The role of writing justification in mathematics concept: the case of trigonometry

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Abstract. This study aims to analyze students' justification in mathematical concepts in the topic of trigonometry. This research was an explorative research conducted on 30 high school students in Bandung, Indonesia. The results showed that 7% of students were able to solve the problem correctly with correct justification, 66% of students solve the problem correctly with wrong justification and 27% of students answered wrong with wrong justification. Further analysis found that students who answered correctly with correct justification are at level 4 where the students provide clear, complete justification and with the right concept. Students who answered correctly with the wrong justification generally are at level 2 and level 3 where the justification given is very short and part of the justification contains a misconception of mathematics so that it can’t be determined whether the student really understand or not. Students who answered incorrectly with wrong justification generally are at level 2 where their justification still contains a misconception of mathematics because of lack of understanding of the concept of trigonometry. On the other hand, the student understands that he must associate the sine concept with the angle of a triangle but he does not know how to do it.

1. Introduction
Mathematics is not just about identifying the truth, but proving that truth [1]. Learning to argue about mathematical ideas is the most basic thing in understanding mathematics and learning mathematical thinking. It is almost undeniable that justification and reasoning about solutions are important goals for students who solve mathematical problems [2]. The ability to justify each step, for example, evidence can be considered a skill that needs to be mastered, at least to some extent, before evidence is introduced. In a broader sense, evidence can even be regarded as justification [3]. In recent years, research has shown that there is a need for attention to the thinking processes of students where there is a rich and creative difference in student approaches to problem solving and how justification supports their proposed solutions [2].

To equip students with the thinking skills described above, today's math learning should be focused on efforts to train students to use their thinking potential. By familiarizing students with answers to the question "Why is that true?" or "Why is it possible to solve the problem?" Can give students insight that statement or activity in problem solving needs to be given arguments or logical reasons [4]. Thus students learn how to justify their results, explain why they think so and to convince teachers and fellow students. The study by [5] also indicates that high school students can develop their justification skills in solving mathematical problems.

But the facts show that mathematics learning tends to emphasize the outcome. Although student work or solutions are sometimes discussed with other students, it is limited to what strategies are used
and the steps taken in solving the problem. If the teacher asks the students to present their answers up front and ask them to explain, then what the students do is to retrace what they have written up front. Teachers have not asked why students are using the strategy and how students believe the answer is true [6]. This agrees with [2] who says that in general, teachers will ask students to explain their reasons if they make a mistake, but ask students to justify issues they solve correctly are usually overlooked. On the other hand, students are not accustomed to providing logical reasons for the problems they solve [7]. Just saying that "My teacher says so" or "I can feel that right" is not enough to show students' reasoning. Likewise, the numbers or numbers that appear in student answers do not indicate student understanding [6]. As a result, we don’t know what the students think, are the correct answers resulted from the right thought process or not. Often, the reasons for pushing forward solutions remain implied [7,8].

In this study, the problems students solved were related to trigonometric topics. Trigonometry is one of the mathematics that students must understand to develop their mathematical understanding [9]. In line with [10] opinion that for most students in higher education, trigonometry is an important piece of analysis in reasoning. The same thing is said by [11] that if students engage extensively in the manipulation of mathematical symbols before they develop the correct conceptual basis then they will not be able to do more manipulation [11]. Thus, students' understanding in trigonometric material is so important that students can solve mathematical problems.

Based on the above description, then in this paper discussed in more detail about the ability of students' justification in understanding the mathematical concepts on the topic trigonometry. Data analysis will be associated with the level of student justification by Jo, Grant and Flower. The results of this study may be used either for a more in-depth explorative study or a study of the development of future trigonometric problems.

2. Theoretical framework

Justification is the process of validating a statement by giving reasons based on a previously proven definition, theorem, or lemma [12]. According to [13] justification is the act of providing a foundation, proof or argument to convince another person (or ask yourself) that a claim or justification is true. Similarly, according to [6] justification is defined as an argument that demonstrates the truth of a claim using previously accepted statements and the mathematical form of reasoning. By analyzing students' justification skills, it allows teachers to study the development of mathematical understanding and create a learning design that helps students how to justify their answers.

The ability of a person to justify is closely related to his reasoning ability because justification means giving reasonably clear reasoning [14]. As one teacher wrote in a student's journal: "Unless you can explain it to me, you do not really understand" [15]. In addition, the justification process can also be used to track student misunderstandings. Students are also expected not only to solve trigonometric problems without knowing why the strategy is used, but also to hone their reasoning skills in enhancing their conceptual understanding and to provide logical reasoning behind the thinking process.

Several studies have attempted to categorize the characteristics of justification and rank them. In this research, the level of justification according to [16] is divided into five levels based on the results of students' answers in writing (Table 1).

| Levels | Criteria for justification |
|--------|---------------------------|
| Level 0 | Written work is missing or does not contain a valid reasoning strategy |
| Level 1 | Justifications are mainly descriptive or illustrative of the steps |
| Level 2 | Parts of the justification are mathematically incorrect or contain insufficient details. |
| Level 3 | Justification is mostly clear and mathematically correct. Students may have |

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3. Research method
In order to analyze students' justification ability on trigonometric topics, the research approach used is explorative approach through written test and field notes [17]. Subjects in this study were high school student X class semester 2 in Bandung, Indonesia. The subjects in question are students who have studied the material about trigonometry on the grounds that at that level students are able to provide justification well.

The instrument used is a description of trigonometric topics. Furthermore, the instrument is theoretically validated by 3 experts namely mathematics lecturers who have a mathematics education background. Thus will be obtained written data that is descriptive about the ability of student justification. Data analysis is done through two stages. The first stage identifies and compiles the strategies students use in solving trigonometric problems. The second stage, further interpreting the factors affecting the students' justification in solving trigonometric problems, is examined from the level of justification of [16]. This grouping is based on a student's approach in explaining the reasons for the justification they make, regardless of the exactness of the reasons. The problem that given in this research shown as follows,

Albert and Beni calculate the sine value of a triangle. From the calculation, Albert got the answer ½ and Beni got the answer -½. Check which answer is correct? Explain your answer.

4. Result and discussion
A total of 30 students participated in solving problems that focused on trigonometric material. Table 2 summarizes the results of this study for case 1 that the students did. Columns 1-3 are then present: examples of student solutions to problem solving, justification reasons made by students, number of students completing tasks based on their type of justification (#C). Corresponding percentages, relative to the total number of participating students.

Judging from how students justify their answers in completing mathematical problems, we identified at least three possible student responses: students who answered correctly with correct justification (7%), students who answered incorrectly with wrong justification (27%) and students who answered true with wrong justification (66%). While based on the tendency of student answers, we identified at least five different answers that students used. The first and third ways produce the correct answer but the justification used is not correct. The second way leads to incorrect answers and justification, and the fourth and fifth ways produce the right answer. The explanation of the type of student justification found in this study and its relation to the theory of student justification level is described in Table 2.

| Examples of student answers | Reasons | #C (%) |
|-----------------------------|---------|--------|
| Albert's answer is correct  | No sine value is negative, all positive. | 11(37) |
| There is no right answer    | There are sine values that are positive and some are negative. | 8(27) |
| Albert's answer is correct  | The value of sin 30° and sin 150° is ½ | 5(17) |
| Albert's answer is correct  | The value of a triangular sine that meets is (½) instead of (-½) | 4(12) |
| Albert's answer is correct  | The number of angles of a triangle is 180° then the angles are < 180° so that the sine value will always be positive (in quadrants I and II) | 2(7) |
The first way was done by 11 students (Figure 1). The student determines that Albert's correct answer was $\frac{1}{2}$ but the student still gives the wrong justification. Students give a short reason that sine value is always marked positive when there is also a negative sign of sine value. The student gives a reason without confirming whether the statement is true. This is due to a lack of understanding of the concept of trigonometric rules that students have. Based on the level of justification of [16] this type of student justification is at level 0 where students provide justification but do not contain valid reasoning strategies.

**Figure 1.** Representative examples of student written work using method 1. Translated version of the reason: Albert's answer is correct because no sine value is negative, all positive.

The second way is done by 8 students (Figure 2) which results in the wrong answer with wrong justification. Students determine that there is a positive signified sine value (located in quadrants I and II) and those of negative-pointed sine (in quadrants III and IV). But the students do not relate it to the number of angles on a triangle that is $180^\circ$. Thus the students make the wrong conclusion that there is no correct answer between Albert and Beni. On the other hand, the likelihood that the student understands that he should associate it with an angle on a triangle but he does not know how to do it. Regardless of the mistake of student answers resulting in inappropriate justification, when viewed from the level of justification [16], this student belonging to level 2 where some of the students' justification contains a false concept and has not contained a clear explanation.

**Figure 2.** Representative examples of student written work using method 2. Translated version of the reason: There is no right answer because there are sine values that are positive and some are negative.

The third way is done by 5 students which produces the correct answer with an untrue justification. The student determines that the value of the sine that meets is $\sin 30^\circ$ and $\sin 150^\circ$ is $\frac{1}{2}$, so the correct answer is Albert's answer (Figure 3). However, we do not know clearly based on this brief statement whether students really understand that the possible angle is only $30^\circ$ and $150^\circ$ (because the angle of a triangle does not exceed $180^\circ$)? The statement if read by a layman then allows them not to understand the reason why the selected corner is only $30^\circ$ and $150^\circ$ although the answer is correct. This is because the students' justification does not begin with an explanation of the concept of the number of angles in the triangle. Judging from the level of justification of [16] these students are classified as being between the level 2 and level 3 justification where the students give the reasons based on the rules with the examples shown are special but the reason given is very short and less informative so it can't be determined whether students really understand or not.
Figure 3. Representative examples of student written work using method 3. Translated version of the reason: Albert's answer is correct because the value of \( \sin 30^\circ \) and \( \sin 150^\circ \) is \( \frac{1}{2} \).

The fourth way is done by 4 students which results in correct answers with less obvious justification. The student determines that the correct answer is Albert's answer because there is a sine value on a triangle result \( \frac{1}{2} \) (Figure 4). These answers seem at first glance to be true and reasonable but if read by a layperson then lets them not understand the reason why. Why there is a possibility that the sine value of an angle on the triangle is \( \frac{1}{2} \) and not \( -\frac{1}{2} \). The student did not give further explanation as to why he responded so. Judging from the level of justification of [16], these students are classified on level 3 justification where the justification given by the students is largely clear and conceptually correct but slightly omits some important aspects.

Figure 4. Representative examples of student written work using method 4. Translated version of the reason: Albert's answer is correct because the value of a triangular sine that meets is \( (\frac{1}{2}) \) instead of \( (-\frac{1}{2}) \).

The fifth way is done by 2 students. Figure 5 shows that students justify by relating everything to the given problem. This can be seen from a more complete and comprehensive justification by utilizing known information in the form of a sine value in all four quadrants and how students relate it to the angles in a triangle. Student's justification begins by explaining that the sine value is marked positive on both I and II consciousness and is negative in quadrants III and IV. Then the students explain the number of angles on a triangle is \( 180^\circ \) so that the corners that meet are the angles in quadrants I and II. Thus the possible sine value obtained in a triangle is always positive. So it is concluded that Albert's correct answer is \( \frac{1}{2} \). Judging from the level of [16] justification, these students are classified on level 4 justification where students clearly justify and clarify concepts they understand by means of their own appropriate language communication.

Figure 5. Representative examples of student written work using method 5. Translated version of the reason: Albert's answer is correct because the number of angles of a triangle is \( 180^\circ \) then the angles are \( < 180^\circ \) so that the sine value will always be positive (in quadrants I and II).
5. Conclusion

Based on the above description, it can be concluded that the students' justification ability when viewed from the way they justify the answers given in solving trigonometric problems can be grouped into three is students who answered correctly with justification correct (7%), students who answered correctly with wrong justification (66%) and students who answered incorrectly with incorrect justification (27%). Students who answer correctly with correct justification justify by relating all information to the given problem. If viewed from the level of justification [16], students are classified level 4 where students explain the overall clear, complete and with the right concept. Students who answer correctly with the wrong justification tend to provide excuses without confirming whether the statement is true, giving a short reason so that it can’t be determined whether the student really understands and gives short answer that is not necessarily understood by the reader in general. If viewed from the level of justification [16], the students are classified level 2 and level 3 where some of the justification provided contains a misleading mathematical concept or has not contain enough explanation and slightly eliminate some important aspects. The main factors that because it is the inability of students to understand the concept of sine and the concept of angles in the triangle and linking the use of both concepts in solving the problems given. Many students are fooled or stalled because of confusion in preparing justification. The wrong student with wrong justification tends to understand that he or she has to tie it to the corner of a triangle but he does not know how to do it. When viewed from the level of justification [16], students are classified level 2 where some of the justification contains a math concept that is wrong or has not contain enough explanation. The lack of this understanding also resulted in students tend to only use certain rules, without understanding why the rules are used. The results of this study can be used either for a more in-depth explorative study or a study of the development of future trigonometric problems.

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7. References

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