WHEN IS A CONTROL SYSTEM MECHANICAL?

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Abstract. In this work we present a geometric setting for studying mechanical control systems. We distinguish a special class: the class of geodesically accessible mechanical systems, for which the uniqueness of the mechanical structure is guaranteed (up to an extended point transformation). We characterise nonlinear control systems that are state equivalent to a system from this class and we describe the canonical mechanical structure attached to them. Several illustrative examples are given.

1. Introduction. Mechanical control systems provide an important and challenging research area whose roots come from classical mechanics and modern nonlinear control. Both mechanics and control theory have independent and rich histories which overlap in a very interesting way: mechanics provides challenging control problems and nonlinear control brings new ideas and results which lead to the progress of mechanics.

An extensive overview of the history of mechanics can be found in [1]. A crucial period in the development of mechanics, which occurred together with the development of mathematics, dates back to Newton, Euler, Lagrange, Laplace, Poisson, Hamilton and Jacobi. After this an important impulse was carried out by Poincaré, who introduced new ideas and methods based on his geometric point of view of mechanics. Mathematical control theory is a more recent subject. Newer developments in nonlinear control theory can be found for example in the books [4, 7, 10, 22, 26, 42, 55]. Bibliographical notes in the overlapping history of mechanics and control theory are described in [7, 10, 28, 45].

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Two aspects make mechanical control systems a very attractive subject of research. On one hand, they are abundant in real life and, on the other hand, they offer very interesting mathematical problems. Examples of mechanical control systems can be found in robotics, automation, autonomous vehicles in marine, aerospace, flight control, problems in nuclear magnetic resonance, micro-electromechanical systems, and fluid mechanics (see [6, 7, 10, 11, 21, 35, 36, 38, 41, 45, 46, 57, 60]). Questions like controllability, accessibility, observability, motion planning, stabilization and tracking of these systems are of major interest for applications and the emerging related questions provide the starting point for very interesting mathematical theories.

Throughout this paper we shall rest on the Lagrangian point of view on mechanical control systems [7, 10, 40, 45]. For an approach based on the Hamiltonian formalism we refer the reader to Chapter 12 of [42] (see also [1, 7, 16]). This paper is a study of mechanical control systems, which we define to be systems described by the second-order differential equations

\[(MS) : \ddot{x}^i = -\Gamma^i_{jk}(x)\dot{x}^j \dot{x}^k + d^i_j(x)\dot{x}^j + g^i_0(x) + \sum_{r=1}^{m} u_r g^i_r(x), 1 \leq i \leq n,\]  

on a smooth manifold \(Q\), called the configuration manifold. Here \(x = (x^1, \ldots, x^n)\) are local coordinates on \(Q\) and \(u = (u_1, \ldots, u_m)\) are inputs (controls) of the system. We use the summation convention throughout the paper, except for terms involving controls. The expression \(\Gamma^i_{jk}(x)\dot{x}^j \dot{x}^k\) corresponds to Coriolis and centrifugal terms, the terms \(d^i_j(x)\dot{x}^j\) correspond to dissipative-type (or gyroscopic-type) forces acting on the system, \(g^i_0\) represents an uncontrolled force (which can be potential or not) and \(g^i_1, \ldots, g^i_m\) represent controlled forces.

Among systems of the above class are mechanical control systems which are neither subject to dissipative-type (or gyroscopic-type) forces nor uncontrolled ones. These systems are known in the literature as affine connection control systems (see for instance, [10, 30, 31, 32, 34]). This subclass of mechanical control systems is particularly interesting and has attracted a lot of attention in recent years. On one hand, many applications fit to this subclass, and, on the other hand, the equations of motion of affine connection control systems are simpler than for general mechanical control systems (1), but still they retain the crucial features of general mechanical control systems.

Given a nonlinear control system of the form

\[\Sigma : \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z),\]

on a smooth manifold \(M\), we will be interested in answering the main question:

(i) When is the control system \(\Sigma\) mechanical, that is, when is it equivalent via a diffeomorphism to a mechanical system \((MS)\) of the form (1)?

Related questions immediately arise:

(ii) Can a control system admit more than one mechanical structure?

(iii) Which types of mechanical control systems do we have?

(iv) How to geometrically characterise different types of mechanical control systems?

Our problem (i) is analogous to the classical inverse problem in the calculus of variations (in Lagrangian mechanics), which is to decide whether the solutions of a
given system of second order ordinary differential equations
\[ \ddot{x}_i = f_i(x, \dot{x}), \quad 1 \leq i \leq n, \]
are the solutions of Euler-Lagrange equations for some Lagrangian function \( \mathcal{L}(x, \dot{x}) \), see, e.g., [5, 14, 17, 37, 39, 53, 54]. On one hand, our problem is more restrictive because we look for a mechanical Lagrangian only (\( \mathcal{L} \) being the kinetic minus potential energy). On the other hand, our problem is more general. First, we do not assume any a priori tangent bundle structure on \( \mathcal{M} \) (in other words, we do not assume that the original equations are of second order) and its construction constitutes a part of the problem. Secondly, we allow for dissipative-like terms. Thirdly, we require that also controlled vector fields take a form compatible with the Lagrangian (mechanical) structure. It is actually the presence of the controlled vector fields (and their interactions with the vector field defining the considered differential equation) that allow us to solve the problem.

Our solution of the problem (i) is the central result of the paper being a source of further investigations and results and for the convenience of the reader we will formulate it now. We will call a zero-velocity point for the mechanical control system (\( \mathcal{M}\mathcal{S} \)) of the form (1) any point of the form \((x_0, \dot{x}_0) = (x_0, 0)\), that is, any point of the zero section of the tangent bundle \( T\mathcal{Q} \). Let \( \mathcal{V} \) denote the smallest vector space, over \( \mathbb{R} \), containing the vector fields \( G_1, \ldots, G_m \) and satisfying
\[ [\mathcal{V}, \text{ad}_F \mathcal{V}] \subset \mathcal{V}, \]
where \( [\mathcal{V}, \text{ad}_F \mathcal{V}] = \{ [V_i, \text{ad}_F V_j] \mid V_i, V_j \in \mathcal{V} \} \).

**Theorem.** Let \( \mathcal{M} \) be a smooth 2n-dimensional manifold. A system \( \Sigma \) is locally, at \( z_0 \in \mathcal{M} \), equivalent via a diffeomorphism to a geodesically accessible mechanical system (\( \mathcal{M}\mathcal{S} \)) of the form (1) around a zero-velocity point \((x_0, 0)\) if and only if
\begin{enumerate}
  \item[(MS0)] \( F(z_0) \in \mathcal{V}(z_0) \),
  \item[(MS1)] \( \dim \mathcal{V}(z) = n \) and \( \dim (\mathcal{V} + [F, \mathcal{V}]) (z) = 2n \),
  \item[(MS2)] \( [\mathcal{V}, \mathcal{V}] (z) = 0 \),
\end{enumerate}
for any \( z \) in a neighborhood of \( z_0 \).

See Section 3 for the definition of geodesic accessibility and an equivalent computable definition of \( \mathcal{V} \) and Section 4 for a detailed geometric and mechanical interpretation of the conditions (MS0)-(MS2). Here we just want to emphasize that they can be easily verified in terms of the control system \( \Sigma \) and that they encode all structure information about the mechanical system equivalent to \( \Sigma \).

The paper is organized as follows. Section 2 provides some preliminary notions, notation, and, in particular, gives the background on mechanical control systems. Also, some illustrative examples of mechanical control systems are presented. In Section 3, we define geodesically accessible mechanical control systems (\( \mathcal{G}\mathcal{AMS} \)) and illustrate this notion with some examples. Then we characterise this special subclass of mechanical control systems establishing our main result, which is Theorem 3.2 in Subsection 3.2. A geometric interpretation of that characterisation is given in Section 4. The uniqueness of the mechanical structure is studied in Section 5. Also a characterisation of mechanical control systems (\( \mathcal{M}\mathcal{S} \)) that are not necessarily geodesically accessible is given. Section 6 is devoted to special forms of (\( \mathcal{G}\mathcal{AMS} \)) systems and Section 7 presents the proof of our main result. Section 8 contains two examples exhibiting a local nature of the results in this paper. Finally, we give conclusions and some future lines of research in Section 9.
2. Preliminaries. In this section we present some mathematical tools and establish some notation that will be used throughout the paper.

2.1. Control-affine systems, vector fields and distributions. We shall consider control-affine systems, which we define as a pair $\Sigma = (M, C)$, where

(i) $M$ is a finite dimensional smooth manifold,

(ii) $C = \{F, G_1, \ldots, G_m\}$ is an $(m+1)$-tuple of smooth vector fields on $M$.

The differential equations describing the evolution of a control-affine system are written as

$$\dot{\gamma}(t) = F(\gamma(t)) + \sum_{r=1}^{m} u_r(t)G_r(\gamma(t)),$$

in which $u : I \to \mathbb{R}^m$ is the control or input vector; $\gamma : I \to M$, $I \subset \mathbb{R}$, is the corresponding trajectory of $\Sigma$ in the state manifold $M$; $G_1, \ldots, G_m$ are the control (or input) vector fields, and finally, $F$ is the drift vector field, which describes the dynamics of the system in the absence of controls.

Given a smooth manifold $M$, we shall denote by $\mathfrak{X}(M)$ the module of smooth vector fields on $M$ and by $C^\infty(M)$ the ring of smooth real-valued functions on $M$. We will denote by $z = (z^1, \ldots, z^n)$ local coordinates on $M$.

Let $f \in \mathfrak{X}(M)$ and $\lambda \in C^\infty(M)$. We denote by $L_f \lambda$ the Lie derivative of $\lambda$ along $f$, which is defined as

$$L_f \lambda(z) = d\lambda(z)f(z), \quad z \in M.$$

Given $f, g \in \mathfrak{X}(M)$, the Lie bracket of $f$ and $g$ is a new smooth vector field denoted by $[f, g]$ and defined, in coordinates, as

$$[f, g](z) = Dg(z)f(z) - Df(z)g(z), \quad z \in M,$$

where $Dg(z)$ and $Df(z)$ denote the Jacobi matrix of $g$ and $f$ in $z$-coordinates, respectively. We will use often the notation

$$ad^0 g = g \quad \text{and, inductively,} \quad ad^j g = [f, ad^{j-1} g], \quad j \geq 0.$$

Let $M$ and $N$ be smooth manifolds of the same finite dimension and consider a diffeomorphism $\Phi : M \to N$. Given a vector field $f$ on $M$, we define the transformed vector field $\Phi_* f$ on $N$ (the image of $f$ under $\Phi$) as

$$(\Phi_* f)(\tilde{z}) = D\Phi(\Phi^{-1}(\tilde{z}))f(\Phi^{-1}(\tilde{z})), \quad \text{with} \quad \tilde{z} = \Phi(z).$$

A distribution $\mathcal{D}$ on $M$ is a map that assigns to each point $p$ in $M$ a linear subspace $\mathcal{D}(p)$ of the tangent space at this point $T_p M$. If $\mathcal{D}$ is a subbundle of the tangent bundle $TM$, that is, if the dimension $\dim \mathcal{D}(p)$ is constant, and equal to $k$, we say that $\mathcal{D}$ is of constant rank $k$. In what follows we will assume that all distributions are of constant rank. Locally a smooth distribution $\mathcal{D}$ of constant rank $k$ can be spanned by $k$ linearly independent smooth vector fields, which we will denote as $\mathcal{D} = \text{span}\{f_1, \ldots, f_k\}$. Given a vector field $f$, we say that $f$ belongs to the distribution $\mathcal{D}$ if $f(p) \in \mathcal{D}(p)$ for all $p \in M$. In this case we write $f \in \mathcal{D}$.

2.2. Affine connections. Let $Q$ be an $n$-dimensional smooth manifold.

An affine connection on $Q$ is a map

$$\nabla : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q), \quad (X, Y) \mapsto \nabla_X Y$$
which satisfies the following conditions:

\( (\text{AC}1) \quad \nabla_{\alpha_1 X + \alpha_2 Y} Z = \alpha_1 \nabla_X Z + \alpha_2 \nabla_Y Z, \)
\( (\text{AC}2) \quad \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z, \)
\( (\text{AC}3) \quad \nabla_X (\alpha Y) = \alpha \nabla_X Y + (L_X \alpha) Y, \)

for all \( X, Y, Z \in \mathfrak{X}(Q) \) and all \( \alpha_1, \alpha_2, \alpha \in C^\infty(Q) \).

The vector field \( \nabla_X Y \) is called the covariant derivative of the vector field \( Y \) with respect to the vector field \( X \).

In a system of local coordinates \((x^1, \ldots, x^n)\) on \( Q \), an affine connection is uniquely determined by its Christoffel symbols \( \Gamma^i_{jk} \), \( 1 \leq i, j, k \leq n \), which are defined by

\[ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^i_{jk} \frac{\partial}{\partial x^k}. \]

Here the summation convention is used with a sum on the repeated index \( i \).

Using the properties of an affine connection, the definition of the Christoffel symbols and writing the vector fields \( X \) and \( Y \) as \( X = X^j \frac{\partial}{\partial x^j} \) and \( Y = Y^k \frac{\partial}{\partial x^k} \), we obtain the covariant derivative \( \nabla_X Y \) written in coordinates as

\[ \nabla_X Y = \left( \frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial x^i}. \]

(3)

Since the covariant derivative \( \nabla_X Y \) at a point \( q \in Q \) depends only on the value of \( X \) at \( q \), we define the covariant derivative of \( Y \) along a smooth curve \( \gamma : I \subset \mathbb{R} \to Q \), as a vector field along \( \gamma \) given by

\[ I \ni t \mapsto \nabla_{\dot{\gamma}(t)} Y(\gamma(t)) \in T_{\gamma(t)} Q. \]

A geodesic of an affine connection \( \nabla \) on \( Q \) is a smooth curve \( \gamma \) on \( Q \) satisfying \( \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \). In local coordinates, \( \gamma(t) = x(t) = (x^1(t), \ldots, x^n(t)) \), a geodesic is given as the solution of the system of second-order differential equations:

\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad 1 \leq i, j, k \leq n. \]

This second-order system is equivalent to a system of first-order equations on the tangent bundle \( TQ \), with the coordinates \((x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)\) on \( TQ \) induced by the \( x \)-coordinates \((x^1, \ldots, x^n)\) on \( Q \):

\[ \dot{x}^i = y^i, \quad \dot{y}^j = -\Gamma^i_{jk} y^j y^k. \]

These equations define a vector field \( S \) on \( TQ \) which is called the \textit{geodesic spray} of \( \nabla \) and is given in local coordinates by

\[ S = y^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} y^j y^k \frac{\partial}{\partial y^i}. \]

The integral curves of \( S \), being curves on \( TQ \), are projected onto the geodesics on \( Q \) under the canonical projection \( \pi : TQ \to Q; \pi(q, v_q) = q \).

Associated with an affine connection, there is a symmetric real-bilinear operation called the \textit{symmetric product}:

\[ \langle X : Y \rangle = \nabla_X Y + \nabla_Y X \quad X, Y \in \mathfrak{X}(Q). \]

(4)

In coordinates, we have

\[ \langle X : Y \rangle = \left( \frac{\partial X^i}{\partial x^j} Y^j + \frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^j Y^k + \Gamma^i_{jk} Y^j X^k \right) \frac{\partial}{\partial x^i}. \]

(5)
A smooth distribution $\mathcal{D}$ on $Q$ is called \textit{geodesically invariant} with respect to a smooth affine connection $\nabla$ if every geodesic $\gamma : I \to Q$, having the property that $\gamma'(t_0) \in D(\gamma(t_0))$ for some $t_0 \in I$, satisfies $\gamma'(t) \in D(\gamma(t))$ for all $t \in I$. The following geometric interpretation of the symmetric product was given by Lewis in [30]. A distribution $\mathcal{D}$ on a manifold $Q$, equipped with an affine connection $\nabla$, is geodesically invariant if and only if

$$(X : Y) \in \mathcal{D}, \quad \text{for every } X, Y \in \mathcal{D}.$$ 

Given an affine connection $\nabla$, we define the maps

$$T : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q), \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

and

$$R : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q), \quad R(X, Y)Z = \nabla_X (\nabla_Y Z) - \nabla_Y (\nabla_X Z) - \nabla_{[X, Y]} Z,$$

which are called, respectively, the \textit{torsion tensor} and the \textit{curvature tensor} of the connection $\nabla$.

We say that $\nabla$ is symmetric if $T(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(Q)$. In this case, for any local coordinates, the corresponding Christoffel symbols are symmetric, that is, $\Gamma^r_{ij} = \Gamma^r_{ji}$, $1 \leq i, j, k \leq n$.

Given a vector field $k$ on $Q$, we define its vertical lift $K = k^{\text{v lift}} \in \mathfrak{X}(TQ)$, which is the vector field on $TQ$, given as

$$K(v_x) = k^{\text{v lift}}(v_x) = \frac{d}{dt} (v_x + tk(x))|_{t=0}, \quad v_x \in TQ.$$ 

In coordinates, if

$$k(x) = k^i(x) \frac{\partial}{\partial x^i},$$

then we have

$$K(v_x) = k^{\text{v lift}}(v_x) = k^i(x) \frac{\partial}{\partial y^i}. \quad (6)$$

Hence $K$ is a vector field which is annihilated when projected by the canonical projection $\pi$.

2.3. \textbf{Mechanical control systems.} We define a \textit{mechanical control system} $(\mathcal{MS})$ as a 4-tuple $(Q, \nabla, g_0, d)$, in which

(i) $Q$ is an $n$-dimensional \textit{configuration manifold};

(ii) $\nabla$ is a symmetric affine connection on $Q$;

(iii) $g_0 = (g_0, g_1, \ldots, g_m)$ is an $(m+1)$-tuple of smooth vector fields on $Q$;

(iv) $d : TQ \to TQ$ is a map sending the fiber $T_qQ$ into the fiber $T_qQ$, for any $q \in Q$, linear on fibers.

A curve $\gamma : I \to Q$, $I \subset \mathbb{R}$, is a trajectory of $(\mathcal{MS})$ if it satisfies the equation

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = g_0(\gamma(t)) + d(\dot{\gamma}(t)) + \sum_{r=1}^m u_r g_r(\gamma(t)). \quad (7)$$

Our definition of $(\mathcal{MS})$ is motivated by the following considerations. We consider a kinetic energy Lagrangian $\mathcal{L} : TQ \to \mathbb{R}$, $\mathcal{L} = K(x, y)$, with $K$ the kinetic energy, that is, a positive definite symmetric bilinear form on $TQ$, defining a Riemannian metric. The tangent bundle $TQ$ is called the \textit{velocity phase space}.

Assume that the system is subject to a field of external forces $\varphi : TQ \to T^*Q$, with $T^*Q$ the cotangent bundle of $Q$, which is, by definition, a smooth map that
sends the fiber $T_qQ$ into the fiber $T^*_qQ$, for any $q \in Q$. Then, the equations of motion (Euler-Lagrange equations with external forces) are given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial x} = \mathcal{F}(x,y).$$

We consider the force field $\mathcal{F}$ of the form

$$\mathcal{F} = \eta^0 + \sum_{r=1}^{m} u_r \eta^r + d,$$

where

(i) $\eta^0$ is a differential 1-form on $Q$, representing a positional external force that cannot be controlled (which can be conservative or not);

(ii) $\eta^1, \ldots, \eta^m$ are differential 1-forms on $Q$ representing positional input forces (i.e., external controlled forces);

(iii) $d : TQ \to T^*Q$ is a map sending $T_qQ$ into $T^*_qQ$ and linear on fibers.

In $(x,y)$-coordinates on $TQ$ the external force $\mathcal{F}$ is then expressed by

$$\mathcal{F}(x,y) = \eta^0(x) + \sum_{r=1}^{m} u_r \eta^r(x) + d(x,y)$$

$$= \eta^0(x)dx^i + \sum_{r=1}^{m} u_r \eta^r(x)dx^i + D_{ij}(x)y^jdx^i.$$

The forces $\eta^0$ and $\eta^r$, $1 \leq r \leq m$, are positional, that is, for any $q \in Q$,

$$\eta^k(q,v_q) = \eta^k(q,w_q), \quad 0 \leq k \leq m, \quad \text{for all } v_q, w_q \in T_qQ.$$

Our force field $\mathcal{F}$ contains thus arbitrary forces that depend on configurations only, and arbitrary forces that are linear in velocities.

We observe that, if $L = K - V$, with $V : Q \to \mathbb{R}$ the potential energy, then the force $\eta^0$ will consist of an external uncontrolled force together with the potential force $-dV$.

The force $d$ generalizes the notion of a dissipative or gyroscopic force. To see this, let $D$ be a $(0,2)$-tensor, that is, a bilinear form on each $T_xQ$, i.e.,

$$D_x(\cdot, \cdot) : T_xQ \times T_xQ \to \mathbb{R}.$$

In coordinates, $D_z(y, \bar{y}) = D_{ij}(x)y^i \bar{y}^j$, where $D_{ij}(x)$ is a matrix of functions. To any bilinear form $B : V \times V \to \mathbb{R}$, we can associate $B^* : V \to V^*$, by the formula $\langle B^*(v), u \rangle = B(u, v)$, for all $v, u \in V$. We put $d = D^*$, and thus $d(x, \cdot) : T_xQ \to T^*_xQ$ is linear on fibers, and in coordinates,

$$d(x,y) = D_{ij}(x)y^jdx^i,$$

with $D_{ij}$ being the same matrix as that defining $D$. We see that, indeed, $d$ generalises the notion of dissipative forces (for which the matrix $D_{ij}$ is supposed to be symmetric negative definite) and that of gyroscopic forces (if $D_{ij}$ is antisymmetric). We do not put any assumption on the matrix $D_{ij}$. The force $d$ will be called a $d$-force.

The generalized Newton law gives

$$\mu(\nabla \gamma(t)) = \mathcal{F},$$
where $\mu : TQ \to T^*Q$, given by $\mu(v_q)(\cdot) = K(v_q,\cdot)$ for all $v_q \in TQ$, is a diffeomorphism associated to the kinetic energy $K$, which is called the Legendre transformation or mass operator. Then, $\mu^{-1} : T^*Q \to TQ$ and we denote
\[d = \mu^{-1}(d), \quad g_0 = \mu^{-1}(\eta^0) \text{ and } g_r = \mu^{-1}(\eta^r), \quad 1 \leq r \leq m.\]
This yields
\[\nabla \dot{\gamma}(t) = g_0(\gamma(t)) + d(\dot{\gamma}(t)) + \sum_{r=1}^{m} u_r g_r(\gamma(t)),\]
which justifies our definition of $(\mathcal{MS})$, given by (7). We observe that $g_0 : Q \to TQ$ and $g_r : Q \to TQ$, $1 \leq r \leq m$, are vector fields on $Q$, whereas $d : TQ \to TQ$ is a map sending $T_qQ$ into $T_qQ$ and linear on fibers, that is, in coordinates
\[d(x,y) = d^i_j(x)y^j, \quad \text{where} \quad d^i_j(x) = \mu^{is}(x)D_{sj}(x).\]
The matrix $D_{sj}(x)$ is that defining $d$ and $(\mu^{is}(x)) = (\mu_{is}(x))^{-1}$.
In local coordinates $(x^1,\ldots,x^n)$ on $Q$, equation (7) is equivalent to the second-order system of differential equations (1) or, to the first-order system of differential equations on $TQ$, equipped with coordinates $(x^1,\ldots,x^n,y^1,\ldots,y^n)$:
\[
\begin{align*}
\dot{x}^i &= y^i, \\
\dot{y}^j &= -\Gamma^j_{ik}(x)y^i y^k + d^j_i(x)y^j + g_0(x) + \sum_{r=1}^{m} u_r g^r(x).
\end{align*}
\]  \hfill \tag{8}

In a condensed form, the latter equation can also be written as
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\hat{\Gamma}(x,y) + d(x,y) + g_0(x) + \sum_{r=1}^{m} u_r g_r(x),
\end{align*}
\]  \hfill \tag{9}
where $\hat{\Gamma}(x,y)$ is an $\mathbb{R}^n$-valued function, homogeneous of degree two with respect to $y$, i.e.,
\[
\hat{\Gamma}(x,y) = (\hat{\Gamma}^1(x,y),\ldots,\hat{\Gamma}^n(x,y))^T, \quad \text{with} \quad \hat{\Gamma}^i(x,y) = y^T \Gamma^i(x)y, \quad 1 \leq i \leq n,
\]
and each element $\Gamma^i$ is a symmetric $n \times n$ matrix with components $\Gamma^i_{jk}(x)$. As we already explained $d(x,y) = d^i_j(x)y^j$ is homogeneous of degree one with respect to $y$.

### 2.4. Some examples of mechanical control systems.

We present now some simple examples to illustrate our class of mechanical control systems. These examples are simplified models constructed in order to have a small number of states and controls. We briefly introduce each example, outlining information about the configuration of the system, the kinetic and potential energy, the forces acting on the system and the equations of motion. Then, we exhibit the objects of the definition of a mechanical control system given in the last subsection.

**Example 2.4.1 (The planar rigid body).** The presentation of this example closely follows [10]. This system can be thought of as a model for a planar rigid body or, equivalently, for a simplified hovercraft, as depicted in Figure 1.

We consider the inertial reference frame $R_{\text{spatial}} = (O_{\text{spatial}},\{s_1,s_2\})$, and the body reference frame $R_{\text{body}} = (O_{\text{body}},\{b_1,b_2\})$, in which $O_{\text{body}}$ coincides with the center of mass of the body. Two movements are allowed: the planar rigid body can translate in the plane and rotate about its center of mass. Let $q = (\theta,x,y)$ be the configuration of the system, with $\theta$ describing the relative orientation of the body reference frame $R_{\text{body}}$ with respect to the inertial reference frame $R_{\text{spatial}}$, and the
Let $\dot{q} = (\dot{\theta}, \dot{x}, \dot{y})$ be the spatial velocity, that is, the velocity of the system with respect to the inertial coordinate system. The state of the system is given by $(q, \dot{q})$, that is, the state consists of the configuration along with the spatial velocity. We assume that the system moves in a plane perpendicular to the direction of the gravitational forces, so that the potential energy is zero. Concerning the kinetic energy, it is given by

$$K = \frac{1}{2} \dot{q}^T G(q) \dot{q}, \quad \text{where} \quad G(q) = \begin{pmatrix} J & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}. $$

Here "$T$" stands for transpose, $m$ is the mass of the body and $J$ denotes its moment of inertia about the center of mass.

We consider a force $F$ applied to a point on the body that is at a distance $h > 0$ from the center of mass, along the body $b_1$-axis, as shown in the figure. Let $u_1$ be the component of $F$ in the body $b_1$-direction and $u_2$ be the component in the $b_2$-direction. The equations of motion for the planar rigid body system are thus given by

$$J\ddot{\theta} = -hu_2,$$
$$m\ddot{x} = u_1 \cos \theta - u_2 \sin \theta,$$
$$m\ddot{y} = u_1 \sin \theta + u_2 \cos \theta,$$

where $q = (\theta, x, y) \in Q = S^1 \times \mathbb{R}^2$ (compare with [10]). The kinetic energy defines the Riemannian structure $G = J d\theta \otimes d\theta + m(dx \otimes dx + dy \otimes dy)$. Thus, the Christoffel symbols of the corresponding Levi-Civita connection $\nabla$ (see e.g., [10, 44]) are $\Gamma^i_{jk} = 0$. Also, $\eta^0 = 0$ and $d = 0$ (implying that, respectively, $g_0 = 0$ and $d = 0$) and the control forces are

$$\eta^1 = \cos \theta dx + \sin \theta dy \quad \text{and} \quad \eta^2 = -hd\theta - \sin \theta dx + \cos \theta dy,$$

\footnote{In agreement with the notation used in the Section 2.3, it would be logical to denote the configuration of the system by $q = (x^1, x^2, x^3)$. However, physically it is more natural to use $q = (\theta, x, y)$.}
giving the input vector fields
\[
 g_1 = \mu^{-1}(\eta^1) = \frac{\cos \theta}{m} \frac{\partial}{\partial x} + \frac{\sin \theta}{m} \frac{\partial}{\partial y}, \quad \text{and} \\
 g_2 = \mu^{-1}(\eta^2) = -\frac{h}{J} \frac{\partial}{\partial \theta} - \frac{\sin \theta}{m} \frac{\partial}{\partial x} + \frac{\cos \theta}{m} \frac{\partial}{\partial y}.
\]

The mechanical control system is then
\[
 \ddot{\theta} = -\frac{h}{J} u_2, \\
 \ddot{x} = \frac{\cos \theta}{m} u_1 - \frac{\sin \theta}{m} u_2, \\
 \ddot{y} = \frac{\sin \theta}{m} u_1 + \frac{\cos \theta}{m} u_2.
\]

Example 2.4.2 (The robotic Leg). The presentation of this example follows closely [10] and [33]. This system consists of a main body which rotates about a fixed point, and at that fixed point is attached an extensible leg with a point mass \( m \) at its tip (see Figure 2). The moment of inertia of the base rigid body about the pivot point is \( J \) and the direction of the gravitational field is assumed to be orthogonal to the plane of motion of the system (see [10]).

We denote \( q = (r, \theta, \psi) \) the configuration of the system, with \( r \) describing the extension of the leg, \( \theta \) the angle of the leg with respect to an inertial reference frame and \( \psi \) the angle of the body. The configuration manifold is \( Q = \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1 \). Two input forces are considered: the torque \( \eta^1 = d\theta - d\psi \) applied at the point of rotation that controls the angle between the body and the leg, and the force \( \eta^2 = dr \) that extends the leg. There exists neither uncontrolled forces nor dissipative-type forces \( (\eta^0 = 0 \text{ and } d = 0) \). The equations of motion describing the system are [10, 33]:
\[
 m\dddot{r} - mr\dddot{\theta}^2 = u_2, \\
 mrr^2\dddot{\theta} + 2mrr\dddot{\theta} = u_1, \\
 J\dddot{\psi} = -u_1.
\]

The potential energy of this system is zero and the kinetic energy is given by
\[
 K = \frac{1}{2} q^T G(q) \dot{q}, \quad \text{where} \quad G(q) = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & J \end{pmatrix},
\]
which defines the Riemannian structure \( G = mdr \otimes dr + mrr^2d\theta \otimes d\theta + Jd\psi \otimes d\psi \). The corresponding affine connection \( \nabla \) is the Levi-Civita connection and the only
nonzero Christoffel symbols for $\nabla$ are computed to be (compare with [10])

$$\Gamma^r_{\theta\theta} = r$$  and  $$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}.$$

The input vector fields are obtained to be

$$g_1 = \mu^{-1}(\eta^1) = \frac{1}{mr^2} \frac{\partial}{\partial \theta} - \frac{1}{J} \frac{\partial}{\partial \psi}$$  and  $$g_2 = \mu^{-1}(\eta^2) = \frac{1}{m} \frac{\partial}{\partial r},$$

the uncontrolled vector field is $g_0 = 0$ and also $d = 0$.

The mechanical control system is then

\[\begin{align*}
\ddot{r} &= r\dot{\theta}^2 + \frac{1}{m} u_2, \\
\ddot{\theta} &= -2 \frac{r}{r} \dot{\theta} + \frac{1}{mr^2} u_1, \\
\ddot{\psi} &= -\frac{1}{J} u_1.
\end{align*}\]

**Example 2.4.3 (The PVTOL Aircraft).** We consider now a simple model of a planar vertical take off and landing (PVTOL) aircraft as depicted in Figure 3. This mechanical control system was introduced in [21] and has attracted a lot of attention in recent years (see for example [36, 43]).

Similar to the rigid body example, the configuration of the system is $q = (\theta, x, y)$, with $\theta$ the angle the PVTOL makes with the horizontal line and $(x, y)$ the position of the aircraft center of mass. Let $u_1$ be the control corresponding to the body vertical force and let $u_2$ be the control corresponding to forces on the tips of the wings.

The dynamics of this simplified version of the PVTOL, after normalisation of $m$ and $J$, are given as (see [36, 43])

\[\begin{align*}
\dot{\theta} &= u_2, \\
\dot{x} &= -u_1 \sin \theta + \epsilon u_2 \cos \theta, \\
\dot{y} &= u_1 \cos \theta + \epsilon u_2 \sin \theta - a_g,
\end{align*}\]

where the constant $a_g$ is the gravity acceleration and $\epsilon \neq 0$ is a fixed constant related to the geometry of the aircraft. We obtain

$$g_1 = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}$$  and  $$g_2 = \frac{\partial}{\partial \theta} + \epsilon \cos \theta \frac{\partial}{\partial x} + \epsilon \sin \theta \frac{\partial}{\partial y}$$

for the control vector fields, $g_0 = -a_g \frac{\partial}{\partial \theta}$ being the uncontrolled vector field and $d = 0$. 
Example 2.4.4 (Pendulum with Damping). All examples that we have presented so far are with no d-forces. Now we will consider the pendulum with damping
\[
\dot{\theta} = \omega,
ml^2 \dot{\omega} = -ma_g l \sin \theta - k \omega + u,
\]
with \(\theta\) the angle and \(\omega\) the angular velocity. We consider \((\theta, \omega) \in TQ = S^1 \times \mathbb{R}\).

By \(m, l, a_g\) we denote, respectively, the mass, the length, and the gravitational acceleration, and \(k\) is the damping-coefficient. The controlled force applied to the pendulum is the torque \(\eta^1 = ud\theta\), the uncontrolled force is the gravitational force \(\eta^0 = -ma_g l \sin \theta d\theta\), and the map \(d : T\theta Q \to T\theta Q\) associates the covector \(-k\omega_\theta\) to the angular velocity \(\omega_\theta\) at \(\theta\). Represented as a mechanical control system, the pendulum takes the form
\[
\dot{\theta} = \omega,
\dot{\omega} = -a_g \frac{g_0}{l} \sin \theta - \frac{k}{ml^2} \omega + \frac{1}{ml^2} u,
\]
where \(g_0 := \frac{a_g}{l} \sin \theta \frac{\partial}{\partial \theta}\) is the uncontrolled vector field, \(g_1 := \frac{1}{ml^2} \frac{\partial}{\partial \theta}\) is the controlled vector field and the fiber-linear map \(d : T\theta Q \to T\theta Q\) associates the vector \(-\frac{k}{ml^2} \omega_\theta\) to the velocity \(\omega_\theta\).

2.5. State equivalence to mechanical control systems. In this paper, we shall consider state equivalence transformations. We will study the problem of when a control system admits a mechanical structure, that is, when it is state equivalent to a mechanical system. Feedback equivalence of a nonlinear control system to a given Hamiltonian system has been considered in [12, 29, 58] and equivalence to a gradient system in [13].

Two control systems are said to be state equivalent if they are related by a diffeomorphism (and then also their trajectories, corresponding to the same controls, are related by that diffeomorphism) [22, 42, 48]. More precisely, consider two control systems
\[
\Sigma : \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z), \quad z \in M, \quad \text{and}
\]
\[
\tilde{\Sigma} : \dot{\tilde{z}} = \tilde{F}(\tilde{z}) + \sum_{r=1}^{m} u_r \tilde{G}_r(\tilde{z}), \quad \tilde{z} \in \tilde{M}.
\]

**Definition 2.1.** We say that \(\Sigma\) and \(\tilde{\Sigma}\) are state equivalent, shortly S-equivalent, if there exists a diffeomorphism \(\Phi : M \to \tilde{M}\) such that
\[
\Phi_* F = \tilde{F} \quad \text{and} \quad \Phi_* G_r = \tilde{G}_r, \quad 1 \leq r \leq m.
\]

Recall that \(\Phi_*\) stands for the tangent map of \(\Phi\) (see Subsection 2.1).

The systems \(\Sigma\) and \(\tilde{\Sigma}\) are called locally S-equivalent, at \(z_0 \in M\) and \(\tilde{z}_0 \in \tilde{M}\), respectively, if there exists neighborhoods \(U\) of \(z_0\) and \(\tilde{U}\) of \(\tilde{z}_0\), such that \(\Sigma\) restricted to \(U\) and \(\tilde{\Sigma}\) restricted to \(\tilde{U}\) are S-equivalent.

The crucial property of state equivalence is that the diffeomorphism \(\Phi\) establishing the S-equivalence yields a one-to-one correspondence between trajectories of \(\Sigma\) and \(\tilde{\Sigma}\) corresponding to the same controls.

If one of the systems is mechanical, we obtain the following definition, fundamental for our considerations. Assume that the dimension of \(M\) is \(2n\).
Definition 2.2. We say that $\Sigma$ is $S$-equivalent to a mechanical control system if there exists a mechanical system of the form
\[
(MS) : \quad \dot{x} = y, \\
\dot{y} = -\hat{\Gamma}(x,y) + d(x,y) + g_0(x) + \sum_{r=1}^m u_r g_r(x),
\]
where $(x,y) = (x^1,\ldots,x^n,y^1,\ldots,y^n)$ are local coordinates on $TQ$, such that $\Sigma$ and $(MS)$ are $S$-equivalent. In this case, we may also say that $\Sigma$ admits a mechanical structure.

In other words, $\Sigma$ is $S$-equivalent to $(MS)$ if the diffeomorphism $\Phi$, establishing the $S$-equivalence, satisfies
\[
\Phi^* F = y^i \frac{\partial}{\partial x^i} + (-\Gamma^i_{jk}(x)y^j y^k + d^i_j(x)y^j + g^i_0(x)) \frac{\partial}{\partial y^i},
\]
(10)
\[
\Phi^* G_r = g^i_r(x) \frac{\partial}{\partial y^i}.
\]
(11)

Analogously, we define local $S$-equivalence of $\Sigma$ at $z_0$ to a mechanical system $(MS)$ at $(x_0,y_0) \in TQ$. In this case, we say that $\Sigma$ locally admits a mechanical structure.

3. Geodesically accessible mechanical systems.

3.1. Definition and examples. Consider the mechanical control system $(MS) = (Q, \nabla, g_0, d)$. Let $\mathcal{SYM}(g_1,\ldots,g_m)$ be the smallest distribution on $Q$ containing the input vector fields $g_1,\ldots,g_m$ and such that it is closed under the symmetric product defined by the connection $\nabla$.

Definition 3.1. The system $(MS)$ is called geodesically accessible at $x_0 \in Q$ if
\[
\mathcal{SYM}(g_1,\ldots,g_m)(x_0) = T_{x_0}Q,
\]
and geodesically accessible if the above equality holds for all $x_0 \in Q$.

A geodesically accessible mechanical system will be denoted by $(GAMS)$.

Example 3.1.1. Let $(x^1, x^2)$ be coordinates of $\mathbb{R}^2$ in which a mechanical system $(MS)$ reads as
\[
\dot{x}^1 = -\frac{1}{2}(\dot{x}^2)^2, \\
\dot{x}^2 = x^1 + \dot{x}^1 + u,
\]
and is defined by the affine connection $\nabla$ on $\mathbb{R}^2$, given by the Christoffel symbols, which in these coordinates are $\Gamma^1_{22} = \frac{1}{2}$ and $\Gamma^i_{jk} = 0$, otherwise. We have $g = \frac{\partial}{\partial x^2}$. By (5), we obtain $\langle g : g \rangle = \frac{\partial}{\partial x^2}$. Thus, $\mathcal{SYM}(g) = \text{span}\{g, \langle g : g \rangle\} = \text{span}\{\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}\}$ and we conclude that this system is a $(GAMS)$.

Example 3.1.2. Consider the linear mechanical system on $\mathbb{R}^2$, written in coordinates $(x^1, x^2)$,
\[
\dot{x}^1 = u, \\
\dot{x}^2 = x^1.
\]
We consider the affine connection determined by the Christoffel symbols which in these coordinates vanish (that is, the affine connection is the Euclidean). For

\[\text{To avoid confusions, we recall that we systematically use upper indices for indexing coordinates, so below $(\dot{x}^2)^2$ stands for the square of the velocity second component.}\]
$g := \frac{\partial}{\partial x}$, we have $\text{SYM}(g) = \text{span}\{g\}$. We conclude that this system is not a (GAMS). It is noteworthy that this linear system is controllable and so, in particular, accessible and strongly accessible (definitions and a detailed analysis of accessibility, strong accessibility and controllability can be found in [10, 22, 42, 56]; see also [4, 7, 26]).

**Example 3.1.3** (Planar rigid body cont’d). As in Example 2.4.1, we consider the configuration coordinates $q = (\theta, x, y) \in Q = S^1 \times \mathbb{R}^2$ and the input vector fields
\[ g_1 = \frac{\cos \theta}{m} \frac{\partial}{\partial x} + \frac{\sin \theta}{m} \frac{\partial}{\partial y} \quad \text{and} \quad g_2 = -\frac{h}{J} \frac{\partial}{\partial \theta} - \frac{\sin \theta}{m} \frac{\partial}{\partial x} + \frac{\cos \theta}{m} \frac{\partial}{\partial y}. \]
We get the symmetric product of these vector fields as (recall that all Christoffel symbols vanish)
\[ \langle g_1 : g_1 \rangle = 0, \quad \langle g_1 : g_2 \rangle = \frac{h}{Jm} (\sin \theta \frac{\partial}{\partial x} - \cos \theta \frac{\partial}{\partial y}), \quad \langle g_2 : g_2 \rangle = \frac{2h}{Jm} (\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}). \]
Then $\text{SYM}(g_1, g_2)(q) = \text{Vect}_\mathbb{R}\{g_1, g_2, \langle g_1 : g_2 \rangle\}(q) = \mathbb{R}^3$, $\forall q \in Q$, from which we conclude that the system is a (GAMS).

**Example 3.1.4** (Robotic Leg cont’d). As observed in Example 2.4.2, the configuration of the system is $q = (r, \theta, \psi) \in Q = \mathbb{R}^+ \times S^1 \times S^1$ and the input vector fields are given by
\[ g_1 = \frac{1}{mr^2} \frac{\partial}{\partial \theta} - \frac{1}{J} \frac{\partial}{\partial \psi} \quad \text{and} \quad g_2 = \frac{1}{m} \frac{\partial}{\partial r}. \]
We obtain (using the Christoffel symbols calculated in Example 2.4.2)
\[ \langle g_1 : g_1 \rangle = -\frac{2}{mr^3} \frac{\partial}{\partial r} \quad \text{and} \quad \langle g_1 : g_2 \rangle = \langle g_2 : g_2 \rangle = 0. \]
Clearly, $\langle g_1 : g_1 \rangle$ is colinear with $g_2$ (compare with [10]). Direct computations show that all symmetric products of higher order are either colinear with $g_2$ or vanish. Therefore, $\text{SYM}(g_1, g_2) = \text{span}\{g_1, g_2\}$ from which we conclude that the system is not a (GAMS), although it is accessible.

**Example 3.1.5** (The PVTOL Aircraft cont’d). Using the notation from Example 2.4.3, we take $q = (\theta, x, y) \in Q = S^1 \times \mathbb{R}^2$ and consider the input vector fields
\[ g_1 = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} \quad \text{and} \quad g_2 = \frac{\partial}{\partial \theta} + \epsilon \cos \theta \frac{\partial}{\partial x} + \epsilon \sin \theta \frac{\partial}{\partial y}. \]
It follows that (recall that all Christoffel symbols vanish)
\[ \langle g_1 : g_1 \rangle = 0, \quad \langle g_1 : g_2 \rangle = -\cos \theta \frac{\partial}{\partial x} - \sin \theta \frac{\partial}{\partial y}, \quad \langle g_2 : g_2 \rangle = -2\epsilon \sin \theta \frac{\partial}{\partial x} + 2\epsilon \cos \theta \frac{\partial}{\partial y}. \]
Clearly, $\text{SYM}(g_1, g_2)(q) = \text{Vect}_\mathbb{R}\{g_1, g_2, \langle g_1 : g_2 \rangle\}(q) = \mathbb{R}^3$, from which we conclude that the system is a (GAMS).
3.2. Statement of the main result. For any vector field \( F \) and any collections \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) of vector fields, we will put
\[
[\mathcal{G}_1, \mathcal{G}_2] = \{ [G_i, G_j] \mid G_i \in \mathcal{G}_1, G_j \in \mathcal{G}_2 \} \quad \text{and} \quad \text{ad}_F \mathcal{G}_1 = \{ [F, G_i] \mid G_i \in \mathcal{G}_1 \}.
\]
Define the following sequence of families of vector fields on \( M \) :
\[
\mathcal{V}_1 = \{ G_r \mid 1 \leq r \leq m \} \quad \text{and} \quad \mathcal{V}_2 = \{ [G_r, \text{ad}_F G_s] \mid 1 \leq r, s \leq m \}
\]
and, inductively,
\[
\mathcal{V}_i = \bigcup_{p+t=i} (\mathcal{V}_p, \text{ad}_F \mathcal{V}_t).
\]
Put
\[
\mathcal{V} := \text{Vect}_R \bigcup_{i=1}^{\infty} \mathcal{V}_i. \tag{12}
\]

A point \( z_0 \in M \) is said to be an equilibrium point for the system \( \Sigma \) if \( F(z_0) = 0 \).

We will call a zero-velocity point for the mechanical control system \((MS)\) any point of the form \( (x_0, y_0) = (x_0, 0) \), that is any point of the zero section of \( TQ \). An equilibrium point for \((MS)\) is a zero-velocity point such that, additionally, we have \( g_0(x_0) = 0 \). (Recall that \( g_0 \) is the uncontrolled vector field in the definition of the mechanical control system \((MS)\)).

**Theorem 3.2.** Let \( M \) be a smooth \( 2n \)-dimensional manifold. A system \( \Sigma \) is locally, at \( z_0 \in M \), S-equivalent to a geodesically accessible mechanical system \((GAM\Sigma)\) around a zero-velocity point \((x_0, 0)\) if and only if
1. \((MS0)\) \( F(z_0) \in \mathcal{V}(z_0) \),
2. \((MS1)\) \( \dim \mathcal{V}(z) = n \) and \( \dim (\mathcal{V} + [F, \mathcal{V}]) (z) = 2n \),
3. \((MS2)\) \( [\mathcal{V}, [\mathcal{V}, \mathcal{V}]] (z) = 0 \),

for any \( z \) in a neighborhood of \( z_0 \).

The condition \((MS0)\) implies that the diffeomorphism establishing the S-equivalence (if it exists) will map \( z_0 \) into a zero-velocity point. A mechanical system (more generally, a control system that is S-equivalent to a mechanical system \((MS)\)) is geodesically accessible around a zero-velocity point if and only if it satisfies \((MS0)\) and \((MS1)\) (see Corollary 4.5 in Section 4). The condition \((MS2)\) is always necessary for S-equivalence to a mechanical system \((MS)\) (see Proposition 5.2 below) and sufficient provided that \((MS1)\) and \((MS2)\) hold. It says that the Lie algebra \( \mathcal{L} = \{ F, G_1, \ldots, G_m \}_{LA} \) contains an abelian subalgebra \( \mathcal{V} \) which is the structural condition reflecting the existence of a mechanical structure of \( \Sigma \).

Observe that the definition of \( \mathcal{V} \), given in Section 1, as the smallest vector space, over \( \mathbb{R} \), containing the vector fields \( G_1, \ldots, G_m \) and satisfying
\[
[\mathcal{V}, \text{ad}_F \mathcal{V}] \subset \mathcal{V},
\]
coincides with the constructive one given by (12). Indeed, let us denote by \( \mathcal{U} \) the smallest vector space, over \( \mathbb{R} \), containing the vector fields \( G_1, \ldots, G_m \) and satisfying \([\mathcal{U}, \text{ad}_F \mathcal{U}] \subset \mathcal{U}\), where \([\mathcal{U}, \text{ad}_F \mathcal{U}] = \{ [U_i, \text{ad}_F U_j] \mid U_i, U_j \in \mathcal{U} \}\) and consider \( \mathcal{V} \) defined by (12). By construction, \( \mathcal{V} \) satisfies \([\mathcal{V}, \text{ad}_F \mathcal{V}] \subset \mathcal{V} \) and, since \( \mathcal{U} \) is the smallest vector space with that property, it follows that \( \mathcal{U} \subset \mathcal{V} \). To prove \( \mathcal{V} \subset \mathcal{U} \), notice that \( \mathcal{V}_1 \subset \mathcal{U} \) implies that \( \mathcal{V}_2 = [\mathcal{V}_1, \text{ad}_F \mathcal{V}_1] \subset [\mathcal{U}, \text{ad}_F \mathcal{U}] \subset \mathcal{U} \) and, by induction argument, \( \mathcal{V}_i \subset [\mathcal{U}, \text{ad}_F \mathcal{U}] \subset \mathcal{U} \), for any \( i \geq 1 \).
For illustration of Theorem 3.2 we consider the next two examples.

**Example 3.2.1.** Consider the control-affine system in $\mathbb{R}^4$

$$
\begin{align*}
\dot{x}^1 &= y^1, \\
\dot{x}^2 &= y^2, \\
\dot{y}^1 &= -y^2\left(\frac{1}{2}y^2 + x^1 + u\right), \\
\dot{y}^2 &= x^1 + u.
\end{align*}
$$

We have

$$
G = -y^2 \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2} \quad \text{and} \quad F = y^1 \frac{\partial}{\partial x^1} - y^2 \left(\frac{1}{2}y^2 + x^1\right) \frac{\partial}{\partial y^1} + x^1 \frac{\partial}{\partial y^2},
$$

in which, as usual, a sum is understood over the index $i$, with $i = 1, 2$. We have

$$
\text{ad}_F G = y^2 \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^1}, \quad [G, \text{ad}_F G] = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^1},
$$

and

$$
[F, [G, \text{ad}_F G]] = -\frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}.
$$

Hence $\mathcal{V}_1 = \{G\}$, $\mathcal{V}_2 = \{[G, \text{ad}_F G]\}$, $\mathcal{V}_3 = 0$, and

$$
\mathcal{V} = \text{Vect}_\mathbb{R}\{G, [G, \text{ad}_F G]\}, \quad [F, \mathcal{V}] = \text{Vect}_\mathbb{R}\{\text{ad}_F G, [F, [G, \text{ad}_F G]]\}.
$$

Thus $\dim \mathcal{V}(z) = 2$ and $\dim (\mathcal{V} + [F, \mathcal{V}]) = 4$, for any $z \in \mathbb{R}^4$, and $[\mathcal{V}, \mathcal{V}] = 0$. By Theorem 3.2, the system is S-equivalent to a $\mathcal{GAMS}$ around a zero-velocity point $(x_0, 0)$. Indeed, the diffeomorphism

$$
\begin{align*}
\tilde{x}^1 &= x^1 - (y^1 + \frac{1}{2}(y^2)^2), \\
\tilde{x}^2 &= x^2, \\
\tilde{y}^1 &= y^1 + \frac{1}{2}(y^2)^2, \\
\tilde{y}^2 &= y^2,
\end{align*}
$$

transforms the above system into the system of Example 3.1.1.

**Example 3.2.2** (Planar rigid body cont’d). We have seen in Example 3.1.3 that the planar rigid body corresponds to a $\mathcal{GAMS}$. We shall see now that, indeed, it satisfies the conditions of Theorem 3.2. Denote the coordinates of $T\mathcal{Q}$ by $(\theta, x, y, \dot{\theta}, \dot{x}, \dot{y})$.

Set

$$
G_1 = \frac{\cos \theta}{m} \frac{\partial}{\partial x} - \frac{\sin \theta}{m} \frac{\partial}{\partial y}, \quad G_2 = -\frac{h}{J} \frac{\partial}{\partial \theta} - \frac{\sin \theta}{m} \frac{\partial}{\partial x} + \frac{\cos \theta}{m} \frac{\partial}{\partial y},
$$

and

$$
F = \dot{\theta} \frac{\partial}{\partial \theta} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y}.
$$

Direct Lie brackets computations show that

$$
\begin{align*}
\text{ad}_F G_1 &= -\frac{\cos \theta}{m} \frac{\partial}{\partial x} - \frac{\sin \theta}{m} \frac{\partial}{\partial y} - \frac{\sin \theta}{m} \frac{\partial}{\partial \theta} + \frac{\cos \theta}{m} \frac{\partial}{\partial y}, \\
\text{ad}_F G_2 &= \frac{h}{J} \frac{\partial}{\partial \theta} + \frac{\sin \theta}{m} \frac{\partial}{\partial x} + \frac{\cos \theta}{m} \frac{\partial}{\partial y} - \frac{\sin \theta}{m} \frac{\partial}{\partial \theta} - \frac{\cos \theta}{m} \frac{\partial}{\partial \theta} - \frac{\sin \theta}{m} \frac{\partial}{\partial y},
\end{align*}
$$

and

$$
\begin{align*}
[G_1, \text{ad}_F G_2] &= [G_2, \text{ad}_F G_1] &= \frac{h}{Jm} \left(\sin \theta \frac{\partial}{\partial \theta} - \cos \theta \frac{\partial}{\partial \theta}\right), \\
[G_1, \text{ad}_F G_1] &= 0, \\
[G_2, \text{ad}_F G_2] &= \frac{2h}{Jm} \left(\cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial \theta}\right).
\end{align*}
$$

\(^3\)To avoid confusions, we recall that we systematically use upper indices for indexing coordinates, so $(y^2)^2$ stands for the square of the second component of $y$. 

\[\text{SANDRA RICARDO AND WITOLD RESPONDEK} \]
Also,

\[ [F, [G_1, \text{ad}_F G_2]] = \frac{h}{J_m} \left( -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} + \cos \theta \dot{\theta} \frac{\partial}{\partial x} + \sin \theta \dot{\theta} \frac{\partial}{\partial y} \right). \]

Hence

\[ V = \text{Vect}_R \{ G_1, G_2, [G_1, \text{ad}_F G_2] \}, \]

\[ [F, V] = \text{Vect}_R \{ \text{ad}_F G_1, \text{ad}_F G_2, [F, [G_1, \text{ad}_F G_2]] \}. \]

Clearly, \( \dim V(z) = 3 \) and \( \dim (V + [F, V])(z) = 6 \), for any \( z \in TQ \) and \( [V, V] = 0 \).

**Remark 3.1.** Theorem 3.2 gives a geometric characterisation of a \((\mathcal{GAMS})\) around a zero-velocity point \((x_0, 0)\) which, in general, may not be an equilibrium. The PVTOL aircraft considered in Examples 2.4.3 and 3.1.5 is an example of a \((\mathcal{GAMS})\), for which \((x_0, 0) = (0, 0)\) is not an equilibrium, but only a zero-velocity point because of the presence of \( a_g \neq 0 \).

4. **Geometric interpretation.** In the last section, we saw that conditions (MS0), (MS1) and (MS2) of Theorem 3.2 give locally a geometric characterisation of a \((\mathcal{GAMS})\) around a zero-velocity point. It is remarkable to see that all structure information is encoded in conditions (MS0), (MS1), and (MS2). Indeed, as we shall see below, under these three conditions, we are able to find the configuration manifold, to define suitable vector fields on it, to find a connection and a map (linear on fibers) defining the d-forces.

Objects defined for the system \( \Sigma \) due to the conditions (MS0)-(MS2) (i.e., in particular, due to the structural condition \([V, V] = 0\)) will be denoted by an upper index \( \Sigma : Q^\Sigma, g^\Sigma, \nabla^\Sigma \) and \( d^\Sigma \).\(^4\) These objects will be obtained locally around a point \( z_0 \in M \).

The proof of the sufficiency part of Theorem 3.2 (see Section 7) shows that, under the conditions (MS0)-(MS2), there exist \( n \) independent and commuting vector fields \( V_1, \ldots, V_n \) in the set \( V \) and we can find a local system of coordinates \((x, y)\) such that in a neighborhood \( O_{z_0} \) of \( z_0 \in M \) the vector fields \( V_j \) are given by \( V_j = \frac{\partial}{\partial y^j}, 1 \leq j \leq n \). We will continue our considerations in that open set \( O_{z_0} \), where the vector fields \( V_j \) are rectified. In particular, \( \text{span } V = \text{span} \left\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\} \).

We will define the configuration manifold as (see Figure 4)

\[ Q^\Sigma = \{ z \in M \mid F(z) \in V(z) \}. \] (13)

We can easily check that \( Q^\Sigma \) is, indeed, an \( n \)-dimensional smooth submanifold of \( M \). To see this, consider, in \((x, y)\)-coordinates, the vector field \( F \) represented as \( F = f^1 \frac{\partial}{\partial x} + f^n \frac{\partial}{\partial y^n} \). Thus \( Q^\Sigma \) is given by

\[ Q^\Sigma = \{(x, y) \in M \mid f^i(x, y) = 0, 1 \leq i \leq n \}. \]

Let us consider the smooth map defined on \( O_{z_0} \):

\[ \phi : O_{z_0} \subset M \rightarrow \mathbb{R}^n \]

\[ (x, y) \mapsto \phi(x, y) = (f^1(x, y), \ldots, f^n(x, y))^T. \]

We observe that the differentials \( df^1(x, y), \ldots, df^n(x, y) \) are independent on \( O_{z_0} \) (since the vector fields \( \text{ad}_F V_1, \ldots, \text{ad}_F V_n \) are independent mod span \( V \)). Thus,

\(^4\)Since the objects \( Q^\Sigma, g^\Sigma, \nabla^\Sigma, d^\Sigma \) are defined by \( V \), it would be appropriate to denote them by \( Q^{\Sigma,V}, g^{\Sigma,V}, \nabla^{\Sigma,V}, d^{\Sigma,V} \), but we will skip the symbol \( V \) for compactness of notation.
The configuration manifold $Q^\Sigma$ consists of points where $F$ is tangent to the leaves $L_c$ of the foliation defined by $V$. 

$$\text{rank } D\phi(x,y) = n$$

and $Q^\Sigma = \phi^{-1}(0) \cap O_{z_0}$ is an embedded submanifold of $O_{z_0} \subset M$ of dimension $n$.

The tangent space of $Q^\Sigma$ at its arbitrary point $q = (x,y)$ is the set

$$T_qQ^\Sigma = \{ v \in T_qM \mid < df^i(q), v >= 0, \ 1 \leq i \leq n \}$$

and the distribution $\text{span } V = \text{span } \left\{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\}$ defines the canonical foliation $F = \{ L_c \}_{c \in \mathbb{R}^n}$, where $L_c = \{ (x,y) \mid x = c \in \mathbb{R}^n \}$.

The configuration manifold $Q^\Sigma$ is transversal to the leaves of $F$. Indeed, the matrix

$$(< df^i(x,y), V_j >)_{i,j}, \quad \text{with} \quad V_j \in V,$$

is of full rank (which is a direct consequence of the independence of the vector fields $\text{ad}_F V_1, \ldots, \text{ad}_F V_n \mod \text{span } V$). Therefore,

$$T_qM = T_qQ^\Sigma \oplus V(q), \quad \text{for any} \quad q \in Q^\Sigma.\quad (14)$$

Define the surjective submersion $\pi : O_{z_0} \rightarrow Q^\Sigma$ by attaching to any $z \in O_{z_0} \subset M$ the unique point $q = \pi(z)$, such that $q \in Q^\Sigma \cap L_z$, where $L_z$ is the leaf passing through $z$.

Any vector field $V \in V$ gives rise to a vector field $v$ on $Q^\Sigma$. Indeed, by definition of $V$, the vector field $\text{ad}_F V$ satisfies the condition $[\text{ad}_F V, V] \subset V$, which implies that $\text{ad}_F V$ projects to the vector field on $Q^\Sigma$:

$$v := -\pi_*(\text{ad}_F V).$$

In particular, since $G_r \in V, 1 \leq r \leq m$, the above formula defines the control vector fields on $Q^\Sigma$ by:

$$g^\Sigma_r := -\pi_*(\text{ad}_F G_r).$$

In this way, given $n$ locally independent vector fields $V_1, \ldots, V_n \in V$ we obtain a local frame $v_1, \ldots, v_n$ on $Q^\Sigma$. Alternatively, given a local frame $v_1, \ldots, v_n$ on $Q^\Sigma$, there exists a unique collection of independent vector fields $V_1, \ldots, V_n \in \text{span } V$ such that $V_j = (v_j)^{\text{vain}}, 1 \leq j \leq n$ (recall the identity (6) for the vertical lift of a vector field on $Q^\Sigma$).

Let $\hat{V}$ denote the module generated by all vector fields belonging to $V$, over the ring of smooth functions on $O_{z_0}$ that are constant on the leaves of the canonical foliation $F$. Clearly, any vector field $V \in \hat{V}$ projects to a vector field $v$ on $Q^\Sigma$ given...
by the above formula \( v = -\pi_*(\text{ad}_F V) \) but, conversely, for any vector field \( v \in Q^\Sigma \) there exists a unique vector field \( V \in \mathcal{V} \) such that \( v = -\pi_*(\text{ad}_F V) \), which defines the vertical lift of \( v \), namely \( \nu_{\text{lift}} = V \).

Using that one-to-one correspondence of the modules \( \mathcal{V} \) and \( \mathfrak{X}(Q^\Sigma) \), we will proceed now to define an affine connection on \( Q^\Sigma \) for any system satisfying (MS1) and (MS2).

Consider the map \( \nabla^\Sigma : \mathfrak{X}(Q^\Sigma) \times \mathfrak{X}(Q^\Sigma) \to \mathfrak{X}(Q^\Sigma) \) given by

\[
\nabla^\Sigma_v : = \frac{1}{2} \pi_* \left( \left[ \text{ad}_F^2 V, W \right] \right),
\]

with \( v, w \) arbitrary vector fields on \( Q^\Sigma \) and \( V, W \) their corresponding vertical lifts.

We will prove that \( \nabla^\Sigma \) defined by the above equality is, indeed, an affine connection.

**Proposition 4.1.** \( \nabla^\Sigma \) defined by equality (15) is an affine connection on \( Q^\Sigma \).

**Proof.** We prove that \( \nabla^\Sigma \) is an affine connection on \( Q^\Sigma \) by verifying that it satisfies the conditions (AC1), (AC2) and (AC3) of the definition of affine connection in Section 2.2. From the linearity of the Lie bracket we immediately conclude that

\[
\nabla^\Sigma_{w + \bar{w}} v = \nabla^\Sigma_w v + \nabla^\Sigma_{\bar{w}} v \quad \text{and} \quad \nabla^\Sigma_{v + \bar{v}} (v + \bar{v}) = \nabla^\Sigma_v v + \nabla^\Sigma_{\bar{v}} \bar{v},
\]

for any \( v, w, \bar{v}, \bar{w} \in \mathfrak{X}(Q^\Sigma) \). It remains to prove that

\[
\nabla^\Sigma_{\alpha w} v = \alpha \nabla^\Sigma_w v \quad \text{and} \quad \nabla^\Sigma_{\alpha v} (\alpha v) = \alpha \nabla^\Sigma_w v + \alpha (L_w \alpha) v,
\]

for all \( v, w \in \mathfrak{X}(Q^\Sigma) \) and all \( \alpha \in C^\infty(Q^\Sigma) \). Clearly, we have \( \alpha w \in \mathfrak{X}(Q^\Sigma) \) and its vertical lift \( (\alpha w)^{\text{lift}} = \bar{a}W \) is a vector field in \( \mathcal{V} \) such that

\[
\alpha w = -\pi_*(\text{ad}_F(\bar{a}W)), \quad \bar{a} \in C^\infty(TQ^\Sigma),
\]

where \( \bar{a} = \pi^*(\alpha) \), with \( \pi^* \) denoting the pullback defined by the projection \( \pi \), and thus \( \bar{a} \) is constant on the leaves of \( \mathcal{V} \). It follows from the definition of pullback that

\[
\bar{a}(v_q) = \pi^*(\alpha)(v_q) = \alpha(\pi(v_q)) = \alpha(q).
\]

This observation, together with the fact that any vector field from \( \mathcal{V} \) is annihilated by \( \pi \), that is \( \pi_* V = 0 \), for all \( V \in \mathcal{V} \), allows to obtain:

\[
\nabla^\Sigma_{\alpha w} v = \frac{1}{2} \pi_* \left( \left[ \text{ad}_F^2 V, \bar{a}W \right] \right) = \frac{1}{2} \pi_* \left( \bar{a} \left[ \text{ad}_F^2 V, W \right] + (L_{\text{ad}_F^2 V} \bar{a}) W \right)
\]

\[
= \alpha \left( \frac{1}{2} \pi_* \left[ \text{ad}_F^2 V, W \right] \right) = \alpha \nabla^\Sigma_w v.
\]

Moreover, we have

\[
[\text{ad}_F^2(\bar{a}V), W] = \bar{a} \left[ \text{ad}_F^2 V, W \right] + 2L_{\text{ad}_F^2 V} \bar{a} V - 2L_{\text{ad}_F^2 V} \bar{a} V
\]

and thus

\[
\nabla^\Sigma_{\alpha v} = \frac{1}{2} \pi_* \left( \left[ \text{ad}_F^2 V, W \right] \right)
\]

\[
= \frac{1}{2} \pi_* \left( \bar{a} \left[ \text{ad}_F^2 V, W \right] + 2(L_{\text{ad}_F^2 W} \bar{a}) \text{ad}_F V \mod \text{span} \mathcal{V} \right)
\]

\[= \alpha \nabla^\Sigma_w v + \pi_*(L_{\text{ad}_F W} \bar{a} \text{ad}_F V) = \alpha \nabla^\Sigma_w v + (L_w \alpha) v.
\]
As we have just seen, the conditions (MS0)-(MS2) encode enough information that allows to define canonically the configuration manifold $Q^\Sigma$, the control vector fields $g_1^\Sigma, \ldots, g_m^\Sigma$ and to define the affine connection $\nabla^\Sigma$. To completely define a mechanical structure, we still have to identify the uncontrolled vector field $g_0$ and the fibers-linear map $d : TQ^\Sigma \to TQ^\Sigma$.

Recall that by $\hat{\mathcal{V}}$ we denote the module of vector fields generated by $\mathcal{V}$ over the ring of smooth functions $\hat{\alpha}$ on $M$ such that $L_V \hat{\alpha} = 0$, for all $V \in \mathcal{V}$. Note that there exists a unique vector field $G_0 \in \hat{\mathcal{V}}$, such that $G_0$ and $F$ coincide on $Q^\Sigma$ (clearly, $F$ is not tangent to $Q^\Sigma$). In $(x,y)$-coordinates, $G_0$ is of the form

$$G_0 = g_0^\Sigma(x) \frac{\partial}{\partial y^i}.$$  

We have $[\text{ad}_F G_0, \mathcal{V}] \subset \mathcal{V}$, which allows to consider the vector field $g_0^\Sigma$ defined on $Q^\Sigma$ as

$$g_0^\Sigma = -\pi_\ast(\text{ad}_F G_0).$$

Take $n$ vector fields $V_1, \ldots, V_n \in \mathcal{V}$, defined on $\mathcal{O}_{x_0}$, and the corresponding projected vector fields $v_i = -\pi_\ast(\text{ad}_F V_i)$, $1 \leq i \leq n$, defined on $Q^\Sigma$. To define the map $d^\Sigma : TQ^\Sigma \to TQ^\Sigma$, we will evaluate $d^\Sigma(q)$, with $q \in Q^\Sigma$, on $n$ independent vectors $v_1(q), \ldots, v_n(q) \in T_q Q^\Sigma$. Recall that, for any $q \in Q^\Sigma$, we have $T_q M = T_q Q^\Sigma \oplus V(q)$ (see (14)). So, for any $q \in Q^\Sigma \subset M$, we decompose $ad_F V(q) = v_1(q) + v_2(q)$, where $v_1(q) \in T_q Q^\Sigma, v_2(q) \in V(q)$, and we define the value of the map $d^\Sigma(q)$ on $v(q) \in T_q Q^\Sigma$ by

$$d^\Sigma(q)v(q) = v_2(q),$$

where $v_2(q) \in V(q)$ is interpreted as an element of $T_q Q$ due to the identification of $\hat{\mathcal{V}}$ with $\mathfrak{X}(Q)$.

In the remaining part of the section, we will express, in terms of the vector fields $F$ and $G_1, \ldots, G_m$, the symmetric product and then we will calculate the torsion and the curvature of the canonical connection $\nabla^\Sigma$.

Recall the definition of the symmetric product (see Section 2.2) and let $\mathfrak{g} = \{g_1, \ldots, g_m\}$ be the collection of input vector fields for system (7). We consider the following sequence of families of vector fields on $Q$:

\begin{align*}
\text{Sym}^1(\mathfrak{g}) &= \{g_r \mid 1 \leq r \leq m\}, \\
\vdots \\
\text{Sym}^i(\mathfrak{g}) &= \{\langle X : Y \rangle \mid X \in \text{Sym}^p(\mathfrak{g}), Y \in \text{Sym}^l(\mathfrak{g}), p + l = i\}, \quad \text{and} \\
\text{Sym}(\mathfrak{g}) &= \bigcup_{i=1}^\infty \text{Sym}^i(\mathfrak{g}). \quad (16)
\end{align*}

Note that the smallest distribution on $Q$ containing the vector fields $g_1, \ldots, g_m$ and closed under the symmetric product is the distribution spanned by $\text{Sym}(\mathfrak{g})$, that is, we have $\text{Sym}^\Sigma(M)(\mathfrak{g}) = \text{span} \text{Sym}(\mathfrak{g})$. For an alternative but equivalent definition of $\text{Sym}(\mathfrak{g})$ see [10].
The following formula generalises the analogous one proved in [33] for the case of $F$ being a geodesic spray (see also [10]).

We recall the notation $k^{\text{vlift}}$ for the vertical lift of a vector field $k$ on $Q$ (see (6)).

**Lemma 4.2.** Let $X$ and $Y$ be arbitrary vector fields on the configuration manifold $Q$ and

$$F = y^i \frac{\partial}{\partial x^i} + (-\Gamma^i_{jk}(x)y^j y^k + d^i_j(x)y^j + g^i_0(x)) \frac{\partial}{\partial y^i}.$$ 

Then

$$\langle X : Y \rangle^{\text{vlift}} = [Y^{\text{vlift}}, \text{ad}_F X^{\text{vlift}}].$$

**Proof.** Given $X$ and $Y$, vector fields on $Q$, written in local coordinates as

$$X = X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = Y^i \frac{\partial}{\partial x^i},$$

where $X^i = X^i(x)$ and $Y^i = Y^i(x)$, their respective vertical lifts $X^{\text{vlift}}$ and $Y^{\text{vlift}}$ are expressed in local coordinates $(x, y)$ of $TQ$, by

$$X^{\text{vlift}} = X^i \frac{\partial}{\partial y^i} \quad \text{and} \quad Y^{\text{vlift}} = Y^i \frac{\partial}{\partial y^i}.$$ 

We calculate

$$\text{ad}_F X^{\text{vlift}} = -X^i \frac{\partial}{\partial x^i} + \left( \frac{\partial X^i}{\partial x^j} y^j + (\Gamma^i_{jk} y^k + \Gamma^i_{kj} y^k - d^i_j(x)) \right) \frac{\partial}{\partial y^i},$$

and (since $d^i_j = d^i_j(x)$)

$$[Y^{\text{vlift}}, \text{ad}_F X^{\text{vlift}}] = \left( \frac{\partial X^i}{\partial x^j} Y^j + \frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^j Y^k + \Gamma^i_{kj} X^k Y^j \right) \frac{\partial}{\partial y^i}.$$ 

The expression at the right-hand side of the last equality corresponds to the coordinate representation of $\langle X : Y \rangle^{\text{vlift}}$ (see (5) and (6)).

**Lemma 4.3.** If $\Phi : U \to TQ$, $U \subset M$, is a local diffeomorphism such that $\Phi$ transforms locally the system $\Sigma$ into $(\mathcal{M}S)$, then

$$v \in \text{Sym}^i(g) \iff V \in \Phi_*(V_i), \quad i \geq 1,$$

with $V = v^{\text{vlift}}$, the vertical lift of $v$.

**Proof.** We proceed by induction on $i$. The result is trivially true for $i = 1$. Now we suppose that the lemma is true for $1 \leq j \leq i - 1$ and $i \geq 2$.

By definition of $\text{Sym}^i(g)$, for any $v \in \text{Sym}^i(g)$ there exist vector fields $v_1 \in \text{Sym}^p(g)$ and $v_2 \in \text{Sym}^l(g)$ such that

$$v = \langle v_1 : v_2 \rangle \quad \text{and} \quad p + l = i, \quad 1 \leq p, l \leq i - 1.$$ 

The induction assumption yields

$$\langle v_1 \rangle^{\text{vlift}} = V_1 \in \Phi_*(V_p) \quad \text{and} \quad \langle v_2 \rangle^{\text{vlift}} = V_2 \in \Phi_*(V_l).$$

Let $\tilde{F} = \Phi_*F$ be the drift of the system $(\mathcal{M}S)$. Then,

$$[V_1, \text{ad}_{\tilde{F}} V_2] \in [\Phi_*(V_p), \text{ad}_{\tilde{F}} \Phi_*(V_l)] = \Phi_* [V_p, \text{ad}_F V_l] \subseteq \Phi_*(V_i),$$

by construction of $V_i$. On the other hand, lemma 4.2 yields

$$[V_1, \text{ad}_{\tilde{F}} V_2] = \langle v_1 : v_2 \rangle^{\text{vlift}}.$$
proving that given $v \in \text{Sym}^1(g)$, we have $V = v^{\text{viff}} = \langle v_1 : v_2 \rangle^{\text{viff}}$ in $\Phi_*(V_i)$ and, conversely, any $V \in \Phi_*(V_i)$ is of the form $V = v^{\text{viff}}$ for some $v \in \text{Sym}^1(g)$. \hfill $\square$

**Proposition 4.4.** The symmetric product $\langle v_i : v_j \rangle = \nabla^\Sigma_{v_i} v_j + \nabla^\Sigma_{v_j} v_i$ is given by

$$\langle v_i : v_j \rangle = -\pi_* ([F, [V_i, \text{ad}_F V_j]]) = -\pi_* ([F, [V_j, \text{ad}_F V_i]]) .$$

**Proof.** We have

$$\langle v_i : v_j \rangle = \nabla^\Sigma_{v_i} v_j + \nabla^\Sigma_{v_j} v_i = \frac{1}{2} \pi_* ([\text{ad}_F^2 V_j, V_i]) + \frac{1}{2} \pi_* ([\text{ad}_F^2 V_i, V_j]).$$

Since

$$[\text{ad}_F^2 V_i, V_j] = [F, [\text{ad}_F V_i, V_j]] - [\text{ad}_F V_i, [\text{ad}_F V_j, V_i]],$$

$$[\text{ad}_F^2 V_j, V_i] = [F, [\text{ad}_F V_j, V_i]] - [\text{ad}_F V_j, [\text{ad}_F V_i, V_j]],$$

summing up we get

$$[\text{ad}_F^2 V_i, V_j] + [\text{ad}_F^2 V_j, V_i] = [F, [\text{ad}_F V_i, V_j]] + [F, [\text{ad}_F V_j, V_i]].$$

But $[\text{ad}_F V_i, V_j] = [\text{ad}_F V_j, V_i]$, since $[V_i, V_j] = 0$ and hence

$$0 = [F, [V_i, V_j]] = [\text{ad}_F V_i, V_j] + [V_i, \text{ad}_F V_j].$$

\hfill $\square$

**Corollary 4.5.** A mechanical system (more generally, a control system that is $S$-equivalent to a mechanical system) is geodesically accessible around a zero-velocity point if and only if it satisfies the conditions (MS0) and (MS1) of Theorem 3.2.

**Proof.** Consider a geodesically accessible mechanical system around a zero-velocity point. Then there exist $n$ independent vector fields $v_1, \ldots, v_n$ in $\text{Sym}(g_1, \ldots, g_m)$, and we have

$$v_i = -\pi_* (\text{ad}_F V_i), \quad 1 \leq i \leq n,$$

for some vector fields $V_i \in \mathring{\mathcal{V}}$, because of Lemma 4.3 and Proposition 4.4. It follows that $\text{ad}_F V_i, 1 \leq i \leq n$, are independent mod span $\mathcal{V}$. Clearly, we have $F(z_0) \in \mathcal{V}(z_0)$, for $z_0 = (x_0, 0) \in TQ$, that is (MS0) holds. Moreover, this condition implies that $V_1, \ldots, V_n$ are independent, which yields (MS1). Indeed, $F(z_0) \in \mathcal{V}(z_0)$, implies that we can represent

$$F = F_1 + F_2, \quad \text{where} \quad F_1(z_0) = 0 \quad \text{and} \quad F_2 \in \mathcal{V}.$$

Consider

$$\sum_{i=1}^n a_i V_i(z_0) + \sum_{i=1}^n b_i \text{ad}_F V_i(z_0) = 0, \quad a_i, b_i \in \mathbb{R}. $$

Since the vector fields $\text{ad}_F V_i, \quad 1 \leq i \leq n$, are independent mod span $\mathcal{V}$, it follows $b_i = 0$ and hence

$$\sum_{i=1}^n a_i V_i(z_0) = 0, \quad a_i \in \mathbb{R}. $$
We have
\[ [F, \sum_{i=1}^{n} a_i V_i](z_0) = [F_1 + F_2, \sum_{i=1}^{n} a_i V_i](z_0) \]
\[ = [F_1, \sum_{i=1}^{n} a_i V_i](z_0) + \sum_{i=1}^{n} a_i [F_2, V_i](z_0) = 0, \]
where the first bracket vanishes since \( F_1(z_0) = 0 \) and \( \sum_{i=1}^{n} a_i V_i(z_0) = 0 \), while the second bracket vanishes because of condition \([\mathcal{V}, \mathcal{V}] = 0\) (which is always necessary for S-equivalence to a mechanical system \((\mathcal{MS})\)). It follows that
\[ \sum_{i=1}^{n} a_i \text{ad}_F V_i(z_0) = 0, \]
implying that \( a_i = 0 \).

Conversely, if a mechanical system satisfies \((\text{MS}1)\), then there exist \( n \) independent vector fields \( V_1, \ldots, V_n \in \mathcal{V} \). Put \( v_i = -\pi_*(\text{ad}_F V_i), \ 1 \leq i \leq n \). By Lemma 4.3 and Proposition 4.4, \( v_1, \ldots, v_n \in \text{Sym}(g_1, \ldots, g_m) \) and they are independent, thus proving geodesic accessibility, which holds around a zero-velocity point because of condition \((\text{MS}0)\).

Recall the definitions of torsion and curvature tensors of a connection \( \nabla \) given in Subsection 2.2.

**Proposition 4.6.**
(i) The torsion of the connection \( \nabla^\Sigma \) is
\[ T(v_i, v_j) = \nabla^\Sigma_{v_i} v_j - \nabla^\Sigma_{v_j} v_i - [v_i, v_j] = 0, \]
that is, the connection \( \nabla^\Sigma \) is symmetric.
(ii) The curvature of the connection is
\[ R(v_i, v_j) v_k = \nabla^\Sigma_{v_i} \left( \nabla^\Sigma_{v_j} v_k \right) - \nabla^\Sigma_{v_j} \left( \nabla^\Sigma_{v_i} v_k \right) - \nabla^\Sigma_{[v_i, v_j]} v_k \]
\[ = \frac{1}{4} \pi_* \left( \text{ad}_F^2 \left( [\text{ad}_F^2 V_k, V_j] \right), V_i \right) - \left[ \text{ad}_F \left( [\text{ad}_F^2 V_k, V_i] \right), V_j \right] - \frac{1}{2} \pi_* \left( \left[ \text{ad}_F V_k, \text{ad}_F V_i, \text{ad}_F V_j \right] \right). \]

**Proof.** We have
\[ \nabla^\Sigma_{v_i} v_j - \nabla^\Sigma_{v_j} v_i = \frac{1}{2} \pi_* \left( \left[ \text{ad}_F^2 V_j, V_i \right] - \left[ \text{ad}_F^2 V_i, V_j \right] \right) = \frac{1}{2} \pi_* (2 \left[ \text{ad}_F V_i, \text{ad}_F V_j \right]) \]
\[ = \left[ \pi_* (\text{ad}_F V_i), \pi_* (\text{ad}_F V_j) \right] = [v_i, v_j], \]
and so, the connection has zero torsion. We get the expression for the curvature by simple application of equality (15). \( \square \)

5. \((\mathcal{MS})\)-structures: uniqueness and characterisation. In Section 3 we have found necessary and sufficient conditions for the system \( \Sigma \) to admit a \((\mathcal{GAMMS})\)-structure around a zero-velocity point; in Section 4 we have shown how to construct canonically a \((\mathcal{GAMMS})\)-structure for systems admitting one. Now, in this section, we are concerned with the uniqueness of a \((\mathcal{MS})\)-structure in general and the uniqueness of a \((\mathcal{GAMMS})\)-structure in particular. We start with a definition.

Let \((\mathcal{MS})\) and \((\mathcal{MS})\) be two mechanical systems. We say that they are mechanically state-equivalent, shortly, \(\text{MS-equivalent}\) (respectively, locally mechanical
state equivalent at points \((x_0, y_0)\) and \((\tilde{x}_0, \tilde{y}_0)\) if they are S-equivalent (respectively, locally S-equivalent at points \((x_0, y_0)\) and \((\tilde{x}_0, \tilde{y}_0)\)) under an extended point transformation \(\Phi = (\phi_1, \phi_2)\):
\[
\tilde{x} = \phi_1(x) \quad \text{and} \quad \tilde{y} = \phi_2(x, y) = D\phi_1(x)y,
\]
where \((x, y)\) and \((\tilde{x}, \tilde{y})\) are local coordinates, respectively, of \((\mathcal{MS})\) and \((\tilde{\mathcal{MS}})\).

5.1. **Uniqueness of the \((\mathcal{GAMS})\) structure.** If we are given a system \(\Sigma\) that is already a \((\mathcal{GAMS}) = (Q, \nabla, g_0, d)\), around a zero-velocity point, then it is clear that it satisfies conditions (MS0)-(MS2), since they are necessary for S-equivalence to a \((\mathcal{GAMS})\). But this means, in view of the considerations of the last section, that we can determine canonically the 4-tuple defining the structure of the system, that is, to construct \(Q^\Sigma, \nabla^\Sigma, g_0^\Sigma\) and \(d^\Sigma\). A natural question is to ask what is the relation between the original structure \((Q, \nabla, g_0, d)\) and the canonical structure \((Q^\Sigma, \nabla^\Sigma, g_0^\Sigma, d^\Sigma)\). It turns out that both structures coincide. More precisely, we have the following result.

**Theorem 5.1.** (i) For any system \(\Sigma\) that is already a \((\mathcal{GAMS}) = (Q, \nabla, g_0, d)\), the original mechanical structure coincides with the canonical one, that is,
\[
(Q, \nabla, g_0, d) = (Q^\Sigma, \nabla^\Sigma, g_0^\Sigma, d^\Sigma).
\]

(ii) If a control system \(\Sigma\) admits a \((\mathcal{GAMS})\)-structure, then it is unique (among \((\mathcal{GAMS})\)-structures) up to an extended point transformation. More precisely, if \(\Sigma\) is S-equivalent to two mechanical structures \((\mathcal{GAMS})_1\) and \((\mathcal{GAMS})_2\), then they are MS-equivalent.

**Proof.** (i) Consider a \((\mathcal{GAMS})\) given by (8) (or, equivalently, by (9)). We have \(\text{span } \mathcal{V} = \text{span} \{\frac{\partial}{\partial v^r}, \ldots, \frac{\partial}{\partial v^s}\}\) (by geodesic accessibility) and hence \(Q^\Sigma = \{y = 0\}\) which coincides with \(Q\).

Now, we will show that \(\nabla = \nabla^\Sigma\). The drift \(F\) of the \((\mathcal{GAMS})\) is
\[
F = y^i \frac{\partial}{\partial x^i} + (-\Gamma^i_{jk}(x)y^j y^k + d^i_j(x)y^j + g_0^i(x)) \frac{\partial}{\partial y^i},
\]
where \(\Gamma^i_{jk}\) are the Christoffel symbols of the connection \(\nabla\), defining the \((\mathcal{GAMS})\).

Let \(V_i\) and \(V_j\) be arbitrary vector fields in \(\mathcal{V}\):
\[
V_i = v_i^r \frac{\partial}{\partial y^r} \quad \text{and} \quad V_j = v_j^s \frac{\partial}{\partial y^s}.
\]

It is straightforward to get
\[
[V_j, \text{ad}_F V_i] = \left(2v_j^r v_i^s \Gamma^k_{rs} + v_j^r \frac{\partial v_i^k}{\partial x^r} + v_i^s \frac{\partial v_j^k}{\partial x^s}\right) \frac{\partial}{\partial y^k}.
\]

We observe that the vector field \([V_j, \text{ad}_F V_i]\) defined by (20) gives rise to the vector field on \(Q:\)
\[
- \pi_* ([F, [V_j, \text{ad}_F V_i]]) = \left(2v_j^r v_i^s \Gamma^k_{rs}(x) + v_j^r \frac{\partial v_i^k}{\partial x^r} + v_i^s \frac{\partial v_j^k}{\partial x^s}\right) \frac{\partial}{\partial x^k}.
\]

We have
\[
- \pi_* ([F, [V_j, \text{ad}_F V_i]]) = - \pi_* ([\text{ad}_F V_j, \text{ad}_F V_i] + [V_j, \text{ad}_F^2 V_i]),
\]
and

\[
\pi_* ([\text{ad}_F V_j, \text{ad}_F V_i]) = [\pi_* (\text{ad}_F V_j), \pi_* (\text{ad}_F V_i)] = [v_j, v_i]
\]

\[
= \left( v_j \frac{\partial v_k}{\partial x^r} - v_i \frac{\partial v_k}{\partial x^s} \right) \frac{\partial}{\partial x^k}.
\]

We obtain

\[
- \frac{1}{2} \pi_* ([F, [V_j, \text{ad}_F V_i]]) + \frac{1}{2} \pi_* ([\text{ad}_F V_j, \text{ad}_F V_i]) = \frac{1}{2} \pi_* ([\text{ad}_F^2 V_i, V_j]) = \nabla^\Sigma_{v_j} v_i,
\]

from which we conclude:

\[
\nabla^\Sigma_{v_j} v_i = \left( v_j v_i^k \Gamma^k_{rs}(x) + v_j \frac{\partial v_i}{\partial x^k} \right) \frac{\partial}{\partial x^k},
\]

which coincides with the formula (3). This shows that the functions \( \Gamma^k_{rs}(x) \), with \( 1 \leq k, r, s \leq n \), are also the Christoffel symbols for the affine connection \( \nabla^\Sigma \). It follows that \( \nabla = \nabla^\Sigma \).

Given \( \gamma : I \to Q \), \( I \subset \mathbb{R} \), any trajectory of \( \Sigma \), it satisfies both equations:

\[
\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = g_0(\gamma(t)) + d(\dot{\gamma}(t)) + \sum_{r=1}^m u_r g_r(\gamma(t)),
\]

because \( \Sigma \) is assumed to be a \( (GAMS) \), see (7), and

\[
\nabla^\Sigma_{\dot{\gamma}(t)} \dot{\gamma}(t) = g_0^\Sigma(\gamma(t)) + d^\Sigma(\dot{\gamma}(t)) + \sum_{r=1}^m u_r g_r^\Sigma(\gamma(t)),
\]

because it satisfies the conditions (MS0)-(MS2) that yield the canonical structure \( (Q^E, \nabla^\Sigma, g_0^\Sigma, d^\Sigma) \). Since \( \nabla = \nabla^\Sigma \), we get

\[
g_0(\gamma(t)) + d(\dot{\gamma}(t)) + \sum_{r=1}^m u_r g_r(\gamma(t)) = g_0^\Sigma(\gamma(t)) + d^\Sigma(\dot{\gamma}(t)) + \sum_{r=1}^m u_r g_r^\Sigma(\gamma(t)),
\]

for any control \( u(t) = (u_1(y), \ldots, u_m(t)) \) on \( I \). It immediately follows that \( g_0 = g_0^\Sigma \) and \( d = d^\Sigma \), which proves (i).

To prove (ii), let us assume that \( (GAMS)_1 \) and \( (GAMS)_2 \) are two \( (GAMS) \)-structures of the system \( \Sigma \). This means, by definition, that \( \Sigma \) is \( S \)-equivalent, via \( \phi_1 \), to \( (GAMS)_1 \) and also \( S \)-equivalent, via \( \phi_2 \), to \( (GAMS)_2 \). Now, by (i), the \( (GAMS)_i \), structure, for \( i = 1, 2 \), is the canonical structure \( (Q^E_i, \nabla^\Sigma_i, g_0^\Sigma_i, d^E_i) \), defined by the image \( \Sigma_i \) of \( \Sigma \) via \( \phi_i \). In particular, the distributions spanned by \( \nu_1 = (\phi_1)_* \nu \) and \( \nu_2 = (\phi_2)_* \nu \) are related via the diffeomorphism \( \phi = (\phi_2 \circ \phi_1^{-1}) \). Since span \( \nu_i = \text{span} \left\{ \frac{\partial}{\partial q_i}, \ldots, \frac{\partial}{\partial p_i} \right\} \), \( i = 1, 2 \), it follows that \( \phi \) is an extended point transformation, of the form (17), thus proving that \( (GAMS)_1 \) and \( (GAMS)_2 \) are \( MS \)-equivalent.

5.2. Non-uniqueness of \((\mathcal{MS}) \) structures. In the last subsection we proved that, if a control system \( \Sigma \) admits a \( (GAMS) \)-structure, then it is unique up to an extended point transformation. The geodesic accessibility assumption plays a crucial role in guaranteeing the uniqueness of the mechanical structure. Indeed, as the next example shows, if the system is not geodesically accessible, then it can admit multiple mechanical structures.
Example 5.2.1. Consider the mechanical control system

\[(\mathcal{MS})_1 : \begin{align*}
\dot{x}_1 &= y_1, \\
\dot{x}_2 &= y_2, \\
\dot{y}_1 &= u, \\
\dot{y}_2 &= x_1,
\end{align*}\]
on \(TQ = T\mathbb{R}^2\), defined by the Euclidean metric, with an uncontrolled vector field

\[g_{\text{vlift}} = (x_1 \frac{\partial}{\partial x_1})_{\text{vlift}} = x_1 \frac{\partial}{\partial y_2} \] and without d-forces. We have \(G = \frac{\partial}{\partial y_2}\) and \(F = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_2} + x_1 \frac{\partial}{\partial y_2}\). Calculate

\[\text{ad}_F G = -\frac{\partial}{\partial x_1}, \quad \text{ad}_F^2 G = \frac{\partial}{\partial y_2}, \quad \text{ad}_F^3 G = -\frac{\partial}{\partial x_2}, \quad \text{ad}_F^4 G = 0, \quad [G, \text{ad}_F G] = 0.\]

The system is accessible (even strongly accessible) since the Lie algebra \(\mathcal{L}\) generated by \(F\) and \(G\) is

\[\mathcal{L} = \text{Vect}_\mathbb{R}\{F, G, \text{ad}_F G, \text{ad}_F^2 G, \text{ad}_F^3 G\},\]
and hence \(\text{dim } \mathcal{L}(p) = 4\), for any \(p \in \mathbb{R}^4\). The system is not geodesically accessible since \(\mathcal{V} = \text{Vect}_\mathbb{R}\{G\}\), which by Lemma 4.3 implies that

\[\mathcal{S} \mathcal{Y} \mathcal{M}(\mathfrak{g}) = \text{span Sym}(\mathfrak{g}) = \text{span}\{g\},\]
in which \(g := -\pi_*(\text{ad}_F G)\).

The local diffeomorphism \(\Psi\) defined as\(^5\)

\[\begin{align*}
\hat{x}_1 &= x_1, \\
\hat{x}_2 &= x_2 + \frac{1}{2}(y_2)^2, \\
\hat{y}_1 &= y_1, \\
\hat{y}_2 &= y_2(x_1 + 1),
\end{align*}\]
transforms \((\mathcal{MS})_1\), restricted to \(TQ_1\), where \(Q_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > -1\}\), into the mechanical control system

\[(\mathcal{MS})_2 : \begin{align*}
\dot{\hat{x}}_1 &= \hat{y}_1, \\
\dot{\hat{x}}_2 &= \hat{y}_2, \\
\dot{\hat{y}}_1 &= u, \\
\dot{\hat{y}}_2 &= \hat{x}_1 \hat{x}_1 + \hat{y}_1 \hat{y}_2, \quad \hat{x}_1 = \hat{x}_1 + \hat{y}_1 \hat{y}_2 + \frac{\hat{y}_1^2}{1 + \hat{x}_1},
\end{align*}\]
on \(TQ_2\), where \(Q_2 = \{(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2 \mid \hat{x}_1 > -1\}\). This shows that \((\mathcal{MS})_1\) and \((\mathcal{MS})_2\) are \(S\)-equivalent (as control systems) around the origin \((0, 0) \in \mathbb{R}^4\).

Now we will prove that they are not \(MS\)-equivalent. To this end, assume that an extended point diffeomorphism \((\hat{x}, \hat{y}) = \Phi(x, y)\) of the form

\[\begin{align*}
\hat{x} &= \phi(x), \\
\hat{y} &= D\phi(x) y,
\end{align*}\]
where \(\phi\) is a (local) diffeomorphism of \(Q_1\) to \(Q_2\), transforms \((\mathcal{MS})_1\) into \((\mathcal{MS})_2\). We have \(\Phi_* G = \Phi_* (\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial y_2}\) implying that the components \((\phi^1, \phi^2)\) of \(\phi\) are \(\phi^1 = x_1 + a(x_2)\) and \(\phi^2 = b(x_2)\), for some smooth functions \(a\) and \(b\). Calculating

\[\Phi_* F = \Phi_* (y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2}) = \hat{y}_1 \frac{\partial}{\partial \hat{x}_1} + \hat{y}_2 \frac{\partial}{\partial \hat{x}_2} + \left(\hat{x}_1 (1 + \hat{x}_1) + \frac{\hat{y}_1^2}{1 + \hat{x}_1}\right) \frac{\partial}{\partial \hat{y}_2},\]

\(^5\)Recall that \((y_2)^2\) stands for the square of the second component of \(y\).
we conclude that
\[ y^1 + a' y^2 = \dot{y}^1, \]
\[ b' y^2 = \ddot{y}^2, \]
\[ a'' y^2 + a' x^1 = 0, \]
\[ b'' y^2 + b' x^1 = \dot{x}^1 (1 + \dot{x}^1) + \frac{\dot{y}_1^1 \ddot{y}^2}{1 + \dot{x}^1}. \]
The third equation implies \( a' = 0 \) thus giving \( \ddot{y}^1 = y^1 \). Now plugging \( \ddot{y}^1 = y^1 \) and \( \ddot{y}^2 = b' y^2 \) into the fourth equation yields a contradiction.

Therefore the above systems are, indeed, not MS-equivalent and we conclude that
\( \Sigma \) is flat (it is defined by the Euclidean metric) thus having zero curvature tensor, whereas the curvature tensor of the connection for \( \Sigma \) is bi-mechanical (and so is \( \Sigma \)). Indeed, it admits two non-MS-equivalent mechanical structures: the original structure \( \Sigma \) and the structure \( \Sigma \). Notice that for both mechanical structures, the configuration manifolds are the same: \( Q_1 = \{ y^1 = y^2 = 0 \} = \{ \dot{y}^1 = y^2 = 0 \} = Q_2 \) but they have two different bundle structures. In the coordinates \((x^1, x^2, y^1, y^2)\) of \( \Sigma \), the fibers \( T_y Q_1 \) of the first structure are defined by \( \{ x^1 = c^1, x^2 = c^2, c^1, c^2 \in \mathbb{R} \} \), while the fibers of the second structure are given by \( \{ x^1 = c^1, x^2 + \frac{1}{2} (y^2)^2 = c^2, c^1, c^2 \in \mathbb{R} \} \) and are mapped, via \( \Psi \), into \( \{ \dot{x}^1 = c^1, \dot{x}^2 = c^2, c^1, c^2 \in \mathbb{R} \} \).

The non-MS-equivalence of the systems \( \Sigma \) and \( \Sigma \) is also seen if we compute the curvature tensors of the connections defining the systems. In fact, the connection defining \( \Sigma \) is flat (it is defined by the Euclidean metric) thus having zero curvature tensor, whereas the curvature tensor of the connection for \( \Sigma \) (defined by \( \Gamma_{12}^2 = \Gamma_{21}^2 = -\frac{1}{2 (1 + x^1)} \), remaining \( \Gamma_{jk}^i \) being zero) is non-zero. Indeed, using the definition of curvature tensor in Subsection 2.2, we calculate (for simplicity, we omit the “tildas”):

\[
R \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right) \frac{\partial}{\partial x^1} = \left( \nabla_{\frac{\partial}{\partial x^1}} \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^1} - \nabla_{\frac{\partial}{\partial x^1}} \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^1} - \nabla_{ \left( \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^2} \right) \frac{\partial}{\partial x^1} } \right)
\]
\[
= \frac{\partial}{\partial x^1} (\Gamma_{12}^1 \frac{\partial}{\partial x^1} + \Gamma_{21}^1 \frac{\partial}{\partial x^2})
\]
\[
= \frac{\partial}{\partial x^1} \left( - \frac{1}{2 (1 + x^1)} \frac{\partial}{\partial x^2} \right)
\]
\[
= \frac{3}{4 (1 + x^1)^2} \frac{\partial}{\partial x^2} = -R(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}) \frac{\partial}{\partial x^1}
\]
\[
R \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right) \frac{\partial}{\partial x^k} = 0, \quad \text{for remaining cases.}
\]

5.3. A characterisation of \( \text{(MS)} \)-structures. Non-uniqueness of a mechanical structure \( \text{(MS)} \), shown in Example 5.2.1, and uniqueness of a geodesically accessible mechanical structure \( \text{(GAMS)} \), proved in Subsection 5.1, lead to two natural questions: first, what is the meaning of (MS1) and (MS2) for a control system equivalent to a (not necessarily geodesically accessible) mechanical system \( \text{(MS)} \)? and, secondly, how to characterise mechanical control systems \( \text{(MS)} \) that are not necessarily \( \text{(GAMS)} \)?

**Proposition 5.2.** Assume that a control system \( \Sigma \) defined on a smooth manifold \( M \) of dimension \( 2n \), is, locally around \( z_0 \), \( S \)-equivalent to a mechanical control system \( \text{(MS)} \). Then, for any \( z \) in a neighborhood of \( z_0 \), we have

(i) \( [\mathcal{V}, \mathcal{V}] (z) = 0 \),

(ii) \( \dim V(z) \leq n \),

(iii) The mechanical system \((\mathcal{M}S)\) is geodesically accessible around a zero-velocity point if and only if it satisfies \((\mathcal{M}S0)\) and \((\mathcal{M}S1)\).

The proof of the above proposition is actually included in the proof of Theorem 3.2 (see Section 7) and will be omitted. Item (i) implies that \([V, V] = 0\) is a structural condition and it is always necessary for the existence of a mechanical structure. If in the abelian Lie algebra \(V\) there are independent vector fields \(V_1, \ldots, V_n\) that, together with \(\text{ad}_F V_i\), are linearly independent everywhere, then (accordingly to Theorem 3.2), the system \(\Sigma\) admits a mechanical structure which is actually a \((\mathcal{GAMS})\) and, as such, unique. In general, however, we may not be able to find \(n\) independent vector fields in \(V\) (see (ii)). In this case, the following result, which describes (not necessarily geodesically accessible) mechanical control systems \((\mathcal{M}S)\), still holds.

**Proposition 5.3.** A control system \(\Sigma\), defined on a \(2n\)-dimensional smooth manifold \(M\), is locally at \(z_0 \in M\), S-equivalent to a mechanical control system \((\mathcal{M}S)\) if and only if there exists a Lie algebra \(W\) of vector fields on \(M\) satisfying \(V \subset W\) and \([W, \text{ad}_F W] \subset W\) and such that

\[
(\text{MS1})' \quad \dim W(z) = n \quad \text{and} \quad \dim (W + [F, W])(z) = 2n,
\]

\[
(\text{MS2})' \quad [W, W](z) = 0,
\]

for any \(z\) in a neighborhood of \(z_0\).

**Proof.** The proof of the sufficiency part follows the same line as that of Theorem 3.2 (see Section 7) and will be omitted. To prove necessity, we start by observing that any \(V \in V\) is of the form

\[V = v^i(x) \frac{\partial}{\partial y^i},\]

with, as usual, a sum is understood over the index \(i\), \(1 \leq i \leq n\), and, by Proposition 5.2, we have \(\dim V(z_0) = k \leq n\). Let us choose \(k\) vector fields \(V_1, \ldots, V_k\) in \(V\) which are independent at \(z_0\) and denote \(W_i = V_i\), for \(1 \leq i \leq k\). We can complete them by \(W_{k+1} = \frac{\partial}{\partial x^1}, \ldots, W_n = \frac{\partial}{\partial x^n}\) (after permuting, if necessary, the coordinates \(x^1, \ldots, x^n, y^1, \ldots, y^n\), accordingly) such that in a neighborhood of \(z_0\) we have

\[\text{span}\{W_1, \ldots, W_n\} = \text{span}\left\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\}.\]

It is clear that \(V \subset W\) and by direct calculations we check that \([W, \text{ad}_F W] \subset W\) and \((\text{MS1})'\) and \((\text{MS2})'\) hold. \(\Box\)

A control system admits thus a mechanical structure \((\mathcal{M}S)\) if and only if the abelian Lie algebra \(V\) can be completed to an abelian Lie algebra \(W\) satisfying \((\text{MS1})\) and \([W, \text{ad}_F W] \subset W\). In general, there can be various ways to perform that completion. To illustrate this, observe that for the system \((\mathcal{M}S)_1\) of Example 5.2.1, we have \(V = \text{Vect}\{\frac{\partial}{\partial y}\}\) which can be completed to \(W_1 = \text{Vect}\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\}\) defining the structure \((\mathcal{M}S)_1\), but also to \(W_2 = \text{Vect}\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial x_1}, \frac{1}{x_1^2}(-y^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y})\}\) which defines (when passing to \((\bar{x}, \bar{y})\)-coordinates) the structure \((\mathcal{M}S)_2\).

6. **When is a \((\mathcal{GAMS})\) of special form?** Theorem 3.2 describes all \((\mathcal{GAMS})\)'s, that is, all control systems that admit a geodesically accessible mechanical structure.
In this section, we analyse systems that exhibit particular forms of the mechanical structure.

We shall use the notations and objects introduced in Section 4. In particular, recall that $Q^\Sigma = \{ z \in M \mid F(z) \in \mathcal{V}(z) \}$ stands for the configuration manifold of the control-affine system $\Sigma$.

**Proposition 6.1.** The system $\Sigma$ is locally, at $z_0 \in M$, $S$-equivalent to a:

(i) ($GAMS$) without $d$-forces, i.e. with $d = 0$, around a zero-velocity point, if and only if,

- (MS0) $F(z_0) \in \mathcal{V}(z_0)$,
- (MS1) $\dim \mathcal{V}(z) = n$ and $\dim (\mathcal{V} + [F, \mathcal{V}]) (z) = 2n$,
- (MS2) $[\mathcal{V}, \mathcal{V}] (z) = 0$,
- (MSND) $\text{ad}_F \mathcal{V}(z) \subset T_z Q^\Sigma$, $z \in Q^\Sigma$, for any $z$ in a neighborhood of $z_0$.

(ii) ($GAMS$) without uncontrolled forces, i.e. with $g_0 = 0$, around an equilibrium point, if and only if,

- (MS0) $F(z_0) = 0$,
- (MS1) $\dim \mathcal{V}(z) = n$ and $\dim (\mathcal{V} + [F, \mathcal{V}]) (z) = 2n$,
- (MS2) $[\mathcal{V}, \mathcal{V}] (z) = 0$,
- (MSNU) $F|_{Q^\Sigma} = 0$,
- for any $z$ in a neighborhood of $z_0$.

(iii) ($GAMS$) with neither $d$-forces nor uncontrolled forces (i.e., $d = 0$ and $g_0 = 0$), around an equilibrium point, if and only if,

- (MS0) $F(z_0) = 0$,
- (MS1) $\dim \mathcal{V}(z) = n$ and $\dim (\mathcal{V} + [F, \mathcal{V}]) (z) = 2n$,
- (MS2) $[\mathcal{V}, \mathcal{V}] (z) = 0$,
- (MSND) $\text{ad}_F \mathcal{V}(z) \subset T_z Q^\Sigma$, $z \in Q^\Sigma$,
- (MSNU) $F|_{Q^\Sigma} = 0$,
- for any $z$ in a neighborhood of $z_0$.

**Remark 6.1.** Notice that the conditions (MSND) and (MSNU) are in a perfect agreement with the invariant description of the map $d$ and that of the vector field $g_0$ given in Section 4. We also observe that for the cases (ii) and (iii) a zero-velocity point is necessarily an equilibrium point (see the remark before Theorem 3.2), which explains the different appearance of condition (MS0) in those cases.

**Remark 6.2.** As mentioned in Section 1, mechanical control systems for which $d = 0$ and $g_0 = 0$ are known in the literature as affine connection control systems and have been the subject of intensive research in recent years, see for instance, [10, 30, 31, 32, 34]. The case (iii) of Proposition 6.1 gives a characterisation of those affine connection control systems that are geodesically accessible. They are studied in detail in [49].

**Proof of Proposition 6.1.** Theorem 3.2 guarantees that (MS0), (MS1) and (MS2) are necessary and sufficient conditions for the local $S$-equivalence between $\Sigma$, around a point $z_0 \in M$, and a ($GAMS$) around a zero-velocity point. Let $\Phi$ be a local diffeomorphism transforming $\Sigma$ into a ($GAMS$), with $\Phi(z) = (x, y)$ such that $\Phi(z_0) = (x_0, 0)$, and let $(x, 0) = q$ be a point in the configuration manifold $Q = \{ y = 0 \}$ of
the obtained \((GAM\alpha S)\). Then, we have

\[ \tilde{F} = \Phi_\ast F = y^i \frac{\partial}{\partial x^i} + (-\Gamma^i_{jk}(x)y^j y^k + d_i^j(x)y^j + g_i^0(x)) \frac{\partial}{\partial y^i}. \]

We recall that an arbitrary element \(\tilde{V} = \Phi_\ast V\) of \(\tilde{V} = \Phi_\ast V\), has the form

\[ \tilde{V} = \tilde{v}^i(x) \frac{\partial}{\partial y^i}. \]

It follows,

\[ \text{ad}_F \tilde{V} = -\tilde{v}^i \frac{\partial}{\partial x^i} + \left( \frac{\partial \tilde{v}^i}{\partial x^j} y^j + 2\Gamma^i_{jk} y^j \tilde{v}^k - d_i^j \tilde{v}^j \right) \frac{\partial}{\partial y^i}. \]

At \(q = (x,0) \in Q\), we get

\[ \text{ad}_F \tilde{V}(q) = -\tilde{v}^i \frac{\partial}{\partial x^i} + (-d_i^j \tilde{v}^j) \frac{\partial}{\partial y^i}. \]

Since \(\text{span} \tilde{V} = \text{span} \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^4} \right\}\) and \(T_q Q = \text{Vect} \left\{ \frac{\partial}{\partial x^1}(q), \ldots, \frac{\partial}{\partial x^4}(q) \right\}\) it follows that \(\text{ad}_F \tilde{V}(z) \in T_z Q^\Sigma\), where \(\Phi(z) = (x,0) = q\) (that is, \(z \in Q^\Sigma\)) if and only if \(d_i^j = 0\), for all \(1 \leq i, j \leq n\), which proves (i). Item (ii) follows immediately from the form of the transformed drift vector field \(\tilde{F}\) and item (iii) results from (i) and (ii). \(\square\)

**Example 6.0.1** (Example 3.2.1 cont’d). It was shown in Example 3.2.1 that the control-affine system in \(M = \mathbb{R}^4\)

\[
\begin{align*}
\dot{x}^1 &= y^1, & \dot{y}^1 &= -y^2(x^1 + y^2), \\
\dot{x}^2 &= y^2, & \dot{y}^2 &= x^1 + u,
\end{align*}
\]

satisfies the conditions (MS1) and (MS2) for any \(z \in \mathbb{R}^4\), thus being S-equivalent to a \((GAM\alpha S)\) around any \(z_0\) satisfying (MS0), that is, \(z_0 = (x_0^1, x_0^2, 0, 0)\). Now we compute

\[
Q^\Sigma = \{ z = (x_1, x_2, y_1, y_2) \in M \mid F(z) \in V(z) \} = \{ y_1 = y_2 = 0 \},
\]

\[
T_{z_0}Q^\Sigma = \text{Vect} \left\{ \frac{\partial}{\partial x^1}(z_0), \frac{\partial}{\partial x^2}(z_0) \right\}.
\]

We have

\[
\text{ad}_F \tilde{V}(z_0) = \text{Vect} \{ \text{ad}_F G(z_0), [F, [G, \text{ad}_F G]](z_0) \}
\]

\[
= \text{Vect} \left\{ -\frac{\partial}{\partial x^2}(z_0), -\left( \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^2} \right)(z_0) \right\}, \quad \text{and}
\]

\[
F|_{Q^\Sigma} = x^1 \frac{\partial}{\partial y^2},
\]

thus showing that the conditions (MSND) and (MSNU) of Proposition 6.1 are not satisfied by this system. This is in accordance with the fact, mentioned in Example 3.2.1, that the above system is S-equivalent around any zero-velocity point to the system of Example 3.1.1, which is a \((GAM\alpha S)\) with d-forces and an uncontrolled vector field.

**Example 6.0.2** (Planar rigid body cont’d). As shown in Example 3.2.2, the planar rigid body satisfies the conditions (MS1) and (MS2) of Proposition 6.1 around any
point $z_0 = (\theta_0, x_0, y_0, \dot{\theta}_0, \dot{x}_0, \dot{y}_0)$ satisfying (MS0). The latter condition implies that $z_0 = (\theta_0, x_0, y_0, 0, 0, 0)$. We compute

$$Q^\Sigma = \{ z = (\theta, x, y, \dot{\theta}, \dot{x}, \dot{y}) \mid F(z) \in \mathcal{V}(z) \} = \{ \dot{\theta} = \dot{x} = \dot{y} = 0 \},$$

$$T_{z_0} Q^\Sigma = \text{Vect}_\mathbb{R} \left\{ \frac{\partial}{\partial \theta}(z_0), \frac{\partial}{\partial x}(z_0), \frac{\partial}{\partial y}(z_0) \right\}.$$  

We have

$$\text{ad}_F \mathcal{V}(z_0) = \text{Vect}_\mathbb{R} \{ \text{ad}_F G_1, \text{ad}_F G_2, [F, [G_1, \text{ad}_F G_2]] \}(z_0)$$

$$= \text{Vect}_\mathbb{R} \left\{ -\frac{\cos \theta_0}{m} \frac{\partial}{\partial x}(z_0) - \frac{\sin \theta_0}{m} \frac{\partial}{\partial y}(z_0), \right.$$  

$$\left. \frac{h}{J} \frac{\partial}{\partial \theta}(z_0) + \frac{\sin \theta_0}{m} \frac{\partial}{\partial x}(z_0) - \frac{\cos \theta_0}{m} \frac{\partial}{\partial y}(z_0), \right.$$  

$$\left. \frac{h}{J m} \left( -\sin \theta_0 \frac{\partial}{\partial x}(z_0) + \cos \theta_0 \frac{\partial}{\partial y}(z_0) \right) \right\} \subset T_{z_0} Q^\Sigma,$$

and $F|_{Q^\Sigma} = 0$. Hence all conditions of Proposition 6.1, item (iii), are satisfied and the planar rigid body is, indeed, a $(G\mathcal{A}MS)$ with neither d-forces ($d = 0$) nor uncontrolled ones ($g_0 = 0$).

7. Proof of the main result.

Proof. (Proof of Theorem 3.2) To prove necessity, let $\Phi : \mathcal{O}_{z_0} \to TQ$, $\mathcal{O}_{z_0} \subset M$, be a local diffeomorphism transforming the system $\Sigma$, restricted to a neighborhood $\mathcal{O}_{z_0}$ of $z_0$, into a $(G\mathcal{A}MS)$, with $\Phi(z_0) = \tilde{z}_0 = (x_0, 0)$. We observe that in coordinates $\Phi(z) = (x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n) \in TQ$, we have for the mechanical system $\tilde{F} = \Phi_*, F$ and $\tilde{G}_r = \Phi_*, G_r$ as given by equalities (10) and (11), respectively.

We construct $\tilde{\mathcal{V}}$ accordingly to (12), with $G_r$ and $F$ being replaced by $\tilde{G}_r$ and $\tilde{F}$, respectively. Clearly, $\tilde{\mathcal{V}} = \Phi_* \mathcal{V}$. Direct Lie-bracket computations show that any element $\tilde{V}$ of $\tilde{\mathcal{V}}$ is of the form

$$\tilde{V} = \tilde{v}^i(x) \frac{\partial}{\partial y^i}; \quad (22)$$

with, as usual, a sum understood over the index $i$, $1 \leq i \leq n$. Therefore, $[\tilde{V}, \tilde{V}] = 0$ and, clearly,

$$[\mathcal{V}, \mathcal{V}] = \left[ \Phi_*^{-1} \tilde{\mathcal{V}}, \Phi_*^{-1} \tilde{\mathcal{V}} \right] = \Phi_*^{-1} \left[ \tilde{\mathcal{V}}, \tilde{\mathcal{V}} \right] = 0. \quad (23)$$

Also, by (22), $\dim \tilde{\mathcal{V}}(x, y) = \dim \mathcal{V}(x) \leq n$, for any $z \in \mathcal{O}_{z_0}$.

Since $\dim M = \dim TQ = 2n$ and the system $(MS)$ is geodesically accessible around $x_0 \in Q$, we have (by definition)

$$\dim \mathcal{S}M(g)(x) = n, \quad x \in \mathcal{O}_{x_0} \subset Q,$$

with $\mathcal{O}_{x_0}$ an open neighborhood of $x_0$. Therefore, there exist $n$ independent vector fields $\tilde{v}_1, \ldots, \tilde{v}_n$ in $\text{Sym}(g)$. Then, the vector fields on $TQ$ given by

$$\tilde{V}_j = (\tilde{v}_j)^{\text{vir}}, \quad 1 \leq j \leq n,$$

are independent and, by Lemma 4.3, of the form $\tilde{V}_j = \Phi_* V_j$, where each $V_j \in \mathcal{V}_i$, for some $i$. This proves that $\dim \mathcal{V}(z) \geq n$. But, again by Lemma 4.3, all elements of $\mathcal{V}$ are of that form, thus proving that, indeed, $\dim \mathcal{V}(z) = n$, for all $z \in \mathcal{O}_{z_0}$. 


Since \( \tilde{F} \) is of the form (10),
\[
[\tilde{F}, V_j] = [\tilde{F}, \tilde{v}_j(x) \frac{\partial}{\partial y^i}] = -\tilde{v}_j(x) \frac{\partial}{\partial x^i} \mathrm{mod} \ \text{span} \left\{ \frac{\partial}{\partial y^i} \right\},
\]
and we have also \( \dim \left( \tilde{V} + [\tilde{F}, \tilde{V}] \right)(\tilde{z}) = \dim (V + [F, V]) (z) = 2n \), for any \( z \), in a neighborhood of \( z_0 \), thus completing the proof of (MS1). Moreover, the form of (10) implies that \( \Phi \cdot r(0) = g_0(x_0) \frac{\partial}{\partial r} \) proving (MS0).

To prove sufficiency, we start, due to condition (MS1), by choosing \( n \) vector fields in \( V \), that are linearly independent at any \( z \) in a neighborhood \( O_{z_0} \) of \( z_0 \). Denote them by \( V_1, \ldots, V_n \). We claim that \( V_1, \ldots, V_n, [F, V_1], \ldots, [F, V_n] \) are linearly independent at any point of \( O_{z_0} \). Indeed, because of \( \dim V(z) = n \), any \( V \in V \) is of the form
\[
V = \sum_{i=1}^n \alpha^i V_i, \quad \text{for some} \quad \alpha^i \in C^\infty(O_{z_0}).
\]
Then
\[
\text{ad}_F V = [F, \sum_{i=1}^n \alpha^i V_i] = \sum_{i=1}^n \alpha^i \text{ad}_F V_i \mod \text{span} \{V_1, \ldots, V_n\},
\]
showing that \( V + \text{ad}_F V \) is spanned, at any point \( z \in O_{z_0} \), by the vector fields \( V_1, \ldots, V_n, [F, V_1], \ldots, [F, V_n] \), thus proving our claim.

By condition (MS2), the vector fields \( V_1, \ldots, V_n \) commute, i.e., \( [V_i, V_j] = 0 \), \( 1 \leq i, j \leq n \), and thus they can be simultaneously rectified, that is, there exists a local change of coordinates \( \phi(z) = (x, y) \), satisfying \( \phi(z_0) = (x_0, 0) \), and such that, locally around \( (x_0, 0) \), we get \( \phi_* V_j = \frac{\partial}{\partial y^j}, 1 \leq j \leq n \) (see, for instance, [9], page 161). Since \( G_r \in V \), \( 1 \leq r \leq m \), (by definition of \( V \)) and \( \dim V(z) = n \) (by (MS1)), it follows
\[
\phi_* G_r = g^i_r(x, y) \frac{\partial}{\partial y^i}, \quad 1 \leq i \leq n.
\]
Moreover, \( [G_r, V_j] \in [V, V] = 0 \), for all \( 1 \leq r \leq m \), \( 1 \leq j \leq n \), and so
\[
\phi_* G_r = g^i_r(x) \frac{\partial}{\partial y^i}.
\]
In the system of local coordinates \( (x, y) \), we express the drift as
\[
\phi_* F = f^i(x) \frac{\partial}{\partial x^i} + \tilde{f}^i \frac{\partial}{\partial y^i},
\]
where \( f^i(x_0, 0) = 0 \), because of (MS0). To simplify notation, we will denote \( \phi_* F, \phi_* G_r \) and \( \phi_* V_j \) by \( F, G_r \) and \( V_j \) respectively. We have
\[
\text{ad}_F V_j = [F, V_j] = -\frac{\partial F}{\partial y^j} \in [F, V], \quad 1 \leq j \leq n,
\]
and
\[
[V_k, \text{ad}_F V_j] = -\frac{\partial^2 F}{\partial y^k \partial y^j} \in [V, [F, V]] \subset V, \quad 1 \leq j, k \leq n,
\]
with the last inclusion coming from the definition of \( V \). Then, by \( [V, V] = 0 \),
\[
[V_i, [V_k, \text{ad}_F V_j]] = 0, \quad 1 \leq j, k, l \leq n,
\]
implying that all third order derivatives \( \partial^3 F / \partial y^i \partial y^j \partial y^k \) vanish. The drift becomes

\[
F = (c^i(x, y) + d^i_j(x) y^j + e^i(x)) \frac{\partial}{\partial x^i} + (\tilde{c}^i(x, y) + \tilde{d}^i_j(x) y^j + \tilde{e}^i(x)) \frac{\partial}{\partial y^i},
\]

with \( c^i(x, y) \) and \( \tilde{c}^i(x, y) \) being homogeneous functions of degree two with respect to \( y \) and everywhere we take sums over repeated indices \( i \) and \( j \).

Since \([V_k, \text{ad}_F V_j] \in \text{span} \mathcal{V} = \text{span} \left\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\} \), for any \( 1 \leq j, k \leq n \), then we get \( c^i(x, y) = 0 \). It follows,

\[
F = (d^i_j(x) y^j + e^i(x)) \frac{\partial}{\partial x^i} + (\tilde{c}^i(x, y) + \tilde{d}^i_j(x) y^j + \tilde{e}^i(x)) \frac{\partial}{\partial y^i},
\]

where \( e^i(x_0) = 0 \). We have

\[
\text{ad}_F V_j = -d^i_j(x) \frac{\partial}{\partial x^i} \text{ mod span} \mathcal{V}, \quad 1 \leq j \leq n.
\]

Because \( V_1, \ldots, V_n, [F, V_1], \ldots, [F, V_n] \) are linearly independent around \( z_0 \), it follows that the matrix \( (d^i_j(x)) \) is of rank \( n \). Thus

\[
\tilde{x}^i = x^i, \quad \tilde{y}^i = d^i_j(x) y^j + e^i(x),
\]

defines a change of coordinates (mapping \((x_0, 0)\) into \((\tilde{x}_0, 0)\), since \( e^i(x_0) = 0 \)), which transforms the drift \( F \) into

\[
\tilde{F} = \tilde{y}^i \frac{\partial}{\partial \tilde{x}^i} + \left( \tilde{c}^i(\tilde{x}, \tilde{y}) + \tilde{d}^i_j(\tilde{x}) \tilde{y}^j + \tilde{e}^i(\tilde{x}) \right) \frac{\partial}{\partial \tilde{y}^i},
\]

where \( \tilde{c}^i(\tilde{x}, \tilde{y}) \) is homogeneous of degree two with respect to \( y \), and the control vector fields into

\[
\tilde{G}_r = \tilde{y}^i \frac{\partial}{\partial \tilde{x}^i}, \quad 1 \leq r \leq m,
\]

Omitting the tilda notation and denoting \( \tilde{c}^i(\tilde{x}, \tilde{y}) = -\Gamma^i_{jk}(x) y^j y^k \), we obtain the system in the form (8), which is geodesically accessible (by assumption (MS1) and Lemma 4.3).

\[\square\]

8. The local nature of our results. The following examples show that the presented results are of local nature. We would like to thank an anonymous reviewer for raising that issue.

Example 8.0.3. Consider the system

\[
\Sigma: \quad \dot{z} = F(z) + uG(z),
\]

where \( M = \{ z = (x, y) \in \mathbb{R}^2 \mid x < 0 \} \) and

\[
F = y \frac{\partial}{\partial x} \quad \text{and} \quad G = -2y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.
\]

By a direct calculation we get \( \text{ad}_F G = -\frac{\partial}{\partial x} \) and hence \([G, \text{ad}_F G] = 0\). It follows that \( \mathcal{V} = \text{Vect}_\mathbb{R} \{ G \} = \text{Vect}_\mathbb{R} \{ -2y \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \} \). Now it is immediate to see that everywhere on \( M \) we have \([V, \mathcal{V}] = 0\), implying the condition (MS2), and that \( \dim \mathcal{V}(z) = 1 \) and \( \dim (\mathcal{V} + [F, \mathcal{V}](z)) = 2 \), for any \( z \in M \), implying (MS1). By Theorem 3.2, the system \( \Sigma \) is locally \( S \)-equivalent to a geodesically accessible mechanical system around any \( z_0 \) satisfying (MS0), that is, \( F(z_0) \in \mathcal{V}(z_0) \). Notice that the latter defines the configuration manifold \( Q^S = \{ z \in M \mid F(z) \in \mathcal{V}(z) \} \), whose open
subsets are configuration manifolds of local mechanical structures of \( \Sigma \), as discussed in Section 4.

Now assume that there exists a tangent bundle \( TQ \), a geodesically accessible mechanical system \( (G\text{AMS}) \) and a global diffeomorphism \( \Phi \), from \( M \) onto \( TQ \), transforming \( \Sigma \) into \( (G\text{AMS}) \). Because of the uniqueness of the geodesically accessible mechanical structure, it is clear that the restriction of \( \Phi \) to \( Q^\Sigma \) would establish a diffeomorphism between \( Q^\Sigma \) and \( Q \). Moreover, \( \Phi \) would transform the integral leaves of the line-distribution (spanned by \( V \)) passing through any \( z_0 \in Q^\Sigma \) into the fibers \( T_qQ \), where \( \Phi(z_0) = (q, 0) \), with \( (q, 0) \) in the zero section of \( TQ \) evaluated at \( q \). In particular, the union of integral leaves passing through all points of \( Q^\Sigma \) should be mapped, via \( \Phi \), onto \( TQ \). Now observe that the integral leaves are of the form \( \{ x + y^2 = c \} \), where \( c \in \mathbb{R} \), and the leaves corresponding to \( c < 0 \) cross \( Q^\Sigma = \{ (x, y) \mid x < 0, y = 0 \} \) while those corresponding to \( c \geq 0 \) do not. It follows that \( \Phi(M^-) = TQ \), where \( M^- = \{ (x, y) \mid x + y^2 < 0 \} \), thus contradicting the injectivity of \( \Phi \) defined globally on \( M \). Notice that the system \( \Sigma \) restricted to \( M^- \) can be globally transformed by the diffeomorphism

\[
\begin{align*}
\hat{x} &= x + y^2, \\
\hat{y} &= y,
\end{align*}
\]

into the linear geodesically accessible mechanical system

\[
\begin{align*}
\hat{x} &= \hat{y}, \\
\hat{y} &= u,
\end{align*}
\]

on \( TQ \), where \( Q = \{ (\hat{x}, \hat{y}) \mid \hat{x} < 0, \hat{y} = 0 \} \). In other words, the manifold \( M \) is too “large”: we are not able to map the leaves \( \{ x + y^2 = c \mid c \geq 0 \} \) into the fibers of \( TQ \), but if we eliminate them, by restricting the manifold \( M \) to \( M^- \), the restriction of \( \Sigma \) to \( M^- \) becomes globally equivalent to a mechanical system.

In the previous example, the topology of leaves of \( V \) is trivial (but there are leaves not intersecting \( Q^\Sigma \)). Now we will give an example of a locally mechanical system whose topology of leaves of \( V \) is nontrivial.

**Example 8.0.4.** Consider the system

\[
\Sigma: \quad \dot{z} = F(z) + uG(z), \quad z = (x, y)^T \in \mathbb{R}^2 = M,
\]

where

\[
F = \frac{\partial}{\partial x} \quad \text{and} \quad G = e^y \sin x \frac{\partial}{\partial x} + e^y \cos x \frac{\partial}{\partial y}.
\]

We have \( \text{ad}_F G = e^y \cos x \frac{\partial}{\partial x} - e^y \sin x \frac{\partial}{\partial y} \) and \( [G, \text{ad}_F G] = 0 \). It follows that the system \( \Sigma \) satisfies everywhere on \( M = \mathbb{R}^2 \) the conditions (MS1) and (MS2) of Theorem 3.2. Therefore, around any point \( z_0 \) satisfying (MS0), that is, any point \( z_0 \) of \( Q^\Sigma = \{ z \in M \mid F(z) \in V(z) \} = \{ x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \} \), the system is locally equivalent to a geodesically accessible mechanical system. Notice that \( Q^\Sigma \), the candidate for the global configuration manifold, is not connected.

Now assume that there exists a tangent bundle \( TQ \), a geodesically accessible mechanical system \( (G\text{AMS}) \), and a global diffeomorphism \( \Phi \) mapping \( M \) onto \( TQ \) and transforming the system \( \Sigma \) into \( (G\text{AMS}) \). By the uniqueness of the geodesically accessible mechanical structure, the diffeomorphism \( \Phi \) should map the leaves of the line-distribution spanned by \( V \) into the fibers \( T_qQ \). As a consequence, the quotient space \( M/\sim \), where \( z \sim \tilde{z} \) if they belong to the same leaf, should be diffeomorphic
to the manifold $Q$, but $M/\sim$ is not a smooth manifold. So $\Phi$ cannot be a global diffeomorphism.

9. Conclusions and further research. In this paper we have studied the geometry of general mechanical control systems ($\mathcal{MS}$), which we define as a 4-tuple $(\mathcal{MS}) = (Q, \nabla, g_0, d)$, where $Q$ is an $n$-dimensional configuration manifold; $\nabla$ is a symmetric affine connection on $Q$; $g_0$ is an $(m + 1)$-tuple of vector fields which includes an uncontrolled vector field $g_0$ and input vector fields $g_1, \ldots, g_m$; and $d : TQ \to TQ$ is a smooth bundle map over $Q$ (i.e., covering the identity) linear on fibers, that physically represents dissipative-type or gyroscopic-type forces (and their generalisations) acting on the system.

The subclass of geodesically accessible mechanical control systems ($\mathcal{GAMS}$) has been of main interest for our considerations. We define a ($\mathcal{GAMS}$) as a mechanical control system for which the smallest distribution on $Q$, containing the input vector fields and closed under the symmetric product coincides with $TQ$, at each point $q \in Q$.

We gave necessary and sufficient conditions for a control-affine system $\Sigma$ to be state equivalent to a ($\mathcal{GAMS}$) around a point $(x_0, 0) \in TQ$. These conditions encode all the necessary information about the mechanical structure of the ($\mathcal{GAMS}$). Studying the class of geodesically accessible mechanical systems reveals to be very natural, since, as we have shown in Section 5, if $\Sigma$ admits a ($\mathcal{GAMS}$)-structure then it is unique (up to an extended point transformation) and if the system is not geodesically accessible it can admit multiple mechanical structures.

Finally, we gave a characterisation of particular classes of ($\mathcal{GAMS}$) systems. Namely, we have characterised ($\mathcal{GAMS}$) that are subject neither to dissipative-type nor gyroscopic-type forces (that is, $d = 0$) and ($\mathcal{GAMS}$) without uncontrolled forces (that is, $g_0 = 0$). A combination of these two cases leads to a characterisation of geodesically accessible affine connection control systems, a class of mechanical control systems that has recently attracted a lot of attention, see, e.g., [10, 30, 31, 32, 34]).

Also a characterisation of mechanical control systems ($\mathcal{MS}$) that are not necessarily ($\mathcal{GAMS}$) is provided.

In our opinion the presented study has opened many potential future directions of research, that form a natural continuation of this work. In an upcoming paper [49], the authors have studied the class of geodesically accessible affine connection control systems, constructing two families of structure functions which are equivariants for the state and mechanical state equivalence. Also, the results in the present paper have been used to characterise systems in a special form, namely a subclass of mechanical control systems subject to second-order nonholonomic constraints that includes, in particular, the second-order chained form [52].

A challenging problem is to use the methods and techniques developed in the paper in order to get a characterisation of Hamiltonian control systems and to relate that characterisation with the work obtained by Crouch and van der Schaft [15, 16, 59]. Another important class to study are mechanical control systems for which the uncontrolled vector field is a gradient vector field. Exploring relations between our approach with a recent characterisation of that class in [13] is a very natural and interesting problem. Refining the results in this paper to systems on Lie groups and their homogeneous spaces and obtaining global results, instead of local ones, are also very challenging future lines of research.
The equivalence studied in this paper is state equivalence and mechanical state equivalence. A very challenging and open problem is that of feedback equivalence \cite{22, 42, 48} of a control-affine system to a mechanical system, in particular, a search of feedback invariants of mechanical systems. For a particular problem of feedback equivalence of a nonlinear control system to a given Hamiltonian system see \cite{12, 29, 58}. Four approaches have been developed in order to study feedback equivalence and feedback invariants. First is based on studying equivalence of the distributions and vector fields defining the control systems and their singularities \cite{23, 47, 51, 61}. The second approach, proposed by Gardner, uses Cartan’s method of equivalence \cite{18, 19, 20}. The third method, inspired by the Hamiltonian formalism of optimal control, has been developed by Agrachev, Bonnard, and Jakubczyk \cite{2, 3, 8, 24, 25}. Finally, a fourth approach was proposed by Kang and Krener \cite{27} (see the survey \cite{50} and references therein) and is based on formal equivalence. Perhaps using not just one of those approaches but combining them will allow to describe and understand nonlinear control systems that are feedback equivalent to a mechanical system.

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**REFERENCES**

\[1\] R. Abraham and J. E. Marsden, “Foundations of Mechanics,” Addison-Wesley, 1978.

\[2\] A. A. Agrachev, *Feedback-invariant optimal control theory and differential geometry. II. Jacobi curves for singular extremals*, J. Dynam. Control Systems, **4** (1998), 583–604.

\[3\] A. A. Agrachev and R. V. Gamkrelidze, *Feedback-invariant optimal control theory and differential geometry. I. Regular extremals*, J. Dynam. Control Systems, **3** (1997), 343–389.

\[4\] A. A. Agrachev and Y. L. Sachkov, “Control Theory from the Geometric Viewpoint,” Springer-Verlag Berlin and Heidelberg, 2004.

\[5\] I. Anderson and G. Thompson, *The inverse problem of the calculus of variations for ordinary differential equations*, Mem. Amer. Math. Soc., **98** (1992), 108–110.

\[6\] H. Arai, K. Tanie and N. Shiroma, *Nonholonomic control of a three-DOF planar underactuated manipulator*, IEEE Trans. Robot. Autom., **14** (1998), 681–695.

\[7\] A. M. Bloch, “Nonholonomic Mechanics and Control,” Springer-Verlag, New York, 2003.

\[8\] B. Bonnard, *Feedback equivalence for nonlinear systems and the time optimal control problem*, SIAM J. Control and Optim., **29** (1991), 1300–1321.

\[9\] W. Boothby, “An Introduction to Differential Manifolds and Riemannian Geometry,” 2nd edition, Academic Press, Inc, 1986.

\[10\] F. Bullo and A. D. Lewis, “Geometric Control of Mechanical Systems,” Springer-Verlag, New York, 2004.

\[11\] F. Bullo and K. M. Lynch, *Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems*, IEEE Trans. Robot. Autom., **17** (2001), 402–412.

\[12\] D. Cheng, A. Astolfi and R. Ortega, *On feedback equivalence to port controlled Hamiltonian systems*, Systems Control Lett., **54** (2005), 911–917.

\[13\] J. Cortés, A. J. van der Schaft and P. E. Crouch, *Characterization of gradient control systems*, SIAM J. Control Optim., **44** (2005), 1192–1214.

\[14\] M. Crampin, G. E. Prince and G. Thompson, *A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics*, J. Phys. A-Math. Gen., **17** (1984), 1437–1447.

\[15\] P. E. Crouch and A. J. van der Schaft, *Hamiltonian and self-adjoint control systems*, Systems & Control Letters, **8** (1987), 289–295.

\[16\] P. E. Crouch and A. J. van der Schaft, “Variational and Hamiltonian Control Systems,” Lectures Notes in Control and Inform. Sci. **101**, Springer-Verlag, New York, 1987.

\[17\] J. Douglas, *Solution of the inverse problem of the calculus of variations*, Trans. Amer. Math. Soc., **50** (1941), 71–128.

\[18\] R. B. Gardner, “The Method of Equivalence and its Applications,” CBMS Regional Conference Series in Applied Mathematics, **58**, SIAM, Philadelphia, PA, 1989.
WHEN IS A CONTROL SYSTEM MECHANICAL?

[19] R. B. Gardner and W. F. Shadwick, The GS algorithm for exact linearization to Brunovský normal form, IEEE Trans. Automat. Control, 37 (1992), 224–230.
[20] R. B. Gardner, W. F. Shadwick and G. R. Wilkens, Feedback equivalence and symmetries of Brunovský normal forms, Contemp. Math., 97 (1989), 115–130.
[21] J. Hauser, S. Sastry and G. Meyer, Nonlinear control design for slightly non-minimum phase systems: Application to V/STOL aircraft, Automatica J. IFAC, 28 (1992), 665–679.
[22] A. Isidori, “Nonlinear Control Systems,” 3rd edition, Springer Verlag, 1995.
[23] B. Jakubczyk, Equivalence and invariants of nonlinear control systems, in “Nonlinear Controllability and Optimal Control” (eds. H.J. Sussmann), Marcel Dekker, New York-Basel, (1990), 177–218.
[24] B. Jakubczyk, Critical Hamiltonians and feedback invariants, in “Geometry of Feedback and Optimal Control,” Marcel Dekker, New York-Basel, (1998), 219–256.
[25] B. Jakubczyk, Feedback invariants and critical trajectories; Hamiltonian formalism for feedback equivalence, in “Nonlinear Control in the Year 2000” 1 (eds. A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek), LNCS vol. 258, Springer, London, (2000) 545–568.
[26] V. Jurdjevic, “Geometric Control Theory,” Cambridge University Press, 1997.
[27] W. Kang and A. J. Krener, Extended quadratic controller normal form and dynamic feedback linearization of nonlinear systems, SIAM J. Control Optim., 30 (1992), 1319–1337.
[28] J. Koiller, Book review of “Analytical Mechanics: A comprehensive treatise on the dynamics of constrained systems for engineers, physicists and mathematicians,” by John G. Papastavridis, Bulletin (New Series) of the American Mathematical Society, 40 (2003), 405–419.
[29] P. Kokkonen, “Energy-Shaping Control of Physical Systems (ESC),” Matematilainen Ja Tilastotieteen Laitos, 2007.
[30] A. D. Lewis, Affine connections and distributions with applications to nonholonomic mechanics, Rep. Math. Phys., 42 (1998), 153–164.
[31] A. D. Lewis, Affine connections control systems, in “Proc. IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear control,” (2000), 128–133.
[32] A. D. Lewis, The category of affine connection control systems, in “Proc. of the 39th IEEE Conf. on Decision and Control, Sydney, Australia,” (2000), 1260–1265.
[33] A. D. Lewis and R. M. Murray, Configuration Controllability of Simple Mechanical Control Systems, SIAM J. Control Optim., 35 (1997), 766–790.
[34] A. D. Lewis and R. M. Murray, Decompositions for control systems on manifolds with an affine connection, Syst. Contr. Lett., 31 (1997), 199–205.
[35] J. E. Marsden and T. Ratiu, “Introduction to Mechanics and Symmetry,” Springer-Verlag, 1994.
[36] P. Martin, S. Devasia and B. Paden, A different look at output tracking: control of a VTOL aircraft, in “Proc. of the 33rd IEEE Int. Conf. on Control Applications,” 1 (1999), 345–351.
[37] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo and C. Rubano, The inverse problem in the calculus of variations and the geometry of the tangent bundle, Physics Reports, 188 (1990), 147–284.
[38] R. M. Murray, Nonlinear control of mechanical systems: A Lagrangian perspective, Annual Reviews in Control, 21 (1997), 31–42.
[39] R. M. Murray, Z. Li and S. S. Sastry, “A Mathematical Introduction to Robotic Manipulation,” Taylor & Francis Ltd, Boca Raton, 1994.
[40] H. Nijmeijer and A. J. van der Schaft, “Nonlinear Dynamical Control Systems,” Springer-Verlag, New York, 1990.
[41] R. Olfati-Saber, Global configuration stabilization for the VTOL aircraft with strong input coupling, IEEE Trans. Automat. Control, 47 (2002), 1949–1952.
[42] W. M. Oliva, “Geometric Mechanics,” Springer-Verlag, Berlin, 2002.
[43] R. Ortega, A. Loria, P. J. Nicklasson and H. Sira-Ramirez, “Passivity-Based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications,” Springer-Verlag, Berlin, 1998.
R. H. Rand and D. V. Ramani, *Nonlinear normal modes in a system with nonholonomic constraints*, Nonlinear Dynamics, 25 (2001), 49-64.

W. Respondek, *Feedback classification of nonlinear control systems in $\mathbb{R}^2$ and $\mathbb{R}^3$*, in “Geometry of Feedback and Optimal Control” 207 (eds. B. Jakubczyk and W. Respondek), Marcel Dekker, New York, (1998), 347–382.

W. Respondek, *Introduction to geometric nonlinear control; linearization, observability and decoupling*, in “Mathematical Control Theory” (ed. A. Agrachev), ICTP Lecture Notes, (2002), 169–222.

W. Respondek and S. Ricardo, *Equivariants of mechanical control systems*, submitted, (2010).

W. Respondek and I. A. Tall, *Feedback equivalence of nonlinear control systems: A survey on formal approach*, in “Chaos in Automatic Control” (eds. J.-P. Barbot et W. Perruquetti), Taylor and Francis, (2006), 137–262.

W. Respondek and M. Zhitomirskii, *Feedback classification of nonlinear control systems on 3-manifolds*, Math. Control Signals Systems, 8 (1995), 299–333.

S. Ricardo and W. Respondek, *Geometry of second-order nonholonomic chained form systems*, submitted, (2010).

W. Sarlet, *The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics*, J. Phys. A-Math. Theor., 15 (1982), 1503–1517.

W. Sarlet, *Geometrical structures related to second-order equations*, Differential Geometry and Its Applications, (1987), 279–299.

S. Sastry, “Nonlinear Systems: Analysis, Stability, and Control,” Springer-Verlag, New York, 1999.

E. D. Sontag, “Mathematical Control Theory: Deterministic Finite Dimensional Systems,” Springer-Verlag, New York, 1998.

M. W. Spong, *Underactuated mechanical systems*, in “Control Problems in Robotics and Automation,” 230, Springer Berlin/Heidelberg, (1998), 135–150.

P. Tabuada and G. Pappas, *From nonlinear to Hamiltonian via feedback*, IEEE Trans. Automat. Control, 48 (2003), 1439–1442.

A. J. van der Schaft, *Symmetries, conservation laws and time-reversibility for Hamiltonian systems with external forces*, J. Math. Phys., 24 (1983), 2095–2101.

J. Vankerschaver, F. Cantrijn, M. de León and D. Martín de Diego, *Geometric aspects of nonholonomic field theories*, Rep. Math. Phys., 56 (2005), 387–411.

M. Zhitomirskii and W. Respondek, *Simple germs of corank one affine distributions*, Banach Center Publications, 44 (1998), 269–276.

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