Gray codes for Fibonacci $q$-decreasing words

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Abstract. A length $n$ binary word is $q$-decreasing, $q \geq 1$, if every of its maximal factors of the form $0^a1^b$ satisfies $a = 0$ or $q \cdot a > b$. We show constructively that these words are in bijection with binary words having no occurrences of $1^{q+1}$, and thus they are enumerated by the $(q + 1)$-generalized Fibonacci numbers. We give some enumerative results and reveal similarities between $q$-decreasing words and binary words having no occurrences of $1^{q+1}$ in terms of frequency of 1 bit. In the second part of our paper, we provide an efficient exhaustive generating algorithm for $q$-decreasing words in the lexicographic order, for any $q \geq 1$, show the existence of 3-Gray codes and explain how a generating algorithm for these Gray codes can be obtained. Moreover, we give the construction of a more restrictive 1-Gray code for $1$-decreasing words, which in particular settles a conjecture stated recently in the context of interconnection networks by Eğecioğlu and Iršič.

1 Introduction and preliminaries

The Fibonacci sequence origins have been traced back to the works of ancient Indian mathematician Ācārya Pāṇiṅgala dealing with rhythmic structure patterns in Sanskrit poetry [24,15, p. 50]. Over time, the study of words and patterns became more abstract and systematic (see for instance Lothaire’s books [16,17,18] and [6]). A considerable amount of questions concerning efficient enumeration and generation of words respecting certain properties (including pattern avoidance) were mathematically formulated and answered only relatively recently, the works closely related to the present study include [2,4,8,9,11,26,27,28].

In this paper we introduce $q$-decreasing words, a novel class of run-restricted binary words enumerated by the $(q + 1)$-generalized Fibonacci numbers, $q \geq 1$. For $q = 1$ the subclass of such words that start with 0 was recently considered in the context of induced subgraphs of hypercubes [8,9]. The $q$-decreasing words could be interesting objects to study in other domains, for example in stringology, the study of string algorithms [7]. In Section 2 we present a bijection between this novel class of words and Fibonacci words, i.e. binary words avoiding consecutive 1s. Section 3 is devoted to the presentation of several generating functions and enumeration results. Finally, in Section 4, we show the existence of a 3-Gray code for any $q \geq 1$, give an efficient exhaustive generating algorithms and a much more involved construction for a 1-Gray code in the special case $q = 1$. 
In particular, the latter Gray code gives a Hamiltonian path in Fibonacci-run graphs whose existence is conjectured in [8].

The following set of notations is adopted. Let \( \mathcal{B} \) denote the set of all finite-length binary words, i.e. strings over the alphabet \{0, 1\}, and \( \mathcal{B}_n, n \geq 0 \), be the set of all binary words of length \( n \). For a given binary word \( w \) we use the notation \( w_i \) to mean the letter at position \( i \).

A nonempty sequence of adjacent letters inside a word is called factor. A factor \( v \) repeated \( k \) times is denoted by \( v^k \), for instance \( 0011 \). For a given length \( n \), the notation \( \mathcal{B} \) is used to repeat factor \( v \) as many times as possible, until the length \( n \) is reached, possibly trimming extra symbols at the end; and the length \( n \) will be understood from the context. For example, if a word \( v \) of length \( n = 7 \) is equal to \( 0011001 \) it means \( v = 0010010 \).

The set of all \( n \)-length binary words containing no occurrences of factor \( v \) is denoted by \( \mathcal{B}_n v \). Let \( \mathcal{B}_p \) be the set of all binary words of length \( n \). For \( a \) given length \( n \), the notation \( \mathcal{B} \) is used to repeat factor \( v \) as many times as possible, until the length \( n \) is reached, possibly trimming extra symbols at the end; and the length \( n \) will be understood from the context. For example, if a word \( v \) of length \( n = 7 \) is equal to \( 0011001 \) it means \( v = 0010010 \).

Following [19] the \( n \)th \( k \)-generalized Fibonacci number is defined as

\[
 f_{n,k} = \begin{cases} 
 0 & \text{if } 0 \leq n \leq k - 2, \\
 1 & \text{if } n = k - 1, \\
 \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} 
\end{cases}
\]  

(1)

As noted in [27], the generating function \( g_k(x) = \sum_{n=0}^{\infty} f_{n,k} x^n \) for \( k \)-generalized Fibonacci numbers is

\[
g_k(x) = \frac{x^{k-1}}{1 - x - x^2 - \ldots - x^k} = \frac{x^{k-1} - x^k}{1 - 2x + x^{k+1}}.
\]  

(2)

Related constructions also appear in [10, p. 42] and in [1, p. 309].

**Classical fact.** The number of words in \( \mathcal{B}_n(1^k) \) equals \( f_{n+k,k} \) for \( k \geq 2 \), moreover

\[
 \mathcal{B}_n(1^k) = \begin{cases} 
 \mathcal{B}_n & \text{if } n < k, \\
 \bigcup_{i=0}^{k-1} 1^i0 \cdot \mathcal{B}_{n-i-1}(1^k) & \text{otherwise.} 
\end{cases}
\]  

(3)

The classical fact comes, for instance, from [14, p. 286]. The binary words avoiding two consecutive 1s are counted by Fibonacci numbers, words without factor 111 are counted by Tribonacci numbers, etc. We call such words (generalized) Fibonacci words. The On-line Encyclopedia of Integer Sequences founded by N.J.A. Sloane [25] contains several corresponding sequences (see for example A000045 and A000073, after taking a binary complement). Gray codes for Fibonacci words are discussed in [27], and more generally, Gray codes for words avoiding a given factor in [5,26].
Lemma 1. For \( q \geq 1 \), the bivariate generating function \( B_q(x, y) = \sum_{n,k \geq 0} b_{n,k} x^n y^k \) where the coefficient \( b_{n,k} \) of \( x^n y^k \) is the number of Fibonacci words of length \( n \) having \( k \) \( 1 \)'s in \( B_n(1^{q+1}) \) is

\[
B_q(x, y) = \frac{y (1 - (xy)^{q+1})}{y - xy^2 - xy + (xy)^{q+2}}.
\]

Proof. Alternative to relation (3), the set \( B(1^{q+1}) \) of (any length) binary words avoiding \( 1^{q+1} \) can be defined recursively as

\[
B(1^{q+1}) = \mathbb{I}_q \cup \bigcup_{i=0}^{q} 1^i0 \cdot B(1^{q+1})
\]

where \( \mathbb{I}_q = \bigcup_{i=0}^{q} \{1^i\} \). It follows that the bivariate generating function \( B_q(x, y) \) satisfies the functional equation

\[
B_q(x, y) = \sum_{i=0}^{q} x^i y^i + B_q(x, y) \sum_{i=0}^{q} x^{i+1} y^i,
\]

and we have \( B_q(x, y) = \frac{y(1 - (xy)^{q+1})}{y - xy^2 - xy + (xy)^{q+2}}. \) \( \square \)

It is not surprising that \( B_q(x, 1) = \frac{2^{n+1}(x) - x^n}{x^{q+1}} \), see relation (2). Indeed, both sides of this equality are generating functions for the sequence \( (f_{n+q+1, q+1})_{n \geq 0} \) counting words in \( B(1^{q+1}) \), which is the \( (q + 1) \)th left shift of the sequence \( (f_{n,q+1})_{n \geq 0} \), see relation (1) and the classical fact following it.

The Hamming distance between two binary words of the same length equals the number of positions at which they differ. A \( k \)-Gray code for a set \( A \subset B_n \) is an ordered list \( A \) for \( A \), such that the Hamming distance between any two consecutive words in \( A \) is at most \( k \), and we say that words in \( A \) are listed in Gray code order. Frank Gray’s patent [11] discusses an early example and application of such a code for the set of \( n \)-length binary words. The concatenation of two ordered lists \( L_1 \) and \( L_2 \) is denoted by \( L_1 \circ L_2 \), and \( \overline{L} \) designates the reverse of the list \( L \). If \( L \) is a list of words, then \( (L)^i = L \) whenever \( i \) is even, and \( (L)^i = \overline{L} \) otherwise. First and last elements of \( L \) are denoted respectively by first(\( L \)) and last(\( L \)). Also, we denote by \( \hat{L} \) the list obtained from \( L \) by deleting its first element.

For example, a list containing elements 00, 01 and 11 will be noted as \( L = 00 \circ 01 \circ 11 \), and \( \hat{L} = 00 \circ 01 \).

Definition 1. A binary word is called \( q \)-decreasing, for \( q \geq 1 \), if any of its \( q \) length maximal factors of the form \( 0^a1^b \), \( a > 0 \), satisfies \( q \cdot a > b \).

The set of \( q \)-decreasing words of length \( n \) is denoted by \( W_n^q \). For example we have \( W_2^1 = \{0000, 0001, 0100, 1000, 1001, 1100, 1110, 1111\} \). See also Table 1 for the sets \( W_2^1 \) and \( W_6^1 \). Let \( W^q = \bigcup_{n=0}^{\infty} W_n^q \).
2 Bijection with classical Fibonacci words

In this section we prove that $q$-decreasing words, $q \geq 1$, are enumerated by $(q + 1)$-generalized Fibonacci numbers defined in relation (1). We start with a definition and several propositions.

**Definition 2.** For any $q \geq 1$, we define the map $\psi^q$ from $\mathcal{B}_n$ to $\mathcal{B}_{n+q+1}$ as

$$\psi^q(w) = \begin{cases} \nu 001^{k+q} & \text{if } w = \nu 01^k, \nu \in \mathcal{B}, k \geq 0, \\ 1^{n+q+1} & \text{otherwise.} \end{cases}$$

Less formally, $\psi^q$ inserts a factor 01$^q$ immediately after the last occurrence of 0, and it adds the suffix 1$^{q+1}$ to the word containing no 0. For example $\psi^1(0) = 001$, $\psi^1(0011) = 0000111$, $\psi^2(0011101) = 0011100111$ and $\psi^3(1) = 1111111$. The value of $q$ will be clear from the context, so by slight abuse of notation $\psi^q$ will be denoted $\psi$ throughout the paper.

**Proposition 1.** For $n \geq 0, q \geq 1$, $\psi$ is an injective map from from $\mathcal{B}_n$ to $\mathcal{B}_{n+q+1}$.

**Proof.** For two $n$-length words $w \neq w'$ we show that $\psi(w) \neq \psi(w')$. It is clear that if one of the given words contains no 0 the injectivity holds. Otherwise we have two cases. If $w = \nu 01^k$ and $w' = \nu' 01^k$ then we have necessarily $\nu \neq \nu'$ and $\nu 001^{k+q} \neq \nu' 001^{k+q}$, so the images are different. If $w = \nu 01^k$ and $w' = \nu' 01^\ell$ with $k \neq \ell$, then $\nu 001^{k+q} \neq \nu' 001^{\ell+q}$ and again $\psi(w) \neq \psi(w')$.

In the following, we will use the restriction of $\psi$ to the set of $q$-decreasing words, namely $\psi : \mathcal{W}_n^q \to \mathcal{W}_{n+q+1}^q$. It is possible due to Proposition 2 below.

**Proposition 2.** For $n \geq 0, q \geq 1$, $\psi(\mathcal{W}_n^q)$ consists of all $q$-decreasing words of length $n + q + 1$ ending with at least $q$ ones.

**Proof.** If $w = 1^a$, then $\psi(w) = 1^{n+q+1}$. Otherwise, we write $w = \nu 0^a 1^b$ where $a > b/q \geq 0$ and the word $\nu$ is either empty or ends with 1. So $\psi(\nu 0^a 1^b) = \nu 0^{a+1} 1^{q+b}$. As we have $1 + a > 1 + b/q = (q + b)/q$, $\psi(w)$ is a $q$-decreasing word ending with at least $q$ 1s. Similarly, any $(n + q + 1)$-length $q$-decreasing word ending with at least $q$ 1s can be obtained from a (unique) word in $\mathcal{W}_n^q$ by $\psi$.

Now, we present a one-to-one correspondence between Fibonacci and $q$-decreasing words. Recall that, for $q \geq 1$, the set $\mathcal{B}_n(1^{q+1})$ of $(q + 1)$-generalized Fibonacci words is the set of binary words of length $n$ with no 1$^{q+1}$ factors, see relation (3) for the recursive definition of these words according to their length.

**Definition 3.** We define the map $\phi : \mathcal{B}_n(1^{q+1}) \to \mathcal{B}_n$ as

$$\phi(w) = \begin{cases} 1^k & \text{if } w = 1^k \text{ and } k \in [0, q], \\ \psi(\phi(v)) & \text{if } w = 1^q 0 v, \\ \phi(v) 01^k & \text{if } w = 1^k 0 v \text{ and } k \in [0, q-1]. \end{cases}$$  \hspace{1cm} (4)
See Table 1(a) for the images of the words in $B_4(111)$ through $\phi$.

**Theorem 1.** For $n \geq 0$, $q \geq 1$, $\phi$ maps bijectively $B_n(1^{q+1})$ into $W_n^q$.

**Proof.** We proceed by induction on $n$. The classical decomposition in relation (3) gives rise to the three cases specified in relation (4), and we have:

(i) If $n < q$, then the word $1^n$ is sent by $\phi$ to $1^n$;

(ii) Words of the form $1^v0v$, where $v \in B_{n-q-1}(1^{q+1})$, are sent to words ending by at least $q$ 1s (see Proposition 2);

(iii) Words of the form $1^w0v$, where $k \in [0, q-1]$ and $v \in B_{n-k-1}(1^{q+1})$, are sent to words ending by at most $q - 1$ 1s.

Using the bijectivity of $\psi$ (see Propositions 1 and 2) and the induction hypothesis, it is routine to check that $\phi(w) \in W_n^q$ for any $w \in B_n(1^{q+1})$, and $\phi(w) \neq \phi(w')$ for any two different words $w, w' \in B_n(1^{q+1})$.

Similarly, by induction on $n$, any word in $W_n^q$ can be obtained by $\phi$ from a word in $B_n(1^{q+1})$, and the statement holds. \qed

It follows that $(q + 1)$-order Fibonacci words and $q$-decreasing words are equinumerous.

## 3 Some enumeration results

Here we provide a bivariate generating function $W_q(x, y) = \sum_{n,k \geq 0} w_{n,k}x^n y^k$, where $w_{n,k}$ is the number of $n$-length $q$-decreasing words having $k$ 1s. This bivariate generating function is of a particular interest since it will help us (see Corollary 1) to prove that $q$-decreasing words satisfy a necessary condition for the existence of 1-Gray code, called parity condition. More precisely, if a set $A$ of binary words admits a 1-Gray code, and $A^+$ (resp. $A^-$) denotes the subset of $A$ having even (resp. odd) number of 1s, then the parity difference $|A^+| - |A^-|$ must be equal to either 0, 1, or $-1$. Indeed, the graph where the vertex set is $A$ and edges connect vertices with Hamming distance one is bipartite with partite sets $A^+$ and $A^-$. A Hamiltonian path in this graph (or equivalently a 1-Gray code for $A$) cannot exist unless it satisfies the parity condition. This parity condition is used for instance in [26] to investigate the possibility of 1-Gray code for a set of words avoiding a given factor.

In order to derive the expression of $W_q(x, y)$, we use the following decomposition of the set $W^q$:

$$W^q = \mathbb{1} \cup W^q \cdot S^q,$$

where $\mathbb{1} = \cup_{n=0}^{\infty} \{1^n\}$ and $S^q$ corresponds to the set of all factors of the form $0^a1^b$ respecting $q$-decreasing property (i.e. $a > b/q \geq 0$) such that none of them is a concatenation of other factors respecting $q$-decreasing property. More precisely, $a$ is the smallest integer strictly greater than $b/q$, i.e. $a = \lfloor b/q \rfloor + 1$. A factor from $S^q$ will be called $q$-prime factor, and thus $S^q$ is the set of such factors. For instance: $S^1 = \{0, 001, 00011, 0000111, \ldots\}$, $S^2 = \{0, 01, 0011, 000111, 0001111, 00011111, \ldots\}$. 
Lemma 2. For \( q \geq 1 \), the bivariate generating function \( S_q(x, y) = \sum_{n,k \geq 0} s_{n,k} x^n y^k \) where the coefficient \( s_{n,k} \) is the number of \( q \)-prime factors of length \( n \) having exactly \( k \) 1s is:

\[
S_q(x, y) = \frac{x(1 - (xy)^q)}{(xy - 1)(x^{q+1}y^q - 1)}.
\]

Proof. Any \( q \)-prime factor is of the form \( 0^a1^b \) with \( a = \lfloor b/q \rfloor + 1 \). So, if \( b = kq + r \) with \( k \geq 0 \) and \( r \in [0, q - 1] \), then \( a + b = k(q + 1) + r + 1 \) and we can write:

\[
S_q(x, y) = \sum_{k=0}^{\infty} \sum_{r=0}^{q-1} x^{k(q+1)+r+1} y^{kq+r}.
\]

A simple calculation results in the claimed formula. \( \square \)

Theorem 2. For \( q \geq 1 \), the bivariate generating function \( W_q(x, y) = \sum_{n,k \geq 0} w_{n,k} x^n y^k \) where the coefficient \( w_{n,k} \) is the number of \( n \)-length \( q \)-decreasing words containing exactly \( k \) 1s is given by:

\[
W_q(x, y) = \frac{1 - x^{q+1}y^q}{1 - xy - x + x^{q+2}y^{q+1}}.
\]

Proof. Due to the decomposition \( W_q = 1 \cup W_q \cdot S_q \), we have \( W_q(x, y) = \frac{1}{1 - x} \), and the result holds after applying Lemma 2. \( \square \)

Corollary 1. For any \( n \geq 0, q \geq 1 \), the set \( W_n^q \) satisfies the parity condition.

Proof. The generating function \( D_q(x) = \sum_{n \geq 0} d_n x^n \) where the coefficient \( d_n \) is the parity difference corresponding to the set \( W_n^q \) is obtained by making the substitution \( y = -1 \) in \( W_q(x, y) \):

\[
D_q(x) = \frac{(-1)^q x^{q+1} - 1}{(-1)^q x^{q+2} - 1}.
\]

When \( q \) is even, \( D_q(x) = \frac{x^{q+1} - 1}{x^{q+2} - 1} = \sum_{n=0}^{\infty} (x^{n(q+2)} - x^{n(q+2)+q+1}) \), otherwise \( D_q(x) = \frac{x^{q+1} + 1}{x^{q+2} + 1} = \sum_{n=0}^{\infty} (-1)^n (x^{n(q+2)} + x^{n(q+2)+q+1}) \). All involved coefficients are from \( \{-1, 0, 1\} \), and the parity condition holds. \( \square \)

The following two corollaries are obtained by respectively calculating the expressions: \( W_q(x, 1), \left. \frac{\partial W_q(x,y)}{\partial y} \right|_{y=1} \) and \( \left. \frac{\partial W_q(x,y,1)}{\partial y} \right|_{y=1} \).

Corollary 2. For \( q \geq 1 \), the generating function \( F_q(x) = \sum_{n \geq 0} f_n x^n \) where the coefficient \( f_n \) is the number of \( n \)-length \( q \)-decreasing words is given by:

\[
F_q(x) = \frac{1 - x^{q+1}}{1 - 2x + x^{q+2}}.
\]
Note that, as predicted by Theorem 1, $F_q(x) = B_q(x, 1)$, see Lemma 1 and the remark following it.

The _popularity_ of a symbol in a set of words is the overall number of the occurrences of the symbol in the words of the set.

**Corollary 3.** For $q \geq 1$, the generating function $P_{q,1}(x) = \sum_{n \geq 0} p_n x^n$ where the coefficient $p_n$ is the popularity of 1s in all $n$-length $q$-decreasing words is

$$P_{q,1}(x) = \frac{x \left( 1 - qx^q + qx^{q+1} - 2x^{q+1} + x^{2q+2} \right)}{(1 - 2x + x^{q+2})^2}.$$ 

Similarly, the generating function for the popularity of 0s in all $n$-length $q$-decreasing words is

$$P_{q,0}(x) = \frac{x \left( 1 - x^q \right)}{(1 - 2x + x^{q+2})^2}.$$ 

The popularity of 1s in $B_n(11)$ is equal to the number of edges in the Fibonacci cube [12] of order $n$, see [13] and comments to the sequence A001629 in [25]. The generating function $P_{1,0}(x)$ allows us to show that the popularity of 0s in $W_n^1$ is a shift of the sequence A006478 enumerating the number of edges in the Fibonacci hypercube [20], i.e. in a polytope determined by the convex hull of the Fibonacci cube.

Despite the $q$-decreasing words and Fibonacci words have quite different definitions, they are equinumerous and share some common features. We end this section by showing that the 1s frequency (defined formally below) of both sets have the same limit when $n$ tends to infinity.

If $\omega_n$ (resp. $\beta_n$) is the ratio between the popularity of 1s and that of 0s in the words in $W_n^1$ (resp. in $B_n(11)$), then $\lim_{n \rightarrow \infty} \omega_n = \lim_{n \rightarrow \infty} \beta_n$. Indeed, extracting the coefficients of $x^n$ in both $P_{1,1}$ and of $P_{1,0}$, their ratio tends to $2 - \varphi \approx 0.3819660113$ when $n$ tends to infinity, where $\varphi$ is the golden ratio; and this is also the limit of $\beta_n$, which is obtained by investigating the ratio of the coefficients of $x^n$ in $\frac{\partial B_q(x,y)}{\partial y}|_{y=1}$ and in $\frac{\partial B_q(xy,1/\varphi)}{\partial y}|_{y=1}$, where $B_q(x,y)$ is from Lemma 1.

The _1s frequency_ of a set of binary words is the ratio between the popularity of 1s and the overall number of bits in the words of the set. Alternatively, it is the expected value when a bit is randomly chosen in the words of the set. With the notations above, we have that the 1s frequency of $W_n^1$ is $\frac{1}{1 + \frac{1}{\omega_n}}$, and that of $B_n(11)$ is $\frac{1}{1 + \frac{1}{\beta_n}}$, and we have the next result.

**Corollary 4.** The 1s frequency of $W_n^1$ and of $B_n(11)$ both tend to $\frac{2 - \varphi}{3 - \varphi}$ when $n$ tends to infinity, where $\varphi = \frac{1 + \sqrt{5}}{2}$. 

More generally, for any $q \geq 1$, the overall number of bits in both sets $W_n^q$ and $B_n(1^{q+1})$ is $n \cdot f_{n+q+1,q+1}$, and due to the second rule in relation (4) defining
the bijection $\phi : B(1^{q+1}) \rightarrow W_n^q$ we have that in $W_n^q$ there are more 1s than in $B_n(1^{q+1})$. However, the next corollary shows that the difference between the 1s frequency of $W_n^q$ and that of $B_n(1^{q+1})$ tends to zero when $n$ tends to infinity.

**Corollary 5.** For any $q \geq 1$, if $u_{n,q}$ (resp. $v_{n,q}$) is the popularity of 1s in $W_n^q$ (resp. in $B_n(1^{q+1})$), then we have

$$\lim_{n \to \infty} \frac{u_{n,q} - v_{n,q}}{n \cdot f_{n+q+1,q+1}} = 0.$$  

**Proof.** Since, for any $q \geq 1$, $u_{n,q} - v_{n,q} \geq 0$, it suffices to prove that we have $u_{n,q} - v_{n,q} \leq f_{n+q+1,q+1}$. Using Corollary 3 and Lemma 1, the generating function $H(x)$ where the coefficient of $x^n$ is $f_{n+q+1,q+1} + v_{n,q+1} - u_{n,q}$ is

$$H(x) = B_q(x,1) + \frac{\hat{\partial} B_q(x,y)}{\partial y} \bigg|_{y=1} - P_{q,1}(x)$$

$$= \frac{1 - 2x^{q+1}}{1 - 2x + x^{q+2}},$$

which satisfies the functional equation $H(x) = 1 - 2x^{q+1} + 2xH(x) - x^{q+2}H(x)$. By a simple observation, $H(x)$ is also the generating function with respect to the length of binary words different from $0^{q+1}$ and $1^{q+1}$ and that do not start with $0^{q+2}$. Then we have $u_{n,q} - v_{n,q} \leq f_{n+q+1,q+1}$. Dividing by $n f_{n+q+1,q+1}$ and taking the limit when $n$ tends to infinity, we obtain the expected result.

Corollary 4 says that, for $q = 1$, the 1s frequency of $W_n^q$ and that of $B_n(1^{q+1})$ have a common limit when $n$ tends to infinity. For $q \geq 2$, Corollary 5 does not ensure that each of the 1s frequency of $W_n^q$ (that is $\frac{u_{n,q}}{n \cdot f_{n+q+1,q+1}}$) and that of $B_n(1^{q+1})$ (that is $\frac{v_{n,q}}{n \cdot f_{n+q+1,q+1}}$) has a limit when $n$ tends to infinity. However, using asymptotic analysis (see for instance [10]) it can be shown that $\frac{u_{n,q}}{n \cdot f_{n+q+1,q+1}}$ converges to a non-zero value when $n$ tends to infinity, and the limit can be approximated by numerical methods. From Corollary 5 it follows that so does $\frac{v_{n,q}}{n \cdot f_{n+q+1,q+1}}$, and the two limits are equal.

Since the proof of this result is beyond the scope of the present paper we state it (including the case $q = 1$ in Corollary 4) without proof.

**Note 1** For $q \geq 1$ the 1s frequency of $W_n^q$ and of $B_n(1^{q+1})$ have a common non-zero limit when $n$ tends to infinity.

### 4 Exhaustive generation and Gray codes for $q$-decreasing words

Here we show that $q$-decreasing words can be efficiently generated in the lexicographical order and explain how the obtained generating algorithm can be turned
into a 3-Gray code generating one. Then, we give a more intricate construction of a 1-Gray code for the particular case $q = 1$. As a byproduct, this construction gives a positive answer for the existence a Hamiltonian path in Fibonacci-run graph conjectured in [8].

### 4.1 3-Gray codes and exhaustive generation

Algorithm in Figure 1 generates prefixes of $q$-decreasing words in the lexicographical order, and eventually all $n$-length $q$-decreasing words. The size $n$, parameter $q$ and the array $w$ of length $n + 1$ are global variables and the main call is 

$$\text{LEXFib}(1, n).$$

For convenience, $w[0]$ is initialized by 1 and the parameter $\delta$ is the number of consecutive 1s that can be added to the current generated prefix without violating the $q$-decreasingness.

It can be seen that this recursive algorithm satisfies Baronigien and Ruskey’s constant amortized time (CAT) principle in [3] stated below. As noticed in [3], by considering the underlying computation tree it follows that any algorithm satisfying it is efficient, and thus so is 

$$\text{LEXFib}.$$

We refer the reader to [22, Section 1.7] for more about CAT exhaustive generating algorithms.

**CAT principle [3]:** A recursive generation algorithm with the following properties runs in constant amortized time. (i) Every call results in the output of at least one object; (ii) Excluding the computation done by recursive calls the amount of computation of any call is proportional to the degree of the call; (iii) The number of calls of degree one is $O(N)$, where $N$ is the number of generated objects.

```plaintext
procedure LEXFib(pos, delta: integer)
if (pos = n + 1) print w;
else w[pos] := 0;
   if (w[pos - 1] = 1) d := q - 1; else d := delta + q; endif
   LEXFib(pos + 1, d);
   if (delta > 0)
      w[pos] := 1; LEXFib(pos + 1, delta - 1);
   endif
endif
end procedure
```

Fig. 1: Lexicographic generation algorithm for $q$-decreasing words.

Generally, the bijection $\phi$ in relation (4) does not preserve Graycodeness, i.e. a Gray code for $(q + 1)$-Fibonacci words is not necessarily mapped by $\phi$ to Gray code for $q$-decreasing words. For instance, when $q = 1$ and $n = 2k + 1$ the Gray code for Fibonacci words in [27] always contains two consecutive words $u =$
\( u \in B_4(111) \) \( \phi(u) \in W^2_4 \)

| \( u \)  | \( \phi(u) \)  |
|--------|--------------|
| 1100   | 0011         |
| 1101   | 1111         |
| 1001   | 0011         |
| 1000   | 0001         |
| 1010   | 0101         |
| 1111   | 1101         |
| 0011   | 1100         |
| 0010   | 0100         |
| 0000   | 0000         |
| 0001   | 1000         |
| 0101   | 1010         |
| 0100   | 0010         |
| 0110   | 1110         |

Words in \( W^2_4 \) in BRGC order

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Words} & \text{Hamming distance} \\
\hline
12 & 111100 & 3 \\
13 & 111111 & 2 \\
14 & 111110 & 1 \\
15 & 111001 & 3 \\
16 & 110000 & 1 \\
17 & 100100 & 3 \\
18 & 100010 & 2 \\
19 & 100011 & 1 \\
20 & 100001 & 1 \\
21 & 100000 & 1 \\
22 & 110001 & 1 \\
23 & 100010 & 2 \\
24 & 100011 & 1 \\
25 & 100001 & 1 \\
26 & 100000 & 1 \\
\hline
\end{array}
\]

Table 1: (a) The images of words in \( B_4(111) \) under the bijection \( \phi \). Words in \( B_4(111) \) are listed in a BRGC-like order, called local reflected order in [27], which yields a 1-Gray code order. (b) The set \( W^q_n \) in BRGC order together with the Hamming distance between consecutive words.

\[(10)^k1 \text{ and } v = (10)^k0, \text{ but their images } \phi(u) = 1^{2k+1} \text{ and } \phi(v) = 0^{k+1}1^{k} \text{ have arbitrarily large Hamming distance for enough large } n. \text{ A similar phenomenon happens when } n = 2k \text{ and } q = 1 \text{ with } u = (10)^{k-2}10 \text{ and } v = (10)^{k-2}00; \phi(u) = 1^{2k} \text{ and } \phi(v) = 0^{k+1}1^{k-1}. \text{ See also Table 1(a) for the image through } \phi \text{ of the 1-Gray code in [27] for } B_4(111). \]

Below we show that BRGC order (that is, the order induced by Binary Reflected Gray Code in [11]) yields a 3-Gray code on \( W^q_n \). Much more interestingly, thanks to Corollary 1, the necessary condition for the existence of a 1-Gray code is satisfied, and we will provide such a Gray code for \( W^q_n \) in the following part.

In [28] the author introduces the notion of absorbent set, which (up to complement) is defined as: a binary word set \( \mathcal{X} \subset \{0, 1\}^n \) is called absorbent if for any \( u \in \mathcal{X} \) and any \( k, 1 \leq k < n, u_1u_2...u_k0^{n-k} \) is also a word in \( \mathcal{X} \). Corollary 1 from the same paper proves that when an absorbent set \( \mathcal{X} \) is listed in BRGC order (that is, restricting BRGC to \( \mathcal{X} \)) the resulting listing yields a 3-Gray code. Clearly, \( W^q_n \) is an absorbent set and we have the following consequence.

**Corollary 6.** For \( n \geq 0, q \geq 1 \), the restriction of BRGC order yields a 3-Gray code for \( W^q_n \).

See for an example Table 1(b).
Applying reversing sublists technique [21] by adding a new parameter to \textsc{LexFib} which keeps track of the parity of the number of 1s in the current generated prefix, procedure \textsc{LexFib} can be turned into one generating the same class of words in BRGC order instead of the lexicographic one, and so according to Corollary 6 producing a 3-Gray code for \( W_n^q \). Obviously the obtained Gray code generating algorithm inherits CAT property.

At this point it is worth mentioning an alternative powerful framework to design Gray codes as sublists of BRGC. In [23] it is given the following definition (up to mirroring each word): a set of same length binary words is a \textit{flip-swap language} (with respect to 1) if it is closed under the operations (i) flip the rightmost 1, and (ii) swap the rightmost 1 with the bit to its left. And a flip-swap language with respect to 0 is defined similarly. Theorem 2 in [23] states that when a flip-swap language is listed in BRGC order the resulting listing yields a 2-Gray code, and in the same paper there is given a variety of combinatorial classes that are flip-swap languages together with efficient generating algorithms for most of them. It is easy to see that absorbent sets coincide with languages closed under flip operation, and thus it is not surprising that, when listed in BRGC order, flip-swap languages yield more restrictive (that is, 2- instead of 3-) Gray codes than absorbent sets do. In particular, for \( n \geq 0 \) and \( q \geq 1 \), \( W_n^q \) is an absorbent set (or equivalently, a flip-closed language) but not a flip-swap language.

### 4.2 1-Gray code for \( W_n^1 \)

In this section, we construct a 1-Gray code for the set \( W_n^q \), \( n \geq 0 \), when \( q = 1 \), which in particular gives a positive answer to a conjecture in [8]. For this purpose, we decompose \( W_n^1 \), \( n \geq 1 \), as

\[
W_n^1 = Z_n \cup 1 \cdot W_{n-1}^1
\]

where \( W_0^1 \) consists of the empty word \( \epsilon \), and \( Z_n \) is the subset of words starting with 0 in \( W_n^1 \). In turn, we decompose \( Z_n \) as

\[
Z_n = \{0^n\} \cup \bigcup_{r=3}^n D_n^r
\]

where \( D_n^r = \bigcup_{j=1}^{r-2} 0^{r-j}1^j \cdot Z_{n-r} \). We refer to Figure 2(a) for a graphical illustration of the decomposition of \( Z_n \) for \( n = 17 \) where the point at coordinates \((i, j)\) corresponds to the set \( 0^i1^j \cdot Z_{n-i-j} \), \( 1 \leq j < i \leq n-1, \ 3 \leq i + j \leq n \), except the lowest point which corresponds to \( \{0^n\} \). The sets \( D_n^r, \ 3 \leq r \leq n \), correspond to the southwest-northeast diagonals of the graphic.

Let \( Z_0 = \epsilon \) be the list containing only the empty word \( \epsilon \), \( Z_1 = 0 \), \( Z_2 = 00 \), \( Z_3 = 000 \cdot 001 \). If \( L \) is a list and \( w \) is a word, then \( w \cdot L \) denotes the list where \( w \) is concatenated to every word from \( L \). According to the above definitions, it is straightforward to check the following lemma.
Lemma 3. For any $1 \leq k < n$, we suppose that $Z_k$ is a 1-Gray code for $Z_k$ with $\text{first}(Z_k) = 0(001)^*$ and $\text{last}(Z_k) = (001)^*$. Given $i$ and $j$ such that $1 \leq j < i \leq n$, then

(i) for $3 \leq i + j < n$, the list $L = 0^i1^j \cdot Z_{n-i-j}$ is a 1-Gray code for $0^i1^j \cdot Z_{n-i-j}$ with $\text{first}(L) = 0^i1^j0(001)^*$ and $\text{last}(L) = 0^i1^j(001)^*$; when $i + j = n$ the list $L$ contains only the word $0^i1^j$;

(ii) for $3 \leq i + j < n - 1$, the list $L = 0^i1^{j+1} \cdot Z_{n-i-j-1} \circ 0^i1^j \cdot Z_{n-i-j}$ is a 1-Gray code for $0^i1^{j+1} \cdot Z_{n-i-j-1} \circ 0^i1^j \cdot Z_{n-i-j}$ with $\text{first}(L) = 0^i1^{j+1}0(001)^*$ and $\text{last}(L) = 0^i1^{j+1}(001)^*$; when $i + j + 1 = n$, the list $L = 0^i1^{j+1} \circ 0^i1^j$;

(iii) for $3 \leq i + j < n$, the list $L = 0^i1^j \cdot Z_{n-i-j} \circ 0^{i-1}1^{j+1} \cdot Z_{n-i-j}$ is a 1-Gray code for $0^i1^j \cdot Z_{n-i-j} \circ 0^{i-1}1^{j+1} \cdot Z_{n-i-j}$ with $\text{first}(L) = 0^i1^j0(001)^*$ and $\text{last}(L) = 0^{i-1}1^{j+1}0(001)^*$; when $i + j = n$, the list $L = 0^i1^j \circ 0^{i-1}1^{j+1}$;
(iv) for $3 \leq i + j \leq n$, the list $L = 0^i1^j \cdot Z_{n-i-j} \circ 0^{i-1}1^{j+1} \cdot Z_{n-i-j}$ is a 1-Gray code for $0^i1^j \cdot Z_{n-i-j} \cup 0^{i-1}1^{j+1} \cdot Z_{n-i-j}$ with first($L$) = $0^i1^j(001)^*$ and last($L$) = $0^{i-1}1^{j+1}(001)^*$.

**Lemma 4.** Let us consider $r = 1 \pmod{4}$, $3 \leq r \leq n$. For any $1 \leq k < n$, we suppose that $Z_k$ is a 1-Gray code for $Z_k$ with first($Z_k$) = 0(001)$^*$ and last($Z_k$) = (001)$^*$.

(i) If $r \neq n - 1$, then there is a 1-Gray code $\Delta_n^r$ for $D_n^r \cup D_n^{r-1}$ such that first($\Delta_n^r$) = $0^{r-2}1(001)^*$ and last($\Delta_n^r$) = $0^{r-1}1(001)^*$.

(ii) If $r = n - 1$, then there is a 1-Gray code $\Delta_n^{n-1}$ for $D_n^n \cup D_n^{n-1} \cup D_n^{n-2}$ such that first($\Delta_n^{n-1}$) = $0^{n-3}100$ and last($\Delta_n^{n-1}$) = $0^n-11$.

**Proof.** For the first assertion (i), it suffices to consider the list

$$\Delta_n^r = \bigcirc_{j=0}^{n-r-1} 0^{-r-1-j}1^j \cdot \langle Z_{n-r+1} \rangle^j \circ \bigcirc_{j=1}^{n-r} 0^{-r-j}1^j \cdot \langle Z_{n-r} \rangle^j.$$

After considering assertions of Lemma 3, it remains to examine the transition between $w_0 = \text{last}(0^{r-1-j_0}1^{j_0}) \cdot \langle Z_{n-r+1} \rangle^{j_0}$ for $j_0 = \frac{r-1}{2}$ and $w_1$ = first($0^{r-j_1}1^{j_1}$) for $j_1 = \frac{r+1}{2}$. Since $r = 1 \pmod{4}$, we necessarily $j_0$ is odd and $j_0 + 1 = j_1$ which implies that $w_0$ = last($0^{r-1-j_0}1^{j_0}$) $\cdot Z_{n-r+1}$ = $0^{r-1-j_0}1^{j_0}0(001)^*$ and $w_1$ = first($0^{r-j_1}1^{j_1}$) $\cdot Z_{n-r}$ = $0^{r-j_1}1^{j_1}(001)^*$, and they differ by exactly one bit.

For the second assertion (ii), we consider the list

$$\Delta_n^{n-1} = \bigcirc_{j=1}^{n-2} 0^{-n-2-j}1^j001 \circ 1^j001 \circ \bigcirc_{j=1}^{n-2} 0^{-n-1-j}1^{j+1} \cdot \langle 0^{-n-1-j}1^{j+1}0 \circ 0^{-n-1-j}1^{j}0 \rangle^{-1} \cdot 0^{-n-1}1.$$

A simple study of each kind of transitions allows us to see that $\Delta_n^{n-1}$ is a 1-Gray code for $D_n^n \cup D_n^{n-1} \cup D_n^{n-2}$ satisfying first($\Delta_n^{n-1}$) = $0^{n-3}100$ and last($\Delta_n^{n-1}$) = $0^n-11$. An illustration of this Gray code for $n = 10$ (and thus $r = 9$) can be found in the last sketch of Figure 5.

In the following, we write $wL$ instead of $w \cdot L$ to be more concise.

**Lemma 5.** Let us consider $r = 3 \pmod{4}$, $3 < n$ and $3 \leq r < n$. For any $1 \leq k < n$, we suppose that $Z_k$ is a 1-Gray code for $Z_k$ with first($Z_k$) = 0(001)$^*$ and last($Z_k$) = (001)$^*$.

(i) If $r = 3$, then there is a 1-Gray code $\Delta_3^n$ for $D_3^n$ such that first($\Delta_3^n$) = $0010(001)^*$ and last($\Delta_3^n$) = (001)$^*$.

(ii) If $r = n - 2$, then there is a 1-Gray code $\Delta_{n-2}^n$ for $D_{n-2}^n \cup D_{n-3}^n$ such that first($\Delta_{n-2}^n$) = $0^{n-4}1000$ and last($\Delta_{n-2}^n$) = $0^{n-3}100$.

(iii) If $r = n - 1$, then there is a 1-Gray code $\Delta_{n-1}^n$ for $D_{n-1}^n \cup D_{n-2}^n \cup D_{n-3}^n \setminus \{0^{n-1}\}$ such that first($\Delta_{n-1}^n$) = $0^{n-3}100$ and last($\Delta_{n-1}^n$) = $0^{n-2}10$.

(iv) If $r = n$, then there is a 1-Gray code $\Delta_n^n$ for $D_n^n \cup D_n^{n-1}$ such that first($\Delta_n^n$) = $0^{n-2}100$ and last($\Delta_n^n$) = $0^n-11$. 

\(\square\)
(v) If \( r \notin \{3, n-2, n-1, n\} \), then there is a 1-Gray code \( \Delta_n^r \) for \( \mathcal{D}_n^r \cup \mathcal{D}_n^{r-1} \) such that first\((\Delta_n^r) = 0^{r-1}10(001)^*\) and last\((\Delta_n^r) = 0^{r-1}1(001)^*\).

Proof. For the case (i), we set: \( \Delta_3 = 0^21\mathbb{Z}_{n-3} \).

For the case (ii), we set: \( \Delta_n^{n-2} = \bigcup_{j=1}^{\frac{n-3}{2}} 0^{n-3-j}1_j / \mathbb{Z}_n \), it is straightforward to see that \( \Delta_n^{n-2} \) is a 1-Gray code with first\((\Delta_n^{n-2}) = 0^{n-4}10000\) and last\((\Delta_n^{n-2}) = 0^{n-3}1000\).

For the case (iii), we set:

\[
\Delta_n^{n-1} = \bigcup_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1_j / \mathbb{Z}_2 \circ \bigcup_{j=1}^{\frac{n-4}{2}} 0^{n-1-j}1_j / \mathbb{Z}_1 \circ \bigcup_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1_j / \mathbb{Z}_2.
\]

Knowing that \( \mathbb{Z}_2 = 00 \) and \( \mathbb{Z}_1 = 000 \), we can easily check that any pair of consecutive words differ by exactly one bit, which proves that \( \Delta_n^{n-1} \) is a 1-Gray code.

For the case (iv), we set: \( \Delta_n = \bigcup_{j=1}^{\frac{n-3}{2}} 0^{n-1-j}1_j / \mathbb{Z}_n \). As previously the result can be obtained easily.

The case (v) is more challenging to handle. The set \( \mathcal{D}_n^r \cup \mathcal{D}_n^{r-1} \) consists of the union of the following subsets: \( K_1 = 0^{r-2}1 \mathbb{Z}_{n-r+1} \), \( K_2 = 0^{r-3}11 \mathbb{Z}_{n-r+1} \), \( K_3 = 0^{r-4}111 \mathbb{Z}_{n-r+1} \), \( L_1 = 0^{r-1}1 \mathbb{Z}_{n-r} \), \( L_2 = 0^{r-2}11 \mathbb{Z}_{n-r} \), \( L_3 = 0^{r-3}111 \mathbb{Z}_{n-r} \) with \( a = \left\lfloor \frac{r-2}{2} \right\rfloor = \frac{r-3}{2} \). Let us denote by \( K_1, K_2, \ldots, K_a \) and \( L_1, L_2, \ldots, L_{a+1} \) the associated Gray codes obtained by replacing \( \mathbb{Z}_n \) with the Gray code \( \mathbb{Z}_k \).

Remark that for \( 1 \leq i \leq a-1 \) (resp. \( 1 \leq i \leq a \)) and for a given \( j \), the \( j \)-th word of \( K_i \) (resp. \( L_i \)) and the \( j \)-th word of \( K_{i+1} \) (resp. \( L_{i+1} \)) differ by exactly one bit. Recall that \( \tilde{A} \) is obtained from a list \( \tilde{A} \) by deleting its first element. The words first\((K_i) \) and last\((L_{i+1}) \) differ by one bit. We have first\((\tilde{L}_{a+1} \circ K_a) \) = first\((\tilde{L}_{a+1}) \) and it differs obviously by one bit from first\((\tilde{L}_{a+1}) \). We also have \( r = 3 \) mod 4, so \( \frac{r-3}{2} = a \) is even. Taking into account all these remarks, the list \( \Delta_n^r \) defined below is a Gray code:

\[
\Delta_n^r = \bigcup_{i=1}^{a+1} \text{first}(L_i) \circ \bigcup_{i=a}^1 \text{last}(\tilde{L}_{i+1} \circ K_i)^i \circ \tilde{L}_1
\]

We refer to Figure 3 for a graphical representation of this Gray code.

\[\Box\]

**Theorem 3.** For any \( n \geq 0 \), there exists a 1-Gray code \( \mathbb{Z}_n \) for \( \mathbb{Z}_n \) such that first\((\mathbb{Z}_n) = 0(001)^*\) and last\((\mathbb{Z}_n) = (001)^*\).

Using initial conditions \( \mathbb{Z}_0 = \epsilon, \mathbb{Z}_1 = 0, \mathbb{Z}_2 = 00, \mathbb{Z}_3 = 000 \circ 001 \), where \( \epsilon \) is the empty word, and the recursive constructions for lists \( \Delta_n^r \) (Lemmas 3 and 4) we define the 1-Gray code \( \mathbb{Z}_n \) as follows:

\[
\mathbb{Z}_n = \begin{cases}
\Delta_3^r \circ \Delta_3^r \circ \cdots \circ \Delta_3^r \circ 0^n \circ \Delta_3^{n-2} \circ \cdots \circ \Delta_3^1 \circ \Delta_3^3, & \text{if } n = 1 \text{ mod 4}, \\
\Delta_3^r \circ \Delta_3^r \circ \cdots \circ \Delta_3^r \circ 0^n \circ \Delta_3^{n-3} \circ \cdots \circ \Delta_3^1 \circ \Delta_3^3, & \text{if } n = 2 \text{ mod 4}, \\
\Delta_3^r \circ \Delta_3^r \circ \cdots \circ \Delta_3^r \circ 0^n \circ \Delta_3^{n-3} \circ \cdots \circ \Delta_3^1 \circ \Delta_3^3, & \text{if } n = 3 \text{ mod 4}, \\
\Delta_3^r \circ \Delta_3^r \circ \cdots \circ \Delta_3^r \circ 0^n \circ \Delta_3^{n-1} \circ \cdots \circ \Delta_3^1 \circ \Delta_3^3, & \text{if } n = 0 \text{ mod 4}.
\end{cases}
\]
Due to Lemmas 4 and 5, last($\Delta_n^{4i+1}$) differs by one bit from first($\Delta_n^{4i+5}$) and first($\Delta_n^{4i+3}$) differs by one bit from last($\Delta_n^{4i+7}$). Using $0^n$ (and $0^n-1$ when $n = 0 \mod 4$) we connected all $\Delta_n^r$ and ensure that $Z_n$ is a 1-Gray code. See Table 2 for a more detailed view on the structure of $Z_n$. □

We refer to Figure 5 for a graphical representation of $Z_n$ for $4 \leq n \leq 10$, see also Figure 2(b) for $n = 17$. An immediate consequence of Theorem 3 is the following.

**Theorem 4.** For any $n \geq 1$, $W_n^1 = 1 \cdot W_{n-1}^1 \circ Z_n$ is a 1-Gray code for $W_n^1$, such that first($W_n^1$) = $1^n$, last($W_n^1$) = (001)$^*$, and where $W_0^1$ is a list containing only the empty word.

However, the efficient generation of this Gray code remains an open problem.

Eğecioğlu and Iršič introduce in [8] the “run-constrained binary strings”. These are binary words, in which every run of 1s is immediately followed by a strictly longer run of 0s. Using these strings of length $n + 2$ as vertices, and connecting two vertices if they differ at only one position, the authors of [8] form the Fibonacci-run graph $R_n$ as the induced subgraph of the hypercube. (As every non-empty run-constrained string must end with 00, authors of [8] actually drop the last 2 zeros, but not we.) Figure 4 gives small examples.
Lemma 9.1 from [8] says that if \( \Delta n \) is a Hamiltonian cycle. Our method gives a Hamiltonian path, which is not a cycle.

Conjecture 1 following extension of Theorem 4.

Theorem 3 gives a Hamiltonian path in the Fibonacci-run graph. The next corollary settles a conjecture in [8].

Finally, the validity of the parity condition stated in Corollary 1 and experimental investigations for small values, 0 \( \leq n \leq 5 \) and 2 \( \leq q \leq 5 \), suggest the following extension of Theorem 4.

Table 2: The structure of \( Z_n \).

![Table 2: The structure of \( Z_n \).](image)

It turns out that the run-constrained binary strings are precisely the reverse of 1-decreasing words beginning with 0. In this light, the Gray code \( Z_n \) in Theorem 3 gives a Hamiltonian path in the Fibonacci-run graph. The next corollary settles a conjecture in [8].

**Corollary 7.** For any \( n \geq 1 \), the Fibonacci-run graph \( R_n \) has a Hamiltonian path.

Lemma 9.1 from [8] says that if \( n \neq 1 \) mod 3, then \( R_n \) does not contain a Hamiltonian cycle. Our method gives a Hamiltonian path, which is not a cycle. The question of whether there is a Hamiltonian cycle for the case \( n = 1 \) mod 3 remains open.

Finally, the validity of the parity condition stated in Corollary 1 and experimental investigations for small values, 0 \( \leq n \leq 5 \) and 2 \( \leq q \leq 5 \), suggest the following extension of Theorem 4.

**Conjecture 1** For any \( n \geq 0 \) and \( q \geq 1 \), there is a 1-Gray code for \( W_n^q \).
Table 3: The Gray code $W^n_6$ for the set $W^n_6$. The Hamming distance between two consecutive words is one.

| Word | Hamming Distance |
|------|------------------|
| 111111 | 8 110010 15 000110 |
| 211111 | 9 100010 16 000010 |
| 311110 | 10 100011 17 000011 |
| 411100 | 11 100001 18 000001 |
| 511100 | 12 100000 19 000000 |
| 611000 | 13 100100 20 001000 |
| 711000 | 14 000100 21 001001 |

Fig. 4: Fibonacci-run graphs for small values of $n$. Vertices correspond to the reverse of words from $W^n_1$ beginning with 0. The Hamiltonian path is provided by Corollary 7. If we read words from right to left, the path starts at $(001)^*$ and ends at $(001)^*$.

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Fig. 5: An illustration of the recursive definition for the Gray codes $Z_n$, $4 \leq n \leq 10$. A point labelled 001...... (that is 0001 followed by seven dots) corresponds to the set of words in 001·$Z_7$.

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