Exact solutions of Dirac equation on a 2D gravitational background

S. K. Moayedi$^a,c$, F. Darabi$^b,c$*

$^a$Department of Physics, Arak University, Arak, Iran.
$^b$Department of Physics, Azarbaijan University of Tarbiat Moallem, Tabriz, Iran.
$^c$Research Institute for Fundamental Sciences, Tabriz, Iran.

March 24, 2022

Abstract

We obtain classes of two dimensional static Lorentzian manifolds, which through the supersymmetric formalism of quantum mechanics admit the exact solvability of Dirac equation on these curved backgrounds. Specially in the case of a modified supersymmetric harmonic oscillator the wave function and energy spectrum of Dirac equation is given explicitly.

PACS: 04., 04.20.Jb, 04.62.+v, 11.30.Pb

*Corresponding author, e-mail: f.darabi@azaruniv.edu
I Introduction

To describe the physics governing the dynamics of scalar and spinor particles the Klein-Gordon and Dirac equations must be solved. In general, solving these equations in (3+1)-dimensional curved background is difficult and a weak field approximation [1] may be required, or asymptotic solutions may be obtained [2]. One can also solve these equations by numerical methods [3], and WKB approximations [4]. An alternative approach is to consider lower-dimensional space-times and obtain exact solutions [5]. This may help us to get a deeper insight into general features of (3+1)-dimensional problems.

In a previous work [6], we have solved exactly the Klein-Gordon equation on a static 2-dimensional space-time, by using the standard techniques of supersymmetric quantum mechanics. The purpose of the present paper is to solve exactly the Dirac equation on a 2-dimensional conformally flat static space-time and then find its energy spectrum. We are interested in solving the Dirac equation through the supersymmetric formalism of quantum mechanics [7, 8].

II Dirac equation on a (1+1)-dimensional Lorentzian manifold

Dirac’s equation in curved space-time requires the use of bein formalism to project the spinors into a Minkowskian local inertial frame [9]. In our notation, the Latin indices refer to local inertial frame with the metric $\eta_{ab}$, while the Greek ones refer to curved space-time with the metric $g_{\mu\nu}$. The zweibeins $e^a_{\mu}$ are used to project the vectors between the two frames, and satisfy the following relation [10]

$$ g_{\mu\nu}(X) = \eta_{ab} e^a_{\mu}(X) e^b_{\nu}(X), $$

(1)
where $X := (t, x)$, and $\eta_{ab} = \text{diag}(+1, -1)$. In flat space-time the Dirac equation is written as

$$i \gamma^a \partial_a \Psi - m \Psi = 0,$$

where the $\gamma$ matrix conventions are

$$\{\gamma^a, \gamma^b\} = 2 \eta^{ab}, \quad \sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b],$$

with $\sigma^{ab}$ being the generator of Lorentz rotations.

Dirac equation is generalized to curved space-time through the spin connections $\omega^b{}_{\mu}$

$$i \gamma^\mu \nabla_\mu \Psi - m \Psi = 0,$$

where

$$\nabla_\mu = \partial_\mu + \frac{1}{2} \sigma^{bc} \omega_{bc\mu}, \quad \omega_{bc\mu} \equiv E^\nu_b E_{c\nu};$$

with the semicolon denoting covariant differentiation, and

$$\gamma^\mu = \gamma^a E^\mu_a, \quad \{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu},$$

with $E^a_\mu$ being the inverse of $e^a_\mu$.

Now, we take the static conformally flat metric $g_{\mu\nu}$

$$ds^2 = e^{\sigma(x)}(dt^2 - dx^2),$$

where $\sigma(x)$ is a dilatonic static field. Using the relation (1), the zweibeins and their inverse corresponding to the metric (4) are respectively obtained

$$e^a_\mu = \begin{pmatrix} e^{\frac{1}{2}\sigma(x)} & 0 \\ 0 & e^{\frac{1}{2}\sigma(x)} \end{pmatrix},$$

$$E^\mu_a = \begin{pmatrix} e^{-\frac{1}{2}\sigma(x)} & 0 \\ 0 & e^{-\frac{1}{2}\sigma(x)} \end{pmatrix}.$$
It is well known that Dirac equation in conformally flat space-time is identical to the Minkowskian one (apart from a conformal factor) once an appropriate transformation is employed on the spinor $\Psi$.

In this regards, one may obtain the (1+1)-dimensional Dirac equation in the curved space-time (4) by employing the following transformations

$$\gamma^a \partial_a \rightarrow e^{-\frac{2}{\sigma^2}} \gamma^a \partial_a,$$

$$\Psi \rightarrow e^{\frac{\sigma}{4}} \Psi,$$

on the Dirac equation (2). Therefore, we obtain

$$[(i\gamma^0 \partial_t + i\gamma^1 \partial_x) + \frac{i}{4} \gamma^1 \frac{d\sigma(x)}{dx} - me^{\frac{1}{2}\sigma(x)}] \Psi(X) = 0,$$

(7)

where we consider $\Psi$ as

$$\Psi(X) = \begin{pmatrix} \Psi_1(X) \\ \Psi_2(X) \end{pmatrix}.$$  

(8)

Following Jackiw and Rebbi [11, 7], we take the following representations for the $\gamma^a$ matrices

$$\gamma^0 = \sigma^1, \quad \gamma^1 = i\sigma^3,$$

(9)

where $\sigma^1, \sigma^3$ are the Pauli spin matrices. If we now operate $\gamma^0$, from left, on both sides of Eq. (7), we have

$$\begin{pmatrix} 0 & -\partial_x - \frac{1}{4} \frac{d\sigma(x)}{dx} + me^{\frac{1}{2}\sigma(x)} \\ \partial_x + \frac{1}{4} \frac{d\sigma(x)}{dx} + me^{\frac{1}{2}\sigma(x)} \\ 0 \end{pmatrix} \Psi(X) = i\partial_t \Psi(X).$$

(10)

III Exact solutions of 2D Dirac equation by SUSY QM methods

In this section, we will find the exact solutions of Dirac equation by using the standard techniques of supersymmetric quantum mechanics. By assuming the time dependence of the two
component spinor $\Psi(X)$ as

$$\Psi(X) = e^{-i\xi t} \psi(x), \hspace{1cm} (11)$$

Eq.(10) can be written as the following eigenvalue equation

$$H_D \psi(x) = \mathcal{E} \psi(x), \hspace{1cm} (12)$$

where $H_D$, and $\psi(x)$ are

$$H_D = \begin{pmatrix} 0 & -\frac{d}{dx} - \frac{1}{4} \frac{d\sigma(x)}{dx} + me^{\frac{1}{2}\sigma(x)} \\ \frac{d}{dx} + \frac{1}{4} \frac{d\sigma(x)}{dx} + me^{\frac{1}{2}\sigma(x)} & 0 \end{pmatrix}, \hspace{1cm} (13)$$

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \hspace{1cm} (14)$$

By defining the transformation matrix

$$U(x) = \begin{pmatrix} e^{\frac{1}{2}\sigma(x)} & 0 \\ 0 & e^{-\frac{1}{2}\sigma(x)} \end{pmatrix}, \hspace{1cm} (15)$$

and doing a similarity transformation on Eq.(12), we have

$$\tilde{H}_D \tilde{\psi}(x) = \mathcal{E} \tilde{\psi}(x), \hspace{1cm} (16)$$

where

$$\tilde{H}_D = U H_D U^{-1} = \begin{pmatrix} 0 & -\frac{d}{dx} + me^{\frac{1}{2}\sigma(x)} \\ \frac{d}{dx} + me^{\frac{1}{2}\sigma(x)} & 0 \end{pmatrix}, \hspace{1cm} (17)$$

and

$$\tilde{\psi}(x) = U \psi(x) = e^{\frac{1}{2}\sigma(x)} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \hspace{1cm} (18)$$

Now, we show that the Dirac equation (16) is equivalent to the spinor field equation which is obtained from the (1+1)-dimensional Lagrangian in the flat space-time

$$\mathcal{L} = i \bar{\Psi} \gamma^{a} \partial_{a} \Psi - \bar{\Psi} \dot{\Psi} W, \hspace{1cm} (19)$$
where $W(x)$ is a static scalar function to be determined later.

The field equation derived from the Lagrangian (19) becomes

$$i\gamma^a \partial_a \tilde{\Psi}(t, x) - W(x) \tilde{\Psi}(t, x) = 0.$$  \hspace{1cm} (20)

By using the representations (9), and considering the time dependence of the spinor as $\tilde{\Psi}(t, x) = e^{-iE_t \tilde{\psi}(x)}$, the equation of motion (20) after multiplying by $\gamma^0$ from left, can be written as

$$\begin{pmatrix}
0 & -\frac{d}{dx} + W(x) \\
\frac{d}{dx} + W(x) & 0
\end{pmatrix} \tilde{\psi}(x) = \mathcal{E} \tilde{\psi}(x).$$ \hspace{1cm} (21)

Now, considering Eqs.(21) and (16), it is seen that they have equivalent mathematical structures. The function $W(x)$ in Eq.(21) is commonly called superpotential [8, 12] in the context of supersymmetric quantum mechanics. By direct comparison of (21) and (16) the conformal factor in the metric (4) is related to the superpotential $W(x)$ through the relation

$$W(x) = me^{\frac{1}{2} \sigma(x)}.$$ \hspace{1cm} (22)

Now we define the operators $\mathcal{A}^\dagger$ and $\mathcal{A}$, respectively as

$$\mathcal{A}^\dagger := -\frac{d}{dx} + W(x),$$ \hspace{1cm} (23)

$$\mathcal{A} := \frac{d}{dx} + W(x).$$ \hspace{1cm} (24)

Therefore, Eq.(21) can be written as

$$\begin{pmatrix}
0 & \mathcal{A}^\dagger \\
\mathcal{A} & 0
\end{pmatrix} \tilde{\psi}(x) = \mathcal{E} \tilde{\psi}(x).$$ \hspace{1cm} (25)

By operating $\tilde{H}_D$, defined by Eq.(17), from left on both sides of Eq.(16) ( or Eq.(25) ) we obtain

$$H_{-} \tilde{\psi}_1(x) = \mathcal{E}^2 \tilde{\psi}_1(x),$$ \hspace{1cm} (26)
where \( H_+ = A^\dagger A \) and \( H_- = AA^\dagger \) are Partner Hamiltonians [8, 12]. Equations (26) and (27) can now be solved by supersymmetric quantum mechanical methods for shape invariant potentials.

For a typical example we study the superpotential associated with the modified harmonic oscillator, namely [8]

\[
W(x) = \frac{1}{2}\omega|x| + c, \tag{28}
\]

where \( \omega \) and \( c \) are real positive quantities. Using equations (22) and (28) the metric (4) is obtained

\[
ds^2 = \left( \frac{\omega|x|}{2m} + \frac{c}{m} \right)^2 (dt^2 - dx^2). \tag{29}
\]

Considering the superpotential (28), the partner Hamiltonians \( H_\pm \) are

\[
\begin{align*}
H_+ &= -\frac{d^2}{dx^2} + \frac{1}{4}\omega^2 x^2 + \frac{1}{2}\omega + \omega xc + c^2 \quad x > 0, \\
H_- &= -\frac{d^2}{dx^2} + \frac{1}{4}\omega^2 x^2 - \frac{1}{2}\omega + \omega xc + c^2 \quad x < 0.
\end{align*}
\]

Now, by inserting \( \tilde{\psi}_i(x) = e^{-\frac{\omega}{2}(x + \frac{2c}{\omega})^2} \phi_i(x) \) \( (i = 1, 2) \) in Eqs.(26), (27) and change of variable \( y = \sqrt{\frac{\omega}{2}}(x + \frac{2c}{\omega}) \) for \( x > 0 \), we find the following differential equations

\[
\begin{align*}
\frac{d^2\phi_1}{dy^2} - 2y\frac{d\phi_1}{dy} + 2\frac{\mathcal{E}^2}{\omega}\phi_1 &= 0, \\
\frac{d^2\phi_2}{dy^2} - 2y\frac{d\phi_2}{dy} + 2\left[\frac{\mathcal{E}^2}{\omega} - 1\right]\phi_2 &= 0.
\end{align*} \tag{31}
\]
respectively. In the same way, inserting \( \tilde{\psi}_i(x) = e^{-\frac{i}{\omega}(x+\frac{2c}{\omega})^2}\phi_i(x) \) \( (i = 1, 2) \) and change of variable \( y = \sqrt{\frac{\omega}{2}}(-x + \frac{2c}{\omega}) \) for \( x < 0 \) leads to

\[
\frac{d^2\phi_1}{dy^2} - 2y\frac{d\phi_1}{dy} + 2\left[ \frac{\mathcal{E}^2}{\omega} - 1 \right]\phi_1 = 0,
\]

\[
\frac{d^2\phi_2}{dy^2} - 2y\frac{d\phi_2}{dy} + 2\frac{\mathcal{E}^2}{\omega}\phi_2 = 0,
\]

respectively. The wave functions \( \tilde{\psi}_1(x) \) and \( \tilde{\psi}_2(x) \) in Eqs.(26) and (27) are given by

\[
\tilde{\psi}_1(x) = \begin{cases} 
  e^{-\frac{i}{\omega}(x+\frac{2c}{\omega})^2}H_n\left(\sqrt{\frac{\omega}{2}}(x + \frac{2c}{\omega})\right) & x > 0 \\
  e^{-\frac{i}{\omega}(x+\frac{2c}{\omega})^2}H_{n-1}\left(\sqrt{\frac{\omega}{2}}(-x + \frac{2c}{\omega})\right) & x < 0,
\end{cases}
\]

\[
\tilde{\psi}_2(x) = \begin{cases} 
  e^{-\frac{i}{\omega}(x+\frac{2c}{\omega})^2}H_{n-1}\left(\sqrt{\frac{\omega}{2}}(x + \frac{2c}{\omega})\right) & x > 0 \\
  e^{-\frac{i}{\omega}(x+\frac{2c}{\omega})^2}H_n\left(\sqrt{\frac{\omega}{2}}(-x + \frac{2c}{\omega})\right) & x < 0.
\end{cases}
\]

where \( H_n \) and \( H_{n-1} \) are Hermite polynomials leading to the following energy spectrum

\[
\mathcal{E}_n = \pm \sqrt{n\omega},
\]

for both \( x < 0 \) and \( x > 0 \), where \( n \) is a non-negative integer number. Now, using Eqs.(11), (18), (33), and (34), the final solution of Dirac equation is obtained

\[
\Psi_n(X) = \begin{cases} 
  a\left(\frac{mx}{2m} + \frac{c}{m}\right)^{-\frac{1}{2}}e^{-\frac{i}{\omega}(x+\frac{2c}{\omega})^2 - i\sqrt{\omega}x} \left( \begin{array}{c} H_n\left(\sqrt{\frac{\omega}{2}}(x + \frac{2c}{\omega})\right) \\ H_{n-1}\left(\sqrt{\frac{\omega}{2}}(-x + \frac{2c}{\omega})\right) \end{array} \right) + C.C & x > 0, \\
  b\left(\frac{mx}{2m} + \frac{c}{m}\right)^{-\frac{1}{2}}e^{-\frac{i}{\omega}(x+\frac{2c}{\omega})^2 - i\sqrt{\omega}x} \left( \begin{array}{c} H_{n-1}\left(\sqrt{\frac{\omega}{2}}(-x + \frac{2c}{\omega})\right) \\ H_n\left(\sqrt{\frac{\omega}{2}}(x + \frac{2c}{\omega})\right) \end{array} \right) + C.C & x < 0,
\end{cases}
\]

where \( a \) and \( b \) are complex constants and \( C.C \) means complex conjugation.

We note that the ansatz (11) does not work for \( n = 0 \) quantum number. Therefore, we restrict ourselves to the positive integer for the quantum numbers, \( n \). This gives rise to a broken supersymmetry [8]. The continuity condition for the spinor wave function (36) at
$x = 0$ implies

\[
\begin{align*}
    a H_n(\lambda) &= b H_{n-1}(\lambda) \\
    a H_{n-1}(\lambda) &= b H_n(\lambda) \\
    a^* H_n(\lambda) &= b^* H_{n-1}(\lambda) \\
    a^* H_{n-1}(\lambda) &= b^* H_n(\lambda),
\end{align*}
\]  

(37)

where the positive constant $\lambda$ is defined as follows

\[
\lambda = c \sqrt{\frac{2}{\omega}}.
\]  

(38)

Eqs.(37) have non-trivial solutions if

\[
\begin{align*}
    H_n(\lambda) &= H_{n-1}(\lambda) \\
    \text{or} \\
    H_n(\lambda) &= -H_{n-1}(\lambda).
\end{align*}
\]  

(39)

For example when $n = 1$, according to Eqs.(37), the allowed value for $\lambda$ is \( \frac{1}{2} \left( c = \frac{1}{2} \sqrt{\frac{2}{\omega}} \right) \), and for $n = 2$ the corresponding allowed values are $\lambda = 1 \left( c = \sqrt{\frac{\omega}{2}} \right)$ and $\lambda = \frac{1}{2} \left( c = \frac{1}{2} \sqrt{\frac{\omega}{2}} \right)$.

Therefore, the continuity of spinor wave function (36) at $x = 0$ requires, for each mode $n$, a relation between the constants $\omega$ and $c$ in Eq.(28).

**Acknowledgment**

The authors would like to thank the referee for useful comments. This work has been supported by the Research Institute for Fundamental Sciences, Tabriz, Iran.

**References**

[1] J. F. Donoghue and B. R. Holstein, Am. J. Phys. 54, 827 (1986).

[2] D. G. Boulware, Phys. Rev. D11, 1404 (1975), Phys. Rev. D12, 350 (1975).

[3] D. N. Page, Phys. Rev. D16, 2402 (1977).
[4] M. Martellini, Phys. Rev. D16, 3418 (1977).

[5] R. B. Mann, S. Morsink, A. E. Sikkema and T. G. Steele, Phys. Rev. D43, 3948 (1991).

[6] S. K. Moayedi and F. Darabi, J. Math. Phys. 42, 1229 (2001).

[7] F. Cooper, A. Khare, R. Musto and A. Wipf, Ann. Phys. 187, 1 (1988).

[8] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).

[9] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[10] R. A. Bertlmann, *Anomalies in Quantum Field Theory* (Oxford University press, New York, 2000).

[11] R. Jackiw and C. Rebbi, Phys. Rev. D13, 3358 (1976).

[12] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics* (Springer-Verlag, Berlin 1996).