A generalized integral transform and an alternative technique for solving linear ordinary differential equations

Nese Dernek\textsuperscript{1} \textsuperscript{*} and Fatih Aylikci\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Marmara University, Istanbul, Turkey
\textsuperscript{2}Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey

March 11, 2014

Abstract

In the present paper authors introduce the $\mathcal{L}_{n}$-integral transform and the $\mathcal{L}_{n}^{-1}$ inverse integral transform for $n = 2^k$, $k \in \mathbb{N}$, as a generalization of the classical Laplace transform and the $\mathcal{L}^{-1}$ inverse Laplace transform, respectively. Applicability of this transforms in solving linear ordinary differential equations is analyzed. Some illustrative examples are also given.

Keywords: The Laplace transform, The $\mathcal{L}_{2}$-transform, The $\mathcal{L}_{n}$-transform, The $\mathcal{L}_{n}^{-1}$-transform and Linear ordinary differential equations.

1. Introduction, definitions and preliminaries

The following Laplace-type the $\mathcal{L}_{2}$ transform

$$\mathcal{L}_{2}\{f(x); y\} = \int_{0}^{\infty} x \exp(-x^2 y^2) f(x) dx$$

was introduced by Yurekli and Sadek \cite{4}. After then Aghili, Ansari and Sedghi \cite{1} derived a complex inversion formula as follows

$$\mathcal{L}_{2}^{-1}\{\mathcal{L}_{2}\{f(x); y\}\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2\mathcal{L}_{2}\{f(x); \sqrt{y}\} \exp(y x^2) dy$$

\textsuperscript{*}Correspondence: ndernek@marmara.edu.tr

2010 AMS Mathematics Subject Classification: 44A10, 44A15, 44A20, 34A30
where $L^2\{f(x); \sqrt{y}\}$ has a finite number of singularities in the left half plane $Re\ y \leq c$.

In this article, we introduce a new generalized integral transform as follows

$$L_n\{f(x); y\} = \int_0^\infty x^{n-1} \exp(-x^ny^n) f(x) \, dx$$

(1.3)

where $n = 2^k, \ k \in \mathbb{N}$.

The $L_n$-transform is related to the Laplace transform by means

$$L_n\{f(x); y\} = \frac{1}{n} L\{f(x^n); y^n\},$$

(1.4)

where the Laplace transform is defined by

$$L\{f(x); y\} = \int_0^\infty \exp(-xy) f(x) \, dx.$$  

(1.5)

First we shall give several examples of the $L_n$-transforms of some elementary and special functions.

**Example 1.1.** We show that

$$L_n\{1; y\} = \int_0^\infty x^{n-1} \exp(-y^nx^n) \, dx = \frac{1}{ny^n}. \quad (1.6)$$

**Example 1.2.** We show that

$$L_n\{x^k; y\} = \frac{1}{ny^n+k} \Gamma\left(\frac{k}{n} + 1\right). \quad (1.7)$$

where $k, n \in \mathbb{N}$ and $k > -n$. Applying the definition of the $L_n$-transform, we have

$$L_n\{x^k; y\} = \int_0^\infty x^{k+n-1} \exp(-y^nx^n) \, dx, \quad (1.8)$$

where $k \in \mathbb{N}$.

The integral on the right hand side may be evaluated by changing the variable of the integration from $x$ to $u$ where $x^ny^n = u$, and using Gamma function’s relation in (1.8), we obtain

$$L_n\{x^k; y\} = \frac{1}{ny^n+k} \int_0^\infty u^{k+n-1} \exp(-u) \, du = \frac{1}{ny^n+k} \Gamma\left(\frac{k}{n} + 1\right). \quad (1.9)$$

**Example 1.3.** We show that

$$L_n\{\cos(ax^n); y\} = \frac{y^n}{n(y^{2n} + a^2)}. \quad (1.10)$$
Using the definition of the $L_n$-transform and calculating the Taylor expansion of the $\cos$ function in (1.10) we get

$$L_n\{\cos(ax^n); y\} = \sum_{m=0}^{\infty} (-1)^m \frac{a^{2m}}{(2m)!} L_n\{x^{2mn}; y\},$$

(1.11)

where from (1.7) we have the following relation

$$L_n\{x^{2mn}; y\} = \frac{2m + 1}{ny^{n+2mn}}$$

(1.12)

and then we obtain (1.10)

$$L_n\{\cos(ax^n); y\} = \frac{1}{ny^n} \sum_{m=0}^{\infty} (-1)^m \frac{a^{2m}}{y^{2mn}} = \frac{y^n}{n(y^n + a^2)}.$$ 

(1.13)

**Example 1.4.** We show that

$$L_n\{\sin(ax^n); y\} = \frac{a}{n(y^n + a^2)}.$$ 

(1.14)

Using the linearity of the $L_n$-transform and calculating the Taylor expansion of the $\sin$ function in (1.14) we get

$$L_n\{\sin(ax^n); y\} = \sum_{m=0}^{\infty} (-1)^m \frac{a^{2m+1}}{(2m + 1)!} L_n\{x^{(2m+1)n}; y\}.$$ 

(1.15)

Using the following relation in (1.7) of Example 1.2,

$$L_n\{x^{(2m+1)n}; y\} = \frac{\Gamma(2m + 2)}{ny^{2m+2n}},$$ 

(1.16)

we have

$$L_n\{\sin(ax^n); y\} = \frac{a}{ny^{2n}} \sum_{m=0}^{\infty} (-a^2)^m \frac{1}{y^{2mn}} = \frac{a}{n(y^n + a^2)}.$$ 

(1.17)

**Example 1.5.** We show that

$$L_n\{\exp(-a^nx^n); y\} = \frac{1}{n(y^n + a^n)}$$ 

(1.18)

where $0 < Re(a) < Re(y)$.

Using the definition of the $L_n$-transform and calculating the Taylor expansion of the exponential function we have

$$L_n\{\exp(-a^nx^n); y\} = \sum_{m=0}^{\infty} (-1)^m \frac{a^{mn}}{m!} L_n\{x^{mn}; y\}.$$ 

(1.19)

Using the value

$$L_n\{x^{mn}; y\} = \frac{\Gamma(m + 1)}{ny^{n+mn}}$$ 

(1.20)

on the right hand side of (1.19) we get

$$L_n\{\exp(-a^nx^n); y\} = \frac{1}{ny^n} \sum_{m=0}^{\infty} (-1)^m \frac{a^{mn}}{y^{mn}} = \frac{1}{n(y^n + a^n)}.$$ 

(1.21)
Example 1.6. We show that
\[
\mathcal{L}_n\{J_0(2a^n x^n); y\} = \frac{1}{ny^n} \exp\left(-\frac{a^n}{y^n}\right) \tag{1.22}
\]
where the function $J_0$ is the Bessel function of the first kind of order zero.

Using the following Taylor expansion of the function $J_0(x)$,
\[
J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}, \tag{1.23}
\]
we obtain
\[
\mathcal{L}_n\{J_0(2a^n x^n); y\} = \sum_{m=0}^{\infty} \frac{(-1)^m a^{mn}}{m!\Gamma(m+1)} \mathcal{L}_n\{x^{mn}; y\}. \tag{1.24}
\]

We know the $\mathcal{L}_n$ transform of $f(x) = x^{mn}$ as
\[
\mathcal{L}_n\{x^{mn}; y\} = \frac{\Gamma(m+1)}{ny^{m+mn}}. \tag{1.25}
\]

Substituting the relation (1.25) into (1.24) we obtain
\[
\mathcal{L}_n\{J_0(2a^n x^n); y\} = \sum_{m=0}^{\infty} \frac{(-1)^m a^{mn}}{m!\Gamma(m+1)} \mathcal{L}_n\{x^{mn}; y\} = \frac{1}{ny^n} \exp\left(-\frac{a^n}{y^n}\right). \tag{1.26}
\]

Example 1.7. We show that
\[
\mathcal{L}_n\{x^{n^v} J_v(2a^n x^n); y\} = \frac{1}{ny^n} \sum_{m=0}^{\infty} \frac{(-a^n)^m}{m!} \mathcal{L}_n\{x^{mn}; y\} = \frac{1}{ny^n} \exp\left(-\frac{a^n}{y^n}\right) \tag{1.27}
\]
where $\Re(a) > 0, \Re v > -1$.

Using the following Taylor expansion of $J_v(x)$, which is the Bessel function of the first kind of order $v$,
\[
J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v}, \tag{1.28}
\]
we obtain
\[
\mathcal{L}_n\{x^{n^v} J_v(2a^n x^n); y\} = \sum_{m=0}^{\infty} \frac{(-1)^m a^{mn+n^v}}{m!\Gamma(m+v+1)} \mathcal{L}_n\{x^{mn+nv}; y\}. \tag{1.29}
\]

We can calculate the $\mathcal{L}_n$ transform of $f(x) = x^{mn+nv}$ function as follows
\[
\mathcal{L}_n\{x^{mn+nv}; s\} = \frac{\Gamma(m+v+1)}{ny^{m+mn+nv}}. \tag{1.30}
\]

Substituting the relation (1.30) into equation (1.29) we obtain the assertion (1.27) of Example 1.7,
Example 1.8. We show that

$$\mathcal{L}_n\{\text{erfc}(\frac{1}{2} a x^{-\frac{n}{2}}); y\} = \frac{1}{n} y^{-n} \exp(-a^\frac{n}{2} y^\frac{n}{2})$$

(1.32)

where \( \text{Re}(a) > 0 \).

Using the definition of the complementary error function \( \text{erfc}(x) \),

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-u^2) \, du,$$

(1.33)

we get

$$\mathcal{L}_n\{\text{erfc}(\frac{1}{2} a x^{-\frac{n}{2}}); y\} = \frac{2}{\sqrt{\pi}} y^{-n} \int_{\frac{a^{n/2}}{2x^{n/2}}}^{\infty} \exp(-y^n x^n) \int_{0}^{\infty} \exp(-u^2) \, du \, dx.$$  \hspace{1cm} (1.34)

Changing the order of integration, we obtain

$$\mathcal{L}_n\{\text{erfc}(\frac{1}{2} a x^{-\frac{n}{2}}); y\} = \frac{2}{\sqrt{\pi} y^n} \int_{0}^{\infty} \exp(-u^2) \int_{-\frac{a^{n/2}}{2x^{n/2}}}^{\infty} \exp(-y^n x^n) \, dx \, du.$$  \hspace{1cm} (1.35)

and using the relation \( \frac{d}{dx}(\exp(-y^n x^n)) = -n y^n x^{n-1} \exp(-y^n x^n) \) we have

$$\mathcal{L}_n\{\text{erfc}(\frac{1}{2} a x^{-\frac{n}{2}}); y\} = \frac{2}{\sqrt{\pi} y^n} \int_{0}^{\infty} \exp(-u^2) \exp(-\frac{y^n a^n}{4u^2}) \, du.$$  \hspace{1cm} (1.36)

Changing the variable from \( u \) to \( x \) according to the transformation \( u = \frac{a^{n/2}}{2x^{n/2}} \) we find that

$$\mathcal{L}_n\{\text{erfc}(\frac{1}{2} a x^{-\frac{n}{2}}); y\} = \frac{a^{n/2}}{2\sqrt{\pi} y^n} \mathcal{L}_n\{x^{-\frac{n}{2}} \exp(-\frac{a^n}{4x^n}); y\}.$$  \hspace{1cm} (1.37)

Using the Taylor expansion of exponential function and the \( \mathcal{L}_n \)-transform of \( f(x) = x^{-mn-\frac{3m}{2}} \) we obtain

$$\frac{a^{\frac{n}{2}}}{2\sqrt{\pi} y^n} \mathcal{L}_n\{x^{-\frac{n}{2}} \exp(-\frac{a^n}{4x^n}); y\} = \frac{a^{\frac{n}{2}}}{2\sqrt{\pi} y^n} \sum_{m=0}^{\infty} \frac{(-1)^m a^{mn}}{m!4^m} \mathcal{L}_n\{x^{-mn-\frac{n}{2}}; y\}$$

$$= \frac{a^{n/2}}{2n\sqrt{\pi} y^n} \sum_{m=0}^{\infty} \frac{(-1)^m a^{mn}}{m!4^m} \frac{\Gamma(-m - \frac{3}{2} + 1)}{y^{-mn-\frac{n}{2}}}.$$  \hspace{1cm} (1.38)

From the following Euler’s reflection formula,

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)},$$  \hspace{1cm} (1.39)
we get
\[
\frac{a^{\frac{n}{2}}}{2n\sqrt{\pi}y^n} \sum_{m=0}^{\infty} \frac{(-1)^m a^{mn} \Gamma(-m - \frac{1}{2})}{m!4^m} \frac{\Gamma(-m - \frac{3}{2})}{y^{mn-\frac{3}{2}}} = \frac{a^{\frac{n}{2}}\pi}{2n\sqrt{\pi}y^n} \sum_{m=0}^{\infty} \frac{(-1)^{2m+1}a^{mn}y^{mn+\frac{1}{2}}}{\Gamma(m+1)\Gamma(m+1+\frac{1}{2})4^m}
\]
and using the following duplication formula for Gamma function
\[
\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\sqrt{\pi} \Gamma(2z)
\]
we obtain
\[
\frac{a^{\frac{n}{2}}\pi}{2n\sqrt{\pi}y^n} \sum_{m=0}^{\infty} \frac{(-1)^{2m+1}a^{mn}y^{mn+\frac{1}{2}}}{\Gamma(m+1)\Gamma(m+1+\frac{1}{2})4^m} = \frac{1}{n}\frac{1}{y^n} \exp(-a^{\frac{n}{2}}\frac{y^{\frac{n}{2}}}{\pi}).
\]
Thus the assertion (1.32) follows from (1.42).

**Example 1.9.** We show that
\[
\mathcal{L}_n\{\text{erf}(a^{\frac{n}{2}}x^{\frac{n}{2}}); y\} = \frac{a^{\frac{n}{2}}}{n}y^{-n}(y^n + a^n)^{-\frac{1}{2}}
\]
where \(-\text{Re}(a) < y, \text{Re}(y) > 0\).

Using the definition of \(\mathcal{L}_n\)-transform and the error function we have
\[
\mathcal{L}_n\{\text{erf}(a^{\frac{n}{2}}x^{\frac{n}{2}}); y\} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} x^{n-1} \exp(-y^n x^n) \int_{0}^{a^{n/2}x^{n/2}} \exp(-u^2)du dx.
\]
Changing the order of integration and evaluating the inner integral we get
\[
\mathcal{L}_n\{\text{erf}(a^{\frac{n}{2}}x^{\frac{n}{2}}); y\} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \exp(-u^2) \int_{\frac{u^{2/n}}{\sqrt{y^n}}}^{\infty} x^{n-1} \exp(-y^n x^n)dx du
\]
\[
= \frac{2}{\sqrt{\pi}ny^n} \int_{0}^{\infty} \exp(-u^2(1 + \frac{y^n}{a^n}))du.
\]
Changing the variable from \(u\) to \(x\) according to transformation \(u\sqrt{1 + \frac{y^n}{a^n}} = x\), we obtain the assertion (1.43),
\[
\frac{2}{\sqrt{\pi}ny^n} \int_{0}^{\infty} \exp(-u^2(1 + \frac{y^n}{a^n}))du = \frac{a^{\frac{n}{2}}}{n}y^{-n}(y^n + a^n)^{-\frac{1}{2}}.
\]

**Example 1.10.** We show that
\[
\mathcal{L}_n\{\exp(-ax^{2n}); y\} = \frac{\sqrt{\pi}}{2n\sqrt{a}} \exp(\frac{y^{2n}}{4a}) \text{erfc}(\frac{y^{2n}}{2\sqrt{a}})
\]
(1.47)
provided that $Re\ a > 0$.

Using the definition of the $\mathcal{L}_n$-transform we get

$$\mathcal{L}_n\{\exp(-ax^{2n}); y\} = \int_0^\infty x^{n-1} \exp(-y^n x^n - ax^{2n}) \, dx.$$  \hfill (1.48)

Writing on the right hand side of (1.48)

$$-y^n x^n - ax^{2n} = -a(x^n + \frac{y^n}{2a})^2 + \frac{y^{2n}}{4a}$$  \hfill (1.49)

and changing the variable

$$a^{1/2}(x^n + \frac{y^n}{2a}) = u,$$  \hfill (1.50)

using the definition of the complementary error function as follows, we deduce the assertion (1.48),

$$\mathcal{L}_n\{\exp(-ax^{2n}); y\} = \sqrt{\pi} \frac{2n}{2n\sqrt{a}} \exp(\frac{y^{2n}}{4a}) \text{erfc}(\frac{y^n}{2\sqrt{a}}).$$  \hfill (1.51)

**Corollary 1.1.** From the definition of the $\mathcal{L}_n$-transform the following identity hold true:

$$\mathcal{L}_n\{\exp(-ax^n) f(x); y\} = \mathcal{L}_n\{f(x); (y^n + a)^{1/2}\}$$  \hfill (1.52)

where $Re\ a > 0$.

We now introduce a new derivative operator for the $\mathcal{L}_n$-transform and apply the operator to solve following ordinary differential equations:

$$xz'' - (2v + n - 3)z' + x^{n-1}z = 0, \quad n = 2^k, \ k \in \mathbb{N}, \ v > n, \ v = 2^m + 1, \ m \in \mathbb{N} \hfill (1.53)$$

$$xz'' - (n^2 - 1)z' + x^{n-1}z = 0, \quad n = 2^k, \ k = 0, 1, 2, ... \hfill (1.54)$$

### 2. Some properties of the $\mathcal{L}_n$-transform

In this section we will give some properties of the $\mathcal{L}_n$-transform that will be used to solve the ordinary differential equations (1.53)-(1.54) given above.

Firstly, we introduce a differential operator $\delta$ (see [7, 8]) that we call the $\delta$-derivative and define as

$$\delta_x = \frac{1}{x^{n-1}} \frac{d}{dx}, \quad n = 2^k, \ k \in \mathbb{N}$$  \hfill (2.1)

we note that

$$\delta^2_x = \delta_x \delta_x = \frac{1}{x^{n-1}} \frac{d}{dx} \left( \frac{1}{x^{n-1}} \frac{d}{dx} \right) = \frac{1}{x^{2n-2}} \frac{d^2}{dx^2} - \frac{(n - 1)}{x^{2n-1}} \frac{d}{dx}.$$  \hfill (2.2)

The $\delta$ derivative operator can be successively applied in a similar fashion for any positive integer power.

Here we will derive a relation between the $\mathcal{L}_n$-transform of the $\delta$-derivative of a function and the $\mathcal{L}_n$-transform of the function itself.
Suppose that \( f(x) \) is a continuous function with a piecewise continuous derivative \( f'(x) \) on the interval \([0, \infty)\). Also suppose that \( f \) and \( f' \) are of exponential order \( \exp(\alpha x^n) \) as \( x \to \infty \) where \( \alpha \) is a constant. By using the definitions of \( \mathcal{L}_n \)-transform and the \( \delta \) derivative and integration by parts, we obtain

\[
\mathcal{L}_n\{\delta_x f(x); y\} = \int_{0}^{\infty} \exp(-y^n x^n) f'(x) dx, \tag{2.3}
\]

\[
\int_{0}^{\infty} \exp(-y^n x^n) f'(x) dx = \lim_{b \to \infty} f(x) \exp(-y^n x^n)|_0^b + ny^n \int_{0}^{\infty} x^{n-1} \exp(-y^n x^n) f(x) dx. \tag{2.4}
\]

Since \( f \) is of exponential order \( \exp(\alpha x^n) \) as \( x \to \infty \), it follows that

\[
\lim_{x \to \infty} \exp(-y^n x^n) f(x) = 0 \tag{2.5}
\]

and consequently,

\[
\mathcal{L}_n\{\delta_x f(x); y\} = ny^n \mathcal{L}_n\{f(x); y\} - f(0^+). \tag{2.6}
\]

Similarly, if \( f \) and \( f' \) are continuous functions with a piecewise continuous derivative \( f'' \) on the interval \([0, \infty)\), and if all three functions are of exponential order \( \exp(\alpha x^n) \) as \( x \to \infty \) we can use (2.6) to obtain

\[
\mathcal{L}_n\{\delta_x^2 f(x); y\} = n^2 y^{2n} \mathcal{L}_n\{f(x); y\} - ny^n f(0^+) - \delta_x f(0^+). \tag{2.7}
\]

Using (2.6) and (2.7) we get

\[
\mathcal{L}_n\{\delta_x^3 f(x); y\} = n^3 y^{3n} \mathcal{L}_n\{f(x); y\} - n^2 y^{2n} f(0^+) - ny^n \delta_x f(0^+) - \delta_x^2 f(0^+). \tag{2.8}
\]

By repeated application of (2.6) and (2.8) we obtain the following theorem.

**Theorem 2.1.** If \( f, f', ..., f^{(k-1)} \) are all continuous functions with a piecewise continuous derivative \( f^{(k)} \) on the interval \([0, \infty)\), and if all functions are of exponential order \( \exp(\alpha x^n) \) as \( x \to \infty \) for some constant \( \alpha \) then

\[
\mathcal{L}_n\{\delta_x^k f(x); y\} = (ny^n)^k \mathcal{L}_n\{f(x); y\} - (ny^n)^{k-1} f(0^+) - (ny^n)^{k-2} \delta_x f(0^+) - \cdots - \delta_x^{k-1} f(0^+) \tag{2.9}
\]

for \( k \geq 1, \ k \) is a positive integer.

The \( \mathcal{L}_n \)-transform defined in (1.3) is an analytic function in the half plane \( \text{Re} \ y > \alpha \). Therefore, \( \mathcal{L}_n\{f(x); y\} \) has derivatives of all orders and the derivatives can be formally obtained by differentiating (1.3). Applying the \( \delta \) with respect to the variable \( y \) we obtain

\[
\delta_y \mathcal{L}_n\{f(x); y\} = \frac{1}{y^{n-1}} \frac{d}{dy} \int_{0}^{\infty} x^{n-1} \exp(-y^n x^n) f(x) dx
\]

8
\[\frac{1}{y^{n-1}} \int_{0}^{\infty} x^{n-1}(-x^n y^{n-1} \exp(-y^n x^n)) f(x) dx = -n \mathcal{L}_n \{x^n f(x); y\}. \quad (2.11)\]

If we keep taking the $\delta$-derivative of (1.3) with respect to the variable $y$, then we deduce
\[\frac{\partial^k}{\partial y^k} \mathcal{L}_n \{f(x); y\} = \int_{0}^{\infty} x^{n-1} \frac{\partial^k}{\partial y^k} \exp(-y^n x^n) f(x) dx \quad (2.12)\]
for $k \in \mathbb{N}$.

\[\int_{0}^{\infty} x^{n-1} \frac{\partial^k}{\partial y^k} \exp(-y^n x^n) f(x) dx = \int_{0}^{\infty} x^{n-1} \frac{\partial^{k-1}}{\partial y^{k-1}} (-n) x^n \exp(-y^n x^n) f(x) dx \]
\[= \int_{0}^{\infty} x^{n-1} \frac{\partial^{k-2}}{\partial y^{k-2}} (-n)^2 x^{2n} \exp(-y^n x^n) f(x) dx \]
\[= \int_{0}^{\infty} x^{n-1} (-n)^k x^{kn} \exp(-y^n x^n) f(x) dx = (-n)^k \mathcal{L}_n \{x^{kn} f(x); y\}. \quad (2.13)\]

As a result we obtain the following theorem.

**Theorem 2.2.** If $f$ is piecewise continuous on the interval $[0, \infty)$ and is of exponential order $\exp(\alpha^n x^n)$ as $x \to \infty$, then
\[\mathcal{L}_n \{x^{kn} f(x); y\} = \frac{(-1)^k}{(n)^k} \frac{\partial^k}{\partial y^k} \mathcal{L}_n \{f(x); y\}. \quad (2.14)\]
for $k \geq 1$, $k$ is a positive integer.

**Theorem 2.3.** Let $\mathcal{L}_n \{f(x); y^{1/n}\}$, $n = 2^k$, $k = 0, 1, 2, \ldots$ be an analytic function of $y$ except at singular points each of which lies to the left of the vertical line $\text{Re} \ y = a$ and they are finite numbers. Suppose that $y = 0$ is not a branch point and $\lim_{y \to \infty} \mathcal{L}_n \{f(x); y^{1/n}\} = 0$ in the left plane $\text{Re} \ y \leq a$ then, the following identity
\[\mathcal{L}_n^{-1} \{\mathcal{L}_n \{f(x); y\}\} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} n \mathcal{L}_n \{f(x); y^{1/n}\} \exp(yx^n) dy \]
\[= \sum_{k=1}^{m} [\text{Res} n \mathcal{L}_n \{f(x); y^{1/n}\} \exp(yx^n); y = y_k] \quad (2.15)\]
hold true for $m$ singular points.
Proof. We take a vertical closed semi-circle as contour of integration. Using residues theorem and boundedness of $L_n\{f(x); y^{1/n}\}$, we show that the identity (2.13) of Theorem 2.3 is valid. When $y = 0$ is a branch point we take key-hole contour instead of simple vertical semi-circle.

We assume that $L_n\{f(x); y^{1/n}\}$ has a finite number of singularities in the left half plane $Re y \leq a$. Let $\gamma = \gamma_1 + \gamma_2$ be the closed contour consisting of the vertical line segment $\gamma_1$, which is defined from $a - iR$ to $a + iR$ and vertical semi-circle $\gamma_2$, that is defined as $|y - a| = R$. Let $\gamma_2$ lie to the left of vertical line $\gamma_1$. The radius $R$ can be taken large enough so that $\gamma$ encloses all the singularities of the $L_n\{f(x); y^{1/n}\}$. Hence, by the residues theorem we have

$$
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} nL_n\{f(x); y^{1/n}\} \exp(yx^n)dy
= \frac{1}{2\pi i} \int_{\gamma_1} nL_n\{f(x); y^{1/n}\} \exp(yx^n)dy
- \frac{1}{2\pi i} \int_{\gamma_2} nL_n\{f(x); y^{1/n}\} \exp(yx^n)dy
$$

\hspace{1cm} (2.16)

where $y_1, y_2, \ldots, y_m$ are all the singularities of $L_n\{f(x); y^{1/n}\}$. Taking the limit from both sides of the relation (2.16) as $R$ tends to $+\infty$, because of the Jordan’s Lemma, the second integral in the right tends to zero.

Even $L_n\{f(x); y^{1/n}\}$ has one branch point at $y = 0$, we can use the identity (2.13). The proof of the proposition is similar to the proof of the Main Theorem in the paper [1], where we take $n = 2^k, k \in \mathbb{N}$ instead of $n = 2$.

If the number of singularities is infinite, we take the semi-circles $\gamma_m$ which is centered at point $a$, with radius $R_m = \pi^2 m^2, m \in \mathbb{N}$.

We illustrate the above Theorem by showing that the following examples.

Example 2.11. We show that

$$
L^{-1}_n\left\{\frac{1}{y^{2n} + a^{2n}}; x\right\} = \frac{n}{a^n} \sin(a^n x^n)
$$

(2.17)

where $Re a > 0$.

Using the assertion (2.15) of Theorem 2.3 we obtain

$$
L^{-1}_n\left\{\frac{1}{y^{2n} + a^{2n}}; x\right\} = \sum_{k=1}^{2} \text{Res}\left[n\frac{1}{y^{2} + a^{2n}} \exp(yx^n); y = y_k\right]
$$

(2.18)

where the singular points are $y_k = \mp ia^n, k = 1, 2$ and then we have

$$
\text{Res}\left[n\frac{1}{y^{2} + a^{2n}} \exp(yx^n); ia^n\right] = n \lim_{y \to ia^n} (y - ia^n)^{x} \frac{\exp(yx^n)}{y^2 + a^{2n}} = n \frac{\exp(ia^n x^n)}{2ia^n}
$$

(2.19)
and similarly we have
\[
\text{Res}[n \frac{1}{y^2 + a^{2n}} \exp(yx^n); -ia^n] = -n \frac{\exp(-ia^n x^n)}{2ia^n}.
\] (2.20)

Using the relations (2.19) and (2.20) we find the formula (2.17) from (2.18) as follows
\[
\mathcal{L}_n^{-1}\{ \frac{1}{y^{2n} + a^{2n}}; x \} = \frac{n}{a^n} \exp(ia^n x^n) - \exp(-ia^n x^n)
\]
\[
= \frac{n}{a^n} \sin(a^n x^n).
\] (2.21)

**Example 2.12.** We show that
\[
\mathcal{L}_n^{-1}\{ \frac{1}{y^n} \exp(-a^n x^n); x \} = n J_0(2a^n x^{n/2})
\] (2.22)
where \( J_0 \) is the Bessel function of order zero.

Using the assertion (2.15) of Theorem 2.3 we have
\[
\mathcal{L}_n^{-1}\{ \frac{1}{y^n} \exp(-a^n y^n) \exp(y x^n), y = y_k \}.
\] (2.23)

From the following Taylor expansions of the exponential functions in (2.23),
\[
n \frac{1}{y^n} \exp(-\frac{a^n}{y^n}) \exp(y x^n) = n \frac{1}{y} \sum_{m=0}^{\infty} (-1)^m \frac{a^{mn}}{m! y^m} \sum_{k=0}^{\infty} \frac{y^k x^{nk}}{k!}
\]
\[
= \frac{n}{y} [1 - \frac{a^n}{1!y} + \frac{a^{2n}}{2!y^2} - \frac{a^{3n}}{3!y^3} + ...] [1 + \frac{x^ny}{1!} + \frac{x^{2n}y^2}{2!} + \frac{x^{3n}}{3!} + ...]
\] (2.24)

we find that \( \text{Res}[n \frac{1}{y^n} \exp(\frac{a^n}{y^n}) \exp(y x^n)] \) as the coefficient of the term \( \frac{1}{y} \) as follows
\[
\text{Res}[n \frac{1}{y} \exp(\frac{a^n}{y^n}) \exp(y x^n)] = n [1 - \frac{a^n x^n}{(1!)^2} + \frac{a^{2n} x^{2n}}{(2!)^2} - \frac{a^{3n} x^{3n}}{(3!)^2} + ...]
\]
\[
= n \sum_{m=0}^{\infty} (-1)^m \frac{(ax)^{mn}}{(n!)^2} = n J_0(2a^{n/2} x^{n/2}).
\] (2.25)

Thus, we obtain from (2.25) and the formula (2.23), the assertion (2.22) of Example 2.12.

### 3. Application of the \( \mathcal{L}_n \)-transform to ordinary differential equations

First we consider the ordinary differential equation (1.53) for \( v > n \) and \( v = 2m + 1, m = 0, 1, 2, ... \)

Dividing (1.53) by \( x^{n-1} \), adding and subtracting the term \( \frac{n+1}{x^n-1}z' \) we obtain
\[
x^n \left( \frac{1}{a^{2n-2}z^n} - \frac{n-1}{x^{2n-1}z'} \right) + \frac{n-1}{x^{n-1}z'} - \frac{2v + n - 3}{x^{n-1}z'} + z = 0.
\] (3.1)
Using the definition of the $\delta$-derivative given in (2.1) and (2.2), we can express (3.1) as
\[
x^n \delta_x^2 z(x) - 2(v - 1) \delta_x z(x) + z(x) = 0. \tag{3.2}
\]
Applying the $L_n$-transform to (3.2) we find
\[
L_n\{x^n \delta_x^2 z; y\} - 2(v - 1) L_n\{\delta_x z; y\} + L_n\{z; y\} = 0. \tag{3.3}
\]
Using Theorem 2.1 for $k = 1$ and $k = 2$ in (3.3) and performing necessary calculations we obtain
\[
- \frac{1}{n} \delta_y L_n\{\delta_x^2 z; y\} - 2(v - 1) L_n\{\delta_x z; y\} + L_n\{z; y\} = 0, \tag{3.4}
\]
\[
- \frac{1}{n} \left[ \frac{d}{dy} (n^2 y^{2n} \pi(y) - n y^n z(0^+) - \delta_x z(0^+)) - 2(v - 1)(n y^n \pi(y) - z(0^+)) \right] \pi(y) = 0 \tag{3.5}
\]
where $\pi(y) = L_n\{z(x); y\}$. We assume that $z(0^+) = 0$. Thus, we obtain the following first order differential equation:
\[
\pi'(y) + (2(n + v - 1) \frac{1}{y} - \frac{1}{ny^{n+1}}) \pi(y) = 0. \tag{3.6}
\]
Solving the first order differential equation (3.6) we have
\[
\pi(y) = C \sum_{m=0}^{\infty} (-1)^m \frac{1}{m! n^m y^{mn+2n+2v-2}}. \tag{3.7}
\]
Applying the $L_n^{-1}$ transform we obtain
\[
z(x) = C \sum_{m=0}^{\infty} (-1)^m \frac{x^{mn+n+2v-2}}{m! \Gamma(m + \frac{n+2v-2}{n} + 1)n^{2m-1}}. \tag{3.8}
\]
where we use the following relations
\[
L_n\{x^k; y\} = \frac{\Gamma(k + \frac{n}{n} + 1)}{ny^{n+k}}, \quad k = mn + n + 2v - 2 \tag{3.9}
\]
and
\[
L_n^{-1}\left\{ \frac{1}{y^{mn+n+2v-2+n}} \right\} = \frac{n x^{mn+n+2v-2}}{\Gamma(m + 1 + \frac{2v-2}{n} + 1)}. \tag{3.10}
\]
Setting $\alpha = \frac{2v+n-2}{n}$, $C = n^{-\frac{2v-2}{n}}$ we obtain the solution of the ordinary differential equation (1.53)
\[
z(x) = x^{\frac{2v+n-2}{n}} J_\alpha \left( \frac{2v+n-2}{n} \right) \tag{3.11}
\]
where $\alpha \in \mathbb{Z}$ because of the inequality $v > n \ (v, n \in \mathbb{N})$ and $J_\alpha$ is the Bessel function of the first kind of order $\alpha$.

In the second step we will use the $L_n$-transform for solving (1.54). Dividing (1.54) by $x^{n-1}$, adding and subtracting the term $\frac{1}{x^{n-1}} z'$ we obtain
\[
x^n \left( \frac{1}{x^{2n-2} z''(x)} - \frac{n-1}{x^{n-1} z'(x)} \right) + n \frac{1}{x^{n-1}} z'(x) - (n^2 - 1) \frac{1}{x^{n-1}} z'(z) + z(x) = 0. \tag{3.12}
\]
Using the definition of the $\delta_x$-derivative (2.1) and (2.2) we can express (3.12) as

$$x^n\delta_x^2 z(x) - n(n-1)\delta_x z(x) + z(x) = 0. \quad (3.13)$$

Considering the following relations

$$L_n\{x^n\delta_x^2 z(x); y\} = -\frac{1}{n} \delta_y L_n\{\delta_x^2 z(x); y\} = -2n^2 y^n \zeta(y) - ny^{n+1} \zeta'(y) + nz(0^+), \quad (3.14)$$

$$n(n-1)L_n\{\delta_x z(x); y\} = n(n-1)(ny^n \zeta(y) - \zeta(0^+)) = n^2 (n-1)y^n \zeta(y) - n(n-1)z(0^+) \quad (3.15)$$

and applying the $L_n$-transform to (3.13) we obtain

$$L_n\{x^n\delta_x^2 z(x); y\} - n(n-1)L_n\{\delta_x z(x); y\} + L_n\{z(x); y\} = 0 \quad (3.16)$$

$$ny^{n+1} \zeta'(y) + [n^2(n+1)y^n - 1] \zeta(y) - n^2 z(0^+) = 0 \quad (3.17)$$

where $\zeta(y) = L_n\{z(x); y\}$.

We may assume

$$z(0^+) = 0. \quad (3.18)$$

Solving the first order differential equation after substituting (3.18) into (3.17) we get

$$\zeta(y) = Cy^{-n^2-n} \exp\left(-\frac{1}{n^2y^n}\right). \quad (3.19)$$

Calculating the Taylor expansion of the exponential function in (3.19) we have

$$\zeta(y) = C \sum_{m=0}^{\infty} \frac{(-1)^m}{m!n^{2m}y^{n+nm+n^2}}. \quad (3.20)$$

Using the following relation

$$L_n^{-1}\{\frac{1}{y^{n+nm+n^2}}\} = \frac{n^2x^{n+nm+n^2}}{\Gamma(m + n + 1)} \quad (3.21)$$

and applying the $L_n^{-1}$ transform to (3.20) we find

$$z(x) = Cn^{n+1}x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + n + 1)} \left(\frac{2x^{n/2}}{2n}\right)^{m+n}. \quad (3.22)$$

Setting $C = n^{-n-1}$ in (3.22) we obtain the solution of the equation (1.54)

$$z(x) = x^{n^2} J_n\{\frac{2}{n}x^{n/2}\} \quad (3.23)$$

where $J_n$ is the Bessel function of the first kind of order $n$.  

13
References

[1] Aghili A, Ansari A, Sedghi A.: An inversion technique for the $L_2$-transform with applications. Int. J. Contemp. Math. Sciences 2.28, 1387-1394 (2007).

[2] Erdelyi, A.: Tables of Integral Transforms vol. 1. New York, NY, USA, McGraw-Hill, (1954).

[3] Erdelyi, A.: Tables of Integral Transforms vol. 2. New York, NY, USA, McGraw-Hill, (1954).

[4] Yürekli O, Sadek I.: A Parseval-Goldstein type theorem on the Widder potential transform and its applications. International Journal of Mathematics and Mathematical Sciences 14.3, 517-524 (1991).

[5] Yürekli O.: Theorems on $L_2$-transform and its applications. Complex Variables and Elliptic Equations 38.2, 95-107 (1999).

[6] Yürekli O.: New identities involving the Laplace and the $L_2$-transforms and their applications. Applied Mathematics and Computation 99.2, 141-151 (1999).

[7] Yürekli O, Wilson S.: A new method of solving Bessel’s differential equation using the $L_2$-transform. Applied Mathematics and Computation 130.2, 587-591 (2002).

[8] Yürekli O, Wilson S.: A new method of solving Hermite’s differential equation using the $L_2$-transform. Applied Mathematics and Computation 145.2, 495-500 (2003).