Abstract

A set of models is considered which, in a certain sense, interpolates between 1+1 free quantum field theories on topologically distinct backgrounds. The intermediate models may be termed free quantum field theories, though they are certainly not local. Their ground state energies are computed and shown to be finite. The possible relevance to changing spacetime topologies is discussed.
I. INTRODUCTION

The suggestion that the topology of spacetime is not fixed \[^{[1]}\], but may change in the course of (quantum) evolution has attracted much interest and has been addressed from many points of view. Perhaps the earliest concrete work on the subject was that of Anderson and DeWitt \[^{[2]}\] (followed by \[^{[3]}\]) who calculated the effect of a topology changing background on a propagating scalar quantum field. Finding an infinite production of energy, they concluded that topology change was heavily suppressed.

However, in their analysis, the background changed instantly from one topology to the other. It is not surprising that a discontinuous process in quantum field theory produces an infinite amount of energy; indeed, this is what one would expect from the naive estimate

\[ E \approx \hbar / T \quad \text{for} \quad T = 0. \]

A number of attempts may be made to smooth out this transition. One idea is to use a region of Riemannian signature spacetime to connect the initial and final topologically different regions of Lorentzian spacetime. Some steps in this direction were accomplished in \[^{[4-7]}\]. Another approach is to consider 2+1 gravity which, when properly formulated, \[^{[8]}\] contains classical solutions which change topology; both in vacuum \[^{[9]}\] and with matter fields \[^{[10]}\]. However, it is not clear how to put quantum fields on such backgrounds and Euclidean arguments in 2+1 quantum gravity have been used both for \[^{[11,12]}\] and against \[^{[13]}\] topology change. As a final remark, the topology of spacetime appears to fluctuate in string theory \[^{[14]}\].

Our goal here is to address this matter from a new perspective by considering a class of theories that, in a certain sense, interpolate between field theories on different manifolds. Our systems will clearly not be local, but they will be ‘Lorentzian’ quantum field theories possessing a Hilbert space of states, unitary dynamics, and finite ground state energies. Each theory will be characterized by some matrix \( M \in U(2) \), with the local free scalar field theories on \( S^1 \times \mathbb{R} \) and on \( (S^1 \times \mathbb{R}) \cup (S^1 \times \mathbb{R}) \) being represented by particular values of \( M \). Since the set \( U(2) \) is connected, our class of quantum fields can be said to interpolate between these two topologies. The interpolating theories are described below.

II. THE INTERPOLATING THEORIES

The original Anderson-DeWitt calculation is reminiscent of turning on a mirror in 1+1 spacetime. In their case, the mirror was turned on quickly, that is, ‘all at once.’ We wish to study the intermediate cases between the initial and final topologies. As a result, our models will be the topology change analogues of partially silvered mirrors.

The models we consider contain a single free scalar field on a 1+1 manifold whose spatial ‘hypersurfaces’ consist of two disconnected line segments. As such, our field lives on the manifold \( \mathcal{M} = (I \cup I) \times \mathbb{R} \) with boundary \( \mathbb{R} \cup \mathbb{R} \cup \mathbb{R} \cup \mathbb{R} \). The various components of the boundary will be thought of as the worldlines of ‘gates’ into which flux from the field is allowed both to enter and to emerge. That is, the dynamics of our system will be defined so that flux entering one of the gates immediately reemerges from another. The details of these connections will be given by a ‘routing’ matrix \( M \) (described below), with different routing matrices corresponding to different kinds of gates. We will take our gates to be independent of both time and frequency.
For the appropriate routing matrices our system is equivalent to a (local) 1+1 field theory on some manifold. For example, if flux entering a gate reemerges from the same gate, perhaps with a phase shift, then we have a field theory on \((I \cup I) \times \mathbb{R}\) with boundary conditions of the usual type (Dirichlet, Neumann, or a combination). By connecting the gates in different ways, we may also construct field theories on closed manifolds.

Of course, to define a satisfactory quantum field theory requires more than just a well-defined dynamics for the field. We also require a notion of unitarity which, in the context of free field theory, is usually taken to be conservation of the Klein-Gordon flux. Since a general gated theory will be non-local, conservation of the Klein-Gordon current may become a bit subtle at the gates. However, for appropriately constructed gates, the Klein-Gordon norm will be conserved when evaluated on any hypersurface that respects the gates’ notion of simultaneity, as we shall explain below. The key point of our study is that the class of gates for which the Klein-Gordon norm is conserved is a several parameter family which interpolates between different spacetime topologies. Furthermore, these field theories will be seen to possess ground states whose finite energy is a continuous function of the parameters. Thus we will see that field theories on different topologies can in fact be connected, in what seems to be a continuous manner.

We will assume that the line segments are oriented towards the top of the page and that the gates respect this orientation. That is, we assume that flux which enters one of the top gates must emerge from the bottom gates, and vice versa. The more general case in which the upward and downward moving parts of the field become mixed by the gates can also be studied and leads to a larger class of field theories. Such theories behave much the same as the orientation-preserving cases but are significantly more complicated. As a result, we will not consider them here.

Our first task will be to consider the corresponding classical field theories and find conditions under which the Klein-Gordon flux is conserved. It is convenient to give both line segments a coordinate \(x\) proportional to proper distance and running from 0 at the bottom to \(\pi\) at the top; thus, both segments are the same length. Similarly, we use a coordinate \(t\) on \(\mathbb{R}\) proportional to proper time. We consider a free 1+1 scalar field \(\phi_I\) in the interior of the first segment and a field \(\phi_{II}\) on the interior of the second segment. These fields may be separated into their upward and downward moving parts \(\phi_{I,I}^+(x-t)\) and \(\phi_{I,I}^-(x+t)\) as usual. Having assumed that our gates respect this decomposition, the details are described by the boundary conditions they impose:

\[
\phi^+(0,t) = \begin{bmatrix} \phi_I^+(x=0,t) \\ \phi_{II}^+(x=0,t) \end{bmatrix} = M_+ \begin{bmatrix} \phi_I^+(x=\pi,t) \\ \phi_{II}^+(x=\pi,t) \end{bmatrix}
\]  

(2.1)

and
\[ \phi^-(\pi, t) \equiv \begin{bmatrix} \phi_I^{-}(x = \pi, t) \\ \phi_{II}^{-}(x = \pi, t) \end{bmatrix} = M_- \begin{bmatrix} \phi_I^{-}(x = 0, t) \\ \phi_{II}^{-}(x = 0, t) \end{bmatrix}, \quad (2.2) \]

where this matrix notation will simplify a number of expressions below. Note that each gate has an inherent notion of simultaneity in that it connects some set of spacetime events. We consider only the case in which these notions are all consistent in the sense that there is some Lorentz frame in which the gates connect only events with the same value of time \( t \). We use coordinates adapted to this frame and the matrices \( M_{\pm} \) are taken to be time independent.

Let us first consider the case where the downward-moving component vanishes. Equating the flux leaving the top gates with the flux entering the bottom gates leads to the condition

\[ 0 = \text{Im} \left\{ \left[ \phi^+(\pi, t) \right]^\dagger \left[ I - M^+_+ M^-_+ \right] \partial_t \phi^+(\pi, t) \right\} \quad (2.3) \]

where \( \text{Im} \) denotes the imaginary part. Since \( \phi^+(\pi, t) \) and \( \partial_t \phi^+(\pi, t) \) can be chosen independently at any time \( t \), upward-moving flux is conserved if and only if \( M_+ \) is unitary. Similarly, conservation of downward moving flux requires \( M_- \) to be unitary. Considering conservation of the Klein-Gordon inner product \( (\phi_+, \phi_-) \) of an upward moving field \( \phi_+ \) with a downward moving field \( \phi_- \) leads to the condition

\[ 0 = \partial_t \left\{ \left[ \phi_+(\pi, t) \right]^\dagger \left( I - M^+_+ M^{-1}_- \right) \phi_-(\pi, t) \right\} \quad (2.4) \]

so that \( M_+ = M_-^{-1} \). These conditions are sufficient to guarantee conservation of an arbitrary Klein-Gordon inner product. Thus, any set of gates for which \( M_+ = M_-^{-1} \) and for which \( M_+ \) is unitary may be considered to define a free (gated) quantum field theory. While in general such theories are not ‘local’ in any way, particular examples include the case

\[ M^{S^1}_+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2.5) \]

which is equivalent to field theory on \( S^1 \times \mathbb{R} \), the case

\[ M^{S^1\cup S^1}_+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.6) \]

which describes field theory on two (smaller) copies of \( S^1 \times \mathbb{R} \), the case

\[ M^{S^1\cup \tilde{S}^1}_+ = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.7) \]

which yields field theory on two copies of \( S^1 \times \mathbb{R} \) with one circle having antiperiodic boundary conditions, as well as examples equivalent to other local theories. Since the unitary matrices are a (path) connected set, any pair of these systems is linked by a continuous path through our space of nonlocal theories.

In order to show that our field theories are ‘reasonable’, we now compute their ground state energies and show that they are finite and depend continuously on the routing matrix \( M_+ \). Actually, we will not compute the ground state energy of all such theories, but we will perform the computation for a large enough set to form a continuous path between field theories on topologically different manifolds.
The reason for our restriction is that the dynamics of a general theory is quite complicated; a generic gated theory does not have periodic solutions of the form $e^{-i\omega(x\pm t)}$ for any frequency omega. However, such solutions do exist when the gates are appropriately "tuned." To see this, note that $2.4$ may be used to express the field $\phi^+_I(0,t)$ at the origin of the second segment in terms of the fields at $x = 0, \pi$ on the first segment. Incrementing $t$ by $\pi$ expresses $\phi^+_I(\pi,t)$ in terms of $\phi^+_I(0,t + \pi)$ and $\phi^+_I(0,t + 2\pi)$. Again using the top line of $2.4$ leads us to the relation

$$
\begin{bmatrix}
\phi^+_I(0,t)
\phi^+_I(\pi,t)
\end{bmatrix} = 
\begin{bmatrix}
-\det M_+ & \text{Tr} M_+ \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\phi^+_I(2\pi,t)
\phi^+_I(\pi,t)
\end{bmatrix}.
$$

(2.8)

Thus, whenever $\text{Tr} M_+$ vanishes $\phi^+_I(x,t)$ is periodic up to a phase under $t \to t + 2\pi$. A corresponding result for the downward-moving component follows by replacing $M_+$ with $M_- = M_+^\dagger$. The traceless unitary matrices are again a connected set which, in particular (see $2.3, 2.7$), interpolate between the free field on $S^1 \times \mathbb{R}$ and the free field on two copies of $S^1 \times \mathbb{R}$ with one copy having periodic boundary conditions and the other antiperiodic.

This set is thus large enough for our purposes.

It is straightforward to calculate the ground state energy for each theory of this kind. Note that $-\det M$ is some phase $e^{i\theta}$ ($0 \leq \theta < 2\pi$). Suppose now that each segment is of proper length $L$. From $2.3$, we see that the positive frequency spectrum of the upward moving modes is $\left\{\frac{2n\pi + \theta}{2L} : n \geq 0\right\}$ while that of the downward moving modes is $\left\{\frac{2(n+1)\pi - \theta}{2L} : n \geq 0\right\}$. The ground state energy is then given by the expression

$$
\lim_{\epsilon \to 0} \text{Re} \sum_{n \geq 0} e^{i\theta} e^{i\left(\frac{2\pi n + \theta}{4L}\right)} e^{-\frac{i\theta}{4L} e^{i\left(\frac{2(n+1)}{4L}\right)}} e^{i\left(\frac{2\pi(n+1) - \theta}{4L}\right)} + \frac{2L}{2\pi \epsilon^2}
$$

$$
= -\frac{\pi}{12L} \left(\frac{3}{2\pi^2} - \frac{1}{2}\right)
$$

(2.9)

where $\text{Re}$ denotes the real part. Note that $2.9$ is a finite and continuous function of $M_+$.

In particular, the cases described by $2.3$ and $2.7$ can be connected by a path along which every routing matrix satisfies $(\det M_+ = -1)$. As a result, the ground state energy is constant along this path. This is consistent with known expressions for the ground state energies of periodic and antiperiodic fields on $S^1 \times \mathbb{R}$ [13] and for taking the ground state energy of a field on a disconnected manifold to be the sum of the corresponding energies for its connected components. This consistency with additivity provides a certain confirmation that the Minkowski vacuum, relative to which $2.3$ is computed, is the correct zero of energy.

If we could consider a field theory in which the routing matrix was not constant but instead was slowly varied as a function of time, we might construct a system that smoothly evolves from a field theory on $S^1 \times \mathbb{R}$ to a field theory on the manifold $(S^1 \cup S^1) \times \mathbb{R}$ and which passes only through theories with ground state energy $-\pi/12L$. In such an adiabatic process, we expect that only a small amount of energy would be created. As opposed to the

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1 The reader may be surprised that the theory $2.4$ on two periodic copies of $S^1 \times \mathbb{R}$ does not satisfy $\text{Tr} M_+ = 0$. This occurs because $\text{Tr} M_+ = 0$ is only a sufficient and not a necessary condition for periodicity. Nevertheless, it captures the generic periodic case.
results of [2], this would then argue that topology change might be possible and that the amount of energy produced is small so long as the change proceeds ‘slowly.’

On the other hand, the construction of a slowly varying gated theory is far from straightforward. In particular, allowing the matrix $M_\pm$ to be time dependent in any way turns out to destroy conservation of the Klein-Gordon flux. As a result, it is not clear whether our ‘continuous path through the space of gated theories’ is in fact continuous in any physically meaningful sense. Allowing the gates to mix upward and downward moving flux does not help matters. It is possible that a frequency dependent gate might be better behaved but the author, at least, has had no success in this direction.

Note Added: Much the same model of topology change is discussed in [16] in the context of single particle quantum mechanics. This reference, however, also considers the matrix that represents the boundary conditions to be a dynamical quantum object in its own right. It would be interesting to see what effect such a treatment would have on the difficulties encountered here for the field theoretic case.

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