The Gauge Technique* for Heavy Quarks

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Abstract

It is possible to determine an off-shell propagator for heavy quarks to order $1/m$ in mass and in any covariant gauge $\xi$, which applies universally to all the quarks, by using the gauge technique. The result for the leading behaviour of the propagator is

$$S(v \cdot k) = \Gamma(1 + 2\alpha_\xi) \frac{1 + \gamma \cdot v}{2v \cdot k} \left( -\frac{v \cdot k}{\Lambda} \right)^{2\alpha_\xi} F_2\left( 1 + \alpha_\xi, 3/2 + \alpha_\xi; -\frac{\alpha_\xi (v \cdot k)^2}{3\pi \Lambda^2} \right),$$

where $v$ is the quark velocity, $\Lambda$ is a QCD mass scale and $\alpha_\xi \equiv \alpha(2+\xi)/3\pi$. It is totally reliable in the infrared limit and accounts for soft-gluon corrections to the fermion in internal loops.

11.10Jj, 11.15Tk, 11.30Ly

*in memory of Abdus Salam, to whom we owe the gauge technique.
I. INTRODUCTION

The QCD quark Lagrangian ($N$ flavors) is endowed with a higher symmetry in the limit of equal quark velocity which applies even when the quark masses are different \cite{1}. Thus it generalizes the old $U(N) \times U(N)$ supersymmetry \cite{2} for the equal mass case. Provided that the momentum transfer to the gluons is not much greater than the QCD scale one can thereby deduce a number of relations between transition amplitudes, which seem to be borne out by experiment. It has become customary \cite{3} to attribute a velocity $v$ to the constituent heavy quark so that the momentum of the quark in a bound state is written $p = mv + k$, where $k$ denotes the residual quark momentum, itself associated with the light material that makes up the hadron. As a result one can show that the “free” quark propagator in the $m \to \infty$ limit is simply given by $S = (1 + \gamma \cdot v)/2v \cdot k$ and one can use this in subsequent leading order calculations of various matrix elements.

In this paper we would like to show that one can improve on $S$ by taking account of soft gluon corrections. The result of the dressing is to provide a propagator which contains the characteristic QCD scale $\Lambda$ and which coincides with the free one in the limit of vanishing gluon coupling $\alpha = g^2/4\pi$. Thus this propagator applies just as well to all the quarks in the heavy mass limit \cite{4} and does not jeopardise the prevailing higher symmetry. In order to derive it we use the gauge technique \cite{5} for QCD, which is known to be a reliable method in the infrared and ultraviolet limit \cite{6}. The technique produces a self-consistent equation for the quark spectral function in any gauge, from which the propagator follows \cite{7}. In the next section we set out the velocity projector decomposition of the propagator. Next we derive the effective vertex for soft gluons and finally we solve the equation in question, obtaining the result quoted in the abstract; there we also compare the result with QED where the scale $\Lambda$ is missing.
II. VELOCITY PROJECTIONS

When one substitutes \( p = mv + k \) in the free quark propagator, the resulting expression,

\[
S(p) = \frac{1}{m(\gamma \cdot v - 1) + \gamma \cdot k},
\]

has to be taken in the limit \( m \to \infty \) in order to discern the resulting (leading) dependence on four-velocity \( v \) (with \( v^2 = 1 \)). Now any quark matrix \( M \) for a particular flavor can be decomposed into projections using \( P_{\pm} = (1 \pm \gamma \cdot v)/2 \) according to

\[
M = P_+ M_{++} P_+ + P_+ M_{+-} P_- + P_- M_{-+} P_+ + P_- M_{--} P_-.
\]

This has the effect of resolving the 4 × 4 matrix into four separate 2 × 2 matrices:

\[
M \Rightarrow \begin{pmatrix} M_{++} & M_{+} \\ M_{-} & M_{--} \end{pmatrix}.
\]

In this basis,

\[
v \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_\mu \Rightarrow \begin{pmatrix} v_\mu & \gamma_\mu \\ \gamma_\mu & -v_\mu \end{pmatrix}.
\]

In particular the inverse free propagator, resolves to

\[
S^{-1}(p) = m(\gamma \cdot v - 1) + \gamma \cdot k \Rightarrow \begin{pmatrix} k \cdot v & \gamma \cdot k \\ \gamma \cdot k & -k \cdot v - 2m \end{pmatrix},
\]

and correspondingly,

\[
S(p) \Rightarrow \frac{1}{k^2 + (k \cdot v)^2 + 2mk \cdot v} \begin{pmatrix} k \cdot v + 2m & \gamma \cdot k \\ \gamma \cdot k & -k \cdot v \end{pmatrix}.
\]

Thus, up to order \( 1/m^2 \), the free \( S \) decomposes into

\[
S(p) \Rightarrow \frac{1}{k \cdot v} \left( 1 - \frac{k^2}{2mk \cdot v} \begin{pmatrix} \gamma \cdot k \\ \gamma \cdot k \end{pmatrix} + O\left( \frac{1}{m^2} \right) \right),
\]

from which one infers that the leading large component is \( S_{++} \sim 1/k \cdot v \).
Let us now consider the full propagator in a covariant gauge, which is best written in the Lehmann-Kallen spectral form for our purposes:

\[ S(p) = \int \frac{\rho(W) dW}{\gamma \cdot p - W + i\eta(W)}, \quad \int dW \equiv (\int_{-\infty}^{-m} + \int_{m}^{\infty}) dW, \]  

(5)

for an ordinary sort of particle. In the heavy quark limit we anticipate that the negative energy cut is ‘far away’ and that the main contribution from soft gluons will arise in the vicinity of \( W = m \), to within a region of order \( \Lambda \). The free propagator is of course obtained just by setting \( \rho(W) = \delta(W - m) \) above. Put \( W = m + \omega, \ p = mv + k \) and take velocity projections as in (4), to obtain

\[ S(p) \Rightarrow \int \frac{\rho(\omega) d\omega}{2m(v \cdot k - \omega) + k^2 - \omega^2} \begin{pmatrix} 2m + k \cdot v + \omega & \gamma \cdot k \\ \gamma \cdot k & -k \cdot v + \omega \end{pmatrix}. \]  

(6)

We see that the dressed propagator is still dominated by its \( S_{++} \) component, which assumes a very simple form, \( \int d\omega \rho(\omega)/(v \cdot k - \omega) \) despite the inclusion of QCD interactions. More generally, to order \( 1/m \), we get

\[ S \Rightarrow \int \frac{\rho(\omega) d\omega}{v \cdot k - \omega} \left( 1 + \frac{(v \cdot k)^2 - k^2}{2m(v \cdot k - \omega)} \frac{\gamma \cdot k}{2m} \right) + O\left( \frac{1}{m^2} \right). \]

**III. APPLICATION OF THE GAUGE TECHNIQUE**

The next stage involves solution of the Dyson-Schwinger (D-S) equation for the propagator, while taking cognizance of the (longitudinal) Ward-Takahashi identity,

\[ (p - p')^\mu S(p') \Gamma_\mu(p', p) S(p) = S(p') - S(p). \]  

(7)

because it can lead to a self-consistent equation for \( S \), if we ignore certain transverse terms in the vertex \( \Gamma_\mu \). If one were to incorporate the transverse Takahashi identity as well, one would in effect be solving the full field theory; but the transverse identity—in anything but two dimensions—brings in other vertices leading to a system of equations which is
actually not closed, unless one makes further drastic truncations \[10\]. Alternatively if one
knew the full solution of the D-S equation in any particular gauge, one would be able to
determine it in any other gauge via the Landau-Khalatnikov-Zumino gauge covariance \[11\]
relations.

We have none of these luxuries. The gauge technique does its best to solve the equation
(7) in the form stated, while making sure that the singularities in the non-truncated Green
function are properly included. It does not solve the inverse form of the equation (7) because
that would give a linear relation between \( \Gamma \) and \( S^{-1} \) and produce a difficult non-linear
equation for the inverse propagator; besides which, it is not obvious how to handle the
heavy quark limit for the inverse propagator—which is dominated by its \( S_{-+}^{-1} \) projection,
conversely to \( S_{++} \). The gauge technique starts off with the obvious solution to the vertex,

\[
S(p')\Gamma_{\mu}^\parallel(p', p)S(p) = \int dW \rho(W) \frac{1}{\gamma \cdot p'} - W \gamma_\mu \frac{1}{\gamma \cdot p - W}
\]  

(8)
as a weighted mass integral. One can readily check that (8) obeys (7) automatically, but
of course (8) is subject to transverse additions \( \Gamma^\perp \); these are unknown unless one has some
knowledge about them through perturbation theory \[12\] or examines equations for higher-
order Green functions \[13\] or makes use of the transverse identity \[8\], which is essentially
equivalent. It is worth pointing out that such transverse terms are soft, vanishing with the
vector meson momentum. For that reason the gauge technique is a clearly reliable tool in the
infrared limit, though it is also gauge-covariant in the ultraviolet regime \[6\] as it happens;
it is only at intermediate energies that transverse corrections to (8) play an important role.

Returning to heavy quarks, let us expand the solution (8) in powers of \( 1/m \) by writing
it as

\[
S(p')\Gamma_{\mu}^\parallel(p', p)S(p) = \int d\omega \rho(\omega) \frac{1}{m(\gamma \cdot v - 1)} + \gamma \cdot k - \omega \gamma_\mu \frac{1}{m(\gamma \cdot v - 1)} + \gamma \cdot k - \omega
\]  

(9)
and taking velocity projections:

\[
\Rightarrow \int \frac{\rho(\omega) d\omega}{(v \cdot k' - \omega)(v \cdot k - \omega)} \left( v_\mu \left[ 1 + \frac{(v \cdot k')^2 - k'^2}{2m(v \cdot k' - \omega)} + \frac{(v \cdot k)^2 - k^2}{2m(v \cdot k - \omega)} \right] + \frac{\gamma \cdot k' \gamma_\mu + \gamma_\mu \gamma \cdot k}{2m} \frac{v_\mu \gamma \cdot k + \gamma_\mu (\omega - k \cdot v)}{2m} \right)
\]  

(10)
up to order $1/m^2$. Note that representations (8) and (10) are *exact* for small $p - p'$ in the complete Green function (7), for the reasons we have already given.

The next step is to use the approximation (8) in the D-S equation, which we write in the renormalized form,

$$Z^{-1} = (\gamma \cdot p - m + \delta m)S(p) + i \frac{g^2}{(2\pi)^4} \frac{\lambda_i}{2} \int d^4q S(p)\Gamma_\mu(p, p-q)S(p)\gamma_\nu D^{\mu\nu}(q)\frac{\lambda_i}{2}. \quad (11)$$

Recalling the connection, $Z^{-1} = \int \rho(W) dW$, the spectral form of the equation is,

$$\int \rho(W) dW[W - m + \delta m + \Sigma(p, W)] = 0, \quad (12)$$

where

$$\Sigma(p, W) = i \frac{g^2}{(2\pi)^4} \frac{\lambda_i}{2} \int d^4q \gamma_\mu \frac{1}{\gamma.(p-q) - W}\gamma_\nu D^{\mu\nu}(q)\frac{\lambda_i}{2} \quad (13)$$

is the self-energy for a quark of mass $W$ due to gluons in first order perturbation theory.

At this point we carry out the heavy quark expansion and take velocity projections to arrive at

$$\int \rho(\omega) \frac{d\omega}{\omega \cdot k - \omega} \left[ (\omega + \delta m) \left( 1 + \frac{(v\cdot k)^2 - k^2}{2m(v\cdot k - \omega)} \right) + \left( \frac{\Sigma_+}{\Sigma_+} (v \cdot k, \omega) \frac{\Sigma_+}{\Sigma_+} (v \cdot k, \omega) \right) \right] = 0, \quad (14)$$

where, after summing over colours (hence the factor of 4/3),

$$\Sigma(v \cdot k, \omega) \Rightarrow i \frac{4g^2/3}{(2\pi)^4} \int d^4q D^{\mu\nu}(q) \left( \frac{v_{\mu}v_{\nu}}{\gamma-k_{\mu}v_{\nu}} + \frac{(\omega-k_{\mu}\gamma_{\mu}v_{\nu})}{2m} \right) \frac{v_{\mu}\gamma_{\nu} + O(1/m)}{\gamma-k_{\mu}\gamma_{\mu} - (\omega-k_{\mu}\gamma_{\mu})} \frac{(\omega-k_{\mu}\gamma_{\mu})}{2m} \quad (15)$$

and up to order $(1/m)$.

**IV. THE SPECTRAL EQUATION**

To make any further progress and determine the spectral function $\rho$ and thence the propagator, we need to make some further approximations/assumptions about the behaviour
of the gluon. It is generally accepted that the gluons are massless so that the propagator $D(q)$ is at least as singular as $1/q^2$; it is also known that in the ultraviolet regime this is subject to well-defined logarithmic damping; the behaviour for small $q^2$, where the strong force enslaves colour, is more mysterious and there have been suggestions that $D(q)$ could be as singular as $1/q^4$, that it plateaus or even that one should not be using QCD at all but an effective field theory incorporating chiral symmetry with real mesons. What is certain is the occurrence of a mass scale $\Lambda$ demarcating the ultraviolet from the infrared regime of $D$. As we are only interested in soft gluon effects on the heavy quark lines, we will adopt a gluon propagator which implies masslessness, which cuts off in the ultraviolet and which introduces the fundamental QCD mass scale. For our purposes it is enough to use an effective

$$D_{\mu\nu}(q) = (-\eta_{\mu\nu} + \xi q_\mu q_\nu) \frac{\Lambda^2}{q^2(\Lambda^2 - q^2)},$$

knowing its limitations full well. It incorporates the main things we want and also includes a covariant gauge parameter $\xi$. If other readers wish to modify $D$ with a more sophisticated and perhaps more realistic expression, they can repeat our calculations below; while that is sure to alter the precise form of our answers, we believe it will not affect the main features of our results in a very significant way.

Returning to the largest component of (15), we have to consider the spectral equation,

$$0 = \int \frac{\rho(\omega) d\omega}{v \cdot k - \omega} \left[ \omega + \delta m + \Sigma_{++}(v \cdot k, \omega) \right],$$

$$\Sigma_{++}(v \cdot k, \omega) = i \frac{4g^2/3}{(2\pi)^4} \int d^4q \frac{\xi(v.q)^2/q^2 - 1}{v.(k-q) - \omega} \left( \frac{1}{q^2} - \frac{1}{q^2 - \Lambda^2} \right).$$

A straightforward but messy calculation gives

$$\Sigma_{++}(\omega, \omega') = \frac{4\alpha}{3\pi} \left[ (\omega - \omega')(1 + \frac{\xi}{2} - \frac{\xi(\omega - \omega')^2}{\Lambda^2}) \ln \frac{2(\omega - \omega')}{\Lambda} + \frac{1}{4} \xi(\omega - \omega') + \frac{1}{2} (1 - \xi(\omega - \omega')^2) \sqrt{(\omega - \omega')^2 - \Lambda^2} \ln \frac{\omega - \omega' - \sqrt{(\omega - \omega')^2 - \Lambda^2}}{\omega - \omega' + \sqrt{(\omega - \omega')^2 - \Lambda^2}} \right],$$

to leading order and in a general gauge. First we solve the spectral equation (17)
\[
0 = \int d\omega' \rho(\omega') \left[ \frac{\omega' + \delta m}{\omega - \omega'} + \frac{4\alpha}{3\pi} \left( \frac{2(\omega - \omega')}{\Lambda} + \frac{1}{2} \sqrt{(\omega - \omega')^2 - \Lambda^2} \ln \frac{\omega - \omega' - \sqrt{(\omega - \omega')^2 - \Lambda^2}}{\omega - \omega' + \sqrt{(\omega - \omega')^2 - \Lambda^2}} \right) \right]
\]

(20)

in the Fermi-Feynman gauge \( \xi = 0 \) to discover what is going on. By taking the imaginary part of (20), we obtain

\[
0 = (\omega + \delta m - 2\alpha\Lambda/3)\rho(\omega) + \frac{4\alpha/3\pi}{\int_\omega^\infty} \rho(\omega') \, d\omega'.
\]

(21)

This has the solution

\[
\rho(\omega) \propto (\omega + \delta m - 2\alpha\Lambda/3)^{-1-4\alpha/3\pi}
\]

and, since the self-mass is given by

\[
\int (\omega + \delta m)\rho(\omega) \, d\omega = 0,
\]

this fixes \( \delta m = 2\alpha\Lambda/3 \). It makes good sense, being governed by the QCD mass scale and gluon coupling. One last matter is the proportionality factor: we must ensure that \( \rho(\omega) \) reduces to \( \delta(\omega) \) when \( \alpha \to 0 \). Hence we choose the overall constant so that the result for the spectral function for \( \xi = 0 \) is neat and compact, namely

\[
\rho_{\xi=0}(\omega) = \frac{1}{\omega \Gamma(-4\alpha/3\pi)} \left( \frac{\Lambda}{\omega} \right)^{-4\alpha/3\pi}.
\]

(22)

A bonus of this choice is that the heavy quark propagator simplifies to the elegant non-perturbative expression,

\[
S_{\xi=0}(v \cdot k) = \Gamma(1 + \frac{4\alpha}{3\pi}) \frac{1 + \gamma \cdot v}{2v \cdot k} \left( \frac{-\Lambda}{v \cdot k} \right)^{-4\alpha/3\pi}.
\]

(23)

In the limit \( \alpha \to 0 \) one recovers the free result \( 1/(v \cdot k) \) for \( S \).

Now we turn to the general gauge \( \xi \). Noting that the \( \xi \)-dependent part of \( \Sigma_{++}(v \cdot k, \omega) \) vanishes at the threshold \( v \cdot k = \omega \), the gauge-dependence of \( \rho \) arises purely from the imaginary part of \( \Sigma_{++} \). In this way (21) gets modified to

\[
0 = (\omega + \delta m - 2\alpha/3\pi)\rho(\omega) + \frac{4\alpha}{3\pi} \int_\omega^\infty d\omega' \rho(\omega') \left[ 1 + \frac{\xi}{2} - \xi \frac{(\omega - \omega')^2}{\Lambda^2} \right].
\]

(24)
Once again the self-mass condition requires $\delta m = 2\alpha \Lambda / 3$, which is satisfyingly gauge-independent. The resulting integral equation for the spectral function is a little bit harder to solve now; nevertheless one may establish that it reduces to a generalised hypergeometric function:

$$
\rho(\omega) = \frac{1}{\omega \Gamma(-\alpha \xi)} \left( \frac{\Lambda}{\omega} \right)^{-2\alpha \xi} {}_0F_2\left(1 + \alpha \xi, 3/2 + \alpha \xi; -\frac{\alpha \xi \omega^2}{3\pi \Lambda^2} \right),
$$

(25)

where $\alpha \xi \equiv \alpha (2 + \xi)/3\pi$. Thereupon the heavy quark propagator becomes

$$
S(v \cdot k) = \Gamma(1 + 2\alpha \xi) \frac{1 + \gamma \cdot v}{2v \cdot k} \left( -\frac{\Lambda}{v \cdot k} \right)^{-2\alpha \xi} {}_0F_2\left(1 + \alpha \xi, 3/2 + \alpha \xi; -\frac{\alpha \xi (v \cdot k)^2}{3\pi \Lambda^2} \right) + O(1/m).
$$

(26)

This is universal to all the quarks and could be used to estimate the gluon corrections in loops which result from dressing fermion lines and their vertices. However, a word of caution: the result (26) does not take account of gluon self-interactions; those will somehow need to be included separately in heavy quark calculations.

There is one further test of our work. One needs to verify that to order $1/m$ the other, ‘small component’ sectors in the propagator velocity projections are correctly determined by (25), because they are fixed in terms of the leading $\rho$. We have indeed checked this out: the $S_{+-}$ sector produces precisely the same equation as (24), while the $S_{--}$ sector is nothing but the self-mass condition, $\int (\omega + \delta m) \rho(\omega) \, d\omega = 0$, which we have already settled.

Finally it is worth comparing the answers with the gauge technique solutions for scalar QED say. Those solutions do not have the benefit of an intrinsic cut-off; rather the source mass itself acts as the cutoff and the results read,

$$m^2 \rho(W) = \frac{(W^2/m^2 - 1)^{1-2\alpha \xi}}{\Gamma(-2a \xi)} {}_2F_1(-a \xi, 1 - a \xi; -2a \xi; 1 - \frac{W^2}{m^2}); \quad a \xi = (\xi + 2)\alpha/4\pi
$$

$$m^2 S(p) = \Gamma(1 + a \xi) \Gamma(2 + a \xi) {}_2F_1(1 + a \xi, 2 + a \xi; 2; p^2/m^2)
$$

We notice a strong similarity with (25) and (26), which becomes greater when one substitutes $p = mv$ and expands to order $1/m$. The only change is that $m$ takes the place of $\Lambda$ as
the argument of the hypergeometric function. It only remains to obtain the non-leading behaviour of $S$ in the various sectors. This entails solving the spectral equation up to order $1/m$ and is where our use of the approximations (9) and (16) start to look a bit suspect because they are connected with gluons which carry off appreciable momentum. It is a nice subject for future research since it portrays the mass-dependence of the heavy quark Lagrangian.

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[15] The renormalization factor \(Z^{-1}\) comes out infinite in our case.