Fluctuations of Wigner-type random matrices associated with symmetric spaces of class DIII and CI*

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Abstract
Wigner-type randomizations of the tangent spaces of classical symmetric spaces can be thought of as ordinary Wigner matrices on which additional symmetries have been imposed. In particular, they fall within the scope of a framework, due to Schenker and Schulz-Baldes, for the study of fluctuations of Wigner matrices with additional dependencies among their entries. In this contribution, we complement the results of these authors by explicit calculations of the asymptotic covariances for symmetry classes DIII and CI and thus obtain explicit CLTs for these classes. On the technical level, the present work is an exercise in controlling the cumulative effect of systematically occurring sign factors in an involved sum of products by setting up a suitable combinatorial model for the summands. This aspect may be of independent interest.

Keywords: fluctuations, Wigner matrices with dependent entries, Riemannian symmetric spaces, asymptotic covariances, normal-superconducting hybrid structures, combinatorial models for sums of products, dihedral noncrossing pair partitions

(Some figures may appear in colour only in the online journal)

1. Introduction

Much of random matrix theory is concerned with probability measures on the spaces of Hermitian, real symmetric, and quaternion real matrices. One of the reasons for this focus is the fact, proved by Freeman Dyson [1], that any Hermitian matrix (thought of as a truncated...
Hamiltonian of a quantum system) that commutes with a group of unitary symmetries and time reversals, breaks down to these three constituents, which are, in structural terms, the tangent spaces to the Riemannian symmetric spaces (RSS) of class A, AI, and AII. However, this ‘threefold way’, as Dyson referred to it, does not contain all RSS of physical interest. In particular, condensed matter theory exhibits examples for all ten infinite series of irreducible RSS, which, by analogy, are sometimes dubbed the ‘tenfold way’. In this contribution we will be concerned with classes DIII and CI, corresponding to the symmetric spaces $\text{SO}(2N)/\text{U}(N)$ and $\text{USp}(2N)/\text{U}(N)$, respectively. They made their appearance as mathematical descriptions of normal-superconducting hybrid structures ([2–4], see also [5, 6] for their role in the theory of topological insulators); a metallic quantum dot is put in contact with two superconducting regions across potential barriers, the temperature being so low that the dot is much smaller than the electrons’ phase coherence length. In the case that spin-rotation symmetry is broken by the presence of a sufficient number of impurities, the system has class DIII symmetry. Otherwise it has class CI symmetry.

Wigner’s famous result of 1958 [7] states that for a symmetric matrix with independent entries on and above the diagonal, the mean empirical spectral distribution converges weakly to the semicircle distribution as matrix size tends to infinity. It is the prototype of a universality result in that it only depends on certain assumptions about the moments of the matrix entries, but not on the specifics of their distributions, and has been extended to the full tenfold way by Hofmann-Credner and the author in [8]. It turns out that the semicircle distribution remains the limit law of the empirical spectral measures for seven of the ten families, while for the ‘chiral’ classes with the Lie-theoretic labels AII, BDI, CII, it has to be replaced with a suitably transformed Marčenko–Pastur distribution.

Wigner’s theorem has been complemented by results about the corresponding fluctuations, initially under the assumption of Gaussianity of the matrix entries [9]. A level of generality comparable to Wigner’s set-up was reached by Kusalik, Mingo, and Speicher in [10], even though that paper treats sample covariance rather than Wigner type matrices. The Wigner case is contained in the article [11] by Schenker and Schulz-Baldes. Actually, their work does much more, in that it substantially weakens the independence assumption on the upper-diagonal entries of standard Wigner matrices, to the effect that, e.g. additional symmetries can be enforced. A full statement of their main result will be given below. In the case of sample covariance matrices, an analogous relaxation of independence assumptions was achieved by Friesen, Löwe, and the author in [12].

While [11, 12] contain results on a high level of generality, additional work is required if one aims at explicit central limit theorems for the fluctuations in models with specific non-trivial symmetries. The present contribution sets out to develop some convenient bookkeeping tools which are mighty enough to guide the explicit evaluation of formulae for the asymptotic variances of Wigner-type matrices associated with the symmetric spaces of class DIII and CI.

Section 2 below contains the definitions of the matrix ensembles of interest, as well as the statement of our Gaussian limit theorem. Section 3 provides as much background on the framework set up by Schenker and Schulz-Baldes as is necessary to make the present work independent of any previous acquaintance with the details of [11]. Section 4 treats the case of class DIII and, along the way, develops the core material of our approach. Section 5 adapts that material to the slightly different circumstances encountered in the case of class CI. Finally, in section 6, we place our contribution within the context of other methods that are used for the asymptotic analysis of random matrix models in physics.
2. Matrix ensembles and results

If \( g \) is the Lie algebra, i.e. the tangent space (at any point) of a compact Lie group \( G \), then \( i g = \sqrt{-1} g \) is a space of Hermitian matrices. If \( g = \mathfrak{k} \oplus \mathfrak{p} \) is its decomposition into the \((+1)\)-eigenspace \( \mathfrak{k} \) and the \((-1)\)-eigenspace \( \mathfrak{p} \) of a (Cartan) involution, and \( \mathfrak{k} \) is the Lie algebra of a compact Lie group \( K \), then \( \mathfrak{p} \) can be viewed as the tangent space of the Riemannian symmetric space \( G/K \), and \( i \mathfrak{p} \) as a convenient proxy if one prefers doing random matrix theory for matrices with real eigenvalues. In a nutshell, this is the rationale for attaching Lie-theoretic labels to the following two spaces of Hermitian matrices.

Class DIII

\[
\mathcal{M}^{\text{DIII}}_n = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : X_i \in (i\mathbb{R})^{n \times n} \text{ skew symmetric} \right\}.
\]

Class CI

\[
\mathcal{M}^{\text{CI}}_n = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : X_i \in \mathbb{R}^{n \times n} \text{ symmetric} \right\}.
\]

These spaces can be turned into Wigner type ensembles of random matrices as follows. Each matrix space induces a finite partition \( \{ B_i : i \in I \} \) (uniquely determined by the requirement that it possess minimal number of blocks) of the set of index pairs in \( \{1, \ldots, 2n\}^2 \) which do not correspond to the diagonals of skew-symmetric blocks, such that for all \( i \in I \) the following property holds: as soon as the matrix entry which corresponds to an index pair from \( B_i \) is determined, the matrix entries corresponding to all index pairs from \( B_i \) are determined as well. Let \( \{(p_i, q_i) : i \in I\} \) be a system of representatives for the blocks, and let \( (a_n(p_i, q_i))_{i \in I} \) be a family of independent centered complex random variables with the following properties:

- For each \( k \in \mathbb{N} \) the \( k \)th absolute moments \( \mathbb{E}|a_n(p_i, q_i)|^k \) are uniformly bounded in \( n \) and \( i \).
- There exists \( \sigma^2 > 0 \) such that

\[
\mathbb{E}|a_n(p_i, q_i)|^2 = \sigma^2
\]

for all \( i \in I \).

Note that we have defined \( I \) in such a way that the latter condition does not lead to conflict with the fact that the diagonal elements of skew-symmetric matrices are zero. With the \( (a_n(p_i, q_i))_{i \in I} \) at hand, for \( (r, s) \in B_i \), we define the matrix entry \( a_n(r, s) \) as an identical copy (or possibly the negative of an identical copy) of \( a_n(p_i, q_i) \), according to which algebraic relations among entries give rise to the class \( B_i \). If \( (r, s) \) corresponds to the diagonal of a skew symmetric block, we set \( a_n(r, s) = 0 \).

With these definitions in place, we define the matrix ensembles of interest as

\[
X_n^C = \frac{1}{\sqrt{2n}} (a_n(p, q))_{p,q=1,\ldots,2n},
\]

where \( C \in \{ \text{DIII}, \text{CI} \} \). If \( C \) is clear from the context, we drop the superscript.

To state our results, denote by \( T_m \) the \( m \)th Chebychev polynomial of the first kind, i.e. the one which satisfies the identity \( T_m(2 \cos(\theta)) = 2 \cos(m\theta) \). Write \( T_m(\cdot, \sigma) \) for the re-scaled version given by
Theorem 2.1. Let $C \in \{\text{DIII, CI}\}$. For each $M \in \mathbb{N}$, the random vector

$$((\text{Tr}(T_1(X^C_n, \sigma)) - E(\text{Tr}(T_1(X^C_n, \sigma)))), \ldots, (\text{Tr}(T_M(X^C_n, \sigma)) - E(\text{Tr}(T_M(X^C_n, \sigma)))))$$

converges, as $n \to \infty$, to a centered Gaussian vector with covariance matrix

$$\text{diag}(V^C(1), \ldots, V^C(M)),$$

where

$$V^C(m) = \begin{cases} 0, & m = 1, \\ m^*, & m = 2, \\ 0, & m \geq 3 \text{ odd}, \\ 4m\sigma^2m, & m \geq 4 \text{ even}. \end{cases}$$

Remark 2.2. In the statement of theorem 2.1, the value of $V^C(2)$ is unspecified, since it depends on the fourth moments of the $a_n(p, q)$, about which, apart from uniform boundedness, no assumptions are made in the present paper. Understanding the special role of $m = 2$ requires substantially more background on the details of the proofs in [11] than will be provided in section 3 below. In a nutshell, the reason is that for small $m$ the reduction to dihedral noncrossing pair partitions, which will be reviewed in section 3, is subject to a few caveats.

3. Implementing symmetries: a review of SSB theory

The objects of study of the present paper, i.e. Wigner type randomizations of the matrix spaces of class DIII and CI, as defined in section 2, can be viewed as matrices which resemble Wigner matrices of the familiar sort, but in which groups of up to four entries are allowed to exhibit deterministic dependence—which may be seen as an extreme case of stochastic dependence. It will be argued in the present section that viewing our objects in this way contains a clue about the large $n$ asymptotic behaviour one may expect.

Throughout this section, a standard Wigner matrix will be an Hermitian matrix

$$X_n = \frac{1}{\sqrt{n}}(a_n(p, q))_{p, q=1, \ldots, n}$$

such that the family $a_n(p, q), p \leq q$ of matrix entries on and above the diagonal consists of independent, but not necessarily identically distributed, random variables. We will assume that the $a_n(p, q)$ are centered, and for each $2 \leq k \in \mathbb{N}$ we require that the $k$th moments $E|a_n(p, q)|^k$ be finite and uniformly bounded in $n, p, q$. It is customary to assume that the variance of the entries be equal to $\sigma^2 > 0$ for all $n, p, q$, and we will make this part of our notion of a standard Wigner matrix.

The seminal papers [11, 13] of Schenker and Schulz-Baldes (henceforth SSB) contain a threefold contribution to our problem. Firstly, they develop a convenient conceptual framework for dealing with stochastic dependence among the entries of what would otherwise be a standard Wigner matrix. Secondly, they use this framework to state conditions on the dependencies under which, as $n \to \infty$, the empirical spectral measure tends to the semicircular
distribution, with Gaussian fluctuations about this limit. And thirdly, under more restrictive, but still quite general conditions, they establish a general formula for the asymptotic covariances of the Gaussian fluctuations (see (3.1) below).

Writing \([n]\) for the set \(\{1, 2, \ldots, n\} \ (n \in \mathbb{N})\), the basic idea of the SSBE approach to dependence is as follows: given an equivalence relation \(\sim_n\) on \([n]^2\), one stipulates that matrix entries \(a_n(p_1, q_1), \ldots, a_n(p_\nu, q_\nu)\) be independent whenever \((p_1, q_1), \ldots, (p_\nu, q_\nu)\) are elements of \(\nu\) different equivalence classes of \(\sim_n\), whereas the joint distribution of a family \(a_n(p_r, q_\ell)\), where \((p_r, q_\ell)\) runs through an equivalence class of \(\sim_n\), is arbitrary. The interpretation we have in mind is that \(\sim_n\) captures the symmetries that define the different matrix spaces from section 2, and that matrix entry random variables that correspond to \(\sim_n\)-equivalent index pairs are, up to a sign, identical (as measurable maps). But note that the scope of the SSBE formalism is much broader.

The fluctuation results of SSBE assume that certain quantitative characteristics of the \(\sim_n\)-equivalence classes do not grow too fast as a function of \(n\). Specifically, it is assumed that the quantity

\[
\alpha_2(n) = \max_{p \neq q \in [n]} \#\{(r, s) \in [n]^2 : (p, q) \sim_n (r, s)\}
\]

is of order \(O(n^\epsilon)\) for all \(\epsilon > 0\), and that \(\hat{\alpha}_0(n)\alpha_2(n)^9 = o(n^2)\) for all \(\eta > 0\), where

\[
\hat{\alpha}_0(n) = \#\{(p, q, r) \in [n]^3 : (p, q) \sim_n (q, r) \text{ and } p \neq r\}.
\]

Inspection of the matrix spaces of classes DIII and CI, as defined in section 2, yields that for the cases of interest, \(\alpha_2(n) \leq 4\) and \(\hat{\alpha}_0(n) = 0\). So these conditions are always satisfied in our applications.

For investigating the fluctuations of the empirical spectral measure of a standard Wigner matrix \(X_n\) about its limit, i.e. the semicircle distribution, a well-established proxy are the fluctuations of vectors of traces of nonnegative integer powers \(\text{Tr}(X_n^\ell)\) (\(\ell\) running through a finite set). In [11], theorem 2.1 that we are going to treat as a black box, it is shown that the condition on the growth of \(\alpha_2(n)\) alone suffices to guarantee that the joint cumulants of order \(\geq 3\) of a family of \(\text{Tr}(X_n^\ell)\) vanish as \(n \to \infty\). This already yields a Gaussian limit theorem for the fluctuations, albeit one with possibly degenerate asymptotic variances.

The aspect of [11] which will be our starting point for explicit computations, and which uses the growth conditions on both \(\alpha_2\) and \(\hat{\alpha}_0\), is concerned with expressions for the asymptotic cumulants of order \(2\), i.e. the asymptotic covariances. Before stating what [11] has to say about these objects on the level of generality it aims at, let us digress a little and explain what computing covariances of traces of powers of standard Wigner matrices amounts to. As a matter of fact,

\[
\text{Tr}(X_n^\ell) = \frac{1}{n^{\ell/2}} \sum a_n(p_1, q_1)a_n(p_2, q_2) \cdots a_n(p_k, q_k),
\]

where the sum is over all sequences of index pairs \((p_\ell, q_\ell) \in [n]^2\) that satisfy the consistency relations

\[
p_2 = q_1, p_3 = q_2, \ldots, p_k = q_{k-1}, p_1 = q_k.
\]

Our next task is to introduce a convenient bookkeeping for the expansion one obtains when one plugs two expansions of the type (4) (corresponding to \(k_1, k_2\), say) into a bilinear function. Denote by

\[
[k_1] \sqcup [k_2] := \{(i, \ell) : i = 1, 2, \ell \in [k_i]\}
\]
the (generalized) disjoint union of \([k_1] \cup [k_2]\). It is sometimes convenient to represent the set \([k_1] \cup [k_2]\) graphically as in figure 1.

We write \(\text{Consist}(k_1, k_2, n)\) for the set of all consistent multi-indices, i.e. maps \(P : [k_1] \cup [k_2] \to [n]^2\) such that the families \(P(1, \bullet)\) and \(P(2, \bullet)\) individually satisfy the consistency relations (5) above, i.e. the set of all maps \(P\) such that for \(i = 1, 2\) one has

\[
\begin{align*}
\text{Proj}_2(P(i, \ell)) &= \text{Proj}_1(P(i, \ell + 1)), & \ell = 1, \ldots, k_i - 1, \\
\text{Proj}_2(P(i, k_i)) &= \text{Proj}_1(P(i, 1)),
\end{align*}
\]

For better readability, we write \(P_{i, \ell}\) in the place of \(P(i, \ell)\). With this notation, we have the expansion

\[
\text{Cov}(\text{Tr}(X_n^{k_1}), \text{Tr}(X_n^{k_2})) = \frac{1}{n^{(k_1+k_2)/2}} \sum_{P \in \text{Consist}(k_1, k_2, n)} \text{Cov} \left( \prod_{\ell=1}^{k_1} a_n(P_{1, \ell}), \prod_{\ell=1}^{k_2} a_n(P_{2, \ell}) \right). 
\]

(6)

To describe the next step, we consider the equivalence relation \(\sim_n\) on \([n]^2\) as fixed. Given \(P \in \text{Consist}(k_1, k_2, n)\), we say that elements \((i, \ell)\) and \((i', \ell')\) of \([k_1] \cup [k_2]\) are \(P\)-related if, and only if, \(P_{i, \ell} \sim_n P_{i', \ell'}\). Loosely speaking, if \((i, \ell)\) and \((i', \ell')\) are \(P\)-related, then the corresponding matrix entries are allowed to be stochastically dependent. Obviously, being \(P\)-related is an equivalence relation on \([k_1] \cup [k_2]\), since \(\sim_n\) is one on \([n]^2\), and thus induces a partition on the elements of \([k_1] \cup [k_2]\). Graphically, \(P\) induces a grouping of the points on the two circles into blocks. These blocks may comprise points on different circles, and we may represent them by joining the points of a block by a line (see figure 2).

We are now ready for the crucial step: we classify the multi-indices from \(\text{Consist}(k_1, k_2, n)\) according to which partition they induce on \([k_1] \cup [k_2]\), so that we may represent the right hand side of equation (6) as follows:

\[
\sum_{\pi \text{ partition of } [k_1] \cup [k_2]} \left\{ \frac{1}{n^{(k_1+k_2)/2}} \sum_{P \in \text{Consist}(k_1, k_2, n; \pi)} \cdots \right\},
\]

(7)

where we have written \(\text{Consist}(k_1, k_2, n; \pi)\) for the set of those elements of \(\text{Consist}(k_1, k_2, n)\) which induce \(\pi\). We will at times informally refer to the elements of \(\text{Consist}(k_1, k_2, n; \pi)\) as admissible (consistent) multi-indices (for \(\pi\).
In first approximation, the analysis of Cov \((\text{Tr}(X_{n}^{k_1}), \text{Tr}(X_{n}^{k_2}))\) in the \(n \to \infty\) limit, as undertaken in [11], proceeds by classifying the partitions \(\pi\) according to the asymptotic behaviour of the corresponding \{\ldots\}. The technical details, however, are somewhat more involved. In fact, there is an intrinsic way to assign to any partition \(\pi\) of \([k_1] \cup [k_2]\) a number \(m = 1, \ldots, \min(k_1, k_2)\) and a partition \(\hat{\pi}\) of \([m] \cup [m]\) with asymptotically equivalent behaviour. Consequently, one can write (7) as

\[
\sum_{m=1}^{\min(k_1, k_2)} \frac{\sigma_{k_1+k_2-2m} A_{k_1}^{m} A_{k_2}^{m}}{m^m} \pi \text{ partition of } [m] \cup [m] \sum_{P \in \text{Consist}(m, m; \pi)} \ldots \bigg \} + o(1),
\]

where, as above, \(\sigma^2\) is the second moment of the matrix entries, and \(A_{k_1}^{m}\) is the multiplicity with which partitions of \([k_1] \cup [k_2]\) correspond to the same partition of \([m] \cup [m]\). As it turns out, this multiplicity is related to the combinatorics of Chebychev polynomials, but we will see that for our present purposes we may circumvent any precise statement of this claim. It is, however, crucial to point out that, up to an error of order \(o(1)\), we may restrict the sum over partitions of \([m] \cup [m]\) to a smaller subset of partitions which is related to the dihedral group of order \(2m\).

Denote by \(\text{Sym}([m])\) the group (w.r.t. composition of maps) of all bijections of \([m]\) into itself. Let \(D\) be the subgroup of \(\text{Sym}([m])\) which is generated by the cycle \(\gamma = (1, 2, 3, \ldots, m)\) and the involution \(\tau = (1, m)(2, m-1)\ldots(\frac{m+1}{2}, \frac{m}{2} + 1)\) if \(m\) is even, or \(\tau = (1, m)(2, m-1)\ldots(\frac{m+1}{2}, \frac{m}{2} + 1)\) if \(m\) is odd. Note that \(\gamma\) maps \(1\) to \(1\), \(2\) to \(2\), and so on, and that \(\tau\) swaps \(1\) with \(2\), \(2\) with \(m - 1\) and so on, and fixes \(\frac{m+1}{2}\) if \(m\) is odd. It is easily verified that \(\tau^{-1} \gamma \tau = \tau \gamma \tau = \gamma^{-1}\), so \(D\) is isomorphic to, and will be identified with, the dihedral group \(D_{2m}\), see, e.g. [14, lemma 2.14].

Each \(g \in D_{2m}\) can be encoded as a pair partition

\[
\{ \{(1, \ell), (2, g(\ell))\} : \ell \in [m]\},
\]

which we denote again by \(g\). Pictorially, the generator \(\gamma \in D_{2m}\) corresponds (for \(m = 7\)) to the partition in figure 3.

On the face of it, the graphical representation of the generator \(\tau\) will involve many intersecting paths. But this can be circumvented by reversing the order of the labels of the nodes of
the inner circle (see right panel in figure 4). Indeed, the pair partitions that arise from $D_{2m}$ as above are dubbed in [11] noncrossing dihedral pair partitions.

As we have already mentioned, the multiplicities $A^m_{k_1k_2}$ in (8) involve the combinatorics of Chebyshev polynomials. In order to obtain neat results, it is expedient to suppress the combinatorial factors by considering scaled Chebyshev polynomials, as defined in section 2, in the Wigner matrices.

The technical result from [11], which we are going to apply subsequently, now reads as follows:

**Proposition 3.1.** [11, Theorem 2.4]

$$\text{Cov}(\text{Tr}(T_m(X_n, \sigma)), \text{Tr}(T_{\mu}(X_n, \sigma))) = \delta_{m\mu} V_n(m) + o(1),$$

with

$$V_n(m) = \begin{cases} \frac{1}{n} \sum_{(p, q) \in \gamma_n(n)} \mathbb{E}(a_n(p, p)a_n(q, q)), & m = 1, \\ \frac{1}{n^2} \sum_{p \neq r, s, t} \text{Cov}(|a_n(p, q)|^2, |a_n(r, s)|^2), & m = 2, \\ \frac{1}{n^m} \sum_{g \in \text{D}^*_m} \sum_{P \in \text{Consist}^*(m, m; n; g)} \prod_{\ell=1}^{m} \mathbb{E}(a_n(P_1, \ell)a_n(P_{2\ell}(\ell))), & m \geq 3. \end{cases}$$

We have written $\text{Consist}^*(m, m, n; g)$ for the set of those multi-indices in $\text{Consist}(m, m, n; g)$ which do not hit a coordinate pair for which the corresponding matrix entry is deterministically zero.

The identically vanishing matrix entries we have in mind are the diagonal elements of the skew symmetric blocks in the matrix space DIII, as defined in section 2. Note that in the statement of proposition 3.1, we have inserted our $\text{Consist}^*(m, m, n; g)$ in the place of a slightly differently defined set of multi-indices in [11]. In fact, the multi-indices of [11] only stay clear of the main diagonal. Since the number of exceptions grows like $n$, and the total number of
matrix entries grows like $n^2$, it is not very surprising that these modifications should do no harm. A more careful argument can be based on remark 4.7 in [11].

4. Class DIII

We know from section 3 that an arbitrary, but fixed, equivalence relation $\sim_n$ on the set $[n]^2$ assigns to a multi-index $P \in \text{Consist}(k_1, k_2, n)$ an equivalence relation $\pi$ on the disjoint union $[k_1] \sqcup [k_2]$. We have also reviewed a result from [11] (see proposition 3.1), according to which, for our present purposes, we may shift our attention to all those multi-indices $P \in \text{Consist}(m, m, n)$, $m \in \{1, 2, \ldots, \min(k_1, k_2)\}$, which induce pair partitions of the form $\{\{(1, \ell), (2, g(\ell))\} : \ell \in [m]\}$ on $[m] \sqcup [m]$, where $g$ runs through the dihedral group $D_{2m}$.

In the present section, we specialize to the case where $\sim_n$ encodes the symmetries of the matrices of class DIII, and undertake the asymptotic evaluation of the formula for $V_n(m)$ (see proposition 3.1) in the case $m \geq 3$. Note that as the class DIII matrices are of size $2n \times 2n$, the $n$ from the general discussion has to be interpreted as $2n$. Everything that follows will be based on the following obvious lemma.

**Lemma 4.1.** Let $a, b \in [n], a \neq b$. Write

$$C_1(a, b) := \{(a, b), (b, a), (n + a, n + b), (n + b, n + a)\},$$

$$C_2(a, b) := \{(n + a, b), (n + b, a), (a, n + b), (b, n + a)\},$$

and observe that $C_i(a, b) = C_i(b, a)$ ($i = 1, 2$). Then the $C_i(a, b)$ ($i = 1, 2, a, b \in [n], a \neq b$) are pairwise disjoint. They capture the symmetries of class DIII matrices in the sense that

- the matrix entry random variables which correspond to index pairs from the same $C_i(a, b)$ are identical up to a sign, and
- the matrix entry random variables which correspond to index pairs from different $C_i(a, b)$ are stochastically independent.

Note that we need not consider the case $a = b$, since the corresponding entries are diagonal entries of the skew symmetric blocks and thus deterministically fixed as zero. Compare also the remarks in the last paragraph of section 3.
4.1. Preliminary discussion

The building blocks of the formula for \( V_{2a}(m) \) are expectations of the form

\[ E(a_{2a}(p_{1,\ell}) a_{2a}(p_{2,\ell})), \]

\( p_{1,\ell} \) and \( p_{2,\ell} \) being shorthands for index pairs \((p_{1,\ell} \cdot q_{1,\ell})\) and \((p_{2,\ell} \cdot q_{2,\ell})\), which must belong to the same \( C_i(a, b) \) for some \((i,a,b)\). If one depicts this pair of pairs as a \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
    p_{1,\ell} & q_{1,\ell} \\
    p_{2,\ell} & q_{2,\ell}
\end{pmatrix},
\]

then one reads from lemma 4.1 that this matrix can be represented as \( n\Delta + \Lambda(a, b) \), where \( \Delta \) is one of the following eight matrices

\[
\begin{pmatrix}
    0 & 0 \\
    0 & 0
\end{pmatrix}, \begin{pmatrix}
    1 & 1 \\
    1 & 1
\end{pmatrix}, \begin{pmatrix}
    0 & 1 \\
    0 & 1
\end{pmatrix}, \begin{pmatrix}
    1 & 0 \\
    1 & 0
\end{pmatrix}
\]

(10)

\[
\begin{pmatrix}
    0 & 0 \\
    1 & 1
\end{pmatrix}, \begin{pmatrix}
    1 & 0 \\
    0 & 0
\end{pmatrix}, \begin{pmatrix}
    0 & 1 \\
    1 & 1
\end{pmatrix}
\]

(11)

and \( \Lambda(a, b) \) is either \( \begin{pmatrix} a & b \\ a & b \end{pmatrix} \) or \( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \) for some \( a, b \in [n], a \neq b \). Since \( \Delta \) has weight \( n \) and \( a, b \) take on values between 1 and \( n \), the indicated decomposition of \( 2 \times 2 \) matrices with integer entries between 1 and \( 2n \) is unique.

**Lemma 4.2.** The matrices in (10) and (11) contain precisely those \( 2 \times 2 \) matrices with entries in \([0,1]\) for which the sum over all entries is \( \equiv 0 \pmod{2} \).

**Proof.** By linear algebra over the field \( \mathbb{Z}/2\mathbb{Z} \), the kernel of the linear map that sums up the matrix entries and then reduces modulo 2 has eight elements and obviously contains the matrices in (10) and (11). \( \square \)

How does the quantity we wish to evaluate, \( E(a_{2a}(p_{1,\ell} \cdot q_{1,\ell}) a_{2a}(p_{2,\ell} \cdot q_{2,\ell})) \), relate to these matrices? We will see in section 4.4 below that this value is \( \pm \sigma^2 \). The sign does depend on the choice for \( \Delta \) and the overall form of \( \Lambda \), but is independent of the specific values of \( a \) and \( b \). Actually, we will have to evaluate not only a single expectation that is based on a single \( 2 \times 2 \) matrix, but a product of expectations that is based on a sequence of length \( m \) of such matrices. Since the multi-indices \( P \) run through the set Consist\(^*(m,m,n;g) \), the sign of these products is governed by two constraints, the cumulative effect of which is not altogether obvious from the outset. On the one hand, the sequences of matrix entry pairs are required to be consistent, i.e. \( p_{1,\ell+1} = q_{1,\ell} \). On the other hand, even if the upper row in the sequence of \( 2 \times 2 \) matrices will reflect the consistency of \((p_{1,\bullet}, q_{1,\bullet})\), in the lower row the consistency of \((p_{2,\bullet}, q_{2,\bullet})\) is interfered with by the action of \( g \).

To get some intuition for what may happen, let us consider two choices for \( g \) in the case \( m = 5 \). It will be important to keep in mind that writing two matrix index pairs as the upper and lower rows of a \( 2 \times 2 \) matrix is meant to indicate that the corresponding matrix entries are coupled together by the symmetries of the class DIII matrices. We start with the example of the cyclic shift \( \gamma \). Then the relevant information for the evaluation of

\[
\prod_{\ell=1}^{5} E(a_{2a}(p_{1,\ell} \cdot q_{1,\ell}) a_{2a}(p_{2,\ell} \cdot q_{2,\ell}))
\]

is contained in the sequence

\[
\begin{pmatrix}
    p_{11} & q_{11} \\
    p_{22} & q_{22}
\end{pmatrix}, \begin{pmatrix}
    p_{12} & q_{12} \\
    p_{23} & q_{23}
\end{pmatrix}, \begin{pmatrix}
    p_{13} & q_{13} \\
    p_{24} & q_{24}
\end{pmatrix}, \begin{pmatrix}
    p_{14} & q_{14} \\
    p_{25} & q_{25}
\end{pmatrix}, \begin{pmatrix}
    p_{15} & q_{15} \\
    p_{21} & q_{21}
\end{pmatrix}
\]
It emerges from this representation that it is actually rather easy to reconcile the action of \( \gamma \) with the consistency requirement on \((p_\ell, q_\ell)\). One has to make sure that the second column of the \( \ell \)th matrix is equal to the first column of the \((\ell + 1)\)th matrix. Writing the \( \ell \)th matrix as \( n\Delta_\ell + \Lambda_\ell \) (as in the case of a single matrix above), this implies a domino like condition on the \( \Delta_\ell \), i.e. if one thinks of the sequence of \( \Delta_\ell \) as a sequence of domino tiles, then adjacent ends must match. As to the \( \Lambda_\ell \), a configuration like
\[
\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} b & c \\ c & b \end{pmatrix}
\]
is perfectly compatible with a domino type condition, but this comes at the price of requiring that \( a = c \), which reduces the number of degrees of freedom for the multi-indices and causes them to become asymptotically negligible. Therefore the relevant case is that all \( \Lambda_\ell \) are of the form
\[
\begin{pmatrix} a & b \\ a & b \end{pmatrix}.
\]
We will see below that the domino type condition, along with the simple structure of the matrices from (10) and (11), which are the admissible domino tiles, provides plenty of structural information. This will make it possible to determine the signs of the various products of expectations, and control the summation over the dihedral partitions and the corresponding admissible multi-indices.

Next we turn to the involution \( \tau \). The evaluation of
\[
\prod_{\ell=1}^{5} E(a_{2\ell}(p_{1,\ell}, q_{1,\ell})a_{2\ell}(p_{2,\tau(\ell)}, q_{2,\tau(\ell)}))
\]
is then guided by the sequence
\[
\begin{pmatrix} p_{11} & q_{11} \\ p_{25} & q_{25} \end{pmatrix} \begin{pmatrix} p_{12} & q_{12} \\ p_{24} & q_{24} \end{pmatrix} \begin{pmatrix} p_{13} & q_{13} \\ p_{23} & q_{23} \end{pmatrix} \begin{pmatrix} p_{14} & q_{14} \\ p_{22} & q_{22} \end{pmatrix} \begin{pmatrix} p_{15} & q_{15} \\ p_{21} & q_{21} \end{pmatrix}.
\]
While in the upper row we have ‘consistency as usual’, the consistency requirement on the sequence \((p_{2\ell}, q_{2\ell}) \ (\ell \in [m])\) translates into an alignment condition for the lower row that may be visualized as in figure 5.

If the sum over \( P \in \text{Consist}^*(5, 5, n; \tau) \) is to give a nontrivial contribution to (9) in the \( n \to \infty \) limit, then first of all we will have to find enough sequences of the form \((n\Delta_\ell + \Lambda_\ell(a_\ell, b_\ell))_{\ell \in [5]}\) that satisfy this double requirement on their upper and lower rows. This is indeed possible, since we may for instance take for all \( \Delta_\ell \) the null matrix and for the sequence of the \( \Lambda_\ell \) arrays of the following type:
\[
\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} b & c \\ c & b \end{pmatrix} \begin{pmatrix} c & d \\ d & c \end{pmatrix} \begin{pmatrix} d & e \\ e & d \end{pmatrix} \begin{pmatrix} e & a \\ a & e \end{pmatrix}.
\]
One verifies at once that the arcs in figure 5 correspond to equality of characters in the lower row of the matrices. Furthermore, one has \( m = 5 \) degrees of freedom, which matches the prefactor \( 1/n^5 \) in (9). As to the \( \Delta_\ell \), besides the sequence of null matrices there are many other sequences whose upper and lower rows will satisfy the double requirement. And if both the \( \Delta_\ell \) and \( \Lambda_\ell(a_\ell, b_\ell) \) sequences satisfy the requirements, then \( n\Delta_\ell + \Lambda_\ell(a_\ell, b_\ell) \) will do as well.

### 4.2. Overview of the remainder of the section

We have seen by way of examples that it is not very difficult to find sequences \((n\Delta_\ell + \Lambda_\ell(a_\ell, b_\ell))_{\ell \in [m]}\) which satisfy the specific consistency requirements on their upper
and lower rows, and thus there will be many admissible consistent multi-indices for dihedral partitions. These multi-indices serve to encode sequences of pairs of matrix entries, and it is through the product of the covariances of the pairs from such a sequence that a multi-index, and thus the associated sequence
\[(n \Delta_\ell + \Lambda_\ell(a_\ell, b_\ell))_{\ell \in [m]},\]
enters the formula (9) for \(V_n(m)\).

We will see in section 4.4 that these covariances are always \(\pm \sigma^2\), and that the signs depend on both the \(\Delta_\ell\) and the \(\Lambda_\ell\), in ways that are a priori rather involved.

The single most effective step to reduce this complexity is lemma 4.8 below, which shows that for the purposes of \(n \to \infty\) asymptotics the effect of the \(\Lambda_\ell\) is rather straightforward to control. To control the effect of the \(\Delta_\ell\), we have found it helpful to think of them as domino tiles which are placed next to each other, with certain rules on the admissibility of configurations that reflect the consistency requirements of the multi-indices. Of course, the crucial task is to classify and count the admissible configurations, and to elucidate their effect on the signs of the products of covariances. This will be undertaken in section 4.4, using a host of terminology to be set up in section 4.3. The results from these subsections will be put together in section 4.5 to determine the \(V_n(m)\) explicitly.

### 4.3. Terminology and fundamental lemmas on multi-indices

To carry out the program which was outlined in the preceding subsection, we first set up some convenient terminology. For \(a, b \in [n]\), \(a \neq b\), denote by \(A(a, b)\) the ‘aligned’ matrix \(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\), and by \(R(a, b)\) the ‘reversed’ matrix \(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\). Any matrix of one of these types will be called a \(\Lambda\)-matrix. Any matrix from the list in (10) and (11) will be called a \(\Delta\)-matrix.

**Definition 4.3.** An \((m-)\)pattern is a sequence \((\Delta_\ell, \Lambda_\ell)_{\ell \in [m]}\), where each \(\Delta_\ell\) is a \(\Delta\)-matrix and (11), and each \(\Lambda_\ell\) is one of the symbols \(A\) or \(R\). For any choice \((a_\ell, b_\ell)_{\ell \in [m]} \in [n]^m\), \((a_\ell, b_\ell)_{\ell \in [m]} \in [n]^m\) such that \(a_\ell \neq b_\ell\) for all \(\ell \in [m]\), we call the sequence \((n \Delta_\ell + \Lambda_\ell(a_\ell, b_\ell))_{\ell \in [m]}\) of \(2 \times 2\) matrices an instance of the pattern \((\Delta_\ell, \Lambda_\ell)_{\ell \in [m]}\).

**Definition 4.4.** Any of the matrices
\[
\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
is a step. Any of the remaining \(\Delta\)-matrices is a plateau.

**Definition 4.5.** Let \((\Theta_\ell)_{\ell \in [m]}\) be a sequence of \(2 \times 2\) matrices with entries in \([2n]\).

(a) We say that \((\Theta_\ell)_{\ell \in [m]}\) satisfies the domino condition, if for each \(\ell \in [m]\) the second column of \(\Theta_\ell\) equals the first column of \(\Theta_{\ell+1}\) (where \(m + 1\) is identified with 1). In this case, we also refer to \((\Theta_\ell)_{\ell \in [m]}\) as a domino sequence.

(b) We say that \((\Theta_\ell)_{\ell \in [m]}\) satisfies the reverse domino condition, if for each \(\ell \in [m]\), writing
\[ \Theta_\ell = \begin{pmatrix} \alpha_\ell & \beta_\ell \\ \gamma_\ell & \delta_\ell \end{pmatrix}, \quad \Theta_{\ell+1} = \begin{pmatrix} \alpha_{\ell+1} & \beta_{\ell+1} \\ \gamma_{\ell+1} & \delta_{\ell+1} \end{pmatrix}. \tag{12} \]

one has \( \alpha_{\ell+1} = \beta_\ell, \delta_{\ell+1} = \gamma_\ell \) (where, again, \( m + 1 \) is identified with 1). In this case, we also refer to \((\Theta_\ell)_{\ell \in [m]}\) as a reverse domino sequence.

(c) If \((\Theta_\ell)_{\ell \in [m]}\) satisfies the (reverse) domino condition and is a \( \Delta \)-sequence (\( \Lambda \)-sequence), we will refer to it as a (reverse) domino \( \Delta \)-sequence or (reverse) domino \( \Lambda \)-sequence, respectively.

(d) If an instance of a pattern \((\Delta_\ell, \Lambda_\ell)_{\ell \in [m]}\) is a (reverse) domino sequence, then we refer to it as a (reverse) domino instance of \((\Delta_\ell, \Lambda_\ell)_{\ell \in [m]}\).

These definitions are not mutually exclusive, and it is illuminating, and useful for the proof of the crucial lemma 4.8 below, to study the overlap between the two classes of \( \Delta \) sequences.

Lemma 4.6. Suppose that \((\Delta_\ell)_{\ell \in [m]}\) is both a domino \( \Delta \)-sequence and a reverse domino \( \Delta \)-sequence.

(a) If \( m \) is odd, then \( \Delta_1 = \Delta_2 = \ldots = \Delta_m \), where \( \Delta_1 \) is a plateau.

(b) If \( m \) is even, then \((\Delta_\ell)_{\ell \in [m]}\) has one of the following forms:

(b1) an \( m \)-fold iteration of a plateau as in case (a).

(b2) an \( \frac{m}{2} \)-fold iteration of the sequence \( \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \) \( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \).

(b3) an \( \frac{m}{2} \)-fold iteration of the sequence \( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \) \( \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \).

(b4) an \( \frac{m}{2} \)-fold iteration of the sequence \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

(b5) an \( \frac{m}{2} \)-fold iteration of the sequence \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Proof. We think of the combined lower rows of the juxtaposed \( \Delta \) matrices as a binary sequence of length \( 2m \). Suppose that the leftmost entry (at position 1) is fixed. Then, as can be visualized by decorating figure 5 with extra arcs stemming from the domino condition (see figure 6), this value will ‘forward propagate’ to position 4 (by the reverse domino condition), then to position 5 (by the domino condition), then to position 8 (by the reverse domino condition), in total to all positions with \( \ell \equiv 0 \) (mod 4) or \( \equiv 1 \) (mod 4). It will also ‘backward propagate’ to position \( 2m \). If \( m \) is odd, then \( 2m \equiv 2 \) (mod 4), and further backward propagation will exhaust all positions with \( \ell \equiv 2 \) (mod 4) or \( \equiv 3 \) (mod 4). If \( m \) is even, then \( 2m \equiv 0 \) (mod 4), and there is nothing that forces the value at position 1 upon positions with \( \ell \equiv 2 \) (mod 4) or \( \equiv 3 \) (mod 4), see figure 7. But the values at these positions must be the same, as one sees by starting forward propagation at position 2. Finally, the claims about the upper rows follow from lemma 4.2. \( \square \)

The next lemma follows immediately from the unique decomposition of an instance of a pattern into its \( \Delta \) and \( \Lambda \) constituents.

Lemma 4.7. Let \((\Theta_\ell)_{\ell \in [m]} = (n\Delta_\ell + \Lambda_\ell(a_\ell, b_\ell))_{\ell \in [m]}\) be an instance of a pattern \((\Delta_\ell, \Lambda_\ell)_{\ell \in [m]}\). Then \((\Theta_\ell)\) is a domino sequence if, and only if, \((\Delta_\ell)\) is a domino-\( \Delta \)-sequence and \((\Lambda_\ell)\) is a domino-\( \Lambda \)-sequence. Mutatis mutandis, the same is true for reverse domino sequences.
Recall that our ultimate goal is to evaluate formula (9). Now, given \( g \in D_{2m}, P \in \text{Consist}^*(m, m, n; g) \), we know that \((P_1, P_2, g(\ell))_{\ell \in [m]}\), when considered as an \( m \)-sequence of \( 2 \times 2 \) matrices, can be written in a unique way as an instance of a pattern. Furthermore, this pattern will satisfy the domino condition or the reverse domino condition (see the examples in section 4.1 above and the general discussion in section 4.5 below). Therefore the next lemma will significantly simplify our task.

**Lemma 4.8.** Let \( P = (\Delta_{\ell}, \Lambda_{\ell})_{\ell \in [m]} \) be a pattern.

(a) Suppose that \((\Delta_{\ell})\) is a domino-\(\Delta\)-sequence. Then \( P \) has at most \( O(n^m) \) instances that are domino sequences, and it has \( o(n^m) \) such instances if, and only if, at least one of the \( \Lambda_{\ell} \) is an \( R \).

(b) Suppose that \((\Delta_{\ell})\) is a reverse domino-\(\Delta\)-sequence. Then \( P \) has at most \( O(n^m) \) instances that are reverse domino sequences, and it has \( o(n^m) \) such instances if, and only if, at least one of the \( \Lambda_{\ell} \) is an \( A \).

**Proof.** In view of the examples that were discussed in section 4.1, it only remains to verify the sufficient conditions for order \( o(n^m) \). To achieve this for (a), we bound the number of ways in which we may construct an instance \((n\Delta_{\ell} + \Lambda_{\ell}(a_{\ell}, b_{\ell}))_{\ell \in [m]}\) of \( P \) as a domino sequence. In the case that all \( \Lambda_{\ell} \) are equal to \( R \), by lemma 4.7 the sequence \((\Lambda_{\ell}(a_{\ell}, b_{\ell}))_{\ell \in [m]}\) to be constructed must be both a domino-\(\Lambda\)-sequence and a reverse domino-\(\Lambda\)-sequence. Considering the combined lower rows and arguing as in the proof of lemma 4.6 yields a bound of order \( n^2 \) on the number of such sequences, and since \( m \geq 3 \) we are done. In the case that not all \( \Lambda_{\ell} \) are equal to \( R \), the sequence \((\Lambda_{\ell}(a_{\ell}, b_{\ell}))_{\ell \in [m]}\) must contain a configuration of the form \( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \) or \( \begin{pmatrix} b & c \\ b & a \end{pmatrix} \). By the domino condition, this means \( a = b \) or \( a = b = c \), respectively, thus reducing the number of free parameters for the construction to \( \leq m - 1 \).

The corresponding verification in the proof of part (b) uses almost exactly the same arguments. \( \square \)

4.4. **Fundamental lemmas on expectations**

For a matrix \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) with entries in \([2n]\), we write \( \mathbb{E}_2(M) := \mathbb{E}(a_{2n}(\alpha, \beta) a_{2n}(\gamma, \delta)) \), where \( a_{2n}(\alpha, \beta) \) and \( a_{2n}(\gamma, \delta) \) are the entries of a class DIII Wigner type matrix with coordinates...
(α, β) and (γ, δ), respectively. Recall from section 2 that class DIII matrices have a 2 × 2 block structure, where all blocks are of size n × n. If we write \( M = n\Delta + \Lambda(a, b) \) as an instance of a pattern, then it should be noted that this representation serves as a way to describe the coordinates of a pair of matrix entries relative to the block structure. For instance, if the upper row of \( \Delta \) is (0, 1), then \( a_{2n}(α, β) \) is in the upper right block of the class DIII matrix. Specifically, it is the entry at position \((a, b)\) of that block, while relative to the full matrix, it is the entry at position \((a, n + b)\).

**Lemma 4.9.** Let \( \Delta \) be one of the 2 × 2 matrices from (10) and (11), a, b ∈ [n], a ≠ b. Then
\[
E_2(n\Delta + R(a, b)) = -E_2(n\Delta + A(a, b)).
\]

**Proof.** This is a consequence of the fact that the blocks of a class DIII matrix are skew symmetric. To elaborate on this statement, write
\[
n\Delta + A(a, b) = \begin{pmatrix} α & β \\ γ & δ \end{pmatrix}, \quad n\Delta + R(a, b) = \begin{pmatrix} α' & β' \\ γ' & δ' \end{pmatrix}.
\]

Then \( a_{2n}(α', β') = a_{2n}(α, β) \). \( a_{2n}(γ', δ') \) and \( a_{2n}(γ, δ) \) belong to the same block, and they are swapped by the transpose operation of that block (though not necessarily by the transpose operation of the full matrix). Skew symmetry of the blocks then yields
\[
E_2(n\Delta + R(a, b)) = E_2(a_{2n}(α', β') a_{2n}(γ', δ')) = -E_2(a_{2n}(α, β) a_{2n}(γ, δ)) = -E_2(n\Delta + A(a, b)).
\]

**Lemma 4.10.** Let a, b ∈ [n], a ≠ b, and \( Δ \) one of the four matrices in (10). Then
\[
E_2(n\Delta + A(a, b)) = -σ^2.
\]

**Proof.** Writing the matrix argument as \( \begin{pmatrix} α & β \\ γ & δ \end{pmatrix} \) as above, we have \( (α, β) = (γ, δ) \). Recalling from section 2 that the entries of class DIII matrices are purely imaginary, and in view of (1), we obtain
\[
E_2(Δ + A(a, b)) = E(a_{2n}(α, β)^2) = -E|a_{2n}(α, β)|^2 = -σ^2.
\]

**Lemma 4.11.** Let a, b ∈ [n], a ≠ b, and \( Δ \) one of the matrices \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then
\[
E_2(n\Delta + A(a, b)) = -σ^2.
\]

**Proof.** Representing the matrix argument as in lemma 4.10, we have in the case of the first matrix \( β, γ \in [n], α = n + γ, δ = n + β \). This means that \( a_{2n}(α, β) \) and \( a_{2n}(γ, δ) \) belong to different blocks along the skew diagonal of the 2 × 2 block structure, but have the same position relative to their respective blocks. The last statement remains true for the second matrix. Since for class DIII the blocks along the skew diagonal are the same, we obtain \( a_{2n}(α, β) = a_{2n}(γ, δ) \) and thus
\[
E_2(n\Delta + A(a, b)) = E(a_{2n}(α, β)^2) = -E|a_{2n}(α, β)|^2 = -σ^2,
\]
again invoking (1) and the fact that class DIII matrices have purely imaginary entries. □
Lemma 4.12. Let \(a, b \in [n], a \neq b\), and \(\Delta\) one of the matrices \(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\). Then
\[
E_2(n\Delta + A(a, b)) = +\sigma^2.
\]

Proof. This can be proven along the lines of lemma 4.11, this time exploiting the fact that the two blocks along the main diagonal are each other’s negatives.

All matrix arguments in lemmas 4.9–4.12, are of the form \(\Delta + \Lambda(a, b)\). It has turned out that the corresponding expectations are independent of \(a\) and \(b\), but that they do depend on whether \(\Lambda\) is the symbol \(A\) or the symbol \(R\). This observation, together with lemma 4.8, motivates the next definition. To state it, we use the following shorthand: if \(\Theta := (\Theta_\ell)_{\ell \in [m]}\) is an \(m\)-sequence of \(2 \times 2\) matrices with entries from \([2n]\), we write
\[
E_2(\Theta) := \prod_{\ell=1}^{m} E_2(\Theta_\ell).
\]

Definition 4.13.
(a) Let \((\Delta_\ell)_{\ell \in [m]}\) be a domino-\(\Delta\)-sequence. Its \(A\)-value is defined as the expectation \(E_2(\Theta)\), where for all \(\ell \in [m]\) we take \(\Theta_\ell = \Delta_\ell + A(a_\ell, b_\ell)\) for arbitrary \(a_\ell, b_\ell \in [n], a_\ell \neq b_\ell, a_{\ell+1} = b_\ell\).
(b) Let \((\Delta_\ell)_{\ell \in [m]}\) be a reverse domino-\(\Delta\)-sequence. Its \(R\)-value is defined as the expectation \(E_2(\Theta)\), where for all \(\ell \in [m]\) we take \(\Theta_\ell = \Delta_\ell + R(a_\ell, b_\ell)\) for arbitrary \(a_\ell, b_\ell \in [n], a_\ell \neq b_\ell, a_{\ell+1} = b_\ell\).

Note that sequences that are both domino and reverse domino (see lemma 4.6) have both an \(A\)-value and an \(R\)-value.

Lemma 4.14. Let \((\Delta_\ell)_{\ell \in [m]}\) be a domino-\(\Delta\)-sequence. If one of the \(\Delta_\ell\) is of the form \((10)\) (i.e. its upper and lower rows are the same), then the same is true for all \(\Delta_\ell\). In this case, the sequence has \(A\)-value \((-1)^m \sigma^{2m}\).

Proof. We may assume that the rows of \(\Delta_1\) are the same. Then the two entries of the first columns of \(\Delta_2\) must coincide by the domino condition. Since the sum of all entries of a \(\Delta\)-matrix is \(\equiv 0 \mod 2\), the entries of the second column of \(\Delta_2\) must coincide as well, and iteration of this argument proves the first claim. The second claim follows from lemma 4.10.

Lemma 4.15. Let \((\Delta_\ell)_{\ell \in [m]}\) be a domino-\(\Delta\)-sequence. If one of the \(\Delta_\ell\) is of the form \((11)\) (i.e. its upper and lower rows differ), then the same is true for all \(\Delta_\ell\). In this case, the sequence has \(A\)-value \(+\sigma^{2m}\).

Proof. The first claim follows from lemma 4.14. To prove the second claim, observe that by lemmas 4.11 and 4.12, among the possible choices for \(\Delta_\ell\), the steps \(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) each contribute a factor \((-1)\), while the other possible choices do not change the sign. If the first column of \(\Delta_1\) is \((1)\), say, by the domino condition the second column of \(\Delta_2\) must be \((1)\), and this is only possible if any downward step is compensated by an upward step later on in the sequence. Consequently, the steps come in pairs, and their contributions to the sign cancel.
Lemma 4.16.

(a) The map that operates on a domino-$\Delta$-sequence $(\Delta_\ell)_{\ell \in [m]}$ by replacing all 0s by 1s and all 1s by 0s in the (combined) lower row is a bijection between the set of all domino-$\Delta$-sequences with all $\Delta_1$ from the list (10) and the set of all domino-$\Delta$-sequences with all $\Delta_1$ from the list (11).

(b) The map that operates on a domino-$\Delta$-sequence $(\Delta_\ell)_{\ell \in [m]}$ by replacing all 0s by 1s and all 1s by 0s in both rows is a bijection between the set of all domino-$\Delta$-sequences with all $\Delta_1$ from (10) such that the first column of $\Delta_1$ consists of 0s and the set of all domino-$\Delta$-sequences with all $\Delta_1$ from (10) such that the first column of $\Delta_1$ consists of 1s. The cardinality of the latter set is $2^{m-1}$.

Proof. It is obvious that the maps defined in (a) and (b) map domino sequences to domino sequences. To see that $\Delta$-sequences are mapped to $\Delta$-sequences, in view of lemma 4.2 it suffices to observe that the sum of the entries of each $\Delta_\ell$ is increased or decreased by $-2, 0$ or 2.

To prove the claim about the cardinality in (b), it suffices to count the number of possibilities for the upper row. This will be done in lemma 4.17.

Lemma 4.17. There exist $2^{m-1}$ sequences $(p_1, q_1, p_2, q_2, \ldots, p_m, q_m) \in \{0, 1\}^2m$ such that

- $p_1 = 1$,
- $p_1 = q_m, p_{\ell+1} = q_\ell$ ($\ell = 1, \ldots, m-1$),
- $(p_\ell, q_\ell) \in \{(0,0), (1,1), (1,0), (0,1)\}$ ($\ell = 1, \ldots, m$).

Proof. Since $p_1 = q_m = 1$, we may interpret this problem as counting the number of words of length $m$ with letters d (down), u (up) and s (stay), such that

- the first occurrence of d goes before the first occurrence of u,
- between two occurrences of d there must be an occurrence of u, and
- d and u occur an equal number of times.

In view of these requirements, it clearly suffices to specify the set (necessarily of even order) of positions at which either d or u occur to uniquely determine such a word. So the cardinality in question equals the number of subsets of even order in a set of order $m$.

Lemma 4.18.

(a) Among the $\Delta$ matrices, as specified in (10) and (11), i.e. among those matrices with entries in $\{0,1\}$ for which the sum over all entries is $\equiv 0 \pmod{2}$ (see lemma 4.2), the matrices

\[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

are characterized by the condition that both the sum of diagonal entries and the sum of skew-diagonal entries be $\equiv 0 \pmod{2}$.

(b) If a reverse domino-$\Delta$-sequence contains one matrix from (13), then all $\Delta_\ell$ are from (13), and the R-value of the sequence is $+\sigma^{2m}$.

(c) Any sequence in $\{0,1\}^{2m}$ that satisfies the assumptions of lemma 4.17 is the (combined) upper row of a unique reverse domino-$\Delta$-sequence $(\Delta_\ell)_{\ell \in [m]}$ such that all $\Delta_1$ are contained in (13). The same is true if in the assumptions of lemma 4.17 the condition $p_1 = 1$ is replaced by the condition $p_1 = 0$. 

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Proof. (a) can be immediately read from the list. To prove (b), consider consecutive $\Delta$ matrices\[(\begin{array}{cc} a & b \\ c & d \end{array}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\]and suppose that they satisfy the reverse domino condition, i.e. $\alpha = b$ and $\delta = c$. If the matrix on the left is from (13) and thus satisfies $b + c \equiv 0 \pmod{2}$, then we trivially obtain $\alpha + \delta \equiv 0 \pmod{2}$. For the skew diagonal of the matrix on the right observe that $\gamma + \beta \equiv \gamma + \beta + \alpha + \delta \pmod{2} \equiv 0 \pmod{2}$ by lemma 4.2. For the claim about the $R$-value observe that by lemmas 4.10 and 4.11, any $\Delta$ from (13) satisfies $E_2(n\Delta + A(a,b)) = -\sigma^2$, hence $E_2(p\Delta + R(a,b)) = +\sigma^2$ by lemma 4.9.

For the proof of (c) we have to demonstrate that for prescribed $a, b, c \in \{0, 1\}$ there exist unique $\alpha, \beta, \gamma \in \{0, 1\}$ such that the sequence\[\begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} b & c \\ \gamma & \alpha \end{pmatrix}\]satisfies the reverse domino condition as well as the congruence relations which characterize $\Delta$ matrices and the matrices from (13). Since for given $x \in \{0, 1\}$ the unique solution of $y + x \equiv 0 \pmod{2}$ with $y \in \{0, 1\}$ is $y = x$, the choice $\beta = a$, $\alpha = b$, $\gamma = c$ is the only candidate for a solution. It is indeed a solution, because $a + b + \alpha + \beta = (a + b) + (a + b) \equiv 0 \pmod{2}$ and $b + c + \alpha + \gamma = (b + c) + (b + c) \equiv 0 \pmod{2}$. \qed

Lemma 4.19.

(a) Among the $\Delta$ matrices, as specified in (10) and (11), the matrices
\[
\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\]
are characterized by the condition that both the sum of diagonal entries and the sum of skew-diagonal entries be $\equiv 1 \pmod{2}$.

(b) If a reverse domino-$\Delta$-sequence contains one matrix from (14), then all $\Delta_t$ are from (14), and the $R$-value of the sequence is $-\sigma^{2m}$ if $m$ is odd, and $+\sigma^{2m}$ if $m$ is even.

(c) Any sequence in $\{0, 1\}^{2m}$ that satisfies the assumptions of lemma 4.17 is the (combined) upper row of a unique reverse domino-$\Delta$-sequence $(\Delta_t)_{t \in [m]}$ such that all $\Delta_t$ are contained in (14). The same is true if in the assumptions of lemma 4.17 the condition $p_1 = 1$ is replaced by the condition $p_1 = 0$.

Proof. Most of the claims are proven exactly like the analogous claims in lemma 4.18. To prove the claim about the $R$-value, observe that by lemmas 4.10 and 4.9 any occurrence of a step\[\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\]contributes $+\sigma^2$ to the $R$-value, while by lemmas 4.12 and 4.9 any plateau\[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\]contributes $-\sigma^2$. Since occurrences of steps come in pairs, the number of occurrences of plateaux is odd if, and only if, $m$ is odd. \qed

4.5. Proof of theorem 2.1 (DIII case)

It remains to put together the pieces of information we have obtained about the ingredients of formula (9) so far. Recall from section 3 the definitions of the generators $\gamma$ and $\tau$ of the dihedral group $D_{2m}$. It follows from the discussion in section 4.1 that, via the assignment
\[P \mapsto \begin{pmatrix} P_{1,t} \\ P_{2d(t)} \end{pmatrix}_{t \in [m]},\]
to each multi-index from \( \text{Consist}^*(m, m, n; \gamma) \) there corresponds a unique domino-sequence of length \( m \) of \( 2 \times 2 \) matrices with entries from \([2n]\), and that each domino-sequence which arises in this way can be written in a unique fashion as an instance of a pattern \((\Lambda_\ell, \Lambda_\tau)_{\ell \in [m]}\).

Writing this instance in the form \((n\Delta_\ell + \Lambda_\ell(a_\ell, b_\ell))_{\ell \in [m]}\), the (uniquely determined) sequences \((\Delta_\ell)_{\ell \in [m]}\) and \((\Lambda_\ell(a_\ell, b_\ell))_{\ell \in [m]}\) necessarily are a domino-\(\Lambda\)-sequence and a domino-\(\Lambda\)-sequence, respectively. Conversely, each pair of a domino-\(\Delta\)-sequence of length \( m \) and a domino-\(\Lambda\)-sequence of length \( m \) arises as an image of a multi-index from \( \text{Consist}^*(m, m, n; \gamma) \) under this correspondence. Analogously, there is a bijective correspondence between multi-indices from \( \text{Consist}^*(m, m, n; \tau) \) and pairs consisting of a reverse domino-\(\Delta\)-sequence of length \( m \) and a reverse domino-\(\Lambda\)-sequence of length \( m \).

Turning to general elements \( g \in D_{2m} \), a complete list of them is

\[
\{\text{id}, \gamma^1, \gamma^2, \ldots, \gamma^{m-1}, \tau, \tau\gamma, \tau\gamma^2, \ldots, \tau\gamma^{m-1}\}.
\]

With the cyclic identification \( m + 1 = 1 \), \( \gamma^\nu \) maps \( \ell \) to \( \ell + \nu \), and \( \tau\gamma^\nu \) maps \( \ell \) to \( \tau(\ell) + \nu \). Note that \( \gamma^\nu \circ \text{id} = \text{id} \).

Recall from the definitions given in section 4.3 that both the domino and the reverse domino conditions identify certain entries from the combined upper row of a sequence of \( 2 \times 2 \) matrices with other entries from the combined upper row, and certain entries from the combined lower row with other entries from the lower row, i.e. these conditions do not couple the lower row to the upper row. Therefore, \((P_{ij, \ell})_{\ell \in [m]}\) will satisfy the (reverse) domino condition if, and only if, \((P_{ij, \ell})_{\ell \in [m]}\) does. (Note, however, that the analogous statement for the \(\Delta\) sequence property is not true.) Together with the description of the action of a general element of \( D_{2m} \) this observation implies that for any \( \nu = 1, \ldots, m \) we have a bijective correspondence between multi-indices from \( \text{Consist}^*(m, m, n; \gamma^\nu) \) and pairs consisting of a domino-\(\Delta\)-sequence and a domino-\(\Lambda\)-sequence (each of length \( m \)), and a bijective correspondence between multi-indices from \( \text{Consist}^*(m, m, n; \tau\gamma^\nu) \) and pairs consisting of a reverse domino-\(\Delta\)-sequence and a reverse domino-\(\Lambda\)-sequence (each of length \( m \)).

Using the shorthands for products of covariances which were introduced in section 4.4, we are now in a position to translate the \( m \geq 3 \) part of formula (9) into the language of section 4.3 as follows:

\[
V_\nu(m) = \frac{1}{(2n)^m} m \sum_{\Theta} E_2(\Theta), \tag{15}
\]

where \( \Theta = (\Theta_\ell)_{\ell \in [m]} \) runs through all domino instances of all patterns \((\Delta_\ell, \Lambda_\ell)_{\ell \in [m]}\) such that \((\Delta_\ell)_{\ell \in [m]}\) is a domino-\(\Delta\)-sequence, and through all reverse domino instances of all patterns \((\Delta_\ell, \Lambda_\ell)_{\ell \in [m]}\) such that \((\Delta_\ell)_{\ell \in [m]}\) is a reverse domino-\(\Delta\)-sequence. Concerning the prefactor, note that for DIII class matrices, \( 2n \) takes the role of \( n \) in the general formula (9). The factor \( m \) captures the multiplicity that comes from the fact we have just proved, namely, that in \( m \) cases \( \text{Consist}^*(m, m, n; g) \) corresponds bijectively to the set of all domino instances, and in \( m \) cases to the set of all reverse domino instances.

Since theorem 2.1 is about the \( n \to \infty \) limit, we may drop from the sum in (15) all contributions of order \( o(n^m) \). By lemma 4.8 this means that we may replace the sum over all domino instances by \( n^m \) times the sum over the \( A \)-values of all domino-\(\Delta\)-sequences and the \( R \)-values of all reverse domino-\(\Delta\)-sequences. The weight \( n^m \) appears because the notions of \( A \) value and \( R \) value have been defined with respect to a single (reverse) domino instance.

For \( m \) odd, by lemmas 4.14 and 4.15, a domino-\(\Delta\)-sequence either contains only matrices from (10), and contributes \(-\sigma^{2m}\) to the sum, or contains only matrices from (11), and
contributes $+\sigma^{2m}$ to the sum. According to lemma 4.16, both cases occur $2^m$ times, so the sum of the $A$-values of all domino-$\Delta$-sequences is zero. By parts (b) and (c) of lemma 4.18, there are $2^m$ reverse domino-$\Delta$-sequences with all matrices from (13), each contributing $+\sigma^{2m}$ to the sum. According to parts (b) and (c) of lemma 4.19, there are $2^m$ reverse domino-$\Delta$-sequences with all matrices from (14), each contributing $-\sigma^{2m}$ to the sum. Since any $\Delta$ matrix is contained in one of (13) and (14), according to the cited lemmas these two cases exhaust all reverse domino-$\Delta$-sequences, and the sum of the $R$-values of all reverse domino-$\Delta$-sequences is thus zero. So we have proven that as $n \to \infty$, $V_n(m)$ tends to zero if $m$ is odd.

In the case that $m$ is even, by the same lemmas, the negative contributions that led to the cancellations in the $m$ odd case become positive contributions of the same size. This means that the sums over the $A$-values of all domino sequences, and over the $R$-values of all reverse domino sequences, are both equal to $2^m\sigma^{2m}$. In total, taking the prefactors into account, we obtain that, as $n \to \infty$, $V_n(m)$ tends to the value $4m\sigma^{2m}$. The proof of theorem 2.1 (DIII case) is thus complete.

5. Class CI

In view of the similarities between the block structures of the CI and DIII classes, the description of the equivalence classes which was given in lemma 4.1 for the DIII case also applies to the CI case. So the set-up of section 4.3, in particular the notion of a (reverse) domino-$\Delta$-sequence as well as all combinatorial lemmas pertaining to these sequences, may be used for the analysis of class CI matrices without any changes. CI and DIII differ, however, in the way in which the sign of a covariance $\mathbb{E}_2(n\Delta + \Lambda(a,b))$ relates to the nature of $\Delta$ and $\Lambda$. So the fundamental formulae, given in section 4.4, about covariances, $A$-values, and $R$-values, need to be restated. We will, however, omit most proofs, because they can be easily adapted from the corresponding proofs in section 4.4. One only has to keep in mind that the $n \times n$ blocks of class CI matrices are symmetric rather than skew symmetric, and that their entries are real rather than purely imaginary.

**Lemma 5.1.** Let $\Delta$ be one of the $2 \times 2$ matrices from (10) and

$$\mathbb{E}_2(n\Delta + R(a,b)) = \mathbb{E}_2(n\Delta + A(a,b)),$$

**Lemma 5.2.** Let $a, b \in [n]$, $a \neq b$, and $\Delta$ one of the four matrices in (10). Then

$$\mathbb{E}_2(n\Delta + A(a,b)) = +\sigma^2.$$

**Lemma 5.3.** Let $a, b \in [n]$, $a \neq b$, and $\Delta$ one of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\mathbb{E}_2(n\Delta + A(a,b)) = +\sigma^2.$$

**Lemma 5.4.** Let $a, b \in [n]$, $a \neq b$, and $\Delta$ one of the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\mathbb{E}_2(n\Delta + A(a,b)) = -\sigma^2.$$

**Lemma 5.5.** Let $(\Delta_t)_{t \in [m]}$ be a domino-$\Delta$-sequence. If one of the $\Delta_t$ is (hence all of them are) of the form (10), then the sequence has $A$-value $+\sigma^{2m}$.
Lemma 5.6. Let \((\Delta_{\ell})_{\ell \in [m]}\) be a domino-\(\Delta\)-sequence. If one of the \(\Delta_{\ell}\) is (hence all of them are) of the form (11), then the sequence has A-value \(-\sigma^2 m\) if \(m\) is odd, and \(+\sigma^2 m\) if \(m\) is even.

Proof. Observe that by lemmas 5.3 and 5.4, among the possible choices for \(\Delta_{\ell}\), the plateaux \((0 0),(1 1)\) each contribute a factor \((-1)\), while the other possible choices, i.e. the steps, do not change the sign. Since occurrences of steps come in pairs, the number of plateaux is odd if, and only if, \(m\) is odd. \(\Box\)

Lemma 5.7. Let \((\Delta_{\ell})_{\ell \in [m]}\) be a reverse domino-\(\Delta\)-sequence. If one of the \(\Delta_{\ell}\) is (hence all of them are) from (13), then the R-value of the sequence is \(+\sigma^2 m\).

Lemma 5.8. Let \((\Delta_{\ell})_{\ell \in [m]}\) be a reverse domino-\(\Delta\)-sequence. If one of the \(\Delta_{\ell}\) is (hence all of them are) from (14), then the R-value of the sequence is \(-\sigma^2 m\) if \(m\) is odd, and \(+\sigma^2 m\) if \(m\) is even.

Turning to the summation of the A-values of all domino-\(\Delta\)-sequences and of the R-values of all reverse domino-\(\Delta\)-sequences, for \(m\) odd, by lemmas 5.5 and 5.6, the \(2^m\) domino-\(\Delta\)-sequences with matrices from (10) contribute \(+\sigma^2 m\), and the \(2^m\) domino-\(\Delta\)-sequences with matrices from (11) contribute \(-\sigma^2 m\). By parts (b) and (c) of lemma 4.18, there are \(2^m\) reverse domino-\(\Delta\)-sequences with all matrices from (13), each contributing \(+\sigma^2 m\) to the sum by lemma 5.7. According to parts (b) and (c) of lemma 4.19, there are \(2^m\) reverse domino-\(\Delta\)-sequences with all matrices from (14), each contributing \(-\sigma^2 m\) to the sum by lemma 5.8. So the contributions cancel, showing that as \(n \to \infty\), \(V_n(m)\) tends to zero if \(m\) is odd. On the other hand, if \(m\) is even, the negative contributions are replaced by positive contributions of the same size. Hence, as in the DIII case, \(V_n(m)\) tends to the value \(4m\sigma^2 m\) as \(n \to \infty\). The proof of theorem 2.1 (CI case) is thus complete.

6. Discussion

By way of concluding, let us try to place the preceding developments within the context of the broad range of asymptotic methods for random matrices that are in use in the mathematical physics literature. To be sure, the technical contribution of the present paper can be viewed from the intrinsic perspective of the Schenker and Schulz-Baldes (SSB) technique, which obtains spectral information about Wigner type matrices with dependent entries from moments and cumulants that are calculated through a combination of sophisticated bookkeeping with remarkably low tech counting arguments. From this point of view, our work has extended the scope of matrix models for which this approach yields explicit information about the asymptotic fluctuations. While corollary 2.6 in SSB’s [11] obtains explicit asymptotic covariances in a situation in which dependent entries are necessarily identical, the present contribution extends this to two geometric set-ups in which dependent entries are sometimes only identical up to a sign flip.

This said, the main thrust of the current paper is the step from ‘dependent matrix entries are identical’ to ‘dependent matrix entries are identical up to a sign, and the signs arise in some systematic fashion’. It will thus be helpful to clearly distinguish between the SSB core approach on the one hand, and the extra toolkit that has been set up on the preceding pages, on the other. As basically everything in the present paper is about bookkeeping, this can be put in more concrete terms as follows: there is a basic layer of formalism, due to SSB, which
revolves around the notion of multi-indices that are admissible for certain types of partitions of some suitable set. As demonstrated in [11], this basic layer enables explicit asymptotic evaluation of fluctuations in the ‘dependent matrix entries are identical’ regime. The present contribution adds a second layer, with patterns and dominoes as keywords, and thus allows for equalities up to signs in some nice geometrical cases.

Comparison with other methodologies is more straightforward for the basic layer. Here the SSB approach has on several occasions demonstrated its ability to reproduce results about the $n \to \infty$ limit of the empirical spectral measure (the normalized eigenvalue counting function) that had hitherto been obtained along different lines. One case in point, which has already been discussed by SSB in [13], is a result, due to Bellissard, Magnen, and Rivaissseau [15], about Poirot’s flip matrix model [16]. This is a simplified two dimensional Anderson model, and it is a case in which dependent matrix entries are identical. While Bellissard et al had used the supersymmetry technique, as outlined in [17] or [18], to prove the convergence of the density of states of the flip matrix model to Wigner’s semicircle distribution, SSB were able to rederive that result using their approach.

Other results of this type have been obtained via resolvents and Stieltjes transforms, an approach which has been in vigorous development ever since the seminal paper [19] by Marčenko and Pastur, and of which a book length treatment is available in [20]. Gaussian matrix ensembles with additional symmetries have been studied in this vein by Khorunzhy [21] and Vasilchuk [22]. The latter reference covers the flip matrix ensemble from [15], as well as three further examples. The symmetries do not involve sign flips, and the spectral measure results for at least two of the extra examples are clearly amenable to the SSB approach in the version [13].

Wigner type ensembles based on tangent spaces of symmetric spaces, many of which involve sign flips, have been treated from an SSB perspective in our previous work [8], and via the supersymmetry technique by Ivanov [23] and by Kalisch and Braak [24]. They are also amenable to the orthogonal polynomial method, as outlined in [25]. This set of references is quite instructive for discerning the relative strengths and weaknesses of the different methods. Both the supersymmetry and orthogonal polynomial approaches are able to reveal information about finite matrix sizes, but this comes at the price of restrictive assumptions about the probability distributions of the matrix entries. On the other hand, [8] needs nothing more than precise information about the first and second moments and a uniform boundedness assumption about the higher moments. But its results are purely asymptotic.

Strictly speaking, the examples discussed up to this point are Wigner type matrices with extra symmetries. The SSB approach is more general than that, because it also covers types of stochastic dependence that are not deterministic, as long as certain restrictions on the number of dependent entries are satisfied. But in view of the abundance of possible dependence structures, it is easy to come up with examples that are not covered by it, while they are amenable to other approaches. To provide but one very recent example, in [26], Krajewski, Tanasa, and Vu study Wigner matrices with a general type of dependence among the entries, which is made tractable by assuming that it satisfies certain decay conditions on higher multivariate cumulants. Here replica and renormalization group techniques yield a result which is clearly outside the scope of the SSB approach.

Concerning the second layer of bookkeeping, the one inhabited by patterns, dominoes, and their friends, it is perhaps preferable not to view it as something that is inextricably linked to the SSB basic layer. Rather, it is but an instance of a general strategy to control the cumulative effect of a large number of minus signs in an involved sum by setting up a combinatorial model for the occurrences of the minus signs. Another, actually quite different, instance of that strategy is the treatment of the symplectic group case in our previous contribution [27].
Viewed in this way, our second layer could possibly coexist with a broad range of techniques. To put it in a somewhat more concrete, if speculative, way: if one tried to study higher order Green functions for class DIII matrices along the lines of a diagrammatic approach, in the spirit of, for instance, [28], then one would probably have to find a way to cope with cancellation effects in order to determine whether or not contributions of certain orders are degenerate. It is conceivable that translating the overall symmetries of the matrix space into a combinatorial model for the occurrences of sign factors could help. An analogous point could plausibly be made about any supersymmetric approach to that task, this time concerning cancellations in the phase functions of the functional integrals on which saddlepoint analyses are performed.

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