ON THE WEAK AND ERGODIC LIMIT OF THE SPECTRAL SHIFT FUNCTION

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ABSTRACT. We discuss convergence properties of the spectral shift functions associated with a pair of Schrödinger operators with Dirichlet boundary conditions at the end points of a finite interval $(0, r)$ as the length of interval approaches infinity.

1. INTRODUCTION

In this note we study the relationship between the spectral shift function $\xi$ associated with the pair $(H_0, H)$ of the half-line Dirichlet Schrödinger operators

\[ H_0 = -\frac{d^2}{dx^2} \quad \text{and} \quad H = -\frac{d^2}{dx^2} + V(x) \quad \text{in} \quad L^2(0, \infty) \]

and the spectral shift function $\xi^r$ associated with the pair $(H^r_0, H^r)$ of the corresponding Schrödinger operators on the finite interval $(0, r)$ with the Dirichlet boundary conditions at the end points of the interval.

Recall that given a pair of self-adjoint operators $(H_0, H)$ in a separable Hilbert space $\mathcal{H}$ such that the difference $(H + iI)^{-1} - (H_0 + iI)^{-1}$ is a trace class operator, the spectral shift function $\xi$ associated with the pair $(H_0, H)$ is uniquely determined (up to an additive constant) by the trace formula

\[ \text{tr}(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(\lambda)f'(\lambda)d\lambda, \]

valid for a wide class of functions $f$ [3, 4, 10, 11, 15].

In the case where both $H_0$ and $H$ are bounded from below, the standard way to fix the undetermined constant is to require that

\[ \xi(\lambda) = 0 \quad \text{for} \quad \lambda < \inf\{\text{spec}(H_0) \cup \text{spec}(H)\}. \]

Under the short range hypothesis on the potential $V$ that

\[ \int_0^\infty (1 + x)|V(x)|\,dx < \infty, \]

the pairs of the half-line Schrödinger operators $(H_0, H)$ and their finite-interval box-approximations $(H^r_0, H^r)$ are resolvent comparable and the spectral shift functions $\xi$ and $\xi^r$ associated with the pairs $(H_0, H)$ and $(H^r_0, H^r)$ satisfying the normalization condition (1.2) are well-defined. The specifics of the one-dimensional case is that each of the spectral shift functions $\xi$ and $\xi^r$ admits a unique left-continuous representative, for which we will keep the same notation.

The main result of this note is the following theorem.

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Theorem 1. Assume that a real-valued function $V$ on $(0, \infty)$ satisfies condition (1.3). Denote by $\xi$ and $\xi^r$ the left-continuous spectral shift functions associated with the pairs $(H_0, H)$ and $(H^r_0, H^r)$ of the Dirichlet Schrödinger operators on the semi-axis $(0, \infty)$ and on the finite interval $(0, r)$, respectively.

Then for any continuous function $g$ on $\mathbb{R}$ with compact support both the (weak) limit

$$\lim_{r \to \infty} \int_{-\infty}^{\infty} \xi^r(\lambda)g(\lambda)d\lambda = \int_{-\infty}^{\infty} \xi(\lambda)g(\lambda)d\lambda$$

and the Cesàro limit

$$\lim_{r \to \infty} \frac{1}{r} \int_0^r \xi^r(\lambda)d\lambda = \xi(\lambda), \quad \lambda \in \mathbb{R} \setminus \text{spec}_d(H) \cup \{0\}$$

exist.

If, in addition, the Schrödinger operator $H = -\frac{d^2}{dx^2} + V(x)$ has no zero energy resonance, then convergence (1.5) takes place for $\lambda = 0$ as well.

We remark that the weak convergence result (1.4) has been obtained in [9] in the case of arbitrary dimension under the assumption that the potential $V$ belongs to the Birman–Solomyak class $\ell^1(L^2)$. In contrast to the Feymann-Kac path integration approach developed in [9], our approach is based on the study of fine properties of the eigenvalue counting function available in the one-dimensional case.

We also remark that the class of potentials satisfying condition (1.3) is slightly different from the Birman–Solomyak class $\ell^1(L^2)$: the Birman–Solomyak condition allows a slower decay at infinity, while condition (1.3) admits a $L^1$-type singularities at finite points in contrast to the fact that only $L^2$-type singularities are allowed in the Birman–Solomyak class.

2. SOME GENERAL CONVERGENCE RESULTS

In this section we will develop the necessary analytic background for proving the convergence results (1.4) and (1.5).

Assume the following hypothesis.

Hypothesis 2.1. Assume that $f$ is a Riemann integrable function on the interval $[0, 1]$ and $g$ is a continuous function on $[0, 1]$. Suppose that a sequence of real-valued measurable functions $\{f_n\}_{n=1}^{\infty}$ on $[0, 1]$ converges to $f$ pointwise on the open interval $(0, 1)$ and that the convergence is uniform on every compact set of the semi-open interval $(0, 1]$. Assume, in addition, that the sequence $\{f_n\}_{n=1}^{\infty}$ has a $|g|$-integrable majorante $F$. That is,

$$|f_n(x)| \leq F(x), \quad x \in [0, 1], \ n \in \mathbb{N},$$

and

$$\int_0^1 F(x)|g(x)|dx < \infty.$$
Then

\[
\lim_{n \to \infty} \int_0^1 \tilde{f}_n(x)g(x)dx = \int_0^1 f(x)g(x)dx,
\]

where the sequence \( \{\tilde{f}_n\}_{n=1}^\infty \) is given by

\[
\tilde{f}_n(x) = [r_n x + f_n(x)] - [r_n x], \quad n \in \mathbb{N}, \quad x \in [0, 1],
\]

and \([\cdot]\) stands for the integer value function.

**Proof.** Without loss of generality one may assume that the function \( g \) in non-negative on \([0, 1]\).

A simple change of variables shows that

\[
\int_0^1 \tilde{f}_n(x)g(x)dx = \frac{1}{r_n} \int_0^{r_n} \left( [t + f_n \left( r_n^{-1}t \right)] - [t] \right) g \left( r_n^{-1}t \right) dt
\]

\[
= \sum_{k=0}^{[r_n]-1} \frac{1}{r_n} \int_k^{k+1} \left( [t + f_n \left( r_n^{-1}t \right)] - [t] \right) g \left( r_n^{-1}t \right) dt + \varepsilon_n,
\]

\[
= \sum_{k=0}^{[r_n]-1} \frac{1}{r_n} \int_0^1 \left( t + k + f_n \left( \frac{t + k}{r_n} \right) \right) g \left( \frac{t + k}{r_n} \right) dt + \varepsilon_n,
\]

where

\[
\varepsilon_n = \int_0^1 \tilde{f}_n(x)g(x)dx.
\]

Taking into account that \([t + k] = [t]\) for any \( t \) whenever \( k \) is an integer, and \([t] = 0\) for \( t \in (0, 1)\), we get

\[
\int_0^1 \tilde{f}_n(x)g(x)dx = \sum_{k=0}^{[r_n]-1} \frac{1}{r_n} \int_0^1 \left[ t + f_n \left( \frac{t + k}{r_n} \right) \right] g \left( \frac{t + k}{r_n} \right) dt + \varepsilon_n.
\]

The uniform bound \(|\tilde{f}_n(x)| \leq \|F\| + 1\) (with \( F \) from (2.1)) shows that

\[
|\varepsilon_n| \leq \int_0^1 (\|F\| + 1) g(x)dx, \quad n \in \mathbb{N},
\]

and therefore

\[
\lim_{n \to \infty} \varepsilon_n = 0.
\]

Combining (2.6) and (2.7) with the estimate

\[
\left| \int_0^1 \left[ t + f_n \left( \frac{t + k}{r_n} \right) \right] \left( g \left( \frac{t + k}{r_n} \right) - g \left( \frac{k}{r_n} \right) \right) dt \right| \leq (\|f_n\|_{\infty} + 1) \omega_g(r_n^{-1}),
\]

where \( \omega_g(\cdot) \) stands for the modulus of continuity of the function \( g \), one concludes that

\[
\lim_{n \to \infty} \int_0^1 \tilde{f}_n(x)g(x)dx = \lim_{n \to \infty} \sum_{k=0}^{[r_n]-1} \frac{g(kr_n^{-1})}{r_n} \int_0^1 \left[ t + f_n \left( \frac{t + k}{r_n} \right) \right] dt.
\]
(provided that the limit in the RHS exists.) Therefore, in order to prove assertion (2.3) it suffices to establish the equality

\[(2.9) \quad \lim_{n \to \infty} \sum_{k=0}^{[r_n]-1} \frac{g(kr_n^{-1})}{r_n} \int_0^1 \left[ t + f_n \left( \frac{t+k}{r_n} \right) \right] dt = \int_0^1 f(x) g(x) dx. \]

To prove (2.9), assume temporarily that the sequence \( \{f_n\}_{n=1}^{\infty} \) converges to \( f \) uniformly on the closed interval \([0, 1]\).

Since the function \( t \mapsto [t] \) is monotone, from the inequality

\[
\left[ t + \inf_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f - \delta_n \right] \leq \left[ t + f_n \left( \frac{t+k}{r_n} \right) \right] \leq \left[ t + \sup_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f + \delta_n \right],
\]

where \( \delta_n = \| f - f_n \|_{\infty} \), one immediately obtains that

\[(2.10) \quad \left[ t + \frac{\inf_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f - \delta_n}{r_n} \right] \leq \left[ t + f_n \left( \frac{t+k}{r_n} \right) \right] \leq \left[ t + \frac{\sup_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f + \delta_n}{r_n} \right], \]

\[t \in [0, 1], \quad k = 0, 1, \ldots, [r_n] - 1.\]

Integrating (2.10) against \( t \) from 0 to 1 and noticing that

\[\int_0^1 [t + a] dt = a \quad \text{for all} \quad a \in \mathbb{R},\]

and that \( g \) is a non-negative function, one obtains the following two-sided estimate

\[(2.11) \quad \sum_{k=0}^{[r_n]-1} \frac{g(kr_n^{-1})}{r_n} \left( \frac{\inf_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f - \delta_n}{r_n} \right) \leq \sum_{k=0}^{[r_n]-1} \frac{g(kr_n^{-1})}{r_n} \int_0^1 \left[ t + f_n \left( \frac{t+k}{r_n} \right) \right] dt \leq \sum_{k=0}^{[r_n]-1} \frac{g(kr_n^{-1})}{r_n} \left( \sup_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f + \delta_n \right). \]

Since \( f \) is a Riemann integrable function by hypothesis and the function \( g \) is continuous on \([0, 1]\), one concludes that

\[(2.12) \quad \lim_{n \to \infty} \frac{1}{r_n} \sum_{k=0}^{[r_n]-1} g(kr_n^{-1}) \inf_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f = \lim_{n \to \infty} \frac{1}{r_n} \sum_{k=0}^{[r_n]-1} g(kr_n^{-1}) \sup_{[\frac{k}{r_n}, \frac{k+1}{r_n}]} f \]

\[= \int_0^1 f(x) g(x) dx. \]

The additional assumption that the sequence \( \{f_n\}_{n=1}^{\infty} \) converges to \( f \) uniformly on the closed interval \([0, 1]\) means that

\[\lim_{n \to \infty} \delta_n = 0, \]

which together with (2.11) and (2.12) proves (2.9). This completes the proof of (2.3) (provided that \( \{f_n\}_{n=1}^{\infty} \) converges to \( f \) uniformly on \([0, 1]\)).

To remove the extra assumption, we proceed as follows.

Given \( 0 < \varepsilon < 1 \), one gets the inequality

\[(2.13) \quad \lim_{n \to \infty} \left| \int_0^1 \tilde{f}_n(x) g(x) dx - \int_0^1 f(x) g(x) dx \right| \]

\[\leq \varepsilon. \]
Suppose that the limit
\[ \lim_{n \to \infty} \int_{\varepsilon}^{1} f_n(x) g(x) \, dx - \int_{\varepsilon}^{1} f(x) g(x) \, dx \]
exists. Then Corollary 2.3, which completes the proof of the lemma.

By hypothesis the sequence \( \{ f_n \}_{n=1}^{\infty} \) converges uniformly on the interval \( (\varepsilon, 1] \) and therefore by the first part of the proof one concludes that
\[ \lim_{n \to \infty} \int_{\varepsilon}^{1} f_n(x) g(x) \, dx - \int_{\varepsilon}^{1} f(x) g(x) \, dx = 0. \]  
Combining (2.13) and (2.14) one obtains the inequality
\[ \lim_{n \to \infty} \int_{0}^{1} f_n(x) g(x) \, dx - \int_{0}^{1} f(x) g(x) \, dx \leq \int_{0}^{\varepsilon} (2F(x) + 1) |g(x)| \, dx. \]
By the second inequality in (2.13), the right hand side of (2.15) can be made arbitrarily small by an appropriate choice of \( \varepsilon \) and hence
\[ \lim_{n \to \infty} \int_{0}^{1} f_n(x) g(x) \, dx - \int_{0}^{1} f(x) g(x) \, dx = 0 \]
which completes the proof of the lemma.

**Corollary 2.3.** Let \( h \) be a real-valued measurable bounded measurable function on \( [0, \infty) \). Suppose that the limit
\[ A = \lim_{x \to \infty} h(x) \]
exists. Then
\[ \lim_{r \to \infty} \frac{1}{r} \int_{0}^{r} ([x + h(x)] - |x|) \, dx = A. \]

**Proof.** We will prove that convergence (2.16) holds as \( r \) approaches infinity along an arbitrary sequence \( \{ r_n \}_{n=1}^{\infty} \) of positive numbers such that \( \lim_{n \to \infty} r_n = \infty \).

Introduce the sequence of functions \( \{ f_n \}_{n=1}^{\infty} \) by
\[ f_n(x) = h(r_n x), \quad x \in [0, 1]. \]
The sequence \( \{ f_n \}_{n=1}^{\infty} \) converges pointwise to a constant function \( f \) given by
\[ A = \lim_{x \to \infty} h(x), \]
and, moreover, the convergence is uniform on every compact set of the semi-open interval \( (0, 1] \). Since, by hypothesis, the function \( h \) is bounded, so is the sequence \( \{ f_n \}_{n=1}^{\infty} \) and hence one can apply Lemma 2.2 to conclude that
\[ \lim_{n \to \infty} \int_{0}^{1} ([r_n x + f_n(x)] - \lfloor r_n x \rfloor) \, dx = \int_{0}^{1} A \, dx = A. \]  
By a change of variables one gets
\[ \frac{1}{r_n} \int_{0}^{r_n} ([x + h(x)] - |x|) \, dx = \int_{0}^{1} ([r_n x + f_n(x)] - \lfloor r_n x \rfloor) \, dx, \]
which together with (2.17) proves that
\[ \lim_{n \to \infty} \frac{1}{r_n} \int_{0}^{r_n} ([x + h(x)] - |x|) \, dx = A \]
and hence (2.16) holds, since \( \{ r_n \}_{n=1}^{\infty} \) is an arbitrary sequence. \( \square \)
3. Proof of Theorem 1

Proof. As it easily follows from the trace formula (1.1), the value of the spectral shift function \( \xi(\lambda) \) on the negative semi-axis associated with the pair \((H_0, H)\) of the (Dirichlet) Schrödinger operators is directly linked to the number of negative eigenvalues of the operator \( H \) that are smaller than \( \lambda \),

\[
\xi(\lambda) = -N(\lambda), \quad \lambda < 0.
\]  

(3.1)

Analogously, the spectral shift function \( \xi^r \) associated with the finite-interval Schrödinger operators \((H^r_0, H^r)\) on the negative semi-axis can be computed via the eigenvalue counting function \( N^r(\lambda) \) for the Schrödinger operator \( H^r \) (cf. [10]),

\[
\xi^r(\lambda) = -N^r(\lambda), \quad \lambda < 0.
\]  

(3.2)

For \( \lambda \geq 0 \), the function \( \xi \) admits the representation in terms of the phase shift \( \delta \), associated with the potential \( V \) (see, e.g., [2], [4], [6], [15]),

\[
\xi(\lambda) = -\pi^{-1} \delta(\sqrt{\lambda}), \quad \lambda \geq 0.
\]  

(3.3)

On the other hand, on the positive semi-axis the function \( \xi^r \) can be represented as

\[
\xi^r(\lambda) = \left[ \pi^{-1} r \sqrt{\lambda} \right] - \left[ \pi^{-1} r \sqrt{\lambda} + \pi^{-1} \delta^r(\sqrt{\lambda}) \right], \quad \lambda \geq 0, \quad r > 0,
\]  

(3.4)

where \( \delta^r \) is the phase shift associated with the cut off potential \( V^r \) given by

\[
V^r(x) = \begin{cases} 
V(x), & 0 \leq x \leq r, \\
0, & x > r.
\end{cases}
\]  

(3.5)

To prove (3.4) one observes that

\[
\xi^r(\lambda) = N^r_0(\lambda) - N^r(\lambda), \quad \lambda \geq 0, \quad r > 0,
\]  

(3.6)

where \( N^r_0 \) is the eigenvalue counting function for the Schrödinger operator \( H^r_0 \) on the finite interval.

Since

\[
N^r_0(\lambda) = \left[ \pi^{-1} r \sqrt{\lambda} \right], \quad \lambda \geq 0,
\]  

and \( N^r \) can be represented as

\[
N^r(\lambda) = \left[ \pi^{-1} r \sqrt{\lambda} + \pi^{-1} \delta^r(\sqrt{\lambda}) \right], \quad \lambda \geq 0,
\]  

(3.7)

(by the well known counting principle that is a direct consequence of the Sturm oscillation theorem (see, e.g., [13] Ch. II, Sec. 6)), one gets (3.4).

In particular, since the counting functions \( N, N^r, \) and \( N^r_0 \) are continuous from the left and under hypothesis (1.3) the phase shift \( \delta \) is continuous on \([0, \infty)\), equations (3.1), (3.3) and (3.4), (3.6) determine the left-continuous representatives for the spectral shift functions \( \xi \) and \( \xi^r \), respectively.

To prove the first assertion (1.4) of the theorem we proceed as follows. Given a continuous function \( g \) with compact support, one splits the left hand side of (1.4) into two parts

\[
\int_{-\infty}^{\infty} \xi^r(\lambda) g(\lambda) d\lambda = I_r + II_r, \quad r > 0,
\]

where

\[
I_r = \int_{-\infty}^{0} \xi^r(\lambda) g(\lambda) d\lambda \quad \text{and} \quad II_r = \int_{0}^{\infty} \xi^r(\lambda) g(\lambda) d\lambda.
\]
First we prove that
\begin{equation}
(3.8) \quad \lim_{r \to \infty} I_r = \int_{-\infty}^{0} \xi(\lambda) g(\lambda) d\lambda.
\end{equation}

Assume that the half-line Schrödinger operator $H$ has $m, m \geq 0$, negative eigenvalues denoted by
\[
\lambda_1(H) < \lambda_2(H) < \ldots < \lambda_m(H).
\]

Then in accordance with a result in [1], there exists a $r_0$ such that for all $r > r_0$ the operator $H^r$ has $m^r, m^r \geq m$, negative eigenvalues
\[
\lambda_1(H^r) < \lambda_2(H^r) < \ldots < \lambda_{m^r}(H^r), \quad r > 0,
\]
and, in addition,
\begin{equation}
(3.9) \quad \lim_{r \to \infty} \lambda_k(H^r) = \lambda_k(H), \quad k = 1, 2, \ldots, m.
\end{equation}

Moreover, for any $\varepsilon > 0$ there exists $r(\varepsilon)$ such that
\[
-\varepsilon < \lambda_k(H^r) < 0, \quad k = m + 1, \ldots, m^r, \quad r > r(\varepsilon).
\]

Taking into account (3.2), from (3.9) one gets that
\begin{equation}
(3.10) \quad \lim_{r \to \infty} \xi^r(\lambda) = \xi(\lambda), \quad \lambda \in (-\infty, 0) \setminus \text{spec}_d(H),
\end{equation}
where $\text{spec}_d(H)$ denotes the discrete (negative) spectrum of the operator $H$.

To deduce (3.8) from (3.10) it suffices to apply a Bargmann-type estimate [14] that provides an upper bound for the number $m^r$ of the negative eigenvalues of the operator $H^r$ as follows
\begin{equation}
(3.11) \quad m^r \leq \int_{0}^{\infty} x V_-(x) dx \leq \int_{0}^{\infty} x V_-(x) dx < \infty, \quad r > 0,
\end{equation}
where $V_-$ stands for the negative part of the potential
\[
V_-(x) = \begin{cases} -V(x), & V(x) < 0 \\ 0, & \text{otherwise}. \end{cases}
\]

Indeed, since
\[
|\xi^r(\lambda)| = N^r(\lambda) \leq m^r \leq \int_{0}^{\infty} x V_-(x) dx \leq \int_{0}^{\infty} (1 + x)|V(x)| dx < \infty, \quad \lambda < 0,
\]
equality (3.8) follows from (3.10) by the dominated convergence theorem.

Next we prove that
\begin{equation}
(3.12) \quad \lim_{r \to \infty} II_r = \int_{0}^{\infty} \xi(\lambda) g(\lambda) d\lambda.
\end{equation}

To get (3.12) it suffices to show that for every sequence $\{r_n\}_{n=1}^{\infty}$ of non-negative numbers $r_n$ such that $\lim_{n \to \infty} r_n = \infty$ the equality
\begin{equation}
(3.13) \quad \lim_{n \to \infty} \int_{0}^{\infty} \xi^{r_n}(\lambda) g(\lambda) d\lambda = \int_{0}^{\infty} \xi(\lambda) g(\lambda) d\lambda
\end{equation}
holds.

From (3.4) one derives that
\[
\int_{0}^{\infty} \xi^r(\lambda) g(\lambda) d\lambda = \int_{0}^{\infty} \left( \left[ \pi^{-1} r \sqrt{\lambda} \right] + \left[ \pi^{-1} r \sqrt{\lambda} - \pi^{-1} \delta^r(\sqrt{\lambda}) \right] \right) g(\lambda) d\lambda, \quad r > 0,
\]
and therefore, after a change of variables, one obtains that
\[
\int_0^\infty \xi r_n(\lambda) g(\lambda) d\lambda = \int_0^1 \left( [t_n \lambda] - [t_n \lambda + f_n(\lambda)] \right) \widehat{g}(\lambda) d\lambda, \quad n \in \mathbb{N},
\]
where \( t_n = \pi^{-1} r_n, \) \( \widehat{g}(\lambda) = 2 \lambda g(\lambda^2), \) \( \lambda \geq 0, \) and the sequence of functions \( \{f_n\}_{n=1}^\infty \) is given by
\begin{equation}
(3.14) \quad f_n(\lambda) = \pi^{-1} \delta^\mu \pi(\lambda), \quad n \in \mathbb{N}.
\end{equation}

Without loss of generality (by rescaling) one may assume that \( g(\lambda) = 0 \) for \( \lambda > 1 \) and hence to prove (3.13) it remains to check that the sequence of functions (3.14) satisfies the hypotheses of Lemma 2.2 with
\[
\begin{align*}
& f(\lambda) = \pi^{-1} \delta(\lambda), \quad \lambda > 0. \\
\end{align*}
\]
Indeed, from the phase equation (\[7, \) p. 11, Eq. (13); \[5])
\begin{equation}
(3.15) \quad \frac{d}{dr} \delta(k) = -k^{-1} V(r) \sin^2 (kr + \delta(k)), \quad k > 0, \ r > 0,
\end{equation}
one easily concludes that
\begin{equation}
|\delta'(\sqrt{\lambda}) - \delta(\sqrt{\lambda})| \leq \frac{1}{\sqrt{\lambda}} \int_r^\infty |V(r')| dr', \quad \lambda > 0, \ r > 0,
\end{equation}
and therefore
\begin{equation}
|f_n(\lambda) - f(\lambda)| \leq \frac{1}{\pi \lambda} \int_{\pi r_n}^\infty |V(r')| dr', \quad \lambda > 0, \ n \in \mathbb{N},
\end{equation}
which proves that
\begin{equation}
\lim_{n \to \infty} f_n(\lambda) = f(\lambda), \quad \lambda > 0,
\end{equation}
and that the convergence in (3.18) takes place uniformly on every compact subset of the semi-open interval \((0, 1].\)

Moreover, the following bound
\begin{equation}
(3.19) \quad |f_n(\lambda)| \leq \sup_{\lambda > 0} |f(\lambda)| + \frac{1}{\pi \lambda} \int_0^\infty |V(r')| dr', \quad \lambda > 0, \ n \in \mathbb{N},
\end{equation}
holds and therefore the sequence (3.14) has a \( \widehat{g} \)-integrable majorante.
Finally, under the short range hypothesis (1.3) the phase shift \( \delta(\lambda) \) is a continuous function on \((0, \infty)\) and by the Levinson Theorem,
\begin{equation}
(3.20) \quad \lim_{\lambda \downarrow 0} \delta(\sqrt{\lambda}) = \pi m, \quad m \in \mathbb{Z}_+,
\end{equation}
if there is no zero-energy resonance and
\[
\lim_{\lambda \downarrow 0} \delta(\sqrt{\lambda}) = \pi \left( m + \frac{1}{2} \right), \quad m \in \mathbb{Z}_+, \quad \text{otherwise}.
\]
Therefore, the function \( f \) is continuous on \([0, 1]\) and hence it is Riemann integrable.
Now one can apply Lemma 2.2 to conclude that
\[
\lim_{n \to \infty} \int_0^\infty \xi r_n(\lambda) g(\lambda) d\lambda = -\int_0^1 f(\lambda) \widehat{g}(\lambda) d\lambda = -\frac{1}{\pi} \int_0^1 \delta(\lambda) 2 \lambda g(\lambda^2) d\lambda
\]
\[
= -\frac{1}{\pi} \int_0^1 \delta(\sqrt{\lambda}) g(\lambda) d\lambda = \int_0^\infty \xi(\lambda) g(\lambda) d\lambda,
\]
which proves (3.13).
The proof of (1.4) is complete.
Now, we will prove the second statement (1.5) of the theorem.
First we remark that equation (3.10) immediately implies (1.5) for \( \lambda < 0, \lambda \notin \text{spec}_d(H) \).
In order to prove convergence (1.5) for a fixed \( \lambda > 0 \), we again use equation (3.4) to obtain
\[
\frac{1}{r} \int_0^r \xi^r(\lambda) \, dr = \frac{1}{r} \int_0^r \left[ \pi^{-1} r \sqrt{\lambda} \right] - \left[ \pi^{-1} r \sqrt{\lambda} + \pi^{-1} \delta^r(\sqrt{\lambda}) \right] \, dr, \quad \lambda > 0, \quad r > 0.
\]
After the change of variables \( x = \pi^{-1} r \sqrt{\lambda} \), we get
\[
(3.21)
\frac{1}{r} \int_0^r \xi^r(\lambda) \, dr = \frac{\pi}{\sqrt{\lambda}} \int_0^{\pi \sqrt{\lambda}} \left( x - \left[ x + \pi^{-1} \delta^x(\sqrt{\lambda}) \right] \right) \, dx, \quad \lambda > 0, \quad r > 0.
\]
From (3.16) it follows that the function \( h \) given by
\[
h(x) = \delta^x(\sqrt{\lambda}), \quad x > 0, \quad \lambda > 0,
\]
is a bounded function and that
\[
\lim_{x \to \infty} h(x) = \delta(\sqrt{\lambda}), \quad \lambda > 0.
\]
Hence, applying Corollary 2.3 one concludes that
\[
(3.22) \quad \lim_{r \to \infty} \frac{\pi}{\sqrt{\lambda}} \int_0^{\pi \sqrt{\lambda}} \left( x - \left[ x + \pi^{-1} \delta^x(\sqrt{\lambda}) \right] \right) \, dx = -\delta(\sqrt{\lambda}), \quad \lambda > 0,
\]
and, therefore,
\[
\lim_{r \to \infty} \frac{1}{r} \int_0^r \xi^r(\lambda) \, dr = -\pi^{-1} \delta(\sqrt{\lambda}), \quad \lambda > 0,
\]
which together with (3.3) proves (1.5) for \( \lambda > 0 \).
Finally, to prove (1.5) for \( \lambda = 0 \) in the case of the absence of a zero energy resonance, notice that the corresponding Jost function \( F \) does not vanish on the closed semi-interval \([0, \infty)\). Since
\[
\lim_{r \to \infty} \delta^F(k) = \delta(k), \quad k \geq 0,
\]
it is obvious that \( \delta^F(k) \) is not zero for \( k \in [0, \infty) \) for all \( r \) large enough and hence
\[
(3.23) \quad \lim_{r \to \infty} \delta(r, 0) = \delta(0)
\]
for \( \delta(r, k) = \arg(\delta^F(k)) \) and \( \delta(k) = \arg(\delta^F(k)) \).
Now equations (3.21), (3.23), and Corollary 2.3 imply
\[
\lim_{r \to \infty} \frac{1}{r} \int_0^r \xi^r(0) \, dr = -\pi^{-1} \delta(0).
\]
By the Levinson Theorem (see (3.20)), the quantity \( \pi^{-1} \delta(0) \) coincides with the number \( N(0) \) of (negative) eigenvalues of the Schrödinger operator \( H \). Since
\[
\xi(0) = \lim_{\epsilon \to 0} \xi(-\epsilon) = -N(0),
\]
one concludes that
\[
\lim_{r \to \infty} \frac{1}{r} \int_0^r \xi^r(0) \, dr = \xi(0)
\]
which completes the proof of Theorem 1. \( \square \)

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