GALOIS GROUPS AND COHOMOLOGICAL FUNCTORS

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Abstract. Let $q = p^s$ be a prime power, $F$ a field containing a root of unity of order $q$, and $G_F$ its absolute Galois group. We determine a new canonical quotient $\text{Gal}(F_3)/F$ of $G_F$ which encodes the full mod-$q$ cohomology ring $H^*(G_F, \mathbb{Z}/q)$ and is minimal with respect to this property. We prove some fundamental structure theorems related to these quotients. In particular, it is shown that when $q = p$ is an odd prime, $F_3$ is the compositum of all Galois extensions $E$ of $F$ such that $\text{Gal}(E/F)$ is isomorphic to $\{1\}$, $\mathbb{Z}/p$ or to the nonabelian group $H_{p,3}$ of order $p^3$ and exponent $p$.

1. Introduction

Let $p$ be a fixed prime number and $q = p^s$ a fixed $p$-power, where $s \geq 1$. For a profinite group $G$, let $H^*(G) = \bigoplus_{i=0}^\infty H^i(G, \mathbb{Z}/q)$ be the cohomology (graded) ring with the trivial action of $G$ on $\mathbb{Z}/q$. We will be mostly interested in the case where $G = G_F$ is the absolute Galois group of a field $F$ which contains a root of unity of order $q$. The ring $H^*(G_F)$, and even its degree $\leq 2$ part, is known to encode important arithmetical information on $F$ (see below). In this paper we ask how much Galois-theoretic information is needed to compute $H^*(G_F)$. More specifically, we characterize the minimal quotient of $G_F$ which determines it. In [CEM12] it was shown that $H^*(G_F)$ is determined by a quite small quotient $G_F^{[3]} = G_F/(G_F)^{(3)}$ of $G_F$. Here $G^{(i)} = G^{(i, q)}$, $i = 1, 2, 3, \ldots$, is the descending $q$-central sequence of a given profinite $G$ (see [2]). Namely, the inflation map $\text{inf}: H^*(G_F^{[3]})_{\text{dec}} \to H^*(G_F)$ is an isomorphism, where $H^*(G_F^{[3]})_{\text{dec}}$ denotes the subring of $H^*(G_F^{[3]})$ generated by degree 1 elements. In the present paper we first find an even smaller quotient $(G_F)^{(3)} = G_F/(G_F)^{(3)}$ of $G_F$ with the same property. Here for a profinite group $G$ we set $G_3 = G^q[G^{(2)}, G]$ if $p > 2$, and $G_3 = G^{2q}[G^{(2)}, G]$ if $p = 2$ (see [12]). Thus, when $q = p$, $G_3 = G^{(3)}$ is the third term in the Zassenhaus $p$-filtration of $G$. Moreover, we prove that this quotient is minimal with respect to this property:

**Theorem A.** Let $N$ be a closed normal subgroup of $G_F$. Then the inflation map $H^*(G_F/N)_{\text{dec}} \to H^*(G_F)$ is an isomorphism if and only if $N \leq (G_F)^{(3)}$. In particular, $(G_F)^{(3)}$ determines $H^*(G_F)$.

Theorem A is proved in [5]. Conversely, in [7] we prove:

**Theorem B.** The degree $\leq 2$ part of $H^*(G_F)$ determines $(G_F)^{(3)}$.

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We also prove the following strengthening of [CEM12, Ths. C, D]. Denote the maximal pro-$p$ Galois group of $F$ by $G_F(p)$.

**Theorem C.** Let $F_1, F_2$ be fields containing a $q$th root of unity. Let $\pi: G_{F_1}(p) \to G_{F_2}(p)$ be a continuous homomorphism and $\pi^*: H^*(G_{F_2}(p)) \to H^*(G_{F_1}(p))$ and $\pi[3]: (G_{F_1})[3] \to (G_{F_2})[3]$ the induced maps. Then $\pi$ is an isomorphism if and only if $\pi[3]$ is an isomorphism.

The proof of Theorem C is given in [6]. Theorem C is in the spirit of the “anabelian phenomena” in Galois theory [NSW08, Ch. XII, §§2–3], as it shows that certain isomorphisms of (small) Galois groups already imply that the much richer structures of the maximal pro-$p$ Galois groups are isomorphic.

Regarding the structure of the quotient $(G_F)[3]$, we prove that it is the Galois group of the compositum of all Galois extensions of $F$ with certain specific Galois groups:

**Theorem D.** Assume that $q = p \neq 2$ is prime and let $F$ be as above. Then $(G_F)[3]$ is the intersection of all normal open subgroups $N$ of $G_F$ such that $G_F/N$ is isomorphic to $\mathbb{Z}/p$ or $H_{p^3}$.

Here $H_{p^3}$ is the nonabelian group of order $p^3$ and exponent $p$ (the Heisenberg group):

$$H_{p^3} = \langle r, s, t \mid r^p = s^p = t^p = [r, t] = [s, t] = 1, [r, s] = t \rangle.$$

In the remaining case $q = p = 2$ the group $(G_F)[3]$ is known to be the intersection of all normal open subgroups $N$ of $G_F$ such that $G_F/N$ is isomorphic to either $\mathbb{Z}/2$, $\mathbb{Z}/4$, or the dihedral group $D_4$ of order 8 [Villegas, Mináč–Spira [MS96, Cor. 2.18]; see also [EM11, Cor. 11.3 and Prop. 3.2]). Moreover, $\mathbb{Z}/2$ can be omitted from this list unless $F$ is Euclidean [EM11, Cor. 11.4]. The proof of Theorem D is given in Example [9,5](1).

Theorem A gives a new restriction on the structure of maximal pro-$p$ Galois groups of fields as above. Indeed, it implies that if the defining relations in such a group $G = G_F(p)$ are changed within $G_{(3)}$ and the cohomology ring is changed, then the resulting group may not be realized as a Galois group in this way (see Corollary [6,3] for a precise statement). In particular, Theorem A directly implies the classical Artin–Schreier theorem, asserting that elements of absolute Galois groups can have only order 1, 2, or $\infty$ (Example [6,4](2)).

Our proofs of Theorems A, C, D are based on the bijectivity of the Galois symbol homomorphism $K^M(F)/qK^M(F) \to H^*(G_F)$, proved by Rost and Voevodsky (with a patch by Weibel), where $K^M(F)$ is the Milnor $K$-ring of $F$; see [Voe03, Voe11, Wei09, Wei08]. The bijectivity of the Galois symbol in degree 2 was proved earlier by Merkurjev and Suslin [MS92]. Specifically, the proofs of Theorems A and the second equivalence in Theorem C use the bijectivity of this map, the proof of the first equivalence in Theorem C uses its bijectivity in degree 2, and that of Theorem D uses only its injectivity in degree 2.

Our approach is purely group-theoretic and is based on a fundamental notion of duality between a pair $(T, T_0)$ of normal subgroups of a profinite group $G$ and a subgroup $A$ of the second cohomology $H^2(G/T)$ (see [43]). When applied to $T = G^{(2)}, T_0 = G^{(3)}$ and $A = H^2(G^{(2)})_{dec}$, with $G = G_F$ as above, it leads to Theorems A–D. Moreover, it can also be applied to other choices of $T, T_0, A$ as well to yield analogous results. Notably, taking $T = G^{(2)}, T_0 = G^{(3)}$ and $A = H^2(G^{(2)})$, 

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[6]: 6
[EM11]: EM11
[NSW08]: NSW08
[MS96]: MS96
[MS92]: MS92
[Villegas]: Villegas
[Voe03]: Voe03
[Voe11]: Voe11
[Wei09]: Wei09
[Wei08]: Wei08
we strengthen the main results of [CEM12]. As a third example we may take \( T = G^{(2)}, T_0 = G^q[G,G] \) and \( A = \text{Im}(\beta_{G^{(2)}}) \), where \( \beta_{G^{(2)}} : H^1(G^{(2)}) \to H^2(G^{(2)}) \) is the Bockstein map (see [2]).

The Galois group \((G_F)^{[3]}\) encodes important arithmetical information on \( F \). For instance, when \( q = 2 \) it was shown in [MS90] and [MS96] that \((G_F)^{[3]}\) and the Kummer element of \(-1\) encode the orderings on \( F \) as well as the Witt ring of quadratic forms over \( F \). Moreover, in [EM12] we use Theorem D to prove that \((G_F)^{[3]}\) also encodes a class of valuations which are important in the pro-\( p \) context, namely, (Krull) valuations \( v \) whose value group is not divisible by \( p \) and whose \( 1 \)-units are \( p \)-powers (this is a weak form of Hensel’s lemma). Specifically, under a finiteness assumption and assuming the existence of a root of unity of order 4 in \( F \) when \( p = 2 \), there exists such a valuation if and only if the center \( Z((G_F)^{[3]}) \) has a nontrivial image in \( G_F^{[2]} \).

From a more general perspective, recovering arithmetical information on a field from the profinite group-structure of its absolute Galois group, and even of smaller canonical Galois groups, is the main theme of the birational anabelian geometry. See for instance the recent informative review [BT12], as well as [Pop12], [Top12] and the introduction of [MST14]. Typically such results are limited to various geometric situations. The fact that considerable crucial arithmetical information on valuations and orderings can be recovered even from the small quotient \((G_F)^{[3]}\) (with the only assumption about having a root of unity) is a surprising consequence of this work.

The first version of this work was posted to the archive arXiv in 2011. Our goal was to develop such techniques which would be adaptable for various other higher \( p \)-descending filtrations of absolute Galois groups. Since then there have indeed been various related subsequent works on the \( p \)-Zassenhaus filtration \( G_{(n)} \) of a profinite group \( G \), Massey products in Galois cohomology (which generalize the cup product), and duality principles between them of this nature. In [Efr14a Th. B] it was shown that when \( G \) is a free profinite group, the subgroup \( A \) of \( H^2(G/G_{(n)}) \) generated by all \( n \)-fold Massey products is dual to \((G_{(n)}, G_{(n+1)})\), in the sense of §3.

Further, let \( U_n(F_p) \) be the group of all \( n \times n \) upper-triangular unipotent matrices over \( F_p \). It was observed in [Efr14a] that Theorem D above can be rephrased as saying that for \( G = G_F \) and \( q = p \) a prime number, one has \( G_{(3)} = \bigcap \text{Ker}(\rho) \), where the intersection is over all continuous homomorphisms \( \rho : G \to U_3(F_p) \). It was shown that for every \( n \geq 3 \) profinite group \( G \) satisfying a certain cohomological condition on \((n-1)\)-fold Massey products, one has

\[
G_{(n)} = \bigcap \{\text{Ker}(\rho) \mid \rho : G \to U_n(F_p)\}.
\]

In particular, this holds when \( G \) is free pro-\( p \) (see also [Efr14b] and [MT Th. 2.6] for direct alternative proofs). Then, in [MT15 §8] and [MT], it was conjectured that the equality (1.1) holds when \( G = G_F \) is an absolute Galois group of a field \( F \) containing a root of unity of order \( p \) (the “kernel \( n \)-unipotent conjecture”). It was shown in [MT] that (1.1) holds for profinite groups related to local fields: odd rigid groups and Demuškin groups.

2. Preliminaries

A) Profinite groups. We work in the category of profinite groups. Thus all homomorphisms of profinite groups will be tacitly assumed to be continuous and
all subgroups will be closed. The descending \( q \)-central filtration \( G^{(i)} = G^{(i,q)} \) of a profinite group \( G \) is defined inductively by

\[
G^{(1)} = G, \quad G^{(i)} = (G^{(i-1)})^q[G^{(i-1)}, G], \quad i \geq 2.
\]

Thus \( G^{(i)} \) is generated by all \( q \)th powers of elements of \( G^{(i-1)} \) and all commutators \( [h, g] = h^{-1}g^{-1}hg \), where \( h \in G^{(i-1)}, g \in G \). Set \( G^{(0)} = G/G^{(1)} \). We also define

\[
\delta = \begin{cases} 1, & p > 2, \\ 2, & p = 2, \end{cases} \quad G_{(3)} = G^{\delta q}[G^{(2)}, G], \quad G_{[3]} = G/G_{(3)}.
\]

**Remarks 2.1.** (1) As \( [h, g] = h^{-2}(hg^{-1})^2g^2 \), we have \([G, G] \leq G^2\). Therefore, when \( q = 2 \) we have \( G^{(2)} = G^2 \), so \( G^{(3)} = G^4[G^{(2)}, G] = G_{(3)} \).

(2) One has \( G_{(3)} \leq G_{(3)} \). Indeed, this is immediate when \( p > 2 \). When \( p = 2 \) we have \( g^{2q} = (g^{q/2})^{2q} \) for every \( g \in G \), so \( (G^q)^q \leq G^{2q}[G^q, G] \). Further, one has the identities (see [Lab66, Prop. 5] and its proof)

\[
(g_1g_2)^q = g_1^qg_2^q[g_2, g_1]^{(\frac{q}{2})}, \quad [g_1g_2, h] \equiv [g_1, h][g_2, h] \pmod{[[G, G], G]}.
\]

It follows that \( [g_1^q, g_2^q]^{(\frac{q}{2})} \in G^{2q}[[G, G], G] \) for any \( g_1, g_2 \in G \). By (1), \([G, G]^q \leq (G^2)^q \leq G^{2q}[[G, G], G] \). Consequently, \((G^{(2)})^q \leq G^{2q}[G^{(2)}, G] = G_{(3)} \), whence our claim.

**B) Profinite cohomology.** We refer to [NSW08], [RZ10], or [Ser02] for basic notions and facts in profinite cohomology. In particular, given a profinite group \( G \), let \( H^i(G) = H^i(G, \mathbb{Z}/q) \) be the \( i \)th profinite cohomology group of \( G \) with respect to its trivial action on \( \mathbb{Z}/q \). Thus \( H^1(G) = \text{Hom}_{\text{cont}}(G, \mathbb{Z}/q) \). Let

\[
H^*(G) = \bigoplus_{r=0}^{\infty} H^r(G)
\]

be the cohomology ring with the cup product \( \cup \). Given a homomorphism \( \pi: G_1 \to G_2 \) of profinite groups, let \( \pi^* : H^*(G_2) \to H^*(G_1) \) and \( \pi^* : H^*(G_2) \to H^*(G_1) \) be the induced homomorphisms. We write res, inf, tr for the restriction, inflation, and transgression maps, respectively. For a normal subgroup \( N \) of \( G \), there is a canonical action of \( G \) on \( H^1(N) \). When \( i = 1 \) it is given by \( \psi \mapsto \psi^G \), where \( \psi^G(n) = \psi(g^{-1}ng) \) for \( g \in G \) and \( n \in N \). We denote the group of all \( G \)-invariant elements of \( H^1(N) \) by \( H^1(N)^G \).

For each \( r \geq 0 \), the cup product induces a homomorphism \( H^1(G)^{\otimes r} \to H^r(G) \) of \( \mathbb{Z}/q \)-modules. Let \( H^r(G)_{\text{dec}} \) be its image (where \( H^0(G)_{\text{dec}} = \mathbb{Z}/q \)), and let \( H^r(G)_{\text{dec}} = \bigoplus_{r=0}^{\infty} H^r(G)_{\text{dec}} \) be the decomposable cohomology ring with the cup product.

Following [CEM12 §3], set \( \overline{H^*}(G) = \bigoplus_{r=0}^{\infty} H^1(G)^{\otimes r}/C_r \), where \( C_r \) is the subgroup of \( H^1(G)^{\otimes r} \) generated by all elements \( \psi_1 \otimes \cdots \otimes \psi_r \) such that \( \psi_i \cup \psi_j = 0 \) for some \( i < j \). It is a graded ring and there is a canonical graded ring epimorphism \( \omega_G : \overline{H^*}(G) \to H^*(G)_{\text{dec}} \). The ring \( \overline{H^*}(G) \) and the map \( \omega_G \) are functorial in the natural sense. We call \( H^*(G) \) **quadratic** (resp., \( r \)-**quadratic**) if \( \omega_G \) is an isomorphism (resp., in degree \( r \)).

**Lemma 2.2.** When \( H^*(G) \) is 2-quadratic, the kernel of \( \text{Inf}_G : H^3(G^{[2]})_{\text{dec}} \to H^2(G) \) is generated by elements of the form \( \psi \cup \psi' \), with \( \psi, \psi' \in H^1(G^{[2]}) \) such that \( \text{Inf}_G(\psi \cup \psi') = 0 \).
Proof. This follows from the commutative diagram
\[
\begin{array}{ccc}
H^1(G^{[2]\otimes 2}) & \overset{\text{inf}}{\sim} & H^1(G)^{\otimes 2} \\
\cup & & \cup \\
H^2(G^{[2]\otimes 2})_{\text{dec}} & \overset{\text{inf}}{\longrightarrow} & H^2(G)_{\text{dec}}
\end{array}
\]
and the definition of $C_2$. \(\square\)

The Bockstein map $\beta_G: H^1(G) \to H^2(G)$ is the connecting homomorphism arising from the short exact sequence of trivial $G$-modules

\[
0 \to \mathbb{Z}/q \to \mathbb{Z}/q^2 \to \mathbb{Z}/q \to 0.
\]

It is functorial in the natural way. The following lemma was proved in [EM11, Cor. 2.11] for $I$ finite and follows in general by a limit argument. See also Proposition 10.3 for an alternative proof.

Lemma 2.3. If $G = (\mathbb{Z}/q)^I$ for some $I$, then $H^2(G) = \text{Im}(\beta_G) + H^2(G)_{\text{dec}}$.

For $r \geq 0$ let $B_r$ be the subgroup of $H^1(G)^{\otimes r}$ generated by all elements $\psi_1 \otimes \cdots \otimes \psi_r$ such that $\beta_G(\psi_i) = 0$ for some $i$. We define a graded ring $H^*(G)_{\text{Bock}} = \bigoplus_{r=0}^\infty H^1(G)^{\otimes r}/B_r$ with the tensor product.

C) Galois cohomology. The following theorem collects the cohomological properties of an absolute Galois group which are needed for this paper.

Theorem 2.4. Let $F$ be a field containing a root of unity $\zeta_q$ of order $q$ and let $G_F$ be its absolute Galois group. Then:

(i) $G_F^{[2]} \cong (\mathbb{Z}/q)^I$;
(ii) there exists $\xi \in H^1(G_F)$ with $\beta_G(\psi) = \psi \cup \xi$ for every $\psi \in H^1(G_F)$;
(iii) $H^*(G_F) = H^*(G_{F,\text{dec}})$;
(iv) $H^*(G_F)$ is quadratic.

Proof. We identify the group $\mu_p$ of $q$th roots of unity in $F$ with $\mathbb{Z}/q$, where $\zeta_q$ corresponds to 1 mod $q$. The obvious map $F^\times/(F^\times)^q \to F^\times/(F^\times)^p$ is surjective, and by the Kummer isomorphism, so is the functorial map $H^1(G_F) = H^1(G_F, \mathbb{Z}/q) \to H^1(G_F, \mathbb{Z}/p)$. This implies (i). In (ii) one takes $\xi$ to be the Kummer element corresponding to $\zeta_q$ (see [EM11 Prop. 3.2]). Condition (iii) (resp., (iv)) follows from the surjectivity (resp., injectivity) of the Galois symbol $K^M_*(F)/q \to H^*(G_F)$, as proved by Rost, Voevodsky, with a patch by Weibel (see [CEM12, §8]). \(\square\)

Remark 2.5. Many of our results do not require the full strength of conditions (iii) and (iv) of Theorem 2.3, but only their validity in degrees 1 and 2. These weaker facts follow from the Kummer isomorphism and the Merkurjev–Suslin theorem ([MSS2], [GS06]), respectively.

Remark 2.6. By [CEM12] Remark 8.2, $\text{inf}: H^*(G_{F}(p)) \to H^*(G_F)$ is an isomorphism. Therefore (i)–(iv) hold also for $G_{F}(p)$. 
3. Duality

Let $G$ be a profinite group and let $T, T_0$ be normal subgroups of $G$ such that $T^q[T, G] \leq T_0 \leq T \leq G^{(2)}$. Let

$$K = \text{Ker}(H^1(T)^G \xrightarrow{\text{res}} H^1(T_0)), \quad K' = \text{Ker}(H^2(G/T) \xrightarrow{\text{inf}} H^2(G/T_0)).$$

Note that if $T_0 = T^q[T, G]$, then $K = H^1(T)^G$. As $T, T_0 \leq G^{(2)}$, the inflations $H^1(G/T) \to H^1(G)$, $H^1(G/T_0) \to H^1(G)$ are isomorphisms. The functoriality of the 5-term sequence [NSW08] pp. 78–79 gives a commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \rightarrow & H^1(T)^G & \xrightarrow{\text{trg}} & H^2(G/T) & \xrightarrow{\text{inf}} & H^2(G) \\
\text{res} & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^1(T_0)^G & \xrightarrow{\text{trg}} & H^2(G/T_0) & \xrightarrow{\text{inf}} & H^2(G).
\end{array}
\]

By the snake lemma, $K, K'$ are therefore isomorphic via transgression.

There is a perfect duality

\[
T/T^q[T, G] \times H^1(T)^G \rightarrow \mathbb{Z}/q, \quad (\bar{\sigma}, \psi) \mapsto \psi(\sigma)
\]

which is functorial in the natural sense [EM11] Cor. 2.2]. It induces a (functorial) perfect duality

\[
T/T^q[T, G] \times \text{trg}(H^1(T)^G) \rightarrow \mathbb{Z}/q, \quad (\bar{\sigma}, \varphi) \mapsto (\text{trg}^{-1}(\varphi))(\sigma).
\]

**Proposition 3.1.** (3.2) and (3.3) induce perfect pairings:

(a) $\langle \cdot, \cdot \rangle : T/T_0 \times K \rightarrow \mathbb{Z}/q, \ (\sigma T_0, \psi) \mapsto \psi(\sigma)$;

(b) $\langle \cdot, \cdot \rangle : T/T_0 \times K' \rightarrow \mathbb{Z}/q, \ (\sigma T_0, \varphi) = (\text{trg}^{-1}(\varphi))(\sigma)$.

**Proof.** (a) A perfect pairing of abelian groups $\langle \cdot, \cdot \rangle : A \times B \rightarrow C$ induces for any $A_0 \leq A$ a perfect pairing $(A/A_0) \times \{ b \in B \mid (A_0, b) = 0 \} \rightarrow C$. Take $A = T/T^q[T, G]$, $B = H^1(T)^G$, $C = \mathbb{Z}/q$, and let $A_0$ be the image of $T_0$ in $A$. By (3.2), the substitution pairing $A \times B \rightarrow \mathbb{Z}/q$ is perfect. Furthermore, for $\psi \in H^1(T)^G$ we have $(A_0, \psi) = 0$ if and only if $\psi \in K$.

(b) follows from (a). \qed

For the connection between the second cohomology and central extensions see, e.g., [NSW08] Th. 1.2.4] or [RZ10] §6.8.

**Proposition 3.2.** Let $A$ be a subgroup of $H^2(G/T)$ and let $A_0$ be a set of generators of $A \cap \text{trg}(H^1(T)^G)$. The following conditions are equivalent:

(a) $K = \text{trg}^{-1}[A]$;

(b) there is an exact sequence

\[
0 \rightarrow K \xrightarrow{\text{trg}} A \xrightarrow{\text{inf}} H^2(G);
\]

(c) there is an exact sequence

\[
0 \rightarrow K' \hookrightarrow A \xrightarrow{\text{inf}} H^2(G);
\]

(d) $\langle \cdot, \cdot \rangle$ induces a perfect pairing

\[
T/T_0 \times \text{Ker}(A \xrightarrow{\text{inf}} H^2(G)) \rightarrow \mathbb{Z}/q;
\]

(e) $T_0$ is the annihilator of $A \cap \text{trg}(H^1(T)^G)$ under $\langle \cdot, \cdot \rangle$;
Proposition 3.4. Suppose that \([\text{LEM}12, \text{Lemma } 5.4]\), so by the 5-term sequence, (b) holds. 

corresponding to \(G\) is necessarily surjective, and \(\text{Ker}(\Psi) = T\).

By \([\text{Hoe}68, 2.1]\), for every \(\psi \in H^1(T)^G\) with \(\text{trg}(\psi) = \varphi\) there is a homomorphism \(\Psi: G \rightarrow C\) such that \(\psi = \Psi|_T\) and the diagram in (f) is commutative. For such \(\Psi, \psi\) and for \(\sigma \in T\) and \(\varphi \in A_0\) we have \(\langle \sigma T_0, \varphi \rangle = \psi(\sigma) = \Psi(\sigma)\). Moreover, \(\text{Ker}(\Psi) \leq T\). We conclude that the annihilator of \(A \cap \text{trg}(H^1(T)^G)\) in \(T\) under \(\langle \cdot, \cdot \rangle\) is \(\cap \text{Ker}(\Psi)\).

When these conditions are satisfied, we say that \(A\) is dual to \((T, T_0)\).

Examples 3.3. (1) In \([\text{EM}11]\) we will show that, when \(G^{[2]} \cong (\mathbb{Z}/q)^I\) for some \(I\), \(H^2(G^{[2]})_{\text{dec}}\) is dual to \((G^{(2)}, G^{(3)})\).

(2) For \(i \geq 2\), \(H^2(G^{[i]})\) is dual to \((G^{(i)}, G^{(i+1)})\). Indeed, here \(K = H^1(G^{(i)})^G\).

(3) For \(G = \mathbb{Z}/q\) and for an isomorphism \(\psi \in H^1(G)\), the central extension corresponding to \(\beta_{\mathbb{Z}/q}(\psi)\) is 

\[
0 \rightarrow \mathbb{Z}/q \rightarrow \mathbb{Z}/q^2 \rightarrow \mathbb{Z}/q \rightarrow 0
\]

(see \([\text{EM}11\) Prop. 9.2]). For \(\bar{G} = (\mathbb{Z}/q)^I\), and for the projection \(\psi: \bar{G} \rightarrow \mathbb{Z}/q\) on the \(i_0\)th coordinate, the extension corresponding to \(\beta_{\bar{G}}(\psi)\) is then 

\[
0 \rightarrow \mathbb{Z}/q \rightarrow \prod_{i \in I} C_i \rightarrow \bar{G} \rightarrow 1,
\]

where \(C_{i_0} = \mathbb{Z}/q^2\) and \(C_i = \mathbb{Z}/q\) for \(i \neq i_0\), with the natural maps \([\text{EM}11\) Lemma 6.2].

Now assume that \(G\) is a profinite group with \(G^{[2]} \cong (\mathbb{Z}/q)^I\) for some \(I\). Take \(T = G^{(2)}\) and \(A_0 = \text{Im}(\beta_{G^{[2]}})\). Note that then every homomorphism \(\Psi\) as in (e) is necessarily surjective, and \(\text{Ker}(\Psi) \leq T = G^{(2)}\). We deduce that the intersection in (f) is 

\[
G^{(2)} \cap \bigcap \{M \leq G \mid G/M \cong \mathbb{Z}/q^2\} = G^{a^2}[G, G].
\]

Thus \(A\) is dual to \((G^{(2)}, G^{a^2}[G, G])\).

Proposition 3.4. Suppose that \(A\) is dual to \((T, T_0)\). Then the kernels of \(\inf_{G/T_0}: A \rightarrow H^2(G/T_0)\) and \(\inf_G: A \rightarrow H^2(G)\) coincide.

Proof. This is straightforward from condition (c) of Proposition \([\text{EM}12]\).
4. Cohomological duality triples

In this section we axiomatize several important cases of functorial subgroups of profinite groups so that we can treat them all in a unified way. For a profinite group $G$ we set $H^\otimes_*(G) = \bigoplus_{r=0}^\infty H^1(G)^\otimes r$, considered as an abelian group. Assume that we are given:

(i) subfunctors $T, T_0$ of the identity functor on the category of profinite groups;
(ii) a natural transformation $\alpha$ from the functor $G \mapsto H^\otimes_*(G)$ to the functor $G \mapsto H^2(G)$ (both from the category of profinite groups to the category of abelian groups).

In other words, for every profinite group $G$ we are given subgroups $T(G), T_0(G)$ of $G$ and a group homomorphism $\alpha_G: H^\otimes_*(G) \to H^2(G)$, and for every homomorphism $\pi: G_1 \to G_2$ of profinite groups there are commutative squares

$$
\begin{array}{ccc}
T(G_1) & \xleftarrow{\scriptstyle T(\pi)} & G_1 \\
\downarrow \pi & & \downarrow \pi \\
T(G_2) & \xleftarrow{\scriptstyle T(\pi)} & G_2
\end{array} \quad \begin{array}{ccc}
T_0(G_1) & \xleftarrow{\scriptstyle T_0(\pi)} & G_1 \\
\downarrow \pi & & \downarrow \pi \\
T_0(G_2) & \xleftarrow{\scriptstyle T_0(\pi)} & G_2
\end{array} \quad \begin{array}{ccc}
H^\otimes_*(G_2) & \xrightarrow{\scriptstyle \alpha_{G_2}} & H^2(G_2) \\
\downarrow \pi^* & & \downarrow \pi^* \\
H^\otimes_*(G_1) & \xrightarrow{\scriptstyle \alpha_{G_1}} & H^2(G_1)
\end{array}
$$

We denote

$$K(G) = \ker(\text{res}: H^1(T(G))^G \to H^1(T_0(G))), \quad A(G) = \text{Im}(\alpha_G).$$

Observe that, in the previous setup, $\pi^*_G(A(G_2)) \subseteq A(G_1)$.

A cover of a profinite group $G$ (relative to $T$) will be an epimorphism $\pi: S \to G$, where $S$ is a profinite group such that $H^2(S) = 0$ and the induced map $S/T(S) \to G/T(G)$ is an isomorphism.

**Example 4.1.** For a profinite group $G$ let $T(G) = G^{(2)}$. The existence of a cover $S \to G$ means that $G^{[2]} \cong (\mathbb{Z}/q)^I$. Indeed, for such $G$ take $S$ to be a free profinite group of the appropriate rank [NSW08, 3.5.4]. Conversely, a cover $\pi: S \to G$ induces an epimorphism $\pi(p): S(p) \to G(p)$ of the maximal pro-$p$ quotients. Then $H^2(S(p)) = 0$ [CEM12, Lemma 6.5], so $S(p)$ is a free pro-$p$ group [NSW08, Prop. 3.5.17]. Thus $G^{[2]} \cong S^{[2]} \cong S(p)^{[2]} \cong (\mathbb{Z}/q)^I$ for some $I$.

We call $(T, T_0, \alpha)$ a cohomological duality triple if for every profinite group $G$ the following conditions hold:

(A1) $T(G), T_0(G)$ are normal subgroups of $G$;
(A2) $T(G)^q[T(G), G] \leq T_0(G) \leq T(G) \leq G^{(2)}$;
(A3) for every epimorphism $\pi: G \to \overline{G}$ one has $T(\overline{G}) = \pi(T(G))$ and $T_0(\overline{G}) = \pi(T_0(G))$;
(A4) if there is a cover $S \to G$, then $A(G/T(G))$ is dual to $(T(G), T_0(G))$.

We list three basic examples of cohomological duality triples. Condition (A2) for example (2) was shown in Remark 2.11, and (A4) for all these examples is just Examples 3.3. The verification of the remaining conditions is straightforward.

**Examples 4.2.** (1) Let $T(G) = G^{(2)}$ and $T_0(G) = G^{(3)}$, and let $\alpha_G$ be the cup product on $H^1(G)^{\otimes 2}$ and the trivial map on $H^1(G)^{\otimes r}$ for $r \neq 2$. Thus $A(G) = H^2(G)_{\text{dec}}$.

(2) Let $T(G) = G^{(2)}$ and $T_0(G) = G^{(3)}$, and let $\alpha_G$ be $\beta_G$ on $H^1(G)$, the cup product on $H^1(G)^{\otimes 2}$, and the trivial map on $H^1(G)^{\otimes r}$ for $r \neq 1, 2$. Then
Let Corollary 5.2. to deduce the following special cases.

(3) Let $T(G) = G^{(2)}$ and $T_0(G) = G^{*2}[G, G]$, and let $\alpha_G = \beta_G$ on $H^1(G)$ and $\alpha_G = 0$ on $H^1(G)^{\otimes r}$ for $r \neq 1$. Thus $A(G) = \text{Im}(\beta_G)$.

Remark 4.3. Let $(T, T_0, \alpha)$ be a cohomological duality triple, let $\pi: G_1 \to G_2$ be an epimorphism of profinite groups, and suppose that there are covers $S \to G_i$, $i = 1, 2$, which commute with $\pi$. Then $\pi$ induces an isomorphism $G_1/T(G_1) \cong G_2/T(G_2)$. As $T(G_i) \leq G_i^{(2)}$, the induced map $H^\otimes(G_2) \to H^\otimes(G_1)$ is therefore an isomorphism. Hence $\pi^+_2(A(G_2)) = A(G_1)$.

For a cohomological duality triple $(T, T_0, \alpha)$ and for $r \geq 0$, $t \geq 1$, let $C_{r,t}(G)$ be the $\mathbb{Z}/q$-submodule of $H^1(G)^{\otimes r}$ generated by all its elements $\psi_1 \otimes \cdots \otimes \psi_t$, such that $\alpha_G(\psi_{i_1} \otimes \cdots \otimes \psi_{i_t}) = 0$ for some $1 \leq i_1 < \cdots < i_t \leq r$. Thus $C_{r,t}(G) = 0$ for $r < t$. We define $H^t_{r,\alpha}(G) = H^1(G)^{\otimes r}/C_{r,t}(G)$ and a graded ring $H^*_{r,\alpha}(G) = \bigoplus_{t=0}^\infty H^t_{r,\alpha}(G)$. In particular, $H^0_{1,\alpha}(G) = \mathbb{Z}/q$. Since $\alpha$ is a natural transformation, $H^*_{r,\alpha}$ is a functor. Note that $\alpha_G$ induces a homomorphism $\bar{\alpha}_G^t: H^t_{1,\alpha}(G) \to A(G)$.

Examples 4.4. (1) In Example 4.2(1), $H^*_1{\alpha}(G) = H^*G(G)$. The map $\bar{\alpha}_G^2: H^2_{1,\alpha}(G) \to A(G) = H^2(G)_{\text{dec}}$ is surjective, and is injective if and only if $H^*(G)$ is 2-quadratic.

(2) In Example 4.2(2), $H^1_{1,\alpha}(G) = H^\text{Bock}_{1,\alpha}(G)$ and $H^*_2{\alpha}(G) = H^*G(G)$.

(3) In Example 4.2(3), $H^1_{1,\alpha}(G) = H^\text{Bock}_{1,\alpha}(G)$ and the map $\bar{\alpha}_G^1: H^1_{1,\alpha}(G) = H^1(G)/\text{Ker}(\beta_G) \to A(G) = \text{Im}(\beta_G)$ is an isomorphism.

5. Quotients that determine cohomology

This section is devoted to proving Theorem A. More generally, let $(T, T_0, \alpha)$ be a cohomological duality triple. We show that, assuming the existence of a cover $S \to G$, the quotient $G/T_0(G)$ determines the graded rings $H^*_{t,\alpha}(G)$, $t \geq 1$, and is in fact the minimal such quotient:

Theorem 5.1. Assume that there is a cover $S \to G$. Let $N$ be a normal subgroup of $G$ contained in $T(G)$, and consider the following conditions:

(a) $N \leq T_0(G)$;
(b) $\inf G: A(G/N) \to A(G)$ is an isomorphism;
(c) $\inf G: H^*_{t,\alpha}(G/N) \to H^*_{t,\alpha}(G)$ is an isomorphism for every $t$.

Then (a)$\iff$(b)$\iff$(c). Moreover, if there exist $r \geq 1$ such that $\bar{\alpha}_G^r: H^r_{r,\alpha}(G) \to A(G)$ is injective and $\bar{\alpha}_G^r: H^r_{r,\alpha}(G/N) \to A(G/N)$ is surjective, then (a)–(c) are equivalent.

Before we proceed with the proof of Theorem 5.1 we apply it to Examples 4.2 to deduce the following special cases.

Corollary 5.2. Let $G$ be a profinite group with $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some $I$. Let $N$ be a normal subgroup of $G$ contained in $G^{(2)}$.

(1) If $H^*(G)$ is 2-quadratic, then the following conditions are equivalent:

(a) $N \leq G^{(3)}$;
(b) $\inf G: H^2(G/N)_{\text{dec}} \to H^2(G)_{\text{dec}}$ is an isomorphism;
(c) $\inf G: H^*(G/N)_{\text{dec}} \to H^*(G)_{\text{dec}}$ is an isomorphism.
Theorem A.

Proof of Theorem A.

Let

\[ \text{with exact rows} \]

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(b).

1(b), it suffices to show that 1(b) implies 1(c). Indeed, by Theorem 5.1 and 1(b), elements 4.4, except for the equivalence with (1)(c). Since 1(c) trivially implies everything follows directly from Theorem 5.1, using Example 4.1 and Ex-

Proof.

Proof.

(a)

Since \[ G \]

morphism \[ H \]

Hence so is \[ \text{the square below is commutative:} \]

\[ \begin{array}{c}
\pi \\
\downarrow \\
\pi \\
\end{array}
\]

\[ \begin{array}{c}
H^*(G/N) \rightarrow \omega_G \\
\downarrow \\
H^*(G/N)_{\text{dec}} \rightarrow \omega_G \\
\end{array}
\]

Since \[ \omega_G \]

is by assumption bijective, the lower inflation is bijective. \( \square \)

Proof of Theorem A. In view of Theorem 5.1 using Example 4.1 and Examples 4.4, except for the equivalence with (1)(c). Since 1(c) trivially implies 1(b), it suffices to show that 1(b) implies 1(c). Indeed, by Theorem 5.1 and 1(b), \[ \text{inf}_G: H^*_t(G/N) \rightarrow H^*_t(G) \]

is an isomorphism for every \( t \). For \( t = 2 \) this means that \[ \text{inf}_G: H^*(G/N) \rightarrow \hat{H}^*(G) \]

is a bijection. Using the functoriality of \( \omega \) we see that the square below is commutative:

\[ \begin{array}{c}
H^*(G/N) \rightarrow \hat{H}^*(G) \\
\downarrow \\
H^*(G/N)_{\text{dec}} \rightarrow \hat{H}^*(G)_{\text{dec}} \\
\end{array}
\]

Note that in (1) the equivalence (a)\( \Leftrightarrow \) (b) holds even without the 2-quadraticness assumption.

For the proof of Theorem 5.1, we first show:

Proposition 5.3. Let \( \pi_i: S \rightarrow G_i, i = 1, 2 \), be covers and \( \pi: G_1 \rightarrow G_2 \) an epimorphism with \( \pi \circ \pi_1 = \pi_2 \). The following conditions are equivalent:

(a) the induced map \( G_1/T_0(G_1) \rightarrow G_2/T_0(G_2) \) is an isomorphism;
(b) \( \text{Ker}(\pi) \leq T_0(G_1) \);
(c) the induced map \( A(\pi): A(G_2) \rightarrow A(G_1) \) is an isomorphism;
(d) the induced map \( A(\pi): A(G_2) \rightarrow A(G_1) \) is a monomorphism.

Proof. (a)\( \Rightarrow \) (b), (c)\( \Rightarrow \) (d): Trivial.

(b)\( \Rightarrow \) (a): By (A3), \( T_0(G_2) = \pi(T_0(G_1)) \). Hence the kernel of the induced epimorphism \( G_1/T_0(G_1) \rightarrow G_2/T_0(G_2) \) is \( \text{Ker}(\pi)T_0(G_1)/T_0(G_1) \), which is trivial by (b).

(d)\( \Rightarrow \) (a): The map \( \tilde{\pi}: G_1/T(G_1) \rightarrow G_2/T(G_2) \) induced by \( \pi \) is an isomorphism. Hence so is \( \pi^*: H^1(G_2) \rightarrow H^1(G_1) \). By (A4), \( \pi \) induces a commutative diagram with exact rows

\[ \begin{array}{ccc}
0 & \rightarrow & K(G_2) \rightarrow A(G_2/T(G_2)) \rightarrow A(G_2) \\
& \downarrow \text{trg} & \downarrow A(\pi) \\
0 & \rightarrow & K(G_1) \rightarrow A(G_1/T(G_1)) \rightarrow A(G_1) \\
\end{array} \]
By the assumptions and the snake lemma, the left vertical map is an isomorphism. Passing to duals using Proposition 5.3(a), we obtain that the induced map $T(G_1)/T_0(G_1) \to T(G_2)/T_0(G_2)$ is an isomorphism. Now apply the snake lemma for the commutative diagram with exact rows:

$$
\begin{array}{c}
1 \longrightarrow T(G_1)/T_0(G_1) \longrightarrow G_1/T_0(G_1) \longrightarrow G_1/T(G_1) \longrightarrow 1 \\
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
1 \longrightarrow T(G_2)/T_0(G_2) \longrightarrow G_2/T_0(G_2) \longrightarrow G_2/T(G_2) \longrightarrow 1.
\end{array}
$$

(a)$\implies$(c): Consider the commutative diagram

$$
\begin{array}{c}
A(G_2/T(G_2)) \xrightarrow{\inf} A(G_2/T_0(G_2)) \xrightarrow{\inf} A(G_2) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \\
A(G_1/T(G_1)) \xrightarrow{\inf} A(G_1/T_0(G_1)) \xrightarrow{\inf} A(G_1),
\end{array}
$$

where the vertical maps are induced by $\pi$. (a) implies that the middle vertical map is an isomorphism. Since $T_0(G_1) \leq T(G_i) \leq G_i^{(2)}$, the inflations $H^1(G_i/T(G_i)) \to H^1(G_i/T_0(G_i)) \to H^1(G_i)$ are isomorphisms, $i = 1, 2$. Hence the horizontal inflation maps are surjective. By Proposition 3.4, in each row the kernel of the left inflation map equals the kernel of the composed inflation map. It follows that $A(\pi)$ is an isomorphism.

\begin{proof}
Since the induced map $G_1^{(2)} \to G_2^{(2)}$ is an isomorphism, so is $\pi^*: H^1(G_2) \to H^1(G_1)$, and we obtain a commutative square

$$
\begin{array}{c}
H^1(G_2) \otimes \pi^* \rightarrow H^1(G_1) \otimes \pi^* \\
\alpha_{G_2} \quad \quad \quad \quad \quad \quad \quad \quad \alpha_{G_1} \\
A(G_2) \xrightarrow{A(\pi)} A(G_1).
\end{array}
$$

Assuming (d), $(\pi^*_1) \otimes t$ maps $C_{r,t}(G_2)$ bijectively onto $C_{r,t}(G_1)$ for every $t \geq 1$, and therefore $\pi^*: H^r_{r,\alpha}(G_2) \to H^r_{r,\alpha}(G_1)$ is an isomorphism.

For the last assertion, assume (e) and consider the commutative square

$$
\begin{array}{c}
H^r_{r,\alpha}(G_2) \xrightarrow{\pi^*} H^r_{r,\alpha}(G_1) \\
\alpha_{G_2} \quad \quad \quad \quad \quad \quad \quad \quad \alpha_{G_1} \\
A(G_2) \xrightarrow{A(\pi)} A(G_1).
\end{array}
$$

The assumptions imply that $A(\pi)$ is injective.
\end{proof}

**Proposition 5.4.** In the setup of Proposition 5.3, conditions (a)–(d) imply:

(e) $\pi$ induces for every $t \geq 1$ an isomorphism $\pi^*: H^r_{t,\alpha}(G_2) \to H^r_{t,\alpha}(G_1)$.

Moreover, if there exists $r \geq 1$ with $\bar{\alpha}_{G_1, t}^r: H^r_{r,\alpha}(G_1) \to A(G_1)$ injective and $\bar{\alpha}_{G_2, t}^r: H^r_{r,\alpha}(G_2) \to A(G_2)$ surjective, then (e) for $t = r$ is equivalent to (a)–(d).
Proof of Theorem C. As $N \leq T(G)$, (A3) gives $T(G/N) = T(G)/N$. Therefore the composed map $S \to G \to G/N$ is a cover. Now apply Propositions 5.3 and 5.4 for the projection $\pi: G \to G/N$.

6. ISOMORPHISMS

We now apply the results of §5 to the case of pro-$p$ groups.

Proposition 6.1. Let $(T, T_0, \alpha)$ be a cohomological duality triple. Let $\pi_i: S \to G_i$, $i = 1, 2$, be covers and $\pi: G_1 \to G_2$ an epimorphism of pro-$p$ groups with $\pi \circ \pi_1 = \pi_2$. Suppose that $A(G_2) = H^2(G_2)$. Then $\pi$ is an isomorphism if and only if the induced map $G_1/T_0(G_1) \to G_2/T_0(G_2)$ is an isomorphism.

Proof. The “only if” part follows from (A3). For the “if” part, recall that by [Ser65, Lemma 2], $\pi$ is an isomorphism if and only if $\pi_2^*: H^r(G_2) \to H^r(G_1)$ is an isomorphism for $r = 1$ and a monomorphism for $r = 2$. As $\pi_1^*$ commutes with the isomorphisms $H^1(G_i) \to H^1(S)$, $i = 1, 2$, it is also an isomorphism. Since the induced map $G_1/T_0(G_1) \to G_2/T_0(G_2)$ is an isomorphism and by Proposition 5.3 $\pi_2^*: A(G_2) = H^2(G_2) \to H^2(G_1)$ is a monomorphism. Hence $\pi$ is an isomorphism.

We deduce the following strengthening of [CEM12, Remark 6.4, Th. D] (which deal with the quotients $G_i^{[3]}$):

Corollary 6.2. Let $\pi: G_1 \to G_2$ be an epimorphism of pro-$p$ groups inducing an isomorphism $\pi^{[3]}: G_1^{[3]} \to G_2^{[3]} \cong (\mathbb{Z}/q)^I$ for some $I$ and such that $H^2(G_2) = H^2(G_2)_{\text{dec}}$. Then $\pi$ is an isomorphism if and only if the induced map $\pi^{[3]}: (G_1)[3] \to (G_2)[3]$ is an isomorphism.

Proof. Choose bases (i.e., generating subsets converging to 1) $\tilde{Z}_i$ of $G_i^{[2]}$, $i = 1, 2$, such that $\pi^{[3]}(\tilde{Z}_1) = \tilde{Z}_2$. Lift $\tilde{Z}_1$ to a subset $Z_1$ of $G_1$, and let $Z_2 = \pi(Z_1)$. By the Frattini argument, $Z_1, Z_2$ generate $G_1, G_2$, respectively. Let $S$ be a free pro-$p$ group with basis $Z_1$. Let $\pi_1: S \to G_1$ be the natural epimorphism, let $\pi_2 = \pi \circ \pi_1$, and note that $\pi_1, \pi_2$ are covers. Now take the triple of Example 4.2(1) and apply Proposition 6.1.

Proof of Theorem C. In the first equivalence, the “only if” part is immediate. For the “if” part note that, by Remark 2.6, $G_{F_1}(p)^{[2]} \cong G_{F_2}(p)^{[2]} \cong (\mathbb{Z}/q)^I$ for some set $I$. Hence we may apply Corollary 6.2.

For the second equivalence, consider the cohomological duality triple of Example 4.2(1), and let $G_i = G_{F_i}(p)$, $i = 1, 2$. Then $\alpha_{G_1}^2: H^2_{G_1}(G_1) \to H^2_{G_1}(G_1)$ is injective and $\alpha_{G_2}^2: H^2_{G_2}(G_2) \to H^2_{G_2}(G_2)$ is surjective (see Example 4.4(1)). Hence we may apply Proposition 5.4 with $r = 2$ to conclude that the induced map $\pi^{[3]}: (G_1)[3] \to (G_2)[3]$ is an isomorphism if and only if the map $\pi^*: H^2_{G_2}(G_2) \to H^2_{G_2}(G_2)$ is surjective (see Example 4.4(1)). Hence we may apply Corollary 6.2(1) and the fact that $H^*(G_i)$ is quadratic.

Next Corollary 5.2(1) and Remark 2.6 give the following refinement of [CEM12, Prop. 9.1].

Corollary 6.3. Let $G_1, G_2$ be profinite groups such that $(G_1)[3] \cong (G_2)[3]$ but $H^*(G_1) \not\cong H^*(G_2)$. Then at most one of $G_1, G_2$ can be isomorphic to the maximal pro-$p$ Galois group $G_F(p)$ of a field $F$ containing a root of unity of order $q$. 
As in [CEM12 §9], Corollary 6.3 can be used to show that various pro-
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transgression \[\text{EMII} (2.2)\], they commute. We get a commutative diagram

\[
\begin{array}{c}
K(S) \downarrow \trg & \longrightarrow & H^1(T(S))^S \downarrow \res & \longrightarrow & H^1(R)^S \\
\trg & \downarrow & \trg & \downarrow & \trg \\
A(S/T(S)) \downarrow \inf & \longrightarrow & H^2(S/T(S)) \downarrow \inf & \longrightarrow & H^2(G)
\end{array}
\]

where the left isomorphism is by (A4). Let \(g\) be the composite map \(K(S) \to H^2(G)\) arising from this diagram. Let \(\text{Ker}(g)\) denote the annihilator of \(\text{Ker}(g)\) in \(T(S)/T_0(S)\) under the lower pairing of (7.1).

**Lemma 7.1.** \(G/T_0(G) \cong (S/T_0(S))/\text{Ker}(g)\).

**Proof.** By (7.3), \(\text{Ker}(g)\) is the kernel of \(\text{res}_R: K(S) \to H^1(R)^S\). By (7.1), \(\text{Ker}(g)\) is isomorphic to the kernel of the induced epimorphism \(S/T_0(S) \to G/T_0(G)\).

Given covers \(S \to G_i, i = 1, 2\), we have isomorphisms \(S^{[2]} \to G_i^{[2]}\). They induce isomorphisms \(H^1(G_i) \cong H^1(S)\) and \(H^\otimes_2(G_1) \cong H^\otimes_2(G_2)\).

**Theorem 7.2.** Let \((T,T_0,\alpha)\) be a cohomological duality triple. Let \(\pi_i: S \to G_i\) be covers, \(i = 1, 2\), and \(\sigma: H^\otimes_2(G_1) \to H^\otimes_2(G_2)\) the induced isomorphism. Suppose that there is a monomorphism \(\tau: H^2(G_1) \to H^2(G_2)\) with \(\alpha_{G_2} \circ \sigma = \tau \circ \alpha_{G_1}\). Then there is an isomorphism \(G_1/T_0(G_1) \cong G_2/T_0(G_2)\) compatible with \(\pi_i, i = 1, 2\).

**Proof.** For \(i = 1, 2\) there is a commutative diagram

\[
\begin{array}{c}
H^\otimes_2(S/T(S)) \downarrow \alpha_{S/T(S)} & \sim & H^\otimes_2(G_i/T(G_i)) \downarrow \sim & \inf & \longrightarrow & H^\otimes_2(G_i) \\
& \alpha_{G_i/T(G_i)} & \downarrow & \alpha_{G_i} & \downarrow \\
A(S/T(S)) \sim & \longrightarrow & A(G_i/T(G_i)) \longrightarrow & \inf & \longrightarrow & A(G_i).
\end{array}
\]

Define a homomorphism \(\hat{g}_i: K(S) \to H^2(G_i)\) as above.

Given \(\gamma \in H^\otimes_2(S/T(S))\) let \(\hat{g}_i\) be the corresponding element in \(H^\otimes_2(G_i)\). Then \(\gamma\) maps trivially to \(A(G_i)\) if and only if \(\alpha_{G_1}(\hat{g}_i) = 0\). Our assumption implies that \(\alpha_{G_1}(\hat{g}_i) = 0\) if and only if \(\alpha_{G_2}(\hat{g}_i) = 0\). Consequently, the kernels of \(\inf: A(S/T(S)) \to A(G_i) \subseteq H^2(G_i), i = 1, 2\), coincide. Their preimages in \(K(S)\) under transgression are \(\text{Ker}(g_i), i = 1, 2\) (see (7.3)), which therefore also coincide. Now use Lemma (7.1).

Applying this to Examples (1.2) we obtain:

**Corollary 7.3.** Assume that \(G^{[2]} \cong (\mathbb{Z}/q)^I\) for some \(I\).

1. \(G^{[2]}\) and \(\cup: H^1(G) \times H^1(G) \to H^2(G)\) determine \(G^{[3]} = G/G(3)\).
2. \(G^{[2]}, \beta_G,\) and \(\cup: H^1(G) \times H^1(G) \to H^2(G)\) determine \(G^{[3]} = G/G(3)\).
3. \(G^{[2]}\) and \(\beta_G\) determine \(G/G^q[G, G]\).

**Proof of Theorem B.** Use Theorem (2.4) and part (1) of Corollary (7.3).
8. Presentations

Let \((T, T_0, \alpha)\) again be a cohomological duality triple. We use the techniques of the previous section to characterize the surjectivity of \(\alpha_G\) in terms of group presentations. Again let \(\pi : S \to G\) be a cover and \(R = \text{Ker}(\pi)\). Note that \(R^q[R, S] \leq R \cap T_0(S)\), by (A2).

**Theorem 8.1.** There is a natural duality between \((R \cap T_0(S))/R^q[R, S]\) and the cokernel of \(\text{inf}_G : A(G/T(G)) \to H^2(G)\).

**Proof.** The induced map \(S/T(S) \to G/T(G)\) is an isomorphism. From [7.3] we obtain a commutative diagram

\[
\begin{array}{ccc}
K(S) & \xrightarrow{\text{res}_R} & H^1(R)^S \\
\downarrow \text{trg} & & \downarrow \text{trg} \\
A(S/T(S)) & \xrightarrow{\text{inf}} & H^2(G).
\end{array}
\]

The right transgression maps \(\text{Coker}(\text{res}_R)\) isomorphically onto the cokernel of \(\text{inf}_G : A(G/T(G)) \to H^2(G)\). Further, by [4.1], \(\text{Coker}(\text{res}_R)\) is dual to \(\text{Ker}(\iota) = (R \cap T_0(S))/R^q[R, S]\).

**Corollary 8.2.** \(\alpha_G\) is onto \(H^2(G)\) if and only if \(R^q[R, S] = R \cap T_0(S)\).

**Proof.** The surjectivity of \(\alpha_G\) is equivalent to the surjectivity of the inflation \(\text{inf}_G : A(G/T(G)) \to H^2(G)\). Now use Theorem 8.1.

Applying this for Examples [4.2] we deduce:

**Examples 8.3.** Assume that \(G^{[2]} \cong (\mathbb{Z}/q)^I\) for some \(I\).

1. \(H^2(G) = H^2(G)_{\text{dec}}\) if and only if \(R^q[R, S] = R \cap S^{(3)}\).
2. \(H^2(G) = H^2(G)_{\text{dec}} + \text{Im}(\beta_G)\) if and only if \(R^q[R, S] = R \cap S^{(3)}\) (compare also [CEM12, Th. 7.1]).
3. \(H^2(G) = \text{Im}(\beta_G)\) if and only if \(R^q[R, S] = R \cap S^{q^2}[S, S]\).

By Theorem 2.4 if \(G = G_F\) is the absolute Galois group of a field \(F\) containing a root of unity of order \(q\), then (1), and therefore (2), is valid.

9. \(T_0(G)\) as an Intersection

Let \((T, T_0, \alpha)\) again be a cohomological duality triple. In this section we present \(T_0(G)\) as the intersection of all open normal subgroups \(M\) of the profinite group \(G\) with \(G/M\) contained in a certain list \(\mathcal{L}(G)\) of finite groups. In all our main examples and assuming, say, that \(G\) is an absolute Galois group of a field containing a root of unity of order \(p\), the list \(\mathcal{L}(G)\) is finite and explicit and does not depend on \(G\). In particular, this will imply Theorem D.

Following [EM11], we say that \(G\) has **Galois relation type** if

1. \(G^{[2]} \cong (\mathbb{Z}/q)^I\) for some set \(I\);
2. there exists \(\xi \in H^1(G)\) with \(\beta_G(\psi) = \psi \cup \xi\) for every \(\psi \in H^1(G)\);
3. the kernel of \(\text{inf} : H^2(G^{[2]})_{\text{dec}} \to H^2(G)\) is generated by cup products \(\psi \cup \psi', \psi, \psi' \in H^1(G^{[2]})\).

By Theorem 2.4 and Lemma 2.2, this holds when \(G = G_F\) is the absolute Galois group of a field \(F\) containing a root of unity of order \(q\) (this was earlier shown in [EM11, Prop. 3.2]).
**Definition 9.1.** A special set for the profinite group $G$ with respect to $(T, T_0, \alpha)$ will be a set $\Sigma$ of pairs $(G, \bar{\varphi})$ such that $G$ is a finite quotient of $G[2]$, $\bar{\varphi} \in H^{\otimes^a}(G)$, and the kernel of $\inf_G : A(G/T(G)) \to H^2(G)$ is generated by the elements $\alpha_{G/T(G)}(\inf_{G/T(G)}(\bar{\varphi}))$ with $(G, \bar{\varphi}) \in \Sigma$.

**Examples 9.2.** Let $G$ be a profinite group of Galois relation type.

1. Consider the cohomological duality triple of Example 4.2(1). Take $\psi, \psi' \in H^1(G[2])$ such that $\psi \cup \psi' = \alpha_{G[2]}(\psi \otimes \psi') \neq 0$ in the kernel of $\inf : H^2(G[2])_{\text{dec}} \to H^2(G)$. Let $G = G[2]/(\text{Ker}(\psi) \cap \text{Ker}(\psi'))$ and take $\bar{\psi}, \bar{\psi}' \in H^1(G)$ with $\psi = \inf_{G[2]}(\bar{\psi})$, $\psi' = \inf_{G[2]}(\bar{\psi}')$. By (iii), the set of all such pairs $(G, \bar{\psi} \otimes \bar{\psi}')$ is a special set for $G$.

2. Consider the triple of Example 4.2(2). By (i) and Lemma 2.3

$$H^2(G[2]) = \text{Im}(\beta_{G[2]}) + H^2(G[2])_{\text{dec}} = A(G/T(G)).$$

We first claim that $K' = \text{Ker}(\inf_G : H^2(G[2]) \to H^2(G))$ is generated by elements of the form $\alpha_{G[2]}(-\psi \oplus (\psi \otimes \psi')) = -\beta_{G[2]}(\psi) + \psi \cup \psi'$, where $\psi, \psi' \in H^1(G[2])$, $\psi \neq 0$, and $-\psi \oplus (\psi \otimes \psi')$ is considered as an element of $H^{\otimes^a}(G[2])$ (compare [EM11] Prop. 4.3). Indeed, for $\xi$ as in (ii) we take $\xi_0 \in H^1(G[2])$ with $\xi = \inf_G(\xi_0)$. Let $\theta = \beta_{G[2]}(\eta) + \sum_{i=1}^{n} \psi_i \cup \psi_i' \in K'$, where $\eta, \psi_i, \psi_i' \in H^1(G[2])$. Then also $\eta \cup \xi_0 + \sum_{i=1}^{n} \psi_i \cup \psi_i'$ is in $K'$, and by (iii), it can be written as $\sum_{j=1}^{m} \chi_j \cup \chi_j'$ with $\chi_j, \chi_j' \in H^1(G[2])$ and $\chi_j \cup \chi_j' \in K'$ for each $j$. Hence

$$\theta = \beta_{G[2]}(\eta) + (-\eta) \cup \xi_0 + \sum_{j=1}^{m} \chi_j \cup \chi_j'$$

$$= (\beta_{G[2]}(\eta) + (-\eta) \cup \xi_0) + \sum_{j=1}^{m} (-\beta_{G[2]}(\chi_j) + \chi_j \cup (\chi_j' + \xi_0))$$

$$+ \sum_{j=1}^{m} (\beta_{G[2]}(\chi_j) + (-\chi_j) \cup \xi_0),$$

and this sum is in $K'$, proving the claim.

Now given $-\psi \oplus (\psi \otimes \psi')$ as above, let $G = G[2]/(\text{Ker}(\psi) \cap \text{Ker}(\psi'))$ and take $\bar{\psi}, \bar{\psi}' \in H^1(G)$ with $\psi = \inf_{G[2]}(\bar{\psi})$, $\psi' = \inf_{G[2]}(\bar{\psi}')$. The set $\Sigma$ of all pairs $(G, -\psi \oplus (\psi \otimes \psi'))$ is special for $G$.

3. Consider the cohomological duality triple of Example 4.2(3). Trivially, the kernel of $\inf_G : \text{Im}(\beta_{G[2]}) \to \text{Im}(\beta_G)$ is generated by elements $\beta_{G[2]}(\psi)$, with $\psi \in H^1(G[2])$. For such $\psi$ let $G = G[2]/\text{Ker}(\psi)$ and take $\bar{\psi} \in H^1(G)$ with $\psi = \inf(\bar{\psi})$. The set $\Sigma$ of all pairs $(G, \bar{\psi})$ is special for $G$.

For the rest of this section we assume that $q = p$ is prime. When $p \neq 2$ let $H_{p^3}$ be the Heisenberg group of order $p^3$ (see the Introduction), and let

$$M_{p^3} = \langle r, s \mid r^{p^2} = s^p = 1, r^p = [r, s] \rangle$$

be the unique nonabelian group of odd order $p^3$ and exponent $p^2$. Let $D_4$ be the dihedral group of order 8.
Given a special set $\Sigma$ for $G$ with respect to $(T, T_0, \alpha)$, we choose for every $(\bar{G}, \bar{\varphi}) \in \Sigma$ a central extension

\begin{equation}
\omega : 0 \to \Bbb{Z}/p \to B \to \bar{G} \to 1
\end{equation}

(9.1)

corresponding to $\alpha_{\bar{G}}(\bar{\varphi}) \in H^2(\bar{G})$. Note that $B$ depends only on $\alpha_{\bar{G}}(\bar{\varphi})$. Let $\mathcal{L}(G)$ be the class of all (isomorphism classes of) finite groups $B$ arising in this way.

**Examples 9.3.** Suppose that $G$ has Galois relation type.

(1) Let $(T, T_0, \alpha)$ be as in Example 4.2(1) and let $\Sigma$ be the special set for $G$ as in Example 9.2(1). Consider $(G, \bar{\psi} \otimes \bar{\psi}') \in \Sigma$. Thus $\bar{\psi}, \bar{\psi}' \in H^1(\bar{G}), \bar{\psi} \cup \bar{\psi}' \neq 0,$ and $\text{Ker}(\bar{\psi}) \cap \text{Ker}(\bar{\psi}') = \{1\}$. Let

\begin{equation}
\omega : 0 \to \Bbb{Z}/p \to B \to \bar{G} \to 1
\end{equation}

be the central extension corresponding to $\bar{\psi} \cup \bar{\psi}'$.

When $p \neq 2$, $\bar{\psi}, \bar{\psi}'$ are $\Bbb{F}_p$-linearly independent, $\bar{G} \cong (\Bbb{Z}/p)^2$ and $B \cong H^{p,3}$ \[\text{EM11 Prop. 9.1(f)}.\] Hence $\bar{L}(G) = \{H^{p,3}\}$.

Next let $p = 2$. When $\bar{\psi} = \bar{\psi}'$ we have $G \cong \Bbb{Z}/2$ and $B \cong \Bbb{Z}/4$ \[\text{EM11 Prop. 9.1(c)}.\] Otherwise $\bar{\psi}, \bar{\psi}'$ are $\Bbb{F}_p$-linearly independent, $\bar{G} \cong (\Bbb{Z}/2)^2$ and $B \cong D_4$ \[\text{EM11 Prop. 9.1(e)}.\] We conclude that $\mathcal{L}(G) = \{\Bbb{Z}/4, D_4\}$.

(2) Let $(T, T_0, \alpha)$ be as in Example 4.2(2) and let $\Sigma$ be the special set for $G$ as in Example 9.2(2). Consider $(G, -\bar{\psi} \otimes (\bar{\psi} \otimes \bar{\psi}')) \in \Sigma$. Thus $\bar{\psi}, \bar{\psi}' \in H^1(\bar{G}), \bar{\psi} \neq 0,$ and $\text{Ker}(\bar{\psi}) \cap \text{Ker}(\bar{\psi}') = \{1\}$. Let

\begin{equation}
\omega : 0 \to \Bbb{Z}/p \to B \to \bar{G} \to 1
\end{equation}

be the central extension corresponding to $-\beta_{\bar{G}}(\bar{\psi}) + \bar{\psi} \cup \bar{\psi}'$.

When $p \neq 2$ and $\bar{\psi}, \bar{\psi}'$ are $\Bbb{F}_p$-linearly independent, $\bar{G} \cong (\Bbb{Z}/p)^2$ and $B \cong M_{p,3}$ \[\text{EM11 Prop. 9.4}.\] When $p \neq 2$ and $\bar{\psi}, \bar{\psi}'$ are $\Bbb{F}_p$-linearly dependent, $\bar{G} \cong \Bbb{Z}/p$ and $B \cong \Bbb{Z}/p^2$ \[\text{EM11 Cor. 9.3}.\]

(3) Let $(T, T_0, \alpha)$ be as in Example 4.2(3), and take $\Sigma$ as in Example 9.2(3). By \[\text{EM11 Prop. 9.2},\] the central extension corresponding to $(G, \bar{\psi}) \in \Sigma$ is

\begin{equation}
0 \to \Bbb{Z}/p \to \Bbb{Z}/p^2 \to \bar{G} \cong (\Bbb{Z}/p) \to 1,
\end{equation}

so $\mathcal{L}(G) = \{\Bbb{Z}/p^2\}$.

**Theorem 9.4.** Suppose that $\Sigma$ is a special set for the profinite group $G$ with respect to the cohomological duality triple $(T, T_0, A)$. Then

\[T_0(G) = T(G) \cap \bigcap \{M \leq G \mid G/M \in \mathcal{L}(G)\}.\]

**Proof.** Let $(G, \bar{\varphi}) \in \Sigma$ and let $\omega$ be a central extension as in \[\text{EM11}.\] Since $\bar{G}$ is a quotient of $G^{[2]}$ it is also a quotient of $G/T(\bar{G})$. Let $\text{pr} : B \times_{\bar{G}} (G/T(\bar{G})) \to B$ be the projection from the fibred product. The central extension

\begin{equation}
\hat{\omega} : 0 \to \Bbb{Z}/p \to B \times_{\bar{G}} (G/T(\bar{G})) \to G/T(\bar{G}) \to 1
\end{equation}

then corresponds to $\inf_{G/T(\bar{G})}(\alpha_{\bar{G}}(\bar{\varphi}))$ \[\text{EM11 Remark 6.1}.\]
Next let $A_0$ be the set of all elements $\alpha_{G/T(G)}(\inf_{G/T(G)}(\varphi))$, where $(\tilde{G}, \varphi) \in \Sigma$. Thus $A_0$ generates the kernel of $\inf : A(G/T(G)) \to H^2(G)$. Consider homomorphisms $\Psi : G \to B$, $\hat{\Psi} : G \to B \times \tilde{G}(G/T(G))$ as in the following diagram. For every $\Psi$ making the lower triangle commutative there is a unique $\hat{\Psi}$ making the upper triangle commutative with $\Psi = \text{pr} \circ \hat{\Psi}$:

\[
\begin{array}{ccccccccc}
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & \mathbb{Z}/p & \rightarrow & B \times \tilde{G}(G/T(G)) & \rightarrow & G/T(G) & \rightarrow & 1 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & \mathbb{Z}/p & \rightarrow & B & \rightarrow & G & \rightarrow & 1.
\end{array}
\]

Note that $\text{Ker}(\hat{\Psi}) = T(G) \cap \text{Ker}(\Psi)$. Furthermore, if $\pi$ maps a proper subgroup $B_0$ of $B$ onto $\tilde{G}$, then $\pi |_{B_0} : B_0 \to \tilde{G}$ is an isomorphism. Since $\omega$ is nonsplit, $\Psi$ is therefore surjective. Now by condition (f) of Proposition 3.2, $T_0(G) = \bigcap \text{Ker}(\hat{\Psi}) = T(G) \cap \bigcap \text{Ker}(\Psi)$ for all $\Psi$ as above ($T_0(G) = T(G)$ when there is no such $\Psi$).

The kernels $\text{Ker}(\Psi)$ are just the normal subgroups $M$ of $G$ such that $G/M \in \mathcal{L}(G)$.

We now apply Theorem 9.4 to Examples 4.2 to derive concrete presentations of the canonical subgroups discussed so far as intersections. The first example below contains in particular the proof of Theorem D.

**Examples 9.5.** Suppose that $G$ has Galois relation type.

1. Let $(T, T_0, \alpha)$ be as in Example 4.2(1) and let $\Sigma$ be the special set for $G$ as in Example 9.2(1). By Example 9.3(1), $\mathcal{L}(G) = \{H_p^3\}$ when $p \neq 2$ and $\mathcal{L}(G) = \{\mathbb{Z}/4, D_4\}$ when $p = 2$. In the first case we obtain from Theorem 9.4 that

$$G_{(3)} = G^{(2)} \cap \bigcap \{M \trianglelefteq G \mid G/M \cong H_p^3\}.$$  

In view of Theorem 2.4 this gives Theorem D.

In the second case $G_{(3)} = G^{(3)}$ (Remark 2.1(1)), and we obtain that

$$G_{(3)} = \bigcap \{M \trianglelefteq G \mid G/M \cong 1, \mathbb{Z}/p, H_p^3\}.$$  

This last fact was proved by Villegas [Vil88] and Mináč and Spira [MS96] Cor. 2.18 when $G$ is an absolute Galois group of a field, and in [EM11] Cor. 11.3 and Prop. 3.2 for general profinite groups of Galois relation type. Moreover, $\mathbb{Z}/2$ can be omitted from this list if $G \not\cong \mathbb{Z}/2$ [EM11] Cor. 11.4.

2. Let $(T, T_0, \alpha)$ be as in Example 4.2(2) and let $\Sigma$ be the special set for $G$ as in Example 9.2(2). We may assume that $p \neq 2$, since otherwise $G^{(3)} = G_{(3)}$, and this subgroup was described in (1). Now by Example 9.3(2), $\mathcal{L}(G) = \{M_p^3, \mathbb{Z}/p^2\}$. By [EM11] Prop. 10.2, every epimorphism $G \to \mathbb{Z}/p$ breaks via $\mathbb{Z}/p^2$ or via $M_p^3$. Consequently, Theorem 9.4 gives

$$G^{(3)} = \bigcap \{M \trianglelefteq G \mid G/M \cong 1, \mathbb{Z}/p^2, M_p^3\}.$$  

This result was earlier proved in [EM11].
\( (3) \) Let \((T, T_0, \alpha)\) be as in Example 4.2(3), and take \(\Sigma\) as in Example 4.2(3). By Example 6.3(2), \(L(G) = \{\mathbb{Z}/p^2\}\). We get the equality (already noted in Example 6.3(3))

\[
G^p [G, G] = \bigcap \{ M \leq G \mid G/M \cong \mathbb{Z}/p, \mathbb{Z}/p^2 \}.
\]

In fact, since a discrete abelian group of finite exponent is a direct sum of cyclic groups [Kap69, Th. 6], we get using Pontrjagin duality that

\[
G^p [G, G] = \bigcap \{ M \leq G \mid G/M \cong \mathbb{Z}/p^j, \ j = 0, 1, \ldots, n \}.
\]

10. Duality in free pro-p groups

In this section we prove the duality mentioned in Example 6.3(1). First we study the pairing in Proposition 3.1(b) when \(G = S\) satisfies \(H^2(S) = 0\) and when \(T = S^{(2)}\) and \(T_0 = S^{(3)}\). Given \(\sigma \in S\), we write \(\bar{\sigma}\) for its image in \(S^{(2)}\). The next two lemmas extend computations in [Lab66, §2.3], [Koc02, §7.8] and [NSW08, 3.9.13]. Let \(\delta = 1, 2\) be as in (6.11).

**Lemma 10.1.** Let \(\chi, \chi' \in H^1(S^{(2)})\) and \(\sigma, \sigma' \in S\).

(a) If there is a homomorphism \(h : S \to \mathbb{Z}/q\) with \(h(\sigma) = 1\), then \(\langle \sigma \chi S^{(3)}, \chi_{\sigma} \rangle = (q/\delta) \chi(\bar{\sigma})\chi'(\bar{\sigma}) = 0\) for \(q\) odd.

(b) \(\langle [\sigma, \sigma'] S^{(3)}, \chi \cup \chi' \rangle = \chi(\bar{\sigma})\chi'(\bar{\sigma}) - \chi(\bar{\sigma})\chi'(\bar{\sigma}')\).

**Proof.** The cohomology class \(\text{inf} \chi_{\sigma} \in H^2(S)\) is represented by the 2-cocycle \(c(\sigma, \tau) = \chi(\bar{\sigma})\chi'(\bar{\tau})\). Since \(H^2(S) = 0\), there exists an inhomogenous 1-cochain \(u : S \to \mathbb{Z}/q\) with \(\partial u = c\). Thus

\[
\chi(\bar{\sigma})\chi'(\bar{\tau}) = u(\sigma) + u(\tau) - u(\sigma \tau)
\]

for all \(\sigma, \tau \in S\). In particular, for \(\sigma \in S\) and \(\tau \in S^{(2)}\) we have \(u(\sigma \tau) = u(\sigma) + u(\tau) = u(\tau \sigma)\). It follows that for \(\tau \in S^{(2)}\) one has

\[
u(\tau) = u(\sigma^{-1}) + u(\sigma \tau) - \chi(\bar{\sigma}^{-1})\chi'(\bar{\sigma})
\]

\[
= u(\sigma^{-1}) + u(\tau \sigma) - \chi(\bar{\sigma}^{-1})\chi'(\bar{\sigma}) = u(\sigma^{-1})\tau \sigma).
\]

Thus the restriction \(v\) of \(u\) to \(S^{(2)}\) belongs to \(H^1(S^{(2)})S\). By the definition of the transgression [NSW08, Prop. 1.6.6], \(\text{trg}_{S^{(2)}} (v) = \chi \cup \chi'\). Consequently, for every \(\rho \in S^{(2)}\) we have

\[
\langle \rho S^{(3)}, \chi \cup \chi' \rangle = \langle \rho S^{(3)}, \text{trg}_{S^{(2)}} (v) \rangle = -v(\rho) = -u(\rho).
\]

(a) When \(\sigma \in S^{(2)}\) both sides are zero. So assume that \(\sigma \notin S^{(2)}\). Our assumption gives a homomorphism \(h : S \to \mathbb{Z}/q\) with \(h(\sigma) = u(\sigma)\). As \(\partial h = 0\), we may replace \(u\) by \(u - h\) to assume that \(u(\sigma) = 0\). Using (10.1) we obtain inductively that \(u(\sigma) = -(q/\delta) \chi(\bar{\sigma})\chi'(\bar{\sigma})\). It remains to observe that \((q/\delta) \equiv q/\delta \pmod{q}\).

(b) Apply (10.1) with \(\tau = 1\) to obtain \(u(1) = 0\). Apply it with \(\tau = \sigma^{-1}\) to further obtain \(u(\sigma^{-1}) + u(\sigma) = -\chi(\bar{\sigma})\chi'(\bar{\sigma})\). This gives

\[
u((\sigma^*)^{-1}) + u(\sigma') = -\chi(\bar{\sigma})\chi'(\bar{\sigma})\bar{\sigma}^{-1},
\]

\[
u(\sigma^*) = u(\sigma) + u(\sigma') - \chi(\bar{\sigma})\chi'(\bar{\sigma})
\]

\[-u(\sigma') = -u(\sigma') - u(\sigma) + \chi(\bar{\sigma})\chi'(\bar{\sigma}).
\]

Adding these equalities we obtain

\[
u((\sigma^*)^{-1}) + u(\sigma') = -\chi(\bar{\sigma})\chi'(\bar{\sigma})\bar{\sigma}^{-1} + \chi(\bar{\sigma})\chi'(\bar{\sigma}) - \chi(\bar{\sigma})\chi'(\bar{\sigma}).
\]
Hence, by (10.1) again,
\[ u([\sigma, \sigma']) = u((\sigma'\sigma)^{-1}) + u(\sigma\sigma') - \chi((\bar{\sigma}\bar{\sigma'})^{-1})\chi'(\bar{\sigma}\bar{\sigma'}) \]
\[ = \chi(\bar{\sigma}')\chi'(\bar{\sigma}) - \chi(\bar{\sigma})\chi'(\bar{\sigma}) \]
as desired. \(\square\)

**Lemma 10.2.** Let \( \chi \in H^1(S^{[2]}) \) and \( \sigma, \sigma' \in S \).

(a) If \( \chi(\bar{\sigma}) = 0 \), then \( \langle \sigma^q S^{[3]}, \beta_{S^{[2]}}(\chi) \rangle = 0 \).

(b) If \( \chi(\bar{\sigma}) = 1 \), then \( \langle \sigma^q S^{[3]}, \beta_{S^{[2]}}(\chi) \rangle = 1 \).

(c) \( \langle [\sigma, \sigma'] S^{[3]}, \beta_{S^{[2]}}(\chi) \rangle = 0 \).

Proof. Define a section \( \iota \) of the projection \( \mathbb{Z}/q^2 \to \mathbb{Z}/q \) by \( \iota(i + q\mathbb{Z}) = i + \mathbb{Z}/q^2 \) for \( 0 \leq i \leq q - 1 \). Let \( \tilde{\chi} = \iota \circ \chi \in \text{Hom}(S^{[2]}, \mathbb{Z}/q^2) \). Then the cohomology class of \( \beta_{S^{[2]}}(\chi) \) in \( H^2(S^{[2]}) \) is represented by the 2-cocycle
\[ c(\bar{\sigma}, \bar{\tau}) = \frac{1}{q}(\tilde{\chi}(\bar{\sigma}) + \tilde{\chi}(\bar{\tau}) - \tilde{\chi}(\bar{\sigma}\bar{\tau})). \]
Inflating to \( H^2(S) = 0 \), we obtain an inhomogenous 1-cochain \( u: S \to \mathbb{Z}/q \) with \( \partial u = c \). Thus
\[ u(\sigma) + u(\tau) - u(\sigma\tau) = \frac{1}{q}(\tilde{\chi}(\bar{\sigma}) + \tilde{\chi}(\bar{\tau}) - \tilde{\chi}(\bar{\sigma}\bar{\tau})) \]
for all \( \sigma, \tau \in S \). Since \( S^{[2]} \) is abelian, this implies that \( u(\sigma\tau) = u(\tau\sigma) \), whence \( u(\sigma^{-1}\tau\sigma) = u(\tau) \), for all \( \sigma, \tau \in S \). It follows that the restriction \( v \) of \( u \) to \( S^{[2]} \) belongs to \( H^1(S^{[2]})^S \). By the definition of the transgression, \( \text{trg}_{S^{[2]}}(v) \) is represented by the 2-cocycle \( \partial u = c \). Hence \( \beta_{S^{[2]}}(\chi) \equiv \text{trg}_{S^{[2]}}(v) \). Consequently, for every \( \rho \in S^{[2]} \) we have
\[ \langle \rho S^{[3]}, \beta_{S^{[2]}}(\chi) \rangle = \langle \rho S^{[3]}, \text{trg}_{S^{[2]}}(v) \rangle = -v(\rho) = -u(\rho). \]

(a) If \( \chi(\bar{\sigma}) = 0 \), then \( \tilde{\chi}(\sigma^i) = 0 \) for every \( i \geq 0 \). It follows inductively from (10.2) that \( u(\sigma^i) = iu(\sigma) \) for \( i \geq 0 \). Thus \( u(\sigma^q) = 0 \), as required.

(b) If \( \chi(\bar{\sigma}) = 1 \), then \( \tilde{\chi}(\sigma^i) = i \) for \( 0 \leq i \leq q - 1 \), while \( \tilde{\chi}(\sigma^{q-1}) = 0 \). It follows from (10.2) by induction that \( u(\sigma^i) = iu(\sigma) \) for \( 0 \leq i \leq q - 1 \) and \( u(\sigma^{q-1}) = qu(\sigma) - 1 = -1 \).

(c) As \( (\sigma, \tau) \mapsto u(\sigma\tau) \) is symmetric, \( u([\sigma, \sigma']) = 0 \) for all \( \sigma, \sigma' \in S \). \(\square\)

Next we also assume that \( S^{[2]} \cong (\mathbb{Z}/q)^I \) for some index set \( I \); e.g., \( S \) is a free profinite (or pro-p group) group. Fix a linear order \( < \) on \( I \). Choose a basis \( \bar{s}_i \), \( i \in I \), of \( S^{[2]} \), and lift it to elements \( \sigma_i, i \in I \), of \( S \). Let \( \chi_i, i \in I \), be the \( \mathbb{Z}/q \)-basis of \( H^1(S^{[2]}) \) dual to \( \sigma_i S^{(2)} \), \( i \in I \).

**Proposition 10.3.**

(a) Every \( \sigma \in S^{[2]} \) can be uniquely written as
\[ \sigma = \prod_i \sigma_i a_i q \prod_{i < j} [\sigma_i, \sigma_j] b_{ij} \pmod{S^{(3)}} \]
for some \( a_i, b_{ij} \in \mathbb{Z}/q \).

(b) The lists
\( \sigma_i S^{(3)} \), \( i \in I \), and \([\sigma_i, \sigma_j] S^{(3)} \), \( i, j \in I \), \( i < j \),
\( \beta_{S^{[2]}}(\chi_i) \), \( i \in I \), and \( \chi_i \cup \chi_j \), \( i, j \in I \), \( i < j \),
form dual bases of \( S^{[2]}/S^{(3)} \) and \( H^2(S^{[2]}) \), respectively, with respect to \( \langle \cdot, \cdot \rangle \).
Proof. (a) Use [NSW08, Prop. 3.9.13(i)] and a standard limit argument.

(b) By the definition of $S^{(2)}$, the list (i) generates $S^{(2)}/S^{(3)}$. By Lemma 10.1 and Lemma 10.2, the two lists are dual. Hence they form bases. □

**Proposition 10.4.** For $\chi \in \mathcal{H}^1(S^{(2)})$ of order $q$ one has $\chi \cup \chi = (q/\delta)\beta_S^{(2)}(\chi)$ ($= 0$ if $p > 2$).

Proof. We may assume that $\chi = \chi_k$ for some $k \in I$. For $i \in I$ Lemma 10.1(a) and Lemma 10.2(a)(b) give
\[ \langle \sigma_i^qS^{(3)}, \chi_k \cup \chi_k \rangle = (q/\delta)\chi_k(\sigma_i) = \langle \sigma_i^qS^{(3)}, (q/\delta)\beta_S^{(2)}(\chi_k) \rangle. \]
For $i, j \in I$, $i < j$, Lemma 10.1(b) and Lemma 10.2(c) give
\[ \langle [\sigma_i, \sigma_j]S^{(3)}, \chi_k \cup \chi_k \rangle = 0 = \langle [\sigma_i, \sigma_j]S^{(3)}, (q/\delta)\beta_S^{(2)}(\chi_k) \rangle. \]
The assertion now follows by duality from Proposition 10.3(b). □

From this and Proposition 10.3 we deduce the following.

**Corollary 10.5.** When $p > 2$ (resp., $p = 2$), the elements $\chi_i \cup \chi_j, i < j$ (resp. $i \leq j$) form a basis of $H^2(S^{(2)})_{\text{dec}}$.

To this end let $S$ again be a profinite group such that $H^2(S) = 0$ and $S^{(2)} \cong (\mathbb{Z}/q)^I$. Let $\chi_i, i \in I$, be bases as in the previous section. Write $S^{(2)} = \prod_{i \in I} C_i$, where $C_i = \langle \sigma_iS^{(2)} \rangle \cong \mathbb{Z}/q$. Then $\text{res}_{C_i}(\chi_i)$ generates $H^1(C_i) \cong \mathbb{Z}/q$, and $\beta_{C_i}(\text{res}_{C_i}(\chi_i))$ generates $H^2(C_i) \cong \mathbb{Z}/q$, e.g., by Corollary 10.5.

**Lemma 10.6.** Let $\alpha \in H^2(S^{(2)})$ and let $d_i, i \in I$, and $d_{ij}, i < j, i, j \in I$, be the unique elements of $\mathbb{Z}/q$ such that
\[ \alpha = \sum_i d_i \beta_S^{(2)}(\chi_i) + \sum_{i < j} d_{ij} \cdot \chi_i \cup \chi_j. \]
Then for every $k \in I$, one has $\text{res}_{C_k}(\alpha) = d_k \beta_{C_k}(\text{res}_{C_k}(\chi_k))$.

Proof. One has $\text{res}_{C_k}(\chi_i) = 0$ for $i \neq k$. Therefore
\[ \text{res}_{C_k}(\beta_{S^{(2)}}(\chi_i)) = \beta_{C_k}(\text{res}_{C_k}(\chi_i)) = 0 \]
for $i \neq k$, as well as
\[ \text{res}_{C_k}(\chi_i \cup \chi_j) = \text{res}_{C_k}(\chi_i) \cup \text{res}_{C_k}(\chi_j) = 0 \]
for all $i < j$, whence the desired equality. □

We deduce the following local-global principle for groups of the form $(\mathbb{Z}/q)^I$.

**Corollary 10.7.** There is an exact sequence
\[ 0 \to H^2(S^{(2)})_{\text{dec}} \to H^2(S^{(2)}) \xrightarrow{\prod \delta \text{res}_{C_i}} \prod C H^2(C), \]
where $C$ ranges over all cyclic subgroups of $S^{(2)}$ of order $q$.

Proof. Let $\alpha \in H^2(S^{(2)})$ and express it as in (10.3). By Corollary 10.5, $\alpha \in H^2(S^{(2)})_{\text{dec}}$ if and only if $\delta d_i \equiv 0 \pmod{q}$ for every $i \in I$. Since $\beta_{C_i}(\text{res}_{C_i}(\chi_i))$ generates $H^2(C_i)$, Lemma 10.6 shows that this is equivalent to $\delta \text{res}_{C_i}(\alpha) = 0$. It remains to note that every cyclic subgroup $C$ of $S^{(2)}$ of order $q$ occurs as $C_k = \langle \sigma_kS^{(2)} \rangle$ for some choice of $\sigma_k, \psi_k$. □
Now let $G$ be a profinite group. Given a subgroup $C$ of $G^{[2]}$, take a normal subgroup $M$ of $G$ containing $G^{(2)}$ with $C = M/G^{(2)}$. Then $H^1(G^{(2)})^G \leq H^1(G^{(2)})^M$, and by the functoriality of the transgression, there is a commutative square

\[
\begin{array}{c}
H^1(G^{(2)})^G \xrightarrow{\text{trg}_{G^{[2]}}} H^2(G^{[2]}) \\
\downarrow \quad \downarrow \text{res}_C \\
H^1(G^{(2)})^M \xrightarrow{\text{trg}_C} H^2(C).
\end{array}
\]

This makes condition (c) of the following proposition meaningful:

**Proposition 10.8.** Let $G$ be a profinite group with $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some index set $I$. The following conditions on $\psi \in H^1(G^{(2)})^G$ are equivalent:

(a) $\text{trg}_{G^{[2]}}(\psi) \in H^2(G^{[2]})_{\text{dec}}$;

(b) $\delta \text{trg}_C(\psi) = 0$ for every cyclic subgroup $C$ of $G^{[2]}$ of order $q$;

(c) for every normal subgroup $M$ of $G$ containing $G^{(2)}$ with $M/G^{(2)} \cong \mathbb{Z}/q$ there exists $\hat{\psi} \in H^1(M)$ with $\delta \psi = \text{res}_{G^{(2)}}(\hat{\psi})$;

(d) $\psi(G^{(3)}) = 0$.

**Proof.** (a)$\Leftrightarrow$(b): Since $G^{[2]} \cong S^{[2]}$ for some free pro-$p$ group $S$, this follows from Corollary 10.7 and 10.3.

(b)$\Leftrightarrow$(c): Use the 5-term sequence associated with $C = M/G^{(2)}$.

(c)$\Leftrightarrow$(d): We may assume that $G \neq \{1\}$. Then $G$ is the union of its subgroups $M$ such that $G^{(2)} \leq M$ and $M/G^{(2)} \cong \mathbb{Z}/q$. Then $M^q \leq G^{(2)}$ and there is a split extension

$$1 \to G^{(2)}/M^q \to M/M^q \to \mathbb{Z}/q \to 0.$$ 

If there is $\hat{\psi} \in H^1(M)$ with $\delta \hat{\psi} = \text{res}_{G^{(2)}}(\hat{\psi})$, then $(\delta \hat{\psi})(M^q) = \hat{\psi}(M^q) = \{0\}$. Conversely, if $\delta \psi$ vanishes on $M^q$, then it induces a homomorphism $\overline{\delta \psi} \in H^1(G^{(2)}/M^q)^M$. Since the above extension splits, $\overline{\delta \psi}$ extends to a homomorphism $M/M^q \to \mathbb{Z}/q$, so there is $\hat{\psi} \in H^1(M)$ as above.

Consequently, (c) is equivalent to $\psi(G^{[2]}) = \{1\}$. But as $\psi \in H^1(G^{(2)})^G$, one always has $\psi([G^{[2]}, G]) = \{1\}$, so $\psi(G^{(3)}) = \psi(G^{[2]}) = \{1\}$.

The equivalence of (a) and (d) implies the following.

**Corollary 10.9.** Let $G$ be a profinite group with $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some index set $I$. Then $H^2(G^{[2]})_{\text{dec}}$ is dual to $(G^{(2)}, G^{(3)})$.

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**References**

[Bec74] E. Becker, *Euklidische Körper und euklidische Hüllen von Körperrn* (German), J. Reine Angew. Math. 268/269 (1974), 41–52. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, II. MR0354625 (50 #7103)

[BT12] F. Bogomolov and Y. Tschinkel, *Introduction to birational anabelian geometry*, Current developments in algebraic geometry, Math. Sci. Res. Inst. Publ., vol. 59, Cambridge Univ. Press, Cambridge, 2012, pp. 17–63. MR2931864
[CEM12] S. K. Chebolu, I. Efrat, and J. Mináč, Quotients of absolute Galois groups which determine the entire Galois cohomology, Math. Ann. 352 (2012), no. 1, 205–221, DOI 10.1007/s00208-011-0635-6. MR2885583

[Efr14a] I. Efrat, The Zassenhaus filtration, Massey products, and representations of profinite groups, Adv. Math. 263 (2014), 389–411, DOI 10.1016/j.aim.2014.07.006. MR3239143

[Efr14b] I. Efrat, Filtrations of free groups as intersections, Arch. Math. (Basel) 103 (2014), no. 5, 411–420, DOI 10.1007/s00013-014-0701-x. MR3281289

[EM11] I. Efrat and J. Mináč, On the descending central sequence of absolute Galois groups, Amer. J. Math. 133 (2011), no. 6, 1503–1532, DOI 10.1353/ajm.2011.0041. MR2863369

[EM12] I. Efrat and J. Mináč, Small Galois groups that encode valuations, Acta Arith. 156 (2012), no. 1, 7–17, DOI 10.4064/aa156-1-2. MR2997568

[FJ05] M. D. Fried and M. Jarden, Field arithmetic, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 11, Springer-Verlag, Berlin, 2005. MR2102046 (2005k:12003)

[GS06] P. Gille and T. Szamuely, Central simple algebras and Galois cohomology, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006. MR2266528 (2007k:16033)

[Hoe68] K. Hoechsmann, Zum Einbettungsproblem (German), J. Reine Angew. Math. 229 (1968), 81–106. MR0244190 (39 #5507)

[Kap69] I. Kaplansky, Infinite abelian groups, revised edition, The University of Michigan Press, Ann Arbor, Mich., 1969. MR0233887 (38 #2208)

[Koc02] H. Koch, Galois theory of $p$-extensions, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. With a foreword by I. R. Shafarevich; Translated from the 1970 German original by Franz Lemmermeyer; With a postscript by the author and Lemmermeyer. MR1930372 (2003f:11181)

[Lab66] J. P. Labute, Demushkin groups of rank $\aleph_0$, Bull. Soc. Math. France 94 (1966), 211–244. MR0222177 (36 #5529)

[MS82] A. S. Merkurjev and A. A. Suslin, $K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 5, 1011–1046, 1135–1136. MR675529 (84i:12007)

[MS90] J. Mináč and M. Spira, Formally real fields, Pythagorean fields, $C$-fields and $W$-groups, Math. Z. 205 (1990), no. 4, 519–530, DOI 10.1007/BF02571260. MR1082872 (91m:11030)

[MS96] J. Mináč and M. Spira, Witt rings and Galois groups, Ann. of Math. (2) 144 (1996), no. 1, 35–60, DOI 10.2307/2118582. MR1405942

[MST14] J. Mináč, J. Swallow, and A. Topaz, Galois module structure of $(p^n)^{th}$ classes of fields, Bull. Lond. Math. Soc. 46 (2014), no. 1, 143–154, DOI 10.1112/blms/bdt082. MR3161770

[MT15] J. Mináč and N. D. Tán, The kernel unipotent conjecture and the vanishing of Massey products for odd rigid fields, Adv. Math. 273 (2015), 242–270, DOI 10.1016/j.aim.2014.12.028. MR3311763

[MT] J. Mináč and N. D. Tán, Triple Massey products and Galois theory, J. Eur. Math. Soc., to appear.

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg, Cohomology of number fields, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR2392026 (2009a:11123)

[Pop12] F. Pop, On the birational anabelian program initiated by Bogomolov I, Invent. Math. 187 (2012), no. 3, 511–533, DOI 10.1007/s00222-011-0331-x. MR2891876

[RZ10] L. Ribes and P. Zalesskii, Profinite groups, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 40, Springer-Verlag, Berlin, 2010. MR2599132 (2011a:20058)

[Ser65] J.-P. Serre, Sur la dimension cohomologique des groupes profinis (French), Topology 3 (1965), 413–420. MR0180619 (31 #4853)

[Ser02] J.-P. Serre, Galois cohomology, Corrected reprint of the 1997 English edition, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author. MR1867431 (2002d:12004)
[Top12] A. Topaz, *Commuting-liftable subgroups of Galois groups II*, J. reine angew. Math. (2012), to appear, available at arXiv:1208.0583v5.

[Vil88] F. R. Villegas, *Relations between quadratic forms and certain Galois extensions*, a manuscript, Ohio State University, 1988, http://www.math.utexas.edu/users/villegas/osu.pdf.

[Voe03] V. Voevodsky, *Motivic cohomology with \( \mathbb{Z}/2 \)-coefficients*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104, DOI 10.1007/s10240-003-0010-6. MR2031199 (2005b:14038b)

[Voe11] V. Voevodsky, *On motivic cohomology with \( \mathbb{Z}/l \)-coefficients*, Ann. of Math. (2) **174** (2011), no. 1, 401–438, DOI 10.4007/annals.2011.174.1.11. MR2811603 (2012j:14030)

[Wei08] C. Weibel, *2007 Trieste lectures on the proof of the Bloch-Kato conjecture*, Some Recent Developments in Algebraic K-Theory, ICTP Lect. Notes, vol. 23, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008, pp. 277–305. MR2509183

[Wei09] C. Weibel, *The norm residue isomorphism theorem*, J. Topol. **2** (2009), no. 2, 346–372, DOI 10.1112/jtopol/jtp013. MR2529300 (2011a:14039)

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