GABOR ANALYSIS FOR A BROAD CLASS OF QUASI-BANACH MODULATION SPACES

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Abstract. We extend the Gabor analysis in [13] to a broad class of modulation spaces, allowing more general mixed quasi-norm estimates and weights in the definition of the modulation space quasi-norm. For such spaces we deduce invariance and embedding properties, and that the elements admit reconstructible sequence space representations using Gabor frames.

0. Introduction

A modulation space is, roughly speaking, a set of distributions or ultra-distributions, obtained by imposing a suitable quasi-norm estimate on the short-time Fourier transforms of the involved distributions. (See Sections 1 and 2 for definitions.)

In [13], Galperin and Samarah establish fundamental continuity and invariance properties for modulation spaces of the form \( M_{\omega}^{p,q} \), when \( \omega \) is a polynomially moderate weight and \( p, q \in (0, \infty] \). More precisely, Galperin and Samarah prove in [13] among others that the following fundamental properties for such modulation spaces hold true:

1. \( M_{\omega}^{p,q} \) is independent of the choice of involved window function in the short-time Fourier transforms;
2. \( M_{\omega}^{p,q} \) increases with respect to the parameters \( p \) and \( q \), and decreases with respect to \( \omega \);
3. \( M_{\omega}^{p,q} \) admit reconstructible sequence space representations using Gabor frames.

Note that in contrast to what is usual in modulation spaces theory, the Lebesgue exponents \( p \) and \( q \) above are allowed to be strictly smaller than 1. This leads to a more comprehensive and difficult analysis of \( M_{\omega}^{p,q} \) when \( p \) and \( q \) are allowed to stay in \( (0, \infty] \), compared to what is needed when \( p \) and \( q \) stays in the smaller interval \([1, \infty]\). In fact, the theory of classical modulation spaces was established and further developed in [6,10,18] by Feichtinger and Gröchenig. In these investigations, Feichtinger and Gröchenig only considered modulation spaces \( M_{\omega}^{p,q} \) with \( p, q \in [1, \infty] \), and the analysis for deducing the properties (1)–(3) above is less comprehensive and less difficult compared to the analysis in [13].
We also remark that the results in [13] have impact on unifications of the modulation space theories in [1, 2, 32, 33, 35]. In fact, in [1, 2, 32, 33, 35], certain restrictions are imposed on the window functions in the definitions of the modulation space quasi-norms. Due to (1) above, $M_{p,q}^{\omega}$ is equal to the corresponding spaces in [1, 2, 32, 33, 35], provided the weight $\omega$ and the exponents $p$ and $q$ agree with those in [1, 2, 32, 33, 35].

The aim of the paper is to deduce general properties for a broad family of modulation spaces, which contains the modulation spaces $M_{p,q}^{\omega}$ when $p, q \in (0, \infty]$ and $\omega$ is an arbitrary moderate weight. In particular, the assumption that $\omega$ should be polynomially moderate is relaxed. More precisely, we use the framework in [13] and show that (1)–(3) above still holds for this extended family of modulation spaces. If the weights are not polynomially moderate, then the involved modulation spaces do not stay between the Schwartz space $\mathcal{S}$ and its dual space $\mathcal{S}'$. In this situation, the theory is formulated in the framework of the Gelfand-Shilov space $\Sigma_1$ and its dual space $\Sigma'_1$ of Gelfand-Shilov ultra-distributions. Furthermore, we allow more general mixed quasi-norm estimates on the short-time Fourier transform, in the definitions of modulation space quasi-norms. (See Proposition 3.6 and Theorem 3.7.)

In the end of Section 3, we use these results to establish identification properties for compactly supported elements in modulation and Fourier Lebesgue spaces. In particular, we extend the assertions in Remark 4.6 in [28] to more general weights and Lebesgue exponents. (See Proposition 3.8.)

The classical modulation spaces $M_{p,q}^{\omega}$, $p, q \in [1, \infty]$ and $\omega$ polynomially moderate weight on the phase (or time-frequency shift) space, were introduced by Feichtinger in [6]. From the definition it follows that $\omega$, and to some extent the parameters $p$ and $q$ quantify the degrees of asymptotic decay and singularity of the distributions in $M_{p,q}^{\omega}$. The theory of modulation spaces was developed further and generalized in several ways, e.g. in [7, 11, 17, 18], where among others, Feichtinger and Gröchenig established the theory of coorbit spaces.

From the construction of modulation spaces, it turns out that these spaces and Besov spaces in some sense are rather similar, and sharp embeddings can be found in [30], which are improvements of certain embeddings in [16]. (See also [29, 35] for verification of the sharpness, and [16, 20, 34] for further generalizations in terms of $\alpha$-modulation spaces.)

During the last 15 years many results appeared which confirm the usefulness of the modulation spaces. For example, in [9, 17, 18], it is shown that all modulation spaces admit reconstructible sequence space representations using Gabor frames. Important reasons for such links
are that $M_{(\omega)}^{p,q}$ may in straightforward ways be considered within the coorbit space theory.

More broad families of modulation spaces have been considered since [6]. For example, in [6], Feichtinger considers general classes of modulation spaces, defined by replacing the $L_{(\omega)}^{p,q}$ norm estimates of the short-time Fourier transforms, by more general norm estimates. Furthermore, in [24, 25, 32, 33], the conditions on involved weight functions are relaxed, and modulation spaces are considered in the framework of the theory of Gelfand-Shilov distributions. In this setting, the family of modulation spaces are broad compared to [6, 13]. For example, in contrast to [6, 13], we may have $S' \subseteq M_{(\omega)}^{p,q}$, or $M_{(\omega)}^{p,q} \subseteq S'$, for some choices of $\omega$ in [24, 25, 32, 33]. Some steps in this direction can be found already in e.g. [17, 18].

Finally we remarks that in [26, 27], Rauhut extends essential parts of the coorbit space theory in [9, 17] to the case of quasi-Banach spaces. Here it is also shown that modulation spaces of quasi-Banach types in [13] fit well in this theory, and we remark that the results in Sections 2 and 3 show that our extended family of modulation spaces also meets the coorbit space theory in [27] well.

1. Preliminaries

In this section we explain some results available in the literature, which are needed later on, or clarify the subject. The proofs are in general omitted. Especially we recall some facts about weight functions, Gelfand-Shilov spaces, and modulation spaces.

1.1. Weight functions. We start by discussing general properties on the involved weight functions. A weight on $\mathbb{R}^d$ is a positive function $\omega \in L_{\text{loc}}^{\infty}(\mathbb{R}^d)$, and for each compact set $K \subseteq \mathbb{R}^d$, there is a constant $c > 0$ such that

$$\omega(x) \geq c \quad \text{when} \quad x \in K.$$ 

A usual condition on $\omega$ is that it should be moderate, or $v$-moderate for some positive function $v \in L_{\text{loc}}^{\infty}(\mathbb{R}^d)$. This means that

$$\omega(x + y) \leq C\omega(x)v(y), \quad x, y \in \mathbb{R}^d,$$ 

(1.1)

for some constant $C$ which is independent of $x, y \in \mathbb{R}^d$. We note that (1.1) implies that $\omega$ fulfills the estimates

$$C^{-1}v(-x)^{-1} \leq \omega(x) \leq Cv(x), \quad x \in \mathbb{R}^d.$$ 

(1.2)

We let $\mathcal{P}_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$. Furthermore, if $v$ in (1.1) can be chosen as a polynomial, then $\omega$ is called polynomially moderate, or a weight of polynomial type. We let $\mathcal{P}(\mathbb{R}^d)$ be the set of all weights of polynomial type.

It can be proved that if $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, then $\omega$ is $v$-moderate for some $v(x) = e^{r|x|}$, provided the positive constant $r$ is large enough. In
particular, (1.2) shows that for any $\omega \in \mathcal{P}_F(\mathbb{R}^d)$, there is a constant $r > 0$ such that
\[ e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x \in \mathbb{R}^d. \]
Here $A \lesssim B$ means that $A \leq cB$ for a suitable constant $c > 0$.

We say that $v$ is submultiplicative if $v$ is even and (1.1) holds with $\omega = v$. In the sequel, $v$ and $v_j$ for $j \geq 0$, always stand for submultiplicative weights if nothing else is stated.

1.2. Gelfand-Shilov spaces. Next we recall the definition of Gelfand-Shilov spaces.

Let $0 < h, s, t \in \mathbb{R}$ be fixed. Then we let $S_{t,h}^s(\mathbb{R}^d)$ be the set of all $f \in C^\infty(\mathbb{R}^d)$ such that
\[
\|f\|_{S_{t,h}^s} \equiv \sup_{x \in \mathbb{R}^d} \frac{|x^\beta \partial^\alpha f(x)|}{h^{\alpha+|\beta|} \alpha! \beta!}
\]
is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. For conveinency we set $S_{s,h}^s = S_{s,th}^s$.

Obviously $S_{t,h}^s \subseteq \mathcal{S}$ is a Banach space which increases with $h, s$ and $t$. Furthermore, if $s, t > 1/2$, or $s, t = 1/2$ and $h$ is sufficiently large, then $S_{t,h}^s$ contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathcal{S}$, it follows that the dual $(S_{t,h}^s)'(\mathbb{R}^d)$ of $S_{t,h}^s(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbb{R}^d)$.

The Gelfand-Shilov spaces $S_t^s(\mathbb{R}^d)$ and $\Sigma_t^s(\mathbb{R}^d)$ are the inductive and projective limits respectively of $S_{t,h}^s(\mathbb{R}^d)$ with respect to $h$. This implies that
\[
S_t^s(\mathbb{R}^d) = \bigcup_{h>0} S_{t,h}^s(\mathbb{R}^d) \quad \text{and} \quad \Sigma_t^s(\mathbb{R}^d) = \bigcap_{h>0} S_{t,h}^s(\mathbb{R}^d), \quad (1.3)
\]
and that the topology for $S_t^s(\mathbb{R}^d)$ is the strongest possible one such that each inclusion map from $S_{t,h}^s(\mathbb{R}^d)$ to $S_t^s(\mathbb{R}^d)$ is continuous. The space $\Sigma_t^s(\mathbb{R}^d)$ is a Fréchet space with semi norms $\|\cdot\|_{S_{t,h}^s}$; $h > 0$. Moreover, $S_t^s(\mathbb{R}^d) \neq \{0\}$, if and only if $s, t > 0$ satisfy $s + t \geq 1$, and $\Sigma_t^s(\mathbb{R}^d) \neq \{0\}$, if and only if satisfy $s + t \geq 1$ and $(s, t) \neq (1/2, 1/2)$.

For convenience we set $S_s^s = S_s^s$ and $\Sigma_s^s = \Sigma_s^s$, and remark that $S_s(\mathbb{R}^d)$ is zero when $s < 1/2$, and that $\Sigma_s(\mathbb{R}^d)$ is zero when $s \leq 1/2$. For each $\varepsilon > 0$ and $s, t > 0$ such that $s + t \geq 1$, we have
\[
\Sigma_t^s(\mathbb{R}^d) \subseteq S_t^s(\mathbb{R}^d) \subseteq \Sigma_{t+\varepsilon}^{s+\varepsilon}(\mathbb{R}^d).
\]
On the other hand, in [23] there is an alternative elegant definition of $\Sigma_{s_1}(\mathbb{R}^d)$ and $S_{s_2}(\mathbb{R}^d)$ such that these spaces agrees with the definitions above when $s_1 > 1/2$ and $s_2 \geq 1/2$, but $\Sigma_{1/2}(\mathbb{R}^d)$ is non-trivial and contained in $S_{1/2}(\mathbb{R}^d)$.

From now on we assume that $s, t > 1/2$ when considering $\Sigma_t^s(\mathbb{R}^d)$.
The Gelfand-Shilov distribution spaces \((\mathcal{S}_s^d)'(\mathbb{R}^d)\) and \((\Sigma_s^d)'(\mathbb{R}^d)\) are the projective and inductive limit respectively of \((\mathcal{S}_{t,h}^d)'(\mathbb{R}^d)\). This means that
\[
(\mathcal{S}_s^d)'(\mathbb{R}^d) = \bigcap_{h>0} (\mathcal{S}_{t,h}^d)'(\mathbb{R}^d) \quad \text{and} \quad (\Sigma_s^d)'(\mathbb{R}^d) = \bigcup_{h>0} (\mathcal{S}_{t,h}^d)'(\mathbb{R}^d). \tag{1.3}
\]

We remark that already in \[14\] it is proved that \((\mathcal{S}_s^d)'(\mathbb{R}^d)\) is the dual of \(\mathcal{S}_s(\mathbb{R}^d)\), and if \(s > 1/2\), then \((\Sigma_s^d)'(\mathbb{R}^d)\) is the dual of \(\Sigma_s(\mathbb{R}^d)\) (also in topological sense).

The Gelfand-Shilov spaces are invariant or posses convenient mapping properties under several basic transformations. For example they are invariant under translations, dilations, tensor product, and to some extent under Fourier transformation. Here tensor products of elements in Gelfand-Shilov distribution spaces are defined in similar ways as for tensor products for distributions (cf. Chapter V in \[21\]). If \(s, s_0, t, t_0 > 0\) satisfy
\[
s_0 + t_0 \geq 1, \quad s \geq s_0 \quad \text{and} \quad t \geq t_0,
\]
and \(f, g \in (\mathcal{S}_{s_0}^d)'(\mathbb{R}^d) \setminus \emptyset\) and then \(f \otimes g \in (\mathcal{S}_s^d)'(\mathbb{R}^{2d})\), if and only if \(f, g \in (\mathcal{S}_s^d)'(\mathbb{R}^d)\). Similar facts hold for any other choice of Gelfand-Shilov spaces of functions or distributions.

From now on we let \(\mathcal{F}\) be the Fourier transform which takes the form
\[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} \, dx
\]
when \(f \in L^1(\mathbb{R}^d)\). Here \((\cdot, \cdot)\) denotes the usual scalar product on \(\mathbb{R}^d\). The map \(\mathcal{F}\) extends uniquely to homeomorphisms on \(\mathcal{F}'(\mathbb{R}^d)\), \(\mathcal{S}'_s(\mathbb{R}^d)\) and \(\Sigma'_s(\mathbb{R}^d)\), and restricts to homeomorphisms on \(\mathcal{S}(\mathbb{R}^d)\), \(\mathcal{S}_s(\mathbb{R}^d)\) and \(\Sigma_s(\mathbb{R}^d)\), and to a unitary operator on \(L^2(\mathbb{R}^d)\). More generally, \(\mathcal{F}\) extends uniquely to homeomorphisms from \((\mathcal{S}_s^d)'(\mathbb{R}^d)\) and \((\Sigma_s^d)'(\mathbb{R}^d)\) to \((\mathcal{S}_s^d)'(\mathbb{R}^d)\) and \((\Sigma_s^d)'(\mathbb{R}^d)\) respectively, and restricts to homeomorphisms from \(\mathcal{S}_s^d(\mathbb{R}^d)\) and \(\Sigma_s^d(\mathbb{R}^d)\) to \(\mathcal{S}_s^d(\mathbb{R}^d)\) and \(\Sigma_s^d(\mathbb{R}^d)\) respectively.

The following lemma shows that functions in Gelfand-Shilov spaces can be characterized by estimates on the functions and their Fourier transform of the form
\[
|f(x)| \lesssim e^{-\varepsilon|x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/s}}. \tag{1.4}
\]

The proof is omitted, since the result can be found in e. g. \[3, 14\].

**Lemma 1.1.** Let \(f \in \mathcal{S}'_{1/2}(\mathbb{R}^d)\), and let \(s, t > 0\). Then the following is true:

1. if \(s + t \geq 1\), then \(f \in \mathcal{S}_s^d(\mathbb{R}^d)\), if and only if \(1.4\) holds for some \(\varepsilon > 0\);
2. if \(s + t \geq 1\) and \((s, t) \neq (1/2, 1/2)\), then \(f \in \Sigma_s^d(\mathbb{R}^d)\), if and only if \(1.4\) holds for every \(\varepsilon > 0\).
The estimates (1.4) are equivalent to
\[ |f(x)| \leq C e^{-\varepsilon|x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \leq C e^{-\varepsilon|\xi|^{1/s}}. \]
In (2) in Lemma 1.1, it is understood that the (hidden) constant \( C > 0 \) depends on \( \varepsilon > 0 \).

Next we recall related characterizations of Gelfand-Shilov spaces, in terms of short-time Fourier transforms. (See Propositions 1.3 and 1.4 below.)

Let \( \phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \) be fixed. For every \( f \in \mathcal{S}'(\mathbb{R}^d) \), the short-time Fourier transform \( V_\phi f \) is the distribution on \( \mathbb{R}^{2d} \) defined by the formula
\[
(V_\phi f)(x, \xi) = \mathcal{F}(f \phi(\cdot - x))(\xi). \tag{1.5}
\]
We note that the right-hand side defines an element in \( \mathcal{S}'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}) \), and that \( V_\phi f \) takes the form
\[
V_\phi f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \overline{\phi(y - x)} e^{-i(y, \xi)} \, dy \tag{1.5'}
\]
when \( f \in L^2_\omega \) for some \( \omega \in \mathcal{P}(\mathbb{R}^d) \).

In order to extend the definition of the short-time Fourier transform we reformulate (1.5) in terms of partial Fourier transforms and tensor products (cf. [12]). More precisely, let \( \mathcal{F}_2 F \) be the partial Fourier transform of \( F(x, y) \in \mathcal{S}'(\mathbb{R}^{2d}) \) with respect to the \( y \)-variable, and let \( U \) be the map which takes \( F(x, y) \) into \( F(y, y - x) \). Then it follows that
\[
V_\phi f = (\mathcal{F}_2 \circ U)(f \otimes \overline{\phi}) \tag{1.6}
\]
when \( f \in \mathcal{S}'(\mathbb{R}^d) \) and \( \phi \in \mathcal{S}(\mathbb{R}^d) \).

The following result concerns the map
\[
(f, \phi) \mapsto V_\phi f. \tag{1.7}
\]

**Proposition 1.2.** The map (1.7) from \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^{2d}) \) is uniquely extendable to a continuous map from \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \times \mathcal{S}'_{1/2}(\mathbb{R}^d) \) to \( \mathcal{S}'_{1/2}(\mathbb{R}^{2d}) \). Furthermore, if \( s \geq 1/2 \) and \( f, \phi \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \setminus \{0\} \), then the following is true:

1. the map (1.7) restricts to a continuous map from \( \mathcal{S}_s(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \) to \( \mathcal{S}_s(\mathbb{R}^{2d}) \). Moreover, \( V_\phi f \in \mathcal{S}_s(\mathbb{R}^{2d}) \), if and only if \( f, \phi \in \mathcal{S}_s(\mathbb{R}^d) \);

2. the map (1.7) restricts to a continuous map from \( \mathcal{S}'_s(\mathbb{R}^d) \times \mathcal{S}'_s(\mathbb{R}^d) \) to \( \mathcal{S}'_s(\mathbb{R}^{2d}) \). Moreover, \( V_\phi f \in \mathcal{S}'_s(\mathbb{R}^{2d}) \), if and only if \( f, \phi \in \mathcal{S}'_s(\mathbb{R}^d) \).

Similar facts hold after \( \mathcal{S}_s \) and \( \mathcal{S}'_s \) are replaced by \( \mathcal{S}_s' \) and \( \mathcal{S}'_s' \), respectively.

**Proof.** The result follows immediately from (1.6), and the facts that tensor products, \( \mathcal{F}_2 \) and \( U \) are continuous on \( \mathcal{S}_s, \mathcal{S}_s' \) and their duals. See also [3] for details. \( \square \)
We also recall characterizations of Gelfand-Shilov spaces and their distribution spaces in terms of the short-time Fourier transform, obtained in [19,32]. The involved conditions are

\[ s \geq s_0 > 0, \quad t \geq t_0 > 0 \quad \text{and} \quad s_0 + t_0 \geq 1 \] (1.8)

\[ |V_\phi f(x, \xi)| \lesssim e^{-\varepsilon(|x|^{1/s} + |\xi|^{1/q})} \] (1.9)

\[ |(\mathcal{F}(V_\phi f))(\xi, x)| \lesssim e^{-\varepsilon(|x|^{1/s} + |\xi|^{1/q})} \] (1.10)

and

\[ |V_\phi f(x, \xi)| \lesssim e^{\varepsilon(|x|^{1/s} + |\xi|^{1/q})}. \] (1.9')

**Proposition 1.3.** Let \( s, t, s_0, t_0 \in \mathbb{R} \) satisfy (1.8), and let \( \phi \in \mathcal{S}_0(\mathbb{R}^d) \) \( \setminus \) 0 and \( f \in (\mathcal{S}_0(\mathbb{R}^d)^\prime) \). Then the following is true:

1. \( f \in \mathcal{S}_t^s(\mathbb{R}^d) \), if and only if (1.9) holds for some \( \varepsilon > 0 \). Furthermore, if \( f \in \mathcal{S}_t^s(\mathbb{R}^d) \), then (1.10) holds for some \( \varepsilon > 0 \);
2. \( f \) in addition \( (s, t) \neq (1/2, 1/2) \) and \( \phi \in \Sigma_s^t(\mathbb{R}^d) \), then \( f \in \Sigma_s^t(\mathbb{R}^d) \), if and only if (1.9) holds for every \( \varepsilon > 0 \). Furthermore, if \( f \in \Sigma_s^t(\mathbb{R}^d) \), then (1.10) holds for every \( \varepsilon > 0 \).

We refer to [19] Theorem 2.7 for the proof of Theorem 1.3. The corresponding result for Gelfand-Shilov distributions is the following, and refer to [32] Theorem 2.5 for the proof. Note that there is a misprint in the second statement [32] Theorem 2.5, where it stays \( f \in \Sigma_s^t(\mathbb{R}^d) \) instead of \( f \in (\Sigma_s^t)^\prime(\mathbb{R}^d) \).

**Proposition 1.4.** Let \( s, t, s_0, t_0 \in \mathbb{R} \) satisfy (1.8) and \( (s, t) \neq (1/2, 1/2) \), and let \( \phi \in \Sigma_s^t(\mathbb{R}^d) \) \( \setminus \) 0 and \( f \in (\mathcal{S}_0(\mathbb{R}^d)^\prime) \). Then the following is true:

1. \( f \in (\mathcal{S}_t^s)^\prime(\mathbb{R}^d) \), if and only if (1.9)' holds for every \( \varepsilon > 0 \);
2. \( f \in (\Sigma_s^t)^\prime(\mathbb{R}^d) \), if and only if (1.9)' holds for some \( \varepsilon > 0 \).

There are several other ways to characterize Gelfand-Shilov spaces. For example, they can easily be characterized by Hermite functions (cf. e. g. [15]).

1.3. Mixed quasi-normed space of Lebesgue types. Let \( p, q \in (0, \infty) \), and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \). A common type of mixed quasi-norm space on \( \mathbb{R}^{2d} \) is \( L^{p,q}_{\omega}(\mathbb{R}^{2d}) \), which consists of all measurable functions \( F \) on \( \mathbb{R}^{2d} \) such that

\[ \|g\|_{L^p(\mathbb{R}^d)} < \infty, \quad \text{where} \quad g(\xi) \equiv \|F(\cdot, \xi)\omega(\cdot, \xi)\|_{L^p(\mathbb{R}^d)}. \]

Next we introduce a broader family of mixed quasi-norm spaces on \( \mathbb{R}^d \), where the pair \( (p, q) \) above is replaced by a vector in \( (0, \infty]^d \) of Lebesgue exponents. If

\[ p = (p_1, \ldots, p_d) \in (0, \infty]^d \quad \text{and} \quad q = (q_1, \ldots, q_d) \in (0, \infty]^d \]
are two such vectors, then we use the conventions \( p \leq q \) when \( p_j \leq q_j \) for every \( j = 1, \ldots, d \), and \( p < q \) when \( p_j < q_j \) for every \( j = 1, \ldots, d \).

Let \( S_d \) be the set of permutations on \( \{1, \ldots, d\} \), \( p \in (0, \infty]^d \), \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), and let \( \sigma \in S_d \). For every measurable and complex-valued function \( f \) on \( \mathbb{R}^d \), let \( g_{j,\omega} \), \( j = 1, \ldots, d \), be defined inductively by the formulas

\[
g_{\omega}(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \equiv \left| f(x_1, \ldots, x_d)\omega(x_1, \ldots, x_d) \right|, \quad (1.11)
g_{1,\omega}(x_2, \ldots, x_d) \equiv \|g_\omega(\cdot, x_2, \ldots, x_d)\|_{L^p(\mathbb{R})},
g_{k,\omega}(x_{k+1}, \ldots, x_d) \equiv \|g_{k-1,\omega}(\cdot, x_{k+1}, \ldots, x_d)\|_{L^p_k(\mathbb{R})}, \quad k = 2, \ldots, d - 1.
\]

The mixed quasi-norm space \( L^p_{\sigma,\omega}(\mathbb{R}^d) \) of Lebesgue type is defined as the set of all complex-valued measurable functions \( f \) on \( \mathbb{R}^d \) such that \( \|f\|_{L^p_{\sigma,\omega}} < \infty \).

The set of sequences \( \ell^p_{\sigma,\omega}(\Lambda) \), for an appropriate lattice \( \Lambda \) is defined in an analogous way. More precisely, let \( \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d_+ \), and let \( T_{\theta} \) denote the diagonal matrix with diagonal elements \( \theta_1, \ldots, \theta_d \). Here \( \mathbb{R}_+ = \mathbb{R} \setminus 0 \) and we interprete \( \mathbb{R}^d_+ \) as \( (\mathbb{R} \setminus 0)^d \). Also let

\[
\Lambda = T_{\theta}Z^d = \{ (\theta_1 j_1, \ldots, \theta_d j_d) ; (j_1, \ldots, j_d) \in \mathbb{Z}^d \}. \tag{1.12}
\]

For any sequence \( a \) on \( T_\theta Z^d \), let \( b_{j,\omega} \), \( j = 1, \ldots, d \), be defined inductively by the formulas

\[
b_\omega(j_{\sigma(1)}, \ldots, j_{\sigma(d)}) \equiv |a(j_1, \ldots, j_d)\omega(j_1, \ldots, j_d)|, \quad (1.13)
b_{1,\omega}(j_2, \ldots, j_d) \equiv \|b_\omega(\cdot, j_2, \ldots, j_d)\|_{\ell^p(\theta, \mathbb{Z})},
b_{k,\omega}(j_{k+1}, \ldots, j_d) \equiv \|b_{k-1,\omega}(\cdot, j_{k+1}, \ldots, j_d)\|_{\ell^p_k(\theta, \mathbb{Z})}, \quad k = 2, \ldots, d - 1
\]

and

\[
\|a\|_{\ell^p_{\sigma,\omega}(\Lambda)} \equiv b_{d,\omega} \equiv \|b_{d-1,\omega}\|_{\ell^p(\theta, \mathbb{Z})}.
\]

The mixed quasi-norm space \( \ell^p_{\sigma,\omega}(\Lambda) \) is defined as the set of all sequences functions \( a \) on \( \Lambda \) such that \( \|a\|_{\ell^p_{\sigma,\omega}(\Lambda)} < \infty \).

We also write \( L^p_\sigma \) and \( \ell^p_\sigma \) instead of \( L^p_{\sigma,\omega} \) and \( \ell^p_{\sigma,\omega} \) respectively when \( \sigma \) is the identity map. Furthermore, if \( \omega \) is equal to 1, then we write

\[
L^p_\sigma, \quad \ell^p_\sigma, \quad L^p \quad \text{and} \quad \ell^p
\]

instead of

\[
L^p_{\sigma,\omega}, \quad \ell^p_{\sigma,\omega}, \quad L^p_\omega \quad \text{and} \quad \ell^p_\omega.
\]
respectively.

For any \( p \in (0, \infty]^d \), let

\[
\max p \equiv \max(p_1, \ldots, p_d) \quad \text{and} \quad \min p \equiv \min(p_1, \ldots, p_d).
\]

We note that if \( \max p < \infty \), then \( \ell_0(\Lambda) \) is dense in \( \ell^p_\sigma(\omega)(\Lambda) \). Here \( \ell_0(\Lambda) \) is the set of all sequences \( \{a(j)\}_{j \in \Lambda} \) on \( \Lambda \) such that \( a(j) \neq 0 \) for at most finite numbers of \( j \).

1.4. Modulation spaces. Next we define modulation spaces. Let \( \phi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus 0 \). For any \( p, q \in (0, \infty] \) and \( \omega \in \mathcal{E}_E(\mathbb{R}^{2d}) \), the standard modulation space \( M^p_{\phi}(\mathbb{R}^d) \) is the set of all \( f \in \mathcal{S}_{1/2}(\mathbb{R}^d) \) such that \( V_\phi f \in L^p_\omega(\mathbb{R}^{2d}) \), and we equip \( M^p_{\phi}(\mathbb{R}^d) \) with the quasi-norm

\[
\|f\|_{M^p_{\phi}} \equiv \|V_\phi f\|_{L^p_\omega}.
\]

We remark that \( M^p_{\phi}(\mathbb{R}^d) \) is one of the most common types of modulation spaces.

More generally, for any \( \sigma \in S_{2d}, p \in (0, \infty]^{2d} \) and \( \omega \in \mathcal{E}_E(\mathbb{R}^{2d}) \), the modulation space \( M^p_{\sigma,\omega}(\mathbb{R}^d) \) is the set of all \( f \in \mathcal{S}_{1/2}(\mathbb{R}^d) \) such that \( V_\sigma f \in L^p_{\omega,\omega}(\mathbb{R}^{2d}) \), and we equip \( M^p_{\sigma,\omega}(\mathbb{R}^d) \) with the quasi-norm

\[
\|f\|_{M^p_{\sigma,\omega}} \equiv \|V_\sigma f\|_{L^p_{\omega,\omega}}.
\]

In the following propositions we list some properties for modulation. The first one follows from the definition of invariant spaces and Propositions 1.3 and 1.4. The other results can be found in [6,9,10,18,31]. The proofs are therefore omitted

**Proposition 1.5.** Let \( \omega \in \mathcal{E}_E(\mathbb{R}^{2d}), \sigma \in S_{2d} \) and \( p \in (0, \infty]^{2d} \). Then the following is true:

1. \( \Sigma_1(\mathbb{R}^d) \subseteq M^p_{\sigma,\omega}(\mathbb{R}^d) \subseteq \Sigma'_1(\mathbb{R}^d) \);

2. if in addition

\[
e^{-|\cdot|} \lesssim \omega \lesssim e^{\varepsilon|\cdot|},
\]

holds for every \( \varepsilon > 0 \), then \( \Sigma_1(\mathbb{R}^d) \subseteq M^p_{\sigma,\omega}(\mathbb{R}^d) \subseteq \Sigma'_1(\mathbb{R}^d) \);

3. if in addition \( \omega \in \mathcal{E}(\mathbb{R}^{2d}) \), then \( \mathcal{E}(\mathbb{R}^d) \subseteq M^p_{\sigma,\omega}(\mathbb{R}^d) \subseteq \mathcal{E}(\mathbb{R}^d) \).

**Proposition 1.6.** Let

\[
p, q \in [1, \infty], \quad p, p_j \in [1, \infty]^{2d}, \quad \omega, \omega_j, v \in \mathcal{E}_E(\mathbb{R}^{2d}), \quad j = 1, 2,
\]

be such that \( p_1 \leq p_2 \), \( \omega_2 \lesssim \omega_1 \), and \( \omega \) is \( v \)-moderate. Also let \( \sigma \in S_{2d} \). Then the following is true:

1. if \( \phi \in M^1_{(v)}(\mathbb{R}^d) \setminus 0 \), then \( f \in M^p_{\sigma,\omega}(\mathbb{R}^d) \), if and only if

\[
\|V_\phi f\|_{L^p_{\omega,\omega}} < \infty.
\]

In particular, \( M^p_{\sigma,\omega}(\mathbb{R}^d) \) is independent of the choice of \( \phi \in M^1_{(v)}(\mathbb{R}^d) \setminus 0 \). Moreover, \( M^p_{\sigma,\omega}(\mathbb{R}^d) \) is a Banach space under the
norm in (1.14), and different choices of \( \phi \) give rise to equivalent norms;

(2) \( M^{p_1}_{\sigma, (\omega_1)}(\mathbb{R}^d) \subseteq M^{p_2}_{\sigma, (\omega_2)}(\mathbb{R}^d) \);

(3) the \( L^2 \)-form on \( S_{1/2}(\mathbb{R}^d) \) extends uniquely to a dual form between \( M^{p,q}_{(\omega)}(\mathbb{R}^d) \) and \( M^{p',q'}_{(1/\omega)}(\mathbb{R}^d) \). Furthermore, if in addition \( p, q < \infty \), then the dual of \( M^{p,q}_{(\omega)} \) can be identified with \( M^{p',q'}_{(1/\omega)}(\mathbb{R}^d) \) through this form.

Next we recall the notion of Gabor expansions. First we recall some facts on sequences and lattices. In what follows we let \( \Lambda, \Lambda_1 \) and \( \Lambda_2 \) be the lattices

\[
\Lambda_1 \equiv \{ x_j \}_{j \in J} \equiv T_\theta \mathbb{Z}^d, \quad \Lambda_2 \equiv \{ \xi_k \}_{k \in J} \equiv T_\vartheta \mathbb{Z}^d, \quad \Lambda \equiv \Lambda_1 \times \Lambda_2 \quad (1.15)
\]

where \( \theta, \vartheta \in \mathbb{R}^d \) and \( J \) is an index set.

**Definition 1.7.** Let \( \Lambda, \Lambda_1 \) and \( \Lambda_2 \) be as in (1.15). Let \( \omega, v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, and let \( \phi, \psi \in M^1_v(\mathbb{R}^d) \).

(1) The analysis operator \( C^{\Lambda}_\phi \) is the operator from \( M^\infty_{(\omega)}(\mathbb{R}^d) \) to \( \ell^\infty_\omega(\Lambda) \), given by

\[
C^{\Lambda}_\phi f \equiv \{ V_\phi f(x_j, \xi_k) \}_{j,k \in J};
\]

(2) The synthesis operator \( D^{\Lambda}_\psi \) is the operator from \( \ell^\infty_\omega(\Lambda) \) to \( M^\infty_{(\omega)}(\mathbb{R}^d) \), given by

\[
D^{\Lambda}_\psi c \equiv \sum_{j,k \in J} c_{j,k} e^{i \langle \cdot, \xi_k \rangle} \phi(\cdot - x_j);
\]

(3) The Gabor frame operator \( S^{\Lambda}_{\phi, \psi} \) is the operator on \( M^\infty_{(\omega)}(\mathbb{R}^d) \), given by \( D^{\Lambda}_\psi \circ C^{\Lambda}_\phi \), i.e.

\[
S^{\Lambda}_{\phi, \psi} f \equiv \sum_{j,k \in J} V_\phi f(x_j, \xi_k) e^{i \langle \cdot, \xi_k \rangle} \psi(\cdot - x_j).
\]

It follows from the analysis in Chapters 11–14 in [18] that the operators in Definition 1.7 are well-defined and continuous.

We finish the section by discussing some consequences of the following result. The proof is omitted since the result follows from Theorem 13.1.1 in [18], which in turn can be considered as a special case of Theorem S in [17].

**Proposition 1.8.** Let \( v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be submultiplicative, and \( \phi \in M^1_v(\mathbb{R}^d) \). Then there is a constant \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0] \), the frame operator \( S^{\Lambda}_{\varepsilon \phi, \varepsilon \psi} \), with \( \Lambda = \varepsilon Z^{2d} \), is a homeomorphism on \( M^1_{(v)}(\mathbb{R}^d) \).

We also recall the following result, and refer to the proof of Corollaries 12.2.5 and 12.2.6 in [15] for the proof.
Proposition 1.9. Let \( v, \phi \) and \( \Lambda \) be the same as in Proposition 1.8, \( \psi = (S_{\phi, \psi})^{-1} \phi \), \( f \in M_{(1/v)}^\infty(\mathbb{R}^d) \), \( p, q \in [1, \infty] \), and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be \( v \)-moderate. Then
\[
f = \sum_{(x_j, \xi_k) \in \Lambda} V_{\phi} f(x_j, \xi_k) e^{i \langle \cdot, \xi_k \rangle} \psi(\cdot - x_j)
\]
(1.16)
where the sums converge in the weak* topology. Furthermore the following conditions are equivalent.

1. \( f \in M_{p,q}^\omega(\mathbb{R}^d) \);
2. \( \{V_{\phi} f(x_j, \xi_k)\}_{(x_j, \xi_k) \in \Lambda} \in \ell_{p,q}^\omega(\Lambda) \);
3. \( \{V_{\psi} f(x_j, \xi_k)\}_{(x_j, \xi_k) \in \Lambda} \in \ell_{p,q}^\omega(\Lambda) \).

Moreover, if (1)–(3) are true, then the sums in (1.16) converge in the weak* topology to elements in \( M_{p,q}^\omega \) when \( p = \infty \) or \( q = \infty \), and unconditionally in norms when \( p,q < \infty \).

Let \( v, \phi \) and \( \Lambda \) be as in Proposition 1.8. Then
\[
(S_{\phi, \psi}^\Lambda)^{-1} \phi
\]
is called the canonical dual window to \( \phi \), with respect to \( \Lambda \). By duality, it follows that \( S_{\phi, \psi}^\Lambda \) extends to \( f \) a continuous operator on \( M_{(1/v)}^\infty(\mathbb{R}^d) \), and
\[
S_{\phi, \psi}^\Lambda(e^{i \langle \cdot, \xi_k \rangle} f(\cdot - x_j)) = e^{i \langle \cdot, \xi_k \rangle} (S_{\phi, \psi}^\Lambda f)(\cdot - x_j),
\]
when \( f \in M_{(1/v)}^\infty(\mathbb{R}^d) \) and \( (x_j, \xi_k) \in \Lambda \). The series in (1.16) are called Gabor expansions of \( f \) with respect to \( \phi \) and \( \psi \).

Now let \( p = [1, \infty]^d \), \( \sigma \in S_{2d} \), and let \( \omega, v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, and choose \( \phi \) and \( \varepsilon_0 \) such that the conclusions in Proposition 1.8 are true. Also let \( f \in M_{\sigma, (\omega)}^p(\mathbb{R}^d) \). Then the right-hand sides of (1.16) converge unconditionally in \( M_{\sigma, (\omega)}^p \) when \( \max p < \infty \), and in \( M_{\sigma, (\omega)}^\infty(\mathbb{R}^d) \) with respect to the weak* topology when \( \max p = \infty \).
(Cf. [9, 11].) For modulation spaces of the form \( M_{\sigma, (\omega)}^p \) with \( \omega \) belonging to the subset \( \mathcal{P} \) of \( \mathcal{P}_E \), these properties were extended in [13] to the quasi-Banach case, allowing \( p \) and \( q \) to be smaller than 1. In Section 3 we extend all these properties to more general \( M_{\sigma, (\omega)}^p \), where \( \sigma \in S_{2d} \), \( \omega \in \mathcal{P}_E \) and \( p \in (0, \infty)^d \), based on the analysis in [13].

Remark 1.10. Let \( r \in (0,1) \), \( v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be submultiplicative, and set
\[
(\Theta_{\rho,v})(x, \xi) = v(x, \xi)/(x, \xi)\rho, \quad \text{where} \quad \rho > 2d(1 - r)/r. \quad (1.17)
\]
Then \( L^1_{(\Theta_{\rho,v})}(\mathbb{R}^{2d}) \) is continuously embedded in \( L^r_{(\omega)}(\mathbb{R}^{2d}) \), giving that \( M^1_{(\Theta_{\rho,v})}(\mathbb{R}^d) \subseteq M^r_{(\omega)}(\mathbb{R}^d) \). Hence if \( \phi \in M^1_{(\Theta_{\rho,v})} \setminus 0 \), \( \varepsilon_0 \) is chosen such
that $S_{\phi,\phi}^\Lambda$ is invertible on $M_{(\Theta,\nu)}^1(\mathbb{R}^d)$ for every $\Lambda = \epsilon \mathbb{Z}^d$, $\epsilon \in (0, \epsilon_0]$, it follows that both $\phi$ and its canonical dual with respect to $\Lambda$ belong to $M_{(\nu)}^1(\mathbb{R}^d)$.

## 2. Convolution estimates for Lebesgue and Wiener spaces

In this section we deduce continuity properties for discrete, semi-discrete and non-discrete convolutions. Especially we discuss such mapping properties for sequence and Wiener spaces.

In what follows we let $T_\theta, \theta \in \mathbb{R}_d$ be the diagonal $d \times d$-matrix, with $\theta_1, \ldots, \theta_d$ as diagonal values as in the previous section. The semi-discrete convolution with respect to $\theta$ is given by

$$(a *_{[\theta]} f)(x) \sum_{j \in \mathbb{Z}^d} a(j) f(x - T_\theta j),$$

when $f \in \mathcal{S}_{1/2}(\mathbb{R}^d)$ and $a \in \ell_0(\mathbb{Z}^d)$.

We have the following proposition.

**Proposition 2.1.** Let $\sigma \in S_d$, $\omega, \nu \in \mathcal{P}_E(\mathbb{R}^d)$ be such that $\omega$ is $\nu$-moderate, $\sigma, \vartheta \in \mathbb{R}_d$, and let $p, \rho \in (0, \infty]^d$ be such that $\vartheta_j = \vartheta_{\sigma(j)}$, $j = 1, \ldots, d$, and 

$$r_k \leq \min_{m \leq k}(1, p_m).$$

Also let $\nu_0 = \nu \circ T_\theta$. Then the map $(a, f) \mapsto a *_{[\theta]} f$ from $\ell_0(\mathbb{Z}^d) \times \mathcal{S}_{1/2}(\mathbb{R}^d)$ to $\mathcal{S}_{1/2}(\mathbb{R}^d)$ extends uniquely to a linear and continuous map from $\ell^r_{\sigma,\nu_0}(\mathbb{Z}^d) \times L^p_{\sigma,\nu}(\mathbb{R}^d)$ to $L^p_{\sigma,\nu}(\mathbb{R}^d)$, and

$$\|a *_{[\theta]} f\|_{L^p_{\sigma,\nu}} \leq C \|a\|_{\ell^r_{\sigma,\nu_0}(\mathbb{Z}^d)} \|f\|_{L^p_{\sigma,\nu}}, \quad (2.1)$$

where the constant $C$ is the same as in (1.1).

**Proof.** We only consider the case $\max p < \infty$. The modifications to the case when at least one $p_j$ equals $\infty$ is straight-forward and is left for the reader.

Let $h$ be defined by

$$h_\omega(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) = (|a| *_{[\theta]} |f|)(x)\omega(x).$$

Then it follows by straight-forward computations that

$$h_\omega \leq C|b_{\nu_0} *_{[\theta]} g_\omega|,$$

where $b_{\nu_0}$ is given by (1.1) with $\omega = \nu_0$ and $\Lambda = \mathbb{Z}^d$, and $g_\omega$ is given by (1.1). Since

$$\|a\|_{\ell^r_{\sigma,\nu_0}(\mathbb{Z}^d)} = \|b_{\nu_0}\|_{\ell^r(\mathbb{Z}^d)}, \quad \|f\|_{L^p_{\sigma,\nu}} = \|g_\omega\|_{L^p}$$

and

$$\|a *_{[\theta]} f\|_{L^p_{\sigma,\nu}} \leq \|h_\omega\|_{L^p},$$

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it follows that we may assume that $f$ and $a$ are non-negative, $\sigma$ is the identity map and that $\omega = v = 1$, giving that $\vartheta = \theta$.

For $x \in \mathbb{R}^d$ and $j \in \mathbb{Z}^d$, we let

$$y_k = (x_{k+1}, \ldots, x_d) \in \mathbb{R}^{d-k}, \quad l_k = (j_{k+1}, \ldots, j_d) \in \mathbb{Z}^{d-k},$$

$k = 1, \ldots, d - 1$. Also let

$$F_x(j) = f(x + T_\theta j), \quad G_{1,x}(l_1) = G_1(x, l_1) \equiv \|F_x(\cdot, l_1)\|_{L^p_\theta (\mathbb{Z})},$$

$$F_{1,y_1}(l_1) = F_1(y_1, l_1) \equiv \|G_1(\cdot, y_1, l_1)\|_{L^p(0, \theta_1)},$$

and define inductively

$$G_{k,y_{k-1}}(l_k) = G_k(y_{k-1}, l_k) \equiv \|F_{k-1,y_{k-1}}(\cdot, l_k)\|_{L^p(\mathbb{Z})},$$

$$F_{k,y_k}(l_k) = F_k(y_k, l_k) \equiv \|G_k(\cdot, y_k, l_k)\|_{L^p(0, \theta_k)}, \quad k = 2, \ldots, d - 1,$$

$$G_d(x_d) \equiv \|F_{d-1,x_d}\|_{L^p(\mathbb{Z})}, \quad \text{and} \quad F_d \equiv \|G_d\|_{L^p(0, \theta_d)}.$$  

In the same way, let

$$A_1(l_1) \equiv \|a(\cdot, l_1)\|_{L^{r_1}(\mathbb{Z})}$$

$$A_k(l_k) = \|A_{k-1}(\cdot, l_k)\|_{L^{r_k}(\mathbb{Z})}, \quad k = 2, \ldots, d - 1,$$

and

$$A_d = \|A_{d-1}\|_{L^{r_d}(\mathbb{Z})},$$

Finally, let $H_{k,y_k}(l_k) = H_k(y_k, l_k)$ and $H_d$ be the same as $F_{k,y_k}(l_k)$ and $F_d$, respectively, $k = 1, \ldots, d - 1$, after $f$ has been replaced by $a *_{[\theta]} f$.

By straight-forward computations it follows that

$$\|f\|_{L^p} = F_d, \quad \|a *_{[\theta]} f\|_{L^p} = H_d \quad \text{and} \quad \|a\|_{L^r} = A_d.$$

We claim that

$$H_{k,y_k}(l_k) \leq \left( (A^*_k *_{[\theta_k]} F^*_{k,y_k})(l_k) \right)^{1/r_k}, \quad k = 1, \ldots, d - 1,$$

$$H_d \leq A_d F_d,$$

where $\theta_k = (\theta_{k+1}, \ldots, \theta_d)$.

In fact, first assume that $k = 1$. We have

$$H_{1,y_1}(l_1) = \left( \int_0^{\theta_1} J(x_1, y_1, l_1) \, dx_1 \right)^{1/p_1},$$

where

$$J(x_1, y_1, l_1) = \sum_{j_1 \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}^{d-1}} \left( a(\cdot, m) *_{[\theta_1]} F_x(\cdot, T_{\theta_1}(l_1 - m)) \right)(j_1) \right)^{p_1}.$$  

Here $*_{[\theta_k]} = *_{[\theta_k]}$ denotes the one-dimensional semi-discrete convolution with respect to $\theta_k$. We shall consider the cases $p_1 \geq 1$ and $p_1 < 1$ separately, and start to consider the former one.
Therefore, assume that \( p_1 \geq 1 \). By applying Minkowski’s inequality on \( J(x_1, y_1, l_1) \) we get

\[
\sum_{j_1 \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}^{d-1}} (a(\cdot, m) * \theta_1 F_x(\cdot, T_{\theta_1}(l_1 - m))) (j_1) \right)^{p_1} 
\leq \left( \sum_{m \in \mathbb{Z}^{d-1}} \|a(\cdot, m) * \theta_1 F_x(\cdot, T_{\theta_1}(l_1 - m))\|_{\ell^{p_1}} \right)^{p_1} 
\leq \left( \sum_{m \in \mathbb{Z}^{d-1}} A_1(m) G_1(x, T_{\theta_1}(l_1 - m)) \right)^{p_1}.
\]

By using this estimate in (2.3) we get

\[
H_{1,y_1}(l_1) \leq \left( \int_{\mathbb{R}^d} \left( \sum_{m \in \mathbb{Z}^{d-1}} A_1(m) G_1(x, T_{\theta_1}(l_1 - m)) \right)^{1/p_1} \right)^{1/(1/p_1)} 
\leq \sum_{m \in \mathbb{Z}^{d-1}} A_1(m) F_1(y_1, T_{\theta_1}(l_1 - m)) \leq (A_1 * [\theta_1] F_{1,y_1})(l_1),
\]

and (2.2) follows in the case \( k = 1 \) and \( p_1 \geq 1 \).

Next assume that \( p_1 < 1 \). Then we get

\[
\sum_{j_1 \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}^{d-1}} (a(\cdot, m) * \theta_1 F_x(\cdot, T_{\theta_1}(l_1 - m))) (j_1) \right)^{p_1} 
\leq \sum_{m \in \mathbb{Z}^{d-1}} \sum_{j_1 \in \mathbb{Z}} \left( (a(\cdot, m) * \theta_1 F_x(\cdot, T_{\theta_1}(l_1 - m))) (j_1) \right)^{p_1} 
\leq \sum_{m \in \mathbb{Z}^{d-1}} A_1(m)^{p_1} G_1(x, T_{\theta_1}(l_1 - m))^{p_1}.
\]

By using this estimate in (2.3) we get

\[
H_{1,y_1}(l_1) \leq \left( \int_{\mathbb{R}^d} \sum_{m \in \mathbb{Z}^{d-1}} A_1(m)^{p_1} G_1(x, T_{\theta_1}(l_1 - m))^{p_1} d x_1 \right)^{1/(1/p_1)} 
\leq \left( \sum_{m \in \mathbb{Z}^{d-1}} A_1(m)^{p_1} \int_{\mathbb{R}^d} G_1(x, T_{\theta_1}(l_1 - m))^{p_1} d x_1 \right)^{1/(1/p_1)} 
\leq \left( (A_1^{p_1} * [\theta_1] F_{1,y_1}^{p_1})(l_1) \right)^{1/(1/p_1)},
\]

and (2.2) follows in the case \( k = 1 \) for any \( p_1 \in (0, \infty] \).

Next we assume that (2.2) holds for \( k < n \), where \( 1 \leq n \leq d \), and prove the result for \( k = n \). The relation (2.2) then follows by induction.
First we consider the case $q_n \equiv p_n/r_{n-1} \geq 1$. Set $y = y_{n-1} = (x_1, \ldots, x_d)$. Then $r_n = r_{n-1}$, and the inductive assumption together with Minkowski’s and Young’s inequalities give

$$
\sum_{j_n \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}^{d-n}} (A_{n-1}(\cdot, m)^{r_{n-1}} * \theta_n F_{n-1,x}(\cdot, T_{\theta_n}(l_n - m)))(j_n) \right)^{q_n} / r_{n-1}^{q_n} 
\leq \left( \sum_{m \in \mathbb{Z}^{d-n}} \left\| A_{n-1}(\cdot, m)^{r_{n-1}} * \theta_n F_{n-1,x}(\cdot, T_{\theta_n}(l_n - m)) \right\|_{\ell_p} \right) / r_{n-1}^{q_n} 
= \left( \sum_{m \in \mathbb{Z}^{d-n}} \left\| A_{n-1}(\cdot, m)^{r_{n-1}} \right\|_{\ell_p} \left\| F_{n-1,x}(\cdot, T_{\theta_n}(l_n - m)) \right\|_{\ell_p} \right) / r_{n-1}^{q_n} 
\leq \left( \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} \left\| F_{n-1,x}(\cdot, T_{\theta_n}(l_n - m)) \right\|_{\ell_p} \right) / r_{n-1}^{q_n} 
\leq \left( \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} G_n(x_n, y_n, T_{\theta_n}(l_n - m)) \right) / r_{n-1}^{q_n} 
,$$  

where the last inequality follows from the facts that $r_n = r_{n-1}$ when $p_n/p_{n-1} \geq 1$. Minkowski’s inequality now gives

$$
H_{n,y_n}(l_n) = \left( \int_0^{\theta_n} \sum_{j_n \in \mathbb{Z}} \left( H_{n-1,y_{n-1}}(j_n, l_n) \right)^{p_n} \, dx_n \right)^{1/p_n} 
\leq \left( \int_0^{\theta_n} \left( \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} G_n(x_n, y_n, T_{\theta_n}(l_n - m)) \right) \, dx_n \right)^{1/p_n} 
\leq \left( \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} \left\| G_n(\cdot, y_n, T_{\theta_n}(l_n - m)) \right\|_{\ell_p}^{r_n} \right)^{1/r_n},
$$

which gives (22) for $k = n$ in this case.

Next we consider the case when $q_n = p_n/r_{n-1} < 1$. Then $r_n = p_n$, and the inductive assumption together with Minkowski’s inequality, Young’s inequality and the fact that $\ell^{q_n}$ is an algebra under convolution,
\[
\sum_{j_n \in \mathbb{Z}} (H_{n-1, y_{n-1}}(j_n, l_n))^{p_n} \leq \sum_{m \in \mathbb{Z}^{d-n}} \left( \sum_{j_n \in \mathbb{Z}} \left( A_{n-1}(\cdot, m)^{r_n-1} \ast_{\theta_n} F_{n-1,y} (\cdot, T_{\theta_n}(l_n - m)) (j_n)^{r_n-1} \right) \right)^{p_n/r_n-1} \\
\leq \sum_{m \in \mathbb{Z}^{d-n}} \sum_{j_n \in \mathbb{Z}} \left( (A_{n-1}(\cdot, m)^{r_n-1} \ast_{\theta_n} F_{n-1,y} (\cdot, T_{\theta_n}(l_n - m))) (j_n)^{r_n-1} \right)^{p_n/r_n-1} \\
\leq \sum_{m \in \mathbb{Z}^{d-n}} \left( \|A_{n-1}(\cdot, m)^{r_n-1}\|_{\ell^{p_n/r_n-1}} \|F_{n-1,y}(\cdot, T_{\theta_n}(l_n - m))\|^{r_n-1}_{\ell^{p_n/r_n-1}} \right)^{p_n/r_n-1} \\
= \sum_{m \in \mathbb{Z}^{d-n}} \|A_{n-1}(\cdot, m)\|_{\ell^{p_n}}^{p_n} \|G_{n,y}(T_{\theta_n}(l_n - m))\|^{r_n}_{\ell^{p_n}} \\
= \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} \|G_{n,(x_n,y_n)}(T_{\theta_n}(l_n - m))\|^{r_n}_{\ell^{p_n}}.
\]

This gives

\[
H_{n,y_n}(l_n) \leq \left( \int_0^{\theta_n} \sum_{j_n \in \mathbb{Z}} (H_{n-1, y_{n-1}}(j_n, l_n))^{p_n} \, dx_n \right)^{1/p_n} \\
\leq \left( \int_0^{\theta_n} \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} \|G_{n,(x_n,y_n)}(T_{\theta_n}(l_n - m))\|^{p_n}_{\ell^{p_n}} \, dx_n \right)^{1/p_n} \\
= \left( \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} \int_0^{\theta_n} \|G_{n,(x_n,y_n)}(T_{\theta_n}(l_n - m))\|^{p_n}_{\ell^{p_n}} \, dx_n \right)^{1/p_n} \\
= \left( \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} F_{n,y_n}(l_n - m)^{p_n} \right)^{1/p_n} \\
= \left( \sum_{m \in \mathbb{Z}^{d-n}} A_n(m)^{r_n} F_{n,y_n}(l_n - m)^{r_n} \right)^{1/r_n}.
\]

This gives (2.2) in this case as well. Hence (2.2) holds for any \( n \leq d \).

By choosing \( n = d \) in (2.2) it follows that \( a \ast_{[\theta]} f \) is uniquely defined and satisfies (2.1) when \( a \in \ell_0(\mathbb{Z}^d) \) and \( f \in L^p_{\sigma, (\omega)}(\mathbb{R}^d) \). Since \( \ell_0 \) is dense in \( L^p_{\sigma, (\omega)} \), the result now follows for general \( a \in L^p_{\sigma, (\omega)}(\mathbb{Z}^d) \). The proof is complete. \( \square \)
By choosing $\theta_1 = \cdots = \theta_d = 1$ and $f$ to be constant on each open cube $j + (0, 1)^d$ in the previous proposition, we get the following extension of Lemma 2.7 in [13]. The details are left for the reader.

**Corollary 2.2.** Let $\sigma \in S_d$, $\omega, \nu \in \mathcal{P}_E(\mathbb{R}^d)$ be such that $\omega$ is $v$-moderate, and let $p, r \in (0, \infty]^d$ be such that

$$r_k \leq \min_{m \leq k} (1, p_m).$$

Then the map $(a, b) \mapsto a * b$ on $\ell_0(\mathbb{Z}^d)$ extends uniquely to a linear and continuous map from $\ell_p^r(\mathbb{Z}^d) \times \ell_p^s(\mathbb{Z}^d)$ to $\ell_p^{\sigma, (\omega)}(\mathbb{Z}^d)$. In particular,

$$\|a * b\|_{\ell_p^{\sigma, (\omega)}} \leq C \|a\|_{\ell_p^{\sigma, (\nu)}} \|b\|_{\ell_p^{\sigma, (\omega)}},$$

for some constant $C$ which is independent of $a \in \ell_p^{r, (\nu)}(\mathbb{Z}^d)$ and $b \in \ell_p^{s, (\omega)}(\mathbb{Z}^d)$.

For the link between modulation spaces and sequence spaces we need to consider a broad family of Wiener spaces.

**Definition 2.3.** Let $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, $p \in (0, \infty]^d$, $q \in [1, \infty]$, $\sigma \in S_d$, and let $\chi$ be the characteristic function of $Q \equiv [0, 1]^d$. Then the Wiener space $W^q(\omega, \ell_p^q(\mathbb{Z}^d))$ consists of all measurable functions $f$ on $\mathbb{R}^d$ such that

$$\|f\|_{W^q(\omega, \ell_p^q)} \equiv \|b_{f, \omega, q}\|_{\ell_p^q},$$

is finite, where $b_{f, \omega, q}$ is the sequence on $\mathbb{Z}^d$, given by

$$b_{f, \omega, q}(j) \equiv \|f\|_{L^q(\omega, j + Q)} = \|f \cdot \chi(\cdot - j)\|_{L^q(\mathbb{R}^d)}.$$

Especially $W^\infty(\omega, \ell_p^q)$ in Definition 2.3 is important (i.e. the case $q = \infty$), and we set

$$W(\omega, \ell_p^q(\mathbb{Z}^d)) = W^\infty(\omega, \ell_p^q(\mathbb{Z}^d)).$$

This space is also called the *coorbit space* of $L_p^q(\mathbb{R}^d)$ with weight $\omega$, and is sometimes denoted by

$$Co(L_p^q(\mathbb{R}^d)) \text{ or } W(L_p^q(\mathbb{R}^d)) = W(L_p^q(\mathbb{R}^d)),$$

in the literature (cf. [18, 27]).

We also use the notation

$$W^q(\ell_p^q(\mathbb{Z}^d)) \text{ and } W(\ell_p^q(\mathbb{Z}^d))$$

instead of

$$W^q(\omega, \ell_p^q(\mathbb{Z}^d)) \text{ and } W(\omega, \ell_p^q(\mathbb{Z}^d)),$$

respectively, when $\omega = 1$.

We have now the following lemma concerning pullbacks of dilations in Wiener spaces. Here we let $\lfloor x \rfloor$ denote the integer part of $x$. 


Lemma 2.4. Let $R \geq 1$, $\sigma \in S_d$, $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, $\theta \in (0, R]^d$, $p \in (0, \infty]^d$, $q \in (0, \infty]$, and let $f \in \mathcal{W}^q(\omega, \ell^q_p(\mathbb{Z}^d))$. Then

$$T_{\theta}^* f \in \mathcal{W}^q(T_{\theta}^* \omega, \ell^q_p(\mathbb{Z}^d)),$$

and

$$\|T_{\theta}^* f\|_{\mathcal{W}^q(T_{\theta}^* \omega, \ell^q_p)} \leq C \left( \prod_{k=1}^{d} |\theta_k|^{-1/q} \left[ 1 + |\theta_k|^{-1} \right]^{1/p_q(\omega)} \right) \|f\|_{\mathcal{W}^q(\omega, \ell^q_p)},$$

for some constant $C$ which only depends on $\omega$ and $R$.

Proof. By considering

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \quad \text{and} \quad \omega(x_{\sigma(1)}, \ldots, x_{\sigma(d)})$$

instead of $f(x_1, \ldots, x_d)$ and $\omega(x_1, \ldots, x_d)$, we reduce ourselves to the case when $\sigma$ is the identity map.

Let $Q = [0, 1]^d$,

$$\Omega_{n,j} = T_{\theta}(j + Q) \cap (n + Q) \subseteq n + Q,$$

$$I_n \equiv \{ j \in \mathbb{Z}^d; \Omega_{n,j} \neq \emptyset \},$$

$$M \equiv \{ (n_1, j_1, \ldots, n_d, j_d) \in \mathbb{Z}^{2d}; (j_1, \ldots, j_d) \in I_n \},$$

$$r \equiv (p_1, p_1, p_2, p_2, \ldots, p_d, p_d) \in (0, \infty)^{2d}$$

and

$$c_1(\theta) \equiv \prod_{k=1}^{d} |\theta_k|^{-1/q}$$

Then

$$\|T_{\theta}^* f\|_{\mathcal{W}^q(T_{\theta}^* \omega, \ell^q_p)} = \left\| \left\{ \|f(T_{\theta} \cdot)\|_{L^q(T_{\theta}(j + Q)) \omega(T_{\theta}j)} \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell^q_p(\mathbb{Z}^d)}$$

$$= c_1(\theta) \left\| \left\{ \|f\|_{L^q(T_{\theta}(j + Q)) \omega(T_{\theta}j)} \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell^q_p(\mathbb{Z}^d)}$$

$$c_1(\theta) \left\| \left\{ \|f\|_{L^q(\Omega_{n,j}) \omega(n)} \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell^q_p(M)}$$

$$\leq C c_1(\theta) \left\| \left\{ \|f\|_{L^q(\Omega_{n,j}) \omega(n)} \right\}_{(n,j) \in \mathbb{M}} \right\|_{\ell^q_p(M)},$$

where

$$C = \sup_{x \in \mathbb{R}^d} \left( \sup_{y \in [-1, R]^d} \omega(x + y)/\omega(x) \right) < \infty.$$

Here we use the convention that for any subset $M$ of $\mathbb{Z}^d$ and sequence $a$ on $M$, then $\|a\|_{\ell^q_p(M)} \equiv \|b\|_{\ell^q_p(\mathbb{Z}^d)}$, where $b(j) = a(j)$ when $j \in M$, and $b(j) = 0$ otherwise.
Since \( \|f\|_{L^q(\Omega_n,\beta)} \leq \|f\|_{L^q(n+Q)} \) and the number of terms in \( I_n \) in direction \( k \) is at most \( \lfloor 1 + \theta_k \rfloor \), we get

\[
\|T_\theta f\|_{W_\theta(T^n_\theta \omega, \ell p)} \leq C_{c1}(\theta)c_2(\theta) \left\{ \|f\|_{L^q(n+Q)\omega(n)} \right\}
\]

where

\[
c_2(\theta) = \prod_{k=1}^d \lfloor 1 + \theta_k \rfloor^{1/p_k}.
\]

This gives the result. \( \square \)

**Proposition 2.5.** Let \( \sigma \in S_d, \theta \in \mathbb{R}_d, \omega_k \in \mathcal{P}_{E}(\mathbb{R}_d), \) and let \( p_k \in (0, \infty]^d, q_k \in (0, \infty], k = 1, 2, 3, \) be such that \( q_0 \geq 1, \)

\[
L^{q_1}(\mathbb{R}_d)^*L^{q_2}(\mathbb{R}_d) \subseteq L^{p_0}(\mathbb{R}_d) \quad \text{and} \quad \ell^\rho_{\sigma,0}(Z^d)^* \ell^\rho_{\sigma,0}(Z^d) \subseteq \ell^{p_0}_{\sigma,0}(Z^d),
\]

with continuous embeddings, and

\[
\max(p_1, q_1) < \infty \quad \text{or} \quad \max(p_2, q_2) < \infty. \tag{2.5}
\]

Then the following is true:

1. The map \((f_1, f_2) \mapsto f_1 f_2\) is continuous from \( W^{\rho_1}(\omega_1, \ell^\rho_{\sigma,0}(Z^d)) \times W^{\rho_2}(\omega_2, \ell^\rho_{\sigma,0}(Z^d)) \) to \( W^{\rho_0}(\omega_0, \ell^\rho_{\sigma,0}(Z^d)) \), and

\[
\|f_1 f_2\|_{W^{\rho_0}(\omega_0, \ell^\rho_{\sigma,0})} \lesssim \|f_1\|_{W^{\rho_1}(\omega_1, \ell^\rho_{\sigma,0})} \|f_2\|_{W^{\rho_2}(\omega_2, \ell^\rho_{\sigma,0})};
\]

2. The map \((a, f) \mapsto a \star[f] f\) is continuous from \( \ell^\rho_{\sigma,0}(T^n_\theta \omega_1)(Z^d) \times W(\omega_2, \ell^\rho_{\sigma,0}(Z^d)) \) to \( W(\omega_0, \ell^\rho_{\sigma,0}(Z^d)) \), and

\[
\|a \star[f] f\|_{W(\omega_0, \ell^\rho_{\sigma,0})} \lesssim \|a\|_{\ell^\rho_{\sigma,0}(\omega_1)} \|f\|_{W(\omega_2, \ell^\rho_{\sigma,0})}
\]

**Proof.** By (2.5) and density argument, it suffices to prove the quasi-norm estimates. Furthermore, by a suitable change of variables, we may assume that \( \sigma \) is the identity map.

1. Let \( Q = [0, 1]^d \) as usual, and let \( a_k, k = 0, 1, 2, \) be the sequences on \( Z^d \), defined by

\[
a_k(j) \equiv \|f_k\|_{L^{\rho_k}(j+Q)}, \quad j \in Z^d
\]
and \( f_0 = f_1 \ast f_2 \). Then

\[
a_0(j) \leq \left( \int_{j+Q} \left( \int_{\mathbb{R}^d} |f_1(x-y)f_2(y)| \, dy \right)^{q_0} \, dx \right)^{1/q_0}
\]

\[
= \left( \int_{j+Q} \left( \sum_{j_0 \in \mathbb{Z}^d} \int_{j_0+Q} |f_1(x-y)f_2(y)| \, dy \right)^{q_0} \, dx \right)^{1/q_0}
\]

\[
\leq \sum_{j_0 \in \mathbb{Z}^d} \left( \int_{j+Q} \left( \int_{j_0+Q} |f_1(x-y)f_2(y)| \, dy \right)^{q_0} \, dx \right)^{1/q_0} \tag{2.6}
\]

Now, if \( x \in j + Q \) and \( y \in j_0 + Q \), then

\[
x - y \in j - j_0 + [-1, 1]^d = \bigcup_{n \in \{0,1\}^d} (j - j_0 - n + Q).
\]

Hence if \( h_k(j, \cdot) = f_k \chi_{j+Q}, k = 1, 2 \), then (2.6) and Young’s inequality give

\[
a_0(j) \leq \sum_{n \in \{0,1\}^d} \sum_{j_0 \in \mathbb{Z}^d} \| h_1(j - j_0 + n, \cdot) \ast h_2(j_0, \cdot) \|_{L^{q_0}}
\]

\[
\leq \sum_{n \in \{0,1\}^d} \sum_{j_0 \in \mathbb{Z}^d} \| h_1(j - j_0 + n, \cdot) \|_{L^{p_1}} \| h_2(j_0, \cdot) \|_{L^{p_2}}
\]

\[
= \sum_{n \in \{0,1\}^d} (a_1 \ast a_2)(j + n).
\]

Here the convolution between \( h_1(j_1, x) \) and \( h_2(j_2, x) \) should be taken with respect to the \( x \)-variable only, considering \( j_1 \) and \( j_2 \) as constants.

Now it follows from the assumptions that

\[
\| f_1 \ast f_2 \|_{W^{\omega_0}(\omega_0, \ell^{p_0})} = \| a_0 \|_{\ell^{p_0}_{(\omega_0)}}
\]

\[
\leq \sum_{n \in \{0,1\}^d} \| (a_1 \ast a_2)(\cdot + n) \|_{\ell^{p_0}_{(\omega_0)}} \lesssim \| a_1 \ast a_2 \|_{\ell^{p_0}_{(\omega_0)}}
\]

\[
\lesssim \| a_1 \|_{\ell^{p_1}_{(\omega_1)}} \| a_2 \|_{\ell^{p_2}_{(\omega_2)}} = \| f_1 \|_{W^{\omega_1}(\omega_1, \ell^{p_1})} \| f_2 \|_{W^{\omega_2}(\omega_2, \ell^{p_2})},
\]

and the result follows in this case. Here the second inequality follows from the fact that \( \omega_0 \) is \( v \)-moderate for some \( v \). This gives (1).

(2) Since \( \omega \) is \( v \)-moderate we get

\[
|a \ast g| \cdot \omega \lesssim |a \cdot (\omega \circ T_\theta)| \ast |f \cdot v|,
\]

which reduce the situation to the case when \( \omega = v = 1 \). Furthermore, since

\[
\| a \ast g \|_{W(\theta v)} \leq \| a \|_{W(\theta v)} \| g \|_{W(\theta v)}, \quad \| a \|_{\ell^p} = \| a \|_{\ell^p}
\]

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and
\[ \|f\|_{W(L^p)} \asymp \|g\|_{W(p^q)} \]
when
\[ T_\vartheta^* g = \sum_{j \in \mathbb{Z}^d} \|f\|_{L^\infty(j+Q)} \chi_{j+Q}, \]
we may assume that \( a \geq 0 \) and \((T_\vartheta^* f)(x) = b(j) \geq 0\) when \( x \in j + Q\).

Let \( \theta_0 = (1, \ldots, 1) \). By Lemma 2.4 we get
\[ \|a * [\omega] f\|_{W(p^q)} \asymp \|T_\vartheta^*(a * [\omega] f)\|_{W(p^q)} = \|a * [\omega_0] (T_\vartheta^* f)\|_{W(p^q)} \]
\[ = \|a * b\|_{p^q} \lesssim \|a\|_{p^q} \|b\|_{p^q} \asymp \|a\|_{p^q} \|f\|_{W(p^q)}, \]
and the result follows. \( \square \)

3. TIME-FREQUENCY REPRESENTATION OF MODULATION SPACES

In this section we extend the Gabor analysis for modulation spaces of the form \( M_{p,q}^\omega (\mathbb{R}^d) \) with \( p, q \in (0, \infty] \) and \( \omega \in \mathcal{P}(\mathbb{R}^d) \) in [13], to spaces of the form \( M_{p,\omega}^\sigma (\mathbb{R}^d) \) with \( \sigma \in S_{2d}, p \in (0, \infty]^{2d} \) and \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \). Especially we deduce invariance properties for \( M_{p,\omega}^\sigma (\mathbb{R}^d) \) concerning the choice of the window function \( \phi \) in [11,14], and that the results on Gabor expansions in [13,18] also hold in this more general situation. As a consequence we deduce that \( M_{p,\omega}^\sigma \) increases with \( p \).

We have now the following proposition.

**Proposition 3.1.** Let \( p \in (0, \infty)^{2d} \), \( r = \min(1, p) \), \( \omega, v \in \mathcal{P}_E(\mathbb{R}^d) \) be such that \( \omega \) is \( v \)-moderate, and let \( \Theta_{v^\omega} \) be the same as in Remark 1.10. Also let \( \sigma \in S_{2d}, \omega_1, \omega_2 \in M_{1,\Theta_{v^\omega}}^\sigma(\mathbb{R}^d), \) \( \omega_1 \in \Sigma^\sigma_1(\mathbb{R}^d), \) and let \( f \in \Sigma^\sigma_1(\mathbb{R}^d) \). Then
\[ \|V_{\omega_1} f\|_{L^p_{p,\omega_1}} \leq C \|V_{\omega_2} f\|_{L^p_{p,\omega_2}}, \]
for some constant \( C \) which is independent of \( f \in \Sigma^\sigma_1(\mathbb{R}^d) \). In particular, the modulation space \( M_{p,\omega}^\sigma(\mathbb{R}^d) \) is independent of the choice of \( \phi \in M_{1,\Theta_{v^\omega}}^\sigma(\mathbb{R}^d) \) \( \setminus 0 \) in [11,14], and different choices of \( \phi \) give rise to equivalent norms.

The proof follows by similar arguments as the proof of Theorem 3.1 in [13]. In order to be self-contained we here present a proof. For the proof we need the following lemma on point estimates for short-time Fourier transforms with Gaussian windows. The result is a slight extension of Lemma 2.3 in [13]. Here and in what follows we let \( B_r(x_0) \) be the open ball in \( \mathbb{R}^d \) with center at \( x_0 \in \mathbb{R}^d \) and radius \( r > 0 \).

**Lemma 3.2.** Let \( p \in (0, \infty], r > 0, (x_0, \xi_0) \in \mathbb{R}^{2d} \) be fixed, and let \( \phi \in S^1_{1/2}(\mathbb{R}^d) \) be a Gaussian. Then
\[ |V_{\phi} f(x_0, \xi_0)| \leq C \|V_{\phi} f\|_{L^p(B_r(x_0, \xi_0))}, \quad f \in S^1_{1/2}(\mathbb{R}^d), \]
and
\[ \|f\|_{W(L^p)} \asymp \|g\|_{W(p^q)} \]
where the constant $C$ is independent of $(x_0, \xi_0)$ and $f$.

When proving Lemma 3.2 we may first reduce ourself to the case that the Gaussian $\phi$ should be centered at origin, by straight-forward arguments involving pullbacks with translations. The result then follows by using the same arguments as in [13, Lemma 2.3.] and its proof, based on the fact that

$$z \mapsto F_w(z) \equiv e^{c_1|z|^2+c_2(z,w)+c_3|w|^2}V_\phi f(x, \xi), \quad z = x + i\xi$$

is an entire function for one choice of the constant $c_1$ (depending on $\phi$).

**Remark 3.3.** We note that Lemma 2.3 and its proof in [13] contains a mistake, which is not important in the applications. In fact, when using the mean-value inequality for subharmonic functions in the proof, a factor of the volume for the ball which corresponds to $B_r(x_0, \xi_0)$ in Lemma 3.2 is missing. This leads to that stated invariance properties of constants in several results in [13] are more dependent of the involved parameters than what are stated.

**Proof of Proposition 3.1.** Let $v_0 = \Theta_{\mu,v}$, and let

$$\Lambda = \varepsilon Z^{2d} = \{ \varepsilon x_j, \varepsilon \xi_k; j, k \in J \},$$

where $J$ is an index set and $\varepsilon > 0$ is chosen small enough such that $\{e^{i(\cdot, \xi_k)}\phi_1(\cdot - x_j)\}_{j, k \in J}$ is a Gabor frame. Since $\phi_2 \in M^1_{(v_0)}$, it follows that its dual window $\psi$ belongs to $M^1_{(v_0)}$, in view of Proposition 1.8. By Proposition 1.6 (3) we have

$$\phi_1 = \sum_{j,k \in J} (V_\psi \phi_1)(x_j, \xi_k)e^{i(\cdot, \xi_k)}\phi_2(\cdot - x_j),$$

with unconditional convergence in $M^1_{(v_0)}$. This gives,

$$|V_\phi f(x, \xi)| = (2\pi)^{-d/2}|(f, e^{i(\cdot, \xi)}\phi_1(\cdot - x))|$$

$$\leq (2\pi)^{-d/2} \sum_{j,k \in J} |(V_\psi \phi_1)(x_j, \xi_k)||(f, e^{i(\cdot, \xi+\xi_k)}\phi_2(\cdot - x - x_j))|$$

$$= (2\pi)^{-d/2} \sum_{j,k \in J} |(V_\psi \phi_1)(x_j, \xi_k)||V_\phi f(x + x_j, \xi + \xi_j)| = (|b|_{\theta_j}||V_\phi f||(x, \xi),$$

where $b(x_j, \xi_k) = |(V_\psi \phi_1)(-\varepsilon x_j, -\varepsilon \xi_k)|$, and $\theta_j = \varepsilon, j = 1, \ldots, 2d$.

By Proposition 2.1 and Lemma 2.4 we get with $r = \min p$,

$$\|V_\phi f\|_{L^p_{\sigma,(v)}} \lesssim \|b\|_{r_{\sigma,(v)}}\|V_\phi f\|_{L^p_{(v)}}$$

$$\lesssim \|b\|_{r_{(v)}}\|V_\phi f\|_{L^p_{\sigma,(v)}} \lesssim \|V_\psi \phi_1\|_{L^1_{(v)}}\|V_\phi f\|_{L^p_{\sigma,(v)}}.$$
Here we have used the fact that \( \| V_{\psi} \phi_1 \|_{L^1_{\psi,\omega}} \times \| \phi_1 \|_{M^1_{\psi,\omega}} < \infty \) by Proposition 12.1.2 in [13], the result follows. □

We have now the following result related to [13, Theorem 3.3].

**Proposition 3.4.** Let \( p \in (0, \infty)^{2d} \), \( \omega, v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, \( \Theta_{p,v} \) be the same as in Proposition [17], \( \phi_1, \phi_2 \in M^1_{(\Theta, v)}(\mathbb{R}^d) \setminus \{0\}, \sigma \in S_{2d} \) and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \). Then \( V_{\phi_1} f \in L^p_{\sigma,\omega} (\mathbb{R}^{2d}) \), if and only if \( V_{\phi_2} f \in W(\omega, \ell^p_\omega(\mathbb{Z}^{2d})) \), and

\[
\| V_{\phi_1} f \|_{L^p_{\sigma,\omega}} \lesssim \| V_{\phi_2} f \|_{W(\omega, \ell^p_\omega)}, \quad f \in S'_{1/2}(\mathbb{R}^d).
\]

For the proof we note that for every measurable function \( F \) on \( \mathbb{R}^{2d} \) we have

\[
\| F \|_{W^1(v, \ell^p_v)} \lesssim \| F \|_{L^1_{\sigma,(\Theta, v)}},
\]

which follows by an application of Hölder’s inequality.

**Proof.** By the definitions it follows that

\[
\| V_{\phi} f \|_{L^p_{\sigma,\omega}} \lesssim \| V_{\phi} f \|_{W(\omega, \ell^p_\omega)},
\]

when \( \phi \in S_{1/2} \).

When proving the reversed inequality we start by considering the case when \( \phi_1 = \phi_2 = \phi \) is a Gaussian. First we need to introduce some notations. We set

\[
X = (X_1, \ldots, X_{2d}) = (x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)
\]

\[
Y = (Y_1, \ldots, Y_{2d}) = (X_{\sigma(1)}, \ldots, X_{\sigma(2d)}),
\]

\[
r = \min p \quad \text{and} \quad F(Y) = |V_{\phi} f(X)| \omega(X)
\]

For every \( k \in \{0, \ldots, 2d\} \) we also set

\[
q_k = (p_1, \ldots, p_k), \quad r_k = (p_{k+1}, \ldots, p_{2d}),
\]

\[
t_k = (Y_{k+1}, \ldots, Y_{2d}), \quad Q_k = [-2, 2]^k,
\]

and

\[
\bar{b}_k(l) \equiv \left( \int_{l+Q_{2d-k}} \| F(\cdot, t_k) \|_{L^q_k} \, dt_k \right)^{1/r}, \quad l \in \mathbb{Z}^{2d-k}, \quad k < 2d
\]

\[
\bar{b}_{2d} \equiv \| F \|_{L^{q_{2d}}} = \| V_{\phi} f \|_{L^p_{\sigma,\omega}}.
\]

We claim that for every \( k \in \{1, \ldots, 2d\} \), the inequality

\[
\| V_{\phi} f \|_{W(\omega, \ell^p_\omega)} \lesssim \| b_k \|_{\ell^q_k}
\]

holds.

In fact, for \( k = 1 \), the result follows from Lemmas [2.13] and [3.2] Hölder’s inequality and the fact that \( \omega \) is moderate.
Assume that the result is true for \( k \in \{1, \ldots, 2d-1\} \), and prove the result for \( k+1 \). For notational convenience we only prove the statement in the case \( p_0 = p_{k+1} < \infty \). The case \( p_{k+1} < \infty \) follows by similar arguments and are left for the reader.

Let \( t = t_{k+1} \) and

\[
c_k(l) = \left( \sum_{j \in \mathbb{Z}} \left( \int_{l+Q_{2d-k-1}} \int_{j+Q_1} \| F(\cdot, z, t) \|_{L_{q_k}}^r \, dz \, dt \right)^{p_0/r} \right)^{1/p_0}.
\]

Then

\[
\| b_k(\cdot, t) \|_{\ell^{p_{k+1}}(\mathbb{Z})} = c_k(l), \quad l \in \mathbb{Z}^{2d-k},
\]
giving that

\[
\| V^{\rho} f \|_{W^{(\omega, \ell^p)}} \lesssim \| b_k \|_{\ell^{p_{k+1}}(\mathbb{Z}^{2d-k})} = \| c_k \|_{\ell^{p_{k+1}}(\mathbb{Z}^{2d-k-1})}.
\] (3.3)

Since \( p_0 \geq r \), Minkowski’s and Hölder’s inequalities give

\[
c_k(l) = \left( \sum_{j \in \mathbb{Z}} \left( \int_{l+Q_{2d-k-1}} \int_{j-2}^{2} \| F(\cdot, z + j, t) \|_{L_{q_k}}^r \, dz \, dt \right)^{p_0/r} \right)^{1/p_0}
\]

\[
\leq \left( \sum_{j \in \mathbb{Z}} \left( \int_{l+Q_{2d-k-1}} \left( \int_{-2}^{2} \| F(\cdot, z + j, t) \|_{L_{q_k}}^{p_0} \, dz \right)^{r/p_0} \right)^{p_0/r} \right)^{1/p_0}
\]

\[
\leq \left( \int_{l+Q_{2d-k-1}} \left( \sum_{j \in \mathbb{Z}} \int_{-2}^{2} \| F(\cdot, z + j, t) \|_{L_{q_k}}^{p_0} \, dz \right)^{r/p_0} \right)^{1/r}
\]

\[
\leq \left( \int_{l+Q_{2d-k-1}} \int_{\mathbb{R}} \| F(\cdot, z, t) \|_{L_{q_k}}^{p_0} \, dz \right)^{r/p_0} \right)^{1/r} = b_{k+1}(l),
\]

and the induction step follows from these estimates, (3.2) and (3.3). This gives the result when \( \phi \) is a Gaussian.

Next assume that \( \phi \in M^1_{(\Theta, \nu)} \setminus 0 \) is arbitrary, and let \( \phi_0 \) be a fixed Gaussian. Then

\[
\| V^{\rho} f \|_{L^p_{(\Theta, \nu)}} \asymp \| V^{\rho} f \|_{L^p_{(\Theta, \nu)}},
\]

by Proposition 3.1 and the result follows if we prove

\[
\| V^{\rho} f \|_{W^{(\omega, \ell^p)}} \lesssim \| V^{\rho} f \|_{W^{(\omega, \ell^p)}}.
\] (3.4)

We have

\[
| V^{\rho} f | \lesssim | V^{\rho} f | * | V^{\rho} \phi_0 |,
\]

(cf. [18, Chapter 11]). An application of Proposition 2.5 gives

\[
\| V^{\rho} f \|_{W^{(\omega, \ell^p)}} \lesssim \| V^{\rho} \phi_0 \|_{W^{(\omega, \ell^p)}} \| V^{\rho} f \|_{W^{(\omega, \ell^p)}} \lesssim \| V^{\rho} f \|_{L^p_{(\Theta, \nu)}}.
\]
Here the last inequality follows from (3.1). This gives (3.4), and the result follows.

The next result is an immediate consequence of the previous proposition and the fact that $\mathcal{P}_p(\omega)$ is increasing with respect to $p$, giving that $W(\omega, \mathcal{P}_p)$ increases with $p$, when $\omega \in \mathcal{P}_E$.

**Proposition 3.5.** Let $\sigma \in S_{2d}$, $p_1, p_2 \in (0, \infty]^{2d}$ and $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $p_1 \leq p_2$ and $\omega_2 \lesssim \omega_1$. Then

$$M_{\sigma, (\omega_1)}^{p_1}(\mathbb{R}^d) \subseteq M_{\sigma, (\omega_2)}^{p_2}(\mathbb{R}^d),$$

and

$$\|f\|_{M_{\sigma, (\omega_2)}^{p_2}} \lesssim \|f\|_{M_{\sigma, (\omega_1)}^{p_1}}, \quad f \in \Sigma_{\sigma}^1(\mathbb{R}^d).$$

Next we extend the Gabor analysis in [13] to modulation spaces of the form $M_{\sigma, (\omega)}^p$, with Lebesgue exponents and weights as before. The first two results show that the analysis and synthesis operators possess the requested continuity properties.

**Proposition 3.6.** Let $\Lambda = T_\theta Z^{2d}$ for some $\theta \in \mathbb{R}^{2d}$, $p \in (0, \infty]^{2d}$, $0 < r \leq \min(1, p)$, and let $\omega, \nu \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $\nu$-moderate. Also let $\phi, \psi \in \Sigma_{\sigma}^r(\mathbb{R}^d)$, and let $C_\phi$ and $D_\psi$ be as in Definition 1.7. Then the following is true:

1. $C_\phi$ is uniquely extendable to continuous map from $M_{\sigma, (\omega)}^p(\mathbb{R}^d)$ to $\ell_{\sigma, (\omega)}^p(\Lambda)$;

2. $D_\psi$ is uniquely extendable to continuous map from $\ell_{\sigma, (\omega)}^p(\Lambda)$ to $M_{\sigma, (\omega)}^p(\mathbb{R}^d)$.

Furthermore, if $\max p < \infty$, $f \in M_{\sigma, (\omega)}^p(\mathbb{R}^d)$ and $c \in \ell_{\sigma, (\omega)}^p(\Lambda)$, then $C_\phi f$ and $D_\psi c$ converge unconditionally and in norms. If instead $\max p = \infty$, then $C_\phi f$ and $D_\psi c$ converge in the weak$^*$ topology in $\ell_{\sigma, (\omega)}^\infty(\Lambda)$ and $M_{\sigma, (\omega)}^\infty(\mathbb{R}^d)$, respectively.

**Proof.** We shall mainly follow the proofs of Theorems 3.5 and 3.6 in [13]. It suffices to prove the desired norm estimates

$$\|C_\phi f\|_{\ell_{\sigma, (\omega)}^p(\Lambda)} \lesssim \|f\|_{M_{\sigma, (\omega)}^p(\mathbb{R}^d)} \quad \text{and} \quad \|D_\psi c\|_{M_{\sigma, (\omega)}^p(\mathbb{R}^d)} \lesssim \|c\|_{\ell_{\sigma, (\omega)}^p(\Lambda)}, \quad (3.5)$$

when $f \in \Sigma_{1}(\mathbb{R}^d)$ and $c \in \ell_0(\Lambda)$.

In fact, if $\max p < \infty$, then the result follows from (3.5) and the fact that $\Sigma_1$ and $\ell_0$ are dense in $M_{\sigma, (\omega)}^p$ and $\ell_{\sigma, (\omega)}^p$, respectively. If instead $\max p = \infty$, then the result follows from the facts that both $M_{\sigma, (\omega)}^p$ and $\ell_{\sigma, (\omega)}^p$ increase with $p$, and that $\Sigma_1$ and $\ell_0$ are dense in $M_{\sigma, (\omega)}^\infty$ and $\ell_{\sigma, (\omega)}^\infty$, respectively, with respect to the weak$^*$-topologies.

In order to prove the first inequality in (3.5), let $\Lambda = \{(x_j, \xi_k)\}_{j,k \in J}$ as before. Then

$$C_\phi f = \{V_\phi f(x_j, \xi_k)\}_{j,k \in J},$$
and Propositions 3.1 and 3.4 gives

$$\|C_\phi f\|_{P_{\sigma_1}(\Lambda)} \lesssim \|V_\phi f\|_{W^{\omega_0}(\Lambda)} \lesssim \|V_\phi f\|_{L^p_{\sigma_1}(\omega)} \lesssim \|f\|_{M^p_{\sigma_1}(\omega)}.$$  

This gives the first estimate in (3.5).

For the second estimate in (3.5), let $\phi_0 \in \Sigma_1$ be fixed. Then

$$|V_{\phi_0}(D_\phi c)(x, \xi)| = \sum_{j,k \in J} c_j \cdot \theta_k \|V_{\phi_0}(e^{i(x,\xi) \psi(\cdot - x_j)})(x, \xi)| \leq (b \ast |c|)(x, \xi),$$

where $b \equiv T_{\phi_0}^\ast |c|$ is a sequence on $\mathbb{Z}^{2d}$. Hence, by letting $p_0 = p_1 = p$ and $p_2 = r$ in (2) in Proposition 2.5, Propositions 2.1 gives

$$\|D_\phi c\|_{M^p_{\sigma_1}(\omega)} \lesssim \|b \ast |c|\|_{L^p_{\sigma_1}(\omega)} \lesssim \|b \ast |c|\|_{W^{\omega_0}(\Lambda)} \lesssim \|c\|_{L^p_{\sigma_1}(\omega)} \lesssim \|c\|_{L^p_{\sigma_1}(\omega)} \|\psi\|_{M^r_{\sigma_1}(\omega)},$$

and the result follows.

As a consequence of the last proposition we get the following.

**Theorem 3.7.** Let $\Lambda = T_{\phi} \mathbb{Z}^{2d} = \{(x_j, \xi_k)\}_{j,k \in J}$, where $\theta \in \mathbb{R}^{2d}$, $p, r \in (0, \infty]^{2d}$, $\sigma \in \mathcal{S}_{2d}$, and let $\omega, v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be the same as in Proposition 3.6. Also let $\phi, \psi \in M^p_{\sigma_1}(\mathbb{R}^{2d})$ be such that

$$\{e^{i(x,\xi) \psi(\cdot - x_j)}\}_{j,k \in J} \text{ and } \{e^{i(x,\xi) \psi(\cdot - x_j)}\}_{j,k \in J}$$

are dual frames to each other. Then the following is true:

1. The operators $S_{\phi,\psi} \equiv D_\psi \circ C_\phi$ and $S_{\psi,\phi} \equiv D_\phi \circ C_\psi$ are both the identity map on $M^p_{\sigma_1}(\mathbb{R}^{2d})$, and

$$f = \sum_{j,k \in J} (V_\phi f)(x_j, \xi_k) e^{i(x,\xi) \psi(\cdot - x_j)}$$

$$= \sum_{j,k \in J} (V_\psi f)(x_j, \xi_k) e^{i(x,\xi) \phi(\cdot - x_j)},$$

with unconditional norm-convergence in $M^p_{\sigma_1}(\omega)$ when $\max p < \infty$, and with convergence in $M^\infty_{\sigma_1}(\omega)$ with respect to the weak* topology otherwise;

2. $\|f\|_{M^p_{\sigma_1}(\omega)} \lesssim \|(V_\phi f) \circ T_{\theta}\|_{P_{\sigma_1}(\omega)} \lesssim \|(V_\psi f) \circ T_{\theta}\|_{P^p_{\sigma_1}(\omega)}$.

**Proof.** By Corollary 12.2.6 in [18], the result follows in the case $M^p_{\sigma_1}(\omega) = M^\infty_{\sigma_1}(\omega)$. Since $M^p_{\sigma_1}(\omega)$ increases with $p$, the identity (3.7) holds for any $f \in M^p_{\sigma_1}(\omega)$. The result now follows from Proposition 3.6 and the facts that $\ell_0$ and $\Sigma_1$ are dense in $\ell^p_{\sigma_1}(\omega)$ and $L^p_{\sigma_1}(\omega)$, respectively, when $\max p < \infty$. □
We shall end the section by applying the latter results to deduce invariance properties of compactly supported elements in $M_{p}^{\omega}$ and in $W_{p}^{\omega}$. The space $W_{p}^{\omega}(\mathbb{R}^{d})$, with $p, q \in (0, \infty]$ and $\omega \in \mathcal{P}_{E}(\mathbb{R}^{2d})$, is the Wiener amalgam related space, defined as the set of all $f \in \mathcal{S}_{1/2}'(\mathbb{R}^{d})$ such that

$$\|f\|_{W_{p}^{\omega}} \equiv \|V_{\phi} f \cdot \omega\|_{L_{p,q}^{p,q}}$$

is finite. Here $L_{p,q}^{p,q}(\mathbb{R}^{2d})$ is the set of all measurable $F$ on $\mathbb{R}^{d}$ such that

$$\|F\|_{L_{p,q}^{p,q}} \equiv \|f_0\|_{L_p} < \infty, \quad \text{where} \quad f_0(x) = \|F(x, \cdot)\|_{L^q}.$$ 

Evidently, $M_{p}^{\omega}$ is independent of $p \in (0, \infty]^{2d}$ and $\sigma \in S_{2d}$.

As a consequence of Remark 4.6 in [28] and its arguments, it follows that

$$M_{p}^{\omega}(\mathbb{R}^{d}) \cap \mathcal{E}'(\mathbb{R}^{d}) = W_{p}^{\omega}(\mathbb{R}^{d}) \cap \mathcal{E}'(\mathbb{R}^{d})$$

(3.8)

when $p_1, p_2, q \in [1, \infty]$ and $t > 1$. Here $\mathcal{E}'(\mathbb{R}^{d})$ is the set of compactly supported elements in $\mathcal{S}'(\mathbb{R}^{d})$, and for any $q \in (0, \infty]$ and $\omega \in \mathcal{P}_{E}(\mathbb{R}^{2d})$, the set $\mathcal{F}L_{q}^{q}(\mathbb{R}^{d})$ consists of all $f \in \mathcal{S}_{1}(\mathbb{R}^{d})$ such that $\hat{f}$ is measurable and belongs to $L_{q}^{q}(\mathbb{R}^{d})$. We set

$$\|f\|_{\mathcal{F}L_{q}^{q}(\mathbb{R}^{d})} = \|f\|_{\mathcal{F}L_{q}^{q}(\mathbb{R}^{d})} \equiv \|\hat{f} \cdot \omega(x, \cdot)\|_{L^q}.$$ 

Note here that if $x \in \mathbb{R}^{d}$ is fixed, then

$$\|\hat{f} \cdot \omega(x, \cdot)\|_{L^q} \asymp \|f \cdot \omega(0, \cdot)\|_{L^q},$$

since $\omega$ is $v$-moderate for some $v$. Consequently, the condition $\|f\|_{\mathcal{F}L_{q}^{q}(\mathbb{R}^{d})} < \infty$ is independent of $x \in \mathbb{R}^{d}$, though the norm $\|f\|_{\mathcal{F}L_{q}^{q}(\mathbb{R}^{d})}$ might depend on $x.$

We have now the following extension of [28, Remark 4.6].

**Proposition 3.8.** Let $\omega \in \mathcal{P}_{E}(\mathbb{R}^{2d})$, $p, q \in (0, \infty]$ and $t > 1$. Then (3.8) holds. In particular,

$$M_{p}^{\omega}(\mathbb{R}^{d}) \cap \mathcal{E}'(\mathbb{R}^{d}) \quad \text{and} \quad W_{p}^{\omega}(\mathbb{R}^{d}) \cap \mathcal{E}'(\mathbb{R}^{d})$$

are independent of $p$.

We need the following lemma for the proof. Here the first part follows from [4, Proposition 4.2].

**Lemma 3.9.** Let $1 < s < t$ and let $f \in \mathcal{E}'(\mathbb{R}^{d})$. Then the following is true:

1. if $\phi \in \mathcal{S}_{s}(\mathbb{R}^{d})$, then

$$|V_{\phi} f(x, \xi)| \lesssim e^{-h|x|^{1/s}} e^{c|x|^{1/t}},$$

for every $h > 0$ and $c > 0.$
(2) \(|\hat{f}(\xi)| \lesssim e^{c|\xi|^{1/\alpha}}, \) for every \(\varepsilon > 0.\)

Proof. The first part follows from \([4, \text{Proposition 4.2}]\).

By choosing \(\phi\) in (1) such that \(\int \phi \, dx = 1\), we get

\[
|\hat{f}(\xi)| = \left| \int V_\phi f(x, \xi) \, dx \right| \leq \|V_\phi f(\cdot, \xi)\|_{L^1} \lesssim e^{c|\xi|^{1/\alpha}},
\]

where the last estimate follows from (1).

\[\square\]

Proof of Proposition 3.8. We use the same notations as in the proofs of Proposition 3.6 and Theorem 3.7. First assume that \(\omega \geq 1/v_s\) for some \(s\) satisfying \(1 < s < t\), where

\[v_s(x, \xi) = e^{r(|x|^{1/s} + |\xi|^{1/s})}\]

and \(r > 0\) is fixed. Let \(\phi, \psi \in \cap_{r > 0} M^r_{(v_s)}(\mathbb{R}^d)\) and \(\{(x_j, \xi_j)\}\) be such that \((3.6)\) are dual Gabor frames, and such that \(\phi\) has compact support. Such frames exists in view of Proposition 1.8, and the fact that \(\mathcal{D}_{s_0}\) is non-trivial and contained in \(\mathcal{S}_{s_0} \subseteq M^1_{(v_s)}\) when \(1 < s_0 < s\). Here the latter inclusion follows from \([32, \text{Theorem 3.9}]\). For convenience we assume that \(0 \in J\) and \(x_0 = \xi_0 = 0\).

By Theorem 3.7 it follows that any \(f \in \mathcal{S}'_t \subseteq M^{p,q}_{(1/v_s)}\) possess the expansions (3.7), and that

\[
\|f\|_{M^{p,q}_{(\omega)}} \asymp \|c\|_{\ell^{p,q}_{(\omega, \lambda)}}, \quad c = \{c(j, k)\}_{j,k \in J}, \tag{3.9}
\]

where \(c(j, k) = (V_\phi f)(x_j, \xi_k)\). Furthermore, if \(\ell^{p,q}_{\ast, (\omega)}\) is the set of all \(b = \{b(j, k)\}_{j,k \in J}\) such that

\[\|b_0\|_{\ell^s} < \infty, \quad b_0(j) = \|b(j, \cdot)\omega(j, \cdot)\|_{\ell^s},\]

then

\[
\|f\|_{\mathcal{W}^{p,q}_{(\omega)}} \asymp \|c\|_{\ell^{p,q}_{\ast, (\omega)}}.
\]

Now assume that in addition \(f \in \mathcal{E}'_t(\Omega)\), for some bounded and open set \(\Omega \subseteq \mathbb{R}^d\). Since both \(f\) and \(\phi\) has compact supports, it follows that there is a finite set \(J_0 \subseteq J\) such that \(c_{j,k} = 0\) when \(j \in J \setminus J_0\). This implies that

\[
\|c\|_{\ell^{p,q}_{\ast, (\omega)}} \asymp \|c\|_{\ell^{p,q}_{\ast, (\omega)}},
\]

for every \(p_1, p_2 \in (0, \infty]\), and the first equality in (3.8) follows in this case.

Next let \(\omega \in \mathcal{P}_E(\mathbb{R}^{2d})\) be general. Since \(\mathcal{E}'(\mathbb{R}^d) \subseteq M^{p,q}_{(1/v_s)}\), it follows that

\[
M^{p,q}_{(\omega)} \mathcal{E}'_{t} = \left( M^{p,q}_{(\omega)} \cap M^{p,q}_{(1/v_s)} \right) \mathcal{E}'_{t} = M^{p,q}_{(\omega + 1/v_s)} \mathcal{E}'_{t}. \tag{3.10}
\]

In the same way it follows that

\[
\mathcal{W}^{p,q}_{(\omega)} \mathcal{E}'_{t} = \mathcal{W}^{p,q}_{(\omega + 1/v_s)} \mathcal{E}'_{t}.
\]
The first equality in (3.8) now follows from these identities, the first part of the proof and the fact that

\[ \frac{1}{v_s} \leq \omega + \frac{1}{v_s} \in \mathcal{P}_E(\mathbb{R}^{2d}). \]

In order to prove the last equality in (3.8) we again start to consider the case when \( \omega \geq 1/v_s \) for some \( s \) satisfying \( 1 < s < t \). Let \( f \in \mathcal{E}_t \), and choose \( \phi \) and \( \psi \) here above such that \( \phi = 1 \) on \( \text{supp} \ f \), \( \psi = 1 \) on \( \text{supp} \ \phi \) and such that \( \phi(\cdot - x_j) = 0 \) on \( \text{supp} \ f \) when \( x_j \neq 0 \). This is possible in view of Section 3 in [22]. Also let \( Q \) be a closed parallelepiped such that

\[ \bigcup_{k \in J} (\xi_k + Q) = \mathbb{R}^d \]

and that the intersection of two different \( \xi_k + Q \) is a zero set.

Then there is a constant \( C > 0 \) such that

\[ C^{-1} \| f \|_{M_{p,q}^\omega} \leq \| e^{i(\cdot \cdot \cdot \psi)} f \|_{M_{p,q}^\omega} \leq C \| f \|_{M_{p,q}^\omega}, \quad \eta \in Q. \quad (3.11) \]

Furthermore, by the support properties of \( \phi \) and \( f \), and using the fact that the Gabor coefficients \( c_\eta(j,k) \) of \( e^{i(\cdot \cdot \cdot \psi)} f \) are given by

\[ c_\eta(j,k) = (V_\phi f)(x_j, \xi_k - \eta), \]

are zero when \( x_j \neq 0 \), and

\[ c_\eta(0,k) = (V_\phi f)(\xi_k - \eta) = \hat{f}(\xi_k - \eta). \quad (3.12) \]

Hence, (3.11) gives

\[ \| f \|_{M_{p,q}^\omega}^q \asymp \| c_\eta \|_{p,q}^q = \| c_\eta(0, \cdot) \|_{p,q}^q. \]

By integrating the last relations with respect to \( \eta \) over \( Q \) it follows from (3.12) that

\[ \| f \|_{M_{p,q}^\omega}^q \asymp \| \{ \| \hat{f} \|_{L_{p,q}^\omega(\xi_k + Q)} \}_{k \in J} \|_{L^q}^q = \| \hat{f} \|_{L_{p,q}^\omega(\mathbb{R}^d)}^q, \]

and last equality in (3.8) follows in this case.

Next assume that \( \omega \) is arbitrary, and let \( 1 < s < t \). By Lemma 3.9 we have

\[ \mathcal{F} L_{p,q}^\omega \bigcap \mathcal{E}'_t = \left( \mathcal{F} L_{p,q}^\omega \bigcap \mathcal{F} L_{(1/v_s)}^q \right) \bigcap \mathcal{E}'_t = \mathcal{F} L_{(\omega + 1/v_s)}^q \bigcap \mathcal{E}'_t. \]

The last equality in (3.8) now follows from these identities, the previous case and (3.10). The proof is complete.

We finish the section by applying the previous result on compactly supported symbols to pseudo-differential operators. (See Sections 1 and 4 in [33] for strict definitions.) Let \( t \in \mathbb{R}, \ p \in (0, \infty] \) and \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \). Then the set \( s_{t,p}(\omega_1, \omega_2) \) consists of all \( a \in \Sigma_1(\mathbb{R}^{2d}) \) such that the operator \( \Omega_p(a) \) from \( \Sigma_1(\mathbb{R}^{2d}) \) to \( \Sigma_1(\mathbb{R}^{2d}) \) extends (uniquely) to a Schatten-von Neumann operator from \( M_{(\omega_1)}^{2,2}(\mathbb{R}^d) \) to \( M_{(\omega_2)}^{2,2}(\mathbb{R}^d) \).

The following result follows immediately from Theorem A.3 in [33] and Proposition 3.8.

\[ 29 \]
Proposition 3.10. Let $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{4d})$ be such that
\[
\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \prec \omega_0((1 - t)x + ty, t\xi + (1 - t)\eta, \xi - \eta, y - x).
\]
Also let $s > 1$, $p \in (0, \infty]$ and $q \in [1, \infty]$. Then
\[
s_{t, q}(\omega_1, \omega_2) \bigcap \mathcal{E}'_s(\mathbb{R}^{2d}) = M_{(\omega_0)}^{p, q}(\mathbb{R}^{2d}) \bigcap \mathcal{E}'_s(\mathbb{R}^{2d})
= \mathcal{F} L^q((\omega_0)\mathbb{R}^{2d}) \bigcap \mathcal{E}'_s(\mathbb{R}^{2d}).
\]

Remark 3.11. Propositions 3.8 and 3.10 remain true if $\mathcal{E}'_t$ are replaced by compactly supported elements in $\Sigma'_t$, for $t > 1$, or by elements in $\mathcal{E}'_t$. We leave the modifications to the reader.

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