On Particle Collisions and Extraction of Energy from a Rotating Black Hole

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INTRODUCTION

In [1] we put the hypothesis that active galactic nuclei can be the sources of ultrahigh energy particles in cosmic rays observed recently by the AUGER group (see [2]) due to the processes of converting dark matter formed by superheavy neutral particles into visible particles — quarks, leptons (neutrinos), photons. If active galactic nuclei are rotating black holes then in [1] we discussed the idea that “This black hole acts as a cosmic supercollider in which superheavy particles of dark matter are accelerated close to the horizon to the Grand Unification energies and can be scattering in collisions.” It was also shown [3] that in Penrose process [4] dark matter particle can decay on two particles, one with the negative energy, the other with the positive one and particles of very high energy of the Grand Unification order can escape the black hole. Then these particles due to interaction with photons close to the black hole will loose energy analogously up to the Greisen-Zatsepin-Kuzmin limit in cosmology [5, 6].

First calculations of the scattering of particles in the ergosphere of the rotating black hole, taking into account the Penrose process, with the result that particles with high energy can escape the black hole, were made in [7, 8]. Recently in [9] it was shown that for the rotating black hole (if it is close to the critical one) the energy of scattering is unlimited. The result of [9] was criticized in [10], where the authors claimed that if the black hole is not a critical rotating black hole so that its dimensionless angular momentum $A \neq 1$ but $A = 0.998$ then the energy is limited.

In this paper we show that the energy of scattering in the centre of mass system can be still unlimited in the cases of multiple scattering. In [31] we calculate this energy, reproduce the results of [9, 10] and show that in some cases (multiple scattering) the results of [10, 11] on the limitations of the scattering energy for nonextremal black holes are not valid.

In §2 we obtain the results for the extraction of the energy after collision in the field of the Kerr’s metric. It occurs that the Penrose process plays important role for getting larger energies of particles at infinity. Our calculations show that the conclusion of [11] about the impossibility of getting at infinity the energy larger than the initial one in particle collisions close to the black hole is wrong.

The system of units $G = c = 1$ is used in the paper.

THE ENERGY OF COLLISION IN THE FIELD OF BLACK HOLES

Let us consider particles falling on the rotating chargeless black hole. The Kerr’s metric of the rotating black hole in Boyer–Lindquist coordinates has the form

$$ds^2 = dt^2 - \frac{2Mr (dt - a \sin^2 \theta d\varphi)^2}{r^2 + a^2 \cos^2 \theta} - (r^2 + a^2) \frac{d\theta^2}{\Delta} - (r^2 + a^2) \sin^2 \theta d\varphi^2,$$

where

$$\Delta = r^2 - 2Mr + a^2,$$  (2)

$M$ is the mass of the black hole, $J = aM$ is angular momentum. In the case $a = 0$ the metric (4) describes the static chargeless black hole in Schwarzschild coordinates. The event horizon for the Kerr’s black hole corresponds to the value

$$r = r_H \equiv M + \sqrt{M^2 - a^2}. \quad (3)$$

The Cauchy horizon is

$$r = r_C \equiv M - \sqrt{M^2 - a^2}. \quad (4)$$

The surface called “the static limit” is defined by the expression

$$r = r_0 \equiv M + \sqrt{M^2 - a^2 \cos^2 \theta}. \quad (5)$$
The region of space-time between the horizon and the static limit is ergosphere. For equatorial ($\theta = \pi/2$) geodesics in Kerr’s metric one obtains (12, § 61):

$$\frac{dt}{d\tau} = \frac{1}{\Delta} \left[ r^2 + a^2 + \frac{2M a^2}{r} \right] \varepsilon - \frac{2Ma}{r} L, \quad (6)$$

$$\frac{d\varphi}{d\tau} = \frac{1}{\Delta} \left[ \frac{2Ma}{r} \varepsilon + \left( 1 - \frac{2M}{r} \right) L \right], \quad (7)$$

$$\left( \frac{dr}{d\tau} \right)^2 = \varepsilon^2 + \frac{2M}{r^3} (a \varepsilon - L)^2 + \frac{a^2 \varepsilon^2 - L^2}{r^2} - \frac{\Delta}{r^2} \delta_1, \quad (8)$$

where $\delta_1 = 1$ for timelike geodesics ($\delta_1 = 0$ for isotropic geodesics), $\tau$ is the proper time of the moving particle, $\varepsilon = \text{const}$ is the specific energy: the particle with rest mass $m$ has the energy $\varepsilon m$ in the gravitational field; $Lm = \text{const}$ is the angular momentum of the particle relative to the axis orthogonal to the plane of movement.

We denote $x = r/M$, $x_H = r_H/M$, $x_C = r_C/M$, $A = a/M$, $l_m = L/m$, and $\Delta_\varepsilon = x^2 - 2x + A^2$. For the energy in the centre of mass frame of two colliding particles with angular momenta $l_1, l_2$, which are nonrelativistic at infinity ($\varepsilon_1 = \varepsilon_2 = 1$) and are moving in Kerr’s metric using (1), (9)–(8) one obtains $\delta$: $$E_{c.m.}^2 = \frac{1}{2m^2} \left[ 2x^2(x - 1) + l_1l_2(2 - x) + 2A^2(x + 1) - 2A(l_1 + l_2) - \sqrt{(2x^2 + 2(l_1 - A)^2 - l_1^2x)(2x^2 + 2(l_2 - A)^2 - l_2^2x)} \right].$$

To find the limit $r \to r_H$ for the black hole with a given angular momentum $A$ one must take in (9) $x = x_H + \alpha$ with $\alpha \to 0$ and do calculations up to the order $\alpha^2$. Taking into account $A^2 = x_H x_C$, $x_H + x_C = 2$, after resolution of uncertainties in the limit $\alpha \to 0$ one obtains

$$E_{c.m.}(r \to r_H) = \sqrt{\frac{(l_1 - l_2)^2}{2x_C(l_1 - l_H)(l_2 - l_H)}}, \quad (10)$$

where $l_H = 2x_H/A$.

For the extremal black hole $A = x_H = 1$, $l_H = 2$ and the expression (10) is divergent when the dimensionless angular momentum of one of the falling into the black hole particles $l \to 2$. The scattering energy in the centre of mass system is increasing without limit $\delta$.

Can one get the unlimited high energy of this scattering energy for the case of nonextremal black hole? Formula (8) leads to limitations on the possible values of the angular momentum of falling particles: the massive particle free falling in the black hole with dimensionless angular momentum $A$ being nonrelativistic at infinity ($\varepsilon = 1$) to achieve the horizon of the black hole must have angular momentum from the interval

$$- \left( 1 + \sqrt{1 + A} \right) = l_L \leq l \leq l_R = \left( 1 + \sqrt{1 - A} \right). \quad \text{(11)}$$

Putting the limiting values of angular momenta $l_L, l_R$ into the formula (10) one obtains the maximal values of the collision energy of particles freely falling from infinity

$$E_{c.m.}^\text{inf}(r \to r_H) = \frac{2m}{\sqrt{1 - A^2}} \sqrt{\frac{1 - A^2 + 2 \sqrt{1 - A^2} + \left( 1 + \sqrt{1 + A} + \sqrt{1 - A} \right)^2}{1 + \sqrt{1 - A^2}}}. \quad \text{(12)}$$

For $A = 1 - \epsilon$ with $\epsilon \to 0$ formula (12) gives:

$$E_{c.m.}^\text{inf}(r \to r_H) \sim 2 \left( \frac{2^{1/4} + 2^{-1/4}}{\epsilon^{1/4}} \right) \frac{m}{\epsilon^{1/4}}. \quad \text{(13)}$$

So even for values close to the extremal $A = 1$ of the rotating black hole $E_{c.m.}^\text{inf}/m$ can be not very large as it is mentioned in (10, 11). So for $A_{\text{max}} = 0.998$ considered as the maximal possible dimensionless angular momentum of the astrophysical black holes (see [12]), from (12) one obtains $E_{c.m.}^\text{max}/m \approx 18.97$.

Does it mean that in real processes of particle scattering in the vicinity of the rotating nonextremal black holes the scattering energy is limited so that no Grand Unification or even Planckian energies can be obtained? Let us show that the answer is no! If one takes into account the possibility of multiple scattering so that the particle falling from the infinity on the black hole with some fixed angular momentum changes its momentum in the result of interaction with particles in the accreting disc and after this is again scattering close to the horizon then the scattering energy can be unlimited.

The limiting value of the angular momentum of the particle close to the horizon of the black hole can be obtained from the condition of positive derivative in (6) $dt/d\tau > 0$, i.e. going “forward” in time. So close to the horizon one has the condition $l < \varepsilon 2x_H/A$ for which $\varepsilon = 1$ gives the limiting value $l_H$.

From (8) one can obtain the permitted interval in $r$ for particles with $\varepsilon = 1$ and angular momentum $l = l_H - \delta$. In the second order in $\delta$ close to the horizon one obtains

$$l = l_H - \delta \Rightarrow \quad x \lesssim x_H + \frac{\delta^2 x_C^2}{4x_H \sqrt{1 - A^2}}. \quad \text{(14)}$$

If the particle falling from the infinity with $l \leq l_R$ arrives to the region defined by (14) and here it interacts with other particles of the accretion disc or it decays on more light particle so that it gets the larger angular momentum $l_1 = l_H - \delta$, then due to (10) the scattering energy in the centre of mass system is

$$E_{c.m.} \approx \frac{m}{\sqrt{\delta}} \sqrt{\frac{2(l_H - l_2)}{1 - \sqrt{1 - A^2}}} \quad \text{(15)}$$
and it increases without limit for $\delta \to 0$. For $A_{\text{max}} = 0.998$ and $l_2 = l_1$, $E_{\text{cm}} \approx 3.85m/\sqrt{\delta}$.

Note that for rapidly rotating black holes $A = 1 - \epsilon$ the difference between $l_H$ and $l_R$ is not large

$$l_H - l_R = 2\frac{\sqrt{1 - A}}{A} \left( \sqrt{1 - A} + \sqrt{1 + A} - A \right)$$

$$\approx 2(\sqrt{2} - 1)\sqrt{\epsilon}, \quad \epsilon \to 0.$$  \hspace{1cm} (16)

For $A_{\text{max}} = 0.998$, $l_H - l_R \approx 0.04$ so the possibility of getting small additional angular momentum in interaction close to the horizon seems much probable.

THE EXTRACTION OF ENERGY AFTER THE COLLISION IN KERR’S METRIC

Let us consider the case when there occurred for some $r > r_H$ interaction between particles with masses $m$, specific energies $\varepsilon_1$, $\varepsilon_2$, specific angular momenta $L_1$, $L_2$ falling into a black hole, so that two new particles with rest masses $\mu$ appeared, one of them (1) moved outside the black hole, the other (2) moved inside it. Denote the specific energies of new particles as $\varepsilon_{1\mu}$, $\varepsilon_{2\mu}$, their angular momenta (in units of $\mu$) as $L_{1\mu}$, $L_{2\mu}$, $v^i = dx^i/ds$ — their 4-velocities. Consider particle movement in the equatorial plane of the rotating black hole.

Conservation laws in inelastic particle collisions for the energy and momentum lead to

$$m(u^i_{(1)} + u^i_{(2)}) = \mu(v^i_{(1)} + v^i_{(2)}).$$  \hspace{1cm} (17)

Equations (17) for $t$ and $\varphi$-components can be written as

$$m(\varepsilon_1 + \varepsilon_2) = \mu(\varepsilon_{1\mu} + \varepsilon_{2\mu}),$$  \hspace{1cm} (18)

$$m(L_1 + L_2) = \mu(L_{1\mu} + L_{2\mu}),$$  \hspace{1cm} (19)

i.e. the sum of energies and angular momenta of colliding particles is conserved in the field of Kerr’s black hole. The initial particles in our case were supposed to be non-relativistic at infinity: $\varepsilon_1 = \varepsilon_2 = 1$ and therefore for the $r$-component from (18) one obtains

$$-m \left[ \frac{2M}{r^3} (a - L_1)^2 + \frac{2M}{r} - \frac{L_1^2}{r^2} \right]$$

$$+ \sqrt{\frac{2M}{r^3} (a - L_2)^2 + \frac{2M}{r} - \frac{L_2^2}{r^2}}$$

$$= \mu \left[ \varepsilon_{1\mu}^2 + \frac{2M}{r^3} (a\varepsilon_{1\mu} - L_{1\mu})^2 + \frac{a^2\varepsilon_{1\mu}^2 - L_{1\mu}^2 - \Delta}{r^2} \right]$$

$$- \left[ \varepsilon_{2\mu}^2 + \frac{2M}{r^3} (a\varepsilon_{2\mu} - L_{2\mu})^2 + \frac{a^2\varepsilon_{2\mu}^2 - L_{2\mu}^2 - \Delta}{r^2} \right].$$  \hspace{1cm} (20)

The signs in (20) are put so that the initial particles and the particle (2) go inside the black hole while particle (1) goes outside the black hole. The values $\varepsilon_{1\mu}$, $\varepsilon_{2\mu}$ are constants on geodesics (12), §61 so the problem of evaluation of the energy at infinity extracted from the black hole in collision reduces to a problem to find these values.

For the case when the collision takes place on the horizon of the black hole ($r \to r_H$) the system (18–20) can be solved exactly

$$\varepsilon_{1\mu} = \frac{AL_{1\mu}}{2r_H}, \quad \varepsilon_{2\mu} = \frac{2m}{\mu} - \frac{AL_{1\mu}}{2r_H}, \quad L_{2\mu} = \frac{m}{\mu}(L_1 + L_2) - L_{1\mu}. $$  \hspace{1cm} (21)

In general case the system of three Eqs. (18–20) for four variables $\varepsilon_{1\mu}$, $\varepsilon_{2\mu}$, $L_{1\mu}$, $L_{2\mu}$ can be solved numerically for a fixed value of one variable (and fixed parameters $m/\mu$, $L_1/M$, $L_2/M$, $a/M$, $r/M$). The example of numerical solution is $\mu/m = 0.3, l_1 = 2.2, l_2 = 2.198, A = 0.99, x = 1.21, l_{1\mu} = 16.35, l_{2\mu} = -1.69, \varepsilon_{1\mu} = 7.215, \varepsilon_{2\mu} = -0.548$. Note that the energy of the second final particle is negative and the energy of the first final particle contrary to the limit obtained in [11] is larger than the energy of initial particles as it must be in the case of a Penrose process [2, 3]. What is the reason of this contradiction? Let us investigate the problem carefully.

Note that if one neglects the states with negative energy in ergosphere energy extracted in the considered process can not be larger than the initial energy of the pair of particles at infinity, i.e. $2m$. The same limit $2m$ for the extracted energy for any (including Penrose process) scattering process in the vicinity of the black hole was obtained in [11]. Let us show why this conclusion is incorrect.

If the angular momentum of the falling particles is the same then (see (20)) one has the situation similar to the usual decay of the particle with mass $2m$ in two particles with mass $\mu$. Due to the Penrose process in ergosphere it is possible that the particle falling inside the black hole has the negative relative to infinity energy and then the extracted particle can have energy larger than $2m$.

The main assumption made in [11] is the supposition of the collinearity of vectors of 4-momenta of the particles falling inside and outside of the black hole (see (9)–(11) in [11]). The authors of [11] say that these vectors are “asymptotically tangent to the horizon generator”.

First note that from (8), (11) for $A < 1$ and $l \leq l_H$ or $A = 1$, but $l < 2$ the limit of $dr/dr$ is not zero at the horizon. This derivative has opposite signs for the falling and outgoing particles. Signs for other components of the 4-momentum are equal.

In the limiting case ($A = 1$, $l_1 = 2$) the expressions $dt/dr$, $d\varphi/dr$ of the components of the 4-velocity of the infalling particle (9), (11) go to infinity when $r \to r_H$, but $dr/dr$ goes to zero. In spite of smallness of $r$-components
in the expression of the square of the 4-momentum vector they have the factor $g_{rr}$ going to infinity at the horizon. So putting them to zero can lead to a mistake. To see if $u_1(1)$ and $v_1(1)$ are collinear it is necessary to put the coordinate $r$ of the collision point to the limit $r_H = M$ and resolve the uncertainties $\infty/\infty$ and $0/0$. For the falling particle $\varepsilon = 1$, $l = 2$ the expressions for components of the 4-vector $u$ can be easily found from (6)–(8). For the particle outgoing from the black hole due to exact solution on the horizon (21) one puts $\varepsilon_1\mu = l_1\mu/2 + \alpha$, where $\alpha$ is some function of $r$ and $l_1\mu$, such that $\alpha \to 0$ when $r \to r_H$. Putting this $\varepsilon_1\mu$ into (6)–(8) one gets for $x = r/M \to 1$

$$\frac{v_t^{(1)}}{u_t^{(1)}} = \frac{v_\phi^{(1)}}{u_\phi^{(1)}} = \frac{\alpha}{x - 1} + \frac{l_1\mu}{2}, \quad (22)$$

$$\frac{v_r^{(1)}}{u_r^{(1)}} = -\sqrt{\frac{2\alpha^2}{(x - 1)^2} + \frac{2l_1\mu\alpha}{x - 1} + \frac{3}{8}l_1^2\mu - \frac{1}{2}}. \quad (23)$$

Due to the condition $dt/d\tau > 0$ (movement forward in time) the necessary condition for collinearity is that both (22) and (23) must be zero, which is not true. This leads to the conclusion that the considerations of the authors of [11] for scattering exactly on the horizon cannot be used for the real situation of particle scattering close to the horizon.

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