Switching properties of time optimal controls for systems of heat equations coupled by constant matrices

Shulin Qin† Gengsheng Wang‡ Huaiqiang Yu§

Abstract

This paper studies the time optimal control problem for systems of heat equations coupled by a pair of constant matrices. The control constraint is of the ball-type, while the target is the origin of the state space. We obtain an upper bound for the number of switching points of the optimal control over each interval with a fixed length. Also, we prove that at each switching point, the optimal control jump from one direction to the reverse direction.

Keywords. Time optimal control, system of heat equations, switching points, maximum principle

2010 Mathematics Subject Classifications. 93C20, 49B22, 49J20

1 Introduction

We start with introducing notation: Write $\mathbb{R}^+ := (0, +\infty)$, $\mathbb{N} := \{0, 1, \ldots\}$ and $\mathbb{N}^+ := \{1, 2, \ldots\}$. Let $\Omega \subset \mathbb{R}^N$ (with $N \in \mathbb{N}^+$) be a bounded domain with a $C^2$ boundary $\partial \Omega$. Let $\omega \subset \Omega$ be a nonempty and open subset, with its characteristic function $\chi_\omega$. Let $\Delta$ be the Laplace operator with its domain $D(\Delta) := H^2_0(\Omega) \cap H^2(\Omega)$. Let $I_k$ (with $k \in \mathbb{N}^+$) be the $k \times k$ identity matrix. Given a square matrix $D$, we write $\sigma(D)$ for its spectral. Denote by $B_m^1(0)$ (with $m \in \mathbb{N}$) the closed unit ball in $L^2(\Omega; \mathbb{R}^m)$ centered at the origin. Given two Banach spaces $K$ and $F$, write $L(K; F)$ for the space of all linear bounded operators from $K$ to $F$. Given $a \in \mathbb{R}$, let $[a]$ be the largest integer less than or equal to $a$. Given a subset $E \subset \mathbb{R}^+$, denote respectively by $\#E$ and $|E|$ the cardinality and the measure (if it is measurable) of $E$. Write $L_{x,t}^\infty$ for the space $L^\infty(\Omega \times \mathbb{R}^+)$. Given $T > 0$, let

$$P\mathcal{C}([0, T]; L^2(\Omega; \mathbb{R}^m)) := \left\{ f : [0, T) \rightarrow L^2(\Omega; \mathbb{R}^m) \mid f \text{ has at most finite discontinuities which are of the first kind} \right\}. \tag{1.1}$$

(Here, a discontinuity $\hat{t}$ of $f$ is called to be of the first kind, if both $\lim_{t \to \hat{t}^-} f(t)$ and $\lim_{t \to \hat{t}^+} f(t)$ exist.)

1.1 Control problem

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m} \setminus \{0\}$ (with $n, m \in \mathbb{N}^+$). Consider the system of coupled heat equations:

$$\begin{cases}
y_t = (I_n \Delta + A)y + \chi_\omega Bu & \text{in } \Omega \times \mathbb{R}^+,
y = 0 & \text{on } \partial \Omega \times \mathbb{R}^+,
y(0) = y_0 \in L^2(\Omega; \mathbb{R}^n),
\end{cases} \tag{1.2}$$

This work was partially supported by the National Natural Science Foundation of China under grants 11601377, 11901432, 11971022.

†School of Science, Tianjin University of Commerce, Tianjin 300134, China (shulinqin@yeah.net)
‡Center for Applied Mathematics, Tianjin University, Tianjin 300072, China (wanggs62@yeah.net)
§School of Mathematics, Tianjin University, Tianjin 300354, China (huaiqiangyu@tju.edu.cn)
where \( u \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^m)) \). We will treat the solution to (1.2) as a function from \([0, +\infty)\) to \(L^2(\Omega; \mathbb{R}^m)\) and denoted it by \( y(\cdot; y_0, u) \). Let

\[
\mathcal{A} := I_n \triangle + A \text{ and } \mathcal{B} := \chi_B.
\]

(1.3)

One can directly check that the operator \( \mathcal{A} \), with its domain \( H^1_0(\Omega; \mathbb{R}^n) \cap H^2(\Omega; \mathbb{R}^n) \), generates an analytic semigroup \( \{e^{tA}\}_{t \geq 0} \) on \( L^2(\Omega; \mathbb{R}^n) \). Then for each \( y_0 \in L^2(\Omega; \mathbb{R}^n) \) and each \( u \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^m)) \),

\[
y(t; y_0, u) = e^{tA}y_0 + \int_0^t e^{(t-s)A}Bu(s)ds, \quad t \geq 0.
\]

We next introduce our time optimal control problem:

\[
(\mathcal{TP})_{y_0} : \quad T^*_{y_0} := \inf \left\{ \hat{t} > 0 : \exists u \in L^\infty(\mathbb{R}^+; B_i^{m}(0)) \text{ s.t. } y(\hat{t}; y_0, u) = 0 \right\},
\]

(1.4)

where \( y_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \) is the initial state, \( B_i^{m}(0) \) is the control constraint set, \( \{0\} \in L^2(\Omega; \mathbb{R}^n) \) is the target set. In the above problem, the number \( T^*_{y_0} \) is called the optimal time; \( u \in L^\infty(\mathbb{R}^+; B_i^{m}(0)) \) is called an admissible control if there is \( t \in \mathbb{R}^+ \) so that \( y(t; y_0, u) = 0 \); \( u^{*}_{y_0} \in L^\infty(\mathbb{R}^+; B_i^{m}(0)) \) is called an optimal control if \( y(T^*_{y_0}; y_0, u^{*}_{y_0}) = 0 \) and \( u^{*}_{y_0}(\cdot) = 0 \) over \( (T^*_{y_0}, +\infty) \). (The effective domain of \( u^{*}_{y_0} \) is \([0, T^*_{y_0}])\).

Thus, the optimal control to \((\mathcal{TP})_{y_0}\) is unique, if any two optimal controls coincide a.e. over \([0, T^*_{y_0}])\.

The main assumption of this paper is as: the initial state \( y_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \) satisfies

* Assumption \((\mathcal{A})_{y_0}\): The problem \((\mathcal{TP})_{y_0}\) has an admissible control.

Several notes on the assumption \((\mathcal{A})_{y_0}\) are given in order.

(a1) The reason that we ask \( y_0 \neq 0 \) is as: when \( y_0 = 0 \), the problem \((\mathcal{TP})_{y_0}\) is trivial.

(a2) The assumption \((\mathcal{A})_{y_0}\) is equivalent to that the problem \((\mathcal{TP})_{y_0}\) has an optimal control. (See Theorem 3.11 in [22, Chapter 3].)

(a3) In many cases, \( y_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \) satisfies \((\mathcal{A})_{y_0}\). For instance, according to [12, Theorem 3.1], any \( y_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \) holds \((\mathcal{A})_{y_0}\), provided that the system \((1.2)\) is null controllable on some \([0, T]\) and \( \|e^{(t_1+t_2)A}\|_{L(L^2(\Omega; \mathbb{R}^n); L^2(\Omega; \mathbb{R}^n))} \leq 1 \) for each \( t \in \mathbb{R}^+ \). For more studies on this issue, we refer the readers to [22, Chapter 3].

We end this subsection with introducing the following subspace:

\[
\mathfrak{L} := L^2(\Omega; \mathfrak{R}) \text{ where } \mathfrak{R} := \left\{ \sum_{j=0}^{n-1} A^jBv_j : \{v_j\}_{j=0}^{n-1} \subset \mathbb{R}^m \right\}.
\]

(1.5)

The space \( \mathfrak{L} \) is indeed the controllable subspace of the system \((1.2)\). Our Corollary 2.3 says that if \( y_0 \) satisfies \((\mathcal{A})_{y_0}\), then \( y_0 \in \mathfrak{L} \).

1.2 Main results

We start with the next definition.

**Definition 1.1.** Let \( T > 0 \) and \( u \in \mathcal{PC}([0, T]; L^2(\Omega; \mathbb{R}^m)) \). The number \( \hat{t} \in (0, T) \) is said to be a switching point of \( u \), if both \( \lim_{t \to \hat{t}^-} u(t) \) and \( \lim_{t \to \hat{t}^+} u(t) \) exist and \( \lim_{t \to \hat{t}^-} u(t) \neq \lim_{t \to \hat{t}^+} u(t) \).

We next introduce two important numbers \( d_A \) and \( q_{A,B} \):

\[
d_A := \min \{ \pi/|\text{Im}\lambda| : \lambda \in \sigma(A) \};
\]

(1.6)
\[ q_{A,B} := \max \left\{ \text{rank}(b, A b, \ldots, A^{n-1} b) : b \text{ is a column of } B \right\}. \tag{1.7} \]

In (1.6), we agree that 1/0 = +∞, consequently, we have \( d_A = +\infty \), when \( \sigma(A) \subset \mathbb{R} \). In (1.7), we have \( q_{A,B} \leq n \). The numbers \( d_A \) and \( q_{A,B} \) were introduced in [15], where the controllability of impulse controlled systems of heat equations coupled by constant matrices was studied.

The main results are now stated as follows:

**Theorem 1.2.** Suppose \( y_0 \in L^2(\Omega; \mathbb{R}^n)\setminus\{0\} \) satisfies the assumption \( (A)_{y_0} \). Then the following conclusions are true:

(i) The problem \((TP)_{y_0}\) has a unique optimal control \( u^*_{y_0} \) satisfying \( \|u^*_{y_0}(t)\|_{L^2(\Omega;\mathbb{R}^n)} = 1 \) for a.e. \( t \in (0, T^*_{y_0}) \) (i.e., it has the bang-bang property). Moreover, the restriction of \( u^*_{y_0} \) over \([0, T^*_{y_0}]\) is in the space \( PC([0, T^*_{y_0}]; B^m_0(0)) \), which is given by (1.1);

(ii) For any open interval \( I \subset (0, T^*_{y_0}) \) with \( |I| \leq d_A \), the optimal control \( u^*_{y_0} \) has at most \( (q_{A,B} - 1) \) switching points in \( I \), where \( d_A \) and \( q_{A,B} \) are given by (1.6) and (1.7), respectively;

(iii) Let \( \hat{t} \in (0, T^*_{y_0}) \) be a switching point of \( u^*_{y_0} \). Then
\[
\lim_{t \to \hat{t}^-} u^*_{y_0}(t) + \lim_{t \to \hat{t}^+} u^*_{y_0}(t) = 0.
\]

Several notes on Theorem 1.2 are listed in order.

(b1) From (i) of Theorem 1.2, we see that \( u^*_{y_0} \in PC([0, T^*_{y_0}]; B^m_0(0)) \). It is natural to ask the behaviour of \( u^*_{y_0} \) at \( T^*_{y_0} \). Unfortunately, this is a very hard problem for us.

(b2) In (ii) of Theorem 1.2, we only give an upper bound for the number of switching points of \( u^*_{y_0} \) in any open subinterval \( I \subset (0, T^*_{y_0}) \) with \( |I| \leq d_A \). How to get a global upper bound over \((0, T^*_{y_0})\) is extremely hard for us. However, for the special case that \( \sigma(A) \subset \mathbb{R} \), we have \( d_A = +\infty \), and thus \( u^*_{y_0} \) has at most \( (q_{A,B} - 1) \) switching points over the whole the interval \((0, T^*_{y_0})\).

(b3) For the pure heat equation on \( \Omega \), i.e., \( A = 0 \) and \( B = I_1 \), we see from (ii) of Theorem 1.2 that the corresponding time optimal control \( u^*_{y_0} \) has no any switching point, and thus \( u^*_{y_0} \) is continuous over \([0, T^*_{y_0}]\). Indeed, in this case, \( d_A = +\infty \) (see the statement in (b2)) and \( q_{A,B} = 1 \). From this, we can say that the coupling causes switching points.

(b4) The conclusion (iii) in Theorem 1.2 says that the optimal control jumps from one direction to its reverse direction at each switching point.

1.3 Comparison with related works

To our best knowledge, the studies on the switching points for time optimal controls governed by PDEs have not been touched upon. There have been some literatures on the related studies for ODEs, for instance, [13, 14, 16, 18, 19] and references therein. We would like mention, in particular, the work [16], where the similar problem was studied and the similar results were obtained for ODEs. However, it is not easy to extend results from finite-dimensional systems to infinite-dimensional systems. Indeed, to study the switching points for the problem \((TP)_{y_0}\), we built up an \( L^\infty \) null controllability for (1.1), used some point-wise unique continuation to the dual system of (1.2), and utilized some results obtained in [15]. With regard to time optimal controls for parabolic equations, we would like mention [2–5, 9, 20, 21, 24–26] and the references therein.

1.4 Plan of this paper

The rest of the paper is organized as follows: Section 2 gives some auxiliary results; Section 3 proves the main theorem; Section 4 presents an example.
2 Auxiliary results

2.1 Decomposition of the system

This subsection presents a decomposition of the system (1.2) from perspective of the controllability. We starts with introducing the following well-known Kalman controllability decomposition for ODEs (see [18, Lemma 3.3, p.93]):

**Lemma 2.1.** Let \( R \) be given in (1.5) with \( k \triangleq \dim \mathcal{R} \). Then there is an invertible matrix \( P \in \mathbb{R}^{n \times n} \) with \( P^T = P^{-1} \) and four matrices \( A_1 \in \mathbb{R}^{k \times k} \), \( A_2 \in \mathbb{R}^{k \times (n-k)} \), \( A_3 \in \mathbb{R}^{(n-k) \times (n-k)} \), \( B_1 \in \mathbb{R}^{k \times m} \) so that

\[
P^{-1}R = \mathbb{R}^k \times \{0\}, \quad P^{-1}AP = \begin{pmatrix}
A_1 & A_2 \\
0 & A_3
\end{pmatrix}
\text{ and } P^{-1}B = \begin{pmatrix}
B_1 \\
0
\end{pmatrix} ,
\]

and so that

\[
\text{rank}(B_1, A_1B_1, \ldots, A_1^{k-1}B_1) = k.
\]

(Here, it is agreed that \( A_2, A_3 \) are not there if \( k = n \).)

With the help of Lemma 2.1, we have the following decomposition for the system (1.2):

**Proposition 2.2.** Let the matrices \( P, \{A_j\}_{j=1}^3 \) and \( B_1 \) be given in Lemma 2.1. Then for each \( t \geq 0 \),

\[
P^{-1}e^{tA}P = \begin{pmatrix}
e^{(t_k \Delta + A_1)} & (M(t)) \\
e^{(t_k \Delta + A_3)} & 0
\end{pmatrix}
\text{ and } P^{-1}e^{tA}B = \begin{pmatrix}
e^{(t_k \Delta + A_1)} & \chi_\omega B_1 \\
e^{(t_k \Delta + A_3)} & 0
\end{pmatrix} ,
\]

where the operator \((M(t))\) is as:

\[
M(t) := e^{t_k \Delta} \int_0^t e^{(t-s)A_1}A_2e^{sA_3}ds.
\]

**Proof.** Arbitrarily fix \( t \geq 0 \). Since \( e^{tA} = e^{t_k \Delta}e^{tA} \) (see for instance [15, Proposition 3.1]), it follows from the first equality in (2.1) that

\[
P^{-1}e^{tA}P = e^{t_k \Delta} \left( \begin{array}{cc}
e^{t_k \Delta} & 0 \\
e^{t_k \Delta} & 0
\end{array} \right) \left( \begin{array}{cc}
e^{t_k \Delta} & 0 \\
e^{t_k \Delta} & 0
\end{array} \right) \left( \begin{array}{cc}
e^{t_k \Delta} & 0 \\
e^{t_k \Delta} & 0
\end{array} \right)
\]

This leads to the first equality in (2.3). The second one in (2.3) can be proved in a very similar way. This finishes the proof of Proposition 2.2.

**Corollary 2.3.** Suppose that \( y_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \) satisfies the assumption \((A)_{y_0} \). Then the following conclusions are true:

(i) It holds that \( 0 < T_{y_0}^* < +\infty \);

(ii) There is a unique \( y_0 \in L^2(\Omega; \mathbb{R}^k) \) so that \( P^{-1}y_0 = (\hat{y}_0, 0)_T \), where \( P \) is given in Lemma 2.1. In particular, \( y_0 \in \mathcal{L} \), where \( \mathcal{L} \) is given by (1.5).

**Proof.** From \((A)_{y_0}\), one can easily derive the conclusion (i) (see \((a_2)\) in Section 1.1). We next prove the conclusion (ii). When \( k = n \), we have \( \mathcal{L} = L^2(\Omega; \mathbb{R}^n) \), which leads to (ii). We now suppose that \( k < n \). Write

\[
P^{-1}y_0 := (y_1, y_2)_T, \quad \text{with } y_1 \in L^2(\Omega; \mathbb{R}^k) \text{ and } y_2 \in L^2(\Omega; \mathbb{R}^{n-k}).
\]
By \((A)_{0}\), we can find \(u \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^m))\) and \(t > 0\) so that

\[
0 = y(t; y_0, u) = e^{tA}y_0 + \int_0^te^{(t-s)A}Bu(s)ds
\]

\[
= P \left[ (P^{-1}e^{tA}P)P^{-1}y_0 + \int_0^t (P^{-1}e^{(t-s)A}B)u(s)ds \right] = P \left( e^{t(\ln n + \Delta + A_0)}y_2 \right).
\]

This yields that

\[
0 = e^{t(\ln n + \Delta + A_0)}y_2.
\]

From this and [15, (ii) of Proposition 3.2], it follows that \(y_2 = 0\). Then the conclusion (ii) follows from (2.4) and the first equation in (2.1). This completes the proof of Corollary 2.3.

2.2 Null controllability of the system

This subsection studies the \(L^\infty_{x,t}\) null controllability for the system (1.2).

**Proposition 2.4.** Let \(A\) and \(B\) be given by (1.3). Then the following three statements are equivalent:

(i) It holds that rank \((B, AB, \cdots, A^{n-1}B) = n\);

(ii) The system (1.2), with \(L^\infty_{x,t}\)-controls, is null controllable, i.e., for each \(T > 0\) and each \(y_0 \in L^2(\Omega; \mathbb{R}^n)\), there is \(u \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^m))\) so that \(y(T; y_0, u) = 0\).

(iii) There is \(C = C(\Omega, \omega, A, B) > 0\) so that for each \(T > 0\),

\[
\|e^{TA^*}z\|_{L^2(\Omega; \mathbb{R}^n)} \leq Ce^{\frac{T}{S^{p'}}} \frac{1}{S} \int_{T-S}^T \|B^*e^{tA^*}z\|_{L^1(\Omega; \mathbb{R}^m)}dt \quad \text{for each} \quad z \in L^2(\Omega; \mathbb{R}^n).
\]

**Remark 2.5.** The following two remarks are worth mentioning.

(i) The \(L^2\)-controllability of coupled parabolic equations was obtained in [8]. Proposition 2.4 improve the corresponding result in [8]. We also mention the work [10] for the \(L^2\)-controllability of an abstract heat-like equation.

(ii) In this paper, we only need the null controllability over each \((0, T)\) for the system (1.2) with \(L^\infty(0, T; L^2(\Omega_0; \mathbb{R}^m))\) controls. This is weaker than the controllability in Proposition 2.4. However, the later may have independent significance. This is the reason we present it here.

The following lemma is used to prove Proposition 2.4.

**Lemma 2.6.** Suppose that \(\text{rank } (B, AB, \cdots, A^{n-1}B) = n\). Let

\[p := \min \left\{ j \in \mathbb{N}^+ : \text{rank } (B, AB, \cdots, A^{j-1}B) = n \right\}.
\]

Then there is \(C = C(\Omega, \omega, A, B) > 0\) so that when \(\theta \in (0, 1)\) and \(T \geq S > 0\),

\[
\|e^{TA^*}z\|_{L^2(\Omega; \mathbb{R}^n)} \leq Ce^{\frac{T}{S^{p'}}} \left( \frac{1}{S} \int_{T-S}^T \|B^*e^{tA^*}z\|_{L^1(\Omega; \mathbb{R}^m)}dt \right)^{1-\theta} \quad \text{for all } z \in L^2(\Omega; \mathbb{R}^n).\quad (2.5)
\]

**Proof.** Write \(\{\lambda_j\}_{j \geq 1}\), with \(\lambda_1 < \lambda_2 \leq \cdots\), for all eigenvalues of the operator \((-\Delta, H^1_0(\Omega) \cap H^2(\Omega))\).

Let \(e_j\) (with \(j = 1, 2, \ldots\)) be the corresponding normalized eigenfunction. We organize the proof by two steps.
Step 1. We show that there is $C = C(\Omega, \omega, A, B) > 0$ so that when $t \in (0, 1)$ and $\lambda > 0$, 

$$
\|e^{tA^*}z\|_{L^2(\Omega; \mathbb{R}^n)} \leq Ce^{C\sqrt{\lambda}} \frac{1}{t} \int_0^t \|B^*e^{sA^*}z\|_{L^1(\Omega; \mathbb{R}^n)} ds
$$

(2.6)

for all $z = \sum_{\lambda_j \leq \lambda} z_je_j$, with $\{z_j\}_{j \in \{i \in \mathbb{N}^+ : \lambda_i \leq \lambda\}} \subset \mathbb{R}^n$.

For this purpose, we arbitrarily fix $t \in (0, 1)$, $\lambda > 0$ and $z = \sum_{\lambda_j \leq \lambda} z_je_j$ with $\{z_j\}_{j \in \{i \in \mathbb{N}^+ : \lambda_i \leq \lambda\}} \subset \mathbb{R}^n$.

First, one can easily see that for each $s \in (0, t)$,

$$
e^{sA^*}z = \sum_{\lambda_j \leq \Lambda} (e^{-\lambda_j s}e^{sA^*}z) e_j
$$

and

$$
B^*e^{sA^*z} = \sum_{\lambda_j \leq \Lambda} (e^{-\lambda_j s}B^Te^{sA^*}z) e_j.
$$

(2.7)

Second, since rank $(B, AB, \cdots , A^{n-1}B) = n$, we can apply [6, Lemma 2] (or [17, Theorem 1]) to find $C = C(A, B) > 0$ (independent of $t \in (0, 1)$) so that

$$
\|e^{tA^*}v\|_{\mathbb{R}^n} \leq C \frac{1}{t} \int_0^t \|B^*e^{sA^*}v\|_{\mathbb{R}^n} ds, \text{ when } v \in \mathbb{R}^n.
$$

From this and the first equation (2.7), it follows that

$$
\|e^{tA^*}z\|_{L^2(\Omega; \mathbb{R}^n)} = \sum_{\lambda_j \leq \Lambda} \|e^{-\lambda_j t}e^{tA^*}z_j\|_{\mathbb{R}^n} \leq \sum_{\lambda_j \leq \Lambda} e^{-\lambda_j t} \left( C \frac{1}{t} \int_0^t \|B^*e^{sA^*}z_j\|_{\mathbb{R}^n} ds \right)
$$

$$
\leq \frac{C}{t} \int_0^t \sum_{\lambda_j \leq \Lambda} \left( \|B^*e^{-\lambda_j s}e^{sA^*}z_j\|_{\mathbb{R}^n} \right) ds = C \frac{1}{t} \int_0^t \|B^*e^{sA^*}z\|_{L^2(\Omega; \mathbb{R}^n)} ds.
$$

(2.8)

Meanwhile, according to [1, Theorems 5,3,8] (see [7] for the original study), there is $C = C(\Omega, \omega) > 0$ so that for each sequence $\{a_j\}_{\lambda \leq \Lambda} \subset \mathbb{R}$,

$$
\sum_{\lambda_j \leq \Lambda} a_j^2 \leq Ce^{C\sqrt{\lambda}} \left( \sum_{\lambda_j \leq \Lambda} a_j e_j \right)^2_{L^1(\omega)}.
$$

Set $v_j(s) := e^{-\lambda_j s}B^Te^{sA^*}z_j$ when $\lambda_j \leq \lambda$. The above (adapted to the vector valued case), along with the second equation in (2.7), yields that for each $s \in (0, t)$,

$$
\|B^*e^{sA^*}z\|_{L^2(\Omega; \mathbb{R}^n)}^2 = \left( \sum_{\lambda_j \leq \Lambda} v_j(s)e_j \right)^2_{L^2(\Omega; \mathbb{R}^n)} = \sum_{\lambda_j \leq \Lambda} \|v_j(s)\|_{\mathbb{R}^n}^2
$$

$$
\leq Ce^{C\sqrt{\lambda}} \sum_{\lambda_j \leq \Lambda} \|v_j(s)e_j\|_{L^1(\omega; \mathbb{R}^n)}^2 = Ce^{C\sqrt{\lambda}} \|B^*e^{sA^*}z\|_{L^1(\omega; \mathbb{R}^n)}^2,
$$

which, together with (2.8), leads to (2.6).

Step 2. We use (2.6) to prove (2.5).

We only need to show (2.5) for $z \neq 0$. Arbitrarily fix $\lambda > 0$, $\theta \in (0, 1), 0 < S \leq T$ and $z \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\}$. Write

$$
z = \sum_{j \geq 1} z_je_j = \sum_{\lambda_j \leq \Lambda} z_je_j + \sum_{\lambda_j > \Lambda} z_je_j := z_{\lambda \leq \Lambda} + z_{> \Lambda},
$$

where $\{z_j\}_{j \in \mathbb{N}^+} \subset \mathbb{R}^n$. Set $S_1 := \min\{S, T/2\}$. It is clear that $S/2 \leq S_1 \leq S$. By (2.6), where $(t, z)$ is replaced by $(S_1, e^{(T-S_1)A^*}z)$, after some simple computations, we can find $C_1$ and $C_2$ (only depending on $\Omega$, $\omega$, $A$ and $B$) so that

$$
\|e^{T\Lambda^*}z\|_{L^2(\Omega; \mathbb{R}^n)} \leq \|e^{T\Lambda^*}z_{\leq \Lambda}\|_{L^1(\Omega; \mathbb{R}^n)} + \|e^{T\Lambda^*}z_{> \Lambda}\|_{L^2(\Omega; \mathbb{R}^n)}
$$

(2.9)

For $z_{> \Lambda}$, we can show that

$$
\|e^{T\Lambda^*}z_{> \Lambda}\|_{L^2(\Omega; \mathbb{R}^n)} \leq Ce^{C\sqrt{\lambda}} \frac{1}{T-S} \int_{T-S}^T \|B^*e^{s\Lambda^*}z_{> \Lambda}\|_{L^1(\Omega; \mathbb{R}^n)} ds
$$

(2.10)

for all $z_{> \Lambda}$ as above. Combining (2.9) and (2.10), we complete the proof of (2.5).
Indeed, if \((7.4)\), implies that 
\[
\lambda > 0
\]

Meanwhile, we have 
\[
\int_{T-S}^{T} \|B^* e^{sA^*} z\|_{L^2(\Omega; R^n)} ds \neq 0. \tag{2.11}
\]

Indeed, if (2.11) were not true, then by (2.10), we would have 
\[
\|e^{TA^*} z\|_{L^2(\Omega; R^n)} \leq \frac{C_2}{S^{p-1}} e^{\frac{C}{S^2} \int_{T-S}^{T} \|B^* e^{sA^*} z\|_{L^1(\Omega; R^n)} ds} \|z\|_{L^2(\Omega; R^n)} \quad (\varepsilon \in (0, 1)).
\]

\[
\text{Letting } \varepsilon \to 0^+ \text{ in the above leads to } \|e^{TA^*} z\|_{L^2(\Omega; R^n)} = 0. \text{ This, together with } [15, (ii) \text{ of Proposition 3.2}], \text{ implies that } z = 0 \text{ which leads to a contradiction. So (2.11) is true.}
\]

In the case that \(\|z\|_{L^2(\Omega; R^n)} > \frac{1}{S} \int_{T-S}^{T} \|B^* e^{sA^*} z\|_{L^1(\Omega; R^n)} ds\), one can directly check that 
\[
\|e^{TA^*} z\|_{L^2(\Omega; R^n)} \leq C_3 \|z\|_{L^2(\Omega; R^n)} \leq C_3 \left( \frac{1}{S} \int_{T-S}^{T} \|B^* e^{sA^*} z\|_{L^1(\Omega; R^n)} ds \right)^{1-\theta} \left( \|z\|_{L^2(\Omega; R^n)} \right)^{\theta}
\]

form some constants \(C_3\) and \(C_4\) depending only on \(A\). (Here we used \(T < 1\).) Thus, (2.5) is true in this case.

In the case that \(\|z\|_{L^2(\Omega; R^n)} > \frac{1}{S} \int_{T-S}^{T} \|B^* e^{sA^*} z\|_{L^1(\Omega; R^n)} ds\), we let 
\[
\varepsilon_0 := \left( \frac{1}{S} \int_{T-S}^{T} \|B^* e^{sA^*} z\|_{L^1(\Omega; R^n)} ds / \|z\|_{L^2(\Omega; R^n)} \right)^{\frac{1}{1+\theta}}.
\]

By (2.11), we have \(\varepsilon_0 \in (0, 1)\). Thus we can use (2.10) where \(\varepsilon = \varepsilon_0\) to find 
\[
\|e^{TA^*} z\|_{L^2(\Omega; R^n)} \leq 2 \frac{C_2}{S^{p-1}} e^{\frac{C}{S^2} \int_{T-S}^{T} \|B^* e^{sA^*} z\|_{L^1(\Omega; R^n)} ds} \left( \|z\|_{L^2(\Omega; R^n)} \right)^{\frac{\theta}{1+\theta}}.
\]

This, along with the fact \(\beta = \frac{\theta}{1+\theta}\), leads to (2.5) for this case.

In summary, we complete the proof of Lemma 2.6.
We now on the position to prove Proposition 2.4.

Proof of Proposition 2.4. We divide the proof by several steps.

Step 1. We show that (i) ⇒ (iii).

Suppose that (i) is true. It suffices to prove (iii) for the case that \( 0 < T < 1 \). To this end, we arbitrarily fix \( T \in (0, 1) \) and \( z \in L^2(\Omega; \mathbb{R}^n) \). By (i), we can apply Lemma 2.6, where \( (T, S, \theta, z) \) is replaced by \( (T/2^{j+1}, T/2^{j+1}, 1/3, e^{T A^*/2^{j+1}} z) \), with \( j \) a nonnegative integer, to find \( C = C(\Omega, \omega, A, B) > 0 \) so that for each \( \varepsilon > 0 \),

\[
\| e^{T A^*/2^j} z \|_{L^2(\Omega; \mathbb{R}^n)} \leq C e^{\varepsilon j} \left( \int_0^{T/2^{j+1}} \| B^* e^{s A^*/2^j} z \|_{L^1(\Omega; \mathbb{R}^m)} ds \right)^{2/3} \left( \| e^{T A^*/2^{j+1}} z \|_{L^2(\Omega; \mathbb{R}^n)} \right)^{1/3}
\]

\[
\leq C e^{\varepsilon j} \left( \int_0^{T/2^j} \| B^* e^{s A^*/2^j} z \|_{L^1(\Omega; \mathbb{R}^m)} ds \right)^{2/3} \left( \| e^{T A^*/2^{j+1}} z \|_{L^2(\Omega; \mathbb{R}^n)} \right)^{1/3}
\]

\[
\leq C e^{-\frac{\varepsilon}{2e}} 2 \frac{2}{3\sqrt{3}} \int_{T/2^{j+1}}^{T/2^j} \| B^* e^{s A^*/2^j} z \|_{L^1(\Omega; \mathbb{R}^m)} ds + \varepsilon \| e^{T A^*/2^{j+1}} z \|_{L^2(\Omega; \mathbb{R}^n)}.
\]

(In the above, we used the Young inequality.) Choosing \( \varepsilon = e^{-\frac{\varepsilon}{2e}} \) in the above leads to that for all \( j \geq 0 \),

\[
e^{-\frac{\varepsilon}{2e}} \| e^{T A^*/2^j} z \|_{L^2(\Omega; \mathbb{R}^n)} - e^{-\frac{\varepsilon}{2e}} \| e^{T A^*/2^{j+1}} z \|_{L^2(\Omega; \mathbb{R}^n)} \leq 2 \frac{2}{3\sqrt{3}} C e^{\frac{j}{2}} \int_{T/2^{j+1}}^{T/2^j} \| B^* e^{s A^*/2^j} z \|_{L^1(\Omega; \mathbb{R}^m)} ds
\]

Summing the above from \( j = 0 \) to \( +\infty \) yields

\[
\| e^{T A^*/2^j} z \|_{L^2(\Omega; \mathbb{R}^n)} \leq 2 \frac{2}{3\sqrt{3}} C e^{\frac{j}{2}} \int_0^T \| B^* e^{s A^*/2^j} z \|_{L^1(\Omega; \mathbb{R}^m)} ds,
\]

which leads to (iii).

Step 2. We show that (iii) ⇒ (ii).

This follows by the classical duality method. We omit the details here.

Step 3. We show that (ii) ⇒ (i).

By contradiction, we suppose that (ii) holds but

\[
\text{rank}(B, AB, \ldots, A^{n-1} B) < n.
\]

Then there is \( \vartheta \in \mathbb{R}^n \setminus \{0\} \) so that

\[
\langle \vartheta, v \rangle_{\mathbb{R}^n} = 0 \quad \text{for all} \quad v \in \mathcal{R}.
\]  \hspace{1cm} (2.12)

(Here, \( \mathcal{R} \) is given by (1.5).) Set

\[
y_0(x) := \vartheta \quad \text{for a.e.} \quad x \in \Omega.
\]  \hspace{1cm} (2.13)

According to (ii), there is a control \( u \in L^\infty(\Omega \times \mathbb{R}^+; \mathbb{R}^m) \) so that \( y(T; y_0, u) = 0 \). Let \( \varepsilon > 0 \) so that \( \varepsilon u \in L^\infty(\mathbb{R}^+; B^1_\varepsilon(0)) \). Then \( \varepsilon u \) is an admissible control to the problem \( (TP)_{\varepsilon u} \). From this and (ii) of Corollary 2.3, we see that \( y_0 \in \mathcal{L} \setminus \{0\} \) (where \( \mathcal{L} \) is given by (1.5)). This contradicts (2.13) and (2.12). Therefore, (i) is true.

Hence, we finish the proof of Proposition 2.4. \( \square \)
2.3 Unique continuation of the dual system

This subsection presents a continuation property for the dual system of (1.2).

**Proposition 2.7.** Let $I \subset \mathbb{R}^+$ be an open interval with $|I| \leq d_A$, where $d_A$ is given by (1.6). Suppose that there is $z \in L^2(\Omega; \mathbb{R}^n)$ and $\{t_j\}_{j=1}^{q_{A,B}} \subset I$ (where $q_{A,B}$ is given by (1.7)) so that

$$B^* e^{t_j A^*} z = 0 \text{ for each } j = 1, \ldots, q_{A,B}. \quad (2.14)$$

Then

$$z(x) \in \text{Ker} \left( \begin{array}{c} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{array} \right) \text{ for a.e. } x \in \Omega. \quad (2.15)$$

Especially, $B^* e^{t A^*} z = 0$ for each $t \geq 0$.

**Remark 2.8.** Suppose that $(A, B)$ satisfies the kalman rank condition. Then the right hand side of (2.15) is $\{0\}$. Thus, by Proposition 2.7, we see that (2.14) implies $z = 0$ over $\Omega$. This is a unique continuation property of the dual system of (1.2).

Before proving Proposition 2.7, we recall the following two lemmas:

**Lemma 2.9.** ([15, Theorem 5.2]) Let $\{t_j\}_{j=1}^{p} \subset (0, +\infty)$ with $p \in \mathbb{N}^+$. Then the following two statements are equivalent:

(i) It holds that $\text{rank} (e^{A t_1} B, e^{A t_2} B, \ldots, e^{A t_p} B) = n$;

(ii) If $z \in L^2(\Omega; \mathbb{R}^n)$ satisfies

$$B^* e^{t_j A^*} z = 0 \text{ in } L^2(\Omega; \mathbb{R}^m) \text{ for all } j \in \{1, 2, \ldots, p\},$$

then $z = 0$.

**Lemma 2.10.** ([15, Theorem 2.2]) Let $d_A$ and $q_{A,B}$ be given by (1.6) and (1.7), respectively. Then for each increasing sequence $\{t_j\}_{j=1}^{q_{A,B}} \subset \mathbb{R}$ with $t_{q_{A,B}} - t_1 < d_A$,

$$\text{rank} (e^{A t_1} B, e^{A t_2} B, \ldots, e^{A t_{q_{A,B}} B}) = \text{rank} (B, AB, \ldots, A^{n-1}B).$$

**Remark 2.11.** (a) We mention that (ii) of Lemma 2.9 can be replaced by an interpolation inequality. See [23, Proposition 1.3] for details; (b) The number $d_A$ is necessary to ensure Lemma 2.10. Besides, the optimality of the number $q_{A,B}$ is stressed in some sense in [15, Theorem 2.2].

We now on the position to prove Proposition 2.7.

**Proof of Proposition 2.7.** Let the matrices $P, \{A_j\}_{j=1}^{3}$ and $B_1$ be given in Lemma 2.1. Let $z_1 \in L^2(\Omega; \mathbb{R}^k)$ and $z_2 \in L^2(\Omega; \mathbb{R}^{n-k})$ satisfy that

$$P^T z = (z_1, z_2)^T. \quad (2.16)$$

By the second equality in (2.3), it follows that for each $t \geq 0$,

$$B^* e^{t A^*} z = B^* e^{t A^*} (P^{-1})^T (z_1, z_2)^T = (P^{-1} e^{t A B})^T z_1, z_2)^T = \chi_{\omega} B_1^T e^{t (I_{A_{\omega}} + A_{\omega}^T)} z_1. \quad (2.17)$$
From this and (2.14), one has that
\[ \chi_{\omega} B^T_{e_{j}} e_{j}(z_1, A^T_{A,B}) z_1 = 0, \quad j = 1, \ldots, q_{A,B}. \]

Using (2.2) and noting that \( d_A \leq d_{A_1} \) and \( q_{A,B} \geq q_{A_1,B_1} \), we can apply Lemmas 2.9, 2.10, where \((A,B)\) is replaced by \((A_1,B_1)\) to get
\[ z_1 = 0. \tag{2.18} \]

This, along with (2.16) and Lemma 2.1, yields that for a.e. \( x \in \Omega \),
\[ \langle v, z(x) \rangle_{\mathbb{R}^n} = (P^{-1} v, (z_1(x), z_2(x))^\top)_{\mathbb{R}^n} = 0, \quad \text{when } v \in \mathcal{R}, \]
(Here, \( \mathcal{R} \) is given by (1.5) which leads to (2.15).)

Finally, from (2.17) and (2.18), it follows that
\[ B^* e^{(T-t)A^*} z = 0 \quad \text{for each } t \geq 0. \]
This ends the proof of Proposition 2.7. \( \square \)

The following result is a direct consequence of Proposition 2.7.

**Corollary 2.12.** Let \( T > 0 \) and \( z \in L^2(\Omega; \mathbb{R}^n) \) satisfy that \( I_{z,T} \neq (0,T) \), where
\[ I_{z,T} := \{ t \in (0,T) : B^* e^{(T-t)A^*} z = 0 \}. \]

Then \( I_{z,T} \) has at most \((|T/d_A| + 1)(q_{A,B} - 1)\) elements, where \( A \) and \( q_{A,B} \) are given by (1.6) and (1.7), respectively.

**Proof.** Since \( I_{z,T} \neq (0,T) \), we have that \( B^* e^{(T-t)A^*} z \) is not a zero function over \( \mathbb{R}^+ \). By contradiction, we suppose that
\[ z[I_{z,T}] > (|T/d_A| + 1)(q_{A,B} - 1). \]

Then there is an open interval \( \hat{I} \subset (0,T) \) with \(|\hat{I}| \leq d_A \) so that \( z[I_{z,T} \cap \hat{I}] \geq q_{A,B} \). Thus, by Proposition 2.7, we have that \( B^* e^{(T-t)A^*} \equiv 0 \), which leads to a contradiction. This finishes the proof of Corollary 2.12. \( \square \)

### 2.4 Local maximum principle

This section deals with the maximum principle of the problem \((TP)_{y_0}\). It deserves mentioning that the standard maximum principle may not hold for \((TP)_{y_0}\) and what we get is the local maximum principle. (see [22, Chapter 4].)

**Proposition 2.13.** Let \( y_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \). Then for each \( T \in (0, T_{y_0}^*) \), there is a multiplier \( \xi_T \in \mathcal{L} \setminus \{0\} \) (where \( \mathcal{L} \) is given by (1.5)), with the property
\[ B^* e^{(T-t)A^*} \xi_T \in L^1(0,T; L^2(\Omega; \mathbb{R}^n)) \setminus \{0\}, \tag{2.19} \]
so that if \( u_{y_0}^* \) is an optimal control of \((TP)_{y_0}\), then
\[ \left\langle u_{y_0}^*(t), B^* e^{(T-t)A^*} \xi_T \right\rangle_{L^2(\Omega; \mathbb{R}^n)} = \max_{v \in B^*_{(T)}} \left\langle v, B^* e^{(T-t)A^*} \xi_T \right\rangle_{L^2(\Omega; \mathbb{R}^n)} \quad \text{for a.e. } t \in (0,T). \tag{2.20} \]

**Remark 2.14.** The multiplier \( \xi_T \) in (2.20) is independent of the choice of the optimal controls to \((TP)_{y_0}\). The equality (2.20) is called the local maximum principle introduced in [22, Section 4.2].

10
Our strategy to prove Proposition 2.13 is as follows: We set up a new time optimal control problem which is equivalent to \((TP)_{\eta_0}\), then build up the local maximum principle for the new problem, and finally go back to to \((TP)_{\eta_0}\). To introduce the new problem, we recall Lemma 2.1 for \(A_1, B_1, k \) and \(P\). Write \(\hat{y}(\cdot; \hat{y}_0, u)\) for the solution to the following reduced control system:
\[
\begin{aligned}
\dot{y}_1 &= (A_1 + B_1) y + B_1 u \quad \text{in } \Omega \times \mathbb{R}^+, \\
\dot{y} &= 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
\dot{y}(0) &= \hat{y}_0 \in L^2(\Omega; \mathbb{R}^k),
\end{aligned}
\] where \(u\) is taken from \(L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^m))\). The new time optimal control problem reads:
\[
(\hat{TP})_{\hat{y}_0} : \quad \hat{T}_{\hat{y}_0}^* := \inf \{ t > 0 : \exists u \in L^\infty(\mathbb{R}^+; B_1^m(0)) \text{ s.t. } \hat{y}(t; \hat{y}_0, u) = 0 \}. \tag{2.22}
\]
Notice that the new problem holds the state system (2.21), where \((A_1, B_1)\) satisfies the Kalman rank condition which plays an important role in getting the local maximum principle.

The following lemma gives the equivalence between \((TP)_{\eta_0}\) and \((\hat{TP})_{\hat{y}_0}\).

**Lemma 2.15.** Let \(\eta_0 \in L^2(\Omega; \mathbb{R}^m) \setminus \{0\}\) and \(\hat{y}_0 \in L^2(\Omega; \mathbb{R}^k) \setminus \{0\}\) satisfy \(P^{-1}\eta_0 = (\hat{y}_0,0)^\top\). Then the problems \((TP)_{\eta_0}\) and \((\hat{TP})_{\hat{y}_0}\) are equivalent, i.e., they share the same optimal time and the same optimal controls (if one of them has an optimal control).

**Proof.** We first claim that for each control \(u \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^m))\),
\[
P^{-1}y(t; \eta_0, u) = (\hat{y}(t; \eta_0, u), 0)^\top \quad \text{for each } t \geq 0. \tag{2.23}
\]
To this end, we use Proposition 2.2 to see that when \(u \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^m))\),
\[
P^{-1}y(t; \eta_0, u) = P^{-1}e^{tA_1}PP^{-1}\eta_0 + P^{-1} \int_0^t e^{(t-s)A}Bu(s)ds \\
= (P^{-1}e^{tA_1}P)(\eta_0,0)^\top + \int_0^t (P^{-1}e^{(t-s)A}B)u(s)ds \\
= (e^{t(A_1 \Delta + A_1)}\eta_0,0)^\top + \int_0^t (e^{(t-s)(A_1 \Delta + A_1)}A_1B_1(0))^\top u(s)ds,
\]
which leads to (2.23).

Next, from (2.23), we find that for each \(u \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^m))\),
\[
y(t; \eta_0, u) = 0 \quad \text{for each } t \geq 0 \quad \text{if and only if } \hat{y}(t; \hat{y}_0, u) = 0 \quad \text{for each } t \geq 0.
\]
This, along with (1.4) and (2.22), shows that \((TP)_{\eta_0}\) and \((\hat{TP})_{\hat{y}_0}\) are equivalent.

Hence, we finish the proof of Lemma 2.15. \(\square\)

The next lemma gives the local maximum principle for the problem \((\hat{TP})_{\hat{y}_0}\).

**Lemma 2.16.** Let \(\hat{y}_0 \in L^2(\Omega; \mathbb{R}^k) \setminus \{0\}\). Then for each \(T \in (0, \hat{T}_{\hat{y}_0}^*)\), there is a nontrivial multiplier \(\eta_T \in L^2(\Omega; \mathbb{R}^k) \setminus \{0\}\), with the property
\[
f_T(\cdot) := \chi_\omega B_1^T e^{(T-\cdot)(A_1 \Delta + A_1^\top)}\eta_T \in L^1(0,T; L^2(\Omega; \mathbb{R}^m)) \setminus \{0\}, \tag{2.24}
\]
so that if \(\bar{u}_{\hat{y}_0}\) is an optimal control to \((\hat{TP})_{\hat{y}_0}\), then
\[
\langle \bar{u}_{\hat{y}_0}(t), f_T(t) \rangle_{L^2(\Omega; \mathbb{R}^m)} = \max_{\nu \in B_1^T(0)} \langle \nu, f_T(t) \rangle_{L^2(\Omega; \mathbb{R}^m)} \quad \text{for a.e. } t \in (0,T). \tag{2.25}
\]
Proof. Our proof is based on the method provided in [22, Theorems 4.3 and 4.4]. Given $0 < t_1 < t_2 < +\infty$, define the following controllable subspace over $(t_1, t_2)$ with constrained controls:

$$\tilde{Y}_C(t_1, t_2) := \left\{ f \in L^2(\Omega; \mathbb{R}^k) : \exists u \in L^\infty(\mathbb{R}^+; B_1^m(0)) \text{ s.t.} \right.$$ 

$$e^{(t_2-t_1)(\xi_k+iA_1)} f + \int_{t_1}^{t_2} \chi_\omega B_1 e^{(t_2-t)(\xi_k+iA_1)} u(t) dt = 0 \right\}.$$ 

Given $t > 0$, define the following reachable set of the system (2.21):

$$\tilde{Y}_R(t; \hat{y}_0) := \{ \hat{y}(t; \hat{y}_0, u) : u \in L^\infty(\mathbb{R}^+; B_1^m(0)) \}.$$ 

Since rank $(B_1, A_1 B_1, \ldots, A_1^{k-1} B_1) = k$ (see (2.2)), it follows by Proposition 2.4 and [22, Theorem 4.4] that for each $t \in (0, \hat{T}_{\hat{y}_0})$, $\tilde{Y}_R(t; \hat{y}_0)$ and $\tilde{Y}_C(t, \hat{T}_{\hat{y}_0})$ are separable in $(L^2(\Omega))^k$. Thus, by [22, Theorem 4.3], for each $T > 0$, there is $\eta_T \in L^2(\Omega; \mathbb{R}^k) \setminus \{0\}$, with (2.24), so that (2.25) holds for any optimal control to the problem $(\overline{T\mathcal{P}})_{\hat{y}_0}$ (if it has an optimal control). This completes the proof of Lemma 2.16. \[\square\]

We are now in the position to prove Proposition 2.13.

Proof of Proposition 2.13. Arbitrarily fix $\hat{y}_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\}$. Let $\hat{y}_0 \in L^2(\Omega; \mathbb{R}^k) \setminus \{0\}$ satisfy $P^{-1}\hat{y}_0 = (\hat{y}_0, 0)^\top$. Then according to Lemma 2.15, the following conclusions are true:

(C1) $T_{\hat{y}_0} = \hat{T}_{\hat{y}_0}$;

(C2) $(T\mathcal{P})_{\hat{y}_0}$ and $(\overline{T\mathcal{P}})_{\hat{y}_0}$ has same optimal controls, if one of them has optimal controls.

Arbitrarily fix $T \in (0, \hat{T}_{\hat{y}_0})$. Then by the above (C1), we have $T \in (0, \hat{T}_{\hat{y}_0})$. By this, we can apply Lemma 2.16 to find a a multiplier $\eta_T \in L^2(\Omega; \mathbb{R}^k) \setminus \{0\}$, with (2.24), so that (2.25) holds for any optimal control to $(\overline{T\mathcal{P}})_{\hat{y}_0}$ (if it has optimal controls).

Let

$$\xi_T := (P^{-1})^\top(\eta_T, 0)^\top = (P(\eta_T, 0)^\top).$$  \hspace{1cm} (2.26)

Then by the second equality in (2.3), we see that for each $t \in (0, T)$,

$$B^* e^{(T-t)A^*} \xi_T = B^* e^{(T-t)A^*} (P^{-1})^\top(\eta_T, 0)^\top = \left( P^{-1} e^{(T-t)A} B \right)^* (\eta_T, 0)^\top$$

$$= \left( \chi_\omega B_1^\top e^{(T-t)(\xi_k+iA_1)} \right) (\eta_T, 0)^\top = \chi_\omega B_1^\top e^{(T-t)(\xi_k+iA_1)} \eta_T. $$ \hspace{1cm} (2.27)

From (2.24) and (2.27), we see that $\xi_T$ satisfies (2.19), while from (2.24) and the above (C2), we find that (2.20), with the above $\xi_T$, holds for any optimal control to $(T\mathcal{P})_{\hat{y}_0}$ (if it has optimal controls).

Finally, by the first equality in (2.1), (2.26) and (1.5), we have $\xi_T \in \mathcal{L} \setminus \{0\}$. This ends the proof of Proposition 2.13. \[\square\]

3 Proof of main results

We are now on the position to prove Theorem 1.2.

Proof of Theorem 1.2. Arbitrarily fix $y_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\}$ which satisfies $(A)_{y_0}$. Then from the note $(a_1)$ in Section 1.1, the problem $(T\mathcal{P})_{y_0}$ has at least one optimal control, from which, one can easily check that $T_{y_0} > 0$. 

12
Before proving the theorem, we will show a key conclusion. To state it, we arbitrarily fix $T \in (0, T_\omega^*)$. According to Proposition 2.13, there is $\xi_T \in \mathcal{L} \setminus \{0\}$ so that (2.20) holds. Then by (2.19) in Proposition 2.13 and Corollary 2.12, we have

$$
\mathbb{P}[\{t \in (0, T) : B^*e^{(T-t)A^*}\xi_T = 0\}] < +\infty.
$$

So we can find $p := p(\xi_T) \in \mathbb{N}$ so that

$$
\mathcal{S}_{\xi_T} := \{t \in (0, T) : B^*e^{(T-t)A^*}\xi_T = 0\} \cup \{t_0 = 0\} := \{t_i\}^p_{i=0}.
$$

The above-mentioned key conclusion is as: there is a unique left-continuous function $f_{\xi_T}$ in $\mathcal{P}(\mathbb{N}; B^m(0))$ so that

$$
f_{\xi_T}(t) = \frac{B^*e^{(T-t)A^*}\xi_T}{\|B^*e^{(T-t)A^*}\xi_T\|_{L^2(\mathbb{R}^m)}}, \quad t \in [0, T) \setminus \mathcal{S}_{\xi_T}.
$$

To show (3.2), it suffices to prove that for each $t_i$ ($0 \leq i \leq p$),

$$
\lim_{t \to t_i^-} \frac{B^*e^{(T-t)A^*}\xi_T}{\|B^*e^{(T-t)A^*}\xi_T\|_{L^2(\mathbb{R}^m)}} \text{ exists,}
$$

and

$$
\lim_{t \to t_i^+} \frac{B^*e^{(T-t)A^*}\xi_T}{\|B^*e^{(T-t)A^*}\xi_T\|_{L^2(\mathbb{R}^m)}} \text{ exists.}
$$

To show (3.3) and (3.4), we arbitrarily fix $0 \leq i \leq p$ and divide the proof into several steps.

Step 1. We prove that

$$
\chi_{\omega e^{(T-t)i\Delta}\xi_T} \neq 0 \text{ for any } t \in [0, T).
$$

Actually, if there exists a $\hat{i} \in [0, T)$ such that $\chi_{\omega e^{(T-t)i\Delta}\xi_T} = 0$, then by Proposition 2.7 (with $B := I_n$ and $A := 0$), we can get that $\xi_T = 0$, which contradicts our assumption. Hence (3.5) holds.

Step 2. We define two numbers.

By (3.5), we have that, for each $i \in \{0, 1, \ldots, p\}$,

$$
\{j \in \mathbb{N} : \chi_{\omega B^\top(A^\top)^j e^{(T-t_i)A^*}\xi_T} \neq 0\} \neq \emptyset.
$$

and

$$
\{j \in \mathbb{N} : B^\top(A^\top)^j e^{(T-t_i)A^*}\xi_T \neq 0\} \neq \emptyset.
$$

Indeed, to show (3.6), by the Hamilton-Cayley theorem, we only need to prove that, for each $\delta \in (0, T)$

$$
B^\top e^{(T-t)A^\top} \chi_{\omega e^{(T-t-i\Delta)\xi_T}} \neq 0 \text{ in } [T - \delta, T).
$$

Since $\xi_T \in \mathcal{L} \setminus \{0\}$ (see Proposition 2.13), there is $\eta_T \in L^2(\Omega; \mathbb{R}^k)$ so that $\xi_T = (P^{-1})^\top(\eta_T, 0)^\top$. Thus, if there are $\delta \in (0, T)$ and $i \in \{0, 1, \ldots, p\}$ so that (3.8) does not hold, then, by (2.1), for each $t \in [T - \delta, T)$,

$$
0 = B^\top e^{(T-t)A^\top} \chi_{\omega e^{(T-t-i\Delta)\xi_T}} = B^\top e^{(T-t)A^\top} \chi_{\omega e^{(T-t-i\Delta)\eta_T}} = (P^{-1} e^{(T-t)A^\top} B^\top)(\chi_{\omega e^{(T-t-i\Delta)\xi_T}} - \chi_{\omega e^{(T-t-i\Delta)\eta_T}}) = B^\top e^{(T-t)A^\top} \chi_{\omega e^{(T-t-i\Delta)\xi_T}}.
$$

Because of rank $(B_1, A_1 B_1, \ldots, A_1^{k-1} B_1) = k$, it follows that $\chi_{\omega e^{(T-t-i\Delta)\xi_T}} = 0$, which implies that $\chi_{\omega e^{(T-t-i\Delta)\eta_T}} = 0$. It contradicts to (3.5). Thus, (3.8) is true. Moreover, it is obvious that, if (3.6) holds, then (3.7) is true.

Thus, we can define two integers in $\mathbb{N}^+$ by

$$
p(i) := \min\{j \in \mathbb{N} : \chi_{\omega B^\top(A^\top)^j e^{(T-t_i)A^*}\xi_T} \neq 0\}
$$

13
and
\[ \hat{p}(i) := \min\{j \in \mathbb{N} : B^T(A^T)^j e^{(T-t_i)A^T} \xi_T \neq 0 \}. \] (3.10)

(By (3.8) and the Hamilton-Cayley theorem, we have \( p(i) \leq n - 1 \) and \( \hat{p}(i) \leq n - 1 \).)

**Step 3.** We prove that
\[ p(i) = \hat{p}(i). \] (3.11)

By (3.9) and (3.10), we have \( \hat{p}(i) \leq p(i) \). We now show that \( p(i) \leq \hat{p}(i) \). For this purpose, we only need to prove that
\[ \chi_\omega B^T(A^T)^{\hat{p}(i)} e^{(T-t_i)A^T} \xi_T \neq 0. \] (3.12)

To show (3.12), we first use (3.10) to see
\[ \xi_T := B^T(A^T)^{\hat{p}(i)} e^{(T-t_i)A^T} \xi_T \neq 0. \] (3.13)

We next observe that
\[ \varphi(\cdot) := B^T(A^T)^{\hat{p}(i)} e^{(T-t_i)A^T} e^{(T-\cdot)B^Tt \xi_T} = e^{(T-\cdot)B^Tt \xi_T}, \]
from which, it follows that \( \varphi(\cdot) \) is the solution the following equation:
\[
\left\{
\begin{array}{ll}
\varphi_t = -I_m \triangle \varphi & \text{in } \Omega \times (0, T), \\
\varphi = 0 & \text{on } \partial \Omega \times (0, T), \\
\varphi(T) = \varphi_T.
\end{array}
\right.
\]

Now, by contradiction, we suppose that (3.12) is not true. Then we have \( \varphi(t_i) = 0 \). This, together with Proposition 2.7 (with \( B := I_m \) and \( A := 0 \)), gives that \( \varphi_T = 0 \), which contradicts (3.13). So, (3.12) is proved and (3.11) is true.

**Step 4.** We give an expression on \( B^T e^{(T-t)A^T} e^{(T-t_i)B^Tt \xi_T} \).

By (3.11) and (3.10), one can directly check that for any \( t \in (-1, T) \),
\[
\begin{align*}
B^T e^{(T-t)A^T} e^{(T-t_i)B^Tt \xi_T} &= \sum_{j=0}^{\infty} (t - t_i)^j \frac{B^T(-A^T)^j e^{(T-t_i)A^T}}{j!} e^{(T-t_i)B^Tt \xi_T} \\
&= (t - t_i)^{\hat{p}(i)} \sum_{j=\hat{p}(i)}^{\infty} (t - t_i)^{j-\hat{p}(i)} \frac{B^T(-A^T)^j e^{(T-t_i)A^T}}{j!} e^{(T-t_i)B^Tt \xi_T} \\
&= (t - t_i)^{\hat{p}(i)} \left[ a_{\hat{p}(i)} + (t - t_i) b_{\hat{p}(i)}(t) \right],
\end{align*}
\] (3.14)

where
\[ a_{\hat{p}(i)} := \frac{B^T(-A^T)^{\hat{p}(i)} e^{(T-t_i)A^T}}{\hat{p}(i)!} e^{(T-t_i)B^Tt \xi_T}, \] (3.15)
and
\[ b_{\hat{p}(i)}(t) := \sum_{j=\hat{p}(i)+1}^{\infty} (t - t_i)^{j-\hat{p}(i)-1} \frac{B^T(-A^T)^j e^{(T-t_i)A^T}}{j!} e^{(T-t_i)B^Tt \xi_T}. \] (3.16)

From (3.9) and (3.11), we have
\[ \chi_\omega a_{\hat{p}(i)} \neq 0, \] (3.17)
while from (3.16), we can find \( C > 0 \) so that
\[ \| b_{\hat{p}(i)}(t) \|_{L^2(\Omega; \mathbb{R}^n)} \leq C \| \xi_T \|_{L^2(\Omega; \mathbb{R}^n)} \text{ for all } t \in [0, T]. \] (3.18)
Step 5. We prove that there is $\delta_0 > 0$ so that

$$B^T e^{(T-t)A} \xi_T = B^T e^{(T-t)A} e^{(T-t_1)\Delta} \xi_T + O((t-t_1)^{p(i)+1}) \text{ for any } t \in (t_i - \delta_0, t_i + \delta),$$

(3.19)

where and in what follows, $O(s^n)$, with $q \in \mathbb{N}^+$, stands for a function $f : \mathbb{R}^+ \to (L^2(\Omega))^m$ so that $\|f(s)\|_{(L^2(\Omega))^m} \leq Cs^n$ for some constant $C > 0$.

Given $\varepsilon \in (0, T)$ so that $t_i \in (-1, T - \varepsilon)$, we take $\delta > 0$ so that $(t_i - \delta, t_i + \delta) \subset (-1, T - \varepsilon)$. Since the operator $I_n \Delta$, with its domain $D(I_n \Delta) = H^1_0(\Omega; \mathbb{R}^n) \cap H^2(\Omega; \mathbb{R}^n)$, generates an analytical semigroup $\{e^{I_n \Delta t}\}_{t \geq 0}$, we can use the properties of analytic semigroups (see [11, Chapter 2, Section 2.5]) to find $C > 0$ so that

$$\|((I_n \Delta)^j e^{(T-t_1)I_n \Delta}) \|_{L^2(\Omega; \mathbb{R}^n), L^2(\Omega; \mathbb{R}^n)} \leq j! \left(\frac{\tilde{C}}{T-t_i}\right)^j \leq j!(\tilde{C}e\varepsilon^{-1})^j \text{ for all } j \in \mathbb{N}.$$  

(3.20)

(In (3.20), we used the fact $T - t_i > \varepsilon$.) Thus, we have that when $t \in (t_i - \delta, t_i + \delta)$, with $\hat{\delta} := \min\{\frac{\varepsilon}{2\tilde{C}e}, \delta\},

$$e^{(T-t)I_n \Delta} \xi_T = e^{(T-t_1)I_n \Delta} \xi_T + \sum_{j=1}^{+\infty} (t - t_i)^j \frac{(-I_n \Delta)^j}{j!} e^{(T-t_1)I_n \Delta} \xi_T.$$  

(3.21)

Notice that (3.20) ensures the convergence of the series in (3.21) in $L^2(\Omega; \mathbb{R}^n).$) Now, by (3.21) and (3.14), we see that for each $t \in (t_i - \hat{\delta}, t_i + \hat{\delta}),$

$$B^T e^{A^T (T-t)I_n \Delta} - e^{(T-t_1)I_n \Delta} \xi_T = \sum_{j=1}^{+\infty} (t - t_i)^p(i) + j \frac{(-I_n \Delta)^j}{j!} [a_{p(i)} + (t - t_i)b_{p(i)}(t)].$$

(3.22)

While by (3.15), (3.16) and (3.20), one can easily check that there is a constant $C_{p(i)} > 0$ so that

$$\|(-I_n \Delta)^j a_{p(i)}\|_{L^2(\Omega; \mathbb{R}^n)} \leq C_{p(i)} j! (\tilde{C}e\varepsilon^{-1})^j \|\xi_T\|_{L^2(\Omega; \mathbb{R}^n)}$$

and that when $t \in (t_i - \hat{\delta}, t_i + \hat{\delta})$ with $\hat{\delta} := \min\{\delta, \frac{\varepsilon}{2\tilde{C}e\varepsilon^{-1}}\},$

$$\|(-I_n \Delta)^j b_{p(i)}(t)\|_{L^2(\Omega; \mathbb{R}^n)} \leq \|B\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \|A\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \|\xi_T\|_{L^2(\Omega; \mathbb{R}^n)} \sum_{p=1}^{+\infty} (t - t_i)^p \|\xi_T\|_{L^2(\Omega; \mathbb{R}^n)} \sum_{p=1}^{+\infty} \left(\frac{1}{2}\right)^p < +\infty.$$  

These imply that when $t \in (t_i - \hat{\delta}, t_i + \hat{\delta}),$

$$\sum_{j=1}^{+\infty} (t - t_i)^p(i) + j \frac{(-I_n \Delta)^j}{j!} [a_{p(i)} + (t - t_i)b_{p(i)}(t)] = O((t - t_i)^{p(i)+1}).$$

This, together with (3.22), yields (3.19), with $\delta_0 := \hat{\delta}.

Step 6. We prove (3.3).
By (3.14) and (3.19), we see that when \( t \in (t_i - \delta_0, t_i + \delta_0) \),

\[
\begin{align*}
B^* e^{(T-t)A^*} \xi_T &= B^* e^{(T-t)A^*} e^{(T-t_i)A^*} \xi_T + B^* e^{(T-t)A^*} (e^{(T-t_i)A^*} - e^{(T-t)A^*}) \xi_T \\
&= (t-t_i) p^{(i)} \chi_\omega [a_p^{(i)} + (t-t_i) h_p^{(i)}(t)] + \chi_\omega O((t-t_i) p^{(i)+1}) \\
&= (t-t_i) p^{(i)} \chi_\omega [a_p^{(i)} + (t-t_i) h_p^{(i)}(t) + O((t-t_i)].
\end{align*}
\]

Hence, for each \( t \in (-1, t_i) \cap O_{\delta_0}(t_i) \),

\[
\begin{align*}
\frac{B^* e^{(T-t)A^*} \xi_T}{\| B^* e^{(T-t)A^*} \xi_T \|_{L^2(\Omega; \mathbb{R}^m)}} &= \frac{(t-t_i) p^{(i)} \chi_\omega [a_p^{(i)} + (t-t_i) h_p^{(i)}(t) + O((t-t_i)]}{\| \chi_\omega [a_p^{(i)} + (t-t_i) h_p^{(i)}(t) + O((t-t_i)] \|_{L^2(\Omega; \mathbb{R}^m)}} \\
&= \begin{cases} 
\frac{\chi_\omega a_p^{(i)}}{\| \chi_\omega a_p^{(i)} \|_{L^2(\Omega; \mathbb{R}^m)}} \quad & \text{if } p(i) \text{ is even}, \\
\frac{\chi_\omega a_p^{(i)}}{\| \chi_\omega a_p^{(i)} \|_{L^2(\Omega; \mathbb{R}^m)}} \quad & \text{if } p(i) \text{ is odd}.
\end{cases}
\end{align*}
\]

By sending \( t \to t_i^{-} \) in (3.24) and using (3.17) and (3.18), we obtain that

\[
\lim_{t \to t_i^{-}} \frac{B^* e^{(T-t)A^*} \xi_T}{\| B^* e^{(T-t)A^*} \xi_T \|_{L^2(\Omega; \mathbb{R}^m)}} = \begin{cases} 
\frac{\chi_\omega a_p^{(i)}}{\| \chi_\omega a_p^{(i)} \|_{L^2(\Omega; \mathbb{R}^m)}} \quad & \text{if } p(i) \text{ is even}, \\
\frac{\chi_\omega a_p^{(i)}}{\| \chi_\omega a_p^{(i)} \|_{L^2(\Omega; \mathbb{R}^m)}} \quad & \text{if } p(i) \text{ is odd}.
\end{cases}
\]

This leads to (3.3).

Step 7. The proof of (3.4).

By (3.23), (3.17) and (3.18), we see that

\[
\lim_{t \to t_i^{-}} \frac{B^* e^{(T-t)A^*} \xi_T}{\| B^* e^{(T-t)A^*} \xi_T \|_{L^2(\Omega; \mathbb{R}^m)}} = \frac{\chi_\omega a_p^{(i)}}{\| \chi_\omega a_p^{(i)} \|_{L^2(\Omega; \mathbb{R}^m)}},
\]

which leads to (3.4).

In summary, we conclude that the key conclusion has been proved.

We now show that \((TP)_{y_0}\) has a unique optimal control, which has the bang-bang property. To this end, we let \( \tilde{u} \) be an optimal control to \((TP)_{y_0}\). Given \( T \in (0, T^*_y) \), let \( \xi_T \) be given by Proposition 2.13. Then by (2.20) and the key conclusion, we have

\[
\tilde{u}|_{(0, T)}(t) = f_{\xi_T}(t) \quad \text{for a.e. } t \in (0, T),
\]

where \( f_{\xi_T} \) is given by (3.2). Since \( f_{\xi_T} \) is in \( PC([0, T]; B^m(0)) \) (This follows from (3.2) and (3.1),) and because (3.27) holds for each \( T \in (0, T^*_y) \), (Notice that when \( T \) varies, \( f_{\xi_T} \), as well as \( \xi_T \), change,.) we have

\[
f_{\xi_T}|_{(0, S)} = f_{\xi_S} \quad \text{when } 0 < S < T < T^*_y,
\]

where \( \xi_S \) is given in Proposition 2.13, where \( T \) is replaced by \( S \).

Meanwhile, it follows from (2.19) in Proposition 2.13 and Corollary 2.12 that

\[
\sup_{0 < S < T^*_y} \sharp \left\{ t \in (0, S) : B^* e^{(S-t)A^*} \xi_S = 0 \right\} < +\infty.
\]
(Here, we note that $T_{y_0}^* < +\infty$. Define a control $u_{y_0}^*$ in the following manner: For each $t \in (0, T_{y_0}^*)$, we arbitrarily take $T \in (t, T_{y_0}^*)$ and then define
\[
\begin{align*}
  u_{y_0}^*(t) := f_{\xi_T}(t).
\end{align*}
\]
By (3.28), we see that $u_{y_0}^*$ is well defined. From (3.30) and (3.27), it follows that
\[
\begin{align*}
  u_{y_0}^*(t) = \hat{u}(t) \quad \text{for a.e. } t \in (0, T_{y_0}^*).
\end{align*}
\]
Thus, $(\mathcal{T}P)_{y_0}$ has a unique optimal control $u_{y_0}^*$. Moreover, from (3.30) and (3.2), $u_{y_0}^*$ has the bang-bang property.

Finally, since $f_{\xi_T} \in \mathcal{PC}([0, T); B^m_0(0))$ for any $T \in (0, T_{y_0}^*)$, it follows from (3.30), (3.29) that the above control $u_{y_0}^*$ is in the space $\mathcal{PC}([0, T_{y_0}^*); B^m_0(0))$. Thus we finish the proof of the conclusion (i).

We next show (ii). Let $I$ be an open subinterval of $(0, T_{y_0}^*)$ with $|I| \leq d_A$. We aim to show that the optimal control $u_{y_0}^*$ defined in (3.30) has at most $q_{A,B} - 1$ switching points over $I$. By contradiction, we suppose that it was not true. Then there would be a set $\{t_j\}_{j=1}^{q_{A,B}} \subset I$ so that each $t_j$ is a switching point of $u_{y_0}^*$. Let $\hat{T} \in (0, T_{y_0}^*)$ so that
\[
\hat{T} > \max_{1 \leq j \leq q_{A,B}} t_j.
\]
Then from (3.30) and (3.2), one has that
\[
\begin{align*}
  u_{y_0}^*|_{(0, \hat{T})}(t) = \frac{B^* e^{(\hat{T}-t)A^*} \xi_{\hat{T}}}{\|B^* e^{(\hat{T}-t)A^*} \xi_{\hat{T}}\|_{L^2((0, \mathbb{R}^m))}} \quad \text{when } t \in (0, \hat{T}) \setminus \mathcal{I}_{\xi_{\hat{T}}}.
\end{align*}
\]
Notice that the function on the right hand side of (3.31) is continuous at each $t \in (0, \hat{T}) \setminus \mathcal{I}_{\xi_{\hat{T}}}$; and each $t_j$, with $j = 1, \ldots, q_{A,B}$ is in $I$. Thus, we see from (3.31) that
\[
\begin{align*}
  B^* e^{(\hat{T}-t)A^*} \xi_{\hat{T}} = 0 \quad \text{for each } t \in \{t_j\}_{j=1}^{q_{A,B}} \subset I.
\end{align*}
\]
This, along with Proposition 2.7, yields
\[
\begin{align*}
  B^* e^{(\hat{T}-t)A^*} \xi_{\hat{T}} = 0 \quad \text{for all } t \in (0, \hat{T}),
\end{align*}
\]
which contradicts (3.29). Thus, the conclusion (ii) is true.

We finally prove (iii). Indeed, by (3.30), we have that for each $\hat{t} \in (0, T_{y_0}^*)$, if it is a switching point of the optimal control $u_{y_0}^*$, then there exist $T > \hat{t}$ and $\xi_T \in \mathcal{L} \setminus \{0\}$ so that $\hat{t} \in \mathcal{I}_{\xi_T}$, i.e., there is an $i^* \in \{1, 2, \ldots, p\}$ so that $\hat{t} = t_{i^*}$, where $\mathcal{I}_{\xi_T}$ is defined by (3.1). By (3.25) and (3.26), we can conclude that $p(i^*)$, which is defined by (3.9), is an odd number and $\lim_{s \to t_{i^*}^+} u^*(s) + \lim_{s \to t_{i^*}^-} u^*(s) = 0$. Thus (iii) is true.

Hence, we complete the proof of Theorem 1.2. \qed

4 An example

In this section, we present an example to show that, in some time optimal control problem of coupled heat system, the switching phenomenon happens. From this perspective, the system (1.2) differs from the pure heat equation, since the optimal control for the later has no any switching point (see the note (b3)).

Example. Let $\omega = \Omega$. Let $n := 2$ and $m := 1$. Let
\[
\begin{align*}
  A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{align*}
\]
From (4.1), we can directly check that \(\text{rank}(B, AB) = 2\) and \(\sigma(A) = \{i, -i\}\), and that
\[
e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{for each } t \in \mathbb{R}. \tag{4.2}
\]
Write \(\lambda_i\) for the \(i\)-th eigenvalue of the operator \(-\Delta\), with its domain \(H^1_0(\Omega) \cap H^2(\Omega)\), and let \(e_i\) be the corresponding normalized eigenvector.

First, according to Proposition 2.4, the system (1.2), with the above \((A, B)\), is \(L^\infty_{2,t}\) null controllable and \(\mathcal{L} = L^2(\Omega; \mathbb{R}^2)\). (Here, \(\mathcal{L}\) is defined by (1.5).)

Second, from (4.2), one can directly check that
\[
\|e^{tA}\|_{\mathcal{L}(L^2(\Omega; \mathbb{R}^2), L^2(\Omega; \mathbb{R}^2))} = \|e^{t(\xi_2+A)}\|_{\mathcal{L}(L^2(\Omega; \mathbb{R}^2), L^2(\Omega; \mathbb{R}^2))} \leq e^{-\lambda_1 t} \quad \text{for all } t \in \mathbb{R}^+. \tag{4.3}
\]

From these and by the note \((a_1)\) in Section 1.1, we can see that the problem \((TP)_{y_0}\) has an admissible control for each \(y_0 \in L^2(\Omega; \mathbb{R}^2)\). Then by \((i)\) in Theorem 1.2, \((TP)_{y_0}\) has a unique optimal control \(u^*_{y_0}\) whose restriction over \([0, T^*_{y_0}]\) is in \(\mathcal{PC}([0, T^*_{y_0}], B_1(0))\).

Let \(\eta := (\eta_1, \eta_2) \in \mathbb{R}^2\) so that
\[
\|\eta\|_{\mathbb{R}^2} = \eta_1^2 + \eta_2^2 > \lambda_1^{-1}(e^{4\pi\lambda_1} - 1). \tag{4.4}
\]
Choose \(y_0 := \eta e_1\). By the optimality of \((u^*_{y_0}, T^*_{y_0})\), we have
\[
0 = e^{T^*_{y_0}A}y_0 + \int_0^{T^*_{y_0}} e^{(T^*_{y_0}-t)A}Bu^*_{y_0}(t)dt = e^{T^*_{y_0}(A-\lambda_1\xi_2)}\eta e_1 + \int_0^{T^*_{y_0}} e^{(T^*_{y_0}-t)A}Be^{(T^*_{y_0}-t)\xi_2}u^*_{y_0}(t)dt. \tag{4.5}
\]
This, along with (4.2) and (4.3), gives that
\[
e^{-\lambda_1 T^*_{y_0}}\|\eta\|_{\mathbb{R}^2} \leq \int_0^{T^*_{y_0}} e^{-\lambda_1 (T^*_{y_0}-t)}dt = \lambda_1^{-1}(1 - e^{-\lambda_1 T^*_{y_0}}).
\]
(Here, we used the facts that \(\|B\|_{\mathcal{L}(\mathbb{R}; \mathbb{R}^2)} \leq 1\) and \(\|e^{At}\|_{\mathcal{L}(\mathbb{R}; \mathbb{R}^2)} = 1\) for each \(t \in \mathbb{R}\).) The above leads to
\[
\|\eta\|_{\mathbb{R}^2} \leq \lambda_1^{-1}(e^{4\pi\lambda_1} - 1).
\]
This, together with (4.4), gives that
\[
T^*_{y_0} \geq \lambda_1^{-1}\ln(\lambda_1\|\eta\|_{\mathbb{R}^2} + 1) > 4\pi. \tag{4.6}
\]
Let \(\widehat{T} := 4\pi\). By Proposition 2.13 and (4.6), there is a \(\xi_\widehat{T} \in L^2(\Omega; \mathbb{R}^2)\setminus\{0\}\) so that (2.19) and (2.20) hold. Thus, by (2.20), we have
\[
u^*_{y_0}|_{(0, \widehat{T})}(t) = \frac{B^*e^{(\widehat{T}-t)A^*}\xi_\widehat{T}}{\|B^*e^{(\widehat{T}-t)A^*}\xi_\widehat{T}\|_{L^2(\Omega)}} \quad \text{for each } t \in (0, \widehat{T}) \setminus \mathcal{G}_\widehat{T}, \tag{4.7}
\]
where
\[
\mathcal{G}_\widehat{T} := \left\{ t \in (0, \widehat{T}) : B^*e^{(\widehat{T}-t)A^*}\xi_\widehat{T} = 0 \right\}. \tag{4.8}
\]
Our aim is to claim that \((i)\) the set \(\mathcal{G}_\widehat{T}\) is not empty; \((ii)\) each \(\hat{t}\) in this set is a switching point of the optimal control \(u^*_{y_0}\).

For this purpose, we first show
\[
u^*_{y_0}(t) = f(t)e_1 \quad \text{for each } t \in (0, T^*_{y_0}), \tag{4.9}
\]
where \( \|f(t)\|_R \leq 1 \) a.e. \( t \in (0, T^*_{\text{to}}) \). Actually, if \( u^*(t) = \sum_{i=1}^{+\infty} f_i(t)e_i \) with \( \sum_{i=1}^{+\infty} \|f_i(t)\|_R^2 \leq 1 \) a.e. \( t \in (0, T^*_{\text{to}}) \), then by (4.5), we have

\[
0 = e^{T^*_{\text{to}}(A-\lambda_1z_2)}e_1 + \sum_{i=1}^{+\infty} \int_0^{T^*_{\text{to}}} e^{(T^*_{\text{to}}-t)(A-\lambda_1z_2)}Bf_i(t)dte_i.
\]

Thus we have

\[
\begin{cases}
e^{T^*_{\text{to}}(A-\lambda_1z_2)}e_1 + \int_0^{T^*_{\text{to}}} e^{(T^*_{\text{to}}-t)(A-\lambda_1z_2)}Bf_1(t)dte_1 = 0, \\
\sum_{i=2}^{+\infty} \int_0^{T^*_{\text{to}}} e^{(T^*_{\text{to}}-t)(A-\lambda_1z_2)}Bf_i(t)dte_i = 0.
\end{cases}
\]

So the control \( \hat{u}^*_{\text{to}} := f_1e_1 \) is also an optimal control to \((\mathcal{T}\mathcal{P})_{\text{to}}\). By the uniqueness of the optimal control to \((\mathcal{T}\mathcal{P})_{\text{to}}\) (see (i) in Theorem 1.2), we find that

\[
u^*_{\text{to}} = \hat{u}^*_{\text{to}}.
\]

Hence (4.9) holds.

Next, we prove that

\[
\xi_{\hat{T}} = \zeta e_1 \quad \text{for some} \quad \zeta \in \mathbb{R}^2 \setminus \{0\}. \tag{4.10}
\]

For this purpose, we suppose that \( \xi_{\hat{T}} = \sum_{i=1}^{+\infty} \zeta_i e_i \) with \( \{\zeta_i\}_{i \in \mathbb{N}^+} \subset \mathbb{R}^2 \) and \( \sum_{i=1}^{+\infty} \|\zeta_i\|_R^2 > 0 \). Thus, by (4.7), we obtain that

\[
u^*_{\text{to}}|_{(0,\hat{T})\setminus\Theta_{\hat{T}}}(t) = \|B^* e^{(\hat{T} - t)A^*} \xi_{\hat{T}}\|^1_{L^2(\Omega)} \sum_{i=2}^{+\infty} B^* e^{(\hat{T} - t)(A^* - \lambda_i z_2)} \zeta_i e_i.
\]

This, along with (4.9), yields that

\[
\|B^* e^{(\hat{T} - t)(A^* - \lambda_i z_2)} \zeta_i e_i = 0 \quad \text{a.e.} \quad t \in (0, \hat{T}). \tag{4.11}
\]

By Corollary 2.12, the set \( \Theta_{\hat{T}} \) has at most finite elements. It follows from (4.11) that

\[
B^* e^{(\hat{T} - t)(A^* - \lambda_i z_2)} \zeta_i = 0 \quad \text{a.e.} \quad t \in (0, \hat{T}) \quad \text{for each} \quad i \geq 2. \tag{4.12}
\]

Since rank \((B, AB) = 2\) (which implies that rank \((B, (A - \lambda_1z_2)B) = 2\) for each \(i \in \mathbb{N}^+\), by (4.12), we can conclude that

\[
\zeta_i = 0 \quad \text{for each} \quad i \geq 2.
\]

Thus, (4.10) holds.

By (4.10), we can write

\[
\xi_{\hat{T}} = (\zeta_1, \zeta_2)^T e_1.
\]

Thus, by (4.1), after some simple computations, we get

\[
B^* e^{(\hat{T} - t)A^*} \xi_{\hat{T}} = e^{-\lambda_i(\hat{T} - t)}e_1 \left( \zeta_1 \cos(\hat{T} - t) - \zeta_2 \sin(\hat{T} - t) \right) = e^{-\lambda_i(\hat{T} - t)}e_1 \sqrt{\zeta_1^2 + \zeta_2^2} \sin(\hat{T} - t + \theta), \tag{4.13}
\]

where \( \theta := \arctan(\zeta_1/\zeta_2) \). (Here, we permit \( \theta = \frac{\pi}{2} \) if \( \zeta_2 = 0 \).) By (4.7) and (4.13), we have

\[
u^*_{\text{to}}|_{(0,\hat{T})}(t) = \frac{e_1 \sqrt{\zeta_1^2 + \zeta_2^2} \sin(\hat{T} - t + \theta)}{\sqrt{\zeta_1^2 + \zeta_2^2} \sin(T - t + \theta)} \quad \text{for each} \quad t \in (0, \hat{T}) \setminus \Theta_{\hat{T}}. \tag{4.14}
\]
Finally, by (4.6), (4.8) and (4.13), one can check easily that
\[ \mathcal{S}_{\hat{T}} = \{ t \in (0, \hat{T}) : \sin(\hat{T} - t + \theta) = 0 \} \neq \emptyset, \]
which leads to the claim (i). While by (4.14), we see that for each \( \hat{t} \in \mathcal{S}_{\hat{T}} \),
\[ \lim_{t \to \hat{t}^+} u^*_y(t) \neq \lim_{t \to \hat{t}^-} u^*_y(t), \]
which leads to the claim (ii).

References

[1] J. Apraiz, L. Escauriaza, G. Wang and C. Zhang, Observability inequalities and measurable sets, J. Eur. Math. Soc., 16 (2014), 2433-2475.
[2] V. Barbu, The dynamic programming equation for the time optimal control problem in infinite dimensions, SIAM J. Control Optim., 29 (2) (1991), 445-456.
[3] O. Cârjă, The minimum time function for semilinear evolutions, SIAM J. Control Optim., 50(3) (2012), 1265-1282.
[4] P. Cannarsa and O. Cârjă, On the Bellman equation for the minimum time problem in infinite dimensions, SIAM J. Control Optim., 43(2) (2004), 532-548.
[5] F. Gozzi and P. Loreti, Regularity of the minimum time function and minimum energy problems: the linear case. SIAM J. Control Optim. 37(4) (1999), 1195-1221.
[6] E. Gyurkovics, Hölder condition for the minimum time function of linear systems. System Modelling and Optimization. Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 59 (1984) 382-392.
[7] G. Lebeau and L. Robbiano, Contrôle exact de l’équation de la chaleur, Comm. Partial Differential Equations, 20 (1995), 335-336.
[8] P. Lissy and E. Zuazua, Internal observability for coupled systems of linear partial differential equations, SIAM J. Control Optim., 57(2) (2019), 832-853.
[9] Q. Lü and G. Wang, On the existence of time optimal controls with constaints of the rectangular type for heat equations, SIAM J. Control Optim., 49(3) (2011), 1124-1149.
[10] L. Miller, A direct Lebeau-Robbiano stratey for the observability of heat-like semigroups, Discrete and Continuous Dynamical Systems Series B, 14 (2010), 1465-1485.
[11] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[12] K. D. Phung, G. Wang and X. Zhang, On the existence of time optimal controls for linear evolution equations, Discrete and Continuous Dynamical Systems Series B, 8(4) (2007), 925-941.
[13] L. Poggiolini, Structural stability of bang-bang trajectories with a double switching time in the minimum time problem, SIAM J. Control Optim., 55(6) (2017), 3779-3798.
[14] L. S. Pontryagin, V. G. Boltyanski and R.V. Gamkrelidze, et el, Mathematical Theory of Optimal Processes, New York, Wiley, 1962.
[15] S. Qin and G. Wang, Controllability of impulse controlled systems of heat equations coupled by-constant matrices, J. Differential Equations, 263 (2017), 6456-6493.
[16] S. Qin, G. Wang and H. Yu, On switching properties of time optimal controls for linear ODEs, arXiv:1911.07475v1.

[17] T. Seidman and J. Yong, How violent are fast controls? II, Mathematics of Control, Signals and Systems, 9 (1996) 327-340.

[18] E. D. Sontag, Mathematical Control Theory: Deterministic Finite-Dimensional Systems, 2nd edition, Springer-Verlag, New York, 1998.

[19] H. J. Sussmann, A bang-bang theorem with bounds on the number of switchings, SIAM J. Control and Optim., 17(5) (1979), 629-651.

[20] G. Wang, $L^\infty$-null controllability for the heat equation and its consequences for the time optimal control problem, SIAM J. Control Optim., 47(4) (2008), 1701-1720.

[21] G. Wang and L. Wang, The bang-bang principle of time optimal controls for the heat equation with internal controls, Systems Control Lett., 56 (2007), 709-713.

[22] G. Wang, L. Wang, Y. Xu and Y. Zhang, Time Optimal Control of Evolution Equations, Progress in Nonlinear Differential Equations and their Applications, 92. Subseries in Control. Birkhauser/Springer, Cham, 2018.

[23] L. Wang, Q. Yan and H. Yu, Constrained approximate null controllability of coupled heat equation with periodic impulse controls, arXiv:2005.07386v1.

[24] G. Wang and G. Zheng, An approach to the optimal time for a time optimal control problem of an internally controlled heat equation, SIAM J. Control Optim., 50(2) (2012), 601-628.

[25] G. Wang and E. Zuazua, On the equivalence of minimal time and minimal norm controls for internally controlled heat equations, SIAM J. Control Optim., 50(5) (2012), 2938-2958.

[26] H. Yu, Approximation of time optimal controls for heat equations with perturbations in the system potential, SIAM J. Control Optim. 52(3) (2014), 1663-1692.