Aperiodic Weighted Automata and Weighted First-Order Logic

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Abstract
By fundamental results of Schützenberger, McNaughton and Papert from the 1970s, the classes of first-order definable and aperiodic languages coincide. Here, we extend this equivalence to a quantitative setting. For this, weighted automata form a general and widely studied model. We define a suitable notion of a weighted first-order logic. Then we show that this weighted first-order logic and aperiodic polynomially ambiguous weighted automata have the same expressive power. Moreover, we obtain such equivalence results for suitable weighted sublogics and finitely ambiguous or unambiguous aperiodic weighted automata. Our results hold for general weight structures, including all semirings, average computations of costs, bounded lattices, and others.

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1 Introduction

Fundamental results of Schützenberger, McNaughton and Papert established that aperiodic, star-free and first-order definable languages, respectively, coincide [35, 28]. In this paper, we develop such an equivalence in a quantitative setting, i.e., for suitable notions of aperiodic weighted automata and weighted first-order logic.

Already Schützenberger [34] investigated weighted automata and characterized their behaviors as rational formal power series. Weighted automata can be viewed as classical finite automata in which the transitions are equipped with weights. These weights could model, e.g., the cost, reward or probability of executing a transition. The wide flexibility of this automaton model soon led to a wealth of extensions and applications, cf. [33, 25, 2, 32, 12] for monographs and surveys. Whereas traditionally weights are taken from a semiring, recently, motivated by practical examples, also average and discounted computations of weights were considered, cf. [6, 5].

In the boolean setting, the seminal Büchi-Elgot-Trakhtenbrot theorem [4, 18, 36] established the expressive equivalence of finite automata and monadic second-order logic (MSO). A weighted monadic second-order logic with the same expressive power as weighted automata was developed in [10, 11]. This led to various extensions to weighted automata and weighted logics on trees [16], infinite words [15], timed words [20], pictures [19], graphs [8], nested words [9], and data words [11], but also for more complicated weight structures including average and discounted calculations [13] or multi-weights [14]. Recently, in [20], a core weighted MSO logic was introduced and shown to be expressively equivalent to weighted
automata, while permitting a uniform approach to semirings and these more complicated weight structures.

Here, we consider the first-order fragment of this core weighted logic. We will show that its expressive power leads to aperiodic weighted automata which, moreover, are polynomially ambiguous. Our core-weighted first-order logic has two weighted layers. First, we have step formulas which consist of constants and if-then-else applications, where the conditions are formulated in boolean first-order logic. The second step builds on this by performing products of step formulas and then applying if-then-else, finite sums, or existential sums. We will denote by \( \text{core-wFO} \) the full class of our core-weighted first-order sentences. For a set \( X \) of connectives, we let \( \text{core-wFO}(X) \) be the sublogic containing all sentences of \( \text{core-wFO} \) which use only the connectives in \( X \). Natural subsets of connectives will correspond to unambiguous or finitely ambiguous aperiodic weighted automata. These various levels of ambiguity are well-known from classical automata theory [21, 37, 23, 22].

Following the approach of [20], we take an arbitrary set \( R \) of weights. A path in a weighted automaton over \( R \) then has the sequence of weights of its transitions as its value. The abstract semantics of the weighted automaton is defined as the function mapping each non-empty word to the multiset of weight sequences of the successful paths executing the given word. Correspondingly, we will define the abstract semantics of \( \text{core-wFO} \) sentences also as functions mapping non-empty words to multisets of sequences of weights. Our main result will be the following.

**Theorem 1.** Let \( \Sigma \) be an alphabet and \( R \) a set of weights. Then the following classes of weighted automata and weighted first-order logics are expressively equivalent:

1. Aperiodic polynomially ambiguous weighted automata and \( \text{core-wFO} \) sentences,
2. Aperiodic finitely ambiguous weighted automata and \( \text{core-wFO} \text{ifthenelse, } \prod_x, + \text{ sentences,} \)
3. Aperiodic unambiguous weighted automata and \( \text{core-wFO} \text{ifthenelse, } \prod_x \text{ sentences.} \)

The above result applies not only to the abstract semantics. As immediate consequence, we obtain corresponding expressive equivalence results for classical weighted automata over arbitrary (even non-commutative) semirings, or with average or discounted calculations of weights, or bounded lattices as in multi-valued logics. In our proofs, for the implication from weighted automata to core-weighted logic, we analyze the fine structure of possible paths of polynomially ambiguous automata, as employed (for different goals) already in [21, 37]. To establish part 2 of Theorem 1 we also prove that for each aperiodic finitely ambiguous weighted automaton we can construct finitely many aperiodic unambiguous weighted automata whose disjoint union has the same semantics. For the implication from weighted formulas to weighted automata, in particular the product operator involves a new automaton construction ensuring the preservation of aperiodicity properties.

All our constructions are effective. In fact, given a \( \text{core-wFO} \) sentence and deterministic aperiodic automata for its boolean subformulas, we can construct an equivalent aperiodic weighted automaton of exponential size.

We give typical examples for our constructions. We also show that the class of arbitrary aperiodic weighted automata and its subclasses of polynomially resp. finitely ambiguous or unambiguous weighted automata form a proper hierarchy for each of the following semirings: natural numbers \( \mathbb{N}^{+, \times} \), the min-plus-semiring \( \mathbb{N}_{\min, +} \) and the max-plus semiring \( \mathbb{N}_{\max, +} \).

**Related work.** In [24], polynomially ambiguous, finitely ambiguous and unambiguous weighted automata (without assuming aperiodicity) over commutative semirings were shown to be expressively equivalent to suitable fragments of weighted monadic second order logic. This was further extended in [29] to cover polynomial degrees and weighted tree automata.
A hierarchy of these classes of weighted automata (again without assuming aperiodicity) over the max-plus semiring was described in \cite{24}. As a consequence of pumping lemmas for weighted automata, a similar hierarchy was obtained in \cite{27} for the min-plus semiring.

We note that in \cite{11} \cite{28} \cite{7}, an equivalence result for full weighted first-order logic was given, but only for very particular classes of semirings or strong bimonoids as weight structures.

A characterization of the full weighted first-order logic with transitive closure by weighted pebble automata was obtained in \cite{3}. An equivalence result for fragments of weighted first-order logic, weighted LTL and weighted counter-free automata over the max-plus semiring with discounting was given in \cite{25}. Various further equivalences to boolean first-order definability of languages were described in the survey \cite{7}. Due to its possible applications for quantitative verification questions, it remains a challenging problem to develop a weighted linear temporal logic for general classes of semirings with sufficiently large expressive power.

\section{Preliminaries}

A non-deterministic automaton is a tuple $\mathcal{A} = (Q, \Sigma, \Delta)$ where $Q$ is a finite set of states, $\Sigma$ is a finite alphabet, and $\Delta \subseteq Q \times \Sigma \times Q$ is the set of transitions. The automaton $\mathcal{A}$ is complete if $\Delta(q, a) \neq \emptyset$ for all $q \in Q$ and $a \in \Sigma$. A run $\rho$ of $\mathcal{A}$ is a nonempty sequence of transitions $\delta_i = (p_i, a_i, q_i)$, $\delta_2 = (p_2, a_2, q_2), \ldots, \delta_n = (p_n, a_n, q_n)$ such that $q_i = p_{i+1}$ for all $1 \leq i < n$. We say that $\rho$ is a run from state $p_1$ to state $q_n$ and that $\rho$ reads, or has label, the word $a_1a_2\ldots a_n \in \Sigma^+$. We denote by $\mathcal{L}(\mathcal{A}_{p,q}) \subseteq \Sigma^*$ the set of labels of runs of $\mathcal{A}$ from $p$ to $q$. When $p = q$, we include the empty word $\varepsilon$ in $\mathcal{L}(\mathcal{A}_{p,q})$ and say that $\varepsilon$ labels the empty run from $p$ to $p$.

An automaton with accepting conditions is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ where $(Q, \Sigma, \Delta)$ is a non-deterministic automaton, $I, F \subseteq Q$ are the sets of initial and final states respectively. The language defined by the automaton is $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_{I,F}) = \bigcup_{p \in I, q \in F} \mathcal{L}(\mathcal{A}_{p,q})$. We also consider below automata with several accepting sets $F, G, \ldots$ so that the same automaton may define several languages $\mathcal{L}(\mathcal{A}_{I,F}), \mathcal{L}(\mathcal{A}_{I,G}), \ldots$. An automaton $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ is deterministic if $I = \{i\}$ is a singleton and the set $\Delta$ of transitions is a partial function: for all $(p, a) \in Q \times \Sigma$ there is at most one state $q \in Q$ such that $(p, a, q) \in \Delta$.

Next, we consider degrees of ambiguity of automata. A run in an automaton is successful, if it leads from an initial to a final state. The automaton $\mathcal{A}$ is called \textit{polynomially ambiguous} if there is a polynomial $p$ such that for each $w \in \Sigma^+$ the number of successful paths in $\mathcal{A}$ for $w$ is at most $p(|w|)$. Then, $\mathcal{A}$ is \textit{finitely ambiguous} if $p$ is a constant. Further, for an integer $k \geq 1$, $\mathcal{A}$ is $k$-ambiguous if $p = k$, and unambiguous means 1-ambiguous. Notice that $k$-ambiguous implies $(k + 1)$-ambiguous. An automaton $\mathcal{A}$ is at most exponentially ambiguous.

A non-deterministic automaton $\mathcal{A} = (Q, \Sigma, \Delta)$ is \textit{aperiodic} if there exists an integer $m \geq 1$ such that for all states $p, q \in Q$ and all words $u \in \Sigma^+$, we have $u^m \in \mathcal{L}(\mathcal{A}_{p,q})$ iff $u^{m+1} \in \mathcal{L}(\mathcal{A}_{p,q})$. We say that $m$ is an aperiodicity index of $\mathcal{A}$. In other words, the non-deterministic automaton $\mathcal{A}$ is aperiodic if its transition monoid $\text{Tr}(\mathcal{A})$ is aperiodic.

It is well-known that aperiodic languages coincide with first-order definable languages, cf. \cite{25} \cite{28} \cite{7}.

The syntax of first-order logic is given in Table \ref{FO}. The semantics is defined by structural induction on the formula and requires an interpretation of the free variables. Let $\mathcal{V} = \{y_1, \ldots, y_n\}$ be a finite set of first-order variables. Given a nonempty word $u \in \Sigma^+$, we let $\text{pos}(u) = \{1, \ldots, |u|\}$ be the set of positions of $u$. A valuation or interpretation is a map $\sigma : \mathcal{V} \rightarrow \text{pos}(u)$ assigning positions of $u$ to variables in $\mathcal{V}$. For a first-order formula $\varphi$
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having free variables contained in \( \mathcal{V} \), we write \( u, \sigma \models \varphi \) when the word \( u \) satisfies \( \varphi \) under the interpretation defined by \( \sigma \). When \( \varphi \) is a sentence, the valuation \( \sigma \) is not needed and we simply write \( u \models \varphi \).

We extend the classical semantics by defining when the empty word \( \varepsilon \) satisfies a sentence. We have \( \varepsilon \models \top \) and if \( \forall \psi \) is a sentence then \( \varepsilon \models \forall \psi \). The semantics \( \varepsilon \models \varphi \) is extended to all sentences \( \varphi \) since they are boolean combinations of the basic cases above. Notice that if \( \varphi \) has free variables then \( \varepsilon \models \varphi \) is not defined. When \( \varphi \) is a sentence we denote by \( \mathcal{L}(\varphi) \subseteq \Sigma^* \) the set of words satisfying \( \varphi \). Notice that \( \mathcal{L}(\forall \psi \bot) = \{ \varepsilon \} \) where \( \bot = \neg \top \).

**Theorem 2** ([28][7]). Let \( A = (Q, \Sigma, \Delta) \) be an aperiodic non-deterministic automaton. For all states \( p,q \) of \( A \) we can construct a first-order sentence \( \varphi_{p,q} \) such that \( \mathcal{L}(A_{p,q}) = \mathcal{L}(\varphi_{p,q}) \).

For the converse of Theorem 2 we need a stronger statement to deal with formulas having free variables. As usual, we encode a pair \( (u, \sigma) \) where \( u \in \Sigma^+ \) is a nonempty word and \( \sigma : \nu \rightarrow \text{pos}(u) \) is a valuation by a word \( \pi \) over the extended alphabet \( \Sigma_\nu = \Sigma \times \{0,1\}^\nu \). A word \( \pi \) over \( \Sigma_\nu \) is a valid encoding if for each variable \( y \in \nu \), its projection on the \( y \)-component belongs to \( 0^*1^* \). Throughout the paper, we identify a valid word \( \pi \) with its encoded pair \( (u, \sigma) \).

**Theorem 3** ([28][7]). For each FO-formula \( \varphi \) having free variables contained in \( \mathcal{V} \), we can build a deterministic, complete and aperiodic automaton \( A_{\varphi,V} = (Q, \Sigma_\nu, \Delta, i, F, G) \) over the extended alphabet \( \Sigma_\nu \) such that for all words \( \pi \in \Sigma_\nu^+ \) we have:

\[
\begin{align*}
\Delta(i, \pi) & \in F \text{ iff } \pi \text{ is a valid encoding of a pair } (u, \sigma) \text{ with } (u, \sigma) \models \varphi, \\
\Delta(i, \pi) & \in G \text{ iff } \pi \text{ is a valid encoding of a pair } (u, \sigma) \text{ with } (u, \sigma) \models \neg \varphi, \\
\Delta(i, \pi) & \notin F \cup G \text{ otherwise, i.e., iff } \pi \text{ is not a valid encoding of a pair } (u, \sigma).
\end{align*}
\]

Given \( u \in \Sigma^+ \) and integers \( k, \ell \), we denote \( u[k, \ell] \) the factor of \( u \) between positions \( k \) and \( \ell \). By convention \( u[k, \ell] = \varepsilon \) is the empty word when \( \ell < k \) or \( \ell = 0 \) or \( k > |u| \).

We will apply the equivalence of Theorem 2 to prefixes, infixes or suffixes of words. Towards this, we use the classical relativization of sentences. Let \( \varphi \) be a first-order sentence and let \( x, y \in \mathcal{V} \) be first-order variables. We define below the relativizations \( \varphi^{<x}, \varphi^{(x,y)} \) and \( \varphi^{>y} \) so that for all words \( u \in \Sigma^+ \), and all positions \( i, j \in \text{pos}(u) = \{1, \ldots, |u|\} \) we have

\[
\begin{align*}
u, x \mapsto i & \models \varphi^{<x} \implies u[1, i-1] \models \varphi \\
u, x \mapsto i, y \mapsto j & \models \varphi^{(x,y)} \implies u[i+1, j-1] \models \varphi \\
u, x \mapsto j & \models \varphi^{>y} \implies u[j+1, |u|] \models \varphi
\end{align*}
\]

Notice that, when \( i = 1 \) or \( j \leq i + 1 \) or \( j = |u| \), the relativization is on the empty word, this is why we had to define when \( \varepsilon \models \psi \) for sentences \( \psi \). The relativization is defined by structural induction on the formulas as follows:

\[
\begin{align*}
\top^{<x} & = \top \\
(P_a(z))^{<x} & = P_a(z) \\
(\neg \psi)^{<x} & = \neg(\psi^{<x}) \\
(\psi_1 \land \psi_2)^{<x} & = \psi_1^{<x} \land \psi_2^{<x} \\
(\forall z \psi)^{<x} & = \forall z(z < x \implies \psi^{<x})
\end{align*}
\]

The relativizations \( \varphi^{(x,y)} \) and \( \varphi^{>y} \) are defined similarly. Notice that when \( \varphi \) is a sentence, i.e., a boolean combination of formulas of the form \( \top \) or \( \forall z \psi \), then the above equivalences hold even when \( i = 1 \) for \( \varphi^{<x} \), or when \( i = |u| \) for \( \varphi^{>x} \), or when \( j \leq i + 1 \) for \( \varphi^{(x,y)} \).
3 Weighted Automata

Given a set $X$, we let $\mathbb{N}(X)$ be the collection of all finite multisets over $X$, i.e., all functions $f : X \to \mathbb{N}$ such that $f(x) \neq 0$ only for finitely many $x \in X$. The multiset union $f \uplus g$ of two multisets $f, g \in \mathbb{N}(X)$ is defined by pointwise addition of functions: $(f \uplus g)(x) = f(x) + g(x)$ for each $x \in X$.

For a set $R$ of weights, an $R$-weighted automaton over $\Sigma$ is a tuple $A = (Q, \Sigma, \Delta, \text{wt})$ where $(Q, \Sigma, \Delta)$ is a non-deterministic automaton and $\text{wt} : \Delta \to R$ assigns a weight to every transition. The weight sequence of a run $\rho = \delta_1\delta_2 \cdots \delta_n$ is $\text{wt}(\rho) = \text{wt}(\delta_1)\text{wt}(\delta_2) \cdots \text{wt}(\delta_n) \in R^+$. The abstract semantics of $A$ from state $p$ to state $q$ is the map $\{A\}_{p,q} : \Sigma^+ \to \mathbb{N}(R^+)$ which assigns to every word $u \in \Sigma^+$ the multiset of weight sequences of runs from $p$ to $q$ with label $u$:

$$\{A\}_{p,q}(u) = \{\text{wt}(\rho) \mid \rho \text{ is a run from } p \text{ to } q \text{ with label } u\}.$$  

Notice that $\{A\}_{p,q}(u) = \emptyset$ is the empty multiset when there are no runs of $A$ from $p$ to $q$ with label $u$, i.e., when $L(A)_{p,q} = \emptyset$. When we consider a weighted automaton $A = (Q, \Sigma, \Delta, \text{wt}, I, F)$ with initial and final sets of states, for all $u \in \Sigma^+$ the semantics $\{A\}$ is defined as the multiset union: $\{A\}(u) = \bigcup_{p \in I,q \in F} \{A\}_{p,q}(u)$. Hence, $\{A\}$ assigns to every word $u \in \Sigma^+$ the multiset of all weight sequences of accepting runs of $A$ reading $u$. The support of $A$ is the set of words $u \in \Sigma^+$ such that $\{A\}(u) \neq \emptyset$, i.e., $\text{supp}(A) = L(A)$.

For instance, consider the weighted automaton $A$ of Figure 1. We have $\text{supp}(A) = a^+a(a + b)^*b^+$. Moreover, consider $w = a^m(ba)^nbp$ with $m > 1$ and $p > 0$. We have $w \in \text{supp}(A)$ and

$$\{A\}(w) = \{2^{k-1} \cdot 1 \cdot 3^{m-k-1} \cdot 5 \cdot (3 \cdot 5)^n \cdot 5^{\ell-1} \cdot 1 \cdot 2^p \mid 1 \leq k < m \text{ and } 1 \leq \ell \leq p\}.$$  

A concrete semantics over semirings, or valuation monoids, or valuation structures can be obtained from the abstract semantics defined above by applying the suitable aggregation operator $\text{aggr} : \mathbb{N}(R^+) \to S$ as explained in [20]. For convenience, we include a short outline.

A semiring is a structure $(S, +, \cdot, 0, 1)$ where $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $0 \times s = s \times 0 = 0$ for each $s \in S$. If the multiplication is commutative, we say that $S$ is commutative. If the addition is idempotent, the semiring is called idempotent. Important examples of semirings include:

- the natural numbers $\mathbb{N}_{+,\cdot} = (\mathbb{N}, +, \cdot, 0, 1)$ with the usual addition and multiplication;
- the Boolean semiring $\mathcal{B} = (\{0, 1\}, \lor, \land, 0, 1)$;
- the min-plus (or tropical) semiring $\mathbb{N}_{\min,+} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$;
- the max-plus (or arctical) semiring $\mathbb{N}_{\max,+} = ((\mathbb{N} \cup \{\infty\}, \max, +, \infty, 0)$;
- the semiring of languages $(P(\Sigma^+), \cup, \lor, 0, \emptyset)$ where $\lor$ denotes concatenation of languages;
- the semiring of multisets of sequences $(\mathbb{N}(R^+), \uplus, \lor, 0, \{\varepsilon\})$.

Here, $\lor$ denotes the concatenation of multisets (Cauchy product), cf. [20].

Let $(S, +, \cdot, 0, 1)$ be a semiring and $A = (Q, \Sigma, \Delta, \text{wt})$ be an $S$-weighted automaton over $\Sigma$. The value of a run $\rho = \delta_1\delta_2 \cdots \delta_n$ is then defined as $\text{val}(\rho) = \text{wt}(\delta_1) \times \text{wt}(\delta_2) \times \cdots \times \text{wt}(\delta_n)$.
The concrete semantics of $\mathcal{A}$ is the function $\llbracket \mathcal{A} \rrbracket : \Sigma^+ \rightarrow S$ given by $\llbracket \mathcal{A} \rrbracket (w) = \sum_{\rho} \text{val}(\rho)$ where the sum is taken over all successful paths $\rho$ executing the word $w$.

Let us define the aggregation function $\text{aggr}_{\text{sp}} : \mathbb{N}(\mathbb{R}^+) \rightarrow S$ by letting $\text{aggr}_{\text{sp}}(f)$ be the sum over all sequences $s_1s_2 \cdots s_k$ in the multiset $f$ of the products $s_1 \times s_2 \times \cdots \times s_k$ in $S$. It follows that the concrete semantics of $\mathcal{A}$ is the composition of the aggregation function and the abstract semantics of $\mathcal{A}$, i.e., $\llbracket \mathcal{A} \rrbracket (w) = \text{aggr}_{\text{sp}}(\llbracket \mathcal{A} \rrbracket (w))$ for all $w \in \Sigma^+$. Also, the abstract semantics $\llbracket \mathcal{A} \rrbracket$ coincides with the concrete semantics of $\mathcal{A}$ over the semiring of multisets of sequences over $\Sigma$, i.e., $\llbracket \mathcal{A} \rrbracket = \llbracket \mathcal{A} \rrbracket$ (since the aggregation function is the identity function).

As another example, assume the weights of $\mathcal{A}$ are taken in $\mathbb{R}_{\geq 0} \cup \{-\infty\}$, the weight of a run $\rho$ is computed as the average $\text{avg}(\rho) = (\text{wt}(\delta_1) + \cdots + \text{wt}(\delta_\ell))/n$ of the weights in $\rho$, and the concrete semantics of $\mathcal{A}$ is defined for $w \in \Sigma^+$ by $\llbracket \mathcal{A} \rrbracket (w) = \max_{\rho} \text{avg}(\rho)$ where the maximum is taken over all successful runs $\rho$ executing $w$, cf. [5, 6, 13]. In this case, we define the aggregation $\text{aggr}_{\text{sp}}(M)$ of a multiset $M$ by taking the maximum of all averages of sequences in $M$. Again, we obtain $\llbracket \mathcal{A} \rrbracket (w) = \text{aggr}_{\text{sp}}(\llbracket \mathcal{A} \rrbracket (w))$ for all $w \in \Sigma^+$. See [20] for further discussion and examples.

Now, consider the natural semiring $(\mathbb{N}, +, \times, 0, 1)$ and the sum-product aggregation operator $\text{aggr}_{\text{sp}}$. We continue the example above with the automaton $\mathcal{A}$ of Figure 1 and the word $w = a^n(ba)^n b^p$ with $m > 1$ and $p > 0$. The concrete semantics is given by

$$\llbracket \mathcal{A} \rrbracket (w) = \text{aggr}_{\text{sp}}(\llbracket \mathcal{A} \rrbracket (w)) = \sum_{1 \leq k < m} \sum_{1 \leq \ell \leq p} 2^{k-1+p-\ell}3^{n-k-1+n\ell}5^{n+\ell}.$$

### 4 Finitely ambiguous Weighted Automata

In this section, we will investigate finitely ambiguous weighted automata. It was shown in [23] that over the max-plus semiring $\mathbb{N}_{\text{max,+}}$ they are expressively equivalent to finite disjoint unions of unambiguous weighted automata. It was also claimed that their construction can be changed to prove the same result for arbitrary semirings. However, it is not clear, given an aperiodic finitely ambiguous weighted automata, that the construction of [23] produces aperiodic unambiguous automata. Therefore, here we give a different proof of the general result which may be of independent interest and which applies also to the class of aperiodic automata.

▶ Theorem 4. Let $K \geq 1$. Given a $K$-ambiguous weighted automaton $\mathcal{A}$, we can construct an unambiguous weighted automata $\mathcal{B}_1, \ldots, \mathcal{B}_K$ such that $\llbracket \mathcal{A} \rrbracket = \llbracket \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_K \rrbracket = \llbracket \mathcal{B}_1 \rrbracket \cup \cdots \cup \llbracket \mathcal{B}_K \rrbracket$. Moreover, if $\mathcal{A}$ is aperiodic then we can construct aperiodic unambiguous weighted automata $\mathcal{B}_1, \ldots, \mathcal{B}_K$ such that $\llbracket \mathcal{A} \rrbracket = \llbracket \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_K \rrbracket = \llbracket \mathcal{B}_1 \rrbracket \cup \cdots \cup \llbracket \mathcal{B}_K \rrbracket$.

For instance, consider the 3-ambiguous weighted automaton $\mathcal{A}$ of Figure 2 over the alphabet $\Sigma = \{a, b\}$ and the semiring $\mathbb{N}_{+, \times}$ of natural numbers. Clearly, the support of $\mathcal{A}$ is...
\(a^* (a^3 + a^2 b)b^*\) and \([A](a^n a^3 b b^p) = 2^n \cdot \{2 \cdot 1 \cdot 4 \cdot 3 + 2 \cdot 2 \cdot 3 \cdot 3 + 2 \cdot 1 \cdot 5 \cdot 3\} \cdot 3^p\) for \(n, p \geq 0\).

We construct in the proof an automaton \(A_{\geq 3}\) which checks that \(A\) has 3 accepting runs on a given word. Hence, we will have \(L(A_{\geq 3}) = a^* a^3 b b^*\). To do so, \(A_{\geq 3}\) will run three copies of \(A\), make sure that the three runs are lexicographically ordered (to be unambiguous) and accept if the three runs are accepting and pairwise distinct. The set of states is \(Q' = Q^3 \times \{0, 1\}^2\) where \(Q = \{1, \ldots, 6\}\) is the set of states of \(A\). The initial state is \((1, 1, 1, 0, 0)\) and the booleans turn to 1 when the runs differ. The accepting state is \((6, 6, 6, 1, 1)\). The unique accepting run of \(A_{\geq 3}\) on the word \(a^3 b^2\) is

\[(1, 1, 1, 0, 0) \xrightarrow{a} (1, 1, 2, 0, 1) \xrightarrow{a} (2, 3, 4, 1, 1) \xrightarrow{a} (5, 5, 6, 1, 1) \xrightarrow{b} (6, 6, 6, 1, 1) \xrightarrow{b} (6, 6, 6, 1, 1).

Proof. Let \(A = (Q, \Sigma, \Delta, I, F, \omega t)\) be an arbitrary weighted automaton. We may assume that \(A\) has a single initial state \(q_0\). For \(k \geq 1\), we construct an automaton \(A_{\geq k} = (Q', \Sigma, \Delta', I', F')\) which accepts the set of words \(w = a_1 a_2 \cdots a_n \in \Sigma^+\) having at least \(k\) accepting runs in \(A\). Moreover, if \(A\) is aperiodic then so is \(A_{\geq k}\).

Fix a strict total order \(\prec\) on \(Q\). We write \(\preceq\) for the induced lexicographic order on \(Q^+\) and \(\prec\) for the strict order. A run of \(A\) on \(w\) induces a sequence of states \(\rho = q_0 a_1 q_2 \cdots q_n \in Q^+\) with \((q_{i-1}, a_i, q_i) \in \Delta\) for all \(1 \leq i \leq n\). Overloading our terminology, such a sequence is also called a run below. Runs of \(A\) on \(w\) are lexicographically ordered. For \(0 \leq j \leq n\), we denote by \(\rho[j] = q_0 a_1 q_2 \cdots q_j\) the prefix of length \(j\) of \(\rho\).

The idea is that \(A_{\geq k}\) will guess \(k\) runs \(\rho^1 \preceq \rho^2 \preceq \cdots \preceq \rho^k\) of \(A\) on \(w\). For \(1 \leq \ell \leq k\), we let \(\rho^\ell = q_0 q^\ell_1 q^\ell_2 \cdots q^\ell_n\). Now, after reading the prefix \(w[j] = a_1 a_2 \cdots a_j\), the state of \(A_{\geq k}\) will consist of the tuple \((q^\ell_1, \ldots, q^\ell_j)\) of states reached by the prefixes \(\rho^\ell[j] \preceq \cdots \preceq \rho^k[j]\) together with a bit vector \((c_1, \ldots, c^{k-1})\) such that for all \(1 \leq k < c_j\), \(c_j = 1\) iff \(\rho^\ell[j] \prec \rho^{\ell+1}[j]\). The automaton \(A_{\geq k}\) will accept if all states \(q^\ell_1 \in F\) are final in \(A\) and the bit vector contains only 1’s. This ensures that \(\rho^1 \prec \rho^2 \prec \cdots \prec \rho^k\) are distinct accepting runs for \(w\) in \(A\).

We turn now to the formal definition of \(A_{\geq k}\). Let \(Q' = Q^k \times \{0, 1\}^{k-1}\), \(I' = \{q_0\}^k \times \{0\}^{k-1}\) and \(F' = F^k \times \{1\}^{k-1}\). We write tuples with superscripts: \((q, \tau) \in Q'\) with \(\tau = (q^1, \ldots, q^k)\) and \(\tau = (c^1, \ldots, c^{k-1})\). Now, \((((q, \tau), a, (q', \tau'))))\) is a transition of \(A_{\geq k}\) if the following conditions hold:

1. \((q', a, q'^\ell) \in \Delta\) for all \(1 \leq \ell \leq k\) (the \(k\) runs are non-deterministically guessed),
2. then the bit vector is deterministically updated: for \(1 \leq \ell \leq k\) we have either \((c^\ell = 0, q'^\ell = q'^{\ell+1}\) and \(c^\ell = 0), or \((c^\ell = 1\) or \(q'^\ell \prec q'^{\ell+1}\)) and \(c^\ell = 1\). Notice that \(c^\ell = 0\) and \(q'^{\ell+1} \prec q'^\ell\) is not allowed.

When \(k = 1\) then the accessible part of \(A_1\) is equal to \(A\). We will now state formally the main properties of \(A_{\geq k}\).

> **Claim 5.** For each \(w \in \Sigma^+\), there is a bijection between the accepting runs \(\overline{\rho}\) of \(A_{\geq k}\) on \(w\) and the tuples \((\rho^1, \ldots, \rho^k)\) of accepting runs of \(A\) on \(w\) such that \(\rho^1 \prec \cdots \prec \rho^k\).

**Proof.** Consider a word \(w = a_1 a_2 \cdots a_n \in \Sigma^+\) and a run \(\overline{\rho}\) of \(A_{\geq k}\) on \(w\) starting from its initial state. Write \((\overline{q}_j, \overline{c}_j)\) the \(j\)th state of \(\overline{\rho}\). For \(1 \leq \ell \leq k\), let \(\rho^\ell\) be the projection of \(\overline{\rho}\) on the \(\ell\)th component: \(\rho^\ell = q^\ell_0 q^\ell_1 q^\ell_2 \cdots q^\ell_n\). Clearly, \(\rho^\ell\) is a run of \(A\) on \(w\). Moreover, we can easily check by induction on \(0 \leq j \leq n\) that for all \(1 \leq \ell \leq k\) we have \(\rho^\ell[j] = q^\ell+1[j]\) if \(c^\ell_j = 0\) and \(\rho^\ell[j] \prec q^\ell+1[j]\) if \(c^\ell_j = 1\). We deduce that if \(\overline{\rho}\) is accepting in \(A_{\geq k}\) then each \(\rho^\ell\) is accepting in \(A\) and \(\rho^1 \prec \cdots \prec \rho^k\). Therefore, every word accepted by \(A_{\geq k}\) admits at least \(k\) accepting runs in \(A\).

Conversely, assume that \(w \in \Sigma^+\) has at least \(k\) accepting runs \(\rho^1 \prec \cdots \prec \rho^k\) in \(A\). We can easily construct an accepting run \(\overline{\rho}\) of \(A_{\geq k}\) on \(w\) such that the \(\ell\)th projection of \(\overline{\rho}\) is \(\rho^\ell\) for each \(1 \leq \ell \leq k\). We deduce that \(A_{\geq k}\) accepts exactly the set of words \(w \in \Sigma^+\) having at least \(k\) accepting runs in \(A\). ▶
We deduce from Claim 5 that if $A$ is $k$-ambiguous then $A_{\geq k}$ is unambiguous and accepts exactly the words accepted by $A$ with ambiguity $k$.

\begin{itemize}
  \item \textbf{Claim 6.} If $A$ is aperiodic with index $m$ then $A_{\geq k}$ is aperiodic with index $k(m+1)$.
\end{itemize}

\textbf{Proof.} Consider a word $w \in \Sigma^+$ and a run $\rho$ of $A_{\geq k}$ reading $w^{k(m+1)}$. The sequence of bit vectors along $\rho$ is monotone component-wise. Hence, its value can change at most $k-1$ times. We deduce that we can write $\rho = \rho' \rho''$ where $\rho''$ reads $w^{m+1}$ with the bit vector unchanged. Let $(\overline{\rho}, \overline{\sigma})$ and $(\overline{\rho}, \overline{\tau})$ be the source and target states of $\rho''$. The projections $\rho^1, \ldots, \rho^k$ of $\rho''$ are runs reading $w^{m+1}$ in $A$. Since $A$ is aperiodic with index $m$, we find runs $\sigma^1, \ldots, \sigma^k$ reading $w^m$ in $A$ from states $p^1, \ldots, p^k$ to $q^1, \ldots, q^k$ respectively. We may assume that for all $1 \leq \ell < k$ we have $\sigma^\ell = \sigma^{\ell+1}$ if $\rho^\ell = \rho^{\ell+1}$. Consider the run $\sigma$ of $A_{\geq k}$ starting from $(\overline{\rho}, \overline{\sigma})$ whose projections are $\sigma^1, \ldots, \sigma^k$. It reaches a state $(\overline{\tau}', \overline{\tau})$. Clearly, we have $\overline{\sigma} = \overline{\sigma}$. We show that $\overline{\tau}' = \overline{\tau}$. Let $1 \leq \ell < k$. If $c^\ell = 1$ then $c^{\ell+1} = 1$ by definition of $A_{\geq k}$. If $c^\ell = 0$ then $\rho^\ell = \rho^{\ell+1}$ by definition of $A_{\geq k}$. We deduce that $\sigma^\ell = \sigma^{\ell+1}$ and $c^{\ell+1} = 0$. Finally, we conclude that $\rho' \rho''$ is a run of $A_{\geq k}$ reading $w^k(m+1)^{-1}$ with the same source (resp. target) state as $\rho$.

By choosing runs $\sigma^1, \ldots, \sigma^k$ reading the word $w^{m+2}$ instead of $w^m$, we obtain a run of $A_{\geq k}$ reading $w^k(m+1)^{-1}$ with the same source (resp. target) state as $\rho$. \hfill \blacksquare

Now, let $A_{\leq k}$ be the minimal automaton for the complement of the language accepted by $A_{k+1}$. Notice that $A_{\leq k}$ is deterministic, complete. Moreover, it is aperiodic if $A$ is aperiodic.

For each $1 \leq \ell \leq k$, define the weighted automaton $A_{\leq k}^\ell = (A_{\leq k}, \text{wt}^\ell)$ where the weight function corresponds to the $\ell$th path computed by $A_{\geq k}$. More precisely, we set $\text{wt}^\ell((\overline{\rho}, \overline{\sigma}), a, (\overline{\tau}', \overline{\tau})) = \text{wt}(q', a, q'')$. Finally, let $A_{\leq k}^\ell = A_{\leq k} \times A_{\leq k}^\ell$. It is not difficult to see that $A_{\leq k}^\ell$ has the following properties.

\begin{itemize}
  \item \textbf{Claim 7.} The automaton $A_{\leq k}^\ell$ is unambiguous. A word $w \in \Sigma^+$ is in the support of $A_{\leq k}^\ell$ if and only if $w$ accepts runs $\rho^1 \prec \cdots \prec \rho^k$ in $A$. Moreover, in this case, $\|A_{\leq k}^\ell\| (w) = \{\text{wt}(\rho^\ell)\}$. Also, if $A$ is aperiodic then so is $A_{\leq k}^\ell$.
\end{itemize}

Finally, to conclude the proof of Theorem 4, we define for each $1 \leq \ell \leq K$ the weighted automaton $B^\ell = A_{\leq k}^\ell \cup \cdots \cup A_{\leq k}^K$. Since the automata $(A_{\leq k}^\ell)_{\ell \leq K}$ have pairwise disjoint supports, we deduce that $B^\ell$ is unambiguous. Moreover, using Claim 7, we can easily show that $\|A\| = \{B_1 \cup \cdots \cup B_K\}$. \hfill \blacksquare

\section{Weighted First-Order Logic}

In this section, we define the syntax and semantics of our weighted first-order logic. As in Section 3, we consider a set $R$ of weights. The syntax of wFO is obtained from core-wMSO as defined in [20] by removing set variables, set quantifications, and set sums. In addition to the classical boolean first-order logic (FO), it has two weighted layers which are defined in Table 1. Step formula are defined in \textbf{[step-wFO]}, and \textbf{[core-wFO]} defines the weighted first-order logic.

The semantics of step-wFO formulas is defined inductively. As above, let $u \in \Sigma^+$ be a nonempty word and $\sigma : V \to \text{pos}(u) = \{1, \ldots, |u|\}$ be a valuation. For step-wFO formulas whose free variables are contained in $V$, we define the $V$-semantics as

\begin{equation}
\|r\|_V(u, \sigma) = r \\
\|\varphi \wedge \Psi_1 \wedge \Psi_2\|_V(u, \sigma) = \begin{cases} \\
\|\Psi_1\|_V(u, \sigma) & \text{if } u, \sigma \models \varphi \\
\|\Psi_2\|_V(u, \sigma) & \text{otherwise.}
\end{cases}
\end{equation}
\[
\varphi ::= T \mid P_a(x) \mid x \leq y \mid \neg \varphi \mid \varphi \land \varphi \mid \forall x \varphi
\]  
\(\text{FO}\)

\[
\Psi ::= r \mid \varphi ? \Psi : \Psi
\]  
\(\text{step-wFO}\)

\[
\Phi ::= 0 \mid \prod_x \Psi \mid \varphi ? \Phi : \Phi \mid \Phi + \Phi \mid \sum_x \Phi
\]  
\(\text{core-wFO}\)

**Table 1** Syntax of core-wFO\((\Sigma, R)\) with \(a \in \Sigma, r \in R\) and \(x, y\) first-order variables.

![Figure 3](image)

**Figure 3** A weighted automaton, which is both aperiodic and unambiguous.

Notice that the semantics of a step-wFO formula is always a single weight from \(R\).

For core-wFO formulas \(\Phi\) whose free variables are contained in \(\mathcal{V}\), we define the \(\mathcal{V}\)-semantics \(\{\Phi\}_{\mathcal{V}} : \Sigma^+_{\mathcal{V}} \rightarrow N(R^+)\). First, we let \(\{\Phi\}_{\mathcal{V}}(\overline{\pi}) = \emptyset\) be the empty multiset when \(\overline{\pi} \in \Sigma^+_{\mathcal{V}}\) is not a valid encoding of a pair \((u, \sigma)\). Assume now that \(\overline{\pi} = (u, \sigma)\) is a valid encoding of a nonempty word \(u \in \Sigma^+\) and a valuation \(\sigma : \mathcal{V} \rightarrow \text{pos}(u)\). The semantics of core-wFO formulas is also defined inductively: \(\{0\}_{\mathcal{V}}(u, \sigma) = \emptyset\) is the empty multiset, and

\[
\begin{align*}
\{\prod_x \Psi\}_{\mathcal{V}}(u, \sigma) & = \{r_1 r_2 \cdots r_{|u|}\} \quad \text{where} \quad r_i = [\Psi]_{\mathcal{V} \cup \{x\}}(u, \sigma[x \mapsto i]) \quad \text{for} \quad 1 \leq i \leq |u| \\
\{\varphi ? \Phi_1 : \Phi_2\}_{\mathcal{V}}(u, \sigma) & = \begin{cases} \\
\{\Phi_1\}_{\mathcal{V}}(u, \sigma) & \text{if} \quad u, \sigma \models \varphi \\
\{\Phi_2\}_{\mathcal{V}}(u, \sigma) & \text{otherwise} \\
\end{cases} \\
\{\Phi_1 + \Phi_2\}_{\mathcal{V}}(u, \sigma) & = \{\Phi_1\}_{\mathcal{V}}(u, \sigma) \uplus \{\Phi_2\}_{\mathcal{V}}(u, \sigma) \\
\{\sum_x \Phi\}_{\mathcal{V}}(u, \sigma) & = \biguplus_{i \in \text{pos}(u)} \{\Phi\}_{\mathcal{V} \cup \{x\}}(u, \sigma[x \mapsto i]).
\end{align*}
\]

The semantics of the product (first line), is a singleton multiset which consists of a weight sequence whose length is \(|u|\). We deduce that all weight sequences in a multiset \(\{\Phi\}_{\mathcal{V}}(u, \sigma)\) have the same length and \(\{\Phi\}_{\mathcal{V}}(u, \sigma) \in \mathbb{N}(R^{(|u|)})\). We simply write \(\{\Psi\}\) and \(\{\Phi\}\) when the set \(\mathcal{V}\) of variables is clear from the context. See [11] for examples of quantitative specifications in weighted logic.

## 6 From Weighted Automata to Weighted FO

We say that a non-deterministic automaton \(A = (Q, \Sigma, \Delta)\) is unambiguous from state \(p\) to state \(q\) if for all words \(u \in \Sigma^*\), there is at most one run of \(A\) from \(p\) to \(q\) with label \(u\).

**Theorem 8.** Let \(A\) be an aperiodic weighted automaton which is unambiguous from state \(p\) to state \(q\). We can construct a core-wFO sentence \(\Phi_{p,q} = \varphi_{p,q} ? \prod_{x} \Psi_{p,q} : 0\) where \(\varphi_{p,q}\) is a first-order sentence and \(\Psi_{p,q}(x)\) is a step-wFO formula with a single free variable \(x\) such that \(\|A_{p,q}\| = \{\Phi_{p,q}\}\).

Before proving Theorem 8, we start with an example. The automaton \(A\) of Figure 3 is unambiguous and it accepts the language \(L(A) = (a^*b + a^*c)^+ = (a + b + c)^*(b + c)\). We define
a core-wFO sentence \( \Phi_{1,3} = \varphi_{1,3} \wedge \prod_x \Psi_{1,3}(x) : 0 \) as follows. The FO sentence \( \varphi_{1,3} \) checks that \( \mathcal{A} \) has a run from state 1 to state 3 on the input word \( w \), i.e., that \( w \in a^*b(a^*b + a^*c)^* \):

\[
\varphi_{1,3} = \exists y \,( P_b(y) \land \forall z \,(z < y \implies P_a(z))) \land \exists y \,(\neg P_b(y) \land \forall z \,(z \leq y))
\]

When this is the case, the step-wFO formula \( \Psi_{1,3}(x) \) computes the weight of the transition taken at a position \( x \) in the input word:

\[
\Psi_{1,3}(x) = (P_b(x) \lor P_c(x)) \land 1 : \exists y \,(x < y \land P_b(y) \land \forall z \,(x < z < y \implies P_a(z))) \land 2 : 3.
\]

Notice that the same formula \( \Psi = \Psi_{2,3} = \Psi_{1,3} \) also allows to compute the sequence of weights for the accepting runs starting in state 2. Therefore, \( \mathcal{A} \) is equivalent to the core-wFO sentence

\[
\Phi = \exists y \,(\neg P_a(y) \land \forall z \,(z \leq y)) \land \prod_x \Psi(x) : 0.
\]

**Proof of Theorem 8.** Let \( \mathcal{A} = (Q, \Sigma, \Delta, \text{wt}) \) be the aperiodic weighted automaton. By Theorem 2, for every pair of states \( r,s \in Q \) there is a first-order sentence \( \varphi_{r,s} \) such that \( \mathcal{L}(\mathcal{A}_{r,s}) = \mathcal{L}(\varphi_{r,s}) \). This gives in particular the first-order sentence \( \varphi_{p,q} \) which is used in \( \Phi_{p,q} \).



\[\text{Claim 9.} \] We can construct a step-wFO formula \( \Psi_{p,q}(x) \) such that for each word \( u \in \mathcal{L}(\mathcal{A}_{p,q}) \) and each position \( 1 \leq i \leq |u| \) in the word \( u \), we have \([\Psi_{p,q}](u, x \mapsto i) = \text{wt}(\delta)\) where \( \delta \) is the \( i \)th transition of the unique run \( \rho \) of \( \mathcal{A} \) from \( p \) to \( q \) with label \( u \).

Before proving this claim, let us show how we can deduce the statement of Theorem 8. Clearly, if a word \( u \in \Sigma^+ \) is not in \( \mathcal{L}(\mathcal{A}_{p,q}) \) then we have \([\mathcal{A}_{p,q}](u) = \emptyset = [\Phi_{p,q}](u)\).

Consider now a word \( u = a_1a_2\cdots a_n \in \mathcal{L}(\mathcal{A}_{p,q}) \) and the unique run \( \rho = \delta_1\delta_2\cdots \delta_n \) of \( \mathcal{A} \) from \( p \) to \( q \) with label \( u \). We have

\[
[\mathcal{A}_{p,q}](u) = \{\text{wt}(\delta_1)\text{wt}(\delta_2)\cdots \text{wt}(\delta_n)\} = ]\prod_x [\Psi_{p,q}](u)
\]

where the second equality follows from Claim 9. We deduce that \([\mathcal{A}_{p,q}] = [\Phi_{p,q}]\).

We turn now to the proof of Claim 9. Let \( \delta = (r,a,s) \in \Delta \) be a transition of \( \mathcal{A} \). We define the FO-formula with one free variable

\[
\varphi_\delta(x) = \varphi_{p,r}^\leq_x \land P_a(x) \land \varphi_{s,q}^>^x.
\]

**Claim 10.** For each word \( u \in \Sigma^+ \) and position \( 1 \leq i \leq |u| \), we have \( u, x \mapsto i \vdash \varphi_\delta \) iff \( u \in \mathcal{L}(\mathcal{A}_{p,q}) \) and \( \delta \) is the \( i \)th transition of the unique run of \( \mathcal{A} \) from \( p \) to \( q \) with label \( u \).

Indeed, assume that \( u, x \mapsto i \vdash \varphi_\delta \). Then, \( u[1,i-1] = \varphi_{p,r} \) and there is a run \( \rho' \) of \( \mathcal{A} \) from \( p \) to \( r \) with label \( u[1,i-1] \). Notice that if \( i = 1 \) then \( p = r \) and \( \rho' \) is the empty run. Similarly, from \( u[i+1,|u|] \vdash \varphi_{s,q} \) we deduce that there is a run \( \rho'' \) of \( \mathcal{A} \) from \( s \) to \( q \) with label \( u[i+1,|u|] \). Finally, \( u, x \mapsto i \vdash P_a(x) \) means that the \( i \)th letter of \( u \) is \( a \). We deduce that \( \rho = \rho'\rho'' \) is a run of \( \mathcal{A} \) from \( p \) to \( q \) with label \( u \), hence \( u \in \mathcal{L}(\mathcal{A}_{p,q}) \). Moreover, \( \rho \) is the unique such run since \( \mathcal{A} \) is unambiguous from \( p \) to \( q \). Now, \( \delta \) is the \( i \)th transition of \( \rho \), which concludes one direction of the proof. Conversely, assume that \( u \in \mathcal{L}(\mathcal{A}_{p,q}) \) and \( \delta \) is the \( i \)th transition of the unique run of \( \mathcal{A} \) from \( p \) to \( q \) with label \( u \). Then, \( u[1,i-1] \vdash \varphi_{p,r} \), \( u[i+1,|u|] \vdash \varphi_{s,q} \), and the \( i \)th letter of \( u \) is \( a \). Therefore, \( u, x \mapsto i \vdash \varphi_\delta \). This concludes the proof of Claim 10.

Now, choose an arbitrary enumeration \( \delta^1, \delta^2, \ldots, \delta^k \) of the transitions in \( \Delta \) and define the step-wFO formula with one free variable

\[
\Psi_{p,q}(x) = \varphi_{\delta^1}(x) \land \text{wt}(\delta^1) : \varphi_{\delta^2}(x) \land \text{wt}(\delta^2) : \cdots \varphi_{\delta^k}(x) \land \text{wt}(\delta^k) : \text{wt}(\delta^k).
\]
We show that this formula satisfies the property of Claim 9. Consider a word $u \in \mathcal{L}(A_{p,q})$ and a position $1 \leq i \leq |u|$. Let $\delta$ be the $i$th transition of the unique run of $A$ from $p$ to $q$ with label $u$. By Claim 10 we have $u, x \mapsto i \equiv \varphi_{\delta}$, iff $\delta^j = \delta$. Therefore, $[\Psi_{p,q}](u, x \mapsto i) = \text{wt}(\delta)$, which concludes the proof of Claim 9.

**Corollary 11.** Let $A = (Q, \Sigma, \Delta, \text{wt}, I, F)$ be an aperiodic and unambiguous weighted automaton. We can construct a core-wFO sentence $\Phi$ which does not use any $\sum_x$ operator or $+$ operator, and such that $[\mathcal{A}] = [\Phi]$.

**Proof.** Since $A$ is unambiguous, it is also unambiguous from $p$ to $q$ for all $p \in I$ and $q \in F$. Therefore, a first attempt is the formula $\Phi' = \sum_{p \in I, q \in F} \Phi_{p,q}$ where the core-wFO sentences $\Phi_{p,q}$ are given by Theorem 8. We have $[\mathcal{A}] = [\Phi]$ and the formula $\Phi'$ does not use any $\sum_x$ operator, but it does use some $+$ operator. One should notice that, since $A$ is unambiguous, for any word $u \in \Sigma^+$ at most one of the $([\Phi_{p,q}](u))_{p \in I, q \in F}$ is nonempty. Therefore, if $(p_1, q_1), (p_2, q_2), \ldots, (p_m, q_m)$ is an enumeration of $I \times F$ then we define

$$\Phi = \varphi_{p_1, q_1} \cdot \varphi_{p_2, q_2} \cdot \ldots \cdot \varphi_{p_m, q_m} : 0.$$ 

We have $[\mathcal{A}] = [\Phi]$ and the formula $\Phi$ does not use any $\sum_x$ or $+$ operator. Notice that in the formula above, we may replace $\Phi_{p_i, q_i}$ by $\prod_x \Psi_{p_i, q_i}(x)$ as given by Theorem 8.

Alternatively, by a standard construction adding a new initial and a new final state and appropriate transitions, we can obtain an aperiodic weighted automaton $A'$ with a single initial and a single final state such that $[\mathcal{A}''] = [\mathcal{A}]$ and, moreover, $A'$ becomes unambiguous because $A$ is unambiguous. Then apply Theorem 8 to $A'$.

Let $A = (Q, \Sigma, \Delta)$ be a non-deterministic automaton. Two states $p, q \in Q$ are in the same strongly connected component (SCC), denoted $p \approx q$, if $p = q$ or there exist a run of $A$ from $p$ to $q$ and also a run of $A$ from $q$ to $p$. Notice that $\approx$ is an equivalence relation on $Q$. We denote by $[p]$ the strongly connected component of state $p$, i.e., the equivalence class of $p$ under $\approx$.

The automaton $A$ is *SCC-unambiguous* if it is unambiguous on each strongly connected component, i.e., $A$ is unambiguous from $p$ to $q$ for all $p, q$ such that $p \approx q$. Notice that a trimmed and unambiguous automaton is SCC-unambiguous.

For instance, the automaton $A$ of Figure 1 has three strongly connected components: \{1\}, \{2, 3\} and \{4\}. It is not unambiguous from 1 to 4, but it is SCC-unambiguous.

**Proposition 12** ([31] 21 and [37] Theorem 4.1). Let $A = (Q, \Sigma, \Delta, I, F)$ be a trimmed non-deterministic automaton. Then $A$ is polynomially ambiguous if and only if $A$ is SCC-unambiguous.

**Theorem 13.** Let $A$ be an aperiodic weighted automaton which is SCC-unambiguous. For each pair of states $p$ and $q$, we can construct a core-wFO sentence $\Phi_{p,q}$ such that $[\mathcal{A}_{p,q}] = [\Phi_{p,q}]$. Moreover, we can construct a core-wFO sentence $\Phi$ such that $[\mathcal{A}] = [\Phi]$.

Before starting the proof of Theorem 13 we give for the weighted automaton $A$ of Figure 1 the equivalent core-wFO formula $\Phi_{1,4} = \sum_y \sum_{y_1} \varphi(y_1, y_2) \cdot \prod_x \Psi(x, y_1, y_2) : 0$ where $\varphi$ and $\Psi$ are defined below. When reading a word $w \in \text{supp}(A)$, the automaton makes two non-deterministic choices corresponding to the positions $y_1$ and $y_2$ at which the transitions *switching* between the strongly connected components are taken, i.e., transition from state 1 to state 2 is taken at position $y_1$, and transition from state 3 to state 4 is taken at position $y_2$. Since the automaton is SCC-unambiguous, given the input word and these two positions,
When this is the case, the \( A \)-phase formula \( \Psi(x, y_1, y_2) \) computes the weight of the transition taken at a position \( x \) in the input word:

\[
\Psi(x, y_1, y_2) = (x < y_1 \lor y_2 < x) \land (x = y_1 \land x = y_2) \land P_a(x + 1) \land 3 : 5.
\]

With these definitions, we obtain \( \langle A \rangle = \langle \Phi_1, a \rangle \).

**Proof of Theorem 13.** Let \( A = (Q, \Sigma, \Delta, \text{wt}) \) be the aperiodic weighted automaton which is SCC-unambiguous. Let \( p, q \in Q \) be a pair of states of \( A \). Assume first that \( p \approx q \) are in the same strongly connected component. Then \( A \) is unambiguous from \( p \) to \( q \) and we obtain the formula \( \Phi_{p, q} \) directly by Theorem 8. So we assume below that \( p \not\approx q \) are not in the same SCC.

Consider a word \( u \in \mathcal{L}(A_{p, q}) \). Let \( \rho \) be a run from \( p \) to \( q \) with label \( u \). This run starts in the SCC of \( p \) and ends in the SCC of \( q \). So it uses some transitions linking different SCCs. More precisely, we can uniquely split the run as \( p \approx \rho_1 \rho_2 \cdots \rho_m \rho_{m+1} \approx q \), with \( m \geq 1 \) such that each subrun \( \rho_i \) stays in some SCC and each transition \( \delta_i = (p_i, a_i, q_i) \) switches to a different SCC:

\[
p \approx p_1 \not\approx q_1 \approx p_2 \not\approx q_2 \approx p_3 \cdots \approx p_m \not\approx q_m \approx q.
\]

This motivates the following definition. A sequence of switching transitions from \( p \) to \( q \) is a tuple \( \vec{\delta} = (\delta_1, \ldots, \delta_m) \) with \( m \geq 1 \) satisfying (1), where \( \delta_i = (p_i, a_i, q_i) \) for \( 1 \leq i \leq m \). A \( \vec{\delta} \)-run from \( p \) to \( q \) is a run from \( p \) to \( q \) using exactly the sequence of switching transitions \( \vec{\delta} \), i.e., a run of the form \( \rho = \rho_0 \delta_1 \rho_2 \cdots \delta_m \rho_m \). Notice that each subrun \( \rho_i \) must stay in some SCC of \( A \).

\[\triangleright\) Claim 14. For each sequence \( \vec{\delta} \) of switching transitions from \( p \) to \( q \), we can construct a core-wFO sentence \( \Phi_{p, \vec{\delta}, q} \) such that for all \( u \in \Sigma^+ \) we have

\[
\langle \Phi_{p, \vec{\delta}, q} \rangle (u) = \{ \langle \text{wt}(\rho) \mid \rho \text{ is a } \vec{\delta}\text{-run from } p \text{ to } q \text{ with label } u \} \}
\]

(2)

During the proof of Claim 14 we fix the sequence \( \vec{\delta} = (\delta_1, \ldots, \delta_m) \) of switching transitions from \( p \) to \( q \), with \( m \geq 1 \) and \( \delta_i = (p_i, a_i, q_i) \) for \( 1 \leq i \leq m \).

By Theorem 8 for every pair of states \( r, s \in Q \) there is a first-order sentence \( \varphi_{r, s} \) such that \( \mathcal{L}(A_{r, s}) = \mathcal{L}(\varphi_{r, s}) \). We will use these formulas and also their relativizations \( \varphi_{r, s}^<z \) and \( \varphi_{r, s}^>z \).

We define the FO formula \( \varphi \) with free variables \( \mathcal{V} = \{ y_1, \ldots, y_m \} \) by

\[
\varphi = y_1 < y_2 < \cdots < y_m \land \bigwedge_{1 \leq i \leq m} P_{a_i}(y_i) \land \varphi_{p, p_1}^{<y_1} \land \bigwedge_{1 \leq i < m} \varphi_{q_i, p_{i+1}}^{(y_i, y_{i+1})} \land \varphi_{q_m, q}^{>y_m}.
\]

Now, we fix a word \( u \in \Sigma^+ \).

\[\triangleright\) Claim 15. There is a bijection between the valuations \( \sigma : \mathcal{V} \to \text{pos}(u) = \{ 1, \ldots, |u| \} \) such that \( u, \sigma \models \varphi \) and the \( \vec{\delta} \)-runs \( \rho \) from \( p \) to \( q \) with label \( u \).
We prove now that Equation (2) holds. By definition, $u, \sigma \models \varphi$. We have $\sigma(y_1) < \sigma(y_2) < \cdots < \sigma(y_m)$. Since $u, \sigma \models \varphi_{\rho_0, p_1}$, there is a (possibly empty) run $\rho_0(\sigma)$ from $p$ to $p_1$ reading the prefix $u_0 = u[1, \sigma(y_1) - 1]$ of $u$. Notice that such a run is unique since $p \approx p_1$ and $A$ is SCC-unambiguous. Similarly, for all $1 \leq i < m$, $u, \sigma \models \varphi_{q_i, p_{i+1}}^{\rho_i(\sigma)}$ implies that there is a unique run $\rho_i(\sigma)$ from $q_i$ to $p_{i+1}$ reading the factor $u_i = u[\sigma(y_i) + 1, \sigma(y_{i+1}) - 1]$ of $u$. Also, $u, \sigma \models \varphi_{q_m, q}^{\rho_m(\sigma)}$ implies that there is a unique run $\rho_m(\sigma)$ from $q_m$ to $q$ reading the suffix $u_m = u[\sigma(y_m) + 1, |u|]$ of $u$. Now, since $u, \sigma \models \bigwedge_{1 \leq i \leq m} P_{A_i}(u_i)$, we deduce that $u = u_0 a_1 u_2 \cdots a_m u_m$ and that $\rho(\sigma) = \rho_0(\sigma) \delta_1 \rho_1(\sigma) \cdots \delta_m \rho_m(\sigma)$ is a $\delta$-run of $A$ from $p$ to $q$ with label $u$.

Finally, the sentence for Claim 14 is defined by

$$\Phi_{\rho, \delta, q} = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_m} (\varphi \land \prod_{\delta} \Psi : 0).$$

We prove now that Equation (2) holds. By definition, $\{\Phi_{\rho, \delta, q}\}(u)$ is the (multiset) union over all valuations $\sigma : V \rightarrow \text{pos}(u)$ of $\{\varphi \land \prod_{\delta} \Psi : 0\}(u, \sigma)$. By Claim 15, there is a bijection between the valuations $\sigma : V \rightarrow \text{pos}(u)$ such that $u, \sigma \models \varphi$ and the $\delta$-runs from $p$ to $q$ with label $u$. Notice that such a run is unique since $p \approx p_1$ and $A$ is SCC-unambiguous. Similarly, for all $1 \leq i < m$, $u, \sigma \models \varphi_{q_i, p_{i+1}}^{\rho_i(\sigma)}$ implies that there is a unique run $\rho_i(\sigma)$ from $q_i$ to $p_{i+1}$ reading the factor $u_i = u[\sigma(y_i) + 1, \sigma(y_{i+1}) - 1]$ of $u$. Also, $u, \sigma \models \varphi_{q_m, q}^{\rho_m(\sigma)}$ implies that there is a unique run $\rho_m(\sigma)$ from $q_m$ to $q$ reading the suffix $u_m = u[\sigma(y_m) + 1, |u|]$ of $u$. Now, since $u, \sigma \models \bigwedge_{1 \leq i \leq m} P_{A_i}(u_i)$, we deduce that $u = u_0 a_1 u_2 \cdots a_m u_m$ and that $\rho(\sigma) = \rho_0(\sigma) \delta_1 \rho_1(\sigma) \cdots \delta_m \rho_m(\sigma)$ is a $\delta$-run of $A$ from $p$ to $q$ with label $u$.

Finally, the core-wFO sentence for Claim 14 is defined by

$$\Phi_{\rho, \delta, q} = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_m} (\varphi \land \prod_{\delta} \Psi : 0).$$

We prove now that Equation (2) holds. By definition, $\{\Phi_{\rho, \delta, q}\}(u)$ is the (multiset) union over all valuations $\sigma : V \rightarrow \text{pos}(u)$ of $\{\varphi \land \prod_{\delta} \Psi : 0\}(u, \sigma)$. By Claim 15, there is a bijection between the valuations $\sigma : V \rightarrow \text{pos}(u)$ such that $u, \sigma \models \varphi$ and the $\delta$-runs from $p$ to $q$ with label $u$.
label $u$. Therefore, it remains to show that for all valuations $\sigma : V \to \text{pos}(u)$ such that $u, \sigma \models \varphi$ with associated $\delta$-run $\rho$ we have

$$\{\text{wt}(\rho)\} = (\prod_{i} \Psi_i)(u, \sigma).$$

Let $i \in \text{pos}(u)$ and let $\delta$ be the $i$th transition of $\rho$. From the definitions above, we deduce easily that $u, \sigma[x \mapsto i] \models \varphi_\delta$, iff $\delta^i = \delta$. Therefore, $[\Psi](u, \sigma[x \mapsto i]) = \text{wt}(\delta)$. The announced equality $\{\text{wt}(\rho)\} = (\prod_{i} \Psi_i)(u, \sigma)$ follows. This concludes the proof of Claim 14.

To conclude the proof of the first part of Theorem 13, we define

$$\Phi_{p,q} = \sum_\delta \Phi_{p,\delta, q}$$

where the sum ranges over all sequences $\delta$ of switching transitions from $p$ to $q$. Recall that we have assumed that $p \neq q$ are not in the same SCC of $A$. Therefore, each run from $p$ to $q$ should go through some sequence of switching transitions. More precisely, given a word $u \in \Sigma^+$, the runs of $A$ from $p$ to $q$ with label $u$ can be partitionned according to the sequence $\delta$ of switching transitions that they use. Therefore, $\{[A_{p,q}](u)\}$ is the multisets union over all sequences $\delta$ of switching transitions from $p$ to $q$ of the multisets $\{\text{wt}(\rho) | \rho$ is a $\delta$-run from $p$ to $q$ with label $u\}$. Using Claim 14 we deduce that $\{[A_{p,q}](u)\} = \{\Phi_{p,q}(u)\}$.

Finally, consider a weighted automaton with acceptance conditions $A = (Q, \Sigma, \Delta, I, F)$ which is aperiodic and SCC-unambiguous. We set $\Phi = \sum_{p \in I, q \in F} \Phi_{p, q}$ where for each pair of states $(p, q) \in I \times F$, the formula $\Phi_{p, q}$ is defined as above.

## 7 From Weighted FO to Weighted Automata

Let $A = (Q, \Sigma, \Delta)$ and $A' = (Q', \Sigma, \Delta')$ be two non-deterministic automata over the same alphabet $\Sigma$. Assuming that $Q \cap Q' = \emptyset$, we define their disjoint union as $A \sqcup A' = (Q \sqcup Q', \Sigma, \Delta \sqcup \Delta')$ and their product as $A \times A' = (Q \times Q', \Sigma, \Delta'')$ where $\Delta'' = \{(p, p'), a, (q, q') \} | (p, a, p') \in \Delta$ and $(q', a, q') \in \Delta'$.

**Lemma 16.** The following holds.

1. If $A$ and $A'$ are aperiodic, then $A \sqcup A'$ and $A \times A'$ are also aperiodic.
2. If $A$ and $A'$ are SCC-unambiguous, then $A \sqcup A'$ and $A \times A'$ are also SCC-unambiguous.

Now let $\varphi$ be an FO-formula with free variables contained in the finite set $V$, and let $A_{\varphi, V} = (Q, \Sigma_V, \Delta, I, F, G)$ be the deterministic, complete, trim and aperiodic automaton given by Theorem 3. For $i = 1, 2$, let $A_i = (Q_i, \Sigma_V, \Delta_i, i, F_i)$ be two weighted automata over $\Sigma_V$ with $Q_1 \cap Q_2 = \emptyset$. We define the weighted automaton $A' = (Q', \Sigma_V, \Delta', \text{wt}', I', F')$ by letting

- $Q' = Q \times Q_2 \sqcup Q_1 \times Q_2$, $I' = \{i\} \times I_1 \cup \{i\} \times I_2$, $F' = F \times F_1 \sqcup G \times F_2$,
- $\Delta' = \{(p, p), a, (q, q)\} | (p, a, q) \in \Delta$ and $(p_i, a, q_i) \in \Delta_i$, for $i = 1, 2$, and
- $\text{wt}'((p, p), a, (q, q)) = \text{wt}_i((p_i, a, q_i))$ for $i = 1, 2$.

We also denote $A'$ by $A_{\varphi, V} \times A_1 \sqcup A_{\varphi, V} \times A_2$. Then we have:

**Lemma 17.** For each $\pi \in \Sigma_V^+$, we have

- $\langle A' \rangle(\pi) = \langle A_1 \rangle(\pi)$, if $\pi$ is valid and $\pi \models \varphi$,
- $\langle A_2 \rangle(\pi)$, if $\pi$ is valid and $\pi \not\models \varphi$,
- $\emptyset$, if $\pi$ is not valid.

Moreover, if $A_1$ and $A_2$ are aperiodic (resp. unambiguous, SCC-unambiguous) then so is $A'$. 

Proof. The first part is immediate by the construction of \( A' \) and Theorem 15. For the final statement, we can argue as for Lemma 16. For the unambiguity part observe that the sets \( F \) and \( G \) of \( A_{\Sigma, \nu} \) are disjoint.

Let \( \mathcal{V} \) be a finite set of first-order variables and let \( \mathcal{V}' = \mathcal{V} \cup \{ y \} \) where \( y \notin \mathcal{V} \). Given a word \( \overline{w} \in \Sigma^+ \) and a position \( i \in \text{pos}(w) \), we denote by \((\overline{w}, y \mapsto i)\) the word over \( \Sigma_{\mathcal{V}'} \) whose projection on \( \Sigma_{\mathcal{V}} \) is \( \overline{w} \) and projection on the \( y \)-component is \( 0^{i-1}10^{\nu_i-1} \), i.e., has a unique \( 1 \) on position \( i \). Given a function \( A : \Sigma_{\mathcal{V}'}^+ \to \mathbb{N}(X) \), we define the function \( \sum_y A : \Sigma_{\mathcal{V}'}^+ \to \mathbb{N}(X) \) for \( \overline{w} \in \Sigma_{\mathcal{V}'}^+ \) by

\[
(\sum_y A)(\overline{w}) = \bigcup_{i \in \text{pos}(w)} A(\overline{w}, y \mapsto i).
\]

Lemma 18. Let \( A \) be a weighted automaton over \( \Sigma_{\mathcal{V}'} \). We can construct a weighted automaton \( A' \) over \( \Sigma_\mathcal{V} \) such that \( \|A'\| = \sum_y \|A\| \). Moreover,

1. If \( A \) is aperiodic then \( A' \) is also aperiodic.
2. If \( A \) is SCC-unambiguous then \( A' \) is also SCC-unambiguous.

Proof. Let \( A = (Q, \Sigma_{\mathcal{V}'}, \Delta, \gamma, I, F) \). We construct \( A' = (Q', \Sigma_{\mathcal{V}}, \Delta', \gamma', I', F') \) as follows:

\[
Q' = Q \times \{0, 1\}, \quad I' = I \times \{0\}, \quad F' = F \times \{1\}
\]

and for \( \overline{w} \in \Sigma_{\mathcal{V}'} \) the transitions and weights are given by:

\[
\begin{align*}
\delta &= (p, (\pi, 0), q) \Delta \triangleq \delta_0 = ((p, 0), \pi, (q, 0)) \Delta' = ((p, 1), \pi, (q, 1)) \Delta' \text{ and} \\
\gamma'(\delta_0) &= \gamma'(\delta_1) = \gamma(\delta) \quad \text{wt}'.
\end{align*}
\]

Claim 19. We have \( \|A'\| = \sum_y \|A\| \).

Consider a word \( \overline{w} \in \Sigma_{\mathcal{V}'}^+ \) and let \( i \in \text{pos}(w) \). It is easy to see that there is a bijection between the accepting runs \( \rho \) of \( A \) on \((\overline{w}, y \mapsto i)\) and the accepting runs \( \rho' \) of \( A' \) on \( \overline{w} \) and switching from \( Q \times \{0\} \) to \( Q \times \{1\} \) on the \( i \)-th transition. Moreover, this bijection preserves the weight sequences: \( \gamma'(\rho') = \gamma(\rho) \). We deduce easily that \( \|A'\| = (\sum_y \|A\|)(\overline{w}) \).

Claim 20. If \( A \) is aperiodic then \( A' \) is also aperiodic.

Assume that \( m \) is an aperiodicity index of \( A \). We claim that \( m' = 2m \) is an aperiodicity index of \( A' \). Let \( \overline{w} \in \Sigma_{\mathcal{V}'}^+ \), let \( k \geq m' \) and \( \rho' \) be a run of \( A' \) reading \( \overline{w}^m \) from some state \( (p, b) \) to some state \( (r, c) \). We distinguish two cases. Either there is a prefix \( \rho'_1 \) of \( \rho' \) reading \( \overline{w}^m \) and staying in \( Q \times \{0\} \), i.e., \( \rho'_1 \) goes from \( (p, b) \) to \( (p, 0) \) to some \( (q, 0) \). We deduce that there is a run \( \rho_1 \) of \( A \) from \( p \) to \( q \) and reading \( (\overline{w}, 0)^m \) (recall that we denote by \( (\overline{w}, 0)^m \) the word over \( \Sigma_{\mathcal{V}'} \) whose projection on \( \Sigma_{\mathcal{V}} \) is \( \overline{w} \) and projection on the last component belongs to \( 0^+ \)). Since \( m \) is an aperiodicity index of \( A \) there is another run \( \rho_2 \) of \( A \) from \( p \) to \( q \) reading \( (\overline{w}, 0)^{m+1} \). We obtain a run \( \rho'_2 \) of \( A' \) from \( (p, 0) \) to \( (q, 0) \) reading \( \overline{w}^m \). Now, replacing the prefix \( \rho'_1 \) of \( \rho' \) by \( \rho'_2 \) we obtain a new run \( \rho'' \) of \( A' \) reading \( \overline{w}^{m+1} \) from state \( (p, 0) \) to \( (r, c) \). In the second case, there is a suffix \( \rho'_1 \) of \( \rho' \) reading \( \overline{w}^m \) from some state \( (q, 1) \) to \( (r, c) \). We can construct as above another run \( \rho'' \) from \( (q, 1) \) to \( (r, 1) \) reading \( \overline{w}^{m+1} \). Replacing the suffix \( \rho'_1 \) of \( \rho' \) by \( \rho'_2 \), we obtain the run \( \rho'' \) from \( (p, b) \) to \( (r, c) \) reading \( \overline{w}^{k+1} \). Finally, when \( k > m' = 2m \), a similar argument allows to construct a run \( \rho'' \) from \( (p, b) \) to \( (r, c) \) reading \( \overline{w}^{k-1} \).

Claim 21. If \( A \) is SCC-unambiguous then \( A' \) is also SCC-unambiguous.

Let \( \overline{w} \in \Sigma_{\mathcal{V}'}^+ \) and let \( (p, b) \approx' (q, c) \) be two states of \( Q' \) which are in the same SCC of \( A' \). Then, \( b = c \) and \( p \approx q \) are in the same SCC of \( A \). Since \( b = c \), there is a bijection between
the runs of \(A'\) from \((p, b)\) to \((q, c)\) reading \(w\) and the runs of \(A\) from \(p\) to \(q\) reading \((\bar{w}, 0)\). Since \(A\) is SCC-unambiguous and \(p \approx q\), there is at most one run of \(A\) from \(p\) to \(q\) reading \((\bar{w}, 0)\). We deduce that there is at most one run of \(A'\) from \((p, b)\) to \((q, c)\) reading \(\bar{w}\).

We turn now to one of our main results: given a step-w\(\text{FO}\) formula \(\varphi\), we can construct a weighted automaton for \(\prod_x \Psi\) which is both aperiodic and unambiguous.

When weights are uninterpreted, a weighted automaton \(A = (Q, \Sigma, \Delta, \text{wt}, I, F)\) is a letter-to-letter transducer from its input alphabet \(\Sigma\) to the output alphabet \(R\). If in addition the input automaton is unambiguous, then we have a functional transducer. In the following lemma, we will construct such functional transducers using the boolean output alphabet \(B = \{0, 1\}\).

\begin{lemma}
Let \(V = \{y_1, \ldots, y_m\}\). Given an FO formula \(\varphi\) with free variables contained in \(V' = V \cup \{x\}\), we can construct a transducer \(B_{\varphi,V}\) from \(\Sigma_V\) to \(B\) which is aperiodic and unambiguous and such that for all words \(\bar{w} \in \Sigma_V^+\):
1. there is a (unique) accepting run of \(B_{\varphi,V}\) on the input word \(\bar{w}\) iff it is a valid encoding of a pair \((w, \sigma)\) where \(w \in \Sigma^+\) and \(\sigma: V \to \text{pos}(w)\) is a valuation,
2. and in this case, for all \(1 \leq i \leq |w|\), the \(i\)th bit of the output is 1 iff \(w, \sigma[x \mapsto i] \models \varphi\).
\end{lemma}

\textbf{Proof.} Notice that \(\Sigma_V = \Sigma_V \times B\) so letters in \(\Sigma_V\) are of the form \((\pi, 0)\) or \((\pi, 1)\) where \(\pi \in \Sigma_V\). Abusing the notations, when \(\pi \in \Sigma_V^+\), we write \((\pi, 0)\) to denote the word over \(\Sigma_V\) whose projection on \(\Sigma_V\) is \(\pi\) and projection on the \(x\)-component consists of 0’s only.

Consider the deterministic, complete and aperiodic automaton \(A_{\varphi,V} = (Q, \Sigma_V, \Delta, I', F, G)\) associated with \(\varphi\) by Theorem\[\text{[3]}\]. We also denote by \(\Delta\) the extension of the transition function to subsets of \(Q\). So we see the deterministic and complete transition relation both as a total function \(\Delta: Q \times \Sigma_V \to Q\) and \(\Delta: 2^Q \times \Sigma_V \to 2^Q\).

We construct now the transducer \(B_{\varphi,V} = (Q', \Sigma_V, \Delta', \text{wt}, I', F')\). The set of states is \(Q' = Q \times 2^Q \times 2^Q \times B\). The unique initial state is \(I' = (\iota, \emptyset, \emptyset, 0)\). The set of final states is \(F' = (Q \times 2^F \times 2^G \times B) \setminus \{I'\}\). Then, we define the following transitions:

\[
\delta = (p, X, Y, b, \bar{\pi}, (p', X', Y', 1)) \in \Delta' \text{ is a transition with weight } \text{wt}(\delta) = 1 \text{ if } \\
p' = \Delta(p, (\pi, 0)), X' = \Delta(X, (\pi, 0)) \cup \{(\Delta(p, (\pi, 1))\}) \text{ and } Y' = \Delta(Y, (\pi, 0)),
\]

\[
\delta = (p, X, Y, b, \bar{\pi}, (p', X', Y', 0)) \in \Delta' \text{ is a transition with weight } \text{wt}(\delta) = 0 \text{ if } \\
p' = \Delta(p, (\pi, 0)), X' = \Delta(X, (\pi, 0)) \text{ and } Y' = \Delta(Y, (\pi, 0)) \cup \{(\Delta(p, (\pi, 1))\}).
\]

Notice that, whenever we read a new input letter \(\pi \in \Sigma_V\), there is a non-deterministic choice. In the first case above, we guess that formula \(\varphi\) will hold on the input word when the valuation is extended by assigning \(x\) to the current position, whereas in the second case we guess that \(\varphi\) will not hold. The guess corresponds to the output of the transition, as required by the second condition of Lemma\[\text{[22]}\]. Now, we have to check that the guess is correct. For this, the first component of \(B_{\varphi,V}\) computes the state \(p = \Delta(I, (\pi, 0))\) reached by \(A_{\varphi,V}\) after reading \((\pi, 0)\) where \(\pi \in \Sigma_V^+\) is the current prefix of the input word. When reading the current letter \(\pi \in \Sigma_V\), the transducer adds the state \(\Delta(p, (\pi, 1)) = \Delta(I, (\pi, 0)(\pi, 1))\) either to the “positive” \(X\)-component or to the “negative” \(Y\)-component of its state, depending on its guess as explained above. Then, the transducer continues reading the suffix \(\pi \in \Sigma_V^+\) of the input word. It updates the \(X\) (resp. \(Y\))-component so that it contains the state \(q = \Delta(I, (\pi, 0)(\pi, 1))\) at the end of the run. Now, the acceptance condition allows us to check that the guess was correct.

1. If \(\bar{w} = \bar{w}_0 \bar{w}_1 \ldots \bar{w}_n\) is not a valid encoding of a pair \((w, \sigma)\) with \(w \in \Sigma^+\) and \(\sigma: V \to \text{pos}(w)\) then \(q \notin F \cup G\) and the run of the transducer is not accepting. Otherwise, let \(i \in \text{pos}(w)\) be the position where the guess was made.
2. If the guess was positive then $q$ belongs to the $X$-component and the accepting condition implies $q \in F$, which means by definition of $A_{\psi,Y}$ that $w, \sigma[x \mapsto i] = \psi$.
3. If the guess was negative then $q$ belongs to the $Y$-component and the accepting condition implies $q \in G$, which means by definition of $A_{\psi,Y}$ that $w, \sigma[x \mapsto i] \neq \psi$.

We continue the proof with several remarks.

First, since the automaton $A_{\psi,Y}$ is complete, after reading a nonempty input word $w \in \Sigma_Y^+$ the transducer cannot be back in its initial state $\iota' = (\iota, \emptyset, \emptyset, 0)$. This is because the second and third components of the state cannot both be empty. Since $\iota' \notin F'$, the support of the transducer consists of nonempty words only.

Second, consider a run of the transducer on some input word $w \in \Sigma_Y^+$ from its initial state $\iota'$ to some state $(p, X, Y, b)$. As explained above, one can check that $X \cup Y \subseteq F \cup G$ iff $w$ is a valid encoding of a pair $(w, \sigma)$. Therefore, the support of the transducer consists of valid encodings only.

Now, consider a valid encoding $w$ of a pair $(w, \sigma)$ and consider a run $\rho$ of $B_{\psi,Y}$ on $w$ from $\iota'$ to some state $(p, X, Y, b)$. This run is entirely determined by the sequence of guesses made at every position of the input word. As explained above, one can check that all guesses are correct iff $X \subseteq F$ and $Y \subseteq G$. Therefore, $B_{\psi,Y}$ admits a unique accepting run on $w$. This shows that the support of $B_{\psi,Y}$ is exactly the set of valid encodings, that the transducer is unambiguous, and that the last condition of the lemma holds, i.e., the $i$th bit of the output is $1$ iff $w, \sigma[x \mapsto i] = \psi$.

To complete the proof, it remains to show that $B_{\psi,Y}$ is aperiodic. Let $m \geq 1$ be an aperiodicity index of $A_{\psi,Y}$. We claim that $m' = 2m + Q|Q|$ is an aperiodicity index of $B_{\psi,Y}$. Let $\alpha = (p, X, Y, b)$ and $\alpha' = (p', X', Y', b')$ be two states of $B_{\psi,Y}$ and let $w \in \Sigma_Y^+$ be a nonempty word.

Assume first that there is a run $\rho$ of $B_{\psi,Y}$ from $\alpha$ to $\alpha'$ reading the input word $w^k$ with $k \geq 2m + 1$. We show that there is another run of $B_{\psi,Y}$ from $\alpha$ to $\alpha'$ reading the input word $w^{k+1}$. We split $\rho$ in three parts: $\rho = \rho_1 \rho_2 \rho_3$ where $\rho_1$ reads the prefix $w^m$, $\rho_2$ reads $w$ and $\rho_3$ reads the suffix $w^{k-m-1}$. Consider the intermediary states $\alpha_i = (q_i, X_i, Y_i, b_i)$ reached after $\rho_i$ $(1 \leq i \leq 3)$: $\alpha \xrightarrow{\rho_1} \alpha_1 \xrightarrow{\rho_2} \alpha_2 \xrightarrow{\rho_3} \alpha_3 = \alpha'$. Since $A_{\psi,Y}$ is deterministic with aperiodicity index $m$ we obtain $\Delta(p, (\psi, 0)^m) = \Delta(p, (\psi, 0)^{m+1}) = \Delta(p, (\psi, 0)^k)$. Therefore, $q_1 = q_2 = q_3 = p'$.

Notice that, by definition of the transitions of $B_{\psi,Y}$, a run is entirely determined by its starting state, its input word, and the sequence of choices which is indicated in the fourth component of the states. Let $\rho_2'$ be the run starting from $\alpha_2$, reading $w$ and following the same sequence of choices as $\rho_2$. Let $\alpha_2' = (q_2', X_2', Y_2', b_2')$ be the state reached after $\rho_2'$. Let also $\rho_3'$ be the run starting from $\alpha_2'$, reading $w^{k-m-1}$ and following the same sequence of choices as $\rho_3$. Let $\alpha_3' = (q_3', X_3', Y_3', b_3')$ be the state reached after $\rho_3'$. Thus, we obtain a run $\rho' = \alpha \xrightarrow{\rho_1} \alpha_1 \xrightarrow{\rho_2} \alpha_2 \xrightarrow{\rho_2'} \alpha_2' \xrightarrow{\rho_3} \alpha_3 \xrightarrow{\rho_3'} \alpha_3'$ reading the input word $w^{k+1}$. It remains to show that $\alpha_3' = \alpha_3$. As above, we have $q_3 = \Delta(p, (\psi, 0)^{k+1}) = \Delta(p, (\psi, 0)^k) = q_3$. Also, $b_3'$ stores the last choice of $\rho_3'$, which is the same as the last choice of $\rho_3$ stored in $b_3$ and we get $b_3' = b_3$. It remains to show that $X_3' = X_3$ and $Y_3' = Y_3$. To this end, we introduce yet another variant of the runs $\rho_2$ and $\rho_3$. Let $\rho_3''$ be the run starting from $(p', \emptyset, \emptyset, 0)$, reading $w$ and following the same sequence of choices as $\rho_2$. Let $\alpha_3'' = (q_3'', X_3'', Y_3'', b_3'')$ be the state reached after $\rho_3''$. It is easy to see that $q_3'' = q_2 = p'$ and $b_3'' = b_2$. Moreover, we have

$$X_2 = X_2'' \cup \Delta(X_1, (\psi, 0))$$
$$Y_2 = Y_2'' \cup \Delta(Y_1, (\psi, 0))$$

$$X_2' = X_2'' \cup \Delta(X_2, (\psi, 0))$$
$$Y_2' = Y_2'' \cup \Delta(Y_2, (\psi, 0))$$.
Similarly, let \( \rho_i' \) be the run starting from \( (p', \emptyset, \emptyset, 0) \), reading \( \overline{w}^{k-m-1} \) and following the same sequence of choices as \( \rho_i \). Let \( \alpha_i'' = (p'', X_2'', Y_3'', b_3) \) be the state reached after \( \rho_i'' \). We have

\[
X_3 = X_3'' \cup \Delta(X_2, (\overline{w}, 0)^{k-m-1}) \quad X_3'' = X_3'' \cup \Delta(X_2', (\overline{w}, 0)^{k-m-1}) \\
Y_3 = Y_3'' \cup \Delta(Y_2, (\overline{w}, 0)^{k-m-1}) \quad Y_3'' = Y_3'' \cup \Delta(Y_2', (\overline{w}, 0)^{k-m-1})
\]

Notice that \( k - m - 1 \geq m \), hence we get \( \Delta(X_2, (\overline{w}, 0)^{k-m-1}) = \Delta(X_2, (\overline{w}, 0)^{k-m}) \) from the aperiodicity of \( \mathcal{A}_\varphi, \nu \). Finally, using \( X_2'' \subseteq X_2 \), we obtain \( \Delta(X_2', (\overline{w}, 0)^{k-m-1}) = \Delta(X_2, (\overline{w}, 0)^{k-m-1}) \) and \( X_3'' = X_3 \). Similarly, we prove that \( Y_3'' = Y_3 \).

Conversely, we assume that there is a run \( \rho \) of \( \mathcal{B}_\varphi, \nu \) from \( \alpha \) to \( \alpha' \) reading the input word \( \overline{w}^k \) with \( k > m' = 2m + 2|Q| \). We show that there is another run \( \rho' \) of \( \mathcal{B}_\varphi, \nu \) from \( \alpha \) to \( \alpha' \) reading the input word \( \overline{w}^{k-1} \). We split \( \rho \) in \( 2|Q| + 3 \) parts: \( \rho = \rho_0 \rho_1 \cdots \rho_{2|Q|+1} \rho_{2|Q|+2} \) where \( \rho_0 \) reads the prefix \( \overline{w}^{2|Q|-m-1} \), each \( \rho_i \) with \( 1 \leq i \leq 2|Q| + 1 \) reads \( \overline{w} \), and \( \rho_{2|Q|+2} \) reads the suffix \( \overline{w}^m \).

Consider the intermediate states \( \alpha_i = (q_i, X_i, Y_i, b_i) \) reached after \( \rho_i \) (\( 0 \leq i \leq 2|Q| + 2 \)). We have

\[
\alpha \xrightarrow{\rho_0} \alpha_0 \xrightarrow{\alpha_1 \cdots \alpha_{2|Q|+1}} \alpha_{2|Q|+2} = \alpha'.
\]

Since \( k - 2|Q| - m - 1 \geq m \) and \( \mathcal{A}_\varphi, \nu \) is deterministic with aperiodicity index \( m \), we deduce that \( q_0 = q \) if \( \alpha_i = \alpha_0 \cdots \alpha_{2|Q|+1} \rho_{2|Q|+2} \alpha_{2|Q|+2} = \alpha' \). As in the previous part of the aperiodicity proof, for each \( 1 \leq i \leq 2|Q| + 2 \), we consider the run \( \rho_i' \) starting from \( (p', \emptyset, \emptyset, 0) \), reading the same input word as \( \rho_i \) and making the same sequence of choices as \( \rho_i \). Let \( \alpha_i' = (p', X_i', Y_i', b_i) \) be the state reached after \( \rho_i' \) (\( 1 \leq i \leq 2|Q| + 2 \)). We have, for all \( 1 \leq i \leq 2|Q| + 1 \):

\[
X_i = X_i' \cup \Delta(X_i-1, (\overline{w}, 0)) \quad X_{2|Q|+2} = X_{2|Q|+2} \cup \Delta(X_{2|Q|+1}, (\overline{w}, 0)^m) \\
Y_i = Y_i' \cup \Delta(Y_i-1, (\overline{w}, 0)) \quad Y_{2|Q|+2} = Y_{2|Q|+2} \cup \Delta(Y_{2|Q|+1}, (\overline{w}, 0)^m).
\]

The states in \( X' = X_{2|Q|+2} \) and \( Y' = Y_{2|Q|+2} \) originate from the initial sets \( X_0 \) and \( Y_0 \) and from the sets \( X_i' \) and \( Y_i' \) created by the subruns \( \rho_i \) (\( 1 \leq i \leq 2|Q| + 2 \)). Intuitively, there is at least one index \( 1 \leq i \leq 2|Q| + 1 \) such that the contribution of \( \rho_i \) is subsumed by other subruns (formal proof below). Removing the subrun \( \rho_i \) yields the desired run \( \rho' \) of \( \mathcal{B}_\varphi, \nu \) from \( \alpha \) to \( \alpha' \) reading the input word \( \overline{w}^{k-1} \) (formal proof below).

For \( 0 \leq i \leq 2|Q| + 1 \), we let \( k_i = 2|Q| + 1 - i + m \). For \( 1 \leq i \leq 2|Q| + 2 \), we define by descending induction on \( i \) the contributions \( X_i'' \) and \( Y_i'' \) to \( X' = X_{2|Q|+2} \) and \( Y' = Y_{2|Q|+2} \) which originate from subruns \( \rho_j \) with \( j \geq i \):

\[
X_{2|Q|+2}'' = X_{2|Q|+2}' \\
Y_{2|Q|+2}'' = Y_{2|Q|+2}'.
\]

We deduce easily that for all \( 1 \leq i \leq 2|Q| + 2 \) we have

\[
X_{2|Q|+2} = X_i'' \cup \Delta(X_{i-1}, (\overline{w}, 0)^{k_i-1}) \quad Y_{2|Q|+2} = Y_i'' \cup \Delta(Y_{i-1}, (\overline{w}, 0)^{k_i-1}).
\]

Let \( 1 \leq i \leq 2|Q| + 1 \) be such that \( X_i'' = X_{i+1}'' \) and \( Y_i'' = Y_{i+1}'' \). Using the monotonicity of the sequences, it is easy to see that such an index \( i \) must exist. We show that we can remove the subrun \( \rho_i \). Let \( \rho'' \) be the run from \( \alpha_{i-1} \) (not \( \alpha_i \)) which reads \( \overline{w}^{k_i} \) and makes the same sequence of choices as \( \rho_{i+1} \cdots \rho_{2|Q|+2} \). Let \( \alpha'' = (q''', X''', Y''', b''') \) be the state reached after \( \rho'' \). It is easy to see that \( q''' = q_{2|Q|+2} = p' \) and \( b''' = b_{2|Q|+2} = b' \). We show that \( X'' = X_{2|Q|+2} \) and \( Y'' \) make the same sequence of choices as \( \rho_{i+1} \cdots \rho_{2|Q|+2} \), we see that the contribution to \( X'' \) coming from \( \rho'' \) is exactly \( X_{i+1}'' \). Therefore,

\[
X'' = X_{i+1}'' \cup \Delta(X_{i-1}, (\overline{w}, 0)^{k_i}) = X_i'' \cup \Delta(X_{i-1}, (\overline{w}, 0)^{k_i-1}) = X_{2|Q|+2} = X'.
\]
where the second equality follows from the hypothesis \(X_{i'} = X_{i+1}'\) and the aperiodicity of \(A_{\Sigma, \cal V}\) with index \(m\) since \(k_{i-1} = k_i + 1 > m\). Similarly, we can prove that \(Y' = Y''\) and we obtain \(\alpha'' = \alpha'\). Therefore, \(\rho' = \rho_0 \cdots \rho_{i-1} \rho''\) is the desired run of \(B_{\varphi, \cal V}\) from \(a\) to \(a'\) reading the input word \(\overline{w}^{n-1}\). This concludes the proof of aperiodicity of \(B_{\varphi, \cal V}\) with index \(m' = 2|Q| + 2m\).

\[\Box\]

**Theorem 23.** Let \(\cal V = \{y_1, \ldots, y_m\}\). Given a step-wFO formula \(\varphi\) with free variables contained in \(\cal V' = \cal V \cup \{x\}\), we can construct a weighted automaton \(A_{\varphi, \cal V}\) over \(\Sigma_{\cal V}\) which is aperiodic and unambiguous and which is equivalent to \(\prod_{\cal V} \varphi\), i.e., such that \(\langle A_{\varphi, \cal V}, \overline{w} \rangle = \prod_{\cal V} \varphi(\overline{w})\) for all words \(\overline{w}\).

**Proof.** In case \(\varphi = r\) is an atomic step-wFO formula, we replace it with the equivalent \(T ? r : r\) step-wFO formula. Let \(\varphi_1, \ldots, \varphi_k\) be the FO formulas occurring in \(\varphi\). By the above remark, we have \(k \geq 1\). Consider the aperiodic and unambiguous transducers \(B_1, \ldots, B_k\) given by Lemma 22. For \(1 \leq i \leq k\), we let \(B_i = (Q_i, \Sigma_{\cal V}, \Delta_i, \text{wt}_i, I_i, F_i)\). The weighted automaton \(A_{\varphi, \cal V}\) is essentially a cartesian product of the transducers \(B_i\). More precisely, we let \(Q = \prod_{i=1}^k Q_i\), \(I = \prod_{i=1}^k I_i\), \(F = \prod_{i=1}^k F_i\), and

\[\Delta = \{(p_1, \ldots, p_k, \overline{\pi}, (q_1, \ldots, q_k)) \mid (p_i, \overline{\pi}, q_i) \in \Delta_i \text{ for all } 1 \leq i \leq k\}.\]

Since the transducers \(B_i\) are all aperiodic and unambiguous, we deduce by Lemma 16 that \(A_{\varphi, \cal V}\) is also aperiodic and unambiguous. It remains to define the weight function \(\text{wt}\).

Given a bit vector \(\overline{b} = (b_1, \ldots, b_k) \in \mathbb{B}^k\) of size \(k\), we define \(\Psi(\overline{b})\) as the weight from \(R\) resulting from the step-wFO formula \(\varphi\) when the FO conditions \(\varphi_1, \ldots, \varphi_k\) evaluate to \(\overline{b}\).

Formally, the definition is by structural induction on the step-wFO formula:

\[r(\overline{b}) = \begin{cases} \varphi_i : \Psi_2(\overline{b}) & \text{if } b_i = 1 \\ \Psi_1(\overline{b}) & \text{if } b_i = 0 \end{cases} \]

Consider a transition \(\delta = ((p_1, \ldots, p_k), \overline{\pi}, (q_1, \ldots, q_k)) \in \Delta\) and let \(\delta_i = (p_i, \overline{\pi}, q_i)\) for \(1 \leq i \leq k\). Let \(\overline{b} = (b_1, \ldots, b_k) \in \mathbb{B}^k\) where \(b_i = \text{wt}(\delta_i) \in \mathbb{B}\) for all \(1 \leq i \leq k\). We define \(\text{wt}(\delta) = \Psi(\overline{b})\).

Let \(\overline{w} \in \Sigma_{\cal V}^+\). If \(\overline{w}\) is not a valid encoding of a pair \((w, \sigma)\) then \(\prod_{\cal V} \Psi(\overline{w}) = \emptyset\) by definition. Moreover, \(\prod_{\cal V} \Psi(\overline{w}) = \emptyset\) since by Lemma 22 \(\overline{w}\) is not in the support of \(B_1\). We assume below that \(\overline{w}\) is a valid encoding of a pair \((w, \sigma)\) where \(w \in \Sigma_{\cal V}^+\) and \(\sigma : \cal V \rightarrow \text{pos}(w)\) is a valuation. Then, each transducer \(B_i\) admits a unique accepting run reading \(\overline{w}\) from the unique accepting run \(\rho\) of \(A_{\varphi, \cal V}\) reading \(\overline{w}\). The projections of \(\rho\) on \(B_1, \ldots, B_k\) are \(\rho_1, \ldots, \rho_k\). Let \(j \in \text{pos}(w) = \{1, \ldots, |w|\}\) be a position in \(\overline{w}\) and let \(\delta_j\) be the \(j\)-th transition of \(\rho\). For \(1 \leq i \leq k\), we denote by \(\delta_{ij}\) the projection of \(\delta_j\) on \(B_i\) and we let \(b_i = \text{wt}(\delta_{ij})\). By Lemma 22 we get \(b_i = 1\) if \(w, \sigma[x \rightarrow j] = \varphi_i\). Finally, let \(\overline{b}' = (b_1', \ldots, b_k')\).

From the above, we deduce that \(\Psi(\overline{w}) = \Psi(\overline{b}') = \text{wt}(\delta)\). Putting things together, we have

\[\prod_{\cal V} \Psi(\overline{w}) = \text{wt}(\rho) = \text{wt}(\rho) = \prod_{\cal V} \Psi(\overline{w}) = \prod_{\cal V} \Psi(\overline{w}).\]

\[\Box\]

**Theorem 24.** Let \(\Phi\) be a core-wFO sentence. We can construct an aperiodic SCC-unambiguous weighted automaton \(A\) such that \(\prod_{\cal V} \Phi = \prod_{\cal V} \Phi\). Moreover, if \(\Phi\) does not contain the sum operations \(+\) and \(\sum_{\cal V}\), then \(A\) can be chosen to be unambiguous. If \(\Phi\) does not contain the sum \(\sum_{\cal V}\), then we can construct \(A\) as a finite union of unambiguous weighted automata.
Proof. We proceed by structural induction on $\Phi$. For $\Phi = 0$ this is trivial. For $\Phi = \prod_x \Psi$ with a step-wFO formula $\Psi$, we obtain an aperiodic unambiguous weighted automaton $A$ by Theorem 23. For formulas $\varphi ? \Phi_1 : \Phi_2$, $\Phi_1 + \Phi_2$ and $\sum_x \Phi$, we apply Lemmas 17, 16 and 18 respectively.

In the proof of Theorem 24 we may obtain the final statement also as a consequence of the preceding one by the following observations which could be of independent interest. Let $\varphi$ be an FO-formula and $\Phi_1$, $\Phi_2$ two core-wFO formulas, each with free variables contained in $\mathcal{V}$. Then,

$$\|\varphi ? \Phi_1 : \Phi_2\|_\mathcal{V} = \|\varphi ? \Phi_1 : 0 + \neg \varphi ? \Phi_2 : 0\|_\mathcal{V},$$

$$\|\varphi ? \Phi_1 + \Phi_2 : 0\|_\mathcal{V} = \|\varphi ? \Phi_1 : 0 + \varphi ? \Phi_2 : 0\|_\mathcal{V}.$$

Hence, given a core-wFO sentence $\Phi$ not containing the sum operation $\sum_x$, we can rewrite $\Phi$ as a sum of 0, $\prod_x \Psi$ and if-then-else sentences of the form $\varphi ? \Phi' : 0$ where $\Phi'$ does not contain the sum operations + or $\sum_x$.

Proof of Theorem 11 Immediate by Theorem 13 Theorem 4 Corollary 11 and Theorem 24.

8 Examples

In this section, we give examples separating the classes of finitely, polynomially and exponentially aperiodic weighted automata for several weight structures including the semiring of natural numbers $\mathbb{N}_{+\times}$, the min-plus semiring $\mathbb{N}_{\min,+}$ and the max-plus semiring $\mathbb{N}_{\max,+}$.

Example 25. Let $\Sigma$ be any alphabet, $R$ a set of weights, and $A = (Q, \Sigma, \Delta, wt, I, F)$ any (possibly aperiodic) weighted automaton over $\Sigma$ and $R$ which is not polynomially ambiguous.

1. Since the size of the multisets $\{|A\}(w)$ is not polynomially bounded with respect to $|w|$, there can be no polynomially ambiguous weighted automaton $B$ with $\{|B\} = \{|A\}$.

2. Assume that $|\Delta| \leq |R|$ and all transitions of $A$ have different weights, and consider $A$ as a weighted automaton over the semiring $(\mathcal{P}_{\text{fin}}(R^\ast), \cup, \cdot, \emptyset, \{\varepsilon\})$, or, equivalently, as a non-deterministic transducer outputting the weights of the transitions. Again, there can be no polynomially ambiguous weighted automaton $B$ with $\{|A\} = \{|B\}$.

3. For each $q \in Q$ and $a \in \Sigma$, the transitions $\delta = (q, a, p) \in \Delta$ ($p \in Q$) are enumerated as $\delta_1, \ldots, \delta_m$ where $m$ is the degree of non-determinism for $q \in Q$ and $a \in \Sigma$. Then put $\text{wt}(\delta_i) = i$, and let $\mathcal{R}$ comprise all these numbers. In comparison to 2., $|\mathcal{R}|$ might be considerably smaller than $|\Delta|$. But, again, over the semiring $(\mathcal{P}_{\text{fin}}(R^\ast), \cup, \cdot, \emptyset, \{\varepsilon\})$ there is no polynomially ambiguous weighted automaton equivalent to $A$.

This shows that for suitable idempotent semirings and also for non-deterministic transducers, there are aperiodic weighted automata for which there is no equivalent polynomially ambiguous weighted automaton. Next we show that this is also the case for the max-plus semiring $\mathbb{N}_{\max,+}$.

Example 26. Let $\Sigma = \{a, b, c\}$ and consider the function $f_{\max} : \Sigma^* \rightarrow \mathbb{N}$ defined as follows. For a word $w = w_1 \cdots w_n$ with $w_0, \ldots, w_n \in \{a, b\}^*$, we let $f_{\max}(w) = \sum_{i=0}^n \max\{|w_{i,a}|, |w_{i,b}|\}$. Over the max-plus semiring $\mathbb{N}_{\max,+}$, this function is realized by the following automaton $A$: 

...
Notice that $A$ is aperiodic and not polynomially ambiguous. We show that $f_{\text{max}}$ cannot be realized over the max-plus semiring by a polynomially ambiguous and aperiodic weighted automaton.

Notice that a similar automaton was considered in [23], the only difference being that $c$-transitions have weight 1. It was shown that the corresponding series cannot be realized over $\mathbb{N}_{\text{max,+}}$ by a finitely ambiguous weighted automaton, be it aperiodic or not. Here we want to separate exponentially ambiguous from polynomially ambiguous. We prove this separation for aperiodic automata which makes some of the arguments in the proof simpler (essentially we have self-loops instead of cycles). The separation also holds if we drop aperiodicity.

Towards a contradiction, assume that there was a polynomially ambiguous and aperiodic weighted automaton $B = (Q, \Sigma, \Delta, wt, I, F)$ which realizes the function $f_{\text{max}}$. We assume $B$ to be trimmed. We start with some easy remarks.

1. If there is a cycle $p \xrightarrow{u^k} p$ in $B$ with $u \in \Sigma^+$ and $k \geq 1$ then $p \xrightarrow{u} p$.

Let $m \geq 1$ be the aperiodicity index of $B$. For $\ell k \geq m$ we have $u^{\ell k}, u^{\ell k+1} \in \mathcal{L}(B_{p,p})$.

Since $B$ is polynomially ambiguous, these cycles lie in some SCC which is unambiguous.

2. Consider a looping transition $\delta = (p, v, p)$ in $B$ with $v \in \Sigma$. Then, $\text{wt}(\delta) \in \{0,1\}$.

Since $B$ is trimmed, there is an accepting run $p \xrightarrow{u} p \xrightarrow{v} p_2$ with $|uv| \leq 2|Q|$. We deduce that for all $\ell \geq 0$ there is an accepting run reading $uv^\ell w$ with weight at least $\text{wt}(\delta) \cdot \ell$.

Since $f_{\text{max}}(uv^\ell w) \leq \ell + |uv|$, we deduce that $\text{wt}(\delta) \in \{0,1\}$.

3. If there is a path $p \xrightarrow{a} p \xrightarrow{q} b \xrightarrow{q}$ in $B$ with $v \in \{a,b\}^*$, then one of the two looping transitions has weight zero: $\text{wt}(p,a,p) = 0$ or $\text{wt}(q,b,q) = 0$.

Since $B$ is trimmed, there are two runs $p_1 \xrightarrow{u} p$ and $q \xrightarrow{w} p_2$ with $p_1 \in I$ initial, $p_2 \in F$ final and $|uv| \leq 2|Q|$. We deduce that for all $\ell \geq 0$ there is an accepting run reading $uv^\ell wb^\ell w$ with weight at least $\ell \cdot (\text{wt}(p,a,p) + \text{wt}(q,b,q))$.

Since $f_{\text{max}}(uv^\ell wb^\ell w) \leq \ell + |uv|$, we deduce that $\text{wt}(p,a,p) + \text{wt}(q,b,q) \leq 1$.

Let $n = |Q|$ be the number of states in $B$. We show below that for each $k \geq 1$, the word $w_k = a^n b^n (a^n b^n)^{k-1}$ admits at least $2^k$ accepting runs in $B$. This implies that $B$ is not polynomially ambiguous, a contradiction.

Let $M = \max(\text{wt}(\Delta))$ be the maximal weight used in $B$. Notice that $M \geq 1$. Fix $k \geq 1$ and let $N \geq 2knM$.

Let $u_0 = a^n b^n$ and $u_1 = a^n b^N$. For each word $x = x_1 \cdots x_k \in \{0,1\}^k$, define $w_x = u_{x_1} cu_{x_2} c \cdots cu_{x_k}$ and consider an accepting run $\rho_x$ of $B$ reading $w_x$ and realizing $f_{\text{max}}(w_x) = kN$. For each $1 \leq j \leq k$, we focus on the subrun $\rho_{x_j}$ of $\rho_x$ reading $w_{x_j}$.

Assume that $x_j = 0$. Using the remarks above, we deduce that the prefix of $\rho_{x_j}$ reading $a^N$ is of the form

\begin{equation}
\begin{array}{c}
p_1 \xrightarrow{a^{\ell_1}} p_1 \xrightarrow{a} p_2 \xrightarrow{a^{\ell_2}} p_2 \xrightarrow{a} \cdots \xrightarrow{a} p_m \xrightarrow{a^{\ell_m}} p_m
\end{array}
\end{equation}

where $p_1, \ldots, p_m$ are pairwise distinct and $N = m - 1 + \ell_1 + \cdots + \ell_m$. Since looping $a$-transitions have weights in $\{0,1\}$, we deduce that $\text{wt}(\rho_{x_j}) \leq N + (2n-1)M$. We claim that in $\rho_{x_j}$ some a-loop has weight 1. If this is not the case, then $\text{wt}(\rho_{x_j}) \leq (2n-1)M$. We deduce that $\text{wt}(\rho_{x_j}) \leq (k-1)(N + (2n-1)M) + (2n-1)M + (k-1)M = (k-1)N + (2nk-1)M$, but
wt(\rho_x) = kN = f_{\text{max}}(w_z)$, a contradiction with $N \geq 2knM$. Let $(p_1, a, p_3)$ be some $a$-loop of weight 1 in $\rho_x^a$. We replace the prefix of $\rho_x^a$ reading $a^N$ with

$$p_1 \xrightarrow{a^{n-i}} \rho_x^a \xrightarrow{a^{n-m+1}} p_1 \xrightarrow{a^{m-i}} p_m$$

3 to obtain a run $\hat{\rho}_x^a$ reading $a^nb^n$. The suffix of $\rho_x^a$ reading $b^n$ has a form similar to [3], having at least one $b$-loop since $n = |Q|$. From the third remark above, all $b$-loops in $\rho_x^a$ have weight 0. We deduce that $\hat{\rho}_x^a$ has one $a$-loop with weight 1 but all its $b$-loops have weight 0.

We proceed similarly when $x_j = 1$ defining a run $\hat{\rho}_x^a$ reading $a^nb^n$ where all $a$-loops have weight 0 and one $b$-loop has weight 1. Now, consider the run $\hat{\rho}_x$ obtained from $\rho_x$ by replacing $\rho_x^b$ with $\hat{\rho}_x^b$ for each $1 \leq j \leq k$. We see that $\hat{\rho}_x$ is an accepting run for $w_k$. Also, if $x, y \in \{0, 1\}^k$ are different then $\hat{\rho}_x \neq \hat{\rho}_y$. Therefore, $B$ has at least $2^k$ accepting runs reading $w_k$, which concludes the proof.

Example 27. Let $\Sigma = \{a, b, c\}$ and consider the function $f_{\text{min}}: \Sigma^* \rightarrow \mathbb{N}$ defined as follows. For a word $w = w_0cw_1c \ldots cw_n$ with $w_0, \ldots, w_n \in \{a, b\}^*$, we let $f_{\text{min}}(w) = \sum_{i=0}^n \min\{|w_i|_a, |w_i|_b\}$. Over the min-plus semiring $\mathbb{N}_{\text{min,+}}$, this function is realized by the automaton $A$ depicted in Example 26 which is aperiodic and not polynomially ambiguous. It was shown in [27] that in the min-plus semiring there is no polynomially ambiguous weighted automaton $B$ with $[A] = [B]$.

Example 28. Let $\Sigma = \{a\}$ and consider the following automaton $A$ over the semiring $\mathbb{N}_{+, \times}$ of natural numbers:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{a} & \bullet \\
\end{array}
\]

Note that the weighted automaton computes the sequence $(F_n)_{n \geq 0}$ of Fibonacci numbers $0, 1, 1, 2, 3, 5, \ldots$. More precisely, for any $n \in \mathbb{N}$, we have $[A](a^n) = F_n$.

Clearly, $A$ is exponentially ambiguous and aperiodic with index 2. We claim that there is no aperiodic polynomially ambiguous weighted automaton $B = (Q, \Sigma, \Delta, \text{wt}, I, F)$ with $[A] = [B]$. Suppose there was such a trimmed automaton $B$.

First, consider any loop $q \xrightarrow{a^k} q$ with $k \geq 1$ of $B$. Since $B$ is aperiodic and SCC-unambiguous, hence unambiguous on the component containing $q$, as in Example 26, it follows that $(q, a, q) \in \Delta$. Next, we claim $\alpha = \text{wt}(q, a, q) = 1$. Indeed, suppose that $\alpha \geq 2$. Choose $m, \ell \geq 2$ minimal such that there is a path reading $a^m$ from $I$ to $q$ and a path for $a^{\ell}$ from $q$ to $F$. Considering, for $n \geq m + \ell$, the path $\rho_n: I \xrightarrow{a^m} q \xrightarrow{a^{n-m-\ell}} q \xrightarrow{a^{\ell}} F$, we obtain $[B](a^n) \geq \text{wt}(\rho_n) \geq 2^{n-m-\ell}$. Since $F_n = o(2^n)$, for $n$ large enough, we get $F_n < 2^{-m-\ell} \cdot 2^n$, a contradiction.

So, in $B$ all loops have weight 1. Hence there exists $K \in \mathbb{N}$ such that $\text{wt}(\rho) \leq K$ for all paths $\rho$ in $B$. Consequently, if $B$ is polynomially ambiguous of degree $d$, we have $[B](a^n) \leq O(n^d)$ for $n \in \mathbb{N}$. This yields a contradiction since $F_n \sim \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n$ grows exponentially.

Next we wish to show that aperiodic polynomially ambiguous weighted automata are strictly more expressive than aperiodic finitely ambiguous weighted automata.

Example 29. Let $\Sigma$ be any alphabet, $\mathcal{R}$ a set of weights and $A$ an aperiodic polynomially ambiguous weighted automaton which is not finitely ambiguous. We may argue as in
Example 25 to show that there is no finitely ambiguous weighted automaton $B$ with $[A] = ([B])$, respectively, under the assumptions of Example 25 with $[A] = ([B])$ for the idempotent semiring $(\mathcal{P}_{\text{idem}}(\mathbb{R}^*), \cup, \emptyset, \{e\})$.

\begin{itemize}
  \item \textbf{Example 30.} Let $\Sigma = \{a, b\}$.
  \begin{enumerate}
    \item Consider the following weighted automaton $A$ over $\Sigma$ and $\mathbb{N}_{\text{min},+}$ from [22] p.558:
      \begin{align*}
      a & \mid 0 & a & \mid 0 \\
      b & \mid 0 & b & \mid 0 \\
      \end{align*}
      \begin{tikzpicture}[node distance=2cm,>=latex,auto]
        \node (q0) [state, initial] {$q_0$};
        \node (q1) [state, below of=q0] {$q_1$};
        \node (q2) [state, right of=q1] {$q_2$};
        \draw [->] (q0) edge node {$b$} (q1);
        \draw [->] (q1) edge node {$b$} (q2);
      \end{tikzpicture}

      \hspace{1cm}
      Here $[A](w)$ is the least $\ell \geq 0$ such that $ba^\ell b$ is a factor of $w$. If $w$ does not admit a factor of this form, than $[A](w) = \infty$. Clearly, $A$ is SCC-unambiguous and aperiodic, but, as shown in [22] Proposition 3.2, $A$ is not equivalent to any finitely ambiguous weighted automaton.
    \item Consider the following weighted automaton $A$ over $\Sigma$ and $\mathbb{N}_{\text{min},+}$ from [27]:
      \begin{align*}
      a & \mid 1 & a & \mid 0 \\
      b & \mid 0 & b & \mid 1 \\
      \end{align*}
      \begin{tikzpicture}[node distance=2cm,>=latex,auto]
        \node (q0) [state, initial] {$q_0$};
        \node (q1) [state, below of=q0] {$q_1$};
        \node (q2) [state, right of=q1] {$q_2$};
        \draw [->] (q0) edge node {$a$} (q1);
        \draw [->] (q1) edge node {$a$} (q2);
      \end{tikzpicture}

      \hspace{1cm}
      Then $[A](w) = \min\{|w|_a + |v|_b \mid w = uv\}$. Clearly, $A$ is aperiodic and polynomially ambiguous. As shown in [27], Example 15, as a consequence of a pumping lemma, $A$ is not equivalent to any finitely ambiguous weighted automaton.
  \end{enumerate}

Next we consider the max-plus semiring $\mathbb{N}_{\text{max},+}$.

\begin{itemize}
  \item \textbf{Example 31.} Consider the following automaton $A$ over $\Sigma = \{a, b\}$ and $\mathbb{N}_{\text{max},+}$.
    \begin{align*}
    a & \mid 1 & a & \mid 0 \\
    b & \mid 0 & b & \mid 1 \\
    \end{align*}
    \begin{tikzpicture}[node distance=2cm,>=latex,auto]
      \node (q0) [state, initial] {$q_0$};
      \node (q1) [state, below of=q0] {$q_1$};
      \node (q2) [state, right of=q1] {$q_2$};
      \draw [->] (q0) edge node {$a$} (q1);
      \draw [->] (q1) edge node {$a$} (q2);
    \end{tikzpicture}

    \hspace{1cm}
    Note that $A$ is almost identical to the automaton of Example 30 part 2. Now for $f = [A]$ we have $f(w) = \max\{|w|_a + |v|_b \mid w = uv\}$ for each $w \in \Sigma^*$. Clearly, $A$ is aperiodic and polynomially ambiguous. But no aperiodic finitely ambiguous weighted automaton is equivalent to $A$.

    Suppose there was such a trimmed aperiodic finitely ambiguous weighted automaton $B = (Q, \Sigma, \Delta, \omega, I, F)$ with $[B] = f$. We make the following observations on the structure of $B$.
  \item \textbf{Remark 1.} If $B$ contains a loop $q \xrightarrow{a^k} q$ for some $q \in Q$ and $k \geq 1$, then $t \in (q, a, q) \in \Delta$, and the loop is a sequence of this transition $t$.

    This follows from the fact that $B$ is aperiodic and unambiguous on the strong component containing $q$ (as in Example 26).
  \item \textbf{Remark 2.} $B$ cannot contain a path of the form $p \xrightarrow{a} p \xrightarrow{a^k} q \xrightarrow{a} q$ with $p \neq q$.

    Indeed, otherwise the word $a^{n+k}$ would have at least $n+1$ different paths from $p$ to $q$.

    Since $B$ is trimmed, this contradicts the finite ambiguity of $B$.
  \item \textbf{Remark 3.} If $(q, a, q) \in T$ and $a = \omega(q, a, q)$, then $a \in \{0, 1\}$.

    Indeed, let $u$ be the label of a path from $I$ to $q$ and $v$ the label of a path from $q$ to $F$. Let $w_n = ua^nv$. Then $f(w_n) \leq |uw| + n$, and $[B](w_n) \geq \alpha \cdot n$ for each $n \in \mathbb{N}$. This shows that $\alpha \leq 1$.
\end{itemize}
Remark 4. \( \mathcal{B} \) cannot contain a path of the form \( p \xrightarrow{b_1} p \xrightarrow{v} a \xrightarrow{a_1} q \) with \( v \in \Sigma^* \).

Indeed, otherwise let \( u \) be a label of a path from \( I \) to \( p \) and \( w \) the label of a path from \( q \) to \( F \). Consider \( w_n = ub^nv^aw^n (n \in \mathbb{N}) \). Then \( f(w_n) \leq |uvw| + n \) but \( |\mathcal{B}(w_n)| \geq 2n \), a contradiction for \( n > |uvw| \).

Lemma 32. Let \( m \geq |Q| \) and \( u,v \in \Sigma^* \). Then \( \mathcal{B} \) contains an accepting path for the word \( u\alpha a^mb^n v \) of the form

\[
\begin{array}{c}
 i & u & a^k_1 & p & a^k_2 b^k_3 & q & b^k_4 v & f \\
 a | \alpha & b | \beta \\
\end{array}
\]

with \( k_1, k_2, k_3, k_4 < |Q| \), and where the transition \((p,a,p)\) is taken \( n - k_1 - k_2 \) times and the transition \((q,b,q)\) is taken \( n - k_3 - k_4 \) times in \( \rho \).

By Remark 3 we have \( \alpha, \beta \in \{0,1\} \). Let \( \rho_{123} \) be the path obtained from \( \rho \) by deleting the loops at \( p \) and at \( q \): \( \rho_1 = i \xrightarrow{u_{a_k_1}} p \), \( \rho_2 = p \xrightarrow{a^{k_2}b^{k_3}} q \), and \( \rho_3 = q \xrightarrow{b^{k_4}} f \). Let \( c = \text{wt}(\rho_{123}) \). Then \( \text{wt}(\rho) \leq c + n \cdot \alpha + n \cdot \beta \).

But \( \text{wt}(\rho) = f(w_n) \geq 2n \). Since \( u,v \in \Sigma^* \) are fixed, there are only finitely many values \( c = \text{wt}(\rho_{123}) \in \mathbb{N} \) which can arise in \( \mathcal{B} \) as above with \( i,p,q,f \in Q \) and \( k_1, k_2, k_3, k_4 < |Q| \).

By choosing \( n \) larger than their maximum, we obtain a path for \( w_n = u\alpha a^mb^nv \) as above and now for this path it follows that \( \alpha = \beta = 1 \). By reducing the number of loops taken at \( p \) and at \( q \), we obtain an accepting path of the prescribed form for \( w_m = u\alpha a^mb^nv \), proving the lemma.

Now, let \( m \geq |Q| \) and consider the word \( w_K = (b^ma^m)^K \) \((K \in \mathbb{N}) \). For all \( 0 < k < K \) we can write \( w_K = u_k a^m b^m v_k \) with \( u_k = (b^ma^m)^{k-1}b^m \) and \( v_k = a^m (b^m a^m)^{K-k-1} \). We apply Lemma 32 to the word \( u_k a^m b^m v_k \) and obtain a path \( \rho_k \) of the form

\[
\begin{array}{c}
 u_k a^k_1 & a^k_2 b^k_3 & b^k_4 v_k \\
 a | 1 & b | 1 \\
\end{array}
\]

We claim that if \( 0 < k < k' < K \), then \( \rho_k \neq \rho_{k'} \). Indeed, if \( \rho_k = \rho_{k'} \), we see that the path \( \rho_{k'} \) must have the form

\[
\begin{array}{c}
 u_k a^k_1 & a^{k_2} b^{k_3} & b^{k_4} v_{k'} \\
 a | 1 & b | 1 \\
\end{array}
\]

contradicting Remark 4. Therefore \( \mathcal{B} \) contains at least \( K - 1 \) accepting paths for \( w_K \) \((K \in \mathbb{N}) \). This contradicts \( \mathcal{B} \) being finitely ambiguous.

We just note that by similar arguments and further analysing the weights of loops, it can be shown that \( \mathcal{A} \) is not equivalent to any finitely ambiguous weighted automaton, even if it is not aperiodic.
Example 33. Consider the following automaton $A$ over $\Sigma = \{a\}$ and $\mathbb{N}_{+, \times}$.

Clearly, $[A](a^n) = n$ for each $n > 0$, and $A$ is aperiodic and polynomially (even linearly) ambiguous. We claim that $A$ is not equivalent to any finitely ambiguous weighted automaton.

Towards a contradiction, suppose there was a trimmed finitely ambiguous weighted automaton $B$ with $[B] = [A]$.

Remark 34. Let $q \xrightarrow{a^n} q$ be a loop in $B$ with weight $\alpha$, where $m \geq 1$. Then $\alpha = 1$.

Indeed, choose a path in $B$ from $I$ to $q$ with label $u$ and a path from $q$ to $F$ with label $v$. Then $[B](u a^n v) \geq \alpha^n$, for each $n \in \mathbb{N}$. On the other hand, $f(u a^n v) = |uv| + m \cdot n$. Hence $\alpha \geq 2$ is impossible, showing $\alpha = 1$.

Consequently, in paths of $B$ we may remove all loops without changing the weight. Hence there is $C \in \mathbb{N}$ such that $\text{wt}(\rho) \leq C$ for each run $\rho$ of $B$. Since $B$ is finitely ambiguous, it follows that $\{[B](w) \mid w \in \Sigma^*\}$ is bounded. This contradicts $[B] = [A]$.

Conclusion

We introduced a model of aperiodic weighted automata and showed that a suitable concept of weighted first order logic and two natural sublogics have the same expressive power as polynomially ambiguous, finitely ambiguous, resp. unambiguous aperiodic weighted automata. For the three semirings $\mathbb{N}_{+, \times}$, $\mathbb{N}_{\max, +}$ and $\mathbb{N}_{\min, +}$ we showed that the hierarchies of these automata classes and thereby of the corresponding logics are strict. Our main result generalizes a classical result of automata theory into the weighted setting. The challenging open problem is to develop similar results for suitable weighted linear temporal logics.
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