ON THE SET OF CHERN NUMBERS IN LOCAL RINGS

HOANG LE TRUONG AND HOANG NGOC YEN

Abstract. This paper purposes to characterize Noetherian local rings \((R, \mathfrak{m})\) such that the Chern numbers of certain \(\mathfrak{m}\)-primary ideals in \(R\) bounded above or range among only finitely many values. Consequently, we characterize the Gorensteinness, Cohen-Macaulayness, generalized Cohen-Macaulayness of local rings in terms of the behavior of its Chern numbers.

1. Introduction

Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring of dimension \(d\), where \(\mathfrak{m}\) is the maximal ideal and \(k = R/\mathfrak{m}\) is the residue field of \(R\). Let \(M\) be a finitely generated \(R\)-module of dimension \(s\). For an \(\mathfrak{m}\)-primary ideal \(I\) of \(M\), it is well-known that there are integers \(\{e_i(I, M)\}_{i=0}^{\infty}\), called the Hilbert coefficients of \(M\) with respect to \(I\) such that for \(n \geq 0\)

\[
\ell_R(M/I^{n+1}M) = e_0(I, M)\binom{n+s}{s} - e_1(I, M)\binom{n+s-1}{s-1} + \cdots + (-1)^se_s(I, M),
\]

where \(\ell_R(N)\) denotes the length of an \(R\)-module \(N\). In particular, the integer \(e_0(I, M) > 0\) is called the multiplicity of \(M\) with respect to \(I\) and has been explored very intensively. Notice that Nagata showed that \(R\) is a regular local ring if and only if \(e_0(\mathfrak{m}, R) = 1\), provided \(R\) is unmixed ([Sam51,Nag62]). Recall that a local ring \(R\) is unmixed, if \(\dim R = \dim R/p\) for every associated prime ideal \(p\) of the \(\mathfrak{m}\)-adic completion \(\hat{R}\) of \(R\). Moreover, the Cohen-Macaulayness of \(R\)-module \(M\) is characterized in terms of \(e_0(q, M)\) of parameter ideals \(q\) of \(M\). On the other hand, S. Goto and other authors in [MMV11,GGH+10,GO11,CGT13,GGH+14] analyzed the behavior of the values \(\{e_i(q, M)\}_{i=1}^{d}\) for parameter ideals \(q\) of modules \(M\) and it is used to characterize Cohen-Macaulayness, Buchsbaumness, generalized Cohen-Macaulayness of modules \(M\). In particular, let

\[\Lambda_i(M) = \{e_i(q, M) \mid q\text{ is a parameter ideal of }M\},\]

for all \(i = 1, \ldots, s\). Then we have the following results as in Table 1. The aim of our paper is to continue this research direction. Concretely, we will give characterizations of some special classes of rings in terms of its Hilbert coefficients with respect to certain non-parameter ideals.

| \(\Lambda_1(M)\subseteq (-\infty, 0]\) | \(\Lambda_1(M)\subseteq (-\infty, 0]\) | \(\Lambda_1(M)\subseteq (-\infty, 0]\) |
|---------------------------------|---------------------------------|---------------------------------|
| \(0 \in \Lambda_1(M), (\ast)\) | \(M\text{ is Cohen-Macaulay}\) | \(M\text{ is Cohen-Macaulay}\) |
| \(|\Lambda_1(M)| < \infty, (\ast)\) | \(M\text{ is generalized Cohen-macaulay}\) | \(M\text{ is generalized Cohen-macaulay}\) |
| \(|\Lambda_i(M)| < \infty, \text{ for all } i = 1, \ldots, d\) | \(R\text{ is generalized Cohen-macaulay}\) | \(R\text{ is generalized Cohen-macaulay}\) |

Table 1. Properties of a finitely generated module \(M\) carried by the behavior of the certain set. A symbol \((\ast)\) requires that the module \(M\) be unmixed.

To state the results of this paper, first of all let us fix our notation and terminology. Let \(b(M) = \bigcap_{i=1}^{s} \text{Ann}(\langle 0 \rangle : M/(x_1, \ldots, x_{i-1}, x_i) M, x_i)\), where \(x = x_1, \ldots, x_s\) runs over all systems of parameters of \(M\). A

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system \(x_1, \ldots, x_s\) of parameters of \(M\) is called a \(C\)-system of parameters of \(M\) if \(x_s \in b(M)\) and \(x_i \in b(M/(x_{i+1}, \ldots, x_s)M)\) for all \(i = s-1, \ldots, 1\) and a parameter ideal \(q\) of \(M\) is called \(C\)-parameter ideal if it is generated by a \(C\)-system of parameters of \(M\) ([MQ17, Definition 2.15]). Notice that, \(C\)-systems of parameters of \(M\) always exist, provided \(R\) is a homomorphic image of a Cohen-Macaulay local ring (see [CC18]). Moreover, the index of reducibility of \(C\)-parameter ideals \(q\) of \(M\), which is independent of the choice of \(q\), is called the stable value of \(M\) and denoted by \(N_R(M)\) (see [CQ20]).

Let

\[
\Xi_i(M) = \{ e_i(q :_R m, M) : q \text{ is a } C\text{-parameter ideal of } M \},
\]

for all \(i = 1, \ldots, s\). Then the main results of this paper are expressed as in the Table 2.

| \(\Xi_1(R) \subseteq (-\infty, N(R)]\), (*) | \(R\) | Corollary 3.7 |
|---|---|---|
| \(N(R) \in \Xi_1(R)\) , (*) | \(R\) is Cohen-Macaulay | Corollary 3.7 |
| \(|\Xi_1(R)| < \infty, (\ast)\) | \(R\) is generalized Cohen-Macaulay | Theorem 4.5 |
| \(\Xi_i(R) \subset (-\infty, \infty)\), for all \(i = 1, \ldots, d\) | \(R\) is generalized Cohen-Macaulay | Theorem 4.6 |

**Table 2.** Properties of a finitely generated module \(M\) carried by the behavior of the certain set. A symbol (\(\ast\)) requires that the ring \(R\) be unmixed.

This paper is divided into four sections. In the next section we recall the notions and prove some preliminary results on Hilbert coefficients, index of reducibility, \(C\)-system of parameters and generalized Cohen-Macaulay modules. In the Section 3, we will explore the relation between the Chern numbers and the stable value of \(R\). This will apply in particular to characterize the Gorensteinness, Cohen-Macaulayness of local rings in term of its Chern numbers and the stable value (Theorem 3.6, Corollary 3.7 and Corollary 3.9). In the last section, we will study the problem of when the sets \(\Xi_i(R)\) are finite, where \(i = 1, \ldots, d\). We shall show that \(R\) is generalized Cohen-Macaulay if and only if \(\Xi_1\) is finite, provided \(R\) is unmixed (Theorem 4.5). Moreover \(R\) is generalized Cohen-Macaulay if and only if \(\Xi_i(R)\) are finite for all \(i = 1, \ldots, d\) (Theorem 4.6).

2. Preliminary

Throughout this paper, let \(R\) denote a Noetherian local ring with maximal ideal \(m\) of dimension \(d > 0\). Assume that \(R\) is a homomorphic image of a Cohen-Macaulay local ring. Let \(H_m^i(\bullet)\) \((i \in \mathbb{Z})\) be the \(i\)-th local cohomology functor of \(R\) with respect to \(m\). For each finitely generated \(R\)-module \(M\) let \(\ell_R(M)\) stand for the length of \(M\). Moreover, if the module \(H_m^i(M)\) is finitely generated, its length is denoted by \(h_i(M)\) and \(e_i(M) = \ell(0 :_{H_m^i(M)} m)\). We also put \(a_i = \text{Ann} H_m^i(M)\) for all \(j \in \mathbb{Z}\), and \(a(M) = a_0(M) \ldots a_{s-1}(M)\). We denote \(b(M) = \bigcap_{i=1}^s \text{Ann}(0 :_{M/(x_1, \ldots, x_{i-1})M} x_i)\) where \(x = x_1, \ldots, x_s\) runs over all systems of parameters of \(M\).

2.1. Hilbert coefficients. It is well-known that there are integers \(\{e_i(I, M)\}_{i=0}^s\), called the **Hilbert coefficients** of \(M\) with respect to \(I\) such that

\[
\ell_R(M/I^{n+1}M) = e_0(I, M) \binom{n+s}{s} - e_1(I, M) \binom{n+s-1}{s-1} + \ldots + (-1)^s e_s(I, M).
\]

for large integer \(n\). In particular, the leading coefficient \(e_0(I)\) is said to be the **multiplicity** of \(M\) with respect to \(I\) and \(e_1(I)\), which Vasconcelos ([Vas08]) refers to as the **Chern number of** \(M\) with respect to \(I\). For inductive arguments that used in this paper we need the following results.

**Lemma 2.1** ([Nag62, 22.6]). For every superficial element \(x \in I\), we have

\[
e_j(I, M/IM) = \begin{cases} 
e_j(I, M) & \text{if } 0 \leq j \leq s-2, \\
& (-1)^j \ell_R(0 :_M x) & \text{if } 0 \leq j = s-1.
\end{cases}
\]

**Lemma 2.2** ([CGT13, Lemma 3.4]). Let \(N\) be a submodule of \(M\) with \(\dim N = t < s\). Then

\[
e_j(I, M) = \begin{cases} 
e_j(I, M/IN) & \text{if } 0 \leq j \leq s-t-1, \\
& (-1)^{s-t} e_0(I, N) & \text{if } j = s-t.
\end{cases}
\]
2.2. **Index of reducibility.** Recall that a proper submodule $N$ of $M$ is called irreducible if $N$ cannot be written as an intersection of two strict larger submodules of $M$. For a submodule $N$ of $M$, the number of irreducible components of an irredundant irreducible decomposition of $N$, which is independent of the choice of the decomposition, is called the index of reducibility of $N$ and denoted by $\text{ir}_M(N)$ ([Noe21]). For an $\mathfrak{m}$-primary ideal $I$, we have

$$\text{ir}_M(I) := \text{ir}_M(IM) = \ell_R([IM :_M \mathfrak{m}]/IM).$$

Moreover, there exists the integers $f_i(I, M)$ such that

$$\text{ir}_M(I^{n+1}) = \sum_{i=0}^{d-1} (-1)^i f_i(I, M) \binom{n + d - 1 - i}{d - 1 - i}.$$

for all large $n$ ([CQT15, Lemma 4.2]). The leading coefficient $f_0(I, M)$ is called the irreducible multiplicity of $M$ with respect to $I$.

A system $x_1, \ldots, x_s$ of parameters of $M$ is called a $C$-system of parameters of $M$ if $x_s \in b(M)^3$ and $x_i \in b(M/(x_{i+1}, \ldots, x_s)M)^3$ for all $i = s - 1, \ldots, 1$. A parameter ideal $q$ of $M$ is called $C$-parameter ideal if it is generated by a $C$-system of parameters of $M$ ([MQ17, Definition 2.15]). Notice that, $C$-systems of parameters of $M$ always exist, provided $R$ is a homomorphic image of a Cohen-Macaulay local ring (see [CC18]) and a $C$-system of parameters forms a $d$-sequence. Moreover, the index of reducibility of $C$-parameter ideals $q$ of $M$, which is independent of the choice of $q$, is called the stable value of $M$ and denoted by $\mathcal{N}_R(M)$ (see [CQ20]). In particular, $\mathcal{N}_R(M) = r_s(M)$ provided $M$ is Cohen-Macaulay.

Furthermore, we have the following results which are useful in this paper.

**Lemma 2.3.** Let $x_1, \ldots, x_s$ be a $C$-system of parameters of $M$. Then

i) $x_1^{n_1}, \ldots, x_s^{n_s}$ is a $C$-system of parameters of $M$ for all $n_1, \ldots, n_s \geq 1$.

ii) $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s$ is a $C$-system of parameters of $M/x_iM$ and

$$\mathcal{N}_R(M) = \mathcal{N}_R(M/x_iM),$$

for all $i = 1, \ldots, s$.

iii) $x_1, \ldots, x_s$ is a $C$-system of parameters of $M/W$ and

$$\mathcal{N}_R(M) = \mathcal{N}_R(M/W) + r_0(M),$$

where $W = H^0_W(M)$.

**Proof.** i) and ii) are followed from [CQ20b, Lemma 2.13].

iii), Since $W$ has finite length, by i) and Lemma 2.4 in [CT08] there exists a $C$-parameter ideal $q$ of $M$ such that $qM \cap W = 0$ and $(qM + W):_M \mathfrak{m} = qM :_M \mathfrak{m} + W$. Therefore, we have the exact sequence

$$0 \to W \to M/qM \to M/W + qM \to 0.$$ 

After applying the functor $\text{Hom}_R(t, \bullet)$ to the above exact sequence, the sequence

$$0 \to (0 :_W \mathfrak{m}) \to (0 :_{M/qM} \mathfrak{m}) \to (0 :_{M/W + qM} \mathfrak{m}) \to 0$$

is exact. Since $q$ is also a $C$-parameter ideal of $M/W$, we have

$$\mathcal{N}_R(M) = \ell((0 :_{M/qM} \mathfrak{m})) = \ell((0 :_{M/W + qM} \mathfrak{m})) + \ell((0 :_W \mathfrak{m})) = \mathcal{N}_R(M/W) + r_0(M),$$

this complete the proof. \hfill \Box

The following results are useful for our inductive technique.

**Lemma 2.4** ([MQ17, Remark 3.3]). Assume that $M$ is unmixed. Let $x_1, \ldots, x_s$ be a $C$-system of parameters of $M$. Then

i) $H^1_W(M)$ is finitely generated provided $s \geq 2$.

ii) $\text{Ass}(M/x_1M) \subseteq \text{Assh}(M/x_1M) \cup \{\mathfrak{m}\}$, where $\text{Assh}M = \{p \in \text{Ass}M \mid \text{dim } R/p = \text{dim } M\}$. 

2.3. Generalized Cohen-Macaulay. Now we recall the notions of generalized Cohen-Macaulay modules and standard parameter ideals in terms of local cohomology ([CST78, Tru86]).

**Definition 2.5.** i) An $R$-module $M$ is said to be general}zed Cohen-Macaulay, if $H^i_m(M)$ are of finite length for all $i < s$.

ii) A parameter ideal $q = (x_1, \ldots, x_s)$ of $M$ is called standard if

$$qH^i_m(M/(x_1, \ldots, x_j)M) = 0,$$

for all $0 \leq i + j < s$.

It is well known that if $M$ is a generalized Cohen-Macaulay module, then there exists a positive integer $n$ such that every parameter ideals of $M$ contained in $m^n$ is standard. Note that a standard system of parameters forms a $d$-sequence.

**Lemma 2.6.** Let $q$ be a $C$-parameter ideal of a generalized Cohen-Macaulay $R$-module $M$. Then we have

i) $q$ is standard and $N_R(M) = \sum_{i=0}^{s} \binom{s}{i} r_i(M)$.

ii) $f_0(q, M) = \sum_{j=1}^{s} \binom{s-1}{j-1} r_j(M)$ (see [Tru17, Lemma 4.2]).

iii) In the case $M = R$,

$$e_i(q :_R m, R) = \begin{cases} (-1)^i \sum_{j=1}^{d-i} \binom{d-i}{j-1} \ell_R(H^i_m(R)) - \sum_{j=1}^{d-i} r_j(R) & \text{if } i = 1, \ldots, d-1, \\ (-1)^{d}\ell_R(H^0_m(R)) - r_1(R) & \text{if } i = d, \\ \end{cases}$$

(see [CQT19, Theorem 5.2]).

3. Characterizations of Cohen-Macaulay rings

The purpose of this section is to present characterizations of Cohen-Macaulay rings in terms of its Chern numbers, irreducible multiplicities, stable value and the Cohen-Macaulay type. As corollaries, we obtained the characterizations of a Gorenstein ring in terms of its Chern numbers, irreducible multiplicity and stable value. We begin with the following result.

**Proposition 3.1.** Assume that $M$ is unmixed of dimension $d \geq 2$. Suppose that

$$N(M) \leq f_0(q, M),$$

for some $C$-parameter ideal $q$ of $M$. Then $M$ is Cohen-Macaulay.

**Proof.** We use induction on the dimension $s$ of $M$. In the case $\dim M = 2$, then $M$ is generalized Cohen-Macaulay because of Lemma 2.4. We get by the Lemma 2.6 that

$$r_1(M) + r_2(M) = f_0(q, M) \geq N(M) = r_2(M) + 2r_1(M).$$

Thus, $r_1(M) = 0$ and therefore $M$ is Cohen-Macaulay.

Suppose that $\dim M \geq 3$ and that our assertion holds true for $\dim M - 1$. Since $M$ is unmixed, $x_1$ is $M$-regular whence $H^0_m(M) = (0) :_M x_1 = (0)$. Let $A = M/x_1M$ and $\overline{x} = (x_2, \ldots, x_s)$. Then by Lemma 2.4 and Lemma 2.3 ii), we have $\text{Ass} A \subset \text{Ass} A \cup \{m\}$ and $x_2, \ldots, x_d$ is a $C$-system of parameters of $A$. Thus $B := A/H^0_m(A)$ is unmixed and $x_2, \ldots, x_d$ is also a $C$-system of parameters of $B$. It follows from Lemma 2.3 that we have

$$N(B) \leq N(A) = N(M).$$

On the other hand, we have

$$f_0(q, M) \leq f_0(\overline{x}, A) \leq f_0(\overline{x}, B),$$

because of Lemma 2.1 and 2.2 in [TT20]. Consequently, we get that $N(B) \leq f_0(\overline{x}, B)$. By the hypothesis of induction, $B$ is Cohen-Macaulay. Thus $H^i_m(A) = 0$ for all $1 \leq i < d$.

Now it follows from the following sequence

$$0 \rightarrow M \xrightarrow{x_1} M \rightarrow A \rightarrow 0$$
that we have the long exact sequence

\[ \cdots \to H^1_\mathfrak{m}(M) \xrightarrow{z_1} H^1_\mathfrak{m}(M) \to H^1_\mathfrak{m}(A) \to 0 \cdots \]

\[ \cdots \to H^1_\mathfrak{m}(M) \xrightarrow{z_2} H^1_\mathfrak{m}(M) \to H^1_\mathfrak{m}(A) \to 0 \cdots \]

Then we have \( H^1_\mathfrak{m}(M) = 0 \) for all \( 2 \leq i \leq d - 1 \) and \( H^1_\mathfrak{m}(M) = x_1 H^1_\mathfrak{m}(M) \). Thus \( H^1_\mathfrak{m}(M) = 0 \) because \( H^1_\mathfrak{m}(M) \) is a finite generated \( R \)-module. Hence \( M \) is Cohen-Macaulay, as required.

For each integer \( n \geq 1 \), we denote by \( x^n \) the sequence \( x^1, x^2, \ldots, x^d \). Let \( K^*(x^n) \) be the Koszul complex of \( R \) generated by the sequence \( x^n \) and let \( H^*(x^n, R) = H^*(\text{Hom}_R(K^*(x^n), R)) \) be the Koszul cohomology module of \( R \). Then for every \( p \in \mathbb{Z} \), the family \( \{ H^p(x^n, R) \}_{n \geq 1} \) naturally forms an inductive system of \( R \), whose limit

\[ H_p^n(R) = \lim_{n \to \infty} H^p(x^n, R) \]

is isomorphic to the local cohomology module

\[ H_p^n(R) = \lim_{n \to \infty} \text{Ext}^p_R(R/m^n, R). \]

For each \( n \geq 1 \) and \( p \in \mathbb{Z} \), let \( \phi^{p,n}_{x^n} : H^p(x^n, R) \to H^n_p(R) \) denote the canonical homomorphism into the limit.

**Definition 3.2** ([GS03, Lemma 3.12]). There exists an integer \( n_0 \) such that for all systems of parameters \( \underline{x} = x_1, \ldots, x_d \) for \( R \) contained in \( m^{n_0} \) and for all \( p \in \mathbb{Z} \), the canonical homomorphisms

\[ \phi^{p,1}_{\underline{x}} : H^p(\underline{x}, R) \to H^n_p(R) \]

into the inductive limit are surjective on the socles. The least integer \( n_0 \) with this property is called a \( g \)-invariant of \( R \) and denote by \( g(R) \).

Now let \( \bigcap_{p \in \text{Ass}_{\mathcal{M}}} N_p = 0 \) be a reduced primary decomposition of the zero submodule of \( M \). Then the submodule \( U_M(0) = \bigcap_{p \in \text{Ass}_{\mathcal{M}}, \dim R/p = s} N_p \), is called the unmixed component of \( M \) and denoted by \( U_M(0) \). We denote \( u \) the unmixed component of \( R \).

**Definition 3.3** ([Sch99]).

1. A filtration \( \mathcal{D} : D_0 = (0) \subseteq D_1 \subseteq D_2 \subseteq \cdots \subseteq D_t = M \) of submodule of \( M \) is said to be a dimension filtration, if \( D_i \) is the largest submodule of \( D_{i+1} \) with \( \dim D_i < \dim D_{i+1} \) for all \( i = 1, \ldots, t - 1 \).
2. A system \( \underline{x} = x_1, x_2, \ldots, x_s \) of parameters of \( M \) is called distinguished, if \( (x_j \mid d_i < j \leq s)D_i = (0) \) for all \( 1 \leq i \leq t \). A parameter ideal \( q \) of \( M \) is called distinguished, if it is generated by a distinguished system of parameters of \( M \). Note that every \( C \)-system of parameter is distinguished, (see [CQ17, Proposition 4.8 and Remark 4.10]).

**Lemma 3.4** ([KOY20, Proposition 3.11]). Suppose that

\[ e_1(q : m, R) - e_1(q, R) \leq r_d(R) \]

for some \( C \)-parameter ideal \( q \subseteq m^{s(R)} \) of \( R \). Then \( R/u \) is Cohen-Macaulay.

**Lemma 3.5.** Assume that \( \dim R \geq 2 \). Then we have

\[ e_1(q : m, R) - e_1(q, R) \leq f_0(R), \]

for all \( C \)-parameter ideals \( q \).

**Proof.** Let \( I = q : R \). In the case that \( e_0(m, R) = 1 \). It follows from \( R \) is unmixed and Theorem 40.6 in [Nag62] that \( R \) is Cohen-Macaulay. We get by Lemma 2.6 that

\[ e_1(I, R) - e_1(q, R) = f_0(q, R). \]

Now suppose that \( e_0(m, R) > 1 \). Since \( \dim R \geq 2 \), by Proposition 2.3 in [GS03], we have \( mI^n = mq^n \) for all \( n \). Therefore \( I^n \subseteq q^n : m \) for all \( n \). Consequence, we obtain

\[ \ell(R/q^{n+1}) = \ell(R/I^{n+1}) \leq \ell((q^{n+1} : m/q^{n+1})). \]

Hence we have

\[ e_1(I, R) - e_1(q, R) \leq f_0(q, R), \]
as required. □

In [Tru14] and [Tru17], the first author provided the characterizations of Cohen-Macaulay rings in terms of its Chern numbers, irreducible multiplicities, index of reducibility of a parameter of \( R \), and the Cohen-Macaulay type, provided \( R \) is unmixed. Notice that the necessary and sufficient conditions of these characterizations need to hold true for all parameter ideals of \( R \). Recently, N. T. T. Tam and the first author in [TT20] gave the characterizations of Cohen-Macaulay rings in terms of its Chern numbers, irreducible multiplicities and the type, which was introduced by S. Goto and N. Suzuki ([GS84]). Let us now state the main results of this section and its corollaries.

**Theorem 3.6.** Assume that \( R \) is unmixed of dimension \( d \geq 2 \). Then the following statements are equivalent.

1. \( R \) is Cohen-Macaulay.
2. For some \( C \)-parameter ideal \( q \) of \( R \), we have
   \[
   \mathcal{N}(R) \leq e_1(q : m) - e_1(q).
   \]
3. For some \( C \)-parameter ideal \( q \) of \( R \), we have
   \[
   \mathcal{N}(R) \leq f_0(q, R).
   \]
4. For some \( C \)-parameter ideal \( q \subseteq \mathfrak{m}^g(R) \) of \( R \), we have
   \[
   f_0(q) \leq r_d(R).
   \]
5. For some \( C \)-parameter ideal \( q \subseteq \mathfrak{m}^g(R) \) of \( R \), we have
   \[
   e_1(q : m) - e_1(q) \leq r_d(R).
   \]

**Proof.** (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (4) follow from Lemma 2.6.

(2) \( \Rightarrow \) (3) and (4) \( \Rightarrow \) (5) are trivial.

(3) \( \Rightarrow \) (1) and (5) \( \Rightarrow \) (1) are immediate from Proposition 3.1 and Lemma 3.4. □

Let
\[
\Xi_i(M) = \{ e_i(q : R, M) \mid q \text{ is a } C \text{-parameter ideal of } M \},
\]
for all \( i = 1, \ldots, s \). Then we have the following results.

**Corollary 3.7.** Assume that \( R \) is unmixed of dimension \( d \geq 2 \). Then
\[
\Xi_1(R) \subseteq (-\infty, \mathcal{N}(R)].
\]
Moreover, \( \mathcal{N}(R) \in \Xi_1(R) \) if and only if \( R \) is Cohen-Macaulay.

**Proof.** Since \( e_1(q, R) \leq 0 \), we get by Theorem 3.6 that
\[
e_1(q : R, m, R) \leq \mathcal{N}(R)
\]
for all \( C \)-parameter ideals \( q \) of \( R \).

Assume that \( R \) is Cohen-Macaulay. We get by Lemma 2.6 that \( e_1(q : R, m, R) = \mathcal{N}(R) \) for all \( C \)-parameter ideals \( q \) of \( R \). Conversely, assume that \( q \) is a \( C \)-parameter ideals of \( R \) such that \( e_1(q : R, m, R) = \mathcal{N}(R) \). Since \( e_1(q, R) \leq 0 \), it follows from Theorem 3.6 that
\[
\mathcal{N}(R) \geq e_1(q : R, m, R) - e_1(q, R) \geq e_1(q : R, m, R) = \mathcal{N}(R).
\]
Therefore \( \mathcal{N}(R) = e_1(q : R, m, R) - e_1(q, R) \). Thus \( R \) is Cohen-Macaulay because of Theorem 3.6. This completes the proof. □

**Corollary 3.8.** Assume that \( \dim R \geq 2 \). Then we have
\[
\mathcal{N}(R/w) \geq f_0(q, R) \geq e_1(q : R, m, R) - e_1(q, R) \geq r_d(R),
\]
for all \( C \)-parameter ideals \( q \subseteq \mathfrak{m}^g(R) \) of \( R \).
Proof. Put \( S = R/u, I_R = q : R m \) and \( I_S = qS :S mS \). Then it follows from Corollary 3.2 in [Sch99] that \( S \) is a unmixed ring of dimension \( d \). Then, applying Theorem 3.6 we have \( \mathcal{N}(S) \geq f_0(q, S) \). In addition, it follows from Lemma 2.1 in [TT20], Lemma 3.5 and Corollary 4.4 in [KOY20] that
\[
f_0(q, S) \geq f_0(q, R) \geq e_1(I_R, R) - e_1(q, R) \geq r_d(R).
\]
Consequently,
\[
\mathcal{N}(S) \geq f_0(q, R) \geq e_1(q : R m, R) - e_1(q, R) \geq r_d(R),
\]
and this complete the proof. \( \square \)

The following consequence of Theorem 3.6 provides a characterization of Gorenstein rings.

**Corollary 3.9.** Assume that \( R \) is unmixed of dimension \( d \geq 2 \). Then the following statements are equivalent.

1. \( R \) is Gorenstein.
2. For some \( C \)-parameter ideal \( q \) of \( R \), we have \( \mathcal{N}(R) = 1 \).
3. For some \( C \)-parameter ideal \( q \) of \( R \), we have \( f_0(q, R) = 1 \).
4. For some \( C \)-parameter ideal \( q \subseteq m^g(R) \), we have \( e_1(q : m) - e_1(q) = 1 \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( q \) be a \( C \)-parameter ideal of \( R \). Since \( R \) is Gorenstein,
\[
\mathcal{N}(R) = \ell(0) :_{H^d_m(R)} m = 1.
\]

(2) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (4). Since \( R \) is Gorenstein, \( r_d(R) = \mathcal{N}(R) = 1 \). Therefore by Corollary 3.8, we have
\[
\mathcal{N}(R) = f_0(q, R) = e_1(q : R m) - e_1(q) = r_d(R) = 1.
\]

(4) \( \Rightarrow \) (1). Let \( q \) be a \( C \)-parameter ideal such that \( e_1(q : R m) - e_1(q) = 1 \) and \( q \subseteq m^g(R) \), then we have \( e_1(q : R m) - e_1(q) \leq r_d(R) \). By Theorem 3.6, \( R \) is Cohen-Macaulay. Therefore, we have
\[
1 = e_1(q : R m) - e_1(q) = r_d(R).
\]
Hence, \( R \) is Gorenstein, as required. \( \square \)

4. The finiteness of the set of Chern numbers

In this section, we analyse the boundness of the values \( e_1(q : R m, R) \) for parameter ideals \( q \) of \( R \) and deduced that the local cohomology modules \( \{ H^i_m(R) \} \) are finitely generated, once \( R \) is unmixed.

For each \( n, i \geq 1 \), we put
\[
\Omega^i_q(R) = \{ e_1(q : R m, R) \mid q \text{ is a } C \text{-parameter ideal of } R \text{ contained in } m^n \}.
\]
Note that \( \Omega^i_q(R) \neq \emptyset \) and \( \Omega^i_q(R) \subseteq \Omega^i_q(R) \subseteq \Xi_i(R) \), for all \( t \geq t' \geq 1 \). Moreover, we get by Lemma 2.6 that if \( R \) is generalized Cohen-Macaulay then \( \Xi_i(R) \) is finite for all \( i \geq 1 \) and \( n \geq 1 \).

Let \( \underline{x} = x_1, x_2, \ldots, x_d \) be a \( C \)-system of parameters of \( R \). Put \( q^{[n]} = (x_1^n, x_2^n, \ldots, x_d^n) \) and \( I^{[n]} = q^{[n]} : R m, \) for all \( n \geq 1 \). For each \( t \geq 1 \), let
\[
\Omega^i_{\underline{x}, 1}(R) = \{ e_1(I^{[n]}, R) \mid n \geq t \}.
\]

Note that \( \Omega^i_{\underline{x}, 1}(R) \neq \emptyset \) and \( \Omega^i_{\underline{x}, 1}(R) \subseteq \Omega^i_{\underline{x}, 1}(R) \) for all \( t \geq t' \geq 1 \).

**Lemma 4.1.** Assume that \( R \) is unmixed of dimension \( d \geq 2 \). Suppose that there exist a \( C \)-system of parameters \( \underline{x} = x_1, \ldots, x_d \) of \( R \) and an integer \( t \) such that the set \( \Omega^i_{\underline{x}, 1}(R) \) is finite. Let
\[
h = \max\{|X| : X \in \Omega^i_{\underline{x}, 1}(R)\} + \mathcal{N}(R).
\]
Then \( m^h H^i_m(R) = 0 \) for all \( i \neq d \).

**Proof.** We use induction on the dimension \( d \) of \( R \). Suppose that \( d = 2 \). Since \( R \) is unmixed, \( R \) is generalized Cohen-Macaulay, and so \( \mathcal{N}(R) = 2r_1(R) + r_2(R) \). We choose \( n \geq t \). Put \( q = (x_1^n, x_2^n) \) and \( I = q : R m \). Then, since \( x_1, x_2 \) is a \( C \)-system of parameters of \( R \), by Lemma 2.6 we have
\[
\ell(H^i_m(R)) = -e_1(I, R) + r_1(R) + r_2(R) \leq |e_1(I, R)| + \mathcal{N}(R) \leq h.
\]
Hence, $m^h H^1_m(R) = 0$.

Suppose that $d \geq 3$ and our assertion holds true for $d - 1$. Let $n \geq t$ be an integer such that $x^n H^0_m(R) = (0)$. Let $y_1 = x^n$ and $A = R/(y_1)$. Then $\dim A = d - 1$ and by the fact 2.4, we have $\text{Ass} A \subset \text{Ass} A \cup \{m\}$. It follows that $U_A(0) = H^0_m(A)$. Let $B = A/H^0_m(A)$. Then $B$ is unmixed.

Now, by Lemma 2.3, $\varepsilon' = x_2, x_3, \ldots, x_d$ is a $C$-system of parameters of $A$. Thus, $\varepsilon'$ is also a $C$-system of parameters of $B$. On the other hand, it follows from Lemma 2.4 in [CT08] that there exists a positive integer $t' \geq t$ such that

$$[(x_{1}^{m}, x_{2}^{m}, \ldots, x_{t'}^{m}) + H^0_m(A)] : A \cdot m A = [(x_{2}^{m}, x_{3}^{m}, \ldots, x_{t'}^{m}) : A \cdot m A] + H^0_m(A),$$

for all $m \geq t'$. Therefore, we have the following.

**Claim 4.2.** $\Omega_{\varepsilon'}^{-1}(B) \subseteq \Omega_{\varepsilon_1}^{-1}(R)$.

**Proof.** For each $m \geq t'$, we put $y_2 = x_2, \ldots, y_d = x_d$, $q = (y_1, y_2, \ldots, y_d)$, $I = q : R \cdot m$, $q' = (y_2, \ldots, y_d)$. Since $\varepsilon$ is a $C$-system of parameters of $R$, it forms a $d$-sequence of $R$. Thus $x_1$ is a superficial element of $R$ with respect to $I$ and $q$. Therefore, by Lemma 2.1, we have

$$e_1(I, R) = e_1(I', A),$$

where $I' = q' : A \cdot m A = (q : R \cdot m) A$. It follows from Lemma 2.2, $d \geq 3$ and $[q' + H^0_m(A)] : A \cdot m A = [q' : A \cdot m A] + H^0_m(A)$ that we have

$$e_1(I', A) = e_1(q' : B \cdot m B, B).$$

Consequently, $e_1(q' : B : B \cdot m B, B) = e_1(I, R) \in \Omega_{\varepsilon_1}^{-1}(R)$. Hence $\Omega_{\varepsilon'}^{-1}(B) \subseteq \Omega_{\varepsilon_1}^{-1}(R)$, as required.

It follows from the above claim that the set $\Omega_{\varepsilon'}^{-1}(B)$ is finite. Thus by the hypothesis inductive on $d$, we have $m^h H^1_m(B) = 0$, for all $i \neq d - 1$, where $h' = \max\{|X| : X \in \Omega_{\varepsilon'}^{-1}(B)\} + \mathcal{N}(B)$. On the other hand, by Lemma 2.3, we have

$$\mathcal{N}(R) = \mathcal{N}(A) = \mathcal{N}(B) + r_0(A).$$

Therefore $h' \leq h$, and so $m^h H^1_m(B) = 0$, for all $i \neq d - 1$. Hence $m^h H^1_m(A) = 0$, for all $1 \leq i \leq d - 2$.

Now since $R$ is unmixed, $y_1$ is $R$-regular. Therefore it follows from the exact sequence

$$0 \longrightarrow R \longrightarrow R \overset{y_1}{\longrightarrow} R \longrightarrow A \longrightarrow 0$$

that we have the surjective maps $H^i_m(A) \longrightarrow (0) : H^{i+1}_m(R) y_1$ for all $i \leq d - 2$ and the injective map $H^1_m(R) : H^1_m(R) y_1 \longrightarrow H^1_m(A)$, since $y_1 H^1_m(R) = 0$. Thus we have

$$m^h[(0) : H^{i+1}_m(R) y_1] = (0)$$

and $m^h H^1_m(R) = 0$

for all $1 \leq i \leq d - 2$. It follows from $H^i_m(R) = \bigcup_{n \geq 1} [(0) : H^i_m(R) m^n]$ that we have $m^h H^i_m(R) = (0)$ for all $i \neq d$, as required.

**Lemma 4.3.** Assume that $R$ is unmixed of dimension $d \geq 2$. Suppose that $R/u$ is generalized Cohen-Macaulay. Then

$$q : R \cdot m = q : R/u \cdot m,$$

for all $C$-parameter ideals $q$ of $R/u$.

**Proof.** Put $S = R/u$. Let $q$ is a $C$-parameter of $S$, $I_R = q : R \cdot m$ and $I_S = q : S \cdot m S$. Then $q$ is also a parameter of $S$. Since $S$ is generalized Cohen-Macaulay, we have $(I_S)^2 = q I_S$. Thus we get by Theorem 1.1 in [GN03] that $e_1(I_S, S) = \ell(S/I_S) - e_0(q, S) + e_1(q, S)$. By Theorem 3.1 in [GN03], we have

$$\ell(S/I_S) - e_0(q, S) = e_1(I_S, S) - e_1(q, S) \geq e_1(I_R, S) - e_1(q, S) \geq \ell(S/I_R S) - e_0(q, S).$$

Then we have $I_S = I_R$ since $I_R \subseteq I_S$. 

**Theorem 4.4.** Assume that $d \geq 2$. Then the following statements are equivalent.
(1) $R/u$ is generalized Cohen-Macaulay and $\dim u \leq d - 2$.

(2) There exists an integer $t$ such that the set $\Omega^d_{\ell}(R)$ is finite.

**Proof.** (1) $\Rightarrow$ (2). Let $q$ be a $C$-parameter ideals of $R$ containing in $m^{g(R)}$. We get by Corollary 3.8 that
\[
\mathcal{N}(R/u) \geq e_1(q : R, m, R) - e_1(q, R) \geq r_d(R).
\]
On the other hand, the set $\{e_1(q, R) \mid q$ is a parameter of $R\}$ is finite because of Theorem 4.5 in [GHH\textsuperscript{+}14]. Thus $\Omega^d_{\ell}(R)$ is finite for all $t \geq g(R)$.

(2) $\Rightarrow$ (1). We put $S = R/u$ and $u = \dim u$. Then there exists a $C$-system $\underline{x} = x_1, \ldots, x_d$ of parameters of $R$ such that
1. $(\underline{x}) \subseteq m^t$,
2. $(x_{u+1}, x_{u+2}, \ldots, x_d)u = 0$,
3. $\underline{x}$ is a $C$-system of parameters of $S$.

Let $q^{[n]} = (x_1^n, \ldots, x_d^n)R$, $I^n_S = q^{[n]} : R$ m and $I^n_S = q^{[n]}S : S$ mS, where $n \geq t$. We have $I^n_R \subseteq I^n_S$, and so we get $\ell(S/(I^n_S)^{n+1}) \leq \ell(S/(I^n_R)S)$ for all $m \geq 0$. Thus $e_1(I^n_S, S) \geq e_1(I^n_R, S)$, because $I^n_S$ is integral over $q^{[n]}S$ by Proposition 2.3 in [GS03]. Therefore by Corollary 3.8, we have
\[
\mathcal{N}(S) \geq e_1(I^n_S, S) - e_1(q^{[n]}, S) \geq e_1(I^n_R, S) - e_1(q^{[n]}, S)
\]
\[
\geq e_1(I^n_R, R) - e_1(q^{[n]}, R) \geq r_d(R) = r_d(S).
\]
Since the set $\Omega^d_{\ell}(R)$ is finite, so is the set $\{e_1(I^n_S, R) \mid n \geq 0\}$. Thus the set $\{e_1(q^{[n]}, R) \mid n \geq 0\}$ is finite. By Lemma 2.2, we have
\[
0 \geq e_1(q^{[n]}, S) \geq e_1(q^{[n]}, R).
\]
Therefore, the set $\{e_1(q^{[n]}, S) \mid n \geq 0\}$ is finite. By (1), the set $\Omega^d_{n+1}(S)$ is finite for some $t'$. It follows from $S$ is unmixed and Lemma 4.1 that $S$ is generalized Cohen-Macaulay.

Now suppose that $u = d - 1$. It follows from Lemma 2.6 that
\[
e_1(I^n_S, S) = -\left(\sum_{j=1}^{d-1} \binom{d-2}{j-1} \ell(H_m^j(S)) - \sum_{j=1}^d r_j(S)\right),
\]
which is independent of the choice of $n$. On the other hand, we have
\[
e_0(I^n_R, u) = e_0(q^{[n]}, u) = e_0(x_1^n, \ldots, x_d^n, u) = n^ue_0(q, u) \geq n^t.
\]
Thus, by Lemma 2.2, we get that
\[
-e_1(I^n_R, R) = -e_1(I^n_R, S) + e_0(I^n_R, u)
\]
\[
= -e_1(I^n_S, S) + n^ue_0(q, u),
\]
which is in contradiction with the finiteness of $\Omega^d_{\ell}(R)$ and $\Omega^d_{n}(S)$. Hence $u \leq d - 2$. 

Applying Theorem 4.4 we obtain the following result.

**Theorem 4.5.** Assume that $R$ is unmixed with $\dim R \geq 2$. Then the following statements are equivalent.

1. $R$ is generalized Cohen-Macaulay.
2. The set $\Xi_1(R)$ is finite.

**Proof.** It is immediate from Theorem 4.4. 

**Theorem 4.6.** Assume that $d \geq 2$. Then the following statements are equivalent.

1. $R/H_m^0(R)$ is Cohen-Macaulay.
2. The sets $\Xi_i(R)$ are finite for all $i \geq 2$ and there exists a $C$-parameter ideal $q \subseteq m^{g(R)}$ of $R$

such that
\[
e_1(q : R, m, R) - e_1(q, R) \leq r_d(R).
\]
Proof. Let $S = R/u$.
1) $\Rightarrow$ 2). Since $S$ is Cohen-Macaulay, $R$ is generalized Cohen-Macaulay. Therefore $\Xi_i(R)$ are finite for all $i \geq 2$. Let $q \subseteq \mathfrak{m}^{k(R)}$ be a $C$-parameter ideals of $R$. Then by Lemma 2.4 in [CT08], we can assume that

$$[u + q] : \mathfrak{m} = u + [q : \mathfrak{m}].$$

It follows from the Lemma 2.6, Lemma 2.2 that

$$e_1(q : R \mathfrak{m}, R) - e_1(q, R) = e_1(q : S \mathfrak{m}, S) - e_1(q, S) = e_1(qS : S \mathfrak{m}S, S) - e_1(q, S) = r_d(S) = r_d(R),$$

as required.
2) $\Rightarrow$ 1). Since

$$e_1(q : R \mathfrak{m}, R) - e_1(q, R) \leq r_d(R)$$

for some $C$-parameter ideals $q \subseteq \mathfrak{m}^{k(R)}$ of $R$, by Lemma 3.4, $S$ is Cohen-Macaulay.

Suppose that $u = \dim u \geq 1$. Then there exists a $C$-system $\underline{x} = x_1, \ldots, x_d$ of parameters of $R$ such that

i) $(\underline{x}) \subseteq \mathfrak{m}^t$,

ii) $(x_{u+1}, x_{u+2}, \ldots, x_d)u = 0$,

iii) $\underline{x}$ is a $C$-system of parameters of $S$.

Let $q^n = (x^n_1, \ldots, x^n_d)R$, $I^n_R[q^n] : R \mathfrak{m}$ and $I^n_S[q^n] : S \mathfrak{m}S$ for all $n \geq 0$. Since $S$ is Cohen-Macaulay, we get by Lemma 4.3 that $I^n_S = I^n_R$. Moreover, it follows Lemma 2.6 that

$$e_{d-u}(I^n_S[S], S) = \begin{cases} 0 & \text{if } u \leq d - 2, \\ r_d(S) & \text{if } u = d - 1. \end{cases}$$

which is independent of the choice of $n$. On the other hand, we have

$$e_0(I^n_R[u], u) = e_0(q^n[u], u) = e_0((x^n_1, \ldots, x^n_d), u) = n^ue_0(q, u) \geq n^u.$$ 

Thus, by Lemma 2.2, we get that

$$(-1)^{d-u}e_{d-u}(I^n_R[R]) = (-1)^{d-u}e_{d-u}(I^n_R[S], S) + e_0(I^n_R[u], u)$$

$$\geq n^u + r_d(S),$$

which is in contradiction with the finiteness of $\Xi_{d-u}(R)$. Hence $u = 0$, and this complete the proof. □

**Theorem 4.7.** Assume that $\dim R \geq 2$. Then the following statements are equivalent.

(i) $R$ is generalized Cohen-Macaulay.

(ii) The sets $\Xi_i(R)$ are finite for all $i = 1, 2, \ldots, d$.

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 2.6.

(ii) $\Rightarrow$ (i). We put $S = R/u$ and $u = \dim u$. Assume that $u > 0$. Since $\Xi_1(R)$ is finite, it follows from Theorem 4.5 that $S$ is generalized Cohen-Macaulay. Then there exists a $C$-system $\underline{x} = x_1, \ldots, x_d$ of parameters of $R$ such that

i) $(\underline{x}) \subseteq \mathfrak{m}^t$,

ii) $(x_{u+1}, x_{u+2}, \ldots, x_d)u = 0$,

iii) $\underline{x}$ is a $C$-system of parameters of $S$.

Let $q^n = (x^n_1, \ldots, x^n_d)R$, $I^n_R[q^n] : R \mathfrak{m}$ and $I^n_S[q^n] : S \mathfrak{m}S$ for all $n \geq 0$. Since $S$ is generalized Cohen-Macaulay, we get by Lemma 4.3 that $I^n_S = I^n_R$. Moreover, it follows Lemma 2.6 that

$$e_{d-u}(I^n_S[S], S) = (-1)^{d-u} \left( \sum_{j=1}^{u+1} \binom{u}{j-1} \ell_R(H^n_{\mathfrak{m}^n}(S)) - \sum_{j=1}^{u+1} r_j(S) \right),$$

which is independent of the integer $n$. On the other hand, we have

$$e_0(I^n_R[u], u) = e_0(q^n[u], u) = e_0((x^n_1, \ldots, x^n_d), u) = n^ue_0(q, u) \geq n^u.$$
Thus, by Lemma 2.2, we get that
\[
(-1)^{d-u}e_{d-u}(I_R^n, R) = (-1)^{d-u}e_{d-u}(I_R^n, S) + e_0(I_R^n, u)
\]
\[
= (-1)^{d-u}e_{d-u}(I_S^n, S) + nu^c_0(q, u),
\]
which is in contradiction with the finiteness of $\Xi_{d-u}(R)$ and $\Xi_{d-u}(S)$. Hence $u = 0$, as required. \(\square\)

**Remark 4.8.** In [KT15], A. Koura and N. Taniguchi proved that the Chern numbers of $m$-primary ideals in $R$ range among only finitely many values if and only if $\dim R = 1$ and $R/H_m^n(R)$ is analytically unramified.

Let us note the following example of non-generated Cohen-Macaulay local rings $R$ with $|\Xi_1(R)| = 1$. Moreover, this example also shows that the statements (1), (4) and (5) in Theorem 3.6 will not be equivalent, if the unmixed condition is removed from the hypothesis.

**Example 4.9.** Let $S = k[[X, Y, Z, W]]$ be the formal power series ring over a field $k$. Put $R = S/[(X, Y, Z) \cap (W)]$. Then

1. $R$ is not unmixed with $\dim R = 3$. Moreover, $R$ is not generalized Cohen-Macaulay.
2. $|\Xi_1(R)| = 1$.
3. For all parameter ideals $q$ in $R$, we have

\[
f(q, R) = e_1(q :_R m) - e_1(q) = r_d(R).
\]

**Proof.** We put $A = S/((W)$ and $B = S/((X, Y, Z)$. Then $A$ and $B$ are Cohen-Macaulay. Then $R$ is not generalized Cohen-Macaulay because $H_m^n(R) = H_m^n(B)$ is not a finitely generated $R$-module. Let $q$ be a parameter ideal in $R$ and put $I = q : m$. Then $I^n = q^n : m$, for all $n \geq 0$. Thanks to the exact sequence $0 \to B \to R \to A \to 0$, the sequence

\[
0 \longrightarrow B/q^n+1B \longrightarrow R/q^n+1R \longrightarrow A/q^n+1A \longrightarrow 0
\]

is exact. By applying the functor $\text{Hom}_R(R/m, \bullet)$ we obtain the following exact sequence

\[
0 \longrightarrow [q^{n+1} :_B m]/q^n+1 \longrightarrow [q^{n+1} : R m]/q^n+1 \longrightarrow [q^{n+1} : A m]/q^n+1 \longrightarrow 0.
\]

Therefore, we have

\[
\ell_R(R/q^n) = \ell_R(A/q^n) + \ell_R(B/q^n)
\]
\[
= \ell_R(A/qA) \binom{n+3}{3} + \ell_R(B/qB) \binom{n+1}{1}
\]

and

\[
\ell_R([q^{n+1} :_B m]/q^n) = \ell_R([q^{n+1} : A m]/q^n) + \ell_R([q^{n+1} : B m]/q^n)
\]
\[
= \binom{n+1}{2} + 1,
\]

for all integers $n \geq 0$. Since $\ell_R(R/I^n) = \ell_R(R/q^n) - \ell_R([q^{n+1} : R m]/q^n)$, we have $e_1(q : R m) = 0 + 1 = 1$. Thus $|\Xi_1(R)| = 1$. Moreover, we have $f(q, R) = e_1(q : R m) - e_1(q) = r_d(R) = 1$, as required. \(\square\)

Now we close this paper with the following open question, which are suggested during the work in this paper, on the characterization of the Cohen-Macaulayness in terms of its Chern numbers and stable value as follows.

**Question 4.10.** Assume that $\dim R \geq 2$. Then is $R$ is Cohen-Macaulay if and only if we have $\mathcal{N}(R) \leq e_1(q : m) - e_1(q)$ for some $C$-parameter ideal $q$ of $R$?
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Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, 10307 Hanoi, Viet Nam

Thang Long Institute of Mathematics and Applied Sciences, Hanoi, Vietnam

Email address: hltruong@math.ac.vn, truonghoangle@gmail.com

The Department of Mathematics, Thai Nguyen University of education, 20 Luong Ngoc Quyen Street, Thai Nguyen City, Thai Nguyen Province, Viet Nam.

Email address: yenhn@tnue.edu.vn