INFINITE-DIMENSIONAL LIE ALGEBRAS RESPONSIBLE FOR ZERO-CURVATURE REPRESENTATIONS OF SCALAR EVOLUTION EQUATIONS

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Abstract. Zero-curvature representations (ZCRs) are well known to be one of the main tools in the theory of integrable PDEs. In particular, Lax pairs for (1 + 1)-dimensional PDEs can be interpreted as ZCRs.

This paper is part of a research program on investigating the structure of ZCRs for PDEs of various types. In this paper we study (1 + 1)-dimensional scalar evolution equations.

For any such equation, we define a sequence of Lie algebras \( F_p \), \( p = 0, 1, 2, 3, \ldots \), which classify all ZCRs of this equation up to local gauge transformations. Representations of the algebra \( F_p \) classify ZCRs whose \( x \)-part depends on jets of order not greater than \( p \).

More precisely, we define a Lie algebra \( F_p(\mathcal{E}, a) \) for each nonnegative integer \( p \) and each point \( a \) of the infinite prolongation \( \mathcal{E} \) of the considered equation. So the full notation for the algebra is \( F_p(\mathcal{E}, a) \).

Using these algebras, one obtains some necessary conditions for integrability of the considered PDEs and necessary conditions for existence of a Bäcklund transformation between two given equations.

In our construction, jets of arbitrary order are allowed. In the case of low-order jets, the algebras \( F_p(\mathcal{E}, a) \) generalize Wahlquist-Estabrook prolongation algebras.

We describe general properties of \( F_p(\mathcal{E}, a) \) for arbitrary (1 + 1)-dimensional scalar evolution equations and study the structure of \( F_p(\mathcal{E}, a) \) for some classes of equations of orders 3, 5, 7, which include KdV, Krichever-Novikov, Kaup-Kupershmidt, Sawada-Kotera type equations. Among the obtained algebras, one encounters infinite-dimensional Lie algebras of certain matrix-valued functions on rational and elliptic algebraic curves.

Applications to obtaining some non-existence results for Bäcklund transformations and proving non-integrability for some scalar evolution equations are also discussed.

1. Introduction and the main results

1.1. Zero-curvature representations and the algebras \( F_p(\mathcal{E}, a) \). Zero-curvature representations and Bäcklund transformations belong to the main tools in the theory of integrable PDEs (see, e.g., \[24, 34\]). This paper is part of a research program on investigating the structure of zero-curvature representations for PDEs of various types. The study of zero-curvature representations leads to some results on Bäcklund transformations and integrability, which are described on the next pages.

Consider an arbitrary PDE

\[
F^l(x_1, \ldots, x_n, u^j, u^j_{x_1}, u^j_{x_1 x_2}, \ldots) = 0, \quad u^j = u^j(x_1, \ldots, x_n), \quad j = 1, \ldots, m, \quad l = 1, \ldots, s,
\]

where \( u^j_{x_1}, u^j_{x_1 x_2}, \ldots \) are partial derivatives of \( u^j = u^j(x_1, \ldots, x_n) \).

A zero-curvature representation (ZCR) for (1) is given by functions

\[
A_q = A_q(x_1, \ldots, x_n, u^j, u^j_{x_1}, u^j_{x_1 x_2}, \ldots), \quad q = 1, \ldots, n,
\]

with values in the space of \( N \times N \) matrices for some \( N \in \mathbb{Z}_{>0} \) such that

\[
D_{x_q}(A_r) - D_{x_r}(A_q) + [A_q, A_r] = 0 \quad \text{modulo (1)} \quad \text{for all } q, r = 1, \ldots, n.
\]

Here \( D_{x_q}, D_{x_r} \) are the total derivative operators corresponding to (1). Note that functions (2) may depend on any finite number of partial derivatives of \( u^j \) of arbitrary order.

1991 Mathematics Subject Classification. 37K30, 37K35.
Let \( \tilde{A}_q \), \( q = 1, \ldots, n \), be another ZCR for (1). The ZCR \( \tilde{A}_q \) is said to be \textit{gauge equivalent} to the ZCR \( A_q \) if there is a function \( G = G(x, t, u_0, u_1, \ldots, u_d) \) with values in the group of invertible \( N \times N \) matrices such that \( \tilde{A}_q = GA_qG^{-1} - D_{xq}(G) \cdot G^{-1} \) for all \( q = 1, \ldots, n \).

The above-mentioned research program concerns the problem of classification of ZCRs up to gauge equivalence. The main ideas can be outlined as follows.

For a very wide class of PDEs, one can prove the following property. For any given PDE from this class, we can define a family of Lie algebras so that representations of these algebras classify all ZCRs (for all \( N \)) of this PDE up to local gauge equivalence, in a certain sense.

If one regards the PDE as a submanifold of the corresponding infinite jet space, it can be shown that these Lie algebras have some coordinate-independent geometric meaning. So these algebras are geometric invariants of the PDE. (The coordinate-independence of the algebras is not shown in the present paper, but is proved in other preprints, see Remark 11 in Section 1.2 for more details.)

Also, these Lie algebras play an important role in the theory of Bäcklund transformations and integrability. Some applications to Bäcklund transformations and integrability are described below.

The present paper can be studied independently of [11]. In the present paper we elaborate the above-mentioned ideas in the case of \((1+1)\)-dimensional scalar evolution equations

\[
\begin{align*}
(3) \quad u_t &= F(x, t, u_0, u_1, \ldots, u_d), \quad u = u(x, t), \quad u_t = \frac{\partial u}{\partial t}, \quad u_k = \frac{\partial^k u}{\partial x^k}, \quad u_0 = u, \quad k \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

Here the number \( d \geq 1 \) is such that the function \( F \) may depend only on \( x, t, u_k \) for \( k \leq d \).

The methods of this paper can be applied also to \((1+1)\)-dimensional multicomponent evolution systems, which are briefly discussed in Remark 2.

PDEs of the form (3) have attracted a lot of attention in the last 40 years and have been a source of many remarkable results in integrability. In particular, some types of equations (3) possessing higher (generalized) symmetries and conservation laws have been classified (see, e.g., [20, 21, 28] and references therein). However, the problem of complete understanding of all integrability properties for equations (3) is still far from being solved.

Examples of integrable PDEs of the form (3) include the Korteweg-de Vries (KdV), Krichever-Novikov [16, 32], Kaup-Kupershmidt [14], Sawada-Kotera [29] (Caudrey-Dodd-Gibbon [2]) equations (these equations are discussed below). Many more examples can be found in [20, 21, 28] and references therein.

In the present paper, integrability is understood in the sense of soliton theory and the inverse scattering method. (This is sometimes called S-integrability.)

It is well known that, in order to investigate possible integrability properties of (3), one needs to consider ZCRs. Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra.

For an equation of the form (3), a \textit{ZCR with values in} \( \mathfrak{g} \) is given by \( \mathfrak{g} \)-valued functions

\[
\begin{align*}
(4) \quad A &= A(x, t, u_0, u_1, \ldots, u_p), \quad B = B(x, t, u_0, u_1, \ldots, u_{p+d-1})
\end{align*}
\]

satisfying

\[
(5) \quad D_x(B) - D_t(A) + [A, B] = 0.
\]

The \textit{total derivative operators} \( D_x, D_t \) in (5) are

\[
\begin{align*}
D_x &= \frac{\partial}{\partial x} + \sum_{k \geq 0} u_{k+1} \frac{\partial}{\partial t_k}, \\
D_t &= \frac{\partial}{\partial t} + \sum_{k \geq 0} D_x^k(F(x, t, u_0, u_1, \ldots, u_d)) \frac{\partial}{\partial u_k}.
\end{align*}
\]

We assume that all considered functions are analytic.

The number \( p \) in (4) is such that the function \( A \) may depend only on the variables \( x, t, u_k \) for \( k \leq p \). Then equation (5) implies that the function \( B \) may depend only on \( x, t, u_{k'} \) for \( k' \leq p + d - 1 \).
Such ZCRs are said to be of order $\leq p$. In other words, a ZCR given by $A$, $B$ is of order $\leq p$ iff $\frac{\partial A}{\partial u_l} = 0$ for all $l > p$.

We study the following problem. How to describe all ZCRs (4), (5) for a given equation (3)?

In the case when $p = 0$ and the functions $F$, $A$, $B$ do not depend on $x$, $t$, a partial answer to this question is provided by the Wahlquist-Estabrook prolongation method (WE method for short). Namely, for a given equation of the form $u_t = F(u_0, u_1, \ldots, u_d)$, the WE method constructs a Lie algebra so that ZCRs of the form

\begin{equation}
A = A(u_0), \quad B = B(u_0, u_1, \ldots, u_{d-1}), \quad D_x(B) - D_t(A) + [A, B] = 0
\end{equation}

(7)

correspond to representations of this algebra (see, e.g., [3, 15, 33]). It is called the Wahlquist-Estabrook prolongation algebra.

To study the general case of ZCRs (4), (5) with arbitrary $p$ for any equation (3), we need to consider gauge transformations.

Without loss of generality, one can assume that $g$ is a Lie subalgebra of $gl_N$, for some $N \in \mathbb{Z}_{>0}$, where $gl_N$ is the algebra of $N \times N$ matrices. Let $G$ be the connected matrix Lie group corresponding to $g \subset gl_N$. A gauge transformation is given by a function $G = G(x, t, u_0, u_1, \ldots, u_l)$ with values in $G$.

For any ZCR (4), (5) and any gauge transformation $G = G(x, t, u_0, u_1, \ldots, u_l)$, the functions

\begin{equation}
\bar{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \bar{B} = GBG^{-1} - D_t(G) \cdot G^{-1}
\end{equation}

(8)

satisfy $D_x(\bar{B}) - D_t(\bar{A}) + [\bar{A}, \bar{B}] = 0$ and, therefore, form a ZCR.

The ZCR (8) is said to be gauge equivalent to the ZCR (4), (5). For a given equation (3), formulas (8) determine an action of the group of gauge transformations on the set of ZCRs of this equation.

The WE method does not use gauge transformations in a systematic way. In the classification of ZCRs (7) this is acceptable, because the class of ZCRs (7) is relatively small.

The class of ZCRs (4), (5) is much larger than that of (7). As is shown in the present paper, gauge transformations play a very important role in the classification of ZCRs (4), (5). Because of this, the classical WE method does not produce satisfactory results for (4), (5), especially in the case $p > 0$.

To overcome this problem, we find a normal form for ZCRs (4), (5) with respect to the action of the group of gauge transformations. Using the normal form of ZCRs, for any given equation (3), we define a Lie algebra $\mathbb{F}^p$ for each $p \in \mathbb{Z}_{\geq 0}$ so that the following property holds.

For every finite-dimensional Lie algebra $g$, any $g$-valued ZCR (4), (5) of order $\leq p$ is locally gauge equivalent to the ZCR arising from a homomorphism $\mathbb{F}^p \to g$.

More precisely, as is discussed below, we define a Lie algebra $\mathbb{F}^p$ for each $p \in \mathbb{Z}_{\geq 0}$ and each point $a$ of the infinite prolongation $E$ of equation (3). So the full notation for the algebra is $\mathbb{F}^p(E, a)$.

Recall that the infinite prolongation $E$ of (3) is the infinite-dimensional manifold with the coordinates $x, t, u_k$ for $k \in \mathbb{Z}_{\geq 0}$. The precise definition of $\mathbb{F}^p(E, a)$ for any equation (3) is presented in Section 2.

In this definition, the algebra $\mathbb{F}^p(E, a)$ is given in terms of generators and relations.

For every finite-dimensional Lie algebra $g$, homomorphisms $\mathbb{F}^p(E, a) \to g$ classify (up to gauge equivalence) all $g$-valued ZCRs (4), (5) of order $\leq p$, where functions $A$, $B$ are defined on a neighborhood of the point $a \in E$. See Section 2 for details.

Using the algebras $\mathbb{F}^p(E, a)$, we obtain some necessary conditions for integrability of equations (3) and necessary conditions for existence of a Bäcklund transformation between two given equations.

To obtain such conditions, one needs to study some properties of ZCRs (4), (5) with arbitrary $p$, and we do this by means of the algebras $\mathbb{F}^p(E, a)$. As has been explained above, the classical WE method (which studies ZCRs of the form (7)) is not sufficient for this.

Applications of $\mathbb{F}^p(E, a)$ to obtaining necessary conditions for integrability of equations (3) are presented in Section 1.3. Applications of $\mathbb{F}^p(E, a)$ to the theory of Bäcklund transformations are briefly discussed in Section 1.2.
Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$. We suppose that the variables $x$, $t$, $u_k$ take values in $\mathbb{K}$. A point $a \in \mathcal{E}$ is determined by the values of the coordinates $x$, $t$, $u_k$ at $a$. Let

$$a = (x = x_0, t = t_0, u_k = a_k) \in \mathcal{E}, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0},$$

be a point of $\mathcal{E}$.

To clarify the definition of $\mathbb{F}^p(\mathcal{E}, a)$, let us consider the case $p = 1$. To this end, we fix an equation (3) and study ZCRs of order $\leq 1$ of this equation.

According to Theorem 6 in Section 2 any ZCR of order $\leq 1$

$$A = A(x, t, u_0, u_1), \quad B = B(x, t, u_0, u_1, \ldots, u_d), \quad D_x(B) - D_t(A) + [A, B] = 0$$
on a neighborhood of $a \in \mathcal{E}$ is gauge equivalent to a ZCR of the form

$$\tilde{A} = \tilde{A}(x, t, u_0, u_1), \quad \tilde{B} = \tilde{B}(x, t, u_0, u_1, \ldots, u_d),$$

$$D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0,$$

(12)

$$\frac{\partial \tilde{A}}{\partial u_1}(x, t, u_0, a_1) = 0, \quad \tilde{A}(x, t, a_0, a_1) = 0, \quad \tilde{B}(x_0, t, a_0, a_1, \ldots, a_d) = 0.$$

In other words, properties (12) determine a normal form for ZCRs (9) with respect to the action of the group of gauge transformations on a neighborhood of $a \in \mathcal{E}$.

A similar normal form for ZCRs (11), (12) with arbitrary $p$ is described in Theorem 6 in Section 2.

Recall that all considered functions are assumed to be analytic. Therefore, on a neighborhood of $a \in \mathcal{E}$, the functions $\tilde{A}$, $\tilde{B}$ from (10), (12) are represented as absolutely convergent power series

$$\tilde{A} = \sum_{l_1, l_2, j_0, j_1 \geq 0} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{j_0}(u_1 - a_1)^{j_1} \cdot \tilde{A}_{l_0, l_1}^{l_1, l_2},$$

(13)

$$\tilde{B} = \sum_{l_1, l_2, j_0, \ldots, j_d \geq 0} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{j_0} \ldots (u_d - a_d)^{j_d} \cdot \tilde{B}_{j_0, \ldots, j_d}^{l_1, l_2}.$$ 

Here $\tilde{A}_{l_0, l_1}^{l_1, l_2}$ and $\tilde{B}_{j_0, \ldots, j_d}^{l_1, l_2}$ are elements of a Lie algebra, which we do not specify yet.

Using formulas (13), (14), we see that properties (12) are equivalent to

$$\tilde{A}_{l_0, l_1}^{l_1, l_2} = \tilde{A}_{0, l_1}^{l_1, l_2}, \quad \tilde{B}_{j_0, \ldots, j_d}^{l_1, l_2} = 0 \quad \forall l_1, l_2, l_0 \in \mathbb{Z}_{\geq 0}.$$

To define $\mathbb{F}^1(\mathcal{E}, a)$, we regard $\tilde{A}_{l_0, l_1}^{l_1, l_2}$, $\tilde{B}_{j_0, \ldots, j_d}^{l_1, l_2}$ from (13), (14) as abstract symbols. By definition, the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is generated by the symbols $\tilde{A}_{l_0, l_1}^{l_1, l_2}$, $\tilde{B}_{j_0, \ldots, j_d}^{l_1, l_2}$ for $l_1, l_2, l_0, j_1, j_0, \ldots, j_d \in \mathbb{Z}_{\geq 0}$. Relations for these generators are provided by equations (11), (15). A more detailed description of this construction is given in Section 2.

According to Section 2, the algebras $\mathbb{F}^p(\mathcal{E}, a)$ for $p \in \mathbb{Z}_{\geq 0}$ are arranged in a sequence of surjective homomorphisms

$$\cdots \to \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a) \to \cdots \to \mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a).$$

**Remark 1.** Somewhat similar (but not the same) ideas were considered in [9] for a few scalar evolution equations of order 3. The theory described in the present paper is much more powerful than that of [9]. See Remark 9 in Section 1.2 for a more detailed discussion of the relations between the present paper and [9].

**Remark 2.** It is possible to introduce an analog of $\mathbb{F}^p(\mathcal{E}, a)$ for multicomponent evolution systems

$$\frac{\partial u^i}{\partial t} = F(x, t, u^1, \ldots, u^m, u_1^1, \ldots, u_1^m, \ldots, u_d^1, \ldots, u_d^m), \quad u^i = u^i(x, t), \quad u_k^i = \frac{\partial k u^i}{\partial x^k}, \quad i = 1, \ldots, m.$$

In this paper we study only the scalar case $m = 1$. For $m > 1$ one gets interesting results as well, but the case $m > 1$ requires much more computations, which will be presented elsewhere. Some results for $m > 1$ are sketched in the preprints [11, 12].
As has been discussed above, the algebra $\mathbb{F}^p(\mathcal{E}, a)$ is defined by a certain set of generators and relations arising from a normal form of ZCRs. In Theorem 8 in Section 3 we describe a smaller subset of generators for $\mathbb{F}^p(\mathcal{E}, a)$.

**Example 1.** Consider the case $p = 1$. According to the above definition of $\mathbb{F}^1(\mathcal{E}, a)$, the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is given by the generators $\bar{A}_{i_0,i_1}^{l_1,l_2}$, $\bar{B}_{i_0,i_1}^{l_1,l_2}$ and the relations arising from (11), (15). Theorem 8 implies that the algebra $\mathbb{F}^1(\mathcal{E}, a)$ coincides with the subalgebra generated by $\bar{A}_{i_0,i_1}^{l_1,0}$ for $l_1, i_0, i_1 \in \mathbb{Z}_{\geq 0}$.

According to Theorem 8 a similar result is valid also for $\mathbb{F}^p(\mathcal{E}, a)$ for every $p$.

This result helps us to describe the structure of $\mathbb{F}^p(\mathcal{E}, a)$ and the homomorphisms (16) more explicitly for some PDEs. Consider first equations of the form $u_t = u_{2q+1} + f(x, t, u_0, u_1, \ldots, u_{2q-1})$ for $q \in \{1, 2, 3\}$. Examples of such PDEs include

- the KdV equation $u_t = u_3 + u_0 u_1$,
- the Kaup-Kupershmidt equation $u_t = u_5 + 10 u_0 u_3 + 25 u_1 u_2 + 20 u_2^2 u_1$,
- the Sawada-Kotera equation $u_t = u_5 + 5 u_0 u_3 + 5 u_1 u_2 + 5 u_2^2 u_1$ (which is sometimes called the Caudrey-Dodd-Gibbon equation).

Many more examples of integrable equations of the form $u_t = u_3 + f(x, u_0, u_1)$ and $u_t = u_5 + f(u_0, u_1, u_2, u_3)$ can be found in [20, 21] and references therein.

**Remark 3.** The paper [20] presents a classification of equations of the form $u_t = u_3 + g(x, u_0, u_1, u_2)$ and $u_t = u_5 + g(u_0, u_1, u_2, u_3, u_4)$ satisfying certain integrability conditions related to generalized symmetries and conservation laws.

We study the problem of describing all ZCRs (4), (5) for a given equation (3). This problem is very different from the problems of describing generalized symmetries and conservation laws.

Theorems 1 and 2 are proved in Sections 4 and 6 respectively.

**Theorem 1 (Section 4).** Let $\mathcal{E}$ be the infinite prolongation of an equation of the form

$$u_t = u_{2q+1} + f(x, t, u_0, u_1, \ldots, u_{2q-1}),$$

where $f$ is an arbitrary function and $q \in \{1, 2, 3\}$. Let $a \in \mathcal{E}$.

For each $p \in \mathbb{Z}_{\geq 0}$, consider the surjective homomorphism $\varphi_p$: $\mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a)$ from (16).

If $p \geq q + \delta_{q,3}$ then

$$[v_1, v_2] = 0 \quad \forall v_1 \in \ker \varphi_p, \quad \forall v_2 \in \mathbb{F}^p(\mathcal{E}, a).$$

In other words, if $p \geq q + \delta_{q,3}$ then the kernel of $\varphi_p$ is contained in the center of the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$.

Here $\delta_{q,3}$ is the Kronecker delta. So $\delta_{q,3} = 0$ if $q \neq 3$, and $\delta_{q,3} = 1$ if $q = 3$.

For each $k \in \mathbb{Z}_{>0}$, let $\psi_k$: $\mathbb{F}^{k+q-1+\delta_{q,3}}(\mathcal{E}, a) \to \mathbb{F}^{q+\delta_{q,3}}(\mathcal{E}, a)$ be the composition of the homomorphisms

$$\mathbb{F}^{k+q-1+\delta_{q,3}}(\mathcal{E}, a) \to \mathbb{F}^{k+q-2+\delta_{q,3}}(\mathcal{E}, a) \to \cdots \to \mathbb{F}^{q+\delta_{q,3}}(\mathcal{E}, a) \to \mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)$$

from (16). Then

$$[h_1, [h_2, \ldots, [h_{k-1}, [h_k, h_{k+1}]] \ldots]] = 0 \quad \forall h_1, \ldots, h_{k+1} \in \ker \psi_k.$$
Remark 4. Theorem 1 implies that, for any equation of the form (17) with \( q \in \{1, 2, 3\} \),
- for every \( p \geq q + \delta_{q,3} \) the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is obtained from \( \mathbb{F}^{p-1}(\mathcal{E}, a) \) by central extension,
- for every \( p \geq q + \delta_{q,3} \) the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is obtained from \( \mathbb{F}^{p-1+\delta_{q,3}}(\mathcal{E}, a) \) by applying several times the operation of central extension.

Theorem 2 (Section 6). Consider the infinite-dimensional Lie algebra \( \mathfrak{sl}_2(\mathbb{K}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{K}) \otimes \mathbb{K}[\lambda] \), where \( \mathbb{K}[\lambda] \) is the algebra of polynomials in \( \lambda \).

Let \( \mathcal{E} \) be the infinite prolongation of the KdV equation \( u_t = u_{3} + u_{0}u_{1} \). Let \( a \in \mathcal{E} \). Then
- the algebra \( \mathbb{F}^0(\mathcal{E}, a) \) is isomorphic to the direct sum of \( \mathfrak{sl}_2(\mathbb{K}[\lambda]) \) and a 3-dimensional abelian Lie algebra,
- for every \( p \in \mathbb{Z}_{>0} \), the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is obtained from \( \mathfrak{sl}_2(\mathbb{K}[\lambda]) \) by applying several times the operation of central extension.

To describe \( \mathbb{F}^0(\mathcal{E}, a) \) for the KdV equation in Theorem 2 we use the following fact. If the function \( F \) in (3) does not depend on \( x, t \), then the algebra \( \mathbb{F}^0(\mathcal{E}, a) \) is isomorphic to a certain subalgebra of the Wahlquist-Estabrook prolongation algebra for (3) (see Theorem 1 in Section 5 for details).

The explicit structure of the Wahlquist-Estabrook prolongation algebra for the KdV equation is given in [15], and this helps us to describe \( \mathbb{F}^0(\mathcal{E}, a) \) for KdV.

Now assume that \( \mathbb{K} = \mathbb{C} \). For any constants \( e_1, e_2, e_3 \in \mathbb{C} \), consider the Krichever-Novikov equation (16)

\[
(18) \quad \text{KN}(e_1, e_2, e_3) = \left\{ u_t = u_{xxx} - \frac{3}{2} u_{xx} u_x + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x}, \quad u = u(x, t) \right\}.
\]

To study \( \mathbb{F}^p(\mathcal{E}, a) \) for this equation, we need some auxiliary constructions.

Let \( \mathbb{C}[v_1, v_2, v_3] \) be the algebra of polynomials in the variables \( v_1, v_2, v_3 \). Let \( e_1, e_2, e_3 \in \mathbb{C} \) be such that \( e_1 \neq e_2 \neq e_3 \neq e_1 \). Consider the ideal \( \mathcal{I}_{e_1, e_2, e_3} \subset \mathbb{C}[v_1, v_2, v_3] \) generated by the polynomials

\[
(19) \quad v_i^2 - v_j^2 + e_i - e_j, \quad i, j = 1, 2, 3.
\]

Set \( E_{e_1, e_2, e_3} = \mathbb{C}[v_1, v_2, v_3]/\mathcal{I}_{e_1, e_2, e_3} \). In other words, \( E_{e_1, e_2, e_3} \) is the commutative associative algebra of polynomial functions on the algebraic curve in \( \mathbb{C}^3 \) defined by the polynomials (19).

Since we assume \( e_1 \neq e_2 \neq e_3 \neq e_1 \), this curve is nonsingular and of genus 1.

We have the natural surjective homomorphism \( \mu : \mathbb{C}[v_1, v_2, v_3] \to \mathbb{C}[v_1, v_2, v_3]/\mathcal{I}_{e_1, e_2, e_3} = E_{e_1, e_2, e_3} \). Set \( \bar{v}_i = \mu(v_i) \in E_{e_1, e_2, e_3} \) for \( i = 1, 2, 3 \).

Consider also a basis \( x_1, x_2, x_3 \) of the Lie algebra \( \mathfrak{so}_3(\mathbb{C}) \) such that \( [x_1, x_2] = x_3, \ [x_2, x_3] = x_1, \ [x_3, x_1] = x_2 \). We endow the space \( \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1, e_2, e_3} \) with the following Lie algebra structure

\[
[z_1 \otimes h_1, z_2 \otimes h_2] = [z_1, z_2] \otimes h_1 h_2, \quad z_1, z_2 \in \mathfrak{so}_3(\mathbb{C}), \quad h_1, h_2 \in E_{e_1, e_2, e_3}.
\]

Denote by \( \mathcal{R}_{e_1, e_2, e_3} \) the Lie subalgebra of \( \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1, e_2, e_3} \) generated by the elements

\[
x_i \otimes \bar{v}_i \in \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1, e_2, e_3}, \quad i = 1, 2, 3.
\]

Since \( \mathcal{R}_{e_1, e_2, e_3} \subset \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1, e_2, e_3} \), we can regard elements of \( \mathcal{R}_{e_1, e_2, e_3} \) as \( \mathfrak{so}_3(\mathbb{C}) \)-valued functions on the elliptic curve in \( \mathbb{C}^3 \) determined by the polynomials (19).

It is known that such \( \mathfrak{so}_3(\mathbb{C}) \)-valued functions on elliptic curves appear in the structure of some ZCRs for the Landau-Lifshitz and Krichever-Novikov equations (see, e.g., [6, 7, 16, 22, 31]).

It is shown in [25] that the Wahlquist-Estabrook prolongation algebra of the anisotropic Landau-Lifshitz equation is isomorphic to the direct sum of \( \mathcal{R}_{e_1, e_2, e_3} \) and a 2-dimensional abelian Lie algebra. The paper [25] describes a basis for \( \mathcal{R}_{e_1, e_2, e_3} \), which implies that the algebra \( \mathcal{R}_{e_1, e_2, e_3} \) is infinite-dimensional.

According to Proposition 1 below, the algebra \( \mathcal{R}_{e_1, e_2, e_3} \) shows up also in the structure of \( \mathbb{F}^p(\mathcal{E}, a) \) for the Krichever-Novikov equation. A proof of Proposition 1 is sketched in [11] (this proof uses some results of [9, 13]).

Proposition 1 (11). For any \( e_1, e_2, e_3 \in \mathbb{C} \), consider the Krichever-Novikov equation \( \text{KN}(e_1, e_2, e_3) \) given by (18). Let \( \mathcal{E} \) be the infinite prolongation of this equation. Let \( a \in \mathcal{E} \). Then
• the algebra $F^{0}(E, a)$ is zero,
• for each $p \geq 2$, the kernel of the surjective homomorphism $F^{p}(E, a) \to F^{1}(E, a)$ from (16) is nilpotent,
• if $e_1 \neq e_2 \neq e_3 \neq e_1$, then $F^{1}(E, a) \cong \mathfrak{R}_{e_1,e_2,e_3}$ and for each $p \geq 2$ the algebra $F^{p}(E, a)$ is obtained from $\mathfrak{R}_{e_1,e_2,e_3}$ by applying several times the operation of central extension.

### Remark 5
For the Burgers and KdV equations, ZCRs of the form

$$A = A(u_0, u_1, u_2, \ldots), \quad B = B(u_0, u_1, u_2, \ldots), \quad D_A(B) - D_t(A) + [A, B] = 0$$

(where $A$ and $B$ may depend on any finite number of the coordinates $u_k$) were studied in [8]. However, gauge transformations were not considered in [8]. Because of this, the paper [8] had to impose some additional constraints on the functions $A$, $B$ in (20).

### Remark 6
Some other approaches to the study of the action of gauge transformations on ZCRs can be found in [17, 18, 19, 26, 27, 30] and references therein. For a given $g$-valued ZCR, the papers [17, 18, 26] define certain $g$-valued functions that transform by conjugation when the ZCR transforms by gauge. Applications of these functions to construction and classification of some types of ZCRs are described in [17, 18, 19, 26, 27, 30].

To our knowledge, the theory of [17, 18, 19, 26, 27, 30] does not produce any infinite-dimensional Lie algebras responsible for ZCRs. So this theory does not contain the algebras $F^{p}(E, a)$.

### 1.2. Necessary conditions for existence of Bäcklund transformations.

The algebras $F^{p}(E, a)$ help to obtain necessary conditions for existence of a Bäcklund transformation between two given evolution equations. In the present paper we do not study Bäcklund transformations. We just briefly mention some results in this direction.

For each $p \in \mathbb{Z}_{> 0}$, consider the surjective homomorphism $\varphi_{p}: F^{p}(E, a) \to F^{p-1}(E, a)$ from (16).

Let $F(E, a)$ be the inverse (projective) limit of the sequence (16). An element of $F(E, a)$ is given by a sequence $(c_0, c_1, c_2, \ldots)$, where $c_p \in F^{p}(E, a)$ and $\varphi_{p}(c_p) = c_{p-1}$ for all $p$.

Since (16) consists of homomorphisms of Lie algebras and $F(E, a)$ is the inverse limit of (16), the space $F(E, a)$ is a Lie algebra as well. If $(c_0, c_1, c_2, \ldots)$ and $(c_0', c_1', c_2', \ldots)$ are elements of $F(E, a)$, where $c_p, c_p' \in F^{p}(E, a)$, then the corresponding Lie bracket is

$$[(c_0, c_1, c_2, \ldots), (c_0', c_1', c_2', \ldots)] = ([c_0, c_0'], [c_1, c_1'], [c_2, c_2'], \ldots) \in F(E, a).$$

For each $k \in \mathbb{Z}_{\geq 0}$ we have the homomorphism

$$\rho_k: F(E, a) \to F^k(E, a), \quad \rho_k((c_0, c_1, c_2, \ldots)) = c_k.$$ 

Since the homomorphisms (16) are surjective, $\rho_k$ is surjective as well.

We define a topology on the algebra $F(E, a)$ as follows. For every $k \in \mathbb{Z}_{\geq 0}$ and every $v \in F^k(E, a)$, the subset $\rho_k^{-1}(v) \subset F(E, a)$ is, by definition, open in $F(E, a)$. Such subsets form a base of the topology on $F(E, a)$.

### Remark 7
Let $\mathcal{L}$ be a Lie algebra endowed with the discrete topology. Then a homomorphism $F(E, a) \to \mathcal{L}$ is continuous iff it is of the form $F(E, a) \xrightarrow{\rho_k} F^k(E, a) \to \mathcal{L}$ for some $k \in \mathbb{Z}_{\geq 0}$ and some homomorphism $F^k(E, a) \to \mathcal{L}$.

A Lie subalgebra $H \subset F(E, a)$ is called tame if there are $k \in \mathbb{Z}_{\geq 0}$ and a subalgebra $\mathfrak{h} \subset F^k(E, a)$ such that $H = \rho_k^{-1}(\mathfrak{h})$. Since $\rho_k: F(E, a) \to F^k(E, a)$ is surjective, the codimension of $H$ in $F(E, a)$ is equal to the codimension of $\mathfrak{h}$ in $F^k(E, a)$.

### Remark 8
It is easy to check that a subalgebra $H \subset F(E, a)$ is tame iff $H$ is open and closed in $F(E, a)$ with respect to the topology on $F(E, a)$.

A proof of Proposition 2 is sketched in [11].

### Proposition 2 ([11])
Let $E_1$ and $E_2$ be evolution equations. Suppose that $E_1$ and $E_2$ are connected by a Bäcklund transformation. Then for each $i = 1, 2$ there are a point $a_i \in E_i$ and a tame subalgebra $H_i \subset F(E_i, a_i)$ such that
• $H_i$ is of finite codimension in $\mathbb{F}(\mathcal{E}_i, a_i)$.
• $H_i$ is isomorphic to $H_2$, and this isomorphism is a homeomorphism with respect to the topology induced by the embedding $H_i \subset \mathbb{F}(\mathcal{E}_i, a_i)$.

The preprint [11] presents also a more general result about PDEs that are not necessarily evolution.

Proposition 2 provides a powerful necessary condition for two given evolution equations to be connected by a Bäcklund transformation (BT).

For example, the following result is obtained in [10] by means of methods similar to Proposition 2. For any $e_1, e_2, e_3 \in \mathbb{C}$, we have the Krichever-Novikov equation $\text{KN}(e_1, e_2, e_3)$ given by (18). Consider also the algebraic curve $C(e_1, e_2, e_3) = \{(z, y) \in \mathbb{C}^2 \mid y^2 = (z - e_1)(z - e_2)(z - e_3)\}$.

Proposition 3 ([10]). Let $e_1, e_2, e_3, e_1', e_2', e_3' \in \mathbb{C}$ be such that $e_1 \neq e_2 \neq e_3 \neq e_1', e_1' \neq e_2' \neq e_3' \neq e_1'$. If the curve $C(e_1, e_2, e_3)$ is not birationally equivalent to the curve $C(e_1', e_2', e_3')$, then the equation $\text{KN}(e_1, e_2, e_3)$ is not connected with the equation $\text{KN}(e_1', e_2', e_3')$ by any Bäcklund transformation.

Also, if $e_1 \neq e_2 \neq e_3 \neq e_1$, then $\text{KN}(e_1, e_2, e_3)$ is not connected with the KdV equation by any BT.

BTs of Miura type (differential substitutions) for (18) were studied in [20, 32]. According to [20, 32], the equation $\text{KN}(e_1, e_2, e_3)$ is connected with the KdV equation by a BT of Miura type iff $e_i = e_j$ for some $i \neq j$.

Propositions 2, 3 consider the most general class of BTs, which is much larger than the class of BTs of Miura type studied in [20, 32].

Note that the present paper can be studied independently of [10, 11].

Remark 9. Somewhat similar (but not the same) ideas on ZCRs and BTs were considered in [9] for a few equations of order 3. The theory described in the present paper is much more powerful than that of [9].

Indeed, in [9] one studied a certain class of transformations of PDEs and considered three examples: the KdV equation, the Krichever-Novikov equation, and the linear equation $u_t = u_{xxx}$. For these equations, the paper [9] introduced Lie algebras $\mathfrak{f}_p$ for $p \in \mathbb{Z}_{\geq 0}$ so that representations of $\mathfrak{f}_p$ classify ZCRs the form

$$A = A(u_0, u_1, \ldots, u_p), \quad B = B(u_0, u_1, \ldots, u_{p+2}), \quad D_x(B) - D_t(A) + [A, B] = 0$$

up to local gauge equivalence. Since the paper [9] assumed that $A, B$ in (21) do not depend on $x, t$, the algebras $\mathfrak{f}_p$ in [9] are different from the algebras $\mathbb{F}^p(\mathcal{E}, a)$ defined in the present paper. (Recall that the algebras $\mathbb{F}^p(\mathcal{E}, a)$ are responsible for ZCRs [11, 15], where dependence on $x, t$ is allowed.)

So in [9] one studied ZCRs and transformations that are invariant with respect to the change of variables $x \mapsto x + \alpha, \quad t \mapsto t + \beta$ for any constants $\alpha, \beta$. In this framework, the paper [9] showed that finite-dimensional quotients of the algebras $\mathfrak{f}_p$ have some coordinate-independent meaning. This helped [9] to prove that the equation $u_t = u_{xxx}$ is not connected with the KdV equation and is not connected with the Krichever-Novikov equation by any BTs.

Concerning necessary conditions for existence of Bäcklund transformations between two given equations, Proposition 2 described above is much more powerful than the theory of [9]. Indeed, Proposition 2 works with the algebras $\mathbb{F}(\mathcal{E}, a)$, which are infinite-dimensional in interesting cases and contain much more information than the finite-dimensional quotients of $\mathfrak{f}_p$ considered in [9].

For example, Proposition 3 is proved in [10] by means of methods similar to Proposition 2. The theory of [9] is not sufficient to obtain the result of Proposition 3.

Using the algebras $\mathbb{F}^p(\mathcal{E}, a)$, in Section 1.3 we present some necessary conditions for integrability of equations (3). Such results on integrability were not considered at all in [9].

Remark 10. Using the theory of infinite jet bundles, one can regard PDEs as certain geometric objects (manifolds with distributions) (see, e.g., [11, 11, 15] and references therein). This is applicable to PDEs satisfying some non-degeneracy conditions, which are satisfied for evolution equations.

In this framework, it is shown in [11] that the algebra $\mathbb{F}(\mathcal{E}, a)$ has some coordinate-independent geometric meaning.
Remark 11. According to Theorem 2, the Lie algebra $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ plays the main role in the description of $\mathbb{F}^p(\mathcal{E}, a)$ for the KdV equation. According to Proposition 1, the Lie algebra $\mathfrak{F}_{e_1,e_2,e_3}$ plays a similar role in the description of $\mathbb{F}^p(\mathcal{E}, a)$ for the Krichever-Novikov equation (18) in the case $e_1 \neq e_2 \neq e_3 \neq e_1$.

As has been discussed in Section 1.1 if $e_1 \neq e_2 \neq e_3 \neq e_1$, we can view elements of $\mathfrak{F}_{e_1,e_2,e_3}$ as certain $\mathfrak{so}_3(\mathbb{C})$-valued functions on the elliptic curve determined by the polynomials (19).

The field $\mathbb{K}$ can be regarded as a rational algebraic curve with coordinate $\lambda$. Then elements of $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ are identified with polynomial $\mathfrak{sl}_2(\mathbb{K})$-valued functions on this rational curve.

The invariant meaning of algebraic curves related to $\mathbb{F}^p(\mathcal{E}, a)$ and $\mathbb{F}(\mathcal{E}, a)$ for the KdV and Krichever-Novikov equations is studied in [10].

1.3. Necessary conditions for integrability. In this subsection, $\mathfrak{g}$ is a finite-dimensional matrix Lie algebra, and $\mathcal{E}$ is the infinite prolongation of an equation of the form (3). ZCRs and gauge transformations are supposed to be defined on a neighborhood of a point $a \in \mathcal{E}$.

Let $\mathcal{G}$ be the connected matrix Lie group corresponding to $\mathfrak{g}$. Recall that a gauge transformation is given by a function $G = G(x, t, u_0, u_1, \ldots, u_k)$ with values in $\mathcal{G}$.

A $\mathfrak{g}$-valued ZCR

$$A = A(x, t, u_0, u_1, \ldots), \quad B = B(x, t, u_0, u_1, \ldots), \quad D_x(B) - D_t(A) + [A, B] = 0$$

is called gauge-solvable if there is a gauge transformation $G$ such that the functions

$$\tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$$

take values in a solvable Lie subalgebra of $\mathfrak{g}$. In other words, a $\mathfrak{g}$-valued ZCR is gauge-solvable iff it is gauge equivalent to a ZCR with values in a solvable Lie subalgebra of $\mathfrak{g}$.

Recall that, in this paper, integrability of PDEs is understood in the sense of soliton theory and the inverse scattering method. In soliton theory, one is interested in ZCRs that are not gauge-solvable. In particular, the inverse scattering method for $(1 + 1)$-dimensional PDEs is based on the use of ZCRs that are not gauge-solvable.

Hence the property

(22) “there is $\mathfrak{g}$ such that equation (3) possesses a $\mathfrak{g}$-valued ZCR that is not gauge-solvable”

can be regarded as a necessary condition$^2$ for integrability of equation (3).

According to Theorem 7 in Section 2, for any $\mathfrak{g}$-valued ZCR of order $\leq p$, there is a homomorphism $\rho: \mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g}$ such that this ZCR is gauge equivalent to a ZCR with values in the Lie subalgebra $\rho(\mathbb{F}^p(\mathcal{E}, a)) \subset \mathfrak{g}$.

Therefore, if for each $p \in \mathbb{Z}_{\geq 0}$ and each $a \in \mathcal{E}$ the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ is solvable then any ZCR is gauge-solvable. This fact implies the following.

Theorem 3. Let $\mathcal{E}$ be the infinite prolongation of an equation of the form (3). If for each $p \in \mathbb{Z}_{\geq 0}$ and each $a \in \mathcal{E}$ the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ is solvable, then this equation is not integrable.

In other words, the property

(23) “there exist $p \in \mathbb{Z}_{\geq 0}$ and $a \in \mathcal{E}$ such that the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ is not solvable”

is a necessary condition for integrability of equation (3).

For some classes of equations (3) one can find a nonnegative integer $r$ such that the kernel of the surjective homomorphism $\mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^p(\mathcal{E}, a)$ from (19) is nilpotent for all $k > r$. Then condition (23) should be checked for $p = r$.

For example, according to Theorem 1 for equations of the form (17) we can take $r = q - 1 + \delta_{q,3}$. According to Proposition 1 for the Krichever-Novikov equation (18) one can take $r = 1$.

Let us show how this works for equations (17).

$^1$Sometimes the inverse scattering method is described by means of Lax pairs, but, in the $(1 + 1)$-dimensional case, Lax pairs can be interpreted as ZCRs.

$^2$This condition is necessary, but is probably not sufficient for integrability of (3). Another necessary condition can be obtained in the consideration of parameter-dependent ZCRs, which are discussed in Remark 12.
**Theorem 4.** Let $\mathcal{E}$ be the infinite prolongation of an equation of the form
\begin{equation}
(24) \quad u_t = u_{2q+1} + f(x, t, u_0, u_1, \ldots, u_{2q-1}), \quad q \in \{1, 2, 3\}.
\end{equation}
Let $a \in \mathcal{E}$. If the Lie algebra $\mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)$ is nilpotent, then $\mathbb{F}^p(\mathcal{E}, a)$ is nilpotent for all $p \in \mathbb{Z}_{\geq 0}$.

If $\mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)$ is solvable, then $\mathbb{F}^p(\mathcal{E}, a)$ is solvable for all $p \in \mathbb{Z}_{\geq 0}$.

**Proof.** According to Theorem 1 and Remark 4 for every $p \geq q + \delta_{q,3}$ the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ is obtained from $\mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)$ by applying several times the operation of central extension.

Since the homomorphisms are surjective, for each $\tilde{p} \leq q - 1 + \delta_{q,3}$ we have a surjective homomorphism $\mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a) \to \mathbb{F}^p(\mathcal{E}, a)$.

Clearly, these properties imply the statement of Theorem 4.$\square$

Combining Theorem 4 with Theorem 3 we obtain the following.

**Theorem 5.** Let $\mathcal{E}$ be the infinite prolongation of an equation of the form (24), where $q \in \{1, 2, 3\}$. If for all $a \in \mathcal{E}$ the Lie algebra $\mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)$ is solvable, then for each $p \in \mathbb{Z}_{\geq 0}$ any ZCR of order $\leq p$
\begin{equation}
A = A(x, t, u_0, u_1, \ldots, u_p), \quad B = B(x, t, u_0, u_1, \ldots), \quad D_x(B) - D_t(A) + [A, B] = 0
\end{equation}
is gauge-solvable. Hence, if $\mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)$ is solvable for all $a \in \mathcal{E}$, then equation (24) is not integrable.

In other words, the property
\begin{equation}
\text{“the Lie algebra } \mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a) \text{ is not solvable for some } a \in \mathcal{E}”
\end{equation}
is a necessary condition for integrability of equations of the form (24).

**Remark 12.** In this remark we briefly discuss (without detailed proof) another necessary condition for integrability of equations of the form (3). In soliton theory and the inverse scattering method for (1 + 1)-dimensional PDEs, one is interested in a ZCR such that
- the ZCR is not gauge-solvable,
- the ZCR depends nontrivially on a parameter which cannot be removed by gauge transformations.

That is, if an equation (3) is integrable then there are a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-valued ZCR
\begin{equation}
(26) \quad A = A(\lambda, x, t, u_0, u_1, \ldots, u_p), \quad B = B(\lambda, x, t, u_0, u_1, \ldots, u_{p+d-1}), \quad D_x(B) - D_t(A) + [A, B] = 0
\end{equation}
such that $A, B$ depend (nontrivially) on a parameter $\lambda$ which cannot be removed by gauge transformations, and the ZCR (26) is not gauge-solvable.

Let $\tilde{\mathfrak{g}}$ be the infinite-dimensional Lie algebra of functions $h(\lambda)$ with values in $\mathfrak{g}$. Then (26) can be regarded as a ZCR with values in $\tilde{\mathfrak{g}}$.

It can be shown that the algebras $\mathbb{F}^p(\mathcal{E}, a)$ are responsible also for ZCRs with values in infinite-dimensional Lie algebras. In particular, the ZCR (26) is gauge equivalent to the ZCR determined by a homomorphism $\tilde{\rho}: \mathbb{F}^p(\mathcal{E}, a) \to \tilde{\mathfrak{g}}$. If $\tilde{\rho}(\mathbb{F}^p(\mathcal{E}, a))$ is finite-dimensional, then the ZCR (26) cannot be used to establish integrability of the considered equation.

Therefore, the property
\begin{equation}
(27) \quad \text{“there exist } p \in \mathbb{Z}_{\geq 0} \text{ and } a \in \mathcal{E} \text{ such that the algebra } \mathbb{F}^p(\mathcal{E}, a) \text{ is infinite-dimensional”}
\end{equation}
is another necessary condition for integrability of equation (3). A more detailed explanation of this fact will be given elsewhere. In the present paper we do not use condition (27).

Recall that equation (3) reads $u_t = F(x, t, u_0, u_1, \ldots, u_d)$. According to Section 2 the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ for this equation is defined in terms of generators and relations. To define the generators and relations for $\mathbb{F}^p(\mathcal{E}, a)$ in Section 2 we use some explicit procedure, which involves the Taylor series of the function $F = F(x, t, u_0, u_1, \ldots, u_d)$ from (3).

\[A \text{ ZCR with values in an infinite-dimensional Lie algebra } L \text{ is given by}
\]
\[
\dot{A} = \dot{A}(x, t, u_0, u_1, \ldots, u_p), \quad \dot{B} = \dot{B}(x, t, u_0, u_1, \ldots, u_{p+d-1}), \quad D_x(\dot{B}) - D_t(\dot{A}) + [\dot{A}, \dot{B}] = 0,
\]
where $\dot{A}, \dot{B}$ are formal power series with coefficients in $L$. 
So \( \mathbb{F}^p(\mathcal{E}, a) \) is determined by the Taylor series of \( F \) in some (rather complicated, but explicit) way. Therefore, in principle, conditions \((23), (27)\) can be understood as some conditions on the function \( F = F(x, t, u_0, u_1, \ldots, u_d) \) from \((3)\).

In general, for a Lie algebra \( \mathfrak{L} \) given in terms of generators and relations, it is not easy to recognize whether \( \mathfrak{L} \) is infinite-dimensional and is not solvable. Because of this, if we do not make any additional assumptions, it is not easy to use conditions \((23), (27)\) for actual computations of integrable equations. However, if we consider a particular class of equations, conditions \((23), (27)\) may show that some equations from this class are not integrable.

For example, consider equations \((24)\) in the case \( q = 2 \). According to Theorem \([10]\) in Section \([4]\) for equations of the form \( u_t = u_5 + f(x, t, u_0, u_1, u_2, u_3) \), if \( \frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3} \neq 0 \) then \( \mathbb{F}^1(\mathcal{E}, a) = 0 \) for all \( a \in \mathcal{E} \).

Combining this with Theorem \([5]\) we get the following. If \( \frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3} \neq 0 \) then the equation \( u_t = u_5 + f(x, t, u_0, u_1, u_2, u_3) \) is not integrable.

Similar results can be obtained for many other classes of \((1 + 1)\)-dimensional evolution equations as well. We plan to study the resulting necessary conditions for integrability in more detail in forthcoming publications.

**Remark 13.** Recall that in this paper we study integrability by means of ZCRs. Another well-known approach to integrability involves symmetries and conservation laws (see, e.g., \([11, 21, 23, 28]\)).

Many remarkable classification results for some types of equations \((3)\) possessing higher (generalized) symmetries or conservation laws are known (see, e.g., \([21, 21, 28]\) and references therein).

The problem of describing ZCRs is very different from the problems of describing generalized symmetries and conservation laws. Since ZCRs play an essential role in the inverse scattering method and the theory of Bäcklund transformations, we think that the problem of ZCRs deserves to be studied in detail.

Since our integrability conditions \((23), (25), (27)\) arise from the study of ZCRs, we do not see any way to deduce these conditions from known results on symmetries and conservation laws.

**Remark 14.** In the study of integrable evolution equations, it often happens that the Wahlquist-Estabrook prolongation algebra is infinite-dimensional and is not solvable. This looks similar to conditions \((23), (25), (27)\), but one should keep in mind the following. The Wahlquist-Estabrook prolongation algebra is responsible for ZCRs of the form \((7)\). Our results concern the most general class of ZCRs \((1), (5)\), which is much larger than the class of Wahlquist-Estabrook ZCRs \((7)\).

### 1.4. Abbreviations, conventions, and notation.

The following abbreviations, conventions, and notation are used in the paper. ZCR = zero-curvature representation, WE = Wahlquist-Estabrook, BT = Bäcklund transformation. All manifolds and functions are supposed to be analytic.

\( \mathbb{K} \) is either \( \mathbb{C} \) or \( \mathbb{R} \). All vector spaces and algebras are supposed to be over the field \( \mathbb{K} \).

### 2. The algebras \( \mathbb{F}^p(\mathcal{E}, a) \)

Recall that \( x, t, u_k \) take values in \( \mathbb{K} \), where \( \mathbb{K} \) is either \( \mathbb{C} \) or \( \mathbb{R} \). Let \( \mathbb{K}^\infty \) be the infinite-dimensional space with the coordinates \( x, t, u_k \) for \( k \in \mathbb{Z}_{\geq 0} \). The topology on \( \mathbb{K}^\infty \) is defined as follows.

For each \( l \in \mathbb{Z}_{\geq 0} \), consider the space \( \mathbb{K}^{l+3} \) with the coordinates \( x, t, u_k \) for \( k \leq l \). One has the natural projection \( \pi_l: \mathbb{K}^\infty \rightarrow \mathbb{K}^{l+3} \) that “forgets” the coordinates \( u_k \) for \( k > l \).

We consider the standard topology on \( \mathbb{K}^{l+3} \). For any \( l \in \mathbb{Z}_{\geq 0} \) and any open subset \( V \subset \mathbb{K}^{l+3} \), the subset \( \pi_l^{-1}(V) \subset \mathbb{K}^\infty \) is, by definition, open in \( \mathbb{K}^\infty \). Such subsets form a base of the topology on \( \mathbb{K}^\infty \). In other words, we consider the smallest topology on \( \mathbb{K}^\infty \) such that the maps \( \pi_l, l \in \mathbb{Z}_{\geq 0} \), are continuous.

Let \( U \subset \mathbb{K}^{d+3} \) be an open subset such that the function \( F(x, t, u_0, u_1, \ldots, u_d) \) from \((3)\) is defined on \( U \). The infinite prolongation \( \mathcal{E} \) of equation \((3)\) can be defined as follows \( \mathcal{E} = \pi_d^{-1}(U) \subset \mathbb{K}^\infty \).

So \( \mathcal{E} \) is an open subset of the space \( \mathbb{K}^\infty \) with the coordinates \( x, t, u_k \) for \( k \in \mathbb{Z}_{\geq 0} \). The topology on \( \mathcal{E} \) is induced by the embedding \( \mathcal{E} \subset \mathbb{K}^\infty \).
A point \( a \in \mathcal{E} \) is determined by the values of the coordinates \( x, t, u_k \) at \( a \). Let

\[
(28) \quad a = (x = x_0, t = t_0, u_k = a_k) \in \mathcal{E}, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0},
\]

be a point of \( \mathcal{E} \).

Let \( \ell \in \mathbb{Z}_{\geq 0} \). For a function \( M = M(x, t, u_0, u_1, u_2, \ldots) \), the notation \( M \bigg|_{u_k = a_k, \ k \geq \ell} \) means that we substitute \( u_k = a_k \) for all \( k \geq \ell \) in the function \( M \).

Also, sometimes we need to substitute \( x = x_0 \) or \( t = t_0 \) in such functions. For example, if \( M = M(x, t, u_0, u_1, u_2, u_3) \), then

\[
M \bigg|_{x = x_0, \ u_k = a_k, \ k \geq 2} = M(x_0, t, u_0, a_1, a_2, a_3).
\]

For every \( N \in \mathbb{Z}_{\geq 0} \), let \( \mathfrak{gl}_N \) be the algebra of \( N \times N \) matrices with entries from \( \mathbb{K} \). Denote by \( \text{Id} \in \mathfrak{gl}_N \) the identity matrix.

**Theorem 6.** Let \( N \in \mathbb{Z}_{\geq 0} \) and \( p \in \mathbb{Z}_{\geq 0} \). Let \( \mathfrak{g} \subset \mathfrak{gl}_N \) be a matrix Lie algebra. Denote by \( \mathcal{G} \) the connected matrix Lie group corresponding to \( \mathfrak{g} \subset \mathfrak{gl}_N \).

Let

\[
(29) \quad A = A(x, t, u_0, u_1, \ldots, u_p), \quad B = B(x, t, u_0, u_1, \ldots, u_{p+d-1}),
\]

\[
(30) \quad D_x(B) - D_t(A) + [A, B] = 0
\]

be a ZCR of order \( \leq p \) such that the functions \( A, B \) are defined on a neighborhood of \( a \in \mathcal{E} \) and take values in \( \mathfrak{g} \).

Then there is a \( \mathcal{G} \)-valued function \( G = G(x, t, u_0, \ldots, u_{p-1}) \) on a neighborhood of \( a \in \mathcal{E} \) such that the functions

\[
(31) \quad \tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}
\]

satisfy

\[
(32) \quad \frac{\partial \tilde{A}}{\partial u_s} \bigg|_{u_k = a_k, \ k \geq \ell} = 0 \quad \forall \ s \geq 1,
\]

\[
(33) \quad \tilde{A} \bigg|_{u_k = a_k, \ k \geq 0} = 0,
\]

\[
(34) \quad \tilde{B} \bigg|_{x = x_0, \ u_k = a_k, \ k \geq 0} = 0.
\]

**Proof.** To explain the main idea, let us consider first the case \( p = 2 \). So \( A = A(x, t, u_0, u_1, u_2) \).

Consider the ordinary differential equation (ODE)

\[
(35) \quad \frac{\partial G_1}{\partial u_1} = G_1 \cdot \left( \frac{\partial A}{\partial u_2} \bigg|_{u_k = a_k, \ k \geq 2} \right)
\]

with respect to the variable \( u_1 \) and an unknown function \( G_1 = G_1(x, t, u_0, u_1) \). The variables \( x, t, u_0 \) are regarded as parameters in this ODE.

Let \( G_1(x, t, u_0, u_1) \) be a local solution of the ODE (35) with the initial condition \( G_1(x, t, u_0, a_1) = \text{Id} \). Since \( \partial A/\partial u_2 \) takes values in \( \mathfrak{g} \), the function \( G_1(x, t, u_0, u_1) \) takes values in \( \mathcal{G} \).

Set

\[
(36) \quad \tilde{A} = G_1AG_1^{-1} - D_x(G_1) \cdot G_1^{-1}, \quad \tilde{B} = G_1BG_1^{-1} - D_t(G_1) \cdot G_1^{-1}.
\]

Since \( G_1 \) takes values in \( \mathcal{G} \), the functions \( \tilde{A}, \tilde{B} \) take values in \( \mathfrak{g} \). Using (36) and (35), we get

\[
(37) \quad \frac{\partial \tilde{A}}{\partial u_2} \bigg|_{u_k = a_k, \ k \geq 2} = G_1 \left( \frac{\partial A}{\partial u_2} \bigg|_{u_k = a_k, \ k \geq 2} \right) G_1^{-1} - \left( \frac{\partial}{\partial u_2} (D_x(G_1)) \right) \bigg|_{u_k = a_k, \ k \geq 2} G_1^{-1} =
\]

\[
= G_1 \left( \frac{\partial A}{\partial u_1} \bigg|_{u_k = a_k, \ k \geq 2} \right) G_1^{-1} - G_1 \left( \frac{\partial A}{\partial u_2} \bigg|_{u_k = a_k, \ k \geq 2} \right) G_1^{-1} - G_1 \left( \frac{\partial A}{\partial u_2} \bigg|_{u_k = a_k, \ k \geq 2} \right) G_1^{-1} = 0.
\]
Now consider the ODE
\[
\frac{\partial G_0}{\partial u_0} = G_0 \cdot \left( \frac{\partial \hat{A}}{\partial u_1} \bigg|_{u_k = a_k, \ k \geq 1} \right)
\]
with respect to the variable \(u_0\) and an unknown function \(G_0 = G_0(x, t, u_0)\), where \(x, t\) are regarded as parameters.

Let \(G_0(x, t, u_0)\) be a local solution of the ODE (33) with the initial condition \(G_0(x, t, a_0) = \text{Id}\). Set
\[
\hat{A} = G_0 \hat{A}G_0^{-1} - D_x(G_0) \cdot G_0^{-1}, \quad \hat{B} = G_0 \hat{B}G_0^{-1} - D_t(G_0) \cdot G_0^{-1}.
\]
Then (37), (38), (39) yield
\[
\frac{\partial \hat{A}}{\partial u_s} \bigg|_{u_k = a_k, \ k \geq s} = 0 \quad \forall s \geq 1.
\]

Let \(\tilde{G} = \tilde{G}(x, t)\) be a local solution of the ODE
\[
\frac{\partial \tilde{G}}{\partial x} = \tilde{G} \cdot \left( \hat{A} \bigg|_{u_k = a_k, \ k \geq 0} \right)
\]
with the initial condition \(\tilde{G}(x_0, t) = \text{Id}\), where \(t\) is viewed as a parameter. Set
\[
\hat{A} = \tilde{G} \hat{A} \tilde{G}^{-1} - D_x(\tilde{G}) \cdot \tilde{G}^{-1}, \quad \hat{B} = \tilde{G} \hat{B} \tilde{G}^{-1} - D_t(\tilde{G}) \cdot \tilde{G}^{-1}.
\]
Then
\[
\frac{\partial \hat{A}}{\partial u_s} \bigg|_{u_k = a_k, \ k \geq s} = 0 \quad \forall s \geq 1, \quad \hat{A} \bigg|_{u_k = a_k, \ k \geq 0} = 0.
\]

Finally, let \(\tilde{G} = \tilde{G}(t)\) be a local solution of the ODE
\[
\frac{\partial \tilde{G}}{\partial t} = \tilde{G} \cdot \left( \hat{B} \bigg|_{x = x_0, \ u_k = a_k, \ k \geq 0} \right)
\]
with the initial condition \(\tilde{G}(t_0) = \text{Id}\). Set
\[
\hat{A} = \tilde{G} \hat{A} \tilde{G}^{-1} - D_x(\tilde{G}) \cdot \tilde{G}^{-1}, \quad \hat{B} = \tilde{G} \hat{B} \tilde{G}^{-1} - D_t(\tilde{G}) \cdot \tilde{G}^{-1}.
\]
Then \(\hat{A}, \hat{B}\) obey (32), (33), (34).

Let \(G = \hat{G} \cdot \tilde{G} \cdot G_0 \cdot G_1\). Then equations (36), (39), (40), (42) imply
\[
\hat{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \hat{B} = GBG^{-1} - D_t(G) \cdot G^{-1}.
\]
Therefore, \(G = \hat{G} \cdot \tilde{G} \cdot G_0 \cdot G_1\) satisfies all the required properties in the case \(p = 2\).

This construction can be easily generalized to the case of arbitrary \(p\). One can define \(G\) as the product \(G = \hat{G} \cdot \tilde{G} \cdot G_0 \cdot G_1 \ldots G_{p-1}\,\), where the \(\mathcal{G}\)-valued functions
\[
G_q = G_q(x, t, u_0, \ldots, u_p), \quad q = 0, 1, \ldots, p - 1, \quad \hat{G} = \hat{G}(x, t), \quad \tilde{G} = \tilde{G}(t).
\]
are defined as solutions of certain ODEs similar to the ODEs considered above. \(\Box\)

**Remark 15.** Since the functions (29) obey (30), the functions (31) satisfy
\[
D_x(\hat{B}) - D_t(\hat{A}) + [\hat{A}, \hat{B}] = 0.
\]
Recall that all functions are supposed to be analytic. Therefore, taking a sufficiently small neighborhood of \(a \in \mathcal{E}\), we can assume that \(\hat{A}, \hat{B}\) are represented as absolutely convergent power series
\[
\hat{A} = \sum_{l_1, l_2, j_0, \ldots, j_p \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \ldots (u_p - a_p)^{j_p} \cdot \hat{A}^{l_1, l_2}_{j_0 \ldots j_p},
\]
\[
\hat{B} = \sum_{l_1, l_2, j_0, \ldots, j_p + d - 1 \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \ldots (u_{p+d-1} - a_{p+d-1})^{j_{p+d-1}} \cdot \hat{B}^{l_1, l_2}_{j_0 \ldots j_p + d - 1},
\]
\[
\hat{A}^{l_1, l_2}_{j_0 \ldots j_p}, \hat{B}^{l_1, l_2}_{j_0 \ldots j_p + d - 1} \in \mathfrak{g}.
\]
For each $k \in \mathbb{Z}_{>0}$, set
\begin{equation}
\mathcal{V}_k = \left\{ (i_0, \ldots, i_k) \in \mathbb{Z}_{\geq 0}^{k+1} \middle| \exists r \in \{1, \ldots, k\} \text{ such that } i_r = 1, \ i_q = 0 \ \forall \ q > r \right\}.
\end{equation}
In other words, for $k \in \mathbb{Z}_{>0}$ and $i_0, \ldots, i_k \in \mathbb{Z}_{\geq 0}$, one has $(i_0, \ldots, i_k) \in \mathcal{V}_k$ iff there is $r \in \{1, \ldots, k\}$ such that $(i_0, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_k) = (i_0, \ldots, i_{r-1}, 1, 0, \ldots, 0)$. Set also $\mathcal{V}_0 = \emptyset$.

Using formulas (44), (45), we see that properties (32), (33), (34) are equivalent to
\begin{equation}
\tilde{A}_{l_0}^{l_1, l_2} = \tilde{B}_{l_0}^{0, l_2} = 0, \quad \tilde{A}_{l_0}^{l_1, l_2} = 0, \quad (i_0, \ldots, i_p) \in \mathcal{V}_p, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}.
\end{equation}

**Remark 16.** Let $\mathfrak{L}$ be a Lie algebra. Consider a formal power series of the form
\begin{equation}
C = \sum_{l_1, l_2, i_0, \ldots, i_m \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^i \ldots (u_m - a_m)^i C_{i_0 \ldots i_m}^{l_1, l_2}, \quad C_{i_0 \ldots i_m}^{l_1, l_2} \in \mathfrak{L}.
\end{equation}
Set
\begin{align}
D_x(C) &= \sum_{l_1, l_2, i_0, \ldots, i_m} D_x ((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^i \ldots (u_m - a_m)^i) C_{i_0 \ldots i_m}^{l_1, l_2}, \\
D_t(C) &= \sum_{l_1, l_2, i_0, \ldots, i_m} D_t ((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^i \ldots (u_m - a_m)^i) C_{i_0 \ldots i_m}^{l_1, l_2}.
\end{align}
The expressions
\begin{equation}
D_x ((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^i \ldots (u_m - a_m)^i), \quad D_t ((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^i \ldots (u_m - a_m)^i)
\end{equation}
are functions of the variables $x$, $t$, $u_k$. Taking the corresponding Taylor series at the point $(28)$, we regard (50) as power series. Then (48), (49) become formal power series with coefficients in $\mathfrak{L}$.

According to (9), one has $D_t = \frac{\partial}{\partial t} + \sum_{k=0}^{d} \frac{\partial}{\partial u_k}$, where $F = F(x, t, u_0, u_1, \ldots, u_d)$ is given in (3). When we apply $D_t$ in (49), we view $F$ as a power series, using the Taylor series of the function $F$.

Consider another formal power series
\begin{equation}
R = \sum_{q_1, q_2, i_0, \ldots, i_m \geq 0} (x - x_0)^{q_1} (t - t_0)^{q_2} (u_0 - a_0)^i \ldots (u_m - a_m)^i R_{i_0 \ldots i_m}^{q_1, q_2}, \quad R_{i_0 \ldots i_m}^{q_1, q_2} \in \mathfrak{L}.
\end{equation}
Then the Lie bracket $[C, R]$ is defined as follows
\begin{equation}
[C, R] = \sum_{l_1, l_2, i_0, \ldots, i_m, q_1, q_2 \geq 0} (x - x_0)^{l_1+q_1} (t - t_0)^{l_2+q_2} (u_0 - a_0)^i \ldots (u_m - a_m)^i C_{i_0 \ldots i_m}^{l_1, l_2} R_{i_0 \ldots i_m}^{q_1, q_2}.
\end{equation}

**Remark 17.** The main idea of the definition of the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ can be informally outlined as follows. According to Theorem 6 and Remark 15 any ZCR (29), (30) of order $\leq p$ is gauge equivalent to a ZCR given by functions $\tilde{A}$, $\tilde{B}$ that are of the form (44), (45) and satisfy (43), (47).

To define $\mathbb{F}^p(\mathcal{E}, a)$, we regard $\tilde{A}_{l_0}^{l_1, l_2}$, $\tilde{B}_{l_0}^{l_1, l_2}$ from (44), (45) as abstract symbols. By definition, the algebra $\mathbb{F}^p(\mathcal{E}, a)$ is generated by the symbols $\tilde{A}_{l_0}^{l_1, l_2}$, $\tilde{B}_{l_0}^{l_1, l_2}$ for $l_1, l_2, i_0, \ldots, i_p, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0}$. Relations for these generators are provided by equations (43), (47). The details of this construction are presented below.

Let $\mathfrak{F}$ be the free Lie algebra generated by the symbols $\tilde{A}_{l_0}^{l_1, l_2}$ and $\tilde{B}_{l_0}^{l_1, l_2}$ for all $l_1, l_2, i_0, \ldots, i_p, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0}$. Consider the following power series with coefficients in $\mathfrak{F}$
\begin{align}
\mathbf{A} &= \sum_{l_1, l_2, i_0, \ldots, i_p \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^i \ldots (u_p - a_p)^i \cdot \tilde{A}_{i_0 \ldots i_p}^{l_1, l_2}, \\
\mathbf{B} &= \sum_{l_1, l_2, j_0, \ldots, j_{p+d-1} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^j \ldots (u_{p+d-1} - a_{p+d-1})^{j_{p+d-1}} \cdot \tilde{B}_{j_0, j_{p+d-1}}^{l_1, l_2}.
\end{align}
Then the power series \( D_x(B) - D_t(A) + [A, B] = \sum_{l_1, l_2, q_0, \ldots, q_{p+d} \geq 0} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{q_0} \ldots (u_{p+d} - a_{p+d})^{q_{p+d}} \cdot Z_{q_0 \ldots q_{p+d}}^{l_1, l_2} \) for some elements \( Z_{q_0 \ldots q_{p+d}}^{l_1, l_2} \in \mathfrak{g} \).

Let \( \mathcal{I} \subset \mathfrak{g} \) be the ideal generated by the elements

\[
\begin{align*}
Z_{q_0 \ldots q_{p+d}}^{l_1, l_2}, & \quad A_{i_0 \ldots i_p}^{l_1, l_2}, & \quad B_{j_0 \ldots j_{p+d-1}}^{l_1, l_2}, & \quad l_1, l_2, q_0, \ldots, q_{p+d} \in \mathbb{Z}_{\geq 0}, \\
(i_0, \ldots, i_p) & \in \mathcal{V}_p, & \quad l_1, l_2 & \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

Set \( \mathbb{F}^p(\mathcal{E}, a) = \mathfrak{g} / \mathcal{I} \). Consider the natural homomorphism \( \psi: \mathfrak{g} \to \mathfrak{g} / \mathcal{I} = \mathbb{F}^p(\mathcal{E}, a) \) and set

\[
\begin{align*}
\psi(A_{i_0 \ldots i_p}^{l_1, l_2}) & = A_{i_0 \ldots i_p}^{l_1, l_2}, & \quad \psi(B_{j_0 \ldots j_{p+d-1}}^{l_1, l_2}) & = B_{j_0 \ldots j_{p+d-1}}^{l_1, l_2}.
\end{align*}
\]

The definition of \( \mathcal{I} \) implies that the power series

\[
\begin{align*}
\mathbb{A} & = \sum_{l_1, l_2, i_0 \ldots i_p \geq 0} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{i_0} \ldots (u_{p+d} - a_{p+d})^{i_p} \cdot A_{i_0 \ldots i_p}^{l_1, l_2}, \\
\mathbb{B} & = \sum_{l_1, l_2, j_0 \ldots j_{p+d-1} \geq 0} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{j_0} \ldots (u_{p+d-1} - a_{p+d-1})^{j_{p+d-1}} \cdot B_{j_0 \ldots j_{p+d-1}}^{l_1, l_2}
\end{align*}
\]

satisfy

\[
D_x(B) - D_t(A) + [A, B] = 0.
\]

**Remark 18.** The Lie algebra \( \mathbb{F}^p(\mathcal{E}, a) \) can be described in terms of generators and relations as follows.

Equation (53) is equivalent to some Lie algebraic relations for \( A_{i_0 \ldots i_p}^{l_1, l_2}, B_{j_0 \ldots j_{p+d-1}}^{l_1, l_2} \). The algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is given by the generators \( A_{i_0 \ldots i_p}^{l_1, l_2}, B_{j_0 \ldots j_{p+d-1}}^{l_1, l_2} \), the relations arising from (53), and the following relations

\[
\begin{align*}
A_{0 \ldots 0}^{0 \ldots 0} & = 0, & \quad A_{i_0 \ldots i_p}^{l_1, l_2} & = 0, & \quad (i_0, \ldots, i_p) & \in \mathcal{V}_p, & \quad l_1, l_2 & \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

Note that, according to Remark 16, the definition of the power series \( D_t(A) \) in (53) uses the Taylor series of the function \( F = F(x, t, u_0, u_1, \ldots, u_d) \) from (3), because \( D_t \) is determined by \( F \).

So the constructed generators and relations for the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) are determined by the Taylor series of the function \( F \) at the point \((28)\).

Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra. A homomorphism \( \rho: \mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g} \) is said to be regular if the power series

\[
\begin{align*}
\tilde{A} & = \sum_{l_1, l_2, i_0 \ldots i_p} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{i_0} \ldots (u_{p+d} - a_{p+d})^{i_p} \cdot \rho(A_{i_0 \ldots i_p}^{l_1, l_2}), \\
\tilde{B} & = \sum_{l_1, l_2, j_0 \ldots j_{p+d-1}} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{j_0} \ldots (u_{p+d-1} - a_{p+d-1})^{j_{p+d-1}} \cdot \rho(B_{j_0 \ldots j_{p+d-1}}^{l_1, l_2})
\end{align*}
\]

are absolutely convergent on a neighborhood of \( a \in \mathcal{E} \). In other words, \( \rho \) is regular iff (55), (56) are analytic functions with values in \( \mathfrak{g} \) on a neighborhood of \( a \in \mathcal{E} \).

Since (51), (52) obey (53), the power series (55), (56) satisfy (43) for any homomorphism \( \rho: \mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g} \). Therefore, if \( \rho \) is regular, the analytic functions (55), (56) form a ZCR. Denote this ZCR by \( \mathbb{Z}(\mathcal{E}, a, p, \rho) \).

Combining this construction with Theorem 6 and Remark 15 we obtain the following result.

**Theorem 7.** Let \( \mathfrak{g} \) be a finite-dimensional matrix Lie algebra. For any \( \mathfrak{g} \)-valued ZCR (29), (30) of order \( \leq p \) on a neighborhood of \( a \in \mathcal{E} \), there is a regular homomorphism \( \rho: \mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g} \) such that the ZCR (29), (30) is gauge equivalent to the ZCR \( \mathbb{Z}(\mathcal{E}, a, p, \rho) \).

The ZCR \( \mathbb{Z}(\mathcal{E}, a, p, \rho) \) takes values in the Lie algebra \( \rho(\mathbb{F}^p(\mathcal{E}, a)) \subset \mathfrak{g} \).
Suppose that \( p \geq 1 \). According to Remark 18, the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is given by the generators \( \mathbb{A}_{i_0 \ldots i_p}^{l_1 l_2}, \mathbb{B}_{j_0 \ldots j_{p+d-1}}^{l_1 l_2} \) and the relations arising from (53), (54). Similarly, the algebra \( \mathbb{F}^{p-1}(\mathcal{E}, a) \) is given by the generators \( \mathbb{A}_{i_0 \ldots i_{p-1}}^{l_1 l_2}, \mathbb{B}_{j_0 \ldots j_{p+d-2}}^{l_1 l_2} \) and the relations arising from (58), (59).

This implies that the map
\[
\mathbb{A}_{j_0 \ldots j_p \ldots p+1}^{l_1 l_2} \mapsto \delta_{0,0}^{l_1 l_2}, \quad \mathbb{B}_{j_0 \ldots j_{p+d-2}}^{l_1 l_2} \mapsto \delta_{0,0}^{l_1 l_2}
\]
determines a surjective homomorphism \( \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a) \). Here \( \delta_{0,0}^{l_1 l_2} \) and \( \delta_{0,0}^{l_1 l_2} \) are the Kronecker deltas.

According to Theorem 7, the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is responsible for ZCRs of order \( \leq p \), and the algebra \( \mathbb{F}^{p-1}(\mathcal{E}, a) \) is responsible for ZCRs of order \( \leq p - 1 \). The constructed homomorphism \( \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a) \) reflects the fact that any ZCR of order \( \leq p - 1 \) is at the same time of order \( \leq p \).

Thus we obtain the following sequence of surjective homomorphisms of Lie algebras
\[
\cdots \to \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a) \to \cdots \to \mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a).
\]

### 3. Some Results on Generators of \( \mathbb{F}^p(\mathcal{E}, a) \)

According to Remark 18, the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is given by the generators \( \mathbb{A}_{i_0 \ldots i_p}^{l_1 l_2}, \mathbb{B}_{j_0 \ldots j_{p+d-1}}^{l_1 l_2} \) and the relations arising from (53), (54). Using (59), we can rewrite equation (53) as
\[
\frac{\partial}{\partial x} (\mathbb{B}) + \sum_{k=0}^{p+1} u_{k+1} \frac{\partial}{\partial u_k} (\mathbb{B}) - \frac{\partial}{\partial t} (\mathbb{A}) - \sum_{k=0}^{p} D_x^k (F(x, t, u_0, u_1, \ldots, u_d)) \frac{\partial}{\partial u_k} (\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0.
\]

We regard \( F = F(x, t, u_0, u_1, \ldots, u_d) \) as a power series, using the Taylor series of the function \( F \) at the point \( \Phi_0 \).

**Theorem 8.** The elements
\[
\mathbb{A}_{i_0 \ldots i_p}^{l_1 l_2}, \quad l_1, i_0, \ldots, i_p \in \mathbb{Z}_{\geq 0},
\]
generate the algebra \( \mathbb{F}^p(\mathcal{E}, a) \).

**Proof.** For each \( l \in \mathbb{Z}_{\geq 0} \), denote by \( \mathfrak{A}_l \subseteq \mathbb{F}^p(\mathcal{E}, a) \) the subalgebra generated by all the elements \( \mathbb{A}_{i_0 \ldots i_p}^{l_1 l_2} \) with \( l_2 \leq l \). To prove Theorem 8 we need several lemmas.

**Lemma 1.** Let \( l_1, l_2, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0} \) be such that \( j_0 + \cdots + j_{p+d-1} > 0 \). Then \( \mathbb{B}_{j_0 \ldots j_{p+d-1}}^{l_1 l_2} \in \mathfrak{A}_l \).

**Proof.** For any \( j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0} \) satisfying \( j_0 + \cdots + j_{p+d-1} > 0 \), denote by \( \Phi(j_0, \ldots, j_{p+d-1}) \) the maximal integer \( r \in \{0, 1, \ldots, p + d - 1\} \) such that \( j_r \neq 0 \). Set also \( \Phi(0, \ldots, 0) = -1 \).

Differentiating (53) with respect to \( u_{p+d} \), we obtain
\[
\frac{\partial}{\partial u_{p+d}} (\mathbb{B}) = \frac{\partial F}{\partial u_d} \cdot \frac{\partial}{\partial u_p} (\mathbb{A}),
\]
which implies \( \mathbb{B}_{j_0 \ldots j_{p+d-1}}^{l_1 l_2} \in \mathfrak{A}_l \) for all \( l_1, l_2, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0} \) obeying \( \Phi(j_0, \ldots, j_{p+d-1}) = p + d - 1 \).
We are going to show that \( \B_{j_0^1, \ldots, j_{p+d}^1} \in \mathfrak{U}_2 \) for all \( l_1, l_2, \ldots, l_{p+d-1} \in \mathbb{Z}_{\geq 0} \) satisfying \( \Phi(j_0^1, \ldots, j_{p+d-1}) > m \).

Combining (58) with (61), we get

\[
\Phi(\tilde{\mathfrak{B}}_{k} S_{}\mathcal{D}) = (63)
\]

From (54) it follows that

\[
\text{In view of (63), (64), for any } A, B, C \text{ of the form (66) obviously, for any } \hat{l}_2 \leq l_2 \text{ one has } \mathfrak{U}_1 \subset \mathfrak{U}_2. \text{ Taking into account assumption (60), we obtain that the elements (65) belong to } \mathfrak{U}_2. \text{ Hence } \B_{j_0^1, \ldots, j_{p+d}^1} \in \mathfrak{U}_2.
\]

The proof of the lemma is completed by induction.

**Lemma 2.** For all \( l_1, l_2 \in \mathbb{Z}_{\geq 0} \), one has \( \B_{0^1, \ldots, 0^1} \in \mathfrak{U}_2 \).

**Proof.** According to (54), we have \( \mathfrak{U}_{0^1, \ldots, 0^1} = 0 \). Therefore, it is sufficient to prove \( \B_{\hat{l}_0^1, \hat{l}_1^1, \ldots, \hat{l}_p^1} \in \mathfrak{U}_2 \) for \( l_1 > 0 \).

Note that property (54) implies

\[
\partial_t^{(\hat{l}_0^1, \hat{l}_1^1, \ldots, \hat{l}_p^1)} \B_{\hat{l}_0^1, \hat{l}_1^1, \ldots, \hat{l}_p^1} = 0.
\]
In view of (52), one has

\[
(67) \quad \frac{\partial}{\partial x}(\mathbb{B}) \bigg|_{u_k = a_k, \ k \geq 0} = \sum_{l_1 > 0, \ l_2 \geq 0} l_1(x - x_0)^{l_1-1}(t - t_0)^{l_2} \cdot \mathbb{B}_{0-0}^{l_1,l_2}.
\]

Substituting \( u_k = a_k \) for all \( k \in \mathbb{Z}_{\geq 0} \) in (58) and using (66), (67), we get

\[
(68) \quad \sum_{l_1 > 0, \ l_2 \geq 0} l_1(x - x_0)^{l_1-1}(t - t_0)^{l_2} \cdot \mathbb{B}_{0-0}^{l_1,l_2} = -\left( \sum_{k=0}^{p+d-1} u_{k+1} \frac{\partial}{\partial u_k}(\mathbb{B}) \right) \bigg|_{u_k = a_k, \ k \geq 0} + \left( \sum_{k=0}^{p} D^k_x(F) \frac{\partial}{\partial u_k}(\mathbb{A}) \right) \bigg|_{u_k = a_k, \ k \geq 0}.
\]

Combining (51), (52), (68), we see that for any \( l_1 > 0 \) and \( l_2 \geq 0 \) the element \( \mathbb{B}_{0-0}^{l_1,l_2} \) is equal to a linear combination of elements of the form

\[
(69) \quad \mathcal{A}_l^{l_1,l_2}, \quad \mathbb{B}_l^{l_1,l_2}, \quad l'_1, i_0, \ldots, i_p, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0}, \quad j_0 + \ldots + j_{p+d-1} = 1.
\]

According to Lemma 1 and the definition of \( \mathfrak{A}_l \), the elements (69) belong to \( \mathfrak{A}_l \). Thus \( \mathbb{B}_{0-0}^{l_1,l_2} \in \mathfrak{A}_l \). \( \square 

**Lemma 3.** For all \( l_1, l, i_0, \ldots, i_p \in \mathbb{Z}_{\geq 0} \), we have \( \mathcal{A}_l^{l_1,l+1} \in \mathfrak{A}_l \).

**Proof.** Using (51), we can rewrite equation (58) as

\[
\sum_{l_1, l, i_0, \ldots, i_p \geq 0} (l + 1)(x - x_0)^{l_1}(t - t_0)^l(u_0 - a_0)^{i_0} \ldots (u_p - a_p)^{i_p} \cdot \mathcal{A}_l^{l_1,l+1} = \frac{\partial}{\partial x}(\mathbb{B}) + \sum_{k=0}^{p+d-1} u_{k+1} \frac{\partial}{\partial u_k}(\mathbb{B}) - \sum_{k=0}^{p} D^k_x(F) \frac{\partial}{\partial u_k}(\mathbb{A}) + [\mathbb{A}, \mathbb{B}].
\]

This implies that \( \mathcal{A}_l^{l_1,l+1} \) is equal to a linear combination of elements of the form

\[
(70) \quad \mathcal{A}_l^{l_1,l_2}, \quad \mathbb{B}_l^{l_1,l_2}, \quad \left[ \mathcal{A}_l^{l_1,l_2}, \mathbb{B}_l^{l_1,l_2} \right], \quad \tilde{l}_1 \leq l, \quad \tilde{l}_2 \leq l, \quad \tilde{i}_0, \ldots, \tilde{i}_p, \tilde{j}_0, \ldots, \tilde{j}_{p+d-1} \in \mathbb{Z}_{\geq 0}.
\]

Using Lemmas 1, 2 and the condition \( \tilde{l}_2 \leq l \), we get \( \mathbb{B}_l^{l_1,l_2} \in \mathfrak{A}_l \subset \mathfrak{A}_l \). Therefore, the elements (70) belong to \( \mathfrak{A}_l \). Hence \( \mathcal{A}_l^{l_1,l+1} \in \mathfrak{A}_l \). \( \square 

Return to the proof of Theorem 8. According to Lemmas 1, 2 and the definition of \( \mathfrak{A}_l \), we have \( \mathcal{A}_l^{l_1,l_2}, \mathbb{B}_l^{l_1,l_2} \in \mathfrak{A}_l \) for all \( l_1, l_2, i_0, \ldots, i_p, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0} \). Lemma 3 implies that

\[ \mathfrak{A}_l \subset \mathfrak{A}_{l-1} \subset \mathfrak{A}_{l-2} \subset \cdots \subset \mathfrak{A}_0. \]

Therefore, \( \mathbb{F}^p(\mathcal{E}, a) \) is equal to \( \mathfrak{A}_0 \), which is generated by the elements (59). \( \square 

4. SOME RESULTS ON \( \mathbb{F}^p(\mathcal{E}, a) \) FOR EQUATIONS (17)

In this section we study the algebras (57) for equations of the form

\[
(71) \quad u_t = u_{2q+1} + f(x, t, u_0, u_1, \ldots, u_{2q-1}), \quad q \in \{1, 2, 3\},
\]

where \( f \) is an arbitrary function. Let \( \mathcal{E} \) be the infinite prolongation of equation (71). Then \( \mathcal{E} \) is an infinite-dimensional manifold with the coordinates \( x, t, u_k \) for \( k \in \mathbb{Z}_{\geq 0} \).

For equation (71), the total derivative operators (6) are

\[
(72) \quad D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{q} u_{k+1} \frac{\partial}{\partial u_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_{k=0}^{q} D^k_x(u_{2q+1} + f(x, t, u_0, u_1, \ldots, u_{2q-1})) \frac{\partial}{\partial u_k}.
\]

Consider an arbitrary point \( a \in \mathcal{E} \) given by

\[
(73) \quad a = (x = x_0, t = t_0, u_k = a_k) \in \mathcal{E}, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0}.
\]
Let $p \in \mathbb{Z}_{>0}$ be such that $p \geq q + \delta_{q,3}$, where $\delta_{q,3}$ is the Kronecker delta. According to Remark 18 the algebra $\mathbb{F}^p(\mathcal{E}, a)$ can be described as follows. Consider formal power series
\begin{align}
\mathbb{A} &= \sum_{l_1,l_2,l_0,\ldots,l_p \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{l_0} \cdots (u_p - a_p)^{l_p} \cdot \mathbb{E}^{l_1,l_2}_{l_0,\ldots,l_p}, \\
\mathbb{B} &= \sum_{l_1,l_2,l_0,\ldots,l_p \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{l_0} \cdots (u_p + 2q - a_p + 2q)^{l_p} \cdot \mathbb{E}^{l_1,l_2}_{l_0,\ldots,l_p + 2q}
\end{align}
satisfying
\begin{align}
\mathbb{A}^{l_1,l_2}_{l_0,\ldots,l_p} &= 0 \text{ if } \exists r \in \{1, \ldots, p\} \text{ such that } i_r = 1, \ i_m = 0 \ \forall \ m > r, \\
\mathbb{A}^{l_1,l_2}_{0,\ldots,0} &= 0 \ \forall l_1, l_2 \in \mathbb{Z}_{\geq 0}, \\
\mathbb{B}^{0,l_2}_{0,\ldots,0} &= 0 \ \forall l_2 \in \mathbb{Z}_{\geq 0}.
\end{align}
Then $\mathbb{A}^{l_1,l_2}_{l_0,\ldots,l_p}$ and $\mathbb{B}^{l_1,l_2}_{l_0,\ldots,l_p + 2q}$ are generators of the algebra $\mathbb{F}^p(\mathcal{E}, a)$, and the equation
\begin{equation}
D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0
\end{equation}
provides relations for these generators (in addition to relations (76), (77), (78)).

Note that condition (76) is equivalent to
\begin{equation}
\frac{\partial}{\partial u_{k}}(\mathbb{A}) \bigg|_{u_{k} = a_{k}, \ k \geq s} = 0 \ \forall \ s \geq 1.
\end{equation}

Using (72), one can rewrite equation (79) as
\begin{equation}
\frac{\partial}{\partial x}(\mathbb{B}) + \sum_{k=0}^{p+2q} u_{k+1} \frac{\partial}{\partial u_{k}}(\mathbb{B}) + [\mathbb{A}, \mathbb{B}] = \frac{\partial}{\partial t}(\mathbb{A}) + \sum_{k=0}^{p} \left( u_{k+2q+1} + D_{x}^{k}(f(x, t, u_{0}, u_{1}, \ldots, u_{2q-1})) \right) \frac{\partial}{\partial u_{k}}(\mathbb{A}).
\end{equation}
Here we regard $f(x, t, u_{0}, u_{1}, \ldots, u_{2q-1})$ as a power series, using the Taylor series of the function $f$ at the point (73). Differentiating (81) with respect to $u_{p+2q+1}$, we obtain
\begin{equation}
\frac{\partial}{\partial u_{p+2q}}(\mathbb{B}) = \frac{\partial}{\partial u_{p}}(\mathbb{A}).
\end{equation}
From (82) it follows that $\mathbb{B}$ is of the form
\begin{equation}
\mathbb{B} = u_{p+2q} \frac{\partial}{\partial u_{p}}(\mathbb{A}) + \mathbb{B}_{0}(x, t, u_{0}, \ldots, u_{p+2q-1}),
\end{equation}
where $\mathbb{B}_{0}(x, t, u_{0}, \ldots, u_{p+2q-1})$ is a power series in the variables
\begin{align}
x - x_0, \ t - t_0, \ u_0 - a_0, \ \ldots, \ u_{p+2q-1} - a_{p+2q-1}.
\end{align}

Differentiating (81) with respect to $u_{p+2q}$ for $i = 1, \ldots, 2q - 1$ and using (83), one gets
\begin{equation}
\frac{\partial^{2}}{\partial u_{p} \partial u_{p}}(\mathbb{A}) + \frac{\partial^{2}}{\partial u_{p+1} \partial u_{p+2q-1}}(\mathbb{B}_{0}) = 0, \ \frac{\partial^{2}}{\partial u_{p+s} \partial u_{p+2q-1}}(\mathbb{B}_{0}) = 0, \ 2 \leq s \leq 2q - 1.
\end{equation}
Therefore, $\mathbb{B}_{0} = \mathbb{B}_{0}(x, t, u_{0}, \ldots, u_{p+2q-1})$ is of the form
\begin{equation}
\mathbb{B}_{0} = u_{p+1} u_{p+2q-1} \left( \frac{1}{2} \delta_{q,1} - 1 \right) \frac{\partial^{2}}{\partial u_{p} \partial u_{p}}(\mathbb{A}) + u_{p+2q-1} \mathbb{B}_{01}(x, t, u_{0}, \ldots, u_{p}) + \mathbb{B}_{00}(x, t, u_{0}, \ldots, u_{p+2q-2}).
\end{equation}
Here $\mathbb{B}_{01}(x, t, u_{0}, \ldots, u_{p})$ is a power series in the variables $x - x_0, t - t_0, u_0 - a_0, \ldots, u_p - a_p$ and $\mathbb{B}_{00}(x, t, u_{0}, \ldots, u_{p+2q-2})$ is a power series in the variables $x - x_0, t - t_0, u_0 - a_0, \ldots, u_{p+2q-2} - a_{p+2q-2}$.

Lemma 4. Recall that $q \in \{1, 2, 3\}$ and $p \geq q + \delta_{q,3}$. We have
\begin{equation}
D_{x} \left( \frac{\partial^{2}}{\partial u_{p} \partial u_{p}}(\mathbb{A}) \right) + \left[ \mathbb{A}, \frac{\partial^{2}}{\partial u_{p} \partial u_{p}}(\mathbb{A}) \right] = 0.
\end{equation}
Proof. Consider first the case $q = 1$. Then, by our assumption, $p \geq 1$.

Equation (71) reads $u_i = u_3 + f(x, t, u_0, u_1)$. According to (83), (84), for $q = 1$ one has

$$B = u_{p+2}(\partial_{u_p}^2(A) - \frac{1}{2}(u_{p+1})^2 \partial_{u_p} A) + u_{p+1}B_01(x, t, u_0, \ldots, u_p) + B_{00}(x, t, u_0, \ldots, u_p).$$

Using (80), one can check that

$$\left(\frac{\partial^2}{\partial u_p \partial u_{p+2}} - \frac{1}{2} \frac{\partial^2}{\partial u_{p+1} \partial u_{p+1}}\right) \left(D_x(B) - D_t(A) + [A, B]\right) = \frac{3}{2} \left(D_x\left(\frac{\partial^2}{\partial u_p \partial u_p}(A)\right) + [A, \frac{\partial^2}{\partial u_p \partial u_p}(A)]\right).$$

Since $D_x(B) - D_t(A) + [A, B] = 0$ by (81), equation (87) implies (85) for $q = 1$.

Now let $q = 2$. Then $p \geq 2$. Equation (71) reads $u_i = u_5 + f(x, t, u_0, u_1, u_2, u_3)$. Using (83), (84), for $q = 2$ one obtains

$$B = u_{p+4} \frac{\partial}{\partial u_p}(A) - u_{p+1}u_{p+3} \frac{\partial^2}{\partial u_p \partial u_p}(A) + u_{p+3}B_01(x, t, u_0, \ldots, u_p) + B_{00}(x, t, u_0, \ldots, u_{p+2}).$$

Applying the operator $\frac{\partial^2}{\partial u_{p+2} \partial u_{p+3}}$ to equation (81) and using (88), we get

$$\frac{\partial^2}{\partial u_{p+2} \partial u_{p+3}}(B_{00}) - \frac{\partial^2}{\partial u_p \partial u_p}(A) = 0.$$

Combining (88) and (89), one can verify that

$$\left(\frac{\partial^2}{\partial u_p \partial u_{p+4}} - \frac{\partial^2}{\partial u_{p+1} \partial u_{p+3}} + \frac{1}{2} \frac{\partial^2}{\partial u_{p+2} \partial u_{p+2}}\right) \left(D_x(B) - D_t(A) + [A, B]\right) = \frac{5}{2} \left(D_x\left(\frac{\partial^2}{\partial u_p \partial u_p}(A)\right) + [A, \frac{\partial^2}{\partial u_p \partial u_p}(A)]\right).$$

Since $D_x(B) - D_t(A) + [A, B] = 0$ by (81), equation (90) yields (85) for $q = 2$.

Finally, consider the case $q = 3$. Then $p \geq 4$.

Equation (71) reads $u_i = u_7 + f(x, t, u_0, u_1, u_2, u_3, u_4, u_5)$. According to (83), (84), for $q = 3$ we have

$$B = u_{p+6} \frac{\partial}{\partial u_p}(A) - u_{p+1}u_{p+5} \frac{\partial^2}{\partial u_p \partial u_p}(A) + u_{p+5}B_01(x, t, u_0, \ldots, u_p) + B_{00}(x, t, u_0, \ldots, u_{p+4}).$$

Differentiating (81) with respect to $u_{p+5}$, $u_{p+i}$ for $i = 2, 3, 4$ and using (91), one gets

$$- \frac{\partial^2}{\partial u_p \partial u_p}(A) + \frac{\partial^2}{\partial u_{p+2} \partial u_{p+4}}(B_{00}) = 0, \quad \frac{\partial^2}{\partial u_{p+3} \partial u_{p+4}}(B_{00}) = \frac{\partial^2}{\partial u_{p+4} \partial u_{p+4}}(B_{00}) = 0.$$

Therefore, $B_{00} = B_{00}(x, t, u_0, \ldots, u_{p+4})$ is of the form

$$B_{00} = u_{p+2}u_{p+4} \frac{\partial^2}{\partial u_p \partial u_p}(A) + u_{p+4}B_{01}(x, t, u_0, \ldots, u_{p+1}) + B_{000}(x, t, u_0, \ldots, u_{p+3})$$

for some $B_{001}(x, t, u_0, \ldots, u_{p+1})$ and $B_{000}(x, t, u_0, \ldots, u_{p+3})$.

Applying the operator $\frac{\partial^2}{\partial u_{p+3} \partial u_{p+4}}$ to equation (81) and using (91), (92), we obtain

$$\frac{\partial^2}{\partial u_{p+3} \partial u_{p+4}}(B_{00}) + \frac{\partial^2}{\partial u_p \partial u_p}(A) = 0.$$

Using (91), (92), (93), one can check that

$$\left(\frac{\partial^2}{\partial u_p \partial u_{p+6}} - \frac{\partial^2}{\partial u_{p+1} \partial u_{p+5}} + \frac{\partial^2}{\partial u_{p+2} \partial u_{p+4}} - \frac{1}{2} \frac{\partial^2}{\partial u_{p+3} \partial u_{p+3}}\right) \left(D_x(B) - D_t(A) + [A, B]\right) =$$
Since $D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}] = 0$ by (91), equation (94) implies (85) for $q = 3$. □

**Lemma 5.** One has

\[ \frac{\partial^3}{\partial u_k \partial u_p \partial u_p}(\mathcal{A}) = 0 \quad \forall k \in \mathbb{Z}_{\geq 0}. \]

**Proof.** Suppose that (95) does not hold. Let $k_0$ be the maximal integer such that \( \frac{\partial^3}{\partial u_k \partial u_p \partial u_p}(\mathcal{A}) \neq 0 \).

Equation (80) for $s = k_0 + 1$ says

\[ \frac{\partial}{\partial u_{k_0+1}}(\mathcal{A}) \bigg|_{u_k = a_k, \ k \geq k_0+1} = 0. \]

Differentiating (85) with respect to $u_{k_0+1}$, we obtain

\[ \frac{\partial^3}{\partial u_{k_0} \partial u_p \partial u_p}(\mathcal{A}) + \left[ \frac{\partial}{\partial u_{k_0+1}}(\mathcal{A}), \frac{\partial^2}{\partial u_p \partial u_p}(\mathcal{A}) \right] = 0. \]

Substituting \( u_k = a_k \) in (97) for all \( k \geq k_0 + 1 \) and using (96), one gets \( \frac{\partial^3}{\partial u_{k_0} \partial u_p \partial u_p}(\mathcal{A}) = 0 \), which contradicts to our assumption. □

Using equation (80) for $s = p$ and equation (95) for all $k \in \mathbb{Z}_{\geq 0}$, we see that $\mathcal{A}$ is of the form

\[ \mathcal{A} = (u_p - a_p)^2 \mathcal{A}_2(x, t) + \mathcal{A}_0(x, t, u_0, \ldots, u_{p-1}), \]

where $\mathcal{A}_2(x, t)$ is a power series in the variables $x - x_0$, $t - t_0$ and $\mathcal{A}_0(x, t, u_0, \ldots, u_{p-1})$ is a power series in the variables $x - x_0$, $t - t_0$, $u_0 - a_0$, $\ldots$, $u_{p-1} - a_{p-1}$.

From (95), (98) it follows that equation (85) reads

\[ 2 \frac{\partial}{\partial x}(\mathcal{A}_2) + 2[\mathcal{A}_0, \mathcal{A}_2] = 0. \]

Note that condition (77) implies

\[ \mathcal{A}_0 \bigg|_{u_k = a_k, \ k \geq 0} = 0. \]

Substituting $u_k = a_k$ in (99) for all $k \geq 0$ and using (100), we get

\[ \frac{\partial}{\partial x}(\mathcal{A}_2) = 0. \]

Combining (101) with (99), one obtains

\[ [\mathcal{A}_2, \mathcal{A}_0] = 0. \]

In view of (74), (98), we have

\[ \mathcal{A}_0 = \sum_{l_0, l_1, \ldots, i_1, \ldots, i_{p-1} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \ldots (u_{p-1} - a_{p-1})^{i_{p-1}} \cdot \mathcal{A}^{l_1, l_2}_{i_0, \ldots, i_{p-1}}. \]

According to (74), (98), (101), one has

\[ \mathcal{A}_2 = \sum_{l \geq 0} (t - t_0)^l \cdot \mathcal{A}^l, \quad \mathcal{A}^l = \mathcal{A}^{0,l}_{0,0,2} \in \mathbb{R}^p(\mathcal{E}, a). \]

Combining (98), (103), (104) with Theorem 8 we see that the elements

\[ \mathcal{A}^0, \quad \mathcal{A}^{l_1, 0}_{i_0, \ldots, i_{p-1}, 0}, \quad l_1, i_0, \ldots, i_{p-1} \in \mathbb{Z}_{\geq 0}, \]
generate the algebra $\mathbb{F}^p(\mathcal{E}, a)$. Substituting $t = t_0$ in (102) and using (103), (104), one gets

$$
[\tilde{A}^0, \tilde{A}^0_{l_1,0,\ldots,i_{p-1},0}] = 0 \quad \forall l_1, i_0, \ldots, i_{p-1} \in \mathbb{Z}_{\geq 0}.
$$

Since the elements (105) generate the algebra $\mathbb{F}^p(\mathcal{E}, a)$, equation (106) yields

$$
[\tilde{A}^0, \mathbb{F}^p(\mathcal{E}, a)] = 0.
$$

**Lemma 6.** One has

$$
[\tilde{A}^l, \mathbb{F}^p(\mathcal{E}, a)] = 0 \quad \forall l \in \mathbb{Z}_{\geq 0}.
$$

**Proof.** We prove (108) by induction on $l$. The property $[\tilde{A}^0, \mathbb{F}^p(\mathcal{E}, a)] = 0$ has been obtained in (107).

Let $r \in \mathbb{Z}_{\geq 0}$ be such that $[\tilde{A}^l, \mathbb{F}^p(\mathcal{E}, a)] = 0$ for all $l \leq r$. Since $\frac{\partial^l}{\partial t^l}(\tilde{A}_2) \bigg|_{t=t_0} = l! \cdot \tilde{A}^l$, we get

$$
\left[ \frac{\partial^l}{\partial t^l}(\tilde{A}_2) \bigg|_{t=t_0}, \frac{\partial^m}{\partial t^m}(\tilde{A}_0) \bigg|_{t=t_0} \right] = 0 \quad \forall l \leq r, \quad \forall m \in \mathbb{Z}_{\geq 0}.
$$

Applying the operator $\frac{\partial^{r+1}}{\partial t^{r+1}}$ to equation (102), substituting $t = t_0$, and using (109), one obtains

$$
0 = \frac{\partial^{r+1}}{\partial t^{r+1}}([\tilde{A}^0_2, \tilde{A}^0_0]) \bigg|_{t=t_0} = \sum_{k=0}^{r+1} \binom{r+1}{k} \cdot \left[ \frac{\partial^k}{\partial t^k}(\tilde{A}^0_2) \bigg|_{t=t_0}, \frac{\partial^{r+1-k}}{\partial t^{r+1-k}}(\tilde{A}^0_0) \bigg|_{t=t_0} \right] =
$$

$$
= \left[ \frac{\partial^{r+1}}{\partial t^{r+1}}(\tilde{A}^0_2) \bigg|_{t=t_0}, \tilde{A}^0_0 \bigg|_{t=t_0} \right] =
$$

$$
= [(r+1)! \cdot \tilde{A}^{r+1}, \sum_{l_1,i_0,\ldots,i_{p-1}} (x-x_0)^{l_1}(u_0-a_0)^{i_0} \ldots (u_{p-1}-a_{p-1})^{i_{p-1}} \cdot \tilde{A}^0_{l_1,i_0,\ldots,i_{p-1},0}],
$$

which implies

$$
[\tilde{A}^{r+1}, \tilde{A}^{l_1,0}_{i_0,\ldots,i_{p-1},0}] = 0 \quad \forall l_1, i_0, \ldots, i_{p-1} \in \mathbb{Z}_{\geq 0}.
$$

Equation (107) yields

$$
[\tilde{A}^0, \tilde{A}^{r+1}] = 0.
$$

Since the elements (105) generate the algebra $\mathbb{F}^p(\mathcal{E}, a)$, from (110), (111) it follows that $[\tilde{A}^{r+1}, \mathbb{F}^p(\mathcal{E}, a)] = 0$. $\square$

**Theorem 9.** Let $\mathcal{E}$ be the infinite prolongation of an equation of the form

$$
u_t = u_{2q+1} + f(x, t, u_0, u_1, \ldots, u_{2q-1}), \quad q \in \{1, 2, 3\}.
$$

Let $a \in \mathcal{E}$. For each $p \in \mathbb{Z}_{>0}$, consider the homomorphism $\varphi_p: \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a)$ from (57).

If $p \geq q + \delta_{q,3}$ then

$$
[v_1, v_2] = 0 \quad \forall v_1, v_2 \in \ker \varphi_p, \quad \forall v_2 \in \mathbb{F}^p(\mathcal{E}, a).
$$

In other words, if $p \geq q + \delta_{q,3}$ then the kernel of $\varphi_p$ is contained in the center of the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$.

For each $k \in \mathbb{Z}_{>0}$, let $\psi_k: \mathbb{F}^{k+q-1+\delta_{q,3}}(\mathcal{E}, a) \to \mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)$ be the composition of the homomorphisms

$$
\mathbb{F}^{k+q-1+\delta_{q,3}}(\mathcal{E}, a) \to \mathbb{F}^{k+q-2+\delta_{q,3}}(\mathcal{E}, a) \to \cdots \to \mathbb{F}^{q+\delta_{q,3}}(\mathcal{E}, a) \to \mathbb{F}^{q-1+\delta_{q,3}}(\mathcal{E}, a)
$$

from (57). Then

$$
[h_1, [h_2, \ldots, [h_{k-1}, [h_k, h_{k+1}]] \ldots]] = 0 \quad \forall h_1, \ldots, h_{k+1} \in \ker \psi_k.
$$

In particular, the kernel of $\psi_k$ is nilpotent.
Proof. Let $p \geq q + \delta_{q,3}$.

Combining formulas (108), (112), (114) with the definition of $\varphi_p : \mathbb{F}^p(E, a) \to \mathbb{F}^{p-1}(E, a)$, we see that ker $\varphi_p$ is generated by the elements $\hat{A}^l, l \in \mathbb{Z}_{\geq 0}$. Then (112) follows from (108).

So we have proved that the kernel of the homomorphism $\varphi_p : \mathbb{F}^p(E, a) \to \mathbb{F}^{p-1}(E, a)$ is contained in the center of the Lie algebra $\mathbb{F}^p(E, a)$ for any $p \geq q + \delta_{q,3}$.

Let us prove (113) by induction on $k$. Since $\psi_1 = \varphi_{q+\delta_{q,3}}$, for $k = 1$ property (113) follows from (112).

Let $r \in \mathbb{Z}_{>0}$ be such that (113) is valid for $k = r$. Then for any $h'_1, h'_2, \ldots, h'_{r+2} \in$ ker $\psi_{r+1}$ we have

\[ \varphi_{r+q+\delta_{q,3}}(h'_1), \varphi_{r+q+\delta_{q,3}}(h'_2), \ldots, \varphi_{r+q+\delta_{q,3}}(h'_{r+1}), \varphi_{r+q+\delta_{q,3}}(h'_{r+2}) \subseteq \ker \psi_r. \]

Equation (114) says that $\varphi_{r+q+\delta_{q,3}}(h'_i) \in$ ker $\psi_r$ for $i = 2, 3, \ldots, r + 2$. Equation (114) says that

\[ [h'_2, h'_3, \ldots, [h'_{r+1}, h'_{r+2}]] \subseteq \ker \varphi_{r+q+\delta_{q,3}}. \]

Since ker $\varphi_{r+q+\delta_{q,3}}$ is contained in the center of $\mathbb{F}^{r+q+\delta_{q,3}}(E, a)$, property (113) yields

\[ [h'_1, h'_2, h'_3, \ldots, [h'_{r+1}, h'_{r+2}]] = 0. \]

So we have proved (113) for $k = r + 1$. Clearly, property (113) implies that ker $\psi_k$ is nilpotent. \qed

Theorem 10. Let $E$ be the infinite prolongation of the equation

\[ u_t = u_5 + f(x, t, u_0, u_1, u_2, u_3) \]

for some function $f = f(x, t, u_0, u_1, u_2, u_3)$ such that $\frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3} \neq 0$. More precisely, we assume that the function $\frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3}$ is not identically zero on any connected component of the manifold $E$.

Then $\mathbb{F}^1(E, a) = 0$ and $\mathbb{F}^p(E, a)$ is nilpotent for all $a \in E$, $p > 1$.

Proof. Consider an arbitrary point $a \in E$ given by (73). According to Remark 18, the algebra $\mathbb{F}^1(E, a)$ for equation (116) can be described as follows. Consider formal power series

\[ A = \sum_{l_1, l_2, j_0, i_1 \geq 0} (x - x_0)^{l_1}(t - t_0)^{i_1}(u_0 - a_0)^{j_0}(u_1 - a_1)^{i_1} \cdot A_{l_1, i_1, j_0}, \]

\[ B = \sum_{l_1, l_2, j_0, \ldots, j_5 \geq 0} (x - x_0)^{l_1}(t - t_0)^{l_2}(u_0 - a_0)^{j_0} \ldots (u_5 - a_5)^{j_5} \cdot B_{l_1, l_2, j_0 \ldots j_5}, \]

satisfying

\[ A_{l_1, i_1, j_0} = A_{0, 0, 0} = B_{0, 0, 0} = 0, \quad l_1, l_2, i_0 \in \mathbb{Z}_{\geq 0}. \]

Then $A_{l_1, i_1, j_0}$, $B_{l_1, i_1, j_0}$ are generators of the algebra $\mathbb{F}^1(E, a)$, and the equation

\[ D_x(B) - D_t(A) + [A, B] = 0 \]

provides relations for these generators (in addition to relations (119)).

Similarly to (88), from (120) we obtain that $B$ is of the form

\[ B = u_5 \frac{\partial}{\partial u_1}(A) - u_2 u_3 \frac{\partial^2}{\partial u_1 \partial u_1}(A) + u_4 B_{0, 0}(x, t, u_0, u_1) + B_{0, 0}(x, t, u_0, u_1, u_2, u_3). \]

where $B_{0, 0}(x, t, u_0, u_1)$ is a power series in the variables $x - x_0$, $t - t_0$, $u_0 - a_0$, $u_1 - a_1$ and $B_{0, 0}(x, t, u_0, u_1, u_2, u_3)$ is a power series in the variables $x - x_0$, $t - t_0$, $u_0 - a_0$, $u_1 - a_1$, $u_2 - a_2$, $u_3 - a_3$.

Differentiating (120) with respect to $u_4, u_3$ and using (121), we get

\[ \frac{\partial^2}{\partial u_3 \partial u_3} B_{0, 0} = \frac{\partial^2}{\partial u_1 \partial u_1}(A) + \frac{\partial^2 f}{\partial u_3 \partial u_3} \cdot \frac{\partial}{\partial u_1}(A). \]

\[ 4 \text{ Usually, the manifold } E \text{ is connected, then our assumption means that } \frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3} \text{ is not identically zero on } E. \]
Using (121) and (122), one can verify that

\[(123) \quad \left(\frac{\partial^3}{\partial u_3 \partial u_1 \partial u_5} - \frac{\partial^3}{\partial u_3 \partial u_2 \partial u_4} + \frac{1}{2} \frac{\partial^3}{\partial u_3 \partial u_3 \partial u_3}\right)\left(D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}]\right) = \frac{1}{2} \frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3} \left(D_x\left(\frac{\partial}{\partial u_1}(\mathbb{A})\right) + [\mathbb{A}, \frac{\partial}{\partial u_1}(\mathbb{A})] - \frac{\partial}{\partial u_0}(\mathbb{A})\right).
\]

Since \(D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0\) by (120), equation (123) implies

\[(124) \quad \frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3} \left(D_x\left(\frac{\partial}{\partial u_1}(\mathbb{A})\right) + [\mathbb{A}, \frac{\partial}{\partial u_1}(\mathbb{A})] - \frac{\partial}{\partial u_0}(\mathbb{A})\right) = 0.
\]

Since the analytic function \(\frac{\partial^3 f}{\partial u_3 \partial u_3 \partial u_3}\) is not identically zero on any connected component of the manifold \(E\), equation (124) yields

\[(125) \quad D_x\left(\frac{\partial}{\partial u_1}(\mathbb{A})\right) + [\mathbb{A}, \frac{\partial}{\partial u_1}(\mathbb{A})] - \frac{\partial}{\partial u_0}(\mathbb{A}) = 0.
\]

Differentiating (125) with respect to \(u_2\), we obtain \(\frac{\partial^2}{\partial u_1 \partial u_1}(\mathbb{A}) = 0\).

Recall that \(\mathbb{A}\) is of the form (117). Since \(A_{l_0,l_1} = 0\) for all \(l_1, l_2, i_0 \in \mathbb{Z}_{\geq 0}\) by (119), equation \(\frac{\partial^2}{\partial u_1 \partial u_1}(\mathbb{A}) = 0\) yields \(\frac{\partial}{\partial u_1}(\mathbb{A}) = 0\). Combining \(\frac{\partial}{\partial u_1}(\mathbb{A}) = 0\) with (125), one gets \(\frac{\partial}{\partial u_0}(\mathbb{A}) = 0\).

Combining the equations \(\frac{\partial}{\partial u_1}(\mathbb{A}) = 0\) yields (119), we get \(A_{l_0,l_1} = 0\) for all \(l_1, l_2, i_0, i_1 \in \mathbb{Z}_{\geq 0}\).

Since, by Theorem 8, the algebra \(\mathbb{F}^1(\mathcal{E}, a)\) is generated by the elements \(A_{l_0,l_1}\) for \(l_1, i_0, i_1 \in \mathbb{Z}_{\geq 0}\), we obtain \(\mathbb{F}^1(\mathcal{E}, a) = 0\). According to Theorem 9 for \(q = 2\), for any \(k \in \mathbb{Z}_{\geq 0}\) the kernel of the homomorphism \(\psi_k: \mathbb{F}^{k+1}(\mathcal{E}, a) \rightarrow \mathbb{F}^1(\mathcal{E}, a)\) is nilpotent. Since \(\mathbb{F}^1(\mathcal{E}, a) = 0\), this implies that \(\mathbb{F}^p(\mathcal{E}, a)\) is nilpotent for all \(p > 1\).

5. Relations between \(\mathbb{F}^0(\mathcal{E}, a)\) and the Wahlquist-Estabrook prolongation algebra

Consider a scalar evolution equation of the form

\[(126) \quad u_t = F(u_0, u_1, \ldots, u_d), \quad u = u(x, t), \quad u_k = \frac{\partial^k u}{\partial x^k}, \quad u_0 = u.
\]

Note that the function \(F\) in (126) does not depend on \(x, t\).

Let \(E\) be the infinite prolongation of (126). Let

\[(127) \quad a = (x = x_0, t = t_0, u_k = a_k) \in E, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0},
\]

be a point of \(E\). The Wahlquist-Estabrook prolongation algebra of equation (126) at the point (127) can be defined in terms of generators and relations as follows. Consider formal power series

\[(128) \quad \mathcal{A} = \sum_{i \geq 0} (u_0 - a_0)^i \cdot \mathcal{A}_i, \quad \mathcal{B} = \sum_{j_0, \ldots, j_d-1 \geq 0} (u_0 - a_0)^{j_0} \cdot (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot B_{j_0 \ldots j_{d-1}},
\]

where

\[(129) \quad \mathcal{A}_i, \quad B_{j_0 \ldots j_{d-1}}, \quad i, j_0, \ldots, j_{d-1} \in \mathbb{Z}_{\geq 0},
\]

are elements of a Lie algebra, which we do not specify yet. The equation

\[(130) \quad D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}] = 0
\]

is equivalent to some Lie algebraic relations for (129). The Wahlquist-Estabrook prolongation algebra (WE algebra for short) is given by the generators (129) and these relations. Denote this algebra by \(\mathcal{W}\).

We are going to show that the algebra \(\mathbb{F}^0(\mathcal{E}, a)\) for equation (126) is isomorphic to some subalgebra of \(\mathcal{W}\).
According to Remark 13 the algebra $\mathbb{P}^0(\mathcal{E}, a)$ is generated by $A_{i_1, i_2}$, $\mathbb{B}_{j_0, \ldots, j_{d-1}}^{l_1, l_2}$. According to (54), one has $A_0^{l_1, l_2} = \mathbb{B}_{0, \ldots, 0}^{l_1, l_2} = 0$ for all $l_1, l_2$.

Since equation (126) is invariant with respect to the change of variables $x \mapsto x - x_0$, $t \mapsto t - t_0$, we can assume $x_0 = t_0 = 0$ in (127). Since $A_0^{l_1, l_2} = \mathbb{B}_{0, \ldots, 0}^{l_1, l_2} = 0$ and $x_0 = t_0 = 0$, in the case $p = 0$ the power series (51), (52), (53) are written as

\begin{equation}
A = \sum_{l_1, l_2 \geq 0, \ i > 0} x^1 t^2 (u_0 - a_0)^i \cdot A_1^{l_1, l_2},
\end{equation}

\begin{equation}
B = \sum_{l_1, l_2, j_0, \ldots, j_{d-1} \geq 0} x^1 t^2 (u_0 - a_0)^{j_0} \cdots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot B_{j_0, \ldots, j_{d-1}}^{l_1, l_2},
\end{equation}

\begin{equation}
\mathbb{E}_{0, \ldots, 0}^{l_1, l_2} = 0.
\end{equation}

D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0, \quad A_1^{l_1, l_2}, \mathbb{B}_{j_0, \ldots, j_{d-1}}^{l_1, l_2} \in \mathbb{P}^0(\mathcal{E}, a).

The next lemma follows from the definition of $\mathbb{P}^0(\mathcal{E}, a)$.

**Lemma 7.** Let $\mathfrak{L}$ be a Lie algebra. Consider formal power series of the form

\begin{equation}
P = \sum_{l_1, l_2 \geq 0, \ i > 0} x^1 t^2 (u_0 - a_0)^i \cdot P_{l_1, l_2}^i, \quad P_{l_1, l_2}^i \in \mathfrak{L},
\end{equation}

\begin{equation}
Q = \sum_{l_1, l_2, j_0, \ldots, j_{d-1} \geq 0} x^1 t^2 (u_0 - a_0)^{j_0} \cdots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot Q_{j_0, \ldots, j_{d-1}}^{l_1, l_2}, \quad Q_{j_0, \ldots, j_{d-1}}^{l_1, l_2} \in \mathfrak{L}, \quad Q_{0, \ldots, 0}^{l_1, l_2} = 0.
\end{equation}

If $D_x(Q) - D_t(P) + [P, Q] = 0$, then the map $A_i^{l_1, l_2} \mapsto P_{l_1, l_2}^i$, $B_{j_0, \ldots, j_{d-1}}^{l_1, l_2} \mapsto Q_{j_0, \ldots, j_{d-1}}^{l_1, l_2}$ determines a homomorphism from $\mathbb{P}^0(\mathcal{E}, a)$ to $\mathfrak{L}$.

Let $\mathfrak{L}$ be a Lie algebra. A formal ZCR of Wahlquist-Estbrook type with coefficients in $\mathfrak{L}$ is given by formal power series

\begin{equation}
M = \sum_{i \geq 0} (u_0 - a_0)^i \cdot M_i, \quad N = \sum_{j_0, \ldots, j_{d-1} \geq 0} (u_0 - a_0)^{j_0} \cdots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot N_{j_0, \ldots, j_{d-1}},
\end{equation}

satisfying

\begin{equation}
D_x(N) - D_t(M) + [M, N] = 0.
\end{equation}

The nextlemma follows from the definition of the WE algebra $\mathfrak{M}$.

**Lemma 8.** Any formal ZCR of Wahlquist-Estbrook type (134), (135) with coefficients in $\mathfrak{L}$ determines a homomorphism $\mathfrak{M} \rightarrow \mathfrak{L}$ given by $A_i \mapsto M_i$, $B_{j_0, \ldots, j_{d-1}} \mapsto N_{j_0, \ldots, j_{d-1}}$.

**Remark 19.** For any Lie algebra $\mathfrak{L}$, there is a (possibly infinite-dimensional) vector space $V$ such that $\mathfrak{L}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(V)$. Here $\mathfrak{gl}(V)$ is the algebra of linear maps $V \rightarrow V$.

For example, one can use the following construction. Denote by $U(\mathfrak{L})$ the universal enveloping algebra of $\mathfrak{L}$. We have the injective homomorphism of Lie algebras

\[ \xi: \mathfrak{L} \hookrightarrow \mathfrak{gl}(U(\mathfrak{L})), \quad \xi(v)(w) = vw, \quad v \in \mathfrak{L}, \quad w \in U(\mathfrak{L}). \]

So one can set $V = U(\mathfrak{L})$.

Denote by $\mathbf{F}$ the vector space of formal power series in variables $z_1, z_2$ with coefficients in $\mathbb{P}^0(\mathcal{E}, a)$. That is, an element of $\mathbf{F}$ is a power series of the form

\[ \sum_{l_1, l_2 \in \mathbb{Z}_{\geq 0}} z_1^{l_1} z_2^{l_2} C_{l_1, l_2}, \quad C_{l_1, l_2} \in \mathbb{P}^0(\mathcal{E}, a). \]

The space $\mathbf{F}$ has the Lie algebra structure given by

\[ \left[ \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} C_{l_1, l_2}, \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \tilde{C}_{l_1, l_2} \right] = \sum_{l_1, l_2, l_1, l_2} z_1^{l_1+l_1} z_2^{l_2+l_2} \left[ C_{l_1, l_2}, \tilde{C}_{l_1, l_2} \right], \quad C_{l_1, l_2}, \tilde{C}_{l_1, l_2} \in \mathbb{P}^0(\mathcal{E}, a). \]
We have also the following homomorphism of Lie algebras

\begin{equation}
\nu: F \to \mathbb{P}^0(E, a), \quad \nu \left( \sum_{l_1, l_2 \in \mathbb{Z}_{>0}} z_1^{l_1} z_2^{l_2} C^{l_1 l_2} \right) = C^{00}.
\end{equation}

For \( i = 1, 2 \), let \( \partial_{z_i}: F \to F \) be the linear map given by \( \partial_{z_i} \left( \sum z_1^{l_1} z_2^{l_2} C^{l_1 l_2} \right) = \sum \partial_{z_i} \left( z_1^{l_1} z_2^{l_2} \right) C^{l_1 l_2} \).

Let \( D \) be the linear span of \( \partial_{z_1}, \partial_{z_2} \) in the vector space of linear maps \( F \to F \). Since the maps \( \partial_{z_1}, \partial_{z_2} \) commute, the space \( D \) is a 2-dimensional abelian Lie algebra with respect to the commutator of maps.

Denote by \( \mathbb{L} \) the vector space \( D \oplus F \) with the following Lie algebra structure

\[ [X_1 + f_1, X_2 + f_2] = X_1(f_2) - X_2(f_1) + [f_1, f_2], \quad X_1, X_2 \in D, \quad f_1, f_2 \in F. \]

An element of \( \mathbb{L} \) can be written as a sum of the following form

\[ (y_1 \partial_{z_1} + y_2 \partial_{z_2}) + \sum z_1^{l_1} z_2^{l_2} C^{l_1 l_2}, \quad y_1, y_2 \in \mathbb{R}, \quad C^{l_1 l_2} \in \mathbb{P}^0(E, a). \]

**Theorem 11.** Let \( \mathcal{R} \subset \mathcal{W} \) be the subalgebra generated by the elements

\begin{equation}
(\text{ad} A_0)^k(A_i), \quad k \in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{>0}.
\end{equation}

Then the map \( (\text{ad} A_0)^k(A_i) \to k! \cdot A_i^{\mathcal{W}} \) determines an isomorphism between \( \mathcal{R} \) and \( \mathbb{P}^0(E, a) \).

**Proof.** We have

\begin{equation}
\sum_{l_1, l_2, j_0, \ldots, j_{d-1}} \partial_{t_1} \left( x_1^{l_1} t_2^{l_2} \right) \left( u_0 - a_0 \right)^{j_0} \cdots \left( u_{d-1} - a_{d-1} \right)^{j_{d-1}} \cdot B_{j_0, \ldots, j_{d-1}}^{l_1, l_2} + \\
\sum_{l_1, l_2, j_0, \ldots, j_{d-1}} x_1^{l_1} t_2^{l_2} D_t \left( \left( u_0 - a_0 \right)^{j_0} \cdots \left( u_{d-1} - a_{d-1} \right)^{j_{d-1}} \cdot B_{j_0, \ldots, j_{d-1}}^{l_1, l_2} \right) - \\
\sum_{l_1, l_2, i} \partial_{t_1} \left( x_1^{l_1} t_2^{l_2} \right) (u_0 - a_0)^i \cdot A_i^{l_1, l_2} - \sum_{l_1, l_2} x_1^{l_1} t_2^{l_2} D_t \left( (u_0 - a_0)^i \right) \cdot A_i^{l_1, l_2} + [A_i, B] = 0.
\end{equation}

We regard the expressions

\begin{equation}
\tilde{A} = \partial_{z_1} + \sum_{i>0} (u_0 - a_0)^i \cdot \left( \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} A_i^{l_1, l_2} \right),
\end{equation}

\begin{equation}
\tilde{B} = \left( \partial_{z_2} + \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} B_{0,0}^{l_1, l_2} \right) + \sum_{j_0, \ldots, j_{d-1} \geq 0, j_0 + \cdots + j_{d-1} > 0} \left( u_0 - a_0 \right)^{j_0} \cdots \left( u_{d-1} - a_{d-1} \right)^{j_{d-1}} \cdot \left( \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} B_{j_0, \ldots, j_{d-1}}^{l_1, l_2} \right)
\end{equation}

as formal power series with coefficients in \( \mathbb{L} \).

Since the function \( F \) in (126) does not depend on \( x \) and \( t \), equation (138) is equivalent to

\[ D_x (\tilde{B}) - D_t (\tilde{A}) + [\tilde{A}, \tilde{B}] = 0, \]

which implies that the power series (139), (140) constitute a formal ZCR of Wahlquist-Estabrook type with coefficients in \( \mathbb{L} \).

Applying Lemma 8 to this formal ZCR, we obtain the homomorphism

\begin{equation}
\varphi: \mathcal{W} \to \mathbb{L}, \quad \varphi(A_0) = \partial_{z_1}, \quad \varphi(A_i) = \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} A_i^{l_1, l_2}, \quad i > 0,
\end{equation}

\begin{equation}
\varphi(B_{0,0}) = \left( \partial_{z_2} + \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} B_{0,0}^{l_1, l_2} \right), \quad \varphi(B_{j_0, \ldots, j_{d-1}}) = \left( \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} B_{j_0, \ldots, j_{d-1}}^{l_1, l_2} \right), \quad j_0 + \cdots + j_{d-1} > 0.
\end{equation}
Clearly, $\mathbf{F}$ is a Lie subalgebra of $\mathbf{L} = \mathbf{D} \oplus \mathbf{F}$. In view of (141), for any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{> 0}$ one has

$$\varphi \left( (\text{ad} \mathcal{A}_0)^k (\mathcal{A}_i) \right) = (\text{ad} \partial_z)_k \left( \sum_{l_1, l_2} \partial^1_{12} \mathcal{A}_{i, l_1, l_2} \right) \in \mathbf{F}. \tag{142}$$

Since $\mathcal{R} \subset \mathfrak{W}$ is generated by the elements (137), property (142) implies $\varphi(\mathcal{R}) \subset \mathbf{F} \subset \mathbf{L}$. Using the homomorphism (136) and property (142), we get

$$\nu \circ \varphi \mid \mathcal{R} : \mathcal{R} \to \mathbb{F}^0(\mathcal{E}, a), \quad (\nu \circ \varphi) \left( (\text{ad} \mathcal{A}_0)^k (\mathcal{A}_i) \right) = k! \cdot \partial_z^{k, 0}, \quad k \in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{> 0}. \tag{143}$$

Using Remark 19, we can assume that $\mathfrak{W}$ is embedded in the algebra $\mathfrak{gl}(\mathcal{V})$ for some vector space $\mathcal{V}$. Let $\mathcal{S}$ be the vector space of power series of the form

$$\sum_{l_1, l_2, i_0, \ldots, i_k \geq 0} x^{l_1} t^{l_2} (u_0 - a_0)^{i_0} \ldots (u_k - a_k)^{i_k} \cdot C_{i_0 \ldots i_k}^{l_1, l_2}, \quad C_{i_0 \ldots i_k}^{l_1, l_2} \in \mathfrak{gl}(\mathcal{V}), \quad k \in \mathbb{Z}_{\geq 0}. \tag{144}$$

Note that $\mathcal{S}$ contains power series (141) for all $k \in \mathbb{Z}_{\geq 0}$. For each $C \in \mathcal{S}$, the power series $D_x(C), D_t(C) \in \mathcal{S}$ are defined according to Remark 16.

Recall that $\mathfrak{gl}(\mathcal{V})$ consists of linear maps $V \rightarrow V$. Since $\mathfrak{gl}(\mathcal{V})$ is an associative algebra with respect to the composition of maps, the space $\mathcal{S}$ is an associative algebra with respect to the standard multiplication of formal power series.

Also, using Remark 16 and the Lie bracket on $\mathfrak{gl}(\mathcal{V})$, we obtain a Lie bracket on the space $\mathcal{S}$. Set $\mathcal{B}_0 = \mathcal{B}_{0 \ldots 0}$, where $\mathcal{B}_{0 \ldots 0}$ is the free term of the power series $\mathcal{B}$ from (128). Since $\mathcal{A}_i, \mathcal{B}_{j_0 \ldots j_{d-1}} \in \mathfrak{W}$ for all $i, j_0, \ldots, j_{d-1} \in \mathbb{Z}_{\geq 0}$, the power series $e^{x \mathcal{A}_0}, e^{t \mathcal{B}_0}$, and (128) belong to $\mathcal{S}$. Set

$$P = -e^{t \mathcal{B}_0} A_0 e^{-t \mathcal{B}_0} + e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} A_0 e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0}, \quad Q = -\mathcal{B}_0 + e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} B_0 e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0}. \tag{145}$$

Using (145), we get

$$D_x(Q) = e^{t \mathcal{B}_0} \left[ A_0, e^{x \mathcal{A}_0} B_0 e^{-x \mathcal{A}_0} \right] e^{-t \mathcal{B}_0} + e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} D_x(B_0) e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0}, \tag{146}$$

$$D_t(P) = -\left[ B_0, e^{t \mathcal{B}_0} A_0 e^{-t \mathcal{B}_0} \right] + \left[ B_0, e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} A_0 e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0} \right] + e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} D_t(A_0) e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0}. \tag{147}$$

Recall that $D_x(\mathcal{B}) - D_t(A) + [\mathcal{A}, \mathcal{B}] = 0$ according to (130). Combining this with (145), (146), (147), one obtains

$$D_x(Q) - D_t(P) + [P, Q] = e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} (D_x(B_0) - D_t(A_0) + [\mathcal{A}, \mathcal{B}]) e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0} = 0. \tag{148}$$

Formulas (128), (145) yield

$$P = -e^{t \mathcal{B}_0} A_0 e^{-t \mathcal{B}_0} + \sum_{i \geq 0} (u_0 - a_0)^i \cdot e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} A_0 e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0} = \sum_{l_1, l_2 \geq 0, \nu > 0} x^{l_1} t^{l_2} (u_0 - a_0)^i \frac{1}{l_1! l_2!} \left( \text{ad} \mathcal{B}_0 \right)^{l_2} \left( (\text{ad} \mathcal{A}_0)^i \mathcal{A}_i \right). \tag{149}$$

$$Q = -\mathcal{B}_0 + \sum_{j_0, \ldots, j_{d-1} \geq 0} (u_0 - a_0)^{j_0} \ldots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot e^{t \mathcal{B}_0} e^{x \mathcal{A}_0} \mathcal{B}_{j_0 \ldots j_{d-1}} e^{-x \mathcal{A}_0} e^{-t \mathcal{B}_0} = \sum_{l_1, l_2, j_0, \ldots, j_{d-1} \geq 0} x^{l_1} t^{l_2} (u_0 - a_0)^{j_0} \ldots (u_{d-1} - a_{d-1})^{j_{d-1}} \frac{1}{l_1! l_2!} \left( \text{ad} \mathcal{B}_0 \right)^{l_2} \left( (\text{ad} \mathcal{A}_0)^{i_1} (\mathcal{B}_{j_0 \ldots j_{d-1}}) \right). \tag{150}$$

From (148), (149), (150) it follows that the power series $P, Q$ satisfy all conditions of Lemma 7. Applying Lemma 1 to $P, Q$ given by (149), (150), we obtain the homomorphism

$$\psi : \mathbb{F}^0(\mathcal{E}, a) \to \mathfrak{W}, \quad \psi \left( \mathcal{A}_i^{l_1, l_2} \right) = \frac{1}{l_1! l_2!} \left( \text{ad} \mathcal{B}_0 \right)^{l_2} \left( (\text{ad} \mathcal{A}_0)^{i_1} (\mathcal{A}_i) \right), \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{> 0},$$

$$\psi \left( \mathcal{B}_{j_0 \ldots j_{d-1}}^{l_1, l_2} \right) = \frac{1}{l_1! l_2!} \left( \text{ad} \mathcal{B}_0 \right)^{l_2} \left( (\text{ad} \mathcal{A}_0)^{i_1} (\mathcal{B}_{j_0 \ldots j_{d-1}}) \right), \quad l_1, l_2, j_0, \ldots, j_{d-1} \in \mathbb{Z}_{\geq 0}, \quad j_0 + \cdots + j_{d-1} > 0,$$
ψ(\((p_{0...0}^{l_1,l_2})\)) = \frac{1}{l_1!l_2!}(\text{ad } A_0)^{l_1}(\text{ad } B_0)^{l_2}((\text{ad } A_0)^{l_1}(\text{ad } B_0)^0), \quad l_1 \in \mathbb{Z}_{>0}, \quad l_2 \in \mathbb{Z}_{\geq0}.

From (151) we get

\[(152) \quad \psi(A_{i}^{l_1,0}) = \frac{1}{l_1!}(\text{ad } A_0)^{l_1}(A_i) \in \mathcal{R}, \quad l_1 \in \mathbb{Z}_{\geq0}, \quad i \in \mathbb{Z}_{>0}.\]

Since, by Theorem 8, the elements \(A_{i}^{l_1,0}, l_1 \in \mathbb{Z}_{>0}, i \in \mathbb{Z}_{>0},\) generate the algebra \(F^0(\mathcal{E}, a),\) property (152) implies \(\psi(F^0(\mathcal{E}, a)) \subset \mathcal{R}.\) Then from (143), (152) it follows that the homomorphisms \(\psi: F^0(\mathcal{E}, a) \to \mathcal{R}\) and \(\nu \circ \varphi|_\mathcal{R}: \mathcal{R} \to F^0(\mathcal{E}, a)\) are inverse to each other. \(\square\)

6. The algebras \(F^p(\mathcal{E}, a)\) for the KdV equation

Consider the infinite-dimensional Lie algebra \(\mathfrak{sl}_2(\mathbb{K}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[\lambda],\) where \(\mathbb{K}[\lambda]\) is the algebra of polynomials in \(\lambda.\)

**Theorem 12.** Let \(\mathcal{E}\) be the infinite prolongation of the KdV equation \(u_t = u_3 + u_0u_1.\) Let \(a \in \mathcal{E}.\)

Then \(F^0(\mathcal{E}, a)\) is isomorphic to the direct sum of \(\mathfrak{sl}_2(\mathbb{K}[\lambda])\) and a 3-dimensional abelian Lie algebra.

For every \(p \in \mathbb{Z}_{>0},\) consider the homomorphism \(\varphi_p: F^p(\mathcal{E}, a) \to F^{p-1}(\mathcal{E}, a)\) from (57) and the homomorphism \(\psi_p: F^p(\mathcal{E}, a) \to F^0(\mathcal{E}, a)\) that is equal to the composition of

\(F^p(\mathcal{E}, a) \to F^{p-1}(\mathcal{E}, a) \to \cdots \to F^1(\mathcal{E}, a) \to F^0(\mathcal{E}, a).\)

Then the kernel of \(\varphi_p\) is contained in the center of the Lie algebra \(F^p(\mathcal{E}, a),\) and the kernel of \(\psi_p\) is nilpotent. Moreover, the algebra \(F^p(\mathcal{E}, a)\) is obtained from \(\mathfrak{sl}_2(\mathbb{K}[\lambda])\) by applying several times the operation of central extension.

**Proof.** Let \(\mathfrak{W}\) be the Wahlquist-Estabrook prolongation algebra of the KdV equation. According to [4, 5], the algebra \(\mathfrak{W}\) is isomorphic to the direct sum of \(\mathfrak{sl}_2(\mathbb{K}[\lambda])\) and a 5-dimensional nilpotent Lie algebra.

Consider the subalgebra \(\mathcal{R} \subset \mathfrak{W}\) defined in Theorem 11. According to Theorem 11, one has \(F^0(\mathcal{E}, a) \cong \mathcal{R}.\) From the description of \(\mathfrak{W}\) in [4, 5] it follows that \(\mathcal{R}\) is isomorphic to the direct sum of \(\mathfrak{sl}_2(\mathbb{K}[\lambda])\) and a 3-dimensional abelian Lie algebra.

The results about the homomorphisms \(\varphi_p: F^p(\mathcal{E}, a) \to F^{p-1}(\mathcal{E}, a)\) and \(\psi_p: F^p(\mathcal{E}, a) \to F^0(\mathcal{E}, a)\) follow from Theorem 9 in the case \(q = 1.\)

Hence for each \(p \in \mathbb{Z}_{>0}\) the algebra \(F^p(\mathcal{E}, a)\) is obtained from \(F^{p-1}(\mathcal{E}, a)\) by central extension. Since \(F^0(\mathcal{E}, a)\) is isomorphic to a central extension of \(\mathfrak{sl}_2(\mathbb{K}[\lambda]),\) we see that \(F^p(\mathcal{E}, a)\) is obtained from \(\mathfrak{sl}_2(\mathbb{K}[\lambda])\) by applying several times the operation of central extension. \(\square\)

**Acknowledgements**

The author would like to thank A.Henriques, I. S. Krasil’shchik, Yu. I. Manin, V. V. Sokolov, and A. M. Verbovetsky for useful discussions.

This work was supported by the Netherlands Organisation for Scientific Research (NWO) grants 613.000.906 and 639.031.515. The author is grateful also to the Max Planck Institute for Mathematics (Bonn, Germany) for its hospitality and support during 02.2006–01.2007 and 06.2010–09.2010, when part of this research was done.

**References**

[1] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor’kova, I. S. Krasil’shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. Symmetries and Conservation Laws for Differential Equations of Mathematical Physics. Amer. Math. Soc., Providence, RI, 1999.

[2] P. J. Caudrey, R. K. Dodd, and J. D. Gibbon. A new hierarchy of Korteweg-de Vries equations. Proc. Roy. Soc. London Ser. A 351 (1976), 407–422.

[3] R. Dodd and A. Fordy. The prolongation structures of quasipolynomial flows. Proc. Roy. Soc. London Ser. A 385 (1983), 389–429.
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[4] H. N. van Eck. The explicit form of the Lie algebra of Wahlquist and Estabrook. A presentation problem. *Nederl. Akad. Wetensch. Indag. Math.* **45** (1983), 149–164.

[5] H. N. van Eck. A non-Archimedean approach to prolongation theory. *Lett. Math. Phys.* **12** (1986), 231–239.

[6] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa. Landau-Lifshitz equation: solitons, quasiperiodic solutions and infinite-dimensional Lie algebras. *J. Phys. A* **16** (1983), 221–236.

[7] L. D. Faddeev and L. A. Takhtajan. *Hamiltonian methods in the theory of solitons.* Springer-Verlag, 2007.

[8] J. D. Finley and J. K. McIver. Prolongations to higher jets of Estabrook-Wahlquist coverings for PDEs. *Acta Appl. Math.* **32** (1993), 197–225.

[9] S. Igonin. Coverings and fundamental algebras for partial differential equations. *J. Geom. Phys.* **56** (2006), 993–998.

[10] S. Igonin. Algebras and algebraic curves associated with PDEs and Bäcklund transformations. Max Planck Institute preprint MPIM2010-120. www.mpim-bonn.mpg.de/preprints/

[11] S. Igonin. Analogues of coverings and the fundamental group for the category of partial differential equations. Preprint at www.math.unl.nl/people/igonin/preprints/

[12] S. Igonin. Higher jet prolongation Lie algebras and Bäcklund transformations for (1+1)-dimensional PDEs. Preprint at arXiv:1212.2199

[13] S. Igonin and R. Martini. Prolongation structure of the Krichever-Novikov equation. *J. Phys. A* **35** (2002), 9801–9810.

[14] D. J. Kaup. On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$. *Stud. Appl. Math.* **62** (1980), 189–216.

[15] I. S. Krasilshchik and A. M. Vinogradov. Nonlocal trends in the geometry of differential equations. *Acta Appl. Math.* **15** (1989), 161–209.

[16] I. M. Krichever and S. P. Novikov. Holomorphic bundles over algebraic curves and nonlinear equations. *Russian Math. Surveys* **55** (1980), 53–79.

[17] M. Marvan. On zero-curvature representations of partial differential equations. *Differential geometry and its applications (Opava, 1992)*, 103–122. Silesian Univ. Opava, 1993. www.emis.de/proceedings/5ICDGA

[18] M. Marvan. A direct procedure to compute zero-curvature representations. The case $sl_2$. *Secondary Calculus and Cohomological Physics (Moscow, 1997)*, 9 pp. www.emis.de/proceedings/SCCP97

[19] M. Marvan. On the spectral parameter problem. *Acta Appl. Math.* **109** (2010), 239–255.

[20] A. G. Meshkov and V. V. Sokolov. Integrable evolution equations with constant separatant. *Ufa Math. J.* **4** (2012), 104–153, arXiv:1302.6010

[21] A. V. Mikhailov, A. B. Shabat, and V. V. Sokolov. The symmetry approach to classification of integrable equations. *What is integrability?,* 115–184. Springer, 1991.

[22] D. P. Novikov. Algebraic-geometric solutions of the Krichever-Novikov equation. *Theoret. and Math. Phys.* **121** (1999), 1567–1573.

[23] P. J. Olver. *Applications of Lie groups to differential equations.* Springer-Verlag, 1993.

[24] C. Rogers and W. F. Shadwick. *Bäcklund transformations and their applications.* Academic Press, New York, 1982.

[25] G. H. M. Roelofs and R. Martini. Prolongation structure of the Landau-Lifshitz equation. *J. Math. Phys.* **34** (1993), 2394–2399.

[26] S. Yu. Sakovich. On zero-curvature representations of evolution equations. *J. Phys. A* **28** (1995), 2861–2869.

[27] S. Yu. Sakovich. Cyclic bases of zero-curvature representations: five illustrations to one concept. *Acta Appl. Math.* **83** (2004), 69–83.

[28] J. A. Sanders and J. P. Wang. Number theory and the symmetry classification of integrable systems. *Integrability*, 89–118, *Lecture Notes in Phys.* **767**. Springer, Berlin, 2009.

[29] K. Sawada and T. Kotera. A method for finding $N$-soliton solutions of the K.d.V. equation and K.d.V.-like equation. *Progr. Theoret. and Math. Phys.* **51** (1974), 1355–1367.

[30] P. Sebestyén. On normal forms of irreducible $sl_n$-valued zero-curvature representations. *Rep. Math. Phys.* **62** (2008), 57–68.

[31] E. K. Sklyanin. On complete integrability of the Landau-Lifshitz equation. *Preprint LOMI E-3-79*, Leningrad, 1979.

[32] S. I. Svinolupov, V. V. Sokolov, and R. I. Yamilov. On Bäcklund transformations for integrable evolution equations. *Soviet Math. Dokl.* **28** (1983), 165–168.

[33] H. D. Wahlquist and F. B. Estabrook. Prolongation structures of nonlinear evolution equations. *J. Math. Phys.* **16** (1975), 1–7.

[34] V. E. Zakharov and A. B. Shabat. Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. *Functional Anal. Appl.* **13** (1979), 166–174.

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