Thermodynamical Phase transitions, the
mean-field theories, and the renormalization
(semi)group: A pedagogical introduction

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“Since ’tis Nature’s law to change, Constancy alone is strange” — Earl of Rochester.

Abstract

While analyzing second order thermodynamical phase transitions, Lev
Landau (the famous Russian physicist) introduced a very vital concept, the concept of an “order parameter”. This not only amalgamated the previous fragmentary theoretical understanding of phase transitions (an arsenal of mean-field theories) but also it put forward the important theory of “spontaneous symmetry breaking”. Today, order parameter concept is a paradigm both in condensed matter physics and in high energy physics, and Landau theory is a pinnacle of all mean-field theories. Mean field theories are good qualitative descriptors of the phase transition behavior. But all mean-field theories (including Landau’s theory) fail at the critical point (the problem of large correlation length). The problems with large correlation length in quantum many-body systems are the hardest problems known in theoretical physics (both in condensed matter and in particle physics). It was Ken Wilson’s physical insights and his powerful mathematical skills that opened a way to the solution of such hard problems.

This manuscript is a perspective on these issues. Starting with simple examples of phase transitions (like ice/water; diamond/graphite etc.) we address the following important questions: Why does non-analyticity (sharp phase transitions) arise when thermodynamical functions (i.e., free energies etc) are good analytic functions? How does Landau’s program unify all the previous mean-field theories? Why do all the mean-field theories fail near the critical point? How does Wilson’s program go beyond all the mean-field theories? What is the origin emergence and universality?

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1 Introduction to Thermodynamical Phase Transitions

Mankind is familiar with various phases of matter and transitions between the phases from antiquity. For example, ice-water-steam; ice is solid that exhibit characteristic rigidity, water is liquid and it flows like other liquids, and steam is a gas, much like other gases. It is only in the 19th century that with the advent of the science of thermodynamics the phases of matter and their transformations were rationalized within the scientific domain. The phase transition of water to steam gave birth to the great discovery of the steam engine by Thomas Newcomen in 1712, and it resulted in the subsequent great industrial revolution first in Britain and then in Europe. It is beyond doubt that man is able to use, for the service of humanity, the great practical potential of the fundamental understanding of the materials and their transformations.

Aim of the the present paper is to give a broad overview of the phase transition phenomena from fundamental perspective starting from early works of Andrews and van der Waals. The Mean-Field Theories (MFTs) were developed to understand thermodynamical phase transitions from atomic/molecular perspective, and van der Waals was the pioneer in this[1]. We discuss various Mean-Field Theories (MFTs) and their reformulation by Lev Landau in 1937[2]. MFTs are good qualitative descriptors of the phase transition behavior. But all mean-field theories fail at the critical point including the Landau’s formulation. At the critical point correlation length (defined precisely in the text) diverges and fluctuations dominate over the average behavior. It is this failure of the MFTs that triggered further investigations that resulted the formulation of renormalization group by Wilson and others in 1970s[3]. The problems with large correlation length in quantum many-body systems are the hardest problems known in theoretical physics (both in condensed matter and in particle physics) and this branch of physics is far from being closed.

The present paper is a pedagogical introduction to this vast but coherent topic. The story of the successes and failures of MFTs and their systematic
replacement at the critical point by the renormalization group theory is given in a clear and pedagogical way. We also address, more specifically, the following questions: Why does non-analyticity (sharp phase transitions) arise when thermodynamical functions (i.e., free energies etc) are good analytic functions? How does Landau’s program unify all the previous mean-field theories? Why do all the mean-field theories fail near the critical point? How does Wilson’s program go beyond all the mean-field theories? What is origin emergence and universality?

The answers of above questions are well known\cite{3, 4, 5}. Presentation here is sufficiently self-contained to address the above questions. Thus it will be useful to the students learning the subject. We also express our standpoint regarding the failure of MFTs, and point out that MFTs faithfully reproduce diverging fluctuations at the critical point and divergence of fluctuations can be understood most clearly by recognizing the fact that the property of Statistical Independence (SI) is violated in systems with long correlation length (when various parts of the system become statistically correlated)\cite{6}. And the fundamental thesis of Boltzmann-Gibbs statistical mechanics–fluctuations converge as $\propto \frac{1}{N}$ in sum-function observables\cite{6}–is no longer applicable in systems with large correlation length (of the order of the system’s size) making the fluctuations to dominate over the average behavior. The reason why MFTs predict wrong values of critical indices is given in\cite{7}, that also clarifies a common misconception.

The paper is organized as follows. In the next subsection we give some examples of phase transitions. The next section (section 2) is devoted to Ehrenfest’s classification and to some mean-field theories. Then in section 3 we present Landau’s formulation of MFTs. Section 4 deals with the problems of MFTs. After recognizing the difficulties when dealing with problems with long correlation length we present Kadanoff’s construction and Wilson’s Renormalization Group (RG) formulation that rectify the problems of MFTs (section 5). This is illustrated by a specific example of $\phi^4$-theory. Although there are many different formulations of renormalization group method and has been applied to many different problems\cite{3, 4, 5, 8, 9, 10, 11}, our point here is to present the essence of the method for pedagogical purposes. In section 6 we also address the important question: Why do all the mean-field theories fail near the critical point and how does Wilson’s program go beyond MFTs? A short biographical sketch of Ken Wilson is given in the appendix.

1.1 Some examples

Phase transitions are ubiquitous in Nature! Most common phase transition is that of water and ice (Figure 1(a)). We see this when we put liquid water in the refrigerator for making ice, a common experience. We know that to make ice from water we need to extract some heat for a given mass of water. This is called the latent heat and for water it is 334 Joules/gram. Thus to make ice from water you have to extract this much amount of heat per gram from water (i.e., by cooling it) and to convert back water from ice you have to supply that amount (i.e., by heating the ice). You also note that when you cool water to convert it into ice, it expands, i.e.,, the volume of ice for a given mass of water is more than the volume of liquid water of that mass, or ice floats on water. These facts can be quantitatively described by introducing
the thermodynamical free energy $F$. Discontinuity of its first order derivatives with respect to thermodynamical parameters leads to above discontinuities. We will explain this in the section devoted to Ehrenfest’s classification.

Figure 1: (a) Three phases of water. (b) Phase diagram of a simple substance like water.

Figure 1(b) depicts a typical phase diagram of a “simple” substance like water for example (only change one has to make is that of a negative slope of the liquid-solid phase line for the case of water). Phase diagram depicts the kind of phases that are taken by a given substance under a given set of external conditions. These external conditions are the thermodynamical parameters of the system, for example, Pressure ($P$) and Temperature ($T$) for a simple substance, or, temperature and external magnetic field for a magnetic substance, like, for example, a ferromagnetic.

The phase diagram, as in figure 1(b), is a two dimensional projection on the $P$-$T$ plane of a three dimensional picture, with free energy along an axis coming out of the plane of the paper (for example, referring to figure 1(b)) and perpendicular to the $P$-$T$ axis. So in this 3-D picture one has three free energy surfaces. A substance is stable in a given region of the phase diagram if its free energy surface takes the minimum value. Consider the gaseous regime, in this, the free energy surface for the gaseous phase takes the minimum value as compared to two other free energy surfaces that of liquid and solid. The phase boundary lines shown in figure 1(b) are the lines at which above mentioned surfaces cut each other. And at each phase line free energies of two substances separated by that line takes the same value. Thus on each phase line both can co-exist. The point at which all can co-exist is called the triple point. Another important point is called the critical point. It is a point in a phase diagram where first order phase transition goes to second order phase transition. The discontinuity in the first order derivatives of $g$ (latent heat and change of volume per unit mass) goes to zero. Thus in liquid-gas transition the density difference (between gas and liquid) becomes zero and it is not possible to distinguish gas from liquid at the critical point $T_c$. However, the second order derivatives $\begin{equation} 1 \end{equation}$But density fluctuations dominate over all length scales at the critical point. Density fluctuations on the length scale of the wavelength of visible light cause strong scattering of light and results in a very spectacular phenomenon called the critical opalescence. Here is
of the Gibbs free energy remains discontinuous (see the section on Ehrenfest’s classification).

Diamond and Graphite: Another example of phase transitions is that of diamond and graphite (figure 2). These are two allotropes of carbon. Pencils write because layers of graphite easily slip past each other and can coat another object, such as paper. Diamonds look very very different from pencils! They are the hardest mineral known!! This is an example of the structural phase transition in which carbon atoms rearrange themselves. Diamonds and graphite are naturally occurring substances. Diamonds formed from carbon deep under earth (∼ 200 km) by geologic processes under larger pressure and temperature environments (as compared to ambient conditions). One can even transform graphite into diamonds—in a lab! This is possible by applying a very large pressure, 100,000 times that of normal atmospheric pressure, with graphite sample kept at 2000 K (figure 2(b)). Thus formed artificial diamonds are smaller in size as compared to diamonds created by geological processes. Nature is a smarter creator! One would like to say!

Superfluidity and Superconductivity: Phase transitions can also lead to quantum behavior at macro-scale. Superconductivity and superfluidity are two prime examples[13]. One interesting example is that of helium. Helium exists in nature in two stable isotopes. One common isotope is $^4\text{He}$. The other rare isotope $^3\text{He}$ is one atom for a million atoms of $^4\text{He}$ in earth’s atmosphere. Consider first the phase diagram of common isotope $^4\text{He}$ (Figure 3(a)). It liquefies at 4.2 K at the ambient pressure (Heike Kamerlingh Onnes was the first to liquefy helium. He liquefied it in 1908 and later discovered superconductivity a quick experiment[12]. Take equal quantities of cyclohexane and acetic anhydride in a test tube. Two liquids do not mix at room temperature, but on heating they mix. If one let the mixture to cool, near 53 C the mixture turns milky white—-A spectacular phenomenon called critical opalescence.
in Mercury in 1911). When cooled further $\sim 2.17 \text{ K}$ it transforms to a new form of liquid (a new thermodynamical phase called the superfluid phase). In this phase liquid Helium flows without friction (zero viscosity) and becomes a “super-heat-conductor”. This superfluid phase of $^4\text{He}$ is also called the HeII phase (the normal one is called HeI). Heat capacity shows an infinite discontinuity at 2.17 K (figure 4). There is no latent heat involved. The shape of heat capacity curve resembles with that of a Greek letter lambda ($\lambda$). Thus it is also called the lambda-transition in liquid Helium literature. Liquid Helium can be made solid by applying a pressure of 25 atm for temperature less than 2 K. Higher temperatures require higher pressure for solidification (figure 3 (a)). Interestingly it does not have a triple point! The rare isotope $^3\text{He}$ behaves exotically (figure 3(b)). If you start from a very low temperature, say 0.0001 K,\footnote{But it is not the second order phase transition! see Ehrenfest’s classification in the next section.}
Heat Capacity $T_c T$ = $-1.7 T / T$

Figure 5: Superconducting phase transition. (a) Heat capacity show a finite discontinuity at $T_c$, (b) Superconducting substances show Meissner effect (Image: public domain/Wikimedia Commons.).

with a sample of $^3He$ at 30 atm pressure, you first have liquid Helium. On reaching 0.18 $K$ it solidifies! It remains solid upto 0.49 $K$ and again turns liquid for higher temperatures. This behavior of the rare isotope of Helium is quite unusual (as common solids remain solid with reducing temperature!).

The superfluidity of the normal isotope $^4He$ is due to the Bose-Einstein condensation below the condensation temperature $\sim 2.17 K$. As $^4He$ atoms are bosons they condense into the ground state with order parameter (the condensate wavefunction) as $\rho e^{i\phi}$. Where $\rho$ is the superfluid density and $\phi$ is phase of the wave function. All the observed properties can be understood using two-fluid picture (see for details[14]). The reason why Helium remains liquid down to the lowest temperatures achievable at normal pressure is the large amplitudes of the zero-point quantum oscillations that disrupt the crystalline order.

The case of the rare isotope is much more interesting. As $^3He$ atoms are fermions they cannot show BEC directly and thus superfluidity. But experimentally superfluidity has been seen. The reason for superfluidity in the rare isotope is the formation of Cooper pairs between Helium atoms much like the BCS (Bardeen-Cooper-Schrieffer) theory of superconductivity[15], but with important differences: As there is no crystalline lattice, phonons are not operative in forming Cooper pairs (in contrast to simple metals explainable by BCS theory). The origin of Cooper pairing in the rare isotope is magnetic and Cooper pairs resides in higher angular momentum states with internal degrees-of-freedom (see Anderson-Morel[16]).

Superconductivity is another example that show macroscopic quantum effects. First discovered by Heike Kamerlingh Onnes in 1911, it took about fifty years of theoretical efforts and finally the microscopic understanding came with BCS theory of superconductivity in 1957[15]. Loosely speaking the basis of superconductivity is the Bose-Einstein condensation of Cooper pairs formed due to mutual attractive interaction mediated by the exchange of lattice phonons. The important difference between rare isotope of Helium and superconductivity in simple metals is that the size of a Cooper pair in metals is much greater than mean particle separation ($\sim 1000$ times the lattice constant) while in Helium it is like forming a bosonic molecule by two fermionic atoms (the size of the

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Cooper pair is mean particle separation. BCS theory quantitatively account for many observed properties of simple superconductors (excluding Cuprate superconductors\cite{13, 17}).

Magnetism of Iron: Iron loses its magnetic properties above 770°C (the Curie temperature). What happens is that due to thermal agitation the ordered arrangement of electronic spins becomes unstable. The origin of magnetism in Iron is due to a dual role played by d-band electrons\cite{18, 19}. They participate in metallic bonding as well as act like localized spins. Iron drives its magnetism from electron-electron repulsive Coloumb interaction due to which they avoid each other in real space and that is effectively executed by keeping spins in the same direction. This a rough picture (see for more accurate and quantitative analysis\cite{19}).

Cosmological Phase Transition: The origin of SU3, Isospin, and Parity! It has been postulated (after the discovery of Hubble expansion) that when our universe was very young (only sub-pico seconds of age) after the Big-Bang, our universe underwent a phase transition from that high energy scale—called the Cosmological phase transition. Below that high energy scale the electroweak symmetry is broken. This the phase transition of vacuum itself in which vacuum expectation value of a scalar field takes non-zero value\cite{20}. This is intimately connected to the Higgs mechanism\cite{21}.

The above are some examples of phase transitions. Obviously, there are many more examples. The above are sufficient for our purpose of illustrating the mean-field theories and renormalization group. Before we discuss the MFTs we first review the work of Ehrenfest on the classifications of phase transitions.

2 Phase transitions without Landau’s paradigm

2.1 Ehrenfest’s classification of phase transitions

Paul Ehrenfest (1880-1933) classified various phase transitions in 1933 according to the behavior of the derivatives of the Gibbs free energy (unfortunately, later that year he killed himself as he was suffering from severe depression). According to this scheme, the order of a phase transition is given by the order of the discontinuous derivative of Gibbs free energy. On the line separating the two phases (say, in the liquid-gas transition) the Gibbs free energies per unit mass $g(T, P)$ of the two phases take the same value. On the left of the line, $g$ is less for the liquid, and on the right of it it is less for the gas. By definition\cite{22}: $g(T, P) = U - Ts + Pv$ or $dg = -sdT + vdP$ with

\begin{align*}
\text{Entropy/mass} & = s = -\frac{\partial g}{\partial T}|_P, \\
\text{Volume/mass} & = v = \frac{\partial g}{\partial P}|_T
\end{align*}

First order phase transitions are those in which first order derivatives of the Gibbs’ Free energy are discontinuous: i.e., if the above derivatives are discon-

\footnote{He was a student of Ludwig Boltzmann.}
tinuous. If we cross the liquid-solid line vertically in up direction (at constant temperature, see figure (b)) clearly:

\[ \frac{\partial g_l}{\partial P} |_T > \frac{\partial g_s}{\partial P} |_T \]

As the system goes from liquid stability to solid stability (an experimental fact!). This implies that:

\[ v_l = \frac{\partial g_l}{\partial P} |_T > v_s = \frac{\partial g_s}{\partial P} |_T \]

In other words, for a given mass, volume of the liquid phase is more than that of the solid phase, or, substances expand on melting.

If we cross the liquid-gas line horizontally (ref. figure (b)) from left to right (at constant pressure) clearly:

\[ \frac{\partial g_s}{\partial T} |_P > \frac{\partial g_l}{\partial T} |_P \]

As system goes from solid stability to liquid stability (again an experimental fact!). Above implies that

\[ -s_s > -s_l \]

or

\[ s_s < s_l, \quad \Rightarrow \frac{g_l - g_s}{T} > 0. \]

Where \( q_l - q_s = l \) is called the latent heat of solid-to-liquid transition, which is positive! as it should be! Thus heat must be absorbed from solid to liquid transition. Examples of first order are: Solid-liquid-vapor, transitions in simple fluids, allotropic transformations (like diamond graphite) etc.

For the second order phase transitions, first order derivative of Gibbs free energy, i.e., \( v, s \), are continuous, but the second order derivatives, i.e., heat capacity \( c \), and compressibility \( \kappa \) are discontinuous. Thus no latent heat and volume change occur during a second order phase transition. But in a second order transition, heat capacity and compressibility show finite discontinuity. An ideal second order transition is the superconducting transition in zero magnetic field[22].

Phase transition in liquid Helium is not second order (as the discontinuity in heat capacity is infinite). This is a special transition called the \( \lambda \)-transition.

Above is called the Ehrenfest’s classification scheme. One can define higher order phase transitions along these lines (third and higher order derivatives of the Gibbs free energy), but these are seldom observed in practice. Ehrenfest’s scheme is based on the fundamental thermodynamical principle: the phase of minimum free energy is the stable one[22]. From this it derives other properties of phase transitions like latent heat, volume changes etc. Importantly, it does not provide more fundamental statistical mechanical formulation of the phase transition behavior. We will see in the next section that MFTs are the first steps in these directions.

We will also see that with the Landau’s paradigm above scheme of Ehrenfests’ can be nicely re-interpreted by using the order parameter concept.

\[ ^4 \text{Note that the case of water is opposite (ice is lighter) as the ice-water (solid-liquid) line has negative slop.} \]
2.2 The Mean-Field Theories (MFTs)

The above is the macroscopic way of understanding/classifying the phase transitions without any microscopic (atomic/molecular) point of view. Attempts to understand the laws of heat from the microscopic (atomic/molecular) perspective resulted in the science of statistical mechanics and Ludwig Boltzmann (1844-1906) was the pioneer \[23\]. Similar to this philosophy, to understand phase transition phenomena from atomic/molecular perspective, early attempts were made by van der Waals \[5\].

Thus the credit of first mean-field theory goes to van der Waals \[1\]. Van der Waals was interested in the notion of continuity of matter, namely, how a gaseous substance, on cooling, transforms into a liquid while chemically, at molecular level things remain intact \[6\]. His main concern was the ideal gas equation and the problems of the Boyle’s law. Here is a rough sketch of his argument.

We know that the ideal gas equation

$$PV = Nk_B T = nRT$$

shows no phase transition for obvious reasons (due to the neglect of intermolecular forces). In contrast, real gases at low temperature and high pressure deviate markedly from the ideal gas behavior. To see this, let $V_m^0$ be the molar volume (volume per mole) of an ideal gas:

$$pV_m^0 = RT.$$  

Define the compression factor $Z$ at a temperature $T$ for a real gas ($PV_m = RT$):

$$Z = \frac{V_m}{V_m^0} = \frac{pV_m}{RT}.$$  

In figure 7(a) compression factors of both the gases are plotted as a function of pressure. In an ideal gas, as there is no interaction between the molecules, the compression factor remains unity with pressure while for a real gas it first decreases and then at higher pressure it starts to rise again (figure 7(a)). This can be easily correlated with the inter-molecular potential plotted in figure 7(b). At high pressure gas is compressed and molecules are relatively near to each other and thus exert repulsive forces on each other as the potential is large positive (figure 7(b)). This makes the compression factor greater than that of an ideal gas. In other words, for the same number of moles of a real gas and an ideal gas in the high pressure limit (when the mean separation between the molecules is less than $r_c$), same pressure will be exerted only if the real gas occupies more volume thus making $Z > 1$. In the opposite limit of low pressure (mean intermolecular separation greater than $r_c$), molecules exert attractive forces and thus smaller molar volume and smaller compression factor for the real gas. To account for deviations of the ideal gas law and to understand the phase

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5 The thought process of understanding macro-phenomena from microscopic perspective is very old one, and is related to one of the very basic aspect of human psychology, namely the habit of understanding phenomena around from the basic “building blocks”. Examples go back to BC era, for example, Democritus was the first to postulate atoms and atomic constitution of matter. Other example is that of statistical mechanics (thermodynamics from atomic/molecular perspective) and so on. And this unique habit of mankind resulted in a body of organized knowledge that we call science. Author do not want to go into the debates of reductionism and emergence philosophies \[24\].

6 There is no chemical reaction when water converts from liquid to steam and vice versa.
Figure 7: (a) Behavior of the compression factor for an ideal gas as compared with a real gas. (b) Inter molecular force as a function of separation in a real gas.

Figure 8: Comparing isotherms obtained from van der Waals’ equation with those of experiment: curve 1 for $T > T_c$; curve 2 at $T = T_c$; curve 3 at $T < T_c$. 
transition of gas-to-liquid, van de Waals postulated a modified equation of state, now called the van der Waals equation of state\cite{1}. This is a remarkable piece of physical intuition and deduction of an equation of state from average properties of molecular interactions without going into formal mathematical development. This is a physical tour de force and he was awarded with Nobel prize (1910) for this.

He argued “....as you are aware the two factors which I specified as reasons why a non-dilute aggregate of moving particles fails to comply with Boyle’s law are firstly the attraction between the particles, secondly their proper volume...” —From Nobel prize lecture—JOHANNES D. VAN DER WAALS.

Thus replacing the actual volume and pressure in the ideal gas equation by an effective ones:

\[
p_{\text{eff}} = \frac{Nk_BT}{V_{\text{eff}}}
\]

\[
p = \frac{Nk_BT}{V - Nb} - a\left(\frac{N}{V}\right)^2
\]

results a mean-field theory. This takes “an average effect” of molecular interactions, and this type of theory is called the mean-field theory. As the real molecules repel at short distance, the effective volume is less \(V_{\text{eff}} = V - Nb\) (where \(N\) is the number of molecules and \(b\) is an effective volume of a single molecule). Similarly effective pressure \(p_{\text{eff}}\) is less due to long range attractive interaction between the molecules. If we compare the isotherms of a real gas with those obtained by van der Waals equation we get a qualitative agreement. Van der Waals isotherms look qualitatively the same as those in the experimental one (figure (8)), but this has two basic defects:

1. Unphysical van der Waals loops.
2. Different critical point behaviour.

The unphysical loops means that if one increases volume pressure increases (for the curve with positive slope in figure (8) in the coexistence regime). This is unphysical and can be rectified with Maxwell construction\cite{7}.

The critical pressure, temperature, and volume can be calculated from:

\[
\frac{dp}{dV} = \frac{d^2p}{dV^2} = 0.
\]

And near the critical point \(|T_c - T| \ll T_c\) and from the above derivatives, it can be easily shown that:

\[
\rho_{\text{liquid}} - \rho_{\text{gas}} \propto (T_c - T)^\beta.
\]

From van der Waals’ theory \(\beta\) comes out to be 1/2 (the mean-field value). But, in actual experiments on liquid-gas systems, \(\beta = 1/3\). This was first experimentally deduced by T. Andrews in 1869! This very clearly showed the contradiction of the mean-field theory with experiment.

\[\text{Figure 9: van der Waals (1837–1923)}.\]

\[\text{Figure 10: Thomas Andrews (1813–1885)}.\]
We will see that this discrepancy is not specific to van der Waals theory but a basic defect of all such theories called the MFTs at the critical point. It took about 100 years for the resolution of this discrepancy! This happened in the hands of K. G. Wilson in early 70s. The culprit is non-negligible fluctuations (density fluctuations in this liquid-gas case) at all length scales that invalidates the application of the mean-field theory itself[7].

The second example of MFT we consider is that of the famous Ising model—a caricature for spontaneous ferromagnetism (like in Iron). A simple two dimensional Ising model is depicted in figure 11(a) in which spins are represented by arrows and these interact with their nearest neighbors. Each spin can have two states, either up, or down. This short range interacting model shows the phenomenon of spontaneous magnetization. Consider that the model is immersed in an external magnetic field $H$. Write the Hamiltonian in dimensionless form:

$$\frac{H}{k_B T} = -K \sum_{<nn>} \sigma_i \sigma_j - h \sum_i \sigma_i$$

Here $K = \frac{J}{k_B T}$ and $h = \frac{\mu H}{k_B T}$. There are two competing tendencies:

1. Lining-up of spins due to mutual spin-spin interaction ($K$) and the external field ($h$).

2. Disruption due to thermal agitation.

At low temperatures first tendency wins over and at high temperature the second one wins over. To calculate mean magnetization (a measure of the lining-up of the spins), one computes the Free Energy in the standard way:

$$F = -k_B T \ln \left( \sum_{\text{all configurations: } c} e^{-H(c)/k_B T} \right).$$

Here comes the big question: Why do statistical mechanical systems show phase transitions (non-analytical behavior like spontaneous magnetization, see figure 11(b); the graph near $T = T_c$) when the system’s free energy is a smooth
function of temperature and of system’s parameters (an analytical function)?
Sum of smooth exponentials is a smooth function!
The origin of non-analyticity (like spontaneous magnetization) is in the thermodynamical limit (infinite system size!): Larger the system size sharper is the transition: As real phase transitions involve thermodynamical systems having very large number of particles of the order of Avogadro number, transitions appear infinitely sharp to human perception! This has been dubbed as “extended singularity theorem” by Leo Kadanoff or better one can say “extended singularity conjecture”[4]. As the proof exists only in one special case[26].

Mean-field theory for the Ising model is done in usual way[4, 27, 28]:

1. First, consider that just one spin is immersed in magnetic field (that can take two values: +1 (up-spin, say), and −1 (down-spin)). Thermal expectation value of the spin is:

\[
\langle \sigma \rangle = \sum_{\sigma = \pm 1} \frac{\sigma e^{h \sigma}}{\sum_{\sigma = \pm 1} e^{h \sigma}} = \tanh(h).
\]

2. Second, when our test spin is interacting with many neighboring spins, all immersed in magnetic field:

\[
\langle \sigma \rangle = \sum_{\sigma = \pm 1} \frac{\sigma e^{h_{\text{eff}} \sigma}}{\sum_{\sigma = \pm 1} e^{h_{\text{eff}} \sigma}} = \tanh(h + K \sum_{\langle \sigma \rangle} \sigma_i) \approx \tanh(h_{\text{eff}}).
\]

\[
h_{\text{eff}} = h + \text{effective field due to other spins} = h + zK \langle \sigma \rangle.
\]

Here the exact field \( K \sum_{\langle \sigma \rangle} \sigma_i \) due to near-neighbor spins is replaced by the effective field \( \sim zK \langle \sigma \rangle \) (\( z \) is called the coordination number, \( = 4 \) in 2D). This constitutes the mean-field approximation.

Thus in MFT: \( \langle \sigma \rangle = \tanh(h + zK \langle \sigma \rangle) \). To see what values \( \langle \sigma \rangle \) can take, one must solve this self-consistently. As \( K \propto \frac{1}{T} \) it turns out that \( zK = \frac{T}{T_c} \) (where \( T_c \) is the critical temperature). In zero external field \( h = 0 \) the equation \( \langle \sigma \rangle = \tanh(\frac{T}{T_c}) \langle \sigma \rangle \) has the following solutions (figure 12):

\[
\langle \sigma \rangle = \begin{cases} 
0 & \text{if } T > T_c \\
0, +1, -1 & \text{if } T < T_c \\
degenerate & \text{if } T = T_c
\end{cases}
\]

The region \( T \leq T_c \) is of particular importance. Near the critical point \( \langle \sigma \rangle \approx 0 \), \( \tanh(x) \) can be expanded for small \( x \) as \( \approx x - \frac{1}{3} x^3 + \ldots \). And it can be easily shown that near the critical point:

\[
\langle \sigma \rangle \propto (T_c - T)^{\beta}.
\]

With \( \beta = \frac{1}{2} \) (the mean field value). Compare this with van der Waals theory! (THE VERY SAME BETA VALUE!)

Experiments on a wide variety of magnetic systems show that \( \beta \approx \frac{1}{4} \). Again, we see the contradiction between mean-field theory and experiments. So, something is wrong with mean-field theory near the critical point! But, as mentioned
before, solution comes much later. However, the great merit of MFTs is that they qualitatively explain the emergence of phases below $T_c$ (refer to the middle graph in figure (12)). One can show that the free energy for $\langle \sigma \rangle = 0$ phase is greater than the other phases with non-zero $\langle \sigma \rangle$, and the phase with minimum free energy is the phase that is realized in a thermodynamical system. Thus one has non-zero $\langle \sigma \rangle$ for temperature less than the critical temperature (figure 11).

3 Landau’s paradigm: symmetry breaking and the order parameter concept

In a landmark paper of 1937 Lev Landau formulated the phase transition problems from the broken symmetry perspective. *All the above phase transition theories can be written in one mathematical language near the critical point.* He noticed that most of the phase transitions involves breaking of discrete or continuous symmetry and introduced a very important concept to capture the nature and extent of this symmetry breaking: the order parameter concept. One can appreciate this rather formal language related to symmetry breaking by considering some concrete examples. Before we do that, let us first consider two important theorems related to symmetry and its breaking. Emmy Noether in 1918 put forward an important theorem: The Noether theorem (1918):

“Every continuous symmetry of a system can be associated with a conserved quantity.” This theorem is the cornerstone of classical mechanics. There are numerous examples, consider for example; if, gravitational potential along x-direction is constant, then linear momentum along that direction is conserved and so on. Another important theorem, for the present context, is the Goldstone theorem (1961): “The spontaneous breaking of a continuous symmetry can be associated with a spinless and massless particle—the Nambu-Goldstone boson (or simply the Goldstone Boson)”. To illustrate this we consider few examples:

Example No. 1:

1. liquid-to-Crystal transition: In the liquid state atoms/molecules are under the effect of mutual potential (like van der Waals potential) which
2. In the crystalline state, a periodic potential emerges, which break the continuous symmetry under translations (however, this leads to a discrete translational symmetry due to the periodicity of the lattice: the crystal momentum) [15].

3. Since in the above example continuous translational symmetry is broken, momentum does not remain a conserved quantity. And this violation of the conservation of momentum in the crystalline state leads a Goldstone boson called the phonon (lattice vibrations).

Example no. 2: Ferromagnetic transition and broken rotational symmetry: In a ferromagnetic system above the transition temperature, spins are randomly oriented in all possible directions. But below the transition temperature, this rotational symmetry is broken and spins align in the same direction. The Goldstone boson in the ferromagnetic case is the spin-wave excitations (or, collective excitation of spin waves).

One can consider many more examples. Thus each phase transition is a manifestation of some kind of broken symmetry: In ferromagnetic rotational symmetry is broken; in crystal formation translational symmetry is broken; in superconductor or superfluid electromagnetic gauge symmetry is broken.... On this basis, Landau introduced the very important concept of an order parameter. For example, in ferromagnetic transition the order parameter is an average magnetization per unit volume (which is zero for $T > T_c$ and non-zero for $T < T_c$); in liquid-gas transition it is the density difference; in superconductor it is the expectation value of the Cooper field $<\psi^+_\sigma \psi^\_\sigma>$; and in electroweak symmetry breaking it is expectation value of Higgs field $<\phi>$. Order parameter is zero in symmetrical phase and is non-zero in the unsymmetrical phase. It is a macroscopic variable, much like a hydrodynamical variable. It has discontinuous jump for first order transitions as the order parameters are the first order derivatives of the free energy (for example, $M = -\frac{\partial G}{\partial H}$). Thus one can connect Landau’s scheme with that of Ehrenfests’ and one can interpret Ehrenfests’ scheme in terms of continuity or discontinuity of order parameters. For the second order transitions order parameter evolve from zero in symmetrical phase to non-zero in the unsymmetrical phase in continuous manner (thus 2nd order transitions are also called continuous transitions). Landau’s great contribution in unifying
the theoretical understanding of wide variety of phase transitions lies in his series expansion of free energy as a function of order parameter near the critical point where the order parameter is small.

\[ G(M, T) = G_0(T) + r(T)M^2 + u(T)M^4 + \ldots \]

Where \( M \) is some order parameter (like magnetization in ferromagnetic case). The fundamental assumption that goes into this is the analyticity of free energy as a function of order parameter (magnetization here) and temperature. This fundamental assumption gets its justification a-posteriori. Most importantly, this is in accord with "extended singularity conjecture" in which free energy per unit volume remains finite in the thermodynamical limit \( (N \to \infty, V \to \infty, \frac{N}{V} = \text{finite}) \). The sharpness of phase transitions (apparent non-analyticity) is something to do with very large number, of the order of Avogadro number, of particles present in the system.

Landau argued, according to the basic principle of thermodynamics, that at a given temperature, the phase with minimum free energy is the stable one: \( \frac{\partial G}{\partial M} = 0 \) leads to:

\[ r(T)M + 2u(T)M^3 = 0. \]

Which implies that either \( M = 0 \) or \( M = \pm \sqrt{-\frac{r(T)}{2u(T)}} \).

One can relate these to the following two possible cases:

1. Case A: If \( r(T) > 0 \): one has only one minimum at \( M = 0 \) (\( u(T) \) has to be positive as free energy is bounded from below). \( M = 0 \) in this case and it corresponds to symmetrical phase \( (T > T_c) \).

2. Case B: If \( r(T) < 0 \): one has two minima at \( M = \pm \sqrt{-\frac{r(T)}{2u(T)}} \) and this corresponds to the unsymmetrical phase \( (T < T_c) \) (see figure 15) with non-zero value of the order parameter.

From above, one can deduce the critical exponent \( \beta \) as follows:

1. As \( r(T) \) is positive for \( T > T_c \) and negative for \( T < T_c \), it must pass through zero at \( T = T_c \).

2. So near the critical temperature \( r(T) \) can be expanded as: \( r(t) = a(T - T_c) + O((T - T_c)^2) \).

3. Thus magnetization in \( T < T_c \) phase can be written as: \( M = \pm \sqrt{\frac{a(T - T_c)}{2u(T)}} \sim (T - T_c)^{\beta} \). Here \( \beta = \frac{1}{2} \).

Again \( \beta \) takes the same value 1/2, as in van der Waals theory and in the mean-field theory of the Ising model. The general disagreement of MFTs with experiment is clear. We will see in the next section that systems near criticality violates an important property of statistical independence and the correlation length diverges at the critical point. Before we embark upon the question of the behaviour of correlation length and renormalization group we first would like to consider the question of the critical point in the next section in the light of some exact solutions.
4 Problems of the Mean-Field Theories and Landau’s paradigm

In the previous section we have seen the contradiction of MFT with experiment near a critical point. The $\beta_{\text{theory}} = \frac{1}{2}$ and $\beta_{\text{experiment}} = \frac{1}{3}$. This was apparent even in the first decade of 20th century. Still, MFTs were developed (like Curi-Weiss, Ising model, Landau’s generalization) and trusted. Nobody attempted to resolve the problems of MFTs at the critical point\[4\]. Then in 1944, with the Onsager’s—a mathematical tour de force—exact solution of the 2-D Ising model, it became very clear that something is very much wrong with MFTs near criticality:

1. C. N. Yang with an extension of Onsager’s solution found that $\beta$ is $1/8$ different from mean-field result $\beta = 1/2$.

2. Onsager’s solution showed that heat capacity diverges as a logarithm of $T - T_c$ whereas MFT shows a finite jump in heat capacity at $T_c$.

3. The correlation length $\xi$ diverges as $\frac{1}{\sqrt{T - T_c}}$ in MFT whereas it diverges as $\frac{1}{T - T_c}$ in the Onsager’s solution. Thus, $\nu$, the critical index of the correlation length, is one half in MFT whereas it is equal to one in the Onsager’s solution!

Here comes the 2nd big question: why do MFTs fail to predict the correct values of critical indices? The answer lies in the fact that mean-field equation $\langle \sigma \rangle = \tanh(h + \frac{T}{\xi} \langle \sigma \rangle)$ fails to take into account the longer length scale fluctuations (as compared to inter-particle and lattice constant). See for details\[7\].

As fluctuations at all length scales (from lattice constant to the correlation length) become important near a critical point, a proper theory in this regime must address this issue. The theory is called the renormalization group (next section).

5 Going beyond Landau’s paradigm—The renormalization (semi)group (RG)

The renormalization theory in general deals with problems of long correlation length. The presence of long correlation length leads to the breakdown of an important property of statistical mechanical systems called Statistical Independence (SI). Thus the results of standard Gibbs-Boltzmann statistical mechanics become unreliable. Near the critical point correlation length diverges as can be easily seen:

Consider that the order parameter is space dependent $M(x)$ and vary very slowly as compared to atomic dimensions: One writes Ginzburg-Landau free
energy functional:\[ F = \int d^3 x \{ (\nabla M(x))^2 + r(T)M(x)^2 + u(T)M(x)^4 - B(x)M(x) \}. \]

Correlation length \( \xi \) is defined as the distance over which the effect of a test spin extends out to other nearby spins. To calculate \( \xi \) consider \( B(x) = B_0 \delta^3(x) \) (magnetic field in a narrow region of space). Neglect the smaller 4th order term and by minimizing \( F \) with a variational calculation, one readily obtains:

\[ -\nabla^2 M(x) + r(T)M(x) = B_0 \delta^3(x). \]

With Fourier transforms one gets an immediate solution:

\[ M(x) \sim e^{\sqrt{r(T)} |x| |x|}. \]

From which one defines a correlation length \( \xi = \frac{1}{r(T)} \propto \frac{1}{(T - T_c)^{1/2}} \). Thus correlation length diverges at the critical point with critical index \( \nu = 1/2 \).

Due to this diverging correlation length various parts of the system respond in a correlated way. This makes the system non-self-averaging! The fundamental hypothesis (actually an important property of ordinary statistical mechanical systems with short range interactions) of statistical independence as highlighted by Landau[6] cannot be applied and fluctuations in the additive observables (sum-functions) do not obey \( \frac{1}{\sqrt{N}} \) law[6]. In other words, fluctuations could be significant as compared to the average behavior. A comparison of physical problems with short and long correlation length is presented in table (1). The problems with large correlation length are among the hardest problems in theoretical physics. For example, thermodynamical systems near criticality; Kondo problem; high temperature superconductivity; relativistic quantum field theory; strong interactions and QCD etc. And most importantly, systems with

| Physical Problems                                      | Problems with short correlation length (\( \xi << L \))                                      | Problems with large correlation length (\( \xi \sim L \))                                      |
|-------------------------------------------------------|-------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------|
| Various parts of the system with size \( < \xi \) behaves statistically independently | As correlations extend throughout the system, various parts of it are statistically correlated! |                                                                              |
| Statistical independence (SI) leads to product distribution functions: \( \rho_{12} = \rho_1 \rho_2 \) | \( \rho_{12} \neq \rho_1 \rho_2 \)                                                       |                                                                                    |
| Entropy and Energy of sub-parts become additive (weakly interacting sub-parts of size \( \xi \)) | non-additivity of entropy and energy of sub-parts                                             |                                                                                      |
| Standard statistical mechanics can be applied         | Results of standard statistical mechanics are not reliable!                                |                                                                                      |
| Fluctuations in observable converge \( \sim \frac{1}{\sqrt{N}} \) | As SI breaks down, fluctuations in observable diverge!                                     |                                                                                      |
| Examples: systems away from criticality: MFTs; Hartree-Fock; Cluster and virial exp... | Thermodynamical systems near criticality; QCD: Kondo problem; cuprate superconductivity etc. |                                                                                      |
large correlation length show a certain kind of cooperative behavior in which the details of the microscopic interactions play secondary role! Before we explore the renormalization group as formulated by Wilson, we first review the essence of a key work due to Kadanoff\cite{29,4} on which Wilson highly relied.

### 5.1 Kadanoff’s great insight

1. Consider for concreteness the 2-D Ising model. Away from critical point ($T = T_c$) the correlation length is small ($\xi \propto \frac{1}{(T - T_c)^\nu}$), fluctuations on a smaller length scale are important and one can reliably use the mean field theory.

2. At $T = T_c$ the correlation diverges, as shown before. A reliable picture can only be obtained if one take all the spins into account in an area $\sim \xi^2$ (A HOPELESS TASK!).

3. Kadanoff approached the problem with a brilliant idea: map the problem of large correlation length to a problem with small correlation length!!!!

4. This is possible by exploiting the fact that on smaller lengths scales $l << \xi$ spins are well correlated!

To execute the point number 3 above, consider the original lattice (figure 16) with lattice constant $= a$ in which we mentally construct blocks, each containing four spins (dashed squares in the figure). We then assign an effective spin to each block—the block spin. The effective spin is an average of the original spins (one can use simple majority rule, if majority is up the effective spin or block is also up). The ease with which we can define block spins is granted by the point 4 above. If all the spins are up in a given block, then our effective spin is four times the original spin. The resulting lattice of this effective block spins will have lattice constant $a_1 = 2a$. The block spins or effective spins will interact with each other with some modified interaction that one has to calculate from the detailed model. The correlation length scales as follows: If the $\xi$ is 1000 $a$
λ = Kat the critical index ν length is infinite. This scale invariance (figure 17(a)) leads to picture T Repetition of this procedure ultimately leads to ξ (AT THE CRITICAL PONIT): relation (AT THE CRITICAL PONIT): the following fundamental fact: “ξ diverges at the critical point” This fundamental fact in an infinite system leads to a very curious behavior of system’s couplings. They tend towards a fixed point as explained below:

Let the Hamiltonian of un-scaled system be \( H = \sum_{n,i} K_0 s_n s_{n+i} \). Here \( K_0 \) is near neighbor interaction in the un-scaled lattice. After first scaling, let \( K_0 \rightarrow K_1^{(1)} \) and \( s_n \rightarrow s_n^{(1)} \) (i.e., block spins \( s_n \) interact with effective coupling \( K_1^{eff} \)). Thus after first scaling:

\[
\sum_{n,i} K_0 s_n s_{n+i} \rightarrow \sum_{n,i} K_1^{eff} s_n^{(1)} s_{n+i}^{(1)}.
\]

After \( m \) repetitions:

\[
H_{\text{effective}}^{(m)} = \sum_{n,i} K_m^{eff} s_n^{(m)} s_{n+i}^{(m)}.
\]

The correlation length behaves as follows:

After first scaling \( \xi^{(1)}(K_1^{eff}) = \frac{1}{2} \xi(K_0) \) (If the ξ is 1000 a (in the original lattice), then it will be 500 a1 (in the scaled lattice)). After \( m \)-scalings : \( \xi^{(m)}(K_m^{eff}) = \frac{1}{2^m} \xi(K_0) \). Now the fundamental assumption that goes in Kadanoff treatment is this: The scaling does not change the physics of the problem, namely, \( \xi^{(m)}(K_m^{eff}) \) diverges as \( K_m^{eff} \rightarrow K_c \) as the original correlation length \( \xi(K_0) \) does at \( K_c \) (\( K_c = \frac{1}{\xi(K_0)} \)). If one knows the functional form \( K_m^{eff} = f(K_m^{eff}) \) for one transformation, the same functional form will be true for all scalings!—precisely at the critical point \( K_0 = K_c \) (where correlation length is infinite). This scale invariance (figure 17(a)) leads to \( m \) independent relation (AT THE CRITICAL PONIT):

\[
K_c = f(K_c).
\]

The self consistent solution to this equation leads to the critical point solution (or fixed point solution) that gives \( T_c \) (figure 17(b)). Thus from the Kadanoff picture \( T_c \) can be calculated if one knows the function \( f(K) \). More importantly, the critical index \( \nu \) can be expressed in terms of the derivative of \( f(K) \) wrt \( K \) at \( K = K_c \): Suppose \( K \) is near to \( K_c \), then \( f(K) = f(K_c) + (K - K_c) \lambda \). Here \( \lambda = \frac{df}{dk} \). As \( K_c = f(K_c) \), implies \( f(K) - K_c = \lambda(K - K_c) \). Now suppose correlation length behaves as \( \xi(K) \sim (K - K_c)^{-\nu} \), then one must have

\[
\frac{\xi(f(K))}{\xi(K)} = \left( \frac{f(K) - K_c}{K - K_c} \right)^{-\nu}
\]

From the Taylor expansion above, it immediately follows that \( \lambda = (\lambda)^{-\nu} \) as the scaling factor is 2. Thus if \( \lambda \) is known then \( \nu \) can be determined (see for other complications[2]).

The problem of the Kadanoff’s procedure is that it does not give the central method to determine the important function \( f(K) \). He used the above argument to derive scaling relations[29]. Important point worth noting is that Kadanoff
5.2 Wilson’s formulation

As promised in the last subsection we would like to answer the following question: How did Wilson change Kadanoff’s qualitative scheme into a powerful computational method? For this we consider the $\phi^4$-theory.

The $\phi^4$ – Theory: Consider again the example of the Ising model (figure 11) in zero external magnetic field:

$$\frac{H}{k_B T} = -K \sum_{<nm>} \sigma_i \sigma_j.$$  

We stress that if one wants to understand the physics of the renormalization group one must work through the mathematical apparatus which is rather standard now; namely, the functional integrals and the method of contractions. Below the logic of the calculation and essential steps are given (see for details the original reference; Wilson-Kogut in [3]). The partition function $Z$ for the Ising model can be written as:

$$\prod_m \int_{-\infty}^{+\infty} ds_m 2\delta(s_m^2-1)e^{K \sum_n \sum_i s_n s_{n+i}} \to \prod_m \int_{-\infty}^{+\infty} ds_m e^{-\frac{1}{2}bs_m^2-uds_n^4} e^{K \sum_n \sum_i s_n s_{n+i}}$$
Smoothing delta functions

![Diagram of smoothing delta functions](image)

Figure 18: Smoothing the delta function peaks into narrow Gaussian peaks.

Where on the resulting expression on the right hand side we have smoothed the sharp delta function peaks into narrow Gaussian peaks (see figure 18). Writing the above in Fourier space with:

\[ s_m = \int_q \sigma_q e^{i q \cdot \mathbf{n}}, \quad \text{where} \quad \int_q = \frac{1}{(2\pi)^d} \int_{-\pi}^{+\pi} dq_1 \ldots \int_{-\pi}^{+\pi} dq_d, \]

transforms the partition function to:

\[ Z = \prod_m \int_{-\infty}^{+\infty} ds_m e^{-\frac{1}{2} \int_q (K \sum_j |e^{iq_j} - 1|^2 + b - 2Kd)\sigma_q \sigma_{-q} - u \int_{q_1} \int_{q_2} \int_{q_3} \sigma_{q_1} \sigma_{q_2} \sigma_{q_3} \sigma_{-q_1 - q_2 - q_3}}. \]

The program is: We will perform one quick step and then the sequence of steps! With this we will end up at two coupled algebraic equations. From there we can compute the critical indices.

One quick step:

Let us assume that the atomic scale fluctuations do not effect the value of the critical indices (an assumption which can only be verified a-posteriori). Thus we will consider wavelengths much larger than the lattice constant. A momentum cut-off \( |q| < q_c \) will be imposed in the integrals (where \( q_c << \frac{2\pi}{a}, \ a = 1, \) a is the lattice constant). Thus one can approximate \( |e^{iq} - 1|^2 \approx q^2 \). But limiting \( |q| < q_c \) harms:

\[ s_m = \int_q \sigma_q e^{i q \cdot \mathbf{n}} \]

One cannot get back the original \( s_m \) (as large momenta \( |q| > q_c \) are excluded in the limits of the integral!) So there is a technical problem. To resolve this one is forced to define the functional integral to count all possible states that the system can realize:

\[ Z = \left[ e^{-\frac{1}{2} \int_q (q^2 + r)\sigma_q \sigma_{-q} - u \int_{q_1} \int_{q_2} \int_{q_3} \sigma_{q_1} \sigma_{q_2} \sigma_{q_3} \sigma_{-q_1 - q_2 - q_3}} \right]^{r = \frac{b - 2Kd}{K}}. \]

Where

\[ \int_q = \prod_j \int_{-\infty}^{+\infty} (...) d\sigma_{q_j} \]

And now

\[ \int_q = \frac{1}{(2q_c)^d} \int_{-q_c}^{+q_c} dq_1 \ldots \int_{-q_c}^{+q_c} dq_d. \]
This is one typical example of defining an Effective Field Theory (EFT)! Our aim is to execute Kadanoff’s idea of “thinning out the degrees-of-freedom”. Here we are doing in momentum space (instead in real space as was done in the previous example of the last subsection). Higher momentum states will be integrated out \textit{iteratively}. The logic of the procedure is given below. The fundamental importance of Wilson’s iterative procedure (the renormalization group) is that we will get the pivotal function \( f(K) \) of the previous example. From that it is possible to get the critical point and all the critical indices etc.

The Procedure: Define the functional integral as: 
\[
Z = \left[ e^{H[\sigma]} \right] \text{with } H[\sigma] = -\frac{1}{2} \int_q \left( q^2 + r \right) \sigma q \sigma - q - u \int_{q_2} \int_q \sigma q_1 \sigma q_2 \sigma q_3 \sigma q_4 \sigma - q_1 - q_2 - q_3 - q_4.
\]

This functional integral can be written as the product of two functional integrals (as is clear from the definition of the functional integral above).

\[
Z = \left[ \begin{array}{c}
\sigma_0 \\
\sigma_1
\end{array} \right] \left( \begin{array}{c}
\int e^{H[\sigma_0 + \sigma_1]}
\end{array} \right)
\]

\text{higher momenta integrals}

Here
\[
\sigma_q = \begin{cases} 
\sigma_{0,q} \equiv \sigma_0 & \text{if } 0 < |q| < \frac{q_c}{2} \\
\sigma_{1,q} \equiv \sigma_1 & \text{if } \frac{q_c}{2} < |q| < q_c.
\end{cases}
\]

We want to integrate out \( \sigma_q \) (the higher momentum shell). Also rename various parts of the Hamiltonian as:

\[
H[\sigma] = \frac{1}{2} \int_q (q^2 + r) \sigma q \sigma - q + u \int_{q_2} \int_q \sigma q_1 \sigma q_2 \sigma q_3 \sigma q_4 \sigma - q_1 - q_2 - q_3.
\]

Let the integration of higher momenta states yields:

\[
e^{H_F[\sigma']} = e^{H_F[\sigma_0]} \left[ e^{H_F[\sigma_1] + H_I[\sigma_0 + \sigma_1]} \right]
\]

Here we define \( \sigma' \equiv \sigma_q' = \frac{1}{2} \sigma_{0,q} \) (i.e., \( \sigma' \) and \( q' \) are rescaled variables with \( 0 < |q'| < q_c \) and \( 0 < |q| < \frac{q_c}{2} \)). \( H_F[\sigma_0] \) can be written in terms of scaled variables trivially as:

\[
-\frac{1}{2} \int_{0 < |q'| < \frac{q_c}{2}} (q'^2 + r) \sigma q' \sigma - q' \quad \Rightarrow \quad -\frac{1}{2} \left( \zeta^{2d-2} \right) \int_{0 < |q'| < q_c} (q'^2 + 4r) \sigma' q' \sigma' - q'.
\]

This trivial scaling cannot be done on the interaction term (due to coupling of momenta). One must use perturbation theory and express:

\[
e^{H_I[\sigma]} \simeq 1 - x + \frac{1}{2} (x)(x) - ...
\]

Where \( x \equiv -H_I[\sigma] \). In this series expansion inside the functional integral one will encounter terms like:

\[
I(q_1, q_2, ..., q_k) = \left[ \begin{array}{c}
\sigma_1, q_1, \sigma_1, q_2, ..., \sigma_1, q_k \end{array} \right] e^{H_F[\sigma_1]} \times \{ \sigma_1, q_1, \sigma_1, q_2, \sigma_1, q_3, \sigma_1, q_4, ..., \text{all combinations} \}.
\]

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Figure 19: (a) Iteration formula for $u_n$. (b) Calculation of $\nu$: if one starts with some initial $u_0, r_0$ at $T = T_c$, after sufficient number of iterations $r \to r^*$, $u \to u^*$ (the fixed point). If one starts at $T \neq T_c$ iterations take the initial conditions to some asymptotic line $D$.

For the proof of it see the original article: Wilson-Kogut in[3]. After doing the functional integral over $\sigma_1 s$ (say up to 2nd order) one must rescale the $\sigma_0$ as before (by expanding the momenta $0 < |q| < \frac{q_c}{2}$ to $0 < |q| < q_c$ etc). Effective Hamiltonian (after integrating up to second order (see for details Wilson-Kogut in[3])) takes the form:

$$H[\sigma'] = -\frac{1}{2} \int q (q'^2 + r') \sigma_{q1}' \sigma_{q2}' - u'\int q_1' \int q_2' \int q_3' \sigma_{q1}' \sigma_{q2}' \sigma_{q3}' - q_1' - q_2' - q_3' \cdot$$

With

$$r' = 4(r + 3c \frac{u}{1 + r})$$

$$u' = 2^{4-d} (u - 9c \frac{u^2}{(1+r)^2}).$$

These are the coupled algebraic equations. One can make analogy to the equation $K_{n+1}^{eff} = f(K_{n+1}^{eff})$ that we derived by using Kadanoff’s real space blocking method. Here calculation is done in momentum space. By invoking the same logic that at the critical point the function $f$ becomes scale independent due to the divergence of correlation length. This divergence of correlation length leads to fixed point solution (as we learn in Kadanoff’s method in last subsection). Thus self-consistent solution of the above contains the physics of the critical point. One can obtain the self consistent solution by iterative procedure i.e., by evolving the above equations as a recursive procedure. This iterative procedure can be depicted by the following diagram (figure 19(a)). Let us express the equations after nth iteration as

$$r_{n+1} = 4(r_n + 3c \frac{u_n}{1 + r_n})$$

$$u_{n+1} = 2^{4-d} (u_n - 9c \frac{u_n^2}{(1+r_n)^2}).$$
After sufficient number of iterations one will end up at the critical point \( u^*, r^* \).
For illustration consider the case of \( u_n \) (figure 19(a)). The two curves (one for \( u \) and other for \( f(u) \)) meet at point \( u^* \). If one starts with some initial value \( u_0 \) (\( u_0 > u^* \) or \( u_0 < u^* \), does not matter) one will end up, after sufficient number of iteration, at \( u^* \).

The great advantage of the Wilson’s formulation is that we are able to know the required function explicitly. The other important point is that the critical fixed point \( r^*, u^* \) does not depend on the initial parameters in the Hamiltonian i.e., \( r_0, u_0 \). Thus the critical point behavior does not depend on the microscopic details of the system. This is called the universality. For example, \( \beta = 1/3 \) both for the magnetic systems and for the liquid gas transitions! (see for details \([3, 4, 5, 11, 27]\)).

This how Wilson changed Kadanoff’s qualitative ideas into a powerful computational procedure, as we know today, RG has been applied to wide variety of problems. Here by considering one example we have covered the basics of the method.

One can also compute the critical indices. Let us take one example of the calculation of \( \nu \). For this we need to linearize the fixed point equations. For \( T \) very near to \( T_c \) (say curves 2, or 3 in figure 19(b)):

\[
\begin{align*}
    r_l(T) &= r_l(T_c) + (T - T_c) \frac{dr_l}{dT} \bigg|_{T_c} + ... \\
    u_l(T) &= u_l(T_c) + (T - T_c) \frac{du_l}{dT} \bigg|_{T_c} + ...
\end{align*}
\]

As \( r = \frac{-2K}{T} \) is a function of temperature through \( K = \frac{J}{\varepsilon^2 T} \), and \( u = \frac{U}{\varepsilon T} \) is also a function of temperature. Thus these can be Taylor expanded for \( T \) near \( T_c \). For very large \( l \):

\[
\begin{align*}
    r_l(T) &= r^* + (T - T_c) \frac{dr_l}{dT} \bigg|_{T_c} + ... \\
    u_l(T) &= u^* + (T - T_c) \frac{du_l}{dT} \bigg|_{T_c} + ...
\end{align*}
\]

Or

\[
\begin{align*}
    r_{l+1}(T) - r^* &= (T - T_c) \frac{dr_{l+1}}{dT} \bigg|_{T_c} + ... \\
    u_{l+1}(T) - u^* &= (T - T_c) \frac{du_{l+1}}{dT} \bigg|_{T_c} + ...
\end{align*}
\]

For \( T \) sufficiently close to \( T_c \). As \( r_{l+1} = f(r_l, u_l) \) and \( u_{l+1} = g(r_l, u_l) \) (our fixed point equations), the temperature derivatives can again be Taylor expanded around the fixed point. This leads to:

\[
\begin{align*}
    r_{l+1} - r^* &= c_1(r_l - r^*) + c_2(u_l - u^*) \\
    u_{l+1} - u^* &= c_3(r_l - r^*) + c_4(u_l - u^*)
\end{align*}
\]

As the functions \( f(r_l, u_l) \) and \( g(r_l, u_l) \) are known one can work out the expansion coefficients explicitly (see for details Wilson-Kogut in \([3]\)):

\[
\begin{pmatrix}
    r_{l+1} - r^* \\
    u_{l+1} - u^*
\end{pmatrix} = M_{2 \times 2} \begin{pmatrix}
    r_l - r^* \\
    u_l - u^* 
\end{pmatrix}
\]
By iterating $n$-times:
\[
\begin{pmatrix}
    r_{l+n} - r^* \\
    u_{l+n} - u^*
\end{pmatrix}
= (M_{2\times 2})^n
\begin{pmatrix}
    r_l - r^* \\
    u_l - u^*
\end{pmatrix}
\]

The purpose of iterating $n$-times is that the matrix $M^n$ is completely dominated by the largest eigenvalue $\lambda^n$ ($\lambda = 4$ for $u = 0$ case). One can find eigensystem of the above matrix and finally (as the largest eigenvalue dominates!):
\[
  r_{l+n} - r^* \simeq \lambda^n w_{11} c_l (T - T_c).
\]

here $\lambda_1$ is the largest eigen value, $w_{11}$ is the corresponding eigenvector, and $c_l$ is a constant (see for details Wilson-Kogut in[3]). And a similar expression holds for the $u_l$. One observe that:
\[
r_{l+n+1}(T) - r^* \text{ at } (T - T_c = \frac{\tau}{\lambda_1}) = r_{l+n}(T) - r^* \text{ at } T - T_c = \tau
\]

Thus the correlation length:
\[
  \xi(r_{l+n+1}, u_{l+n+1})|_{T=T_c+\tau/\lambda_1} = \xi(r_{l+n}, u_{l+n})|_{T=T_c+\tau}.
\]

Now
\[
  \frac{1}{2^{l+n+1}} \xi(T_c + \tau/\lambda_1) = \frac{1}{2^{l+n}} \xi(T_c + \tau), \quad \text{as} \quad \xi(r_{l+n}, u_{l+n}) = \frac{1}{2^{l+n}} \xi(r_0, u_0).
\]

If correlation length scales as:
\[
  \xi(T_c + \tau) \propto \tau^{-\nu}
\]

Then from above:
\[
  \left(\frac{\tau}{\lambda_1}\right)^{-\nu} = 2^{-\nu} \implies \nu = \frac{\ln 2}{\ln \lambda_1}.
\]

If $u = 0$ (simple Gaussian model), then $\lambda_1 = 4$ which leads to $\nu = 0.5$ (the MFT result). With explicit calculation of $\lambda_1$ for $u \neq 0$ (see for details Wilson-Kogut in[3]), $\nu$ comes out to be:
\[
  \nu \simeq 0.58
\]

This is very close to the experimentally observed value $\nu = 0.6$ (in MFT it is 0.5!!!!). Similarly other critical indices can be calculated and are quite close to what is observed experimentally[11, 27].

### 6 Why do MFTs fail and how does RGT rectify that?

Thus, by exploiting scale invariance at the critical point and by finding the critical point from fixed point equations by iterative procedure one is able to calculated correct values of critical indices. The important point that goes implicit in the above statement is that iterative procedure is not only a mathematical device to reach to the fixed point but also at each step of iteration one is averaging over smaller length scale (or higher momentum scale) fluctuations and with this appropriate averaging over increasingly longer length scales one is able calculate
the correct values of critical indices. With this sophistication of renormalization group method we immediately see the drawback of MFTs. In MFTs we do an abrupt step in dealing with the fluctuations: \( \langle \sigma \rangle = \tanh(h + \frac{T}{k_B} \langle \sigma \rangle) \) i.e., we average out only the smaller length scale (of the order of lattice constant) fluctuations (in calculating the effective field seen by the test spin). Near the critical point just below the critical temperature, ensemble averages over smaller length scales (of the order of lattice constant) lead to zero value of the order parameter (here mean magnetization). Only when one averages over a large length scale (of the order of correlation length) can one see some tiny magnetization just below criticality. Thus our assumption of tiny mean magnetization over a smaller length scale (of the order of lattice constant) in MFT breaks down! And MFT predicts wrong values of critical indices (see also [7]).

7 Appendix: K. G. Wilson—a short biographical sketch

Ken Wilson earned his Ph.D. from the California Institute of Technology (CalTech) in 1961, studying under Murray Gell-Mann. Elementary Particle Physics (now called high energy physics) was a fashionable subject in those days (today also it is one of the mainstream topics along with condensed matter physics) and students would like to opt for it. But Wilson rebelled, and worked for a while on plasma physics. Something that did not go very well [3]. During that time he used to discuss his problems with S. Chandrasekhar (cousin of sir C. V. Raman). Here is an interesting story [12]:

His father had told him: ’When you go to Caltech, be sure to meet the two great physicists there namely, Feynman and Gell-Mann’. K. Wilson says that when he called on Feynman, the latter seemed to be gazing at the ceiling. He asked Feynman, ’Professor, what are you working on at present?’ Feynman replied, ’Nothing! Wilson then went to Gell-Mann and became his student!!!

He worked on the solutions of Gell-Mann and Low equation \( \lambda^2 \frac{d(e^2)}{d\lambda^2} = \phi(e^2, m^2/\lambda^2) \) and on fixed-source meson theory for his PhD Thesis. Around 1963 he read Onsager’s paper and became interested in critical phenomena,
and after that studying other works on critical phenomena he developed the
renormalization group theory, using tricks and techniques that he had learnt in
particle physics.

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