SU(2)_k \times SU(2)_l / SU(2)_{k+l} Coset Conformal Field Theory and Topological Minimal Model on Higher Genus Riemann Surface

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ABSTRACT

We consider the Feigin-Fuchs-Felder formalism of the SU(2)_k \times SU(2)_l / SU(2)_{k+l} coset minimal conformal field theory and extend it to higher genus. We investigate a double BRST complex with respect to two compatible BRST charges, one associated with the parafermion sector and the other associated with the minimal sector in the theory. The usual screened vertex operator is extended to the BRST invariant screened three string vertex. We carry out a sewing operation of these string vertices and derive the BRST invariant screened g-loop operator. The latter operator characterizes the higher genus structure of the theory. An analogous operator formalism for the topological minimal model is obtained as the limit l = 0 of the coset theory. We give some calculations of correlation functions on higher genus.
1. Introduction

The $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ minimal coset theory\cite{1,2} is known as a rational conformal field theory, which contains a series of minimal conformal models. The cases $k = 1, 2, 4, \ldots$ provide the Belavin-Polyakov-Zamolodchikov (BPZ) minimal model\cite{3}, the N=1 super minimal model\cite{4}, the $S_3$ symmetric minimal model\cite{5}, and so on.

It is also a remarkable fact that the theory describes the critical behavior of the series of exactly solvable restricted solid on solid (RSOS) models\cite{6,7}. The underlying mathematical structures which connect the RSOS models and the $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ coset theory are, however, unknown. It is, however, suggestive that the one point functions in the RSOS models are given by the branching coefficients in the coset theory. In other words, there is a correspondence between the reduced one dimensional configuration sum in the corner transfer matrix method and the trace over the irreducible highest weight representation (IHWR) space in the coset theory. One of the aim of this paper is to clarify the structure of IHWR of the $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ minimal coset theory in the Feigin-Fuchs construction\cite{8}.

On the other hand, it is known that the $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ coset theory is deeply related to the topological conformal minimal model. The latter theory was first obtained by Eguchi and Yang\cite{9} by twisting the N=2 super conformal field theory. As they showed, the same theory is also realized as the case $l = 0$ in the $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ minimal coset theory. Calculation of correlation functions in the topological minimal model has been discussed by Witten\cite{10} and extended by Dijkgraaf and Verlinde at the tree level\cite{11} and by Vafa to an arbitrary genus\cite{12}. However, since their method totally depends on the correspondence of the chiral ring structure of the topological conformal field theory to the one of the super potential in the topological Landau-Ginzburg theory, its relation to the Feigin-Fuchs construction, a standard method in rational conformal field theories, is unclear. Our second aim is to formulate the topological field theory in the
Feigin-Fuchs scheme on Riemann surfaces with arbitrary genus.

In the next section, we begin with reviewing the Feigin-Fuchs construction of the $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ minimal coset theory.\textsuperscript{[2,13,16]} The theory is represented by the $Z_k$ parafermion theory\textsuperscript{[14]} and a scalar boson field. There exist two types of screening charges, one of which is just the one in the $Z_k$ parafermion theory and the other is associated mainly with the minimal sector in the theory.

In section 3, we discuss the BRST formalism of the theory according to Felder.\textsuperscript{[15]} We treat a double BRST complex associated with the above two types of screening charges. We construct the singular vectors explicitly and obtain the BRST cohomology groups.

In section 4, we discuss an extension of the above BRST formalism by introducing the Caneschi-Schwimmer-Veneziano (CSV) vertex,\textsuperscript{[17]} a kind of vertex describing the interaction of three strings. The CSV vertex connects three different Fock spaces. Therefore, one can make a multi loop calculation by taking traces and sewing of a set of these vertices.\textsuperscript{[18]} One point to be clarified in this extension is a definition of contours for the screening operators. The contour should be a homologically nontrivial cycle on the three punctured sphere. We construct a contour explicitly according to the general method discussed by Felder and Silvotti.\textsuperscript{[19]} We then define a screened CSV vertex. The BRST relations on the usual vertex operator is hence extended to the one on the screened CSV vertex operator.

In section 5, we calculate a BRST invariant $g$-loop operator in the coset theory. A $g$-loop operator is, in general, used to define a conformal field theory on a genus $g$ Riemann surface.\textsuperscript{[20]} This is possible, because a $g$-loop operator maps a conformal field to the one on the corresponding genus $g$ Riemann surface by the Bogoliubov transformation. Various Ward identities on a genus $g$ Riemann surface can also be derived by calculating the action of the currents on the $g$-loop operator.

Our $g$-loop operator is a generalization of those in the nondegenerate conformal field theories in the following two points.\textsuperscript{[21]} The first point is that, in a process of
making loops (i.e. handles), traces are taken over the subspace of the Fock space corresponding to IHWR. This is, of course, necessary to guarantee the BRST invariance, or in other words, the decoupling of the null states from the $g$-loop operator. The second point is the fact that our $g$-loop operator is properly screened. This is carried out by sewing the above screened CSV vertices. Our screened $g$-loop operator hence provides a consistent higher genus extension of the Feigin-Fuchs-Felder formalism.

In section 6, we demonstrate some higher genus calculations in the coset theory. We give a vacuum amplitude of arbitrary genus.

The final section is devoted to a discussion of the topological limit of the coset theory. Using the results in the coset theory, we formulate a similar operator formalism in the topological minimal model. We also give a possible picture changing operator, which makes the evaluation of correlators easy.

2. Free Field Realization

In this section we review a free field realization of the $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ minimal coset theory.\cite{2,13,16} Let us begin with the Wakimoto construction of the $su(2)_k$ affine Kač-Moody algebra.\cite{22} In the bosonized form, the currents are given by\cite{23}

\begin{align*}
J_+ &= -e^\phi \partial e^\chi, \\
J_0 &= \partial \phi - \sqrt{\frac{k+2}{2}} \partial \Phi, \\
J_- &= \left[(k+2)\partial \phi - (k+1)\partial \chi - \sqrt{2(k+2)}\partial \Phi \right] e^\phi e^{-\chi},
\end{align*}

(2.1)

where $\phi, \chi$ and $\Phi$ are scalar bosons. These fields satisfy the operator product expansion (OPE) $< \chi(z)\chi(w) > = < \Phi(z)\Phi(w) > = \ln(z-w) = - < \phi(z)\phi(w) >.$ Note that, in (2.1), spin 1 conjugate bosons $(\beta, \gamma)$ in the usual Wakimoto construction are bosonized as $\beta = -e^{-\phi} \partial e^\chi$ and $\gamma = e^\phi e^{-\chi}.\cite{24}$
The energy-momentum (EM) tensor defined by the Sugawara form is obtained as:

\[ T^{(k)} = \frac{1}{2} (\partial \chi)^2 + \frac{1}{2} \partial^2 \chi - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} \partial^2 \phi + \frac{1}{2} (\partial \Phi)^2 + \frac{1}{2} \sqrt{\frac{2}{k+2}} \partial^2 \Phi. \] (2.2)

This generates the Virasoro algebra with central charge \( c^{(k)} = \frac{3k}{k+2} \). The semidirect product of this Virasoro and the \( su(2)_k \) affine Kac-Moody algebra generates a symmetry in the \( SU(2)_k \) Wess-Zumino-Witten (WZW) theory.\(^{[25]}\)

The primary field \( \Phi_{j,j}(z) \) in the \( SU(2)_k \) WZW theory, whose conformal dimension is \( h_j = \frac{j(j+2)}{k+2} \), is given by \( \Phi_{j,j} = \Phi_{j,m}|_{j=m} \), where

\[ \Phi_{j,m} = e^{(j-m)(\phi-\chi)} e^{-j\sqrt{\frac{2}{k+2}}\Phi}. \] (2.3)

In the unitary representation, the spin \( j \) is characterized by an integer \( n \) as \( 2j+1 = n \), \( 1 \leq n \leq k+1 \). In the following paragraph, we consider the unitary representation only.

The screening operator in the \( SU(2)_k \) WZW theory is given by

\[ S^{(k)} = e^{-\phi} e^{\sqrt{\frac{2}{k+2}}\Phi} e \partial \chi. \] (2.4)

Since the conformal dimension of \( S^{(k)} \) is one and its OPEs with all the currents in (2.1) yield only regular or total derivative terms, the screening charge \( \oint S^{(k)} \) is commutative with all the currents.

Now, let us consider the \( SU(2)_k \times SU(2)_l/SU(2)_{k+l} \) coset theory. Let \( (\phi^{(a)}, \chi^{(a)}, \Phi^{(a)}) \) \( a = k, l \) be the two sets of the fields realizing \( SU(2)_a \) WZW model \( a = k, l \) in the way as (2.1). Let us define new sets of fields \( (\varphi, \chi, \phi_0) \) and \( (\phi^H, \chi^H, \Phi^H) \) by

\[ \phi^H = \phi^{(l)}, \quad \chi^H = \chi^{(l)}, \quad \Phi^H = \sqrt{\frac{k+2}{p'}} \Phi^{(k)} + \sqrt{\frac{p}{p'}} \Phi^{(l)} - \sqrt{\frac{2}{p'}} \phi^{(k)} , \]

5
\[
\varphi = \sqrt{\frac{k + 2}{k}} \phi^{(k)} - \sqrt{\frac{2}{k}} \Phi^{(k)}, \quad \chi = \chi^{(k)},
\]
\[
\phi_0 = \sqrt{\frac{2(k + 2)}{kp'}} \phi^{(k)} - \sqrt{\frac{(k + 2)p}{kp'}} \Phi^{(k)} + \sqrt{\frac{k}{p'}} \Phi^{(l)},
\]

where \( p = l + 2 \) and \( p' = k + l + 2 \). These new fields satisfy \(< \chi^H(z)\chi^H(w) > = < \phi^H(z)\phi^H(w) > = < \phi_0(z)\phi_0(w) > = \ln(z-w) = - < \phi^H(z)\phi^H(w) > = - < \varphi(z)\varphi(w) > \). The other OPEs vanish. Then, the coset theory is realized by a set \((\varphi, \chi, \phi_0)\), whereas the diagonal \( SU(2)_{k+l} \) WZW model is realized by \((\phi^H, \chi^H, \Phi^H)\) \cite{13}.

The change of fields (2.5) makes the total EM tensor \( T^{(k)} + T^{(l)} \) in the \( SU(2)_k \times SU(2)_l \) WZW theory split into two parts \( T^{(k)} + T^{(l)} = T^{(k+l)} + T \), where

\[
T^{(k+l)} = \frac{1}{2} (\partial \chi^H)^2 + \frac{1}{2} \partial^2 \chi^H - \frac{1}{2} (\partial \phi^H)^2 - \frac{1}{2} \partial^2 \phi^H + \frac{1}{2} (\partial \Phi^H)^2 + \frac{1}{2} \sqrt{\frac{2}{p'}} \partial^2 \Phi^H,
\]

\[
T = T_{Z_k} + \frac{1}{2} (\partial \phi_0)^2 + \frac{1}{2} (2\sqrt{2\alpha_0}) \partial^2 \phi_0
\]

(2.6)

with \( 2\alpha_0 = \sqrt{\frac{k}{pp'}} \) and \( T_{Z_k} \) being the EM tensor in the \( Z_k \) parafermion theory \cite{27}:

\[
T_{Z_k} = \frac{1}{2} (\partial \chi)^2 + \frac{1}{2} \partial^2 \chi - \frac{1}{2} (\partial \varphi)^2 - \frac{1}{2} \sqrt{\frac{k}{k+2}} \partial^2 \varphi.
\]

(2.7)

The expression of \( T \) indicates that the coset theory is realized by the \( Z_k \) parafermion theory and the scalar \( \phi_0 \) boson theory. The central charge associated with \( T \) is given by

\[
c = \frac{3kl(k + l + 4)}{(k + 2)(l + 2)(k + l + 2)}. \quad (2.8)
\]

One should note the relation: \( c^{(k)} + c^{(l)} = c^{(k+l)} + c \).
Next let us consider the primary fields and the screening operators. Let the primary field in the $SU(2)_k \times SU(2)_l$ tensor theory be $\Phi^{(k)}_{j,j}(z)\Phi^{(l)}_{j',j'}(z)$. By the field redefinitions in (2.5), one can derive

$$\Phi^{(k)}_{j,j}(z)\Phi^{(l)}_{j',j'}(z) = \Phi^{(k+l)}_{j+j,j'+j}(z)\Psi_{j,j}(z), \quad (2.9)$$

where

$$\Psi_{j,j} = \phi_{j,j} : \exp\{(-\sqrt{2}\alpha_- \tilde{j} - 2\sqrt{2}\alpha_0 j)\phi_0\} : \quad (2.10)$$

$$\phi_{j,m} = \exp\left(-\sqrt{\frac{k+2}{k}}(m - \frac{k}{k+2}j)\varphi\right)\exp\{-(j - m)\chi\} \quad (2.11)$$

with $\alpha_- = -\sqrt{\frac{p}{kp}}$. Since $\Phi^{(k+l)}_{j+j,j+j}$ is the primary field in the $SU(2)_{k+l}$ WZW model, the field $\Psi_{j,j}$ in (2.10) can be identified with the primary field in the coset theory. In fact, in the unitary representation, the spin $\tilde{j}$ and $j$ are given by $2\tilde{j} + 1 = \tilde{n}$ and $2j + 1 = n$, with integers $\tilde{n}$ and $n$, $1 \leq \tilde{n} \leq k+1$ and $1 \leq n \leq l+1$. Introducing the constant $\alpha_+ = \sqrt{\frac{p}{kp}}$ satisfying $\alpha_+ + \alpha_- = 2\alpha_0$ and $\alpha_+\alpha_- = -1/k$, we obtain

$$\Psi_{J,n',n} \equiv \Psi_{j,j} = \phi_{J,j} : \exp \sqrt{2\alpha_{n',n}}\phi_0 : \quad (2.12)$$

with $\alpha_{n',n} = \frac{1-n'}{2}\alpha_- + \frac{1-n}{2}\alpha_+$ and $J \equiv 2\tilde{j} = n' - n \mod 2k$, where $n' = n + \tilde{n} - 1$ satisfying $1 \leq n' \leq k + l + 1$ so that $0 \leq J \leq k$. This expression of the primary field has been obtained in Ref.[2].

The Virasoro highest weight states (the primary states) satisfying

$$L_0|J;n',n> = h_{J;n',n}|J;n',n>,$$

$$L_n|J;n',n> = 0, \quad \text{for} \quad n \geq 1 \quad (2.13)$$

with the highest weights

$$h_{J;n'n} = \frac{J(k-J)}{2k(k+2)} + \frac{(np'-n'p)^2 - k^2}{4kpp'} \quad (2.14)$$
are created by the primary fields $\Psi_{J;n'}$ on the $SL(2,C)$ invariant vacuum $|0>$ as

$$|J; n', n> = \lim_{z \to 0} \Psi_{J;n', n}(z)|0> = |J> \otimes \lim_{z \to 0} e^{\sqrt{2}a_{n', n} \phi_0(z)}|0>.$$  \hfill (2.15)

Here $|J>$ is the parafermion highest weight state given by

$$|J> = \lim_{z \to 0} \phi_{J, J}(z)|0>.$$  \hfill (2.16)

Let us denote by $F_{J;n', n}$ the Fock spaces created by the action of the creation operators of the fields $\varphi, \chi$ and $\phi_0$ on the primary states (2.15). Here the mode expansion of the field $X(z)$ is defined by

$$X(z) = ix + \alpha_{X,0} \ln z + \sum_{n \geq 1} \frac{1}{\sqrt{n}}(a_{X,n}^\dagger z^n - a_{X,n} z^{-n}),$$  \hfill (2.17)

and the following canonical commutation relations are imposed.

$$[x, \alpha_{X,0}] = i \epsilon_X, \quad [a_{X,n}, a_{X,m}^\dagger] = \epsilon_X \delta_{n,m},$$  \hfill (2.18)

where $\epsilon_X = 1$ or -1 for $X = \chi, \phi_0$ or $\varphi$, respectively.

For later convenience (see §3), we formally introduce the field

$$\Psi_{J,m_{n', n}; n', n} = \phi_{J, \frac{m_{n', n}}{2}} e^{\sqrt{2}a_{n', n} \phi_0}$$  \hfill (2.19)

with $m_{n', n} = n' - n \mod 2k$ as well as the state

$$|J, m_{n', n}; n', n> = \lim_{z \to 0} \Psi_{J,m_{n', n}; n', n}(z)|0>.$$  \hfill (2.20)

The conformal dimension of the field $\Psi_{J,m_{n', n}; n', n}$ is formally given by $h_{J,m_{n', n}; n', n} = h_{J,m_{n', n}} + h_{n', n}$, where $h_{J,m} = \frac{J(J+2)}{4(k+2)} - \frac{m^2}{4k}$ and $h_{n', n} = \frac{(n'-n')^2 - k^2}{4kpp'}$. The following
relations are satisfied.

\[
\begin{align*}
\alpha_{\varphi,0} |J, m; n, n > & = 0 \quad \text{for} \quad s \geq 1, \\
\alpha_{\varphi,0} |J, m; n, n > & = \frac{1}{2} \sqrt{\frac{k}{k+2}(\frac{k+2}{k}m - J)} |J, m; n, n> , \\
\alpha_{\chi,0} |J, m; n, n > & = -\frac{1}{2} (J - m) |J, m; n, n>, \\
\alpha_{\phi_0,0} |J, m; n, n > & = \sqrt{2} \alpha_{\phi_0,0} |J, m; n', n>. 
\end{align*}
\]  

(2.21)

One should also note that the state \(|J >\) plays a similar role to the spin state in the Ramond sector in the N=1 super minimal model. In fact, in the case \(k = 2\), which corresponds to the N=1 super conformal minimal model, the conformal dimension \(h_J \equiv \frac{J(k-J)}{2k(k+2)}\) of the parafermion primary field \(\phi_{\frac{J}{2}, \frac{J}{2}}\) takes 0 for \(J = 0\) mod 2, or \(\frac{1}{16}\) for \(J = 1\) mod 2. As is well known, the two sets of primary fields of conformal dimension (2.14) with these \(h_J\)s define the Neveu-Schwarz and the Ramond sectors, respectively.\(^4\) Similarly, in the case \(k = 4\), which corresponds to the \(S_3\) symmetric model, \(h_J\) takes three values, 0 for \(J = 0\) mod 4, \(\frac{1}{16}\) for \(J = 1\) mod 4, or \(\frac{1}{12}\) for \(J = 2\) mod 4. In this case, there exist three sectors of representation labeled by \(h_J\).\(^5\) In the case of generic \(k\), \(h_J\) takes \(\left[\frac{k}{2}\right] + 1\) different values as \(J\) varies in the range \(0 \leq J \leq k\), where \(\left[\frac{k}{2}\right]\) denotes a maximum integer less than \(\frac{k}{2}\). Hence the representation space, in this case, has \(\left[\frac{k}{2}\right] + 1\) sectors labeled by \(h_J\).

In the same way, the screening operator \(S^{(k)}(z)S^{(l)}(z)\) in the tensor theory is factorized into \(S^{(k+l)}(z)S(z)\). One thus obtains the conformal dimension one operator, up to the identity operator \(e^{2\sqrt{2}\alpha_0\phi_0}\),

\[
S = e^{-\sqrt{\frac{k}{4\pi}\varphi}}\varphi e^{\chi}. 
\]  

(2.22)

This operator is nothing but the screening operator in the \(Z_k\) parafermion theory, as well as in the \(SU(2)_k\) WZW theory.\(^{26,27}\) We hence regard this operator as a screening operator in the coset theory.
As discussed in [2], there exist another set of screening operators, namely

\[ S_+ = \Psi e^{\sqrt{2} \alpha + \phi_0}, \quad S_- = \Psi^\dagger e^{\sqrt{2} \alpha - \phi_0}. \]  \hspace{1cm} (2.23)

Here \( \Psi \) and \( \Psi^\dagger \) are the parafermion currents given by

\[
\Psi = -\frac{1}{\sqrt{k}} e^{-\sqrt{2k} \varphi} \partial e^\chi, \\
\Psi^\dagger = \frac{1}{\sqrt{k}} (\sqrt{k(k+2)} \partial \varphi - (k+1) \partial \chi) e^{\sqrt{2k} \varphi} e^{-\chi}. \hspace{1cm} (2.24)
\]

The three screening operators \( S_\pm \) and \( S \) satisfy the following OPEs.

\[
S_+(z)S_-(w) = (\frac{1}{(z-w)^2} + \frac{1}{z-w} \sqrt{2} \alpha_+ \partial \phi_0(w) ) e^{2\sqrt{2} \alpha_0 \phi_0(w)} + RT, \\
S_+(z)S(w) = RT, \\
S_-(z)S(w) = -\frac{k+2}{\sqrt{k}} \partial w (\frac{1}{z-w} e^{2\sqrt{2} \alpha \varphi(w)} e^{\sqrt{2} \alpha - \phi_0(w)}) + RT. \hspace{1cm} (2.25)
\]

By making use of the primary field \( \Psi_{J;n',n} \) and these screening operators \( S \) and \( S_\pm \), we obtain the following fusion rules:

\[
\left[ \Psi_{J_1;n_1',n_1} \right] \times \left[ \Psi_{J_2;n_2',n_2} \right] = \sum_{n_3 = |n_1 - n_2| + 1}^{\min(n_1 + n_2 - 1, 2p - 1, 1 - n_1 - n_2)} \sum_{n_3' = |n_1' - n_2'| + 1}^{\min(n_1' + n_2' - 1, 2p' - 1, 1 - n_1' - n_2')} \left[ \Psi_{J_1;n_1',n_1} \right]\hspace{1cm} (2.26)
\]
3. BRST Cohomology

In this section we give the definition of the BRST charges associated with $S$ and $S_+$ and investigate their cohomologies.

For convenience, let us introduce the Fock space $F_{J,m,n',n}^{\phi_0}$ defined on the primary state $|J, m, n', n>$. The Fock space $F_{J,m;n',n}$ is the tensor product of the parafermion sector $F_{J,m}^{PF}$ and the boson sector $F_{n',n}^{\phi_0}$:

$$F_{J,m;n',n} = F_{J,m}^{PF} \otimes F_{n',n}^{\phi_0}. \quad (3.1)$$

Now let us define the BRST charges $Q_{J+1}$ and $Q_n^+$ by

$$Q_{J+1} = \frac{1}{J+1} \frac{e^{2i\pi(J+1)/k+2} - 1}{e^{2i\pi/k+2} - 1} \oint \prod_{i=1}^{J+1} du_i \prod_{i=1}^{J+1} S(u_i), \quad (3.2)$$

$$Q_n^+ = \frac{1}{n} \frac{e^{2i\pi n/p} - 1}{e^{2i\pi/p} - 1} \oint \prod_{i=1}^{n} dv_i \prod_{i=1}^{n} S_+(v_i), \quad (3.3)$$

where the integration contours are taken in the same way as in Ref.[15]. Note that the BRST charge $Q_{J+1}$ is the same one as in the $Z_k$ parafermion theory.\cite{26,29}

It is shown that these charges are nilpotent \cite{15,16}

$$Q_{J+1} Q_{k+2-(J+1)} = Q_{k+2-(J+1)} Q_{J+1} = 0, \quad (3.4)$$

$$Q_n^+ Q_{p-n}^+ = Q_{p-n}^+ Q_n^+ = 0 \quad (3.5)$$

and from the OPEs in (2.25), these are commuting with each other: $[Q_{J+1}, Q_n^+] = 0$.

The action of $Q_{J+1}$ and $Q_n^+$ on the Fock space $F_{J,m,n',n'};n',n$ give rise to the
following double BRST complex.

\[
\begin{array}{c}
\vdots \\
Q^+_J \rightarrow F_{-J-2,m_{n',-n+2p}} \rightarrow Q^+_{J+1} \rightarrow F_{-J-2,m_{n',n'n}} \rightarrow Q^+_J \\
\downarrow Q_{k+2-(J+1)} \\
\vdots \\
\end{array}
\]

Let us first consider the cohomologies of the vertical complexes. These complexes are characterized by the $Z_k$ parafermion sector only.

Since the BRST operator $Q_{J+1}$ in the $Z_k$ parafermion theory is common in the $SU(2)_k$ WZW theory, the BRST cohomology groups in the parafermion sector are obtained by the following theorem.

**Theorem (Bernard and Felder)**[26,29]

\[
Ker Q^{[s]}_r / Im Q^{[s-1]}_r = \begin{cases} 0 & \text{for } s \neq 0 \\ \mathcal{H}^{PF}_{J_{r',m}} & \text{for } s = 0 \end{cases} \quad (3.7)
\]

where $\mathcal{H}^{PF}_{J_{r',m}}$ is the IHWR with highest weight $h_{J_{r',m}}$, $J_{r',m} = r - r'(k+2) - 1$. In (3.7), $Q^{[2s]}_r = Q_r$ and $Q^{[2s+1]}_r = Q_{k+2-r}$ act on the parafermion Fock spaces $F^{[2s]}_{J_{r',m}} = F^{PF}_{J_{r-2s(k+2),r',m}}$ and $F^{[2s+1]}_{J_{r',m}} = F^{PF}_{J_{r'-2s(k+2),r',m}}$, respectively. As a corollary, we get the trace formula

\[
\text{Tr}_{\mathcal{H}^{PF}_{J_{r',m}} \otimes F^{\phi_0}_{J_{r',n'}}} \mathcal{O} = \sum_{s \in \mathbb{Z}} (-)^s \text{Tr}_{F^{[s]}_{J_{r',m}} \otimes F^{\phi_0}_{J_{r',n'}}} \mathcal{O}^{[s]}, \quad (3.8)
\]

12
where $\mathcal{O}^{[s]}$ is an arbitrary operator on $F^{[s]}_{J,m}$, and is obtained recursively by

$$Q^{[s]}_{J+1} \mathcal{O}^{[s]} = \mathcal{O}^{[s+1]} Q^{[s]}_{J+1}$$  \hspace{1cm} (3.9)$$

with $\mathcal{O}^{[0]} = \mathcal{O}$.

In (3.8), $\tilde{F}^{[s]}_{J,m}$ is the Fock space obtained from $F^{[s]}_{J,m}$ by fixing picture and dropping redundant zero-modes. This reduction process is necessary because the Fock space of scalar boson fields $\phi$ and $\chi$ is much larger than the one of $\beta$ and $\gamma$ due to the bosonization.

In order to find the relation between $\tilde{F}$ and $F$, one has to impose the constraint $\alpha_{\phi,0} + \alpha_{\chi,0} = 0$ and consider the cohomology of the nilpotent operator $Q_V = \oint dz e^{-x}$ in the trace.$^{29,30,31}$ $Q_V$ commutes with both $Q_{J+1}$ and $Q_n^+$, and maps the Fock space $F^{[u]}_{J,m} \equiv F^{PF}_{J+(k+2)u,m+ku}$ to $F^{[u+1]}_{J,m}$. It is easy to show

$$\text{Ker} Q^{[u]}_V / \text{Im} Q^{[u-1]}_V = 0,$$

for all $u \in \mathbb{Z}$. From this, we have the trace formulae

$$\text{Tr}_{F^{[u]}_{J,m}} \mathcal{O} = \sum_{u \geq 0} (-)^u \text{Tr}_{F^{[0]}_{J,m}} \mathcal{O} \bigg|_{\alpha_{\phi,0} + \alpha_{\chi,0} = 0}$$  \hspace{1cm} (3.10)$$

and

$$0 = \sum_{u \in \mathbb{Z}} (-)^u \text{Tr}_{F^{[u]}_{J,m}} \mathcal{O}.$$  \hspace{1cm} (3.11)$$

Combining (3.8) and (3.10), we get

$$\text{Tr}_{\mathcal{H}^{PF}_{J,m} \otimes F^{[0]}_{n,n'}} \mathcal{O} = \sum_{s \in \mathbb{Z}} \sum_{u \geq 0} (-)^{s+u} \text{Tr}_{F^{[s,u]}_{J,m} \otimes F^{[s]}_{n,n'}} \mathcal{O} \bigg|_{\alpha_{\phi,0} + \alpha_{\chi,0} = 0}.$$  \hspace{1cm} (3.12)$$
By making use of the resolution in the vertical complexes, the double complex in (3.6) is now reduced to the following complex

\[ \ldots Q^+_{n} \to H^{PF}[n] \to \cdots \]

\[ \to H^{PF}[0] \to \cdots \to H^{PF}[1] \to \cdots \]

where

\[ H^{PF}[2t] \otimes F_{n',n}^{\phi_0}[2t] = H^{PF}_{J,m,n',n-2tp} \otimes F_{n',n-2tp}^{\phi_0} \]

\[ H^{PF}[2t+1] \otimes F_{n',n}^{\phi_0}[2t+1] = H^{PF}_{J,m,n',n-2tp} \otimes F_{n',n-2tp}^{\phi_0}. \] (3.14)

Let us consider the singular vectors in the Fock space \( H^{PF}_{J,m,n',n} \otimes F_{n',n}^{\phi_0} \). According to Ref.[28] and the remark given below (2.21), the singular fields, in the \( h_{J,m} = 0 \) sector, are given by

\[ \chi_{-n',n}(z) = \oint \prod_{i=1}^{n} dv_i \prod_{i=1}^{n} S_+(v_i) e^{\sqrt{2}a_{-n',n} \phi_0(z)} \] (3.15)

with positive integers \( n' \) and \( n \). In the generic sector with nonzero \( h_{J,m} \), the singular fields are obtained by

\[ \chi_{J;-n',n}(z) = \chi_{-n',n}(z) \phi_{J,m_{n',n}}(z) \] (3.16)

with \( J = n' - n \mod 2k \). They exist at the levels

\[ h_{J,m,n',n} - h_{J,m,n',n} = \frac{4n'n + m_{n',n}^2 - m_{-n',n}^2}{4k}. \] (3.17)

From (3.15) and (3.16), we find the following chain of the singular vectors in \( H^{PF}_{J,m,n',n} \otimes F_{n',n}^{\phi_0} \).

\[ \Psi_{J,m,n',n} \]

\[ \vdash \begin{array}{c} \chi_{J,n',-n} \to \chi_{J,n',n+2p} \to \chi_{J,n',-n-2p} \to \chi_{J,n',n+4p} \to \cdots \\ \chi_{J,n',-n+2p} \to \chi_{J,n',n-2p} \to \chi_{J,n',-n+4p} \to \chi_{J,n',n+4p} \to \cdots \end{array} \] (3.18)

where an arrow or a chain of arrows from one vector to another means that the second vector is in the submodule generated by the first vector.
From (3.15) and (3.16), the primary state \( |J, m' n; n', n> \) belongs to the kernel of \( Q_n^+ \). In addition, all the singular vectors in the lower sequence in (3.18) belong to the kernel of \( Q_n^+ \), because these states are the images of \( Q_{p-n}^+ \) due to

\[
\begin{align*}
\chi J; n', n+2ip & \sim Q_{p-n}^+ \Psi J, m_{n'-2ip'}, p-n ; -n' -2ip', p-n, \\
\chi J; n', -n+2ip & \sim Q_{p-n}^+ \Psi J, m_{n'-2ip'}, p-n ; -n' -2ip', p-n,
\end{align*}
\]

(3.19)

for \( i \in \mathbb{N} \). We therefore claim that the IHWR \( \mathcal{H}_{J,m, n'} G/H \) with highest weight \( h_{J,m, n'} \) in the coset theory is given by

\[
\text{Ker} Q_n^+/\text{Im} Q_{p-n}^+ = \mathcal{H}_{J,m, n'} G/H.
\]

(3.20)

For the singular vectors in the Fock spaces \( \mathcal{H}_{J,m, n'} G/H \otimes F_{[t]} F \) with \( t \neq 0 \), we find the same chains as (3.18). For \( t < 0 \), the relations in (3.19) as well as the relations

\[
\begin{align*}
\chi J; n', n+2ip & \sim Q_n^+ \Psi J, m_{n'-2ip'}, n' ; -n' -2ip', n, \\
\chi J; n', -n+2ip & \sim Q_n^+ \Psi J, m_{n'-2ip'}, n' ; -n' -2ip', n,
\end{align*}
\]

(3.21)

for \( i \in \mathbb{N} \) give rise to the following trivial cohomologies:

\[
\text{Ker} Q_n^+[t]/\text{Im} Q_{n}^+[t-1] = 0,
\]

(3.22)

where \( Q_n^+[2t] = Q_n^+ \) and \( Q_n^+[2t+1] = Q_{p-n}^+ \). Furthermore, for \( t > 0 \), we have

\[
\begin{align*}
\Psi J, m_{n'-2ip'}, n' ; -n-2ip & \sim Q_n^+ \chi J; n', -n-2ip, \quad \text{for} \quad i \geq 0, \\
\Psi J, m_{n'-2ip'}, n'; -n-2ip & \sim Q_n^+ \chi J; n', n-2(i-1)p', \quad \text{for} \quad i \geq 1.
\end{align*}
\]

(3.23)

These relations are derived as follows. In the Fock space \( F_{-J-2, -m-n', -n+2ip; -n' n+2ip} \) dual to \( F_{J,m_n', -n', n' -2ip} \), there exists a singular vector

\[
\chi -J-2; -n' +2ip \sim Q_n^+ \Psi -J-2, -m-n', n+2ip
\]

(3.24)
at the level $h_{-J-2,-m_{-n'},-n';n+2ip;} - h_{-J-2,-m_{-n'},-n'+2ip;} - n',-n+2ip$. Therefore there must be a vector, say $w_0$, in $F_{J,m_{n'},n-2ip;} - n',-n-2ip; \Psi_{J,m_{n'},n-2ip;} - n',-n-2ip$. Since $(Q_n^+)^t = Q_n^+$, this indicates the existence of a vector $Q_n^+w_0 (\neq 0)$ in $F_{J,m_{n'},n-2ip;} - n',-n-2ip$. This vector is a level zero vector in $F_{J,m_{n'},n-2ip;} - n',-n-2ip$. In addition, since the vector $w_0$ is a level $h_{J,m_{n'},n-2ip;} - n',-n-2ip$ vector in $F_{J,m_{n'},n-2ip;} - n',-n-2ip$, $w_0$ is identified with $\chi_{J,n';-n-2ip}$. Hence we obtain the first relation in (3.23). The similar argument implies the second relation. Combining (3.19), (3.21) and (3.23), we claim that the cohomology groups for $t > 0$ are trivial, too.

We thus conclude that, for the whole complex in (3.13), the BRST cohomology groups are given by

$$Ker Q_n^{[t]} / Im Q_n^{[t-1]} = \begin{cases} 0 & \text{for } t \neq 0 \\ \mathcal{H}_{J,m_{n'},n'} & \text{for } t = 0 \end{cases} \quad (3.26)$$

As a corollary, we get a trace formula, which relates the trace over $\mathcal{H}_{J,m_{n'},n'}^{G/H}$ to those over the Fock spaces $\mathcal{H}_{J,m_{n'},n'}^{PF[t]} \otimes F_{n',n}^{\phi_0[t]}$, as follows:

$$\text{Tr}_{\mathcal{H}_{J,m_{n'},n'}^{G/H}} \mathcal{O} = \sum_{t \in \mathbb{Z}} (-)^t \text{Tr}_{\mathcal{H}_{J,m_{n'},n'}^{PF[t]} \otimes F_{n',n}^{\phi_0[t]}} \mathcal{O}^{[t]}, \quad (3.27)$$

where $\mathcal{O}^{[t]}$ are defined recursively by

$$Q_n^{[t]} \mathcal{O}^{[t]} = \mathcal{O}^{[t+1]} Q_n^{[t]} \quad (3.28)$$

with $\mathcal{O}^{[0]} = \mathcal{O}$.
Combining (3.12) and (3.27), we finally get

\[
\text{Tr}_{\mathcal{H}^{G/H}} O = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \sum_{u \geq 0} (-)^{s+t+u} \text{Tr}_{F_{J,m,n';n,n}^{[s,t,u]} \otimes F_{n,n}^{e[0]}} O^{[s,t]} \bigg|_{\alpha_{\varphi,0} + \alpha_{\chi,0} = 0}.
\]

Here the Fock space \( F_{J,m,n';n,n}^{[s,t,u]} \) is defined by the relations

\[
\begin{align*}
F_{J,m,n';n,n}^{[2s,2t,u]} &= F_{J,m,n';n,n}^{[u]} \\
F_{J,m,n';n,n}^{[2s+1,2t,u]} &= F_{J,m,n';n,n}^{[u]} - 2 pt \\
F_{J,m,n';n,n}^{[2s,2t+1,u]} &= F_{J,m,n';n,n}^{[u]} - 2 pt.
\end{align*}
\]

We also denote an arbitrary physical operator, which acts on the Fock space \( F_{J,m,n';n,n}^{[s,t,u]} \), by \( O^{[s,t]} \). Its defining relations are

\[
\begin{align*}
O^{[s,0]} &= O^{[s]}, \\
Q_{n}^{[t]} O^{[s,t]} &= O^{[s,t+1]} Q_{n}^{[t]}.
\end{align*}
\]

4. Three String Vertex

In this section, we discuss the screened vertex operator and its BRST properties. We next extend it to the screened three string vertex. This vertex allows us to extend the BRST structure discussed in the previous section to the one on higher genus Riemann surfaces by the sewing operation.

The screened vertex operator was first introduced by Felder \cite{15} in the BPZ minimal model. Its counterpart in the coset theory is given as follows

\[
\psi^{(r,r_+,r_-)}_{J,m,n'}(z) = \oint_{c} \prod_{i} du_{i} \oint_{c_+} \prod_{j} dv_{j} \oint_{c_-} \prod_{k} dw_{k} \psi_{J,m,n}(z) \prod_{i} S(u_{i}) \prod_{j} S_{+}(v_{j}) \prod_{k} S_{-}(w_{k}),
\]

where the integration contours are taken in the same way as in Ref.\cite{15}.

17
An important property of the screened vertex (4.1) is the following BRST invariance:

\[
Q_{J+1} \Psi^{(r,r_+,r_-)}_{A;m',m}(z) = e^{i\pi(J+1)(-\frac{A}{e+2}-1)}\Psi^{(r,r_+,r_-)}_{A;m',m}(z)Q_{J+1},
\]
\[
Q_{k+1-J} \Psi^{(r,r_+,r_-)}_{A;m',m}(z) = e^{i\pi(k+1-J)(-\frac{A}{e+2}-1)}\Psi^{(r,r_+,r_-)}_{A;m',m}(z)Q_{k+1-J},
\]
\[
Q^n \Psi^{(r,r_+,r_-)}_{A;m',m}(z) = e^{i\pi n(-\frac{r-d}{e+2}-1+2\alpha_0\alpha_{m',m})}\Psi^{(r,r_+,r_-)}_{A;m',m}(z)Q^n,
\]
\[
Q^p_{p-n} \Psi^{(r,r_+,r_-)}_{A;m',m}(z) = e^{i\pi(p-n)(-\frac{r-d}{e+2}+2\alpha_0\alpha_{m',m})}\Psi^{(r,r_+,r_-)}_{A;m',m}(z)Q^p_{p-n}.
\]

Let us extend this screened vertex to a screened three string vertex. There are several realizations of three string vertex. We here take the CSV realization. The CSV vertex for the fields $\varphi, \chi$ and $\phi_0$ are given by

\[
<V_{123}| = <V_{123}^\varphi <V_{123}^\chi <V_{123}^\phi_0 >.
\]
\[
<V_{123}| = \left[ \prod_{r=1,2,3} <\varphi_X| \right] \exp(-\epsilon_X) \left\{ (\alpha_X^{(1)}|\alpha_X^{(2)} + (\alpha_X^{(2)}|\alpha_X^{(3)} + (\alpha_X^{(3)}|\alpha_X^{(1)})
\right.
\]
\[
+ \frac{1}{2} \sum_{r,s=1}^{3} (\alpha_X^{(r)}|D(U_r V_s)|\alpha_X^{(s)}) \right\} \delta \left( \sum_{r=1}^{3} \alpha_X^{(r)} + Q_X \right),
\]

where $\langle \alpha|\alpha_0 \rangle = \sum_{n=1}^{\infty} \alpha_n \alpha_0$, $\langle \alpha|D|\alpha \rangle = \sum_{n,m=1}^{\infty} \alpha_n D_{nm} \alpha_m$ and $<\varphi_X| = \sum_{n_X^r < n_X^r;0}|$ with $<\varphi_X^r;0| satisfying

\[
<V_{123}| = <V_{123}^\varphi <V_{123}^\chi <V_{123}^\phi_0 >.
\]
\[
<V_{123}| = \left[ \prod_{r=1,2,3} <\varphi_X| \right] \exp(-\epsilon_X) \left\{ (\alpha_X^{(1)}|\alpha_X^{(2)} + (\alpha_X^{(2)}|\alpha_X^{(3)} + (\alpha_X^{(3)}|\alpha_X^{(1)})
\right.
\]
\[
+ \frac{1}{2} \sum_{r,s=1}^{3} (\alpha_X^{(r)}|D(U_r V_s)|\alpha_X^{(s)}) \right\} \delta \left( \sum_{r=1}^{3} \alpha_X^{(r)} + Q_X \right),
\]

where $\langle \alpha|\alpha_0 \rangle = \sum_{n=1}^{\infty} \alpha_n \alpha_0$, $\langle \alpha|D|\alpha \rangle = \sum_{n,m=1}^{\infty} \alpha_n D_{nm} \alpha_m$ and $<\varphi_X| = \sum_{n_X^r < n_X^r;0}|$ with $<\varphi_X^r;0| satisfying

\[
<V_{123}| = <V_{123}^\varphi <V_{123}^\chi <V_{123}^\phi_0 >.
\]
\[
<V_{123}| = \left[ \prod_{r=1,2,3} <\varphi_X| \right] \exp(-\epsilon_X) \left\{ (\alpha_X^{(1)}|\alpha_X^{(2)} + (\alpha_X^{(2)}|\alpha_X^{(3)} + (\alpha_X^{(3)}|\alpha_X^{(1)})
\right.
\]
\[
+ \frac{1}{2} \sum_{r,s=1}^{3} (\alpha_X^{(r)}|D(U_r V_s)|\alpha_X^{(s)}) \right\} \delta \left( \sum_{r=1}^{3} \alpha_X^{(r)} + Q_X \right),
\]

for $X = \varphi, \chi$ and $\phi_0$. The anomalous charge $Q_X$ takes values $\sqrt{\frac{k}{k+d+2}}, -1$ and $-2\sqrt{2}\alpha_0$ for $X = \varphi, \chi$ and $\phi_0$, respectively. The coefficients $D_{nm}$ are the $(0,0)$ representation matrix elements of the projective group.

By using the CSV vertex as a bare vertex, we define the screened three string vertex as follows.

\[
<V_{123}^{(r,r_+,r_-)} > = \oint_{\mathcal{C}} \prod_{i=1}^{r} dv_i \oint_{\mathcal{C}_+} \prod_{j=1}^{r_+} dv_j \oint_{\mathcal{C}_-} \prod_{k=1}^{r_-} dv_k <V_{123}||R_{3^3} >
\]
\[
\times \prod_r S^{(1)}(u_i) \prod_j S^{(1)}(v_j) \prod_k S^{(1)}(w_k), \quad (4.8)
\]

where \(|R_{3:3} >\) is the reflection operator.\(^{[21]}\) In the expression (4.8), the integration contours \(C, C_+\) and \(C_-\) should be taken as the \(r, r_+\) and \(r_-\)-homology cycles defined on the three punctured sphere. These cycles can be derived by following the argument given by Felder and Silvotti.\(^{[19]}\) Their expressions are given in Appendix A.

The delta function constraints in (4.8) give rise to the following constraints on the (eigenvalues of) momentum operators.

\[
\begin{align*}
\alpha^{(3)}_{\phi,0} &= \alpha^{(1)}_{\phi,0} + \alpha^{(2)}_{\phi,0} + \alpha_+ r_+ + \alpha_- r_-,
\alpha^{(3)}_{\varphi,0} &= \alpha^{(1)}_{\varphi,0} + \alpha^{(2)}_{\varphi,0} + \sqrt{\frac{k}{k+2}} r + \sqrt{\frac{k+2}{k}} (r_+ - r_-), \\
\alpha^{(3)}_{\chi,0} &= \alpha^{(1)}_{\chi,0} + \alpha^{(2)}_{\chi,0} + r + r_+ - r_-.
\end{align*}
\quad (4.9)
\]

Taking account of these constraints and the overlapping condition for the bare vertices,\(^{[21]}\) one can derive the BRST relations for the screened three string vertex.

\[
\begin{align*}
Q^{(3)}_{J+1} < V_{123}^{(r,r_+,r_-)} > &= < V_{123}^{(r,r_+,r_-)} > Q^{(1)}_{J+1} e^{-\pi i(J+1)(\sqrt{\frac{k}{k+2}}\alpha^{(2)}_{\varphi,0} - \alpha^{(2)}_{\chi,0})} \\
&\quad + e^{\pi(A-J)(A+1)/k+2} < V_{123}^{(r_+-A,r_+,r_-)} > Q^{(2)}_{A+1},
\end{align*}
\quad (4.10)
\]

\[
\begin{align*}
Q^{(3)}_{n} < V_{123}^{(r,r_+,r_-)} > &= < V_{123}^{(r,r_+,r_-)} > Q^{(1)}_{n} e^{-\pi in(\sqrt{\frac{k+2}{k}}\alpha^{(2)}_{\varphi,0} - \alpha^{(2)}_{\chi,0} - \sqrt{2}\alpha^{(2)}_{\phi,0})} \\
&\quad + e^{2\pi(n-m)(n-1)/p} < V_{123}^{(r_+ + n-m,r_-)} > Q^{(2)}_{m},
\end{align*}
\quad (4.11)
\]

where the momentum operator \(\alpha^{(3)}_{\varphi,0}, \alpha^{(3)}_{\chi,0}\) and \(\alpha^{(3)}_{\phi,0}\) are replaced with their eigenvalues (see Appendix B). When the second Fock space in (4.10) and (4.11) is saturated with the highest weight state \(|A; m', m >\), (4.10) and (4.11) coincide with the BRST relations for the screened vertex operator (4.2) and (4.4).
5. Screened Multi-Loop Operators

In this section, we derive the screened $g$-loop operator in the coset theory.

Let us first consider the screened one loop operator. From the BRST relations (4.10) , (4.11) and the trace formula (3.29) , the screened one-loop operator is defined by

\[ < T_{J,m,n',n}^{1(r^+, r^-)} > \frac{\Omega^{(1)}_{123} q^{L_0} - \frac{\pi i}{12} \Omega^{(1)}}{H_{J,m,n,n',n}} \]

where \( q^{2\pi i} \), \( \Omega = e^{L-1(-)L_0} \) and \( \text{Tr}^{13} \) denotes the trace over the first and the third Fock spaces.\[^{[21]}\]

Carrying out the trace \( \text{Tr}^{13} \) explicitly, we obtain the expressions for the screened one-loop operator as follows.

\[ < T_{J,m,n',n}^{1(r^+, r^-)} > | = \oint_{c} \prod_{i=1}^{r} du_i \oint_{c_+} \prod_{j=1}^{r^+} dv_j \oint_{c_-} \prod_{k=1}^{r_-} dw_k < T_{J,m,n',n}^{(1)} > | S(u_i) \prod_{j=1}^{r^+} S_+(v_j) \prod_{k=1}^{r_-} S_-(w_k) \]

\[ \times \delta(\alpha_{\phi_0,0} + \sqrt{2} \alpha_+ r_+ + \sqrt{2} \alpha_- r_-) \]

\[ \times \delta(\alpha_{\varphi,0} + \frac{k}{k+2} r_+ + \frac{k+2}{k} (r_+ - r_-)) \]

\[ \times \delta(\alpha_{\chi,0} + r_+ + r_-), \]

where

\[ < T_{J,m',n}^{(1)} > | = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \sum_{u \geq 0} (-)^{s+t+u} < T_{J}^{(1) \varphi[s,t,u]} > \]

\[ < T_{J,m,n'}^{(1) \varphi[s,t,u]} > \]

\[ = \eta(\tau)^{-1} q^{B_{\chi}[s,t,u]} < 0_{\chi} | \exp \{ 2\pi i [ p_{\varphi}^{[s,t]}(\alpha_{\varphi}) + p_{\chi}^{[s,t,u]}(\alpha_{\chi}) ] | A \}
\]

\[ + \left[ Q_{\varphi}(\alpha_{\varphi}) + Q_{\chi}(\alpha_{\chi}) \right] | C \} + \frac{1}{2} \left[ (\alpha_{\chi} | Q | \alpha_{\chi}) - (\alpha_{\varphi} | Q | \alpha_{\varphi}) \right] \]

\[ < T_{m',n}^{(1) \phi_0[t]} > \]

\[ = \eta(\tau)^{-1} q^{B_{\phi_0}[t]} < 0_{\phi_0} | \exp \{ 2\pi i p_{\phi_0}^{[t]}(\alpha_{\phi_0}) | A \} + Q_{\phi_0}(\alpha_{\phi_0} | C \} + \frac{1}{2} (\alpha_{\phi_0} | Q | \alpha_{\phi_0}) \}
\]

with \( \theta^{[2s,2t]} = 0 \), \( \theta^{[2s+1,2t+1]} = \theta^{[2s+1,2t]} + \theta^{[2s,2t+1]} \) and

\[ \theta^{[2s+1,2t]} = (J + 1) \left( \sqrt{\frac{k}{k+2} \alpha_{\varphi,0} - \alpha_{\chi,0}} \right), \]

\[ (5.6) \]
\[ \theta^{2s,2t+1} = n \left( \sqrt{\frac{k+2}{k}} \alpha_{s,0} - \alpha_{s,0} - \sqrt{2} \alpha_0 \right), \]  
(5.7)

\[ B^{[s,t,u]}_{\chi} = (k+2) \left( \left[ \frac{s}{2} \right] - \frac{u}{2} - \frac{J^{[s]}}{2(k+2)} + \frac{1}{2} \right)^2 - k \left( \frac{m^{[t]}}{2k} + \frac{u}{2} \right)^2, \]  
(5.8)

\[ B^{[t]}_{\phi_0} = \frac{1}{4kpp'} \left( n'p - (-)^{s} np' + 2pp' \left[ \frac{t}{2} \right] \right)^2, \]  
(5.9)

\[ p^{[s,t]}_{\phi} = -\frac{1}{2} \sqrt{k+2} \left( \left[ \frac{k+2}{k} \right] - \frac{m^{[t]}}{2} - J^{[s]} + 2(k+2) \left[ \frac{s}{2} \right] \right), \]  
(5.10)

\[ p^{[s,t,u]}_{\phi} = -\frac{1}{2} \left( \frac{J^{[s]}}{2} - 2(k+2) \left[ \frac{s}{2} \right] - \frac{m^{[t]}}{2} + 2u \right), \]  
(5.11)

\[ p^{[t]}_{\phi_0} = \sqrt{2} \alpha_{n',(-)^{s}n-2p',\frac{t}{2}}. \]  
(5.12)

We also use the notations \( J^{[2s]} = J, J^{[2s+1]} = -J - 1 \) and \( m^{[t]} = m_{n',(-)^{s}n-2p',\frac{t}{2}} \).

The coefficients \( A, C \) and \( Q \) in (5.4) and (5.5) are given by

\[ A_n = \frac{1}{n!} \frac{\partial^n}{\partial_x^n} \int d\omega \bigg|_{x=0}, \]  
(5.13)

\[ C_n = \frac{1}{n!} \frac{\partial^n}{\partial_x^n} \ln \sigma(x) \bigg|_{x=0}, \]  
(5.14)

\[ Q_{nm} = \frac{1}{n!m!} \frac{\partial^n}{\partial_x^n} \frac{\partial^m}{\partial_y^m} \ln \frac{E(x,y)}{x-y} \bigg|_{x=y=0}, \]  
(5.15)

where \( d\omega, \sigma(x) \) and \( E(z,w) \) are the first Abelian differential, the \( \frac{g}{2} \)-differential and the prime form on the genus \( g \) Riemann surface ( \( g=1 \) for the above case ), respectively. The expressions for these functions in terms of the Schottky variables are given in Ref.[18].

From (4.10) and (4.11), the BRST relation for the screened loop operator is given by

\[ < \mathcal{T}_{J,n',n}^{-1(r,r+1)} | Q^+_{\tilde{J}} = 0, \]  
(5.16)

\[ < \mathcal{T}_{J,n',n}^{-1(r,r+1)} | Q^-_{\tilde{n}} = 0 \]  
(5.17)

with \( \tilde{J} = r \) and \( \tilde{n} = 2r+1 \). These relations indicate the decoupling of the null states from the screened loop operator.
The $g$ loop extension is obtained by sewing the $g$ screened one loop operators (5.2) by the $g$-1 screened three string vertices (4.8). The result is

\[
< T_{[J; N', N]}^g(r, r_+, r_-) > = \prod_{a=1}^{g} \left[ \prod_{i=1}^{r_a^a} du_i^a \prod_{j=1}^{r_b^a} dv_j^a \prod_{k=1}^{r_c^a} dw_k^a \right] \\
\times < T_{[J; N', N]}^g(s_1, s_2, \ldots, s_{g-1}) > \prod_{i=1}^{r_+^a} S(u_i^a) \prod_{k=1}^{r_-^a} S_-(w_k^a) \\
\times \delta(\alpha_{\phi_0, 0} + \sqrt{2\alpha_+} + \sqrt{2\alpha_-} + 2\sqrt{2\alpha_0}(g - 1)) \\
\times \delta(\alpha_{\phi_0, 0} + \sqrt{\frac{k}{k+2}} + \sqrt{\frac{k+2}{k}}(r_+ - r_-) + \sqrt{\frac{k}{k+2}}(g - 1)) \\
\times \delta(\alpha_{\chi, 0} + r_+ - r_-),
\]

(5.18)

where \( r = \sum_{a=1}^{g} r_a, \quad r_+ = \sum_{a=1}^{g} r_+^a, \quad r_- = \sum_{a=1}^{g} r_-^a \) and

\[
< T_{[J; N', N]}^g(s_1, s_2, \ldots, s_{g-1}) > = \prod_{a=1}^{g} \left[ \sum_{s_a \in \mathbb{Z}} \sum_{t_a \in \mathbb{Z}} \sum_{u_a \geq 0} (-1)^{s_a + t_a + u_a} \right] < T_{[J]}^g(\varphi)[S, T, U] < T_{[N', N]}^g(T) \prod_{a=1}^{g} e^{-i\pi \theta s_a, t_a} \]

\[
< T_{[J]}^g(\varphi)[S, T, U] = F[G(g)]^{-2} < \varphi_{\chi, \chi} > \exp \left\{ \sum_{X=\varphi, \chi} \left[ \pi i \epsilon_X \sum_{a,b=1}^{g} P_X^{s_a, t_a, u_a} \tau_{ab} P_X^{s_b, t_b, u_b} \right. \\
+ Q_X(\alpha_X | C) + \epsilon_X \frac{1}{2}(\alpha_X | Q | \alpha_X) \\
- 2\pi i \sum_{a=1}^{g} P_X^{s_a, t_a, u_a} \left[ (\alpha_{\phi} | A^a) + Q_X(\Delta(g) a + \frac{1}{2}) \right] \right\},
\]

(5.19)

\[
< T_{[J]}^g(\phi_0) = F[G(g)]^{-1} < \phi_0 > \exp \left\{ \pi i \sum_{a,b=1}^{g} P_{\phi_0}^{t_a, t_b} \tau_{ab} P_{\phi_0}^{t_b, t_a} + Q_{\phi_0}(\alpha_{\phi_0} | C) + \frac{1}{2}(\alpha_{\phi_0} | Q | \alpha_{\phi_0}) \\
- 2\pi i \sum_{a=1}^{g} P_{\phi_0}^{t_a, t_a} \left[ (\alpha_{\phi_0} | A^a) + Q_{\phi_0}(\Delta(g) a + \frac{1}{2}) \right] \right\}
\]

(5.20)

with \( \theta^{[2s_a, 2t_a]} = 0, \theta^{[2s_a+1, 2t_a+1]} = \theta^{[2s_a+1, 2t_a]} + \theta^{[2s_a, 2t_a+1]} \) and

\[
\theta^{[2s_a+1, 2t_a]} = \frac{2(J + 1)r_a}{k + 2}, \quad \theta^{[2s_a, 2t_a+1]} = \frac{2nr_+}{p}.
\]

(5.21)
In the above, $\mathcal{J} = \{ J_a \}$, $N^{()} = \{ n_a^{()} \}$, $S = \{ s_a \}$, $T = \{ t_a \}$ and $U = \{ u_a \}$, $a = 1, 2, \ldots, g$. We also denote by $\tau$ the period matrix, by $\triangle$ the Riemann constant and by $F[G(g)]$ the partition function associated with the Schottky group $G(g)$. $F[G(g)]$ is related to the scalar determinant as follows:

$$F[G(g)] = (\det \tilde{\theta}_0)^{1/2}.$$  \hfill (5.22)

The coefficients $A, C$ and $Q$ are given by (5.13) $\sim$ (5.14) with using the functions $\varphi, \sigma$ and $E$ defined on the genus $g$ Riemann surface.

Taking account of the BRST invariance of the screened CSV vertex (4.10) and (4.11) as well as those for the screened one loop operator (5.16) and (5.17), one can show the BRST invariance relations for the screened $g$-loop operator analogous to (5.16) and (5.17).

6. Conformal Blocks on Higher Genus

The loop operator maps the fields on the sphere to the one on the corresponding higher genus Riemann surface. This allows one to calculate any higher genus conformal block functions.

In general, a genus $g$ conformal block function can be evaluated by

$$< \prod_{i=1}^{L} \Psi_{J_i;n_i',n_i}(z_i) >_{g,[\mathcal{J},N',N]} = \langle \mathcal{T}_{[\mathcal{J},N',N]}^{g(r_+,r_-)} \prod_{i=1}^{N} \Psi_{J_i;n_i',n_i}(z_i) | 0 >,$$  \hfill (6.1)

up to an insertion of certain identity operators (see later paragraphs). The delta function constraints on the screened loop operator give rise to the following selection rules for this conformal block function.

$$\sum_{i=1}^{L} \alpha_{n_i',n_i} + \alpha_+ r_+ + \alpha_- r_- + 2\alpha_0 (g - 1) = 0,$$
\[
\sum_{i=1}^{L} \frac{J_i}{\sqrt{k(k+2)}} + \sqrt{\frac{k}{k+2}} r + \sqrt{\frac{k+2}{k}} (r_+ - r_) + \sqrt{\frac{k}{k+2}} (g-1) = 0, \quad (6.2)
\]
\[
r + r_+ - r_- = 0.
\]

These relations determine \( r, r_+ \) and \( r_- \).

Let us consider the \( g=1 \) zero point function, i.e. the character of the Virasoro algebra in the coset theory. By making use of (5.1), we obtain

\[
\chi_{J,n',n}(\tau) = \langle T_{g(0,0,0)}^{(0,0,0)} || 0 >
= \eta(\tau)^{-3} \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \sum_{u \geq 0} (-)^{s+t+u} q^{B_{s,t,u}^{[2]} + B_{s,t,u}^{[0]}}
= \sum_{m \in \mathbb{Z}} C_{J,m}(\tau) \left( \sum_{t \in \mathbb{Z}} \delta_{m,m_{n',n-2pt}} q^{B_{s,t,u}^{[2]}} - \sum_{t \in \mathbb{Z}} \delta_{m,m_{n',n-2pt}} q^{B_{s,t,u}^{[2]+1}} \right). \quad (6.3)
\]

where \( C_{J,m} \) is the string function given by

\[
C_{J,m}(\tau) = \sum_{s \in \mathbb{Z}} \sum_{u \geq 0} (-)^{u} \left( q^{B_{s,t,u}^{[2]}} - q^{B_{s,t,u}^{[2]+1}} \right). \quad (6.4)
\]

The result (6.3) is nothing but the branching coefficient.\(^{[2,16]}\)

Let us next extend this result to higher genus. Let us call a genus \( g \) vacuum (zero point) amplitude as a genus \( g \) character. The selection rules for the genus \( g \) character are given by

\[
\alpha_+ r_+ + \alpha_- r_- + 2\alpha_0 (g-1) = 0,
\]
\[
\sqrt{\frac{k}{k+2}} r + \sqrt{\frac{k+2}{k}} (r_+ - r_) + \sqrt{\frac{k}{k+2}} (g-1) = 0, \quad (6.5)
\]
\[
r + r_+ - r_- = 0.
\]

Solving these equations, one gets

\[
r = \frac{k}{2} (g-1), \quad r_+ = \frac{l}{2} (g-1), \quad r_- = \frac{k+l}{2} (g-1). \quad (6.6)
\]

Since the total number of screening charges must be an integer, these conditions are satisfied only for \( g \) being an odd integer. The same phenomenon has been
discussed in the BPZ minimal model\textsuperscript{[19]} as well as in the \( SU(2)_k \) WZW model\textsuperscript{[21,29]}. For the even genus character, one has to insert the identity operator

\[
I = e^{-\sqrt{\frac{k}{k+2}} \varphi} e^{2\sqrt{2} \alpha_0 \phi_0}.
\] (6.7)

This shifts the \( \phi_0 \) and the \( \varphi \) charge so that one gets the new conditions

\[
r = \frac{k}{2} g, \quad r_+ = \frac{l}{2} g, \quad r_- = \frac{k+l}{2} g.
\] (6.8)

Under these conditions, one obtains the expressions for the genus \( g \) characters as follows.

\[
\chi^{(g)}_{[J;N,N']} ([J,N,N'] \Omega | Q) = \left< T^g (r,r_+ r_-) | I(Q) | 0 \right>
\]

\[
\times \mathcal{E}^g (u,v,w) \sum_{K=K_+ \cup K_-} \prod_{I} \frac{\partial}{\partial \rho_I} \prod_{J} \frac{\partial}{\partial \lambda_J} \prod_{K_+} \frac{\partial}{\partial \mu_{K_+}} \prod_{K_-} \frac{\partial}{\partial \nu_{K_-}} I^g (\rho, \lambda, \mu, \nu) \bigg| 0 \bigg>.
\] (6.9)

where \( \{u_I\}_{I=1,2,...r} = \{u_1, u_2, u_3, ..., u_{r-1}, u_r\} \), \( \{v_J\}_{J=1,2,...r_+} = \{v_1, v_2, v_3, ..., v_{r_+}\} \), and \( \{w_K\}_{K=1,2,...r_-} = \{w_1, w_2, w_3, ..., w_{r_-}\} \). In (6.9), the sum \( \sum_{K=K_+ \cup K_-} \) should be taken over all the ways of decomposition of the index set \( K \) into two disjoint sets \( K_+ \) and \( K_- \), and \( I(Q) \) should be inserted only in the even genus case.

We also use the notation

\[
\int d\Omega = \prod_{a=1,...,g} \left[ \phi \prod_{i=1}^{r_1^a} du_i^a \int_{c_{+}^a} \prod_{j=1}^{r_2^a} dv_j^a \int_{c_{-}^a} \prod_{k=1}^{r_3^a} dw_k^a \right].
\] (6.10)

For odd genus, we obtain

\[
\Delta^{(g;\text{odd})} ([J,N,N';S,T,U] [\tau])
\]
\[\exp\left\{ i\pi \sum_{a,b=1}^{g} \left[p_{\phi_0}^{[t_a]} \tau_{ab} p_{\phi_0}^{[t_b]} + p_{\chi}^{[s_a,t_a,u_a]} \tau_{ab} p_{\chi}^{[s_b,t_b,u_b]} - p_{\varphi}^{[s_a,t_a]} \tau_{ab} p_{\varphi}^{[s_b,t_b]} \right]\right.\]
\[+ 2\pi i \sum_{a=1}^{g} \left[ p_{\chi}^{[s_a,t_a,u_a]} + \sqrt{\frac{k+2}{k}} p_{\varphi}^{[s_a,t_a]} \right] \sum_{I}^{u_I} \int_{P_0} d\omega^a\]
\[- \left( \sqrt{2}\alpha_+ p_{\phi_0}^{[t_a]} + p_{\chi}^{[s_a,t_a,u_a]} + \sqrt{\frac{k+2}{k}} p_{\varphi}^{[s_a,t_a]} \right) \sum_{J}^{v_J} \int_{P_0} d\omega^a\]
\[+ \left( \sqrt{2}\alpha_- p_{\phi_0}^{[t_a]} - p_{\chi}^{[s_a,t_a,u_a]} - \sqrt{\frac{k+2}{k}} p_{\varphi}^{[s_a,t_a]} \right) \sum_{K}^{w_K} \int_{P_0} d\omega^a\]
\[+ \left( -2\sqrt{2}\alpha_0 p_{\phi_0}^{[t_a]} + p_{\chi}^{[s_a,t_a,u_a]} - \sqrt{\frac{k}{k+2}} p_{\varphi}^{[s_a,t_a]} \right) \int_{(g-1)P_0} d\omega^a\}\]

\[E(g;\text{odd})(u, v, w) = \left[ \prod_{I,J} E(u_I, v_J) \prod_{I,I'} E(u_I, u_{I'}) \right]^{\frac{1}{2}} \prod_{J,J'} \sigma(v_J)^{-\frac{1}{2}} \prod_{K<K'} E(w_K, w_{K'})^{\frac{\pi}{2} - \frac{1}{2}} \quad \tag{6.12}\]

\[I(g;\text{odd})(\rho, \lambda, \mu, \nu) = \exp\left\{ 2\pi i \sum_{a=1}^{g} \left[ \sqrt{\frac{k+2}{k}} p_{\varphi}^{[s_a,t_a]} \sum_{K_+} \mu_{K_+} \omega^a(w_{K_+}) \right.\right.\]
\[- \left. p_{\chi}^{[s_a,t_a,u_a]} \sum_{I} \rho_I \omega^a(u_I) + \sum_{J} \lambda_J \omega^a(v_J) - \sum_{K_-} \nu_{K_-} \omega^a(w_{K_-}) \right]\}
\times \prod_{I} e^{-\rho_I \partial \ln \sigma(u_I)} \prod_{J} e^{-\lambda_J \partial \ln \sigma(v_J)} \prod_{K_+} e^{-\mu_{K_+} \partial \ln \sigma(w_{K_+})} \prod_{K_-} e^{-\nu_{K_-} \partial \ln \sigma(w_{K_-})}\]
\times \prod_{I,K_+} e^{(k+2)\mu_{K_+} \partial w_{K_+} - \rho_I \partial w_{K_+})} \ln E(u_I, w_{K_+}) \prod_{J,K_+} e^{(k+2)\mu_{K_+} \partial w_{K_+} - \lambda_J \partial w_{K_+})} \ln E(v_J, w_{K_+})\]
\times \prod_{K_-,K_+} e^{-(k+2)\mu_{K_+} \partial w_{K_+} + \nu_{K_-} \partial w_{K_-})} \ln E(w_{K_-}, w_{K_+})\]

26
\[ \prod_{I,J} e^{D(p_I,p_J;u_I,u_J)} \ln E(u_I,u_J) \prod_{J<K'} e^{D(\lambda_J,\lambda_{J'};v_J,v_{J'})} \ln E(v_J,v_{J'}) \prod_{I,J} e^{D(p_I,\lambda_{J'};u_I,v_J)} \ln E(u_I,v_J) \]

\[ \times \prod_{I,K} e^{D(p_I,\nu_{K};u_I,w_{K})} \ln E(u_I,w_{K}) \prod_{J,K} e^{D(\lambda_J,\nu_{K};w_J,w_{K})} \ln E(w_J,w_{K}) \]

\[ \times \prod_{K_-<K'_-} e^{D(\nu_{K_-},\nu_{K'_-};w_{K_-},w_{K'_-})} \ln E(w_{K_-},w_{K'_-}) \prod_{K_+<K'_+} e^{-\frac{1}{k} 2D(\mu_{K_+},\mu_{K'_+};w_{K_+},w_{K'_+})} \ln E(w_{K_+},w_{K'_+}) \]

where

\[ D(\mu, \nu; x, y) = \mu \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial y} + \mu \nu \frac{\partial}{\partial x} \frac{\partial}{\partial y}. \]  

For even genus, we obtain

\[ \Delta^{(g;\text{even})}_{[J,N',N';S,T,U]}(\tau) = \Delta^{(g;\text{odd})}_{[J,N',N';S,T,U]}(\tau) \]

\[ \times \exp \left\{ -2\pi i \sum_{a=1}^{g} \left( 2\sqrt{2} \alpha_0 p_{[t_a]} + \sqrt{\frac{k}{k+2}} p_{[s_a,t_a]} \right) \int_{P_0}^Q d\tau \right\}. \]

\[ \mathcal{E}^{(g;\text{even})}(u,v,w) = \mathcal{E}^{(g;\text{odd})}(u,v,w) \]

\[ \times \frac{\sigma(Q)^c \prod_I E(v_I,Q)^{2-1}}{\prod_I E(u_I,Q) \prod_K E(w_K,Q)^{2-1}}, \]

\[ I^{(g;\text{even})}(\rho, \lambda, \mu, \nu) = I^{(g;\text{odd})}(\rho, \lambda, \mu, \nu) \exp \left\{ \sum_{K_+} \mu_{K_+} \partial_{w_{K_+}} \ln E(w_{K_+},Q) \right\}. \]

It is worthwhile to note the equation

\[ p_{\phi_0}^{[t_a]} r_{ab} p_{\phi_0}^{[t_b]} + p_X^{[s_a,t_a,u_a]} r_{ab} p_X^{[s_b,t_b,u_b]} - p_{\phi_0}^{[s_a,t_a]} r_{ab} p_{\phi_0}^{[s_b,t_b]} = \frac{1}{2 \sqrt{k p p}} \left( 2 p p^{\left[ \frac{t_a}{2} \right]} + (n_a' p - (-)^{t_a} n_a p') + k \right) r_{ab} \left( 2 p p^{\left[ \frac{t_b}{2} \right]} + (n_b' p - (-)^{t_b} n_b p') + k \right) \]

\[ + 2(k+2) \left( \left[ \frac{s_a}{2} \right] - \frac{u_a}{2} - \frac{J[s_a]}{2(k+2)} \right) r_{ab} \left( \left[ \frac{s_b}{2} \right] - \frac{u_b}{2} - \frac{J[s_b]}{2(k+2)} \right) \]

\[ - 2k \left( \frac{m[t_a]}{2k} + \frac{u_a}{2} \right) r_{ab} \left( \frac{m[t_b]}{2k} + \frac{u_b}{2} \right). \]
7. Topological Limit

We here discuss a topological limit, i.e. \( l = 0 \), in the operator formalism given in the previous sections.

First of all, the central charge \( c \) (2.8) vanishes, in this limit, and the integer \( n \), which label the primary field \( \Psi_{J,n',n} \) is restricted to one due to (2.12). Hence, only the primary field of the type \( \Psi_{J,n',1} \) survives. Explicitly, such a field is given by

\[
\Phi_J \equiv \Psi_{J,n',1} = e^{-\frac{J}{\sqrt{k(k+2)}}(\varphi-\varphi_0)}
\]

(7.1)

with \( J = n' - 1 \), \( 0 \leq J \leq k \). This field gives rise to the chiral primary field in the \( N=2 \) super conformal theory. The chiral primary ring is as usual

\[
\Phi_J \Phi_{J'} = \begin{cases} 
\Phi_{J+J'} & \text{for } 0 \leq J + J' \leq k \\
0 & \text{otherwise}.
\end{cases}
\]

(7.2)

Due to the same restriction \( n = 1 \), the \( Q^+ \) type BRST charge becomes just the screening charge \( Q^+_1 = \oint dv S_+ (v) \). One should note that from (2.23), the screening operator \( S_+ \) becomes one of the super generator \( G \) in the \( N=2 \) theory.

\[
\lim_{l \to 0} S_+ = \Psi e^{\varphi_0} \equiv \sqrt{\frac{k+2}{k}} G.
\]

(7.3)

Hence, the BRST charge \( Q^+_1 \) is identified with the one discussed by Eguchi and Yang.\(^9\)

The EM tensor \( T \) is now BRST exact in the sense

\[
T = T_{Z_k} + \frac{1}{2} (\partial \varphi_0)^2 - \frac{1}{2} Q_{\varphi_0} \partial^2 \varphi_0 = \{ \tilde{G}(z), Q^+_1 \},
\]

(7.4)

where \( Q_{\varphi_0} = -\sqrt{\frac{k}{k+2}} \) and

\[
\tilde{G} = \sqrt{\frac{2k}{k+2}} \Psi e^{-\sqrt{\frac{k}{k+2}} \varphi_0}.
\]

(7.5)
The super partner of the chiral primary field $\Phi_J$ is given by

$$\Phi_J^{(1)} = \oint dz G(z) \Phi_J(w) = \sqrt{\frac{2}{k+2}} J e^{-\sqrt{\frac{2}{k+2}}(\varphi-\phi_0)} e^{-\chi}. \quad (7.6)$$

This satisfies the relation

$$\delta_B \Phi_J^{(1)} = \partial \Phi_J. \quad (7.7)$$

Collecting the chiral and the antichiral sector, one obtains a set of BRST invariant observables: $\Phi_J$, $\oint dw \Phi_J^{(1)}(w)$, $\oint d\bar{w} \Phi_J^{(1)}(\bar{w})$ and $\int dw \wedge d\bar{w} \Phi_J^{(1)}(w) \Phi_j^{(1)}(\bar{w})$.

The other BRST charge $Q_{J+1}$ is unchanged in the limit. Hence the BRST cohomology groups as well as the trace formulae for the topological minimal model are obtained by setting $l = 0$ in (3.26) and (3.29).

It is instructive to make a connection to the manifestly supersymmetric form of the model.$^{[36,37]}$ Let us define a pair of fields $\omega$ and $\sigma$ by the relations

$$\varphi - \phi_0 = \sqrt{\frac{k}{k+2}} (\omega - \sigma), \quad (7.8)$$

$$\varphi + \phi_0 = \sqrt{\frac{k+2}{k}} (\omega + \sigma). \quad (7.9)$$

Then the bosonic pair $(\beta, \gamma)^*$ and the fermionic pair $(b, c)$ defined by

$$\beta = e^{-\omega} \partial e^{\chi}, \quad \gamma = e^{\omega} e^{-\chi}, \quad (7.10)$$

$$b = e^{-\sigma}, \quad c = e^{\sigma} \quad (7.11)$$

give rise to a supersymmetric multiplet of spin $\lambda = \frac{k+1}{k+2}$ conjugate first order systems.$^{[24]}$ In terms of these fields, the EM tensor (7.4) and the super generators

---

* Do not confuse this with $(\beta, \gamma)$ in §2 and §3.
\(G\) and \(\bar{G}\) are reexpressed as

\[
T = -\lambda b \partial c + (1 - \lambda)(\partial b)c - \lambda \beta \partial \gamma + (1 - \lambda)(\partial \beta)c, \quad (7.12)
\]

\[
G \sim \beta c, \quad (7.13)
\]

\[
\bar{G} \sim \lambda (\partial \gamma)b - (1 - \lambda)\gamma \partial b. \quad (7.14)
\]

Let us next consider the calculation of correlation functions. In the operator formalism discussed in the previous sections, we have to consider the following changes. First, the screening charge of the type \(\oint S_+\) can no longer be inserted into any correlators. Otherwise it causes vanishing results. Therefore, in the screened three string vertex and the screened \(g\)-loop operator, we set \(r_+ = 0\). Secondly, we have to consider a coupling with the super partner \(\Phi^{(1)}_J\). However, for the correlator with any nonzero number of \(\Phi^{(1)}_J\), it is not possible to screen the whole background charges by making use of the screening operators \(S\) and \(S_-\) only. Hence one has to add another set of screening operators. A candidate is the one appearing in the \(Z_k\) parafermion theory:\[27,29,30\]

\[
V_+ = e^{-\chi}, \quad V_- = e^{\sqrt{k(k+2)\xi} \phi^{-(k+1)}\chi}
\]

The charge \(Q_V = \oint V_+\) has been used in section 3 to subtract redundant zero modes of \(\eta\). There, the Fock space of the \(Z_k\) parafermion sector is projected onto the kernel of \(Q_V\). Therefore only \(V_-\) can be used in the correlators.

By adding the screening charge \(\oint V_-\), \(r_V\) times, to the screened \(g\)-loop operator with properly chosen integration contours, the selection rules for the correlator

\[
< g^{(r,0,r_-)} | \prod_i^{N} \Phi_{J_i}(z_i) \prod_j^{M} \oint dy_j \Phi^{(1)}_{J_j}(y_j) | 0 >
\]

are given by

\[
\sum_i^{N} \frac{J_i}{\sqrt{k(k+2)}} + \sum_j^{M} \sqrt{\frac{k+2}{k}(-1 + \frac{J_j}{k + 2})} - \frac{2r_-}{\sqrt{k(k+2)}} + \sqrt{\frac{k}{k+2}(g-1)(3)}(\mathbb{F}, \mathbb{G})
\]
\[ \sum_{i} \frac{J_i}{\sqrt{k(k+2)}} + \sum_{j} \sqrt{\frac{k+2}{k}} \left( -1 + \frac{J_j}{k+2} \right) + \sqrt{\frac{k}{k+2}} r - \sqrt{\frac{k+2}{k}} r_{-} \]
\[ - \sqrt{k(k+2)} r_{V} + \sqrt{\frac{k}{k+2}} (g-1) = 0, \quad (7.16) \]
\[ r - r_{-} - M - (k+1) r_{V} = 0. \quad (7.17) \]

Solving these equations, we obtain the equivalent conditions: \( r_{V} = M \), \( r - r_{-} = (k+2) r_{V} \) and (7.15).

Note that if one sets \( r_{-} = 0 \), the condition (7.15) coincides with the one obtained by Li\(^{[32]}\), whereas the scheme using nonzero \( r_{-} \) has been discussed by Kawai et.al\(^{[35]}\).

Let us take the minimal choice of screening operators, \( r_{-} = 0, r_{V} = M \) and \( r = (k+2) M \), following Li. Making use of the operator formalism with \( l = 0 \) discussed in the previous sections, one can now in principle calculate any correlators in the topological minimal model.

It is, however, a formidable task to carrying out the whole integrations associated with \( \Phi_{J}^{(1)} \) and screening charges. The root of this troublesome feature is however simple. Namely, it lies in the extra \( \chi \)-charge in \( \Phi_{J}^{(1)} \), which requires nonzero \( r \) and \( r_{V} \). Hence, if a picture of \( \Phi_{J}^{(1)} \) can be changed to the one with \( \chi \)-charge zero, it becomes possible to make a calculation of arbitrary genus correlator without any screening charges.

One possible picture-changing operation is as follows.

\[ \Phi_{J}^{(1)'}(w) = \{ Q_{1}^{+}, \bar{I}(w) \Phi_{J}^{(1)}(w) \} = \frac{J}{J-k-2} \partial \left( e^{-\sqrt{\frac{k}{k+2}} (-1 + \frac{r_{V}}{k+2}) (\varphi-\phi_{0})(w)} \right), \quad (7.18) \]

where \( \bar{I} \) is an identity operator given by

\[ \bar{I}(w) = e^{\sqrt{\frac{k}{k+2}} (\varphi-\phi_{0})(w)}. \quad (7.19) \]

The operator \( \bar{I} \) has just the same amount of \( \varphi \)- and \( \phi_{0} \)-charges as \( Q_{1}^{+} \) with the opposite sign so that the picture-changed field \( \Phi_{J}^{(1)'}(w) \) has the same \( \varphi \)- and \( \phi_{0} \)-charges as \( \Phi_{J}^{(1)} \).
Note that, in (7.18), the picture-changing operator is identified with
g(z)\tilde{I}(w) \sim \partial e^{\chi(w)}. \tag{7.20}

A similar picture changing operator has been discussed by Distler\textsuperscript{[38]} in the context of two dimensional topological gravity.\textsuperscript{*}

Since the $\chi$-charge zero picture operator $\Phi^{(1)'}_j$ is a total derivative, a naive calculation using $\Phi^{(1)'}_j$ gives rise to a vanishing result. However, as discussed by Distler, an insertion of the picture changing operator causes a singularity so that one has to consider a certain regularization of the integral. If a proper regularization scheme is found, one could expect that the integral of the above total derivative term gives rise to a representative of a certain cohomology class associated with the topological minimal model. To find such a regularization scheme is an open problem.

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\section*{APPENDIX A}

We here give a derivation of the homology cycles $\mathcal{C}, \mathcal{C}_+, \mathcal{C}_-$ in (4.8). Let us consider a form dual to $\mathcal{C} \wedge \mathcal{C}_+ \wedge \mathcal{C}_-$:

$$\omega = \prod_{i=1}^{r} du_i \prod_{j=1}^{r_+} dv_j \prod_{k=1}^{r_-} dw_k \times \langle J_3; n'_3, n_3 | \langle V_{l23} > \prod_{i=1}^{r} S^{(1)}(u_i) \prod_{j=1}^{r_+} S^{(1)}_{+}(v_j) \prod_{k=1}^{r_-} S^{(1)}_{-}(w_k) | J_1; n'_1, n_1 \rangle_1 > | J_2; n'_2, n_2 \rangle_2 >$$

\textsuperscript{*} Naively, the fields in Distler's 2D topological gravity are obtained by (7.10) and (7.11) with the level number $k$ being analytically continued to -3.
\[
\frac{r}{r_+} \prod_{i} du_i \prod_{j} dv_j \prod_{k} dw_k f(u_i, v_j, w_k) \prod_{i,k} (u_i - w_k)^{-2} \prod_{j,k} (v_j - w_k)^{-2} \\
\times \frac{r}{r_-} \prod_{i=1}^{r_-} \frac{u_i - j_i}{k+2} - 1 \left(1 - u_i\right)^{-\frac{j_i}{r+2} - 1} \prod_{i\neq i'} \frac{r}{i'} \left(u_i - u_i'\right)^{\frac{2}{1+2}} \\
\times \frac{r}{r_+} \prod_{j=1}^{r_+} \frac{1-n_{1j}}{p} - 1 \left(1 - v_j\right)^{-\frac{1-n_{1j}}{p} - 1} \prod_{j < j'} \frac{r}{j'} \left(v_j - v_{j'}\right)^{2 \frac{2}{p}} \\
\times \frac{r}{r_-} \prod_{k=1}^{r_-} \frac{1-n_{2k}}{p} - 1 \left(1 - w_k\right)^{-\frac{1-n_{2k}}{p} - 1} \prod_{k < k'} \frac{r}{k'} \left(w_k - w_{k'}\right)^{-\frac{2}{p} - 2}, \tag{A.1}
\]

with \(r = r_- - r_+, 2r_+ = n_1 + n_2 - n_3 - 1\) and \(2r_- = n'_1 + n'_2 - n'_3 - 1\). In (A.1), the source points are chosen as \(z_1 = 0, z_2 = 1\) and \(z_3 = \infty\), and \(f\) is a holomorphic, symmetric and single-valued function. A relevant homology group dual to the cohomology of the forms in (A.1) is defined on a pair of circles, one of which winds the source points 0 (=\(z_1\)) and the other does 1 (=\(z_2\)) once, and contact each other at one point. Defining the \(r\)-chains \(C_j^{(\pm)}\) and \((r-1)\)-chains \(S_j^{(\pm)}\), \(j = 0, 1, \ldots, r(\pm)\) according to Ref.[19], we find that the boundaries of the \(j\)th \(r(\pm)\)-chains \(C_j^{(\pm)}\) are given by

\[
\partial C_j^+ = (q^{r-J_2-J_j} - 1)S_j^{+} - (q^{J_1-J_j+1} - 1)S_{j-1}^{+} \tag{A.2}
\]

\[
\partial C_j^- = (q^{r-n_2-n_j} - 1)S_j^{-} - (q^{n_1-J_j} - 1)S_{j-1}^{-} \tag{A.3}
\]

\[
\partial C_j^{\pm} = (q^{r-n_2-n_j} - 1)S_j^{\pm} - (q^{n_1-J_j} - 1)S_{j-1}^{\pm} \tag{A.4}
\]

The desired cycles are thus obtained by

\[
\mathcal{C} = \sum_{j=0}^{r} A_j C_j, \quad \mathcal{C}_+ = \sum_{j=0}^{r_+} A_j^{+} C_j^{+}, \quad \mathcal{C}_- = \sum_{j=0}^{r_-} A_j^{-} C_j^{-}, \tag{A.5}
\]

with \(A_0 = A_0^{+} = A_0^{-} = 1\) and

\[
A_j = \prod_{k=1}^{j} \frac{q^{r-J_2-J_j-k} - 1}{(q^{J_1-J_j+k+1} - 1)}, \tag{A.6}
\]
\[ A^+_j = \prod_{k=1}^{j} \frac{(q^{r_{+}^j - n_{2}^j - k + 1} - 1)}{(q^{n_{1}^j - k} - 1)}, \quad (A.7) \]

\[ A^-_j = \prod_{k=1}^{j} \frac{(q^{r_{-}^j - n_{2}^j - k + 1} - 1)}{(q^{n_{1}^j - k} - 1)}. \quad (A.8) \]

One can easily check \( \partial \mathcal{C} = 0, \partial \mathcal{C}_+ = 0 \) and \( \partial \mathcal{C}_- = 0 \) from (A.2)~(A.4).

**APPENDIX B**

We here give a proof of the BRST relations in (4.10) and (4.11).

The line of proof is the same as that in the \( SU(2)_k \) WZW model\(^{[21]}\). Using the overlapping condition, we obtain

\[
Q^{(3)}_{J+1} < V_{123}^{(r,r_{+},r_{-})} > = - < V_{123}^{(r,r_{+},r_{-})} > C^J_1 P_J P_{Jr} Q^{(1)}_{J+1} - (C^A_3 P_A)^{-1} < V_{123}^{(r+J-A,r_{+},r_{-})} > Q^{(2)}_{J+1} \quad (B.1)
\]

where the phase factors \( C^J_a, a = 1, 3, P_J \) and \( P_{Jr} \), arise from the overlapping condition, from the commutation among \( J+1 \) \( S \)'s and from the one between \( J+1 \) and \( r \) \( S \)'s, respectively;

\[
C^J_1 = e^{i\pi(J+1)(-\sqrt{\frac{k}{\pi^2}+\alpha_{\phi,0}^{(a)}+\alpha_{\chi,0}^{(a)})}}, \quad \phantom{=} \quad (B.2) \\
P_J = e^{i\pi J(J+1)(-\frac{k}{\pi^2}+1)/2}, \quad \phantom{=} \quad (B.3) \\
P_{Jr} = e^{i\pi r(J+1)(-\frac{k}{\pi^2}+1)}. \quad \phantom{=} \quad (B.4)
\]

Similarly, we obtain for the BRST charge \( Q^+_n \)

\[
Q^{+(3)}_n < V_{123}^{(r,r_{+},r_{-})} > = - < V_{123}^{(r,r_{+},r_{-})} > C^+_1 P_n P_{nr_+} Q^{+(1)}_n - (C^+_3 P_m)^{-1} < V_{123}^{(r,r_{+},n-m,r_{-})} > Q^{+(2)}_m \quad (B.5)
\]

34
where

\[ C_a^{+n} = e^{i\pi n(-\sqrt{\frac{k}{k+2}}\alpha_{\phi,0}^{(a)}+\alpha_{\chi,0}^{(a)}+\sqrt{2}\alpha_{\phi_0,0}^{(a)})}, \]  

(B.6)

\[ P_n = e^{i\pi n(n-1)(\alpha_+^2 - \frac{1}{k})}, \]  

(B.7)

\[ P_{nr_+} = e^{i\pi nr_+\left(-\frac{k+2}{k}+1+2\alpha_+^2\right)}, \]  

(B.8)

Replacing the momentum operator \( \alpha_{\phi,0}^{(3)}, \alpha_{\chi,0}^{(3)} \) and \( \alpha_{\phi_0,0}^{(3)} \) with their eigenvalues

\[ \frac{1}{2}\sqrt{\frac{k}{k+2}}(\frac{k+2}{k}m_{n',n} - J), \quad -\frac{1}{2}(J - m_{n',n} \right) \text{ and } \sqrt{2}\alpha_{n',n} \], and using the charge conservation law (4.9), one gets the BRST relations (4.10) and (4.11).

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