On the PIR capacity of MSR codes

Adway Patra  Navin Kashyap

Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore, India

Email: {adwaypatra, nkashyap} @ iisc.ac.in

January 15, 2019

Abstract

We consider the problem of private information retrieval (PIR) from multiple storage nodes when the underlying database is encoded using regenerating codes, i.e., the database has the ability to recover from individual node failures. We seek optimality in terms of download rates when a user wants to retrieve a message privately (in an information-theoretic sense) from the database. We focus mostly on the more extensively explored Minimum Storage Regenerating (MSR) codes and formally state and prove the PIR capacity of such codes. We also give a simple scheme to achieve this capacity which is based on the similarly motivated construction for MDS codes. Furthermore, we give a PIR scheme for a specific class of Minimum Bandwidth Regenerating (MBR) codes that achieves the same rate as its minimum storage counterpart.

1 Introduction

Private Information Retrieval (PIR) is the problem of retrieval of a message among several messages stored in a storage system without revealing any information about the identity of the desired message to the database. The problem of PIR was first introduced in the seminal paper by Chor et al. [1]. Since then, a considerable amount of research has been done in the field of PIR. In the classical PIR problem, a user trying to download one message out of many contacts multiple non-communicating storage nodes (or servers) that store the messages in a replicative format (using repetition coding) such that no information about the identity of the message is leaked to any of the nodes. The requirement of multiple storing entities is justified by the observation that for the case of a single-server database, the only possible privacy-preserving download scheme is the trivial one of downloading everything. On the other hand, the efficiency of a retrieval scheme while maintaining user privacy becomes a non-trivial concern in case of a database consisting of multiple servers in a distributed storage setting. A well-accepted measure of efficiency for information-theoretic PIR schemes is the download rate which is the number of desired bits retrieved per downloaded bit.

It was shown in [2] that the maximum rate of download for such a scheme termed the PIR capacity is \( \frac{1}{1- \frac{1}{N}} \) for \( m \) equal-length messages stored in a replicative manner across \( N \) nodes. The authors also presented an achievable scheme based on the idea of blind interference alignment technique which was introduced for wireless networks in [3]. Although, the redundancy introduced by the repetition coding scheme allows the system to tolerate the maximum possible number of simultaneous node failures, it is somewhat wasteful of the storage resources. To offer more efficient use of storage space, Maximum Distance Separable (MDS) coded databases were considered for the PIR setting in [4] and the capacity expression was derived to be \( \frac{1 - R_c}{1 - R_c} \) where \( R_c \) is the code-rate \( \frac{k}{n} \) of the underlying MDS \([n, k]\) code.

Since distributed storage systems are prone to frequent node failures, regenerating codes were proposed in [5]. These codes have the capability of replacing a failed node with an exact or functional replica by contacting a subset of the surviving nodes while minimizing the amount of data downloaded for repair. In this work, we have considered the problem of PIR while the underlying distributed storage system is of a regenerating nature. We will show that for storage-optimal regenerating codes (i.e., the Minimum Storage Regenerating codes to be formally defined later) the PIR capacity is exactly that of an MDS coded database. Hence, we establish the fact that having additionally the regenerating property does not incur any penalty in terms of privacy.

2 Related Work

Having been introduced in [1], the basic PIR problem has seen a lot of practical variations throughout the past few years. As stated before, the PIR capacity for replicative database was found in [2] and the same for the MDS coded database in [4]. The problem of PIR with colluding servers (TPIR)
where at most $T$ of the total number of servers may collude with each other, was considered in \cite{9} and later extended to byzantine and adversarial servers with possible node collisions in \cite{7} and \cite{8}. The idea of symmetric PIR (SPIR), which ensures the privacy of the messages other than the desired one from the retriever as well as the privacy of the desired message from the individual nodes, was investigated in \cite{10} for replicated databases and was later extended to MDS coded databases in \cite{11}. A further extension of this SPIR problem was considered in \cite{12} with different types of adversaries and the capacity was derived in each scenario. In \cite{13}, the authors consider the problem of PIR with available private side information (PIR-PSI) at the user end. The capacity of PIR-PSI was derived in \cite{14}. Further variations of this work are cache-aided PIR with uncoded partial side information \cite{15}, single-server single-message PIR with coded side information \cite{16}, single-server multi-message PIR with side information \cite{17} and PIR-PSI under storage constraints \cite{18}. Two recent variations of the PIR scheme are asymmetrical PIR schemes \cite{19} which download at different rates from different servers and staircase PIR \cite{20} which presents an universally robust solution of the problem of unresponsive servers.

Regenerating codes were introduced in \cite{21} to address the issue of frequent node failures in erasure coded databases and to expedite the subsequent repairs. An $[n, k, d]$ regenerating code for $n$ nodes allows the user to reconstruct the data by contacting any $k$ of the $n$ nodes and allows any failed node to be regenerated by contacting any $d$ of the remaining $n - 1$ nodes. Two types of node repairs are considered in the literature: functional repair and exact repair. The problem of functional repair was shown to be connected to network multi-casting problem in a directed graph \cite{22}. On the other hand, the exact repair problem has a trade-off between the amount of storage per node and the amount of download required for repair of a node. Codes at the two extreme points: the Maximum Storage Regenerating (MSR) codes and the Maximum Bandwidth Regenerating (MBR) codes are of fair interest and have been vastly studied. The achievability of the MSR point was first shown in \cite{23} for $k = 2$ and $d = n - 1$. An MBR construction with $d = n - 1$ and an MSR code for $d = k + 1$ performing approximately-exact regeneration was given in \cite{24}. A combination of functional and exact regeneration i.e. a hybrid regeneration at the MSR point was proposed in \cite{25}. All these constructions require large field sizes. The Product-Matrix (PM) construction for both the MSR and MBR points, proposed in \cite{26}, provides codes for almost all parameters (all $n, k, d$ for MBR and all $n, k, d \geq 2k - 1$ for MSR) and requires considerably smaller field sizes. In \cite{27}, the authors showed that all other points except the MSR and MBR on the storage-bandwidth trade-off are not achievable under exact repair.

The problem of PIR for regenerating codes was first considered in \cite{28}. However, the construction requires the number of nodes in the system to be exponential in the number of messages, which, in real life scenarios, is not practical to implement. Very recently, there has been a considerable body of work on this topic. In \cite{29}, the authors have given an explicit PIR scheme for the MSR PM code that achieves the rate $1 - \frac{d}{n}$ independent of the number of messages, for an $[n, k, d]$ MSR PM code. The idea was extended to multi-message PIR for both MSR and MBR PM codes in \cite{30}. In \cite{31}, the authors have given PIR schemes that outperform all previously suggested schemes for both MSR and MBR PM codes. Table \ref{tab:rates} gives a summary of all the rates of all schemes up until now (The notation is for an $[n, k, d, \alpha, \beta, B]$ regenerating codes and $p$ in the second row denotes the number of desired messages in multi-message PIR setting).

| MBR Parameters | MBR PIR Rate | MSR Parameters | MSR PIR Rate |
|----------------|--------------|----------------|--------------|
| \cite{26}      | -            | $n = 3k - 3, k, d = 2k - 2,$ | $1 - \frac{d}{n}$ |
|                | $\alpha = k - 1, \beta = 1, B = k(k - 1)$ | \cite{27} | $n = pk + d, k, d,$ |
|                | $\alpha = k - 1, \beta = 1,$ | $1 - \frac{d}{n}$ |
|                | $B = k(k - 1)$ | \cite{28} | $n, k, d,$ |
|                | $\alpha = k - 1, \beta = 1,$ | $1 - \frac{d}{n}$ |
|                | $B = k(k - 1)$ |

Table 1: Rates of different PIR schemes for PM Codes

To the best of our knowledge, the optimality (in the sense of download rates) of any proposed PIR scheme for the broader setting of general MSR or MBR codes has not been considered in the literature. In this work, we try to address this for general MSR codes. While all previous schemes deal with the specific PM construction of regenerating codes, our scheme works for any MSR code. The following sections are organized as follows. Section 3 gives some preliminaries about regenerating codes that are useful in further sections. We introduce our model in Section 4. In Section 5, we state and prove our main theorem which gives the PIR capacity for all MSR codes. Next in Section 6, we give a scheme applicable to any MSR code that achieves the capacity and also provide a same rate achieving scheme for the MBR PM codes. Finally, we draw conclusion in Section 7.
3 Preliminaries

3.1 Regenerating Codes

Distributed Storage Systems (DSS) are prone to frequent node failures. To combat this random phenomenon, controlled redundancy is introduced in the system such that even under the event of a certain number of node failures, the performance of the DSS from the user perspective is not affected. Erasure codes such as Reed-Solomon codes are widely used in DSS. An \([n,k]\) erasure code allows the user to retrieve a message allowing at most \(n-k\) arbitrary nodes to fail. To keep such failure tolerance property invariant with time, there is a natural necessity of regeneration of the failed nodes. For MDS codes such as Reed-Solomon codes, a trivial technique for regeneration of a failed node can be to download the data contained in any \(k\) surviving nodes, decode the message from the downloaded data and then re-encode it for the failed nodes. This asks for huge communication overheads and hence more efficient regeneration is expected. In their seminal paper, Dimakis et al. [5] first showed, by use of network coding arguments, that such efficient codes do indeed exist and perform strictly superior to existing techniques.

A regenerating code \([n,k,d,\alpha,\beta,B]\) is used to store a message of size \(B\), whose symbols belong to a certain finite field \(F_q\) among \(n\) identical servers storing \(\alpha\) coded symbols each such that the following two conditions are satisfied.

- The original message can be retrieved by contacting any \(k\) out of \(n\) nodes and downloading their stored data.
- If a node fails, it can be regenerated by contacting any \(d\) out of \(n-1\) remaining nodes and downloading \(\beta\) (\(\leq \alpha\)) symbols from each of them. Hence, we have the parametric relation

\[
k \leq d \leq n - 1
\]

Using the cut set bound of network coding, the authors in [5] showed that the parameters of regenerating codes are constrained by the equation

\[
B \leq \sum_{i=0}^{k-1} \min\{\alpha, (d-i)\beta\}
\]

This inequality not only provides an upper bound on the maximum possible information that can be stored using a fixed regenerating code, but also gives a trade-off between storage per node \(\alpha\) and regeneration bandwidth \(\gamma = d\beta\). Along the storage-bandwidth trade-off curve, the two extremal points are of significant interest. The point corresponding to the Minimum Storage Regenerating (MSR) code has parameters

\[
(\alpha_{MSR},\beta_{MSR}) = \left(\frac{B}{k}, \frac{B}{k(d-k+1)}\right)
\]

It is clear from the expressions that having \(d = k\) at the MSR point does not provide us with any savings in terms of download. To repair a failed node, a newcomer, restricted to contact only \(k\) nodes, can only perform the trivial technique of regeneration for MDS codes. The regeneration bandwidth turns out to be a decreasing function of \(d\) and hence is minimized when all the other nodes are contacted for repairing a failed node, i.e., at \(d = n - 1\), having the value of

\[
\gamma_{\text{min}} = \alpha \left(\frac{n-1}{n-k}\right) = \left(\frac{B}{k}\right) \left(\frac{n-1}{n-k}\right)
\]

A bandwidth expansion factor is necessary for the reliability-redundancy optimality of the MDS property.

The point on the other extreme of the curve, the Minimum Bandwidth Regenerating (MBR) code has the parameters

\[
(\alpha_{MBR},\beta_{MBR}) = \left(\frac{2Bd}{2kd-k^2+k}, \frac{2B}{2kd-k^2+k}\right)
\]

At this operating point, we have \(\alpha = d\beta\) and so MBR codes do not incur any repair bandwidth expansion and download exactly the amount of information stored in the failed node. However, at the point \(d = n - 1\), we have

\[
\gamma_{\text{min}} = \alpha_{\text{MBR}} = \left(\frac{B}{k}\right) \left(\frac{2n-2}{2n-k-1}\right)
\]

which shows that a storage expansion has been incurred for minimizing the bandwidth and the codes are no longer optimal in the MDS sense.
3.2 Product-Matrix (PM) Codes

The product-matrix framework [23] provides regenerating code constructions for almost all parameters of interest at the two extreme points of the trade-off curve (all \([n, k, d]\) values at the MBR point and all \([n, k, d \geq 2k - 1]\) values at the MSR point). An \([n, k, d, \alpha, \beta, B]\) MSR PM code is described by an \(n \times \alpha\) code matrix \(C\) whose \(i^{th}\) row corresponds to the \(\alpha\) symbols stored by the \(i^{th}\) node. The code matrix is the product of an \(n \times d\) encoding matrix \(\psi\) and \(d \times \alpha\) message matrix \(M\):

\[
C = \psi M
\]  

Note that \(M\) possesses certain symmetry properties to facilitate regeneration and does not necessarily contains \(d\alpha\) independent message symbols. If a user contacts any set \(K\) of nodes \({i_1, i_2, \ldots, i_k}\) such that the cardinality, \(|K|\), of the set equals \(k\) and downloads the stored data, the information available to the user can be represented as \(\psi_{\bar{K}} M\) where the matrix \(\psi_{\bar{K}}\) is a sub-matrix of \(\psi\) corresponding to these rows. The user must be able to decode the original message from these symbols. If a user contacts any set \(D\) of surviving nodes \({h_1, h_2, \ldots, h_d}\) such that \(|D| = d\). A node \(h_j\) sends back \(\psi_{h_j} M\mu_f\) where \(\mu_f\) is a length \(\alpha\) column vector that is dependent upon the failed node index \(f\). From the data sent back by the \(d\) helper databases, the newcomer should be able to exactly reproduce the lost \(\alpha\) symbols of the node \(f\). In this framework, \(\beta\) is assumed to be 1 and the authors in [23] showed that this assumption is without loss of generality.

3.2.1 Minimum Storage Regenerating Codes

The parameter set of the MSR PM code can be expressed as

\[
[n, k, d = 2k - 2, \alpha = d - k + 1 = k - 1, \beta = 1, B = k\alpha]
\]  

The message matrix \(M\) is a \(d \times \alpha\) matrix

\[
M = \begin{bmatrix}
S_1 \\
S_2
\end{bmatrix}
\]  

where each of \(S_1\) and \(S_2\) is a \(\alpha \times \alpha\) symmetric matrix containing \(\binom{\alpha + 1}{2}\) distinct independent symbols each. Hence the total message symbols in the two matrices together is \(\alpha(\alpha + 1)\) which is exactly \(B\).

The encoding matrix \(\psi\) is defined as

\[
\psi = \begin{bmatrix}
\phi & \Delta \phi
\end{bmatrix}
\]  

where \(\phi\) is an \(n \times \alpha\) matrix having any \(\alpha\) rows linearly independent and \(\Delta\) is an \(n \times n\) diagonal matrix with distinct elements. Also the matrix \(\psi\) needs to be such that any \(d\) rows are linearly independent. It was shown in [23] that one can construct codes for all \(n - 1 \geq d \geq 2k - 2\) by using an MSR code at \(d = 2k - 2\) (Theorem 6 in [23]).

3.2.2 Minimum Bandwidth Regenerating Codes

The parameter set of the MBR PM code can be expressed as

\[
[n, k, d \geq k, \alpha = d, \beta = 1, B = \binom{k + 1}{2} + k(d - k)]
\]  

The message matrix \(M\) is defined as

\[
M = \begin{bmatrix}
S \\
T
\end{bmatrix}
\]  

where \(S\) is a symmetric matrix of dimension \(k \times k\) containing the first \(\binom{k + 1}{2}\) symbols of the message and the matrix \(T\) is of dimension \(k \times (d - k)\) containing the other \(k(d - k)\) message symbols. The encoding matrix \(\psi\) has the structure

\[
\psi = \begin{bmatrix}
\phi & \Delta
\end{bmatrix}
\]  

where \(\phi\) is an \(n \times k\) matrix having any \(k\) rows linearly independent and \(\Delta\) is an \(n \times (d - k)\) matrix such that, overall, any \(d\) rows of \(\psi\) are linearly independent.

4 System Model

In this section we describe the system model that we will be following throughout the rest of the paper. Let \([a]\), for some positive integer \(a\), denote the set \(\{1, 2, \ldots, a\}\) and \([a_1 : a_2]\), for some positive integers \(a_1, a_2\) and \(a_2 \geq a_1\), denote the set \(\{a_1, a_1 + 1, \ldots, a_2 - 1, a_2\}\). For a set of random variables \(\{A_i : i \in [a]\}\), \(A_{j_1:k}\) for \(1 \leq j_1 \leq j_2 \leq a\) denotes the set of random variables \(\{A_{j_1}, A_{j_1+1}, A_{j_1+2}, \ldots, A_{j_2}\}\) and \(A_K\) for \(K \subseteq [a]\) denotes \(\{A_i : i \in K\}\). For any matrix \(X\), the \(r^{th}\) row of matrix \(X\) is denoted by \(X(r)\) and for
a set of indices \( K \), \( X_K \) denotes the sub-matrix of \( X \) restricting its rows to those with indices in \( K \). The transpose of the matrix \( X \) is denoted by \( X^T \).

We have \( n \) identical servers denoted by the set \( [n] = \{1, 2, \ldots, n\} \) and they store \( m \) messages \( \{W_1, W_2, \ldots, W_m\} \). We fix an \( [n, k, d, \alpha, \beta, B] \) regenerating code and take the size of each message to be \( l = LB \) where \( L \) is some positive integer. The \( L \) symbols of each message are independently chosen at random from a finite field \( \mathbb{F}_q \):

\[
H(W_i) = L \quad \forall \ i \in [m] \tag{14}
\]

\[
H(W_1, W_2, \ldots, W_m) = \sum_{i=1}^{m} H(W_i) = mL \tag{15}
\]

where \( H(\cdot) \) is the entropy function with logarithm taken to the base \( q \). The encoding of a message of \( L \) symbols is done in the following way. Each block of \( B \) message symbols is separately encoded using the same encoding procedure of the above regenerating code. Since the encoding of each such block is carried out independently of the other blocks, the regeneration and reconstruction (i.e., decoding) of each such block can also be carried out independently of all the other blocks of the same message. The resulting encoding of the message with \( L \) symbols can be thought of as the encoding obtained using an \([n, k, d, \alpha, \beta, B = L]\) regenerating code that stores \( \alpha = L\alpha \) symbols per node and requires \( \beta = L\beta \) downloads from each helper node during node regeneration. This idea is inspired by the concept of “striping of data” as discussed in Section I-C of [23]. Such an encoding scheme gives us the flexibility to increase \( L \) in multiples of \( B \) by simply increasing \( L \). This causes the parameters \( \alpha, \beta, B \) to increase accordingly while the underlying regenerating code parameters \( \alpha, \beta, \tilde{B} \) do not change. In the limit, the file size \( L \) can be made to go to infinity simply by increasing \( L \). As \( L \to \infty \), the parameters \( \alpha, \beta, L (= B) \to \infty \) while the underlying regenerating code parameters \([n, k, d, \alpha, \beta, \tilde{B}] \) remain fixed.

Let \( Y_i \) denote the data stored at node \( i \). Since the messages themselves are independent and the encoding of each message is performed independently of the others, it follows that \( Y_i \) consists of \( m \) independent components \( \{Y_{i1}, Y_{i2}, \ldots, Y_{in}\} \) with \( Y_{ij} \) being the component corresponding to the \( j \)th message.

\[
H(Y_{ij}) = \alpha \quad \forall \ j \in [m] \tag{16}
\]

\[
H(Y_i) = m\alpha \tag{17}
\]

A user should be able to recover a message by accessing any subset \( K \subset N \) of cardinality \( k \) and downloading the data corresponding to that message. So,

\[
H(W_j|Y_K) = 0 \quad \forall \ j \in [m] \tag{18}
\]

which when combined with Equations (14)-(17) gives:

\[
H(W_1, W_2, \ldots, W_m|Y_K) = 0 \tag{19}
\]

In the event of a node failure, the newcomer node contacts a subset \( D \subset N \) of nodes of cardinality \( d \) and each of these helper nodes sends \( \beta \) symbols for the repair. Let us use \( S_{h\to f} \) to denote the random variable sent by helper node \( h \) for the repair of node \( f \). Again using independence arguments we can say that the variable \( S_{h\to f} \) can be split into \( m \) independent components \( \{S_{h\to f}^1, S_{h\to f}^2, \ldots, S_{h\to f}^m\} \) where \( S_{h\to f}^i \) is sent by helper node \( h \) for regeneration of the data stored at node \( f \) corresponding to the \( j \)th message. From the regenerating code parameters it follows that:

\[
H(S_{h\to f}^j) = \beta \quad \forall \ j \in [m] \tag{20}
\]

\[
I(S_{h\to f}^j; S_{h\to f}^{j'}) = 0 \quad \forall \ j_1, j_2 \in [m], \ j_1 \neq j_2 \tag{21}
\]

which implies

\[
H(S_{h\to f}) = m\beta \tag{22}
\]

The regeneration property further states that

\[
H(S_{h\to f}^j|Y_K) = 0 \quad \forall \ j \in [m] \tag{23}
\]

\[
H(Y_i^j|S_{D\to f}) = 0 \quad \forall \ j \in [m] \tag{24}
\]

where \( S_{D\to f} \equiv \{S_{h\to f}^i : h \in D\} \). Now let us suppose that an user is interested in retrieving a message \( W_i \) privately from the \( n \) nodes. The user has a realization of \( j \) private to him. A set of queries \( \{Q_{1}^{(j)}, Q_{2}^{(j)}, \ldots, Q_{m}^{(j)}\} \) is prepared by the user such that individually the queries reveal no information about the desired index \( j \) to any of the nodes:

\[
I(j; Q_{i}^{(j)}) = 0 \quad \forall \ i \in [n] \tag{25}
\]
Since the queries are prepared with no information about the original messages, we have
\[
I(Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_m^{(j)}; W_1, W_2, \ldots, W_m) = 0 \quad \forall \ j \in [m]
\] (26)
The \(i\)th node, upon receiving the query \(Q_i^{(j)}\) returns an answer variable \(A_i^{(j)}\) which is a deterministic function of the data stored \(Y_i\) and the received query \(Q_i^{(j)}\):
\[
H(A_i^{(j)}|Q_i^{(j)}, Y_i) = 0 \quad \forall \ i \in [n]
\] (27)
Getting back all the answers \((A_1^{(j)}, A_2^{(j)}, \ldots, A_n^{(j)})\) from the \(n\) nodes, the user should be able to recover the desired message with very small probability of error. From Fano’s inequality it follows that:
\[
H(W_j|A_1^{(j)}, \ldots, A_n^{(j)}, Q_1^{(j)}, \ldots, Q_m^{(j)}) = o(L) \quad \forall \ j \in [m]
\] (28)
where \(o(L) \to 0\) as \(L \to \infty\).

The rate of the PIR scheme is defined as the ratio of the size of the desired message to the total download cost under the two constraints of equations (25) and (28) of privacy and reliability respectively. The rate \(R\) is defined as
\[
R = \frac{H(W_j)}{\sum_{i=1}^{n} H(A_i^{(j)})}
\] (29)
The PIR capacity \(C\) is the supremum of the rate over all PIR schemes. As mentioned before, since we are following the standard setting of information-theoretic PIR, we will be neglecting the upload cost related to the queries and only focus on the download cost.

5 PIR Capacity of MSR Codes

In this section, we derive the PIR capacity for all MSR codes. It turns out that the capacity expression is the same as that of an \([n,k]\) scalar MDS code as derived in [4].

**Theorem 1.** For an \([n,k,d,\alpha,\beta,B]\) MSR coded distributed storage system, storing \(m\) files of equal size \(L = B\), the PIR capacity is
\[
C_{MSR} = \frac{1 - \left(\frac{\alpha}{k}\right)}{1 - \left(\frac{\beta}{k}\right)^m}
\] (30)
i.e., for a fixed number of files, the capacity depends only on the ratio \(\frac{\beta}{n}\) and is independent of the regenerating parameters \(d,\alpha,\beta\).

Intuitively, this follows simply from the fact that MSR codes are, in fact, vector MDS codes. We formally prove this by first proving, in this section, that the RHS expression of Equation (30) is indeed an upper bound on the download rate of any such PIR scheme for MSR coded databases. The achievability scheme described in Section 6 establishes equality, proving Theorem 1. We proceed by first observing the following property:

**Lemma 5.1:** For any PIR scheme for an \([n,k,d,\alpha,\beta,B]\) MSR coded database with \(m\) independent messages each of size \(L = B\), for any set \(K = \{i_1, i_2, \ldots, i_k\} \subset [n]\) with \(|K| = k\) and any \(j \in [m]\), the following property is satisfied
\[
H(A_K^{(j)}|Q_K^{(j)}) = \sum_{a=1}^{k} H(A_{a|a}^{(j)}|Q_{a|a}^{(j)})
\] (31)

**Proof:** Following the system model and parameter definitions of Section 4, without loss of generality we assume \(K\) to be the set of the first \(k\) nodes \(\{1,2,\ldots,k\}\). We first prove that,
\[
H(Y_K) = \sum_{i=1}^{k} H(Y_i)
\] (32)
Due to the mutual independence of messages (equations (16)-(17)), we need only focus on the variables \(\{Y_i^j : i \in [n]\}\) for a specific \(j \in [m]\). Note that, \(H(Y_i^j) = \alpha \quad \forall \ i \in [n]\), i.e., each node stores \(\alpha\) coded symbols corresponding to message \(W_j\). From the variables \(\{Y_i^j : i \in K\}\), the user should be able to retrieve the message \(W_j\). So
\[
L = H(W_j) \leq H(Y_K^j) \leq \sum_{i=1}^{k} H(Y_i^j) \leq k\alpha
\] (33)
But for MSR codes, \( L = k\alpha \). This means all inequalities are in fact satisfied with equality and hence

\[
H(Y_k^j) = \sum_{i=1}^{k} H(Y_i^j)
\]  

(34)

which gives

\[
H(Y_K) = \sum_{i=1}^{k} H(Y_i)
\]  

(35)

Since the answer strings \( A_i^{(b)} \) are functions of \( Y_i, Q_i^{(b)} \), the variables \( \{A_i^{(b)} : i \in K\} \) are conditionally independent conditioned on \( \{Q_i^{(b)} : i \in K\} \).

\[
H(A_K^{(b)}|Q_K^{(b)}) = \sum_{i=1}^{k} H(A_i^{(b)}|Q_i^{(b)})
\]  

(36)

With this result, we proceed to the capacity derivation in the same way as [3]. We reproduce the derivation here just for the sake of completeness. We state the two important lemmas that will be used. The proof of these two lemmas is directly from [3].

**Lemma 5.2:** The interference from undesired messages is upper bounded as

\[
L \left( \frac{1}{R} - 1 + \frac{o(L)}{L} \right) \geq I(W_{2;m};Q_{1:n}^{(1)}, A_{1:n}^{(1)}|W_1)
\]

**Proof:**

\[
I(W_{2;m};Q_{1:n}^{(1)}, A_{1:n}^{(1)}|W_1)
= I(W_{2;m};Q_{1:n}^{(1)}, A_{1:n}^{(1)}, W_1)
= I(W_{2;m};Q_{1:n}^{(1)}, A_{1:n}^{(1)}|W_1) + I(W_{2;m};W_1|Q_{1:n}^{(1)}, A_{1:n}^{(1)})
= I(W_{2;m};Q_{1:n}^{(1)}|W_1) + I(W_{2;m};A_{1:n}^{(1)}|Q_{1:n}^{(1)}) + o(L)
= I(W_{2;m};A_{1:n}^{(1)}|Q_{1:n}^{(1)}) + o(L)
= H(A_{1:n}^{(1)}) - H(A_{1:n}^{(1)}|Q_{1:n}^{(1)};W_1)
\leq \sum_{i=0}^{n} H(A_i^{(1)}) - H(A_i^{(1)}|Q_i^{(1)}, W_1)
\leq L \left( \frac{1}{R} - 1 + \frac{o(L)}{L} \right)
\]

where (37) follows from the independence of messages, (38) follows from (28), (39) follows from (26), (40) follows from chain rule of entropy, (41) follows from (28) and (42) follows from (26), (27) and message independence.

**Lemma 5.3:**

\[
I(W_{m};Q_{1:n}^{(m-1)}, A_{1:n}^{(m-1)}|W_{1:m-1}) \geq \frac{k}{n} I(W_{m+1};Q_{1:n}^{(m)}, A_{1:n}^{(m)}|W_{1:m})
\]

\[
+ \frac{kL(1 - \frac{o(kL)}{L})}{n}
\]

**Proof:**

\[
I(W_{m};Q_{1:n}^{(m-1)}, A_{1:n}^{(m-1)}|W_{1:m-1})
= \frac{1}{\binom{n}{k}} \binom{n}{k} I(W_{m};Q_{1:n}^{(m-1)}, A_{1:n}^{(m-1)}|W_{1:m-1})
\geq \frac{1}{\binom{n}{k}} \binom{n}{k} \sum_{K \leq n; |K|=k} I(W_{m};Q_{K}^{(m-1)}, A_{K}^{(m-1)}|W_{1:m-1})
\]

(43)
\[
\begin{align*}
&= \frac{1}{k} (\frac{n}{k}) \sum_{K \subseteq N \setminus \{\hat{k}\}} I(W_{\hat{k}; m}; A^{(\hat{k})}_{1:n} | Q^{(\hat{k})}_{1:n}, W_{1:m-1}) \\
&= \frac{1}{k} (\frac{n}{k}) \sum_{K \subseteq N \setminus \{\hat{k}\}} [H(A^{(\hat{k})}_{K} | Q^{(\hat{k})}_{K}, W_{1:m}) - H(A^{(\hat{k})}_{K} | Q^{(\hat{k})}_{K}, W_{1:m-1})] \\
&= \frac{1}{k} (\frac{n}{k}) \sum_{K \subseteq N \setminus \{\hat{k}\}} \sum_{k \in K} H(A^{(\hat{k})}_{k} | Q^{(\hat{k})}_{k}, W_{1:m-1}) \\
&= \frac{1}{k} (\frac{n}{k}) \sum_{K \subseteq N \setminus \{\hat{k}\}} \sum_{k \in K} [H(A^{(\hat{k})}_{k} | Q^{(\hat{k})}_{k}, W_{1:m-1})] \\
&\geq \frac{1}{k} (\frac{n}{k}) \sum_{K \subseteq N \setminus \{\hat{k}\}} [H(A^{(\hat{k})}_{k} | Q^{(\hat{k})}_{k}, W_{1:m-1})] \\
&\geq \frac{k}{n} [I(W_{\hat{k}; m}; A^{(\hat{k})}_{1:n} | Q^{(\hat{k})}_{1:n}, W_{1:m-1})] \\
&= \frac{k}{n} [I(W_{\hat{k}; m}; A^{(\hat{k})}_{1:n} | Q^{(\hat{k})}_{1:n}, W_{1:m-1}) - o(L)] \\
&= \frac{k}{n} [I(W_{\hat{k}; m}; W_{\hat{k}}; A^{(\hat{k})}_{1:n} | Q^{(\hat{k})}_{1:n}, W_{1:m-1}) - o(L)] \\
&= \frac{k}{n} [I(W_{\hat{k}; m}; W_{\hat{k}}; A^{(\hat{k})}_{1:n} | Q^{(\hat{k})}_{1:n}, W_{1:m-1}) - o(L)] \\
&= \frac{k}{n} [L + I(W_{\hat{k}; m}; A^{(\hat{k})}_{1:n} | Q^{(\hat{k})}_{1:n}, W_{1:m-1}) - o(L)] \\
&= \frac{k}{n} [I(W_{\hat{k}; m}; A^{(\hat{k})}_{1:n} | Q^{(\hat{k})}_{1:n}, W_{1:m-1}) + kL(1 - \frac{o(L)}{L})]
\end{align*}
\]

where \([13]\) follows from properties of mutual information, \([14]\) follows from \([28]\), \([15]\) follows from Lemma 5.1, \([16]\) follows from \([25]\) and \([27]\), \([17]\) again follows from Lemma 5.1, \([18]\) follows from Han’s inequality, \([19]\) follows from \([27]\), \([20]\) follows from \([26]\), \([21]\) follows from \([28]\) and message independence.

**Proof of the upper bound on rate:** With these lemmas we proceed to prove the upper bound on the PIR download rate. We start with Lemma 5.2:

\[
L \left( \frac{1}{R} - 1 + \frac{o(L)}{L} \right) \geq I(W_{2:m}; Q^{(1)}_{1:n}, A^{(2)}_{1:n} | W_{1:n}) (52)
\]

\[
\geq \frac{k}{n} I(W_{3:m}; A^{(2)}_{1:n}, Q^{(2)}_{1:n} | W_{1:2}) + \frac{kL(1 - \frac{o(L)}{L})}{n} \geq \ldots \geq \left( \frac{k}{n} \right)^{m-2} I(W_{m;m}; A^{(m-1)}_{1:n}, Q^{(m-1)}_{1:n} | W_{1:m-1}) + \left( \frac{k}{n} \right)^{m-2} L(1 - \frac{o(L)}{L}) \geq \left( \frac{k}{n} \right)^{m-2} + \ldots + \left( \frac{k}{n} \right)^{m-1} L(1 - \frac{o(L)}{L})
\]

where \((52)\) is from Lemma 5.2 and subsequent steps are by using Lemma 5.3 successively. By rearranging we have

\[
\frac{1}{R} \geq (1 + \frac{k}{n} + \left( \frac{k}{n} \right)^{2} + \ldots + \left( \frac{k}{n} \right)^{m-1})(1 - \frac{o(L)}{L}) (53)
\]

By taking \(L \to \infty\), \(\frac{o(L)}{L} \to 0\) and we have

\[
R \leq \frac{1}{\sum_{j=0}^{m-1} \left( \frac{k}{n} \right)^{j}} \leq \frac{1}{1 - \left( \frac{k}{n} \right)^{m}} (54)
\]
6 Achievability of the PIR Capacity (based on [4])

In this section we propose a PIR scheme that achieves the PIR rate upper bound of MSR codes with equality and hence completes the proof of Theorem 1. Since the rate of this scheme is equal to capacity, this scheme performs strictly better than any algorithms previously suggested for such codes. The algorithm possesses the properties of a PIR scheme described in [2] i.e. symmetry across messages, symmetry across databases and efficient exploitation of side information and are, in intuition, similar to the algorithm for MDS coded databases described by Banawan et al. in [4]. To build the intuition, we will first discuss an example of the original scheme of [4] for MDS coded databases and then extend the techniques to MSR codes by using the fact that MSR codes are essentially vector MDS codes. Afterwards, we shall give a similar scheme for MBR PM codes.

6.1 PIR Scheme for MDS Codes

For an \([n,k]\) MDS coded database, with \(m\) messages each of size \(L = kn^m\), each “stripe” of length \(k\) is encoded using the \([n,k]\) scalar MDS code and stored across the \(n\) nodes. For every such coded stripe, downloading any \(k\) out of \(n\) coded symbols is sufficient to decode that stripe due to the MDS property. For the undesired messages, such a stripe, once decoded, acts as side information that can be exploited while downloading desired coded symbols from the \(n - k\) nodes that did not originally participate in the download of the stripe (of the undesired message). This is the fundamental idea for the efficient use of side information that leads to the capacity-achieving property of the scheme of [4].

Example 1: We present an example with small parameter values: \(n = 3, k = 2, m = 3, L = kn^m = 2 \cdot 3^3 = 54, m = 1\). The \(L\) symbols (each belonging to \(\mathbb{F}_q\)) of each message is divided into \(L' = n^m\) stripes of length \(k\) and coded using a \([3,2]\) MDS code. For each message there is a unique permutation \(\pi_j, j \in [m]\), over the \(L'\) indices of the stripes which is chosen randomly over all such permutations and is known only at the user end. We denote by \(C_{i,j}^{(r)}\) as the \(r\)th coded stripe (after permutation is applied to the indices) of the \(j\)th message for \(r \in [L'], j \in [m]\). The \(i\)th coded symbol of \(C_{i,j}^{(r)}\) stored in node \(i, i \in [n]\), is denoted by \(C_{i,j}^{(r)}\). If the query \(Q_i\) for the \(i\)th node contains \(C_{i,r}^{(j)}\) that means the node is asked to return the \(\pi_j^{-1}(r)\)th coded symbol of the \(j\)th message. The following Table 2 gives the queries to the \(3\) nodes for this example that achieve the PIR capacity of MDS codes.

Note that, for any message \(j \in [m]\), any stripe index \(r \in [L']\) appears at most once in a query \(Q_i\), whether as an individual symbol or in a sum of symbols. From a node’s perspective, the total number of symbols requested from a message (both as individual symbol and as part of sum) is the same for all messages and this holds for all nodes. The actual stripe indices of a message requested from a node is determined by the random permutation on the user end which is unknown to the node. Finally another private randomly chosen permutation is applied to each query itself to remove any possibility of inference of the desired index from the order of \(C_{i,j}^{(r)}\)’s in the query.

6.2 PIR Scheme for MSR codes

With the idea of the MDS PIR scheme in mind, we move to constructing schemes for MSR codes. The simple intuition for this is that MSR codes have the vector MDS property and the PIR scheme discussed in the last section is not restricted to scalar symbols. We shall follow the general MSR code framework rather than work with a more specific setting like the Product-Matrix structure. We take an \([n,k,d,\bar{\alpha},\beta,\bar{B} = k\bar{a}]\) MSR code for some feasible parameter values \(\bar{\alpha}, \beta\). Let \(x\) be a \(k\bar{a}\) length message vector: \(x = [x_1, x_2, \ldots, x_k]^\top\) where each \(x_i \in \mathbb{F}_q^\bar{a}\), \(i \in [k]\) is a row vector of dimension \(\bar{a}\). The \(n\bar{a} \times k\bar{a}\) systematic encoding matrix \(G\) can be written as

\[
G = \begin{bmatrix}
I_{\bar{a}} & 0 & \cdots & 0 \\
0 & I_{\bar{a}} & \cdots & 0 \\
A_{1,1} & A_{1,k} & \cdots & A_{1,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n-k,1} & A_{n-k,k} & \cdots & A_{n-k,k-1}
\end{bmatrix}
\tag{55}
\]

where \(I_{\bar{a}}\) is the \(\bar{a} \times \bar{a}\) identity matrix and \(A_{u,v}\) for \(u \in [n-k], v \in [k]\), are \(\bar{a} \times \bar{a}\) encoding matrices such that the desired MSR properties are satisfied. The encoded vector \(y\) of length \(n\bar{a}\) is simply \(y = Gx = [y_1, y_2, \ldots, y_{n\bar{a}}]^\top\) where \(y_i \in \mathbb{F}_q^\bar{a}\) is the row vector of \(\bar{a}\) coded symbols stored in the \(i\)th node, i.e., \(y_i = G_i x\) where \(G_i\) is the matrix \(G\) restricted to the rows \((i - 1)\bar{a} + 1\) to \(i\bar{a}\). The assumption of systematic encoding is without loss of generality.

We take the message sizes to be \(L = \bar{a}kn^m\). We divide the long string of message symbols into stripes of \(k\bar{a}\) message symbols. Each stripe is encoded independently using the above MSR code to
| $Q_{1}^{[1]}$ | $Q_{2}^{[1]}$ | $Q_{3}^{[1]}$ |
|-----------|-----------|-----------|
| $c_{1,1}$ | $c_{1,1}$ | $c_{1,1}$ |
| $c_{1,2}$ | $c_{1,2}$ | $c_{1,2}$ |
| $c_{1,4}$ | $c_{1,4}$ | $c_{1,4}$ |
| $c_{1,5}$ | $c_{1,5}$ | $c_{1,5}$ |
| $c_{1,7}$ | $c_{1,7}$ | $c_{1,7}$ |
| $c_{1,8}$ | $c_{1,8}$ | $c_{1,8}$ |
| $c_{1,10}$ | $c_{1,10}$ | $c_{1,10}$ |
| $c_{1,11}$ | $c_{1,11}$ | $c_{1,11}$ |
| $c_{2,1}$ | $c_{2,1}$ | $c_{2,1}$ |
| $c_{2,2}$ | $c_{2,2}$ | $c_{2,2}$ |
| $c_{2,4}$ | $c_{2,4}$ | $c_{2,4}$ |
| $c_{2,5}$ | $c_{2,5}$ | $c_{2,5}$ |
| $c_{2,7}$ | $c_{2,7}$ | $c_{2,7}$ |
| $c_{2,8}$ | $c_{2,8}$ | $c_{2,8}$ |
| $c_{2,10}$ | $c_{2,10}$ | $c_{2,10}$ |
| $c_{2,11}$ | $c_{2,11}$ | $c_{2,11}$ |
| $c_{3,1}$ | $c_{3,1}$ | $c_{3,1}$ |
| $c_{3,2}$ | $c_{3,2}$ | $c_{3,2}$ |
| $c_{3,3}$ | $c_{3,3}$ | $c_{3,3}$ |
| $c_{3,4}$ | $c_{3,4}$ | $c_{3,4}$ |
| $c_{3,5}$ | $c_{3,5}$ | $c_{3,5}$ |
| $c_{3,6}$ | $c_{3,6}$ | $c_{3,6}$ |
| $c_{3,7}$ | $c_{3,7}$ | $c_{3,7}$ |
| $c_{3,8}$ | $c_{3,8}$ | $c_{3,8}$ |
| $c_{3,9}$ | $c_{3,9}$ | $c_{3,9}$ |
| $c_{3,10}$ | $c_{3,10}$ | $c_{3,10}$ |
| $c_{3,11}$ | $c_{3,11}$ | $c_{3,11}$ |
| $c_{3,12}$ | $c_{3,12}$ | $c_{3,12}$ |

| $c_{1,1}$ + $c_{1,2}$ | $c_{1,1}$ + $c_{1,2}$ | $c_{1,1}$ + $c_{1,2}$ |
| $c_{1,1}$ + $c_{1,3}$ | $c_{1,1}$ + $c_{1,3}$ | $c_{1,1}$ + $c_{1,3}$ |
| $c_{1,1}$ + $c_{1,4}$ | $c_{1,1}$ + $c_{1,4}$ | $c_{1,1}$ + $c_{1,4}$ |
| $c_{1,1}$ + $c_{1,5}$ | $c_{1,1}$ + $c_{1,5}$ | $c_{1,1}$ + $c_{1,5}$ |
| $c_{1,1}$ + $c_{1,7}$ | $c_{1,1}$ + $c_{1,7}$ | $c_{1,1}$ + $c_{1,7}$ |
| $c_{1,1}$ + $c_{1,8}$ | $c_{1,1}$ + $c_{1,8}$ | $c_{1,1}$ + $c_{1,8}$ |
| $c_{1,1}$ + $c_{1,10}$ | $c_{1,1}$ + $c_{1,10}$ | $c_{1,1}$ + $c_{1,10}$ |
| $c_{1,1}$ + $c_{1,11}$ | $c_{1,1}$ + $c_{1,11}$ | $c_{1,1}$ + $c_{1,11}$ |
| $c_{1,2}$ + $c_{1,3}$ | $c_{1,2}$ + $c_{1,3}$ | $c_{1,2}$ + $c_{1,3}$ |
| $c_{1,2}$ + $c_{1,4}$ | $c_{1,2}$ + $c_{1,4}$ | $c_{1,2}$ + $c_{1,4}$ |
| $c_{1,2}$ + $c_{1,5}$ | $c_{1,2}$ + $c_{1,5}$ | $c_{1,2}$ + $c_{1,5}$ |
| $c_{1,2}$ + $c_{1,7}$ | $c_{1,2}$ + $c_{1,7}$ | $c_{1,2}$ + $c_{1,7}$ |
| $c_{1,2}$ + $c_{1,8}$ | $c_{1,2}$ + $c_{1,8}$ | $c_{1,2}$ + $c_{1,8}$ |
| $c_{1,2}$ + $c_{1,10}$ | $c_{1,2}$ + $c_{1,10}$ | $c_{1,2}$ + $c_{1,10}$ |
| $c_{1,2}$ + $c_{1,11}$ | $c_{1,2}$ + $c_{1,11}$ | $c_{1,2}$ + $c_{1,11}$ |

Table 2: Queries for MDS coded database with $n = 3, k = 2$
nα coded symbols and stored across the n nodes as described above. Since, the regeneration property of each such stripe is independently preserved, we can think of this as each message of size L being encoded using an \( [n, k, d, α = αn^m, β = βn^m, B = Bn^m = α kn^m] \) MSR code. For message j, we denote the \( r \)-th stripe of size \( kα \) as the vector \( B^{(j)}_r \) for \( r \in [n^m] \). For this stripe the corresponding \( α \) coded symbols stored in node \( i \) are \( G_i B^{(j)}_r \).

**Initial Random Permutation:** For each message \( j \in [m] \), we choose a random permutation \( π_r \) over the stripe indices \([n^m]\) such that

\[
G_i B^{(j)}_{π_r(r)} = C^{(j)}_{i,r} \quad \forall j \in [m], i \in [n], r \in [n^m]
\]  

(56)

This permutation for each message is privately and independently chosen by the user and the database has no information regarding this. Notice the difference from the notation of the previous section.

Instead of a single coded symbol in \( F_q \), \( C^{(j)}_{i,r} \) now corresponds to a vector (in \( F_q^m \)) of coded symbols of the \( π_i(r)^{th} \) stripe of message \( W_j \) stored in the \( i^{th} \) node. Additionally, we introduce the term ‘\( z\)-sum’, for \( 1 \leq z \leq m \), to denote a sum of the form \( C_{i,r_1}^{(j)} + C_{i,r_2}^{(j)} + \ldots + C_{i,r_z}^{(j)} \), where addition is defined over the vector space \( F_q^m \) and the message indices \( \{j_1, j_2, \ldots, j_z\} \) are all distinct i.e., each component of the sum belongs to a different message. The type of a \( z\)-sum is described by the \( z\)-tuple \((j_1, j_2, \ldots, j_z)\) involved in the \( z\)-sum. For a fixed value of \( z \in [m] \), there are exactly \( \binom{m}{z} \) different types of \( z\)-sum. A \( 1\)-sum is trivially defined as simply \( C^{(j)}_{i,r} \). An instance of a \((j_1, j_2, \ldots, j_z)\)-type \( z\)-sum corresponds to a possible value of the tuple \((r_1, r_2, \ldots, r_z)\) where \( r_i \in [n^m], \forall i \in [z] \). To increment an instance means to increment each component of it by 1 i.e., incrementing \((r_1, r_2, \ldots, r_z)\) gives us \((r_1 + 1, r_2 + 1, \ldots, r_z + 1)\). For a type \((j_1, j_2, \ldots, j_z)\) not containing the desired message index, the set of \( z\)-tuples \((r_1, r_2, \ldots, r_z)\) corresponding to the instances that are downloaded from node \( i \) is denoted by \( Λ_z \). It will be clear later that \( Λ_z \) depends only on \( i \) and \( z \) and does not depend on the specific values of \((j_1, j_2, \ldots, j_z)\).

A query \( Q^{(i)}_z \), for node \( i \), consists of a collection of sub-queries where each sub-query is a request to node \( i \) to return a specific instance of a \( z\)-sum. With this initial setup and the notation explained, we now give the algorithm for retrieval of message \( W_α \).

1. **Initial download of desired message:** For the desired message (with index \( ˆm \)), starting from node 1, we download the \( α \)-dimensional vector related to stripe 1 from subsequent \( k \) nodes i.e. we download \( C^{(j)}_{(1)} \) such that \( C^{(j)}_{(1)} = C^{(j)}_{i,1} \) from nodes 1,2,\ldots, \( k \) respectively. The \( α \)-dimensional vectors corresponding to stripe 2 of the desired message are downloaded starting from node \( k + 1 \) and proceeding to another \( k \) nodes in a round-robin fashion i.e., download \( C^{(j)}_{(k+1)} \mod n^2, C^{(j)}_{(k+2)} \mod n^2, \ldots, C^{(j)}_{(k+n^2)} \mod n^2 \) from the nodes \( (k + 1) \mod n, (k + 2) \mod n, \ldots, (k + n^2 - 1) \mod n \) respectively. This is continued up to stripe \( i = n \cdot k \cdot m \). After completing this procedure, for each stripe \( i \in [n \cdot k \cdot m - 1] \), we shall have exactly \( k \) \( α \)-dimensional vectors from \( k \) different nodes from which we can decode the original \( r\)-th stripe of length \( kα \) using the vector MDS property.

2. **Preserving message symmetry:** To preserve message symmetry property for every node, we download equal number of symbols (equal number of \( α \)-dimensional vectors) from each of the other messages from each node. The above procedure is repeated for each \( j \in [m] \setminus \{ ˆm \} \). Note that, due to the initial permutation, the original ids of the stripes which are decoded can be different for different messages i.e., the true ids are not necessarily 1,2,\ldots, \( n \cdot k \cdot m \). Further, it can be verified that, for each message, exactly \( k \) \( α \)-dimensional vectors have been downloaded from each node so far.

3. **Exploitation of side information in next round:** From node 1, exactly \( k \) \( α \)-dimensional stripe ids for a message index \( i \in [m] \) have been queried. But we know that, cumulatively from all the \( n \) nodes, the first \( n \cdot k \cdot m - 1 \) stripe ids have been queried and decoded for all the messages. Hence, for a node \( i \), \((n - k) \cdot k \cdot m - 1 \) stripe ids for a message index \( j \in [m] \) were not queried from that node and by our definition this corresponds to the set \([n \cdot k \cdot m - 1] \setminus Λ_j \). For each message index \( j \in [m] \setminus \{ ˆm \} \) we download the 2-sum of type \(( ˆm, j)\) from node \( i \) with instances \(( ˆr_α, r_j)\), \( ∀ r_j \in [n \cdot k \cdot m - 1] \setminus Λ_j \), and \( r_α \) is a stripe id that was not queried in the first step. The value of \( r_α \) is gradually incremented starting from \( n \cdot k \cdot m - 1 + 1 \) in such a way that for every value of \( r_α \), an instance of \(( ˆr_α, r_j)\) is downloaded from exactly \( k \) nodes, keeping the constraint on \( r_j \) as described above for all \( j \in [m] \setminus \{ ˆm \} \). Up until now, we have been only concerned with 2-sums with types that included the desired message index. To preserve message symmetry, we need to download instances of 2-sums of the type \((j_1, j_2)\), \( ∀ j_1, j_2 \in [m] \setminus \{ ˆm \}, j_1 \neq j_2 \). For each such type, the instance \(( ˆr_1, r_2)\), is gradually incremented, starting from \((n \cdot k \cdot m - 1 + 1, n \cdot k \cdot m + 1)\), such that each such instance is downloaded from exactly \( k \) nodes. This goes on until equal number of instance downloads for all types of 2-sums for all nodes is established. Since, each instance of the form \( C^{(j_1)}_{i,r_1} + C^{(j_2)}_{i,r_2} \) where \( j_1, j_2 \in [m] \setminus \{ ˆm \} \), is downloaded from exactly \( k \) nodes, the sum of the \( r\)-th stripe of \( W_α \) and \( r_α \)th stripe of \( W_α \) is decodable by linearity of the encoding scheme. These are used as side information in subsequent rounds.
4. *Subsequent rounds:* In the $z^{th}$ round, we download $z$-sums. For the types of $z$-sums including the desired message index, we use the side information of the $(z-1)^{th}$ round. To be more specific, for a type $(\hat{m}, j_1, j_2, \ldots, j_z)$, from node $i$, we download instances $(r_{\hat{m}}, r_1, r_2, \ldots, r_{z-1})$, \( \forall (r_1, r_2, \ldots, r_{z-1}) \in \{n^{z-2}, k^{m-z+2}+1 : n^{z-1}, k^{m-z+1}\} \setminus I_{z-1} \) and $r_{\hat{m}}$ is chosen to be a new stripe id starting from $n \cdot k^{m-1} + \sum_{a=1}^{z-2} (m-1) \cdot n \cdot k^{m-z-1} \cdot (n-k)^{n-1} + 1$ and incremented in a similar fashion as before. For a type $(j_1, j_2, \ldots, j_z)$ not containing the desired message index, the instance $(r_1, r_2, \ldots, r_z)$ is gradually incremented such that each instance is downloaded from exactly $k$ nodes. This goes on till equal number of instance downloads for all types of $z$-sums, whether containing the desired message index or not, have been established.

5. *Permutation in the order of the queries:* Finally, for each node, a permutation over the sub-queries of the query for that node is randomly chosen by the user and applied without the database’s knowledge. The purpose is to preserve the privacy even in the event of complete knowledge of the retrieval scheme by the nodes. If not for this permutation, the nodes could have inferred the index of the desired message by simply observing the first round of sub-queries.

The following Algorithm [3] gives an explicit technique of building the queries for the above scheme. Note that, the initial and final permutations have not been explicitly written in the algorithm. The analysis of the algorithm including the explanation of notation follows.

---

**Algorithm 1: Determining queries \( \{Q_i\} \)**

**Result:** Set of Queries \( \{Q_i : i \in N\} \) for desired message \( \hat{m} \)

**Input:** Number of databases \( n \), Number of messages \( m \), Reconstruction parameter \( k \), Desired message \( \hat{m} \)

1. **Initialization:**
   2. For each message in \([m]\) initialize a block index
   3. counter \( count_j \) to 1;
   4. For desired message \( \hat{m} \) define \( S \equiv [m] \setminus \hat{m} \);

   // LOOP1

5. **for** node in \( 1 : n \cdot k^m \)**
   6. Add \( C^{(\hat{m})}_{((\text{node}-1) \mod n)+1, count_{\hat{m}}} \) to \( Q_{((\text{node}-1) \mod n)+1} \)
   7. if node \( \mod k == 0 \) then
   8. \( count_{\hat{m}} \leftarrow count_{\hat{m}} + 1 \)
   9. end

10. end

   // LOOP2

11. **for** \( z \) \( m \) \( 1 : m - 1 \)**
   12. \( y \leftarrow k^{m-z+1} \cdot (n-k)^{z-1} \)

   // SUB-LOOP1

13. **for Each subset \( R \) of \( S \) such that \( |R| = z \)**

   // SUB-SUB-LOOP1

14. **for** node in \( 1 : n \cdot y \)
   15. Add \( C^{R}_{((\text{node}-1) \mod n)+1, count_{R}} \) to \( Q_{((\text{node}-1) \mod n)+1} \)
   16. if node \( \mod k == 0 \) then
   17. \( count_{R} \leftarrow count_{R} + 1 \)
   18. end

19. end

20. \( count_{R} \leftarrow count_{R} - \frac{ny}{k} \)

21. \( a \leftarrow 1; \; x \leftarrow a \cdot k \)

   // SUB-SUB-LOOP2

22. **for** node in \( 1 : \frac{ny(n-k)}{k} \)

   23. Add \( C^{(\hat{m})}_{(x \mod n)+1, count_{\hat{m}}} + C^{R}_{(x \mod n)+1, count_{R}} \) to \( Q_{(x \mod n)+1} \)

   24. \( x \leftarrow x + 1 \)

   25. if node \( \mod (n-k) == 0 \) then

   26. \( count_{R} \leftarrow count_{R} + 1 \)

   27. \( a \leftarrow a + 1; \; x \leftarrow a \cdot k \)

   28. end

   29. if node \( \mod k == 0 \) then

   30. \( count_{\hat{m}} \leftarrow count_{\hat{m}} + 1 \)

   31. end

32. end

33. end

34. end
Analysis of Algorithm 1: The algorithm takes as input the parameter values of \([n, k, m]\) and the desired index \(\hat{m}\) and outputs the queries \(Q_i (Q_i)\) to be consistent with our previous notation) for each node \(i \in [n]\). As stated before, the queries are basically collection of sub-queries where each sub-query denotes an instance of a type of \(z\)-sum being requested from that node. With all the counters initialized to index 1, we proceed as follows:

- **LOOP1** visits each node, starting from node 1, in a round-robin fashion and increments \(\text{count}_{\hat{m}}\) every \(k\) iterations. This ensures that for each of the first \(n \cdot k^{m-1}\) stripe ids of the desired message, exactly \(k\) \(z\)-dimensional vectors are downloaded from \(k\) different nodes and hence that corresponding stripe is decodable due to the vector MDS property. After this initial download phase, we enter LOOP2 which progresses in subsequent rounds from 1 to \(m - 1\).

- Each iteration of LOOP2 operates for a value of \(z \in [m - 1]\). **SUB-LOOP1** iterates, for a fixed value of \(z\), on all possible types of \(z\)-sums not containing the desired message index. For each such type, we do the following:

In **SUB-SUB-LOOP1**, \(z\)-sums of the form \(C^{R}_{\text{count}_{\hat{m}}} \equiv C^{(j_{1})}_{\text{count}_{\hat{m}}}, C^{(j_{2})}_{\text{count}_{\hat{m}}} + \ldots + C^{(j_{k})}_{\text{count}_{\hat{m}}}\) where \(R = (j_{1}, j_{2}, \ldots, j_{k})\) is the type of the current iteration of **SUB-LOOP1** and \(\text{count}_{\hat{m}} = (\text{count}_{\hat{m}}, \text{count}_{\hat{m}}, \ldots, \text{count}_{\hat{m}})\) is an instance and each \(j_{i} \in [m] \setminus \{m\} l \in [z]\), are downloaded. The logic in Line 16-18 ensures that each such instance is downloaded from exactly \(k\) nodes and hence decodable (as a sum, not individually, for \(z > 1\) as explained before). For \(z = 1\), **SUB-LOOP1** simply replicates the task performed by **LOOP1**, but for all the undesired messages. In this way, it establishes message symmetry. The task of using the side information to obtain new desired message symbols is performed by **SUB-SUB-LOOP2**. Firstly, Line 20 resets the increments done in **SUB-SUB-LOOP1**. For message \(W_{j}\), with \(j \in [n] \setminus \hat{m}\), if the instance \(\text{count}_{\hat{m}}\) was downloaded from nodes \(\{i_{1}, i_{2}, \ldots, i_{k}\}\) in **SUB-SUB-LOOP1**, then the instance \(\text{count}_{\hat{m}} + 1\) is downloaded from nodes \(\{i_{1}, i_{2}, \ldots, i_{k}\}\). As explained before, \(\text{count}_{\hat{m}}\) is a new stripe id which was not queried before and is incremented in a \(k\)-round-robin fashion. For \(z > 1\), the algorithm proceeds in a sort of recursive manner. **SUB-SUB-LOOP1** downloads new instances of types containing only undesired messages, ensuring decodability (as a sum) of these instances for use as side information. For each such instance \(\text{count}_{\hat{m}}\), that is downloaded from a set of nodes \(\{i_{1}, i_{2}, \ldots, i_{k}\}\), **SUB-SUB-LOOP2** downloads \((z + 1)\)-sum instance \(\text{count}_{\hat{m}}, \text{count}_{j}\) from nodes \([n] \setminus \{i_{1}, i_{2}, \ldots, i_{k}\}\). This proceeds till \(z = m - 1\), when the **SUB-SUB-LOOP2** downloads \(z\)-sums with \(z = m\).

So in the \(z\)th round exactly \(\frac{1}{k}(n-k)\) \(z\)-dimensional vectors are downloaded due to **SUB-SUB-LOOP2** from each node for each \((z+1)\)-sum containing the desired message and in the \((z+1)\)th round these many \(z\)-dimensional vectors are downloaded due to **SUB-SUB-LOOP1** from each node for each \((z+1)\)-sum not containing the desired message. This ensures message symmetry and database symmetry.

**Claim 6.2.1:** The rate of this scheme is \(1 - \frac{1}{k}\) for any \(z\) and \(k\).

**Proof:** To calculate the rate, first we show that the number of stripes of desired message downloaded by this scheme is \(n^{m}\). To do that, we simply need to keep track of the variable \(\text{count}_{\hat{m}}\) which gets initialized to 1. This variable is incremented in Lines 8 and 30 of Algorithm 1. **LOOP1** starting at Line 5 runs \(n \cdot k^{m-1}\) times incrementing \(\text{count}_{\hat{m}}\) every \(k\) iterations. So the number of increments is \(n \cdot k^{m-1}\). Next we proceed to **LOOP2**, having **SUB-LOOP1** at Line 13 running for each possible subset of size \(z\) which contains **SUB-SUB-LOOP2** at Line 22 which contains the increment operation at Line 30. Careful calculation easily gives us the number of increments of \(\text{count}_{\hat{m}}\) in the \(z\)th round, which turns out to be \(n \cdot \left(\frac{m-1}{z}\right) \cdot k^{m-z+1} \cdot (n-k)^{z-1} \cdot \frac{n-k}{k} \cdot \frac{1}{z} = n \cdot \left(\frac{m-1}{z}\right) \cdot k^{m-z+1} \cdot (n-k)^{z}\). So the final value of \(\text{count}_{\hat{m}}\)

\[
\begin{align*}
\text{count}_{\hat{m}} &= n \cdot k^{m-1} + \sum_{z=1}^{m-1} \left(\frac{m-1}{z}\right) \cdot n \cdot k^{m-z-1} \cdot (n-k)^{z} \\
&= n \cdot \sum_{z=0}^{m-1} \left(\frac{m-1}{z}\right) \cdot k^{m-z-1} \cdot (n-k)^{z} \\
&= n^{m} \tag{57}
\end{align*}
\]

For each value of \(\text{count}_{\hat{m}}\), exactly \(k\) \(z\)-dimensional vectors, with same stripe id, from \(k\) different nodes are downloaded. By the vector MDS property, the corresponding stripe of message is decodable. So, indeed the complete message is decodable. Now in a similar manner the number of undesired stripes downloaded can be calculated by observing **SUB-SUB-LOOP1** at Line 14 of the algorithm. The total number of \(z\)-dimensional vectors downloaded from all nodes having no component from the desired message is

\[
\sum_{z=1}^{m-1} \left(\frac{m-1}{z}\right) \cdot n \cdot k^{m-z+1} \cdot (n-k)^{z-1} = \frac{n \cdot k^{m+1}}{n-k} \tag{58}
\]

13
Now we can calculate the rate to be

\[
R = \frac{\text{Size of message}}{\text{Total number of symbols downloaded}}
\]

\[
= \frac{\tilde{a}kn^m}{\tilde{a}kn^m + \tilde{\alpha}n^{m-k}n^{k(m+1)} - k \cdot n^m}
\]

\[
= \frac{n^m + \frac{n^{m+1}}{k \cdot n^m}}{n^m + \frac{n^{m+1}}{k \cdot n^m}}
\]

\[
= \frac{n^m + \frac{k \cdot n^m}{n}}{n^m + \frac{k \cdot n^m}{n}}
\]

\[
= \frac{1 - \frac{k}{n}}{1 - \left(\frac{k}{n}\right)^m}
\]

\[\text{(59)}\]

**Claim 6.2.2:** The scheme is private in the sense of Section 4.

**Proof:** It is clear from the algorithm that for any message a particular stripe id appears at most once in query \(Q_i\) of a node, whether as a single vector or z-sum of vectors. Symmetry across messages and across databases are also established since the same number of instances are downloaded for each z-sum irrespective of whether it contains the desired message index or not, and each node contributes equally to each z-sum. Due to the initial permutation over block ids, the true realizations of block ids for a query \(Q_i\) is completely random and hence independent of the desired message index. The final permutation of each query further ensures that the order of sub-queries in which they arrive at a node \(i\) reveals no information about the desired message index. Hence the condition of privacy in Section 4 i.e., \(I(j; Q_i^{(j)}) = 0\) is indeed satisfied for this scheme for every \(i \in [n]\).

\[\text{\blacksquare}\]

### 6.3 A Similar PIR Scheme for PM MBR Codes

The scheme described in the previous section is optimal in terms of rate and works on any MSR code parameters \([n, k, d, \tilde{\alpha}, \tilde{\beta}]\). In this section, we propose a similar scheme for the product-matrix MBR code that achieves the same rate of \(\frac{1 - \frac{k}{n}}{1 - \left(\frac{k}{n}\right)^m}\). Since the MBR code also allows you to reconstruct the original message by contacting any \(k\) out of \(n\) nodes, a simple trivial PIR scheme would be to just repeat Algorithm 2 for PM MBR parameters \([n, k, d, \alpha = \tilde{\alpha}, \beta = \tilde{\beta}, B = k(d - \frac{k}{n})]\) and file sizes \(L = Bn^m\). But this causes unnecessary downloads since \(k\alpha = kd \geq k(d - \frac{k}{n})\). Intuitively, this is because the MBR codes, unlike MSR codes, are not storage optimal.

By using the symmetric structure of the message matrix in the PM MBR encoding procedure described by Equation (12), we can reduce this overhead. The idea is that for an \([n, k, d, \tilde{\alpha}, \tilde{\beta}, B]\) PM MBR code, since there are exactly \(B\) independent message symbols, we need exactly \(B\) coded symbols out of the total \(kd\) coded symbols stored in the \(k\) nodes. The following claim formally states and proves this property.

**Claim 6.3.1:** If a message of size \(B = k(d - \frac{k}{n})\) is encoded using a \([n, k, d, \tilde{\alpha}, \tilde{\beta}, B]\) PM MBR code and stored across \(n\) nodes, there exists a (possibly asymmetric) download strategy that downloads exactly \(B\) coded symbols from any \(k\) out of \(n\) nodes such that, the user is able to decode the original message.

**Proof:** For an \([n, k, d, \tilde{\alpha}, \tilde{\beta}, B]\) PM MBR code the coded symbols can be represented by a \(d \times n\) matrix \(C\)

\[
C = M\psi^t
\]

The coded symbols stored at the \(i^{th}\) node for \(i \in [n]\) can be written as a \(d \times 1\) vector \(C_i\)

\[
C_i = M\psi((i)^t
\]

\[
= \begin{bmatrix}
S & T
\end{bmatrix}
\begin{bmatrix}
\phi(i)
\Delta(i)
\end{bmatrix}^t
\]

\[
= \begin{bmatrix}
S\phi(i)^t + T\Delta(i)^t
\end{bmatrix}
\begin{bmatrix}
\phi(i)^t
\end{bmatrix}
\]

\[\text{(61)}\]

Observe that the last \((d - k)\) entries of \(C_i\) are functions of the \(T\) matrix only. Each row of the \(T\) matrix contains \(k\) independent message symbols. And so any such row of \(k\) symbols is encoded into \(n\) symbols by the matrix \(\phi^t\) (of dimension \(k \times n\)). Since any \(k\) rows of \(\phi\) are linearly independent by construction, the encoding is MDS and hence to retrieve the \(k\) message symbols it is sufficient to download any \(k\) out of \(n\) symbols. This property holds for all the last \((d - k)\) rows of \(C\). So we need \(k(d - k)\) downloads for retrieving the complete matrix \(T\).

Once we have the matrix \(T\), by using symmetry of \(M\), each of the top \(k\) rows of \(C\) also becomes an \([n, k]\) MDS code. But we can use the symmetry of the matrix \(S\) to further reduce the amount of download. Without loss of generality, we start from the first row. We need \(k\) coded symbols to retrieve
the $k$ corresponding message symbols of the first row of $S$. Now considering the second row, notice that due to symmetry we already know one message symbol of this row. Hence we need only download $k - 1$ coded symbols for this row to completely decode this row of $S$. For successive rows we can have further reduction in required downloads until finally at the $k$th row we only need to download a single coded symbol. This way the complete matrix $S$ can be recovered. The total required download for this strategy is $\sum_{i=1}^{n}(k - i + 1) + k(d - k) = k(d - \frac{k+1}{2}) + k(d - k) = \tilde{B}$. ■

To describe a specific download strategy, we use $k$ $d$-dimensional binary vectors for each node participating in the reconstruction process. The position of ‘1’s in each vector denotes which coded symbols we need to download from that node. We give an algorithm that finds the necessary vectors for an $[n,k,d,\tilde{\alpha},\tilde{\beta},\tilde{B}]$ PM MBR code.

**Algorithm 2: Determining download strategy**

**Result:** Set of Vectors $\{x_l : l \in [k]\}$ of length $d$.

```plaintext
1. **Initialization:** Set the last $(d - k)$ entries of each vector to 1 and all other entries to 0.
2. $r \leftarrow 1$, $l \leftarrow 1$, $\text{count} \leftarrow l$, $\text{thres} \leftarrow k$;
3. **while** $\text{thres} > 0$ **do**
   4. Set the $r$th entry of $x_l$ to 1;
   5. $l \leftarrow (l \mod k) + 1$;
   6. $\text{count} \leftarrow \text{count} + 1$;
   7. if $\text{count} == \text{thres}$ then
      8. $\text{thres} \leftarrow \text{thres} - 1$;
      9. $\text{count} \leftarrow 1$;
     10. $r \leftarrow r + 1$;
7. **end**
```

Note that, there is not one unique optimal download strategy. Algorithm 2 just gives one of these. Also, for a fixed $n$, the download strategy depends on the two parameters $k$ and $d$. After we have the download strategy for the specific $[n,k,d,\tilde{\alpha},\tilde{\beta},\tilde{B}]$ PM MBR code, we follow the similar steps as described in the previous section on MSR codes. Each file is now of size $L = B n^m = k(d - \frac{k+1}{2}) n^m$ and each block (in the previous section, we used “stripe” to be consistent with the general MSR encoding scheme, but in this section the word “block” is more appropriate with regard to the PM framework) of size $\tilde{B}$ is encoded using the PM MBR code. To generate the queries, the same Algorithm 1 applies but with a slight modification. Recall from the previous section that in the $z$th round of Algorithm 1, a z-sum of the form $C_{i,\text{count}_1}^{(j_1)} + C_{i,\text{count}_2}^{(j_2)} + \ldots + C_{i,\text{count}_z}^{(j_z)}$ was downloaded where $C_{i,\text{count}_j}^{(j)}$ denoted the $\tilde{\alpha}$-dimensional coded vector stored in node $i$ corresponding to the block id (after permutation) $\text{count}_j$. ‘+’ denoted element-wise vector sum over $F_2^d$ and $j_1, j_2, \ldots, j_z \in [m]$ are all distinct. For an instance $(\text{count}_1, \text{count}_2, \ldots, \text{count}_z)$, the download was carried out from exactly $k$ nodes such that the decodability could be ensured.

The modification is as follows. For an instance, instead of downloading all $\tilde{\alpha}$ symbols of the resulting vector sum from all the $k$ nodes, we download a subset of symbols from each node. That subset is determined by the vectors $\{x_l : l \in [k]\}$. To be more specific, assume that according to Algorithm 1 an instance $(\text{count}_1, \text{count}_2, \ldots, \text{count}_z)$ of a type $(j_1, j_2, \ldots, j_z)$ is to be downloaded from a subset $K = \{i_1, i_2, \ldots, i_{\tilde{\alpha}}\} \subset [n]$. Instead of downloading all the $\tilde{\alpha}$ symbols of the $z$-sums from some node $i_a \in K$, we only download those symbols that are indexed by having a ‘1’ in the corresponding vector $x_{i_a}$ for $a \in [k]$, as given by Algorithm 2. With this simple modification, we are able to reduce the download requirements without compromising in the decodability.

**Claim 6.3.2:** The rate achieved by this scheme is also $\frac{1 - \frac{1}{k}}{1 - \frac{1}{(d-1)m}}$.

**Proof:** The rate calculation trivially follows from that of Claim 6.2.1 as the numerator and denominator of Equation (59) now are multiples of $k(d - \frac{k+1}{2})$ instead of $k\tilde{\alpha}$. Hence this scheme achieves the same rate of $\frac{1 - \frac{1}{k}}{1 - \frac{1}{(d-1)m}}$. ■

**Claim 6.3.3:** The scheme is private in the sense of Section 1.

**Proof:** The only modification to the PIR scheme of MSR codes is that of following a download strategy instead of downloading all $\tilde{\alpha}$ symbols of a $z$-sum. Since this modification is applied irrespective of whether the $z$-sum contains the desired message index or not, the modification does not hurt the privacy arguments. ■

**Remark:** For odd values of $k$, all the vectors generated by Algorithm 2 have equal Hamming weights i.e., the number of ‘1’s in each vector is the same. Hence, in this case the number of downloaded symbols across nodes is the same. But, in case of even values of $k$, we can observe that the Hamming weights of these vectors are different. So, it may seem that an asymmetrical download traffic arises in this case. But, due to the round-robin structure of the algorithm, the overall traffic still remains
symmetrical.

Rate comparison with previous schemes: While the rate of $\frac{1 - \frac{k}{n}}{1 - \frac{k}{n}}$ for MSR codes is proved to be optimal, no such claim can be made for MBR codes. In fact, we would like to point out that the scheme in [28] outperforms our PM MBR PIR scheme when the number of messages in the system becomes large. Figure 1 gives a comparison of the rates of the two schemes for fixed parameters of $[n = 6, k = 3, d = 4]$ and varying number of messages.

7 Conclusion

In this work, we have tried to find an answer to the problem of optimality of PIR schemes when the underlying database has node regeneration properties. The retrieval scheme of Section 6.2, to the best of our knowledge, outperforms the schemes available in the literature since it achieves the PIR capacity of MSR codes. We would also like to point out that this is the first PIR scheme to be applicable to all MSR codes. An explicit expression for the PIR capacity of general class of MBR codes or even that of the Product Matrix MBR code and the achievability of such capacity still remain open problems.

References

[1] B. Chor, E. Kushilevitz, O. Goldreich, and M. Sudan, “Private information retrieval,” J. ACM, vol. 45, no. 6, pp. 965–981, November 1998. [Online]. Available: http://doi.acm.org/10.1145/293347.293350

[2] H. Sun and S. A. Jafar, “The capacity of private information retrieval,” IEEE Transactions on Information Theory, vol. 63, no. 7, pp. 4075–4088, July 2017.

[3] S. A. Jafar, “Blind interference alignment,” IEEE Journal of Selected Topics in Signal Processing, vol. 6, no. 3, pp. 216–227, June 2012.

[4] K. Banawan and S. Ulukus, “The capacity of private information retrieval from coded databases,” IEEE Transactions on Information Theory, vol. 64, no. 3, pp. 1945–1956, March 2018.

[5] A. G. Dimakis, P. B. Godfrey, Y. Wu, M. J. Wainwright, and K. Ramchandran, “Network coding for distributed storage systems,” IEEE Transactions on Information Theory, vol. 56, no. 9, pp. 4539–4551, Sep. 2010.

[6] H. Sun and S. A. Jafar, “The capacity of robust private information retrieval with colluding databases,” IEEE Transactions on Information Theory, vol. 64, no. 4, pp. 2361–2370, April 2018.

[7] K. A. Banawan and S. Ulukus, “Private information retrieval from byzantine and colluding databases,” 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 1091–1098, 2017.

[8] R. Tajeddine, O. W. Gnilke, D. A. Karpuk, R. Freij, and C. Hollanti, “Robust private information retrieval from coded systems with byzantine and colluding servers,” 2018 IEEE International Symposium on Information Theory (ISIT), pp. 2451–2455, 2018.

[9] H. Sun and S. A. Jafar, “The capacity of symmetric private information retrieval,” in 2016 IEEE Globecom Workshops (GC Wkshps), Dec 2016, pp. 1–5.
Q. Wang and M. Skoglund, “Symmetric private information retrieval for MDS coded distributed storage,” in 2017 IEEE International Conference on Communications (ICC), May 2017, pp. 1–4.

——, “Secure symmetric private information retrieval from colluding databases with adversaries,” 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 1083–1090, 2017.

S. Kadhe, B. Garcia, A. Heidarzadeh, S. E. Rouayheb, and A. Sprintson, “Private information retrieval with side information: The single server case,” in 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Oct 2017, pp. 1099–1106.

Z. Chen, Z. Wang, and S. Jafar, “The capacity of private information retrieval with private side information,” 2017. [Online]. Available: http://arxiv.org/abs/1709.03022

Y. Wei, K. Banawan, and S. Ulukus, “Cache-aided private information retrieval with partially known uncoded prefetching: Fundamental limits,” IEEE Journal on Selected Areas in Communications, vol. 36, no. 6, pp. 1126–1139, June 2018.

A. Heidarzadeh, F. Kazemi, and A. Sprintson, “Capacity of single-server single-message private information retrieval with coded side information,” 2018. [Online]. Available: http://arxiv.org/abs/1806.00661

A. Heidarzadeh, B. Garcia, S. Kadhe, S. Y. E. Rouayheb, and A. Sprintson, “On the capacity of single-server multi-message private information retrieval with side information,” 2018. [Online]. Available: http://arxiv.org/abs/1807.09908

Y. Wei and S. Ulukus, “The capacity of private information retrieval with private side information under storage constraints,” 2018. [Online]. Available: http://arxiv.org/abs/1806.01253

K. A. Banawan and S. Ulukus, “Asymmetry hurts: Private information retrieval under asymmetric traffic constraints,” 2018. [Online]. Available: http://arxiv.org/abs/1801.03079

R. Bitar and S. E. Rouayheb, “Staircase-PIR: Universally robust private information retrieval,” 2018. [Online]. Available: http://arxiv.org/abs/1806.08825

Y. Wu and A. G. Dimakis, “Reducing repair traffic for erasure coding-based storage via interference alignment,” in 2009 IEEE International Symposium on Information Theory, June 2009, pp. 2276–2280.

K. V. Rashmi, N. B. Shah, P. V. Kumar, and K. Ramchandran, “Explicit construction of optimal exact regenerating codes for distributed storage,” 2009 47th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 1243–1249, 2009.

Y. Wu, “A construction of systematic MDS codes with minimum repair bandwidth,” IEEE Transactions on Information Theory, vol. 57, no. 6, pp. 3738–3741, June 2011.

K. V. Rashmi, N. B. Shah, and P. V. Kumar, “Optimal exact-regenerating codes for distributed storage at the MSR and MBR points via a product-matrix construction,” IEEE Transactions on Information Theory, vol. 57, no. 8, pp. 5227–5239, Aug 2011.

N. B. Shah, K. V. Rashmi, P. V. Kumar, and K. Ramchandran, “Distributed storage codes with repair-by-transfer and nonachievability of interior points on the storage-bandwidth tradeoff,” IEEE Transactions on Information Theory, vol. 58, no. 3, pp. 1837–1852, 2012.

N. B. Shah, K. V. Rashmi, and K. Ramchandran, “One extra bit of download ensures perfectly private information retrieval,” in 2014 IEEE International Symposium on Information Theory, June 2014, pp. 856–860.

C. Dorkson and S. Ng, “Private information retrieval using product-matrix minimum storage regenerating codes,” 2018. [Online]. Available: http://arxiv.org/abs/1805.07190

——, “Multi-message private information retrieval using product-matrix MSR and MBR codes,” 2018. [Online]. Available: http://arxiv.org/abs/1808.02023

J. Lavauzelle, R. Tajeddine, R. Freij-Hollanti, and C. Hollanti, “Private information retrieval schemes with regenerating codes,” 2018. [Online]. Available: http://arxiv.org/abs/1811.02898