Mapping Among Manifolds II

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Abstract
In a previous paper we built a modified Hamiltonian formalism to make possible explicit maps among manifolds. In this paper the modified formalism was generalized. As an application, we have built maps among spaces associated to spinors, as well as maps among Kaehler spaces.

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1 Introduction

The Jacobi fields are very important to the Riemannian Geometry [1] and to the singularity theorems [2],[3]. These fields were used to study a free falling particle motion in a Schwarzschild spacetime [4], and a charged particle motion in Kaluza-Klein manifolds [5]. In a previous paper we have modified the Hamiltonian formalism to build maps among manifolds [6]. In this paper we present a second map building method among manifolds. As an application, we have built maps among complex spaces, and also among spaces associated to spinors.

This paper is organized as follows. In Sec. 2 we give a brief overview of the modified Hamiltonian formalism which we call the first modified Hamiltonian formalism. In Sec. 3 we build the second modified Hamiltonian formalism. In Sec. 4 we apply this formalism to spaces associated to spinors. In Sec. 5 we apply it to complex spaces. In Sec. 6 we summarize the main results of this work.

2 The First Modified Hamiltonian Formalism

It is well-known that in the Hamiltonian formalism the Hamilton equations and the Poisson brackets are conserved only by a canonical or sympletic transformation. In [4] we changed the non-relativistic time-dependent harmonic oscillator [7],[8] to a general relativistic approach. In the first modified Hamiltonian formalism only Hamilton equations will be kept, in the sense that they will be transformed into other Hamilton equations by a non-canonical or non-sympletic transformation, and the Poisson brackets will not be invariant.

We now give a brief overview of the first modified Hamiltonian formalism [6]. Consider a time-dependent Hamiltonian $H(\tau)$ where $\tau$ is an affine parameter, in this case, the proper-time of the particle. Let us define $2n$ variables that will be called $\xi^j$ with index $j$ running from 1 to $2n$ so that we have $\xi^j \in (\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ where $q^j$ and $p^j$ can be or not coordinates and momenta, respectively. We now define the
Hamiltonian by

\[ H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j, \quad (2.1) \]

where \( H_{ij} \) is a symmetric matrix. We consider that the Hamiltonian obeys the Hamilton equation

\[ \frac{d\xi^i}{d\tau} = J^{ik} \frac{\partial H}{\partial \xi^k}. \quad (2.2) \]

The equation (2.2) introduces the sympletic \( J \), given by

\[ \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \quad (2.3) \]

where \( O \) and \( I \) are the \( nxn \) zero and identity matrices, respectively. We now make a linear transformation from \( \xi^j \) to \( \eta^j \) given by

\[ \eta^j = T^j_k \xi^k, \quad (2.4) \]

where \( T^j_k \) is a non-sympletic matrix, and the new Hamiltonian is given by

\[ \bar{H} = \frac{1}{2} C_{ij} \eta^i \eta^j, \quad (2.5) \]

where \( C_{ij} \) is a symmetric matrix. The matrices \( H, C, \) and \( T \) obey the following system

\[ \frac{dT^i_j}{d\tau} + \frac{dt}{d\tau} T^i_k J^{kl} X_{lj} = J^{im} Y_{ml} T^j_k, \quad (2.6) \]

where \( 2X_{lj} = \frac{\partial H_{ij}}{\partial \xi^i} \xi^j + 2H_{ij} \) and \( 2Y_{ml} = \frac{\partial C_{ij}}{\partial \eta^i} \eta^j + 2C_{ml} \), \( t \) and \( \tau \) are the proper-times of the particle in two different manifolds. We note that (2.6) is a first order linear differential equation system in \( T^i_k \), and it is the response for what we looked for because the non-linearity in the Hamilton equations were transferred to their coefficients [6]. Consider \( \frac{d}{d\tau} X_{lj} = Z_{lj} \) and write (2.6) in the matrix form

\[ \frac{dT}{d\tau} + T J Z = JYT, \quad (2.7) \]

where \( T, Z \) and \( Y \) are \( nxn \) matrices as

\[ \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \quad (2.8) \]
with similar expressions for \( Z \) and \( Y \). Let us write (2.7) as follows

\[
\dot{T}_1 = Y_3 T_1 + Y_4 T_3 + T_2 Z_1 - T_1 Z_3, \quad (2.9)
\]
\[
\dot{T}_2 = Y_3 T_2 + Y_4 T_4 + T_2 Z_2 - T_1 Z_4, \quad (2.10)
\]
\[
\dot{T}_3 = -Y_1 T_1 - Y_2 T_3 + T_4 Z_1 - T_3 Z_3, \quad (2.11)
\]
\[
\dot{T}_4 = -Y_1 T_2 - Y_2 T_4 + T_4 Z_2 - T_3 Z_4. \quad (2.12)
\]

Now consider

\[
\dot{S}_1 = Y_3 S_1 + Y_4 S_3, \quad (2.13)
\]
\[
\dot{S}_2 = Y_3 S_2 + Y_4 S_4, \quad (2.14)
\]
\[
\dot{S}_3 = -Y_1 S_1 - Y_2 S_3, \quad (2.15)
\]
\[
\dot{S}_4 = -Y_1 S_2 - Y_2 S_4, \quad (2.16)
\]

and

\[
\dot{R}_1 = R_2 Z_1 - R_1 Z_3, \quad (2.17)
\]
\[
\dot{R}_2 = R_2 Z_2 - R_1 Z_4, \quad (2.18)
\]
\[
\dot{R}_3 = R_4 Z_1 - R_3 Z_3, \quad (2.19)
\]
\[
\dot{R}_4 = R_4 Z_2 - R_3 Z_4. \quad (2.20)
\]

From the theory of first order differential equation systems [9], it is well-known that each system in (2.13)-(2.20) has a solution in the region where \( Z_{lj} \) and \( Y_{ml} \) are continuous functions. In this case, the solution for (2.6) or (2.7) is given by

\[
T_1 = (S_1 a + S_2 b) R_1 + (S_1 d + S_2 c) R_3, \quad (2.21)
\]
\[
T_2 = (S_1 a + S_2 b) R_2 + (S_1 d + S_2 c) R_4, \quad (2.22)
\]
\[
T_3 = (S_3 a + S_4 b) R_1 + (S_3 d + S_4 c) R_3, \quad (2.23)
\]
\[
T_4 = (S_3 a + S_4 b) R_2 + (S_3 d + S_4 c) R_4. \quad (2.24)
\]

where a, b, c and d are constant \( nxn \) matrices, and using (2.21)-(2.24) in (2.4) we will have completed the mapping among manifolds. In many situations where it is not possible to consider \( \frac{dX}{d\tau}, X_{lj}, Y_{lj} \) as explicit functions of one of the two parameters, \( t \) or \( \tau \), we should expand them in series of \( \tau \), for example [9], so that with the modified Hamiltonian formalism we can map one differential equation system into another.
The Second Modified Formalism

In the first modified Hamiltonian formalism only Hamilton equations will be conserved, in the sense that they will be transformed into other Hamilton equations by a non-canonical or non-sympletic transformation, and the Poisson brackets will not be invariant. The second modified formalism differs from the first because we will have a set of signs in the Hamilton equations. We use one class of functions that include usual and unusual Hamiltonians in both formalisms. We will maintain part of the usual notation. Consider a time-dependent function \( H(\tau) \) where \( \tau \) is an affine parameter. Let us define 2n variables that will be called \( \xi^j \) with index \( j \) running from 1 to 2n so that we have \( \xi^j \in (\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n) \) where \( q^i \) and \( p^i \) can be or not the usual coordinates and momenta, respectively. We now define the function by

\[
H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j, \tag{3.1}
\]

where \( H_{ij} \) is a symmetric matrix. Consider the following system

\[
\frac{d\xi^i}{d\tau} = I_{1k}^i \frac{\partial H}{\partial \xi^k}. \tag{3.2}
\]

The equation (3.2) introduces the \( I_1 \), given by

\[
\left( \begin{array}{cc}
O & A \\
B & O
\end{array} \right) \tag{3.3}
\]

where O, A and B are the \( nxn \), with O as the zero matrix, and \( A = \epsilon_1 I, B = \epsilon_2 I \) are proportional to identity matrix, with \( \epsilon_i = -1, +1 \) and \( i = 1 \) or 2. We now make a linear transformation from \( \xi^j \) to \( \eta^j \) given by

\[
\eta^j = T^j_{\ k} \xi^k, \tag{3.4}
\]

where \( T^j_{\ k} \) is a non-sympletic matrix, and the new function is given by

\[
\bar{H} = \frac{1}{2} C_{ij} \eta^i \eta^j, \tag{3.5}
\]

where \( C_{ij} \) is a symmetric matrix. Consider that (3.5) obeys the following equation

\[
\frac{d\eta^i}{d\tau} = T^i_{\ k} \frac{\partial H}{\partial \eta^k}. \tag{3.6}
\]
where $I_2$ is given by
\[
\begin{pmatrix}
O & E \\
D & O
\end{pmatrix}
\] (3.7)
and $O$, $E$ and $D$ are $n\times n$, with $O$ as the zero matrix, and $E = \epsilon_3 I$, $D = \epsilon_4 I$ are proportional to the identity matrix, with $\epsilon_j = -1, +1$ and $j = 3$ or $4$. The functions $A$, $B$, $E$, and $D$ could be chosen as arbitrary diagonal matrices, however, such possibility will not be used in this paper. The matrices $H$, $C$, and $T$ obey the following system
\[
\frac{dT_{ij}}{d\tau} + \frac{dt}{d\tau} T_{ik} I_1^{kl} X_{lj} = I_2^{lm} Y_{ml} T_{kj},
\] (3.8)
where $2X_{lj} = \frac{\partial H_{ij}}{\partial \xi^l} \xi^i + 2H_{lj}$ and $2Y_{ml} = \frac{\partial C_{ij}}{\partial \eta^l} \eta^i + 2C_{ml}$, $t$ and $\tau$ are the proper-times of the particle in two different manifolds. We note that (3.8) is a first order linear differential equation system in $T_{ik}$, and that the non-linearity in the Hamiltonians were transferred to their coefficients. Consider $\frac{dt}{d\tau} X_{ij} = Z_{ij}$ and write (3.8) in the matrix form
\[
\frac{dT}{d\tau} + T I_1 Z = I_2 Y T,
\] (3.9)
where $T$, $Z$ and $Y$ are $2n\times 2n$ matrices as
\[
\begin{pmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{pmatrix}
\] (3.10)
with similar expressions for $Z$ and $Y$. Let us write (3.9) as follows
\[
\dot{T}_1 = \epsilon_3 (Y_3 T_1 + Y_4 T_3) - \epsilon_2 T_2 Z_1 - \epsilon_1 T_1 Z_3,
\] (3.11)
\[
\dot{T}_2 = \epsilon_3 (Y_3 T_2 + Y_4 T_4) - \epsilon_2 T_2 Z_2 - \epsilon_1 T_2 Z_4,
\] (3.12)
\[
\dot{T}_3 = \epsilon_4 (Y_1 T_1 + Y_2 T_3) - \epsilon_2 T_4 Z_1 - \epsilon_1 T_3 Z_3,
\] (3.13)
\[
\dot{T}_4 = \epsilon_4 (Y_1 T_2 + Y_2 T_4) - \epsilon_2 T_4 Z_2 - \epsilon_1 T_3 Z_4.
\] (3.14)
Now consider
\[
\dot{S}_1 = \epsilon_3 (Y_3 S_1 + Y_4 S_3),
\] (3.15)
\[
\dot{S}_2 = \epsilon_3 (Y_3 S_2 + Y_4 S_4),
\] (3.16)
\[
\dot{S}_3 = \epsilon_4 (Y_1 S_1 + Y_2 S_3),
\] (3.17)
\[ \dot{S}_4 = \epsilon_4(Y_1S_2 + Y_2S_4), \tag{3.18} \]

and
\[ \begin{align*}
\dot{R}_1 &= -\epsilon_2 R_2 Z_1 - \epsilon_1 R_1 Z_3, \\
\dot{R}_2 &= -\epsilon_2 R_2 Z_2 - \epsilon_1 R_1 Z_4, \\
\dot{R}_3 &= -\epsilon_2 R_4 Z_1 - \epsilon_1 R_3 Z_3, \\
\dot{R}_4 &= -\epsilon_2 R_4 Z_2 - \epsilon_1 R_3 Z_4. 
\end{align*} \tag{3.19-3.22} \]

From the theory of first order differential equation systems \[9\], it is well-known that each system in (3.15)-(3.22) has a solution in the region where \(Z_{lj}\) and \(Y_{ml}\) are continuous functions. In this case, the solution for (3.8) or (3.9) is given by
\[ \begin{align*}
T_1 &= (S_1 a + S_2 b) R_1 + (S_1 d + S_2 c) R_3, \\
T_2 &= (S_1 a + S_2 b) R_2 + (S_1 d + S_2 c) R_4, \\
T_3 &= (S_3 a + S_4 b) R_1 + (S_3 d + S_4 c) R_3, \\
T_4 &= (S_3 a + S_4 b) R_2 + (S_3 d + S_4 c) R_4. 
\end{align*} \tag{3.23-3.26} \]

where \(a, b, c\) and \(d\) are constant nxn matrices, and using (3.23)-(3.26) into (3.4) we will have completed the mapping among manifolds. Using the first or the second formalism we can build maps among manifolds. They are not equivalent maps among manifolds and the choice of one of them is not a preference matter, but the second formalism can be reduced to the first one by an appropriate choice of the constants \(\epsilon_i\) and \(\epsilon_j\). As in the first formalism, it is important to note that the same particle has different proper-times in different manifolds, so that line elements are not preserved by local non-sympletic maps among manifolds. The derivative \(\frac{dt}{d\tau}\) increases the difficulty in (3.8), so that we assume the condition \(\frac{dt}{d\tau} = 1\). It implies in a decrease on mapped regions. The local non-sympletic maps are well-defined for equal proper-times and time intervals. In this paper, for the same particle in different manifolds with different proper-times, we use the proper-time of one of the manifolds, so that (3.8) assumes the following form
\[ \frac{dT_{ij}}{d\tau} + T^{i}_{ik}T^{kl}X_{lj} = I^{im}_{2}Y_{ml}T^{ij}_{k}. \tag{3.27} \]

As a consequence (3.23)-(3.26) will be simplified. It is important to note that all that we call Hamiltonian, sometimes are not true Hamiltonians because they are not usual functions of coordinates and momenta.
4 Spaces Associated to Spinors

In this section we assume the convention used in [10] and we will use the second modified formalism, although we could use the first one. In this and in the following sections, what we call Hamiltonian are not true Hamiltonians because they are not usual functions of coordinates and momenta. In other words, we have generic spaces. We note that in the first and second formalisms $\tau$ can be one parameter without association to a particle or to any physics question. Let us consider

$$H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j = \frac{1}{2} (\bar{X}^i M^i X + X^i M \bar{X}) \quad (4.1)$$

where $\xi^j \in (\xi^0, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n+2}) = (X^0, X^1, \ldots, X^n, \bar{X}^1, \ldots, \bar{X}^{n+1})$, and $X^0, X^j$ and $\bar{X}^j$ are coordinates in a $2n+2$-dimensional manifold. Sometimes they can be identified as complex and complex-conjugated coordinates, respectively. We have that $H_{ij}$ is a symmetric matrix given by

$$\begin{pmatrix} O & M^t \\ M & O \end{pmatrix} \quad (4.2)$$

where $O$ is the $(n+1)\times(n+1)$ zero matrix, $M^t$ is the $(n+1)\times(n+1)$ transposed matrix of $M$. Explicitly

$$H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j = \bar{X}^i M_{ij} X^i, \quad (4.3)$$

where $M_{ij}$ can be complex, having or not a defined symmetry. Using the Hamiltonian (4.1) in (3.2), we obtain

$$\frac{dX^i}{d\tau} = \epsilon_1 \partial H / \partial \bar{X}^i, \quad (4.4)$$

and

$$\frac{d\bar{X}^i}{d\tau} = \epsilon_2 \partial H / \partial X^i. \quad (4.5)$$

Let us consider

$$F = \frac{1}{2} C_{ij} \eta^i \eta^j = \frac{1}{2} (\bar{x}^i N^t x + x^i N \bar{x}) \quad (4.6)$$
where $\eta^j \in (\eta^0, \eta^1, \ldots, \eta^n, \eta^{n+1}, \ldots, \eta^{2n+2}) = (x^0, x^1, \ldots, x^n, \bar{x}^0, \bar{x}^1, \ldots, \bar{x}^n)$ and $x^j$ and $\bar{x}^j$ are coordinates in another 2n+2-dimensional manifold. We have that $C_{lk}$ is a symmetric matrix given by

\[
\begin{pmatrix}
O & N \\
N^t & O
\end{pmatrix}
\]  

(4.7)

where $O$ is the $(n+1)\times(n+1)$ zero matrix, $N^t$ is the $(n+1)\times(n+1)$ transposed matrix of $N$, and $N_{lk} = \delta_{kl}$. Explicitly

\[F = \bar{x}^0 x^0 + \bar{x}^1 x^1 + \ldots + \bar{x}^n x^n,\]  

(4.8)

where (4.8) is a 2n+2-dimensional manifold. Using (3.23)-(3.26) into (3.4) we will have a map between (4.1) and (4.6). For $\bar{x}^0 = x^0$ we have a 2n+1-dimensional manifold

\[\tilde{F} = (x^0)^2 + \bar{x}^1 x^1 + \ldots + \bar{x}^n x^n,\]  

(4.9)

where associated spinors can be built [10]. For $\bar{x}^0 = x^0 = 0$, we have the 2n-dimensional manifold

\[\tilde{F} = \bar{x}^1 x^1 + \ldots + \bar{x}^n x^n.\]  

(4.10)

As in (4.9), we can build spinors associated to (4.10). We note that (4.8) and (4.10) are not usual Hamiltonians, they are the special forms chosen by Cartan [10].

5 Kaehler Manifolds

In this section we present some facts about Kaehler manifolds and use the second formalism to build maps among manifolds.

Let us consider a real 2n-dimensional manifold. We will denote the coordinates of a point $P$ by $(x^1, \ldots, x^n, \bar{x}^1, \ldots, \bar{x}^n)$, and build a n-dimensional complex manifold, where $P$ has the following complex and complex-conjugated coordinates

\[z^\alpha = x^\alpha + i\bar{x}^\alpha,\]  

(5.1)

\[\bar{z}^\alpha = x^\alpha - i\bar{x}^\alpha,\]  

(5.2)
where $\alpha \in (1, \ldots, n)$. Consider a symmetric tensor $g_{ij}$. It is self-adjoint if it satisfies

$$g_{\alpha\bar{\sigma}} = g_{\bar{\sigma}\alpha} = g_{\bar{\alpha}\sigma} = g_{\sigma\bar{\alpha}} \quad (5.3)$$

Moreover, if the inverse contravariant tensor $g^{ij}$ defined by

$$g_{ik}g^{kj} = \delta_i^j \quad (5.4)$$

the usual Christoffel symbols, the Riemann-Christoffel curvature tensor, the Ricci tensor, and the scalar curvature are all self-adjoint \[11\]. We assume now that, in this complex manifold, there is a positive definite line element

$$ds^2 = g_{ij}dz^i \bar{dz}^j, \quad (5.5)$$

where the symmetric tensor $g_{ij}$ is self-adjoint and satisfies

$$g_{\alpha\sigma} = g_{\bar{\alpha}\bar{\sigma}} = 0, \quad (5.6)$$

and (5.4). In this case the metric tensor is called a Hermitian metric and the line element has the following expression

$$ds^2 = 2g_{\alpha\bar{\sigma}}dz^\alpha d\bar{z}^\sigma. \quad (5.7)$$

The Christoffel symbols are given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\bar{\sigma}} \left( \frac{\partial g_{\sigma\mu}}{\partial z^\nu} + \frac{\partial g_{\sigma\nu}}{\partial z^\mu} \right), \quad (5.8)$$

$$\Gamma^\alpha_{\mu\bar{\nu}} = \frac{1}{2}g^{\alpha\bar{\sigma}} \left( \frac{\partial g_{\mu\sigma}}{\partial \bar{z}^\nu} - \frac{\partial g_{\mu\bar{\nu}}}{\partial \bar{z}^\sigma} \right), \quad (5.9)$$

$$\Gamma^\alpha_{\mu\bar{\nu}} = 0, \quad (5.10)$$

and other components are given by symmetry and self-adjointness. The components (5.9) transform as tensors. Kaehler made the following choice

$$\Gamma^\alpha_{\mu\bar{\nu}} = 0. \quad (5.11)$$

It is the same as

$$\frac{\partial g_{\mu\bar{\sigma}}}{\partial z^\nu} = \frac{\partial g_{\mu\bar{\sigma}}}{\partial \bar{z}^\sigma}, \quad (5.12)$$

$$\frac{\partial g_{\mu\sigma}}{\partial z^\nu} = \frac{\partial g_{\mu\sigma}}{\partial \bar{z}^\sigma}, \quad (5.13)$$
so that
\[ g_{\bar{\sigma}\alpha} = \frac{\partial^2 \phi}{\partial z^\alpha \partial \bar{z}^\sigma}. \] (5.14)

The self-adjointness of \( g_{\bar{\sigma}\alpha} \) implies that \( \phi \) is a real valued function. The (5.14) is called Kaehler condition and a metric satisfying (5.4), (5.6) and (5.14) will be called a Kaehler metric. Thus, in a Kaehler metric we have

\[ \Gamma^\alpha_{\mu\nu} = g^{\alpha\bar{\sigma}} \frac{\partial g_{\bar{\sigma}\mu}}{\partial z^\nu}, \] (5.15)

\[ \Gamma^{\bar{\sigma}}_{\bar{\mu}\nu} = g^{\bar{\sigma}\alpha} \frac{\partial g_{\alpha\bar{\nu}}}{\partial \bar{z}^\mu}, \] (5.16)

and Riemann-Christoffel tensor components will be simplified. For a n-dimensional Kaehler manifold, if at every point, the sectional curvature is the same for all possible 2-dimensional sections, then the curvature tensor is identically zero. The same is not true for the holomorphic sectional curvature, thus, if we assume that at all points of the manifold they are all the same, we have

\[ R_{\alpha\bar{\sigma}\mu\bar{\nu}} = \frac{K}{2} (g_{\alpha\bar{\sigma}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\sigma}}). \] (5.17)

From (5.17)
\[ R_{\alpha\bar{\sigma}} = R_{\bar{\sigma}\alpha} = \frac{(n + 1)K}{2} g_{\alpha\bar{\sigma}}, \] (5.18)

where (5.18) is an Einstein manifold.

Let us consider
\[ H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j = \frac{1}{2} (\bar{x}^i M^i x + x^i M \bar{x}) \] (5.19)

where \( \xi^i \in (\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n}) = (x^1, \ldots, x^n, \bar{x}^1, \ldots, \bar{x}^n) \) and \( x^i \) and \( \bar{x}^j \) are real coordinates in a 2n-dimensional manifold. We have that \( H_{ij} \) is a symmetric matrix given by

\[ \begin{pmatrix} O & M \\ M^t & O \end{pmatrix} \] (5.20)

where \( O \) is the \( n \times n \) zero matrix, \( M^t \) is the \( n \times n \) transposed matrix of \( M \). Using the Hamiltonian (5.19) in (3.2), we obtain

\[ \frac{dx^i}{d\tau} = \epsilon_1 \frac{\partial H}{\partial \bar{x}^i}, \] (5.21)
and

\[ \frac{d\bar{x}^i}{d\tau} = \epsilon^2 \frac{\partial H}{\partial \bar{x}^i}. \] (5.22)

We can present a new Hamiltonian similar to (5.19) and build a map between them, and consider that they are real representations of two complex manifolds. We still have another possibility where we present two Hamiltonians with complex and complex-conjugated coordinates given by (5.1) and (5.2). For this last option we can build a map between (5.7) and another Kaehler line element.

6 Concluding Remarks

The objective of this paper is twofold. Firstly, it presents a second modified formalism as an option to and a generalization of the first one [6]. Secondly, it shows, through maps, the use of this second formalism in some areas of mathematics such spaces associated to spinors and complex spaces. Explicit applications will be presented in a next paper. We could have presented a section with maps among Finsler spaces. For such, it would be necessary the usual choice of a metric tensor with appropriate momentum dependence. However a more general or not momentum dependence can be introduced directly in (2.7) or (3.9). It would be a repetitive procedure, therefore, we decided not to include it. Invariance is a fundamental property in many theories, as in general relativity. However, if we want to build maps among manifolds, the first or the second modified formalism can be useful, and both can be considered additional mathematical resources for research.

References

[1] J. Cheeger and D. G. Ebin, Comparison Theorems in Riemannian Geometry (North-Holland Publishing Company, Amsterdam, 1975)

[2] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge, 1973).

[3] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (W. H. Freeman and Company, San Francisco, 1973).
[4] A. C. V. V. de Siqueira, I. A. Pedrosa, and E. R. Bezerra de Mello, \textit{arXiv:hep-th/9709094v1}.

[5] A. C. V. V. de Siqueira, \textit{arXiv:hep-th/0710.1824v1}

[6] A. C. V. V. de Siqueira, \textit{arXiv:math-ph/0802.2299v1}

[7] K. R. Meyer and G. R. Hall, \textit{Introduction to Hamiltonian Dynamical Systems and the N-Body Problem} (Springer-Verlag, New York, 1991)

[8] P. G. L. Leach, \textit{J. Math. Phys.} 18, 1902, (1977).

[9] E. A. Coddington and N. Levinson, \textit{Theory of Ordinary Differential Equations} (McGraw-Hill, New York, 1955)

[10] E. Cartan, \textit{The Theory of Spinors} (Dover Publications, New York, 1981).

[11] S. Bochner, \textit{J. Indian Math. Soc. XI} (1947) 1-21.