Zap Q-Learning With Nonlinear Function Approximation

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Abstract

The Zap stochastic approximation (SA) algorithm was introduced recently as a means to accelerate convergence in reinforcement learning algorithms. While numerical results were impressive, stability (in the sense of boundedness of parameter estimates) was established in only a few special cases. This class of algorithms is generalized in this paper, and stability is established under very general conditions. This general result can be applied to a wide range of algorithms found in reinforcement learning. Two classes are considered in this paper:

(i) The natural generalization of Watkins’ algorithm is not always stable in function approximation settings. Parameter estimates may diverge to infinity even in the linear function approximation setting with a simple finite state-action MDP. Under mild conditions, the Zap SA algorithm provides a stable algorithm, even in the case of nonlinear function approximation.

(ii) The GQ algorithm of Maei et. al. 2010 is designed to address the stability challenge. Analysis is provided to explain why the algorithm may be very slow to converge in practice. The new Zap GQ algorithm is stable even for nonlinear function approximation.

Keywords: Reinforcement learning, Q-learning, Stochastic optimal control

1 Introduction

The theory of Q-learning with function approximation has not caught up with the famous success stories in applications. Counter examples appeared in the 1990’s, shortly following the seminal work of Watkins and Dayan, establishing consistency of the Q-learning algorithm in the tabular setting [50]. These examples demonstrate failure of the natural generalization of Watkins’ algorithm, even in very simple settings such as linear function approximation in a simple finite state-action Markov decision process (MDP) [3, 28]. Even when convergence holds, it is found in practice that convergence of Q-learning can be extremely slow.

This paper focuses on algorithm design to ensure stability of the algorithm, consistency, and techniques to obtain at least qualitative insight on the rate of convergence. The framework for algorithm design is the theory of stochastic approximation (SA), and the ODE approximation that is central to that theory. There is a long history of application of SA tools in the analysis of reinforcement learning (RL) algorithms [44, 46, 23, 7, 28].
An explanation for the slow convergence of Watkins’ Q-learning is given in [13, 14]. The starting point is the recognition that this RL algorithm can be represented as a $d$-dimensional SA recursion

$$\theta_{n+1} = \theta_n + \alpha_{n+1} f(\theta_n, \Phi_{n+1})$$

(1)
in which $\Phi$ is a Markov chain on a finite state space $Z$, $\{\alpha_n\}$ is a non-negative gain sequence, and $f: \mathbb{R}^d \times Z \to \mathbb{R}^d$. In the tabular Q-learning algorithm of Watkins, the dimension $d$ is equal to the number of state-action pairs. The definition of $\Phi$ for this case is given in Section 2. We assume throughout that this Markov chain has a unique invariant probability mass function (pmf).

It is known that the evolution of (1) can be approximated by the solution to the ODE: $\frac{d}{dt} \theta(t) = \overline{f}(\theta(t))$, in which $\overline{f}$ denotes the SA vector field $\overline{f}(\theta) = E[f(\theta, \Phi_{n+1})]$, where the expectation is in steady-state. This is the essence of the SA algorithms that have been refined over nearly 70 years since the original work of Robbins and Monro [35]. See [6] for an accessible treatment. If the ODE is stable in a suitably strong sense; in particular, if solutions converge to some limit $\theta^* \in \mathbb{R}^d$ from each initial condition, then the same is true for (1): $\lim_{n \to \infty} \theta_n = \theta^*$ with probability one.

SA theory also provides tools to understand the rate of convergence. This is based on an approximation of (1) via the linear recursion,

$$\mathcal{E}_{n+1} = \mathcal{E}_n + \alpha_{n+1} [A_s \mathcal{E}_n + \Delta_{n+1}], \quad \mathcal{E}_0 = \theta_0 - \theta^*$$

(2)

where $A_s = \partial \overline{f}(\theta^*)$ is called the linearization matrix, and $\Delta_{n+1} = f(\theta^*, \Phi_{n+1})$. This is intended to approximate the error dynamics: $\mathcal{E}_n \approx \tilde{\theta}_n := \theta_n - \theta^*$; see the standard textbooks [25, 6, 4].

The sequence $\Delta_n$ is zero mean for the stationary version of the Markov chain $\Phi$. Its asymptotic covariance (appearing in the Central Limit Theorem) is denoted

$$\Sigma_\Delta = \sum_{k=-\infty}^{\infty} E[\Delta_k \Delta_0^T]$$

(3)

where the expectation is in steady state. For a fixed but arbitrary initial condition for $(\Phi_0, \mathcal{E}_0)$, denote $\Sigma_n = \text{Cov}(\mathcal{E}_n)$. We obtain the following remarkable conclusions; The proof is in Appendix A.

**Proposition 1.1.** Suppose that the matrix $A_s$ is Hurwitz: $\text{Real}(\lambda) < 0$ for every eigenvalue $\lambda$ of $A_s$, and $\alpha_n = 1/n$ for $n \geq 1$. Then, for the linear recursion (2),

(i) If $\text{Real}(\lambda) < -\frac{1}{2}$ for every eigenvalue $\lambda$ of $A_s$, then rate of convergence is $O(1/n)$:

$$\Sigma_n = n^{-1} \Sigma_\infty + O(n^{-1-\delta})$$

where $\delta > 0$ and $\Sigma_\infty \geq 0$ is the solution to the Lyapunov equation

$$(A_s + \frac{1}{2}I) \Sigma_\infty + \Sigma_\infty (A_s + \frac{1}{2}I)^T + \Sigma_\Delta = 0$$

(4)

(ii) Suppose there is an eigenvalue satisfying $\text{Real}(\lambda) > -\frac{1}{2}$, let $v \neq 0$ denote a corresponding left eigenvector, and suppose that $\Sigma_\Delta v \neq 0$. Then, with $\varrho_0 = |\text{Real}(\lambda)|$,

$$\lim_{n \to \infty} n^\theta E[(v^T \mathcal{E}_n)^2] = 0, \quad \varrho < \varrho_0 \quad \lim_{n \to \infty} n^\theta E[(v^T \mathcal{E}_n)^2] = \infty, \quad \varrho > \varrho_0 \quad \Box$$

The slow convergence for Watkins’ algorithm can be explained by the fact that many eigenvalues of $A_s$ may be close to zero, and $\lambda = -(1 - \beta)$ is always an eigenvalue in the case of tabular Q-learning [13, 14], so that the convergence rate can be as slow as $n^{-(1-\beta)}$. It is shown in Section 2.3 that the situation can be far worse for the GQ-learning algorithm of [28]: When implemented using
a tabular basis, it is shown that the linearization matrix will have an eigenvalue that is at least 
\(-(1 - \beta)^2\), implying a convergence rate as slow as \(n^{-1} \beta^2\).

What is remarkable is that to know if the convergence rate is \(O(1/n)\), it is sufficient to analyze
only the deterministic ODE: Provided \(A_*\) is Hurwitz, the \(O(1/n)\) convergence rate is guaranteed
by using a modified gain \(\alpha_n = g/n\), with \(g > 0\) chosen so that the matrix \(\frac{1}{n} I + g A_*\) is Hurwitz. We can obtain much more reliable algorithms by turning to matrix gain algorithms.

The main contributions of the present paper are summarized as follows:

(i) A significant generalization of the Zap SA algorithm of \([13, 14, 12]\) is proposed:

\[\begin{align*}
\hat{A}_{n+1} &= \hat{A}_n + \gamma_{n+1} [A_{n+1}(\theta_n) - \hat{A}_n], \\
\hat{A}_{n+1}(\theta) &= \partial f(\theta, \Phi_{n+1}) \\
\theta_{n+1} &= \theta_n + \alpha_{n+1} G_n f(\theta_n, \Phi_{n+1}),
\end{align*}\]

with \(\{\alpha_n\}\), \(\{\gamma_n\}\) defined in (9). The algorithm is designed so that it approximates the ODE:

\[\frac{d}{dt} \theta(t) = -[\varepsilon I + A(\theta(t))^\top A(\theta(t))]^{-1} A(\theta(t))^\top \bar{f}(\bar{f}(t)), \quad A(\theta) = \partial \bar{f}(\theta)\]

It is shown in Prop. 2.1 that (6) is stable and consistent under mild assumptions. In particular,
if \(V = \frac{1}{2} ||\bar{f}||^2\) is a coercive function on \(\mathbb{R}^d\), then it serves as a Lyapunov function for (6).

(ii) This new class of SA algorithms are used to propose a new class of Zap-RL algorithms. Specifically, we generalize the Zap Q-learning of [14] to a nonlinear function approximation setting. Stability and convergence of this algorithm are proved under mild conditions.

(iii) We analyze the slow convergence of GQ-learning of [28] and use motivation from Zap-SA
techniques to propose a new class of Zap GQ-learning algorithms, which is stable even for
cancel{nonlinear function approximation.

Literature review The Newton-Raphson flow, introduced for deterministic control applications
in [38, 48], is the special case of (6) obtained with \(\varepsilon = 0\). In this special case, this ODE was
studied in the context of tabular Q-learning in [13, 14]. Stability and convergence of the ODE were
established using ideas similar to the proof of Prop. 2.1.

The covariance approximation in Prop. 1.1 is the basis of algorithms designed to optimize the
asymptotic covariance \(\Sigma_\infty\). Matrix gain algorithms to optimize the covariance were proposed in
[25, 36, 24], and alternative approaches based on two time-scale SA in [37, 32, 33, 24].

There are many versions of Prop. 1.1 in the literature, such as [25, 24], with most results
couched in terms of the Central Limit Theorem rather than finite \(n\) bounds (an exception is [16]).
A complete proof of the proposition for general state space Markov chains is contained in the
supplementary material. Analogous results for the nonlinear recursion (1) are obtained through
linearization (subject to additional conditions on \(f\); e.g. [16]).

Significant progress has been obtained very recently on finite-\(n\) bounds for SA and RL. Finite-\(n\)
error bounds are obtained in [39] for the linear recursion (2) with fixed step-size \(\alpha_n \equiv \epsilon > 0\), and
with \(A = A_n\) also a function of a geometrically ergodic Markov chain. In [8, 11] the authors obtain
concentration bounds for two time-scale SA algorithms, under a martingale difference sequence
noise assumption.

However, Q-learning with function approximation has remained a challenge for several years,
with counterexamples dating back to the famous paper of [3] (also see [45, 40, 18]).

A significant part of literature on RL with function approximation deals with this issue by
formulating an optimization problem, with the objective being mean square projected Bellman

error [41, 28, 10]. Classical first order method such as stochastic gradient descent can not be
directly applied to solve this problem due to the double sampling issue [3, 10]. Most recent works
that aim to optimize this objective take an alternative, primal-dual approach to solve this problem
[29, 27, 47].
The GQ-learning algorithm of [28] is of particular interest in this work. It can be interpreted
as a matrix-gain algorithm in which the gain is chosen for an entirely different purpose: to ensure
stability of Q-learning in a linear function approximation setting, and to ensure that the estimates
converge to the minimum of a projected Bellman error loss function\footnote{The explicit matrix gain representation is hidden in the algorithm because the recursions tend to estimate matrix-vector products, rather than the matrices itself.}. The algorithm is discussed
in detail in Section 2.3.

2 Zap Q-Learning with Nonlinear Function Approximation

2.1 Guidelines for Algorithm Design

Consider the $d$-dimensional SA recursion (1) with matrix gain:

$$\theta_{n+1} = \theta_n + \alpha_{n+1}G_nf(\theta_n, \Phi_{n+1})$$

(7)

The Markov chain is assumed to be irreducible, so there is a unique invariant pmf denoted \(\pi\), which
is used to define the SA vector field \(\mathcal{J}(\theta) = \sum \pi(z)f(\theta, z), \theta \in \mathbb{R}^d\). The goal of SA is to find a
vector \(\theta^* \in \mathbb{R}^d\) satisfying \(\mathcal{J}(\theta^*) = 0\).

As a part of algorithm design, the \(d \times d\) matrix sequence \(\{G_n\}\) is chosen so that it approximates
the ODE, \(d/dt \psi(t) = G(\psi(t))\mathcal{J}(\psi(t))\) for a function \(G: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}\).

Based on SA theory surveyed in the introduction, we arrive at two guidelines for algorithm
design:

\textbf{G1.} The solutions to the ODE converge to the desired limit \(\theta^*\)

\textbf{G2.} The matrix \(1/2I + G(\theta^*)A_*\) is Hurwitz, with \(A_* = \partial \mathcal{J}(\theta^*)\).

The Zap SA algorithm introduced in this paper is designed to achieve these two goals, and in
addition achieve \(\mathcal{J}(\theta^*) \approx -A_*^{-1}\).

It is assumed that \(f\) is \(C^1\) in its first variable. Fix \(\varepsilon \geq 0\) (assumed small), and for \(\theta \in \mathbb{R}^d\) denote

$$A(\theta) = \sum \pi(z)\partial_{\theta} f(\theta, z) \quad \mathcal{G}(\theta) = -[\varepsilon I + A(\theta)^T A(\theta)]^{-1} A(\theta)^T$$

(8)

A two time-scale algorithm are used in the definition of the Zap SA algorithm (5). The step-size
sequences \(\{\alpha_n\}\) and \(\{\gamma_n\}\) are assumed to satisfy standard requirements for two-time-scale SA
algorithms [6]: \(\gamma_n/\alpha_n \rightarrow \infty\) as \(n \rightarrow \infty\). For concreteness we fix throughout:

$$\alpha_n = 1/n, \quad \gamma_n = 1/n^\rho, \quad n \geq 1, \quad \text{with} \quad \rho \in (0.5, 1)$$

(9)

The approximations \(\tilde{A}_{n+1} = A(\theta_n)\) and \(G_n = \mathcal{G}(\theta_n)\) will hold for large \(n\) under general conditions –
this is the basis of two time-scale SA theory [6], and commonly applied in RL analysis [7, 24, 11, 21].

The proof of the following proposition is contained in Appendix B.

\textbf{Proposition 2.1.} Consider the following conditions for the function \(f\):

(a) \(f\) is globally Lipschitz continuous and continuously differential in its first variable. Hence
\(A(\cdot)\) is a bounded matrix-valued function.

(b) \(\|J\|\) is coercive. That is, \(\{\theta : \|J(\theta)\| \leq n\}\) is compact for each \(n\).

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The function $f$ has a unique zero $\theta^*$, and $A^\top(\theta)f(\theta) \neq 0$ for $\theta \neq \theta^*$. Moreover, the matrix $A_* = A(\theta^*)$ is non-singular.

The following hold for solutions to the ODE (6) under increasingly stronger assumptions:

(i) If (a) holds then for each $t$, and each initial condition

$$\frac{d}{dt}f(\varphi(t)) = -[\varepsilon I + A(\varphi(t))^\top A(\varphi(t))]^{-1}A(\varphi(t))^\top f(\varphi(t))$$

(ii) If in addition (b) holds, then the solutions to the ODE are bounded, and

$$\lim_{t \to \infty} A(\varphi(t))^\top f(\varphi(t)) = 0$$

(iii) If (a)–(c) hold, then (6) is globally asymptotically stable.

Implications of Prop. 2.1 to the Zap SA Algorithm: We must first understand what is meant by the term “ODE approximation” (6). A precise definition can be found in [6], but we recall the basic ideas here. A change of time-scale is required: denote $t_n = \sum_{k=1}^n \alpha_k$ for $n \geq 1$. A continuous-time process is defined via $\varphi(\cdot) = \theta_n$ for $t = t_n$, and by piecewise linear interpolation to obtain a continuous function on $\mathbb{R}_+$. A variation on the law of large numbers for Markov chains is then used to obtain the approximation, for any $T_0, t > 0$,

$$\varphi(T_0 + t) = \varphi(T_0) + \int_{T_0}^{T_0+t} G(\varphi(r)) f(\varphi(r)) \, dr + \mathcal{E}(T_0, t)$$

where the error term satisfies for any $T > 0$,

$$\sup_{0 \leq t \leq T} \| \mathcal{E}(T_0, t) \| = o(\| \varphi(T_0) \|)$$

This is by definition the ODE approximation, and is the basis of convergence theory for SA [25, 6, 4].

Based on the ODE approximation we anticipate that we can obtain the following conclusions under Assumptions (a)–(c), perhaps under slightly stronger assumptions on the function $f$. The following results are presented as conjectures, listed in order in increasing level of difficulty for the proofs.

I. If $\lim \sup_{n \to \infty} \| \theta_n \| < \infty$, then the Zap SA algorithm (5) is consistent, $\lim_{n \to \infty} \theta_n = \theta^*$ a.s.

This almost follows from [6, Theorem 6.2] (the martingale noise assumption is imposed in [6] only for convenience – much of the work in SA on single time-scales allows for Markovian noise, such as [1, 4] or the more recent [15, 21, 34]).

II. The sequence $\{\theta_n\}$ is bounded. That is $\lim \sup_{n \to \infty} \| \theta_n \|$ is finite, and this result only requires Assumption (a) of the proposition.

The Lyapunov function $V$ used in the proof of stability of the ODE satisfies the conditions of [1, Theorem 2.3] or [15, Theorem 2.1]. These results establish stability for single time-scale SA algorithms. We believe they can be generalized to the two time-scale setting of this section [26].

III. The covariance is almost optimal: Let $\hat{\theta}_n = \theta_n - \theta^*$. Its covariance satisfies

$$\text{Cov}(\hat{\theta}_n) = n^{-1} \Sigma^c + o(n^{-1}), \quad \Sigma^c = \Sigma^* + O(\varepsilon^2), \quad \Sigma^* = A_*^{-1} \Sigma_\Delta (A_*^\top)^{-1}$$

with $\Sigma_\Delta$ defined in (3). The optimal “asymptotic covariance” $\Sigma^*$ is found in all of the aforementioned papers [25, 24, 13, 14, 12]; in particular, [25, Ch. 10, eq. 2.7(a)].
The proof of the covariance approximation requires a Taylor series approximation of the error dynamics, as discussed in the introduction. The approximation of \( \Sigma^\varepsilon \) is immediate by optimality of \( \Sigma^* \). The bound can be refined through a second Taylor series expansion:

\[
\Sigma^\varepsilon = \Sigma^* + \varepsilon^2 \Sigma^{(2)} + o(\varepsilon^2)
\]

where \( \Sigma^{(2)} = (A_* A_*^T A_*)^{-1} \Sigma_\Delta (A_* A_*^T A_*^T)^{-1} \). The proof is contained in Appendix E.

The final two statements are truly conjectures — they require substantial additional effort, and stronger assumptions:

IV. Extension to parameter-dependent Markovian noise. Rather than a time-homogeneous Markov chain, the transition matrix for \( \Phi^N \) at time \( n \) depends on the parameter \( \theta_n \). There has been significant recent work on stochastic approximation with state dependent noise that can be applied [15, 20, 34]. The challenge is to construct algorithms to estimate \( A(\theta_n) \).

V. Finite time bounds. This is the topic of the very recent work [11, 8, 39, 43], which may provide tools to obtain bounds for Zap stochastic approximation.

In the following subsections we introduce new Q-learning algorithms motivated by this theory, and show how Prop. 2.1 can be extended to these algorithms.

2.2 Zap Q-Learning

We restrict to a discounted reward optimal control problem, with finite state space \( X \), finite action space \( U \), reward function \( r : X \times U \rightarrow \mathbb{R} \), and discount factor \( \beta \in (0, 1) \). Extensions to other criteria, such as average cost or weighted shortest path are obtained by substituting the corresponding formulation of the Bellman error.

The joint state-action process \((X, U)\) is adapted to a filtration \( \{ F_n : n \geq 0 \} \), so that \( F_n \) is intended to model the information available to the controller at time \( n \). The Q-function is defined as the maximum over all possible input sequences \( \{ U_k : k \geq 1 \} \) of the total discounted reward: For each \( x \in X \) and \( u \in U \),

\[
Q^* (x, u) := \max_{U} \sum_{k=0}^{\infty} \beta^k \mathbb{E}[r(X_k, U_k) | X_0 = x, U_0 = u]
\]

Let \( P_u \) denote the state transition matrix when action \( u \in U \) is taken. It is known that the Q-function is the unique solution to the Bellman equation [5]:

\[
Q^* (x, u) = r(x, u) + \beta \sum_{x' \in X} P_u (x, x') Q^* (x'), \quad x \in X, u \in U,
\]

where \( Q(x) := \max_{u \in U} Q(x, u) \) for any function \( Q : X \times U \rightarrow \mathbb{R} \).

For any such function \( Q \) there is a corresponding stationary policy \( \phi(x) \in \arg \max_u Q(x, u) \) (the greedy policy induced by \( Q \)). To avoid ambiguities when the maximizer is not unique, we enumerate all stationary policies as \( \{ \phi^i : 1 \leq i \leq \ell_\phi \} \), and specify

\[
\phi := \phi^{(\kappa)}, \quad \text{where} \quad \kappa := \min_{i} \{ i : \phi^i (x) : \arg \max_u Q(x, u), \text{for all} x \in X \}
\]

The fixed point equation (14) is the basis for Watkins’ Q-learning algorithm and its extensions [49, 2, 14]. In general, the goal of Q-learning algorithms are to best approximate the solution to (14).

Most of these algorithms are based on a Galerkin relaxation [42, 13, 51]. Consider a (possibly nonlinear) parameterized family of approximators \( \{ Q^\theta : \theta \in \mathbb{R}^d \} \), wherein \( Q^\theta : X \times U \rightarrow \mathbb{R} \) for each
Algorithm 1 Zap Q-learning Algorithm

Input: Initial $\theta_0 \in \mathbb{R}^d$, $\hat{A}_0 \in \mathbb{R}^{d \times d}$, $\rho \in (0.5, 1)$, $\alpha_n, \gamma_n$ using (9), $\varepsilon > 0$ small, # iterations $N$, $n = 0$, and a randomized stationary policy to determine $\{U_n\}$

1: repeat
2: $\hat{A}_{n+1} = \hat{A}_n + \gamma_{n+1}[A_{n+1}(\theta_n) - \hat{A}_n]$; with $A_{n+1}(\theta)$ defined in (19)
3: $G_n = -\varepsilon I + \hat{A}_n^{-1} \hat{A}_{n+1}^{-1}$
4: $\theta_{n+1} = \theta_n + \alpha_{n+1} G_n f(\theta_n, \Phi_{n+1})$; with $f(\theta_n, \Phi_{n+1})$ defined in (17)
5: $n = n + 1$
6: until $n \geq N$

Output: $\theta = \theta_N$

$\theta$. The Galerkin relaxation is then obtained by specifying a $d$-dimensional stochastic process $\{\zeta_n\}$ that is adapted to $\mathcal{F}_n$, and setting the goal: Find $\theta^*$ such that

$$\bar{f}(\theta^*) = 0, \quad \text{with} \quad \bar{f}(\theta) := \mathbb{E}[(r(X, U) + \beta Q^\theta(X_{n+1}) - Q^\theta(X, U))] \quad (16)$$

where the expectation is with respect to the steady state distribution of the Markov chain.

The root finding problem (16) is an ideal candidate for stochastic approximation. The matrix gain algorithm (7) is obtained on specifying $\Phi_{n+1} := (X, U, X_{n+1})$, and

$$f(\theta_n, \Phi_{n+1}) := (r(X, U) + \beta Q^\theta(X_{n+1}) - Q^\theta(X, U)) \zeta_n \quad (17)$$

It is assumed that $\zeta_n := \zeta(X, U, n) \geq 0$, for some function $\zeta: \mathbb{R} \times U \to \mathbb{R}^d$.

At points of differentiability, the derivative of $\bar{f}$ has a simple form:

$$A(\theta) := \partial_\theta \bar{f}(\theta) = \mathbb{E}[\zeta_n (\beta \partial_\theta Q^\theta(X_{n+1}, \phi^\theta(X_{n+1})) - \partial_\theta Q^\theta(X, U))] \quad (18)$$

where $\phi^\theta$ denotes the greedy policy induced by $Q^\theta$ (defined in (15), with $Q$ replaced by $Q^\theta$). The definition of $A(\theta)$ is extended to all of $\mathbb{R}^d$ through eq. (18), in which $\phi^\theta$ is uniquely determined using (15). Under this notation, $A(\theta)$ can be interpreted as a weak derivative of $\bar{f}(\theta)$ [9].

The Zap SA algorithm for Q-learning is exactly as described in (5) with $f$ defined in (17), and $A_{n+1}(\theta)$ defined to be the term inside the expectation (18):

$$A_{n+1}(\theta) = \partial_\theta f(\theta, \Phi_{n+1}) = \zeta_n [\beta \partial_\theta Q^\theta(X_{n+1}, \phi^\theta(X_{n+1})) - \partial_\theta Q^\theta(X, U)] \quad (19)$$

These recursions are collected together in Algorithm 1. Observe that it is assumed that $U$ is defined using a randomized stationary policy. This requires that In future work we will consider parameter dependent policies such as $\varepsilon$-greedy. It is assumed that the joint process $(X, U)$ is an irreducible Markov chain, with unique invariant pmf denoted $\pi$.

If the parameterization is linear, we have: $Q^\theta = \sum_{i=1}^d \theta_i \psi_i$, where each $\psi_i: X \times U \to \mathbb{R}$, $1 \leq i \leq d$ is a basis function. In tabular Q-learning [49], the basis functions are indicator functions: $\psi(x, u) = \mathbb{I}\{x = x^k, u = u^k\}$, $1 \leq k \leq \ell_x, \ell_u$, where $\ell_x = |X|$ and $\ell_u = |U|$, and $\zeta \equiv \psi$. The parameterization makes large scale MDP problems tractable and also invites use of prior knowledge of the structure of the value function. But stability is not guaranteed when $Q^\theta$ is nonlinear in $\theta$, or even in a linear setting with a general set of basis functions [3, 18].

**Assumption Q1:** $Q^\theta$ is continuously differentiable, and Lipschitz continuous with respect to $\theta$; $\bar{f}(\theta)$ defined in (16) satisfies the coercivity property: $\{\theta: ||\bar{f}(\theta)|| \leq n\}$ is compact for each $n$.

The following result extends Prop. 2.1 to Zap Q-learning. The extension is non-trivial because the function $\bar{f}(\theta)$ for Q-learning (defined in (16)) is only piece-wise smooth. The proof is contained in Appendix C.
Theorem 2.2. Consider the functions $\overline{f}(\theta)$ and $A(\theta)$ defined in (16, 18). Suppose Assumption Q1 holds. Then, the differential inclusion (6) admits at least one solution from each initial condition, and for any solution
\[
\lim_{t \to \infty} A(\vartheta(t))^\top \overline{f}(\vartheta(t)) = 0 \tag{20}
\]
If in addition $\overline{f}(\theta)$ has a unique zero at $\theta^*$, $A(\theta^*)$ is non-singular, and $A(\theta)^\top \overline{f}(\theta) \neq 0$ for $\theta \neq \theta^*$, then the ODE (10) is globally asymptotically stable. \hfill \Box

The main step in the proof of the theorem is to establish the ODE (10), and this rests on convexity in $\theta$ of the “inverse reward function” defined for $\theta \in \mathbb{R}^d$ by
\[
r^\theta(x, u) := \beta \sum_{x' \in X} P_u(x, x') Q^\theta(x') - Q^\theta(x) \quad x \in X, u \in U.
\]

Implications of Thm. 2.2 to Algorithm 1: Non-smoothness of $\overline{f}(\theta)$ presents a tougher challenge to establish the ODE approximation of $\{\theta_n\}$ compared to the arguments made following Prop. 2.1;

Since $A(\theta) = \partial_\theta \overline{f}(\theta)$ is discontinuous, and the setting is Markovian, standard tools to analyze the SA recursion for $\{A_n\}$ in Algorithm 1 cannot be applied. The authors in [14] make the technical assumption $\sum_n \gamma_n \mathbb{I}\{\phi_n \neq \phi_{n+1}\} < \infty$ to deal with this issue. A discontinuous vector field is also encountered in the GQ algorithm [28]. The authors obtain the ODE approximation only under the assumption that $\{\Delta_n\}$ is a martingale difference. Unfortunately, this assumption typically fails in function approximation settings.

It is believed that the techniques of [15] can be extended to establish the ODE approximation of $\{\theta_n\}$ for the two time-scale Zap Q-learning algorithm. We leave this to future work.

2.3 GQ-learning and Zap GQ-learning

We now take a close look at the GQ-learning algorithm of [28]. The algorithm is based on a linear function approximation setting, but here we consider a generalized version of the algorithm to fit a non-linear function approximation setting.

GQ-learning can be interpreted as a stochastic approximation algorithm that is designed to solve a particular optimization problem. With $\overline{f}(\theta)$ defined in (16), and for a given function $\zeta : X \times U \to \mathbb{R}^d$, and $\zeta_n = \zeta(X_n, U_n)$, the objective in [28] is the following $^2$:
\[
\min_{\theta} J(\theta) = \frac{1}{2} \overline{f}(\theta)^\top M \overline{f}(\theta), \quad \text{with} \quad M = \mathbb{E}[\zeta_n^\top \zeta_n]^{-1} \tag{21}
\]
where the expectation is in steady state. Using (18), we have: $-\nabla J(\theta) = -A(\theta)^\top M \overline{f}(\theta)$, and under the assumption made in [28] that $A(\theta)$ is nonsingular for any $\theta$, the two time scale SA algorithm GQ-learning aims to approximate the solution to the following ODE:
\[
\frac{d}{dt} \vartheta(t) = \overline{f}_{GQ}(\vartheta(t)) \quad \overline{f}_{GQ}(\theta) := -A(\theta)^\top M \overline{f}(\theta) \tag{22}
\]

The eigenvalue test G2 fails when $(1 - \beta)^2 \leq \frac{1}{2}$ in one special case:

Proposition 2.3. The linearization matrix for GQ-learning is given by $A_{GQ} = -A(\theta^*)^\top M A(\theta^*)$, with $M$ and $A(\theta)$ defined in (18, 21). In the special case of a linear function approximation, and with a tabular basis, there is an eigenvalue $\lambda_{GQ}$ of $A_{GQ}$ satisfying
\[
\lambda_{GQ} \geq - (1 - \beta)^2 \max_{x,u} \varpi(x, u)
\]

$^2$In [28], $\zeta \equiv \psi$, the basis functions for the linearly parameterized Q-function.
Prop. 2.3 combined with Prop. 1.1 implies that the convergence rate of GQ-learning algorithm can be as slow as $n^{-(1-\beta)^2}$. The tabular case is of course uninteresting from the point of view of the motivation of this paper or [28], but the proposition serves as a warning that the eigenvalue test may fail in GQ learning without care in choosing the basis function.

Following the steps in Q-learning we obtain

**Zap GQ-learning:** Initialize $\theta_0 \in \mathbb{R}^d$, $\hat{A}_0 \in \mathbb{R}^{d \times d}$, $\rho \in (0.5, 1)$, $\alpha_n$, $\gamma_n$ using (9), $\varepsilon > 0$ small, $M \in \mathbb{R}^{d \times d}$ positive definite; Update for $n \geq 0$:

$$
\hat{A}_{n+1} = \hat{A}_n + \gamma_{n+1} [A_{n+1}(\theta_n) - \hat{A}_n], \quad \text{with } A_{n+1}(\theta) \text{ defined in (19)} \quad (23a)
$$

$$
G_n := -[\varepsilon I + \hat{A}_{n+1}^\top M \hat{A}_{n+1}]^{-1} \hat{A}_{n+1}^\top M \quad (23b)
$$

$$
\theta_{n+1} = \theta_n + \alpha_{n+1} G_n f(\theta_n, \Phi_{n+1}), \quad \text{with } f(\theta_n, \Phi_{n+1}) \text{ defined in (17)} \quad (23c)
$$

The matrix $M$ in (23) can either be as defined in (21), in which case the expectation is approximated using Monte-Carlo, or it could be any other positive definite matrix. It is interesting to note that if $M \equiv I$, the recursion (23) is the same as Zap Q-learning algorithm in Alg. 1.

The GQ ODE (22) has a discontinuous right hand side, which has prevented an extension of Prop. 2.1 to this case. In preliminary experiments it is observed in numerical results that the Zap GQ algorithm with $M$ defined in (21) has similar performance to Zap-Q learning.
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Appendix

A Proof for Prop. 1.1

We prove the proposition for a general state space Markov chain rather than finite state space. Recall that we are considering the linear recursion (2), presented again here for convenience:

\[ E_{n+1} = E_n + \alpha_n [A_n E_n + \Delta_{n+1}], \quad E_0 = \theta - \theta^* \]

where \( A_n = \partial f (\theta^*) \) and \( \Delta_{n+1} = f (\theta^*, \Phi_{n+1}) \). The \( d \times d \) matrix \( A_n \) is Hurwitz, and the following are assumed throughout:

Assumptions: \( \Phi \) is \( V \)-uniformly ergodic on a locally compact and metrizable state space \( Z \) (the conditions of [31]), with unique invariant measure denoted \( \varpi \), and \( \| f (\theta^*, \cdot) \|^2 \in L^\infty \).

The reader is referred to [31] for definitions, except for a few clarifications and consequences: to say that \( g \in L^\infty \) means that \( g : Z \to \mathbb{R} \) is measurable, and that the norm is finite:

\[ \| g \|^V = \sup_{z \in Z} \frac{|g(z)|}{V(z)} \]

It is assumed throughout [31] that \( V : Z \to [1, \infty) \). The \( V \)-uniform ergodic theorem (Theorem 16.0.1 of [31]) gives the following conclusions. Part (iii) is a simple consequence of Jensen’s inequality and the drift criterion that characterizes \( V \)-uniform ergodicity in [31, Theorem 16.0.1].

**Theorem A.1.** The following hold for a \( V \)-uniformly ergodic Markov chain

(i) There is \( \rho \in (0, 1) \) and \( B_V < \infty \) such that for each \( g \in L^\infty \), and each \( z \) and \( n \),

\[ \left| E[g(\Phi_n) \mid \Phi_0 = z] - \varpi(g) \right| \leq B_V \| g \|^V \rho^n V(z) \]

where \( \varpi(g) = \int g(z) \varpi(dz) \).

(ii) Consider the function \( \hat{g} \in L^\infty \) defined by

\[ \hat{g}(z) = \sum_{n=0}^{\infty} \left[ E[g(\Phi_n) \mid \Phi_0 = z] - \varpi(g) \right], \quad z \in Z \]

This solves Poisson’s equation:

\[ E[\hat{g}(\Phi_{k+1}) \mid \Phi_k = z] = \hat{g}(z) - g(z) + \varpi(g) \]

(iii) The Markov chain is also \( V^\delta \)-uniformly ergodic for any \( \delta \in (0, 1] \). In particular, if \( g^2 \in L^\infty \),

then \( g_z^2 \in L^\infty \).

The proof of Prop. 1.1 is composed of the following steps: The sequence can be expressed as the sum of three terms:

\[ E_n = E_n^{(1)} + E_n^{(2)} + E_n^{(3)} \]

each of which is a linear SA recursion (described in (31)) differentiated by initial condition and “noise” input: the first has martingale difference input, the second zero input (driven only by the initial condition, and the input for the third is a telescoping sequence based on a solution to Poisson’s equation. Lemma A.2 shows how the telescoping input is converted into a zero-mean input input of the form \( \hat{f}(\Phi_{n+1})/n \), with \( \hat{f} \in L^\infty \) a solution to Poisson’s equation.

Lemmas A.3 and A.4 imply the conclusions of Prop. 1.1 for \( \{E_n^{(1)}\} \); Lemma A.5 implies the desired conclusions for \( \{E_n^{(2)}\} \); and Lemma A.6 implies the desired conclusions for \( \{E_n^{(3)}\} \).
A.1 Noise statistics and Poisson’s equation

Under the assumptions of this section, the sequence \( \{\Delta_n\} \) appearing in (24) is zero mean for the stationary version of the Markov chain \( \Phi \). This is because 
\[
E_\omega[\Delta_n] = f(\theta^*) = 0.
\] Its asymptotic covariance (appearing in the Central Limit Theorem) is denoted
\[
\Sigma_\Delta = \sum_{k=-\infty}^{\infty} E_\omega[\Delta_k \Delta_0^T]
\] (28)
where the expectations are in steady state.

A more useful representation of \( \Sigma_\Delta \) is obtained through a decomposition of the noise sequence based on Poisson’s equation. This now standard technique was introduced in the SA literature in the 1980s [30]. Two Poisson equation solutions are used in the analysis that follows:
\[
P^\hat{f}(z) = \hat{f}(z) - f(\theta^*, z), \quad P^{\hat{\hat{f}}}(z) = \hat{\hat{f}}(z) - \hat{f}(z),
\] (29)
It is assumed for convenience that the solutions are normalized so \( \hat{f} \) and \( \hat{\hat{f}} \) have zero steady-state mean. The existence of zero-mean solutions follows from (26), and the fact that \( \hat{g} - \omega(\hat{g}) \) also solves (27) for \( g \in L_\infty^V \). Bounds on solutions can be obtained under slightly weaker assumptions: see the main result of [17], and also [31, Theorem 17.4.2]. The bounds \( \hat{f}^2, \hat{\hat{f}}^2 \in L_\infty^V \) follows from Thm. A.1 (iii).

We then have this representation, for \( n \geq 1 \),
\[
\Delta_n = \Delta_{n+1}^m + Z_n - Z_{n+1}
\]
where \( Z_n = \hat{f}(\Phi_n) \) and \( \Delta_{n+1}^m = Z_{n+1} - E[Z_{n+1} \mid F_n] \) is a martingale difference sequence. Each of the sequences is bounded in \( L_2 \), and the asymptotic covariance is expressed
\[
\Sigma_\Delta = E_\omega[\Delta_n^m \Delta_n^m]\n\] (30)
where the expectation is taken in steady-state. The equivalence of (30) and (28) appears in [31, Theorem 17.5.3] for the case in which \( \Delta_n \) is scalar valued; the generalization to vector valued processes involves only notational changes.

A.2 Decomposition of the parameter sequence

The solution of the linear recursion (24) can be decomposed into three terms
\[
\mathcal{E}_n = \mathcal{E}_n^{(1)} + \mathcal{E}_n^{(2)} + \mathcal{E}_n^{(3)}
\]
each evolving as stochastic approximation sequence with different noise and initial conditions:
\[
\begin{align*}
\mathcal{E}_{n+1}^{(1)} &= \mathcal{E}_n^{(1)} + \alpha_{n+1}[A \mathcal{E}_n^{(1)} + \Delta_n^m] , & \mathcal{E}_0^{(1)} = 0 \\
\mathcal{E}_{n+1}^{(2)} &= \mathcal{E}_n^{(2)} + \alpha_{n+1} \mathcal{E}_n^{(2)} , & \mathcal{E}_0^{(2)} = \mathcal{E}_0 \\
\mathcal{E}_{n+1}^{(3)} &= \mathcal{E}_n^{(3)} + \alpha_{n+1}[A \mathcal{E}_n^{(3)} + Z_{n+1} - Z_{n+2}] , & \mathcal{E}_0^{(3)} = 0
\end{align*}
\] (31)

The second recursion admits a more tractable realization through a change of variables, \( \Xi_n = \mathcal{E}_n^{(3)} - \alpha_n Z_{n+1}, \ n \geq 1 \).
Lemma A.2. The sequence $\{\Xi_n : n \geq 1\}$ evolves as the SA recursion

$$\Xi_{n+1} = \Xi_n + \alpha_{n+1} [A\Xi_n - \alpha_n[I + A]Z_{n+1}], \quad \Xi_1 = \mathcal{E}_1^{(3)} - Z_2 \quad (32)$$

Proof. Recall the summation by parts formula: for scalar sequences $\{a_k, b_k\}$,

$$\sum_{k=0}^{N} a_{k+1}[b_{k+1} - b_k] = a_{k+1}b_{k+1} - a_1b_0 - \sum_{k=1}^{N} [a_{k+1} - a_k]b_k \quad (33)$$

This is applied to (31c), beginning with

$$\mathcal{E}_{N+1}^{(3)} = \sum_{n=0}^{N} \alpha_{n+1}A\mathcal{E}_n^{(3)} + \sum_{n=0}^{N} \alpha_{n+1}[Z_{n+1} - Z_{n+2}]$$

Hence with $a_k = \alpha_k$ and $b_k = Z_{k+1}$, the identity (33) implies

$$\sum_{n=0}^{N} \alpha_{n+1}[Z_{n+1} - Z_{n+2}] = Z_1 - \alpha_{N+1}Z_{N+2} + \sum_{n=1}^{N} [\alpha_{n+1} - \alpha_n]Z_{n+1}$$

$$= Z_1 - \alpha_{N+1}Z_{N+2} - \sum_{n=1}^{N} \alpha_{n+1}\alpha_nZ_{n+1}$$

By substitution, and using $\mathcal{E}_n^{(3)} = 0$,

$$\mathcal{E}_{N+1}^{(3)} = Z_1 - \alpha_{N+1}Z_{N+2} + \sum_{n=1}^{N} \alpha_{n+1}[A\mathcal{E}_n^{(3)} - \alpha_nZ_{n+1}]$$

With $\Xi_n := \mathcal{E}_n^{(3)} + \alpha_nZ_{n+1}$ for $n \geq 1$ we finally obtain for $N \geq 1$,

$$\Xi_{N+1} = Z_1 + \sum_{n=1}^{N} \alpha_{n+1}[A\Xi_n - \alpha_n[I + A]Z_{n+1}]$$

which is equivalent to (32).

A.3 Scaled parameter sequence.

For any $\varrho \in (0, 1/2]$ consider the scaled error sequence $\mathcal{E}_n^\varrho = n^\varrho \mathcal{E}_n$. To obtain a recursion for this sequence, consider the Taylor series expansion:

\[
\frac{(n + 1)^\varrho}{n^\varrho} = (1 + n^{-1})^\varrho = 1 + \varrho n^{-1} - \frac{1}{2} \varrho(1 - \varrho)n^{-2} + O(n^{-3})
\]

\[= 1 + \varrho(n + 1)^{-1} + \varrho n^{-1}(n + 1)^{-1} - \frac{1}{2} \varrho(1 - \varrho)n^{-2} + O(n^{-3})\]

where the second equation uses $n^{-1} - (n + 1)^{-1} = n^{-1}(n + 1)^{-1}$. With $\alpha_n = 1/n$, the following bound follows:

\[(n + 1)^\varrho = n^\varrho\left[1 + \alpha_{n+1}(\varrho + \varepsilon(n, \varrho))\right]\]

where $\varepsilon(n, \varrho) = O(n^{-1})$, and $\varepsilon(\varrho, n) > 0$ for all sufficiently large $n$. 

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Case 1: The matrix $\Sigma$.

We begin with an analysis of $\Sigma$.

Lemma A.2 combined with (31) gives the decomposition

$$
\mathcal{E}_n^\varrho = n^\varrho \mathcal{E}_n^{(1)} + n^\varrho \mathcal{E}_n^{(2)} + n^\varrho \mathcal{E}_n^{(3)} + n^\varrho \alpha_n Z_{n+1}
$$

(35)

where $\mathcal{E}_n = \Xi_n$ for $n \geq 1$. Denote $\mathcal{E}_n^{(i)} = n^\varrho \mathcal{E}_n^{(i)}$ for each $i$. These three sequences evolve as solutions to the linear equations

$$
\mathcal{E}_{n+1}^{(1)} = \mathcal{E}_{n}^{(1)} + \alpha_{n+1} \left[ [\varrho_n I + A_n] \mathcal{E}_n^{(1)} + (n+1)^\varrho \Delta_{n+2}^m \right],
$$

(36a)

$$
\mathcal{E}_{n+1}^{(2)} = \mathcal{E}_{n}^{(2)} + \alpha_{n+1} \left[ [\varrho_n I + A_n] \mathcal{E}_n^{(2)} \right],
$$

(36b)

$$
\mathcal{E}_{n+1}^{(3)} = \mathcal{E}_{n}^{(3)} + \alpha_{n+1} \left[ [\varrho_n I + A_n] \mathcal{E}_n^{(3)} - (n+1)^\varrho \alpha_n [I + A] Z_{n+1} \right]
$$

(36c)

The autocorrelation matrix for $\mathcal{E}_n^\varrho$ is denoted $\Sigma_n^\varrho$, and $\Sigma_n^{\varrho(i)}$ for each of (36).

A.4 The martingale term

We begin with an analysis of $\Sigma_n^{\varrho(1)}$.

Case 1: The matrix $\frac{1}{2} I + A_*$ is Hurwitz. In this case we take $\varrho = 1/2$ in the definition of $\Sigma_n^{\varrho(1)}$.

Lemma A.3. If $\text{Real}(\lambda) < -1/2$ for every eigenvalue then $\lim_{n \to \infty} \Sigma_n^{\varrho(1)} = \Sigma$, the solution to the Lyapunov equation

$$
\Sigma + A\Sigma + \Sigma A^T + \Sigma \Delta = 0
$$

Moreover, $\|\Sigma_n^{\varrho(1)} - \Sigma\| = O(n^{-\delta})$, where $\delta = \delta(A, \Sigma \Delta) > 0$.

Recall that $\{\Delta_n^m\}$ is a martingale difference sequence. It is thus an uncorrelated sequence for which $\Sigma_n^{\varrho(1)}$ and $\Delta_{n+k}^m$ are uncorrelated for $k \geq 1$. The following recursion is obtained from these facts and (36a):

$$
\Sigma_{n+1}^{\varrho(1)} = \Sigma_n^{\varrho(1)} + \alpha_{n+1} \left[ \Sigma_n^{\varrho(1)} + A \Sigma_n^{\varrho(1)} + \Sigma_n^{\varrho(1)} A^T + \Sigma \Delta + \mathcal{E}_n^{\varrho(1)} \right]
$$

where the error matrix $\mathcal{E}_n^{\varrho(1)}$ consists of two terms: one is from the error $|\varrho - \varrho_n| + \|A - A_n\| = O(n^{-1})$, and the other is $\Sigma_{\Delta_{n+1}} - \Sigma_{\Delta}$, which converges to zero at a geometric rate.

This is a deterministic SA recursion, which may be regarded as a perturbed Euler approximation to the LTI system

$$
\frac{d}{dt} \mathcal{X}(t) = \mathcal{X}(t) + A \mathcal{X}(t) + \mathcal{X}(t) A^T + \Sigma \Delta
$$

(37)

Let $t_n = \sum_{k=1}^{n} \alpha_k$ and let $\mathcal{X}^n(t)$ denote the solution to this ODE on $[t_n, \infty)$ with $\mathcal{X}^n(t_n) = \Sigma_n^{\varrho(1)}$, $t \geq t_n$, for any $n \geq 1$. We can prove using standard SA arguments that

$$
\lim \sup_{n \to \infty} \|\mathcal{X}^n(t_k) - \Sigma_k^{\varrho(1)}\| = 0
$$

with rate $O(1/n^\delta)$ for some $\delta > 0$. Exponential convergence of $\mathcal{X}$ to $\Sigma$ implies convergence of $\{\Sigma_n^{\varrho(1)}\}$ at rate $O(1/n^\delta)$ (for a possibly smaller value of $\delta$).
Case 2: Suppose that there is a left eigenvalue \( v \) for which \( v^\top A_* = \lambda A_* \), and \( \text{Real}(\lambda) \geq -1/2 \). Suppose moreover that \( \Sigma_{\Delta} v \neq 0 \).

Lemma A.4. Denote \( e^\theta_n = v^\top \Sigma^\theta_n v = E[(v^\top E_n^\theta v)^2]n^{2\theta} \).

Case 2a: \( \text{Real}(\lambda) > -1/2 \)

\[
0 < \liminf_{n \to \infty} e^\theta_n \leq \limsup_{n \to \infty} e^\theta_n < \infty
\]

Case 2b: \( \text{Real}(\lambda) = -1/2 \)

\[
\lim_{n \to \infty} e^\theta_n = \infty
\]

Proof. We begin with the proof that

\[
\liminf_{n \to \infty} e^\theta_n > 0 \quad (38)
\]

This holds whenever \( \text{Real}(\lambda) \geq -1/2 \) (cases 2a and 2b).

With \( \theta = |\text{Real}(\lambda)| \) we have \( v^\top [I \varrho + A] = 0 \), and hence also \( v^\top [I \varrho_n + A_n] = \varepsilon_v(n, \varrho)v \), with \( \varepsilon_v(n, \varrho) = O(n^{-1}) \). Applying (36a) gives

\[
v^\top E_{n+1}^\theta = v^\top E_n^\theta + \alpha_{n+1}v^\top E_n^\theta + (n + 1)\varepsilon_v(n, \varrho)\varepsilon_v(n, \varrho) + (n + 1)^{2\theta - 2}b_n
\]

Consequently, with \( b_n = v^\top \Sigma_{\Delta_n} v \),

\[
e^\theta_{n+1} = [1 + \varepsilon_v(n, \varrho)/(n + 1)]^2e^\theta_n + (n + 1)^{2\theta - 2}b_n
\]

By assumption, \( e^\theta_n \to v^\top \Sigma_{\Delta} v > 0 \) as \( n \to \infty \). Fix \( n_0 > 0 \) so that \( b_{n_0} > 0 \), and hence also \( e^\theta_{n_0 + 1} > 0 \).

We also assume that \( 1 + \varepsilon_v(n, \varrho)/(n + 1) > 0 \) for \( n \geq n_0 \), which is possible since \( \varepsilon_v(n, \varrho) = O(n^{-1}) \).

For \( N > n_0 \) we obtain the uniform bound

\[
\log(e^\theta_N) \geq \log(e^\theta_{n_0 + 1}) + 2 \sum_{n = n_0 + 2}^{\infty} \log[1 - |\varepsilon_v(n, \varrho)|/(n + 1)] > -\infty
\]

which proves that \( \liminf_{n \to \infty} v^\top \Sigma^\theta_n v > 0 \).

The proof of an upper bound for \( \varrho < 1/2 \): by concavity of the logarithm,

\[
\log(e^\theta_{n+1}) \leq \log([1 + \varepsilon_v(n, \varrho)/(n + 1)]^2e^\theta_n) + K(n + 1)^{2\theta - 2}
\]

where \( K = \sup_{n > n_0}[1 + \varepsilon_v(n, \varrho)/(n + 1)]^{-2}[e^\theta_n]^{-1}b_n \). Using concavity of the logarithm once more gives

\[
\log(e^\theta_{n+1}) \leq \log(e^\theta_n) + 2\varepsilon_v(n, \varrho)/(n + 1) + K(n + 1)^{2\theta - 2}
\]

which gives the uniform upper bound

\[
\log(e^\theta_N) \leq \log(e^\theta_{n_0 + 1}) + \sum_{n = n_0 + 2}^{\infty} \left(2|\varepsilon_v(n, \varrho)|/(n + 1) + K(n + 1)^{2\theta - 2}\right) < \infty
\]

The marginal case \( \text{Real}(\lambda) = -1/2 \) is slightly more complex, and is omitted. \( \square \)
A.5 Initial condition response

Analysis of the initial condition response (36b) is via an ODE approximation similar to (37).

Lemma A.5. (i) If \( \text{Real}(\lambda) < -1/2 \) for every eigenvalue then \( \lim_{n \to \infty} \Sigma_n^{1/2}(2) = 0 \), where the convergence is at a rate \( n^{-\delta} \) for some \( \delta > 0 \).

(ii) Suppose that there is a left eigenvalue \( v \) for which \( v^\top A = \lambda A \), and \( \text{Real}(\lambda) > -1/2 \). Then, \( \lim_{n \to \infty} v^\top \Sigma_n^{\theta}(2) v = 0 \), for any \( \theta < |\text{Real}(\lambda)| \), where the convergence is at a rate \( n^{-\delta} \) for some \( \delta > 0 \).

Proof. Consider the vector ODE

\[
\frac{d}{dt} \mathcal{X}(t) = [I \theta + A] \mathcal{X}(t)
\]

Let \( t_n = \sum_{k=1}^{n} \alpha_k \) and let \( \mathcal{X}^n(t) \) denote the solution to this ODE on \( [t_n, \infty) \) with \( \mathcal{X}^n(t_n) = \mathcal{E}^n_{\theta}(2) \), \( t \geq t_n \), for any \( n \geq 1 \). That is, \( \mathcal{X}^n(t) = \exp([I \theta + A](t-t_n))\mathcal{E}^n_{\theta}(2) \) for \( t \geq t_n \). We have as before,

\[
\lim_{n \to \infty} \sup_{k \geq n} \| \mathcal{X}^n(t_k) - \mathcal{E}^\theta_k(2) \| = 0
\]

And as previously, it follows that \( \mathcal{E}^\theta_k(2) \to 0 \) at rate \( k^{-\delta} \) for some \( \delta(A) > 0 \).

A.6 Telescoping sequence term

Lemma A.6. (i) If \( \text{Real}(\lambda) < -1/2 \) for every eigenvalue then \( \lim_{n \to \infty} \Sigma_n^{1/2}(3) = 0 \), where the convergence is at a rate \( n^{-\delta} \) for some \( \delta > 0 \).

(ii) Suppose that there is a left eigenvalue \( v \) for which \( v^\top A = \lambda A \), and \( \text{Real}(\lambda) > -1/2 \). Then, \( \lim_{n \to \infty} v^\top \Sigma_n^{\theta}(3) v = 0 \), for any \( \theta < |\text{Real}(\lambda)| \), where the convergence is at a rate \( n^{-\delta} \) for some \( \delta > 0 \).

Proof. It is convenient to apply the Poisson decomposition a second time. Recall that \( Z_n = \hat{f}(\Phi_n) \) is assumed to have zero steady-state mean. With \( \hat{Z}_n = \hat{f}(\Phi_n) \) we have by (29) and familiar arguments

\[
Z_n = \hat{\Delta}_n^{m+1} + \hat{Z}_n - \hat{Z}_{n+1}
\]

with \( \hat{\Delta}_n^{m+1} = \hat{Z}_{n+1} - E[\hat{Z}_{n+1} \mid F_n] \) a martingale difference sequence. We then transform (36c):

\[
\mathcal{E}^\theta_{n+1}^{(3)} = \mathcal{E}^\theta_{n+1}^{(3)} + \alpha_{n+1} \left[ (\varphi_n I + A_n)\mathcal{E}^\theta_{n+1}^{(3)} - (n+1)^{\theta} \alpha_n \left[I + A \right] \hat{\Delta}^{m+1}_n \right]
\]

which can be expressed as a sum of two recursions. Express \( \mathcal{E}^\theta_{n+1}^{(3)} = \mathcal{E}^\theta_{n+1}^{(3a)} + \mathcal{E}^\theta_{n+1}^{(3b)} \) with

\[
\mathcal{E}^\theta_{n+1}^{(3a)} = \mathcal{E}^\theta_{n+1}^{(3a)} + \alpha_{n+1} \left[ (\varphi_n I + A_n)\mathcal{E}^\theta_{n+1}^{(3a)} - (n+1)^{\theta} \alpha_n \left[I + A \right] \hat{\Delta}^{m+1}_n \right]
\]

\[
\mathcal{E}^\theta_{n+1}^{(3b)} = \mathcal{E}^\theta_{n+1}^{(3b)} + \alpha_{n+1} \left[ (\varphi_n I + A_n)\mathcal{E}^\theta_{n+1}^{(3b)} - (n+1)^{\theta} \alpha_n \left[I + A \right] \hat{Z}_n - \hat{Z}_{n+1} \right]
\]

The covariance of the first sequence is analyzed as in Lemmas A.3 and A.4.

The covariance of \( \mathcal{E}^\theta_{n+1}^{(3b)} \) is analyzed through a second application of Lemma A.2: the covariance is approximated by that of the sequence

\[
\Xi^{(b)}_{n+1} = \Xi^{(b)}_{n+1} + \alpha_{n+1} \left[ (\varphi_n I + A_n)\Xi^{(b)}_{n+1} - (n+1)^{\theta} \alpha_n \left[I + A \right] \hat{Z}_{n+1} \right]
\]

The covariance is easily bounded since \( \sum \alpha_n^2 < \infty \).
B Proof of Prop. 2.1

The result (i) follows from the chain rule and the definitions.

The proof of (ii) is based on the Lyapunov function $V(\vartheta) = \frac{1}{2}\|f(\vartheta)\|^2$ combined with (a):

$$\frac{d}{dt} V(\vartheta(t)) = -f(\vartheta(t))^\top A(\vartheta(t)) [\varepsilon I + A(\vartheta(t))^\top A(\vartheta(t))]^{-1} A(\vartheta(t))^\top f(\vartheta(t))$$

The right hand side is non-positive when $\vartheta(t) \neq \theta^*$. Integrating each side gives for any $T > 0$,

$$V(\vartheta(T)) = V(\vartheta(0)) - \int_0^T f(\vartheta(t))^\top A(\vartheta(t))[\varepsilon I + A(\vartheta(t))^\top A(\vartheta(t))]^{-1} A(\vartheta(t))^\top f(\vartheta(t)) \, dt \quad (39)$$

so that $V(\vartheta(T)) \leq V(\vartheta(0))$ for all $T$. Under the coercive assumption, it follows that solutions to (6) are bounded. Also, letting $T \to \infty$, we obtain from (39) the bound

$$\int_0^\infty f(\vartheta(t))^\top A(\vartheta(t))[\varepsilon I + A(\vartheta(t))^\top A(\vartheta(t))]^{-1} A(\vartheta(t))^\top f(\vartheta(t)) \, dt \leq V(\vartheta(0))$$

This combined with boundedness of $\vartheta(\cdot)$ implies that $\lim_{t \to \infty} A(\vartheta(t))^\top f(\vartheta(t)) = 0$.

We next prove (iii). Global asymptotic stability of (6) requires that solutions converge to $\theta^*$ from each initial condition, and also that $\theta^*$ is stable in the sense of Lyapunov [22]. Assumption (c) combined with (ii) gives the former, that $\lim_{t \to \infty} \vartheta(t) = \theta^*$. A convenient sufficient condition for the latter is obtained by considering $A_\infty = \partial\vartheta \mathcal{G}(\theta^*, f(\theta))]_{\theta = \theta^*}$. Stability in the sense of Lyapunov holds if this matrix is Hurwitz (all eigenvalues are in the strict left half plane in $\mathbb{C}$) [22, Thm. 4.7]. Apply the definitions, we obtain $A_\infty = -[\varepsilon I + M]^{-1} M$ with $M = A(\theta^*)^\top A(\theta^*) > 0$ (recall that $A(\theta^*)$ is assumed to be non-singular). The matrix $A_\infty$ is negative definite, and hence Hurwitz. □

C Proof for Thm. 2.2

With $\overline{f}(\theta)$, $A(\theta)$ defined in (16, 18), we now consider the corresponding differential equation (6) and its integral form:

$$\vartheta(t) = \vartheta(0) - \int_0^t [\varepsilon I + A(\vartheta(\tau))^\top A(\vartheta(\tau))]^{-1} A(\vartheta(\tau))^\top f(\vartheta(\tau)) \, d\tau, \quad 0 \leq t \leq T \quad (40)$$

Lemma C.1. There exists a solution $\vartheta^0$ to (40).

The proof of existence is obtained by first considering a smoothed vector field. We take a particular smoothing kernel used in standard functional analysis applications [9]. Define a probability density $\eta$ on $\mathbb{R}^d$ via

$$\eta(x) := \begin{cases} k \exp\left(\frac{1}{\|x\|^2 - 1}\right) & \|x\| < 1, \\
0 & \|x\| \geq 1, \end{cases} \quad (41)$$

where $k > 0$ is a normalization constant: $\int \eta(x) \, dx = 1$. It can be verified that $\eta$ is $C^\infty(\mathbb{R}^d)$, and furthermore, it can be used to define the $C^\infty$ vector field as follows. For each $\delta > 0$, define

$$\overline{f}_\delta(x) = \frac{1}{\delta d} \int \overline{f}(x - z) \eta(z/\delta) \, dz \quad (42)$$

It is easily shown that the family of functions $\{\overline{f}_\delta : 0 < \delta \leq 1\}$ is globally uniformly Lipschitz continuous, with the same Lipschitz constant as $\overline{f}$. It is also evident that $\lim_{\delta \to 0} \overline{f}_\delta = \overline{f}$ point-wise. The uniform Lipschitz continuity implies that the convergence is uniform on compact sets.
Proof. (i) Consider the ODE (40) with $\bar{f}$ replaced by its smooth approximation:

$$\frac{d}{dt} \varrho^\delta(t) = -[\varepsilon I + A^\delta(T_0)A^\delta(T_0)]^{-1} A^\delta(T_0)\bar{f}(\varrho^\delta(t)), \quad \varrho^\delta(0) = \varrho(0) \quad (43)$$

The following functional equation holds for any $\delta > 0$:

$$\varrho^\delta(t) = \varrho(0) - \int_0^t [\varepsilon I + A^\delta(T_0)A^\delta(T_0)]^{-1} A^\delta(T_0)\bar{f}(\varrho^\delta(\tau)) d\tau \quad (44a)$$

For any $\delta \in (0, 1]$, $\bar{f}$ is continuously differentiable and Lipschitz continuous with Lipschitz constant $L$. For $[\varepsilon I + A^\delta(T_0)A^\delta(T_0)]^{-1}$, its spectral norm is bounded by $1/\varepsilon^2$:

$$||[\varepsilon I + A^\delta(T_0)A^\delta(T_0)]^{-1}||_2 = \frac{1}{\sigma_{\text{min}}(\varepsilon I + A^\delta(T_0)A^\delta(T_0))} \leq \frac{1}{\varepsilon^2}$$

Therefore, for any matrix norm, there exists a constant $C_0$ such that $||[\varepsilon I + A^\delta(T_0)A^\delta(T_0)]^{-1}|| < C_0$. Consequently,

$$||\varrho^\delta(t)|| \leq ||\varrho(0)|| + \int_0^t ||[\varepsilon I + A^\delta(T_0)A^\delta(T_0)]^{-1}|| \cdot ||A^\delta(T_0)|| \cdot ||\bar{f}(\varrho^\delta(\tau))|| d\tau$$

$$\leq ||\varrho(0)|| + \int_0^t C_0 L A ||\bar{f}(\varrho^\delta(\tau))|| d\tau$$

$$\leq ||\varrho(0)|| + \int_0^t \left[C_0 L A (||\bar{f}(\varrho^\delta(\tau)) - \bar{f}(\varrho^\delta(0))|| + C_0 L A ||\bar{f}(\varrho^\delta(0))||)\right] d\tau$$

$$\leq ||\varrho(0)|| + T C_0 L A (||\bar{f}(\varrho^\delta(0))|| + \int_0^t C_0 L A L ||\varrho^\delta(\tau) - \varrho^\delta(0)|| d\tau$$

$$\leq ||\varrho(0)|| + T C_0 L A (||\bar{f}(\varrho^\delta(0))|| + L ||\varrho^\delta(0)||) + \int_0^t C_0 L A L ||\varrho^\delta(\tau)|| d\tau$$

$\varrho^\delta(0) = \varrho(0)$ and $||\bar{f}(\varrho(0))||, \delta \in (0, 1]$ is bounded by $C_1:=\max_{y \in \mathcal{B}(\varrho(0), 1)} ||\bar{f}(y)||$, where $\mathcal{B}(\varrho(0), 1)$ denotes the closed unit ball centered at $\varrho(0)$. By Gronwall’s inequality, there exists some constants $C_1$ and $C_2$ such that

$$||\varrho^\delta(t)|| \leq C_1 + C_2 e^{C_0 L A L T} < \infty, \quad t \in [0, T]$$

Provided boundedness of $\varrho^\delta(t)$, the vector field in (43) is uniformly bounded for any $\delta \in (0, 1]$. Hence $\{\varrho^\delta(\cdot) : \delta \in (0, 1]\}$ is an equicontinuous, bounded family, and is therefore pre-compact. For any convergent subsequence $\{\delta_n\}$ such that $\delta_n \to 0$, and $\varrho^\delta_n(\cdot) \to \varrho^\delta(\cdot)$ with uniform norm in $C([0, T]; \mathbb{R}^d)$:

$$\lim_{\delta \downarrow 0} \sup_{t \in [0, T]} ||\varrho^\delta(t) - \varrho^0(t)|| = 0$$

So (44) holds in the limit, and this limit $\varrho^0(\cdot)$ is a solution of (40).

□

Lemma C.2. Let $G(\theta) := \max_{1 \leq i \leq \ell_u} G_i(\theta)$ where each $G_i : \mathbb{R}^d \to \mathbb{R}$ is $C^1(\mathbb{R}^d)$, and Lipschitz continuous. Let $\varrho(t) : [0, T] \to \mathbb{R}^d$ be a Lipschitz continuous function, and denote $g(t) := G(\varrho(t))$. Then,

(i) $g$ is Lipschitz continuous.
(ii) At any time \( t_0 \in (0, T) \) such that the derivative of \( g \) and \( \theta \) exist,

\[
\frac{d}{dt} g(t) \bigg|_{t=t_0} = \partial_\theta G_k(\vartheta(t)) \cdot \frac{d}{dt} \vartheta(t) \bigg|_{t=t_0} \text{ for each } k \in \arg \max_i G_i(\vartheta(t)). \tag{45}
\]

Proof. Denote \( g_k(t) = G_k(\vartheta(t)) \), so that \( g(t) = \max_k g_k(t) \). Let \( L \) denote a Lipschitz constant for each of these functions:

\[
|g_k(t_1) - g_k(t_0)| \leq L|t_1 - t_0|, \quad t_0, t_1 \in [0, T], \quad 1 \leq k \leq \ell_u
\]

For any \( t_0, t_1 \in [0, T] \),

\[
g(t_1) - g(t_0) \leq g_k(t_1) - g_k(t_0), \quad k \in \arg \max_i g_i(t_0)
\]

\[
\leq L|t_1 - t_0|
\]

The same bound holds, for \( g(t_0) - g(t_1) \), with \( k \in \arg \max_i g_i(t_1) \) in the first inequality. This proves (i).

The proof of (ii) is also straightforward: The difference \( g(t) - g_k(t) \) has a global minimum at \( t_0 \) if \( k \in \arg \max g_i(t_0) \), and consequently

\[
0 = \frac{d}{dt} [g(t) - g_k(t)] \bigg|_{t=t_0}
\]

\( \square \)

Given a parameter vector \( \theta \in \mathbb{R}^d \), we denote by \( r^\theta : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \) the reward function that satisfies (14), with \( Q^* \) replaced by \( Q^\theta \): For each \( x \in \mathcal{X} \) and \( u \in \mathcal{U} \),

\[
r^\theta(x,u) := \beta \sum_{x' \in \mathcal{X}} P_u(x,x') Q^\theta(x') - Q^\theta(x,u) \tag{46}
\]

Lemma C.3. Under Assumption Q1, and that \( \vartheta : [0, T] \to \mathbb{R}^d \) is Lipschitz continuous, \( r^{\vartheta(t)}(x,u) \) is Lipschitz continuous for each \( x, u \) and at any point \( t_0 \) of differentiability

\[
\frac{d}{dt} r_t(x,u) \bigg|_{t=t_0} = \left[ \beta \sum_{x' \in \mathcal{X}} P_u(x,x') \partial_\theta Q^{\vartheta(t_0)}(x', \varphi^{\vartheta(t_0)}(x')) - \partial_\theta Q^{\vartheta(t_0)}(x,u) \right] \frac{d}{dt} \vartheta(t) \bigg|_{t=t_0} \tag{47}
\]

with any policy \( \varphi^{(k)} \) that is \( Q^{\vartheta(t_0)} \)-greedy.

Proof. From definition (46), it is sufficient to establish the derivative formula

\[
\frac{d}{dt} Q^{\vartheta(t)}(x') \bigg|_{t=t_0} = \partial_\theta Q^{\vartheta(t_0)}(x', \varphi^{(k)}(x')) \cdot \frac{d}{dt} \vartheta(t) \bigg|_{t=t_0}
\]

This is immediate from Lemma C.2. \( \square \)

Lemma C.4. If \( \vartheta(t) \) is any solution to (40), then for all \( t \), \( \vartheta(\vartheta(t)) \) satisfies

\[
\frac{d}{dt} \vartheta(\vartheta(t)) = -A(\vartheta(t)) [\varepsilon I + A(\vartheta(t))^T A(\vartheta(t))]^{-1} A(\vartheta(t))^T \vartheta(\vartheta(t)). \tag{48}
\]

Suppose conditions (a)-(c) from Prop. 2.1 hold for \( \vartheta \). Then, the ODE (40) is globally asymptotically stable.
Proof. Suppose \( \vartheta : [0, T] \rightarrow \mathbb{R}^d \) is any solution of (40). The derivative of \( \overline{f}(\vartheta(t)) \), \( 0 \leq t \leq T \), is given by

\[
\frac{d}{dt} \overline{f}(\vartheta(t)) = \mathbb{E}\left[ \frac{d}{dt} \{ \beta Q^{\vartheta(t)}(X_{n+1}, \varphi^{\vartheta(t)}(X_{n+1})) - Q^{\vartheta(t)}(X_n, U_n) \} \zeta_n \right]
\]

(49)

Within the expectation, \( r_t(X_n, U_n) \) is a function of \( t \) whose Lipschitz continuity and chain rule are established in Lemma C.3. Consequently

\[
\frac{d}{dt} \overline{f}(\vartheta(t)) = \mathbb{E}\left[ \zeta_n \frac{d}{dt} r^{\vartheta(t)}(X_n, U_n) \right]
\]

\[
= \mathbb{E}\left[ \zeta_n \beta \partial_\vartheta Q^{\vartheta(t)}(X_{n+1}, \varphi^{\vartheta(t)}(X_{n+1})) - \partial_\vartheta Q^{\vartheta(t)}(X_n, U_n) \right] \frac{d}{dt} \vartheta(t)
\]

\[
= -A(\vartheta(t)) [\varepsilon I + A(\vartheta(t))^\top A(\vartheta(t))]^{-1} A(\vartheta(t))^\top \overline{f}(\vartheta(t))
\]

We define the Lyapunov function as \( V(\vartheta(t)) = \frac{1}{2} \| \overline{f}(t) \|^2 \). The following holds:

\[
\frac{d}{dt} V(\vartheta(t)) = -\overline{f}(\vartheta(t))^\top A(\vartheta(t))[\varepsilon I + A(\vartheta(t))^\top A(\vartheta(t))]^{-1} A(\vartheta(t))^\top \overline{f}(\vartheta(t))
\]

Using the same argument as in Prop. 2.1, the ODE in (48) is globally asymptotically stable. \( \square \)

D Proof of Prop. 2.3

In tabular case, the Q-function \( Q^\phi \) is represented as \( Q^\phi = \sum_{k=1}^d \psi_k \theta_k \) with \( d = \ell_x \cdot \ell_u \), where the basis functions are indicator functions: \( \psi(x, u) = \{ x = x^k, u = u^k \}, 1 \leq k \leq d \) and \( \zeta_n = \psi(X_n, U_n) \). We obtain the *Galerkin relaxation* for this case

\[
\overline{f}(\vartheta) := \mathbb{E}[\psi(X_n, U_n)(r(X_n, U_n) + \beta \psi(X_{n+1}, \varphi^\vartheta(X_{n+1}))^\top \theta - \psi(X_n, U_n)^\top \theta)]
\]

For any deterministic stationary policy \( \varphi \), let \( S_\varphi \) denote the substitution operator, defined for any function \( q : X \times U \rightarrow \mathbb{R} \) by \( S_\varphi q(x) = q(x, \varphi(x)) \). Let \( P \) be a single matrix with \( \ell_x \cdot \ell_u \) rows and \( \ell_x \) columns defined as follows:

\[
P(k, j) = P_{u,k}(x^k, x^j), \quad 1 \leq k \leq d, 1 \leq j \leq \ell_x
\]

\( PS_\varphi \) can be interpreted as the transition matrix for the joint process \( (X, U) \) when \( U \) is defined using policy \( \varphi \) [13]. Under this convention, \( \overline{f}(\vartheta) \) can be written in matrix form

\[
\overline{f}(\vartheta) = \Pi r + \Pi [\beta PS_\varphi - I] \theta
\]

where \( \Pi \) is the \( d \times d \) diagonal matrix with entries: \( \Pi(k, k) := \varpi(x^k, u^k) \) and \( r \) is the vector with entries: \( r(k) := r(x^k, u^k) \). By the definition in (18), the derivative of \( \overline{f}(\vartheta) \) is given by

\[
A(\vartheta) = \mathbb{E}[\psi(X_n, U_n)(\beta \psi(X_{n+1}, \varphi^\vartheta(X_{n+1})) - \psi(X_n, U_n))^\top] = \Pi[\beta PS_\varphi - I]
\]

Proof of Prop. 2.3. The matrix \( A_{GGQ} \) is the derivative of \( \overline{f}_{GGQ}(\vartheta) \) at \( \theta^* \), where \( \overline{f}_{GGQ}(\vartheta) = -A(\vartheta)M \overline{f}(\vartheta) \) is defined in (22).

Recall that \( M = \mathbb{E}[\zeta_n \zeta_n^\top]^{-1} = \mathbb{E}[\psi(X_n, U_n)\psi(X_n, U_n)^\top]^{-1} = \Pi^{-1} \). It is equivalent to show that the matrix \( A(\theta^*)^\top \Pi^{-1} A(\theta^*) \) has a positive eigenvalues less than \((1 - \beta)^2 \max_{x,u} \varpi(x, u)\). We rewrite the matrix as:

\[
A(\theta^*)^\top \Pi^{-1} A(\theta^*) = [\beta PS_{\theta^*} - I]^\top \Pi [\beta PS_{\theta^*} - I]
\]
Denote \( H := \Lambda [\beta P S_{\phi^*} - I] \) with \( \Lambda = \Pi^{1/2} \), by Perron-Frobenius theory of positive matrix and [13, Theorem 3.3], matrix \( H \) has a negative eigenvalue \( \lambda_H \) such that
\[
\lambda_H \geq -(1 - \beta) \max_{x,u} \sqrt{\varpi(x,u)}
\]
The singular values of \( H \) are the eigenvalues of \( A(\theta^*)^\top \Pi^{-1} A(\theta^*) \). Let \( \lambda_{\min}(H) \) denote the eigenvalue of \( H \) with smallest modulus, the following holds by standard arguments in matrix theory
\[
|\lambda_H| \geq |\lambda_{\min}(H)| \geq \sigma_{\min}(H) = \lambda_{\min}(A(\theta^*)^\top \Pi^{-1} A(\theta^*))
\]
Consequently,
\[
\lambda_{\min}(A(\theta^*)^\top \Pi^{-1} A(\theta^*)) \leq (1 - \beta)^2 \max_{x,u} \varpi(x,u)
\]
\[ \square \]

**E Proof of Covariance Approximation**

For any matrix \( C \in \mathbb{R}^{d \times d} \), let \( \lambda(C) \) denote the set of eigenvalues of \( C \), \( \sigma(C) \) the set of singular values of \( C \), and \( \sigma_{\min}(C) \) the smallest singular value of \( C \).

**Proposition E.1.** Let \( A_* \) be a non-singular matrix, \( 0 < \varepsilon < \sigma_{\min}(A) \), and \( G_\varepsilon = -[\varepsilon I + A^\top A]^{-1} A^\top \), then the following holds:

(i) The eigenvalues of \( \frac{1}{2} I + G_\varepsilon A_* \) are strictly negative. Consequently, there exists a unique solution \( \Sigma^\varepsilon \geq 0 \) to the Lyapunov equation
\[
(G_\varepsilon A_* + \frac{1}{2} I) \Sigma^\varepsilon + \Sigma^\varepsilon (G_\varepsilon A_* + \frac{1}{2} I)^\top + G_\varepsilon \Sigma \Delta G_\varepsilon^\top = 0 \quad (50)
\]

(ii) With \( A_*^\top A_*^{-1} \Sigma \Delta (A_*^\top)^{-1} \), and \( \Sigma^{(2)} = (A A^\top)^{-1} \Sigma \Delta (A^\top A A^\top)^{-1} \), the follow approximation holds for \( \tilde{\Sigma}_\varepsilon := \Sigma^\varepsilon - \Sigma_*^\varepsilon \):
\[
\tilde{\Sigma}_\varepsilon = \varepsilon^2 \Sigma^{(2)} + o(\varepsilon^2)
\]

**Proof.** The proof of (i) is based on standard matrix theory ("=" in this equation denotes set equality):
\[
\lambda(\frac{1}{2} I + G_\varepsilon A) = \{ \frac{1}{2} - \lambda' : \lambda' \in \lambda([\varepsilon I + A^\top A]^{-1} A^\top A) \}
\]
\[
= \{ \frac{1}{2} - \frac{1}{\lambda'} : \lambda' \in \lambda(\varepsilon (A^\top A)^{-1} + I) \}
\]
\[
= \{ \frac{1}{2} - \frac{1}{\varepsilon \lambda' + 1} : \lambda' \in \lambda((A^\top A)^{-1}) \}
\]
\[
= \{ \frac{1}{2} - \frac{1}{\varepsilon / \sigma' + 1} : \sigma' \in \sigma(A) \}
\]
Therefore, eigenvalues of \( \frac{1}{2} I + G_\varepsilon A_* \) are strictly negative if \( \varepsilon < \sigma_{\min}(A) \). The fact that the solution \( \Sigma^\varepsilon \geq 0 \) exists and is unique follows from linear system theory [19].

We next prove (ii). It is known that \( \tilde{\Sigma}_\varepsilon \) and \( \tilde{G}_\varepsilon := G_\varepsilon + A_* \) satisfy the following Lyapunov equation [4]:
\[
(\frac{1}{2} I + G_\varepsilon A) \tilde{\Sigma}_\varepsilon + \tilde{\Sigma}_\varepsilon (\frac{1}{2} I + G_\varepsilon A)^\top + \tilde{G}_\varepsilon \Sigma \Delta \tilde{G}_\varepsilon^\top = 0 \quad (51)
\]
Since \( \frac{1}{2} I + G_\varepsilon A_* \) is Hurwitz, the solution of (51) exists and is given by the following matrix integral
\[
\tilde{\Sigma}_\varepsilon = \int_0^\infty \exp([\frac{1}{2} I + G_\varepsilon A]^\top \tau) \tilde{G}_\varepsilon \Sigma \Delta \tilde{G}_\varepsilon^\top \exp([\frac{1}{2} I + G_\varepsilon A]^\top \tau) d\tau \quad (52)
\]
Standard Neumann series results in the following approximations for $G_\varepsilon A_*$ and $\tilde{G}_\varepsilon$:

$$
G_\varepsilon A = -[\varepsilon I + A^T A]^{-1} A^T = -[I - \varepsilon A_*^{-1} (A^T)^{-1}] + o(\varepsilon)
$$

$$
\tilde{G}_\varepsilon = G_\varepsilon + A_* = \varepsilon A_*^{-1} (A^T)^{-1} A_*^{-1} + o(\varepsilon)
$$

Denote $A_* = A_*^{-1} (A^T)^{-1} A_*^{-1}$, the integral in (52) satisfies

$$
\tilde{\Sigma}_\varepsilon = \varepsilon^2 \int_0^\infty \exp(-[\frac{1}{2} I - \varepsilon A_*^{-1} (A^T)^{-1}] \tau) A_* \Sigma A_*^T \exp(-[\frac{1}{2} I - \varepsilon A_*^{-1} (A^T)^{-1}]^T \tau) d\tau + o(\varepsilon^2)
$$

By Taylor series expansion for matrix exponential, we obtain the following approximation

$$
\exp(-[\frac{1}{2} I - \varepsilon A_*^{-1} (A^T)^{-1}] \tau) = \exp(-\frac{\tau}{2} I) + \varepsilon \tau A_*^{-1} (A^T)^{-1} \exp(-\frac{\tau}{2} I) + o(\varepsilon)
$$

Consequently the integral in (53) is simplified into

$$
\tilde{\Sigma}_\varepsilon = \varepsilon^2 \int_0^\infty \exp(-\frac{\tau}{2} I) A_* \Sigma \Delta A_*^T \exp(-\frac{\tau}{2} I) d\tau + o(\varepsilon^2)
$$

$$
= \varepsilon^2 A_* \Sigma \Delta A_*^T + o(\varepsilon^2)
$$

$$
= \varepsilon^2 \Sigma^{(2)} + o(\varepsilon^2)
$$