Graded Off-diagonal Bethe ansatz solution of the $SU(2|2)$ spin chain model with generic integrable boundaries

Xiaotian Xu$^a$, Junpeng Cao$^{a,b,c,d}$, Yi Qiao$^{a,e}$, Wen-Li Yang$^{d,e,f,g}$, Kangjie Shi$^e$ and Yupeng Wang$^{a,h}$

$^a$ Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
$^b$ Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China
$^c$ School of Physical Sciences, University of Chinese Academy of Sciences, Beijing, China
$^d$ Peng Huanwu Center for Fundamental Theory, Xian 710127, China
$^e$ Institute of Modern Physics, Northwest University, Xian 710127, China
$^f$ Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xian 710127, China
$^g$ Physics school, Northwest University, Xian 710127, China
$^h$ The Yangtze River Delta Physics Research Center, Liyang, Jiangsu, China

Abstract

The graded off-diagonal Bethe ansatz method is proposed to study supersymmetric quantum integrable models (i.e., quantum integrable models associated with superalgebras). As an example, the exact solutions of the $SU(2|2)$ vertex model with both periodic and generic open boundary conditions are constructed. By generalizing the fusion techniques to the supersymmetric case, a closed set of operator product identities about the transfer matrices are derived, which allows us to give the eigenvalues in terms of homogeneous or inhomogeneous $T - Q$ relations. The method and results provided in this paper can be generalized to other high rank supersymmetric quantum integrable models.

PACS: 75.10.Pq, 02.30.Ik, 71.10.Pm
Keywords: Bethe Ansatz; Lattice Integrable Models; $T - Q$ Relation

1Corresponding author: wlyang@nwu.edu.cn
2Corresponding author: yupeng@iphy.ac.cn
1 Introduction

Quantum integrable models \[1\] play important roles in fields of theoretical physics, condensed matter physics, field theory and mathematical physics, since exact solutions of those models may provide useful benchmarks to understand a variety of many-body problems. During the past several decades, much attention has been paid to obtain exact solutions of integrable systems with unusual boundary conditions. With the development of topological physics and string theory, study on off-diagonal boundaries becomes an interesting issue. Many interesting phenomena such as edge states, Majorana zero modes, and topological excitations have been found.

Due to the existence of off-diagonal elements contained in boundaries, particle numbers with different intrinsic degrees of freedom are not conserved anymore and the usual $U(1)$ symmetry is broken. This leads to absence of a proper reference state which is crucial in the conventional Bethe ansatz scheme. To overcome this problem, several interesting methods \[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\] are proposed. A remarkable one is the off-diagonal Bethe ansatz (ODBA) \[16, 17\], which allow us to construct the exact spectrum systematically. The nested ODBA has also been developed to deal with the models with different Lie algebras such as $A_n$ \[22, 23\], $A_2^{(2)}$ \[24\], $B_2$ \[25\], $C_2$ \[26\] and $D_3^{(1)}$ \[27\]. Nevertheless, there exists another kind of high rank integrable models which are related to superalgebras \[28\] such as the $SU(m|n)$ model, the Hubbard model, and the supersymmetric $t – J$ model. The $SU(m|n)$ model has many applications in AdS/CFT correspondence \[29, 30\], while the Hubbard and $t – J$ model have many applications in the strongly correlated electronic theory. These models with $U(1)$ symmetry have been studied extensively \[31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41\]. A general method to approach such kind of models with off-diagonal boundaries is still missing.

In this paper, we develop a graded version of nested ODBA to study supersymmetric integrable models (integrable models associated with superalgebras). As an example, the $SU(2|2)$ model with both periodic and off-diagonal boundaries is studied. The structure of the paper is as follows. In section 2, we study the $SU(2|2)$ model with periodic boundary condition. A closed set of operator identities is constructed by using the fusion procedure. These identities allow us to characterize the eigenvalues of the transfer matrices in terms of homogeneous $T – Q$ relation. In section 3, we study the model with generic open boundary
conditions. It is demonstrated that similar identities can be constructed and the spectrum can be expressed in terms of inhomogeneous \( T - Q \) relation. Section 4 is attributed to concluding remarks. Some technical details can be found in the appendices.

2 \( SU(2|2) \) model with periodic boundary condition

2.1 The system

Let \( V \) denote a 4-dimensional graded linear space with an orthogonal basis \( \{|i\} | i = 1, \cdots, 4\} \), where the Grassmann parities are \( p(1) = 0, p(2) = 0, p(3) = 1 \) and \( p(4) = 1 \), which endows the fundamental representation of the \( SU(2|2) \) Lie superalgebra. For the matrix \( A_j \in \text{End}(V_j) \), \( A_j \) is a super embedding operator in the \( \mathbb{Z}_2 \)-graded tensor space \( V_1 \otimes V_2 \otimes \cdots \), which acts as \( A \) on the \( j \)-th space and as identity on the other factor spaces. For the matrix \( R_{ij} \in \text{End}(V_i \otimes V_j) \), \( R_{ij} \) is a super embedding operator in the \( \mathbb{Z}_2 \) graded tensor space, which acts as identity on the factor spaces except for the \( i \)-th and \( j \)-th ones. The super tensor product of two operators is the graded one satisfying the rule \( (A \otimes B)_{\alpha \gamma}^{\beta \delta} = (-1)^{[p(\alpha)+p(\beta)]p(\gamma)}A_{\beta \delta}^{\alpha \gamma} \) [42].

The supersymmetric \( SU(2|2) \) model is described by the 16 \( \times \) 16 \( R \)-matrix

\[
R_{12}(u) = \begin{pmatrix}
  u + \eta & u & \eta & \eta \\
  \eta & u & \eta & u \\
  \eta & \eta & u - \eta & u \\
  \eta & \eta & \eta & u - \eta \\
\end{pmatrix},
\]

(2.1)

where \( u \) is the spectral parameter and \( \eta \) is the crossing parameter. The \( R \)-matrix (2.1) enjoys
the following properties

\[ R_{12}(0) = \eta P_{12}, \]

unitarity : \[ R_{12}(u)R_{21}(-u) = \rho_1(u) \times \text{id}, \]

crossing – unitarity : \[ R_{12}^{st_1}(-u)R_{21}^{st_2}(u) = \rho_2(u) \times \text{id}, \]

where \( P_{12} \) is the \( Z_2 \)-graded permutation operator with the definition

\[ P_{\beta_1,\beta_2}^{\alpha_1,\alpha_2} = (-1)^{p(\alpha_1)p(\alpha_2)}\delta_{\alpha_1,\beta_2}\delta_{\beta_1,\alpha_2}, \]

(2.2)

\( R_{21}(u) = P_{12}R_{12}(u)P_{12}, \) \( st_i \) denotes the super transposition in the \( i \)-th space \( (A_{st_i})_{ij} = A_{ji}(-1)^{p(i)p(i)+p(j)} \), and the functions \( \rho_1(u) \) and \( \rho_2(u) \) are given by

\[ \rho_1(u) = -(u - \eta)(u + \eta), \quad \rho_2(u) = -u^2. \]

(2.3)

The \( R \)-matrix \((2.1)\) satisfies the graded Yang-Baxter equation (GYBE) \[(23)\]

\[ R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \]

(2.4)

In terms of the matrix entries, GYBE \((2.4)\) reads

\[ R(u - v)^{\alpha_1,\alpha_2}_{\beta_1,\beta_2} R(u)^{\beta_1,\beta_2}_{\gamma_1,\gamma_2} R(v)^{\beta_2,\beta_3}_{\gamma_2,\gamma_3}(-1)^{(p(\beta_1)+p(\gamma_1))p(\beta_2)} \]

\[ = R(v)^{\alpha_2,\alpha_3}_{\beta_2,\beta_3} R(u)^{\alpha_1,\beta_2}_{\beta_1,\gamma_3} R(u - v)^{\beta_1,\beta_2}_{\gamma_1,\gamma_2}(-1)^{(p(\alpha_1)+p(\beta_1))p(\beta_2)}. \]

(2.5)

For the periodic boundary condition, we introduce the “row-to-row” (or one-row) monodromy matrix \( T_0(u) \)

\[ T_0(u) = R_{01}(u - \theta_1)R_{02}(u - \theta_2) \cdots R_{0N}(u - \theta_N), \]

(2.6)

where the subscript 0 means the auxiliary space \( V_0 \), the other tensor space \( V^{\otimes N} \) is the physical or quantum space, \( N \) is the number of sites and \( \{\theta_j\}_{j=1,\cdots,N} \) are the inhomogeneous parameters. In the auxiliary space, the monodromy matrix \( (2.6) \) can be written as a \( 4 \times 4 \) matrix with operator-valued elements acting on \( V^{\otimes N} \). The explicit forms of the elements of monodromy matrix \( (2.6) \) are

\[ \left\{ T_0(u) \right\}^{\alpha_1 \cdots \alpha_N}_{\beta_1 \cdots \beta_N} = R_{0N}(u)_{\epsilon_N}^{\alpha_N} \cdots R_{0j}(u)_{\epsilon_j,\beta_j}^{\epsilon_{j+1},\alpha_j} \cdots R_{01}(u)_{\epsilon_1,\beta_1}^{\epsilon_2,\alpha_1} \]

\[ \times (-1)^{\sum_{j=2}^{N}(p(\alpha_j)+p(\beta_j))\sum_{i=1}^{j-1}p(\alpha_i)}. \]

(2.7)
The monodromy matrix $T_0(u)$ satisfies the graded Yang-Baxter relation
\[ R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v). \] (2.8)

The transfer matrix $t_p(u)$ of the system is defined as the super partial trace of the monodromy matrix in the auxiliary space
\[ t_p(u) = \text{str}_0\{T_0(u)\} = \sum_{\alpha=1}^{4} (-1)^{p(\alpha)}[T_0(u)]^\alpha_{\alpha}. \] (2.9)

From the graded Yang-Baxter relation (2.8), one can prove that the transfer matrices with different spectral parameters commute with each other, $[t_p(u), t_p(v)] = 0$. Thus $t_p(u)$ serves as the generating functional of all the conserved quantities, which ensures the integrability of the system. The model Hamiltonian is constructed by
\[ H_p = \frac{\partial \ln t_p(u)}{\partial u} \bigg|_{u=0,\{\theta_j\}=0}. \] (2.10)

2.2 Fusion

One of the wonderful properties of $R$-matrix is that it may degenerate to the projection operators at some special points, which makes it possible to do the fusion procedure [44, 45, 46, 47, 48]. It is easy to check that the $R$-matrix (2.1) has two degenerate points. The first one is $u = \eta$. At which, we have
\[ R_{12}(\eta) = 2\eta P_{12}^{(8)}, \] (2.11)

where $P_{12}^{(8)}$ is a 8-dimensional supersymmetric projector
\[ P_{12}^{(8)} = \sum_{i=1}^{8} |f_i\rangle\langle f_i|, \] (2.12)

and the corresponding basis vectors are
\[ |f_1\rangle = |11\rangle, \quad |f_2\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \quad |f_3\rangle = |22\rangle, \]
\[ |f_4\rangle = \frac{1}{\sqrt{2}}(|34\rangle - |43\rangle), \quad |f_5\rangle = \frac{1}{\sqrt{2}}(|13\rangle + |31\rangle), \quad |f_6\rangle = \frac{1}{\sqrt{2}}(|14\rangle + |41\rangle), \]
\[ |f_7\rangle = \frac{1}{\sqrt{2}}(|23\rangle + |32\rangle), \quad |f_8\rangle = \frac{1}{\sqrt{2}}(|24\rangle + |42\rangle), \]

with the corresponding parities
\[ p(1) = p(2) = p(3) = p(4) = 0, \quad p(5) = p(6) = p(7) = p(8) = 1. \]
The operator $P_{12}^{(8)}$ projects the original 16-dimensional tensor space $V_1 \otimes V_2$ into a new 8-dimensional projected space spanned by $\{|f_i\rangle | i = 1, \cdots, 8\}$. Taking the fusion by the operator (2.12), we construct the fused $R$-matrices

\[
R_{(12)3}(u) = (u + \frac{1}{2} \eta)^{-1} P_{12}^{(8)} R_{23}(u - \frac{1}{2} \eta) R_{13}(u + \frac{1}{2} \eta) P_{12}^{(8)} \equiv R_{13}(u),
\]

(2.13)

\[
R_{3(21)}(u) = (u + \frac{1}{2} \eta)^{-1} P_{21}^{(8)} R_{32}(u - \frac{1}{2} \eta) R_{31}(u + \frac{1}{2} \eta) P_{21}^{(8)} \equiv R_{31}(u),
\]

(2.14)

where $P_{21}^{(8)}$ can be obtained from $P_{12}^{(8)}$ by exchanging $V_1$ and $V_2$. For simplicity, we denote the projected space as $V_{\bar{1}} = V_{(12)} = V_{(21)}$. The fused $R$-matrix $R_{12}(u)$ is a $32 \times 32$ matrix defined in the tensor space $V_{\bar{1}} \otimes V_2$ and has the properties

\[
R_{12}(u) R_{21}(-u) = \rho_3(u) \times \text{id},
\]

\[
R_{12}(u)^{st_1} R_{21}(-u)^{st_1} = \rho_4(u) \times \text{id},
\]

(2.15)

where

\[
\rho_3(u) = -(u + \frac{3}{2} \eta)(u - \frac{3}{2} \eta), \quad \rho_4(u) = -(u + \frac{1}{2} \eta)(u - \frac{1}{2} \eta).
\]

(2.16)

From GYBE (2.4), one can prove that the following fused graded Yang-Baxter equations hold

\[
R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v).
\]

(2.17)

It is easy to check that the elements of fused $R$-matrices $R_{12}(u)$ and $R_{21}(u)$ are degree one polynomials of $u$.

At the point of $u = -\frac{3}{2} \eta$, the fused $R$-matrix $R_{12}(u)$ can also be written as a projector

\[
R_{12}(-\frac{3}{2} \eta) = -3 \eta P_{12}^{(20)},
\]

(2.18)

where $P_{12}^{(20)}$ is a 20-dimensional supersymmetric projector

\[
P_{12}^{(20)} = \sum_{i=1}^{20} |\phi_i\rangle \langle \phi_i|,
\]

(2.19)

with the basis vectors

\[
|\phi_1\rangle = \frac{1}{\sqrt{3}} (\sqrt{2} |f_1\rangle \otimes |2\rangle - |f_2\rangle \otimes |1\rangle), \quad |\phi_2\rangle = \frac{1}{\sqrt{3}} (|f_2\rangle \otimes |2\rangle - \sqrt{2} |f_3\rangle \otimes |1\rangle),
\]

\[
|\phi_3\rangle = \frac{1}{\sqrt{3}} (\sqrt{2} |f_2\rangle \otimes |1\rangle - |f_3\rangle \otimes |2\rangle - \sqrt{2} |f_1\rangle \otimes |1\rangle), \quad |\phi_4\rangle = \frac{1}{\sqrt{3}} (|f_3\rangle \otimes |1\rangle - \sqrt{2} |f_1\rangle \otimes |2\rangle + \sqrt{2} |f_2\rangle \otimes |1\rangle),
\]

\[
|\phi_5\rangle = \frac{1}{\sqrt{3}} (\sqrt{2} |f_3\rangle \otimes |2\rangle - |f_1\rangle \otimes |1\rangle - \sqrt{2} |f_2\rangle \otimes |2\rangle), \quad |\phi_6\rangle = \frac{1}{\sqrt{3}} (|f_1\rangle \otimes |1\rangle - \sqrt{2} |f_3\rangle \otimes |2\rangle - \sqrt{2} |f_2\rangle \otimes |1\rangle),
\]

\[
|\phi_7\rangle = \frac{1}{\sqrt{3}} (\sqrt{2} |f_1\rangle \otimes |2\rangle - |f_3\rangle \otimes |1\rangle - \sqrt{2} |f_2\rangle \otimes |2\rangle), \quad |\phi_8\rangle = \frac{1}{\sqrt{3}} (|f_3\rangle \otimes |1\rangle - \sqrt{2} |f_1\rangle \otimes |2\rangle - \sqrt{2} |f_2\rangle \otimes |1\rangle),
\]

(5.20)
The elements of fused $R$-matrix defined in the tensor space the projected subspace as $V$, the second degenerate point of $P$

Taking the fusion by the projector $\phi$

\[ |\phi_3\rangle = \frac{1}{\sqrt{6}}(2|f_6\rangle \otimes |3\rangle + |f_5\rangle \otimes |4\rangle + |f_4\rangle \otimes |1\rangle), \quad |\phi_4\rangle = \frac{1}{\sqrt{2}}(|f_5\rangle \otimes |4\rangle - |f_4\rangle \otimes |1\rangle), \]
\[ |\phi_5\rangle = \frac{1}{\sqrt{6}}(|f_8\rangle \otimes |3\rangle + 2|f_4\rangle \otimes |2\rangle - |f_7\rangle \otimes |4\rangle), \quad |\phi_6\rangle = \frac{1}{\sqrt{2}}(|f_7\rangle \otimes |4\rangle + |f_8\rangle \otimes |3\rangle), \]
\[ |\phi_7\rangle = |f_5\rangle \otimes |3\rangle, \quad |\phi_8\rangle = |f_7\rangle \otimes |3\rangle, \quad |\phi_9\rangle = |f_6\rangle \otimes |4\rangle, \quad |\phi_{10}\rangle = |f_8\rangle \otimes |4\rangle, \]
\[ |\phi_{11}\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|f_1\rangle \otimes |3\rangle - |f_5\rangle \otimes |1\rangle), \quad |\phi_{12}\rangle = \frac{1}{\sqrt{2}}(|f_1\rangle \otimes |4\rangle - |f_6\rangle \otimes |1\rangle), \]
\[ |\phi_{13}\rangle = \frac{1}{\sqrt{6}}(|f_7\rangle \otimes |1\rangle + |f_2\rangle \otimes |3\rangle - 2|f_5\rangle \otimes |2\rangle), \quad |\phi_{14}\rangle = \frac{1}{\sqrt{2}}(|f_2\rangle \otimes |3\rangle - |f_7\rangle \otimes |1\rangle), \]
\[ |\phi_{15}\rangle = \frac{1}{\sqrt{6}}(|f_8\rangle \otimes |1\rangle + |f_2\rangle \otimes |4\rangle - 2|f_6\rangle \otimes |2\rangle), \quad |\phi_{16}\rangle = \frac{1}{\sqrt{2}}(|f_2\rangle \otimes |4\rangle - |f_8\rangle \otimes |1\rangle), \]
\[ |\phi_{17}\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|f_3\rangle \otimes |3\rangle - |f_7\rangle \otimes |2\rangle), \quad |\phi_{18}\rangle = \frac{1}{\sqrt{2}}(|f_3\rangle \otimes |4\rangle - |f_8\rangle \otimes |2\rangle), \]
\[ |\phi_{19}\rangle = |f_4\rangle \otimes |3\rangle, \quad |\phi_{20}\rangle = |f_4\rangle \otimes |4\rangle. \]

The corresponding parities of the basis vectors are

\[ p(1) = p(2) = \cdots = p(10) = 0, \quad p(11) = p(12) = \cdots = p(20) = 1. \]

The operator $P_{12}^{(20)}$ projects the 32-dimensional product space $V_1 \otimes V_2$ into a 20-dimensional subspace spanned by \{\(|\phi_i\rangle, i = 1, \cdots, 20\}\).

Taking the fusion by the projector $P_{12}^{(20)}$, we obtain another new fused $R$-matrix

\[
R_{(12)3}(u) = (u - \eta)^{-1}P_{21}^{(20)}R_{13}(u + \frac{1}{2}\eta)R_{23}(u - \eta)P_{21}^{(20)} \equiv R_{13}(u), \tag{2.20}
\]
\[
R_{3(21)}(u) = (u - \eta)^{-1}P_{12}^{(20)}R_{31}(u + \frac{1}{2}\eta)R_{32}(u - \eta)P_{12}^{(20)} \equiv R_{31}(u), \tag{2.21}
\]

where $P_{21}^{(20)}$ can be obtained from $P_{12}^{(20)}$ by exchanging $V_1$ and $V_2$. For simplicity, we denote the projected subspace as $V_1 = V_{(12)} = V_{(21)}$. The fused $R$-matrix $R_{12}(u)$ is a 80 $\times$ 80 matrix defined in the tensor space $V_1 \otimes V_2$ and satisfies following graded Yang-Baxter equations

\[
R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \tag{2.22}
\]

The elements of fused $R$-matrix $R_{12}(u)$ are also degree one polynomials of $u$.

The second degenerate point of $R$-matrix (2.1) is $u = -\eta$. At which we have

\[
R_{12}(-\eta) = -2\eta \bar{P}_{12}^{(8)} = -2\eta(1 - P_{12}^{(8)}), \tag{2.23}
\]
where \( \bar{P}^{(8)}_{12} \) is an 8-dimensional supersymmetric projector in terms of
\[
\bar{P}^{(8)}_{12} = \sum_{i=1}^{8} |g_i\rangle \langle g_i|, \tag{2.24}
\]
with
\[
|g_1\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \quad |g_2\rangle = |33\rangle, \quad |g_3\rangle = \frac{1}{\sqrt{2}}(|34\rangle + |43\rangle), \\
|g_4\rangle = |44\rangle, \quad |g_5\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle), \quad |g_6\rangle = \frac{1}{\sqrt{2}}(|14\rangle - |41\rangle) \\
|g_7\rangle = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle), \quad |g_8\rangle = \frac{1}{\sqrt{2}}(|24\rangle - |42\rangle). \tag{2.25}
\]
The corresponding parities are
\[
p(1) = p(2) = p(3) = p(4) = 0, \quad p(5) = p(6) = p(7) = p(8) = 1.
\]
The operator \( \bar{P}^{(8)}_{12} \) projects the 16-dimensional product space \( V_1 \otimes V_2 \) into a new 8-dimensional projected space spanned by \( \{|g_i\rangle | i = 1, \cdots , 8\} \).

Taking the fusion by the projector \( \bar{P}^{(8)}_{12} \), we obtain the fused \( R \)-matrices
\[
R_{(12)\gamma3}(u) = (u - \frac{1}{2} \eta)^{-1} \bar{P}^{(8)}_{12} R_{23}(u + \frac{1}{2} \eta) R_{13}(u - \frac{1}{2} \eta) \bar{P}^{(8)}_{12} \equiv R_{1\gamma3}(u), \tag{2.26}
\]
\[
R_{3(21)\gamma}(u) = (u - \frac{1}{2} \eta)^{-1} \bar{P}^{(8)}_{21} R_{32}(u + \frac{1}{2} \eta) R_{31}(u - \frac{1}{2} \eta) \bar{P}^{(8)}_{21} \equiv R_{31\gamma}(u). \tag{2.27}
\]
For simplicity, we denote the projected space as \( V_{1\gamma} = V_{(12)\gamma} = V_{(21)\gamma} \). The fused \( R \)-matrix \( R_{1\gamma2}(u) \) is a 32 \( \times \) 32 matrix defined in the product space \( V_{1\gamma} \otimes V_2 \) and possesses the properties
\[
R_{1\gamma2}(u) R_{2\gamma1}(u) = \rho_5(u) \times \text{id}, \\
R_{1\gamma2}(u) R_{2\gamma1}(u)^{st} = \rho_6(u) \times \text{id}, \\
R_{1\gamma2}(u - v) R_{1\gamma3}(u) R_{23}(v) = R_{23}(v) R_{1\gamma3}(u) R_{1\gamma2}(u - v), \tag{2.28}
\]
where
\[
\rho_5(u) = -(u - \frac{3}{2} \eta)(u + \frac{3}{2} \eta), \quad \rho_6(u) = -(u - \frac{1}{2} \eta)(u + \frac{1}{2} \eta). \tag{2.29}
\]
Now, we consider the fusions of \( R_{1\gamma2}(u) \), which include two different cases. One is the fusion in the auxiliary space \( V_1 \) and the other is the fusion in the quantum space \( V_2 \). Both are necessary to close the fusion processes.
We first introduce the fusion in the auxiliary space. At the point $u = \frac{3}{2} \eta$, we have

$$R_{1/2}(\frac{3}{2} \eta) = 3\eta P_{1/2}^{(20)},$$

(2.30)

where $P_{1/2}^{(20)}$ is a 20-dimensional supersymmetric projector with the form of

$$P_{1/2}^{(20)} = \sum_{i=1}^{20} |\tilde{\phi}_i\rangle \langle \tilde{\phi}_i|,$$

(2.31)

and the corresponding vectors are

$$|\tilde{\phi}_1\rangle = |g_1\rangle \otimes |1\rangle, \quad |\tilde{\phi}_2\rangle = |g_1\rangle \otimes |2\rangle,$$

$$|\tilde{\phi}_3\rangle = \frac{1}{\sqrt{2}}(|g_3\rangle \otimes |1\rangle - |g_5\rangle \otimes |4\rangle), \quad |\tilde{\phi}_4\rangle = \frac{1}{\sqrt{6}}(|g_5\rangle \otimes 4 + |g_3\rangle \otimes |1\rangle - 2|g_6\rangle \otimes |3\rangle),$$

$$|\tilde{\phi}_5\rangle = \frac{1}{\sqrt{2}}(|g_8\rangle \otimes |3\rangle - |g_7\rangle \otimes |4\rangle), \quad |\tilde{\phi}_6\rangle = \frac{1}{\sqrt{6}}(2|g_3\rangle \otimes |2\rangle - |g_7\rangle \otimes |4\rangle - |g_8\rangle \otimes |3\rangle),$$

$$|\tilde{\phi}_7\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|g_2\rangle \otimes |1\rangle - |g_5\rangle \otimes |3\rangle), \quad |\tilde{\phi}_8\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|g_2\rangle \otimes |2\rangle - |g_7\rangle \otimes |3\rangle),$$

$$|\tilde{\phi}_9\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|g_4\rangle \otimes |1\rangle - |g_6\rangle \otimes |4\rangle), \quad |\tilde{\phi}_{10}\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|g_4\rangle \otimes |2\rangle - |g_8\rangle \otimes |4\rangle),$$

$$|\tilde{\phi}_{11}\rangle = |g_5\rangle \otimes |1\rangle, \quad |\tilde{\phi}_{12}\rangle = |g_6\rangle \otimes |1\rangle,$$

$$|\tilde{\phi}_{13}\rangle = \frac{1}{\sqrt{2}}(|g_7\rangle \otimes |1\rangle - |g_1\rangle \otimes |3\rangle), \quad |\tilde{\phi}_{14}\rangle = \frac{1}{\sqrt{6}}(|g_7\rangle \otimes |1\rangle + 2|g_5\rangle \otimes |2\rangle + |g_1\rangle \otimes |3\rangle)$$

$$|\tilde{\phi}_{15}\rangle = \frac{1}{\sqrt{2}}(|g_8\rangle \otimes |1\rangle - |g_1\rangle \otimes |4\rangle), \quad |\tilde{\phi}_{16}\rangle = \frac{1}{\sqrt{6}}(|g_6\rangle \otimes |2\rangle + 2|g_8\rangle \otimes |1\rangle + |g_1\rangle \otimes |4\rangle),$$

$$|\tilde{\phi}_{17}\rangle = |g_7\rangle \otimes |2\rangle, \quad |\tilde{\phi}_{18}\rangle = |g_8\rangle \otimes |2\rangle,$$

$$|\tilde{\phi}_{19}\rangle = \frac{1}{\sqrt{3}}(|g_3\rangle \otimes |3\rangle - \sqrt{2}|g_2\rangle \otimes |4\rangle), \quad |\tilde{\phi}_{20}\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|g_4\rangle \otimes |3\rangle - |g_3\rangle \otimes |4\rangle).$$

The parities read

$$p(1) = p(2) = \cdots = p(10) = 0, \quad p(11) = p(12) = \cdots = p(20) = 1.$$

The operator $P_{1/2}^{(20)}$ projects the 32-dimensional product space $V_1 \otimes V_2$ into a 20-dimensional projected space spanned by $\{|\tilde{\phi}_i\rangle, i = 1, \cdots, 20\}$. Taking the fusion by the projector $P_{1/2}^{(20)}$, we obtain the following fused $R$-matrices

$$R_{1(1/2)3}(u) = (u + \eta)^{-1} P_{1/2}^{(20)} R_{1/3}(u - \frac{1}{2} \eta) R_{23}(u + \eta) P_{1/2}^{(20)} \equiv R_{1/3}(u),$$

(2.32)

$$R_{3(21)}(u) = (u + \eta)^{-1} P_{1/2}^{(20)} R_{31}(u - \frac{1}{2} \eta) R_{32}(u + \eta) P_{1/2}^{(20)} \equiv R_{31}(u).$$

(2.33)
For simplicity, we denote the projected space as \( V_1 = V_{(1^2)} = V_{(2^1)} \). The fused \( R \)-matrix \( R_{12}(u) \) is a \( 80 \times 80 \) one defined in the product spaces \( V_1 \otimes V_2 \) and satisfies the following graded Yang-Baxter equation

\[
R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \tag{2.34}
\]

A remarkable fact is that after taking the correspondences

\[
|\phi_i\rangle \longrightarrow |\psi_i\rangle, \quad |\tilde{\phi}_i\rangle \longrightarrow |\tilde{\psi}_i\rangle, \quad i = 1, \ldots, 20, \tag{2.35}
\]

the two fused \( R \)-matrices \( R_{12}(u) \) given by (2.20) and \( R_{12}(u) \) given by (2.32) are identical,

\[
R_{12}(u) = R_{12}(u), \tag{2.36}
\]

which allows us to close the recursive fusion process.

The fusion of \( R_{12}(u) \) in the quantum space is carried out by the projector \( P_{23}^{(8)} \), and the resulted fused \( R \)-matrix is

\[
R_{12(23)}(u) = (u + \eta)^{-1}P_{23}^{(8)}R_{12}(u - \frac{1}{2}\eta)R_{12}(u + \frac{1}{2}\eta)P_{23}^{(8)} = R_{12}(u), \tag{2.37}
\]

which is a \( 64 \times 64 \) matrix defined in the space \( V_1 \otimes V_2 \) and satisfies the graded Yang-Baxter equation

\[
R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v), \tag{2.38}
\]

which will help us to find the complete set of conserved quantities.

### 2.3 Operator product identities

Now, we are ready to extend the fusion from one site to the whole system. From the fused \( R \)-matrices given by (2.13), (2.20), (2.26) and (2.32), we construct the fused monodromy matrices as

\[
T_0(u) = R_{01}(u - \theta_1)R_{02}(u - \theta_2) \cdots R_{0N}(u - \theta_N),
\]

\[
T_{\theta'}(u) = R_{\theta'1}(u - \theta_1)R_{\theta'2}(u - \theta_2) \cdots R_{\theta'N}(u - \theta_N),
\]

\[
T_{\tilde{0}}(u) = R_{\tilde{0}1}(u - \theta_1)R_{\tilde{0}2}(u - \theta_2) \cdots R_{\tilde{0}N}(u - \theta_N),
\]

\[
T_{\tilde{\theta}'}(u) = R_{\tilde{\theta}'1}(u - \theta_1)R_{\tilde{\theta}'2}(u - \theta_2) \cdots R_{\tilde{\theta}'N}(u - \theta_N), \tag{2.39}
\]
where the subscripts 0, 0', 0 and 0' mean the auxiliary spaces, and the quantum spaces in all
the monodromy matrices are the same. By using the graded Yang-Baxter equations (2.17),
(2.22), (2.28), (2.34) and (2.38), one can prove that the monodromy matrices satisfy the
relations among the monodromy matrices

\[ R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v), \]
\[ R_{1v2}(u - v)T_{1v}(u)T_2(v) = T_2(v)T_{1v}(u)R_{1v2}(u - v), \]
\[ R_{1v2}(u - v)T_{1v}(u)T_2(v) = T_2(v)T_{1v}(u)R_{1v2}(u - v), \]
\[ R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v), \]
\[ R_{1v2}(u - v)T_{1v}(u)T_2(v) = T_2(v)T_{1v}(u)R_{1v2}(u - v). \]  

(2.40)

According to the property that the R-matrices in above equations can degenerate into the
projectors \( P^{(8)}_{12}, \bar{P}^{(8)}_{12}, P^{(20)}_{12}, \bar{P}^{(20)}_{12} \) and using the definitions (2.39), we obtain following fusion
relations among the monodromy matrices

\[ P^{(8)}_{12}T_2(u)T_1(u + \eta)P^{(8)}_{12} = \prod_{l=1}^{N}(u - \theta_l + \eta)T_1(u + \frac{1}{2}\eta), \]
\[ P^{(8)}_{12}T_2(u)T_1(u - \eta)P^{(8)}_{12} = \prod_{l=1}^{N}(u - \theta_l - \eta)T_1(u - \frac{1}{2}\eta), \]
\[ P^{(20)}_{21}T_1(u + \frac{1}{2}\eta)T_2(u - \eta)P^{(20)}_{21} = \prod_{l=1}^{N}(u - \theta_l - \eta)T_1(u), \]
\[ P^{(20)}_{21}T_{1v}(u - \frac{1}{2}\eta)T_2(u + \eta)P^{(20)}_{21} = \prod_{l=1}^{N}(u - \theta_l + \eta)T_{1v}(u). \]  

(2.41)

The fused transfer matrices are define as the super partial traces of fused monodromy
matrices in the auxiliary space

\[ t^{(1)}_p(u) = \text{str}_0 T_0(u), \ t^{(2)}_p(u) = \text{str}_{0'} T_{0'}(u), \ \bar{t}^{(1)}_p(u) = \text{str}_{\bar{0}} T_{\bar{0}}(u), \ \bar{t}^{(2)}_p(u) = \text{str}_{\bar{0'}} T_{\bar{0'}}(u). \]

From Eq. (2.41), we know that these fused transfer matrices with certain spectral difference
must satisfy some intrinsic relations. We first consider the quantity

\[ t_p(u)t_p(u + \eta) = \text{str}_{12}\{T_1(u)T_2(u + \eta)\} \]
\[ = \text{str}_{12}\{(P^{(8)}_{12} + \bar{P}^{(8)}_{12})T_1(u)T_2(u + \eta)(P^{(8)}_{12} + \bar{P}^{(8)}_{12})\} \]

11
Here we give some remarks. Both $V_1$ and $V_2$ are the 4-dimensional auxiliary spaces. From Eq. (2.42), we see that the 16-dimensional auxiliary space $\bar{V}$, those in $\bar{V}$, during the derivation, we have used the relations $u$ at the point of $t$. From Eq. (2.42), we also know that the product of two transfer matrices with fixed spectral difference can be written as the summation of two fused transfer matrices.

$$
\begin{align*}
str_{12}\{P_{12}^{(8)}T_1(u)T_2(u+\eta)P_{12}^{(8)}\} & = str_{12}\{\bar{P}_{12}^{(8)}\bar{P}_{12}^{(8)}T_1(u)T_2(u+\eta)\bar{P}_{12}^{(8)}\} \\
& = \prod_{j=1}^{N}(u-\theta_j+\eta)t_p^{(1)}(u+\frac{1}{2}\eta) + \prod_{j=1}^{N}(u-\theta_j)t_p^{(2)}(u+\frac{1}{2}\eta). \\
\end{align*}
(2.42)
$$

During the derivation, we have used the relations $P_{12}^{(8)} + \bar{P}_{12}^{(8)} = 1$, $P_{12}^{(8)} \bar{P}_{12}^{(8)} = 0$.
From Eq. (2.43), we see that the 32-dimensional auxiliary space $V_1 \otimes V_2$ can be projected into a 20-dimensional subspace $V_{1(2)} \equiv V_1$ by the projector $P_{12}^{(20)}$ and a 12-dimensional subspace $V_{1(2)}$ by the projector $\tilde{P}_{12}^{(12)}$, $V_1 \otimes V_2 = V_{1(2)} \oplus V_{1(2)}$. The vectors in $P_{12}^{(20)}$ and $\tilde{P}_{12}^{(12)}$ are the complete and orthogonal basis. Eq. (2.43) also gives that the quantity $t_p^{(1)}(u + \frac{1}{2}\eta) = \tilde{t}_p^{(1)}(u)$ is the summation of two new fused transfer matrices $\tilde{t}_p^{(1)}(u)$ and $\tilde{t}_p^{(2)}(u)$ with some coefficients.

In Eq. (2.41), the 32-dimensional auxiliary space $V_1 \otimes V_2$ is projected into a 20-dimensional and a 12-dimensional subspaces by the operators $P_{1/2}^{(20)}$ and $\tilde{P}_{1/2}^{(12)}$, respectively. Thus the quantity $t_p^{(2)}(u - \frac{1}{2}\eta) = \tilde{t}_p^{(2)}(u)$ is the summation of two fused transfer matrices $\tilde{t}_p^{(1)}(u)$ and $\tilde{t}_p^{(2)}(u)$ with some coefficients. At the point of $u = \theta_j - \eta$, the coefficient of $\tilde{t}_p^{(1)}(u)$ in Eq. (2.43) and that of $\tilde{t}_p^{(2)}(u)$ in Eq. (2.43) are zero. While at the point of $u = \theta_j$, the coefficient of $\tilde{t}_p^{(1)}(u)$ in Eq. (2.43) and that of $\tilde{t}_p^{(2)}(u)$ in Eq. (2.44) are zero. Here, the explicit forms of $\tilde{P}_{12}^{(12)}$, $\tilde{P}_{1/2}^{(12)}$, $\tilde{t}_p^{(1)}(u)$ and $\tilde{t}_p^{(2)}(u)$ are omitted because we donot use them.

Combining the above analysis, we obtain the operator product identities of the transfer matrices at the fixed points as

\begin{equation}
  t_p(\theta_j) t_p(\theta_j + \eta) = \prod_{l=1}^{N} (\theta_j - \theta_l + \eta)) t_p^{(1)}(\theta_j + \frac{1}{2}\eta),
\end{equation}

\begin{equation}
  t_p(\theta_j) t_p(\theta_j - \eta) = \prod_{l=1}^{N} (\theta_j - \theta_l - \eta)) t_p^{(2)}(\theta_j - \frac{1}{2}\eta),
\end{equation}

\begin{equation}
  \tilde{t}_p^{(1)}(\theta_j + \frac{1}{2}\eta) t_p(\theta_j - \eta) = \prod_{l=1}^{N} (\theta_j - \theta_l - \eta)) \tilde{t}_p^{(1)}(\theta_j),
\end{equation}

\begin{equation}
  \tilde{t}_p^{(2)}(\theta_j - \frac{1}{2}\eta) t_p(\theta_j + \eta) = \prod_{l=1}^{N} (\theta_j - \theta_l + \eta)) \tilde{t}_p^{(2)}(\theta_j), \quad j = 1, \ldots, N.
\end{equation}

From the property (2.36), we obtain that the fused transfer matrices $\tilde{t}_p^{(1)}(u)$ and $\tilde{t}_p^{(2)}(u)$ are equal

\begin{equation}
  \tilde{t}_p^{(1)}(u) = \tilde{t}_p^{(2)}(u).
\end{equation}

With the help of Eqs. (2.49), (2.47) and (2.48), we can obtain the constraint among $t_p(u)$, $\tilde{t}_p^{(1)}(u)$ and $\tilde{t}_p^{(2)}(u)$,

\begin{equation}
  t_p^{(1)}(\theta_j + \frac{1}{2}\eta) t_p(\theta_j - \eta) = \prod_{l=1}^{N} (\theta_j - \theta_l - \eta)) t_p^{(2)}(\theta_j - \frac{1}{2}\eta) t_p(\theta_j + \eta).
\end{equation}
Then Eqs. (2.45), (2.46) and (2.50) constitute the closed recursive fusion relations. From the
definitions, we know that the transfer matrices $t_p(u)$, $t_p^{(1)}(u)$ and $t_p^{(2)}(u)$ are the operator
polynomials of $u$ with degree $N - 1$. Then, the $3N$ conditions (2.45), (2.46) and (2.50) are
sufficient to solve them.

From the graded Yang-Baxter relations (2.40), the transfer matrices $t_p(u)$, $t_p^{(1)}(u)$ and
$t_p^{(2)}(u)$ commutate with each other, namely,

$$[t_p(u), t_p^{(1)}(u)] = [t_p(u), t_p^{(2)}(u)] = [t_p^{(1)}(u), t_p^{(2)}(u)] = 0. \quad (2.51)$$

Therefore, they have common eigenstates and can be diagonalized simultaneously. Let $|\Phi\rangle$
be a common eigenstate. Acting the transfer matrices on this eigenstate, we have

$$t_p(u)|\Phi\rangle = \Lambda_p(u)|\Phi\rangle, \quad t_p^{(1)}(u)|\Phi\rangle = \Lambda_p^{(1)}(u)|\Phi\rangle, \quad t_p^{(2)}(u)|\Phi\rangle = \Lambda_p^{(2)}(u)|\Phi\rangle,$$

where $\Lambda_p(u)$, $\Lambda_p^{(1)}(u)$ and $\Lambda_p^{(2)}(u)$ are the eigenvalues of $t_p(u)$, $t_p^{(1)}(u)$ and $t_p^{(2)}(u)$, respectively.

Meanwhile, acting the operator product identities (2.45), (2.46) and (2.50) on the state $|\Phi\rangle$,
we have the functional relations among these eigenvalues

$$\Lambda_p(\theta_j)\Lambda_p(\theta_j + \eta) = \prod_{l=1}^{N} (\theta_j - \theta_l + \eta)\Lambda_p^{(1)}(\theta_j + \frac{1}{2}\eta),$$

$$\Lambda_p(\theta_j)\Lambda_p(\theta_j - \eta) = \prod_{l=1}^{N} (\theta_j - \theta_l - \eta)\Lambda_p^{(2)}(\theta_j - \frac{1}{2}\eta),$$

$$\Lambda_p^{(1)}(\theta_j + \frac{1}{2}\eta)\Lambda_p(\theta_j - \eta) = \prod_{l=1}^{N} \frac{\theta_j - \theta_l - \eta}{\theta_j - \theta_l + \eta} \Lambda_p^{(2)}(\theta_j - \frac{1}{2}\eta)\Lambda_p(\theta_j + \eta), \quad (2.52)$$

where $j = 1, 2, \cdots N$. Because the eigenvalues $\Lambda_p(u)$, $\Lambda_p^{(1)}(u)$ and $\Lambda_p^{(2)}(u)$ are the polynomials
of $u$ with degree $N - 1$, the above $3N$ conditions (2.52) can determine these eigenvalues
completely.

### 2.4 $T - Q$ relations

Let us introduce the $z$-functions

$$z_p^{(l)}(u) = \begin{cases} 
(-1)^{p(l)} Q_p^{(0)}(u) \frac{Q_p^{(l-1)}(u+\eta)Q_p^{(l)}(u-\eta)}{Q_p^{(l)}(u)Q_p^{(l-1)}(u)}, & l = 1, 2, \\
(-1)^{p(l)} Q_p^{(0)}(u) \frac{Q_p^{(l-1)}(u-\eta)Q_p^{(l)}(u+\eta)}{Q_p^{(l)}(u)Q_p^{(l-1)}(u)}, & l = 3, 4, 
\end{cases} \quad (2.53)$$
where the $Q$-functions are
\[ Q_p^{(0)}(u) = \prod_{l=1}^{N}(u - \theta_l), \quad Q_p^{(m)}(u) = \prod_{j=1}^{L_m}(u - \lambda_j^{(m)}), \quad m = 1, 2, 3, \quad Q_p^{(2)}(u) = 1, \]
and $\{L_m|m = 1, 2, 3\}$ are the numbers of the Bethe roots $\{\lambda_j^{(m)}\}$.

According to the closed functional relations (2.52), we construct the eigenvalues of the transfer matrices in terms of the homogeneous $T - Q$ relations
\[
\Lambda_p(u) = \sum_{l=1}^{4} z_p^{(l)}(u), \\
\Lambda_p^{(1)}(u) = \left[ Q_p^{(0)}(u + \frac{1}{2}\eta) \right]^{-1} \left\{ \sum_{l=1}^{2} z_p^{(l)}(u + \frac{1}{2}\eta)z_p^{(l)}(u - \frac{1}{2}\eta) \\
+ \sum_{l=2}^{4} \sum_{m=1}^{l-1} z_p^{(l)}(u + \frac{1}{2}\eta)z_p^{(m)}(u - \frac{1}{2}\eta) \right\}, \\
\Lambda_p^{(2)}(u) = \left[ Q_p^{(0)}(u - \frac{1}{2}\eta) \right]^{-1} \left\{ \sum_{l=3}^{4} z_p^{(l)}(u + \frac{1}{2}\eta)z_p^{(l)}(u - \frac{1}{2}\eta) \\
+ \sum_{l=2}^{4} \sum_{m=1}^{l-1} z_p^{(l)}(u - \frac{1}{2}\eta)z_p^{(m)}(u + \frac{1}{2}\eta) \right\}.
\]

The regularities of the eigenvalues $\Lambda_p(u)$, $\Lambda_p^{(1)}(u)$ and $\Lambda_p^{(2)}(u)$ give rise to the constraints that the Bethe roots $\{\lambda_j^{(m)}\}$ should satisfy the Bethe ansatz equations (BAEs)
\[
\frac{Q_p^{(0)}(\lambda_j^{(1)} + \eta)}{Q_p^{(0)}(\lambda_j^{(1)})} = -\frac{Q_p^{(1)}(\lambda_j^{(1)} + \eta)Q_p^{(2)}(\lambda_j^{(1)} - \eta)}{Q_p^{(2)}(\lambda_j^{(1)})Q_p^{(1)}(\lambda_j^{(1)} - \eta)}, \quad j = 1, \cdots, L_1, \\
\frac{Q_p^{(1)}(\lambda_j^{(2)} + \eta)}{Q_p^{(1)}(\lambda_j^{(2)})} = \frac{Q_p^{(3)}(\lambda_j^{(2)})}{Q_p^{(3)}(\lambda_j^{(2)})}, \quad j = 1, \cdots, L_2, \\
\frac{Q_p^{(2)}(\lambda_j^{(3)} - \eta)}{Q_p^{(2)}(\lambda_j^{(3)})} = -\frac{Q_p^{(3)}(\lambda_j^{(3)} - \eta)}{Q_p^{(3)}(\lambda_j^{(3)} + \eta)}, \quad j = 1, \cdots, L_3.
\]

We have verified that the above BAEs indeed guarantee all the $T - Q$ relations (2.54) are polynomials and satisfy the functional relations (2.52). Therefore, we arrive at the conclusion that $\Lambda_p(u)$, $\Lambda_p^{(1)}(u)$ and $\Lambda_p^{(2)}(u)$ given by (2.54) are indeed the eigenvalues of the transfer matrices $t_p(u)$, $t_p^{(1)}(u)$, $t_p^{(2)}(u)$, respectively. The eigenvalues of the Hamiltonian (2.10) are
\[
E_p = \frac{\partial \ln \Lambda_p(u)}{\partial u} \bigg|_{u=0,\{\theta_j\}=0}.
\]
3 $SU(2|2)$ model with off-diagonal boundary reflections

3.1 Boundary integrability

In this section, we consider the system with open boundary conditions. The boundary reflections are characterized by the reflection matrix $K^-(u)$ at one side and $K^+(u)$ at the other side. The integrability requires that $K^-(u)$ satisfies the graded reflection equation (RE) \[49, 50\]

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \quad (3.1)$$

while $K^+(u)$ satisfies the graded dual RE

$$R_{12}(v-u)K_1^+(u)R_{21}(-u-v)K_2^+(v) = K_2^+(v)R_{12}(-u-v)K_1^+(u)R_{21}(v-u). \quad (3.2)$$

The general solution of reflection matrix $K_0^-(u)$ defined in the space $V_0$ satisfying the graded RE (3.1) is

$$K_0^-(u) = \xi + uM, \quad M = \begin{pmatrix} 1 & c_1 & 0 & 0 \\ c_2 & -1 & 0 & 0 \\ 0 & 0 & -1 & c_3 \\ 0 & 0 & c_4 & 1 \end{pmatrix}, \quad (3.3)$$

and the dual reflection matrix $K^+(u)$ can be obtained by the mapping

$$K_0^+(u) = K_0^-(-u)|_{\xi, c_i \rightarrow \xi, \tilde{c}_i}, \quad (3.4)$$

where the $\xi$, $\tilde{\xi}$ and $\{c_i, \tilde{c}_i|i = 1, \cdots, 4\}$ are the boundary parameters which describe the boundary interactions, and the integrability requires

$$c_1c_2 = c_3c_4, \quad \tilde{c}_1\tilde{c}_2 = \tilde{c}_3\tilde{c}_4.$$

The reflection matrices (3.3) and (3.4) have the off-diagonal elements, thus the numbers of “quasi-particles” with different intrinsic degrees of freedom are not conserved during the reflection processes. Meanwhile, the $K^-(u)$ and $K^+(v)$ are not commutative, $[K^-(u), K^+(v)] \neq 0$, which means that they cannot be diagonalized simultaneously. Thus it is quite hard to derive the exact solutions of the system via the conventional Bethe ansatz because of the absence of a proper reference state. We will develop the graded nested ODBA to solve the system exactly.
For the open case, besides the standard “row-to-row” monodromy matrix \( T_0(u) \) specified by (2.6), one needs to consider the reflecting monodromy matrix

\[
\hat{T}_0(u) = R_{N0}(u + \theta_N) \cdots R_{20}(u + \theta_2) R_{10}(u + \theta_1),
\]

which satisfies the graded Yang-Baxter relation

\[
R_{12}(u - v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u - v).
\]

The transfer matrix \( t(u) \) is defined as

\[
t(u) = \text{str}_0\{K_0^+(u)\hat{T}_0(u)K_0^-(u)\hat{T}_0(u)\}.
\]

The graded Yang-Baxter relations (2.8), (3.6) and reflection equations (3.1), (3.2) lead to the fact that the transfer matrices with different spectral parameters commute with each other, \([t(u), t(v)] = 0\). Therefore, \( t(u) \) serves as the generating function of all the conserved quantities and the system is integrable. The model Hamiltonian with open boundary condition can be written out in terms of transfer matrix (3.7) as

\[
H = \frac{1}{2} \frac{\partial \ln t(u)}{\partial u} \big|_{u=0, \{\theta_j\}=0}.
\]

The hermiticity of Hamiltonian (3.8) further requires \( c_1 = c_2^* \) and \( c_3 = c_4^* \).

### 3.2 Fused reflection matrices

In order to solve the eigenvalue problem of the transfer matrix (3.7), we should study the fusion of boundary reflection matrices \([51, 52]\). The main idea of the fusion for reflection matrices associated with a supersymmetric model is expressed in Appendix A. Focusing on the supersymmetric \( SU(2|2) \) model with the boundary reflection matrices (3.3) and (3.4), we can take fusion according to Eqs.(A.3)-(A.6) or (A.7)-(A.8). The two 8-dimensional fusion associated with the super projectors \( P_{12}^{(8)} \) (2.12) and \( \hat{P}_{12}^{(8)} \) (2.21) gives

\[
K_1^-(u) = (u + \frac{1}{2}\eta)^{-1}P_{21}^{(8)}K_1^-(u - \frac{1}{2}\eta)R_{21}(2u)K_2^-(u + \frac{1}{2}\eta)P_{12}^{(8)},
\]

\[
K_1^+(u) = (u - \frac{1}{2}\eta)^{-1}P_{12}^{(8)}K_2^+(u + \frac{1}{2}\eta)R_{12}(-2u)K_1^+(u - \frac{1}{2}\eta)P_{21}^{(8)},
\]

\[
K_1^-\prime(u) = (u - \frac{1}{2}\eta)^{-1}\hat{P}_{21}^{(8)}K_1^-\prime(u + \frac{1}{2}\eta)R_{21}(2u)K_2^-(u - \frac{1}{2}\eta)\hat{P}_{12}^{(8)},
\]

\[
K_1^+\prime(u) = (u + \frac{1}{2}\eta)^{-1}\hat{P}_{12}^{(8)}K_2^+(u - \frac{1}{2}\eta)R_{12}(-2u)K_1^+(u + \frac{1}{2}\eta)\hat{P}_{21}^{(8)}.
\]
By specific calculation, we know that all the fused $K$-matrices are the $8 \times 8$ ones and their matric elements are the polynomials of $u$ with maximum degree two. The fused reflection $K$-matrices (3.39) satisfy the resulting graded reflection equations. We can further use the reflection matrices $K_{1}^{\pm}(u)$ [or $K_{1}^{\pm}(u)$] and $K_{2}^{\pm}(u)$ to obtain the the 20-dimensional projector $P_{12}^{(20)}$ (2.19) [or $P_{12}^{(20)}$ (2.31)]. The resulted new fused reflection matrices are

$$K_{1}^{-}(u) = (u - \eta)^{-1}P_{21}^{(20)}K_{1}^{-}(u) + \frac{1}{2}\eta)R_{21}(2u - \frac{1}{2}\eta)K_{2}^{-}(u - \eta)P_{12}^{(20)},$$

$$K_{1}^{+}(u) = (2u + \eta)^{-1}P_{21}^{(20)}K_{2}^{+}(u - \eta)R_{12}(-2u + \frac{1}{2}\eta)K_{1}^{+}(u + \frac{1}{2}\eta)P_{12}^{(20)},$$

$$K_{1}^{-}(u) = (u + \eta)^{-1}P_{21}^{(20)}K_{1}^{-}(u) - \frac{1}{2}\eta)R_{21}(2u + \frac{1}{2}\eta)K_{2}^{-}(u + \eta)P_{12}^{(20)},$$

$$K_{1}^{+}(u) = (2u - \eta)^{-1}P_{12}^{(20)}K_{2}^{+}(u + \eta)R_{12}(-2u)K_{1}^{+}(u)P_{12}^{(20)}. \quad (3.10)$$

It is easy to check that the fused reflection matrices (3.10) are the $20 \times 20$ ones where the matric elements are polynomials of $u$ with maximum degree three. Moreover, keeping the correspondences (2.35) in mind, we have the important relations that the fused reflection matrices defined in the projected subspace $V_{1}$ and that defined in the projected subspace $V_{1}'$ are equal

$$K_{1}^{-}(u) = K_{1}'^{-}(u), \quad K_{1}^{+}(u) = K_{1}'^{+}(u), \quad (3.11)$$

which will be used to close the fusion processes with boundary reflections.

### 3.3 Operator production identities

For the model with open boundary condition, besides the fused monodromy matrices (2.39), we also need the fused reflecting monodromy matrices, which are constructed as

$$\hat{T}_{0}(u) = R_{N0}(u + \theta_{N}) \cdots R_{20}(u + \theta_{2})R_{10}(u + \theta_{1}),$$

$$\hat{T}_{0}^{'}(u) = R_{N0}^{'}(u + \theta_{N}) \cdots R_{20}^{'}(u + \theta_{2})R_{10}^{'}(u + \theta_{1}). \quad (3.12)$$

The fused reflecting monodromy matrices satisfy the graded Yang-Baxter relations

$$R_{12}(u - v)\hat{T}_{1}(u)\hat{T}_{2}(v) = \hat{T}_{2}(v)\hat{T}_{1}(u)R_{12}(u - v),$$

$$R_{12}^{'}(u - v)\hat{T}_{1}(u)\hat{T}_{2}^{'}(v) = \hat{T}_{2}^{'}(v)\hat{T}_{1}(u)R_{12}^{'}(u - v),$$

$$R_{12}^{'}(u - v)\hat{T}_{1}(u)\hat{T}_{2}^{'}(v) = \hat{T}_{2}^{'}(v)\hat{T}_{1}(u)R_{12}^{'}(u - v). \quad (3.13)$$
The fused transfer matrices are defined as

\[ t^{(1)}(u) = \text{str}_0 \{ K_0^+ T_0(u) K_0^- T_0(u) \}, \]
\[ t^{(2)}(u) = \text{str}_0 \{ K_0^+ T_0(u) K_0^- T_0(u) \}. \]

(3.14)

Using the method we have used in the periodic case, we can obtain the operator product identities among the fused transfer matrices as

\[ t(\pm \theta_j) t(\pm \theta_j + \eta) = -\frac{1}{4} (\pm \theta_j)(\pm \theta_j + \eta) \]
\[ \times \prod_{l=1}^{N} (\pm \theta_j - \theta_l + \eta)(\pm \theta_j + \theta_l + \eta) t^{(1)}(\pm \theta_j + \frac{1}{2} \eta), \]

(3.15)

\[ t(\pm \theta_j) t(\pm \theta_j - \eta) = -\frac{1}{4} (\pm \theta_j)(\pm \theta_j - \eta) \]
\[ \times \prod_{l=1}^{N} (\pm \theta_j - \theta_l - \eta)(\pm \theta_j + \theta_l - \eta) t^{(2)}(\pm \theta_j - \frac{1}{2} \eta), \]

(3.16)

\[ t(\pm \theta_j - \eta) t^{(1)}(\pm \theta_j + \frac{1}{2} \eta) = \frac{(\pm \theta_j + \frac{1}{2} \eta)^2(\pm \theta_j - \eta)}{(\pm \theta_j + \eta)(\pm \theta_j - \frac{1}{2} \eta)^2} \]
\[ \times \prod_{l=1}^{N} (\pm \theta_j - \theta_l - \eta)(\pm \theta_j + \theta_l - \eta) t(\pm \theta_j + \eta) t^{(2)}(\pm \theta_j - \frac{1}{2} \eta). \]

(3.17)

The proof of the above operator identities is given in Appendix B.

From the definitions, we know that the transfer matrix \( t(u) \) is a operator polynomial of \( u \) with degree \( 2N + 2 \) while the fused ones \( t^{(1)}(u) \) and \( t^{(2)}(u) \) are the operator polynomials of \( u \) both with degree \( 2N + 4 \). Thus they can be completely determined by \( 6N + 13 \) independent conditions. The recursive fusion relations (3.15), (3.16) and (3.17) gives \( 6N \) constraints and we still need 13 ones, which can be achieved by analyzing the values of transfer matrices at some special points. After some direct calculation, we have

\[ t(0) = 0, \quad t^{(1)}(0) = 0, \quad t^{(2)}(0) = 0, \quad t^{(1)}(\frac{\eta}{2}) = -2\xi \tilde{\xi} t(\eta), \]
\[ t^{(2)}(\frac{\eta}{2}) = -2\xi \tilde{\xi} t(-\eta), \quad t^{(1)}(\frac{-\eta}{2}) = 2\xi \tilde{\xi} t(\eta), \quad t^{(2)}(\frac{-\eta}{2}) = 2\xi \tilde{\xi} t(-\eta), \]
\[ \frac{\partial t^{(1)}(u)}{\partial u} |_{u=0} + \frac{\partial t^{(2)}(u)}{\partial u} |_{u=0} = 0. \]

(3.18)
Meanwhile, the asymptotic behaviors of \( t(u) \), \( t^{(1)}(u) \) and \( t^{(2)}(u) \) read

\[
\begin{align*}
t(u)|_{u \to \infty} &= -[c_1 \tilde{c}_2 + \tilde{c}_1 c_2 - c_3 \tilde{c}_4 - \tilde{c}_3 c_4] u^{2N+2} \times \text{id} - \eta \hat{U} u^{2N+1} + \cdots, \\
t^{(1)}(u)|_{u \to \infty} &= -4\{2[c_3 c_4 \tilde{c}_3 \tilde{c}_4 - \tilde{c}_3 c_4 - c_3 \tilde{c}_4 - 1] + (1 + c_1 \tilde{c}_2)^2 + (1 + \tilde{c}_1 c_2)^2 \\
&\quad - (c_1 \tilde{c}_2 + \tilde{c}_1 c_2)(c_3 \tilde{c}_4 + \tilde{c}_3 c_4)\} u^{2N+4} \times \text{id} - 4\eta \hat{Q} u^{2N+3} + \cdots, \\
t^{(2)}(u)|_{u \to \infty} &= -4\{2[c_1 c_2 \tilde{c}_1 \tilde{c}_2 - \tilde{c}_1 c_2 - c_1 \tilde{c}_2 - 1] + (1 + c_3 \tilde{c}_4)^2 + (1 + \tilde{c}_3 c_4)^2 \\
&\quad - (c_1 \tilde{c}_2 + \tilde{c}_1 c_2)(c_3 \tilde{c}_4 + \tilde{c}_3 c_4)\} u^{2N+4} \times \text{id} + \cdots. \tag{3.19}
\end{align*}
\]

Here we find that the operator \( \hat{U} \) related to the coefficient of transfer matrix \( t(u) \) with degree \( 2N+1 \) is given by

\[
\hat{U} = \sum_{i=1}^{N} \hat{U}_i = \sum_{i=1}^{N} (M_i \bar{M}_i + \bar{M}_i M_i), \tag{3.20}
\]

where \( M_i \) is given by (3.3), \( \bar{M}_i \) is determined by (3.4) and the operator \( \hat{U}_i \) is

\[
\hat{U}_i = \left( \begin{array}{cccc}
2 + c_1 \tilde{c}_2 + \tilde{c}_1 c_2 & 0 & 0 & 0 \\
0 & 2 + c_1 \tilde{c}_2 + \tilde{c}_1 c_2 & 0 & 0 \\
0 & 0 & 2 + c_3 \tilde{c}_4 + \tilde{c}_3 c_4 & 0 \\
0 & 0 & 0 & 2 + c_3 \tilde{c}_4 + \tilde{c}_3 c_4
\end{array} \right). \tag{3.21}
\]

We note that \( \hat{U}_i \) is the operator defined in the \( i \)-th physical space \( V_i \) and can be expressed by a diagonal matrix with constant elements. The summation of \( \hat{U}_i \) in Eq.(3.20) is the direct summation and the representation matrix of operator \( \hat{U} \) is also a diagonal one with constant elements. Moreover, we find that the operator \( \hat{Q} \) related to the coefficient of the fused transfer matrix \( t^{(1)}(u) \) with degree \( 2N+3 \) is given by

\[
\hat{Q} = \sum_{i=1}^{N} \hat{Q}_i, \tag{3.22}
\]

where the operator \( \hat{Q}_i \) is defined in \( i \)-th physical space \( V_i \) with the matrix form of

\[
\hat{Q}_i = \left( \begin{array}{cccc}
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & \beta & 0 \\
0 & 0 & 0 & \beta
\end{array} \right)_i,
\]

\[
\alpha = 2 - 2c_1 \tilde{c}_2 + 4c_1 \tilde{c}_2 + (c_1 \tilde{c}_2)^2 + 4\tilde{c}_1 c_2 - 2c_1 c_2 + (\tilde{c}_1 c_2)^2, \\
\beta = 2 - 2\tilde{c}_3 c_4 - (c_1 \tilde{c}_2)^2 - (\tilde{c}_1 c_2)^2 - 4c_1 c_2 \tilde{c}_1 \tilde{c}_2 + 4c_3 \tilde{c}_4 - 2c_1 \tilde{c}_2 c_3 \tilde{c}_4 \\
+ 2\tilde{c}_1 c_2 c_3 \tilde{c}_4 + 4\tilde{c}_3 c_4 + 2c_1 \tilde{c}_2 \tilde{c}_3 c_4 + 2\tilde{c}_1 c_2 \tilde{c}_3 c_4 - 2c_3 c_4.
\]

20
Again, the operator $\hat{Q}_i$ is a diagonal matrix with constant elements and the summation of $\hat{Q}_i$ in Eq. (3.22) is the direct summation.

So far, we have found out the $6N + 13$ relations (3.15), (3.16), (3.17), (3.18)-(3.22), which allow us to determine the eigenvalues of the transfer matrices $t(u), t^{(1)}(u)$ and $t^{(2)}(u)$.

### 3.4 Functional relations

From the graded Yang-Baxter relations (2.40), (3.13) and graded reflection equations (3.1) (3.2), one can prove that the transfer matrices $t(u), t^{(1)}(u)$ and $t^{(2)}(u)$ commutate with each other, namely,

$$[t(u), t^{(1)}(u)] = [t(u), t^{(2)}(u)] = [t^{(1)}(u), t^{(2)}(u)] = 0. \quad (3.23)$$

Therefore, they have common eigenstates and can be diagonalized simultaneously. Let $|\Phi\rangle$ be a common eigenstate. Acting the transfer matrices on this eigenstate, we have

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle,$$

$$t^{(1)}(u)|\Psi\rangle = \Lambda^{(1)}(u)|\Psi\rangle,$$

$$t^{(2)}(u)|\Psi\rangle = \Lambda^{(2)}(u)|\Psi\rangle,$$

where $\Lambda(u), \Lambda^{(1)}(u)$ and $\Lambda^{(2)}(u)$ are the eigenvalues of $t(u), t^{(1)}(u)$ and $t^{(2)}(u)$, respectively. It is easy to check that the eigenvalue $\Lambda(u)$ is a polynomial of $u$ with degree of $2N + 2$, and both $\Lambda^{(1)}(u)$ and $\Lambda^{(2)}(u)$ are the polynomials of $u$ with degree $2N + 4$. Thus $\Lambda(u), \Lambda^{(1)}(u)$ and $\Lambda^{(2)}(u)$ can be determined by $6N + 13$ independent conditions.

Acting the operator product identities (3.15), (3.16) and (3.17) on the state $|\Phi\rangle$, we obtain the functional relations among the eigenvalues

$$\Lambda(\pm \theta_j)\Lambda(\pm \theta_j + \eta) = -\frac{1}{4} \frac{(\pm \theta_j)(\pm \theta_j + \eta)}{(\pm \theta_j + \frac{1}{2} \eta)^2} \times \prod_{l=1}^{N} (\pm \theta_j - \theta_l + \eta)(\pm \theta_j + \theta_l + \eta)\Lambda^{(1)}(\pm \theta_j + \frac{1}{2} \eta),$$

$$\Lambda(\pm \theta_j)\Lambda(\pm \theta_j - \eta) = -\frac{1}{4} \frac{(\pm \theta_j)(\pm \theta_j - \eta)}{(\pm \theta_j - \frac{1}{2} \eta)^2} \times \prod_{l=1}^{N} (\pm \theta_j - \theta_l - \eta)(\pm \theta_j + \theta_l - \eta)\Lambda^{(2)}(\pm \theta_j - \frac{1}{2} \eta),$$

21
\[ \Lambda(\pm \theta_j - \eta) \Lambda^{(1)}(\pm \theta_j + \frac{1}{2} \eta) = \frac{(\pm \theta_j + \frac{1}{2} \eta)(\pm \theta_j - \etā)}{(\pm \theta_j - \eta)(\pm \theta_j + \frac{1}{2} \eta)^2} \times \prod_{i=1}^{N} \frac{(\pm \theta_j - \eta)(\pm \theta_j + \frac{1}{2} \eta)}{(\pm \theta_j - \theta_i + \eta)(\pm \theta_j + \theta_i + \frac{1}{2} \eta)} \Lambda(\pm \theta_j + \eta) \Lambda^{(2)}(\pm \theta_j - \frac{1}{2} \eta), \] (3.24)

where \( j = 1, 2, \ldots, N \). Acting Eqs. (3.18) and (3.19) on the state \(|\Phi\rangle\), we have

\[
\Lambda(0) = 0, \quad \Lambda^{(1)}(0) = 0, \quad \Lambda^{(2)}(0) = 0, \quad \Lambda^{(1)}(\frac{\eta}{2}) = -2\xi \bar{\xi} \Lambda(\eta), \\
\Lambda^{(1)}(-\frac{\eta}{2}) = -2\xi \bar{\xi} \Lambda(-\eta), \quad \Lambda^{(2)}(\frac{\eta}{2}) = 2\xi \bar{\xi} \Lambda(\eta), \quad \Lambda^{(2)}(-\frac{\eta}{2}) = 2\xi \bar{\xi} \Lambda(-\eta), \\
\frac{\partial \Lambda^{(1)}(u)}{\partial u}\bigg|_{u=0} + \frac{\partial \Lambda^{(2)}(u)}{\partial u}\bigg|_{u=0} = 0, \\
\Lambda(u)|_{u\to\infty} = -[c_1 \bar{c}_2 + \bar{c}_1 c_2 - c_3 \bar{c}_4 - \bar{c}_3 c_4]u^{2N+2}, \\
\Lambda^{(1)}(u)|_{u\to\infty} = -4\{2[c_3 \bar{c}_4 \bar{c}_3 \bar{c}_4 - \bar{c}_3 c_4 - c_3 \bar{c}_4 - 1] + (1 + c_1 \bar{c}_2)^2 + (1 + \bar{c}_1 c_2)^2 \\
- (c_1 \bar{c}_2 + \bar{c}_1 c_2)(c_3 \bar{c}_4 + \bar{c}_3 c_4)\}u^{2N+4}, \\
\Lambda^{(2)}(u)|_{u\to\infty} = -4\{2[c_1 \bar{c}_2 \bar{c}_1 c_2 - \bar{c}_1 c_2 - c_1 \bar{c}_2 - \bar{c}_1 c_2 - 1] + (1 + \bar{c}_3 c_4)^2 + (1 + \bar{c}_3 c_4)^2 \\
- (c_1 \bar{c}_2 + \bar{c}_1 c_2)(c_3 \bar{c}_4 + \bar{c}_3 c_4)\}u^{2N+4}. \] (3.25)

Because the operators \( \hat{U} \) given by (3.20) and \( \hat{Q} \) given by (3.22) can be expressed by the constant diagonal matrices, they commutate with each other and commutate with all the fused transfer matrices. Thus the state \(|\Phi\rangle\) also is the eigenvalues of \( \hat{U} \) and \( \hat{Q} \). After detailed calculation, the operator \( \hat{U} \) has \( N + 1 \) different eigenvalues

\[ N(2 + c_1 \bar{c}_2 + \bar{c}_1 c_2) + k(c_3 \bar{c}_4 + \bar{c}_3 c_4 - c_1 \bar{c}_2 - \bar{c}_1 c_2), \quad k = 0, 1, \ldots, N. \] (3.26)

Eq. (3.26) gives all the possible values of coefficients of the polynomial \( \Lambda(u) \) with the degree \( 2N + 1 \). Acting the operator \( \hat{U} \) on the state \(|\Phi\rangle\), one would obtain one of them. With direction calculation, we also know the operator \( \hat{Q} \) has \( N + 1 \) different eigenvalues

\[ N[2 - 2\bar{c}_1 \bar{c}_2 + 4c_1 \bar{c}_2 + (c_1 \bar{c}_2)^2 + 4\bar{c}_1 c_2 - 2c_1 c_2 + (\bar{c}_1 c_2)^2] \\
+ k[2(c_1 \bar{c}_2 + \bar{c}_1 c_2)(c_3 \bar{c}_4 + \bar{c}_3 c_4) - 2(c_1 \bar{c}_2 + \bar{c}_1 c_2)^2 \\
+ 4(c_3 \bar{c}_4 + \bar{c}_3 c_4 - c_1 \bar{c}_2 - \bar{c}_1 c_2)], \quad k = 0, 1, \ldots N. \] (3.27)

Eq. (3.27) indeed gives all the possible values of coefficients of polynomial \( \Lambda^{(1)}(u) \) with the degree \( 2N + 3 \). The operator \( \hat{Q} \) acting on the state \(|\Phi\rangle\) gives one of them. Then we arrive
at that the above $6N + 13$ relations (3.24)-(3.27) enable us to completely determine the eigenvalues $\Lambda(u)$, $\Lambda^{(1)}(u)$ and $\Lambda^{(2)}(u)$ which are expressed as the inhomogeneous $T - Q$ relations in the next subsection.

3.5 Inhomogeneous $T - Q$ relations

For simplicity, we define $z^{(l)}(u)$, $x_1(u)$ and $x_2(u)$ functions

$$z^{(l)}(u) = \begin{cases} (-1)^{l} \alpha_l(u) Q^{(0)}(u) K^{(l)}(u) \frac{Q^{(l-1)}(u + \eta)Q^{(0)}(u - \eta)}{Q^{(l)}(u)Q^{(l-1)}(u)}, & l = 1, 2, \\ (-1)^{l} \alpha_l(u) Q^{(0)}(u) K^{(l)}(u) \frac{Q^{(l-1)}(u - \eta)Q^{(0)}(u + \eta)}{Q^{(l)}(u)Q^{(l-1)}(u)}, & l = 3, 4, \end{cases}$$

$$x_1(u) = u^2 Q^{(0)}(u + \eta)Q^{(0)}(u)\frac{f^{(1)}(u)Q^{(2)}(-u - \eta)}{Q^{(1)}(u)},$$

$$x_2(u) = u^2 Q^{(0)}(u + \eta)Q^{(0)}(u)Q^{(0)}(-u)\frac{f^{(2)}(u)Q^{(2)}(-u - \eta)}{Q^{(3)}(u)}.$$  

Here the structure factor $\alpha_l(u)$ is defined as

$$\alpha_l(u) = \begin{cases} \frac{u}{u + \frac{1}{2} \eta}, & l = 1, 4, \\ \frac{u^2}{(u + \frac{1}{2} \eta)(u + \eta)}, & l = 2, 3. \end{cases}$$

The $Q$-functions are

$$Q^{(0)}(u) = \prod_{l=1}^{N} (u - \theta_l)(u + \theta_l), \quad Q^{(m)}(u) = \prod_{j=1}^{L_m} (u - \lambda_j^{(m)})(u + \lambda_j^{(m)} + m\eta), \quad m = 1, 2,$$

$$Q^{(3)}(u) = \prod_{j=1}^{L_3} (u - \lambda_j^{(m)})(u + \lambda_j^{(m)} + \eta), \quad Q^{(4)}(u) = 1,$$  

(3.28)

where $L_1$, $L_2$ and $L_3$ are the non-negative integers which describe the numbers of Bethe roots $\lambda_j^{(1)}$, $\lambda_j^{(2)}$ and $\lambda_j^{(3)}$, respectively. The forms of functions $K^{(l)}(u)$ are related with the boundary reflections and given by

$$K^{(1)}(u) = (\xi + \sqrt{1 + c_1 c_2 u})(\tilde{\xi} + \sqrt{1 + \tilde{c}_1 \tilde{c}_2 u}),$$

$$K^{(2)}(u) = (\xi - \sqrt{1 + c_1 c_2 (u + \eta)})(\tilde{\xi} - \sqrt{1 + \tilde{c}_1 \tilde{c}_2 (u + \eta)}),$$

$$K^{(3)}(u) = (\xi + \sqrt{1 + c_1 c_2 (u + \eta)})(\tilde{\xi} + \sqrt{1 + \tilde{c}_1 \tilde{c}_2 (u + \eta)}),$$

$$K^{(4)}(u) = (\xi - \sqrt{1 + c_1 c_2 u})(\tilde{\xi} - \sqrt{1 + \tilde{c}_1 \tilde{c}_2 u}).$$  

(3.29)
The polynomials \( f^{(l)}(u) \) in the inhomogeneous terms \( x_1(u) \) and \( x_2(u) \) are
\[
f^{(l)}(u) = g_l u(u + \eta)(u - \eta)(u + \frac{1}{2}\eta)^2(u + \frac{3}{2}\eta)(u - \frac{1}{2}\eta)(u + 2\eta), \quad l = 1, 2, \tag{3.30}
\]
where \( g_l \) are given by
\[
g_1 = -2 - c_1 c_2 - c_1 \tilde{c}_2 - 2 \sqrt{(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)},
\]
\[
g_2 = 2 + c_3 \tilde{c}_4 + \tilde{c}_3 c_4 + 2 \sqrt{(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)}. \tag{3.31}
\]

By using the above functions and based on Eqs. \((3.24)-(3.27)\), we construct the eigenvalues \( \Lambda(u), \Lambda^{(1)}(u) \) and \( \Lambda^{(2)}(u) \) as following inhomogeneous \( T - Q \) relations
\[
\Lambda(u) = \sum_{l=1}^{4} z^{(l)}(u) + x_1(u) + x_2(u),
\]
\[
\Lambda^{(1)}(u) = -4u^2[Q^{(0)}(u + \frac{1}{2}\eta)(u + \frac{1}{2}\eta)(u - \frac{1}{2}\eta)]^{-1}\left\{\sum_{l=1}^{4} \sum_{m=1}^{2} \tilde{z}^{(l)}(u + \frac{1}{2}\eta)z^{(m)}(u - \frac{1}{2}\eta)ight\},
\]
\[
\Lambda^{(2)}(u) = -4u^2[Q^{(0)}(u - \frac{1}{2}\eta)(u + \frac{1}{2}\eta)(u - \frac{1}{2}\eta)]^{-1}\left\{\sum_{l=1}^{4} \sum_{m=3}^{4} \tilde{z}^{(l)}(u + \frac{1}{2}\eta)z^{(m)}(u - \frac{1}{2}\eta)ight\}, \tag{3.32}
\]
where
\[
\tilde{z}^{(1)}(u) = z^{(1)}(u) + x_1(u), \quad \tilde{z}^{(2)}(u) = z^{(2)}(u), \quad \tilde{z}^{(3)}(u) = z^{(3)}(u), \quad \tilde{z}^{(4)}(u) = z^{(4)}(u) + x_2(u).
\]

Since all the eigenvalues are the polynomials, the residues of Eq.\((3.32)\) at the apparent poles should be zero, which gives the Bethe ansatz equations
\[
1 + \lambda^{(l)}_1 \frac{K^{(2)}(\lambda^{(1)}_1)Q^{(0)}(\lambda^{(1)}_1)}{K^{(1)}(\lambda^{(1)}_1)Q^{(1)}(\lambda^{(1)}_1)} \frac{Q^{(1)}(\lambda^{(1)}_1 + \eta)Q^{(2)}(\lambda^{(1)}_1 - \eta)}{Q^{(1)}(\lambda^{(1)}_1 - \eta)Q^{(2)}(\lambda^{(1)}_1)} = -\frac{\lambda^{(1)}_1 (\lambda^{(1)}_1 + \frac{1}{2}\eta) f^{(l)}(\lambda^{(1)}_1) Q^{(0)}(\lambda^{(1)}_1) Q^{(2)}(-\lambda^{(1)}_1 - \eta)}{K^{(1)}(\lambda^{(1)}_1) Q^{(1)}(\lambda^{(1)}_1 - \eta)}, \quad l = 1, \cdots, L_1,
\]
\[
\frac{K^{(3)}(\lambda^{(2)}_1) Q^{(3)}(\lambda^{(2)}_1 + \eta)}{K^{(2)}(\lambda^{(2)}_1) Q^{(3)}(\lambda^{(2)}_1)} = \frac{Q^{(1)}(\lambda^{(2)}_1 + \eta)}{Q^{(1)}(\lambda^{(2)}_1)}), \quad l = 1, \cdots, L_2,
\]

24
\[
\frac{\lambda_i^{(3)}(\lambda_i^{(3)} + \frac{1}{2}\eta)Q^{(0)}(\lambda_i^{(3)} + \eta)Q^{(0)}(-\lambda_i^{(3)})f^{(2)}(\lambda_i^{(3)})Q^{(2)}(-\lambda_i^{(3)} - \eta)}{K^{(4)}(\lambda_i^{(3)})Q^{(3)}(\lambda_i^{(3)} - \eta)}
= 1 + \frac{\lambda_i^{(3)}K^{(3)}(\lambda_i^{(3)})Q^{(2)}(\lambda_i^{(3)} - \eta)Q^{(3)}(\lambda_i^{(3)} + \eta)}{\lambda_i^{(3)} + \eta K^{(4)}(\lambda_i^{(3)})Q^{(2)}(\lambda_i^{(3)})Q^{(3)}(\lambda_i^{(3)} - \eta)}, \quad l = 1, \cdots, L_3. \tag{3.33}
\]

From the analysis of asymptotic behaviors and contributions of second higher order of corresponding polynomials, the numbers of Bethe roots should satisfy

\[
L_1 = L_2 + N + 4, \quad L_3 = 2N + L_2 + 4, \quad L_2 = k, \quad k = 0, 1, \cdots, N. \tag{3.34}
\]

Some remarks are in order. The coefficient of term with \(u^{2N+1}\) in the polynomial \(\Lambda(u)\) and that of term with \(u^{2N+3}\) in the polynomial \(\Lambda^{(1)}(u)\) are not related with Bethe roots. The constraints (3.26) and (3.27) require \(L_2 = k\), where \(k = 0, \cdots, N\) is related to the eigenvalues of the operators \(\hat{U}\) and \(\hat{Q}\). Then the Bethe ansatz equations (3.33) can describe all the eigenstates of the system. The second set of Bethe ansatz equations in Eq. (3.33) are the homogeneous ones. This is because that the reflection matrices \(K^{(\pm)}(u)\) are the blocking ones. The matrix elements involving both bosonic (where the parity is 0) and fermionic (where the parity is 1) bases are zero. The integrability of the system requires that the reflection processes from bosonic basis to fermionic one and vice versa are forbidden. We note that the Bethe ansatz equations obtained from the regularity of \(\Lambda(u)\) are the same as those obtained from the regularities of \(\Lambda^{(1)}(u)\) and \(\Lambda^{(2)}(u)\). Meanwhile, the functions \(Q^{(m)}(u)\) has two zero points, which should give the same Bethe ansatz equations.

We have checked that the inhomogeneous \(T - Q\) relations (3.32) satisfy the above mentioned \(6N + 13\) conditions (3.24)-(3.27). Therefore, \(\Lambda(u), \Lambda^{(1)}(u)\) and \(\Lambda^{(2)}(u)\) are the eigenvalues of transfer matrices \(t(u), t^{(1)}(u)\) and \(t^{(2)}(u)\), respectively. Finally, the eigenvalues of Hamiltonian (3.8) are obtained from \(\Lambda(u)\) as

\[
E = \frac{\partial \ln \Lambda(u)}{\partial u} \bigg|_{u=0,\{\theta_i\}=0}. \tag{3.35}
\]

4 Conclusion

In this paper, we develop a graded nested off-diagonal Bethe ansatz method and study the exact solutions of the supersymmetric \(SU(2|2)\) model with both periodic and off-diagonal boundary conditions. After generalizing fusion to the supersymmetric case, we obtain the closed sets of operator product identities. For the periodic case, the eigenvalues are given in
terms of the homogeneous $T - Q$ relations \((2.54)\). While for the open case, the eigenvalues are given by the inhomogeneous $T - Q$ relations \((3.32)\). This scheme can be generalized to other high rank supersymmetric quantum integrable models.

**Acknowledgments**

The financial supports from the National Program for Basic Research of MOST (Grant Nos. 2016YFA0300600 and 2016YFA0302104), the National Natural Science Foundation of China (Grant Nos. 11934015, 11975183, 11947301, 11774397, 11775178 and 11775177), the Major Basic Research Program of Natural Science of Shaanxi Province (Grant Nos. 2017KCT-12, 2017ZDJC-32), Australian Research Council (Grant No. DP 190101529), the Strategic Priority Research Program of the Chinese Academy of Sciences (Grant No. XDB33000000), the National Postdoctoral Program for Innovative Talents (BX20180350) and the Double First-Class University Construction Project of Northwest University are gratefully acknowledged.

**Appendix A: Fusion of the reflection matrices**

The general fusion procedure of the reflection matrices was given \([51, 52]\). We will generalize the method developed in \([24]\) to study the fusion of the reflections matrices for supersymmetric models (taking the $SU(2|2)$ model as an example). The (graded) reflection equation at special point gives

$$ R_{12}(-\alpha)K_1(u - \alpha)R_{21}(2u - \alpha)K_2^-(u) = K_2^-(u)R_{12}(2u - \alpha)K_1^-(u - \alpha)R_{21}(-\alpha), \quad (A.1) $$

where $R_{12}(-\alpha) = P^{(d)}_{12} S_{12}$ as we defined perviously. Multiplying Eq. \((A.1)\) with the projector $P^{(d)}_{12}$ from left and using the property $P^{(d)}_{12} R_{12}(-\alpha) = R_{12}(-\alpha)$, we have

$$ R_{12}(-\alpha)K_1(u - \alpha)R_{21}(2u - \alpha)K_2^-(u) = P^{(d)}_{12} K_2^-(u)R_{12}(2u - \alpha)K_1^-(u - \alpha)R_{21}(-\alpha). \quad (A.2) $$

Comparing the right hand sides of Eqs. \((A.1)\) and \((A.2)\), we obtain

$$ P^{(d)}_{12} K_2^-(u)R_{12}(2u - \alpha)K_1^-(u - \alpha)P^{(d)}_{21} = K_2^-(u)R_{12}(2u - \alpha)K_1^-(u - \alpha)P^{(d)}_{21}. \quad (A.3) $$

Which give the general principle of fusion of the reflection matrices. If we define $P^{(d)}_{12} K_2^-(u) R_{12}(2u - \alpha)K_1^-(u - \alpha)P^{(d)}_{21}$ as the fused reflection matrix $K^\langle_{(12)}^\rangle(u) \equiv K_1^-(u)$, where the integrability requires that the inserted $R$-matrix with determined spectral parameter is necessary,
we can prove the the fused $K$-matrix $K_1^-(u)$ also satisfies the (graded) reflection equation

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = P_{00'}^{(d)}R_{00'}(u-v)R_{02}(u-v-\alpha)P_{00'}^{(d)}$$

$$\times P_{00'}^{(d)}K_0^-(u)R_{00'}(2u-\alpha)K_0^-(u-\alpha)P_{00'}^{(d)}P_{00'}^{(d)}R_{02}(u+v)R_{20}(u+v-\alpha)P_{00'}^{(d)}$$

$$= P_{00'}^{(d)}R_{00'}(u-v)R_{02}(u-v-\alpha)K_0^-(u)R_{00'}(2u-\alpha)K_2^-(v)$$

$$\times K_0^-(u-\alpha)R_{20}(u+v)R_{20}(u+v-\alpha)K_2^-(v)P_{00'}^{(d)}$$

$$= P_{00'}^{(d)}R_{00'}(u-v)K_0^-(u)R_{02}(u-v-\alpha)R_{00'}(2u-\alpha)R_{20}(u+v)$$

$$\times K_0^-(u-\alpha)R_{20}(u+v-\alpha)K_2^-(v)P_{00'}^{(d)}$$

$$= P_{00'}^{(d)}R_{00'}(u-v)K_0^-(u)R_{02}(u-v-\alpha)R_{00'}(2u-\alpha)K_0^-(u)R_{20}(u+v-\alpha)P_{00'}^{(d)}$$

$$\times K_2^-(v)R_{02}(u+v-\alpha)K_0^-(u-\alpha)R_{20}(u-v-\alpha)P_{00'}^{(d)}$$

$$= P_{00'}^{(d)}R_{00'}(u-v)K_0^-(u)R_{02}(u+v-\alpha)K_2^-(v)$$

$$\times R_{00'}(2u-\alpha)R_{02}(u+v-\alpha)K_0^-(u-\alpha)R_{20}(u-v-\alpha)P_{00'}^{(d)}$$

$$= K_2^-(v)P_{00'}^{(d)}R_{02}(u+v-\alpha)K_0^-(u)R_{02}(u+v-\alpha)R_{00'}(2u-\alpha)R_{20}(u-v-\alpha)$$

$$\times K_0^-(u-\alpha)R_{20}(u+v-\alpha)P_{00'}^{(d)}$$

$$= K_2^-(v)P_{00'}^{(d)}R_{02}(u+v-\alpha)R_{02}(u+v-\alpha)K_0^-(u)R_{00'}(2u-\alpha)K_0^-(u-\alpha)$$

$$\times R_{20}(u+v-\alpha)R_{20}(u-v-\alpha)P_{00'}^{(d)}$$

$$= K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v). \quad (A.4)$$

In the derivation, we have used the relation

$$P_{21}^{(d)}R_{32}(u)R_{31}(u-\alpha)P_{21}^{(d)} = R_{32}(u)R_{31}(u-\alpha)P_{21}^{(d)} = R_{31}(u). \quad (A.5)$$

From the dual reflection equation (3.2), we obtain the general construction principle of fused dual reflection matrices

$$P_{12}^{(d)}K_2^+(u)R_{12}(-2u-\alpha)K_1^+(u+\alpha)P_{21}^{(d)} = K_2^+(u)R_{12}(-2u-\alpha)K_1^+(u+\alpha)P_{21}^{(d)}. \quad (A.6)$$

If $R_{12}(-\beta) = S_{12}P_{12}^{(d)}$, the corresponding fusion relations are

$$P_{12}^{(d)}K_1^-(u-\beta)R_{21}(2u-\beta)K_2^-(u)P_{21}^{(d)} = P_{12}^{(d)}K_1^-(u-\beta)R_{21}(2u-\beta)K_2^-(u), \quad (A.7)$$
We introduce the reflection monodromy matrices

\[ P_{12}^{(d)} K_1^+(u + \beta) R_{21}(-2u - \beta) K_2^+(u) P_{21}^{(d)} \]

\[ = P_{12}^{(d)} K_1^+(u + \beta) R_{21}(-2u - \beta) K_2^+(u). \]  \quad (A.8)

Finally the fused K-matrices in subsection 3.2 can be carried out according to Eqs. (A.3)-(A.6) or (A.7)-(A.8).

**Appendix B: Proof of the operator product identities**

We introduce the reflection monodromy matrices

\[ \hat{T}_{\theta}^0(u) = R_{N\theta}(u + \theta_N) \cdots R_{2}(u + \theta_2) R_{10}(u + \theta_1), \]

\[ \hat{T}_{N\theta}(u) = R_{N\theta}(u + \theta_N) \cdots R_{2}(u + \theta_2) R_{10}(u + \theta_1), \]  \quad (B.1)

which satisfy the graded Yang-Baxter equations

\[ R_{1\hat{2}}(u - v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{1\hat{2}}(u - v), \]

\[ R_{1\hat{2}}(u - v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{1\hat{2}}(u - v). \]  \quad (B.2)

In order to solve the transfer matrix \( t(u) \) \((3.7)\), we still need the fused transfer matrices which are defined as

\[ \bar{\ell}^{(1)}(u) = str_0\{K_0^+(u)T_0^v(u)K_0^-(u)\hat{T}_0^v(u)\}, \]

\[ \bar{\ell}^{(2)}(u) = str_0\{K^+_0(u)T_0^v(u)K^-_0(u)\hat{T}_0^v(u)\}. \]  \quad (B.3)

Similar with periodic case, from the property that above R-matrices can degenerate into the projectors and using the definitions \((3.12)\) and \((B.1)\), we obtain following fusion relations among the reflecting monodromy matrices

\[ P_{21}^{(8)}\hat{T}_2(u)\hat{T}_1(u + \eta)P_{21}^{(8)} = \prod_{l=1}^{N}(u + \theta_l + \eta)\hat{T}_1(u + \frac{1}{2}\eta), \]

\[ \bar{P}_{21}^{(8)}\hat{T}_2(u)\hat{T}_1(u - \eta)\bar{P}_{21}^{(8)} = \prod_{l=1}^{N}(u + \theta_l - \eta)\hat{T}_1(u - \frac{1}{2}\eta), \]

\[ P_{12}^{(20)}\hat{T}_1(u + \frac{1}{2}\eta)\hat{T}_2(u - \eta)P_{12}^{(20)} = \prod_{l=1}^{N}(u + \theta_l - \eta)\hat{T}_1(u), \]

\[ P_{V2}^{(20)}\hat{T}_V(u - \frac{1}{2}\eta)\hat{T}_2(u + \eta)P_{V2}^{(20)} = \prod_{l=1}^{N}(u + \theta_l + \eta)\hat{T}_V(u). \]  \quad (B.4)
From the definitions, we see that the auxiliary spaces are erased by taking the super partial traces and the physical spaces are the same. We remark that these transfer matrices are not independent. Substituting Eqs. (2.36) and (3.11) into the definitions (B.3), we obtain that the fused transfer matrices \( \tilde{t}^{(1)}(u) \) and \( \tilde{t}^{(2)}(u) \) are equal

\[
\tilde{t}^{(1)}(u) = \tilde{t}^{(2)}(u). \quad \text{(B.5)}
\]

Consider the quantity

\[
t(u)t(u + \eta) = \text{str}_{12}\{K^+_1(u)T_1(u)K^-_1(u)\hat{T}_1(u)
\times[T_2(u + \eta)K^-_2(u + \eta)\hat{T}_2(u + \eta)]^{stz}[K^+_2(u + \eta)]^{stz}\}
= [\rho_2(2u + \eta)]^{-1}\text{str}_{12}\{K^+_1(u)T_1(u)K^-_1(u)\hat{T}_1(u)
\times[T_2(u + \eta)K^-_2(u + \eta)\hat{T}_2(u + \eta)]^{stz}\text{R}^{stz}_{21}(2u + \eta)\text{R}^{stz}_{12}(-2u - \eta)[K^+_2(u + \eta)]^{stz}\}
= [\rho_2(2u + \eta)]^{-1}\text{str}_{12}\{K^+_2(u + \eta)R_{12}(-2u - \eta)K^+_1(u)T_1(u)T_2(u + \eta)
\times K^-_1(u)R_{21}(2u + \eta)K^-_2(u + \eta)\hat{T}_1(u)\hat{T}_2(u + \eta)\}
= [\rho_2(2u + \eta)]^{-1}\text{str}_{12}\{[P^{(8)}_{12} + P^{(8)}_{21}]K^+_2(u + \eta)R_{12}(-2u - \eta)K^+_1(u)\}
\times(P^{(8)}_{21} + \tilde{P}^{(8)}_{12})T_1(u)T_2(u + \eta)(P^{(8)}_{21} + \tilde{P}^{(8)}_{12})K^-_1(u)
\times R_{21}(2u + \eta)K^-_2(u + \eta)(P^{(8)}_{12} + \tilde{P}^{(8)}_{21})\hat{T}_1(u)\hat{T}_2(u + \eta)(P^{(8)}_{12} + \tilde{P}^{(8)}_{21})\}
= [\rho_2(2u + \eta)]^{-1}\text{str}_{12}\{[P^{(8)}_{21}K^+_1(u)R_{21}(2u + \eta)K^-_2(u + \eta)P^{(8)}_{12}][P^{(8)}_{12}\hat{T}_1(u)\hat{T}_2(u + \eta)P^{(8)}_{12}]\}
+ [\rho_2(2u + \eta)]^{-1}\text{str}_{12}\{[\tilde{P}^{(8)}_{21}K^+_1(u)R_{21}(2u + \eta)K^-_2(u + \eta)\tilde{P}^{(8)}_{12}][\tilde{P}^{(8)}_{21}\hat{T}_1(u)\hat{T}_2(u + \eta)\tilde{P}^{(8)}_{21}]\}
= t_1(u) + t_2(u). \quad \text{(B.6)}
\]

The first term is the fusion by the 8-dimensional projectors and the result is

\[
t_1(u) = [\rho_2(2u + \eta)]^{-1}(u + \eta)(u)\prod_{j=1}^{N}(u - \theta_j + \eta)(u + \theta_j + \eta)
\]

29
\[
\times \text{str}_{(12)} \left\{ K^+_{(12)} (u + \frac{1}{2} \eta) T_{(12)}^{(8)} (u + \frac{1}{2} \eta) K^-_{(12)} (u + \frac{1}{2} \eta) \hat{T}^{(8)}_{(12)} (u + \frac{1}{2} \eta) \right\}
\]

\[
= [\rho_2 (2u + \eta)]^{-1} (u + \eta) N \prod_{j=1}^{N} (u - \theta_j + \eta)(u + \theta_j + \eta) t^{(1)} (u + \frac{1}{2} \eta). 
\]

(B.7)

The second term is the fusion by the other 8-dimensional projectors. Detailed calculation gives

\[
t_2 (u) = [\rho_2 (2u + \eta)]^{-1} \text{str}_{12} \{ \hat{P}^{(8)}_{21} [\hat{P}^{(8)}_{21} K^+_{2} (u + \eta) R_{12} (-2u - \eta) K^+_{1} (u)] \hat{P}^{(8)}_{12}
\]

\[
\times \hat{P}^{(8)}_{12} [\hat{P}^{(8)}_{12} T_1 (u) T_2 (u + \eta)] \hat{P}^{(8)}_{12}
\]

\[
\times \hat{P}^{(8)}_{12} [\hat{P}^{(8)}_{12} K^-_{2} (u) R_{21} (2u + \eta) K^-_{1} (u)] \hat{P}^{(8)}_{21}
\]

\[
\times \hat{P}^{(8)}_{21} [\hat{P}^{(8)}_{21} \hat{T}_1 (u) \hat{T}_2 (u + \eta)] \hat{P}^{(8)}_{21} \}
\]

\[
= [\rho_2 (2u + \eta)]^{-1} \text{str}_{12} \{ [\hat{P}^{(8)}_{21} K^+_{1} (u) R_{21} (-2u - \eta) K^+_{2} (u + \eta)] \hat{P}^{(8)}_{12}
\]

\[
\times [\hat{P}^{(8)}_{12} T_2 (u + \eta) T_1 (u) \hat{P}^{(8)}_{12}]
\]

\[
\times [\hat{P}^{(8)}_{12} K^-_{2} (u + \eta) R_{12} (2u + \eta) K^-_{1} (u)] \hat{P}^{(8)}_{21}
\]

\[
\times [\hat{P}^{(8)}_{21} \hat{T}_2 (u + \eta) \hat{T}_1 (u) \hat{P}^{(8)}_{21}]
\}
\]

\[
= [\rho_2 (2u + \eta)]^{-1} (u + \eta) N \prod_{j=1}^{N} (u - \theta_j + \eta)(u + \theta_j + \eta) t^{(2)} (u + \frac{1}{2} \eta). 
\]

(B.8)

In the derivation, we have used the relations

\[
\text{str}_{12} \{ A_{12}^{st_1} B_{12}^{st_1} \} = \text{str}_{12} \{ A_{12}^{st_2} B_{12}^{st_2} \} = \text{str}_{12} \{ A_{12} B_{12} \},
\]

\[
\hat{T}_1 (u) R_{21} (2u + \eta) T_2 (u + \eta) = T_2 (u + \eta) R_{21} (2u + \eta) \hat{T}_1 (u),
\]
In addition,

\[
P_{12}^{(8)} + \tilde{P}_{12}^{(8)} = 1, \quad P_{21}^{(8)} + \tilde{P}_{21}^{(8)} = 1, \quad P_{12}^{(8)}\tilde{P}_{12}^{(8)} = P_{21}^{(8)}\tilde{P}_{21}^{(8)} = 0, \quad P_1^{(8)} = \tilde{P}_1^{(8)}, \quad \tilde{P}_2^{(8)} = \tilde{P}_2^{(8)}.
\]

\[
t^{(1)}(u + \frac{1}{2}\eta)t(u - \eta) = str_{12}\{K_1^+(u + \frac{1}{2}\eta)T_1(u + \frac{1}{2}\eta)K_1^-(u + \frac{1}{2}\eta)\hat{T}_1(u + \frac{1}{2}\eta)
\]
\[
\times[T_2(u - \eta)K_2^-(u - \eta)\hat{T}_2(u - \eta)]^{stz}[K_2^+(u - \eta)]^{stz}\}
\]
\[
= \rho_4^{-1}(2u - \frac{1}{2}\eta)str_{12}\{K_1^+(u + \frac{1}{2}\eta)T_1(u + \frac{1}{2}\eta)K_1^-(u + \frac{1}{2}\eta)\hat{T}_1(u + \frac{1}{2}\eta)
\]
\[
\times[T_2(u - \eta)K_2^-(u - \eta)\hat{T}_2(u - \eta)]^{stz}[R_{21}(2u - \frac{1}{2}\eta)]^{stz}
\]
\[
\times[R_{12}(-2u + \frac{1}{2}\eta)]^{stz}[K_2^+(u - \eta)]^{stz}\}
\]
\[
= \rho_4^{-1}(2u - \frac{1}{2}\eta)str_{12}\{K_2^+(u - \eta)R_{12}(-2u + \frac{1}{2}\eta)K_1^+(u + \frac{1}{2}\eta)T_1(u + \frac{1}{2}\eta)
\]
\[
\times[T_2(u - \eta)K_1^-(u + \frac{1}{2}\eta)R_{21}(2u - \frac{1}{2}\eta)K_2^+(u - \eta)\hat{T}_2(u - \eta)]^{stz}[R_{21}(2u - \frac{1}{2}\eta)]^{stz}
\]
\[
\times[12(-2u + \frac{1}{2}\eta)]^{stz}[K_2^+(u - \eta)]^{stz}\}
\]
\[
= \rho_4^{-1}(2u - \frac{1}{2}\eta)str_{12}\{(P_{12}^{(20)} + \tilde{P}_{12}^{(12)})K_2^+(u - \eta)R_{12}(-2u + \frac{1}{2}\eta)K_1^+(u + \frac{1}{2}\eta)
\]
\[
\times[P_{21}^{(20)} + \tilde{P}_{21}^{(12)}]T_1(u + \frac{1}{2}\eta)T_2(u - \eta)(P_{21}^{(20)} + \tilde{P}_{21}^{(12)}]
\]
\[
\times[K_1^-(u + \frac{1}{2}\eta)R_{21}(2u - \frac{1}{2}\eta)K_2^+(u - \eta)]^{stz}[P_{12}^{(20)} + \tilde{P}_{12}^{(12)}]
\]
\[
\times[\hat{T}_1(u + \frac{1}{2}\eta)\hat{T}_2(u - \eta)(P_{12}^{(20)} + \tilde{P}_{12}^{(12)}])
\]
\[
= \rho_4^{-1}(2u - \frac{1}{2}\eta)str_{12}\{(P_{12}^{(20)}K_2^+(u - \eta)R_{12}(-2u + \frac{1}{2}\eta)K_1^+(u + \frac{1}{2}\eta)P_{21}^{(20)}
\]
\[
\times[T_1(u + \frac{1}{2}\eta)T_2(u - \eta)P_{21}^{(20)}K_1^-(u + \frac{1}{2}\eta)R_{21}(2u - \frac{1}{2}\eta)K_2^-(u - \eta)
\]
\[
\times[P_{12}^{(20)}\hat{T}_1(u + \frac{1}{2}\eta)\hat{T}_2(u - \eta)P_{12}^{(20)}]
\]
\[
+ \rho_4^{-1}(2u - \frac{1}{2}\eta)str_{12}\{\tilde{P}_{12}^{(20)}K_2^+(u - \eta)R_{12}(-2u + \frac{1}{2}\eta)K_1^+(u + \frac{1}{2}\eta)\tilde{P}_{21}^{(12)}
\]
\[
\times[T_1(u + \frac{1}{2}\eta)T_2(u - \eta)\tilde{P}_{21}^{(12)}K_1^-(u + \frac{1}{2}\eta)R_{21}(2u - \frac{1}{2}\eta)K_2^-(u - \eta)
\]
\[
\times[\tilde{P}_{12}^{(12)}\hat{T}_1(u + \frac{1}{2}\eta)\hat{T}_2(u - \eta)\tilde{P}_{12}^{(12)}]
\]
\[
= \rho_4^{-1}(2u - \frac{1}{2}\eta)(2u + \eta)(u - \eta)\prod_{j=1}^{N}(u - \theta_j - \eta)(u + \theta_j - \eta)
\]
\[
\times str_{12}\{K_{(12)}^+(u)T_{(12)}(u)K_{(12)}^-(u)\hat{T}_{(12)}(u)\}
\]
\begin{align*}
&+ \rho_4^{-1}(2u - \frac{1}{2}\eta)(2u + \eta)(u - \eta) \prod_{j=1}^{N}(u - \theta_j)(u + \theta_j) \\
&\times \text{str}_{(12)} \{K_{(12)}^{+}(u)T_{(12)}(u)K_{(12)}^{-}(u)\hat{T}_{(12)}(u)\} \\
&= \rho_4^{-1}(2u - \frac{1}{2}\eta)(2u + \eta)(u - \eta) \prod_{j=1}^{N}(u - \theta_j - \eta)(u + \theta_j - \eta)\tilde{t}^{(1)}(u) \\
&+ \rho_4^{-1}(2u - \frac{1}{2}\eta)(2u + \eta)(u - \eta) \prod_{j=1}^{N}(u - \theta_j)(u + \theta_j)\tilde{t}^{(1)}(u),
\end{align*}

\begin{align*}
\tilde{t}^{(2)}(u - \frac{1}{2}\eta)(u + \eta) &= \rho_6^{-1}(2u + \frac{1}{2}\eta) \text{str}_{1\{2\}} \{ \frac{1}{2}\eta) + \eta) \prod_{j=1}^{N}(u - \theta_j + \eta)(u + \theta_j + \eta) \\
&\times \{K_{1\{2\}}^{+}(u + \eta)R_{1\{2\}}(-2u - \frac{1}{2}\eta) \\
&\times K_{1\{2\}}^{+}(u - \frac{1}{2}\eta)T_{1\{2\}}(u - \frac{1}{2}\eta)T_{2}(u + \eta)K_{1\{2\}}^{-}(u - \frac{1}{2}\eta) \\
&\times R_{2\{1\}}(2u + \frac{1}{2}\eta)K_{2\{1\}}^{-}(u + \eta)T_{1\{2\}}(u - \frac{1}{2}\eta)\hat{T}_{2}(u + \eta)\}
\end{align*}

\begin{align*}
&= \rho_6^{-1}(2u + \frac{1}{2}\eta)(2u - \eta)(u + \eta) \prod_{j=1}^{N}(u - \theta_j + \eta)(u + \theta_j + \eta) \\
&\times \text{str}_{1\{2\}} \{K_{1\{2\}}^{+}(u)T_{1\{2\}}(u)K_{1\{2\}}^{-}(u)\hat{T}_{1\{2\}}(u)\} \\
&\times \{K_{1\{2\}}^{+}(u + \eta)R_{1\{2\}}(-2u - \frac{1}{2}\eta) \\
&\times K_{1\{2\}}^{+}(u - \frac{1}{2}\eta)T_{1\{2\}}(u - \frac{1}{2}\eta)T_{2}(u + \eta)K_{1\{2\}}^{-}(u - \frac{1}{2}\eta) \\
&\times R_{2\{1\}}(2u + \frac{1}{2}\eta)K_{2\{1\}}^{-}(u + \eta)T_{1\{2\}}(u - \frac{1}{2}\eta)\hat{T}_{2}(u + \eta)\}
\end{align*}

\begin{align*}
&= \rho_6^{-1}(2u + \frac{1}{2}\eta)(2u - \eta)(u + \eta) \prod_{j=1}^{N}(u - \theta_j + \eta)(u + \theta_j + \eta)\tilde{t}^{(2)}(u) \\
&+ \rho_6^{-1}(2u + \frac{1}{2}\eta)(2u - \eta)(u + \eta) \prod_{j=1}^{N}(u - \theta_j)(u + \theta_j)\tilde{t}^{(2)}(u),
\end{align*}

(B.9)
where we have used the relations

\[
\hat{T}_1(u + \frac{1}{2}\eta)R_{21}(2u - \frac{1}{2}\eta)T_2(u - \eta) = T_2(u - \eta)R_{21}(2u - \frac{1}{2}\eta)\hat{T}_1(u + \frac{1}{2}\eta),
\]

\[
P_{12}^{(20)} + \tilde{P}_{12}^{(12)} = 1, \quad P_{21}^{(20)} + \tilde{P}_{21}^{(12)} = 1, \quad P_{12}^{(20)}\tilde{P}_{12}^{(12)} = 0, \quad P_{21}^{(20)}\tilde{P}_{21}^{(12)} = 0,
\]

\[
\hat{T}_1'(u - \frac{1}{2}\eta)R_{21}'(2u + \frac{1}{2}\eta)T_2'(u + \eta) = T_2'(u + \eta)R_{21}'(2u + \frac{1}{2}\eta)\hat{T}_1'(u - \frac{1}{2}\eta),
\]

\[
P_{12}^{(20)} + \tilde{P}_{12}^{(20)} = 1, \quad P_{21}^{(20)} + \tilde{P}_{21}^{(20)} = 1, \quad P_{12}^{(20)}\tilde{P}_{12}^{(20)} = 0, \quad P_{21}^{(20)}\tilde{P}_{21}^{(20)} = 0.
\]

Focusing on the special points introduced in the main text, we have

\[
t(\pm \theta_j - \eta)\tilde{t}^{(1)}(\pm \theta_j + \frac{1}{2}\eta) = -\frac{1}{2} \frac{(\pm \theta_j + \frac{1}{2}\eta)(\pm \theta_j - \eta)}{(\pm \theta_j)(\pm \theta_j - \frac{1}{2}\eta)}\prod_{l=1}^{N}(\pm \theta_j - \theta_l - \eta)(\pm \theta_j + \theta_l - \eta)\tilde{t}^{(1)}(\pm \theta_j), \quad (B.11)
\]

\[
t(\pm \theta_j + \eta)\tilde{t}^{(2)}(\pm \theta_j - \frac{1}{2}\eta) = -\frac{1}{2} \frac{(\pm \theta_j - \frac{1}{2}\eta)(\pm \theta_j + \eta)}{(\pm \theta_j)(\pm \theta_j + \frac{1}{2}\eta)}\prod_{l=1}^{N}(\pm \theta_j - \theta_l + \eta)(\pm \theta_j + \theta_l + \eta)\tilde{t}^{(2)}(\pm \theta_j), \quad j = 1, 2, \cdots, N. \quad (B.12)
\]

With the help of Eqs. (B.5), (B.11) and (B.12), we can derive the relation (3.17). Finally, we have proven the identities (3.15)-(3.17).

**References**

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982.

[2] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *J. Phys. A* **20** (1987), 6397.

[3] E. K. Sklyanin, *J. Phys. A* **21** (1988), 2375.

[4] R. I. Nepomechie, Nucl. Phys. B **622** (2002), 615.

[5] J. Cao, H. -Q. Lin, K. -J. Shi and Y. Wang, Nucl. Phys. B **663** (2003), 487.

[6] W. -L. Yang, Y. -Z. Zhang and M. D. Gould, *Nucl. Phys. B* **698** (2004), 503.

[7] J. de Gier, A. Nichols, P. Pyatov and V. Rittenberg, Nucl. Phys. B **729** (2005), 387.
[8] W.-L. Yang, R.I. Nepomechie and Y.-Z. Zhang, *Phys. Lett.* B 633 (2006), 664.

[9] P. Baseilhac, *Nucl. Phys.* B 754 (2006), 309.

[10] P. Baseilhac and K. Koizumi, *J. Stat. Mech.* (2007), P09006.

[11] P. Baseilhac and S. Belliard, *Lett. Math. Phys.* 93 (2010), 213.

[12] P. Baseilhac and S. Belliard, *Nucl. Phys.* B 873 (2013), 550.

[13] H. Frahm, A. Seel and T. Wirth, *Nucl. Phys.* B 802 (2008), 351.

[14] H. Frahm, J. H. Grelik, A. Seel and T. Wirth, *J. Phys.* A 44 (2011), 015001.

[15] G. Niccoli, *J. Stat. Mech.* (2012), P10025; *Nucl. Phys.* B 870 (2013), 397; *J. Phys.* A 46 (2013), 075003.

[16] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Phys. Rev. Lett.* 111 (2013), 137201.

[17] Y. Wang, W.-L. Yang, J. Cao and K. Shi, *Off-Diagonal Bethe Ansatz for Exactly Solvable Models*, Springer Press, 2015.

[18] S. Belliard and N. Crampé, *SIGMA* 9 (2013), 072.

[19] S. Belliard, *Nucl. Phys.* B 892 (2015), 1.

[20] S. Belliard and R. A. Pimenta, *Nucl. Phys.* B 894 (2015), 527.

[21] J. Avan, S. Belliard, N. Grosjean and R. A. Pimenta, *Nucl. Phys.* B 899 (2015), 229.

[22] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *JHEP* 04 (2014) 143.

[23] J. Cao, S. Cui, W.-L. Yang, K. Shi and Y. Wang, *JHEP* 02 (2015), 036.

[24] K. Hao, J. Cao, G.-L. Li, W.-L. Yang, K. Shi and Y. Wang, *JHEP* 06 (2014), 128.

[25] G.-L. Li, J. Cao, P. Xue, K. Hao, P. Sun, W.-L. Yang, K. Shi and Y. Wang, *Nucl. Phys.* B 946 (2019), 114719.

[26] G.-L. Li, J. Cao, P. Xue, Z.-R. Xin, K. Hao, W.-L. Yang, K. Shi and Y. Wang, *JHEP* 05 (2019), 067.
[27] G.-L. Li, J. Cao, P. Xue, K. Hao, P. Sun, W.-L. Yang, K. Shi and Y. Wang, \textit{JHEP} \textbf{12} (2019), 051.

[28] L. Frappat, P. Sorba and A. Sciarrino, \textit{Dictionary on Lie algebras and Superalgebras}, Academic Press, 2000.

[29] J. M. Maldacena, \textit{Int. J. Theor. Phys.} \textbf{38} (1999), 1113.

[30] N. Beisert et al., \textit{Lett. Math. Phys.} \textbf{99} (2012), 3.

[31] V. E. Korepin, N. M. Boliubov and A. G. Izergin, \textit{Quantum Inverse Scattering Method and Correlation Functions}, Cambridge University Press, 1993.

[32] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper and V. Korepin, \textit{The one-dimensional Hubbard Model}, Cambridge University Press, 2005.

[33] J. H. Perk and C. L. Schultz, \textit{Phys. Lett} \textbf{A 84} (1981), 407.

[34] H. J. de Vega and E. Lopes, \textit{Phys. Rev. Lett.} \textbf{67} (1991), 489.

[35] R. H. Yue, H. Fan and B. Y. Hou, \textit{Nucl. Phys.} \textbf{B 462} (1996), 167.

[36] F. H. L. Essler, V. E. Korepin and K. Schoutens, \textit{Phys. Rev. Lett.} \textbf{68} (1992), 2960.

[37] F. H. L. Essler, V. E. Korepin and K. Schoutens, \textit{Nucl. Phys.} \textbf{B 372} (1992), 559.

[38] F. H. L. Essler and V. E. Korepin, \textit{Phys. Rev.} \textbf{B 46} (1992), 9147.

[39] F. H. L. Essler, V. E. Korepin and K. Schoutens, \textit{Nucl. Phys.} \textbf{B 384} (1992), 431.

[40] A. Foerster and M. Karowski, \textit{Nucl. Phys.} \textbf{B 396} (1993), 611.

[41] M. J. Martins and P. B. Ramos, \textit{Nucl. Phys.} \textbf{B 522} (1998), 413.

[42] A. M. Grabinski and H. Frahm, \textit{New J. Phys.} \textbf{15} (2013), 043026.

[43] P. P. Kulish, \textit{J. Sov. Math.} \textbf{35} (1986), 2648.

[44] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, \textit{Lett. Math. Phys.} \textbf{5} (1981), 393.

[45] P. P. Kulish and E. K. Sklyanin, \textit{Lecture Notes in Physics} \textbf{151} (1982), 61.
[46] M. Karowski, *Nucl. Phys.* **B 153** (1979), 244.

[47] A. N. Kirillov and N. Yu. Reshetikhin, *J. Sov. Math.* **35** (1986), 2627.

[48] A. N. Kirillov and N. Yu. Reshetikhin, *J. Phys. A* **20** (1987), 1565.

[49] I. Cherednik, *Theor. Math. Phys.* **61** (1984), 977.

[50] A. Bracken, X.-Y. Ge, Y.-Z. Zhang and H. -Q. Zhang, *Nucl. Phys.* **B 516** (1998), 588.

[51] L. Mezincescu and R. I. Nepomechie, *Nucl. Phys.* **B 372** (1992), 597.

[52] Y.-K. Zhou, *Nucl. Phys.* **B 458** (1996), 504.