Global solutions to an initial boundary problem for the compressible 3D MHD equations with Navier-slip and perfectly conducting boundary conditions in exterior domains

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Abstract
An initial boundary value problem for compressible magnetohydrodynamics (MHD) is considered on an exterior domain (with the vanishing first Betti number) in $\mathbb{R}^3$ in this paper. The global existence of smooth solutions near a given constant state for compressible MHD with the boundary conditions of Navier-slip for the velocity filed and perfect conduction for the magnetic field is established. Moreover the explicit decay rate is given. In particular, the results obtained in this paper also imply the global existence of classical solutions for the full compressible Navier–Stokes equations with Navier-slip boundary conditions on exterior domains in three dimensions, which was not available in literature prior to the work in this paper, to the best of knowledge of the authors’.

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1. Introduction and main theorems

Magnetohydrodynamics (MHD) mainly investigates the dynamics of compressible quasineutrally ionized fluids under the influence of electromagnetic fields. It is well-known that the applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas. In this paper, the three-dimensional compressible full magnetohydrodynamic equations will be considered, which take the following form ([19, 24]):

\[
\begin{cases}
 \rho_t + \text{div}(\rho u) = 0, \\
 \rho(u_t + u \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \nabla p = \text{curl} H \times H, \\
 \partial_t(\rho e) + \text{div}(\rho eu) + p \text{div} u = \Psi : \nabla u + \text{div}(\kappa \nabla T) + \eta |\text{curl} H|^2, \\
 H_t - \eta \Delta H = \text{curl}(u \times H), \quad \text{div} H = 0,
\end{cases}
\]

(1.1)

where \(\rho, u = (u_1, u_2, u_3), H = (H_1, H_2, H_3), e, T\) denote the density, the velocity, the magnetic field, the internal energy and temperature, respectively. The viscosity coefficients \(\mu\) and \(\lambda\) of the fluid should satisfy

\[2\mu + 3\lambda > 0, \quad \mu > 0,
\]

due to physical considerations. The constant \(\kappa > 0\) is the heat conductivity, and the constant \(\eta > 0\) is the magnetic diffusion coefficient. The viscous stress tensor \(\Psi\) is given by

\[\Psi = \mu (\nabla u + \nabla u^T) + \lambda \text{div} u I.
\]

And \(\Psi : \nabla u\) denotes the scalar product of two matrices. A calculation gives that

\[\Psi : \nabla u = \lambda (\text{div} u)^2 + \sum_{i,j=1}^{3} \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 = \lambda (\text{div} u)^2 + 2\mu|S(u)|^2.
\]

For simplicity, we study the typical case with the equations of state:

\[e = c_v T, \quad p = R \rho T,
\]

where \(c_v\) and \(R\) are positive constants. In this case, we may write (1.1) as

\[c_v \rho (T_t + u \cdot \nabla T) + p \text{div} u = \kappa \Delta T + \Psi : \nabla u + \eta |\text{curl} H|^2.
\]

Let \(U\) be a simply connected bounded smooth domain in \(\mathbb{R}^3\), and \(\Omega \equiv \mathbb{R}^3 \setminus \overline{U}\) be the exterior domain. In this paper, we study the initial-boundary value problem of (1.1) in \(\Omega\) with the initial condition

\[(\rho, u, T, H)|_{t=0} = (\rho_0, u_0, T_0, H_0)(x), \quad \text{in } \Omega,
\]

(1.2)

where \(H_0\) satisfies the constraint \(\text{div} H_0 = 0\), and the boundary conditions:

\[u \cdot n = 0, \quad \text{curl} u \times n = 0, \quad \text{on } x \in \partial \Omega,
\]

(1.3)

\[H \cdot n = 0, \quad \text{curl} H \times n = 0, \quad \frac{\partial T}{\partial n} = 0, \quad \text{on } x \in \partial \Omega,
\]

(1.4)
where $n$ stands for the outward unit normal to $\partial \Omega$. Moreover, besides the compatibility condition of the initial data with the boundary condition, we also assume that the initial data satisfy the condition,

$$\rho_0, u_0, T_0, H_0)(x) \to (1, 0, 1, 0), \quad \text{as} \ |x| \to \infty. \quad (1.5)$$

The boundary condition (1.3) means the Navier-slip boundary conditions for the velocity field, conditions in (1.4) stand for the perfectly conducting boundary condition for the magnetic field, and Neumann boundary condition for the temperature.

The Navier-slip boundary conditions for the velocity field were first introduced by Navier in [30] and have been used in many applications, for instance, in the large eddy simulations of turbulent flows to compute the large eddies of a turbulent flow accurately by neglecting small flow structure. For this, the slip boundary conditions are more suitable than the Dirichlet boundary condition [16].

The system of compressible MHD equations have been studied extensively by physicists and mathematicians because of its physical importance, rich and complex phenomena and mathematical challenges. We mainly review the results on the global existence of solutions related to the main theme of this paper, for which one may refer to [7, 22, 35] for one-dimensional case and [14, 19, 20, 25, 26, 31, 39] for higher dimensions for example. The results for the global existence and large time behavior of solutions in 3D for the Cauchy problem, can be found, for instance, the classical solutions in [17, 25, 26, 31, 39]. While for the initial boundary value problem in bounded domains, the global variational weak solutions have been investigated in [14, 19, 20], concerned with the homogenous Dirichlet condition for the velocity.

In the presence of physical boundaries, there have been extensive studies on the global existence and large time behavior of solutions to the initial boundary value problems for compressible MHD equations in 3D for the homogenous Dirichlet condition for both the velocity filed $u$ and the magnetic field $H$, i.e., $u = 0$ and $H = 0$ on the boundary. It is more physically relevant to consider the case that the magnetic field is non-zero on the boundary. In this direction, we noticed that the global existence of classical solutions for 3D compressible isentropic MHD was proved in [4], a little bit before our work, in a bounded domain (see also [5, 13] for the related results in 2D and [3] for the related results for compressible isentropic Navier–Stokes equations in 3D).

For the full compressible Navier–Stokes equations in 3D with Navier-slip boundary conditions, there are only few results prior to the work in this paper on the global solutions for the initial boundary value problem for the Navier-slip boundary conditions in 3D, though results are available for some small physical parameter limits for local time, for instance, see the related zero viscosity limit [2, 29, 36–38] and [9, 18] for low Mach number limit.

Compared with the problem for 3D compressible isentropic MHD equations studied in [4] in a bounded domain which we noticed a little bit prior to the work in this paper, the difficulty in proving the global existence of classical solution for the initial boundary value problem of (1.1)–(1.4) in an exterior domain in $\mathbb{R}^3$ is due to the unboundedness of the exterior domain, for which the Poincare type inequality to use the $L^2$-norm of the derivatives to control the $L^2$-norm of the solution itself is not available. For example, in a bounded domain, the dissipation estimates (the $L^2(\Omega \times [0, \infty)$ space-time estimates) of the velocity and magnetic fields can be obtained via the corresponding estimates of the derivatives which follow from the dissipation of the viscosity and magnetic diffusion in the basic energy estimates. This is an important decay mechanism for the problem on a bounded domain for which various exponential decay estimates are obtained in [4]. However, for the problem on an exterior domain studied in the present paper, only algebraic decay can be expected in general. Another difference from the case studied in [4] is that we can handle the case of the full compressible MHD system with variable
entropy, while only isentropic case is studied in [4] for which possibly large oscillations and vacuum states (with small energy) are allowed. It should be noted that, for non-isentropic flows, even in the case without magnetic fields \((H = 0)\), there have been no global-in-time theory available for the problems with physical boundaries under the Navier-slip boundary conditions.

For the boundary conditions of Navier-slip for the velocity field and the perfect conduction of magnetic field, it is more suitable to use the \(L^2\)-norms of \(\text{div}\) and \(\text{curl}\) to control that of the derivatives of the velocity and magnetic fields. This control usually depends on the topology of the domain, which relates to the first Betti number. A key observation is that first Betti number of the exterior domain \(\Omega = \mathbb{R}^3 \setminus \overline{U}\) vanishes when \(U\) is a bounded simply connected open set. In this case, one can apply the refined \(\text{div–curl}\) estimate (see proposition 2.3) to obtain the dissipation estimates of \(\|\nabla u\|\) with the help of those for \(\|\text{curl} u\|\) and \(\|\text{div} u\|\). This observation is important in obtaining various dissipation estimates.

It should be noted that it is rather interesting and challenging to obtain the global existence and decay estimates of classical solutions of the initial boundary value problem for the boundary conditions (1.3)–(1.4), compared with the homogeneous Dirichlet boundary conditions \(u = H = 0\) on the boundary, various estimates on \(\text{div} u\), \(\text{curl} u\) and \(\text{curl} H\) are crucial for the problem investigated in this paper.

The main results of this paper are stated as follows.

**Theorem 1.1.** Suppose that the initial data satisfy the compatibility condition with the boundary conditions (1.3)–(1.4). There exists some constant \(\delta^* > 0\) such that for any \(\delta_1 \leq \delta^*\), if

\[
\|\rho_0 - 1\|_3 + \|(u_0, T_0 - 1, H_0)\|_3 \leq \delta_1,
\]

then the initial boundary value problem (1.1)–(1.4) admits a unique strong solution \((\rho, u, T, H)\) globally in time satisfying

\[
\rho - 1 \in C^0(0, \infty, H^2(\Omega)) \cap C^1(0, \infty, H^1(\Omega)),
\]

\[
(u, T - 1, H) \in C^0(0, \infty, H^3(\Omega)) \cap C^1(0, \infty, H^1(\Omega)),
\]

and

\[
\left(\|\rho - 1\|_3 + \|\rho - 1\|_2^2 + \|(\rho, u, H, T)\|_3^2\right)(t)
\]

\[
+ \int_0^t \left(\|\nabla u\|_3^2 + \|\nabla T\|_2^2 + \|\text{curl} H\|_3^2 + \|\text{curl} u\|_2^2 + \|\text{curl} H\|_2^2\right)(s)\,ds
\]

\[
\leq C\left(\|\rho - 1\|_2^2 + \|(\rho, u, H, T)\|_3^2\right)(0)
\]

where \(C\) is a positive constant independent of \(t\).

**Remark 1.** The compatibility condition of the initial data with the boundary conditions is standard to ensure the regularity of the solutions. The data

\[
\|(\rho_0, u_0, H_0, T_0)\|_1^2(0)
\]

is determined by the initial data \((\rho_0, u_0, T_0, H_0)\) through the equation (1.1).

For the initial data close to the constant state \((1, 0, 1, 0)\) in higher-order Sobolev norm, we can improve the regularity of solutions in theorem 1.1. Precisely,
Remark 2. Note that for the IBVP of the Navier–Stokes equations with Dirichlet conditions (1.3) and (1.4). There exists some constant $\delta^*>0$ such that for any $\delta \leq \delta^*$, if
\[
\|\rho_0 - 1\|_{1} + \|(u_0, \mathcal{T}_0 - 1, H_0)\|_4 \leq \delta,
\]
then the initial boundary value problem (1.1)–(1.4) admits a unique smooth solution $(\rho, u, \mathcal{T}, H)$ globally in time satisfying
\[
\rho - 1 \in C^0(0, \infty, H\dot{1}(\Omega)) \cap C^1(0, \infty, H^2(\Omega)),
\]
and
\[
(u, \mathcal{T} - 1, H) \in C^0(0, \infty, H^4(\Omega)) \cap C^1(0, \infty, H^2(\Omega)),
\]
and
\[
\begin{align*}
\left(\|\rho - 1\|^2_3 + \|(u, \mathcal{T} - 1, H)\|^2_3 + \|(\rho_t, u_t, \mathcal{T}_t, H_t)\|^2_3 + \|(\rho_{tt}, u_{tt}, \mathcal{T}_{tt}, H_{tt})\|^2_3\right)(t) \\
+ \int_0^t \left(\|\nabla u\|_2^2 + \|\Delta \rho\|_2^2 + \|(\rho_{tt}, u_{tt}, \mathcal{T}_{tt}, H_{tt})\|^2_3 + \|\mathcal{T}\|_3^2\right) ds \\
\leq C \left(\|\rho - 1\|^2_3 + \|(u, \mathcal{T} - 1, H)\|^2_3 + \|(\rho_t, u_t, \mathcal{T}_t, H_t)\|^2_3\right)(0),
\end{align*}
\]
where $C$ is positive constant independent of $t$.

In addition, we shall show that the solution in theorem 1.2 approaches the stationary state as $t \to \infty$, and give the explicit decay rate. More precisely,

**Theorem 1.3.** Let $(\rho, u, \mathcal{T}, H)$ be the global solution obtained in theorem 1.2, and $(1, u_i, 1, H_i)$ with $u_i = H_i = (0, 0, 0)$ be a stationary state. Then it holds:
\[
\begin{align*}
\|(\rho_t, u_t, \mathcal{T}_t, H_t)(t)\| &= O(t^{-1/2}), \quad \text{as } t \to \infty, \\
\|\nabla \rho, \nabla u, \nabla \mathcal{T}, \nabla \mathcal{H}(t)\| &= O(t^{-1/4}), \quad \text{as } t \to \infty, \\
\|\nabla^2 \rho, \nabla^2 u, \nabla^2 \mathcal{T}, \nabla^2 \mathcal{H}(t)\| &= O(t^{-1/4}), \quad \text{as } t \to \infty, \\
\|(u, \mathcal{T} - 1, H)(t)\|_{C^0(\Omega)} &= O(t^{-1/4}), \quad \text{as } t \to \infty, \\
\|(\rho - 1)(t)\|_{C^0(\Omega)} &= O(t^{-1/8}), \quad \text{as } t \to \infty.
\end{align*}
\]

**Remark 2.** Note that for the IBVP of the Navier–Stokes equations with Dirichlet conditions $u|_{\partial \Omega} = 0$ on exterior domains, the decay estimates of solutions to stationary state can be found in [10, 11, 21, 23, 28]. By contrast, the big difference in this paper is that we consider the Navier-slip boundary conditions for $u$, instead of the homogeneous Dirichlet condition. Inspired by [10], we establish a differential inequalities as in lemma 2.7. Different from [10], lemma 5.3 is used to obtain the estimate for $|\nabla \mathcal{C} \nabla u|$, which is important for us to derive the decay estimates of $\|\nabla^2 u, \nabla \rho\|$ (lemma 5.4). Moreover, some classical elliptic estimates will be applied for $H$ to find the corresponding decay rates.
Remark 3. The algebraic decay for Cauchy problem of compressible MHD equations was shown in [6, 25, 39] by careful analysis of the linearized equations using the Fourier transformation. However, due to the presence of the physical boundary, the approach of obtaining the decay for the Cauchy problem used in [6, 25, 39] cannot be applied to the problem studied in this paper. Moreover, the conditions required for initial data are assumed to be in $H^s$ only in the present paper.

We sketch now the main strategy and ideas for the proofs of theorems 1.1–1.3. Indeed, compared with the previous results for Cauchy problem ([26, 31]), the Navier-slip boundary conditions cause additional difficulties in developing a priori estimates for solutions of the initial boundary value problem (1.1)–(1.4). Overcoming these difficulties relies on two observations. First, thanks to [34] (see proposition 2.3 below), for the exterior domain $\Omega$ (with the vanishing first Betti number), the inequality $\|\nabla v\|^2 \leq C(\|\text{curl} v\|^2 + \|\text{div} v\|^2)$ holds for vector fields $v$ satisfying $\nabla v \in L^2(\Omega)$ with $v \cdot n = 0$ on $\partial \Omega$. We point out that $\|\text{curl} u, \text{div} u, \text{curl} H\|$ will be used to estimate $\|\text{curl} u, \nabla H\|$ instead of estimating $\|\nabla u, \nabla H\|$ directly, due to the boundary conditions for $(u, H)$ and the structure of (2.4). Another fact is that the boundary condition $\text{curl} u \times n = 0$ yields $\text{curl}^2 u \cdot n|_{\partial \Omega} = 0$. As a direct consequence of this observation, we can apply lemma 2.2 to obtain the dissipation estimate $\int_0^t \|\nabla^2 \text{curl}^2 u\|^2(\sigma)ds$, see (4.6) and lemma 4.2 for details.

Theorems 1.1 and 1.2 are proved by energy estimates required to be valid in the exterior domain. First, we obtain the estimates for basic energy and the first-order spatial derivatives with the help of the refined $\text{div} - \text{curl}$ estimate (see lemmas 3.2–3.4). Second, we differentiate system (1.1) with respect to time $t$, noting that time-differentiations of solutions also satisfy the boundary conditions (1.3) and (1.4), we obtain the estimates of $\|[(\rho, u, T, H)]\|$ in a similar manner to that of $\|[(\rho - 1, u, T - 1, H)]\|$ (see lemmas 3.6–3.8). We then derive the estimates for third-order derivatives by using some elliptic estimates for $(u, H)$ (see lemmas 3.10 and 3.11). It should be noted that, in order to get estimates for the higher-order spatial derivatives of $(u, T, H)$, we apply lemmas 2.2–2.6 of elliptic estimates according to different boundary conditions and the structure of equations. Finally, we prove the estimate of $\|\nabla^2 q\|$ (where and in the following, $q = \rho - 1$, the perturbation of the density $\rho$ to the constant 1) (lemma 3.12) to close a priori estimates (see proposition 3.1), which enable us to extend the local strong solution to a global one. Furthermore, we improve the regularity of the strong solutions in theorem 1.1 to obtain smooth solutions.

In order to obtain explicit convergence rates in theorem 1.3, we first establish a differential inequality for $\|[(\rho, u, T, H)]\|^2$, and apply lemma 2.7 (a generalized Gronwall-type inequality) to obtain the algebraic decay rate for $\|[(\rho, u, T, H)]\|$ (lemma 5.1). We then control $\|[(\nabla u, \nabla \theta, \nabla H)]\|^2$ and $\|[(\nabla^2 \theta, \nabla^2 H)]\|^2$ through $\|[(\rho, u, T, H)]\|$ (lemma 5.2). Next, combining some estimates about $\text{curl} u$ and lemma 2.7, we derive the convergence rates for $\|\text{curl}^2 u\|$ (lemma 5.3). Finally, according to the boundary conditions for $u$, one can employ the results of Stokes problem (lemma 2.5) to get the convergence rate for $\|\nabla^2 u\| + \|\nabla q\|$. However, it is a difficult point to estimate $\|\nabla q\|^2$ because there is no boundary condition for $q$. To overcome this, we use the level surface of the distance function $d(x) := \text{dist}(x, \partial \Omega)$ to extend the unit normal on the boundary to a smooth vector field in a neighborhood of the boundary, and estimate the tangential and normal components of $\text{curl} q$, see (5.36), respectively.

This paper is organized as follows. Section 2 contains some notations and basic lemmas for later use. In section 3 we prove the global existence of strong solutions. The improved regularity of the strong solutions in theorem 1.1 will be proved in section 4. Finally, section 5 is devoted to obtaining the decay rates.
2. Notations and basic lemmas

In this section, we introduce some notations and list some lemmas which will be frequently used throughout this paper.

We use $C^0(\bar{\Omega})$ to denote the Banach space of bounded continuous function on $\bar{\Omega}$. Let $C(t_1, t_2, B) = \{ \phi(x, t) : l-\text{time continuously differential function of } t \in [t_1, t_2] \text{ with values in a Banach space } B \}$ with the following norm

$$\max_{0 \leq k \leq l} \sup_{t_1 \leq t \leq t_2} \left\| \frac{\partial^k \phi}{\partial t^k} \right\|_B,$$

where $\| \cdot \|_p$ is the norm for Banach space $B$. The $L^p$-norm on $\Omega$ for $\phi(x, t)$ is given by

$$\| \phi(\cdot, t) \|_{L^p} \equiv \| \phi \|_{L^p} \equiv \| \phi \|_{L^p(\Omega)} = \left( \int_{\Omega} |\phi|^p(x, t) dx \right)^{1/p},$$

and $H^m$ is used to denote the standard Sobolev space with the following norm

$$\| \phi(\cdot, t) \|_m \equiv \| \phi \|_m \equiv \| \phi \|_{H^m(\Omega)} = \left( \sum_{l=0}^m \| \nabla^l \phi \|_2^2 \right)^{1/2}.$$

Moreover, $H^{m,p}$ is applied to express the standard Sobolev space with the following norm

$$\| \phi(\cdot, t) \|_{H^{m,p}} \equiv \| \phi \|_{H^{m,p}} \equiv \left( \sum_{l=0}^m \| \nabla^l \phi \|_p^p \right)^{1/p}.$$

Let

$$\| \phi \|^2_{L^2(H^m)} \equiv \| \phi \|^2_{L^2(\Omega, H^m(\Omega))} = \int_0^t \| \phi \|^2_{H^m(s)} ds,$$

in particularly,

$$\| \phi \|^2_{L^2(H^m)} \equiv \| \phi \|^2_{L^2(\Omega, L^2(\Omega))} = \int_0^t \| \phi \|^2(s) ds.$$

Besides, $C$ will be used as a generic constant independent of time $t$.

Next, we recall some inequalities of Sobolev type.

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary. Then

(i) $\| f \|_{C^0(\Omega)} \leq C \| f \|_{H^{m,p}}$ for $f \in H^{m,p}(\Omega), \quad mp > 3 + (m-1)p$.

(ii) $\| f \|_{L^p} \leq C \| f \|_1$ for $f \in H^1(\Omega), \quad 2 \leq p \leq 6$.

(iii) $\| f \|_{L^6} \leq C \| \nabla f \|$ for $f \in H^1(\Omega)$.

The following lemma allows one to control the $H^m$-norm of a vector valued function $v$ by its $H^{m-1}$-norm of curl $v$ and div $v$ (see [37]).

**Lemma 2.2.** Let $\Omega$ be a domain in $\mathbb{R}^3$ with smooth boundary $\partial \Omega$ and outward normal $n$. Then there exists a constant $C > 0$, such that

$$\| v \|_{H^m(\Omega)} \leq C \left( \| \text{div } v \|_{H^{m-1}(\Omega)} + \| \text{curl } v \|_{H^{m-1}(\Omega)} + |v \cdot n|_{H^{m-1/2}(\partial \Omega)} + \| v \| \right), \quad (2.1)$$

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\[
\|v\|_{H^s(\Omega)} \leq C\left(\|\text{div} \ v\|_{H^{s-1}(\Omega)} + \|\text{curl} \ v\|_{H^{s-1}(\Omega)} + |v \times n|_{H^{s-1/2}(\partial\Omega)} + \|v\|_{H^{s}(\Omega)}\right), \tag{2.2}
\]
for any \( v \in H^s(\Omega) \), \( s \geq 1 \).

In addition, since the first Betti number of the exterior domain \( \Omega \) in our paper vanishes, we can apply theorem 3.2 in [34] to obtain the following refined estimate of \( \|\nabla v\| \) which is crucial in the case of exterior domain. We also note that the topological property of \( \Omega \) is necessary for the following proposition.

**Proposition 2.3 ([34]).** Let \( \nabla v \in L^2(\Omega) \), and \( v \cdot n|_{\partial\Omega} = 0 \). The estimate \( (C_\Omega = \text{constant that is independent of } v) \)
\[
\|\nabla v\| \leq C_\Omega (\|\text{div} \ v\| + \|\text{curl} \ v\|) \tag{2.3}
\]
is true for all \( v \) as above if and only if \( \Omega \) has a first Betti number of zero.

Next, we list some elliptic estimates for elliptic equations with Navier-slip or Neumann boundary conditions in the smooth exterior domain \( \Omega \).

**Lemma 2.4 ([32]).** Let \( s \) be a nonnegative integer. Suppose that \( v \) satisfy the Lamé equation:
\[
\begin{cases}
-\mu \Delta v - (\mu + \lambda) \nabla \text{div} \ v = f, & \text{in } \Omega, \\
v \cdot n = 0, & \text{curl} \ v \times n = 0 \text{ on } \partial\Omega
\end{cases}
\]
Then, it holds
\[
\|\nabla^2 v\|_{H^s(\Omega)} \leq C(\|f\|_{H^s(\Omega)} + \|\nabla v\|_{L^2(\Omega)}).
\]
In particular, the same conclusion also holds for Laplace operator, that is, for
\[
v \cdot n = 0, \quad \text{curl} \ v \times n = 0 \text{ on } \partial\Omega,
\]
it holds
\[
\|\nabla^2 v\|_{H^s(\Omega)} \leq C(\|\Delta v\|_{H^s(\Omega)} + \|\nabla v\|_{L^2(\Omega)}).
\]

**Lemma 2.5 ([12]).** Let \( s \) be a nonnegative integer. Suppose that \((v, p)\) is the solution of the Stokes problem:
\[
\begin{cases}
\text{div} \ v = g, & \text{in } \Omega, \\
-\Delta v + \nabla p = f, & \text{in } \Omega, \\
v \cdot n = 0, & \text{curl} \ v \times n = 0 \text{ on } \partial\Omega
\end{cases}
\]
Then, it holds
\[
\|\nabla^2 v\|_{H^s(\Omega)} + \|\nabla p\|_{H^s(\Omega)} \leq C(\|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1}(\Omega)} + \|\nabla v\|_{L^2(\Omega)}).
\]

**Lemma 2.6 ([1]).** For any function \( \theta \in H^s(\Omega) \) with \( \frac{\partial \theta}{\partial n}|_{\partial\Omega} = 0 \), then it holds
\[
\|\nabla^2 \theta\|_{H^s(\Omega)} \leq C(\|\Delta \theta\|_{H^s(\Omega)} + \|\nabla \theta\|_{L^2(\Omega)}).
\]
Finally, we recall the following generalized Gronwall-type inequality which is the main argument in proving theorem 1.3. The reader is referred to [27] for a proof.

**Lemma 2.7 (i)** Let \( g \in C^1([t_0, \infty)) \) such that \( g \geq 0 \), \( E = \int_{t_0}^{\infty} g(t)dt < \infty \) and

\[
g'(t) \leq a(t)g(t) \quad \text{for all} \quad t \geq t_0
\]

where \( a \geq 0 \), \( M = \int_{t_0}^{\infty} a(t)dt < \infty \). Then

\[
g(t) \leq \left((t_0g(t_0) + 1)\exp(E + M) - 1\right)t^{-1} \quad \forall \ t \geq t_0.
\]

(ii) Let \( g \in C^1([t_0, \infty)) \) such that \( g \geq 0 \) and

\[
g'(t) + c_0g(t) \leq c_1t^{-\alpha} \quad \text{for all} \quad t \geq t_0
\]

with positive constants \( c_0, c_1 \) and \( \alpha \). Then there exists a \( t_1 = t_1(c_0, \alpha) \geq t_0 \) such that

\[
g(t) \leq C t^{-\alpha}, \quad \text{for} \quad t \geq t_1.
\]

At the last of this section, we will recall and explain the main steps towards the construction of local solutions to problem (1.1)–(1.5). Let

\[
\rho = 1 + q, \quad T = 1 + \theta,
\]

then \( (q, u, \theta, H) \) satisfy the following system

\[
\begin{aligned}
q_t + \text{div } u &= -\text{div}(qu), \\
\rho(u_t + u \cdot \nabla u) - \mu \Delta u - (\mu + \lambda)\nabla \text{div } u + R\nabla q + R\nabla \theta &= -R\nabla (q\theta) + \text{curl } H \times H, \\
c_v \rho(\theta_t + u \cdot \nabla \theta) - \kappa \Delta \theta + R \text{div } u &= \lambda(\text{div } u)^2 + 2\mu|\text{curl } u|^2 - R(\rho\theta + q)\text{div } u + \eta|\text{curl } H|^2, \\
H_t - \eta \Delta H &= \text{curl } (u \times H), \quad \text{div } H = 0,
\end{aligned}
\]

(2.4)

with the initial data

\[
(q, u, \theta, H)|_{t=0} = (q_0, u_0, \theta_0, H_0)(x), \quad \text{div } H_0 = 0, \quad \text{in } \Omega,
\]

(2.5)

and the boundary conditions

\[
\begin{aligned}
u \cdot n &= 0, \quad \text{curl } u \times n = 0, \quad H \cdot n = 0, \quad \text{curl } H \times n = 0, \quad \frac{\partial \theta}{\partial n} = 0, \quad \text{on } \ x \in \partial \Omega.
\end{aligned}
\]

(2.6)

Now, we explore the local-in-time existence result of strong solutions to the initial boundary value problem (2.4)–(2.6). To that purpose, we set \( E_1(t) \) and \( D_1(t) \) as follows:

\[
E_1(t) = \|(u, \theta, H)(\cdot, t)\|_3 + \|q(\cdot, t)\|_2 + \|(q, u, H, \theta)(\cdot, t)\|_1
\]

and

\[
D_1(t) = \|(\nabla u, \nabla \theta, \nabla H)(\cdot, t)\|_2 + \|\nabla q(\cdot, t)\|_1 + \|(u, \theta, H)(\cdot, t)\|_2 + \|q(\cdot, t)\|_1 + \|(q_0, u_0, \theta_0, H_0)(\cdot, t)\|.
\]
Lemma 2.8. Let \( q_0(x) \in H^2(\Omega), (u_0, \theta_0, H_0)(x) \in H^1(\Omega), \) and \( \inf q_0(x) > -1, \) \( \text{div} H_0 = 0. \) Assume that the initial data satisfy the compatibility condition with the boundary conditions (2.6). Then there exist constants \( c_0 > 0 \) and \( T > 0 \) only depending on \( E_1(0) \) such that problem (2.4)–(2.6) admits a unique solution in \( \Omega \times (0, T], \) satisfying the following estimates

\[
E_1(t) \leq 2E_1(0), \quad \left( c_0 \int_0^t \mathcal{D}_1^2(s) ds \right)^{1/2} \leq 2E_1(0), \quad t \in [0, T].
\]

Proof. The local-in-time well-posedness of problem (2.4)–(2.6) can be proved by using the linearization and iteration technique in a similar manner to that in [8, 33]. We will explain the main steps towards the construction of local solutions.

Step 1. For given \( T^* > 0, \) consider the following linearized problem

\[
q_t + v \cdot \nabla q + (q + 1) \text{div} v = 0,
\]

\[
H_t - \eta \Delta H = \text{curl}(v \times H), \quad \text{div} H = 0,
\]

\[
c_\varepsilon \rho(\theta_t + v \cdot \nabla \theta) - \kappa \Delta \theta = \lambda (\text{div} v)^2 + 2\mu [S(v)]^2 - \kappa_0 (q + 1)(\theta + 1) \text{div} v + \eta \text{curl} H^2,
\]

\[
\rho u_t + v \cdot \nabla u - \mu \Delta u + (\mu + \lambda) \text{div} u = -R \nabla q - R \nabla \theta - R \nabla (q \theta) + \text{curl} H \times H,
\]

where \( v \in C(0, T^*; H^1(\Omega)) \cap C^1(0, T^*; H^1(\Omega)) \) is a known vector field in \( (0, T^*) \times \Omega \) such that \( v(x, 0) = u_0(x) \) and \( v \cdot n |_{\partial \Omega} = 0. \)

Firstly, it is well-known that the hyperbolic problem (2.7) has a unique solution \( q \) satisfying

\[
q(x, t) > -1,
\]

\[
q \in C^0(0, T^*; H^2(\Omega)), \quad \text{and} \quad q_t \in C^0(0, T^*; H^1(\Omega)).
\]

Secondly, we consider the linear parabolic problems (2.8)–(2.10) with the initial-boundary condition (2.5) and (2.6). Applying the Galerkin method, we can prove the existence results of \( H, \theta \) and \( u \) in turn.

Furthermore, we will get some \textit{a priori} estimates about the solution \( (q, H, \theta, u) \) of problems (2.7)–(2.10) with the initial-boundary conditions (2.5), (2.6). To that purpose, we fix a positive constant \( b_0 \) large enough such that

\[
2 + \|q_0\|_2 + \|(H_0, \theta_0, u_0)\|_3 \leq b_0
\]

and

\[
\sup_{0 \leq t \leq T} \|v\|_1^2(t) + \int_0^T (\|\nabla v\|_1^2 + \|v_t\|_2^2) dt \leq b_1^2,
\]

\[
\sup_{0 \leq t \leq T} (\|\nabla^2 v\|_1^2(t) + \|v_t\|_2^2(t)) + \int_0^T (\|\nabla^3 v\|_1^2 + \|\nabla^2 v_t\|_2^2) dt \leq b_2^2
\]

and

\[
\sup_{0 \leq t \leq T} (\|\nabla^3 v\|_1^2(t) + \|\nabla^2 v_t\|_2^2(t)) + \int_0^T \|\nabla^2 v_t\|_2^2(t) dt \leq b_3^2
\]
for some time $T \in (0, T^*)$ and constants $b_i (i = 1, 2, 3)$ such that $2 < b_0 \leq b_1 \leq b_2 \leq b_3$. The constants $b_i (i = 1, 2, 3)$ and time $T$ can be explicitly determined by $b_0$ and $T^*$ due to \textit{a priori} estimates. This can be done by a slight modification of the arguments in [8], we just give \textit{a priori} estimates about $H$ for example because of the exterior domain and the boundary conditions in present paper. It is clear that

$$\|H\|^2(t) + \int_0^t (\|\nabla H\|^2 + \|H_t\|)^2(s)ds \leq Cb_0^2.$$  \hfill (2.11)

Indeed, multiplying (2.8) by $H$ and integrating by parts with $\nabla H \times n|_{\partial \Omega} = 0$, $(v \times H) \times n|_{\partial \Omega} = 0$ give

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |H|^2 dx + \eta \int_{\Omega} |\nabla H|^2 dx = \int_{\Omega} \nabla (v \times H) \cdot H dx = \int_{\Omega} (v \times H) \cdot \nabla H dx \leq \|v\|_{L^\infty} \|H\| \|\nabla H\|.$$  

Next, multiplying (2.8) by $\nabla^2 H$ and integrating over $\Omega$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla H|^2 dx + \eta \int_{\Omega} |\nabla^2 H|^2 dx = \int_{\Omega} \nabla (v \times H) \cdot \nabla^2 H dx \leq (\|v\|_{L^\infty} \|\nabla H\| + \|\nabla v\|_{L^2} \|H\|_{L^3}) \|\nabla^2 H\|.$$  

Then, using Young’s inequality and Gronwall’s inequality, the estimates $\|\nabla H\| \leq C\|\nabla H\|$ and $\|\nabla^2 H\| \leq C(\|\nabla^2 H\| + \|\nabla H\|)$ yield the desired estimate (2.11) for $0 < t \leq \min\{T^*, b_2^{-2}\}$.

Similarly, differentiating (2.8) with respect to $t$, multiplying the resulted equation by $H_t$ and integrating over $\Omega$, one can obtain

$$\|H_t\|^2(t) + \int_0^t \|\nabla H_t\|^2(s)ds \leq Cb_0^4, \quad \text{for } 0 < t \leq \min\{T^*, b_3^{-2}\},$$

and

$$\|\nabla^2 H\| \leq \left(\|H_t\| + \|\nabla (v \times H)\| + \|\nabla H\| \right) \leq Cb_0b_2, \quad \text{for } 0 < t \leq \min\{T^*, b_3^{-2}\}$$

thanks to the elliptic estimate of $H$ from lemma 2.4.

Finally, multiplying $\partial_t (2.8)$ by $H_{tt}$ and integrating over $\Omega$, we have

$$\|\nabla H_{tt}\|^2(t) + \int_0^t \|H_{tt}\|^2(s)ds \leq Cb_0^4, \quad \text{for } 0 < t \leq \min\{T^*, b_2^{-2}b_3^{-2}\},$$

and

$$\|\nabla^3 H\| \leq \left(\|H_t\| + \|\nabla (v \times H)\|_{L^1} + \|\nabla H\| \right) \leq Cb_0b_2^2, \quad \text{for } 0 < t \leq \min\{T^*, b_2^{-2}b_3^{-2}\}$$

owing to the elliptic estimate of $H$ again.

Step 2. Let $u^2$ be the solution to the linear parabolic problem.
\[
\begin{aligned}
w_t - \Delta w &= 0 \text{ in } (0, \infty) \times \Omega, \\
w(x, 0) &= u_0(x) \in \Omega, \\
w \cdot n &= 0, \text{ curl } w = 0, \text{ on } \partial \Omega.
\end{aligned}
\]

It then follows from the results in step 1 that there exists a unique solution \((q^i, H^i, \theta^i, u^i)\) to the linearized problem \((2.7)\)–\((2.10)\), \((2.5)\) and \((2.6)\) with \(v\) replaced by \(u^0\). Similarly, we construct approximate solutions \((q_i, H_i, \theta_i, u_i)\), defined for \(0 \leq i \leq N\), to the linearized problem \((2.7)\)–\((2.10)\), \((2.5)\) and \((2.6)\) with \(v\) replaced by \(u^0\). Then we can show that the full sequence \((q_i, H_i, \theta_i, u_i)\) converges to a limit \((q, H, \theta, u)\) in a strong sense, and \((q, H, \theta, u)\) is a solution to the original nonlinear problem \((2.4)\)–\((2.6)\) in \((0, T) \times \Omega\). This can be done by a slight modification of the arguments in [15]. We omit its details. This completes the proof of lemma 2.8. \(\square\)

3. Global existence of strong solutions

In this section, we derive some \textit{a priori} estimates to extend the local in time solutions to global ones under some smallness condition \((1.6)\) to prove theorem 1.1.

Recall
\[
\mathcal{E}_1(t) = \|u, \theta, H\|_3 + \|q(\cdot, t)\|_2 + \|(q, u, H, \theta)(\cdot, t)\|_1
\]
\[
\mathcal{D}_1(t) = \|\nabla u, \nabla \theta, \nabla H\|_2 + \|q(\cdot, t)\|_1 + \|\theta_t, H_t\|_2 + \|\theta_{tt}, u_{tt}, H_{tt}\|_1
\]

We assume there exists a constant \(\delta_1\) sufficiently small such that

\[
\sup_{0 \leq t \leq T} \mathcal{E}_1(t) \leq \delta_1,
\]

then one can see, for \(t \in [0, T]\),
\[
\|q, u, \theta, H\|_2 \leq C \|\nabla q, \nabla u, \nabla \theta, \nabla H\|_1 \leq C \delta_1
\]

and
\[
\|\nabla u, \nabla \theta, \nabla H\|_2 \leq C \|\nabla^2 u, \nabla^2 \theta, \nabla^2 H\|_1 \leq C \delta_1
\]

by the Sobolev’s embedding theorem and the definition of \(\mathcal{E}_1(t)\). Moreover, the lower and upper positive bounds \(\rho\) is given by, for \(t \in [0, T]\),
\[
1 - C \delta_1 \leq 1 - \|q(\cdot, t)\|_1 \leq \rho(x, t) \leq 1 + \|q(\cdot, t)\|_1 \leq 1 + C \delta_1, \quad x \in \Omega.
\]

In particular, there exists some \(\delta^* > 0\) such that if \(\delta_1 \leq \delta^*\), then
\[
\frac{1}{2} \leq \rho(x, t) \leq \frac{3}{2}, \quad x \in \Omega, \quad t \in [0, T].
\]

(3.3)
**Proposition 3.1** (a priori estimates). Let \( \delta^* > 0 \) be given as in (3.3), and suppose that for some \( T > 0 \), \((q, u, \theta, H)\) is a solution to the initial boundary value problem (2.4)–(2.6) on \( t \in [0, T] \). Then there exist positive constants \( \delta_1 \) and \( C_0 = C_0(\delta_1) \) which are independent of \( t \), such that if \( \delta_1 \leq \delta^* \) and
\[
\sup_{0 \leq t \leq T} E_1(t) \leq \delta_1,
\]
then there holds, for any \( t \in [0, T] \),
\[
E_1^2(t) + \int_0^t D_1^2(s) ds \leq C_0 E_1^2(0).
\]

Combining lemma 2.8 and proposition 3.1, we can obtain the global existence result by using the continuity method. The proof is standard. For reader’s convenience, we present it here.

**Proof of theorem 1.1.** Choose the initial data \((\rho_0 - 1, u_0, T_0 - 1, H_0)\) so small that
\[
E_1(0) < \min \left\{ \frac{\delta^*}{2}, \frac{\delta^*}{2\sqrt{C_0}} \right\}.
\]

Then based on lemma 2.8, there exists a constant \( T^* \) such that there is a unique solution \((q, u, \theta, H)\) of (2.4)–(2.6) satisfying
\[
E_1^{(1)} := \sup_{0 \leq t \leq T^*} E_1(t) \leq 2E_1(0) < \delta^*. \tag{3.4}
\]

On the other hand, proposition 3.1 implies that
\[
E_1^{(1)} \leq \sqrt{C_0 E_1(0)} < \frac{\delta^*}{2}. \tag{3.5}
\]

It should be noted that \( T^* \) depends only on \( E_1(0) \) in lemma 2.8, so starting from \( T^* \) and regarding \((q, u, \theta, H)(T^*)\) as a new initial data, then system (2.4)–(2.6) has a unique solution on \([T^*, 2T^*] \), which meets
\[
E_1^{(2)} := \sup_{T^* \leq t \leq 2T^*} E_1(t) \leq 2E_1(T^*) \leq 2E_1^{(1)} < \delta^*
\]
due to (3.5). Since \( C_0 \) does not depend on \( t \), we can apply proposition 3.1 again, to obtain
\[
\sup_{0 \leq t \leq 2T^*} E_1(t) \leq \sqrt{C_0 E_1(0)} < \frac{\delta^*}{2}.
\]

Repeating the above procedure for \( 0 \leq t \leq NT^*, \) \( N = 1, 2, 3, \ldots \), then we can extend the local solution to infinity as far as the initial data are small enough such that \( E_1(0) < \min \left\{ \frac{\delta^*}{2}, \frac{\delta^*}{2\sqrt{C_0}} \right\} \). It is easy to see that \( E_1(0) \) is equivalent to \( ||(u_0, T_0 - 1, H_0)||_3 + ||\rho_0 - 1||_2 \), this completes the proof of theorem 1.1 once proposition 3.1 is proved.

In the following, we shall prove proposition 3.1. Here and in what follows we will use the following facts frequently, so we list them as follows. From the definitions of curl and div, it is clear that
\[
\| \text{curl } f \|_s \leq \| \nabla f \|_s, \quad \| \text{div } f \|_s \leq \| \nabla f \|_s
\]
for any vector-valued function \( f \in H^{s+1} \), where \( s \) is a nonnegative integer. Now, we shall divide the proof of proposition 3.1 into several lemmas.
3.1. Estimates for lower-order derivatives

To begin with, we have the following basic energy estimate.

**Lemma 3.2.**
\[
\|(q, u, \theta, H)\|^2(t) + c_0 \int_0^t \|(\nabla u, \nabla \theta, \nabla H)\|^2(s) \, ds \leq C \|(q, u, \theta, H)\|^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds,
\]
where \(c_0\) and \(C\) are positive constants independent of \(t\).

**Proof.** Computing the following integral
\[
\int_\Omega \left\{ (2.4)_1 Rq + (2.4)_2 \cdot u + (2.4)_3 \theta + (2.4)_4 \cdot H \right\} \, dx,
\]
and noting that \(\Delta u = -\text{curl} u + \text{div} u\), after integrating by parts, one has
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( Rq^2 + \rho |u|^2 + c_v \rho \theta^2 + |H|^2 \right) \, dx + \mu \|\text{curl} u\|^2 + (2\mu + \lambda) \|\text{div} u\|^2 \\
+ \kappa \|\nabla \theta\|^2 + \eta \|\text{curl} H\|^2 \\
= \int_\Omega \left( -\frac{1}{2} Rq^2 - R \rho \theta^2 \right) \text{div} u + \left( \lambda (\text{div} u)^2 + 2\mu |S(u)|^2 + \eta \|\text{curl} H\|^2 \right) \theta \, dx \\
\leq C E_1(t) D_1^2(t), \tag{3.6}
\]
where we have used the boundedness of \(\rho\), Hölder’s inequality and Sobolev’s inequality. Applying proposition 2.3 to \(u\) and \(H\), it holds that
\[
\|\nabla u\|^2 \leq C (\|\text{curl} u\|^2 + \|\text{div} u\|^2), \tag{3.7}
\]
and
\[
\|\nabla H\|^2 \leq C \|\text{curl} H\|^2. \tag{3.8}
\]
Therefore, integrating (3.6) over \([0, t]\), and noting that \(2\mu + \lambda > 0\), we prove the basic energy estimate. \qed

Next, we prove estimates of the first-order derivatives \((\nabla u, \nabla \theta, \nabla H)\).

**Lemma 3.3.**
\[
\|(\nabla u, \nabla \theta, \nabla H)\|^2(t) + c \int_0^t \|(u_t, q_t, \theta_t, H_t)\|^2(s) \, ds \leq C E_1^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds,
\]
where \(c\) and \(C\) are positive constants independent of \(t\).

**Proof.** Computing the following integral
\[
\int_\Omega \left\{ (2.4)_2 \cdot u_t + (2.4)_1 q_t + (2.4)_3 \theta_t + (2.4)_4 \cdot H_t \right\} \, dx,
\]
and integrating by parts, one has

\[
\frac{1}{2} \frac{d}{dt} \left\{ \mu \| \text{curl} \, u \|^2 + (2\mu + \lambda) \| \text{div} \, u \|^2 + \kappa \| \nabla \theta \|^2 + \eta \| \text{curl} \, H \|^2 \right\} 
- \frac{d}{dt} \int_{\Omega} R(q + \theta) \text{div} \, u \, dx + \| \sqrt{\rho} \omega \|^2 + \| q \|_2^2 + c_v \| \sqrt{\rho} \theta \| + \| H_t \|^2
= - \int_{\Omega} \left[ (R + 1)q_t + 2R\theta_t \right] \text{div} \, u \, dx - \int_{\Omega} \text{div}(q u)q_t \, dx
+ \int_{\Omega} (-\rho u \cdot \nabla u - R\nabla (q\theta) + \text{curl} \, H \times H) \cdot u, \, dx
+ \int_{\Omega} (-c_v \rho u \cdot \nabla \theta + \lambda \text{div} \, u^2 + 2\mu |\Delta u|^2 - R(\rho\theta + q) \text{div} \, u
\]
\[+ \eta \| \text{curl} \, H \|^2 \theta_t \, dx + \int_{\Omega} \text{curl} (u \times H) \cdot H_t \, dx\]
\[= : I_1 + I_2 + I_3 + I_4 + I_5.\]

First, employing the Sobolev’s embedding theorem, Cauchy’s inequality and the smallness of \( \delta_1 \), we control \( I_1, I_2 \) and \( I_3 \) as follows

\[
I_1 \leq \frac{1}{8} (\| q_t \|^2 + c_v \| \sqrt{\rho} \theta \|^2) + C \| \text{div} \, u \|^2.
\]

\[
I_2 \leq \frac{1}{8} \| q_t \|^2 + C (\| u \|_2^\infty \| \nabla q \|_2 + \| q \|_2^2 \| \text{div} \, u \|^2)
\leq \frac{1}{8} \| q_t \|^2 + C \| \nabla q \|_2 \| \nabla u \|_2^2
\leq \frac{1}{8} \| q_t \|^2 + CE_2^2(t)D_1^2(t),
\]

\[
I_3 \leq \frac{1}{8} \| H_t \|^2 + C (\| H \|_2^\infty \| \nabla u \|^2 + \| u \|_2^\infty \| \nabla H \|^2)
\leq \frac{1}{8} \| H_t \|^2 + C \| \nabla u \|_2 \| \nabla H \|_2^2
\leq \frac{1}{8} \| H_t \|^2 + CE_2^2(t)D_1^2(t),
\]

Next, \( I_3 \) and \( I_4 \) can be estimated as

\[
I_3 \leq \frac{1}{8} \| u_t \|^2 + C (\| \rho \|_2^\infty \| u \|_2^\infty \| \nabla u \|^2 + \| q \|_2^\infty \| \nabla q \|_2^2 + \| \theta \|_2^\infty \| \nabla \theta \|^2 + \| H \|_2^\infty \| \text{curl} \, H \|^2)
\leq \frac{1}{8} \| u_t \|^2 + C (\| \nabla u \|_2^2 \| \nabla u \|^2 + \| q \|_2^2 \| \nabla q \|_2^2 + \| \nabla \theta \|_2^2 \| \nabla q \|_2^2 + \| \nabla H \|_2^2 \| \text{curl} \, H \|^2)
\leq \frac{1}{8} \| u_t \|^2 + CE_2^2(t)D_1^2(t),
\]

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\[ I_4 \leq \frac{1}{8} \| \theta_t \|^2 + C(\| \rho \|_2^2 \| u \|_2 \| \nabla \theta \|^2 + \| \nabla u \|_2 \| \nabla u \|^2) \\
+ C(\| \rho \|_2^2 \| \theta \|_2 \| \text{div} u \|^2 + \| q \|_2 \| \text{div} u \|^2 + \| \| H \|_2 \| \text{curl} H \|^2) \\
\leq \frac{1}{8} \| \theta_t \|^2 + C \left( \| \nabla u \|_2 \| \nabla \theta \|^2 + \| \nabla u \|_2 \| \nabla u \|^2 + \| \nabla \theta \|_2 \| \text{div} u \|^2 \\
+ \| \nabla q \|_2 \| \text{div} u \|^2 + \| \nabla^2 H \|_2 \| \text{curl} H \|^2 \right) \\
\leq \frac{1}{8} \| \theta_t \|^2 + C\epsilon_1^2(t)D_1^2(t), \]

where we have used the Sobolev’s embedding theorem, Cauchy’s inequality, the smallness of \( \delta_1 \) and the boundedness of \( \rho \). Combining the above estimates together, we arrive at

\[
\frac{1}{2} \frac{d}{dt} \{ \mu \| \text{curl} u \|^2 + (2\mu + \lambda)\| \text{div} u \|^2 + \kappa \| \nabla \theta \|^2 + \eta \| \text{curl} H \|^2 \} \\
- \frac{d}{dt} \int_{\Omega} R(q + \theta) \text{div} u \, dx + \frac{1}{2} \left( \| \sqrt{\mu} \eta \|_2^2 + \| q_t \|_2^2 + c_v \| \sqrt{\mu} \| + \| H_t \|_2^2 \right) \\
\leq C \| \text{div} u \|^2 + \delta_1 D_1^2(t). \tag{3.9} \]

Integrating (3.9) over [0, 1], using lemma 3.2, (3.7) and (3.8), lemma 3.3 is proved. \( \square \)

Now, we prove the estimates for \( \nabla q, \text{div} u, \text{and} \nabla \theta \).

**Lemma 3.4.**

\[
\| \nabla q, \text{div} u, \nabla \theta \|^2(t) + c \int_0^t \| \text{div} u \|_2^2(s)ds \leq C\| \nabla q, \text{div} u, \nabla \theta \|^2(0) + C\delta_1 \int_0^t D_1^2(s)ds, \tag{3.10} \]

where \( c \) and \( C \) are positive constants independent of \( t \). Moreover,

\[
\int_0^t \| \nabla^2 \theta \|^2(s)ds \leq C\epsilon_1^2(0) + C\delta_1 \int_0^t D_1^2(s)ds. \tag{3.11} \]

**Proof.** Computing the following integral

\[
\int_{\Omega} \{(2.4)_2 \cdot \nabla \text{div} u + \nabla (2.4)_4 \cdot R \nabla q + \nabla (2.4)_3 \cdot \nabla \theta \} \, dx, \]

and integrating by parts, noting that \( \int_{\Omega} \text{curl}^2 u \cdot \text{div} u \, dx = 0 \) thanks to \( \text{curl} u \times n |_{\partial \Omega} = 0 \), one has

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho |\text{div} u|^2 + R |\nabla q|^2 + \rho |\nabla \theta|^2 \right) \, dx + (2\mu + \lambda)\| \nabla \text{div} u \|^2 + \kappa \| \Delta \theta \|^2 \\
= \frac{1}{2} \int_{\Omega} \rho \| \text{div} u \|^2 dx + c_v \int_{\Omega} \| \nabla \theta \|^2 dx - \int_{\Omega} u \cdot \nabla q \, \text{div} u \, dx \\
+ \int_{\Omega} \left( \rho u \cdot \nabla u + R \nabla (q\theta) - \text{curl} H \times H \right) \cdot \nabla \text{div} u \, dx \\
- \int_{\Omega} \text{div}(q\theta_t) \cdot R \nabla q \, dx - c_v \int_{\Omega} (\nabla q \theta_t + u \cdot \nabla \theta) + \rho \nabla (u \cdot \nabla \theta) \cdot \nabla \theta \, dx \\
- \int_{\Omega} \nabla (R \rho \theta + q) \text{div} u \cdot \nabla \theta \, dx + \int_{\Omega} \nabla (\lambda |\text{div} u|^2 + 2\mu |u|^2 + \eta |\text{curl} H|^2) \cdot \nabla \theta \, dx. \]
The nonlinear terms on the right-hand side can be easily controlled by $CE_{i}(t)D_{j}^{2}(t)$. Therefore, integrating the above equality over $[0, t]$ implies the desired estimate (3.10). Furthermore, applying lemma 2.6 to $\theta$, we get

$$
\int_{0}^{t} \int_{\Omega} \left( \| \nabla^2 \theta \| \right)^2 \, dx \, dt \leq C \int_{0}^{t} \left( \| \Delta \theta \| + \| \nabla \theta \| \right)^2 \, dt
$$

$$
\leq C E_{i}^{2}(0) + C \delta_{i} \int_{0}^{t} D_{j}^{2}(s) \, ds,
$$
due to lemma 3.2 and (3.10). Thus the proof of lemma 3.4 is completed.

In order to obtain the dissipation estimates of $\| \nabla^2 u \|^2$ and $\| \nabla^2 H \|^2$, in view of lemma 2.4, we need to control $\| \text{curl}^2 u \|_{L^2(\Omega)}$ and $\| \text{curl}^2 H \|_{L^2(\Omega)}$, which is given in the following lemma.

**Lemma 3.5.**

$$
\| \left( \sqrt{\rho} \text{ curl} u, \text{ curl} H \right) \|^2 (t) + c \int_{0}^{t} \| \left( \text{curl}^2 u, \text{ curl}^2 H \right) \|^2 \, ds \leq C \| \left( \sqrt{\rho} \text{ curl} u, \text{ curl} H \right) \|^2 (0) + C \delta_{i} \int_{0}^{t} D_{j}^{2}(s) \, ds,
$$

(3.12)

where $c$ and $C$ are positive constants independent of $t$. Moreover,

$$
\int_{0}^{t} \left( \| \nabla^2 u \|^2 + \| \nabla^2 H \|^2 \right) \, ds \leq C E_{i}^{2}(0) + C \delta_{i} \int_{0}^{t} D_{j}^{2}(s) \, ds.
$$

(3.13)

**Proof.** Let $\text{curl} u = w$, then one has the identity

$$
\text{curl}(u \cdot \nabla)u = u \cdot \nabla w - w \cdot \nabla u + w \text{ div} u.
$$

(3.14)

Taking operator $\text{curl}$ to equation (2.4)$_{2}$, we easily derive the following equation for $w$:

$$
\rho(w_{t} + u \cdot \nabla w) - \mu \Delta w = K + \text{curl}(\text{curl} \times H),
$$

(3.15)

where

$$
K = -\rho w \cdot \nabla u - \rho w \text{ div} u - \nabla q \times (u_{t} + u \cdot \nabla u).
$$

Multiplying (3.15) by $w$, noting that $\Delta w = -\text{curl} \text{ curl} w$, integrating by parts by using the fact that $w \times n|_{\partial \Omega} = 0$, we obtain

$$
\frac{1}{2} \int_{\Omega} \rho |w|^{2} \, dx + \mu \int_{\Omega} \text{curl} \text{ curl} \, w \cdot w \, dx = \int_{\Omega} K \cdot w \, dx + \int_{\Omega} \text{curl}(\text{curl} \times H) \cdot w \, dx.
$$

(3.16)

We estimate the right-hand side as follows:

$$
\int_{\Omega} K \cdot w \, dx \leq C \left( \| \rho \|_{L^{\infty}} \| w \|_{L^{2}} \| \nabla u \|_{L^{2}} + \| \nabla q \|_{L^{2}} \| u_{t} \|_{L^{2}} \| w \| + \| \nabla q \|_{L^{2}} \| u_{t} \|_{L^{2}} \| w \| + \| \nabla q \|_{L^{2}} \| \nabla w \|_{L^{2}} \| \nabla u \|_{L^{2}} \| \nabla u \| \right)
$$

$$
\leq C \left( \| w \|_{L^{2}} \| \nabla u \|_{L^{2}} \| w \| + \| \nabla q \|_{L^{2}} \| \nabla u \|_{L^{2}} \| w \| + \| \nabla q \|_{L^{2}} \| \nabla w \|_{L^{2}} \| \nabla u \|_{L^{2}} \| \nabla u \| \right)
$$

$$
\leq \frac{\mu}{4} \| \text{curl} \text{ curl} \, w \|^{2} + C \delta_{i} \left( \| \nabla u \|_{L^{2}}^{2} + \| \nabla u \|_{L^{2}}^{2} \right),
$$

(3.16)
where we have used Hölder’s inequality, Sobolev’s inequality, the boundedness of $\rho$ and the estimate $\|w\|_{L^6}^2 \leq C\|\nabla w\|^2 + C\|\text{curl} w\|^2$ thanks to lemma 2.2 and $w \times n_{|\partial \Omega} = 0$. Integrating by parts and using the boundary condition $w \times n_{|\partial \Omega} = 0$ again to obtain

$$\int_{\Omega} \text{curl}(\text{curl} H \times H) \cdot w \, dx = \int_{\Omega} (\text{curl} H \times H) \cdot \text{curl} w \, dx$$

by Sobolev embedding inequality $\|H\|_{L^\infty} \leq C\|\nabla H\|_1 \leq \delta_1$. Putting these into (3.16) yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho \|w\|^2 + \frac{\mu}{2} \|\text{curl} w\|^2 \leq C\|\nabla u\|^2 + C\|\nabla u\|^2 + \|\nabla H\|^2). \quad (3.17)$$

Similarly, letting $\text{curl} H = \phi$, taking curl to equation (2.4), we obtain

$$\phi_t + \eta \text{curl}^2 \phi = \text{curl}^2(u \times H). \quad (3.18)$$

Multiplying (3.18) by $\phi$, integrating by parts with $\phi \times n_{|\partial \Omega} = 0$, one has

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \|\phi\|^2 \, dx + \eta \|\text{curl} \phi\|^2 = \int_{\Omega} \text{curl}(u \times H) \cdot \text{curl} \phi \, dx$$

by Cauchy’s inequality and $\|H\|_{L^\infty} \leq \delta_1$ again. Therefore,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \|\phi\|^2 \, dx + \frac{\eta}{2} \|\text{curl} \phi\|^2 \leq C\|\nabla u\|^2 + C\|\nabla H\|^2. \quad (3.19)$$

Summing up (3.17) and (3.19), integrating over $[0, t]$ give (3.12).

Furthermore, applying lemma 2.4 with $v = H$, and noting that $\Delta H = -\text{curl}^2 H$, we get

$$\int_0^t \|\nabla^2 H\|^2(s) \, ds \leq C\int_0^t (\|\Delta H\|^2 + \|\nabla H\|^2) \, ds$$

$$\leq C\int_0^t (\|\text{curl}^2 H\|^2 + \|\nabla H\|^2) \, ds$$

$$\leq C\varepsilon_1^2(0) + C\delta_1 \int_0^t D_1^2(s) \, ds,$$

thanks to the basic energy estimate of lemma 3.2 and (3.12).
Similarly, applying lemma 2.4 with \( v = u \), we obtain
\[
\int_0^t \|\nabla^2 u\|^2(s) ds \leq C \int_0^t \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right)(s) ds \\
\leq C \int_0^t \left( \|\text{curl}^2 u\|^2 + \|\nabla \text{div} u\|^2 + \|\nabla u\|^2 \right)(s) ds \\
\leq C E_1^2(t) + C \delta_1 \int_0^t D_1^2(s) ds,
\]
thanks to lemmas 3.2, 3.4 and (3.12).

Next, we prove estimates for the first-order temporal derivatives \( (q_t, u_t, H_t) \).

**Lemma 3.6.**
\[
\|(q_t, u_t, \theta_t, H_t)\|^2(t) + c \int_0^t \|(\nabla u_t, \nabla \theta_t, \nabla H_t)\|^2(s) ds \leq C \|(q_t, u_t, \theta_t, H_t)\|^2(0) + C \delta_1 \int_0^t D_1^2(s) ds,
\]
where \( c \) and \( C \) are positive constants independent of \( t \).

**Proof.** Computing the following integral
\[
\int_\Omega \left\{ \partial_t (2.4) R q_t + \partial_t (2.4) \cdot u_t + \partial_t (2.4) \theta_t + \partial_t (2.4) H_t \right\} dx,
\]
and noting that differentiation of the system (2.4) with respect to \( t \) will keep the boundary conditions (2.6), it is easy to the prove lemma 3.6, so we omit it.

**3.2. Estimates for the second-order derivatives**

Firstly, we prove estimates of the second-order derivatives \( (\nabla u_t, \nabla \theta_t, \nabla H_t) \).

**Lemma 3.7.**
\[
\|(\nabla u_t, \nabla \theta_t, \nabla H_t)\|^2(t) + c \int_0^t \|(u_{tt}, q_{tt}, \theta_{tt}, H_{tt})\|^2(s) ds \leq C E_1^2(0) + C \delta_1 \int_0^t D_1^2(s) ds,
\]
where \( c \) and \( C \) are positive constants independent of \( t \).

**Proof.** Computing the following integral
\[
\int_\Omega \left\{ \partial_t (2.4) \cdot u_t + \partial_t (2.4) q_t + \partial_t (2.4) \theta_t + \partial_t (2.4) H_t \right\} dx
\]
and integrating by parts, one has
\[
\frac{1}{2} \frac{d}{dt} \left\{ \mu \|\nabla u_t\|^2 + (2\mu + \lambda) \|\text{div} u_t\|^2 + \kappa \|\nabla \theta_t\|^2 + \eta \|\text{curl} H_t\|^2 \right\} \\
- \frac{d}{dt} \int_\Omega R(q_t + \theta_t) \text{div} u_t dx + \|\sqrt{\rho} u_t\|^2 + \|q_t\|^2 + c_1 \|\sqrt{\rho} \theta_t\|^2 + \|H_t\|^2 \\
= \int_\Omega ((R + 1) q_{tt} + 2R \theta_{tt}) \text{div} u_t dx + \int_\Omega \text{div}(qu_t) q_t dx \\
+ \int_\Omega (-q_t (u_t + u \cdot \nabla u_t) + \rho (u \cdot \nabla u_t) - R \nabla (q \theta_t) + (\text{curl} H_t) \cdot u_t) dx.
\]
Proof. Taking (3.20) over \( \Omega \), one has

\[
\begin{align*}
&\leq \frac{1}{2} \left( \| \sqrt{\rho} u_t \|_2^2 + \| q_t \|_2^2 + c_v \| \sqrt{\rho} \theta_t \| + \| H_t \|_2^2 \right) + C \| \text{div} u_t \|_2^2 + C \mathcal{E}_1(t) D_1^2(t).
\end{align*}
\]

(3.20)

On the other hand, applying proposition 2.3 to \( u_t \) and \( H_t \) with \( u_t \cdot n_{\partial \Omega} = 0, H_t \cdot n_{\partial \Omega} = 0 \), it holds

\[
\| \nabla u_t \|_2 \leq C \left( \| \text{curl} u_t \|_2 + \| \text{div} u_t \|_2 \right), \quad \text{and} \quad \| \nabla H_t \|_2 \leq C \| \text{curl} H_t \|_2.
\]

(3.21)

Then, integrating (3.20) over \([0, t]\), recalling lemma 3.6, and using (3.21), we prove lemma 3.7.

Next, we prove the estimate of the second-order space-time derivative \( \nabla q_t \).

**Lemma 3.8.**

\[
\begin{align*}
\|( \nabla q_t, \sqrt{\rho} \text{div} u_t, \sqrt{\rho} \nabla \theta_t)\|_2^2(t) + c \int_0^t \|( \nabla \text{div} u_t, \Delta \theta_t)\|_2^2(s)ds \\
\leq \|( \nabla q_t, \sqrt{\rho} \text{div} u_t, \sqrt{\rho} \nabla \theta_t)\|_2^2(0) + C \delta \int_0^t D_1^2(s)ds,
\end{align*}
\]

(3.22)

where \( c \) and \( C \) are positive constants independent of \( t \). Moreover,

\[
\int_0^t \| \nabla^2 \theta_t \|_2^2(s)ds \leq C \mathcal{E}_1^2(0) + C \delta \int_0^t D_1^2(s)ds.
\]

(3.23)

**Proof.** Taking \( \partial_t \) to (2.4), multiplying the resulting identity by \( \nabla \text{div} u_t \), and integrating over \( \Omega \), one has

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\text{div} u_t|^2 dx + (2 \mu + \lambda) \| \nabla \text{div} u_t \|_2^2 - R \int_{\Omega} (\nabla q_t + \nabla \theta_t) \cdot \nabla \text{div} u_t dx \\
= \frac{1}{2} \int_{\Omega} q_t |\text{div} u_t|^2 dx + \int_{\Omega} u_t \cdot \nabla q \text{div} u_t dx + \int_{\Omega} (\rho (u \cdot \nabla u)_t \\
+ q_t (u_t + u \cdot \nabla u) + R \nabla (q \theta_t) - (\text{curl} H \times H_t) \cdot \nabla \text{div} u_t dx \\
\leq \| q_t \|_{L^\infty} \| \text{div} u_t \|_2^2 + \| u_t \|_2 \| \nabla q \|_{L^2} \| \text{div} u_t \|_2 \\
+ C \left( \| (u, q, \theta, H) \|_2 + \| q_t \|_1 + \| u_t \|_2 \right) \| q_t \|_1 \\
\times \left( \| \nabla \text{div} u_t \|_2^2 + \| \nabla u_t \|_2^2 + \| \nabla^2 u \|_2^2 + \| \nabla q_t \|_2^2 + \| \nabla \theta_t \|_2^2 + \| \nabla H_t \|_2^2 \right),
\end{align*}
\]

by the boundedness of \( \rho \), Hölder’s inequality, Sobolev’s inequality and \( \| \text{curl} H_t \| \leq \| \nabla H_t \| \), which yields
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (\rho |\nabla u|)^2 \, dx + \frac{\mu + \lambda}{2} |\nabla \nabla u| u - \int_\Omega (\nabla q_t + \nabla \theta_t) \cdot \nabla \nabla u \, dx \\
= C \delta_1 \left( |\nabla u|^2 + |\kappa u|^2 + |\nabla \nabla u|^2 + |\nabla u|^2 + |\nabla \nabla u|^2 + |\nabla q_t|^2 \right) + |\nabla \theta_t|^2 + |\nabla H_t|^2 \right). 
\]

To eliminate the singular terms on the left-hand side of (3.24), we take \( \partial_t \) to (2.4), multiply the resulting identity by \( \Delta \theta_t \), and integrate over \( \Omega \), then it holds
\[
\frac{c}{\Omega} \frac{d}{dt} \int_\Omega (\rho |\nabla \theta|^2 \, dx + \kappa |\Delta \theta|^2 + R \int_\Omega \nabla \theta \cdot \nabla \nabla u \, dx \\
= \frac{c}{\Omega} \int_\Omega q_t |\nabla \theta|^2 \, dx + c_v \int_\Omega \theta u \cdot \nabla \theta_t \, dx + \int_\Omega (\rho (u \cdot \nabla \theta) + q_t (\theta_t + u \cdot \nabla \theta)) \Delta \theta_t \, dx \\
- \int_\Omega \partial_t \left( \lambda (\nabla u)^2 + 2 \mu |S(u)|^2 - R (\rho \theta + q) \nabla u + \eta |\nabla H|^2 \right) |\Delta \theta_t| \, dx \\
\leq |\nabla q_t|_{\Omega} \|\nabla \theta\|_{L^2}^2 \|\nabla \theta_t\|_{L^2} + |\nabla q_t|_{\Omega} \|\nabla \theta\|_{L^2} \|\theta_t\|_{L^2} + c \left( |(u, \theta, H)|_3^3 + |(q, q_0)|_1 \right) \left\{ |\Delta \theta_t|^2 + |\nabla u_{\Omega}|^2 + |\nabla \theta_t|^2 \right. \\
+ |\nabla q_t|^2 + |\nabla \nabla u_{\Omega}|^2 + |\nabla q_t|^2 + |\nabla H_{\Omega}|^2 \right\}.
\]

Therefore, by using the elliptic estimate \(|\nabla^2 \theta_t|^2 \leq C \left( |\Delta \theta_t|^2 + |\nabla \theta_t|^2 \right)\), one has
\[
\frac{c}{\Omega} \frac{d}{dt} \int_\Omega (\rho |\nabla \theta|^2 \, dx + \kappa |\Delta \theta|^2 + R \int_\Omega \nabla \theta \cdot \nabla \nabla u \, dx \\
\leq C \delta_1 \left( |\nabla u|^2 + |\nabla \theta|^2 + |\nabla \theta_t|^2 + |\nabla q_t|^2 + |\nabla \nabla u_{\Omega}|^2 + |\nabla q_t|^2 + |\nabla H_{\Omega}|^2 \right). 
\]

Similarly, taking \( \partial_t \nabla \) to (2.4), multiplying the resulting identity by \( R \nabla q_t \), and integrating over \( \Omega \), we get
\[
\frac{R}{2} \frac{d}{dt} \int_\Omega |\nabla q_t|^2 \, dx + R \int_\Omega \nabla \theta \cdot \nabla \nabla u \, dx = R \int_\Omega \partial_t \nabla \nabla (q u) \cdot \nabla q_t \, dx. 
\]

The integral on the right-hand side contains the term \( \nabla^2 q_t \) which is not included in \( D_1(t) \), we shall work on it carefully.
\[
\int_\Omega \partial_t \nabla \nabla (q u) \cdot \nabla q_t \, dx \\
= \int_\Omega u \cdot \nabla^2 q_t \cdot \nabla q_t \, dx + \int_\Omega (\nabla u \nabla q_t + q_t \nabla \nabla u + \nabla \nabla u \cdot \nabla q_t + \nabla q_t \cdot \nabla q_t) \cdot \nabla q_t \, dx \\
\leq \int_\Omega u \cdot \nabla^2 q_t \cdot \nabla q_t \, dx + \left( \|u\|_{L^\infty} + \|\nabla \nabla u\|_{L^3} + \|\nabla q_t\|_{L^3} + \|q_t\|_{L^\infty} \right) \left\{ |\nabla q_t|^2 + |\nabla \theta_t|^2 \right. \\
+ \left| \nabla^2 q_t \right| \cdot \left\{ |\nabla q_t|^2 + |\nabla u_{\Omega}|^2 \right\} \left. = \int_\Omega u \cdot \nabla^2 q_t \cdot \nabla q_t \, dx + C \delta_1 \left( |\nabla q_t|^2 + |u_{\Omega}|^2 \right). \right. 
\]
In addition, integrating by parts with \( u \cdot n_{|\partial \Omega} = 0 \) gives
\[
\int_{\Omega} u \cdot \nabla^2 q_t \cdot \nabla q_t \, dx = -\frac{1}{2} \int_{\Omega} \text{div} \, u |\nabla q_t|^2 \, dx \leq C \delta_1 |\nabla q_t|^2.
\]
Therefore, plugging the above two inequalities into (3.26), we obtain
\[
\frac{d}{dt} \int_{\Omega} |\nabla q_t|^2 \, dx + \int_{\Omega} \nabla q_t \cdot \nabla \text{div} \, u_t \, dx \leq C \delta_1 \left( |\nabla q_t|^2 + |u_t|^2 \right).
\] (3.27)

Finally, summing up (3.24), (3.25) and (3.27), and integrating the resulting inequality over \([0, t]\), we obtain the desired estimate (3.22).

Apply lemma 2.6 to \( \theta \) to obtain
\[
\frac{d}{dt} \int_{0}^{t} |\nabla^2 \theta|^2(s) \, ds \leq C \int_{0}^{t} \left( |\Delta \theta|^2 + |\nabla \theta|^2 \right) \, ds.
\] (3.28)

This, together with lemma 3.6 and (3.22), implies the estimate (3.23), which completes the proof of lemma 3.8.

**Lemma 3.9.**
\[
\| (\sqrt{\rho} \, \text{curl} \, u_t, \text{curl} \, H_t) \|^2(t) + c \int_{0}^{t} \| (\text{curl}^2 \, u_t, \text{curl}^2 \, H_t) \|^2(s) \, ds
\]
\[
\leq \| (\sqrt{\rho} \, \text{curl} \, u_t, \text{curl} \, H_t) \|^2(0) + C \delta_1 \int_{0}^{t} D_1^2(s) \, ds,
\] (3.29)

where \( c \) and \( C \) are positive constants independent of \( t \). Moreover, it holds
\[
\int_{0}^{t} \left( \| \nabla^2 H_t \|^2 + \| \nabla^2 u_t \|^2 \right) \, ds \leq C \| \rho \| D_1^2(0) + C \delta_1 \int_{0}^{t} D_1^2(s) \, ds.
\] (3.30)

**Proof.** First, we note that \( w_t \times n_{|\partial \Omega} = 0 \) and \( \phi_t \times n_{|\partial \Omega} = 0 \). Similar to the proof of lemma 3.5, computing the following two integrals
\[
\int_{\Omega} \partial_t(3.15) \cdot w_t \, dx \quad \text{and} \quad \int_{\Omega} \partial_t(3.18) \cdot \phi_t \, dx
\]
gives
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |w_t|^2 \, dx + \frac{\mu}{2} \| \text{curl} \, w_t \|^2 \leq C \| \rho \| D_1^2(t),
\] (3.30)

and
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \| \phi_t \|^2 \, dx + \frac{\eta}{2} \| \text{curl} \, \phi_t \|^2 \leq C \| \rho \| D_1^2(t).
\] (3.31)

Then, summing up (3.30) and (3.31), integrating over \([0, t]\), one obtains (3.28).

Furthermore, applying lemma 2.4 with \( v = H_t \) and noting that \( \Delta H_t = -\text{curl}^2 H_t \), we get
\[
\int_{0}^{t} \| \nabla^2 H_t \|^2(s) \, ds \leq C \int_{0}^{t} \left( \| \Delta H_t \|^2 + \| \nabla H_t \|^2 \right) \, ds
\]
\[
\leq C \int_{0}^{t} \left( \| \text{curl}^2 H_t \|^2 + \| \nabla H_t \|^2 \right) \, ds
\]
\[
\leq C \| \rho \| D_1^2(0) + C \delta_1 \int_{0}^{t} D_1^2(s) \, ds.
\]
thanks to lemma 3.6 and (3.28).

Similarly, applying lemma 2.4 with \(v = u_t\), we get

\[
\int_0^t \|\nabla^2 u_t\|^2(s) ds \leq C \int_0^t (\|\Delta u_t\|^2 + \|\nabla u_t\|^2)(s) ds \\
\leq C \int_0^t (\|\text{curl}^2 u_t\|^2 + \|\nabla \text{div} u_t\|^2 + \|\nabla u_t\|^2)(s) ds \\
\leq CE_1^2(0) + C\delta_1 \int_0^t D_2^1(s) ds,
\]
due to lemmas 3.8, 3.6 and (3.28).

Next, we shall derive the estimates of \((\nabla^2 u, \nabla^2 H, \nabla^2 \theta)\) from the following two lemmas.

**Lemma 3.10.**

\[
\| (\nabla \text{div} u, \Delta \theta) \|_2^2(t) + c \int_0^t \| (\nabla q_t, \nabla \theta_t) \|_2^2(s) ds \leq C \| (\nabla \text{div} u, \Delta \theta) \|_2^2(0) + C\delta_1 \int_0^t D_2^1(s) ds,
\]

(3.32)

where \(c\) and \(C\) are positive constants independent of \(t\).

Moreover,

\[
\| \nabla^2 \theta \|_2^2(t) \leq CE_1^2(0) + C\delta_1 \int_0^t D_2^1(s) ds.
\]

(3.33)

**Proof.** Taking the inner product of \(\partial_t (2.4)_2\) with \(\nabla \text{div} u\) in \(L^2(\Omega)\), and using the fact

\[
\int_\Omega \text{curl}^2 u_t \cdot \nabla \text{div} u \ dx = 0,
\]

we get

\[
\frac{2\mu + \lambda}{2} \frac{d}{dt} \int_\Omega |\nabla \text{div} u|^2 \ dx - R \int_\Omega (\nabla q_t + \nabla \theta_t) \cdot \nabla \text{div} u \ dx \\
= \int_\Omega \rho(u_t + u \cdot \nabla u_t) \cdot \nabla \text{div} u \ dx + \int_\Omega (q_t u_t + (\rho u)_t) \cdot \nabla u + R\nabla (q\theta)_t \\
- (\text{curl} H \times H_t) \cdot \nabla \text{div} u \ dx \\
\leq CE_1(t)D_2^1(t).
\]

(3.34)

To eliminate the singular terms on the left-hand side of (3.34), we apply operator \(\nabla\) to equations (2.4)_1 and (2.4)_3, then multiply the resulting equality by \(R\nabla q_t\) and \(\nabla \theta_t\) in \(L^2(\Omega)\) to get

\[
R |\nabla q_t|^2 + R \int_\Omega \nabla \text{div} u \cdot \nabla q_t \ dx \\
= -R \int_\Omega \nabla (\text{div}(qu)) \cdot \nabla q_t \ dx \\
\leq CE_1(t)D_2^1(t),
\]

(3.35)
and
\[
\frac{\kappa}{2} \frac{d}{dt} \int_{\Omega} |\Delta \theta|^2 \, dx + c_i \| \sqrt{\rho} \nabla \theta_i \|^2 + R \int_{\Omega} \nabla \text{div} u \cdot \nabla \theta_i \, dx
= \int_{\Omega} (c_i \nabla q \cdot (\theta_i + u \cdot \theta) + c_i \rho \nabla (u \cdot \nabla \theta)) \cdot \nabla \theta_i \, dx
+ \int_{\Omega} \nabla \left( \lambda \left( \text{div} u \right)^2 + 2\mu |\nabla u|^2 - R(\rho \theta + q) \text{div} u + \eta |\nabla \theta|^2 \right) \cdot \nabla \theta_i \, dx
\leq C E_1(t) D_2^2(t). \quad (3.36)
\]

Then, (3.32) follows from (3.34)–(3.36).

Apply lemma 2.6 to $\theta$ to obtain
\[
\| \nabla^2 \theta \|_2 \leq C \left( \| \Delta \theta \|_2 + \| \nabla \theta \|_2 \right),
\]
which, together with lemma 3.4 and (3.32), yields estimate (3.33). The proof of the lemma is completed. \hfill \Box

**Lemma 3.11.**

\[
\| ( \sqrt{\rho} \text{curl}^2 u, \text{curl}^2 H ) \|_2^2(t) + c \int_0^t \| ( \text{curl}^3 u, \text{curl}^3 H ) \|_2^2(s) \, ds
\leq \| ( \sqrt{\rho} \text{curl}^2 u, \text{curl}^2 H ) \|_2^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds,
\quad (3.37)
\]

where $c$ and $C$ are positive constants independent of $t$. Moreover,

\[
\| \nabla^2 H \|_2^2(t) + \int_0^t \| \nabla^3 H \|_2^2(s) \, ds \leq C E_2^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds \quad (3.38)
\]

and

\[
\| \nabla^2 u \|_2^2(t) + \int_0^t \| \nabla \text{curl}^2 u \|_2^2(s) \, ds \leq C E_2^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds.
\quad (3.39)
\]

**Proof.** First, integrating by parts, in view of the fact $w_i \times n|_{\partial \Omega} = 0$, we have

\[
\int_{\Omega} \rho w_i \cdot \text{curl}^2 w \, dx = \int_{\Omega} \text{curl}(\rho w_i) \cdot \text{curl} w \, dx
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho \| \text{curl} w \|^2 \, dx - \frac{1}{2} \int_{\Omega} q_i |\text{curl} w|^2 \, dx + \int_{\Omega} (\nabla q \times w_i) \cdot \text{curl} w \, dx.
\]

Taking the inner product of (3.15) with $\text{curl}^2 w$, and integrating over $\Omega$, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho \| \text{curl} w \|^2 \, dx + \mu \| \text{curl}^2 w \|^2
= \frac{1}{2} \int_{\Omega} q_i |\text{curl} w|^2 \, dx - \int_{\Omega} (\nabla q \times w_i) \cdot \text{curl} w \, dx
+ \int_{\Omega} (K + \text{curl}(\text{curl} H \times H) - \rho u \cdot \text{curl}^2 w) \cdot \text{curl}^2 w \, dx
\leq C E_1(t) D_1^2(t). \quad (3.40)
\]
Similarly, taking the inner product of (3.18) with \( \text{curl}^2 \phi \) in \( L^2(\Omega) \), one has

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\text{curl} \phi|^2 \, dx + \eta \| \text{curl}^2 \phi \|^2 = \int_{\Omega} \text{curl}^2 (u \times H) \cdot \text{curl}^2 \phi \, dx
\]

\[
\leq C \mathcal{E}_1(t) D_2^2(t). \tag{3.41}
\]

Therefore, integrating (3.40) and (3.41) over \([0, t]\) gives (3.37).

Moreover, since \( \text{curl}^2 u \cdot n |_{\partial \Omega} = 0 \), applying lemma 2.2, it holds

\[
\int_0^t \| \nabla \text{curl}^2 \phi \|^2(s) \, ds \leq C \int_0^t (\| \text{curl}^3 u \|^2 + \| \text{curl}^2 u \|^2)(s) \, ds
\]

\[
\leq C \mathcal{E}_1^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds,
\]
due to lemma 3.5 and (3.37).

Noting that \( \text{curl}^2 H \cdot n |_{\partial \Omega} = 0 \), applying elliptic estimates and lemma 2.2 again, we have

\[
\int_0^t \| \nabla^3 H \|^2(s) \, ds \leq C \int_0^t (\| \Delta H \|^2 + \| \nabla H \|^2)(s) \, ds
\]

\[
\leq C \int_0^t (\| \text{curl}^2 H \|^2 + \| \nabla H \|^2)(s) \, ds
\]

\[
\leq C \int_0^t (\| \text{curl}^3 H \|^2 + \| \nabla H \|^2 + \| \nabla^2 H \|^2)(s) \, ds
\]

\[
\leq C \mathcal{E}_1^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds,
\]
thanks to lemmas 3.2, 3.5 and (3.37).

Furthermore, applying the elliptic estimate in lemma 2.4 and the fact \( \text{div} H = 0 \), it yields

\[
\| \nabla^2 H \|^2 \leq C (\| \Delta H \|^2 + \| \nabla H \|^2)
\]

\[
\leq C (\| \text{curl}^2 H \|^2 + \| \nabla H \|^2)
\]

\[
\leq C \mathcal{E}_1^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds,
\]

owing to lemma 3.3 and (3.37). Applying the elliptic estimate again, it holds

\[
\| \nabla^2 u \|^2 \leq C (\| \Delta u \|^2 + \| \nabla u \|^2)
\]

\[
\leq C (\| \text{curl}^2 u \|^2 + \| \nabla \text{div} u \|^2 + \| \nabla u \|^2)
\]

\[
\leq C \mathcal{E}_1^2(0) + C \delta_1 \int_0^t D_1^2(s) \, ds,
\]
due to lemmas 3.3, 3.10 and (3.37). \(\square\)
Now, we prove the estimate of $\nabla^2 q$.

**Lemma 3.12.**

\[
\|\nabla^2 q\|^2(t) + c \int_0^t \|\nabla^2 \text{div } u\|^2(s) \, ds \leq C \varepsilon_1^2(0) + \varepsilon \int_0^t \|\nabla^2 q\|^2(s) \, ds + C \delta_1 \int_0^t D_1^2(s) \, ds
\]  

(3.42)

where $c$, $C$ and $\varepsilon$ are positive constants independent of $t$.

**Proof.** Taking $\partial_i$ to (2.4), multiplying the resulting identities by $\partial_i \nabla \text{div } u$, $i = 1, 2, 3$, and summing them up, integrating over $\Omega$, we obtain

\[
(2\mu + \lambda)\|\nabla^2 \text{div } u\|^2 - R \int_\Omega (\nabla^2 q + \nabla^2 \theta) \cdot \nabla^2 \text{div } u \, dx \]

\[
= -\mu \int_\Omega \nabla \text{curl}^2 u \cdot \nabla^2 \theta \, dx + \int_\Omega (\nabla (\rho (u_t + u \cdot \nabla u)) + \nabla (\rho (u_t + u \cdot \nabla u)) + R \nabla^2 (q\theta))
\]

\[
- \nabla (\text{curl } H \times H) \cdot \nabla^2 \text{div } u \, dx.
\]

To eliminate the singular terms on the left-hand side of the above equation, we compute the integral

\[
\int_\Omega \{\partial_i \nabla (2.4) \cdot R (\partial_i \nabla q + \partial_i \nabla \theta)\} \, dx
\]

to obtain

\[
\frac{R}{2} d \frac{d}{dt} \int_\Omega |\nabla^2 q|^2 \, dx + R \frac{d}{dt} \int_\Omega \nabla^2 q \cdot \nabla^2 \theta \, dx + R \int_\Omega (\nabla^2 q + \nabla^2 \theta) \cdot \nabla^2 \text{div } u \, dx
\]

\[
= R \int_\Omega \nabla^2 q \cdot \nabla^2 \theta_t \, dx - R \int_\Omega \nabla^2 \text{div } (qu) \cdot (\nabla^2 q + \nabla^2 \theta) \, dx.
\]

Summing up the above two equalities yields

\[
\frac{R}{2} d \frac{d}{dt} \int_\Omega |\nabla^2 q|^2 \, dx + R \frac{d}{dt} \int_\Omega \nabla^2 q \cdot \nabla^2 \theta \, dx + (2\mu + \lambda)\|\nabla^2 \text{div } u\|^2
\]

\[
= -\mu \int_\Omega \nabla \text{curl}^2 u \cdot \nabla^2 \theta \, dx + \int_\Omega (\nabla (\rho (u_t + u \cdot \nabla u)) + R \nabla^2 (q\theta))
\]

\[
- \nabla (\text{curl } H \times H) \cdot \nabla^2 \text{div } u \, dx + R \int_\Omega \nabla^2 q \cdot \nabla^2 \theta_t \, dx + R \int_\Omega (\nabla^2 q + \nabla^2 \theta) \cdot \nabla^2 \text{div } (qu) \, dx
\]

\[
\equiv K_1 + K_2 + K_3 + K_4.
\]  

(3.43)

By Hölder’s inequality, Sobolev’s inequality and $\frac{1}{2} \leq \|\rho\|_{L^\infty} \leq \frac{3}{2}$, it holds that

\[
K_1 \leq \frac{2\mu + \lambda}{4} \|\nabla^2 \text{div } u\|^2 + C \|\nabla \text{curl}^2 u\|^2,
\]  

(3.44)
\[ K_2 \leq \frac{2\mu + \lambda}{4} \|\nabla^2 \text{div} \, u\|^2 + C\left(\|\nabla (\rho(u_t + u \cdot \nabla u))\|^2 + \|\nabla^2 (q\theta)\|^2 + \|\nabla (\text{curl} \, H \times H)\|^2\right) \]

\[ \leq \frac{2\mu + \lambda}{4} \|\nabla^2 \text{div} \, u\|^2 + C\|\rho(u, q, \theta, H)\|^2 \left(\|\nabla^2 u\|^2 + \|\nabla^2 q\|^2 + \|\nabla^2 \theta\|^2 + \|\nabla \text{curl} \, H\|^2\right) + C\|\nabla u_t\|^2, \]

and

\[ K_3 \leq \varepsilon \|\nabla^2 q\|^2 + C\varepsilon \|\nabla^2 \theta_t\|^2. \] (3.45)

The term \( K_4 \) becomes

\[ K_4 = R \int_{\Omega} \left( \nabla^2 q + \nabla^2 \theta \right) \cdot (\nabla^2 q \text{div} \, u + 2\nabla q \cdot \nabla \text{div} \, u + q \nabla^2 \text{div} \, u + \nabla^2 u \cdot \nabla q + u \cdot \nabla^3 q + 2\nabla u \cdot \nabla^2 q\) \, dx. \]

For the term \( \nabla^3 q \) included in \( K_4 \), by using integration by parts with \( u \cdot n|_{\partial \Omega} = 0 \), we obtain

\[ R \int_{\Omega} \left( \nabla^2 q + \nabla^2 \theta \right) u \cdot \nabla^3 q \, dx \]

\[ = -R \int_{\Omega} \text{div} \, u |\nabla^2 q|^2 \, dx - R \int_{\Omega} \nabla^3 \theta u \cdot \nabla^2 q \, dx - R \int_{\Omega} \text{div} \, u |\nabla^2 \theta|^2 \, dx \]

\[ \leq \left( \|\text{div} \, u\|_{L^\infty} + \|u\|_{L^\infty}\right) \left(\|\nabla^2 q\|^2 + \|\nabla^2 \theta\|^2 + \|\nabla \theta\|^2\right) \]

\[ \leq C\delta_1 (\|\nabla^2 q\|^2 + \|\nabla^2 \theta\|^2 + \|\nabla \theta\|^2). \]

While the other terms in \( K_4 \) is controlled by \( \delta (\|\nabla^2 q\|^2 + \|\nabla^2 \theta\|^2 + \|\nabla^3 u\|^2) \). So

\[ K_4 \leq C\delta_1 (\|\nabla^2 q\|^2 + \|\nabla^2 \theta\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^3 u\|^2). \] (3.47)

Put estimates (3.44)–(3.47) into (3.43) to get

\[ \frac{d}{dt} \int_{\Omega} \|\nabla^2 q\|^2 + \frac{d}{dt} \int_{\Omega} \|\nabla^2 \theta\|^2 + \frac{2\mu + \lambda}{2} \|\nabla^2 \text{div} \, u\|^2 \]

\[ \leq C\|\nabla \text{curl}^2 u\|^2 + \varepsilon \|\nabla^2 q\|^2 + C\varepsilon \|\nabla^2 \theta_t\|^2 + \delta_1 (\|\nabla^2 q\|^2 + \|\nabla^2 \theta\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^3 u\|^2). \] (3.48)

Integrating the above inequality over \([0, t]\) yields

\[ \frac{R}{4} \|\nabla^2 q\|^2(t) + \frac{2\mu + \lambda}{2} \int_0^t \|\nabla^2 \text{div} \, u\|^2(s) \, ds \]

\[ \leq C\|\nabla^2 \theta\|^2(t) + \varepsilon \int_0^t \|\nabla^2 q\|^2(s) \, ds + C \int_0^t \|\nabla \text{curl}^2 u\|^2 + \|\nabla^2 \theta_t\|^2 \, ds \]

\[ + C\delta_1 \int_0^t (\|\nabla^2 q\|^2 + \|\nabla^2 \theta\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^3 u\|^2) \, ds. \]
which, together (3.33), (3.23) and (3.39), implies the desired result (3.42).

At last, with lemmas 3.2–3.12 at hand, we can use lemma 2.4 to obtain the energy estimates of \( \| (\nabla^3 u, \nabla^3 H) \| \), and apply lemma 2.6 to (2.4) to get the estimate for \( \| \nabla^3 \theta \| \). Moreover, we complete the proof of proposition 3.1 by lemmas 3.2–3.12, and using equation (2.4) to obtain the dissipation estimate of \( \| \nabla^2 u \|_1^2 + \| \nabla q \|_1^2 \) by lemma 2.5 and choosing \( \varepsilon \) small in lemma 3.12. We omit the detail for simplicity.

4. Smooth solution

In this section, we shall prove theorem 1.2, improving the regularity of the strong solutions in theorem 1.1. Let

\[
E(t) = \| (u, \theta, H)(\cdot, t) \|_4 + \| q(\cdot, t) \|_3 + \| (q_t, u_t, H_t, \theta_t)(\cdot, t) \|_2 + \| (q_{tt}, u_{tt}, H_{tt}, \theta_{tt})(\cdot, t) \|_1 \tag{4.1}
\]

and

\[
D(t) = \| (\nabla u, \nabla \theta, \nabla H)(\cdot, t) \|_3 + \| \nabla q(\cdot, t) \|_2 + \| (q_t, u_t, H_t)(\cdot, t) \|_2 \\
+ \| \theta_t(\cdot, t) \|_3 + \| (u_{tt}, \theta_{tt}, H_{tt})(\cdot, t) \|_1 + \| q_{tt}(\cdot, t) \|. \tag{4.2}
\]

We show \textit{a priori} estimates of \( H^4 \) norm in the following.

**Proposition 4.1 (a priori estimates of \( H^4 \) norm).** Let \((q, u, \theta, H)\) be a solution to the initial boundary value problem (2.4)–(2.6) in \( t \in [0, T] \). There exists a positive constant \( \delta \) independent of \( T \), such that if

\[
\sup_{0 \leq t \leq T} E(t) \leq \delta,
\]

then it holds, for any \( t \in [0, T] \),

\[
E^2(t) + \int_0^t D^2(s) ds \leq C E^2(0),
\]

for some positive constant \( C \) independent of \( t \).

We divide the proof of proposition 4.1 into several lemmas. Since the proof of proposition 4.1 is based on proposition 3.1, we take \( \delta \leq \delta_1 \). Similarly, as in section 3, we have, \( \frac{1}{2} \leq \| \rho \|_\infty \leq \frac{3}{2} \),

\[
\| (q_t, u_t, H_t)(\cdot, t) \|_L^\infty \leq C \| (\nabla q, \nabla u, \nabla H)(\cdot, t) \|_1 \leq C \delta,
\]

\[
\| (\nabla u_t, \nabla \theta_t, \nabla H_t)(\cdot, t) \|_L^\infty \leq C \| (\nabla^2 u_t, \nabla^2 \theta_t, \nabla^2 H_t)(\cdot, t) \|_1 \leq C \delta
\]

and

\[
\| (q_{tt}, u_{tt}, \theta_{tt}, H_{tt})(\cdot, t) \|_L^\infty \leq C \delta
\]

by the assumption of \( E(t) \) and Sobolev’s inequality. First, with the results in proposition 3.1, using elliptic estimates, one can obtain the following dissipation estimates.

**Lemma 4.2.**

\[
\int_0^t \| \nabla^2 \text{curl}^2 u \|_1^2(s) ds \leq C E^2(0), \tag{4.3}
\]
\[ \int_0^t \| \nabla^4 \theta \|^2(s) \, ds \leq CE_1^2(0), \]  
\text{(4.4)}

and

\[ \int_0^t \| \nabla^4 H \|^2(s) \, ds \leq CE_1^2(0), \]  
\text{(4.5)}

where \( C \) is positive constant independent of \( t \).

**Proof.** Applying lemma 2.2 with \( v = \text{curl}^2 u \), we have

\[ \| \nabla^2 \text{curl}^2 u \|^2 \leq \| \text{curl}^3 u \|_1^2 + \| \text{curl}^2 u \|^2 + |\text{curl}^2 u \cdot n|_{W^{1/2}(\Omega)} \]
\[ \leq \| \text{curl}^3 u \|_1^2 + \| \text{curl}^2 u \|^2, \]  
\text{(4.6)}

thanks to \( \text{curl}^2 u \cdot n|_{\partial \Omega} = 0 \). Moreover, noting that

\[ \| \text{curl}^3 u \|_1^2 = \| \text{curl}^2 w \|_1^2 = \| \Delta w \|_1^2, \]

and using the equation of \( w \) (3.15), we obtain

\[ \| \Delta w \|_1^2 \leq \| \rho w_t + \rho(u \cdot \nabla w - w \cdot \nabla u) + \nabla q \times (u_t + u \cdot \nabla u) + \rho w \text{div} u \]
\[ - H \cdot \nabla \text{curl} H + \text{curl} H \cdot \nabla H \|_1^2 \]
\[ \leq C \| w_t \|_1^2 + C \left( \| \nabla q \|_1^2 + \| u \|_3^2 + \| H \|_2^2 \right) \cdot \left( \| \nabla w_t \|^2 + \| \nabla w \|_1^2 + \| u_t \|_3^2 \right)
\[ + \| \nabla w_t \|_1^2 + \| \nabla H \|_2^2 \).

Plugging the above inequality back in (4.6), one gets

\[ \int_0^t \| \nabla^2 \text{curl}^2 u \|^2(s) \, ds \leq C \int_0^t \left( \| w_t \|_1^2 + \| \text{curl}^2 u \|^2 \right)(s) \, ds
\[ + C \delta \int_0^t \left( \| \nabla w_t \|^2 + \| \nabla w \|_1^2 + \| u_t \|_3^2 + \| \nabla u \|_3^2 + \| \nabla H \|_3^2 \right)(s) \, ds
\[ \leq C(1 + \delta) \int_0^t D_1^2(s) \, ds \leq CE_1^2(0). \]

For the magnetic field \( H \), applying lemma 2.4 and using equation (2.4), we get

\[ \int_0^t \| \nabla^4 H \|^2(s) \, ds \leq \int_0^t \left( \| \Delta H \|_3^2 + \| \nabla H \|^2 \right)(s) \, ds
\[ \leq \int_0^t \left( \| H_t \|_5^2 + \| \text{curl}(u \times H) \|_3^2 + \| \nabla H \|^2 \right)(s) \, ds
\[ \leq C \int_0^t \left( \| H_t \|_3^2 + \| H \|_3^2 \| \nabla u \|_3^2 + \| u_t \|_3^2 \| \nabla H \|_2^2 + \| \nabla H \|^2 \right)(s) \, ds
\[ \leq C(1 + \delta) \int_0^t D_1^2(s) \, ds \leq CE_1^2(0). \]
Finally, we estimate the fourth-order derivatives of $\theta$. Rewrite equation (2.4) as

$$\kappa \Delta \theta = c_v \rho (\theta_t + u \cdot \nabla \theta) + R \nabla \text{div} u - \lambda (\text{div} u)^2 - 2 \mu |\nabla \theta|^2 + R (\rho \theta + q) \text{div} u$$

and

$$- \eta |\nabla \theta|^2 \equiv N.$$ (4.7)

then, applying the elliptic estimates to $\theta, \theta_t$ (lemma 2.6), it is easy to obtain the estimates for $\|\nabla^4 \theta\|_{L^2_t(L^2)}$ and $\|\nabla^3 \theta_t\|_{L^2_t(L^2)}$. Precisely,

$$\int_0^t \|\nabla^4 \theta\|^2(s) ds \leq C \int_0^t \|\nabla^2 q\| \|\nabla^2 q\|_{L^2_s} \|\theta\|^2(s) ds$$

$$+ C \int_0^t \|\nabla \theta, \nabla^2 \text{div} u\|^2(s) ds + C\delta \int_0^t D_1^2(s) ds$$

$$\leq C \int_0^t \|\nabla \theta, \nabla^2 \text{div} u\|^2(s) ds + C\delta \int_0^t D_1^2(s) ds$$

$$\leq C(1 + \delta) \int_0^t D_1^2(s) ds \leq C \mathcal{E}^2(0).$$

The proof of lemma 4.2 is completed. \hfill \square

Next, we prove estimates of the second-order temporal derivatives ($q_{tt}, u_{tt}, \theta_{tt}, H_{tt}$).

**Lemma 4.3.**

$$\|(q_{tt}, u_{tt}, \theta_{tt}, H_{tt})\|^2(t) + C \int_0^t \|\nabla u, \nabla \theta_t, \nabla H_{tt}\|^2(s) ds \leq C \mathcal{E}^2(0).$$ (4.8)

where $c$ and $C$ are positive constants independent of $t$. Moreover,

$$\|(u_t, \theta_t, H_t)\|^2(t) \leq C \mathcal{E}^2(0),$$ (4.9)

and

$$\int_0^t \|\nabla^3 \theta_t\|^2(s) ds \leq C \mathcal{E}^2(0).$$ (4.10)

**Proof.** Computing the following integral

$$\int_{\Omega} \left\{ \partial_t (2.4) R q_{tt} + \partial_t (2.4) \cdot u_{tt} + \partial_t (2.4) \cdot \theta_{tt} + \partial_t (2.4) \cdot H_{tt} \right\} dx,$$

and noting that the time-derivatives of the related quantities also satisfy the boundary conditions (2.6), after integrating by parts, one has

$$\frac{1}{2} \int_{t_0}^t \frac{d}{dt} \int_{\Omega} \left\{ R |q_{tt}|^2 + \rho |u_{tt}|^2 + c_v \rho |\theta_{tt}|^2 + |H_{tt}|^2 \right\} dx + \mu \|\nabla u\|^2$$

$$+ (2\mu + \lambda) \|\nabla u\|^2 + \kappa \|\nabla \theta_t\|^2 + \eta \|\nabla H_t\|^2$$

$$= -R \int_{\Omega} (\text{div}(qu)) u_{tt} dx - \int_{\Omega} (q_{tt} (u_t + u \cdot \nabla u) + 2q_t (u_t + u \cdot \nabla u))$$
\[-\rho u_t \cdot \nabla u + 2\rho u_t \cdot \nabla u_t \cdot u_t \, dx - R \int_\Omega (\nabla (q\theta))_n \cdot u_n \, dx \]
\[+ \int_\Omega (\text{curl } H \times H)_n \cdot u_n \, dx - c_v \int_\Omega (q_t(\theta_i + u \cdot \nabla \theta) + 2q_t(\theta_i + u \cdot \nabla \theta)_t - \rho u_t \cdot \nabla \theta + 2\rho \theta_t \cdot \nabla \theta_t \cdot \nabla u_t d \Omega + \int_\Omega (\lambda (\text{div } u)^2 + 2\mu |S(u)|^2) - R(\rho \theta + q) \text{div } u + \eta \|\text{curl } H\|^2_\Omega \theta_n \, dx + \int_\Omega (\text{curl}(u \times H))_n \theta_n \, dx \]
\[= : \sum_{i=1}^7 J_i. \quad (4.11)\]

The term involving third-order mixed derivations of \( q \) can be estimated as follows:

\[J_1 = \int_\Omega (\text{div}(q u)u)q \, dx \]
\[= \int_\Omega u \cdot \nabla q_n q_n \, dx + \int_\Omega (\text{div } u q + \text{div } u q_t + \text{div } u q_n)q_n \, dx \]
\[= -\frac{1}{2} \int_\Omega \text{div } u q^2_n \, dx + \int_\Omega (\text{div } u q + \text{div } u q_t + \text{div } u q_n)q_n \, dx \]
\[\leq \frac{2\mu + \lambda}{4} |\text{div } u_n|^2 + (||\text{div } u||_{L^\infty} + ||q||_{L^\infty} + ||q_t||_{L^3}) \]
\[\times (||\text{div } u_n||^2 + ||\nabla \text{div } u||^2 + ||q_n||^2) \]
\[\leq \frac{2\mu + \lambda}{4} |\text{div } u_n|^2 + C \delta D^2_1(t) \quad (4.12)\]

and

\[J_3 = -R \int_\Omega (\nabla (q\theta))_n \cdot u_n \, dx \]
\[= R \int_\Omega (q\theta) n \cdot \text{div } u_n \, dx \]
\[\leq \frac{2\mu + \lambda}{4} |\text{div } u_n|^2 + C ||(q\theta)_n||^2 \]
\[\leq \frac{2\mu + \lambda}{4} |\text{div } u_n|^2 + (||\theta, q||_2 + ||q_t||_1) (||q_n||^2 + ||\theta_n||^2 + ||\nabla \theta_t||^2) \]
\[\leq \frac{2\mu + \lambda}{4} |\text{div } u_n|^2 + C \delta D^2_1(t). \quad (4.13)\]

While for \( J_2 \), we obtain that

\[J_2 \leq ||u_t||_{L^\infty} ||q_n|| ||u_n|| + ||u||_{L^\infty} ||\nabla u||_{L^\infty} ||q_n|| ||u_n|| + ||q_t||_{L^\infty} ||u_n||^2 + ||\rho||_{L^\infty} ||\nabla u||_{L^\infty} ||u_n||^2 + C \delta D^2_1(t) \]
\[\leq C \delta D^2_1(t). \]
Similarly, one has

\[ J_5 \leq C \delta D_1^2(t), \quad J_6 \leq C \delta D_1^2(t), \]

It is easy to see that

\[ J_4 \leq \frac{\eta}{8} \| \text{curl} \: H_n \|^2 + C \delta D_1^2(t). \]

Noting that

\[ (u \times H) \times n|_{\partial \Omega} = 0, \]

due to the vector identity \((u \times H) \times n = (u \cdot n)H - (H \cdot n)u\) and \(u \cdot n|_{\partial \Omega} = 0, H \cdot n|_{\partial \Omega} = 0\), we have

\[ (u \times H)n \times n|_{\partial \Omega} = 0. \]

Then the last term on the right-hand side of (4.11) becomes

\[ J_7 = \int_{\Omega} (\text{curl}(u \times H) \times H_n) \cdot H_n \, dx = \int_{\Omega} (u \times H)_n \cdot \text{curl} \: H_n \, dx \]
\[ \leq \frac{\eta}{4} \| \text{curl} \: H_n \|^2 + C \|(u \times H)_n\|^2 \]
\[ \leq \frac{\eta}{4} \| \text{curl} \: H_n \|^2 + C \delta D_1^2(t) \quad (4.14) \]

by using Hölder’s inequality and Sobolev’s inequality directly. Therefore, (4.11) yields

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( R \frac{q_n^2}{\rho} + \rho |u_n|^2 + c_v \theta_n^2 + |H_n|^2 \right) \, dx + \mu \| \text{curl} \: u_n \|^2 \]
\[ + \frac{2 \mu + \lambda}{2} \| \text{div} \: u_n \|^2 + \kappa \| \nabla \theta_n \|^2 + \frac{\eta}{2} \| \text{curl} \: H_n \|^2 \]
\[ \leq C \delta D_1^2(t). \quad (4.15) \]

Finally, integrating (4.15) over \([0, t]\) and applying proposition 2.3 to \(u_n\) and \(H_n\), respectively, we obtain

\[ \|(q_n, u_n, \theta_n, H_n)\|^2(t) + c_{11} \int_0^t \|(\nabla u_n, \nabla \theta_n, \nabla H_n)\|^2(s) \, ds \]
\[ \leq C \|(q_n, u_n, \theta_n, H_n)\|^2(0) + C \delta \int_0^t D_1^2(s) \, ds \]
\[ \leq CE^2(0), \]

where we have used proposition 3.1 in the last inequality.

Furthermore, applying lemma 2.4 to \(u_n, \theta_n\) and \(H_n\), one obtains (4.9).
Moreover, recalling (4.7), we see that
\[
\int_0^t \| \nabla^3 \theta_i \|^2(s) ds \leq \int_0^t \| N_i \|^2(s) ds + \int_0^t \| \nabla \theta_i \|^2(s) ds
\leq \int_0^t \| \nabla \theta_n \|^2(s) ds + C(1 + \delta) \int_0^t D_i^2(s) ds
\leq CE^2(0).
\]
This completes the proof of lemma 4.3.

In order to obtain the estimate of \( \| \nabla^2 q \|_{L^2_t(L^2)} \), we use the equation \( q_t + \text{div} \ u + \text{div}(qu) = 0 \), to obtain the estimates for \( \| \nabla^3 q \|_{L^2_t(L^2)} \) and \( \| \nabla^2 \text{div} \ u \|_{L^2_t(L^2)} \) (see lemma 3.12). To obtain the estimate for \( \| \nabla^3 q \|_{L^2_t(L^2)} \), one can utilize the elliptic estimate of Stokes-type equation (2.4)2. In order to apply lemma 2.5 to control \( \| \nabla^3 q \|_{L^2_t(L^2)} \), we estimate \( \| \text{div} \ u \|_{L^2(\Omega)} \) first.

\textbf{Lemma 4.4.} For any \( \varepsilon > 0 \), it holds that
\[
\frac{R}{4} \| \nabla^3 q \|^2(t) + \frac{2\mu + \lambda}{2} \int_0^t \| \nabla^3 \text{div} \ u \|^2(s) ds
\leq CE^2(0) + \varepsilon \int_0^t \| \nabla^3 q \|^2(s) ds + C(\varepsilon) \int_0^t (\| \nabla^3 q \|^2 + \| \nabla^4 u \|^2)(s) ds,
\]
(4.16)
where \( C(\varepsilon) \) is a positive constant independent of \( t \).

\textbf{Proof.} Taking \( \partial_i \partial_j \) to (2.4)2, multiplying the resulting identities by \( \partial_i \partial_j \nabla \text{div} \ u \), \( i = 1, 2, 3 \), \( j = 1, 2, 3 \), summing them up, and integrating over \( \Omega \), we obtain
\[
(2\mu + \lambda) \| \nabla^3 \text{div} \ u \|^2 - R \int_{\Omega} (\nabla^3 q + \nabla^3 \theta) \cdot \nabla^3 \text{div} \ u \ dx
= -\mu \int_{\Omega} \nabla^2 \text{curl}^2 u \cdot \nabla^3 \text{div} \ u \ dx + \int_{\Omega} \left( \nabla^2 (\rho(\text{div} \ u + \text{div} u)) + R \nabla^2 (q \theta) 
\right. \\
- \left. \nabla \left( H \cdot \nabla H - \frac{1}{2} |H|^2 \right) \right) \cdot \nabla^3 \text{div} \ u \ dx.
\]
Computing the integral \( \int_{\Omega} (\partial_i \partial_j \nabla \text{div}(\Omega) \cdot (\partial_i \partial_j \nabla q + \partial_i \partial_j \nabla \theta) ) \ dx \), one has
\[
\frac{R d}{dt} \int_{\Omega} |\nabla^3 q|^2 \ dx + \frac{d}{dt} \int_{\Omega} \nabla^3 q \cdot \nabla^3 \theta \ dx + R \int_{\Omega} (\nabla^3 q + \nabla^3 \theta) \cdot \nabla^3 \text{div} \ u \ dx
= R \int_{\Omega} \nabla^3 q \cdot \nabla^3 \theta \ dx - R \int_{\Omega} \nabla^3 \text{div}(qu) \cdot (\nabla^3 q + \nabla^3 \theta) \ dx.
\]
Summing up the above two equations yields
\[
\frac{R d}{dt} \int_{\Omega} |\nabla^3 q|^2 \ dx + \frac{d}{dt} \int_{\Omega} \nabla^3 q \cdot \nabla^3 \theta \ dx + (2\mu + \lambda) \| \nabla^3 \text{div} \ u \|^2
\leq -\mu \int_{\Omega} \nabla^2 \text{curl}^2 u \cdot \nabla^3 \text{div} \ u \ dx + \int_{\Omega} \left( \nabla^2 (\rho(\text{div} \ u + \text{div} u)) + R \nabla^2 (q \theta) \right.
\]
Using Cauchy’s inequality, we have

\[-\nabla^2(\text{curl } H \times H) \cdot \nabla^3 \text{div } u \, dx + R \int_{\Omega} \nabla^3 q \cdot \nabla^3 \theta, \, dx\]

\[- R \int_{\Omega} (\nabla^3 q + \nabla^3 \theta) \cdot \nabla^3 \text{div}(q u) \, dx\]

\[\equiv J_1 + J_2 + J_3 + J_4. \quad (4.17)\]

Using Cauchy’s inequality, we have

\[J_1 \leq \frac{2\mu + \lambda}{4} ||\nabla^3 \text{div } u||^2 + C ||\nabla^2 \text{curl}^2 u||^2, \quad (4.18)\]

and

\[J_3 \leq \varepsilon ||\nabla^3 q||^2 + C_\varepsilon ||\nabla^3 \theta||^2. \quad (4.19)\]

Now \(J_2\) could be dominated as

\[J_2 \leq \frac{2\mu + \lambda}{4} ||\nabla^3 \text{div } u||^2 + C(\nabla^2 (\rho(u + u \cdot \nabla u))||^2 + ||\nabla^2 (q \theta)||^2\]

\[+ ||\nabla^2 (\text{curl } H \times H)||^2)\]

\[= \frac{2\mu + \lambda}{4} ||\nabla^3 \text{div } u||^2 + J_2^{(1)} + J_2^{(2)} + J_2^{(3)}\]

\[\leq \frac{2\mu + \lambda}{4} ||\nabla^3 \text{div } u||^2 + C ||\nabla^2 u||^2\]

\[+ C\delta (||\nabla^3 q||^2 + ||\nabla^2 u||^2 + ||\nabla^3 q||^2 + ||\nabla^3 \theta||^2 + ||\nabla^3 H||^2) \quad (4.20)\]

by Cauchy’s inequality and the following inequalities

\[J_2^{(1)} \leq C (||\nabla^2 q||^2_{L^p} ||u||_{L^6} ||u||_{L^6} + ||\nabla q||_{L^6} ||\nabla u||^2_{L^6} ||\nabla q|| + ||\rho||^2_{L^2} ||\nabla^2 u||^2\]

\[+ ||\nabla u||^2_{L^6} ||\nabla^2 u||^2 + ||u||^2_{L^6} ||\nabla^3 u||^2)\]

\[\leq C\delta (||\nabla^3 q||^2 + ||\nabla^2 u||^2 + C ||\nabla^2 u||^2, \]

\[J_2^{(2)} \leq C (||\theta||^2_{L^6} ||\nabla^3 q||^2 + ||\nabla \theta||^2_{L^6} ||\nabla^2 q|| ||\nabla^2 q|| + ||\nabla q||_{L^6} ||\nabla q|| ||\nabla^2 \theta||^2_{L^6} \]

\[+ ||q||^2_{L^6} ||\nabla^3 \theta||^2)\]

\[\leq C\delta (||\nabla^2 q||^2 + ||\nabla^3 q||^2 + ||\nabla^3 \theta||^2), \]

\[J_2^{(3)} \leq C (||H||^2_{L^6} ||\nabla^2 \text{curl } H||^2 + ||\nabla \text{curl } H||^2_{L^6} ||\nabla \text{curl } H|| + \]

\[||\nabla \text{curl } H||_{L^6} ||\nabla \text{curl } H||_{L^6} \]

\[\leq C\delta ||\nabla^3 H||^2\]

where we have used Hölder’s inequality, Sobolev’s inequality, the boundedness of \(\rho\) and the smallness of \(\delta\). The term \(J_4\) becomes

\[J_4 = R \int_{J_1} (\nabla^3 q + \nabla^3 \theta) \cdot (\nabla^3 q \text{div } u + 3 \nabla^2 q \cdot \nabla \text{div } u + 3 \nabla q \cdot \nabla^2 \text{div } u\]

\[+ q \nabla^3 \text{div } u + \nabla^3 u \cdot \nabla q + 3 \nabla^2 u \cdot \nabla^2 q + 3 \nabla u \cdot \nabla^3 q + u \cdot \nabla^4 q) \, dx.\]
In order to estimate the term $\nabla^4 q$ involved in $J_4$, we do integration by parts with $u \cdot n|_{\partial \Omega} = 0$ to obtain

$$R \int_\Omega (\nabla^3 q + \nabla^3 \theta) u \cdot \nabla^4 q \, dx$$

$$= -R \int_\Omega \text{div} u |\nabla^3 q|^2 \, dx - R \int_\Omega \nabla^4 \theta u \cdot \nabla^3 q \, dx - R \int_\Omega \text{div} u \nabla^3 \theta \cdot \nabla^3 q \, dx$$

$$\leq (\|\text{div} u\|_{L^\infty} + \|u\|_{L^\infty}) \cdot (\|\nabla^3 q\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^4 \theta\|^2)$$

$$\leq C\delta (\|\nabla^3 q\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^4 \theta\|^2).$$

While the other terms in $J_4$ could be controlled by $C\delta (\|\nabla^3 q\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^4 u\|^2)$ due to Hölder’s inequality, Sobolev’s inequality and the assumption of $E(t)$. So

$$J_4 \leq C\delta (\|\nabla^3 q\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^4 \theta\|^2 + \|\nabla^4 u\|^2).$$

Putting estimates (4.18)–(4.21) into (4.17) implies

$$\frac{R}{2} \frac{d}{dt} \int_\Omega |\nabla^3 q|^2 \, dx + \frac{\mu + \lambda}{2} \|\nabla^3 \text{div} u\|^2$$

$$\leq C\|\nabla^2 \text{curl}^2 u\|^2 + C\|\nabla^3 \theta\|^2 + C\|\nabla^2 u\|^2$$

$$+ C\delta (\|\nabla^3 q\|^2 + \|\nabla^3 \theta\|^2 + \|\nabla^4 \theta\|^2 + \|\nabla^4 u\|^2 + \|\nabla^3 H\|^2).$$

Then, integrating the above inequality over $[0, t]$, and using Cauchy inequality and proposition 3.1, lemma 4.2 implies the desired estimate (4.16).}

Finally, we use lemma 2.4 to obtain the energy estimates for $\|(|\nabla^4 u, \nabla^4 \theta, \nabla^4 H)|$, and apply lemma 2.6 with (2.4)_1 to derive the estimates for $\|\nabla^3 \theta\|$. On the other hand, we apply the elliptic estimates of Stokes-type system (lemma 2.5) to obtain the estimate for $\left(\|\nabla^3 q\| + \|\nabla^4 u\|\right)_{L^2(\Omega)}$. Therefore, the high-order energy is closed, i.e. proposition 4.1 is proved.

**Proof of proposition 4.1.** Applying the standard elliptic estimates (i.e. lemma 2.4) to $u, \theta$ and $H$, one obtains the estimates for $\|(|\nabla^4 u, \nabla^4 \theta, \nabla^4 H)|$.

Next, we rewrite (2.4)_2 as

$$-\mu \Delta u + R \nabla q = -\rho (u_t + u \cdot \nabla u) + (\mu + \lambda) \nabla \text{div} u - R \nabla \theta - R \nabla (q \theta)$$

$$+ \text{curl} H \times H \equiv f,$$

Applying lemma 2.5 with $g = \text{div} u$, one has

$$\int_0^t (\|\nabla^2 u\|_2^2 + \|\nabla q\|_2^2) (s) \, ds$$

$$\leq C \left( \int_0^t \|f\|_2^2 (s) \, ds + \int_0^t \|\text{div} u\|_2^2 (s) \, ds \right)$$

$$\leq C(1 + \delta) \int_0^t D_q (s) \, ds + C_\ast \int_0^t \|\nabla^3 \text{div} u\|_2^2 (s) \, ds$$

$$\leq CE_1^2 (0) + C_\ast \int_0^t \|\nabla^3 \text{div} u\|_2^2 (s) \, ds,$$  \hspace{1cm} (4.22)
where we have used the result of proposition 3.1. Choosing \( \varepsilon \) small in lemma 4.4, such that \( \varepsilon \ll \frac{2\mu + \lambda}{4} \), then, from (4.22), we have

\[
\frac{R}{4} \| \nabla^3 q \|^2(t) + \frac{2\mu + \lambda}{4} \int_0^t \| \nabla^3 \text{div } u \|^2(s) ds \leq CE^2(0) + C\delta \int_0^t (\| \nabla^3 q \|^2 + \| \nabla^4 u \|^2) (s) ds.
\]

(4.23)

Combining (4.22) with (4.23), we conclude, for \( \delta \) small, it holds that

\[
\int_0^t (\| \nabla^2 u \|^2 + \| \nabla q \|^2) (s) ds \leq CE^2(0).
\]

This completes the proof of proposition 4.1.

\[\square\]

5. Decay rates

For the solution \((q, u, \theta, H)\) of theorem 1.2, one has

\[
\int_0^\infty \| (\nabla q, \nabla u, \nabla \theta, \nabla H) \|_1(t) dt < \infty, \quad \int_0^\infty \frac{d}{dt} \| (\nabla q, \nabla u, \nabla \theta, \nabla H) \|_1(t) dt < \infty
\]

where \( q = \rho - 1, \theta = T - 1 \), it yields that

\[
\| (\nabla q, \nabla u, \nabla \theta, \text{curl } H) \|_1(t) \to 0, \quad \text{as } t \to \infty.
\]

(5.1)

In this section, we will prove theorem 1.3 by the following lemmas. Let

\[
X(0, \infty; \mathcal{E}_0) := \left\{ (q, u, \theta, H) | q \in C^0(0, \infty, H^1(\Omega)) \cap C^1(0, \infty, H^2(\Omega)), \right. \\
\left. (u, \theta, H) \in C^0(0, \infty, H^1(\Omega)) \cap C^1(0, \infty, H^2(\Omega)), \right. \\
\left. \mathcal{E}^2(t) + \int_0^t \mathcal{D}^2(s) ds \leq CE_0^2 \right\}
\]

where \( \mathcal{E} \) and \( \mathcal{D} \) are the same as in (4.1) and (4.2). It is observed that the solution of IBVP (2.4)–(2.6) is sought in the space \( X(0, \infty; \mathcal{E}_0) \), where \( 0 \leq \mathcal{E}_0 \leq \mathcal{C} \tilde{\delta} \) and \( \tilde{\delta} \) is given in theorem 1.2.

The first and most important key point to prove theorem 1.3 is to obtain the large-time behavior of the time derivatives \((q_t, u_t, \theta_t, H_t)\).

**Lemma 5.1.** For every solution \((q, u, \theta, H) \in X(0, \infty; \mathcal{E}_0)\) of the problem (2.4)–(2.6), there exists a time \( T_1 > 0 \), depending on \((q, u, \theta, H)\), such that

\[
\| (q_t, u_t, \theta_t, H_t) \|_1(t) \leq Ct^{-1/2}, \quad \forall \ t \geq T_1.
\]

(5.2)

**Proof.** Differentiate (2.4)1 with respect to \( t \), multiply the resulting identity by \( Rq_t \) and integrate over \( \Omega \), to obtain

\[
\frac{R}{2} \frac{d}{dt} \int_\Omega q_t^2 dx + R \int_\Omega \text{div } u_t q_t dx = -R \int_\Omega (\text{div } (qu)) q_t dx
\]

\[
= -R \int_\Omega q_t u_t \cdot \nabla q dx - R \int_\Omega q_t u_t \cdot \nabla q dx
\]

\[\text{6191}\]
\[- R \int_\Omega q_i^2 \text{div } u \, dx - R \int_\Omega q_i q_j \, dx \]
\[ \equiv \sum_{i=1}^{4} \mathcal{J}_i. \]  
(5.3)

Utilizing lemma 2.1 and Hölder’s inequality, we deduce that, for \( \varepsilon > 0 \),
\[ |\mathcal{J}_1| \leq \|u_i\|_{L^\infty} \|\nabla q_i\|_{L^1} \|q_j\| \leq \varepsilon \|\nabla u_i\| + C \|\nabla q_i\|_2^2 \|q_j\|, \]
\[ |\mathcal{J}_2| \leq \|\text{div } u_i\| \|q_i\| \|q_j\| \leq \varepsilon \|\nabla u_i\| + C \|\nabla q_i\|_2^2 \|q_j\| \]

Integration by parts with boundary condition \( u \cdot n|_{\partial \Omega} = 0 \) yields
\[ \mathcal{J}_2 = \frac{R}{2} \int_\Omega q_i^2 \text{div } u \, dx, \]
which implies that
\[ \mathcal{J}_2 + \mathcal{J}_3 = -\frac{R}{2} \int_\Omega q_i^2 \text{div } u \, dx. \]

In view of equation (2.4)_1, it holds that
\[ \mathcal{J}_2 + \mathcal{J}_3 = \frac{R}{2} \int_\Omega q_i^2 (q_j + q \text{ div } u + u \cdot \nabla q) dx \]
\[ = \frac{R}{2} \frac{d}{dt} \int_\Omega q_i^2 \, dx - R \int_\Omega q_i q_j \, dx + \frac{R}{2} \int \int q_i^2 q \, dx + \frac{R}{2} \int q_i^2 u \cdot \nabla q \, dx \]
\[ \leq \frac{R}{2} \frac{d}{dt} \int_\Omega q_i^2 \, dx - R \int_\Omega q_i q_j \, dx + \|\nabla u_i\| + C \|\nabla q_i\|_2^2 \|q_j\|^2. \]

For the term \( \int \int q_i q_j \, dx \), using (2.4)_1 again, one has
\[ -R \int \int q_i q_j \, dx = R \int \int q_i (\text{div } u + \text{div } (qu)) \, dx \]
\[ \leq \varepsilon \|\nabla u_i\|^2 + C \|\nabla q_i\|_2^2 \|q_j\|^2. \]

Plugging all above inequalities into (5.3), we obtain
\[ \frac{R}{2} \frac{d}{dt} \int_\Omega q_i^2 \, dx - \frac{R}{2} \frac{d}{dt} \int_\Omega q_i^2 \, dx + R \int \text{div } u_i \, dx \]
\[ \leq \varepsilon \|\nabla u_i\|^2 + C \|\nabla q_i\|_2^2 \|q_j\|^2. \]  
(5.4)

In a similar way, we get the following inequalities, with the help of (2.4)_2,
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u_i|^2 \, dx + R \int u_i \cdot \nabla q_i \, dx + R \int u_i \cdot \nabla \theta_i \, dx + \mu \|\text{curl } u_i\|^2 + (2 \mu + \lambda) \|\text{div } u_i\|^2 \]
\[ \leq (\varepsilon + \|\nabla u_i\|^2) (\|\nabla u_i\|^2 + \|\text{curl } H_i\|^2) + C \|\text{curl } H_i\|_2^2 + \|\nabla q_i, \nabla \theta_i\|_1^2 \]
\[ + \|\text{curl } H_i\|_2^2 \cdot (\|q_i\|^2 + \|u_i\|^2 + \|\theta_i\|^2 + \|H_i\|^2). \]  
(5.5)
From (2.4)_3, one has
\[
\frac{c_v}{2} \frac{d}{dt} \int_{\Omega} \rho |\theta_t|^2 \, dx + R \int_{\Omega} \theta_t \text{div} u_t \, dx + \kappa \| \nabla \theta_t \|^2 \\
\leq (\varepsilon + \| \nabla u_t \|^2) (\| \nabla \theta_t \|^2 + \| \nabla u_t \|^2 + \| \text{curl} \, H_t \|^2) \\
+ C_v (\| \theta_t \|^2 + \| \nabla q, \nabla \theta \|^2 + \| (\nabla u, \text{curl} \, H) \|^2) \\
\cdot (\| q_t \|^2 + \| u_t \|^2 + \| \theta_t \|^2).
\] (5.6)

It follows from (2.4)_4 that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |H_t|^2 \, dx + \eta \| \text{curl} \, H_t \|^2 \leq \varepsilon (\| \nabla u_t \|^2 + \| \text{curl} \, H_t \|^2) \\
+ C_v \| (\nabla u, \text{curl} \, H) \|^2 \| H_t \|^2.
\] (5.7)

Summing up the estimates (5.4)–(5.7), and using proposition 2.3, we have
\[
\frac{1}{2} \frac{d}{dt} \left\{ R \| q_t \|^2 + \| \sqrt{\rho} u_t \|^2 + c_v \| \sqrt{\rho} \theta_t \|^2 + \| H_t \|^2 \right\} - R \frac{d}{dt} \int_{\Omega} q q_t^2 \, dx \\
+ c_v \| \nabla u_t \|^2 + \kappa \| \nabla \theta_t \|^2 + \eta \| \text{curl} \, H_t \|^2 \\
\leq (\varepsilon + \| \nabla u_t \|^2) (\| \nabla \theta_t \|^2 + \| \nabla u_t \|^2 + \| \text{curl} \, H_t \|^2) \\
+ C_v (\| u_t \|^2 + \| \theta_t \|^2 + \| (\nabla q, \nabla u, \text{curl} \, H) \|^2 + \| \nabla \theta \|^2) \\
\cdot (\| q_t \|^2 + \| u_t \|^2 + \| \theta_t \|^2 + \| H_t \|^2),
\] (5.8)

where \( c_v > 0 \) is some positive constant depending on \( \mu, \lambda \) and \( C_{\Omega} \), and \( C_{\Omega} \) is given as in proposition 2.3. Define
\[
\phi(t) = \frac{1}{2} \left\{ R \| q_t \|^2 + \| \sqrt{\rho} u_t \|^2 + c_v \| \sqrt{\rho} \theta_t \|^2 + \| H_t \|^2 \right\} - R \frac{1}{2} \int_{\Omega} q q_t^2 \, dx \\
+ \frac{1}{2} \left\{ \| \sqrt{\rho} u_t \|^2 + c_v \| \sqrt{\rho} \theta_t \|^2 + \| H_t \|^2 \right\},
\]
then it is obvious that
\[
\| (q_t, u_t, \theta_t, H_t) \|^2 \leq C \phi(t).
\]

Moreover, in view of (5.1), there exists a positive constant \( T_1 \) such that
\[
\varepsilon + \| \nabla u \|^2 \leq \frac{1}{2} \min(c_v, \kappa, \eta), \quad \forall \ t \geq T_1.
\]

Therefore,
\[
\phi'(t) \leq a(t) \phi(t), \quad \forall \ t \geq T_1,
\]
where
\[
a(t) = C_v (\| u_t \|^2 + \| \theta_t \|^2 + \| (\nabla q, \nabla u, \text{curl} \, H) \|^2 + \| \nabla \theta \|^2).
\]
We have
\[
\int_0^\infty \phi(t) \, dt < \infty, \quad \int_0^\infty a(t) \, dt < \infty.
\]
due to \((q, u, \theta, H) \in X(0, \infty; E_0)\). Lemma 2.7(i) then implies the assertion of lemma 5.1. □

**Lemma 5.2.** For every solution \((q, u, \theta, H) \in X(0, \infty; E_0)\) of the problem \((2.4)\)–\((2.6)\), there exists a time \(T_2 = T_2(q, u, \theta, H) \geq T_1 > 0\) such that

\[
\|(\nabla u, \nabla \theta, \nabla H)\|^2(t) \leq C\|(q, u, \theta, H)\|(t), \quad \forall \ t \geq T_2, \tag{5.9}
\]

\[
\|(\nabla^2 \theta, \nabla^2 H)\|^2(t) \leq C\|(q, u, \theta, H)\|(t), \quad \forall \ t \geq T_2. \tag{5.10}
\]

**Proof.** Multiplying \(q_t + \text{div}(\rho u) = 0\) by \(R(\theta + 1)\ln(q + 1)\) and integrating over \(\Omega\), after integration by parts with the boundary condition \(u \cdot n|_{\partial \Omega} = 0\), we have

\[
- \int_\Omega R(\theta + 1)u \cdot \nabla q \, dx = - \int_\Omega Rq(\theta + 1)\ln(q + 1) \, dx
\]

\[
+ \int_\Omega R\rho \ln(q + 1)u \cdot \nabla \theta \, dx
\]

\[
\leq C\|q\| + C\|q\|_{L^3}\|u\|_{L^6}\|\nabla \theta\|, \tag{5.11}
\]

where we have used the fact

\[
|\ln(q + 1) - \ln 1| \leq C[q].
\]

Let us rewrite \((2.4)_2\) in the following form

\[
\rho(u_t + u \cdot \nabla u) - \mu \Delta u - (\mu + \lambda)\nabla \text{div} u + R(\theta + 1)\nabla q + R\nabla \theta
\]

\[
= -Rq\nabla \theta + \text{curl} \ H \times H.
\]

Multiply this equation by \(u\), it then follows from and integration by parts that

\[
\mu\|\text{curl} \ u\|^2 + (2\mu + \lambda)\|\text{div} u\|^2 + \int_\Omega R(\theta + 1)u \cdot \nabla q \, dx - \int_\Omega R \text{div} u \theta \, dx
\]

\[
- \int_\Omega (\text{curl} \ H \times H) \cdot u \, dx
\]

\[
= - \int_\Omega \rho u_t \cdot u \, dx - \int_\Omega \rho(u \cdot \nabla)u \cdot u \, dx - \int_\Omega Rqu \cdot \nabla \theta \, dx.
\]

By Hölder’s inequality, it leads to

\[
\mu\|\text{curl} \ u\|^2 + (2\mu + \lambda)\|\text{div} u\|^2 + \int_\Omega R(\theta + 1)u \cdot \nabla q \, dx - \int_\Omega R \text{div} u \theta \, dx
\]

\[
- \int_\Omega (\text{curl} \ H \times H) \cdot u \, dx
\]

\[
\leq C\|u_t\| + C\|(u, q)\|_{L^3}\|(\nabla u, \nabla \theta)\|^2. \tag{5.12}
\]
Similarly, multiplying (2.4) by \( \theta \), we have
\[
\kappa \| \nabla \theta \|^2 + R \int_\Omega \text{div } u \theta \, dx = \int_\Omega (-c_i \rho(\theta_i + u \cdot \nabla \theta) - R(\rho \theta + q) \text{div } u \\
+ \lambda (\text{div } u)^2 + 2\mu |S(u)|^2 + \eta |\text{curl } H|^2) \theta \, dx,
\]
which implies
\[
\kappa \| \nabla \theta \|^2 + R \int_\Omega \text{div } u \theta \, dx \leq C \| \theta_t \| + C \| \theta \|_{L^\infty} + C \| (u, q, \theta) \|_{L^3}) \times \| (\nabla u, \nabla \theta, \text{curl } H) \|^2. \tag{5.13}
\]
And, multiplying (2.4) by \( H \), and integrating by parts with the boundary condition \( \text{curl } H \times n \mid_{\partial \Omega} = 0 \), it shows
\[
\eta \| \text{curl } H \|^2 - \int_\Omega \text{curl}(u \times H) \cdot H \, dx = -\int_\Omega H_t \cdot H \, dx \leq C \| H_t \|. \tag{5.14}
\]
Putting (5.11)–(5.14) together, we arrive at
\[
\mu \| \text{curl } u \|^2 + (2\mu + \lambda) \| \text{div } u \|^2 + \kappa \| \nabla \theta \|^2 + \eta \| \text{curl } H \|^2 \\
\leq C \| (q_t, u_t, \theta_t, H_t) \| + C \| \theta \|_{L^\infty} + C \| (u, q, \theta) \|_{L^3}) \times \| (\nabla u, \nabla \theta, \text{curl } H) \|^2.
\]
where we have used the fact
\[
\int_\Omega (\text{curl } H \times H) \cdot u \, dx + \int_\Omega \text{curl}(u \times H) \cdot H \, dx = 0.
\]
The same argument as (5.8) of lemma 5.1 implies the desired estimate (5.9).
In order to prove (5.10), we employ (2.4) to get the estimate for \( \Delta \theta \) first. For this, we write
\[
\kappa \Delta \theta = c_i \rho(\theta_i + u \cdot \nabla \theta) + R \text{div } u - \lambda (\text{div } u)^2 - 2\mu |S(u)|^2 + R(\rho \theta + q) \text{div } u \\
- \eta |\text{curl } H|^2
\]
which implies
\[
\| \Delta \theta \|^2 \leq C (\| \theta_t \|^2 + \| \nabla u, \nabla \theta, \text{curl } H \|)^2. \tag{5.15}
\]
Recall the standard elliptic estimate in lemma 2.6 to obtain
\[
\| \nabla^2 \theta \|^2 \leq C (\| \Delta \theta \|^2 + \| \nabla \theta \|^2) \quad \text{for } \theta \in H^2(\Omega) \quad \text{with } \left. \frac{\partial \theta}{\partial n} \right|_{\partial \Omega} = 0, \tag{5.16}
\]
which, together with (5.9) and (5.15), induces that
\[
\| \nabla^2 \theta \|^2 \leq C \| (q_t, u_t, \theta_t, H_t) \|, \quad \forall \, t \geq T_2. \tag{5.17}
\]
Similarly, we have the following equation from (2.4):
\[
\eta \Delta H = H_t - \text{curl}(u \times H),
\]
which implies that
\[ \| \Delta H \|^2 \leq \| H_t \|^2 + \| \nabla H \|^2 + \| H \|_{L^\infty} \| \nabla u \|^2 \]
\[ \leq C \| (q_t, u_t, \theta, H_t) \| . \]

Furthermore, by the standard elliptic estimate, it holds
\[ \| \nabla^2 H \|^2 \leq C \left( \| \Delta H \|^2 + \| \nabla H \|^2 \right) \leq C \| (q_t, u_t, \theta, H_t) \|. \]

Therefore, the proof of lemma 5.2 is completed. □

Lemma 5.3. For every solution \( (q, u, \theta, H) \in X(0, \infty; E_0) \) of the problem (2.4)–(2.6), there exists a time \( T_3 = T_3(q, u, \theta, H) \geq T_2 \) such that
\[ \| \text{curl}^2 u \|_{(t)} \leq Ct^{-1/4}, \quad \forall t \geq T_3 \] (5.18)

Proof. Recall the vorticity equation:
\[ \rho (w_t + u \cdot \nabla w) - \mu \Delta w = K + \text{curl}(\text{curl} \ H \times H), \] (5.19)

where
\[ K = -\nabla q \times (u + u \cdot \nabla u) - \rho (w \cdot \nabla) u - \rho w \text{ div} u. \]

Taking the \( L^2 \) inner product of (5.19) with \( w \), using integration by parts with \( w \times n | \partial \Omega = 0 \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} w \|^2 + \mu \| \text{curl} \ w \|^2 = \int_{\Omega} K \cdot w \ dx + \int_{\Omega} (\text{curl} \ H \times H) \cdot \text{curl} \ w \ dx. \] (5.20)

It is easy to check
\[ \int_{\Omega} K \cdot w \ dx \leq C \| (q_t, u_t, H_t, \theta_t) \| \]
and
\[ \int_{\Omega} (\text{curl} \ H \times H) \cdot \text{curl} \ w \ dx \leq \frac{\mu}{2} \| \text{curl} \ w \|^2 + C \| \text{curl} \ H \times H \|^2 \]
\[ \leq \frac{\mu}{2} \| \text{curl} \ w \|^2 + C \| (q_t, u_t, H_t, \theta_t) \|, \]
after utilizing Sobolev’s inequality and lemma 5.2. Hence, (5.20) becomes
\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} w \|^2 + \frac{\mu}{2} \| \text{curl} \ w \|^2 \leq C \| (q_t, u_t, H_t, \theta_t) \|. \] (5.21)

On the other hand, multiplication (5.19) by \( w_t \) followed by integration over \( \Omega \) gives that
\[ \frac{\mu}{2} \frac{d}{dt} \| \text{curl} \ w \|^2 + \| \sqrt{\rho} w \|_t \|^2 = -\int_{\Omega} \rho u \cdot \nabla w \cdot w_t \ dx + \int_{\Omega} K \cdot w_t \ dx \]
\[ + \int_{\Omega} \text{curl}(\text{curl} \ H \times H) \cdot w_t \ dx. \] (5.22)
Applying Sobolev’s inequality and lemma 5.2, we obtain

\begin{align*}
\int_{\Omega} \rho u \cdot \nabla w \cdot w \, dx & \leq C \| u \|_{L^\infty} \| \nabla w \| \| \sqrt{\rho} w_t \| \\
& \leq \frac{1}{8} \| \sqrt{\rho} w_t \|^2 + C \| \nabla u \|_1^2 \| \nabla w \|^2 \\
& \leq \frac{1}{8} \| \sqrt{\rho} w_t \|^2 + C \| \nabla u \|_1^2 \| \text{curl} \ w \|^2 + C \| \nabla u \|_1^2 \| w \|^2 \\
& \leq \frac{1}{8} \| \sqrt{\rho} w_t \|^2 + C \| \nabla u \|_1^2 \| \text{curl} \ w \|^2 + C \|(q_t, u_t, H_t, \theta_t)\|. 
\end{align*}

(5.23)

\begin{align*}
\int_{\Omega} K \cdot w_t \, dx & \leq \frac{1}{8} \| \sqrt{\rho} w_t \|^2 + C \|(q_t, u_t, H_t, \theta_t)\|, 
\end{align*}

(5.24)

and

\begin{align*}
\int_{\Omega} \text{curl} (\text{curl} \ H \times H) \cdot w_t \, dx & \leq \frac{1}{8} \| \sqrt{\rho} w_t \|^2 + C \| \nabla H \|_1^2 \| \nabla^2 H \|^2 \\
& \leq \frac{1}{8} \| \sqrt{\rho} w_t \|^2 + C \|(q_t, u_t, H_t, \theta_t)\|. 
\end{align*}

(5.25)

Substituting these into (5.22), we derive that

\begin{align*}
\frac{\mu}{2} \frac{d}{dt} \| \text{curl} \ w \|^2 + \frac{5}{8} \| \sqrt{\rho} w_t \|^2 & \leq C \| \nabla u \|_1^2 \| \text{curl} \ w \|^2 + C \|(q_t, u_t, H_t, \theta_t)\|. 
\end{align*}

(5.26)

Summing (5.21) and (5.26) up, one gets

\begin{align*}
\frac{d}{dt} \left\{ \| \sqrt{\rho} w \|^2 + \mu \| \text{curl} \ w \|^2 \right\} + c_0 \left\{ \| \text{curl} \ w \|^2 + \| \sqrt{\rho} w_t \|^2 \right\} \\
& \leq C \| \nabla u \|_1^2 \| \text{curl} \ w \|^2 + C \|(q_t, u_t, H_t, \theta_t)\|. 
\end{align*}

(5.27)

for some positive constant $c_0$. In view of (5.1), there exists $T_3'$ such that if $t \geq T_3'$, then

\[ C \| \nabla u \|_1^2 \leq \frac{c_0}{2}, \]

which yields

\[ \frac{d}{dt} \left\{ \| \sqrt{\rho} w \|^2 + \mu \| \text{curl} \ w \|^2 \right\} + \frac{c_0}{2} \left\{ \| \text{curl} \ w \|^2 + \| \sqrt{\rho} w_t \|^2 \right\} \]

\[ \leq C \|(q_t, u_t, H_t, \theta_t)\|. \]

So,

\[ \frac{d}{dt} \left\{ \| \sqrt{\rho} w \|^2 + \mu \| \text{curl} \ w \|^2 \right\} + \frac{c_0}{2} \| \text{curl} \ w \|^2 \leq C \|(q_t, u_t, H_t, \theta_t)\|. \]
Putting (5.9) together with the above inequality induces that
\[
\frac{d}{dt}\left\{ \| \sqrt{\rho} w \|_2^2 + \mu \| \text{curl} \ w \|_2^2 \right\} + \frac{c_0}{2} \left\{ \| \text{curl} \ w \|_2^2 + \| \sqrt{\rho} w \|_2^2 \right\} \\
\leq C \|(q_t, u_t, H_t, \theta_t)\|, \quad \forall \ t > T_3 = \max\{T'_3, T_2\}.
\]
It follows from lemma 5.1 that
\[
g'(t) + \frac{c_0}{2} g(t) \leq C t^{-1/2}, \quad \forall \ t > T_3,
\]
where \( g(t) \) is defined as
\[
g(t) \equiv \| \sqrt{\rho} w \|_2^2 + \mu \| \text{curl} \ w \|_2^2.
\]
An application of lemma 2.7(ii) implies
\[
g(t) \leq C t^{-1/2} \quad \text{for} \quad t \geq T_3.
\]
Therefore, this completes the proof of lemma 5.3.

Lemma 5.4. For every solution \((q, u, \theta, H) \in X(0, \infty; E_0)\) of the problem (2.4)–(2.6), there exists a time \(T_4 = T_4(q, u, \theta, H) \geq T_3\), such that
\[
\| \nabla^2 u(t) \| + \| \nabla q(t) \| \leq C t^{-1/4}, \quad \forall \ t \geq T_4.
\]

Proof. First, we note that
\[
\begin{cases}
-\mu \Delta u + R \nabla q = G & \text{in} \quad \Omega \\
u \cdot n|_{\partial \Omega} = 0, \quad \text{curl} \ u \times n|_{\partial \Omega} = 0
\end{cases}
\]
where
\[
G = (\mu + \lambda) \nabla \text{div} u - \rho(ut + u \cdot \nabla u) - R \nabla \theta - R \nabla (q\theta) + H \cdot \nabla H - \frac{1}{2} \nabla |H|^2
\equiv (\mu + \lambda) \nabla \text{div} u + G_1.
\]
In view of H"older’s inequality, Sobolev’s inequality and lemma 5.2, it is easy to see that
\[
\|G_1\|^2 \leq C \|(q_t, u_t, \theta_t, H_t)\| + \|\theta\|_{L^\infty}^2 \|\nabla q\|^2,
\]
which implies that
\[
\|G\|^2 \leq C \|\nabla \text{div} u\|^2 + C \|(q_t, u_t, \theta_t, H_t)\| + \|\theta\|_{L^\infty}^2 \|\nabla q\|^2.
\]
Then elliptic estimate (see lemma 2.5) for the Stoke problem yields
\[
\|\nabla^2 u\|^2 + \|\nabla q\|^2 \leq C \left( \|G\|^2 + \|\text{div} u\|_2^2 + \|\nabla u\|_2^2 \right)
\leq C \|\nabla \text{div} u\|^2 + C \|(q_t, u_t, \theta_t, H_t)\| + \|\theta\|_{L^\infty} \|\nabla q\|^2.
\]
By (5.1) and lemma 2.1, there exists a time \(T_4 > T_3\) such that
\[
\|\theta\|_{L^\infty} \leq C \|\nabla \theta\|_2^2 \leq \frac{1}{4},
\]
Therefore, we obtain the following inequality

\[ \| \nabla^2 u \|^2 + \frac{3}{4} \| \nabla q \|^2 \leq C \| \nabla \text{div} \, u \|^2 + C \| (q_t, u_t, \theta_t, H_t) \| \]  

(5.32)

for \( t \geq T_4 \). Next, we shall estimate the term \( \| \nabla \text{div} \, u \| \). For this purpose, we take the derivative of the continuity equation with respect to \( x_i, 1 \leq i \leq 3 \) as follows

\[
(\text{div} \, u)_{x_i} = -\frac{1}{\rho} (q_t x_i + q x_i \text{div} \, u + u \cdot \nabla q x_i + u x_i \cdot \nabla q)
\]

\[ \equiv -\frac{1}{\rho} q x_i - \frac{1}{\rho} F_i. \]  

(5.33)

It is clear that

\[ \| F \|^2 \leq C \| \nabla q \|^2 \| \nabla u \|^2 \leq C \| (q_t, u_t, H_t, \theta_t) \|. \]  

(5.34)

which together with (5.33) implies that

\[ \| \nabla \text{div} \, u \|^2 \leq C \int_\Omega |\nabla q_t|^2 \, dx + C \| (q_t, u_t, H_t, \theta_t) \|. \]  

(5.35)

In order to deal with \( \| \nabla q_t \|^2 \), we follow the method given in [10]. Let \( d(x) := \text{dist}(x, \partial \Omega) \). There exists \( h > 0 \) such that

\[ d \in C^2(\overline{\Omega}), \quad \Gamma_h = \{ x \in \Omega \mid 0 \leq d(x) < h \}. \]

Let \( \psi \in C^\infty(\mathbb{R}^+) \) be a non-increasing function with \( \psi(d) = 1 \) for \( 0 \leq d \leq \frac{1}{4} h \), and \( \psi(d) = 0 \) for \( d \geq \frac{1}{4} h \). For \( 1 \leq i, k \leq 3 \), we define

\[ e_{ik}(x) := \delta_{ik} - \psi(d(x))^2 d_{x_i}(x) d_{x_k}(x). \]

For any smooth function \( f : \Omega \to \mathbb{R} \), one has

\[ f_{x_i} = e_{ik} f_{x_k} + \psi(d)^2 d_{x_i} d_{x_k} f_{x_k}, \quad 1 \leq i \leq 3, \]

\[ |\nabla f|^2 = e_{ik} f_{x_i} f_{x_k} + \psi(d)^2 (\nabla f \cdot \nabla d)^2. \]  

(5.36)

Since \( \psi(0) = 1 \) and \( \nabla d(x) = n(x) \) for any \( x \in \partial \Omega \), where \( n \) denotes the outer unit normal, we have

\[ e_{ik} n_{x_i} \big|_{\partial \Omega} = 0, \quad 1 \leq i \leq 3, \]

\[ e_{ik} f_{x_i} n_{x_i} \big|_{\partial \Omega} = 0. \]  

(5.37)

Therefore, we have the following equivalent form

\[ \int_\Omega |\nabla q_t|^2 \, dx = \int_\Omega e_{ik} q_{t, x_i} q_{t, x_k} \, dx + \int_\Omega \psi(d)^2 (\nabla q_t \cdot \nabla d)^2 \, dx. \]  

(5.38)
The first term on the right-hand side of (5.38) can be controlled as
\[
\int_{\Omega} e_{ik}q_{i,t}q_{i,k} \, dx = -\int_{\Omega} (e_{ik})_{x_i}q_{i,t}q_{i,k} \, dx - \int_{\Omega} e_{ik}n_{i}q_{i,t}q_{i,k} \, dx
\]
\[
+ \int_{\partial \Omega} e_{ik}n_{i}q_{i,t}q_{i,k} \, dx
\]
\[
= -\int_{\Omega} (e_{ik})_{x_i}q_{i,t}q_{i,k} \, dx - \int_{\Omega} e_{ik}q_{i,t}q_{i,k} \, dx
\]
\[
\leq C \| q_t \| \| \nabla q_t \| \leq C \| q_t \|
\]
by integrating by parts with the boundary condition (5.37). Therefore, combining (5.32) and (5.35) with (5.38) and (5.39) gives that
\[
\| \nabla^2 u \| + \| \nabla q \| \leq C \| \psi(d)(\nabla q_f \cdot \nabla d) \| ^2 + C \| (q_t, u_t, \theta_t, H_t) \| , \quad \forall \, t \geq T_3.
\]
(5.40)

Multiply (5.33) by \((2\mu + \lambda)\psi(d)d_{x_i}\), and sum up all these equations for \(i = 1, 2, 3\), to obtain
\[
\frac{2\mu + \lambda}{\rho} \psi(d)\nabla q_f \cdot \nabla d + (2\mu + \lambda)\psi(d)\nabla \div u \cdot \nabla d = \frac{2\mu + \lambda}{\rho} \psi(d)F \cdot \nabla d.
\]
(5.41)

In order to cancel the second term on the left-hand side on (5.41), we rewrite (5.29) as
\[
-(2\mu + \lambda)\nabla \div u + R\nabla q = -\mu \curl u + G_1,
\]
then multiply this identity by \(\psi(d)\nabla d\), and add the result to (5.41), to deduce
\[
\frac{2\mu + \lambda}{\rho} \psi(d)\nabla q_f \cdot \nabla d + \psi(d)\nabla q \cdot \nabla d
\]
\[
= \frac{2\mu + \lambda}{\rho} \psi(d)F \cdot \nabla d - \mu \psi(d)\curl u \cdot \nabla d + \psi(d)G_1 \cdot \nabla d \equiv Q.
\]
(5.42)

Moreover, (5.34), (5.30), lemmas 5.1 and 5.3 imply that
\[
\| Q \|^2 \leq C \| \psi d \| \| \nabla d \| ^2 + \| F \|^2 + \| G_1 \|^2
\]
\[
\leq C \| \theta \|^2 \| \nabla \theta \| ^2 + Ct^{-1/2},
\]
(5.43)

where we have used the definition of \(\psi(d)\) and \(\| \nabla n \| = 1\). Therefore, it follows from (5.31) and (5.40) that
\[
\| \nabla^2 u \|^2 + \frac{1}{2} \| \nabla q \|^2 \leq C \| \psi(d)\nabla q \cdot \nabla d \|^2 + Ct^{-1/2},
\]
(5.44)
for $t \geq T_4 > T_3$. On the other side, multiplying (5.42) by $\psi(d) \nabla q \cdot \nabla d$ yields

\[
\frac{d}{dt} \int_{\Omega} \frac{2\mu + \lambda}{\rho} (\psi(d) \nabla q \cdot \nabla d)^2 dx + R \|\psi(d) \nabla q \cdot \nabla d\|^2 \\
= \int_{\Omega} (\psi(d) \nabla q \cdot \nabla d) Q dx + \int_{\Omega} \left( \frac{2\mu + \lambda}{\rho} \right) \psi(d) \nabla q \cdot \nabla d dx \\
\leq \varepsilon \|\psi(d) \nabla q \cdot \nabla d\|^2 + C_1 (|Q|^2 + \|q_i\|^2).
\]

Therefore, from lemma 5.1, (5.1), (5.43) and (5.44), we conclude that

\[
\frac{d}{dt} \int_{\Omega} \frac{2\mu + \lambda}{\rho} (\psi(d) \nabla q \cdot \nabla d)^2 dx + R \|\psi(d) \nabla q \cdot \nabla d\|^2 \\
\leq \varepsilon \|\psi(d) \nabla q \cdot \nabla d\|^2 + C_1 (|Q|^2 + \|q_i\|^2).
\]

One can apply (ii) of lemma 2.7 to obtain

\[
\int_{\Omega} \frac{2\mu + \lambda}{\rho} (\psi(d) \nabla q \cdot \nabla d)^2 dx \leq C t^{-1/2},
\]

which, together with (5.44), finishes the proof of lemma 5.4. \qed

Now, we are ready to prove theorem 1.2.

**Proof of theorem 1.2.** Lemmas 5.2 and 5.4 yield

\[
\|(u, \theta, H)(\cdot, t)\|_{C^0(\Omega)} \leq C \|(\nabla u, \nabla \theta, \text{curl } H)(\cdot, t)\|_1 \leq C t^{-1/4}.
\]

Noting that

\[
\|q(\cdot, t)\|_{L^4}^4 \leq \|q(\cdot, t)\|_{L^4}^3 \leq C \|\nabla q(\cdot, t)\|^3 \leq C t^{-3/4},
\]

and

\[
\|\nabla q(\cdot, t)\|_{L^4}^4 \leq \|\nabla q(\cdot, t)\|_{L^\infty}^2 \|\nabla q(\cdot, t)\|_2^2 \leq C t^{-1/2},
\]

we have

\[
\|(\rho - 1)(\cdot, t)\|_{C^0(\Omega)} = \|q(\cdot, t)\|_{C^0(\Omega)} \leq C \|q(\cdot, t)\|_{H^1} \leq C t^{-3/16 + t^{-1/8}} \leq C t^{-1/8}.
\]

Hence theorem 1.2 is proved. \qed

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