Weakly Chained Spaces

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Abstract

We introduce “weakly chained spaces”, which need not be locally connected or path connected, but for which one has a reasonable notion of generalized fundamental group and associated generalized universal cover. We show that all path connected spaces, and all connected boundaries of proper, geodesically complete CAT(0) spaces are weakly chained. In the compact metric case, “weakly chained” is equivalent to the notion of “pointed 1-movable” from classical shape theory. From this we derive the following result: If $G$ is a group that acts properly and co-compactly by isometries on a geodesically complete CAT(0) space then $G$ is semi-stable at infinity. This gives a partial answer to a long-standing problem in geometric group theory. In contrast to the rather complex definition of pointed 1-movable, “weakly chained” can be defined in a single paragraph using only the definition of metric space. This simple definition facilitates many proofs.

1 Introduction

In this paper we introduce weakly chained spaces, which we will show are the most general metrizable spaces (Corollary 31) for which there is a generalized universal cover in the sense of Berestovskii-Plaut ([1], [4]). By “space” we mean a one of the following: a metrizable topological space (to understand topological invariants), a metric space (to understand uniform invariants), or a uniform space (the most natural setting for these ideas). In this Introduction we will discuss the situation for metric spaces first, followed by uniform spaces, followed by topological spaces. In the compact metric case, weakly chained is equivalent to the notion of “pointed 1-movable” from classical shape theory (Remark 33), but since we are interested in covering spaces it is essential to move beyond the compact case. “Pointed 1-movable” also involves rather complex definitions ([20]) that we will not recall in this paper. In contrast, for metric spaces “weakly
A single point (excluding endpoints). The ε-chains denoted by \([\alpha]\) is a finite sequence of ε-chains \(\eta = [\alpha_0, ..., \alpha_m = \beta]\) all with the same endpoints such that each \(\alpha_i\) differs from \(\alpha_{i+1}\) by adding or removing a single point (excluding endpoints). The ε-homotopy equivalence class of \(\alpha\) is denoted by \([\alpha]_\varepsilon\). A metric space \(X\) is called \(\varepsilon\)-homotopic to \([x, y]_\varepsilon\); that is, \(\alpha\) is \(\varepsilon\)-homotopic to the two-point chain consisting of its endpoints.

In almost as simple terms we can define the uniform fundamental group \(\pi_U(X)\) ([4]), which for compact weakly chained spaces is naturally isomorphic to the classical shape group (Remark 33). Fixing a basepoint * in a chain connected metric space \(X\), for any \(\varepsilon > 0\) the group \(\pi_\varepsilon(X)\) consists of all \([\lambda]_\varepsilon\), where \(\lambda = \{\ast = x_0, ..., x_n = \ast\}\) is an \(\varepsilon\)-loop, with group operation induced by concatenation. When \(0 < \delta < \varepsilon\) there is a natural homomorphism \(\pi_\delta(X) \to \pi_\varepsilon(X)\) defined by \(\phi_{\varepsilon\delta}([\lambda]_\delta) = [\lambda]_\varepsilon\). These maps form an inverse system, and the inverse limit is by definition \(\pi_U(X)\). If \(X\) is weakly chained and \(\pi_U(X)\) is trivial then \(X\) is called universal; we show (Corollary 30) that any path connected, simply connected space is universal.

Discrete homotopy theory was first developed by Berestovskii-Plaut in [1] for topological groups. We produced covering spaces of topological groups using a construction discovered independently by Schreier in 1925 ([27]) and Mal’tsev in 1941 ([19]). The inverse limit of these covering groups is a kind of generalized universal covering group. The Schreier-Mal’tsev construction required an underlying local group structure, but in [1] Berestovskii-Plaut found an equivalent construction that uses discrete chains and homotopies and does not require a local group structure. This construction could be used more generally for uniform spaces ([4]). In the metric case, the Berestovskii-Plaut construction may be described roughly as follows ([23]). Imitating the standard construction of the universal covering map, fix a basepoint * and let \(X_\varepsilon\) denote the set of all \([\alpha]_\varepsilon\), where \(\alpha\) is an \(\varepsilon\)-chain starting at *. Then \(\phi_\varepsilon : X_\varepsilon \to X\) is the endpoint map; this is a traditional regular covering map (except \(X_\varepsilon\) may not be connected) with deck group \(\pi_\varepsilon(X)\). When \(0 < \varepsilon < \delta\), a mapping \(\phi_{\varepsilon\delta} : X_\varepsilon \to X_\delta\) is defined by \(\phi_{\varepsilon\delta}([\alpha]_\varepsilon) = [\alpha]_\delta\). These mappings form an inverse system called the fundamental inverse system and the inverse limit is called \(\bar{X}\). This construction always exists, and the question becomes what are the properties of \(\bar{X}\) and the natural projection \(\phi : \bar{X} \to X\)?

We will give precise details later, but for simplicity in this Introduction we will give only an intuitive definition of generalized regular covering map as defined by Plaut in [21] (perhaps confusingly just called “covers” in that paper): A
traditional regular covering map of a topological space may be roughly described as “the quotient map of a discrete action by a group of homeomorphisms”. Analogously, a generalized regular covering map of a uniform space may be roughly described as “the quotient map of a prodiscrete action by a group of uniform homeomorphisms”. In either case, as is traditional in geometry, we will call the group in question the “deck group” of the generalized covering map. If $\phi : \tilde{X} \to X$ is a generalized regular covering map such that $\tilde{X}$ is universal, then following [4] we will call it the uniform universal cover (UU-cover) of $X$. The UU-cover has the following properties analogous to those of traditional universal covers. **Universal** (Theorem 3 in [4]): If $g : Y \to X$ is a generalized regular covering map then $\phi$ factors through it, i.e. there is a unique (up to basepoint) generalized regular covering map $h : \tilde{X} \to Y$ such that $\phi = g \circ h$. In particular, the UU-cover is unique up to uniform homeomorphism. **Lifting** (Theorem 5 in [4]): If $Y$ is universal and $f : Y \to X$ is uniformly continuous then there is a unique (up to basepoint) uniformly continuous map $\tilde{f} : \tilde{Y} \to \tilde{X}$ such that $f = \phi \circ \tilde{f}$. **Functorial** (Theorem 2 in [4]): If $f : Y \to X$ is uniformly continuous then there is a unique (up to basepoint) uniformly continuous map $\tilde{f} : \tilde{Y} \to \tilde{X}$ that commutes with $f$ and the respective UU-covers. This map also induces a homomorphism $f_# : \pi_U(X) \to \pi_U(Y)$. **Deck Group** (Theorem 6 in [4]): The deck group of the UU-cover is naturally isomorphic to $\pi_U(X)$ (via liftings of discrete loops).

In [4] these properties were proved under the assumption that the uniform spaces in question satisfied a condition called “coverable”, which we will describe after more background. The next result shows that all of these theorems apply to metrizable weakly chained spaces.

**Theorem 1** Every metrizable weakly chained uniform space is coverable, and every coverable uniform space is weakly chained.

In [4] we showed that metrizable coverable (hence weakly chained) spaces can be totally disconnected, and also include all Peano continua and some compact spaces that are not path connected, such as the closed Topologist’s Sine Curve (or Warsaw Circle). In this paper we extend this list of weakly chained spaces with the next two theorems:

**Theorem 2** Every path connected uniform space is weakly chained.

**Remark 3** For locally compact topological groups with their unique invariant uniform structures, Berestovskii-Plaut showed that coverable is equivalent to path connected (Theorem 7, [2]). On the other hand, the character group $G$ of the discrete group $\mathbb{Z}^N$ was shown by Dixmier in 1957 ([12]) to be a compact (non-metrizable!), connected, locally connected topological group that is not path connected. Therefore $G$ is not coverable. On the other hand, since $G$ is compact and locally connected, it is weakly chained (Remark [14]). On the other hand, as shown in the proof of Theorem 9 in [2], $\phi : \tilde{G} \to G$ is not surjective. This shows
that “metrizable” cannot be removed from Theorem 1 or Theorem 2. Nonetheless, since $G$ is a compact, connected group, $G$ has a (compact!) universal cover in yet another sense ([3]).

Recall that a CAT(0) space is a geodesic space that has non-positive curvature in the sense of Alexandrov triangle comparisons (see [6] for more details). The space is called proper if closed, bounded subsets are compact. CAT(0) spaces have a notion of boundary that we will describe in more detail later. It is known that connected boundaries of Gromov Hyperbolic groups ([6]) must be Peano continua ([5], [30]). On the other hand, arbitrary metric compacta can be realized as boundaries of CAT(0) spaces (attributed to Gromov with a proof sketched in [16], Proposition 2). Nonetheless, with some restrictions there are constraints on boundaries of CAT(0) spaces. For example, if $X$ is a proper CAT(0) space that is co-compact in the sense that there is an action by a group of isometries with compact quotient, then there are the following known constraints: According to Swenson ([31]), $X$ must be finite dimensional. According to Geoghegan-Ontaneda ([16]) if the dimension of $X$ is $d$ then the $d$-dimensional Čech cohomology with integer coefficients is non-trivial. In the same paper, the authors show that co-compact proper CAT(0) spaces must be “almost geodesically complete” in a sense attributed to Mike Mihalik that extends the following notion of geodesically complete (also known as the geodesic extendability property): Every geodesic extends to a geodesic defined for all $\mathbb{R}$.

**Theorem 4** If $X$ is a proper CAT(0) space such that for some $x_0 \in X$ and $r > 0$ every geodesic starting at $x_0$ of length $r$ extends to a geodesic ray (in particular if $X$ is geodesically complete) then $\partial X$ is weakly chained, hence pointed 1-movable.

Upon receiving a preprint of this paper, Kim Ruane almost immediately pointed out the following corollary, which gives one answer to a long-standing problem in geometric group theory. Recall that a group $G$ is said to be CAT(0) if it acts properly and co-compactly by isometries on some CAT(0) space $X$. $G$ is said to be semistable at infinity if for some $X$, $\partial X$ is semistable at infinity. We do not need to define the latter property for this paper, but it is known to be a quasi-isometric invariant of $X$ (see for example [14]). That is, if it is true for one such $X$ then it is true for every such $X$ and it follows that being semistable at infinity is a property of the group independent of $X$. According to the second remark in [13], a proper CAT(0) space is semistable at infinity if and only if its boundary is pointed 1-connected. Putting these observations together with Theorem 4 we obtain:

**Corollary 5** If a group $G$ acts properly and co-compactly on a geodesically complete CAT(0) space, then $G$ is semistable at infinity.

Here is a sketch of the proof of Theorem 4. For $x_0 \in X$ and $r > 0$, denote the metric sphere of radius $r$ at $x_0$ by $S_r(x_0) := \{y \in X : d(x_0, y) = r\}$. Let $\Sigma_r(x_0)$ be the set of all $y \in S_r(x_0)$ such that there is a geodesic ray from $x_0$
through $y$. We regard $\partial X$ as the inverse limit of the spaces $\Sigma_r(x_0)$. We show that if $\Sigma_r(x_0) = S_r(x_0)$ (which the assumptions of the theorem imply is true for large $r$) then the induced geodesic metric on $\Sigma_r(x_0)$ is finite and compatible with the subspace topology (Proposition 48). We then exploit the fact that point preimages of the projections $\psi_{rs}: \Sigma_s \to \Sigma_r$ have small diameter when $r$ and $s$ are close, and apply a general theorem about inverse limits of weakly chained spaces (Theorem 41). In fact, the proof really only needs that the induced geodesic metric on $\Sigma_r(x_0)$ be finite and compatible (Theorem 47). We also show that for any proper CAT(0) space $X$, a necessary condition for $\partial X$ to be weakly chained is that $\Sigma_r(x_0)$ is weakly chained for all $x \in X_0$ and $r > 0$ (Proposition 43). The usefulness of our simple definition of “weakly chained” should be evident in the proofs of these results.

It would be interesting to know whether every metrizable weakly chained compact metric space is the boundary of a proper, geodesically complete CAT(0) space. Since solenoids are not weakly chained (Example 24), it follows that they cannot be boundaries of such spaces. But for example suspensions of Cantor sets occur as boundaries of products of trees and the real line ([6]), which are proper, geodesically complete CAT(0) spaces. More strongly, Conner-Mihalik-Tschchantz ([10]) and Croke-Kleiner (unpublished, attributed in [10]) established the existence of boundaries of proper, geodesically complete CAT(0) spaces that are not path connected. One of these examples is the well-known Croke-Kleiner example that established the non-uniqueness of boundaries of CAT(0) groups ([11]).

We do not know whether Theorem 4 is true with a weakened hypothesis of almost geodesically complete, or simply assuming co-compactness. If even the latter is true this would prove that every CAT(0) group is semistable at infinity, completely solving the above-mentioned open problem. Note that every finitely presented groups acts on a geodesically complete geodesic space, namely the standard combinatorial 2-complex associated with the Cayley graph of the group (see Lemma 8.9, Chapter 1 ([6])). CAT(0) spaces that are homology manifolds (e.g. Hadamard manifolds) must be geodesically complete ([6], Proposition 5.12 and footnote).

For topological spaces one can do the following. Every metrizable (more generally completely regular) topological space has a unique finest uniform structure compatible with its topology, called the fine uniformity. This result springs out of some category theory in [18], but we give a more concrete statement and proof for the metrizable case in Proposition 9. As is stated in [18], but as also immediately follows from Proposition 9 if topological spaces have the fine uniformity, then maps between them are continuous if and only if they are uniformly continuous.

We will say that a topological space $X$ is “weakly chained” if $X$ is metrizable and weakly chained with the fine uniformity. The UU-cover and uniform fundamental group are the (now topological!) invariants with the same names associated with the fine uniformity, and all of the above theorems have appropriate analogs for topological spaces, with “continuous” replacing “uniformly continuous”. The only issue to be clarified is that if $X$ is a weakly chained
topological space then $\widetilde{X}$ has the fine uniformity, which is shown in Proposition 35. No doubt there are results for more general topological spaces (e.g. completely regular), but our primary interest at this point metrizable spaces.

We have the following obvious corollary of Theorem 2:

**Corollary 6** If $X$ is a path connected, metrizable topological space then $X$ is weakly chained.

**Remark 7** We do not know whether there exist weakly chained topological spaces having compatible uniform structures that are not weakly chained.

In the same year that [4] appeared, Fischer and Zastrow ([13]) also defined what they called a “generalized universal cover” for topological spaces, which we will distinguish from ours by referring to it as the “Fischer-Zastrow simply connected cover”. They defined it to be a continuous function $p : \widetilde{X}_{FZ} \to X$ where $\widetilde{X}_{FZ}$ is path connected and simply connected, such that maps from path and locally path connected, simply connected spaces into $X$ lift uniquely (up to base-point) to $\widetilde{X}_{FZ}$. They worked with more general topological spaces, but showed that for metrizable spaces, if the classical homomorphism $\kappa : \pi_1(X) \to \check{\pi}_1(X)$ from the fundamental group into the shape group is injective (a property referred to as “shape injective” in [8]), then the Fischer-Zastrow simply connected cover exists. At the same time, Berestovskii-Plaut showed in [4], Proposition 80, that if a naturally defined homomorphism $\lambda : \pi_1(X) \to \pi_U(X)$, which in the compact metrizable case may be identified with $\kappa$ (Remark 33), is injective and $X$ is path connected, then the UU-cover is simply connected! It easily follows that in this case, the restriction of the UU-cover to the path component of $\widetilde{X}$ is the Fischer-Zastrow simply connected cover. This provides an alternate proof of their existence theorem under the assumption that $X$ is coverable (in the sense of [4]). Corollary 6 and Theorem 27 now show that this alternate proof requires only path connectedness.

On the other hand, suppose that $X$ is a path connected space with a Fischer-Zastrow simply connected cover. If one takes the fine uniformity on $\widetilde{X}_{FZ}$, then Corollary 30 implies that the uniform space $\widetilde{X}_{FZ}$ is universal in the sense of the present paper. According to the lifting property of $\widetilde{X}$ described above, there is a unique base-point preserving uniformly continuous lift $\tau : \widetilde{X}_{FZ} \to \widetilde{X}$, which maps onto the path component of the basepoint of $\widetilde{X}$, and which commutes with the two generalized universal covering maps. But the lifting properties of the Fischer-Zastrow map produce an inverse function to $\tau$. We obtain the following version of their main theorem with a completely different proof.

**Theorem 8** Let $X$ be a path connected metrizable space. If the map $\lambda : \pi_1(X) \to \pi_U(X)$ is injective (e.g. if $X$ is compact and shape injective) then the Fischer-Zastrow simply connected covering map of $X$ exists and is naturally identified with the restriction of the UU-cover of $X$ to its path component.
From Theorem 79 in [4] we also know that with the above assumptions, \( \tilde{X} \) is path connected if and only if the map \( \lambda \) (equivalently \( \kappa \) in the compact metrizable case) is surjective. In that case, the Fischer-Zastrow simply connected cover must be the UU-cover. If \( \lambda \) is not surjective, then in choosing between these two generalized universal covers, one must decide whether path connectedness or the regular structure and completeness of the UU-cover is more appropriate for the given problem. The Hawaiian Earring \( H \) is an example of a Peano continuum such that the map \( \lambda \) (equivalently \( \kappa \)) is not surjective (this is known to topologists for \( \kappa \), but specific calculations involving \( \lambda \) for \( H \) and other Peano continua are given in Section 7 of [4]). In general if \( X \) is any Peano continuum then by uniqueness, the UU-cover can be obtained by taking the completion as a uniform space ([18]) of the fine uniformity on the Fischer-Zastrow simply connected cover, because the path component of \( \tilde{X} \) is dense ([4], Proposition 82).

There is another connection between these two constructions related to an earlier construction of Sormani-Wei from 2001 ([28]) for metric spaces. Fischer-Zastrow in their existence proof use a construction of Spanier ([29]) involving existence of covering maps that are determined, by open coverings of a space. In the case of Sormani-Wei, they used the same construction of Spanier applied to the covering of a metric space by \( \varepsilon \)-balls. Specifically, call a path loop “\( \varepsilon \)-small” if it is of the form \( \alpha \ast \tau \ast \overline{\alpha} \), where the loop \( \tau \) is contained in an \( \varepsilon \)-ball. Spanier’s construction then provides a covering map determined by the subgroup of the fundamental group generated by \( \delta \)-small loops, which they called the \( \delta \)-cover of a metric space. Despite the completely different construction, Plaut-Wilkins showed that for compact geodesic spaces, the Plaut-Wilkins \( \varepsilon \)-cover is equivalent to the Sormani-Wei \( \delta \)-cover when \( \delta = \frac{3}{2} \varepsilon \) ([24]). Sormani-Wei did not consider the inverse limit of their \( \delta \)-covers (their are many applications of this construction to Riemannian geometry without doing so), but the inverse limit, of course, must be \( \tilde{X} \).

Since \( \varepsilon \)-small path loops are trivially \( \varepsilon \)-null in the sense of [25], they lift as loops to \( X_{\varepsilon} \). It follows that if a path loop \( \lambda \) is homotopic to arbitrarily small loops, then it lifts as a loop to every \( X_{\varepsilon} \) and hence as a loop to \( \tilde{X} \). In the compact case, the contrapositive statement gives in an alternative proof that shape injective spaces are homotopically Hausdorff in the sense of Cannon-Conner ([8]). The converse is not true. Virk-Zastrow ([32]) constructed a path and locally path connected space \( RX \) that is homotopically Hausdorff but for which the Fischer-Zastrow construction does not produce a generalized universal cover because uniqueness of path liftings fails. In other words, from the above discussion, the space is not shape injective. But according to Corollary 6 the Berestovskii-Plaut UU-cover \( RX \) nonetheless exists. In this case \( RX \) is not simply connected, but it has all the universal, lifting, and functorial properties described earlier.
2 Background

We first recall a few concepts from uniform spaces, sometimes with notation not used by classical authors, among whom notation varies somewhat. To help with the exposition and our notation, we will give a couple of proofs of very basic concepts but claim no originality for those results. Basic statements that we do not prove can be found in standard texts such as [18]. A uniform space \( X \) consists of a topological space together with a collection of symmetric subsets of \( X \times X \) called entourages, each of which contains an open set containing the diagonal. Entourages form a uniform structure if they satisfy the “triangle inequality” property that for any entourage \( E \) there is an entourage \( F \) such that \( F^2 \subset E \). A collection \( \mathcal{B} \) of entourages such that every entourage contains an element of \( \mathcal{B} \) is called a basis. The most important examples of uniform spaces are topological groups, metric spaces (with a basis of metric entourages \( E_\varepsilon := \{ (x, y) : d(x, y) < \varepsilon \} \)) and compact topological spaces. The latter have a unique uniform structure compatible with the topology, consisting of all symmetric subsets containing an open set containing the diagonal. For a topological space, being “uniformizable” (having a uniform structure compatible with a given topology) is equivalent to being completely regular. We do not know of a direct proof in the literature of the next result:

\[ \text{Proposition 9} \]
Let \( X \) be a metrizable topological space. The collection of all symmetric sets containing open sets containing the diagonal in \( X \times X \) is a uniform structure compatible with the topology that contains every uniform structure compatible with the topology.

\[ \text{Proof.} \] Let \( X \) have any metric. Let \( E \) be a symmetric open set containing the diagonal in \( X \times X \). For each \( x \in X \) there is some \( \varepsilon_x > 0 \) such that \( B(x, \varepsilon_x) \times B(x, \varepsilon_x) \subset E \). Define \( F := \bigcup_{x \in X} [B(x, \frac{\varepsilon_x}{2}) \times B(x, \frac{\varepsilon_x}{2})] \). \( F \) is clearly a symmetric open set containing the diagonal, and we claim that \( F^2 \subset E \). If \( (a, c) \in F^2 \), this by definition means that there is some \( b \in X \) such that \( (a, b) \in F \) and \( (b, c) \in F \). That is, there exist \( x, y \in X \) such that \( d(a, x) \), \( d(b, x) < \frac{\varepsilon_x}{2} \) and \( d(b, y), d(c, y) < \frac{\varepsilon_x}{2} \). Without loss of generality, \( \varepsilon_x \geq \varepsilon_y \). By the triangle inequality, \( d(c, x) < \frac{\varepsilon_x}{2} + \frac{\varepsilon_y}{2} \leq \varepsilon_x \). Since \( d(a, x) < \frac{\varepsilon_x}{2} \), \( (a, c) \in B(x, \varepsilon_x) \times B(x, \varepsilon_x) \subset E \). The last statement of the proposition is obvious, since entourages are by definition symmetric sets containing open sets containing the diagonal.

As mentioned in the Introduction, the above mentioned uniformity is called the “fine uniformity” and when topological spaces are equipped with this uniformity, maps are continuous if and only if they are uniformly continuous. In this paper we assume all spaces are Hausdorff, which is equivalent to the intersection of all entourages being the diagonal. A uniform space has a compatible metric if and only if it has a countable basis. When \( A \subset X \), we assume the subspace uniformity on \( A \) unless otherwise stated. The subspace uniformity consists of the intersections of entourages in \( X \) with \( A \times A \).
It is useful to interpret many concepts involving uniform spaces using the notion of $E$-chain, which is a sequence $\alpha = \{x_0, \ldots, x_n\}$ such that $(x_i, x_{i+1}) \in E$ for all $E$. So in a metric space, the $\varepsilon$-chains previously mentioned are precisely $E_\varepsilon$-chains. If $f$ is a function, the notation $f(\alpha)$ will always refer to the chain \( \{f(x_0), \ldots, f(x_n)\} \). It is a useful exercise to check that the classical product $E^n \subset X \times X$ is precisely the set of all $(x, y)$ such that there is an $E$-chain $\{x = x_0, \ldots, x_n = y\}$.

In [4], we called a uniform space $X$ chain connected (equivalent to what is often called “uniformly connected” in the literature) if every $x, y$ can be joined by an $E$-chain for every entourage $E$, which as previously mentioned for metric spaces we will shorten as “$x, y$ can be joined by arbitrarily fine chains”. For any $x \in X$, the chain component of $X$ containing $x$ is the largest chain connected set containing $x$, consisting of all points that can be joined to $x$ by arbitrarily fine chains. Like components, chain components are closed but need not be open.

We define the $E$-ball at $x$ to be $B(x, E) := \{y : (x, y) \in E\}$; in metric spaces, $B(x, \varepsilon) = B(x, E\varepsilon)$. A subset $U$ of a uniform space is called uniformly $F$-open for an entourage $F$ if for any $x \in U$, $B(x, F) \subset U$. If $F$ is unspecified we will simply call $U$ uniformly open. It is easy to check that uniformly open sets are both open and closed. The following lemma is useful to understand these concepts.

**Lemma 10** Let $X$ be a uniform space, $E$ be an entourage, and $x \in X$. The set $U_x^E := \{y \in X : \text{there is an $E$-chain from $x$ to $y$}\}$ is the smallest uniformly $E$-open set in $X$ containing $x$.

**Proof.** Suppose $y \in U_x^E$, i.e. there is an $E$-chain $\alpha = \{x = x_0, \ldots, x_n = y\}$. Let $z \in B(y, E)$. Since $(y, z) \in E$, $\{x = x_0, \ldots, x_n = y, z\}$ is an $E$-chain, showing $z \in U_x^E$ and hence that $U_x^E$ is uniformly open. Now suppose that $V$ is a uniformly $E$-open set containing $x$ that doesn’t contain $y$. Since $x = x_0 \in V$ there is some $i$ such that $x_i \in V$ but $x_{i+1} \notin V$. Since $V$ is uniformly $E$-open and $x_{i+1} \in B(x_i, E)$, this is a contradiction. \[\square\]

Note that by definition the chain component of $x$ is the intersection of all the sets $U_x^E$. From the above lemma also follows the classical fact that $X$ is chain connected if and only if the only uniformly open subsets of $X$ are $X$ and $\emptyset$.

A simple example to illustrate these concepts is the following: Let $\gamma_1$ be the graph of $y = \frac{1}{x}$ and $\gamma_2$ be the $x$-axis, and $X$ be the union of these sets, with the uniformity of the subspace metric (which coincides with the subspace uniformity). The sets $\gamma_i$ are open in $X$ but not uniformly open in $X$. $X$ is not connected, but is chain connected, and in particular $X$ has two components but only one chain component. One can make this example totally disconnected by restricting to only rational values of $x$, yet this new space is still chain connected.

**Remark 11** Although we will not need it in this paper, it is a straightforward exercise to show that if a uniform space has a single chain component that is not uniformly open, then it must have infinitely many chain components.
We will abuse notation involving images and inverse images of subsets of $X \times X$, for example writing $f(E)$ rather than $(f \times f)(E)$. In this notation one may take the definition of uniform continuity of $f : X \to Y$ between uniform spaces to be that for any entourage $E$ in $Y$, $f^{-1}(E)$ is an entourage in $X$. Equivalently, for every entourage $E$ in $Y$ there is an entourage $F$ in $X$ such that $f(F) \subset E$. Following [4], we say that $f$ is bi-uniformly continuous if $f$ is uniformly continuous and for every entourage $E$ in $X$, $f(E)$ is an entourage in $f(X)$. A bijective bi-uniformly continuous function is called a uniform homeomorphism. It is easy to check that if $f$ is uniformly continuous, the inverse image (resp. image) of any uniformly open (resp. chain connected) set is uniformly open (resp. chain connected).

We now very briefly recall the basics of discrete homotopy theory for uniform spaces, much of which is from [4], but [23] has additional results and uses our current notation; [23] has an exposition in the more familiar setting of metric spaces. Let $X$ be a uniform space. A discrete homotopy between $E$-chains $\alpha_0$ and $\alpha_n$ with the same endpoints is a finite sequence $\eta = \{\alpha_0, ..., \alpha_n\}$ of $E$-chains such that $\alpha_i$ differs from $\alpha_{i-1}$ by adding or removing a single point (excluding endpoints). If $\alpha_0$ is a loop and $\alpha_n$ is a single point then $\alpha_0$ is called $E$-null. We denote by $\overline{\alpha}$ the reversal of $\alpha$ and $\alpha * \beta$ denotes the concatenation of $\beta$ followed by $\alpha$ (assuming the endpoint of $\alpha$ is the first point of $\beta$). Fixing a basepoint $*$, the set of all equivalence classes $[\alpha]_E$ of $E$-chains starting at $*$ is denoted by $X_E$; the endpoint mapping is denoted $\phi_E : X_E \to X$, which is surjective if and only if every pair of points in $X$ may be joined by an $E$-chain; more strongly if $X$ is chain connected, for the metric entourage $E$, in a metric space, we denote $X_E$ simply by $X_E$ and $\phi_E$ by $\phi$. We will denote $[[x_0, ..., x_n]]_E$ simply by $[x_0, ..., x_n]_E$. There are two basic moves that are useful: adding or taking away a repeated point. For example, by adding or taking away repeated points we can always assume, up to $E$-homotopy, that two $E$-chains with the same endpoints have the same number of points. One can then work with “corresponding” points, which is useful for some arguments. Also, removing a duplicate basepoint is always the final step in a non-trivial null $E$-homotopy. As long as $X$ is chain connected, choice of basepoint does not matter up to uniform homeomorphism ([4]), and we may always choose (and we will always assume) the maps in these constructions to be basepoint preserving. For example, if $*$ is any basepoint in $X$, we choose $[*]_E$ to be the basepoint of $X_E$.

For any entourage $F \subset E$, define $F^*$ to be the set of all $([\alpha]_E, [\beta]_E)$ such that $[[\alpha * \beta]]_E = [a, b]_E$, where $a, b$ are the endpoints of $\alpha, \beta$, respectively, and $(a, b) \in F$. This is equivalent to the slightly more cumbersome definition in [4]. The set of all such $F^*$ is the basis for a uniform structure on $X_E$. If $X$ is chain connected then $\phi_E$ is a uniformly continuous regular covering map (with $X_E$ possibly not chain connected!) having deck group $\pi_E(X)$ consisting of equivalence classes of loops with operation induced by concatenation. Moreover, the entourages $F^*$ are invariant with respect to the action of $\pi_E(X)$ and $\phi_E$ is a uniform homeomorphism restricted to any $B(x, F^*)$ onto $B(\phi_E(x), F^*)$ (with $F \subset E$). Note also that by definition $\phi_E(F^*) \subset F$, and if $X$ is chain connected then $\phi_E(F^*) = F$. To see this, suppose $(x, y) \in X$. Since $X$ is chain connected,
there is an $E$-chain $\alpha$ from $*$ to $x$. Letting $\beta := \alpha * \{x, y\}$, $([\alpha]_E, [\beta]_E) \in F^*$ by definition, and $\phi_\alpha \beta = x$ and $\phi_\beta \gamma = y$.

The first part of the next Lemma is from [25] (it was hinted at but not explicitly stated in [4]):

**Lemma 12 (Chain Lifting)** Let $X$ be a uniform space and $E$ be an entourage. Suppose that $\beta := \{x_0, \ldots, x_n\}$ is an $E$-chain and $[\alpha]_E$ is such that $\phi_\beta ([\alpha]_E) = x_0$. Let $y_i := [\alpha * \{x_0, \ldots, x_i\}]_E$. Then $\tilde{\beta} := \{y_0 = [\alpha]_E, y_1, \ldots, y_n = [\alpha * \beta]_E\}$ is the unique "lift" of $\beta$ at $[\alpha]_E$. That is, $\tilde{\beta}$ is the unique $E^*$-chain in $X_E$ starting at $[\alpha]_E$ such that $\phi_\alpha (\tilde{\beta}) = \beta$. Moreover,

1. If $\beta$ is an $F$-chain then $\tilde{\beta}$ is an $F^*$-chain.
2. If $[\beta]_E = [\gamma]_E$ then $[	ilde{\beta}]_{E^*} = [\tilde{\gamma}]_{E^*}$.

**Proof.** The only new parts are the numbered statements. The first one is immediate from the construction and definition of $F^*$. For the second one, by induction we need only verify that if $\beta$ and $\gamma$ differ by a basic move, then the conclusion holds. Let $\beta := \{\bar{x}_0, \ldots, \bar{x}_n\}$, where $\phi_\beta (\bar{x}_i) = x_i$. Suppose $\gamma = \{x_0, \ldots, x_j, x, x_{j+1}, \ldots, x_n\}$. By the first part of the lemma, the $E$-chain $\{x_j, x, x_{j+1}\}$ has a unique lift to an $E^*$-chain $\kappa = \{\tilde{x}_j, \tilde{x}, \tilde{z}\}$ starting at $\tilde{x}_j$. Since $[x_j, x, x_{j+1}]_E = [\tilde{x}_j, \tilde{x}, \tilde{z}]_E$, by the first part of this lemma, $\kappa$ and the unique lift of $\{x_j, x_{j+1}\}$ must end in the same point. That is, $\tilde{z} = \tilde{x}_{j+1}$. Since $(\tilde{z}, \tilde{x}) \in E^*$, $(\tilde{x}, \tilde{x}_{j+1}) \in E^*$. That is, adding $\tilde{x}$ is a basic move. By uniqueness, $\{\tilde{x}_0, \ldots, \tilde{x}_j, \tilde{x}, \tilde{x}_{j+1}, \ldots, \tilde{x}_n\}$ is the lift of $\gamma$. Removing a point is simply the reverse operation, so the proof of the lemma is complete. 

When $F \subseteq E$ there is a natural mapping $\phi_{EF} : X_F \to X_E$ defined by $\phi_{EF}([\alpha]_F) = [\alpha]_E$. The restriction $\theta_{EF}$ of $\phi_{EF}$ to $\pi_F(X)$ is a homomorphism into $\pi_E(X)$, which is surjective or injective if and only if $\phi_{EF}$ is, respectively. In fact, $\phi_{EF}$ is a special case of the induced mapping defined in [4]: Suppose that $f : X \to Y$ is a (possibly not even continuous!) function and $E, F$ are entourages in $X, Y$, respectively, such that $f(E) \subseteq F$. Then the function $f_\#: X \to X_F$ defined by $f_\# ([\alpha]_E) = [f(\alpha)]_F$ is well-defined. Put another way (which we will use frequently without reference), if $\alpha$ and $\beta$ are $E$-homotopic in $X$ then $f(\alpha)$ and $f(\beta)$ are $F$-homotopic in $Y$. If $X$ and $Y$ are metric spaces and $f$ is 1-Lipschitz, then as a special case we have: If $\alpha$ and $\beta$ are $\varepsilon$-homotopic in $X$ then $f(\alpha)$ and $f(\beta)$ are $\varepsilon$-homotopic in $Y$.

The Chain Lifting Lemma gives a simple way to describe the natural identification defined in [4], $i_{EF} : X_F \to (X_E)_F$, whenever $F \subseteq E$ are entourages. If $[\alpha]_F \in X_F$ then $i_{EF}([\alpha]_F) = [\beta]_{F^*}$, where $\beta$ is the unique lift of $\alpha$ to the basepoint in $X_E$. We have the following commutative diagram:

$$
\begin{array}{ccc}
X_F & \xrightarrow{i_{EF}} & (X_E)_F \\
\downarrow \phi_{EF} & & \uparrow \phi_{F^*} \\
X_E & & \\
\downarrow \phi_E & & \\
X & & \\
\end{array}
$$

(1)
This identification of \( \phi_F \cdot \) with \( \phi_{EF} \) is useful because it allows one to apply theorems about \( X \) to obtain theorems about \( X_E \). As a simple example, one immediately obtains that if \( X_E \) is chain connected then \( \phi_{EF} : X_F \to X_E \) is surjective, because we already know this about \( \phi_F : (X_E)_{F'} \to X_E \).

By definition, if \( D \subset E \subset F \) are entourages, \( \phi_{DE} \circ \phi_{EF} = \phi_{DF} \) and therefore the collection \( \{ X_E, \phi_{EF} \} \) forms an inverse system, which we refer to as the fundamental inverse system of \( X \). When \( X \) is metrizable, the fundamental inverse system has a countable cofinal sequence, which is useful because elements of the inverse limit, denoted by \( \tilde{X} \), can be constructed by iteration. In this case, \( \tilde{X} \) is also metrizable. For any entourage \( E \), we will denote the projection by \( \phi^E : \tilde{X} \to X_E \). Note that \( X = X_E \) for \( E := X \times X \); in this case we denote the projection by \( \phi : \tilde{X} \to X \). When the projections are surjective, they are bi-uniformly continuous (by definition of the inverse limit uniformity).

We now recall some definitions and results from [21] concerning what we will call in this paper generalized regular covering maps. For a uniform space \( X \) we denote by \( H_X \) the group of uniform homeomorphisms of \( X \). \( H \) has a neighborhood basis at the identity making it a (Hausdorff) topological group consisting of the following sets, for all entourages \( E \): \( U(E) \) is the set of all \( g \in H_X \) such that \( (x, g(x)) \in E \) for all \( x \in X \). That is, elements of \( U(X) \) “do not move any elements of \( X \) more than \( E \”).

For any subgroup \( G \subset H_X \), we let \( N_E(G) \) denote the normal closure of \( G \) (i.e. normal subgroup generated by) the set of all \( g \in G \) such that \( (x, g(x)) \in E \) for some \( x \in X \). We say that \( G \) acts prodiscretely if for every entourage \( E \) there exists an entourage \( F \) such that \( N_E(G) \subset U(E) \). If for some \( F, N_E(G) \) is trivial, we say that \( G \) acts discretely. That is, “the only element of \( G \) that moves some element of \( X \) no more than \( E \)” is the identity function. Clearly in this case the orbit of any point in \( X \) is (uniformly) discrete. We say that \( G \) acts isomorphically if \( X \) has a basis consisting of invariant entourages, that is, entourages \( E \) such that for all \( g \in G \), \( g(E) = E \). In this language we see from the above discussion that \( \pi_E(X) \) acts discretely and isomorphically on \( X_E \), and as one might expect, taking the inverse limit, \( \pi_U(X) \) acts prodiscretely and isomorphically on \( \tilde{X} \). A quotient via such an action is precisely the definition of a generalized regular covering map in the sense of [21]. Although we do not need it here, to help with the reader’s intuition we recall the satisfying result that, conversely, any generalized regular covering map resolves as an inverse limit of discrete covering maps induced by the quotients \( G/N_E(G) \) ranging over all entourages \( E \) (Theorem 48 in [21]). This implies that the deck group of a generalized regular covering map is a prodiscrete topological group.

3 Weakly Chained Spaces

If \( X \) is a uniform space and \( F \subset E \), we will denote arbitrary chain components of \( X_E \) by \( X_E^K \), and the restriction of \( \phi_E \) to \( X_E^K \) by \( \phi_E^K \). For any entourage \( F \subset E \), we denote \( F^* \cap (X_E^K \times X_E^K) \) by \( F^K \). Note that the collection of all \( F^K \) is a basis for the uniform structure of \( X_E^K \). For the chain component of the identity
we will use “c” rather than “K”, e.g. \( \phi_E : X_E \to X \). We let \( \pi_E^c(X) \subset \pi_E(X) \) denote the stabilizer of \( X_E \) (i.e. the subgroup that leaves \( X_E \) invariant).

**Lemma 13** Let \( X \) be a uniform space and \( E \) be an entourage. Then \( [\alpha]_E \in X_E^c \) if and only if there are arbitrarily fine chains \( \beta \) such that \( [\alpha]_E = [\beta]_E \).

**Proof.** That \( [\alpha]_E \in X_E^c \) is equivalent to: for every entourage \( F \subset E \) there is an \( F^* \)-chain \( \hat{\beta} \) from \([*]_E \) to \([\alpha]_E \). If such a \( \hat{\beta} \) exists then \( \hat{\beta} \) is the unique lift of the \( F \)-chain \( \beta := \phi_E(\hat{\beta}) \), and since \( \hat{\beta} \) ends at \([\alpha]_E \), by the Chain Lifting Lemma, \([\beta]_E = [\alpha]_E \). Conversely, if there is such a \( \beta \), then by the Chain Lifting Lemma the lift of \( \beta \) to \([*]_E \) is an \( F^* \)-chain that ends at \([\alpha]_E \).

**Lemma 14** If \( X \) is a uniform space and \( F \subset E \) are entourages then \( \pi^c_E(X) = \pi_E(X) \cap X_E^c \). In particular, \( F^c \) is invariant with respect to \( \pi^c_E(X) \).

**Proof.** Let \( [\alpha]_E \in X_E^c \) and \( g = [\lambda]_E \in \pi_E(X) \); that is \( g \) is the uniform homeomorphism of \( X_E \) induced by pre-concatenation by \( \lambda \). Then \( [\alpha]_E \in X_E^c \) if and only if for every entourage \( F \subset E \) there is an \( F^* \)-chain \( \beta \) from \([*]_E \) to \([\alpha]_E \). If \( g \in \pi_E(X) \cap X_E^c \) then there is an \( F^* \)-chain \( \beta' \) from \([*]_E \) to \([\lambda]_E \). Since \( g(F^*) = F^* \), \( g(\beta') \) is an \( F^* \)-chain from \([\lambda]_E \) to \( g([\alpha]_E) \). Then \( \beta' + \beta \) is an \( F^* \)-chain from \([*]_E \) to \( g([\alpha]_E) \). Since \( F \) was arbitrary, \( g([\alpha]_E) \in X_E^c \) and \( g \in \pi^c_E(X) \). The proof of the opposite inclusion is similar.

**Definition 15** We say that an entourage \( E \) in a uniform space \( X \) is weakly \( F \)-chained if there exists some entourage \( F \subset E \) such that for every \((x, y) \in F \) there are arbitrarily fine chains \( \alpha \) joining \( x, y \) such that \([\alpha]_E = [x, y]_E \). If \( F \) is not specified we simply say that \( E \) is weakly chained. We say that \( X \) is weakly chained if \( X \) is chain connected and the uniform structure of \( X \) has a basis of weakly chained entourages.

**Remark 16** Let \( D \supset E \supset F \supset G \) be entourages. It is immediate from the definition that if \( F \) is weakly \( E \)-chained then \( G \) is weakly \( D \)-chained. In particular, if \( F \) is weakly chained then \( G \) is weakly chained. As a consequence, if \( X \) is weakly chained then every entourage in \( X \) is weakly chained. In the opposite direction, to prove that a space is not weakly chained one need only find a single entourage that is not.

**Remark 17** We have assumed that \( X \) is chain connected in the definition of weakly chained for simplicity. If one removes this assumption then the rest of the definition implies that the chain components of \( X \) are uniformly open. This means that there is, in a sense, no uniform topological connection among the chain components, and one should simply consider them separately.

**Remark 18** For metric spaces, the above definition is equivalent to the one given in the Introduction, which is equivalent to the statement that every metric entourage \( E_{\epsilon} \) is weakly \( E_{\delta} \)-chained for some \( \delta > 0 \).
Remark 19 In [25] an entourage $E$ was called chained if whenever $(x, y) \in E$ there is a chain connected set $C$ containing $x, y$ contained in $B(x, E) \cap B(y, E)$. By Lemma 32 in [22], chained entourages are weakly chained. But chained entourages must have chain connected balls, whereas weakly chained spaces need not (for example the suspension of a Cantor set). The main examples of chained spaces are Peano continua, but any compact, locally connected space is uniformly locally connected (Proposition 68, [4]), hence weakly chained. A useful way to see this for Peano Continua is to apply the Bing-Moise Theorem to obtain a compatible geodesic metric. Then if $d(x, y) < \varepsilon$, any chain $\alpha$ lying on a geodesic $\gamma$ from $x$ to $y$ is an $\varepsilon$-chain, since the distance between any pair of points on $\gamma$ is less than $\varepsilon$. Then $\alpha$ is trivially $\varepsilon$-homotopic to $\{x, y\}$ because removing any point except an endpoint is a legal move.

Theorem 20 Let $X$ be a chain connected uniform space, $F \subset E$ be a entourages. The following are equivalent:

1. $E$ is weakly $F$-chained.
2. Every $X^F_E$ is uniformly $F^*$-open.
3. The image of any $F^K$-ball is an $E$-ball.
4. Any map $\phi^K_E$ is surjective.
5. Any map $\phi^K_E$ is a covering map in which the $F$-balls are evenly covered by unions of $F^K$-balls.

Proof. $1 \Rightarrow 2$: Suppose that $E$ is weakly $F$-chained, and there exist $([\alpha]_E, [\gamma]_E) \in F^*$ such that $[\gamma]_E$ does not lie in the same chain component of $X_E$ as $[\alpha]_E$. That is, there is some entourage $D \subset F$ such that $[\alpha]_E$ and $[\gamma]_E$ are not connected by a $D^*$-chain in $X_E$. But letting $x := \phi_E([\alpha]_E)$ and $y = \phi_E([\gamma]_E)$, we have $(x, y) \in F$. By assumption there is some $D$-chain $\beta$ joining $x$ and $y$ such that $[\beta]_E = [x, y]_E$. Applying the Chain Lifting Lemma, the unique lift $\tilde{\beta}$ of $\beta$ starting at $[\alpha]_E$ is a $D^*$-chain from $[\alpha]_E$ to $[\alpha * \beta]_E = [\alpha * \{x, y\}]_E$. But since $\phi_E$ restricted to any $F^*$-ball is 1-1, the unique lift of $\{x, y\}$ starting at $[\alpha]_E$ must be $\{[\alpha]_E, [\gamma]_E\}$. That is, $[\alpha * \beta]_E = [\gamma]_E$, a contradiction.

$2 \Rightarrow 1$: Let $(x, y) \in F$. Since $X$ is chain connected, then as mentioned in the background section, $(x, y) \in \phi_E(F^*)$. This means that there exist $\tilde{x}, \tilde{y} \in F^*$ such that $\phi_E(\tilde{x}) = x$ and $\phi_E(\tilde{y}) = y$. Then $\tilde{x}$ lies in some chain component $X^K_{\tilde{E}}$, and since $X^K_{\tilde{E}}$ is uniformly $F^*$-open, $\tilde{y} \in X^K_{\tilde{E}}$. Since $X^K_{\tilde{E}}$ is chain connected, for any entourage $D \subset F$ there is a $D^*$-chain $\beta$ joining $\tilde{x}$ and $\tilde{y}$ in $X^K_{\tilde{E}}$. Then $\phi_E(\beta)$ is a $D$-chain joining $x$ and $y$. Moreover, the unique lift of $\{x, y\}$ to $\tilde{x}$ in $X_E$ is $\{\tilde{x}, \tilde{y}\}$, which has the same endpoint as $\beta$. Since $\beta$ is the unique lift of $\phi_E(\beta)$ to $\tilde{x}$, it follows from the Chain Lifting Lemma that $[x, y]_E = [\phi_E(\beta)]_E$.

$2 \Rightarrow 3$: Let $\tilde{x} \in X^K_{\tilde{E}}$, and $x := \phi_E(\tilde{x})$. Since $\phi_E(B(\tilde{x}, F^*)) = B(x, F)$, $\phi_E(B(\tilde{x}, F^K)) \subset B(x, F)$ and we need only show the opposite inclusion. If $(x, y) \in F = \phi_E(F^*)$, there exist $x', y' \in X_E$ such that $(x', y') \in F^*$, $\phi_E(x') = x$ and $\phi_E(y') = y$. Since $\phi_E(x') = \phi_E(\tilde{x})$, there is some $g \in \pi_E(X)$ such that

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therefore $\hat{y} = g(y')$ we have that $(\hat{x}, \hat{y}) \in F^*$. Since $X_E^K$ is uniformly $F^*$-open, $\hat{y} \in X_E^K$, showing that $y \in \phi_E(B(x, F^K))$, as needed.

3 $\Rightarrow$ 4: We will show that $\phi_E^K$ is surjective by showing that $\phi_E(X_E^K)$ is uniformly $F$-open—this is sufficient since $X$ is chain connected. Suppose that $x \in \phi_E(X_E^K)$ and $(x, y) \in F$. Then there is some $\hat{x} \in X_E^K$ such that $\phi_E(\hat{x}) = x$. By assumption, $\phi_E(B(\hat{x}, F^K)) = B(x, F)$. That is, there is some $\hat{y} \in B(\hat{x}, F^K) \subset X_E^K$ such that $\phi_E(\hat{y}) = y$.

4 $\Rightarrow$ 5: We already know that $\phi_E : X_E \rightarrow X$ is covering map such that $E$-balls are evenly covered by unions of $E^*$-balls. Note that $(\phi_E^K)^{-1}(B(x, F))$ is the union of the intersections of $F^*$-balls with $X_E^K$. Since $\phi_E^K$ is surjective, the only remaining question is whether $\phi_E^K$ restricted to an $F^K$-ball is surjective onto an $F$-ball, which is precisely what the fourth statement gives us.

5 $\Rightarrow$ 2: Let $\hat{x} \in X_E^K$ and suppose that $(\hat{x}, \hat{y}) \in F^*$. Letting $x := \phi_E(\hat{x})$ and $y := \phi_E(\hat{y})$, we have that $y \in B(x, F)$. Now the restriction of $\phi_E$ to $B(\hat{x}, F^K)$ is surjective onto $B(x, F)$ and therefore there is some $y' \in B(\hat{x}, F^K)$ such that $\phi_E(y') = y$. But $\phi_E$ is injective on $B(\hat{x}, F^*)$, which means that $y' = \hat{y}$ and therefore $\hat{y} \in X_E^K$.

\begin{remark}
The implication 2 $\Rightarrow$ 4 was also proved in Lemma 5, \[22\].
\end{remark}

\begin{corollary}
If $X$ is a uniform space and $E$ is weakly $F$-chained in $X$, then $E^c$ is weakly $F^c$-chained in $X_E^K$.
\end{corollary}

\begin{proof}
By definition, $X_E^K$ is chain connected. Suppose $(x, y) \in F^c$. Then $(f(x), f(y)) \in F$, so for any $D \subset F$ there is a $D$-chain $\alpha$ from $f(x)$ to $f(y)$ such that $[\alpha]_E = [f(x), f(y)]_E$. Let $\hat{\alpha}$ be the unique lift of $\alpha$ starting at $x$. By Theorem 20, $X_E^K$ is uniformly $F^*$-open, so the $D'$-chain $\hat{\alpha}$ stays inside $X_E^K$. Since $\phi_E$ is 1-1 on any $F^*$-ball, $(\hat{x}, \hat{y})$ is the unique lift of $(f(x), f(y))$ starting at $x$. By the Chain Lifting Lemma, since $[\hat{\alpha} E^c = [f(x), f(y)]_E$, $\hat{\alpha}$ and $(\hat{x}, \hat{y})$ end at the same point, namely $y$. From the Chain Lifting Lemma, $[\hat{\alpha}]_{E^c} = [x, y]_{E^c}$, and since these chains are in $X_E^K$, $[\hat{\alpha}]_{E^c} = [x, y]_{E^c}$. Since $D$ was arbitrary, this completes the proof.
\end{proof}

\begin{proof}[Proof of Theorem 2]
For any entourage $E$ in $X$, $\phi_E : X_E \rightarrow X$ is a covering map, hence has the unique path lifting property. Since $X$ is path connected, this implies that the restriction of $\phi_E$ to the path component of the identity, hence $\phi_E^1$, is surjective. The proof is now complete by Theorem 20.
\end{proof}

\begin{corollary}
Let $X$ be a chain connected uniform space and $E$ be an entourage. Then the following are equivalent:

1. $E$ is weakly chained.

2. The chain components of $X_E$ are uniformly open.

3. The restriction of $\phi_E$ to any chain component is surjective.
\end{corollary}
4. The map \( \phi_E : X_E \to X \) is a regular covering map with deck group \( \pi_E(X) \).

**Example 24** It is useful to see why some of these statements fail for the 2-adic solenoid \( \Sigma \). For this purpose we regard \( \Sigma \) as a compact topological group, namely the inverse limit of circles with their normal group structure and bonding maps that are double covers (which are also homomorphisms). As is well-known from the theory of compact, connected groups, the topology of \( \Sigma \) has an open set \( U \) of the identity in \( \Sigma \) that is locally isomorphic as a local group to \( K \times I \), where \( K \) is a Cantor set and \( I \) is an open interval in \( \mathbb{R} \). The set \( U \) uniquely determines an invariant entourage \( E_U \) in \( \Sigma \) using the rule \( (x,y) \in E_U \) if and only if \( xy^{-1} \in U \). As was shown in [1], this local isomorphism leads to the fact that \( \Sigma_{E_U} = K \times \mathbb{R} \). Intuitively speaking, \( \Sigma_{E_U} \) is the simplest topological group that can be reconstructed from relations only contained in \( U \) (which was the original idea of Schreier and Malcev), and that obviously should just be the global product \( K \times \mathbb{R} \). At any rate, the chain components of \( \Sigma_{E_U} \) are copies of \( \mathbb{R} \), and since \( K \) has no isolated points, the chain components cannot be uniformly open. Also, since \( \Sigma \) is not path connected, the restriction to chain components cannot be surjective. Since this single entourage is not weakly chained, \( \Sigma \) cannot be (Remark 16). It is also classically known that \( \Sigma \) is not pointed 1-movable (20) from which it also follows that \( \Sigma \) is not weakly chained (see Remark 33).

**Lemma 25** Suppose that \( X \) is a weakly chained uniform space and \( F \subset E \) are entourages. Then the restriction \( \phi_{EF} \) of \( \phi_E \) to \( X_F \) is onto \( X_E \).

**Proof.** Note that by Remark 16 \( E \) and \( F \) are weakly chained. We refer now to the Diagram [1] which in this case provides a uniform homeomorphism \( \iota_{EF} : X_F \to (X_E)_{F'} \). Since all maps are basepoint-preserving, the restriction of \( \iota_{EF} \) to \( X_F \) identifies the restriction \( \phi_{EF} : X_F \to X_E \) with \( \phi_{F'} : (X_E)_{F'} \to X_E \). By Corollary 22 \( X_F \) is is weakly chained. By Theorem 20 \( \phi_{F'} \) is surjective.

For a weakly chained uniform space, we now have a new inverse system \( \{X_F, \phi_{EF}\} \) in which the bonding maps \( \phi_{EF} \) are surjective. The inverse limit of this system is a subset of \( \hat{X} \). The next proposition provides a new way to produce \( \hat{X} \) and \( \pi_U(X) \).

**Proposition 26** If \( X \) is a weakly chained, then \( \lim X_E = \hat{X} \) and \( \lim \pi_E^{\xi}(X) = \pi_U(X) \). If \( X \) is also metrizable then the maps \( \phi_E : \hat{X} \to X_E \) are onto \( X_E \) for all \( E \).

**Proof.** Let \( \hat{X} \) denote \( \lim X_E \); by the prior comments, for the first part we need only show that \( \hat{X} \subset \hat{X} \). For this it suffices to show that if \( \hat{\alpha} \in \hat{X} \) then \( \phi_X(\hat{\alpha}) \in X_F \) for any \( E \). We will show the contrapositive: if \( [\alpha]_E \notin X_F \) then \( [\alpha]_E \) is not in the image of \( \phi_E \). For this, in turn, it suffices to show that there exists some \( F \subset E \) such that \( [\alpha]_E \) is not in the image of \( \phi_{EF} \). By Theorem 20 \( X_F \) is weakly \( F^* \)-open for some \( F \subset E \). Since \( [\alpha]_E \notin X_F \), this means that there is no \( F^* \)-chain from the basepoint * to \( [\alpha]_E \). On the other hand, suppose there
is some $[\beta]_F \in X^*_E$ such that $\phi_{EF}([\beta]_F) = [\alpha]_E$, i.e. $[\beta]_E = [\alpha]_E$. By the Chain Lifting Lemma, the unique lift of $\beta$ to the basepoint is an $F^*$-chain that ends at $[\alpha]_E$, a contradiction. The same proof holds for elements of $\pi_U(X) \subset \tilde{X}$.

The second statement follows, as we have already observed in a different context, by taking a countable totally ordered cofinal sequence. □

**Theorem 27** If $X$ is a metrizable uniform space then the following are equivalent.

1. $X$ is weakly chained.
2. $X$ is coverable.
3. The map $\phi : \tilde{X} \to X$ is surjective.

**Proof.** Suppose that $X$ is weakly chained. By Proposition 26, the image of each $\phi^E$ is $X^*_E$, which is uniformly open in $X_E$. It now follows from Proposition 10 in [22] that $\tilde{X}$ is universal. In fact, tracing through the proof of that result, one sees that for any entourage $E$ in $X$, $G_E := (\phi^E)^{-1}(E^*)$ is a “universal entourage” in the sense that the map $\phi_{GE} : \tilde{X}_{GE} \to \tilde{X}$ is a uniform homeomorphism. That $X$ is coverable follows from Proposition 10 and Lemma 11 in [22].

That the second part implies the third was shown in [4]. Suppose that the third statement is true. Note that since $\phi$ is surjective, $\phi_E : X_E \to X$ is surjective for every entourage $E$, and therefore $X$ is chain connected. We next show that $\phi^E$ is surjective for any entourage $E$, and it will follow from Corollary 23 that $X$ is weakly chained. Let $x \in X$, and let $E_i$ be a countable, nested basis for the uniform structure of $X$. Since $x$ is in the image of $\phi$, there exist $[\alpha_i]_E$, such that $\phi_{E,E_i}([\alpha_{i+1}]_{E,E_i}) = [\alpha_i]_E$, and all of these $\alpha_i$ are from $*\to x$. For some $n$, $E_n \subset E$, and let’s consider the lifts of the chains $\alpha_i$ to $X_E$ for all $i \geq n$. All of them end at $[\alpha_n]_E$, which implies that $[\alpha_n]_E \in X^*_E$. Since $\phi_E([\alpha_n]_E) = x$, the proof is complete. □

**Remark 28** Theorem 27 gives a strong answer to Problem 106 in [4], which asks if a space is coverable (hence universal) if $\phi : \tilde{X} \to X$ is a uniform homeomorphism.

**Proof of Theorem 1** Theorem 27 proves the statement about the metrizable case. Suppose that $X$ is a coverable uniform space. By definition, there is a basis for the uniform structure of $X$ such that for any $E$ in that basis, $\phi^E : \tilde{X} \to X_E$ is surjective. By Theorem 51 in [4], $\tilde{X}$ is chain connected, and therefore the chain component $X^*_E$ of $X_E$ contains $\phi^E(\tilde{X})$. Since $\phi : \tilde{X} \to X$ is surjective and $\phi = \phi_{E} \circ \phi^E$, the restriction of $\phi_E$ to $\phi^E(\tilde{X})$, hence $\phi^E$ is surjective. The proof is now complete by Corollary 23. □

**Remark 29** In [3], a coverable space $X$ was called “universal” if there is a basis for the uniform structure of $X$ such that for any $E$ in that basis, $\phi_E : X_E \to X$ is a uniform homeomorphism. In Corollary 52 of that paper, we showed that
for a coverable space this is equivalent to the triviality of $\pi_U(X)$ (in that paper called $\delta_1(X)$). From Theorem 27 it now follows that the definition of “universal” for metric spaces in the Introduction to the present paper is equivalent to the one in [4]. We also note that even for a universal space there may be arbitrarily small entourages $E$ such that $\pi_E(X)$ is non-trivial. In this situation, $X_E$ is not chain connected, and there are elements of $\pi_E(X)$ that “permute” the chain components of $X_E$. There are concrete examples in [9], [33] such that this occurs for metric entourages in compact metric spaces.

The next Corollary answers for metrizable spaces Problem 107 in [4].

**Corollary 30** Any simply connected, path connected metrizable space $X$ is universal.

**Proof.** Note that “space” here refers to a uniform or topological space as per the Introduction. Since $X$ is path connected, it is weakly chained. Therefore, for any entourage $E$, $\phi_{cE} : X_{cE} \to X$ is a traditional covering map (since $X_{cE}$ is connected). Since $X$ is simply connected, the deck group $\pi_{cE}(X)$ is trivial. The proof is now finished by Proposition 26.

**Corollary 31** If a uniform space $X$ has a UU-cover then it is weakly chained.

**Proof.** Lemma 11 in [22] says that if $f : X \to Y$ is a quotient via an action on a uniform space $X$ and $X$ has a universal basis that is invariant with respect to the action then $Y$ is coverable. Since this hypothesis is true by definition when $X$ has a UU-cover, the theorem is proved.

**Remark 32** Theorem 27 ultimately derives from four papers ([1], [4], [22] and the present one). Starting from scratch with metrizable spaces, in retrospect it would be more efficient to simply work with $\{X_{cE}, \phi_{cEF}\}$ from the beginning and rework the main results of [4] for this system rather than the fundamental inverse system itself. That is, start from the beginning with weakly chained spaces rather than coverable spaces, check that the maps $\phi_{cEF}$ are discrete covers and then essentially apply the original proof in [4] that $\tilde{X}$ is universal by applying similar arguments directly to inverse system $\{X_{cE}, \phi_{cEF}\}$.

**Remark 33** In [7], the authors made a simple translation of the Berestovskii-Plaut construction into the language of Rips complexes. Their construction involves a notion of “generalized paths” from shape theory of the 1970’s due to Krasinkiewicz-Minc ([17]). It is natural to consider the space of all generalized paths starting at a basepoint and imitate the traditional construction of the universal cover. Why this was done sooner is not clear, but evidently the new ingredient, “inspired by” the Berestovskii-Plaut paper, is to give this space a uniform structure. However, Section 7 of [7], billed as a “comparison” with the Berestovskii-Plaut construction, does not completely describe the situation. In fact, their construction is precisely the same as the Berestovskii-Plaut construction via a simple and natural identification of elements of $\tilde{X}$ with generalized
paths starting at the basepoint, as is discussed in the two paragraphs after Example 17 in \[22\]. The authors of [7] did consider the construction with an a priori weaker assumption than coverable, known as “uniformly joinable”, but Theorem 27 shows that this condition is, in the end, equivalent to the a priori weaker condition of weakly chained, hence coverable. The authors also proved, for compact metrizable spaces, that Theorem 27.2 and Theorem 27.3 are equivalent. Their translation into classical concepts does provide some good connections to classical shape theory that Berestovskii-Plaut were not aware of. For example, from [7], Corollary 6.6 and Theorem 27 in the present paper, one may conclude that for compact metrizable spaces, weakly chained is equivalent to the classical condition of pointed 1-moveable. It also follows that in this case, \( \pi_1(X) \) is naturally identified with the classical shape group (Corollary 6.5 in [7]), and sorting through the correspondence mentioned above, one sees that the natural mapping \( \lambda : \pi_1(X) \to \pi_U(X) \) defined by Berestovskii-Plaut is identified with the classical mapping from the fundamental group into the shape group, referred to as \( \kappa \) in the Introduction. For this reason, \( \pi_U(X) \), for a path connected metrizable topological space, can be considered as a generalized shape group. We also note that, unlike shape theory, which depends on extrinsic approximations of a space by, or embeddings into, nicer spaces, discrete homotopy theory is purely intrinsic to the space. For example, the proof of Corollary 6.5 in [7] begins with “Embed \( X \) in the Hilbert Cube \( Q \).”

Proposition 34 If \( X \) is a weakly chained uniform space and \( f : X \to Y \) is a bi-uniformly continuous surjection then \( Y \) is weakly chained.

**Proof.** Let \( E \) be an entourage in \( Y \). Then \( E' := f^{-1}(E) \) is an entourage in \( X \). Since \( X \) is weakly chained, there is some entourage \( F' \) such that \( E' \) is weakly \( F' \)-chained. Since \( f \) is bi-uniformly continuous, \( F := f(F') \) is an entourage in \( Y \). We claim that \( E \) is weakly \( F \)-chained. Suppose that \( (x, y) \in E \). By definition there exist \( x' \in f^{-1}(x) \) and \( y' \in f^{-1}(y) \) such that \( (x', y') \in E' \). Let \( D \) be an entourage in \( Y \). Then \( D' := f^{-1}(D) \) is an entourage in \( X \). Since \( E' \) is weakly \( F' \)-chained, there is a \( D' \)-chain \( \alpha \) joining \( x' \) and \( y' \) such that \( [\alpha]_{E'} = [x', y']_{E'} \). Since \( f(D') = D, f(E') = E \) and \( f(F') = F \), \( \alpha := f(\alpha') \) is a \( D \)-chain such that

\[
[\alpha]_E = [f(\alpha)]_E = [f(x'), f(y')]_E = [x, y]_E.
\]

\[\Box\]

Proposition 35 If \( X \) is a metrizable topological space that is weakly chained with the fine uniformity then \( \tilde{X} \) has the fine uniformity.

**Proof.** By Proposition 34 it suffices to show that if \( U \) is an open subset containing the diagonal in \( X \times \tilde{X} \) then \( U = \phi^{-1}(E) \) for some entourage \( E \) in \( X \). But \( \phi \) is an open mapping, so \( E := \phi(U) \) is an open subset of \( X \times X \) containing the diagonal, hence is an entourage. Since \( \phi \) is a uniformly continuous surjection, \( U := \phi^{-1}(E) \) is an entourage. \[\Box\]
Example 36 Let $X$ be the surface obtained by revolving the graph of $y = e^x$ about the $x$-axis. Then $X$ is universal as a uniform space \([3]\) but is not universal as a topological space. Roughly speaking this is true because for any $\varepsilon > 0$, any $\varepsilon$-loop that wraps around the topological cylinder can be $\varepsilon$-homotoped down until the hole in the cylinder is small enough that a basic $\varepsilon$-move can cross it, so the loop must be $\varepsilon$-null. This shows that every $\varepsilon$-cover is trivial, and hence $X$ is its own $UU$-cover. This example also shows (not unexpectedly) that the $UU$-cover of a uniform space need not in general have the fine uniformity. It is easy to check that the fine uniformity in this case is the one equivalent to the flat metric on the cylinder, with which the space of course is not universal as a uniform space. In fact when $\varepsilon$ is smaller than $\frac{1}{2}$ the circumference of the circle then $\varepsilon$-loops can never be $\varepsilon$-homotoped across the hole, and the $\varepsilon$-cover is the plane.

4 Connected Boundaries of Geodesically Complete CAT(0) Spaces

In this section we will consider inverse systems \(\{X_r, \psi_{rs}\}_{r,s \in \Lambda}\) with inverse limit $X$. In this section we will assume that each $X_r$ is a metric space, $\Lambda$ is an unbounded subset of $\mathbb{R}^+$, and the bonding maps $\psi_{rs}$ are 1-Lipschitz and surjective. Since the indexing set has a countable, totally ordered cofinal set, it follows (by induction) that the projection maps $\psi_r : X \to X_r$ are also surjective. We will denote elements of $X = \varprojlim X_r$ by $\hat{x}$ and $\psi_r(\hat{x})$ by $x_r$. When no confusion will result, we will simply denote the subspace metric in any $X_r$ by $d$, e.g. writing $d(x_r, y_r)$ in the next proof.

Lemma 37 A basis for the inverse limit uniformity on $X = \varprojlim X_r$ consists of the set of all $E_{r,\varepsilon} := \{(\hat{x}, \hat{y}) : d(x_r, y_r) < \varepsilon\}$. Moreover, $E_{r,\varepsilon} \subset E_{s,\delta}$ if $r \geq s$ and $\varepsilon \leq \delta$.

Proof. A standard basis element for the inverse limit uniformity consists of entourages

\[
E(\varepsilon_1, ..., \varepsilon_n; r_1, ..., r_n) := \{(\hat{x}, \hat{y}) : d(x_{r_i}, y_{r_i}) < \varepsilon_i \text{ for all } i\}.
\]

Since each $E_{r,\varepsilon}$ is of this form, we need only show that an arbitrary $E(\varepsilon_1, ..., \varepsilon_n; r_1, ..., r_n)$ contains some $E_{r,\varepsilon}$. Let $\varepsilon := \min\{\varepsilon_1, ..., \varepsilon_n\}$ and $r := \max\{r_1, ..., r_n\}$. If $(\hat{x}, \hat{y}) \in E_{r,\varepsilon}$ then $d(x_r, y_r) < \varepsilon$. By the 1-Lipschitz assumption, for any $i$,

\[
d(x_{r_i}, y_{r_i}) = d(\psi_{r,r}(x_r), \psi_{r,r}(y_r)) \leq d(x_r, y_r) < \varepsilon \leq \varepsilon_i.
\]

That is, $E_{r,\varepsilon} \subset E(\varepsilon_1, ..., \varepsilon_n; r_1, ..., r_n)$. The second statement is simply a special case of what we just proved.

By definition, $\hat{\gamma} = \{\hat{x}_0, ..., \hat{x}_m\}$ is an $E_{r,\varepsilon}$-chain in $X$ if and only if $\gamma_r = \{(x_{r_0}, ..., x_{r_n})\}$ is an $\varepsilon$-chain in $X_r$. In this circumstance, $\hat{\gamma}$ will be called a lift of $\gamma$. When $\gamma$ is a loop we will always require that $\hat{x}_0 = \hat{x}_m$ so that $\hat{\gamma}$ is also
a loop. Note that for $\varepsilon$-chains in $X_r$, lifts always exist due to the surjectivity of the maps $\psi_r$. Consider a basic move adding $x \in X_r$ between points $x_i$ and $x_{i+1}$ in an $\varepsilon$-chain $\gamma = \{x_0, \ldots, x_m\}$ in $X$ which has a given lift $\hat{\gamma} = \{\hat{x}_0, \ldots, \hat{x}_m\}$. Let $\hat{x}$ be such that $\hat{x}_r = x$. Since $d(x_i, x) < \varepsilon$, $d(x, x_{i+1}) < \varepsilon$, $(\hat{x}_i, \hat{x})$, $(\hat{x}, \hat{x}_{i+1}) \in E_{r,\varepsilon}$ and in particular, $\{\hat{x}_0, \ldots, \hat{x}, \hat{x}_{i+1}, \ldots, \hat{x}_m\}$ is an $E_{r,\varepsilon}$-chain. That is, adding $\hat{x}$ is a basic move. The basic move of removing a point $\hat{x}_i$ from $\hat{\gamma}$ leaves an $E_{r,\varepsilon}$-chain if and only if removing $x_i$ leaves an $\varepsilon$-chain in $X_r$. It now follows by induction that if $\eta = \{\gamma_0, \ldots, \gamma_m\}$ is an $\varepsilon$-homotopy in $X_r$ then there are lifts $\hat{\gamma}_i$ of $\gamma_i$ such that $\hat{\gamma}_i = \{\hat{\gamma}_0, \ldots, \hat{\gamma}_m\}$ is an $E_{r,\varepsilon}$-homotopy. Then $\hat{\gamma}$ will be called a lift of $\eta$. Clearly we can always specify in advance $\hat{\gamma}_0 \in \psi_r^{-1}(\gamma_0)$. What if we have also specified $\hat{\gamma}_m$ in advance? When the $E_{r,\varepsilon}$-homotopy construction above is finished, we have some particular, possibly different lift $\hat{\gamma}_m = \{\hat{y}_0, \ldots, \hat{y}_k\}$ of $\gamma_m = \{y_0, \ldots, y_k\}$. Note that since the endpoints in the chains of $\hat{\eta}$ are never changed, $\hat{\gamma}_0 = \hat{y}_0$ and $\hat{\gamma}_k = \hat{y}_k$. Proceeding inductively, observe that for any $i$, $d((\hat{y}_i^i), (\hat{y}_i^i)) = d(y_i, y_i) < \varepsilon$ and therefore $(\hat{y}_i^i, \hat{y}_i^i) \in E_{r,\varepsilon}$. Likewise, $(\hat{y}_i, \hat{y}_{i+1}) \in E_{r,\varepsilon}$, and we have the following basic moves:

\[
\{\hat{y}_0, \ldots, \hat{y}_{i-1}, \hat{y}_i, \hat{y}_{i+1}, \ldots, \hat{y}_k\} \rightarrow \{\hat{y}_0, \ldots, \hat{y}_{i-1}, \hat{y}_i, \hat{y}_i, \hat{y}_{i+1}, \ldots, \hat{y}_k\}
\]

Therefore we may extend $\hat{\eta}$ to an $E_{r,\varepsilon}$-homotopy from $\hat{\gamma}_0$ to $\hat{\gamma}_m$. We will call such a homotopy a lift of $\eta$ "with specified endpoints". To summarize:

**Lemma 38** Let $\{X_r, \psi_r\}_{r \in \Lambda}$ be an inverse system of metric spaces with surjective $1$-Lipschitz bonding maps, where $\Lambda$ is an unbounded subset of $\mathbb{R}$. Suppose that $E_{r,\varepsilon}$ is an entourage in $X = \lim X_r$. If $\eta = \{\gamma_0, \ldots, \gamma_k\}$ is an $\varepsilon$-homotopy in $X_r$ then for any choice of lifts $\hat{\gamma}_0, \hat{\gamma}_k$ of $\gamma_0, \gamma_k$ there is a lift $\hat{\eta}$ of $\eta$, where $\hat{\eta}$ is an $E_{r,\varepsilon}$-homotopy from $\hat{\gamma}_0$ to $\hat{\gamma}_k$ (i.e. with specified endpoints). In particular, any lift of an $\varepsilon$-null $\varepsilon$-loop in $X_r$ is $E_{r,\varepsilon}$-null in $X$.

**Definition 39** Let $f : X \rightarrow Y$ be a uniformly continuous surjection between metric spaces, $0 < \delta < \varepsilon$. Then $f$ is said to be $(\varepsilon, \delta)$-refining if whenever $d(a, b) < \delta$ in $Y$, $a' \in f^{-1}(a)$ and $b' \in f^{-1}(b)$, there are arbitrarily fine chains $\alpha$ in $X$ from $a'$ to $b'$ such that $[f(\alpha)]_\varepsilon = [a, b]_\varepsilon$. When $\delta$ exists but is not specified we will simply say that $f$ is $\varepsilon$-refining. If $f$ is $\varepsilon$-refining for every $\varepsilon > 0$ then $f$ is simply called refining.

**Lemma 40** Let $f : X \rightarrow Y$ be a uniformly continuous surjection between metric spaces, $0 < \delta < \varepsilon$. Then $f$ is $(\varepsilon, \delta)$-refining if and only if for every $\delta$-chain $\beta$ in $Y$ from $a$ to $b$ and $a' \in f^{-1}(a)$ and $b' \in f^{-1}(b)$, there are arbitrarily fine chains $\alpha$ in $X$ from $a'$ to $b'$ such that $[f(\alpha)]_\varepsilon = [\beta]_\varepsilon$.

**Proof.** Necessity is obvious. Suppose that $f$ is $(\varepsilon, \delta)$-refining. The proof is by induction on the length of the $\delta$-chain $\beta = \{x_0, \ldots, x_n\}$. The $n = 1$ case is simply the definition of $(\varepsilon, \delta)$-refining. Suppose the statement is true for a
\[ \delta \text{-chain } \beta_i := \{x_0, \ldots, x_i\} \] with \( 0 < i < n \) and let \( x'_0 \in f^{-1}(x_0), x'_i \in f^{-1}(x_i) \) and \( x'_{i+1} \in f^{-1}(x_{i+1}) \). By assumption there are arbitrarily fine chains \( \alpha' \) from \( x'_0 \) to \( x'_i \) such that \( \lfloor f(\alpha') \rfloor_\varepsilon = [\beta_i]_\varepsilon \). Since \( f \) is \((\varepsilon, \delta)\)-refining there are arbitrarily fine chains \( \alpha'' \) from \( x'_i \) to \( x'_{i+1} \) such that \( \lfloor f(\alpha'') \rfloor_\varepsilon = [x_i, x_{i+1}]_\varepsilon \). Then \( \alpha := \alpha' \ast \alpha'' \) is the desired chain.

**Theorem 41** Let \( \{X_r, \psi_{rs}\}_{r,s \in \Lambda} \) be an inverse system of weakly chained metric spaces with surjective 1-Lipschitz bonding maps, where \( \Lambda \) is a closed, unbounded subset of \( \mathbb{R}^+ \). Suppose that for all \( r < t \in \mathbb{R}^+ \) and \( \varepsilon > 0 \) there exist \( s, s' \) such that \( r \leq s < t < s' \) and both \( \psi_{st} \) and \( \psi_{ts'} \) are \( \varepsilon \)-refining. Then \( X = \lim_{\varepsilon \to 0} X_r \) is weakly chained.

**Proof.** That \( X \) is chain connected follows from Lemma 11 in [4]. We will show that if \( E_\varepsilon \) is weakly \( E_\delta \)-chained in \( X_r \) then \( E := E_{r,\varepsilon} \) is weakly \( F = E_{r,\delta} \)-chained in \( X \). Since each \( X_r \) is weakly chained, the proof is complete by Lemma 37.

Suppose that \( (\tilde{x}, \tilde{y}) \in F \), meaning that \( d(x_r, y_r) < \delta \). We need to find, for any \( t \geq r \) and \( \kappa > 0 \), an \( E_{t,\kappa} \)-chain \( \tilde{\alpha} \) joining \( \tilde{x}, \tilde{y} \) in \( X \) such that \( [\tilde{\alpha}]_E = [\tilde{x}, \tilde{y}]_E \). If \( \alpha \) is a \( \kappa \)-chain in \( X_t \), any lift \( \tilde{\alpha} \) of \( \alpha \) with specified endpoints \( \tilde{x} \) to \( \tilde{y} \) is an \( E_{t,\kappa} \)-chain that is also a lift of \( \psi_{rt}(\alpha) \). Moreover, if \( [\psi_{rt}(\alpha)]_E = [x_r, y_r]_E \) then by Lemma 38 \( [\tilde{\alpha}]_E = [\tilde{x}, \tilde{y}]_E \). Therefore the proof will be complete if we show that the following statement is true for all \( t \). \( J(t) \): There are arbitrarily fine chains \( \alpha \) in \( X_t \) from \( x_t \) to \( y_t \) such that \( [\psi_{rt}(\alpha)]_E = [x_r, y_r]_E \). Note that \( J(r) \) is true by definition, because \( d(x_r, y_r) < \delta \), and \( E_\varepsilon \) is weakly \( E_\delta \)-chained in \( X_r \).

Let \( T := \sup \{t : J(t) \text{ is true}\} \); we will show \( T = \infty \).

We will first show that \( J(t) \) implies \( J(t + u) \) for some \( u > 0 \). In fact, by assumption, for some \( u > 0 \), \( \psi_{t,t+u} \) is \((\varepsilon, \delta)\)-refining for some \( \delta > 0 \). Since \( J(t) \) is true there is a \( \delta \)-chain \( \beta \) in \( X_t \) from \( x_t \) to \( y_t \) such that \( [\psi_{rt}(\beta)]_E = [x_r, y_r]_E \). Since \( \psi_{t,t+u} \) is \((\varepsilon, \delta)\)-refining there exist arbitrarily fine chains \( \alpha \) from \( x_{t+u} \) to \( y_{t+u} \) such that \( [\psi_{t,t+u}(\alpha)]_E = [\beta]_E \). By the 1-Lipschitz assumption (for the second equality below),

\[
[\psi_{r,t+u}(\alpha)]_E = [\psi_{rt}(\psi_{r,t+u}(\alpha)))]_E = [\psi_{rt}(\beta)]_E = [x_r, y_r]_E.
\]

This shows in particular that \( T > r \), and the proof will now be complete if we show that if \( T < \infty \) then \( J(T) \) is true. By assumption there is some \( r \leq s < T \) such that \( \psi_{sT} \) is \((\varepsilon, \tau)\)-refining for some \( \tau > 0 \). Since \( t < T \) there is some \( \tau \)-chain \( \beta \) in \( X_t \) such that \( [\psi_{rt}(\beta)]_E = [x_r, y_r]_E \). Now let \( \alpha \) be an arbitrarily fine chain in \( X_T \) such that \( [\psi_{rT}(\alpha)]_E = [\beta]_E \). Since \( \psi_{rT} \) is 1-Lipschitz, \( \psi_{rT}(\alpha) \) is also an arbitrarily fine chain such that

\[
[\psi_{rT}(\alpha)]_E = [\psi_{rt}(\psi_{rT}(\alpha))]_E = [\psi_{rt}(\beta)]_E = [x_r, y_r]_E.
\]

**Example 42** We will revisit the solenoid \( \Sigma \) discussed earlier, to see why the hypotheses of Theorem 41 fail in this case— as they must since \( \Sigma \) is not weakly chained. Considering \( \Sigma \) as the inverse limit of circles \( C_i \), we may give the circles
their standard Riemannian metrics, with the diameter of the $i$th circle equal to $2^i$; that is, the double covers $\phi_{i,i+1}: C_{i+1} \to C_i$ are local isometries and hence 1-Lipschitz. However, the double covers are not $\varepsilon$-refining for small enough $\varepsilon$. To simplify the notation, simply consider the double cover $\phi: C' \to C$, where $C$ has circumference 1 and $C'$ has circumference 2. For points $x_1, x_2 \in C$, let $x_1'$ be one of the two points in $\phi^{-1}(x_1)$ and $x_2'$ be the point in $\phi^{-1}(x_2)$ closest to the antipodal point of $x_1$. If $d(x_1, x_2) < \varepsilon = \frac{1}{3}$ then $d(x_1', x_2') > \frac{2}{3}$. If $\alpha$ is an arbitrarily fine chain between $x_1'$ and $x_2'$, its image must wrap more than $2/3$ of the way around the circle, and therefore cannot be $\varepsilon$-homotopic to its endpoints. It is not hard to see that this must be true for small enough $\varepsilon > 0$ — a basic move cannot “cross” the circle. The number $\frac{2}{3}$ more precisely is the single “homotopy critical value” of $C$ (see [23]). This is, $\frac{2}{3}$ precisely the largest number at which the $\varepsilon$-cover of $C$ “unrolls” into a line.

For more details about boundaries of CAT(0) spaces, see [6], Chapter II.8. Here is a summary to establish our notation. If $X$ is a CAT(0) space, for any $x_0 \in X$ one has the projections $p_{r,s}: \overline{B}(x_0, s) \to \overline{B}(x_0, r)$ between closed metric balls defined as follows: If $d(x_0, x) > r$ then $p_{r,s}(x) = \gamma_x(r)$, where $\gamma_x$ is the unit parameterized geodesic from $x_0$ to $x$; otherwise (i.e. $x \in \overline{B}(x_0, r)$), $p_{r,s}(x) = x$. These projections are the bonding maps for an inverse system, the inverse limit of which is denoted by $\hat{X}$. By definition, elements of $\hat{X}$ are contained in $\prod_{r \in \mathbb{R^+}} \overline{B}(x_0, r)$ and denoted by $\hat{x} = (x_r)$. There is a topological embedding of $X$ into $\hat{X}$ defined as follows $\iota(x) = \hat{x}$, where $x_r = x$ whenever $r \geq d(x_0, x)$. The boundary $\partial X$ at $x_0$ (which is topologically independent of $x_0$) is defined to be $\hat{X} \setminus \iota(X)$. On the other hand, consider the restrictions $\psi_{r,s}$ of $p_{r,s}$ to $\Sigma_s(x_0)$, which as defined in the Introduction consists of all $x \in S_r(x_0)$ such that there is a geodesic ray from $x_0$ through $x$. We again have an inverse system $(\Sigma_r(x_0), \psi_{r,s})$, and we claim that $\hat{B} := \lim_{r \to \infty} \Sigma_r(x_0) = \partial X$. First note that since each $\Sigma_r(x_0)$ is contained in $\overline{B}(x_0, r)$, $B \subset \hat{X}$. Moreover, since elements of $B$ do not have constant coordinates, $B \subset \partial X$. Therefore we need only show the opposite inclusion. There is a natural bijection between elements of $\partial X$ and unit parameterized geodesic rays starting at $x_0$, which takes a geodesic ray $\gamma$ to the element of $\hat{x} \in \partial X$ with $x_r = \gamma(r)$. But $x_r \in \Sigma_r(x_0)$ by definition, showing that $\partial X \subset B$. Note that by the CAT(0) condition, $d(\psi_{r,s}(x), \psi_{r,s}(y)) \leq \frac{2}{3} d(x, y)$ and in particular $\psi_{r,s}$ is 1-Lipschitz. Note also that the projections $\psi_{r}: \partial X \to \Sigma_r(x_0)$ are surjective open maps (by definition of the inverse limit topology). If $X$ is proper then $\partial X$ is compact and hence these maps are in fact bi-uniformly continuous with respect to the unique uniform structures on $\partial X$ and $\Sigma_r(x_0)$. From Proposition 44 we conclude:

**Proposition 43** If $X$ is a proper CAT(0) space with weakly chained boundary then $\Sigma_r(x_0)$ is weakly chained for all $x_0 \in X$ and $r > 0$.

**Definition 44** Let $f: X \to Y$ be a function between metric spaces. The preimage diameter $PD(f)$ of $f$ is defined to be the supremum of the diameters of the sets $f^{-1}(y)$ for all $y \in Y$. 

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Proposition 45 Let $f : X \to Y$ be a continuous function between compact metric spaces, and let $\varepsilon > PD(f)$. Then there exists a $\delta > 0$ such that if $d(x, y) < \delta$ then the diameter of $f^{-1}(\{x, y\})$ is less than $\varepsilon$. Moreover, if $X$ is a geodesic space and $f$ is $1$-Lipschitz then $f$ is $\varepsilon$-refining.

Proof. By compactness, if not, there exist $x_i, y_i \to x \in Y$ such that the diameter of $f^{-1}\{(x_i, y_i)\}$ is at least $\varepsilon$ for all $i$. This means there must be $x'_i \in f^{-1}(x_i)$ and $y'_i \in f^{-1}(y_i)$ such that $d(x'_i, y'_i) \geq \varepsilon$. Applying compactness again we may assume that $x'_i \to x'$ and $y'_i \to y'$ in $X$. By continuity, $d(x', y') \geq \varepsilon > PD(f)$, but $f(x') = f(y') = x$, a contradiction.

For the second statement, suppose that $d(x, y) < \delta$, where $\delta$ is as in the previous statement. Then for any $x' \in f^{-1}(x)$ and $y' \in f^{-1}(y)$, $d(x', y') < \varepsilon$. Let $\gamma$ be any geodesic from $x'$ to $y'$. As mentioned in Remark 24, any $\kappa$-chain $\alpha$ on $\gamma$ from $x'$ to $y'$ is $\varepsilon$-homotopic to $\{x', y'\}$. Since $f$ is $1$-Lipschitz, $f(\alpha)$ is also a $\kappa$-chain that is $\varepsilon$-homotopic to $\{x, y\}$. ■

Proposition 46 If $X$ is a proper $\text{CAT}(0)$ space then the projection $\psi_{rt} : \Sigma_t(x_0) \to \Sigma_r(x_0)$ is a bi-uniformly continuous $1$-Lipschitz surjection. If $\Sigma_t(x_0)$ is a geodesic space then $\psi_{rt}$ is $\varepsilon$-refining whenever $2(t-r) < \varepsilon$.

Proof. By definition, $\psi_{rt}$ is a $1$-Lipschitz surjection. We first show that for any $\varepsilon > 0$, $\psi_{rt}(E_\varepsilon)$ contains some $E_\delta$ in $X_r$. If not there exist $x_i, y_i \in \Sigma_t(x_0)$ such that $x_i, y_i \to x$ and $(x_i, y_i) \notin \psi_{rt}(E_\varepsilon)$. By definition of $\Sigma_r(x_0)$ there are geodesic rays $\gamma_i, \eta_i$ from $x_0$ through $x_i, y_i$, respectively. Since $X$ is proper, we can assume that $\gamma_i(t), \eta_i(t) \to z$ in $\Sigma_t$. Then for large enough $i$, $\gamma_i(t), \eta_i(t) \in E_\varepsilon$, i.e. $(x_i, y_i) \in \psi_{rt}(E_\varepsilon)$, a contradiction.

For the second statement, note that for any $x \in \Sigma_t(x_0)$, $x$ and $\psi_{rt}(x)$ lie on a geodesic of length $t-r$. By the triangle inequality, $PD(\psi_{rt}) \leq 2(t-r)$ and the proof is finished by Proposition 45. ■

Combining Theorem 41 and Proposition 46 we obtain the following:

Theorem 47 If $X$ is a $\text{CAT}(0)$ space such that for some $x_0 \in X$ and all sufficiently large $r$, the induced geodesic metric on $\Sigma_r(x_0)$ is finite and compatible with the subspace topology, then $\partial X$ is weakly chained.

The following is a classical construction; see for example Rนow’s book [20], Section 15, for more details. Let $X$ be a metric space. Classically speaking, the induced length (or intrinsic, i.e. inner, in [20]) metric on $X$ is defined by letting $d_I(x, y)$ be equal to the infimum of lengths of curves from $x$ to $y$ measured with respect to the original metric $d$. Since points may not be joined by rectifiable (or any!) curves, the induced length metric may be infinite even if the original is finite. The identity map from $d$ to $d_I$ is distance non-decreasing. When $d_I$ is finite, it is a length metric; that is, $d_I(x, y)$ is the infimum of the lengths of curves joining $x, y$ (as measured with $d_I$—so there is something to be proved!). However, the topology of $d_I$ may be strictly finer than the topology of $d$; the topology is the same when “close points are joined by short curves”, i.e. the space is ein Raum ohne Umwege as in [20]. For example, since spaces with
length metrics are locally path connected, the new topology must be strictly finer whenever the original topology is not locally path connected. Finally, recall that the Rinow part of the Hopf-Rinow Theorem states that any length metric on a compact space is a geodesic metric. In this case we will refer to $d_I$ the “induced geodesic metric”.

**Proposition 48** Let $x_0 \in X$, where $X$ is a proper $\text{CAT}(0)$ space with connected boundary, and suppose that every geodesic starting at $x_0$ of length $r$ extends as a geodesic ray. Then the induced geodesic metric on $S_r(x_0) = \Sigma_r(x_0)$ is finite and topologically compatible with the subspace metric.

**Proof.** We will first show that if $x_i \rightarrow x$ in $S_r(x_0)$ then for any $\varepsilon > 0$ and large enough $i$ there are curves $c_i$ from $x_i$ to $x$ of length less than $\varepsilon$. Let $\gamma_i$ be any geodesic ray from $x_0$ through $x_i$. Since $X$ is proper, we may assume that $\gamma_i$ converges uniformly on compact sets to a geodesic ray $\gamma$ from $x_0$ through $x$. Let $R > r$. Since $\gamma_i(R) \rightarrow \gamma(R)$, for large enough $i$, the geodesic $\alpha$ from $\gamma_i(R)$ to $\gamma(R)$ has length less than $\min\{R - r, \varepsilon\}$. This implies that $\alpha$ stays outside $S_r(x_0)$ and therefore the the projection $\psi_{r,R} \circ \alpha$ onto $S_r(x_0)$ is defined. By the $\text{CAT}(0)$ condition, the projection is $1$-Lipschitz, so $L(\psi_{r,R} \circ \alpha) \leq L(\alpha) < \varepsilon$. It now follows that for any $x \in S_r(x_0)$ and $\varepsilon > 0$ there is some $\delta > 0$ such that if $d(x, y) < \delta$ then $x, y$ are joined by a curve of length less than $\varepsilon$. This is precisely the definition of *ein Raum ohne Umwege*, and topological compatibility of the metrics follows from Statement 15.4 in [26].

We next show that there is some $\delta > 0$ such that if $d(x, y) < \delta$ in $S_r(x_0)$ then $x, y$ are joined by a rectifiable curve in $S_r(x_0)$. If not then there exist $x_i, y_i \rightarrow z$ in $S_r(x_0)$ such that $x_i, y_i$ are not joined by a rectifiable curve. But from the above argument we see that for large enough, $x_i$ and $y_i$ are joined to $z$ by rectifiable curves in $S_r(x_0)$. The concatenation of these curves is a rectifiable curve from $x_i$ to $y_i$ in $S_r(x_0)$, a contradiction. Since $S_r(x_0)$ is connected, we may join any two points in $S$ by a $\delta$-chain $\alpha$ in $S$. Each point and its successor may be joined by a rectifiable curve in $S_r(x_0)$, and the concatenation of these curves is also rectifiable. That is, the induced geodesic metric between the original two points is finite.

Theorem 4 now follows from Theorem 47 and Proposition 48.

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