A CLT FOR WEIGHTED TIME-DEPENDENT UNIFORM
EMPIRICAL PROCESSES

YUPING YANG

ABSTRACT. For a uniform process \( \{X_t : t \in E\} \) (by which \( X_t \) is uniformly distributed on \((0,1)\) for \( t \in E \)) and a function \( w(x) > 0 \) on \((0,1)\), we give a sufficient condition for the weak convergence of the empirical process based on \( \{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0,1]\} \) in \( \ell^\infty(E \times [0,1]) \). When specializing to \( w(x) \equiv 1 \) and assuming strict monotonicity on the marginal distribution functions of the input process, we recover a result of [9]. In the last section, we give an example of the main theorem.

1. Introduction

Given a sequence of independent uniform \((0,1)\) random variables \( X_1, X_2, \ldots, \) if let \( G_n(x) = n^{-1/2} \sum_{i=1}^n (1_{X_i \leq x} - x) \) be the uniform empirical process, then Donsker’s theorem ([1]) says \( G_n(x) \) converges weakly to the Brownian bridge process, \( B(x) \), on \([0,1]\). Weighted empirical processes consider suitable weight functions \( w(x) \) such that \( w(x)G_n(x) \) converges weakly to the weighted Brownian bridge process \( w(x)B(x) \); in the literature, such a theorem is called the Chibisov-O’Reilly theorem; see [2], [12], [3] etc. [9] considered a time dependent empirical process \( G_n(t,y) := n^{-1/2} n \sum_{i=1}^n w(y)(1_{Y_i(t) \leq y} - P(Y_i(t) \leq y)), \ t \in E, \ y \in \mathbb{R}, \) for independent and identically distributed (iid) stochastic processes \( Y_1(t), Y_2(t), \ldots \) for \( t \in E \). Under a condition the authors call the L-condition, this empirical process converges weakly in \( \ell^\infty(E \times \mathbb{R}) \). In [5], the authors proved a CLT for weighted tail empirical processes under a small oscillation condition as the L-condition guarantees.

We consider a time dependent weighted uniform empirical process. For a process \( X(t) \) for \( t \in E \) and a “weight function” \( w(x) \) on \((0,1)\), we are interested in conditions on the process and the weight function so that the empirical process \( \nu_n(t,y) := n^{-1/2} \sum_{i=1}^n w(y)(1_{X_i(t) \leq y} - y)), \ t \in E, \ y \in [0,1], \) where \( X(t), X_1(t), X_2(t), \ldots \) are iid, converges weakly in \( \ell^\infty(E \times [0,1]) \). We give a sufficient condition in Section 3 for a Central limit theorem (CLT) for this empirical process.

This paper is organized as following. In Section 2, we give some definitions and results about weak convergence (CLT) for empirical processes. Section 3 contains

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the main result. The proof is to use Theorem 4.4 in [1]. In particular, the pre-
Gaussian condition and the local modulus condition are to be checked under the
assumptions. An example of the main theorem is given at the last section.

2. Preliminaries

Given a centered stochastic process \( \{X(t) : t \in T\} \), we define the empirical
process based on it by

\[
\nu_n(t) := n^{-1/2} \sum_{j=1}^{n} X_j(t), \quad t \in T,
\]

(2.1)

where \( \{X_j(t) : t \in T\} \) for \( j = 1, 2, \cdots \) are independent and identically distributed
as \( \{X(t) : t \in T\} \).

On a probability space \((\Omega, \mathcal{A}, P)\), recall the outer expectation of an arbitrary
function \( f : \Omega \to \mathbb{R} \)

\[
E^*(f) := \inf \{ Eg : g \geq f, g \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R})) \text{ measurable} \}.
\]

**Definition 2.1.** Let \( X := \{X(t) : t \in T\} \) be a centered stochastic process on a
parameter set \( T \), and sample paths in \( \ell^\infty(T) \). Assume \( E|X(t)|^2 < \infty \) for \( t \in T \).
The empirical process based on \( X \), \( \nu_n(t) \) in (2.1), satisfies the central limit theorem,
– for short \( X \in \text{CLT} \) – if there exists a centered Radon measure \( \gamma \) on \( \ell^\infty(T) \) such
that for all \( H : \ell^\infty(T) \to \mathbb{R} \) bounded and continuous, we have

\[
\lim_{n \to \infty} E^*(H(\nu_n)) = \int_{\ell^\infty(T)} H \, d\gamma.
\]

**Definition 2.2.** A centered stochastic process \( \{X_t : t \in T\} \) is pregaussian if
its covariance coincides with the covariance of a centered Gaussian process \( G \) on
\( T \) with bounded and uniformly \( d_G \)-continuous sample paths, where \( d_G(s,t) := (E(G(s) - G(t))^2)^{1/2} \).

**Theorem 2.3** (cf. [9], Proposition 1). Let \( H_1 \) and \( H_2 \) be zero mean Gaussian
processes with \( L_2 \) distances \( d_{H_1}, d_{H_2} \), respectively, on \( T \). Furthermore, assume \( T \)
is countable, and \( d_{H_1}(s,t) \leq d_{H_2}(s,t) \) for all \( s, t \in T \). Then, \( H_2 \) sample bounded and uniformly continuous
on \((T,d_{H_2})\) with probability one, implies \( H_1 \) is sample bounded and uniformly continuous
on \((T,d_{H_1})\) with probability one.

When \( T = [0,1] \), this is Lemma 2.1 in [11].

The assumption that \( T \) is countable can be removed if \( T \) is given a totally
bounded metric.

**Lemma 2.4.** Let \( \{G(t) : t \in T\} \) be a zero mean Gaussian process. Further assume
\( \sup_{t \in T} EG(t)^2 < \infty \). Let \( d_G(s,t) := (E(G(s) - G(t))^2)^{1/2} \). Then, if \( T_0 \) is a dense
set in \((T,d_G)\) and the restricted process \( \{G(t) : t \in T_0\} \) is sample bounded and uniformly \( d_G \)-continuous, then \( \{G(t) : t \in T\} \) has a version with bounded and uniformly \( d_G \)-continuous sample paths.

The proof of this lemma is given in the appendix.

We will use the following theorem to prove our main result.

**Theorem 2.5** ([1], Theorem 4.4). Let \( \{X(t) : t \in T\} \) be a sample bounded process
on a set \( T \) such that \( EX(t) = 0 \) and \( EX(t)^2 < \infty \) for all \( t \in T \). Assume:

(i) \( u^2 P^*[\|X\|_\infty > u] \to 0 \) as \( u \to \infty \),
(ii) $X$ is pregaussian, and
type there is pseudometric $\rho$ on $T$ dominated by the pseudometric $d_G$ corresponding to a centered Gaussian process $G$ on $T$ with bounded and uniformly $d_G$-continuous paths such that for some $K$ and for all $t \in T$ and $\varepsilon > 0$,
\[ \sup_{u > 0} u^2 P^*(\sup_{s \in B_\rho(t, \varepsilon)} |X(t) - X(s)| > u) \leq K\varepsilon^2. \]

Then $X \in \text{CLT}$ as a $\ell^\infty(T)$-valued random element.

**Definition 2.6.** Let $F(x)$ be a distribution function (df) on $\mathbb{R}$. The (randomized) distributional transform of $F(x)$ as defined in [13] is
\[ \tilde{F}(x) := \tilde{F}(x, V) := F(x-) + (F(x) - F(x-))V, \]
where $V$ is a uniform random variable on $[0, 1]$.

Next we give some simple properties of the distributional transform.

**Lemma 2.7.** (i) $\tilde{F}(x) \leq F(x)$ for all $x \in \mathbb{R}$.

(ii) If $x < y$, then $F(x) \leq \tilde{F}(y)$.

(iii) If $x \leq y$, then $\tilde{F}(x) \leq \tilde{F}(y)$.

(iv) If $x < y$ and $F(\cdot)$ is strictly increasing, then $F(x) < \tilde{F}(y)$.

**Proof.** By definition, (i) is obvious. For (ii), take $x < z < y$, hence $F(x) \leq F(z)$.
Since $F(z) \leq F(y)$ and $F(y-) \leq \tilde{F}(y)$, hence $F(x) \leq \tilde{F}(y)$. For (iii), if $x = y$, there is nothing to prove; assume $x < y$. By (i) and (ii), we get (iii). For (iv), take $x < z < y$. Since $F(\cdot)$ is strictly increasing, $F(x) < F(z)$. But by (iii), $F(z) \leq \tilde{F}(y)$. Hence $F(x) < \tilde{F}(y)$. \qed

For a continuous df $F$ of a random variable $X$, the random variable $F(X)$ is uniform on $[0, 1]$; but for a general df $F$, this might not be the case. However using the (randomized) distributional transform overcomes this.

**Lemma 2.8.** If $F(x)$ is the distribution function of a random variable $X$, then $\tilde{F}(x) := \tilde{F}(X, V)$ is uniform on $[0, 1]$. Here $V$ is a uniform random variable on $[0, 1]$ independent of $X$.

**Proof.** For a proof, see [13]. \qed

**Definition 2.9.** We say a (pseudo) distance $\rho$ on a set $T$ is a continuous Gaussian distance if there is a zero mean Gaussian process $\{G(t) : t \in T\}$ with bounded and uniformly $d_G$-continuous sample paths where $d_G(s, t) := (E(G(s) - G(t))^2)^{1/2}$ and $\rho(s, t) = d_G(s, t)$ for all $s, t \in T$.

For notation, we write $X$ for a process $\{X(t) : t \in E\}$ and $X_t$ for $X(t)$. We recall from [9]

**Definition 2.10 (L-condition for a stochastic process).** Let $X := \{X_t : t \in E\}$ be a stochastic process. The process $X$ satisfies the L-condition if there exists a continuous Gaussian distance $\rho$ on $E$ such that for every $\varepsilon > 0$
\[ \sup_{t \in E} P^*(\sup_{s \in B_\rho(s, t) \leq \varepsilon} |\tilde{F}_t(X_t) - \tilde{F}_t(X_s)| > \varepsilon^2) \leq L\varepsilon^2, \]
where $\tilde{F}_t(\cdot)$ is the distributional transform of the distribution function $F_t(\cdot)$ of $X_t$. 

Theorem 2.11 ([9], Theorem 3). Let $X(t)$ be a process on $E$. Let $\rho$ be given by $\rho(s, t)^2 = E(H(s) - H(t))^2$, for some centered Gaussian process $H$ that is sample bounded and uniformly continuous on $(E, \rho)$ with probability one. Further, assume that for some $L < \infty$, and all $\varepsilon > 0$, the L-condition holds for $X$, and $D(E)$ is a collection of real valued functions on $E$ such that $P(X(\cdot) \in D(E)) = 1$. If

$$ C = \{C_{s, x} : s \in E, x \in \mathbb{R}\}, $$

where

$$ C_{s, x} = \{z \in D(E) : z(s) \leq x\} $$

for $s \in E$, $x \in \mathbb{R}$, then $C \in \text{CLT}(X)$.

In this case, we say the empirical process based on $\{1_{Y(t) \leq y} - P(Y(t) \leq y) : t \in E, y \in \mathbb{R}\}$ satisfies the CLT or write $C \in \text{CLT}(X)$ in $\ell^\infty(E \times \mathbb{R})$.

3. Weak convergence of the time dependent weighted empirical process

In view of Theorem 2.11 and the classical weighted empirical process, a natural question is to consider the time dependent weighted (uniform) empirical process,

$$ \alpha_n(t, y) := n^{-1/2} \sum_{i \leq n} w(y)(1_{X_i(t) \leq y} - y), t \in E, y \in [0, 1] $$

where $\{X(t), X_1(t), X_2(t), \cdots \}$ are iid uniform processes (see the definition below). Under the WL-condition (below) and some regularity conditions on the weight function $w(\cdot)$, we prove a CLT for the empirical process $\alpha_n$.

Definition 3.1. We call a process $X = \{X(t) : t \in E\}$ a uniform process if for each $t \in E$, $X(t)$ is uniformly distributed on (0, 1).

We call the main condition in our theorem the WL-condition.

Definition 3.2. [WL-condition for $(X; w)$] Given a uniform process $X := \{X_t : t \in E\}$ and a function $w := w(x) > 0$ on (0, 1), we say $(X; w)$ satisfies the WL-condition if for some constant $L$ (depending on $w$, but not on $x$), some continuous Gaussian distance $\rho$ on $E$ and all $\varepsilon > 0$, $0 < x < 1$, we have

$$ \sup_t P^*(\sup_{s, p(s, t) \leq x} 1_{X_t \leq x < X_s > 0}) \leq \frac{L\varepsilon^2}{w(x)^2} $$

$$ \sup_t P^*(\sup_{s, p(s, t) \leq x} 1_{X_s \leq x < X_t > 0}) \leq \frac{L\varepsilon^2}{w(x)^2} $$

The following is the main result of this paper.

Theorem 3.3. Let $X := \{X_t : t \in E\}$ be a uniform process on a parameter set $E$. Let $w := w(x) > 0$, $0 < x < 1$ be continuous and symmetric about $x = 1/2$ for which there exists $\gamma \in (0, 1/2]$ such that $w$ is non-increasing and $xw(x)^2$ is non-decreasing on $(0, \gamma)$ and such that $w$ is uniformly bounded on $[\gamma, 1/2]$. Further, assume that $w(x)$ is regularly varying in a neighborhood of zero and satisfies the integral condition

$$ \int_0^\gamma s^{-1} \exp[-c/(sw(s)^2)] ds < \infty \text{ for all } c > 0. $$
If
\[ \lim_{\alpha \to \infty} \alpha^2 P^*(\sup_{t \in E} w(X_t) > \alpha) = 0 \]
and the WL-condition for \((X; w)\) is satisfied, then the empirical process based on
\{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0, 1]\} converges weakly in \(\ell^\infty(E \times [0, 1])\).

Remark 3.4. (1) We require that the function \(w(x)\) be symmetric about 1/2 is no loss of generality. As the Brownian bridge has the same behavior at 0 and 1. Moreover we only give the proof of the theorem for 0 < \(x < 1/2\). Indeed, if let \(\tilde{X}_t := 1 - X_t\), then \((\tilde{X}; w)\) satisfies the WL-condition. The result for \(\tilde{X}\) for 0 < \(x < 1/2\) gives a result of \(X\) for 1/2 < \(x < 1\). The fact (cf. [8], Corollary 1.6, p. 61) that if \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are Donsker classes, then \(\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2\) is a Donsker class gives the result for \(\mathcal{F} = E \times [0, 1]\).

(2) For a general process \(Y := \{Y_t : t \in E\}\), if we define \(X := X_t := \tilde{F}_t (Y_t)\), where \(\tilde{F}_t (\cdot)\) is the (randomized) distributional transform of the df \(F_t\) of \(Y_t\), then \(X\) is a uniform process (see Lemma 2.8). Such a process \(X\) is called a copula process. If we have a CLT for the \(X\) process, then we have a CLT for the \(Y\) process; see Proposition 3.5 for precise statement. In case of \(w = 1\), this theorem gives a proof of Theorem 2.11 provided that \(F_t(\cdot)\) for each \(t \in E\) is strictly increasing; see Corollary 3.6.

(3) The integral condition (3.3) is necessary and sufficient for one dimensional weighted uniform empirical process under regularity of the weight function; see [1], Example 4.9.

The proof of the theorem is given at the end of this section.

The following is a possible way that a CLT for the time dependent empirical process for \(Y\) can be obtained from proving a CLT for the process \(X\).

Proposition 3.5. Let \(w(x)\) be any function on \((0, 1)\). Let \(\{Y_t : t \in E\}\) be a process and \(F_t(\cdot)\) is the df of \(Y_t\). Let \(X_t := \tilde{F}_t (Y_t)\). Then the following hold:

(i) If \(F_t(\cdot)\) is strictly increasing for each \(t \in E\), then
\[ \{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0, 1]\} \in \text{CLT in } \ell^\infty(E \times [0, 1]) \]
implies
\[ \{w(F_t(y))(1_{Y_t \leq y} - F_t(y)) : t \in E, y \in \mathbb{R}\} \in \text{CLT in } \ell^\infty(E \times \mathbb{R}). \]

(ii) Without assuming that \(F_t(\cdot)\) is strictly increasing for each \(t \in E\), we have
\[ \{w(x)(1_{X_t \leq x} - x) : t \in E, x \in [0, 1]\} \in \text{CLT in } \ell^\infty(E \times [0, 1]) \]
implies
\[ \{w(F_t(y))(1_{Y_t \leq y} - F_t(y)) : (t, y) \in T_0\} \in \text{CLT in } \ell^\infty(T_0), \]
where \(T_0\) is any countable subset of \(E \times \mathbb{R}\).

Proof. Proof of (i). Recall that \(\tilde{F}(x) \leq \tilde{F}(y)\) for \(x \leq y\) and \(\tilde{F}(x) \leq F(x)\) for all \(x \in \mathbb{R}\) and for any df \(F\) (see Lemma 2.7). Hence \(Y_t \leq y\) implies that \(\tilde{F}_t(Y_t) \leq \tilde{F}_t(y)\); i.e.
\[ 1_{Y_t \leq y} \leq 1_{\tilde{F}_t(Y_t) \leq \tilde{F}_t(y)} \]
uniformly in \(t \in E, y \in \mathbb{R}\).
Recalling that \( \tilde{F}(\cdot) \) is strictly increasing, by the same lemma if \( x < y \), then \( F(x) < \tilde{F}(y) \). Now if \( \tilde{F}(Y_t) \leq F_t(y) \) and \( Y_t > y \) for some \( t \in E \) and \( y \in \mathbb{R} \), then \( F_t(y) < \tilde{F}(Y_t) \). We have a contradiction: \( F_t(y) < F_t(y) \). Thus \( \tilde{F}(Y_t) \leq F_t(y) \) implies \( Y_t \leq y \); i.e.

\[
1_{Y_t \leq y} \geq 1_{F_t(Y_t) \leq F_t(y)}, \quad \text{uniformly in } t \in E, y \in \mathbb{R}.
\]

Combining the two displays, we have

\[
1_{Y_t \leq y} = 1_{F_t(Y_t) \leq F_t(y)}, \quad \text{uniformly in } t \in E, y \in \mathbb{R}.
\]

Since \( \{F_t(y) : t \in E, y \in \mathbb{R}\} \) is a subset of \([0, 1]\), thus if the empirical process based on \( \{w(x)(1_{\tilde{F}(Y_t) \leq x} - x) : t \in E, x \in [0, 1]\} \) satisfies CLT in \( \ell^\infty(E \times [0, 1]) \), then, by substituting \( x \) with \( F_t(y) \) and using \( \text{(3.5)} \), the empirical process based on \( \{w(F_t(y))(1_{Y_t \leq y} - F_t(y)) : t \in E, y \in \mathbb{R}\} \) satisfies the CLT in \( \ell^\infty(E \times \mathbb{R}) \).

Proof of \( \text{(ii)} \). Fix \( t \in E \) and \( y \in \mathbb{R} \). If \( \tilde{F}(Y_t) \leq F_t(y) \), since \( \tilde{F}(Y_t) = F_t(y) \) has probability zero, then, after throwing out this null set, \( \tilde{F}(Y_t) < F_t(y) \), which will imply \( Y_t \leq y \). If not, then \( Y_t > y \), by Lemma \( \text{(2.7)} \) hence \( F_t(y) \leq \tilde{F}(Y_t) \). Again we have a contradiction \( F_t(y) < F_t(y) \). Thus almost surely \( 1_{\tilde{F}(Y_t) \leq F_t(y)} \leq 1_{Y_t \leq y} \).

Combining this with \( \text{(3.4)} \) gives, almost surely,

\[
1_{Y_t \leq y} = 1_{F_t(Y_t) \leq F_t(y)}, \quad \text{uniformly in } (t, y) \in T_0,
\]

where \( T_0 \) is any countable set in \( E \times \mathbb{R} \). Restricting to the countable set, we have the stated implication as in \( \text{(i)} \).

\[\square\]

**Corollary 3.6** (cf. \[9\], Theorem 3). Let \( Y := \{Y_t : t \in E\} \) be a process. Let \( F_t \) be the df of \( Y_t \). In addition, assume that \( F_t(\cdot) \) is strictly increasing for each \( t \in E \) and that \( Y \) satisfies the L-condition:

\[
\sup_{t \in E} \mathbb{P}^*( \sup_{s, \rho(s,t) \leq \epsilon} |\tilde{F}(Y_t) - \tilde{F}(Y_s)| > \epsilon^2 | \leq L \epsilon^2,
\]

for a constant \( L \) and a continuous Gaussian metric \( \rho(s,t) \) on \( E \). Then

\[
\{1_{Y_t \leq y} - \mathbb{P}(Y_t \leq y) : t \in E, y \in \mathbb{R}\} \in \text{CLT in } \ell^\infty(E \times \mathbb{R}).
\]

**Remark 3.7.** Under the L-condition, we will see from the proof of Theorem \( \text{(3.12)} \) that there is a countable dense set in \( E \times \mathbb{R} \) with respect to the \( L_2 \) distance of the limiting Gaussian process. Hence without the restriction that \( F_t(\cdot) \) is strictly increasing, we still have a CLT but on a countable dense set.

**Proof of Corollary 3.6** By part \( \text{(i)} \) of Proposition \( \text{(3.3)} \) we only need to check the conditions in Theorem \( \text{(3.3)} \) with \( w(x) \equiv 1 \).

Under the L-condition, we have (cf. \[9\], Lemma 1)

\[
\sup_x |F_t(x) - F_s(x)| \leq 2(L + 1)\rho(s, t)^2.
\]

Consequently by passing to the limit,

\[
\sup_x |F_t(x) - F_s(x)| \leq 2(L + 1)\rho(s, t)^2.
\]

Recalling that \( \tilde{F}_s(x) = F_s(x) + V(F_s(x) - F_s(x)) \), we obtain

\[
\sup_x |\tilde{F}_t(x) - \tilde{F}_s(x)| \leq \sup_x |F_t(x) - F_s(x)| + \sup_x |V(F_t(x) - F_s(x))| + \sup_x |V(F_t(x) - F_s(x))| \leq 6(L + 1)\rho(s, t)^2.
\]
For $t \in E$ fixed, let $A := \{ \sup_{s \in \mathbb{R}: |s| \leq \varepsilon} |\hat{F}_t(Y_s) - \hat{F}_t(Y_t)| > \varepsilon^2 \}$. On the complement, $A^c$, of $A$, we have for all $s$ with $|s| \leq \varepsilon$,

$$|\hat{F}_s(Y_s) - \hat{F}_t(Y_t)| \leq |\hat{F}_s(Y_s) - \hat{F}_s(Y_t)| + |\hat{F}_s(Y_t) - \hat{F}_t(Y_t)| \leq 6(L + 1)\rho(s, t)^2 + \varepsilon^2 \leq (6L + 7)\varepsilon^2.$$ 

Hence

$$P^*(\sup_{s \in \mathbb{R}: |s| \leq \varepsilon} 1_{\hat{F}_s(Y_s) \leq x < \hat{F}_t(Y_t) > 0}) = P^*(A^c, \sup_{s \in \mathbb{R}: |s| \leq \varepsilon} 1_{\hat{F}_s(Y_s) \leq x < \hat{F}_t(Y_t) > 0}) + P^*(A, \sup_{s \in \mathbb{R}: |s| \leq \varepsilon} 1_{\hat{F}_s(Y_s) \leq x < \hat{F}_t(Y_t) > 0}) \leq P(A^c, 1_{\hat{F}_t(Y_t) - (6(L + 1)\varepsilon^2 + \varepsilon^2) \leq x < \hat{F}_t(Y_t) > 0} + L\varepsilon^2)

$$

Keeping in mind that $\hat{F}_t(Y_t) \overset{d}{=} U(0, 1)$, we have

$$\leq (7L + 7)\varepsilon^2.$$ 

Similarly,

$$P^*(\sup_{s \in \mathbb{R}: |s| \leq \varepsilon} 1_{\hat{F}_s(Y_s) \leq x < \hat{F}_t(Y_t) > 0}) \leq (7L + 7)\varepsilon^2.$$ 

In addition, obviously for $w(x) \equiv 1$

$$\lim_{\alpha \to \infty} \alpha^2P^*(\sup_{t \in E} w(\hat{F}_t(Y_t)) > \alpha) = 0.$$ 

Thus we have verified the conditions in Theorem $3.3$.

We will prove Theorem $3.3$ only for $0 < x < 1/2$ as explained in Remark $3.4$. We will check the pre-Gaussian condition (ii) and the local modulus condition (iii) in Theorem $2.5$.

3.1. Pre-Gaussian. Let $\{G_0((s, x)) : s \in E, x \in [0, 1]\}$ be the zero mean Gaussian process with covariance

$$(3.8) \quad EG_0(s, x)G_0(t, y) = w(x)w(y)P(X_s \leq x, X_t \leq y).$$

Under the assumptions of Theorem $3.3$, we will prove $G_0(s, x)$ has a version with bounded and uniformly continuous sample paths with its $L_2$ distance $d_{G_0}$ by comparing it with some other continuous Gaussian distance; consequently by another comparison the centered Gaussian process with covariance

$$(3.9) \quad EG(s, x)\tilde{G}(t, y) := w(x)w(y)[P(X_s \leq x, X_t \leq y) - xy]$$

has a version with bounded and uniformly continuous sample paths with its $L_2$ distance $d_{G}$, which is equivalent to say the process $\{w(y)(1_{X_t \leq y} - y) : t \in E, y \in [0, 1]\}$ is pre-Gaussian.

**Lemma 3.8** (see [1], Example 4.8). Let $W(y)$ be a Brownian motion and $w(y)$ as in Theorem $3.3$. Then the Gaussian process $\{w(y)W(y) : y \in [0, 1]\}$ is sample bounded and uniformly continuous w.r.t. its $L_2$ distance, which is given by

$$(3.10) \quad d(x, y)^2 := E(w(y)W(y) - w(x)W(x))^2 = w(x)w(y)|y - x| + (x \wedge y)(w(x) - w(y))^2.$$
Lemma 3.9. If $xw(x)^2$ is non-decreasing and $w(x)$ is non-increasing for $0 < x < \delta$, then 
\[ d(x, y) \leq d(x, z) \]
for $0 < x \leq y \leq z \leq \delta$.

Proof. Let $0 < x \leq y \leq z \leq \delta$. Using definition (3.10) and the monotonicity of $xw(x)^2$ and $w(x)$, we obtain
\[
\begin{align*}
d(x, y)^2 &= w(y)^2(y - x) + x(w(y) - w(x))^2 \\
&= xw(x)^2 + yw(y)^2 - 2xw(x)w(y) \\
&\leq xw(x)^2 + zw(z)^2 - 2xw(x)w(z) \\
&= d(x, z)^2.
\end{align*}
\]

Next we give an upper bound for $d_{G_0}$ under WL-condition in Theorem 3.3.

Lemma 3.10. Let \(d(x, y)\) be as in (3.10) and \(d_{G_0}((s, x), (t, y))\) the \(L_2\) distance of the Gaussian process \(G_0\) in (3.8). Then under the WL-condition, we have
\[
d_{G_0}^2((s, x), (t, y)) \leq 2d^2(x, y) + 4L\rho(s, t)^2.
\]

Proof. First observe that for \(t \in E\)
\[
d(x, y)^2 = E[w(y)W(y) - w(x)W(x)]^2 = E[w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}]^2.
\]
Using, by the WL-condition for fixed \(s\) and \(t\),
\[
\begin{align*}
P(X_s \leq x < X_t) &\leq \frac{L\rho(s, t)^2}{w(x)^2} \quad \text{and} \quad P(X_t \leq x < X_s) \leq \frac{L\rho(s, t)^2}{w(x)^2},
\end{align*}
\]
we obtain
\[
\begin{align*}
d_{G_0}^2((s, x), (t, y)) &\leq E[w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}]^2 \\
&= E[w(x)1_{X_t \leq x} - w(x)1_{X_t \leq x} + w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}]^2 \\
&\leq 2E[w(x)1_{X_t \leq x} - w(x)1_{X_t \leq x}]^2 + 2E[w(x)1_{X_t \leq x} - w(y)1_{X_t \leq y}]^2 \\
&= 2w(x)^2E[1_{X_t \leq x} - 1_{X_t \leq x}]^2 + 2d(x, y)^2 \quad \text{by (3.11)} \\
&\leq 2w(x)^2P(X_s \leq x < X_t) + P(X_t \leq x < X_s) + 2d(x, y)^2 \\
&\leq 4L\rho(s, t)^2 + 2d(x, y)^2 \quad \text{by (3.12)}.
\end{align*}
\]

Corollary 3.11. Under the WL-condition, the process \(G_0(t, y)\) is sample bounded and uniformly continuous with respect to its \(L_2\) distance; the same is true for a zero mean Gaussian process with covariance
\[
(3.14) \quad EG(s, x)G(t, y) := w(x)w(y)[P(X_s \leq x, X_t \leq y) - xy].
\]

Proof. By assumption, \(\rho\) is the \(L_2\) distance of a zero mean Gaussian process on \(E\), say \(\{H_0(t) : t \in E\}\), with bounded and uniformly \(\rho\)-continuous sample paths. Let the metric \(d\) on \([0, 1]\) as given in (3.10) with the corresponding Gaussian process \(w(x)W(x)\), which is sample bounded and uniformly \(d\)-continuous. Let \(H_2((t, y)) := 2^{1/2}w(y)W(y) + 2L^{1/2}H_0(t) : t \in E, y \in [0, 1]\), where \(W\) and \(H_0\) are independent. Then the \(L_2\) distance, \(d_{H_2}((s, x), (t, y))\), of \(H_2\) is \(2^{1/2}d(x, y) + 2L^{1/2}\rho(s, t)\). Total boundedness of \(d\) and \(\rho\) implies that of \(d_{H_2}\). Thus let \(T_0\) be a dense subset in \((E \times [0, 1], d_{H_2})\); since \(d_{G_0} \leq d_{H_2}\) by (3.12), \(T_0\) is also a dense subset in \((E \times
Using the comparison Theorem 2.3 with $H_1 := G_0$ and that $d_{G_0} \leq d_{H_2}$, the Gaussian process \{\(G_0 : (s, x) \in T_0\)\} is sample bounded and uniformly $d_{G_0}$ continuous. By Lemma 2.3 \{\(G_0 : (s, x) \in E \times R\)\} is sample bounded and uniformly $d_{G_0}$-continuous; the second statement in the Lemma is straightforward. \(\square\)

For the pre-Gaussian property of the empirical process considered in [9], we give a different proof rather than the constructive one in [9] using the generic chaining [15].

**Theorem 3.12.** Let \(\{Y(t) : t \in E\}\) be a process and satisfies the L-condition, then the centered Gaussian process on \(E \times R\) with covariance either
\[
P(Y_s \leq x, Y_t \leq y) - P(Y_s \leq x)P(Y_t \leq y)
\]
or
\[
P(Y_s \leq x, Y_t \leq y)
\]
has a version, which is sample bounded and uniformly continuous with respect to its \(L_2\) distance.

**Proof.** Let \(\{G_1(t, y) : t \in E, y \in R\}\) and \(\{G_2(t, y) : t \in E, y \in R\}\) be the Gaussian processes on \(E \times R\) with covariance \(P(Y_s \leq x, Y_t \leq y) - P(Y_s \leq x)P(Y_t \leq y)\) and \(P(Y_s \leq x, Y_t \leq y)\), respectively. Let \(d_{G_1}\) and \(d_{G_2}\) be their \(L_2\) distances, respectively; i.e,
\[
d_{G_1}((s, x), (t, y))^2 = E(1_{Y_s \leq x} - 1_{Y_t \leq y})^2 - (E(1_{Y_s \leq x} - 1_{Y_t \leq y}))^2,
\]
\[
d_{G_2}((s, x), (t, y))^2 = E(1_{Y_s \leq x} - 1_{Y_t \leq y})^2.
\]
And,
\[
d_{G_2}((s, x), (t, y))^2 = E(1_{Y_s \leq x} - 1_{Y_t \leq y})^2
\]
\[
= E(1_{Y_s \leq x} - 1_{Y_t \leq y} + 1_{Y_t \leq x} - 1_{Y_s \leq y})^2
\]
\[
\leq 2E(1_{Y_s \leq x} - 1_{Y_t \leq x})^2 + E(1_{Y_t \leq x} - 1_{Y_s \leq y})^2
\]
\[
\leq 2[\{P(Y_s \leq x < Y_t) + P(Y_t \leq x < Y_s)\} + |F_t(y) - F_s(x)|]
\]
\[
\leq 6(L + 1)\rho(s, t)^2 + |F_t(y) - F_s(x)|,
\]
where in the last line of the above display, we used Lemma 1 in [9].

Let \(W(\cdot)\) be a Brownian motion on \([0, \infty)\). Define the centered Gaussian process
\[
H_2(t, y) := W(F_t(y)) : t \in E, y \in R,
\]
where \(F_t(\cdot)\) be the df of \(Y_t\). Then its \(L_2\) distance \(d_{H_2}((s, x), (t, y)) = |F_t(y) - F_s(x)|^{1/2}\). By the uniform continuity of the sample paths of \(W(\cdot)\) on \([0, 1]\), it follows that \(H_2\) is sample bounded and uniformly continuous with respect to \(d_{H_2}\). By the L-condition, let \(\{H_1(t) : t \in E\}\), independent from \(H_2\), be a Gaussian process with bounded and uniformly continuous sample paths with it’s \(L_2\) distance \(\rho\). Define \(H(t, y) = H_2(t, y) + (6L + 6)^{1/2}H_1(t)\). Then \(\{H(t, y) : t \in E, y \in R\}\) is sample bounded and uniformly continuous with respect to it’s \(L_2\) distance \(d_H\). Total boundedness of \(d_{H_1}\) and \(d_{H_2}\) implies that of \(d_H\) as can be seen from the equation
\[
d_H((t_1, y_1), (t_2, y_2))^2 = d_{H_2}((t_1, y_1), (t_2, y_2))^2 + (6L + 6)d_{H_1}(t_1, t_2)^2.
\]
Thus let \(T_0\) be a countable dense subset in \((E \times R, d_H)\). Since \(d_{G_1} \leq d_H\) in view of (3.15) and (3.16), by the comparison theorem 2.3 \{\(G_1(s, x) : (s, x) \in T_0\)\} is sample
bounded and uniformly continuous with respect to $d_{G_1}$. Since $T_0$ is also dense in $(E \times \mathbb{R}, d_{G_1})$, by Lemma 2.4, \{\mathcal{G}_1(s, x) : (s, x) \in E \times \mathbb{R}\} has a version which is sample bounded and uniformly $d_{G_1}$-continuous. □

3.2. Local modulus. Recall that a positive function $L(x)$ defined on $(0, \infty)$ is slowly varying at infinity (in a neighborhood of zero) if $L(\lambda x)/L(x) \to 1, x \to \infty (x \to 0)$ for every $\lambda > 0$ (see [7, p. 276]). One says a function $U(x)$ is regularly varying at infinity (in a neighborhood of zero) if $U(x) = x^\rho L(x)$ for some $-\infty < \rho < \infty$, and some slowly varying at infinity (in a neighborhood of zero) function $L(x)$; $\rho$ is called the exponent (see [7, p. 275]).

Lemma 3.13. Let $w(x) > 0$ for $0 < x \leq 1/2$ and is regularly varying in a neighborhood of 0 with nonzero exponent $\alpha$. Let $\theta_0 > 0$ be small enough such that $w(x)$ is non-increasing for $0 < x < \theta_0$. Then for $0 < \theta < \theta_0$

$$\sum_{k=0}^{\infty} \frac{1}{w(2^{-k}\theta)^2} \leq \frac{C}{w(\theta)^2},$$

where $C$ depends only on the weight function $w(x)$, but not on the argument $x$.

Proof. Since $w(x)$ is non-increasing for $0 < x < \theta_0$,

$$(\ln 2) \sum_{k=1}^{\infty} \frac{1}{w(2^{-k}\theta)^2} \leq \int_0^\theta \frac{1}{w(y)^2} \frac{dy}{y} \leq (\ln 2) \sum_{k=0}^{\infty} \frac{1}{w(2^{-k}\theta)^2}.$$

By Theorem 1 in [7, p. 281], we have

$$\frac{1}{w(\theta)^2} \int_0^\theta \frac{1}{w(y)^2} \frac{dy}{y} \to \alpha, \quad \text{as } \theta \to 0,$$

where $\alpha > 0$ is the exponent of the regularly varying function $1/w(x)^2$ (note that if $w(x)$ is regularly varying, so is $1/w(x)^2$). Therefore, there is a constant $C(w)$ such that

$$\left| \int_0^\theta \frac{1}{w(y)^2} \frac{dy}{w(y)^2} \right| \leq C(w), \quad 0 < \theta < \theta_0.$$

Lemma 3.14. Given $\varepsilon > 0$, under the assumptions of Theorem Y.3, we have for $0 < a < b < 1$ and $t$ fixed

$$P^*(\exists s, \rho(s, t) \leq \varepsilon, \exists x \in (a, b] : X_s \leq x < X_t) \leq \frac{C \varepsilon^2}{w(b)^2} + (b-a),$$

and

$$P^*(\exists s, \rho(s, t) \leq \varepsilon, \exists x \in (a, b] : X_t \leq x < X_s) \leq \frac{C \varepsilon^2}{w(b)^2} + (b-a),$$

where $C$ is a constant depending only on the function $w(x)$. 

Proof. Let $N \geq 0$ be the biggest integer such that $b/2^N \geq a$. Then,
\[
P^*(\exists s, \rho(s, t) \leq \epsilon, \exists x \in (a, b] : X_s \leq x < X_t) \\
\leq \sum_{k=0}^{N-1} P^*(\exists s, \rho(s, t) \leq \epsilon, \exists x \in (2^{-k-1}b, 2^{-k}b] : X_s \leq x < X_t) \\
+ P^*(\exists s, \rho(s, t) \leq \epsilon : X_s \leq x < X_t)
\]
\[
\leq \sum_{k=0}^{N-1} P^*(\exists s, \rho(s, t) \leq \epsilon : X_s \leq 2^{-k}b < X_t) + \sum_{k=0}^{N-1} P(2^{-k-1}b < X_t \leq 2^{-k}b) \\
+ P^*(\exists s, \rho(s, t) \leq \epsilon : X_s \leq 2^{-N}b < X_t) + P(a < X_t \leq 2^{-N}b)
\]
\[
\leq \sum_{k=0}^{N-1} P^*(\exists s, \rho(s, t) \leq \epsilon : X_s \leq 2^{-k}b < X_t) + \sum_{k=0}^{N-1} (2^{-k}b - 2^{-k-1}b) \\
+ P^*(\exists s, \rho(s, t) \leq \epsilon : X_s \leq 2^{-N}b < X_t) + 2^{-N}b - a
\]
\[
\leq \sum_{k=0}^{N} P^*(\exists s, \rho(s, t) \leq \epsilon : X_s \leq 2^{-k}b < X_t) + \sum_{k=0}^{N-1} (2^{-k}b - 2^{-k-1}b) + 2^{-N}b - a
\]
\[
\leq \sum_{k=0}^{\infty} \frac{L\varepsilon^2}{w(2^{-k}b)^2} + (b - a) \quad \text{using WL-condition to bound the probabilities}
\]
\[
\leq C\frac{\varepsilon^2}{w(b)^2} + (b - a) \quad \text{by Lemma 3.14!}
\]
The proof for the second part is similar; just change from $X_t \leq x < X_s$ for $2^{-k-1}b < x < 2^{-k}b$ to $X_t \leq 2^{-k}b < X_s$, with the same exceptional probability $(2^{-k}b - 2^{-k-1}b)$.

For the following, we use $C$ to denote a constant which may change from line to line and depends only on the weight function $w(x)$. Let the distance $d$ be as in (3.10). Then,
\[
ed((s, x), (t, y)) := \max\{d(x, y), \rho(s, t)\}
\]
is bounded by the Gaussian distance $(d(x, y)^2 + \rho(s, t)^2)^{1/2}$ on $E \times (0, 1)$ and will be used as the ‘$ho$’ in (iii) of Theorem 2.5.

Lemma 3.15. For $t \in E$, $y \in (0, 1)$, let $x_0 := \inf\{x : \text{for some } s, e((s, x), (t, y)) < \epsilon\}$, then
\[
d(x_0, y) \leq \epsilon.
\]

Proof. Indeed there exist a sequence $(s_n, x_n)_{n \in \mathbb{N}}$ in the set over which the infimum is taken such that $|x_n - x_0| \rightarrow 0$ as $n \rightarrow \infty$ and that $d(x_n, y) \leq \epsilon$. By the sample continuity of the weighted Wiener process $w(x)W(x)$, we have $d(x_n, y) \rightarrow d(x_0, y)$ as $n \rightarrow \infty$. Hence we have obtained $d(x_0, y) \leq \epsilon$.

Remark. The finiteness of $d(x_0, y)$ implies that $x_0$ can’t be zero in view of (3.10) since $w(x) \rightarrow \infty$ and $xw(x)^2 \rightarrow 0$ as $x \rightarrow 0$.

Lemma 3.16. For $t \in E$, $y \in (0, 1)$, let $x_1 := \sup\{x : \text{for some } s, e((s, x), (t, y)) < \epsilon\}$, then
\[
d(y, x_1) \leq \epsilon.
\]
Lemma 3.17. Under the assumptions of Theorem 3.3, we have for all $(s, x) ∈ E$ be as in Lemma 3.15. Then,

\[ w(y)^2P^*(\sup_{(s,x);e<\varepsilon,x\leq y}|1_{x_r\leq x} - 1_{X_{t}\leq x}| > 0) \leq C\varepsilon^2. \]

Proof. Let \( x_0 \) be as in Lemma 3.15. Then,

\[ w(y)^2P^*(\sup_{(s,x);e<\varepsilon,x\leq y}|1_{x_r\leq x} - 1_{X_{t}\leq x}| > 0) \]

\[ = w(y)^2\left( P^*(\exists(s,x),e((s,x), (t,y)) < \varepsilon, x \leq y : X_s \leq x < X_t) \right. \]

\[ + P^*(\exists(s,x), e((s,x), (t,y)) < \varepsilon, x \leq y : X_t \leq x < X_s) \]

\[ = w(y)^2\left( P^*(\exists(s,x), e((s,x), (t,y)) < \varepsilon, x \in (x_0,y) : X_s \leq x < X_t) \right. \]

\[ + P^*(\exists(s,x), e((s,x), (t,y)) < \varepsilon, x \in (x_0,y) : X_t \leq x < X_s) \]

\[ \leq w(y)^2(C\varepsilon^2/w(y)^2 + (y - x_0)) \] by Lemma 3.14

\[ \leq C\varepsilon^2. \]

For the last inequality, we used

\[ w(y)^2(y - x_0) \leq d(x_0,y)^2 \leq \varepsilon^2 \] by (3.17). □

Lemma 3.18. Under the assumptions of Theorem 3.3, we have for all \( \varepsilon > 0 \) and \((t, y) \in E \times [0,1]\),

\[ w(x_1)^2P^*(\sup_{(s,x);e<\varepsilon,x>y}|1_{x_r\leq x} - 1_{X_{t}\leq x}| > 0) \leq C\varepsilon^2. \]

Proof. Let \( x_1 \) be as in Lemma 3.16. Then,

\[ w(x_1)^2P^*(\sup_{(s,x);e<\varepsilon,x>y}|1_{x_r\leq x} - 1_{X_{t}\leq x}| > 0) \]

\[ = w(x_1)^2\left( P^*(\exists(s,x), e((s,x), (t,y)) < \varepsilon, x > y : X_s \leq x < X_t) \right. \]

\[ + P^*(\exists(s,x), e((s,x), (t,y)) < \varepsilon, x > y : X_t \leq x < X_s) \]

\[ = w(x_1)^2\left( P^*(\exists(s,x), e((s,x), (t,y)) < \varepsilon, x \in (y,x_1) : X_s \leq x < X_t) \right. \]

\[ + P^*(\exists(s,x), e((s,x), (t,y)) < \varepsilon, x \in (y,x_1) : X_t \leq x < X_s) \]

\[ \leq w(x_1)^2(C\varepsilon^2/w(x_1)^2 + (x_1 - y)) \] by Lemma 3.13

\[ \leq C\varepsilon^2. \]

For the last inequality, we used

\[ w(x_1)^2(x_1 - y) \leq d(y,x_1)^2 \leq \varepsilon^2 \] by (3.18). □
In the following lemma, for fixed \((t, y) \in E \times [0, 1]\), we write \(\sup_{(s, x) \in <t} \sup_{\{(s, x) : ((s, x), (t, y)) < \varepsilon\}}\)
and the same applies to other similar quantities.

**Lemma 3.19.** Under the assumptions of Theorem 3.3, we have for all \(\varepsilon > 0\) and \((t, y) \in E \times [0, 1]\),
\[
\sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t} |w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}| > \alpha \right) \leq C \varepsilon^2.
\]

**Proof.** We split the quantity:
\[
w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y} = [w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}] + [w(x)(1_{X_s \leq x} - 1_{X_t \leq x})].
\]
Consider the weak \(L_2\) norms of the components:
\[(3.19) \quad A := \sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t} |w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}| > \alpha \right)\]
\[(3.20) \quad B := \sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right).
\]
First we estimate A. Since
\[
\sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t} |w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}| > \alpha \right)
\leq \sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{x : d(x, y) < \varepsilon} |w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}| > \alpha \right)
\]
and \(t\) is fixed, this is the case in Example 4.9 in [1]. Hence we have
\[(3.21) \quad A := \sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t} |w(x)1_{X_s \leq x} - w(y)1_{X_t \leq y}| > \alpha \right) \leq C \varepsilon^2.
\]
Now we consider B. Since
\[
\sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right) \leq \sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t, x \leq y} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right)
\leq \sup_{\alpha > 0} \alpha^2 P^* \left( \sup_{(s, x) \in <t, x \leq y} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \leq C \varepsilon^2 \text{ by Lemma } 3.17.
\]
It suffices to consider bounds of the last two quantities. Without loss of generality, we assume \(w(x)\) is monotone on \((0, 1/2]\). For \(\alpha > 0\), let
\[x_\alpha = \sup \{x \in [0, 1/2] : w(x) > \alpha\}.
\]
Case \(x \leq y\).
Recall \(x_0 = \inf \{x : e((s, x), (t, y)) < \varepsilon\}\). First we consider the extreme cases for \(x_\alpha\).
(1) By continuity of \(w(\cdot)\), if \(x_\alpha > y\), then \(\alpha < w(y)\), consequently
\[
\sup_{\alpha < w(y)} \alpha^2 P^* \left( \sup_{(s, x) \in <t, x \leq y} w(x)|1_{X_s \leq x} - 1_{X_t \leq x}| > \alpha \right)
\leq w(y)^2 P^* \left( \sup_{(s, x) \in <t, x \leq y} |1_{X_s \leq x} - 1_{X_t \leq x}| > 0 \right) \leq C \varepsilon^2 \text{ by Lemma } 3.17
\]
(2) If \(x_\alpha \leq x_0\), then \(w(x_0) \leq \alpha\), hence \(w(x) \leq \alpha\) for \(x_0 \leq x\). For \(\alpha\) such that \(x_\alpha \leq x_0\), the event under the probability of (3.20) is empty.
(3). Now \( x_0 < x_0 \leq y \). In this case, \( w(y) \leq \alpha < w(x_0) \). Take \( \varepsilon > 0 \). We have

\[
B := \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) |1_{x_0 \leq x} - 1_{x_0 \leq x}| > \alpha) \\
\leq \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) 1_{x_0 \leq x} > \alpha) \\
+ \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) 1_{x_0 \leq x} < x_0 > \alpha) \\
= I + II.
\]

For \( I \),

\[
I = \sup_{w(y) \leq \alpha < w(x_0)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) 1_{x_0 \leq x} > \alpha) \\
\leq \sup_{x_0 < x_0 \leq y} w(x_0)^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) 1_{x_0 \leq x} > \alpha) \\
\leq \sup_{x_0 < x_0 \leq y} w(x_0)^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} 1_{x_0 \leq x} > x_0 > 0) \\
\leq \sup_{x_0 < x_0 \leq y} w(x_0)^2 (C\varepsilon^2 / w(x_0)^2 + (x_0 - x_0)) \text{ using Lemma 3.14} \\
\leq C\varepsilon^2.
\]

For the last inequality, we used

\[
w(x_0)^2(x_0 - x_0) \leq d(x_0, x_0)^2 \leq d(x_0, y)^2 \leq \varepsilon^2
\]

of Lemma 3.9 and Lemma 3.15.

\( II \) can be handled in the same way.

Case \( x > y \).

Recall \( x_1 = \sup \{x : e((s,x),(t,y)) < \varepsilon\} \). First we consider the extreme cases for \( x_0 \).

(1). By continuity of \( w(.) \), if \( x_0 > x_1 \), then \( \alpha \leq w(x_1) \), consequently

\[
\sup_{\alpha < w(x_1)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon,y} w(x) |1_{x_0 \leq x} - 1_{x_0 \leq x}| > \alpha) \\
\leq w(x_1)^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon,y} |1_{x_0 \leq x} - 1_{x_0 \leq x}| > 0) \leq C\varepsilon^2.
\]

by Lemma 3.18 consider \( \alpha \geq w(y) \), i.e. \( x_0 \leq y \).

(2). If \( x_0 \leq y \), then \( w(y) \leq \alpha \), hence \( w(x) \leq \alpha \) for \( y \leq x \). For \( \alpha \) such that \( x_0 \leq y \), the event under the probability of \( 3.20 \) is empty.

(3). Now \( y < x_0 \leq x_1 \). In this case, \( w(x_1) \leq \alpha < w(y) \). Take \( \varepsilon > 0 \). We have

\[
B := \sup_{w(x_1) \leq \alpha < w(y)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) |1_{x_0 \leq x} - 1_{x_0 \leq x}| > \alpha) \\
\leq \sup_{w(x_1) \leq \alpha < w(y)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) 1_{x_0 \leq x} > \alpha) \\
+ \sup_{w(x_1) \leq \alpha < w(y)} \alpha^2 \mathbb{P}^*(\sup_{(s,x):\varepsilon < \varepsilon} w(x) 1_{x_0 \leq x} < x_0 > \alpha) \\
= I + II.
\]
For $I$,

$$I = \sup_{w(x) \leq \alpha < w(y)} \alpha^2 P^*(\sup_{(s,x) : \epsilon < \epsilon} w(x)1_{X_s \leq x < X_t} > \alpha)$$

$$\leq \sup_{y < x_\alpha \leq x_1} w(x_\alpha)^2 P^*(\sup_{(s,x) : \epsilon < \epsilon} w(x)1_{X_s \leq x < X_t} > \alpha)$$

$$\leq \sup_{y < x_\alpha \leq x_1} w(x_\alpha)^2 P^*(\sup_{(s,x) : \epsilon < \epsilon} 1_{X_s \leq x < x_\alpha} > 0)$$

$$\leq \sup_{y < x_\alpha \leq x_1} w(x_\alpha)^2 (C\varepsilon^2/w(x_\alpha)^2 + (x_\alpha - y))$$

$$\leq \sup_{y < x_\alpha \leq x_1} w(x_\alpha)^2 (C\varepsilon^2/w(x_\alpha)^2)$$

using Lemma 3.14

$$\leq C\varepsilon^2.$$

For the last inequality, we used

$$w(x_\alpha)^2(x_\alpha - y) \leq d(y, x_\alpha)^2 \leq d(y, x_1)^2 \leq \varepsilon^2$$

of Lemma 3.9 and Lemma 3.16.

$II$ can be handled in the same way. Hence we have

$$B \leq C\varepsilon^2.$$

This together with (3.21) completes the proof. $\square$

Proof of Theorem 3.3. We apply Theorem 2.5 to the process \(\{w(y)(1_{X_s \leq y} - y) : t \in E, y \in [0,1]\}\).

Since for each \(s \in E\), \(X(s)\) takes values on \((0,1)\) and \(xw(x) \to 0\) as \(x \to 0\), almost surely

$$\sup_{s \in E, x \in [0,1/2]} w(x)|1_{X_s \leq x} - x| < \infty.$$

Also we observe for each \(s \in E, x \in [0,1/2]\)

$$P(w(x)(1_{X_s \leq x} - x))^2 < \infty.$$

Since \(w(x)\) is decreasing near 0,

$$\lim_{\alpha \to \infty} \alpha^2 P^*(\sup_{s \in E, x \in [0,1/2]} w(x)1_{X_s \leq x} > \alpha) \leq \lim_{\alpha \to \infty} \alpha^2 P^*(\sup_{s \in E} w(X_s) > \alpha)$$

$$= 0$$

by assumption of Theorem 3.3

which in turn implies

$$\lim_{\alpha \to \infty} \alpha^2 P^*(\sup_{s \in E, x \in [0,1/2]} w(x)|1_{X_s \leq x} - x| > \alpha) = 0.$$

This verifies (i) in Theorem 2.5. Corollary 3.11 verifies the pre-Gaussian condition (ii).

In view of Lemma 3.19 and the inequality

$$A_{2,\infty}(f + g) \leq C(A_{2,\infty}(f) + A_{2,\infty}(g))$$

where \(A_{2,\infty}(f) := \sup_{t>0} t^2 P(\{|f| > t\})^{1/2}\) for some constant \(C\), to verify the local modulus condition (iii) in Theorem 2.5 for the functions \(w(x)(1_{X_s \leq x} - x)\), it is enough to have

$$\sup_{\alpha > 0} \alpha^2 P^*(\sup_{d(x,y) \leq \epsilon} |w(x)x - w(y)y| > \alpha) \leq K\varepsilon^2$$

(3.22)
for some constant $K$. W.o.l.g, assume $x < y$. Inequality $3.22$ follows from
\[ |xw(x) - yw(y)|^2 \leq 2x^2(w(x) - w(y))^2 + 2w(y)^2(y - x)^2 \]
\[ \leq 2x(w(x) - w(y))^2 + 2w(y)^2(y - x) \]
\[ = 2d(x, y)^2 \text{ by } 3.10 \]
\[ \leq 2x^2. \]

4. An example

A special class of uniform processes (copula processes) can be obtained from distributional transforms. Specifically, given a process $Y := \{Y_t : t \in E\}$, define $X := X_t := \tilde{F}_t(Y_t)$, where $\tilde{F}_t(\cdot)$ is the distributional transform of the df of $Y_t$. Now, we give an example as an application of Theorem 3.3 when $\{Y_t : t \in E\} = \{B_t : t \in [1, 2]\}$, where $B_t$ is a Brownian motion.

**Theorem 4.1.** Let $\{B_t : t \geq 0\}$ be a Brownian motion and $F_t(x)$ be the distribution function of $B_t$. Let $w(x) = x^{-\alpha}L(x)$, for $0 < x < 1/2$, $0 < \alpha < 1/2$, and $L(x)$ slowly varying at 0 and assume $w(x)$ is symmetric about 1/2. Further assume that $w(x)$ is non-increasing and $xw(x)^2$ non-decreasing near 0. Then
\[ \{w(F_t(y))(1_{B_{t \leq y}} - F_t(y)) : t \in [1, 2], y \in \mathbb{R}\} \in \text{CLT in } \ell^\infty([1, 2] \times \mathbb{R}). \]

**Remarks 4.2.** The interval $[1, 2]$ can be replaced by any interval $[a, b]$ provided $a > 0$, which can be seen from the proof of the above theorem; also a priori, we need $F_t(\cdot)$ be strictly increasing.

We will verify the conditions in Proposition 3.3 to prove this theorem at the end of this section. To this end, we start with some lemmas. For the following, let $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $\Phi(y) := (2\pi)^{-1/2} \int_{-\infty}^y e^{-s^2/2} ds$.

**Lemma 4.3** ([6], p. 175). For $y > 0$,
\[ y^{-1}(1 - y^{-2})(2\pi)^{-1/2}e^{-y^2/2} \leq \Phi(-y) \leq y^{-1}(2\pi)^{-1/2}e^{-y^2/2}. \]
In particular, for $y > \sqrt{2}$,
\[ 2^{-1}y^{-1}(2\pi)^{-1/2}e^{-y^2/2} \leq \Phi(-y) \leq y^{-1}(2\pi)^{-1/2}e^{-y^2/2}. \]

**Lemma 4.4** ([14], p. 18). Let $L(x)$ be a slowly varying function at 0, then for any $\gamma > 0$,
\[ x^\gamma L(x) \to 0, x^{-\gamma}L(x) \to \infty \text{ as } x \to 0. \]
Consequently, for $0 < \gamma_1 < 2\alpha < \gamma_2 < 1$ and a function $L(x)$ slowly varying (at 0), there are constants $c_1, c_2$,
\[ c_1x^{\gamma_2} \leq x^{2\alpha}/L(x) \leq c_2x^{\gamma_1}, \quad 0 < x < 1/2. \]

For $c > 0$, let $L_c(x) = \exp(c\sqrt{\ln(1/x)})$.

**Lemma 4.5.** The function $L_c(x)$ is slowly varying at 0; that is for all $\lambda > 0$
\[ \lim_{x \to 0} \frac{L_c(\lambda x)}{L_c(x)} = 1. \]

**Proof.** By definition.
Lemma 4.6. For $0 < x < 1/4$, let $y = -\Phi^{-1}(x)$. Then
\[ y \leq \sqrt{2\ln(1/x)} \]
and
\[ \phi(-\Phi^{-1}(x) + c) \leq CxLc(x) \quad \text{for } c < 0, \]
\[ \phi(-\Phi^{-1}(x) + c) \leq 2^{1/2}x\ln(1/x) \quad \text{for } c \geq 0, \]
where $C$ depends only on $c$.

Proof. By Lemma 4.3 for $y > (2\pi)^{-1/2}$, $x \leq e^{-y^2/2}$; hence $y \leq \sqrt{2\ln(1/x)}$.
\[
\phi(-\Phi^{-1}(x) + c) = (2\pi)^{-1/2} \exp\left(-\frac{(y+c)^2}{2}\right) \\
= (2\pi)^{-1/2} \exp\left(-\frac{y^2}{2}\right) \exp(-yc) \exp(-c^2/2) \\
\leq 2y\Phi(-y) \exp(-yc) \quad \text{by Lemma 4.3} \\
\leq 2xy\exp(-yc).
\]
The statement for $c > 0$ follows from that $\exp(-yc) \leq 1$ and $y \leq \sqrt{2\ln(1/x)}$. For $c \leq 0$ the statement follows from that $y \leq C\exp(yC)$ for some constant $C$. \hfill $\Box$

Theorem 4.7 (Borell, see also [10]. Theorem 7.1). Let $G = (G_t)_{t \in T}$ be a centered Gaussian process indexed by a countable set $T$ such that $\sup_{t \in T} G_t < \infty$ almost surely. Then, $E(\sup_{t \in T} G_t) < \infty$ and for every $r > 0$
\[
P\left(\sup_{t \in T} G_t \geq E(\sup_{t \in T} G_t) + r\right) \leq e^{-r^2/2\sigma^2},
\]
where $\sigma = \sup_{t \in T}(EG_t)^{1/2}$.

For the following, let $B_t$ be a Brownian motion and $F_t(x)$ the distribution function of $B_t$, which is $\Phi\left(\frac{x}{\sqrt{t}}\right)$. Also for $1 \leq t \leq 2$, $0 < \varepsilon < 1/2$, set
\[
D := D(t, \varepsilon) := \sup_{t < s \leq t+\varepsilon} \frac{B_s - B_t}{\sqrt{s}} , \\
m := m(t, \varepsilon) := \sup_{t < s \leq t+\varepsilon} \frac{B_s - B_t}{\sqrt{s}} , \\
m_0 := \sup\{m(t, \varepsilon) : 1 \leq t \leq 2, 0 < \varepsilon < 1/2\}.
\]
We use $C$ to denote a constant, which may vary in each occurrence.

Lemma 4.8. For $1 \leq t \leq 2$, $0 < \varepsilon < 1/2$
\[
m \leq 2(2/\pi)^{1/2}\varepsilon^{1/2}.
\]

Proof. By the maximal inequality for Brownian motion,
\[
m := E\sup_{t < s \leq t+\varepsilon} \frac{B_s - B_t}{\sqrt{s}} \\
\leq E\sup_{t < s \leq t+\varepsilon} \frac{|B_s - B_t|}{\sqrt{t}} \\
\leq Ee^{1/2}2|\mathcal{N}(0,1)| \\
\leq 2(2/\pi)^{1/2}\varepsilon^{1/2}. \hfill \Box
\]

Lemma 4.9. Let $d := E(\sup_{1 \leq t \leq 2} \frac{B_t}{\sqrt{t}})$. Then, $d > 0$ and
\[
P\left(\inf_{1 \leq t \leq 2} F_t(B_t) \leq x\right) \leq (2\pi)^{1/2}\phi(-\Phi^{-1}(x) - d).
\]
Proof.

\[ P\left( \inf_{1 \leq t \leq 2} F_t(B_t) \leq x \right) = P\left( \inf_{1 \leq t \leq 2} \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) \right) \]
\[ = P\left( \sup_{1 \leq t \leq 2} \frac{B_t}{\sqrt{t}} \geq -\Phi^{-1}(x) \right) \]
\[ = P\left( \sup_{1 \leq t \leq 2} \frac{B_t}{\sqrt{t}} \geq d - \Phi^{-1}(x) - d \right) \]

which, by Theorem 4.7 and for \( x \) such that \( -\Phi^{-1}(x) - d > 0 \), is

\[ \leq \exp\left( -\frac{(\Phi^{-1}(x) - d)^2}{2} \right) \]
\[ = (2\pi)^{1/2} \phi(-\Phi^{-1}(x) - d). \]

Note that here \( \sigma^2 = \sup_{1 \leq t \leq 2} E\left( \frac{B_t}{\sqrt{t}} \right)^2 = 1. \]

\( \Box \)

**Lemma 4.10.** Let \( w(x) = x^{-\alpha} L(x) \), \( 0 < \alpha < 1/2 \) and \( L(x) \) be a slowly varying function (growing to infinity as \( x \downarrow 0 \)). Assume \( w(x) \) is decreasing near 0. Then

\[ \lim_{\lambda \to \infty} \lambda^2 P^*(\sup_{1 \leq t \leq 2} w(F_t(B_t)) > \lambda) = 0. \]

**Proof.** Let \( \lambda = w(x) \). Then, by Lemma 4.6 and Lemma 4.9

\[ \lim_{\lambda \to \infty} \lambda^2 P^*(\sup_{1 \leq t \leq 2} w(F_t(B_t)) > \lambda) = \lim_{\lambda \to \infty} \lambda^2 P^*(w(F_t(B_t)) > \lambda) \]
\[ = \lim_{x \to 0} w(x)^2 P^*(\inf_{1 \leq t \leq 2} F_t(B_t) \leq x) \]
\[ \leq \lim_{x \to 0} w(x)^2 (2\pi)^{1/2} \phi(-\Phi^{-1}(x) - d) \]
\[ \leq \lim_{x \to 0} x^{-2\alpha} L(x)^2 (2\pi)^{1/2} C x L(x) \]
\[ = 0. \]

\( \Box \)

**Lemma 4.11.** For \( 1 \leq t \leq 2 \), \( 0 < \varepsilon < 1/2 \), and \( l > m \),

\[ P\left( \frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t + \varepsilon} \frac{B_s}{\sqrt{s}} \right) \leq C_t \varepsilon^{1/2} \phi(l - m) \frac{\varepsilon^{1/2}}{t^{1/2}}, \]

where \( C_t \) is a constant depending only on \( t \). In particular, if we let \( C := \sup_{1 \leq t \leq 2} C_t \), and recall \( m_0 := \sup\{m(t, \varepsilon) : 1 \leq t \leq 2, 0 < \varepsilon < 1/2\} \), then for \( l > m_0 \), we have \( C < \infty \) and

\[ \left( \frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t + \varepsilon} \frac{B_s}{\sqrt{s}} \right) \leq C \varepsilon^{1/2} \phi(l - m_0) \frac{\varepsilon^{1/2}}{t^{1/2}}. \]

**Proof.** Since \( \sigma^2 := \sup_{t < s \leq t + \varepsilon} E\left( \frac{B_s - B_t}{\sqrt{s - t}} \right)^2 = \frac{\varepsilon}{t \varepsilon}, \) by Borell's concentration inequality Theorem 4.7 (since the process \( (B_s - B_t)/s^{1/2} \) is continuous in \( s \), we can take supremum over a countable set in the definition of \( D \)) it follows that for \( r > 0 \)

\[ P(D > m + r) \leq e^{-r^2(t + \varepsilon)/(2\varepsilon)}. \]

Hence, conditioning on \( \frac{B_t}{\sqrt{t}} \),

\[ P\left( \frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t + \varepsilon} \frac{B_s}{\sqrt{s}} \right) \leq P\left( \frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t + \varepsilon} \left( \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}} \right) + \sup_{t < s \leq t + \varepsilon} \frac{B_s}{\sqrt{s}} \right) \]
\[ = E\left( \frac{B_t}{\sqrt{t}} < l \leq \sup_{t < s \leq t + \varepsilon} \left( \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}} \right) + \sup_{t < s \leq t + \varepsilon} \frac{B_s}{\sqrt{s}} \right). \]
by independence of \( \{B_s - B_t : s > t\} \) and \( B_t \\
\int_{-\infty}^{\infty} P\left(y < l \leq D + \sup_{t < s \leq l + \varepsilon} \{(t/s)^{1/2}y\}\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
= \int_{0}^{l} P\left(y < l \leq D + \sup_{t < s \leq l + \varepsilon} \{(t/s)^{1/2}y\}\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
+ \int_{l}^{l-m} P\left(D \geq y-l\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
+ \int_{l}^{l-m} P\left(D \geq y-l\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
= I + II. \\
(4.3)

Note that \\
\sup_{t < s \leq l + \varepsilon} \{(t/s)^{1/2}y\} = y \quad \text{for } y > 0, \\
\sup_{t < s \leq l + \varepsilon} \{(t/s)^{1/2}y\} = ((t/(t + \varepsilon))^{1/2}y =: ay \quad \text{for } y \leq 0.

Therefore,

\[ I = \int_{0}^{l} P\left(y < l \leq D + y\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
= \int_{0}^{l-m} P\left(D \geq y-l\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy + \int_{l-m}^{l} P\left(D \geq y-l\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
\]

by inequality (4.2) for the first summand and noting \( r := l - y - m > 0 \)

\[ \leq \int_{0}^{l-m} e^{-\frac{(l-m)^2(l+\varepsilon)}{2\varepsilon}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy + \int_{l-m}^{l} P\left(D \geq y-l\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
\]

by completing the square in \( y \) for the first summand

\[ \leq (\frac{r}{l+\varepsilon})^{1/2} e^{-\frac{(l-m)^2(l+\varepsilon)}{2\varepsilon}} + m\phi(l-m) \]

bounding \( m \) using Lemma 4.8

(4.4)

\[ \leq (\frac{r}{l+\varepsilon})^{1/2}(2\pi)^{1/2} \phi(l-m)^{l+\varepsilon} + 2(2/\pi)^{1/2} e^{1/2}\phi(l-m). \]

For \( II, \)

\[ II = \int_{-\infty}^{0} P\left(y < l \leq D + ay\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
\leq \int_{-\infty}^{0} P\left(D \geq l-ay\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \\
\]

by equation (4.2)

\[ \leq \int_{-\infty}^{0} e^{-\frac{(l-ay-m)^2(l+\varepsilon)}{2\varepsilon}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy \]
by completing the square in \( y \)

\[
\leq \left( \frac{t-s}{t-s} \right)^{1/2} e^{-\frac{(t-m)^2}{2}}
\]

(4.5)

Combining (4.3), (4.4), and (4.5) completes the proof.

\[ \square \]

Lemma 4.12. For \( 1 \leq t \leq 2, \ 0 < \varepsilon \leq 1/2 \), there is a universal constant \( C \) such that for \( 0 < x < 1/4 \)

\[
P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} \right) \leq C\varepsilon^{1/2}(x\ln \frac{1}{x}).
\]

Proof.

\[
P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} \right) = P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \left[ \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}} \right] + \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} \right)
\]

letting \( D = \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}} \) and noting \( B_t < 0 \) inside the probability above

\[
\leq E_D P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) \right) \leq D + \frac{B_t}{\sqrt{t}} |D|
\]

by independence of \( \{B_s - B_t : s > t\} \) and \( B_t \)

\[
eq E_D P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) \right) \leq \frac{B_t}{\sqrt{t}} + D
\]

\[
eq E_D P\left( \left( \frac{t-s}{t-s} \right)^{1/2} (\Phi^{-1}(x) - D) \leq \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) \right)
\]

bounding the density of \( \frac{B_t}{\sqrt{t}} \) from above by \( \phi(\Phi^{-1}(x)) \)

\[
\leq E_D \phi(\Phi^{-1}(x)) \left[ (1 - (\frac{t-s}{t-s} \right)^{1/2} \Phi^{-1}(x) + (\frac{t-s}{t-s} \right)^{1/2} D \right]
\]

\[
\leq \phi(\Phi^{-1}(x))(-\Phi^{-1}(x)) \varepsilon/t + \phi(-\Phi^{-1}(x))(\frac{t-s}{t-s} \right)^{1/2} E_D D
\]

\[
\leq C(x \ln \frac{1}{t})(\varepsilon/t) + Cx(\ln \frac{1}{t})^{1/2} \varepsilon^{1/2} \text{ by Lemma 4.6 and Lemma 4.8}
\]

\[
\leq C\varepsilon^{1/2}(x \ln \frac{1}{t}).
\]

\[ \square \]

Proposition 4.13. For \( 1 \leq t \leq 2, \ 0 < \varepsilon \leq 1/2 \), there is a universal constant \( C \) such that for \( 0 < x < 1/4 \)

\[
P(F_t(B_t) \leq x < \sup_{s:|s-t|\leq \varepsilon} F_s(B_s)) \leq C\varepsilon^{1/2}(x \ln \frac{1}{z}) + C\varepsilon^{1/2} \phi(-\Phi^{-1}(x) - m_0) \frac{t}{t+\varepsilon}.
\]

Proof.

\[
P(F_t(B_t) \leq x < \sup_{s:|s-t|\leq \varepsilon} F_s(B_s))
\]

\[
= P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{s:|s-t|\leq \varepsilon} \frac{B_s}{\sqrt{s}} \right)
\]

\[
= P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \left[ \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}} \right] + \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} \right)
\]

\[
\leq P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t<s\leq t+\varepsilon} \frac{B_s}{\sqrt{s}} \right) + P\left( \frac{B_t}{\sqrt{t}} \leq \Phi^{-1}(x) < \sup_{t-\varepsilon\leq s\leq t} \frac{B_s}{\sqrt{s}} \right)
\]

\[
= I + II.
\]
By Lemma 4.12

\[ I \leq C\varepsilon^{1/2}(x \ln \frac{1}{x}). \]

Now we consider II.

\[ II = P(\frac{B_t}{\sqrt{t}} < \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

\[ = P(\frac{B_t}{\sqrt{t}} < \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

\[ + P(\frac{B_t}{\sqrt{t}} < \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

\[ \leq P(\frac{B_t}{\sqrt{t}} < \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

\[ \leq P(\frac{B_t}{\sqrt{t}} < \Phi^{-1}(x) < \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

(4.7) \[ \leq C\varepsilon^{1/2}(x \ln \frac{1}{x}) + C\varepsilon^{1/2}\phi(-\Phi^{-1}(x) - m_0) \frac{t}{t+\varepsilon} \] by Lemmas 4.12 and 4.11.

\[ \square \]

**Proposition 4.14.** For 1 \( \leq t \leq 2 \), 0 < \( \varepsilon \leq 1/2 \), there is a universal constant \( C \) such that for 0 < \( x < 1/4 \)

\[ P( \sup_{s \leq t} F_s(B_s) \leq x < F_t(B_t)) \leq C\varepsilon^{1/2}\phi(-\Phi^{-1}(x) - m_0) \frac{t}{t+\varepsilon} + C\varepsilon^{1/2}(x \ln \frac{1}{x}). \]

**Proof.** First we consider the case \( \{s > t : |s-t| \leq \varepsilon\} \). Let \( D = \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}} - \frac{B_t}{\sqrt{t}} \).

\[ P( \inf_{t < s \leq t+\varepsilon} F_s(B_s) \leq x < F_t(B_t)) \]

\[ = P( \inf_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}} \leq \Phi^{-1}(x) < \frac{B_t}{\sqrt{t}}) \]

\[ = P(\Phi^{-1}(x) \leq \sup_{t < s \leq t+\varepsilon} \frac{B_s}{\sqrt{s}}) \]

\[ \leq C\varepsilon^{1/2}\phi(-\Phi(x) - m_0) \frac{t}{t+\varepsilon} \] by Lemma 4.11.

For the case \( \{s < t : |s-t| \leq \varepsilon\} \),

\[ P( \inf_{t-\varepsilon \leq s < t} F_s(B_s) \leq x < F_t(B_t)) \]

\[ = P(\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

\[ + P(\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

\[ = P(\Phi^{-1}(x) \leq \sup_{t-\varepsilon \leq s < t} \frac{B_s}{\sqrt{s}}) \]

\[ \leq C\varepsilon^{1/2}\phi(-\Phi(x) - m_0) \frac{t}{t+\varepsilon} + C\varepsilon^{1/2}(x \ln \frac{1}{x}) \] by Lemmas 4.11 and 4.12. \[ \square \]
Proof of Theorem 4.1. Let $0 < \varepsilon < 1/2$ and $1 \leq t \leq 2$. Choose $\theta > 4$ big enough such that $\frac{t}{t+\varepsilon} > 2\alpha$ uniformly in $t$ and $\varepsilon$. Let $\rho(s,t) = |s-t|^{1/\theta}$. Then $\rho(s,t)$ is a continuous Gaussian metric on $[0,1]$ (indeed it is the $L_2$ distance of the fractional Brownian motion with Hurst index $1/\theta$). By Lemmas 4.4, 4.5 and 4.6 it follows that for $0 < x < 1/4$ (for $1/4 \leq x \leq 1/2$, the proof is trivial as $w(\cdot)$ is uniformly bounded on it)

$$\phi(-\Phi^{-1}(x) - m_0)^{1+\varepsilon_0} \leq |C x L_C(x)|^{1+\varepsilon_0} \leq C x^{2\alpha}/L(x) = \frac{C}{w(x)^2}.$$ 

Hence Propositions 4.13 and 4.14 verify the WL-condition in Theorem 3.3 and Lemma 4.10 verifies the envelope function condition therein. Hence by part (i) of Proposition 3.5 and noting the distribution functions $F_i$ of $B_i$ are strictly increasing, we conclude the proof.

□

APPENDIX

In this appendix, we give the proof of Lemma 2.4.

Proof of Lemma 2.4. We denote the restricted process $\{G(t): t \in T_0\}$ by $G_0$. Then almost surely its sample paths are uniformly continuous on $T_0$. Each sample path can be extended to a uniformly continuous sample path on $T$. Indeed, if we let $G_0(\omega)$ be a sample path and $t \in T$, then there is a sequence, say $(t_m) \subset T_0$, such that $d_G(t_m, t) \to 0$ as $m \to \infty$ and define $\tilde{G}(t)(\omega) := \lim_{m \to \infty} G(t_m)(\omega)$. It’s easy to see it’s well defined and is uniformly $d_G$ continuous on $T$. Moreover, in view of its characteristic function, $G(t)$ is normal. Let $\tilde{\rho}$ be the covariance of $G$. It remains to show $\rho = \tilde{\rho}$. But that $\rho$ and $\tilde{\rho}$ coincide on $T_0 \times T_0$ implies they coincide on $T \times T$. Indeed, for any $s, t \in T$, we can find sequences $(s_m)$ and $(t_m)$ in $T_0$, such that $d_G(s_m, s) \to 0$ and $d_G(s, t_m) \to 0$. Then $|\rho(s,t) - \tilde{\rho}(s_m, t_m)| \to 0$.

□

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School of Mathematics and Statistics, Southwest University, Chongqing 400715, People’s Republic of China

E-mail address: yangyuping@gmail.com