Analysis of Shear Lag Effect in Twin-cell Box Girders

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Abstract. For the twin-cell box girder, considering the difference of the shear lag between each cantilever plate, the shear lag warping displacement function of each wing of the box girder is defined. Based on the basis of the variation principle, the governing differential equations for considering the shear lag effect of twin-cell box girder are established. For a typical simply supported beam of twin-cell box girder, according to the three dimensional numerical methods used for plate and shell and the analytical solution method in this paper, the analytical solution of the shear lag effect is established by using the principle of energy. The results show that the shear lag warping displacement model proposed in this paper can reflect the difference of shear lag between each cantilever plate. The analytical solution is in good agreement with the finite element numerical solution. The shear lag effect at the top and bottom of the web site of the twin-cell box girder is different from that of the edge web site. In this paper, the stress at the top and bottom of the middle web is smaller than that at the side web.

1. Introduction

Curved box girder subjected to symmetrical loads, in its vertical bending plane, the normal stress caused by shear deformation is non-uniformed distributed along the flange width, the phenomenon, known as "shear lag effect". For the analysis of shear lag effect, there are usually variational method, the bar simulation method, finite strip method, folded plate theory and finite element method[1]. By introducing the warping displacement function, the shear lag warping of the displacement pattern is established, and the analytical solution of the shear lag effect is established by using the principle of energy[2][3]. Based on the variational method, the finite element analysis method for one-dimensional beam segment with shear lag effect can be further constructed[4-6]. At present, with the construction of large traffic flow Bridges, the single-box twin-cell girder or single-box multi-cell girder appears. Because of the difference of the distribution of shear flow of the single-box multi-cell girder in bending, the distribution of shear lag effect of the flange plate must be different from the single-cell box girder. There are studies focusing on single-cell box girder, so it is necessary to establish a corresponding analytical method for the study of single-box multi-cell girder.

In this paper, the single-box twin-cell girder is studied, considering the difference of shear lag warping between each flange plate, and combined with the axial force equilibrium condition of the whole section, the new shear-lag warping displacement function is defined which concerning each flange plate of box girder. The governing differential equations of equilibrium considering the shear lag for single-box twin-cell girder are established on the basis of the variation principle. For a typical simply supported beam of single-box twin-cell girder, according to the three dimensional numerical methods used for plate and shell and the analytical solution method in this paper, the shear lag distribution law of uniform load and concentrated force has been studied.
2. The selection of the warp displacement function of twin-cell box girder

To the single-box twin-cell girder shown in figure 1, the longitudinal displacement function of the cross section is shown below[3].

\[ u(x, y, z) = -z\omega(x) + f(y)U(x) \]  

(1)

Where \( \omega(x) \) = vertical displacement (deflection) of the box girder, \( \omega'(x) \) = around the corner of the box girder, \( u(x, y, z) \) = longitudinal displacement at any point of the cross section, \( U(x) \) = generalized displacement for shear lag, and \( f(y) \) = warping displacement function for shear lag. According to the box girder section construction, \( f(y) \) can be expressed.

\[
f(y) = \begin{cases} 
-\frac{\cos\pi(y-h_0)}{2b_0} + D & \text{for top slab} \\
-\frac{\alpha_1\cos\pi(y-2h_0 - h_0)}{2b_0} + D & \text{for cantilever slab} \\
-\frac{\alpha_2\cos\pi(y-h_0)}{2b_2} + D & \text{for bottom slab} \\
D & \text{for web slab}
\end{cases}
\]

(2)

Where \( D \) is the additional axial displacement of meeting the axial force balance of the whole section, according to the axial force of the bending member must be equal to zero, that is \( \int A f(y)dA = 0 \), therefore, the expression of \( D \) as follows.

\[
D = \frac{2(A_1 + \alpha_1 A_1 + \alpha_2 A_2)}{\pi} \]

(3)

Where \( A \) = area of cross-sectional of the box girder, \( A_1 \) = area of the top slab, \( A_2 \) = area of the bottom slab, \( A_3 \) = area of cantilever slabs at both sides, \( \alpha_1 \) and \( \alpha_2 \) are the coefficients that reflect the difference of warping between different flange plates, the expression see below.

![Box Girder Section](image)

(a) Section of box girder  
(b) Beam longitudinal

Figure 1. Single-box twin-cell girder

3. The differential equation and its solution

3.1. The total potential energy of the bending beam body

When the warp displacement function is determined, the elastic strain of the cross section can be obtained according to the longitudinal displacement of the section[3].

\[
\varepsilon = \frac{\partial u}{\partial x} = -z\omega'(x) + f(y)U'(x) \\
\gamma = \frac{\partial u}{\partial y} = f'(y)U(x)
\]

(4)

Considering the bending moment \( M(x) \) acts on the beam body \( (x_1, x_2) \), the total potential energy expression of considering the strain energy and the potential energy force can be obtained as follows.

\[
\Pi = \frac{1}{2}E \int f^2 dx - 2I_{Qv} \omega^2 + I_{E} U^2 + \frac{G}{E} A_0 U^2 dx + \int_{x_1}^{x_2} M(x)\omega dx
\]

(5)

Where \( E \) = young’s modulus, \( G \) = shear modulus, \( Q(x) \) = denotes shear force, \( M(x) \) = bending moment, \( I_{v} \) = moment of inertia for the flange plates, \( I \) is that for the whole section, \( I_{A}u = \) product of intertia for the flange plates, \( A_0 = \) the area intertia for the flange plates[3], its expression is[3],

\[
I_v = \int f^2(y)dA \quad I_{A} = \int f(y)z dA \quad A_0 = \int (f'(y))^2 dA
\]
3.2. The calculation of $\alpha_1$ and $\alpha_2$

Let’s say that the span is $l$ of the simple support beam have an approximate deflection curve

$$\omega = \omega_0 \sin \frac{\pi x}{l},$$ according to equation (1)

$$u(x, y) = -z \omega(x) + f(y) \cos \frac{\pi x}{l}$$

The arbitrary cross section of the inner top slab has axial displacement along the X-axis.

$$u(x, y, z) = \omega_0 \frac{\pi x}{l} \cos \frac{\pi x}{l} + U_0 \cos \frac{\pi(y-b)}{2b_0} \cos \frac{\pi x}{l} \tag{6}$$

Where $\omega_0 \frac{\pi x}{l} \cos \frac{\pi x}{l}$ is the uniform displacement of vertical bending in plane-section assumption, $U_0 \cos \frac{\pi(y-b)}{2b_0} \cos \frac{\pi x}{l}$ is the generalized displacement for shear lag.

Use the principle of minimum potential energy, for inner top slab

$$\frac{\partial \Pi}{\partial U_0} = 0 \tag{7}$$

Considering $G = \frac{E}{2(1+\mu)}$, can be get

$$U_0 = -\frac{32(1+\mu)b_0^2 z_0}{l(8b_0^2(1+\mu)+l^2)} \omega_0 \tag{8}$$

Similar to the inner top slab, the shear lag displacement of the cantilever slab and the bottom slab can be obtained.

- cantilever slab $U_{01} = -\frac{32(1+\mu)b_0^2 z_0}{l(8b_0^2(1+\mu)+l^2)} \omega_0$ \tag{9}
- bottom slab $U_{02} = \frac{32(1+\mu)b_0^2 z_0}{l(8b_0^2(1+\mu)+l^2)} \omega_0$ \tag{10}

If $U_0$ is a benchmark, the following proportion can be established.

$$\frac{U_{01}}{U_0} = \frac{8b_0^2(1+\mu)+l}{8b_0^2(1+\mu)+l^2}, \quad \frac{U_{02}}{U_0} = \frac{1}{b_0^2} \quad \frac{b_0^2}{b_0^2} = \alpha_1 \tag{11}$$

$$\frac{1}{b_0^2} \quad \frac{b_0^2}{b_0^2} = \beta_1 = \beta_2 = 1 \tag{12}$$

The study formula (11) and (12) can be seen that when $l$ is larger, $\beta_1 \approx \beta_2 \approx 1$. In other words, the warp displacement is basically proportional to the square of the width, also proportional to the distance between flange plate and neutral axis. According to the study of single-cell box girder, making $\beta_1 = \beta_2 = 1$ can satisfy the requirement of calculation accuracy for general span box girder.[3]

3.3. The differential equation of shear-lag is established in Beam segment element

The displacement function of shear lag is known, the minimum energy principle of the beam segment is applied, can obtain the following governing differential equations and the boundary conditions, according to hooke law, the stress formula of the cross section considering the shear lag effect is obtained[3].

$$\sigma(x, y, z) = E \varepsilon = -Ez \omega(x) + Ef(y)U'(x) = \frac{zM(x)}{I} + [f(y) - \frac{I_{yz} z}{I}]EU'(X) \tag{13}$$

4. The analytical solution of Single-box twin-cell girder of simply supported

Combination with the load and boundary conditions of the simply supported beam, the longitudinal stress expression of the shear lag effect is obtained by using equation (13). The normal stress of cross section is obtained when the load $P$ of concentrated load acts on the mid-span section[3].

$$\sigma_z = -\frac{M_z}{I} + \frac{nP}{kl} \sinh \frac{kl}{2} \left( \frac{\sinh \frac{kl}{2}}{\tanh kl} - \cosh \frac{kl}{2} \right) (f(y) - \frac{I_{yz} z}{I}) \tag{14}$$

The corresponding shear lag coefficient is follows.

$$\lambda = 1 - \frac{4a}{z kl} \sinh \frac{kl}{2} \left( \frac{\sinh \frac{kl}{2}}{\tanh kl} - \cosh \frac{kl}{2} \right) (f(y) - \frac{I_{yz} z}{I}) \tag{15}$$
In the same way, the normal stress of the mid-span section under the load $q$ of the uniformly distributed load can be obtained according to the load and boundary conditions.

$$\sigma_n = -\frac{M_z}{I} + \frac{nq}{k^2} \left( \frac{1}{\cosh \frac{kI}{2}} - 1 \right) (f(y) - \frac{I_{yz}}{I})$$  \hspace{1cm} (16)

The shear lag coefficient is follows.

$$\lambda = 1 - \frac{8n}{z^2 k^2 l^2} \left( \frac{1}{\cosh \frac{kI}{2}} - 1 \right) (f(y) - \frac{I_{yz}}{I})$$  \hspace{1cm} (17)

5. The example analysis

5.1. Basic information

Take the example of a simple box beam with a span length of 10m, the size of the model section is shown in figure 2, the young’s modulus of the material is $E=3.45 \times 10^5$ Mpa, the poisson ratio is $\mu=0.375$. There are two forms of load acting on the model, the load $P=1 \times 10^5$ N of concentrated load acts on the mid-span section and the load $q=2 \times 10^4$ N/m of the uniformly distributed load acts on the all across. In order to compare the reliability of the analytical method, the finite element numerical model of space plate shell was established by using the ANSYS finite element of shell63 unit, as shown in figure 3. The uniformly distributed load and the concentrated load are respectively applied to the three web slab, as shown in figure 4.

![Figure 2. 1/2 section of box girder (unit: m)](image)

![Figure 3. Finite element model](image)

(a) Concentrated load $P$  \hspace{1cm} (b) Uniform load $q$

Figure 4. Load cross section distribution pattern

5.2. Shear lag effect across the middle section

The shear coefficient of 1/2 and 1/4 Section of the single box girder with the numerical solution of this paper analytical solution and the ANSYS plate shell is shown in figure 5 and 6.
It can be seen from figure 5 and 6, the analytical solutions are in good agreement with the finite element results. Two kinds of working conditions of results show that although the beam body under symmetrical vertical bending load, but edge web plate stress and middle web plate in the top and bottom there is a certain difference, the edge web plate stress were greater than middle web plate in the top and bottom plate, and the maximum shear lag of the two is 12%. This indicates that the shear lag coefficient of single-box twin-cell girder in uniform bending is different between each plates.

5.3. The shear lag coefficient varies along the span
In order to further analysis the changes of the shear lag coefficients of web and edge web in top, bottom plate, select the distribution of the shear lag coefficients of 1, 2 and 3 and 4 points of the top and bottom plate of the section, as shown in figure 7.

It can be seen that the simply supported beam of twin-cell box girder under the concentrated load and uniform load, the shear lag coefficient of the 1 points of edge web plane are larger than the 2 points of the the middle web plane in top plate. Similarly, the shear lag coefficient of the 3 points of edge web plane are larger than the 4 points of the the middle web plane in bottom plate. The difference between the two parts is the L/4 section, and the maximum difference is nearly 12%.

6. Conclusion
The main conclusions are as follows.
(1) The result shows that the shear lag warping of the displacement pattern of this paper can reflect the difference of shear lag warping between each flange plate and the results obtained the analytical
solutions which are in good agreement with the finite element results. It is shown that the analytical solution of this paper has better precision.

(2) For the simply supported beam of single-box twin-cell girder under the concentrated load and uniform load, although it is a uniform vertical symmetry bending force state, the shear lag coefficient is different between the each plates.

(3) In this paper, the analytical solution method is general and can be applied in the analysis of shear lag effect of multi-cell box girder.

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