DIFFEOMORPHISM TYPE OF SYMPLECTIC FILLINGS OF
UNIT COTANGENT BUNDLES

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Abstract. We prove uniqueness, up to diffeomorphism, of symplectically aspherical fillings of certain unit cotangent bundles, including those of higher-dimensional tori.

1. Introduction

We consider the cotangent bundle $T^*L$ of a closed, connected manifold $L$ with its Liouville form $p\, dq$, which induces a contact form on the unit cotangent bundle $ST^*L$ (with respect to any Riemannian metric on $L$). In what follows, it will always be understood that $ST^*L$ is endowed with this natural contact structure. The unit disc bundle $(DT^*L, dp \wedge dq)$, equipped with its canonical symplectic form, is a strong symplectic (in fact, Stein) filling of $ST^*L$. For basic notions of different types of fillings see [5, Chapter 5].

It is a fundamental question in symplectic topology in how many ways the contact manifold $ST^*L$ can be written as the boundary of a symplectic filling $(W, \omega)$. For $ST^*S^2 = \mathbb{R}P^3$ it was shown by McDuff [15] that the unit disc bundle is the unique symplectically aspherical (or minimal) strong filling up to diffeomorphism, and up to symplectomorphism if one fixes the cohomology class of the symplectic form. For Stein fillings of $ST^*S^2$, uniqueness up to Stein homotopy was established by Hind [10]. Stipsicz [20] showed that all Stein fillings of $ST^*T^2 = T^3$ are homeomorphic to the unit disc bundle $DT^*T^2 = D^2 \times T^2$. Wendl [22] strengthened that last result to uniqueness up to diffeomorphism. In fact, he proved uniqueness up to deformation equivalence for all minimal symplectic fillings of $T^3$. Regarding higher genus surfaces, Sivek and Van Horn-Morris [19] showed that every Stein filling of $ST^*\Sigma_g, g \geq 2$, is $s$-cobordant rel boundary to the unit disc bundle $DT^*\Sigma_g$; see also [13].

These 4-dimensional results are typically based on techniques not available in higher dimensions, such as foliations by holomorphic curves.

The question about the analogue in higher dimensions of the result for $T^2$ was posed to us by Otto van Koert and András Stipsicz, and we answer it in the following theorem. From now on, ‘filling’ without any further specification always means strong symplectic filling.

Theorem 1.1. The diffeomorphism type of symplectically aspherical fillings of $ST^*T^n$ is unique.

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Remark 1.2. While the present paper was in preparation, Theorem 1.1 has independently been obtained by Bowden–Gironella–Moreno [2] as an application of their extensive study of Bourgeois contact structures, which they use to replace some parts of Wendl’s argument and then combine with results from [1]. Our proof, by contrast, is based directly on a refinement of the techniques developed in [1]. As we shall explain, our approach establishes uniqueness of fillings of unit cotangent bundles for a considerably larger class of base manifolds. In the latest version of their paper, Bowden–Gironella–Moreno more closely follow our approach and also use holomorphic spheres rather than punctured holomorphic discs; see [2, Remark 3].

The earliest results about the diffeomorphism type of symplectic fillings in higher dimensions (after Gromov’s work [9] in dimension four) are due to Eliashberg–Floer–McDuff [10]. By treating evaluation maps on moduli spaces of holomorphic curves with methods from algebraic topology, they proved that every symplectically aspherical filling of the sphere $S^{2n-1}$ equipped with the standard contact structure is diffeomorphic to the disc $D^{2n}$. Uniqueness of these fillings up to symplectomorphism is known only for $n = 1, 2$.

Starting point for the classification of fillings is an understanding of homological restrictions. Oancea–Viterbo [18] showed that the inclusion of a subcritically Stein fillable contact manifold into any of its symplectically aspherical fillings induces a surjection in homology. Ghiggini–Niederkrüger–Wendl [8] found obstructions on the relative homology of semi-positive symplectic fillings in terms of belt spheres of subcritical handles.

The degree method from [1] systematically combines the filling by holomorphic curves technique with the s-cobordism theorem and yields uniqueness, up to diffeomorphism, of subcritical Stein fillings for a wide range of contact manifolds. In [11] it was shown that the subcriticality assumption can be dropped if instead one requires the existence of a complex hypersurface in the filling having a suitable intersection behaviour. As an application, it was shown there that for $ST^*S^{2d-1}$ such types of fillings are unique up to diffeomorphism. Critical fillings were also studied by Lazarev [12], who proved uniqueness results up to symplectomorphism (after completion) for certain classes of flexible fillings.

In the present paper we show that the subcriticality assumption on the filling made in [1] can be dropped in situations where suitable topological information is available on the manifold that is to be filled. Our arguments apply to unit cotangent bundles $ST^*L$ of closed, connected manifolds $L$ that admit a Lagrangian embedding into a subcritical Stein manifold. Assuming that the second relative homology of a symplectic filling $(W, \omega)$ of $ST^*L$ vanishes, we first prove homological uniqueness of the filling, i.e. that $H_*(W)$ is isomorphic to $H_*(DT^*L)$. This is the content of Theorem 2.2 where the situation is analysed in a slightly more general setting. If in addition $L = Q \times S^1$ with $\chi(Q) = 0$, we show in Theorem 2.7 that this homology isomorphism is induced by an embedding $DT^*L \to W$.

By a result of Chekanov [3], the fundamental group of a Lagrangian submanifold in a subcritical Stein manifold contains an element of infinite order, see Proposition 2.5. So it is quite natural to consider manifolds $L = Q \times S^1$ that split off a circle factor. There is an obvious Lagrangian embedding of $L$ into the subcritical manifold $T^*Q \times \mathbb{C}$ that allows filling by holomorphic curves. Using a filling by
holomorphic annuli we show that the inclusion \( L \to W \) is surjective on \( \pi_1 \), see Theorem 3.1. A filling by infinitely long holomorphic strips can be used to show that the lifted inclusion \( \tilde{L} \to \tilde{W} \) of universal covers — which exists when the inclusion \( ST^*L \to W \) is \( \pi_1 \)-injective — is surjective on \( H_\ast \), see Proposition 4.1. Arguments parallel to [1] then lead us to the main result of the present paper.

**Theorem 1.3.** (a) Suppose that \( Q \) is a closed manifold of dimension at least three, with Euler characteristic \( \chi(Q) = 0 \), satisfying one of the following assumptions:

(i) \( Q \) is a product manifold of the form \( Q = N \times F \), where \( F \) is a surface different from \( S^2 \) and \( \mathbb{R}P^2 \).

(ii) \( Q \) is aspherical.

Then any Stein filling of \( ST^*(Q \times S^1) \) is homotopy equivalent to \( DT^*(Q \times S^1) \).

(b) If \( Q \) is a product of unitary groups and spheres, including at least one \( S^1 \)-factor, then any symplectically aspherical filling of \( ST^*(Q \times S^1) \) is diffeomorphic to \( DT^*(Q \times S^1) \).

Part (a) of this theorem will be proved in Section 4.1; part (b), in Section 4.2. Theorem 1.1 is an obvious corollary of part (b). Actually, we are going to prove more general statements in Theorems 4.3 and 4.4. In the theorem above we only listed the most obvious examples illustrating those theorems.

The main technical innovations in this paper that allow us to go beyond the results of [1] are the moving complex hypersurface argument in Section 3.1 and the analysis of infinite holomorphic strips in Section 3.2.

A brief word on notation: We write \( D_r \) for the closed disc of radius \( r \) in the complex plane \( \mathbb{C} \), centred at 0. The open disc will be denoted by \( B_r \), that is, \( B_r = \text{Int}(D_r) \).

2. **Domains in subcritical Stein manifolds**

The homology of symplectically aspherical fillings of contact manifolds that are subcritically Stein fillable is unique, see [1, Theorem 1.2]. As we shall see in this section, uniqueness of homology holds also for all symplectically aspherical fillings of contact type hypersurfaces in subcritical Stein manifolds that are not necessarily a level set of a corresponding plurisubharmonic function. Examples are given by the boundaries of Weinstein tubular neighbourhoods of closed Lagrangian submanifolds, which by [1, Proposition 3.9] are not subcritically Stein fillable.

2.1. **The Oancea–Viterbo argument revisited.** Let \((M_Z, \xi_Z)\) be a \((2n - 1)\)-dimensional contact manifold that admits a subcritical Stein filling \((Z, \omega_Z)\). This means that the plurisubharmonic Morse function given by the Stein structure does not have any critical points of index \( n \).

Consider a closed, connected contact type hypersurface \((M, \xi)\) in \((Z, \omega_Z)\), disjoint from \( \partial Z \). Observe that \( M \) is separating because \( H_{2n-1}(Z) = 0 \). Denote by \( D_Z \subset Z \) the closure of the component of \( Z \setminus M \) not containing \( \partial Z \).

**Remark 2.1.** As shown in [1, Theorem 3.4], the contact manifold \((M, \xi)\) is a convex boundary of the symplectic manifold \((D_Z, \omega_Z|_{D_Z})\), so the latter constitutes a symplectic filling. Alternatively, one may appeal to [21, Remark 3.3], which gives an elementary argument. In order to apply either reference in the present setting, one needs to appeal to Cieliebak’s splitting theorem for subcritical Stein manifolds [1, Section 14.4].
The symplectic form \( \omega_Z|_{D_Z} \) is exact, but it need not be of Liouville type, i.e. there need not be a primitive 1-form of \( \omega_Z|_{D_Z} \) that restricts to a contact form for \( \xi \) on the boundary \( M \).

**Theorem 2.2.** Let \( (W, \omega) \) be a symplectic filling of \( (M, \xi) \) that satisfies one of the following conditions:

(i) \( H_2(W, M) = 0 \);

(ii) \( M \) is simply connected and \( (W, \omega) \) symplectically aspherical;

(iii) \( H^1(M; \mathbb{R}) = 0 \) and \( \omega \) is exact.

Then \( W \) and \( D_Z \) have isomorphic homology.

**Remark 2.3.** (1) Under assumption (i) or (iii) of the theorem, \( (W, \omega) \) is symplectically aspherical for obvious (co-)homological reasons.

(2) Under assumption (i) or (ii), Oancea–Viterbo proved in [18] that the inclusion map \( M \rightarrow W \) induces a surjection in homology, cf. [18] Theorem 3.2.

(3) It was shown in [1, Proposition 3.5] that the normal subgroup in \( \pi_1(W) \) generated by the image of \( \pi_1(M) \) in \( \pi_1(W) \) is equal to \( \pi_1(W) \). This group-theoretical property by itself does not imply that the inclusion map \( M \rightarrow W \) induces a surjection of fundamental groups. For example, the normal closure of any non-trivial subgroup of the alternating group of degree five \( A_5 \), such as the Klein four-group, is equal to the full group \( A_5 \), because \( A_5 \) is a simple group.

**Proof of Theorem 2.2.** We define a new symplectic manifold

\[
(W_Z, \Omega_Z) := (W, \omega) \cup_{(M, \xi)} (Z \setminus \text{Int}(D_Z), \omega_Z)
\]

by replacing \( D_Z \) in \( Z \) with \( W \). Any of the assumptions (i)–(iii) in Theorem 2.2 implies that \( (W_Z, \Omega_Z) \) is a symplectically aspherical filling of \( (M_Z, \xi_Z) \): The argument for assumptions (i) and (ii) is given in [1] Remark 3.3 (2) and Lemma 3.4; in case (iii) a straightforward argument in de Rham theory shows that \( \Omega_Z \) is globally exact. In fact, in this last case one can find a global primitive for \( \Omega_Z \) that restricts to a contact form for \( \xi \) on \( M \), so \( (M, \xi) \) is of restricted contact type in \( (W_Z, \Omega_Z) \).

Therefore, in all three cases we can appeal to [1] Theorem 1.2 to conclude that \( H_k(W_Z) \cong H_k(Z) \) for all \( k \in \mathbb{Z} \).

Denote by \( \ell \) the dimension of the CW complex obtained from \( Z \) by following the negative gradient flow of the plurisubharmonic Morse function of the Stein structure on \( Z \). Since the Stein structure is assumed to be subcritical, we have \( \ell \leq n - 1 \). Then \( H_k(W_Z) \) is trivial for \( k \geq \ell + 1 \), and a free abelian group for \( k = \ell \), since there are no \( n \)-cells. The Mayer–Vietoris sequence yields

\[
H_k(W) \oplus H_k(W \setminus W) \cong H_k(M) \cong H_k(D_Z) \oplus H_k(Z \setminus D_Z) \quad \text{for } k \geq \ell + 1.
\]

Since \( W_Z \setminus W = Z \setminus D_Z \) it follows that \( H_k(W) \cong H_k(D_Z) \) for \( k \geq \ell + 1 \).

It remains to prove \( H_k(W) \cong H_k(D_Z) \) for \( k = 0, 1, \ldots, \ell \). We first observe, by combining Poincaré duality and excision in cohomology, that

\[
H_k(W) \cong H^{2n-k}(W_Z, W_Z \setminus W).
\]

Secondly, since \( \ell \leq n - 1 \), the condition \( k \leq \ell \) translates into \( 2n - 1 - k \geq \ell + 1 \). By using the universal coefficient theorem in the form \( H^i = FH_i \oplus TH_{i-1} \), with \( F, T \) denoting the free and the torsion part, respectively, we see that the cohomology groups of \( W_Z \) vanish in degree \( 2n - 1 - k \) and \( 2n - k \). Hence, the connecting homomorphism

\[
H^{2n-1-k}(W_Z \setminus W) \rightarrow H^{2n-k}(W_Z, W_Z \setminus W)
\]
of the cohomology long exact sequence for the pair \((W_Z, W_Z \setminus W)\) is an isomorphism.

Combining these two observations, we have

\[ H_k(W) \cong H^{2n-1-k}(W_Z \setminus W) \quad \text{for } k = 0, 1, \ldots, \ell. \]

The same argument yields \( H_k(D_Z) \cong H^{2n-1-k}(Z \setminus D_Z) \) for \( k = 0, 1, \ldots, \ell \). Since \( W_Z \setminus W \) and \( Z \setminus D_Z \) coincide, we conclude that \( H_k(W) \cong H_k(D_Z) \) for \( k = 0, 1, \ldots, \ell \).

\[ \square \]

2.2. Weinstein neighbourhoods. Examples to which Theorem 2.2 applies can be obtained by intrinsic Weinstein surgery. But the most prominent applications of Theorem 2.2 are Weinstein tubular neighbourhoods of closed Lagrangian submanifolds \( L \subset Z \) in a subcritical Stein manifold \( Z \). The contact type hypersurface in question is the unit cotangent bundle \( M = ST^*L \) for some metric on \( L \), with corresponding domain \( D_Z = DT^*L \). In this situation Theorem 2.2 (i) implies that the homology type of any symplectic filling \( W \) of \( M \) is the one of \( L \), provided \( H_2(W, M) = 0 \). Notice that for \( \dim L = n \geq 3 \) we have \( H_2(DT^*L, ST^*L) = 0 \). The following proposition says that such fillings \( W \) do not exist for \( n = 2 \).

**Proposition 2.4.** For \( L \) a closed (possibly non-orientable) surface and \( M = ST^*L \subset Z \) as described, there is no symplectic filling \( W \) of \( M \) with \( H_2(W, M) = 0 \).

**Proof.** Suppose \( M \) admitted a symplectic filling \( W \) with \( H_2(W, M) = 0 \). Then, by Theorem 2.2 we have

\[ 0 = H_2(W, M) \cong H^2(W) \cong FH_2(W) \oplus TH_1(W) \cong FH_2(L) \oplus TH_1(L). \]

The condition \( FH_2(L) = 0 \) would force \( L \) to be non-orientable, but then \( TH_1(L) = \mathbb{Z}_2 \), so this is not possible. \[ \square \]

Concerning cases (ii) and (iii) of Theorem 2.2, the following proposition says that these are irrelevant for the filling of unit cotangent bundles. In particular, there are specific topological restrictions on manifolds that can be realised as Lagrangian submanifolds in a subcritical Stein manifold.

**Proposition 2.5.** Conditions (ii) and (iii) in Theorem 2.2 are never satisfied for \( M = ST^*L \).

**Proof.** By a result of Chekanov [3] there exists a non-constant holomorphic disc \( \Delta \) in \((Z, \omega_Z)\) with boundary \( \partial \Delta \) on \( L \). The reason is the following. The Lagrangian submanifold \( L \) is displaceable in the completion \( \hat{Z} \) of \( Z \), since \( \hat{Z} \) symplectically splits off a \( \mathbb{C} \)-factor by Cieliebak’s splitting theorem, see [4] Section 14.4]. There are no non-constant holomorphic spheres in \( \hat{Z} \) by exactness of the Stein symplectic form. If there were no non-constant holomorphic disc in \((Z, \omega_Z)\) with boundary on \( L \), the energy bound on non-displacing symplectomorphisms in the main theorem of [3] would be infinite, i.e. \( L \) would not be displaceable under any symplectomorphism.

Let \( \lambda \) be a primitive 1-form for the symplectic form \( \omega_Z \). Then the 1-form \( \lambda_L := \lambda_{|T_L} \) is closed (\( L \) being Lagrangian), and it represents a non-trivial class in \( H^1(L; \mathbb{R}) \), since it integrates non-trivially over \( \partial \Delta \) by the theorem of Stokes. Thus, condition (iii) is violated.

Further, the circle \( \partial \Delta \subset L \) represents an element of infinite order in \( \pi_1(L) \), and this lifts to an element of infinite order in \( \pi_1(M) \), which violates condition (ii). \[ \square \]
2.3. **The split situation.** The isomorphism between the homology of $D_Z$ and that of $W$ in Theorem 2.2 is in some sense natural on a formal algebraic level, but it is not, in general, induced by a map between these two manifolds. In this and the following section we describe a situation where it is.

Let $Q$ be a closed, connected $(n-1)$-dimensional manifold. The manifold $L = Q \times S^1$ embeds as a Lagrangian submanifold into $T^*Q \times \mathbb{C} =: (Z, \omega_Z)$, by first embedding $L$ as the zero section into $T^*L = T^*Q \times T^*S^1$, followed by the embedding of the factor $T^*S^1$ into $\mathbb{C}$ as a neighbourhood of the unit circles $S^1 \subset \mathbb{C}$. In $T^*L \subset Z$ we consider the contact type hypersurface $M = ST^*L = \partial(DT^*L)$, and we form the symplectic manifold $(W \subset Z, \Omega_Z)$ as in (1) by replacing $DT^*L =: D_Z$ with a given symplectic filling $(W, \omega)$ of $M$. This is illustrated in Figure 1. where the lightly shaded solid torus represents $DT^*L$, which is being replaced by $W$. Beware that the latter no longer has a rotational symmetry, in general, hence the note of warning in Figure 1.

![Figure 1. The setting of Theorem 2.7.](image)

Thanks to the splitting $L = Q \times S^1$, we have global sections of $T^*L$ of the form $Q \times S^1 \times \{t\}$, where $Q$ is identified with the zero section in $T^*Q$, and $S^1 \times \{t\} \subset S^1 \times \mathbb{R} = T^*S^1$. This allows us to find inside $DT^*L$ a smaller copy $W_0$ of itself which intersects $ST^*L$ in a copy of $Q \times S^1$. This copy $W_0$ may be assumed to sit inside a thin tubular neighbourhood of $ST^*L \subset DT^*L$, and hence can also be found inside a collar of the manifold $W$ replacing $DT^*L$, as indicated in Figure 1.

2.4. **A homology equivalence in the split situation.** For better reference, we formulate the main hypothesis of the following theorem separately. We say that hypothesis (H) is satisfied if one of the following conditions holds, where we recall that $M = ST^*(Q \times S^1)$.

(H-i) $H_2(W, M) = 0$ and $\chi(Q) = 0$;
(H-ii) $Q = S^1 \times N$ with $N$ a manifold of dimension at least 1, and $(W, \omega)$ is symplectically aspherical.

**Remark 2.6.** Under condition (H-i), $\dim Q \geq 2$ follows from Proposition 2.4.
Theorem 2.7. In the situation as just described, and under the assumption that hypothesis (H) is satisfied, the embedding $W_0 \to W$ induces isomorphisms on homology.

Proof. Up to homotopy, it may be assumed that $W_0$ touches the boundary of $W$ from the inside along a tubular neighbourhood of $Q \times S^1 \subset ST^*L$. This $Q \times S^1$ bounds $Q \times D^2$ in $Q \times \mathbb{C}$. By gluing in a smaller and a larger thickening of this $Q \times D^2$ inside $T^*Q \times \mathbb{C}$ we obtain the manifolds $W'_0$ and $W'$ as shown in Figure 2. The manifold $W'_1$ is obtained as a further thickening of $W'$ in the radial direction of $\mathbb{C}$ and the fibres of $T^*Q$.

Thanks to $\chi(Q) = 0$ (or even $Q = S^1 \times N$) we have a nowhere vanishing section of $T^*Q$, which allows us to isotope $W'_0$ inside $W'_1$ to a position as shown in Figure 3 by first shrinking it in the $\mathbb{C}$-direction such that it becomes positioned inside the ‘neck’ formed by $DT^*(Q \times S^1)$ in $T^*Q \times \mathbb{C}$.

Remark 2.8. Hypothesis (H) is required to guarantee that in the relevant moduli space of holomorphic spheres there is no bubbling off. For further details see Remark 3.3 below.
Now, in Figure 2, consider the splittings (up to homotopy)

\[ W'_0 = W_0 \cup T^*Q \times S^1 \times D^2 \]

and

\[ W' = W \cup T^*Q \times S^1 \times D^2. \]

This leads to the commutative diagram

\[
\begin{array}{ccc}
H_{k+1}(W'_0) & \rightarrow & H_k(T^*Q \times S^1) \\
\downarrow & & \downarrow \\
H_k(W_0) & \oplus & H_k(T^*Q \times D^2) \\
\downarrow & & \downarrow \\
H_{k+1}(W') & \rightarrow & H_k(W) \oplus H_k(T^*Q \times D^2)
\end{array}
\]

of Mayer–Vietoris sequences (continuing horizontally on either side). By the five-lemma, the homomorphism \( i_* : H_k(W_0) \rightarrow H_k(W) \) induced by inclusion is an isomorphism for all \( k \in \mathbb{Z} \).

\[ \square \]

2.5. **A cobordism.** Still under the assumptions of Theorem 2.7, and after pushing \( W_0 \) a little into the interior of \( W \), we set \( X = W \setminus \text{Int}(W_0) \). This defines a cobordism \( M_0, X, M \) between two copies \( M_0 \backsimeq M \) of \( ST^*L \).

**Lemma 2.9.** The boundary inclusions \( M_0 \rightarrow X \) and \( M \rightarrow X \) induce isomorphisms on homology.

**Proof.** The excision isomorphism gives \( H_k(X, M_0) \cong H_k(W, W_0) \), and the latter relative homology groups vanish for all \( k \) by Theorem 2.7. The relative cohomology groups \( H^k(X, M) \) then vanish by Poincaré duality, and hence so do the relative homology groups \( H_k(X, M) \) by the universal coefficient theorem.

\[ \square \]

The proof of [1, Lemma 5.1] shows that if the inclusion \( M \rightarrow W \) is \( \pi_1 \)-isomorphic, then so are the inclusions \( M_0, M \rightarrow X \). The bundle projection \( ST^*L \rightarrow L \) is \( \pi_1 \)-isomorphic by the homotopy long exact sequence of the fibration. Moreover, since this bundle has a section by our assumption that \( L \) equals \( Q \times S^1 \), we have an inclusion \( L \rightarrow W \) that factors through \( ST^*L = M = \partial W \). Hence, the \( \pi_1 \)-isomorphism of the inclusion \( M \rightarrow W \) reduces to the inclusion \( L \rightarrow W \) being \( \pi_1 \)-isomorphic.

3. **Holomorphic curves analysis**

We continue with the setup as described in Section 2.3 and illustrated in Figure 1.

3.1. **The fundamental group.** As explained in Remark 2.3, the fundamental group of a symplectically aspherical filling \( (W, \omega) \) of a given contact manifold \( (M, \xi) \) cannot, in general, be determined without some specific assumptions on \( (M, \xi) \). The following theorem deals with the situation at hand, i.e. the case \( (M, \xi) = ST^*L \), where \( L = Q \times S^1 \).

**Theorem 3.1.** Let \( Q \) be a closed, connected \((n - 1)\)-dimensional manifold, \( n \geq 3 \). Set \( L = Q \times S^1 \). Let \( (W, \omega) \) be a symplectic filling of \( (M, \xi) = ST^*L \) satisfying hypothesis (H). Then the inclusion \( M \rightarrow W \) induces a surjection of fundamental groups.
Remark 3.2. (1) For this and other statements concerned only with $\pi_1$, one may drop the condition $\chi(Q) = 0$ from hypothesis (H-i). For all statements about higher homology groups, it will be required.

(2) If $\pi_1(Q)$ (and hence $\pi_1(M)$) is abelian, then under the assumptions of the theorem the inclusion $M \to W$ is even $\pi_1$-isomorphic, since $\pi_1$-surjectivity guarantees that $\pi_1(W)$ is likewise abelian, so the fundamental group of $M$ and $W$ coincides with the respective first homology group. On $H_1$, the inclusion $M \to W$ is an isomorphism by Theorem 2.7, since for $n \geq 3$ the first homology group of $ST^*L$ coincides with that of $L \simeq DT^*L = W_0$.

Proof of Theorem 3.1. Equip $T^*Q$ with a compatible almost complex structure that is cylindrical in the complement of $Int(DT^*Q)$. Other choices are possible; all that matters to us is that we have an almost complex structure for which the maximum principle holds in fibre direction. As before, we embed $DT^*L$ into $T^*Q \times \mathbb{C}$, and we compactify the $\mathbb{C}$-factor to $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ with the Fubini–Study symplectic form and its natural complex structure.

Define a symplectic manifold

\[(\hat{Z}, \hat{\Omega}) := (W, \omega) \cup_{(M, \zeta)} ((T^*Q \times \mathbb{CP}^1) \setminus Int(DT^*L)).\]

(2) Compared with the definition of $WZ$ in [1], this amounts to a partial compactification by the hyperplane $H_\infty := T^*Q \times \{\infty\}$. Choose a compatible almost complex structure on $(\hat{Z}, \hat{\Omega})$ that is generic on $Int(W)$, and the product almost complex structure coming from $T^*Q \times \mathbb{CP}^1$ on the complement of $Int(W)$.

Consider the moduli space $\mathcal{M}$ of holomorphic spheres $u: \mathbb{CP}^1 \to \hat{Z}$ subject to the following conditions, where we fix a real number $\varrho \gg 1$:

- (M1) $[u] = [v \times \mathbb{CP}^1]$ in $H_2(\hat{Z})$ for some $v \in T^*Q \setminus DT^*Q$;
- (M2) $u(z) \in H_z := T^*Q \times \{z\}$ for $z \in \{0, \varrho, \infty\} \subset \mathbb{CP}^1$.

This places us in the situation of [1] Section 2, except for an inessential difference in the choice of the hyperplanes in the 3-point condition (M2).

Remark 3.3. The moduli space $\mathcal{M}$ is an oriented $(2n - 2)$-dimensional manifold, cf. [6] [17] and [1] Section 2.2. If $H_2(W, M) = 0$, i.e. under condition (H-i), the manifold $(WZ, \Omega_Z)$ is symplectically aspherical thanks to [1] Lemma 3.4, where it is shown that the gluing of a symplectically aspherical manifold with vanishing relative $H_2$ (here: $W$) and an exact symplectic manifold (here: $(T^*Q \times \mathbb{C}) \setminus Int(DT^*L)$) is symplectically aspherical. Then one argues exactly as in [1] Proposition 2.3 to see that there is no bubbling off. In Lemma 3.4 below we shall give an argument that excludes bubbling off under assumption (H-ii).

Therefore, the evaluation map

$$ev: \mathcal{M} \times \mathbb{CP}^1 \to \hat{Z}, \quad (u, z) \mapsto u(z)$$

is proper. Moreover, the degree of the evaluation map is 1, because any $u \in \mathcal{M}$ whose image $u(\mathbb{CP}^1)$ is disjoint from $Int(W)$ is of the form $u(z) = (v_0, z)$, $z \in \mathbb{CP}^1$, for some $v_0 \in T^*Q$ by the maximum principle.

Let $Z^*$ be the space obtained by removing the complex hypersurfaces $H_0$ and $H_\infty$ from $\hat{Z}$ (or by removing $H_0$ from $WZ$), i.e.

\[(3) \quad Z^* = \hat{Z} \setminus (H_0 \cup H_\infty) = WZ \setminus H_0.\]
Observe that $Z^*$ deformation retracts onto $W$. By positivity of intersections, we have $u^{-1}(H_{\infty}) = \{\infty\}$ and $u^{-1}(H_0) = \{0\}$ for all $u \in \mathcal{M}$. Therefore, the restriction of the evaluation map to $\mathbb{C}^*$, i.e. the map $\mathcal{M} \times \mathbb{C}^* \to Z^*$, is well defined, proper and of degree 1.

Since the intersection number $\langle u \rangle H_0$ equals 1, positivity of intersections tells us that each $u \in \mathcal{M}$ intersects $H_0$ transversely, so that the vector space

$$T_{(u,0)} \text{ev}(\{0\} \oplus T_0\mathbb{CP}^1) = T_0\text{ev}(T_0\mathbb{CP}^1)$$

is a complex line in $T_{u(0)} \mathbb{Z}$ transverse to $T_{u(0)} H_0$. Even though $\mathcal{M}$ is not compact, the spheres $u \in \mathcal{M}$ that intersect $H_0$ outside a certain compact region are standard. Therefore, for $\varepsilon > 0$ sufficiently small, the preimage $\text{ev}^{-1}(T^*Q \times D_{\varepsilon})$, where $D_{\varepsilon} \subset \mathbb{C}$ denotes the closed disc of radius $\varepsilon$ centred at the origin, is a product tubular neighbourhood of $\text{ev}^{-1}(H_0) = \mathcal{M} \times \{0\}$ in $\mathcal{M} \times \mathbb{CP}^1$. It follows that the inclusion map

$$\text{ev}^{-1}(T^*Q \times \partial D_{\varepsilon}) \to \mathcal{M} \times \mathbb{C}^*$$

is a homotopy equivalence.

Now consider the following commutative diagram, where the vertical arrows are inclusion maps:

$$\begin{array}{ccc}
\text{ev}^{-1}(T^*Q \times \partial D_{\varepsilon}) & \xrightarrow{\text{ev}} & T^*Q \times \partial D_{\varepsilon} \\
\uparrow \cong & & \uparrow \iota \\
\mathcal{M} \times \mathbb{C}^* & \xrightarrow{\text{ev}} & Z^*.
\end{array}$$

As a map of degree 1, the evaluation map at the bottom of the diagram is surjective on $\pi_1$, cf. [1 Section 2.5]. It follows that $\iota$ is surjective on $\pi_1$. Up to homotopy, this map $\iota$ may be regarded as the inclusion of a section $Q \times S^1 \subset ST^*(Q \times S^1) = M$ into $W \subset Z^*$, which factors through the inclusion $M \to W$. Hence, that last inclusion must likewise be $\pi_1$-surjective.

Observe that by [1 Lemma 2.1], any $u \in \mathcal{M}$ with $u(\mathbb{CP}^1) \cap \text{Int}(W) = \emptyset$ is of the form $u(z) = (v_0, z)$, $z \in \mathbb{CP}^1$, for some $v_0 \in T^*Q$. Therefore, in order to show that the evaluation map $ev : \mathcal{M} \times \mathbb{CP}^1 \to \mathbb{Z}$ is proper under assumption (H-ii) of Theorem [3.1] (which then completes the proof of that theorem), it suffices to establish the following statement.

**Lemma 3.4.** Under assumption (H-ii) of Theorem [3.1] any sequence of holomorphic spheres in $\mathcal{M}$ that intersect $\text{Int}(W)$ non-trivially has a $C^\infty$-converging subsequence.

**Proof.** There exists a subsequence of the given sequence that Gromov-converges to a stable map $(u^\alpha)_{1 \leq \alpha \leq N}$. We need to show that $N = 1$.

We may assume that $u^1 \cdot H_{\infty} = 1$ and $u^j \cdot H_{\infty} = 0$ for $j = 2, \ldots, N$. If in addition $u^j \cdot H_0 = 1$, then $u^j \cdot H_0 = 0$ for $j = 2, \ldots, N$, which by positivity of intersection implies that these latter spheres are disjoint from $H_0$, and hence homotopic to spheres in $W$. In fact, the maximum principle forces these spheres to be contained in $W$. As $(W, \omega)$ is symplectically aspherical, it does not contain spheres of positive $\omega$-energy, which implies $N = 1$. 


The only other possibility is that, after reindexing the bubble sphere \( s \), we have \( u^2 \cdot H_0 = 1 \) and \( u^j \cdot H_0 = 0 \) for \( j \neq 2 \), see Figure 4. This leads to the conclusion \( N = 2 \) by a similar argument.

Our aim is to rule out this second case. Arguing by contradiction, we assume that we have such a stable map \((u^1, u^2)\). We then observe the following consequences of the maximum principle (in \( T^*Q \times \mathbb{C} \) or separately in the factors \( T^*Q \) and \( \mathbb{C} \)). Notice that because of \( u^2 \cdot H_\infty = 0 \) and positivity of intersection, the sphere \( u^2 \) is disjoint from the hyperplane \( H_\infty \).

(1) The sphere \( u^2 \) intersects \( W \), since there are no non-constant holomorphic sphere in \( T^*Q \times \mathbb{C} \).

(2) The sphere \( u^2 \) is disjoint from \((T^*Q \setminus DT^*Q) \times \mathbb{C}P^1\).

(3) The sphere \( u^2 \) does not intersect \( T^*Q \times (\mathbb{C} \setminus B_r) \) for \( r \in (1, \infty) \) so large that \( DT^*(Q \times S^1) \subset T^*Q \times B_r \).

Therefore, and because of \( u^2 \cdot H_0 = 1 \) and \( u^2 \cdot H_\infty = 0 \), it is not possible to extend \((T^*Q \times [0, \infty]) \setminus \text{Int}(DZ)\), where \([0, \infty] \subset \mathbb{RP}^1 \subset \mathbb{CP}^1\), through \( W \) to a chain with boundary \( H_\infty - H_0 \).

On the other hand, we shall now use the splitting \( Q = S^1 \times N \) for the construction of precisely such a chain, which is the desired contradiction. To this end, we slightly modify the definition of \((\hat{Z}, \hat{\Omega})\) in this split setting, without invalidating what we have said so far.

We compactify the \( T^*S^1\)-factor in \( T^*S^1 \times T^*N \times \mathbb{CP}^1 \), analogous to the last factor. The hypersurfaces \( H_z \) are now read as

\[ H_z = \mathbb{CP}^1 \times T^*N \times \{z\} \]

With this convention understood, the conclusions (1) to (3) above are still valid.

Denote by \( \mathcal{M}' \) the moduli space of all holomorphic spheres \( u: \mathbb{CP}^1 \to \hat{Z} \) with the following properties:

\((\mathcal{M}'1)\) \( |u| = [\mathbb{CP}^1 \times \{v\} \times \{\infty\}] \) in \( H_2(\hat{Z}) \) for some \( v \in T^*N \setminus DT^*N \);

\((\mathcal{M}'2)\) \( u(z) \in H'_z := \{z\} \times T^*N \times \mathbb{CP}^1 \) for \( z \in \{0, g, \infty\} \).

The situation is illustrated in Figure 5, where the \( T^*N\)-factor is not shown. The shaded region is meant to indicate a neighbourhood of \( S^1 \times S^1 \subset \mathbb{C}^2 \), obtained by rotating the picture around a horizontal and a vertical axis.
Figure 5. The space $\tilde{Z}$ and the hyperplanes $H$ and $H^\vee$.

**Remark 3.5.** Notice that each of the hyperplanes $H_\zeta$, for $\zeta \in \mathbb{C}P^1$ sufficiently close to 0 or $\infty$, is foliated by standard spheres in $M^\vee$ of the form $z \mapsto (z, v_0, \zeta)$, $z \in \mathbb{C}P^1$, where $v_0 \in T^*N$. By positivity of intersection, no other sphere in $M^\vee$ intersects these hyperplanes.

By an *a priori* perturbation of the almost complex structure over $\text{Int}(W)$, the moduli space $\mathcal{M}^\vee$ will be an oriented $(2n-2)$-dimensional manifold, cf. [6, Proposition 6.1]. Here are the key arguments to establish this fact. All $u \in M^\vee$ intersect $H_0^\vee$ with intersection number 1, so that all $u \in M^\vee$ are simple. Furthermore, all $u \in M^\vee$ with image disjoint from $\text{Int}(W)$ are standard spheres of the form $u(z) = (z, v_0, z_0)$, $z \in \mathbb{C}P^1$, for some $v_0 \in T^*Q$ and $z_0 \in \mathbb{C}P^1$, so the freedom to choose the almost complex structure over $\text{Int}(W)$ suffices to achieve regularity for all holomorphic spheres in $M^\vee$ by [17, Remark 3.2.3].

Spheres in $\mathcal{M}^\vee$ that do not intersect $\text{Int}(W)$ are standard, so to all intents and purposes this source of non-compactness can be ignored. However, as with the original moduli space $\mathcal{M}$, *a priori* it cannot be ruled out that the limit of a Gromov-convergent sequence in $\mathcal{M}^\vee$ happens to be a 1-nodal stable map $(v^1, v^2)$ representing the homology class $[v^1] + [v^2] = [\mathbb{C}P^1 \times \{v\} \times \{\infty\}]$; simply replace the hyperplanes $H_0, H_\infty$ in the argument at the beginning of this proof by $H_0^\vee, H_\infty^\vee$. The two spheres $v^1, v^2$ can be characterised by $v^1 \bullet H_\infty^\vee = 1$ and $v^2 \bullet H_0^\vee = 1$, and no intersections with the other respective hyperplane. This intersection behaviour implies that $v^1, v^2$ are simple, and $(v^1, v^2)$ is a simple stable map in the sense of [17, Definition 6.1.1].

Even so, we shall see presently that we can still use $\mathcal{M}^\vee$ to define a pseudocycle, which then allows us to exclude the purported bubble sphere $u^2$ coming from a Gromov-convergent sequence in $\mathcal{M}$. By the symmetry of the situation, Gromov-limits of sequences in $\mathcal{M}^\vee$, *a posteriori*, likewise lie in $\mathcal{M}^\vee$. 

Write $S$ for the moduli space of unparametrised 1-nodal simple stable spheres $(v^1, v^2)$, representing the given homology class, with one marked point and the described intersection behaviour with the hyperplanes $H_0, H_\infty$, and with non-trivial intersection of both components with $\text{Int}(W)$. By repeating the intersection argument for a sequence of such spheres, we see that no further bubbling is possible. So $S$ is Gromov-compact if we include the 1-nodal spheres where the marked point coincides with the nodal point — these become stable maps by introducing a ghost bubble at the nodal point [17, p. 116]. As we shall see, these additional elements play no role for the definition of the desired pseudocycle.

By [17, Theorem 6.2.6], the set of regular almost complex structures, for which $S$ is a smooth oriented manifold, is residual. As before, by [17, Remark 3.2.3] this remains true even if we require the almost complex structure to be fixed outside $\text{Int}(W)$. Notice that in our situation we have an energy bound on the holomorphic spheres, so there are only finitely many homology classes represented by holomorphic spheres [17, Proposition 4.1.5]. This means that there are only finitely many splittings $[v^1] + [v^2]$ in homology to consider. Without the energy bound there could be countably many splittings, which would still be fine for an application of [17, Theorem 6.2.6], see the discussion preceding that theorem.

Later on, we shall find further residual sets of almost complex structures for which certain moduli spaces are manifolds, or evaluation maps transverse to some submanifold. An almost complex structure in the non-empty intersection of these residual sets will then settle all transversality problems simultaneously.

Again by [17, Theorem 6.2.6], the dimension of $S$ is

$$\dim S = 2n + 2c_1(\mathbb{CP}^1) + 2 \cdot 1 - 6 - 2 \cdot 1 = 2n - 2,$$

where the first 1 is the number of marked points, the second, the number of edges in the bubble tree. We can derive this formula \emph{ad hoc}, which will be useful for the subsequent discussion. We may think of $S$ as the moduli space of triples $((v^1, z_1), (v^2, z_2), z)$, where $v^1(z_1) = v^2(z_2)$ and $z$ is the marked point (different from the nodal point), modulo reparametrisations. The moduli space of pairs of parametrised spheres representing the class of $\mathbb{CP}^1$ has dimension $(2 + 2)n + 2c_1(\mathbb{CP}^1) = 4n + 4$. The marked point $z$ adds 2 dimensions, the automorphisms fixing $z_1$ and $z_2$, respectively, have dimension 4 for each sphere, and the nodal condition $v^1(z_1) = v^2(z_2)$ means that the evaluation map $(ev_1, ev_2)$ at the nodal points, that is, $ev_i(v^i) = v^i(z_i)$, $i = 1, 2$, maps (transversely) to the diagonal in $\hat{Z} \times \hat{Z}$, which has codimension $2n$. In total, this yields

$$\dim S = 4n + 4 + 2 \cdot 4 - 2n = 2n - 2,$$

as before.

Now we are in the position to apply what we should like to call the \emph{moving complex hypersurface argument}. Observe that the evaluation map

$$\text{ev}^\vee: \mathcal{M}^\vee \times \mathbb{CP}^1 \to \hat{Z}, \quad (u, z) \mapsto u(z),$$

which maps $z = \infty$ to $H_\infty$, is transverse to the complex hyperplanes $H_0$ and $H_\infty$ by Remark 3.5.

We now appeal to [17, Theorem 6.6.1] to see that, for a residual set of almost complex structures, the evaluation map $\text{ev}^\vee$ defines a $2n$-dimensional pseudocycle. For this we may again ignore the non-compactness coming from the ‘ends’ of our
moduli space, which are made up entirely of standard spheres. The limit set of the evaluation map in the sense of [17, p. 178] is contained in \( ev_S(S) \), where \( ev_S \) denotes the evaluation at the marked point.

By the transversality results for curves with pointwise constraints in [17, Section 3.4], cf. [7, Section 5.2] and [23, Remark 3.6], we may assume — again for a residual set of almost complex structures — that \( ev \) and \( ev_S \) are transverse to \( \mathbb{C}P^1 \times T^*N \times A \), where \( A \) is an arc in \( \mathbb{C}P^1 \) containing the interval \([0, \infty] \subset \mathbb{R}P^1\) in its interior. Then the preimage

\[
\mathcal{N} := (ev^\vee)^{-1}(\mathbb{C}P^1 \times T^*N \times [0, \infty]) \subset \mathcal{M}^\vee \times \mathbb{C}P^1
\]

is a \((2n-1)\)-dimensional oriented submanifold of \( \mathcal{M}^\vee \times \mathbb{C}P^1 \) with boundary given by the preimage of \( \mathcal{C}P^1 \times T^*N \times \{0, \infty\} = H_0 \cup H_\infty \); the evaluation map \( ev^\vee \) defines a foliation of these two hyperplanes by standard spheres. Similarly, the preimage

\[
\mathcal{N}_S := ev_S^{-1}(\mathbb{C}P^1 \times T^*N \times [0, \infty]) \subset S
\]

is a submanifold of dimension \( 2n - 3 \). It follows that

\[
ev_N := ev^\vee \vert_N : \mathcal{N} \to \hat{Z}
\]

is a pseudocycle whose limit set is contained in \( ev_S(\mathcal{N}_S) \).

In order to arrive at the desired contradiction, we now consider the intersection of this pseudocycle with the purported bubble sphere \( u^2 \), or rather a perturbation of \( u^2 \) into a smooth (but not necessarily holomorphic) embedding \( w^2 : \mathbb{C}P^1 \to \hat{Z} \). Because of \( u^2 \cdot H_0 = 1 \), the sphere \( u^2 \) is transverse to \( H_0 \) and an embedding near it. This means that \( w^2 \) may be chosen to coincide with \( u^2 \) on

\[
(u^2)^{-1}(\mathbb{C}P^1 \times T^*N \times B_\varepsilon)
\]

for some sufficiently small \( \varepsilon > 0 \). Moreover, since \( u^2 \cdot H_\infty = 0 \), we may assume that \( w^2 \) is likewise disjoint from \( H_\infty \).

Once we again wish to apply the transversality theory for curves with pointwise constraints, now to \( ev_N \) and transversality to \( w^2(\mathbb{C}P^1) \). This works fine, as long as we do not run into the situation that the nodal point of \((v^1, v^2)\) is mapped to \( w^2(\mathbb{C}P^1) \). This situation can indeed be avoided, as a dimension count shows. The space of pairs of unparametrised simple spheres with one marked point each, representing in sum the class \([\mathbb{C}P^1 \times \{v\} \times \{\infty\}]\), has dimension

\[
(2 + 2)n + 2c_1(\mathbb{C}P^1) - 2 \cdot 4 = 4n - 4
\]

We now look at the evaluation map \((ev_1, ev_2)\) at the marked points, and we make this transverse, by a generic choice of almost complex structure, to the diagonal \( \Delta_{w^2} \) in \( w^2(\mathbb{C}P^1) \times w^2(\mathbb{C}P^1) \subset \hat{Z} \times \hat{Z} \). Since \( \Delta_{w^2} \) has codimension \( 4n - 2 \) in \( \hat{Z} \times \hat{Z} \), this means that its preimage under \((ev_1, ev_2)\) will be empty, that is, in the 1-nodal spheres the nodal point never maps to \( w^2(\mathbb{C}P^1) \).

By a further generic choice of almost complex structure we also achieve transversality to \( w^2(\mathbb{C}P^1) \) of the evaluation maps \( ev_N \), that is, our pseudocycle, and \( ev_S \vert_{\mathcal{N}_S} \), whose image is the limit of the pseudocycle. Since \( w^2(\mathbb{C}P^1) \) has codimension \( 2n - 2 \) in \( \hat{Z} \), this actually implies that the limit set, as the image of the \((2n - 3)\)-dimensional manifold \( \mathcal{N}_S \), is disjoint from \( w^2(\mathbb{C}P^1) \), and \( ev_N^{-1}(w^2(\mathbb{C}P^1)) \) is a 1-dimensional submanifold of the \((2n - 1)\)-dimensional manifold \( \mathcal{N} \), in other words, a finite, disjoint union of properly embedded circles and closed intervals. Precisely one boundary point of the intervals belongs to the boundary component \((ev^\vee)^{-1}(H_0)\) of \( \mathcal{N} \), again
because of $u^2 \cdot H_0 = 1$ and Remark 3.5. So there would have to be a boundary point in $(ev^\vee)^{-1}(H_\infty)$. But this would mean that $w^2 \cdot H_\infty \neq 0$, contradicting the fact that $w^2(CP^1)$ is disjoint from $H_\infty$.

So the bubble $u^2$ cannot have existed in the first place. □

3.2. Infinite holomorphic strips. We continue to assume that hypothesis (H) is satisfied. In addition, we now assume that $\pi_1(Q)$ is abelian, so that the homomorphism $\pi_1(M) \to \pi_1(W)$ induced by inclusion is an isomorphism, as explained in Remark 3.2. This allows us to pass to the universal covers. More generally, it would suffice to require that the homomorphism $\pi_1(M) \to \pi_1(W)$ is injective, since surjectivity is guaranteed by Theorem 3.1.

Recall the definition of $Z^*$ from (3). We consider the proper degree 1 evaluation map $ev: \mathcal{M} \times \mathbb{C}^* \to Z^*$. Define

$$Z_\varepsilon = W_Z \setminus (T^*Q \times B_\varepsilon),$$

i.e. $Z_\varepsilon$ is obtained from $Z^*$ by removing $T^*Q \times (B_\varepsilon \setminus \{0\})$. Observe that $\partial Z_\varepsilon = T^*Q \times \partial D_\varepsilon$.

The universal cover of $Z_\varepsilon$ is

$$\tilde{Z}_\varepsilon = \tilde{W} \cup (\tilde{T}^*Q \times [\varepsilon, \infty) \times \mathbb{R}) \setminus DT^*\tilde{L},$$

where $\tilde{L} = \tilde{Q} \times \mathbb{R}$, and the universal cover of $\partial Z_\varepsilon$ equals

$$\partial \tilde{Z}_\varepsilon = T^*\tilde{Q} \times \{\varepsilon\} \times \mathbb{R}.$$

Set $\mathbb{C}_\varepsilon := \mathbb{C} \setminus B_\varepsilon$. After reparametrising the non-standard discs in $\mathcal{M}$ so as to make them look like standard disc on $D_\varepsilon \setminus \{0\}$, we may assume that the evaluation map restricts to a proper degree 1 map

$$ev_{\varepsilon}: (\mathcal{M} \times \mathbb{C}_\varepsilon, \mathcal{M} \times \partial D_\varepsilon) \to (Z_\varepsilon, \partial Z_\varepsilon).$$

Notice the slight abuse of notation: elements in $\mathcal{M}$ are now understood to be these reparametrised curves.

We now want to construct, on a suitable covering space $\mathcal{M}'$ of $\mathcal{M}$, a proper degree 1 map to $(Z_\varepsilon, \partial Z_\varepsilon)$ that covers $ev_{\varepsilon}$. Observe that for every element $u: \mathbb{C}_\varepsilon \to Z_\varepsilon$ of $\mathcal{M}$ we have $u(\varrho) \in H_\varrho$ by construction, where $\varrho$ is the marking from (M2). Moreover, $u$ and $H_\varrho$ intersect uniquely at $\varrho \in \mathbb{C}$. Therefore, writing

$$u(\varrho) =: (v, \varrho) \in T^*Q \times \{\varrho\} = H_\varrho,$$

we obtain for each lift $\tilde{v} \in T^*\tilde{Q}$ of $v$ a lift

$$u' \equiv u'_\varepsilon: [\varepsilon, \infty) \times \mathbb{R} \to \tilde{Z}_\varepsilon$$

of $u \in \mathcal{M}$ with

$$u'(\varrho, 0) = (\tilde{v}, \varrho, 0) \in T^*\tilde{Q} \times \{\varepsilon, \infty\} \times \mathbb{R}.$$  

We define $\mathcal{M}'$ as the set of all such lifts, equipped with $C^\infty_{loc}$-topology to make it a covering space of $\mathcal{M}$. The map $ev_{\varepsilon}'$ lifts to

$$ev_{\varepsilon}': (\mathcal{M}' \times [\varepsilon, \infty) \times \mathbb{R}, \mathcal{M}' \times \{\varepsilon\} \times \mathbb{R}) \to (\tilde{Z}_\varepsilon, \partial \tilde{Z}_\varepsilon).$$

Proposition 3.6. The evaluation map $ev_{\varepsilon}'$ is proper of degree 1.
Proof. Once properness of $ev'_\nu$ has been established, the mapping degree is well defined. By looking at standard holomorphic strips of the form $(r, \theta) \mapsto (\tilde{v}, r, \theta)$, with $\tilde{v} \in T^*\mathcal{Q}$ sufficiently large (in fibre direction), we see that this mapping degree equals 1.

Regarding properness, we need to show that the preimage
\[(ev'_\nu)^{-1}(\tilde{K}) \subset \mathcal{M}' \times [\varepsilon, \infty) \times \mathbb{R}\]
of any given compact subset $\tilde{K} \subset \tilde{Z}_\varepsilon$ is compact. Observe that $\tilde{K}$ projects to a compact set $K \subset Z_\varepsilon$, and
\[(ev_\varepsilon)^{-1}(K) \subset \mathcal{M} \times \mathbb{C}_\varepsilon\]
is compact by the properness of $ev_\varepsilon$.

Let $(u'_\nu, r_\nu, \theta_\nu)$ be a sequence in $(ev'_\nu)^{-1}(\tilde{K})$. After selecting a subsequence, we may assume that $u'_\nu(r_\nu, \theta_\nu)$ converges to some point $\tilde{p} \in \tilde{Z}_\varepsilon$, and that the projection
\[(u_\nu, z_\nu = r_\nu e^{i\theta_\nu}) \in (ev_\varepsilon)^{-1}(K) \subset \mathcal{M} \times \mathbb{C}_\varepsilon\]
of $(u'_\nu, r_\nu, \theta_\nu)$ converges to a pair $(u_0, z_0)$. In particular, writing $z_0 = r_0 e^{i\theta_0}$, this means that $\theta_\nu$ converges to $\theta_0$ modulo $2\pi$, and $r_\nu$ converges to $r_0$. The point $\tilde{p}$ projects to $p = u_0(z_0)$. The situation is summarised in the following diagram:

\[\begin{array}{c}
(u'_\nu, r_\nu, \theta_\nu) \to (?, r_0, \theta_0 \mod 2\pi) \in \mathcal{M}' \times \mathbb{C}_\varepsilon \\
\downarrow \hspace{2cm} \downarrow \\
(u_\nu, r_\nu e^{i\theta_\nu}) \to (u_0, z_0) \in \mathcal{M} \times \mathbb{C}_\varepsilon \\
\downarrow \hspace{2cm} \downarrow \\
\tilde{Z}_\varepsilon \supset \tilde{K} \ni \tilde{p} \quad \text{ev}_\varepsilon \\
\end{array}\]

Choose $k_\nu \in \mathbb{Z}$ such that
\[\theta_\nu - 2\pi k_\nu \longrightarrow \theta_0 \quad \text{for} \quad \nu \to \infty.\]

Write $\ell_\nu$ for the line segment in $[\varepsilon, \infty) \times \mathbb{R}$ connecting $(\varrho, 2\pi k_\nu \varrho)$ with $(r_\nu, \theta_\nu)$. The arc $u'_\nu(\ell_\nu)$ projects to $u_\nu(\ell_\nu)$. Since $u_\nu \to u_0$ in $\mathcal{M}$, the distance
\[d(u'_\nu(\varrho, 2\pi k_\nu \varrho), u'_\nu(r_\nu, \theta_\nu))\]
in $\tilde{Z}_\varepsilon$ is uniformly bounded from above by a constant times the length of $\ell_\nu$, and hence stays bounded as $\nu \to \infty$. Recall that $u'_\nu(\varrho, 2\pi k_\nu \varrho)$ may be written as $(\tilde{v}_\nu, \varrho, 2\pi k_\nu \varrho)$. Then, from $u'_\nu(r_\nu, \theta_\nu) \to \tilde{p}$, we deduce that $(\tilde{v}_\nu, \varrho, 2\pi k_\nu \varrho)$ stays at bounded distance from $\tilde{p}$ in $\tilde{Z}_\varepsilon$.

Hence, after passing to a subsequence we may assume that the sequence $(\tilde{v}_\nu)$ converges to some point $\tilde{v}_0 \in T^*\mathcal{Q}$, and $k_\nu = k_0$ for all $\nu$. Then the sequence $(u'_\nu)$ of lifted holomorphic curves converges to $u'_0 \in \hat{\mathcal{M}}$, characterised by $u'_0(\varrho, 0) = (\tilde{v}_0, \varrho, 0)$.

\[\square\]

4. Proof of the Main Result

We are now in the position to prove our main result Theorem 1.3 about the uniqueness of certain fillings up to homotopy equivalence or up to diffeomorphism, respectively.
4.1. Homotopy type. In this section we prove part (a) of Theorem 1.3.

Proposition 4.1. Let $Q$ be a closed, connected manifold and $(W, \omega)$ a symplectically aspherical filling of $M = ST^*(Q \times S^1)$ satisfying hypothesis (H). Assume that $\pi_1(Q)$ is abelian or, more generally, that the inclusion $M \to W$ is $\pi_1$-injective. Then the composition $\tilde{Q} \to \tilde{M} \to \tilde{W}$ of inclusion maps of universal covers is surjective on homology.

Proof. The evaluation map $ev'_\varepsilon$ from Proposition 3.6 fits into the following diagram, where the horizontal map at the top is the obvious restriction of $ev'_\varepsilon$, and the vertical maps are inclusions:

$$
\begin{array}{c}
\mathcal{M}' \times \{\varepsilon\} \times \mathbb{R} \xrightarrow{ev'_\varepsilon} \partial \tilde{Z}_\varepsilon \\
\cong \\
\mathcal{M}' \times [\varepsilon, \infty) \times \mathbb{R} \xrightarrow{ev'_\varepsilon} \tilde{Z}_\varepsilon.
\end{array}
$$

A homological argument as in [1, Proposition 2.4] shows that $ev'_\varepsilon$ at the bottom is surjective on homology. It follows that the inclusion map $\partial \tilde{Z}_\varepsilon \to \tilde{Z}_\varepsilon$ must likewise be surjective on homology. Observe that $\partial \tilde{Z}_\varepsilon = T^*\tilde{Q} \times \{\varepsilon\} \times \mathbb{R}$ strongly deformation retracts onto $\tilde{Q} \times \{\varepsilon\} \times \{0\}$, where $\tilde{Q}$ is regarded as the zero section of $T^*\tilde{Q}$. Also, $\tilde{Z}_\varepsilon$ strongly deformation retracts onto $\tilde{W}$. The latter retraction may be assumed to send $\tilde{Q} \times \{\varepsilon\} \times \{0\}$ to the copy of $\tilde{Q}$ in $\tilde{M}$ coming from the inclusion $Q \times S^1 \subset M$ described in Section 2.3. This means that the inclusion $\partial \tilde{Z}_\varepsilon \to \tilde{Z}_\varepsilon$ retracts to the composition $\tilde{Q} \to \tilde{M} \to \tilde{W}$ of inclusions. This proves the proposition. □

Recall from Section 2.3 the definition of the copy $W_0 \subset W$ of $DT^*(Q \times S^1)$. This $W_0$ retracts to $Q \times S^1 \subset M \subset W$, so the inclusion $\tilde{W}_0 \to \tilde{W}$ is likewise surjective on homology.

The following theorem is the analogue of results in [1, Section 7] in the present more general setting.

Theorem 4.2. Let $Q$ be a closed, connected manifold and $(W, \omega)$ a symplectically aspherical filling of $M = ST^*(Q \times S^1)$ satisfying hypothesis (H). Assume further that one of the following conditions is satisfied:

$(\alpha)$ $Q$ is aspherical and $\pi_1(Q)$ abelian;
$(\beta)$ $M$ is a simple space.

Then $W$ has the homotopy type of $DT^*(Q \times S^1)$.

Since the fundamental group of an aspherical manifold is torsion-free, and the topological Borel conjecture holds for the torus [14], $(\alpha)$ implies that $Q$ is homeomorphic to a torus. However, as in Proposition 4.1, we may replace the requirement in $(\alpha)$ that $\pi_1(Q)$ be abelian by the less restrictive assumption that the inclusion $M \to W$ is $\pi_1$-injective.

Recall that a topological space is called simple if its fundamental group acts trivially on all its homotopy group. The action of $\pi_1$ on itself is given by conjugation, so a simple space has an abelian fundamental group.
Proof of Theorem 4.3. Under assumption (α), we argue as follows. The universal cover \( \widetilde{Q} \) is contractible, hence so is \( W \) by Proposition 4.2. So the inclusion \( W_0 \to \widetilde{W} \) of universal covers is a homotopy equivalence (of contractible spaces). This means that \( H_k(\widetilde{W}, W_0) = 0 \) for all \( k \).

The remaining argument is analogous to the proof of [1] Theorem 7.2. Here are the details. By excision, and with the cobordism \( \{M_0, X, M\} \) as defined in Section 2.4, we have \( H_k(\widetilde{X}, \widetilde{M}_0) = 0 \) for all \( k \). The relative Hurewicz theorem implies \( \pi_k(\widetilde{X}, \widetilde{M}_0) = 0 \) for all \( k \). Since the inclusion \( M_0 \to X \) is an isomorphism on \( \pi_1 \), as remarked after the proof of Lemma 2.9 we also have \( \pi_k(X, M_0) = 0 \) for all \( k \). Whitehead’s theorem then implies that \( M_0 \) is a strong deformation retract of \( X \). Hence \( W = W_0 \cup_{M_0} X \cong W_0 \).

Under assumption (β), the fundamental group \( \pi_1(Q) \) must be abelian, since the fundamental groups of \( M = ST^*(Q \times S^1) \) and \( Q \times S^1 \) are isomorphic, and the former is abelian thanks to \( M \) being a simple space.

As before, we need to show that all relative homotopy groups \( \pi_k(X, M_0) \) are zero. This is precisely the content of [1] Lemma 8.1; we only need to verify that the subsidiary results cited in the proof of that lemma are available in our more general setting discussed here.

First of all, we need the inclusion \( M_0 \to X \) to be \( \pi_1 \)-isomorphic, which is guaranteed by Theorem 5.1 Remark 5.2 and the comments at the end of Section 2.5. Secondly, we need the inclusion \( M_0 \to W \) to be surjective on homology, which is guaranteed by Proposition 4.1.

For Stein fillings \((W, \omega)\) of dimension at least 6 we see from the handlebody structure that \( H_2(W, M) = 0 \) and \( \pi_2(W, M) = 0 \) by general position. Hence, the inclusion \( M \to W \) is \( \pi_1 \)-injective, and thus \( \pi_1 \)-isomorphic by Theorem 5.1. This observation leads to the following theorem.

Theorem 4.3. Let \( Q \) be a closed, connected manifold of dimension \( n - 1 \geq 2 \) with \( \pi_1(Q) \) infinite and \( H_{n-2}(Q) = 0 \). Let \((W, \omega)\) be a Stein filling of \( M = ST^*(Q \times S^1) \). Assume that \( \chi(Q) = 0 \), so that hypothesis (H-i) is satisfied. Then \( W \) is homotopy equivalent to \( DT^*(Q \times S^1) \).

Proof. The 2n-dimensional Stein manifold \( W \) has the homotopy type of a cell complex of dimension at most \( n \). In fact, there are no subcritical fillings of unit cotangent bundles, see [1] Proposition 3.9, so the cellular dimension of \( W \) is actually equal to \( n \). From

\[
H_k(W, M) \cong H^{2n-k}(W) \cong FH_{2n-k}(W) \oplus TH_{2n-k-1}(W)
\]

we conclude that \( H_k(W, M) = 0 \) for \( k \leq n - 1 \), since \( FH_\ell(W) = 0 \) for \( \ell \geq n + 1 \), and \( TH_\ell = 0 \) for \( \ell \geq n \), as there are no \((n+1)\)-cells in \( W \).

Similarly one sees that \( \pi_k(W, M) = 0 \) for \( k \leq n - 1 \), since any relative \( k \)-disc with \( k \leq n - 1 \) can be made disjoint from the \( n \)-dimensional cellular skeleton of \( W \). It follows that \( \pi_k(\widetilde{W}, \widetilde{M}) = 0 \) and \( H_k(\widetilde{W}, \widetilde{M}) = 0 \) for \( k \leq n - 1 \).

As earlier, we regard \( L = Q \times S^1 \) as a section of \( M = ST^*(Q \times S^1) \). By general position, we have \( \pi_k(M, L) = 0 \) for \( k \leq n - 2 \), since in this range a relative \( k \)-disc can be made disjoint from the section antipodal to \( L \), and then be pushed into \( L \). It then follows that \( H_k(\widetilde{M}, \widetilde{L}) = 0 \) for \( k \leq n - 2 \).
From the homology long exact sequence of the triple $(\widetilde{W}, \widetilde{M}, \tilde{L})$ we then deduce that $H_k(\tilde{W}, \tilde{W}_0) = H_k(\tilde{W}, \tilde{L}) = 0$ for $k \leq n - 2$. So the inclusion $\tilde{W}_0 \to \tilde{W}$ induces isomorphisms on homology in degrees $k \leq n - 3$.

On the other hand, the assumption on $\pi_1(Q)$ being infinite implies that $\widetilde{Q}$ is not compact, and hence $H_{n-1}(\widetilde{Q}) = 0$. With the further homological assumption on $\tilde{Q}$ in the theorem, we have $H_k(\tilde{W}) = 0$ for $k \geq n - 2$. With Proposition 4.1 we find that $H_k(\widetilde{W}) = 0$ for $k \geq n - 2$. The same is obviously true for $\tilde{W}_0$.

Thus, the inclusion $\tilde{W}_0 \to \tilde{W}$ induces an isomorphism on all homology groups, and hence $H_k(\tilde{W}, \tilde{W}_0) = 0$ for all $k$. The argument now concludes as in case $(\alpha)$ of Theorem 4.2.

**Proof of Theorem 1.3 (a).** The manifolds in (a-i) and (a-ii) satisfy the assumptions of Theorem 4.3.

### 4.2. Diffeomorphism type

In this section we prove part (b) of Theorem 1.3. The key point is to show that the cobordism $\{M_0, X, M\}$, under appropriate assumptions, is an $h$-cobordism. Compared with the previous discussion, one needs to ensure that the ‘upper’ inclusion $M \to X$ is likewise a homotopy equivalence. An additional assumption on the vanishing of the Whitehead group $\text{Wh}(\pi_1(M))$ then guarantees $\{M_0, X, M\}$ to be an $s$-cobordism, and hence diffeomorphic to a product $M \times [0, 1]$ by the $s$-cobordism theorem.

**Theorem 4.4.** Let $Q$ be a closed, connected manifold and $(W, \omega)$ a filling of $M = ST^*(Q \times S^1)$ satisfying hypothesis (H). If $M$ is a simple space and $\text{Wh}(\pi_1(M)) = 0$, then $W$ is diffeomorphic to $DT^*(Q \times S^1)$.

**Proof.** The ‘lower’ inclusion $M_0 \to X$ is a homotopy equivalence by the proof of Theorem 4.2 under assumption $(\beta)$. The ‘upper’ inclusion $M \to X$ is a homotopy equivalence thanks to [1, Lemma 8.2], which only uses the simplicity of $M_0$ (and hence that of $X$).

**Remark 4.5.** Under assumption $(\alpha)$ of Theorem 4.2, the above argument breaks down, because the corresponding argument in [1] is based on Lemma 5.2 in that paper, for which the assumption that the filling is subcritical is crucial.

**Proof of Theorem 1.3 (b).** The manifolds $Q$ in Theorem 1.3 (b) satisfy hypothesis (H-ii). The remaining assumptions of Theorem 4.4 are satisfied thanks to [1, Example 9.3.(2)].

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