Investigating new forms of gravity-matter couplings in the gravitational field equations

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This paper proposes a toy model where, in the Einstein equations, the right-hand side is modified by the addition of a term proportional to the symmetrized partial contraction of the Ricci tensor with the energy-momentum tensor, while the left-hand side remains equal to the Einstein tensor. Bearing in mind the existence of a natural length scale given by the Planck length, dimensional analysis shows that such a term yields a correction linear in $\hbar$ to the classical term, that is instead just proportional to the energy-momentum tensor. One then obtains an effective energy-momentum tensor that consists of three contributions: pure energy part, mechanical stress and thermal part. The pure energy part has the appropriate property for dealing with the dark sector of modern relativistic cosmology. Such a theory coincides with general relativity in vacuum, and the resulting field equations are here solved for a Dunn and Tupper metric, for departures from an interior Schwarzschild solution as well as for a Friedmann-Lemaitre-Robertson-Walker universe.

I. INTRODUCTION

At the time when Einstein assumed that gravity couples to the energy-momentum tensor of matter, it was not yet known that matter fields are quantum fields in the first place, and no attempt had been made to understand the physical implications of the Planck length

$$\ell_P \equiv \sqrt{\frac{G \hbar}{c^3}}$$  \hspace{1cm} (1.1)

Einstein obtained his field equations in the well-known form \cite{1,2}

$$E_{\mu
u} \equiv R_{\mu
u} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = \frac{8\pi G}{c^4}$$  \hspace{1cm} (1.2)

whose contracted covariant differentiation leads therefore to the local relation

$$\nabla^\nu T_{\mu\nu} = 0,$$  \hspace{1cm} (1.3)

which however does not yield any integral conservation law unless the spacetime manifold $(M, g)$ admits Killing vector fields, which is not necessarily the case in a generic spacetime.

When the renaissance of general relativity and cosmology \cite{3} began in the sixties, several approaches were developed along the years in order to modify the classical field equations \cite{1,2}:

(i) Quantum field theory in curved space-time \cite{4,9}, where the classical energy-momentum tensor is replaced by the expectation value $(T_{\mu\nu})$ of its regularized and renormalized form in a quantum state (the choice of quantum state being taken not to affect the result). With the help of point-split regularization or heat-kernel methods, one can therefore obtain a number of correction terms quadratic in the curvature \cite{5,10,11}. This is certainly relevant as one approaches the quantum era, which affects the very early universe.

(ii) Full quantum gravity via functional integrals is studied \cite{12,14}, writing down the functional equations obeyed by the effective action, possibly allowing for supersymmetry and supergravity \cite{15,17}.

(iii) One resorts to string theory where, at perturbative level, spacetime is described by a set of coupling constants in a two-dimensional quantum field theory, whereas at non-perturbative level spacetime must be reconstructed from a holographic dual theory \cite{18}.

(iv) One studies instead the most general family of classical, relativistic Lagrangians for the gravitational field \cite{19,20}.

The latter has given rise to the so-called $f(R)$ theories \cite{21,24} and their many variants, where the Lagrangian is no longer linear in the trace of the Ricci tensor. This is certainly relevant for the analysis of classical phenomena such as the expansion of the universe. However, it is then difficult to develop a rigorous theory of the Cauchy problem of the same standard of rigor now available in general relativity \cite{23,20}. Moreover, it is unclear how to achieve a smooth transition towards general relativity in the solar system, where Einstein’s theory has been successfully tested \cite{27}, showing no compelling need for alternative classical theories \cite{28}. In other words, as far
as the large-scale structure of the universe is concerned, the discovery of the acceleration of the universe cannot be understood by using general relativity, and hence one resorts to alternative classical Lagrangians. But the smooth transition to general relativity, at least on solar-system scale, deserves further work, as far as we can see.

In light of all the above well known properties or open problems, we have been led to consider a modification of Eqs. (1.2) that fulfills the following requirements:

1. The Einstein-Hilbert Lagrangian is not modified.

2. The modified theory coincides with general relativity when the energy-momentum tensor source of the gravitational field vanishes. The modified field equations correct only the right-hand side of Eqs. (1.2), by means of an additional term that is a symmetrized partial contraction of the Ricci tensor with the energy-momentum tensor of matter.

Because of these goals, we assume a tensor equation reading as (we use summation over repeated indices, and the symbol $\alpha$ to denote that the index $\alpha$ among the others is not affected by symmetrization over its adjacent index)

$$E_{\mu\nu} = \kappa T_{\mu\nu} + BR_{\alpha} \cdot T_{\alpha[\nu]}.$$  
(1.4)

Note that the left-hand side is classical and results from variation of the Einstein-Hilbert action, while the right-hand side is tensorial but phenomenological, since it is affected by possible quantum laws of coupling gravity to matter fields. The coefficient $B$ should be therefore dimensionful, and in such a way that the dimension of $B$ times the dimension of Ricci equals the dimension of $\frac{\kappa}{\kappa}$, in the Planck length $l_P$. on denoting by $b$ an arbitrary real number, we write

$$B = b\kappa(l_P)^2.$$  
(1.5)

as well as

$$A = \frac{B}{\kappa} = b(l_P)^2.$$  
(1.6)

where we have introduced the related quantities $b$ (dimensionless) and $A$ (with the dimension of a length squared) for convenience and a later use. Both these quantities can be either positive or negative. Hence Eq. (1.4) can be re-expressed in the form

$$E_{\mu\nu} = \kappa \left[ T_{\mu\nu} + AR_{\alpha} \cdot T_{\alpha[\nu]} \right].$$  
(1.7)

Furthermore, at this stage, nothing can be said about the real-valued parameter $b$, but Eq. (1.7) tells us that the classical Einstein equations (1.2) can be viewed as the zeroth-order in $b$ of a richer scheme. By virtue of the arbitrariness of $b$, it appears desirable to work with finite values of $b$, without imposing the limit as $b$ approaches 0.

Obviously, the proposed modification/extension of the field equations is just one of the many conceivable modifications. In fact, along the same lines, one could have equally considered coupling terms of the type (cf. Appendix A)

$$RT g_{\mu\nu}, \quad RT_{\mu\nu}, \quad E_{\alpha} T_{\alpha[\nu]}, \ldots$$

This circumstance (i.e., the possibility to consider other choices) is not crucial in the present study. In fact, we are only interested here in analyzing the consequences of one of the choices in such a family. It is also worth noting that we are not changing the gravitational Lagrangian, at the price of introducing the coupling

$$b \frac{8\pi G}{c^2} (l_P)^2 R_{\mu} \cdot T_{\alpha|\nu|}.$$  

By virtue of Eq. (1.7) and of the Bianchi identity, the local relation (1.3) is now replaced by

$$\nabla_{\mu} T_{\mu\nu} = 0,$$  
(1.8)

having defined the effective energy-momentum tensor

$$\tau_{\mu\nu} = T_{\mu\nu} + AR_{\alpha} \cdot T_{\alpha[\nu]},$$  
(1.9)

which by rescaling the Ricci tensor by a natural length scale associated with it, say $L$,

$$R_{\mu\nu} = L^{-2} \tilde{R}_{\mu\nu},$$  
(1.10)

($\tilde{R}_{\mu\nu}$ dimensionless) becomes

$$\tau_{\mu\nu} = T_{\mu\nu} + \epsilon \tilde{R}_{\mu} \cdot T_{\alpha[\nu]}, \quad \epsilon = b \left( \frac{l_P}{L} \right)^2 = \frac{A}{L^2}.$$  
(1.11)

In all the explicit examples studied in the rest of the paper we will explore the case $\epsilon \ll 1$, but evidently $\epsilon$ does not need to be considered small at all.

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1 We note incidentally that, in a rather different context, the full contraction of the Ricci tensor with the energy-momentum tensor (whereas we consider their partial contraction in Eq. (1.7)) is met in quantum Yang-Mills theory. Indeed, as pointed out in Ref. [12], the presence of $m^2 R_{\mu\nu}$ in the $a_2$ heat-kernel coefficient means that, although the Yang-Mills coupling constant gets renormalized, the finite part of the effective action now depends on the auxiliary mass in a way that cannot be absorbed into a running coupling constant. Each choice of auxiliary mass corresponds to a different theory. Thus, the coupling to the gravitational field destroys the perturbative renormalizability of the Yang-Mills field, even in the purely Yang-Mills sector.
Section II studies in detail our effective energy-momentum tensor. Sections III, IV and V are devoted to modifications of perfect fluid spacetimes, spherically symmetric static spacetimes sourced by a perfect fluid and FLRW spacetimes sourced by a perfect fluid and with nonvanishing cosmological constant, respectively. Concluding remarks are made in Sec. VI, and relevant technical details are provided in Appendix A.

II. STRUCTURE OF THE EFFECTIVE ENERGY-MOMENTUM TENSOR

Let us consider matter sources of the Einstein’s field equations, i.e., a perfect fluid described by the energy-momentum tensor

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \] (2.1)

with \( u \) the four-velocity vector corresponding to the rest frame of the fluid, \( \rho \) the (proper) energy density and \( p \) the proper isotropic pressure. Let us introduce the following convenient notation for contractions of a tensor, say \( S \), with a vector, say \( X \),

\[ S_X = X^\beta S_{\beta\alpha}, \] (2.2)

as well as the standard \(^{1+3}\) decomposition of the Ricci tensor, \( R_{\mu\nu} \), parallel and orthogonal to \( u \),

\[ R_{\mu\nu} = \delta_\mu^\alpha \delta_\nu^\beta R_{\alpha\beta} \]

\[ = [\Pi(u)_{\mu}^\alpha - u_\mu u^\alpha][\Pi(u)^\beta_\nu - u_\nu u^\beta]R_{\alpha\beta} \]

\[ = [\Pi(u) R_{\mu\nu} - \Pi(u)^\beta_\nu]u_\mu u^\alpha R_{\alpha\beta} \]

\[ - \Pi(u)^\beta_\nu u_\mu u^\alpha u_\delta u^\beta R_{\alpha\beta} \]

\[ = R_{\mu\nu}^{\perp\perp} + 2R_{\mu\nu}^{\perp\parallel} + R_{\mu\nu}^{\parallel\parallel} u_\mu u_\nu, \] (2.3)

where

\[ \Pi(u)^\mu_\nu \equiv g_{\mu\nu} + u_\mu u_\nu, \] (2.4)

projects orthogonally onto \( u \) and we have defined

\[ R_{\mu\nu}^{\perp\perp} \equiv [\Pi(u) R_{\mu\nu}] = \Pi(u)_{\mu}^\alpha \Pi(u)^\beta_\nu R_{\alpha\beta}, \]

\[ R_{\mu\nu}^{\perp\parallel} \equiv -\Pi(u)^\beta_\nu u_\mu u^\alpha R_{\alpha\beta}, \]

\[ R_{\mu\nu}^{\parallel\parallel} \equiv u_\mu u^\beta R_{\alpha\beta} = R_{uu}. \] (2.5)

Upon inserting these splitted components into Eq. (1.9) one finds

\[ \tau_{\mu\nu} = [\rho + A\rho R_{\mu\nu}^{\parallel\parallel}] u_\mu u_\nu + p[\Pi(u)_{\mu\nu} + AR_{\mu\nu}^{\perp\perp}] \]

\[ - A(p - \rho)R_{\mu\nu}^{\perp\parallel} u_\nu, \] (2.6)

which gives for \( \tau_{\mu\nu} \) a pure energy part

\[ [\tau_{\text{en}}]_{\mu\nu} = \rho \left( 1 - AR_{\mu\nu}^{\parallel\parallel} \right) u_\mu u_\nu \equiv \rho_{\text{eff}} u_\mu u_\nu, \] (2.7)

a mechanical stress part

\[ [\tau_{\text{mec}}]_{\mu\nu} = p[\Pi(u)_{\mu\nu} + AR_{\mu\nu}^{\perp\perp}], \] (2.8)

and a thermal part

\[ [\tau_{\text{th}}]_{\mu\nu} = -A(p - \rho)R_{\mu\nu}^{\perp\parallel} u_\nu, \] (2.9)

so that

\[ \tau_{\mu\nu} = [\tau_{\text{en}} + \tau_{\text{mec}} + \tau_{\text{th}}]_{\mu\nu}. \] (2.10)

For the original perfect fluid energy-momentum tensor one could have written the equivalent expression

\[ T_{\mu\nu} = \rho u_\mu u_\nu + p \Pi(u)_{\mu\nu}, \] (2.11)

showing the absence of thermal stresses in the proper reference frame, coherent with the definition of perfect fluid. Note that since there are no a priori sign restrictions on \( R_{\mu\nu}^{\parallel\parallel} \), it is legitimate to expect either positive or negative values for \( \rho_{\text{eff}} \), a property which is of basic importance to model dark matter, dark energy or even exotic types of matter. Moreover, the constant \( A \) can be replaced by \( \epsilon \) if one rescales the Ricci tensor by a squared length scale.

In the case \( p = 0 \) the mechanical stress term cancels out

\[ \tau_{\mu\nu} = \rho_{\text{eff}} u_\mu u_\nu + A\rho R_{\mu\nu}^{\perp\parallel} u_\nu, \] (2.12)

while the thermal stress disappears only in a frame where \( R_{\mu\nu}^{\perp\parallel} u_\nu \) vanishes.

Things are much simpler when using an adapted frame to \( u \), i.e., such that \( e_\alpha = u \) and \( e_\alpha \), \( a = 1, 2, 3 \), span the local rest space of \( u \) (Note that while \( e_0 \) is supposed to be orthogonal to \( e_\alpha \), the latter spatial vectors are not necessarily orthonormal). Similarly, it is useful to introduce the standard \(^{1+3}\) decomposition of the Riemann tensor into its electric (\( \mathcal{E}(u)_{\alpha\beta} \)), magnetic (\( \mathcal{H}(u)_{\alpha\beta} \)) and mixed (\( \mathcal{F}(u)_{\alpha\beta} \)) parts, respectively, given by

\[ \mathcal{E}(u)_{\alpha\beta} = \Pi(u)^\beta_\nu u^\alpha u_\nu, \]

\[ \mathcal{H}(u)_{\alpha\beta} = -R_{\alpha\gamma\beta\delta} u^\gamma u^\delta, \]

\[ \mathcal{F}(u)_{\alpha\beta} = \left[ R^* \right]_{\alpha\mu\beta\delta} u^\mu u^\delta, \] (2.13)

where \( \mathcal{E}(u)_{\alpha\beta} = 0 = \mathcal{F}(u)_{\alpha\beta} \) and \( \mathcal{H}(u)_{\alpha\beta} = 0 \). \( \mathcal{E}(u)_{\alpha\beta} \), \( \mathcal{H}(u)_{\alpha\beta} \) and \( \mathcal{F}(u)_{\alpha\beta} \) are called tidal fields. The 20 independent components of the Riemann tensor are then summarized by the 6 independent components of the electric part (spatial and symmetric tensor), the 8 independent components of the magnetic part (spatial and trace-free tensor) and the 6 independent components of the mixed part (spatial and symmetric tensor). Then, in terms of frame components, all the above spatial quantities can be written as

\[ \mathcal{E}(u)_{ab} = R_{a0b0}, \]

\[ \mathcal{H}(u)_{ab} = -R^*_{a0b0} = \frac{1}{2} \eta_{(a} u^c d \eta_{b)} R_{a0cd}, \]

\[ \mathcal{F}(u)_{ab} = \left[ R^* \right]_{a0b0} = \frac{1}{4} \eta_{(a} u^c d \eta_{b)} e^f R_{a0c0f} \]
and can be inverted to give
\[
R_{\alpha \beta} = \mathcal{H}(u)_{ab} \eta(u)_{bc} \\
R^{\alpha \beta} = \eta(u)^{ab} \eta(u)^{cd} F(u)_{ac},
\]
where \( \eta(u)_{abc} = u^\alpha \eta_{abc} \) is the unit volume (spatial) three-form. By using these relations one also has the frame components of the Ricci tensor \( R^\alpha_{\beta \mu \nu} = R^\alpha_{\mu \nu \beta} \),
\[
R^\alpha_{\beta 0} = -\mathcal{E}(u)^\alpha c_c, \\
R^\alpha_{\beta a} = \eta(u)_{abc} \mathcal{H}(u)^{bc}, \\
R^\alpha_{\beta b} = -\mathcal{E}(u)^\alpha b - F(u)^\alpha b + \delta^\alpha_b F(u)^c c,
\]
so that the curvature scalar takes the form
\[
R = R^\alpha_{\beta 0} + R^\alpha_{\beta a} = -2(\mathcal{E}(u)^c c_c - F(u)^c c_c). \tag{2.17}
\]
On converting into the previous language, one writes
\[
R^\mu_{\alpha \nu} \to R_{ab}, \\
R^\mu_{||} \to R_{0a}, \\
R^\mu_{\parallel} \to R_{00}.
\tag{2.18}
\]
Therefore
\[
\tau_{00} = [\tau_{bc}]_{00} = \rho (1 - AR_{00}), \\
\tau_{ab} = [\tau_{bac}]_{ab} = p[\Pi(u)_{abc} + A R_{abc}], \\
\tau_{0a} = [\tau_{h0}]_{0a} = A(p - \rho) R_{0a}, \tag{2.19}
\]
in turn re-expressible in terms of the tidal fields \( \mathcal{E}(u), \mathcal{H}(u) \) and \( F(u) \).

In the following sections we are going to write and possibly solve Eqs. \( \mathbf{(1.7)} \) (analytically, or numerically when analytic treatments are very difficult) for various geometrically meaningful backgrounds. Let us further note that Eq. \( \mathbf{(1.7)} \) can also be written as
\[
E_{\mu \nu} = B R^\alpha_{\mu \nu} T_{[a][a]} = \kappa T_{\mu \nu}. \tag{2.20}
\]
This equation can be used to re-express \( T_{\mu \nu} \) in the exact form (recalling that \( A \equiv \frac{B}{\kappa} \))
\[
T_{\mu \nu} = \frac{1}{\kappa} E_{\mu \nu} - \frac{A}{2} R^\alpha_{\mu \nu} T_{\alpha \nu} - \frac{A}{2} R^\alpha_{\nu \mu} T_{\alpha \nu}, \tag{2.21}
\]
By re-inserting it into the left-hand side of Eq. \( \mathbf{(2.20)} \) and recalling that
\[
R^\alpha_{\nu \mu} = E^\alpha_{\nu \mu} + \frac{1}{2} \delta^\alpha_{\nu \mu} R - \Lambda \delta^\alpha_{\nu \mu},
\]
that is \( R = E + 2R - 4\Lambda \) \( \tag{2.22} \)
we finally obtain the full Einstein tensor to linear order in \( A \):
\[
E_{\mu \nu} = \kappa T_{\mu \nu} + T^A_{\mu \nu}, \tag{2.23}
\]
where \( E \equiv g^{\mu \nu} E_{\mu \nu} \) and we have defined
\[
T^A_{\mu \nu} \equiv A \left[ E^\alpha_{\mu \nu} E_{\alpha \nu} - \frac{1}{2} E_{\mu \nu} (E - 2\Lambda) \right]. \tag{2.24}
\]
The result is then either a \( f(R) \) gravity theory or a modification of the Einstein’s field equations by the addition of an extra energy-momentum tensor, completely geometrically motivated and small (see Eq. \( \mathbf{(2.24)} \)). The latter is not the main interest in the present study, which as stated above, assumes finite values of the dimensionless parameter \( b \) occurring in \( B \) or \( \epsilon \) if one uses the rescaled version of \( B \) termed as \( A \). However, when working with a finite value of \( b \) will imply excessive difficulties, we will also explore the case of infinitesimal \( b \).

The case of a constant curvature spacetime is also relevant
\[
R^\alpha_{\beta \mu \nu} = \frac{R}{12} \delta^\alpha_{\mu \nu}, \\
R_{\mu \nu} = \frac{R}{4} g_{\mu \nu}, \tag{2.25}
\]
with \( R \) constant. Equations \( \mathbf{(1.7)} \) become then
\[
\Lambda - \frac{1}{1 + \frac{R}{2}} R_{\mu \nu} = \kappa T_{\mu \nu}. \tag{2.26}
\]
In the case of a perfect fluid this equation is compatible with
\[
p = -\rho = \frac{\Lambda - \frac{R}{1 + \frac{R}{2}}}{1 + \frac{R}{2}}. \tag{2.27}
\]
III. MODIFYING PERFECT FLUID SPACETIMES

Let us consider the Dunn and Tupper spacetime (see [30], and Chapter 12 of [31]). This was discovered by looking for solutions of the Einstein-Maxwell equations for source-free electromagnetic fields [32]. The spacetime metric of this solution reads as

\[ ds^2 = -2 \, du \, dr + u^{-2n} r^{-2m} dy^2 + u^{-2m} r^{-2n} dz^2, \quad (3.1) \]

where

\[ m = \frac{\sqrt{3} - 1}{4}, \quad n = \frac{\sqrt{3} + 1}{4}. \quad (3.2) \]

This solution has the property that the principal null congruences of the electromagnetic field are geodesic, and the corresponding null tetrad is parallelly propagated along these congruences. It is the unique twist-free solution (for \( m \neq n \)) from which it is clear that the new coordinate transformation

\[ t = \sqrt{2} u, \quad x = \frac{(m-n)}{2} \log \left( \frac{r}{u} \right), \]

\[ y \rightarrow 2^{(m+n)} y, \quad z \rightarrow 2^{(m+n)} z, \quad (3.3) \]

the metric (3.1) takes the form

\[ ds^2 = -dt^2 + \frac{t^2}{(m-n)^2} dx^2 + t^{-2(m+n)} [e^{-2x} dy^2 + e^{2z} dz^2], \quad (3.4) \]

(for \( m \neq n \)) from which it is clear that the new coordinate system is comoving. Last, but not least, one investigates the possibility of adapting the metric (3.4), so that it represents a perfect-fluid matter distribution. For this purpose, one can no longer impose the particular values (3.2), but a restriction on the admissible values of \( m \) and \( n \) (see below) is still necessary in order to obtain a solution of the Einstein equations.

Unlike our Secs. I and II, where we needed physical dimensions, here the coordinates \( t, x, y, z \) are all dimensionless and we may assume that the physical coordinates scale with the same constant \( \mathcal{L} \),

\[ x^\alpha_{\text{phys}} = \mathcal{L} x^\alpha. \quad (3.5) \]

The metric (3.4) is an exact solution of the Einstein equations in absence of cosmological constant and sourced by a perfect fluid with four-velocity \( u = \partial_t \) (i.e., at rest with respect to the space coordinates) and

\[ \mathcal{L}^2 \kappa p_0 = \frac{m^2 + mn + n^2}{t^2}, \quad \mathcal{L}^2 \kappa p_0 = -\frac{4mn}{t^2}, \quad (3.6) \]

provided the dimensionless constants \( m \) and \( n \) satisfy the additional constraint

\[ m(2m+1) + n(2n+1) = 0. \quad (3.7) \]

We will conveniently work in the rest of this section with the dimensionless coordinates, but restoring the physical ones with the length scale \( \mathcal{L} \) when necessary.

Note that the conditions \( \rho_0 > 0 \) and \( \rho_0 \geq 0 \) require \( mn \leq 0 \). The solution for a dust fluid (i.e., with \( p_0 = 0 \)) corresponds to either \( m = 0 \) or \( n = 0 \), but not both of them vanishing since in that case the spacetime would be flat. Also the strong energy conditions

\[ \rho_0 + p_0 \geq 0, \quad \rho_0 + 3p_0 \geq 0, \quad (3.8) \]

are always satisfied.\(^2\)

Equation (3.10) sets the relative dependence of \( m \) and \( n \). Other parametrizations can be found in order to satisfy automatically the constraint (3.7) which represents a circle in the space of the parameters \( m \) and \( n \), e.g.,

\[ m = -\frac{1}{4} + \frac{1}{2\sqrt{2}} \cos \alpha, \quad n = -\frac{1}{4} + \frac{1}{2\sqrt{2}} \sin \alpha, \quad (3.9) \]

with \( \alpha \in [0, 2\pi] \), \( \alpha \neq \pi/4 \). The associated “sound speed,”

\[ v_s(m,n) = \sqrt{\frac{\rho_0}{\rho_0}} = \sqrt{\frac{-4mn}{m^2 + mn + n^2}} = 2 \sqrt{\frac{m}{1 + \frac{m}{m + n} \alpha}}, \quad (3.10) \]

is then a constant dependent on the parameters \( m \) and \( n \) (actually it is a function of the ratio \( n/m \)). In terms of the parameter \( \alpha \) the above relation becomes

\[ v_s(\alpha) = 2 \sqrt{\frac{\sqrt{2(\sin \alpha + \cos \alpha) - 1 - \sin(2\alpha)}}{5 + \sin(2\alpha) - 3\sqrt{2(\sin \alpha + \cos \alpha)}}} \quad (3.11) \]

It is easy to see that \( v_s(m,n) \) vanishes at \( n = 0 \) (or \( m \to \infty \)), or equivalently at \( \alpha = 3\pi/4, 7\pi/4 \). This velocity reaches its maximum value \( v_s(m,n) = 2 \) at \( m = n \), i.e., \( \alpha = \pi/4 \), so that forcing it to stay in the physical region would restrict the range of allowed parameters. For example \( v_s(m,n) = 1 \) corresponds to \( m/n = \left( 5 \pm \sqrt{21} \right)/2 \).

Let us shortly review some geometrical properties of the metric (3.4) which has not received much attention in the recent literature. A Lorentz frame (adapted to \( u = \partial_t \)) reads

\[ e_0 = u, \quad e_1 = \frac{(m-n)}{t} \partial_x, \quad e_2 = e^x t^{m+n} \partial_y, \quad e_3 = e^{-x} t^{m+n} \partial_z. \quad (3.12) \]

\(^2\) One should require, however, \( m \neq n \), as already assumed in Eq. (3.4). Relaxing this condition is possible, but one should revert to the original form of the metric.
When expressed with respect to the frame (3.12), the Riemann tensor components simplify as
\[
R_{0000} = R_{0i0i} = -\frac{(m+n)(n+m+1)}{t^2},
\]
\[
R_{0i1} = -R_{0i10} = -\frac{(m-n)(n+m+1)}{t^2},
\]
\[
R_{2323} = -\frac{(m+n)}{t^2},
\]
\[
R_{1i12} = R_{1i21} = -\frac{(m+n-4mn)}{2t^2},
\]
and are assembled in \( \mathcal{E}(u), \mathcal{H}(u) \) and \( \mathcal{F}(u) \) as
\[
\mathcal{E}(u)_{ab} = \frac{(m+n)(n+m+1)}{t^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\mathcal{H}(u)_{ab} = \frac{(m-n)(n+m+1)}{t^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\mathcal{F}(u)_{ab} = -\frac{1}{t^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Similarly, the nonvanishing frame components of the Ricci tensor are
\[
R_{00} = -\frac{2(m+n)(n+m+1)}{t^2},
\]
\[
R_{11} = -\frac{2(m^2-2mn+n^2+m+n)}{t^2},
\]
\[
R_{22} = R_{33} = \frac{2(m+n)^2}{t^2},
\]
and the Ricci scalar is then given by
\[
R = \frac{2m-n+8mn}{t^2}.
\]
Moreover, the metric (3.3) is in general of Petrov type I. In fact, in a standard Newman-Penrose frame built by using the Lorentz frame (3.12), with
\[
l = \frac{1}{\sqrt{2}}(e_0+e_1), \quad n = \frac{1}{\sqrt{2}}(e_0-e_1), \quad m = \frac{1}{\sqrt{2}}(e_2+ie_3),
\]
the nonvanishing Weyl scalars are
\[
\psi_0 = -\psi_4 = \frac{(m^2-n^2+m+n)}{t^2},
\]
\[
\psi_2 = -\frac{(m-n)^2}{t^2}.
\]
The speciality of the metric would imply the relation
\[
I^3 = 27J^2
\]
or, introducing the speciality index
\[
S = \frac{I^3}{27J^2} = 1,
\]
where, in the present case with \( \psi_1 = 0 = \psi_3 \)
\[
I = 3\psi_2^2 + \psi_0\psi_4, \quad J = \psi_2(\psi_0\psi_4 - \psi_2^2).
\]
It is convenient to introduce the (dimensionless) ratio
\[
\gamma = -\frac{\psi_2^2}{\psi_0\psi_4} = \left(\frac{m-n}{3(m+n+1)}\right)^2,
\]
such that
\[
S = \frac{(3\gamma-1)^3}{27(1+\gamma)^2}.
\]
Equation (3.23) shows that the condition of being algebraically special \( (S = 1) \) is approached as soon as \( \gamma \to \infty \), or \( m = -n-1 \). In that case the spacetime becomes of Petrov type D, where only the Weyl scalar survives and equals
\[
\psi_2 = -\frac{(2n+1)^2}{t^2}.
\]
Moreover, \( S = 0 \) at \( \gamma = 1/3 \), that is for
\[
m = \frac{(6+7n)}{(7+4n)}.
\]
Finally, the geodesic equations (for any causality condition) read
\[
\frac{d^2t}{d\lambda^2} = -\frac{t}{(m-n)^2} \left(\frac{dx}{d\lambda}\right)^2 + (m+n)(P_y^2e^{2x} + P_z^2e^{-2x})t^{m+2n-1}
\]
\[
\frac{d^2x}{d\lambda^2} = -\frac{2}{t} \frac{dt}{d\lambda} \left(\frac{dx}{d\lambda}\right)^2 + (m-n)(P_y^2e^{2x} - P_z^2e^{-2x})t^{m+2n-2}
\]
\[
\frac{dy}{d\lambda} = P_y e^{2x}t^{2(m+n)}
\]
\[
\frac{dz}{d\lambda} = P_z e^{-2x}t^{2(m+n)},
\]
where \( \lambda \) is an affine parameter\(^3\). It is immediate to recognize that a particle at rest with respect to the coordinates, i.e., with \( x = x_0, \ y = y_0 \) and \( z = z_0 \) \( (x_0, y_0 \) and \( z_0 \) constant, implying \( P_y = P_z = 0 \) \( (y = y_0 \) and \( z = z_0 \) with the above equations reducing to
\[
\frac{d^2t}{d\lambda^2} = -\frac{t}{(m-n)^2} \left(\frac{dx}{d\lambda}\right)^2,
\]
\[
\frac{d^2x}{d\lambda^2} = \frac{2}{t} \frac{dt}{d\lambda} \left(\frac{dx}{d\lambda}\right)^2.
\]
\(^3\) To the best of our knowledge this study is absent in the literature.
The $x$-equation implies then
\[
\frac{dx}{d\lambda} = \frac{C_1^2}{t^2}, \quad (3.28)
\]

$[C_1$ can be assumed as positive without any loss of generality] which once inserted in the first one gives
\[
\frac{d^2t}{dx^2} = -\frac{C_1^4}{(m-n)^2t^4}, \quad (3.29)
\]

and can be easily reduced to a first-order equation multiplying both sides by $2\frac{dt}{dx}$,
\[
\left(\frac{dt}{d\lambda}\right)^2 = \frac{C_1^4}{(m-n)^2t^2} + \text{const.} \quad (3.30)
\]

Let us re-name the constant term in the above equation as $C_1^2C_2/4$,
\[
\left(\frac{dt}{d\lambda}\right)^2 = \frac{C_1^4}{(m-n)^2t^2} + \frac{C_1^2C_2}{4}, \quad (3.31)
\]

and introduce the new variable $T = C_1t$, so that
\[
\left(\frac{dT}{d\lambda}\right)^2 = \frac{1}{(m-n)^2T^2} + \frac{C_2}{4}. \quad (3.32)
\]

We will assume hereafter $C_2 > 0$: otherwise Eq. (3.32) would be valid only in a bounded interval of the temporal coordinate, a case which we are not interested in here. The last equation can be rewritten as
\[
4T^2\left(\frac{dT}{d\lambda}\right)^2 = \left(\frac{dT}{d\lambda}\right)^2 = \frac{4}{(m-n)^2} + C_2T^2, \quad (3.33)
\]

that is
\[
\frac{1}{C_2} \left(\frac{dT}{d\lambda}\right)^2 = \frac{4}{(m-n)^2} + C_2T^2. \quad (3.34)
\]

On introducing $T = \frac{1}{(m-n)^2} + C_2T^2$ one finds
\[
\frac{1}{C_2} \left(\frac{dT}{d\lambda}\right)^2 = T, \quad (3.35)
\]

and hence
\[
T(\lambda) = \left(\pm \frac{C_2}{2} + C_3\right)^2, \quad (3.36)
\]
or
\[
C_2T^2 = \left(\frac{C_2}{2} + C_3\right)^2 - \frac{4}{(m-n)^2}
= \left(\frac{C_2}{2} + C_3 - \frac{2}{m-n}\right) \left(\frac{C_2}{2} + C_3 + \frac{2}{m-n}\right), \quad (3.37)
\]

The $\pm$ sign choice in front of $\lambda$ should be set as a plus sign if one wants the orbit to be future-oriented, $dt/d\lambda > 0$:
\[
C_2T^2 = \left(\frac{C_2}{2} + C_3 - \frac{2}{m-n}\right) \left(\frac{C_2}{2} + C_3 + \frac{2}{m-n}\right). \quad (3.38)
\]

Moreover, if we require
\[
t(0) = 0, \quad (3.39)
\]

we may then choose the constant in such a way that
\[
C_2^2 = \frac{4}{(m-n)^2}. \quad (3.40)
\]

Let us define
\[
A_{\pm}(\lambda) = \frac{1}{C_1\sqrt{C_2}} \left(\frac{C_2}{2} + C_3 \pm \frac{2}{m-n}\right). \quad (3.41)
\]

We have eventually
\[
t(\lambda) = \sqrt{A_+ A_-}, \quad (3.42)
\]

and hence
\[
x(\lambda) = C_4 + \frac{128}{C_2^2(m-n)^3(A_+ - A_-)^2} \ln \left(\frac{A_+}{A_-}\right). \quad (3.43)
\]

For large values of $\lambda$ we see that $A_{\pm}(\lambda) \to \frac{C_2}{2C_1} \pm \lambda$ and $x \to C_4$ ($A_+ - A_-$ does not depend on $\lambda$), while $t \sim \frac{1}{C_2} \lambda$. $C_4$ can be therefore identified with $x_{\infty} = \lim_{\lambda \to \infty} x(\lambda)$.

We will specialize our considerations below to the case of a dust fluid ($p_0 = 0$), with
\[
m = 0, \quad n = -\frac{1}{2}. \quad (3.44)
\]

Working at linear order in $A$ and restoring the physical length scale $\mathcal{L}$ so that in this case
\[
\epsilon = \frac{A}{\mathcal{L}^2}. \quad (3.45)
\]

one modifies this solution to satisfy the new equations by changing simply the $tt$ component of the metric as
\[
g_{tt} = -1 + \epsilon \frac{4C_p}{5\mathcal{L}^2} \quad (3.46)
\]

and the energy and pressure of the fluid
\[
\rho = \rho_0 + \epsilon \rho_1, \quad p = p_0 + \epsilon p_1 \quad (3.47)
\]

with
\[
\kappa \mathcal{L}^2 \rho_0 = \frac{1}{t^2}, \quad p_0 = 0, \quad (3.48)
\]

and
\[
\kappa \mathcal{L}^2 \rho_1 = \frac{C_p}{t^4}, \quad \kappa \mathcal{L}^2 p_1 = \frac{C_p}{t^4} \quad (3.49)
\]
with the constraint
\[ C_ρ = -\frac{1}{2} + C_p. \] (3.50)

Note that the additional piece (linearizing in \( \epsilon \), i.e. in \( A \)) results in the following tensor:
\[
\kappa \mathcal{L}^4 R_{(\mu} T_{(\nu|\rho)} = -\frac{1}{2t^2} \rho \delta^0 \delta^\rho + \frac{\epsilon}{t^4} \text{diag}\left[ -\frac{(5 + 42 C_p)}{20 t^2}, 2 C_p, C_p \frac{e^{-2x}}{2 t}, C_p \frac{e^{2x}}{2 t} \right].
\] (3.51)

If instead of coupling the Ricci tensor to the energy-momentum tensor one uses the Einstein tensor (see Appendix) the result is simply
\[
\kappa \mathcal{L}^4 E_{(\mu} T_{(\nu|\rho)} = -\frac{1}{t^4} \left[ 1 + \epsilon \left( -\frac{5 + 12 C_p}{10 t^2} \right) \delta^0 \delta^\rho \right].
\] (3.52)

Interestingly, one can look for exact solutions also in the general case, i.e., not considering the linear expansion in \( \epsilon \). In the case \( m = 0, n = -1/2 \) discussed above, still modifying only the \( tt \) metric component as
\[ g_{tt} = -1 + \epsilon f_{tt}, \] (3.53)

with energy density and pressure
\[ \kappa \mathcal{L}^2 \rho(t) = \frac{1}{t^2} + \epsilon \kappa \mathcal{L}^2 \rho_1(t), \quad p(t) = \epsilon p_1(t), \] (3.54)

and the exact solution is
\[
f_{tt} = \frac{1}{e} + \frac{1}{C_1 - \frac{11 \epsilon + 12 t^2}{33 e^{3/2}}},
\kappa \mathcal{L}^2 \rho_1(t) = -\frac{(\epsilon - t^2)}{t^2 \epsilon^2},
\kappa \mathcal{L}^2 p_1(t) = -\frac{1}{t^2},
\] (3.55)

where \( C_1 \) is an integration constant and \( \epsilon \) is not necessarily small (both \( t \) and \( \epsilon \) are dimensionless). In this exact solution, Eq. (3.55), negative pressure and sign-changing energy density during the evolution are evident, confirming the expectations of the linearized model for the presence of dark or exotic matter in a universe modeled as in this toy model.

However, finding -as is this case- an explicit, exact solution is never a trivial task. It requires always some care, even when using algebraic manipulation systems like MapleTM and MathematicaTM. As a first attempt one may try to use the same symmetries of the background metric and then proceed by relaxing some hypothesis. In the present case we have been looking for simple conditions on the fluid source of the spacetime, like constant energy density and/or pressure with a minimal backreaction on the modified metric. However, this may not be enough in general, and one should then analyze the system of coupled equations, looking for some simplifications. The risk is to end up with a purely mathematical solution, devoid of physical meaning, or that any improvement is reached only by trial and error, without a clear understanding of the intermediate steps.

IV. MODIFYING SPHERICALLY SYMMETRIC STATIC SPACETIMES SOURCED BY A PERFECT FLUID

In order to investigate the role of the curvature-energy coupling discussed above the simplest arena is that associated with an internal solution of the Schwarzschild spacetime. The latter is a spherically symmetric spacetime, with metric written in the form
\[
ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (4.1)

(with \( g_{tt} \) and \( g_{rr} \) depending only on \( r \)) sourced by a perfect fluid with a constant energy density and in absence of cosmological constant. Let us introduce the notation
\[
F(x, y) \equiv \sqrt{1 - \frac{x^2}{y^2}},
\] (4.2)

with \( F \) dimensionless.

The interior Schwarzschild solution, for example, corresponds to
\[
\begin{align*}
g_{tt}^i &= \frac{1}{4} \left[ 3F(r_s, R) - F(r, R) \right]^2 \\
g_{rr}^i &= F^2(r, R) \\
\rho_0 &= \frac{3}{\kappa R}\frac{F(r, R) - F(r_s, R)}{3F(r_s, R) - F(r, R)},
\end{align*}
\] (4.3)

where \( r_s = 2M \) denotes the Schwarzschild radius and \( R = \sqrt{r_{\text{body}}/r_s} \) is a length scale built with the radius of the interior “body.” In order to shorten equations we will introduce the notation
\[
F_r = F(r, R), \quad F_s = F(r_s, R).
\] (4.4)

The \( \epsilon \)-modifications to this rather simple solution are not simple at all, and in general one is left only with the numerical integration of the associated equations. One can look at linear perturbations of the interior Schwarzschild solution, i.e.
\[
\begin{align*}
g_{tt} &= g_{tt}^i + \epsilon f_{tt} \\
g_{rr} &= g_{rr}^i + \epsilon f_{rr} \\
\rho &= \rho_0 + \epsilon \rho_1 \\
p &= \rho_0 + \epsilon \rho_1,
\end{align*}
\] (4.5)

assuming spherical symmetry (all \( B \)-corrections depend only on the radial variable and \( B \) itself is given by \( B = \))
kR^2\epsilon)$. Formally, one can introduce a vector notation for the unknown functions

$$\mathbf{X} = [X_1, X_2, X_3, X_4], \quad (4.6)$$

with $X_1 = f_{tt}$, $X_2 = f_{rr}$, $X_3 = p_1$ and $X_4 = p_1$ and write down a system of coupled linear equations

$$\frac{d}{dr} X_i = A_{ij} X_j + C_i, \quad (4.7)$$

with $A_{ij}$ and $C_i$ depending on $r$ and whose explicit expression does not involve an equation for $d\rho_1/dr$ (the perturbation equations are actually three). It is summarized by the only nonvanishing components listed below

\[
\begin{align*}
A_{11} &= -\frac{2r}{R^2 F_r(F_r - 3F_s)}, \\
A_{12} &= -\frac{(3F_s - F_r)F_r(3F_s F_r - 3F_r^2 + 2)}{4r}, \\
A_{22} &= -\frac{(F_r R - 2r)(F_r R + 2r)}{R^2 F_r^2}, \\
A_{22} &= \frac{3F_s F_r(F_r^2 - 1)(3F_s F_r - 3F_r^2 + 2)}{\kappa r^4(3F_s - F_r)^2}, \\
A_{23} &= \frac{r\kappa}{F_r}, \\
A_{33} &= \frac{(F_r^2 - 1)}{F_r r(3F_s - F_r)}, \\
A_{44} &= -\frac{(3F_s + F_r)r}{R^2(3F_s - F_r)F_r^2},
\end{align*}
\]

and

\[
\begin{align*}
C_1 &= \frac{9(2F_s - F_r)(F_s - F_r)r}{4R^2 F_r^2}, \\
C_2 &= \frac{9r}{R^2 F_r^2(3F_s - F_r)}, \\
C_4 &= \frac{18F_s^2(F_r^2 - 1)^2}{\kappa (3F_s - F_r)^2 F_r^2 r^3}.
\end{align*}
\]

Of course, in this case the integration of the full system can only be carried out numerically and an example is given in Fig. II.

The form of the system is such that the evolution equation for $\rho_1$ is implicit in the compatibility of the system. Consequently $\rho_1 = \text{constant}$ is a natural choice to cast the perturbation equations in their normal form. This is what has been done in the case of Fig. II. [Actually, it is worth mentioning that an exact solution for $f_{rr}$ can also be found in this case. We will not display it here because of its length and because it does not add much to the present discussion.] Numerical integration shows that the perturbed pressure can be negative during its evolution, leaving also in this case the possibility open for the birth of dark or exotic matter during the evolution.

\[
\begin{align*}
\frac{d^2 s^4}{dt^2} + a^2(t) \left[ \frac{dr^2}{(1 - kr^2)} + f^2(r)(d\theta^2 + \sin^2 \theta \, d\phi^2) \right],
\end{align*}
\]

with $k = [-1, 0, 1]$, corresponding to a closed ($k = -1$), spatially flat ($k = 0$), open ($k = 1$) $t = \text{constant} \, 3$-spaces.
and

\[
f(r) = \begin{cases} 
\sinh r & k = -1 \\
\cosh r & k = 0 \\
\sin r & k = 1.
\end{cases}
\]  

(5.2)

Here \( t \) and \( a \) have the dimensions of a length while \( r, \theta \) and \( \phi \) are dimensionless.

The perfect fluid, source of this spacetime, is assumed at rest with respect to the chosen coordinate system, i.e., it is associated with a four-velocity field \( u = \partial_t \) and its energy-momentum tensor reads in general as

\[
T_{\mu \nu} = \rho_0(t)u_\mu u_\nu + p_0(t)\Pi(u)_{\mu \nu},
\]

(5.3)

with

\[
\Pi(u)_{\mu \nu} = g_{\mu \nu} + u_\mu u_\nu,
\]

(5.4)

and the Einstein’s field equations are given by

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2} + \frac{\kappa}{3} \rho_0,
\]

\[
\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{\kappa}{2} \left(\frac{1}{3}\right)^2 \rho_0 + p_0,
\]

(5.5)

plus the compatibility condition

\[
\dot{\rho}_0 = -3\frac{\dot{a}}{a}(\rho_0 + p_0).
\]

(5.6)

The modified equations become

\[
\frac{\dot{a}}{a} = \frac{2}{3} \frac{A_2 \kappa p (\kappa + \Lambda) - \kappa (p + 3p) + 2A_2 \kappa p}{2 - A_2 \kappa (p + \gamma) + 2A^2 \kappa^2 p},
\]

\[
\frac{\ddot{a}}{a^2} = \frac{[\kappa^2 p - \kappa 3(\rho + p)] A + 2A_2 \kappa p + 2A_2 \kappa^2 p}{3[2 - A_2 \kappa (p + \gamma) + 2A^2 \kappa^2 p]} - \frac{k}{a^2},
\]

(5.7)

Let us consider for simplicity the spatially flat case \( k = 0 \) and let us assume \( \Lambda = 0 \). The solution of the Einstein’s field equation is termed Friedmann-Lemaitre solution and is given by

\[
\kappa t_0^2 \rho_0 = \frac{4\rho_0^2}{3T^2},
\]

\[
\rho_0 = (\gamma - 1)\rho_0,
\]

\[
a(t) = t_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}},
\]

(5.8)

where \( \gamma \) is dimensionless parameter and \( t_0 \) is an arbitrary length scale\(^4\) associated with \( a(t) \). When \( \gamma = 1 \) it becomes the Einstein-de Sitter Universe solution. It is convenient to introduce the rescaled, dimensionless time variable,

\[
T = \frac{t}{t_0}.
\]

(5.9)

Looking for perturbative solutions at the first order in \( B = \kappa L^2 \epsilon \) (\( \epsilon \) dimensionless; in this way the perturbed quantities have the same dimensions of the corresponding original ones), i.e.,

\[
\rho = \rho_0 + \epsilon \rho_1,
\]

\[
p = p_0 + \epsilon p_1
\]

\[
a = a_0 + \epsilon a_1,
\]

(5.10)

it is straightforward to identify the following solution

\[
\kappa t_0^2 \rho_1(t) = -\frac{8(7\gamma - 6)}{9\gamma^3(3\gamma - 2)T^4} - \frac{2p_1 \kappa t_0^2}{(2 + \gamma)} + T^{-\frac{2(2 + \gamma)}{3}},
\]

\[
p_1(t) = p_1,
\]

\[
a_1(t) = C_1 T^{-1} - \frac{\gamma \kappa^2 \rho_1}{4(2 + \gamma)} - \frac{2A_2}{3(2 + \gamma)} - \frac{k}{t_0} + \frac{2(\gamma - 1)(\gamma - 2)}{9(3\gamma - 2)\gamma^3} T^{\frac{2}{3}},
\]

(5.11)

where \( C_1 \) in an integration constant. Note that here we have been looking for solutions with \( p_1(t) \) constant. This simplifying condition (which is enough for the purposes of the present discussion) can be eventually relaxed. Let us assume \( \gamma = 1 \) and \( C_1 = 0 \), for a practical purpose. We find

\[
\kappa t_0^2 \rho_1(t) = -\frac{8}{9T^4} - \frac{2A_2 \kappa p_1}{3} + \frac{1}{T^3},
\]

\[
p_1(t) = p_1,
\]

\[
a_1(t) = -\kappa T^{\frac{2}{3}} - \frac{1}{4} T^{-\frac{4}{3}},
\]

(5.12)

showing that \( p_1(t) \) diverges asymptotically, whereas \( a_1(t) \) diverges in general. The special case \( p_1 = 0 \) avoids such a divergence and gives an asymptotic damping of the perturbation, i.e.,

\[
\kappa t_0^2 \rho_1(t) = -\frac{8}{9T^4} + \frac{1}{T^3},
\]

\[
p_1(t) = 0,
\]

\[
a_1(t) = -\frac{1}{4} T^{-\frac{4}{3}},
\]

(5.13)

to be compared with the unperturbed values (for \( \gamma = 1 \),

\[
\kappa t_0^2 \rho_0 = \frac{4}{3T^2},
\]

\[
p_0(t) = 0,
\]

\[
a_0(t) = t_0 T^{\frac{2}{3}}.
\]

(5.14)

We have then

\[
\kappa t_0^2 \rho = \frac{4}{3T^2} + \epsilon \left( -\frac{8}{9T^4} + \frac{1}{T^3} \right),
\]

\[
\frac{a(t)}{t_0} = T^{\frac{2}{3}} - \frac{\epsilon}{4} T^{-\frac{4}{3}}.
\]

(5.15)

\(^4\) The choice \( t_0 = 1 \) makes the form of the solution (5.3) more familiar.
FIG. 2: The behavior of $a(t)$ (in units of $t_0$) vs $\rho(t)$ (in units of $\kappa t_0^2$) of the solution (5.15) is shown for different values of $\epsilon = 0, \pm 1, \pm 3, \pm 6$ showing the displacement of the perturbed quantities from the unperturbed ones.

Therefore, the perturbation changes both the background geometry and the mass-energy content of the spacetime. Moreover, $\rho_1(t)$ may change sign during the evolution and it is therefore not an arbitrary speculation to admit that a minimal modification of the Einstein’s field equation may allow for the theoretical existence of dark or exotic matter in some spacetime region.

VI. CONCLUDING REMARKS

We have explored the features of a toy model where, in the Einstein equations, the right-hand side is modified by the addition of a term proportional to the symmetrized partial contraction of the Ricci tensor with the energy-momentum tensor, while the left-hand side remains equal to the Einstein tensor. Indeed, one can modify the Einstein’s field equations in a number of ways, grounded on geometrical reasons or on physical reasons. Our choice of including an “R-T” correction is in between but, for the purpose of the present study, any particular choice is valid. Thinking of “small corrections” we have argued that the coupling constant in the “R-T” term might have a quantum origin, by virtue of the existence of a natural length scale given by the Planck length. This remark, supplemented by dimensional analysis, shows that such a term yields a correction linear in $\hbar$ to the classical term, that is instead just proportional to the energy-momentum tensor. A nice feature of this model is that it coincides with general relativity in vacuum and can be related to various $f(R)$ theories already studied in the recent literature.

Motivated by the analysis of the subsequent corrections on the background geometry and the background energy-momentum tensor source of the spacetime curvature, we have studied linear perturbations by using as unperturbed situation some special, non-vacuum exact solutions. These are the Dunn and Tupper metric, the interior Schwarzschild solution and a Friedmann-Lemaitre cosmological solution, besides some general considerations concerning the simple case of constant curvature spacetimes.

All the studied situations are interesting and the Friedmann-Lemaître case is also illuminating: it is far from being an arbitrary conjecture that the dark or exotic matter may form in some spacetime region, even with a small temporal duration. This is made clear by our Eqs. (2.7), (2.12), (3.55) and (5.13). Instead of postulating new forms of matter or new gravitational lagrangians, we have allowed for a novel way of coupling gravity, i.e., its geometrical description, to matter fields, showing that we might need both conceptual ingredients at once in order to overcome the apparent shortcomings of general relativity on large scales. In other words, the net separation of geometry (curvature tensors) and physics (matter energy-momentum tensor), which is implicit in the Einstein equations, is “called into question” in the toy model presented here, and modified by the addition of a direct (i.e., the simplest possible one) coupling among these two ingredients.

However, since the right-hand side of our field field equations (see also (A1)) is tensorial but not variational, the model we have introduced and discussed in this paper remains a toy model, unless one can find a stronger foundation for field equations whose right-hand side is not variational. This open problem deserves further attention, since not all partial differential equations of interest are variational (see, e.g., Ref. [33]).

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Appendix A: Nonlinear coupling of Einstein’s tensor to the energy-momentum tensor

The field equation (1.4) that we have postulated results from considering a nonlinear coupling of gravity to matter, and hence suggests considering also the following alternative:

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} + BE^a_{\mu\nu} T_{[a]\nu}.$$ (A1)
Now we write explicitly the symmetrization on the right-hand side, finding therefore

\[ \left( \delta^\alpha_{\nu} - \frac{B}{2} T^\alpha_{\nu} \right) E_{\alpha\mu} - \frac{B}{2} T^\alpha_{\mu} E_{\alpha\nu} = \kappa T_{\mu\nu}. \quad (A2) \]

This form of the field equation suggests defining the tensor

\[ U^\alpha_{\nu} \equiv \delta^\alpha_{\nu} - \frac{B}{2} T^\alpha_{\nu}, \quad (A3) \]

whose inverse \( W^\nu_{\beta} \) should fulfill the condition

\[ U^\alpha_{\nu} W^\nu_{\beta} = \delta^\alpha_{\beta}. \quad (A4) \]

At this stage, bearing in mind the definition (A3), we multiply both sides of Eq. (A2) by \( W^\nu_{\beta} \) and sum over repeated indices. Hence we find, exploiting the symmetry of Einstein’s tensor,

\[ E_{\mu\beta} = \frac{B}{2} T^\alpha_{\mu} E_{\alpha\nu} W^\nu_{\beta} = \kappa T_{\mu\nu} W^\nu_{\beta}. \quad (A5) \]

We can point out that, upon inserting (A3) into the condition (A4) one finds the recursive algorithm

\[ W^\nu_{\beta} = \delta^\nu_{\beta} + \frac{B}{2} T^\nu_{\beta} W^\nu_{\beta} = \delta^\nu_{\beta} + \frac{B}{2} T^\nu_{\beta} \left( \delta^\nu_{\beta} + \frac{B}{2} T^\nu_{\gamma} W^\gamma_{\beta} \right) \]

\[ = \delta^\nu_{\beta} + \frac{B}{2} T^\nu_{\beta} + \left( \frac{B}{2} \right)^2 T^\nu_{\alpha} T^\alpha_{\gamma} W^\gamma_{\beta} = \ldots. \quad (A6) \]

If the dimensionless parameter \( b \) introduced in (1.5) approaches 0, we can therefore deal with finitely many powers of the energy-momentum tensor in Eq. (A5), by truncating the sum of terms on the right-hand side of Eq. (A6).

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