Bogomolny equations and conformal transformations in curved space

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The coupling of the Higgs field through the Ricci tensor, put forward by Balakrishna and Wali, is derived using a conformal rescaling of the metric. Earlier results on “Bogomolny-type” equations in curved space, by Comtet, and others, are recovered. The procedure can be generalized to any static background metric.

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I. INTRODUCTION

In the study of solitons in flat space, the ‘Bogomolny’, or ‘self-duality’ equations [1], \( D_i \Phi = \pm \epsilon_{ijk} F^{jk} \), play two, complementary roles. On the one hand, they provide the absolute minima of the energy, yielding static solutions. But they also allow us to reduce the second-order field equations to first-order ones.

In curved space, an obstruction arises, though: a self-dual field may fail to solve the field equations.

To be specific, consider a purely magnetic, static Yang-Mills-Higgs system \( (A_i, \phi) \) in 3 space dimensions, where the YM potential \( A_i \) takes its values in the Lie algebra of the gauge group \( G \) (a compact Lie group), \( F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \) is the YM field strength and \( \phi \), the Higgs field, belongs to the adjoint representation of the gauge group. Let \( g_{\mu\nu} = (g_{oo}, -g_{ij}) \) be a static background metric. For vanishing Higgs potential, the Lagrangian is

\[
L_{YMH} = \mathrm{tr} \left( -\frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} D_i \phi D^i \phi \right) \sqrt{g},
\]

where \( g = g_{oo}\hat{g} \) is the determinant of metric with \( \hat{g} = \det(g_{ij}) \) the determinant of the space-metric alone. The associated field equations read

\[
\begin{align*}
\frac{1}{\sqrt{g}} D_i (\sqrt{g} F^{ij}) &= [\phi, D^j \phi], \\
\frac{1}{\sqrt{g}} D_i (\sqrt{g} D^i \phi) &= 0.
\end{align*}
\]

The natural generalization to curved space of the Bogomolny equations would be

\[
D_i \phi = \pm \sqrt{g} \epsilon_{ijk} F^{jk}
\]

However, inserting Eq. (3) into the r.h.s. of the second equation in (2) yields, using the Bianchi identities,

\[
\frac{1}{\sqrt{g}} \partial^i (\ln g_{oo}) D_i \phi, \text{ which does not vanish, unless } g_{oo} \text{ is a constant.}
\]

A clever way of removing this obstruction is to add a suitable curvature term to the Lagrangian [2, 3]. One assumes that the metric is of the Papapetrou-Majumdar form

\[
V^2 dt^2 - \frac{d\hat{s}^2}{V^2},
\]

and one seeks static fields \( \phi, A_i \) which extremize

\[
S = \int \text{tr} \left[ -\frac{R}{4} \psi^2 - \frac{1}{4} (F_{ij} F^{ij}) + \frac{1}{2} (D_i \psi D^i \psi) \right] V^{-2} d^4 x,
\]

where \( R \) is the Ricci scalar of the background metric. This expression differs from (1) in the term \((R/4) \text{tr} \psi^2\), which makes it indefinite. The associated field equations,

\[
\begin{align*}
D_i (V^2 F_{ij}) &= [\psi, D_j \psi], \\
V^2 D^2 \psi &= \frac{1}{2} R \psi,
\end{align*}
\]

can be obtained by solving instead [10],

\[
F_{ij} = \pm \frac{1}{V^2} \epsilon_{ijk} D^k (V \psi).
\]

To explain why this works, we note first that (7) are in fact (9) for the conformally rescaled metric \( G_{\mu\nu} = V^{-2} g_{\mu\nu} = (1, -V^{-4} \mathbb{I}_3) \), for which no obstruction arises. But such a rescaling, implemented on the fields \( (A_i, \phi) \) as \( A_i \to A_i, \phi \to \psi = V^{-1} \phi \), changes (1) written in the rescaled metric \( G_{ij} \) into

\[
\text{tr} \left[ -\frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} D_i \psi D^i \psi \right] \sqrt{-g} + \partial^i \left[ \frac{1}{\sqrt{g}} \partial^i \ln \sqrt{g_{oo}} \text{tr} \psi^2 \right] - \frac{1}{\sqrt{g}} \partial^i \ln \sqrt{g_{oo}},
\]

where \( \hat{\Delta} = \frac{1}{\sqrt{g}} \partial^i (\sqrt{g} \partial^i) \) is the Laplacian associated with the space metric \( \hat{g}_{ij} \). Our clue is now that, for the class [4] of metrics,

\[
\hat{\Delta} \ln \sqrt{g_{oo}} = \frac{1}{2} R,
\]
one half of the Ricci scalar. In the last term in (8) we recognize, hence, precisely the curvature term in (5). The rescaled expression, (8), only differs from the density (5) by a surface term; they are, therefore, equivalent. At last, the self-duality equations are conformally invariant.

Our results here shed some new light on those, obtained earlier by Comtet, and others [7]. In that approach, the original field equations (2), are kept unchanged but the Bogomolny equations are modified.

Then for the YMH theory (1) with the rescaled metric \( g_{\mu\nu} \), the Bogomolny (alias self-dual) equations (3) work. In the Reissner-Nordström case, at last, the self-duality equations are conformally invariant.

It is worth mentioning that the procedure considered here can be further generalized. Let indeed \( g_{\mu\nu} = (g_{oo}, -\hat{g}_{ij}) \) be an arbitrary static spacetime, and consider the rescaled metric \( \hat{G}_{\mu\nu} = (1, -\hat{G}_{ij}), \hat{G}_{ij} = \hat{g}_{ij}/g_{oo} \). Then for the YMH theory (1) with the rescaled metric \( \hat{G}_{\mu\nu} \), the Bogomolny (alias self-dual) equations (3) work. Then implementing the rescaling as \( \psi = (g_{oo})^{-1/2}\phi \) provides us with a YMH theory on the original space with Lagrangian

\[
\text{tr} \left[ \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} D_i \psi D^i \psi + \frac{1}{2} \Omega \psi^2 \right] \sqrt{\hat{g}}, \tag{12}
\]

where a surface term has been dropped. The Bogomolny equations (3) become “Bogomolny-type”, Eqns. (10).

In the Papapetrou-Majumdar case, \( \Omega = -R/2 \), and we recover the results in [2].

Another interesting case can be that of AdS space [11],

\[
ds^2 = \left(1 - \frac{\Lambda}{3} r^2\right) dt^2 - \frac{dr^2}{1 - \frac{r_s^2}{r^2}} - r^2 d\omega^2,
\]

where \( d\omega = d\theta^2 + \sin^2 \theta d\phi^2 \).

For large \( r \), (15) becomes approximately \( \Omega \approx R/6 \), where \( R = 4\Lambda \) is the scalar curvature.

Similarly in Schwarzschild space,

\[
ds^2 = \left(1 - \frac{r_c}{r}\right) dt^2 - \frac{1}{\left(1 - \frac{r_c}{r}\right)} dr^2 - r^2 d\omega^2,
\]

we have

\[
\Omega = \frac{1}{4} \frac{r_c^2}{r^4(1 - \frac{r_c}{r})}.
\]

In the Reissner-Nordström case, at last,

\[
ds^2 = (1 - \frac{r_s}{r} + \frac{r_s^2}{r^2}) dt^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r} + \frac{r_s^2}{r^2}\right)^2} - r^2 d\omega^2,
\]

we find

\[
\Omega = \frac{1}{4} \frac{r_s^2 - 4r_s^2 Q^2}{r^4(1 - \frac{r_s}{r} + \frac{r_s^2}{r^2})},
\]

which, in the extreme case \( r_s^2 = 4r_s^2 Q \), vanishes, leaving us with the result of [7].

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[11] Monopoles in AdS space were considered by Lugo et al. [9]. The do not have Bogomolny equations, however, since their model is different. In particular, they don’t have our $\Omega$-term here.