Self-Similarity in Geometry, Algebra and Arithmetic

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Abstract

We define the concept of self-similarity of an object by considering endomorphisms of the object as ‘similarity’ maps. A variety of interesting examples of self-similar objects in geometry, algebra and arithmetic are introduced. Self-similar objects provide a framework in which, one can unite some results and conjectures in different mathematical frameworks. In some general situations, one can define a well-behaved notion of dimension for self-similar objects. Morphisms between self-similar objects are also defined and a categorical treatment of this concept is provided. We conclude by some philosophical remarks.

Introduction

The goal of this manuscript is to study the concept of self-similarity in the abstract sense of the word. For example, an abstract notion of equality can be implemented by fixing an equivalence relation. In the same way, given any concept of similarity, we consider the associated concept of a self-similar object. For now, consider the following general definition for self-similar objects:

Definition 0.1 Let $X$ be a space and let $f_i$ for $i = 1$ to $n$ denote appropriate endomorphisms of $X$. A subset $F \subseteq X$ is called ‘self-similar with respect to $f_i$’ if $F$ is disjoint union of its images under the endomorphisms $f_i$. We write $F = \sqcup f_i(F)$. Endomorphisms $f_i$ are called ‘similarity maps’. $F$ is called ‘self-similar’ if it is self-similar with respect to a finite number of similarity maps. Evidently, choice of $f_i$ is not unique. If finite intersection of similar images are allowed, $F$ is called an ‘almost self-similar’ subset of $X$.

This formulation works for both geometric and arithmetic settings and is even fit for an algebraic treatment after minor changes. Several examples are introduced in the first section, and many old results and conjectures are reformulated using this language in the second section. A more abstract categorical formulation paves the way for a functorial understanding of self-similarity which form the content of the third chapter. Philosophical remarks conclude this manuscript.
1 Examples of self-similar objects

In order to provide grounds for better understanding of the scope of generality of results proposed in the following sections, we start by providing a list of self-similar objects. In geometry, there is a rich gallery of spaces and endomorphisms. Some classical self-similar affine fractals are already introduced in the literature. In algebra, considering automorphisms of self-similar objects and associated algebraic objects lead to some self-similar objects. In algebraic geometry, algebraic endomorphisms of varieties could be considered as similarity maps. In arithmetic, there are fairly general frame-works in which we have an understanding of the asymptotic behavior of points of bounded norm or bounded height. We will see that, these are natural places to look for self-similar arithmetic objects. Some of these arithmetic self-similar objects can be formulated in the language of arithmetic geometry.

1.1 Self-similarity in geometry

Thinking of self-similar geometric objects as geometric spaces would lead to an unfamiliar concept of geometric intuition. Here are some examples:

1. Cantor set, Serpinski carpet, von Koch curve are all examples of affine self-similar or almost self-similar objects. To obtain such objects, one starts from a simple geometric object and finitely many non-intersecting similar copies inside. By iteration of this process one always gets a self-similar object with respect to similarity maps one starts with. To obtain Serpinski carpet and von Koch curve one allows one point of intersection. These are examples of affine self-similar objects, for which one starts from an affine ambient space and considers finitely many affine maps as similarity maps. Standard box formulas provide a notion of fractal dimension [Falc].

2. Take the projective space $\mathbb{P}^n(\mathbb{C})$ as ambient space and projective endomorphisms as similarity maps. This way one could embed affine fractals in projective ambient spaces. There are examples of projective self-similar objects which are not induced by affine objects. For example, the subset

$$\{(2^i; 2^j) \in \mathbb{P}^1(\mathbb{C}) | i, j \in \mathbb{N} \cup \{0\}\} \cup \{(0; 1), (1; 0)\}$$

is a projective self-similar set with $f(x; y) = (x; 2y)$ as a single similarity map.

3. The set of periodic points and the set of pre-periodic points of an endomorphism of a geometric space could also be thought as a self-similar object. Kawaguchi proved that for a holomorphic endomorphism of degree $\geq 2$ of the projective space $\mathbb{P}^n(\mathbb{C})$ or of a compact Riemann surface there are only finitely many periodic points [Ka]. This brings finite dynamical systems into attention. There are many other finiteness results in this direction. For example, Fakhruddin proved that the set of all possible periodicity exponents for an endomorphism of a proper projective variety is a finite set [Fakh].
1.2 Self-similarity in algebra

Algebraic objects associated to self-similar geometric objects usually express a self-similar nature:

1. Consider the vector space \( V \) of continuous real-valued functions on a geometric self-similar object with injective similarity maps. Any similarity map induces a self-map of this vector space if we extend by zero. The vector space \( V \) is generated by finitely many non-intersecting copies of its similar images which makes a self-similar vector space.

2. Take a ring \( R \) as ambient space and injective endomorphisms as similarity maps. Suppose that \( R \) can be generated by finitely many of its own copies. This makes \( R \) to be a self-similar object. For example, \( \mathbb{Z} \) can be generated by finitely many subrings isomorphic to itself.

3. Consider the Cayley graph of a finitely generated free group with respect to a free generating set. Remove the identity vertex and edges landing on it. Let \( G \) denote the automorphism group of one of the connected components of the remaining graph. \( G \) is generated by a finite number of subgroups isomorphic to \( G \) together with finitely many of their cosets. Therefore, \( G \) is a self-similar object.

1.3 Self-similarity in algebraic geometry

In algebraic geometry, an algebraic variety plays the role of the ambient space and algebraic endomorphisms of varieties could be taken as similarity maps:

1. Take the complex projective space \( \mathbb{P}^n(\mathbb{C}) \) as ambient space and polynomial endomorphisms \( f_i = (\phi_0, \phi_1, ..., \phi_n) \) which are induced by \( n + 1 \) homogeneous polynomials in \( n + 1 \) variables of degree \( m_i \), as similarity maps. For example, the subset \( \{(2^i; 2^j) \in \mathbb{P}^1(\mathbb{C}) | i, j \in \mathbb{N} \cup \{0\}\} \) is self-similar with respect to \( f_1(x; y) = (x^2; y^2) \) and \( f_2(x; y) = (2x^2; y^2) \).

2. Let \( A \) denote an abelian variety over \( \mathbb{C} \). Fix a natural number \( n \). Take \( A \) for an ambient space and translated multiplications by \( n \) as similarity endomorphisms. Then, for every integer \( n > 1 \), the set of torsion elements of \( A \) is a self-similar subset with respect to all translations of multiplication by \( n \) via \( n \)-torsion elements.

3. Let \( A \) denote an abelian variety over \( \mathbb{C} \). A finitely generated subgroup of \( A \) is self-similar with respect to a few translations of multiplication by \( n \) taken as similarity maps if \( n \) is appropriately chosen. The following result of Neron addresses the growth of the number of points of bounded height [Ne]:

Let \( A \subset \mathbb{P}^n \) denote an abelian variety defined over a number field \( K \) and let \( r = r(A, K) \) denote the rank of the group of \( K \)-rational points in \( A \), then there exists a constant \( c_{A, K} \) such that

\[
N(A(K), x) \sim c_{A,K}(\log x)^{r/2}.
\]

where \( N(A(K), x) \) denotes the number of points of height bounded by \( x \). One can see that the rate of growth mentioned above is related to fractal dimension of the set of \( K \)-rational points as a self-similar subset of \( A \) [Ra].
1.4 Self-similarity in arithmetic

The setting of arithmetic self-similarity allows us to define a well-defined and well-behaved concept of fractal dimension for these arithmetic objects. This concept is intimately related to the concept of arithmetic height [Ra]:

1. Take a ring $R$ as ambient space and polynomial maps as similarity maps. A self-similar subset is affine, in case these polynomials are of degree one. An example of affine self-similar subset of $\mathbb{Z}$ is the set of integers with only certain digits appearing in their decimal expansion. The fractal dimension of an affine self-similar subset of $\mathbb{Z}$ is defined by the box formula $\sum_i a_i^{-s} = 1$ where $a_i$ are the leading coefficients of degree one similarity maps. This is a well-defined and well-behaved notion of fractal dimension [Ra].

2. By a $t$-module of dimension $N$ and rank $d$ defined over the algebraic closure $\overline{k} = \mathbb{F}_q(t)$ we mean, fixing an additive group $(\mathbb{G}_a)^N(\overline{k})$ and an injective homomorphism $\Phi$ from the ring $\mathbb{F}_q[t]$ to the endomorphism ring of $(\mathbb{G}_a)^N$ which satisfies

$$\Phi(t) = a_0 F^0 + \ldots + a_d F^d$$

where $a_i$ are $N \times N$ matrices with coefficients in $\overline{k}$ with $a_d$ non-zero, and $F$ is a Frobenius endomorphism on $(\mathbb{G}_a)^N$. Polynomials $P_i \in \mathbb{F}_q[t]$ of degrees $r_i$ for $i = 1$ to $n$ could be taken as similarity maps. Denis defines a canonical height $\hat{h}$ on these modules for which one has

$$\hat{h}[\Phi(P)(\alpha)] = q^{dr}.\hat{h}[\alpha]$$

for all $\alpha \in (\mathbb{G}_a)^N$, where $P$ is a polynomial in $\mathbb{F}_q[t]$ and degree $r$ [De]. We define the fractal dimension of a self-similar subset with respect to $P_i$ to be the real number $s$ such that $\sum_i (r_i d)^{-s} = 1$.

3. Start from a linear semi-simple algebraic group $G$ and a rational representation $\rho : G \to GL(W)\mathbb{Q}$ defined over $\mathbb{Q}$. Let $w_0 \in W\mathbb{Q}$ be a point whose orbit $V = w_0 \rho(G)$ is Zariski closed. Then the stabilizer $H \subset G$ of $w_0$ is reductive and $V$ is isomorphic to $H \setminus G$. By a theorem of Borel-Harish-Chandra $V(\mathbb{Z})$ breaks up to finitely many $G(\mathbb{Z})$ orbits [Bo-HC]. Thus the points of $V(\mathbb{Z})$ are parametrized by cosets of $G(\mathbb{Z})$. Fix an orbit $w_0 G(\mathbb{Z})$ with $w_0$ in $G(\mathbb{Z})$. Then the stabilizer of $w_0$ is $H(\mathbb{Z}) = H \cap G(\mathbb{Z})$. Duke-Rudnick-Sarnak [D-R-S] putting some extra technical assumptions, have determined the asymptotic behavior of

$$N(V(\mathbb{Z}), x) = o\{\gamma \in H(\mathbb{Z}) \setminus G(\mathbb{Z}) : ||w_0 \gamma|| \leq x\}.$$ 

They prove that there are constants $a \geq 0, b > 0$ and $c > 0$ such that

$$N(V(\mathbb{Z}), x) \sim cx^a (log x)^b.$$ 

The additive structure of $G$ allows one to define self-similar subsets of $V(\mathbb{Z})$ and study their asymptotic behavior using the idea of fractal dimension. The whole set $V(\mathbb{Z})$ could not be a fractal of finite dimension, since the asymptotic behavior of its points is not polynomial [Ra].
1.5 Self-similarity in arithmetic geometry

In arithmetic geometry, the geometric nature of self-similar objects is combined with arithmetic and height function formalism:

1. Take the complex projective space $\mathbb{P}^n(\mathbb{C})$ as ambient space and polynomial endomorphisms $f_i = (\phi_0, \phi_1, ..., \phi_n)$ which are induced by $n+1$ homogeneous polynomials in $n+1$ variables of degree $m_i$, as similarity maps. $\mathbb{P}^n(K)$ is self-similar for any given number field $K$, but with respect to infinitely many similarity maps. Schanuel describes the asymptotic behavior of points in $\mathbb{P}^n(K)$ [Scha]: Let $h, R, w, r_1, r_2, d_K, \zeta_K$ denote class number, regulator, number of roots of unity, number of real and complex embeddings, absolute discriminant and the zeta function associated to the number field $K$. Then the asymptotic behavior of points in $\mathbb{P}^n(K)$ of non-logarithmic height bounded by $x$ is given by

$$\frac{hR}{w\zeta_K(n+1)} \left( \frac{2^{r_1}(2\pi)^{r_2}}{d_K^{1/2}} \right)^{n+1} (n+1)^{r_1+r_2-1} x^{n+1}.$$

One can show that fractal dimension of $\mathbb{P}^n(K)$ is related to the above asymptotic behavior which implies that $\mathbb{P}^n(K)$ is not a fractal.

2. Schmidt [Schm] (in case $K = \mathbb{Q}$) and Thunder [Th] for general number field $K$ generalized the above theorem to Grassmanian varieties, and proved that

$$C(G(m, n)(K), x) \sim c_{m,n,K} x^n$$

where $C$ denotes the number of points of bounded non-logarithmic height and $c_{m,n,K}$ is an explicitly given constant. This implies that $G(m, n)(K)$ is not a fractal either. The following is a generalization to flag manifolds proved by Franke-Manin-Tschinkel [Fr-Ma-Tsh]:

Let $G$ be a semi-simple algebraic group over $K$ and $P$ a parabolic subgroup and $V = P\backslash G$ the associated flag manifold. Choose an embedding of $V \subset \mathbb{P}^n$ such that the hyperplane section $H$ is linearly equivalent to $-sK_V$ for some positive integer $s$, then there exists an integer $t \geq 0$ and a constant $c_V$ such that

$$C(V(K), x)^s = c_V x(log x)^t.$$

3. Let $\mathbb{F}_q(X)$ denote the function field of an absolutely irreducible projective variety $X$ which is non-singular in codimension one, defined over a finite field $\mathbb{F}_q$ of characteristic $p$. One can consider the logarithmic height on $\mathbb{P}^n(\mathbb{F}_q(X))$ defined by Neron [La-Ne] and study the asymptotic behavior of $N(\mathbb{P}^n(\mathbb{F}_q(X)), d)$. Serre and Wan proved the following [Wa]: Let $n(\mathbb{P}^n(\mathbb{F}_q(X)), d)$ denote the number of points in $\mathbb{P}^n(\mathbb{F}_q(X))$ with logarithmic height equal $d$ then

$$n(\mathbb{P}^n(\mathbb{F}_q(X)), d) = \frac{hq^{(n+1)(1-g)}}{\zeta_X(n+1)(q-1)} q^{(n+1)d} + O(q^{d/2+\epsilon})$$

for $d \to \infty$. Therefore $N(\mathbb{P}^n(\mathbb{F}_q(X)), d) \sim c q^{(n+1)d}$ for a constant $c$, which confirms that $\mathbb{P}^n(\mathbb{F}_q(X))$ could be thought of as a fractal.
2 Unification results and conjectures

The general formulation of self-similarity introduces a new perspective to many old results. Many seemingly unrelated results can be unified in a single self-similar framework. In this section, we provide perspectives and formulate some results and conjectures in this new setting.

2.1 Prospects of self-similarity in geometry

Let us start with self-similar vector spaces. Algebraic operations on vector spaces such as direct sum, tensor product, wedge and symmetric powers induce new self-similar vector spaces. The concept of linear endomorphism, should be restricted to linear maps commuting with all similarity maps. To define a morphism between different self-similar vector spaces, one should associate to any similarity map of the origin a similarity map of the target such that these two commute the linear map between self-similar vector spaces. This gives us a perspective towards a formulation of self-similarity in the categorical setting.

The concept of pairing and duality and other related concepts of linear algebra could be extended to the world of self-similar vector spaces. As usual, pairings take values in the base field and are not supposed to be compatible with similarity maps. Dual of a self-similar vector space is again self-similar and fits in a nondegenerate pairing with the original vector space.

Suppose that a self-similar vector space can be generated by finitely many vectors and their images under all combinations of similarity maps. This introduces the notions of basis and dimension, and brings us to the realm of matrix theory. Although one can not determine a self-similar linear map between self-similar vector spaces with finitely many numbers, a beautiful theory of determinants survives the new complications.

Self-similar vector spaces introduce a new perspective to the concept of space. The new formulation of linear algebra brings up a whole new world of geometrical objects. Self-similar manifolds could be defined as spaces which are locally isomorphic to self-similar open subspaces of self-similar vector spaces. Self-similar tangent spaces and self-similar vector bundles and the corresponding geometric set up could be studied. Self-similar cohomology theories could also be introduced which satisfy Poincare duality. These cohomology groups are self-similar vector spaces with similarity maps induced by the similarity maps of the corresponding space. Self-similar intersection theory and self-similar $K$-theory could also be studied in this new setting.

Self-similar spaces could be used as mathematical models for the flow of time and continuous movement. This brings up a new perspective towards space-time and possibility to formulate the infinitesimal notions of quantum theory in the new geometric setting. In fact, real line which appeared first as a model for the flow of time is the origin of the Euclidean concept of similarity.
2.2 Prospects of self-similarity in algebra

Since a module is nothing but a family of vector spaces, the concept of a self-similar vector space, when considered in families, defines a self-similar module. A self-similar module is generated by finitely many non-intersecting submodules each isomorphic to the full module. A morphism between self-similar modules is defined similar to a morphism between self-similar vector spaces. Direct sum, tensor product, wedge and symmetric powers of self-similar modules induce new self-similar modules.

Main concepts of homological algebra could also be formulated in the new setting of self-similar modules. The main obstacle is that a self-similar module can not be free. We shall replace the concept of self-similar free resolution by almost self-similar free resolution where an almost self-similar free module with respect to similarity maps \( \phi_1, \ldots, \phi_n \) is defined as the direct sum \( F(n) \) of a free module \( F \) and all formal copies \( \phi_{i_1} \circ \ldots \circ \phi_{i_k}(F) \). It is evident how to define self-similarity maps \( \phi_i : F(n) \to F(n) \).

The automorphism groups of finite dimensional self-similar vector spaces which are self-similar analogue of classical groups could also be studied. The automorphism group of a self-similar vector space is a self-similar group. It is a beautiful challenge to develop the whole theory of Lie groups and Lie algebras and their representations in the new self-similar setting. A self-similar representation is nothing but an action of a self-similar object on a self-similar vector space.

The concept of self-similar classical groups brings up the concept of self similar Hopf algebras. This motivates how to define a self-similar ring and a self-similar algebra. A self-similar ring is a ring generated as a ring by finitely many subrings each isomorphic to itself. Self-similar rings and self-similar modules are intimately related. Self-similar ideals induce self-similar rings after taking the quotient. Morphisms between self-similar rings is defined as usual.

Let a group act on a self-similar object by self-similar endomorphisms. The fixed subring will also be self-similar. One could even consider the action of a self-similar group on a self-similar ring. To get an idea how to define a self-similar action, one shall consider the action of the automorphism group \( G \) of a self-similar vector space \( V \) on the vector space itself. To each self-map \( \psi_i \) of \( G \) one shall associate a self-map \( \phi_j \) of \( V \) such that for all \( g \in G \) and \( v \in V \) we have \( \phi_j(g.v) = \psi_i(g).\phi_j(v) \). The corresponding invariant theory could also be developed in the setting of self-similar actions.

Self-similar deformation of self-similar algebraic structures is also an important topic. This makes a bridge between self-similar geometric spaces and self-similar algebraic structures. One should preserve the self-similarity structure while deforming, but choice of self-similar maps is not unique. This is why the deformation space of a self-similar algebraic object is a self-similar space.
2.3 Prospects of self-similarity in algebraic geometry

Arithmetic fractals provide a common framework in which similar theorems in Diophantine geometry could be united in a single context. Consider the following results of Raynaud and Faltings:

**Theorem 2.1** (Raynaud) Let $A$ be an abelian variety over an algebraically closed field $\overline{K}$ of characteristic zero, and $Z$ a reduced subscheme of $A$. Then every irreducible component of the Zariski closure of $Z(\overline{K}) \cap A(\overline{K})_{\text{tor}}$ is a translation of an abelian subvariety of $A$ by a torsion point.

**Theorem 2.2** (Faltings) Let $A$ be an abelian variety over an algebraically closed field $\overline{K}$ of characteristic zero and $\Gamma$ be a finitely generated subgroup of $A(\overline{K})$. For a reduced subscheme $Z$ of $A$, every irreducible component of the Zariski closure of $Z(\overline{K}) \cap \Gamma$ is a translation of an abelian subvariety of $A$.

Considering the fact that one could think of $A(\overline{K})_{\text{tor}}$ and $A(\mathbb{K})$ as fractals in $A$, the theorems of Raynaud [Ra] and Faltings [Fa] can be united in the following general conjecture:

**Conjecture 2.3** Let $X$ be an irreducible variety over a finitely generated field $K$ and let $F \subset X(\overline{K})$ denote an arithmetic fractal on $X$ and let $Z$ be a reduced subscheme of $X$. The Zariski closure of $Z(\overline{K}) \cap F$ is union of finitely many components $B_i$ for which, either $B_i$ is a point, or $B_i(\overline{K}) \cap F$ is an arithmetic fractal with respect to some induced endomorphisms of $B_i$.

A consequence of the above conjecture would be the generalized Lang’s conjecture which is confirmed by results of Raynaud [Ray], Laurent [Lau], Zhang [Zh]:

**Conjecture 2.4** (Lang) Let $X$ be an algebraic variety defined over a number-field $K$ and let $f : X \rightarrow X$ be a surjective endomorphism defined over $K$. Suppose that the subvariety $Y$ of $X$ is not pre-periodic in the sense that the orbit $\{Y, f(Y), f^2(Y), ...\}$ is not finite, then the set of pre-periodic points in $Y$ is not Zariski-dense in $Y$.

We also present another conjecture in the same lines for quasi-fractals in an algebraic variety $X$, where self-similarities are allowed to be induced by geometric self-correspondences on $X$ instead of self-maps. This time, we drop the requirement that similar images shall be almost-disjoint.

**Conjecture 2.5** Let $X$ be an irreducible variety defined over a finitely generated field $K$ and let $Q \subset X(\overline{K})$ denote a quasi-fractal on $X$ with respect to correspondences $Y_i$ in $X \times X$ with both projection maps finite and surjective. Then, for any reduced subscheme $Z$ of $X$ defined over $K$ the Zariski closure of $Z(\overline{K}) \cap Q$ is union of finitely many points and finitely many components $B_i$ such that for each $i$ the intersection $B_i(\overline{K}) \cap Q$ is a quasi-fractal in $B_i$ with respect to correspondences induced by $Y_i$.

The above conjecture is confirmed by special cases of Andre-Oort conjecture which are proved by Edixhoven [Ed].
2.4 Prospects of self-similarity in arithmetic

The idea of considering self-similar subsets of $\mathbb{Z}$ is due to O. Naghshineh who proposed the following problem for "International Mathematics Olympiad" held in Scotland in July 2002.

**Problem 2.6** Let $F$ be an infinite subset of $\mathbb{Z}$ such that $F = \bigcup_{i=1}^{n} a_iF + b_i$ for integers $a_i$ and $b_i$ where $a_iF + b_i$ and $a_jF + b_j$ are disjoint for $i \neq j$ and $|a_i| > 1$ for each $i$. Prove that

$$\sum_{i=1}^{n} \frac{1}{|a_i|} \geq 1.$$ 

In [Na], he explains his ideas about self-similar subsets of $\mathbb{Z}$ and suggests how to define their dimension:

**Definition 2.7** Let $\phi_i : \mathbb{Z} \to \mathbb{Z}$ for $i = 1$ to $n$ denote linear maps of the form $\phi_i(x) = a_i x + b_i$ where $a_i$ and $b_i$ are integers with $|a_i| > 1$. Let $F \subseteq \mathbb{Z}$ be a fractal with respect to $\phi_i$. The fractal dimension of $F$ is defined to be the real number $s$ such that

$$\sum_{i=1}^{n} |a_i|^{-s} = 1.$$ 

**Theorem 2.8** (Mahdavifar, Naghshineh) Let $F_1 \subseteq F_2 \subseteq \mathbb{Z}$ be fractals. Then the notion of fractal dimension is well-defined and $\dim(F_1) \leq \dim(F_2)$.

The proof of the above theorem works for $\mathbb{Z}[i]$ as well. This motivates us to consider the general case of arbitrary number field. Let $K$ be a number field and let $O_K$ denote its ring of integers. One can take $O_K$ as ambient space and polynomial maps $\phi_i : O_K \to O_K$ of degrees $n_i$ with coefficients in $O_K$ as self-similarities. Let $a_i$ denote the leading coefficient of $\phi_i$, and $n_i$ denote the degree of $\phi_i$. Fix an embedding $\rho : K \hookrightarrow \mathbb{C}$. Assume $\text{Norm}(a_i) > 1$ in case $\phi_i$ is linear. Let $F \subseteq O_K$ be self-similar with respect to $\phi_i$ for $i = 1$ to $n$. One can define the fractal dimension of $F$ to be the real number $s$ for which

$$\sum_{i=1}^{n} \text{Norm}(a_i)^{-\frac{s}{n_i}} = 1.$$ 

**Conjecture 2.9** The above notion of fractal dimension for self-similar subsets of $O_K$ is well-defined and well-behaved with respect to inclusion, i.e. fractal dimension is independent of the choice of self-similarities and compatible with inclusion of self-similar subsets.
2.5 Prospects of self-similarity in arithmetic geometry

Our main theorems are extensions of Siegel’s and Falting’s theorems on finiteness of integral points, which are special cases of our general Diophantine conjecture. Here is our version of Siegel’s theorem:

**Theorem 2.10** Let $X$ be an affine irreducible curve defined over a number field $K$ and let $F \subset \mathbb{A}^n(\bar{K})$ denote an arithmetic fractal in the affine ambient space of $X$. If genus of $X$ is $\geq 1$, then $X(K) \cap F$ is finite.

Our version of Faltings’ theorem is as follows:

**Theorem 2.11** Let $A$ be an abelian variety defined over a number field. Let $W$ be an affine open subset of $A$ and $F$ be an arithmetic fractal contained in $\mathbb{A}^n(\bar{K}) \supset W(\bar{K})$ where $K$ is a number field. Then $F \cap W(K)$ is finite.

Extension of Siegel’s theorem is proved using the following strong version of Roth’s theorem [Ra]:

**Theorem 2.12** Fix a number-field $K$ and $\sigma : K \hookrightarrow \mathbb{C}$ a complex embedding. Let $V$ be a smooth projective algebraic variety defined over $K$ and let $L$ be an ample line-bundle on $V$. Denote the arithmetic height function associated to the line-bundle $L$ by $h_L$. Suppose $F \subset V(K)$ is a fractal subset with respect to finitely many height-increasing self-endomorphisms $\phi_i : V \to V$ defined over $K$ such that for all $i$ we have

$$h_L(\phi_i(f)) = m_i h_L(f) + 0(1)$$

where $m_i > 1$. Fix a Riemannian metric on $V_\sigma(\mathbb{C})$ and let $d_\sigma$ denote the induced metric on $V_\sigma(\mathbb{C})$. Then for every $\delta > 0$ and every choice of an algebraic point $\alpha \in V(\bar{K})$ which is not a critical value of any of the $\phi_i$’s and all choices of a constant $C$, there are only finitely many fractal points $\omega \in F$ approximating $\alpha$ in the following manner

$$d_\sigma(\alpha, \omega) \leq C \exp(-\delta h_L(\omega)).$$

In fact an extended version of Siegel’s theorem could be proved using the above version of Roth’s theorem [Ra]:

**Theorem 2.13** Fix a number-field $K$. Let $V$ be a smooth affine algebraic variety defined over $K$ with smooth projectivization $\bar{V}$ and let $L$ be an ample line-bundle on $\bar{V}$. Denote the arithmetic height function associated to the line-bundle $L$ by $h_L$. Suppose $F \subset V(\bar{K})$ is an fractal subset with respect to finitely many height-increasing polynomial self-endomorphisms $\phi_i : V \to V$ defined over $K$ such that for all $i$ we have

$$h_L(\phi_i(f)) = m_i h_L(f) + 0(1)$$

where $m_i > 1$. One could also replace this assumption with norm analogue. For any affine hyperbolic algebraic curve $X$ embedded in $V$ defined over a number field $X(\bar{K}) \cap F$ is at most a finite set.
3 self-similarity in category theory

In category theory, one tries to translate properties of mathematical objects by their external behavior, i.e., in terms of morphisms to other objects. In some sense, category theory is sociology of mathematical concepts, where dealing with internal structures of mathematical objects is regarded as psychology of these objects. It is believed that there is a duality between sociological and psychological interpretations of mathematical phenomena.

3.1 Self-similar objects and self-similar morphisms

Let us keep the example of self-similar vector spaces and functions on Cantor fractal in mind. For objects $X$ and $Y$ in an abelian category $\mathcal{C}$, one can restrict morphisms $X \to Y$ to images of self-similarity maps $\phi_i : X \to X$. Therefore, for self-similar objects $X$, we get maps

$$\text{Hom}(X,Y) \cong \text{Hom}(\phi_i(X),Y) \cong \text{Hom}(X,Y)$$

such that the natural induced maps

$$\prod_i \text{Hom}(\phi_i(X),Y) \to \text{Hom}(X,Y) \to \prod_i \text{Hom}(\phi_i(X),Y)$$

are one-to-one correspondences which are inverse to each other. There are also maps in the other direction:

$$\text{Hom}(Y,X) \cong \text{Hom}(Y,\phi_i(X)) \cong \text{Hom}(Y,X)$$

such that the natural maps

$$\prod_i \text{Hom}(Y,\phi_i(X)) \to \text{Hom}(Y,X) \to \prod_i \text{Hom}(Y,\phi_i(X))$$

are also one-to-one correspondences which are inverse to each other. Although, it seems that working with abelian categories is essential, one can axiomatize self-similarity in arbitrary category as follows:

**Definition 3.1** An object $X$ of a category $\mathcal{C}$ is called self-similar if for any object $Y$ there are finitely many similarity maps

$$\psi_i : \text{Hom}(X,Y) \to \text{Hom}(X,Y)$$

such that the induced map

$$\prod_i \psi_i : \text{Hom}(X,Y) \to \prod_i \text{Hom}(X,Y)$$

is a one-to-one correspondence. Similarly, there should exist co-similarity maps

$$\mu_j : \text{Hom}(Y,X) \to \text{Hom}(Y,X)$$
such that the induced map

$$\prod_j \mu_j : Hom(Y, X) \to \prod_j Hom(Y, X)$$

is a one-to-one correspondence.

Morphisms between self-similar vector spaces motivate how to define morphisms between self-similar objects in a category. But we need an abelian category to have a one-to-one correspondence between similarity maps and co-similarity maps. In this case, for self-similar objects $X$ and $Y$ with similarity maps indexed by index sets $I$ and $J$ respectively, we have one-to-one correspondences

$$\prod_i Hom(X, Y) \to Hom(X, Y) \to \prod_j Hom(X, Y).$$

Suppose one fixes a surjective map $\pi : I \to J$. Elements of $Hom(X, Y)$ are self-similar with respect to $\pi$ if the combination of the above one-to-one correspondences respects the indices according to $\pi$.

3.2 Self-similar categories

In order to formulate self-similar categories, the natural choice for similarity maps would be endo-functors of the category. Here is the first candidate coming to mind.

**Definition 3.2** A category $C$ is called a self-similar category with respect to finitely many similarity functors $\phi_i : C \to C$ if $C$ is disjoint union of its images under similarity maps:

$$C = \bigsqcup_i \phi_i(C).$$

A self-similar morphism between self-similar categories is defined similar to morphisms between self-similar vector spaces.

On the other hand, a natural example for a self-similar category should be the category of self-similar vector-spaces over a field or the category of modules over a self-similar ring. In the first case, there is no ambient concept of similarity maps. In the second case, all similarity maps are surjective and thus they have intersecting images.

4 Some philosophical remarks

Self-similar objects represent a form of co-finiteness which embeds in the realm of finite mathematics because of their recursive self-similarity. Box formulas let us think of self-similar objects as finite dimensional structures, which confirms finiteness of these structures. Parallel geometric, algebraic and arithmetic formulations suggests
that these objects are fundamental in understanding mathematical structures and
glossary of examples show that they naturally appear in everyday mathematics.
Fractal dimension of arithmetic self-similar objects encodes the asymptotic behavior of points of bounded height. It does not escape the eyes that for self-similar arithmetic objects, points of bounded height are of polynomial growth. It would be interesting if one could introduce an interpretation of the coefficient of the asymptotic in terms of self-similarity concepts.

Considering similarity as an abstract concept is confirmed to be a natural abstraction. Because, it appears naturally in a number of different mathematical frameworks and helps in generalizing and unifying many seemingly unrelated results and conjectures. This would be the main philosophical contribution of this manuscript.

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