Feedback game on Eulerian graphs

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Abstract

In this paper, we introduce a two-player impartial game on graphs, called a feedback game, which is a variant of the generalized geography. The feedback game can be regarded as the undirected edge geography with an additional rule that the first player who goes back to the starting vertex wins the game. We consider the feedback game on Eulerian graphs since the game ends only by going back to the starting vertex. We first show that deciding the winner of the feedback game on Eulerian graphs is PSPACE-complete in general even if its maximum degree is at most 4. In the latter half of the paper, we discuss the feedback game on two subclasses of Eulerian graphs, triangular grid graphs and toroidal grid graphs.

Keywords: Feedback game; Edge geography; Eulerian graph; Triangular grid graph; Toroidal grid graph.

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1 Introduction

All graphs considered in this paper are finite, loopless and undirected unless otherwise mentioned. A graph \(G\) is Eulerian if each vertex of \(G\) has even degree. For other basic terminology in graph theory, we refer to [6].

In combinatorial game theory, an impartial game has been well studied for a long time. So far, many interesting impartial graphs have been found; e.g., Nim [4], Kayles [8] and Poset game [16]. The most famous result in this area is the Sprague-Grundy theorem [12, 17] stating that every impartial game (under the normal play convention) is equivalent to the Grundy value (or Nimber) which plays an important role to determine whether the player can win the game from a given position. There are also many interesting games played on graphs; Vertex Nim [7], Ramsey game [9], Voronoi game [18] and so on. For more details and other topics, we refer the reader to survey several books and articles [1] [2] [3] [5].

One of most popular impartial games on graphs is the generalized geography. The generalized geography is a two-player game played on a directed graph \(D\) whose vertices are words and \(xy \in \mathcal{A}(D)\) if and only if the end character of a word \(x\) is the first one of \(y\),

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where $A(D)$ is the set of arcs of $D$. For example, if $x$ is “Japan” and $y$ is “Netherlands”, then $xy \in A(D)$ but $yx \notin A(D)$. In this setting, the game begins from some starting word and both players alternately extend a directed path using unused words. The first player unable to extend the directed path loses. It is known that deciding the winner of the generalized geography is PSPACE-complete \[14\]. Moreover, several variants of the generalized geography have been considered, e.g., the planar generalized geography \[14\], edge geography \[15\] and undirected geography \[11\]. It is also known that for each above variant, the decision problem which player wins the game is PSPACE-complete in general, except the undirected vertex geography.

In this paper, we consider a new impartial game on a graph, called a feedback game, which is a variant of the undirected edge geography. (We sometimes call it a game for the simplicity of the paper.)

**Definition 1** (Feedback game). There are two players; Alice and Bob, starting with Alice. For a given connected graph $G$ with a starting vertex $s$, a token is put on $s$. They alternately move the token on a vertex $u$ to a neighbor $v$ of $u$ and then delete an edge $uv$. The first player who able to move the token back to $s$ or to an isolated vertex (after removing an edge used by the last move) wins the game.

In this paper, we investigate the feedback game on Eulerian graphs. Note that if a given connected graph $G$ is Eulerian, then the game does not end until the token goes back to the starting vertex $s$, and further observe that Bob always wins the feedback game on any connected bipartite Eulerian graph (cf. \[11\]): Let $G$ be a connected bipartite Eulerian graph, and so, all vertices of $G$ are properly colored by two colors, black and white. Without loss of generality, we may suppose that the starting vertex is colored by black. Throughout the game on $G$, a token is always moved to a white (resp., black) vertex by Alice (resp., Bob). Thus Bob necessarily wins the game.

On the other hand, for a given connected Eulerian graph $G$, the decision problem which player wins the feedback game on $G$ is PSPACE-complete even if the maximum degree of $G$ is at most 4 (Theorem \[3\]). Therefore, a main study on the feedback game is to determine the winner of the game on a connected Eulerian graph with more additional restrictions.

The remaining of the paper is organized as follows. In the next section, we prove the PSPACE-completeness of the feedback game. In Section \[3\], we introduce an even kernel (resp., an even kernel graph), first introduced in \[11\], which is an useful subset (resp., subgraph) guaranteeing the existence of a winning strategy of the second player. In Sections \[1\] and \[5\], focusing on triangular grid graphs and toroidal grid graphs, we determine the winner of the feedback game on several subclasses of them.

## 2 Complexity of the feedback game

Because the feedback game can be seen as a variant of the undirected edge geography, it is a simple idea to construct a reduction from the undirected edge geography to the feedback game.
**Definition 2** (Undirected/Directed edge geography). There are two players; Alice and Bob, starting with Alice. For a given connected undirected/directed graph $G$ with a starting vertex $s$, a token is put on $s$. They alternately move the token on a vertex $u$ to a neighbor/out-neighbor $v$ of $u$ and then delete an edge/arc $uv$. The first player who can move the token to an isolated vertex (after removing an edge/arc used by the last move) wins the game.

The directed edge geography is known as PSPACE-complete [15] via a reduction from TQBF, and the undirected edge geography is also known as PSPACE-complete [11] via a reduction from the directed edge geography. Here TQBF (true quantified Boolean formula) is, given a quantified formula, the determination of whether there exists an assignment to the input variables such that the formula shows $1$.

The feedback game is different from these edge geographies on the winning rule. Since a player wins when a token reaches the starting vertex, it is difficult to reduce from an instance of undirected edge geographies to that of the feedback game. To avoid this difficulty we use the same idea about reduction from TQBF to the directed edge geography, and add a gadget before making the graph undirected.

**Theorem 3.** It is PSPACE-complete to determine whether there exists a strategy that the first player wins a feedback game, even if the given graph is Eulerian.

**Proof.** We can see that this determination is in PSPACE, since we can check the winner using a DFS-like algorithm that recurs $O(|E|)$ times and uses $O(|E|)$ spaces on each recursion.

Now we reduce any instance of TQBF to an instance of determining the winner on the feedback game. The first step is the same as the famous reduction from TQBF to the directed edge geography [15] and we obtain a graph $H$ as an instance. Note that $\Delta(H) = 3$ [14], where $\Delta(H)$ denotes the maximum degree of $H$, and that the obtained graph $H$ has only one vertex $s$ with in-degree 0 and out-degree 2.

By the definition of the feedback game, the winner can also win in the view of the “directed version” of the feedback game on $H$. (Note that any player wins without going back to $s$; the player wins only when the other cannot move anymore.) From now on, as shown in Figure 1, we use pseudo-arcs to make a reduction to the “undirected version” of the feedback game [11].

![Figure 1: Replacing an arc with a pseudo-arc](image)

We make the undirected graph $H'$ obtained as above be Eulerian. Let $D = \{x_1, x_2, \ldots, x_{2p}\}$ for $p \geq 1$ be the set of vertices in $V(H')$ of odd degree. First, we add a path $abc$ and two edges $as$ and $cs$, that is, $sabc$ forms a 4-cycle. Note that the first player does not use the edge $sa$ nor $sc$ at the start on the game; that immediately
leads to a suicide. Next, for each \( x_i \) where \( 1 \leq i \leq 2p \), we make a path \( P_i = x_iy_i z_i \) of length 2 with adding two vertices \( y_i \) and \( z_i \). Finally, we add edges \( z_1 a \), \( z_2 a \), \( z_{2i-1} y_{2i-3} \) and \( z_{2i} y_{2i-3} \), where \( 2 \leq i \leq p \). Clearly, the resulting graph \( G \) is Eulerian. Furthermore, it is not difficult to see that the winner of the feedback game on \( G \) is the same as that of \( H' \); note that the player who moves the token from a vertex \( x_i \) (which is an odd in \( H' \)) to \( y_i \) loses the game.

Note that, a graph we obtain from these reductions has no vertex degree greater than 3. When we discuss Eulerian graphs, a graph can be added vertices and edges and can have vertices degree only 2 or 4. Thus, we obtain the following corollary.

**Corollary 4.** It is PSPACE-complete to determine whether there exists a strategy that the first player wins a feedback game, even if the given graph is a connected graph with maximum degree at most 3 or a connected Eulerian graph with maximum degree at most 4.

### 3 Even kernel graph

Remember that Bob wins the feedback game on every connected bipartite Eulerian graph. Focusing on this fact, Fraenkel et al. [11] introduced a good concept, called an **even kernel**.

**Definition 5 (Even kernel).** Let \( G \) be a connected graph with a starting vertex \( s \). An even kernel of \( G \) with respect to \( s \) is a non-empty subset \( S \subseteq V(G) \) such that

1. \( s \in S \),
2. no two elements of \( S \) are adjacent, and
3. every vertex not in \( S \) is adjacent to an even number (possibly 0) of vertices in \( S \).

It is known in [10] that to find an even kernel of a given graph is NP-complete even if the graph is bipartite or its maximum degree is at most 3. To make the even kernel be easy to handle, we introduce a graph version concept of the even kernel, called an **even kernel graph**. For a graph \( G \) and two disjoint subsets \( A, B \subseteq V(G) \), \( E_G(A, B) \) denotes the set of edges between \( A \) and \( B \) (i.e., one of ends of the edge in the set belongs to \( A \) and the other belongs to \( B \)).

**Definition 6 (Even kernel graph).** Let \( G \) be a connected Eulerian graph with a starting vertex \( s \). An even kernel graph with respect to \( s \) is a bipartite subgraph \( H_s \) with the bipartition \( V(H_s) = B \cup W \) and \( E(H_s) = E_G(B, W) \), where \( B \) is an even kernel of \( G \) and \( W \) is a superset of the set \( N_G(B) = \{ v \in V(G) \setminus B : v \text{ is adjacent to a vertex } u \in B \} \).

For example, see Figure 2. The right of the figure, the graph \( H_s \), is an even kernel graph of the graph \( G \) with a starting vertex \( s \). The bold lines are edges of \( H_s \) and dotted lines are ones in \( E(G) \setminus E(H_s) \), and black vertices in \( B \) (where \( s \in B \)) and white ones in \( W \). Observe that for every vertex \( v \in B \), all edges incident to \( v \) in \( G \) belong to \( E(H_s) \).

**Remark.** If \( G \) has an even kernel, then \( G \) always has an even kernel graph. In Figure 2 \( H_s \) is a spanning subgraph of \( G \), but an even kernel graph is not necessarily spanning in
general. Furthermore, the existence of even kernel graphs depends on the position of a starting vertex \( s \). In fact, it is easy to see that the graph \( G \) shown in Figure 2 has no even kernel graph if its starting vertex is of degree 4.

By the definition, we see the existence of an even kernel (graph) of a connected Eulerian graph \( G \) guaranteeing that Bob wins the game on \( G \).

**Lemma 7** ([11]). Let \( G \) be a connected graph with a starting vertex \( s \). If \( G \) has an even kernel with respect to \( s \), then Bob can win the game on \( G \).

We conclude this section with showing that the converse of Lemma 7 is not true even if \( G \) is Eulerian, that is, a connected Eulerian graph \( G \) does not necessarily have an even kernel graph even if Bob can win the game on \( G \).

**Proposition 8.** There exist infinitely many connected Eulerian graphs without an even kernel graph on which Bob wins the game (with respect to a prescribed starting vertex).

**Proof.** We first give a construction of desired connected Eulerian graphs. Prepare two even cycles \( C_{2k} = u_0u_1u_2\ldots u_{2k-1} \) and \( C_{4k} = v_0v_1v_2\ldots v_{4k-1} \) for some \( k \geq 2 \). Add edges \( u_0v_2i \) and \( u_0v_{2i+1} \) for any \( i \in \{0, 1, \ldots, 2k - 1\} \). Finally, we add a starting vertex \( s \) so that \( s \) and \( v_j \) are adjacent for any \( j \in \{0, 1, \ldots, 4k - 1\} \). The resulting graph is denoted by \( G_k \); for example, see Figure 3.

![Figure 2: An even kernel graph \( H_s \) of a connected Eulerian graph \( G \)](image)

![Figure 3: The graph \( G_2 \)](image)

We next show that Bob can win the game on \( G_k \). Without loss of generality, we may suppose that Alice first moves the token to \( v_0 \) and that Bob moves it from \( v_0 \) to \( u_0 \). If Alice
moves the token to $v_1$, then Bob wins the game. Thus we may assume that Alice moves it to $u_1$, and then Bob moves it to $u_2$. After that, Alice (resp., Bob) moves the token from $u_{2i}$ to $u_{2i+1}$ (resp., from $u_{2i+1}$ to $u_{2i+2}$), where subscripts are modulo $2k$. Therefore, Bob finally moves the token to $u_0$, that is, Alice has to move it to $v_1$. Thus, Bob wins the game on $G_k$.

Finally, we claim that $G_k$ has no even kernel graph with respect to $s$. Suppose to the contrary that $G_k$ has an even kernel graph $H_s$ with bipartite sets $B$ and $W$ where $s \in B$. By the definition of an even kernel graph, $sv_i \in E(H_s)$ for all $i \in \{0, 1, \ldots, 4k-1\}$, that is, $v_i \in W$. Since $H_s$ is bipartite, $v_iv_{i+1} \notin E(H_s)$ where subscripts are modulo $4k$. Thus all edges between two cycles $C_{4k}$ and $C_{2k}$ belong to $E(H_s)$, and hence, $u_j \in B$ for any $j \in \{0, 1, \ldots, 2k-1\}$. However, $u_0$ and $u_1$ must be adjacent in $H_s$, which contradicts the bipartiteness of $H_s$. \(\square\)

4 Triangular grid graphs

At first, we give a recursive definition of triangular grid graphs.

**Definition 9** (Triangular grid graph). A triangular grid graph $T_n$ with $n \geq 0$ is recursively constructed as follows.

- $T_0 (= P^0)$ consists of an isolated vertex $v_0^0$ and no edge.
- $T_n$ with $n \geq 1$ is obtained from $T_{n-1}$ by adding a path $P^n = v_0^nv_1^n \ldots v_n^n$ and edges $v_0^nv_0^{n-1}$, $v_n^nv_{n-1}^n$, $v_i^nv_{i-1}^n$ and $v_i^nv_i^{n-1}$ for any $i \in \{1, \ldots, n-1\}$.

For example, see Figure 4. It is easy to see that every triangular grid graph is connected and Eulerian and that its maximum degree is at most 6. Moreover, it has high symmetry as we know. Thus the class of triangular grid graphs seems to be a reasonable subclass of connected Eulerian graphs for considering the feedback game.

For triangular grid graphs, we have the following setting $v_0^0$ as a starting vertex (where note that the vertex $v_0^0$ can be regarded as $v_0^n$ and $v_n^n$ by symmetry).

**Theorem 10.** If $n \neq 2^m - 3$ with $m \geq 2$, then Bob wins the game on the triangular grid graph $T_n$ with a starting vertex $v_0^0$.

**Proof.** We prove the theorem by induction on $n$. For the base case, we can easily find that each of $T_2$ (the left of Figure 5), $T_3$ (the right of Figure 2), $T_4, T_6$ (Figure 5) has at
least one even kernel graph, i.e., Bob wins the game on these triangular grid graphs by Lemma 7.

For an induction rule, we assume that each of $T_{2i-2}, T_{2i-1}, \ldots, T_{2i+1-4}, T_{2i+1-2}$ has at least one even kernel graph. Here we construct even kernel graphs on triangular grid graphs using those even kernel graphs. Using four even kernel graphs on $T_\alpha$, we can construct an even kernel graph on $T_{2\alpha+3}$; for example, see Figure 6.

From the assumption and this fact, each of $T_{2^{i+1}-1}, T_{2^{i+1}+1}, \ldots, T_{2^{i+2}-5}, T_{2^{i+2}-1}$ has at least one even kernel graph. For triangular grid graphs $T_{2^{i+1}-2}, T_{2^{i+1}+1}, \ldots, T_{2^{i+2}-4}, T_{2^{i+2}-2}$, it is clear that they have an even kernel graph with bipartite sets $B = \{v_{i+k}^j : j \equiv k \equiv 0 \pmod{2}\}$ and $W = \{v_{i+k}^j : j \equiv 1 \pmod{2} \text{ or } k \equiv 1 \pmod{2}\}$ since their height is even (as shown in Figure 5); note that in every even kernel graph constructed above,
all vertices of degree 2 are in the same partite set. Then, all triangular grid graphs $T_{2^i+1-2}, T_{2^i+1-1}, \ldots, T_{2^i+2-4}, T_{2^i+2-2}$ have at least one even kernel graph. By induction, all triangular grid graph $T_n$ has at least one even kernel graph when $n \neq 2^m - 3$. This together with Lemma 7 leads to that Bob wins the game on $T_n$ when $n \neq 2^m - 3$.

Theorem 11 shows that Bob can win the game when the starting vertex is $v_0^0$. This is a common case, that is, we can see that every even kernel graphs in $T_n$ must include $v_0^0$.

**Lemma 11.** There is no even kernel graph on $T_n$ that does not include $v_0^0$ when $n > 1$.

**Proof.** We prove the lemma by contradiction and induction on the distance of $v_0^0$ and another vertex. If an even kernel graph $H$ does not include $v_0^0$, then neither $v_0^1$ nor $v_1^0$ is contained in $H$ by the definition. Therefore, all vertices whose distance from $v_0^0$ is 1 must not be in $H$.

Assume that no vertex whose distance from $v_0^0$ is at most $k$ is in an even kernel graph $H$ with bipartite sets $B$ and $W$, where $B$ contains a starting vertex, we can see that any $v_i^{k+1}(0 \leq i \leq k + 1)$ cannot be in $B$ by definition; because any $v_i^j(0 \leq j \leq k)$ is not a member of $W$ from the assumption. If $v_i^{k+1}$ is a member of $W$, by definition, $v_i^{k+1}$ must have two or four edges in $H$. This condition and local restrictions show that both $v_i^{k+2}$ and $v_i^{k+2}$ are a member of $B$. This violates the definition for $B$. Therefore, $v_i^{k+1}$ cannot be a member of $W$.

By induction on $k$, if $H$ does not include $v_0^0$, any vertex is not a member of $H$, a contradiction. Therefore, all even kernel graphs of $T_n$ must includes $v_0^0$.

For the case when $n = 2^m - 3$ with $m \geq 2$, we have checked that Alice wins the game on $T_n$ with a starting vertex $v_0^0$ for small cases $n = 1, 5$. Furthermore, we confirm that there exists no even kernel graph of $T_n$ if $n = 2^m - 3$ with $m \geq 2$, as follows.

**Theorem 12.** If $n = 2^m - 3$ with $m \geq 2$, then there exists no even kernel graph of the triangular grid graph $T_n$.

**Proof.** Suppose to the contrary that $T_n$ has a even kernel graph $H_n$. From Lemma 11, any even kernel graph $H_n$ of $T_n$ contains $v_0^0, v_0^1$ and $v_0^n$. Furthermore, these three vertices are in $B \subset V(H_n)$, which is a subset containing a starting vertex.

By symmetry, let $i$ be the smallest number such that $v_0^{2i} \notin B$, i.e., if $v_2^j \notin B$ with $j < i$, then we relabel $v_0^0, v_1^0, \ldots, v_k^0$ as $v_k^0, v_k^1, \ldots, v_k^n$ for any $k \in \{1, 2, \ldots, n\}$. Then $v_0^i, \ldots, v_0^{2i-1}$ are in $W$. By definition and local restrictions, it must be hold that $v_j^k \in B$ when $j, k$ is even, otherwise $v_j^k \in W$. We say this pattern on $\Delta v_0^0 v_0^{2i-2} v_0^{2i-2} \Delta v_0^0 v_0^{2i-2} v_0^{2i-2}$ is called a close-packed triangle, where $\Delta abc (a,b,c \in V(T_n))$ denotes a triangular grid graph $T_p$ for some $p \in \{0, 1, \ldots, n\}$ which is contained in $T_n$ as a subgraph.

Since $v_0^{2i-1}, v_0^{2i-1}$ and $v_0^1$ are in $W$ and $v_0^{2i-2} \in B$, we have $v_0^{2i} \in B$, and this leads to $v_0^{2i+1} \in W$ and $v_0^{2i+1} \in B$. Observe that $v_0^{2i} \notin B$, since otherwise, $v_0^{2i}$ is also in $B$ by the observation in the second paragraph, which contradicts $v_0^{2i} \in B$. Moreover, with the similar observation, $v_2^{2i-1}, v_2^{2i+1} \in B$ and $v_2^{2i+1} \in W$.

Two black vertices $v_1^{2i}$ and $v_2^{2i-1}$ and white vertices $v_2^{2i-1}$, where $0 \leq j \leq 2i - 1$, force that all $v_2^{2i} \in B$ if $j$ is odd and all $v_2^{2i} \in W$ if $j$ is even. By local restrictions, $\Delta v_2^{2i}, v_2^{2i-1} v_2^{4i-2} \Delta v_2^{2i}$ is close-packed; note that $v_2^{2i+2}, v_2^{2i+2} \notin B$, since if $v_2^{2i+2} \in B$ (resp., $v_2^{2i+2} \in B$), then $v_1^{2i+1}$...
(resp., \(v^{2i+1}_{2i}\)) in \(W\) must be of degree 3, a contradiction. Furthermore, \(\Delta v^{2i+1}_{2i}v^{4i+2}_{2i-1}v^{4i-2}_{2i-1}\) forces \(v^{2i+j}_j\), where \(0 \leq j \leq 2i - 1\) and \(v^{2i+j}_j\), where \(0 \leq j \leq 2i - 1\) are in \(W\). Note that whether \(v^{2i}_i\) is in \(B\) or \(W\) is not revealed yet under above discussions. (Since the degree of a vertex in \(W\) may be zero, every vertex of \(T_n\) can be a member in \(V(H_n)\).) Now we have:

1. White vertices \(v^{2i+j}_j\), where \(0 \leq j \leq 2i - 1\), and black one \(v^{2i+1}_0\) generate a close-packed triangle \(\Delta v^{2i+1}_0v^{4i-1}_0v^{4i-1}_{2i-2}\).

2. White vertices \(v^{2i+j}_j\), where \(0 \leq j \leq 2i - 1\), and black one \(v^{2i+1}_0\) generate a close-packed triangle \(\Delta v^{2i+1}_0v^{4i-1}_0v^{4i-1}_{2i-1}\).

These successive generation can stop if \(n = 4i - 1\). If \(n = 4i - 1\), \(H_n\) is constructed by four close-packed triangles. If \(n < 4i - 1\), above generation are not satisfied. Therefore there does not exist such \(i\) under that \(n\). Otherwise, \(n > 4i - 1\), above generation must continue, as follows:

Two close-packed triangles \(\Delta v^{2i+1}_0v^{4i-1}_0v^{4i-1}_{2i-2}\) and \(\Delta v^{2i+1}_0v^{4i-1}_0v^{4i-1}_{2i-1}\) force that \(v^{4j}_j\) is in \(W\), where \(0 \leq j \leq 4i\) and \(j \neq 2i\). We next focus on the fact that \(v^{4i-1}_{2i-1}, v^{4i-1}_{2i-1}, v^{4i}_{2i-1}\) and \(v^{4i}_{2i+1}\) must be in \(W\). This fact implies \(v^{4i}_0\) cannot be a member of \(H_n\) since local constrains force \(v^{4i+1}_{2i-1}, v^{4i+1}_{2i+1}, v^{4i+2}_{2i+1} \in B\). These new black vertices generate new three close-packed triangles \(\Delta v^{4i+1}_0v^{4i+1}_0v^{4i+1}_{2i-1}\), \(\Delta v^{4i+2}_2v^{4i+1}_2v^{4i+1}_{2i-1}\), and \(\Delta v^{4i+1}_0v^{4i+1}_0v^{4i+1}_{2i-1}\), and these new close-packed triangles force two extra close-packed triangles \(\Delta v^{4i+2}_0v^{4i+2}_0v^{4i+2}_{2i-2}\), \(\Delta v^{4i+2}_0v^{4i+2}_0v^{4i+2}_{2i-2}\). In this case, these successive generation can stop if \(n = 6i\), and also this discussion can continue recursively if \(n > 6i\).

Let \(r\) be the number of recursion on the above discussion, i.e., \(H_n\) contains \(r^2\) close-packed triangles. By the hypothesis, there can exist such \(i\) on \(T_n\) if \(n = r(2i + 1) - 3\), where \(i, r \geq 1\), which implies that only if \(n\) can be represented as \(n = r(2i + 1) - 3\), where \(i, r \geq 1\), \(H_n\) can exist. Therefore, by the assumption that \(n = 2^m - 3\), there must not exist an even kernel graph for \(T_n\) when \(m > 1\) since \(2^{m-j}\) cannot be represented as \(2^i + 1\) for any \(i \geq 1\) and \(j \leq m\), a contradiction.

Thus we propose the following conjecture which implies that for every triangular grid graph \(T_n\) with a starting vertex \(v^0_0\), Bob wins the game on \(T_n\) if and only if \(T_n\) contains an even kernel graph with respect to \(v^0_0\).

**Conjecture 13.** If \(n = 2^m - 3\) with \(m \geq 2\), then Alice wins the game on the triangular grid graph \(T_n\) with a starting vertex \(v^0_0\).

### 5 Toroidal grid graphs

In this section, we investigate the feedback game on toroidal grid graphs. The undirected edge geography on a grid graph (which is the Cartesian product of two paths) is completely solved [11], and the directed edge geography on a directed toroidal grid graph is also investigated in [13].
**Definition 14** (Toroidal grid graph). A toroidal grid graph $Q(m, n)$ is the Cartesian product of two cycles $C_m = u_0u_1\ldots u_{m-1}$ and $C_n = v_0v_1\ldots v_{n-1}$ with $m \geq 2$ and $n \geq 2$, that is,

- $V(Q(m, n)) = \{(u_i, v_j) : i \in \{0, 1, \ldots, m-1\}, j \in \{0, 1, \ldots, n-1\}\}$.
- $(u_i, v_j)(u_{i'}, v_{j'}) \in E(Q(m, n))$ if and only if
  - $i = i'$ and $v_jv_{j'} \in E(C_n)$ or
  - $j = j'$ and $u_iu_{i'} \in E(C_m)$.

In other words, $Q(m, n)$ is a 4-regular quadrangulation embedded on the torus, which is a graph on a surface with each face quadrangular. For example, see Figure 7; by identifying the top and bottom (resp., right and left) sides along the direction of arrows, we have the toroidal grid graph $Q(3, 4)$. Note that $Q(m, n)$ is vertex-transitive, that is, there exists an automorphism of the graph mapping a vertex into any other vertex. Thus the feedback game on $Q(m, n)$ does not depend on the choice of a starting vertex, and hence, toroidal grid graphs seem to be a reasonable subclass of connected Eulerian graphs with maximum degree at most 4 for considering the feedback game.

![Figure 7: The toroidal grid graph $Q(3, 4)$](image)

For several combinations of $m$ and $n$, we have determined a winner of the game, as follows. In particular, if the greatest common divisor of $m$ and $n$, denoted by $\gcd(m, n)$, is bigger than one, then Bob can win the game on $Q(m, n)$, and otherwise it seems to be that Alice can win the game.

**Theorem 15.** If $\gcd(m, n) = c > 1$, then Bob can win the game on $Q(m, n)$.

*Proof.* By the assumption, let $m = ck$ and $n = ck'$ for some positive integers $k$ and $k'$. The toroidal grid graph $Q(c, c)$ with a starting vertex $s = (u_0, v_0)$ has an even kernel graph $H_c$ with bipartite sets $B$ and $W$ such that $(u_i, v_i) \in B$, $(u_i, v_{i+1}), (u_{i+1}, v_i) \in W$ and edges $(u_i, v_i)(u_i, v_{i+1}), (u_i, v_i)(u_{i+1}, v_i), (u_i, v_{i+1})(u_{i+1}, v_{i+1})$ and $(u_{i+1}, v_i)(u_{i+1}, v_{i+1})$ are in $E(H_c)$ for any $i \in \{0, 1, \ldots, c-1\}$, where subscripts are modulo $c$ (see Figure 8).

Note that $Q(m, n)$ can be “covered” by $Q(c, c)$’s, and hence, we can obtain an even kernel graph of $Q(m, n)$ by combining that of $Q(c, c)$, as shown in Figure 9. (Figure 9 represents $Q(6, 9)$ covered by six $Q(3, 3)$’s with an even kernel graph shown in Figure 8.)
Therefore, since $Q(m, n)$ has an even kernel graph if $\gcd(m, n) = c > 1$, the theorem holds by Lemma 7.

**Theorem 16.** If $\gcd(2, n) = 1$, then Alice can win the game on $Q(2, n)$.

**Proof.** Without loss of generality, we set $(u_0, v_0)$ be a starting vertex. Since $\gcd(2, n) = 1$, $n$ is odd. Alice first moves the token to $(u_0, v_1)$. After that, Alice plays the game according to Bob’s move as follows:

(i) If Bob moves the token to $(u_1, v_i)$ through an edge $(u_0, v_i)(u_1, v_i)$, Alice moves it to $(u_0, v_i)$ using $(u_1, v_i)(u_0, v_i)$.

(ii) If Bob moves the token to $(u_0, v_{i+1})$, then Alice moves it to $(u_0, v_{i+2})$, where subscripts modulo $n$.

Observe that the strategy (i) can be always applied and that after the strategy (i) is applied, Bob must move the token from $(u_0, v_i)$ to $(u_0, v_{i+1})$. Note that the index $i + 1$ is always even when Alice uses the strategy (ii). Therefore, since $n$ is odd, Alice finally moves the token from $(u_0, v_{n-1})$ to $(u_0, v_0)$, that is, she wins the game.

**Theorem 17.** If $\gcd(3, n) = 1$, then Alice can win the game on $Q(3, n)$. 


Proof. Without loss of generality, a starting vertex \( s \) is \((u_0, v_0)\). Moreover, by Theorem 16, we may assume that \( n \geq 4 \).

Alice first moves the token to \((u_1, v_0)\). If Bob moves it to \((u_2, v_0)\), then Alice wins the game. Thus Bob moves the token to \((u_1, v_1)\) by symmetry and then Alice moves it to \((u_2, v_1)\). Next Bob has to move the token to \((u_2, v_2)\) (otherwise Alice can move it back to \( s \)) and then Alice moves it to \((u_0, v_2)\). After that, Alice plays the game according to Bob’s move until the token is moved to \((u_1, v_{n-2})\) for some \( i \in \{0, 1, 2\} \) by herself, as follows (where \( i \) and \( j \) in the following are modulo 3 and \( n \), respectively):

(i) If Bob moves the token on \((u_i, v_j)\) to \((u_i, v_{j-1})\), then Alice moves it to \((u_i, v_{j-2})\).

(ii) If Bob moves the token on \((u_i, v_j)\) to \((u_{i+1}, v_j)\), then Alice moves it to \((u_{i+1}, v_{j-1})\).

(iii) If Bob moves the token on \((u_i, v_j)\) to \((u_i, v_{j+1})\) then Alice moves it to \((u_{i+1}, v_{j+1})\).

Observe that in the above beginning moves from the starting vertex to \((u_0, v_2)\), Alice applies only the strategy (iii) twice except her first move.

In the strategy (i), after Alice’s move, \((u_i, v_{j-2})\) is incident to the unique edge \((u_{i-1}, v_{j-2})\) \((u_{i+1}, v_{j-2})\) unless \((u_i, v_{j-2}) = (u_0, v_0)\), since two edges \((u_i, v_{j-3})\) \((u_{i+1}, v_{j-2})\) and \((u_i, v_{j-2})\) are used by the moves in (iii). Similarly, for the strategy (ii), \((u_{i+1}, v_{j-1})\) is incident to the unique edge \((u_i, v_{j-1})\) \((u_{i+1}, v_{j-1})\). Thus, after Alice’s move by the strategy (i) (resp., (ii)), Bob must move the token to \((u_{i-1}, v_{j-2})\) (resp., \((u_{i+1}, v_{j-1})\)). Hereafter, Alice moves the token to \((u_{i-1}, v_{j-3})\) (resp., \((u_{i+1}, v_{j-2})\)) and then the same situation occurs of the current vertex. Hence by applying the above move repeatedly, the token is finally carried to \( s \) from \((u_0, v_1)\) by Alice.

Therefore, we may suppose that until Alice moves the token to \((u_i, v_{n-2})\) for some \( i \in \{0, 1, 2\} \) by herself, she always applies the strategy (iii), that is, two indices \( i \) and \( j \) of a current vertex \((u_i, v_j)\) are alternately increased one by one by Alice and Bob, respectively. Therefore, we may assume that Alice finally moves the token to \((u_0, v_{n-2})\) (resp., \((u_1, v_{n-2})\)) from \((u_2, v_{n-2})\) (resp., \((u_0, v_{n-2})\)) depending on \( n \); otherwise, i.e., if Alice finally moves the token \((u_2, v_{n-2})\) from \((u_1, v_{n-2})\), then \( n - 2 \equiv 1 \pmod{3} \) and hence \( n \equiv 0 \pmod{3} \), contrary to \( \gcd(3, n) = 1 \).

Thus the token is now put on \((u_0, v_{n-2})\) or \((u_1, v_{n-2})\). In the former case, Bob moves to \((u_0, v_{n-1})\) and then Alice wins the game by moving it back to \( s \). In the latter case, Bob moves to \((u_1, v_{n-1})\) and then Alice moves it to \((u_{i+1}, v_0)\). After that, since Bob must move the token to \((u_2, v_0)\), Alice wins the game by moving it from \((u_2, v_0)\) to \( s \). Therefore, the theorem holds.

\[ \square \]

**Theorem 18.** If \( \gcd(m, n) = 1 \), then there exists no even kernel graph of \( Q(m, n) \).

**Proof.** Let \( \text{Ev}(m, n) \subseteq Q(m, n) \) be an even kernel graph of \( Q(m, n) \). From the definition, any vertex in the white part of \( \text{Ev}(m, n) \), denoted by \( W(m, n) \), has two or four neighbours and they are in the black part of \( \text{Ev}(m, n) \), denoted by \( B(m, n) \). A stopgap of \( \text{Ev}(m, n) \) is a vertex in \( W(m, n) \) of degree 2 such that its neighbours lie on the same row or column. When we ignore all stopgaps, \( \text{Ev}(m, n) \) has several components surrounded by vertices in \( W(m, n) \). Note that any vertex in \( B(m, n) \) cannot be adjacent to vertices not in \( W(m, n) \).

\[ ^1 \text{A vertex in } W(m, n) \text{ can have no neighbour, but in this case we can remove it from } \text{Ev}(m, n). \]
We denote a component and stopgaps which are its neighbours (if exist) as a cluster (see Figure 10). In Figure 10 black vertices are in $B(m, n)$, gray vertices with bold circle are in $W(m, n)$, and gray vertices without edges are not in $\text{Ev}(m, n)$.

Every cluster looks a rectangle rotated 45 degrees. This means that a cluster has four sides consisting of diagonally consecutive vertices in $W(m, n)$. For clusters, we have following claims.

**Claim 1.** Every clusters are rectangles unless $\text{Ev}(m, n) = Q(m, n)$.

**Proof.** Assume that a cluster $C$ is not a rectangle. Then there must exist a vertex in $W(m, n) \subset C$ which is not a stopgap, and is adjacent to a vertex not in $\text{Ev}(m, n)$ and odd number of vertices in $B(m, n)$ (since all vertices in $B(m, n)$ are of degree 4). This contradicts the definition of $\text{Ev}(m, n)$.

**Claim 2.** Any component in $Q(m, n) \setminus \text{Ev}(m, n)$ induces a rectangular region.

**Proof.** This follows from the similar discussion in the proof of Claim 1. (For an example of such regions, see a white region shown in Figure 10.)

Using these claims, $\text{Ev}(m, n)$ must be $Q(m, n)$ when $\gcd(m, n) = 1$ if exists. On the other hand, $Q(m, n)$ is not bipartite when $\gcd(m, n) = 1$. This contradicts the definition of $\text{Ev}(m, n)$. Therefore there exists no even kernel graph of $Q(m, n)$ if $\gcd(m, n) = 1$.

Under results obtained above, we conclude the paper with proposing the following conjecture which implies that Alice can win the feedback game on $Q(m, n)$ if and only if $\gcd(m, n) = 1$.

**Conjecture 19.** Alice can win the game on $Q(m, n)$ if $\gcd(m, n) = 1$. 

Figure 10: An even kernel graph $\text{Ev}(10, 10)$ of $Q(10, 10)$ and its clusters denoted by shaded regions
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References

[1] B. Allen and M.A. Nowak, Games on graphs, Eur. Math. Soc. 1 (2014), 113–151.
[2] C. Berge, Combinatorial games on a graph, Discrete Math. 151 (1996), 59–65.
[3] E.R. Berlekamp, J.H. Conway and R.K. Guy, Winning Ways for Your Mathematical Plays, A K Peters/CRC Press, 4 vols. (1982 – 2004).
[4] C.L. Bouton, Nim, a game with a complete mathematical theory, Ann. of Math. 3 (1901), 35–39.
[5] J.H. Conway, On Numbers and Games, A K Peters/CRC Press (2000).
[6] R. Diestel, Graph Theory (fifth edition), Graduate Texts in Mathematics 173, Springer, 2016.
[7] E. Duchêne and G. Renault, Vertex Nim played on graphs, Theoret. Comput. Sci. 516 (2014), 20–27.
[8] H.E. Dudeney, The Canterbury puzzles, Dover, puzzle 73 (2002), 118–119 (Originally published in 1908).
[9] P. Erdös and J.L. Selfridge, On a combinatorial game, J. Combin. Theory Ser. A 14 (1973), 298–301.
[10] A.S. Fraenkel, Even kernels, Electron. J. Combin. 1 (1994), #R5.
[11] A.S. Fraenkel, E.R. Scheinerman and D. Ullman, Undirected edge geography, Theoret. Comput. Sci. 112 (1993), 371–381.
[12] P.M. Grundy, Mathematics and games, Eureka 2 (1939), 6–9.
[13] M.S. Hogan and D.G. Horrocks, Geography played on an n-cycle times a 4-cycle, Integers 3 (2003), #G02.
[14] D. Lichtenstein and M. Sipser, GO is Polynomial-Space Hard, J. ACM 27 (1980), 393–401.
[15] T.J. Schaefer, On the complexity of some two-person perfect-information games, J. Comput. System Sci. 16 (1978), 185–225.
Appendix

Since it is clear that Alice wins the feedback game on $T_1$, we shall prove that Alice wins the game on $T_5$ with starting vertex $s = v_0^0$.

Without loss of generality, Alice first moves the token to $v_0^0$, and then Bob moves it to either (i) $v_0^2$ or (ii) $v_1^0$. In the case (i) (resp., (ii)), Alice next moves the token to $v_0^5$ (resp., $v_2^2$). For the case (i), as shown the left of Figure 11 we can construct a “good” bipartite subgraph for Alice; note that Alice can move the token to a black vertex in the remaining game as in the argument of the even kernel. Therefore, Alice can finally move the token back to the starting vertex $s$.

We divide the case (ii) to two subcases; (a) Bob moves the token to $v_3^0$, or (b) he moves the token to $v_3^5$. In the former case, as shown in the right of Figure 11 Alice wins the game similarly to the case (i). In the latter case, Alice moves the token to $v_3^4$. If Bob moves the token to $v_4^1$ or $v_4^5$, then Alice can move it back to $v_3^4$ along a 4-cycle $v_3^4 v_4^1 v_5^2 v_4^5$. Moreover, if Bob moves the token to $v_3^0$, then Alice can win the game by moving it to $v_1^2$ (since Bob must move it to $v_1^1$ in his next move). Such a vertex $u$ (to which a player loses the game by moving the token) is called a dead vertex (see Figure 12 a dead vertex is marked by ‘d’ and colored by gray). Thus Bob moves the token from $v_3^4$ to either (1) $v_3^5$ or (2) $v_2^2$.

The proof of the case (1): Alice moves the token to $v_3^5$. By observing that $v_3^5$ and $v_1^7$ are dead, Bob must move the token to $v_4^1$, and then Alice moves it to $v_3^5$. Since $v_3^5$ is also dead.
now, Bob moves the token to $v_3^3$. Therefore, Alice can force Bob to move the token to a dead vertex, by moving it from $v_3^3$ to $v_4^3$.

The proof of the case (2): Alice moves the token to $v_5^5$. Similarly to the previous case, Bob must move the token to $v_4^4$ since $v_3^5$ and $v_3^5$ are dead. After that, Alice can force Bob to move the token to a dead vertex by using one of the following two patterns ($\rightarrow_A$ (resp., $\rightarrow_B$) means a move of the token by Alice (resp., Bob)):

1. $v_4^4 \rightarrow_A v_2^4 \rightarrow_B v_1^3 \rightarrow_A v_1^2 \rightarrow_B v_2^3 \rightarrow_A v_2^4$

2. $v_4^4 \rightarrow_A v_2^4 \rightarrow_B v_2^3 \rightarrow_A v_1^2 \rightarrow_B v_3^3 \rightarrow_A v_2^4$

Therefore, Alice wins the feedback game on the triangular grid graph $T_5$. 

Figure 12: The case (ii)-(b)