Geometric symmetries in light nuclei

R Bijker
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A.P. 70-543, 04510 México, D.F., México
E-mail: bijker@nucleares.unam.mx

Abstract. The algebraic cluster model is applied to study cluster states in the nuclei $^{12}\text{C}$ and $^{16}\text{O}$. The observed level sequences can be understood in terms of the underlying discrete symmetry that characterizes the geometrical configuration of the $\alpha$-particles, i.e. an equilateral triangle for $^{12}\text{C}$, and a regular tetrahedron for $^{16}\text{O}$. The structure of rotational bands provides a fingerprint of the underlying geometrical configuration of $\alpha$-particles.

1. Introduction

Ever since the early days of nuclear physics the structure of $^{12}\text{C}$ has been extensively investigated both experimentally and theoretically [1, 2, 3, 4]. In recent years, the measurement of new rotational excitations of both the ground state [5, 6, 7] and the Hoyle state [8, 9, 10, 11] has generated a lot of renewed interest to understand the structure of $^{12}\text{C}$ and that of $\alpha$-cluster nuclei in general. Especially the (collective) nature of the $0^+$ Hoyle state at 7.65 MeV which is of crucial importance in stellar nucleosynthesis to explain the observed abundance of $^{12}\text{C}$, has presented a challenge to nuclear structure calculations, such as $\alpha$-cluster models [12], antisymmetrized molecular dynamics [13], fermionic molecular dynamics [14], BEC-like cluster model [15], (no-core) shell models [16, 17], ab initio calculations based on lattice effective field theory [18, 19], and the algebraic cluster model [7, 20, 21].

In this contribution, I discuss some properties of the $\alpha$-cluster nuclei $^{12}\text{C}$ and $^{16}\text{O}$ in the framework of the algebraic cluster model.

2. Algebraic Cluster Model

The Algebraic Cluster Model (ACM) describes the relative motion of the $n$-body clusters in terms of a spectrum generating algebra of $U(\nu + 1)$ where $\nu = 3(n - 1)$ represents the number of relative spatial degrees of freedom. For the two-body problem the ACM reduces to the $U(4)$ vibron model [22], for three-body clusters to the $U(7)$ model [20, 23] and for four-body clusters to the $U(10)$ model [21, 24]. In the application to $\alpha$-cluster nuclei the Hamiltonian has to be invariant under the permutation group $S_n$ for the $n$ identical $\alpha$ particles. Since one does not consider the excitations of the $\alpha$ particles themselves, the allowed cluster states have to be symmetric under the permutation group.

The potential energy surface corresponding to the $S_n$ invariant ACM Hamiltonian gives rise to several possible equilibrium shapes. In addition to the harmonic oscillator (or $U(3n - 3)$ limit) and the deformed oscillator (or $SO(3n - 2)$ limit), there are other solutions which are of special interest for the applications to $\alpha$-cluster nuclei. These cases correspond to a geometrical...
Table 1. Algebraic Cluster Model for two-, three- and four-body clusters

|        | 2α   | 3α   | 4α   |
|--------|------|------|------|
| ACM    | U(4) | U(7) | U(10) |
| Point group | C_2  | D_{3h} | T_d  |
| Geometry | Linear | Triangle | Tetrahedron |
| G.s. band | 0^+  | 0^+  | 0^+  |
|         | 2^+  | 2^+  |      |
|         | 3^-  | 3^-  |      |
|         | 4^+  | 4^±  | 4^+  |
|         | 5^-  |      |      |
|         | 6^+  | 6^±± | 6^±  |

configuration of α particles located at the vertices of an equilateral triangle for $^{12}$C and of a regular tetrahedron for $^{16}$O. Even though they do not correspond to dynamical symmetries of the ACM Hamiltonian, one can still obtain approximate solutions for the rotation-vibration spectrum

$$E = \begin{cases} \omega_1(v_1 + \frac{1}{2}) + \omega_2(v_2 + 1) + \kappa L(L + 1) & \text{for } n = 3 \\ \omega_1(v_1 + \frac{1}{2}) + \omega_2(v_2 + 1) + \omega_3(v_3 + \frac{3}{2}) + \kappa L(L + 1) & \text{for } n = 4 \end{cases}$$

The rotational structure of the ground-state band depends on the point group symmetry of the geometrical configuration of the α particles and is summarized in Table 1.

The triangular configuration with three α particles has point group symmetry $D_{3h}$ [20]. Since $D_{3h} \sim D_3 \times P$, the transformation properties under $D_{3h}$ are labeled by parity $P$ and the representations of $D_3$ which is isomorphic to the permutation group $S_3$. The corresponding rotation-vibration spectrum is that of an oblate top: $v_1$ represents the vibrational quantum number for a symmetric stretching $A$ vibration, $v_2$ denotes a doubly degenerate $E$ vibration. The rotational band structure of $^{12}$C is shown in the left panel of Fig. 1.

The tetrahedral group $T_d$ is isomorphic to the permutation group $S_4$. In this case, there are three fundamental vibrations: $v_1$ represents the vibrational quantum number for a symmetric stretching $A$ vibration, $v_2$ denotes a doubly degenerate $E$ vibration, and $v_3$ a three-fold degenerate $F$ vibration. The right panel of Fig. 1 shows the rotational band structure of $^{16}$O.

3. Electromagnetic transitions

For transitions along the ground state band the transition form factors are given in terms of a product of a spherical Bessel function and an exponential factor arising from a Gaussian distribution of the electric charges, $F(0^+ \to L^P; q) = c_LJ_L(q\beta)e^{-q^2/4\alpha}$ [20]. The charge radius can be obtained from the slope of the elastic form factor in the origin $\langle r^2 \rangle^{1/2} = \sqrt{\beta^2 + 3/2\alpha}$. The transition form factors depend on the parameters $\alpha$ and $\beta$ which can be determined from the first minimum in the elastic form factor and the charge radius.

The transition probabilities $B(EL)$ along the ground state band can be extracted from the form factors in the long wavelength limit

$$B(EL; 0^+ \to L^P) = \frac{(Ze)^2}{4\pi} c_L^2 \beta^{2L}$$
Figure 1. (Color online) Rotational band structure of the ground-state band, the Hoyle band (or $A$ vibration) and the bending vibration (or $E$ vibration) in $^{12}\text{C}$ (left) [7], and the ground-state band (closed circles), the $A$ vibration (closed squares), the $E$ vibration (open circles) and the $F$ vibration (open triangles) in $^{16}\text{O}$ (right) [21].

Table 2. $B(EL)$ values in $^{12}\text{C}$ (top) and $^{16}\text{O}$ (bottom).

|          | $^{12}\text{C}$ | $^{16}\text{O}$ |
|----------|-----------------|-----------------|
| $B(E2; 2^+_1 \rightarrow 0^+_1)$ | $8.4 \pm 0.4\text{ e}^2\text{fm}^4$ | $215 \pm 10\text{ e}^2\text{fm}^6$ |
| $B(E3; 3^-_1 \rightarrow 0^+_1)$ | $44 \pm 17\text{ e}^2\text{fm}^6$ | $425 \pm 133\text{ e}^2\text{fm}^8$ |
| $B(E4; 4^+_1 \rightarrow 0^+_1)$ | $73\text{ e}^2\text{fm}^8$ | $9626\text{ e}^2\text{fm}^{12}$ |
| $M(E0; 0^+_2 \rightarrow 0^+_1)$ | $0.4\text{ fm}^2$ | $0.54\text{ fm}^2$ |

with

$$c_L^2 = \begin{cases} \frac{2L+1}{3} \left[ 1 + 2P_L(-\frac{1}{2}) \right] & \text{for } n = 3 \\ \frac{2L+1}{4} \left[ 1 + 3P_L(-\frac{3}{2}) \right] & \text{for } n = 4 \end{cases}$$

The good agreement for the $B(EL)$ values for the ground band in Table 2 shows that both in $^{12}\text{C}$ and in $^{16}\text{O}$ the positive and negative parity states merge into a single rotational band. Moreover, the large values of $B(EL; L^P \rightarrow 0^+_1)$ indicate a collectivity which is not predicted for simple shell model states. The large deviation for the $E0$ between the first excited $0^+_1$ (Hoyle) state and the ground state indicates that the $0^+_2$ state cannot be interpreted as a simple vibrational excitation of a rigid triangular ($^{12}\text{C}$) or tetrahedral ($^{16}\text{O}$) configuration, but rather corresponds to a more floppy configuration with large rotation-vibration couplings. A more
detailed study of the electromagnetic properties of $\alpha$-cluster nuclei in the ACM for non-rigid configurations is in progress.

4. Summary and conclusions

In this contribution, the cluster states in $^{12}$C and $^{16}$O were interpreted in the framework of the ACM as arising from the rotations and vibrations of a triangular and tetrahedral configuration of $\alpha$ particles, respectively. In both cases, the ground state band consist of positive and negative parity states which coalesce to form a single rotational band. This interpretation is validated by the observance of strong $B(EL)$ values. The rotational sequences can be considered as the fingerprints of the underlying geometric configuration (or point-group symmetry) of $\alpha$ particles.

For the Hoyle band in $^{12}$C there are several interpretations for the geometrical configuration of the three $\alpha$ particles. In order to determine whether the geometrical configuration of the $\alpha$-particles for the Hoyle band is linear, bent or triangular, the measurement of a possible 3− Hoyle state is crucial, since its presence would indicate a triangular configuration, just as for the ground state band.

Finally, the results presented here for $^{12}$C and $^{16}$O emphasize the occurrence of $\alpha$-cluster states in light nuclei with $D_{3h}$ and $T_d$ point group symmetries, respectively.

Acknowledgments

This work was supported in part by research grant IN107314 from PAPIIT-DGAPA.

References

[1] Wheeler J A 1937 Phys. Rev. 52 1083
[2] Hafstad L R and Teller E 1938 Phys. Rev. 54 681
[3] Von Oertzen W, Freer M and Kanada-En’yo Y 2006 Phys. Rep. 432 43
[4] Freer M and Fynbo H O U 2014 Prog. Part. Nucl. Phys. 78 1
[5] Freer M et al 2007 Phys. Rev. C 76 034320
[6] Kirsebom O S et al 2010 Phys. Rev. C 81 064313
[7] Marín-Lámarbarri D J, Bijker R, Freer M, Gai M, Kokalova T, Parker D J and Wheldon C 2014 Phys. Rev. Lett. 113 012502
[8] Itoh M et al 2011 Phys. Rev. C 84 054308
[9] Freer M et al 2012 Phys. Rev. C 83 034314
[10] Zimmerman W R et al 2013 Phys. Rev. Lett. 110 152502
[11] Freer M et al 2007 Phys. Rev. C 76 034320
[12] Robson D 1978 Nucl. Phys. A 308 381
[13] Kanada-En’yo Y 2007 Prog. Theor. Phys. 117 655
[14] Chernykh M, Feldmeier H, Neff H, Von Neumann-Cosel P and Richter A 2007 Phys. Rev. Lett. 98 032501
[15] Funaki Y, Horiuchi H, Von Oertzen W, Ropke G, Schuck P, Tohsaki A and Yamada T 2009 Phys. Rev. C 80 044326
[16] Roth R, Langhammer J, Calci A, Binder S and Navrtil J Phys. Rev. Lett. 107 072501
[17] Dreyfuss A C, Launey K D, Dytrych T, Draayer J P and Bahri C 2013 Phys. Lett. B 727 511
[18] Epelbaum E, Krebs H, Lee D and Meissner U G 2011 Phys. Rev. Lett. 106 192501
[19] Epelbaum E, Krebs H, Lähde T, Lee D and Meissner U G 2012 Phys. Rev. Lett. 109 252501
[20] Bijker R and Iachello F 2000 Phys. Rev. C 61 067305
[21] Bijker R and Iachello F 2002 Ann. Phys. (N.Y.) 298 334
[22] Bijker R and Iachello F 2014 Phys. Rev. Lett. 112 152501
[23] Iachello F 1981 Chem. Phys. Lett. 78 581
[24] Bijker R, Iachello F and Leviatan A 1994 Ann. Phys. (N.Y.) 236 69
[25] Bijker R, Iachello F and Leviatan A 2000 Ann. Phys. (N.Y.) 284 89
[26] Bijker R 2010 AIP Conference Proceedings 1323 28
[27] Bijker R 2012 J. Phys.: Conf. Series 380 012003
[28] Ajzenberg-Selove F 1990 Nucl. Phys. A 506 1
[29] Reuter W, Fricke G, Merle K and Miska H 1982 Phys. Rev. C 26 806
[30] Streih P and Schucan Th H 1968 Phys. Lett. B 27 641
[31] Tilley D R, Weller H R and Cheves C M 1993 Nucl. Phys. A 564 1