Global analysis of GG systems

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Abstract

This paper deals with some analytic aspects of GG system introduced by I.M.Gelfand and M.I.Graev. We compute the dimension of the solution space of GG system over the field of functions meromorphic and periodic with respect to a lattice. We describe the monodromy invariant subspace of the solution space. We give a connection formula between a pair of bases consisting of $\Gamma$-series solutions of GG system associated to a pair of regular triangulations adjacent to each other in the secondary fan.

1 Introduction

In the 80’s and 90’s, the general study of hypergeometric functions made progress in a series of papers by I.M.Gelfand, M.M.Kapranov, and A.V.Zelevinsky ([GZK89], [GKZ90], [GKZ94]). One of the new perspectives of their study is that there is a combinatorial structure of convex polytopes behind hypergeometric systems. GKZ system is a system of linear partial differential equations determined by two inputs: an $n \times N$ ($n < N$) integer matrix $A = (a_{ij})$ and a parameter vector $c \in \mathbb{C}^{n \times 1}$. GKZ system $M_A(c)$ is defined by

$$
M_A(c) : \begin{cases} 
E_i \cdot f(z) = 0 & (i = 1, \ldots, n) \\
\Box_u \cdot f(z) = 0 & (u = (u_1, \ldots, u_N) \in L_A = \text{Ker}(A \times : \mathbb{Z}^N \to \mathbb{Z}^n)),
\end{cases}
$$

where $E_i$ and $\Box_u$ are differential operators defined by

$$
E_i = \sum_{j=1}^{N} a_{ij} z_j \frac{\partial}{\partial z_j} + c_i, \quad \Box_u = \prod_{u_j > 0} \left( \frac{\partial}{\partial z_j} \right)^{u_j} \prod_{u_j < 0} \left( \frac{\partial}{\partial z_j} \right)^{-u_j}.
$$

We write $a(j)$ for the $j$-th column vector of $A$. A fundamental property that GKZ system $M_A(c)$ enjoys is the holonomicity ([Ado94, THEOREM 3.9]) and in particular, the finiteness of the dimension of the solution space. Many classical hypergeometric systems are realized as particular examples of GKZ system for special choices of the configuration matrices $A$. GKZ system also appears naturally in various contexts of applications such as mirror symmetry ([Bat93], [Sti98]) and algebraic statistics ([Kur18], [YG]).

An important open question is to understand the monodromy representation of GKZ system. More concretely, one hopes to find explicit monodromy matrices with respect to a given basis of solutions and a set of generators of the fundamental group of the complement of the singular locus. However, there are two difficulties: Firstly, GKZ system $M_A(c)$ behaves in a singular way when the parameter $c$ takes a special value. A typical example of this phenomenon is the discontinuity of the rank ([SST00], [MMW05]). Secondly, the structure of the fundamental group is not yet fully understood. It is known that the singular locus of the GKZ system is (contained

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in) a product of principal $A$-determinants \([BZAMW15]\). However, the degree of the defining equation of the principal $A$-determinant can be too big even if $A$ is relatively small (\([GZK90\] Chapter 1\)). Note that there is an interesting recent work \([IF]\) on the fundamental group of the complement of the principal $A$-determinant.

Despite these difficulties, several people made progress on the global analysis of GKZ system. Let us review some of the preceding results on the monodromy of GKZ system when the parameter $c$ is generic. The genericity of the parameter $c$ plays a key role when we take a specific basis of the solution space. Let us also assume that the GKZ system $M_A(c)$ in question is regular holonomic. Note that $M_A(c)$ is regular holonomic if and only if the configuration matrix $A$ is homogeneous, i.e., there is a linear function $\phi : \mathbb{Z}^n \to \mathbb{Z}$ such that $\phi(a(j)) = 1$ for any $j = 1, \ldots, N$ (\([Hot], [SW08], [PP10]\)). Recall that the totality of regular polyhedral subdivisions has the structure of a convex polyhedral fan, which is called the secondary fan denoted by $\text{Fan}(A)$ in this paper (\([GKZ94\] Chapter 7\)). To each regular triangulation $T$, we can associate a basis $\Phi_T$ of series solutions of $M_A(c)$ convergent on a suitable open subset $U_T$ of $\mathbb{C}^N$ (\([GZK89], [SST00]\)).

When the configuration matrix $A$ comes from a product of simplices $\Delta_1 \times \Delta_{n-1}$, Mutsumi Saito and Nobuki Takayama gave an explicit description of the connection matrices among bases $\Phi_T$ in \([ST94]\). They fully utilized the fact that the open cones of the secondary fan and elements of the permutation group $\mathfrak{S}_n$ are in one-to-one correspondence. In this way, the connection matrices they obtained are labeled by the permutation group $\mathfrak{S}_n$. Another approach to the connection problem was proposed by Frits Beukers in \([Ben16]\). Under a suitable assumption on the configuration matrix $A$, he constructed a special basis $\Phi$ of solutions in terms of Mellin-Barnes integral. Taking residues of the integrand in one direction, he succeeded in showing a relation between the $\Phi_T$ and $\Phi_{T'}$ for each regular triangulation $T$, which is utilized to compute the connection matrices among bases $\Phi_T$. Though this method is useful, a basis consisting of Mellin-Barnes integral does not always exist. In fact, GKZ systems corresponding to Appell-Lauricella’s $F_A, F_B$ and $F_D$ have such a basis while $F_C$ does not.

In this paper, we address the problem of computing connection matrices among bases $\Phi_T$ of GKZ system $M_A(c)$ with a generic parameter from another point of view. Namely, we give a combinatorial description of the connection matrix between $\Phi_T$ and $\Phi_{T'}$ for any regular triangulations $T$ and $T'$. For this purpose, we only need to discuss the case when $T$ is adjacent to $T'$, that is, when the cone of the secondary fan corresponding to $T$ shares a facet with that corresponding to $T'$. When $T$ is adjacent to $T'$, $T$ and $T'$ are related to each other by a combinatorial operation called modification (\([GKZ94]\ Chap.7, \S 2.C\)). Therefore, it is natural to expect that the connection matrix between $\Phi_T$ and $\Phi_{T'}$ is described in terms of modification. A useful method of performing an analytic continuation of a solution of a system of partial differential equations with regular singularities is the method of boundary value problem (\([Hec87], [KO77]\)). The use of this method in the context of GKZ system has already been indicated in \([ST94\] Theorem 1.3\]. When $T$ is adjacent to $T'$, \([ST94\] Theorem 1.4\] shows that the boundary value problem along a particular coordinate subspace is naturally defined and the connection matrix for boundary values gives rise to that between $\Phi_T$ and $\Phi_{T'}$.

In this paper, we provide another perspective to the boundary value problem by employing the viewpoint of GG system, a system of linear partial difference-differential equations on $\mathbb{C}^N \times \mathbb{C}^n$ (\([GG99]\)):

$$GG(A) : \begin{cases} E_i \cdot f(z; c) = 0 \quad & (i = 1, \ldots, n) \\ \frac{\partial}{\partial z_j} f(z; c) = f(z; c + a(j)) \quad & (j = 1, \ldots, N). \end{cases} \tag{1.3a}$$

Solutions of GKZ system $M_A(c)$ with a generic parameter $c$ are naturally regarded as those of GG system (\([GG99\] Theorem 4\]). A crucial difference, however, is that the parameter vector $c$
is now regarded as an independent variable. For this reason, it is natural to regard the solution space $\text{Sol}_{GG(A)}$ of $GG(A)$ as a vector space over a field $\mathcal{M}_A(\mathbb{C}^n)$ of $A$-periodic meromorphic functions in $c \in \mathbb{C}^n$. In this sense, $\Phi_T$ is also a basis of $\text{Sol}_{GG(A)}$. Let us also remark that the viewpoint of GG system naturally encodes the contiguity structure of hypergeometric functions. Therefore, saying that a function $f(z; c)$ is a solution of GG system has more information than saying that it is a solution of a GKZ system.

In the setting of GG system, we can prove a unique solvability of the boundary value problem (Theorem 2.8) on the level of formal solutions, which plays the role of [ST94, Theorem 1.3]. Thanks to the appearance of the variable $c$, we can also describe the inverse of the boundary value map as a difference operator of infinite order $\mathcal{D}$. Since this operator $\mathcal{D}$ may produce a divergent series, we should establish some estimates to justify our argument which will be carried out in §3. We also construct a path of analytic continuation on which our estimate is valid. The main result of ours Theorem 3.10 is given in §3.4.

Another usage of the boundary value problem is the construction of a monodromy invariant subspace. It was conjectured in [FP19] that an irregular GKZ system has a monodromy invariant subspace associated to any facet of its Newton polytope which does not contain the origin. In §2, we prove this conjecture on the level of GG system.

This paper consists of two parts. §2 is devoted to a preliminary study of GG system. Throughout this section, we do not assume that the configuration matrix is homogeneous. In §2.1 we deal with series solutions of GG systems. After we recall the relation between series solutions and the combinatorics of the secondary fan, we provide the formula of the dimension of the solution space (Theorem 2.4). This formula can be seen as an analytic counterpart of [OT09, Theorem 3]. In §2.2 we establish the unique solvability of the boundary value problem (Theorem 2.8). §2.3 is independent of the discussion of the next section, but we give an interesting application of Theorem 2.8. Namely, we prove that the combinatorics of the Newton polytope gives rise to a decomposition of the solution space $\text{Sol}_{GG(A)}$ into monodromy invariant subspaces (Theorem 2.10). §3 is devoted to the formulation and the proof of the main theorem of this paper, a connection formula. In this section, we assume that $A$ is homogeneous. In §3.2 we recall the notion of modification and the well-known method of analytic continuation by means of Mellin-Barnes integral in our setting ([Sla00, Chap.4]). In §3.3 we give some combinatorial lemmata related to the secondary fan. In §3.3 we establish estimates of difference operators utilizing an integral operator called Erdélyi-Kober operator. In §3.4 we prove the connection formula in Theorem 3.10.

Throughout this paper, we use the following notation: for any vectors $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{C}^n$, we set $|a| = a_1 + \cdots + a_n$, $e^{2\pi \sqrt{-1}a} = (e^{2\pi \sqrt{-1}a_1}, \ldots, e^{2\pi \sqrt{-1}a_n})$, $ab = (a_1 b_1, \ldots, a_n b_n)$, and $a^b = a_1^{b_1} \cdots a_n^{b_n}$. For any univariate function $F$, we write $F(a)$ for the product $F(a_1) \cdots F(a_n)$. For any $1 \times n$ row vector $z$ and any $n \times m$ matrix $B = (b(1) | \cdots | b(m))$, we write $z^B$ for the row vector $(z^{b(1)}, \ldots, z^{b(m)})$. The symbol $\mathbb{Z}B$ denotes the lattice in $\mathbb{Z}^n$ generated by column vectors of $B$.

2 Basic properties of GG system

In this section, we establish some basic properties of GG system. Throughout this section, $A$ denotes an $n \times N$ integer matrix with $n < N$. For any subset $\sigma \subset \{1, \ldots, N\}$, we write $A_{\sigma}$ for the matrix whose column vectors consist precisely of $j$-th column vectors of $A$ for all $j \in \sigma$. We set $\overline{\sigma} := \{1, \ldots, N\} \setminus \sigma$. Note that $A_{\sigma}$ (resp. $A_{\overline{\sigma}}$) is regarded as a matrix whose rows are labeled by the set $\{1, \ldots, n\}$ (resp. $\sigma$) and whose columns are labeled by the set $\overline{\sigma}$ (resp. $\{1, \ldots, n\}$). We say that $\sigma$ is a simplex if its cardinality $|\sigma|$ is $n$ and $\det A_{\sigma} \neq 0$. For any simplex $\sigma$, $i \in \sigma$ and a column vector $v \in \mathbb{C}^n$, we write $p_{\sigma,i}(v)$ for the $i$-th entry of the vector $A_{\sigma}^{-1} v$. 

3
2.1 The dimension of the solution space

In this subsection, we assume that the column vectors of $A$ generate the lattice $\mathbb{Z}^n$. We fix a simplex $\sigma \subset \{1, \ldots, N\}$. For any partition $\sigma = \sigma^u \sqcup \sigma^d$, we put

$$\psi_{\sigma}^u(z; c) = \sum_{m \in \mathbb{Z}_{\geq 0}^n} \prod_{i \in \sigma^u} \frac{\Gamma(p_{\sigma_i}(c + A_{\sigma}m))}{\Gamma(1 - p_{\sigma_i}(c + A_{\sigma}m))} \prod_{i \in \sigma^d} (e^{\sqrt{-1} z_i} - p_{\sigma_i}(c + A_{\sigma}m)) \prod_{i \in \sigma^u} z_i^{-p_{\sigma_i}(c + A_{\sigma}m)} z_{\sigma_i}^{-m}. \quad (2.1)$$

For any choice of $k \in \mathbb{Z}^n$, we set $\psi_{\sigma}^u(z; c) := \psi_{\sigma}^u(e^{2\pi \sqrt{-1} k} z_{\sigma}, z_{\sigma}; c)$.

**Proposition 2.1.** $\psi_{\sigma}^u(z; c)$ is a formal solution of $GG(A)$.

**Proof.** It is enough to prove the proposition for $\psi_{\sigma}^u(z; c)$. For any $t \in (\mathbb{C}^*)^{n \times 1}$, we can easily see that $\psi_{\sigma}^u(tA \cdot z; c) = t^{-c} \psi_{\sigma}^u(z; c)$. Taking the partial derivative in the variable $t_i$ and substituting $t_1 = \cdots = t_n = 1$, we obtain the equation (1.1a).

Suppose that $j \in \sigma$. We write $e_j \in \mathbb{Z}^n$ for the vector whose entries are 0 except for the $j$-th position and $j$-th entry of which is 1. Then,

$$\frac{\partial}{\partial z_j} \psi_{\sigma}^u(z; c) \quad (2.2)$$

$$= \sum_{m - e_j \in \mathbb{Z}_{\geq 0}^n} \prod_{i \in \sigma^u} \frac{\Gamma(p_{\sigma_i}(c + a(j) + A_{\sigma}m - e_j))}{\Gamma(1 - p_{\sigma_i}(c + a(j) + A_{\sigma}m - e_j))} \times \prod_{i \in \sigma^d} (e^{\sqrt{-1} z_i} - p_{\sigma_i}(c + a(j) + A_{\sigma}m - e_j)) \prod_{i \in \sigma^u} z_i^{-p_{\sigma_i}(c + a(j) + A_{\sigma}m - e_j)} z_{\sigma_i}^{-m - e_j} \quad (2.3)$$

$$= \psi_{\sigma}^u(z; c + a(j)). \quad (2.4)$$

Next, let us fix any $i_0 \in \sigma^u$. We have

$$\frac{\partial}{\partial z_{i_0}} \psi_{\sigma}^u(z; c) \quad (2.5)$$

$$= \sum_{m \in \mathbb{Z}_{\geq 0}^n} \prod_{i \in \sigma^u} \frac{\Gamma(p_{\sigma_i}(c + a(i_0) + A_{\sigma}m))}{\Gamma(1 - p_{\sigma_i}(c + A_{\sigma}m))} \prod_{i \in \sigma^d} (e^{\sqrt{-1} z_i} - p_{\sigma_i}(c + a(i_0) + A_{\sigma}m)) \prod_{i \in \sigma^u} z_i^{-p_{\sigma_i}(c + a(i_0) + A_{\sigma}m)} z_{\sigma_i}^{-m} \quad (2.6)$$

$$= \psi_{\sigma}^u(z; c + a(i_0)). \quad (2.7)$$

The case when $i_0 \in \sigma^d$ can be proved in a similar way. \hfill \Box

We briefly recall the definition of a regular polyhedral subdivision. In general, for any subset $\sigma$ of $\{1, \ldots, N\}$, we write cone($\sigma$) for the positive span of $\{a(1), \ldots, a(N)\}$ i.e., cone($\sigma$) = $\sum_{i \in \sigma} \mathbb{R}_{\geq 0} a(i)$. We often identify a subset $\sigma \subset \{1, \ldots, N\}$ with the corresponding set of vectors $\{a(i)\}_{i \in \sigma}$ or with the set cone($\sigma$). A collection $T$ of subsets of $\{1, \ldots, N\}$ is called a triangulation if $\{\text{cone}(\sigma) \mid \sigma \in T\}$ is the set of cones in a simplicial fan whose support equals cone($A$). We write $(\mathbb{Z}^N)^\vee$ for the dual lattice of $\mathbb{Z}^N$. We write $\pi_A : (\mathbb{Z}^N)^\vee \to L_A^\vee$ for the dual of the natural inclusion $L_A \to \mathbb{Z}^N$ where $L_A^\vee$ is the dual lattice $\text{Hom}_{\mathbb{Z}}(L_A, \mathbb{Z})$. By abuse of notation, we still write $\pi_A : (\mathbb{R}^N)^\vee \to L_A^\vee \otimes \mathbb{R}$ for the linear map $\pi_A \otimes \text{id}_\mathbb{R}$ where $\text{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}$ is the identity map.

For any cone $C \subset \mathbb{R}^N$, the symbol $C^\vee$ denotes its dual cone. We often identify $(\mathbb{R}^n)^\vee$ with the
set of row vectors via dot product. Then, for any choice of a vector \( \omega \in \pi^{-1}_A(\pi_A((\mathbb{R}^N)')) \), we can define a polyhedral subdivision \( S(\omega) \) as follows: A subset \( \{1, \ldots, N\} \) belongs to \( S(\omega) \) if there exists a vector \( n \in \mathbb{R}^{1\times n} \) such that \( n \cdot a(i) = \omega_i \) if \( i \in \sigma \) and \( n \cdot a(j) < \omega_j \) if \( j \notin \sigma \). A polyhedral subdivision \( S \) is called a regular polyhedral subdivision if \( S = S(\omega) \) for some \( \omega \).

Given a regular polyhedral subdivision \( S \), we write \( C_S \subset (\mathbb{R}^N)' \) for the cone consisting of vectors \( \omega \) such that \( S(\omega) = S \). If any maximal (with respect to inclusion) element \( \sigma \) of a regular polyhedral subdivision \( T \) is a simplex, we call \( T \) a regular triangulation. For a fixed regular triangulation \( T \), we say that the parameter vector \( c \) is very generic if \( A^{-1}_\sigma(c + k) \) has no integral entry for any simplex \( \sigma \in T \) and any \( k \in \mathbb{Z}^n \). Let us put \( H_\sigma = \{ j \in \{1, \ldots, N\} \mid |A^{-1}_\sigma a(j)| = 1 \} \). Here, \( |A^{-1}_\sigma a(j)| \) denotes the sum of all entries of the vector \( A^{-1}_\sigma a(j) \). We set

\[
U_\sigma = \left\{ z \in (\mathbb{C}^*)^N \left| \text{abs} \left( z_\sigma^{-1} a(j) z_j \right) < R, \text{for all } a(j) \in H_\sigma \setminus \sigma \right. \right\},
\]

where \( R > 0 \) is a small positive real number and \( \text{abs} \) stands for the absolute value. Recall that a regular triangulation \( T \) is said to be convergent if for any \( n \)-simplex \( \sigma \in T \) and for any \( j \in \sigma \), one has the inequality \( |A^{-1}_\sigma a(j)| \leq 1 \) ([MM Definition 5.2]).

**Proposition 2.2.** Fix a convergent regular triangulation \( T \), a partition \( \sigma = \sigma^u \sqcup \sigma^d \) for each \( \sigma \in T \), and a complete set of representatives \( \{k(i)\}_{i=1}^{r_\sigma} \) of \( \mathbb{Z}^n / \mathbb{Z}A_\sigma \). Then, a set of functions

\[
\bigcup_{\sigma \in T} \left\{ \psi^{\sigma^u}_{\sigma^d, k(i)}(z; c) \right\}_{i=1}^{r_\sigma}
\]

is a set of linearly independent holomorphic solutions of \( GG(A) \) on

\[
\bigcap_{\sigma \in T} U_\sigma \neq \emptyset
\]

where \( r_\sigma \) is the cardinality of the group \( \mathbb{Z}^n / \mathbb{Z}A_\sigma \).

**Proof.** For any \( k \in \mathbb{Z}^n \), put \( \Lambda_k = \{ k + m \in \mathbb{Z}^n \mid A_{\sigma} m \in \mathbb{Z}A_\sigma \} \) and

\[
\varphi^{\sigma^u}_{\sigma^d, k}(z; c) = \sum_{k+m \in \Lambda_k} \frac{\prod_{i \in \sigma^u} \Gamma(p_{\sigma^u i}(c + A_{\sigma^u}(k + m)))}{\prod_{i \in \sigma^d} \Gamma(1 - p_{\sigma^d i}(c + A_{\sigma^d}(k + m))) (k + m)!} \prod_{i \in \sigma^u} (z_i - p_{\sigma^u i}(c + A_{\sigma^u}(k + m))) \prod_{i \in \sigma^d} z_i^{-p_{\sigma^d i}(c + A_{\sigma^d}(k + m))} e^{k \cdot m}. \tag{2.9}
\]

For any column vector \( k \in \mathbb{Z}^n \), we easily see that the identity

\[
\psi^{\sigma^u}_{\sigma^d, k}(z; c) = e^{-2\pi \sqrt{-1} k A_{\sigma}^{-1}} \sum_{j=1}^{r_\sigma} e^{-2\pi \sqrt{-1} k A_{\sigma}^{-1} A_{\sigma^d} k(j)} \varphi^{\sigma^u}_{\sigma^d, k(j)}(z; c) \tag{2.10}
\]

holds. Here, \( {}^t k \) denotes the transpose of the column vector \( k \). Thus, we have

\[
\begin{pmatrix}
\psi^{\sigma^u}_{\sigma^d, k(1)}(z; c) \\
\vdots \\
\psi^{\sigma^u}_{\sigma^d, k(r_\sigma)}(z; c)
\end{pmatrix} = \text{diag} \left( e^{-2\pi \sqrt{-1} {}^t k A_{\sigma}^{-1}} \right)^{r_\sigma} \begin{pmatrix}
\varphi^{\sigma^u}_{\sigma^d, k(1)}(z; c) \\
\vdots \\
\varphi^{\sigma^u}_{\sigma^d, k(r_\sigma)}(z; c)
\end{pmatrix}. \tag{2.11}
\]

Since it can be readily seen that the set

\[
\bigcup_{\sigma \in T} \left\{ \varphi^{\sigma^u}_{\sigma^d, k(i)}(z; c) \right\}_{i=1}^{r_\sigma}
\]

is linearly independent as in [FF10 §3] and that

\[
\frac{1}{\sqrt{\sigma}} \left( e^{-2\pi \sqrt{-1} {}^t k A_{\sigma}^{-1}} A_{\sigma^d} k(j) \right)_{i,j}
\]

is a unitary matrix, [2.11] shows the proposition. \(\Box\)

**Remark 2.3.** We set \( \varphi_{\sigma, k}(z; c) := \varphi^{\sigma^u}_{\sigma^d, k}(z; c) \). Using the reflection formula of Gamma function, it is straightforward to write down any \( \varphi^{\sigma^u}_{\sigma^d, k}(z; c) \) as a linear combination of \( \varphi_{\sigma, k}(z; c) \). The series \( \varphi_{\sigma, k}(z; c) \) is called a \( \Gamma \)-series ([GZK89 §1]). We also call the series \( \psi^{\sigma^u}_{\sigma^d, k}(z; c) \) a \( \Gamma \)-series.
Let us prove a dimension formula of the solution space. First of all, we need to specify the function space on which we take solutions of GG system. Let \( D(2 \varepsilon) \) be an open ball with radius \( \varepsilon > 0 \) with center at the point \( z_0 \). For any point \( z_0 \in \mathbb{C}^n \), we write \( f(z; c) \in \mathcal{G}(D(z_0; \varepsilon) \times \mathbb{C}^n) \) if for any point \( c_0 \in \mathbb{C}^n \), there are holomorphic functions \( g(c), h(z; c) \) defined on \( D(c_0; \varepsilon) \) and \( D(z_0; \varepsilon) \times D(c_0; \varepsilon') \) for some \( \varepsilon' > 0 \) respectively such that \( f(z; c) = \frac{h(z; c)}{g(c)} \). We call such a function \( f \) a function meromorphic in \( c \in \mathbb{C}^n \). For any pair of positive numbers \( \varepsilon_1 < \varepsilon_2 \), we can define a natural restriction morphism \( \mathcal{G}(D(z_0; \varepsilon_2) \times \mathbb{C}^n) \to \mathcal{G}(D(z_0; \varepsilon_1) \times \mathbb{C}^n) \). The symbol \( \mathcal{G}(z_0) \) denotes the inductive limit \( \lim_{\varepsilon \to 0} \mathcal{G}(D(z_0; \varepsilon) \times \mathbb{C}^n) \). We write \( R_A \) for the ring of difference-differential operators \( \mathcal{C}(z_1, \ldots, z_N, \partial_1, \ldots, \partial_N, c_1, \ldots, c_n, \sigma_1^\pm \ldots, \sigma_n^\pm) \) with relations \( \sigma_ic_i = (c_i + 1)\sigma_i, \sigma_i^{-1}c_i = (c_i - 1)\sigma_i^{-1} \) and \( \partial_i z_i = z_i\partial_i + 1 \) and other types of product of two generators commute. If \( I_A \) denotes the left ideal of \( R_A \) generated by \( E_i \) \( (i = 1, \ldots, n) \) and \( \partial_j - \sigma_k(j) (j = 1, \ldots, N) \), we set \( \mathcal{G}(A) := R_A/I_A \). Since \( \mathcal{G}(z_0) \) admits a natural structure of a left \( R_A \)-module, we can define the set \( \text{Sol}_{\mathcal{G}(A), z_0} \) of local solutions at \( z_0 \) by \( \text{Sol}_{\mathcal{G}(A), z_0} := \text{Hom}_{\mathcal{G}(A)}(\mathcal{G}(A), \mathcal{G}(z_0)) \). A function \( g \) on \( \mathbb{C}^n \) is said to be \( A \)-periodic if \( g(c) = g(c + a(j)) \) for any \( j = 1, \ldots, N \). We write \( \mathfrak{M}(\mathbb{C}^n) \) for the field of \( A \)-periodic meromorphic functions on \( \mathbb{C}^n \). It is clear that \( \text{Sol}_{\mathcal{G}(A), z_0} \) is an \( \mathfrak{M}(\mathbb{C}^n) \)-vector space.

We write \( \Delta_A \) for the convex hull of the column vectors of \( A \) and the origin. We write \( \text{Sing}(A) \) for the zero set of principal \( A \)-determinant, which is defined to be a product of \( A \)-discriminants \( D_A \) for any face \( \Gamma \) of \( \Delta_A \) which does not contain the origin ([GKZ94 Chapter 9]). If \( z_0 \notin \text{Sing}(A) \), any \( f(z; c) \in \text{Sol}_{\mathcal{G}(A), z_0} \) is a solution of \( M_A(c) \) for any generic \( c \in \mathbb{C}^n \). Since we can show that the singular locus of \( M_A(c) \) is contained in \( \text{Sing}(A) \) by [SW08, Theorem 2.14] any local solution \( f \in \text{Sol}_{\mathcal{G}(A), z_0} \) admits an analytic continuation along any path \( \gamma \) in \( \mathbb{C}^n \setminus \text{Sing}(A) \). This observation shows that, for any \( z_1, z_2 \in \mathbb{C}^n \setminus \text{Sing}(A) \), we have an isomorphism of solution spaces \( \text{Sol}_{\mathcal{G}(A), z_1} \simeq \text{Sol}_{\mathcal{G}(A), z_2} \) given by an analytic continuation along a path connecting \( z_1 \) and \( z_2 \). Let \( \text{vol}_\mathbb{R} \) be the Lebesgue measure on \( \mathbb{R}^n \) and set \( \text{vol}_\mathbb{E} := \frac{1}{n!} \text{vol}_\mathbb{R} \).

**Theorem 2.4.** For any \( z \in \mathbb{C}^n \setminus \text{Sing}(A) \), one has an identity

\[
\dim \mathfrak{M}(\mathbb{C}^n) \text{Sol}_{\mathcal{G}(A), z} = \text{vol}_\mathbb{E}(\Delta_A).
\]

**Proof.** Let \( D_N \) denote the Weyl algebra in \( N \) variables \( z_1, \ldots, z_N \). By an algorithmic method, we easily see that the GKZ system \( M_A(c) \) has a Pfaffian representation, i.e., if we denote by \( \mathcal{C}(z) \) the field of rational functions in \( z \), one has a \( \mathcal{C}(z) \)-basis \( \{\partial^n\} \) of \( \mathcal{C}(z) \otimes \mathbb{C}[z]/M_A(c) \) ([Tak13 §6.1.6.2]). We may assume that there is an index \( \alpha = 0 \). By the algorithm, we can also see that the corresponding Pfaffian system \( d_z Y = PY \) has rational coefficients, namely we have \( P \in \sum_{j=1}^N \mathcal{C}(z, c)^{r_xr_y}dz_j \).

We take a convergent regular triangulation \( T \). Without loss of generality, we may assume \( z \in U_T \). Then, we get a corresponding basis of \( T \)-series solutions \( \Psi_T = \{\psi_i\}_{i=1}^r \) \((r = \text{vol}_\mathbb{E}(\Delta_A))\) of \( M_A(c) \) which is also a set of linearly independent (over \( \mathfrak{M}(\mathbb{C}^n) \)) solutions of \( \mathcal{G}(A) \) as in Proposition 2.2. Then, \( \Psi = \left(\partial^n\psi\right)_{\alpha, i} \) is a fundamental solution matrix of the system of \( d_z Y = PY \) for any generic \( c \). Let us take any \( \phi(z; c) \in \text{Sol}_{\mathcal{G}(A), z} \). We put \( \Phi = (\partial^n\phi)_{\alpha, i} \) and regard it as a row vector. Then, for any generic \( c \), there is a column vector \( S(c) \) such that the relation \( \Phi = \Psi S(c) \) holds. Since \( S(c) = \Psi^{-1}\Phi, S(c) \) has globally defined meromorphic entries. Therefore, if we write \( S_i(c) \) for the \( i \)-th entry of \( S(c) \), we have \( \phi(z; c) = \sum_{i=1}^r S_i(c)\psi_i(z; c) \). By taking the partial derivative \( \frac{\partial}{\partial z_i} \), we easily see that \( S_i(c + a(j)) = S_i(c) \), which shows \( S_i(c) \in \mathfrak{M}(\mathbb{C}^n) \). We have shown that \( \text{Sol}_{\mathcal{G}(A), z} = \text{span} \mathfrak{M}(\mathbb{C}^n) \{\psi_i\}_{i=1}^r \).

\[ \square \]

\(^1\)In [SW08], the configuration matrix \( A \) is assumed to be pointed, i.e., it is assumed that there is a linear functional \( \phi \in (\mathbb{Z}^n)^\vee \) so that \( \phi(a(j)) > 0 \) for any column vector \( a(j) \) of \( A \). However, a careful reading of [SW08 Theorem 2.14] is true without the assumption that \( A \) is pointed. See also [FF10 §5].
Remark 2.5. Theorem 2.4 can be modified to give a formula of the dimension of the solution space of GG system when the column vectors of $A$ do not generate the lattice $\mathbb{Z}^n$. Suppose that the $\mathbb{Q}$-span of the column vectors of $A$ generates $\mathbb{Q}^n$. Let $Q \in \mathbb{Z}^{n \times n}$ be a matrix such that $ZA = ZQ$. We put $A' = Q^{-1}A \in \mathbb{Z}^{n \times N}$ and $c' = Q^{-1}c$. By definition, we have $\mathbb{Z}A' = \mathbb{Z}^n$. It can readily be seen that $M_A(c) = M_{A'}(c')$ and $GG(A) = GG(A')$. Replacing $A$ and $c$ by $A'$ and $c'$, the argument of Theorem 2.4 gives a formula

$$\dim \mathcal{M}_A(C^n) = \frac{\text{vol}_2(D_A)}{\dim \mathbb{Q}^n} = \text{rank} GG(A).$$

The formula (2.13) is an analytic counterpart of the result of [OT09].

Remark 2.6. Let us write $c'_j$ for the $j$-th entry of $c'$. Then, we have an identity

$$\mathcal{M}_A(C^n) = \mathbb{C} \left( e^{2\pi \sqrt{-1}c'_1}, \ldots, e^{2\pi \sqrt{-1}c'_r} \right).$$

We remark that, working on the level of GG system, we can show that monodromy matrices or connection matrices are translation invariant. Let us take a fundamental system of solutions $\Phi_1(z; c)$ and $\Phi_2(z; c)$ regarded as row vectors which are defined near the points $z_1$ and $z_2$ respectively. We take a path $\gamma$ of analytic continuation in $z$-space connecting $z_1$ and $z_2$ to obtain a relation

$$\gamma \Phi_1(z; c) = \Phi_2(z; c)M_c(c).$$

Here, $\gamma \Phi_1(z; c)$ denotes the analytic continuation of $\Phi_1(z; c)$ along the path $\gamma$. Clearly, $M_c(c)$ is a meromorphic function in $c$-variables. In view of the equation (1.3b), we obtain an identity

$$\Phi_2(z; c + a(j))M_c(c + a(j)) = (\partial_2 \Phi_2(z; c))M_c(c) = \Phi_2(z; c + a(j))M_c(c)$$

which gives

$$M_c(c + a(j)) = M_c(c).$$

We summarize the statement above as a

Proposition 2.7. A monodromy matrix or a connection matrix of GG system belongs to $\mathcal{M}_A(C^n)^{r \times r}$ with $r = \frac{\text{dim} \mathbb{Q}^n}{\text{rank} \mathbb{Q}^n}$.

2.2 Unique solvability of the boundary value problem

A simple, but important observation is that GG system has a structure of boundary value problem. Let $a(n + 1) \in \mathbb{Z}^n$ be a lattice vector and put $\hat{A} = (Aa(n + 1))$. We consider a formal solution $f(z, z_{N+1}; c)$ of $GG(\hat{A})$. Then, we can easily see that its boundary value map $bv_{N+1}(f)(z; c) := f(z, 0; c)$ along $\{ z_{N+1} = 0 \}$ (if it exists) gives rise to a solution of $GG(\hat{A})$. Thus, the boundary value map $bv_{N+1}$ takes the space of formal solutions of $GG(\hat{A})$ to that of $GG(A)$. It is now natural to ask if this boundary value map gives rise to a bijection.

Let us make this observation more precise. In this section, we do not assume that the column vectors of $A$ generate the lattice $\mathbb{Z}^n$ but we assume that they generate $\mathbb{Q}^n$ over $\mathbb{Q}$. We consider a left $R_A$-module $M$. We set $M[[z_{N+1}]] := M \otimes_\mathbb{C} \mathbb{C}[[z_{N+1}]]$. The natural action of $\partial_{N+1}$ onto $\mathbb{C}[[z_{N+1}]]$ induces an action of $R_A$ onto $M[[z_{N+1}]]$. Any $R_A$-morphism $F : GG(\hat{A}) \to M[[z_{N+1}]]$ is determined by the value of $[1] \in GG(\hat{A})$. By abuse of notation, we write $F(z, z_{N+1}; c) := \sum_{m=0}^\infty f_m(z; c)z_{N+1}^m$ for the morphism $F$ where $f_m(z; c) \in M$. We can naturally associate the boundary value $bv_{N+1}(F(z, z_{N+1}; c))$ to the hyperplane $\{ z_{N+1} = 0 \}$ by setting $bv_{N+1}(F(z, z_{N+1}; c)) := f_0(z; c) \in M$. The symbol $F(z, 0; c)$ also denotes the boundary
value \( bv_{N+1}(F(z, z_{N+1}; c)) = f_0(z; c) \). It can readily be seen that there is a unique \( R_A\)-morphism from \( GG(A) \) to \( M \) which sends \([1]\) to \( f_0(z; c) \in M \). This \( R_A\)-morphism is also denoted by the symbols \( F(z, 0; c), bv_{N+1}(F(z, z_{N+1}; c)) \) or \( f_0(z; c) \). Lastly, for any \( R_A\)-morphism \( f : GG(A) \to M \) and a vector \( v \in \mathbb{Z}^n \), we write \( f(z; c + v) \) for the image of \([\sigma^v] \).

**Theorem 2.8.** The map \( bv_{N+1} \) induces a linear isomorphism

\[
\begin{align*}
\text{bv}_{N+1} : \ Hom_{R_A}(GG(\tilde{A}), M[[z_{N+1}]]) & \quad \mapsto \quad Hom_{R_A}(GG(A), M) \\
F(z, z_{N+1}; c) & \quad \mapsto \quad F(z, 0; c),
\end{align*}
\]

(2.18)

whose inverse is given by the formula

\[
\mathcal{D}_{N+1} : f(z; c) \mapsto \sum_{m=0}^{\infty} f(z; c + ma(N + 1)) \frac{z^{m+1}}{m!}
\]

(2.19)

for any \( f(z; c) \in Hom_{R_A}(GG(A), M) \).

**Proof.** For an element \( f(z; c) \in Hom_{R_A}(GG(A), M) \), we set \( F(z, z_{N+1}; c) := \sum_{m=0}^{\infty} f(z; c + ma(N + 1)) \frac{z^{m+1}}{m!} \). First, we show that \( F(z, z_{N+1}; c) \) is a solution of \( GG(\tilde{A}) \), i.e., \( F \) is an \( R_A\)-morphism. In view of the fact that \( \mathcal{D}_{N+1} \circ \partial_j = \partial_j \circ \mathcal{D}_{N+1} \) for any \( j = 1, \ldots, N \), we have \( \partial_j F(z, z_{N+1}; c) = F(z, z_{N+1}; c + a(j)) \). On the other hand, we have

\[
\partial_{N+1} F(z, z_{N+1}; c) = \sum_{n=1}^{\infty} f(z; c + na(N + 1)) \frac{z^{n-1}}{(n-1)!} z^{N+1}_n
\]

(2.20)

\[
= \sum_{n=0}^{\infty} f(z; c + a(N + 1) + na(N + 1)) \frac{z^n}{n!} z^{N+1}_n
\]

(2.21)

\[= F(z, z_{N+1}; c + a(N + 1)). \]

(2.22)

As for Euler equations, for any \( i = 1, \ldots, n \), we have

\[
\sum_{j=1}^{N} a_{ij} \partial_j + a_i N_1 + \theta_{N+1} + c_i \left( \sum_{m=0}^{\infty} \frac{f(z; c + ma(N + 1))}{n!} z^{m+1}_n \right)
\]

(2.23)

\[= \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!} \{-c_i + na_i N_1 + na_i N_1 + c_i\} f(z; c + a(N + 1))
\]

(2.24)

\[= 0.
\]

(2.25)

This shows that \( F(z, z_{N+1}; c) \) is a solution of \( GG(\tilde{A}) \) and that \( \mathcal{D}_{N+1} : Hom_{R_A}(GG(A), M) \to Hom_{R_A}(GG(\tilde{A}), M[[z_{N+1}]]) \) is well-defined. It is easy to see that \( \text{bv}_{N+1} \circ \mathcal{D}_{N+1} = id \).

Next, we suppose \( F(z, z_{N+1}; c) \in Hom_{R_A}(GG(\tilde{A}), M[[z_{N+1}]]) \). We expand it as \( F(z, z_{N+1}; c) = \sum_{m=0}^{\infty} \frac{1}{m!} [(\partial_{N+1})^m F] (z, 0; c) z^{m+1}_n \). Since \( (\partial_{N+1})^m F(z, z_{N+1}; c) = F(z, z_{N+1}; c + ma(N + 1)) \), if we put \( f(z; c) = \text{bv}_{N+1}(F(z, z_{N+1}; c)) \), we obtain the identity \( F(z, z_{N+1}; c) = \mathcal{D}_{N+1} f(z; c) \). This argument shows the identity \( \mathcal{D}_{N+1} \circ \text{bv}_{N+1} = id \).

**Remark 2.9.** Let \( \sigma \subset \{1, \ldots, N\} \) be a simplex and fix a partition \( \sigma = \sigma^v \sqcup \sigma^d \). It is easily verified that we have an equality \( \text{bv}_{N+1}(\psi_{\sigma^v}^\circ(z, z_{N+1}; c)) = \psi_{\sigma^d}^\circ(z; c) \) and therefore, \( \mathcal{D}_{N+1} \psi_{\sigma^v}^\circ(z; c) = \psi_{\sigma^d}^\circ(z, z_{N+1}; c) \).
2.3 Monodromy invariant subspaces

In this subsection, we assume that the column vectors of $A$ generate the lattice $\mathbb{Z}^n$. We take a facet $F$ of $\Delta_A$ which does not contain the origin. We fix $n \times n$ integer matrix $Q$ such that $Z\mathbb{A}_F = ZQ$. For any local solution $f(z; c) \in \text{Sol}_{GG(A_F)}$, and $k \in \mathbb{Z}^n/\mathbb{Z}^lQ$, we have $e^{2\pi \sqrt{-1}kQ^{-1}c}f(z; c) \in \text{Sol}_{GG(A_F), z}$. Therefore, the function $\prod_{j \notin F} \mathbb{D}_j \left( e^{2\pi \sqrt{-1}kQ^{-1}c}f(z; c) \right)$ is a formal solution of $GG(A_F)$. Let us take a fundamental basis of solutions $\Phi_{AF}(z; c)$ of $GG(A_F)$ and put $m = \text{rank } GG(A_F)$ and a complete system of representatives $\left\{ k(i) \right\}_{i=1}^r$ of $\mathbb{Z}^n/\mathbb{Z}^lQ$. Regarding $\Phi_{AF}$ as a row vector, we put

$$
\Psi_{AF}(z; c) = \left( e^{2\pi \sqrt{-1}k(1)Q^{-1}c}\Phi_{AF}(z; c), \ldots, e^{2\pi \sqrt{-1}k(r)Q^{-1}c}\Phi_{AF}(z; c) \right). \quad (2.26)
$$

We consider a path $\gamma$ of analytic continuation along which one has a relation $\gamma_s^*\Phi_{AF}(z; c) = \Phi_{AF}(z; c)M(c)$. Then, one has a relation $\gamma_s^*\Psi_{AF}(z; c) = \Psi_{AF}(z; c)\begin{pmatrix} M(c) & \cdots & M(c) \end{pmatrix}$. Now we put $\Psi_F(z; c) := \prod_{j \notin F} \mathbb{D}_j \Psi_{AF}(z; c)$. We define the tensor product $A \otimes B$ of $m_1 \times m_1$ matrix $A = (a_{ij})$ and $m_2 \times m_2$ matrix $B$ by the formula

$$
A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m_1}B \\
\vdots & \ddots & \vdots \\
a_{m_1}B & \cdots & a_{m_1m_1}B \end{pmatrix}. \quad (2.27)
$$

Note that $A \otimes B$ is invertible if and only if both $A$ and $B$ are invertible and the inverse in this case is given by $A^{-1} \otimes B^{-1}$ Then,

$$
\Psi_F(z; c) = (\Psi_1, \ldots, \Psi_r) = (F_1(z; c), \ldots, F_r(z; c))^t C \otimes I_m. \quad (2.28)
$$

Here, we have put

$$
F_i(z; c) = \sum_{[A_Fm]=[A_Fk(l)]} \Phi_{AF}(z, c + A_Fk[l]) \frac{\pi^m}{\pi^m!} \quad (2.30)
$$

and

$$
C = \text{diag} \left( e^{2\pi \sqrt{-1}k(i)Q^{-1}c} \right)_{i=1}^r \left( e^{2\pi \sqrt{-1}k(i)Q^{-1}A_Fk[j]} \right)_{i,j=1}^r, \quad (2.31)
$$

where the symbol $[ ]$ denotes the equivalence class in the group $\mathbb{Z}^n/\mathbb{Z}A_F = \mathbb{Z}^n/\mathbb{Z}Q$. Note that we have

$$
^t C^{-1} = \frac{1}{r} \text{diag} \left( e^{2\pi \sqrt{-1}k(i)Q^{-1}c} \right)_{i=1}^r \left( e^{2\pi \sqrt{-1}k(i)Q^{-1}A_Fk[j]} \right)_{i,j=1}^r. \quad (2.32)
$$

Putting $M_l = M(c + A_Fk[l])$ and ignoring the problem of convergence for the moment, we have

$$
\gamma_s^* \Psi_F = \prod_{j \notin F} \mathbb{D}_j \gamma_s^* \Phi_{AF} \quad (2.33)
$$

$$
= \left( \sum_{i=1}^r e^{2\pi \sqrt{-1}k(1)Q^{-1}(c + A_Fk[l])} F_1M_l, \ldots, \sum_{i=1}^r e^{2\pi \sqrt{-1}k(r)Q^{-1}(c + A_Fk[l])} F_rM_l \right) \quad (2.34)
$$

$$
= \frac{1}{r} \Psi_F \left( ^t C^{-1} \otimes I_m \right) \begin{pmatrix} C_{11}M_1 & \cdots & C_{r1}M_1 \\
\vdots & \ddots & \vdots \\
C_{1r}M_r & \cdots & C_{rr}M_r \end{pmatrix} \quad (2.35)
$$

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\[= \frac{1}{r} \Psi_F \left( \sum_{l=1}^{r} C_{l1}^{-1} C_{l1} M_l \cdots \sum_{l=1}^{r} C_{rl}^{-1} C_{rl} M_l \right). \]  

(2.36)

Note that \( C_{ij}^{-1} \) stands for the fraction \( \frac{1}{C_{ij}} \). If we showed that each entry of \( \Psi_F \) is convergent for any \( z_F \in \mathbb{C}^F \) as a power series, (2.36) shows that the entries of \( \Psi_F \) span a monodromy invariant subspace of \( \text{Sol}_{GG(A),z} \).

**Theorem 2.10.** Let \( z \in \mathbb{C}^N \setminus \text{Sing}(A) \) be a point. We have a decomposition of the solution space of the GG system into monodromy invariant subspaces

\[ \text{Sol}_{GG(A),z} = \bigoplus_{0 \notin F < \Delta_A \atop F: \text{facet}} S_F. \]  

(2.37)

Here, \( S_F \) is a subspace of \( \text{Sol}_{GG(A),z} \) canonically isomorphic to \( \text{Sol}_{GG(A,F),z_F}^{\mathbb{Z}_n \times \mathbb{Z}_A} \) as \( \mathcal{M}_A(F) (\mathbb{C}^n) \)-vector spaces through the boundary value map \( \prod_{j \notin F} \mathcal{D}_j f \).

**Proof.** We only need to ensure that \( \Psi_F \) is well-defined, convergent in \( z_F \in \mathbb{C}^F \) and \( \text{Sol}_{GG(A),z} \) is spanned by these functions. We first take a row vector \( l_1 \in \mathbb{Z}^{1 \times n} \) such that \( l_1(Q^{-1}A) = (1, \ldots, 1) \). We prolong \( l_1 \) to a basis \( \{ l_j \}_{j=1}^n \) of \( \mathbb{Z}^{1 \times n} \) and put \( L = \left( \begin{array}{c} l_1 \\ \vdots \\ l_n \end{array} \right) \). Then, we put \( A'_F = LQ^{-1}A_F = \left( 1 \cdots \frac{1}{\tilde{A}_F} \right) \), \( A' = LQ^{-1}A \), and \( c' = LQ^{-1}c = \left( \frac{\gamma}{\tilde{c}} \right) \). We show that for any solution \( f(z_F; c) \) of \( GG(A_F) \), the function \( \prod_{j \notin F} \mathcal{D}_j f(z_F; c) \) is convergent in \( z_F \in \mathbb{C}^F \). We write \( \tilde{a}(j) \) for the \( j \in F \)-th column vector of the matrix \( \tilde{A}_F \). By the general theory of Euler integral representation (GKZ90 Theorem 2.14)), any solution of the GKZ system has the form

\[ f(z_F; c) = e^{\pi \sqrt{-1} \gamma} \Gamma(\gamma) \int_C h_F(x; z)^{-\gamma} x^\gamma \frac{dx}{x}. \]  

(2.38)

where \( h_F(x; z) = \sum_{j \in F} z_j x^{\tilde{a}(j)}, x = (x_2, \ldots, x_n) \) is a variable on \((n-1)\)-dimensional complex torus, \( \frac{dx}{x} = \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n} \) and \( C \) is a cycle in a twisted homology group. Note that (2.38) is also a solution of \( GG(A_F) \). Based on this formula, we have a relation

\[ \prod_{j \notin F} \mathcal{D}_j f(z_F; c) \]

\[ = \sum_{m \in \mathbb{Z}^F_{m \leq 0}} e^{\pi \sqrt{-1} (\gamma + \sum_{j \in F} a'_j m_j)} \Gamma \left( \gamma + \sum_{j \in F} a'_j m_j \right) \frac{z^F_F}{m!} \left( \int_C h_{\tilde{A}_F,F}(x; z_F)^{-\gamma - \sum_{j \in F} a'_j m_j} x^{\gamma + \tilde{A}_F m} \frac{dx}{x} \right), \]  

(2.39)

where we write \( a'_j \) for the \((i,j)\)-entry of the matrix \( A' \). We set \( \tilde{F}_+ = \{ j \notin F \mid a'_j \geq 0 \} \) (resp. \( \tilde{F}_- = \{ j \notin F \mid a'_j < 0 \} \)). By the definition of beta function, we have

\[ \frac{\Gamma \left( \gamma + \sum_{j \in F} a'_j m_j \right)}{\Gamma \left( \gamma + \sum_{j \in \tilde{F}_+} a'_j m_j \right) \Gamma \left( \sum_{j \in \tilde{F}_-} a'_j m_j \right)} = \frac{1}{1 - e^{-2\pi \sqrt{-1} (\gamma + \sum_{j \in \tilde{F}_-} a'_j m_j)}} \]
\[ \int_{C'} t^{\gamma + \sum_{j \notin F} a'_{i,j} m_j - 1} (1 - t)^{\sum_{j \in F} a'_{i,j} m_j - 1} dt, \quad (2.40) \]

where the contour \( C' \) begins from \( t = 1 \), approaches \( t = 0 \), turns around the origin in the negative direction, and goes back to \( t = 1 \). We easily see that the inequality

\[ \left| \Gamma \left( \gamma + \sum_{j \notin F} a'_{i,j} m_j \right) \right| \leq C^i_1 \Gamma \left( \sum_{j \in F} a'_{i,j} m_j \right) \quad (2.41) \]

holds for some \( C_1 > 0 \). Let us observe that \( a'_{i,j} < 1 \) for any \( j \in \bar{F} \) since \( F \) is a facet of \( \Delta_A \). Taking into account that the contour \( C \) can be taken so that it does not meet the vanishing locus of \( h_F(x; z) \), we have an estimate

\[ e^{\pi \sqrt{-1} (\gamma + \sum_{j \notin F} a'_{i,j} m_j)} \Gamma \left( \gamma + \sum_{j \notin F} a'_{i,j} m_j \right) \left( \int_C h_F(x; z)^{-\gamma - \sum_{j \in F} a'_{i,j} m_j x} e^{+ \bar{A}_F m \bar{x}} \right) \leq C^i_2 \Gamma \left( \sum_{j \in F} a'_{i,j} m_j \right), \quad (2.42) \]

which ensures that \( \prod_{j \notin F} \mathcal{D}_j f(z_F; c) \) is convergent for any \( z_j \in \mathbb{C} \) with \( j \notin F \).

We claim that for a basis of \( \Phi_{A_F}(z_F; c) \) of \( GG(A_F) \) for each facet \( F \) which does not contain the origin, \( \bigcup_{\bar{F} \text{ facet}} \{ \Psi_F(z; c) \} \) is a basis of solutions of \( GG(A) \). For this purpose, we take a convergent regular triangulation \( T \). Then, \( T \) induces a regular triangulation to each facet \( F \) which does not contain the origin, i.e., if the symbol \( T_F \) denotes the set \( \{ \sigma \in T \mid \sigma \subset F \} \), \( T_F \) is a regular triangulation. We set \( A'_\sigma = Q^{-1} A_\sigma \) and fix a complete system of representatives \( \{ [k'_j(j)] \}_{j=1}^r \) of \( \mathbb{Z}^r / \mathbb{Z}^r A'_\sigma \).

In view of Proposition 2.2, we may set \( \Phi_{A_F}(z_F; c) = \bigcup_{\sigma \in T_F} \{ \psi_{\sigma, k}(j)(z_F; c) \} \}_{j=1}^r \). By a direct computation, we obtain a relation \( \prod_{j \notin F} \mathcal{D}_j \left( e^{\pi \sqrt{-1} k Q^{-1} \psi_{\sigma, k}(j)(z_F; c)} \right) = \psi_{\sigma, A'_\sigma k + k'(j)}(z; c). \)

Combining this fact with the exact sequence

\[ 0 \to \mathbb{Z}^n / \mathbb{Z}^r Q^{1 A'_\sigma} \mathbb{Z}^r A_\sigma \to \mathbb{Z}^r / \mathbb{Z}^r A'_\sigma \to 0, \quad (2.44) \]

we can see that \( \bigcup_{\bar{F} \text{ facet}} \{ \Psi_F(z; c) \} \) is identical to the basis consisting of \( \Gamma \)-series discussed in Proposition 2.2.

Let us discuss the case when \( c \) is fixed. For any \( \bar{l} = 1, \ldots, r \), and for any column vector \( v \), we have

\[ \frac{1}{r} \begin{pmatrix} \sum_{l=1}^r C^{-1}_{i_l} C_{i_l M_l} & \cdots & \sum_{l=1}^r C^{-1}_{i_l} C_{i_l M_l} \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^r C^{-1}_{i_l} C_{i_l M_l} & \cdots & \sum_{l=1}^r C^{-1}_{i_l} C_{i_l M_l} \end{pmatrix} \begin{pmatrix} C^{-1}_{i_l} v \\ \vdots \\ C^{-1}_{i_l} v \end{pmatrix} = \begin{pmatrix} C^{-1}_{i_l} M_{l} v \\ \vdots \\ C^{-1}_{i_l} M_{l} v \end{pmatrix}. \quad (2.45) \]

Here, we have used the formula \( \sum_{l=1}^r C_{i_l C^{-1}_{j_l}} = r \delta_{i_l j_l} \). Let us write \( \Psi_i \) as \( \Psi_i = (\Psi_{i_1}, \ldots, \Psi_{i_r}) \).

The computation above shows that for each \( l = 1, \ldots, r \), the space \( \text{span}_\mathbb{C} \{ \sum_{i=1}^r C^{-1}_{i l} \Psi_{i j} \} \) is monodromy invariant and is isomorphic to \( \text{Sol}_{M_{A_F}(c+k(l)), z} \). Therefore, we have the following theorem.
Corollary 2.11. Suppose $z \notin \text{Sing}(A)$ and $c$ is very generic with respect to a convergent regular triangulation $T$. Then, one has the following canonical decomposition of the monodromy representation.

$$\text{Sol}_{M_A,c,z} = \bigoplus_{0 \in F \in \Delta_A} S_F,$$

where $S_F$ is a subspace of $\text{Sol}_{M_A,c,z}$ non-canonically isomorphic to $\bigoplus_{k \in R} \text{Sol}_{M_{A_F}(c+k)}$. Here, $R$ is a complete system of representatives of $\mathbb{Z}^n / \mathbb{Z} A_F$.

Remark 2.12. Corollary 2.11 is compatible with the description of the derived restriction of GKZ system obtained in [FFW11].

Example 2.13. We consider GG system for the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}$. Any solution $f(z;c)$ of $GG(A)$ is of the form

$$f(z;c) = z_1^{-c_1} z_2^{-c_2} z_3^{-c_3} F\left( \frac{z_3 z_4}{z_2}, \frac{z_3 z_5}{z_1}, c_1, c_2 \right),$$

where $F(z,\zeta; c_1, c_2)$ is subject to a system of difference-differential equations

$$(\theta_z + \theta_{\zeta} + c_1) F(z, \zeta; c_1, c_2) = -F(z, \zeta; c_1+1, c_2)$$

$$(\theta_z + c_2) F(z, \zeta; c_1, c_2) = -F(z, \zeta; c_1, c_2+1)$$

$$(\theta_z - c_3) F(z, \zeta; c_1, c_2) = F(z, \zeta; c_1, c_2+1)$$

$$\partial_{\zeta} F(z, \zeta; c_1, c_2) = F(z, \zeta; c_1+1, c_2).$$

Here, we have set $\theta_z = z \frac{\partial}{\partial z}$ and $\theta_{\zeta} = \zeta \frac{\partial}{\partial \zeta}$. It is easy to see that the Newton polytope $\Delta_A$ has two facets which do not contain the origin (Figure 1). According to Theorem 2.10, we see that the facet 1234 corresponds to a 2-dimensional monodromy invariant subspace. It is straightforward to see that the boundary value $F(z, \zeta; c_1, c_2) := F(z, 0; c_1, c_2)$ is subject to a system of difference-differential equations

$$(\theta_z + c_1) F(z, \zeta; c_1, c_2) = -F(z, \zeta; c_1+1, c_2)$$

$$(\theta_z + c_2) F(z, \zeta; c_1, c_2) = -F(z, \zeta; c_1, c_2+1)$$

$$(\theta_z - c_3) F(z, \zeta; c_1, c_2) = F(z, \zeta; c_1, c_2+1)$$

$$\partial_{\zeta} F(z, \zeta; c_1, c_2) = F(z, \zeta; c_1+1, c_2).$$

Setting $\alpha := c_1, \beta := c_2, \gamma := 1 - c_3$ and $2F_1(z; a, b, c) := \sum_{m=0}^{\infty} \frac{\Gamma(a + m) \Gamma(b + m)}{\Gamma(c + m) m!} z^m$, it is easy to see that the function $e^{-\pi \sqrt{-1} \alpha} \Gamma(\alpha + \beta) 2F_1(z; a, \beta) \zeta^n$ is a solution of the system (2.53)-(2.56). Since the function $2F_1(z; a, \beta)$ is essentially the Gauß’ hypergeometric function, the analytic continuations of it give rise to a two dimensional space of functions over the field $\mathbb{C}(e^{2\pi \sqrt{-1} \alpha}, e^{2\pi \sqrt{-1} \beta}, e^{2\pi \sqrt{-1} \gamma})$. Thus, the analytic continuations of the function $e^{-\pi \sqrt{-1} \alpha} \Gamma(\alpha + \beta) \sum_{n=0}^{\infty} 2F_1(z; a + n, \beta + n) \zeta^n$ define a two dimensional monodromy invariant subspace of the system (2.48)-(2.52). Note that this function is, up to a multiplication by a function in parameters $\alpha, \beta, \gamma$, equal to an analytic continuation.
of Horn’s $\Phi_1$ function ([EMOT53, Vol.1, §5.7.1])

$$\Phi_1(z, \zeta; \alpha, \beta, \gamma) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_m (\gamma)_m}{m! n!} z^m \zeta^n.$$ (2.57)

![Figure 1: Newton polytope](image)

3 Connection formula associated to a modification

In this section, we establish a connection formula among particular bases of GG system. Throughout this section, we assume that the matrix $A$ is homogeneous and the column vectors of it generate the lattice $\mathbb{Z}^n$. Our aim is to construct a path along which we can perform an analytic continuation of the basis discussed in §2.

3.1 Connection formula of a Mellin-Barnes integral

We first recall the notion of modification (perestroika). Let $\Sigma(A)$ be the secondary polytope, i.e., $\Sigma(A)$ is a convex polytope in $\mathbb{R}^N$ the dual fan of which is identical to the secondary fan ([GKZ94, Chapter 7, §1.D]). We can conclude that for each regular triangulation $T$, there is a unique vertex $v_T$ of $\Sigma(A)$ such that the normal cone $N_{\Sigma(A)}(v_T)$ of $\Sigma(A)$ at $v_T$ is equal to the cone $C_T \subseteq (\mathbb{R}^N)^\vee$. For any pair of regular triangulation $T$ and $T'$, we say $T$ is adjacent to $T'$ if the corresponding vertices $v_T$ and $v_{T'}$ are connected by an edge of $\Sigma(A)$. The adjacency can be interpreted in a combinatorial way. We say $Z \subseteq \{1, \ldots, N\}$ is a circuit if \{$a(i)\}_{i \in Z}$ is a minimal linearly dependent subset of \{$a(j)\}_{j=1}^N$. If $Z$ is a circuit, the corresponding subconfiguration \{$a(i)\}_{i \in Z}$ has only two regular triangulations. They are denoted by $T_+$ and $T_-$. This choice is not canonical but depends on the choice of the generator $u$ of $L_Z = \text{Ker}(A_Z \times : \mathbb{Z}^I \rightarrow \mathbb{Z}^n)$. If we fix a generator $u$ of $L_Z$, no entry of $u$ is zero by definition. We put $Z_+ = \{i \mid u_i > 0\}$ and $Z_- = \{i \mid u_i < 0\}$. Then $T_+$ (resp. $T_-$) is defined by $\{Z \setminus \{i\}\}_{i \in Z_+}$ (resp. $\{Z \setminus \{i\}\}_{i \in Z_-}$). A subconfiguration $I \subseteq \{1, \ldots, N\}$ is called a corank 1 configuration if the rank of $\text{Ker}(A_I \times : \mathbb{Z}^I \rightarrow \mathbb{Z}^n)$ is 1. We say that a regular polyhedral subdivision $Q$ of $A$ is an almost triangulation if any refinement of $Q$ is a triangulation. The following propositions are standard ([GKZ94, Chap.7, §2]).

**Proposition 3.1.** A regular polyhedral subdivision $Q$ of $A$ is an almost triangulation if and only if each cell of $Q$ has at most corank 1 and there is a unique circuit $Z$ such that any corank 1 cell contains $Z$.

**Proposition 3.2.** Let $T$ and $T'$ be a pair of regular triangulations such that $T$ is adjacent to $T'$. Let $e$ be the edge of $\Sigma(A)$ connecting $v_T$ and $v_{T'}$. Any weight vector $\omega$ in the relative interior of
the cone $N_{\Sigma(A)}(e)$ defines the same regular polyhedral subdivision $S$. Moreover, $S$ is an almost triangulation whose refinements are given by $T$ and $T'$.

One also has a precise description of the change of adjacent regular triangulations as follows. This is what we call the modification of a regular triangulation. Let $T$ and $T'$ be a pair of adjacent regular triangulations. For any corank 1 configuration $I$ which contains a circuit $Z$, there are only two triangulations. Namely, they are $T_+ = \{I \setminus \{i\}\}_{i \in Z_+}$ (resp. $T_- = \{I \setminus \{i\}\}_{i \in Z_-}$). Let $Q$ be the intermediate regular polyhedral subdivision of Proposition 3.1. We decompose regular triangulations. For any corank 1 configuration, there is an element $A$ of Proposition 3.1 and 3.2. Since $T$ (or $T'$) is a refinement of $Q$ and $T$ is a triangulation, we see that each $I_s$ has a maximal space dimension $n$, i.e., the convex hull of the origin and the points $\{a(i)\}_{i \in I_s}$ has a non-zero Euclidian volume. This implies that $|I_s| = n + 1$. Thus, if we write $T_+(I_s)$ and $T_-(I_s)$ for the pair of regular triangulations coming from $I_s$, we have $T = T_{irr} \cup \{T_+(I_s)\}_s$ and $T' = T_{irr} \cup \{T_-(I_s)\}_s$. This is also denoted by $T = T_{irr} \cup T_+(Z)$ and $T' = T_{irr} \cup T_-(Z)$. Note that $T_{irr}$ can be empty.

Now, let us concentrate on the GG system for a corank 1 configuration $I$. By abuse of notation, we write $A = (a(1) \ldots | a(n + 1))$ for a corank 1 configuration which may not generate the ambient lattice $\mathbb{Z}^n$ but span the vector space $\mathbb{Q}^n$. We put $I = \{1, \ldots, n + 1\}$. By the definition of corank 1 configuration, there is an element $u \in L_A$ such that $L_A = Zu$. We put $I_{\geq 0} = \{j \in I \mid u_j \geq 0\}$, $I_0 = \{j \in I \mid u_j = 0\}$, $Z_+ = \{j \in I \mid u_j > 0\}$, and $Z_- = \{j \in I \mid u_j < 0\}$. Note that the circuit $Z$ contained in $I$ is given by $Z = Z_+ \cup Z_-$. We fix an element $j_0 \in Z_+$ and put $\sigma = I \setminus \{j_0\}$. Let $\{e_i\}_{i \in \sigma}$ be the standard basis of the lattice $\mathbb{Z}^\sigma$. We set $1_\sigma = \sum_{i \in Z_-} e_i$. Consider an integral

$$I_\sigma(z_I; c) = \frac{1}{2\pi \sqrt{-1}} \int_{C} \prod_{i \in \sigma; j \geq 0} \Gamma(p_{si}(c + a(j_0)s)) \prod_{i \in \sigma; j \geq 0} \Gamma(1 - p_{si}(c + a(j_0)s)) (e^{\pi \sqrt{-1} z_\sigma} - a_\sigma^{-1} a(j_0)s)(e^{\pi \sqrt{-1} z_{j_0}})^s ds,$$

where $C$ is a vertical contour from $-\sqrt{-1} \infty$ to $+\sqrt{-1} \infty$ separating two spirals of poles of Gamma functions in the integrand (ibid). Note that $e^{\pi \sqrt{-1} z_\sigma} - a_\sigma^{-1} a(j_0)s$ depends only on circuit variables $z_+ := (z_j)_{j \in Z_+}$ and $z_- := (z_j)_{j \in Z_-}$. By Stirling’s formula, one can easily prove that this integral is convergent if

$$|\arg\left(e^{\pi \sqrt{-1} z_\sigma} - a_\sigma^{-1} a(j_0)s(e^{\pi \sqrt{-1} z_{j_0}})\right)| < \pi. \tag{3.2}$$

We rewrite the convergence condition (3.2). Observe that $\left(-A_\sigma^{-1} a(j_0)\right)1_\sigma = u_{j_0}^{-1} u$. Therefore, we have

$$|\arg\left(e^{\pi \sqrt{-1} z_\sigma} - a_\sigma^{-1} a(j_0)(e^{\pi \sqrt{-1} z_{j_0}})\right) = u_{j_0}^{-1} |\arg(e^{\pi \sqrt{-1} z_\sigma}, e^{\pi \sqrt{-1} z_{j_0}}) \cdot u|, \tag{3.3}$$

where the symbol $\cdot$ denotes the dot product. Thus, the convergence condition can be written as

$$-\pi < \sum_{i \in Z_-} \arg(e^{\pi \sqrt{-1} z_i})u_{j_0}^{-1} u_i + \sum_{j \in Z_+ \cap \sigma} \arg(z_j)u_{j_0}^{-1} u_j + \arg(e^{\pi \sqrt{-1} z_{j_0}}) < \pi \tag{3.4}$$

\(^2\)To be precise, the notation $A$ here should be replaced by $A_t$. However, we use a slightly confusing notation here in order to avoid complication.
A direct computation employing (3.7) and (3.8) shows that
\[ \text{arg}(e^{\pi \sqrt{-1} z_i}) u_i + \sum_{j \in Z_+} \text{arg}(z_j)^2 u_j < 0. \]  
(3.5)

Note that the last condition depends only on the circuit variable.

We consider an analytic continuation of (3.1) via Mellin-Barnes contour throw. This method has been previously discussed by several authors in various settings ([Sla66, Chap.4]). We first recall that \((e^{\pi - z_\sigma}) A_{\sigma}^{-1} I_{\sigma}(z; c)\) is a univariate function of a complex variable
\[ \zeta = (e^{\pi - z_\sigma}) A_{\sigma}^{-1} a(j_0) e^{\pi \sqrt{-1} z_{j_0}} = e^{\pi \sqrt{-1} (e^{\pi - z_\sigma}) A_{\sigma}^{-1} (u_{j_0} - (z_\sigma) u_{j_0})}, \]  
(3.6)

where \(u_+\) (resp. \(u_-\)) is the vector consisting of entries of \(u\) labeled by the set \(Z_+\) (resp. \(Z_-\)). The distribution of poles are as in the Figure 2:

If we evaluate the integral along the poles \(s = 0, 1, 2, \ldots\), we have \(I_{\sigma}(z_1; c) = \psi_{\sigma_\sigma}^Z(z; c)\).

Let us fix an element \(i_0 \in Z_-\) and consider the evaluation of \(I_{\sigma}(z_1; c)\) along a negative spiral \(s = \frac{u_{j_0}}{u_{i_0}} (p_{\sigma i_0}(c) + m)\) \((m \in Z\geq 0)\). We put \(\sigma' = I \setminus \{i_0\}\). We put \(v_{\sigma} = \begin{pmatrix} -A_{\sigma}^{-1} c \\ 0 \end{pmatrix}\) so that we have an equality \(Av_{\sigma} = -c\). In view of the equality \(A_{\sigma}^{-1} Av_{\sigma} = -A_{\sigma}^{-1} c\), we obtain relations
\[ p_{\sigma' j_0}(a(i_0)) \cdot p_{\sigma i_0}(c) = p_{\sigma' i}(c) \]  
(3.7)
\[ p_{\sigma' i}(a(i_0)) \cdot p_{\sigma i_0}(c) + p_{\sigma i}(c) = p_{\sigma' i}(c) \]  
(i \(\in\) \(\sigma\) \(\setminus\) \{\(j_0\)\}).
(3.8)

A direct computation employing (3.7) and (3.8) shows that
\[ I_{\sigma}(z_1; c) = \sum_{i \in Z_-} p_{\sigma i}(a(j_0)) \frac{1}{\psi_{\sigma' \sigma}^Z(i \setminus \{j_0\}) \psi_{\sigma_\sigma}^{Z_\sigma}(i \setminus \{j_0\})} I_{\sigma}(z_1; c). \]  
(3.9)

In the same way, we choose any \(\tilde{k}_{\sigma} = (\tilde{k}_i)_{i \in \sigma} \in \mathbb{Z}^\sigma\) and consider a function \(I_{\sigma}(e^{2\pi \sqrt{-1} z_\sigma}, z_{j_0}; c)\). The integral is convergent if
\[ -2\pi u_{j_0} < 2\pi u_{j_0} \sum_{i \in \sigma} \tilde{k}_i p_{\sigma i}(a(j_0)) + \sum_{i \in Z_-} \text{arg}(e^{\pi \sqrt{-1} z_i}) u_i + \sum_{j \in Z_+} \text{arg}(z_j)^2 u_j < 0. \]  
(3.10)

By a direct computation, we have a formula
\[ I_{\sigma}(e^{2\pi \sqrt{-1} z_\sigma}, z_{j_0}; c) = \sum_{i \in Z_-} \frac{1}{p_{\sigma i}(a(j_0))} \psi_{\sigma_\sigma}^{Z_\sigma}(i \setminus \{j_0\}) I_{\sigma}(z_1; c) \]  
(3.11)

Figure 2: distribution of poles
3.2 Some lemmata needed for the construction of the path of analytic continuation

Suppose that $T$ and $T'$ are adjacent regular triangulations of the configuration matrix $A$. We inherit the notation of the previous subsection. First, we prove a

**Proposition 3.3.** Let $Q$ be the intermediate almost triangulation and $Z$ be the common circuit. Then, one can choose a complete system of representatives $\{\tilde{k}_\sigma\}$ of $\mathbb{Z}^n/\mathbb{Z}^1A_\sigma$ for any corank 1 configuration $I$ in $Q$ and for any $\sigma = I \setminus \{j_0\}$ with $j_0 \in \mathbb{Z}_+$ so that the inequalities (3.10) define a non-empty open subset in the space of circuit variables $\mathbb{C}^Z$.

**Proof.** We set $1 = \sum_{i \in \sigma} e_i \in \mathbb{Z}^\sigma$. Then, the equivalence class $[1]$ in the group $\mathbb{Z}^\sigma/\mathbb{Z}^1A_\sigma$ is zero. Indeed, since $A$ is homogeneous, for any element $j \in \sigma$, we have $(1)A_{\sigma}^{-1}a(j) = 1$. This computation combined with the fact that a pairing $(\bullet, \bullet) : \mathbb{Z}^\sigma/\mathbb{Z}^1A_\sigma \times \mathbb{Z}^n/\mathbb{Z}A_\sigma \ni (u, v) \mapsto t_uA_{\sigma}^{-1}v \in \mathbb{Q}/\mathbb{Z}$ is perfect and that $\mathbb{Z}^n/\mathbb{Z}A_\sigma = \sum_{j \in \sigma} \mathbb{Z}[a(j)]$ implies that $[1] = 0$. Therefore, for any $\tilde{k}_\sigma \in \mathbb{Z}^n/\mathbb{Z}^1A_\sigma$ and $r \in \mathbb{Z}$, we have $[\tilde{k}_\sigma] = [\tilde{k}_\sigma + r1]$. On the other hand, if we replace $\tilde{k}_\sigma$ by $\tilde{k}_\sigma + r1$, the sector defined by the inequality (3.10) is translated by $2\pi i_{j_0} \times r$. Thus, we can choose a complete system of representatives as in the statement of the proposition.

Proposition 3.3 gives the information of the argument of the path of analytic continuations. Now, we control other directions $z_I$ for a corank 1 configuration in $Q$. The important point is that, if we consider a modification, there can be several corank 1 configurations in general. So we have to control $z_I$ simultaneously. Let us identify the space of row vectors $\mathbb{R}^{1 \times N}$ with the dual lattice $(\mathbb{R}^N)^\vee$ via dot product. We put

$$\tilde{C}_{I,+} = \{ \omega \in \mathbb{R}^{1 \times N} \mid \omega_{\Gamma \setminus \{j\}} A_{\Gamma \setminus \{j\}}^{-1} a(k) < \omega_k, \text{ for any } j \in \mathbb{Z}_+ \text{ and } k \in \tilde{I} \}$$

and

$$\tilde{C}_+ = \bigcap_{I, \text{corank 1 configurations in } Q} \tilde{C}_{I,+}.$$  

**Proposition 3.4.** $C_Q \cup C_T \subset \tilde{C}_+$.

**Proof.** Let us take an element $\omega \in C_T$. Since we have $I \setminus \{j\} \in T$ for any $j \in \mathbb{Z}_+$, there exists a row vector $n \in \mathbb{Q}^{1 \times n}$ such that

$$\begin{cases} n \cdot a(i) = \omega_i & (i \in I \setminus \{j\}) \\ n \cdot a(k) < \omega_k & (k \in \tilde{I} \cup \{j\}). \end{cases}$$  

From the first inequality, we can derive the equality $n = \omega_{\Gamma \setminus \{j\}} A_{\Gamma \setminus \{j\}}^{-1}$. Substituting this equality to the inequality above, for any $k \in \tilde{I} \cup \{j\}$, we obtain

$$\omega_{\Gamma \setminus \{j\}} A_{\Gamma \setminus \{j\}}^{-1} a(k) < \omega_k.$$  

This implies the inclusion $C_T \subset \tilde{C}_+$.

Next, we take $\omega \in C_Q$. Since $I \in Q$, there exists a row vector $n \in \mathbb{Q}^{1 \times n}$ such that

$$\begin{cases} n \cdot a(i) = \omega_i & (i \in I) \\ n \cdot a(k) < \omega_k & (k \in \tilde{I}). \end{cases}$$  

Again, from the first equality, we can derive the equality $n = \omega_{\Gamma \setminus \{j\}} A_{\Gamma \setminus \{j\}}^{-1}$ for any $j \in \mathbb{Z}_+$. Therefore, from the second inequality, for any $j \in \mathbb{Z}_+$ and $k \in \tilde{I}$, we obtain

$$\omega_{\Gamma \setminus \{j\}} A_{\Gamma \setminus \{j\}}^{-1} a(k) < \omega_k.$$  

This implies the inclusion $C_Q \subset \tilde{C}_+$.\qed
We conclude this section with a lemma on the existence of a good direction.

**Proposition 3.5.** For any corank 1 configuration $I$ in $Q$, there is a weight vector $\omega^{(I)} = (0_I, \omega_I)$ which belongs to the relative interior of the cone $C_Q$ such that $\omega_I > 0$.

**Proof.** We take a weight vector $\omega$ in the relative interior of the cone $C_Q$. Since $I$ belongs to $Q$, there is a row vector $n \in \mathbb{Q}^{1 \times n}$ such that

$$\begin{cases} n \cdot a(i) = \omega_i & (i \in I) \\ n \cdot a(k) < \omega_k & (k \in \bar{I}). \end{cases}$$

(3.18)

If we put $\omega^{(I)} = \omega - nA$, this satisfies the desired properties. \(\square\)

**Corollary 3.6.** There exists a weight vector $\omega_Q$ in the relative interior of the cone $C_Q$ such that $\omega_Q = (0_{\text{core}(Q)}, \omega_{\text{core}(Q)})$ and $\omega_{\text{core}(Q)} > 0$. Here, $\text{core}(Q) = \bigcap_{I: \text{corank 1 configurations in } Q} I$.

### 3.3 Estimate of a difference operator of infinite order

In this subsection, we consider hypergeometric function of the following type:

$$F(\frac{a,b}{\bar{a},\bar{b}}, z) = \sum_{n=0}^{\infty} \frac{\Gamma(a + bn)}{\Gamma(bn)} z^n. \quad (3.19)$$

Here, parameters are $a \in \mathbb{C}^p$, $\bar{a} \in \mathbb{C}^q$, $b \in \mathbb{Z}_p^{\geq 0}$, $\bar{b} \in \mathbb{Z}_q^{\geq 0}$ and we assume that for any $n \in \mathbb{Z}_{>0}$, $a + bn$ has no non-positive integer entry. Note that if $b \in \mathbb{Q}_\leq 0$ or $\bar{b} \in \mathbb{Q}_\leq 0$, there is a positive integer $k$ so that $kb \in \mathbb{Z}_p^{\geq 0}$, $\bar{kb} \in \mathbb{Z}_q^{\geq 0}$. Therefore, we have a decomposition

$$F(\frac{a,b}{\bar{a},\bar{b}}, z) = \sum_{i=0}^{k-1} z^i F(\frac{a+ib,kb}{\bar{a}+ib,\bar{b}}, z^k). \quad (3.20)$$

Thus, the consideration is reduced to the case when $b \in \mathbb{Z}_p^{\geq 0}$ and $\bar{b} \in \mathbb{Z}_q^{\geq 0}$. Now we consider the following operator which is sometimes referred to as Erdélyi-Kober integral:

$$(I^0(\alpha,\beta,\kappa)f)(z) = \frac{1}{\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}f(t^\kappa z)dt \quad (\text{Re} \, \alpha, \text{Re} \, \beta > 0, \kappa \in \mathbb{R}_{>0}). \quad (3.21)$$

Here, $f(z)$ is a germ of univariate holomorphic function defined around the origin. In terms of power series, this operator is well-understood. Indeed, if we write $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$(I^0(\alpha,\beta,\kappa)f)(z) = \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} a_n \left( \int_0^1 t^{\alpha + \kappa n - 1}(1-t)^{\beta - 1}dt \right) z^n$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \kappa n)}{\Gamma(\alpha + \beta + \kappa n)} z^n \quad (3.22)$$

From the formula (3.22), shifting the complex parameters $a, \bar{a}$ in (3.19) can be expressed in terms of the integral operator $I^0(\alpha,\beta,\kappa)$. We want to generalize this operator $I^0(\alpha,\beta,\kappa)$ to $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, $\beta \in \mathbb{C}$, when $\kappa \in \mathbb{Z}_{>0}$.

**case 1: $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$**

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In this case, we have

\[(I_{\alpha,\beta}^{(\kappa)} f)(z) = \frac{1}{\Gamma(\beta)(1 - e^{-2\pi\sqrt{-1}\alpha})(1 - e^{-2\pi\sqrt{-1}\beta})} \int_{P} t^{\alpha-1}(1 - t)^{\beta-1} f(t^\kappa z) dt, \tag{3.23}\]

where \(P\) is the Pochhammer cycle connecting \(t = 0\) and \(t = 1\).

case 2: \(\alpha \in \mathbb{C} \setminus \mathbb{Z}\) and \(\text{Re} \beta > 0\)

In this case, we have

\[(I_{\alpha,\beta}^{(\kappa)} f)(z) = \frac{1}{\Gamma(\beta)(1 - e^{-2\pi\sqrt{-1}\alpha})} \int_{Q} t^{\alpha-1}(1 - t)^{\beta-1} f(t^\kappa z) dt, \tag{3.24}\]

where \(Q\) is the cycle which starts from and ends at 1 as in the following figure.

![Figure 3: cycle Q](image)


case 3: \(\alpha \in \mathbb{C} \setminus \mathbb{Z}\) and \(\beta = -s\) \((s \in \mathbb{Z}_{\geq 0})\)

In this case, we first take \(\beta\) to be a generic complex number and \(\text{Re} \alpha > 0\)

\[(I_{\alpha,s}^{(\kappa)} f)(z) = \frac{1}{\Gamma(\beta)(1 - e^{-2\pi\sqrt{-1}\alpha})} \int_{R} t^{\alpha-1}(1 - t)^{\beta-1} f(t^\kappa z) dt, \tag{3.25}\]

where \(R\) is the cycle which starts from and ends at 0 as in the following figure.

![Figure 4: cycle R](image)

If let \(\beta\) tend to \(-s\), we obtain

\[(I_{0}^{(\alpha,-s)} f)(z) = \frac{(-1)^{s+1}s!}{2\pi \sqrt{-1}} \int_{\partial \Delta(1; \varepsilon)} t^{\alpha-1} f(t^\kappa z) dt. \tag{3.26}\]

The formula above is valid even when \(\alpha \in \mathbb{C} \setminus \mathbb{Z}\).

Now we are in a position to apply the integral representation of the difference operator to the key Gevrey estimate. For any \(0 < \varepsilon < R, 0 < \varepsilon_{\theta}, \) we put \(S_{\varepsilon,\varepsilon_{\theta},R} = \Delta(0; \varepsilon) \cup \{ |\arg z - \theta| \leq \varepsilon_{\theta}, |z| \leq R\} \).

**Lemma 3.7.** Let \(a \in \mathbb{C}, \{h_{s}\} \subset \mathbb{Q}\) satisfy \(a + \sum_{s} l_{s} h_{s} \notin \mathbb{Z}\) for any non-negative integers \(l_{s}\) and let \(\kappa \in \mathbb{Z}_{\geq 0}\). Then, for any \(0, \varepsilon' < \varepsilon, 0 < \varepsilon'_{\theta} < \varepsilon_{\theta}, 0 < R' < R\), there exists a constant \(0 < C\) and \(0 < r_{s}\) which only depends on \(a, h_{s}, \kappa, \varepsilon', \varepsilon'_{\theta}, R'\) such that for any holomorphic function \(f\) in a neighbourhood of \(S_{\varepsilon,\varepsilon_{\theta},R}\), the inequality

\[|I_{0}^{(a + \sum_{s} l_{s}, h_{s}, \kappa, \sum_{s} l_{s} h_{s})} f(z)| \leq C \prod_{s} r_{s}^{l_{s}} |h_{s}|^{l_{s}} \sup_{z \in S_{\varepsilon,\varepsilon_{\theta},R}} |f(z)| \tag{3.27}\]

holds for any \(z \in S_{\varepsilon,\varepsilon_{\theta},R'}\) and \(l_{s} \in \mathbb{Z}_{\geq 0}\).
Proof. By homotopy, we can choose integration contours \(P, Q, \partial \Delta(1; \tilde{\varepsilon})\) so that for any \(t\) in one of these contours and for any \(z \in S_{\varepsilon', \varepsilon''} R\), we have \(t^\varepsilon z \in S_{\varepsilon, \varepsilon''} R\). By the assumption, we have \(a + \sum_s l_h s \notin \mathbb{Z}\). Therefore, we have

\[
(I_0^{(a + \sum_s l_h s, -\sum_s l_h s)} f)(z) = \frac{e^{-\varepsilon n \sqrt{-1} \Theta} \Gamma(1 + \sum_s l_h s)}{2 \pi n \sqrt{-1} (1 - e^{-2 \pi n \sqrt{-1} (a + \sum_s l_h s)\Theta})} \int_P e^{a + \sum_s l_h s - 1 (1 - t) - \sum_s l_h s - 1} |f(t^\varepsilon z)| dt \quad (\sum_s l_h s \notin \mathbb{Z}) \tag{3.28}
\]

\[
\frac{1}{\Gamma(-\sum_s l_h s) (1 - e^{-2 \pi n \sqrt{-1} (1 + \sum_s l_h s)\Theta})} \int_Q e^{a + \sum_s l_h s - 1 (1 - t) - \sum_s l_h s - 1} |f(t^\varepsilon z)| dt \quad (\sum_s l_h s \in \mathbb{Z}) \tag{3.29}
\]

Since each \(h_s\) is rational, we see that the desired estimate exists for the first and the third case since \(\sum_s l_h s (1 - t) - \sum_s l_h s = \prod_s (h_s (1 - t) - h_s)\). As for the second case, if we put \(r = \max\{|1 - t| \mid t \in Q\}\) and \(r' = \inf\{|t| \mid t \in Q\}\), we have, for any small positive number \(\delta\), an estimate

\[
\frac{1}{\Gamma(-\sum_s l_h s) r' \sum_s l_h s - \sum_s l_h s - \delta} \sup \{|f(z)| \mid z \in S_{\varepsilon, \varepsilon''} R\} \tag{3.30}
\]

\[
\leq C \prod_s t^{|l_h s|} \sup \{|f(z)| \mid z \in S_{\varepsilon, \varepsilon''} R\} \tag{3.31}
\]

By the assumption, we have

\[
(I_0^{(\tilde{a}, \sum_s l_{\tilde{h}} s)} \tilde{f})(z) = \frac{1}{\Gamma(-\sum_s l_{\tilde{h}} s) (1 - e^{-2 \pi n \sqrt{-1} (\tilde{a} + \sum_s l_{\tilde{h}} s)\Theta})} \int_Q (\tilde{a} + \sum_s l_{\tilde{h}} s - 1 (1 - t) - \sum_s l_{\tilde{h}} s - 1) |f(t^\varepsilon z)| dt \quad (\sum_s l_{\tilde{h}} s \notin \mathbb{Z}) \tag{3.32}
\]

\[
\frac{1}{\Gamma(-\sum_s l_{\tilde{h}} s) (1 - e^{-2 \pi n \sqrt{-1} (1 + \sum_s l_{\tilde{h}} s)\Theta})} \int_Q (\tilde{a} + \sum_s l_{\tilde{h}} s - 1 (1 - t) - \sum_s l_{\tilde{h}} s - 1) |f(t^\varepsilon z)| dt \quad (\sum_s l_{\tilde{h}} s \in \mathbb{Z}) \tag{3.33}
\]

The rest of the proof is similar to that of Lemma 3.7.

Lemma 3.8. Let \(\tilde{a} \in \mathbb{C} \setminus \mathbb{Z}, \{\tilde{h}_s\} \subset \mathbb{Q}\) be arbitrary and \(\tilde{\kappa} \in \mathbb{Z}_{\geq 0}\). Then, for any \(0, \varepsilon' < \varepsilon, 0 < \varepsilon'' < \varepsilon, 0 < R' < R\), there exists a constant \(0 < \tilde{C} < \tilde{C} > 0\) which only depends on \(\tilde{a}, \tilde{h}_s, \tilde{\kappa}, \varepsilon', \varepsilon'', R'\) such that for any holomorphic function \(f\) in a neighbourhood of \(S_{\varepsilon, \varepsilon''} R\), the inequality

\[
|I_0^{(\tilde{a}, \sum_s l_{\tilde{h}} s)} \tilde{f})(z)| \leq \tilde{C} \prod_s t^{|l_{\tilde{h}} s|} \sup \{|f(z)| \mid z \in S_{\varepsilon, \varepsilon''} R\} \tag{3.34}
\]

holds for any \(z \in S_{\varepsilon', \varepsilon''} R\) and any \(l_s \in \mathbb{Z}_{\geq 0}\).

Proof. In this case, we have

\[
(I_0^{(\tilde{a}, \sum_s l_{\tilde{h}} s)} \tilde{f})(z) = \frac{1}{\Gamma(-\sum_s l_{\tilde{h}} s) (1 - e^{-2 \pi n \sqrt{-1} (\tilde{a} + \sum_s l_{\tilde{h}} s)\Theta})} \int_Q (\tilde{a} + \sum_s l_{\tilde{h}} s - 1 (1 - t) - \sum_s l_{\tilde{h}} s - 1) |f(t^\varepsilon z)| dt \quad (\sum_s l_{\tilde{h}} s \notin \mathbb{Z}) \tag{3.35}
\]

\[
\frac{1}{\Gamma(-\sum_s l_{\tilde{h}} s) (1 - e^{-2 \pi n \sqrt{-1} (1 + \sum_s l_{\tilde{h}} s)\Theta})} \int_Q (\tilde{a} + \sum_s l_{\tilde{h}} s - 1 (1 - t) - \sum_s l_{\tilde{h}} s - 1) |f(t^\varepsilon z)| dt \quad (\sum_s l_{\tilde{h}} s \in \mathbb{Z}) \tag{3.36}
\]

The rest of the proof is similar to that of Lemma 3.7.

By applying the lemmata above repeatedly, we obtain the desired

Theorem 3.9. Let \(\{h_s\} \subset \mathbb{Q}^p, \{\tilde{h}_s\} \subset \mathbb{Q}^q\) be finite vectors such that for any \(l_s, n \in \mathbb{Z}_{\geq 0}, a + \sum_s l_h s + bn \) and \(\tilde{a} + \sum_s l_{\tilde{h}} s \) does not have any integer entry. If \(F (a, b; z)\) is holomorphic in a neighbourhood of \(S_{\varepsilon, \varepsilon''} R\), then, for any \(0, \varepsilon' < \varepsilon, 0 < \varepsilon'' < \varepsilon, 0 < R' < R\), there exist constants \(0 < C < \tilde{C} < \tilde{C} > 0\) which only depend on \(a, \tilde{a}, h, \tilde{h}, \varepsilon', \varepsilon''\) such that

\[
|F (a, b; z) - F (a, b; z)| \leq C \prod_s t^{|l_h s|} \left\{|h_s| \right\} \sup \{|f(z)| \mid z \in S_{\varepsilon, \varepsilon''} R\} \tag{3.37}
\]

holds for any \(z \in S_{\varepsilon', \varepsilon''} R\) and any \(l_s \in \mathbb{Z}_{\geq 0}\).
Proof. Note that we have an equality

\[ F(a + \sum_i l_i, h_i, b, z) = \prod_{i=1}^p f_0^{(a_i + \sum_i l_i, h_i)} \cdot \prod_{j=1}^q f_0^{(a_j, \sum_i l_i)} \cdot \left( F\left( \tilde{a}, \tilde{b}, z \right) \right). \]  

(3.35)

Theorem follows from a successive application of Lemma 3.7 and Lemma 3.8.

3.4 Construction of a path and a proof of a connection formula

Let \( T \) and \( T' \) be adjacent regular triangulations. We use the notation of §3.1. For any \( z = (z_1, \ldots) \in (\mathbb{C}^*)^N \), we set \(-\log |z| := (-\log |z_1|, \ldots) \in \mathbb{R}^N\). We first take a point \( z_{\text{start}} \in \mathbb{C}^N \) so that \(-\log |z_{\text{start}}| \in (\omega_T + C_T)\). Here, \( \omega_T \in C_T \) is taken so that \( \omega_T + C_T \subset -\log |U_T| \) is true. Then, we choose a suitable positive real number \( l \) so that \(-\log |z_{\text{end}}| \overset{\text{def}}{=} -\log |z_{\text{start}}| + l(v_T - v_T) \in (\omega_T + C_T) \cap \tilde{C}_+. \) Note that Proposition 3.4 and the fact that \( \tilde{C}_+ \) is an open set implies that \( C_T \cap \tilde{C}_+ \neq \emptyset \). Here, \( v_T \) and \( v_T' \) are vertices of the secondary polytope \( \Sigma(A) \) corresponding to \( T \) and \( T' \). We take a path \( \gamma(t) \) (\( 0 \leq t \leq 1 \)) in \( \mathbb{C}^N \) so that an equality \(-\log |\gamma(t)| := -\log |z_{\text{start}}| + l(t(v_T' - v_T) \) is true. We choose a complete system of representatives \( \{ \tilde{k}_\sigma \} \) of \( \mathbb{Z}^N / \mathbb{Z}^N A_\sigma \) for any \( \sigma = I \setminus \{ j_0 \} \) and \( j_0 \in \mathbb{Z}_+ \) as in Proposition 3.3. Then, we choose \( \arg z \) along this path \( \gamma \) so that the inequalities (3.10) are valid. When \( z \) runs over this path \( \gamma \), the circuit variables runs over a set of the form \( S_{\varepsilon, \varepsilon, R} \). Therefore, by Theorem 3.9 combined with Proposition 3.4 and Corollary 3.6 we can conclude that there exists a positive real number \( r \) and a vector \( \omega_Q = (\omega_{\text{core}}(Q), \omega_{\text{core}}(Q)) \) so that if \(-\log |z| \in r\omega_Q - \log |\gamma|\), for any corank 1 configuration \( I \) in \( Q \), and for any \( j_0 \in \mathbb{Z}_+ \), the function

\[ \prod_{j \in I} \Omega_j I_\sigma(e^{2\pi \sqrt{-1} k_\sigma z_\sigma, z_{j_0}; c}) \]  

(3.36)
Theorem 3.10. Let $T$ and $T'$ be adjacent regular triangulations of $A$. Suppose that for any corank 1 configuration $I$ appearing in the modification of $T$ and $T'$ and for any $j_0 \in Z_+$, a complete set of representatives $\{k_\sigma\}$ of $\mathbb{Z}^\sigma/\mathbb{Z}^\sigma A_\sigma$ with $\sigma = I \setminus \{j_0\}$ is given as in Proposition 3.3. Then, along the path $\gamma$ constructed above, for any corank 1 configuration $I$ and $j_0 \in Z_+$, we have a connection formula

$$
\psi_{I_{\mathcal{L}_0}(\gamma \cup \{j_0\})}^{Z_+}(z;c) = \sum_{i \in Z_+} \frac{1}{p_{\mathcal{L}_0}(a(j_0))} \psi_{I_{\mathcal{L}_0}(\gamma \cup \{j_0\})}^{Z_+ \setminus \{j_0\}}(z;c) \tag{3.37}
$$

along $\gamma$. Moreover, $\Gamma$-series corresponding to $\sigma \in T_{irr}$ are invariant after analytic continuation.

Example 3.11. Let us consider a $4 \times 6$ matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$. The secondary fan is a complete fan in $(\mathbb{R}^6)^\vee$. The projected image of $\Sigma(A)$ through the projection $\pi_A : (\mathbb{R}^6)^\vee \to L_A^\vee \otimes \mathbb{R}$ is shown in Figure 6. Here, we use the isomorphism $L_A \simeq \mathbb{Z}^2$ specified by choosing a basis

$$
\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \ \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}
$$

of $L_A$.

Figure 6: The projected image of the secondary fan

The basis of solutions at $T = \{1234, 1236, 1256\}$ is given as follows:

$$
\phi_{T,1234}(z;c) = e^{-\sqrt{-1}(c_1+c_3)} \sin \pi c_2 \Gamma(c_1) \Gamma(c_2) \Gamma(c_3) \frac{\sin \pi c_2 \pi(c_1+c_3) \Gamma(c_1) \Gamma(c_2) \Gamma(c_3) G_1}{\pi \Gamma(1 - c_4) z_2 z_3 z_4} F_1 \left( c_1, c_2, c_3, z_4 z_5, z_6 \right) \tag{3.38}
$$

$$
\phi_{T,1236}(z;c) = e^{-\sqrt{-1}(c_1+c_3+2c_4)} \sin \pi c_2 \sin \pi(c_4) \Gamma(c_1 + c_4) \Gamma(c_3 + c_4) \Gamma(c_2) \Gamma(-c_4) z_2 z_3 z_4 \frac{G_2(c_1+c_4,c_2,-c_4,c_3+c_4)}{z_2 z_3 z_4 z_5} \tag{3.39}
$$

$$
\phi_{T,1256}(z;c) = \frac{\sin \pi(c_1+c_4) \sin \pi(c_2+c_3+c_4) \sin \pi c_3 \Gamma(c_1+c_3) \Gamma(c_2+c_3+c_4) \Gamma(c_3)}{\pi \Gamma(1 + c_3 + c_4) z_2 z_3 z_4 z_5 z_6 F_1 \left( c_1, c_2, c_3, c_4, \frac{z_2 z_5}{z_1 z_3}, \frac{z_4 z_6}{z_2 z_3} \right)} \tag{3.40}
$$
where \( F_1 \left( \frac{\alpha, \beta, \beta'; z_1, z_2}{} \right) \) and \( G_2 \left( \alpha, \alpha', \beta, \beta'; z_1, z_2 \right) \) are Appell’s \( F_1 \) and Horn’s \( G_2 \) series defined by

\[
F_1 \left( \frac{\alpha, \beta, \beta'; z_1, z_2}{} \right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!} z_1^m z_2^n \quad (3.41)
\]

and

\[
G_2 \left( \alpha, \alpha', \beta, \beta'; z_1, z_2 \right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m}(\alpha')_n(\beta)_{m-n}(\beta')_{m-n}}{m!n!} z_1^m z_2^n. \quad (3.42)
\]

Note that these functions \( \phi_{T', \sigma}(z; c) \) are convergent when \( z_4 \) and \( z_5 \) are small enough when other variables \( z_1, z_2, z_3, z_6 \) are fixed.

On the other hand, we have

\[
\phi_{T', 1246}(z; c) = e^{-\pi \sqrt{-1} c_3} \frac{\sin \pi (c_1 + c_4) \sin \pi c_2 \Gamma(c_1) \Gamma(c_2) \Gamma(c_1 + c_4)}{\pi \sin \pi (c_3 - c_1) \Gamma(1 + c_1 - c_3)} \times 
\]

\[
z_2^{-c_2} z_3^{c_1 - c_3} z_4^{-c_1 - c_4 - c_5} z_6^{-c_3} F_1 \left( \frac{c_1 + c_4, 1 + c_1 - c_3, z_1 z_2, z_3 z_5}{z_4 z_6} \right)
\]

\[
\phi_{T', 2346}(z; c) = e^{-\pi \sqrt{-1} c_1} \frac{\sin \pi c_2 \sin \pi (c_3 + c_4) \Gamma(c_1 - c_3) \Gamma(c_2) \Gamma(c_3) \Gamma(c_3 + c_4)}{\pi^2} \times 
\]

\[
z_1^{c_1 - c_3} z_2^{-c_2} z_4^{-c_1 - c_4 - c_5} z_5^{c_3} G_2 \left( \frac{c_3, c_2, c_1 - c_3, c_3 + c_4 - \frac{z_1 z_3}{z_4 z_6}}{z_4 z_6} \right)
\]

\[
\phi_{T', 1256}(z; c) = \frac{\sin \pi (c_1 + c_4) \sin \pi (c_2 + c_3 + c_4) \sin \pi c_5 \Gamma(c_1 + c_3) \Gamma(c_2 + c_3 + c_4) \Gamma(c_3)}{\pi^3 \Gamma(1 + c_3 + c_4)} \times 
\]

\[
z_1^{-c_1 - c_4} z_2^{-c_2 - c_4 - c_5} z_3^{c_3} z_4^{-c_1 - c_3 - c_4} z_5^{c_2} z_6^{-c_3} F_1 \left( \frac{c_3, c_1 + c_4, 1 + c_3 + c_4, z_1 z_2}{z_4 z_6} \right)
\]

We see that the modification from \( T \) to \( T' \) is controlled by the corank 1 configuration 12346 and that \( Z_+ = 46, Z_- = 13 \) and \( I_0 = 2 \). In view of Theorem 3.10, solving the boundary value problem for \( \{ z_5 = 0 \} \) yields the connection formulae

\[
\phi_{T, 1234}(z; c) \sim \phi_{T', 1246}(z; c) + \phi_{T', 2346}(z; c)
\]

and

\[
\phi_{T, 1236}(z; c) \sim e^{-\pi \sqrt{-1} c_4} \frac{\sin \pi c_1}{\sin \pi (c_1 + c_4)} \phi_{T', 1246}(z; c) + e^{-\pi \sqrt{-1} c_4} \frac{\sin \pi c_3}{\sin \pi (c_3 + c_4)} \phi_{T', 2346}(z; c).
\]

Note that the connection coefficients are translation invariant, i.e., they belong to the field \( \mathcal{M}_A(C^1) = \mathbb{C} \left( e^{2\pi \sqrt{-1} c_1}, e^{2\pi \sqrt{-1} c_2}, e^{2\pi \sqrt{-1} c_3}, e^{2\pi \sqrt{-1} c_4} \right) \). On the other hand, we have

\[
\phi_{T', 1256}(z; c) = \phi_{T', 1256}(z; c).
\]

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