A symmetry result on Reinhardt domains

Vittorio Martino

Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato 5, 40127 Bologna, Italy.
E-mail address: martino@dm.unibo.it

Abstract. We show the following symmetry property of a bounded Reinhardt domain $\Omega$ in $\mathbb{C}^{n+1}$: let $M = \partial \Omega$ be the smooth boundary of $\Omega$ and let $h$ be the Second Fundamental Form of $M$; if the coefficient $h(T,T)$ related to the characteristic direction $T$ is constant then $M$ is a sphere. In Appendix we state the result from an hamiltonian point of view.

1 Introduction

A Reinhardt domain $\Omega$ (with center at the origin) is by definition an open subset of $\mathbb{C}^{n+1}$ such that

$$\text{if } (z_1, \ldots, z_{n+1}) \in \Omega \text{ then } (e^{i\theta_1}z_1, \ldots, e^{i\theta_{n+1}}z_{n+1}) \in \Omega$$

(1)

for all the real numbers $\theta_1, \ldots, \theta_{n+1}$. These domains naturally arise in the theory of several complex variables as the logarithmically convex Reinhardt domains are the domains of convergence of power series (see for instance [4], [7]). We will suppose from now on that the Reinhardt domain $\Omega$ has a smooth boundary ($C^2$ would be enough). The boundary $M := \partial \Omega$ is then a smooth real hypersurface in $\mathbb{C}^{n+1}$ and thus a CR-manifold of CR-codimension equal to one, with the standard CR structure induced by the holomorphic structure of $\mathbb{C}^{n+1}$. Thus for every $p \in M$ the tangent space $T_pM$ splits in two subspaces: the $2n-$dimensional horizontal subspace $H_pM$, the largest subspace in $T_pM$ invariant under the action of the standard complex structure $J$ of $\mathbb{C}^{n+1}$ and the vertical one-dimensional subspace generated by the characteristic direction $T_p := J \cdot N_p$, where $N_p$ is the unit normal.
at $p$. Moreover, if $\tilde{g}$ is the standard metric on $\mathbb{C}^{n+1}$, then it holds

$$T_p M = H_p M \oplus \mathbb{R} T_p$$

and the sum is $\tilde{g}$-orthogonal.

Let us consider the complexified horizontal space

$$H^C M := \{ Z = X - iJ \cdot X : X \in HM \}$$

The Levi Form $l$ is then the sesquilinear and hermitian operator on $H^C M$ defined in the following way: $\forall Z_1, Z_2 \in H^C M$

$$l(Z_1, Z_2) = \tilde{g}(\tilde{\nabla} Z_1, \bar{Z}_2, N)$$ (2)

where $\tilde{\nabla}$ is the Levi-Civita connection for $\tilde{g}$. Moreover by a direct computation it holds

$$l(Z, Z) = \tilde{g}(\tilde{\nabla} Z, N) = \tilde{g}([X, Y], T)$$ (3)

where $Y = J \cdot X$. We will say $M$ be (strictly) pseudoconvex if $l$ is (strictly) positive definite as quadratic form.

In analogy with classical curvatures defined in terms of elementary symmetric functions of the eigenvalues of the Second Fundamental Form, one defines the $j$-th Levi curvatures $L^j$ in terms of elementary symmetric functions of the eigenvalues of the Levi Form

$$L^j = \frac{1}{\binom{n}{j}} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \lambda_{i_1} \cdots \lambda_{i_j},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $l$. In particular when $j = n$ we have the Total-Levi Curvature and when $j = 1$ we have the Levi-Mean Curvature $L$.

Being hypersurfaces in $\mathbb{C}^{n+1}$ real hypersurfaces in $\mathbb{R}^{2n+2}$, one can also compare the Levi Form with the Second Fundamental Form $h$ of $M$ by using the identity [3]

$$l(Z, Z) = h(X, X) + h(J(X), J(X)), \quad \forall X \in HM$$

Thus, a direct calculation leads to the relation between the classical Mean Curvature $H$ and the Levi-Mean Curvature $L$ [12]:

$$H = \frac{1}{2n+1} (2nL + h(T, T))$$ (4)

where $h(T, T) = \tilde{g}(\tilde{\nabla} T, N)$ is the coefficient of the Second Fundamental Form related to the characteristic direction $T$. 
Definition 1.1. We will call $h(T,T)$ the characteristic curvature of $M$.

By (4) the characteristic curvature is a sort of complementary of the Levi-Mean Curvature in computing the Mean Curvature. Moreover, for every hypersurface in $\mathbb{C}^{n+1}$, $h(T,T)$ is invariant under a biholomorphic (rigid) transformation, as the Levi curvatures are.

Following the pioneering result due to Alexandrov [1] on the classical Mean Curvature of Euclidean surface, the problem of characterizing compact hypersurfaces with positive constant Levi-Mean Curvature has recently received a great amount of attention. Klingenberg in [8] gave a first positive answer to this problem by showing that if the characteristic direction is a geodesic and the Levi Form is diagonal, then $M$ is a sphere. Monti and Morbidelli in [13] proved a Darboux-type theorem for $n \geq 2$: the unique Levi umbilical hypersurfaces in $\mathbb{C}^{n+1}$ with all constant Levi curvatures are spheres or cylinders. Later on Montanari and the author proved two results of this type: in [11] they relaxed Klingenberg conditions and they proved that if the characteristic direction is a geodesic, then Alexandrov Theorem holds for hypersurfaces with positive constant Levi-Mean Curvature; in [10] they proved some integral formulas for compact hypersurfaces, of independent interest, and then they follow the Reilly approach [14], [15], [16] to prove Isoperimetric estimates and a Alexandrov type theorem, namely: let $M$ be a closed smooth real hypersurface bounding a star-shaped domain in $\mathbb{C}^{n+1}$, if the $j$-Levi curvature is a positive constant $K$ and the maximum of the Mean Curvature of $M$ is bounded from above by $K$ then $M$ is a sphere.

In a couple of recent papers Hounie and Lanconelli proved Alexandrov type theorems for Reinhardt domains in $\mathbb{C}^2$ first and for Reinhardt domain in $\mathbb{C}^{n+1}$, $n \geq 1$, with an additional rotational symmetry then. In [5] they showed the result for bounded Reinhardt domain of $\mathbb{C}^2$, i.e. for domains $\Omega$ such that if $(z_1,z_2) \in \Omega$ then $(e^{i\theta_1}z_1,e^{i\theta_2}z_2) \in \Omega$ for all real $\theta_1, \theta_2$. Under this hypothesis, in a neighborhood of a point, there is a defining function $F$ only depending on the radii $r_1 = |z_1|$, $r_2 = |z_2|$, $F(r_1,r_2) = f(r_2^2) - r_1^2$ with $f$ the solution of the ODE

$$sff'' = sf'^2 - k(f + sf'^2)^{3/2} - ff'$$

Alexandrov Theorem follows from uniqueness of the solution of (5). Their technique has then been used in [6] to prove an Alexandrov Theorem for bounded Reinhardt domains in $\mathbb{C}^{n+1}$ with an additional rotational symmetry in two complementary sets of variables, for every $n$.

Here we prove a similar result of symmetry for Reinhardt domains in $\mathbb{C}^{n+1}$ starting from the characteristic curvature rather than the Levi ones.
Theorem 1.2. Let $M := \partial \Omega$ be the smooth boundary of a bounded Reinhardt domain $\Omega$ in $\mathbb{C}^{n+1}$. If the characteristic curvature $h(T, T)$ is constant then $M$ is a sphere of radius equal to $1/h(T, T)$.

Let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$, with $Y_k = J \cdot X_k$, be an orthonormal basis of the horizontal space $HM$; keeping in mind the structure of the Second Fundamental Form

$$
h = \begin{pmatrix}
    h(X_k, X_k) & h(X_k, Y_j) & h(X_k, T) \\
    h(Y_j, X_k) & h(Y_j, Y_j) & h(Y_j, T) \\
    h(T, X_k) & h(T, Y_k) & h(T, T)
\end{pmatrix}
$$

with $k$ and $j$ running in $1, \ldots, n$, we are making assumption only on the one-dimensional characteristic subspace of the tangent space rather than on the $2n$-dimensional horizontal one $HM$: moreover when in addition one assumes that one of the Levi curvatures is non zero (as in the Alexandrov type results) then $HM$ spans the whole tangent space; in fact the vector fields $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ satisfy the Hörmander rank condition.

When there exists a defining function $f : \mathbb{C}^{n+1} \rightarrow \mathbb{R}$

$$
\Omega = \{ z \in \mathbb{C}^{n+1} : f(z) < 0 \}, \quad M = \partial \Omega = \{ z \in \mathbb{C}^{n+1} : f(z) = 0 \}
$$
such that $f(z) = g(r)$ depends only on the radii $r = (r_1, \ldots, r_{n+1})$, where

$$
r_k = z_k \bar{z}_k, \quad k = 1, \ldots, n+1
$$

then we can find an explicit formula to compute the characteristic curvature $h(T, T)$. In fact by using the following identities

$$
\begin{align*}
    f_k &= \bar{z}_k g_k, \\
    f_k &= z_k g_k, \\
    f_{jk} &= \delta_{jk} g_k + z_j \bar{z}_k g_{jk} \\
    |\partial f|^2 &= \sum_k r_k g_k^2
\end{align*}
$$

the unit normal $N$ is

$$
N = -\frac{1}{|\partial f|} \sum_k (z_k g_k \partial z_k + \bar{z}_k g_k \partial \bar{z}_k)
$$

and the characteristic direction $T$ reads as

$$
T = J \cdot N = -\frac{i}{|\partial f|} \sum_k (z_k g_k \partial z_k - \bar{z}_k g_k \partial \bar{z}_k)
$$

Then by a direct computation we have that

$$
h(T, T) = \tilde{g}(\nabla_T T, N) = \sum_k \frac{r_k g_k^3}{|\partial f|^3}
$$

(6)
Example 1.3 (characteristic curvature of the sphere). Let
\[ g(r_1, \ldots, r_{n+1}) = r_1 + \ldots + r_{n+1} - R^2 \]
be the defining function of the sphere of radius equal to \( R \) in \( \mathbb{C}^{n+1} \). By the formula (6) we have that the characteristic curvature of the sphere is
\[ h(T, T) = \frac{1}{R} \]

Example 1.4 (characteristic curvature of ellipsoidal type domains). Let
\[ g(r_1, \ldots, r_{n+1}) = \frac{r_1}{a_1^2} + \ldots + \frac{r_{n+1}}{a_{n+1}^2} - 1 \]
be the defining function of an ellipsoid in \( \mathbb{C}^{n+1} \) with \((a_1, \ldots, a_{n+1})\) positive constants. By the formula (6) we have that at a point \( p = (r_1, \ldots, r_{n+1}) \in M \) its characteristic curvature is
\[ h_p(T, T) = \frac{\sum_{k} \frac{r_k}{a_k^2}}{\left( \sum_{k} \frac{r_k}{a_k^4} \right)^{3/2}} \]

In the next section we will prove the Theorem 1.2, then in the Appendix we will show an Hamiltonian point of view of the result.

Acknowledgement The paper was completed during the year spent at the Mathematics Department of Rutgers University: the author wishes to express his gratitude for the hospitality and he is grateful to the Nonlinear Analysis Center for its support.

2 Proof of Theorem 1.2

Let us identify \( \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \simeq \mathbb{C}^{n+1} \) so that \( z = (x, y) \). First we prove a property of independent interest.

Lemma 2.1. Let \( \Omega \) be a Reinhardt domain in \( \mathbb{C}^{n+1} \) and let
\[ p = (z_1, \ldots, z_{n+1}) = (x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) \]
the “position vector” of a point on $M := \partial \Omega$. If $T_p$ is the characteristic direction at $p \in M$ then it holds identically
\[ \tilde{g}(p, T_p) \equiv 0 \] (7)

**Proof.** If $M$ is any smooth hypersurface bounding a domain $\Omega$ in $\mathbb{C}^{n+1}$ with defining function $f : \mathbb{C}^{n+1} \to \mathbb{R}$ such that
\[ \Omega = \{ z \in \mathbb{C}^{n+1} : f(z) < 0 \}, \quad M = \partial \Omega = \{ z \in \mathbb{C}^{n+1} : f(z) = 0 \} \]
then the unit normal $N$ is:
\[ N = -\frac{1}{|\partial f|} \sum_{k=1}^{n+1} (f_k \partial z_k + f_k \partial \bar{z}_k) \]
where $f_k = \frac{\partial f}{\partial z_k}$, with $k = 1, \ldots, n + 1$. Thus the characteristic direction $T$ is:
\[ T = J \cdot N = -\frac{i}{|\partial f|} \sum_{k=1}^{n+1} (f_k \partial z_k - f_k \partial \bar{z}_k) \]

By identifying $f(z) = f(x, y)$, from the real point of view we have:
\[ N = -\frac{1}{|\nabla f|} \sum_{k=1}^{n+1} (f_{x_k} \partial x_k + f_{y_k} \partial y_k) \]
\[ T = \frac{1}{|\nabla f|} \sum_{k=1}^{n+1} (f_{y_k} \partial x_k - f_{x_k} \partial y_k) \]

Now, if $\Omega$ is a Reinhardt domain (with center at the origin) in $\mathbb{C}^{n+1}$ then we can find (at least locally) a defining function $f(z) = g(r)$ depending only on the radii $r = (r_1, \ldots, r_{n+1})$ where
\[ r_k = z_k \bar{z}_k = x_k^2 + y_k^2, \quad k = 1, \ldots, n + 1 \]
So if $g_k = \frac{\partial g}{\partial r_k}$ we obtain
\[ f_{x_k} = 2x_k g_k, \quad f_{y_k} = 2y_k g_k \]
with $k = 1, \ldots, n + 1$. In vectorial notation then we have
\[ T = \frac{1}{|\nabla f|} (f_{y_1}, \ldots, f_{y_{n+1}}, -f_{x_1}, \ldots, -f_{x_{n+1}}) = \]
\[ \left( y_1 g_1, \ldots, y_{n+1} g_{n+1}, -x_1 g_1, \ldots, -x_{n+1} g_{n+1} \right) \]

and thus it holds identically

\[ \tilde{g}(p, T_p) = \frac{2}{|\nabla f(p)|} \sum_{k=1}^{n+1} \left( x_k y_k g_k(p) - y_k x_k g_k(p) \right) \equiv 0 \]

for every \( p \in M \)

In other words, the position vector \( p \) has generally a normal component and a tangential component; in turn, the tangential component has an horizontal component and a characteristic component: for Reinhardt domains the characteristic component of the position vector \( p \) identically vanishes.

Now we can prove the main result.

**Proof.** (of Theorem 1.2) Let us consider the function:

\[ \varphi : M \to \mathbb{R}, \quad \varphi(p) = \frac{|p|^2}{2} = \tilde{g}(p, p) \]

that represents one half the squared distance of the manifold from the origin. If \( V \in TM \) is a tangent vector field to \( M \) then the derivative of \( \varphi \) along \( V \) is

\[ V(\varphi(p)) = \frac{1}{2} V(\tilde{g}(p, p)) = \tilde{g}(p, V_p) \]

and by Lemma 2.1 we have

\[ T(\varphi) = \tilde{g}(p, T) \equiv 0 \]

Thus, if \( \hat{p} \) is a critical value of \( \varphi \), then

\[ X_k(\varphi)_{\hat{p}} = Y_k(\varphi)_{\hat{p}} = 0 \]

Moreover, \( \varphi \) evaluated at a critical value is

\[ \varphi(\hat{p}) = \frac{|\hat{p}|^2}{2} \quad (8) \]

and the position vector of any critical value \( \hat{p} \) is parallel to the (inner) unit normal direction \( N \) at \( \hat{p} \)

\[ \hat{p} = \tilde{g}(\hat{p}, N_{\hat{p}}) N_{\hat{p}} = -|\hat{p}| N_{\hat{p}} \]
Differentiating again \( \varphi \) along the characteristic direction \( T \) we obtain
\[
0 \equiv T^2(\varphi) = T(\tilde{g}(p,T)) = \tilde{g}(T,T) + \tilde{g}(p,\tilde{\nabla}_T T) = 1 + \tilde{g}(p,\tilde{\nabla}_T T)
\]
and if \( \hat{p} \) is a critical value for \( \varphi \) then we get
\[
1 - |\hat{p}|\tilde{g}(N_{\hat{p}},\tilde{\nabla}_T T) = 1 - |\hat{p}|h_{\hat{p}}(T,T) = 0 \quad (9)
\]
where \( h_{\hat{p}}(T,T) \) is the characteristic curvature of \( M \) at \( \hat{p} \).
Since \( M \) is a smooth compact hypersurface, then \( \varphi \) admits maximum and minimum which are critical values for \( \varphi \). If \( h(T,T) \) is constant then by (9) we have
\[
|\hat{p}| = \frac{1}{h_{\hat{p}}(T,T)} = \frac{1}{h(T,T)} = \text{const}.
\]
Then by (8) \( \varphi \) is constant on \( M \) and it holds
\[
(2\varphi(p))^{1/2} = |p| = \frac{1}{h(T,T)} = \text{const}.
\]
for every \( p \in M \), and it means that \( M \) is a sphere of radius \( \frac{1}{h(T,T)} \).

The boundedness hypothesis is crucial as the next example shows.

**Example 2.2** (characteristic curvature of a cylinder type domain). *Let*
\[
g(r_1, r_2) = r_1 - R^2
\]
*be the defining function of a cylinder type domain in \( \mathbb{C}^2 \). By the formula (6) we have that its characteristic curvature is constant:*
\[
h(T,T) = \frac{1}{R}
\]

**3 Appendix**

Here we want to look at the Reinhardt domains from an hamiltonian point of view. First we recall that for every hypersurface \( M \) in \( \mathbb{C}^{n+1} \), with \( f \) as defining function, the characteristic direction of \( M \) is exactly the (normalized) hamiltonian vector field for the hamiltonian function \( f \). In fact let us consider a dynamic system with hamiltonian function (smooth enough) depending on position and momentum variables
\[
H : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}, \quad z = (q,p) \mapsto H(q,p)
\]
and define the Action functional

$$A(z) = \int_{t_0}^{t_1} \left( \langle p, \dot{q} \rangle - H(q, p) \right) dt, \quad z : [t_0, t_1] \to \mathbb{R}^{2n+2}$$

The first variation of $A$ on a suitable space of curves leads to the following system of differential equations (Hamilton)

$$\begin{cases}
\dot{q}_k = \frac{\partial H}{\partial p_k}(q, p) \\
\dot{p}_k = -\frac{\partial H}{\partial q_k}(q, p)
\end{cases} \quad k = 1, \ldots, n + 1 \quad (10)$$

Now, a Least Action Principle states that trajectories of motion (in the generalized phase space $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$) are solutions of $(10)$. The isoenergetic surface of $H$ of energy $E$ is the following hypersurface in $\mathbb{R}^{2n+2}$: $M = \{ z \in \mathbb{R}^{2n+2} : H(z) = E \}$. The conservation of energy principle ensures that if $z$ is a critical point for $A$, then $z(t) \in M, \forall t \in [t_0, t_1]$. The hamiltonian vector field for $H$ is the tangent vector field to $M$

$$X^H_z := \left( \frac{\partial H}{\partial p}(q, p), -\frac{\partial H}{\partial q}(q, p) \right) = J \cdot \nabla H(q, p)$$

where

$$J = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}$$

is the canonical symplectic matrix in $\mathbb{R}^{2n+2}$ and in our case it coincides with the standard complex structure in $\mathbb{C}^{n+1}$.

The Hamilton system $(10)$ rewrites as

$$\dot{z} = X^H_z$$

Now, if one identifies

$$\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}, \quad z = (z_1, \ldots, z_{n+1}), \quad z_k = x + iy \simeq (x_k, y_k)$$

then the hypersurface $M$ defined by

$$M = \{ z \in \mathbb{C}^{n+1} : f(z) = 0 \}, \quad f : \mathbb{C}^{n+1} \to \mathbb{R}$$

is exactly the isoenergetic surfaces of $H = f + E$. Thus the hamiltonian vector field on $M$ is

$$X^H_z = J \cdot \nabla H(z) = J \cdot \nabla f(z) = J \cdot N = T$$
where $N = \nabla f$ is the normal direction to $M$ and $T$ is the (not normalized) characteristic direction. Moreover the integral curves of $X^H$ (the orbits in the phase space) coincide with that ones of $T$, eventually reparametrized. In this situation the characteristic curvature $h(T, T)$ is the normal curvature of the hamiltonian trajectories on the isoenergetic surface in the generalized phase space $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Now, we recall that if $\Omega$ is a Reinhardt domain (with center at the origin) in $\mathbb{C}^{n+1}$ then we can find (at least locally) a defining function $f(z) = g(r)$ depending only on the radii $r = (r_1, \ldots, r_{n+1})$ where

$$r_k = z_k \bar{z}_k = x_{2k}^2 + y_{2k}^2, \quad k = 1, \ldots, n + 1$$

This means that the hamiltonian function depends only on the quantities $r_k = q_k^2 + p_k^2$ that represent the actions in the pair of variables action-angle. Thus the angle variables are cyclic and then the actions $r_k$ (and all the functions depending on them) are conserved quantities along the trajectories of motion. In fact we have that the characteristic direction $T$ is:

$$T = -\frac{i}{|\partial f|} \sum_k (z_k g_k \partial z_k - \bar{z}_k g_k \partial \bar{z}_k)$$

then it holds

$$T(r_k) = 0, \quad k = 1, \ldots, n + 1$$

Moreover the system (10) reads as

$$\dot{z}_k = -i f_{\bar{z}_k} = -iz_k g_k$$

and since $g_k(t) = g_k(0)$, then the curve

$$z(t) = z_k(0)e^{-ig_k(0)t}$$

is an explicit solution of (11) with initial condition $z_k(0)$. In particular, we have that the following curves

$$z(t) = z_k(0)e^{-i g_k(0) |\partial f(0)| t}$$

are integral curves of the characteristic direction $T$.

We explicitly note that the trajectories of the characteristic direction belong to a $(n + 1)$-dimensional torus $T^{n+1}$ (eventually degenerate) identified by

$$T^{n+1} = S^1 \times \ldots \times S^1 = \{z \in \Omega : |z_1| = c_1 \geq 0, \ldots, |z_{n+1}| = c_{n+1} \geq 0\} \quad (12)$$
and this is a particular case of the well-known Liouville-Arnold Theorem [2].
In other words we have a symplectic toric action group on $\mathbb{C}^{n+1}$ with a fixed point at the origin.
Let us now consider the following explicit formula to compute the $j$-th Levi curvature of $M$ in term of a defining function $f$ (see [9]):

$$L^j = -\frac{1}{n} \frac{1}{|\partial f|^{j+2}} \sum_{1 \leq i_1 < \cdots < i_{j+1} \leq n+1} \Delta_{(i_1, \ldots, i_{j+1})}(f)$$

(13)

for all $j = 1, \ldots, n$, where

$$\Delta_{(i_1, \ldots, i_{j+1})}(f) = \det \begin{pmatrix} 0 & f_{i_1} & \cdots & f_{i_{j+1}} \\ f_{i_1} & f_{i_1, i_1} & \cdots & f_{i_1, i_{j+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_{j+1}} & f_{i_{j+1}, i_1} & \cdots & f_{i_{j+1}, i_{j+1}} \end{pmatrix}$$

(14)

If $f(z) = g(r)$ depends only on the radii $r = (r_1, \ldots, r_{n+1})$ then by a direct computation we have that $\Delta_{(i_1, \ldots, i_{j+1})}(g)$ depends only on $(r_{i_1}, \ldots, r_{i_{j+1}})$. Thus all the $j$-th Levi curvatures are conserved quantities on every fixed $(n + 1)$-dimensional torus $\mathbb{T}^{n+1}$: in particular they are constant along the trajectories of the characteristic direction $T$.
Moreover by the formula (6) also the characteristic curvature $h(T, T)$ is constant on every fixed $(n + 1)$-dimensional torus. We explicitly recall that $h(T, T)$ (and all the conserved quantities as well) is constant along the trajectories of the characteristic direction $T$ but the value of the constant changes accordingly to the initial condition of the equation (11).
Then our main result Theorem (1.2) states that if the value of the constant $h(T, T)$ is the same on all the trajectories of the characteristic direction $T$ then $M$ is a sphere.

References

[1] A.D. Alexandrov, A characteristic property of spheres, Ann. Mat. Pura Appl., (4) 58, pag. 303-315, 1962

[2] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag (1989)

[3] A. Bogges, CR Manifolds and the Tangential Cauchy-Riemann Complex, Studies in Advanced Mathematics, 1991
[4] L. Hörmander, An introduction to complex analysis in several variables, North-Holland (1973)

[5] J.G. Hounie, E. Lanconelli, An Alexander type Theorem for Reinhardt domains of $\mathbb{C}^2$. Recent progress on some problems in several complex variables and partial differential equations, Contemp. Math., Amer. Math. Soc., Providence, RI, 400,129–146, 2006.

[6] J.G. Hounie, E. Lanconelli, A sphere theorem for a class of Reinhardt domains with constant Levi curvature, Forum Mathematicum 2008 20:4 , 571-586

[7] M. Jarnicki, P. Pflug, First steps in several complex variables: Reinhardt domains, European Mathematical Society, 2008

[8] W.Klingenberg, Real hypersurfaces in Kähler manifolds, Asian J. Math. 5, no. 1, pag. 1-17, 2001

[9] E. Lanconelli, A. Montanari, Pseudoconvex Fully Nonlinear Partial Differential Operators. Strong Comparison Theorems, J. Differential Equations 202 (2004), no. 2, 306-331

[10] V. Martino, A. Montanari, Integral formulas for a class of curvature PDE’s and applications to isoperimetric inequalities and to symmetry problems, Forum Mathematicum, vol. 22/2 (2010); p. 255 - 267

[11] V. Martino, A. Montanari, On the characteristic direction of real hypersurfaces in $\mathbb{C}^{n+1}$ and a symmetry result, Advances in Geometry, Vol. 10/3, 2010, 371 - 377

[12] V. Martino, La forma di Levi per ipersuperfici in $\mathbb{C}^{N+1}$ e l’equazione di pseudocurvatura media per grafici reali, PhD Thesis

[13] R. Monti and D. Morbidelli, Levi umbilical surfaces in complex space, J. Reine Angew. Math. Math. 603 (2007) 113-131

[14] R. C. Reilly, Applications of the Hessian operator in a Riemann manifold Indiana Univ. Math. J., 26 (1977) 459-472.

[15] R. C. Reilly, Mean curvature, the Laplacian, and Soap Bubbles Amer. Math. Monthly 89 (1982), no. 3, 180–188, 197–198.

[16] R. C. Reilly, On the Hessian of a function and the curvatures of its graph. Michigan Math. J. 20 (1973), 373–383.