Threshold for weak saturation stability

M.R. Bidgoli\textsuperscript{1}  A. Mohammadian\textsuperscript{2}  B. Tayfeh-Rezaie\textsuperscript{1}  M. Zhukovskii\textsuperscript{3,4,5}

\textsuperscript{1}School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran
\textsuperscript{2}School of Mathematical Sciences, Anhui University, Hefei 230601, Anhui, China
\textsuperscript{3}Laboratory of combinatorial and geometric structures, MIPT Moscow Region 141701, Russian Federation
\textsuperscript{4}Adyghe State University, Caucasus mathematical center Republic of Adygea, 385000, Russian Federation
\textsuperscript{5}RANEPA, Moscow, 119571, Russian Federation

bd@ipm.ir ali_m@ahu.edu.cn tayfeh-r@ipm.ir zhukmax@gmail.com

Abstract

We study $\text{wsat}(G(n, p), K_s)$, the weak $K_s$-saturation number of the Erdős–Rényi random graph $G(n, p)$, where $K_s$ is the complete graph on $s$ vertices. Korándi and Sudakov proved that the weak $K_s$-saturation number of $K_n$ is stable, i.e. it remains the same after removing edges with constant probability. We prove that there exists a threshold function for this stability property and give upper and lower bounds on the threshold. Having this, we simplify the proof of Korándi and Sudakov and make their result more general. A general upper bound for $\text{wsat}(G(n, p), K_s)$ is also provided.

Keywords: Weak saturation, Random graph, Stability.
AMS Mathematics Subject Classification (2020): 05C80, 05C35, 60K35.
1 Introduction

Given a graph $F$, an $F$-bootstrap percolation process is a sequence of graphs $H_0 \subset H_1 \subset \cdots \subset H_m$ such that, for every $i \in \{1, \ldots, m\}$, $H_i$ is obtained from $H_{i-1}$ by adding an edge that belongs to a copy of $F$ in $H_i$. The $F$-bootstrap processes were introduced by Bollobás 50 years ago [3] and can be seen as special cases of the ‘cellular automata’ introduced by von Neumann [14] after a suggestion of Ulam [17]. The $F$-bootstrap percolation is also similar to the $r$-neighborhood bootstrap percolation model having applications in physics (see, e.g., [1, 6, 13]).

A fundamental question about bootstrap percolation is, how large is the smallest possible percolating set in the $F$-bootstrap process? Formally, given two graphs $G$ and $F$, a spanning subgraph $H$ of $G$ is said to be a weakly $F$-saturated subgraph of $G$ if $H$ contains no subgraph isomorphic to $F$ and there exists an $F$-bootstrap percolation process $H = H_0 \subset H_1 \subset \cdots \subset H_m = G$. The minimum number of edges in a weakly $F$-saturated subgraph of $G$ is called weak $F$-saturation number of $G$, denoted by $\text{wsat}(G, F)$.

As usual, we denote by $G(n, p)$, the binomial random graph on vertex set $[n] := \{1, \ldots, n\}$ with the edge probability $p$ (recall that this graph contains every edge with probability $p$ independently of all the others). Korándi and Sudakov initiated the study of weak saturation number of random graphs in [10]. They proved that, for every fixed $p \in (0, 1)$ and integer $s \geq 3$, $\text{wsat}(G(n, p), K_s) = \text{wsat}(K_n, K_s)$ whp (this is the abbreviation for the common notion ‘with high probability’ meaning with a probability approaching 1 as $n \to \infty$), which is in turn $\binom{s-2}{2} + (n-s+2)(s-2)$ by a classic result proved by Lovász [12]. They also noticed that the same is true for $p \geq n^{-\varepsilon}$ for small enough $\varepsilon > 0$ and asked about smaller $p$ and about possible threshold probability [9, Chapter 1.5] for the property of having the weak $K_s$-saturation number exactly $\binom{s-2}{2} + (n-s+2)(s-2)$, which we denote by $\mathcal{A}_s$. In this paper, we prove that this threshold exists and present lower and upper bounds on that.

Before proceeding with the results, let us list the notation we use in the paper. For a graph $G$, we denote the vertex set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. For $S \subset V(G)$, we denote the induced subgraph of $G$ on $S$ by $G[S]$. For a vertex $v \in V(G)$, set $N_G(v) = \{x \in V(G) \mid v \text{ is adjacent to } x\}$. Also for a set $U \subset V(G)$ let $N_G(U) = \cap_{u \in U} N_G(u)$, and $N_G[U] = U \cup N_G(U)$.

2 A lower bound on $p$ and the existence of threshold

In this section, we prove the existence of threshold and present a lower bound.
Chapter 3 of [9]) that, for most $L$

by asymptotically Poisson random variable (for a constant $L$ also not concentrated in a unit set since the number of triangles is bounded from above vertices and at least two cycles; see Theorem 3.4 of [9]). The latter random variable is

$\sum_{i=1}^{n} X_i$. Finally, if $\lambda = \sum_{i=1}^{n} X_i$, then whp the number of triangles in $G(n, p)$ equals the difference between the number of edges and the number of triangles (whp all

Let $\lambda = \sum_{i=1}^{n} X_i$, then whp the number of triangles in $G(n, p)$ equals the difference between the number of edges and the number of triangles (whp all

(e.g., Chapter 3.1 of [9]).

To finish the proof, we need the following technical lemma about the property $\mathcal{B}_s$ saying that every edge belongs to $K_s$.

**Lemma 2.2.** Let

$$p(n) = c_s q_s(n) \left[ 1 + \frac{2 \ln n}{(s-2)^2(s^2+s) \ln n} + \frac{w_n}{\ln n} \right].$$

Then the following hold:

(i) There exists a threshold probability for the property $\mathcal{A}_s$.

(ii) If $p(n) \leq c_s q_s(n)$, then whp the property $\mathcal{A}_s$ does not hold in $G(n, p)$.

**Proof.** Below, we assume that $p > \frac{1}{n \ln n}$ since, otherwise, whp there is no $K_s$ in $G(n, p)$ and the number of edges is asymptotically smaller than $n$ (it follows from Markov’s inequality, see, e.g., Chapter 3.1 of [9]).

Let $\varepsilon \in (0, 1)$. Then, whp the number of edges of $G(n, p)$, hereinafter denoted by $e(G(n, p))$, belongs to $(n^2 p, n^2 p \frac{1}{(1+\varepsilon)})$ since it has binomial distribution with parameters $(n^2, p)$ and $p$ (it follows, for example, from Chebyshev’s inequality).

Let $X_s$ be the number of $s$-cliques in $G(n, p)$ and let $\varepsilon > 0$. It is well known (see, e.g., Chapter 3 of [9]) that, for $p \ll n^{-\frac{2}{s+1}}$, whp $X_s = o(n^2 p)$, implying $\text{wsat}(G(n, p), K_s) = e(G(n, p))(1 + o(1))$ whp since $e(G(n, p)) - X_s \leq \text{wsat}(G(n, p), K_s) \leq e(G(n, p))$. From this, we immediately get that whp $\text{wsat}(G(n, p), K_s) \neq \text{wsat}(K_n, K_s)$ for $p \ll n^{-\frac{2}{s+1}}$ such that $p \notin I := (\frac{1}{n} \frac{2k-1}{n}, \frac{1}{n} \frac{2k+1}{n})$. For $p \in I$ and $s \geq 4$, whp $X_s = 0$ and, therefore, the weak saturation number is exactly $e(G(n, p))$ that is not concentrated on a single value. Finally, if $p \in (\frac{1}{n} \frac{2k-1}{n}, \frac{1}{n} \frac{2k+1}{n})$, then whp the weak $K_3$-saturation number equals the difference between the number of edges and the number of triangles (whp all the triangles in $G(n, p)$ are disjoint since whp there are no subgraphs with at most 5 vertices and at least two cycles; see Theorem 3.4 of [9]). The latter random variable is also not concentrated in a unit set since the number of triangles is bounded from above by asymptotically Poisson random variable (for a constant $L$, the property of having at most $L$ triangles is decreasing [9, Chapter 1.3] while the asymptotical distribution of the number of triangles in $G(n, 2(1+\varepsilon)/n)$ is Poisson; see Lemma 1.10 and Theorem 3.19 of [9]). Summing up, for $p \ll n^{-\frac{2}{s+1}}$, whp $G(n, p)$ does not have $\mathcal{A}_s$.

Now, let $\gamma > 0$ be fixed and small enough, and $p \geq n^{-\frac{2}{s+1} - \gamma}$. To finish the proof, we need the following technical lemma about the property $\mathcal{B}_s$ saying that every edge belongs to $K_s$.

**Theorem 2.1.** Let $s \geq 3$ be fixed,

$$q_s(n) = n^{-\frac{2}{s+1}}(\ln n)^{\frac{2}{s^2+2s+1}}, \quad c_s = \left[ 2 \left( 1 - \frac{1}{s+1} \right) (s-2)! \right]^{\frac{2}{s+1}(s-2)}.$$
(i) If \( w_n \to \infty \), then whp \( G(n, p) \) has \( B_s \).

(ii) If \( w_n \to -\infty \), then whp \( G(n, p) \) does not have \( B_s \).

Proof. In [16], it is proven that there exists \( R \) such that, for \( p > R q_s(n) \), whp every pair of vertices have \( s - 2 \) adjacent common neighbors. Therefore, we may assume that \( w_n \leq R \ln n \).

Below, we, as usual, apply, so called, second moment method and the Janson inequality [8]. Let \( N_s \) be the number of edges that do not belong to any \( K_s \).

Let \( u, v \in [n] \) be distinct, and \( W := \{w_1, \ldots, w_{s-2}\} \subset [n] \setminus \{u, v\} \). Consider an event \( K[W] \) saying that each \( w_i \) is adjacent to both \( u, v \) and every other \( w_j \) in \( G(n, p) \). Let \( \mu(u, v) \) counts the number of sets \( W \) as above such that \( K[W] \) happens. Then,

\[
\mathbb{E} N_s = \binom{n}{2} p \mathbb{P}(\mu(1, 2) = 0). \tag{1}
\]

If \( s = 3 \), then \( \mathbb{P}(\mu(1, 2) = 0) = (1 - p^2)^{n-2} \). For \( s \geq 4 \), compute

\[
\sum_W \mathbb{P}(K[W]) = \binom{n-2}{s-2} p^{\frac{(s+1)(s-2)}{2}} =: \lambda,
\]

\[
\sum_{W_1, W_2} \mathbb{P}(K[W_1] \cap K[W_2]) = \sum_{\ell=1}^{s-3} \binom{n-2}{s-2} \binom{s-2}{\ell} \binom{n-s}{s-2-\ell} p^{\ell+1} = \Delta,
\]

where the first summation is over \( W \in \binom{[n]\setminus\{u,v\}}{s-2} \), and the second one is over distinct \( W_1, W_2 \in \binom{[n]\setminus\{u,v\}}{s-2} \) having a non-empty intersection. Then, the Janson inequality implies that

\[
\exp \left[ -\lambda \left( 1 + O \left( p^{\frac{(s+1)(s-2)}{2}} \right) \right) \right] \leq \mathbb{P}(\mu(1, 2) = 0) \leq \exp \left[ -\lambda + \Delta/2 \right].
\]

For the considered values of \( p \), \( \Delta = o(\lambda \cdot n^{-1/(s+1)} \ln n) \). Therefore,

\[
\mathbb{P}(\mu(1, 2) = 0) = \exp \left[ -\lambda \left( 1 + o \left( n^{-1/(s+1)} \ln n \right) \right) \right] = e^{-\lambda} \left( 1 + o \left( n^{-1/(s+1)} \ln^2 n \right) \right). \tag{2}
\]

If \( w_n \to \infty \), then

\[
\mathbb{E} N_s \leq \binom{n}{2} p e^{-\lambda + \Delta/2} \leq \exp[2 \ln n + \ln p - \lambda + O(1)] \to 0.
\]

Part (i) follows.
If \( w_n \to -\infty \), then
\[
\mathbb{E}N_s \geq \left( \frac{n}{2} \right)p \exp \left[ -\lambda \left( 1 + O(n^{-s-2}) \ln n \right) \right] \geq \exp \left[ 2 \ln n + \ln p - \lambda + o(1) \right] \to \infty.
\]

To finish the proof of Part (ii), we need the second moment. Similarly, using the Janson inequality, we estimate \( P(\mu(1, 2) = 0, \mu(3, 4) = 0) \) and \( P(\mu(1, 2) = 0, \mu(2, 3) = 0) \).

For \( W := \{w_1, \ldots, w_{s-2}\} \subset [n] \setminus [4] \), consider an event \( K'[W] \) saying that each \( w_i \) is either adjacent to both 1, 2, or adjacent to both 3, 4 and every other \( w_j \) in \( G(n, p) \). Let \( \mu' \) counts the number of sets \( W \) as above such that \( K'[W] \) happens. Then,
\[
P(\mu(1, 2) = 0, \mu(3, 4) = 0) \leq P(\mu' = 0). \tag{3}
\]
If \( s = 3 \), then \( P(\mu' = 0) = (1 - p^2)^{2(n-4)} \). For \( s \geq 4 \), compute
\[
\sum_{W} P(K'[W]) = \binom{n-4}{s-2} p^{\frac{(s+1)(s-2)}{2}} (2 - p^{2(s-2)}) =: \lambda' = 2\lambda(1 + O(1/n)),
\]
\[
\sum_{W_1, W_2} P(K'[W_1] \cap K'[W_2]) = \sum_{\ell=1}^{s-3} \binom{n-4}{s-2} \binom{s-2}{\ell} \binom{n-s-2}{s-2-\ell} 	imes p^{(s-2)(s-3)-\frac{(s-1)\ell}{2}} 2p^{2\ell} \left[ 2p^{2(s-2)-2\ell} \right]^2 (1 + O(p)) =: \Delta' = 8\Delta(1 + O(q_s(n)))).
\]
where the first summation is over \( W' \in \binom{[n]\setminus[4]}{s-2} \), and the second one is over distinct \( W_1, W_2 \in \binom{[n]\setminus[4]}{s-2} \) having a nonempty intersection. Then, the Janson inequality implies that
\[
P(\mu' = 0) \leq \exp \left[ -\lambda' + \Delta'/2 \right] = \exp \left[ -2\lambda(1 + O(1/n)) + 4\Delta(1 + O(q_s(n))) \right] = \exp \left[ -2\lambda + o \left( n^{-1/(s+1)} \ln n \right) \right] = e^{-2\lambda} \left( 1 + o \left( n^{-1/(s+1)} \ln^2 n \right) \right). \tag{4}
\]
In the same way, let us estimate \( P(\mu(1, 2) = 0, \mu(2, 3) = 0) \). For \( W := \{w_1, \ldots, w_{s-2}\} \subset [n] \setminus [3] \), consider an event \( K''[W] \) saying that each \( w_i \) is either adjacent to both 1, 2, or adjacent to both 2, 3 and every other \( w_j \) in \( G(n, p) \). Let \( \mu'' \) counts the number of sets \( W \) as above such that \( K''[W] \) happens. Then,
\[
P(\mu(1, 2) = 0, \mu(2, 3) = 0) \leq P(\mu'' = 0). \tag{5}
\]
If \( s = 3 \), then \( P(\mu'' = 0) = (1 - p + p(1-p)^n)^{n-3} \). For \( s \geq 4 \), compute
\[
\sum_{W} P(K''[W]) = \binom{n-3}{s-2} p^{\frac{(s+1)(s-2)}{2}} (2 - p^{s-2}) =: \lambda'' = 2\lambda(1 + o(1/\sqrt{n})),
\]

\[\sum_{W_1, W_2} P(K''[W_1] \cap K''[W_2]) = \sum_{\ell=1}^{s-3} \binom{n-3}{s-2} \binom{s-2}{\ell} \binom{n-s-1}{s-2-\ell} \times p^{(s-2)(s-3)-\ell(\ell+1)/2} p^{2(s-2)-\ell(2s-2-\ell)} (1 + O(p)) =: \Delta'' = 8\Delta(1 + O(q_s(n))),\]

where the first summation is over \([n] \setminus [s-2]\), and the second one is over distinct \(W_1, W_2 \in [n] \setminus [s-2]\) having a non-empty intersection. Then, the Janson inequality implies that

\[P(\mu'' = 0) \leq \exp \left[ -\lambda'' + \Delta''/2 \right] = \exp \left[ -2\lambda(1 + O(1/\sqrt{n})) + 4\Delta(1 + O(q_s(n))) \right] = \exp \left[ -2\lambda \left( 1 + o \left( n^{-1/(s+1)} \ln n \right) \right) \right]. \tag{6}\]

Then, relations (1)–(3) imply

\[
\text{Var}_s = EN_s^2 - \langle EN_s \rangle^2 = EN_s + \binom{n}{2} \left( \binom{n-2}{2} p^2 P(\mu(1, 2) = \mu(3, 4) = 0) + n(n-1)(n-2)p^2 P(\mu(1, 2) = \mu(2, 3) = 0) - \left( \binom{n}{2} pP(\mu(1, 2) = 0) \right)^2 \leq \text{EN}_s + n^2 p^2 e^{-2\lambda} \left( 1 + o \left( n^{-1/(s+1)} \ln^2 n \right) \right) + \right. \]

\[\left. n^3 p^2 e^{-2\lambda} \left( 1 + o \left( n^{-1/(s+1)} \ln^2 n \right) \right) - \binom{n}{2}^2 p^2 e^{-2\lambda} \left( 1 + o \left( n^{-1/(s+1)} \ln^2 n \right) \right) = o(EN_s)^2. \]

Part (ii) follows from Chebyshev’s inequality. \(\square\)

Let \(p(n)\) be as in Lemma 2.2, and \(w_n \to -\infty\). As far as \(G(n, p)\) contains an edge which is not contained in any copy of \(K_s\), we cannot hope for \(A_s\) to be held. Indeed, assuming that there exists an edge \(e\) in \(G(n, p)\) that is not contained in any \(K_s\), we immediately get that \(e\) should belong to a weakly \(K_s\)-saturated graph. Assume that \(F\) is a weakly \(K_s\)-saturated subgraph of \(G(n, p)\) with \(\binom{s-2}{2} + (n - s + 2)(s - 2)\) edges. Then \(F \setminus e\) has \(\binom{s-2}{2} + (n - s + 2)(s - 2) - 1\) edges and is weakly \(K_s\)-saturated in \(G(n, p) \setminus e\). But this is impossible since whp \(G(n, p) \setminus \{e\}\) is weakly \(K_s\)-saturated in \(K_n\) (the threshold for \(G(n, p)\) to be weakly \(K_s\)-saturated in \(K_n\) is at most \(n^{-2/(s+1)\gamma}\) \(\text{[2]}\), and the same argument works for \(G(n, p) \setminus \{e\}\)). Part (ii) of Theorem 2.1 follows.

To prove Part (i), consider \(B_s^*\), the graph property that every pair of vertices have \(s - 2\) adjacent common neighbors. This property is increasing \([9]\, \text{Chapter 1.3}\) with the sharp threshold probability \([9]\, \text{Chapter 1.6}\) \(q_s^*(n) = \left(2(s-2)!\right)^{s/(s-2)} q_s(n)\) (in the same
way as in the proof of Lemma 2.2, this fact follows from the second moment method and the Janson inequality, see details in [16]). Clearly, $A_s \cap B^*_s$ increases as well. Therefore, it has a threshold probability $r_s(n) \geq q^*_s(n)$. The following lemma concludes the proof of Theorem 2.1. □

**Lemma 2.3.** $r_s(n)$ is threshold for $A_s$.

**Proof.** Clearly, whp $G(n, p)$ has the property $A_s$ if $p \gg r_s(n)$. It remains to prove the opposite for $p \ll r_s(n)$.

Let us first assume that there exists a sequence $n_k$ and a constant $C$ such that $r_s(n_k) < Cq^*_s(n_k)$ for all large enough $k$. Then, for this sequence $n_k$, the result immediately follows from Part (ii) of Theorem 2.1.

Therefore, we may assume that $r_s(n) > w_n q^*_s(n)$ for all $n$ and some $w_n \to \infty$ as $n \to \infty$. If $p(n) < c_s q_s(n)$ for infinitely many $n$, then whp (for these $n$) $G(n, p)$ does not have the property $A_s$ by Part (ii) of Theorem 2.1. If $(1 + \varepsilon)q^*_s(n_k) \leq p(n_k) \ll r_s(n_k)$ for some $\varepsilon > 0$ and an infinite sequence $n_k$, then

$$P(G(n_k, p) \in A_s) \leq P(G(n_k, p) \notin B^*_s) + P(G(n_k, p) \in A_s \cap B^*_s) \to 0, \quad k \to \infty.$$  

Finally, assume that there exists an $\varepsilon > 0$ and an infinite sequence $n_k$ (below, we omit $k$ for simplicity of notations) such that

$$p_s(n_k) := c_s q_s(n_k) \leq p(n_k) \leq (1 + \varepsilon)q^*_s(n_k)$$

and $P(G(n_k, p(n_k)) \in A_s) \geq \delta > 0$.

Let $p_0(n) = p_s(n) \left[1 + \frac{1}{\sqrt{\ln n \ln \ln n}}\right]$. If $p < p_0$, then $G(n, p_0)$ can be obtained from $G(n, p)$ by drawing every missing edge with probability $\frac{p_0 - p}{1 - p}$. Then, by (1) and (2), the probability that there exists an edge $xy$ in $G(n, p_0) \setminus G(n, p)$ such that $x, y$ do not have $s - 2$ adjacent common neighbors in $G(n, p)$ is at most

$$\binom{n}{2} (p_0 - p) \frac{1 - p}{1 - p} \exp \left[ - \binom{n - 2}{s - 2} \frac{(s - 1)^2}{p_0} + o(1) \right] = O \left( \frac{1}{\ln \ln n} \right).$$

Therefore,

$$P(G(n, p_0) \in A_s) \geq P(G(n, p) \in A_s) - O \left( \frac{1}{\ln \ln n} \right) \geq \delta - o(1).$$

So we may assume that $p \geq p_0$. In the same manner, $G(n, (1 + \varepsilon)q^*_s(n))$ can be obtained from $G(n, p)$ by drawing every missing edge. Then, the probability that there exists an edge $xy$ in $G(n, (1 + \varepsilon)q^*_s(n)) \setminus G(n, p)$ such that $x, y$ do not have $s - 2$ adjacent common neighbors in $G(n, p)$ is at most

$$\binom{n}{2} (1 + \varepsilon)q^*_s(n) - p \frac{1 - p}{1 - p} \exp \left[ - \binom{n - 2}{s - 2} \frac{(s - 1)^2}{p_0} + o(1) \right] = O \left( \frac{\ln n}{\ln \ln n} \right).$$
Therefore,
\[ P(G(n, (1 + \varepsilon)q_s(n)) \in \mathcal{A}_s) \geq P(G(n, p) \in \mathcal{A}_s) - o(1) \geq \delta - o(1), \]
a contradiction. □

3 Upper bound on \( p \)

**Theorem 3.1.** Let \( s \geq 3 \) be an integer. If \( p \geq n^{-\frac{1}{2s-3}}(\ln n)^2 \), then \( G(n, p) \) has the property \( \mathcal{A}_s \) whp.

**Proof.** Since for \( p \) in the range, whp \( G(n, p) \) is weakly \( K_s \)-saturated in \( K_n \) (see [2]), we get that whp \( \text{wsat}(G(n, p), K_s) \geq \text{wsat}(K_n, K_s) \).

It remains to prove that whp there exists a weakly \( K_s \)-saturated subgraph of \( G(n, p) \) with \((\frac{(s-2)}{2}) + (n - s + 2)(s - 2)\) edges. Below, we define two graph properties that imply such weak saturation ‘stability’. To define these properties, let us recall the following definition. The \( k \text{-th power of a graph} G \) is the graph \( G^k \) on the same vertex set such that \( x, y \) are adjacent in \( G^k \) if and only if the distance between them in \( G \) is at most \( k \).

Let \( \text{EXT} \) be the graph property ‘every set of \( s \) vertices has \( s - 2 \) common neighbors inducing a complete graph’ and \( \text{HAM} \) be the graph property ‘for every set of \( s - 1 \) vertices, its common neighborhood induces a graph containing \((s - 2)\)-th power of a Hamiltonian path’.

The following two lemmas conclude the proof of Theorem 3.1. □

**Lemma 3.2.** If \( \text{EXT} \) and \( \text{HAM} \) hold for a graph \( G \) with \( n \) vertices, then \( \text{wsat}(G, K_s) \leq \text{wsat}(K_n, K_s) \).

**Lemma 3.3.** If \( p \geq n^{-\frac{1}{2s-3}}(\ln n)^2 \), then \( P(G(n, p) \in \text{EXT} \cap \text{HAM}) \to 1 \) as \( n \to \infty \).

**Proof of Lemma 3.3.** Assume that \( G \) is a graph satisfying \( \text{EXT} \) and \( \text{HAM} \). Let \( H_0 \) be a copy of \( K_{s-2} \) in \( G \). Define a weakly \( K_s \)-saturated spanning subgraph \( H \subset G \) as follows. \( H \) contains all edges of \( H_0 \), and also all edges of \( G \) between \( H_0 \) and \( N_G(H_0) \). We still have to add some other edges going outside \( N_G(H_0) \). For every \( v \in V(G) \setminus N_G[H_0] \), we add \( s - 2 \) edges adjacent to \( v \) described below. By the property \( \text{HAM} \), the graph \( F_v \) induced on \( N_G(V(H_0) \cup \{v\}) \) contains a copy of \((s - 2)\)-th power of a Hamiltonian path. Starting from an arbitrary vertex, denote the vertices of \( F_v \) going in the natural order induced by the Hamiltonian path, by \( x_1^v, x_2^v, \ldots, x_{f_v}^v \), where \( f_v = |V(F_v)| \). We add edges \( vx_1^v, vx_2^v, \ldots, vx_{s-2}^v \) to \( H \). It is easy to see that \( H \) is of size \((\frac{(s-2)}{2}) + (n - s + 2)(s - 2)\), so it suffices to prove that \( H \) is weakly \( K_s \)-saturated in \( G \).

First, all edges between the vertices of \( N_G(H_0) \) can be ‘infected’ since they belong to \( K_s \) containing \( H_0 \). Second, for each \( v \in V(G) \setminus N_G[H_0] \), we may ‘infect’ edges
\(v_{x_{s-1}}v_s, \ldots, xv_v\) one by one since every such edge belongs to the \(s\)-clique containing previous \(s-2\) vertices of the \((s-2)\)-th power of the Hamiltonian path. Finally, each edge \(xy\) between vertices of \(V(G) \setminus N_G[H_0]\) can be ‘infected’, since by the property EXT, \(N_G(\{x, y\} \cup V(H_0))\) contains a copy of \(K_{s-2}\), say \(K^*\). But note that \(V(K^*) \subset N_G(V(H_0))\), so \(xy\) is the last edge of the \(s\)-clique \(G[V(K^*) \cup \{x, y\}]\) and hence can be ‘infected’ as well. \(\square\)

**Proof of Lemma 3.3.** It is known \([10]\) that, for \(p \gg n^{-\frac{2}{3(\gamma-1)}} [\ln n]^{\frac{2}{3(\gamma-1)}}\), whp \(G(n, p)\) has the property EXT.

To prove the second property, notice that, for \(k \in \mathbb{N}, r \in \mathbb{N}\), if \(p(m) \gg m^{-\frac{4}{3}} (\ln m)^4\), then \(G(m, p)\) contains the \(k\)-th power of a Hamiltonian path with probability at least \(1 - m^{-r}\). This follows from the fact (see \([15]\), Theorem 1.6 of \([5]\), and Theorem 1.1 of \([11]\)) that there exists \(C\) such that if \(p^*(m) > Cm^{-\frac{1}{2}} (\ln m)^3\), then \(G(m, p^*)\) contains the \(k\)-th power of a Hamiltonian path with probability at least \(1 - 1/e\). To boost this probability to \(1 - m^{-r}\) it suffices to take the union of \(r \ln m\) independent copies of \(G(m, p^*)\) on \([m]\) and set \(p = 1 - (1 - p^*)^r \ln m\).

Fix \(W_0 \subset \{n\}\) of size \(s - 1\) and set \(m = \frac{np^{s-1}}{2}\). Then, by the Chernoff bound, the probability that the common neighborhood of \(W_0\) does not contain the \((s-2)\)-th power of a Hamiltonian path is at most

\[
P(|N_{G(n, p)}(W_0)| < m) + m^{-3s}P(|N_{G(n, p)}(W_0)| \geq m) \leq e^{-\frac{m}{4}} + n^{-s}.
\]

From the union bound, the result follows. \(\square\)

**Remark 3.4.** The power in the log factor in Theorem 3.1 is not the best possible. Using best known estimations of threshold probability for containing the \(k\)-th power of a Hamiltonian path, we may replace \([\ln n]^2\) with \([\ln n]^{1/3+\varepsilon}\) for \(s = 3\), \([\ln n]^{s/5}\) for \(s = 4\) and \([\ln n]^{1/2}\) for \(s \geq 5\).

### 4 Upper bound on \(\text{wsat}(G(n, p), K_s)\)

From the previous arguments, we know that whp \(\text{wsat}(G(n, p), K_s) = e(G(n, p))(1 + o(1))\) when \(p \ll n^{-\frac{2}{s+1}}\) and \(\text{wsat}(G(n, p), K_s) = (s-2) + (n-s+2)(s-2)\) when \(p \geq n^{-\frac{2}{3\gamma-3}} (\ln n)^3\), where \(\beta = 1/3 + \varepsilon\) for \(s = 3\), \(\beta = 8/5\) for \(s = 4\) and \(\beta = 1/2\) for \(s \geq 5\). In this section, we prove an upper bound for \(\text{wsat}(G(n, p), K_s)\) for all the remaining values of \(p\) (the trivial lower bound equals \((s-2) + (n-s+2)(s-2)\) whp).

**Theorem 4.1.** Let \(\gamma = 1\) for \(s = 3\), \(\gamma = 6\) for \(s = 4\) and \(\gamma = 0\) for \(s \geq 5\). Let \(w_n \to \infty\) as \(n \to \infty\). Then whp

\[
\text{wsat}(G(n, p), K_s) \leq n(s-2) + \frac{w_n(\ln n)^{2(\gamma+s-2)}p^{2s-3}}{p^{2s-3}}.
\]

9
Proof. For $p \leq \left((\ln n)^{\gamma+s-2}/n\right)^{1/(s-1)}$, the result is immediate since whp $|E(G(n,p))| \ll \frac{w_n(\ln n)^{2(\gamma+s-2)}}{p^{s-3}}$. For $p \geq n^{-\frac{1}{2s-3}}(\ln n)^{3}$ the result follows from Remark 3.4. Let 

$$((\ln n)^{\gamma+s-2}/n)^{1/(s-1)} < p < n^{-\frac{1}{2s-3}}(\ln n)^{3}.$$ 

Let $G$ be a graph and let $X$ be a subset of $V(G)$ having two properties:

(i) For every vertex $v \in \overline{X} = V(G) \setminus X$ the subgraph of $G$ induced on $N_G(v) \cap X$ contains an $(s-2)$-th power of a Hamiltonian path.

(ii) Any two distinct vertices $v, w \in \overline{X}$ have $s-2$ pairwise adjacent common neighbors in $X$.

Let $H$ be a spanning subgraph of $G$ containing all edges with both endpoints in $X$ and $(s-2)|\overline{X}|$ more edges such that every vertex $v \in \overline{X}$ belongs to exactly $s-2$ edges \{v, w_1\}, \ldots, \{v, w_{s-2}\}$ where $w_1, \ldots, w_{s-2}$ are initial vertices of an $(s-2)$-th power of a Hamiltonian path of $G[N_G(v) \cap X]$.

Let us show that $H$ is weakly $K_s$-saturated in $G$. In the same way, as in the proof of Lemma 3.2, all edges having one vertex in $X$ can be ‘infected’. Let $\tilde{H}$ have all edges of $H$ and all edges between $X$ and $\overline{X}$ that are presented in $G$. Let $v, w \in \overline{X}$ be adjacent in $\tilde{H}$. Then, the property (ii) ensures that adding the edge \{v, w\} to $\tilde{H}$ creates a copy of $K_s$. Thus, $\text{wsat}(G, K_s) \leq (s-2)|\overline{X}| + e(G[X])$.

Let $G = G(n, p)$. Let $m = \left\lfloor \frac{(\ln n)^{\gamma+s-2}}{p^{s-1}}\sqrt{w_n} \right\rfloor$, $X = [m]$. Since whp $e(G[X]) \leq m^2 p$, it remains to show that whp (i) and (ii) hold for $G$ with our choice of $X$.

As we noticed in the proof of Lemma 3.3, for any $\ell \geq \left\lceil \frac{(\ln n)^{\gamma+s-2}}{p^{s-1}} \right\rceil$, the relation $p \gg \ell^{-\frac{1}{2s-3}}(\ln n)^{\gamma/(s-2)+1}$ implies that $G(n, p)[\ell]$ contains the $(s-2)$-th power of a Hamiltonian path with probability at least $1 - o(1/n)$ (actually, the power of log-factor in the proof of Lemma 3.3 is 4, but it can be improved in this way due to best known estimations of threshold probability for containing the $k$-th power of a Hamiltonian path, see [7]). Given a vertex $v \in [n] \setminus [m]$, with probability $o(1/n)$, it has less than $\frac{(\ln n)^{\gamma+s-2}}{p^{s-1}}$ neighbors in $[m]$ by the Chernoff bound. Property (i) follows.

Finally, by the Janson inequality [8], the probability that, there exist distinct $v, w \in \overline{X}$ without $s-2$ pairwise adjacent common neighbors in $[\tilde{m}]$, $\tilde{m} = \left\lfloor \frac{\ln n}{p^{s+1}/2} \right\rfloor < m$, is at most

$$\binom{n}{2} \exp \left(-\left(\frac{\tilde{m}}{s-2}\right)^{p(s+1)/2}(s-2)^{-1} + \sum_{t=1}^{s-3} \binom{\tilde{m}}{s-2} \binom{s-2}{t} \binom{\tilde{m} - s + 2}{s-2-t} \binom{\tilde{m} - s + 2}{s-2-t} p(s+1)(s-2)-2t-(t^2/2) \right) \leq \exp \left(2\ln n - \frac{1}{(s-2)!} \tilde{m}^{s-2} p(s+1)(s-2)/2 (1 - o(1)) \right) \to 0, \quad n \to \infty.$$
Acknowledgments

Maksim Zhukovskii gratefully acknowledge the financial support from the Ministry of Educational and Science of the Russian Federation in the framework of MegaGrant no 075-15-2019-1926

References

[1] J. Adler and U. Lev, Bootstrap percolation: visualizations and applications, Braz. J. Phys. 33 (2003), 641–644.

[2] J. Balogh, B. Bollobás and R. Morris, Graph bootstrap percolation, Random Structures Algorithms 41 (2012), 413–440.

[3] B. Bollobás, Weakly k-saturated graphs, 1968 Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967), pp. 25–31, Teubner, Leipzig.

[4] B. Bollobás and A. Thomason, Threshold functions, Combinatorica 7 (1987), 35–38.

[5] M. Fischer, N. Škorić, A. Steger and M. Trujić, Triangle resilience of the square of a Hamilton cycle in random graphs, arXiv:1809.07534 (2018).

[6] L.R. Fontes, R.H. Schonmann and V. Sidoravicius, Stretched exponential fixation in stochastic Ising models at zero temperature, Comm. Math. Phys. 228 (2002), 495–518.

[7] A. Frieze, Hamilton cycles in random graphs: a bibliography, arXiv:1901.07139 (2019).

[8] S. Janson, T. Łuczak and A. Rucinński, An exponential bound for the probability of nonexistence of a specified subgraph in a random graph, Random graphs '87, (Poznań, 1987), 73–87 Wiley, Chichester, 1990.

[9] S. Janson, T. Łuczak and A. Rucinński, Random Graphs, John Wiley & Sons, Inc., 2000.

[10] D. Korándi and B. Sudakov, Saturation in random graphs, Random Structures Algorithms 51 (2017), 169–181.

[11] D. Kühn and D. Osthus, On Pósas conjecture for random graphs, SIAM J. Discrete Math. 26 (2012), 1440–1457.
[12] L. Lovász, Flats in matroids and geometric graphs, Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977), pp. 45–86, Academic Press, London, 1977.

[13] R. Morris, Zero-temperature Glauber dynamics on $\mathbb{Z}^d$, Probab. Theory Related Fields 149 (2011), 417–434.

[14] J. von Neumann, Theory of Self-Reproducing Automata, Univ. Illinois Press, Urbana, 1966.

[15] L. Pósa, Hamiltonian circuits in random graphs, Discrete Math. 14 (1976), 359–364.

[16] J.H. Spencer, Threshold functions for extension statements, J. Combin. Theory Ser. A 53 (1990), 286–305.

[17] S. Ulam, Random processes and transformations, Proceedings of the International Congress of Mathematicians, Vol. 2, Cambridge, Mass., 1950, pp. 264–275, Amer. Math. Soc., Providence, R. I., 1952.