MIMO Multiway Relaying with Pairwise Data Exchange: A Degrees of Freedom Perspective

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Abstract—In this paper, we study achievable degrees of freedom (DoF) of a multiple-input multiple-output (MIMO) multiway relay channel (mRC) where \( K \) users, each equipped with \( M \) antennas, exchange messages in a pairwise manner via a common \( N \)-antenna relay node. A novel and systematic way of joint beamforming design at the users and at the relay is proposed to align signals for efficient implementation of physical-layer network coding (PNC). It is shown that, when the user number \( K = 3 \), the proposed beamforming design can achieve the DoF capacity of the considered mRC for any \((M, N)\) setups. For the scenarios with \( K > 3 \), we show that the proposed signaling scheme can be improved by disabling a portion of relay antennas so as to align signals more efficiently. Our analysis reveals that the obtained achievable DoF is always piecewise linear, and is bounded either by the number of user antennas \( M \) or by the number of relay antennas \( N \). Further, we show that the DoF capacity can be achieved for \( \frac{M}{N} \in \left(0, \frac{K-1}{K(K-2)}\right) \) and \( \frac{M}{N} \in \left[\frac{1}{K(K-2)}, \frac{2}{K}, \infty\right) \), which provides a broader range of the DoF capacity than the existing results. Asymptotic DoF as \( K \to \infty \) is also derived based on the proposed signaling scheme.

Index Terms—Multiple-input multiple-output (MIMO), physical-layer network coding (PNC), multiway relay channel (mRC), signal space alignment.

I. INTRODUCTION

During the past decade, an exponential increase in the demands of wireless service has imposed a significant challenge on the design of wireless networks. Advanced techniques, such as physical layer network coding (PNC), are developed to achieve high spectrum efficiency [1], [2]. The simplest model for PNC is the two-way relay channel (TWRC) where two users exchange messages with the help of a relay node. With the well-known two-phase PNC protocol, the relay node receives a combination of the signals transmitted from the two users in the first phase, and then broadcasts a network-coded message in the second phase. The desired message is then extracted at each user end by exploiting the knowledge of the self-message. As compared to conventional one-way relaying where four phases are required in one round of information exchange, PNC potentially achieves 100% improvement in spectrum efficiency over TWRCs.

Abundant progresses have been made on the PNC design for TWRCs; see [3]–[7] and the references therein. In particular, it was shown in [4] that, with nested lattice coding, the capacity of the TWRC can be achieved within \( \frac{1}{2} \) bit. Later, the authors in [5]–[7] introduced the multiple-input multiple-output (MIMO) technique into TWRCs. It was revealed that the space-division based network coding scheme proposed in [7] achieves the asymptotic capacity of the MIMO TRWC at high signal-to-noise ratio (SNR) within \( \frac{1}{2} \log_2(\frac{M}{2}) \) bit per relay spatial dimension for an arbitrary antenna configuration.

A natural generalization of TWRC is a multiway relay channel (mRC), where multiple users exchange messages with the help of a single relay. Several mRC models have been studied in the literature recently. For example, the authors in [10] studied a cellular two-way relaying model where a base station intends to exchange private messages with multiple mobile users via a relay node; the authors in [8], [9] investigated mRCs in which users are grouped into pairs and the two users in each pair exchange information with each other; more generally, the authors in [11] studied clustered mRCs, in which the users in the network are grouped into clusters and each user in a cluster wants to exchange information with the other users in the same cluster. Approximate capacities of these mRC models were studied in [11] and [12], although the exact capacity characterizations are still far from being well understood. Also, these initial works on mRC are limited to the single-antenna setup, i.e., each node in the network is equipped with one antenna.

The MIMO technique has been introduced into mRCs to allow spatial multiplexing. In a MIMO mRC, as each user in general transmits multiple spatial streams, a new challenging issue to be addressed is to mitigate the inter-stream interference at the relay and at the user ends. Degrees of freedom (DoF) is a critical metric in characterizing the fundamental capacity of a wireless multi-terminal network [18], [19]. The DoF of the MIMO mRC has been previously studied in [13]–[17]. For example, the authors in [14] investigated the DoF capacity of the MIMO Y channel (a special case of the MIMO mRC with three users) and showed that the DoF capacity can be achieved when \( \frac{M}{N} \geq \frac{2}{3} \), where \( M \) denotes the number of antennas at each user and \( N \) denotes the number of antennas at the relay. Later, the work in [15] generalized the channel model in [14] to the case of \( K \) users and studied the achievability conditions of the DoF capacity. Very recently, the authors in [16], [17] studied more general scenarios in which the users in the network are grouped into clusters, and each user in a cluster exchanges information only with the other users in the same cluster. In particular, the authors in [16] derived sufficient conditions on the antenna configuration to achieve the DoF capacity of a clustered mRC with pairwise data exchange model, in which each user in a cluster sends a different message to each of the other users in the same cluster. Note
that the data exchange models considered in [14] and [15] can be regarded as the one-cluster case of the model studied in [16]. Moreover, the author in [17] derived an achievable DoF for a clustered MIMO mRC with full data exchange, i.e., each user in a cluster delivers a common message to all the other users in the same cluster.

In this work, we study a symmetric MIMO mRC with pairwise data exchange, and derive an achievable DoF for an arbitrary setup of antenna numbers \((M, N)\) and user number \(K\). In general, the DoF of a network can be seen as the number of independent spatial streams that can be supported by the network. In the MIMO mRC of concern, multiple users are simultaneously served by a common relay. To ensure that multiple spatial streams are still separable at every user end, the number of relay antennas is usually the bottleneck of the network to achieve a higher DoF. Therefore, the challenge is how to align the user and relay signals to efficiently utilize the relay’s signal space. To this end, we propose a novel and systematic way of beamforming design to align signals for efficient implementation of PNC. Specifically, we refer to a bunch of \(K(K - 1)\) spatial streams as a *unit*, in which each pair of users only contributes two spatial streams; each spatial stream corresponds to a directed vector in the relay’s signal space; the \(K(K - 1)\) spatial streams in a unit form a spatial structure, referred to as a *pattern*. The dimension of the space spanned by the spatial streams in a pattern gives a metric to evaluate the efficiency of this pattern. Then, the signal alignment problem becomes to construct units with the most efficient patterns to occupy the overall relay’s signal space. An achievable DoF can be obtained by counting the number of units constructed for any given given a specific antenna setup of \((M, N)\).

The main contributions of this work are summarized as follows:

- We show that, for the considered MIMO mRC with \(K = 3\), the proposed signal alignment scheme achieves the DoF capacity for an arbitrary \((M, N)\) setup. This finding improves the existing DoF capacity result in [14], where the achievability of the DoF capacity in [14] is limited in the range of \(\frac{M}{N} \in [\frac{3}{4}, \infty)\).
- For the case of \(K \geq 3\), we derive the DoF capacity of the MIMO mRC for \(\frac{M}{N} \in \left\{0, \frac{K-1}{K} \right\}\) and \(\frac{M}{N} \in \left[\frac{1}{K(K-1)} + \frac{1}{2}, \infty\right)\). This result is stronger than the previous result obtained in [16], where the achievable capacity is limited in the range of \(\frac{M}{N} \in \left(0, \frac{1}{K} \right)\) and \(\frac{M}{N} \in \left(\frac{1}{K(K-1)} + \frac{1}{2}, \infty\right)\).
- For \(K \geq 3\), we also derive an achievable DoF for an arbitrary setup of antenna numbers \((M, N)\) satisfying \(\frac{M}{N} \in \left(\frac{K-1}{K}, \frac{K}{K+1}\right)\). Our analysis reveals that the achievable DoF is piecewise linear with respect to the \(\frac{M}{N}\) ratio and is bounded either by the number of antennas at each user or by the number of antennas at the relay. This piecewise linearity is similar to the DoF capacity obtained for the MIMO interference channel in [19].
- Finally, based on the proposed signal alignment scheme, we derive an asymptotic achievable DoF when \(K\) tends to infinity.

The rest of the paper is organized as follows. In Section II, we present the system model. The DoF capacity of the considered MIMO mRC with three users is presented in Section III. In Section IV, we generalize the results to the case of an arbitrary number of users. In Section V, an improved DoF result is presented by disabling a portion of relay antennas. Finally, we conclude the paper in Section VI.

**Notation**: Scalars, vectors, and matrices are denoted by lowercase regular letters, lowercase bold letters, and uppercase bold letters, respectively. For a matrix \(A\), \(A^T\) and \(A^H\) denote the transpose and the Hermitian transpose of \(A\), respectively; \(\text{tr}(A)\) and \(A^{-1}\) stand for the trace and the inverse of \(A\), respectively; \(\text{diag}(A_1, A_2, \cdots, A_n)\) denote a block-diagonal matrix with the \(i\)-th diagonal block specified by \(A_i\); \(\text{null}(A)\) denotes the column space and the nullspace of \(A\), respectively; \(I_n\) denotes an \(n \times n\) identity matrix; \(\text{dim}(S)\) denotes the dimension of a space \(S\); \(S \cap U\) and \(S \oplus U\) denote the intersection and the direct sum of two spaces \(S\) and \(U\), respectively; \(\mathbb{R}^{n \times m}\) and \(\mathbb{C}^{n \times m}\) denote the \(n \times m\) dimensional real space and complex space, respectively; \(\log(\cdot)\) denotes the logarithm with base 2; \([\cdot]^+\) denotes max\{\cdot, 0\}; \(\mathcal{CN}(\mu, \sigma^2)\) denotes the distribution of a circularly symmetric complex Gaussian random variable with mean \(\mu\) and variance \(\sigma^2\).

**II. SYSTEM MODEL**

A. Channel Model

Consider a discrete memoryless symmetric MIMO mRC \((M, N, K)\), where \(K\) users, each equipped with \(M\) antennas, exchange messages in a pairwise manner with the help of a common \(N\)-antenna relay node, as illustrated in Fig. 1. The direct links between users are ignored due to physical impairments such as shadowing and path loss of wireless fading channels.
Each round of information exchange is implemented in two phases with equal time duration $T$. In the first phase (termed the uplink phase), all the users simultaneously transmit signals to the relay node. The received signal at the relay node can be written as

$$Y_R = \sum_{k=1}^{K} H_k X_k + Z_R, \quad k \in \mathcal{I}_K \triangleq \{1, 2, \cdots, K\}$$  \hspace{1cm} (1)$$

where $H_k \in \mathbb{C}^{N \times M}$ denotes the channel matrix from user $k$ to the relay; $X_k \in \mathbb{C}^{M \times T}$ is the transmitted signals from user $k$; similarly, $Y_R \in \mathbb{C}^{N \times T}$ denotes the received signal at the relay node; $Z_R \in \mathbb{C}^{N \times T}$ is the additive white Gaussian noise (AWGN) matrix at the relay node and the elements are independently drawn from the distribution of $\mathcal{CN}(0, \sigma_R^2)$. The transmit signal $X_k$ at user $k$ satisfies the power constraint of

$$\frac{1}{T} \text{tr}(X_k X_k^H) \leq P_k, \quad k \in \mathcal{I}_K$$

where $P_k$ is the maximum transmission power allowed at user $k$.

In the second phase (termed the downlink phase), the relay sends the processed signals to all user ends. The received signal at user $k$ is denoted by

$$Y_k = G_k X_R + Z_k, \quad k \in \mathcal{I}_K$$  \hspace{1cm} (2)$$

where $G_k \in \mathbb{C}^{M \times N}$ denotes the channel matrix from the relay to user $k$; $X_R \in \mathbb{C}^{N \times T}$ is the transmitted signal at the relay node; $Z_k \in \mathbb{C}^{N \times T}$ is the AWGN noise matrix at user $k$ with the elements independently drawn from $\mathcal{CN}(0, \sigma_R^2)$. The transmitted signal $X_R$ satisfies the power constraint of

$$\frac{1}{T} \text{tr}(X_R X_R^H) \leq P_R,$$

where $P_R$ is the maximum transmission power allowed at the relay.

We assume that the elements of the channel matrices $H_k$ and $G_k$, $\forall k$, are drawn from a continuous distribution, which implies that these channel matrices are of full column or row rank, whichever is smaller, with probability one. The channel state information is assumed to be perfectly known at all nodes, following the convention in [13]–[16]. It is worth noting that the considered MIMO mRC reduces to the MIMO two-way relay channel (TWRC) when $K = 2$, and to the MIMO Y channel when $K = 3$. As the DoF capacity of the MIMO TWRC is well understood, we henceforth focus on the case of $K \geq 3$.

B. Linear Signaling Scheme

In the considered mRC, each user $k$, $k \in \mathcal{I}_K$, intends to send a private message $W(k,k')$ to the user $k'$, $\forall k' \neq k$. The message $W(k,k')$ is then encoded as $f(W(k,k')) = \{s_1^{(k,k')}, s_2^{(k,k')}, \cdots, s_T^{(k,k')}\}$, where $f(\cdot)$ is an encoding function; $s_l^{(k,k')} \in \mathbb{C}^{1 \times T}$ denotes the signal stream transmitted in unit $l$. The goal of this work is to analyze the DoF of the considered MIMO mRC. Linear processing is assumed to be applied at the transmitter, relay, and receiver sides. The transmit signal at the user $k$ can be denoted by

$$X_k = \sum_{l=1}^{L} U_{k,l} S_{k,l},$$

where $k$ denotes the user index; $l$ denotes the unit index and $L$ is the number of the units which can be supported by the network; $U_{k,l} = [u_{l}^{(k,1)}, u_{l}^{(k,2)}, \cdots, u_{l}^{(k,k-1)}, u_{l}^{(k,k)}] \in \mathbb{C}^{M \times (K-1)}$ denotes the beamforming matrix applied at user $k$ for the $l$-th unit; $S_{k,l} = [s_{l}^{(k,1)}, s_{l}^{(k,2)}, \cdots, s_{l}^{(k,k-1)}, s_{l}^{(k,k)}] \in \mathbb{C}^{(K-1) \times T}$ denotes the transmit spatial streams over $T$ channel uses; $u_{l}^{(k,k')}$ corresponds to the beamformer of data stream $s_{l}^{(k,k')}$. Note that the maximum number of spatial streams in a unit is $K(K - 1)$. But this number can be reduced to $K'(K’ - 1)$, where $K'$ is the number of active users in the unit.

During the uplink, the equivalent channel matrix from user $k$ to the relay can be expressed by

$$H_k U_{k,l} = \left[ h_j^{(k,1)}, h_j^{(k,2)}, \cdots, h_j^{(k,k-1)}, h_j^{(k,k)} \right], \quad \forall j.$$  \hspace{1cm} (3)$$

In the following, we will see that by properly designing beamformer $u_{l}^{(k,k')}$, the subspace spanned by $\{h_l^{(k,k')}\}_{k,k',k \neq k'}$ forms a spatial structure, referred to as a pattern, such that a higher DoF can be achieved.

The transmit signal at the relay node can be written as

$$X_R = F Y_R,$$  \hspace{1cm} (4)$$

where $F$ denotes the linear beamforming matrix used at the relay. Similar to the uplink, by using linear receiving matrix

$$V_{k,l} = \left[ v_j^{(k,1)}, v_j^{(k,2)}, \cdots, v_j^{(k,k-1)}, v_j^{(k,k)} \right] \in \mathbb{C}^{(K-1) \times M},$$

the equivalent channel matrix in the downlink is given by

$$V_{k,l} G_k = \left[ g_j^{(k,1)}, g_j^{(k,2)}, \cdots, g_j^{(k,k-1)}, g_j^{(k,k)} \right]^T.$$ \hspace{1cm} (5)$$

Later, we will show that due to symmetry between the uplink and the downlink, the uplink design straightforwardly carries over to the downlink. Thus, we mostly focus on the uplink design in this paper.

C. Degrees of Freedom

Let $R(k,k')$ be the information rate carried in $W(k,k')$, and $W_r(k,k')$ be the estimate of $W(k,k')$ at user $k$. We say that user $k$ achieves a sum rate of $C_k = \sum_{k=1,k \neq k'}^{K} R(k,k')$, if $P_T \{W(k,k') \neq W_r(k,k')\}$ tends to zero as $T \rightarrow \infty$.

We assume a symmetric mRC with $P_1 = P_2 = \cdots = P_K = P_R = P$ and $\sigma_R^2 = \sigma_0^2 = \cdots = \sigma_K^2 = \sigma^2$. Denote $\text{SNR} = \frac{P}{\sigma^2}$. Let $C_k(\text{SNR})$, $k \in \mathcal{I}_K$, be an achievable rate of user $k$. The total DoF of the mRC is defined as

$$d_{\text{sum}} \triangleq \lim_{\text{SNR} \to \infty} \frac{\sum_{k=1}^{K} C_k(\text{SNR})}{\log \text{SNR}}.$$
Also, we define the DoF per user and the DoF per relay dimension respectively as
\[
d_{\text{user}} = \frac{1}{K} d_{\text{sum}} \quad \text{and} \quad d_{\text{relay}} = \frac{1}{N} d_{\text{sum}}.
\] (6)

D. A DoF Outer Bound

An outer bound on the total DoF of the MIMO mRC is given in [16] as
\[
d_{\text{sum}} \leq d_{\text{outer}} \triangleq \min\left(\frac{KM}{2}, 2N\right),
\] (7a)
or equivalently
\[
d_{\text{user}} \leq \frac{1}{K} d_{\text{outer}} = \min\left(M, \frac{2N}{K}\right).
\] (7b)

The above outer bound can be intuitively explained as follows. On one hand, the total number of independent spatial data streams supported by the MIMO mRC cannot exceed \(2N\), as the relay signal space has \(N\) dimensions and thus the relay can only decode and forward \(N\) network-coded messages. On the other hand, the number of independent spatial data streams transmitted or received by each user cannot exceed \(M\), as each user only has \(M\) antennas. The outer bound in (7) will be used as a benchmark in the following system design.

III. MIMO mRC with \(K = 3\)

In this section, we focus the DoF of the MIMO mRC with \(K = 3\). We propose a signal alignment scheme to achieve the DoF capacity of the MIMO mRC with \(K = 3\) for an arbitrary antenna setup of \((M,N)\).

A. Preliminaries

We give some intuitions of the signal alignment by considering only one unit. For brevity, we omit the unit index \(l\) in this subsection. Recall that \(s^{(k,k')}\) and \(s^{(k',k)}\) are exchanged in a pairwise manner for any \(k \neq k'\). For convenience, we refer to \(s^{(k,k)}\) and \(s^{(k',k)}\) as the signal pair \((k,k')\). Denote by \(h^{(k,k)}\) and \(g^{(k,k)}\) the equivalent channels in the uplink and the downlink, respectively. The system model in (1) and (2) reduces to
\[
Y_{R} = \sum_{k=1}^{K} \sum_{k' = 1, \ k' \neq k}^{K} h^{(k,k')} s^{(k,k')} + Z_R
\] (8a)
\[
y_k^{(k,k')} = g^{(k,k')} X_R + Z_k^T, \quad k \in I_K.
\] (8b)

The principle of PNC is applied in relay decoding. Specifically, for each user pair \((k,k')\), the relay decodes a linear mixture of \(s^{(k,k)}\) and \(s^{(k',k)}\) as follows. Denote by \(H^{(k,k')}\) the matrix formed by all the uplink channel vectors except \(h^{(k,k')}\) and \(h^{(k',k)}\). Then, define the projection matrix of pair \((k,k')\) as \(P^{(k,k')} = I_N - H^{(k,k')} H^{(k,k')} H^{(k',k)} H^{(k',k')}\), \(\in \mathbb{C}^{N \times N}\). For each pair \((k,k')\), the relay projects the received signal vector \(Y_{R}\) onto the nullspace of \(span(H^{(k,k')}))\), yielding
\[
P^{(k,k')} Y_{R} = P^{(k,k')} \left( \sum_{k=1}^{K} \sum_{k' = 1, \ k' \neq k}^{K} h^{(k,k')} s^{(k,k')} + Z_R \right)
\[
= P^{(k,k')} \left( h^{(k,k')} s^{(k,k')} + h^{(k,k)} s^{(k',k)} \right) T + P^{(k,k')} Z_R.
\] (9)

We now move to the relay-to-user phase modeled in (5b). Similarly to \(H^{(k,k')}\), we denote \(G^{(k,k')} \in \mathbb{C}^{N \times 4}\) for the matrix formed by all the downlink channel vectors except \(g^{(k,k')}\) and \(g^{(k',k)}\). The projection matrix of pair \((k,k')\) in the downlink is then defined as \(W^{(k,k')} = I_N - G^{(k,k')} (G^{(k,k')} H^{(k,k')} H^{(k,k')} G^{(k,k')})^{-1} G^{(k,k')} H^{(k,k')} \in \mathbb{C}^{N \times N}\). The relay sends out \(FY_{R}\) with \(F\) defined in [4] given by
\[
F = \alpha \sum_{k = 1}^{K} \sum_{k' = k+1}^{K} W^{(k,k')} P^{(kk')},
\] (10)
where \(\alpha\) is a scaling factor to meet the relay’s power constraint. In [10], the index \(k'\) start from \(k + 1\) since a project matrix \(P^{(kk')}\) is used to extract the signals \(s^{(k,k')}\) and \(s^{(k',k)}\) simultaneously. The received signal at user \(k\) is given by
\[
y_k^{(k,k')} = g^{(k,k')} T \sum_{i = 1}^{K} W^{(kk')} P^{(kk')} \left( h^{(k,k')} s^{(k,k')} + h^{(k',k)} s^{(k',k)} + Z_R \right) + \beta Z_k^T.
\] (11)

We note that \(g^{(k,k')}\), \(W^{(k,k')}\), \(P^{(kk')}\), and \(h^{(k',k)}\) are independent of each other. Therefore, the equivalent user-to-user channel coefficient \(g^{(k,k')} T W^{(k,k')} P^{(kk')} h^{(k',k)}\) is non-zero with probability one, provided that \(W^{(k,k')}\) and \(P^{(kk')}\) are of at least rank one. Then, each user \(k\) receives one linear combination of the two signals in pair \((k,k')\). By subtracting the self-interference, each user can decode the desired messages from the other two users, which achieves a per-user DoF \(d_{\text{user}} = 2\) for each user, or equivalently, a total DoF of \(d_{\text{sum}} = 6\) can be achieved. From (11), we see that the symmetry exists between the design of the uplink and the design of the downlink. Given the design of the beamformer \(u^{(k,k')}\) and the projection matrix \(P^{(kk')}\), the receive vectors \(v^{(k,k')}\) and the projection matrix \(W^{(k,k')}\) in the downlink can be designed similarly, since \(g^{(k,k')} T W^{(k,k')} P^{(kk')}\) can be simply regarded as the transpose of \(P^{(kk')} \cdot h^{(k',k)}\). Therefore, it suffices to focus on design of the uplink in what follows.

We now describe four patterns involved (with a different \(d_{\text{relay}}\)) in achieving the DoF capacity of the MIMO mRC with \(K = 3\). Denote \(U \triangleq \{h^{(k,k')}| k \in I_K, k' \in I_K, k' \neq k\} \). Let \(U \setminus \{h^{(k,k')}, h^{(k',k)}\}\) be the vector set obtained by excluding \(h^{(k,k')}\) and \(h^{(k',k)}\) from \(U\).

1) Pattern 1.1: \(U\) spans a subspace with dimension 6 (dim-6).

2) Pattern 1.2: \(U\) spans a subspace with dim-5; for any pair \((k,k')\), \(U \setminus \{h^{(k,k')}, h^{(k',k)}\}\) spans a subspace of dim-4.

3) Pattern 1.3: \(U\) spans a subspace with dim-4; the intersection of \(span(h^{(1,2)}), span(h^{(2,3)}), span(h^{(3,2)})\), and \(span(h^{(1,3)}, h^{(3,1)})\) is of dim-1, i.e., these three planes go through a common line, such that \(U\) spans a subspace of dim-4.

4) Pattern 1.4: \(U\) spans a subspace with dim-3; for any pair \((k,k')\), \(h^{(k,k')}\) and \(h^{(k',k)}\) spans a subspace of dim-1.

The above four patterns are geometrically illustrated in Fig. 2. It can be verified that the projection matrices \(P^{(kk')}\),
∀k, k′, k′ ≠ k, for Patterns 1.1-1.4 are of at least rank one for sure. For example, P(k,k′) for Pattern 1.1 is of at least rank two for sure. Hence the proposed signaling scheme achieves a total DoF of 6. However, a different pattern spans a subspace with a different number of dimensions, which yields a different d_{relay} as shown in Table I. In general, a pattern with a greater d_{relay} is more efficient in utilizing the relay’s signal space, and hence is more desirable in the signal alignment design. The requirement on \( \frac{M}{N} \) to realize each specific pattern is given in the last column of Table I. Note that these requirements will be discussed in detail in Subsection III-C. It is also worth noting that Pattern 1.2 and Pattern 1.3 have the same requirement on \( \frac{M}{N} \), but Pattern 1.3 achieves a higher d_{relay} than Pattern 1.2. Thus, Pattern 1.2 is ruled out by Pattern 1.3 in the proposed signal alignment scheme.

### B. Main Result

We now consider the general case that each user transmits multiple spatial data streams over a MIMO mRC, i.e., multiple units co-exist in the relay’s signal space with each unit consisting of \( K(K-1) \) spatial streams. Our goal is to construct units with the most efficient patterns to occupy the relay’s signal space. The main result is summarized in the following theorem.

**Theorem 1.** For the MIMO mRC \((M, N, K)\) with \( K = 3 \), the optimal DoF per user is given by

\[
d_{\text{user}} = \begin{cases} 
M, & \frac{M}{N} < \frac{2}{3} \\
2N, & \frac{M}{N} \geq \frac{2}{3} 
\end{cases}
\]  

(12)

The optimal per-user DoF with respect to \( \frac{M}{N} \) is shown in Fig. 3. We see that the per-user DoF \( d_{\text{user}} = M \) is achieved for \( \frac{M}{N} < \frac{2}{3} \), which means that the DoF is bounded by the number of antennas at the user ends. On the other hand, when \( \frac{M}{N} \geq \frac{2}{3} \), the DoF is bounded by the number of relay antennas. Note that the optimal per-user DoF obtained in [14] is only for the range of \( \frac{M}{N} \geq \frac{2}{3} \). In this work, we show that the proposed signal alignment scheme achieves the DoF capacity for all the \((M, N)\) setups.

### C. Proof of Theorem [1]

We first note that \( d_{\text{user}} \) in (12) coincides with the DoF outer bound in (7) with \( K = 3 \). Thus, to prove Theorem [1] it suffices to show the achievability of (12). We start with a brief description of the overall transceiver design. We need to jointly design the transmit beamformers \( \{u^{(k,k')}\} \), the receive vectors \( \{v_i^{(k,k')}\} \), and the relay projection matrices \( \{P_i^{(k,k')}\} \) and \( \{W_i^{(k,k')}\} \) to efficiently utilize the relay’s signal space. As different from (10), here the relay’s projection matrices \( P_i^{(k,k')} \) and \( W_i^{(k,k')} \) null the interference not only from the other pairs in unit \( l \) but also from the other units. Taking \( P_i^{(k,k')} \) as an example, it projects a vector into the null space of span\( \{H_i^{(k,k')}, H_i^{(k',k')}\} \). Hence the relay beamforming matrix \( F \) given in (4) is expressed as

\[
F = \alpha \sum_{l=1}^{L} \sum_{k=1}^{K} \sum_{k'=k+1}^{K} W_i^{(k,k')} P_i^{(k,k')},
\]

(13)
where $L$ denotes the total number of units. Based on that, in each unit, each user can achieve a DoF of two, provided that the projection matrices $P_{l}^{(kk)}$ and $W_{l}^{(kk)}$ are at least of rank one.

1) Case of $\frac{M}{N} \leq \frac{1}{3}$: In this case, the number of relay antennas $N$ is no less than the total number of antennas of all the users, i.e., $\text{span}(\{H_{l}\})$ is of dim-$3M$ with probability one. This implies that the relay’s signal space has enough dimensions to support full multiplexing at the user end, i.e., each user transmits $M$ spatial streams. $M$ units with Pattern 1.1 can be constructed. As the directions of signals with Pattern 1.1 are randomly drawn from the relay’s signal space, the independence of different units can be guaranteed with probability one. Clearly, the projection matrix $P_{l}^{(kk)}$ is of at least rank one. Thus, each unit achieves a DoF of 6. Considering all the $M$ units, we obtain that the achievable per-user DoF is $M$.

2) Case of $\frac{1}{3} < \frac{M}{N} \leq \frac{1}{2}$: From Table 1, this case corresponds to Patterns 1.2 and 1.3. As Pattern 1.3 is more efficient than Pattern 1.2 (i.e., the former achieves a higher $d_{\text{relay}}$ than the latter), we focus on the construction of units following Pattern 1.3. Denote the intersection of $\text{span}(H_{1}, H_{2})$ and $\text{span}(H_{3})$ by $S^{(1,3)}$. The dimension of $S^{(1,3)}$ is $3M - N > 0$. We choose two vectors $u_{l,3}$ and $u_{l,3}^{(3,1)}$ such that $H_{3}u_{l,3}$ and $H_{3}u_{l,3}^{(3,1)}$ are two linearly independent vectors in $S^{(1,3)}$. By definition, both $H_{3}u_{l,3}$ and $H_{3}u_{l,3}^{(3,1)}$ belong to $\text{span}(H_{1}, H_{2}) = \text{span}(H_{1}) \oplus \text{span}(H_{2})$. Thus, there uniquely exist $\{u_{l,1}^{(3)}, u_{l}^{(3,1)}, u_{l,1}^{(1,3)}\}$ and $\{u_{l,1}, u_{l,1}^{(2,1)}\}$ satisfying

\begin{align}
H_{1}u_{l}^{(3,1)} + H_{2}u_{l}^{(2,3)} + H_{3}u_{l,3} = 0 \quad (14a) \\
H_{1}u_{l,1} + H_{2}u_{l}^{(2,1)} + H_{3}u_{l,3}^{(3,1)} = 0. \quad (14b)
\end{align}

Let $u_{l,1}^{(3,2)} = u_{l,3} - u_{l,3}^{(3,1)}$ and $u_{l,1}^{(1,2)} = u_{l,1} - u_{l,1}^{(1,3)}$. Together with (14), we obtain

\begin{align}
H_{1}u_{l}^{(3,1)} + H_{2}u_{l}^{(2,3)} + H_{3}(u_{l}^{(3,2)} + u_{l,1}^{(3,1)}) = 0 \quad (15a) \\
H_{1}(u_{l,1}^{(1,2)} + u_{l,1}^{(1,3)}) + H_{2}u_{l}^{(2,1)} + H_{3}u_{l,3}^{(3,1)} = 0. \quad (15b)
\end{align}

Subtracting (15a) by (15b), we obtain

\begin{align}
H_{1}u_{l,1}^{(1,2)} + H_{2}(u_{l}^{(2,1)} - u_{l}^{(2,3)}) - H_{3}u_{l,3}^{(3,2)} = 0. \quad (15c)
\end{align}

We now show that there are three signal direction pairs $\{H_{1}u_{l,1}^{(1,2)}, H_{2}u_{l}^{(2,1)}\}$, $\{H_{2}u_{l}^{(2,3)}, H_{3}u_{l,3}^{(3,2)}\}$, $\{H_{3}u_{l,3}, H_{3}u_{l,3}^{(3,1)}\}$ forms a unit with Pattern 1.3. From (15), $\{H_{l}u_{l}^{(kk)}, \forall k, k' \neq k\}$ span a subspace of dim-4. Further, from (15a), two signal pairs (1, 3) and (2, 3) span a subspace of dim-3. Thus, the dimension of $\text{null}(\{H_{1}u_{l,1}^{(1,2)}, H_{3}u_{l,3}^{(3,1)}, H_{2}u_{l}^{(2,3)}, H_{3}u_{l,3}^{(3,2)}\}) \cap \text{span}(\{H_{l}u_{l}^{(kk)}, \forall k, k' \neq k\})$ is of dim-1. Similarly, from (15b) and (15c), the intersection of nullspace of any two of the three pairs in unit $l$ and the subspace spanned by signals in unit $l$ is of dim-1. Therefore, a linear combination for each signal pair can be decoded at the relay without interference.

We now describe how to construct multiple linearly independent units following Pattern 1.3. Let the columns of $U_{H} \in \mathbb{C}^{3M \times (3M - N)}$ give a basis of $\text{null}(\{H_{1}, H_{2}, H_{3}\})$.

Partition $U_{H}$ as $U_{H} = [U_{H}^{(1,3)}, U_{H}^{(2,3)} U_{H}^{(2,3)}]^{T}$ with $U_{H}^{(1,3)} \in \mathbb{C}^{M \times (3M - N)}$. Then, $u_{l}^{(1,3)}, u_{l}^{(2,3)},$ and $u_{l,1}^{(3,1)}$ in (14a), are respectively chosen as the $(2l - 1)$-th column of $U_{H}^{(1,3)}, U_{H}^{(2,3)}$, and $U_{H}^{(2,3)}$. Further, $u_{l,1}^{(1,2)}, u_{l,1}^{(2,1)},$ and $u_{l,1}^{(3,1)}$ in (14b) are respectively chosen as the $(2l)$-th column of $U_{H}^{(1,3)}, U_{H}^{(2,3)}$, and $U_{H}^{(2,3)}$. From (14), we see that $\text{span}(H_{1}u_{l,1}^{(1,3)}, H_{1}u_{l}^{(2,3)}, H_{3}u_{l,3}, H_{1}u_{l,1}, H_{2}u_{l}^{(2,1)}), H_{3}u_{l,3}^{(3,1)})$ is of dim-4, and

\begin{align}
\text{span}(H_{1}u_{l,1}^{(1,3)}, H_{1}u_{l}^{(2,3)}, H_{3}u_{l,3}, H_{1}u_{l,1}, H_{2}u_{l}^{(2,1)}, H_{3}u_{l,3}^{(3,1)}) \\
= \text{span}(H_{1}u_{l,1}^{(1,3)}, H_{1}u_{l}^{(1,2)}, H_{2}u_{l}^{(1,2)}, H_{2}u_{l}^{(2,3)}, H_{3}u_{l,3}^{(3,1)}, H_{3}u_{l,3}^{(3,2)}),
\end{align}

by noting $u_{l,3}^{(3,2)} = u_{l,1} - u_{l,3}^{(3,1)}$ and $u_{l,1}^{(1,2)} = u_{l,1}, u_{l,1}^{(1)} = u_{l,1}^{(1,3)}$. Thus each unit $l$ spans a subspace of dim-4.

Recall that the directions of signals with Pattern 1.1 are randomly drawn from the relay’s signal space, the independence of the units with Pattern 1.3 and the units with Pattern 1.1 can be guaranteed with probability one. If the overall relay’s signal space is occupied by Pattern 1.3, the maximum per-user DoF is $\frac{2M - 2}{2} = M$. Therefore, the achievable per-user DoF is given by

\begin{align}
d_{\text{user}} = 3M - N + \frac{2(N - 2(3M - N))}{6} = M. \quad (17)
\end{align}

3) Case of $\frac{M}{N} > \frac{1}{2}$: In this case, $\frac{M}{N}$ is large enough to construct units with Pattern 1.4. The intersection of $\text{span}(H_{k})$ and $\text{span}(H_{k'})$ is of $2M - N > 0$. Let $H_{k}u_{l}^{(k,k')}$ be the intersection of $\text{span}(H_{k})$ and $\text{span}(H_{k'})$. There exists $\{u_{l}^{(k,k')}, u_{l}^{(k',k)}\}$ satisfying

\begin{align}
H_{k}u_{l}^{(k,k')} = H_{k'}u_{l}^{(k',k)}, \quad \forall k, k' \neq k',
\end{align}

which implies that the two signals of pair $(k, k')$ in a unit span a subspace of dim-1. In this way, we can construct $2M - N$ units with Pattern 1.4, in total spanning a subspace of dim-$3(2M - N)$. According to Lemma 4 in Appendix A, we obtain that $\{u_{l}^{(k,k')}, \forall k, k' \neq k\}$ are linear independent with probability one. Further, due to the randomness of $H_{l}$, the independence of $\{u_{l}^{(k,k')}, \forall l, k, k', k' \neq k\}$ are guaranteed with probability one. Again, the remaining $N - 3(2M - N)$ dimensions can be used

\footnote{Here we assume that $\frac{3M - N}{2}$ is an integer. Otherwise, we use symbol extension to ensure that the dimension of the above intersection is dividable by two; see Appendix B for details. Note that the symbol extension is used to achieved a fractional DoF throughout of the rest of this paper without further explicit notification.}
for constructing $\frac{N - 3(2M - N)}{3}$ units with Pattern 1.3. Thus an achievable per-user DoF is given by

$$d_{\text{user}} = 2(2M - N) + \frac{2(N - 3(2M - N))}{4} = M.$$ 

The independence of the units with Pattern 1.4 and the units with Pattern 1.3 can be proven similar to Lemma 4 in Appendix A. When the overall relay’s signal space is occupied by the units with Pattern 1.4, we have the maximum per user DoF of $\frac{2N}{3} \times 2 = \frac{4N}{3}$. Therefore, the maximum per-user DoF is given by $\min(M, \frac{4N}{3})$, or equivalently

$$d_{\text{user}} = \begin{cases} M, & \frac{1}{2} \leq \frac{M}{N} \leq \frac{2}{3} \\ 2N, & \frac{2}{3} > \frac{M}{N}. \end{cases}$$

This completes the proof of Theorem 1.

IV. MIMO mRC with $K > 3$

In this section, we generalize Theorem 1 to the case of an arbitrary number of users. We start with the case of $K = 4$.

A. Preliminaries

Again, we start with some intuitions of the signal alignment by assuming that each user transmits one independent spatial stream to the each of the other users in a unit. The relay’s precoder is still given by (10).

The following patterns are involved in deriving the achievable DoF to be presented later. Also we omit the unit index $l$ for brevity in this subsection. Denote $U \triangleq \{h^{(k,k')}|k \in \mathcal{I}_K, k' \in \mathcal{I}_K; k \neq k'\}$ with $\mathcal{I}_K = \{1,2,3,4\}$, and $U\setminus\{h^{(k,k')}, h^{(k',k)}\}$ is the vector set obtained by excluding $h^{(k,k')}$ and $h^{(k',k)}$ from $U$. Let $U_{i} \triangleq \{h^{(k,k')}|k \in \mathcal{I}_K, k' \in \mathcal{I}_K; k \neq k'\}$ and $U\setminus\{h^{(k,k')}, h^{(k',k)}\}$ be the vector set obtained by excluding $h^{(k,k')}$ and $h^{(k',k)}$ from $U$.

1. Pattern 2.1: $U_{i}$ spans a subspace with dim-12.
2. Pattern 2.2: $U_{i}$ spans a subspace with dim-9; for any pair $(k, k')$, $U\setminus\{h^{(k,k')}, h^{(k',k)}\}$ spans a subspace with dim-8.
3. Pattern 2.3: For each $i$, $U_{i}$ spans a subspace of dim-4 following Pattern 1.3.
4. Pattern 2.4: $U_{i}$ spans a subspace with dim-6; for any pair $(k, k')$, $h^{(k,k')}$ and $h^{(k',k)}$ spans a subspace with dim-1.

It can be readily shown that the projection matrix $P^{(k,k')}$ corresponding to Patterns 2.1 to 2.4 are of at least rank one with probability one. Thus, Patterns 2.1, 2.2, and 2.4 achieve a total DoF of 12, while Pattern 2.3 achieves a total DoF of 24.

Corresponding antenna requirement for each pattern is given in Table III which will be discussed in details in the Subsection IV.C. It is worth mentioning that for a same requirement on $\frac{M}{N}$, some other patterns may possibly be constructed. However, they are ruled out due to a relatively low $d_{\text{relay}}$, i.e., less efficiency in utilizing the relay’s signal space. Again, the downlink patterns are omitted due to the uplink/downlink symmetry.

| Pattern | Dimension | $d_{\text{sum}}$ | $d_{\text{relay}}$ | Requirement |
|---------|-----------|------------------|-------------------|-------------|
| 2.1     | 12        | 12               | 1                 | $\frac{M}{N} > 0$ |
| 2.2     | 9         | 12               | 4                 | $\frac{M}{N} > \frac{1}{4}$ |
| 2.3     | 4         | 6                | 3                 | $\frac{M}{N} > \frac{1}{3}$ |
| 2.4     | 6         | 12               | 2                 | $\frac{M}{N} > \frac{1}{2}$ |

![Achievable per-user DoF vs. ratio of M/N](image)

Fig. 4. An achievable per-user DoF for the MIMO mRC with $K = 4$ at different $(M, N)$ configurations.

B. Main Result

**Proposition 1.** For the MIMO mRC $(M, N, K)$ with $K = 4$, the per-user DoF capacity of $d_{\text{user}} = M$ is achieved when $0 < \frac{M}{N} \leq \frac{3}{8}$, and the per-user DoF capacity of $d_{\text{user}} = \frac{N}{7}$ is achieved when $\frac{M}{N} \geq \frac{7}{12}$. For $\frac{M}{N} \in (\frac{3}{8}, \frac{7}{12})$, an achievable per-user DoF is given by

$$d_{\text{user}} = \begin{cases} 3M, & \frac{3}{8} < \frac{M}{N} \leq \frac{1}{2} \\ 3M - \frac{3N}{8}, & \frac{1}{2} < \frac{M}{N} \leq \frac{7}{12}. \end{cases}$$

The achievable per-user DoF for MIMO mRC with $K = 4$ is illustrated in Fig. 4. We observe that, different from the case of $K = 3$, the DoF bound given in Subsection II-D can only be achieved in the ranges of $\frac{4N}{3} \in (0, \frac{3}{8})$ and $\frac{2N}{3} \in (\frac{7}{12}, \infty)$; for $\frac{M}{N} \in (\frac{3}{8}, \frac{7}{12})$, there is a certain DoF gap between the achievable DoF and the capacity outer bound. The achievable DoF of the considered MIMO mRC with $K = 4$ has also be considered in [14], where the authors derived that for an antenna setup $(M, N) = (4, 7)$, a total DoF of 12 can be achieved. Here we see that our proposed scheme achieves a total DoF of $\frac{27}{2}$, which is higher than the DoF of 12 obtained in [14].
C. Proof of Proposition 1

To prove Proposition 1, we consider four cases detailed below.

1) Case of $\frac{M}{N} \leq \frac{1}{2}$: In this case, since $N \geq 4M$, the relay’s signal space has enough dimensions to support full multiplexing at the users, which implies that each user can transmit $M$ independent spatial streams, or equivalently, $M$ units with Pattern 2.1 can be constructed. Therefore, an achievable per-user DoF of $M$ can be achieved.

2) Case of $\frac{2}{3} < \frac{M}{N} \leq \frac{1}{2}$: As shown in Table I, this case corresponds to Pattern 2.2. As $4M - N > 0$, the nullspace of span($H_1, H_2, H_3, H_4$) is of dim-$4M - N$. Let the columns of $U_H \in \mathbb{C}^{N \times (4M - N)}$ give a basis of null($H_1, H_2, H_3, H_4$). Partition $U_H$ as $U_H = [U_H^T, U_H^T, U_H^T, U_H^T]^T$. From Lemma 3, span($H_1, U_H, H_2, U_H, H_3, U_H, H_4, U_H$) is of $3(4M - N)$ for sure. Arbitrarily choose three columns of $U_H$, denoted by $[(u_1)^T, (u_2^{(1)})^T, (u_3^{(1)})^T, (u_4^{(1)})^T]^T$, $[(u_1^{(2)})^T, (u_2^{(l)})^T, (u_3^{(1)})^T, (u_4^{(2)})^T]^T$, and $[(u_1^{(3)})^T, (u_2^{(3)})^T, (u_3^{(3)})^T, (u_4^{(3)})^T]^T$, we have

$$H_1u_1 + H_2u_2^{(1)} + H_3u_3^{(1)} + H_4u_4^{(1)} = 0 \quad (18a)$$
$$H_1u_1^{(2)} + H_2u_2^{(l)} + H_3u_3^{(2)} + H_4u_4^{(2)} = 0 \quad (18b)$$
$$H_1u_1^{(3)} + H_2u_2^{(3)} + H_3u_3^{(3)} + H_4u_4^{(3)} = 0. \quad (18c)$$

Let $u_1^{(1)} = u_1 - u_2^{(1)} - u_3^{(1)}, u_2^{(2)} = u_2^{(1)} + u_3^{(2)}$, and $u_3^{(3)} = u_3^{(1)} + u_4^{(2)}$. Subtracting $18a$ by $18b$ and $18c$, we have

$$H_1u_1^{(4)} + H_2u_2^{(4)} + H_3u_3^{(3)} + H_4u_4^{(3)} = 0. \quad (19)$$

We now show that each unit spans a subspace of dim-12 corresponding to Pattern 2.2. Eqs. $18a$ and $19$ imply that one dimension is saved for the subspace spanned by the signals related to one user. The signal alignment given in $18a$ and $19$ is similar to the one used by Pattern 1.3 shown in $15$. Then, the set $\{H_ku_{(k,k')}^{(k')}| \forall k, k' \neq k\}$ span a subspace of dim-9, while $\{H_ku_{(k,k')}^{(k)}| \forall k, k' \neq k\}$ span a subspace of dim-8. Hence we always have dim-1 nullspace to decode the linear combination of each pair at the relay. In this way, we can construct $4M - N$ units with Pattern 2.2, occupying $3(4M - N)$ dimensions of the relay’s signal space. The remaining $N - 3(4M - N)$ dimensions of relay’s signal space is used to construct $N - 3(4M - N)$ units with Pattern 2.1. Thus, an achievable per-user DoF is given by

$$\frac{4M - N}{3} \times \frac{12}{4} + \frac{3(N - 3(4M - N))}{12} = M. \quad (20)$$

If the overall relay’s signal space is occupied with Pattern 2.3, the maximum achievable per-user DoF of $\frac{2}{3} < \frac{M}{N} \leq \frac{1}{2}$ is achieved. Therefore, the achievable per-user DoF is given by

$$\min(M, \frac{N^3}{N^2 - M}) = M$$

3) Case of $\frac{1}{3} < \frac{M}{N} \leq \frac{1}{2}$: In this case, $\frac{M}{N}$ is large enough to construct the units with Pattern 2.3 shown in Table I. We form the following four three-user groups: $\{H_1, H_2, H_3\}$, $\{H_1, H_2, H_4\}$, $\{H_2, H_3, H_4\}$, and $\{H_1, H_3, H_4\}$, and align the signals within each three-user group. For each group, the signal alignment is conducted as Pattern 1.3 for $K = 3$, where 6 spatial streams occupy a relay’s signal subspace of dim-4. For each group, similarly to Pattern 1.3, $\frac{3M - N}{2}$ units with Pattern 2.3 are constructed and span a subspace of $4 \times \frac{3M - N}{2} = 2(3M - N)$ dimensions. Considering the units from four groups, we have $2(3M - N)$ units which span a relay’s subspace of dim-$8(3M - N)$. The independence of units can be proven by using the result given in Lemma 3 in Appendix A. The remaining $N - 8(3M - N)$ dimension of relay’s signal space are used to construct the units with Pattern 2.2, the achievable per-user DoF can be expressed as

$$\frac{3M - N}{2} \times 3 \times 2 + \frac{3(N - 8(3M - N))}{9} = M.$$

If the overall relay’s signal space is occupied by Pattern 2.3, we achieve the maximum per-user DoF with $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. Hence, the achievable per-user DoF can be denoted by $\min(M, \frac{4}{9})$.

4) Case of $\frac{M}{N} > \frac{1}{2}$: In this case, as $4M - N > 0$, the signals in each pair can be aligned together to occupy a subspace of dim-1. In total, $2M - N$ units with Pattern 2.4 can be constructed, which span a subspace of dim-2($2M - N$). Similarly, the remaining $N - 2M - N$ dimensions of relay’s signal space are used to construct the units with Pattern 2.3. The achievable per-user DoF can be expressed as

$$3(2M - N) + \frac{3(N - 6(2M - N))}{8} = 3M - \frac{3N}{8}.$$ 

While the whole relay’s signal space is occupied by the units with Pattern 2.4, we have the maximum per-user DoF of $\frac{2}{3} \times 3 = \frac{2}{3}$. The achievable per-user DoF can be given by $\min(3M - \frac{3}{2}, \frac{N}{3})$, which is equivalent to

$$d_{user} = \left\{ \begin{array}{ll}
\frac{3M - 3N}{2} & \frac{1}{2} < M \leq \frac{12}{7} \\
\frac{N}{2} & M > \frac{12}{7}.
\end{array} \right.$$

This completes the proof of Proposition 1.

D. Achievable DoF for a General $K$

We now generalize the achievable DoF result to an arbitrary $K$. Denote

$$\alpha_t = \left(\frac{K - 1}{t - 1}\right)(t - 1) \quad and \quad \beta_t = \left(\frac{K}{t - 1}\right)(t - 1)^2. \quad (20)$$

PROPOSITION 2. For the MIMO mRC with $(M, N, K)$, the per-user DoF capacity of $d_{user} = M$ is achieved when $\frac{M}{N} \in \left(0, \frac{K - 1}{K(K - 2)}\right]$ and the per-user DoF capacity of $d_{user} = \frac{2N}{K}$ is achieved when $\frac{M}{N} \in \left[\frac{1}{K - 1} + \frac{1}{2}, \infty\right)$. Further, for the remaining range of $\frac{M}{N}$, an achievable per-user DoF is given by

$$d_{user} = \min\left(\frac{\alpha_t(M - N)}{t - 1} + \frac{\alpha_{t+1}(N - \frac{\beta_t(M - N)}{t - 1})}{\beta_{t+1}}, \frac{M}{N}\right). \quad (21)$$
where $t = 2, 3, \cdots, K - 1$.  

Proof: For the case of $\frac{M}{N} < \frac{1}{K}$, it is easy to obtain that $d_{\text{user}} = M$, which is also the DoF capacity. On the other hand, for $\frac{M}{N} \in (1, \infty)$, the achievable DoF capacity is equal to the one achieved at $\frac{M}{N} = 1$ as the number of relay antennas $N$ is the bottleneck. Thus, we only focus on the range of $\frac{M}{N} \in \left(\frac{1}{K}, 1\right]$ in the following proof. Moreover, we partition the range of $\left(\frac{1}{K}, 1\right]$ into intervals of $\left(\frac{1}{K}, \frac{1}{K-1}\right)$ with $t = 2, 3, \cdots, K$, and discuss the signal alignment design on $\frac{M}{N} \in \left(\frac{1}{K}, \frac{1}{K-1}\right]$ with an arbitrary $t$.

Note that, $\frac{M}{N} \in \left(\frac{1}{K}, \frac{1}{K-1}\right]$ implies $N < tM$, which implies that only the spatial streams from $t$ users can be aligned together. Based on that, the most efficient way is to split all the users into different $t$-user groups and perform signal alignment in each group. In this way, we have $\binom{K}{t}$ number of different $t$-user groups. Further, each user is included in $\binom{K-1}{t-1}$ number of different $t$-user groups. Denote the channel matrices for an arbitrarily chosen $t$-user group as $\{H_{1,i}, H_{2,i}, \cdots, H_{t,i}\}$, the nullspace of span $\{H_{1,i}, H_{2,i}, \cdots, H_{t,i}\}$ is of dim $(tM - N)$ with probability one. Similarly to Pattern 2.3, the efficient way to align signals in one unit is to let the spatial streams related to the relay's signal space is used to construct the units with Pattern 3.3. The beamformers in unit $l$ can be designed to satisfy the conditions given in (22a)-(22c) shown at the top of next page, where

$$d_{l(i,j)} = \begin{cases} 1, & i = 1 \text{ or } j = 1 \\ -1, & \text{otherwise.} \end{cases}$$

Subtracting (22a) by equations from (22b) to (22c), we obtain the equation given in (22d) shown at the top of next page. Based on (22), we see that total $t - 1$ dimensions can be saved for the subspace spanned by the signals in unit $l$. In total, we can construct $\frac{tM-N}{t-1}$ units with Pattern 3.t in each group and the achievable per-user DoF in one unit is $t - 1$. Thus, the achievable per-user DoF is $\frac{\alpha_{t}(tM-N)}{t-1}$. Note that the linear independence of the subspaces spanned by the units with Pattern 3.t can be found in Lemma 3 in Appendix A. Moreover, for each $t$-user group, one unit spans a subspace of $t(t-1) = (t-1)^2$ dimensions. Considering all $\binom{K}{t}$ number of groups, all the units span a subspace of $\frac{\beta_{t}(tM-N)}{t-1}$ dimensions. The remaining $N - \frac{\beta_{t}(tM-N)}{t-1}$ dimensions of relay's signal space is used to construct the units with Pattern 3.($t+1$). Then, the achievable per-user DoF can be expressed as

$$d_{\text{user}} = \frac{\alpha_{t}(tM-N)}{t-1} + \frac{N - \frac{\beta_{t}(tM-N)}{t-1}}{\beta_{t+1}}.$$  \hspace{1cm} (23)

When the relay's signal space is wholly occupied by Pattern 3.3, we achieve a maximum per-user DoF of $\frac{\alpha_{K}N}{\beta_{t}}$. Thus, an achievable per-user DoF is obtained as (22).

When $t = K$, we have only one group, which leads to $\alpha_{K} = K - 1$ and $\beta_{K} = (K - 1)^2$. Note that for $\frac{M}{N} \in \left(\frac{1}{K}, \frac{1}{K-1}\right)$, we cannot do any signal alignment, $\alpha_{K+1}$ and $\beta_{K+1}$ defined in (23) should be equal to $K - 1$ and $K(K - 1)$, respectively. Substitute them into (21), we have $d_{\text{user}} = \min(M, \frac{N}{K-1}) = M$, which is the DoF capacity.

When $t = K - 1$, we have $\alpha_{K-1} = K - 1$ and $\beta_{K-1} = (K - 1)^2$. Substitute $\alpha_{K}, \beta_{K}, \alpha_{K-1}$, and $\beta_{K-1}$ into (21), we obtain $d_{\text{user}} = \min(M, \frac{K-1}{K}) = M$, which implies that the per-user DoF capacity of $d_{\text{user}} = M$ can be achieved for $\frac{M}{N} \in \left(\frac{1}{K-1}, \frac{1}{K-2}\right)$. Similarly, we can also verify that the per-user DoF capacity of $\frac{N}{2}$ can be achieved in $\frac{M}{N} \in \left[\frac{1}{K-1} + \frac{1}{2}, 1\right]$ by letting $t = 2$. Then, we complete the proof Proposition 2.

When the number of users $K$ tends to infinity, the following asymptotic DoF can be obtained from Proposition 2.

Corollary 1. For the MIMO mRC $(M, N, K)$ with $K \rightarrow \infty$, the total DoF of $d_{\text{sum}} = 2N$ is achieved when $\frac{M}{N} > \frac{2}{K}$ and $d_{\text{sum}} = N$ is achieved as $\frac{M}{N} \rightarrow 0$. For $\frac{M}{N} \in (0, \frac{2}{K})$, the achievable total DoF has a ladder-like shape with respect to $\frac{M}{N}$ which contains discontinuities at the points of $\frac{M}{N} = \frac{1}{t}, t = 2, 3, 4, \cdots$. Specifically, when $\frac{M}{N} \in \left(\frac{1}{t}, \frac{1}{t-1}\right)$, a total DoF of $\frac{tN}{t-1}$ is achieved.

Remark 4.1: When $K \rightarrow \infty$, the number of spatial streams $K(K-1)$ also tends infinity. Based on that, the total achievable DoF is always bounded by the number of relay antennas $N$, which further implies that only a portion of the users can realize data exchange. The overall achievable total DoF with respect to $\frac{M}{N}$ is illustrated in Fig. 5. From Corollary 1, we see that for each antenna setup $\frac{M}{N} = \frac{1}{t}$, the total achievable DoF jumps from $\frac{tN}{t-1}$ to $\frac{t}{1}$. It is interesting to verify that the discontinuous points $(\frac{1}{t}, \frac{1}{t+1})$ are went through by the line $y = N + M$, while the discontinuous points $(\frac{1}{t}, \frac{1}{t-1})$ are enveloped by the curve $y = \frac{N^2}{N - M}$. In this case, the achievable total DoF is nicely bounded by these two curves as shown in Fig. 5.  

1When $K \rightarrow \infty$, $d_{\text{user}}$ tends to 0. Thus, we consider the total achievable DoF $d_{\text{sum}}$ here.
Therefore, without loss of generality, we refer to \( y = \frac{N^2}{N + M} \) as an upper bound of the total achievable DoF and refer to \( y = N + M \) as a lower bound of the total achievable DoF.

V. IMPROVED DOF USING RELAY ANTENNA DEACTIVATION

In previous sections, we show that the proposed beamforming design achieves the DoF capacity for MIMO mRC with \( K = 3 \). However, certain gap occurs in the range of \( \left( \frac{N^2}{N + M}, \frac{N}{K(1-K)} \right] \) when \( K > 3 \). In this section, we show that the achievable per-user DoF given in Proposition 2 can be enhanced by disabling a portion of relay antennas disabled in the uplink and downlink transmissions.

**LEMMA 1.** For a general MIMO mRC \((M, N, K)\), assume that a certain DoF of \( d_{\text{user}} = d_0 \) is achievable at \( \frac{M}{N} = a_0 \). Then every point of \( \left( \frac{M}{N}, d_{\text{user}} \right) \) on the line segment of \( y = \frac{d_0}{a_0} x \) for \( x \in (0, a_0] \) is achievable by disabling \( N - \frac{M}{a_0} \) relay antennas.

**Proof:** In Proposition 2 any achievable DoF \( d_{\text{user}} = d_0 \) at \( \frac{M}{N} = a_0 \) can be expressed as \( d_0 = \beta N \). Then for any antenna setup \((M, N)\) with \( \frac{M}{N} < a_0 \), we can always apply the signal alignment scheme at \( \frac{M}{N} = a_0 \) by disabling certain relay antennas to obtain a new achievable DoF. Assume that the number of relay antennas after antenna deactivation is \( N = \alpha N \) with \( 0 < \alpha \leq 1 \). For any \( \frac{M}{N} < a_0 \), by setting \( \frac{M}{N} = M N = a_0 \) with \( \alpha = \frac{M}{a_0 N} \). Using the signal alignment scheme same with the point at \( \frac{M}{N} = a_0 \), a new achievable per-user DoF is given by

\[
d_{\text{user}} = \beta \frac{M}{a_0 N} = \beta \frac{M}{a_0} M.
\]

which is indeed on the line segment \( y = \frac{d_0}{a_0} x \). Thus we complete the proof of Lemma 1.

**Lemma 2** offers an efficient way to further improve the achievable DoF, i.e., for each achievable DoF \( d_i \) obtained in Proposition 2 at \( \frac{M}{N} = a_i \), we get a line segment \( y = \frac{d_i}{a_i} x \) containing new achievable DoF in the range of \( \frac{M}{N} \in (0, a_i] \). By comparing all the line segments \( y = \frac{d_i}{a_i} x \) for \( \forall i \), we finally obtain an improved achievable DoF in the range of \( \frac{M}{N} \in (0, 1) \).

We next give a simpler way to obtain the above improved achievable DoF. Before that, we define

\[
\gamma_{1,1} = \frac{\alpha_1 t M - N}{t - 1} + \frac{\alpha_{t+1} (N - \frac{M}{a_t} (M - N))}{\beta_{t+1}}
\]

\[
\gamma_{1,2} = \frac{\alpha_1 N}{\beta_t}
\]

\[
\theta_t = t - 1 + 1 + t, \ t = 2, 3, \cdots, K - 1.
\]

In (25), \( \gamma_{1,1} \) and \( \gamma_{1,2} \) are two terms given in (21). In addition, at \( \frac{M}{N} = \theta_t \), we have \( \gamma_{1,1} = \gamma_{1,2} \); \( (\theta_1, \gamma_{1,2}) \) can be regarded as a corner point, implying the overall relay’s signal space is occupied by one pattern, i.e., Pattern 3.t. When \( \frac{M}{N} \in \left( \frac{1}{\theta_t}, \theta_t \right] \), we have \( \gamma_{1,1} > \gamma_{1,2} \).

We have the following lemma.

**LEMMA 2.** For \( \frac{M}{N} \in (\theta_{t+1}, \theta_t] \), the achievable DoF obtained in Proposition 2 can be improved in the range of \( \frac{M}{N} \in (\alpha_{t+1} (t - 1 + \beta_{t+1}), \theta_t) \). The highest improved achievable DoF is given by \( \frac{M}{N} = \frac{M \alpha_t}{t - 1 + \beta_t} \), which is on the line segment of \( y = \frac{\gamma_{1,2}}{\theta_t} x \).

**Proof:** To obtain the highest improved achievable DoF in the range of \( \frac{M}{N} \in (\theta_{t+1}, \theta_t] \), we need to consider all the achievable DoF on line segments of \( y = \frac{d_i}{a_i} x \) with \( a_t \in \left( \frac{1}{\theta_t}, 1 \right] \). To prove Lemma 2 we next treat two ranges, \( a_t \in \left( \frac{1}{\theta_t}, \theta_t \right] \) and \( a_t \in (\theta_t, 1] \), separately.

We first deal with the range of \( a_t \in \left( \frac{1}{\theta_t}, \theta_t \right] \). Note that the achievable per-user DoF in the range of \( \frac{M}{N} \in \left( \frac{1}{\theta_t}, \theta_t \right] \) is \( \gamma_{1,1} \). It can be verified that \( \gamma_{1,1} \) has a form of \( \gamma_{1,1} = a M + b N \) with \( a > 0 \) and \( b < 0 \). Thus, the line segment of \( y = \frac{\gamma_{1,1}}{\theta_t} x \) is always higher than other line segments of \( y = \frac{d_i}{a_i} x \) with \( a_t \in \left( \frac{1}{\theta_t}, \theta_t \right] \). We denote the improved achievable DoF on \( y = \frac{\gamma_{1,1}}{\theta_t} x \) by \( d_{\text{user}}^{++} = \frac{M \alpha_t}{t - 1 + \beta_t} \).

Next, we deal with the range of \( a_t \in (\theta_t, 1] \). Note that, in this range, it suffices to compare \( d_{\text{user}}^{++} \)with the achievable per-user DoF on line segment of \( y = \frac{\theta_{t-1}}{\theta_{t-1} - 2} x \), which goes through the corner point of \( (\theta_{t-1}, \gamma_{t-1,2}) \). The achievable DoF on this line segment can be expressed as

\[
d_{\text{user}}^{++} = \frac{M \gamma_{t-1}}{t - 2 + \beta_{t-1}} \frac{N \alpha_t}{\theta_{t-1}} = \frac{M (t - 1) \alpha_{t-1}}{t - 2 + \beta_{t-1}}.
\]
It is easy to verify that in the range of \( \{\theta_{t-1}, \theta_{t}\} \), \( d^\uparrow_{\text{user}} \) is always higher than \( d^\uparrow_{\text{user}} \), as \( y = \frac{\theta_{t-1}}{\theta_{t}} x \) has a decreased slope compared to \( y = \frac{\theta_{t}}{\theta_{t-1}} x \).

Finally, by comparing \( d^\uparrow_{\text{user}} \) with the achievable DoF in Proposition 2, we obtain that in the range of \( M/N \in (\theta_{t-1}, \theta_{t}] \), the original achievable DoF can be improved in the range of \( M/N \in (\frac{\alpha_{t+1}(t-1)+\beta_{t}}{t\alpha_{t}+\beta_{t+1}}, \theta_{t}] \). We thus complete the proof.

Note that Lemma 2 implies that the range of \( \left(\frac{1}{K-1}, 1\right) \) can be re-partitioned into intervals of \( \left(\frac{1}{K-1}, \frac{1}{K-2}\right) \cup (\theta_{K-1}, \theta_{K-2}] \cup \cdots \cup (\theta_{2}, 1] \) and the achievable DoF at each interval of \( (\theta_{t}, \theta_{t-1}] \) with \( t = 3, 4, \ldots, K-1 \) can be improved by disabling a portion of relay antennas. Take \( K = 4 \) as an example, as illustrated in Fig. 6 we see that, in the range of \( (\theta_{3}, \theta_{2}] \), the original achievable DoF can be improved for \( (\frac{3}{7}, \theta_{2}] \) and the improved achievable DoF is on the line \( y = \frac{\theta_{2}}{\theta_{3}} x = \frac{3}{7} x \).

Based on Lemma 1 and Lemma 2, we have the following improved achievable per-user DoF.

**Theorem 2.** For the MIMO mRC \((M, N, K)\) with \( K > 3 \), the per-user DoF of \( d^\uparrow_{\text{user}} = M \) is achieved when \( M/N \in (0, K/(K-2)] \) and the per-user DoF of \( d^\uparrow_{\text{user}} = M/N \) is achieved when \( M/N \in \left(\frac{2N}{K(K-1)}\right) \), \( \frac{1}{K-1} < \frac{2N}{K(K-1)} < \frac{1}{2} \). For \( M/N \in \left(\frac{1}{K-1}, \frac{2N}{K(K-1)}\right) \), an achievable per-user DoF is given by

\[
d^\uparrow_{\text{user}} = \begin{cases} \frac{N\alpha_{t+1}+t\alpha_{t}}{\beta_{t+1}} & M/N \in \left(\frac{t}{t+1}\beta_{t+1} + \frac{1}{t}, \frac{1}{t}\right], \\ \frac{M\alpha_{t}}{t+1} & M/N \in \left(\frac{1}{t}, \frac{1}{t+1}\right], \end{cases}
\]

where \( \alpha_{t}, \beta_{t} \) are defined in (20), \( \tau_{t} = \frac{\alpha_{t}(t+1)+\beta_{t}}{t\alpha_{t}+\beta_{t+1}} \), and \( t \in [2, K-2] \) is an integer.

**Corollary 2.** For the MIMO mRC \((M, N, K)\) with \( K = 4 \), the per-user DoF of \( d^\uparrow_{\text{user}} = M \) is achieved when \( 0 < M/N \leq \frac{3}{8} \) and the per-user DoF capacity of \( d^\uparrow_{\text{user}} = M \) is achieved when \( M/N \geq \frac{7}{12} \). For \( M/N \in \left(\frac{3}{8}, \frac{7}{12}\right) \), an achievable per-user DoF is given by

\[
d^\uparrow_{\text{user}} = \begin{cases} \frac{3N}{8} & M/N \leq \frac{7}{16}, \\ \frac{6M}{7} & \frac{7}{16} < M/N < \frac{7}{12}, \\ \frac{10M}{9} & M/N \geq \frac{7}{12}. \end{cases}
\]

**Corollary 2** is obtained from Theorem 2 by letting \( K = 4 \), with the DoF curve illustrated in Fig. 6. It is interesting to see that the obtained DoF curve is piecewise linear, depending on the number of user antennas \( M \) and the number of relay antennas \( N \) alternately. This result is similar to the DoF capacity of the interference channel obtained in [19]. The piecewise linearity implies antenna redundancy. Specifically, for \( M/N \in \left(0, \frac{3}{8}\right] \cup \left(\frac{7}{16}, \frac{7}{12}\right] \), the DoF is bounded by the number of relay antennas, implying antenna redundancy at the user side; for \( M/N \in \left(\frac{3}{8}, \frac{7}{12}\right] \cup \left(\frac{7}{12}, 1\right] \), the DoF is bounded by the number of user antennas, which implies that the antenna redundancy occurs at the relay.

The achievable DoF of MIMO mRC with \( K = 4 \) has also be considered in [14], where the author derived that for an antenna setup \((M, N) = (4, 7)\), a total DoF of 12 can be achieved. Here we see that our proposed scheme achieves a total DoF of \( \frac{27}{7} \), which is higher than a DoF of 12 DoF obtained in [13].

When the user number \( K \to \infty \), we obtain an asymptotic DoF given in the following corollary.

**Corollary 3.** For the MIMO mRC \((M, N, K)\) with \( K \to \infty \), the total DoF of \( d^\uparrow_{\text{sum}} = 2N \) is achieved when \( M/N \to \frac{1}{2} \) and \( d^\uparrow_{\text{sum}} = N \) is achieved as \( M/N \to 0 \). For \( M/N \in (0, \frac{1}{2}) \), the achievable total DoF is piecewise linear with respect to \( M/N \).

Specifically, for \( t = 2, 3, \ldots, \infty \), we have \( d^\uparrow_{\text{sum}} = \frac{t(N)}{t+1} \) for \( M/N \in \left(\frac{1}{t+1}, \frac{1}{t}\right] \), and \( d^\uparrow_{\text{sum}} = \frac{tM/N}{t+1} \) for \( M/N \in \left(\frac{(t+1)(t-1)}{t}, \frac{1}{t+1}\right] \).

The DoF curve for Corollary 3 is illustrated in Fig. 7. We see that the achievable DoF for \( M/N \in (0, \frac{1}{2}) \) is enveloped by the curve of \( y = \frac{N^2}{M+N} \). Further, we can partition the range of \( M/N \in (0, \frac{3}{8}) \) into an infinite number of intervals, namely, \( \frac{M}{N} \in \left(\frac{t(t+1)}{r(t+1)}, \frac{t+1}{t}\right] \) for \( t = 2, 3, \ldots, \infty \); implying that a new pattern arises for efficient signal alignment. Similarly, the achievable DoF in the ranges of \( M/N \in \left(\frac{(t+2)t}{r(t+1)}, \frac{1}{t+1}\right] \) and \( M/N \in \left(\frac{1}{r(t+1)}, \frac{(t+1)(t-1)}{t}\right] \) indicates antenna redundancy at the user side and at the relay side, respectively.

**VI. Conclusion**

In this paper, we studied an achievable DoF of the MIMO mRC for an arbitrary number of users with any antenna setup. A novel and systematic way of beamforming design was proposed realize different kind of signal space alignments, which were then to used to implement PNC. It was shown that the proposed signal alignment scheme achieves the DoF capacity of the MIMO mRC with \( K = 3 \). For the case of \( K > 3 \), we showed that our proposed signal alignment scheme achieves the DoF capacity of the MIMO mRC for...
The future research of interest include the optimal precoding bound of the DoF capacity of the considered MIMO mRC.

Lemma 4. The subspace of span(A_1) ∩ span(A_2), i.e., the intersection of span(A_1) and span(A_2), has a dimension of (2M − N)^+ with probability one.

Proof: This result has been proven in Lemma 1, e.g., of [14]. We omit the details here for brevity.

Lemma 5. Assume we randomly choose t entries from the set of \{ A_i | i = 1, 2, \ldots, K \} to form a t-entry subset, denoted by W_j = \{ A_i | i \in \{ j_1, j_2, \ldots, j_t \} \}. We have total J = \binom{K}{t} number of different t-entry subsets W_j.

Let \( S_j = \text{span} \left( \{ A_i^q | q = 1, 2, \ldots, t-1 \} \right) \cap \text{span} (A_j) \). Then, \( S_1 \oplus S_2 \oplus \cdots \oplus S_J \) is a subspace with dimension of \( \min \{ J(tM - N)^+, N \} \) with probability one.

Proof: To prove Lemma 5, it suffices to show that the subspaces \( S_i \), for \( i = 1, 2, \ldots, J \), are linearly independent of each other if \( J(tM - N)^+ \leq N \). Note that the proof for the case of \( tM - N \leq 0 \) is trivial, we next only consider \( tM - N > 0 \). For notation convenience, we focus on the case of \( K = 4 \) and \( t = 3 \). The proof can be readily extended to the case of an arbitrary \( K \) and \( t \).

Let \( S_1 = \text{span}(A_1, A_2) \cap \text{span}(A_3) \), \( S_2 = \text{span}(A_2, A_3) \cap \text{span}(A_4) \), \( S_3 = \text{span}(A_3, A_4) \cap \text{span}(A_1) \), and \( S_4 = \text{span}(A_1, A_4) \cap \text{span}(A_2) \). Consider an \( N \times 1 \) vector \( a_{1,j} \in S_i \), for \( i = 1, 2, 3, 4 \) and \( j = 1, 2, \ldots, 3M - N \). Then, taking the intersection subspace \( S_i \) as example, there exist \( q_{1,1}, q_{1,2} \) and \( q_{1,3} \) such that

\[
\begin{bmatrix}
I_N & A_1 & A_2 & 0 \\
0 & 0 & 0 & A_3 \\
\end{bmatrix}
\begin{bmatrix}
a_{1,j} \\
q_{1,j} \\
q_{2,j} \\
q_{3,j} \\
\end{bmatrix}
= 0,
\]

Note that \( S_i \), for \( i = 1, 2, 3, 4 \), are linearly dependent, if and only if there exist \( \{ q_{m,j} \} \) for \( i = 1, 2, 3, 4, m = 1, 2, 3 \) and \( j = 1, 2, \ldots, 3M - N \), such that

\[
\sum_{i=1}^{4} \sum_{j=1}^{3M-N} a_{i,j} = 0.
\]

Appendix A
some useful lemmas

Let \( A_i \in \mathbb{C}^{N \times M} \), for \( i = 1, 2, \ldots, K \), be independent random matrices with \( M \leq N \), for all \( i \). Assume \( K M > N \).

Denote \( \tilde{A} = [A_1, \ldots, A_K] \), and let the columns of \( U \in \mathcal{C}^{K M \times (K M - N)} \) be bases of \( \text{null}(\tilde{A}) \). Further, we represent \( \tilde{A} \) as \( U = [U_1^T, \ldots, U_K^T] \), where \( U_i \in \mathbb{C}^{M \times (K M - N)} \). Denote \( \hat{A} = [A_1 U_1, \ldots, A_K U_K] \in \mathbb{C}^{N \times K M - N} \). We have the following result.

Lemma 3. The rank of matrix \( \hat{A} \) is

\[
\min ((K - 1)(K M - N), N) \text{ with probability one.}
\]

Proof: We first assume \((K - 1)(K M - N) \leq N\), or equivalently, \((K - 1)M \leq N\). Let \( v = [v_1^T, \ldots, v_K^T]^T \) be an arbitrary vector in \( \text{null}(\hat{A}) \), where \( v_i \in \mathbb{C}^{(K M - N) \times 1} \). Then, we obtain \( 0 = \hat{A} v = A \text{diag}(U_1, \ldots, U_K) v \). Thus, \( \text{diag}(U_1, \ldots, U_K) v \) belongs to \( \text{null}(A) \). As \( \tilde{A} \) spans \( \text{null}(A) \), there exists \( x \in \mathbb{C}^{(K M - N) \times 1} \) such that \( \text{diag}(U_1, \ldots, U_K) v = U x \), or equivalently, \( U_i v_i = U_i x \), for all \( i \). From the randomness of \( A_i \), \( U_i \) is of rank \( (K M - N) \) with probability one. By assumption of \((K - 1)(K M - N) \leq N\), we obtain \( K M - N \leq M \). Thus, the left inverse of \( U_i \) exists with probability one. Then, \( U_i v_i = U_i x \) implies \( v_i = x \), for all \( i \), or equivalently, \( v = [x^T, \ldots, x^T]^T \). Since \( x \) can be any vector in \( \mathbb{C}^{(K M - N) \times 1} \), the nullspace of \( \hat{A} \) is of rank \( K M - N \) with high probability. By using the rank-nullity theorem of linear algebra, we conclude that \( \tilde{A} \) is of rank \( K(K M - N) - (K M - N) = (K - 1)(K M - N) \) with probability one. What remains is the case of \((K - 1)(K M - N) > N\). Note that, when \((K - 1)(K M - N) = N\), \( \text{span}(\tilde{A}) \) is of dimension \( N \) with probability one. Further increasing \( M \) cannot increase the dimension of \( \text{span}(\tilde{A}) \) (as \( \tilde{A} \) only has \( N \) rows), which concludes the proof.

Fig. 7. The improved achievable DoF for the MIMO mRC with infinite number of users at different \((M, N)\) configurations.

\[
\frac{M}{N} \in \left[ 0, \frac{K - 1}{K M - 2}\right] \quad \text{and} \quad \frac{M}{N} \in \left[ \frac{1}{K M - 1}, \frac{1}{2}, \infty \right).
\]

This result has a broader range of \( \frac{M}{N} \) compared to the existing result in [16]. The asymptotic achievable DoF when the number of users tending to infinity was also analyzed. The derived achievable DoF in this work can in general serve as a lower bound of the DoF capacity of the considered MIMO mRC. The future research of interest include the optimal precoding design of the MIMO mRC in finite SNR and extension our DoF analysis to more complex scenarios, such as clustered MIMO mRCs.
\[
\begin{bmatrix}
I_N & A_1 & 0 \\
I_N & 0 & -A_2
\end{bmatrix}, \quad \text{and } C = [I_N, 0, 0, 0]. \quad \text{Let}
\]
\[
D_1 = \text{diag}\left( B_1, B_2, \cdots, B_3 \right). \tag{29}
\]

Equations (27) and (28) can be combined in a matrix form as
\[
\begin{bmatrix}
D_1 & 0 & 0 & 0 \\
0 & D_2 & 0 & 0 \\
0 & 0 & D_3 & 0 \\
C & C & C & C
\end{bmatrix} q = Mq = 0,
\]
where \( q = [q_1^T, q_2^T, q_3^T, q_4^T]^T \) with \( q_i = [n_i, 1, (q_{i,2}^T, q_{i,3}^T, q_{i,4}^T, \cdots, a_{i,3M-4}, q_{i,3M-3}, q_{i,3M-2}, q_{i,3M-1})]^T, \)
\( (q_{i,3M-3}, q_{i,3M-2}, q_{i,3M-1})]^T \). Then, to see that \( S_i \), for
\( i = 1, 2, 3, 4 \), are linearly independent, it suffices to show that \( M \) is of full column rank. Recall that \( A_i, \) for \( i = 1, 2, 3, 4 \),
are independent random matrices. Thus, \( M \) is indeed of full column rank with probability one, provided that
\( 4(2M - N) + 3 \leq N \), which completes the proof. ■

**APPENDIX B**

**DETAILS OF SYMBOL EXTENSION**

We suppose that there is an MIMO mRC with \( K = 3 \),
\( M = 2 \), and \( N = 5 \), which results in \( \frac{3}{2} < \frac{M}{N} < \frac{1}{2} \). Then
the whole relay’s signal space should be occupied by the subspaces
constructed by Pattern 1.3 and Pattern 1.1. However, as \( \frac{3M - N}{2} \) is not
an integer, we need to use two-symbol extension to make \( \frac{3M - N}{2} \) an integer. By using two-symbol extension, we rewrite
the received signal at the relay node as
\[
\hat{y}_R = \sum_{k=1}^{3} H_k \tilde{x}_k + z_R,
\]
where \( H_k = \begin{bmatrix} H_k & 0 \\ 0 & H_k \end{bmatrix} \in \mathbb{C}^{10 \times 4} \) and \( \tilde{x}_k = [x_k(2t + 1), x_k(2t + 2)]^T \in \mathbb{C}^{4 \times 1} \) denote the combined two-symbol
channel matrix and transmit signal at user \( k \), respectively. Note
that the time-variant channel at two symbols are unnecessary here. After the symbol extension, based on (17), we can transmit 2 spatial streams with Pattern 1.3 and 2 spatial
streams with Pattern 1.1 for each user. Let
\[
\tilde{x}_k = U_k s_k
\]
\[
= \sum_{k'=1, k' \neq k}^{3} u_{k,k'}^{(k,k')} \gamma_{k,k'} + \sum_{k'=1, k' \neq k}^{3} \hat{u}_{k,k'}^{(k,k')} \gamma_{k,k'}, \tag{30}
\]
where \( U_k \in \mathbb{C}^{4 \times 4} \) is the beamforming matrix and \( s_k \in \mathbb{C}^{4 \times 1} \)
is the transmit symbol vector before the beamforming. Further in (30), two beamforming vectors \( u_{k,k'}^{(k,k')} \), \( k' \in \{1, 2, 3\} \), and \( k' \neq k \) are constructed with Pattern 1.3, i.e.
\[
\begin{align*}
\tilde{H}_{1} u_{1}^{(1,3)} + \tilde{H}_{2} u_{1}^{(2,3)} + \tilde{H}_{3} u_{2}^{(3,2)} + u_{3}^{(3,1)} &= 0 \\
\tilde{H}_{1} u_{1}^{(1,2)} + u_{1}^{(1,3)} + \tilde{H}_{2} u_{2}^{(2,1)} + \tilde{H}_{3} u_{3}^{(3,1)} &= 0.
\end{align*}
\]

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