (the function not its value) has to change upon a change of the renormalization scale in order to have the bare $h_B(\cdot)$ scale independent. One finds

$$\lambda \beta_h(h/\lambda) = (2 - d) h(\rho) - h(\rho) \int du h(u) \frac{\delta H(\rho, \lambda)}{\delta h(u)}.$$  (3.15)

Inserting (3.9) and setting $d = 2$ for simplicity yields the announced result (3.13), (3.14).

### 3.3 Beta function coefficients and examples

The coset spaces $G/H$ arising by dimensional reduction from 4 dimensions fall into 5 infinite series associated with the ‘classical’ groups and 10 isolated cases associated with exceptional groups. The leading beta function coefficient $\zeta_1$ turns out to be given by the simple formula $\zeta_1 = -(k + 2)/2$, where $k$ is the number of vector fields in (2.1).

We postpone its derivation to section 4 where $\zeta_1$ will play a pivotal role. The next coefficient $\zeta_2$ is still scheme independent while the higher orders depend on the choice of the renormalization scheme; as described we adopt the minimal subtraction scheme throughout. Explicit results for $\zeta_2$ and $\zeta_3$ can in principle be obtained by evaluating the expressions (B.2) in the vielbein frame. On account of (A.7) this reduces the computation to a group theoretical one. In contrast to $\zeta_1$ however details about the embedding $H \subset G$ enter, which is somewhat cumbersome to describe because $H$ is in general not simple even though $G$ is. It is therefore more convenient to rely on the known results for the compact symmetric spaces [25, 26] and to translate them into the situation needed by using the fact that compact and non-compact symmetric spaces come in dual pairs [21].
In brief, a symmetric space $G_d/H_d$ is said to be dual to $G/H$ if the commutation relations in $\mathfrak{h}$ are preserved but the bilinear form flips sign and vice versa for the $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ relations and the bilinear form on $\mathfrak{m}$. In the notation of appendix A

$$df^c_{\tilde{a}\tilde{b}} = f^c_{\tilde{a}\tilde{b}}, \quad \langle \tilde{t}_a, \tilde{t}_b \rangle_{G_d} = -\langle \tilde{t}_a, \tilde{t}_b \rangle_{G},$$

$$df^c_{\tilde{a}\tilde{b}} = -f^c_{\tilde{a}\tilde{b}}, \quad \langle \tilde{t}_a, \tilde{t}_b \rangle_{G_d} = \langle \tilde{t}_a, \tilde{t}_b \rangle_{G},$$

(3.16)

where $df^c_{\tilde{a}\tilde{b}}$ are the structure constants of $\mathfrak{g}_d$, the Lie algebra of $G_d$. For example the cosets $\text{SL}(n)/\text{SO}(n)$ and $\text{SU}(n)/\text{SO}(n)$ are dual in this sense. Using (A.7) and (B.1) an immediate consequence is

$$R_{ijkn}(m_d) = -R_{ijkn}(m), \quad d\zeta l = (-)^ld\zeta l, \ l \geq 1,$$

(3.17)

where $m_d$ is the semi-riemannian metric on $G_d/H_d$ and $d\zeta l$ are the associated coefficients in (B.3). From (3.17) the table of dual pairs in [21] and Hikami’s results for compact cosets [26] one obtains table 1 below.

| $G/H$                      | $k$                  | $2\zeta_2$                          | $-3\zeta_3$                          |
|---------------------------|----------------------|-------------------------------------|--------------------------------------|
| $\text{SL}(n+2)/\text{SO}(n+2)$ | $n$                  | $\frac{1}{3}(n+2)(n+4)$             | $\frac{1}{64}(n+2)(3n^2 + 22n + 40)$ |
| $\text{SU}(n+1, m+1)$      | $n+m$                | $\frac{1}{2}(k+nm+2)$               | $\frac{1}{16}(k+2)(3k + 3nm + 10)$  |
| $\text{SO}(n+2, m+2)$      | $n+m$                | $\frac{1}{4}(3k+2nm+4)$             | $\frac{1}{32}[7(k+4)(k+2)+2nm(3k+14)-16]$ |
| $\text{SO}^*(2n+4)/\text{U}(n+2)$ | $2n$                | $\frac{1}{2}(n^2 + n + 2)$          | $\frac{1}{8}(3n^3 + 4n^2 + 15n + 10)$ |
| $\text{Sp}(2n+2, \mathbb{R})/\text{U}(n+1)$ | $n$                  | $\frac{1}{8}(n^2 + 5n + 8)$         | $\frac{1}{64}(3n^3 + 23n^2 + 72n + 80)$ |

Table 1: $l \leq 3$ loop beta function coefficients for all non-exceptional cosets arising from (2.1). Here $k > 0$ except for $\text{SL}(2)/\text{SO}(2)$ and $\zeta_1 = -(k + 2)/2$.

In the table the riemannian version of the cosets ($\epsilon_1 = +1$ in Eq. (2.5)) was used throughout. The coefficients for the pseudo-riemannian version ($\epsilon_1 = -1$) are identical

$$\zeta_l(\epsilon_1 = +1) = \zeta_l(\epsilon_1 = -1), \quad l \geq 1,$$

(3.18)

For the proof we interpret both symmetric spaces, the $\epsilon_1 = +1$ and the $\epsilon_1 = -1$ version, as different real sections of a complex manifold (which may no longer be a symmetric
space) and show that the $\zeta_l$ are constant for the complex manifold. To this end we promote the initially real and positive parameter $q$ in the metric (2.5) to a complex one, $q = |q|e^{i\alpha}$, $|q| \neq 0$, $0 \leq \alpha < 2\pi$. Simultaneously we allow the vector fields $A$ to be complex. The complexified metric might no longer be that of a symmetric space because the symmetry enhancement described after Eq. (2.5) might cease to work. However the generic $\dim G + 2k + 2$ isometries of the metric (2.5) will still be present, just that the $2k$ constant translations $A \rightarrow A + a$ now refer to complex constants $a$. These Killing vectors are enough to infer from part (i) of the Lemma in appendix B that the curvature scalars $n\zeta_l = m_{ij}T_{ij}^{(1,l)}(m)$, $n = \dim G/H$, in (B.3) are field independent also for the complexified manifold. In principle they could depend on the parameter $q$ but since the field redefinition $A \rightarrow |q|^{-1/2}e^{-i\alpha/2}A$ removes any $q$-dependence in the line element the $\zeta_l$ are $q$-independent. On the other hand $A \rightarrow -A$ is an (involutive) isometry and the combined sign flip $q \rightarrow -q, A \rightarrow -A$ is equivalent to flipping the sign of $\epsilon_1$ in (2.5). Since the former flip leaves the $\zeta_l$ unaffected the latter must do so too, which establishes Eq. (3.18).

For the exceptional cosets the coefficient $\zeta_1$ directly follows from the general formula (4.13) quoted earlier:

$$\zeta_1 = -2, -\frac{9}{2}, -6, -9, -15 \text{ for } G_2, F_4, E_6, E_7, E_8,$$

(3.19)

respectively. For each of $G = E_6, E_7, E_8$, several symmetric spaces arise [13, 14] differing by the non-compact version of $G$ and the subgroup $H$ used. Notably $\zeta_1$ is the same for the different versions. We have not tried to compute the higher order coefficients and are not aware of results in the literature on them.

Let us also briefly mention the special role of hermitian symmetric spaces, i.e. those which admit a complex structure. In table 1 these are the $n = 0$ member of the first series, the $m = 0$ members of the third series, and all others [21]. In the present context the characteristic feature of hermitian symmetric spaces is that they have their ‘oxidation endpoint’ in four dimensions [14], i.e. the highest possible dimension for a gravity theory which upon dimensional reduction gives a 2D theory of the form (2.6) with such a coset $G/H$ (and $h(\rho) = \rho$) is four. The complex structure also allows for the introduction of Ashtekar-type variables [18].

We proceed by illustrating the above features for the lowest members of the first three series in table 1, which arguably are also the physically most interesting ones. The main datum characterizing its 2-Killing vector reduction is the metric $m$ on the symmetric space $G/H$. This metric is always of the form (2.5) so that only the constituents of the
latter have to be specified. Retroactively one can verify that a particular metric (2.5) is indeed that of a particular coset space by examining the Lie algebra generated by its Killing vectors; see e.g. [17] for the SO(k + 2, 2)/SO(k + 2) × SO(2) series. A constructive approach starts by picking a unique (e.g. triangular) representative on the coset G/H. After choosing an explicit parameterization for it one can use the definition of \( M \) and (A.3) to compute the metric. The result will be of the form (2.5) but with the isometry group already identified; see [13] for more details on this.

5 dimensional gravity: The cosets are SL(3)/SO(3) for \( \epsilon_1 = +1 \) and SL(3)/SO(1, 2) for \( \epsilon_1 = -1 \). The Kaluza-Klein vector gives rise to two potentials \( B, C \); the scalar is denoted by \( \varphi \). The scalar-vector coupling is \( \mu = \varphi^2 \) and \( \nu = 0 \) with \( q = 1 \); the induced metric for \( \varphi \) is \( \bar{m} = \frac{1}{3} \varphi^{-2} \). Further \( \Upsilon = i \sigma_2 \) in terms of the Pauli matrix \( \sigma_2 \). In coordinates \( (\Delta, \psi, \varphi, B, C) \) the metric is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{1}{2} C & -\frac{1}{2} B \\
0 & 0 & \frac{4}{3} \Delta^2 & 0 & 0 \\
0 & \frac{1}{2} C & 0 & \frac{1}{4} C^2 + \epsilon_1 \Delta \varphi^2 & -\frac{1}{4} BC \\
0 & -\frac{1}{2} B & 0 & -\frac{1}{4} BC & \frac{1}{4} B^2 + \epsilon_1 \frac{\Delta}{\varphi^2}
\end{pmatrix}
\]

The first three counter term coefficients in (3.2) can now be computed directly from (B.2). They come out as \( \zeta_1 = -3/2, \zeta_2 = 15/16, \zeta_3 = -65/64 \), for both signs \( \epsilon_1 = \pm 1 \), in agreement with table 1 and Eq. (3.18).

Einstein-Maxwell theory: The relevant cosets are SU(2,1)/S[U(2) × U(1)] for \( \epsilon_1 = +1 \) and SU(2,1)/S[U(1,1) × U(1)] for \( \epsilon_1 = -1 \). The scalars \( \varphi \) are absent and the Maxwell field is parameterized by two potentials \( B, C \). The scalar-vector coupling is trivial, \( \mu = \mathbb{1}_2 \) and \( \nu = 0 \) with \( q = 4 \); further \( \Upsilon = i \sigma_2 \) as above. In coordinates \( (\Delta, \psi, B, C) \) the metric reads

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 2 C & -2 B \\
0 & 2 C & 4(C^2 + \epsilon_1 \Delta) & -4 B C \\
0 & -2 B & -4 B C & 4(B^2 + \epsilon_1 \Delta)
\end{pmatrix}
\]

In contrast to the full theory [9] the 2-Killing vector reduction remains strictly renormalizable at one loop. Beyond one loop both the full and the reduced theory are non-
renormalizable. Inserting (3.21) into (B.2) gives \( \zeta_1 = -3/2, \zeta_2 = 3/4, \zeta_3 = -13/16, \) for both signs \( \epsilon_1 = \pm 1. \)

**Einstein-Maxwell-dilaton-axion theory:** The cosets are \( \text{SO}(3,2)/\text{SO}(3) \times \text{SO}(2) \) for \( \epsilon_1 = +1 \) and \( \text{SO}(3,2)/\text{SO}(2,1) \times \text{SO}(2) \) for \( \epsilon_1 = -1. \) This is the first member of the third series in table 1 because the \( n = m = 0 \) system describing the 2-Killing vector reduction of dilaton-axion gravity corresponds to the decomposable coset \( \text{SO}(2,2)/\text{SO}(2) \times \text{SO}(2). \) It will be treated in section 5. In addition to being hermitian symmetric spaces all \( \text{SO}(k + 2,2)/\text{SO}(k + 2) \times \text{SO}(2), k \geq 0, \) have a Kähler structure detailed in [17].

In the \( k = n = 1 \) system exemplified here the scalar vector couplings \( \mu \) and \( \nu \) in (2.1) can be identifies with the dilaton \( \varphi \) and the pseudo-scalar axion field \( \chi, \) respectively. They also parameterize the scalar coset \( \mathcal{G}/\mathcal{H} = \text{SL}(2)/\text{SO}(2) \) with metric \( m_{ij} = \text{diag}(\varphi^{-2}, \chi^{-2}) \) in coordinates \( \varphi^1 = \varphi, \varphi^2 = \chi. \) In addition there are gauge potentials \( (B,C). \) With \( \Upsilon = i\sigma_2 \) as in the previous examples and \( q = 1 \) the metric in (2.5) is completely determined. Explicitly \( m_{ij} \) is a 6 \times 6 symmetric matrix whose lower 2 \times 2 block reads

\[
\begin{pmatrix}
\frac{\epsilon_1}{\Delta} \left( \varphi + \frac{\chi^2}{\varphi} \right) + \frac{C^2}{4\Delta^2} & \frac{\epsilon_1 \chi}{\Delta} \varphi - BC \frac{4\Delta}{4\Delta^2} \\
\frac{\epsilon_1 \chi}{\Delta} \varphi - BC \frac{4\Delta}{4\Delta^2} & \frac{\epsilon_1}{\Delta} \varphi + B^2 \frac{4\Delta}{4\Delta^2}
\end{pmatrix},
\]

(3.22)

in coordinates \( \phi^5 = B, \phi^6 = C. \) Proceeding as before one finds \( \zeta_1 = -3/2, \zeta_2 = 7/8, \zeta_3 = -89/96, \) once more independent of \( \epsilon_1 \) and in agreement with table 1.
4. UV stable fixed point

Recapitulating: one can achieve strict cut-off independence in the renormalization of dimensionally reduced gravity theories at the expense of introducing infinitely many essential couplings – thereby identifying the systems as non-renormalizable ones. Fortunately all these couplings can be arranged into a single scalar function $h(\cdot)$ of one real variable, whose flow is governed by the beta functional (3.14). We adopt S. Weinberg’s terminology and call a non-renormalizable quantum field theory asymptotically safe if the flow of the essential couplings has an ultraviolet stable fixed point. We now show that dimensionally reduced gravity theories are asymptotically safe in this sense.

Fixed points of the functional $h$-flow correspond to zeros of the $\beta_h$ function. They are conveniently found by converting the condition $\beta_h(h) = 0$ into the differential equation

$$\frac{\lambda}{2\pi} \rho \frac{\partial}{\partial \rho} h = C(\lambda) h^2 \frac{\lambda}{h} \frac{\beta_\lambda}{h} \left( \frac{\lambda}{h} \right),$$

for some $C(\lambda) = \sum_{l \geq 0} C_l (\frac{\lambda}{2\pi})^l$ with constant $C_l$. Here $\beta_\lambda = \lambda^2 \partial_\lambda B_\lambda$ is the conventional beta function (in the minimal subtraction scheme) of the symmetric space sigma-model without coupling to gravity. The solution corresponding to a $\lambda$-independent $C(\lambda) = C_0 = p/\zeta_1$ is

$$h_{\text{beta}}(\rho, \lambda) = \rho^p - \frac{\lambda}{2\pi} \frac{2\zeta_2}{\zeta_1} \rho^p - \left( \frac{\lambda}{2\pi} \right)^2 \frac{3\zeta_3}{2\zeta_1} \rho^{p-2} + \ldots.$$  

(4.2)

The fixed point is non-trivial in the sense that the gravitational degrees of freedom remain self-interacting at the fixed point and remain coupled to the matter degrees of freedom. It is also unique in that all other solutions of (4.1) violate the boundary conditions. Note that the leading quantum correction has the scheme independent coefficient $-\frac{2\zeta_2}{\zeta_1}$.

In the previous computation $\rho$ was treated as if it was a real variable. On the other hand $\rho$ is really the renormalized field $\rho = \phi^{n+1}$ in Eq. (3.4). Being an another operator valued field one would not expect that nonlinear functions $f(\rho)$ thereof can be built by pointwise multiplication, but rather that the latter have to be declared as normal products $[f(\rho)]$ by ‘subtracting’ the additional short distance singularities. Fortunately this is not necessary here as

$$[f(\rho)] = \mu^{d-2} f(\rho),$$

holds for an arbitrary function $f(\cdot)$. That is, up to the trivial $\mu$-prefactor the function $x \to f(\rho(x))$ and the composite operator can be identified. (Of course this is not true for any other field $\phi^{j}, j \neq n + 1$.) Equivalently pointwise multiplication of the operator...
valued field \( \rho(x) \) is legitimate. The derivation of (4.3) parallels that in [7] where also the precise definition of the normal product can be found. The upshot here is that \( \rho \) can be treated as if it was a classical field, rendering both the above and the subsequent computations well-defined.

To proceed we consider the linearization of the flow equation (3.13) around the fixed point function \( h^\text{beta} \). Since the lowest order term is fixed by strict renormalizability an appropriate parameterization is

\[
\overline{h}(\rho, \lambda, \mu) = h^\text{beta}(\rho, \lambda) + \delta h(\rho, \lambda, \mu), \quad \delta h(\rho, \lambda, \mu) = \frac{\lambda}{2\pi} \overline{s}_1(\rho, t) + \left( \frac{\lambda}{2\pi} \right)^2 \overline{s}_2(\rho, t) + \ldots.
\]

(4.4)

where the \( \overline{s}_l(\rho, t) \) are functions of \( \rho := \rho^p \) and \( t = \frac{1}{2\pi} \ln \mu / \mu_0 \), which vanish for \( \rho \to \infty \) uniformly in \( t \). This boundary condition adheres to the ‘freezing’ of the full nonlinear chart at \( \rho = \infty \). For the \( t \)-dependent quantities the scaling dimensions no longer match the powers in \( \lambda \). A natural grading can be introduced as follows: We assign to the renormalization time \( t \) scaling dimension 1, and to the functions \( \overline{s}_l \) a scaling dimension \( 1 - l \) in order to match the scaling dimensions of the \( h^\text{beta} \) under constant rescalings of \( h_0 = \rho \). Since the \( \overline{s}_l \) also appear under integrals the appropriate scaling transformation is

\[
(\rho, t) \quad \longrightarrow \quad (\Lambda \rho, \Lambda t),
\]

\[
\overline{s}_l(\cdot, \cdot) \quad \longrightarrow \quad \Lambda^{1-l} \overline{s}_l(\Lambda^{-1} \cdot, \Lambda^{-1} \cdot).
\]

(4.5)

We decompose the linearized flow equations into pieces transforming with weight \( \Lambda^{-l} \), \( l \geq 1 \), under (4.5). Retroactively one can also restore powers of \( \lambda \) according to this grading. The result is

\[
\frac{d}{dt} \overline{h} = 2\pi \int du \overline{\delta h}(u) \frac{\delta \beta h}{\delta h}(h^\text{beta}/\lambda)(u)
\]

\[
= -\frac{\zeta_1}{p^h} \rho \partial_\rho \overline{h} - \frac{\overline{h}}{h} \sum_{l \geq 2} l(l-1) \zeta_l \left( \frac{\lambda}{2\pi h} \right)^{l-1}
\]

\[
+ \rho \partial_\rho h \sum_{l \geq 1} l^2 \zeta_l \left( \frac{\lambda}{2\pi} \right)^{l-1} \int_\rho^\infty \frac{du}{u} \frac{\overline{h}(u)}{h(u)^{l+1}},
\]

(4.6)

where in the explicit expression \( h \) refers to \( h^\text{beta} \) and we omitted the \( \lambda \) and/or \( \mu \) arguments. Decomposing (4.6) according to the above grading yields a recursive system of inhomogeneous integro-differential equations for the \( \overline{s}_l \), \( l \geq 1 \),

\[
\frac{d}{dt} \overline{s}_l = \zeta_1 \partial_\rho \overline{s}_l(u, t) - \zeta_1 \partial_\rho \overline{s}_l + R_l[\overline{s}_{l-1}, \ldots, \overline{s}_1], \quad l \geq 1.
\]

(4.7)
Notably the homogeneous parts always have the same form, the inhomogeneities however get more complicated with increasing \( l \). They are differentiable functions of \( \rho \) and \( t \) for which we write \( R_l = R_l(\rho, t) \). Moreover they come out to have the following additive structure: \( R_1 = 0 \) and

\[
R_l[\vec{s}_{l-1}, \ldots, \vec{s}_1] = \sum_{k=2}^{l} R_{k1}[\vec{s}_{l+1-k}] \quad \text{for} \quad l \geq 2 \quad \text{with}
\]

\[
R_{k1}[\vec{s}] = \rho^{-k}(\alpha_k \vec{s} + \beta_k \rho \partial_\rho \vec{s}) + \rho \int_0^\infty \frac{du}{u^{k+2}} \left( \sum_{q=0}^{k-1} (u/\rho)^q \gamma_{k,q} \right) \vec{s}(u, t), \quad (4.8)
\]

where \( \alpha_k, \beta_k, \gamma_{k,0}, \ldots, \gamma_{k,k-1} \) are real constants, with \( \gamma_{k,1} = 0 \). For example

\[
\begin{align*}
\alpha_2 &= -2\zeta_2, & \beta_2 &= -2\zeta_2, & \gamma_{2,0} &= 8\zeta_2, \\
\alpha_3 &= -\frac{1}{\zeta_4}(8\zeta_2^2 + 6\zeta_1 \zeta_3), & \beta_3 &= -\frac{1}{2\zeta_4}(8\zeta_2^2 + 3\zeta_1 \zeta_3), & \gamma_{3,0} &= 12\zeta_3 + 36\zeta_2^2, & \gamma_{3,2} &= \frac{3}{2}\zeta_3.
\end{align*}
\]

Using (4.8) one can recursively construct a solution of (4.7) satisfying the desired boundary condition. The instrumental formula expresses \( \vec{s}_l \) in terms of \( \vec{s}_1, \ldots, \vec{s}_{l-1} \) as

\[
\vec{s}_1(\rho, t) = \rho \int_\rho^\infty \frac{du}{u} r_1(u - \zeta_1 t), \quad (4.10a)
\]

\[
\vec{s}_l(\rho, t) = \rho \int_\rho^\infty \frac{du}{u} r_l(u - \zeta_1 t) + \sum_{k=2}^{l} \int_0^t ds F_k[\vec{s}_{l+1-k}](\rho, t; s), \quad l \geq 2, \quad (4.10b)
\]

\[
F_k[\vec{s}](\rho, t; s) = \rho \int_\rho^\infty \frac{du}{u^2(u - \zeta_1 s)^{k+1}} \left( \gamma_k u + s \zeta_1 (\alpha_k + (k+2)\beta_k) - 2\beta_k(s\zeta_1)^2 u^{-1} \right)
\]

\[
+ \frac{1}{(\rho - \zeta_1 s)^k} \left( \beta_k (\rho - \zeta_1 s) \partial_\rho + \alpha_k - \beta_k \zeta_1 s/\rho \right) \vec{s}(\rho - \zeta_1 s, t - s). \quad (4.10c)
\]

Here \( r_1 \) and \( r_l \) are smooth functions of one variable satisfying \( u r_k(u) \to 0 \) for \( u \to \infty \); the constants \( \alpha_k, \beta_k \) and \( \gamma_k := \sum_{q=0}^{k-1} \gamma_{k,q} \) refer to the coefficients in (4.8). Since \( \vec{s}_1 \) is known explicitly Eq. (4.10) in principle allows one to compute all \( \vec{s}_l, l \geq 2 \), recursively. At each iteration step \( l-1 \to l \) a new function \( r_l \) enters via the solution of the homogeneous Eq. (4.7). Alternatively they can be viewed as parameterizing the initial configuration via \( r_l(\rho) = -\rho \partial_\rho[\vec{s}_l(\rho, t = 0)/\rho] \). Eventually thus \( \vec{s}_l \) is parameterized by \( l \) functions of
Based on the recursive solution formula (4.10) one can establish the following result.

**Theorem (UV stability):** Given a smooth initial configuration \(s_l(\rho) = s_l(\rho, 0), l \geq 1,\) let \(s_l(\rho, t)\) be a solution of (4.7) such that for \(\rho \to \infty\) both \(s_l(\rho, t)\) and \(\rho \partial_\rho s_l(\rho, t)\) vanish uniformly in \(t\). Then, for all \(l \geq 1,\) the solution is unique, smooth, and satisfies

\[
s_l(\rho, t) \to 0 \text{ for } t \to \infty \text{ if } \zeta_1 < 0 ,
\]

where the convergence is uniform in \(\rho\), for all \(\rho\) bounded away from zero. The situation is illustrated in Fig. 1 below.

We do not require that the initial data remain bounded as \(\rho \to 0\). For example one might wish to take \(\overline{h}(\rho, t = 0) = \rho - \frac{\lambda}{2\pi} \zeta_1^2\) as initial condition for \(\overline{h}\). Then \(s_1 \equiv 0\) and \(s_l(\rho, t = 0) \sim \rho^{-l+1}, r_l(u) \sim u^{-l}, l \geq 2.\)

![Figure 1: Decay of linearized perturbations \(s_l(\rho, t)\) for \(\zeta_1 < 0\): The initial configuration at \(t = 0\) and the vanishing at \(\rho = \infty\) are prescribed. The vanishing for \(t \to \infty\) is a dynamical property shown in appendix C.](image-url)
If \( \zeta_1 > 0 \) the same result formally holds for \( t \to -\infty \). However since we are concerned here with UV renormalizability this ‘infrared stability’ of the UV renormalization flow is of little significance. The proof of the theorem is somewhat technical and is relegated to appendix C.

Since \( \zeta_1 \) enters through the relation \( R_{ij}(m) = \zeta_1 m_{ij} \), its value depends on the normalization of \( m \). The relation itself is valid with \( \zeta_1 \neq 0 \) for indecomposable symmetric spaces \( \mathbb{G}/\mathbb{H} \) with \( \mathbb{G} \) semi-simple (c.f. appendix A), but for the higher order analogues in (3.2) \( \mathbb{G} \) has to be simple in general (c.f. appendix B). If \( m \) has euclidean signature only positive rescalings of \( m \) are allowed and the sign of \( \zeta_1 \) is intrinsic. Then \( \zeta_1 < 0 \) is a necessary and sufficient condition for the symmetric space to be non-compact [21]. If the metric \( m \) is indefinite the sign of \( \zeta_1 \) is unambiguously defined only once overall sign flips of \( m \) are prohibited. For the non-compact symmetric spaces arising by dimensional reduction this turns out to be the case, moreover

\[
\zeta_1 = -\frac{k + 2}{2}, \quad k = \# \text{vector fields} = \frac{1}{4}(\dim \mathbb{G} - \dim \mathbb{G} - 3). \tag{4.13}
\]

Importantly this is the same for the riemannian \((\epsilon_1 = +1)\) and the semi-riemannian \((\epsilon_1 = -1)\) version of the metric in (2.5). This means the UV stability of the fixed point holds irrespective of the signature of the Killing vectors used in the reduction.

For the derivation of (4.13) it is enough to note that both the overall sign of \( m \) and the overall normalization of the invariant bilinear form \( \langle \cdot, \cdot \rangle_g \) are unambiguously fixed in the symmetric spaces arising by dimensional reduction. The result then follows from Eq. (A.7). The overall sign of \( m \) in (2.5) is fixed by the requirement of positive energy. In terms of the Lagrangian (2.1) this entails that the Kaluza-Klein scalars \( \varphi^i \) interact via the positive definite metric \( m_{ij}(\varphi) \) and it also constrains the coupling matrix \( \mu \) to be positive definite. If there are no Kaluza-Klein scalars \( \varphi^i \) the two purely gravitational degrees of freedom \( \Delta, \psi \) serve to fix the overall sign of \( m \). The corresponding \( 2 \times 2 \) block of the metric (2.5) always has eigenvalues \((+,+)\), irrespective of the signatures \( \epsilon_1, \epsilon_2 \) of the two Killing vectors, provided the 3D duality transformation (2.3) is performed.

For a similar reason also the overall normalization of the invariant form \( \langle \cdot, \cdot \rangle_g \) is fixed. This is because in all cases \( \mathbb{G} \) is simple and contains the purely gravitational \( SL(2) \) as a subgroup. Thus up to an overall constant \( \langle \cdot, \cdot \rangle_g \) must coincide with the Killing form. Requiring that the restriction of \( \langle \cdot, \cdot \rangle_g \) to the \( sl_2 \) subalgebra coincides with the Killing form on \( sl_2 \) fixes \( \langle t_a, t_b \rangle_g = \frac{1}{k+2} f^d f_{bc}^c \) [13]. In the notation of appendix A this gives \((k + 2)\langle t_{\hat{a}}, t_{\hat{b}} \rangle_g = 2 f_{\hat{a}c}^e f_{\hat{b}d}^c = -2 R_{\hat{a}c}^e f_{\hat{b}}^c \), which establishes (4.13).
5. Augmented scalar matter

The scalar-vector couplings in (2.1) typically arise from dimensional reduction of $D \geq 4$ dimensional (super-)gravity theories and are largely dictated by the requirement of on-shell covariance. Of course one can also couple 4D scalars transforming under some group $\tilde{G}$ minimally to 4D gravity without regard to any vectors. One obvious case not covered by the systems (2.1) is when a set of 4D massless free scalars is minimally coupled to 4D gravity. Another example is 4D dilaton-axion gravity whose 2-Killing vector reduction gives rise to the decomposable symmetric space $SO(2,2)/SO(2) \times SO(2)$ [16]. Further, as explained in section 2 the ‘symmetry enhancement’ from the $\tilde{G}/\tilde{H}$ to the $G/H$ coset in the target space metric (2.5) hinges on the presence of vector fields and certain properties of $\tilde{G}/\tilde{H}$. If these conditions are not satisfied one can just as well switch off the vectors and consider 4D gravity minimally coupled to a 4D sigma-model with a coset $\tilde{G}/\tilde{H}$, which need no longer be non-compact. For positivity reasons $\tilde{G}/\tilde{H}$ should have riemannian signature and we may take $\tilde{G}$ to be simple or abelian. The 2-Killing vector reduction of these 4D theories gives rise to 2D systems of the form (2.6) where the coset $G/H$ is replaced by the direct product of SL(2)/SO(2) with $\tilde{G}/\tilde{H}$.

5.1 Renormalization: coupling to free vs interacting scalars

Since it comes at little extra price we consider more generally 2D sigma-models with target space metric

$$
\mathcal{G}_{ij} = \begin{pmatrix}
\mathbf{g}_{ij}^{1 \leq i,j \leq n+2} & 0 \\
0 & k(\rho)(\mathbf{m}_{ij})^{1 \leq i,j \leq \bar{n}}
\end{pmatrix},
$$

where \( \mathbf{g}_{ij} \) is that of Eq. (3.1) and \( \mathbf{m}_{ij} \) is the metric of some riemannian symmetric space $\tilde{G}/\tilde{H}$ of dimension $\bar{n}$ with $\tilde{G}$ simple or abelian (but not necessarily non-compact). The total dimension of the target space thus is $n + 2 + \bar{n}$. The function $k(\rho)$ is assumed to be of the form

$$
k(\rho, \lambda) = k_0(\rho) + \frac{\lambda}{2\pi} k_1(\rho) + \left(\frac{\lambda}{2\pi}\right)^2 k_2(\rho) + \ldots, \quad k_0(\rho) \sim \rho^p,
$$

where the condition on $k_0(\rho)$ (with the same $p \neq 0$ as in (3.6)) is motivated by the classical limit and the dimensional reduction procedure. The scalar curvature of $\mathcal{G}$ is

$$
R(\mathcal{G}) = \frac{R(\mathbf{m})}{h(\rho)} + \frac{R(\mathbf{m})}{k(\rho)},
$$
where \( R(m) = \zeta_1 n \) and \( \tilde{R}(\tilde{m}) = \tilde{\zeta}_1 \tilde{n} \) are the constant curvatures of \( G/H \) and \( \tilde{G}/\tilde{H} \), respectively.

Proceeding similarly as in appendix B one finds that the counter terms have to all loop orders the structure

\[
T^{(1,l)}_{ij}(\mathcal{G}) = \begin{pmatrix} (T^{(1,l)}_{ij}(\mathcal{G}))_{1 \leq i,j \leq n+2} & 0 \\ 0 & k(\rho)^{1-l} \tilde{\zeta}_l (\tilde{m}_{ij})_{1 \leq i,j \leq \tilde{n}} \end{pmatrix}, \quad l \geq 1, \quad (5.4)
\]

where in the lower block \( l \tilde{\zeta}_l \) are the beta function coefficients associated with the \( \tilde{G}/\tilde{H} \) sigma-model, again defined in minimal subtraction. The upper block has the same form as in (3.2) but the \( \rho\rho \) components are now differential polynomials in \( k(\rho) \) and \( h(\rho) \). For example:

\[
T^{(1,1)}_{\rho\rho}(\mathcal{G}) = \frac{n}{\rho^2} S_1(h) + \frac{\tilde{n}}{\rho^2} S_1(k) + \frac{n}{2} \partial_\rho \ln k \partial_\rho \ln h/k,
\]

\[
T^{(1,2)}_{\rho\rho}(\mathcal{G}) = 0,
\]

\[
T^{(1,3)}_{\rho\rho}(\mathcal{G}) = \frac{n}{\rho^2 h^2} S_3(h) + \frac{\tilde{n}}{\rho^2 k^2} S_3(k) + \frac{\tilde{n}\tilde{\zeta}_2}{4k^2} \partial_\rho \ln k \partial_\rho \ln h/k,
\]

with \( S_l \) as in (3.3). We also prepare shorthands for the sums \( T^{(1)}_{ij}(\mathcal{G}) := \sum_{l \geq 1} \frac{1}{2\pi l} T^{(1,l)}_{ij}(\mathcal{G}) \), and in particular \( \mathcal{G}(\rho, \lambda) := \frac{\lambda}{\rho} \sum_{l \geq 1} \frac{1}{2\pi l} T^{(1,l)}_{ij}(\mathcal{G}) \) which generalizes \( S(\rho, \lambda) \) in (3.10).

For the absorption of these counterterms we adopt a similar strategy as before, that is we allow for nonlinear field renormalizations of the form (3.4) a functional renormalization (3.5) of \( h \) as well as for its \( k \) counterpart, i.e.

\[
k_B(\rho) = \mu^{d-2} k(\rho, \lambda) \left[ 1 + \frac{1}{2-d} K(\rho, \lambda) + \ldots \right], \quad K(\rho, \lambda) = \sum_{l \geq 1} \left( \frac{\lambda}{2\pi} \right)^l K_l(\rho). \quad (5.6)
\]

In terms of the diagonal matrix \( \mathcal{H} = \text{diag}(H^{n+2}, K^{\tilde{n}}) = \mathcal{H}(\rho, \lambda) \) the finiteness condition reads

\[
L_\Xi \mathcal{G}_{ij} + (\mathcal{H} \mathcal{G})_{ij} = \lambda T^{(1)}_{ij}(\mathcal{G}/\lambda).
\]

We search for a solution with \( \Xi^j = (0, \ldots, 0, \xi^\rho(\rho, \lambda), \xi^\sigma(\rho, \lambda), 0, \ldots, 0) \), where the number of zeros is \( n \) and \( \tilde{n} \), respectively. Again we use \( \rho = n + 1 \), \( \sigma = n + 2 \) for the index labeling. The Lie derivative term with this \( \Xi^j \) is

\[
L_\Xi \Xi^j = \begin{pmatrix} (L_\Xi \xi^j)_{1 \leq i,j \leq n+2} & 0 \\ 0 & \Xi^\rho (h/\lambda) \partial_\rho k (\tilde{m}_{ij})_{1 \leq i,j \leq \tilde{n}} \end{pmatrix}, \quad (5.8)
\]
where the upper block equals (3.9). The finiteness condition (5.7) then reproduces the differential equations leading to (3.9) with \( S(\rho, \lambda) \) replaced by \( \tilde{S}(\rho, \lambda) \) and one extra equation. Accordingly the solutions for \( H \) and \( \Xi^\rho \) in terms of \( h \) are the same as in Eq. (3.9) and the solution for \( \Xi^\sigma \) is that in (3.9) with \( S(\rho, \lambda) \) replaced by \( \tilde{S}(\rho, \lambda) \). The extra equation determines – for given \( k \) and \( h \) – the counter terms \( K \) as

\[
K \left( \frac{h}{\lambda}, \frac{k}{\lambda} \right) = \tilde{B}_\lambda(\lambda/k) - \Xi^\rho \partial_\rho \ln k. \tag{5.9}
\]

Here \( \tilde{B}_\lambda(\lambda) := \sum_{l \geq 1} (\frac{\lambda}{2\pi})^l \tilde{\zeta}_l \) is the counterpart of \( B_\lambda(\lambda) \) in Eq. (3.10) for the \( \tilde{G}/\tilde{H} \) sigma-model. Further we adjusted the notation for \( K \) to stress the functional dependence on \( h \) and \( k \).

This completes the construction of the basic renormalized Lagrangian at some fixed renormalization scale \( \mu \). Strict cut-off independence can be achieved in terms of the two-fold infinite set of couplings \( h \) and \( k \). The deviation from conventional renormalizability in the matter sector is parameterized by \( K(h/\lambda, k/\lambda) \) in (5.6). In contrast to \( H \) in (3.5), (3.9) \( K \) may be non-zero already at one loop. Indeed there are two cases:

(a) For \( \tilde{\zeta}_1 \neq 0 \) one has \( K_1(\rho) = 0 \) with \( k_0(\rho) = \frac{\tilde{\zeta}_1}{\zeta_1} \rho^p \). The matter extended system remains strictly renormalizable at one loop.

(b) For \( \tilde{\zeta}_1 = 0 \) one can achieve \( k_0(\rho) \sim \rho^p, p \neq 0, \) only by taking \( K_1(\rho) = -\zeta_1 \rho^{-p} \).

Thus already at the 1-loop level standard renormalizability cannot be maintained.

In case (b) the 4D matter always consists of a

**Collection of free scalar fields:** This is because we assumed \( \tilde{G} \) to be either simple or abelian. If \( \zeta_1 = 0 \) the former possibility is ruled out by the results surveyed in appendix A. But if \( \tilde{G} \) is abelian not just the Ricci tensor but the full Riemann tensor vanishes \[22\]; in particular then \( \tilde{G}_l = 0, l \geq 1 \). The breakdown of standard renormalizability in this situation already at one loop seems to reflect the well-known feature of full Einstein gravity coupled to scalars \[8\].

Case (a) corresponds to self-interacting scalar matter in 4D arranged into a \( \tilde{G}/\tilde{H} \) sigma-model. The latter can either be put in by hand or arise itself via dualization and field redefinitions. A typical example for the second possibility is

**Dilaton-axion gravity:** The 2-Killing vector reduction is described e.g. in \[16\] and leads to a \( \text{SO}(2, 2)/\text{SO}(2) \times \text{SO}(2) \) coset. Since \( so(2, 2) \simeq so(1, 2) \times so(2, 1) \) is semi-simple rather than simple this theory cannot directly be subsumed into the class of systems in section...
2 (though it can be viewed as a limiting case e.g. of the \( \text{SO}(k+2,2)/\text{SO}(k+2) \times \text{SO}(2) \) series in table 1 with \( k = 0 \) vectors). Denoting the dilaton by \( \varphi \) and the pseudo-scalar ‘axion’ field by \( \chi \) the target space metric is of the form (5.1) with
\[
G/H = \text{SL}(2)/\text{SO}(2) : \quad \mathbf{m} = \text{diag}(\Delta^{-2}, \Delta^{-2}),
\]
\[
\tilde{G}/\tilde{H} = \text{SL}(2)/\text{SO}(2) : \quad \tilde{\mathbf{m}} = \text{diag}(\varphi^{-2}, \varphi^{-2}),
\]
(5.10)
in coordinates \( (\Delta, \psi, \rho, \sigma, \varphi, \chi) \). In particular \( \zeta_l = \tilde{\zeta}_l \), \( l \geq 1 \), which entails that one can consistently take \( h(\rho) = k(\rho) \); c.f. below. The counter term formulas (3.2) and (5.4) then coincide, which is a special case of part (iii) of the Lemma in appendix B. Explicitly the first three coefficients are \( \zeta_1 = -1, \zeta_2 = 1/2, \zeta_3 = -5/12 \), and coincide with the ones for pure gravity.

The qualitative differences between self-interacting scalars (case (a), \( \tilde{\zeta}_1 \neq 0 \)) and free scalars (case (b), \( \tilde{\zeta}_1 = 0 \)) become more pronounced if one considers the renormalization flow of the generalized coupling \( k \).

### 5.2 \( k \)-flow and its fixed point structure

For functionals \( X \) of \( h \) and \( k \) it is convenient to introduce the scaling operator
\[
\hat{X}(h, k) = \int du \left( h(u) \frac{\delta X}{\delta h(u)} + k(u) \frac{\delta X}{\delta k(u)} \right).
\]
(5.11)
In particular we prepare
\[
\hat{\Xi}^{\rho}(h/\lambda) = -\rho \int_{\rho}^{\infty} \frac{du}{u} \frac{h(u)}{\lambda} \beta_\lambda \left( \frac{\lambda}{h(u)} \right).
\]
(5.12)
Using (5.9) in (5.6) one obtains the flow equation
\[
\mu \frac{d}{d\mu} \frac{\tilde{K}}{k} = \lambda \beta_k \left( \frac{h}{h}, \frac{k}{\lambda} \right),
\]
\[
\lambda \beta_k \left( \frac{h}{h}, \frac{k}{\lambda} \right) = (2 - d)k - k \hat{K} = (2 - d)k + \frac{k^2}{\lambda} \tilde{\beta}_\lambda \left( \frac{\lambda}{k} \right) + \hat{\Xi}^{\rho}(h/\lambda) \frac{\partial}{\partial \rho} k,
\]
(5.13)
where \( \tilde{\beta}_\lambda = \lambda^2 \partial_\lambda \tilde{B}_\lambda \) is the beta function of the \( \tilde{G}/\tilde{H} \) model. Since we insist on \( k_0(\rho) \sim \rho^p \) a natural boundary condition for Eq. (5.13) is \( k(\rho, t)/\rho^p \rightarrow \tilde{\zeta}_1/\zeta_1 \), as \( \rho \rightarrow \infty \).

The \( \hat{\Xi}^{\rho} \) term in (5.13) describes the impact of the gravitational \( G/H \) model on the scalar matter in \( \tilde{G}/\tilde{H} \). For comparison note that using \( H = -\Xi^{\rho} \partial_\rho \ln h + B_\lambda (\lambda/h) \)
from (3.9) the $\beta_h$ function can be written in a similar form: 

$$-h\dot{H} = \frac{k^2}{\lambda} \beta_h(\frac{\lambda}{k}) + \dot{\bar{h}} \partial_\rho \bar{h}.$$ 

Evidently the key difference is that the $\bar{h}$-flow equation is autonomous while the $\bar{k}$-flow is triggered by $\bar{h}$. In the special case when $\zeta_l = \tilde{\zeta}_l$, $l \geq 1$, one sees that $\bar{h} = \bar{k}$ is a solution of the coupled system of flow equations. In particular this entails that the results in sections 3 and 4 generalize from cosets $G/H$ with $G$ simple to product manifolds $G/H = \tilde{G}/\tilde{H} \times \tilde{G}/\tilde{H}$ containing two identical factors (or factors with the same coefficients $\tilde{\zeta}_l$, $l \geq 1$) with $\tilde{G}$ simple. An example where this is relevant is the before mentioned dilaton-axion gravity which may be subsumed either under the present framework with $G/H = \tilde{G}/\tilde{H} = SL(2)/SO(2)$, or under the generalized framework of sections 3,4 with coset $G/H = SO(2,2)/SO(2) \times SO(2)$. Both descriptions yield the same results.

Returning to the general case one would expect that the relevant stationary point of the $\bar{k}$-flow is the one where also $\bar{h}$ is at its fixed point. Indeed, as we shall see later, the trace anomaly vanishes modulo an improvement term iff $\beta_h(h) = 0 = \beta_k(h,k)$. At $d = 2$ the latter gives the condition

$$\frac{\lambda}{2\pi} \rho \partial_\rho \bar{k} = C(\lambda) h_{\beta} \frac{k^2}{\lambda} \beta_{\lambda} \left( \frac{\lambda}{k} \right),$$

where $C(\lambda)$ is the same constant as in (4.1) and we used

$$\dot{\bar{\varphi}}(h_{\beta}/\lambda) = -\frac{\lambda}{2\pi C(\lambda) h_{\beta}(\rho)}.$$

As before we take $C(\lambda) = \rho/\zeta_1$ and write $k_{\beta}(\rho, \lambda) = \sum_{l \geq 1} (\frac{\lambda}{2\pi})^l h_l(\rho)$ for the solution. For theories with $\tilde{\zeta}_1 \neq 0$ the first two terms are

$$k_0(\rho) = \frac{\zeta_1}{\zeta_1} \rho^p, \quad k_1(\rho) = 2 \left( \frac{\zeta_2}{\zeta_1} - \frac{2\tilde{\zeta}_1}{\zeta_1} \right) \ln \rho^p - \frac{2\zeta_2}{\zeta_1}.$$ (5.16)

The additive constant in $k_1$ is chosen such that for $\tilde{\zeta}_2/\zeta_1^2 = \zeta_2/\zeta_1^2$ it coincides with $h_1$. Generally the $k_l$ contain terms of the same form as $h_l$ with coefficients which for generic $\tilde{\zeta}_l$, $\zeta_l$ are not fully determined. We fix them by requiring that in the special case where $\tilde{\zeta}_l = \zeta_l$, $l \geq 1$, one has $h_{\beta} = k_{\beta}$. Inserting (4.2), (5.16) into $K$ one finds

$$K(\rho, \lambda) = \left( \frac{\lambda}{2\pi} \right)^2 \left( \frac{\zeta_2}{2} - \frac{\tilde{\zeta}_2}{\zeta_1^2} \right) \rho^{-2p} + O(\lambda^3), \quad \text{for} \quad h = h_{\beta}, \quad k = k_{\beta}. $$ (5.17)
In general therefore conventional renormalizability is lost at two loops even at the fixed point.

On the other hand for theories with \( \tilde{\zeta}_1 = 0 \) the \( \tilde{\beta}_\lambda \) function vanishes identically, one is dealing with free scalars, and \( (5.14) \) implies \( k = \text{const} \). However this is in conflict with the requirement \( k_0(\rho) \sim \rho^p, p \neq 0 \), needed to allow for an interpretation as a dimensionally reduced system. In fact for constant \( k \) the matter sector decouples from the gravitational sector. We conclude that when a collection of free massless fields is minimally coupled to 4D Einstein gravity the 2-Killing vector reduction has only a degenerate fixed point where the scalars decouple from gravity.

This illustrates that the non-trivial gravitational fixed point found in \([7]\) – and its matter generalizations obtained here – are not automatic consequences of the relaxed notion of (‘conformal’) renormalizability. For example if the ‘backreaction’ terms proportional to \( \dot{\Xi}_\rho \) were dropped in the \( \overline{h} \) and \( \overline{k} \) flow equations only the trivial fixed point \( k = h = 0 \) would exist, presumably corresponding to the absence of interaction. Notably this does not happen for gravity coupled to abelian gauge fields and self-interacting scalars. It is only for free scalars that the flow equation \( (5.13) \) becomes linear and the \( \dot{\Xi}_\rho \) term precludes the existence of a genuine fixed point. Moreover, if one relies on the principle \([7]\) that an UV fixed point in the reduced theory should be a prerequisite for an UV fixed point in the full theory, the present result suggests a similar ‘triviality’ result for 4D Einstein gravity minimally coupled to free massless scalars – that is the gravity-matter coupling is expected to vanish at the fixed point. Although preliminary investigations in the full theory based on non-perturbative flow equations are available \([10]\) the issue so far does not seem to have been studied.

Let us return to the case where the augmented 4D scalars are self-interacting and its 2-Killing vector reduction has \( \tilde{\zeta}_1 \neq 0 \) in which case a non-trivial fixed point exists. In order to investigate its stability properties we linearize \( (5.13) \) around \( (h^{\beta}, k^{\beta}) \). We parameterize \( \delta h \) as in \((4.1)\) and \( \delta k \) analogously as

\[
\overline{k}(\rho, \lambda, \mu) = k^{\beta}(\rho, \lambda) + \delta k(\rho, \lambda, \mu), \quad \overline{\delta k}(\rho, \lambda, \mu) = \frac{\lambda}{2\pi} \overline{\kappa}_1(\varrho, t) + \left( \frac{\lambda}{2\pi} \right)^2 \overline{\kappa}_2(\varrho, t) + \ldots
\]

Here \( \varrho = \rho^p \) and \( t = \frac{1}{2\pi} \ln \mu/\mu_0 \), as before. The perturbations \( \overline{\kappa}_l(\varrho, t) \) are supposed to vanish for \( \varrho \to \infty \) uniformly in \( t \) and are assigned scaling dimension \( 1 - l \) as in \((4.5)\). Decomposing the linearized \( \overline{k} \)-flow according to this grading and restoring powers of \( \lambda \)
yields

\[
\frac{d}{dt} k = 2\pi \int du \frac{\delta f_k(h^{\beta}/\lambda, k^{\beta}/\lambda)}{\delta k(u)} \delta k(u)
\]

\[
= -\frac{\zeta_1}{p} h \frac{\partial \rho}{\partial k} - \sum_{l \geq 2} l(l-1) \zeta_l \left( \frac{\lambda}{2\pi} \right)^{l-1} k^{-l} \delta \kappa
\]

\[
+ \rho \partial_k \sum_{l \geq 1} l^2 \zeta_l \left( \frac{\lambda}{2\pi} \right)^{l-1} \int_0^{\infty} \frac{du}{u} \frac{\delta h(u)}{h(u)^{l+1}},
\]

where in the explicit expression \( h \) and \( k \) refer to \( h^{\beta} \) and \( k^{\beta} \). Comparing with (4.6) one sees that in the special case when \( \zeta_l = \tilde{\zeta}_l, \ l \geq 1 \), and hence \( h^{\beta} = k^{\beta} \), also \( \delta h = \delta k \) is a solution of the combined linearized flow equations. In the following we exclude this trivial situation and consider \( \delta h \) and \( \delta k \) as independent perturbations, where \( \delta h \) is supposed to be known through the solution of (4.6).

Then Eq. (5.19) converts into a recursive system of inhomogeneous partial differential equations for the \( \kappa_l, l \geq 1 \). The first equation is

\[
\frac{d}{dt} \kappa_1 + \zeta_1 \partial \rho \kappa_1 = \tilde{\zeta}_1 \rho \int_0^{\infty} \frac{du}{u^3} \tilde{s}_1(u, t).
\]

Specifying initial data by some function \( j_1 \) of one variable vanishing at infinity the solution of (5.20) is

\[
\kappa_1(\rho, t) = j_1(\rho - \zeta_1 t) + \frac{\tilde{\zeta}_1}{\zeta_1} [\tilde{s}_1(\rho, t) - \tilde{s}_1(\rho - \zeta_1 t, 0)].
\]

Evidently the condition \( \zeta_1 < 0 \) ensures that \( \kappa_1(\rho, t) \) vanishes for \( t \to \infty \) uniformly in \( \rho \). Notably this holds irrespective of the sign of \( \tilde{\zeta}_1 \), only \( \tilde{\zeta}_1 \neq 0 \) is required. The higher order equations are of the form

\[
\frac{d}{dt} \kappa_l + \zeta_1 \partial \rho \kappa_l = \tilde{\zeta}_l \rho \int_0^{\infty} \frac{du}{u^3} \tilde{s}_l(u, t) + Q_l[\kappa_{l-1}, \ldots, \kappa_1, \tilde{s}_{l-1}, \ldots, \tilde{s}_1], \ l \geq 2,
\]

where \( Q_l \) has a similar additive structure as \( R_l \) in (4.8). Regarding the right hand side as an in principle known function \( Q_l(\rho, t) \) the solution with initial data \( \kappa_l(\rho, 0) = j_l(\rho) \) is

\[
\kappa_l(\rho, t) = j_l(\rho - \zeta_1 t) + \frac{\tilde{\zeta}_l}{\zeta_1} [\tilde{s}_l(\rho, t) - \tilde{s}_l(\rho - \zeta_1 t, 0)]
\]

\[
+ \int_0^t ds \left( Q_l - \frac{\tilde{\zeta}_l}{\zeta_1} R_l \right)(\rho - \zeta_1 s, t - s).
\]

(5.23)
From here it is relatively straightforward to establish the desired uniform decay properties for $g \to \infty$ (boundary condition) and $t \to \infty$ (renormalization group dynamics). The analysis is simpler than the one in appendix C because the $\kappa_p, p = 1, \ldots, l-1$, enter without integrals in $Q_l$ and the $s_p$ are already known to decay. We thus omit the details and just state that $\kappa_l(g,t) \to 0$ for $t \to \infty$ for $\zeta_1 < 0$, uniformly in $g$ and irrespective of the sign of $\tilde{\zeta}_1 \neq 0$. The case $\tilde{\zeta}_1 = 0$ has already been discussed and we can summarize the results:

**Theorem (free vs interacting 4D scalars)**

(a) For 4D scalar matter corresponding to a $\tilde{G}/\tilde{H}$ sigma-model with $\tilde{\zeta}_1 \neq 0$ in the reduced theory a non-trivial fixed point $(h^{\beta}, k^{\beta})$ exists where gravity remains self-interacting and coupled to the scalars. The fixed point is UV stable under the two-fold infinite set of perturbations $(s_l, \kappa_l), l \geq 1$. An exact analogue of the theorem in section 4 holds for both $s_l$ and $\kappa_l$, irrespective of the sign of $\tilde{\zeta}_1 \neq 0$.

(b) If $\tilde{\zeta}_1 = 0$ the 4D matter consists of a collection of free scalar fields. Only a ‘trivial’ fixed point exists where the scalars decouple from gravity.
6. Conformal invariance co-exists with local running couplings

So far we took for granted that the vanishing conditions $\beta_h(h) = 0$ and $\beta_k(h, k) = 0$ are related to an intrinsic property of the system. Indeed, as we verify here, the simultaneous vanishing of both beta functionals is a necessary and sufficient condition for the trace anomaly to vanish modulo an improvement term. Specifically the result is:

$$[T^\mu]\partial_\mu\partial_\mu[\Phi] \iff \beta_h(h) = 0 = \beta_k(h, k),$$

with

$$\Phi = \frac{\lambda}{2\pi C} (a \ln \rho + b \sigma) - \int_{\rho}^{\infty} du W_\rho(\rho) - b \int_{\rho}^{\infty} \frac{du}{u} \dot{\nu}(u) \dot{\Xi}(u). \quad (6.1)$$

Here $[T_{\mu\nu}]$ is the renormalized energy momentum tensor based on (2.9) and $W_i$ is a co-vector induced by operator mixing which is built from the curvature tensors of $\mathcal{G}_{ij}$ in (5.1) and their covariant derivatives. $W_i$ can be shown to have the form

$$W_i(\phi) = (0, \ldots, 0, W_\rho(\rho), 0, 0, \ldots, 0),$$

$$W_\rho(\rho) = \left(\frac{\lambda}{2\pi}\right)^3 \frac{1}{8} \partial_\rho \left(\frac{n\zeta_2}{h^2} + \tilde{n}\zeta_2 \frac{k^2}{k^2}\right) + O(\lambda^4), \quad (6.2)$$

where the number of zeros is $n$ and $\tilde{n} + 1$, respectively.

On the other hand the renormalized energy momentum tensor is unique only up to addition of an improvement term $[\tau_{\mu\nu}]$ with potential $\tau(\phi)$. In order to qualify as an improvement potential compatible with the “conformal renormalizability” requirement adopted, $\tau$ has to be a function of $\rho$ and $\sigma$ only with $\sigma$ entering linearly [7], viz

$$[\tau_{\mu\nu}] = (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2)[\tau(\phi)],$$

$$\tau(\phi) = f(\rho, \lambda) + \tau_0(\lambda)\sigma. \quad (6.3)$$

Clearly its trace is of the same form as (6.1) so by defining $T_{\mu\nu}^{\text{imp}} := T_{\mu\nu} + \tau_{\mu\nu}$, with $\tau := \Phi$, one can render the improved energy momentum tensor traceless. Inserting the expression for $\dot{\Xi}$ derived from (3.9) (with $S$ replaced by $\mathcal{G}$) this fixes

$$\tau_0(\lambda) = \frac{\lambda b}{2\pi C(\lambda)}, \quad \partial_\rho f = \frac{\lambda a}{2\pi C} + W_\rho(\rho) - \frac{h}{2\rho} \int_{\rho}^{\infty} \frac{du}{u} \mathcal{G}(u, \lambda). \quad (6.4)$$

With the previous choice $C(\lambda) = p/\zeta_1$ the improvement potential (6.3) thus is completely determined at the fixed point up to some inessential integration constants. Away from the fixed point the function $f(\cdot)$ can be shown to be subject to a renormalization flow
equation which in principle determines it in terms of the running \( h \) and \( k \). We omit the details, c.f. [7].

It remains to derive Eq. (6.1). To this end one operates with \( 1 - h \frac{\delta}{\delta h} - k \frac{\delta}{\delta k} \) on the finiteness condition (5.7). Using the scaling properties of the constituents

\[
\hat{T}_{ij}^{(1,0)}(\mathfrak{O}) = (1 - l)T_{ij}^{(1,0)}(\mathfrak{O}), \quad \phi_i^j = -l \phi_i^j, \quad \hat{H}_i = -l H_i, \quad \hat{K}_i = -l K_i, \quad (6.5)
\]
one finds

\[
\sum_{l \geq 1} \left( \frac{\lambda}{2\pi} \right)^l (1 - l)T_{ij}^{(1,0)}(\mathfrak{O}) = -\hat{\mathcal{H}} \mathfrak{O} - \mathcal{L}_\mathfrak{O} \mathfrak{O}, \quad (6.6)
\]
with \( \hat{\mathcal{H}} = \text{diag}(\hat{H} \mathbb{1}_{n+2}, \hat{K} \mathbb{1}_{\tilde{n}}) \). The left hand side enters the general formula for the trace anomaly of a riemannian sigma-model [31, 32, 30]. Inserting (6.6) the anomaly can be rewritten as

\[
[T^\mu_\mu] = -\frac{1}{2} [\hat{\mathcal{H}} \mathfrak{O}]_{ij} \partial^\mu \phi^i \partial_\mu \phi^j + \partial^\mu [\partial_\mu \phi^i (W_i - \hat{\xi}_i)]. \quad (6.7)
\]
This is of the form (6.1) iff \( \hat{H} = \hat{K} = 0 \) and \( W_i - \hat{\xi}_i \) is the gradient of a scalar. Combining (6.2) with the form of \( \hat{\xi}_i = \mathfrak{O}_{ij} \hat{\xi}^j \) and (5.15) one finds that at the fixed point indeed \( W_i - \hat{\xi}_i = \partial_i \Phi \), with \( \Phi \) as in (6.1).

The improved energy momentum tensor is traceless at the fixed point so that its components \( [\mathcal{H}_0] := [T^\text{imp}_{00}] = \epsilon_1 \epsilon_2 [T^\text{imp}_{11}] \) and \( [\mathcal{H}_1] := [T^\text{imp}_{01}] \) can be interpreted as quantum versions of the hamiltonian and 1D diffeomorphism constraints, respectively, see Eq. (14). The linear combinations \( [\mathcal{H}_0 \pm \mathcal{H}_1] \) are thus expected to generate two commuting copies of a Virasoro algebra with formal central charge \( c = 2 + \text{dim} G/H \). This central charge is only formal because it refers to a state space with indefinite norm. The construction of quantum observables commuting with the constraints and the exploration of the physical state space are major desiderata.

There is a conceptual tension between the expectation that a quantum theory of gravity should in some sense be diffeomorphism invariant and the intuition that in a matter coupled theory the conventional running couplings should leave a remnant at the gravitational fixed point. An interesting lesson to be learned from the dimensionally reduced gravity theories is that both intuitions can elegantly be reconciled.

To this end one combines the flow equations for the essential couplings \( h \) and \( k \) with the flow for the inessential \( \rho \) ‘coupling’ [7]

\[
\mu \frac{d}{du} \rho = -\hat{\mathcal{H}} \rho [h/\lambda](\rho), \quad (6.8)
\]
which generalizes the scale dependence carried by the wave function renormalization constant in a multiplicatively renormalizable quantum field theory. Evaluating the running coupling functions $\mathcal{h}$ and $\mathcal{k}$ at the ‘comoving’ field $\mathcal{p}$ yields quantities

$$
\mathcal{\lambda}_h(\mu) := \frac{1}{\mathcal{h}(\mathcal{p}, \mu)}, \quad \mathcal{\lambda}_k(\mu) := \frac{1}{\mathcal{k}(\mathcal{p}, \mu)},
$$

(6.9)

that depend parametrically on the value of $h(\rho(x))$ and $k(\rho(x))$ – and hence on $x$ – and which describe the running of these (inverse) values. Combining (3.13), (5.13) with (6.8) one obtains

$$
\mu \frac{d}{d\mu} \lambda_h = -\beta_\lambda(\lambda_h), \quad \mu \frac{d}{d\mu} \lambda_k = -\tilde{\beta}_\lambda(\lambda_k).
$$

(6.10)

These are the usual flow equations for the one coupling $G/H$ and $\tilde{G}/\tilde{H}$ sigma-models, respectively! In other words the ‘gravitationally dressed’ functional flows for $\mathcal{h}$ and $\mathcal{k}$ have been ‘undressed’ by reference to the scale dependent ‘rod field’ $\mathcal{p}$. The equations (6.10) are not by themselves useful for renormalization purposes – which requires determination of the flow of $\mathcal{h}(\cdot, \mu)$ and $\mathcal{k}(\cdot, \mu)$ with respect to a fixed set of field coordinates. Moreover in the technical sense $\mathcal{\lambda}_h$ and $\mathcal{\lambda}_k$ are “inessential” couplings. However since Eqs. (6.10) are valid for any $\mathcal{h}$ and $\mathcal{k}$, in particular for their fixed points $h^{\beta}$ and $k^{\beta}$, they display how diffeomorphism (i.e. here 2D conformal) invariance can co-exist with conventional scale dependent running parameters.

7. Conclusions

The asymptotic safety property offers for a non-renormalizable quantum field theory similar rewards as asymptotic freedom does for a renormalizable one. Here a large class of such theories – arising as the 2-Killing vector reduction of 4D Einstein gravity coupled to scalars and abelian gauge fields – has been perturbatively constructed. The non-renormalizability manifests itself in the presence of infinitely many essential couplings $\lambda_n$, $n \geq 0$, which (morally though not technically) parameterize the function $\mathcal{h}(\varrho, t) = \varrho + \sum_{n \geq 0} \lambda_n(t) \varrho^{-n}$ featuring in section 4 (and similarly for $\mathcal{k}$ used in section 5). Here $\varrho$ can be identified with the “area radius”, at some fixed renormalization scale $t_0$, defined locally by the two Killing vector fields. The running couplings $\lambda_n$ approach a non-trivial fixed point in the ultraviolet, i.e. $\lambda_n(t) \to \lambda^*_n$ as $t \to \infty$. As mentioned in the introduction this is a surprising feature for which no obvious explanation exists in the
reduced theories. On the other hand the existence of this fixed point can be argued to be a prerequisite for the full theories to have an UV stable fixed point.

A running coupling is not in itself a physical quantity, although the nature of the running of course has an impact on the latter. In a diffeomorphism invariant theory it is not obvious to which quantities one should attribute the status of ‘physical quantity’. Uncontroversial candidates are “classical observables”, i.e. quantities weakly Poisson commuting with the hamiltonian and the diffeomorphism constraints. Remarkably, in the above reduced theories an infinite set of such observables can classically be constructed. An important open problem therefore is to investigate their fate in the quantum theory (e.g. in the spirit of [37]) and the impact the asymptotic safety property potentially has on their relations. This might help to characterize the asymptotic safety property also in the full theories in terms of physical quantities, and thus in a way that is independent of a particular computational and conceptual approach to quantum gravity.

Acknowledgments: I wish to thank M. Reuter, P. Forgács and T. Duncan for discussions. This work was completed while visiting the University of Pittsburgh, where it was supported in part by NSF grant PHY00-88946.
Appendix A: Symmetric space sigma-models

Symmetric space sigma-models describe the dynamics of generalized harmonic maps from a base manifold $\Sigma$ to an indecomposable symmetric space (as defined below) of the form $G/H$, where $G$ is the real form of a semi-simple Lie group with Lie algebra $\mathfrak{g}$ and $H$ is a maximal subgroup of $G$ with Lie algebra $\mathfrak{h}$ [23, 11]. For the exposition in this appendix the nature of the base manifold $\Sigma$ is inessential, for definiteness we take $\Sigma = \mathbb{R}^d, d \geq 1$, and denote the coordinates by $x^\alpha$. There are two useful action principles for these coset sigma-models. The first is a gauge theoretical one

$$S[V, Q] = \frac{1}{2} \int d^d x \langle D_\alpha V V^{-1}, D_\alpha V V^{-1} \rangle_{\mathfrak{g}},$$

(A.1)

where $V$ is a group-valued field transforming as $V \to V h$ under an $H$-valued gauge transformation and $Q_\alpha$ is the associated connection ensuring that $D_\alpha V = \partial_\alpha V - V Q_\alpha$ transforms covariantly. The currents $D_\alpha V V^{-1}$ take values in the Lie algebra $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is a non-degenerate bilinear form on it invariant under $\text{Ad} G$ and the (differential of the) involution $\tau$ below. Clearly the gauge symmetry removes $\dim H$ degrees of freedom leaving $\dim G/H$ physical ones. One can fix a gauge and work with preferred representatives $V_\ast$ intersecting each orbit once. Then $G$ acts nonlinearly via $V_\ast \mapsto g V_\ast h(V_\ast, g), g \in G,$ on the representatives because a gauge transformation is needed to bring $g V_\ast$ back into the original section.

Alternatively one can obtain a non-redundant parameterization of the coset space by matrices $M \in G$ obeying a suitable quadratic constraint $M \tau_0(M) = \pm \mathbb{1}$, where $\tau_0$ is an involutive outer automorphism of $G$. The subgroup $H$ can then be characterized as the set of fixed points of a related involutive automorphism $\tau$ of $G$, given by $\tau(g) = g_0^{-1} \tau_0(g) g_0$, for some fixed $g_0 \in G$ which likewise satisfies $g_0 \tau_0(g_0) = \pm \mathbb{1}$. (More precisely $\{G_\tau\}_0 \subset H \subset G_\tau$, where $G_\tau$ is the fixed point set of $\tau$ and $(G_\tau)_0$ its identity component.) Explicitly, the matrices $M$ can be constructed as $M = V g_0^{-1} \tau_0(V^{-1})$; they are gauge invariant and parameterize the coset space as $V$ runs through $G$. A symmetric space of the above type is called compact if $G$ is compact and non-compact otherwise; in the latter case $H$ may be both compact or non-compact. For non-compact symmetric spaces one can always take $g_0 = \mathbb{1}$, so that $\tau$ and $\tau_0$ coincide. In all cases one has $\partial_\alpha M M^{-1} = 2 D_\alpha V V^{-1}$, with $Q_\alpha = \frac{1}{2} (V^{-1} \partial_\alpha V + \tau(V^{-1} \partial_\alpha V))$. The action (A.1) becomes

$$S[M] = \frac{1}{2} \int d^d x \langle \partial_\alpha M M^{-1}, \partial_\alpha M M^{-1} \rangle_{\mathfrak{g}}, \quad M \tau_0(M) = \pm \mathbb{1}.$$ 

(A.2)
Choosing local coordinates $\varphi^i$ on $G/H$ the Lagrangian (A.2) is just the pull-back of the line element

$$\frac{1}{4} \langle dMM^{-1}, dMM^{-1} \rangle_g = m_{ij}(\varphi) d\varphi^i d\varphi^j, \quad i, j = 1, \ldots, n := \dim G/H, \quad (A.3)$$

with respect to $\varphi : \mathbb{R}^d \to G/H$. The matrix $m_{ij}$ defines a semi-riemannian metric on $G/H$ which has riemannian signature iff $H$ is compact [22]. By construction $m$ admits $\dim G$ Killing vectors (generating $g$) $n$ of which are algebraically independent. Further $m$ enjoys the important properties

$$\nabla_p R_{ijkl}(m) = 0, \quad R_{ij}(m) = \zeta_1 m_{ij}, \quad (A.4)$$

where the metric connection and the Riemann tensor refer to $m_{ij}$. The first property defines the larger class of locally symmetric semi-riemannian spaces, sufficient conditions for the second relation to hold are best formulated in terms of the Lie algebra $g$.

The involution $\tau$ of $G$ induces a decomposition of the Lie algebra $g = m \oplus h$, where $h$ and $m$ are even and odd under (the differential of) $\tau$, respectively. Further $[h, h] \subset h$, $[h, m] \subset m$, $[m, m] \subset h$. In terms of a basis $\{t_a\}$ of $g$ with structure constants $[t_a, t_b] = f_{abc} t_c$ one has a corresponding decomposition into even generators $t_{\hat{a}}$, $\hat{a} = 1, \ldots, \dim h$, and odd generators $t_{\hat{a}} = 1, \ldots, \dim m = n$. Since $\langle \cdot, \cdot \rangle_g$ is invariant under $\tau$ the subspaces $h$ and $m$ are orthogonal with respect to it. Indices are lowered and raised with $\eta_{ab} := \langle t_a, t_b \rangle_g$ and its inverse. The structure constants $f_{abc}$ then are completely antisymmetric. The symmetric space $G/H$ is called indecomposable if $m$ does not contain a proper subspace that is non-singular with respect to $\langle \cdot, \cdot \rangle_g$ and invariant under $\text{Ad} H$. This implies that $H$ is a maximal subgroup of $G$. Further one then has a non-degenerate $\text{Ad} H$ invariant bilinear form $\hat{\eta}_{\hat{a}\hat{b}} = \langle t_{\hat{a}}, t_{\hat{b}} \rangle_g$ induced on $m$. For $\exp t_{\hat{c}} \in H$ the $\text{Ad} H$ invariance amounts to $\langle [t_{\hat{a}}, t_{\hat{c}}], t_{\hat{b}} \rangle_g = \langle t_{\hat{a}}, [t_{\hat{c}}, t_{\hat{b}}] \rangle_g$, so that, conversely, the matrices $(\text{ad} t_{\hat{c}})_{\hat{a}\hat{b}} = f_{c\hat{c}}^{\hat{a}\hat{b}}$ form a subalgebra of $\text{so}(n, \hat{\eta})$. Note also that – compatible with the complete antisymmetry of the mixed structure constants $f_{c\hat{c}}^{\hat{a}\hat{b}}$ – ‘hatted’ indices can be lowered with $\hat{\eta}_{\hat{a}\hat{b}}$. In this situation the Ricci tensor $R_{ij}(m)$ is non-degenerate iff $g$ is semi-simple [22]. We consider indecomposable symmetric spaces with $g$ semi-simple throughout. Then Eq. (A.4) holds with $\zeta_1 \neq 0$; c.f. below. For indecomposable symmetric spaces with riemannian signature a number of stronger statements hold: First the semi-simplicity of $g$ does not have to be assumed but is a consequence. Further the sign $\zeta_1 > 0$ then characterizes compact spaces while $\zeta_1 < 0$ characterizes non-compact spaces [21].
The curvature tensors associated with the semi-riemannian metric (A.3) will be frequently needed. They are best computed in the vielbein frame.* To introduce the vielbein we decompose the left invariant Maurer-Cartan form according to \( \mathbf{g} = \mathbf{m} \oplus \mathbf{h} \)

\[
V^{-1} dV = L_i^a(\varphi) t_a d\varphi^i = L_i^\hat{a}(\varphi) t_{\hat{a}} d\varphi^i + L_i^{\hat{a}}(\varphi) t_{\hat{a}} d\varphi^i .
\]  

Since \( \partial_i M M^{-1} = 2 L_i^{\hat{a}}(\varphi) V t_{\hat{a}} V^{-1} \) one infers from (A.3) that \( L_i^{\hat{a}}(\varphi) \) is a vielbein for \( \mathbf{m}_{ij}(\varphi) \), where the flat metric is \( \hat{\eta}_{\hat{a}\hat{b}} := \langle t_{\hat{a}}, t_{\hat{b}} \rangle_\mathbf{g} \). In particular it follows that there is a one-to-one correspondence between \( G \)-invariant semi-riemannian metrics on \( G/H \) and \( \text{Ad}H \) invariant bilinear forms on \( \mathbf{m} \). The integrability condition for (A.5) gives rise to

\[
\begin{align*}
\partial_i L_j^{\hat{a}} - \partial_j L_i^{\hat{a}} - (L_j^b L_i^c - L_i^b L_j^c) f_{b\hat{c}}^{\hat{a}} &= 0 , \\
\partial_i L_j^{\hat{a}} - \partial_j L_i^{\hat{a}} - L_j^b L_i^c f_{\hat{b}\hat{c}} - L_j^b L_i^c f_{b\hat{c}}^{\hat{a}} &= 0 .
\end{align*}
\]  

The first equation can be interpreted as the Cartan structure equation for the spin connection associated with the vielbein \( L_i^{\hat{a}}(\varphi) \). Thus \( \omega_i^{\hat{a}} = L_i^c f_{\hat{b}\hat{c}}^{\hat{a}} \) in the conventions declared in the footnote. Using this and the second equation (A.6) one readily computes the Riemann tensor in the vielbein frame

\[
R_{\hat{a}\hat{b}\hat{c}\hat{d}} = f_{\hat{c}\hat{a}}^d f_{\hat{b}\hat{d}}^e , \quad \text{i.e.} \quad t_{\hat{a}} R_{\hat{b}\hat{c}\hat{d}} = -[\langle t_{\hat{a}}, t_{\hat{d}} \rangle, t_{\hat{b}}] .
\]  

Converting back to the coordinate basis yields the covariant constancy \( \nabla_p R_{ijkl}(\mathbf{m}) = 0 \) announced in (A.3). Further upon contraction one obtains from (A.7) \( R_{\hat{a}\hat{b}} = -\frac{\tilde{\eta}_{\hat{a}\hat{b}}}{2} \), if \( \langle t_a, t_b \rangle_\mathbf{g} = \frac{1}{\tilde{\xi}} f_{ac} d f_{bd} e. \) When \( G \) is simple this follows because then \( \langle \cdot , \cdot \rangle_\mathbf{g} \) is unique up to an overall constant and is proportional to the Killing form; in the above normalization this induces \( \hat{\eta}_{\hat{a}\hat{b}} = \frac{\tilde{\xi}}{2} f_{\hat{c}\hat{a}} f_{\hat{b}\hat{e}} \), and hence \( \tilde{\xi} = -\tilde{\xi}/2 \), in agreement with [11] in the compact case. When \( G \) is semi-simple a choice of proportionality constant is required for each simple factor. The ratios of these constants can then be adjusted such that \( R_{\hat{a}\hat{b}} \sim \eta_{\hat{a}\hat{b}} \) is recovered. Converting back to the coordinate basis results in \( R_{ij}(\mathbf{m}) = \zeta_1 \mathbf{m}_{ij} \), \( \zeta_1 \neq 0 \), for all indecomposable symmetric spaces with \( \mathbf{g} \) semi-simple.

---

*Our conventions are: \( \nabla e^k = \partial_i e^k + \Gamma^k_{ij} v^j \), with \( \Gamma^k_{ij} = \frac{1}{2} \mathbf{g}^{kl} [\partial_j \mathbf{g}_{kl} + \partial_k \mathbf{g}_{lj} - \partial_l \mathbf{g}_{jk}] \). The Riemann tensor is defined by \( (\nabla_i \nabla_j - \nabla_j \nabla_i) v^k = R^k_{ij} v^j \), so that \( R^k_{ij} = \partial_i \Gamma^k_{lj} - \partial_j \Gamma^k_{li} + \Gamma^k_{im} \Gamma^m_{lj} - \Gamma^k_{jm} \Gamma^m_{li} \). The Ricci tensor is \( R_{ij} = R^m_{imj} \). Let \( e^a_i \) be a vielbein for \( \mathbf{g}_{ij} = e^a_i e^b_j \eta_{ab} \) with inverse \( e_i^a \). The spin connection coefficients \( \omega_i^{ab} \) can be read off from the structure equation \( \partial_i e^a_j - \partial_j e^a_i = \omega_i^{ab} e_j^b - \omega_j^{ab} e_i^b \). The Riemann tensor in the vielbein frame is \( R_{abcd} = e^e_i e^d_j [\partial_j \omega_{cba} - \partial_j \omega_{bab} - \omega_{bac} \omega_{jeb} + \omega_{jbe} \omega_{iae}] \).
Appendix B: Structure of the counter terms

Here we derive the result (3.2) on the structure of the $l$-loop counter terms. We repeatedly exploit two generic properties of them: First, they are sums of monomials built from the Riemann tensor of $g_{ij}$ and its covariant derivatives such that in each monomial the number of derivatives is even and at $l$-loops adds up with the power of the Riemann tensor according to

$$\text{power of Riemann } + \frac{1}{2}(\#\nabla) = l.$$  \hfill (B.1)

Second they transform as $T_{ij}^{(1,l)}(\Lambda^{-1}g) = \Lambda^{l-1}T_{ij}^{(1,l)}(g)$, $\Lambda \in \mathbb{R}$, under constant rescalings of the metric. For illustration and later use let us quote the explicit results \[31, 32, 33, 34\] for $l \leq 3$:

$$T_{ij}^{(1,1)}(g) = R_{ij},$$
$$T_{ij}^{(1,2)}(g) = \frac{1}{4}R_{iklm}R_{j}^{kdm}.$$ \hfill (B.2)
$$T_{ij}^{(1,3)}(g) = \frac{1}{6}R_{imn}R_{ipqk}R_{j}^{kmpnq} - \frac{1}{8}R_{iklm}R_{j}^{kmpn}R_{lmnp} - \frac{1}{12}\nabla_nR_{iklm}\nabla_kR_{j}^{lmn}.$$

We adopt the specifications and notations of appendix A throughout and begin with the

Lemma B: (i) Killing vectors of $g_{ij}$ are Killing vectors of $T_{ij}^{(1,l)}(g)$.
(ii) For a symmetric space $G/H$ with $G$ simple and with metric $m_{ij}$, $i, j = 1, \ldots, n$, one has

$$T_{ij}^{(1,l)}(m) = \zeta_l m_{ij}, \quad \zeta_l \in \mathbb{R}.$$ \hfill (B.3)

(iii) Let $G/H$ be a direct product of symmetric spaces as in (ii) with identical coefficients $\zeta_l$, $l \geq 1$. Then Eq. (B.3) also holds for the canonical metric on the product space.

Proof. (i) This is an application of the well-known integrability conditions arising from repeated differentiation of the Killing equation $\mathcal{L}_v g_{ij} = 0$. One obtains

$$\mathcal{L}_v(\nabla_{i_1} \ldots \nabla_{i_d} R_{ijkl}(g)) = 0, \quad d \geq 0,$$ \hfill (B.4)

for any Killing vector $v^k$ of $g_{ij}$ (where $R_{ijkl}(g)$ is the Riemann tensor of $g_{ij}$). Since the Lie derivative is a derivation that preserves tensor type and commutes with contractions, the result follows. (ii) For a maximally symmetric space (which is uniquely characterized by its $n(n+1)/2$ Killing vectors and the signature) this is a direct consequence of (i). For a generic symmetric space one uses the fact that $T_{ij}^{(1,l)}(m)$ contains contractions of the Riemann tensor only, but no derivatives on account of Eq. (A.4). Further, in the
The vielbein frame the Riemann tensor is constant so that $T^{(1,l)}_{ij}(m)$ in the vielbein frame must likewise be constant. On account of the $G$-invariance and the indecomposability it must be in one-to-one correspondence to an AdH invariant second rank tensor on $m$ in the decomposition $g = m \oplus h$ of the Lie algebra. Since $G$ is simple this tensor must be proportional to $\hat{\eta}_{ab}$. Converting back to the coordinate basis yields the result. (iii) This is a direct consequence of (ii), concluding the proof of Lemma B.

We remark that the set of coefficients $\zeta_l$, $l \geq 1$, does not uniquely characterize a symmetric space. A simple (counter-) example are maximally symmetric spaces which are uniquely determined by their constant sectional curvature $K$ and the number of positive and negative eigenvalues of the metric $m_{ij}$. From $R_{ijkl}(m) = K(m_{ik}m_{jl} - m_{il}m_{jk})$ one sees that the curvature scalars $\zeta_l$ depend on $K$ and the dimension $n$ but on the signature of the metric at most through the sign of $K$. For example one computes from (B.2)

$$
\zeta_1 = K(n - 1) , \quad \zeta_2 = \frac{K^2}{2}(n - 1) , \quad \zeta_3 = \frac{K^3}{12}((n + 1)^2 - 4) , \quad (B.5)
$$
in agreement with [25, 26].

The target space metric (3.1) of course is not that of a symmetric space. In order to be able to apply the Lemma we need

$$
T^{(1,l)}_{ij}(g) = \frac{1}{h^{l-1}} T^{(1,l)}_{ij}(m) , \quad i, j = 1, \ldots, n , \quad (B.6a)
$$

$$
T^{(1,l)}_{\sigma\rho}(g) = 0 , \quad T^{(1,l)}_{\sigma i}(g) = 0 , \quad T^{(1,l)}_{\rho i}(g) = 0 . \quad (B.6b)
$$

To show this one consecutively computes the Christoffel symbols, the Riemann tensor and its covariant derivative for the metric (3.1). The independent non-vanishing components are:

$$
\Gamma^k_{ij} = \Gamma^k_{ij}(m) , \quad \Gamma^k_{i\rho} = \frac{1}{2} \partial_\rho \ln h \delta^k_i , \quad \Gamma^\rho_{\rho\rho} = \partial_\rho \ln(h/\rho) . \quad (B.7a)
$$

$$
\Gamma^\sigma_{ij} = -\frac{1}{2b} \rho \partial_\rho \ln h m_{ij} , \quad \Gamma^\sigma_{\rho\rho} = -\frac{a}{2b} \partial_\rho \ln h , \quad (B.7b)
$$

$$
R_{ijkl} = h(\rho) R_{ijkl}(m) , \quad R_{\rho i\rho j} = \frac{h(\rho)}{\rho^2} S_1(\rho) m_{ij} , \quad (B.7b)
$$

$$
\nabla_\rho R_{ijkl} = 2 \nabla_i R_{\rho j k l} = -\partial_\rho h R_{ijkl}(m) , \quad \nabla_\rho R_{\rho i\rho j} = \frac{h^3}{\rho^2} \partial_\rho \left( \frac{S_1}{h^2} \right) m_{ij} , \quad (B.7c)
$$

with $S_1(\rho)$ as in (3.3) and $i, j, k, l \in \{1, \ldots, n\}$. All components not related to these by symmetries of the Riemann tensor vanish. In Eq. (B.7c) the covariant constancy (A.4) enters.
For \( l \leq 3 \) one can now verify the asserted structure (3.2) by explicit computation, using Eqs. (A.4), (B.2) and Lemma B. In particular one thereby obtains the expressions for \( S_l(\rho) \), \( l = 1, 2, 3 \), anticipated in (3.3).

Proceeding with the general analysis one shows by induction from (B.7) and (A.4)

\[
\nabla_{i_1} \ldots \nabla_{i_d} R_{ijkl} \neq 0 \quad \text{only if} \quad \left\{ \begin{array}{c} \#\rho \text{ indices} \neq 0, \\ \#\rho \text{ indices} + \# \text{ derivatives} = \text{even}. \end{array} \right. \tag{B.8}\]

In particular all components containing one or more ‘lower’ \( \sigma \) index vanish. Further \( \rho \) cannot appear as a summation index since an ‘upper’ \( \rho \) index amounts to a ‘lower’ \( \sigma \) index. Combining this with (B.7b) establishes Eq. (B.6a) and

\[
T^{(1, l)}_{\rho i}(g) = 0. \tag{B.9}\]

To show that also \( T^{(1, l)}_{\rho i}(g) \) vanishes for \( i = 1, \ldots, n \), one can proceed as follows. Since \( \rho \) cannot appear as a summation index the monomials in \( T^{(1, l)}_{\rho i}(g) \) must have the following form

\[
\nabla_{i_1} \ldots \nabla_{i_d} R_{i_{d+1} \ldots i_{d+4}} Q(R)^{i_1 \ldots i_r \ldots i_s}_{i_{d+1} \ldots i_{d+4}}, \quad \text{with} \quad \rho = i_r, \quad \text{or},
\]

\[
\nabla_{i_1} \ldots \nabla_{i_d} R_{i_{d+1} \ldots i_{d+4}} Q(R)^{i_1 \ldots i_s \ldots i_r}_{i_{d+1} \ldots i_{d+4}}, \quad \text{with} \quad \rho = i_r, i = i_s, \tag{B.9}\]

for some \( r, s \in \{1, \ldots, d+4\} \), \( d \geq 0 \), and the summation is over all but the ‘hatted’ indices, which are omitted. Here \( Q(R) \) is a tensor which by (B.8) cannot contain derivatives of the Riemann tensor. Thus all derivatives must be carried by the first term and \( d \) in (B.9) equals the total number of derivatives. On account of (B.1) \( d \) must be even, and so must be the number of \( \rho \) indices in view of Eq. (B.8). However this contradicts Eq. (B.9) which allows only a single \( \rho \) index. Hence no non-zero candidate monomial exists and we infer \( T^{(1, l)}_{\rho i}(g) = 0 \). This concludes the derivation of Eq. (B.6). We note that the result hinges on the covariant constancy in Eq. (A.4); in particular the block diagonal form of the counter terms is not a trivial consequence of the block diagonal form of the metric (3.1). The rest is straightforward: Applying part (ii) of Lemma B to Eq. (B.6a) gives \( T^{(1, l)}_{ij}(g) = h^{1-l}\zeta_i m_j \). The remaining matrix element \( T^{(1, l)}_{\rho \rho}(g) \) has to scale with weight \( 1-l \) under constant rescalings of \( h \). Extracting an explicit power of \( h \) one can write \( T^{(1, l)}_{\rho \rho}(g) = nh^{1-l}S_l(\rho)/\rho^2 \), for some differential polynomial \( S_l \) in \( h \) invariant under constant rescalings of \( h \). For constant \( h \) the metric (3.1) is that of a direct product of the (irreducible) symmetric space \( G/H \) with \( \mathbb{R}^{1,1} \), and it is not hard to see that \( S_l(\rho) \) then vanishes. This concludes the derivation of Eq. (3.2).
Appendix C: Proof of UV stability

Here we establish the result on the UV stability described in section 4. It is sufficient to consider the case $\zeta_1 < 0$ and $t \geq 0$. The flow equation (4.10) is invariant under $t \rightarrow -t$, $\zeta_l \rightarrow -\zeta_l$, $l \geq 1$. Thus if indeed, as the theorem asserts, $\zeta_1 < 0$ implies decay of the perturbations for $t \rightarrow \infty$ irrespective of the values of $\zeta_2, \zeta_3, \ldots$, a positive $\zeta_1$ will imply decay for $t \rightarrow -\infty$, i.e. (formal) infrared stability of the fixed-point. We therefore assume $\zeta_1 < 0$ throughout this appendix and set $t_1 := -\zeta_1 t \geq 0$. In the following we first outline the derivation of Eq. (4.10) and then show that the solutions recursively constructed thereby have the announced properties.

Since integro-differential equations are cumbersome we convert Eqs. (4.10) into a system of partial differential equations

$$\partial_t \mathbf{S}_l + \zeta_1 \partial_\varrho \mathbf{S}_l = \varrho \partial_\varrho [R_l/\varrho], \quad l \geq 1,$$

(C.1)

where $\mathbf{S}_l = \varrho \partial_\varrho (s_l/\varrho)$. The general solution of Eq. (C.1) at given $l$ is a sum of the generic solution to the homogeneous equation and a particular solution of the inhomogeneous equation. The former can be described in terms of a (smooth) function $r_l$ of one variable as $-r_l(\varrho - \zeta_1 t)$. Regarding $\varrho \partial_\varrho [R_l/\varrho]$ as an (in principle) known function of $\varrho$ and $t$ the full solution of (C.1) is

$$\mathbf{S}_l (\varrho, t) = -r_l (\varrho - \zeta_1 t) + \int_0^t ds \left( \varrho \partial_\varrho [\varrho^{-1} R_l] \right) (\varrho - \zeta_1 s, t - s).$$

(C.2)

The solutions for $s_l$ are obtained by integration

$$s_l (\varrho, t) = -\varrho \int_\varrho^\infty \frac{du}{u} \mathbf{S}_l (u, t),$$

(C.3)

where the upper integration boundary is chosen with regard to the boundary condition at $\varrho = \infty$ aimed at (though of course it does not in itself entail it). Inserting (C.2) into (C.3) with

$$\varrho \partial_\varrho [R_l/\varrho] = \sum_{k=2}^l \varrho^{-k-1} [\beta_k \varrho^2 \partial_\varrho^2 + (\alpha_k - k \beta_k) \varrho \partial_\varrho - (\gamma_k + (k+1) \alpha_k)] s_{l+1-k} (\varrho, t),$$

(C.4)

and performing some integrations by parts, one arrives at the recursive solution Eq. (4.10).
A simple consequence is that there can be at most one solution of (4.7) with given (smooth) initial data \( \mathfrak{s}_l(\rho, t = 0) \). From Eq. (4.10) one infers: \( \mathfrak{s}_l(\rho, t) = 0 \) for all \( t \) if \( r_k(\rho) = -\rho \partial_\rho [\mathfrak{s}_k(\rho, t = 0)/\rho] = 0 \) for \( k \leq l \) (*). Thus if there were two distinct solutions with the same initial data, their difference \( \mathfrak{s}_l^{\text{diff}}(\rho, t) \) would solve (4.7) by linearity. Then (*) applies and entails \( \mathfrak{s}_l^{\text{diff}}(\rho, t) \equiv 0 \), i.e. both solutions coincide. A similar argument shows that the system (4.7) does not admit ‘static’ solutions satisfying the required boundary conditions. Indeed, a \( t \)-independent solution for \( \mathfrak{s}_l \) is proportional to \( \rho \ln \rho \) which satisfies the boundary condition only if the proportionality constant vanishes. Hence \( \mathfrak{s}_1 = 0 \) and \( R_2[\mathfrak{s}_1] = 0 \), so that the \( l = 2 \) equation now gives \( \mathfrak{s}_2 = 0 \). Since \( R_l \) vanishes if \( \mathfrak{s}_1, \ldots, \mathfrak{s}_{l-1} \) vanish the absence of static solutions follows by iteration. Finally it is also clear that the solutions produced by (4.10) from smooth initial data will the smooth in both \( t \) and \( \rho \) (for \( \rho \) bounded away from zero) as long as the integrals involved converge absolutely. The latter will be a byproduct of the inductive bounds shown below.

For any smooth function \( r(u) \) on \( \mathbb{R}^+ \) satisfying \( u r(u) \to 0 \) for \( u \to \infty \) we set

\[
\mathfrak{s}_l^{\text{hom}}(\rho, t) = \rho \int_{\rho}^{\infty} du \frac{u r(u - \zeta_1 t)}{u},
\]

which describes the homogeneous parts of the solutions (4.10). They are readily seen to satisfy both the premise and the conclusion of the theorem: By definition \( M_x := \max_{u \geq x > 0} u |r(u)| \) is a finite positive constant, decreasing with increasing \( x \) such that \( M_\infty = 0 \). For \( t_1 = -\zeta_1 t > 0 \) one verifies the bounds

\[
|\mathfrak{s}_l^{\text{hom}}(\rho, t)| \leq M_\rho + t_1, \quad |\rho \partial_\rho \mathfrak{s}_l^{\text{hom}}(\rho, t)| \leq 2M_\rho + t_1,
\]

using \( \rho \partial_\rho \mathfrak{s}_l^{\text{hom}} = \mathfrak{s}_l^{\text{hom}} - \rho r(\rho + t_1) \) in the second case. For all \( t_1 \geq 0 \) then \( M_{\rho + t_1} \leq M_\rho \), so that for \( \rho \) large the bound is uniform in \( t_1 \). Likewise for \( t_1 \) large one has \( M_{\rho + t_1} \leq M_{t_1} \) for all \( \rho > 0 \), giving a bound uniform in \( \rho \).

**Strategy for inductive bounds:** The aim in the following is to show by induction on \( l \) that the solutions recursively produced by (4.10) indeed have the announced uniform decay properties, both for \( \rho \to \infty \) (boundary condition) and for \( t \to \infty \) (renormalization group dynamics). We decompose each \( \mathfrak{s}_l \) as \( \mathfrak{s}_l = \mathfrak{s}_l^{\text{hom}} + \mathfrak{s}_l^{\text{inh}} \), where \( \mathfrak{s}_l^{\text{hom}} \) is of the form (4.5) with \( r = r_l \) and \( \mathfrak{s}_l^{\text{inh}} \) is the remainder in (4.10). The homogeneous parts are already known to have the desired properties. For the inhomogeneous parts we seek to establish
uniform bounds for each of the terms on the right hand side of
\[ |\mathcal{S}_i^{inh}(\varrho, t)| \leq \sum_{k=2}^l \int_0^t ds \left| F_k[\mathcal{S}_{i+1-k}^{\hom}] \right| + \sum_{k=2}^l \int_0^t ds \left| F_k[\mathcal{S}_{i+1-k}^{inh}] \right|. \]  
(C.7)

It turns out that for all but the term this is relatively straightforward. When estimated using only the induction hypothesis for \( s_{i-1}^{inh} \) (i.e. that \( s_{i-1}^{inh} \) decays for \( t \to \infty \) uniformly in \( \varrho \), but the decay may be arbitrarily soft) this term seems to diverge logarithmically for \( t \to \infty \) at fixed \( \varrho \). In order to overcome this problem we establish in Lemma 2 below that the inhomogeneous parts \( s_{i+1-k}^{inh} \) actually decay faster, like an inverse power of \( t \). For the homogeneous parts, in contrast, no such result holds and they may decay arbitrarily soft for \( t \to \infty \). Clearly both statements are compatible only if the linear functional of the \( s_{i+1-k}^{hom} \) appearing in (C.7) decays like an inverse power of \( t \). This is what we show first:

**Lemma C1:** Let \( s^{hom}(\varrho, t) \) be of the form (C.5) for some smooth function \( r(u) \) satisfying \( ur(u) \to 0, \) for \( u \to \infty \). Then there exists \( 0 < \epsilon < 1 \) such that for all \( k = 2, \ldots, l \)
\[ t^\epsilon \int_0^t ds F_k[s^{hom}](\varrho, t; s) \to 0 \quad \text{for} \quad t \to \infty, \]  
(C.8)

where the convergence is uniform in \( \varrho \), for all \( \varrho \) bounded away from zero. For \( \varrho \to \infty \) the left hand side also vanishes uniformly in \( t \).

**Proof.** For \( k \geq 2 \) we rewrite the integrand as follows
\[ F_k[s^{hom}] = \varrho \int_0^\infty du \frac{P_k(u, s)}{u - \zeta_1 s} r(u + t_1) - \frac{\beta_k}{(\varrho - \zeta_1 s)^k} r(\varrho + t_1) \]
\[ + \frac{s^{hom}(\varrho - \zeta_1 s, t - s)}{\varrho - \zeta_1 s} \left[ \alpha_k + \beta_k - \beta_k \zeta_1 s/\varrho \right] - \varrho P_k(\varrho, s), \]
\[ P_k(\varrho, s) = -\int_\varrho^\infty du \frac{au + sb + cs^2/u}{u^2(u - \zeta_1 s)^k}. \]  
(C.9)

The constants \( a, b, c \) entering \( P_k \) can be read off from (4.10) but except for \( k = 2 \) their values are inessential. For \( k \geq 3 \) we shall later use the function symbol \( P_k \) also when the values are different.

For \( k = 2 \) specifically the relations \( \alpha_2 = \beta_2 = -\gamma_2/4 \) among the coefficients in (4.9)
imply that (C.9) simplifies as follows

\[
\int_0^t ds F_2[S_{\text{hom}}](\varrho, t; s) = -\frac{2\zeta_2}{\zeta_1} \varrho \int_\varrho^\infty du r(u + t_1) \partial_u \left[ \frac{1}{u} \ln \left( 1 + \frac{t_1}{u} \right) \right] - \frac{2\zeta_2}{\zeta_1} r(\varrho + t_1) \ln(1 + t_1/\varrho),
\]

using

\[
P_2(\varrho, s) = \frac{2\zeta_2 \zeta_1 s - 2 \varrho}{\varrho^2 - \zeta_1 s}, \quad \int_0^t ds P_2(\varrho, s) = -\frac{2\zeta_2}{\zeta_1} \varrho \left[ \frac{1}{\varrho} \ln \left( 1 + \frac{t_1}{\varrho} \right) \right].
\]

A straightforward estimate then is

\[
\int_0^t ds \left| F_2[S_{\text{hom}}](\varrho, t; s) \right| \leq \frac{M_{\varrho+t_1}}{\varrho} \left| \frac{4\zeta_2}{\zeta_1} \ln(1 + t_1/\varrho) \right| \left( 1 + t_1/\varrho \right).
\]

Here \( M_x := \max_{u \geq x > 0} |u| r(u) \) as before. Observing that for \( 0 < \epsilon < 1 \) and \( x \geq 0 \) one has \((1 + x)^{\epsilon^{-1}} \ln(1 + x) \leq [(1 - \epsilon) e]^{-1} \), the required uniform bounds follow: For all \( t_1 \geq 0 \) and \( \varrho \) large have \( \text{lhs} \leq |4\zeta_2/(e \zeta_1)| M_\varrho/\varrho \). Likewise for given \( \varrho_0 > 0 \) and \( t_1 \) large have \( t_1^* \times \text{lhs} \leq |4\zeta_2/((1 - \epsilon) e \zeta_1)| M_{t_1}/\varrho_0 \), for all \( \varrho > \varrho_0 \).

For \( k \geq 3 \) we prepare the bounds\(^\dagger \)

\[
\left| F_{\text{hom}}(\varrho - \zeta_1 s, t - s) \right| \leq \frac{M_{\varrho+t_1}}{t_1} \ln \left( 1 + \frac{t_1}{\varrho} \right),
\]

\[
\int_0^t ds \left| P_k(\varrho, s) \right| \leq \frac{C}{\varrho^k} \left[ 1 - \left( \frac{\varrho}{\varrho + t_1} \right)^{k-1} \right], \quad k \geq 3,
\]

\[
\int_0^t ds \left| P_k(\varrho, s) \right| \leq \begin{cases} 
\frac{C_1}{\varrho^2} \left[ 1 + C_2 \ln \left( 1 + \frac{t_1}{\varrho} \right) \right], & k = 3, \\
\frac{C}{\varrho^{k-1}}, & k \geq 4.
\end{cases}
\]

A simple computation then gives

\[
\int_0^t ds \left| F_k[S_{\text{hom}}](\varrho, t; s) \right| \leq \frac{M_{\varrho+t_1}}{\varrho + t_1} \frac{C_1}{\varrho^{k-2}}
\]

\[
+ \frac{M_{\varrho+t_1}}{t_1} \frac{C_2}{\varrho^{k-2}} \ln \left( 1 + \frac{t_1}{\varrho} \right) \times \begin{cases} 
1 + C_3 \ln \left( 1 + \frac{t_1}{\varrho} \right), & k = 3, \\
1 & k \geq 4.
\end{cases}
\]

\( ^\dagger \)We use \( C, C_1, C_2, \ldots \), as placeholders for positive \( k \)-dependent constants; in different formulas usually different constants appear.
From here the required uniform bounds follow. This concludes the proof of Lemma C1.

Having the homogeneous part in (C.7) under control one can bound the inhomogeneous part with an inductive argument.

**Lemma C2:** For \( l \geq 2 \) let \( \mathfrak{s}_l^{\text{inh}} \) be the inhomogeneous part of the solution recursively generated by Eq. (4.10). Then there exists \( 0 < \epsilon < 1 \) such that

\[
\forall \epsilon > 0 \; \; \exists C > 0 \; \; \text{for} \; \; t \rightarrow \infty \; \text{uniformly in} \; \; \rho \; \; \text{for} \; \; \rho \; \text{bounded away from zero.}
\]

**Proof.** For \( l = 2 \) the result follows from Eq. (C.12) with \( \mathfrak{s}_1^{\text{hom}} = \mathfrak{s}_1 \). For \( l \geq 3 \) we use Eq. (C.7) where the homogeneous part is taken care of by Lemma 1 and proceed by induction. Assuming that \( \mathfrak{s}_2^{\text{inh}}, \ldots, \mathfrak{s}_{l-1}^{\text{inh}} \) are already known to satisfy Eq. (C.15) we have to bound \( \int_0^t ds |F_k[\mathfrak{s}_l^{\text{inh}}]_k] \) in Eq. (C.7). For \( k = 2, \ldots, l-1 \), define

\[
M_k(\rho, t) = \max_{0 \leq s \leq t} \left\{ \left| \mathfrak{s}_k^{\text{inh}}(\rho - \zeta_1 s, t - s) \right|, \left| u \partial_s \mathfrak{s}_k^{\text{inh}}(\rho - \zeta_1 s, t - s) \right| \right\} \quad \text{(C.16)}
\]

By the induction hypothesis \( M_k(\rho, t) \) is finite for all \( \rho, t > 0 \), vanishes for \( \rho \rightarrow \infty \) uniformly in \( t \), and satisfies \( t^\epsilon M_k(\rho, t) \rightarrow 0 \), as \( t \rightarrow \infty \) uniformly in \( \rho \) for all \( \rho \) bounded away from zero. This stronger induction hypothesis now makes the rest of the argument straightforward.

For \( k = 2, \ldots, l-1 \), one verifies

\[
|F_k[\mathfrak{s}_l^{\text{inh}}]_k] \leq M_{l-k}(\rho, t) \left\{ \rho |P_{k+1}(\rho, s)| + \frac{|\alpha_k| + |\beta_k|}{\rho^{k-1} \rho - \zeta_1 s} \right\} \quad \text{(C.17)}
\]

Using the last Eq. in (C.13) this yields

\[
\int_0^t ds |F_k[\mathfrak{s}_l^{\text{inh}}]_k]_k| \leq M_{l-k}(\rho, t) C_1 \rho^{k-1} \left\{ \begin{array}{ll} 1 & \quad k = 2, \\ C_2 \ln \left( 1 + \frac{t}{\rho} \right), & \quad k \geq 3 \end{array} \right. \quad \text{(C.18)}
\]

From here the required uniform bounds readily follow, completing the proof of Lemma C2. Lemmas C1, C2 and Eq. (C.6) imply the theorem.
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