Information, Estimation, and Lookahead in the Gaussian channel

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Abstract

We consider mean squared estimation with lookahead of a continuous-time signal corrupted by additive white Gaussian noise. We show that the mutual information rate function, i.e., the mutual information rate as function of the signal-to-noise ratio (SNR), does not, in general, determine the minimum mean squared error (MMSE) with fixed finite lookahead, in contrast to the special cases with 0 and infinite lookahead (filtering and smoothing errors), respectively, which were previously established in the literature. We also establish a new expectation identity under a generalized observation model where the Gaussian channel has an SNR jump at $t = 0$, capturing the tradeoff between lookahead and SNR.

Further, we study the class of continuous-time stationary Gauss-Markov processes (Ornstein-Uhlenbeck processes) as channel inputs, and explicitly characterize the behavior of the minimum mean squared error (MMSE) with finite lookahead and signal-to-noise ratio (SNR). The MMSE with lookahead is shown to converge exponentially rapidly to the non-causal error, with the exponent being the reciprocal of the non-causal error. We extend our results to mixtures of Ornstein-Uhlenbeck processes, and use the insight gained to present lower and upper bounds on the MMSE with lookahead for a class of stationary Gaussian input processes, whose spectrum can be expressed as a mixture of Ornstein-Uhlenbeck spectra.

Index Terms

Mutual information, mean squared error (MSE), Brownian motion, Gaussian channel, additive white Gaussian noise (AWGN), causal/filtering error, non-causal/smoothing error, lookahead/delay, Ornstein-Uhlenbeck process, stationary process, power spectral density, signal-to-noise ratio (SNR)

I. INTRODUCTION

Mean squared estimation of a signal in the presence of Gaussian noise has been a topic of considerable importance to the communities of information theory and estimation theory. There have further been discoveries tying the two fields together, through identities between fundamental informational quantities and the squared estimation loss.

Consider the continuous-time Gaussian channel. In [1], Duncan established the equivalence between the input-output mutual information and the integral of half the causal mean squared error in estimating the signal based on

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the observed process. In [2], Guo et al. present what is now known as the I-MMSE relationship, which equates the derivative of the mutual information rate to half the average non-causal squared error. These results were extended to incorporate mismatch at the decoder for the continuous-time Gaussian channel by Weissman in [3], building upon similar relations for the scalar Gaussian channel by Verdú in [4]. In [3], it was shown that when the decoder assumes an incorrect law for the input process, the difference in mismatched filtering loss is simply given by the relative entropy between the true and incorrect output distributions. Recently, in [5], a unified view of the aforementioned identities was presented. In particular, pointwise analogues of these and related results were developed, characterizing the original relationships as identities involving expectations of random objects with information-estimation significance of their own.

In this work, we shall focus on stationary continuous-time signals corrupted by Gaussian noise. Let \( X = \{ X_t, t \in \mathbb{R} \} \) denote a continuous-time signal, which is the channel input process. We assume square integrability of the stochastic process \( X \), which says that for any finite interval \([a, b] \subset \mathbb{R}, E[\int_a^b X_t^2 \, dt] < \infty\).

The output process \( Y \) of the continuous-time Gaussian channel with input \( X \) is given by the following model

\[
dY_t = \sqrt{\text{snr}} X_t \, dt + dW_t,
\]

for all \( t \), where \( \text{snr} > 0 \) is the channel signal-to-noise ratio (SNR) and \( W \) is a standard Brownian Motion (cf. [6] for definition and properties of Brownian motion) independent of \( X \). For simplicity let us assume the input process to be stationary. Throughout, we will denote \( X^b_a = \{ X_t : a \leq t \leq b \} \).

Let \( I(\text{snr}) \) denote the mutual information rate, given by

\[
I(\text{snr}) = \lim_{T \to \infty} \frac{1}{T} I(X^T_0; Y^T_0).
\]

Let \( \text{mmse}(\text{snr}) \) denote the smoothing squared error

\[
\text{mmse}(\text{snr}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T E[(X_t - E[X_t|Y^T_0])^2] \, dt,
\]

which, in our setting of stationarity, can equivalently be defined as

\[
\text{mmse}(\text{snr}) = E[(X_0 - E[X_0|Y^{+\infty}_0])^2].
\]

Similarly, the causal squared error is given by

\[
\text{cmmse}(\text{snr}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T E[(X_t - E[X_t|Y^T_0])^2] \, dt = E[(X_0 - E[X_0|Y^{-\infty}_0])^2].
\]

From [1] and [2], we know that the above quantities are related according to

\[
\frac{2I(\text{snr})}{\text{snr}} = \text{cmmse}(\text{snr}) = \frac{1}{\text{snr}} \int_0^{\text{snr}} \text{mmse}(\gamma) \, d\gamma
\]

for all signal-to-noise ratios, \( \text{snr} > 0 \), and for all choices of the underlying input distribution for which the process \( X \) is square integrable. Thus, the mutual information, the causal error and the non-causal error are linked together.

\[1\] Indeed, [6] continues to hold in the absence of stationarity of the input process, in which case the respective quantities are averaged over a fixed, finite interval \([0, T]\).
in a distribution independent fashion for the Gaussian channel under mean-squared loss. Having understood how the time averaged causal and non-causal errors behave, and in particular the relation (7) implying that they are completely characterized by the mutual information rate function - the current work seeks to address the question - what happens for lookahead between 0 and \( \infty \)?

The mean squared error with finite lookahead \( d \geq 0 \) is defined as

\[
\text{lmmse}(d, \text{snr}) = E[(X_0 - E[X_0|Y_{-\infty}^d])^2],
\]

where it is instructive to note that the special cases \( d = 0 \) and \( d = \infty \) in (8) yield the filtering (6) and smoothing (4) errors respectively. Let us be slightly adventurous, and rewrite (7) as

\[
\frac{2I(\text{snr})}{\text{snr}} = E[\text{lmmse}(0, \Gamma_0)] = E[\text{lmmse}(\infty, \Gamma_\infty)],
\]

where the random variables \( \Gamma_0, \Gamma_\infty \) are distributed according to \( \Gamma_0 = \text{snr} \) a.s., and \( \Gamma_\infty \sim U[0, \text{snr}] \), where \( U[a,b] \) denotes the uniform distribution on \([a,b]\). This characterization of the mutual information rate may lead one to conjecture the existence of a random variable \( \Gamma_d \), whose distribution depends only on \( d, \text{snr} \), and some features of the process (such as its bandwidth and spectrum), and satisfies the identity

\[
\frac{2I(\text{snr})}{\text{snr}} = E[\text{lmmse}(d, \Gamma_d)],
\]

regardless of the distribution of \( X \). I.e., there exists a family of distributions \( \Gamma_d \) ranging from one extreme of a point mass at \( \text{snr} \) for \( d = 0 \), to the uniform distribution over \([0, \text{snr}]\) when \( d = \infty \). One of the corollaries of our results in this work is that such a relation cannot hold in general, even when allowing the distribution of \( \Gamma_d \) to depend on distribution-dependent features such as the process spectrum. In fact, in Section VII we show that in general, the mutual information rate function of a process is not sufficient to characterize the estimation error with finite lookahead. Therefore, one would need to consider features of the process that go beyond the mutual information rate and the spectrum - in order to characterize the finite lookahead estimation error. Motivated by this question, however, in Section VI we establish an identity which relates the filtering error to a double integral of the mean squared error, over lookahead and SNR, for the Gaussian channel with an SNR jump at \( t = 0 \).

In the literature, finite lookahead estimation has been referred to as fixed-lag smoothing (cf., for example, [7] and references therein). Note that (8) is well defined for all \( d \in \mathbb{R} \). For \( d < 0 \), \( \text{lmmse}(d, \text{snr}) \) denotes the finite horizon average prediction mean square error, which is as meaningful as its finite lookahead counterpart, i.e. the case when \( d > 0 \). Motivated by the desire to understand the tradeoff between lookahead and mmse, in Section II we explicitly characterize this trade-off for the canonical family of continuous-time stationary Gauss-Markov (Ornstein-Uhlenbeck) processes. In particular, we find that the rate of convergence of \( \text{lmmse}(\cdot, \text{snr}) \) with increasing lookahead (from causal to the non-causal error) is exponential, with the exponent given by the inverse of the non-causal error, i.e. \( \frac{1}{\text{mmse}(\text{snr})} \) itself.

The rest of the paper is organized as follows. In Section II we explicitly characterize the lookahead-MSE trade-off for the canonical family of continuous-time stationary Gauss-Markov (Ornstein-Uhlenbeck) processes. In particular, we find that the convergence of \( \text{lmmse}(\cdot, \text{snr}) \) with increasing lookahead (from causal to the non-causal error) is
exponentially fast, with the exponent given by the inverse of the non-causal error, i.e. $\frac{1}{\text{mmse}(\text{snr})}$ itself. In Section III, we extend our results to a larger class of processes, that are expressible as a mixture of Ornstein-Uhlenbeck processes. We then consider stationary Gaussian input processes in Section IV and characterize the MMSE with lookahead via spectral factorization methods. In Section V, we consider the information utility of infinitesimal lookahead for a general stationary input process and relate it to the squared filtering error. In Section VI, we introduce a generalized observation model for a stationary continuous-time signal where the channel has a non-stationary SNR jump at $t = 0$. For this model, we establish an identity relating the squared error with finite lookahead and the causal filtering error. In Section VII, we show by means of an example that a distribution-independent identity of the form in (10) cannot hold in general. We conclude with a summary of our findings in Section VIII.

II. ORNSTEIN-ULHENBECK PROCESS

A. Definitions and Properties

The Ornstein-Uhlenbeck process [8] is a classical continuous-time stochastic process characterized by the following stochastic differential equation

$$dX_t = \alpha(\mu - X_t)dt + \beta dB_t,$$

(11)

where $\{B_t\}_{t \geq 0}$ is a Standard Brownian Motion and $\alpha$, $\mu$, $\beta$ are process parameters. The reader is referred to [9] for a historical perspective on this process. Below, we present some of its important statistical properties. The mean and covariance functions are given by:

$$\mathbb{E}(X_t) = \mu,$$

(12)

and,

$$\text{Cov}(X_t, X_s) = \frac{\beta^2}{2\alpha} e^{-\alpha|t-s|}.$$  

(13)

We can further denote the autocorrelation function and power spectral density of this process by

$$R_X(\tau) = \frac{\beta^2}{2\alpha} e^{-\alpha|\tau|},$$

(14)

$$S_X(\omega) = \frac{\beta^2}{\alpha^2 + \omega^2}.$$  

(15)

In all further analysis, we consider the process mean $\mu$ to be 0. We also note that all expressions are provided assuming $\alpha > 0$ which results in a mean-reverting evolution of the process.

B. Mean Squared Estimation Error

Recall the mean squared error with finite lookahead $d$ defined in (8) from which one can infer the filtering (6) and smoothing (4) errors respectively. We now compute these quantities for the Ornstein-Uhlenbeck process at a fixed signal to noise ratio $\gamma$. Define, for the Ornstein-Uhlenbeck process corrupted by the Gaussian channel (1), the error in estimating $X_0$ based on a finite lookahead $d$ into the future and a certain lookback $l$ into the past:

$$\nu(l, d, \gamma) = \text{Var}(X_0|Y^{d}_{\omega l}) \quad d, l \geq 0.$$  

(16)
Before we explicitly characterize $\nu(l, d, \gamma)$, we present the following intermediate result, which is proved in Appendix A.

**Lemma 1:** Let $e_d = \nu(0, d, \gamma) = \nu(d, 0, \gamma)$ Then,

$$e_d = \frac{\Lambda_d \sqrt{\alpha^2 + \gamma \beta^2} - \alpha}{\gamma},$$

where

$$\Lambda_d = \frac{e^{2d\sqrt{\alpha^2 + \gamma \beta^2}} \rho + 1}{e^{2d\sqrt{\alpha^2 + \gamma \beta^2}} \rho - 1}$$

and $\rho = \frac{\gamma R X(0) + \sqrt{\alpha^2 + \gamma \beta^2 + \alpha}}{\gamma R X(0) - \sqrt{\alpha^2 + \gamma \beta^2 + \alpha}}$

Using Lemma 1, we now compute the more general quantity $\nu(l, d, \gamma)$ defined in (16), which denotes the loss in estimating $X_t$ based on the observations $Y_{t-d}^t$, for a stationary process.

**Lemma 2 (estimation error with finite window of observations in the past and future):**

$$\nu(l, d, \gamma) = \frac{e_l e_d R X(0)}{R X(0)(e_l + e_d) - e_l e_d}.$$  

The proof of Lemma 2 is straightforward, upon noting (i) the Markov Chain relationship $Y_{0-l} - X_0 - Y_{0-d}$, and (ii) the joint Gaussianity of $(X_0, Y_{0-d})$. Thus, one can use the conditional variances $\text{Var}(X_0 | Y_{0-l})$ and $\text{Var}(X_0 | Y_{0-d})$ to obtain the required expression for $\text{Var}(X_0 | Y_{0-d})$.

Using Lemma 1 and Lemma 2, we explicitly characterize the filtering (6) and smoothing (4) (mean squared error) MSE’s for the noise corrupted Ornstein-Ulhenbeck process, in the following Lemma.

**Lemma 3 (MSE for the Ornstein-Ulhenbeck process):** For the Ornstein-Ulhenbeck process corrupted by white Gaussian noise at signal-to-noise ratio $\text{snr}$, the filtering and smoothing errors are given by:

- Filtering Error
  \[
  \text{cmmse}(\text{snr}) = \nu(\infty, 0, \text{snr}) = \frac{\sqrt{\alpha^2 + \text{snr} \beta^2} - \alpha}{\text{snr}}
  \]

- Smoothing Error
  \[
  \text{mmse}(\text{snr}) = \nu(\infty, \infty, \text{snr}) = \frac{\beta^2}{2 \sqrt{\alpha^2 + \text{snr} \beta^2}}
  \]

The expressions in (19) and (20) recover the classical expressions for the optimal causal and smoothing errors (for Gaussian inputs), due to Yovits and Jackson in [12], and Wiener in [13], respectively.

**Remark:** (19) and (20) can be easily seen to verify the following general relationship between the filtering and smoothing error for the continuous time Gaussian channel, established in [2], i.e.

$$\text{cmmse}(\text{snr}) = \frac{1}{\text{snr}} \int_0^{\text{snr}} \text{mmse}(\gamma) \, d\gamma.$$  

**C. Estimation with Lookahead**

Having introduced the Ornstein-Ulhenbeck process and its properties, we now study the behavior of the MMSE with finite lookahead.

**Note:** For the rest of this paper we will set $\beta = 1$ in (11), as it only involves an effective scaling of the channel SNR parameter.

\(^2\)where the latter equality holds due to time-reversibility of the Ornstein-Ulhenbeck process
Recall that,
\[
\text{lmmse}(d, \gamma) = \text{Var}(X_0|Y_{d-\infty})
\]
\[
= \nu(\infty, d, \gamma),
\]
where (23) follows from the definition of \(\nu(\cdot, \cdot, \gamma)\) in Lemma 2. The following lemma explicitly characterizes the finite lookahead MMSE for the Ornstein-Ulhenbeck process with parameter \(\alpha\), corrupted by the Gaussian channel at signal-to-noise ratio \(\gamma\).

**Lemma 4 (MMSE with Lookahead for OU(\(\alpha\))):**
\[
\text{lmmse}(d, \gamma) = \begin{cases} 
(1 - e^{-2d\sqrt{\alpha^2 + \gamma}})\text{mmse}(\gamma) + e^{-2d\sqrt{\alpha^2 + \gamma}}\text{cmmse}(\gamma) & \text{if } d \geq 0 \\
 e^{-2\alpha|d|}\text{cmmse}(\gamma) + \frac{1}{2\alpha}(1 - e^{-2\alpha|d|}) & \text{if } d < 0.
\end{cases}
\]

**Proof:** We will establish the expression for positive and negative values lookahead separately.

For \(d \geq 0\): We combine (22) and Lemma 2 to obtain the closed form expression for \(\text{lmmse}(d, \gamma)\). In addition, from Lemma 3 we note that
\[
\text{cmmse}(\gamma) = \sqrt{\alpha^2 + \gamma - \alpha \gamma},
\]
\[
\text{mmse}(\gamma) = \frac{1}{2\alpha}\sqrt{\alpha^2 + \gamma}.
\]
Using the above expressions, we can express the estimation error with finite lookahead \(d \geq 0\) in the following succinct manner
\[
\text{lmmse}(d, \gamma) = (1 - e^{-2d\sqrt{\alpha^2 + \gamma}})\text{mmse}(\gamma) + e^{-2d\sqrt{\alpha^2 + \gamma}}\text{cmmse}(\gamma).
\]

For \(d < 0\): We denote the absolute value of \(d\) as \(|d|\), and note that
\[
Y_{-|d|} - X_{-|d|} - X_0
\]
forms a Markov triplet. We further note that \((X_0, X_{-|d|}, Y_{-|d|})\) are jointly Gaussian. In particular, we know that
\[
X_0|X_{-|d|} \sim \mathcal{N}(X_{-|d|}e^{-\alpha|d|}, \frac{1}{2\alpha}(1 - e^{-2\alpha|d|})),
\]
\[
\text{Var}(X_{-|d|}|Y_{-\infty}) = \text{cmmse}(\gamma)
\]
From the above relations, it is easy to arrive at the required quantity \(\text{Var}(X_0|Y_{-|d|})\) for \(d < 0\),
\[
\text{lmmse}(d, \gamma) = e^{-2\alpha|d|}\text{cmmse}(\gamma) + \frac{1}{2\alpha}(1 - e^{-2\alpha|d|}).
\]

This completes the proof.

A plot of the estimation error with finite lookahead for the Ornstein-Ulhenbeck process is shown in Fig. 1. It is seen from the expression in Lemma 4 that the asymptotes at \(d = -\infty\) and \(d = +\infty\) correspond to the values \(R_X(0)\) and \(\text{mmse}(\text{snr})\), respectively, for a fixed SNR level snr.
We now focus on the case where \( d \geq 0 \), which corresponds to estimation with finite delay. We define, for \( d \geq 0 \),

\[
    p_d = \frac{\text{mmse}(d, \gamma) - \text{mmse}(\gamma)}{\text{cmmse}(\gamma) - \text{mmse}(\gamma)}.
\]  

(32)

From Lemma 4, we observe that for the Ornstein-Uhlenbeck process,

\[
    p_d = e^{-2d\sqrt{\alpha^2 + \gamma}}
\]  

(33)

\[
    = e^{-\frac{d}{\text{mmse}(\gamma)}},
\]  

(34)

where (34) follows from (26). In other words, the causal error approaches the non-causal error exponentially fast with increasing lookahead. This conveys the importance of lookahead in signal estimation with finite delay. We can state this observation as follows:

**Observation 5:** For any AWGN corrupted Ornstein-Uhlenbeck process, the mean squared error with finite positive lookahead, approaches the non-causal error exponentially fast, with decay exponent given by \( \frac{1}{\text{mmse}(\gamma)} \).

**D. SNR vs Lookahead tradeoff**

One way to quantify the utility of lookahead in finite delay estimation is to compute the corresponding gain in SNR. Specifically, we compute the required SNR level which gives the same mmse with lookahead \( d \) as the filtering error at a fixed SNR level \( \text{snr} > 0 \).
Let \( \gamma_d^*(\text{snr}) \) be the value of ‘signal to noise ratio’ that provides the same mean square error as the causal filter with zero lookahead. I.e.

\[
\text{lmmse}(d, \gamma_d^*(\text{snr})) = \text{cmmse}(\text{snr}),
\]

whenever a solution \( \gamma_d^*(\text{snr}) > 0 \) exists.

We now present some general observations about \( \gamma_d^*(\text{snr}) \).

- \( \gamma_d^*(\text{snr}) \) is a monotonically decreasing function of \( d \).
- \( \gamma_d^*(\text{snr}) = \text{snr} \).
- Let \( \gamma_\infty(\text{snr}) = \lim_{d \to \infty} \gamma_d^*(\text{snr}) \). Then, we have \( \text{mmse}(\gamma_\infty(\text{snr})) = \text{cmmse}(\text{snr}) \). For the Ornstein-Uhlenbeck process parameterized by \( \alpha \), we get the following closed form expression for \( \gamma_\infty(\text{snr}) \):

\[
\gamma_\infty(\text{snr}) = \frac{1}{2(2\text{cmmse}(\text{snr}))^2 - \alpha^2}.
\]

- \( \gamma_d^*(\text{snr}) \) has a vertical asymptote at \( d = d_{\text{snr}}^* < 0 \), where \( d_{\text{snr}}^* \) is defined as the solution to the equation \( \text{Var}(X_0 | X_{d_{\text{snr}}^*}) = \text{cmmse}(\text{snr}) \). Then we have,

\[
\lim_{d \to d_{\text{snr}}^*} \gamma_d^*(\text{snr}) = \infty
\]

In Fig. 2 we illustrate the behavior of \( \gamma_d(\text{snr}) \) as a function of lookahead, for the Ornstein-Uhlenbeck process corrupted by Gaussian noise according to the channel in [1].

III. A MIXTURE OF ORNSTEIN-UHLENBECK PROCESSES

Having presented the finite lookahead MMSE for the Ornstein-Ulhenbeck process in Lemma 4, in this section we obtain the MMSE with lookahead for the class of stochastic processes that are mixtures of Ornstein-Ulhenbeck processes. We then proceed to establish a general lower bound on the MMSE with lookahead for the class Gaussian processes whose spectra can be decomposed as a mixture of spectra of Ornstein-Ulhenbeck processes. For the same class of Gaussian processes, we also present an upper bound for the finite lookahead MMSE, in terms of a mismatched estimation loss.

Let \( \mu(\alpha) \) be a probability measure defined on \( [0, \infty) \). Let \( P(\alpha) \) be the law of the Ornstein-Uhlenbeck process with parameter \( \alpha \). Note that each \( P(\alpha) \) is the law of a stationary ergodic stochastic process. We define the stationary distribution generated by taking a \( \mu \)-mixture of these processes:

\[
P = \int P(\alpha) \, d\mu(\alpha).
\]

Note that \( P \) need not be Gaussian in general.

Lemma 6: Let \( X \) be a stationary stochastic process governed by law \( P \) which is a \( \mu \) mixture of Ornstein-Ulhenbeck processes, and is corrupted by AWGN at SNR \( \gamma \). The MMSE with finite lookahead \( d \) is given by

\[
\text{lmmse}_P(d, \gamma) = \int \text{lmmse}_\alpha(d, \gamma) \, d\mu(\alpha),
\]
Fig. 2: Plot of $\gamma_d^*(\text{snr})$ (snr = 1) as a function of $d$ for the Ornstein Uhlenbeck process with parameter $\alpha = 0.2$. In this case $\gamma_\infty = 0.3320$ and $d^* = -0.9935$.

where $\text{lmmse}_\alpha(d, \gamma)$ is (as defined in Lemma 4) the corresponding quantity for estimating an Ornstein-Ulhenbeck process with parameter $\alpha$.

The proof of Lemma 6 follows in one line, upon observing that the underlying ‘active mode’ is eventually precisely learned from the infinitely long observation of the noisy process. This relation allows us to compute the minimum mean squared error with finite lookahead for the large class of processes that can be expressed as mixtures of Ornstein-Uhlenbeck processes.

As another important corollary of this discussion, consider any Gaussian process $G$ whose spectrum $S_G$ can be expressed as a mixture of spectra of Ornstein-Uhlenbeck processes, for some appropriate mixing measure $\mu(\alpha)$, i.e.

$$S_G = \int S_\alpha d\mu(\alpha), \quad (40)$$

where $S_\alpha$ denotes the spectrum of the Ornstein-Ulhenbeck process with parameter $\alpha$. The approach outlined above provides us with a computable lower bound on the minimum mean squared error with fixed lookahead $d$ (under AWGN) for the process $G$, which we state in the following Lemma.

**Lemma 7:** For a Gaussian process $G$ with spectrum as in (40), the finite lookahead mmse has the following
lower bound:

\[ \text{Imms}e_G(d, \gamma) \geq \int \text{Imms}e_\alpha(d, \gamma) \, d\mu(\alpha), \quad (41) \]

where \( \text{Imms}e_\alpha(d, \gamma) \) is characterized explicitly in Lemma 4.

To see why (41) holds note that its right hand side represents the mmse at lookahead \( d \) associated with the process whose law is expressed in (38), while the left side corresponds to this mmse under a Gaussian source with the same spectrum.

In the following example, we will illustrate the bound in (41). In Section IV-A, we discuss the computation of the mmse with lookahead for any stationary Gaussian process, of a known spectrum. This computation, based on Wiener-Kolmogorov theory, relies on the spectral factorization of the spectrum of the AWGN corrupted process. This factorization is, in general, difficult to perform. Thus, a bound on the mmse with lookahead, such as in (41), for a Gaussian process whose spectrum can be expressed as a mixture of spectra of Ornstein-Ulhenbeck processes, is quite useful.

A natural question that emerges from the above discussion is, what functions can be decomposed into mixtures of spectra of Ornstein-Ulhenbeck processes? To answer this question, note that one can arrive at any spectrum \( S_G \) which can be expressed (upto a multiplicative constant) as

\[ S_G(\omega) = \int_0^\infty \frac{1}{\alpha^2 + \omega^2} \, d\mu(\alpha) \quad (42) \]

where we use (15) to characterize \( S_\alpha(\omega) \). Equivalently, in the time domain, the desired auto-correlation function is expressible as

\[ R_G(\tau) = \int_0^\infty e^{-\alpha|\tau|} \frac{d\mu(\alpha)}{\alpha}, \quad (43) \]

which can be viewed as a real-exponential transform of the function \( \mu \). However, as can be seen from (42), the spectrum \( S_G(\omega) \) is always constrained to be monotonically decreasing with \( \omega \). This shows that the space of functions that can be candidates for the spectrum of the process \( G \), is not exhausted by the class of functions decomposable as spectra of Ornstein-Ulhenbeck processes.

A. Illustration of bound in Lemma 7

We consider the simple scenario of a process which is the equal mixture of two Ornstein-Ulhenbeck processes, \( X_1 \) and \( X_2 \), parametrized by \( \alpha_1 \) and \( \alpha_2 \) respectively. Specifically, let us define a stationary continuous-time process \( X \) as

\[ X = \begin{cases} 
X_1 & \text{if w.p. } \frac{1}{2} \\
X_2 & \text{if w.p. } \frac{1}{2}
\end{cases} \quad (44) \]

Note that the process \( X \) will not be Gaussian in general. The spectrum \( S_X(\cdot) \) of the mixture, will be given by the mixture of the spectra of the constituent Ornstein-Ulhenbeck processes. I.e.

\[ S_X(\cdot) = \frac{1}{2} S_{X_1}(\cdot) + \frac{1}{2} S_{X_2}(\cdot), \quad (45) \]
where $S_{X_i}$, denotes the spectrum of $X_i$, $i = 1, 2$. Now consider a stationary Gaussian process $G$ with the same spectrum, i.e.

$$S_G(\cdot) = S_X(\cdot).$$  \hspace{1cm} (46)

We will consider the noise corruption mechanism in (11) at a signal-to-noise ratio $\text{snr}$. Note that Lemma 4, in conjunction with (39), allow us to compute the mmse with lookahead $\text{lmms}_{X}(d, \text{snr})$, for the $X$ process:

$$\text{lmms}_{X}(d, \text{snr}) = \frac{1}{2}\text{lmms}_{X_1}(d, \text{snr}) + \frac{1}{2}\text{lmms}_{X_2}(d, \text{snr}),$$  \hspace{1cm} (47)

for all $d$. Note that when $d = -\infty$, the quantity $\text{lmms}(-\infty, \text{snr})$ will be given by the stationary variance of the underlying process. Since this quantity depends only on the spectrum of the stationary process, the lower bound in (41) will coincide with the estimation error for the $G$ process at $d = -\infty$. For the $G$ process, we can analytically compute the mmse with finite lookahead (116) as well, using the method outlined in Section IV-A. The reader is referred to Appendix B for a detailed discussion on computing the estimation error with finite lookahead for the $G$ process under any $\alpha_1, \alpha_2$. For the purposes of this example, we illustrate the bound (41) for the case when $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$. In Fig. 3, we present the mmse with lookahead for process $G$, as well as the lower bound, given by the corresponding quantity for $X$. For the given choice of parameters, at $\text{snr} = 1$: $\text{lmms}_{X}(-\infty, \text{snr}) = \text{lmms}_{G}(-\infty, \text{snr}) = 1.333; \text{lmms}_{X}(+\infty, \text{snr}) = 0.4425; \text{lmms}_{G}(+\infty, \text{snr}) = 0.4568.$

**B. Towards deriving an Upper Bound**

Note that Lemma 7 gives us a computable lower bound for the mmse with lookahead for a Gaussian process $G$ whose spectrum can be expressed as a mixture of spectra of Ornstein-Ulhenbeck processes, under an appropriate mixture $\mu$.

Define, the mismatched mmse with lookahead for a filter that assumes the underlying process has law $P^{(\beta)}$ [Ornstein-Ulhenbeck process with parameter $\beta$], whereas the true law of the process is $P^{(\alpha)}$

$$\text{lmms}_{\alpha,\beta}(d, \gamma) = \mathbb{E}_{\alpha}[(X_0 - \mathbb{E}_{\beta}[X_0|Y^{d}_{-\infty}])^2],$$  \hspace{1cm} (48)

where the outer expectation is with respect to the true law of the underlying process $P^{(\alpha)}$, while the inner expectation is with respect to the mismatched law $P^{(\beta)}$. Note that the mismatched filter which assumes that the underlying signal is an Ornstein-Ulhenbeck process with parameter $\beta$, is in particular, a linear filter. The process $G$ is Gaussian, and hence the optimal filter for $G$ is also linear. Thus, for any fixed $\beta$, the mismatched filter thus defined, will be suboptimal for process $G$. This yields the following natural upper bound, which in conjunction with Lemma 7 can be stated as:

**Lemma 8 (Upper and lower bound in terms of mixture of Ornstein-Ulhenbeck spectra):**

$$\int \text{lmms}_{\alpha}(d, \gamma) d\mu(\alpha) \leq \text{lmms}_{G}(d, \gamma) \leq \min_{\beta > 0} \int \text{lmms}_{\alpha,\beta}(d, \gamma) d\mu(\alpha).$$  \hspace{1cm} (49)

We should note that the upper bound stated in Lemma 8 can be hard to compute for continuous time processes, even though the filter applied is linear. In summary, analysis of $\text{lmms}(-, \gamma)$ for the Ornstein-Ulhenbeck process
provides us with a rich class of processes for which we can compute the finite lookahead estimation error. And for a class of Gaussian processes (43), we have a lower and upper bound on this quantity. These characterizations may be used to approximate the behavior of the MMSE with finite lookahead for a large class of Gaussian processes, instead of obtaining the optimal solution which relies on spectral factorization, and can be cumbersome.

IV. Stationary Gaussian Processes

In this section, we focus exclusively on Gaussian processes having a known spectrum. We are interested in obtaining the MMSE with finite lookahead in estimating such a process when corrupted by AWGN at a fixed SNR level $\gamma$.

A. Calculation of MMSE with finite lookahead via Spectral Factorization

Let $\{X_t, t \in \mathbb{R}\}$ stationary and Gaussian, with power spectral density $S_X(\omega)$, be the input to the continuous-time Gaussian channel at SNR $\gamma$. $Y(\cdot)$, the noise corrupted version of $X(\cdot)$, is stationary, Gaussian and has PSD $S_Y(\omega) = 1 + \gamma S_X(\omega)$. Let $S_Y^+(\omega)$ be the Wiener-Hopf factorization of $S_Y(\omega)$, i.e. a function satisfying $|S_Y^+(\omega)|^2 = S_Y(\omega)$.
with \(1/S_Y^+(\omega)\) being the transfer function of a stable causal filter. This factorization exists whenever the Paley-Wiener conditions are satisfied for \(S_Y(\omega)\). Denote by \(\hat{Y}_t\) the output of the filter \(1/S_Y^+(\omega)\) applied on \(Y_t\), then \(\hat{Y}_t\) is a standard Brownian motion. This theory is well developed and the reader is encouraged to peruse any standard reference in continuous-time estimation for details (cf., e.g., [10] and references therein).

Let \(h(t)\) be the impulse response of the filter with transfer function
\[
H(\omega) = \frac{\sqrt{\gamma}S_X(\omega)}{S_Y(\omega)},
\]
with \(S_Y(\omega) = S_Y^+(-\omega)\). The classical result by Wiener and Hopf [11], applied to non-causal filtering of stationary Gaussian signals can be stated as,
\[
\mathbb{E} \left[ X_0 | Y_{-\infty}^d \right] = \mathbb{E} \left[ X_0 | Y_{-\infty}^\infty \right] = \int_{-\infty}^{\infty} h(-t) d\hat{Y}_t.
\]
Moreover, an expression for the finite-lookahead MMSE estimator of \(X_0\) can be immediately derived, using the fact that \(\hat{Y}_t\) is both a standard Brownian motion and a reversible causal function of the observations:
\[
\mathbb{E} \left[ X_0 | Y_{-\infty}^d \right] = \mathbb{E} \left[ X_0 | \hat{Y}_{-\infty}^d \right] = \mathbb{E} \left[ \mathbb{E} \left\{ X_0 | \hat{Y}_{-\infty}^\infty \right\} | \hat{Y}_{-\infty}^d \right] = \mathbb{E} \left[ \int_{-\infty}^{\infty} h(-t) d\hat{Y}_t | \hat{Y}_{-\infty}^d \right] = \int_{-\infty}^{\infty} h(-t) d\hat{Y}_t.
\]

Using again the fact that \(\hat{Y}_t\) is a brownian motion, as well as the orthogonality property of MMSE estimators, we can obtain a simple expression for \(\text{Immse}(d, \gamma)\):
\[
\text{Immse}(d, \gamma) = \mathbb{E} \left( X_0 - \mathbb{E} \left[ X_0 | Y_{-\infty}^d \right] \right)^2 = \mathbb{E} \left( X_0 - \mathbb{E} \left[ X_0 | \hat{Y}_{-\infty}^d \right] \right)^2 = \mathbb{E} \left( X_0 - \mathbb{E} \left[ X_0 | Y_{-\infty}^\infty \right] \right)^2 + \mathbb{E} \left( \mathbb{E} \left[ X_0 | Y_{-\infty}^\infty \right] - \mathbb{E} \left[ X_0 | \hat{Y}_{-\infty}^d \right] \right)^2
\]
\[
= \text{mmse}(\gamma) + \mathbb{E} \left( \int_{-\infty}^{d} h(-t) d\hat{Y}_t \right)^2 = \text{mmse}(\gamma) + \int_{-\infty}^{d} h^2(t) dt.
\]
This classical formulation of the lookahead problem for stationary Gaussian processes shows us that the MMSE with lookahead behaves gracefully with the lookahead \(d\), and is intimately connected to the impulse response \(h(\cdot)\) of the filter induced by the Wiener spectral factorization process. In particular, that the solution to the lookahead problem for each value of \(d\), can be approached in this single unified manner as shown in (56), is quite satisfying.

B. Processes with a rational spectrum

Let \(S_X(\omega)\) be a rational function, i.e. of the form \(\frac{P(\omega)}{Q(\omega)}\) with \(P, Q\) being finite order polynomials in \(\omega\). In this case \(S_Y(\omega)\) is also a rational function, and the Wiener-Hopf factorization can be preformed simply by factorizing the numerator and denominator of \(S_Y(\omega)\) and composing \(S_Y^+(\omega)\) only from the stable zeros and poles.

Recall the definition of \(p_d\) in [32],
\[
p_d = \frac{\text{Immse}(d, \gamma) - \text{mmse}(\gamma)}{\text{cmmse}(\gamma) - \text{mmse}(\gamma)}
\]
Clearly, $p_0 = 1$ and $\lim_{d \to \infty} p_d = 0$. The value $p_d$ goes to zero in the same rate as $\text{lmmse}$ converges to $\text{mmse}$. For the Ornstein-Ulhenbeck process we observed that $p_d$ converges exponentially to zero. In the following lemma, we generalize this result to include all Gaussian input processes with rational spectra.

**Lemma 9:** For any Gaussian input process with a rational spectrum, $\text{lmmse}(d, \text{snr})$ approaches $\text{mmse}(\text{snr})$ exponentially fast as $d \to \infty$. Equivalently, $p_d \to 0$ exponentially, as $d \to \infty$.

**Proof:** Since $S_X^+(\omega)$ is a rational function, so is $H(\omega)$ defined in (50). Therefore, $h(t)$ must be a finite sum of exponentially decreasing functions [should use a ref here] and consequently $\int_{-\infty}^{-d} h^2(t)dt$ must also decrease exponentially as $d \to \infty$. This in conjunction with the relation in (56) concludes the proof.

As an illustration, we consider an equal mixture of two Ornstein-Ulhenbeck processes with parameters $\alpha_1$ and $\alpha_2$, as in the example of the previous section. In Appendix B the $\text{mmse}$ with lookahead is explicitly computed for a Gaussian process with this spectrum, acting as input to the channel. From these calculations, one can observe that the corresponding expression for $p_d$ in (119) vanishes exponentially with increasing lookahead. However, it is important to note that there exist input spectra for which the decay of $p_d$ is not exponentially fast.

As an example, consider the input power spectrum
\[ S_{X}^{\text{triang}}(\omega) = (1 - |\omega|) 1\{|\omega| \leq 1\}, \] (58)
and in Figure 4 we plot the behaviour of $p_d$ as a function of $d$ for input power spectrum $S_X^{\text{triang}}(\omega)$ and different SNR’s. This demonstrates the polynomial rate of convergence of the finite lookahead estimation error towards non-causal MMSE. The values of $\text{lmmse}(d, \text{snr})$ were found by numerically approximating $h(t)$, using FFT in order to perform the factorization of $S_Y(\omega)$.

V. INFORMATION UTILITY OF LOOKAHEAD

Consider a stationary process $X_t$ observed through the continuous-time Gaussian channel (1) at SNR level $\gamma$. In the previous sections we have tried to ascertain the benefit of lookahead in mean squared estimation. We now address the question, how much information does lookahead provide in general, and whether this quantity has any interesting connections with classical estimation theoretic objects.

For $\tau > 0$, let us define the Information Utility $U(\cdot)$ as a function of lookahead $\tau$, to be
\[ U(\tau) = I(X_0; Y^\tau_0|Y^0_{-\infty}). \] (59)

When the input process is Gaussian,
\[ U(\tau) = h(X_0|Y^0_{-\infty}) - h(X_0|Y^\tau_{-\infty}) = \frac{1}{2} \log (2\pi e \text{Var}(X_0|Y^0_{-\infty})) - \frac{1}{2} \log (2\pi e \text{Var}(X_0|Y^\tau_{-\infty})) \] (60)
\[ = \frac{1}{2} \log \left( \frac{\text{cmmse}(\text{snr})}{\text{lmmse}(\tau, \text{snr})} \right). \] (61)
Rearranging the terms, we have
\[ \text{lmmse}(\tau, \text{snr}) = \text{cmmse}(\text{snr}) \exp(-2U(\tau)). \] (62)
The finite lookahead MMSE is seen to converge to the non-causal MMSE at an approximately cubic rate.

Furthermore, when the input is non-Gaussian but $h(X_0|Y_{-\infty}^\tau)$ is well-defined for every $\tau \geq 0$,

$$h(X_0|Y_{-\infty}^\tau) \leq \mathbb{E}_{y_{-\infty}^\tau} \left[ \frac{1}{2} \log (2\pi e \text{Var}(X_0|Y_{-\infty}^\tau = y_{-\infty}^\tau)) \right] \leq \frac{1}{2} \log (2\pi e \text{Immse}(\tau, \text{snr})).$$

The first inequality is due to the fact that the Gaussian distribution has maximum entropy under a variance constraint, and the second inequality follows from Jensen’s inequality. Rearranging the terms once more, we have that for every stationary input process,

$$\text{Immse}(\tau, \text{snr}) \geq N(X_0|Y_{-\infty}^0) \exp(-2U(\tau)),$$

where $N(Z) = \frac{1}{2\pi e} \exp(2h(Z))$ is the entropy power functional.

We now present a relation between $U(\cdot)$ and the MMSE via a differential equation. Consider an infinitesimal increment $d\tau$ in $U(\tau)$,

$$U(\tau + d\tau) - U(\tau) = I(X_0; Y_{\tau + d\tau}^\tau|Y_{-\infty}^\tau) = I(X_{\tau + d\tau}; Y_{\tau + d\tau}^\tau|Y_{-\infty}^\tau) - I(X_{\tau + d\tau}; Y_{\tau + d\tau}^\tau|Y_{-\infty}^\tau, X_0).$$

The last equality is due to the Markov Chain relationship $X_0 - (X_{\tau + d\tau}, Y_{-\infty}^\tau) - Y_{\tau + d\tau}^\tau$. Using the time-incremental channel argument introduced in [2, Section III.D], we are able to write,

$$I(X_{\tau + d\tau}^\tau; Y_{\tau + d\tau}^\tau|Y_{-\infty}^\tau) = \frac{1}{2} d\tau \cdot \text{snr} \cdot \text{Var}(X_{\tau + d\tau}^\tau|Y_{-\infty}^\tau) + o(d\tau),$$

where $\text{Var}(X_{\tau + d\tau}^\tau|Y_{-\infty}^\tau)$ is an upper bound of $\text{Var}(X_{\tau + d\tau}|Y_{-\infty}^\tau)$.

Fig. 4: Plot of $p_d$ as defined in (32), vs. lookahead in various SNRs, for input power spectrum $(1 - |\omega|) 1_{(|\omega| \leq 1)}$. The finite lookahead MMSE is seen to converge to the non-causal MMSE at an approximately cubic rate.
and
\[
I(X^t_{\tau + d\tau}; Y^t_{\tau + d\tau} | Y^{\infty}_{\tau}, X^0) = \frac{1}{2} d\tau \cdot \text{snr} \cdot \text{Var}(X^t_{\tau} | Y^{\infty}_{\tau}, X^0) + o(d\tau).
\] (67)

The expression for the time derivative of the information utility function is therefore,
\[
U' (\tau) = \frac{\text{snr}}{2} \left( \text{Var}(X^t_{\tau} | Y^{\infty}_{\tau}) - \text{Var}(X^t_{\tau} | Y^{\infty}_{\tau}, X^0) \right).
\] (68)

Since the input is assumed stationary, \( \text{Var}(X^t_{\tau} | Y^{\infty}_{\tau}) = \text{Var}(X^0 | Y^{\infty}_{0}) = \text{cmmse} (\text{snr}) \). We notice that \( \text{Var}(X^t_{\tau} | Y^{\infty}_{\tau}, X^0) \) is the causal MMSE in estimating \( X^t_{\tau} \) when \( X^0 \) is known. The value of \( U' (\tau) \) is therefore intimately connected to the effect of initial conditions on the causal MMSE estimation of the process. In particular,
\[
U'(0) = \frac{\text{snr}}{2} \text{cmmse}(\text{snr}),
\] (69)

meaning that the added value in information of an infinitesimal lookahead is proportional to the causal MMSE. Noticing that \( U(0) = 0 \), we may integrate (68) to obtain,
\[
U(\tau) = \frac{\text{snr}}{2} \left[ \tau \text{cmmse}(\text{snr}) - \int_0^\tau \text{Var}(X_s | Y^s_{\infty}, X^0) \, ds \right].
\] (70)

Specializing to stationary Gaussian input processes, and joining equations (62) and (70), we obtain
\[
\text{lmmse}(\tau, \text{snr}) = \text{cmmse}(\text{snr}) \exp \left[ -\text{snr} \left( \tau \text{cmmse}(\text{snr}) - \int_0^\tau \text{Var}(X_s | Y^s_{\infty}, X^0) \, ds \right) \right].
\] (71)

For the Ornstein-Uhlenbeck process, \( \text{Var}(X_s | Y^s_{\infty}, X^0) \) can be calculated explicitly. This, in conjunction with (71), provides an alternative derivation of the finite lookahead MMSE in the Ornstein-Uhlenbeck process, that is based on Information-Estimation arguments. The reader is referred to Appendix C for more details.

VI. GENERALIZED OBSERVATION MODEL

In this section, we present a new observation model to understand the behavior of estimation error with lookahead. Consider a stationary continuous-time stochastic process \( X_t \). The observation model is still additive Gaussian noise, where the SNR level of the channel has a jump at \( t = 0 \). Letting \( Y_t \) denote the channel output, we describe the channel as given below.
\[
dY_t = \begin{cases} \sqrt{\text{snr}} X_t \, dt + dW_t & t \leq 0 \\ \sqrt{\gamma} X_t \, dt + dW_t & t > 0 \end{cases}
\] (72)

where, as usual, \( W_t \) is a standard Brownian motion independent of \( X_t \). Note that for \( \gamma \neq \text{snr} \), the \( (X_t, Y_t) \) process is not jointly stationary. Letting \( d, l \geq 0 \), we define the finite lookahead estimation error at time \( d \) with lookahead \( l \) as
\[
f(\text{snr}, \gamma, d, l) = \text{Var}(X_d | Y^{l+d}_{\infty}).
\] (73)

We call this a generalized observation model, as for \( \gamma = \text{snr} \), we recover the usual time-invariant Gaussian channel. For instance, we note that the error \( f \) reduces to the filtering, smoothing or finite lookahead errors, depending on
the parameters $\gamma$, $l$ and $d$ as:

\[
c_{\text{mmse}}(\text{snr}) = f(\text{snr}, \text{snr}, d, 0) \tag{74}
\]

\[
n_{\text{mmse}}(\text{snr}) = f(\text{snr}, \text{snr}, d, \infty) \tag{75}
\]

\[
l_{\text{mmse}}(l, \text{snr}) = f(\text{snr}, \text{snr}, t, l) \tag{76}
\]

In the following we relate the estimation error with finite lookahead for the observation model described in (72), with the original filtering error.

**Theorem 10:** Let $X_t$ be any finite variance continuous time stationary process which is corrupted by the Gaussian channel in (72). Let $f$ be as defined in (73). For fixed $\text{snr} > 0$ and $T > 0$ let $\Gamma \sim U[0, \text{snr}]$ and $L \sim U[0, T]$ be independent random variables. Then

\[
c_{\text{mmse}}(\text{snr}) = \mathbb{E}_{\Gamma, L}[f(\text{snr}, \Gamma, T - L, L)] \tag{77}
\]

**Proof:** Before proving this result, we take a detour to consider a stationary continuous-time stochastic process $X_{0}^{T}$ which is governed according to law $P_{X_{0}^{T}}$ and is observed in the window $t \in [0, T]$ through the continuous-time Gaussian channel (1) at SNR level $\text{snr}$. We define the operators which evaluate the filtering and smoothing estimation errors for this process model, as follows:

\[
c_{\text{mmse}}(P_{X_{0}^{T}}, \text{snr}) = \int_{0}^{T} \mathbb{E}_{\text{snr}}[(X_{t} - \mathbb{E}[X_{t}])^2] \, dt \tag{78}
\]

\[
n_{\text{mmse}}(P_{X_{0}^{T}}, \text{snr}) = \int_{0}^{T} \mathbb{E}_{\text{snr}}[(X_{t} - \mathbb{E}[X_{t}])^2] \, dt \tag{79}
\]

Further, [2] gives us the celebrated relationship between the causal and non-causal estimation errors:

\[
c_{\text{mmse}}(P_{X_{0}^{T}}, \text{snr}) = \frac{1}{\text{snr}} \int_{0}^{\text{snr}} n_{\text{mmse}}(P_{X_{0}^{T}}, \gamma) \, d\gamma. \tag{80}
\]

We now return to the auxiliary observation model introduced above. We note that the causal and non-causal estimation errors of process $X_t$ at signal to noise ratio $\text{snr}$, can be written in terms of

\[
c_{\text{mmse}}(\text{snr}) = \frac{1}{T} \mathbb{E}_{\text{snr}} \left[ c_{\text{mmse}}(P_{X_{0}^{T} \mid Y_{-\infty}^{0}}, \text{snr}) \right] \tag{81}
\]

\[
n_{\text{mmse}}(\text{snr}) = \frac{1}{T} \mathbb{E}_{\text{snr}} \left[ n_{\text{mmse}}(P_{X_{0}^{T} \mid Y_{-\infty}^{0}}, \text{snr}) \right], \tag{82}
\]

where $\mathbb{E}_{\text{snr}}$ denotes expectation over $Y_{-\infty}^{0}$. And, now employing the relation between the two quantities above, we get

\[
\frac{1}{T} \mathbb{E}_{\text{snr}} \left[ c_{\text{mmse}}(P_{X_{0}^{T} \mid Y_{-\infty}^{0}}, \text{snr}) \right] = \frac{1}{T} \mathbb{E}_{\text{snr}} \left[ \frac{1}{\text{snr}} \int_{0}^{\text{snr}} n_{\text{mmse}}(P_{X_{0}^{T} \mid Y_{-\infty}^{0}}, \gamma) \, d\gamma \right] \tag{83}
\]

Using the definition of $f$ and inserting in above, we get the following integral equation which holds for every stationary process $X_t$ observed through the continuous Gaussian channel as described in (72)

\[
f(\text{snr}, \text{snr}, x, 0) = \frac{1}{T \cdot \text{snr}} \int_{0}^{\text{snr}} \int_{0}^{T} f(\text{snr}, \gamma, t, T - t) \, dt \, d\gamma, \tag{84}
\]
Lookahead and Mutual Information

New Observation Model

Information Utility of Lookahead

Theorem

In other words

cmmse(snr) = \frac{1}{T} \cdot \text{snr} \int_0^T \int_0^{\text{snr}} \text{Var}(X|Y_0^1)(\text{snr})^2 \text{d} \Gamma \text{d} T

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Fig. 5: Region in the Lookahead-SNR plane over which the finite lookahead MMSE given by (73) is averaged over in Theorem 10 to yield the filtering error at level snr.

for all T and x. Note that the left hand side of (84) is nothing but the causal squared error at SNR level snr. Note, in particular, that for independent random variables $\Gamma \sim U[0, \text{snr}]$ and $L \sim U[0, T]$, (84) can be expressed as

$$cmmse(snr) = E_{\Gamma, L}[f(snr, \Gamma, T - L, L)].$$

(85)

It is interesting to note that the double integral over lookahead and SNR are conserved in such a manner by the filtering error, for any arbitrary underlying stationary process. Note that one way of interpreting this result, is to take the average of the finite lookahead mmse (under the given observation model), over a rectangular region in the Lookahead vs. Signal-to-Noise Ratio plane, as depicted in Fig. 5. Theorem 10 tells us that for all underlying processes, this quantity is always the filtering error at level snr. Thus, we find the classical estimation theoretic quantity described by the causal error to emerge as a bridge between the effects of varying lookahead and the signal to noise ratio.

Given Theorem 10, it is natural to investigate whether the relation (77) can be inverted to determine the function $f$, from the causal mmse. If that were the case, then, in particular, the filtering error would completely determine the estimation loss with finite lookahead. To pose an equivalent question, let us recall Duncan’s result in [1] establishing the equivalence of the mutual information rate and the filtering error. On the other hand, by [2], [15] the mutual information rate function $I(\cdot)$, determines the non-causal mmse. Might the same mutual information rate function $I(snr)$ can completely determine the estimation loss for a fixed finite lookahead $d$ as well? This question is addressed in the following section.

VII. CAN $I(\cdot)$ RECOVER FINITE-LOOKAHEAD MMSE?

In Fig. 6 we present a characteristic plot of the mmse with lookahead for an arbitrary continuous time stationary process, corrupted by the Gaussian channel (1) at SNR level snr. In particular, we note three important features
Fig. 6: Characteristic behavior of the minimum mean squared error with lookahead for any process.

of the process, namely (i) the asymptote at \( d = \infty \) is \( \text{mmse}(\text{snr}) \), (ii) \( \text{lmmse}(0, \text{snr}) = \text{cmmse}(\text{snr}) \) and (iii) the asymptote at \( d = -\infty \) is the variance of the stationary process, \( \text{Var}(X_0) \). Further, the curve is non-decreasing for all \( d \).

From [1] and [2], we know that the mutual information, the causal, and the non-causal mmse determine each other according to

\[
\frac{2I(\text{snr})}{\text{snr}} = \text{cmmse}(\text{snr}) = \frac{1}{\text{snr}} \int_{0}^{\text{snr}} \text{mmse}(\gamma) \, d\gamma.
\] (86)

In other words, the mutual information rate function is sufficient to characterize the behavior of the causal and smoothing errors as functions of snr. In particular, (86) also determines the three important features of the \( \text{lmmse}(\cdot, \text{snr}) \) curve discussed above, i.e. the value \( 3 \) at \( d = 0 \), and the asymptotes at \( d = \pm \infty \). It is natural to ask whether it can actually characterize the entire curve \( \text{lmmse}(\cdot, \text{snr}) \). The following theorem implies that the answer is negative.

**Theorem 11:** For any finite \( d < 0 \) there exist stationary continuous-time processes which have the same mutual information rate \( I(\text{snr}) \) for all \( \text{snr} \), but have different minimum mean squared errors with (negative) lookahead \( d \).

\(^3\)Note that \( \text{Var}(X_0) = \text{cmmse}(0) \), which \( I(\cdot) \) determines.
One of the corollaries of Theorem 11, which we demonstrate by means of an example, is that the mutual information rate as a function of snr does not in general determine \( \text{lmmse}(d, \text{snr}) \) for any finite non-zero \( d \).

Thus, if the triangular relationship between mutual information, the filtering, and the smoothing errors is to be extended to accommodate finite lookahead, one will have to resort to distributional properties that go beyond mutual information. To see why this must be true in general, we note that the functional \( \text{lmmse}(\cdot, \text{snr}) \) may depend on the time-dynamical features of the process, to which the mutual information rate function is invariant. For example, we note the following.

**Observation 12:** Let \( X_t \) be a stationary continuous time process. Define \( X_t^{(a)} = X_{at} \) for a fixed constant \( a > 0 \). Let \( Y_t \) and \( Y_t^{(a)} \), denote respectively, the outputs of the Gaussian channel (1) with \( X_t \) and \( X_t^{(a)} \) as inputs. Then, for all \( d \) and \( \text{snr} > 0 \),

\[
\text{lmmse}_{X^{(a)}}(d, \text{snr}) = \text{lmmse}_{X}(ad, \text{snr}/a),
\]

where the subscript makes the process under consideration explicit. Note that for the special cases when \( d \in \{0, \pm \infty\} \), the error of the scaled process results in a scaling of just the SNR parameter, i.e.

\[
\text{mmse}_{X^{(a)}}(\text{snr}) = \text{mmse}_{X}(\text{snr}/a),
\]

and

\[
\text{cmmse}_{X^{(a)}}(\text{snr}) = \text{mmse}_{X}(\text{snr}/a).
\]

For all other values of \( d \), we see that the error depends on the error at a scaled lookahead, in addition to a scaled SNR level. This indicates, the general dependence of the mmse with finite, non-zero lookahead on the time-dynamics of the underlying process.

One of the consequences of Duncan’s result in [1] is that the causal and anti-causal errors as functions of snr are the same (due to the mutual information acting as a bridge, which is invariant to the direction of time). Let \( X_t \) be a continuous-time stationary stochastic process and \( Y_t \) be the output process of the Gaussian channel (1) with \( X_t \) as input. We can write,

\[
2I(\gamma) = \text{cmmse}(\gamma) = \text{acmmse}(\gamma),
\]

or, writing the rightside equality explicitly,

\[
\text{Var}(X_0|Y_{\infty}^{-d}) = \text{Var}(X_0|Y_{\infty}^{-\infty}).
\]

It is now natural to wonder whether \([91]\) carries over to the presence of lookahead, which would have to be the case if the associated conditional variances are to be determined by the mutual information function, which is invariant to the direction of the flow of time. In the following we present an explicit construction of a process for which

\[
\text{Var}(X_0|Y_{\infty}^{d}) \neq \text{Var}(X_0|Y_{\infty}^{-d})
\]

\(^4\)the anti-causal error denoted as \( \text{acmmse}(\text{snr}) \) denotes the filtering error for the time-reversed input process.
for some values of \( d \). Note that the left and right sides of (92) are the mmse’s with lookahead \( d \) associated with the original process, and its time reversed version, respectively. Thus, mutual information alone does not characterize these objects.

A. Construction of a continuous-time process

In this section, we construct a stationary continuous time process from a stationary discrete time process. This process will be the input to the continuous time Gaussian channel in (1).

Let \( \tilde{X} = \{\tilde{X}_t\}_{t=-\infty}^{+\infty} \) be a discrete time stationary process following a certain law \( P_{\tilde{X}} \). Let us define a piecewise constant continuous-time process \( X_t \) such that

\[
X_t \equiv \tilde{X}_i, \quad t \in (i - 1, i]
\]

We now apply a random shift \( \Delta \sim U[0,1] \) to the \( \{X_t\} \) process to transform the non-stationary continuous time process into a stationary one. Let us denote this stationary process by \( X \). The process \( X \) is observed through the Gaussian channel in (1) at snr = 1, with \( Y \) denoting the channel output process. This procedure is illustrated in Fig. 7.

1) Time Reversed Process: Consider the discrete-time process \( \tilde{X}^{(R)} \), denoting the time reversed version of the stationary process \( \tilde{X} \). The reversed process will, in general, not have the same law as the forward process (though it will of course inherit its stationarity from that of \( \tilde{X} \)). Let us construct an equivalent continuous time process corresponding to \( \tilde{X}^{(R)} \), using the procedure described above for the process \( \tilde{X} \), and label the resulting stationary continuous-time process by \( X^{(R)} \). In the following example, we compare the minimum mean square errors with finite lookahead for the processes \( X \) and \( X^{(R)} \), for a certain underlying discrete-time Markov process.
B. Proof of Theorem 11, Examples and Discussions

Define a stationary discrete time Markov chain \( \tilde{X} \), on a finite alphabet \( \mathcal{X} \), characterized by the transition probability matrix \( P \).

The intuition behind the choice of alphabet and transition probabilities, is that we would like the Discrete Time Markov Chain (DTMC) to be highly predictable (under mean square error criterion) in the forward direction compared to the time reversed version of the chain. Note that the time reversed Markov chain will have the same stationary distribution, but different transition probabilities. We transform this discrete time Markov chain to a stationary continuous time process by the transformation described above. Because of the difference in the state predictability of the forward and reverse time processes in the DTMC, the continuous time process thus created will have predictability that behaves differently for the forward and the time reversed processes and, in turn, the MSE with finite lookahead will also depend on whether the process or its time-reversed version is considered.

1) Infinite SNR scenario: We now concentrate on the case when the signal-to-noise ratio of the channel is infinite. The input to the Gaussian channel is the continuous time process \( X \), which is constructed based on \( \tilde{X} \) as described above. Let us consider negative lookahead \( d = -1 \). For infinite SNR, the filter will know what the underlying process is exactly, so that

\[
\operatorname{immse}(-1, \infty) = \operatorname{Var}(X_0|Y_{-\infty}^{-1})
\]

(94)

\[
= \operatorname{Var}(X_0|X_{-\infty}^{-1})
\]

(95)

\[
= \operatorname{Var}(\tilde{X}_0|\tilde{X}_{-1})
\]

(96)

where (96) follows from the Markovity of \( \tilde{X} \). Note that the quantity in (96) is the prediction variance of the DTMC \( \tilde{X} \). Let \( \nu : \mathcal{X} \rightarrow \mathbb{R} \) be any probability measure on the finite alphabet \( \mathcal{X} \) and \( V[\nu] \) be the variance of this distribution. Let \( \mu, P(\tilde{X}_{i+1} | \tilde{X}_i) \) denote respectively, the stationary distribution and the probability transition matrix of the Markov chain \( \tilde{X} \). Then the prediction variance is given by

\[
\operatorname{Var}(\tilde{X}_0|\tilde{X}_{-1}) = \sum_{x \in \mathcal{X}} \mu(x) V[P(\cdot|x)].
\]

(97)

In the infinite SNR setting, it is straightforward to see that

\[
\operatorname{immse}(d, \infty) = 0, \quad \forall d \geq 0.
\]

(98)

Since the process \( X \) is constructed by a uniformly random shift according to a \( U[0,1] \) law, for each \( -1 \leq d \leq 0 \), we have

\[
\operatorname{immse}(d, \infty) = |d| \operatorname{Var}(\tilde{X}_0|\tilde{X}_{-1}), \quad -1 \leq d \leq 0.
\]

(99)

For fixed \( d \in [-1,0] \), with probability \( 1 - |d| \), the process \( Y_{\infty}^d \) will sample \( \tilde{X}_0 \) in which case the resulting mean squared error is 0 at infinite SNR. Alternately, with probability \( |d| \) the error will be given by the prediction variance, i.e \( \operatorname{Var}(\tilde{X}_0|\tilde{X}_{-1}) \), which gives rise to (99). A similar analysis can be performed for the time-reversed DTMC \( \tilde{X}^{(R)} \).

Having found analytic expressions for the mmse with lookahead for the infinite SNR case, we show a characteristic
plot of the same in Fig. 8. Note the difference in the curves for negative lookahead arises simply due to a difference in prediction variance for the forward and time-reversed DTMC’s $\tilde{X}$ and $\tilde{X}^{(R)}$. Since the mmse with lookahead are different for $d < 0$ at infinite SNR, they would also be different at a large enough, finite signal-to-noise ratio. This completes the proof of Theorem 11 as stated for $d < 0$.

To further argue the existence of processes where lmmse$(d, \text{snr})$ are different, also for positive $d$, we provide the following explicit construction of the underlying processes. We then provide plots of the mmse with lookahead for this process, based on Markov Chain Monte Carlo simulations. The associated plots make it clear that lmmse$(d, \cdot)$ are distinct for both processes when $d$ is finite and non-zero.

2) Simulations: Let $\tilde{X}$ be a Discrete Time Markov Chain with the following specifications: The alphabet is $X = \{5, 0, -5\}$. The probability transition matrix $P$ is given by:

$$
P = \begin{bmatrix}
0.6 & 0.4 & 0 \\
0 & 0.2 & 0.8 \\
0.875 & 0 & 0.125
\end{bmatrix}
$$

where $P_{ij} = P(\tilde{X}_{k+1} = x_j | \tilde{X}_k = x_i)$. Note that $x_1 = 5$, $x_2 = 0$, $x_3 = -5$, in the above example. For this markov chain, we can compute the prediction variance according to (97) to obtain $\text{Var}(\tilde{X}_0 | \tilde{X}_{-1}) = 6.6423$. The stationary prior for this DTMC is $\mu = (0.5109, 0.2555, 0.2336)$. For the reversed process $\tilde{X}^{(R)}$, the probability transition matrix is given by

$$
P^{(R)} = \begin{bmatrix}
0.6 & 0 & 0.4 \\
0.8 & 0.2 & 0 \\
0 & 0.875 & 0.125
\end{bmatrix}
$$

and the prediction variance is 13.9234.

We performed monte carlo simulations to obtain the MSE with finite lookahead (and lookback) for the forward and time reversed continuous-time processes thus formed. A brief description of the simulation is provided in Appendix D.

3) Discussion: In Fig. 9 we present the MSE with finite lookahead and lookback for the continuous time process $X$ and $X^{(R)}$ denoting the forward and time-reversed stationary noise-free processes, respectively. From Duncan’s result, we know that the causal and the anti-causal errors must coincide. This is observed (and highlighted in Fig. 10) by the same values for the MSE with 0 lookahead for the forward and reversed processes. Indeed, for both positive and negative lookahead, the MSE’s are different.

Note that we know the asymptotic behavior of the MSE’s with lookahead. As $d \to -\infty$, the forward(and reverse) MSE will converge to the variance $\text{Var}(\tilde{X}_0)$ of the corresponding underlying DTMC (similarly for the time-reversed chain). For $d \to \infty$, the MSE’s converge to the non-causal errors respectively (which are naturally equal for the forward and reversed processes). Note that the forward and reversed processes have the same mutual information
rate function, as well as the same spectrum. This indicates, that such measures are not capable of characterizing the MMSE with finite lookahead.

The above construction illustrates the complicated nature of lookahead in its role as a link between estimation and information for the Gaussian channel. While answering several important questions, it also raises new ones - Do there exist other informational measures of the input-output laws that are sufficient to characterize the estimation error with lookahead? Such directions remain for future exploration.

**VIII. Conclusions**

This work can be viewed as a step towards understanding the role of lookahead in information and estimation. We investigate the benefit of finite lookahead in mean squared estimation under additive white Gaussian noise. We study the class of Ornstein-Uhlenbeck processes and explicitly characterize the dependence of squared estimation loss on lookahead and SNR. We extend this result to the class of processes that can be expressed as mixtures of Ornstein-Uhlenbeck processes, and use this characterization to present bounds on the finite lookahead MMSE for a class of Gaussian processes. We observe that Gaussian input processes with rational spectra have the finite lookahead MMSE converging exponentially rapidly from the causal, to the non-causal MMSE. We define and obtain relationships for the information utility of lookahead. We then present an expectation identity for a generalized observation model.
Fig. 9: Comparison of the Estimation Loss with finite lookahead for the Forward and Reverse processes, \( X \) and \( X^{(R)} \) respectively at SNR=1. For reference, the infinite SNR curves (dashed) are also shown.

which presents the filtering error as a double integral, over lookahead and SNR - of the estimation error with finite lookahead. Finally, we illustrate through means of an example that the mutual information rate function does not uniquely characterize the behavior of estimation error with lookahead, except in the special cases of 0 and infinite lookahead. Our example shows that the finite lookahead MMSE depends on features of the process that are not captured by the information rate function, or indeed, the entire spectrum.

APPENDIX A

In this appendix we prove Lemma [1] Employing the continuous-time Kalman-Bucy filtering framework (cf. [14] for a detailed presentation), we obtain the following differential equation for the error covariance \( e_t = \text{Var}(X_t|Y_t^0) \)

\[
\frac{de_t}{dt} = -2\alpha e_t - \gamma e_t^2 + \beta^2, \quad (100)
\]

where \( e_0 = R_X(0) = \frac{\beta^2}{2\alpha} \). We integrate over limits 0 to \( d \) and re-arrange terms to get the desired result,

\[
\int_0^d \frac{de_t}{-2\alpha e_t - \gamma e_t^2 + \beta^2} = \int_0^d dt \\
\]

\[
d = \frac{1}{2\sqrt{\alpha^2 + \gamma^2}} \log \left| \frac{-\gamma e_d - \alpha - \sqrt{\alpha^2 + \gamma^2}}{-\gamma e_d - \alpha + \sqrt{\alpha^2 + \gamma^2}} \right| - \frac{1}{2\sqrt{\alpha^2 + \gamma^2}} \log |\rho|, \quad (102)
\]
which upon simplification yields the following expression for $e_d$

$$e_d = -\frac{\alpha}{\gamma} + \frac{\sqrt{\alpha^2 + \gamma \beta^2}}{\gamma} \left( e^{2d\sqrt{\alpha^2 + \gamma \beta^2} \rho + 1} - e^{2d\sqrt{\alpha^2 + \gamma \beta^2} \rho - 1} \right),$$

(103)

where $\rho$ is as defined as

$$\rho = \frac{|\gamma R_X(0) + \sqrt{\alpha^2 + \gamma \beta^2} + \alpha|}{\gamma R_X(0) - \sqrt{\alpha^2 + \gamma \beta^2} + \alpha}.$$

(104)

**APPENDIX B**

In this appendix, we illustrate the computation of the MMSE with finite lookahead for a Gaussian process, whose spectrum is an equal mixture of the spectra of two Ornstein-Ulhenbeck processes, parametrized by $\alpha_i$, $i = 1, 2$. In this computation, we use the equations developed in Section IV-A.

For simplicity, we will operate in the ‘$s$’ domain for the spectra, where the region of convergence of the Laplace transform will be implicit. Recall from (14), that the spectrum of an Ornstein-Ulhenbeck process with parameter $\alpha$ is given by

$$S_\alpha(s) = \frac{1}{\alpha^2 - s^2}.$$

(105)
Thus, the spectrum of the input process $X_t$, which is a mixture of two Ornstein-Ulhenbeck processes is given by

$$S_X(s) = \frac{1}{2} \cdot \frac{1}{\alpha_1^2 - s^2} + \frac{1}{2} \cdot \frac{1}{\alpha_2^2 - s^2}, \quad (106)$$

Under the observation model in (1) at a fixed signal-to-noise ratio $\text{snr} > 0$, the output spectrum is given by:

$$S_Y(s) = 1 + \frac{\text{snr}}{2} \cdot \left[ \frac{1}{\alpha_1^2 - s^2} + \frac{1}{\alpha_2^2 - s^2} \right], \quad (107)$$

Performing the Wiener-Hopf decomposition of the output spectrum in (107), we obtain the following spectral factors:

$$S_Y^+(s) = \frac{(s + p_1)(s + p_2)}{(\alpha_1 + s)(\alpha_2 + s)}, \quad (108)$$

and

$$S_Y^-(s) = \frac{(s - p_1)(s - p_2)}{(\alpha_1 - s)(\alpha_2 - s)}, \quad (109)$$

where $p_1, p_2 > 0$ are such that $p_i^2, i = 1, 2$ are solutions of $k(x) = 0$, where

$$k(x) = x^2 - (\alpha_1^2 + \alpha_2^2 + \text{snr})x + \frac{\text{snr}}{2} (\alpha_1^2 + \alpha_2^2) + \alpha_1^2 \alpha_2^2. \quad (110)$$

Note that $S_Y^+(s) = S_Y^-(s)$. Invoking (50), the transfer function of the optimal non-causal filter $H(\cdot)$ is characterized by

$$H(s) = \sqrt{\text{snr}} \frac{S_X(s)}{S_Y(s)} \quad (111)$$

$$= \frac{\sqrt{\text{snr}}}{2} \frac{(\alpha_1^2 + \alpha_2^2) - \sqrt{\text{snr}}^2}{(s - p_1)(s - p_2)(\alpha_1 + s)(\alpha_2 + s)} \quad (112)$$

$$= \frac{u_1}{s - p_1} + \frac{u_2}{s - p_2} + \frac{v_1}{\alpha_1 + s} + \frac{v_2}{\alpha_2 + s}, \quad (113)$$

where (112) follows from (106) and (109); and (113) denotes the partial fraction decomposition of the expression in (112). Having derived an exact expression for the Wiener filter, we are now in a position to compute the desired quantity, i.e. $\text{Immse}(d, \text{snr})$. From (56) we have,

$$\text{Immse}(d, \text{snr}) = \text{mmse}(\text{snr}) + \int_{-\infty}^{-d} h^2(t) \, dt. \quad (114)$$

From the classical result by Wiener [13], we know that the mmse of a Gaussian process is given by

$$\text{mmse}(\text{snr}) = \int_{-\infty}^{\infty} S_X(j\omega) \frac{d\omega}{1 + \text{snr} S_X(j\omega) 2\pi}. \quad (115)$$

Using (113) and (114), we are now in a position to explicitly compute the mmse with finite lookahead for the $X$ process corrupted by the Gaussian channel:

$$\text{Immse}_X(d, \text{snr}) = \begin{cases} 
\text{mmse}(\text{snr}) + \frac{u_1^2}{2p_1} e^{-2p_1d} + \frac{u_2^2}{2p_2} e^{-2p_2d} + \frac{u_1 u_2}{p_1 + p_2} e^{-(p_1 + p_2)d} & \text{if } d \geq 0 \\
\text{mmse}(\text{snr}) + C + \frac{v_1^2}{2\alpha_1} e^{2\alpha_1d} + \frac{v_2^2}{2\alpha_2} e^{2\alpha_2d} + \frac{v_1 v_2}{\alpha_1 + \alpha_2} e^{(\alpha_1 + \alpha_2)d} & \text{if } d < 0 
\end{cases} \quad (116)$$

where

$$C = \frac{u_1^2}{2p_1} + \frac{u_2^2}{2p_2} + \frac{u_1 u_2}{p_1 + p_2} - \frac{v_1^2}{2\alpha_1} - \frac{v_2^2}{2\alpha_2} - \frac{v_1 v_2}{\alpha_1 + \alpha_2}. \quad (117)$$
Recall the rate of convergence of the mmse with lookahead, from causal to non-causal error, was defined as

\[ p_d = \frac{\text{imms}(d, \text{snr}) - \text{mmse}(\text{snr})}{\text{cmms}(\text{snr}) - \text{mmse}(\text{snr})}. \]  

For the Gaussian input process with spectrum as given in (106), we can use (116) to compute the quantity \( p_d \):

\[ p_d = \frac{1}{C} \left[ \frac{u_1^2}{2p_1} e^{-2p_1 d} + \frac{u_2^2}{2p_2} e^{-2p_2 d} + \frac{u_1 u_2}{p_1 + p_2} e^{-(p_1 + p_2) d} \right] \]  

Clearly \( p_d \to 0 \) for large \( d \), in accordance with Lemma 9.

**APPENDIX C**

Let \( \hat{e}_t \equiv \text{Var}(X_t | Y^d_t, X_0) \), and notice that \( \hat{e}_t \) satisfies the Kalman-Bucy differential equation (100), with the initial condition \( \hat{e}_0 = 0 \). Applying a similar integration procedure to the one described above, we find that

\[ \hat{e}_d = \frac{-\alpha}{\gamma} + \sqrt{\frac{\alpha^2 + \gamma \beta^2}{\gamma}} \left( e^{2\tau \sqrt{\alpha^2 + \gamma \beta^2}} \frac{1}{\hat{\rho} + 1} \right), \]

with

\[ \hat{\rho} = \sqrt{\frac{\alpha^2 + \gamma \beta^2 + \alpha}{\alpha^2 + \gamma \beta^2 - \alpha}}. \]

We define the quantity,

\[ \tau = \sqrt{\alpha^2 + \gamma \beta^2}. \]

Equations (19) and (20) can be rewritten as

\[ \text{cmms}(\gamma) = \frac{\tau - \alpha}{\gamma}, \quad \text{mmse}(\gamma) = \frac{\tau^2 - \alpha^2}{2\tau \gamma} = \text{cmms}(\gamma) \frac{\tau + \alpha}{2\tau}. \]

Using the above relations, we have

\[ \text{Var}(X_d | Y^d_0, X_0) = \frac{-\alpha}{\gamma} + \frac{\tau}{\gamma} \left( \frac{e^{2\tau \sqrt{\alpha^2 + \gamma \beta^2}} (\tau + \alpha) - (\tau - \alpha)}{e^{2\tau \sqrt{\alpha^2 + \gamma \beta^2}} (\tau + \alpha) + (\tau - \alpha)} \right) \]

\[ = \text{cmms}(\gamma) - \frac{2\tau}{\gamma} \left( \frac{e^{2\tau \sqrt{\alpha^2 + \gamma \beta^2}} (\tau + \alpha) + (\tau - \alpha)}{e^{2\tau \sqrt{\alpha^2 + \gamma \beta^2}} (\tau + \alpha) + (\tau - \alpha)} \right). \]

Integrating, we obtain

\[ \int_0^t \text{Var}(X_s | Y^s_0, X_0) ds = t \text{cmms}(\gamma) - \frac{2\tau}{\gamma} \left[ t - \frac{1}{2\tau} \log \left( \frac{e^{2\tau t (\tau + \alpha) + (\tau - \alpha)}}{e^{2\tau t (\tau + \alpha) - (\tau - \alpha)}} \right) \right]. \]

Since the Ornstein-Uhlenbeck process is Markov, it satisfies \( \text{Var}(X_t | Y^t_\infty, X_0) = \text{Var}(X_t | Y^t_0, X_0) \) for any \( t \geq 0 \). We may therefore use the relation in (70) in conjunction with (126) to write an expression for the information utility of lookahead for this process:

\[ 2U(t) = 2\tau t - \log \left( \frac{e^{2\tau t (\tau + \alpha) + (\tau - \alpha)}}{2\tau} \right) \]

\[ = - \log \left( e^{-2\tau t} + (1 - e^{-2\tau t}) \frac{\tau + \alpha}{2\tau} \right). \]
Since the Ornstein-Uhlenbeck process is also stationary and Gaussian, we may plug our expression for \( U(\cdot) \) into (71) and obtain the finite lookahead MMSE,

\[
\text{Imms}(d, \gamma) = \text{cmmse}(\gamma) \cdot e^{-2U(d)} = \text{cmmse}(\gamma) \left( e^{-2\tau_d} + \left(1 - e^{-2\tau_d}\right) \frac{\tau + \alpha}{2\tau} \right)
\]

(129)

\[
= e^{-2\tau_d} \text{cmmse}(\gamma) + \left(1 - e^{-2\tau_d}\right) \text{mmse}(\gamma),
\]

(130)

which recovers Lemma 4 for positive lookahead.

**APPENDIX D**

Let \( \tilde{X} \) be a stationary discrete time Markov chain, with known probability transition matrix \( P \). Let \( \tilde{Y}(\gamma) \) denote the noisy observation of \( \tilde{X} \), corrupted via independent Gaussian noise with the signal to noise ratio of the measurement being \( \gamma \). I.e.,

\[
\tilde{Y}(\gamma) = \gamma \tilde{X} + \tilde{W}^{(i)}(\gamma),
\]

where \( \{\tilde{W}^{(i)}\} \) are independent standard Brownian motions, which are further independent of the \( \tilde{X} \)'s. Let \( X \) denote the corresponding stationary continuous time process generated by the process described in Section VII-A and let \( Y \) denote the noisy process generated via the continuous time Gaussian channel (1) with input \( X \), i.e.

\[
dY_t = X_t dt + dW_t,
\]

(132)

where \( W \) is a standard Brownian motion independent of the \( X \) process.

Define

\[
h(\gamma_1, \gamma_2) = \text{Var} (\tilde{X}_0|Y_{-\infty}(1), Y_0(\gamma_1), Y_1(\gamma_2))
\]

(133)

Let, as before

\[
\text{Imms}(d) = \text{Var}(X_0|Y^d_{-\infty}).
\]

(134)

For \( 0 < d < 1 \), note that

\[
\text{Imms}(d) = \int_0^d h(1, d - u) \, du + \int_1^d h(1 + d - u, 0) \, du.
\]

(135)

**A. MCMC approach to estimating \( h(\cdot, \cdot) \)**

Note that \( \{\tilde{X}_i, \tilde{Y}_i\}_{i \geq 1} \) for a Hidden Markov process. By using state estimation for HMP's, the following computation can be performed for any \( i \),

\[
\hat{X}_i = \mathbb{E}[\tilde{X}_i|Y^{i-1}_1(1), \tilde{Y}_i(\gamma_1), \tilde{Y}_{i+1}(\gamma_2)]
\]

(136)

Also, one can observe that

\[
\frac{1}{M} \sum_{i=1}^{M} (\tilde{X}_i - \hat{X}_i)^2 \to h(\gamma_1, \gamma_2),
\]

(137)
as $M \to \infty$. In the simulations, we chose $M = 10000$. For this value of $M$, the quantity in the l.h.s. of (137) was used to approximate $h(\cdot, \cdot)$, which was then used in the expression for $\text{lmmse}(d)$ in (135) to obtain the desired values of MSE with finite lookahead, via a monte carlo approach.

Based on a similar approach, it is also possible to compute via MCMC, $\text{lmmse}(d)$ for $-1 < d < 0$.

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