MÖBIUS-IN Variant SELF-avoidance EnerGies FOR
non-smooth SETS IN ARBITRARY DIMENSIONS

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Abstract. In the present paper we investigate generalizations of O’Hara’s Möbius energy on curves [37], to Möbius-invariant energies on non-smooth subsets of $\mathbb{R}^n$ of arbitrary dimension and co-dimension. In particular, we show under mild assumptions on the local flatness of an admissible possibly unbounded set $\Sigma \subset \mathbb{R}^n$ that locally finite energy implies that $\Sigma$ is, in fact, an embedded Lipschitz submanifold of $\mathbb{R}^n$—sometimes even smoother (depending on the a priori given additional regularity of the admissible set). We also prove, on the other hand, that a local graph structure of low fractional Sobolev regularity on a set $\Sigma$ is already sufficient to guarantee finite energy of $\Sigma$. This type of Sobolev regularity is exactly what one would expect in view of Blatt’s characterization [5] of the correct energy space for the Möbius energy on closed curves. Our results hold in particular for Kusner and Sullivan’s cosine energy $E_{KS}$ [35] since one of the energies considered here is equivalent to $E_{KS}$.

1. Introduction

1.1. Motivation and outline. One of the most prominent examples of a repulsive energy on curves is the Möbius energy introduced by J. O’Hara [37], which can be written as

$$E_{\text{Möb}}(\gamma) = \int_\gamma \int_\gamma \left( \frac{1}{|x-y|^2} - \frac{1}{d_\gamma(x,y)^2} \right) dxdy,$$  \hspace{1cm} (1.1)

where $d_\gamma(x,y)$ denotes the intrinsic distance of two points $x, y \in \gamma$ along the curve $\gamma$. Ever since the seminal work of M. Freedman, Zh.-Xu He, and Zh. Wang [19] it is clear that the Möbius energy can be used as a fundamental tool in Geometric Knot Theory, and since then a lot of geometric and analytic work has been done. Variational and gradient formulas were derived and analyzed [22, 43, 5, 24, 25, 26, 44], the regularity of minimizers and critical points was established in [19, 22, 43, 12, 13], and the $L^2$-gradient flow was studied in [22, 6, 8]. Various discrete versions of the Möbius energy were examined [35, 41, 45, 9, 10], and one knows that the round circle is the absolutely minimizing closed curve [19, 1], whereas the stereographic projection of the standard Hopf link uniquely minimizes the corresponding version of the Möbius energy on non-split links in $\mathbb{R}^3$ [2].

Apart from the very recent contribution by O’Hara on the self-repulsiveness of Riesz potentials on smooth immersions [39], comparatively little is known for higher-dimensional versions of the Möbius energy such as the ones discussed in [4, 35], and

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\footnote{This name originates in the energy’s invariance under Möbius transformations of the ambient space as first shown in [19, Theorem 2.1].}
It is the aim of this paper to initiate a systematic study of a family of Möbius-invariant energies on non-smooth subsets $\Sigma \subset \mathbb{R}^n$ of arbitrary dimension and codimension. Such an investigation has been carried out for other (not Möbius-invariant) self-avoiding energies such as integral Menger curvature or tangent-point energies in arbitrary dimensions \cite{47,48,30,7,32,31,11}.

Following the ideas of R. B. Kusner and J. H. Sullivan in \cite{35} we first describe in an informal way how to use only first order information encoded in the class of admissible sets to define Möbius-invariant energies. As observed by P. Doyle and O. Schramm \cite{35}, \cite{38} Chapter 3.4] the Möbius energy for closed curves $\gamma : S^1 \to \mathbb{R}^n$ can be rewritten -- up to an additive constant -- in terms of the conformal angle $\vartheta_{\gamma}(x,y)$ as

$$E_{\text{Möb}}(\gamma) = \int_{\gamma} \int_{\gamma} \frac{1 - \cos \vartheta_{\gamma}(x,y)}{|x-y|^2} \, dx \, dy.$$  (1.2)

The conformal angle $\vartheta_{\gamma}(x,y)$ is defined as the angle between the circle $S^1(x,y)$ through the points $x,y \in \gamma$ and tangent to $\gamma$ at $x$, and the circle $S^1(y,y,x)$ also containing $x,y$ but now tangent to $\gamma$ at $y$.

With that idea in mind Kusner and Sullivan \cite{35} created Möbius-invariant energies defined on embedded, oriented $m$-dimensional $C^1$-submanifolds $\mathcal{M}^m \subset \mathbb{R}^n$ of the form

$$E_L(\mathcal{M}) = \int_{\mathcal{M}} \int_{\mathcal{M}} L(\vartheta_{\mathcal{M}}(x,y)) \frac{d\text{vol}_{\mathcal{M}}(x)}{|x-y|^{2m}} \, d\text{vol}_{\mathcal{M}}(y),$$  (1.3)

where now the conformal angle $\vartheta_{\mathcal{M}}(x,y)$ is the angle between the two unique $m$-dimensional spheres $S^m(x,y)$ and $S^m(y,y,x)$, tangent to $\mathcal{M}$ at $x$ and $y$, respectively, and both containing $x$ and $y$. In principle, $L$ in the numerator of the Lagrangian could be any non-negative function vanishing sufficiently fast at zero to balance the singularity of the denominator. Kusner and Sullivan, however, investigate more closely, in particular, numerically, the specific energy $E_{KS} := E_{LKS}$ with the numerator

$$L_{KS}(\vartheta) := (1 - \cos \vartheta)^m.$$  (1.4)

Notice that any choice of $L$ in (1.3) requires first order information about the submanifold, and Kusner and Sullivan provide in \cite{35} Section 11] a convenient method to calculate the angle $\vartheta_{\mathcal{M}}(x,y)$ in terms of the tangent $m$-planes $T_x\mathcal{M}$ and $T_y\mathcal{M}$ by a simple reflection. We adopt this idea in Definition 1.2 below, but aiming at non-smooth sets $\Sigma \subset \mathbb{R}^n$ we need to replace classic tangent planes at points $p \in \Sigma$ by suitably approximating $m$-planes $H(p)$ that serve as “mock tangent planes”, similarly as in previous collaborations of the second author on various geometric curvature energies \cite{47,48,31}. These mock tangent planes enter our definition of admissible sets; see Definition 1.1.

We then introduce in Definition 1.2 a family of Möbius-invariant energies $E_\tau \equiv E_{L\tau}$ parametrized by a scalar $\tau \in \mathbb{R}$, on non-smooth admissible sets $\Sigma \subset \mathbb{R}^n$ by replacing the numerator $L$ in (1.3) by functions $L_\tau = L_\tau(x,y,H(x),H(y))$ that roughly correspond to $\tau$-dependent powers of the conformal angle $\vartheta_{\Sigma}(x,y)$. Instead of principal angles as in \cite{35}, however, we prefer to work with the angle metric

$$\vartheta(F,G) := \| \Pi_F - \Pi_G \|$$  (1.5)

for two $m$-planes $F,G$, where $\Pi_F$ denotes the orthogonal projection onto the subspace $F$, and $\| \cdot \|$ stands for the operator norm, so that our new energies turn out to resemble the “sin-energies”
that are discussed only briefly in §35 Section 4. Nevertheless, we show in Appendix A that the choice $\tau = 1$ for our numerator $L_{\tau}$ generates a Lagrangian bounded from above and below by constant multiples of $L_{KS}$ in (1.4). So all of the results described in Section 1.2 below also hold for the specific Möbius energy $E_{KS}$ studied by Kusner and Sullivan.

1.2. Main results. Let $\mathcal{G}(n, m)$ denote the Grassmannian consisting of all $m$-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the angle metric defined in (1.3). For $x \in \mathbb{R}^n$ and $F \in \mathcal{G}(n, m)$ the orthogonal projection onto the affine $m$-plane $x + F$ is defined by

$$\Pi_{x + F}(z) := x + \Pi_F(z - x) \quad \text{for} \quad z \in \mathbb{R}^n.$$  \hfill (1.6)

In addition, let

$$C_x(\beta, F) := \{z \in \mathbb{R}^n : |\Pi_{F^\perp}(z - x)| \leq \beta |\Pi_F(z - x)|\}$$  \hfill (1.7)

be the cone around the affine $m$-plane $x + F$, centered at $x$ with opening angle $2 \arctan \beta$. Throughout the paper, $B_r(x)$ denotes the open ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$.

We start with the definition of admissible sets.

**Definition 1.1** (Admissible sets). Let $m, n \in \mathbb{N}$, $1 \leq m \leq n$, $\alpha > 0$, $M > 0$, and define the admissibility class $\mathcal{A}^m(\alpha, M)$ to be the set of all subsets $\Sigma \subset \mathbb{R}^n$ satisfying the following two properties.

(i) $\Sigma$ is closed, and there exists a function; $H : \Sigma \to \mathcal{G}(n, m)$.

(ii) There exists a dense subset $\Sigma^* \subset \Sigma$, with the following property: For all compact sets $K \subset \Sigma$, there exist a radius $R_K > 0$ and a constant $c_K > 0$, such that for all $p \in \Sigma^* \cap K$, there is a dense subset $D_p \subset (p + H(p)) \cap B_{R_K}(p)$, such that for all $x \in D_p$, there exists a point $\eta_x \in \Sigma \cap C_p(\alpha, H(p))$ with $\Pi_{p + H(p)}(\eta_x) = x$, and

$$\mathcal{H}^m \left( E_{\alpha, M}(p) \cap B_r(\eta_x) \right) \geq c_K r^m \quad \text{for all} \quad r \in \left(0, R_K / 10^3\right],$$

where $E_{\alpha, M}(p) := \{\mu \in \Sigma : \mathcal{H}(\mu, H(p)) < M\alpha\}$.

As an immediate consequence of that definition we notice the monotonicity relation

$$\mathcal{A}^m(\alpha_1, M_1) \subset \mathcal{A}^m(\alpha_2, M_2) \quad \text{for all} \quad 0 < \alpha_1 \leq \alpha_2, 0 < M_1 \leq M_2.$$  \hfill (1.8)

Intuitively speaking, an admissible set $\Sigma$ possesses a dense subset $\Sigma^*$ of “good points” $p$ such that the conical portion $\Sigma \cap C_p(\alpha, H(p))$ projects densely onto the affine plane $p + H(p)$ locally near $p$, and in each of the fibres under this projection there is at least one $\Sigma$-point $q$ near which there is sufficient mass of other $\Sigma$-points (possibly stretching beyond the cone) whose mock tangent planes are close to $H(q)$. We should emphasize that we neither require $\Sigma$ to be contained in the cone near $p$, nor a topological linking condition as postulated, e.g., in the admissibility class described in [48, Section 2.3], nor do we assume any relation between the mock tangent planes $H(q)$ of various points $q \in \Sigma \cap C_p(\alpha, H(p))$, or between $H(q)$ and $H(p)$. On the other hand, we do require a uniform local radius $R_K$ once we have fixed a compact subset $K \subset \Sigma$, which excludes some of the admissible example sets of previous work such as in [48, Example 2.14 & Figure 1] with smaller and smaller structures accumulating locally.

It is easy to see that embedded $m$-dimensional submanifolds $\Sigma := \mathcal{M}^m \subset \mathbb{R}^n$ of class $C^2$ without boundary are admissible, since one can choose $H(p) := T_p \Sigma$ for all
Figure 1. Admissible sets. Finite unions of embedded $C^2$-submanifolds (a) and the annulus (b) as the uncountable union of circles with positively bounded radii are contained in $\mathcal{A}^m(\alpha, M)$. c. An unbounded set with fine structures accumulating within a compact subset is in the modified class $\mathcal{A}^m_*(\alpha, M)$ for a certain $\alpha$. d. Finite unions of smooth manifolds with and without boundary may also be in $\mathcal{A}^m_*(\alpha, M)$ for any positive $\alpha$ and $M$.

$p \in \Sigma^* := \mathcal{M}$, so that $\mathcal{M} \subset \mathcal{A}^m(\alpha, M)$ for all $\alpha > 0$ and $M > 0$. Also finite unions $\Sigma := \bigcup_{i=1}^N \mathcal{M}_i$ of such $C^2$-submanifolds are admissible for any $\alpha, M > 0$ (see Figure 1 a), since one can define $H(p) := T_p \mathcal{M}_i(p)$, where $i(p) := \min\{j \in \{1, \ldots, N\} : p \in \mathcal{M}_j\}$.

For these examples one can allow $c_K = \omega_m/2$ in Definition 1.1, where $\omega_m$ denotes the volume of the $m$-dimensional unit ball $B_1(0) \subset \mathbb{R}^m$. Higher-dimensional sets that are foliated by lower-dimensional $C^2$-submanifolds with a uniform curvature bound can also be admissible, such as the two-dimensional planar annulus generated by uncountably many circles of varying radius depicted in Figure 1 b. Lipschitz submanifolds, countable collections of Lipschitz graphs, and the images of compact $C^1$-manifolds under $C^1$-immersions, as well as finite unions of those, also turn out to be admissible; the detailed proofs of these last statements are carried out in Section 2.

It is possible to relax condition (ii) in Definition 1.1 requiring only sufficiently large projection of $\Sigma \cap C_p(\alpha, H(p))$ onto an affine halfspace $p + H_p(p)$, at the cost of adding a condition relating different measures of flatness, as done for the definition of $m$-fine sets in [29, 31]. This generates a new admissibility class $\mathcal{A}^m_*(\alpha, M)$ that also contains non-smooth examples with accumulation zones or certain unions of manifolds with and without boundaries; see the bottom of Figure 1. All results mentioned below also hold for this modified admissibility class $\mathcal{A}^m_*(\alpha, M)$; for its precise definition and the necessary modifications in the proofs we refer to Remark 3.14 at the end of Section 3.3.

The conformal angle between the tangential spheres $S^m(x, x, y)$ and $S^m(y, y, x)$ is defined as the angle between the spheres’ tangent planes $T_x S^m(x, x, y)$ and $T_y S^m(y, y, x)$ at an arbitrary point $z$ contained in the intersection of the two spheres, so, e.g., at $z := x$, where one has $T_x S^m(x, x, y) = H(x)$. This leaves us to compute the tangent plane $T_x S^m(y, y, x)$ for which it suffices to reflect $T_y S^m(y, y, x) = H(y)$ at the subspace
$(x - y) \perp$ by virtue of the mapping
\[
R_{xy}: \mathbb{R}^n \to \mathbb{R}^n, \quad z \mapsto z - \frac{2}{|x - y|^2} (z, x - y) \cdot (x - y).
\]
Therefore, we can express the conformal angle $\vartheta_\Sigma$ of an admissible set $\Sigma$ as
\[
\vartheta_\Sigma(x, y) = \frac{\langle H(x), R_{xy}(H(y)) \rangle}{|x - y|^{2m}}.
\]
This leads to the following definition of energies $E^\tau$.

**Definition 1.2** (Möbius energies). Let $\alpha > 0$ and $M > 0$. For $\tau \in \mathbb{R}$ and $\Sigma \in \mathcal{A}^m(\alpha, M)$ we define the energy
\[
E^\tau(\Sigma) = \int \int_{\Sigma} \frac{\langle H(x), R_{xy}(H(y)) \rangle}{|x - y|^{2m}} d\mathcal{H}^m(x) d\mathcal{H}^m(y). \quad (1.10)
\]

In Lemma A.5 it is shown that the angle in the numerator coincides with the sine of the largest principal angle, which is invariant under Möbius transformations, so that according to [35, Section 2] all these energies $E^\tau$ are Möbius invariant. For $\tau = 1$ we prove in Corollary A.7 that $E^1$ is equivalent to the Kusner-Sullivan energy $E_{KS}$ with the numerator (1.4).

Since unbounded sets are not excluded in the admissibility class we cannot expect finite energy of the whole set. So, we say $\Sigma$ has locally finite energy $E^\tau$ if and only if
\[
E^\tau(\Sigma \cap B_N(0)) < \infty \quad \text{for all } N \in \mathbb{N}. \quad (1.11)
\]

For admissible sets with locally finite energy we prove the following self-avoidance result, under a smallness condition on the product of the parameters $\alpha$ and $M$, which balances locally the degree of flatness with the mass of regions of mildly varying mock tangent planes.

**Theorem 1.3** (Self-avoidance). For fixed dimensions $2 \leq m \leq n$ there is a universal constant $\delta = \delta(m)$ such that for any $\alpha, M > 0$ with
\[
\alpha(M + 1) < \delta/50 \quad (1.12)
\]
every admissible set $\Sigma \in \mathcal{A}^m(\alpha, M)$ with locally finite Möbius energy $E^\tau$, $\tau \in (-1, \infty)$, is an embedded Lipschitz submanifold of $\mathbb{R}^n$.

In fact, we prove in Theorem 3.11 that every compact subset $K \subset \Sigma$ possesses a local graph representation by Lipschitz functions, which according to [36] is even slightly stronger than $\Sigma$ being a Lipschitz submanifold.

In Section 2.1 we prove that the image $\Sigma := f(M)$ of a compact abstract $m$-dimensional $C^1$-manifold $M$ under an immersion $f : M \to \mathbb{R}^n$ is admissible for any $\alpha > 0$ and $M > 0$. Consequently, one can guarantee that assumption (1.12) of Theorem 1.3 holds true so that finite Möbius energy implies that $\Sigma$ is an embedded $C^{0,1}$-submanifold of $\mathbb{R}^n$. But the initially granted regularity of $M$ and $f$ leads to the corresponding smoothness of the embedded submanifold.

**Corollary 1.4.** Let $k \in \mathbb{N}$ and suppose $\Sigma = f(M)$ satisfies $E^\tau(\Sigma) < \infty$ for $\tau \in (-1, \infty)$, where $M$ is an $m$-dimensional compact $C^k$-manifold, and $f : M \to \mathbb{R}^n$ is a $C^k$-immersion. Then $\Sigma$ is an embedded $m$-dimensional $C^k$-submanifold of $\mathbb{R}^n$. 

So, finite Möbius energy yields embedded submanifolds, which then inherit some additional presupposed regularity of the admissible set. This effect of a transferred initial regularity can also be observed in the Lipschitz category; see Corollary 3.12 and 3.13.

Very recently O’Hara proved the self-repulsiveness of the Kusner-Sullivan energy $E_{KS}$ in the $C^2$-topology on the class of embedded $C^2$-submanifolds with uniform curvature bounds; see [39, Theorem 3.3]. The equivalence of $E_{KS}$ and $E^\tau$ for $\tau = 1$ proven in Corollary A.7 in Appendix A together with Corollary 1.4 might help to generalize O’Hara’s result to self-repulsiveness on suitably normalized $C^1$-submanifolds in the $C^1$-topology.

While Theorem 1.3 and Corollary 1.4 describe self-avoidance effects of the Möbius energies $E^\tau$, one may ask, on the other hand, under which regularity assumptions on (topological) embedded submanifolds one obtains finite energy. It is easy to show that $C^2$-regularity implies finite energy $E^\tau$ for any $\tau > 0$; see Lemma 4.1 and Corollary 4.2. But we prove, in addition, that a relatively mild fractional Sobolev regularity of the local graph representations is already sufficient to produce finite energy. To state the precise result we recall the notion of these fractional spaces (see, e.g., [15]): For an open set $\Omega \subset \mathbb{R}^m$, and parameters $k \in \mathbb{N} \cup \{0\}$, $s \in (0, 1)$, and $\varrho \in [1, \infty)$, the Sobolev-Slobodeckiǐ-space $W^{k+s, \varrho}(\Omega)$ is the set of all Sobolev functions $f \in W^{k,p}(\Omega)$ such that

$$[\partial^\beta f]_{s, \varrho} := \int_\Omega \int_\Omega \frac{|\partial^\beta f(x) - \partial^\beta f(y)|^\varrho}{|x - y|^{m+\varrho}} \, dx \, dy < \infty$$

for all multi-indices $\beta$ with $|\beta| = k$.

**Theorem 1.5 (Sufficient fractional Sobolev regularity).** If $\mathcal{M}^m \subset \mathbb{R}^n$ is an embedded compact submanifold with local graph representations of class $C^{0,1} \cap W^{3/2,2m}(1+\tau)m$ for some $\tau \in (0, \infty)$, then $E^\tau(\mathcal{M}) < \infty$.

By Corollary A.7 the Kusner-Sullivan energy $E_{KS}$ is bounded from above by a constant multiple of $E^\tau$ for $\tau \in (0, 1)$ and even equivalent to $E^1$, so that $C^{0,1} \cap W^{3/2,2m}$-regularity suffices to guarantee finite $E_{KS}$. This fractional Sobolev regularity corresponds to the exact regularity that characterizes finite Möbius energy in dimension $m = 1$ by the work of S. Blatt [5, Theorem 1.1]. For arbitrary dimensions $m \geq 2$, however, Theorem 1.5 establishes only one direction of such a characterization. It is open at this point – even for $\tau = 1$ – if the energies $E^\tau$ exhibit sufficiently strong regularizing effects \(^2\) to guarantee that finite $E^\tau$-energy yields embedded $W^{2+\tau,2m}(1+\tau)m$-submanifolds, even if we add the extra assumption that the admissible set is already an embedded $C^1$-submanifold. Such a characterization, however, holds true in arbitrary dimensions for the scale-invariant tangent point energy introduced in [48] but analyzed there only in the regime above scale-invariance. In an upcoming note we prove the self-avoidance property on a wider class of admissible non-smooth sets and use Blatt’s technique developed in [6] to prove the characterization of the exact energy space for the scale-invariant tangent-point energy. It seems, however, that this energy is slightly more singular than $E^1$.

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\(^2\)That the energies $E^\tau$ do regularize at least to a certain extent is reflected in the fact that certain types of singularities, like a wedge-shaped crease, lead to infinite $E^\tau$-energy for any $\tau > -1$; see Remark A.8 in Appendix A.
1.3. Strategy of proofs. The crucial step to prove the self-avoidance property stated in Theorem 1.3 is to bound for any given \( \delta \in (0, 1) \) the beta number

\[
\beta_\Sigma(x, r) := \inf_{F \in \mathcal{F}(n,m)} \sup_{y \in \Sigma \cap B_r(x)} \frac{\text{dist}(y, (x + F) \cap B_r(x))}{r} \quad \text{for } x \in \Sigma \tag{1.13}
\]

from above by \( \delta \) on small scales \( r \). This is carried out in Theorem 3.3 with an indirect argument as follows. Assuming the contrary one finds points \( p \in \Sigma \) and \( q \in \Sigma \cap B_r(p) \) such that \( \text{dist}(q, p + H(p)) > \delta r \). This geometric situation contributes substantially to the energy in a way that depends on the angle \( \angle(H(p), H(q)) \) between the mock tangent planes at \( p \) and \( q \). If that angle happens to be small then, loosely speaking, a sufficient amount of mass of \( \Sigma \) near \( q \) interacts a lot with strands of \( \Sigma \) through suitable points contained in the cone \( C_p(\alpha, H(p)) \) near \( p \), because according to Part (ii) in Definition 1.1 the deviation of the mock tangent planes from \( H(p) \) is also small on these \( \Sigma \)-strands near \( p \). These “almost parallel” sheets of \( \Sigma \) thus generate a certain quantum of energy basically through many almost identical but mutually shifted tangent-point spheres so that the respective conformal angles are large; see Lemma 3.4. If \( \angle(H(p), H(q)) \) is large, on the other hand, then for each mock tangent plane at points near \( q \) there is at least one basis vector that deviates substantially from any basis of \( H(p) \). That basis vector can be used to define a controlled macroscopic shift orthogonal to its projection onto \( H(p) \) to find sufficiently many pairs of tangent-point spheres with a fairly large conformal angle; see Lemma 3.5. It is interesting to note that the explicit estimates in those lemmas reveal that two close-by almost parallel sheets of \( \Sigma \) seem to contribute a lot more energy than transversal sheets close to self-intersection. Such a phenomenon was first observed for the suitably desingularized Möbius energy on immersed planar curves \( (m = 1) \) with self-intersections by R. Dunning [16]; see also the work of D. Kube [34] who derived a limit energy depending only on the angle between two self-intersecting arcs, and which is uniquely minimized by the intersection angle \( \pi/2 \). Similarly, O’Hara observed in [39, Section 3.1.2] different energy contributions to the regularized Riesz energies comparing tangential with orthogonal self-intersections of smooth surfaces \( (m = 2) \).

Combining the bounds of the beta numbers with a uniform estimate on \( \text{dist}(\xi, \Sigma \cap B_r(p)) \) for points \( \xi \in (p + H(p)) \cap B_r(p) \) which can actually be derived for all sets in \( \mathcal{A}^m(\alpha, M) \), one establishes Reifenberg flatness of \( \Sigma \) (Corollary 3.2). This implies by virtue of Reifenberg’s famous topological disk lemma [42, 46, 23] that \( \Sigma \) is a topological manifold locally by-Hölder homeomorphic to the open unit ball in \( \mathbb{R}^m \) as stated in Corollary 3.7. But we do not rely on Reifenberg’s deep result, we can take an easier and more direct route instead to prove the better Lipschitz regularity of local graph representations in Theorem 1.3. For that it suffices to show that the orthogonal projections onto approximating planes restricted to sufficiently small balls are bijective; see Lemmas 3.8 and 3.9. This approach partly inspired by the proof of [14, Proposition 9.1] also leads to the improved \( C^1 \)-regularity of the graph representations in Corollary 1.4 as well as to improved Lipschitz constants as stated in Corollary 3.12.

In Section 4 we estimate the integrand, first assuming \( C^2 \)-smoothness (Lemma 4.1), and then assuming only fractional Sobolev regularity (Lemma 4.3), which proves Theorem 1.5.
In Appendix A we express the angle metric (1.5) in terms of principal angles and demonstrate how our Möbius-invariant energies relate to the ones considered by Kusner and Sullivan, in particular to \(E_{KS}\); see Corollary A.7 In Appendix B we prove various results on general Lipschitz graphs. Of particular and independent interest is Lemma B.5 stating in a quantitative way that the intersection of two \(m\)-dimensional Lipschitz graphs in \(\mathbb{R}^n\) is contained in a lower-dimensional Lipschitz graph as long as the \(m\)-planar domains of the graph functions intersect in an angle that is sufficiently large compared to the Lipschitz constants. This, in some way, generalizes the well-known fact that the intersection of two transversal \(C^1\)-submanifolds is a lower-dimensional \(C^1\)-submanifold, which is usually proven using the implicit function theorem; see, e.g., [21, p. 30].

2. EXAMPLES OF ADMISSIBLE SETS

We already mentioned in the introduction that the class of admissible sets contains immersed compact \(C^1\)-manifolds as well as countable collections of Lipschitz graphs, which we prove now.

2.1. Immerged compact \(C^1\)-manifolds.

**Proposition 2.1.** Let \(\mathcal{M}\) be an \(m\)-dimensional, compact \(C^1\)-manifold and \(f: \mathcal{M} \rightarrow \mathbb{R}^n\) a \(C^1\)-immersion. Then, \(\Sigma := f(\mathcal{M}) \in \mathcal{A}^m(\alpha, M)\) for all \(\alpha > 0\) and all \(M > 0\).

The proof of this proposition is based on the well-known fact that the images of sufficiently small portions of the manifold \(\mathcal{M}\) under \(f\) can be expressed as graph patches with arbitrarily small \(C^1\)-norm; a detailed proof of the following lemma is carried out in [28, Section 4.3].

**Lemma 2.2 (Local graph representation of immersed coordinate patches).** Let \(\mathcal{M}\) be an \(m\)-dimensional \(C^1\)-manifold for \(1 \leq m \leq n\) and \(f \in C^1(\mathcal{M}, \mathbb{R}^n)\) a \(C^1\)-immersion. Then for every \(\beta > 0\) and \(x \in \mathcal{M}\) there exist a radius \(r_x = r_x(\beta, f, \mathcal{M}) > 0\) and a function \(u_x \in C^1(T_x f, T_x f^\perp)\) satisfying \(u_x(0) = 0\), \(Du_x(0) = 0\), and
\[
\|Du_x\|_{C^0} < \beta,
\]
(2.1)

such that
\[
f(\mathcal{U}_{x,r_x}) = (f(x) + \text{graph } u_x) \cap B_{r_x}(f(x)),
\]
(2.2)

where we set \(T_x f := Df(x)(T_x \mathcal{M})\), and
\[
\mathcal{U}_{x,r_x} \subset \mathcal{M}
\]
(2.3)
is defined as the largest connected open subset of the preimage \(f^{-1}(B_r(f(x)))\) containing the point \(x \in \mathcal{M}\).

In order to define \(H: \Sigma \rightarrow \mathcal{G}(n, m)\) when proving Proposition 2.1 we use a covering with these local graph patches. Notice that \(H\) depends on the choice of the covering and of the finite subcovering below, and also on the ordering of the finite index set. Any such choice will lead to an admissible set \(\Sigma = f(\mathcal{M})\). Note that for compact \(\mathcal{M}\) the set \(\Sigma\) is compact, hence closed.
**Proof of Proposition 2.1.** By virtue of the monotonicity (1.8) we may assume $\alpha \leq 1$. Now, fix
\[
\beta := M \alpha \cdot (M+1)^{-1}(\alpha + 1)^{-1}/5
\] (2.4)
and consider the open covering $\mathcal{M} \subset \bigcup_{k \in \mathcal{M}} \mathcal{U}_{x_k,r_k}/4$ for the sets $\mathcal{U}_{x_k,r_k}/4$ as defined in (2.3) with positive radii $r_k = r_k(\beta, f, \mathcal{M})$. The manifold $\mathcal{M}$ is compact, so that there is a finite subcover
\[
\mathcal{M} \subset \bigcup_{i=1}^{N} \mathcal{U}_{x_i,r_i}/4 \subset \mathcal{M}
\] (2.5)
for distinct points $x_1, \ldots, x_N \in \mathcal{M}$ and radii $r_i := r_i(\beta, f, \mathcal{M}) > 0$ for $i = 1, \ldots, N$. Set
\[
R := R(\alpha, M, f, \mathcal{M}) := \min\{r_1, \ldots, r_N\}/4,
\] (2.6)
where we note that $R$ depends on $\alpha$ and $M$ via (2.4). Observe that for any $y \in \mathcal{M}$ there is at least one $k = k_y \in \{1, \ldots, N\}$ such that $y \in \mathcal{U}_{x_k,r_k}/4$ implying $f(y) \in B_{r_k}(f(x_k))$; hence $B_r(f(y)) \subset B_{r_k}(f(x_k))$ for all $r \in (0, 3R]$. Using (2.2) of Lemma 2.2 we therefore find
\[
f(\mathcal{U}_{x_k,r_k}) \cap B_r(f(y)) = (f(x_k) + \text{graph } u_k(x)) \cap B_r(f(y)) \text{ for all } r \in (0, 3R].
\] (2.7)
Define $H : \Sigma \to \mathcal{M}(n, m)$ as $H(p) := T_p(F(x_{i(p)}) + \text{graph } u_{x_{i(p)}})$ for $p \in \Sigma$, where $i(p)$ is the smallest index $i \in \{1, \ldots, N\}$ such that $p \in f(\mathcal{U}_{x_i,r_i}/4)$ which is well-defined by virtue of (2.5). Denote $p_k := f(x_k)$ and $F_k := T_{x_k}F$ and notice that $F_k = T_{p_k}(p_k + \text{graph } u_k)$ for $k = 1, \ldots, N$. Set $\Sigma^* := \Sigma$ and fix any $p \in \Sigma$ and abbreviate $i := i(p) \in \{1, \ldots, N\}$, $u_i := u_{x_{i(p)}} : F_i \to F_i^\perp$, so that by definition $H(p) = T_{p-p_i} \text{graph } u_i$. Now use (2.7) for $y \in \mathcal{U}_{x_i,r_i}/4$ with $f(y) = p$ and for the radius $r = 3R$ to obtain
\[
f(\mathcal{U}_{x_i,r_i}) \cap B_{3R}(p) = (p_i + \text{graph } u_i) \cap B_{3R}(p).
\] (2.8)
In particular, $p = p_i + \xi + u_i(\xi)$ for some $\xi \in F_i$. Any other graph point $q = p_i + \mu + u_i(\mu)$ with $\mu \in F_i$, is contained in the cone $C_p(\beta, F_i)$ since (2.1) implies
\[
|\Pi_{F_i^\perp}(q - p)| = |u_i(\mu) - u_i(\xi)| \leq \beta |\mu - \xi| = \beta |\Pi_{F_i}(q - p)|.
\] (2.9)
The Cone Lemma A.2 applied to $F := F_i$, $G := H(p)$, $\chi := \beta$, $\sigma := \beta$, and $\kappa := \alpha$ implies that any such $q \in p_i + \text{graph } u_i$ is also contained in $C_p(\alpha, H(p))$ since Lemma B.3 guarantees
\[
\hat{\chi}(H(p), F_i) = \hat{\chi}(T_{p-p_i} \text{graph } u_i, T_0 \text{graph } u_i) \leq \|Du_i(\xi) - Du_i(0)\|_{C^0} \leq \beta
\] (2.10)
with $\beta < \frac{\alpha}{5} \leq \frac{1}{5}$ by (2.4) and, therefore, $\frac{\beta + (1+\beta)\beta}{1-(1+\beta)^2} < \frac{1}{19}\alpha$. We deduce from (2.8)
\[
(p_i + \text{graph } u_i) \cap B_{3R}(p) = f(\mathcal{U}_{x_i,r_i}) \cap B_{3R}(p) \subset \Sigma \cap C_p(\alpha, H(p)) \cap B_{3R}(p).
\] (2.11)
Recall that $p = p_i + \xi + u_i(\xi)$ for $\xi \in F_i$, so that we can use the Shifting Lemma B.2 to find the shifted function $\tilde{u} : F_i \to F_i^\perp$ satisfying $\tilde{u}(0) = 0$, Lip $\tilde{u} = \text{Lip } u_i \leq \beta$, and $p_i + \text{graph } u_i = p + \text{graph } \tilde{u}$. Therefore, (2.11) yields
\[
(p_i + \text{graph } u_i) \cap B_{3R}(p) = (p + \text{graph } \tilde{u}) \cap B_{3R}(p) \subset \Sigma \cap C_p(\alpha, H(p)) \cap B_{3R}(p).
\] (2.12)
Applying the Tilting Lemma \(B.3\) to \(u := \tilde{u}, F := F_i, G := H(p)\) with \(\tilde{\lambda}(H(p), F_i) \leq \beta\) by \((2.10)\), so \(\gamma := \beta\) and \(\sigma = \chi(1 + \text{Lip } u) \leq \beta(1 + \beta) < 1\), we find

\[
B_{\frac{1 - \beta(1 + \beta)}{\sqrt{1 + \beta^2}}}(0) \cap H(p) \subset \Pi_{|H(p)}(\text{graph } \tilde{u} \cap B_\beta(0)) \quad \text{for all } q > 0. \tag{2.13}
\]

For the set \(D_p := B_{\frac{1 - \beta(1 + \beta)}{\sqrt{1 + \beta^2}}}(p) \cap (p + H(p))\) we obtain in particular, by \((2.12)\)

\[
D_p \subset \Pi_{p + H(p)}((p_i + \text{graph } u_i) \cap B_R(p)) \subset \Pi_{p + H(p)}(\Sigma \cap C_p(\alpha, H(p))) , \tag{2.14}
\]

so that we can choose a uniform radius

\[
R_K = R_\Sigma(\alpha, M, f, \mathcal{M}) := \frac{1 - \beta(1 + \beta)}{\sqrt{1 + \beta^2}} R \quad \text{for all compact } K \subset \Sigma , \tag{2.15}
\]

where \(R\) is defined as in \((2.6)\) and \(\beta\) as in \((2.4)\). In particular, for all \(x \in D_p\) there exists a point

\[
\eta_x \in (p_i + \text{graph } u_i) \cap B_R(p) \subset \Sigma \cap C_p(\alpha, H(p)) \quad \text{with } \Pi_{p + H(p)}(\eta_x) = x . \tag{2.16}
\]

Let \(\mathcal{L} \subset \{1, \ldots, N\}\) be the set of indices \(l\) such that \(\tilde{\lambda}(T_0 \text{graph } u_i, T_0 \text{graph } u_i) \geq M\alpha/2\). Notice, of course, \(i \notin \mathcal{L}\). Then, by Lemma \(B.1\) and \((2.4)\), one has

\[
\tilde{\lambda}(T_{q-p_i} u_i, T_{q-p_i} \text{graph } u_i) = \tilde{\lambda}(T_0 \text{graph } u_i, T_{q-p_i} \text{graph } u_i) 
- \tilde{\lambda}(T_0 \text{graph } u_i, T_{q-p_i} \text{graph } u_i) \geq M\alpha/2 - \|D_0 u_i(0) - D_0 u_i(\mu)\|_{C^0} - \|D_0 u_i(0) - D_0 u_i(\xi)\|_{C^0} \geq M\alpha/2 - 2\beta > 0 , \tag{2.17}
\]

for all \(l \in \mathcal{L}\) and all points \(q = p_1 + \mu + u_i(\mu) = p_i + \xi + u_i(\xi)\) contained in the intersection \(S_l := (p_i + \text{graph } u_i) \cap (p_i + \text{graph } u_i)\), which means that the two \(C^1\)-graphs intersect transversally, so that \(\dim \mathcal{H}(S_l) \leq m - 1\) for all \(l \in \mathcal{L}\). In particular, for any \(x \in D_p\) with \(\eta_x\) as in \((2.16)\), one finds for all \(r > 0\)

\[
\mathcal{H}^m \left( \left[ (p_i + \text{graph } u_i) \cap B_r(\eta_x) \right] \setminus \bigcup_{l \in \mathcal{L}} (p_i + \text{graph } u_i) \right) = \mathcal{H}^m \left( (p_i + \text{graph } u_i) \cap B_r(\eta_x) \right) . \tag{2.18}
\]

Notice, finally, that all points \(q \in \left( (p_i + \text{graph } u_i) \setminus \bigcup_{l \in \mathcal{L}} (p_i + \text{graph } u_i) \right) \cap B_{3R}\) are contained in the set \(E_{\alpha, M}(p)\) introduced in Definition \(1.1\). Indeed, by \((2.12)\) all such points \(q\) are contained in \(\Sigma\) and, by definition, the \(m\)-plane \(H(q)\) is contained in the set \(\{T_{q-p_i} \text{graph } u_k : k \in \{1, \ldots, N\} \setminus \mathcal{L}\}\), so that there is a \(k = k(q) \in \{1, \ldots, N\} \setminus \mathcal{L}\) with

\[
\tilde{\lambda}(H(q), H(p)) \leq \tilde{\lambda}(H(q), T_0 \text{graph } u_k, T_0 \text{graph } u_k) + \tilde{\lambda}(T_0 \text{graph } u_k, H(p))
+ \tilde{\lambda}(T_0 \text{graph } u_k, H(p)) < \|Du_k(\eta) - Du_k(0)\|_{C^0} + M\alpha/2 + \|Du_k(0) - Du_k(\xi)\|_{C^0} \leq 2\beta + M\alpha/2 < M\alpha ,
\]

where we wrote \(q = p_k + \mu + u_k(\mu)\) for some \(\mu \in F_k\), and as before, \(p = p_i + \xi + u_i(\xi)\) for \(\xi \in F_i\), and used Lemma \(B.1\) and \((2.4)\). Consequently,

\[
\left[ (p_i + \text{graph } u_i) \setminus \bigcup_{l \in \mathcal{L}} (p_i + \text{graph } u_i) \right] \cap B_{3R}(p) \subset E_{\alpha, M}(p) \cap B_{3R}(p) . \tag{2.19}
\]

Now, for \(x \in D_p = (p + H(p)) \cap B_{R_k}(p)\) with \(\eta_x\) as in \((2.16)\) we estimate

\[
|\eta_x - p|^2 = |\Pi_{H(p)}(\eta_x - p)|^2 + |\Pi_{H(p)}(\eta_x - p)|^2 \leq (1 + \alpha^2)|x - p|^2 ,
\]
so that $|\eta_x - p| < \sqrt{1 + \alpha^2} R_K < 2R_K < 2R$, since we assumed $\alpha \leq 1$. This implies, by definition of $R_K$ in (2.6),

$$B_r(\eta_x) \subset B_{3R}(p) \text{ for all } r \in (0, R_K].$$

Combining (2.19), (2.18), and (2.20) we arrive at

$$\mathcal{H}^m(E_{\alpha,M}(p) \cap B_r(\eta_x)) \geq \mathcal{H}^m\left([ (p_i + \text{graph } u_i) \setminus \bigcup_{i \in \mathbb{N}} (p_i + \text{graph } u_i) \right] \cap B_r(\eta_x))$$

$$= \mathcal{H}^m\left( (p_i + \text{graph } u_i) \cap B_r(\eta_x) \right) \geq \mathcal{H}^m\left( \Pi_{F_i}((p_i + \text{graph } u_i) \cap B_r(\eta_x)) \right)$$

$$\geq \omega_m(r/\sqrt{1 + \beta^2})^m$$

for all $r \in (0, R_K]$.

Notice that we used (2.16), i.e., $\eta_x = p_i + x + u_i(x)$, so that any other point $q = p_i + \mu + u_i(\mu) \in p_i + \text{graph } u_i$ with $|\mu - x| < r/\sqrt{1 + \beta^2}$ satisfies $q \in B_r(\eta_x)$, which implies $B_r/\sqrt{1 + \beta^2}(x) \cap (p_i + F_i) \subset \Pi_{F_i}(B_r(\eta_x) \cap (p_i + \text{graph } u_i))$. \hfill \qed

2.2. Countable unions of Lipschitz graphs. The following considerations can also be localized to study countable unions of pieces of Lipschitz graphs using similar arguments as in Section 2.1. For simplicity we restrict here to collections of entire Lipschitz graphs.

**Proposition 2.3.** Suppose $\Sigma = \bigcup_{i \in \mathbb{N}} (p_i + \text{graph } u_i)$, where $u_i \in C^{0,1}(F_i, F_i^\perp)$, $F_i \in \mathcal{G}(n, m)$, $u_i(0) = 0$, and $\text{Lip } u_i \leq \beta$ for all $i \in \mathbb{N}$. If

$$0 \leq \beta \leq M\alpha/16(M + 1)$$

(2.21) for given $\alpha \in (0, 1)$ and $M > 0$, then one finds $\Sigma \in \mathcal{A}^m(\alpha, M)$.

Notice that the monotonicity property (1.8) implies that $\Sigma \in \mathcal{A}^m(\alpha, M)$ for all $\alpha, M > 0$ as long as $\beta$ satisfies (2.21) with $\alpha$ replaced by some constant $\tilde{\alpha} < \min\{\alpha, 1\}$.

**Proof.** Fix an arbitrary $m$-plane $F_0 \in \mathcal{G}(n, m)$ as a “dummy plane”. Notice that for every point $p$ contained in the union $\bigcup_{i \in \mathbb{N}} (p_i + \text{graph } u_i)$ there exists a unique smallest index $i(p) \in \mathbb{N}$, such that $p \in p_{i(p)} + \text{graph } u_{i(p)}$, that is, for every $1 \leq j < i(p)$ with $i(p) > 1$ one has $p \notin p_j + \text{graph } u_j$. Now define the map $H : \Sigma \to \mathcal{G}(n, m)$ by setting $H(p) := T_{p_{i(p)}}(p_{i(p)} + \text{graph } u_{i(p)})$ if $p = p_{i(p)} + \xi + u_{i(p)}(\xi)$ for some $\xi \in F_i$ such that $D_{u_{i(p)}}(\xi)$ exists. In all other cases set $H(p) := F_0$, which happens either if $D_{u_{i(p)}}(\xi)$ does not exist or if $p$ is not contained in any of the graphs $p_i + \text{graph } u_i$.

Let $\Sigma_* := \{ p \in \Sigma : p = p_{i(p)} + \xi + u_{i(p)}(\xi) \text{ and } D_{u_{i(p)}}(\xi) \text{ exists} \}$. Notice that $\Sigma$ is closed by definition, and that $\Sigma_* \subset \Sigma$ is dense, since for any $\varepsilon > 0$ and any $q \in \Sigma$, there is a point $q_\varepsilon \in \bigcup_{i = 1}^\infty (p_i + \text{graph } u_i)$, such that $|q - q_\varepsilon| < \varepsilon/2$. For $q_\varepsilon$ there exists $i_\varepsilon := i(q_\varepsilon)$ such that $q_\varepsilon = p_{i_\varepsilon} + x + u_{i_\varepsilon}(x)$ for some $x \in F_{i_\varepsilon}$. By Rademacher’s Theorem applied to the Lipschitz function $u_{i_\varepsilon}$ there exists $\xi_\varepsilon \in F_{i_\varepsilon}$ such that $D_{u_{i_\varepsilon}}(\xi_\varepsilon)$ exists and $|x - \xi_\varepsilon| < \varepsilon/\sqrt{2 + \beta^2}$ so that the corresponding graph point $\tilde{q}_\varepsilon := p_{i_\varepsilon} + \xi_\varepsilon + u_{i_\varepsilon}(\xi_\varepsilon)$ satisfies $|q_\varepsilon - \tilde{q}_\varepsilon| < \varepsilon/\sqrt{2 + \beta^2}$. We may assume that $i_\varepsilon = i(\tilde{q}_\varepsilon)$, since for every point $\xi \in F_{i_\varepsilon}$ with $|x_\varepsilon - \xi| < \varepsilon/\sqrt{2 + \beta^2}$ such that $D_{u_{i_\varepsilon}}(\xi)$ exists, the corresponding graph point $\sigma := p_{i_\varepsilon} + \xi + u_{i_\varepsilon}(\xi)$ had smallest index $i(\sigma) < i_\varepsilon$, we could select a sequence $\xi_k \to x_\varepsilon$ with graph points $\sigma_k := p_{i_\varepsilon} + \xi_k + u_{i_\varepsilon}(\xi_k)$ satisfying $i(\sigma_k) = j$ for some fixed $1 \leq j < i_\varepsilon$. But this would imply
implies for all $\varrho > 0$.

To check the remaining conditions of Definition 1.1 fix $p \in \Sigma^*$ and set $i := i(p)$. Then $p = p_i + \xi + u_i(\xi)$ for some $\xi \in F_i$, and we can use the Shifting Lemma B.2 to find the shifted function $\tilde{u} \in C^{0,1}(F_i, F_i^\perp)$ satisfying

$$p_i + \text{graph } u_i = p + \text{graph } \tilde{u} \quad \text{with } \tilde{u}(0) = 0 \quad \text{and } \Lip \tilde{u} = \Lip u_i \leq \beta.$$  \hspace{1cm} (2.22)

In particular, similarly as in (2.9)

$$\text{graph } \tilde{u} \cap B_\varrho(0) \subset C_0(\beta, F_i) \quad \text{for all } \varrho > 0.$$  \hspace{1cm} (2.23)

Now, $\hat{\varrho}(H(p), F_i) = \hat{\varrho}(T_{p-p_i} \text{graph } u_i, F_i) \leq ||Du_i(\xi)|| \leq \beta$, by Lemma B.1. The Cone Lemma A.2 applied to $F := F_i, G := H(p), \chi = \sigma := \beta$, and $\kappa := \alpha$, where condition (A.1) is satisfied due to (2.21), implies $\text{graph } \tilde{u} \cap B_\varrho(0) \subset C_0(\alpha, H(p))$, and therefore,

$$(p_i + \text{graph } u_i) \cap B_\varrho(p) \supset (p + \text{graph } \tilde{u}) \cap B_\varrho(p) \subset C_p(\alpha, H(p)).$$  \hspace{1cm} (2.24)

The Tilting Lemma B.3 applied to $F := F_i, G := H(p), \hat{\varrho}(F, G) \leq \beta \equiv \chi, u := \tilde{u}$ with $\Lip u \leq \beta$ satisfying $\sigma := \chi(1 + \Lip u) \leq \beta(1 + \beta) < 1$, implies

$$B_{1/(\beta + 1)} \subset \varrho \subset H(p) \subset \Pi_{H(p)}(\text{graph } \tilde{u} \cap B_\varrho(0)) \quad \text{for all } \varrho > 0.$$  \hspace{1cm} (2.25)

Define for arbitrary $\varrho > 0$ the flat $m$-dimensional disks $D^\varrho_p := B_{1/(\beta + 1)}(\varrho) \cap (p + H(p))$ so that

$$D^\varrho_p \subset \Pi_{p + H(p)}(p + \text{graph } \tilde{u}) \cap B_\varrho(p) \subset \Pi_{p + H(p)}(\Sigma \cap C_p(\alpha, H(p))),$$  \hspace{1cm} (2.26)

since $C_p(\alpha, H(p))$ is closed. Now let $\mathcal{L} \subset \mathbb{N}$ be the set of indices $l \in \mathbb{N}$, such that

$$\hat{\varrho}(F_i, F_i) \geq M\alpha/2.$$  \hspace{1cm} (2.27)

For any fixed $l \in \mathcal{L}$ consider the graph $p_l + \text{graph } u_l$. If there exists a point $q \in (p_l + \text{graph } u_l) \cap (p_i + \text{graph } u_i)$ use the Shifting Lemma B.2 to find the shifted functions $\tilde{u}_l \in C^{0,1}(F_i, F_i^\perp)$ and $\tilde{u}_i \in C^{0,1}(F_i, F_i^\perp)$ with $\tilde{u}_l(0) = 0, \Lip \tilde{u}_l \leq \beta$, and $\tilde{u}_i(0) = 0, \Lip \tilde{u}_i \leq \beta$, satisfying as in (2.22)

$$p_l + \text{graph } u_l = q + \text{graph } \tilde{u}_l \quad \text{and } p_i + \text{graph } u_l = q + \text{graph } \tilde{u}_i.$$  \hspace{1cm} (2.28)

We can now use (2.21) and (2.26) to find $\hat{\varrho}(F_i, F_i) \geq M\alpha/2 > 8\beta$, which allows us to apply the Lemma of Intersecting Lipschitz Graphs B.5 for $\sigma := \beta$ and $\chi := M\alpha/2$ to conclude by means of (2.27)

$$\dim_{\mathcal{L}}((p_i + \text{graph } u_i) \cap (p_l + \text{graph } u_l)) = \dim_{\mathcal{L}}((q + \text{graph } \tilde{u}_l) \cap (q + \text{graph } \tilde{u}_i))$$  \hspace{1cm} (2.29)

which is true, of course, also if $(p_l + \text{graph } u_l) \cap (p_i + \text{graph } u_i) = \emptyset$, so that (2.28) holds for all $l \in \mathcal{L}$. If $k \in \mathbb{N} \setminus \mathcal{L}$, on the other hand, we have for any point $q \in p_k + \text{graph } u_k$, due to (2.21) and Lemma B.1

$$\hat{\varrho}(T_{q-p_i} \text{graph } u_k, H(p)) \leq \hat{\varrho}(F_k, F_i) + 2\beta < M\alpha,$$

which implies for all $\varrho > 0$

$$B_\varrho(p) \cap ( (p_i + \text{graph } u_i) \setminus \bigcup_{l \in \mathcal{L}} (p_l + \text{graph } u_l) ) \subset B_\varrho(p) \cap E_{\alpha, M}(p).$$  \hspace{1cm} (2.29)
For all $x \in D_{\beta}^{\theta} = B_{1/(\beta+1)}(x) \cap (p + H(p))$ there is by virtue of (2.25) and (2.24) a point $\eta_x \in (p_i + \text{graph } u_i) \cap B_{\theta/2}(p) \subset \Sigma \cap C_p(\alpha, H(p))$, such that $\Pi_{p + H(p)}(\eta_x) = x$, and $|\eta_x - p| < \sqrt{1 + \alpha^2} \theta/2 < \theta$, since $\alpha < 1$. This implies $B_r(\eta_x) \subset B_{2\theta}(p)$ for all $r \in (0, \theta]$, which combined with (2.29) and (2.28) yields

$\mathcal{H}^m(E_{\alpha, M}(p) \cap B_r(\eta_x)) \geq \mathcal{H}^m((\{ (p_i + \text{graph } u_i) \} \cap B_r(\eta_x)) \geq \mathcal{H}^m(\Pi_F((p_i + \text{graph } u_i) \cap B_r(\eta_x))) \geq \mathcal{H}^m(B_{\sqrt{1 + \beta^2}/r}(0) \cap F_i) = \omega_m(r/\sqrt{1 + \beta^2})^m \quad \text{for all} \quad r \in (0, \theta], \theta > 0.$

□

3. Finite energy sets are manifolds

3.1. Good approximating planes for admissible sets. In addition to the $\beta$-numbers (1.13) measuring the local flatness of a set we introduce here the corresponding $\beta$-number with respect to a fixed plane, as well as the corresponding bilateral flatness parameter $\theta$.

**Definition 3.1.** For $p \in \Sigma \subset \mathbb{R}^m$, an $m$-plane $F \in \mathcal{F}(n, m)$, and a radius $r > 0$, the $\beta$- and $\theta$-numbers of $\Sigma$ with respect to $F$ are given by

$$\beta_\Sigma(p, F, r) := \sup_{y \in \Sigma \cap B_r(p)} \frac{\text{dist}(y, (p + F) \cap B_r(p))}{r}, \quad (3.1)$$

$$\theta_\Sigma(p, F, r) := \text{dist}_{\mathcal{F}}(\Sigma \cap B_r(p), (p + F) \cap B_r(p))/r. \quad (3.2)$$

We say $\Sigma$ is an $(m, \delta)$-Reifenberg-flat set if for all compact subsets $K \subset \Sigma$ there is a radius $r_0 = r_0(K) > 0$, such that for all $p \in K$ and $r \in (0, r_0(K))$, there exists a plane $F_p(r, \delta) \in \mathcal{F}(n, m)$ with $\theta_\Sigma(p, F_p(r, \delta), r) \leq \delta$. Minimizing over all $m$-planes one obtains analogously to (1.13) the $\theta$-number

$$\theta_\Sigma(p, r) := \inf_{F \in \mathcal{F}(n, m)} \frac{\theta_\Sigma(p, F, r)}{r}. \quad (3.3)$$

Recall that the Hausdorff-distance of two sets $A, B \subset \mathbb{R}^n$ is given by $\text{dist}_{\mathcal{H}}(A, B) := \max\{\text{sup}_{a \in A} \text{dist}(a, B), \text{sup}_{b \in B} \text{dist}(b, A)\}$ so that

$$\theta_\Sigma(p, F, r) = \max\{\beta_\Sigma(p, F, r), \sup_{\xi \in (p + F) \cap B_r(p)} \text{dist}(\xi, \Sigma \cap B_r(p))/r\}. \quad (3.4)$$

The second term of the right hand side of (3.4) can be bounded uniformly for any admissible set in $\mathcal{F}^m(\alpha, M)$.

**Lemma 3.2.** Let $\delta \in (0, 1)$ and $\Sigma \in \mathcal{F}^m(\alpha, M)$ for $1 \leq m \leq n$, $M > 0$, and $\alpha > 0$. For all compact subsets $K \subset \Sigma$, there exists a radius $\theta_K \in (0, \theta_K]$, such that for all $p \in K \cap \Sigma^*$, we have

$$\sup_{\xi \in (p + H(p)) \cap B_r(p)} \text{dist}(\xi, \Sigma \cap B_r(p))/r < 2\alpha/\sqrt{1 + \alpha^2} \quad \text{for all} \quad r \in (0, \theta_K].$$
Moreover, for \( q \in \mathbb{K} \setminus \Sigma^* \) and any sequence \((p_i)_{i \in \mathbb{N}} \subset \Sigma^* \) with \( \lim_{i \to \infty} H(q) = F \in \mathcal{G}(n, m) \) exists, one also finds

\[
\sup_{\xi \in (q + F) \cap B_r(q)} \operatorname{dist}(\xi, \Sigma \cap B_r(p)) / r \leq 2\alpha / \sqrt{1 + \alpha^2} \quad \text{for all } r \in (0, qK].
\]

Notice that for every \( q \in \mathbb{K} \setminus \Sigma^* \) there is a sequence \((p_i)_{i}\), as stated in the last part of Lemma 3.2 by density of \( \Sigma^* \) in \( \Sigma \) combined with the compactness of the Grassmannian \( \mathcal{G}(n, m) \).

Proof. First, assume \( p \in \Sigma^* \cap B_{R_K/10}(K) \) and define \( \tilde{K} := \Sigma \cap \overline{B_{R_K/10}(K)} \). Then \( \tilde{K} \) is compact, since \( \Sigma \) is closed, and the constants \( R_K \in (0, R_K] \) and \( c_K \in (0, c_K] \) of Definition 1.1 applied to \( \tilde{K} \) are solely determined by \( K \) itself. So, there exists a dense subset \( D_p \subset (p + H(p)) \cap B_{R_K}(p) \) such that for all \( x \in D_p \) one finds a point \( \eta_x \in \Sigma \cap C_p(\alpha, H(p)) \) with

\[
x = \Pi_{p + H(p)}(\eta_x) = p + \Pi_{H(p)}(\eta_x - p),
\]

so that

\[
\eta_x - p = \Pi_{H(p)}(\eta_x - p) + \Pi_{H(p)}(\eta_x - p) = x - p + \Pi_{H(p)}(\eta_x - p),
\]

or \( \eta_x - x = \Pi_{H(p)}(\eta_x - p) \), which implies by the fact that \( \eta_x \in C_p(\alpha, H(p)) \) and by (3.5)

\[
|\eta_x - x| \leq \alpha \|\Pi_{H(p)}(\eta_x - p)\| = \alpha \|x - p\|.
\]

In particular, setting \( q_K := R_K \leq R_K / 10 \), we obtain for \( r \in (0, q_K] \) and \( x \in D_p \cap B_{r/\sqrt{1 + \alpha^2}}(p) \)

\[
|\eta_x - x| < \alpha r / \sqrt{1 + \alpha^2}
\]

by means of (3.7), and also \( |\eta_x - p|^2 = |x - p|^2 + |\eta_x - x|^2 \leq (1 + \alpha^2)|x - p|^2 \), so that \( |\eta_x - p| < r \). Combined with (3.8) one finds

\[
\operatorname{dist}(x, \Sigma \cap B_r(p)) \leq |\eta_x - x| < \alpha r / \sqrt{1 + \alpha^2} \quad \text{for all } x \in D_p \cap B_{r/\sqrt{1 + \alpha^2}}(p), \ r \in (0, q_K].
\]

By density of \( D_p \) in \( (p + H(p)) \cap B_{R_K}(p) \), we find for each \( \xi \in (p + H(p)) \cap B_{r/\sqrt{1 + \alpha^2}}(p) \) and each \( l \in \mathbb{N} \) some point \( x_l = x_l(\xi) \in D_p \cap B_{r/\sqrt{1 + \alpha^2}}(p) \), such that \( |\xi - x_l| < 1/l \), and such that there is a point \( \eta_{x_l} \in \Sigma \cap C_p(\alpha, H(p)) \cap B_r(p) \) with \( \Pi_{p + H(p)}(\eta_{x_l}) = x_l \). Consequently, by means of (3.8) with \( x \) replaced by \( x_l \), one obtains

\[
\operatorname{dist}(\xi, \Sigma \cap B_r(p)) \leq |\xi - x_{l}| < 1/l + |x_l - \eta_{x_l}| < 1/l + \alpha r / \sqrt{1 + \alpha^2}
\]

for all \( l \in \mathbb{N} \).

Taking the limit \( l \to \infty \), we find

\[
\operatorname{dist}(\xi, \Sigma \cap B_r(p)) \leq \alpha r / \sqrt{1 + \alpha^2} \quad \text{for all } \xi \in (p + H(p)) \cap B_{r/\sqrt{1 + \alpha^2}}(p).
\]

For \( \mu \in p + H(p) \) with \( r / \sqrt{1 + \alpha^2} < |\mu - p| < r \), we set \( \xi_\mu := \frac{r}{\sqrt{1 + \alpha^2}} \cdot \frac{\mu - p}{|\mu - p|} \in (p + H(p)) \cap B_{r/\sqrt{1 + \alpha^2}}(p) \), and estimate

\[
\operatorname{dist}(\mu, \Sigma \cap B_r(p)) \leq \operatorname{dist}(\xi_\mu, \Sigma \cap B_r(p)) + |\mu - \xi_\mu| < \frac{\alpha r}{\sqrt{1 + \alpha^2}} + \left(1 - \frac{1}{\sqrt{1 + \alpha^2}}\right) r.
\]
\[ \left( \alpha + \sqrt{1 + \alpha^2} - 1 \right) r < \left( \alpha + (1 + \alpha) - 1 \right) r = \frac{2\alpha r}{\sqrt{1 + \alpha^2}}, \]  

so that the combination of (3.9) and (3.10) gives

\[ \text{dist} (\mu, \Sigma \cap B_r(p)) < 2\alpha r/\sqrt{1 + \alpha^2} \text{ for all } \mu \in (p + H(p)) \cap B_r(p), \ r \in (0, q_K] \]  

(3.11)

if \( p \in \Sigma^* \cap B_{R_K/10}(K) \). Now assume \( q \in K \setminus \Sigma^* \) and take a sequence \( (p_i) \in N \subset \Sigma^* \) with \( \lim_{i \to \infty} p_i = q \) and \( \lim_{i \to \infty} H(p_i) =: F \in \mathcal{G}(n, m) \). Thus, \( p_i \in \Sigma^* \cap B_{R_K/10}(K) \) for all \( i \gg 1 \). We claim that

\[ \sup_{x \in (q + F) \cap B_{r}(q)} \text{dist} (x, \Sigma \cap B_r(q)) \leq 2\alpha r/\sqrt{1 + \alpha^2} \text{ for all } r \in (0, q_K]. \]  

(3.12)

Indeed, by virtue of the Hausdorff convergence of the \( m \)-planar disks \( (p_i + H(p_i)) \cap B_r(p_i) \) to the closed disk \( (q + F) \cap B_r(q) \) as \( i \to \infty \) we can find for any \( x \in (q + F) \cap B_r(q) \) a sequence \( x_l \to x \) as \( l \to \infty \) with \( x_l \in (p_i + H(p_i)) \cap B_r(p_i) \), and by (3.11) applied to \( p \equiv p_i \) one finds points \( \mu_l \in \Sigma \cap B_r(p_l) \), such that \( |x_l - \mu_l| < 2\alpha r/\sqrt{1 + \alpha^2} \) for all \( l \in \mathbb{N} \). Since \( \Sigma \) is closed and by the Hausdorff-convergence we may assume \( \mu_l \to \mu \in \Sigma \cap B_r(q) \) as \( l \to \infty \). Hence, \( |x - \mu| \leq 2\alpha r/\sqrt{1 + \alpha^2} \), which implies (3.12).

Finally, let \( r \in (0, q_K] \). For arbitrary \( \xi \in (q + F) \cap B_r(q) \) use (3.12) to estimate

\[ \text{dist} (\xi, \Sigma \cap B_r(q)) \leq \sup_{x \in (q + F) \cap B_{r}(q)} \text{dist} (x, \Sigma \cap B_r(q)) \]

\[ \leq \sup_{x \in (q + F) \cap B_{r}(q)} \text{dist} (x, \Sigma \cap B_{r}(q)) \leq \frac{2\alpha}{\sqrt{1 + \alpha^2}} |\xi - q| < \frac{2\alpha r}{\sqrt{1 + \alpha^2}}. \]

\( \square \)

### 3.2. Finite energy yields Reifenberg flatness

We have seen in Lemma 3.2 that the second term in bilateral \( \theta \)-number (3.4) is automatically controlled for admissible sets in \( \mathcal{A}^m(\alpha, M) \), but in order to control the first term, the \( \beta \)-number, we need locally finite energy. To begin with, notice that the numerator \( L_r(x, y, H(x), H(y)) := \mathcal{H}(H(x), R_{xy}(H(y)))^{(1+\gamma)m} \) of our Lagrangian in (1.10) involving the angle metric defined in (1.5) can be rewritten as \( L_r(x, y, H(x), H(y)) = \sup_{\epsilon \in H(x) \cap \mathcal{S}^{n-1}} F_r(x, y, \epsilon) \) with

\[ F_r(x, y, \epsilon) := |\Pi_{H(y)}(\epsilon)|^2 (e, x - y) \Pi_{H(y)}(x - y)/|x - y|^2 |(1 + \tau)|^m, \]  

(3.13)

by means of the explicit formula (1.9) for the reflection \( R_{xy} \).

**Theorem 3.3** (\( \beta \)-number estimate). Let \( \delta \in (0, 1), \Sigma \in \mathcal{A}^m(\alpha, M) \) for \( 2 \leq m \leq n, M > 0, \) and \( \alpha > 0 \) satisfy

\[ \alpha(M + 1) < \delta/50 \]  

(3.14)

and assume that \( \Sigma \) has locally finite energy \( E^\tau \) as in (1.11) for some \( \tau > -1 \). Then, for all compact subsets \( K \subset \Sigma \) there exists a radius \( r_K = r_K(\delta, \tau, m, K) \in (0, R_K] \), such that for all \( p \in K \) there is an \( m \)-plane \( G_p \in \mathcal{G}(n, m) \) with

\[ \Sigma \cap B_r(p) \setminus B_{r\delta r}(p + G_p) = \emptyset \text{ for all } r \in (0, r_K]. \]  

(3.15)
In particular, \( \sup_{p \in K} \beta_\Sigma(p, G_p, r) \leq \delta \) for all \( r \in (0, r_K] \). Moreover, \( G_p = H(p) \) for all \( p \in \Sigma^* \), and for \( q \in \Sigma \setminus \Sigma^* \), there exists a sequence \( (p_i)_{i \in \mathbb{N}} \subset \Sigma^* \) with \( \lim_{i \to \infty} p_i = q \) and \( G_q = \lim_{i \to \infty} H(p_i) \).

**Proof. Step 1.** By density it suffices to find for a given compact set \( K \subset \Sigma \) a radius \( r_K \in (0, R_K] \) such that

\[
\Sigma \cap B_r(p) \setminus B_{\delta r}(p + H(p)) = \emptyset \quad \text{for all} \quad p \in \Sigma^* \cap B_{R_K/(10+R_K)}(K), \quad r \in (0, r_K],
\]

(3.16)
since an arbitrary point \( p \in K \) may be approximated by points \( p_l \in \Sigma^* \cap B_{R_K(10+R_K)}(K) \), for which we may assume w.l.o.g. that there exist \( H(p_l) \in \mathcal{G}(n, m) \), such that \( H(p_l) \) converges to \( G_p \in \mathcal{G}(n, m) \) as \( l \to \infty \) by compactness of the Grassmannian. The Hausdorff-convergence of the balls \( B_r(p_l) \) to \( B_r(p) \) and of the \( m \)-planar disks \( B_{R_K}(p_l) \setminus (p_l + H(p_l)) \) to \( B_{R_K}(p) \cap (p + G_p) \) as \( l \to \infty \) implies that any point \( q \in \Sigma \cap B_r(p) \setminus B_{\delta r}(p + G_p) \) satisfies \( q \in \Sigma \cap B_r(p_l) \) for \( l \gg 1 \), as well as \( \text{dist}(q, p_l + H(p_l)) \geq \text{dist}(q, p + G_p) - \text{dist}_{\mathcal{G}}((p + G_p) \cap B_{R_K}(p), (p_l + H(p_l)) \cap B_{R_K}(p_l)) > \delta r \) for all \( l \gg 1 \), thus contradicting (3.16) for \( p \equiv p_l \) and \( l \gg 1 \). Before moving on to the second step notice that all \( p_l \) are contained in the compact set \( \tilde{K} := \Sigma \cap \overline{B_1(K)} \), so that the constants \( R_{\tilde{K}} \) and \( c_{\tilde{K}} \) of Definition 1.1 applied to the compact set \( \tilde{K} \) satisfy \( 0 < R_{\tilde{K}} \leq R_K \) and \( 0 < c_{\tilde{K}} \leq c_K \). However, since the original compact set \( K \) completely determines \( \tilde{K} \) we think of \( R_{\tilde{K}} \) and \( c_{\tilde{K}} \) as depending on \( K \) only.

**Step 2.** Assuming for contradiction that there is a point \( p \in \Sigma^* \cap B_{R_K/(10+R_K)}(K) \) such that (3.16) does not hold for some \( r \in (0, r_K] \) for a constant \( r_K \in (0, R_K] \) to be determined later, then there is some point \( \tilde{q} \in \Sigma \cap B_r(p) \) with \( \text{dist}(\tilde{q}, p + H(p)) > \delta r \). By density of \( \Sigma^* \) in \( \Sigma \) we find a point \( q \in \Sigma^* \cap B_r(p) \) such that

\[
\text{dist}(q, p + H(p)) > \delta r.
\]

(3.17)

We quantify the energy contribution of a geometric situation like in (3.17) in the following two lemmas, depending on the size of the angle \( \phi(H(p), H(q)) \). The proofs of these auxiliary lemmas are postponed to the end of this subsection.

**Lemma 3.4 (Almost parallel strands).** Let \( \tau > -1, \delta \in (0, 1), \varepsilon \in (0, \delta/500), \) and \( \Sigma \in \mathcal{A}^m(\alpha, M) \) for \( 1 \leq m \leq n \), where the constants \( M, \alpha > 0 \) satisfy (3.14). Assume that there is a compact subset \( K \subset \Sigma \) and points \( p \in \Sigma^* \cap B_{R_K/10}(K) \) and \( q \in \Sigma^* \cap B_{\varepsilon R}(p) \) for some radius \( R \in (0, \min\{R_{\tilde{K}}, 1\}] \) for \( \tilde{K} = \Sigma \cap \overline{B_1(K)} \), such that

\[
\text{dist}(q, p + H(p)) > \delta \varepsilon R,
\]

\[
\phi(H(p), H(q)) < \omega(\delta) + 2M\alpha \quad \text{for} \quad \omega(\delta) := 153 \cdot \delta/50^3.
\]

(3.18)

(3.19)

Then,

\[
\int_{\Sigma \cap B_{\tau R}(q)} \int_{\Sigma \cap B_{\tau R}(p)} \frac{L_\tau(\mu, \eta, H(\mu), H(\eta))}{|\mu - \eta|^{2m}} d\mathcal{H}^m(\eta) d\mathcal{H}^m(\mu) > c_1
\]

(3.20)

where \( c_1 = c_1(K, \varepsilon, \delta, \tau, m) := c_1^2(1 + \tau)^m \cdot \frac{1}{255^{2m}} \cdot \frac{\delta^{(1+\tau)m}}{10^{2(1-\tau)m}} > 0. \)
Lemma 3.5 (Transversal strands). Suppose in addition to $m \geq 2$ that all assumptions of Lemma 3.4 hold true except 3.19 then
\[
\int_{\Sigma \cap B_{\varepsilon}(q)} \int_{\Sigma \cap B_{\varepsilon}(p)} \frac{L_r(\mu, \eta, H(\mu), H(\eta))}{|\mu - \eta|^{2m}} \, d\mathcal{H}^m(\eta) \, d\mathcal{H}^m(\mu) > c_2, \tag{3.21}
\]
where $c_2 = c_2(K, \varepsilon, \delta, \tau, m) := c_2^2 \cdot c^4 \cdot (1.9)^{(3+\tau)m} \cdot \delta^{(1+\tau)m} > 0$.

To apply these lemmas, we fix $\varepsilon_0 = \varepsilon_0(\delta) := \frac{\delta}{500}$ and obtain
\[
c_2 < c_1 \quad \text{for all } \tau > -1. \tag{3.22}
\]

Step 3. In order to deduce from (3.17) a contradiction notice first that there is some $N = N(K) \in \mathbb{N}$ such that $B_2(K) \cap \Sigma \subset B_N(0)$, so that $E^r(\Sigma \cap B_2(K)) \leq E^r(\Sigma \cap B_N(0)) < \infty$ since $\Sigma$ has locally finite energy; see (1.11). Therefore, we can use the absolute continuity of the double integral defining $E^r$ to find: For all $c > 0$, there exists a radius $R_K^w(c) > 0$, such that
\[
\int_{\Sigma \cap B_{\varepsilon}(x)} \int_{\Sigma \cap B_{\varepsilon}(y)} \frac{L_r(\mu, \eta, H(\mu), H(\eta))}{|\mu - \eta|^{2m}} \, d\mathcal{H}^m(\mu) \, d\mathcal{H}^m(\eta) < c \tag{3.23}
\]
for all $x, y \in K$ and all $r \in (0, \min\{R_K^w(c), 1\}]$. In particular, $R_K^w(c_2) \leq R_K^w(c_1)$ by means of (3.22).

Choose $r_K := \varepsilon_0 \cdot \min\{R_K^w(c_2), R_K^w(1)\} < R_K^w/500 \leq R_K/500$ and distinguish two cases.

Case I: $\nabla (H(p), H(q)) < \omega(\delta) + 2M\alpha$. Then apply Lemma 3.4 for $R := \frac{r_K}{\varepsilon_0} \leq \min\{R^w_K, 1\}$ and $\varepsilon := \varepsilon_0$ to find by (3.23), (3.20), and (3.22) for $x = q, y = p$
\[
c_2 > \int_{\Sigma \cap B_{\tau}^w(q)} \int_{\Sigma \cap B_{\tau}^w(p)} \frac{L_r(\mu, \eta, H(\mu), H(\eta))}{|\mu - \eta|^{2m}} \, d\mathcal{H}^m(\eta) \, d\mathcal{H}(\mu)
\geq \int_{\Sigma \cap B_{\tau}^w(q)} \int_{\Sigma \cap B_{\tau}^w(p)} \frac{L_r(\mu, \eta, H(\mu), H(\eta))}{|\mu - \eta|^{2m}} \, d\mathcal{H}^m(\eta) \, d\mathcal{H}(\mu) > c_1 > c_2,
\]
which provides a contradiction.

Case II: $\nabla (H(p), H(q)) \geq \omega(\delta) + 2M\alpha$. Then apply Lemma 3.5 for $R := \frac{r_K}{\varepsilon_0}$ and $\varepsilon := \varepsilon_0$ to find by (3.21) and (3.23)
\[
c_2 > \int_{\Sigma \cap B_{\tau}^w(q)} \int_{\Sigma \cap B_{\tau}^w(p)} \frac{L_r(\mu, \eta, H(\mu), H(\eta))}{|\mu - \eta|^{2m}} \, d\mathcal{H}^m(\eta) \, d\mathcal{H}(\mu)
\geq \int_{\Sigma \cap B_{\tau}^w(q)} \int_{\Sigma \cap B_{\tau}(p)} \frac{L_r(\mu, \eta, H(\mu), H(\eta))}{|\mu - \eta|^{2m}} \, d\mathcal{H}^m(\eta) \, d\mathcal{H}(\mu) > c_2,
\]
which is a contradiction as well. \qed

It remains to prove the two auxiliary lemmas.

Proof of Lemma 3.4. Fix $\delta \in (0, 1)$ and $\varepsilon \in (0, \delta/500]$. Notice that both points $p, q$ are contained in $K$. Let $r \in (0, R/10^5)$. Definition 1.1 applied to $K$ guarantees the existence
of a dense subset $D_q \subset (q + H(q)) \cap B_R(q)$, so that for any given $\varrho < r/\sqrt{1 + \alpha^2}$ we can find a point $y(\varrho) \in D_q \cap B_{\varrho}(q)$ and a corresponding point $\eta_y(\varrho) \in \Sigma \cap C_q(\alpha, H(q))$ with $\Pi_{q+H(q)}(\eta_y(\varrho)) = y(\varrho)$, which implies $|\eta_y(\varrho) - q| \leq \sqrt{1 + \alpha^2} \varrho < r$. Consequently, for any such $\varrho \in (0, r/\sqrt{1 + \alpha^2})$,

$$\mathcal{H}^m(E_{\alpha,M}(q) \cap B_r(q)) \geq \mathcal{H}^m(E_{\alpha,M}(q) \cap B_{r - \sqrt{1 + \alpha^2}}(\eta_y(\varrho))) \geq c_K(r - \sqrt{1 + \alpha^2} \varrho)^m.$$  

Taking the limit $\varrho \to 0$ guarantees $\mathcal{H}^m(E_{\alpha,M}(q) \cap B_r(q)) \geq c_K r^m$. In particular, for the set $M_q(\varepsilon) := E_{\alpha,M}(q) \cap B_{\varepsilon R}(q)$ we obtain

$$\mathcal{H}^m(M_q(\varepsilon)) \geq c_K (\varepsilon^2 R)^m. \quad (3.24)$$  

For any point $\mu \in M_q(\varepsilon)$ and an arbitrary vector $e \in H(\mu) \cap S^{n-1}$ set

$$w \equiv w(\mu, e) := \frac{\Pi_H(\varepsilon)}{|\Pi_H(\varepsilon)|} \in H(p) \cap S^{n-1}, \quad (3.25)$$

which is well-defined since

$$\hat{\xi}(H(\mu), H(p)) \leq \hat{\xi}(H(\mu), H(q)) + \hat{\xi}(H(q), H(p)) < \omega(\delta) + 3 \alpha \omega, \quad (3.26)$$

so that by Lemma A.1 we obtain by definition of $\omega(\delta)$ in (3.19) and assumption (3.14)

$$|\Pi_{H(p)}(\varepsilon)| = |e - \Pi_{H(p)}(\varepsilon)| \geq 1 - \hat{\xi}(H(p), H(q)) > 1 - (\omega(\delta) + 3 \alpha \omega) > 0. \quad (3.27)$$

Furthermore, the definition of the set $M_q(\varepsilon)$ implies

$$|\Pi_{p+H(p)}(\mu + \varepsilon R w/2 - p)| = |\Pi_{H(p)}(\mu + \varepsilon R w/2 - p)| \leq |\mu - q| + |q - p| + \varepsilon R/2$$

$$< (3/2 + \varepsilon) \varepsilon R < R. \quad (3.28)$$

By means of Definition 1.1 we can find a point $x \equiv x(\mu, e) \in D_p$ such that we obtain $|x - \Pi_{p+H(p)}(\mu + \varepsilon R w/2)| < \varepsilon^2 R$, and there exists a corresponding point $\eta_x = \eta_x(\mu, e) \in \Sigma \cap C_p(\alpha, H(p))$ with $x = \Pi_{p+H(p)}(\eta_x) = p + \Pi_{H(p)}(\eta_x)$. Consequently,

$$\Pi_{H(p)}(\eta_x - p) = x - p = f_x + \Pi_{H(p)}(\mu + \varepsilon R w/2 - p) \quad (3.29)$$

for $|f_x| = |x - \Pi_{p+H(p)}(\mu + \varepsilon R w/2)| < \varepsilon^2 R$, which implies, on the one hand, the estimate

$$|\Pi_{H(p)}(\eta_x - p)| \leq |f_x| + |\Pi_{H(p)}(\mu + \varepsilon R w/2 - p)| < (3/2 + 2 \varepsilon) \varepsilon R \quad (3.30)$$

due to (3.28), and, on the other hand, the identity

$$\Pi_{H(p)}(\eta_x - \mu) = \Pi_{H(p)}(\varepsilon R w/2) + f_x. \quad (3.31)$$

Since $\eta_x \in C_p(\alpha, H(p))$ we find by (3.30)

$$|\Pi_{H(p)}(\eta_x - p)| \leq \alpha |\Pi_{H(p)}(\eta_x - p)| < \alpha (3/2 + 2 \varepsilon) \varepsilon R; \quad (3.32)$$

hence $|\eta_x - p|^2 = |\Pi_{H(p)}(\eta_x - p)|^2 + |\Pi_{H(p)}(\eta_x - p)|^2 < (1 + \alpha^2) (3/2 + 2 \varepsilon)^2 \varepsilon^2 R^2$, i.e.,

$$|\eta_x - p| < \sqrt{1 + \alpha^2} (3/2 + 2 \varepsilon) \varepsilon R < R, \quad (3.33)$$

since $\delta < 1$ and by virtue of (3.14). The set $M_p(\mu, e, \varepsilon) := E_{\alpha,M}(\mu) \cap B_{\varepsilon R}(\eta_x(\mu, e))$ satisfies

$$M_p(\mu, e, \varepsilon) \subset \Sigma \cap B_{2 \varepsilon R}(p), \quad (3.34)$$
since for all $\eta \in M_p(\mu, e, \varepsilon)$ one has
\[
|\eta - p| \leq |\eta - \eta_x| + |\eta_x - p| < \varepsilon^2 R + \sqrt{1 + \alpha^2} (3/2 + 2\varepsilon) R < 2\varepsilon R, \tag{3.35}
\]
where we also used \((3.33)\) and $\sqrt{1 + \alpha^2} \leq 1 + \alpha < 51/50$ by \((3.14)\). Moreover,
\[
\mathcal{E}^m \left( M_p(\mu, e, \varepsilon) \right) \geq c_R (\varepsilon^2 R)^m \tag{3.36}
\]
by virtue of Definition 1.1 applied to the compact set $\tilde{K} \subset \Sigma$. For the fixed point $\eta_x \in \Sigma \cap C_p(\alpha, H(p)) \cap B_R(p)$, we estimate by means of the identity $\text{Id} = \Pi_{H(p)}^+ + \Pi_{H(p)}^-$
\[
|\langle e, \mu - \eta_x \rangle| \geq |\langle \Pi_{H(p)}(e), \Pi_{H(p)}(\mu - \eta_x) \rangle| - |\langle \Pi_{H(p)}(e), \Pi_{H(p)}(\mu - \eta_x) \rangle|
\]
\[
\geq |\langle \Pi_{H(p)}(e), \varepsilon R \cdot \Pi_{H(p)}(w)/2 \rangle| - |f_x| - |\Pi_{H(p)}(e)| \cdot |\Pi_{H(p)}(\mu - \eta_x)|
\]
\[
> \varepsilon R |\Pi_{H(p)}(e)|/2 - \varepsilon^2 R - (\omega(\delta) + 3M\alpha) |\Pi_{H(p)}(\mu - \eta_x)|,
\]
due to \((3.31)\), \((3.26)\), $|e| = 1$, and $|f_x| < \varepsilon^2 R$. Consequently, with \((3.27)\) we obtain
\[
|\langle e, \mu - \eta_x \rangle| > (1 - (\omega(\delta) + 3M\alpha)) \varepsilon R/2 - \varepsilon^2 R - (\omega(\delta) + 3M\alpha) |\Pi_{H(p)}(\mu - \eta_x)|.
\]
The estimate \((3.32)\) implies
\[
|\langle e, \mu - \eta_x \rangle| \leq |\langle e - q, \mu - \eta_x \rangle| + \alpha (3/2 + 2\varepsilon) \varepsilon R
\]
\[
< (\varepsilon + 1 + \alpha (3/2 + 2\varepsilon)) \varepsilon R;
\]
hence
\[
|\langle e, \mu - \eta_x \rangle| > (1/2 - (\omega(\delta) + 3M\alpha) (3/2 + 2\varepsilon) (1 + \alpha) - \varepsilon) \varepsilon R.
\]
For arbitrary $\eta \in M_p(\mu, e, \varepsilon)$ one therefore has
\[
|\langle e, \mu - \eta \rangle| \geq |\langle e, \mu - \eta_x \rangle| - |\eta_x - \eta|
\]
\[
> (1/2 - (\omega(\delta) + 3M\alpha) (3/2 + 2\varepsilon) (1 + \alpha) - 2\varepsilon) \varepsilon R. \tag{3.37}
\]
Moreover, for all $\mu \in M_q(\varepsilon)$, $e \in H(\mu) \cap \mathbb{S}^{n-1}$ and $\eta \in M_p(\mu, e, \varepsilon)$ we estimate by means of \((3.35)\)
\[
|\mu - \eta| \leq |\mu - q| + |q - p| + |p - \eta| < 2\varepsilon^2 R + \varepsilon R + \sqrt{1 + \alpha^2} (3/2 + 2\varepsilon) \varepsilon R
\]
\[
< (2\varepsilon + 1) (1 + 3\sqrt{1 + \alpha^2}/2) \varepsilon R. \tag{3.38}
\]
For the remaining term in the energy density we simply write $|\Pi_{H(p)}(\mu - \eta)| = |\Pi_{H(p)}(\mu - p) - \Pi_{H(p)}(\eta_x - p)| + \Pi_{H(p)}(\eta_x - \eta) - \Pi_{H(p)}(\mu - \eta) + \Pi_{H(p)}(\mu - \eta)|$, which – using \((3.32)\) and \((3.38)\) – can be bounded from below by
\[
|\Pi_{H(p)}(\mu - p)| - |\Pi_{H(p)}(\eta_x - p)| - \alpha (3/2 + 2\varepsilon) \varepsilon R - \varepsilon^2 R - \mathcal{H}(H(p), H(\eta))(2\varepsilon + 1)(1 + 3\sqrt{1 + \alpha^2}/2) \varepsilon R
\]
\[
\geq |\Pi_{H(p)}(q - p)| - |q - \mu| - \alpha (3/2 + 2\varepsilon) \varepsilon R - \varepsilon^2 R - M\alpha(2\varepsilon + 1)(1 + 3\sqrt{1 + \alpha^2}/2) \varepsilon R.
\]
Since $|q - \mu| < \varepsilon^2 R$ and $|\Pi_{H(p)}(q - p)| > \delta \varepsilon R$ by assumption, we obtain
\[
|\Pi_{H(p)}(\mu - \eta)| > (\delta - 2\varepsilon - \alpha (3/2 + 2\varepsilon) - M\alpha(2\varepsilon + 1)(1 + 3\sqrt{1 + \alpha^2}/2)) \varepsilon R \tag{3.39}
\]
for all $\mu \in M_q(\varepsilon)$, $e \in H(\mu) \cap \mathbb{S}^{n-1}$, and $\eta \in M_p(\mu, e, \varepsilon)$. Finally,
\[
\mathcal{H}(H(\eta), H(\mu)) \leq \mathcal{H}(H(\eta), H(p)) + \mathcal{H}(H(p), H(q)) + \mathcal{H}(H(q), H(\mu))
\]
\[
< M\alpha + \omega(\delta) + 2M\alpha + M\alpha = \omega(\delta) + 4M\alpha, \tag{3.40}
\]
since we have \((3.19)\) and $\mu \in M_q(\varepsilon)$ as well as $\eta \in M_p(\mu, e, \varepsilon)$. 
For \( \mu \in M_\eta(\varepsilon) \) and \( e \in H(\mu) \cap S^{n-1} \) the numerator \( L_{\text{r}} \) of the energy density of \( E^r \) satisfies \( L_{\text{r}}(\mu, \eta, \mu, \mu) \geq F^r(\mu, \eta, e) \) for all \( \eta \in B_3(p) \), where \( F_\text{r} \) is given by (3.13). In particular, for \( \eta \in M_p(\mu, \varepsilon, e) \), we may use (3.37), (3.38), (3.39), and (3.40) to conclude that the energy density \( L_{\text{r}}(\mu, \eta, H(\mu), H(\eta))/|\mu - \eta|^{2m} \) is bounded from below by

\[
|\mu - \eta|^{-2m} (2|e, \mu - \eta|)|\mu - \eta|^{-2} \cdot |\Pi_{H(\mu)}(\mu - \eta) - |\Pi_{H(\mu)}(\mu - \eta)\rangle (1 + \tau)^m
\]

\[
\geq ((2\varepsilon + 1)(1 + 3\sqrt{1 + \alpha^2 / 2})\varepsilon R)^{-2m} \cdot \left[ (1 - (\omega(\delta) + 3M\alpha)(3 + 4\varepsilon)(1 + \alpha) - 4\varepsilon)e^2R^2 \right] \frac{((2\varepsilon + 1)(1 + 3\sqrt{1 + \alpha^2 / 2})\varepsilon R)^{-2m} \cdot \{ \delta - \varepsilon - \alpha(3/2 + 2\varepsilon) - M\alpha(2\varepsilon + 1)(1 + 3\sqrt{1 + \alpha^2 / 2}) \} - (\omega(\delta) + 4M\alpha)\right]^{(1 + \tau)^m}.
\]

We define \( f(\varepsilon, \delta)/(\varepsilon R)^{2m} \) to be the right hand side of the last estimate. Since \( \varepsilon \leq \delta/500 \) one obtains by means of (3.14) \( f(\varepsilon, \delta) > \left( \frac{100}{253} \right)^{2m} \cdot \left( \frac{\delta}{100} \right)^{(1 + \tau)^m} \). Integrating the energy density over the Cartesian product of \( M_\eta(\varepsilon) \subset \Sigma \cap B_{2R}(q) \) and \( M_p(\mu, e, \varepsilon) \subset \Sigma \cap B_{2\varepsilon R}(p) \) (see (3.34)), we find by virtue of (3.24) and (3.36) the strict inequality (3.20) with the lower bound \( c_1(K, \varepsilon, \delta, \tau, m) \) as stated in Lemma 3.4.

**Proof of Lemma 3.7.** Again fix \( \delta \in (0, 1) \) and \( \varepsilon \in (0, \delta/500) \). Analogously to the setting of Lemma 3.4, we obtain \( p, q \in \bar{K} \), and for \( M_\eta(\varepsilon) := E_{\alpha, M}(q) \cap B_{2\varepsilon R}(q) \) we have

\[
\mathcal{H}^{2m}(M_\eta(\varepsilon)) \geq c_\varepsilon (\varepsilon R)^m.
\]

Since \( \varepsilon (H(p), H(q)) \geq \omega(\delta) + 2M\alpha \), we have for arbitrary \( \mu \in M_\eta(\varepsilon) \)

\[
\varepsilon (H(\mu), H(p)) \geq \varepsilon (H(p), H(q)) + \varepsilon (H(q), H(\mu)) > \omega(\delta) + M\alpha,
\]

so that there exists by Lemma A.1 a vector \( e^* = e^*(\mu) \in H(\mu) \cap S^{n-1} \), such that

\[
|\Pi_{H(p)}(e^*)| > \omega(\delta) + M\alpha.
\]

We claim that there is a vector \( v = v(\mu, e^*) \in H(p) \cap S^{n-1} \), with

\[
\langle \Pi_{H(p)}(e^*), v \rangle = 0.
\]

Indeed, if \( \Pi_{H(p)}(e^*) = 0 \) then any \( v \in H(p) \cap S^{n-1} \) satisfies identity (3.43), and if \( \Pi_{H(p)}(e^*) \neq 0 \) then we have \( (\mathbb{R}e^*)^\perp \cap H(p) \neq \{0\} \) because \( \dim(\mathbb{R}e^*)^\perp = n - 1 \) and \( \dim H(p) = m \geq 2 \), so that one can choose \( v \in (\mathbb{R}e^*)^\perp \cap H(p) \cap S^{n-1} \) to satisfy (3.43).

Now compute

\[
|\Pi_{p + H(p)}(\mu + Rv/2) - p| = |\Pi_{p + H(p)}(\mu + Rv/2 - p)| \leq |\mu - q| + |q - p| + R|\Pi_{H(p)}(v)|/2 < (1/2 + \varepsilon e^2) R < R.
\]

By Definition 1.1 there is a point \( y \in D_p \), such that \( |y - \Pi_{p + H(p)}(\mu + Rv/2)| < \varepsilon^2 R \), together with a corresponding point \( \eta_y = \eta_y(\mu, e^*) \in \Sigma \cap C_p(\alpha, H(p)) \) with \( \Pi_{p + H(p)}(\eta_y) = y = p + \Pi_{H(p)}(\eta_y - p) \). Therefore,

\[
\Pi_{H(p)}(\eta_y - p) = y - p = f_y + \Pi_{H(p)}(\mu + Rv/2 - p),
\]

with \( f_y := y - \Pi_{H(p)}(\mu + Rv/2) \). In other words,

\[
\Pi_{H(p)}(\eta_y - p) = \Pi_{H(p)}(Rv/2) + f_y = Rv/2 + f_y.
\]
In particular, by (3.44) and (3.45) one finds
\[
|\Pi_{H(p)}(\eta_y - p)| \leq |f_y| + |\Pi_{H(p)}(\mu + R\nu/2 - p)| < (1/2 + \varepsilon + 2\varepsilon^2) R, \tag{3.47}
\]
and
\[
|\Pi_{H(p)}(\eta_y - p)| \geq |\Pi_{H(p)}(\mu + R\nu/2 - p)| - |f_y| \geq R|\Pi_{H(p)}(\nu)/2| - |\mu - q| - |p - q| - |f_y| > (1/2 - \varepsilon - 2\varepsilon^2) R. \tag{3.48}
\]
Inequality (3.47) together with the fact that \(\eta_y \in C_p(\alpha, H(p)) \) implies
\[
|\Pi_{H(p)}(\eta_y - p)| \leq \alpha|\Pi_{H(p)}(\eta_y - p)| < \alpha(1/2 + \varepsilon + 2\varepsilon^2) R. \tag{3.49}
\]
With \(\eta_y - p = \Pi_{H(p)}(\eta_y - p) + \Pi_{H(p)}(\eta_y - p)\) one arrives at
\[
(1/2 - \varepsilon - 2\varepsilon^2) R < |\eta_y - p| < \sqrt{1 + \alpha^2}(1/2 + \varepsilon + 2\varepsilon^2) R. \tag{3.50}
\]
Define \(M_p(\mu, e^*, \varepsilon) := E_{\alpha,M}(p) \cap B_\varepsilon R(\eta_y(\mu, e^*))\) which satisfies
\[
M_p(\mu, e^*, \varepsilon) \subset \Sigma \cap B_R(p), \tag{3.51}
\]
because \(|\eta - p| \leq |\eta - \eta_y| + |\eta_y - p| < \sqrt{1 + \alpha^2}(1/2 + \varepsilon + 3\varepsilon^2) R < R\) since \(\sqrt{1 + \alpha^2} \leq 1 + \alpha < 51/50\) by (3.14). By virtue of Definition 1.1 one has also
\[
\mathcal{H}^m(M_p(\mu, e^*, \varepsilon)) \geq c_R (\varepsilon^2 R)^m. \tag{3.52}
\]
Now, by \(|e^*| = 1\), (3.43), and (3.46) one uses \(\text{Id} = \Pi_{H(p)} + \Pi_{H(p)}\) to estimate
\[
|\langle e^*, \mu - \eta_y \rangle| \leq |\langle \Pi_{H(p)}(e^*), \Pi_{H(p)}(\mu - \eta_y) \rangle| + |\langle \Pi_{H(p)}(e^*), \Pi_{H(p)}(\mu - \eta_y) \rangle| \leq |f_y| + |\mu - q| + |q - p| + |\Pi_{H(p)}(p - \eta_y)|,
\]
so that we obtain by means of (3.49) \(|\langle e^*, \mu - \eta_y \rangle| < (2\varepsilon + \varepsilon)(1 + \alpha + \alpha/2) R\). Consequently, for an arbitrary \(\eta \in M_p(\mu, e^*, \varepsilon)\) one finds
\[
|\langle e^*, \mu - \eta \rangle| \leq |\langle e^*, \mu - \eta_y \rangle| + |\eta_y - \eta| < (3\varepsilon^2 + \varepsilon)(1 + \alpha + \alpha/2) R. \tag{3.53}
\]
On the one hand, we obtain
\[
|\mu - \eta| \leq |\mu - q| + |q - p| + |p - \eta| < \varepsilon^2 R + \varepsilon R + \sqrt{1 + \alpha^2}(1/2 + \varepsilon + 3\varepsilon^2) R < \sqrt{1 + \alpha^2}(1/2 + 2\varepsilon + 4\varepsilon^2) R, \tag{3.54}
\]
and, on the other hand,
\[
|\mu - \eta| \geq |\eta_y - p| - |\eta_y - \eta| - |p - q| - |\mu - q| > (1/2 - \varepsilon - 2\varepsilon^2) R - \varepsilon^2 R - \varepsilon R - \varepsilon^2 R = (1/2 - 2\varepsilon - 4\varepsilon^2) R, \tag{3.55}
\]
due to (3.50). Moreover, according to (3.49) and (3.54),
\[
|\Pi_{H(p)}(\mu - \eta)| \leq |\Pi_{H(p)}(\mu - \eta)| + |\Pi_{H(p)}(\mu - p)| + |\Pi_{H(p)}(p - \eta_y)| + |\Pi_{H(p)}(\eta_y - \eta)| \leq A(H(\eta), H(p)) |\mu - \eta| + |\mu - q| + |q - p| + |\Pi_{H(p)}(p - \eta_y)| + |\eta_y - \eta| < M\alpha \sqrt{1 + \alpha^2(1/2 + \varepsilon + 4\varepsilon^2) R + \varepsilon^2 R + \varepsilon R + \alpha(1/2 + \varepsilon + 2\varepsilon^2) R + \varepsilon^2 R + \varepsilon^2 R}
\]
\[
< \alpha(M \sqrt{1 + \alpha^2 + 1}(1/2 + 2\varepsilon + 4\varepsilon^2) R + (2\varepsilon^2 + \varepsilon) R). \tag{3.56}
\]
Finally, since $\eta \in M_p(\mu, e^*, \varepsilon)$ and by (3.42) we obtain
\[
|\Pi_{H(\eta)}(e^*)| \geq |\Pi_{H(p)}(e^*)| - 3\|H(\eta), H(p)\| + M\alpha - M\alpha = \omega(\delta).
\] (3.57)
For fixed $\mu \in M_q(\varepsilon) \subset B_{2R}(q)$ the numerator $L_\tau$ of the energy density satisfies
$L_\tau(\mu, \eta, H(\mu), H(\eta)) \geq F^\tau(\mu, \eta, e^*\rangle$ for all $e \in H(\mu) \cap S^{n-1}$ and all $\eta \in B_R(p)$. In particular, for $\eta \in M_p(\mu, e^*, \varepsilon) \subset B_R(p)$, we can use (3.53), (3.54), (3.55), (3.56), and (3.57) in order to conclude that the energy density $L_\tau(\mu, \eta, H(\mu), H(\eta))/|\mu - \eta|^{2m}$ is bounded from below by
\[
|\mu - \eta|^{-2m} \left( |\Pi_{H(\eta)}(e^*)| - 2 \langle e^*, \mu - \eta \rangle |\mu - \eta|^{-2} : |\Pi_{H(\eta)}(\mu - \eta)\right)^{(1+\tau)m} 
\geq \left( \sqrt{1 + \alpha^2} \right)^{2(1/2 + 2\varepsilon) + 4\varepsilon^2R)^{-2m} \left[ \omega(\delta) - (1/2 - 2\varepsilon - 4\varepsilon^2 R)^{-2} 
\right]^{(1+\tau)m}.
\]
We define $g(\varepsilon, \delta)/R^{2m}$ to be the right hand side of this inequality. Then, as $\varepsilon \leq \delta/500$ one obtains by means of (3.14) the estimate $g(\varepsilon, \delta) > \left( \frac{(1.9)^{(1+\tau)m}}{10^{-m}\tau} \cdot \alpha^{(1+\tau)m} \cdot \delta(1+\tau)m \right)$. Now we can integrate the energy density over the Cartesian product of $M_p(\varepsilon) \subset \Sigma \cap B_{2R}(q)$ and $M_p(\mu, e^*, \varepsilon) \subset \Sigma \cap B_R(p)$ (see 3.53), in order to establish with the help of (3.41) and (3.52) the desired inequality 3.41 with the constant $c_2(K, \varepsilon, \delta, \tau, m)$ as stated in Lemma 3.41.

**Corollary 3.6.** Let $\delta \in (0, 1)$, $\alpha, M > 0$ satisfy (3.14), and suppose $\Sigma \in a^m(\alpha, M)$, $2 \leq m \leq n$, has locally finite energy $E^\tau$ for some $\tau > -1$. Then, for all compact sets $K \subset \Sigma$ and all $p \in K$, we have
\[
\theta(\Sigma, G_p, \varepsilon) \leq \delta \quad \text{for all} \quad r \in (0, \min\{r_K, q_K\}],
\] (3.58)
where $q_K$ denotes the radius of Corollary 3.2 and $r_K$ as well as $G_p$ are as stated in Theorem 3.3.

**Proof.** According to Theorem 3.3 one finds $\sup_{p \in K} \beta(\Sigma, G_p, r) \leq \delta$ for all $r \in (0, r_K)$. Moreover, we have $G_p = H(p)$ for all $p \in \Sigma^*$ and $G_q = \lim_{t \to \infty} H(p_i)$ for a sequence $(p_i)_{i \in \mathbb{N}} \subset \Sigma^*$ approximating $q \in \Sigma \setminus \Sigma^*$. Consequently, Lemma 3.2 guarantees
\[
\sup_{\xi \in (p+G_p) \cap B_r(p)} \text{dist}(\xi, \Sigma \cap B_r(p)) \leq 2\alpha\tau/\sqrt{1 + \alpha^2} < \delta r \quad \text{for all} \quad r \in (0, q_K],
\]
where we used (3.14) for the last inequality. In view of (3.4), this finishes the proof.

Now Reifenberg’s famous topological disk lemma [12, 46, 23] implies that finite energy sets are topological manifolds, a result that we do not rely on in the following sections.

**Corollary 3.7.** Let $m, n \in \mathbb{N}$ with $2 \leq m \leq n$. For any $\kappa \in (0, 1)$ there is a constant $\delta_0(m, \kappa) \in (0, 1)$ such that any set $\Sigma \in a^m(\alpha, M)$ with locally finite energy $E^\tau$, where $\alpha, M > 0$ satisfy (3.14) for $\delta = \delta_0$, is an $m$-dimensional topological manifold which is locally bi-Hölder homeomorphic to the $m$-dimensional flat unit disk $B_1(0) \subset \mathbb{R}^m$ with Hölder constant $\kappa$. 
3.3. **Lipschitz and \(C^1\)-submanifolds.** The local structure of the admissibility class \(\mathcal{A}^m(\alpha, M)\) allows us to directly derive the stronger Lipschitz manifold statement of Theorem 3.3 *without* using Reifenberg’s topological disk theorem at all. In order to do so, we take a closer look at the projection of \(\Sigma \cap B_r(p)\) onto the affine plane \(p + G_p\). The following arguments are based solely on the Reifenberg-flatness of \(\Sigma\).

First, we notice injectivity of the orthogonal projection onto approximating planes for \((m, \delta)\)-Reifenberg-flat sets.

**Lemma 3.8 (Injectivity of projection).** Suppose \(\delta \in (0, 1/2)\), and \(\Sigma \subset \mathbb{R}^n\) is an \((m, \delta)\)-Reifenberg-flat set. Let \(p \in K \subset \Sigma\), where \(K\) is compact, and suppose that \(\eta \in K \cap B_r(p)\) for some \(r \in (0, r_0(K))\) satisfies

\[
\frac{1}{2} (F_\eta(q, \delta), F_p(r, \delta)) + \delta < 1 \quad \text{for all} \quad q \in (0, r),
\]

3.59

for the radius \(r_0(K)\) and the \(m\)-planes \(F_\eta(q, \delta)\) and \(F_p(r, \delta)\) as in Definition 3.1. Then for every \(\xi \in \Sigma \cap B_r(p) \setminus \{\eta\}\) one has \(\Pi_{p + F_p(r, \delta)}(\xi) \neq \Pi_{p + F_p(r, \delta)}(\eta)\).

Notice that if (3.59) is satisfied for all \(\eta \in \Sigma \cap B_r(p)\) then the orthogonal projection \(p + \Pi_{F_p(r, \delta)}\) restricted to \(\Sigma \cap B_r(p)\) is injective.

**Proof.** Assuming the contrary for some \(\xi \in \Sigma \cap B_r(p) \setminus \{\eta\}\), we can estimate for \(F := F_p(r, \delta) \in \mathcal{G}(n, m)\)

\[
|\eta - \xi| = |\Pi_F(\eta - \xi)| = |\Pi_{F,\perp}(\eta - \xi)| = |\Pi_{F,\perp}(\eta - \xi)| + \|\Pi_{F,\perp}(\eta - \xi)\|_\infty < \frac{1}{2} r.
\]

3.60

Since \(\delta < 1/2\), therefore, there exists an integer \(N \in \mathbb{N}\), such that

\[
r_n := (1 + 1/n) \cdot |\eta - \xi| < r \leq r_0(K) \quad \text{for all} \quad n \geq N.
\]

3.61

Obviously, \(\xi \in \Sigma \cap B_{r_n}(\eta)\), and therefore, by the Reifenberg-flatness of \(\Sigma\), used in the point \(\eta\) for the radius \(r_n\) with the approximating plane \(Q := F_\eta(q, r, \delta)\)

\[
\text{dist} (\xi, (\eta + Q) \cap B_{r_n}(\eta)) \leq \delta r_n,
\]

3.62

so that we can estimate using (3.60) and (3.61)

\[
|\eta - \xi| = |\Pi_{F,\perp}(\eta - \xi)| = |\Pi_{F,\perp}(\eta - \xi) - \Pi_{Q,\perp}(\eta - \xi)| + \|\Pi_{Q,\perp}(\eta - \xi)\|_\infty \leq \left[ \frac{1}{2} (F, Q) \cdot |\eta - \xi| + \text{dist} (\xi, (\eta + Q) \cap B_{r_n}(\eta)) \right] \cdot |\eta - \xi|,
\]

3.63

and the right-hand side is by assumption (3.59) for \(q := r_n\) strictly less than \(|\eta - \xi|\) for \(n\) sufficiently large, which yields a contradiction. \(\Box\)

For sufficiently small \(\delta > 0\) and radius \(r > 0\), the orthogonal projection of \(\Sigma \cap B_r(p)\) onto good approximating affine planes contains a whole \(m\)-dimensional disk.

**Proposition 3.9 (Surjectivity of projection).** There exists a \(\delta_1 = \delta_1(m) > 0\), such that for all closed \((m, \delta)\)-Reifenberg-flat sets \(\Sigma \subset \mathbb{R}^n\) with \(\delta \leq \delta_1\) and all compact subsets \(K \subset \Sigma\) there exists a radius \(q_1(K) \in (0, r_0(K))\) such that \((p + F_p(r, \delta)) \cap B_{r/4}(p) \subset \Pi_{p + F_p(r, \delta)}(\Sigma \cap B_{r/2}(p))\) for all \(p \in K\) and \(r \in (0, q_1(K))\).
Proof. Up to isometry the composition \( \Psi := \Pi_{p+F_p(r,\delta)} \circ \tau : p + F_p(r,\delta) \to p + F_p(r,\delta) \) satisfies the assumptions of Proposition 2.5 to yield the claim. Here, \( \tau \) is the mapping constructed in the following technical lemma proven in [24]. \( \square \)

Lemma 3.10 ([24] Lemma 3.7). There exists a \( \delta_* = \delta_*(m) > 0 \) such that for every closed, \( m \)-dimensional \((\delta, m)\)-Reifenberg-flat set \( \Sigma \subset \mathbb{R}^m \) with \( \delta \leq \delta_* \) and \( x \in \Sigma \) there is a radius \( R_0 = R_0(x, \delta, \Sigma) > 0 \) and a constant \( C_* = C_*(m) \) such that for all \( F \in \mathscr{G}(n, m) \) with \( \theta(x, F, r) \leq \delta \) for \( r \leq R_0 \) there exists a continuous function \( \tau: (x+F) \cap B_{15r/16}(x) \to \Sigma \cap B_r(x) \) with \( |\tau(y) - y| \leq C_* \delta \leq 5r/144 \) for all \( y \in (x+F) \cap B_r(x) \).

We can now use the local bijectivity of the projection established in Lemma 3.8 and Proposition 3.9 to prove that every admissible set with locally finite energy possesses a local graph representation\(^3\), which in particular implies Theorem 1.3.

Theorem 3.11. Let \( 2 \leq m \leq n \). There exist constants \( C = C(m) \) and \( \delta_2 = \delta_2(m) \in (0, \min\{\delta_1, 1/C\}) \) such that for every \( \Sigma \in \mathscr{G}^m(\alpha, M) \) with \( \alpha, M > 0 \) satisfying (3.14) for some \( \delta \in (0, \delta_1) \), with locally finite energy \( E^r \) for any \( \tau > -1 \) the following holds. For all compact sets \( K \subset \Sigma \) there is a radius \( q_2 = q_2(K) \in (0, \min\{r_K, \rho_K, \eta_1(K)\}) \) such that for all \( p \in K \) there is a function \( u_p \in C^{0,1}(G_p, G_p^\perp) \) with \( u_p(0) = 0 \) and \( \text{Lip} u_p \leq C_\delta/(1-C_\delta) \), such that

\[
\Sigma \cap B_{q_2/4}(p) = (p + \text{graph } u_p) \cap B_{q_2/4}(p),
\]

where \( q_2 \) denotes the radius of Corollary 3.2, and \( r_K \) as well as \( G_p \) are as stated in Theorem 3.3. In particular, \( \Sigma \) is a \( C^{0,1}\)-submanifold of \( \mathbb{R}^n \).

Proof. We define \( \tilde{K} := \Sigma \cap B_{R_K/2}(K) \) and \( q_2 = q_2(K) := \min\{r_{\tilde{K}}, \rho_{\tilde{K}}, \eta_1(\tilde{K})\} \leq \min\{r_K, \rho_K, \eta_1(K)\} \).

Applying Corollary 3.6, one finds

\[
\theta_\Sigma(\tilde{p}, G_p, r) \leq \delta \quad \text{for all } r \in (0, q_2], \quad \tilde{p} \in \tilde{K}.
\]

For \( \delta \leq \delta_1, p \in \tilde{K} \) and \( r \leq q_2 \leq \eta_1(\tilde{K}) \), by Proposition 3.9 one has

\[
(p + G_p) \cap B_{r/4}(p) \subset \Pi_{p+G_p}(\Sigma \cap B_{r/2}(p)),
\]

which guarantees for each \( x \in (p+G_p) \cap B_{r/4}(p) \) the existence of a point \( q_x \in \Sigma \cap B_{r/2}(p) \) with \( \Pi_{p+G_p}(q_x) = x \).

In particular, for \( p \in K \), one finds \( \Sigma \cap B_{r/2}(p) \subset \tilde{K} \), since \( r \leq q_2 \leq r_K \leq R_K \), so that by (3.63) and Lemma A.3 for \( r_1 = r_2 := r, F_1 := G_p, F_2 := G_p, p_1 := p, \) and \( p_2 := q \), we can estimate

\[
\gamma(G_p, G_q) \leq 6d\tilde{C}/(1-2d) \quad \text{for all } q \in \Sigma \cap B_{r/2}(p),
\]

with \( \tilde{C} = \tilde{C}(m) > \sqrt{2} \). Choosing \( \delta \) sufficiently small, one finds a constant \( C = C(m) > 0 \) such that

\[
\gamma(G_p, G_q) + \delta \leq 6d\tilde{C}/(1-2d) + \delta \leq C\delta < 1 \quad \text{for all } q \in \Sigma \cap B_{r/2}(p).
\]

\(^3\)Independent of the admissibility class, such a statement holds true for every \((m, \delta)\)-Reifenberg-flat set \( \Sigma \subset \mathbb{R}^n \) with \( \delta \leq \delta_1 \) such that for all \( x \in (p+G_p) \cap B_{r/2}(p) \) there is an \( \eta \in \Sigma \cap B_{r/2}(p) \) with \( \Pi_{p+G_p}\eta(x) = x \) and the approximating planes \( F_\eta(\eta, \delta) \) allow an estimate like (3.59) with a uniform upper bound \( C < 1 \) for all such \( \eta \) and \( q \in (0, r) \); see [23] Ch. 3.1.

\(^4\)Notice as before that the compact set \( \tilde{K} \) is solely determined by \( K \) itself, so that all the constants for \( \tilde{K} \) can be considered to actually depend on \( K \).
Lemma 3.8 for \( \eta := q_\sharp, F_\eta(q, \delta) := G_q \), for all \( q \leq r \), and \( F_p(\eta, \delta) := G_p \) implies together with (3.69) that \( \Pi_{p+G_p} \) is injective and \( \Pi_{p+G_p}|_{\Sigma_p(r)} \) is surjective, where

\[
\Sigma_p(r) := \left\{ \xi \in \Sigma \cap B_{r/2}(p) : \Pi_{p+G_p}(\xi) \in (p + G_p) \cap B_{r/4}(p) \right\}. \tag{3.67}
\]

Therefore \( \Pi_{p+G_p}|_{\Sigma_p(r)} : \Sigma_p(r) \to (p + G_p) \cap B_{r/4}(p) \) is bijective and

\[
\left( \Pi_{p+G_p}|_{\Sigma_p(r)} \right)^{-1} : (p + G_p) \cap B_{r/4}(p) \to \Sigma_p(r) \tag{3.68}
\]

is well-defined. Obviously,

\[
\Sigma \cap B_{r/4}(p) \subset \Sigma_p(r) \quad \text{for all} \quad r \leq q_2. \tag{3.69}
\]

With (3.68), we can define the function \( u_p : G_p \cap B_{r/4}(0) \to G_p^\perp \) by means of \( u_p(x) := \Pi_{G_p^\perp} \left( \left( \Pi_{p+G_p}|_{\Sigma_p(r)} \right)^{-1} (x + p) - p \right) \) satisfying \( u_p(0) = 0 \). Then for all \( q \in \Sigma_p(r) \) and \( x_q := \Pi_{G_p}(q - p) \in G_p \cap B_{r/4}(0) \), one finds

\[
qu = p + \Pi_{G_p}(q - p) = p + x_q + \Pi_{G_p^\perp} \left( \left( \Pi_{p+G_p}|_{\Sigma_p(r)} \right)^{-1} (x_q + p) - p \right) = p + x_q + u_p(x_q).
\]

Consequently,

\[
\Sigma_p(r) \subset (p + \text{graph } u_p) \cap B_{r/2}(p),
\]

\[
\Sigma_p(r) \cap B_{r/4}(p) = (p + \text{graph } u_p) \cap B_{r/4}(p),
\]

which by (3.69) for \( r = q_2 \) implies (3.62). Moreover, for \( \eta, \mu \in \Sigma_p(q_2) \) and corresponding points \( x_\eta, x_\mu \in G_p \cap B_{r/4}(0) \), one has \( \eta \in B_{(1+1/n)}(\eta - \mu) \), so that (3.63) implies

\[
|u_p(x_\eta) - u_p(x_\mu)| = \left| \Pi_{G_p^\perp}(\eta - \mu) - \Pi_{G_p^\perp}(\eta - \mu) \right| \\
\leq \frac{1}{2} \left( \Pi_{G_p}(\eta - \mu) + \text{dist} (\eta, (\mu + G_p) \cap B_{(1+1/n)}(\eta - \mu)) \right) \\
\leq \frac{1}{2} \left( \Pi_{G_p}(\eta - \mu) + \delta (1 + 1/n) \right) \leq C \delta \cdot |\eta - \mu|
\]

for \( n \) sufficiently large. By taking the limit \( n \to \infty \) and (3.66) for \( r = q_2 \), we obtain

\[
|u_p(x_\eta) - u_p(x_\mu)| \leq C \delta \cdot |\eta - \mu| \leq C \delta \cdot \left( |\Pi_{G_p}(\eta - \mu)| + |\Pi_{G_p^\perp}(\eta - \mu)| \right)
\]

\[
= C \delta \cdot \left( |x_\eta - x_\mu| + |u_p(x_\eta) - u_p(x_\mu)| \right).
\]

Absorbing the last term yields for sufficiently small \( \delta \) the Lipschitz continuity of \( u_p \) on the disk \( G_p \cap B_{q_2/4}(0) \) with Lipschitz constant \( \text{Lip } u_p \leq C \delta / (1 - C \delta) \). Finally, extending \( u_p \) to the whole plane \( G_p \) by Kirszbraun’s theorem [18, 2.10.43] concludes the proof.

Applying Theorem 3.11 to the examples mentioned in the introduction and discussed in Sections 2 one finds that all such sets with locally finite energy are embedded Lipschitz submanifolds of \( \mathbb{R}^n \) as long as the constants \( \alpha, M > 0 \) defining their admissibility class \( \mathcal{A}^\alpha_M(\alpha, M) \) are sufficiently small. In particular, countable collections of sufficiently flat Lipschitz graphs, and all \( C^1 \)-immersions of compact \( m \)-dimensional \( C^1 \)-manifolds with locally finite Möbius energy, are \( C^{0,1} \)-submanifolds. For both classes, however, we already have a local graph structure to improve this statement.

**Corollary 3.12 (Lipschitz graphs with small Lipschitz constant).** Suppose that the set \( \Sigma = \bigcup_{i \in \mathbb{N}} (p_i + \text{graph } u_i) \) has locally finite energy \( E^\tau \) for some \( \tau > -1 \), where \( u_i \in C^{0,1}(\mathcal{F}_i, F_i^\perp) \), \( F_i \in \mathcal{G}(n, m) \) for \( 2 \leq m \leq n \), \( u_i(0) = 0 \), and \( \text{Lip } u_i \leq \beta \) for all
i \in \mathbb{N}. If \( \beta < \delta_2/800 \) then for all compact subsets \( K \subset \Sigma \) and \( p \in K \), there exists an index \( i = i(p) \in \mathbb{N} \), such that

\[
\Sigma \cap B_{\rho_2/8}(p) = (p_i + \text{graph } u_i) \cap B_{\rho_2/8}(p).
\]

Here, the constant \( \delta_2 = \delta_2(m) \) and the radius \( \rho_2 = \rho_2(K) \) are taken from Theorem 3.11.

Notice that the Lipschitz constants \( \text{Lip } u_i \) inherited from the initially given Lipschitz functions \( u_i \) are in general much smaller than the Lipschitz constants of the functions \( u_p \) produced by Theorem 3.11.

**Proof of Corollary 3.14** Given a compact subset \( K \subset \Sigma \) we may assume by density that \( p \in K \cap \Sigma^* \). Since \( \beta < \delta_2/800 \), we can find \( \alpha, M > 0 \) such that

\[
\beta \leq M \alpha/16(M + 1) < (M + 1)\alpha/16 < \delta_2/(16 \cdot 50) = \delta_2/800.
\]

Therefore, \( \Sigma \subset \mathcal{A}^m(\alpha, M) \), by Proposition 2.3. Moreover, since \( \beta < \delta_2 \) is sufficient for \( \delta := \delta_2 \), we can apply Theorem 3.11 for \( \delta = \delta_2 \) in order to obtain a radius \( \rho_2 = \rho_2(K) \) such that there is a function \( u_p \in C^{0,1}(G_p, G_p^+) \) with \( u_p(0) = 0 \) such that

\[
\Sigma \cap B_{\rho_2/4}(p) = (p + \text{graph } u_p) \cap B_{\rho_2/4}(p).
\]

Using the structure of \( \Sigma \) we can find a smallest index \( i = i(p) \) such that \( p \in p_i + \text{graph } u_i \), so that by virtue of (3.73)

\[
(p_i + \text{graph } u_i) \cap B_{\rho_2/4}(p) \subset (p + \text{graph } u_p) \cap B_{\rho_2/4}(p).
\]

We claim equality in (3.74) for the smaller radius \( \rho_2/8 \), thus proving the corollary. For that, notice first that since \( p \in \Sigma^* \) we have \( G_p = H(p) = T_{p-p_i} \text{graph } u_i \) (cf. the proof of Proposition 2.3), so that by Lemma B.1

\[
\text{Lip } u_i \leq \delta.
\]

Now shift graph \( u_i \) by means of the Shifting Lemma B.2 to obtain a Lipschitz function \( \tilde{u}_i : F_i \to F_i^+ \) with \( \text{Lip } \tilde{u}_i = \text{Lip } u_i, \tilde{u}_i(0) = 0 \), such that

\[
p + \text{graph } \tilde{u}_i = p_i + \text{graph } u_i.
\]

Now assume for contradiction that there exists a point \( z \in B_{\rho_2/8}(p) \cap (p + \text{graph } u_p) \) that is *not* contained in \( p + \text{graph } \tilde{u}_i \). For the projection \( \zeta := \Pi_{F_i}(z - p) \in F_i \) there exists a graph point \( q := p + \zeta + \tilde{u}_i(\zeta) \in p + \text{graph } \tilde{u}_i \), which differs from \( z \) by assumption. If \( |q - p| < \rho_2/2 \) were true we would find by (3.75)

\[
|\Pi_{G_p}(q - p)| \leq |\Pi_{F_i}(q - z)| + \beta|q - p|
\]

\[
\leq |\Pi_{F_i}(q - z)| + |\Pi_{F_i}(z - p)| + \beta|q - p|
\]

\[
< |\Pi_{F_i}(q - p) - \Pi_{F_i}(z - p)| + \rho_2/8 + \beta \rho_2/2 < \rho_2/4,
\]

which implies \( q \in \Sigma_p \) as defined in (3.67); hence by means of (3.70) \( q \in (p + \text{graph } u_p) \cap B_{\rho_2/2}(p) \setminus \{z\} \). But this contradicts the Quasi-normal Planes Lemma B.4 applied to \( G := G_p, F := F_i, and u := u_p \). Therefore we have shown that \( |q - p| \geq \rho_2/2 \). But this contradicts \( |q - p|^2 = |\zeta|^2 + |\tilde{u}_i(\zeta)|^2 \leq (1 + \beta^2)|\zeta|^2 \leq (1 + \beta^2)|z - p|^2 < (1 + \beta^2)(\rho_2/8)^2 \). □
Proof of Corollary 3.12. As in Section 2.1 we can find for any given $\beta > 0$ the representation $\Sigma = \bigcup_{i=1}^N (p_i + \text{graph } u_i) \cap B_{r_i}(p_i)$, where $u_i \in C^k \left( F_i, F_i^1 \right)$, $F_i \in \mathcal{G}(n,m)$, $u_i = 0$, $Du_i(0) = 0$, and $\|Du_i\|_{C^0} \leq \beta$. In particular, $\text{Lip } u_i \leq \beta$ for all $i \in \{1, \ldots, N\}$. Since $\beta$ can be chosen arbitrarily, we can locally argue analogously to Corollary 3.12 and find for all $p \in \Sigma$ a radius $r = r(p) > 0$ and an $i \in \{1, \ldots, N\}$ such that $\Sigma \cap B_r(p) = (p_i + \text{graph } u_i) \cap B_r(p)$. As the $u_i$ are of class $C^k$, the set $\Sigma$ is a $C^k$-submanifold of $\mathbb{R}^n$.

The graph structure allows us to improve the result of Corollary 3.12 in that we can admit moderate Lipschitz constants that do not allow a direct application of Theorem 3.11 since $\delta_2(m)$ in that theorem is already very small.

Corollary 3.13 (Lipschitz graphs with moderate Lipschitz constant). The claim of Corollary 3.12 holds true for every $\beta \in [0,1/1600]$.

Proof. Since $\beta < 1/1600$, we can find $\alpha, M > 0$ such that $\beta \leq M \alpha/(16(M + 1)) < (M + 1)\alpha/16 < 1/1600$, hence $\Sigma \in \mathcal{A}^m(\alpha, M)$ and $\alpha, M$ satisfy (3.14) for a $\delta < 1/2$.

For $\Sigma = \bigcup_{i \in \mathbb{N}} (p_i + \text{graph } u_i)$, we can prove the existence of $u_p$ in Theorem 3.11 without applying Proposition 3.9 and Lemma A.3 by arguing iteratively for each $i \in \mathbb{N}$. If $i = 1$, then $G_p = H(p) = T_{p-p_i} \text{graph } u_1$ for all $p \in (p_1 + \text{graph } u_1) \cap \Sigma$. For all remaining points $q \in (p_1 + \text{graph } u_1) \setminus \Sigma'$ we choose the approximating sequence $\Sigma' \ni q_i \to q$ defining the $m$-planes in Theorem 3.3 to be contained in the first graph as well. Then, Lemma B.1 implies $\mathfrak{q}(G_p, G_q) \leq 2\beta$ for all $p, q \in p_1 + \text{graph } u_1$. Moreover, the surjectivity of the projection onto $G_p$ is provided by the Tilting Lemma B.3 applied for $F = F_1$, $G = G_{p_1}$, $u = u_1$, and $\chi = \beta$. Since $\delta < 1/2$, the angle bound is sufficient for Lemma 3.8. Hence, similarly as in the proof of Theorem 3.11 for all compact subsets $K \subset \Sigma$, we obtain a graph representation as (3.62) for all $p \in p_1 + \text{graph } u_1$. In particular, $(p_1 + \text{graph } u_1) \cap (p_j + \text{graph } u_j) = \emptyset$ for all $j \in \mathbb{N}$ such that $p_1 + \text{graph } u_1 \neq p_j + \text{graph } u_j$, and we can repeat this argument for the next integer $j = 2$. Iteratively, we obtain local graph representations for all points of $\Sigma$ and can proceed as in Corollary 3.12.

Remark 3.14. If we require in condition (ii) of Definition 1.1 only that the set $D_p$ is a dense subset of an affine $m$-dimensional half space $(p + H_u(p)) \cap B_R(u)(p)$, where $H_u(p) := \{ x \in H(p) \colon \langle x, \nu_p \rangle \geq 0 \}$ for some vector $\nu_p \in H(p) \cap \mathbb{S}^{m-1}$, then all results of this section also hold true if we add the following extra condition on the $\theta$- and $\beta$-numbers defined in (3.3) and (1.13): There is a constant $M_\Sigma$ such that for all compact subsets $K \subset \Sigma$ one has $\beta_2(p, r) \leq M_\Sigma \beta_2(p, r)$ for all $p \in K$ and $r \in (0, R_K]$. We denote this modified class by $\mathcal{A}^m(\alpha, M)$, and briefly indicate the necessary modifications in the proofs of the $\beta$-number estimate in Theorem 3.3 and of the Reifenberg flatness in Corollary 3.6 so that then Theorem 3.11 is still applicable. In order to establish Theorem 3.3 the main problem is to guarantee the existence of $\eta_x$ and $\eta_y$, respectively, satisfying (3.29) and (3.45). For almost parallel strands one can choose $e \in H(\mu)$ such that $w = \Pi_{H(p)}(e)/\|\Pi_{H(p)}(e)\| = \nu_p$ and replace $\mu + \varepsilon R_u/2$ by $\mu + 2\varepsilon R_u$ in (3.28).

\footnote{Such a condition was also used in \cite{29, 31} Definition 1.1] to define the so-called $m$-fine sets.}
guaranteeing $\Pi_{p+H(p)}(\mu + 2\varepsilon Rv_p) \in (p + H_p(p)) \cap B_r(p)$ and hence the existence of the desired $\eta_p$. For transversal strata $e^\perp \in H(\mu)$ is fixed. However, we can choose the sign of $v$ such that $\langle v, \nu_p \rangle \geq 0$ and consider $\mu + Rv + 2\varepsilon Rv_p$ instead of $\mu + Rv$ in (3.44) so that one finds $\eta_p$ analogously to the first case. Then the remaining parts of Lemma 3.4 and therefore Theorem 3.3 can be adopted if we set $\varepsilon_0$ sufficiently small. For Corollary 3.6 we first find $F_p(r, \delta) \in \mathcal{G}(n, m)$ satisfying $\theta_{\Sigma}(p, F_p(r, \delta), r) \leq M_\Sigma \delta$ by virtue of the additional condition defining $\mathcal{A}_{\varepsilon}^m(\alpha, M)$ and the already proven $\beta$-number estimate. Bounding the angle $\hat{\alpha}(F_p(r, \delta), G_p)$ by means of Lemma A.3 one finds a constant $C = C(m, M_\Sigma)$ such that $\theta_{\Sigma}(p, G_p, r) \leq C\delta$ for all $p \in K$ and all $r \leq r_K$. To establish the condition of $(m, \delta)$-Reifenberg-flatness on a fixed $K \subset \Sigma$ one finally has to choose $\delta = \delta/C$. Then we obtain the local graph representation of Theorem 3.11 for all $K \subset \Sigma \in \mathcal{A}_{\varepsilon}^m(\alpha, M)$ with sufficiently small $\alpha$ and $M$ and locally finite energy $E^\tau$.

4. SUFFICIENT REGULARITY FOR FINITE ENERGY

First we assume $C^2$-regularity and prove that this implies finite energy.

**Lemma 4.1** ($C^2$-regularity). Let $1 \leq m \leq n$, $F \in \mathcal{G}(n, m)$, $u \in C^2(F, F^\perp)$, $\Sigma := \text{graph } u$, and $p, q \in \Sigma \cap B_N(0)$ for some $N \in \mathbb{N}$. Then for each $\tau \geq (1/m) - 1$ there is a constant $C = C(\tau, m) > 0$ such that

$$L_\tau(p, q, T_p \Sigma, T_q \Sigma) |p - q|^{-2m} \leq C\|D^2u\|_{C^0(F \cap B_N(0))} \cdot |p - q|^{(\tau - 1)m}. \quad (4.1)$$

In particular, $E^\tau(\Sigma \cap B_N(0)) < \infty$ for $\tau > 0$.

**Proof.** Recall that $L_\tau(p, q, T_p \Sigma, T_q \Sigma) = \sup_{e \in T_p \Sigma \cap S^{n-1}} F_\tau(p, q, e)$, which according to the explicit formula [3.13] and by means of $\tau \geq (1/m) - 1$ can be bounded from above by

$$2^{(1+\tau)m-1} \sup_{e \in T_p \Sigma \cap S^{n-1}} \left[ \|\Pi_{T_p \Sigma}(e)\|^2 + (2|p - q|^2 \|\Pi_{T_q \Sigma}(p - q)\|)(1+\tau)^m \right]$$

$$\leq 2^{(1+\tau)m-1} \left( \hat{\alpha}(T_p \Sigma, T_q \Sigma) \right)^m + (2|p - q|\|\Pi_{T_q \Sigma}(p - q)\|)^{(1+\tau)m}, \quad (4.2)$$

where we also used Lemma A.1. By Lemma [3.1] the angle can be estimated as

$$\hat{\alpha}(T_p \Sigma, T_q \Sigma) \leq \|Du(x) - Du(y)\| \leq \|D^2u\|_{C^0(F \cap B_N(0))} \cdot |x - y|$$

$$\leq \|D^2u\|_{C^0(F \cap B_N(0))} \cdot |p - q|, \quad (4.3)$$

for $x, y \in F \cap B_N(0)$ with $p = x + u(x)$ and $q = y + u(y)$. For the projection in the second term in (4.2) we use the notation $g(\xi) := \xi + u(\xi)$ for $\xi \in F$ to write $p - q = g(x) - g(y) = \int_0^1 Dg(tx + (1-t)y)(x - y)dt$, so that by virtue of $Dg(y)(x - y) \in T_q \Sigma$ and $Dg(\xi) = Dg(\eta) = Du(\xi) - Du(\eta)$ for all $\xi, \eta \in F$ we derive the identity

$$\Pi_{T_q \Sigma}(p - q) = \int_0^1 \Pi_{T_q \Sigma}((Du(tx + (1-t)y) - Du(y))(x - y)) \cdot dt$$

$$= \int_0^1 \int_0^1 \Pi_{T_q \Sigma}(\frac{d}{d\sigma}Du(\sigma(tx + (1-t)y) + (1-\sigma)y)|_{\sigma=s}(x - y)) \cdot ds \cdot dt,$$
which yields the estimate $|\Pi T_{\Sigma}(p - q)| \leq \|D^2 u\|_{C^0(F \cap B_N(0))} \cdot |q - p|^2$. Combining this with (4.2) and (4.3) leads to the proof of (4.1). Finally, integrating (4.1) for $\tau > 0$ over $\Sigma \cap B_N(0) \times \Sigma \cap B_N(0)$ one obtains $E^\tau(\Sigma \cap B_N(0)) < \infty$. \hfill \Box

With a covering argument (as in the proof of Theorem 1.5 below) one can show the following corollary.

**Corollary 4.2.** Embedded locally compact $C^2$-submanifolds have locally finite energy $E^\tau$ for all $\tau > 0$.

A closer look on the proof of Lemma 4.1 reveals that a substantially lower regularity already guarantees finite Möbius energy.

**Lemma 4.3 (Fractional Sobolev regularity).** Let $F \in \mathcal{G}(n, m)$, $1 \leq m \leq n$, $\tau > 0$, $u \in C^{0,1} \cap W^{1+1/(1+\tau), (1+\tau)m}(F, F^\perp)$, $\Sigma_u := \text{graph } u$, and fix a point $z = \zeta + u(\zeta) \in \Sigma_u$. Then for all $\tau > 0$ there is a constant $C = C(\tau, m, \text{Lip } u) > 0$ such that $E^\tau(\Sigma_u \cap B_r(z)) \leq C[Du]^{(1+\tau)m}C^r(\Sigma_u \cap B_r(z)) \leq C(Du)_{1/(1+\tau)-1, (1+\tau)m}$.

**Proof.** Applying the Area Formula [17, Theorem 3.9 & Section 3.3.4B] we find a constant $C > 0$ depending on $m$ and on Lip $u$ such that

$$E^\tau(\Sigma_u \cap B_r(z)) \leq C \int_{F \cap B_r(z)} \frac{L^\tau(g(x, g(y), T_{g(x)}\Sigma_u, T_{g(y)}\Sigma_u))}{|g(x) - g(y)|^{2m}} d\mathcal{H}^m(x) d\mathcal{H}^m(y),$$

where we used the notation $g(\xi) := \xi + u(\xi)$ for $\xi \in F$ as in the proof of Lemma 4.1. As in that proof we can adjust the constant $C$ (now depending also on $\tau$) to bound the numerator of the integrand from above by

$$C \cdot \left[ \frac{1}{n} \left( T_{g(x)}\Sigma_u, T_{g(y)}\Sigma_u \right)^{(1+\tau)m} + (2|g(x) - g(y)|^{-1}\Pi T_{g(y)}\Sigma_u (g(x) - g(y))) \right]^{(1+\tau)m}.$$ 

For the second term one estimates $|\Pi T_{g(y)}\Sigma_u (g(x) - g(y))| \leq \int_0^1 |Du(tx + (1-t)y) - Du(y)| \cdot |x - y| dt$, which together with Lemma [B.1] and $|x - y| \leq |g(x) - g(y)|$ implies

$$E^\tau(\Sigma_u \cap B_r(z)) \leq C \int_{F \cap B_r(z)} \frac{\|Du(x) - Du(y)\|^{(1+\tau)m}}{|x - y|^{2m}} d\mathcal{H}^m(x) d\mathcal{H}^m(y)$$

$$+ C \int_0^1 \int_{F \cap B_r(z)} \frac{\|Du(tx + (1-t)y) - Du(y)\|^{(1+\tau)m}}{|x - y|^{2m}} d\mathcal{H}^m(x) d\mathcal{H}^m(y) dt.$$ 

Changing variables to $\tilde{x} = tx + (1-t)y$ yields $|x - y| = |	ilde{x} - y|/t$, so that

$$E^\tau(\Sigma_u \cap B_r(z)) \leq C (1 + 1/(m + 1)) [Du]^{(1+\tau)m}_{(1+\tau)-1, (1+\tau)m}. \quad (4.4)$$

**Proof of Theorem 1.5.** Consider an embedded compact submanifold of $\mathbb{R}^n$ with local graph representations of class $C^{0,1} \cap W^{1+1/(1+\tau)-1, (1+\tau)m}$. Then, we can find $z_1, \ldots, z_n \in \Sigma$ and radii $r_i > 0$, such that $\Sigma \subseteq \bigcup_{i=1}^N \Sigma \cap B_{r_i/2}(z_i)$ and $\Sigma \cap B_{r_i}(z_i) = (z_i + \text{graph } u_i) \cap B_{r_i}(z_i)$ with $u_i \in C^{0,1}(F_i, F_i^\perp) \cap W^{1+1/(1+\tau)-1, (1+\tau)m}(F_i, F_i^\perp)$ where $F_i \in \mathcal{G}(n, m)$ for
which implies after absorbing the last summand (1 − \(\chi(1 + \sigma)\))\(|\Pi_G(z - p)| ≤ (\sigma + (1 + \sigma)\chi)|\Pi_G(z - p)|

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\[\int_{\Sigma} \int_{\Sigma \cap B_r(q)} L_r(p, q, T_p \Sigma, T_q \Sigma) |p - q|^{-2m} d\mathcal{H}^m(p) d\mathcal{H}^m(q)\]

\[\leq \sum_{i=1}^{N} E^r(\Sigma_i \cap B_{r_i}(z_i)) \leq C (1 + 1/(m + 1)) \sum_{i=1}^{N} |\Sigma_i| (1 + \tau)\frac{m}{(1 + \tau)}\frac{m}{(1 + \tau)} + \mathcal{H}^m(\Sigma_i) 2r^{-2m} < \infty.\]

\[\boxdot\]

\section*{Appendix A. Angles}

We first recall in Lemma \[\text{A.1}\] some general identities for the angle metric \[\text{(1.5)}\] on the Grassmannian \(\mathcal{G}(n, m)\), demonstrate a simple inclusion for two cones around different \(m\)-planes and with different opening angles (Lemma \[\text{A.2}\]), and estimate in Lemma \[\text{A.3}\] the angle between two planes approximating a set in two different ways on different scales. Then we introduce principal angles and relate these to the angle metric (see Definition \[\text{A.4}\] - Lemma \[\text{A.6}\]), which finally allows us to compare in Corollary \[\text{A.7}\] our Möbius invariant energies \(E^r\) to the Kusner-Sullivan energy \(E_{KS}\).

\textbf{Lemma A.1} (8.9 (3) in \[3\]). Let \(F, G \in \mathcal{G}(n, m)\), then

\[\|\Pi_F - \Pi_G\| = \|\Pi_{F^\perp} - \Pi_{G^\perp}\| = \|\Pi_{F^\perp} \circ \Pi_G\| = \|\Pi_G \circ \Pi_{F^\perp}\| = \|\Pi_G \circ \Pi_{F^\perp}\|.\]

Recall from \[\text{(1.7)}\] our notation \(C_\chi(\beta, F)\) for a cone around \(F \in \mathcal{G}(n, m)\), centered at \(x\) with opening angle \(2\arctan \beta\).

\textbf{Lemma A.2} (Cone Lemma). Let \(p \in \mathbb{R}^n\), \(F, G \in \mathcal{G}(n, m)\) with \(\mathcal{G}(F, G) \leq \chi\) for some \(\chi \geq 0\), and assume that \(\sigma, \kappa \geq 0\) satisfy

\[\frac{\sigma + (1 + \sigma)\chi}{1 - (1 + \sigma)\chi} \leq \kappa.\]

Then we have \(C_p(\sigma, F) \subset C_p(\kappa, G)\).

\textbf{Proof.} Let \(z \in C_p(\sigma, F)\) and estimate

\[|\Pi_{G^\perp}(z - p)| \leq |\Pi_{F^\perp}(z - p)| + \chi|z - p| \leq \sigma|\Pi_{F^\perp}(z - p)| + \chi|\Pi_{G^\perp}(z - p)| + \chi|\Pi_{G^\perp}(z - p)| \leq \sigma|\Pi_{G}(z - p)| + (\sigma + 1)\chi(|\Pi_{G}(z - p)| + |\Pi_{G^\perp}(z - p)|),\]

which implies after absorbing the last summand \((1 - \chi(1 + \sigma))|\Pi_{G^\perp}(z - p)| \leq (\sigma + (1 + \sigma)\chi)|\Pi_{G}(z - p)|\), proving the claim by means of assumption \[\text{(A.1)}\]. \(\boxdot\)
For the following estimate between two approximating planes recall Definition (3.2) of the theta-number with respect to a fixed plane.

**Lemma A.3.** Let \( p_1, p_2 \in \Sigma \subset \mathbb{R}^n \) and \( 0 < r_1 \leq r_2, \ d_1, d_2 \in (0,1/2) \), and \( F_1, F_2 \in \mathcal{G}(n,m) \) such that \( |p_1 - p_2| < r_1/2 \), and
\[
\text{dist} \left( \xi, \Sigma \cap B_r(p_1) \right) \leq d_1 r_1 \text{ for all } \xi \in (p_1 + F_1) \cap B_r(p_1), \quad (A.2)
\]
\[
\beta_{\Sigma}(p_2, F_2; r_2) \leq d_2. \quad (A.3)
\]
Then there is a constant \( \tilde{C} = \tilde{C}(m) > \sqrt{2} \) with \( \zeta(F_1, F_2) \leq 2\tilde{C}(d_1 + 2d_2 r_2/r_1)/(1 - 2d_1) \).

**Proof.** For an orthonormal basis \( \{e_1, \ldots, e_m\} \) of \( F_1 \) and an arbitrary \( \varepsilon \in (0,(1-2d_1)/2) \) we set \( x_0 := p_1 \) and
\[
x_i := p_1 + (1-2d_1-2\varepsilon)r_1 \cdot e_i/2 \in (p_1 + F_1) \cap B_{r_1}(p_1) \text{ for } i = 1, \ldots, m, \quad (A.4)
\]
By means of (A.2) we can find \( q_i \in \Sigma \cap B_{r_1}(p_1) \) such that
\[
|q_i - x_i| \leq (d_1 + \varepsilon)r_1 \text{ for all } i = 1, \ldots, m. \quad (A.5)
\]
Additionally, we define \( q_0 := p_1 \) so that \( |q_i - p_1| \leq |q_i - x_i| + |x_i - p_1| \leq (d_1 + \varepsilon)r_1 + (1-2d_1-2\varepsilon)r_1/2 = r_1/2 \) for all \( i = 1, \ldots, m \). Consequently, \( |q_i - p_2| \leq |q_i - p_1| + |p_1 - p_2| < r_1 \leq r_2 \) for all \( i \) and \( |q_0 - p_2| = |p_1 - p_2| < \frac{r_1}{2} < r_2 \). Hence, \( q_i \in \Sigma \cap B_{r_1}(p_2) \) for all \( i \in \{0, \ldots, m\} \). Now we can use (A.3) to obtain points \( y_i \in (p_2 + F_2) \cap B_{r_2}(p_2) \) with
\[
|y_i - q_i| \leq d_2 r_2 \text{ for all } i = 0, \ldots, m. \quad (A.6)
\]
For \( i \in \{1, \ldots, m\} \) we can define \( \tilde{x}_i := \frac{x_i - x_0}{|x_i - x_0|} = e_i \in F_1 \) and \( \tilde{y}_i := \frac{y_i - y_0}{|y_i - y_0|} \in F_2 \). Then (A.4)–(A.6) imply for all \( i = 1, \ldots, m \)
\[
|\tilde{x}_i - \tilde{y}_i| = \frac{2}{(1-2d_1-2\varepsilon)r_1} |x_i - x_0 - y_i + y_0| \leq \frac{2}{1-2d_1-2\varepsilon} \left( d_1 + \varepsilon + 2r_2 d_2/r_1 \right).
\]
Consequently, if
\[
2 \left( d_1 + 2r_2 d_2/r_1 \right)/(1-2d_1) < 1/\sqrt{2}, \quad (A.7)
\]
then we can choose \( 0 < \varepsilon \ll 1 \) such that \( \text{dist} \left( e_i, F_2 \right) \leq |\tilde{x}_i - \tilde{y}_i| < 1/\sqrt{2} \) for all \( i = 1, \ldots, m \). Applying [31 Prop. 2.5] then yields a constant \( \tilde{C} = \tilde{C}(m) > \sqrt{2} \) with \( \zeta(F_1, F_2) \leq 2\tilde{C} \left( d_1 + \varepsilon + 2r_2 d_2/r_1 \right)/(1-2d_1-2\varepsilon) \). We conclude this case by taking the limit \( \varepsilon \to 0 \). If (A.7) does not hold, then the desired inequality for \( \zeta(F_1, F_2) \) trivially holds true for the same constant \( \tilde{C} \).

Now we recall the definition of principal angles.

**Definition A.4** (Ch. 12.4.3 in [20]). For two \( m \)-planes \( F, G \in \mathcal{G}(n,m) \), the principal angles \( \vartheta_1, \ldots, \vartheta_m \in [0, \pi/2] \) are given by \( \cos \vartheta_1 := \sup_{x \in F} \sup_{y \in S^{n-1}} |\langle x, y \rangle| \) and
\[
\cos \vartheta_k := \sup_{x \in F \setminus \text{span}(x_1, \ldots, x_{k-1})} \sup_{y \in G \setminus \text{span}(y_1, \ldots, y_{k-1})} |\langle x, y \rangle| \text{ for } k \in \{2, \ldots, m\}.
\]
Here, for each \( k \in \{1, \ldots, m\} \) the principal vectors \( x_k \in (F \setminus \text{span}(x_1, \ldots, x_{k-1})) \cap S^{n-1} \) and \( y_k \in (G \setminus \text{span}(y_1, \ldots, y_{k-1})) \cap S^{n-1} \) are chosen to satisfy \( \langle x_k, y_k \rangle = \cos \vartheta_k \).
It is mentioned in [20, Ch. 12.4.3] that one can find principal vectors such that
\[
\langle x_i, y_j \rangle = \begin{cases} \cos \vartheta_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (A.8)
\]
by virtue of the singular value decomposition of a matrix. Moreover, each set of principal vectors forms an orthonormal basis of the corresponding plane.

**Lemma A.5** (Angle metric vs. principal angle). For \( F, G \in \mathcal{G}(n, m) \) one has \( \sin \vartheta_m = \vartheta(F,G) \), where \( \vartheta_m \in [0, \pi/2] \) denotes the largest principal angle for \( F \) and \( G \).

**Proof.** For a collection of principal vectors of \( F \) and \( G \) satisfying (A.8), we define \( X_1 := (x_1 \ldots | x_m) \in \mathbb{R}^{n \times m} \) and \( Y_1 := (y_1 \ldots | y_m) \in \mathbb{R}^{n \times m} \). With an orthonormal basis \((x_{m+1}, \ldots, x_n)\) of \( F^\perp \) and \((y_{m+1}, \ldots, y_n)\) of \( G^\perp \), we set \( X_2 := (x_{m+1} \ldots | x_n) \in \mathbb{R}^{n \times (n-m)} \) and \( Y_2 := (y_{m+1} \ldots | y_n) \in \mathbb{R}^{n \times (n-m)} \). Then, \( X := X_1X_2 \) and \( Y := Y_1Y_2 \) are orthogonal matrices, and so is \( X^T \cdot Y \). In particular, for \( z = (\zeta, 0) \in (\mathbb{R}^m \times \{0\}^{n-m}) \cap S^{m-1} \), one finds \( 1 = |X^T \cdot Y \cdot z|^2 = |X^T \cdot Y_1 \cdot \zeta|^2 + |X^T \cdot Y_2 \cdot Y_1 \cdot \zeta|^2 \).

Therefore,
\[
\|X^T \cdot Y_1\| := \sup_{\zeta \in \mathbb{R}^m \cap S^{m-1}} |X^T \cdot Y_1 \cdot \zeta| = \sqrt{1 - \inf_{\zeta \in \mathbb{R}^m \cap S^{m-1}} |X^T \cdot Y_1 \cdot \zeta|^2}. \quad (A.9)
\]
By choice of \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \) one finds \( X_1^T \cdot Y_1 = \text{diag}(\cos \vartheta_1, \ldots, \cos \vartheta_m) \).

Consequently, the right hand side of (A.9) is equal to \( \sqrt{1 - \cos^2 \vartheta_m} = \sin \vartheta_m \). On the other hand, we have \( \Pi_{F^\perp}(z) = X_2 \cdot X_2^T \cdot z \) and \( \Pi_{G}(z) = Y_1 \cdot Y_1^T \cdot z \) for all \( z \in \mathbb{R}^n \).

Together with Lemma A.1 and the orthonormality of \( X \) and \( Y \), one finds
\[
\vartheta(F,G) = \|\Pi_{F^\perp} \circ \Pi_{G}\| = \|X_2 \cdot X_2^T \cdot Y_1 \cdot Y_1^T\| = \|X^T \cdot X_2 \cdot X_2^T \cdot Y_1 \cdot Y_1^T \cdot Y\| = \|X^T \cdot Y_1 \| = \|X^T \cdot Y_2\|, \text{ which coincides with the left hand side of (A.9).}
\]

\(\square\)

**Lemma A.6** (Combined angle). For \( F, G \in \mathcal{G}(n, m) \) and \( \cos \vartheta_1 := \Pi_i m \vartheta_i \), where \( \vartheta_1, \ldots, \vartheta_m \) are the principal angles for \( F \) and \( G \), one has \( \sin \vartheta_m \leq \sin \vartheta \leq \sqrt{m} \sin \vartheta_m \), and
\[
0 \leq (1 - \cos \vartheta)^m \leq (\sin \vartheta)^{(1+\tau)m} \quad \forall \tau \in [0,1), \quad (A.10)
\]
\[
0 \leq 2^{-m}(\sin \vartheta)^{2m} \leq (1 - \cos \vartheta)^m \leq (\sin \vartheta)^{2m}, \quad (A.11)
\]
\[
0 \leq c \cdot (\sin \vartheta)^{(1+\tau)m} \leq (1 - \cos \vartheta)^m \quad \forall \tau \in (1, \infty), \quad (A.12)
\]
where \( c = c(\tau, m) := \min\{1, (1 - \frac{1}{\tau}) m^{-\tau(1+\tau)/2} \cdot \left( 1 - \frac{1}{\tau} \right)^{-\tau}\} \).

**Proof.** The ordering \( 0 \leq \vartheta_1 \leq \cdots \leq \vartheta_m \leq \pi/2 \) implies \( (\cos \vartheta_m)^m \leq \cos \vartheta \leq \cos \vartheta_m \), so that
\[
\sin \vartheta_m = \sqrt{1 - \cos^2 \vartheta_m} \leq \sqrt{1 - \cos^2 \vartheta} = \sin \vartheta. \quad (A.13)
\]
On the other hand, by the formula for the geometric series, one has \( 1 - (\cos \vartheta_m)^{2m} = (1 - \cos^2 \vartheta_m) \cdot \sum_{i=0}^{m-1} (\cos \vartheta_m)^{2i} \leq (1 - \cos^2 \vartheta_m) \cdot m \).

Hence, we obtain
\[
\sin \vartheta = \sqrt{1 - \cos^2 \vartheta} \leq \sqrt{1 - (\cos \vartheta_m)^{2m}} \leq \sqrt{m} \cdot \sqrt{1 - \cos^2 \vartheta_m} = \sqrt{m} \sin \vartheta_m. \quad (A.14)
\]
Combining (A.13) and (A.14) yields \( \sin \vartheta_m \leq \sin \vartheta \leq \sqrt{m} \sin \vartheta_m \). For the remaining inequalities, we define \( h_\tau(\vartheta) := (1 - \cos \vartheta) \cdot (\sin \vartheta)^{-(1+\tau)} \) and compute

\[
\lim_{\vartheta \to \pi/2} h_\tau(\vartheta) = 1 \quad \text{and} \quad \lim_{\vartheta \to 0} h_\tau(\vartheta) = \begin{cases} 
0 & \text{for } \tau \in [0, 1), \\
1/2 & \text{for } \tau = 1, \\
\infty & \text{for } \tau \in (1, \infty).
\end{cases} \tag{A.15}
\]

For \( \vartheta \in (0, \pi/2) \) we can differentiate \( h_\tau \) to obtain \( h'_\tau(\vartheta) = (\sin \vartheta)^{-\tau} \cdot [1 - (1 + \tau) \cos \vartheta \cdot (1 + \cos \vartheta)^{-1}] \). In the case \( \tau \leq 1 \), one finds \((1 + \tau) \cos \vartheta / (1 + \cos \vartheta) \leq (1 + \tau)/2 \leq 1\) guaranteeing that \( h_\tau \) is non-decreasing on \((0, \pi/2)\). Together with (A.10) this implies (A.11). In the case \( \tau > 1 \), one finds that \( h'_\tau(\vartheta) = 0 \) if and only if \( \cos \vartheta = 1/\tau \), and therefore (A.12) holds true. \( \square \)

Since the numerator (1.4) in the energy density of the Kusner-Sullivan energy \( E_{KS} \) uses the combined angle defined in Lemma A.6, it is easy to conclude from the inequalities in that lemma the following comparison result for \( E^r \) and \( E_{KS} \).

**Corollary A.7.** For \( \Sigma \in \mathcal{A}^m(\alpha, M) \) one finds

\[
0 \leq E_{KS}(\Sigma) \leq \sqrt{m}^{(1+\tau)m} E^r(\Sigma) \quad \text{for all} \quad \tau \in [0, 1), \tag{A.16}
\]

\[
0 \leq 2^{-m} E^r(\Sigma) \leq E_{KS}(\Sigma) \leq m^m E^r(\Sigma) \quad \text{for} \quad \tau = 1, \tag{A.17}
\]

\[
0 \leq c E^r(\Sigma) \leq E_{KS}(\Sigma) \quad \text{for all} \quad \tau \in (1, \infty), \tag{A.18}
\]

where \( c = c(\tau, m) > 0 \) is the constant defined in Lemma A.6.

**Remark A.8.** We observe that a wedge-shaped singularity leads to infinite \( E^r \)-energy for any \( \tau > -1 \). Indeed, consider for simplicity the set \( \Sigma_\beta = \{ xe_1 + \beta [x e_2 + ye_3] : x, y \in \mathbb{R} \} \subset \mathbb{R}^3 \) for \( \beta \in (0, \infty) \) and an orthonormal basis \( \{e_1, e_2, e_3\} \), and set \( H(p) \) to be tangential to the set for all \( p = p(x, y) \in \Sigma_\beta \) with \( x \neq 0 \). For \( i \in \mathbb{N} \) let \( p_i := (e_1 + \beta e_2)/2^i \) and \( q_i := (-e_1 + \beta e_2 + e_3)/2^i \). Then we can compute \( |p_i - q_i| = \sqrt{5}/2^i \), \( |\langle e_3, p_i - q_i \rangle| = 2^{-i} \) and \( ||H(p_i) - (p_i - q_i)|| = 2\beta/(2^i \sqrt{1 + \beta^2}) \). Therefore, we find for \( \kappa(\beta) := \beta/(4 \sqrt{1 + \beta^2}) \) and \( \varepsilon_i := \kappa/2^i \) a constant \( C = C(\beta, m, \tau) > 0 \) such that \( F_\tau(\mu, \eta, e_3) / |\mu - \eta|^{2m} \geq 2^{2m} C \) for all \( \eta \in B_{\varepsilon_i}(p_i) \cap \Sigma_\beta \) and \( \mu \in B_{2\varepsilon_i}(q_i) \cap \Sigma_\beta \), where we used that \( e_3 \in H(\eta) = H(p_i) \) for all such \( \eta \). Moreover, for \( \eta_1 \in B_{\varepsilon_i}(p_{i_1}) \) and \( \eta_2 \in B_{\varepsilon_i}(p_{i_2}) \) with \( i_1 < i_2 \), one finds \( |\eta_1 - \eta_2| > (1 - \kappa)/2^{i_1} - (1 + \kappa)/2^{i_2} \geq (1 - 3\kappa)/2^{i_1+1} > 0 \) since \( \kappa < 1/4 \). Hence, the \( \varepsilon_i \) neighbourhoods of the \( p_i \) are disjoint. Analogously, the same holds true for \( q_i \). Finally, for any \( N \in \mathbb{N} \) there is an index \( i_0 \in \mathbb{N} \) such that

\[
E^r(\Sigma_\beta \cap B_N(0)) \geq \sum_{i=i_0}^{\infty} \int_{\Sigma_\beta \cap B_{\varepsilon_i}(q_i)} \int_{\Sigma_\beta \cap B_{\varepsilon_i}(p_i)} \frac{F_\tau(\mu, \eta, e_3)}{|\mu - \eta|^{2m}} d\mathcal{H}^m(\eta) \ d\mathcal{H}^m(\mu) \geq \sum_{i=i_0}^{\infty} \omega^2_m \cdot (\kappa/2^i)^{2m} \cdot 2^{2m} C = \infty.
\]

**APPENDIX B. LIPSCHITZ GRAPHS**

First, in Lemma B.1 we estimate the deviation of a Lipschitz graph’s tangent plane from its domain plane. Then in Lemma B.2 we shift Lipschitz functions without changing the trace of the graph, and in the Tilting Lemma B.3 we provide a lower bound
on the size of the projection of a Lipschitz graph onto a plane slightly tilted from its domain plane. Similarly, we prove in Lemma [B.4] that almost orthogonal planes intersect Lipschitz graphs in exactly one point, and finally, we prove that the intersection of two Lipschitz graphs that are sufficiently flat in comparison to the angle between their domain planes is contained in a lower-dimensional graph; see Lemma [B.5] This quantitative result generalizes the well-known fact that the transversal intersection of two $C^1$-submanifolds constitutes a lower-dimensional $C^1$-submanifold as, e.g., proven in [21] p. 30].

**Lemma B.1.** Let $\beta \in [0, 1)$, $F \in \mathcal{G}(n,m)$, and assume $u \in C^{0,1}(F,F^\perp)$ satisfies $\Lip u \leq \beta$. For $x, y \in F$, with $p = x + u(x)$ and $q = y + u(y)$, such that $Du(x)$ and $Du(y)$ exist, one finds $\varepsilon(T_p \graph u, F) \leq \|Du(x)\| \leq \beta$, and

$$\varepsilon(T_p \graph u, T_q \graph u) \leq \|Du(x) - Du(y)\| \leq \sqrt{\frac{1 + \beta^2}{1 - \beta^2}} \varepsilon(T_p \graph u, T_q \graph u).$$

*Proof.* This is an immediate implication of 8.9 (5) in [3].

**Lemma B.2** (Shifting Lemma). For any $x \in p + \graph u$, where $u \in C^{0,1}(F,F^\perp)$, $F \in \mathcal{G}(n,m)$, there exists a function $\tilde{u} \in C^{0,1}(F,F^\perp)$ with $\Lip \tilde{u} = \Lip u$ and $\tilde{u}(0) = 0$, such that $x + \graph \tilde{u} = p + \graph u$.

*Proof.* For $x = p + x + u(x), x \in F$, set $\tilde{u}(y) := u(y + \xi) - u(\xi).$ Then the claim follows from the following identity for an arbitrary point $q \in p + \graph u$:

$$q = p + \eta + u(\eta) = x + \eta - \xi + u(\eta) - u(\xi) = x + \eta - \xi + \tilde{u}(\eta - \xi).$$

**Lemma B.3** (Tilting Lemma). Let $F,G \in \mathcal{G}(n,m)$ satisfy $\varepsilon(F,G) \leq \chi$ for some $\chi \in [0, 1)$, and suppose $u \in C^{0,1}(F,F^\perp)$ with $u(0) = 0$ and whose Lipschitz constant $\Lip u$ satisfies $\sigma := \chi(1 + \Lip u) < 1$. Then we have

$$B_{\frac{(1 - \chi)u}{\sqrt{1 - \sigma \|u\|}}}(0) \cap G \subset \Pi_G(\graph u \cap B_\rho(0)) \text{ for all } \rho > 0.$$

*Proof.* For $p \in \graph u$ set $x := \Pi_F(p), z := \Pi_G(p)$ and estimate

$$|x - z| = |(\Pi_F - \Pi_G)(p)| \leq \chi|p| = \chi|x + u(x)| \leq \chi(1 + \Lip u)|x| = \sigma|x|, \quad \text{(B.1)}$$

since $u(0) = 0$. Define $\phi_1 : F \to \graph u$, $x \mapsto x + u(x)$ and $\phi_2 : \graph u \to G$ by $p \mapsto \Pi_G(p)$, and look at the composition $\phi := \phi_2 \circ \phi_1 : F \to G$, then (B.1) implies

$$|x - \phi(x)| \leq \sigma|x| \text{ for all } x \in F. \quad \text{(B.2)}$$

Taking the linear isometry $I_F : F \to \mathbb{R}^m$ and define $\Psi \in C^0(\mathbb{R}^m, \mathbb{R}^m)$ by $\Psi := I_F \circ \Pi_{|F|} \circ \phi \circ I_F^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ we infer from (B.2) the inequality

$$|\xi - \Psi(\xi)| = |I_F(x) - \Psi(I_F(x))| = |I_F(x) - I_F \circ \Pi_{|F|} \circ \phi(x)| = |x - \Pi_{|F|} \circ \phi(x)| = |\Pi_F(x - \phi(x))| \leq |x - \phi(x)| \leq \sigma|x| = \sigma|I_F^{-1}(\xi)| = \sigma|\xi| \text{ for all } \xi = I_F(x) \in \mathbb{R}^m. \quad \text{(B.3)}$$
Applying [33, Prop. 2.5] to $F := \Psi$, we find for any given $\delta > 0$ that for all $\eta \in B_{(1-\sigma)(0)} \subset \mathbb{R}^m$, there exists a $\xi \in B_{(1-\sigma)(0)}$, such that $\Psi(\xi) = \eta$. Moreover, for each $y \in F \cap B_{(1-\sigma)(0)}$, there is a unique $\eta \in B_{(1-\sigma)(0)} \subset \mathbb{R}^m$, such that $y = \Pi_F^{-1}(\eta)$, so that with $x := \Pi_F^{-1}(\xi) \in F \cap B_{(1-\sigma)(0)}$ for $\xi$ as above one finds $\Pi_F \circ \phi(x) = \Pi_F^{-1} \circ \Psi(\xi) = \Pi_F^{-1}(\eta) = y$, which implies that $\phi : F \to G$ is surjective since $\Pi_{F|G}$ is bijective, due to $\angle(F,G) \leq \chi < 1$, see [33, Lem. 2.2]. Notice that $\phi(0) = \phi_2 \circ \phi_1(0) = \phi_2(0 + u(0)) = \Pi_G(0) = 0$, and (B.2) implies that $(1 + \sigma)|x| \geq |\phi(x)| \geq (1 - \sigma)|x|$ for all $x \in F$, so that

$$G \cap B_r(0) \subset \phi(B_{\frac{1}{1-\sigma}} (0) \cap F) = \Pi_G \circ \phi_1(B_{\frac{1}{1-\sigma}}(0) \cap F) \subset \Pi_G(\text{graph } u \cap B_{\sqrt{1 + (\text{Lip } u)^2}}(0)),$$

because $|x + u(x)|^2 = |x|^2 + |u(x)|^2 \leq (1 + (\text{Lip } u)^2) \, |x|^2$ for all $x \in F$. \hfill $\square$

**Lemma B.4** (Quasi-normal Planes). Let $\beta, \tau > 0$ satisfy $\beta + \tau < 1$, and suppose there are $m$-planes $G, F \in \mathcal{G}(n, m)$ with $\angle(G,F) \leq \tau$, and a function $u \in C^{0,1}(G, G^\perp)$ with $\text{Lip } u \leq \beta$. Then for any point $p \in \mathbb{R}^n$ one has

$$(p + \text{graph } u) \cap (z + F^\perp) = \{z\} \quad \text{for all} \quad z \in p + \text{graph } u.$$

**Proof.** For $z = p + \xi + u(\xi), \xi \in G$, and any other graph point $x = p + \xi + u(\xi), \xi \in G$, one has $|\Pi_{G^\perp}(x-z)| = |u(\xi) - u(\xi)| \leq \beta|\xi - \xi| = \beta|\Pi_G(x-z)|$ which proves that the graph $p + \text{graph } u$ is contained in the cone $C_z(\beta, G)$ for any graph point $z$. For arbitrary $y \in (z + F^\perp)$ one has $\Pi_F(y - z) = 0$, so that

$$|\Pi_{G^\perp}(y-z)| \geq |\Pi_{F^\perp}(y-z)| - \tau|y-z| = (1 - \tau)|y-z|$$

$$\geq (1 - \tau)|\Pi_G(y-z)| \geq \beta|\Pi_G(y-z)| \quad \text{for all} \quad y \in (z + F^\perp) \setminus \{z\},$$

i.e., $(z + F^\perp) \setminus \{z\} \cap C_z(\beta, G) = \emptyset$. \hfill $\square$

**Lemma B.5** (Intersecting Lipschitz Graphs). Assume

$$0 \leq \sigma < \frac{\chi}{8} < 1/8. \quad \text{(B.4)}$$

Then for any two $m$-planes $F, G \in \mathcal{G}(n, m)$ with $\angle(F,G) \geq \chi$ and for functions $f \in C^{0,1}(F, F^\perp), g \in C^{0,1}(G, G^\perp)$ satisfying $f(0) = g(0)$, $\text{Lip } f \leq \sigma$, $\text{Lip } g \leq \sigma$, the intersection of their graphs is contained in the graph of a Lipschitz function with an at most $(m-1)$-dimensional domain. More precisely, there exists a $j$-plane $X \in G(n,j)$ for some $0 \leq j \leq m-1$, and a Lipschitz function $S \in C^{0,1}(X, X^\perp)$, such that the intersection graph $f \cap \text{graph } g$ is contained in graph $S$. In particular, $\dim_{\mathcal{M}}(\text{graph } f \cap \text{graph } g) \leq j \leq m - 1$.

**Proof.** Let $\{x_1, \ldots, x_n\} \subset F$ and $\{y_1, \ldots, y_m\} \subset G$ be two sets of orthonormal principal vectors satisfying (A.8). Then by Lemma A.3 we find a minimal $j \in \{0, \ldots, m-1\}$ such that $\sin \vartheta_k \geq \chi$ for all $k \geq j + 1$. Here, $\vartheta_k$ denotes the $k$-th principal angle as defined in Definition A.2. Then, for any $v \in \text{span}\{x_{j+1}, \ldots, x_m\}$, i.e., $v = \sum_{i=j+1}^m a_i x_i$, we obtain

$$|\Pi_G(v)|^2 = \sum_{k=1}^m |\langle v, y_k \rangle y_k|^2 = \sum_{k=j+1}^m |a_k|^2 \cdot |\langle x_k, y_k \rangle|^2 = \sum_{k=j+1}^m |a_k|^2 \cos^2 \vartheta_k$$

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Combining (B.12) and (B.13) yields that this intersection contains no other point, thus proving the lemma in this simple case.

\[
C := 5(v_i^2 \cos^2 \theta_{j+1}) \leq |v|^2 \cos^2 \theta_{j+1},
\]
where we used the monotonicity of \(\theta_k\) and (A.8). Consequently,

\[
|\Pi_{G^\perp}(v)|^2 = |v|^2 - |\Pi_{G}(v)|^2 \geq |v|^2 \left(1 - \cos^2 \theta_{j+1}\right) = |v|^2 \sin^2 \theta_{j+1} \geq |v|^2 \cdot \chi^2. \tag{B.5}
\]

First, we investigate the case \(j = 0\) which can only occur if \(F \cap G = \{0\}\) and, therefore, \(n \geq 2m\). Since \(f(0) = 0 = g(0)\) the origin is contained in graph \(f \cap graph \, g\), and we claim that this intersection contains no other point, thus proving the lemma in this simple situation. Assume contrariwise that there is a point \(p \in graph \, f \cap graph \, g\). Then the Lipschitz continuity of \(f\) implies \(\|\Pi_{F^\perp}(q)\| \leq \sigma |\Pi_{F}(q)|\), so that\( |q| \leq (1 + \sigma) |\Pi_{F}(q)|\), from which we infer by means of (B.5) applied to \(v := \Pi_{F}(q)\)

\[
|\Pi_{G^\perp}(q)| \geq |\Pi_{G^\perp}(\Pi_{F}(q))| - |\Pi_{G^\perp}(\Pi_{F^\perp}(q))| \geq \chi |\Pi_{F}(q)| - |\Pi_{F^\perp}(q)|
\]

\[
\geq (\chi - \sigma) |\Pi_{F}(q)| \geq (\chi - \sigma)(1 + \sigma)^{-1} |q|.
\] \tag{B.6}

The Lipschitz continuity of \(g\), on the other hand, yields \(\|\Pi_{G^\perp}(q)\| \leq \sigma |\Pi_{G}(q)| \leq \sigma |q|\), which can be combined with (B.6) to find \((\chi - \sigma)(1 + \sigma)^{-1} \leq \sigma\) contradicting (B.4).

If \(j > 0\), then we define \(Z := F \cap G = \text{span}\{x_1, \ldots, x_i\}\), where we allow \(i = 0\) if \(F \cap G = \{0\}\). \(Y := \text{span}\{x_{j+1}, \ldots, x_m\}\), \(W := F \cap \text{span}\{Z, Y\}^\perp = \{x_{i+1}, \ldots, x_j\}\), and \(X := F \cap Y^\perp = \text{span}\{x_1, \ldots, x_i\}\). We claim that for all \(q_1, q_2 \in graph \, f \cap graph \, g\), and \(C := 5(\chi - 8\sigma)^{-1}\), we have

\[
|\Pi_{Y}(q_2 - q_1)| \leq C |\Pi_{X}(q_2 - q_1)|. \tag{B.7}
\]

Assuming the contrary, one finds \(q_1, q_2 \in graph \, f \cap graph \, g\) with

\[
|\Pi_{X}(q_2 - q_1)| < |\Pi_{Y}(q_2 - q_1)|/C. \tag{B.8}
\]

As in the first case, the Lipschitz continuity of \(f\) and \(g\) yields

\[
|\Pi_{F^\perp}(q_2 - q_1)| \leq \sigma |\Pi_{F}(q_2 - q_1)| \leq \sigma |q_2 - q_1|, \tag{B.9}
\]

\[
|\Pi_{G^\perp}(q_2 - q_1)| \leq \sigma |\Pi_{G}(q_2 - q_1)| \leq \sigma |q_2 - q_1|.
\] \tag{B.10}

Since \(\text{Id}_{\mathbb{R}^n} = \Pi_{X} + \Pi_{Y} + \Pi_{F^\perp}\) and \(\Pi_{F} = \Pi_{X} + \Pi_{Y}\), (B.10) and (B.8) guarantee

\[
|q_2 - q_1| = |(\Pi_{X} + \Pi_{Y} + \Pi_{F^\perp})(q_2 - q_1)| \leq (1 + \sigma) (|\Pi_{X}(q_2 - q_1)| + |\Pi_{Y}(q_2 - q_1)|)
\]

\[
< (1 + \sigma) (1 + 1/C) |\Pi_{Y}(q_2 - q_1)|. \tag{B.11}
\]

Setting \(p_i := \Pi_{X}(q_i)\) for \(i = 1, 2\), we compute \(\Pi_{G^\perp}(q_2 - q_1) = \Pi_{G^\perp}(\Pi_{X}(q_2 - q_1)) + \Pi_{G^\perp}(p_2 - p_1)\). Consequently, (B.8) and (B.10) imply

\[
|\Pi_{G^\perp}(p_2 - p_1)| \leq |\Pi_{G^\perp}(q_2 - q_1)| + |\Pi_{G^\perp}(\Pi_{X}(q_2 - q_1))| \leq (\sigma + 1/C)|q_2 - q_1|. \tag{B.12}
\]

On the other hand, since \(\Pi_{X} = \Pi_{Y} + \Pi_{F^\perp}\), (B.5), (B.9), and (B.11) guarantee

\[
|\Pi_{G^\perp}(p_2 - p_1)| \geq |\Pi_{G^\perp}(\Pi_{Y}(q_2 - q_1))| - |\Pi_{G^\perp}(\Pi_{F^\perp}(q_2 - q_1))|
\]

\[
\geq \chi |\Pi_{Y}(q_2 - q_1)| - |\Pi_{F^\perp}(q_2 - q_1)| > (\chi(1 + \sigma)^{-1} (1 + 1/C)^{-1} - \sigma) |q_2 - q_1|. \tag{B.13}
\]

Combining (B.12) and (B.13) yields \(\chi(1 + \sigma)^{-1} (1 + 1/C)^{-1} - \sigma < \sigma + 1/C\), contradicting (B.4). Hence, (B.7) holds true. Therefore, for \(q_1, q_2 \in graph \, f \cap graph \, g\) with \(\Pi_{X}(q_1) = \Pi_{X}(q_2)\), one finds \(|\Pi_{Y}(q_2 - q_1)| \leq C |\Pi_{X}(q_2 - q_1)| = 0\). Due to (B.9) and \(\Pi_{F} = \Pi_{X} + \Pi_{Y}\), we obtain \(|\Pi_{F^\perp}(q_2 - q_1)| \leq \sigma |\Pi_{F}(q_2 - q_1)| = 0\). Consequently, \(q_1 = q_2\),
i.e., for all \( x \in X \), there exists at most one \( q_x \in \text{graph} f \cap \text{graph} g \) such that \( \Pi_X(q_x) = x \).

To define the map \( S \) set \( M := \{ x \in X : \text{ graph } f \cap \text{graph } g \cap (x + X^\perp) \neq \emptyset \}. \) Then,

\[
\text{graph } f \cap \text{graph } g \cap (x + X^\perp) = \{ q_x \} \quad \text{for all } x \in M,
\]

and we obtain the well-defined map \( S_M : M \to X^\perp, \ x \mapsto q_x - x \). By \( (\text{B.14}) \) one finds

\[
\text{graph } f \cap \text{graph } g = \bigcup_{x \in X} \left( \text{graph } f \cap \text{graph } g \cap (x + X^\perp) \right) = \bigcup_{x \in M} q_x = \text{graph } S_M.
\]

Moreover, for \( \xi_1, \xi_2 \in M \), we can use the fact that \( q_{\xi_i} - \xi_i \in X^\perp \) and \( \xi_i \in X \) for \( i = 1, 2 \), to write

\[
|S_M(\xi_1) - S_M(\xi_2)| = |q_{\xi_1} - \xi_1 - (q_{\xi_2} - \xi_2)|, \text{ which equals } \|\Pi_X(q_{\xi_1} - \xi_1) + \Pi_X(q_{\xi_2} - \xi_1) - \Pi_X(q_{\xi_1} - \xi_1) - \Pi_X(q_{\xi_2} - \xi_2) - \Pi_X(q_{\xi_1} - \xi_1)\| = \|\Pi_X(q_{\xi_1} - q_{\xi_2})\|. \]

This expression can be bounded from above by \( |\Pi_Y(q_{\xi_1} - q_{\xi_2})| + |\Pi_{X^\perp}(q_{\xi_1} - q_{\xi_2})| \) since \( \Pi_X + \Pi_{X^\perp} = \Pi_Y + \Pi_{X^\perp} \). With \( (\text{B.16}) \) and \( \Pi_F = \Pi_X + \Pi_Y \) we can compute \( |\Pi_F(q_{\xi_1} - q_{\xi_2})| \leq \sigma(|\Pi_X(q_{\xi_1} - q_{\xi_2})| + |\Pi_Y(q_{\xi_1} - q_{\xi_2})|) \).

Consequently, \( |S_M(\xi_1) - S_M(\xi_2)| \) is bounded from above by \( ((1 + \sigma) C + \sigma) |\Pi_X(q_{\xi_1} - q_{\xi_2})| \), where we additionally applied \( (\text{B.7}) \). Finally, using \( q_{\xi_1} - \xi_1 \in X^\perp \) to derive the identity \( |\Pi_X(q_{\xi_1} - q_{\xi_2})| = |\xi_1 - \xi_2 + \Pi_X(q_{\xi_1} - \xi_1) - \Pi_X(q_{\xi_2} - \xi_2)| = |\xi_1 - \xi_2| \), we obtain

\[
|S_M(\xi_1) - S_M(\xi_2)| \leq ((1 + \sigma) C + \sigma) |\xi_1 - \xi_2|.
\]

Hence, \( S_M \) is Lipschitz continuous with \( \text{Lip } S_M \leq (1 + \sigma) C + \sigma \). Due to Kirszbraun’s theorem [1] 2.10.43 we can find a Lipschitz continuous extension \( S : X \to X^\perp \) with \( \text{Lip } S = \text{Lip } S_M \). Hence, we have

\[
\text{graph } f \cap \text{graph } g = \text{graph } S_M \subset \text{graph } S.
\]

In particular, by virtue of [17] 2.4.2 Thm. 2 (ii) and \( \text{dim } X = j \), we finally conclude

\[
\text{dim}_E \left( \text{graph } f \cap \text{graph } g \right) \leq \text{dim}_E \left( \text{graph } S \right) = j.
\]

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