Isospin-breaking corrections to nucleon electroweak form factors in the constituent quark model

V. Dmitrašinović and S.J. Pollock

Nuclear Physics Laboratory, Physics Department,
University of Colorado, P.O.Box 446, Boulder, CO 80309-0446
Email addresses: dmitra@godot.colorado.edu, pollock@lucky.colorado.edu

Abstract

We examine isospin breaking in the nucleon wave functions due to the $u-d$ quark mass difference and the Coulomb interaction among the quarks, and their consequences on the nucleon electroweak form factors in a nonrelativistic constituent quark model. The mechanically induced isospin breaking in the nucleon wave functions and electroweak form factors are exactly evaluated in this model. We calculate the electromagnetically induced isospin admixtures by using first-order perturbation theory, including the lowest-lying resonance with nucleon quantum numbers but isospin $3/2$. We find a small ($\leq 1\%$), but finite correction to the anomalous magnetic moments of the nucleon stemming almost entirely from the quark mass difference, while the static nucleon axial coupling remains uncorrected. Corrections of the same order of magnitude appear in charge, magnetic, and axial radii of the nucleon. The correction to the charge radius in this model is primarily isoscalar, and may be of some significance for the extraction of the strangeness radius from e.g. elastic forward angle parity violating electron-proton asymmetries, or elastic $^4He(e,e')$ experiments.

PACS numbers: 13.40.-f, 13.40.Ks, 14.20.Dh
I. INTRODUCTION

Parity-violating electroweak lepton-nucleon scattering such as \( N(\vec{e}, e')N \), \( N(\nu, \nu')N \) is of particular interest to subnucleon physics because it is sensitive to the Standard Model fundamental coupling constants \( \alpha, \sin^2 \theta_W \) and various polar- and axial-vector current form factors of the nucleon \([1]\). The standard analysis, that led to the current interest in these processes, is based on the following assumptions: (i) Lorentz + translational invariance, (ii) Standard Model of lepton and quark electroweak interactions, (iii) one boson exchange approximation, (iv) significance of only \( u \) and \( d \) quarks in the nucleon states, (v) good parity of the nucleon states, and (vi) good isospin of the nucleon states. In view of pending \( N(\vec{e}, e')N \) experiments \([2]\), and of pending and completed elastic \( N(\nu, \nu')N \) \([3]\) measurements, it seems worthwhile to explore the corrections due to the relaxation of some, or all, of these premises. So far, assumptions (ii-v) have been relaxed and their consequences explored \([1,4]\). In this note we will relax assumption (vi) in a specific quark model of the nucleon and look at its effects on electroweak form factors of the nucleon. The results are of immediate relevance to the interpretation of the isoscalar axial coupling as measured e.g. in elastic neutrino-nucleon scattering \([3,5,6]\). They will also be relevant for isoscalar vector couplings extracted in future neutrino and parity violation experiments. This is because intrinsic isospin breaking, although expected to be small, modifies observables in much the same way as nonzero strangeness content of the nucleon does.

A common present-day picture of the nucleon \([7–9]\) is one of the ground state of three quarks bound by gluon exchange according to quantum chromodynamics (QCD). Given the intractable nature of this strong coupling few-body problem, we resort to a nonrelativistic constituent quark model with harmonic oscillator confining quark-quark interactions, chosen for its simplicity. The isospin breaking (IB) in our model is due to the following two sources: quark mass differences, and electroweak interactions, which is essentially the same as in QCD \([10]\). Since both of these corrections are expected to be small relative to the isospin-conserving components, we will evaluate each one in its own right without considering the cross-terms. The results of our analysis will justify this assumption \( \text{ex post facto} \). Furthermore, we will confine ourselves to the investigation of electroweak interactions among quarks in the nucleon and neglect their self-interactions. The results will turn out to be particularly sensitive to the shift in the nucleon’s center of mass, which is faithfully reflected in this model.

In this paper, we begin (Section II) with a general discussion of isospin breaking and the connection to electroweak form factors. The purpose is to formalize and clarify the extent to which e.g. the strangeness content of the nucleon can be distinguished from IB modifications in a model independent way. In section III, we describe the specific quark model we use, and outline the calculation of IB effects due to electromagnetic quark-quark interactions, and quark mass differences. In section IV, we present the results of these calculations and discuss them. We find that some IB effects are significantly larger than one might naively expect. Only one of the corrections (the weak neutral current charge radius) we have found appears to present possible qualitative problems for extracting non-trivial strangeness content, if that should turn out to be present. The calculations presented in sections III and IV are model dependent, and are manifestly just a part of a larger set of corrections. We conclude section IV with a brief discussion of future directions to be pursued. In section V, we summarize
our results and draw our conclusions.

II. MODEL-INDEPENDENT FORMALISM

One can define isospin at two different levels: (a) at the level of quarks, where the Standard Model is constructed; and (b) at the hadron (here, nucleon) level, where all of experimental data originate. We will need both levels to define IB in our formalism. That fact limits the applicability of the present formalism to quark models of the nucleon and leaves out nucleon models with purely hadronic degrees of freedom such as Skyrme’s. Attempts to define isospin-breaking admixtures to the nucleon wave functions exclusively at level (b) can easily lead to tautologies and circular arguments. In order to avoid that we will first review the consequences of exact isospin, and then introduce notation which should clearly denote the origin and character of the various contributions to the total (physical) form factors.

In the isospin-symmetric limit the isospin transformation properties of the operators are identical to those of the matrix elements, as a direct consequence of the Wigner-Eckart theorem. The main consequence of isospin breaking in the nucleon wave functions is the loss of these, previously exact, isospin transformation properties of the electroweak current matrix elements as compared with those of the current operators themselves. A simple example will best clarify this statement. The third component of the isovector axial current in the “nuclear domain”, where only $u, d$ quarks are kept, reads

$$ J^z_{\mu5} = \bar{q} \frac{\tau^z}{2} \gamma_5 q. \quad (1) $$

When sandwiched between two nucleon states $|N\rangle$ with “impure” isospin, this does not yield just $+\frac{1}{2} F_A$ for the proton and $-\frac{1}{2} F_A$ for the neutron, but rather it induces a small but finite effective isoscalar axial form factor. To see the relevance of these comments, we turn to the weak neutral current (NC) $J_{\mu NC} = J_{\mu NC} - J_{\mu5}^{NC}$. The axial neutral current, in the presence of three flavors, is

$$ J_{\mu5}^{NC} = \frac{1}{2} \left( \bar{u} \gamma_{\mu5} u - \bar{d} \gamma_{\mu5} d - \bar{s} \gamma_{\mu5} s \right). \quad (2) $$

$J_{\mu5}^{NC}$ consists of the nuclear domain isovector current and a strange quark part. It is these “strange” (hidden strangeness-induced) terms that are a topic of considerable recent interest. They would also yield an effective isoscalar axial form factor. The hidden strangeness and intrinsic isospin breaking effects cannot be separated by experiment, thus any nontrivial isospin breaking could potentially affect a determination of the strange form factors.

Similar, but not identical comments hold for the charge and magnetic form factors, because both the isoscalar and isovector current operators enter the EM current. In the following we will define the effects of such isospin-breaking in the nucleon wave functions on

$^{1}$Note that our conventions on the definitions of electroweak currents differ slightly from ref. [1]. We adopt the Bjorken and Drell (BD) [11] metric and conventions, with the exception of the normalization of the Pauli form factor which we take to be $F_2(0) = \kappa$. 

3
electroweak form factors. We define the axial-vector current form factors for real nucleon states as follows:

\[
\langle N(p') | \frac{1}{2} (\bar{u} \gamma_\mu \gamma_5 u - \bar{d} \gamma_\mu \gamma_5 d) | N(p) \rangle \equiv \bar{u}(p') \left[ \frac{1}{2} (u^{-d} F^{p+n}_A + u^{-d} F^{p-n}_A) \gamma_\mu \gamma_5 \right] u(p) ,
\]

(3)

where \( \pm = \langle N | \tau \gamma_5 | N \rangle \) for proton or neutron matrix elements, respectively, and \( q^2 = q_0^2 - \mathbf{q}^2 \)
is the invariant momentum transfer. The notation for the form factor superscripts is as follows: the upper left-hand index refers to the quark operator on the left-hand side of the equation. The upper right-hand index refers to the nucleon isospin operator. In the above case, the quark operator is \( u - d \) and hence quark-isovector. It induces a large nucleon-isovector \( u^{-d} F^{p-n}_A \) axial form factor and a small nucleon-isoscalar axial form factor \( u^{-d} F^{p+n}_A \) correction. We also define the “strange” form factors \( S_{1,2,A} \) as follows

\[
\langle N(p') | (\bar{s} \gamma_\mu \gamma_5 s) | N(p) \rangle \equiv \bar{u}(p') \left[ S_A(q^2) \gamma_\mu \gamma_5 \right] u(p) ,
\]

(4)

\[
\langle N(p') | (\bar{s} \gamma_\mu s) | N(p) \rangle \equiv \bar{u}(p') \left[ S_1(q^2) \gamma_\mu + \frac{i \sigma_{\mu\nu} q^\nu}{2 M_N} S_2(q^2) \right] u(p) ,
\]

(5)

which are all isoscalar to first approximation. We ignore here any (isovector) IB corrections to the strange form factors of the nucleon, since the strange form factors themselves can be viewed as a consequence of flavor SU(3) symmetry breaking admixtures and hence are expected to be small relative to the \( u, d \) induced ones, thus making the IB corrections to them “doubly small” i.e. a second order effect. \( S_1(0) = 0 \) follows from the fact that the nucleon has zero total strangeness.

With our conventions, the EM current operator reads

\[
J_{\mu}^{\text{EM}} = \frac{1}{2} (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) + \frac{1}{6} (\bar{u} \gamma_\mu u + \bar{d} \gamma_\mu d - 2 \bar{s} \gamma_\mu s) .
\]

(6)

The polar-vector part of the weak neutral current operator is

\[
J_{\mu}^{\text{NC}} = \frac{1}{2} (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) - \frac{1}{2} \bar{s} \gamma_\mu s - 2 \sin^2 \theta_W J_{\mu}^{\text{EM}} ,
\]

(7)

where \( \theta_W \) is the weak mixing angle. We define polar-vector form factors for real nucleon states as follows:

\[
\langle N(p') | \frac{1}{2} (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) | N(p) \rangle \equiv \bar{u}(p') \left[ \frac{1}{2} (u^{-d} F^{p+n}_1 + u^{-d} F^{p-n}_1) \gamma_\mu + \frac{1}{2} (u^{-d} F^{p+n}_2 + u^{-d} F^{p-n}_2) \frac{i \sigma_{\mu\nu} q^\nu}{2 M_N} \right] u(p) .
\]

(8)

\[
\langle N(p') | \frac{1}{2} (\bar{u} \gamma_\mu u + \bar{d} \gamma_\mu d) | N(p) \rangle \equiv \bar{u}(p') \left[ \frac{1}{2} (u^{-d} F^{p+n}_1 + u^{-d} F^{p-n}_1) \gamma_\mu + \frac{1}{2} (u^{-d} F^{p+n}_2 + u^{-d} F^{p-n}_2) \frac{i \sigma_{\mu\nu} q^\nu}{2 M_N} \right] u(p) .
\]

(9)

In this notation, if isospin were a good symmetry of the nucleon states, and there were no strangeness content, then \( u^{-d} F^{p+n}(q^2) = u^{-d} F^{p-n}(q^2) = 0 \) (by the Wigner-Eckart theorem), and \( u^{-d} F^{p+n}(q^2) \) would be the usual nucleon isoscalar form factor, \( u^{-d} F^{p-n}(q^2) \) would be
the usual nucleon isovector form factor. With the above definitions, we can write weak matrix elements in terms of the above electromagnetic and the strange ones with some, as yet unknown, nucleon IB corrections. For example,

\[ F_{i,N}^{NC} = \frac{1}{2} (u-d F_{i}^{p+n} \pm u-d F_{i}^{p-n}) - 2 \sin^2 \theta_W F_{i,N}^{EM} - \frac{1}{2} S_i , \]

where \( i = 1, 2 \), i.e. for polar-vector neutral current form factors. Similarly, we have for the axial-vector NC form factors

\[ F_{A,N}^{NC} = \frac{1}{2} (u-d F_{A}^{p+n} \pm u-d F_{A}^{p-n}) - \frac{1}{2} S_A . \]

Taking the difference between the proton and the neutron polar-vector NC form factors, we find

\[ F_{i,p}^{NC} - F_{i,n}^{NC} = (1 - 2 \sin^2 \theta_W)(F_{i,p}^{EM} - F_{i,n}^{EM}) - u+d F_{i}^{p-n} . \]

Here the isospin breaking form factor \( u+d F_{i}^{p-n} \) appears as a correction to the usual isovector form factor relation, and strangeness content cancels out. In principle, this means that the IB isovector correction \( u+d F_{i}^{p-n} \) is measurable to this order in perturbation theory, given a complete set of experiments, i.e. if we had both the proton and the neutron experiments. Similarly, we find for the isoscalar part

\[ F_{i,p}^{NC} + F_{i,n}^{NC} = -2 \sin^2 \theta_W (F_{i,p}^{EM} + F_{i,n}^{EM}) - S_i + u-d F_{i}^{p+n} . \]

The final term again represents isospin breaking effects. For the axial form factors we find

\[ F_{A,p}^{NC} - F_{A,n}^{NC} = u-d F_{A}^{p-n} , \]

\[ F_{A,p}^{NC} + F_{A,n}^{NC} = u-d F_{A}^{p+n} - S_A . \]

As discussed above, the isospin correction terms in Eqs. (13) and (14) enter exactly like the strange form factors do. All of these IB corrections are expected to be extremely small, of course, and will be estimated in this paper.

Finally, there are some constraints on the IB admixture-induced form factors at zero momentum transfer stemming from exact symmetries. Gauge invariance says that the electric charge of the proton must be unity, so \( F_{1,p}(0) = 1 \), and \( F_{1,n}(0) = 0 \). In fact, both \( u+d F_{1}^{p-n}(0) \) and \( u-d F_{1}^{p+n}(0) \) vanish identically at zero momentum transfer to leading order in IB interactions; the first relation is due to the Ademollo-Gatto \[12\] theorem, the second due to baryon number conservation. The latter result follows from the orthogonality of excited states used in the first-order perturbation theory and the fact that the charge form factor reduces to the norm of the ground state, at zero momentum transfer. This statement does not apply to the magnetic or axial form factors at any \( q^2 \), nor to the electric one at nonvanishing momentum transfers. Such correction terms will be typically of order \( \alpha \). In

\[ ^2\text{Note that we have absorbed a factor of} \ 1/2 \text{ into the definition of the quark isovector, “u-d”, superscript, and a factor of} \ 1/6 \text{ into the quark isoscalar, “u+d”, notation.} \]
the next section, we calculate, in a simple constituent quark model, the IB corrections to the Walecka-Sachs [13] form factors \( u^+dG_{E}^{p-n}(q^2) \) and \( u^-dG_{E}^{p+n}(q^2) \), \( i = E, M \), which are related to the Dirac and Pauli form factors \( F_{1,2} \) by

\[
G_{E}(q^2) = F_{1}(q^2) + \frac{q^2}{4M_N^2}F_{2}(q^2)
\]

\[
G_{M}(q^2) = F_{1}(q^2) + F_{2}(q^2)
\]

(16)

III. THE MODEL

We use the constituent quark model [7,8] to calculate both of the above mentioned isospin-breaking corrections and use the harmonic oscillator Hamiltonian for that purpose. This three-body problem can be reduced to two uncoupled harmonic oscillators by application of the (equal mass) Jacobi three-body coordinates

\[
\rho = \frac{1}{\sqrt{2}}(r_1 - r_2)
\]

(17a)

\[
\lambda = \frac{1}{\sqrt{6}}(r_1 + r_2 - 2r_3)
\]

(17b)

\[
R = \frac{1}{3}(r_1 + r_2 + r_3)
\]

(17c)

Then, the Hamiltonian consists of two independent harmonic oscillators with equal mass \( m \) and a freely moving center of momentum with mass \( M = 3m \),

\[
H = \frac{P_{\rho}^2}{2M} + \frac{P_{\lambda}^2}{2m} + \frac{P_{R}^2}{2m} + \frac{3k}{2}(\rho^2 + \lambda^2)
\]

(18)

Solutions to the Schrödinger equation with this Hamiltonian are well known and have been tabulated in [7,8,14] for low-lying nucleon resonances. A simple confining interaction such as the harmonic oscillator leads to spatial wave functions that are Gaussians, in the case of the ground state, or that decay as Gaussians at large distances, for any other state of the system. That, in turn, leads to Gaussian EM and axial form factors which fall off far too rapidly at high values of momentum transfer compared with experiment. This model is clearly rather simple, but should be adequate for the purposes of identifying qualitative features, and making first estimates of small IB effects. In this spirit we have neglected the strong-hyperfine-interaction interference with the IB terms in the Hamiltonian [10,14]. The former is an important part of the extended constituent quark model [13], and its effects have been evaluated for simple IB observables such as the hadron mass differences [10,14]. But, in our case it complicates the evaluation of the observables to such an extent that we relegate its inclusion to the future.

A. Mechanical Corrections

The notion of quark mass is an ill-defined one: free quarks have not been observed. Traditionally one distinguishes between two kinds of quark masses: (1) the current quark
mass $m_q$ which is one that the quarks would have in the absence of all strong interactions; 
(2) the constituent quark mass $m_Q$ which is the measure of inertia of a quark moving within hadrons. There is of course considerable difficulty in precisely pinning down the values of the current quark masses \[16\], and even more in connecting these to constituent quark masses. Weinberg \[17\] has used chiral symmetry and empirical meson masses to argue that $m_u \simeq 5\text{MeV}; m_d \simeq 9\text{MeV}; m_s \simeq 140\text{MeV}$ and $m_Q = m_q + \text{const}$, where $\text{const} \simeq 330\text{MeV}$. This, in turn, implies

$$\Delta m_q \equiv m_u - m_d = \Delta m_Q \equiv m_U - m_D \simeq -4\text{MeV},$$  \hspace{1cm} (19)$$

and those are the values that we use\[3\]. Consequently, we expect the leading isospin breaking corrections due to explicit quark mass differences (which we call “mechanical corrections”) to be of $\mathcal{O}(\Delta m_Q/m_Q)$, where $\Delta m_Q/m_Q \simeq -1/85$ i.e. rather small. They can be exactly evaluated if one assumes harmonic oscillator confining quark-quark interactions with \textit{unequal} masses,

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m_i} + \frac{k}{2} \sum_{i<j}^{3} (r_i - r_j)^2,$$  \hspace{1cm} (20)$$

where $m = m_1 = m_2 \neq m_3 = m'$. The first two ($\rho$ and $\lambda$) of the three-body Jacobi coordinates in this case are the same as Eqs. (17a) and (17b), but now carry quark number indices e.g.

$$\rho_3 = \frac{1}{\sqrt{2}} (r_1 - r_2)$$  \hspace{1cm} (21a)$$

$$\lambda_3 = \frac{1}{\sqrt{6}} (r_1 + r_2 - 2r_3) \ ,$$  \hspace{1cm} (21b)$$

and the center of mass is shifted to

$$R = \frac{1}{(2m + m')} \left( m r_1 + m r_2 + m' r_3 \right) \ .$$  \hspace{1cm} (22)$$

This allows a separation of variables and an \textit{exact} solution to the problem in terms of two harmonic oscillator wave functions with two \textit{different} masses,

$$m_\lambda = \frac{3mm'}{2m + m'}; \ m_\rho = m,$$  \hspace{1cm} (23)$$

and a freely moving center of momentum with mass $M_N = 2m + m'$ (where $m = m_u, m' = m_d$ in the proton, $m = m_d, m' = m_u$ in the neutron):

$$H = \frac{P_\rho^2}{2m_\rho} + \frac{P_\lambda^2}{2m_\lambda} + \frac{3k}{2} \left( \rho^2 + \lambda^2 \right).$$  \hspace{1cm} (24)$$

\[3\]Isgur \[10\] finds $\Delta m_q = -6\text{MeV}$ in his extended version of the present model that includes the strong hyperfine interaction effects.
The total wave function factors into the CM plane wave solution and an internal motion wave function

\[ | \Psi_N(P_i) \rangle = \left( \frac{1}{3} \right)^{3/4} \exp\left( i(P_i \cdot R - E_i t) \right) |N(940)\rangle, \quad (25) \]

where \( E_i^N = \frac{P_i^2}{2M_N} + \frac{3}{2} (\omega_\rho + \omega_\lambda) \), \( \omega \equiv \sqrt{3k/m} \), and the factor \( \left( \frac{1}{3} \right)^{3/4} \) is the square root of the inverse Jacobian for the above three-body Jacobi coordinates. The spring constant \( k \) is an adjustable parameter of this model.

For spin- and isospin-independent quark potentials the complete internal wave function factorizes into a product of the spin, isospin, and spatial wave functions. Once the isospin-dependent terms are introduced into the Hamiltonian, however, the factorization property breaks down and the spatial and isospin parts of the internal wave function become entangled

\[ |N(940)\rangle = \frac{1}{\sqrt{2}} \left[ \chi^\rho \Phi^\rho + \chi^\lambda \Phi^\lambda \right], \quad (26) \]

The spin parts are given in a standard notation:

\[ \chi^\rho_\uparrow = \frac{1}{\sqrt{2}} (\alpha\beta - \beta\alpha) \alpha \]
\[ \chi^\lambda_\uparrow = \frac{1}{\sqrt{6}} (2\alpha\alpha\beta - \alpha\beta\alpha - \beta\alpha\alpha), \quad (27) \]

while the spatial-isospin parts for the proton are given by

\[ \Phi^\rho_p = \frac{1}{\sqrt{2}} (\psi_0(udu) - \psi_0(duu)) \]
\[ \Phi^\rho_n = \frac{1}{\sqrt{2}} (\psi_0(udd) - \psi_0(ddu)) \]
\[ \Phi^\lambda_p = \frac{1}{\sqrt{6}} (2\psi_0(uud) - \psi_0(udu) - \psi_0(duu)) \]
\[ \Phi^\lambda_n = \frac{1}{\sqrt{6}} (2\psi_0(ddu) - \psi_0(ddd) - \psi_0(udd)), \quad (28a) \]

where\n
\[ \psi_0(udu) = \left( \frac{m_\rho \omega_\rho}{\pi} \right)^{3/4} \left( \frac{m_\lambda \omega_\lambda}{\pi} \right)^{3/4} \exp \left( -\frac{1}{2} \left( \frac{\rho_\rho^2}{R_\rho^2} + \frac{\lambda_\lambda^2}{R_\lambda^2} \right) \right) \otimes udu \]
\[ \approx \left( \frac{m_u \omega_u}{\pi} \right)^{1/2} \left( \frac{m_d \omega_d}{\pi} \right)^{1/2} \left( \frac{m_o \omega_o}{\pi} \right)^{1/2} \otimes udu \]
\[ \times \exp \left( -\frac{1}{2} \left( \omega_u m_u (r_1 - R)^2 + \omega_d m_d (r_2 - R)^2 + \omega_o m_o (r_3 - R)^2 \right) \right), \quad (29) \]

the second line of Eq. (29) is meant in Faiman and Hendry’s symbolic sense only and is not to be used in actual calculations. Here \( R_\alpha^2 = m_\alpha \omega_\alpha = \sqrt{3km_\alpha} \), \( \alpha = \rho, \lambda \). The corresponding neutron expression is found by switching \( u \) with \( d \) everywhere. The simpler isospin-symmetric wave function \[ \langle N(940) | \] can be read off directly by setting \( m_u = m_d \equiv m \).
It factorizes into three parts as expected. The complete wavefunction is crucial to our subsequent work, however, as straightforward use of the standard factorized nucleon wave functions fails to correctly treat isospin breaking.

We are primarily interested in the elastic charge and magnetic form factors $G_E(q^2)$, $G_M(q^2)$ defined as the Fourier transforms of the charge and current densities

$$
\int dR \langle \Psi_N(P') | J_{EM}(R) | \Psi_N(P) \rangle \exp(iq \cdot R) =
$$

$$
= (2\pi)^3 \delta(P' - q - P)e \left[ \frac{\langle P + P' \rangle}{2M_N} \langle 1 \rangle_N G_E(q^2) + i \left( \frac{\langle \Sigma \rangle_N \times q}{2M_N} \right) G_M(q^2) \right]
$$

in the nonrelativistic limit, where $|\Psi_N\rangle$ is the ground state of the exact Hamiltonian Eq. (24), $P$ and $P'$ are the CM momenta of the initial and final state nucleons, respectively, and $q = P' - P$ is the three-momentum transfer. We work in the Breit frame defined by $q = 2P' = -2P$, which ensures that $E = E'$. In the Eq. (30) above, $R$ is the photon position vector and not the CM position vector, $\Sigma$ and $I$ are the Pauli and unit matrices, respectively, operating in the nucleon spin space and $\langle \Sigma \rangle_N$, $\langle 1 \rangle_N$ are their matrix elements taken between nucleon spinors. The quark EM current density operator is the nonrelativistic reduction of the Dirac fermion current

$$
J_{EM}(R) = \sum_{i=1}^{3} \frac{e_i}{2m_i}(-i \{\nabla_{r_i}, \delta (r_i - R)\} + \sigma_i \times [\nabla_R, \delta (r_i - R)]) , \tag{31}
$$

where $e_i = \frac{1}{2}(\frac{1}{3} + \tau_i^z)$ is the quark electric charge operator. Similarly, for the axial current in the isospin-symmetric limit, we have

$$
\int dR \langle \Psi_N(P') | J^a_A(R) | \Psi_N(P) \rangle \exp(iq \cdot R) = (2\pi)^3 \delta(P' - q - P) \langle \Sigma \rangle_N \langle \frac{1}{2} \tau^a_N \rangle_N F_A(q^2) . \tag{32}
$$

Here, $\tau_N$ are the isospin matrices for the nucleons, and the quark axial current reads

$$
J^a_A(R) = \sum_{i=1}^{3} \frac{\tau_i^a}{2} \sigma_i \delta (r_i - R) . \tag{33}
$$

Thus we have made yet another assumption: our constituent quark electroweak interactions are identical to those of the corresponding current quarks. Specifically, this means that we assume vanishing anomalous magnetic moments and no form factors for constituent quarks. For a discussion of this point see section IV.C.

Despite the exact solvability of the Schrödinger equation with the Hamiltonian Eq. (24), we will expand the exact form factors in powers of $q^2$. The reason for this is that a simple confining interaction such as the harmonic oscillator leads to Gaussian spatial wave functions, which in turn lead to an unrealistically rapid form factor fall-off at high $q^2$. Therefore, it only makes sense to talk about the leading terms in the expansion in powers of momentum transfer.
B. Electromagnetic Corrections

The electroweak interactions among quarks are readily divided into parity-conserving electromagnetic (EM) terms that are of $O(\alpha)$, where $\alpha \simeq 1/137$ is the fine-structure constant, and the parity-violating weak interactions that are of $O(G_F)$ where $G_F \simeq (1.023 \pm 0.002) \times 10^{-5} M_p^2$ is the Fermi weak coupling constant. The latter terms lead to parity-admixture corrections that have already been analyzed in Ref. [4]. Since we are working in the nonrelativistic approximation, we expand the invariant Möller operator in powers of momentum over mass, which leads to the Coulomb interaction as the leading EM term

$$ V_{EM} = \sum_{i<j}^3 \frac{e_i e_j}{4\pi |r_i - r_j|}. $$

The hyperfine and the spin-orbit coupling interactions are “higher-order” corrections in $\frac{p}{m_Q}$. The Coulomb interaction $V_{EM}$ induces a small $O(\alpha)$ abnormal isospin admixture in the nuclear wave function. To determine this admixture we use the first-order (Rayleigh-Schrödinger) perturbation theory in the perturbing Coulomb potential, as applied to the Hamiltonian

$$ H = H_0 + V_{EM}. $$

The ground state of the nucleon $|\Psi_0\rangle$, to $O(\alpha)$, is given by

$$ |\Psi_0\rangle = |\Phi_0\rangle + \sum_{n\neq 0} |\Phi_n\rangle \frac{\langle \Phi_n | V_{EM} | \Phi_0 \rangle}{E_0 - E_n} + O(\alpha^2), $$

where $|\Phi_n\rangle$ are the exact eigenstates of the isospin-symmetric Hamiltonian $H_0$: $H_0 |\Phi_n\rangle = E_n |\Phi_n\rangle$. This involves evaluating the isospin-breaking admixture to the nucleon wave function defined by the admixture coefficients $\varepsilon_n$:

$$ \varepsilon_n = \frac{\langle \Phi_n | V_{EM} | \Phi_0 \rangle}{E_0 - E_n}. $$

The sum extends over all $n$, i.e., over infinitely many excited states of the nucleon.

The isospin-admixtures in the wave function generate isospin breaking contributions to the elastic nucleon electroweak current matrix elements. These are determined by using the first-order perturbation theory parameters $\varepsilon_n$ and the vector and axial current transition matrix elements between the nucleon and its excited states calculated in the non-relativistic impulse approximation. The definition of the isospin-violating corrections to the non-relativistic current matrix elements to $O(\alpha)$ is thus

$$ \langle \Psi_0 | J | \Psi_0 \rangle = \langle \Phi_0 | J | \Phi_0 \rangle + \sum_{n\neq 0} \frac{1}{E_0 - E_n} \left( \langle \Phi_0 | J | \Phi_n \rangle \langle \Phi_n | V_{EM} | \Phi_0 \rangle + \langle \Phi_0 | V_{EM} | \Phi_n \rangle \langle \Phi_n | J | \Phi_0 \rangle \right), $$

and similarly for the charge density elastic matrix element $\langle \Psi_0 | \rho | \Psi_0 \rangle$. These formulae have a simple Feynman diagrammatic interpretation shown in Fig. [9]. It is important to remember...
that the above $\mathcal{O}(\alpha)$ corrections do not constitute a complete set. There are other graphs, e.g. those shown in Fig. 2 that contribute to the same order in $\alpha$, but are not included in Eq. 38. Our class of corrections, however, is gauge-invariant and therefore a physically sensible subset.

IV. RESULTS

The first and foremost question is: do we observe any change in the static properties of the nucleon, i.e., in the zero momentum transfer values of the form factors? Then, the second question is: how do the effective radii of the nucleon change? Both questions will be answered separately in each case.

A. Mechanical Corrections

1. Corrections to static nucleon moments

It turns out that two, the charge and the axial, out of three static couplings remain unchanged. In the case of the isoscalar “electric” form factor (convection part of the vector current) at zero momentum transfer this is a straightforward consequence of baryon number conservation. The isovector electric form factor at zero momentum transfer is unrenormalized as well, in agreement with the Ademollo-Gatto theorem [12]. The axial coupling constant also remains unchanged, but there does not seem to be as deep a reason for that as for the charge conservation. In other words, this quark model as it stands, does not predict the existence of an additional isovector or any isoscalar axial couplings due to the Coulomb interaction between the quarks and/or different quark masses. The magnetic moments, however, are corrected.

The change of variables from the equal mass three-body Jacobi coordinates Eqs. (17a-c) to the unequal mass case (Eq. 22) amounts to a shift of the center of mass (CM) of the system. That shift provides the main source of mechanical corrections. The change of the oscillator frequencies plays a secondary role (see tables I and II). E.g., the anomalous magnetic moments are renormalized due to the change in the quark and nucleon masses, via the definition of the magnetic moments:

$$\langle N \uparrow | \frac{eG_{M,N}(0)}{2M_N} \Sigma | N \uparrow \rangle = \sum_{i=1}^{3} \langle N \uparrow | \frac{e_i}{2m_i} \sigma_i | N \uparrow \rangle,$$

where $eG_{M,N}(0) = e_N + e\kappa_N$. It is important to note that the quark mass $m_i$ is now an operator in isospin space $m_i = \bar{m} + \Delta m \frac{1}{2} \tau_i$. Then we find

$$1 + \kappa_p = \frac{8m_d + m_u}{3m_{\lambda,p}} = \frac{1}{9} \left( 1 + \frac{2m_u}{m_d} \right) \left( 1 + \frac{8m_d}{m_u} \right) \tag{40a}$$

$$\kappa_n = -\left( \frac{2}{3} \right) \frac{m_d + 2m_u}{m_{\lambda,n}} = -\frac{2}{9} \left( 1 + \frac{2m_d}{m_u} \right) \left( 1 + \frac{2m_u}{m_d} \right). \tag{40b}$$

Corrections to nucleon static electroweak moments are summarized in Table I. Since the magnetic moments are the only static nucleon properties that do receive corrections in this
model, we will separate them into various parts according to the nomenclature defined in Sec. II:

\[ u^+dG_M(0)p^n = \frac{1}{9} \left[ 7 + \frac{m_u}{m_d} + \frac{m_d}{m_u} \right] \approx 1.0 \] (41a)

\[ u^-dG_M(0)p^n = \left[ 3 + \frac{m_u}{m_d} + \frac{m_d}{m_u} \right] \approx 5.0 \] (41b)

\[ u^-dG_M(0)p^n = \frac{1}{3} \left[ \frac{m_d}{m_u} - \frac{m_u}{m_d} \right] \approx 0.008 \] (41c)

\[ u^+dG_M(0)p^n = \frac{1}{3} \left[ \frac{m_d}{m_u} - \frac{m_u}{m_d} \right] \approx 0.008. \] (41d)

We can immediately use the results from Eqs. (12, 13) to find

\[ G_{NC}^{M,p}(0) = \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W)G_{EM}^{M,p}(0) - G_{EM}^{M,n}(0) - S_M(0) - u^+dG_M(0)p^n + u^-dG_M(0)p^n \right] \]

\[ \approx \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W)G_{EM}^{M,p}(0) - G_{EM}^{M,n}(0) - S_M(0) \right] \] (42a)

\[ G_{NC}^{M,n}(0) = \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W)G_{EM}^{M,n}(0) - G_{EM}^{M,p}(0) - S_M(0) + u^-dG_M(0)p^n + u^+dG_M(0)p^n \right] \]

\[ \approx \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W)G_{EM}^{M,n}(0) - G_{EM}^{M,p}(0) - S_M(0) + 2u^-dG_M(0)p^n \right] , \] (42b)

where the second lines of these two formulas represent this model’s results. Using the values in Eq. (41c-d), the proton weak magnetic form factor is essentially unaffected, while the neutron weak magnetic form factor gets a small but nonzero isospin-breaking correction of approximately 0.3%. We conclude that, in this model, the proton magnetic moment measurements in elastic parity-violating electron-nucleon scattering can be directly interpreted as nucleon strangeness content, whereas the neutron ones are subject to the above small IB correction. It is not clear, however, to what degree the above cancellation of IB terms in \(G_{NC}^{M,p}\) is model dependent.

2. Corrections to nucleon radii

The \(q^2\) dependent isospin-breaking effects are present in all of the form factors. A few words about their evaluation are in order. The nucleon wave function does not factor into three coherent parts due to isospin breaking: the isospin and the spatial wave functions are now coupled (see Eq. 28a) and that has to be taken into account. With proper book-keeping of quark flavors in the spatial wave functions, one derives the following formulae:

\[ eG_{E,p}(q^2) = 2e_u\langle u\rangle_p + e_d\langle d\rangle_p \] (43a)

\[ eG_{E,n}(q^2) = 2e_d\langle d\rangle_n + e_u\langle u\rangle_n \] (43b)

\[ \frac{e}{2M_p}G_{EM,p}(q^2) = \frac{e}{18} \left[ \frac{8}{m_u} \langle u\rangle_p + \frac{1}{m_d} \langle d\rangle_p \right] \] (43c)

\[ \frac{e}{2M_n}G_{EM,n}(q^2) = -\frac{e}{9} \left[ \frac{1}{m_u} \langle u\rangle_n + \frac{2}{m_d} \langle d\rangle_n \right] \] (43d)
The parity-violating (PV) elastic strange form factor, it is of immediate experimental interest and will be measured in e.g. the nucleon strange radius is then the first term in the Taylor expansion of the nucleon and neutron electric form factors in the static limit: the weak NC charge radius. Since form factors.

It is hence important that we know the relevant IB corrections. As with the magnetic moment in the previous subsection, we break up the total correction to the nucleon EM charge radius into its isoscalar and isovector components

\[
F_{A,p}^{NC}(q^2) = \frac{1}{3} [4\langle u \rangle_p + \langle d \rangle_p] \\
F_{A,n}^{NC}(q^2) = -\frac{1}{3} [4\langle d \rangle_n + \langle u \rangle_n].
\]

Here \( \langle u \rangle_p \) is defined as the Fourier transform of the spatial wave function matrix element of a \( u \) quark in the proton i.e. \( \langle u \rangle_p = \langle \exp i \mathbf{q} \cdot \mathbf{r}_u \rangle_p \), etc.. They are evaluated as

\[
\begin{align*}
\langle u \rangle_p &= \langle \exp i \mathbf{q} \cdot \mathbf{r}_u \rangle_p = \exp \left[ -\frac{q^2}{8} \left( R_{pp}^2 + 3 \left( \frac{m_d}{M_p} \right)^2 R_{lp}^2 \right) \right] \\
\langle d \rangle_p &= \langle \exp i \mathbf{q} \cdot \mathbf{r}_d \rangle_p = \exp \left[ -\frac{3q^2}{2} \left( \frac{m_u}{M_p} \right)^2 R_{lp}^2 \right] \\
\langle u \rangle_n &= \langle \exp i \mathbf{q} \cdot \mathbf{r}_u \rangle_n = \exp \left[ -\frac{3q^2}{2} \left( \frac{m_d}{M_n} \right)^2 R_{ln}^2 \right] \\
\langle d \rangle_n &= \langle \exp i \mathbf{q} \cdot \mathbf{r}_d \rangle_n = \exp \left[ -\frac{q^2}{8} \left( R_{pm}^2 + 3 \left( \frac{m_u}{M_n} \right)^2 R_{ln}^2 \right) \right].
\end{align*}
\]

Upon inserting these into Eqs. (43a-f) and expanding in Taylor series, we find the results that are shown in Table 2. They all turn out to be small (\( \leq 1\% \)), and hence are not likely to be distinguishable from the experimental uncertainties in the “known” EM form factors form factors.

One case is of particular importance, though, because of the vanishing of the strange and neutron electric form factors in the static limit: the weak NC charge radius. Since the nucleon strange radius is then the first term in the Taylor expansion of the nucleon strange form factor, it is of immediate experimental interest and will be measured in e.g. parity-violating (PV) elastic \( ^4He(\vec{e},\vec{e}') \) and forward angle \( p(\vec{e},\vec{e}')p \) experiments. It is hence important that we know the relevant IB corrections. As with the magnetic moment in the previous subsection, we break up the total correction to the nucleon EM charge radius into its isoscalar and isovector components

\[
\begin{align*}
{u+d} \langle r_E^2 \rangle_p &= \frac{3}{2} \left( \frac{m_u}{M_p} \right)^2 R_{lp}^2 + \frac{1}{4} \left( R_{pp}^2 + 3 \left( \frac{m_d}{M_p} \right)^2 R_{lp}^2 \right) \\
{u-d} \langle r_E^2 \rangle_p &= -\frac{9}{2} \left( \frac{m_u}{M_p} \right)^2 R_{lp}^2 + \frac{3}{4} \left( R_{pp}^2 + 3 \left( \frac{m_d}{M_p} \right)^2 R_{lp}^2 \right) \\
{u+d} \langle r_E^2 \rangle_n &= \frac{3}{2} \left( \frac{m_d}{M_n} \right)^2 R_{ln}^2 + \frac{1}{4} \left( R_{pm}^2 + 3 \left( \frac{m_u}{M_n} \right)^2 R_{ln}^2 \right) \\
{u-d} \langle r_E^2 \rangle_n &= \frac{9}{2} \left( \frac{m_d}{M_n} \right)^2 R_{ln}^2 - \frac{3}{4} \left( R_{pm}^2 + 3 \left( \frac{m_u}{M_n} \right)^2 R_{ln}^2 \right).
\end{align*}
\]

(In the isospin symmetric limit, e.g., \( \langle r_E^2 \rangle_p = {u+d} \langle r_E^2 \rangle_p + {u-d} \langle r_E^2 \rangle_p \rightarrow R_{E}^2 \).) Inserting the numerical values into these formulae gives

\[
{u+d} \langle r_E^2 \rangle_{p+n} / R_{E}^2 - 1 \simeq 0.0\% \tag{46a}
\]
\[ u-d \langle r_E^2 \rangle_{p-n}/R^2 - 1 \simeq 0.0\% \] (46b)
\[ u-d \langle r_E^2 \rangle_{p+n}/R^2 \approx 2.1\% \] (46c)
\[ u+d \langle r_E^2 \rangle_{p-n}/R^2 \approx 0.1\% \] (46d)

There is a simple physical picture which roughly explains these numbers. The bulk of the effect arises from the shift in the position of the center of mass when the constituent quarks are given different masses. In the case of the EM charge radius, we are evaluating a weighted sum of the quark positions squared. The isovector rms charge radius is weighted by the sign of the quark charges, and thus in the neutron, although the shift of the center of mass is in the opposite direction from that of the proton, the sign of the quark charges is also switched, resulting in the same increase in radius for proton and neutron. For the same reason the isoscalar rms charge radius is very small. Thus the bulk effect is primarily due to \( u-d \langle r_E^2 \rangle_{p+n} \). This simple geometrical picture yields \( \Delta \langle r^2 \rangle / R^2 \approx 2(\Delta m_Q/3m_Q) \approx .8\% \) for proton and neutron separately, and these are then additive. The remainder arises primarily from the modification of oscillator frequencies due to reduced mass effects, see Eq. (23). We see that the complete IB in the rms charge radius is due to fairly simple kinematical effects, which, however, can only be evaluated if the CM motion is correctly accounted for.

When we include strange form factors, we find
\[
\langle r_{E_p}^2 \rangle^\text{NC} = \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W) \langle r_{E_p}^2 \rangle_{EM} - \langle r_{E_n}^2 \rangle_{EM} - \langle r_{E_n}^2 \rangle_{s} - u+d \langle r_{E}^2 \rangle_{p-n} + u-d \langle r_{E}^2 \rangle_{p+n} \right], \tag{47a}
\]
\[
\langle r_{E_n}^2 \rangle^\text{NC} = \frac{1}{2} \left[ (1 - 4 \sin^2 \theta_W) \langle r_{E_n}^2 \rangle_{EM} - \langle r_{E_p}^2 \rangle_{EM} - \langle r_{E_p}^2 \rangle_{s} + u-d \langle r_{E}^2 \rangle_{p+n} + u+d \langle r_{E}^2 \rangle_{p-n} \right]. \tag{47b}
\]

Using the numbers from Eqs. 16a through d and Table 2, the IB contribution is apparently not negligible in either proton or neutron weak radii, and is indistinguishable from the strangeness radius contribution. The fact that both of the leading terms in Eq. 47a are suppressed, i.e. that the weak neutral charge radius of the proton is naturally small, makes the forward angle proton asymmetry an attractive place to look for strangeness content. However, the relatively large size of the isospin breaking correction in this model, \( u-d \langle r_{E}^2 \rangle_{p+n} \) is more than 2% of the electromagnetic charge radius) shows that isospin corrections may prove to be a nontrivial effect which should be carefully taken into account. The shift in our calculated value of \( \langle r_{E}^2 \rangle^\text{NC} \) due to mechanical IB is approximately +0.5% \( R^2 \) for both proton and neutron. For the proton, this shift is in fact +13% of the (very small) uncorrected Born approximation value \( 0.5(1 - 4 \sin^2 \theta_W) \). Even if our model result is viewed merely as an indication of the uncertainty introduced by isospin breaking, an extraction of \( \langle r_{E}^2 \rangle \) by this means will be difficult at or below a level of \( \approx 2\% \) of \( R^2 \). This is equally relevant to the case of elastic \( ^4\text{He} (e, e') \) experiments, where the isoscalar radius is measured. Here again, the isoscalar IB radius contributions add constructively (as would any intrinsic nucleon strangeness radii, of course). Note that the IB corrections due to the nuclear structure of \( ^4\text{He} \), i.e. without the intrinsic nucleon IB breaking, were recently calculated in Ref. [19] and were found to be comparatively small.

**B. Electromagnetic Corrections**

As the first step in this analysis we take the lowest-lying baryon resonances with all the quantum numbers equal to those of the nucleon, but with “wrong” isospin as the leading
source of isospin admixtures. A number of such resonances can be found in e.g. Table 1.1 of Ref. [7] all at the $N = 2$ level in the harmonic oscillator model. Since the Coulomb potential does not carry orbital or spin angular momentum, each of these is separately conserved. That immediately eliminates all $L = 2$ states from this consideration. We are left with only one multiplet of $L = 0$ resonances, the $\Delta(1550\text{MeV})$, with the required properties\footnote{In fact, this resonance does not appear in the latest Particle Data Group tables\cite{16}, as it has been observed only by a single group\cite{20}.}.

The mass of $\Delta(1550)$ is anomalously low, i.e., at about the same mass as the $N = 1$ negative-parity resonances. Irrespective of its present poorly confirmed experimental status, this is the only $N = 2$, SU(6) quark model state with the necessary quantum numbers; hence we will use it here. Its isospin-symmetric wave function is

\[ |\Delta(1550)\rangle = \frac{1}{\sqrt{3}} \left[ \chi^\rho \psi_{20}^\rho + \chi^\lambda \psi_{20}^\lambda \right] \phi_S \]

\[ \psi_{20}^\rho = \frac{m\omega}{\sqrt{3}} \left( \frac{m\omega}{\pi} \right)^{3/2} \left[ \rho^2 - \lambda^2 \right] \exp \left( -\frac{1}{2R^2} (\rho^2 + \lambda^2) \right) \]

\[ \psi_{20}^\lambda = \frac{m\omega}{\sqrt{3}} \left( \frac{m\omega}{\pi} \right)^{3/2} 2(\rho \cdot \lambda) \exp \left( -\frac{1}{2R^2} (\rho^2 + \lambda^2) \right) \]

\[ \phi^\Delta_+ = \frac{1}{\sqrt{3}} (uud + udu + duu) \]

\[ \phi^\Delta_0 = \frac{1}{\sqrt{3}} (udd + dud + ddu) \]

The next admixed terms occur at the $N = 4$ level, which ought to be suppressed due to the twice-as-large energy denominator. After some straightforward algebra we find the admixture coefficients from Eq. (37) are

\[ \varepsilon_p = -\frac{\alpha}{3R\Delta E \sqrt{3\pi}} = 4.1 \times 10^{-4} \] \hspace{1cm} (49a)

\[ \varepsilon_n = -\frac{1}{2} \varepsilon_p = -2.0 \times 10^{-4} , \] \hspace{1cm} (49b)

where $\Delta E = E_N - E_{\Delta(1550)} = -612 \text{ MeV}$ is the experimental mass difference.

At this stage we must say a few words about the choice of free parameters in this model. The constituent quark mass is fixed by fitting the nucleon magnetic moments, and is then entirely consistent with the mass of the nucleon. The oscillator frequency can be fixed in several ways, the two best known ones being: (a) using the observed spectrum of the nucleon resonances \cite{15}, or (b) using the observed nucleon EM radius \cite{7,8}. The former leads to reasonable values of $\omega$ only in the full-blown version of the model with strong hyperfine interaction and it underestimates the charge radius of the nucleon. The latter underestimates the energy gap between the ground and the excited states. While this discrepancy may be a serious cause of concern in attempts to make this model as realistic as possible, it is only mildly worrying in our case. Here, we are interested in a first estimate of isospin-breaking effects on electroweak form factors. We have therefore taken the former approach, and set...
\[ \omega \approx 300 \text{MeV}, \] which leads to \( \langle r^2 \rangle = R^2 = (0.62 \text{fm})^2, \) an underestimate of the experiment by about 30%. This does correctly yield the observed \( \Delta(1550) - N \) mass difference, however. As one can see from Eqs. (49a-b), an attempt to fix this up would somewhat decrease the admixture coefficients. In this sense, we are taking a conservative point of view, and our results might be viewed as a rough upper limit of the EM-induced isospin mixing effects.

The appropriate EM transition matrix elements are all isoscalar, i.e., they are identical for the charged and neutral members of the multiplet

\[
\langle \Delta | \rho(q) | N \rangle = \langle \Delta(1550) | \sum_{i=1}^{3} e_i \exp(iq \cdot r_i) | N \rangle \\
= -\frac{e}{12} \sqrt{2} \left( \frac{q^2}{m \omega} \right) \exp \left( -\frac{q^2}{6m \omega} \right) 
\]

\[
\langle \Delta \uparrow | e | \Sigma^z \Sigma^z \langle q) | N \uparrow \rangle = \langle \Delta(1550) \uparrow | \sum_{i=1}^{3} \frac{1}{2m_i} \sigma_i^z e_i \exp(iq \cdot r_i) | N \uparrow \rangle \\
= \frac{e}{72m} \sqrt{2} \left( \frac{q^2}{m \omega} \right) \exp \left( -\frac{q^2}{6m \omega} \right) 
\]

\[
\langle \Delta \uparrow | J_{a=3}^{(a=3),z} \langle q) | N \uparrow \rangle = \langle \Delta(1550) \uparrow | \sum_{i=1}^{3} \frac{1}{2} \sigma_i^z r_i^{a=z} \exp(iq \cdot r_i) | N \uparrow \rangle \\
= \frac{1}{36} \sqrt{2} \left( \frac{q^2}{m \omega} \right) \exp \left( -\frac{q^2}{6m \omega} \right).
\]

None of these matrix elements survive the taking of the threshold limit. Hence, these corrections do not renormalize the static nucleon electroweak couplings. The only exception is the neutron anomalous magnetic moment which is renormalized by a small neutron EM mass shift (see below). The corrections to the effective radii are shown in Table III. The main conclusion of this section is that all of the explicit electromagnetic corrections due to admixtures with the \( \Delta(1550) \) are extremely small, due to the smallness of \( \alpha \), coupled with the relatively small nuclear state overlaps, and relatively large energy denominator.

In order to assess the reliability of our model, and that of our other assumptions, we apply them to a well-understood case of isospin breaking in nucleons: most notably the nucleon mass difference \( \Delta M_N = M_p - M_n \approx -1.3 \text{MeV} \). We calculate this mass difference with all free parameters in the model already fixed, compare it with experiment and then discuss other possible approaches (schemes) to fixing the free parameters. We find

\[
\Delta M_N = m_u - m_d + \delta E_C
\]

where \( \delta E_C \equiv E_C^p - E_C^n = \frac{\alpha}{3} \left( \frac{1}{|r_1 - r_2|} \right) \) is the difference of the Coulomb energies of the proton and the neutron.

Using our Gaussian spatial wave functions, with \( R = 0.62 \text{fm} \), the Coulomb energy \( \delta E_C = \frac{\alpha}{3R} \sqrt{\frac{2}{\pi}} \approx 0.6 \text{MeV} \). This reproduces the correct (negative) sign and order of magnitude (a few MeV) of the mass difference, but quantitatively our computed value of -3 MeV is more than two times too large. Our assumptions are also well known to lead to other unrealistic features of this model, such as Gaussian form factors. One can in principle increase the small Coulomb energy by turning on the strong hyperfine interaction between quarks, [L4]
but that would take us far outside of the intended scope of this article. We could also just reverse the procedure and fix the constituent quark mass difference from this calculation and the observed nucleon mass difference, but that would open new questions with regard to the applicability of those quark masses in the light meson sector. As far as the IB effects in the nucleon are concerned, that procedure would decrease their size as compared with our “direct” procedure. In that sense, as discussed earlier, we are again taking a conservative point of view. The above discussion ought to be an indication of the expected range of validity of this model: we may expect to have calculated the correct sign and order of magnitude of the effects of interest. We leave the investigation of next-order effects such as strong hyperfine interactions, and more realistic nucleon models as a task for the future.

C. Discussion

We would like to make a few comments with regard to the place of this investigation in the big picture of all “radiative corrections” that have to be evaluated. Our corrections form only one separately gauge-invariant class of purely EM radiative corrections. We have completely ignored EM quark vertex corrections and the associated self-energies and Bremsstrahlung effects. These form another separately gauge-invariant class of corrections that must be dealt with. If one assumes elementary i.e. Dirac constituent quarks, then one sub-part of such corrections, the anomalous magnetic moment of the quarks, are finite and gauge-invariant by themselves and are given by Schwinger’s QED formula. This correction is manifestly of the same order of magnitude as the above-found IB nucleon-structure-induced effects. It is not clear to us whether they, or indeed any of the additional classes of graphs we have neglected, ought to be treated as intrinsic IB effects, or as pure radiative corrections. Furthermore, even if we were to include them into the former, the calculation would not be as straightforward as in free space, due to the strong interaction (“binding”) effects. Similar considerations hold for the EM corrections to the axial current vertex.

On the formal side, the present model is subject to the criticism that chiral symmetry is absent from it. If we assume that confinement and chiral symmetry breaking are two essentially independent phenomena, in accord with lattice QCD, then the situation can be remedied. The chiral symmetry and its spontaneous breakdown, which endows constituent quarks with their mass, can be installed into our model following e.g. Ref. [8], or Ref. [21]. These are, of course, just two of many chirally symmetric models for the constituent quark sector. Such chiral models allow a consistent evaluation of new kinds of diagrams such as the EM correction to the constituent quark axial current vertex shown in Fig. 4. Diagrams of this sort create a constituent quark isoscalar axial current coupling constant $g^S_{A,Q}$, which leads to an isoscalar nucleon axial current coupling constant $g^S_{A,N}$ according to

$$g^S_{A,N} = \frac{g^S_{A,Q}}{2} \sum_i \langle N \uparrow | g^S_{A,Q} \sigma^z_i | N \uparrow \rangle = \sum_i \langle N \uparrow | \frac{g^S_{A,Q}}{2} \sigma^z_i | N \uparrow \rangle = \frac{g^S_{A,Q}}{2}. \quad (52)$$

Manifestly, the result will depend on the model of the constituent quarks that one adopts. Should the vanishing of this class of IB corrections persist even at that next level of approximation, then we would be allowed the straightforward interpretation of a nonzero isoscalar
nucleon axial coupling result, e.g. that of Ref. [3]. in terms of strange quark contributions.

V. CONCLUSIONS

In summary, we have evaluated the IB admixture effects on the nucleon electroweak form factors in the nonrelativistic constituent quark model. We used the harmonic oscillator Hamiltonian as the confining interaction. The results were classified according to a model-independent formalism developed in Sect. II.

If we are ultimately interested in either extracting strange nucleon form factors from electroweak data, or performing precision moderate energy Standard Model tests, we must first eliminate corrections such as the isospin breaking effects discussed here. These are certainly more “conventional” than the strangeness content, although also interesting and potentially important in their own right. A direct experimental measure of them, via e.g. Eq. (12), would be clearly valuable as well. Strange quark form factor estimates range considerably [1], but typically yield from a small fraction of a percent up to $\geq 10\%$ corrections to electroweak structure. A priori, IB effects also modify electroweak structure of the nucleon, but only at a level of $\mathcal{O}(\alpha)$. The issue we have addressed here is effectively one of finding the coefficient in front of $\alpha$ in these corrections. If one based one’s judgement solely on the observed nucleon mass splitting, then one would naively expect the correction to be very small, i.e. $\mathcal{O}(\Delta M_N/M_N) \approx 0.1\%$, and not $1\%$.

Our results show that the u-d mass difference can yield noticeably larger modifications to electroweak form factors than naively expected from the nucleon mass splitting. Nevertheless, most of the corrections, with one possible exception, appear to be safely below the levels of what might be considered “interesting” strangeness content. Indeed, our results agree with the original expectation of $\mathcal{O}(\alpha)$ corrections, as well as with our assumption that the EM-mechanical interference (cross-) terms may be neglected. In the isoscalar charge radius, we find corrections of several percent. The main source of this IB is simply the shift of the nucleon’s center of mass. It is in this regard that the present model is very well suited to the task at hand, in contrast to e.g. the bag models. We have, however, chosen the model parameters to maximize these IB effects, so within the subclass of IB corrections considered here, our numbers should represent an upper limit. E.g., choosing $\Delta m_Q$ to reproduce the observed $M_p - M_n$ splitting in our model would halve the predicted mechanical isospin corrections to the charge radius. Since IB breaking can, in principle, interfere with the extraction of a relatively modest strangeness-carrying nucleon electric form factor from lepton-nucleon scattering experiments, our results indicate the need for careful checking of these results in other nucleon models.

There are no odd parity admixture corrections to leading order in the experiment [3], since that is a neutrino scattering experiment.
ACKNOWLEDGMENTS

We would like to thank Volker Burkert for helpful comments and Susan Gardner for emphasizing the importance of the discrepancy between the two possible ways of fixing the oscillator frequency in this model. This work was supported by the US DOE. SJP acknowledges the support of a Sloan Foundation Fellowship.
REFERENCES

[1] M.J. Musolf, T.W. Donnelly, J. Dubach, S.J. Pollock, S. Kowalski and E.J. Beise, Phys. Rep. 239, 1-178 (1994).
[2] CEBAF proposals ER-91-017 (1991) D.H. Beck spokesperson; ER-91-004 (1991) E.J. Beise spokesperson; ER-91-010, J.M Finn and P. Souder, spokespersons; MIT/Bates experiment 91-09 (1991), R. Alarcon and J. van den Brand spokespersons. See also Ref [1] for a summary of completed and pending experiments.
[3] L.A. Ahrens et al., Phys. Rev. D35, 785 (1987).
[4] V. Dmitrašinović, Nucl. Phys. A 537, 551 (1992).
[5] C.J. Horowitz, Hungchong Kim, D.P. Murdock, S. Pollock, Phys. Rev C48, 3078 (1993).
[6] G.T. Garvey, W.C. Louis, and D.H. White, Phys. Rev. C48, 761 (1993).
[7] R.K. Bhaduri, Models of the Nucleon, Addison-Wesley, Redwood City CA (1988).
[8] A. LeYaounac, LL. Oliver, O. Pene and J.-C. Raynal, Hadron Transitions in the Quark Model, Gordon and Breach, New York (1988).
[9] J.D. Walecka, Theoretical Nuclear and Subnuclear Physics, Oxford University Press, New York (1995).
[10] N. Isgur, Phys. Rev. D21, 779 (1980).
[11] J.D. Bjorken and S.D. Drell Relativistic Quantum Mechanics (McGraw-Hill, 1964).
[12] M. Ademollo and R. Gatto, Phys. Rev. Lett. 13, 264 (1964).
[13] J.D. Walecka, Nuovo Cimento 11, 821 (1959) and F.J. Ernst, R.G. Sachs, and K.C. Wali, Phys. Rev. 119, 1105 (1960).
[14] J.-M. Richard, Phys. Rep. 212, 1-76 (1992).
[15] N. Isgur and G. Karl, Phys. Rev. D18, 4187 (1979).
[16] Particle Data Group, L. Montanet et al., Phys. Rev. D50, 1173 (1994) Review of Particle Properties.
[17] S. Weinberg, in Festschrift for I.I. Rabi, ed. L. Motz, Transactions of the N.Y. Academy of Sciences 38, 185 (1977).
[18] D. Faiman, and A.W. Hendry, Phys. Rev. 173, 1720 (1968).
[19] S. Ramavataram, E. Hadjimichael and T.W. Donnelly, Phys. Rev. C50, (1994).
[20] R.S. Longacre et al., Nucl. Phys. B122, 493 (1977).
[21] U. Vogl and W. Weise, Prog. Part. Nucl. Phys. 27, 195 (1991).
FIGURES

FIG. 1. Feynman diagrams describing the EM isospin-breaking contribution to the polar- and axial-vector currents. The solid line denotes the quarks, wavy solid line is the photon and wiggly solid line is the neutral intermediate vector boson $Z^0$. The shaded blob together with the three solid lines and one double solid line leading to it denotes the nucleon wave function.

FIG. 2. An example of an $\mathcal{O}(\alpha)$ EM correction to the quark EM and axial current vertices of the type that may renormalize the nucleon magnetic moment and the axial coupling constant. These diagrams are not evaluated in this paper.
TABLE I. The shift of nucleon static properties due only to mechanical IB corrections in our model. Here $M_p = 2m_u + m_d$; $M_n = 2m_d + m_u$ and $x$ is defined by $x = \sin^2 \theta_W$. Numerical values used are given before Eq. 19 in the text.

| mechanical IB | relative correction |
|---------------|---------------------|
| $G_{E,p}^{EM}(0)$ | 1 | 0 |
| $G_{E,n}^{EM}(0)$ | 0 | 0 |
| $G_{M,p}^{EM}(0)$ | $\frac{1}{9} \frac{M_p}{m_d} \left(1 + \frac{8m_d}{m_u}\right)$ | 0.3% |
| $G_{M,n}^{EM}(0)$ | $-\frac{2}{9} \frac{M_n}{m_u} \left(1 + \frac{2m_u}{m_d}\right)$ | < 0.01% |
| $F_{A,p}^{NC}(0)$ | $\pm 5$ | 0 |
| $F_{A,n}^{NC}(0)$ | $\pm 5$ | 0 |
| $G_{M,p}^{NC}(0)$ | $\frac{1}{2} \left(1 - 4x\right)G_{M,p}^{EM}(0) - G_{M,n}^{EM}(0)$ | 0.03% |
| $G_{M,n}^{NC}(0)$ | $\frac{1}{2} \left(1 - 4x\right)G_{M,n}^{EM}(0) - G_{M,p}^{EM}(0) + \frac{3}{2} \left(\frac{m_u}{m_d} - \frac{m_d}{m_u}\right)$ | -0.25% |

TABLE II. Theoretical nucleon radii $\langle r^2 \rangle$ with mechanical IB corrections, as well as the relative change. Notation is defined in the text, after Eqs. (14a - d), except $x = \sin^2 \theta_W$. The isospin symmetric nucleon radius in each case is $R^2$ (except $\langle r^2 \rangle^{EM} = 0$) where we use $R = 0.62$ fm.

| mechanical IB | $\delta\langle r^2 \rangle^{EM}$ |
|---------------|-------------------------------|
| $\langle r^2 \rangle^{EM}_{E,p}$ | $R_{p,p}^2 + 3 \left(\frac{m_u^2 - m_d^2}{M_p^2}\right) R_{\lambda,p}^2$ | 1.1% |
| $\langle r^2 \rangle^{EM}_{E,n}$ | $-\frac{1}{2} R_{p,n}^2 + 3 \left(\frac{4m_u^2 - m_d^2}{2M_p^2}\right) R_{\lambda,n}^2$ | 1.0% |
| $\langle r^2 \rangle^{EM}_{M,p}$ | $\left(\frac{8m_u m_d}{8m_d + m_u}\right) \frac{1}{3m_u} \left\{ R_{p,p}^2 + 3 \left(\frac{m_u}{M_p}\right)^2 R_{\lambda,p}^2 \right\} + \frac{1}{2m_d} \left(\frac{m_u}{M_p}\right)^2 R_{\lambda,n}^2$ | 0.4% |
| $\langle r^2 \rangle^{EM}_{M,n}$ | $\left(\frac{8m_u m_d}{8m_d + m_u}\right) \frac{1}{4m_d} \left\{ R_{p,n}^2 + 3 \left(\frac{m_d}{M_n}\right)^2 R_{\lambda,n}^2 \right\} + \frac{3}{2m_u} \left(\frac{m_d}{M_n}\right)^2 R_{\lambda,n}^2$ | -0.1% |
| $\langle r^2 \rangle^{NC}_{A,p}$ | $\frac{3}{5} R_{p,p}^2 + 3 R_{\lambda,p}^2 \left(\frac{m_u^2 + m_d^2}{M_p^2}\right)$ | 0.3% |
| $\langle r^2 \rangle^{NC}_{A,n}$ | $\frac{3}{5} R_{p,n}^2 + 3 R_{\lambda,n}^2 \left(\frac{m_u^2 + m_d^2}{M_n^2}\right)$ | -0.3% |

TABLE III. The shift of nucleon radii $\delta\langle r^2 \rangle$ due to purely electromagnetic IB corrections, as calculated in section 4B. Here $\varepsilon_p = 4.1 \times 10^{-4}$; $\varepsilon_n = -1/2 \varepsilon_p$.

| $\delta\langle r^2 \rangle^{EM}$ | $\delta\langle r^2 \rangle^{NC}$ | $\delta\langle r^2 \rangle^{NC}$ |
|-------------------------------|-------------------------------|-------------------------------|
| n | p | n | p |
| $\delta\langle r^2 \rangle^{EM}_{E}$ | $\varepsilon_p \sqrt{\frac{2}{3}} R^2$ | $3 \times 10^{-4}$ | $\varepsilon_n \sqrt{\frac{2}{3}} R^2$ | $-2 \times 10^{-4}$ |
| $\delta\langle r^2 \rangle^{EM}_{M}$ | $-\varepsilon_p \sqrt{\frac{2}{3}} R^2$ | $-1 \times 10^{-4}$ | $-\varepsilon_n \sqrt{\frac{2}{3}} R^2$ | $-8 \times 10^{-5}$ |
| $\delta\langle r^2 \rangle^{NC}_{E}$ | $-\frac{2}{5} \varepsilon_p \sqrt{\frac{2}{3}} R^2$ | $-1 \times 10^{-4}$ | $-\frac{2}{5} \varepsilon_n \sqrt{\frac{2}{3}} R^2$ | $-7 \times 10^{-5}$ |

22
