Lyapunov-Sylvester Computational Method for Two-Dimensional Boussinesq Equation

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Abstract
A numerical method is developed leading to algebraic systems based on generalized Lyapunov-Sylvester operators to approximate the solution of two-dimensional Boussinesq equation. It consists of an order reduction method and a finite difference discretization. It is proved to be uniquely solvable, stable and convergent by using Lyapunov criterion and manipulating Lyapunov-Sylvester operators. Some numerical implementations are provided at the end of the paper to validate the theoretical results.

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Key words: Boussinesq equation, Finite difference, Lyapunov-Sylvester operators.

1 Introduction
In the present work we propose to serve of algebraic operators to approximate the solutions of some PDEs such as Boussinesq one in higher dimensions without adapting classical developments based on separation of variables, radial solutions, ...etc. The crucial idea is to prove that even simple methods of discretization of PDEs such as finite difference, finite volumes, can be transformed to well adapted algebraic systems such as Lyapunov-Sylvester ones developed hereafter. In the present paper, fortunately, we were confronted with more complicated but fascinating method to prove the invertibility of the algebraic operator yielded in the numerical scheme. Instead of using classical methods such as tri-diagonal transformations we applied a topological method to prove the invertibility. This is good as it did not necessitate to compute eigenvalues and precisely bounds/estimates of eigenvalues or direct inverses which remains a complicated problem in general linear algebra and especially for generalized Lyapunov-Sylvester operators. Recall that even though, bounds/estimates of eigenvalues can already be efficient in studying stability. Recall also that block tridiagonal systems for classical methods can be already used here also and can be solved for example using iterative techniques, or highly structured bandwidth...
solvers, Kronecker-product techniques, etc. These methods have been subjects of more general discretizations. See [10], [11], [12], [14], [13], [19] for a review on tridiagonal and block tridiagonal systems, their advantages as well as their disadvantages. In the present paper our principal aim is to apply other algebraic methods to investigate numerical solutions for PDEs in multi-dimensional spaces. We aim to prove that Lyapunov-Sylvester operators can be good candidates for such aim and that they may give best solvers compared to tridiagonal and/or block tridiagonal ones. The present paper is devoted to the development of a numerical method based on two-dimensional finite difference scheme to approximate the solution of the nonlinear Boussinesq equation in \( \mathbb{R}^2 \) written on the form

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} = \Delta u + qu_{xxxx} + (u^2)_{xx}, & \quad ((x, y), t) \in \Omega \times (t_0, +\infty) \\
\frac{\partial u}{\partial t} = \varphi(x, y), & \quad (x, y) \in \Omega
\end{align*}
\]

with initial conditions

\[
\begin{align*}
u(x, y, t_0) = u_0(x, y) & \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, t_0) = \varphi(x, y), & \quad (x, y) \in \Omega
\end{align*}
\]

and boundary conditions

\[
\frac{\partial u}{\partial \eta}(x, y, t) = 0, & \quad ((x, y), t) \in \partial \Omega \times (t_0, +\infty).
\]

In order to reduce the derivation order, we set

\[
v = qu_{xx} + u^2.
\]

We have to solve the system

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} = \Delta u + v_{xx}, & \quad ((x, y), t) \in \Omega \times (t_0, +\infty) \\
v = qu_{xx} + u^2, & \quad ((x, y), t) \in \Omega \times (t_0, +\infty) \\
(u, v)(x, y, t_0) = (u_0, v_0)(x, y), & \quad (x, y) \in \Omega \\
\frac{\partial u}{\partial \eta}(x, y, t) = \varphi(x, y), & \quad (x, y) \in \Omega \\
\frac{\partial v}{\partial \eta}(u, v)(x, y, t) = 0, & \quad (x, y, t) \in \partial \Omega \times (t_0, +\infty)
\end{align*}
\]

on a rectangular domain \( \Omega = L_0 \times L_0 \times L_1 \) in \( \mathbb{R}^2 \), \( t_0 \geq 0 \) is a real parameter fixed as the initial time, \( \Delta \) is the second order partial derivative in time, \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplace operator in \( \mathbb{R}^2 \), \( q \) is a real constant, \( u_{xx} \) and \( u_{xxxx} \) are respectively the second order and the fourth order partial derivative according to \( x \). \( \frac{\partial}{\partial \eta} \) is the outward normal derivative operator along the boundary \( \partial \Omega \).

Finally, \( u \), \( u_0 \) and \( \varphi \) are real valued functions with \( u_0 \) and \( \varphi \) are \( C^2 \) on \( \Omega \). \( u \) (and consequently \( v \)) is the unknown candidates supposed to be \( C^4 \) on \( \Omega \).

The Boussinesq equation has a wide reputability in both theoretic and applied fields such as hydrodynamics, traveling-waves. It governs the flow of ground water, heat conduction, natural convection in thermodynamics for both volume and fluids in porous media, etc. For this reason, many studies have been
developed discussing the solvability of such equations. There are works dealing with traveling-wave solutions, self-similar solutions, scattering method, mono and multi dimensional versions, reduction of multi dimensional equations with respect to algebras, etc. In [7], several finite difference schemes such as three fully implicit finite difference schemes, two fully explicit finite difference techniques, an alternating direction implicit procedure and the Barakat and Clark type explicit formula are discussed and applied to solve the two-dimensional Schrödinger equation with Dirichlets boundary conditions. In [8], the solution of a generalized Boussinesq equation has been developed by means of the homotopy perturbation method. It consisted in a technique method that avoids the discretization, linearization, or small perturbations of the equation and thus reduces the numerical computations. In [26], a collocation and approximation of the numerical solution of the improved Boussinesq equation is obtained based on radial bases. A predictor-corrector scheme is provided and the Not-a-Knot method is used to improve the accuracy in the boundary. Next, [9], a boundary-only meshfree method has been applied to approximate the numerical solution of the classical Boussinesq equation in one dimension. See for instance [1, 3, 6, 20, 21, 22, 23, 24, 27, 28, 29, 30, 31] and the references therein for backgrounds on theses facts.

The method developed in this paper consists in replacing time and space partial derivatives by finite-difference approximations in order linear Lyapunov systems. An order reduction method is adapted leading to a system of coupled PDEs which is transformed by the next to a discrete algebraic one. The motivation behind the idea of applying Lyapunov operators was already evoked in our work [3]. We recall in brief that such a method leads to fast convergent and more accurate discrete algebraic systems without going back to the use of tridiagonal and/or fringe-tridiagonal matrices already used when dealing with multidimensional problems especially in discrete PDEs.

In the organization of the present paper, the next section is concerned with the introduction of the finite difference scheme. Section 3 is devoted to the discretization of the continuous reduced system obtained from (1)-(3) by the order reduction method. Section 4 deals with the solvability of the discrete Lyapunov equation obtained from the discretization developed in the section 3. In section 5, the consistency of the method is shown and next, the stability and convergence of are proved based on Lyapunov method. Finally, a numerical implementation is provided in section 6 leading to the computation of the numerical solution and error estimates.

2 Discrete Two-Dimensional Boussinesq Equation

Consider the domain \( \Omega = [L_0, L_1] \times [L_0, L_1] \subset \mathbb{R}^2 \) and an integer \( J \in \mathbb{N}^* \). Denote \( h = \frac{L_1 - L_0}{J} \) for the space step, \( x_j = L_0 + jh \) and \( y_m = L_0 + mh \) for all \( (j, m) \in I^2 = \{0, 1, \ldots, J\}^2 \). Let \( l = \Delta t \) be the time step and \( t_n = t_0 + nl \),
Similarly, the following discrete equation is obtained from equation (4).

\[ u_t \approx \frac{U_{j,m}^{n+1} - U_{j,m}^{n-1}}{2t} \quad \text{and} \quad u_t \approx \frac{U_{j,m}^{n+1} - 2U_{j,m}^{n} + U_{j,m}^{n-1}}{t^2} \]

and for space derivatives, we shall use

\[ u_x \approx \frac{U_{j+1,m}^{n} - U_{j-1,m}^{n}}{2h} \quad \text{and} \quad u_y \approx \frac{U_{j,m+1}^{n} - U_{j,m-1}^{n}}{2h} \]

for first order derivatives and

\[ u_{xx} \approx \frac{U_{j+1,m}^{n,\alpha} - 2U_{j,m}^{n,\alpha} + U_{j-1,m}^{n,\alpha}}{h^2}, \quad u_{yy} \approx \frac{U_{j,m+1}^{n,\alpha} - 2U_{j,m}^{n,\alpha} + U_{j,m-1}^{n,\alpha}}{h^2} \]

for second order ones, where for \( n \in \mathbb{N}^* \) and \( \alpha \in \mathbb{R} \),

\[ u^{n,\alpha} = \alpha U^{n+1} + (1 - 2\alpha)U^n + \alpha U^{n-1}. \]

Finally, we denote \( \sigma = \frac{l^2}{h^2} \) and \( \delta = \frac{q}{h^2} \).

For \( (j,m) \in \mathbb{I} \), an interior point of the grid \( \mathbb{I}^2 \), \( \mathbb{I} = \{1, 2, \ldots, J - 1\} \), and \( n \geq 1 \), the following discrete equation is deduced from the first equation in the system (4).

\[
U_{j,m}^{n+1} - 2U_{j,m}^{n} + U_{j,m}^{n-1} = \sigma \alpha \left( (U_{j-1,m}^{n+1} - 4U_{j,m}^{n+1} + U_{j+1,m}^{n+1} + U_{j,m-1}^{n+1} + U_{j,m+1}^{n+1})
+ \sigma (1 - 2\alpha) \left( (U_{j-1,m}^{n} - 4U_{j,m}^{n} + U_{j+1,m}^{n} + U_{j,m-1}^{n} + U_{j,m+1}^{n})
+ \sigma (1 - 2\alpha) \left( (U_{j-1,m}^{n-1} - 4U_{j,m}^{n-1} + U_{j+1,m}^{n-1} + U_{j,m-1}^{n-1} + U_{j,m+1}^{n-1})
+ \sigma (V_{j-1,m}^{n+1} - 2V_{j,m}^{n+1} + V_{j+1,m}^{n+1})
+ \sigma (V_{j-1,m}^{n} - 2V_{j,m}^{n} + V_{j+1,m}^{n})
+ \sigma (V_{j-1,m}^{n-1} - 2V_{j,m}^{n-1} + V_{j+1,m}^{n-1}) \right) \right) \right) \right). \quad (6)
\]

Similarly, the following discrete equation is obtained from equation (4).

\[
V_{j,m}^{n+1} + V_{j,m}^{n-1} = 2\delta \alpha \left( (U_{j-1,m}^{n+1} - 4U_{j,m}^{n+1} + U_{j+1,m}^{n+1})
+ \delta (1 - 2\alpha) \left( (U_{j-1,m}^{n} - 4U_{j,m}^{n} + U_{j+1,m}^{n})
+ \delta (1 - 2\alpha) \left( (U_{j-1,m}^{n-1} - 4U_{j,m}^{n-1} + U_{j+1,m}^{n-1})
+ 2F(U_{j,m}^{n}) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right). \quad (7)
\]

where

\[ F(u) = u^2, \quad F^n = F(u^n) \quad \text{and} \quad \tilde{F}^n = \frac{F^{n-1} + F^n}{2}. \]

The discrete boundary conditions are written for \( n \geq 0 \) as

\[ U_{1,m}^0 = U_{1,m}^n \quad \text{and} \quad U_{J-1,m}^0 = U_{J-1,m}^n. \quad (8)\]
The parameter $q$ is related to the equation and has the role of a viscosity-type coefficient and thus it is related to the physical domain of the model. The barycenter parameter $\alpha$ is used to calibrates the position of the approximated solution around the exact one. Of course, these parameters affect surely the numerical solution as well as the error estimates. This fact will be recalled later in the numerical implementations part. In a future work in progress now, we are developing results on numerical solutions of 2D Schrödinger equation on the problem $(1)-(3)$ or its discrete equivalent system $(6)-(9)$. Denote Lyapunov-Sylvester operators to approximate the solution of the continuous solution as well as the error estimates. This fact will be recalled later in the numerical implementations part. In a future work in progress now, we are developing results on numerical solutions of 2D Schrödinger equation on the error estimates as a function on the barycenter calibrations by using variable coefficients $\alpha_n$ instead of constant $\alpha$. The use of these calibrations permits the use of implicit/explicit schemes by using suitable values. For example for $\alpha = \frac{1}{2}$, the barycentre estimation

\[ V^{n,\alpha} = \alpha V^{n+1} + (1 - 2\alpha)V^n + \alpha V^{n-1} = \frac{V^{n+1} + V^{n-1}}{2} \]

which is an implicit estimation that guarantees an error of order 2 in time.

Next, as it is motioned in the introduction, the idea consists in applying Lyapunov-Sylvester operators to approximate the solution of the continuous problem $(1)-(3)$ or its discrete equivalent system $(6)-(9)$. Denote

\[ a_1 = \frac{1}{2} + 2\alpha \sigma, \quad a_2 = -\alpha \sigma, \]
\[ b_1 = 1 - 2(1 - 2\alpha)\sigma, \quad b_2 = (1 - 2\alpha)\sigma, \]
\[ c_1 = (1 - 2\alpha)\delta \quad \text{and} \quad c_2 = \alpha \delta. \]

Equation $(6)$ becomes

\[ a_2 U^{n+1}_{j-1,m} + a_1 U^{n+1}_{j,m} + a_2 U^{n+1}_{j+1,m} + a_3 U^{n+1}_{j,m-1} + a_1 U^{n+1}_{j,m} + a_2 U^{n+1}_{j,m+1} + a_2 \left( V^{n+1}_{j-1,m} - 2V^{n+1}_{j,m} + V^{n+1}_{j+1,m} \right) = b_2 U^{n+1}_{j-1,m} + b_1 U^{n+1}_{j,m} + b_2 U^{n+1}_{j+1,m} + b_1 U^{n+1}_{j,m-1} + b_2 U^{n+1}_{j,m+1} - a_2 U^{n-1}_{j-1,m} - a_1 U^{n-1}_{j,m} - a_2 U^{n-1}_{j+1,m} - a_1 U^{n-1}_{j,m-1} - a_2 U^{n-1}_{j,m+1} + b_2 \left( V^{n-1}_{j-1,m} - 2V^{n-1}_{j,m} + V^{n-1}_{j+1,m} \right) - a_2 \left( V^{n-1}_{j-1,m} - 2V^{n-1}_{j,m} + V^{n-1}_{j+1,m} \right). \]

Equation $(7)$ becomes

\[ V^{n+1}_{j,m} - 2c_2 \left( U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m} \right) = 2c_1 \left( U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m} \right) + 2c_2 \left( U^{n-1}_{j-1,m} - 2U^{n-1}_{j,m} + U^{n-1}_{j+1,m} \right) - V^{n-1}_{j,m} + 2F(U^{n}_{j,m}). \]
Denote $A$, $B$ and $R$ the matrices defined by

$$A = \begin{pmatrix} a_1 & 2a_2 & 0 & \ldots & \ldots & 0 \\ a_2 & a_1 & a_2 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_2 \\ 0 & \ldots & \ldots & 0 & 2a_2 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 2b_2 & 0 & \ldots & \ldots & 0 \\ b_2 & b_1 & b_2 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & b_2 \\ 0 & \ldots & \ldots & 0 & 2b_2 & b_1 \end{pmatrix}$$

and

$$R = \begin{pmatrix} -2 & 2 & 0 & \ldots & \ldots & 0 \\ 1 & -2 & 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & \ldots & 0 & 2 & -2 \end{pmatrix}$$

The system (8)-(11) can be written on the matrix form

$$\begin{cases} \mathcal{L}_A(U^{n+1}) + a_2RV^{n+1} = \mathcal{L}_B(U^n) - \mathcal{L}_A(U^{n-1}) + R(b_2V^n - a_2V^{n-1}), \\ V^{n+1} - 2c_2RU^{n+1} = 2R(c_1U^n + c_2U^{n-1}) - V^{n-1} + 2F^n \end{cases} \quad (12)$$

for all $n \geq 1$ where

$$U^n = (U^n_{j,m})_{0 \leq j, m \leq J}, \quad V^n = (V^n_{j,m})_{0 \leq j, m \leq J} \text{ and } F^n = (F(U^n_{j,m}))_{0 \leq j, m \leq J}$$

and for a matrix $Q \in \mathcal{M}_{(J+1)^2}(\mathbb{R})$, $\mathcal{L}_Q$ is the Lyapunov operator defined by

$$\mathcal{L}_Q(X) = QX + XQ^T, \quad \forall X \in \mathcal{M}_{(J+1)^2}(\mathbb{R}).$$

Remark that $V$ is obtained from the auxiliary function $v$ that is applied to reduce the order of the original PDEs in $u$. This reduction yielded the Lyapunov-Syslvester system (12) above. A natural question that can be raised here turns around the ordering of $U$ and $V$. So, we stress the fact that no essential idea is fixed at advance but, this is strongly related to the system obtained. For example, in (12) above, it is easy to substitute the second equation into the first to omit the unknown matrix $V^{n+1}$ from the first equation. But in the contrary, it is not easier to do the same for $U^{n+1}$, due to the difficulty to substitute it from $\mathcal{L}_A(U^{n+1})$. It is also not guaranteed that the part $a_2RV^{n+1}$ in the first equation is invertible to substitute $V^{n+1}$. So, it is essentially the final system that shows the ordering in $U$ and $V$. 
3 Solvability of the Discrete Problem

In [3], the authors have transformed the Lyapunov operator obtained from the discretization method into a standard linear operator acting on one column vector by juxtaposing the columns of the matrix $X$ horizontally which leads to an equivalent linear operator characterized by a fringe-tridiagonal matrix. We used standard computation to prove the invertibility of such an operator. Here, we do not apply the same computations as in [3], but we develop different arguments. The first main result is stated as follows.

**Theorem 1** The system (12) is uniquely solvable whenever $U_0$ and $U_1$ are known.

**Proof.** It reposes on the inverse of Lyapunov operators. Consider the endomorphism $\Phi$ defined on $\mathcal{M}_{(J+1)^2} (\mathbb{R}) \times \mathcal{M}_{(J+1)^2} (\mathbb{R})$ by $\Phi(X,Y) = (AX + XA^T + a_2RY, \frac{1}{2}Y - c_2RX)$. To prove Theorem 1 it suffices to show that $\ker \Phi$ is reduced to 0. Indeed,

$$\Phi(X,Y) = 0 \iff (AX + XA^T + a_2RY, \frac{1}{2}Y - c_2RX) = (0,0)$$

or equivalently,

$$Y = 2c_2RX \quad \text{and} \quad (A + 2a_2c_2R^2)X + XA^T = 0.$$ 

So, the problem is transformed to the resolution of a Lyapunov type equation of the form

$$L_{W,A}(X) = WX + XA^T = 0$$

where $W$ is the matrix given by $W = A + 2a_2c_2R^2$. Denoting

$$\omega = 2a_2c_2, \ \omega_1 = a_1 + 6\omega, \ \omega = \omega_1 + \omega \quad \text{and} \quad \omega_2 = a_2 - 4\omega$$

the matrix $W$ is explicitly given by

$$W = \begin{pmatrix}
\omega_1 & 2\omega_2 & 2\omega & 0 & \ldots & \ldots & \ldots & 0 \\
2\omega_2 & \omega_1 & \omega_2 & \omega & \ldots & \ldots & \ldots & 0 \\
\omega & \omega & \omega_1 & \omega_2 & \omega & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\omega & \ldots & \omega_1 & \omega_2 & \omega & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
& & & & & & & & \\
\omega & \ldots & \omega & \omega_1 & \omega_2 & \omega & \ldots & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\omega_1 & \omega_2 & \omega_1 & \omega_2 & \omega & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
2\omega & \omega_1 & \omega_2 & \omega & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & 0 & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}$$

Next, we use the following preliminary result of differential calculus (See [16] for example).
Lemma 2 Let $E$ be a finite dimensional ($\mathbb{R}$ or $\mathbb{C}$) vector space and $(\Phi_n)_n$ be a sequence of endomorphisms converging uniformly to an invertible endomorphism $\Phi$. Then, there exists $n_0$ such that, for any $n \geq n_0$, the endomorphism $\Phi_n$ is invertible.

The proof is simple and can be found anywhere in differential calculus references such as [10]. We recall it here for the convenience and clearness of the paper. Recall that the set $\text{Isom}(E)$ (the set of isomorphisms on $E$) is already open in $L(E)$ (the set of endomorphisms of $E$). Hence, as $\Phi \in \text{Isom}(E)$ there exists a ball $B(\Phi, r) \subset \text{Isom}(E)$. The elements $\Phi_n$ are in this ball for large values of $n$. So these are invertible.

Assume now that $l = o(h^{2+s})$, with $s > 0$ which is always possible. Then, the coefficients appearing in $A$ and $W$ will satisfy as $h \to 0$ the following.

$$A_{i,i} = \frac{1}{2} + \varepsilon h^{2+2s} \to \frac{1}{2},$$

For $1 \leq i \leq J - 1$,

$$A_{i,i-1} = A_{i,i+1} = \frac{A_{0,1}}{2} = \frac{A_{J,J-1}}{2} = -\varepsilon h^{2+2s} \to 0.$$

For $2 \leq i \leq J - 2$,

$$W_{i,i} = W_{0,0} = W_{J,J} = \frac{1}{2} + 2\alpha\varepsilon h^{2+2s} - 2\alpha^2 \varepsilon h^{2s} \to \frac{1}{2}.$$

Similarly,

$$W_{1,1} = W_{J-1,J-1} = \frac{1}{2} + 2\alpha\varepsilon h^{2+2s} - 14\alpha^2 \varepsilon h^{2s} \to \frac{1}{2}$$

and

$$W_{i,i-1} = W_{i,i+1} = \frac{W_{0,1}}{2} = \frac{W_{J,J-1}}{2} = -\alpha\varepsilon h^{2+2s} + 8\alpha^2 \varepsilon h^{2s} \to 0$$

Finally,

$$W_{i,i-2} = W_{i,i+2} = \frac{W_{0,2}}{2} = \frac{W_{J,J-2}}{2} = -2\alpha^2 \varepsilon h^{2s} \to 0.$$

Recall that the technique assumption $l = o(h^{2+s})$ is a necessary requirement for the resolution of the present problem and may not be necessary in other PDEs. See for example [2], [3] and [4] for NLS and Heat equations. Next, observing that for all $X$ in the space $\mathcal{M}_{(J+1)^2}(\mathbb{R}) \times \mathcal{M}_{(J+1)^2}(\mathbb{R})$,

$$\| (\mathcal{L}_{W,A} - I)(X) \| = \| (W - \frac{1}{2} I) X + X (A^T - \frac{1}{2} I) \| \leq \| W - \frac{1}{2} I \| + \| A^T - \frac{1}{2} I \| \| X \|,$$

it results that

$$\| \mathcal{L}_{W,A} - I \| \leq \| W - \frac{1}{2} I \| + \| A^T - \frac{1}{2} I \| \leq C(\alpha)h^{2s}. \quad (14)$$

Consequently, the Lyapunov endomorphism $\mathcal{L}_{W,A}$ converges uniformly to the identity $I$ as $h$ goes towards 0 and $l = o(h^{2+s})$ with $s > 0$. Using Lemma 2, the operator $\mathcal{L}_{W,A}$ is invertible for $h$ small enough.
4 Consistency, Stability and Convergence of the Discrete Method

The consistency of the proposed method is done by evaluating the local truncation error arising from the discretization of the system

\[
\begin{align*}
    & u_{tt} - \Delta u - v_{xx} = 0, \\
    & v = q u_{xx} + u^2.
\end{align*}
\]

The principal part of the first equation is

\[
\begin{align*}
    L_{u,v}^1(t,x,y) &= \frac{l^2}{12} \frac{\partial^4 u}{\partial t^4} - \frac{h^2}{12} \left( \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right) - \alpha \frac{l^2}{t^2} \frac{\partial^2 (\Delta u)}{\partial t^2} \\
    &- \frac{h^2}{12} \frac{\partial^2 v}{\partial x^4} - \alpha \frac{l^2}{t^2} \frac{\partial^4 v}{\partial t^2 \partial x^2} + O(l^2 + h^2).
\end{align*}
\]

The principal part of the local error truncation due to the second part is

\[
\begin{align*}
    L_{u,v}^2(t,x,y) &= \frac{l^2}{2} \frac{\partial^2 v}{\partial t^2} + \frac{l^4}{24} \frac{\partial^4 v}{\partial t^4} - \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4} \\
    &- \alpha \frac{l^2}{t^2} \frac{\partial^4 u}{\partial t^2 \partial x^2} + O(l^2 + h^2).
\end{align*}
\]

It is clear that the two operators \(L_{u,v}^1\) and \(L_{u,v}^2\) tend toward 0 as \(l\) and \(h\) tend to 0, which ensures the consistency of the method. Furthermore, the method is consistent with an order 2 in time and space.

We now proceed by proving the stability of the method by applying the Lyapunov criterion. A linear system \(L(x_{n+1}, x_n, x_{n-1}, \ldots) = 0\) is stable in the sense of Lyapunov if for any bounded initial solution \(x_0\) the solution \(x_n\) remains bounded for all \(n \geq 0\). Here, we will precisely prove the following result.

Lemma 3 \(\mathcal{P}_n\): The solution \((U^n, V^n)\) is bounded independently of \(n\) whenever the initial solution \((U^0, V^0)\) is bounded.

We will proceed by recurrence on \(n\). Assume firstly that \(||(U^0, V^0)|| \leq \eta\) for some \(\eta\) positive. Using the system (12), we obtain

\[
\begin{align*}
    & L_{W,A}(U^{n+1}) = L_{B,B}(U^n) + b_3 R V^n - L_{W,A}(U^{n-1}) - a_3 R (F^{n-1} + F^n), \\
    & V^{n+1} = 2c_2 R U^{n+1} + 2R(c_1 U^n + c_2 U^{n-1}) - V^{n-1} + 2\hat{F}^n,
\end{align*}
\]

where \(\tilde{B} = B - 2a_2 c_1 R^2\). Consequently,

\[
||L_{W,A}(U^{n+1})|| \leq ||L_{B,B}|| ||U^n|| + 2|b_3| ||V^n|| \\
+ ||L_{W,A}|| ||U^{n-1}|| + 2|a_2| (||F^{n-1}|| + ||F^n||)
\]

and

\[
||V^{n+1}|| \leq 4|c_2| ||U^{n+1}|| + 4(|c_1| ||U^n|| + |c_2| ||U^{n-1}||) \\
+ ||V^{n-1}|| + ||F^{n-1}|| + ||F^n||.
\]
Next, recall that, for \( l = o(h^{s+2}) \) small enough, \( s > 0 \), we have
\[
\begin{align*}
a_1 &= \frac{1}{2} + 2\alpha h^{2s+2} \to \frac{1}{2}, \quad a_2 = -\alpha h^{2s+2} \to 0, \\
b_1 &= 1 - 2(1 - 2\alpha)h^{2s+2} \to 1, \quad b_2 = (1 - 2\alpha)h^{2s+2} \to 0, \\
c_1 &= (1 - 2\alpha)h^{-2} \to \infty \quad \text{and} \quad c_2 = \alpha h^{2s+2} \to \infty, \\
a_2c_1 &= -\alpha(1 - 2\alpha)h^{2s} \to 0.
\end{align*}
\]
As a consequence, for \( h \) small enough,
\[
\|L_{\tilde{B},B}\| \leq 2\|B\| + 2|a_2c_1|\|R\| \leq 2\max(|b_1|,|b_2|) + 4|a_2c_1| \leq 2 + 4 = 6, \quad (21)
\]
and the following lemma deduced from (14).

**Lemma 4** For \( h \) small enough, it holds for all \( X \in \mathcal{M}_{(J+1)^2}(\mathbb{R}) \) that
\[
\frac{1}{2}\|X\| \leq (1 - C(\alpha)h^{2s})\|X\| \leq \|L_{W,A}(X)\| \leq (1 + C(\alpha)h^{2s})\|X\| \leq \frac{3}{2}\|X\|.
\]
Indeed, recall that equation (14) affirms that \( \|L_{W,A} - I\| \leq C(\alpha)h^{2s} \) for some constant \( C(\alpha) > 0 \). Consequently, for any \( X \) we get
\[
(1 - C(\alpha)h^{2s})\|X\| \leq \|L_{W,A}(X)\| \leq (1 + C(\alpha)h^{2s})\|X\|.
\]
For \( h \leq \frac{1}{(C(\alpha))^{1/2s}} \), we obtain
\[
\frac{1}{2} \leq (1 - C(\alpha)h^{2s}) < (1 + C(\alpha)h^{2s}) \leq \frac{3}{2}
\]
and thus Lemma 4. As a result, (19) yields that
\[
\frac{1}{2}\|U^{n+1}\| \leq 6\|U^n\| + 2\|V^n\| + \frac{3}{2}\|U^{n-1}\| + 2(\|F^{n-1}\| + \|F^n\|)). \quad (22)
\]
For \( n = 0 \), this implies that
\[
\|U^1\| \leq 12\|U^0\| + 4\|V^0\| + 3\|U^{-1}\| + 4(\|F^{-1}\| + \|F^0\|)). \quad (23)
\]
Using the discrete initial condition
\[
U^0 = U^{-1} + l\varphi.
\]
Here we identify the function \( \varphi \) to the matrix whom coefficients are \( \varphi_{j,m} = \varphi(x_j,y_m) \). We obtain
\[
\|U^{-1}\| \leq \|U^0\| + l\|\varphi\|. \quad (24)
\]
Observing that
\[
F_{j,m}^{-1} = F(U_{j,m}^{-1}) = (U_{j,m}^0 - l\varphi_{j,m})^2,
\]
\[
\|F_{j,m}^{-1}\| \leq (U_{j,m}^0 - l\varphi_{j,m})^2 \leq 4(U_{j,m}^0)^2 + 4l^2\|\varphi\|^2.
\]
it results that
\[ |F_{j,m}^{-1}| \leq |U_{j,m}^0|^2 + 2l|\varphi_{j,m}| |U_{j,m}^0| + l^2|\varphi_{j,m}|^2 \]
and consequently,
\[ \|F^{-1}\| \leq \|U^0\|^2 + 2l\|\varphi\|.\|U^0\| + l^2\|\varphi\|^2. \tag{25} \]
Hence, equation (23) yields that
\[ \|U_1\| \leq (15 + 8l\|\varphi\|)\|U_0\| + 4\|V_0\| + 8\|F_0\| + 3l\|\varphi\| + 4l^2\|\varphi\|^2. \tag{26} \]
Now, the Lyapunov criterion for stability states exactly that
\[ \forall \varepsilon > 0, \exists \eta > 0 \text{ s.t.} \quad \|(U_0, V_0)\| \leq \eta \Rightarrow \|(U^n, V^n)\| \leq \varepsilon, \quad \forall n \geq 0. \tag{27} \]
For \( n = 1 \) and \( \|(U^1, V^1)\| \leq \varepsilon \), we seek an \( \eta > 0 \) for which \( \|(U_0, V_0)\| \leq \eta \). Indeed, using (26), this means that, it suffices to find \( \eta \) such that
\[ 8\eta^2 + (19 + 8l\|\varphi\|)\eta + 3l\|\varphi\| + 4l^2\|\varphi\|^2 - \varepsilon < 0. \tag{28} \]
The discriminant of this second order inequality is
\[ \Delta(l, h) = (19 + 8l\|\varphi\|)^2 - 32(3l\|\varphi\| + 4l^2\|\varphi\|^2 - \varepsilon) > 0. \tag{29} \]
For \( h, l \) small enough, this is estimated as
\[ \Delta(l, h) \sim 361 + 32\varepsilon > 0. \]
Thus there are two zeros of the second order equality above \( \eta_1 = \frac{\sqrt{\Delta(l, h) - (19 + 8l\|\varphi\|)}}{16} > 0 \) and a second zero \( \eta_2 < 0 \) rejected. Consequently, choosing \( \eta \in [0, \eta_1] \) [we obtain \( \Delta \). Finally, (26) yields that \( \|U^1\| \leq \varepsilon \). Next, equation (20), for \( n = 0 \), implies that
\[ \|V^1\| \leq A(l, h, \varphi)\|U^0\|^2 + B(l, h, \varphi)\|U^0\| + C(l, h, \varphi) + 16|c_2|\|V^0\|, \tag{30} \]
where
\[ A(l, h, \varphi) = 3 + 32|c_2|, \]
\[ B(l, h, \varphi) = 4 \left(|c_1| + 8|c_2|(2 + l\|\varphi\|) + l\|\varphi\| + \frac{1}{h^2}\right), \]
and
\[ C(l, h, \varphi) = 2(1 + 8|c_2|)l^2\|\varphi\|^2 + 4l(4|c_2| + \frac{1}{h^2})\|\varphi\|. \]
Choosing \( \|(U_0, V_0)\| \leq \eta \), it suffices to study the inequality
\[ A(l, h, \varphi)\eta^2 + (B(l, h, \varphi) + 16|c_2|) \eta + C(l, h, \varphi) - \varepsilon \leq 0. \tag{31} \]
Its discriminant satisfies for $h, l$ small enough,

$$
\Delta(l, h) \sim \frac{16}{h^4} (1 + 20\alpha + |1 - 2\alpha|)^2 + \frac{128\alpha|q|}{h^2} \varepsilon > 0.
$$

(32)

Here also there are two zeros, $\eta'_1 = \frac{\sqrt{\Delta(l, h)} - (B(l, h, \varphi) + 16|c_2|)}{2A(l, h, \varphi)} > 0$ and a second one $\eta' < 0$ and thus rejected. As a consequence, for $\eta \in [0, \eta'_0]$ we obtain $\|V^1\| \leq \varepsilon$. Finally, for $\eta \in [0, \eta_0]$ with $\eta_0 = \min(\eta_1, \eta'_1)$, we obtain $\|(U^1, V^1)\| \leq \varepsilon$ whenever $\|(U^0, V^0)\| \leq \eta$. Assume now that the $(U^k, V^k)$ is bounded for $k = 1, 2, \ldots, n$ (by $\varepsilon_1$) whenever $(U^0, V^0)$ is bounded by $\eta$ and let $\varepsilon > 0$. We shall prove that it is possible to choose $\eta$ satisfying $\|(U^{n+1}, V^{n+1})\| \leq \varepsilon$. Indeed, from (22), we have

$$
\|U^{n+1}\| \leq 19\varepsilon_1 + 8\varepsilon_1^2.
$$

(33)

So, one seeks, $\varepsilon_1$ for which $8\varepsilon_1^2 + 19\varepsilon_1 - \varepsilon \leq 0$. Its discriminant $\Delta = 361 + 32\varepsilon$, with one positive zero $\varepsilon_1 = \frac{16}{361 + 32\varepsilon}$. Then $\|U^{n+1}\| \leq \varepsilon$ whenever $\|(U^k, V^k)\| \leq \varepsilon_1$, $k = 1, 2, \ldots, n$. Next, using (20) and (33), we have

$$
\|V^{n+1}\| \leq (4|c_1| + 80|c_2| + 1) \varepsilon_1 + (32|c_2| + 2) \varepsilon_1^2.
$$

(34)

So, it suffices as previously to choose $\varepsilon_1$ such that

$$(32|c_2| + 2) \varepsilon_1^2 + (4|c_1| + 80|c_2| + 1) \varepsilon_1 - \varepsilon \leq 0.
$$

Then $\|V^{n+1}\| \leq \varepsilon$ whenever $\|(U^k, V^k)\| \leq \varepsilon_1$, $k = 1, 2, \ldots, n$. Next, it holds from the recurrence hypothesis for $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_1')$ that there exists $\eta > 0$ for which $\|(U^0, V^0)\| \leq \eta$ implies that $\|(U^k, V^k)\| \leq \varepsilon_0$, for $k = 1, 2, \ldots, n$, which by the next induces that $\|(U^{n+1}, V^{n+1})\| \leq \varepsilon$.

**Lemma 5** As the numerical scheme is consistent and stable, it is then convergent.

This lemma is a consequence of the well known Lax-Richtmyer equivalence theorem, which states that for consistent numerical approximations, stability and convergence are equivalent. Recall here that we have already proved in (16) and (17) that the used scheme is consistent. Next, Lemma 4 Lemma 4 and equation (27) yields the stability of the scheme. Consequently, the Lax equivalence Theorem guarantees the convergence. So as Lemma 5

5 Numerical implementations

We propose in this section to present some numerical examples to validate the theoretical results developed previously. The error between the exact solutions
and the numerical ones via an $L_2$ discrete norm will be estimated. The matrix norm used will be
\[ \|X\|_2 = \left( \sum_{i,j=1}^{N} |X_{ij}|^2 \right)^{1/2} \]
for a matrix $X = (X_{ij}) \in \mathcal{M}_{N \times 2}$\mathbb{C}$. Denote $u^n$ the net function $u(x,y,t^n)$ and $U^n$ the numerical solution. We propose to compute the discrete error
\[ Er = \max_n \|U^n - u^n\|_2 \] (35)
on the grid $(x_i, y_j), 0 \leq i, j \leq J + 1$ and the relative error between the exact solution and the numerical one as
\[ RelativeEr = \max_n \frac{\|U^n - u^n\|_2}{\|u^n\|_2} \] (36)
on the same grid.

5.1 A Polynomial-Exponential Example
We develop in this part a classical example based on polynomial function with an exponential envelop. We consider the inhomogeneous problem
\[
\begin{aligned}
&u_{tt} = \Delta u + v_{xx} + g(x,y,t), \quad (x,y,t) \in \Omega \times (t_0,T), \\
v = qu_{xx} + u^2, \quad (x,y,t) \in \Omega \times (t_0,T), \\
(u,v)(x,y,t_0) = (u_0,v_0)(x,y), \quad (x,y,t) \in \Omega \times (t_0,T), \\
\frac{\partial u}{\partial n}(x,y,t_0) = \varphi(x,y), \quad (x,y) \in \Omega, \\
\nabla(u,v) = 0, \quad (x,y,t) \in \partial \Omega \times (t_0,T)
\end{aligned}
\] (37)
where $\Omega = [-1,1]^2$ and where the right hand term is
\[ g(x,y,t) = \left[ (x^2 - 1)^2 (x^4 - 58x^2 + 9) - 48 (35x^4 - 30x^2 + 3) + y^4 - 14y^2 + 5 \right] e^{-t} - 16(x^2 - 1)^2 \left[ (x^2 - 1)^4 (15x^2 - 1) + (y^2 - 1)^2 (7x^2 - 1) \right] e^{-2t} \]
The exact solution is
\[ u(x,y,t) = \left[ (x^2 - 1)^4 + (y^2 - 1)^2 \right] e^{-t}. \] (38)
In the following tables, numerical results are provided. We computed for different space and time steps the discrete $L_2$-error estimates defined by (35). The time interval is $[0,1]$ for a choice $t_0 = 0$ and $T = 1$. The following results are obtained for different values of the parameters $J$ (and thus $h$), $l$ (and thus $N$). The parameters $q$ and $\alpha$ are fixed to $q = 0.01$ and $\alpha = 0.25$. We just notice that some variations done on these latter parameters have induced an important variation in the error estimates which explains the effect of the parameter $q$ which has the role of a viscosity-type coefficient and the barycenter parameter.
α which calibrates the position of the approximated solution around the exact one. Finally, some comparison with our work in [3] has proved that Lyapunov type operators already result in fast convergent algorithms with a maximum time of execution of 2.014 sd for the present one. The classical tri-diagonal algorithms associated to the same problem with the same discrete scheme and the same parameters yielded a maximum time of 552.012 sd, so a performance of $23 \times 10^{-4}$ faster algorithm for the present one. We recall that the tests are done on a Pentium Dual Core CPU 2.10 GHz processor and 250 Mo RAM.

Table 1.

| J  | t   | Er   | Relative Er |
|----|-----|------|-------------|
| 10 | 1/100 | 4.0 \times 10^{-3} | 0.1317 |
| 16 | 1/120 | 3.3 \times 10^{-3} | 0.1323 |
| 20 | 1/200 | 2.0 \times 10^{-3} | 0.1335 |
| 24 | 1/220 | 1.8 \times 10^{-3} | 0.1337 |
| 30 | 1/280 | 1.4 \times 10^{-3} | 0.1340 |
| 40 | 1/400 | 9.8 \times 10^{-4} | 0.1344 |
| 50 | 1/500 | 7.8 \times 10^{-4} | 0.1346 |

5.2 A 2-Particles Interaction Example

The example developed hereafter is a model of interaction of two particles or two waves. We consider the inhomogeneous problem

\[
\begin{align*}
\begin{cases}
    u_{tt} &= \Delta u + v_{xx} + g(x,y,t), \quad (x,y,t) \in \Omega \times (t_0,T), \\
v &= q u_{xx} + u^2, \quad (x,y,t) \in \Omega \times (t_0,T), \\
(u,v)(x,y,t_0) &= (u_0,v_0)(x,y), \quad (x,y,t) \in \Omega \times (t_0,T), \\
\n\end{cases}
\end{align*}
\]

where

\[
g(x,y,t) = (4 - 6\psi^2(y))u^2 - \psi^2(x)u.
\]

and $u$ is the exact solution given by

\[
u(x,y,t) = 2\psi^2(x)\psi^2(y)\theta(t)
\]

with

\[
\psi(x) = \cos\left(\frac{x^2}{2}\right), \quad \theta(t) = e^{-it}
\]

and

\[
\varphi(x,y) = -2i\psi^2(x)\psi^2(y)
\]

As for the previous example, the following tables shows the numerical computations for different space and time steps the discrete $L_2$-error estimates defined by (35) and the relative error (36). The time interval is $[-2\pi, +2\pi]$ for a choice
$t_0 = 0$ and $T = 1$. The following results are obtained for different values of the parameters $J$ (and thus $h$), $l$ (and thus $N$). The parameters $q$ and $\alpha$ are fixed here—also the same as previously, $q = 0.01$ and $\alpha = 0.25$. Compared to the tri-diagonal scheme the present one leads a faster convergent algorithms

$$
\begin{array}{|c|c|c|c|}
\hline
J & l & \varepsilon r & \text{Relative } \varepsilon r \\
\hline
10 & 1/100 & 4.10^{-3} & 0.2311 \\
16 & 1/120 & 4.10^{-3} & 0.2372 \\
20 & 1/200 & 2.10^{-3} & 0.2506 \\
24 & 1/220 & 2.10^{-3} & 0.2671 \\
30 & 1/280 & 2.10^{-3} & 0.3074 \\
40 & 1/400 & 1.10^{-3} & 0.3592 \\
50 & 1/500 & 7.10^{-4} & 0.2355 \\
\hline
\end{array}
$$

**Remark 6** For the convenience of the paper, we give here some computations of the determinants $\Delta(l, h)$ for different values of the parameters of the discrete scheme. Firstly, for both examples above, we can easily see that $\|\varphi\|$ = and thus, equation (29) yields that

$$
\Delta(l, h) = 361 + 32\varepsilon + 416l - 256l^2.
$$

For the different values of $l$ as in the tables 1 and 2, we obtain a positive discriminant leading two zeros with a rejected one. For the discriminant of equation (32) we obtain

$$
\Delta(l, h) = \frac{676}{h^4} + \frac{8\varepsilon}{h^2}.
$$

Hence, the results explained previously hold.

## 6 Conclusion

This paper investigated the solution of the well-known Boussinesq equation in two-dimensional case by applying a two-dimensional finite difference discretization. The Boussinesq equation in its original form is a 4-th order partial differential equation. Thus, in a first step it was recasted into a system of second order partial differential equations using a reduction order idea. Next, the system has been transformed into an algebraic discrete system involving Lyapunov-Sylvester matrix terms by using a full time-space discretization. Solvability, consistency, stability and convergence are then established by applying well-known methods such as Lax-Richtmyer equivalence theorem and Lyapunov Stability and by examining the Lyapunov-Sylvester operators. The method was finally improved by developing numerical examples. It was shown to be efficient by means of error estimates as well as time execution algorithms compared to classical ones.
7 Appendix

7.1 The Tridiagonal Associated System

Consider the lexicographic mesh $k = j(J + 1) + m$ for $0 \leq j, m \leq J$, and denote $N = J(J + 2)$, and

$$
\Lambda_N = \{nJ + n - 1 ; \ n \in \mathbb{N}\}, \ \tilde{\Lambda}_N = \{n(J + 1) ; \ n \in \mathbb{N}\} \ \text{and} \ \Theta_N = \Lambda_N \cup \tilde{\Lambda}_N.
$$

We obtain a tri-diagonal block system on the form

$$
\begin{cases}
\tilde{\Lambda}_N = \Lambda_N + 1) - \text{vectors and the matrices} \\
\Lambda_N = \Lambda_N + 1 \text{; and } \Theta_N = \Lambda_N \cup \tilde{\Lambda}_N.
\end{cases}
$$

\[ (40) \]

The numerical solutions' matrices $U^n$ and $V^n$ are identified here as one-column $(N + 1)$-vectors and the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{R}$ are evaluated as follows.

The matrix $\tilde{A}$

$$
\tilde{A}_{j,j} = 2a_1, \ \forall j ; 0 \leq j \leq N,
$$

$$
\tilde{A}_{j,j+1} = \frac{1}{2}A_{0,1} = a_2, \ \forall j ; 1 \leq j \leq N ; \ j \notin \Theta_N, \ \text{and} \ 0 \ \text{on } \Lambda_N,
$$

$$
\tilde{A}_{j-1,j} = \frac{1}{2}\tilde{A}_{N,N-1} = a_2, \ \forall j ; 1 \leq j \leq N ; \ j \notin \Theta_N, \ \text{and} \ 0 \ \text{on } \tilde{\Lambda}_N,
$$

$$
\tilde{A}_{j,j+J+1} = 2a_2, \ \forall j, \ 0 \leq j \leq J,
$$

$$
\tilde{A}_{j,j+1} = a_2, \ \forall j, \ J + 1 \leq j \leq N - J - 1 \text{ on } \Lambda_N,
$$

$$
\tilde{A}_{j-1,j} = a_2, \ \forall j, \ J + 1 \leq j \leq N - J - 1, \ \text{on } \tilde{\Lambda}_N,
$$

$$
\tilde{A}_{j-1,j} = 2a_2, \ \forall j, \ N - J \leq j \leq N.
$$

The matrix $\tilde{B}$

$$
\tilde{B}_{j,j} = 2b_1, \ \forall j ; 0 \leq j \leq N,
$$

$$
\tilde{B}_{j,j+1} = \frac{1}{2}\tilde{B}_{0,1} = b_2, \ \forall j ; 1 \leq j \leq N ; \ j \notin \Theta_N, \ \text{and} \ 0 \ \text{on } \Lambda_N,
$$

$$
\tilde{B}_{j-1,j} = \frac{1}{2}\tilde{B}_{N,N-1} = b_2, \ \forall j ; 1 \leq j \leq N ; \ j \notin \Theta_N, \ \text{and} \ 0 \ \text{on } \tilde{\Lambda}_N,
$$

$$
\tilde{B}_{j,j+J+1} = 2b_2, \ \forall j, \ 0 \leq j \leq J,
$$

$$
\tilde{B}_{j,j+1} = b_2, \ \forall j, \ J + 1 \leq j \leq N - J - 1 \text{ on } \Lambda_N,
$$

$$
\tilde{B}_{j-1,j} = b_2, \ \forall j, \ J + 1 \leq j \leq N - J - 1, \ \text{on } \tilde{\Lambda}_N,
$$

$$
\tilde{B}_{j-1,j} = 2b_2, \ \forall j, \ N - J \leq j \leq N.
$$

The matrix $\tilde{R}$

$$
\tilde{R}_{j,j} = -2, \ \forall j ; 0 \leq j \leq N,
$$

$$
\tilde{R}_{j,j+J+1} = 2, \ \forall j, \ 0 \leq j \leq J.
$$

$$
\tilde{R}_{j,j+J+1} = 2, \ \forall j, \ N - J \leq j \leq N.
$$

$$
\tilde{R}_{j,j+J+1} = \tilde{R}_{j-1,j} = 1, \ \forall j, \ J + 1 \leq j \leq N - J - 1.
$$

\[ (40) \]
7.2 Headlines of the algorithm applied

- Compute the matrices of the system
- Initialisation: Compute the matrices $U^0$, $U^1$, $V^0$ and $V^1$
- for $n \geq 2$, 
  \[ U^n = l_y a p(W, A, L_{R, B}(U^{n-1})+b_2RV^{n-1}-L_{W, A}(U^{n-2})-a_2R(F^{n-2}+F^{n-1}), \]
  and 
  \[ V^{n+1} = 2c_2RU^n + 2R(c_1U^{n-1}+c_2U^{n-2}) - V^{n-2} + 2\hat{F}^{n-1}. \]

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