On nonexistence of continuous families of stationary nonlinear modes for a class of complex potentials

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Abstract
There are two cases when the nonlinear Schrödinger equation with an external complex potential is well known to support continuous families of localized stationary modes: the $PT$-symmetric potentials and the Wadati potentials. Recently, Kominis et al. have suggested that the continuous families can be also found in complex potentials of the form $W(x) = W_1(x) + iCW_{1,x}(x)$, where $C$ is an arbitrary real and $W_1(x)$ is a real-valued and bounded differentiable function. Here we study in detail nonlinear stationary modes that emerge in complex potentials of this type (for brevity, we call them $W$-dW potentials). First, we assume that the potential is small and employ asymptotic methods to construct a family of nonlinear modes. Our asymptotic procedure stops at the terms of the $\epsilon^2$ order, where small $\epsilon$ characterizes amplitude of the potential. We therefore conjecture that no continuous families of authentic nonlinear modes exist in this case, but “pseudo-modes” that satisfy the equation up to $\epsilon^2$-error can indeed be found in W-dW potentials. Second, we consider the particular case of a W-dW potential well of finite depth and support our hypothesis with qualitative and numerical arguments. Third, we simulate the nonlinear dynamics of found pseudo-modes and observe that, if the
amplitude of W-dW potential is small, then the pseudomodes are robust and display persistent oscillations around a certain position predicted by the asymptotic expansion. Finally, we study the authentic stationary modes that do not form a continuous family, but exist as isolated points. Numerical simulations reveal dynamical instability of these solutions.

**KEYWORDS**
absorption, dissipation, localized, nonlinear Schrödinger, solitary wave, soliton family

### INTRODUCTION

One of the fundamental differences in properties of nonlinear conservative and dissipative systems is the structure of their stationary modes. It is typical of conservative systems to support continuous families of localized nonlinear modes that result from the combined effect of the linear broadening, inhomogeneity of the conservative medium, and the nonlinear self-action. However, for stationary modes that appear in a dissipative medium, the situation becomes more complicated, because an additional balance between gain and loss of energy is required to sustain the steady-state propagation.\(^1\)–\(^4\) In view of this new requirement, the structure of dissipative stationary modes is usually scarcer than in the conservative case, and, instead of continuous families, dissipative stationary modes exist only as isolated points.

A prominent example of this dissimilarity is provided by the nonlinear Schrödinger equation (NLSE) with an additional, real- or complex-valued potential (alias the Gross-Pitaevskii equation). A spatially one-dimensional version of this equation reads

\[ i\Phi_t + \Phi_{xx} - W(x)\Phi + \sigma|\Phi|^2\Phi = 0, \]

where \( W(x) \) is the potential, and \( \sigma \) is the real nonlinearity coefficient. Equation (1) arises in various areas of present-day physics. In optics, it describes the laser beam propagation in nonlinear media with the refractive index modulated in the transverse direction.\(^5\) In the theory of Bose-Einstein condensate (BEC) Equation (1) models the dynamics of a cigar-shaped cloud of ultracold quantum gas trapped by an external field.\(^6\) Solitary-wave stationary modes for this equation correspond to the substitution \( \Phi(x, t) = e^{i\mu t} \phi(x) \), where \( \mu \) is a real parameter and \( \phi(x) \) is the localized stationary wavefunction. If the potential \( W(x) \) is real-valued, then the model is conservative. It is well known that it supports continuous one-parametric families of nonlinear modes that can be obtained by the continuous change of \( \mu \). This situation has been comprehensively documented for various types of the external potential, including periodic, parabolic, double-well one, and for either sign in front of the nonlinear term–see, for instance, Refs. \(^7\)–\(^14\) where this aspect has received the special emphasis. In the meantime, the situation can change drastically when one switches from real to complex potentials in (1), that is, assumes \( W(x) = W_1(x) + iW_2(x) \). In the optical context, the imaginary part of the potential describes the transverse distribution of gain and losses along
the guiding medium, and in the BEC theory it demarcates spatial regions where the particles are absorbed from or pumped in the condensate.

The model with a complex potential is no longer conservative, and it is expected in general that its nonlinear modes will only exist as some “isolated” points that cannot be continued in \( \mu \). However, it is known that there exist at least two situations when Equation (1) with a complex potential supports continuous families of stationary modes. The first example corresponds to PT-symmetric potentials, \(^{15-17}\) when \( W_1(x) \) and \( W_2(x) \) are even and odd functions, respectively. Physically, the continuous families in PT-symmetric potentials can be understood as a result of the synergy between symmetry of the potential and that of solution itself, which facilitates the gain-and-loss balance. Rigorous analyses of bifurcations of continuous PT-symmetric families have been reported on recently.\(^ {18,19}\) The second class of complex potentials that enable continuous families of nonlinear modes corresponds to the so-called Wadati potentials, \(^ {20}\) where real and imaginary parts of \( W(x) \) are expressed through an auxiliary real-valued function \( w(x) \) as follows

\[
W_1(x) = -w^2(x), \quad W_2(x) = -w_x(x).
\]

Function \( w(x) \) is required to be differentiable, but is not supposed to bear any special symmetry. Existence of continuous families in Wadati potentials can be qualitatively explained by the fact that the ODE system that describes the shape of stationary modes has a conserved quantity that effectively decreases the order of associated dynamical system.\(^ {21,22}\) Formal asymptotic expansions for families of nonlinear modes bifurcating from linear eigenstates of Wadati potentials have been recently obtained in Ref. 23.

It should be also added that unusual properties of PT-symmetric and Wadati potentials appear not only for nonlinear modes but in the linear case, too. In particular, complex eigenvalues that eventually emerge in the spectra of the corresponding non-Hermitian Schrödinger operators (obtained from Equation (1) with \( \sigma = 0 \)) always form complex-conjugate pairs.\(^ {24,25}\) A generic complex potential does not have this property.

In recent paper\(^ {26}\) by Kominis et al., it has been suggested that there exists yet another class of complex potentials that support continuous families of stationary modes. Real and imaginary parts of those potentials are related as

\[
W_{1,x}(x) = CW_2(x), \tag{2}
\]

where \( C \) is an arbitrary real constant, and the subscript \( x \) means the derivative. The relation (2) was arrived in Ref. 26 by means of the Melnikov-vector technique. In contrast to PT-symmetric potentials that relies on the parity of real and imaginary parts of the potential, condition (2) only involves a spatially local relation between the real and imaginary parts. This can potentially offer additional flexibility in the experimental realization of complex potentials of this form. As the specific shapes of \( W_1(x) \) and \( W_2(x) \) are not constrained, relations (2) can be used to create complex potentials of different forms: localized or extended, periodic or quasiperiodic, etc. Peculiar nonlinear dynamics in potentials (2) have been studied in several earlier studies.\(^ {27,28}\)

Let us briefly overview the outcomes of Ref. 26. Authors consider Equation (1) with the focusing (attractive) nonlinearity, in the situation when the complex potential is characterized by a small amplitude proportional to \( \varepsilon \ll 1 \):

\[
i \Phi_t + \Phi_{xx} - \varepsilon(W_1(x) + iW_2(x))\Phi + 2|\Phi|^2\Phi = 0. \tag{3}
\]
Stationary modes have the form \( \Phi(x, t) = e^{i\mu t} \phi(x) \), where \( \mu \) is a real coefficient whose optical meaning corresponds to the propagation constant. Continuously differentiable function \( \phi(x) \) satisfies the solitary-wave boundary conditions \( \lim_{x \to \infty} \phi(x) = \lim_{x \to -\infty} \phi(x) = 0 \). The shape of \( \phi(x) \) is described by the stationary equation

\[
\phi_{xx} - \mu \phi - \varepsilon(W_1(x) + iW_2(x))\phi + 2|\phi|^2\phi = 0. 
\] (4)

Equation (4) is invariant with respect to the phase rotation \( \phi(x) \to \phi(x)e^{i\theta}, \theta \in \mathbb{R} \). Separating \( \phi(x) \) into real and imaginary parts, \( \phi(x) = u(x) + i v(x) \), one transforms Equation (4) into a system

\[
u_{xx} - \mu v + 2(u^2 + v^2)u - \varepsilon(W_1(x)u - W_2(x)v) = 0,
\] (5)

\[u_{xx} - \mu u + 2(u^2 + v^2)u - \varepsilon(W_1(x)u + W_2(x)v) = 0.
\] (6)

In the limit \( \varepsilon = 0 \), these equations can be considered as a Hamiltonian system with two degrees of freedom. At \( \varepsilon = 0 \), this system has a homoclinic orbit corresponding to the bright soliton solution \( u(x) = p_0(x - x_0; \mu) \cos \theta, v(x) = p_0(x - x_0; \mu) \sin \theta \), where \( p_0(x; \mu) = \sqrt{\mu} \text{sech}(\sqrt{\mu} x) \), and \( x_0 \) and \( \theta \) are arbitrary reals that, respectively, reflect translational and rotational symmetries of the system with \( \varepsilon = 0 \). The terms proportional to \( \varepsilon \) are considered as small spatially inhomogeneous perturbations to the Hamiltonian system. To check the persistence of the homoclinic orbit in the perturbed system, authors of Ref. 26 construct a two-component Melnikov vector \( \vec{M} = [M_1(x_0; \theta, \mu), M_2(x_0; \theta, \mu)] \). Using formalism from Refs. 29, 30, the result is found in the form

\[
M_1(x_0; \theta, \mu) = -\int_{-\infty}^{\infty} \frac{dW_1(x - x_0)}{dx} p_0^2(x; \mu) dx,
\] (7)

\[
M_2(x_0; \theta, \mu) = \int_{-\infty}^{\infty} W_2(x - x_0) p_0^2(x; \mu) dx.
\] (8)

Authors of Ref. 26 argue that the localized stationary state is expected to persist under the perturbation if the entries of the Melnikov vector have a simple zero, that is, if for some \( x_0, \theta, \) and \( \mu \), one has \( M_1(x_0; \theta, \mu) = M_2(x_0; \theta, \mu) = 0 \). Then relation (2) emerges as a compatibility condition for both entries of \( \vec{M} \) to vanish simultaneously: if (2) holds, then \( M_1 \) and \( M_2 \) can be both made zero by adjusting the value of \( x_0 \).

Although the Melnikov function is a well-known tool in the study of dynamical system perturbed by a driving force, its applicability to system (5)-(6) raises several doubts. First, in Refs. 29, 30 and, apparently, in most of the literature on the Melnikov theory (e.g., Refs. 31–33), it is assumed that the perturbation is a periodic function of \( x \), which is obviously not the case of system (5)-(6) with potential \( W(x) \) of generic form. At a less superficial level, we note that the zero of the Melnikov function can be a necessary, but not sufficient condition for the persistence of the homoclinic orbit under the perturbation. The sufficient conditions include several more subtle constraints that, in particular, involve the derivatives of the Melnikov vector entries with respect to their arguments. 29 However, in the case at hand, we observe that the Melnikov vector in (7)-(8) does not depend on \( \theta \) (which is a natural consequence of the rotational symmetry). This suggests that the situation may be, in some sense, degenerate, and a simple zero of the Melnikov vector may not yet be sufficient. As these issues are not fully discussed in Ref. 26, the results of
this paper may not be fully rigorous and conclusive, but rather provide an analytical indication at
the possible existence of continuous families. To confirm the predictions of the Melnikov-vector
analysis, authors of Ref. 26 numerically compute families of stationary modes in periodic and
quasiperiodic potentials of small, but finite amplitude $\varepsilon$. Authors notice that the use of different
numerical methods yields results of different accuracy, and therefore it is desirable to search for
a more efficient method to deepen the numerical analysis of found solutions.

Motivated by the intriguing outcomes of Ref. 26, in the present paper we use a different com-
bination of analytical and numerical approaches to continuous families of nonlinear modes in
complex potentials that satisfy (2). For brevity, in what follows we call the potentials that satisfy
$W$-$dW$ potentials. Assuming that a $W$-$dW$ potential is small, in Section 2 we employ asymp-
totic methods to construct the nonlinear modes starting from the limit of zero potential where
the stationary solution is readily given in the form of a bright soliton. We seek for the profile of a
nonlinear mode in the form of a power series and show that the Melnikov-vector conditions (2)
enable only the first order of the perturbation theory. For a generic (asymmetric) $W$-$dW$ poten-
tial, the asymptotic procedure stops at the second-order terms. To check this prediction, in Sec-
tion 3 we consider a specific example of asymmetric $W$-$dW$ potential. In contrast to Ref. 26,
where sophisticated periodic and quasiperiodic potentials have been considered, we address the case of
a more simple finite-depth well potential that decays exponentially as $x \to \pm \infty$. Using a trans-
parent numerical shooting method, we confirm that the numerical solutions that bifurcate from
the family of bright solitons satisfy the equation only up to $\varepsilon^2$-accuracy. The combination of these
results allows to conjecture that the continuous families in $W$-$dW$ potentials are in fact formed
by approximate solutions that satisfy the equation only up to $O(\varepsilon^2)$ accuracy. A similar suggestion
has been recently formulated in Ref. 23. We call such approximate solutions pseudo-modes. In
Section 4, we use numerical simulations to demonstrate that, even though the pseudo-modes do
not correspond to exact stationary modes, they play a distinctive role in the nonlinear dynamics
governed by the time-dependent NLSE. Numerical dynamics simulations show that for small-
amplitude potentials the pseudo-modes exhibit nearly perfect oscillations of the center of mass
around the position predicted by the asymptotic expansion. Finally, in Section 5 we extend the
numerical shooting method to compute authentic stationary modes that do not form a continu-
ous family and, in contrast to the solutions in Ref. 26, can be found only if the propagation constant
$\mu$ is tuned to a certain isolated value.

2 SMALL-AMPLITUDE W-DW POTENTIAL: ASYMPTOTIC
EXPANSIONS

In this section, we study the persistence of the continuous family of solitary-wave solutions in
a complex potential of small amplitude described by the formal parameter $\varepsilon \ll 1$. In contrast to
the analysis of Ref. 26, where the Melnikov theory has been employed for this purpose, we take
a somewhat blunter approach and try to construct asymptotic power series that directly describe
the deformation of the continuous family as the amplitude of the complex potential increases
departing from zero. Unsurprisingly, for a complex potential of general form, the asymptotic pro-
cedure terminates already in the first order of amplitude. However, if the complex potential is of
$W$-$dW$ form, then the asymptotic procedure can proceed beyond the first order. Therefore, the
$W$-$dW$ relations (2) appear as a necessary condition for the persistence of the continuous fam-
ily under the perturbation by a non-$PT$-symmetric small-amplitude complex potential. However,
even if the potential is of $W$-$dW$-type, the asymptotic procedure generically terminates at the
second order. This outcome leads us to a conjecture that the W-dW relation \textit{per se} is not sufficient for the persistence of the continuous family.

For $\varepsilon = 0$ system, (5)-(6) has a well-known family of bright soliton solutions

$$u_0(x) = \sqrt{\mu} \text{sech}(\sqrt{\mu} (x - x_0)) \cos \theta, \quad v_0(x) = \sqrt{\mu} \text{sech}(\sqrt{\mu} (x - x_0)) \sin \theta,$$

where $x_0$ and $\theta$ are arbitrary reals. Due to the rotational symmetry of the NLSE, without loss of generality one can fix $\theta = 0$. We therefore set

$$u_0(x) = \sqrt{\mu} \text{sech}(\sqrt{\mu} (x - x_0)), \quad v_0(x) = 0.$$

Assume that for nonzero, but fixed $\varepsilon \ll 1$, system (5)-(6) has a family of solitary wave solution, that is, there exits functions $u(x; \mu)$ and $v(x; \mu)$, where $\mu$ changes continuously inside some interval. Our strategy in this section is to approach solutions $u(x; \mu)$ and $v(x; \mu)$ from the limit $\varepsilon = 0$. We therefore fix $\mu > 0$ and seek for solutions of (5)-(6) in the form of asymptotic expansions

$$u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots,$$

$$v(x) = \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \cdots.$$

(As $\mu$ is fixed in our computations, hereafter we do not indicate the dependence on $\mu$ explicit and write $u(x)$ instead of $u(x; \mu)$, $v(x)$ instead of $v(x; \mu)$, etc.)

Balance of the terms of the order $O(\varepsilon)$ yields

$$\mathcal{L}_6 u_1 = W_1(x)u_0(x), \quad (9)$$
$$\mathcal{L}_2 v_1 = W_2(x)u_0(x), \quad (10)$$

where we have introduced operators

$$\mathcal{L}_n : = \frac{d^2}{dx^2} - \mu + n u_0^2(x), \quad n \in \{2, 6\}.$$

As $\mathcal{L}_6 u_{0,x}(x) = 0$, the operator $\mathcal{L}_6$ has nonempty kernel. This implies that Equation (9) has a solitary-wave solution if the orthogonality condition holds:

$$\int_{-\infty}^{\infty} W_1(x)u_0(x)u_{0,x}(x) \, dx = -\frac{1}{2} \int_{-\infty}^{\infty} W_{1,x}(x)u_0^2(x) \, dx = 0. \quad (11)$$

The kernel of operator $\mathcal{L}_2$ is also nonempty, because $\mathcal{L}_2 u_0 = 0$. Therefore Equation (10) has a solitary wave solution if

$$\int_{-\infty}^{\infty} W_2(x)u_0^2(x) \, dx = 0. \quad (12)$$

Therefore, two nontrivial conditions (Equations 11 and 12) emerge already in the first order of the asymptotic theory. They cannot be satisfied for a complex potential of general shape. However, if
the potential is of W-dW form, that is,

\[ W_{1,x}(x) = CW_2(x), \]  

(13)

where \( C \) is an arbitrary real, then the conditions (11) and (12) coincide. This agrees completely with the result of. In this case, the admissible values of \( x_0 \) are determined at the first step of the asymptotic procedure by the equation

\[ \int_{-\infty}^{\infty} W_2(x) \text{sech}^2(\sqrt{\mu(x-x_0)}) \, dx = 0. \]  

(14)

If the solvability conditions of (11)-(12) are fulfilled, then the general solitary-wave solutions of system (11)-(12) have the form

\[ u_1(x) = \bar{u}_1(x) + C_1 u_{0,x}(x), \]  

(15)

\[ v_1(x) = \bar{v}_1(x) + C_2 u_0(x), \]  

(16)

where \( C_{1,2} \in \mathbb{R} \) are arbitrary constants and \( \bar{u}_1(x) \) and \( \bar{v}_1(x) \) are some fixed solitary wave solutions of (11) and (12), respectively. Thus the functions \( u_1(x) \) and \( v_1(x) \) are not yet uniquely defined.

To specify the constants \( C_{1,2} \), let us analyze the terms of order \( O(\epsilon^2) \). Balance of the terms yields the system

\[ \mathcal{L}_0 u_2 = -6u_0 u_1^2 - 2u_0 v_1^2 + W_1 u_1 - W_2 v_1, \]  

(17)

\[ \mathcal{L}_2 v_2 = -4u_0 u_1 v_1 + W_1 v_1 + W_2 u_1, \]  

(18)

(we simplify the notations taking \( u_k(x) := u_k, k = 0, 1, 2, v_k(x) := v_k, W_k(x) := W_k, k = 1, 2 \) and \( W_{k,x}(x) := W_{k,x}, k = 1, 2 \)). The solvability conditions for (17)-(18) are

\[ \int_{-\infty}^{\infty} ( -6u_0 u_1^2 - 2u_0 v_1^2 + W_1 u_1 - W_2 v_1 ) u_{0,x} \, dx = 0, \]  

(19)

\[ \int_{-\infty}^{\infty} ( -4u_0 u_1 v_1 + W_1 v_1 + W_2 u_1 ) u_0 \, dx = 0. \]  

(20)

These conditions should be satisfied by the proper choice of constants \( C_{1,2} \) in (15)-(16). Substituting (15)-(16) into (19)-(20) and collecting the terms with \( C_1 \) and \( C_2 \) separately, one arrives at the system of linear equations

\[ A_{11} C_1 + A_{12} C_2 + F_1 = 0, \]  

(21)

\[ A_{21} C_1 + A_{22} C_2 + F_2 = 0. \]  

(22)
Taking into account condition (13), it is straightforward to show that (see the Appendix for detailed calculations)

\[ A_{12} = A_{22} = 0, \]

\[ A_{21} = 8 \int_{-\infty}^{\infty} u_0^2 u_{0,x} \vartheta_1 \, dx, \quad A_{11} = -\frac{C}{2} A_{12}, \]

\[ F_1 = C \int_{-\infty}^{\infty} W_2 u_0 \vartheta_1 \, dx + \int_{-\infty}^{\infty} \vartheta_1 (2W_2 u_{0,x} + W_2, x u_0) \, dx, \]

\[ F_2 = -2 \int_{-\infty}^{\infty} W_2 \vartheta_1 u_0 \, dx. \]

Therefore \( C_2 \) in fact does not enter equations (21) and (22). This means that the system (21)-(22) has a solution if the following condition holds:

\[ I := F_2 C + 2F_1 = 2 \int_{-\infty}^{\infty} \vartheta_1 (2W_2 u_{0,x} + W_2, x u_0) \, dx = 0. \]

Generically, \( I \neq 0 \) and the asymptotic procedure terminates. One can try to make \( I \) equal to zero by adjusting the value of the parameter \( \mu \). However, even if the condition \( I = 0 \) is satisfied for some isolated values of \( \mu \), this still contradicts to the assumption of existence of a continuous family. Moreover, even if \( I = 0 \), the procedure is still not self-consistent, because \( C_2 \) cannot be determined unambiguously.

Note that if the potential is \( PT \) symmetric, that is, \( W_1(x) = W_1(-x) \) and \( W_2(x) = -W_2(-x) \), then, due to the parity of the solutions, the solvability conditions are automatically satisfied either at \( \varepsilon \)- and \( \varepsilon^2 \)-order by setting \( x_0 = 0 \).

The upshot of our analysis is that for a generic (i.e., non-\( PT \)-symmetric) small-amplitude \( W \)-d\( W \) potential the asymptotic procedure allows to construct an approximation that satisfies the stationary equation (4) with \( O(\varepsilon^2) \)-accuracy. However, the procedure fails to produce a more exact result. Strictly speaking, this may be due to the prescribed analytic form of expansion (power series with respect to \( \varepsilon \)) that may be not appropriate for the stationary solution. Therefore in Section 3 we employ another approach for the problem.

3 \hspace{1em} \textbf{W-dW WELL OF FINITE DEPTH: A NUMERICAL STUDY}

In this section, we support the results of our asymptotic analysis with a numerical study of a \( W \)-d\( W \) potential of finite amplitude. In contrast to the analysis of Ref. 26, where the Levenberg-Marquardt algorithm and MATLAB boundary-value solver have been employed for numerical search of stationary states, here we use a shooting-type approach. Various modifications of this method have been previously applied for real,\textsuperscript{10,34} dissipative,\textsuperscript{35} \( PT \)-symmetric\textsuperscript{36} and Wadati\textsuperscript{22} potentials. Advantages of this method consist in its transparency and geometric visualization, because the stationary modes can be searched as intersections of certain two-dimensional curves. We argue that the system of equations that determines a continuously differentiable nonlinear mode is, generically speaking, overdetermined (i.e., the number of equations in the system is larger than the number of unknowns). Therefore it is generically impossible to find an exact
solution to this system, and only approximate solutions with nonzero residual are possible. We call such nonzero-residual solutions pseudo-modes, to distinguish them from the authentic continuously differentiable modes. Varying the amplitude of the W-dW potential, we numerically confirm that for the pseudo-modes of the simplest form the residual behaves as $O(\varepsilon^2)$. This outcome confirms the finding of the asymptotic analysis in the previous section. There also exist pseudomodes of more complex shapes that have not been captured by our asymptotic procedure. For these pseudomodes, the residual is also generically different from zero, but depends on $\varepsilon$ according to a more complex law.

Consider now Equation (4) with a potential that is a W-dW well of finite depth

$$\lim_{x \to -\infty} W(x) = \lim_{x \to \infty} W(x) = \lim_{x \to -\infty} W_x(x) = \lim_{x \to \infty} W_x(x) = 0.$$  

We also assume that $W(x)$ and its derivative decay exponentially when $x \to \pm \infty$. A prototypical example is $W(x) = W_1(x) + iW_2(x)$, where

$$W_1(x) = -\frac{A}{e^{\alpha x} + Be^{-\beta x}}, \quad W_2(x) = W_{1,x}(x), \quad \alpha, \beta, A, B > 0. \quad (23)$$

This shape of $W_1(x)$ guarantees that, for generic values of parameters $\alpha, \beta, A, B$, the resulting complex W-dW potential is neither $PT$ symmetric nor Wadati-type. Its real part $W_1(x)$ has the unique local minimum situated at $x = (\alpha + \beta)^{-1} \ln(\alpha^{-1} \beta B)$.

Fix $\mu > 0$. Let $S^+$ be the class of solutions for Equation (4) that tend to zero when $x \to +\infty$, that is,

$$S^+ = \{\phi(x) \mid \phi(x) \to 0, \quad x \to +\infty\}.$$  

Then $\phi(x) \in S^+$ has the asymptotic behavior

$$\phi(x) = e^{-\sqrt{\mu} x} (C^+ + o(1)), \quad x \to +\infty, \quad (24)$$

where $C^+$ is a complex constant. We assume that any $\phi(x) \in S^+$ uniquely defines $C^+$ in the asymptotic relation (24) and vice versa, for any $C^+$ there exists the unique $\phi(x) \in S^+$ that obeys (24) (for real potentials the existence of this one-to-one correspondence was proven in10). Note that if $\phi(x) \in S^+$ with constant $C^+ = |C^+| e^{i\theta^+}$ in asymptotic relation (24), then the phase-rotated solution $\phi(x) e^{-i\theta^+} \in S^+$ is associated with real constant $|C^+|$ in (24). Similarly, let $S^-$ be the class of solutions for Equation (4) that tend to zero when $x \to -\infty$, that is,

$$S^- = \{\phi(x) \mid \phi(x) \to 0, \quad x \to -\infty\}.$$  

Then $\phi(x) \in S^-$ has the asymptotic behavior

$$\phi(x) = e^{\sqrt{\mu} x} (C^- + o(1)), \quad x \to -\infty. \quad (25)$$

and any $\phi(x) \in S^-$ with complex constant $C^- = |C^-| e^{i\theta^-}$ can be phase-rotated such that $\phi(x) e^{-i\theta^-} \in S^-$ corresponds to real constant $|C^-|$ in (25).

If $\phi(x)$ is a localized solution for Equation (4), then $\phi(x) \in S^+ \cap S^-$. The constants $C^+$ and $C^-$ that uniquely define the behavior of $\phi(x)$ at $x \to \pm \infty$ are generically complex. As the solution $\phi(x)$
is physically indistinguishable from its phase-rotated counterpart $\phi(x)e^{-i\theta}$, we can assume that one of the constants (either $C^+$ or $C^-$) is real. However, the second constant is generically complex.

Consider solutions of Equation (4) on semi-axes, $\mathbb{R}^+$ and $\mathbb{R}^-$. Let a solution $\phi^+(x) \in S^+$ be defined on $\mathbb{R}^+$ having real constant $C^+$ in (24). Also, let a solution $\phi^-(x) \in S^-$ be defined on $\mathbb{R}^-$ with real constant $C^-$ in (25). In order to get a solution that is continuously differentiable on the entire axis $\mathbb{R}$, one has to find two phases $\theta^+$ and $\theta^-$ such that the matching conditions hold

$$e^{i\theta^-} \phi^-(0) = e^{i\theta^+} \phi^+(0),$$
$$e^{i\theta^-} \phi_x^-(0) = e^{i\theta^+} \phi_x^+(0),$$

or, alternatively

$$\phi^-(0) = e^{i\theta} \phi^+(0),$$
$$\phi_x^-(0) = e^{i\theta} \phi_x^+(0),$$

where $\theta = \theta^+ - \theta^-$. We note that the system (26)-(27) implies that

$$|\phi^-(0)| = |\phi^+(0)|,$$
$$|\phi_x^-(0)| = |\phi_x^+(0)|,$$

$$\arg \phi^-(0) = \arg \phi^+(0) + \theta,$$
$$\arg \phi_x^-(0) = \arg \phi_x^+(0) + \theta.$$ 

This is a system of four real equations that includes only three unknowns $C^+$, $C^-$ and $\theta$. Generically, it does not have solutions. It might have solutions in the presence of some additional symmetries or integrals (in particular, in the PT-symmetric case the shooting approach can be reduced to solution of only one equation with respect to one real unknown, while for Wadati potential the situation reduces to a system of three equations with respect to three real unknowns). However, for a generic complex potential, the existence of localized solutions for Equation (4) is dubious.

To check whether the system (28)-(31) has a solution for a given W-dW potential, we use the following strategy.

1. For fixed $\varepsilon$ and $\mu$, compute the values of $C^\pm \in \mathbb{R}$ such that the equations (28)-(29) hold. Algorithmically this was done as follows. Denote $|\phi^-(0)| = R^-$, $|\phi^+_x(0)| = r^-$, $|\phi^+(0)| = R^+$, $|\phi^+_x(0)| = r^+$. In view of (24) and (25), $R^- \equiv R^-(C^-)$, $r^- \equiv r^-(C^-)$, $R^+ \equiv R^+(C^+)$, $r^+ \equiv r^+(C^+)$. Plot on the plane $(R, r)$ two curves: $\gamma^- = \{(R^-(C^-), r^-(C^-))|C^- \in (0; C^-_{\text{max}})\}$, parameterized by $C^-$ and $\gamma^+ = \{(R^+(C^+), r^+(C^+))|C^+ \in (0; C^+_{\text{max}})\}$ parameterized by $C^+$. At the point of intersection of these curves, $R^-(C^-) = R^+(C^+)$ and $r^-(C^-) = r^+(C^+)$. If this point is determined, the values of $C^+$ and $C^-$ are found such that (28)-(29) are satisfied. The procedure involves computation of $\phi^\pm(0)$ and $\phi^\pm_x(0)$ by given $C^\pm$. This can be done by standard Runge-Kutta method that solves ODE
with initial conditions
\[ \phi^+(x_+) = C^+ e^{-\sqrt{\mu}x_+}, \quad \phi^-_x(x_+) = -\sqrt{\mu}C^+ e^{-\sqrt{\mu}x_+} \]
for \( \phi^+(x) \), and
\[ \phi^-(x_-) = C^- e^{\sqrt{\mu}x_-}, \quad \phi^-_x(x_-) = \sqrt{\mu}C^- e^{\sqrt{\mu}x_-} \]
for \( \phi^-(x) \). The value \( x_+ > 0 \) has to be chosen large enough in such a way that the correction \( o(1) \) can be neglected safely in (24). Similarly, the value \( x_- < 0 \) has to be chosen large negative such that \( o(1) \) can be neglected in (25). If \( x_+ \) and \( x_- \) are chosen properly, then the further increase (respectively, decrease) of \( x_+ \) (respectively, \( x_- \)) does not affect the values of \( C^+ \) and \( C^- \) corresponding to the intersection point.

2. Having \( \phi^+(0), \phi^-(0), \phi^+_x(0), \phi^-_x(0) \) that correspond to the intersection \( \gamma^+ \cap \gamma^- \), compute the values
\[ \theta = -\arg \phi^+(0) + \arg \phi^-(0), \tag{32} \]
\[ \bar{\theta} = -\arg \phi^+_x(0) + \arg \phi^-_x(0). \tag{33} \]
The condition for solvability of (28)-(31) is
\[ \delta \equiv \theta - \bar{\theta} = 0. \tag{34} \]
If this condition is satisfied, then the piecewise-defined function
\[ \phi(x) = \begin{cases} \phi^-(x)e^{-i\theta}, & x \leq 0 \\ \phi^+(x), & x \geq 0 \end{cases} \tag{35} \]
solves all four equations of system (28)-(31) and therefore corresponds to an authentic stationary mode that is continuously differentiable. However, if \( \delta \neq 0 \), then function (35) solves only three of four equations. In what follows, we will say that such a function with \( \delta \neq 0 \) corresponds to a pseudo-mode.

As an example, we chose a nonsymmetric W-dW potential of the class (23) having
\[ W_1(x) = -\frac{1}{e^x + e^{-3x}}, \quad W_2(x) = \frac{e^x - 3e^{-3x}}{(e^x + e^{-3x})^2}. \tag{36} \]
Figure 1 presents two plots of curves \( \gamma^\pm \) computed for \( \mu = 1 \) and two different values of \( \epsilon \). We observe that the curves can have multiple intersection points, and for different \( \epsilon \) the shapes of the curves can be significantly different, that is, the intersection points can emerge or disappear as \( \epsilon \) changes. Here we focus on three first intersection points that are labeled as P1-P3 in Figure 1a. Note that with the increase of \( \epsilon \) points P2 and P3 merge and then disappear as illustrated in Figure 1b. To visualize the pseudo-modes that correspond to the chosen intersection points, we introduce
The curves $\gamma^+$ (blue) and $\gamma^-$ (red) and their intersections (labeled as P1–P3) for $\varepsilon = 1$ (panel a) and $\varepsilon = 1.5$ (panel b). For all curves, $C^\pm \in [0, 40]$. Here $\mu = 1$.

The shapes of the pseudo-modes corresponding to the intersection points P1–P3 in Figure 1.

Real-valued piecewise-defined functions

$$\rho(x) = \begin{cases} |\phi^-(x)|, & x \leq 0, \\ |\phi^+(x)|, & x \geq 0, \end{cases} \quad j(x) = \begin{cases} i(\overline{\phi_+^+}\phi^--\overline{\phi^-}^{++}), & x \leq 0, \\ i(\overline{\phi_+^+}\phi^+-\overline{\phi^+}^{++}), & x \geq 0. \end{cases}$$

These functions do not depend on the rotation $\theta$ and are therefore especially convenient. Hereafter the overline means complex conjugation. Note that in the physical context function $j(x)$ can be interpreted as energy flux across the (pseudo)-mode. By construction, for each pseudo-mode $\rho(x)$ is continuous, but it is not necessarily smooth; function $j(x)$ must be continuous for authentic stationary modes, but may have a discontinuity at $x = 0$ for pseudo-modes.

Figure 2 presents the pseudo-modes corresponding to intersections P1-P3. We observe that for each shown pseudo-mode, the corresponding function $j(x)$ has a jump at $x = 0$. Additionally, for P2 the cusp of $\rho(x)$ is well-visible at $x = 0$. The pseudo-mode at the first intersection point P1 resembles the bright soliton, that is, corresponds to the approximate solution constructed above in Section 2 by means of the asymptotic expansions. Solutions at the next intersections P2 and P3 have more sophisticated shapes and therefore cannot be captured by the asymptotic expansions developed above.
Dependencies of $\delta$ plotted with linear scale (a) and with log-log scale (b). For the reference, in (b) we also plot the straight dotted line with the slope equal to 2. In the linear scale, this line corresponds to $\delta \propto \varepsilon^2$. Note that the vertical axis is broken in (a). Labels 1–3 correspond to the intersection points P1–P3 in Figure 1. Here $\mu = 1$

Discontinuous shapes of pseudo-modes plotted in Figure 2 suggest that those solutions do not correspond to authentic continuously differentiable stationary modes. Indeed, evaluating the solvability indicator $\delta$, we observe that it is generically different from zero. Dependencies $\delta$ on $\varepsilon$ are presented in Figure 3 in linear and log-log scales. For the simple pseudo-mode corresponding to the first intersection point P1, we observe that the dependence of $\log \delta$ on $\log \varepsilon$ is well approximated by linear function with the slope close to 2. This again agrees with the above asymptotic analysis and suggests that this pseudo-mode solves Equation (4) for all $x$ except for $x = 0$, where the derivative of function $\phi(x)$ has a jump that is of order $O(\varepsilon^2)$.

For the pseudo-modes corresponding to P2 and P3, the dependencies $\delta(\varepsilon)$ are more sophisticated and cannot be described by a simple quadratic law. In the meantime, it is remarkable that for $\varepsilon \approx 0.8$ the $\delta(\varepsilon)$-function corresponding to P3 has a zero (in the log-log plot in Figure 3b it corresponds to a spike). This suggests that for some isolated value of $\varepsilon$ close to 0.8, an authentic continuously differentiable solution can potentially be found. Solutions of this type will be discussed below in Section 5. Summarizing the analysis of the present section, we have to conclude that for an arbitrarily chosen value of $\varepsilon$, Equation (4) with potential (36) only admits pseudomodes and no stationary modes.

4 | NONLINEAR DYNAMICS IN W-dW POTENTIALS

The goal of this section is to elucidate nonlinear dynamics governed by the time-dependent equation (3) with a W-dW potential. One of important analytical results of this section consists in the approximate conservation law that constrains the nonlinear dynamics in W-dW potentials. A dynamical invariant of motion for solitons in W-dW potentials has been earlier reported on in.27,28 Our result differs from the previous in two important aspects. First, the dynamical invariant from27,28 is obtained with a qualitative approach that treats a soliton as a particle with some mass, velocity, and position. However, in the framework of the time-dependent Equation (3), some of the effective quantities (namely, the soliton position and velocity) are not well-defined. Our conservation law is obtained in terms of the wavefunction $\Phi$. It does not rely on the effective-particle
formalism and is therefore valid for localized nonlinear waves of arbitrary shape (not necessarily single-soliton-shaped). Second, our conservation law is only approximate. Our dynamical invariant is exactly time-independent only in the limit of zero amplitude of W-dW potential, while for nonzero ε the temporal derivative of our invariant is of the ε²-order. In this section, we also perform numerical dynamical runs of the time-dependent equation (3) and observe that even though the pseudo-modes do not correspond to authentic stationary states, they feature meaningful nonlinear dynamics associated with the persistent oscillations of the soliton center around the position \( x₀ \) obtained from the above asymptotic analysis.

**4.1 Approximate conservation law and necessary steady-state conditions**

A natural question emerges on whether the pseudo-modes encountered in the previous sections have any signature in the nonlinear time-dependent dynamics governed by the non-stationary equation (3). This issue will be addressed in the present section. However, let us first outline some general features of nonlinear dynamics in W-dW potentials. Let \( \Phi(x, t) \) be a localized wavepacket whose dynamics is governed by Equation (3). We introduce the squared \( L² \)-norm of the solution (in the optical context it can be interpreted as the beam power) and the location of the center of the wavepacket:

\[
N(t) = \int_{-\infty}^{\infty} |\Phi|^2 dx,
\quad X(t) = N^{-1}(t) \int_{-\infty}^{\infty} x|\Phi|^2 dx.
\]

(37)

Computing the temporal derivative of \( N(t) \), we obtain the standard “balance equation”

\[
N_t = 2 \varepsilon \int_{-\infty}^{\infty} W_2 |\Phi|^2 dx.
\]

(38)

For a shape-preserving stationary mode \( \Phi = e^{i\mu t} \phi(x) \), this gives an obvious condition

\[
2 \varepsilon \int_{-\infty}^{\infty} W_2 |\phi|^2 dx = -\frac{2 \varepsilon}{C} \int_{-\infty}^{\infty} W_1 \left( \frac{d}{dx} |\phi|^2 \right) dx = 0.
\]

(39)

This condition generalizes that derived above in the first-order perturbation theory (see Equation 12).

Additionally, introducing the momentum \( P(t) = i \int_{-\infty}^{\infty} (\overline{\Phi_x} \Phi - \Phi_x \overline{\Phi}) dx \), from Equation (3), we compute

\[
P_t = -2C \varepsilon \int_{-\infty}^{\infty} W_2 |\Phi|^2 dx + 2i \varepsilon \int_{-\infty}^{\infty} W_2 (\Phi^* \Phi_x - \Phi_x \Phi^*) dx.
\]

(40)

An additional calculation yields

\[
\dot{\varepsilon} \int_{-\infty}^{\infty} W_1 |\Phi|^2 dx = i \varepsilon C \int_{-\infty}^{\infty} W_2 (\Phi^* \Phi_x - \Phi_x \Phi^*) dx + 2 \varepsilon^2 \int_{-\infty}^{\infty} W_1 W_2 |\Phi|^2 dx.
\]

(41)
Combining the latter relations with (38) and (40), we obtain

\[
\frac{d}{dt}\left(P + CN - \frac{2\varepsilon}{C} \int_{-\infty}^{\infty} W_1 |\Phi|^2 dx\right) = -\frac{4\varepsilon^2}{C} \int_{-\infty}^{\infty} W_1 W_2 |\Phi|^2 dx. \tag{42}
\]

For small \( \varepsilon \), the latter equality can be considered as an “approximate” conservation law that is specific to small-amplitude W-dW potentials. In the line with findings of Section 2, this result indicates that a careful attention to the \( \varepsilon^2 \)-order-behavior is crucial for the most precise description of nonlinear waves in W-dW potentials.

For a stationary mode, the left-hand side of (42) is zero, which leads to another necessary condition for the shape of the solitary state:

\[
-\frac{4\varepsilon^2}{C} \int_{-\infty}^{\infty} W_1 W_2 |\phi|^2 dx = \frac{2\varepsilon^2}{C^2} \int_{-\infty}^{\infty} W_1^2 \left(\frac{d}{dx} |\phi|^2\right) dx = 0. \tag{43}
\]

Introducing the transverse current \( j(x) \) across the stationary state

\[
j(x) = i(\bar{\phi} x \phi - \bar{\phi} \phi x), \tag{44}
\]

we obtain the standard result that interrelates the derivative of the current and the gain-and-loss distribution:

\[
j_x = 2\varepsilon W_2 |\phi|^2. \tag{45}\]

More interestingly, for W-dW potentials, we obtain

\[
C \int_{-\infty}^{\infty} j(x) W_2(x) dx + 2\varepsilon \int_{-\infty}^{\infty} W_1 W_2 |\phi|^2 dx = 0. \tag{46}
\]

Therefore, condition (43) is equivalent to

\[
\int_{-\infty}^{\infty} j(x) W_2(x) dx = 0. \tag{47}
\]

Comparing (39) and (47), we observe that for a stationary mode the imaginary part of the potential \( W_2 \) must be orthogonal not only to the squared modulus of the wavefunction but also to the shape the transverse current distribution.

### 4.2 Numerical simulations of nonlinear dynamics

Let us now turn to dynamics of the pseudo-mode solitary waves that satisfy Equation (4) with \( \varepsilon^2 \)-accuracy (see Section 2). As a model example, we again choose the W-dW potential (23). First, we solve the Cauchy problem with the initial condition \( \Phi(x, 0) = \text{sech}(x - x_0) \), where \( x_0 \) is chosen to satisfy the compatibility condition (12) that emerges in the first order of the perturbation procedure. Numerical solution of Equation (12) gives \( x_0 \approx 0.4640 \). Representative examples of our dynamical simulations are shown in Figure 4 for the squared norm \( N(t) \) and center of mass \( X(t) \).
For sufficiently small $\varepsilon$, we observe that the plotted characteristics feature small-amplitude nearly periodic oscillations. For small $\varepsilon$, the periodicity is almost perfect, whereas for larger $\varepsilon$ a slow drift appears. Amplitude of the oscillations and the drift velocity naturally become stronger with the increase of amplitude of the potential $\varepsilon$.

Next, we address the situation when the initial condition $\Phi(x,0) = \text{sech}(x - \bar{x})$ is situated at a different position than that prescribed by the asymptotic analysis, that is, $\bar{x} \neq x_0$. The results plotted in Figure 5 show that for small $\varepsilon$ the center of initially displaced wavepacket performs nearly perfect periodic oscillations around $x_0$, and the amplitude of the oscillations increases with the increase of the initial displacement $|\bar{x} - x_0|$. This suggests that even though there is no authentic stationary mode existing at $x = x_0$, this asymptotically predicted position still plays an important role in the nonlinear dynamics. For extremely large initial positions $\bar{x}$, the initial soliton is situated in an effectively homogeneous medium, because the numerical value of the exponentially decaying potential becomes zero for large $x$. In this situation, the periodic oscillations naturally disappear.

Representative plots illustrating evolution of the amplitude $|\Phi(x,t)|$ are presented in Figure 6.

5 | NONLINEAR MODES FOR ISOLATED VALUES OF $\varepsilon$

In this section, we complement our study by numerical computing the authentic stationary modes. They are found using the extension of the numerical shooting approach described above in Section 3. In contrast to the main outcome of Ref. 26, we find that stationary nonlinear modes
FIGURE 5 Results of the nonlinear dynamics simulation for
$\Phi(x, 0) = \text{sech}(x - \tilde{x})$ when the initial center of the wavepacket $X(0) = \tilde{x}$ is different from the value $x_0$ predicted by the asymptotic analysis. Three curves correspond to $\tilde{x} = x_0 \pm x_0$ (blue and red curves, respectively) and to $\tilde{x} = -x_0$ (green curve). In all cases, $\varepsilon = 0.01$. Horizontal dashed line corresponds to $X = x_0$.

FIGURE 6 Plot of the amplitude $|\Phi(x, t)|$ for initial condition $\Phi(x, 0) = \text{sech}(x - \tilde{x})$ when $\varepsilon = 0.05$ and $\tilde{x} = x_0$ (left panel) and $\varepsilon = 0.01$ and $\tilde{x} = -x_0$ (right panel).

in a generic $W$-$dW$ potential of fixed amplitude do not form a continuous family, but exist only at isolated values of the propagation constant $\mu$. The found authentic nonlinear modes have complex two-hump shapes. Their dynamics is unstable.

As explained above, for a continuously differentiable stationary mode $\phi(x)$ the resulting system of matching conditions (28)-(31) cannot be generically solved if the values of $\varepsilon$ and $\mu$ are fixed. However, if $\varepsilon$ or $\mu$ is treated as another unknown, then the system of four equations is no longer overdetermined, and a numerical solution can be found by Newton iterations. A good initial guess for the iterative procedure is given by the pseudo-mode with $\mu = 1$ and $\varepsilon = 0.8$ that corresponds to the intersection point $P3$ (see the corresponding panel in Figure 2). In Figure 7, we illustrate
a numerically found branch of nonlinear modes in terms of dependencies $\varepsilon(\mu)$ and $N(\mu)$, where $N = \int_{-\infty}^{\infty} |\phi(x)|^2 dx$. We stress that the shown dependencies do not represent a continuous family, because they can exist only if $\mu$ and $\varepsilon$ are varied simultaneously. For instance, for $\mu = 1$, the nonlinear mode can only be found at the isolated value $\varepsilon \approx 0.809$.

Representative shapes of the stationary modes are exemplified in Figure 8a,b in terms of the amplitude $\rho = |\phi|$ and the current $j(x)$. We observe that the amplitude has a distinctively double-hump shape and features a local minimum approximately at the minimum of the real part of the potential. On the other hand, the current $j(x)$ is dominantly negative, which agrees with the spatial distribution of the gain-and-losses (see the imaginary part of the potential plotted in Figure 8a). The maximal negative value of the current is approached at the local minima of the real part of the potential. For small values of $\varepsilon$, the form of the stationary state resembles a bound state of two elementary nonlinear modes.

The eccentric shape of the obtained nonlinear modes suggests that they can hardly be stable. The instability was indeed confirmed using the linear stability analysis. Following the standard procedure, we consider a perturbed solution $\Phi(x, t) = e^{i\mu t} [\phi(x) + u(x)e^{i\omega t} + \tilde{v}(x)e^{-i\tilde{\omega}t}]$, where $u(x)$ and $v(x)$ are small perturbations. Linearization of Equation (3) with respect to $u(x)$ and $v(x)$...
FIGURE 8  Shapes of stationary modes for $\mu = 1$, $\varepsilon \approx 0.091$ (a) and $\mu \approx 3.830$, $\varepsilon \approx 0.068$ (b). Red and blue curves correspond to the modulus $\rho(x) = |\phi(x)|$ and current $j(x)$, respectively. Dashed curves in (a) plot real ($W_1$) and imaginary ($W_2$) parts of the potential. Lower panels show the linear stability eigenvalues.

Unstable modes correspond to eigenvalues $\omega$ with negative imaginary parts.

Numerical solution of the linear stability eigenvalue problem reveals several unstable eigenvalues in the spectrum, see the eigenvalue portraits in Figure 8c,d. A closer inspection indicates that, in the contrast with situation that takes place for real-valued potentials, $PT$-symmetric potentials, and Wadati potentials, in the case at hand the eigenvalues that emerge in linearization spectra do not form quartets $(\omega, \bar{\omega}, -\omega, -\bar{\omega})$. This fact provides another validation of the essentially dissipative nature of W-dW potentials.

Simulating nonlinear dynamics of found stationary modes [exemplified in Figure 9], we observe that for larger $\varepsilon$ the mode breaks up into two solitary waves that move in opposite directions and eventually leave the domain where the complex potential is localized. In the meanwhile, for

\[
\begin{bmatrix}
\delta_x^2 - \mu - \varepsilon(W_1 + iW_2) + 4|\phi|^2 \\
-2\phi^2 \\
\delta_x^2 + \mu - \varepsilon(W_1 - iW_2) - 4|\phi|^2
\end{bmatrix}
\begin{bmatrix}
u \\
u
\end{bmatrix}
= \omega
\begin{bmatrix}
u \\
u
\end{bmatrix}.
\]

(48)
smaller $\epsilon$ only one solitary wave escapes, while the second one performs periodic movement that resembles the oscillations of pseudo-modes observed in Section 4.

6  |  CONCLUSION

In our study, we have examined the peculiar features of a recently discovered class of complex potentials. More specifically, we have considered the class of $W$-$dW$ potentials that by definition have the form $W(x) = W_1(x) + iCW_{1,x}(x)$, where $W_1(x)$ is a differentiable real-valued function, and $C$ is a real. It has been suggested, that the nonlinear Schrödinger equation (NLSE) with a $W$-$dW$ potential can support continuous families of stationary solitary-wave nonlinear modes. These objects have been in the focus of the present study. Assuming that the potential is small, of $\epsilon$-order, we have employed asymptotic methods to search for the stationary nonlinear modes, seeking them in the form of formal power series with respect to $\epsilon$. The asymptotic procedure stops at the terms of the $\epsilon^2$-order, which leads us to a conjecture that no continuous families of nonlinear modes exist in generic $W$-$dW$ potentials. In order to validate this hypothesis, we have considered a particular example of the $W$-$dW$ potential whose real part is a finite-depth well. The prediction of the asymptotic approach has been confirmed by numerical arguments, because instead of any authentic nonlinear mode we have been able to find only a pseudo-mode that solves the equation with $O(\epsilon^2)$-accuracy. At the same time, with numerical simulations of nonlinear dynamics in $W$-$dW$ potentials, we have demonstrated that the pseudo-modes can be dynamically robust in small-amplitude $W$-$dW$ potentials. More specifically, the dynamics of pseudo-modes reveals persistent oscillations of the center-of-mass around the specific position that characterizes the center of the pseudo-mode in the asymptotic expansion. So, even not being authentic stationary nonlinear modes in the mathematical sense, these objects can be regarded as meaningful physical entities. Finally, we have also computed authentic stationary modes that only exist if the parameters of the equation and of the solution itself are tuned precisely. These stationary modes are unstable, and their dynamical instability reveals several distinctive behaviors.
Examples of physically meaningful “pseudo-modes” (or, more generically, “pseudo-solutions”) are not unheard in the previous literature. For instance, a great number of models where “asymptotics beyond all orders” occurs (see, for numerous examples,39) provide physical objects that cannot be described by idealized mathematical models. One of them is the famous example of $\phi^4$ breather that is nonexistent in mathematical sense40,41 but that may have “decay timescale ... longer than the predicted lifetime of the universe”.42 Another type of pseudo-solutions that is especially well-known in PT-symmetric potentials corresponds to the so-called ghost states, which can be found from the stationary nonlinear equation with complex-valued propagation constants $\mu$.43–45 Even though a ghost state is not a true solution of the time-dependent NLSE, several studies report on that if the imaginary part of $\mu$ is small, then it can dynamically persist as a metastable entity, even if the coexisting authentic nonlinear modes are unstable. We surmise that the pseudomodes reported herein can be probably also interpreted as ghost states after the consideration is extended to allow $\mu$ to be complex-valued.

To conclude, the search of new complex potentials that admit continuous families of nonlinear modes remains a challenging problem for the future studies. Regarding the particular class of W-dW potentials, we believe that an interesting task is related to a more systematic analysis of oscillating patterns encountered in the nonlinear dynamics. A more systematic study of stationary modes in W-dW potentials is also in order. An especially intriguing issue is the search for stable stationary modes that can eventually exist in W-dW potentials of the form different from that considered herein. Although only the case of focusing nonlinearity has been considered herein, the behavior of (pseudo)modes can also be addressed under the defocusing (repulsive) nonlinearity.

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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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**APPENDIX A: CALCULATION OF THE COEFFICIENTS FOR THE SYSTEM (21) and (22)**

Direct substitution of (15) and (16) into (19) and (20) yields the system (21) and (22) where (for compactness in what follows we write $\int$ instead of $\int_{-\infty}^{\infty}$, bearing in mind that the integration is always over the whole real axis):

$$A_{11} = \int (12u_0\bar{u}_1 - W_1)u_{0,x}^2 \, dx,$$  \hspace{1cm} (A1)

$$A_{12} = \int (4u_0\bar{v}_1 + W_2)u_0u_{0,x} \, dx,$$  \hspace{1cm} (A2)

$$A_{21} = \int (4u_0\bar{v}_1 - W_2)u_0u_{0,x} \, dx,$$  \hspace{1cm} (A3)

$$A_{22} = \int (4u_0\bar{u}_1 - W_1)u_0^2 \, dx,$$  \hspace{1cm} (A4)

$$F_1 = \int \left(6u_0\bar{u}_1^2 + 2u_0\bar{v}_1^2 - W_1\bar{u}_1 + W_2\bar{v}_1 \right)u_{0,x} \, dx,$$
\[ F_2 = \int (4u_0\dddot{u}_1 - W_1\dddot{v}_1 - W_2\dddot{u}_1)u_0\, dx. \]

Let us simplify these expressions. In the space of rapidly decreasing functions (Schwartz space), define the inner product of \(a(x)\) and \(b(x)\) as

\[ \langle a(x), b(x) \rangle := \int a(x)b(x)\, dx. \]

The operators \(L_2\) and \(L_6\) are self-adjoint, therefore

\[ \langle L_2a(x), b(x) \rangle = \langle a(x), L_2b(x) \rangle, \quad \langle L_6a(x), b(x) \rangle = \langle a(x), L_6b(x) \rangle. \]

Also we make use of the fact that by construction (see Equations 9-10 and 15-16]

\[ L_6\dddot{u}_1 = W_1u_0, \quad L_2\dddot{v}_1 = W_2u_0. \]

1. Consider \(A_{12}\).

\[ A_{12} = \int (4u_0\dddot{v}_1 + W_2)u_0u_{0,x}\, dx = 4 \int u_0^2u_{0,x}\dddot{v}_1\, dx + \langle u_{0,x}, L_2\dddot{v}_1 \rangle. \]

The last term can be transformed as follows

\[ \langle u_{0,x}, L_2\dddot{v}_1 \rangle = \langle L_2u_{0,x}, \dddot{v}_1 \rangle = -4\langle u_0^2u_{0,x}, \dddot{v}_1 \rangle = -4 \int u_0^2u_{0,x}\dddot{v}_1\, dx. \]

Here we make use of the formula \(L_6u_{0,x} = 0\) that implies that

\[ L_2u_{0,x} = -4u_0^2u_{0,x}. \] (A5)

Therefore \(A_{12} = 0\).

2. Consider \(A_{22}\).

\[ A_{22} = \int (4u_0\dddot{u}_1 - W_1)u_0^3\, dx = 4 \int u_0^3\dddot{u}_1\, dx - \langle L_6\dddot{u}_1, u_0 \rangle \]

\[ = 4 \int u_0^3\dddot{u}_1\, dx - \langle \dddot{u}_1, L_6u_0 \rangle. \]

Since \(L_2u_0 = 0\), then \(L_6u_0 = 4u_0^3\). Therefore

\[ A_{22} = 4 \int u_0^3\dddot{u}_1\, dx - \langle \dddot{u}_1, 4u_0^3 \rangle = 0. \]
3. Consider $A_{21}$.

\[ A_{21} = A_{12} - 2 \int W_2 u_0 u_{0,x} \, dx = -2 \langle \mathcal{L}_2 \tilde{v}_1, u_{0,x} \rangle = 8 \int \tilde{v}_1 u_0^2 u_{0,x} \, dx, \]

where we have again used (A5).

4. Consider $A_{11}$.

\[
A_{11} = 12 \int u_0 \tilde{u}_1 u_{0,x}^2 \, dx - \int W_1 u_{0,x}^2 \, dx = 12 \int u_0 \tilde{u}_1 u_{0,x}^2 \, dx + \int u_0 (W_1 u_{0,x} + W_1 u_{0,xx}) \, dx \\
= 12 \int u_0 \tilde{u}_1 u_{0,x}^2 \, dx + C \int u_{0,x} W_2 u_0 \, dx + \int W_1 u_0 (u_0 - 2u_0^3) \, dx \\
= 12 \int u_0 \tilde{u}_1 u_{0,x}^2 \, dx + C \int u_{0,x} W_2 u_0 \, dx + \int W_1 u_0 \tilde{u}_1 \, dx \\
= 12 \int u_0 \tilde{u}_1 u_{0,x}^2 \, dx - 4C \int \tilde{v}_1 u_0^2 u_{0,x} \, dx + 4 \int \tilde{u}_1 u_0^3 \, dx - 2 \langle \tilde{u}_1, \mathcal{L}_3 u_0^3 \rangle. \quad (A6)
\]

Straightforward computation yields $\mathcal{L}_3 u_0^3 = 6u_0 u_{0,x}^2 + 2u_0^3$. This implies that the first, third, and fourth summands in (A6) annihilate, and we finally obtain

\[ A_{11} = -4C \int \tilde{v}_1 u_0^2 u_{0,x} \, dx. \]

5. Consider $F_2$.

\[
F_2 = 4 \int u_0^2 \tilde{u}_1 \tilde{v}_1 \, dx - \int W_1 u_0 \tilde{v}_1 \, dx - \int W_2 u_0 \tilde{u}_1 \, dx \\
= 4 \int u_0^2 \tilde{u}_1 \tilde{v}_1 \, dx - \langle \mathcal{L}_6 \tilde{u}_1, v_1 \rangle - \int W_2 u_0 \tilde{u}_1 \, dx \\
= 4 \int u_0^2 \tilde{u}_1 \tilde{v}_1 \, dx - \int \tilde{u}_1 (W_2 u_0 + 4u_0^2 \tilde{v}_1) \, dx - \int W_2 u_0 \tilde{u}_1 \, dx = -2 \int W_2 u_0 \tilde{u}_1 \, dx,
\]

where we have used the equality $\mathcal{L}_6 \tilde{v}_1 = W_2 u_0 + 4u_0^2 \tilde{v}_1$ that can be derived easily.

6. Consider $F_1$. As its calculation is a bit more involved, we decompose $F_1$ into four summands representing $F_1 = I_1 + I_2 + I_3 + I_4$, where

\[
I_1 = 6 \int u_0 \tilde{u}_1^2 u_{0,x} \, dx, \quad I_2 = 2 \int u_0 \tilde{u}_1^2 u_{0,x} \, dx, \quad I_3 = - \int W_1 \tilde{u}_1 u_{0,x} \, dx, \quad I_4 = \int W_2 \tilde{u}_1 u_{0,x} \, dx.
\]

The calculation proceeds as follows:

\[ I_3 = \int u_0 (W_{1,x} \tilde{u}_1 + W_1 \tilde{u}_{1,x}) \, dx = C \int u_0 W_2 \tilde{u}_1 + \langle \mathcal{L}_6 \tilde{u}_1, \tilde{u}_{1,x} \rangle. \quad (A7)\]
Straightforward differentiation yields \( L_0 \tilde{u}_{1,x} = W_{1,x} u_0 + W_1 u_{0,x} - 12 u_0 u_{0,x} \tilde{u}_1 \), which after substitution in (A7) eventually leads to

\[
I_3 = 2C \int u_0 W_2 \tilde{u}_1 dx - I_3 - 2I_1, \tag{A8}
\]

and hence

\[
I_1 + I_3 = C \int u_0 W_2 \tilde{u}_1 dx = -\frac{C}{2} F_2. \tag{A9}
\]

In a similar manner, using that \( L_2 \tilde{v}_{1,x} = W_{2,x} u_0 + W_2 u_{0,x} - 4 u_0 u_{0,x} \tilde{v}_1 \), we deduce

\[
I_2 + I_4 = \int \tilde{v}_1 (2u_0 u_{0,x} \tilde{v}_1 + W_2 u_{0,x}) dx = \frac{1}{2} \int \tilde{v}_1 (-L_2 \tilde{v}_{1,x} + W_{2,x} u_0 + 3W_2 u_{0,x}) dx
\]

\[
= \frac{1}{2} \left(- \int W_2 u_0 \tilde{v}_{1,x} dx + \int \tilde{v}_1 (W_{2,x} u_0 + 3W_2 u_0) dx \right) = \int \tilde{v}_1 (W_{2,x} u_0 + 2W_2 u_{0,x}) dx.
\]

Combining the latter result with (A9), we obtain the final expression for \( F_1 \).