Supersymmetric, Integrable Boundary Field Theories

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Quantum integrable models that possess $N = 2$ supersymmetry are investigated on the half-space. Conformal perturbation theory is used to identify some $N = 2$ supersymmetric boundary integrable models, and the effective boundary Landau-Ginzburg actions are constructed. It is found that $N = 2$ supersymmetry largely determines the boundary action in terms of the bulk, and in particular, the boundary bosonic potential is $|W|^2$, where $W$ is the bulk superpotential. Supersymmetry is also discussed from the perspective of the affine quantum group symmetry of exact scattering matrices, and exact $N = 2$ supersymmetry preserving boundary reflection matrices are described.

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1. INTRODUCTION

Apart from being an intrinsically interesting subject, the study of conformal and quantum integrable models in two-dimensional systems with a boundary also provides another avenue by which one can relate integrable models to physically relevant systems and, in particular to situations that are of interest in three- or four-dimensional field theory. Some of these applications are discussed by other speakers at this workshop.

In this paper, I will consider \((1+1)\)-dimensional models defined upon the half-line \((x < 0)\) with a spatial boundary at \(x = 0\). The field theory in the bulk \((i.e. \text{ for } x < 0)\) will be required to be \( N = 2 \) supersymmetric, and either conformal or quantum integrable \((i.e. \text{ possess higher spin conserved charges})\). The boundary conditions, and boundary dynamics, will be chosen so as to preserve quantum integrability along with as much supersymmetry as possible. In general, for boundary models to be integrable there are stringent constraints upon the bulk and boundary sectors (for example, see [1, 2]). As will be seen here, there are \( N = 2 \) supersymmetric boundary integrable models whose boundary dynamics is quite non-trivial, with the boundary superpotential determined by the bulk superpotential, but with a boundary mass scale that is independent of the bulk mass scale.

There are several motivations for considering \( N = 2 \) supersymmetric boundary integrable models. First, if one has \( N = 2 \) supersymmetry on the plane or cylinder, then one can obtain quite a number of exact quantum properties of such models via semi-classical analysis [5–10]. One might hope that some of these results can be generalized to the half-space. At a more formal level, the supersymmetric models, and their topological counterparts, provide the simplest examples of Coulomb gas methods, along with the associated action of the affine quantum group on the soliton spectrum and \( S \)-matrices. Once again, one would like to know how much of these structures survive for quantum integrable models on the half-space. Finally, from the point of view of higher dimensional field theories, if one considers monopoles or strings in a supersymmetric field theory, and treats them as impurity problems, then the result will be supersymmetric \((1 + 1)\)-dimensional boundary field theories.

The starting point in this paper is \( N = 2 \) superconformal models on the plane. In section 2 I briefly review some pertinent facts about conformal field theory and extended chiral algebras on the half-space. In section 3, I review the conditions under which boundary and bulk perturbations of conformal models lead to quantum integrable models, and section 4 contains an analysis of the \( N = 2 \) supersymmetric boundary in-
integrable models that can be obtained from perturbations of the $N = 2$ superconformal minimal models. In particular, it is shown how families of such models can be obtained from the bulk perturbations that lead to bulk superpotentials consisting of Chebyshev polynomials. In section 5, effective boundary Landau-Ginzburg actions are constructed for these, and more general models. It is shown that if the bulk superpotential is $W(\phi)$, then the boundary superpotential, $V$, satisfies $\frac{\partial V}{\partial \phi} = W$, and hence the boundary bosonic potential is $|W|^2$. Finally, section 6 contains a brief discussion of $N = 2$ supersymmetry preserving, exact boundary reflection matrices.

2. CONFORMAL SYSTEMS WITH A BOUNDARY

Consider a conformal field theory on the complete complex plane, and suppose that the theory is symmetric between the left-moving and right-moving sectors. This means that every left-moving operator, $O(z)$, in the chiral algebra $\mathcal{A}$, has a right-moving counterpart, $\bar{O}(\bar{z})$, in the chiral algebra $\bar{\mathcal{A}} \equiv \mathcal{A}$. As is familiar in open string theory, the introduction of a boundary requires that the left-movers and right-movers be locked together. As a result, the two chiral algebras, $\mathcal{A}$ and $\bar{\mathcal{A}}$, become identified, producing a single copy, $\tilde{\mathcal{A}}$, of $\mathcal{A}$ in the system with the boundary. Perhaps the simplest way to think of this is as a generalized method of images: That is, $\tilde{\mathcal{A}}$-preserving boundary conditions require \(1\):

$$O(z)\big|_{x=0} = \bar{O}(-\bar{z})\big|_{x=0}. \quad (1)$$

One can then think of $\bar{O}(-\bar{z})$ as the analytic continuation of $O(z)$ into $x = Re(z) > 0$.

In an $N = 2$ superconformal theory the chiral algebra consists of a spin-1 current, $J$, two spin-$\frac{3}{2}$ supercurrents, $G^\pm$, and the energy momentum tensor, $T$. For $N = 2$ superconformal boundary conditions, one requires:

$$J(z)\big|_{x=0} = \tilde{J}(-\bar{z})\big|_{x=0},$$
$$T(z)\big|_{x=0} = \tilde{T}(-\bar{z})\big|_{x=0},$$
$$G^\pm(z)\big|_{x=0} = \tilde{G}^\pm(-\bar{z})\big|_{x=0}. \quad (2)$$

The result is a single $N = 2$ superconformal algebra on the half-space. The choice of pairing $G^\pm(z)$ with $\tilde{G}^\pm(-\bar{z})$, or with $G^\mp(-\bar{z})$, is a matter of convention, but changing this pairing will introduce a relative negative sign in the pairing of $J(z)$ and $\tilde{J}(z)$.

3. BOUNDARY INTEGRABLE MODELS

Given a boundary conformal model, there are two natural types of massive perturbation. The first is a (relevant) bulk perturbation of the form:

$$\Delta A_{bulk} = g \int_{Re(z)<0} \psi(z, \bar{z}) \ d^2z, \quad (3)$$

where $g$ is a coupling constant, and $\psi(z, \bar{z})$ is generically a sum of products of holomorphic and anti-holomorphic operators: $\psi(z, \bar{z}) = \sum_j \psi_j(z) \bar{\psi}_j(\bar{z})$. The coupling $g$ introduces the bulk mass scale.

There are also boundary perturbations of the form:

$$A_{bdry} = \mu \int_{-\infty}^{\infty} \chi(t) \ dt, \quad (4)$$

where $\chi(t)$ is some operator defined upon the boundary at $x = 0$. To be relevant (or marginal), the operator $\chi$ must have dimension less than (or equal to) one. For a relevant boundary operator, the coupling constant, $\mu$, introduces a boundary mass scale. This mass scale is a priori independent of the bulk mass scale.

3.1. Conformal perturbation theory

When there is no boundary, it is well known how to analyze whether a bulk perturbation of a conformal field theory leads to an integrable model \([\text{2R}]\). The most direct method is to compute the corrections to the conformal Ward identity $\partial O(z) = 0$ for a current, $O(z)$, in the presence of a perturbation, $\psi(z, \bar{z}) = \psi(z) \bar{\psi}(\bar{z})$. The corrected identity is of the form $\partial \tilde{O}(z) = \mathcal{Y}$, and to have a conserved current in the perturbed theory, one must have $\mathcal{Y} = \partial \bar{Q}$, for some $Q$. The form of the correction $\mathcal{Y}$ is easily computed (to lowest order in $g$) and depends solely upon the operator $O$, and the representation of the chiral algebra to
which $\psi$ belongs. (The null vectors are usually crucial.)

If the system has a boundary, then there can be boundary contributions to a conserved charge. The perturbative boundary corrections to the bulk conservation laws can be computed much as in the bulk theory \( \mathcal{I}_3 \). The result is once again a matter of representation theory, and can be expressed in the following manner:

If a bulk perturbation, $\psi$, generates a representation, $\mathcal{R}$, of the bulk chiral algebra, and leads to a quantum integrable field theory in the system without boundary, then a boundary operator $\chi$ in precisely the same representation, $\mathcal{R}$, of the boundary chiral algebra will lead to an integrable boundary field theory. Moreover, the conserved charges of the boundary theory are constructed from those of the bulk integrable theory by pairing left-moving and right-moving currents as in (1), and adding boundary contributions to the charge.

This result has been established only to first order in perturbation theory. Under some circumstances one can argue that there are no higher order corrections \( \mathcal{I}_2 \). Even when such arguments cannot be made, experience with integrable models in systems without boundary suggests that first order perturbation theory is usually sufficient to establish the existence of conserved charges.

While the foregoing may seem straightforward prescription for the construction of boundary integrable models, there can be subtleties in obtaining the requisite boundary operator, $\chi$. Specifically, in a given conformal model with conformal boundary conditions, the boundary spectrum might not contain a boundary operator in the required representation, $\mathcal{R}$ \( \mathcal{I}_1 \).

### 3.2. Spin-1 currents and topological charges

There are two exceptions to the result described above. The first is trivial: The holomorphic and anti-holomorphic parts, $J(z)$ and $\bar{J}(\bar{z})$, of a spin-1 current in a conformal model are separately conserved. If one makes a bulk perturbation, then $J(z)$ and $\bar{J}(\bar{z})$ give rise to a single conserved $U(1)$ current if and only if the perturbing operator is neutral with respect to some linear combination of the associated left-moving and right-moving charges. While this is obvious, the computation in terms of conformal perturbation theory is substantially different from the one employed in obtaining the result above, and most importantly, even though a bulk perturbation can preserve a $U(1)$ charge, the corresponding boundary perturbation may well destroy it (even at first order).

The second exception is a little more subtle: if the bulk theory has charges that are conserved at first order in perturbation theory, but these charges do not commute with one another, then their commutators will generate new charges that may not be conserved at the boundary, even at first order in perturbation theory. This can happen with the topological charges that appear in massive theories with extended supersymmetry (for example, see \( \mathcal{I}_4 \)). These topological charges appear in the anti-commutator of two supersymmetries, and generically consist of integrals of total derivatives of bosonic fields. As a result, the anti-commutators of supersymmetry generators will pick up boundary terms that could break the superalgebra, unless one does one of the following:

(i) Enforces bosonic boundary conditions that cause the boundary terms to vanish.

(ii) Keeps the bulk conformally invariant, and only perturbs on the boundary. There are no topological charges in a bulk superconformal algebra.

(iii) Tries to compensate for the topological charge terms by using boundary degrees of freedom, and making re-definitions of the boundary supersymmetry and hamiltonian.

To illustrate possibility (iii), suppose that a bulk topological charge term appears in the square of a supersymmetry generator and gives rise to a boundary topological term $\mathcal{X}$. One can cancel this term by introducing boundary fermions $b$ and $\bar{b}^\dagger$, with $\{b, \bar{b}^\dagger\} = 1$, and then adding boundary correction of $b - \mathcal{X}b^\dagger$ to the supercharge. Indeed,
it will be seen later that it is precisely this mechanism that enables one to construct \( N = 2 \) supersymmetry boundary integrable models with non-trivial boundary interactions.

4. SUPERSYMMETRY PRESERVING BOUNDARY PERUBRATIONS

I will, for simplicity, focus on the \( N = 2 \) superconformal minimal models with \( A \)-type modular invariants, and whose central charge is \( c = 3k/(k + 2) \). The relevant bulk perturbations of these models that lead to \( N = 2 \) supersymmetric integrable field theories are well known [13-17]. There are three distinct such perturbations, but the one of importance here is the Chebyshev perturbation. That is, one perturbs the bulk using

\[
\lambda \int G_{z}^\frac{k}{2} \hat{G}_{\bar{z}}^\frac{k}{2} \phi_k^+(z, \bar{z}) \, dz + \lambda \int G_{\bar{z}}^\frac{k}{2} \hat{G}_{z}^\frac{k}{2} \phi_k^-(z, \bar{z}) \, d\bar{z} ,
\]

where \( \lambda \) is a coupling constant, \( \phi^+_k \) is the chiral primary field with charge \( q = \frac{k}{k+2} \) and conformal weight \( \hat{h} = \frac{h}{2(k+2)} \), and \( \phi^-_k \) is the antichiral conjugate of \( \phi^+_k \). This perturbation leads to a massive \( N = 2 \) supersymmetric integrable model whose effective Landau-Ginzburg potential is the Chebyshev polynomial of degree \( (k+2) \) (see, for example [18]). This may be thought of as the \( N = 2 \) superconformal analogue of the energy perturbation of the ordinary minimal series.

Suppose that \( \psi^\pm(t) \) are boundary operators that transform in exactly the same representations as the holomorphic operators \( G_{z}^\frac{k}{2} \phi_k^+(z) \) under the action of the \( N = 2 \) superconformal algebra. In particular this implies that the operators \( \psi^\pm(t) \) may be written

\[
\psi^\pm(t) = \hat{G}_{z}^\frac{k}{2} \left( \phi_k^+(t) \right) ,
\]

where \( \hat{G}_{z}^\frac{k}{2} \) are the supercharges of the boundary theory, and \( \phi_k^+(t) \) are boundary operators in the same \( N = 2 \) superconformal representation as \( \phi_k^+(z) \) and \( \phi_k^-(\bar{z}) \).

In order to use \( \psi^\pm(t) \) in a boundary perturbation, it is necessary to show that these operators are in the boundary spectrum of some conformally invariant boundary condition. This is straightforward, and is completely parallel with the analogous situation in the Ising model. Let \( \phi^+_1(z, \bar{z}) \) be the most relevant chiral primary field (with \( q = \frac{1}{2} \) and conformal weight \( h = \frac{1}{2(k+2)} \)). This field is the basic order parameter of the massive, bulk integrable model [21,22]. If we send this bulk field to the boundary, the result is an operator in the fusion product of \( \phi^+_1(z) \) with itself. There are two fields in this fusion product: (i) \( \phi^+_2 \), the chiral primary with \( q = \frac{2}{k+2} \) and conformal weight \( \hat{h} = \frac{1}{2(k+2)} \), and (ii) the field \( \hat{G}^{-\frac{k}{2}} \phi^-_1 \), with \( q = 1 - \frac{k}{k+2} = \frac{k+1}{k+2} \) and conformal weight \( h = \frac{1}{2} + \frac{k}{2(k+2)} = \frac{k+1}{k+2} \). If one starts with the \( N = 2 \) superconformal model with free boundary conditions in the Landau-Ginzburg formulation, then the expectation value of \( \phi^+_1 = (\phi^+_1)^2 \) will vanish at the boundary, and one will get the sub-leading operator \( \psi^-(t) = \hat{G}^{-\frac{k}{2}} \phi^-_1(t) \).

Similarly, one obtains \( \psi^+(t) = \hat{G}^{-\frac{k}{2}} \phi^+_1(t) \) by sending the anti-chiral conjugate, \( \phi^-_1(z, \bar{z}) \), of \( \phi^+_1(z, \bar{z}) \), to the boundary. Conformal perturbation theory can now be invoked to show that (at least to first order) one can obtain a boundary integrable model from a simultaneous bulk perturbation using (5) and boundary perturbation using (6) with:

\[
\chi(t) = \nu \psi^+(t) + \bar{\nu} \psi^-(t) .
\]

The constant, \( \nu \), is a complex phase, with complex conjugate \( \bar{\nu} \). If there is no bulk perturbation then one can absorb this phase \( \nu \) into a re-definition of \( \phi_k^+ \). If there is a bulk perturbation, then this freedom of re-definition can be used to adjust the phase of \( \lambda \) in (5) or the phase, \( \nu \). Thus there are three independent couplings: the bulk mass scale, determined by \( |\lambda| \), the boundary mass scale, determined by \( \mu \) in (6), and the relative phase between \( \lambda / \bar{\lambda} \) and \( \nu / \bar{\nu} \).

It is well known that the bulk perturbation (5), in the infinite domain, preserves all of the supersymmetry [13-17,18,24], and provides one with a massive \( N = 2 \) supersymmetric model. There are two ways of seeing the latter: One can either explicitly verify the result using conformal
perturbation theory, or, more generally, one can appeal to the theory of supersymmetric actions, which states that perturbations that involve only top components of superfields (as does (8)) preserve the supersymmetry. The same arguments can also be naively applied to the boundary perturbation (8), with similar conclusions. However, the massive bulk model has topological charges, and so the supersymmetry variation of the bulk action generates boundary terms via total derivatives. As described earlier, unless the bulk is massless, the boundary conditions or dynamics need to be fixed with care if one is going to preserve $N = 2$ supersymmetry. I will return to this issue later, but first I think it useful to present a simple and fairly complete example.

4.1. An example: The sine-Gordon model

The simplest example of an $N = 2$ superconformal minimal model is the $k = 1, c = 1$ minimal model that can be realized by a single free boson compactified at the “supersymmetric” radius. The superconformal generators can be written in terms of the holomorphic part of a canonically normalized boson, $\varphi(z)$, as:

$$T(z) = -\frac{i}{2} (\partial \varphi(z))^2 \quad ; \quad J(z) = \frac{1}{\sqrt{3}} \partial \varphi(z) \quad ; \quad G^\pm(z) = e^{\pm i \sqrt{3} \varphi(z)}.$$  \hspace{1cm} (8)

If $\bar{\varphi}(\bar{z})$ denotes the anti-holomorphic part of the boson, then the $N = 2$ superconformal boundary condition implies that $\varphi(z)|_{x=0} = \bar{\varphi}(\bar{z})|_{x=0}$. The order parameter, and its conjugate, are given by $\phi^\pm_1(z, \bar{z}) = e^{\pm \sqrt{2} \varphi(\varphi(z)+\bar{\varphi}(\bar{z}))}$, and as $x \to 0$ this becomes $e^{\pm \sqrt{2} \varphi(z)}|_{x=0} = G^\mp_+ \phi^\pm_1(z)|_{x=0}$. The bulk integrable perturbations are $G^\mp_+ G^\mp_0 \phi^\pm_1 = e^{\mp \sqrt{2} \varphi(\varphi(z)+\bar{\varphi}(\bar{z}))}$. We are thus dealing with the boundary sine-Gordon theory described in [3], with action:

$$\int_{-\infty}^{\infty} \int_{x<0} \left( \frac{i}{2} (\partial \Phi)^2 - \frac{3}{2} (\partial \Phi)^2 \right) - \frac{M}{\beta} \left[ \cos(\beta \Phi) - 1 \right] dx \, dt - \frac{2m}{\beta} \int_{-\infty}^{\infty} \cos \frac{\beta}{2} (\Phi - \Phi_0) \, dt \quad (9)$$

From this one finds that $\Phi$ satisfies the usual sine-Gordon equation with boundary condition:

$$\partial \Phi|_{x=0} = m \sin \frac{\beta}{2} (\Phi - \Phi_0)|_{x=0}. \quad (10)$$

The boson, $\Phi$, has the standard normalization of sine-Gordon theory, for which the supersymmetry point corresponds to $\beta^2 = \frac{16\pi}{3}$. The parameters $M, m$ and $\Phi_0$ coincide with the three parameters described above. It is known that this model is indeed integrable for arbitrary choices of $M, m$ and $\Phi_0$, and so the boundary and bulk perturbations are simultaneously and independently integrable.

5. MANIFESTLY SUPERSYMMETRIC ACTIONS

In the sine-Gordon model, the action is by no means manifestly supersymmetric. Indeed, the supersymmetry is obtained from vertex operators and appears as an “accident” of the choice of the coupling, $\beta$. To obtain manifestly supersymmetric actions for the $N = 2$ superconformal minimal models, perhaps the easiest way to proceed is to use the Landau-Ginzburg approach in which one looks for an effective field theory of the order parameter $\phi_1$ [21]. The idea here is to generalize this to boundary field theories.

The first step is to make the elementary observation that the boundary perturbations, defined by (8) and (9), must be fermionic operators, and so there is no way that they can be added by themselves to this action. The only option is to introduce a dimensionless boundary fermions, $b$ and $b^\dagger$, with $\{b, b^\dagger\} = \{b^\dagger, b\} = 0$, and $\{b, b^\dagger\} = 1$. One then considers boundary perturbations of the form:

$$\nu \ b^\dagger \ \psi^+ \ + \ \bar{\nu} \ \bar{\psi}^- \ b.$$  \hspace{1cm} (11)

Note that if $\psi^\pm$ transforms under a $U(1)$ charge, then one can arrange that the action preserve this $U(1)$ by making $b$ and $b^\dagger$ transform appropriately.

(It was for this reason that I did not use a real fermion, $b$, with $\{b, b\} = 1$.)

A more standard way to think of the boundary operators, $b$ and $b^\dagger$, is as introducing a boundary spin degree of freedom exactly as one does in the Kondo problem. Indeed, the quantization rules for $b$ and $b^\dagger$ imply that they generate
some (possibly time dependent) representation of the gamma matrix algebra. In the sine-Gordon model, the boundary interaction \( \mathcal{W} \) corresponds to coupling vertex operators to spin raising and lowering operators.

For the sine-Gordon model, one can also see rather explicitly that one should add these complex boundary degrees of freedom. One considers the model as one approaches the limit of free boundary conditions. As discussed in [1], any boundary degrees of freedom will not disappear in the free limit, but they will simply decouple from the bulk and will become massless boundary excitations. These will appear as poles at \( \theta = i\pi/2 \) in the boundary reflection matrix. One indeed finds such poles in all channels of the reflection matrix. Since the soliton and the anti-soliton have fermion numbers \( \pm 1/2 \), the off-diagonal poles in the reflection matrix indicate that there will be boundary excitations that carry fermion number \( \pm 1 \). The corresponding operators are \( b \) and \( b^\dagger \).

5.1. Non-trivial, supersymmetric boundary interactions

The free \( N = 2 \) superconformal model with a complex boson and a complex Dirac fermion has central charge \( c = 3 \), and the (Euclidean) action is:

\[
\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[ - \left( \partial_x \phi \right) \left( \partial_x \bar{\phi} \right) - \left( \partial_y \phi \right) \left( \partial_y \bar{\phi} \right) + \frac{i}{2} \left( \lambda \gamma^\mu \partial_x \lambda - \left( \partial_x \lambda \right) \gamma^\mu \lambda \right) \right] + \int_{-\infty}^{\infty} i \left( b^\dagger \frac{d}{dy} b \right) - \frac{i}{2} \left( \lambda \gamma^\nu \lambda \right) \bigg|_{x=0} \ dy \ , \tag{12}
\]

where \( x = x^0 \), \( y = x^1 \), \( \bar{\lambda} \) is the hermitian conjugate of \( \lambda \), and

\[
\gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^* = -i \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\]

The boundary term in (12) is motivated by the boundary action for the Ising model (see, for example, [2]), and has the property that the variation of the action cleanly enforces free or fixed boundary conditions on the bulk fermion.

To the free action one adds bulk Landau-Ginzburg superpotential terms:

\[
\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[ \left( \frac{\partial^2 \mathcal{W}}{\partial \phi^2} \right) \lambda_1 \lambda_2 - \left( \frac{\partial^2 \mathcal{W}}{\partial \bar{\phi}^2} \right) \bar{\lambda}_1 \bar{\lambda}_2 - \left| \frac{\partial \mathcal{W}}{\partial \phi} \right|^2 \right] , \tag{13}
\]

for some scalar superpotential, \( \mathcal{W} \).

The bulk action is invariant, up to boundary terms, under the following \( N = 2 \) supersymmetry transformations:

\[
\delta \phi = - (\lambda_1 \bar{\alpha}_1 + \lambda_2 \bar{\alpha}_2) \ , \quad \delta \bar{\phi} = (\bar{\lambda}_1 \alpha_1 + \bar{\lambda}_2 \alpha_2) \ ,
\]

\[
\delta \lambda_i = e^{ij} \left[ - (\partial_x \phi - i \partial_y \bar{\phi}) \alpha_j + \left( \frac{\partial \mathcal{W}}{\partial \phi} \right) \bar{\alpha}_j \right] ,
\]

\[
\delta \bar{\lambda}_i = e^{ij} \left[ - (\partial_x \bar{\phi} + i \partial_y \phi) \bar{\alpha}_j + \left( \frac{\partial \mathcal{W}}{\partial \bar{\phi}} \right) \alpha_j \right] , \tag{14}
\]

where \( e^{ij} = -e^{ji} \) and \( e^{j2} = -1 \).

To have supersymmetry in the boundary theory, one must cancel, or otherwise cause to vanish, all of the boundary terms in the supersymmetry variation of (12) and (13). One must also consider the ordinary variation of the action, leading to the Euler-Lagrange equations of motion, and require that all the boundary terms generated in this variation also vanish. There are several ways to accomplish this, but if one wants \( N = 2 \) supersymmetry, and trivial boundary dynamics, then there are two choices. The first is to take

\[
\alpha_1 = \alpha_2 = \alpha \ ; \quad \bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha} , \tag{15}
\]

along with Dirichlet boundary conditions:

\[
\phi_{x=0} = \phi_0 \ ; \quad (\lambda_1 + \lambda_2)_{x=0} = 0 . \tag{16}
\]

The second choice results in a “real” form of \( N = 2 \) supersymmetry. One must first add the following to the boundary action:

\[
\int_{-\infty}^{\infty} \left( \mathcal{W}(\phi) + \bar{\mathcal{W}}(\bar{\phi}) \right) \bigg|_{x=0} \ dy . \tag{17}
\]

Then one takes:

\[
\alpha_j = \bar{\alpha}_j ; \quad \left( \lambda_j + \bar{\lambda}_j \right)_{x=0} = 0 ; \quad j = 1, 2 ;
\]

\[
\left( \phi - \bar{\phi} \right)_{x=0} = 2i \phi_0 ;
\]

\[
\left[ \partial_x (\phi + \bar{\phi}) \right]_{x=0} = \left( \frac{\partial \mathcal{W}}{\partial \phi} + \frac{\partial \mathcal{W}}{\partial \bar{\phi}} \right)_{x=0} . \tag{18}
\]
The foregoing identifications of the $\alpha_j$ and $\bar{\alpha}_j$ are simply a manifestation of the identifications given in (13). The boundary term, (17), can be viewed as the boundary part of the bulk topological charge. This term was anticipated in section 3, and is needed to preserve supersymmetry with the more general boundary conditions. However, this term is not necessary for the Dirichlet conditions since such conditions are sufficiently stringent to kill off the boundary terms arising from the topological charge.

To obtain a theory with non-trivial boundary dynamics (and with couplings of positive dimension), one can add the boundary interaction:

$$\frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{\partial^2 V}{\partial \phi^2} \right) b^\dagger (\lambda_1 + \lambda_2) - \left( \frac{\partial^2 V}{\partial \phi^2} \right) b (\lambda_1 + \lambda_2) + \left| \frac{\partial V}{\partial \phi} \right|^2 dy ,$$

(19)

where $V$ is some scalar potential for $\phi|_{x=0}$.

Consider the model consisting of (12), (13) and (17) (but not (11)). This model has $N = 2$ supersymmetry provided one imposes (13) and

$$\left( \frac{\partial V}{\partial \phi} \right)|_{x=0} = \mu W|_{x=0} .$$

(20)

The boundary fermion fields transform according to:

$$\delta b = 2\mu^{-1} \bar{\alpha} - \mu \alpha W|_{x=0} ;$$

$$\delta b^\dagger = 2\mu^{-1} \bar{\alpha} - \mu \alpha W|_{x=0} .$$

(21)

The parameter $\mu$ is an arbitrary complex number whose modulus can be thought of as the ratio of the boundary and bulk mass scales. The real and imaginary parts of $\mu$, along with the bulk mass scale are precisely the three independent parameters identified in the conformal perturbation theory of section 4.

It is interesting to observe that $N = 2$ supersymmetry requires that the boundary potential be fixed completely by the bulk potential, and that if $W$ is a polynomial of degree $n$ then $V$ is a polynomial of degree $n + 1$. If one wants to have a theory in which the bulk and boundary potentials are independent, then one can only have $N = 1$ supersymmetry. Given the results of section 3, such independence is also probably inconsistent with integrability. Having said this, one should also note that more species of boundary fermions can be added, with whatever potentials one desires, and one can still preserve $N = 2$ supersymmetry provided one of the boundary fermions has the action described above. The choice of bosonic potentials for the other fermion species is arbitrary, but only a small subset of these models will be quantum integrable.

It is also important to observe that if $W$ is quasi-homogeneous: that is, if $W(\alpha^i \phi) = a W(\phi)$, for some $\omega$ and any value of $a$, then the theory has an $R$-symmetry. Namely, it is invariant under: $\phi \rightarrow e^{i\omega \theta} \phi$, $\bar{\phi} \rightarrow e^{-i\omega \theta} \bar{\phi}$, $\lambda_j \rightarrow e^{-i(1-2\omega)\theta} \lambda_j$, $\bar{\lambda}_j \rightarrow e^{i(1-2\omega)\theta} \bar{\lambda}_j$, $(j = 1, 2)$, $\alpha \rightarrow e^{-i\theta} \alpha$, $\bar{\alpha} \rightarrow e^{i\theta} \bar{\alpha}$, $b \rightarrow e^{i\theta} b$, $b^\dagger \rightarrow e^{-i\theta} b^\dagger$, where $\theta$ is some parameter. Such a symmetry is a critical element of establishing that the quasi-homogeneous potential generates the superconformal model. In particular, this $R$-symmetry provides the $N = 2$ superconformal $U(1)$ current. This observation highlights an, as yet, unresolved subtlety: the form of the effective action when the bulk is conformal but the boundary is massive. If the bulk is conformal then $W$ is quasi-homogeneous, and thus (21) implies that the boundary potential must be similarly scale invariant. On the other hand, one knows that the boundary sine-Gordon model has independent bulk and boundary masses, and has $W(\phi) = \phi^3 + \phi$. Presumably the resolution of this lies in taking the careful massless limit of the bulk massive model.

6. EXACT SCATTERING MATRICES

The $N = 2$ supersymmetric integrable models arising from perturbations of the $N = 2$ superconformal minimal series are fairly well understood. The bulk $S$-matrices are known, and some TBA computations have been performed [1, 13, 21, 22]. In these models, the supersymmetry generators are realized as charges that commute with the $S$-matrix, and these charges can be interpreted as a part of an affine quantum group symmetry of the model. It is thus natural to ask about the exact boundary reflection matrices and their behaviour with respect to the supersymmetry. Unfortunately, very few of the appropriate
boundary reflection matrices are known: Of the \( N = 2 \) supersymmetric models, only the boundary sine-Gordon model has been studied in sufficient detail, and so I will focus on this. As will be seen, this model is sufficient to highlight some interesting open problems. A relatively detailed discussion may be found in [14], and here I will merely summarize the ideas and conclusions.

The bulk \( S \)-matrix for sine-Gordon has been known for some time [24], and as explained in [1], this \( S \)-matrix commutes with the generators of \( U_q(SU(2)) \), the affine quantum group based upon \( SU(2) \). At the supersymmetry point, \( \beta^2 = 16\pi/3 \) one has \( q^2 = -1 \), and the four quantum group generators satisfy the fermionic anticommutation relations of the massive \( N = 2 \) supersymmetry algebra.

The boundary is incorporated by finding an exact boundary reflection matrix, which gives the amplitudes for solitons and anti-solitons to reflect into one another. As described in [24, 25, 1, 27], such matrices can be determined by boundary analogues of the Yang-Baxter equations, crossing, unitarity and the bootstrap. For supersymmetry to persist in the presence of the boundary, one seeks linear combinations of the generators that commute appropriately with the reflection matrix. One finds that for general boundary conditions (i.e. general \( M, m, \) and \( \Phi_0 \) in [3]), there is only one supersymmetry remaining, and not two as one might expect from the analysis above.

On the other hand, two supersymmetries survive at two special surfaces in the parameter space: one corresponds to Dirichlet boundary conditions while the other is like a free boundary condition [3]. At these two surfaces in parameter space, the two \( N = 2 \) superalgebras differ in the same manner as the two superalgebras described in section 5. The fact that one does not get \( N = 2 \) supersymmetry, except at special points, can be tracked directly to the problems of bulk topological charges.

The resolution of the conflict between this result and the analysis above is relatively straightforward. One of the assumptions in the derivation of the boundary reflection matrix in [3] is that the boundary has no structure, and in particular, can store no charge. As we saw, the construction of supersymmetric actions explicitly required the introduction of boundary fermions and non-trivial boundary dynamics. Thus to get the correct boundary reflection matrix one will have to allow for this possibility.

There are straightforward ways of decorating the boundary reflection matrix of [3]: One can formally glue a particle to the boundary [1, 13]. That is, one thinks of the new boundary as a combination of the known boundary and a particle of formal rapidity, \( \zeta \), running parallel to the boundary. The rapidity of the boundary particle is formal since this particle is never considered as hitting the boundary – the particle is simply “glued” to the boundary forever. The new boundary reflection matrix, \( \tilde{R} \), has the schematic form:

\[
\tilde{R}(\theta) = S(\theta - \zeta) \ R(\theta) \ S(\theta + \zeta) ,
\]

where \( S \) is the bulk \( S \)-matrix and \( R \) is the original reflection matrix. The matrix, \( \tilde{R} \), satisfies boundary Yang-Baxter, unitarity, crossing and bootstrap as a consequence of the fact that \( S \) and \( R \) satisfy such conditions.

The new boundary reflection matrix inherits the structure of the states of the boundary particle, and also has a free parameter in the rapidity, \( \zeta \). If one chooses the parameters in \( \tilde{R} \) so as to preserve two supersymmetries, then the new boundary reflection matrix will also preserve two supersymmetries since the \( S \)-matrices basically commute with the supercharges. Thus one can easily construct \( N = 2 \) supersymmetry preserving boundary reflection matrices, and these matrices have an independent boundary scale parameter, \( \zeta \).

One can easily imagine gluing more and more particles to the boundary, and getting ever more exotic boundary reflection matrices. In the field theory, this would correspond to adding more and more boundary fermions, and in terms of physics it would correspond to coupling the boson to impurities of higher and higher spin.

2This boundary condition has the property that solitons always reflect as anti-solitons, and vice-versa. This situation corresponds to free boundary conditions only when the bulk is massless [14, 13].

3There are some subtleties here concerning the co-product, see [14].
Thus, there are natural conjectures for the boundary reflection matrices of the $N = 2$ supersymmetric boundary integrable models. These conjectures still need to be tested via Bethe Ansatz since there may be some other way of decorating the known boundary reflection matrices, or some other more general solution to the boundary Yang-Baxter, crossing, unitarity and bootstrap conditions.

7. FINAL COMMENTS

Conformal perturbation theory, along with the results of exact $S$-matrix theory provide a fairly compelling body of evidence that there are $N = 2$ supersymmetric, boundary integrable models, and that these models should have independent boundary and bulk mass scales. There remain some obvious issues to be investigated. First, it would be useful to know if the $N = 2$ supersymmetry preserving, boundary reflection matrices described here are the only such matrices for the sine-Gordon model at the supersymmetry point. In addition, some TBA computations need to be done to check the correspondence between the boundary actions and the boundary reflection matrices. For example, one could compute the boundary entropy and check it against the semi-classical arguments based upon the form of the boundary potential, as in [29].

More generally, it would be valuable to have more examples of $N = 2$ supersymmetry preserving boundary reflection matrices. The obvious models to investigate are those based upon the Chebyshev perturbations of the higher members of the $N = 2$ superconformal minimal series.

Finally, one has seen that the effective Landau-Ginzburg field theory of the bulk extends naturally to the boundary theory, and already gives some semi-classical insight into the structure of the quantum integrable model. It remains to be seen how much further this can be taken, and whether one can get as much quantum exact information from the boundary Landau-Ginzburg theory as one gets in the bulk theory from knowledge of the bulk Landau-Ginzburg potential.

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REFERENCES

1. S. Ghoshal and A.B. Zamolodchikov, Int. J. Mod. Phys. A9 (1994) 3841, hep-th/9306002.
2. N. Andrei, K Furuya and J. Lowenstein, Rev. Mod. Phys. 55 (1983) 331; A.M. Tsvelik and P.B. Wiegmann, Adv. Phys. 32 (1983) 453.
3. E. Corrigan, P.E. Dorey, R.H. Rietdijk and R. Sasaki, Phys. Lett. 333B (1994) 83, hep-th/9501098; P. Bowcock, E. Corrigan, P.E. Dorey and R.H. Rietdijk, Nucl. Phys. B445 (1995) 469, hep-th/9501098.
4. S. Penati and D. Zanon “Quantum Integrability in Two–Dimensional Systems with Boundary,” IFUM-490-FT, hep-th/9501105.
5. E. Witten and D. Olive, Phys. Lett. 78 (1978) 97.
6. P. Fendley, S. Mathur, C. Vafa and N.P. Warner, Phys. Lett. 243B (1990) 257.
7. W. Lerche and N.P. Warner, Nucl. Phys. B358 (1991) 571.
8. S. Cecotti, P. Fendley, K. Intriligator and C. Vafa, Nucl. Phys. B386 (1992) 405; P. Fendley and H. Saleur, Nucl. Phys. B388 (1992) 609.
9. E. Witten, Int. J. Mod. Phys. A9 (1994) 4783; P. di Francesco and S. Yankielowicz, Nucl. Phys. B409 (1993) 18; P. di Francesco, O. Aharony and S. Yankielowicz, Nucl. Phys. B411 (1994) 584; T. Kawai, Y. Yamada and S.-K. Yang, Nucl.
14. N.P. Warner, \textit{Phys. Lett.} B414 (1994) 191;
15. D. Bernard and A. LeClair, \textit{Phys. Lett.} B340 (1990) 721;
16. P. Mathieu and M.A. Walton, \textit{Phys. Lett.} B346 (1990) 409.

17. P. Mathieu and M.A. Walton, \textit{Nucl. Phys.} B433 (1995) 311;
18. N.P. Warner, \textit{Nucl. Phys.} B359 (1991) 125;
19. P. Fendley, W. Lerche, S.D. Mathur and N.P. Warner, \textit{Nucl. Phys.} B348 (1991) 66.

20. A. LeClair, D. Nemeschansky and N.P. Warner, \textit{Phys. Lett.} B390 (1993) 653.
21. E. Martinec, \textit{Phys. Lett.} B217B (1989) 431;
22. A. Berkovich, C. Gomez and G. Sierra, \textit{Nucl. Phys.} B415 (1994) 681, \texttt{hep-th/9302001};
23. A. Fring and R. Körberle, \textit{Nucl. Phys.} B419 (1994) 589, \texttt{hep-th/9312032}.

24. J.L. Cardy and D.C. Lewellen, \textit{Phys. Lett.} B240 (1989) 514;
25. E. Sklyanin, \textit{JETP Letters} B326 (1989) 73;
26. A.B. Zamolodchikov, \textit{Nucl. Phys.} B390 (1993) 739.

27. A. Fring, P.R. Johnson, M.A.C. Kneipp and D.I. Olive, \textit{Nucl. Phys.} B430 (1994) 597.
28. S. Ghoshal, \textit{Int. J. Mod. Phys.} A9 (1994) 4801, \texttt{cond-mat/9304031}.

29. P. Fendley, \textit{Phys. Rev. Lett.} B382 (1992) 265.
30. A.B. Zamolodchikov and Al.B. Zamolodchikov, \textit{Ann. Phys.} B419 (1994) 253.
31. I.V. Cherednik, \textit{Theor. Math. Phys.} 61 (1984) 977.
32. E. Martinec, \textit{Nucl. Phys.} B218B (1989) 51;
33. A. Fring and R. Körberle, \textit{Nucl. Phys.} B419 (1994) 647;
34. C. Gomez and G. Sierra, \textit{Nucl. Phys.} B419 (1994) 589, \texttt{hep-th/9312032}.
35. A.B. Zamolodchikov and Al.B. Zamolodchikov, \textit{Ann. Phys.} B419 (1994) 253.
36. L. Mezincescu and R. Nepomechie, \textit{Int. J. Mod. Phys.} A6, (1991) 5231.
37. A. Fring and R. Körberle, \textit{Nucl. Phys.} B419 (1994) 647.
38. C. Ahn, D. Bernard, and A. LeClair, \textit{Nucl. Phys.} B340 (1990) 721;
39. \textit{Commun. Math. Phys.} 142 (1991) 99;
40. \textit{Int. J. Mod. Phys.} A10 (1995) 739;
41. \texttt{hep-th/9312032}.
42. A. Fring, P.R. Johnson, M.A.C. Kneipp and D.I. Olive, \textit{Nucl. Phys.} B430 (1994) 597.
43. S. Ghoshal, \textit{Int. J. Mod. Phys.} A9 (1994) 4801; \texttt{cond-mat/9304031}.