Surveying the solar system by measuring angles and times: from the solar density to the gravitational constant

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Abstract

A surprisingly large amount of information on our solar system can be gained from simple measurements of the apparent angular diameters of the sun and the moon. This information includes the average density of the sun, the distance between earth and moon, the radius of the moon, and the gravitational constant. In this note it is described how these and other quantities can be obtained by simple earthbound measurements of angles and times only, without using any explicit information on distances between celestial bodies. The pedagogical and historical aspects of these results are also discussed briefly.
I. INTRODUCTION

All standard textbooks of astronomy and classical mechanics show how the great insights of Kepler and Newton can be used to determine the geometry of our solar system and the physical properties of its constituent bodies (Refs. 1–3 are representative examples). Such calculations are normally based on simple relations between masses and distances, which follow directly from Newton’s or Kepler’s laws. The purpose of the present note is to discuss a little noticed similar relation that allows us to determine the densities and other properties of celestial bodies, as well as the gravitational constant, from entirely earthbound and very simple observations of angles and times.

Section II of this paper shows how the average solar density can be obtained from knowing nothing more than the apparent angle under which the sun appears as seen from the earth and the duration of a year. Section III applies the same idea to the moon. Although the details are slightly more complicated in this case, it still proves possible to calculate the moon’s average distance from the earth, and its radius, by measuring the density of the earth. Section IV shows how the relations derived can also be used to calculate the gravitational constant (which was not known to Newton) by using only data available at Newton’s time.

Although, of course, nothing new is added to current research in astronomy by these considerations, it seems that the resulting curious relations are well suited to stimulate a student’s interest in the subject. They might also be interesting from the point of view of the history of science, because they show how Newton could, e.g., have obtained the gravitational constant long before the celebrated Cavendish experiment. Another aspect of the relations discussed here is that they serve as vivid illustrations of how indirect measurements, together with the assumed universal validity of the laws of physics, allow us to obtain information on quantities that are completely inaccessible for direct measurements. These pedagogical and historical issues are taken up again in the final Section V of this little note.
II. PHYSICAL PROPERTIES OF THE SUN

In order to derive an expression for the average density of the sun, \( \rho_s \), we start by writing this density as the ratio of total mass to total volume,

\[
\rho_s = \frac{M}{V} = \frac{3M}{4\pi R_s^3},
\]

where the sun is assumed to be a perfect sphere of radius \( R_s \). The mass can be eliminated from this equation by equating the centripetal force the earth experiences on its (approximately circular) orbit with the gravitational attraction of the sun, which is assumed to be so much heavier than the earth that the center of mass of the system earth-sun coincides with that of the sun alone. Hence,

\[
-m\omega^2 r = -G \frac{mM}{r^2},
\]

where \( m \) is the earth’s mass, \( G \) the gravitational constant, and \( r \) the distance from earth to sun. Expressing the angular velocity of the earth, \( \omega \), in terms of its rotation period, \( T = \frac{2\pi}{\omega} \), and solving Eq. (2) for \( M \), one readily finds

\[
M = \frac{4\pi^2 r^3}{GT^2},
\]

which is essentially Kepler’s third law.

As seen from Fig. 1 the distance \( r \) between earth and sun is related to the angle under which the sun appears from the center of the earth by

\[
\sin(\beta) = \frac{R_s}{r},
\]

where \( 2\beta \) is the apparent angular diameter of the sun. Since the radius of the earth is much smaller than the distance \( r \) between earth and sun, this equation still holds approximately when \( \beta \) is measured from the surface of the earth, and not its center. (In technical terms this means that surface parallax can be neglected.) Since the solar radius \( R_s \) is also much smaller than \( r \), \( \beta \) is a very small angle and we could replace \( \sin(\beta) \) by \( \beta \) with negligible
error, but for generality and future applicability to other systems we keep the trigonometric functions here and below.

Substituting Eq. (4) into Eq. (3) and the result into Eq. (1) one finds that the solar radius $R_s$ cancels, and what remains is an expression for $\rho_s$ that does not explicitly involve any distances,

$$\rho_s = \frac{3\pi}{GT^2 \sin^3(\beta)}.$$  

(5)

To the best of the author’s knowledge Eq. (5) does not appear explicitly in any of the standard English-language textbooks (although, as the author learned after completing this work, its derivation is assigned as a problem in Ref. 5). The remainder of this note is dedicated to exploration of a few interesting consequences of this result.

A remarkable feature of Eq. (5) is that all quantities referring explicitly to diameters and distances have disappeared. The remaining quantities, $T$ and $\beta$, are accessible via purely earthbound and very simple measurements. $T$ is simply the duration of a year, and $2\beta$, the angle subtended by the sun as seen from the earth, is found to be about half a degree. This value varies slightly during a year because the earth’s orbit is not exactly circular, but this variation is neglected below, where $\beta = 0.25^\circ$ is adopted for convenience. (Suggestions how to measure $\beta$ using simple equipment are made in Sec. V, below). The gravitational constant $G = 6.7 \times 10^{-11} \text{ m}^3\text{s}^{-2}\text{kg}^{-1}$ must be known to evaluate Eq. (5). This does not impede its use in the classroom, but would have had interesting consequences if Newton had known that equation. Some of these consequences are explored in the following sections. Plugging the above numbers in Eq. (5) one obtains

$$\rho_s = 1.7 \times 10^3 \text{ kg m}^{-3} = 1.7 \text{ g cm}^{-3},$$  

(6)

where the second equation expresses the result in units more common in astronomy.

The curious little formula (5) never fails to surprise students and even more mature scientists. The surprise is not really that the value found in Eq. (6) agrees rather well with the literature value $\rho_s = 1.4 \text{ g cm}^{-3}$ (the main source of error doubtlessly being the imprecise
measurement of the angular diameter of the sun\textsuperscript{7}, but that it has been obtained without measuring any distances. It thus provides us with a rather different type of information about our solar system than do the classical measurements of distances and diameters, which usually treat celestial objects as point masses without internal properties. While these determine the scale of the solar system, knowledge of the density allows us to draw conclusions concerning the internal composition of its constituents.

If one, in an admittedly arbitrary way, takes solid iron as representative of the earth’s metals ($\rho_{\text{Fe}} = 7.87 \text{ g cm}^{-3}$),\textsuperscript{6} silicon dioxide as representative for rocks and sand ($\rho_{\text{SiO}_2} = 2.65 \text{ g cm}^{-3}$),\textsuperscript{6} and both of these as representative for the composition of the earth as a whole, it immediately follows that the sun is not composed primarily of these solid materials, and thus of a different physical nature than the earth. Furthermore, by comparing with the densities of other solids, liquids, and gases one concludes that the sun is, due to its low average density, most likely not solid at all.\textsuperscript{8}

Newton himself, by using a related but much more complicated method, based on observations of the orbit of the planets around the sun and of the moon around the earth, arrived at a very similar conclusion. In the \textit{Principia} he writes: \textit{The sun, therefore, is a little denser than Jupiter, and Jupiter than Saturn, and the earth four times denser than the sun; for the sun, by its great heat, is kept in a sort of rarefied state.}\textsuperscript{9} A little later he provides his estimate of the density of the earth: \textit{Since, therefore, the common matter of our earth on the surface thereof is about twice as heavy as water, and a little lower, in mines, is found about three, or four, or even five times heavier, it is probable that the quantity of the whole matter of the earth may be five or six times greater than if it consisted all of water; ...}.\textsuperscript{10}

This estimate of the density of the earth, five or six times that of water, is remarkably close to the modern value\textsuperscript{6} of 5.5 g cm\textsuperscript{-3}. Together with the factor of four by which, according to him, the earth is denser than the sun, one finds that the solar density is inbetween about 1.3 g cm\textsuperscript{-3} and 1.5 g cm\textsuperscript{-3}, which is even closer to the modern value than that found from Eq. (5) (but obtained with considerably more labour).
It is tempting to apply Eq. (5) to the moon as well. After all, the moon’s apparent angular diameter is almost exactly the same as that of the sun (as evidenced by solar eclipses or direct measurement), and its orbital period is also well known. One could thus determine the average density of ... of what? Of the earth or the moon? The answer is that this tentative procedure does not directly determine the average density of either of these bodies. We can again use Fig. 1, which we already used in our determination of the solar density, to understand why. Quite generally, this figure depicts angles and distances characterizing the geometry of the revolution of a lighter body orbiting around a heavier one. However, when we are dealing with the system earth-moon instead of sun-earth, our point of view has shifted from the orbiting to the central body. This means that Eq. (4) is not directly applicable here: the mass \( M \) in Eq. (1) is that of the central body, so that the angular diameter and the density in Eq. (4) are that of the central body as seen from the accompanying body.

This is not the end of the story, though. Eq. (5) determines the average density of the central body, i.e., in the case of the system earth-moon that of the earth. This density is reasonably well approximated by that of the typical materials mentioned above, and can be obtained without requiring any input from celestial mechanics. This enables us to turn the argument around and obtain information on the moon from Eq. (5), by treating the earth’s density as a known quantity. Let us denote the apparent angle of the moon as seen from the earth by \( 2\alpha_m \) and that of the earth as seen from the moon by \( 2\beta_e \). This latter angle seems hard to obtain in preastronautical times, but from Fig. 1, reinterpreted now as depicting the system earth-moon, it follows that

\[
\sin(\beta_e) = \frac{R_e}{R_m} \sin(\alpha_m),
\]

where \( R_e \) and \( R_m \) denote the radii of earth and moon, respectively. This relation allows us to eliminate the unknown angle \( \beta_e \) from the equations. Eq. (5), rewritten for the system earth-moon, reads
\[ \rho_e = \frac{3\pi}{G T_m^2 \sin^3(\beta_e)}, \]  

(8)

where \( T_m \) is the duration of the orbital period of the moon (about 27 days), and \( \rho_e \) the average density of the earth. From this expression we obtain with the help of Eq. (7) that

\[ \rho_e = \frac{3\pi}{G T_m^2} \frac{R^3_m}{R^3_e} \frac{1}{\sin(\alpha_m)}. \]  

(9)

Assuming again that the average density of the earth is close to the average of that of the typical materials iron and silicon dioxide, given above, and treating the radius of the earth as a known quantity (which it certainly was at Newton’s times), we can calculate the radius of the moon from this equation. Putting in the numbers yields

\[ R_m = \left( \frac{G T_m^2 \rho_e}{3\pi} \right)^{1/3} R_e \sin(\alpha_m) \approx 1600 \text{ km}, \]  

(10)

which is to be compared with the literature value \(^3\) of 1738 km. The difference between both values has several sources. First, there is the rather arbitrary choice of using the average density of iron and silicon dioxide to represent that of the earth. Second, the angular diameter of the moon is only imprecisely known, and also changes slightly during the course of a month. Third, the parallax resulting from the fact that the observer is on the surface of the earth and not at its center, has been neglected. Finally, the derivation of Eq. (5) is only correct for circular orbits around a stationary central body, an assumption which is less well satisfied for the motion of the moon around the earth than for that of the earth around the sun. It can be instructive to discuss with students which of these is the dominant source of error. Discussions of how to improve on some of these aspects of the above procedure can make for rewarding science projects of high-school level students, or be used as homework problems for more advanced ones.

Since the ratios between many distances in the solar system were already known to Newton and his contemporaries, the determination of a single absolute value, like \( R_m \), enables one to calculate the other distances explicitly. As an example, we can now work backwards from the counterpart to Eq. (4) for the system earth-moon,
\[
\sin(\alpha_m) = \frac{R_m}{r_{e-m}}, \tag{11}
\]

where \( r_{e-m} \) stands for the distance earth-moon, and find \( r_{e-m} \approx 3.7 \times 10^5 \) km. The literature value\(^6\) for the average distance is \( 3.8 \times 10^5 \) km. Students may find it rewarding to reflect about how it was possible to come this close to today’s value for the distance of the moon by measuring the density of the earth, and what margin of error the various approximations made imply for the final value.

**IV. THE GRAVITATIONAL CONSTANT**

Newton himself did not know the numerical value of the gravitational constant. Following the style of reasoning common in his days he expressed his results in terms of *ratios* between masses, distances and densities. From such ratios the constant prefactor \( G \) of course disappears. Hence, in Newton’s days there seems to have been little interest in determining this and similar universal prefactors. The first reasonably precise determination of \( G \) was made possible in 1798 by Henry Cavendish, more than hundred years after Newton developed his theory of gravitation, and even Cavendish’s experiment was not explicitly recognized as a determination of \( G \) until another hundred years later.\(^4\)

Newton’s lack of knowledge concerning the value of \( G \) is particularly surprising in view of the fact that he could, for example, have obtained this value from the equation of motion for a test particle of mass \( m_t \) in the gravitational field of the earth,

\[
m_t g = G \frac{m_t M_e}{R_e^2}, \tag{12}
\]

where \( g \) is the gravitational acceleration at the earth’s surface. By expressing the mass of the earth, \( M_e \), in terms of its radius and density, and using \( g = 9.81 \text{ ms}^{-2} \) one readily obtains \( G \).

Apparently Newton did not do this simple calculation. He did, however, work hard to obtain his estimate of the density of the sun quoted above. This estimate, in conjunction with Eq. (5), opens up another path for determining the gravitational constant, which would
have also been available to Newton. By solving Eq. (5) for $G$, substituting the numerical values for $T$ and $\beta$ (which were both available at Newton’s times), and using Newton’s own estimate for $\rho_s$, quoted in Sec. II, one finds

$$G = \frac{3\pi}{\rho_s T^2 \sin^3(\beta)} \approx 8.1 \times 10^{-11} \, \text{m}^3\text{s}^{-2}\text{kg}^{-1},$$

(13)

which is surprisingly close to the modern value of $6.7 \times 10^{-11} \, \text{m}^3\text{s}^{-2}\text{kg}^{-1}$, and could have been obtained by Newton and his contemporaries, or later by Cavendish, without having to perform difficult measurements of the mutual attraction of masses.\textsuperscript{4}

V. DISCUSSION

From a pedagogical point of view, the above calculations demonstrate, in a very simple case, the power of physics. Measuring nothing more than the duration of a year and the apparent angular diameter of the sun we can obtain the solar density, a number which is not related to these two quantities in any obvious way. Surprises like this may be a valuable pedagogical tool, since they illustrate vividly how the universal validity of the laws of physics allows us to obtain information on properties of nature which are totally inaccessible by means of direct measurement. Apart from this more philosophical aspect, the above little calculations are also well suited as classroom exercises in an introductory astronomy course on the undergraduate or high-school level, since they require nothing but the most basic classical mechanics, a measurement of the angular diameter of the sun, and simple algebra.

This angle can be measured directly if a telescope with a cross wire eye piece and a solar filter is available. Using the fact that the sun traverses 360 degrees in one day, and measuring the time it takes the sun to traverse its own apparent diameter, one immediately obtains $2\beta$. Due to the great intensity of the sunlight such a direct measurement is somewhat dangerous and it may be preferable to image the sun with a lense of known focal length instead. Alternative ways of measurement, accessible to high-school or undergraduate students, include estimating the angular diameter of the sun from the duration of a
sunset, or directly measuring the angular diameter of the moon. As pointed out above, this
diameter is almost identical to that of the sun, a fact that is most impressively demonstrated
by showing pictures of solar eclipses. A discussion with the students of the advantages and
disadvantages of the various procedures for obtaining $2\beta$ can be very instructive.

From the point of view of history of science, the above considerations demonstrate that
Newton (or his successors) could have obtained more detailed information concerning the na-
ture of our solar system than they seem to have done. These remarks are in no way meant to
diminish Newton’s tremendous intellectual achievements, but they show that purely earth-
bound and very simple measurements allow to obtain much more detailed information than is
often thought. What makes this possible is precisely the generality of Newtonian mechanics,
the universal validity of which implies relations between quantities measured on earth and
quantities pertaining to other celestial bodies.

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REFERENCES

1. B. W. Carroll and D. A. Ostlie, *Modern Astrophysics* (Addison-Wesley, Reading, 1996).

2. J. M. Knudsen and P. G. Hjorth, *Elements of Newtonian Mechanics* (Springer-Verlag, Berlin, 1995).

3. K. R. Lang, *Astrophysical Formulae*, 2nd ed. (Springer-Verlag, Berlin, 1980).

4. B. E. Clotfelter, *The Cavendish experiment as Cavendish knew it*, Am. J. Phys. 55(3) p. 210 (1987).

5. H. M. Nussenzveig, *Curso de Física Básica-1, Mecânica* (Edgard Blücher Ltda. São Paulo, 1996).

6. *Handbook of Chemistry and Physics*, ed. D. R. Lide 78th ed. (CRC Press LCC, Boca Raton, 1997).

7. The value $\beta = 0.267^\circ$, which to the precision of the simple measurement techniques discussed here is virtually indistinguishable from $\beta = 0.25^\circ$, adopted in the main text, reproduces the literature value $\rho_s = 1.4 \text{ g cm}^{-3}$.

8. Of course there are solid materials with densities comparable to that of the sun, but these are typical neither for the earth’s composition nor for solid materials in general.

9. Isaac Newton, *Mathematical Principles of Natural Philosophy*, Book III, Proposition 8, Corrolary III (Quoted from the reprinting by Encyclopaedia Britannica Inc., Chicago, 1952).

10. Isaac Newton, *Mathematical Principles of Natural Philosophy*, Book III, Proposition 10 (Quoted from the reprinting by Encyclopaedia Britannica Inc., Chicago, 1952)
FIGURES

FIG. 1. Geometry of the motion of an orbiting lighter body of radius $R_e$, revolving about a heavier stationary body of radius $R_s$, at distance $r$. In Sec. II the heavier body is the sun and the lighter the earth, while in Sec. III they are earth and moon, respectively. The angle $\beta$ is half the apparent angular diameter of the central body, as seen from the orbiting body (neglecting parallax), while $\alpha$ is that of the orbiting body, as seen from the central body.
