Interpolation of analytic functions of moderate growth in the unit disc and zeros of solutions of a linear differential equation

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January 7, 2014

Abstract

In 2002 A. Hartmann and X. Massaneda obtained necessary and sufficient conditions for interpolation sequences for classes of analytic functions in the unit disc such that \( \log M(r, f) = O((1 - r)^{-\rho}), 0 < r < 1, \rho \in (0, +\infty) \), where \( M(r, f) = \max\{|f(z)| : |z| = r\} \). Using another method, we give an explicit construction of an interpolating function in this result. As an application we describe minimal growth of the coefficient \( a \) such that the equation \( f'' + a(z)f = 0 \) possesses a solution with a prescribed sequence of zeros.

MathSubjClass 2010: 30C15, 30H05, 30H99, 30J99.

Keywords: analytic function, moderate growth, unit disc, interpolation, canonical product, growth

1 Introduction and results

1.1 Interpolation in the unit disc

Let \((z_n)\) be a sequence of different complex numbers in the unit disc \( D = \{z : |z| < 1\} \), and let \( \sigma(z, \zeta) = \frac{|z - \zeta|}{1 - \bar{z}\zeta} \) denote the pseudohyperbolic distance in \( D \).

Let \( U(z, t) = \{\zeta \in \mathbb{C} : |\zeta - z| < t\} \). In the sequel, the symbol \( C \) stands for positive constants which depend on the parameters indicated, not necessarily the same at each occurrence. We say that the sequence \((z_n)\) is uniformly discrete or separated, if \( \inf_{j \neq k} \sigma(z_k, z_j) > 0 \). L. Carleson (\[2], \[8\]) consider the problem of description of so-called universal interpolation sequences or interpolation sets for the class \( H^\infty \) of bounded analytic functions in \( D \), i.e. those sequences \((z_k)\) in \( D \) that \( \forall (b_k) \in l^\infty \) there exists \( f \in H^\infty \) with

\[
\begin{align*}
    f(z_k) = b_k.
\end{align*}
\]

He proved that \((z_k)\) is a universal interpolation sequence for \( H^\infty \) if and only if

\[
\exists \delta > 0 : \prod_{j \neq k} \sigma(z_j, z_k) \geq \delta, \quad k \in \mathbb{N}.
\]

For the similar problems in \( H^p \) see \[8\] Chap. 9].
For the Banach space $A^{-n}$, $n > 0$, of analytic functions such that $\|f\|_{\infty} = \sup_{z \in \mathbb{D}} (1 - |z|)^n |f(z)| < \infty$, an interpolation set is defined by the condition that for every sequence $(b_k)$ with $(b_k(1 - |z_k|)^n) \in l^\infty$ there is a function $f \in A^{-n}$ satisfying (1). These sets were described by K. Seip in [21]. Namely, necessary and sufficient that $(z_k)$ be an interpolation set for $A^{-n}$ is that $(z_n)$ be separated and $D^+(Z) < n$ where

$$D^+(Z) = \limsup_{r \uparrow 1} \sup_{z \in \mathbb{D}} \frac{\sum_{\frac{1}{2} < \sigma(z_j) < r} \ln \frac{1}{\sigma(z_j)}}{\ln \frac{1}{1-r}}. \quad (3)$$

We note that the condition (2) implies boundedness of the numerator in (3).

For an analytic function $f$ in $D$ we denote $M(r,f) = \max_{|z|=r} |f(z)|$, $r \in (0,1)$. Let $n_{\zeta}(t) = \sum_{|z_k - \zeta| \leq t} 1$ be the number of the members of the sequence $(z_k)$ satisfying $|z_k - \zeta| \leq t$. We write

$$N_{\zeta}(r) = \int_0^r \frac{(n_{\zeta}(t) - 1)^+}{t} dt.$$

The results mentioned above cannot be applied to analytic functions $f$ such that $\ln \frac{1}{1-r} = o(\ln M(r,f))$ ($r \uparrow 1$). In 1956 A. G. Naftalevich [18] described interpolation sequences for the Nevanlinna class. On the other hand, a description of interpolation sets in the class of analytic functions in the unit disc and of infinite order of the growth satisfying

$$\exists C > 0 \forall r \in (0; 1) : \ln \ln M(r,f) \leq C \ln \gamma(\frac{C}{1-r}),$$

where $\ln \gamma(t)$ is a convex function in $\ln t$ and $\ln t = o(\ln \gamma(t))$ ($t \to \infty$), was found by B. Vynnytskyi and I. Sheparovych in 2001 [28].

Consider the class of analytic functions such that

$$\exists C > 0 \forall r \in (0; 1) : \ln M(r,f) \leq C \eta(\frac{C}{1-r}), \quad (4)$$

where $\eta: [1, +\infty) \to (0, +\infty)$ is an increasing convex function in $\ln t$ such that $\ln t = o(\eta(t))$ ($t \to \infty$). In 2001 in the PhD thesis of the second author [23] Theorem 3.1 (see also [29]) it was proved that given a sequence $(z_n)$ in $\mathbb{D}$, in order that for every $(b_n)$ such that

$$\exists C > 0 \forall n \in \mathbb{N} : \log |b_n| \leq C \eta(\frac{C}{1-|z_n|})$$

there exist an analytic function from the class (4) satisfying (1), it is necessary that

$$\exists \delta \in (0, 1) \exists C > 0 \forall n \in \mathbb{N} : N_{z_n}(\delta(1-|z_n|)) \leq \eta(\frac{C}{1-|z_n|}) \quad (5)$$

In 2002 A. Hartmann and X. Massaneda [11] proved that condition (5) is actually necessary and sufficient for a class of growth functions $\eta$ containing all power functions. They also describe interpolation sequences in the unit ball in $C^n$ in the similar situation. Note that the proofs of necessity in both [29], [11]
used similar methods of functional analysis. On the other hand, the proof of sufficiency in [11] is based on $L^2$-estimate for the solution to a $\partial$-equation and is non-constructive.

In 2007 A. Borichev, R. Dhuez and K. Kellay [11] solved an interpolation problem in classes of functions of arbitrary growth in both the complex plane and the unit disc. Following [11] let $h: [0, 1) \to [0, +\infty)$ such that $h(0) = 0$, $h(r) \uparrow \infty$ ($r \uparrow 1$). Denote by $A_h$ and $A^p_h$, $p > 0$ the Banach spaces of analytic functions on $\mathbb{D}$ with the norms

$$
\|f\|_h = \sup_{z \in \mathbb{D}} |f(z)| e^{-h(z)} < +\infty, \quad \|f\|_{h,p} = \left( \int_{\mathbb{D}} |f(z)|^p e^{-ph(|z|)} dm_2(z) \right)^{1/p},
$$

respectively. We then suppose that $h \in C^3([0, 1])$, $\rho(r) := \left( \frac{e^{h(r)}}{(\log r)\beta} \right)^{-\frac{1}{p}} \searrow 0$, and $\rho'(r) \to 0$ as $r \uparrow 1$, for all $K > 0$: $\rho(r + x) \sim \rho(r)$ for $|x| \leq K\rho(r)$, $r \uparrow 1$ provided that $K\rho(r) < 1 - r$, and either $\rho(r)(1 - r)^{-c}$ increases for some finite $c$ or $\rho'(r) \ln \rho(r) \to 0$ as $r \uparrow 1$. Note that these assumptions imply $h(r)/\ln \frac{1}{r} \to +\infty$ ($r \uparrow 1$).

Given such an $h$ and a sequence $Z = (z_k)$ in $\mathbb{D}$ denote by

$$
\mathcal{D}^\perp_{h}(Z) = \limsup_{R \to \infty} \limsup_{|z| \to 1} \frac{\text{card}(Z \cap U(z, R\rho(z)))}{R^2}.
$$

**Theorem 1** (Theorem 2.3 [11]). A sequence $Z$ is an interpolating set for $A_h(\mathbb{D})$ if and only if $\mathcal{D}^\perp_{h}(Z) < \frac{1}{2}$ and

$$
\inf_{k \neq n} \frac{|z_k - z_n|}{\min\{\rho(|z_k|), \rho(|z_n|)\}} > 0.
$$

The similar description holds for interpolation sets for the classes $A^p_h(\mathbb{D})$, $p > 0$ (11).

We give an explicit construction of a function solving the interpolation problem in the class $\bigcup_{0 < \rho} A^\infty_h \cup A^p_h$ when $h$ grows not faster than $(1 - r)^{-\rho}$, $\rho > 0$. Note that the restrictions on $h$ posed in [11] Section 5, Def.2 do not allow growth smaller than that of a power function. In particular, it does not admit the choices $h(r) = (\log r)^{\alpha}$, $\alpha \geq 1$, $h(r) = \exp((\log r)^{\beta})$, $0 < \beta < 1$. On the other hand, Theorem 1 works in these cases.

In particular, we show (Theorem 4) that condition (5) is also sufficient in the case when $\eta$ is a power function.

### 1.2 Zeros of solutions of $f'' + a(z)f = 0$

One of possible applications of the mentioned results is description of zero sequences of solutions of the differential equation

$$
f'' + a(z)f = 0, \quad (6)
$$

where $a(z)$ is an analytic function in $\mathbb{D}$.

We deal with the following problem (cf. [12] Problem 2).

**Problem.** Let $(z_k)$ be a sequence of distinct points in $\mathbb{D}$ without limit points there. Find a function $a(z)$, analytic in $\mathbb{D}$ such that (5) possesses a solution having zeros precisely at the points $z_k$. Estimate the growth of the resulting function $a(z)$.
In the case when $a$ is entire such investigation was initiated by V. Šeda [20] (we address the reader to the paper [15] for further references). A survey of results devoted to the case when $A$ is analytic in the unit disc, and zeros of $f$ form a Blaschke sequence, i.e. $\sum_n (1 - |z|) < \infty$, is given in [13]. The case of an arbitrary domain is treated in [27].

In particular, in [19] it is shown that one can always find a solution to this problem. For an entire function $F(z) = \sum_{n=0}^{\infty} F_n z^n$ define the maximal term

$$\mu(r,F) = \max\{|F_n| r^n : n \geq 0\}.$$

Theorem 2 ([19]). Let $(z_k)$ be a Blaschke sequence of distinct points, $F$ be an entire function such that

$$\mu\left(\frac{1}{1-|z_k|}, F\right) \geq 1 |B'(z_k)|^2,$$

where $B$ is the Blaschke product constructed by $(z_k)$. Then there exists a function $a(z)$ analytic in $D$ such that the equation (6) possesses a solution $f$ whose zero sequence is $(z_k)$ and for some $c > 0$

$$M(r,f) \leq \exp\left(\frac{c}{1-r} \mu\left(\frac{2}{1-r}, F\right)\right), \quad r \in [0,1).$$

Unfortunately, it is hard to check (7) for a given sequence. On the other hand, additional restrictions on the sequence allow to obtain explicit growth estimates of the coefficient $a$ or a solution $f$. In particular, in [12] it is proved that if $(z_k)$ is a Blaschke sequence satisfying (2), then there exists an $a(z)$ analytic in $D$ such that (6) possesses a bounded analytic solution with zero sequence $(z_k)$.

Recently J. Gröhn and J. Heittokangas have also proved the following theorem.

Theorem 3 ([10, Theorem 6(a)]). Let $(z_k)$ be a non-zero sequence in $D$ such that

$$\inf_{k \in \mathbb{N}} (1 - |z_k|)^{-q} \prod_{j \neq k} \sigma(z_j, z_k) > 0,$$

for some $q > 0$, and $\sum_{k=1}^{\infty} (1 - |z|)^{\alpha} < \infty$ for some $\alpha \in (0,1]$. Then there exists a function $a \in A^{2(\alpha + q)}$ satisfying $\limsup |a(z)| (1 - |z|)^{2} > 1$ such that (6) possesses a solution whose zero sequence is $(z_k)$.

Our result (Theorem 3) complements those of Gröhn and Heittokangas in the case when a zero sequence need not to satisfy the Blaschke condition, but still has a finite exponent of convergence. We also show (Theorem 4) that our estimate of the growth of the coefficient $a$ is sharp in some sense.

1.3 Main results

Let $\psi : [1, +\infty) \to \mathbb{R}_+$ be a nondecreasing function. We define

$$\tilde{\psi}(x) = \int_1^x \psi(t) \frac{dt}{t}.$$

Let, in addition, $\psi$ have finite order in the sense of Pólya, i.e.

$$\psi(2x) = O(\psi(x)), \quad x \to +\infty. \quad (8)$$
Remark 1. Polya’s order $\rho^*[\psi]$ of $\psi$ (7) is characterized by the condition that for any $\rho > \rho^*[\psi]$, we have

$$\psi(Cx) \leq C^\rho \psi(x), \quad x, C \to \infty.$$  \hspace{1cm} (9)

Note that (9) implies

$$\tilde{\psi}(C) \leq C^\rho \tilde{\psi}(x) + \tilde{\psi}(C) \leq 2 C^\rho \tilde{\psi}(x), \quad x, C \to \infty.$$  \hspace{1cm} (10)

so $\rho^*[\tilde{\psi}] \leq \rho^*[\psi]$. Also, it is known that (9) implies that $\psi$ has finite order of growth, i.e.

$$\psi(x) = O(x^\rho), \quad x \to +\infty,$$

but not vice versa.

Remark 2. In the case $\psi(x) = \ln p x$, $p \geq 0$ we get $\tilde{\psi}(x) = \frac{1}{p+1} \ln p+1 x$, and in the case $\psi(x) = x^\rho$, $\rho > 0$ we have $\tilde{\psi}(x) = \frac{1}{\rho}(x^\rho - 1)$.

The following theorem gives sufficient conditions for interpolation sequences in classes of analytic functions of moderate growth in the unit disc.

**Theorem 4.** Let $(z_n)$ be a sequence of distinct complex numbers in $\mathbb{D}$. Assume that for some nondecreasing unbounded function $\psi : [1, +\infty) \to \mathbb{R}_+$ satisfying (8) we have

$$\exists C > 0 : \forall n \in \mathbb{N} \quad \mathbb{N}_u \left( \frac{1 - |z_n|}{2} \right) \leq C \psi \left( \frac{1}{1 - |z_n|} \right)$$ \hspace{1cm} (10)

Then for any sequence $(b_n)$ satisfying

$$\exists C > 0 : \ln |b_n| \leq C \tilde{\psi} \left( \frac{1}{1 - |z_n|} \right), \quad n \in \mathbb{N}$$

there exists an analytic function $f$ in $\mathbb{D}$ with the property (11) and

$$\exists C > 0 : \ln M(r, f) \leq C \tilde{\psi} \left( \frac{1}{1 - r} \right).$$ \hspace{1cm} (11)

In order to prove a criterion we define a class of ‘regularly’ growing functions. The class $\mathcal{R}$ consists of functions $\psi : [1, +\infty) \to \mathbb{R}_+$ which are nondecreasing, and such that $\psi(r) = O(\psi(r))$ as $r \to +\infty$. We note that the power function $x^\rho$, $\rho > 0$, belongs to $\mathcal{R}$. Also, given a positive nondecreasing function $\psi$, if for some $C > 1$ there exists $t_0$ such that $\psi(Ct) \geq 2\psi(t)$ for all $t > t_0$, then $\psi \in \mathcal{R}$ (see [24, p.50-51]).

Combining Theorem 4 with the aforementioned result from [29] we are able to prove the following criterion, which essentially coincides with a result from [11].

**Theorem 5.** Let $(z_n)$ be a sequence of distinct complex numbers in $\mathbb{D}$, and let $\psi \in \mathcal{R}$ satisfy (8). The following conditions are equivalent:

(i) $\forall (b_n)$ such that

$$\exists C > 0 : \ln |b_n| \leq C \psi \left( \frac{1}{1 - |z_n|} \right), \quad n \in \mathbb{N}$$

there exists an analytic function $f$ in $\mathbb{D}$ satisfying (11) and

$$\exists C > 0 : \ln M(r, f) \leq C \psi \left( \frac{1}{1 - r} \right).$$
(ii) condition (10) holds;

(iii) \[ \exists C > 0 \forall n \in \mathbb{N} \sum_{0 < |z_n - z_j| < \frac{1}{2}(1 - |z_n|)} \frac{\ln \frac{1}{\sigma(z_n, z_j)}}{\sigma(z_n, z_j)} \leq C \psi \left( \frac{1}{1 - |z_n|} \right). \]

Remark 3. As it was proved in [22], condition (10) is equivalent to that

\[ \exists \delta_1 \in (0, 1) \exists C \in (0, 1) \forall z \in D : \sum_{0 < |z_n - z_j| < \frac{1}{2}(1 - |z|)} \frac{\ln \frac{1}{\sigma(z_n, z_j)}}{\sigma(z_n, z_j)} \leq C \psi \left( \frac{1}{1 - |z|} \right). \]

Remark 4. Repeating the arguments from the proof of the equivalence (ii) \(\iff\) (iii) on p.12, one can prove that (10) holds if and only if

\[ \exists C > 0 \forall z \in D : \sum_{0 < |z_n - z_j| < \frac{1}{2}(1 - |z|)} \frac{\ln \frac{1}{\sigma(z_n, z_j)}}{\sigma(z_n, z_j)} \leq C \psi \left( \frac{1}{1 - |z|} \right). \]

Next results concern oscillation of solutions of equation (6).

Theorem 6. Let conditions of Theorem 4 be satisfied. Then there exists an analytic function \(a\) in \(D\) satisfying

\[ \exists C > 0 : \ln M(r, a) \leq C \psi \left( \frac{1}{1 - r} \right), \quad r \in (0, 1) \]

such that (6) possesses a solution \(f\) having zeros precisely at the points \(z_k\), \(k \in \mathbb{N}\).

Corollary 1. If for some \(\rho > 0\) a sequence \((z_k)\) satisfies the condition

\[ \exists C > 0 : N_k \left( \frac{1 - |z_k|}{2} \right) \leq C \left( \frac{1}{1 - |z_k|} \right)^\rho, \]

then there exists a function \(a\) analytic in \(D\) satisfying \(\ln M(r, a) = O((1-r)^{-\rho})\), 
\(r \in (0, 1)\) such that possesses a solution \(f\) having zeros precisely at the points \(z_k\), \(k \in \mathbb{N}\).

The following theorem is based on an example due to J. Gröhn and J. Heittokangas [10].

Theorem 7. For arbitrary \(\rho > 0\) there exists a sequence of distinct numbers \((z_n)\) in \(D\) with the following properties:

i) \(N_k \left( \frac{1 - |z_k|}{2} \right) \leq C \left( \frac{1}{1 - |z_k|} \right)^\rho, \quad k \in \mathbb{N}; \)

ii) \((z_k)\) cannot be the zero sequence of a solution of (6), where \(\ln M(r, a) = O((1-r)^{-\rho + \varepsilon})\) for any \(\varepsilon > 0\).
2 Preliminaries

2.1 Some auxiliary results

For \( s = [\rho] + 1 \), where \( \rho = \rho^*[\psi] \), we consider a canonical product of the form

\[
P(z) = P(z, Z, s) = \prod_{n=1}^{\infty} E(A_n(z), s),
\]

where \( E(w, 0) = 1 - w \), \( E(w, s) = (1 - w) \exp\{ w + w^2/2 + \cdots + w^s/s \} \), \( s \in \mathbb{N} \), and \( A_n(z) = 1 - |z_n|^2 \). This product is an analytic function in \( \mathbb{D} \) with the zero sequence \( Z = (z_n) \) provided \( \sum_{z_n \in Z} (1 - |z_n|)^{s+1} < \infty \).

The following two lemmas play a key role in the proofs of the theorems. The first one is a generalization of Linden’s lemma from [17], which gives an upper estimate of the canonical product via \( n_z \frac{1 - |z|^2}{2} \).

**Lemma 1.** Let a sequence \( Z = (z_n) \) in the unit disc be such that

\[
n_z \frac{1 - |z|^2}{2} \leq \psi \left( \frac{1}{1 - |z|^2} \right), \quad z \in \mathbb{D},
\]

where the function \( \psi \) satisfies (8). Then for \( s > \rho^*[\psi] \) the canonical product \( P(z) = P(z, Z, s) \) of the form (12) admits the estimate

\[
\log |P(z)| \leq 2^{s+2} \sum_{n=1}^{\infty} \big| A_n(z) \big|^{s+1} \leq C \tilde{\psi} \left( \frac{1}{1 - |z|^2} \right), \quad z \in \mathbb{D},
\]

for some constant \( C > 0 \).

**Remark 5.** Note that any canonical subproduct of \( P \) satisfies (13) with the same constant \( C \). In fact, let \( Z_1 \subset Z \), and \( P_1(z) = P(z, Z_1, s) \). Then

\[
\log |P_1(z)| \leq 2^{s+2} \sum_{z_n \in Z_1} \big| A_n(z) \big|^{s+1} \leq 2^{s+2} \sum_{n=1}^{\infty} \big| A_n(z) \big|^{s+1} \leq C \tilde{\psi} \left( \frac{1}{1 - |z|^2} \right).
\]

We write \( B_k(z) = \frac{P(z)}{E(A_k(z), s)} \).

**Lemma 2.** For an arbitrary \( \delta \in (0, 1) \), any sequence \( Z \) in \( \mathbb{D} \) satisfying \( \sum_{z_k \in Z} (1 - |z_k|)^{s+1} < \infty \), \( s \in \mathbb{Z}_+ \) there exists a positive constant \( C(\delta, s) \)

\[
| \ln |B_k(z_k)| + N_{z_k}(\delta(1 - |z_k|)) | \leq C(\delta, s) \sum_{n=1}^{\infty} |A_n(z_k)|^{s+1}, \quad k \to +\infty.
\]

The next proposition compares some conditions frequently used in interpolation problems.

**Proposition 1.** Given a function \( \psi \in \mathcal{R} \) for

\[
\exists C > 0 \forall z \in \mathbb{D} : N_z \left( \frac{1 - |z|^2}{2} \right) \leq C \psi \left( \frac{1}{1 - |z|^2} \right)
\]
it is necessary and sufficient that
\[ \exists C > 0 \forall z \in \mathbb{D} : n_z \left( \frac{1-|z|}{2} \right) \leq C \psi \left( \frac{1}{1-|z|} \right), \] (14)
and
\[ \forall n \in \mathbb{N} : |\ln((1-|z_n|)|P'(z_n)|)| \leq C \psi \left( \frac{1}{1-|z_n|} \right), \] (15)
where \( P \) is the canonical product defined by [12], \( s = \lfloor \rho \rfloor + 1 \), where \( \rho \) is Polya’s order of \( \psi \).

### 2.2 Proofs of the lemmas

**Proof of Lemma 1** The proof repeats, in general, the original Linden’s one (17), therefore we only sketch it, emphasizing distinctions. Without loss of generality we may assume that \( \frac{1}{2} \leq |z_n| < 1 \), \( n \in \mathbb{N} \). The first inequality (13) is the assertion of Tsuji’s theorem [25]. To prove the second one we denote
\[ \square (re^{i\varphi}) = \{ \rho e^{i\theta} : r \leq \frac{1+r}{2}, |\theta - \varphi| \leq \frac{1-r}{4} \}, \]
and \( \nu(re^{i\varphi}) \) being the number of members of the sequences \( Z = (z_n) \) in \( \square (re^{i\varphi}) \); \( S_{h,k}(\varphi) = \square \left( (1-2^{-k})e^{i(\varphi + 2\pi h(2k+1))} \right) \). Note that the conditions \( n_z \left( \frac{1-|z|}{2} \right) = O(\psi\left(\frac{1}{1-|z|}\right)) \) and \( \nu(z) = O(\psi\left(\frac{1}{|z|}\right)) \) are equivalent as \( |z| \uparrow 1 \).

Then, for \( z_n \in S_{h,k}(\varphi), z = re^{i\varphi} \) we have (see [17, p.24])
\[ \left( \frac{1-|z_n|^2}{|1-z\bar{z}|} \right)^{s+1} \leq \frac{1}{(1-r + r^{2k-1})^2 + \hbar^2 2^{2-2k} h^2}. \]

Thus, similar to that as one deduces formula (18) from [17], we obtain
\[ \sum_{\substack{2^{-k-1} < |z_n| \leq 2^{-k} \atop h=0}} \frac{2\psi(2^k)}{(1-r + r^{2k-1})^2 + \hbar^2 2^{2-2k} h^2} \leq \frac{2\psi(2^k)(8 + B(\frac{1}{2}, \frac{s-1}{2})))}{2^{k}s(1-r + r^{2k-1})^s}, \] (16)
where \( B(x, y) \) is the Beta-function.

Let \( r = |z| \in \left[ 1-2^{-\nu}, 1-2^{-2\nu-1} \right) \). It follows from (16) and (9) that
\[ \sum_{k=\nu+1}^{\infty} \sum_{|z_n| \leq 2^{-k}} \left( \frac{1-|z_n|^2}{|1-z\bar{z}|} \right)^{s+1} \leq C(s) \sum_{k=\nu+1}^{\infty} \frac{\psi(2^k)}{2^{k}s(1-r)^s} \leq \left( \frac{C(s)\psi(2^{\nu+1})}{(1-r)^s2^{\nu+1}s} \right) \sum_{k=\nu+1}^{\infty} 2^{(s-\rho)(k-\nu-1)} \leq C(s, \rho) \psi \left( \frac{1}{1-r} \right). \]

Further, (16) implies
\[ \sum_{k=1}^{\nu} \sum_{1-2^{-k} < |z_n| \leq 2^{-k}} \left( \frac{1-|z_n|^2}{|1-z\bar{z}|} \right)^{s+1} \leq C(s) \sum_{k=1}^{\nu} \psi(2^k) \leq C(s, \rho) \sum_{k=1}^{\nu} \int_{1-2^{-k}}^{1-2^{-k+1}} \psi \left( \frac{1-t}{1-t} \right) dt \leq C \psi(2^\nu) \leq C \psi \left( \frac{1}{1-r} \right). \]
It is well-known that $\psi(x) = O(\tilde{\psi}(2x))$ $(x \to +\infty)$, so the assertion of the lemma follows from the two latter estimates.

**Proof of Lemma** Without loss of generality, we assume that $Z$ is an infinite sequence, $|z_n| \geq \frac{1}{2}$, $n \in \mathbb{N}$, and $s \in \mathbb{N}$. We denote $\delta(1 - |z_k|) = \eta_k$, and note that

$$N_{z_k}(\eta_k) = \sum_{0 < |z_n - z_k| \leq \eta_k} \ln \frac{\eta_k}{|z_n - z_k|} = \int_0^{\eta_k} \frac{n_{z_k}(x) - 1}{x} \, dx.$$  

Then

$$\ln |B_k(z_k)| + N_{z_k}(\eta_k) =$$  

$$= \sum_{0 \neq k} \left( \ln \left| \frac{z_k - z_n}{1 - z_n z_k} \right| + \text{Re} \sum_{j=1}^{s} \frac{1}{j} (A_n(z_k))^j \right) + \sum_{0 < |z_n - z_k| \leq \eta_k} \ln \frac{\eta_k}{|z_n - z_k|} =$$  

$$= \sum_{0 < |z_n - z_k| \leq \eta_k} \left( \ln \frac{\eta_k |z_n|}{1 - z_n z_k} + \text{Re} \sum_{j=1}^{s} \frac{1}{j} (A_n(z_k))^j \right) +$$  

$$+ \sum_{|z_n - z_k| > \eta_k} \ln |E(A_n(z_k), s)|.$$  

(17)

It is easy to see (31 p. 528) that $|1 - \frac{z_k}{z_n}| \leq (2 + \delta)(1 - |z_k|)$ for $|z_n - z_k| \leq \eta_k$. Taking into account that $|A_n(z)| \leq 2$, $z \in \mathbb{D}$, we obtain for $|z_n - z_k| \leq \eta_k$

$$C'(\delta) \leq |A_n(z_k)| \leq C''(\delta), \quad \frac{\delta}{2(2 + \delta)} \leq \frac{\eta_k |z_n|}{1 - z_n z_k} \leq \delta.$$  

Therefore for the first sum in the right-hand side of (17) we get

$$\left| \sum_{0 < |z_n - z_k| \leq \eta_k} \left( \ln \frac{\eta_k |z_n|}{1 - z_n z_k} + \text{Re} \sum_{j=1}^{s} \frac{1}{j} (A_n(z_k))^j \right) \right| \leq$$  

$$\leq \sum_{0 < |z_n - z_k| \leq \eta_k} C(s, \delta) \leq C(s, \delta) \sum_{0 < |z_n - z_k| \leq \eta_k} |A_n(z_k)|^{s+1}.$$  

(18)

Tsuji (see 25 p.8) proved that for an appropriate branch of the logarithm

$$\sum_{|A_n(z)| < \frac{1}{2}} |\ln E(A_n(z), s)| \leq 2 \sum_{|A_n(z)| < \frac{1}{2}} |A_n(z)|^{s+1}.$$  

(19)

It can be checked that for the pseudohyperbolic disc $\mathcal{D}(z, s) = \left\{ \xi : \sigma(z, \xi) < s \right\}$ (29 1.1) the inclusion

$$\mathcal{D}\left( z, \frac{\delta}{2 + \delta} \right) \subset U(z, (1 - |z|)\delta)$$  

holds (see 3 p.529 for details). Thus, for $z_n \notin U(z_k, \eta_k)$ we have

$$0 \geq \ln \left| \frac{\bar{z}_n(z_n - z_k)}{1 - z_n z_k} \right| \geq \ln \frac{\delta}{2(2 + \delta)}.$$  

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where \(0 < \alpha \delta < 1\). Relations (17)–(19) and the last inequality give the assertion of the lemma. \(\square\)

3 Proofs of interpolation theorems and Proposition 1

Proof of Theorem 4. First of all, we note that the estimate

\[
\left( n z \left( \delta \left( 1 - |z| \right) \right) - 1 \right) \ln \alpha \leq \int_{\delta \left( 1 - |z| \right)}^{\delta \left( 1 - |z| \right)} \frac{(n z(x) - 1)^{+}}{x} \, dx \leq N z \left( \delta \left( 1 - |z| \right) \right),
\]

where \(0 < \alpha \delta < 1\), yields

\[
\max_{\theta} n_{r_{e}} s_{e} \left( \delta \left( 1 - r \right) \right) \leq C \left( \frac{1}{1 - r} \right). \tag{20}
\]

It follows from the estimate (20) and Lemma 1 that

\[
\sum_{n=1}^{\infty} |A_{n}(z)|^{s+1} \leq C(s) \tilde{\psi} \left( \frac{1}{1 - |z|} \right), \quad z \in \mathbb{D}.
\]

From this estimate and Lemma 2 we deduce

\[
|\ln |B_{n}(z_{k})|| \leq C(\delta, s) \tilde{\psi} \left( \frac{1}{1 - |z_{k}|} \right), \quad k \to +\infty. \tag{21}
\]

Consider the interpolation function

\[
f(z) = \sum_{n=1}^{\infty} \frac{b_{n}}{z - z_{n}} \frac{P(z)}{P'(z_{n})} \left( \frac{1 - |z_{n}|^{2}}{1 - z_{n}} \right)^{s_{n}-1},
\]

where an increasing sequence of natural numbers \((s_{n})\) will be specified below. It is not hard to check that (11) holds.

Moreover, taking into account Remark 5 and the inequality \(|A_{n}(z)| \leq 2\), we have the following estimates

\[
\left| \frac{1 - \bar{z}_{n} z}{z - z_{n}} \frac{P(z)}{z - z_{n}} \right| = \left| B_{n}(z) \exp \left\{ \sum_{j=1}^{s} \frac{1}{j} (A_{n}(z))^{j} \right\} \right| \leq \exp \left\{ C(s) \sum_{n=1}^{\infty} |A_{n}(z)|^{s+1} \right\} \leq \exp \left\{ C \tilde{\psi} \left( \frac{1}{1 - |z|} \right) \right\}, \tag{22}
\]

\[
\frac{P'(z_{n})(1 - |z_{n}|^{2})}{z_{n}} = - B_{n}(z_{n}) \exp \left\{ 1 + \frac{1}{2} + \cdots + \frac{1}{s} \right\}. \tag{23}
\]
Therefore, using our assumption on \((b_n), (22), \) and \(23\), we deduce
\[
|f(z)| = \sum_{n=1}^{\infty} b_n P(z)(1 - \bar{z}_n z) \frac{\bar{z}_n}{z_n (z - z_n)} \left( \frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right)^{s_n} \leq \\
\leq \sum_{n=1}^{\infty} \exp \left\{ C\psi \left( \frac{1}{1 - |z_n|} \right) \right\} \exp \left\{ C\bar{\psi} \left( \frac{1}{1 - |z|} \right) \frac{1}{|B_n(z_n)|} \left( \frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right)^{s_n} \right\} \leq \\
\leq \exp \left\{ C\tilde{\psi} \left( \frac{1}{1 - |z|} \right) \sum_{n=1}^{\infty} \exp \left\{ C''\tilde{\psi} \left( \frac{1}{1 - |z_n|} \right) \left( \frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right)^{s_n} \right\} \right\}.
\tag{24}
\]

If \(\psi\) is nondecreasing then \(\tilde{\psi}\) is convex with respect to the logarithm. The condition \(\psi(t) \to +\infty\) yields \(\ln x = o(\psi(x))\) \((x \to +\infty)\). By Clunie-Kővari’s theorem \([6]\), given a positive constant \(C_0\) there exists an entire function \(\Phi(z) = \sum_{n=0}^{\infty} \varphi_n z^n\) such that \(\ln M(t, \Phi) = (C_0 + o(1))\tilde{\psi}(C_0 t)\) \((t \to \infty)\). Then there is \(t_0 > 0\) such that the following estimates of the maximal term \(\mu(t, \Phi) = \max\{ |\varphi_n|^t : n \in \mathbb{Z}_+ \}\) are valid:
\[
\mu(t, \Phi) \leq \exp\left\{ 2\tilde{\psi}(C_0 t) \right\}, \quad \mu(t, \Phi) \geq \exp\left\{ \frac{1}{4} \tilde{\psi} \left( \frac{C_0 t}{2} \right) \right\} t \geq t_0.
\]

Let \(\tilde{\Phi}(z) = \sum_{n=0}^{\infty} \tilde{\varphi}_n z^n\) be Newton’s majorant for the function \(\Phi(z)\) (see \[26\], Chap. IX, §68). Then \(\mu(r, \Phi) = \mu(r, \tilde{\Phi}),\) and the sequence \(x_n = \varphi_{n-1}/\tilde{\varphi}_n,\) \(x_0 = 0\) is unbounded and increasing. We choose the sequence \((s_n)\) such that \(x_{s_n} \leq \frac{1}{1 - |z_n|} < x_{s_{n+1}}.\) Then
\[
\mu\left( \frac{1}{1 - |z_n|}, \Phi \right) = \varphi_{s_n} \left( \frac{1}{1 - |z_n|} \right)^{s_n}, \quad \mu(t, \Phi) \geq \tilde{\varphi}_{s_n} t^{s_n}.
\tag{25}
\]

Using the obtained inequalities and choosing \(C_0 > \max\{8C'', 2\},\) we deduce
\[
\sum_{n=1}^{\infty} \exp \left\{ C''\tilde{\psi} \left( \frac{1}{1 - |z_n|} \right) \left( \frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right)^{s_n} \right\} \leq \\
\leq \sum_{n=1}^{\infty} \exp \left\{ C''\tilde{\psi} \left( \frac{1}{1 - |z_n|} \right) \right\} \tilde{\varphi}_{s_n} \left( \frac{2}{1 - |z|} \right)^{s_n} \leq \\
\leq \sum_{n=1}^{\infty} \exp \left\{ C''\tilde{\psi} \left( \frac{1}{1 - |z_n|} \right) \right\} \frac{\mu\left( \frac{2}{1 - |z_n|}, \tilde{\Phi} \right)}{\mu\left( \frac{1}{1 - |z_n|}, \Phi \right)} \leq \\
\leq \sum_{n=1}^{\infty} \exp \left\{ C''\tilde{\psi} \left( \frac{1}{1 - |z_n|} \right) + 2C_0 \tilde{\psi} \left( \frac{2C_0}{1 - |z|} \right) - \frac{C_0}{2(1 - |z_n|)} \right\} \leq \\
\leq \exp \left\{ 2C_0 \tilde{\psi} \left( \frac{2C_0}{1 - |z|} \right) \right\} \sum_{n=1}^{\infty} \exp \left\{ - \frac{C_0}{8} \tilde{\psi} \left( \frac{C_0}{2(1 - |z_n|)} \right) \right\} \leq \\
\leq \exp \left\{ 2C_0 \tilde{\psi} \left( \frac{2C_0}{1 - |z|} \right) \right\} \sum_{n=1}^{\infty} \exp \left\{ -(s + 1) \ln \left( \frac{1}{1 - |z_n|} \right) \right\} \leq \\
\leq C(s) \exp \left\{ 2C_0 \tilde{\psi} \left( \frac{2C_0}{1 - |z|} \right) \right\}.
\tag{26}
\]
Substituting estimate (26) in (24), we get (11).

Proof of Theorem 5. The implications (i) \(\Rightarrow\) (ii) and (ii) \(\Rightarrow\) (i) follow from the theorem of B. V. Vynnytskyi, I. B. Sheparovych (29), and Theorem 4, respectively.

We then show that (ii) and (iii) are equivalent.

As it was proved, for \(|z - z_n| \leq \delta(1 - |z|)\) we have \(1 - |z| \leq |1 - \bar{z}_n z| \leq (2 + \delta)(1 - |z|)\), i.e. \(0 \leq \ln \frac{|1 - \bar{z}_n z|}{1 - |z|} \leq \ln(2 + \delta)\). Hence,

\[
0 \leq \sum_{0 < |z_k - z_n| \leq \delta(1 - |z_k|)} \left( \ln \frac{1}{\sigma(z_n, z_k)} - \ln \frac{1 - |z_k|}{|z_n - z_k|} \right) = \sum_{|z_k - z_n| \leq \delta(1 - |z_k|)} \ln \frac{1 - |z_k z_n|}{1 - |z_k|} \leq N_{z_k}(\alpha\delta(1 - |z_k|)) \ln(2 + \delta) \leq N_{z_k}(\alpha\delta(1 - |z_k|)) \ln(2 + \delta) \ln \alpha.
\]

Further,

\[
\sum_{0 < |z_k - z_n| \leq \delta(1 - |z_k|)} \ln \frac{1 - |z_k|}{|z_k - z_n|} = \int_{0}^{\delta(1 - |z_k|)} \ln \frac{1 - |z_k|}{\tau} d(n_{z_k}(\tau) - 1) = \ln \frac{1}{\delta} \cdot (n_{z_k}(\delta(1 - |z_k|)) - 1) + N_{z_k}(\delta(1 - |z_k|)).
\]

Therefore

\[
N_{z_k}(\delta(1 - |z_k|)) \leq \sum_{0 < |z_k - z_n| \leq \delta(1 - |z_k|)} \ln \frac{1 - |z_k|}{|z_k - z_n|} \leq N_{z_k}(\alpha\delta(1 - |z_k|))(1 + \ln \alpha \ln \frac{1}{\delta}).
\]

The latter inequality together with (27) proves the equivalence between (ii) and (iii).

Proof of Proposition 1. In fact, the necessity of (14) has been already established in the proof of Theorem 4. Necessity of (15) and sufficiency follow from Lemmas 1, 2, formula (23), and Remark 3.

4 Proofs of the oscillation theorems

Proof of Theorem 6. Let \(f(z) = P(z)e^{g(z)}\), be analytic in \(D\) where \(P\) is the canonical product defined by (12) with the zeros sequence \(Z = (z_k)\). We can rewrite (9) as

\[
P'' + 2P'g' + (g'^2 + g'' + a)P = 0,
\]

and, consequently

\[
g'(z_k) = -\frac{P''(z_k)}{2P'(z_k)} =: b_k, \quad k \in \mathbb{N}.
\]
Therefore, in order to find a solution of \((6)\) with the zero sequence \(Z\) we have to find an analytic function \(h = g'\) solving the interpolation problem \(h(z_k) = b_k, k \in \mathbb{N}\). Using Cauchy’s integral theorem and Lemma \(1\) we deduce

\[
|P''(z_k)| \leq \frac{8}{(1 - |z_k|)^2} \max_{|z| = \frac{1}{2} |z_k|} |P(z)| \leq \frac{8}{(1 - |z_k|)^2} e^{C \psi(\frac{1}{1 - r_{z_k}})}.
\]

On the other hand, \((23)\) and \((21)\) imply (cf. \((15)\)) that

\[
\frac{1}{|P''(z_k)|} \leq (1 - |z_k|) e^{C \psi(\frac{1}{1 - r_{z_k}})}.
\]

Hence

\[
|b_k| = \left| \frac{P''(z_k)}{2P'(z_k)} \right| \leq \frac{4}{1 - |z_k|} e^{C \psi(\frac{1}{1 - r_{z_k}})} = e^{C \psi(\frac{1}{1 - r_{z_k}}) + \ln (1 - |z_k|)} \leq e^{C \psi(\frac{1}{1 - r_{z_k}})},
\]

because \(\psi(t)/\ln t \to +\infty (t \to +\infty)\). Since the assumptions of Theorem \(4\) are satisfied there exists a function \(h\) analytic in \(D\) such that \(h(z_k) = b_k\) and \(\ln M(r, h) \leq C \tilde{\psi}(\frac{1}{1 - r}), r \uparrow 1\), i.e. \(\ln M(r, g') \leq C \tilde{\psi}(\frac{1}{1 - r}), r \uparrow 1\).

Then, applying Cauchy’s theorem once more, we get that

\[
M(r, g'') \leq \frac{2}{1 - r} M \left( \frac{1 + r}{2}, g' \right) \leq e^{C \tilde{\psi}(\frac{1}{1 - r})},
\]

\(r \uparrow 1\).

From \((28)\) we obtain

\[
|a(z)| \leq \left| \frac{P''(z)}{P(z)} \right| + 2 |g'(z)| \left| \frac{P'(z)}{P(z)} \right| + |g''(z)| + |g''(z)|.
\]

It follows from results of \([4]\) or \([5]\) that for any \(\delta > 0\) there exists a set \(E_\delta \subset [0, 1)\) such that

\[
\max \left\{ \left| \frac{P''(z)}{P(z)} \right|, \left| \frac{P'(z)}{P(z)} \right| \right\} \leq \frac{1}{(1 - |z|)^\eta}, \quad |z| \in [0, 1) \setminus E_\delta,
\]

where \(\eta \in (0, +\infty)\), and \(m_1(E_\delta \cap [r, 1)) \leq \delta (1 - r) as r \uparrow 1\). Thus,

\[
|a(z)| \leq e^{C \tilde{\psi}(\frac{1}{1 - r})}, \quad |z| \in [0, 1) \setminus E.
\]

Since \(M(r, a)\) increases, condition \((S)\) and Lemma 4.1 from \([5]\) imply that inequality \((30)\) holds for all \(z \in D\) for an appropriate choice of \(C\).

**Proof of Theorem \([7]\)** Let \(\rho > 0\) be given. Let \(\varepsilon_n = \frac{1}{2} e^{-2^{\rho^2}}, n \in \mathbb{N}\). Let \((z_n)\) be the sequence defined by

\[
z_{2n-1} = 1 - 2^{-n}, \quad z_{2n} = 1 - 2^{-n} + \varepsilon_n.
\]

Then for \(m \in \{2n-1, 2n\}\) we have as \(n \to \infty\)

\[
N_{z_m} \left( \frac{1 - |z_m|}{2} \right) = \sum_{0 < |z_k - z_m| \leq \frac{1}{4} (1 - |z_m|)} \ln \frac{1 - |z_k|}{2 |z_k - z_m|} = \ln \frac{2^{-n} + O(\varepsilon_n)}{2 \varepsilon_n} + O \left( \ln \frac{2^{-n} + O(\varepsilon_n)}{2^{-n} + O(\varepsilon_n)} \right) = 2^{n \rho} + O(n) \sim \left( \frac{1}{1 - |z_m|} \right)^\rho.
\]
Thus, assertion i) is proved.

To prove assertion ii) we assume on the contrary that there exists a solution $f = Be^\varphi$ of (6) having the zero sequence $(z_n)$, where $B$ is the Blaschke product, and such that

$$\ln M(r, a) \leq C(1-r)^{-\rho + \varepsilon_0}, \quad r \in [0, 1), \varepsilon_0 > 0.$$  \hfill (31)

Repeating the arguments from the proof of Theorem 5 [12] (the only difference is that we have smaller $\varepsilon_n$), one can show that

Therefore

$$|g'(z_{2n})| = \left| \frac{B''(z_{2n})}{2B'(z_{2n})} \right| \geq \frac{1}{6} e^{2n\rho} \geq C \exp \frac{1}{(1-|z_{2n}|)^\rho}, \quad n \to +\infty.$$  

Hence,

$$\ln M(|z_{2n}|, g') \geq (1-|z_{2n}|)^{-\rho}, \quad n \to +\infty.$$  \hfill (32)

But (31) implies (see e.g. [14]) that $\ln \ln M(r, f) \leq (1-r)^{-\rho + \varepsilon_0}$, $r \uparrow 1$. Repeating the arguments from the proof of Lemma 2 we get for $R_n = 1 - 3 \cdot 2^{-n-1}$ and $\delta = \frac{1}{4}$ that $(n \to +\infty)$

$$\text{Re} g(R_n e^{i\theta}) \leq \ln M(R_n, f) + |\ln |B(R_n e^{i\theta})|| \leq \ln M(R_n, f) + N_{R_n} e^{\varphi} \left( \frac{1 - R_n}{4} \right) + C \left( \frac{1}{4}, 1 \right) \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|1 - z_k R_n e^{i\theta}|} \leq \exp \left\{ \left( \frac{1}{1 - R_n} \right)^{\varphi - \varepsilon_0} \right\} + O \left( \frac{1 - R_n}{1 - R_n} \right)^{\max(\varphi, 1)} \leq \exp \left\{ \left( \frac{C}{1 - r} \right)^{\varphi - \varepsilon_0 / 2} \right\}. \hfill (33)$$

Since $\text{Re} g$ is harmonic, $B(r, \text{Re} g) = \max \{ \text{Re} g(re^{i\theta}) : \theta \in [0, 2\pi] \}$ is an increasing function. It follows from (33) and the relation $1 - R_n \precsim 1 - R_{n+1}$

$$B(r, \text{Re} g) \leq \exp \left\{ \left( \frac{C}{1 - r} \right)^{\varphi - \varepsilon_0 / 2} \right\}, \quad r \to \infty.$$  

The last estimate and Caratheodory’s inequality ([16, Chap.1, §6]) imply

$$\ln M(r, g) \leq C(1-r)^{-\rho + \varepsilon_0 / 2}, \quad r \uparrow 1.$$  

This contradicts to (32). The theorem is proved. \hfill \square

References

[1] Borichev A., Dhuez R., Kellay K. Sampling and interpolation in large Bergman and Fock space, J. Funct. Analysis 242 (2007), 563–606.

[2] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math. 80 (1958) 921–930.

[3] I. Chyzhykov, Argument of bounded analytic functions and Frostman’s type conditions, Ill. J. Math. 53 (2009), no.2, 515–531.

[4] I. Chyzhykov, G. Gundersen, J. Heittokangas, Linear differential equations and logarithmic derivative estimates, Proc. London Math. Soc. 86 (2003), No.3, 735–754.
[5] Chyzhykov I., J. Heittokangas, J. Rättyä, Sharp logarithmic derivative estimates with applications to ODE’s in the unit disc, J. Australian Math. Soc. 88 (2010), 145-167.

[6] J. Clunie, T. Kövari, On integral functions having prescribed asymptotic growth. II, Can. J. Math., 20 (1968), 7–20.

[7] D. Drasin, D. Shea, Pólya peaks and the oscillation of positive functions, Proc. Amer. Math. Soc. 34 (1972), 403–411.

[8] P. L. Duren, Theory of $H^p$ spaces, Academic press, NY and London, 1970, 258 pp.

[9] J. B. Garnett, Bounded analytic functions, Academic Press, NY and London, 1981, 467 pp.

[10] J. Gröhn, J. Heittokangas, New findings on Bank-Sauer approach in oscillatory theory, Constr. Approx. 35 (2012), 345–361.

[11] A. Hartmann, X. Massaneda, Interpolating sequences for holomorphic functions of restricted growth, Ill. J. Math. 46 (2002), no.3, 929–945.

[12] J. Heittokangas, Solutions of $f'' + A(z)f = 0$ in the unit disc having Blaschke sequence as zeros, Comp. Meth. Funct. Theory 5 (2005), no.1, 49–63.

[13] J. Heittokangas, A survey on Blaschke-oscillatory differential equations, with updates, in Blaschke products and their applications, Fields Institute Communications, Vol. 65, J.Mashreghi, E.Fricain (eds.), 2012, 43–98.

[14] J. Heittokangas, R. Korhonen and J. Rättyä, Fast growing solutions of linear differential equations in the unit disc, Result. Math. 49 (2006), 265–278.

[15] J. Heittokangas, I. Laine, Solutions of $f'' + A(z)f = 0$ with prescribed zeros, Acta. Math. Univ. Comenianae LXXIV (2005), no.2, 287–307.

[16] B. Ja. Levin Distribution of zeros of entire functions, revised edition, Transl. Math. Monographs, Volume 5, translated by R. P. Boas et al (Amer. Math. Soc., Providence, 1980).

[17] C. N. Linden, The representation of regular functions, J. London Math. Soc. 39 (1964), 19–30.

[18] A. G. Naftalevich, On interpolation by functions of bounded characteristic, Vilnius Valst. Univ. Moksly Darbai. Mat. Fiz. Chem. Moksly Ser.5 (1956), 5-27. (in Russian)

[19] O. V. Shavala, On the holomorphic solutions of the equation $f'' + a_0f = 0$, the zeros of which satisfy the Blaschke condition, Mat. Stud. 28 (2007), no. 2, 213–216. (in Ukrainian)

[20] V. Šeda, On some properties of solutions of the differential equation $y'' = Q(z)y$, where $Q(z) \neq 0$ is an entire function, Acta. Fac. Nat. Univ. Comenian math. 4 (1959), 223–253. (in Slovak)

[21] K. Seip, Beurling type density theorems in the unit disc, Invent. math. 113 (1993), 21–39.

[22] I. B. Sheparovych, On interpolation sequences of some classes of analytic functions, Visnyk Kharkiv National Univ. Ser. Math., Appl.Math.,Mech. Part 1, (2000), no.475, 204–207. (in Ukrainian)

[23] I. B. Sheparovych, Interpolation sequences of some classes of functions analytic in the disk, PhD thesis, Lviv, 2001. (in Ukrainian)

[24] A. A. Shkalikov, Zero distribution for pairs of holomorphic functions with applications to eigenvalue distribution, Trans. Amer. Math. Soc. 281 (1984), no.1, 49–63.
[25] M. Tsuji, Canonical product for a meromorphic function in a unit circle, J. Math. Soc. Japan 8 (1956), no.1, 7–21.

[26] G. Valiron, Fonctions analytiques, Presses Univ. de France, Paris, 1954.

[27] B. Vynnyts’kyi, O. Shavala, Remarks on Šeda theorem, Acta. Math. Univ. Comenianae, LXXXI (2012), no.1, 55–60.

[28] B. V. Vynnyts’kyi and I. B. Sheparovych, On the interpolation sequences of the some class of functions analytic in the unit disk, Ukr. Math. J. 53 (2001), no.7, 879–886.

[29] B. V. Vynnyts’kyi and I. B. Sheparovych, Interpolation sequences for the class of functions of finite $\eta$-type analytic in the unit disk, Ukr. Math. J. 56 (2004), no.3, 520–526.

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