Introduction to total dominator edge chromatic number

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Abstract

We introduce the total dominator edge chromatic number of a graph $G$. A total dominator edge coloring (briefly TDE-coloring) of $G$ is a proper edge coloring of $G$ in which each edge of the graph is adjacent to every edge of some color class. The total dominator edge chromatic number (briefly TDEC-number) $\chi'_{td}(G)$ of $G$ is the minimum number of color classes in a TDE-coloring of $G$. We obtain some properties of $\chi'_{td}(G)$ and compute this parameter for specific graphs. We examine the effects on $\chi'_{td}(G)$ when $G$ is modified by operations on vertex and edge of $G$. Finally, we consider the $k$-subdivision of $G$ and study TDEC-number of this kind of graphs.

Keywords: total dominator edge chromatic number; vertex removal; $k$-subdivision

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1 Introduction

Let $G = (V, E)$ be a simple graph and $k \in \mathbb{N}$. A mapping $f : V(G) \rightarrow \{1, 2, ..., k\}$ is called a $k$-proper coloring of $G$, if $f(u) \neq f(v)$ whenever the vertices $u$ and $v$ are adjacent in $G$. A color class of this coloring, is a set consisting of all those vertices assigned the same color. If $f$ is a proper coloring of $G$ with the coloring classes $V_1, V_2, ..., V_k$ such that every vertex in $V_i$ has color $i$, then sometimes write simply $f = (V_1, V_2, ..., V_k)$. The chromatic number $\chi(G)$ of $G$ is the minimum number of colors needed in a proper coloring of a graph.

The total dominator coloring, abbreviated TD-coloring studied in [4, 5, 6, 7]. Let $G$ be a graph with no isolated vertex, the total dominator coloring is a proper coloring of $G$ in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number, abbreviated TDC-number, $\chi_{td}(G)$ of $G$ is the minimum number of color classes in a TD-coloring of $G$. Computation of the TDC-number is NP-complete ([4]). The TDC-number of some graphs has computed

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in [4]. Also Henning in [3] established the lower and the upper bounds on the TDC-number of a graph in terms of its total domination number \( \gamma_t(G) \). He has shown that, every graph \( G \) with no isolated vertex satisfies \( \gamma_t(G) \leq \chi_d'(G) \leq \gamma_t(G) + \chi(G) \). The properties of TD-colorings in trees has studied in [3, 4]. Trees \( T \) with \( \gamma_t(T) = \chi_d'(T) \) has characterized in [3]. We have examined the effects on \( \chi_d'(G) \) when \( G \) is modified by operations on the vertex and the edge of \( G \), and the TDC-number of some operations on two graphs studied in [2].

Motivated by TDC-number of a graph, we consider the proper edge coloring of \( G \) and introduce the total dominator edge chromatic number (TDEC-number) of \( G \), \( \chi_d^n(G) \), obtain some properties of \( \chi_d^n(G) \) and compute this parameter for specific graphs, in the next section. In Section 3, we examine the effects on \( \chi_d'(G) \) when \( G \) is modified by operations on vertex and edge of \( G \). Finally in Section 4, we study the TDEC-number of \( k \)-subdivision of graphs.

2 Introduction to total dominator edge chromatic number

In this section, we state the definition of total dominator edge chromatic number and obtain this parameter for some specific graphs.

**Definition 2.1** A total dominator edge coloring, briefly TDE-coloring, of a graph \( G \) is a proper edge coloring of \( G \) in which each edge of the graph is adjacent to every edge of some color class. The total dominator edge chromatic number (TDEC-number) \( \chi_d^n(G) \) of \( G \) is the minimum number of color classes in a TDE-coloring of \( G \). A \( \chi_d^n(G) \)-coloring of \( G \) is any total dominator edge coloring of \( G \) with \( \chi_d^n(G) \) colors.

**Remark 2.2** For every graph \( G \) with maximum degree \( \Delta(G) \), \( \chi_d^n(G) \geq \Delta(G) \). This inequality is sharp. As an example, for the star graph \( K_{1,n} \), \( \chi_d^n(K_{1,n}) = n \).

The following theorem gives the total dominator edge chromatic number of a path.

**Theorem 2.3** If \( P_n \) is the path graph of order \( n \geq 9 \), then

\[
\chi_d^n(P_n) = \begin{cases} 
2k + 2 & \text{if } n = 4k + 1, \\
2k + 3 & \text{if } n = 4k + 2, \\
2k + 4 & \text{if } n = 4k + 3, 4k + 4.
\end{cases}
\]

Also \( \chi_d^n(P_3) = \chi_d^n(P_4) = 2, \chi_d^n(P_5) = 3, \chi_d^n(P_6) = \chi_d^n(P_7) = 4 \) and \( \chi_d^n(P_8) = 5 \).

**Proof.** It is easy to show that \( \chi_d^n(P_3) = \chi_d^n(P_4) = 2, \chi_d^n(P_5) = 3, \chi_d^n(P_6) = \chi_d^n(P_7) = 4 \) and \( \chi_d^n(P_8) = 5 \). Suppose that \( n \geq 9 \). First we show that in a TDE-coloring, for each four consecutive edges we shall use at least two new colors. We consider two cases. If an used color assign to edge \( e_{i+1} \), then we need to assign a new color to the edge \( e_{i+2} \)
and $e_{i+3}$ to have a TDE-coloring (see Figure 1). If a new color assign to the edge $e_{i+1}$, then we have to assign a new color to $e_{i+2}$ or $e_i$ to have a TDE-coloring. So we need at least two new colors in every four consecutive vertices.

If $n = 4k + 1$, for some $k \in \mathbb{N}$, then we give a TDE-coloring for $P_{4k+1}$ which use only two new colors in every four consecutive edges. Define a function $f_0$ on $E(P_{4k})$ such that for any edge $e_i$,

$$f_0(e_i) = \begin{cases} 
1 & \text{if } i = 1 + 4s, \\
2 & \text{if } i = 4s. 
\end{cases}$$

where $s$ is a natural number and for any $e_i$, $i \neq 4s$ and $i \neq 4s + 1$, $f_0(e_i)$ is a new number. Then this coloring is a TDE-coloring of $P_{4k+1}$ with the minimum number $2k + 2$ colors.

If $n = 4k + 2$, for some $k \in \mathbb{N}$, then we first color the $4k - 4$ edges using $f_0$. Now for the rest of edges we define $f_1$ as $f_1(e_{4k-3}) = 1$, $f_1(e_{4k-2}) = 2k + 1$, $f_1(e_{4k-1}) = 2k + 2$, $f_1(e_{4k}) = 2k + 3$ and $f_1(e_{4k+1}) = 2$. Since for every five consecutive edges we have to use at least three new colors, so this edge coloring is a TDE-coloring of $P_{4k+1}$ with the minimum number $2k + 3$ colors.

If $n = 4k + 3$, for some $k \in \mathbb{N}$, then using $f_0$ we color the $4k - 4$ edges. Now for the rest of edges define $f_2$ as $f_2(e_{4k-3}) = 1$, $f_2(e_{4k-2}) = 2k + 1$, $f_2(e_{4k-1}) = 2k + 2$, $f_2(e_{4k}) = 2k + 3$, $f_2(e_{4k+1}) = 2k + 4$ and $f_2(e_{4k+2}) = 2$. Since for every six consecutive edges we have to use at least four new colors, so this edge coloring is a TDE-coloring of $P_{4k+2}$ with the minimum number $2k + 4$ colors.

If $n = 4k + 4$, for some $k \in \mathbb{N}$, then using $f_0$ we color the $4k - 4$ edges and for the rest of edges define $f_3$ as $f_3(e_{4k-3}) = 1$, $f_3(e_{4k-2}) = 2k + 1$, $f_3(e_{4k-1}) = 2k + 2$, $f_3(e_{4k}) = 2$, $f_3(e_{4k+1}) = 2k + 3$, $f_3(e_{4k+2}) = 2k + 4$ and $f_3(e_{4k+2}) = 2$. This coloring is a TDE-coloring of $P_{4k+2}$ with the minimum number $2k + 4$ colors. So we have the result. □

**Theorem 2.4** If $C_n$ is the cycle graph of order $n \geq 8$, then

$$\chi''_d(C_n) = \begin{cases} 
2k + 2, & \text{if } n = 4k, \\
2k + 3, & \text{if } n = 4k + 1, \\
2k + 4, & \text{if } n = 4k + 2, 4k + 3.
\end{cases}$$
Figure 2: TDE-coloring of $P_4$, $P_5$ and $P_6$.

Figure 3: Cycle graph of order $n$, $C_n$.

Also $\chi'_d(C_3) = 3$, $\chi'_d(C_4) = 2$, $\chi'_d(C_5) = \chi'_d(C_6) = 4$ and $\chi'_d(C_7) = 5$.

Proof. It is similar to the Proof of Theorem 2.3. □

The following corollary is an immediate consequence of Theorems 2.3 and 2.4.

Corollary 2.5 For every $n \geq 6$, $\chi'_d(P_n) = \chi'_d(C_{n-1})$.

The following theorem present a lower bound for TDEC-number of graphs $G$ which have the graph $P_6$ as induced subgraph.

Theorem 2.6 If $G$ is a connected graph containing $P_6$ as an induced subgraph, then $\chi'_d(G) \geq \Delta(G) + 2$. More generally, if the path graph $P_n$ is an induced subgraph of $G$, then $\chi'_d(G) \geq \Delta(G) + \chi'_d(P_{n-2})$.

Proof. We assign $\Delta(G)$ colors to the edges which are incident to the vertex with maximum degree $\Delta(G)$. Now we consider $P_6$ as induced subgraph of $G$. As we have seen in the Proof of Theorem 2.3 we need at least two new colors for each four consecutive edges. So we have $\chi'_d(G) \geq \Delta(G) + 2$. The proof of inequality $\chi'_d(G) \geq \Delta(G) + \chi'_d(P_{n-2})$ is similar. □

Remark 2.7 The graph $G$ in Figure 4 and its coloring shows that the lower bound in Theorem 2.6 is sharp.
Theorem 2.8 For every $n \in \mathbb{N}$, $2n - 1 \leq \chi''_d(K_{2n}) \leq 4n - 2$ and $2n \leq \chi''_d(K_{2n+1}) \leq 4n - 1$.

Proof. The lower bounds follow from Remark 2.2. To obtain the upper bound, suppose that $V(K_{2n+1}) = \{u_1, \ldots, u_{2n+1}\}$. By removing the vertex $u_{2n+1}$, we have the complete graph $K_{2n}$. We know that $\chi'(K_{2n}) = 2n - 1$. So we color the edges of $K_{2n}$ with $2n - 1$ colors. Now we add the vertex $u_{2n+1}$ and make $K_{2n+1}$ and assign the new colors $2n, 2n + 1, \ldots, 4n - 1$ to new edges. This is a TDE-coloring for $K_{2n+1}$. Therefore $\chi''_d(K_{2n+1}) \leq 4n - 1$. By the similar method we have $\chi''_d(K_{2n+1}) \leq 4n - 1$. □

Theorem 2.9 (i) For every $n \neq m$, $\max\{n, m\} \leq \chi''_d(K_{n,m}) \leq m + n - 1$.

(ii) For every $n \in \mathbb{N}$, $n \leq \chi''_d(K_{n,n}) \leq 2n$.

Proof.

(i) The lower bounds follow from Remark 2.2. To obtain the upper bound, suppose that $V(K_{n,m}) = X \cup Y$, where $X = \{u_1, \ldots, u_m\}$ and $Y = \{u_{m+1}, \ldots, u_{m+n}\}$ and $m \geq n$. We have the following cases:

Case 1) $m = n + 1$. By removing the vertex $u_1$, we have the complete bipartite graph $K_{n,n}$. We know that $\chi'(K_{n,n}) = n$. So we color the edges of $K_{n,n}$ with $n$ colors. Now we add the vertex $u_1$ and make $K_{n+1,n}$ and assign the new colors $n + 1, n + 2, \ldots, 2n$ to new edges. This is a TDE-coloring for $K_{n,m}$ and we have $\chi''_d(K_{n,m}) \leq 2n = m + n - 1$.

Case 2) $m > n + 1$. By removing the vertex $u_1$, we have the complete bipartite graph $K_{m-1,n}$. We know that $\chi'(K_{m-1,n}) = m - 1$. So we color the edges of $K_{m-1,n}$ with $m - 1$ colors. Now we add the vertex $u_1$ and make $K_{m,n}$ and assign the new colors $m, m + 1, \ldots, m + n - 1$ to new edges. This is a TDE-coloring for $K_{m,n}$ and we have $\chi''_d(K_{m,n}) \leq m + n - 1$.

(ii) In this part we have $m = n$. By removing the vertex $u_1$, we have the complete bipartite graph $K_{n-1,n}$. We know that $\chi'(K_{n-1,n}) = n$. So we color the edges of
$K_{n-1,n}$ with $n$ colors. Now we add the vertex $u_1$ and make $K_{n,n}$ and assign the new colors $n+1, n+2, \ldots, 2n$ to new edges. This is a TDE-coloring for $K_{n,n}$ and we have $\chi^t_d(K_{n,n}) \leq 2n$. \hfill \square

**Remark 2.10** The lower bounds in parts (i) and (ii) of Theorem 2.9 are sharp. It suffices to consider $K_{3,2}$ and $K_{2,2} = C_4$, respectively. Note that $\chi^t_d(K_{3,2}) = 3$ and $\chi^t_d(C_4) = 2$. Also the upper bound of part (i) is sharp. It suffices to consider the star graph $K_{1,6}$. Note that $\chi^t_d(K_{1,6}) = 1 + 6 - 1 = 6$.

Let $n$ be any positive integer and $F_n$ be the friendship graph with $2n+1$ vertices and $3n$ edges, formed by the join of $K_1$ with $nK_2$. By Remark 2.2 and TDE-coloring which has shown in Figure 5, we have the following result for the wheel of order $n$, $W_n$ and the friendship graph $F_n$.

**Theorem 2.11** (i) For any $n \geq 3$, $\chi^t_d(W_n) = n - 1$.

(ii) For $n \geq 2$, $\chi^t_d(F_n) = 2n$.

![Figure 5: TDE-coloring of wheel and friendship graph of order $n$](image)

3 \hspace{1em} TDEC-number of some operations on a graph

The graph $G - v$ is a graph that is made by deleting the vertex $v$ and all edges incident to $v$ from the graph $G$ and the graph $G - e$ is a graph that obtained from $G$ by simply removing the edge $e$. In this section we present bounds for TDEC-number of $G - v$ and $G - e$. We begin with $G - e$.

**Theorem 3.1** If $G$ is a connected graph, and $e \in E(G)$ is not a bridge of $G$, then

$$\chi^t_d(G) - 2 \leq \chi^t_d(G - e) \leq \chi^t_d(G) + 2.$$
Figure 6: Cases which has considered in the proof of Theorem 3.1.  

**Proof.** First we prove the right inequality. Suppose that the edge $e$ in a TDE-coloring of $G$ has color $i$. If no edges of $G$ use the color class $i$, then TDE-coloring of $G$ is a TDE-coloring of $G - e$, too. So $\chi_d'(G - e) \leq \chi_d'(G)$. If some edges of $G$ use the color class $i$ in TDE-coloring, then we have at most two edges with color $i$. If two edges of $G$ have color $i$, then removing $e$ does not effect on TDE-coloring and any edge uses the old color class in TDE-coloring of $G$. So $\chi_d'(G - e) \leq \chi_d'(G)$. If only one edge $e$ has the color $i$, then we change the color of some edges in $G - e$ to have a TDE-coloring for $G - e$. In this case the edge $e$ uses some color class, say $k$, and is adjacent to all color class $k$. We can not have more than two $k$ in this case. Suppose that we have two $k$. Then we have only two cases for the graph $G$ as we see in Figure 6. In Figure 6 the colors $l$ and $m$ are new colors and we only change the color of some edges in $G - e$ and assign the other edges their old color in $G$. This coloring is a TDE-coloring for $G - e$. In any case we do not use more than two new colors. Therefore we have $\chi_d'(G - e) \leq \chi_d'(G) + 2$. Now suppose that we have only one color $k$. Then we have only two cases for the graph $G$ as we see in Figure 7. In Figure 7 the colors $l$ and $m$ are new colors and we only change the color of some edges in $G - e$ and assign the other edges their old color in $G$. This kind of coloring is a TDE-coloring for $G - e$. In any case we do not use more than two new colors. So we have $\chi_d'(G - e) \leq \chi_d'(G) + 2$.  

Now we prove that $\chi_d'(G) - 2 \leq \chi_d'(G - e)$. To do this, first we color $G - e$ and then we add edge $e$. We assign new color $i$ to $e$ and new color $j$ to one edge which is adjacent to $e$. So we have a TDE-coloring for $G$ and $\chi_d'(G) \leq \chi_d'(G - e) + 2$. Therefore we have the result. □

**Theorem 3.2** If $G$ is a connected graph, and $v \in V(G)$ is not a cut vertex of $G$, then  

$$\chi_d'(G) - \deg(v) \leq \chi_d'(G - v) \leq \chi_d'(G) + \deg(v).$$

**Proof.** First we prove the left inequality. We give a TDE-coloring to $G - v$, add $v$ and all the corresponding edges. Then we assign $\deg(v)$ new colors to these edges.
and do not change the color of other edges. So this is a TDE-coloring of $G$ and $\chi''_d(G) \leq \chi''_d(G - v) + \text{deg}(v)$.

For the right inequality, first we give a TDE-coloring to $G$. In this case, since $v$ is not a cut vertex, each edge which is adjacent to an edge with endpoint $v$ has another adjacent edge too. We change the color of this edge to a new color and do this $\text{deg}(v)$ times and do not change the color of the other edges. So this is a TDE-coloring of $G - v$ and $\chi''_d(G - v) \leq \chi''_d(G) + \text{deg}(v)$. Therefore we have the result.

The following theorem is an immediate consequence of Theorems 2.4 and 2.11.

**Theorem 3.3** There is a connected graph $G$, and a vertex $v \in V(G)$ which is not a cut vertex of $G$ such that $|\chi''_d(G) - \chi''_d(G - v)|$ can be arbitrarily large.

In a graph $G$, contraction of an edge $e$ with endpoints $u, v$ is the replacement of $u$ and $v$ with a single vertex such that edges incident to the new vertex are the edges other than $e$ that were incident with $u$ or $v$. The resulting graph $G/e$ has one less edge than $G$. We denote this graph by $G/e$. We end this section with the following theorem which gives bounds for $\chi''_d(G/e)$.

**Theorem 3.4** If $G$ is a connected graph and $e = uv \in E(G)$, then

$$\chi''_d(G) - 2 \leq \chi''_d(G/e) \leq \chi''_d(G) + \min\{\text{deg}(u), \text{deg}(v)\} - 1.$$

**Proof.** First we prove the left inequality. We give a TDE-coloring to $G/e$, add $e$ and assign it a new color, say $i$ and change the color of one of its adjacent edges to new
color \( j \) and do not change other colors. This is a TDE-coloring of \( G \). So we have \( \chi_d'(G) \leq \chi_d'(G/e) + 2 \). For the right inequality, we give a TDE-coloring to \( G \). Suppose that \( \min\{\deg(u), \deg(v)\} = \deg(u) \). Now we make \( G/e \) and change the color of adjacent edges of \( e \) with the endpoint \( u \) to new colors. So we have the result. □

**Remark 3.5** The lower bound in Theorem 3.4 is sharp. It suffices to consider the cycle graph \( C_5 \) as \( G \). Note that \( \chi_d'(C_5) = 4 \) and \( \chi_d'(C_4) = 2 \).

### 4 TDEC-number of \( k \)-subdivision of a graph

The \( k \)-subdivision of \( G \), denoted by \( G_1^k \), is constructed by replacing each edge \( u_iv_j \) of \( G \) with a path of length \( k \), say \( P_{u_iv_j} \). These \( k \)-paths are called superedges, any new vertex is an internal vertex, and is denoted by \( x_{i,j} \) if it belongs to the superedge \( P_{u_iv_j} \), \( i < j \) with distance \( l \) from the vertex \( v_i \), where \( l \in \{1, 2, \ldots, k - 1\} \). Note that for \( k = 1 \), we have \( G_1^1 = G \). If the graph \( G \) has \( v \) vertices and \( e \) edges, then the graph \( G_1^k \) has \( v + (k - 1)e \) vertices and \( ke \) edges. The total dominator chromatic number of a graph has studied in [1]. In this section we study TDEC-number of \( k \)-subdivision of a graph. In particular, we obtain some bounds for \( \chi_d'(G_1^k) \) and prove that for any \( k \geq 2 \), \( \chi_d'(G_1^k) \leq \chi_d'(G_1^{k+1}) \).

**Theorem 4.1** If \( G \) is a graph with \( m \) edges, then \( \chi_d'(G_1^k) \geq m \), for \( k \geq 3 \).

**Proof.** For \( k = 3 \), in any superedge \( P_{\{u,v\}} \) such as \( \{v, x_1^{(v,w)}, x_2^{(v,w)}, w\} \). The edge \( x_1^{(v,w)}, x_2^{(v,w)} \) need to use a new color in at least one of its adjacent edges, and we cannot use this color in any other superedges. So we have the result. □

**Theorem 4.2** If \( G \) is a connected graph with \( m \) edges and \( k \geq 2 \), then

\[
\chi_d'(P_{k+1}) \leq \chi_d'(G_1^k) \leq m\chi_d'(P_{k+1}).
\]

**Proof.** First we prove the the right inequality. Suppose that \( e = uu_1 \) be an arbitrary edge of \( G \). This edge is replaced with the super edge \( P_{\{u,u_1\}} \) in \( G_1^k \), with vertices \( \{u, x_1^{(u,u_1)}, \ldots, x_k^{(u,u_1)}, u_1\} \). We color this superedge with \( \chi_d'(P_{k+1}) \) colors as a total dominator edge coloring of \( P_{k+1} \). We do this for all superedges. Thus we need at most \( m\chi_d'(P_{k+1}) \) new colors for a total dominator edge coloring of \( G_1^k \).

For the left inequality, if \( G \) is a path then the result is true. So we suppose that \( G \) is a connected graph which is not a path. Let \( P_{\{v,w\}} \) be an arbitrary superedge of \( G_1^k \) with vertex set \( \{v, x_1^{(v,w)}, \ldots, x_k^{(v,w)}, w\} \). Since \( G \) is not a path, so at least one of \( v \) and \( w \) is adjacent to some vertices of \( G_1^k \) except \( x_1^{(v,w)} \) and \( x_{k-1}^{(v,w)} \), respectively. Let \( e \) be a total dominator edge coloring of \( G_1^k \). The two following cases can be occured: either the restriction of \( e \) to edges of \( P_{\{v,w\}} \) is a total dominator edge coloring and
so we have the result, or not. If the restriction of \( c' \) to edges of \( P_{\{v,w\}} \) is not a total dominator coloring then since \( c' \) is a total dominator edge coloring of \( G^\perp \), we conclude that at least one of edges \( vx_1^{\{v,w\}} \) and \( wx_k^{\{v,w\}} \), as the edges of the induced subgraph \( P_{\{v,w\}} \), are not adjacent to every edge of some color class. Without loss of generality we assume that the edge \( vx_1^{\{v,w\}} \), as the edge of the induced subgraph \( P_{\{v,w\}} \), is not adjacent to every vertex of some color class. But \( c' \) is a total dominator coloring of \( G^\perp \) so the edge \( vx_1^{\{v,w\}} \) is adjacent to every edge of some color class, as the edge of \( G^\perp \). Hence there is a new color for an adjacent edge of \( vx_1^{\{v,w\}} \), except the edge \( x_1^{\{v,w\}}x_2^{\{v,w\}} \). Thus if we use this new color for the edge \( x_1^{\{v,w\}}x_2^{\{v,w\}} \) and consider the restriction of \( c' \) for the remaining edges of superedge \( P_{\{v,w\}} \), then \( P_{\{v,w\}} \) has a total dominator edge coloring. Therefore the total edge coloring \( c' \) has at least \( \chi''_d(P_{k+1}) \) colors.

The lower bound of Theorem 4.2 is sharp for \( P_2 \) and by the following Proposition we show that the upper bound of this Theorem is sharp for \( G = K_{1,n} \) and \( k = 3 \).

**Proposition 4.3** For every \( n \geq 3 \), \( \chi''_d(K_{1,n}^\perp) = 2n \).

**Proof.** Let \( e_1, \ldots, e_n \) be the pendant edges of \( K_{1,n}^\perp \). The adjacent edges to \( e_i \) is denoted by \( f_i \), and the adjacent edge to \( f_i \) is denoted by \( g_i \) for any \( 1 \leq i \leq n \). Since edge \( f_i \) is the only edge adjacent to \( e_i \), so the color of \( f_i \) should not be used for any other edges of graph, where \( 1 \leq i \leq n \). Thus we color the edges \( f_1, \ldots, f_n \) with colors \( 1, \ldots, n \), respectively, and do not use these colors any more. For every \( 1 \leq i \leq n \), the edge \( f_i \) is adjacent to \( e_i \) and \( g_i \), thus we need a new color for at least one of \( e_i \) and \( g_i \). So we need at least \( 2n \) color to have a TDE-coloring of \( K_{1,n}^\perp \). Now for every \( e_i \) and \( g_i \) we use the new color \( i + n \). Obviously this is a TDE-coloring of \( K_{1,n}^\perp \) and we have the result. \( \square \)

Here we improve the lower bound of Theorem 4.2 for \( k \geq 10 \).

**Theorem 4.4** If \( G \) is a connected graph with \( m \) edges and maximum degree \( \Delta(G) \) and \( k \geq 10 \), then

\[
m(\chi''_d(P_{k-1}) - 2) + 2 \leq \chi''_d(G^\perp).
\]

**Proof.** Let \( e = vw \) be an edge of \( G \). We consider the superedge \( P_{\{v,w\}} \) with vertex set \( \{v, x_1^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}, w\} \). It is clear that \( P_{\{v,w\}} \setminus \{v, w\} \) is the path graph \( P_{k-1} \). Since we use repetitious colors for the edges \( x_1^{\{v,w\}}x_2^{\{v,w\}} \) and \( x_k^{\{v,w\}}x_{k-1}^{\{v,w\}} \) in the TDE-coloring of paths, so we need at least \( \chi''_d(P_{k-1}) - 2 \) colors for each superedges and we cannot use these colors anymore. Also we need two colors for edges \( x_1^{\{v,w\}}x_2^{\{v,w\}} \) and \( x_k^{\{v,w\}}x_{k-1}^{\{v,w\}} \) and some other edges hence the result follows. \( \square \)
Theorem 4.5 If $G$ is a connected graph with $m$ edges and maximum degree $\Delta(G)$ and $k \geq 10$, then
\[
\chi''_d(G^\uparrow) \geq \begin{cases} 
 m\left(\frac{k}{2}\right) + 2 & k \equiv 0 \pmod{4} \\
 m\left(\frac{k-1}{2}\right) + 2 & k \equiv 1 \pmod{4} \\
 m\left(\frac{k-2}{2}\right) + 2 & k \equiv 2 \pmod{4} \\
 m\left(\frac{k-1}{2}\right) + 2 & k \equiv 3 \pmod{4}.
\end{cases}
\]

Proof. It follows by Theorems 2.3 and 4.4. \( \Box \)

Theorem 4.6 If $G$ is a connected graph with $m$ edges with maximum degree $\Delta(G)$ and $k \geq 10$, then
\[
\chi''_d(G^\uparrow) \leq m\left(\chi''_d(P_{k+1}) - 2\right) + \Delta(G).
\]

Proof. As we see in the TDE-coloring of paths, we can use the same color for the pendant edges. So we assign the colors 1, 2, ..., $\Delta(G)$ to all the edges incident to the vertices belong to $G$ and we color other edges of any superedges with $\chi''_d(P_{k+1}) - 2$ colors. This is a TDE-coloring for $G^\uparrow$ and hence the result follows. \( \Box \)

Theorem 4.7 If $G$ is a connected graph with $m$ edges and $k \geq 10$, then
\[
\chi''_d(G^\uparrow) \leq \begin{cases} 
 \frac{mk}{2} + \Delta(G), & k \equiv 0 \pmod{4} \\
 m\left(\frac{k-1}{2}\right) + \Delta(G), & k \equiv 1 \pmod{4} \\
 m\left(\frac{k-2}{2}\right) + \Delta(G), & k \equiv 2 \pmod{4} \\
 m\left(\frac{k-1}{2}\right) + \Delta(G), & k \equiv 3 \pmod{4}.
\end{cases}
\]

Proof. It follows by Theorems 2.3 and 4.6. \( \Box \)

Theorem 4.8 For any $k \geq 4$, $\chi''_d(G^\uparrow) \leq \chi''_d(G^\uparrow_{k+1})$.

Proof. First we give a TDE-coloring to the edges of $G^\uparrow_{k+1}$. Let $P_{\{v,w\}}$ be an arbitrary superedge of $G^\uparrow_{k+1}$ with vertex set $\{v, x_1^{\{v,w\}}, \ldots, x_k^{\{v,w\}}, w\}$. We have the following cases:

Case 1) There exists an edge $u \in \{x_1^{\{v,w\}}, x_2^{\{v,w\}}, \ldots, x_k^{\{v,w\}}\}$ such that other edges of graph are not adjacent to all edges with color class of the edge $u$. Consider the graph in Figure 8. Suppose that the edge $u$ has the color $i$ and the edge $n$ has the color $\alpha$. The edge $m$ is adjacent to all edges with color class $j$ and $j \neq i$ and the edge $n$ is adjacent to all edges with color class $k$ and $k \neq i$. Since $k \geq 4$, without loss of generality, suppose that $m \neq vx_1^{\{v,w\}}$. We have two subcases:

Subcase i) The color of the edge $m$ is not $\alpha$. In this case, we make $G/u$ and do not change the color of any edges. So without adding a new color we have a TDE-coloring for this new graph.
Subcase ii) The color of the edge \( m \) is \( \alpha \). Since the edge \( u \) is adjacent to color class \( \alpha \), so any other edges does not have color \( \alpha \). In this case, by making \( G/m \) and keeping the color of any edges as before, we have a TDE-coloring for this new graph. Because the edge \( t \) is adjacent to color class which is not \( \alpha \), the color of the edge \( t \) is not \( i \) (because if the color of the edge \( t \) is \( i \) it has contradiction with our assumptions), the edge \( n \) is adjacent to all edges with color class \( k \) and the edge \( u \) is adjacent to all edges with color class \( \alpha \).

Case 2) For every edge \( u \in \{ x_1^{\{v,w\}}, x_2^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}, x_k^{\{v,w\}} \} \), there exists an edge such that is adjacent to all edges with color of edge \( u \). Consider the graph in Figure 8. Suppose that the edge \( u \) has the color \( i \) and the edge \( p \) has the color \( j \) and the edge \( p \) is adjacent to all edges with color \( i \). We have two subcases:

Subcase i) The color of the edge \( q \) is not \( i \). We make \( G/r \) and do not change the color of any edges. So without adding a new color we have a TDEC for this new graph since there is no other edges with color \( i \).

Subcase ii) The color of the edge \( q \) is \( i \). In this case the edge \( r \) is adjacent to color class of edge \( s \) and the color of the edge \( s \) does not use for other edges. Now we make \( G/u \) and do not change the color of any edges. Now we consider the color of edge \( r \). If the color of \( r \) is \( j \), then we change it to \( i \) and since obviously the edge \( s \) was adjacent to a color class except \( j \), so we have a TDE-coloring. If the color of the edge \( r \) is not \( j \) we do not change the color of that and we have a TDE-coloring again.

Now we do the same algorithm for all superedges. So we have \( \chi_d^n(G_t^\uparrow) \leq \chi_d^n(G_{t+1}^{\uparrow}) \). □

Figure 8: A part of a superedge in the proof of Theorem 4.8.

**Theorem 4.9** For any graph \( G \), \( \chi_d^n(G_t^\uparrow) \leq \chi_d^n(G_1^\uparrow) \).

**Proof.** First we give a TDE-coloring to the edges of \( G_t^\uparrow \). Let \( P^{(i,j)} \) be an arbitrary superedge of \( G_1^\uparrow \) with edge set \( \{ s, v, u, w \} \) (see Figure 9) and suppose that the edge \( v \) has the color \( \alpha \). We have the following cases:

Case 1) The edges \( u \) and \( s \) are adjacent with an edge with a color class which is not \( \alpha \). we have two subcases:

Subcase i) The color of edges \( u \) and \( s \) are different. In this case, we make \( G/v \) and don’t change the color of any edges. So we have a TDE-coloring for this new graph. Because two edges \( u \) and \( s \) are adjacent with an edge with color class which is not \( \alpha \).
Subcase ii) The color of edges $u$ and $s$ are the same. Suppose that $u$ and $s$ have color $\beta$. In this case $\beta$ does not use for any other edges. So $w$ is adjacent with an edge with color class except $\beta$. Now we make $G/u$. So we have a TDE-coloring for this new graph.

Case 2) The edge $u$ is adjacent to all edges with color class $\alpha$. we have two subcases:

Subcase i) The color of the edge $w$ is not $\alpha$. Suppose that the edge $u$ has color $\gamma$. If the edge $v$ is adjacent with all edges with color $\gamma$, and if the color of $s$ is $\gamma$, we make $G/u$. But if the color of edge $s$ is not $\gamma$, then we make $G/u$ and assign the color $\gamma$ to the edge $w$. So we have a TDE-coloring for this new graph. If the edge $v$ is adjacent to all edges with color except $\gamma$ (edge $s$), then we make $G/u$. So we have a TDE-coloring for this new graph.

Subcase ii) The color of the edge $w$ is $\alpha$. We have two new cases. First, the edge $v$ is adjacent to an edge with color class $\gamma$. Any adjacent edge with $w$ is not adjacent to edge with color class $\alpha$ (except $u$). So we make $G/u$ and assign the color $\gamma$ to $w$. This is a TDE-coloring for this new graph. Second, $v$ is not adjacent with color class $\gamma$. So the color of the edge $s$ does not use any more. Also the edge $s$ is not adjacent to edge with color class $\alpha$. So we make $G/v$. This is a TDE-coloring for this new graph.

Case 3) The edge $s$ is adjacent to all edges with color class $\alpha$. We have two subcases:

Subcase i) If $v$ is the only edge which has color $\alpha$, then we make $G/u$ when $v$ is adjacent with color class of edge $s$ and make $G/s$ when $v$ is adjacent with color class of edge $u$. So this is a TDE-coloring for this new graph.

Subcase ii) If there exist some edges with color $\alpha$, then the edge $u$ is adjacent with color class except $\alpha$. So we make $G/v$. This is a TDE-coloring for this new graph.

We apply this TDE-coloring for all superedges. So we obtain a TDE-coloring for $G^{\frac{1}{2}}$. Therefore we have $\chi''_{cd}(G^{\frac{1}{2}}) \leq \chi''_{cd}(G^{\frac{1}{2}})$. □

**Theorem 4.10** For any graph $G$, $\chi''_{cd}(G^{\frac{1}{2}}) \leq \chi''_{cd}(G^{\frac{1}{2}})$.

**Proof.** First we give a TDE-coloring to the edges of $G^{\frac{1}{2}}$. Let $P^{(ij)}$ be an arbitrary superedge of $G^{\frac{1}{2}}$ with edge set $\{s, v, u\}$ (see Figure 10) and suppose that the edge $v$ has the color $\alpha$. We have the following cases:

Case 1) The edges $u$ and $s$ are adjacent with an edge with a color class which is not $\alpha$. we have two subcases:
Subcase i) The color of edges $u$ and $s$ are different. In this case, we make $G/v$ and don’t change the color of any edges. So we have a TDE-coloring for this new graph. Because two edges $u$ and $s$ are adjacent with an edge with color class which is not $\alpha$.

Subcase ii) The color of edges $u$ and $s$ are the same. Suppose that $u$ and $s$ have color $\beta$. In this case any other edges is not adjacent with color class $\beta$, because $i$ and $j$ are not adjacent vertices (Because of the definition of $G_{13}^{i}$). Now we make $G/u$. So we have a TDE-coloring for this new graph.

Case 2) The edge $s$ is adjacent to all edges with color class $\alpha$. We have two subcases:

Subcase i) If $v$ is the only edge which has color $\alpha$, then we make $G/u$ when $v$ is adjacent with color class of edge $s$ and make $G/s$ when $v$ is adjacent with color class of edge $u$. So this is a TDE-coloring for this new graph.

Subcase ii) If there exist some edges with color $\alpha$, then the edge $u$ is adjacent with color class except $\alpha$. If the edges $u$ and $s$ have the same color then we make $G/u$ and if $u$ and $s$ have different colors, then we make $G/v$. This is a TDE-coloring for this new graph.

Case 3) The edges $u$ and $s$ are adjacent to all edges with color class $\alpha$. So there is no other edge with color $\alpha$. We have two subcases:

Subcase i) The edges $u$ and $s$ have the same color then we make $G/u$.

Subcase ii) The edges $u$ and $s$ have different colors, then we make $G/u$ when $v$ is adjacent with color class of edge $s$ and make $G/s$ when $v$ is adjacent with color class of edge $u$.

We apply this TDE-coloring for all superedges. So we obtain a TDE-coloring for $G_{12}^{i}$. Therefore we have $\chi^{n}_{d}(G_{12}^{i}) \leq \chi^{n}_{d}(G_{13}^{i})$. 

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Figure 10: A superedge in $G_{13}^{i}$. 
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