Families of Singular Chern–Ricci Flat Metrics

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Abstract
We prove uniform a priori estimates for degenerate complex Monge–Ampère equations on a family of hermitian varieties. This generalizes a theorem of Di Nezza–Guedj–Guenancia to hermitian contexts. The main result can be applied to study the uniform boundedness of Chern–Ricci flat potentials in conifold transitions.

Keywords
Complex Monge–Ampère equations · Families of complex spaces · Calabi–Yau manifolds · Singular Chern–Ricci flat metrics

Mathematics Subject Classification 14D06 · 32Q25 · 32U05 · 32W20

1 Introduction
Let \( \pi : \mathcal{X} \to \mathbb{D} \) be a family of hermitian varieties (i.e. irreducible, reduced, complex analytic spaces). Recently, Di Nezza–Guedj–Guenancia [14] developed the first steps of a pluripotential theory in families of Kähler spaces and proved uniform bounds for Kähler–Einstein potentials in several cases. The main purpose of this article is to generalize their theory and establish uniform estimates for complex Monge-Ampère equations in families of hermitian varieties.

The complex Monge–Ampère equation is a powerful tool in complex geometry. Many interesting geometric problems (e.g. the Kähler–Einstein equation) can be reduced to such type of equations. Yau’s celebrated resolution of the Calabi conjecture [46] and the resolution of Yau–Tian–Donaldson conjecture on the Fano manifolds by Chen–Donaldson–Sun [7] are landmarks in smooth Kähler–Einstein problems.

In recent decades, following the works of Yau [46] and Tsuji [44], degenerate complex Monge–Ampère equations have been intensively studied. The breakthrough results of Kołodziej [31] and Eyssidieux–Guedj–Zeriahi [16] led to many important
advances. In [16], Yau’s theorem has been generalized to compact Kähler varieties with log terminal singularities. For varieties with ample canonical divisor and semi-log canonical singularities, Berman–Guenancia [4] applied the variational approach developed in [2] to extend Aubin–Yau’s result [1, 46] on stable varieties. On singular Fano varieties, Berman–Boucksom–Jonsson [3], Li–Tian–Wang [36], and Li [34] built a connection between singular Kähler–Einstein metrics and $K$-stability.

In hermitian contexts, the construction of hermitian Calabi–Yau metrics (i.e. Chern–Ricci flat metrics) is more difficult because the metrics are no longer closed. A Chern–Ricci flat hermitian metric on a complex manifold $X$ can be constructed by solving the complex Monge–Ampère equation:

$$\left(\omega + \dd c \varphi\right)^n = cf dV_X,$$

and $\sup_X \varphi = 0$

where

- $\omega$ is a smooth $(1, 1)$-form,
- $dV_X$ is a smooth volume form on $X$,
- $f \in L^p(X, dV_X)$ with $p > 1$,

and the pair $(\varphi, c) \in (\text{PSH}(X, \omega) \cap L^\infty(X)) \times \mathbb{R}_{\geq 0}$ is the unknown. When $\omega$ is a hermitian metric and $f$ is a smooth positive function, Tosatti–Weinkove [45] first showed the existence and uniqueness of the pair $(\varphi, c)$ with a smooth $\varphi$ to the above equation. For $L^p$ densities $f$, Dinew–Kołodziej [15] used pluripotential techniques to obtain uniform $L^\infty$-estimates. The solvability was further established by Kołodziej–Nguyen [30] via a stability estimate. Recently, Guedj–Lu [24] established uniform estimates and proved the existence of solution when the $(1, 1)$-form $\omega$ is merely big. As a consequence, they generalized Tosatti–Weinkove’s theorem to hermitian $Q$-Calabi–Yau varieties.

It is important to study non-Kähler objects and how special hermitian metrics evolve when complex structures vary. For example, to understand moduli spaces of Calabi–Yau manifolds, a large class of non-Kähler Calabi–Yau threefolds was built via conifold transitions introduced by Clemens and Friedman [9, 19]. Reid [39] speculated that all Calabi–Yau threefolds should form a connected family by conifold transitions. Since then, these models attracted a lot of attention (cf. [8, 10, 17, 20, 40–43] and references therein). This is our motivation to study families of singular Chern–Ricci flat metrics.

### 1.1 Uniform $L^\infty$-Estimate

Before stating our results, we first fix some geometric setting for families $\pi : \mathcal{X} \to \mathbb{D}$.

**Geometric setting (GS)** Let $\mathcal{X}$ be an $(n + 1)$-dimensional variety. Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a proper surjective holomorphic map with connected fibres $X_t := \pi^{-1}(t)$ which are $n$-dimensional varieties. Let $\omega$ be a hermitian metric on $\mathcal{X}$ in the sense of Definition 2.1. For every $t \in \mathbb{D}$, we define a hermitian metric $\omega_t$ on the fibre $X_t$ by restriction (i.e. $\omega_t = \omega|_{X_t}$).

In the sequel, we always assume that families of hermitian varieties $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ satisfy the geometric setting 1.1. We also impose a sup-$L^1$ comparison (see Conjec-
tured 1.2). Under such conditions, we establish a uniform bound for solutions to complex Monge-Ampère equations in families of hermitian varieties:

**Theorem A** Let \( \pi : (X, \omega) \to \mathbb{D} \) be a family of compact, locally irreducible, hermitian varieties and \( 0 \leq f_t \in L^p(X_t, \omega^n_t) \) be a family of densities. Assume that \( \pi : (X, \omega) \to \mathbb{D} \) fits into Conjecture 1.2 and \((f_t)_{t \in \mathbb{D}}\) satisfies the following integral bounds: there exist constants \( c_f, C_f > 0 \) such that for all \( t \in \mathbb{D} \),

\[
c_f \leq \int_{X_t} f_t^{1/2} \omega^n_t \quad \text{and} \quad \| f_t \|_{L^p(X_t, \omega^n_t)} \leq C_f.
\]

For each \( t \in \mathbb{D} \), let the pair \( (\varphi_t, c_t) \in (\text{PSH}(X_t, \omega_t) \cap L^\infty(X_t)) \times \mathbb{R}_{>0} \) be a solution to the complex Monge–Ampère equation:

\[
(\omega_t + \ddc \varphi_t)^n = c_t f_t \omega^n_t, \quad \text{and} \quad \sup_{X_t} \varphi_t = 0.
\]

Then there exists a constant \( C_{MA} = C_{MA}(c_f, C_f, C_{SL}, X, \omega) \) such that for all \( t \in \mathbb{D}_{1/2} \),

\[
c_t + c_t^{-1} + \| \varphi_t \|_{L^n} \leq C_{MA}.
\]

**1.2 Sup-L¹ Comparison Conjecture**

In pluripotential theory, there is a conjecture proposed by Di Nezza–Guedj–Guenancia [14, Conjecture 3.1] which says that if \( X_0 \) is irreducible then one has the following sup-L¹ comparison:

**Conjecture** (SL) There exists a constant \( C_{SL} > 0 \) such that: the inequality

\[
\forall \varphi_t \in \text{PSH}(X_t, \omega_t), \quad \sup_{X_t} \varphi_t - C_{SL} \leq \frac{1}{V_t} \int_{X_t} \varphi_t \omega^n_t \leq \sup_{X_t} \varphi_t
\]

holds for all \( t \in \mathbb{D}_{1/2} \), where \( V_t \) is the volume of \( X_t \) with respect to the hermitian metric \( \omega_t \).

In the Kähler setting, Di Nezza–Guedj–Guenancia [14, Proposition 3.3] established Conjecture 1.2 in the following cases:

(i) The map \( \pi \) is locally trivial or projective;
(ii) The fibres \( X_t \) are smooth for \( t \neq 0 \);
(iii) The fibres \( X_t \) have only isolated singularities for every \( t \in \mathbb{D} \).

One should notice that the irreducibility of all the fibres is a necessary condition for the left-hand side inequality in Conjecture 1.2 (cf. [14, Example 3.5]), and it is the reason why we always assume that the fibres are irreducible in the geometric setting 1.1.

To establish Conjecture 1.2 in hermitian setting, we impose the following assumptions:
**Geometric assumption** (GA) Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a family of hermitian varieties which satisfies the geometric setting 1.1 and one of the following conditions:

(i) $\pi$ is locally trivial;

(ii) $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing of $X_0$ and $X_0$ has only isolated singularities.

Note that both conditions are naturally exclusive unless $X_0$ is smooth. Also, if $\mathcal{X}$ is smooth and $\pi$ is a submersion, then (i) holds. Thus, the geometric assumption 1.2 includes families of smooth hermitian manifolds. Then we prove that, under the geometric assumption 1.2, Conjecture 1.2 is fulfilled:

**Proposition B** If $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ is a family of hermitian varieties satisfying the geometric assumption 1.2, then there exists a uniform constant $C_{SL}$ such that Conjecture 1.2 holds.

### 1.3 Families of Calabi–Yau Varieties

A Calabi–Yau variety $X$ is a normal variety with canonical singularities and trivial canonical bundle $K_X$. Reid [39] has conjectured that all Calabi–Yau threefolds should form a connected family, provided one allows conifold transitions. Roughly speaking, the construction of a conifold transition goes as follows: contracting a collection of disjoint $(-1, -1)$-curves from a Kähler Calabi–Yau threefold $X$ to get a singular Calabi–Yau variety $X_0$ and then smoothing singularities of $X_0$, one obtains a family of Calabi–Yau threefolds $(X_t)_{t \neq 0}$ which are non-Kähler for a general $t$.

In the model of conifold transitions, the central fibre $X_0$ has only ordinary double point singularities which are canonical. Based on these geometric models, it is, thus, legitimate to study a smoothing family of Calabi–Yau varieties where the central fibre has only isolated singularities.

Now, we consider a reasonable "good" family of Calabi–Yau varieties and ask how the bound on the Chern-Ricci potentials vary in families. Assume that $\mathcal{X}$ is a normal variety, $K_\mathcal{X}$ is trivial and $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing. Moreover, we suppose that $X_0$ has only isolated canonical singularities. One can find a trivializing section $\Omega$ of $K_{\mathcal{X}/\mathbb{D}}$. The restriction on each fibre $\Omega_t := \Omega|_{X_t}$ defines a trivialization of $K_{X_t}$. Following from [24, Theorem E], for each $t$, there is a bounded solution to the corresponding complex Monge–Ampère equation of canonical density. Then we show a uniform estimate in families:

**Theorem C** Suppose that $\mathcal{X}$ is normal, $K_\mathcal{X}$ is trivial, and $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing of a variety $X_0$ whose singularities are canonical and isolated. For each $t \in \mathbb{D}$, let $(\varphi_t, c_t) \in (\text{PSH}(X_t, \omega_t) \cap L^\infty(X_t)) \times \mathbb{R}_{>0}$ be a pair solving the complex Monge–Ampère equation:

$$(\omega_t + dd^c \varphi_t)^n = c_t \Omega_t \wedge \overline{\Omega}_t \quad \text{and} \quad \sup_{X_t} \varphi_t = 0.$$ 

Then there is a uniform constant $C_{\text{MA}}$ such that for all $t \in \mathbb{D}_{1/2}$

$$c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty} \leq C_{\text{MA}}.$$
1.4 Structure of the Article

The paper is organized as follows:

– In Sect. 2, we recall basic notions in pluripotential theory and singular spaces.
– Section 3 is a recap on methods to obtain $L^\infty$-estimates in local and global cases.
– In Sect. 4, we study the local and global uniform Skoda estimates.
– In Sect. 5, we deal with the volume-capacity comparison stated in Sect. 3.
– In Sect. 6, we establish Conjecture 1.2 under the geometric assumption 1.2.
– In Sect. 7, we focus on families of Calabi–Yau varieties and show Theorem C.

2 Preliminaries

In this section, we recall some definitions, notations, and conventions which will be used in the sequel. We define the twisted exterior derivative by $d^c = \frac{i}{2}(\bar{\partial} - \partial)$ and we then have $dd^c = i\partial\bar{\partial}$. We denote by

– $\mathbb{D}_r := \{ z \in \mathbb{C} | |z| < r \}$ the open disc of radius $r$;
– $\mathbb{D}_r^* := \{ z \in \mathbb{C} | 0 < |z| < r \}$ the punctured disc of radius $r$.

When $r = 1$, we simply write $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{D}^* := \mathbb{D}_1^*$.

2.1 Smooth Forms and Currents on Singular Spaces

Let $X$ be a reduced complex analytic space of pure dimension $n \geq 1$. We denote by $X^{\text{reg}}$ the complex manifold of regular points of $X$ and $X^{\text{sing}} := X \setminus X^{\text{reg}}$ the singular set of $X$. Now, we recall definitions of smooth forms and currents on a complex analytic space $X$:

Definition 2.1 We say that

(i) A smooth form $\alpha$ on $X$ is the data of a smooth form on $X^{\text{reg}}$ such that given any local embedding $X \hookrightarrow \mathbb{C}^N$, $\alpha$ extends smoothly to $\mathbb{C}^N$;

(ii) A smooth hermitian metric $\omega$ on $X$ is a smooth $(1, 1)$-form which locally extends to a hermitian metric on $\mathbb{C}^N$;

(iii) $\mathcal{D}_{p,q}(X)$ (resp. $\mathcal{D}_{p,p}(X)^{\mathbb{R}}$) is the space of compactly supported (resp. real) smooth forms of bidegree $(p, q)$;

(iv) The notion of currents, $\mathcal{D}_{p,q}'(X)$ (resp. $\mathcal{D}_{p,p}'(X)^{\mathbb{R}}$), is defined by acting on (resp. real) smooth forms with compact support.

The operators $d$, $d^c$, and $dd^c$ are well defined by duality (see [13] for details).

2.2 Plurisubharmonic Functions

Let $\Omega$ is an open domain in $\mathbb{C}^n$. We say that $u$ is a plurisubharmonic function (psh for short) on $\Omega$ if it is upper semicontinuous and satisfies the sub-mean inequality on
each complex line through every point $x \in \Omega$:

$$u(x) \leq \frac{1}{2\pi} \int_{0}^{2\pi} u(x + \zeta e^{i\theta}) d\theta, \quad \forall x \in \Omega \text{ and } \forall \zeta \in \mathbb{C}^n \text{ such that } |\zeta| < \text{dist}(x, \partial\Omega).$$

We denote by $\text{PSH}(\Omega)$ the space of all psh functions on $\Omega$.

Suppose that $X$ is a reduced complex analytic space equipped with a hermitian metric $\omega$.

**Definition 2.2** Let $u : X \to [-\infty, \infty)$ be a given function. We say that

(i) $u$ is a psh function on $X$ if it is locally the restriction of a psh function under local embeddings of $X$ into $\mathbb{C}^N$;

(ii) $u$ is quasi-plurisubharmonic (qpsh for short) on $X$ if it can be locally written as the sum of a psh and a smooth function;

(iii) $\text{PSH}(X, \omega)$ is the set of all $\omega$-plurisubharmonic (abbreviated to $\omega$-psh), namely, the set of all qpsh functions $u$ which satisfies $\omega + dd^c u \geq 0$ in the sense of currents.

**Remark 2.3** There is a weaker notion of (quasi-)plurisubharmonic functions that are defined only as being a locally bounded function on a variety $X$ and its restriction to the complex manifold $X^\text{reg}$ is (quasi-)plurisubharmonic. On a locally irreducible variety, the stronger notion given above and the weaker notion are equivalent (see [13, Théorème 1.7]). In this article, we assume that $X$ also is locally irreducible in some places in order to make sense of the envelope constructions that might not be (quasi-)plurisubharmonic (in the strong sense) on locally reducible complex spaces.

### 2.3 Lelong Numbers

Lelong numbers describe the local behaviour of currents or psh functions near a point at which it has a log pole. Here we recall a generalized definition given by Demailly:

**Definition 2.4** ([12, Définition 3]) Let $X$ be a complex analytic space. If $T$ is a closed positive $(p, p)$-current on $X$ and if $x \in X$ is a fixed point, then the Lelong number of $T$ at $x$ is defined as the decreasing limit:

$$\nu(T, x) := \lim_{r \to 0} \frac{1}{r^{2(n-p)}} \int_{|\psi| < r} T \wedge (dd^c \psi)^{n-p} = \int_{\{x\}} T \wedge (dd^c \log \psi)^{n-p}$$

where $\psi = \sum_{i \in I} |g_i|^2$ and $(g_i)_{i \in I}$ is any finite system of generators of the maximum ideal $m_{X,x} \subset \mathcal{O}_{X,x}$.

### 2.4 Monge–Ampère Capacities

The notion of Monge–Ampère capacities was given by Bedford and Taylor [6]. Using Monge–Ampère capacities, they proved that the negligible sets are pluripolar.
**Definition 2.5** Let \( E \subset \Omega \) be a Borel subset. The Bedford–Taylor capacity (or Monge–Ampère capacity) is defined by

\[
\text{Cap}(E; \Omega) := \sup \left\{ \int_E (ddc u)^n \mid u \in \text{PSH}(\Omega) \text{ and } -1 \leq u \leq 0 \right\}.
\]

**Theorem 2.6** [6] A subset \( E \subset \Omega \) is pluripolar if and only if \( \text{Cap}(E; \Omega) = 0 \).

For global versions, Kołodziej [33] first defined the Monge–Ampère capacity on a given compact Kähler manifold \((X, \omega)\). The definition is analogous to the local cases:

**Definition 2.7** Let \( E \subset X \) be a Borel subset. Define

\[
\text{Cap}_\omega(E) := \left\{ \int_E (\omega + dd^c u)^n \mid u \in \text{PSH}(X, \omega) \text{ and } -1 \leq u \leq 0 \right\}.
\]

These definitions can also be extended to non-closed or non-positive form \( \omega \) and singular complex analytic space \( X \) (cf. [13, 16, 21, 28]).

### 3 Uniform \( L^\infty \)-Estimate

In this section, we mainly pay attention to \( L^\infty \)-estimates of complex Monge-Ampère equations on pseudoconvex domains and compact hermitian varieties. We shall follow the method given by Guedj–Kołodziej–Zeriahi [22] and Guedj–Lu [24] to produce a priori estimates. We also compute the precise dependence of these \( L^\infty \)-estimates on background data.

#### 3.1 Local \( L^\infty \)-Estimate

In this section, our goal is to establish a refined version of Kołodziej’s a priori estimate [31] of complex Monge-Ampère equation on a singular strongly pseudoconvex domain. We recall the definition of a strongly pseudoconvex domain on a Stein space as in [21, Sect. 1]. Let \( S \) be a singular Stein space which is reduced and locally irreducible, of complex dimension \( n \geq 1 \). There is a proper embedding \( S \hookrightarrow \mathbb{C}^N \) for some \( N \) large. A domain \( \Omega \subset S \) is strongly pseudoconvex if it admits a negative smooth psh exhaustion, i.e., a function \( \rho \) smooth strongly psh in a neighborhood \( \Omega' \) of \( \Omega \) such that \( \Omega := \{ x \in \Omega' \mid \rho(x) < 0 \} \) and for any \( c < 0, \Omega_c := \{ x \in \Omega' \mid \rho(x) < c \} \subset \Omega \) is relatively compact. Fix a hermitian metric \( \beta \) on \( \mathbb{C}^N \) and define a volume form on \( S \) by taking \( dV = \beta^n|_S \).

First, we note that the following estimate always holds:

**Volume-capacity comparison (VC)** For every \( k > 1 \), there exists a constant \( C_{VC,k} > 0 \) such that

\[
\forall K \subset \Omega, \quad \text{Vol}(K) \leq C_{VC,k} \text{Cap}^k(K; \Omega).
\]
The proof of the volume-capacity comparison will be given in Sect. 5 not only in a fixed pseudoconvex set $\Omega$ but also for families.

Fix a density $0 \leq f \in L^p(\Omega, dV)$. Suppose that $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ is the solution to the following Dirichlet problem of complex Monge–Ampère equation

$$
\begin{aligned}
(ddc \varphi)^n &= f \, dV \quad \text{in } \Omega, \\
\varphi &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
$$

(\text{locMA})

In smooth setting, the existence, uniqueness and the $L^\infty$-estimate of the continuous solution of (locMA) has been constructed by Kołodziej [31]. The existence and uniqueness have been extended by Guedj–Guenancia–Zeriahi [21, Theorem A] to singular contexts. Now, for singular setup, we prove the following refined version of Kołodziej’s $L^\infty$-estimate:

**Theorem 3.1** (Kołodziej’s $L^\infty$-estimate) Fix $0 \leq f \in L^p(\Omega, dV)$ with $p > 1$. Suppose that $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ is the solution to the complex Monge–Ampère equation (locMA). Then

$$\|\varphi\|_{L^\infty} \leq C_{\text{Kol}, p} \|f\|_{L^p}^{\frac{1}{n}}$$

where

$$C_{\text{Kol}, p} := \left[ 1 + \left( \frac{e}{1 - e^{-1}} \right) C_{V, C, 2q}^{1/q_n} 2^{1+\frac{1}{q}} \text{Vol}^{\frac{1}{nq}}(\Omega) \left( n! C_\rho \|\rho\|_{L^\infty}^n \right)^{\frac{1}{nq}} \right].$$

$1/p + 1/q = 1$, $dV \leq C_\rho (ddc \rho)^n$, and $C_{V, C, 2q} > 0$ is a constant in the volume-capacity comparison 3.1 such that $\text{Vol}(K) \leq C_{V, C, 2q} \text{Cap}^{2q}(K; \Omega)$ for all $K \Subset \Omega$.

**Proof** The idea of proof comes from the work of Guedj–Kołodziej–Zeriahi [22, Section 1].

We are going to prove the following statement: given $\varepsilon > 0$, we have $\|\varphi\|_{L^\infty(\Omega)} \leq M_\varepsilon$ where

$$M_\varepsilon = \varepsilon + \left( \frac{e}{1 - e^{-1}} \right) C_{V, C, 2q}^{1/q_n} \left( \frac{2}{\varepsilon} \right)^{1+\frac{1}{q}} \|f\|_{L^p}^{\frac{2}{n} + \frac{1}{nq}} \text{Vol}^{\frac{1}{nq^2}}(\Omega) \left( n! C_\rho \|\rho\|_{L^\infty}^n \right)^{\frac{1}{nq}}.$$

In particular, when $\varepsilon = \|f\|_{L^p}^{\frac{1}{n}}$, one get the desired estimate. Before explaining the proof, we recall some useful facts. For simplicity, we denote by $\text{Cap}(\bullet) = \text{Cap}(\bullet; \Omega)$. First, we recall some basic lemmas:

**Lemma 3.2** Fix $\varphi, \psi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $\lim \inf_{z \to \partial \Omega} (\varphi - \psi) \geq 0$. Then for all $t, s > 0$ we have

$$t^n \text{Cap}(\{\varphi - \psi < -s - t\}) \leq \int_{\{\varphi - \psi < -s\}} (ddc \varphi)^n.$$

By definition, for all $u \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$, complex Monge-Ampère measures $(ddc u)^n$ put zero mass on $\Omega^{\text{sing}}$; hence we still have the comparison principle on $\Omega$ (cf.
Then one can follow exactly the same argument in [22, Lemma 1.3] to obtain Lemma 3.2.

Using the volume-capacity comparison 3.1 and Hölder’s inequality, one has the estimate as follows

**Lemma 3.3** For all $\tau > 1$, there exists a constant $D_\tau := C_{V_C,k}^{\frac{p-1}{p}} \| f \|_{L^p}$ where $k = k(\tau, p) := \frac{\tau p}{(p-1)} = \tau q$ such that

$$\forall K \subset \Omega, \quad 0 \leq \int_K f \, dV \leq D_\tau \text{Cap}^\tau(K).$$

The following classical lemma is due to De Giorgi and the reader is referred to [22, Lemma 1.5] and [16, Lemma 2.4] for the proof.

**Lemma 3.4** Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a decreasing right-continuous function and satisfy $\lim_{s \to \infty} g(s) = 0$. Assume that there exists $\tau > 1$, $B > 0$ such that $g$ satisfies

$$\forall s, t > 0, \quad tg(s + t) \leq B(g(s))^\tau.$$

Then $g(s) = 0$ for all $s \geq s_\infty$, where

$$s_\infty = \frac{e B(g(0))^{\tau-1}}{1 - e^{1-\tau}}.$$

By Lemmas 3.2 and 3.3, we have

$$t^n \text{Cap}(\{\varphi < -s - t\}) \leq \int_{\{\varphi < -s\}} (dd^c \varphi)^n = \int_{\{\varphi < -s\}} f \, dV \leq D_2 \text{Cap}^2(\{\varphi < -s\})$$

where $D_2 = C_{V_C,2q}^{1/q} \| f \|_{L^p}$. Thus,

$$t \text{Cap}^{1/n}(\{\varphi < -s - t\}) \leq D_2^{1/n} \text{Cap}^{2/n}(\{\varphi < -s\}).$$

Let $g(s) := \text{Cap}^{1/n}(\{\varphi < -s - \varepsilon\})$. Then we have $tg(s + t) \leq Bg(s)^2$ where $B = D_2^{1/n}$. Using Lemma 3.4, one obtains $\text{Cap}(\{\varphi < -s - \varepsilon\}) = 0$ for all $s \geq s_\infty = \frac{e Bg(0)}{1 - e^{1-\tau}}$. This implies that $\varphi \geq -s_\infty - \varepsilon$ almost everywhere and thus everywhere by plurisubharmonicity. Therefore, one has

$$\sup_{\Omega} (-\varphi) \leq \varepsilon + s_\infty = \varepsilon + \frac{e Bg(0)}{1 - e^{-\tau}}.$$

Now, we need to control $g(0) = \text{Cap}^{1/n}(\{\varphi < -\varepsilon\})$. By Lemma 3.2 and Chebyshev inequality for a fixed constant $r > 0$, we have

$$\left(\frac{\varepsilon}{2}\right)^n \text{Cap}\left(\left\{\varphi < -\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right\}\right) \leq \int_{\{\varphi < -\varepsilon/2\}} (dd^c \varphi)^n = \int_{\{\varphi < -\varepsilon/2\}} f \, dV.$$
\[
\leq \int_{\Omega} \left( \frac{-2\varphi}{\varepsilon} \right)^r f \, dV
\leq \|f\|_{L^p} \left( \int_{\Omega} \left( \frac{-2\varphi}{\varepsilon} \right)^{rq} \, dV \right)^{1/q}.
\]

Put \( r = \frac{n}{q} \). Note that integration by parts is legitimate in our setting (see \[14, Lemma 2.11\]). By Błocki’s estimate of integration by parts \[5, Theorem 2.1\] and Hölder inequality, one can derive that

\[
\int_{\Omega} (-\varphi)^n \, dV \leq C_\rho \int_{\Omega} (-\varphi)^n (dd^c \rho)^n \leq n! C_\rho \|\rho\|_{L^\infty}^n \int_{\Omega} (dd^c \varphi)^n \leq n! C_\rho \|\rho\|_{L^\infty}^n \text{Vol}^{\frac{1}{q}}(\Omega) \|f\|_{L^p}.
\]

One can infer

\[
g(0)^n = \text{Cap}(\{\varphi < -\varepsilon\}) \leq \left( \frac{2}{\varepsilon} \right)^{n+\frac{n}{q}} \|f\|_{L^p}^{1+\frac{1}{q}} \text{Vol}^{\frac{1}{q}}(\Omega) \left( n! C_\rho \|\rho\|_{L^\infty}^n \right)^{\frac{1}{q}}.
\]

All in all, we obtain the desired estimate:

\[
\|\varphi\|_{L^\infty} \leq \varepsilon + \left( \frac{e}{1 - e^{-1}} \right) C_{\text{Kol},p} \frac{1}{C_{\text{Kol},2q}} \left( \frac{2}{\varepsilon} \right)^{1+\frac{1}{q}} \|f\|_{L^p}^{\frac{2}{q} + \frac{1}{mq}} \text{Vol}^{\frac{1}{mq}}(\Omega) \left( n! C_\rho \|\rho\|_{L^\infty}^n \right)^{\frac{1}{mq}}.
\]

\[ \square \]

### 3.2 Global \( L^\infty \)-Estimate

Suppose that \((X, \omega)\) is an \(n\)-dimensional compact locally irreducible hermitian variety. Fix a function \(0 \leq f \in L^p(X, \omega^n)\) with \(p > 1\). First, we fix some notations:

(i) Denote the volume of \(X\) with respect to \(\omega^n\) by \(V\);
(ii) A constant \(B' > 0\) is such that \(-B' \omega^n \leq dd^c \omega^{n-1} \leq B' \omega^n\);
(iii) Fix a finite double cover \((\Omega_j := \{\rho_j < 0\})_{1 \leq j \leq N}\) and \((\Omega'_j := \{\rho_j < -c_j\})_{1 \leq j \leq N}\) of \(X\) where for each \(j\), the function \(\rho_j\) is smooth on \(X\), strictly psh near \(\Omega_j\), and \(0 \leq \rho_j \leq 1\) on \(X \setminus \Omega_j\), and \(c_j > 0\) is a constant;
(iv) \(C_{\text{Kol},p}\) is a constant such that \(\|u_j\|_{L^\infty(\Omega_j)} \leq C_{\text{Kol},p} \|f\|_{L^p(\Omega_j, \omega^n)}^{\frac{1}{q}}\) where for each \(1 \leq j \leq N\), the function \(u_j\) is the solution to the Dirichlet problem

\[
\begin{cases}
(dd^c u_j)^n = f \omega^n & \text{in } \Omega_j, \\
u_j = 0 & \text{on } \partial \Omega_j.
\end{cases}
\]

(v) \(A_\rho\) is a constant such that \(A_\rho \omega + dd^c \rho > 0\) on \(X\) for all \(1 \leq j \leq N\);
(vi) \(c_\rho = \min_{1 \leq j \leq N} c_j > 0\).
In this section, we fix some geometric constants and impose two integral bounds on the density $f$:

**Geometric constants** (SL) There is a constant $C_{SL} > 0$ such that the following inequality holds

$$\forall \varphi \in \text{PSH}(X, \omega), \sup_X \varphi - C_{SL} \leq \frac{1}{V} \int_X \varphi \omega^n \leq \sup_X \varphi;$$

**Geometric constants** (Skoda) There exist $\alpha > 0$ and $A_{\alpha} > 0$ such that

$$\forall u \in \text{PSH}(X, \omega), \int_X e^{\alpha \left( \sup_X u - u \right)} \omega^n \leq A_{\alpha}.$$  

**Analytic constants** (AC) Let $0 \leq f \in L^p(X, \omega)$ for some $p > 1$. There are two constants $c_f, C_f > 0$ such that

$$c_f \leq \int_X f^{1/n} \omega^n \quad \text{and} \quad \|f\|_{L^p} \leq C_f.$$  

Following the strategy in [24], we shall prove an a priori $L^\infty$-estimate of complex Monge-Ampère equations on hermitian varieties.

**Theorem 3.5** Let $(\varphi, c) \in (\text{PSH}(X, \omega) \cap L^\infty(X)) \times \mathbb{R}_{>0}$ be a pair solving the complex Monge–Ampère equation

$$(\omega + dd^c \varphi)^n = cf \omega^n \quad \text{and} \quad \sup_X \varphi = 0.$$  

There exists a uniform positive constant $C_{MA} > 0$ such that

$$c + c^{-1} + \|\varphi\|_{L^\infty(X)} \leq C_{MA}$$

and the constant $C_{MA}$ depends only on $n, V, B', N, A_\rho, c_\rho, \alpha, A_\alpha, C_{Kol}, p, c_f, C_f,$ and $C_{SL}.$

**Remark 3.6** Suppose that $X$ is a compact normal variety and $\mu : \tilde{X} \to X$ is a resolution of singularities. Following [24, Theorem B], there exists a pair $(\varphi, c)$ solving the corresponding complex Monge-Ampère equation on $\tilde{X}$. Let $E = \text{Exc}(\mu)$ be the exceptional divisor of $\mu$ which is analytic and thus pluripolar. Since the complex Monge–Ampère measure $(\mu^* \omega + dd^c \varphi)^n$ charges no mass on $E$, one can descend the solution to $X^{\text{reg}}$. By the extension theorem of Grauert and Remmert [25, Satz 4] and normality of $X$, the $\mu_* \varphi$ induces a $\omega$-psh function on $X$ and it solves the complex Monge–Ampère equation.
3.2.1 Upper Bound of \( c \)

Following the same idea as in [30, Lemma 5.9], one can find an upper bound of \( c \) simply using the arithmetic-geometric mean inequality:

**Lemma 3.7** With the geometric constant \( C_{SL} \) in 3.2, one has

\[
\begin{align*}
\int_X (\omega + dd^c u) \wedge \omega^{n-1} &= \int_X \omega^n + \int_X (u - \sup u) dd^c \omega^{n-1} \\
&\leq V(1 + B'C_{SL}) =: C_{\text{Lap}}.
\end{align*}
\]

**Proof** For all \( u \in \text{PSH}(X, \omega) \), we compute

\[
\frac{1}{n} \int_X (\omega + dd^c u) \wedge \omega^{n-1} = \frac{1}{n} \int_X \left( \frac{\omega^n}{\omega^{n}} \right) \chi^\frac{1}{n} \omega^n \leq \frac{1}{n} \int_X (\omega + dd^c \varphi) \wedge \omega^{n-1} \leq \frac{1}{n} C_{\text{Lap}}.
\]

Rearranging the inequality, we obtain an upper bound of \( c \) as desired. \( \square \)

3.2.2 Domination Principle

The domination principle has been proved under several setup (cf. [23, 24, 35, 37] and references therein). Here we establish the following domination principle on singular varieties:

**Lemma 3.8** (Domination principle) Let \( u, v \in \text{PSH}(X, \omega) \cap L^\infty(X) \). Then we have the following properties

(i) if \((\omega + dd^c u)^n \leq c(\omega + dd^c v)^n \), then \( c \geq 1 \);

(ii) if \( e^{-\lambda u} (\omega + dd^c u)^n \leq e^{-\lambda v} (\omega + dd^c v)^n \) for some \( \lambda > 0 \), then \( v \leq u \).

**Sketch of proof.** Let \( \mu : \tilde{X} \to X \) be a resolution of singularities. The \( (1,1) \)-form \( \mu^* \omega \) is semi-positive and big (cf. [24, Definition 1.6]). For all functions \( u \in \text{PSH}(X, \omega) \cap L^\infty(X) \), the Monge-Ampère measure \((\mu^* \omega + dd^c \mu^* u)^n \) puts no mass on the exceptional divisor \( E := \text{Exc}(\mu) \) since \( E \) is a pluripolar set. On the other hand, by definition, \((\omega + dd^c u)^n \) charges no mass on \( X^{\text{sing}} \) for all \( u \in \text{PSH}(X, \omega) \cap L^\infty(X) \). Hence, one can descend the domination principles in [24, Corollary 1.13 and 1.14] to \( X \). Then one can conclude Lemma 3.8. \( \square \)

3.2.3 Subsolution Estimate

In [24, Theorem 2.1], Guedj and Lu constructed a subsolution \((\psi, m)\) to the complex Monge–Ampère equation with a given \( L^p \)-density \( g \). We follow the same method to get subsolution estimate on singular varieties while also carefully keeping track of the dependence of data.
Proposition 3.9 For all $p > 1$, there exist uniform constants $m_p, M_p > 0$ such that for every $0 \leq g \in L^p(X, \omega)$ with $\|g\|_{L^p} = 1$, there is a function $\psi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ satisfying

$$(\omega + \ddc \psi)^n \geq m_p g \omega^n \quad \text{and} \quad \text{osc}_X \psi \leq M_p.$$ 

Precisely, the constants $m_p$ and $M_p$ can be taken as

$$m_p = \frac{c_\rho}{NA_\rho \left(C_{\text{Kol}, p} + 1\right)} \quad \text{and} \quad M_p = C_{\text{Kol}, p} \left(\frac{1}{A_\rho} + 1\right).$$

Proof For each $j$, let the function $u_j \in \text{PSH}(\Omega_j) \cap L^\infty(\Omega_j)$ be the unique solution to the following complex Monge–Ampère equation

$$
\left\{
\begin{aligned}
(dd^c u_j)^n &= g \omega^n \quad \text{in} \quad \Omega_j, \\
u_j &= -1 \quad \text{on} \quad \partial \Omega_j.
\end{aligned}
\right.
$$

From Theorem 3.1, there is a constant $C_{\text{Kol}, p} > 0$ such that for all $j$,

$$\|u_j\|_{L^\infty} \leq C_{\text{Kol}, p} \|g\|^{1/n}_{L^p(\Omega_j, \omega^n)} + 1 \leq C_{\text{Kol}, p} + 1.$$

Now, we consider the psh functions $(v_j)_j$ defined by $v_j := \max\left\{u_j, \frac{C_{\text{Kol}, p} + 1}{c_j} \rho_j \right\}$ for each $j$. One can see that the following properties are satisfied

- $v_j = u_j$ in $\Omega_j' \text{ and } (dd^c v_j)^n = g \omega^n \text{ in } \Omega_j'$;
- $v_j = \frac{C_{\text{Kol}, p} + 1}{c_j} \rho_j$ on $X \setminus \Omega_j$ \text{ and near the boundary of } $\partial \Omega_j$.

From the construction, one can check that for every $j$, \(\left(\frac{A_\rho(C_{\text{Kol}, p} + 1)}{c_\rho}\right) \omega + dd^c v_j \geq 0\) as well. We define the subsolution $\psi$ as follows

$$\psi = \frac{c_\rho}{NA_\rho \left(C_{\text{Kol}, p} + 1\right)} \sum_{j=1}^N v_j.$$

Note that in $\Omega_j'$, we have

$$(\omega + dd^c \psi)^n = \left(\frac{c_\rho}{NA_\rho \left(C_{\text{Kol}, p} + 1\right)} \sum_{j=1}^N \left[A_\rho \left(\frac{C_{\text{Kol}, p} + 1}{c_\rho}\right) \omega + dd^c v_j \right]\right)^n \geq \frac{c_\rho}{NA_\rho \left(C_{\text{Kol}, p} + 1\right)} (dd^c u_j)^n = \frac{c_\rho}{NA_\rho \left(C_{\text{Kol}, p} + 1\right)} g \omega^n.$$

Then we derive that

$$\psi \leq \frac{1}{A_\rho} \quad \text{and} \quad \psi \geq -C_{\text{Kol}, p} - 1 \implies \text{osc}_X \psi \leq \frac{1}{A_\rho} + C_{\text{Kol}, p} + 1.$$

\qed
3.2.4 $L^\infty$-Estimate

We now prove an a priori $L^\infty$-estimate following the method in [24, Theorem 2.1].

**Theorem 3.10** Let $(X, \omega)$ be a compact hermitian variety with $\dim \mathbb{C} X = n$. Fix a density $0 \leq f \in L^p(X, \omega^n)$. Assume that $c_f \leq \int_X f^{\frac{1}{n}} \omega^n$ for a constant $c_f > 0$. Let the pair $(\varphi, c) \in (\text{PSH}(X, \omega) \cap L^\infty(X)) \times \mathbb{R}_{>0}$ be a solution to (MA). Then one has

$$c \geq \frac{m_{p+1}}{A_{\alpha}^{\frac{p-1}{p+1}} c_f}, \quad c \leq \left( \frac{C_{\text{Lap}}}{nc_f} \right)^n,$$

and $\|\varphi\|_{L^\infty} \leq M_{p+1} + \frac{2p(p+1)}{\alpha(p-1)} \left( \log \frac{A_{\alpha}^{\frac{p-1}{p+1}} c_f}{m(nc_f)^n} \right)^{\alpha}.$

**Proof** We consider a twisted function $g' = e^{-\varepsilon \varphi} f$ for some $\varepsilon > 0$. Fix constants $p' = \frac{p+1}{2} \in (1, p)$ and $\varepsilon = \frac{\alpha(p-1)}{2p(p+1)} = \frac{\alpha(p-p')}{2pp'} \in \left(0, \frac{\alpha(p-p')}{pp'}\right)$. One can derive that $g' \in L^{p'}$. Indeed, by Hölder inequality, we have

$$\|g'\|_{L^{p'}} \leq \|e^{-\varepsilon \varphi}\|_{L^{p'}} \|f\|_{L^p} \leq A_{\alpha}^{\frac{p-p'}{pp'}} \|f\|_{L^p}.$$

Put $g = g' / \|g'\|_{L^{p'}}$. From Proposition 3.9, we have a bounded $\omega$-psh function $\psi$ with $\sup_X \psi = 0$ such that

$$(\omega + \ddc \psi)^n \geq m_{p'} g \omega^n = m_{p'} \frac{e^{-\varepsilon \varphi} f}{c \|g'\|_{L^{p'}}} \omega^n = \frac{m_{p'} e^{-\varepsilon \varphi}}{c \|g'\|_{L^{p'}}} c_f \omega^n$$

$$= \frac{m_{p'} e^{-\varepsilon \varphi}}{c \|g'\|_{L^{p'}}} (\omega + \ddc \varphi)^n$$

$$\geq \frac{m_{p'}}{c \|g'\|_{L^{p'}}} (\omega + \ddc \varphi)^n.$$

and $\|\psi\|_{L^\infty} \leq M_{p+1}$. By Lemma 3.8, one get $\frac{m_{p'}}{c \|g'\|_{L^{p'}}} \leq 1$; hence $c$ has a lower bound,

$$c \geq \frac{m_{p'}}{A_{\alpha}^{\frac{p-p'}{pp'}} \|f\|_{L^p}}.$$ 

Also, we see that

$$e^{-\varepsilon \psi} (\omega + \ddc \psi) \geq \frac{m_{p'} e^{-\varepsilon \varphi}}{c \|g'\|_{L^{p'}}} (\omega + \ddc \varphi)^n$$

$$= \exp \left( -\varepsilon \left( \varphi - \frac{1}{\varepsilon} \log \frac{m_{p'}}{c \|g'\|_{L^{p'}}} \right) \right) (\omega + \ddc \varphi)^n.$$

Applying the domination principle again, one can infer

$$M_{p'} \leq \psi \leq \varphi - \frac{1}{\varepsilon} \left( \log \frac{m_{p'}}{c \|g'\|_{L^{p'}}} \right).$$
Finally, this provides a uniform $L^\infty$-estimate as follows

$$\|\varphi\|_{L^\infty} \leq M_{p'} - \frac{1}{\varepsilon} \left( \log \frac{m}{c \|g'\|_{L^{p'}}} \right) \leq M_{p'} + \frac{1}{\varepsilon} \left( \log \frac{A_{p-p'}^p \|f\|_{L^p} C_{\text{Lap}}^n}{m(n c f)^n} \right).$$

\[\square\]

### 3.3 Proof of Theorem A

Recall that the constant $C_{\text{MA}}$ depends only on $n, V, B', N, A_\rho, c_\rho, \alpha, A_\alpha, C_{\text{Kol}}, p, c_f, C_f,$ and $C_{\text{SL}}$. We shall control these constant to get Theorem A. The following data are included in the assumption of Theorem A:

- $n$ is fixed in the geometric setting 1.1;
- $p, c_f, C_f$ are data in the integral bounds (IB);
- $C_{\text{SL}}$ is given by Conjecture 1.2.

Then fixing some choices of background data, we have uniform control of the following constants:

- $A_\rho, c_\rho, N$: After shrinking $D$, we can cover $\mathcal{X}$ by finitely many pseudoconvex double cover $(\mathcal{U}_j = \{\rho_j < 0\})_j$ and $(\mathcal{U}'_j = \{\rho_j < -c_j\})_j$. Then the slices $\Omega_j := \mathcal{U}_j \cap X_t = \{\rho_j |_{X_t} < 0\}$ and $\Omega'_j := \mathcal{U}'_j \cap X_t = \{\rho_j |_{X_t} < -c_j\}$ form a pseudoconvex double cover of $X_t$ for all $t \in \mathbb{D}_{1/2}$. Hence, these constants are fixed under such choice of a double covering;
- $B'$: It can be obtained easily by restriction on each fibres;
- $V$: It follows from continuity of the total mass of the currents $(\omega^n \land [X_t])_{t \in \mathbb{D}_{1/2}}$ (cf. [38, Section 1.4]).

The remaining data is the main focus of Sects. 4 and 5:

- $\alpha, A_\alpha$: These would be established in Proposition 4.3 assuming Conjecture 1.2;
- $C_{\text{Kol}, p}$: A uniform version of the volume-capacity comparison 3.1 will be treated by Proposition 5.3 and thus using Theorem 3.1 and uniform control of $\rho_j |_{X_t}$ for all $t \in \mathbb{D}_{1/2}$, one can obtain a uniform constant $C_{\text{Kol}, p}$.

These complete the proof of Theorem A.

### 4 Uniform Skoda’s Integrability Theorem

In this section, we follow the ideas in [14, 47] to establish a local version of uniform Skoda’s integrability theorem in families. Then we prove a uniform global version of Skoda’s integrability theorem (i.e. geometric constants in 3.2) in a family which has a uniform $C_{\text{SL}} > 0$. 

\[\mathbb{ Springer}\]
4.1 Local Uniform Skoda’s Estimate

Recall that $\pi : \mathcal{X} \rightarrow \mathbb{D}$ is a proper surjective holomorphic map and satisfies the geometric setting 1.1. Then we fix some notations:

(i) $\mathcal{U}$ is a strongly pseudoconvex domain in $\mathcal{X}$ and $\mathcal{U}$ which is contained in a larger strongly pseudoconvex domain $\tilde{\mathcal{U}} \subset \mathcal{X}$ such that $\pi(\tilde{\mathcal{U}}) \subseteq \mathbb{D}$;

(ii) $\rho$ (resp. $\tilde{\rho}$) is a smooth strictly psh function defined in a neighbourhood of $\overline{\mathcal{U}}$ (resp. $\overline{\tilde{\mathcal{U}}}$) such that $\mathcal{U} := \{ \rho < 0 \}$ (resp. $\tilde{\mathcal{U}} := \{ \tilde{\rho} < 0 \}$) and $\mathcal{U}_c = \{ \rho < -c \}$ is relatively compact in $\mathcal{U}$ for each $c > 0$;

(iii) Fix a relatively compact subdomain $\mathcal{U}' \subseteq \mathcal{U}$ such that $\mathcal{U}' := \{ \rho < -c \}$ for some generic $c > 0$ with $d\rho$ non-vanishing on $\partial \mathcal{U}'$;

(iv) Define the slices by $\Omega_t := \mathcal{U} \cap X_t$, $\Omega'_t := \mathcal{U}' \cap X_t$, and $\tilde{\Omega}_t := \tilde{\mathcal{U}} \cap X_t$, which are strongly pseudoconvex domains in $X_t$.

(v) Fix $\omega$ a hermitian metric which comes from a restriction of a hermitian metric in $\mathbb{C}^N$. Let $dV_t = \omega^n|_{X_t}$ be a smooth volume form defined on $\tilde{\Omega}_t$.

Under such setup, we show the following Skoda-type estimate, independently of $t$.

**Theorem 4.1** Let $\mathcal{F}_t$ be a family of negative psh functions defined on $\tilde{\Omega}_t$. Fix $r > 0$ such that $\mathbb{D}_r \Subset \pi(\mathcal{U}) \subseteq \mathbb{D}$. Assume that there is a uniform constant $C_F > 0$ such that for all $u_t \in \mathcal{F}_t$,\[
\int_{\Omega_t} (-u_t) dV_t \leq C_F.
\]

Then there exist positive constants $\alpha$ and $A_\alpha$ which depend on $C_F$ such that for each $u_t \in \mathcal{F}_t$,

\[
\int_{\Omega'_t} e^{-\alpha u_t} dV_t \leq A_\alpha.
\]

for all $t \in \mathbb{D}_r$.

**Remark 4.2** We mostly use $\mathcal{U}$ and $\mathcal{U}'$ in the proof. However, in order to approximate a negative psh function by a decreasing sequence of smooth psh functions, we need to shrink the domain in the very beginning of the proof. That is the reason why we choose $\mathcal{U}$ lying in a bigger pseudoconvex domain $\tilde{\mathcal{U}}$.

**Proof** We proceed in several steps:

**Step 0: Good covers.** First of all, we may assume that $u_t$ are smooth. It follows indeed from a result of Fornaess–Narasimhan [18, Theorem 5.5], that one can approximate $u_t$ by a decreasing sequence of smooth non-positive psh functions on $\mathcal{U} \cap X_t$.

We set

\[
v_t := u_t + \rho_t
\]
where $\rho_t = \rho|_{\Omega_t}$. Since the masses of $\beta^n \wedge [X_t]$ are continuous in $t$, $\text{Vol}(\Omega_t, dV_t)$ is uniformly bounded by a constant $V'$ up to shrinking $\mathbb{D}$ (cf. [38, Section 1.4]). By adding $\|\rho\|_{L^\infty(\mathcal{U})} V'$ to $C_F$, we can also assume that

$$\forall u_t \in \mathcal{F}_t, \quad \int_{\Omega_t} (-v_t) dV_t \leq C_F.$$

Choose a finite collection of balls of radius 2, $B_{CN}(p, 2) \subset \mathbb{C}^N$, centred at some point $p \in X_0$. We denote by $\mathcal{B}_R := X \cap B_{CN}(p, R) \in \mathcal{U}$ for all $R \leq 2$. One may assume that the collection of balls of radius 1/2, $\mathcal{B}_{1/2}$, covers $\Omega'_t$ for all $t \in \mathbb{B}_R$. For convenience, in this section, we fix constants $C_\rho, C_\omega > 0$ satisfying $C_\rho^{-1} d\bar{d}^c \rho \leq d\bar{d}^c |z|^2 \leq C_\rho d\bar{d}^c \rho$ and $C_\omega^{-1} \omega \leq d\bar{d}^c |z|^2 \leq C_\omega \omega$ on each $\mathbb{B}_2$.

**Step 1: Poisson–Szegő inequality.** We first recall the following inequality

$$v_t(x) \geq \int_{\mathbb{B}} v_t(d\bar{d}^c G_x)^n \wedge [X_t] = \int_{\mathbb{B} \cap X_t} v_t(d\bar{d}^c G_x)^n = \int_{\mathbb{B} \cap X_t} G_x(d\bar{d}^c v_t) \wedge (d\bar{d}^c G_x)^{n-1} + \int_{\partial \mathbb{B} \cap X_t} v_t d\bar{d}^c G_x \wedge (d\bar{d}^c G_x)^{n-1}.$$

where $G_x(z) := \log|\Phi_x(z)|$ and $\Phi_x(z)$ is the automorphism of the unit ball $\mathbb{B}$ that sends $x$ to the origin. The reader is referred to [14, page 22–23] for more details.

**Step 2: Control $J_t$, $I_t$ and Lelong numbers.** Following the same proof in [14, middle of page 23], we have $|J_t| \leq C_F C_1$ for some uniform $C_1 > 0$. Now, we are going to treat the other more singular term

$$I_t(x) = \int_{\mathbb{B}} G_x(d\bar{d}^c v_t) \wedge (d\bar{d}^c G_x)^{n-1} \wedge [X_t].$$

In global Kähler setting, this part can be controlled by cohomology class of given Kähler metrics but it is not the case here. The spirit goes back to the local strategy in [47] and Chern–Levine–Nirenberg inequality.

Consider the mass of the measure $d\bar{d}^c v_t \wedge (d\bar{d}^c G_x)^{n-1} \wedge [X_t]$

$$\gamma_t(x) := \int_{\mathbb{B}} d\bar{d}^c v_t \wedge (d\bar{d}^c G_x)^{n-1} \wedge [X_t]$$

$$= \int_{D(x, r)} d\bar{d}^c v_t \wedge (d\bar{d}^c G_x)^{n-1} \wedge [X_t] + \int_{\mathbb{B} \setminus D(x, r)} d\bar{d}^c v_t \wedge (d\bar{d}^c G_x)^{n-1} \wedge [X_t]$$

where $D(x, r) := \{ \zeta \in \mathbb{B} \mid |\Phi_x(\zeta)| < r \}$ for some $r \in (0, 1)$. Note that $\mu_t = \frac{1}{\gamma_t} d\bar{d}^c v_t \wedge (d\bar{d}^c G_x)^{n-1} \wedge [X_t]$ is a probability measure.

\[ \square \]
From the definition of $G_x(z)$, direct computation provides that

$$dd^c G_x(z) \leq C_2 \frac{dd^c |z|^2}{|\Phi_x(z)|^2} \quad (4.1)$$

for some uniform constant $C_2 > 0$. Choose a cutoff function $\chi$ supported in $B_2$ and satisfying $\chi \equiv 1$ on $B$ and $-C_3 dd^c |z|^2 \leq dd^c \chi \leq C_3 dd^c |z|^2$ for some constant $C_3 > 0$. Then we have

$$I''_t(x) \leq \frac{1}{2n-2} \int_B dd^c v_t \wedge (dd^c |z|^2)^{n-1} \wedge [X_t]$$

$$\leq \frac{1}{2n-2} \int_B \chi dd^c v_t \wedge (dd^c |z|^2)^{n-1} \wedge [X_t]$$

$$= \frac{1}{2n-2} \int_B v_t dd^c \chi \wedge (dd^c |z|^2)^{n-1} \wedge [X_t]$$

$$\leq \frac{C_3}{2n-2} \int_{\mathcal{U}} (-v_t)(dd^c |z|^2)^n \wedge [X_t] = \frac{C_3}{2n-2} \int_{\Omega_t} (-v_t)(dd^c |z|^2)^n \leq \frac{C_3 C_F}{2n-2}.$$ 

Because $|x| < 1/2$ in the setting, one may assume $D(x, r_0) \subset B_{3/4}$ for some uniform $r_0 > 0$ sufficiently small. Hence, one get $I''_t(x) \leq \frac{C_3 C_F}{r_0^{2n-2}}$.

Consider cutoffs $(\chi_j)_{j=1}^n$ which are compactly supported on $B$ and satisfy: $\chi_1 \equiv 1$ on $B_{3/4}$, supp$(\chi_1) \subset B$, $\chi_{j+1} \equiv 1$ on supp$(\chi_j)$ for every $j \in \{1, \ldots, n-1\}$ and $-C_4 dd^c |z|^2 \leq dd^c \chi_j \leq C_4 dd^c |z|^2$ for some constant $C_4 > 0$. Using the trick in Chern–Levine–Nirenberg inequality, one can see that

$$I'_t(x) \leq \int_{B_{3/4}} dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t] \leq \int_{\text{supp}(\chi_1)} \chi_1 dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t]$$

$$\leq \int_{\text{supp}(\chi_1)} G_x dd^c v_t \wedge dd^c \chi_1 \wedge (dd^c G_x)^{n-2} \wedge [X_t]$$

$$\leq \frac{C_4}{2n-2} \int_{B_{3/4}} |G_x(z)| \int_{\text{supp}(\chi_1)} dd^c v_t \wedge (dd^c |z|^2)^{n-2} \wedge [X_t]$$

$$\leq \frac{C_4}{2n-2} \int_{B_{3/4}} |G_x(z)| \int_{\text{supp}(\chi_2)} \chi_2 dd^c v_t \wedge (dd^c |z|^2)^{n-2} \wedge [X_t]$$

$$\leq \cdots$$

$$\leq C_4^{n-1} \left( \sup_{z \in B \setminus B_{3/4}} |G_x(z)| \right)^{n-1} \int_{\text{supp}(\chi_{n-1})} dd^c v_t \wedge (dd^c |z|^2)^{n-1} \wedge [X_t]$$

$$\leq C_4^n \left( \sup_{z \in B \setminus B_{3/4}} |G_x(z)| \right)^{n-1} \int_{B} (-v_t) \wedge (dd^c |z|^2)^{n} \wedge [X_t]$$

$$\leq C_4^n \left( \sup_{z \in B \setminus B_{3/4}} |G_x(z)| \right)^{n-1} \int_{\Omega_t} (-v_t) \wedge (dd^c |z|^2)^{n} \leq C_4^n C_F \left( \sup_{z \in B \setminus B_{3/4}} |G_x(z)| \right)^{n-1}.$$
Since $|x| < 1/2$ and $\Phi_x(z)$ moves smoothly in $x$ and $z$, there is a uniform constant $C_5$ such that

$$\left( \sup_{z \in B \setminus B_{3/4}} |G_x(z)| \right) \leq C_5.$$}

Thus, $I_t'$ is uniformly bounded from above by the constant $C_5^n C_F C_4^{n-1}$.

Estimates of $I'$ and $I''$ yield a constant $\nu = C_5 C_4^{n-1}$ such that $\gamma_t(x)$ is bounded by $\nu$ from above.

On the other hand, we have the lower bound of $\gamma_t \geq C_F C_3^{n-1}$ by similar computation in [14, bottom of page 23].

Therefore, we obtain a two-sided bound of $\gamma_t$:

$$\forall x \in B_{1/2} \text{ and } \forall t \in B_r, \quad C_\rho^{-1} \leq \gamma_t(x) \leq \nu. \quad (4.2)$$

**Step 3: Conclusion.** We closely follow the strategy in [14, page 24–25] to conclude. Combining (4.1), (4.2) and Jensen’s inequality, we derive

$$e^{-\alpha I_t(x)} = \exp \left( \int_{x \in B} -\alpha \gamma_t(x) G_x d\mu_t \right) \leq C_\rho \int_{x \in B} \frac{\ddc v_t \wedge (\ddc |x|^2)^{n-1} \wedge [X_t]}{|\Phi_x(z)|^{|\alpha v + 2n - 2|}}.$$

Integrating $x \in B_{1/2}$ and using Fubini’s theorem, one can infer the following inequality

$$\int_{x \in B_{1/2}} e^{-\alpha w} (\ddc |x|^2)^n \wedge [X_t] \leq \int_{x \in B_{1/2}} e^{-\alpha w} (\ddc |x|^2)^n \wedge [X_t]$$

$$\leq e^{\alpha C_F C_1} C_\rho \int_{z \in B} \left( \int_{x \in B_{1/2}} (\ddc |x|^2)^n \wedge [X_t] \right) \ddc v_t(z) \wedge (\ddc |z|^2)^{n-1} \wedge [X_t]$$

Fix a constant $\alpha$ sufficiently small such that $\alpha v < 2$ and put $\beta = \frac{2 - \alpha v}{2n} > 0$. Following the similar proof of [14, Lemma 2.13], we have

$$C_\beta^{-1} (\ddc_x |\Phi_x(z)|^{2\beta})^n \leq \frac{(\ddc |x|^2)^n}{|\Phi_x(z)|^{|\alpha v + 2n - 2|}} \leq C_\beta (\ddc_x |\Phi_x(z)|^{2\beta})^n.$$

Now, using the same trick in the proof of Chern–Levine–Nirenberg inequality, one has

$$\int_{x \in B_{1/2}} \frac{(\ddc |x|^2)^n \wedge [X_t]}{|\Phi_x(z)|^{|\alpha v + 2n - 2|}} \leq C_\beta \int_{x \in B_{1/2}} (\ddc_x |\Phi_x(z)|^{2\beta})^n \wedge [X_t]$$

$$\leq C_\beta \int_{x \in \text{supp}(\chi_1)} \chi_1 (\ddc_x |\Phi_x(z)|^{2\beta})^n \wedge [X_t].$$
\[ C_4^\beta \int_{x \in \mathbb{B}} (dd^c |x|^2)^n \wedge [X_t] \leq C_4^n C_{\omega}^n C_\beta \text{Vol}_{\omega}(\Omega_t). \]

Note that in global Kähler cases, the above estimate is controlled by cohomology classes. Finally, we get the estimate on each \( \mathbb{B}_{1/2} \)

\[
\int_{x \in \mathbb{B}_{1/2}} e^{-a u_t} (dd^c |x|^2)^n \wedge [X_t] \leq e^{\alpha C_F C_1} \int_{\mathbb{B}} (C_4^n C_{\omega}^n C_\beta \text{Vol}_{\omega}(\Omega_t)) (dd^c |z|^2)^{n-1} \wedge [X_t] \\
\leq e^{\alpha C_F C_1} C_4^n C_{\omega}^n C_\beta \text{Vol}_{\omega}(\Omega_t) \int_{\mathbb{B}_{2}} |x| (dd^c |z|^2) \wedge [X_t] \\
\leq e^{\alpha C_F C_1} C_4^n C_{\omega}^n C_\beta \text{Vol}_{\omega}(\Omega_t) \int_{\Omega_t} (-v_t) (dd^c |z|^2)^n \\
\leq e^{\alpha C_F C_1} C_F C_4^n C_{\omega}^n C_\beta \text{Vol}_{\omega}(\Omega_t).
\]

Note that \( t \mapsto \text{Vol}_{\omega}(\Omega_t) \) is continuous. One has a uniform control \( \int_{x \in \mathbb{B}_{1/2}} e^{-a u_t} (dd^c |x|^2)^n \wedge [X_t] \) for all \( t \) close to 0. Summing the integration on every \( \mathbb{B}_{1/2} \) in the collection, we obtain the estimate in Theorem 4.1 as desired.

### 4.2 Global Skoda’s Estimate

Now, we assume that there is a uniform constant \( C_{SL} > 0 \) such that \( X_t \) satisfies 3.2 for all \( t \in \mathbb{D} \). As a consequence, we have the following uniform global version of Skoda’s estimate:

**Proposition 4.3** Assume that there is a uniform constant \( C_{SL} > 0 \) such that for all \( t \in \mathbb{D} \) and for every \( u_t \in \text{PSH}(X_t, \omega_t) \) with \( \sup_{X_t} u_t = 0 \),

\[
\frac{1}{V_t} \int_{X_t} (-u_t) \omega_t^n \leq C_{SL}
\]

where \( V_t := \text{Vol}_{\omega_t}(X_t) \). Then there exists constants \( \alpha, A_\alpha \) such that for all \( t \in \mathbb{D}_{1/2} \) and for all \( u_t \in \text{PSH}(X_t, \omega_t) \) with \( \sup_{X_t} u_t = 0 \),

\[
\int_{X_t} e^{-\alpha u_t} \omega_t^n \leq A_\alpha.
\]

**Proof** Without loss of generality, we just treat the proof for \( t \) in a small neighbourhood near 0 \( \in \mathbb{D} \). Let \( (\mathcal{U}_j)_{j \in J} \) and \( (\mathcal{U}_j')_{j \in J} \) be a strongly pseudoconvex finite double cover of \( \pi^{-1}(\mathbb{D}_r) \) for some \( r > 0 \) sufficiently small. We write \( \mathcal{U}_j := \{ \rho_j < -c_j \} \subseteq \mathcal{U}_j = \{ \rho_j < 0 \} \) for some \( c_j > 0 \). For simplicity, we may assume that \( dd^c \rho_j \geq \omega \) for all \( j \in J \). Also, we set the slices \( \Omega_{t,j} := X_t \cap \mathcal{U}_j \) and \( \Omega_{t,j}' := X_t \cap \mathcal{U}_j' \). For all \( u_t \in \text{PSH}(X_t, \omega_t) \) with \( \sup_{X_t} u_t = 0 \), it is obvious that \( u_t + \rho_{t,j} \in \text{PSH}(\Omega_{t,j}) \). Note that for all \( j \in J \),

\[
\int_{\Omega_{t,j}} -(u_t + \rho_{t,j}) \omega_t^n \leq \int_{X_t} -u_t \omega_t^n + V_t \| \rho_j \|_{L^\infty(\mathcal{U}_j)} \leq C_F
\]

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for some uniform constant $C_F$. By Theorem 4.1, we obtain

$$\forall j \in J, \, \int_{\Omega_{i,j}} e^{-\sigma_j u_i} \omega_i^n \leq \int_{\Omega'_{i,j}} e^{-\sigma_j (u_i + \rho_{i,j})} \omega_i^n \leq A_{\alpha_j,j}.$$ 

Since $J$ is a finite set of indices, one can easily derive the desired estimate. \qed

This ensures that one can find uniform geometric constants 3.2 in a family which fulfils Conjecture 1.2.

\section{5 Uniform Volume-Capacity Comparison}

This section aims to deal with the volume-capacity comparison 3.1 in a family $\pi : \mathcal{X} \to \mathbb{D}$ with locally irreducible fibres. Let $\mathcal{U} = \{\rho < 0\}$ be a strongly pseudoconvex domain in $\mathcal{X}$. We also assume that the closure of $\mathcal{U}$ can be contained in a larger strongly pseudoconvex set $\tilde{\mathcal{U}} = \{\tilde{\rho} < 0\}$ in $\mathcal{X}$. Denote the slices by $\Omega_t := \mathcal{U} \cap X_t$ and $\tilde{\Omega}_t := \tilde{\mathcal{U}} \cap X_t$.

\subsection{5.1 Subextensions and Relative Extremal Functions}

We first recall some useful facts in pluripotential theory:

\subsection*{5.1.1 Subextensions}

Fix a strongly pseudoconvex domain $\Omega = \{\rho < 0\}$ and a strongly pseudoconvex neighbourhood $\tilde{\Omega} = \{\tilde{\rho} < 0\}$ containing $\bar{\Omega}$. We define

$$\mathcal{E}^0(\Omega) := \left\{ u \in \text{PSH}(\Omega) \cap L^\infty(\Omega) \mid u|_{\partial \Omega} = 0 \text{ and } \int_\Omega (dd^c u)^n < +\infty \right\}.$$

We recall some properties of subextension of the plurisubharmonic functions in $\mathcal{E}^0(\Omega)$.

\begin{lemma} \label{lemma:subextension}
([11, Theorem 2.2] and [21, Lemma 1.7]) Suppose that $\varphi \in \mathcal{E}^0(\Omega)$. The subextension of $\varphi$ is defined as:

$$\tilde{\varphi} := \sup \left\{ u \in \text{PSH}(\tilde{\Omega}) \mid u|_{\partial \tilde{\Omega}} \leq 0 \text{ and } u \leq \varphi \text{ in } \Omega \right\}.$$ 

Then the subextension $\tilde{\varphi}$ satisfies the following properties:

\begin{enumerate}
\item $\tilde{\varphi} \in \mathcal{E}^0(\tilde{\Omega})$;
\item $\tilde{\varphi} \leq \varphi$ on $\Omega$;
\item $\int_{\tilde{\Omega}} (dd^c \tilde{\varphi})^n \leq \int_{\Omega} (dd^c \varphi)^n$.
\end{enumerate}

\end{lemma}
5.1.2 Relative Extremal Functions

Next, we review the definition and some basic properties of relative extremal functions (cf. [28, Chapter 3]):

**Definition and Proposition 5.2** Let $E$ be a Borel subset in a strongly pseudoconvex domain $\Omega$. The relative extremal function with respect to $(E, \Omega)$ is defined as follows

$$h_{E; \Omega}(z) = \sup \{ u(z) \mid u \in \text{PSH}(\Omega), u \leq 0 \text{ and } u|_E \leq -1 \}.$$ 

Suppose that $E$ is a relatively compact Borel subset in $\Omega$. Then we have the following facts:

(i) The function $h^*_{E; \Omega}$ is psh in $\Omega$ and $h^*_{E; \Omega} = -1$ on $E$ off a pluripolar subset;
(ii) The boundary value of $h^*_{E; \Omega}$ is zero (i.e. $\lim_{z \to \partial \Omega} h^*_{E; \Omega}(z) = 0$);
(iii) The Monge–Ampère measure $(ddc h^*_{E; \Omega})^n$ puts no mass on $\Omega \setminus E$;
(iv) The capacity can be expressed by the integration of Monge–Ampère measure of the relative extremal function: $\text{Cap}(E; \Omega) = \int_{\Omega} (ddc h^*_{E; \Omega})^n$.

5.2 Volume-Capacity Comparison

Then we explain the volume-capacity comparison in families:

**Proposition 5.3** For every $k > 1$, there exists a constant $C_{VC,k}$ such that

$$\forall K_t \Subset \Omega_t, \quad \text{Vol}(K_t) \leq C_{VC,k} \text{Cap}^k(K_t; \Omega_t),$$

for all $t \in \mathbb{D}_{1/2}$

In smooth setting, the proof of the volume-capacity comparison was first given by Kołodziej [32]. For global semi-positive setup, Guedj and Zeriahi [27] provided a proof simply using Skoda’s integrability theorem and the comparison of Bedford–Taylor and Alexander–Taylor capacities. For local cases, in [21, Lemma 1.9], Guedj–Guenancia–Zeriahi gave an easy proof for us to chase the depending constants. We also compute explicitly the constant $C_{VC,k}$ to verify that it is independent of $t$.

**Proof** Without loss of generality, we may assume that $K_t$ is non-pluripolar. Otherwise, both sides of the inequality are zeros. We define

$$u_{K_t} = \frac{h^*_{K_t; \Omega_t}}{\text{Cap}^{1/n}(K_t; \Omega_t)}.$$ 

According to Proposition 5.2, one has $u_{K_t} \in \mathcal{E}^0(\Omega_t)$ and $\int_{\Omega_t} (ddc u_{K_t})^n = 1$. Recall that $\tilde{U} = \{ \tilde{\rho} < 0 \}$ is a strongly pseudoconvex neighbourhood of $\overline{U}$ and $\tilde{\Omega}_t = \tilde{U} \cap X_t$. Let $C_{\tilde{\rho}} > 0$ be a constant such that $dV \leq C_{\tilde{\rho}}(ddc \tilde{\rho})^n$ on $\tilde{U}$. Consider the subextension of $u_{K_t}$:

$$\tilde{u}_{K_t} = \sup \{ u \in \text{PSH}(\tilde{\Omega}_t) \cap L^\infty(\tilde{\Omega}_t) \mid u|_{\partial \tilde{\Omega}_t} \leq 0 \text{ and } u \leq u_{K_t} \text{ in } \Omega_t \}.$$
By Lemma 5.1, we have \( \tilde{u}_{K_t} \in E^0(\Omega_t) \), \( \tilde{u}_{K_t} \leq u_{K_t} \) in \( \Omega_t \), and \( \int_{\Omega_t} (ddc\tilde{u}_{K_t})^n = 1 \). Using the integration by parts and the condition \( \int_{\Omega_t} (ddc\tilde{u}_{K_t})^n \leq 1 \), one can see that \( \|\tilde{u}_{K_t}\|_{L^1(\Omega_t)} \) is uniformly bounded independent of \( K_t \). Indeed,

\[
\int_{\Omega_t} (-\tilde{u}_{K_t}) dV \leq \text{Vol} \left( \frac{n-1}{n}(\Omega_t) \left( \int_{\Omega_t} (-\tilde{u}_{K_t})^n dV \right)^{1/n} \right) \\
\leq \text{Vol} \left( \frac{n-1}{n}(\Omega_t) C_\rho \left( \int_{\Omega_t} (-\tilde{u}_{K_t})^n (ddc\tilde{\rho})^n \right)^{1/n} \right) \\
\leq \text{Vol} \left( \frac{n-1}{n}(\Omega_t) C_\rho \|\tilde{\rho}\|_{L^\infty(\Omega_t)} \left( \int_{\Omega_t} (ddc\tilde{u}_{K_t})^n \right)^{1/n} \right) \\
\leq \text{Vol} \left( \frac{n-1}{n}(\Omega_t) C_\rho \|\tilde{\rho}\|_{L^\infty(\Omega_t)} \right).
\]

According to Theorem 4.1, there exists constants \( \alpha, A_\alpha > 0 \) such that for all \( K_t \in \Omega_t \) non-pluripolar,

\[
\int_{\Omega_t} e^{-\alpha \tilde{u}_{K_t}} dV \leq A_\alpha.
\]

Recall that \( h^*_{K_t} = -1 \) on \( K_t \) almost everywhere. By the definition of \( u_{K_t} \) and \( \tilde{u}_{K_t} \leq u_{K_t} \), we have

\[
\text{Vol}(K_t) \cdot \exp \left( \frac{\alpha}{\text{Cap}^{1/n}(K_t; \Omega_t)} \right) = \int_{K_t} e^{-\alpha u_{K_t}} dV \leq \int_{\Omega_t} e^{-\alpha u_{K_t}} dV \leq A_\alpha
\]

and this implies

\[
\text{Vol}(K_t) \leq A_\alpha \exp \left( -\frac{\alpha}{\text{Cap}^{1/n}(K_t; \Omega_t)} \right) \leq A_\alpha \frac{b_k}{\alpha^{kn}} \text{Cap}^k(K_t; \Omega_t).
\]

where \( b_k \) is a numerical constant such that \( \exp(-1/x) \leq b_k x^{kn} \) for all \( x > 0 \). \( \Box \)

### 5.3 Global Volume-Capacity Comparison

In this section, we show the uniform volume-capacity comparison in a given family of compact hermitian varieties \( \pi : (X, \omega) \to \mathbb{D} \) with locally irreducible fibres.

Let \( (X, \omega) \) be a compact hermitian variety. We define the similar concept of Monge-Ampère capacity with respect to \( \omega \)-psh functions by

\[
\text{Cap}_\omega(K) := \sup \left\{ \int_K (\omega + dd^c u)^n \mid u \in \text{PSH}(X, \omega), \ 0 \leq u \leq 1 \right\}.
\]
On the other hand, fixing a pseudoconvex finite double cover \((\Omega_j)\) and \((\Omega'_j)\) of \(X\) such that \(\Omega'_j \subset \Omega_j\), we define the Bedford-Taylor capacity by

\[
\text{Cap}_{\text{BT}}(K) := \sum_j \text{Cap}(K \cap \Omega_j; \Omega_j).
\]

To prove the global volume-capacity comparison, we should first compare the Bedford-Taylor capacity and the capacity of \(\omega\)-psh functions.

**Lemma 5.4** There exists a constant \(C_{\text{BT},\omega} > 0\) such that

\[
\forall \text{ compact subset } K_t \subset X_t, \quad C_{\text{BT},\omega}^{-1} \text{Cap}_{\text{BT}}(K_t) \leq \text{Cap}_{\omega_t}(K_t) \leq C_{\text{BT},\omega} \text{Cap}_{\text{BT}}(K_t)
\]

for all \(t \in \mathbb{D}_{1/2}\).

**Proof** On a fixed compact Kähler manifold, a similar version of Lemma 5.4 was provided by Kołodziej [33, Section 1]. The proof of Lemma 5.4 is similar to Kołodziej’s proof. For the reader’s convenience, we include the proof here.

After shrinking \(\mathbb{D}\), we may assume that \((U_j)\) and \((U'_j)\) form a pseudoconvex double cover of \(X\) such that \(U_j = \{\rho_j < 0\}\) for some strictly psh function \(\rho_j\) and \(U_j = \{\rho_j < -c_j\}\). Multiplying a positive constant, we may assume that \(\frac{1}{C} \text{dd}^c X \rho_j \leq \omega \leq C \text{dd}^c X \rho_j\). Let \(C' > 0\) be a constant so that \(0 \leq \rho_j \leq C'\) for all \(j\).

Write \(\Omega_{j,t} = U_j \cap X_t\) and \(\Omega'_{j,t} = U'_j \cap X_t\). Fix \(u_t \in \text{PSH}(X_t, \omega_t)\) with \(0 \leq u_t \leq 1\). Let \(K_t\) be a compact subset of \(X_t\) and \(K_{j,t} := K_t \cap \overline{\Omega'_{j,t}}\). Then we have

\[
\int_{K_t} (\omega_t + \text{dd}^c u_t)^n \leq \sum_j \int_{K_{j,t}} (\text{dd}^c(C\rho_j + u_t))^n = \sum_j \int_{K_{j,t}} (C' + 1)^n \left( \text{dd}^c \left( \frac{C\rho_j + u_t}{C' + 1} \right) \right)^n \leq (C' + 1)^n \sum_j \text{Cap}(K_{j,t}; \Omega_j) = (C' + 1)^n \text{Cap}_{\text{BT}}(K_t)
\]

and hence \(\text{Cap}_{\omega_t}(K_t) \leq (C' + 1)^n \text{Cap}_{\text{BT}}(K_t)\).

On the other hand, we shall use the glueing argument to prove the local capacity is bounded by the global capacity. Suppose \(u_t \in \text{PSH}(\Omega_{j,t})\) and \(0 \leq u_t \leq 1\). Consider a smooth function \(\chi_j\) defined on \(U_j\) such that

\[
\chi(z) = \begin{cases} 
-1 & \text{when } z \in U'_j \\
2 & \text{when } z \text{ is in a neighbourhood of } \partial U_j
\end{cases}
\]

We can find a small \(\delta_j \in (0, \frac{1}{2})\) such that \(\delta_j \chi_j\) can be extended to a \(\omega\)-psh function on \(X\). Now, we use the glueing argument to define a \(\omega_t\)-psh function \(\psi\) and it is identically
equal to $\delta_j u_t$ in $\Omega_{j,t}'$.

$$\psi(z) = \begin{cases} 
\delta_j u(z) & \text{when } z \in \Omega'_{j,t}, \\
\max \{\delta_j \chi_j|_{X_t}, \delta_j u_t\} & \text{when } z \in \Omega_{j,t} \setminus \Omega'_{j,t}, \\
\delta_j \chi_j & \text{when } z \in X_t \setminus \Omega_{j,t}.
\end{cases}$$

Obviously, $\tilde{\psi} = \psi + 1/3$ is a $\omega_t$-psh function and $0 \leq \tilde{\psi} \leq 1$. Then, we obtain

$$\int_{K_{j,t}} (\dd c^j u_t)^n \leq \frac{1}{\delta_j^n} \int_{K_{j,t}} (\dd c^j \delta_j u_t)^n \leq \frac{1}{\delta_j^n} \int_{K_t} (\omega_t + \dd c^j \tilde{\psi})^n \leq \frac{1}{\delta_j^n} \text{Cap}_{\omega_t}(K_t).$$

Combining Proposition 5.3 and Lemma 5.4, we have the global volume-capacity comparison in families:

**Proposition 5.5** Given $k > 1$, there exists a uniform constant $C_{GVC,k} \geq 1$ such that

$$\forall \text{ compact subset } K_t \subset X_t, \quad \Vol_{\omega_t}(K_t) \leq C_{GVC,k} \Cap_k^{\omega_t}(K_t)$$

for all $t \in \mathbb{D}_{1/2}$.

### 6 Sup-$L^1$ Comparison in Families

In this section, we pay a special attention to Conjecture 1.2. We shall follow the strategy of proof in [14, Section 3]. First of all, we recall the assumption which will be used in this section:

**Assumption 6.1** (=Geometric assumption 1.2) Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a family of hermitian varieties satisfies the geometric setting 1.1 and one of the following conditions:

(a) $\pi$ is locally trivial;
(b) $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing of $X_0$ and $X_0$ has only isolated singularities.

Under such assumption, we establish a uniform $L^1$-estimate of $\omega_t$-psh function:

**Proposition 6.2** (=Proposition B) Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ satisfies the geometric assumption 1.2. Then there exists a uniform constant $C > 0$ such that for all $t \in \mathbb{D}_{1/2}$

$$\forall \varphi_t \in \text{PSH}(X_t, \omega_t), \quad \sup_{X_t} \varphi_t - C \leq \frac{1}{V_t} \int_{X_t} \varphi_t \omega_t^n \leq \sup_{X_t} \varphi_t,$$

where $V_t := \Vol_{\omega_t}(X_t)$. 
6.1 Proof of Proposition B

6.1.1 Locally Trivial Families

In locally trivial cases, one can follow the similar strategy in [14, Section 3.2] for the proof. The only difference is to replace the local potentials of $\omega_0$ by the local inequalities $0 < C_\rho^{-1} dd^c \rho \leq \omega_0 \leq C_\rho dd^c \rho$ for some local psh function $\rho$ and some constant $C_\rho > 0$ on $X_0$.

6.1.2 Smoothing of Varieties with Isolated Singularities

Before diving into the main goal, we recall the uniform boundedness of Laplacian.

**Lemma 6.3** Suppose that $\pi : (X, \omega) \to D$ satisfies the geometric setting 1.1 and it is a smoothing of a compact variety $X_0$. There is a uniform constant $C_{Lap} > 0$ such that

$$\forall \varphi_t \in \text{PSH}(X_t, \omega_t), \quad \int_{X_t} (\omega_t + dd^c \varphi_t) \wedge \omega_t^{n-1} \leq C_{Lap},$$

for all $t \in D_{1/2}$.

**Proof** Recall that from [38, Theorem A], there is a uniform constant $C_G > 0$ such that for all $t \in D^*_{1/2}$ the normalized Gauduchon factor $g_t$ with respect to $(X_t, \omega_t)$ (i.e. $dd^c_t (g_t \omega_t^{n-1}) = 0$) is bounded between 1 and $C_G$. Then we have

$$\int_{X_t} (\omega_t + dd^c \varphi_t) \wedge \omega_t^{n-1} \leq \int_{X_t} (\omega_t + dd^c \varphi_t) \wedge g_t \omega_t^{n-1} \leq \int_{X_t} g_t \omega_t^{n-1} + \int_{X_t} \varphi_t dd^c_t (g_t \omega_t^{n-1}) \leq C_G \text{Vol}_{\omega_t}(X_t)$$

for all $t \in D^*_{1/2}$. Since $(\text{Vol}_{\omega_t}(X_t))_{t \in D_{1/2}}$ is uniformly bounded from above (cf. [38, Section 1.4]), we have a desired estimate for all $t \in D^*_{1/2}$. On the central fibre $X_0$, there always exists a constant $C_{SL,0} > 0$ such that $\frac{1}{\text{Vol}_{\omega_0}(X_0)} \int_{X_0} -\varphi_0 \omega_0^n \leq C_{SL,0}$ for all $\varphi_0 \in \text{PSH}(X_0, \omega_0)$ with $\sup_{X_0} \varphi_0 = 0$. For all $\varphi_0 \in \text{PSH}(X_0, \omega_0)$, we take $\tilde{\varphi}_0 = \varphi_0 - \sup_{X_0} \varphi_0$. Then one can use the argument as in Lemma 3.7 to find a constant $C'_{Lap} > 0$ such that $\int_{X_0} (\omega_0 + dd^c_0 \varphi_0) \wedge \omega_0^{n-1} \leq C'_{Lap}$. We have thus obtained the desired estimate.

We follow the idea in [14, Section 3.3] to get the proof. Note that we only need to take care of sequences of sup-normalized $\omega_{t_k}$-psh functions $(\varphi_{t_k})_k$ where $t_k \to +\infty$.

**Step 1: Choose a good covering and a test function.** Following the same argument in [14, page 30, Step 2], up to shrinking $D$, we can find a finite open covering $(V_i)_{i \in I}$ of $\mathcal{X}$ such that

(i) each point of $\mathcal{Z} := \mathcal{X}^{\text{sing}} = X_0^{\text{sing}}$ belongs to exactly one element of $V_i$ of the covering, we denote by $J$ the collection of indices of these open subsets;
(ii) on each $V_i$, we have a smooth strictly psh function $\rho_i$ such that
\[ C^{-1}_\rho \ddc \rho_i \leq \omega \leq C_\rho \ddc \rho_i \quad \text{and} \quad 0 \leq \rho_i \leq C_\rho \]
for a uniform constant $C_\rho > 0$;
(iii) for each $i \in J$, there is a relatively compact open subset $W_i \Subset V_i$ with $W_i \cap Z \neq \emptyset$.

Define
\[ \delta := \frac{1}{2} \min_{i \in J} \left\{ \text{dist}_\omega (\partial W_i, W_i \cap X_0^{\text{sing}}) \right\} > 0. \]

Let $\chi_i$ be a cutoff function supported in $V_i$ and $\chi_i \equiv 1$ in a neighbourhood of $W_i$. Set $\rho = \sum_{i \in I} \chi_i \rho_i$. Obviously, we have $\omega \leq C_\rho \ddc \rho$ on $W := \bigcup_{i \in I} W_i$. Furthermore, we may assume $-C_\rho \rho \leq \ddc \rho \leq C_\rho \omega$ on $X$ by choosing larger $C_\rho$.

**Step 2: Uniform $L^1$-estimate away from singularities.** Define a set
\[ \mathcal{R} := \left\{ p \in X \mid \text{dist}_\omega (p, X_0^{\text{sing}}) > \delta/2 \right\}. \]

Since $\mathcal{R}^c$ lies in $W$, after shrinking $D$, we can cover $\mathcal{R}$ by finitely many open subsets $(U_i)_{i}$ away from the singular locus. We may assume that $\pi$ is locally trivial on $\mathcal{R}$ with respect to $(U_i)$, because $\pi$ is a submersion on $\mathcal{R}$.

Following the argument in [14, page 31, Step 3], one can prove that there is a constant $C > 0$ and a subsequence of $(t_k)_k$ such that
\[ \sup_{\mathcal{R} \cap X_{t_k}} \varphi_{t_k} \geq -C. \]

By the irreducibility of $X_0$, $\mathcal{R}$ is connected. Then one can use the same proof in locally trivial cases to show that
\[ \int_{\mathcal{R} \cap X_{t_k}} (-\varphi_{t_k}) \omega^n_{t_k} \leq C_\mathcal{R} \]
for some uniform constant $C_\mathcal{R} > 0$.

**Step 3: Conclusion.** Recall that on $W$ we have $\omega \leq C_\rho \ddc \rho$. Define a smooth $(n, n)$-form $\Omega := \omega^n - C_\rho \ddc \rho^n$. It is easy to see that $\Omega|_{\mathcal{W} \cap X} \leq 0$ and $\Omega_i := \Omega|_{X_i} \leq C_\Omega \omega^n_i$ for some uniform constant $C_\Omega > 0$. Note that $\mathcal{R}^c \subset \mathcal{W}$. We have
\[ \int_{X_{t_k}} (-\varphi_{t_k}) \Omega_{t_k} = \int_{\mathcal{R} \cap X_{t_k}} (-\varphi_{t_k}) \Omega_{t_k} + \int_{\mathcal{R}^c \cap X_{t_k}} (-\varphi_{t_k}) \Omega_{t_k} \]
\[ \leq C_\Omega \int_{\mathcal{R} \cap X_{t_k}} (-\varphi_{t_k}) \omega^n_{t_k} \leq C_\mathcal{R} C_\Omega. \]

On the other hand, we have
\[ \int_{X_{t_k}} (-\varphi_{t_k}) (\ddc \rho)^n = \int_{X_{t_k}} -\rho \ddc \varphi_{t_k} \wedge (\ddc \rho)^{n-1} \]
\[= - \int_{X_{t_k}} \rho (\omega_{t_k} + dd^c \varphi_{t_k}) \wedge (dd^c \rho)^{-1} + \int_{X_{t_k}} \rho \omega_{t_k} \wedge (dd^c \rho)^{-1} \]
\[\leq C^n \rho \int_{X_{t_k}} (\omega_{t_k} + dd^c \varphi_{t_k}) \wedge \omega_{t_k}^{-1} + C^n \rho \operatorname{Vol}_{\omega_{t_k}} (X_{t_k}) \]
\[\leq C^n C_{\text{Lap}} + C^n \rho \operatorname{Vol}_{\omega_{t_k}} (X_{t_k}). \]

The fourth line comes directly from Lemma 6.3. All in all, we obtain a uniform \(L^1\)-estimate of sup-normalized \(\omega_{t_k}\)-psh functions:

\[\int_{X_{t_k}} (-\varphi_{t_k}) \omega_{t_k}^n = C^n \rho \int_{X_{t_k}} (-\varphi_{t_k}) (dd^c \rho)^n + \int_{X_{t_k}} (-\varphi_{t_k}) \Omega_{t_k} \leq C^{2n} C_{\text{Lap}} \]
\[+ C^{2n} \rho \operatorname{Vol}_{\omega_{t_k}} (X_{t_k}) + C_{\mathcal{R}} C_{\Omega}. \]

7 Families of Calabi–Yau Varieties

In this final section, we apply our results to families of Calabi–Yau varieties. By Calabi–Yau variety, we mean a normal variety with canonical singularities and trivial canonical bundle.

7.1 Canonical Singularities

We first recall the notion of canonical singularities. The reader is referred to [16, Section 5] for more details. Let \(X\) be a normal variety. We say that \(X\) has canonical singularities if the pluricanonical sheaf \(K_X^{[N]}\) is locally free for some \(N \in \mathbb{N}\), and for any resolution of singularities \(p: \tilde{X} \rightarrow X\), for any local generator \(\alpha\) of \(K_X^{[N]}\), the meromorphic pluricanonical form \(p^* \alpha\) is holomorphic.

**Lemma 7.1** Suppose that \(\pi: \mathcal{X} \rightarrow \mathbb{D}\) is a proper surjective holomorphic map satisfying the geometric setting 1.1. In addition, assume that

- \(\mathcal{X}\) is normal and \(K_{\mathcal{X}}\) (or equivalently \(K_{\mathcal{X}/\mathbb{D}}\)) is locally free;
- for each \(t \in \mathbb{D}\), \(X_t\) has only canonical singularities.

Then the following are equivalent

- \(K_{\mathcal{X}/\mathbb{D}}\) (or \(K_{\mathcal{X}}\)) is trivial up to shrinking \(\mathbb{D}\);
- \(K_{X_t}\) is trivial for all \(t\) small (i.e. \(X_t\) is Calabi–Yau for all \(t\) small).

**Proof** We include some arguments here for the reader’s convenience. Suppose that \(K_{\mathcal{X}/\mathbb{D}}\) (or \(K_{\mathcal{X}}\)) is trivial. Then we have

\[K_{X_t}^{\text{reg}} \simeq K_{\mathcal{X}^{\text{reg}}/\mathbb{D}}|_{X_t^{\text{reg}}} = (K_{\mathcal{X}^{\text{reg}}} - \pi^* K_{\mathbb{D}})|_{X_t^{\text{reg}}} \simeq \mathcal{O}_{\mathcal{X}^{\text{reg}}}|_{X_t^{\text{reg}}} \simeq \mathcal{O}_{X_t^{\text{reg}}} \]

When \(t\) is close to 0, since \(X_t\) is normal and \(K_{X_t}\) is reflexive, we have

\[K_{X_t} \simeq (j_t)_*(K_{X_t}^{\text{reg}}) \simeq (j_t)_*(\mathcal{O}_{X_t}^{\text{reg}}) \simeq \mathcal{O}_{X_t}. \]
where $j_t : X_t^{\text{reg}} \hookrightarrow X_t$ is the inclusion map for each $t$.

Now, assume that $K_{X_t}$ is trivial for all $t$ close to 0. Since $(X_t)_{t \in \mathbb{D}}$ are Calabi–Yau varieties, the map $t \mapsto h^0(X_t, K_{X_t})$ is constantly equal to 1. As a direct consequence of Grauert’s theorem [26], the direct image sheaf $\pi_*(K_{X/\mathbb{D}})$ is locally free and

$$\pi_*(K_{X/\mathbb{D}}) \otimes k(0) \cong H^0(X_0, K_{X_0}), \quad (7.1)$$

where $k(0)$ is the residue field at 0. Let $\Omega_0$ be a nowhere vanishing trivialization of $K_{X_0}$ on $X_0$. By the isomorphism (7.1), $\Omega_0$ descends to an element $s_0$ of $\pi_*(K_{X/\mathbb{D}}) \otimes k(0) = \pi_*(K_{X/\mathbb{D}})|_0$. Since every line bundle over $\mathbb{D}$ is trivial, we can extend the vector $s_0 \in \pi_*(K_{X/\mathbb{D}})|_0$ to a non-vanishing section $s$ of $\pi_*(K_{X/\mathbb{D}})$ after shrinking $\mathbb{D}$. Then $s$ is identified with a section $\Omega$ of $K_{X/\mathbb{D}}$ which has the relation $\Omega|_{X_0} = \Omega_0$. Since $\Omega_0$ is nowhere vanishing, the section $\Omega$ is nowhere zero on a neighbourhood of the central fibre.

\[ \square \]

**Remark 7.2** One can prove that under the assumption $X_0$ has canonical singularities, then $X$ is normal, $\mathbb{Q}$-Gorenstein in a neighbourhood of $X_0$ and moreover $X_t$ has canonical singularities for $t$ close to 0. However, we will not use that result.

### 7.2 Families of Calabi–Yau Varieties

It is thus legitimate to work in the following setting:

**Setting (CY)** Let $\pi : (X, \omega) \to \mathbb{D}$ be a family of compact hermitian varieties satisfying the geometric setting 1.1. Suppose that

(i) $X$ is normal and $K_X$ is trivial;

(ii) for all $t \in \mathbb{D}$, $X_t$ has only canonical singularities.

From Lemma 7.1 and the inversion of adjunction (cf. [29, Theorem 5.50]), Setting 7.2 implies following properties:

(a) $X$ has canonical singularities;

(b) For all $t \in \mathbb{D}$, $K_{X_t}$ is trivial.

Let $\Omega$ be a trivializing section of $K_{X/\mathbb{D}}$. Define the function $\gamma_t$ on $X_t$ by the equation

$$\Omega_t \wedge \overline{\Omega_t} = e^{-\gamma_t \omega_t^n}$$

and $\gamma_t$ also induces a function $\gamma$ on $X$ near $X_0$. From [14, Lemma 4.4], we have the following uniform integrability property:

**Proposition 7.3** Up to shrinking $\mathbb{D}$, there exists $p > 1$ and $C > 0$ such that for all $t \in \mathbb{D}$, we have

$$\int_{X_t} e^{-p\gamma_t \omega_t^n} < C.$$
On the other hand, if $K \in \mathcal{X}^{\text{reg}}$, then $\gamma \in C^0(K)$ and it implies that
\[
\int_{X_t} e^{-n/\gamma} \omega_t^n \geq e^{-\sup_{\gamma} \frac{n}{\gamma}} \frac{1}{\gamma} \int_{K_t} \omega_t^n > c
\]
for some constant $c > 0$ independent of $t$ close to 0. Therefore the canonical densities satisfy the integral bound (IB) in Theorem A.

We are now ready to establish a uniform control of Chern–Ricci flat potentials in a family of Calabi–Yau varieties satisfying Setting 7.2 and the geometric assumption 1.2.

**Proof of Theorem C** For all $t \in D$, $K_{X_t}$ is trivial and $K_{X_t} = K_X|_{X_t}$ as well. Therefore, one can find a non-vanishing section $\Omega_t$ of $K_{X_t}$ satisfying $\Omega_t = \Omega|_{X_t}$. According to a recent work of Guedj and Lu [24, Theorem E], for each $t \in D$, there exists a solution $(\varphi_t, c_t) \in (\text{PSH}(X_t, \omega_t) \cap L^\infty(X_t)) \times [0, \infty)$ which solves the complex Monge–Ampère equation
\[
(\omega_t + dd^c \varphi_t)^n = c_t \Omega_t \wedge \overline{\Omega_t}, \quad \text{and} \quad \sup_{X_t} \varphi_t = 0.
\]
According to Theorem A, Proposition B, and Proposition 7.3, there is a uniform constant $C_{\text{MA}}$ such that for all $t \in D_{1/2}$, one has
\[
c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty} \leq C_{\text{MA}}
\]
and this complete the proof of Theorem C. \qed

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