Spectral asymptotics of a strong $\delta'$ interaction on a planar loop

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Abstract
We consider a generalized Schrödinger operator in $L^2(\mathbb{R}^2)$ with an attractive strongly singular interaction of $\delta'$ type characterized by the coupling parameter $\beta > 0$ and supported by a $C^4$ smooth closed curve $\Gamma$ of length $L$ without self-intersections. It is shown that in the strong-coupling limit, $\beta \to 0_+$, the number of eigenvalues behaves as $\frac{L}{\beta} + O(|\ln \beta|)$, and furthermore, that the asymptotic behavior of the $j$th eigenvalue in the same limit is $\frac{L}{\beta^2} + \mu_j + O(\beta |\ln \beta|)$, where $\mu_j$ is the $j$th eigenvalue of the Schrödinger operator on $L^2(0, L)$ with periodic boundary conditions and the potential $-\frac{1}{4}\gamma^2$, where $\gamma$ is the signed curvature of $\Gamma$.

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1. Introduction

Schrödinger operators with singular interactions supported by manifolds of a lower dimension have been studied for several decades beginning with the early works [Ku78, BT92]. In recent years they have attracted attention as a model of a quantum particle confined to sets of nontrivial geometry and a possible alternative to usual quantum graphs [BK13], having two advantages over the latter. The first is that they lack the abundance of free parameters associated with the vertex coupling. The second, perhaps physically more important, is that the confinement is not strict and certain tunneling between parts of the graph is allowed. One usually speaks about ‘leaky’ quantum graphs and describes them using Hamiltonians which can be formally written as $-\Delta - \alpha \delta(\cdot - \Gamma)$, $\alpha > 0$, where $\Gamma$ is the support of the attractive singular interaction. A discussion of such operators and a survey of their properties can be found in [Ex08].

One can think of the singular interaction as a $\delta$ potential in the direction perpendicular to $\Gamma$, at least at the points where the manifold supporting the interaction is smooth. If the codimension
of $\Gamma$ is one, however, there are other singular interactions which can be considered, a prime example being the one coming from the one-dimensional $\delta'$ interaction \cite{AGH05}, that is, operators which can be formally written as
\[ H = -\Delta - \beta^{-1}\delta'(-\Gamma). \]

The formal expression has to be taken with a substantial pinch of salt, of course, because in contrast to the $\delta$ interaction, which can be approximated by naturally scaled regular potentials, the problem of approximating $\delta'$ is considerably more complicated—see \cite{Se86, CS98, ENZ01} and also \cite{CAZ03+, GH10}. What is important for our present purpose, however, is that irrespective of the meaning of such an interaction, there is a mathematically sound way of defining the above operator through boundary conditions, and moreover, one can also specify it using the associated quadratic form \cite{BLL13}.

Apart from the definition, one is naturally interested in spectral properties of such operators, in particular, in relation to the geometry of $\Gamma$. In the case of the $\delta$-type singular interaction, we know, for instance, that $\Gamma$ in the form of a broken or bent line gives rise to a nontrivial discrete spectrum \cite{Ei01} and a similar result can also be proven for the $\delta'$ interaction \cite{BEI13}. In this paper we want to demonstrate another manifestation of the relation between eigenvalues of $H$ and the shape of $\Gamma$. It is inspired by \cite{EY02} in which it was shown how the eigenvalues coming from a $\delta$ interaction supported by a $C^4$ Jordan curve $\Gamma$ behave in the strong-coupling regime, $\sigma \to \infty$, namely, that after a renormalization consisting of subtracting the $\Gamma$-independent divergent term they are in the leading order given by the respective eigenvalue of a one-dimensional Schrödinger operator with a potential determined by the curvature of $\Gamma$.

Here we are going to show that in the $\delta'$ case, where the strong-coupling limit is $\beta \to 0_+$, we have an analogous result, namely that the asymptotic expansion of the eigenvalues starts from a $\Gamma$-independent divergent term followed by the appropriate eigenvalues of a one-dimensional Schrödinger operator, the same as in the $\delta$ case. We will also be able to derive an asymptotic expression for the number of eigenvalues dominated by a natural Weyl-type term.

In the next section we state the problem properly and formulate the indicated results; the next two sections are devoted to the proofs. The technique is similar to that of \cite{EY02}, however, the argument is slightly more complicated because the present form of the associated quadratic form does not allow one to estimate the operator in question using operators with separated variables. In conclusion, we shall comment briefly on possible extensions of the results.

2. Formulation of the problem and main results

We consider a closed curve $\Gamma$ without self-intersections identified with the graph of
\[ \Gamma : [0, L] \to \mathbb{R}^2, \quad s \mapsto (\Gamma_1(s), \Gamma_2(s)), \]
with the component functions $\Gamma_1, \Gamma_2 \in C^4(\mathbb{R})$. We assume conventionally that the curve is parameterized by its arc length, in other words, $\Gamma_1^2 + \Gamma_2^2 = 1$. The operator we are interested in acts as the Laplacian outside the interaction support,
\[ (H_\beta \psi)(x) = -\nabla \psi(x) \]
for $x \in \mathbb{R}^2 \setminus \Gamma$, and its domain is $\mathcal{D}(H_\beta) = \{ \psi \in H^2(\mathbb{R}^2 \setminus \Gamma) \mid \partial_{n_\gamma} \psi(x) = \partial_{-n_\gamma} \psi(x) = \psi'(x)|_{\partial \Gamma} - n_\gamma \psi(x)|_{\partial \Gamma} = \psi(x)|_{\partial \Gamma} - \psi(x)|_{\partial \Gamma}, \partial \Gamma \text{ normal to } \Gamma, \text{ which for definitiveness}
\]

4 It is important to bear in mind that $\delta'$ is not approximated by squeezed potentials of zero mean \cite{CAZ03+, GH10} which illustrates that the name, invented originally by Grossmann, Høegh-Krohn, and Mekhkhout, is unfortunate and can lead a reader unfamiliar with the concept to false conclusions. Some authors proposed alternative terms, see e.g. \cite{CS99}, but the name stuck and we shall use it, keeping in mind that $\delta'$ is not a distributional potential.
is supposed to be the outer one, and $\psi(x)|_{\partial_\Gamma}$ are the traces of the function $\psi$ in the regions separated by the curve. The quadratic form associated with this operator is well known [BLL13, proposition 3.15]; it is bounded from below for any $\beta > 0$. In order to write it, we employ the locally orthogonal curvilinear coordinates $(s, u)$ in the vicinity of the curve introduced in relation (3.1) below. With an abuse of notation we write the value of a function $\psi \in H^1(\mathbb{R}^3 \setminus \Gamma)$ as $\psi(s, u)$; then we have

$$h_\beta[\psi] = \|\nabla \psi\|^2 - \beta^{-1} \int_\Gamma |\psi(s, 0_+) - \psi(s, 0_-)|^2 \, ds.$$  

To state our main theorem we introduce the following operator:

$$S = -\frac{d^2}{ds^2} - \frac{1}{4} \gamma(s)^2,$$

where $\gamma$ denotes the signed curvature of the loop, $\gamma(s) := (\Gamma_1'\Gamma_2' - \Gamma_1'\Gamma_2')(s)$. The domain of this operator is $\mathcal{D}(S) = \{\psi \in H^2(0, L) \mid \psi(0) = \psi(L), \psi'(0) = \psi'(L)\}$. We denote by $\mu_j$ the $j$th eigenvalue of $S$ with the multiplicity taken into account.

**Theorem 2.1.** One has $\sigma_{\text{ess}}(H_\beta) = [0, \infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that

$$\#\sigma_{\text{disc}}(H_\beta) \geq n$$

holds for $\beta \in (0, \beta_n)$.

For any such $\beta$ we denote by $\lambda_j(\beta)$ the $j$th eigenvalue of $H_\beta$, again counted with its multiplicity. Then the asymptotic expansions

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + O(\beta \ln \beta) \quad j = 1, \ldots, n,$$

are valid in the limit $\beta \to 0_+$. The error term here depends on $j$ and the eigenvalues of the two operators are numbered in ascending order.

**Theorem 2.2.** The counting function $\beta \mapsto \#\sigma_{\text{disc}}(H_\beta)$ admits the asymptotic expansion

$$\#\sigma_{\text{disc}}(H_\beta) = \frac{2L}{\pi \beta} + O(|\ln \beta|) \quad \text{as} \quad \beta \to 0_+.$$

### 3. Proof of Theorem 2.1

The essential spectrum of $H_\beta$ is found in [BLL13, theorem 3.16]. To prove the claim about the discrete one we first need a few auxiliary results. To begin with, we introduce locally orthogonal curvilinear coordinates $s$ and $u$ which allow us to write points $(x_1, x_2)$ in the vicinity of the curve as

$$(x_1, x_2) = (\Gamma_1(s) - u \Gamma_2'(s), \Gamma_2(s) + u \Gamma_1'(s)).$$

Since $\Gamma$ is supposed to be a $C^4$ smooth closed Jordan curve, it is not difficult to establish that the map (3.1) is injective for all $u$ small enough; for a detailed proof see [EY02].

We choose a strip neighborhood $\Omega_a := \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma) < a\}$ of $\Gamma$ with the half-width $a$ small enough to ensure the injectivity of (3.1) on $\Omega_a$, and use bracketing to get a two-sided estimate of the operator $H_\beta$ by imposing the Dirichlet and Neumann conditions at the boundary of $\Omega_a$, i.e.

$$H_N(\beta) \preceq H_\beta \preceq H_D(\beta),$$

where both the estimating operators correspond to the same differential expression and $\mathcal{D}(H_N(\beta)) = \{\psi \in \mathcal{D}(H_\beta) \mid \partial_n \psi(s, a) = \partial_n \psi(s, -a) = 0\}$, while the other is $\mathcal{D}(H_D(\beta)) = \{\psi \in \mathcal{D}(H_\beta) \mid \psi(s, a) = \psi(s, -a) = 0\}$. The operators $H_D(\beta)$ and $H_N(\beta)$ are obviously direct sums of operators corresponding to the parts of the plane separated by the
boundary conditions, and since their parts referring to $\mathbb{R}^2 \setminus \Omega_\alpha$ are positive, we can neglect them when considering the discrete spectrum. The parts of $H_N(\beta)$ and $H_D(\beta)$ referring to the strip $\Omega_\alpha$ are associated with the following quadratic forms,

$$h_{N,\beta}[f] = \|\nabla f\|^2 - \beta^{-1} \int_\Gamma \left| f(s, 0_+) - f(s, 0_-) \right|^2 \mathrm{d}s,$$

$$h_{D,\beta}[f] = \|\nabla f\|^2 - \beta^{-1} \int_\Gamma \left| f(s, 0) - f(s, 0_-) \right|^2 \mathrm{d}s,$$

respectively, the former being defined on $H^1(\Omega_\alpha \setminus \Gamma)$, the latter on $H^1_0(\Omega_\alpha \setminus \Gamma)$ understood as a set of functions which are locally $H^1$ and vanish at the boundary of $\Omega_\alpha$. Our first task is to rewrite these forms in terms of the curvilinear coordinates $s$ and $u$.

**Lemma 3.1.** Quadratic forms $h_{N,\beta}$, $h_{D,\beta}$ are unitarily equivalent to quadratic forms $q_{N,\beta}$ and $q_{D,\beta}$ which can be written as

$$q_{D}[f] = \left\| \frac{\partial_x f}{g} \right\|^2 + \left\| \frac{\partial_y f}{g} \right\|^2 + (f, Vf) - \beta^{-1} \int_0^L \left( f(s, 0_+) - f(s, 0_-) \right)^2 \mathrm{d}s$$

$$+ \frac{1}{2} \int_0^L \gamma(s) \left( |f(s, 0_+)|^2 - |f(s, 0_-)|^2 \right) \mathrm{d}s$$

$$q_{N}[f] = q_{D}[f] - \int_0^L \gamma(s) \left( \frac{1}{2(1 + ay(s))} |f(s, 0)|^2 \right) \mathrm{d}s$$

defined on $H^1_0((0, L) \times ((-\alpha, 0) \cup (0, \alpha)))$ and $H^1((0, L) \times ((-\alpha, 0) \cup (0, \alpha)))$, respectively, with periodic boundary conditions in the variable $s$. The geometrically induced potential in these forms is given by $V = \frac{\alpha y}{2\pi} - \frac{5(\alpha y')^2}{4\pi} - \frac{\gamma^2}{4\pi}$ with $g(s, u) : = 1 + uy(s)$, and we employ here the conventional shorthands, $\partial_s = \frac{\mathrm{d}}{\mathrm{d}s}$ etc.

**Proof.** We express $\partial_x$ and $\partial_y$ as linear combinations of $\partial_{s_1}$ and $\partial_{s_2}$ with the coefficients $\partial_{s_1} x_1 = \Gamma'_1 - u\Gamma''_2$, $\partial_{s_1} x_2 = \Gamma'_2 + u\Gamma''_1$, $\partial_{s_2} x_1 = \Gamma'_2$, and $\partial_{s_2} x_2 = -\Gamma'_1$. Working out the inverse coordinate transformation we get

$$\partial_{s_1} = g^{-1}(-\Gamma'_1 \partial_s - (\Gamma'_2 + u\Gamma''_1) \partial_u), \quad \partial_{s_2} = g^{-1}(-\Gamma'_2 \partial_s + (\Gamma'_1 - u\Gamma''_2) \partial_u),$$

where $g = (\Gamma'_1 - u\Gamma''_2)\Gamma'_1 + (\Gamma'_2 + u\Gamma''_1)\Gamma'_2 = 1 + uy$ because $\Gamma''_1 + \Gamma''_2 = 1$ holds by assumption. The last relation gives $\Gamma''_1 \Gamma''_2 = 0$ which in turn implies $\gamma^2 = \Gamma''_1 + \Gamma''_2$. Using these identities we can check by a direct computation [E89] that

$$q_{j,\beta}[Uf] = h_{j,\beta}[f]$$

where $(Uf)(s, u) : = \sqrt{1 + uy(s)} f(x_1(s, u), x_2(s, u))$ holds for $j = D, N$ and all functions $f \in \mathcal{D}(h_{j,\beta})$, which proves the claim. \hfill \Box

The forms $q_{N,\beta}$ and $q_{D,\beta}$ are still not easy to handle and we are going to replace the estimate (3.2) by a cruder one in terms of the following forms associated with operators. As for the upper bound, we introduce the quadratic form $q_{\alpha,\beta}^+$ acting as

$$q_{\alpha,\beta}^+[f] = \left\| \frac{\partial_x f}{\gamma} \right\|^2 + (1 - a\|y\|_{\infty})^{-2} \left\| \frac{\partial_y f}{\gamma} \right\|^2 + (f, V^{(+)} f)$$

$$- \beta^{-1} \int_0^L \left( f(s, 0) - f(s, 0_-) \right)^2 \mathrm{d}s + \frac{1}{2} \int_0^L \gamma(s) \left( |f(s, 0)|^2 - |f(s, 0_-)|^2 \right) \mathrm{d}s,$$

where $V^{(+)} : = \frac{\alpha y}{4(1 - a\|y\|_{\infty})} - \frac{g^2}{4(1 - a\|y\|_{\infty})}$ with $y_+ : = (y')_+$ and the positive (negative) part given by the standard convention, $f_+ : = \frac{1}{2}(|f| + f)$; we have neglected here the non-positive term $-\frac{1}{2}(a\gamma')^2 g^{-4}$. In contrast to the argument used in the $\delta$ interaction case [EY02] the
We are interested in the asymptotic behavior of the solution as $Q \rightarrow \infty$. J. Phys. A: Math. Theor. second part associated with the form

$$t_{a,\beta}(s) := \|f\|^2 - \frac{1}{\beta} |f(0_+) - f(0_-)|^2 + \frac{1}{2} \gamma(s)(|f(s, 0_+)|^2 - |f(s, 0_-)|^2)$$

is independent of $s$. The operator itself acts as $T_{a,\beta}(s)f = -f''$ with the domain

$$D(T_{a,\beta}(s)) = \{f \in H^2((-a, a) \setminus \{0\}) | f(a) = f(-a) = 0, f'(0_+) = f'(0_-) = -\beta^{-1}(f(0_+)-f(0_-)) + \frac{1}{2} \gamma(s)(f(0_+)+f(0_-)) \}.$$

**Lemma 3.2.** The operator $T_{a,\beta}(s)$ has exactly one negative eigenvalue $t_+ = -\kappa_+^2$ provided $\frac{a}{\beta} > 2$ which is independent of $s$ and such that

$$\kappa = \frac{2}{\beta} \tanh(\kappa a). \quad (3.3)$$

We are interested in the asymptotic behavior of the solution as $\beta \rightarrow 0_+$. Let us rewrite the condition as $\beta = \frac{2}{\kappa} \tanh(\kappa a)$; since the right-hand side is monotonous as a function of $\kappa > 0$ it is clear that there is at most one eigenvalue and that this happens if $\beta < 2a$. Furthermore, the right-hand side is less than $\frac{2}{\kappa}$ which means that $\kappa < 2\beta^{-1}$ and the inequality turns to an equality as $\beta \rightarrow 0$ and $\kappa \rightarrow \infty$. Next we employ the Taylor expansion

$$\frac{2}{\beta} \tanh(\kappa a) = \frac{2}{\beta} \left( 1 - 2 e^{-4\kappa a} + 2 e^{-4\kappa a} + O(e^{-6\kappa a}) \right),$$

and since $\kappa \rightarrow \frac{2}{\beta}$ as $\beta \rightarrow 0$, relation (3.3) yields the sought result.

Next we estimate in a similar fashion the operator with the Neumann boundary condition which we need to get a lower bound. To this aim we employ the quadratic form $q_{a,\beta}$, defined as

$$q_{a,\beta}[f] = \|\partial_s f\|^2 + (1 + a\|\gamma\|_\infty)^{-2}\|\partial_s f\|^2 + (f, \mathcal{V}^{(-)} f)$$

with

$$-\beta^{-1} \int_0^L |f(s, 0_+)-f(s, 0_-)|^2 \, ds - \frac{1}{2} \int_0^L \gamma(s)(|f(s, 0_+)|^2 - |f(s, 0_-)|^2) \, ds$$

$$-\|\gamma\|_\infty \int_0^L |f(s, a)|^2 \, ds - \|\gamma\|_\infty \int_0^L |f(s, -a)|^2 \, ds,$$

where $\mathcal{V}^{(-)} = -\frac{a}{2(1-|a|\|\gamma\|_\infty)} - \frac{\bar{V}(a\gamma)^2}{4(1-|a|\|\gamma\|_\infty)^2} - \frac{\bar{V}(\gamma)^2}{4(1-|a|\|\gamma\|_\infty)^2}$. As before, the operator associated with the quadratic form can be written as $Q_{a,\beta}^* = U_a \otimes I + \int_{[0,L]} T_{a,\beta}^*(s) \, ds$, where the operator $T_{a,\beta}^*(s)$ referring to the transverse variable acts for any $s \in [0, L]$ as $T_{a,\beta}^*(s)f = -f''$ with the domain.
\[ \mathcal{D}(T_{a,\beta}(\kappa)) = \{ f \in H^2((-a, a) \setminus \{0\}) \mid \exists \gamma \|f\|_\infty^\infty f(\pm a) = f'(\pm a), \] 
\[ f'(0_+) = f'(0_+) = -\beta^{-1}(f(0_+) - f(0_-) + \frac{\gamma}{2}(s)(f(0_+) + f(0_-))). \] (3.4)

We are going to estimate the spectrum of \( T_{a,\beta}(\kappa) \) and check its independence of \( s \).

**Lemma 3.3.** The operator \( T_{a,\beta}(\kappa) \) has exactly one negative eigenvalue \( \kappa_\gamma = -\kappa_0^2 \) for any \( \beta \) small enough; the latter is independent of \( s \) and for \( \beta \to 0 \) we have 
\[ \kappa = \frac{2}{\beta} + \frac{4}{2 - \beta} \|\gamma\|_\infty^\infty + \mathcal{O}\left( -\frac{4}{\beta} \left( \frac{2}{2 + \beta} \|\gamma\|_\infty^\infty \right)^2 e^{-4a/\beta} \right). \]

**Proof.** The function satisfying \( f''(x) = -\kappa^2 f(x) \) for \( x \neq 0 \) together with the boundary conditions \( \exists \|\gamma\|_\infty^\infty f(\pm a) = f'(\pm a) \), which has its derivative continuous at \( x = 0 \), is of the form 
\[ f(x) = \begin{cases} A e^{\kappa x} + B e^{-\kappa x} & \text{if } x \in (-a, 0) \\ C e^{\kappa x} + D e^{-\kappa x} & \text{if } x \in (0, a) \end{cases} \]
The constant \( A \) is arbitrary, while for the others the requirements imply \( B = A Z e^{-2\kappa a} \) with \( Z := \frac{\kappa - \|\gamma\|_\infty^\infty}{\kappa + \|\gamma\|_\infty^\infty} \) and \( D = -A, C = -B \). The remaining property from (3.4) leads to 
\[ \kappa(A - B) = \frac{1}{\beta}(A + B - C - D) + \frac{1}{2}(s)(A + B + C + D), \]
and since the last term vanishes we can rewrite the spectral condition as 
\[ \kappa = \frac{2}{\beta} + Z e^{-2\kappa a}. \]

As before, we are interested in the regime \( \beta \to 0_+ \). Note that as long as \( Z > 0 \) we have \( \kappa > 2\beta^{-1} \), hence \( \kappa \) is large and \( \xi = Ze^{-2\kappa a} \) is small and the expansion 
\[ \kappa = \frac{2}{\beta} + \xi = \frac{2}{\beta}(1 + \xi)(1 + \xi + \xi^2 + \mathcal{O}(\xi^3)) \]
yields the stated behavior of \( \kappa \) as \( \beta \to 0_+ \). Since we are interested in the strong-coupling situation, we may assume \( Z > 0 \) without loss of generality. This assumption is satisfied for \( 2\beta^{-1} > \|\gamma\|_\infty^\infty \), and the uniqueness of the eigenvalue is a consequence of the above spectral condition and the monotonicity of the function \( \kappa \mapsto \frac{1}{\beta} + \xi \) which can be checked by a direct computation. \( \square \)

Next we estimate the eigenvalues of the operators \( U^\pm_\kappa \), referring to the longitudinal part in the expressions for \( Q^k_{a,\beta} \) in a similar way to [EY02]; they correspond to the second and third terms in the definition of the quadratic forms \( \bar{q}^{\pm}_{a,\beta} \).

**Lemma 3.4.** There is a positive \( C \) independent of \( a \) and \( j \) such that 
\[ |\mu_j^\pm(a) - \mu_j| \leq Ca^2 \]
holds for \( j \in \mathbb{N} \) and \( 0 < a < \frac{1}{2\kappa + 1} \), where \( \mu_j^\pm(a) \) are the eigenvalues of \( U^\pm_a \), respectively, with the multiplicity taken into account.

**Proof.** We employ the operator \( S_0 = -\partial_x^2 \) with the periodic boundary conditions, i.e. the domain \( \mathcal{D}(S_0) = \{ f \in L^2((0, L)) \mid f(0) = f(L), f'(0) = f'(L) \} \); its eigenvalues, counting multiplicity, are \( 4\pi^2 j^2 \pi^2 \), \( j = 1, 2, \ldots \). where \([\cdot]\) as usual denotes the entire part. Its difference from our comparison operator (2.1) on \( L^2(0, L) \) is easily estimated, 
\[ \|S - S_0\| \leq \frac{1}{2} \|\gamma\|_\infty^\infty. \]
and consequently, by the min–max principle we have
\[ |\mu_j - 4 \left( \frac{f_j^2}{\gamma} \right) | \leq \frac{1}{4} \|\gamma\|_\infty^2 \]  
for \( j \in \mathbb{N} \). Next we can use another simple estimate,
\[ U^+_a - \frac{1}{(1-a\|\gamma\|_\infty^2)^2}S = -\frac{ay''_+}{2(1-a\|\gamma\|_\infty^4)^3} - \frac{4a^2\|\gamma\|_\infty^2}{4(1-a\|\gamma\|_\infty^2)^2}, \]
and since the last two terms combine to \( a\|\gamma\|_\infty\gamma^2(1-a^2\|\gamma\|_\infty^2)^{-2} \), we infer that
\[ \left| \mu_j^+ - \frac{\mu_j}{(1-a\|\gamma\|_\infty^2)^2} \right| \leq c_0a \]
holds for some \( c_0 > 0 \) and any \( j \in \mathbb{N} \). Combining now (3.5) and (3.6) we get
\[ |\mu_j^+ - \mu_j| \leq \left| \mu_j^+ - \frac{\mu_j}{(1-a\|\gamma\|_\infty^2)^2} \right| + |\mu_j| \cdot \frac{1 - (1-a\|\gamma\|_\infty^2)^2}{(1-a\|\gamma\|_\infty^2)^2} \]
with suitable constants. The second inequality is checked in a similar way: we use
\[ U^-_a - \frac{1}{(1+a\|\gamma\|_\infty^2)^2}S = -\frac{ay''_+}{2(1-a\|\gamma\|_\infty^4)^3} - \frac{5a^2\|\gamma\|_\infty^2}{4(1-a\|\gamma\|_\infty^2)^2} \]
which implies
\[ \left\| U^-_a - \frac{1}{(1+a\|\gamma\|_\infty^2)^2}S \right\| \leq \tilde{c}_0a + \tilde{c}_1a^2 \leq c_2a, \]
where in the second inequality we employed the fact that \( a \) is bounded. With help of the min–max principle we then get
\[ \left| \mu_j^- - \frac{\mu_j}{(1+a\|\gamma\|_\infty^2)^2} \right| \leq c_2a, \]
hence finally we arrive at the inequality
\[ |\mu_j^- - \mu_j| \leq c_2a + |\mu_j| \frac{1 - (1+a\|\gamma\|_\infty^2)^2}{(1+a\|\gamma\|_\infty^2)^2} \leq c_2a + c_3a|\mu_j| \leq Ca^2 \]
valid for a suitable \( C \) which completes the proof. \( \square \)

Now we are ready to prove our first main result.

We define \( a(\beta) = -\frac{1}{2}\beta \ln \beta \) and denote the eigenvalues of the operators \( T_{a(\beta),\beta}^\pm \) as \( t_{\pm,\beta} \), respectively, taking their multiplicities into account. From Lemmata 3.2 and 3.3 we know that \( t^-_{\pm,\beta} = t^-_\beta \) for small enough \( \beta \), while \( t^+_{\pm,\beta} \geq 0 \) holds for \( j > 1 \). Collecting the estimates worked out above we have
\[ Q_{a(\beta),\beta}^- = U^-_{a(\beta)} \otimes I + \int_{(0,L)} T^-_{a(\beta),\beta}(s) \, ds \leq H_N(\beta) \leq H_\beta \]
\[ \leq H_\beta(\beta) \leq U^-_{a(\beta)} \otimes I + \int_{(0,L)} T^-_{a(\beta),\beta}(s) \, ds = Q_{a(\beta),\beta}^+ \]  
and the eigenvalues of the operators \( Q_{a(\beta),\beta}^\pm \) between which we squeeze our singular Schrödinger operator \( H_\beta \) are naturally \( t^\pm_{\pm,\beta} + \mu_j^\pm(a(\beta)) \) with \( k, j \in \mathbb{N} \). Those with \( k \geq 2 \) and \( j \in \mathbb{N} \) are uniformly bounded from below in view of the inequality
\[ t^\pm_{\pm,\beta} + \mu_j^\pm(a(\beta)) \geq \mu_1 + O(-\beta \ln \beta), \]

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hence we can focus on \( k = 1 \) only. For \( j \in \mathbb{N} \) we denote
\[
\omega^j_{\pm, \beta} = t^j_{\pm, \beta} + \mu_j^+ (\alpha (\beta)).
\]
With our choice of \( \alpha (\beta) \) we have \( e^{-4a_{\alpha}} = \beta^3 \), so from the above lemma we get \( \kappa_{\pm} = \frac{2}{\beta} + O(\beta) \) and \( \mu_j^+ (\alpha (\beta)) \) differ from \( \mu_j \) by \( O(-\beta^2 \ln \beta) \); putting these estimates together we can conclude that
\[
\omega^j_{\pm, \beta} = -\frac{4}{\beta^2} + \mu_j + O(-\beta \ln \beta) \quad \text{as} \quad \beta \to 0^+_+
\]
with the error term in general dependent on \( j \). Combining (3.8) and (3.9) we can conclude that to any \( n \in \mathbb{N} \) there is a \( \beta(n) > 0 \) such that
\[
\omega^a_{\pm, \beta} \leq 0, \quad \omega^b_{\pm, \beta} < t^b_{\pm, \beta} + \mu_j^+ (\alpha (\beta)) \quad \text{and} \quad \omega^c_{\pm, \beta} < t^c_{\pm, \beta} + \mu_j^+ (\alpha (\beta))
\]
holds for \( \beta \leq \beta(n), k \geq 2, \) and \( j \geq 1 \). Hence the \( j \)th eigenvalue of \( Q_{\alpha_1 (\beta), \beta}^j \), counting multiplicity, is \( \omega^a_{\pm, \beta} \) for all \( j \leq n \) and \( \beta \leq \beta(n) \). Furthermore, for \( \beta \leq \beta(n) \) we denote \( \xi^j (\beta) \) and \( \xi^j (\beta) \) as the \( j \)th eigenvalue of \( H_D (\beta) \) and \( H_N (\beta) \), respectively; then from (3.7) and the min–max principle we obtain
\[
\omega^a_{\pm, \beta} \leq \xi^j (\beta), \quad \xi^j (\beta) \leq \omega^a_{\pm, \beta}
\]
for \( j = 1, 2, \ldots, n \), which in particular implies \( \xi_n^a (\beta) < 0 \). Using the min–max principle once again we conclude that \( H_D \) has at least \( n \) eigenvalues in the interval \( (-\infty, \xi_n^a (\beta)) \) and for any \( 1 \leq j \leq n \) we have \( \xi^j (\beta) \leq \lambda_j \leq \xi^j (\beta) \) which completes the proof.

4. Proof of theorem 2.2

For a self-adjoint operator \( A \) with \( \inf \sigma_{\text{ess}} (A) = 0 \) we put \( N^- (A) := \# \{ \sigma_d (A) \cap (-\infty, 0) \} \). In view of (3.7) the eigenvalue number of \( H_D \) can be estimated as
\[
N^- (Q_{\alpha_1 (\beta), \beta}^j) \leq N^- (H_D (\beta)) \leq N^- (H_D (\beta)) \leq N^- (Q_{\alpha_1 (\beta), \beta}^j)
\]
In order to use this estimate we define
\[
K^\pm_\beta = \{ j \in \mathbb{N} | \omega^j_{\pm, \beta} < 0 \}
\]
and derive the following asymptotic expansions of these quantities.

Lemma 4.1. In the strong-coupling limit, \( \beta \to 0^+_+ \), we have
\[
\# K^\pm_\beta = \frac{2L}{\pi \beta} + O(|\ln \beta|).
\]

Proof. We choose \( K \) such that \( \beta^{-1} > K > 0 \) and \( (\beta^{-1} - K)^2 < \beta^{-2} - 4\beta - 16^{-1} \| \gamma \|_{\infty}^2 \); this can be obviously done for all sufficiently small \( \beta \). With the preceding proof in mind we can write
\[
K^\pm_\beta = \{ j \in \mathbb{N} | t^j_{\pm, \beta} + \mu_j^+ (\alpha (\beta)) < 0 \}.
\]
Lemma 3.2 allows us to make the following estimate,
\[
K^+_\beta \supset \left\{ j \in \mathbb{N} | \mu_j + Ca (\beta) j^2 < \frac{4}{\beta^2} - \frac{16}{\beta^2} e^{-4a_{\alpha} \beta / \beta} = \frac{4}{\beta^2} - 16 \beta \right\};
\]
using further (3.5) and the indicated choice of $K$ we infer that

$$K^{+}_\beta \supset \begin{cases} j \in \mathbb{N} | 4 \left( \frac{j^2 \pi^2}{L^2} + C a(\beta) j^2 \right) < \frac{4}{\beta^2} - 16 \beta - \frac{1}{4} \|\gamma\|_\infty^2 \end{cases}$$

$$\supset \begin{cases} j \in \mathbb{N} | j^2 \pi^2 - \frac{3}{4} C \beta \ln \beta j^2 < 4 \left( \frac{1}{\beta} - K \right)^2 \end{cases}$$

$$\supset \begin{cases} j \in \mathbb{N} | j < 2 \left( \frac{1}{\beta} - K \right) \left( \frac{\pi^2}{L^2} - \frac{3}{4} C \beta \ln \beta \right)^{-1/2} \end{cases}.$$  

We employ the Taylor expansion $(M + x)^{-1/2} = M^{-1/2} - \frac{1}{2} x M^{-3/2} + O(x^2)$; since we are interested in the asymptotics $\beta \to 0_+$, we rewrite the right-hand side of the last inequality as

$$2 \left( \frac{1}{\beta} - K \right) \left( \frac{\pi^2}{L^2} - \frac{3}{4} C \beta \ln \beta \right)^{-1/2} \approx 2 \left( \frac{1}{\beta} - K \right) \left[ L \pi + \frac{3}{8} C \beta \ln \beta \left( \frac{L}{\pi} \right)^3 \right],$$

which allows us to infer that

$$\#K^+_\beta \geq \frac{2L}{\pi \beta} + O(\|\ln \beta\|) \quad (4.3)$$

holds as $\beta \to 0_+$. In a similar way we estimate $\#K^-_\beta$. First we choose a number $K'$ satisfying $0 < K' < (4\beta + \frac{4\|\ln \beta\|}{\beta^2})^{1/2}$ and note that $\frac{1}{\beta^2} + 4\beta + \frac{4\|\ln \beta\|}{\beta^2} < (\frac{1}{\beta} + K')^2$. Then we have

$$K^-_\beta = \begin{cases} j \in \mathbb{N} | \mu_j - C a(\beta) j^2 < 0 \end{cases}$$

$$\supset \begin{cases} j \in \mathbb{N} | \mu_j - C a(\beta) j^2 < 4 \beta^2 + \frac{16}{\beta^2} - 2 + \beta \|\gamma\|_\infty e^{-4 a(\beta)/\beta} \end{cases}$$

$$\supset \begin{cases} j \in \mathbb{N} | \mu_j - 4 C \beta \ln \beta j^2 < 4 \beta^2 + 16 \beta \frac{2 - \beta \|\gamma\|_\infty}{2 + \beta \|\gamma\|_\infty} \end{cases}.$$  

With the help of the fact that $2(j - 1) > j$ for $j > 1$ we further have

$$K^-_\beta \subset \{ j \geq 2 \} \left( \frac{(j - 1) \pi}{L} \right)^2 + \frac{3}{4} C \beta \ln \beta (j - 1)^2 < \frac{4}{\beta^2} + 16 \beta + \frac{\|\gamma\|_\infty^2}{4} \right)$$

$$\subset \{ j \geq 2 \} \left( (j - 1)^2 < \left( \frac{4}{\beta^2} + 16 \beta + \frac{\|\gamma\|_\infty^2}{4} \right) \left( \frac{\pi}{L} \right)^2 + \frac{3}{4} C \beta \ln \beta \right)^{-1} \right)$$

$$\subset \{ j \geq 2 \} \left( j < 1 + 2 \frac{1}{\beta} + K' \right) \left( \frac{\pi}{L} \right)^2 + \frac{3}{4} C \beta \ln \beta \right)^{-1/2} \right).$$

Now we can estimate the expression on the right-hand side of the last inequality in the asymptotic regime $\beta \to 0_+$ as

$$2 \left( \frac{1}{\beta} + K' \right) \left( \frac{\pi}{L} \right)^2 + \frac{3}{4} C \beta \ln \beta \right)^{-1/2} \approx \frac{2L}{\pi \beta} + O(\|\ln \beta\|).$$

In combination with the above inclusions this leads to

$$\#K^-_\beta \leq \frac{2L}{\pi \beta} + O(\|\ln \beta\|) \quad (4.4)$$

as $\beta \to 0_+$. Finally, we know that $\mu_{j,\beta} < \mu_{j,\beta}$ which implies $K^+_\beta \subset K^-_\beta$, and this together with (4.3) and (4.4) concludes the proof.\[\square\]

We also need to estimate the second eigenvalue of the operators $T_{\sigma(\beta),\beta}(s)$.  

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Lemma 4.2. $T_{a,\beta}(s)$ with a fixed $s \in [0, L]$ has no eigenvalues in $[0, \min\left\{\frac{\|y\|_\infty}{2a}, \left(\frac{\pi}{2}\right)^2\right\}]$ provided $0 < \beta < 2a$.

Proof. Let us check first that zero is not an eigenvalue. The corresponding eigenfunction should have to be linear and the conditions $\mp\|y\|_\infty f(\pm a) = f'(\pm a)$ and $f'(0_\pm) = f'(0_\mp)$ would require $f(x) = \pm A(\mp\|y\|_\infty x + 1 + \|y\|_\infty a)$ for $\pm x \in (0, a)$, and as in lemma 3.3 the spectral condition would read $-\|y\|_\infty = \frac{2}{\beta}(1 + \|y\|_\infty a)$ which cannot be true because the right-hand side is positive. Furthermore, the spectral condition for an eigenvalue $k^2 > 0$ is found again as in lemma 3.3; after a simple calculation we find that it reads

$$\frac{1}{2} \beta = \frac{1}{\|y\|_\infty} \tan ka + k$$

The right-hand side can be estimated by $\frac{1+\|y\|_\infty}{\|y\|_\infty} k a$ provided that $ka < \frac{\pi}{2}$ and at the same time $\|y\|_\infty - k \tan ka > 0$; by finding the value for which this expression equals $\frac{1}{2} \beta$ we would obviously get a lower bound to $k$. Rewriting the condition as

$$-ka^2 = (1 + \|y\|_\infty a)^2 \frac{2}{\beta} - \|y\|_\infty$$

we see that the left-hand side is negative while the right-hand side is positive under our assumption, hence one has to determine the restriction coming from the condition $\|y\|_\infty - k \tan ka > 0$. In particular, for $ka < \frac{1}{2} \pi$ this is true provided $\|y\|_\infty - 2k^2 a > 0$, which means that the spectral problem has no solution if $k^2$ is smaller than either $\frac{\|y\|_\infty}{2a}$ or $\left(\frac{\pi}{2}\right)^2$, which concludes the argument.

Now we are ready to prove our second main result. We begin by showing that the relation

$$N^-(Q_{a,\beta}^-) = \#K^-_\beta$$

holds for any sufficiently small $\beta > 0$. We know that all the eigenvalues of $Q_{a,\beta}^-$ can be written as $\{t_{j,\beta}^- + \mu_j^- (a(\beta))\}_{j \in \mathbb{N}}$ with the multiplicity taken into account. From the previous lemma we have $t_{j,\beta}^- > \min\left\{\frac{\|y\|_\infty}{2a}, \left(\frac{\pi}{2}\right)^2\right\}$ which together with $|\mu_j^- (a) - \mu_j^-| \leq C a j^2$ implies the existence of a $\beta_0$ such that

$$t_{j,\beta}^- + \mu_j^- (a(\beta)) > 0$$

holds for $j > 1$, $k \geq 1$, and $\beta \in (0, \beta_0)$. This implies

$$N^- (Q_{a,\beta}^-) = \#\{(k, j) \in \mathbb{N}^2 | t_{j,\beta}^- + \mu_j^- (a(\beta)) < 0\}$$

$$= \#\{j \in \mathbb{N} | t_{j,\beta}^- + \mu_j^- (a(\beta)) < 0\} =: K^-_\beta,$$

i.e. the relation (4.5); combining it with (4.1) we obtain

$$\#K^-_\beta \leq \#\sigma (H_\beta) \leq N^- (Q_{a,\beta}^-) = \#K^-_\beta,$$

which by virtue of lemma 4.1 concludes the proof.

5. Concluding remarks

We have seen that, despite very different eigenfunctions, the $\delta'$ ‘leaky loops’ behave in the strong-coupling regime similarly to their $\delta$ counterparts: the number of negative eigenvalues is given in the leading order by a Weyl-type term, and the eigenvalues themselves are after a natural renormalization determined by the one-dimensional Schrödinger operator with the known curvature-induced potential.
The question is whether and how the current results can be extended. The bracketing technique we used would work for infinite smooth curves \( \Gamma \) without ends, provided that suitable regularity assumptions were imposed. If, on the other hand, the curve is finite or semi-infinite the situation becomes more complicated because one has to impose appropriate boundary conditions at the endpoints of the interval on which the comparison operator (2.1) is defined. One can modify the present argument to get an estimate on the number of eigenvalues because there those boundary conditions play no role, the counting functions in the Dirichlet and Neumann case differing by an \( O(1) \) term. For an eigenvalue position estimate, on the other hand, this is not sufficient and one conjectures that the Dirichlet comparison operator has to be used. For a two-dimensional open arc \( \Gamma \) supporting a \( \delta \) interaction this conjecture has recently been proved [EP12]; the argument is more complicated because one cannot use operators with separated variables. We believe that the same method could work in the \( \delta' \) case too—however, the question is not simple and we postpone discussing it to another paper.

On the other hand, finding the asymptotics in the case when \( \Gamma \) is not smooth, or even has branching points, represents a much harder problem and the answer is not known even in the \( \delta \) case, although some inspiration can be found in the squeezing limits of Dirichlet tubes; see, e.g., [CE07].

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