Axion inflation with gauge field production and primordial black holes

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Abstract

We study the process of primordial black hole (PBH) formation at the beginning of radiation era for the cosmological scenario in which the inflaton is a pseudo-Nambu-Goldstone boson (axion) and there is a coupling of the inflaton with some gauge field. In this model inflation is accompanied by the gauge quanta production and a strong rise of the curvature power spectrum amplitude at small scales (along with non-Gaussianity) is predicted. We show that data on PBH searches can be used for a derivation of essential constraints on the model parameters in such an axion inflation scenario. We compare our numerical results with the similar results published earlier, in the work by Linde et al.

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I. INTRODUCTION

It is well known that inflationary models which predict prolonged inflation are very sensitive to Planck-scale physics (see, e.g., recent reviews [1, 2]). This sensitivity is especially important for large field models when one needs to protect the inflationary potential from a possible large effect of an infinite number of higher dimension operators. Even in supersymmetric models of inflation this protection is not guaranteed, because the supersymmetry is broken by the inflationary background at the Hubble scale.

It had been shown very long ago that the simplest and most natural solution of this problem is to assume that the inflaton $\varphi$ is a pseudo-Nambu-Goldstone boson (PNGB) [3–13], because in this case there is the shift symmetry, $\varphi \rightarrow \varphi + \text{const}$, broken by instanton effects (or explicitly). In the limit when this symmetry is exact, the potential is flat, and the corrections to slow-roll parameters are under control due to the smallness of the symmetry breaking.

If PNGB is pseudoscalar (e.g., it is axion), it is natural to assume that there is a coupling of it with some gauge field. This coupling is not forbidden by the shift symmetry and, in general, is phenomenologically favourable (e.g., it can lead to successful reheating). This coupling is essential if the axion decay constant, $f$, is not too large (because the interaction term is inversely proportional to $f$, see Eq. (1) below). In UV-complete models of axion inflation (e.g., those based on the string theory [8]) one has $f \ll M_P$ and, at the same time, large excursion of the axion field is allowed. The inflationary potential in these models is similar with the potential in large field models.

The main feature of the axion inflation with inflaton-gauge field coupling is that such a coupling leads to a production of gauge quanta, and, through the inverse decay of these quanta into inflaton perturbations, to a rise of non-Gaussianity effects and violation of scale-invariance. In particular, a rather essential formation of primordial black holes (PBHs) becomes possible [19, 20].

In the present work we consider a process of PBH formation and PBH constraints for the axion inflation models in which the inflationary expansion is accompanied by the gauge quanta production. Our consideration differs from the consideration carried out in the recent work [20] in two respects. Firstly, we checked the hypothesis that a probability distribution function (PDF) for curvature fluctuations produced in axion inflation model has the same form as in $\chi^2_n$-models. Secondly, for calculation of the $\beta_{PBH}$-functions describing the fraction of the Universe’s mass in PBHs, at their formation time, we use the full machinery of the Press-Schechter [23] formalism rather than the simple integral over the PDF of the curvature field (see, in this connection, works [24, 25]).

The plan of the paper is as follows. In the second section we review the main assumptions and formulas of the axion inflation model in which there is a coupling of the inflaton with the gauge field. In the third section, we discuss the choice of a suitable PDF for the $\zeta$-field in our scenario. In the fourth section we, using the Press-Schechter formalism, derive PBH mass spectra needed for an obtaining the PBH constraints. Last section contains our conclusions. In Appendix A we study a time evolution of the curvature perturbation power spectrum behind the Hubble horizon. In Appendix B we study the shape of the $\zeta$-bispectrum in our

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1 Non-Gaussian effects in processes of PBH formation had been studied in several pioneering works [14–18].
2 Inflation models with PNG-fields coexisting with inflaton, and subsequent PBH production processes, had been considered in [21, 22].
II. AXION INFLATION WITH GAUGE FIELD PRODUCTION

A. Outline of the model

We consider the model of axion inflation in which there is a coupling of the pseudoscalar inflaton (axion) to gauge fields of the form

\[ \mathcal{L}_{\text{int}} = -\frac{\alpha}{4f} \varphi F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{1} \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength corresponding to some \( U(1) \) gauge field \( A_\mu \), and \( \tilde{F}^{\mu\nu} = \epsilon^{\mu
u\omega\theta} F_{\omega\theta}/(2\sqrt{-g}) \) is the dual strength, \( f \) is the axion decay constant, \( \alpha \) is the dimensionless parameter.

It had been shown in [10] that the evolution (rolling) of the inflaton leads to a generation of the field \( A_\mu \) and to a subsequent amplification (due to tachyonic instability) of its modes. The solutions for the amplified modes are well parameterized by the formula (index + means the circular polarization of quanta)

\[ \tilde{A}_+(k, \tau) \approx \frac{1}{\sqrt{2k}} \left( \frac{k}{2\xi aH} \right)^{1/4} \exp \left[ \pi \xi - 2\sqrt{2\xi k/aH} \right], \tag{2} \]

where

\[ \xi \equiv \frac{\alpha \dot{\varphi}}{2fH}, \tag{3} \]

and \( \tau \approx -1/(aH) \). During inflationary expansion the value of \( \xi \) changes with time. If \( \xi \) is larger than 1, the amplification factor \( e^{\pi \xi} \) is essential. The production of gauge field quanta can affect the inflationary process. In general, it prolongs inflation [10] because it sources inflaton perturbations through the inverse decay: \( \delta A + \delta \dot{A} \rightarrow \delta \varphi \) [26].

The tachyonic amplification of gauge field modes leads to a characteristic evolution of the power spectrum of primordial curvature perturbations. The production of gauge quanta causes strong increasing of the spectrum amplitude. To put constraints on this increase from PBHs one must study the behavior of \( \xi \)-parameter as a function of time during inflationary expansion. The cosmological evolution equations for the inflaton with extra contributions from the gauge field are [10]

\[ \ddot{\varphi} + 3H \dot{\varphi} + V' = \frac{\alpha}{f} \langle \tilde{E} \cdot \tilde{B} \rangle, \tag{4} \]

\[ 3H^2 M_P^2 = \frac{1}{2} \dot{\varphi}^2 + V + \frac{1}{2} \langle \tilde{E}^2 + \tilde{B}^2 \rangle. \tag{5} \]

Here,

\[ \tilde{B} \equiv \frac{1}{a^2} \vec{\nabla} \times \vec{A}, \quad \tilde{E} \equiv -\frac{1}{a^2} \ddot{A}. \tag{6} \]

The connection of \( \langle \tilde{E} \cdot \tilde{B} \rangle \) and \( \langle \tilde{E}^2 + \tilde{B}^2 \rangle \) with \( \xi \) is given by [10]

\[ \langle \tilde{E} \cdot \tilde{B} \rangle \approx -2.4 \times 10^{-4} \frac{H^4}{\xi^4} e^{2\pi \xi}, \quad \frac{1}{2} \langle \tilde{E}^2 \tilde{B}^2 \rangle \approx 1.4 \times 10^{-4} \frac{H^4}{\xi^3} e^{2\pi \xi}. \tag{7} \]
For a calculation of the curvature power spectrum one needs the evolution equation for inflaton field perturbation, $\delta \varphi$. Deriving this equation one must take into account the back-reaction effects [10, 27]. The approximate accounting of these effects leads to the (operator) equation [10, 27]

$$\delta \ddot{\varphi} + 3\beta H \delta \dot{\varphi} - \nabla^2 a^2 \delta \varphi + V'' \delta \varphi = \frac{\alpha}{f} (\vec{E} \cdot \vec{B} - \langle \vec{E} \cdot \vec{B} \rangle),$$

where $\beta$ is defined by the expression

$$\beta \equiv 1 - 2\pi \frac{\alpha}{f} \frac{\langle \vec{E} \cdot \vec{B} \rangle}{3H \dot{\varphi}}.$$  

Equations (4) and (5) are solved numerically, giving the solutions $\varphi(t)$ and $H(t)$ with initial conditions for $\varphi(0)$ and $H(0)$, where $t = 0$ corresponds, in our case, to the moment when CMB scales exit the horizon. As a byproduct one obtains the function $\xi(t)$.

**B. Axion potential**

A typical axion inflationary potential which is exploited in natural inflation models [3, 4] is given by the formula

$$V(\varphi) = \Lambda^4 \left[1 - \cos \left(\frac{\varphi}{f}\right)\right].$$

In UV-complete models of axion inflation, the axion action is shift-symmetric, i.e., the shift symmetry $\varphi \to \varphi + \text{const}$ is broken only non-perturbatively. In particular, in closed string models with type IIB Calabi-Yau orientifold compactifications such axions are available (see, e.g., the review paper [28]). The inflationary potential in such models is periodic, due to instanton effects, but it is flat enough for driving inflation only in the case when the axion decay constant is larger than $M_P$. It is well known, however, that it is difficult to obtain such large values of $f$ in UV-complete theories [29, 30]. So, the potential of single axion, Eq. (10), cannot provide the large field inflation with long slow-roll evolution and a large value of the field excursion.

There are several groups of models in which the large field inflation is possible with sub-Planckian axion decay constants: "Racetrack inflation" models [31], $N$-flation models [6], assisted inflation models [32, 33], axion monodromy inflation models [8, 9, 11–13]. The latter approach looks very promising and we used it in the present paper for numerical calculations. In particular, it had been shown in [8] that, in IIB string theory, the presence of space-filling $D_p$-branes wrapping some two-cycles of the compact internal space leads to a breaking of the shift symmetry and to the monodromy phenomenon: the potential energy for the axion arising from integrating two-form fields over these two-cycles is not periodic and increases with an increase of the axion field. As a result, one has the additional component of the axion potential,

$$V(\varphi) = V_{sr}(\varphi) + V_{\text{inst}}(\varphi).$$

Here, the abbreviation "sr" means slow-roll, and "inst" means instanton. In the concrete model [8], with the $C_2$-axion and $NS5$-brane wrapping $\Sigma_2$ (see [28] for notations), the potential $V_{sr}$ is given by the expression

$$V_{sr}(\varphi) = \frac{\epsilon}{g_s^2 (2\pi)^6 \alpha' \alpha^2} \sqrt{L^4 + g_s^2 \frac{\varphi^2}{f^2}}.$$  

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Here, $L$ is the dimensionless modulus ($L^2$ is the size of the 2-cycle $\Sigma_2$), $g_s$ is the string coupling constant, $1/(2\pi\alpha')$ is the string tension, $\epsilon$ is the warp-factor [8]. At large values of $\varphi/f$ one has the linear potential,

$$V_{sr}(\varphi) \approx \mu^3 \varphi.$$  

(13)

The different realization of the monodromy idea (which is not based on the string theory) had been suggested in [11, 12]. In these works, the axion potential is generated by modification of the action introducing there the coupling of the axion to a 4-form. This new coupling leads to a spontaneous breaking of the shift symmetry and to appearing (in the simplest case) the quadratic axion potential just like in the original chaotic inflation scenario [34].

In this work we will consider both cases: the axion inflation with the quadratic potential

$$V(\varphi) = \frac{m^2 \varphi^2}{2}$$  

(14)

(PBH constraints for axion inflationary model with such a potential have been considered in work [20]) and the inflation with the linear potential given by Eq. (13). We assume that effects from the presence of $V_{\text{inst}}$ are subdominant and neglect this term.

Using the expressions for axion potentials, Eqs. (4) and (5) can be solved. The initial conditions for $t = 0$ corresponding to the moment of time when the scale with the comoving wave number $k = k_* = 0.002 \text{ Mpc}^{-1}$ enters horizon are

$$\varphi(t = 0) = \varphi_0, \quad \dot{\varphi}(t = 0) = -\frac{V'(\varphi_0)}{3H_0}, \quad H(t = 0) = H_0 = \frac{1}{M_P \sqrt{3}} V(\varphi_0)^{1/2}.$$  

(15)

The constant $m$ (or $\mu$) is fixed by the requirement that the curvature perturbation power spectrum $P_{\zeta}$ reaches the observed value $[35]$ at cosmological scales, $P_{\zeta}(k) \approx 2.4 \times 10^{-9}$. For the linear potential (13), we obtained $\mu \approx 6.3 \times 10^{-4} M_P$ and the following set of initial conditions: $\varphi_0 \approx 10.6 M_P$, $|\dot{\varphi}_0| \approx 2.8 \times 10^{-6} M_P^2$, $H_0 \approx 2.9 \times 10^{-5} M_P$. For quadratic potential (14), we have $m \approx 6.8 \times 10^{-6} M_P$ and $\varphi_0 \approx 15 M_P$, $|\dot{\varphi}_0| \approx 5.6 \times 10^{-6} M_P^2$, $H_0 \approx 4.2 \times 10^{-5} M_P$. In Fig. 1 we show the results of our numerical calculations: the dependence of $\xi$ on $N$, the number of e-folds before an end of inflation, for different values of $\xi$ at CMB scales.

One should note, closing this subsection, that axion monodromy inflation with potentials given by Eqs. (13) and (14) predict rather large values of tensor-to-scalar ratio: $r = 0.07$ for the linear potential, and $r = 0.14$ for the quadratic one. The latter value is not excluded by the Planck [35] and BICEP2 [36] data.

C. Curvature power spectrum

In the limit of very small backreaction one has $\beta \to 1$. In this limit, the solution of Eq. (8) is [26, 37]

$$P_{\zeta}(k) = P_{\zeta,sr}(k) \left(1 + P_{\zeta,sr}(k) f_2(\xi)e^{4\pi \xi}\right),$$  

(16)

$$P_{\zeta,sr}(k) = \left(\frac{H^2}{2\pi \dot{\varphi}}\right)^2.$$  

(17)

The function $f_2(\xi)$ is defined in [26, 37].
FIG. 1: The value of $\xi(N)$ for different values of $\xi$ at CMB scales and different choice of model potential. Solid curves are for the case of potential (14). Dashed curves are for potential (13). The curves are labeled with the value of $\xi(N_{CMB})$.

Near horizon crossing one has the approximate solution of Eq. (8) (everywhere below we omit the contribution of the vacuum part, i.e., the solution of the homogenous equation):

$$\delta \varphi \approx \frac{\alpha}{f} \frac{(\vec{E} \cdot \vec{B} - \langle \vec{E} \cdot \vec{B} \rangle)}{3 \beta H^2},$$  \hspace{1cm} (18)

and, correspondingly, one has for the curvature (see Appendix A):

$$\zeta \approx \frac{\alpha}{f} \frac{(\vec{E} \cdot \vec{B} - \langle \vec{E} \cdot \vec{B} \rangle)}{3 \beta H \dot{\varphi}},$$  \hspace{1cm} (19)
The variance of the curvature power spectrum is [20]

\[ \langle \zeta(x)^2 \rangle = \frac{H^2}{\varphi^2} \langle \delta \varphi^2 \rangle \approx \frac{\alpha^2 \langle \vec{E} \cdot \vec{B} \rangle^2}{f^2 (3\beta H \dot{\varphi})^2}. \tag{20} \]

From this equation, in the limit of large backreaction, when \( \beta \gg 1 \), the simple approximate formula for the power spectrum is obtained [10, 20, 27]:

\[ P_\zeta(k) \approx \langle \zeta(x)^2 \rangle = \frac{1}{(2\pi \xi)^2}. \tag{21} \]

Some examples of the power spectrum solutions are shown in Fig. 2. For the calculations we used the approximate formula (20) which takes into account backreaction (at latest stages of inflation the backreaction effects are quite essential). Everywhere we add the contribution of the vacuum part which is dominant at small values of \( k \). The connection of the comoving wave number \( k \) with \( N \) is given by

\[ k = a_e H(N)e^{-N}, \tag{22} \]

where \( a_e \) is the scale factor at the end of inflation.

III. PDFS AND NON-GAUSSIANITY

For a derivation of the PBH constraints we need an expression for the PDF of the \( \zeta \)-field. Evidently, this is a technical problem in non-Gaussian case because, for a calculation of the PDF one must know, in principle, all cumulants (moments) contributing to its series expansion.

In our case, the simplest assumption which we can use in this concrete model is the following [20]: \( \zeta \)-field is distributed as a square of some Gaussian field \( \chi \),

\[ \zeta = \chi^2 - \langle \chi^2 \rangle, \tag{23} \]

having in mind that non-Gaussianity of fluctuations \( \delta \varphi \), described by, e.g., Eq. (8), arises just from the fact that the particular solution of this equation is bilinear in the field \( A_\mu \) (the latter is assumed to be Gaussian).

If Eq. (23) holds (in this case, we have so-called \( \chi^2 \)-model), the PDF of \( \zeta \) is given by (see, e.g., [38, 39])

\[ p_\zeta(\zeta) = \frac{1}{\sqrt{\zeta + \langle \chi^2 \rangle}} p_\chi \left( \sqrt{\zeta + \langle \chi^2 \rangle} \right), \tag{24} \]

\[ p_\chi(\chi) = \frac{1}{\sigma_ch \sqrt{2\pi} \sigma_\chi} e^{-\frac{\chi^2}{2\sigma_\chi^2}}, \quad \sigma_\chi^2 \equiv \langle \chi^2 \rangle. \tag{25} \]

Variance and skewness of the \( \zeta \)-field are, respectively,

\[ \langle \zeta^2 \rangle = 2\langle \chi^2 \rangle^2, \quad \langle \zeta^3 \rangle = 8\langle \chi^2 \rangle^3, \tag{26} \]

so that the first non-trivial reduced cumulant is

\[ D_3 = \frac{\langle \zeta^3 \rangle}{\langle \zeta^2 \rangle^{3/2}} = \sqrt{8}. \tag{27} \]
FIG. 2: The curvature perturbation power spectrum $P_\zeta(k)$ calculated for different values of $\xi_{CMB}$, for two shapes of inflaton potential. The curves are labeled with the value of $\xi(N_{CMB})$.

More generally, one can use $\chi^2_n$-model, in which the $\zeta$-field is written as a sum of $n$ squares of Gaussian fields,

$$\zeta = \sum_{i=1}^{n} \chi^2_i - n\langle \chi^2_i \rangle.$$  \hspace{1cm} (28)

In this case, the PDF of $\zeta$ is \cite{40, 41}:

$$p_\nu(\nu) = \frac{(1 + \nu\sqrt{\frac{2}{n}})^{\frac{2}{n}-1}}{\left(\frac{2}{n}\right)^{\frac{2}{n}} \Gamma\left(\frac{n}{2}\right)} \exp \left( -\frac{n}{2} \left( 1 + \sqrt{\frac{2}{n}}\nu \right) \right),$$  \hspace{1cm} (29)

$$\nu \equiv \frac{\zeta}{\sqrt{\langle \zeta^2 \rangle}}, \hspace{1cm} p_\nu(\nu) d\nu = p_\zeta(\zeta) d\zeta.$$  \hspace{1cm} (30)
The cumulants of $\chi^2_n$-distribution are given by the simple formula,

$$D_m = (m - 1)! \left( \frac{2}{n} \right)^{\frac{m}{n} - 1}. \quad (31)$$

It is tempting to assume that the best choice in our case is the $\chi^2_2$-model, i.e., $n = 2$, in accordance with the fact that the photon has two polarizations. The expression for the corresponding PDF follows from Eq. (29):

$$p_{\nu}(\nu) = e^{-(1 + \nu)}, \quad (32)$$

and the PDF for the $\zeta$-field is

$$p_{\zeta}(\zeta) = \frac{1}{\sqrt{\langle \zeta^2 \rangle}} p_{\nu}(\tilde{\nu}), \quad (33)$$

with properties

$$\int_{-\sqrt{\langle \zeta^2 \rangle}}^{\infty} \zeta p_{\zeta}(\zeta) d\zeta = 0; \quad \int_{-\sqrt{\langle \zeta^2 \rangle}}^{\infty} p_{\zeta}(\zeta) d\zeta = 1; \quad \int_{-\sqrt{\langle \zeta^2 \rangle}}^{\infty} \zeta^2 p_{\zeta}(\zeta) d\zeta = \langle \zeta^2 \rangle. \quad (34)$$

If a PDF of the $\zeta$-field is known one can calculate not only the reduced cumulants $D_m$ but also shapes of $\zeta$-polyspectra (e.g., shapes of $\zeta$-bispectra). From the other side, some of these functions can be calculated in our inflation model directly, without using the PDF. In particular, the reduced cumulant $D_3$ is given by the simple relation \[20\] (in the region where the backreaction is large):

$$D_3 = \frac{\langle \zeta^3 \rangle}{\langle \zeta^2 \rangle^{3/2}} \cong \frac{1/(4\pi^3\xi^3)}{(1/(2\pi\xi)^2)^{3/2}} = 2. \quad (35)$$

This value coincides with the $D_3$ following from Eq. \[31\] for $n = 2$ (compare it with the $D_3$ value given by Eq. \[27\]). So, the choice of $\chi^2_2$-model for a description of PDF seems to be appropriate.

Some results of $\zeta$-bispectrum calculations in our axion inflation model and a comparison with corresponding $\chi^2$-model predictions are given in the Appendix \[13\].

**IV. PBH CONSTRAINTS**

For calculations of PBH constraints we need PDF for the *smoothed* $\zeta$-field, $\zeta_R$ ($R$ is the smoothing radius). We assume, using the argumentation of works \[41\,43\] (see also \[39\]) that PDF of the smoothed $\zeta$-field can be expressed in the form

$$p_{\zeta,R}(\zeta_R) = \frac{1}{\sqrt{\langle \zeta_R^2 \rangle}} \bar{p}_{\nu}(\tilde{\nu}), \quad \tilde{\nu} = \frac{\zeta_R}{\sqrt{\langle \zeta_R^2 \rangle}}. \quad (36)$$

Besides, we assume, following conclusions of \[43\,44\] that cumulants of PDF are approximately equal in smoothing and non-smoothing cases,

$$D_{m,R} \approx D_m. \quad (37)$$
It follows from Eq. (37) that PDF of the smoothed $\zeta$-field can be written as

$$p_{\zeta,R}(\zeta_R) = \frac{1}{\sqrt{\langle \zeta_R^2 \rangle}} p_\nu(\tilde{\nu}), \quad (38)$$

where $p_\nu(\tilde{\nu})$ is given by Eq. (29) with $n = 1$ for $\chi^2$-model and $n = 2$ for $\chi^2_2$-model, with a substitution $\nu \rightarrow \tilde{\nu}$. In this approximation the effects of the smoothing come only through the variance $\sqrt{\langle \zeta_R^2 \rangle}$ while the shape of the PDF is the same as in the non-smoothing case. The variance of $\zeta_R$ is given by the formula

$$\langle \zeta_R^2 \rangle = \int_0^\infty \tilde{W}^2(kR) P_\zeta(k) \frac{dk}{k}, \quad (39)$$

where $\tilde{W}(kR)$ is a Fourier transform of the window function, and we use a Gaussian one, $\tilde{W}^2(kR) = e^{-k^2R^2}$.

One can show that the energy density fraction of the Universe contained in PBHs which form near the time of formation, $t = t_f$ (at this time the horizon mass is equal to $M_h(t_f) = M_f^h$) is given by the integral

$$\Omega_{PBH}(M_f^h) \approx \frac{1}{\rho_i} \left( \frac{M_f^h}{M_i} \right)^{1/2} \int n_{BH}(M_{BH}) M_{BH}^2 d\ln M_{BH} \approx \frac{(M_f^h)^{5/2}}{\rho_i M_i^{1/2}} n_{BH}(M_{BH}) \bigg|_{M_{BH} = f_h M_f^h}. \quad (40)$$

Here, $n_{BH}(M_{BH})$ is the PBH mass spectrum, $\rho_i$ and $M_i$ are, correspondingly, the energy density and horizon mass at the beginning of radiation era (if the reheating is fast, it coincides with an end of inflation). $f_h$ is the constant [equal to $(1/3)^{1/2}$] which connects the value of PBH mass forming at the moment $t_f$ with the horizon mass at that moment (see, e.g., [24]). The PBH mass spectrum in Press-Schechter [23] formalism is proportional to the derivative $\partial P/\partial R$, where $P$ is the integral over the $\zeta$-PDF,

$$P(R) = \int_{\zeta_c}^\infty p_\zeta d\zeta. \quad (41)$$

Approximately, one has

$$\Omega_{PBH}(M_f^h) \approx \beta_{PBH}(M_f^h), \quad (42)$$

where $\beta_{PBH}$ is, by definition, the fraction of the Universe’s mass in PBHs at their formation time,

$$\beta_{PBH}(M_f^h) \equiv \frac{\rho_{PBH}(t_f)}{\rho(t_f)}. \quad (43)$$

Now, having Eqs. (40, 42), one can use the experimental limits on the value of $\beta_{PBH}$ [47, 48] to constrain parameters of models used for PBH production predictions. The PBH mass
FIG. 3: Primordial black hole mass spectra corresponding to curvature perturbation power spectra shown in Fig. 2. Solid curves are for the case $\zeta_c = 0.75$ while dashed curves are for $\zeta_c = 1$. The curves are labeled with the value of $\xi(N_{\text{CMB}})$. Thick line schematically shows existing constraints on PBH abundance [47, 48].

spectrum needed for a derivation of $\Omega_{\text{PBH}}$ in Eq. (41) depends on the amplitude of the curvature power spectrum $P_\zeta$ (see [38, 39, 46] for details).

The results of PBH mass spectra calculation for the considered model are given in Fig. 3 for several values of the parameter $\xi_{\text{CMB}} \equiv \xi(N_{\text{CMB}})$ and for two choices of the parameter $\zeta_c$, which is a model-dependent PBH formation threshold (see, e.g., [46]). For a calculation of the $\zeta$-PDF entering Eq. (41) we used $\chi^2$-model.
The PBH mass value, as a function of $N$, in our model is given by the formula

\[ M_{BH} = \frac{f_h M_{eq} k_{eq}^2}{a_e^2} \frac{e^{2N}}{H(N)^2}, \]  

(44)

where $H(N)$ is the Hubble constant during inflation at the epoch determined by the value of $N$, $a_e$ is the scale factor at the end of inflation, $M_{eq}$ and $k_{eq}$ are horizon mass and wave number corresponding to the moment of matter-radiation equality. The result of the calculation using Eq. (44) is shown in Fig. 4 together with the result of the calculation using the more simple formula suggested in \[20\] (namely, $M_{BH} = 10 e^{2N} g$). It is seen that the curves start at almost the same value at $N = 0$. The difference at larger $N$ is due to the fact that Eq. (44) takes into account the dependence of $H$ on $N$.

V. RESULTS AND DISCUSSION

The main results of the paper are shown in Figs. 2 and 3. Fig. 2 illustrates the fact that due to tachyonic instability of gauge field, an amplitude of the curvature power spectrum is very large (up to $10^{-3}$) at small scales, $k \sim (10^{15} - 10^{20})\text{Mpc}^{-1}$, for a broad range of $\xi_{CMB}$ values. Fig. 3 shows the PBH mass spectra for definite values of the parameter $\xi_{CMB}$. On the vertical axis of Fig. 3 the combination $M_i^{-1/2} \rho_i^{-1} M_{BH}^{5/2} n_{BH}(M_{BH})$ is shown; just this combination is approximately equal to $\beta_{PBH}$, as it follows from Eq. (40). We compare these spectra with PBH data \[47, 48\], in which we consider only data for $M_{BH} > 10^9 \text{g}$, as most reliable ones. For such a comparison we drew in Fig. 3 the zigzag line representing, schematically, the well known $\beta_{PBH}$-constraint summary curve (see Fig. 9 in Ref. \[48\]). If some of our curves crosses this zigzag line, the corresponding $\xi$-value is, according to our logic, forbidden. Finally we obtain the constraint on the value of $\xi_{CMB}$, for quadratic potential (14),

\[ \xi_{CMB} < 1.8. \]  

(45)

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This constraint can be compared with the corresponding result of the work\cite{20}, $\xi_{CMB} < 1.5$. In terms of $\alpha$ and $f$ constants, the limit \(\ref{eq:45}\) corresponds to $\alpha/f < 26M_P^{-1}$.

We performed similar analysis for the case of linear potential \(\ref{eq:13}\), and in this case the constraint on $\xi_{CMB}$ turns out to be more strong,

$$\xi_{CMB} < 1.5,$$

\(\ref{eq:46}\)

corresponding to $\alpha/f < 36M_P^{-1}$.

For a derivation of these results, we used the assumption that $\zeta$-field has a $\chi^2$-distribution. For a comparison, we also performed the same calculations for a simple $\chi^2$-model (with one degree of freedom) and obtained the following PBH limits on the parameters: for the quadratic potential $\xi_{CMB} < 1.75$, and for the linear potential $\xi_{CMB} < 1.65$. Luckily, the constraints weakly depend on a choice of PDF ($n=1$ or $n=2$).

One should note, in conclusion, that PBH constraints are stronger than those from CMB scales\cite{2} and forthcoming constraints from gravity wave experiments\cite{49}.

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Appendix A: Curvature power spectrum behind the Hubble horizon

It is well known that, in general, the curvature perturbation amplitude $\zeta$ doesn’t stay constant in time after its scale exits the horizon during inflation. It is so even in the standard single-field inflation model if, in particular, slow-roll is temporarily violated in a process of the inflationary expansion\cite{50,52}. It had been shown in\cite{51,52} (see also\cite{53}) that in such models the modes can have a very complicate evolution and can be strongly amplified on super-horizon scales. As a result of such amplification, in particular, the perturbation amplitudes at horizon re-entry can differ rather strongly from amplitudes at a time of the exit.

In this Appendix we derive the curvature perturbation power spectrum following, closely, the work\cite{10}. Two main differences are: \(i)\) authors of\cite{10} assume that $\alpha$ is very large ($\sim 10^2$ or larger), and \(ii)\) they considered a case of the cosine potential [given by Eq. (10)]. In contrast with this, we considered the case when $f/\alpha \ll M_P$, $\alpha \sim 1$ and our potentials are nonperiodic. We show in this Appendix that, nevertheless, the resulting spectrum formula in our case is just the same as in\cite{10} if we limit ourselves to a consideration of the small scales, exiting the horizon at final stages of inflationary expansion. Just these scales are of interest for us because we study the PBH production processes.

The Eq. (8) which takes into account the back-reaction effects can be simplified using the slow roll approximation in the background equation (4). We assume that the slow roll regime is supported, mainly, by the dissipation into gauge field modes, i.e.,

$$3H\dot{\phi} \ll V',$$

\(\text{A1}\)

$$V' \cong \frac{\alpha}{f} \langle E \cdot B \rangle.$$

\(\text{A2}\)
The inequality (A1) holds if \( f/\alpha \) is small compared with \( M_P \). Using the definition of \( \xi \) [Eq. (3)] and the approximate relation \( 3H^2 M_P^2 \approx V \) one can rewrite (A1) in a form

\[
2\xi \cdot \frac{f}{\alpha} \cdot \frac{V}{V'} \ll 1. \quad (A3)
\]

For the quadratic potential, \( V = \frac{1}{2} m^2 \varphi^2 \), one obtains from (A3):

\[
\xi \cdot \frac{f}{\alpha} \varphi \ll 1, \quad (A4)
\]

and, for the linear potential, \( V = \mu^3 \varphi \),

\[
2\xi \cdot \frac{f}{\alpha} \varphi \ll 1. \quad (A5)
\]

We are interested in the final stage of inflation when small scales exit the horizon \( (N \sim 10) \). During this stage \( \xi \sim 5 \) (see Fig. 1) and \( \varphi \sim M_P \) [20, 27]. Substituting in Eqs. (A4) and (A5) our limiting values of \( f/\alpha \) (see Sec. V) one can see that inequalities (A4) and (A5) really hold.

To obtain the approximate equation (A2) one must show that the term \( \ddot{\varphi} \) in Eq. (4) is small in comparison with \( V' \). The proof of it is easily performed in complete analogy with the proof of (A1). Now, using (A2) and the relation following from Eq. (9),

\[
\beta = 1 - \frac{\pi}{\alpha} \frac{V'}{3H^2 f^2}, \quad (A6)
\]

one can rewrite Eq. (8) in the form (changing the time variable on \( \tau \equiv -1/(aH) \) and going over in a \( k \)-space) [10]

\[
\delta \varphi''(\vec{k}) - \frac{2}{\tau} \left( 1 + \frac{\pi \alpha V'}{2fH^2} \right) \delta \varphi'(\vec{k}) + \left( k^2 + \frac{V''}{H^2 \tau^2} \right) \delta \varphi(\vec{k}) = -\frac{\alpha}{f} a^2 \mathcal{J}(\tau, \vec{k}), \quad (A7)
\]

\[
\mathcal{J}(\tau, \vec{k}) = \int \frac{d^3 x}{(2\pi)^3/2} e^{-i k x} \left[ \vec{E} \cdot \vec{B} - (\vec{E} \cdot \vec{B}) \right]. \quad (A8)
\]

We can treat \( V'/H^2 \) and \( V''/H^2 \) as adiabatically evolving parameters, as well as \( H \) and \( \xi \) (e.g., for the quadratic potential, \( V'/H^2 \sim (V'/V) M_P^2 \sim M_P^2/\varphi, V''/H^2 \sim M_P^2/\varphi^2 \)), because \( \Delta \varphi \ll \varphi \) over \( \Delta t \sim H^{-1} \) [20, 27]. Due to this, we neglect their time dependence during the essential part of the inflationary evolution of each mode. In this case the homogenous equation (A7) (i.e., the equation (A7) with \( \mathcal{J}(\tau, \vec{k}) = 0 \)) can be written in a form

\[
\tau^2 \delta \varphi'' + b\tau \delta \varphi' + (c\tau^2 + d) \delta \varphi = 0, \quad (A9)
\]

\[
b = -\frac{\pi \alpha V'}{f H^2} - 2, \quad c = k^2, \quad d = \frac{V''}{H^2}. \quad (A10)
\]

The solution of this equation is expressed through the cylindrical functions (see, e.g., [54]):

\[
\delta \varphi = \tau^{\frac{1-b}{2}} Z_\nu(k\tau), \quad \nu = \frac{1}{2}\sqrt{(1-b)^2 - 4d}, \quad (A11)
\]

\[
Z_\nu(k\tau) = e^{i \nu \tau} I_\nu(k\tau) + e^{-i \nu \tau} K_\nu(k\tau), \quad (A12)
\]

\[
I_\nu(z) = \frac{1}{2} \left[ \frac{e^{iz}}{z^\nu} + \frac{e^{-iz}}{z^{-\nu}} \right], \quad (A13)
\]

\[
K_\nu(z) = \frac{1}{2} \left[ \frac{e^{iz}}{z^\nu} - \frac{e^{-iz}}{z^{-\nu}} \right]. \quad (A14)
\]
\[ Z_\nu(k\tau) = C_1 J_\nu(k\tau) + C_2 N_\nu(k\tau), \quad (A12) \]
\[ N_\nu(k\tau) = \frac{J_\nu(k\tau) \cos(\pi \nu) - J_{-\nu}(k\tau)}{\sin(\pi \nu)}. \quad (A13) \]

One can check using estimates given above that \(|b| \gg 1, \ d \ll |b|\), so
\[ \nu \approx \frac{1}{2}(1-b) \sqrt{1 - \frac{4d}{b^2}} \approx \frac{1-b}{2} - \frac{d}{1-b}. \quad (A14) \]

We are interested in the power spectrum at \(k \ll aH\), i.e., at \(k|\tau| \ll 1\), so, one can use the approximation
\[ J_\nu(x) \approx \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu + 1)}. \quad (A15) \]

The solution of the full equation \((A7)\) is obtained by the variation of constants method (or, that is technically the same, by the method of Green functions) and is given by the integration over the source function \(J(\tau, \vec{k})\). Using the approximation \((A15)\) one obtains, finally,
\[ \delta \varphi \sim -\frac{\alpha}{f} \int_{-\infty}^{\tau} d\tau' \tau' \left\{ \left(\frac{\tau}{\tau'}\right)^{\nu + \frac{1}{2} - \frac{b}{2}} - \left(\frac{\tau}{\tau'}\right)^{-\nu + \frac{1}{2} - \frac{b}{2}} \right\} a^2(\tau') J(\tau', \vec{k}). \quad (A16) \]

Since \(|\tau| < |\tau'|\) one can neglect the first term in figure brackets, because \(\nu + \frac{1}{2} - \frac{b}{2} \approx 1-b \gg 1, \ -\nu + \frac{1}{2} - \frac{b}{2} \approx \frac{d}{1-b} \ll 1\). It leads, with using \((A14)\), to
\[ \delta \varphi \sim \frac{\alpha}{f} \int_{-\infty}^{\tau} d\tau' \left(\frac{\tau}{\tau'}\right)^{\frac{d}{|b|}}, \quad (A17) \]
\[ \frac{d}{|b|} \approx \frac{V''(f)}{\pi \alpha V} \ll 1. \quad (A18) \]

Using this expression and the relation \(\zeta = H(\delta \varphi/\dot{\varphi})\), a formula for the curvature perturbation power spectrum is obtained straightforwardly \([10]\), with the result:
\[ P_\zeta \approx \frac{10^{-2}}{\xi^2} \left(\frac{\xi k}{aH}\right)^{2d/|b|}, \quad k \ll aH. \quad (A19) \]

We see from this formula that the power spectrum at super-horizon scales has no amplification, on the contrary, it decreases with a time when the scale moves away from the horizon. Due to a small value of \(d/|b|\) the time dependence is rather mild. Further, we see from Eq. \((A19)\) that in a limit of small \(d/|b|\) which corresponds to a limit of the large back-reaction, the curvature spectrum is almost scale invariant in a region of small scales, in accordance with the results shown in Fig. 2. We come to a conclusion that our estimates of the spectrum amplitude based on the approximate solution of Eq. \((8)\) \([given by Eq. \(19)\]) are reliable.

**Appendix B: The shape of the \(\zeta\)-bispectrum**

The bispectrum of the non-Gaussian \(\zeta\)-field is defined by the expression
\[ \langle \zeta(k_1) \zeta(k_2) \zeta(k_3) \rangle = (2\pi)^3 \delta(k_1 + k_2 + k_3) B(k_1, k_2, k_3). \quad (B1) \]
FIG. 5: Shape functions $S(k_1, k_2, k_3)$ (arbitrarily normalized) for $\chi^2$-model (upper panel) and axion inflation model (lower panel).

If $\zeta = \chi^2 - \langle \chi^2 \rangle$, the formula for $B$ is

$$B(k_1, k_2, k_3) = \frac{8}{3} \left[ \int \frac{d^3 k'}{(2\pi)^3} P_G(|k_1 - k'|) P_G(|k_2 + k'|) P_G(k') + 2 \text{ perm.} \right], \quad (B2)$$

where $P_G(k)$ is the curvature power spectrum of the Gaussian $\chi$-field, $P_G(k) \sim k^n$. The shape $S$ of the bispectrum, which is defined by the formula

$$S(k_1, k_2, k_3) = (k_1 k_2 k_3)^2 B(k_1, k_2, k_3) \quad (B3)$$

has a characteristic “squeezed” form, shown in Fig. 5 (upper panel; $n = -2.9$).

The bispectrum in our axion inflation model is calculated using the formula

$$B(k_1, k_2, k_3) = \frac{3}{10} P_{\zeta, \text{sr}} e^{6\pi} \frac{k_1^3 + k_2^3 + k_3^3}{k_1^3 k_2^3 k_3^3} f_3 \left( \xi, \frac{k_2}{k_1}, \frac{k_3}{k_1} \right). \quad (B4)$$

Here, the function $f_3$ is defined in \[\text{[37, 56]}\]. The example of the calculation of the corresponding shape function (for $\xi = 6$) is shown in Fig. 5 (lower panel).

Comparing two shape functions, one can see that the shape function of our model differs rather strongly from the typical equilateral shape function (see, e.g., \[\text{[57]}\] for examples of
equilateral shapes). At the same time, there is some similarity with the \( \chi^2 \)-model prediction (on both figures there is some concentration of points along the diagonal line).

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