WEAK ORDER AND DESCENTS FOR MONOTONE TRIANGLES

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Abstract. Monotone triangles are a rich extension of permutations that biject with alternating sign matrices. The notions of weak order and descent sets for permutations are generalized here to monotone triangles, and shown to enjoy many analogous properties. It is shown that any linear extension of the weak order gives rise to a shelling order on a poset, recently introduced by Terwilliger, whose maximal chains biject with monotone triangles; among these shellings are a family of EL-shellings.

The weak order turns out to encode an action of the 0-Hecke monoid of type $A$ on the monotone triangles, generalizing the usual bubble-sorting action on permutations. It also leads to a notion of descent set for monotone triangles, having another natural property: the surjective algebra map from the Malvenuto-Reutenauer Hopf algebra of permutations into quasisymmetric functions extends in a natural way to an algebra map out of the recently-defined Cheballah-Giraudo-Maurice algebra of alternating sign matrices.

1. Introduction

Permutations in the symmetric group $S_n$ on $n$ letters, when thought of as $n \times n$ permutation matrices, are special cases of fascinating objects known as alternating sign matrices (ASMs). The latter have been intensely studied since their introduction by Mills, Robbins and Rumsey [12], and turn out to be connected with such areas as statistical mechanics, representation theory, and number theory—see Bressoud [3] and Brubaker, Bump and Friedberg [7] for more history and context. We recall their definition here, as well as their bijection with the equivalent objects known as monotone triangles.

A vector in $\{0, \pm 1\}^n$ is called alternating if its $\pm 1$ values alternate in sign, beginning and ending with $+1$. Denote by $A_n$ the set of all such alternating vectors of length $n$. An $n \times n$ alternating sign matrix is one whose row and column vectors all lie in $A_n$. Denote by ASM$_n$ the set of all such matrices. For example, we depict here on the left a matrix $A$ in ASM$_6$, abbreviating "$+$" and $-$ for entries $+1$ and $-1$:

\[
\begin{pmatrix}
0 + 0 0 0 0 0 & 0 0 + 0 0 0 \\
0 0 + 0 0 0 & 0 0 0 + 0 0 \\
0 0 0 + 0 0 & 0 0 0 + 0 0 \\
0 + 0 0 + 0 & 0 + 0 0 + 0 \\
0 + 0 0 + 0 & 0 0 + 0 0 \\
0 + 0 0 + 0 & 0 0 0 + 0 0 \\
\end{pmatrix}
= A
\leftrightarrow
T =
\begin{pmatrix}
2 & 1 & 3 & 6 \\
2 & 4 & 1 & 3 & 4 & 6 \\
1 & 2 & 3 & 5 & 6 \\
\end{pmatrix}
\]

There is a simple bijection between ASM$_n$ and the set MT$_n$ of monotone triangles of size $n$. A monotone triangle of size $n$ is a sequence $T = (T_0, T_1, \ldots, T_{n-1}, T_n)$ of subsets of $[n] := \{1, 2, \ldots, n\}$ where $\#T_m = m$, with the extra property that $T_{m+1}$ interlaces $T_m$ in this sense: if one list entries of $T_m, T_{m+1}$ in increasing order as

- $T_m = \{i_1 < i_2 < \cdots < i_m\}$,
- $T_{m+1} = \{j_1 < j_2 < \cdots < j_m < j_{m+1}\}$,

then one has

\[
j_1 \leq i_1 \leq j_2 \leq i_2 \leq j_3 \leq \cdots \leq j_m \leq i_m \leq j_{m+1}.
\]

One depicts $T$ as a triangular array having $T_m$ as its $m^{th}$ row from the top, omitting $T_0 = \emptyset, T_n = [n]$. For example, $T = (\emptyset, \{2\}, \{2, 4\}, \{1, 3, 6\}, \{1, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, [6]) \in$ MT$_6$ is shown on the right in (1). For the sake of defining the bijections ASM$_n \leftrightarrow$ MT$_n$, first introduce the indicator vector $1_S$ in $\{0, 1\}^n$ for a subset $S \subseteq [n]$, having coordinates $(1_S)_i = 1$ for $i \in S$ and $(1_S)_i = 0$ for $i \notin S$. Then given $A$ in ASM$_n$,
one maps \( A \mapsto T = (T_0, \ldots, T_n) \) in \( MT_n \) whose \( m \)-th row \( T_m \) is the unique subset for which \( 1_{T_m} \) is the sum of the first \( m \) rows of \( A \). The inverse bijection sends \( T \mapsto A \) the \( m \)-th row of \( A \) is \( 1_{T_m} - 1_{T_{m-1}} \). For example, the matrix \( A \) in ASM_6 shown on the left in (1) above has corresponding monotone triangle \( T \) in \( MT_6 \) shown to its right.

It is not hard to check (see Terwilliger \[17\] Thm. 3.2) that an \((m+1)\)-subset \( I \subseteq [n] \) interlaces an \( m \)-set \( J \subseteq [n] \) if and only if the difference of the indicator vectors \( 1_J - 1_I \) lies in \( \text{Alt}_n \). Thus \( MT_n \) is in bijection with the maximal chains of a partial order on the subsets of \([n]\) that is the transitive closure of the relation \( I \subset J \) when \( J \) interlaces \( I \); Terwilliger denotes this partial order \( \Phi_n \). Note that this partial order \( \Phi_n \) is stronger than the usual \( Boolean \) algebra poset \( 2^{[n]} \), whose order relation is given by inclusion \( \subseteq \), and whose maximal chains are the monotone triangles of the form \( T(w) := (\emptyset, \{w_1\}, \{w_1, w_2\}, \ldots, \{w_1, w_2, \ldots, w_{n-1}\}, [n]) \), which correspond to the permutations \( w = (w_1, w_2, \ldots, w_n) \) in \( S_n \). This monotone triangle \( T(w) \) also corresponds to the usual permutation matrix of \( w^{-1} \), thinking of permutation matrices as a subset of \( ASM_n \). The Hasse diagram for the poset \( \Phi_3 \) on subsets of \( [3] \) is shown below, with solid edges indicating the weaker \( Boolean \) algebra \( 2^{[3]} \) ordering, and the unique extra order relation \( \{2\} \subset \{1, 3\} \) from \( \Phi_3 \) shown dotted:

Section \[2\] explores properties of the order \( \Phi_n \), including characterizing it via a generalization of interlacing.

One of our original goals was to show that \( \Phi_n \) is a \textit{shellable} poset, a notion that we review here. Say that an abstract simplicial complex \( \Delta \) is \textit{pure} if all of its facets (=inclusion-maximal simplices) have the same number of vertices. In this case, say that an ordering \( F_1, F_2, \ldots \) of the facets of \( \Delta \) is a \textit{(pure) shelling} if for every \( j \geq 2 \), the intersection of the boundary of \( F_j \) with the subcomplex generated by the facets \( F_1, \ldots, F_{j-1} \) forms a pure subcomplex of codimension one within the boundary of \( F_j \); said differently, for any pair \( 1 \leq i < j \), there exists \( k < j \) such that \( F_i \cap F_j \subseteq F_k \cap F_j \) with \( \#F_k \cap F_j = \#F_j - 1 \). Having a shelling for \( \Delta \) imposes strong topological properties for its \textit{geometric realization} \( ||\Delta|| \), and strong algebraic properties for its \textit{Stanley-Reisner ring} \( k[\Delta] \); see Björner \[1\] Appendix and \[3\] §1. Here we are starting with a partially ordered set \( P \) having both a bottom element \( 0 \) and top element \( 1 \), such as the \textit{Boolean algebra} \( 2^{[n]} \) with inclusion order on subsets of \([n]\), or the order \( \Phi_n \) on subsets, where in either case, \( 0 = \emptyset \) and \( 1 = [n] \). In this setting, one often removes the bottom and top elements, and associates an abstract simplicial complex called the \textit{order complex} to its \textit{proper part}, so that \( \Delta \) has vertex set \( P \setminus \{0, 1\} \), and simplices for each totally ordered subset of \( P \setminus \{0, 1\} \). This means that facets of \( \Delta \) biject with maximal chains of \( P \).

As mentioned above, for \( P = \Phi_n \) and its subposet the \textit{Boolean algebra} \( 2^{[n]} \), these facets or maximal chains are naturally labeled by the monotone triangles \( MT_n \) and permutations \( S_n \), respectively. We illustrate this here for \( n = 3 \), depicting the order complex \( \Delta(\Phi_3 \setminus \{0, 1\}) \), with one extra facet (edge) shown dotted, whose removal gives the subcomplex \( \Delta(2^{[3]} \setminus \{0, 1\}) \).
For the Boolean algebra \(2^{[n]}\), this order complex \(\Delta(2^{[n]} \setminus \{0,1\})\) is isomorphic to the Coxeter complex of type \(A_{n-1}\), and a result of Björner [3, Thm. 2.1] shows that it is shellable, with a shelling order on its facets provided by any linear ordering on the permutations \(S_n\) that extends the (right) weak order \(<_W\). This weak order is the transitive closure of the relation in which \(ws_i <_W w\) if \(w = (w_1, \ldots, w_n)\) has \(w_i > w_{i+1}\), where \(s_i = (i, i+1)\) is an adjacent transposition. One can view this weak order as induced from the action of the bubble-sorting operators \(\pi_1, \ldots, \pi_{n-1}\) on \(S_n\)

\[
\pi_i(w) = \begin{cases} 
ws_i & \text{if } w_i > w_{i+1}, \\
 w & \text{if } w_i < w_{i+1},
\end{cases}
\]

which satisfy the relations of the 0-Hecke monoid of type \(A_{n-1}\):

\[
\begin{align*}
\pi_i \pi_j &= \pi_j \pi_i \text{ if } |j - i| \geq 2, \\
\pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1}, \\
\pi_i^2 &= \pi_i.
\end{align*}
\]

One may then define the weak order by \(w \leq_W w'\) if and only if \(w\) lies in the 0-Hecke orbit of \(w'\).

Section 3 extends this 0-Hecke action from \(S_n\) to \(MT_n\), by letting \(\pi_i(T)\) replace the \(i\)th-row of the monotone triangle \(T\) with the componentwise smallest row that still forms a monotone triangle with the remaining rows. One can then extend the weak order \(<_W\) from \(S_n\) to \(MT_n\) by setting \(T \leq T'\) whenever \(T\) lies in the 0-Hecke orbit of \(T'\). For \(n = 3\), these actions of \(H_3(0)\) on \(S_3\) and \(MT_3\) look as follows, illustrating the weak order posets \(<_W\) on both:

Section 4 then uses this to prove our first main result.

**Theorem 1.1.** Linear extensions of \(<_W\) on \(MT_n\) give shelling orders on \(\Phi_n\).

There is another sense in which the terminology weak order is appropriate. Lascoux and Schützenberger [10] showed that the componentwise order on \(MT_n\) is a distributive lattice, one that turns out to be the MacNeille completion of the (strong) Bruhat order \(<_B\) on \(S_n\); we therefore refer to this componentwise order on \(MT_n\) as its (strong) Bruhat order \(<_B\). Depicted below is the poset \((MT_3,<_B)\), with the usual Bruhat order \((S_3,<_B)\) as a subposet, and dotted edges indicating the order relation to the unique element \(T\) in \(MT_3 \setminus S_3\):

\[
\begin{align*}
3 & \quad 2 \quad 3 \\
\pi_2 & \quad \pi_1 & \quad \pi_3 \\
1 & \quad \pi_1 & \quad \pi_2 \\
\pi_2 & \quad \pi_1 & \quad \pi_3 \\
1 & \quad \pi_2 & \quad \pi_1 \\
1 & \quad 2 & \quad 3
\end{align*}
\]

It turns out (see Remark 3.5) that this Bruhat order \(<_B\) on \(MT_n\) is stronger than the weak order \(<_W\) defined above; in particular, any linear extension of the componentwise order gives rise to a shelling of \(\Phi_n\).
The weak order shellings provided by Theorem 1.1 have another tight analogy to the weak order shellings of the Boolean posets $(2^{[n]}, \subseteq)$, in that they contain as a special case certain EL-shellings, a notion which we recall here. Given a poset $P$, with $C(P) = \{x < y : x, y \in P\}$ its set of cover relations ($x < y$ means $x < y$ but $\not\exists z$ with $x < z < y$), an EL-labeling of $P$ is a function $\lambda : C(P) \to \Lambda$ where $(\Lambda, <_\Lambda)$ is any poset, having these properties:

- for every interval $[x, y] \subset P$, there is a unique maximal chain $(x = x_0 < x_1 < \cdots < x_k = y)$, that has **weakly rising** labels $\lambda(x_0, x_1) \leq_\Lambda \lambda(x_1, x_2) \leq_\Lambda \cdots \leq_\Lambda \lambda(x_{k-1}, x_k)$

- if $x < z < y$, with $z \neq x_1$, then $\lambda(x, x_1) <_\Lambda \lambda(x, z)$.

For example, the Boolean algebras $(2^{[n]}, \subseteq)$ have a very simple EL-labeling. It assigns a covering relation between subsets $I \subset J$ with $\#J = \#I + 1$ the unique integer $\lambda(I, J) := j$ such that $J = I \cup \{j\}$; here the labels come from the poset $\Lambda = \{1, 2, \ldots, n\}$ with the usual ordering on integers. A poset is EL-shellable or lexicographically shellable if it admits an EL-labeling. Björner [1, Thm. 2.3] showed that for a poset with an EL-labeling, one obtains a shelling order on its maximal chains via any linear extension of the lexicographic extension of $\Lambda$ to sequences of edge labels. In Section 5, we prove the following.

**Theorem 1.2.** There is a partial order on $\text{Alt}_n$ so that the edge-labeling $\lambda$ which assigns $\lambda(I \lessdot J) = 1^j - 1_I$ in $\text{Alt}_n$ becomes an EL-labeling of $\Phi_n$. Furthermore, any of the EL-shelling orders associated with this labeling will be a linear order on $\text{MT}_n$ that extends the weak order $<_W$.

The weak order shellings and EL-shellings in Theorems 1.1 and 1.2 show that $\Phi_n$ is a Cohen-Macaulay poset, and allow one to combinatorially re-interpret its flag $f$-vector $f(\Phi_n) := (f_J)_{J \subset [n-1]}$; here $f_J$ is the number of chains in $\Phi_n$ that pass through the ranks in $J$. One can instead consider the flag $h$-vector $h(\Phi_n) = (h_J)_{J \subset [n-1]}$, defined by an inclusion-exclusion relation:

$$
\begin{align*}
  f_J &= \sum_{I \subseteq J} h_I, \quad \text{or equivalently,} \\
  h_J &= \sum_{J \subseteq I} (-1)^{\# J \backslash I} f_I.
\end{align*}
$$

General shelling theory then implies this combinatorial interpretation for $h_J$:

$$
  h_J(\Phi_n) = \# \{ T \in \text{MT}_n : \text{Des}(T) = J \}.
$$

Here one is led to define the descent set $\text{Des}(T)$ for a monotone triangle $T$ as follows via the following generalization of the usual descent set $\text{Des}(w) = \{ i \in [n-1] : w_i > w_{i+1} \}$, that is, $\pi_i(w) \neq w$ for permutations $w$ in $S_n$:

$$
  \text{Des}(T) := \{ i \in [n-1] : \pi_i(T) \neq T \}.
$$

Section 4 discusses this descent set $\text{Des}(T)$, and collects some data on its distribution over $\text{MT}_n$.

There is a further way in which this notion of a descent set for monotone triangles extends a pleasant property of descents for permutations. Recall that Malvenuto and Reutenauer [11] defined a graded Hopf algebra, sometimes denoted $\text{FQSym} = \bigoplus_{n \geq 0} \text{FQSym}_n$, where $\text{FQSym}_n$ has $\mathbb{Z}$-basis elements $w$ indexed by permutations $w$ in $S_n$. The ring structure is determined by a shuffle product for $u, v$ in $S_n, S_m$ defined as

$$
  uv = \sum_{w \in u \shuffle v[n]} w
$$

in which the sum runs over all shuffles $w$ of $u = (u_1, \ldots, u_n)$, and $v[n] = (v_1 + n, \ldots, v_m + n)$. This shuffle product was introduced in such a way as to make a ring (and Hopf algebra) morphism into the quasisymmetric functions $\text{QSym}$, defined by

$$
  \begin{align*}
    \text{FQSym} &\to \text{QSym} \\
    w &\mapsto L_\alpha(\text{Des}(w)).
  \end{align*}
$$

Here $L_\alpha$ denotes Gessel’s fundamental quasisymmetric function associated to a composition $\alpha$, and $\alpha(\text{Des}(w))$ is the composition whose partial sums give the elements of $\text{Des}(w)$; see [16, §7.19] and Section 7 below.
Recently, Cheballah, Giraudo and Maurice embedded FQSym inside a larger graded Hopf algebra $ASM$ whose $n^{th}$-graded component has a basis $\{A\}$ indexed by $A$ in $ASM_n$ [8], and whose product and coproduct extend that of FQSym. Section 7 proves the following.

**Theorem 1.3.** The map $FQSym \rightarrow QSym$ in (6) extends to an algebra (but not a coalgebra) morphism

$$ASM \rightarrow QSym$$

$$A \mapsto L_{\alpha(Des(A))}$$

where $Des(A) = Des(T(A))$ for an alternating sign matrix $A$ is the descent set of its monotone triangle $T(A)$.

Sections 8 concludes by comparing poset properties of the weak order on $MT_n$ with analogous properties for the weak order on $\mathfrak{S}_n$, including a conjecture for the homotopy type of open intervals in $(MT_n, <_W)$.

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2. Interlacing, monotone trapezoids, and the order $\Phi_n$

The goal here is to relate Terwilliger’s order $\Phi_n$ with the notions of interlacing and monotone trapezoids.

**Definition 2.1.**
Start with the componentwise order $\leq_{\text{comp}}$ on subsets $I, I' \subseteq [n]$ of the same cardinality $k$ for $0 \leq k \leq n$,

$$I = \{i_1 < i_2 < \cdots < i_k\},$$

$$I' = \{i'_1 < i'_2 < \cdots < i'_k\},$$

defined by setting $I \preceq_{\text{comp}} I'$ if $i_m \leq i'_m$ for $m = 1, 2, \ldots, k$.

For $J = \{j_1 < \cdots < j_k\} \subseteq [n]$ with $\# J = \ell \geq k = \# I$, say that $J$ interlaces $I$, written $I \leq_{\text{lace}} J$, if

$$\{j_1, j_2, \ldots, j_k\} \preceq_{\text{comp}} \{j_{\ell-k+1}, j_{\ell-k+2}, \ldots, j_{\ell-1}, j_\ell\}.$$ 

Note that when $\# J = k + 1 = \# I + 1$, this condition $I \leq_{\text{lace}} J$ is the usual definition of $J$ interlacing $I$, as given in [2] earlier. One then has the following proposition which is easily checked (or see [17] §3).

**Proposition 2.2.** If $\# J = \# I + 1$, then $I \leq_{\text{lace}} J$ if and only if $1_{J} - 1_{I}$ lies in $\text{Alt}_n$. □.

One can also readily check that $\leq_{\text{lace}}$ is a partial order, that is, $I \leq_{\text{lace}} J \leq_{\text{lace}} K$ implies $I \leq_{\text{lace}} K$. This partial order $\leq_{\text{lace}}$ is closely related to monotone trapezoids and Terwilliger’s order $\Phi_n$, as we now explain.

**Definition 2.3.**
An $(I, J)$-monotone trapezoid is a sequence of subsets $T = (I_k, I_{k+1}, \ldots, I_{\ell-1}, I_{\ell})$ of $\{1, 2, \ldots\}$ with

- $I_k = I, I_{\ell} = J$,
- $\# I_m = m$, and
- $I_m \leq_{\text{lace}} I_{m+1}$ for $k \leq m < \ell$.

In other words, an $(I, J)$-monotone trapezoid is a saturated chain in $\leq_{\text{lace}}$ from $I$ to $J$. When $(I, J) = (\emptyset, [n])$, one calls $T$ a monotone triangle of size $n$.

**Proposition 2.4.** The following are equivalent for subsets $I, J \subseteq [n]$:

(a) There exists at least one $(I, J)$-monotone trapezoid.
(b) $I \leq_{\Phi_n} J$.
(c) $I \leq_{\text{lace}} J$.

In proving this proposition, and in the sequel, the following construction will be useful.
Definition 2.5.
For $I \leq \text{lance} J$ with $\# I = k$ and $\# J \geq k + 2$, define $H_{\text{min}}(I, J) := \{h_1, h_2, \ldots, h_{k+1}\}$ by the rule

$$h_m := \max(i_{m-1}, j_m),$$

and convention $i_p := 0$ for $p = 0$. Thus when $k = 0$, so that $I = \emptyset$, then $H_{\text{min}}(\emptyset, J) = \{j_1\}$.

Lemma 2.6. The set $H_{\text{min}}(I, J)$ has these properties:

(i) It is a $(k+1)$-subset, that is, $h_1 < \cdots < h_{k+1}$.
(ii) It lies in the family $\{H \in \binom{[n]}{k+1} : I \leq \text{lance} H \leq \text{lance} J\}$.
(iii) Every $H'$ in this family has $H_{\text{min}}(I, J) \leq \text{comp} H'$.

Proof. Assertion (i). The definition of $H_{\text{min}}(I, J)$ implies $h_m < h_{m+1}$ since

$$h_m = \max(i_{m-1}, j_m) \leq \max(i_m - 1, j_{m+1} - 1) = \max(i_m, j_{m+1}) - 1 = h_{m+1} - 1.$$

Assertion (ii). We must show two $\leq \text{lance}$-inequalities, or equivalently, four $\leq \text{comp}$-inequalities.
- Two of the four come from $i_{m-1}, j_m \leq \max(i_{m-1}, j_m) = h_m$ for $m = 1, 2, \ldots, k+1$, which shows both that $I \leq \text{comp} \{h_2, \ldots, h_{k+1}\}$ and also that $\{j_1, \ldots, j_{k+1}\} \leq \text{comp} H_{\text{min}}(I, J)$.
- The inequality $\{h_1, \ldots, h_k\} \leq \text{comp} I$ comes from

$$h_m = \max(i_{m-1}, j_m) \leq \max(i_m, j_{m+1}) = i_m$$

which uses $i_{m-1} < i_m$ and the fact that $\{j_1, \ldots, j_{k+1}\} \leq \text{comp} I$ since $I \leq \text{lance} J$.
- The last inequality $H_{\text{min}}(I, J) \leq \{j_{k-\ell}, j_{k-\ell+1}, \ldots, j_k, j_{k+1}\}$ comes from

$$h_m = \max(i_{m-1}, j_m) \leq j_{k-\ell+(m-1)}$$

which uses $j_{m-1} < j_{k-\ell+(m-1)}$ (as $\ell - k \geq 2$) and $i_{m-1} \leq j_{k-\ell+(m-1)}$ (as $I \leq \text{lance} J$).

Assertion (iii). Any such $H' = \{h'_1 < \cdots < h'_{k+1}\}$ has $I \leq \text{lance} H' \leq \text{lance} J$, implying for $1 \leq m \leq k+1$ that
- $h'_m \geq i_{m-1}$, coming from $I \leq \text{comp} \{h'_2, h'_3, \ldots, h'_{k+1}\}$.
- $h'_m \geq j_m$, coming from $\{j_1, \ldots, j_m\} \leq \text{comp} H'$.

Thus $h'_m \geq \max(i_{m-1}, j_m) = h_m$, that is, $H_{\text{min}}(I, J) \leq \text{comp} H'$, as desired. \qed

With the construction $H_{\text{min}}(I, J)$ and its properties in hand, one can now prove Proposition 2.4. Proof of Proposition 2.4. Note (a) $\Leftrightarrow$ (b) via Proposition 2.2 and definition of $\Phi_n$. Then (a) $\Rightarrow$ (c) from the transitivity of $\leq \text{lance}$, while (c) $\Rightarrow$ (a) follows by induction on $\# J - \# I$ via Lemma 2.6. \qed

Remark 2.7.
It is worth pointing out an involutive poset symmetry in $\Phi_n$, coming from the action of the longest permutation $w_0 = (n, n-1, \ldots, 2, 1)$ in $S_n$. This permutation $w_0$ acts on subsets as follows:

$$I = \{i_1 < i_2 < \cdots < i_k\} \xrightarrow{w_0} w_0(I) := \{n+1-i_k < \cdots < n+1-i_2 < n+1-i_1\}.$$

Since $i \leq j$ if and only if $n+1-i \geq n+1-j$, this action of $w_0$ preserves the interlacing inequalities that define the covering relations $I \ll \Phi_n J$. Thus it is an involutive automorphism of the poset $\Phi_n$, and therefore also gives an involution on monotone triangles

$$T = (T_0, T_1, \ldots, T_n) \xrightarrow{w_0} w_0(T) := (w_0(T_0), w_0(T_1), \ldots, w_0(T_n)).$$

Passing through the bijection $\text{ASM}_n \leftrightarrow \text{MT}_n$, the corresponding involution $w_0$ acting on a matrix $A = (a_{ij})$ in $\text{ASM}_n$ simply reflects it through a vertical axis: $w_0(A) := (a_{i,n+1-j})$.

Due to this $w_0$-symmetry, for $I \ll \text{lance} J$ with $\# J - \# I \geq 2$, instead of defining the set $H_{\text{min}}(I, J)$ as in Definition 2.3, we could have defined a set $H_{\text{max}}(I, J)$ as in two equivalent formulas:

$$h'_m = \min(i_{m-1}, j_{m-\ell-k})$$

for $m = 1, 2, \ldots, k+1$, with convention $i_{k+1} := \infty$, or

$$H_{\text{max}}(I, J) = w_0(H_{\text{min}}(w_0(I), w_0(J))).$$

(8)
One would then have the corresponding properties as in Lemma 2.6 namely that $H_{\text{max}}(I, J)$ is actually a $(k+1)$-subset, that it lies between $I$ and $J$ in the order $<_\text{lasc}$, and that it is the componentwise maximum among all such $(k+1)$-subsets between $I$ and $J$. We simply chose here to use $H_{\text{min}}(I, J)$, not $H_{\text{max}}(I, J)$.

The key property that we will need for shellability of $\Phi_n$ is that, for any pair $I \leq_{\text{lasc}} J$, there is a componentwise smallest $(I, J)$-monotone trapezoid, and that it can be characterized locally.

**Lemma 2.8.** Fixing $I \leq_{\text{lasc}} J$, the following are equivalent for an $(I, J)$-monotone trapezoid

$$T := ((I =) I_k, I_{k+1}, \ldots, I_{\ell-1}, I_{\ell}(= J)) :$$

(a) $I_m = H_{\text{min}}(I_{m-1}, J)$ for $m = k + 1, k + 2, \ldots, \ell - 1$.
(b) $I_{m+1} = H_{\text{min}}(I_{m+1}, I_m)$ for $m = k + 1, k + 2, \ldots, \ell - 1$.
(c) The elements of $I_m = \{h^{(m)}_1 < h^{(m)}_2 \cdots < h^{(m)}_m\}$ are $h^{(m)}_p = \max(j_p, i_p + k - (m - 1))$ with $i_q = 0$ for $q \leq 0$.
(d) $T$ is the componentwise smallest among all $(I, J)$-monotone trapezoids.

**Proof.** First check that if $T$ satisfies (a), then its entries have the formula from (c), using induction on $m$. The base case $m = k + 1$ comes from the definition of $H_{\text{min}}(I_k, J)$. The inductive step is this calculation:

$$h^{(m)}_p = \max(j_p, h^{(m-1)}_{p-1}) = \max(j_p, \max(j_p, i_{p+1+k-(m-1)})) = \max(j_p, i_{p+k-m}).$$

Next check that if $T$ satisfies (b), then its entries obey the formula from (c), this time using induction on $\#J - \#I = \ell - k$. Assume that (b) holds for the trapezoid $T$, so

$$I_{m+1} = \{h^{(m-1)}_1 < h^{(m-1)}_2 \cdots < h^{(m-1)}_m\} = H_{\text{min}}(I^{(m-1)}_1, I^{(m-1)}_m).$$

This means that

$$(9) \quad h^{(m)}_p = \max(h^{(m-1)}_p, h^{(m-1)}_{p-1}).$$

By restriction, condition (b) also holds for the smaller trapezoid $(I_m, I_{m+1}, \ldots, I_{\ell-1}, I_\ell = J)$, and hence by induction, one has $h^{(m-1)}_{p-1} = \max(j_p, h^{(m-1)}_{p-1})$. Similarly, by restriction, condition (b) also holds for the smaller trapezoid $(I = I_k, I_{k+1}, \ldots, I_{m-1}, I_m)$, and hence by induction, one has $h^{(m-1)}_{p-1} = \max(h^{(m-1)}_{p-1}, i_{p+1+k-(m-1)})$. Plugging these last two expressions into (9), one concludes that

$$h^{(m)}_p = \max(\max(j_p, h^{(m-1)}_p), \max(h^{(m-1)}_{p-1}, i_{p+1+k-(m-1)}))$$

$$= \max(j_p, h^{(m-1)}_p, i_{p+k-m}) = \max(j_p, i_{p+k-m})$$

since $h^{(m-1)}_{p-1} < h^{(m-1)}_p$. This last expression is the one from (c), as desired.

Thus since (a) does define a monotone trapezoid having $I, J$ as its bottom, top rows, then $T$ satisfying (b) or (c) is equivalent to $T$ being the one defined by (a).

To see (c) $\Leftrightarrow$ (d), let $T' = ((I =) I'_1, I'_{k+1}, \ldots, I'_{\ell-1}, I'_\ell(= J))$ be an $(I, J)$-monotone trapezoid, with $I'_m = \{i'_1 < \ldots < i'_m\}$. Then $i'_p \geq \max(j_p, i_{p+k-m})$ by the inequalities defining monotone trapezoids. Since the sets defined using (c) form an $(I, J)$-monotone trapezoid, we see they must form the minimal $(I, J)$-monotone trapezoid and vice versa. 

**Remark 2.9.**

It should not be surprising that there exists a componentwise smallest $(I, J)$-monotone trapezoid, as in Lemma 2.8 since Lascoux and Schützenberger [10, §5] showed that the componentwise order on $\mathbb{M}_n$ has meet and join operations given by componentwise minimum and maximum. Similarly, there is a componentwise largest such $(I, J)$-monotone trapezoid, having similar properties, which can be built in a analogous fashion by iterating the $H_{\text{max}}(I, J)$ construction from Remark 2.7.
3. Action of $H_n(0)$ and the weak order

Recall from the Introduction \(5\) that the 0-Hecke monoid $H_n(0)$ for the symmetric group $\mathfrak{S}_n$ (or type $A_{n-1}$) is the monoid with $n - 1$ generators $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the usual braid relations

\[
\begin{align*}
\pi_i \pi_j &= \pi_j \pi_i \quad \text{for } |i - j| \geq 2, \\
\pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} \quad \text{for } i = 1, 2, \ldots, n - 2,
\end{align*}
\]

together with the quadratic relations

\[
\pi_i^2 = \pi_i \quad \text{for } i = 1, 2, \ldots, n - 1.
\]

See Norton \([13]\) for background on $H_n(0)$ and the associated monoid algebra, called a 0-Hecke algebra.

Definition 3.1.

Define maps $\pi_i : \text{MT}_n \rightarrow \text{MT}_n$ for $i = 1, 2, \ldots, n - 1$ sending $T \mapsto \pi_i(T)$, where $\pi_i(T)$ is obtained from $T$ by replacing its $i$th row $T_i$ with $H_{\text{min}}(T_{i-1}, T_{i+1})$.

Proposition 3.2. The operators $\pi_i$ on $\text{MT}_n$ satisfy the braid and quadratic relations \([10]\), \([11]\), and hence define an action of $H_n(0)$ on $\text{MT}_n$.

Proof. The relations $\pi_i^2 = \pi_i$ and $\pi_i \pi_j = \pi_j \pi_i$ for $|i - j| \geq 2$ should be clear; only $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ requires verification. We can check this locally in rows $i - 1, i, i + 1, i + 2$ of a monotone triangle $T$, by tracking two generic entries in rows $i, i + 1$ shown in bold below. Here, we are using concatenation of sets of entries to abbreviate their maximum:

\[
\begin{array}{cccccccccccc}
a & b & c & d & e & f & g & h & i & j & k & l \\
\pi_i \rightarrow & ? & a & b & c & d & e & f & g & h & i & j & k & l \\
\end{array}
\]

Thus it only remains to check these equalities

\[
\begin{align*}
\text{max}(a, b, f, i) &\equiv \text{max}(b, d, i), \\
\text{max}(a, f, i) &\equiv \text{max}(a, c, h, i),
\end{align*}
\]

which both follow, since

- $a \leq d \leq b$ and $f \leq i$ implies that the two sides in \([12]\) are both equal to $\text{max}(b, i)$,
- $c, h \leq f \leq i$ implies that the two sides in \([13]\) are both equal to $\text{max}(a, i)$. \qed

Once one knows that the operators $\pi_i$ satisfy the braid relations, one can define operators $\pi_w$ for every permutation $w$ in $\mathfrak{S}_n$ as follows: pick any factorization $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ for $w$ that is shortest possible (i.e., reduced) as a product of the adjacent transpositions $\{s_1, s_2, \ldots, s_{n-1}\} =: S$, and then let

\[
\pi_w := \pi_{i_k} \pi_{i_{k-1}} \cdots \pi_{i_1}.
\]

As a consequence of satisfying the relations of $H_n(0)$, one could equivalently define $\pi_w$ recursively as follows:

\[
\pi_i \pi_w := \begin{cases} 
\pi_{w_s}, & \text{if } w(i) < w(i + 1), \text{ that is, if } i \notin \text{Des}(w), \\
\pi_w & \text{if } w(i) > w(i + 1), \text{ that is, if } i \in \text{Des}(w), 
\end{cases}
\]

starting with the initial condition $\pi_e := 1$. 

Remark 3.3.
It is worth noting in the case where $T$ has $T_i \subset T_{i+1}$ for all $i$, so that
$$T = T(w) := (\emptyset, \{w_1\}, \{w_1, w_2\}, \ldots, \{w_1, w_2, \ldots, w_{n-1}\}, [n])$$
for some permutation $w = (w_1, w_2, \ldots, w_n)$ in $\mathfrak{S}_n$, then one has
$$\pi_i(T(w)) =
\begin{cases}
T(w) & \text{if } w_i < w_{i+1}, \text{ that is, if } i \notin \text{Des}(w), \\
T(ws_i) & \text{if } w_i > w_{i+1}, \text{ that is, if } i \in \text{Des}(w).
\end{cases}
$$
Here $s_i = (i, i + 1)$ is the adjacent transposition, so that
$$ws_i = (w_1, w_2, \ldots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \ldots, w_n).$$
Thus the action of $H_n(0)$ on $MT_n$ extends its action on $\mathfrak{S}_n$ via (bubble-)sorting operators as mentioned in the Introduction. We let $\pi_i(w)$ denote the permutation corresponding to $\pi_i(T(w))$, so that $\pi_i(T(w)) = T(\pi_i(w))$.

Definition 3.4.
Extend the weak order $<_W$ on the symmetric group $\mathfrak{S}_n$ to a weak order $<_W$ on monotone triangles $MT_n$ as the transitive closure of the relations $\pi_i(T) \leq T$ where $i$ is any index in the range $1, 2, \ldots, n - 1$. Equivalently, $T \leq_W T'$ means that $T$ lies in the $H_n(0)$-orbit of $T'$.

Remark 3.5.
The name weak order is appropriate here, since $(MT_n, <_W)$ is indeed weaker than the componentwise order $(MT_n, <_B)$, and we view the latter as the appropriate extension of (strong) Bruhat order on $\mathfrak{S}_n$ to a strong Bruhat order on $MT_n$, via MacNeille completion. To see that $(MT_n, <_W)$ is weaker than the componentwise order, note that it is the transitive closure of the relations $\pi_i(T) \leq_W T$, where $\pi_i(T)$ is obtained from $T$ by replacing the $i$th row of $T$ with $H_{\min}(T_{i-1}, T_{i+1})$, the latter being componentwise smaller by Lemma 2.8.

4. Proof of Theorem 1.1

Recall the statement of the theorem.

Theorem 1.1 Any linear extension $T^{(1)}, T^{(2)}, \ldots, T^{(N)}$ of $<_W$ on $MT_n$ gives a shelling order on $\Phi_n$.

Before proving the theorem, we note in the next proposition a useful reinterpretation of Lemma 2.8 generalizing the definition of the $\pi_i(T)$ on monotone triangles. Given any subset $J \subseteq S := \{s_1, \ldots, s_{n-1}\}$, recall there is a unique longest permutation $w_0(J)$ in the Young or parabolic subgroup $\langle J \rangle$ of $\mathfrak{S}_n$ generated by $J$. This $w_0(J)$ is an involution, characterized within $\langle J \rangle$ by the property that
$$J = \text{Des}(w_0(J)) = \text{Des}(w_0(J)^{-1})$$
(here we identify $J = \{s_{j_1}, \ldots, s_{j_k}\}$ with $\{j_1, \ldots, j_k\}$). For example, if $n = 9$ and $J = \{s_1, s_2, s_4, s_5, s_6, s_8\} \subseteq \{s_1, s_2, \ldots, s_8\} = S$, then the parabolic subgroup $\langle J \rangle$ inside $\mathfrak{S}_9$ is the subgroup isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_4 \times \mathfrak{S}_2$ that stabilizes the blocks of the partition $\{1, 2, 3\}, \{4, 5, 6, 7\}, \{8, 9\}$. Its longest permutation is $w_0(J) = (3, 2, 1, 7, 6, 5, 4, 9, 8)$.

Proposition 4.1. Given any monotone triangle $T$ and $J \subseteq S$, then $\pi_{w_0(J)}(T)$ is the unique componentwise smallest monotone triangle $T^{\min}$ having the same rows $T_m$ as in $T$ for all $s_m \notin J$.

Proof. Lemma 2.8(b) shows that this componentwise smallest triangle $T^{\min}$ is uniquely characterized by
$$T^{\min}_m =
\begin{cases}
T_m & \text{for } s_m \notin J, \\
H_{\min}(T_{m-1}^{\min}, T_{m+1}^{\min}) & \text{for } s_m \in J.
\end{cases}$$
On the other hand, we claim that the triangle $T' = \pi_{w_0(J)}(T)$ has these same properties:
- $T' = \pi_{w_0(J)}(T)$ shares the same rows $T'_m = T_m$ for $s_m \notin J$ since $w_0(J)$ lies in $\langle J \rangle$.
- For any $s_m \in J$ one has $\pi_m(T') = \pi_m(\pi_{w_0(J)}(T)) = \pi_{w_0(J)}(T) = T'$ combining (14) with the fact that $s_m$ lies in $J = \text{Des}(w_0(J))$ by (15). This means that $T_m = H_{\min}(T_{m-1}^{\min}, T_{m+1}^{\min})$. \(\square\)
Proof of Theorem 1.1. Thinking of each monotone triangle \(T^{(i)}\) as corresponding to a facet, we identify it with its subset of \(n + 1\) vertices, namely

\[ T^{(i)} = \{ \emptyset = T_0^{(i)}, T_1^{(i)}, \ldots, T_{n-1}^{(i)}, T_n^{(i)} = [n] \}. \]

Shellability, as defined in the Introduction, requires that for each pair \(i, j\) with \(1 \leq i < j \leq N\), we must exhibit some \(k < j\) satisfying \(\#T^{(k)} \cap T^{(j)} = n\) (including \(\emptyset\) and \([n]\)) and \(T^{(i)} \cap T^{(j)} \subseteq T^{(k)} \cap T^{(j)}\).

Given \(i < j\), let \(J := \{ m : T_m^{(i)} \neq T_m^{(j)} \}\). We claim that \(\pi_m T^{(j)} \neq T^{(j)}\) for at least one \(m \in J\), otherwise Proposition 4.1 implies the two equalities here

\[ T^{(j)} = \pi_{w_0(J)} T^{(j)} = \pi_{w_0(J)} T^{(i)} \leq_W T^{(i)}, \]

but then the inequality \(T^{(j)} \leq_W T^{(i)}\) would contradict \(i < j\).

Given such an \(m\), one checks that the index \(k\) defined by \(\pi_m T^{(j)} = T^{(k)}\) does the job:

- \(T^{(k)} = \pi_m T^{(j)} <_W T^{(j)}\) implies that \(k < j\).
- \(\#(T^{(k)} \cap T^{(j)}) = \#(\pi_m T^{(j)}) \cap T^{(j)} = n - 1\), since \(\pi_m T^{(j)} \neq T^{(j)}\).
- \(T^{(i)} \cap T^{(j)} \subseteq T^{(k)} \cap T^{(j)}\) because \(s_m\) lies in \(J\). \(\Box\)

We close this section with two remarks about the above shelling.

Remark 4.2. Since the \(\pi_i\) operators on \(MT_n\) restrict to the usual bubble-sorting operators on the symmetric group \(\mathfrak{S}_n\) embedded inside \(MT_n\) via \(w \mapsto T(w)\), one finds that the subposet \((\mathfrak{S}_n, <_W)\) is actually an order ideal inside \((MT_n, <_W)\); it is even the principal order ideal below \(T(w_0)\) where \(w_0 = (n, n - 1, \ldots, 2, 1)\).

As a consequence, it is possible to pick a linear extension of \(<_W\) on \(MT_n\) which contains all of the elements of the order ideal \(\mathfrak{S}_n\) as an initial segment. This then gives a shelling order on the facets of \(\Delta(\Phi_n \setminus \{0, 1\})\) which sheils the Coxeter complex \(\Delta(2^n) \setminus \{0, 1\}\) first, before continuing on to shell the remaining facets of \(\Delta(\Phi_n \setminus \{0, 1\})\) that do not correspond to permutations.

Remark 4.3. Shellability implies that the \((n - 2)\)-dimensional simplicial complex \(\Delta(\Phi_n \setminus \{0, 1\})\) has the homotopy type of a bouquet of \((n - 2)\)-spheres. The Coxeter complex \(\Delta(2^n) \setminus \{0, 1\}\) inside it is homeomorphic to a single \((n - 2)\)-sphere, and this sphere has well-known easy embeddings into \(\mathbb{R}^{n-1}\). For example, it is isomorphic to the barycentric subdivision of the boundary of a simplex with vertex set \(\{1, 2, \ldots, n\}\). Alternatively one can embed it within the hyperplane \(x_1 + \cdots + x_n = 0\) inside \(\mathbb{R}^n\) by extending piecewise-linearly the map that sends its vertices to the \(\mathfrak{S}_n\)-images of the fundamental dominant weights of type \(A_{n-1}\): the vertex indexed by a subset \(I\) with \(\emptyset \subsetneq I \subseteq [n]\) is sent to the vector \(\sum_{i \in I} e_i - \frac{\#I}{n} (e_1 + \cdots + e_n)\) where \(e_i\) is the \(i\)th standard basis vector of \(\mathbb{R}^n\).

After looking at the picture [3] of \(\Delta(\Phi_3 \setminus \{0, 1\})\), which embeds it in \(\mathbb{R}^2\), one might wonder whether \(\Delta(\Phi_n \setminus \{0, 1\})\) embeds in some simple way into \(\mathbb{R}^{n-1}\). We are doubtful. For example, when \(n = 4\), one can check that if one takes either of the two vertex coordinates for embedding \(\Delta(2^4) \setminus \{0, 1\}\) into \(\mathbb{R}^3\) as described in the previous paragraph, when one extends this piecewise-linearly over the extra simplices in \(\Delta(\Phi_4 \setminus \{0, 1\})\), it leads to self-intersections, and not an embedding.

5. EL-labeling and proof of Theorem 1.2

Recall the statement of the theorem.

**Theorem 1.2.** There is a partial order on \(\text{Alt}_n\) so that the edge-labeling \(\lambda\) which assigns \(\lambda(I < J) = \mathbb{1}_J - \mathbb{1}_I\) in \(\text{Alt}_n\) becomes an EL-labeling of \(\Phi_n\). Furthermore, any of the EL-shelling orders associated with this EL-labeling is a linear order on \(MT_n\) which extends the weak order \(<_W\).

We will define the partial order on \(\text{Alt}_n\) via its identification with a Boolean algebra \(2^{[n-1]}\). Note that a vector \(v\) in \([0, \pm 1]^n\) lies in \(\text{Alt}_n\) exactly when each of its tail sums \(v \cdot \mathbb{1}_{[i, n]} = v_i + v_{i+1} + \cdots + v_n\) lies in \([0, +1]\), with \(\sum_{i=1}^n v_i = +1\). The following proposition is straightforward to verify.
Proposition 5.1. One has mutually-inverse bijections
\[ \varphi \colon \text{Alt}_n \to 2^{[n-1]} \]
\[ v \mapsto S(v) := \{ i \in [n-1] : v \cdot 1_{[i+1,n]} = +1 \} \]
\[ e_1 + \sum_{i \in S} (e_{i+1} - e_i) \to S. \]

Definition 5.2.
Put a partial order \( \preceq_{EL} \) on \( \text{Alt}_n \) that pulls back the inclusion order on \( 2^{[n-1]} \) via the above bijection \( \varphi \), that is, \( v \preceq_{EL} w \) if and only if \( S(v) \subseteq S(w) \). Equivalently, \( v \preceq_{EL} w \) if and only for every \( i = 1, 2, \ldots, n \) one has dot product \( (w - v) \cdot (e_i + e_{i+1} + \cdots + e_n) \geq 0 \).

Example 5.3.
Here is the order \( \preceq_{EL} \) on \( \text{Alt}_n \) for \( n = 3, 4, 5 \):

\[ \includegraphics[width=\textwidth]{example_diagram} \]

Next, we show that \( \lambda : C(\Phi_n) \to \text{Alt}_n \) defined by \( \lambda(I \preceq J) := 1_J - 1_I \) is an EL-labeling of \( \Phi_n \) with respect to \( \preceq_{EL} \) on \( \text{Alt}_n \). For the rest of this section, fix a pair \( I \prec_{lace} J \) in \( \Phi_n \) with \( H_{\min}(I, J) \) as in Definition 2.5.

Lemma 5.4. Assume \( I \prec_{lace} H \prec_{lace} J \) with \#H = \#I + 1. Then \( 1_{H_{\min}(I, J)} - 1_I \preceq_{EL} 1_H - 1_I \).

Proof. Recall \( \preceq_{EL} \) can be rephrased as follows: \( A \preceq_{EL} B \) if and only if \( (1_B - 1_A) \cdot 1_{[\ell,n]} \geq 0 \) for all \( \ell \).

Thus, since \( H_{\min}(I, J) \leq_{\text{comp}} H \) according to Lemma [2.5 (iii)], for all \( \ell \) one will have
\[ (1_H - 1_I) - (1_{H_{\min}(I, J)} - 1_I)) \cdot 1_{[\ell,n]} = (1_H - 1_{H_{\min}(I, J)}) \cdot 1_{[\ell,n]} \geq 0. \]

It turns out that one can characterize \( H_{\min}(I, J) \) in terms of \( \preceq_{EL} \).

Lemma 5.5. Assume \( I \prec_{lace} H \prec_{lace} J \) with \#J = \#I + 2. Then
\[ 1_H - 1_I \preceq_{EL} 1_J - 1_K \quad \text{if and only if} \quad H = H_{\min}(I, J). \]

Proof. Name the elements of \( I, H, J \) as follows:
\[ I = \{ i_1 < \cdots < i_p \}, \]
\[ H = \{ h_1 < \cdots < h_p < h_{p+1} \}, \]
\[ J = \{ j_1 < \cdots < j_p < j_{p+1} < j_{p+2} \}. \]

(\( \Leftarrow \)): Assume \( H = H_{\min}(I, J) \). We check for each \( \ell \) that \( (1_H - 1_I) \cdot 1_{[\ell,n]} \leq (1_J - 1_H) \cdot 1_{[\ell,n]} \), or equivalently,
\[ \#J \cap [\ell,n] + \#I \cap [\ell,n] - 2\#H \cap [\ell,n] \geq 0. \]
If $H \cap [\ell, n] = \emptyset$, this is clear. Otherwise, let $H \cap [\ell, n] = \{h_k, h_{k+1}, \ldots, h_{p+1}\}$, so that $\# H \cap [\ell, n] = p+2-k$.

Then the interlacing $I <_\text{lace} H <_\text{lace} J$ along with $h_k = \max(i_{k-1}, j_k)$ imply that

$$I \cap [\ell, n] = \begin{cases} \{i_k, i_{k+1}, \ldots, i_p\} & \text{if } h_k > i_{k-1}, \\ \{i_{k-1}, i_k, i_{k+1}, \ldots, i_p\} & \text{if } h_k = i_{k-1}, \end{cases}$$

$$J \cap [\ell, n] = \begin{cases} \{j_{k+1}, j_{k+2}, \ldots, i_{p+1}\} & \text{if } h_k > j_k, \\ \{j_k, j_{k+1}, j_{k+2}, \ldots, j_{p+2}\} & \text{if } h_k = j_k. \end{cases}$$

From this one can calculate that

$$\# J \cap [\ell, n] + \# I \cap [\ell, n] - 2 \# H \cap [\ell, n] = \begin{cases} 0 & \text{if } h_k = j_k > i_{k-1} \text{ or } h_k = i_{k-1} > j_k, \\ +1 & \text{if } h_k = i_{k-1} = j_k. \end{cases}$$

($\Rightarrow$): Assume $1_H - 1_I \leq_{EL} 1_J - 1_{\hat{H}}$.

**Claim:** One cannot have both strict inequalities $i_{k-1} < h_k < i_k$, nor a strict inequality $i_p < h_{p+1}$.

To see this claim, note that in either case $(i_{k-1} < h_k < i_k$ or $i_p < h_{p+1})$, it would imply $h_k \in H \setminus I$. Then since $I <_\text{lace} H$, this would imply $(1_H - 1_I) \cdot 1_{[h_k, n]} = +1$. But then $h_k \in H$ and $H <_\text{lace} J$ implies $(1_J - 1_H) \cdot 1_{[h_k, n]} = 0 < +1 = (1_H - 1_I) \cdot 1_{[h_k, n]}$, a contradiction to our assumption.

By Lemma 2.8(c) and (d), $I <_\text{lace} H <_\text{lace} J$ implies $h_k \geq \max(i_{k-1}, j_k)$ for $k = 1, 2, \ldots, p+1$. We must now show that these are all equalities, not inequalities. For the sake of contradiction, assume not and pick $k$ maximal such that $h_k > \max(i_{k-1}, j_k)$.

The Claim above then forces $k \leq p$ and $h_k = i_k$ (else $i_{k-1} < h_k < i_k$ or $k = p+1$ and $i_p < h_{p+1}$). Then $h_{k+1} > h_k = i_k$ and the maximality of $k$ forces $h_{k+1} = \max(i_k, j_{k+1}) = \max(h_k, j_{k+1}) = j_{k+1}$. And again the Claim forces $k+1 \leq p$ and $i_{k+1} = h_{k+1} (= j_{k+1})$.

We now repeat this argument to show by induction that for all $m = k+1, k+2, \ldots$, one has both $m \leq p$ and this triple coincidence $j_m = h_m = i_m$: this would contradict finiteness of $p$. The inductive step again notes that $h_{m+1} > h_m = i_m$ and maximality of $k$ forces $h_{m+1} = \max(i_m, j_{m+1}) = \max(h_m, j_{m+1}) = j_{m+1}$. But then the Claim forces $m+1 \leq p$ and $i_{m+1} = h_{m+1} (= j_{m+1})$, recreating the inductive hypothesis.

**Proof of Theorem 1.2** We first check our edge-labeling $\lambda$ satisfies the two conditions for an EL-labeling:

- for every interval $[x, y] \subset \Phi_n$, there is a unique maximal chain $(x = x_0 < x_1 < \cdots < x_k = y)$, that has *weakly rising* labels $\lambda(x_0, x_1) \leq_L \lambda(x_1, x_2) \leq_L \cdots \leq_L \lambda(x_{k-1}, x_k)$.

- if $x < x < y$, with $z \neq x_1$, then $\lambda(x, z) <_L \lambda(x_1, z)$.

The first condition follows by combining Lemma 2.8(b) and Lemma 5.4, which show that for any $I <_\text{lace} J$, the unique maximal chain in the interval $[I, J]$ corresponds to the $(I, J)$-monotone trapezoid $T_{\min}(I, J)$.

Then the second condition comes from Lemma 5.5.

For the second assertion of the theorem, it suffices to check that if $T, T'$ are monotone triangles with $T' <_W T$, then any of the above EL-shellings, which come from linearly extending the lexicographic ordering of $<_{EL}$ on edge labels, will have $T'$ earlier than $T$. By definition of the weak order $<_W$, it suffices to check this holds when $T' = \pi_i(T)$ for some $i$. In this case, it follows because Lemma 5.4 shows that $T$ will have lexicographically earlier edge label sequence than $T'$: the two sequences first differ in replacing the label $1_{T_{i+1}} - 1_{T_i}$ with the $<_{EL}$-smaller label $1_{H_{min}(T, T_{i+1})} - 1_{T_i}$. 

**6. Descents, h-vectors and flag h-vectors**

Recall from the Introduction the usual *descent set* for a permutation $w = (w_1, \ldots, w_n)$ in $\mathfrak{S}_n$

$$\text{Des}(w) := \{k \in [n-1] : w_k > w_{k+1}\} = \{k \in [n-1] : \pi_k(w) = ws_k <_W w\}.$$ 

It has a natural extension to monotone triangles $T$, motivated by the weak order $<_W$ and our shelling results.
**Definition 6.1.**
Define the descent set $\text{Des}(T)$ for $T = (T_0, T_1, \ldots, T_n)$ in $\text{MT}_n$ by

$$\text{Des}(T) := \{k \in [n-1] : \pi_k(T) <_W T\} = \{k \in [n-1] : T_k \neq H_{\min}(T_{k-1}, T_{k+1})\}.$$  

There is another way to define $\text{Des}(T)$.

**Lemma 6.2.** For $T$ in $\text{MT}_n$, one has

$$\text{Des}(T) := \{k \in [n-1] : \text{there does not exist some } T' \neq T \text{ with } \pi_k(T') = T\}.$$  

In particular, $T$ is one of the maximal elements of the weak order $<_W$ if and only if $\text{Des}(T) = [n-1]$.

**Proof.** Since $\pi^2_k = \pi_k$, if there exists $T'$ with $\pi_k(T') = T$, then $\pi_k(T) = \pi^2_k(T') = \pi_k(T') = T$, so $k \notin \text{Des}(T)$.

Conversely, if $k \notin \text{Des}(T)$, so that $\pi_k(T) = T$, we wish to exhibit at least one $T' \neq T$ having $\pi_k(T') = T$.

From $\pi_k(T) = T = (T_0, T_1, \ldots, T_n)$ we know that $T_k = H_{\min}(I, J)$ where $I := T_{k-1}, J := T_{k+1}$, so that if we construct $T'$ from $T$ by replacing $T_k$ with $H_{\max}(I, J)$ as defined in $\text{S}$, then it will certainly have $\pi_k(T') = T$.

It only remains to show that $T' \neq T$, that is $H_{\max}(I, J) \neq H_{\min}(I, J)$. To check this, name elements:

$I = \{i_1 < i_2 < \cdots < i_{k-1}\}$,

$$H_{\min}(I, J) = \{h_1 < h_2 < \cdots < h_{k-1} < h_k\},$$

$$H_{\max}(I, J) = \{h'_1 < h'_2 < \cdots < h'_{k-1} < h'_k\},$$

$J = \{j_1 < j_2 < \cdots < j_{k-1} < j_k\}$.

Then the formulas defining $H_{\min}(I, J), H_{\max}(I, J)$ are $h_m = \max(i_m-j_m), h'_m = \min(i_m, j_{m+1})$, implying that $h'_m = h_m$ if and only if $i_m = j_m$ or $i_m-1 = j_{m+1}$. Since $\#I \cap J \leq \#I = k-1$, such an equality occurs at most $k-1$ times, and hence $h'_m \neq h_m$ for at least one $m = 1, 2, \ldots, k$.$\square$

**Remark 6.3.**
Embedded in the previous proof are operators $\pi_k' : T \mapsto T'$ on $\text{MT}_n$ for $k = 1, 2, \ldots, n-1$, where $T'$ is obtained from $T$ by replacing $T_k$ with $T_k' = H_{\max}(T_{k-1}, T_{k+1})$. Because of the relation between the $H_{\min}$ and $H_{\max}$ constructions described in Remark 2.7, the operators $\{\pi_k'\}_{k=1,2,\ldots,n-1}$ satisfy the same braid and quadratic relations as $\{\pi_k\}$, giving a (different) action of the 0-Hecke monoid $H_n(0)$ on $\text{MT}_n$.

One can check that this other action, in fact, extends the (right-)regular action of $H_n(0)$ on itself, when one identifies the monotone triangle $T(w)$ in $\text{MT}_n$ with $\pi_n'$ in $H_n(0)$. One could use it to define a different version of a weak order on $\text{MT}_n$, having a unique top element $T(w_0)$, but several different minimal elements. One reason that we instead chose the action by $\{\pi_k\}$ and their resulting weak order $<_W$ is so that the monotone triangle $T(e)$ corresponding to $e = (1, 2, \ldots, n)$ in $S_n$ labels the first facet in all of the shellings.

As mentioned in the Introduction, descent sets conveniently encode the flag f-vector $f(\Phi_n) := (f_J)_{J \subseteq [n-1]}$, where $f_J$ counts the number of chains that pass through the ranks in $J$. One instead considers the flag h-vector $h(\Phi_n) = (h_J)_{J \subseteq [n-1]}$, defined by these inclusion-exclusion relations:

$$f_J = \sum_{I : I \subseteq J} h_I, \text{ or equivalently, } h_J = \sum_{I : I \subseteq J} (-1)^{|J \setminus I|} f_I.$$  

General shelling theory (e.g., Björner [3, §1(B)]) then implies this combinatorial interpretation for $h_J$:  

$$h_J(\Phi_n) = \#\{T \in \text{MT}_n : \text{Des}(T) = J\}.$$  

The usual f-vector $f = (f_{-1}, f_0, f_1, \ldots, f_{n-2})$ and h-vector $h = (h_0, h_1, \ldots, h_{n-1})$ for $\Delta(\Phi_n \setminus \{0, 1\})$ can then be obtained by grouping the terms in $(f_J), (h_J)$ as follows:

$$f_i = \sum_{J \in \binom{[n-1]}{i}} f_J, \text{ and } h_i = \sum_{J \in \binom{[n-1]}{i}} h_J.$$  

In particular, $h_i(\Phi_n) = \#\{T \in \text{MT}_n : \#\text{Des}(T) = i\}$. See Table 1 for the h-vector $h(\Phi_n)$ and flag h-polynomial for small values of $n$.  

1
\[ h(\Phi_n) = (h_0, h_1, \ldots, h_{n-1}) \]

| \( n \) | \( \sum_{J\subseteq[n-1]} h_J(\Phi_n) x_J \) where \( x_J := \prod_{i\in J} x_i \) |
|---|---|
| 2 | \( x_1 + 1 \) |
| 3 | \( 2x_1x_2 + 2x_1 + 2x_2 + 1 \) |
| 4 | \( 9x_1x_2x_3 + 7x_1x_2 + 7x_1x_3 + 7x_2x_3 + 3x_1 + 5x_2 + 3x_2 + 1 \) |
| 5 | \( 80x_1x_2x_3x_4 + 52x_1x_2x_3 + 44x_1x_3x_4 + 44x_1x_3x_4 + 52x_2x_3x_4 + 16x_1x_2 + 26x_1x_3 + 32x_2x_3 + 14x_1x_4 + 26x_2x_4 + 16x_1x_2 + 16x_1x_4 + 4x_1 + 9x_2 + 9x_3 + 4x_1 + 1 \) |
| 6 | \( 1321x_1x_2x_3x_4x_5 + 745x_1x_2x_3x_4x_5 + 562x_1x_2x_3x_4x_5 + 487x_1x_2x_3x_4x_5 + 562x_1x_2x_3x_4x_5 + 745x_2x_3x_4x_5 + 180x_1x_2x_3 + 251x_1x_2x_4 + 298x_1x_3x_4 + 405x_2x_3x_4 + 120x_1x_2x_5 + 215x_1x_3x_5 + 298x_2x_3x_5 + 120x_1x_4x_5 + 251x_2x_3x_5 + 180x_3x_4x_5 + 30x_1x_2 + 65x_1x_3 + 92x_2x_3 + 58x_1x_4 + 125x_2x_4 + 92x_3x_4 + 23x_1x_5 + 58x_5x_5 + 65x_3x_5 + 30x_1x_5 + 5x_1 + 14x_2 + 19x_3 + 14x_4 + 5x_5 + 1 \) |

Table 1. The \( h \)-vectors of \( \Phi_n \) for \( n \leq 8 \) and flag \( h \)-polynomials of \( \Phi_n \) for \( n \leq 6 \). All data computed using \textsc{Sage}.

We remark on some features of this data. Note the sequence of values 1, 2, 9, 80, 1321, 39026, 2016716 for \( h_{n-1} = \# \{ T \in MT_n : \text{Des}(T) = [n-1] \} = \# \{ \text{maximal elements in the poset } (MT_n, \prec \text{w}) \} \), appearing at the right in Table 1 which is not in the Online Encyclopedia of Integer Sequences (OEIS).

The data invites comparison with the Boolean algebra \( 2^{[n]} \), which has \( h \)-vector \( h(2^{[n]}) = (h_0, h_1, \ldots, h_{n-1}) \) given by the \textit{Eulerian numbers}, that is, \( h_i(2^{[n]}) = \# \{ w \in \mathcal{S}_n : \# \text{Des}(w) = i \} \). The Eulerian numbers are well-behaved in many ways (see Petersen [14]). For example, they satisfy recurrences and have the \textit{symmetry} \( h_i = h_{n-1-i} \). They also have the very strong property that the \textit{h-polynomial}

\[ h(2^{[n]}, t) := \sum_{i=0}^{n-1} h_it^i = \sum_{w\in\mathcal{S}_n} t^{\# \text{Des}(w)} \]

has only real zeroes. This implies \textit{log-concavity} \( h_i^2 \geq h_{i+1}h_{i-1} \), which then implies \textit{unimodality}, meaning that there is some \( k \) (in this case \( k = \left\lceil \frac{n-1}{2} \right\rceil \) works) for which \( h_0 \leq h_1 \leq \cdots \leq h_k \geq \cdots \geq h_{n-2} \geq h_{n-1} \). From the data in Table 1, the reader can check that for \( \Phi_n \), the \( h \)-polynomial

\[ h(\Phi_n, t) := \sum_{i=0}^{n-1} h_it^i = \sum_{T\in\text{MT}_n} t^{\# \text{Des}(T)} \]

is irreducible in \( \mathbb{Q}[t] \) with only real zeroes for \( n \leq 8 \), hence is log-concave for those values.

\textbf{Question 6.4.} Does \( h(\Phi_n, t) \) have only real zeroes? If not, is its coefficient sequence log-concave, or at least unimodal?

\textbf{Question 6.5.} What is the largest entry in the \( h \)-vector of \( \Phi_n \)? Is it always \( h_{n-2} \)?
As described in the Introduction, the map $w \mapsto \text{Des}(w)$ that sends a permutation $w$ in $S_n$ to its descent set was pleasingly reinterpreted in the work of Malvenuto and Reutenauer \[11\] as a morphism of Hopf algebras. We wish to explain here how this extends to the map $T \mapsto \text{Des}(T)$ sending a monotone triangle to its descent set, giving at least an algebra (but not coalgebra) morphism out of the Hopf algebra of ASMs recently defined by Cheballah, Giraudo and Maurice \[3\].

Let us start by recalling the algebra structures on quasisymmetric functions, permutations, and ASMs.

**Definition 7.1.**

The ring of quasisymmetric functions $\text{QSym}$ can be defined as the subalgebra of the algebra $\mathbb{Z}[[x_1, x_2, \ldots]]$ of formal power series that has $\mathbb{Z}$-basis given by the monomial quasisymmetric functions $M_\alpha := \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ as $\alpha = (\alpha_1, \ldots, \alpha_k)$ runs through all (ordered) compositions having $\alpha_i \in \{1, 2, \ldots\}$ and any length $k \geq 0$.

The ring $\text{QSym}$ was introduced by Gessel \[9\]. He observed that if one defines the unitriangularly related $\mathbb{Z}$-basis of fundamental quasisymmetric functions $(16)$ $L_\alpha := \sum_{\beta : \beta \text{ coarsens } \alpha} M_\beta$ then results from Stanley’s theory of $P$-partitions \[16\] Cor. 7.19.5 imply the following expansion for products of $L_\alpha$’s. Given a subset $J = \{j_1 < \cdots < j_\ell\} \subseteq [n-1]$, define its associated composition of $n$ to be

$$\alpha(J) := (j_1, j_2 - j_1, j_3 - j_2, \ldots, j_\ell - j_{\ell-1}, n - j_\ell).$$

In other words, $\alpha(J)$ is the composition whose partial sums are the elements of $J$.

For $u, v$ in $\mathbb{S}_a, \mathbb{S}_b$, let $u \sqcup v[a]$ be the set of all shuffles $w = (w_1, w_2, \ldots, w_{a+b})$ of the sequences $u = (u_1, \ldots, u_a)$ and $v[a] = (a + v_1, a + v_2, \ldots, a + v_b)$. In other words, $w \in \mathbb{S}_{a+b}$ is in $u \sqcup v[a]$ if $(u_1, \ldots, u_a)$ and $(a + v_1, \ldots, a + v_b)$ are subsequences of $w$.

**Proposition 7.2.** Given $u, v$ in $\mathbb{S}_a, \mathbb{S}_b$,

$$L_\alpha(\text{Des}(u)) \cdot L_\alpha(\text{Des}(v)) = \sum_{w \in u \sqcup v[a]} L_\alpha(\text{Des}(w)).$$

This was part of Malvenuto and Reutenauer’s motivation for the following definition.

**Definition 7.3.**

The Malvenuto-Reutenauer (Hopf) algebra of permutations is a graded free abelian group

$$\text{FQSym} = \bigoplus_{n \geq 0} \text{FQSym}_n,$$

in which $\text{FQSym}_n$ has $\mathbb{Z}$-basis elements $\{w\}_{w \in \mathbb{S}_n}$. As an algebra, its multiplication is extended $\mathbb{Z}$-linearly from this rule: for $u, v$ in $\mathbb{S}_a, \mathbb{S}_b$,

$$u \cdot v = \sum_{w \in u \sqcup v[a]} w$$

in which the sum runs over the same set of $w$ as in $(17)$.

Thus the algebra structure on $\text{FQSym}$ was defined so that this map is a (surjective) algebra morphism:

$$\text{FQSym} \xrightarrow{\varphi} \text{QSym} \xrightarrow{\text{w}} L_\alpha(\text{Des}(w)).$$
Definition 7.4.
Cheballah, Giraudo and Maurice [8] embedded FQSym inside a larger graded Hopf algebra

\[(20) \quad \text{ASM} = \bigoplus_{n \geq 0} \text{ASM}_n,\]

whose \(n^{th}\)-graded component \(\text{ASM}_n\) has a \(Z\)-basis \(\{A\}\) indexed by \(A\) in \(\text{ASM}_n\). Its algebra structure generalizes that of FQSym to the following row-shuffle\(^1\) product. Given ASMs \(A, B\) of sizes \(a \times a\) and \(b \times b\), define \(A \circ b\) to be the \(a \times (a + b)\) matrix with first \(a\) columns \(A\) and last \(b\) columns all 0-vectors. Likewise, \(a \circ B\) is the \(b \times (a + b)\) matrix with last \(b\) columns \(B\) and first \(a\) columns all 0-vectors. Then define

\[(21) \quad A \cdot B = \sum_{C \in (A \circ b) \cup (a \circ B)} C\]

where \(C\) runs through all the \((a + b) \times (a + b)\) matrices obtained by shuffling the rows of \(A \circ b\) and of \(a \circ B\).

**Example 7.5.**
If \(A = \begin{vmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{vmatrix}\) and \(B = \begin{vmatrix} 0 + 0 \\ 0 + 0 \end{vmatrix}\), then \(A \circ b = \begin{vmatrix} 0 + 0 0 0 0 \\ 0 + 0 0 0 \end{vmatrix}\), and \(a \circ B = \begin{vmatrix} 0 0 0 0 + 0 \\ 0 0 0 + 0 \end{vmatrix}\). One then has

\[
A \cdot B = \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} \cdot \begin{bmatrix} 0 + 0 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} + \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} & \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} + \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} + \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} \\
\begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} + \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} & \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} + \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} + \begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 0 & 0 + 0 \end{bmatrix} \end{bmatrix}.
\]

Note that when one restricts the product formula (21) to the elements of the form \(w := A(w)\) where \(A(w)\) is the permutation matrix corresponding to \(w^{-1}\), it agrees with the multiplication rule for \(u \cdot v\) given in (15). We also wish to recast the formula (21) in terms of monotone triangles. The following proposition is straightforward using the bijection \(\text{ASM}_n \rightarrow \text{MT}_n\) described in the Introduction.

**Proposition 7.6.** Fix \(A, B\) in \(\text{ASM}_a, \text{ASM}_b\), with corresponding monotone triangles \(T(A), T(B)\) in \(\text{MT}_a, \text{MT}_b\). Let \(C\) in \((A \circ b) \cup (a \circ B)\) have

* \(S \subset [a + b]\) the \(a\)-element subset indexing the rows of \(C\) that come from \(A \circ b\), and
* \([a + b] \setminus S\) the \(b\)-element subset indexing the rows of \(C\) that come from \(a \circ B\).

Then \(T(C)\) in \(\text{MT}_{a+b}\) has as its \(k^{th}\) row the set

\[T(C)_k = T(A)_i \cup (a + T(B))_j,\]

where

* \(i = \#S \cap [k]\), and
* \(j = \#([a + b] \setminus S) \cap [k]\) \((= k - i)\).

**Example 7.7.**
For \(A = \begin{vmatrix} 0 + 0 \\ 0 + 0 \end{vmatrix}\) and \(B = \begin{vmatrix} 0 + 0 \\ 0 + 0 \end{vmatrix}\) as in Example 7.5, one has \(T(A) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\) and \(a + T(B) = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}\). Hence the terms \(C\) appearing in the product \(A \cdot B\) correspond to these monotone triangles \(T(C)\):
In this case, Proposition 7.6 implies that $k$ where $T$

Case 1. Both computed via the formula (7) as the maximum of its two neighboring entries to the northwest and southwest. Each of the entries $b_i$ themselves. As notation, let $\alpha$ here the second equality used the claim, while the third equality used (19). Theorem 1.3. The map $\text{FQSym} \xrightarrow{\varphi} \text{QSym}$ in (6) extends to an algebra (but not a coalgebra) morphism

$$\text{ASM} \xrightarrow{\varphi} \text{FQSym}$$

Here the second equality used the claim, while the third equality used (19).

Recall the statement of the theorem.

**Theorem 1.3** The map $\text{FQSym} \xrightarrow{\varphi} \text{QSym}$ in (6) extends to an algebra (but not a coalgebra) morphism

where $\text{Des}(A) = \text{Des}(T(A))$ for $A$ in $\text{ASM}_n$ is the descent set of its monotone triangle $T(A)$.

**Proof of Theorem 1.3** Given $A, B$ in $\text{ASM}_n$, $\text{ASM}_m$, we claim that the multiset of descent sets $\{\text{Des}(T(C))\}$ as $C$ runs through the elements of $(A \circ b) \cup (a \circ B)$ depends only upon the descent sets $\text{Des}(T(A))$, $\text{Des}(T(B))$, not on $A, B$ themselves. Assuming this claim for the moment, one finishes the proof by picking arbitrary $u, v$ in $\mathcal{S}_a, \mathcal{S}_b$ having $\text{Des}(u) = \text{Des}(T(A))$ and $\text{Des}(v) = \text{Des}(T(B))$, and calculating

$$\varphi(A \cdot B) = \sum_{C \in (A \circ b) \cup (a \circ B)} L_{\alpha(\text{Des}(T(C)))} = \sum_{w \in (a \circ B)} L_{\alpha(\text{Des}(w))}$$

$$= L_{\alpha(\text{Des}(u))} L_{\alpha(\text{Des}(v))} = L_{\alpha(\text{Des}(A))} L_{\alpha(\text{Des}(B))} = \varphi(A) \varphi(B).$$

Here the second equality used the claim, while the third equality used (19).

To prove the claim, note that each $C$ in $(A \circ b) \cup (a \circ B)$ is determined by the $a$-subset $S \subseteq [a+b]$ indexing the rows of $C$ that come from $A \circ b$. We give rules in cases below that decide whether some $k \in [a+b-1]$ lies in $\text{Des}(T(C))$, based only on the subset $S$ and the descent sets $\text{Des}(T(A))$ and $\text{Des}(T(B))$, not on $A, B$ themselves. As notation, let $i := \#S \cap [k-1], j := \#(([a+b] \setminus S) \cap [k-1])$, and name these elements:

$$T(A)_{i+2} = \{a_1 < \cdots < a_i < a_{i+1} < a_{i+2}\},$$

$$T(B)_{j+2} = \{b_1 < \cdots < b_j < b_{j+1} < b_{j+2}\}.$$  

Note that deciding whether $k$ lies in $\text{Des}(T(C))$ simply means checking whether any of the entries of $T'_k$, where $T' := \pi_k(T(C))_k = H_{\min}(T(C)_{k-1}, T(C)_{k+1})$, differs from the corresponding entry of $T(C)_k$, when computed via the formula (7) as the maximum of its two neighboring entries to the northwest and southwest.

Case 1. Both $k, k+1$ lie in $S$.

In this case, Proposition 7.6 implies that $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

$$\begin{array}{cccccc}
  a_1 & \cdots & a_i & b_1 & \cdots & b_j \\
  a_1 & \cdots & a_i & a_{i+1} & b_1 & \cdots & b_j \\
  a_1 & \cdots & a_i & a_{i+1} & a_{i+2} & b_1 & \cdots & b_j
\end{array}$$

Each of the entries $b_m$ in $T(C)_k$ equals its northwest neighbor, so is unchanged in $\pi_k(T(C))$. This implies that $k \in \text{Des}(T(C))$ if and only if $k \in \text{Des}(T(A)).$
Case 2. Both $k, k + 1$ lie in $[a + b] \setminus S$.
Here $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

\[
\begin{array}{cccc}
  a_1 & \cdots & a_i & b_1 \cdots b_j \\
  a_1 & \cdots & a_i & b_1 \cdots b_j & b_{j+1} \\
  a_1 & \cdots & a_i & b_1 & \cdots & b_j & b_{j+1} & b_{j+2}
\end{array}
\]

Similarly to Case 1, each entry $a_m$ in $T(C)_k$ equals its southwest neighbor, so is unchanged in $\pi_k(T(C))$. This implies $k \in \text{Des}(T(C))$ if and only if $k \in \text{Des}(T(B))$.

Case 3. $k$ lies in $S$, but $k + 1$ lies in $[a + b] \setminus S$.
Here $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

\[
\begin{array}{cccc}
  a_1 & \cdots & a_i & b_1 \cdots b_j \\
  a_1 & \cdots & a_i & a_{i+1} & b_1 \cdots b_j \\
  a_1 & \cdots & a_i & a_{i+1} & b_1 & \cdots & b_j & b_{j+1}
\end{array}
\]

We claim that in this case, $k \not\in \text{Des}(T(C))$, since each entry $a_m$ of $T(C)_k$ equals its southwest neighbor, while each entry $b_m$ of $T(C)_k$ equals its northwest neighbor.

Case 4. $k + 1$ lies in $S$, but $k$ lies in $[a + b] \setminus S$.
Here $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

\[
\begin{array}{cccc}
  a_1 & \cdots & a_i & b_1 \cdots b_j \\
  a_1 & \cdots & a_i & b_1 \cdots b_j & b_{j+1} \\
  a_1 & \cdots & a_i & b_1 & \cdots & b_j & b_{j+1}
\end{array}
\]

In this case $k \in \text{Des}(T(C))$, since the entry $b_1$ of $T(C)_k$ has $b_1 > a \geq \max(a_i, a_{i+1})$.

To see that $A \xrightarrow{\varphi} L_{\alpha(\text{Des}(A))}$ is not a coalgebra morphism, for example, one can check from the coproduct formula of Cheballah, Giraudo and Maurice [16, (1.3.5)] that the alternating sign matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has coproduct $\Delta(A) = 1 \otimes A + A \otimes 1$, that is, $A$ is primitive. Meanwhile, its image $\varphi(A) = L_{(1,1,1)}$ has

$\Delta(\varphi(A)) = \Delta(L_{(1,1,1)}) = 1 \otimes L_{(1,1,1)} + L_{(1,1,1)} \otimes L_{(1,1,1)} + L_{(1,1,1)} \otimes L_{(1,1,1)} + L_{(1,1,1)} \otimes 1$,

which is not the same as $(\varphi \otimes \varphi)(\Delta(A)) = 1 \otimes L_{(1,1,1)} + L_{(1,1,1)} \otimes 1$. That is, $\varphi(A)$ is not primitive. \qed

Remark 7.8.
It is well-known, and not hard to see (e.g., as a special case of [16, Thm. 7.19.7]), that applying $\varphi$ to the sum of all of the basis elements $\{w\}_{w \in S_n}$ for FQSym$_n$ gives a readily-identifiable symmetric function

\[\varphi \left( \sum_{w \in S_n} w \right) = \sum_{w \in S_n} L_{\alpha(\text{Des}(w))} = (x_1 + x_2 + \cdots)^n.\]

This fails for ASM$_n$, e.g., the data in Table [1] for $n = 4$ together with [16] shows that

\[\varphi \left( \sum_{A \in \text{ASM}_n} A \right) = \sum_{A \in \text{ASM}_4} L_{\alpha(\text{Des}(T(A)))} = L_{(4)} + 3L_{(1,3)} + 5L_{(2,2)} + 3L_{(3,1)} + 7L_{(1,1,2)} + 7L_{(1,2,1)} + 9L_{(1,1,1,1)} = M_{(4)} + 4M_{(1,3)} + 6M_{(2,2)} + 4M_{(3,1)} + 16M_{(1,1,2)} + 14M_{(1,2,1)} + 16M_{(2,1,1)} + 42M_{(1,1,1,1)}\]

which is not a symmetric function, because its coefficient on $M_n$ is not constant for all compositions $\alpha$ within the same rearrangement class. It would be interesting to find natural subcollections $\{A\}$ of ASM$_n$, not contained entirely in $S_n$, for which $\varphi(\sum A)$ is a symmetric function.

18
8. Poset properties of weak order on $\text{MT}_n$

The weak order $<_W$ on the symmetric group $\mathfrak{S}_n$ has many pleasant poset-theoretic properties:

- It has bottom and top elements $\hat{0} = e = (1, 2, \ldots, n - 1, n)$ and $\hat{1} = w_0 = (n, n - 1, \ldots, 2, 1)$.
- It is a lattice.
- It is ranked with rank function given by the cardinality $\# \text{Inv}(w)$ of the (left-)inversion set of $w$:
  \[ \text{Inv}(w) = \{(w_j, w_i) : 1 \leq i < j \leq n \text{ and } w_i > w_j\} \]

- It has an encoding via inclusion of these (left-)inversion sets: $u <_W v$ if and only if $\text{Inv}(u) \subset \text{Inv}(v)$.
- The Möbius function $\mu(u, v)$ for $u <_W v$ only takes on values in $\{0, +1, -1\}$.
- More precisely, the homotopy type of the order complex $\Delta(u, v)$ of any of its open intervals $(u, v)$ is contractible or homotopy-spherical. Specifically, one can phrase this in terms of $H_*(\text{S}_n)$-action on $S_n$ as follows: $\Delta(u, v)$ is contractible unless $u = \pi_{w_0(J)}v$ for some subset $J \subset \text{Des}(u)$, in which case, $\Delta(u, v)$ is homotopy-equivalent to a $(\#J - 2)$-dimensional sphere; see Björner [2, Theorem 6].

Only a few of these properties extend to the weak order $<_W$ to $\text{MT}_n$. It is still true that $(\text{MT}_n, <_W)$ has a bottom element $\hat{0} = T(e) = (\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{n\})$, but it no longer has a top element $\hat{1}$, as there are many maximal elements.

Since $\text{MT}_n$ is finite, and has no top element, it cannot be a lattice, but it is also true that its intervals fail to be lattices. For example, the lower interval shown on the left in Figure 1 is not a lattice, because, for example,

\[
\begin{align*}
1 &< 2 < 3 < 123 \\
1 &< 2 < 3 < 123
\end{align*}
\]

are minimal upper bounds. Note that this same lower interval is not ranked since there are maximal chains of lengths four and five.

Alternating sign matrices $A = (a_{ij})$ have a well-established inversion number $\text{inv}(A) := \sum_{l<k, j > l} a_{ij} a_{kl}$, introduced by Mills, Robbins and Rumsey [12, p344], which generalizes the rank function $\# \text{Inv}(w)$ for
(S_n, <_W) of permutations. However, it is not clear that it relates to chains in the weak order (MT_n, <_W). For example, one might hope that the length of the shortest saturated chain from 0 to T in weak order might correspond to the inversion number of the alternating sign matrix of T. However, Roger Behrend noted that this fails for the first time in MT_4, where one can check that

\[ T = \begin{array}{cccc}
3 & 2 & 4 \\
1 & 3 & 4
\end{array} \quad \leftrightarrow \quad A = \begin{bmatrix}
0 & 0 & 0 \\
0 & + & 0 \\
+ & + & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \text{inv}(A) = 5
\]

but the shortest saturated chain from 0 to T has length 4. Additionally, in MT_5 one can check that

\[ T = \begin{array}{cccc}
3 & 4 \\
1 & 4 & 5 \\
1 & 2 & 4 & 5
\end{array} \quad \leftrightarrow \quad A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & + \\
0 & + & 0 & 0 \\
+ & 0 & 0 & 0 \\
0 & 0 & + & 0
\end{bmatrix}, \quad \text{inv}(A) = 5
\]

but all saturated chains in the weak order from 0 to T have length at least 6.

**Question 8.1.** Is there a generalization of the notion of the (left-)inversion set Inv(w) for permutations to an inversion set Inv(T) for monotone triangles, encoding the weak order (MT_n, <_W) via inclusion, that is, \( T <_W T' \) if and only if \( \text{Inv}(T) \subset \text{Inv}(T') \)?

In spite of some of the above shortcomings, the Möbius function and homotopy type of open intervals in (MT_n, <_W) may be just as simple to describe as for weak order on S_n.

**Conjecture 8.2.** For two monotone triangles \( T' \leq_W T \), the order complex \( \Delta(T', T) \) of their open interval in \( <_W \) is contractible unless \( T' = \pi_{w_0(J)}(T) \) for some \( J \subset \text{Des}(T) \), namely, \( J := \{ m : T'_m \neq T_m \} \), in which case \( \Delta(T', T) \) is homotopy equivalent to a \((#J - 2)\)-dimensional sphere.

Conjecture 8.2 would imply that \( \mu(T', T) = 0 \) in the contractible case, and \((-1)^{#J} \) when \( T' = \pi_{w_0(J)}(T) \).

**Example 8.3.**
An interesting example is the non-lattice lower interval \([\hat{0}, y]\) on the left in Figure 1 which has the order complex \( \Delta(\hat{0}, y) \) of its open interval homotopy equivalent to a 1-sphere (circle). Meanwhile, its subinterval \([x, y]\) shown to its right has \( \Delta(x, y) \) contractible.

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