A DIFFUSION LIMIT FOR A TEST PARTICLE IN A RANDOM DISTRIBUTION OF SCATTERERS

G. BASILE, A. NOTA, AND M. PULVIRENTI

ABSTRACT. We consider a point particle moving in a random distribution of obstacles described by a potential barrier. We show that, in a weak-coupling regime, under a diffusion limit suggested by the potential itself, the probability distribution of the particle converges to the solution of the heat equation. The diffusion coefficient is given by the Green-Kubo formula associated to the generator of the diffusion process dictated by the linear Landau equation.

1. INTRODUCTION

The evolution of the density of a test particle moving in a configuration of obstacles is described at mesoscopic level by linear kinetic equations. They are obtained from the microscopic Hamiltonian dynamics under a kinetic scaling of space and time, namely $t \to \varepsilon t$, $x \to \varepsilon x$ and a suitable rescaling of the density of the obstacles and the intensity of the interaction. Accordingly to the resulting frequency of collisions, the mean free path of the particle can have or not macroscopic length and different kinetic equations arise. Typical examples are the linear Boltzmann equation and the linear Landau equation.

The first rigorous result appeared in 1969 in the paper of Gallavotti [8], who derived a linear Boltzmann equation starting from a random distribution of fixed hard scatterers in the Boltzmann-Grad limit (low density), namely when the number of collisions is small, thus the mean free path of the particle is macroscopic. The result was improved by Spohn [11].

In the weak-coupling regime, when there are very many but weak collisions, a linear Landau equation appears

$$ (\partial_t + v \cdot \nabla_x) f(x, v, t) = B \Delta_v f(x, v, t), $$

where $\Delta_v$ is the Laplace-Beltrami operator on the $d$-dimensional sphere of radius $|v|$. It describes a momentum diffusion, i.e. the velocity process is a Brownian motion on the (kinetic) energy sphere. This intuitively follows from the facts that there are many elastic collisions with obstacles isotropically distributed. The diffusion coefficient $B$ is proportional to the variance of the transferred momentum in a single collision and depends on the shape of the interaction potential. The first result in this direction was obtained by Kesten and Papanicolaou in 1978 for a particle in $\mathbb{R}^d$ and by Dürr, Goldstein and Lebowitz in 1987 for a particle in $\mathbb{R}^2$ for sufficiently smooth interaction potentials.

The linear Landau equation yields also in an intermediate scale between low density and weak-coupling regime, namely when the (smooth) interaction potential $\phi$ rescales according to $\phi \to \varepsilon^\alpha \phi$, $\alpha \in (0, 1/2)$ and the density of the obstacles is of order $\varepsilon^{-2\alpha-(d-1)}$ ([5], [9]). The limiting cases $\alpha = 0$ and $\alpha = 1/2$ correspond respectively to the low density limit and the weak-coupling limit.
In the present paper we want to investigate the limit \( \varepsilon \to 0 \) in the intermediate case, namely when \( \alpha > 0 \) but sufficiently small, for an interaction potential no more smooth given by a circular potential barrier, in dimension two. The physical interest of this problem is connected to the geometric optics since the trajectory of the test particle is that of a light ray traveling in a medium (say water) in presence of circular drops of a different substance with smaller refractive index (say air). The opposite situation, namely drops of water in a medium of air, can be described as well by the circular well potential. Our analysis applies also to this case with minor modifications, but we consider only the case of potential barrier for sake of concreteness.

The novelty of this choice is that in this case the diffusion coefficient \( B \) diverges logarithmically. Roughly speaking, the asymptotic equation for the density of the Lorentz particle reads

\[
(\partial_t + v \cdot \nabla_x) f(x,v,t) \sim |\log \varepsilon| B \Delta_{|x|} f(x,v,t),
\]

which suggests to look at a longer time scale \( t \to |\log \varepsilon| t \). As expected, a diffusion in space arises.

The proof follows the original constructive idea, due to Gallavotti [8], for the low-density limit of a hard-sphere system. This approach is based on a suitable change of variables which leads to a Markovian approximation described by a linear Boltzmann equation. This presents some technical difficulties since some of the random configurations lead to trajectories that “remember” too much preventing the Markov property of the limit. In the two-dimensional case the probability of those bad behaviors producing memory effects (correlation between the past and the present) is nontrivial. Thus we need to control the unphysical trajectories: we estimate explicitly the set of bad configurations of the scatterers (such as the set of configurations yielding recollisions or interferences) showing that it is negligible in the limit (see [4]). The control of memory effects still holds for a longer time scale \( |\log \varepsilon| \) which allows to get the heat equation from the rescaled linear Boltzmann equation.

We remark that the diffusive limit analyzed in the present paper is suggested by the divergence of the diffusion coefficient for the particular choice of the potential we are considering. However the same techniques could work in presence of a smooth, radial, short-range potential \( \phi \). Also in this case we obtain a diffusive equation as longer time scale limit of a linear Boltzmann equation (Section 5). This is in the same spirit of [10] and [6].

2. Main results

Consider a point particle of mass one in \( \mathbb{R}^2 \), moving in a random distribution of fixed scatterers whose center are denoted by \( c_1, \ldots, c_N \in \mathbb{R}^2 \). The equation of motion are

\[
\begin{cases}
\dot{x} = v \\
\dot{v} = - \sum_{i=1}^{N} \nabla \phi(|x - c_i|)
\end{cases}
\]

where \((x,v)\) denote position and velocity of the test particle, \( t \) the time and, as usual, \( \dot{A} = \frac{dA}{dt} \) indicates the time derivative for any time dependent variable \( A \). Finally \( \phi : \mathbb{R}^2 \to \mathbb{R} \) is a given spherically symmetric potential.

To outline a kinetic behavior of the particle, we usually introduce a scale parameter \( \varepsilon > 0 \), indicating the ratio between the macroscopic and the microscopic
variables, and rescale according to

\[ x \to \varepsilon x, \ t \to \varepsilon t, \ \phi \to \varepsilon^\alpha \phi \]

with \( \alpha \in [0, 1/2] \). Then Eqs. (2.1) become

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\varepsilon^{\alpha-1} \sum_i \nabla \phi(\|x-c_i\|/\varepsilon) 
\end{align*}
\]

(2.2)

We assume the scatterers \( \mathbf{c}_N = (c_1, \ldots, c_N) \) distributed according to a Poisson distribution of intensity \( \mu_\varepsilon = \mu \varepsilon^{-\delta} \), where \( \delta = 1 + 2\alpha \). This means that the probability density of finding \( N \) obstacles in a bounded measurable set \( \Lambda \subset \mathbb{R}^2 \) is given by

\[
P_\varepsilon(d\mathbf{c}_N) = e^{-\mu_\varepsilon |\Lambda|} \mu_\varepsilon^N N! \, dc_1, \ldots, dc_N
\]

(2.3)

where \( |\Lambda| = \text{meas}\Lambda \).

Now let \( T^\varepsilon_{\mathbf{c}_N}(x,v) \) be the Hamiltonian flow solution of Eq. (2.2) with initial datum \((x,v)\) in a given sample \( \mathbf{c}_N = (c_1, \ldots, c_N) \) of obstacles (skipping the \( \varepsilon \) dependence for notational simplicity) and, for a given initial probability distribution \( f_0 = f_0(x,v) \), consider the quantity

\[
f_\varepsilon(x,v,t) = E_\varepsilon[f_0(T^\varepsilon_{\mathbf{c}_N}(x,v))]
\]

(2.4)

where \( E_\varepsilon \) is the expectation with respect to the measure \( P_\varepsilon \) given by (2.3).

In the limit \( \varepsilon \to 0 \) we expect that the probability distribution (2.4) solves a linear kinetic equation depending on the value of \( \alpha \). More precisely if \( \alpha = 0 \) (low-density or Boltzmann-Grad limit) then \( f_\varepsilon \) converges to \( f \), the solution of the following linear Boltzmann equation

\[
(\partial_t + v \cdot \nabla_x) f(x,v,t) = Lf(x,v,t)
\]

(2.5)

where

\[
Lf(x,v,t) = \mu |v| \int_1^{-1} d\rho \{ f(v') - f(v) \}
\]

(2.6)

and where

\[
v' = v - 2(\omega \cdot v)\omega.
\]

(2.7)

Here we are assuming \( \phi \) of range one i.e. \( \phi(r) = 0 \) if \( r > 1 \), and \( \omega = \omega(\rho, |v|) \) is the unit vector obtained by solving the scattering problem associated to \( \phi \). This result was proven and discussed in \([2],[4],[8],[11]\).

On the other hand, if \( \alpha = 1/2 \), the corresponding limit, called weak-coupling limit, yields the linear Landau equation (see \([3]\) and \([9]\))

\[
(\partial_t + v \cdot \nabla_x) f(x,v,t) = Lf(x,v,t)
\]

(2.8)

where

\[
Lf(v) = B \Delta_{|v|},
\]

(2.9)

and

\[
B = \frac{\pi \mu}{|v|} \int_0^\infty r^2 \phi(r)^2 \, dr.
\]

(2.10)

Note that \( \hat{\phi} \) is real and spherically symmetric.
In the present paper we want to investigate the limit $\varepsilon \to 0$, in case $\alpha > 0$ sufficiently small, when the diffusion coefficient $B$ given by (2.10) is diverging. Actually we consider the specific example

$$\phi(r) = \begin{cases} 1 & \text{if } r < 1 \\ 0 & \text{otherwise} \end{cases},$$

namely a circular potential barrier.

For a potential of the form (2.11) a simple computation shows that $B$ defined in (2.10) diverges logarithmically. Therefore we are interested in characterizing the asymptotic behavior of $f_\varepsilon(x,v,t)$, given by (2.4), under the scaling illustrated above. The main result of the present paper can be summarized in the following theorem.

**Theorem 2.1.** Suppose $f_0 \in C_0(\mathbb{R}^2 \times \mathbb{R}^2)$ a continuous, compactly supported initial probability density. Suppose also that $|D^k_x f_0| \leq C$, where $D_x$ is any partial derivative with respect to $x$ and $k = 1, 2$. Finally assume $\alpha \in (0, 1/8)$. The following statements hold

1) if $\mu_\varepsilon = \varepsilon^{-2\alpha - 1}$, for all $t \in (0,T]$, $T > 0$,

$$\lim_{\varepsilon \to 0} f_\varepsilon(x,v,t) = (f_0) := \frac{1}{2\pi} \frac{1}{|v|} \int_{S_{|v|}} f_0(x,v) \, dv. \quad \text{The convergence is in } L^2(\mathbb{R}^2 \times S_{|v|}).$$

2) if $\mu_\varepsilon = \frac{\varepsilon^{-2\alpha - 1}}{\log \varepsilon}$, for all $t \in (0,T]$, $T > 0$,

$$\lim_{\varepsilon \to 0} f_\varepsilon(x,v,t) = f(x,v,t),$$

where $f$ solves the Landau equation (2.8) with a renormalized diffusion coefficient

$$B := \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon}{2} \varepsilon |v| \int_{-1}^{1} \theta^2(\rho) \, d\rho. \quad \text{The convergence is in } L^2(\mathbb{R}^2 \times S_{|v|}).$$

3) if $\mu_\varepsilon = \varepsilon^{-2\alpha - 1}$, defining $F_\varepsilon(x,v,t) := f_\varepsilon(x,v,t|\log \varepsilon|)$, for all $t \in [0,T)$, $T > 0$,

$$\lim_{\varepsilon \to 0} F_\varepsilon(x,v,t) = \rho(x,t),$$

where $\rho$ solves the following heat equation

$$\begin{cases} \partial_t \rho = D \Delta \rho \\ \rho(x,0) = \langle f_0 \rangle, \end{cases}$$

with $D$ given by the Green-Kubo formula

$$D = \frac{1}{\mu |v|} \int_{S_{|v|}} v \cdot ( - \Delta_{S_{|v|}}^{-1}) v \, dv = \frac{2\pi}{\mu} |v| \int_0^\infty \mathbb{E}[v \cdot v(t,v)] \, dt,$$

where $v(t,v)$ is the stochastic process dictated by the generator of the Landau equation starting from $v$ and $\mathbb{E}[\cdot]$ denotes the expectation with respect to the invariant measure on $S_{|v|}$. The convergence is in $L^2(\mathbb{R}^2 \times S_{|v|}).$
Some comments to Theorem 2.1 are in order. As we shall prove in Section 4, the asymptotic behavior of the mechanical system we are considering is the same as the Markov process ruled by the linear Landau equation with a diverging factor in front of $L$. This is equivalent to consider the limit in the Euler scaling of the linear Landau equation, which is trivial. The system quickly thermalizes to the local equilibrium just given by $\langle f_0 \rangle$. This is point 1).

To detect something non-trivial we have to exploit longer times in which the local equilibrium starts to evolve (according to the diffusion equation), see point 3). Note however that, rescaling differently the density of the Poisson process, we can recover the kinetic picture given by Landau equation (with a renormalized diffusion coefficient $B$) as in [5], see point 2).

We finally remark that this picture is made possible because the recollisions set (see below for the precise definition) is negligible, as established in Section 4. We believe that the present result could be recovered also in high-density regimes $\alpha \in \left(\frac{1}{8}, \frac{1}{2}\right]$, namely also when the recollisions are not negligible anymore. However in this case different ideas and techniques are indeed necessary.

The plan of the paper is the following. In the next Section we illustrate our strategy and establish some preliminary results. In Section 3 we prove Theorem 1.1. Finally in Section 4 we prove a basic Lemma showing that our non-Markovian system can indeed be approximated by a Markovian one, easier to handle with.

3. Strategy

We follow the explicit approach in [8], [4] and [5]. By (2.4) we have, for $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $t > 0$,

$$f_\varepsilon(x,v,t) = e^{-\mu \varepsilon |B_t(x,v)|} \sum_{N \geq 0} \frac{\mu^N}{N!} \int_{B_t(x,v)^N} dc_N \int_{T^N_{\varepsilon,0}(x,v)} f_0(T^N_{\varepsilon,0}(x,v))$$

where $T^N_{\varepsilon,0}(x,v)$ is the Hamiltonian flow generated by the Hamiltonian

$$\frac{1}{2} v^2 + \varepsilon \sum_j \phi \left( \frac{|x - c_j|}{\varepsilon} \right)$$

where $\phi$ is given by (2.11), and initial datum $(x,v)$. Finally $B_t(x,v) = B(x,|v|t)$, where here and in the following, $B(x,R)$ denotes the disk of center $x$ and radius $R$.

The explicit solution to the equation of motion is obtained by solving the single scattering problem by using the energy and angular momentum conservation (see figure below).

Here we represent the scattering of a particle entering in the ball $B(0,1) = \{x \text{ s.t. } |x| < 1\}$ toward a potential barrier of intensity $\phi(x) = \varepsilon \alpha$.

We have an explicit expression for the refractive index

$$n_\varepsilon = \frac{\sin \alpha \sin \beta}{\sin \beta} = \frac{|\bar{v}|}{|v|} = \sqrt{1 - \frac{2 \varepsilon \alpha}{v^2}},$$

where $v$ is the initial velocity, $\bar{v}$ the velocity inside the barrier, $\alpha$ the angle of incidence and $\beta$ the angle of refraction. The scattering angle is $\Theta = \pi - 2 \varphi_0 = \cdots$
2(\beta - \alpha) \text{ and the impact parameter is } \rho = \sin \alpha. (\text{See Appendix A for a detailed analysis of the scattering problem.})

Remark 3.1. Formula (3.3) makes sense if \( \frac{2\epsilon \alpha}{v^2} < 1 \) or \( \rho = \sin \alpha < \sqrt{1 - \frac{2\epsilon \alpha}{v^2}} \). When one of such two inequalities is violated, the outgoing velocity is the one given by the elastic reflection.

After the scaling
\[ x \to \epsilon x, \quad t \to \epsilon t \]
the scattering process takes place in a disk of radius \( \epsilon \), but the velocities (and hence the angles) are invariant. A picture of a typical trajectory is given as in Figure 3. Here we are not considering possible overlappings of obstacles. The scattering
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Figure 3. A typical trajectory

The process can be solved in this case as well. However, as we shall see, this event is negligible because of the moderate densities we are considering.

Coming back to Eqn \((3.1)\), we distinguish the obstacles of the configuration \(c_N = c_1 \ldots c_N\) which, up to the time \(t\), influence the motion, called internal obstacles, and the external ones. More precisely, \(c_i\) is internal if

\[
\inf_{-t \leq s \leq 0} |x_\varepsilon(s) - c_i| < \varepsilon,
\]

while \(c_i\) is external if

\[
\inf_{-t \leq s \leq 0} |x_\varepsilon(s) - c_i| \geq \varepsilon.
\]

Here \((x_\varepsilon(s), v_\varepsilon(s)) = T^*_c(x, v)\).

Note that the integration over the external obstacles can be done so that

\[
f_\varepsilon(x, v, t) = \sum_{Q \geq 0} \frac{\mu^Q}{Q!} \int_{B_t(x, v)^Q} d\mathbf{b}_Q e^{-\mu \varepsilon |T(\mathbf{b}_Q)|} f_0(T^-_t(x, v)) \chi(\{\text{the } \mathbf{b}_Q \text{ are internal}\}).
\]

Here and in the sequel \(\chi(\{\ldots\})\) is the characteristic function of the event \(\{\ldots\}\).

Moreover, \(T(\mathbf{b}_Q)\) is the tube:

\[
T(\mathbf{b}_Q) = \{y \in B_t(x, v) \text{ s.t. } \exists s \in (-t, 0) \text{ s.t. } |y - x_\varepsilon(s)| < \varepsilon\}.
\]

Note that

\[
|T(\mathbf{b}_Q)| \leq 2\varepsilon |v| t.
\]
Instead of considering $f_\varepsilon$ we introduce

$$\tilde{f}_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2} |v| t} \sum_{Q \geq 0} \frac{\mu^Q}{Q!} \int_{B_i(x, v)^Q} dB_Q$$

(3.9)

$$\chi(\{\text{the } b_Q \text{ are internal}\}) \chi_1(b_Q) f_0(T_{b_Q}^{-1}(x, v)).$$

where

$$\chi_1(b_Q) = \chi\{b_Q \text{ s.t. } b_i \notin B(x, \varepsilon) \text{ and } b_i \notin B(x(-t), \varepsilon) \text{ for all } i = 1, \ldots, Q\}$$

(3.10)

Obviously

$$f_\varepsilon \geq \tilde{f}_\varepsilon.$$

(3.11)

Following [8], [4], [5] we would like to perform the following change of variables

$$0 \leq t_1 < t_2 < \cdots < t_Q \leq t$$

(3.12)

$$b_1, \ldots, b_Q \rightarrow \rho_1, t_1, \ldots, \rho_Q, t_Q$$

where, after ordering the obstacles $b_1, \ldots, b_Q$ according to the scattering sequence, $\rho_i$ and $t_i$ are the impact parameter and the entrance time of the light particle in the protection disk around $b_i$. 
More precisely, fixed an impact parameter $\rho$ and an entrance time $t$ we construct $b = b(\rho, t)$, the center of the obstacle. Then we perform the backward scattering and iterate the procedure to construct a trajectory $(\xi, \eta(s))$. However $(\xi, \eta(s)) = (x(s), v(s))$ (therefore the mapping (3.12) is one-to-one) only outside the following pathological situations.

i) **Overlapping.**
If $b_i$ and $b_j$ are both internal then $B(b_i, \varepsilon) \cap B(b_j, \varepsilon) \neq \emptyset$.

ii) **Recollisions.**
There exists $b_i$ such that for $s \in (t_j, t_{j+1})$, $j > i$, $\xi(s) \notin B(b_i, \varepsilon)$.

iii) **Interferences.**
There exists $b_i$ such that $\xi(s) \notin B(b_i, \varepsilon)$ for $s \in (t_j, t_{j+1})$, $j < i$.

We simply skip such events by setting

- $\chi_{ov} = \chi(\{b_Q \text{ s.t. i is realized}\})$,
- $\chi_{rec} = \chi(\{b_Q \text{ s.t. ii is realized}\})$,
- $\chi_{int} = \chi(\{b_Q \text{ s.t. iii is realized}\})$, 

and defining

\[
\bar{f}_\varepsilon(x,v,t) = e^{-2\varepsilon^2 |v|^t} \sum_{Q \geq 0} \mu^Q \int_0^t dt_1 \ldots \int_0^{t_{Q-1}} dt_Q 
\int_{-\varepsilon}^{\varepsilon} d\rho_1 \ldots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \chi_1(1 - \chi_{ov})(1 - \chi_{rec})(1 - \chi_{int}) f_0(\xi_\varepsilon(t), \eta_\varepsilon(t)).
\]

(3.13)

Note that $\bar{f}_\varepsilon \leq \tilde{f}_\varepsilon \leq f_\varepsilon$. Note also that in (3.13) we have used the change of variables (3.12) for which, outside the pathological sets i), ii), iii), $T_{\varepsilon}^{-t}(x,v) = (x_\varepsilon(-t), v_\varepsilon(-t))$.

Next we remove $\chi_1(1 - \chi_{ov})(1 - \chi_{rec})(1 - \chi_{int})$ by setting

\[
\bar{h}_\varepsilon(x,v,t) = e^{-2\varepsilon^2 |v|^t} \sum_{Q \geq 0} \mu^Q \int_0^t dt_1 \ldots \int_0^{t_{Q-1}} dt_Q 
\int_{-\varepsilon}^{\varepsilon} d\rho_1 \ldots \int_{-\varepsilon}^{\varepsilon} d\rho_Q f_0(\xi_\varepsilon(t), \eta_\varepsilon(t)).
\]

(3.14)

We can prove:
Proposition 3.2.
\[ \bar{f}_\varepsilon(t) = \bar{h}_\varepsilon(t) + \varphi_1(\varepsilon, t) \]
where \( \|\varphi_1(\varepsilon, t)\|_{L^1} \to 0 \) as \( \varepsilon \to 0 \) for all \( t \in [0, T] \).

Remark 3.3. Proposition 3.2 still holds for longer times, namely:
\[ \|\varphi_1(\varepsilon, t)\|_{L^1} \to 0 \quad \forall t \in [0, |\log \varepsilon|T], \quad T > 0. \]

We postpone the proof of the above Proposition in the last Section.

Next we consider the limiting trajectory \( \bar{\xi}_\varepsilon(s), \bar{\eta}_\varepsilon(s) \) obtained by considering the collision as instantaneous.

More precisely, for the sequence \( t_1, \ldots, t_Q, \rho_1, \ldots, \rho_Q \) consider the sequence \( v_1, \ldots, v_Q \) of incoming velocities before the \( Q \) collisions. Then
\[
\left\{ \begin{array}{l}
\bar{\xi}_\varepsilon(s) = x - v(t - t_1) - v_1(t_1 - t_2) - \cdots - v_Q t_Q \\
\bar{\eta}_\varepsilon(s) = v_Q.
\end{array} \right.
\]

We define
\[
h_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2|v|}t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \cdots \int_0^{t-Q-1} dt_Q
\int_{-\varepsilon}^{\varepsilon} dp_1 \cdots \int_{-\varepsilon}^{\varepsilon} dp_Q f_0(\bar{\xi}_\varepsilon(-t), \bar{\eta}_\varepsilon(-t)).
\]

Due to the Lipschitz continuity of \( f_0 \) we can assert that
\[
\tilde{h}_\varepsilon(x, v, t) = h_\varepsilon(x, v, t) + \varphi_2(x, v, t)
\]
where
\[
\sup_{x, v, t \in [0, T]} |\varphi_2(x, v, t)| \leq C\varepsilon^{1-2\alpha}T.
\]

For more details see [4], Section 3. As matter of facts, since we realize that \( h_\varepsilon \) is the solution of the following Boltzmann equation
\[
(\partial_t + v \cdot \nabla_x) h_\varepsilon(x, v, t) = \mathcal{L}_\varepsilon h_\varepsilon(x, v, t),
\]
where
\[
\mathcal{L}_\varepsilon h(v) = \mu \varepsilon^{-2\alpha|v|} \int_{-1}^1 d\rho [h(v') - h(v)],
\]
we have reduced the problem, thanks to Proposition 1, to the analysis of a Markov process which is an easier task.

4. Proof of the main theorem

Let be \( \eta_\varepsilon = |\log \varepsilon| \). We rewrite the linear Boltzmann equation (3.19) in the following way
\[
(\partial_t + v \cdot \nabla_x) h_\varepsilon(x, v, t) = \eta_\varepsilon \tilde{\mathcal{L}}_\varepsilon h_\varepsilon(x, v, t),
\]
where \( \tilde{\mathcal{L}}_\varepsilon = \mathcal{L}/\eta_\varepsilon \), namely
\[
\tilde{\mathcal{L}}_\varepsilon f(v) = \mu |v| \frac{\varepsilon^{-2\alpha}}{|\log \varepsilon|} \int_{-1}^1 d\rho [f(v') - f(v)], \quad f \in L^1(\mathbb{R}^2).
\]
We will show that for $\eta \to \infty$ we get a trivial result (Theorem 2.1 item 1), then we should look at the solution for times $\eta \epsilon t$, namely in the diffusive scaling. Denoting by $\tilde{h}_\epsilon := h_\epsilon(x,v,\eta \epsilon t)$, where $h_\epsilon$ solves (4.1), $\tilde{h}_\epsilon$ solves
\begin{equation}
(\partial_t + \eta \epsilon v \cdot \nabla_x)\tilde{h}_\epsilon = \eta^2 \tilde{L}_\epsilon \tilde{h}_\epsilon.
\end{equation}

It is convenient to introduce the Cauchy problem associated to the following rescaled Landau equation:
\begin{equation}
\begin{cases}
(\partial_t + \eta v \cdot \nabla_x) g_\eta(x,v,t) = \eta^2 \mathcal{L} g_\eta(x,v,t), \\
g_\eta(t = 0) = f_0,
\end{cases}
\end{equation}
where $\mathcal{L} = \frac{\mu}{2} |\nabla| \Delta_{S_{|v|}}$. We observe preliminarily that eq. (4.4) propagates the regularity of the derivatives with respect to the $x$ variable and, due to the presence of $\mathcal{L}$, gains regularity with respect to the transverse component of the velocity. Indeed, for any fixed $|v|$, denoting by $S_{|v|}$ the circle of radius $|v|$, under the hypothesis of Theorem 2.1 on $f_0$, the solution $g_\eta: \mathbb{R}^2 \times S_{|v|} \to \mathbb{R}^+$ satisfies the bounds
\begin{equation}
|D_x^k g_\eta(x,v)| \leq C, \quad |D_v^h g_\eta(x,v)| \leq C \quad \forall k \leq 2, \ h \geq 0,
\end{equation}
for all $t \in (0,T]$, where $C = C(f_0, T)$ and $D_v$ is the derivative with respect to the transverse component of the velocity. In particular, the solutions of (4.4) are considered classical.

Before analyzing the asymptotic behavior of the solution of (4.4) we first need a preliminary Lemma.

**Lemma 4.1.** Let $\langle g_\eta \rangle$ be the average of $g_\eta$ with respect to the invariant measure $\nu$, namely $\langle g_\eta \rangle := \frac{1}{2\pi |v|} \int_{S_{|v|}} dv \, g_\eta(x,v)$. Under the hypothesis of Theorem 2.1

1. $\eta - \langle g_\eta \rangle \to 0$ \(\eta \to \infty\) in $L^\infty((0,T]; L^2(\mathbb{R}^2 \times S_{|v|}))$.

Moreover, setting $t_\eta = \frac{1}{\eta^2}$ for $\omega > 2$ then

2. $g_\eta(t_\eta) - \langle f_0 \rangle \to 0$ \(\eta \to \infty\) in $L^2(\mathbb{R}^2 \times S_{|v|})$.

where $\langle f_0 \rangle = \frac{1}{2\pi |v|} \int_{S_{|v|}} dv \, f_0$.

**Proof.** Let $R_\eta = g_\eta - \langle g_\eta \rangle$. We have
\begin{equation}
(\partial_t + \eta v \cdot \nabla_x) R_\eta(x,v,t) = \eta^2 \mathcal{L} R_\eta(x,v,t) + \varphi,
\end{equation}
where
\begin{equation}
\varphi = - \left( \eta v \cdot \nabla_x \langle g_\eta \rangle + \partial_t \langle g_\eta \rangle \right)
\end{equation}
\begin{equation}
= \eta \left( \frac{1}{2\pi |v|} \int_{S_{|v|}} \frac{1}{|v|} \int_{S_{|v|}} v' \cdot \nabla_x g_\eta \, dv' - v \cdot \nabla_x \langle g_\eta \rangle \right).
\end{equation}

We can estimate the last quantity by (4.5): \(\sup_{t \leq T} \| \varphi \|_{L^2} \leq C T \| \nabla_x g_\eta \|_{L^2} \leq C T \eta\).

Therefore by (4.6) we have
\begin{equation}
\frac{1}{2} \| R_\eta(t) \|_{L^2}^2 = \eta^2 \langle R_\eta, \mathcal{L} R_\eta \rangle + (R_\eta, \varphi)
\leq -\eta^2 \lambda \| R_\eta \|_{L^2}^2 + \| R_\eta \|_{L^2} \| \varphi \|_{L^2}.
\end{equation}
where \( \lambda \) is the first positive eigenvalue of \( \mathcal{L} \). Hence we used that \( R_\eta \perp 1 \) in \( L^2 \). Hence
\[
\| R_\eta(t) \|_{L^2} \leq e^{-\eta^2 \lambda t} \| R_\eta(0) \|_{L^2} + \int_0^t ds \, e^{-\eta^2 \lambda (t-s)} \| \varphi(s) \|_{L^2} \leq e^{-\eta^2 \lambda t} \| R_\eta(0) \|_{L^2} + \frac{C}{\eta} (1 - e^{-\eta^2 \lambda t}),
\]
so that (1) is proven.

To prove (2) observe that, thanks to the fact \( \mathcal{L} \) is negative, we have
\[
\frac{1}{2} \frac{d}{dt} \| g_\eta(t) - f_0 \|_{L^2}^2 \leq -\eta (g_\eta - f_0, v \cdot \nabla_x f_0) + \eta^2 (g_\eta - f_0, \mathcal{L} f_0) \leq \| g_\eta - f_0 \|_{L^2} \eta |v| \| \nabla_x f_0 \| + \eta^2 \| \mathcal{L} f_0 \|.
\]
Therefore
\[
(4.8) \quad \| g_\eta(t_\eta) - f_0 \|_{L^2} \leq \frac{1}{\eta^2} \eta |v| \| \nabla_x f_0 \| + \eta^2 \| \mathcal{L} f_0 \|,
\]
which vanishes as \( \eta \to \infty \). Finally, recalling that \( \langle f_0 \rangle = \frac{1}{2\pi |v|} \rho_0 \), we have
\[
\| g_\eta(t_\eta) - \langle f_0 \rangle \|_{L^2} \leq \sup_{t \in (0,T]} \| g_\eta - \langle g_\eta \rangle \|_{L^2} + \| \langle g_\eta(t_\eta) \rangle - \langle f_0 \rangle \|_{L^2} \leq \sup_{t \in (0,T]} \| g_\eta - \langle g_\eta \rangle \|_{L^2} + c \| g_\eta(t_\eta) - f_0 \|_{L^2}.
\]
By (4.8) and (1) we conclude the proof.

### Lemma 4.2
Let \( g_\eta \) be the solution of (4.4). Under the hypothesis of Theorem 2.1 for the initial datum \( f_0 \), for \( \eta \to \infty \) \( g_\eta \) converges to the solution of the diffusion equation

\[
(4.9) \quad \begin{cases} \frac{\partial}{\partial t} \varrho = D \Delta \varrho \\ \varrho(x,0) = \langle f_0 \rangle, \end{cases}
\]
where \( \langle f_0 \rangle = \frac{1}{2\pi |v|} \int_{S|v|} dv f_0 \) and
\[
(4.10) \quad D = \frac{1}{\mu} \int_{S|v|} v \cdot \Delta^{-1} v dv.
\]
Convergence is in \( L^\infty([0,T]; L^2(\mathbb{R}^2 \times S|v|)) \).

**Proof.** The proof of the above Lemma is rather straightforward (see e.g. [7]).
Suppose for the moment that the initial datum depends only on the position variables, namely the initial datum has the form of a local equilibrium. We assume that \( g_\eta \) has the following form
\[
g_\eta(x,v,t) = g^{(0)}(x,t) + \frac{1}{\eta} g^{(1)}(x,v,t) + \frac{1}{\eta^2} g^{(2)}(x,v,t) + \frac{1}{\eta} R_\eta,
\]
where \( g^{(i)}, i = 0,1,2 \) are the first three coefficient of a Hilbert expansion in \( \eta \), and \( R_\eta \) is the reminder. Comparing terms of the same order in \( \eta \) we obtain the following...
equations:

\[(i)\ v \cdot \nabla_x g^{(0)} = \mu \frac{1}{2} \frac{1}{|v|} \Delta_{|s|} g^{(1)}\]

\[(ii)\ \partial_t g^{(0)} + v \cdot \nabla_x g^{(1)} = \mu \frac{1}{2} \frac{1}{|v|} \Delta_{|s|} g^{(2)}\]

\[(iii)\ \partial_t g^{(0)} + \eta v \cdot \nabla_x R_\eta = \eta^2 \mu \frac{1}{2} \frac{1}{|v|} \Delta_{|s|} R_\eta - A_\eta(t),\]

with \(A_\eta(t) = A_\eta(x, v, t) = \partial_t g^{(1)} + \frac{1}{\eta} g^{(2)} + v \cdot \nabla_x g^{(2)}\). Since \(v \cdot \nabla_x g^{(0)}\) is an odd function of \(v\), the integral with respect to \(v\) of the left hand side of (i) vanishes. Then we can invert the operator \(\Delta_{|s|}\) and set \(g^{(1)} = \frac{2}{\mu} |v| \Delta_{|s|}^{-1} v \cdot \nabla_x g^{(0)}\), where \(g^{(1)}\) is an odd function of the velocity. Now we integrate the second equation with respect to the velocity. By observing that \(\int_{S|v|} dv \Delta_{|s|} g^{(2)} = 0\), since \(dv_{|s|}\) is proportional the invariant measure, we obtain

\[\partial_t g^{(0)} + \frac{2}{\mu} |v| \int_{S|v|} dv v \cdot \nabla_x (\Delta_{|s|}^{-1} v \cdot \nabla_x g^{(0)}) = 0.\]

We define the \(2 \times 2\) matrix \(D\) as \(D_{ij} = -\frac{2}{\mu} |v| \int_{S|v|} v_i \Delta_{|s|}^{-1} v_j\) and we observe that \(D_{ij} = 0\) for \(i \neq j\) and \(D_{11} = D_{22} = D\), where \(D = \frac{1}{\mu} |v| \int_{S|v|} dv v \cdot (-\Delta_{|s|}^{-1}) v\).

Therefore

\[\partial_t g^{(0)} - D \Delta_x g^{(0)} = 0,\]

where \(g^{(0)}\) satisfies the initial condition \(g^{(0)}(x, 0) = g(t = 0)\). Moreover, the \(L^2\)-norm of \(g^{(1)}\) is bounded. If we show that also the \(L^2\)-norm of \(g^{(2)}\) and \(R_\eta\) are bounded, we deduce that \(g_\eta\) converges to \(g^{(0)}\) for \(\eta \to \infty\).

From equation (ii) and the diffusion equation for \(g^{(0)}\) we derive that the integral with respect to \(v\) of the left hand side of (ii) vanishes. Therefore we can invert the operator \(\Delta_{|s|}\) and obtain

\[g^{(2)} = \frac{2}{\mu} |v| \Delta_{|s|}^{-1} \left(\partial_t g^{(0)} + v \cdot \nabla_x (v \cdot \nabla_x g^{(0)})\right)\]

\[= \frac{2}{\mu} |v| \sum_{i,j} \partial_{x_i} \partial_{x_j} g^{(0)} \Delta_{|s|}^{-1} \left[v_i v_j + D_{ij}\right].\]

Therefore the \(L^2\)-norm of \(g^{(2)}\) is bounded.

We derive from equation (iii)

\[\frac{1}{2} \partial_t ||R_\eta||^2 = -\eta^2 \left(\partial_t R_\eta - \Delta_{|s|} R_\eta - (R_\eta, A_\eta(t))\right),\]

where \((\cdot, \cdot)\) denotes the scalar product in \(L^2\). Using positivity of \(-\Delta_{|s|}\) and Cauchy-Schwartz we deduce \(\partial_t ||R_\eta|| \leq ||A_\eta||\). Recall the explicit expression for \(A_\eta\), namely \(A_\eta = \partial_t g^{(1)} + \frac{1}{\eta} g^{(2)} + v \cdot \nabla_x g^{(2)}\). By direct computation

\[\partial_t g^{(1)} = \frac{2}{\mu} |v| \sum_{i,j,k} \partial_{x_i} \partial_{x_j} \partial_{x_k} g^{(0)} \left[\frac{2}{\mu} |v| v_i \Delta_{|s|}^{-1} v_j - (v_i v_j + D_{i,j})\right] \Delta_{|s|}^{-1} v_k,\]
from which we deduce that the $L^2$-norm of $\partial_t g^{(1)}$ is bounded. Similarly, one can easily show that the $L^2$-norm of $v \cdot \nabla_x g^{(2)}$ is bounded, and then $\|A_\eta\|$ is uniformly bounded in $[0, T]$ and $\|R_\eta\| \leq CT$.

To complete the proof we consider more general initial data $f_0$ depending also on the velocity variable. Let $A := L - \eta \nu \cdot \nabla_x$. We compare $g_\eta$ with $\bar{g}_\eta$, the solution (4.4) with initial datum $\langle f_0 \rangle$. By the same argument as in Lemma 4.1 item (2), we have that $\forall t \geq t_\eta$

$$\|\bar{g}_\eta(t - t_\eta) - \bar{g}_\eta(t)\|_{L^2} \leq \frac{C}{\nu^\rho - 2},$$

where $C$ depends on the $L^2$-norm of $\langle f_0 \rangle$ and $\nabla(f_0)$. Since $g_\eta(t) = e^{A(t-t_\eta)}g_\eta(t_\eta)$ and $\bar{g}_\eta(t - t_\eta) = e^{A(t-t_\eta)}\langle f_0 \rangle$ we derive

$$\|g_\eta(t) - \bar{g}_\eta(t - t_\eta)\|_{L^2} \leq C\|g(t_\eta) - \langle f_0 \rangle\|_{L^2}.$$

Thus, by Lemma 4.1 item (2), we obtain that $g_\eta(t)$ and $\bar{g}_\eta(t)$ have the same asymptotics and this concludes the proof of Lemma (4.2).

**Proposition 4.3.** Let $f_0$ be an initial datum for $\hat{h}_\varepsilon$ solution of (4.3). Under the hypothesis of Theorem 2.1 $\hat{h}_\varepsilon$ converges to $\varrho$ as $\varepsilon \to 0$, where $\varrho : \mathbb{R}^2 \times [0, T] \to \mathbb{R}_+$ is the solution of the diffusion equation

$$(4.11) \quad \begin{cases} \partial_t \varrho = D \Delta \varrho \\ \varrho(x, 0) = \langle f_0 \rangle, \end{cases}$$

with $\langle f_0 \rangle = \frac{1}{2\pi} \int_{|v|} f_0 \, dv$. The diffusion coefficient is $D$ given by the Green-Kubo formula. Convergence is in $L^2(\mathbb{R}^2 \times S_{|v|})$ uniformly in $t \in (0, T)$.

**Proof.** Let $g_{\eta_0}$ be solution of (4.4) with $\eta = \eta_0 := |\log \varepsilon|$ and initial condition $f_0$. We look at the evolution of $\hat{h}_\varepsilon - g_{\eta_0}$, namely

$$(\partial_t + \eta_0 \nu \cdot \nabla_x)(\hat{h}_\varepsilon - g_{\eta_0}) = \eta_0^2 \left( \tilde{L}_\varepsilon \hat{h}_\varepsilon - \mathcal{L} g_{\eta_0} \right),$$

where $\mathcal{L} := \frac{\mu}{2} \Delta_{|S_{|v|}}$. Then we obtain

$$\frac{1}{2} \partial_t \|\hat{h}_\varepsilon - g_{\eta_0}\|^2 = -\eta_0^2 \left( \hat{h}_\varepsilon - g_{\eta_0}, -\tilde{L}_\varepsilon \left[ \hat{h}_\varepsilon - g_{\eta_0} \right] \right) + \eta_0^2 \left( \hat{h}_\varepsilon - g_{\eta_0}, \left[ \tilde{L}_\varepsilon - \mathcal{L} \right] g_{\eta_0} \right),$$

from which, using positivity of $-\tilde{L}_\varepsilon$ and Cauchy-Schwartz,

$$\frac{1}{2} \partial_t \|\hat{h}_\varepsilon - g_{\eta_0}\| \leq \eta_0^2 \| \left( \tilde{L}_\varepsilon - \mathcal{L} \right) g_{\eta_0} \|.$$ 

Recalling that

$$\tilde{L}_\varepsilon g_{\eta_0} = \mu |v| \varepsilon^{-2\alpha} \int_{-1}^1 d\rho \left[ g_{\eta_0}(x, v', t) - g_{\eta_0}(x, v, t) \right],$$

we set

$$g_{\eta_0}(v') - g_{\eta_0}(v)$$

$$= (v' - v) \cdot \nabla_{|S_{|v|}} g_{\eta_0}(v)$$

$$+ \frac{1}{2} (v' - v)(v' - v) \nabla_{|S_{|v|}} g_{\eta_0}(v)$$

$$+ \frac{1}{6} (v' - v) \otimes (v' - v) \nabla_{|S_{|v|}} g_{\eta_0}(v) + R_{\eta_0},$$
with $R_{\eta_x} = O(|v - v'|^4)$. Integrating with respect to $v$ and using symmetry arguments we obtain
\[
\tilde{L}_x g_{\eta_x} = \mu |v| \frac{\varepsilon^{-2\alpha}}{\log \varepsilon} \left\{ \frac{1}{2} \Delta_{|v|} g_{\eta_x} \int_{-1}^{1} d\rho |v'|^2 + \int_{-1}^{1} d\rho R_{\eta_x} \right\}.
\]
Observe that $|v' - v|^2 = 4\sin^2 \frac{\theta}{2}$, then by direct computation (see Appendix)
\[
\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-2\alpha}}{\log \varepsilon} \int_{-1}^{1} d\rho |v' - v|^2 = 2 \frac{\alpha}{|v|^2}
\]
and
\[
\frac{\varepsilon^{-2\alpha}}{\log \varepsilon} \int_{-1}^{1} d\rho |v - v'|^4 = \varepsilon^{\alpha} |\log \varepsilon|^\beta, \quad -1 < \beta < \frac{5}{2} \alpha - 1.
\]
Therefore
\[
\left\| (\tilde{L}_x - \mathcal{L}) g_{\eta_x} \right\| \leq \varepsilon^{\alpha} |\log \varepsilon|^\beta \| \Delta_{|v|} g_{\eta_x} \| \leq \varepsilon^{\alpha} |\log \varepsilon|^\beta C,
\]
which vanishes for $\varepsilon \rightarrow 0$.

In order to complete the proof of the item 3) of Theorem 2.1, we need to show that $f_\varepsilon(\eta, t)$ converges to $h_\varepsilon(t)$ in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$, for every $t \in [0, T]$. By Proposition 3.2 and Remark 3.3 we have that $f_\varepsilon(\eta, t)$ defined in (3.13) converges to $h_\varepsilon(\eta, t)$, (3.14), in $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, for every $t \in [0, T]$. Moreover, using (3.18) and the fact that the initial datum has compact support, we have that $h_\varepsilon(\eta, t)$ converges to $\tilde{h}_\varepsilon(t)$ in $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, for every $t \in [0, T]$. Under hypothesis of Theorem 2.1 convergence in $L^1$ norm implies convergence in $L^2$. Since $f_\varepsilon \leq \tilde{f}_\varepsilon$ and using the fact that at $t = 0$ the equality holds and the linear Boltzmann equation 4.3 preserves the total mass, then also $f_\varepsilon(\eta, t)$ converges to $\tilde{h}_\varepsilon(t)$ in $L^2(\mathbb{R}^2 \times S_{|v|})$, for every $t \in [0, T]$.

Now we go back to equation (4.1). Using the same strategy of the proof of Proposition 4.3 we can replace $L_x$ with $\mathcal{L}$, and we denote $\tilde{g}_\eta$ the solution of
\[
(\partial_t + v \cdot \nabla_x) \tilde{g}_\eta = \eta \mathcal{L} \tilde{g}_\eta,
\]
with initial datum $f_0$. By the same arguments as in Lemma 4.1 item (i), one can prove that for $\eta \rightarrow \infty \tilde{g}_\eta \rightarrow \langle \tilde{g}_\eta \rangle$ and $\nabla_x \tilde{g}_\eta \rightarrow \nabla_x \langle \tilde{g}_\eta \rangle$. We observe that
\[
\partial_t \langle \tilde{g}_\eta \rangle + \nabla_x \int dv (\tilde{g}_\eta - \langle \tilde{g}_\eta \rangle) v = 0,
\]
therefore $\langle \tilde{g}_\eta \rangle$ converges to $\langle \tilde{f}_0 \rangle$ as $\eta \rightarrow \infty$, which concludes the proof of item 1).

Proof of item 2) is included in the proof of Proposition 4.3

5. THE CONTROL OF THE PATHOLOGICAL SETS

In this section we prove Proposition 3.2

Clearly

\[
(5.1) \quad 1 - \chi_1(1 - \chi_{ov})(1 - \chi_{rec})(1 - \chi_{int}) \leq (1 - \chi_1) + \chi_{ov} + \chi_{rec} + \chi_{int}
\]
and we estimate separately all the events in the right hand side of (5.1).

We denote by $\xi_\varepsilon(s), \eta_\varepsilon(s)$ the backward Markov process defined, for $s \in (-t, 0)$,
A similar argument can be used to estimate \( \chi \eta \), namely in Section 2 and we set

\[
\begin{align*}
\mathbb{E}^M_{x,v}(u) &= e^{-2|v|\varepsilon^{-2\alpha t}t} \sum_{Q \geq 0} \frac{(2|v|t)^Q}{Q!} \int_0^t dt_1 \cdots \int_0^{t_{Q-1}} dt_Q \\
&\int_{-\varepsilon}^{\varepsilon} d\rho_1 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho_Q u(\xi, \eta),
\end{align*}
\]

for any measurable function \( u \) of the process \( (\xi, \eta) \). We have

\[
\begin{align*}
\mathbb{E}^M_{x,v}((1 - \chi_1)f_0(\xi(-t), \eta(-t))) \\
&\leq \frac{2\varepsilon}{|v|} e^{-2|v|\varepsilon^{-2\alpha t}t} \sum_{Q > 0} (2|v|\varepsilon^{-2\alpha})^Q (Q - 1)! t^{Q-1} \\
&\leq 4\|f_0\|_{L^\infty} \varepsilon^{1-2\alpha} t \leq C \varepsilon^\gamma t,
\end{align*}
\]

for \( \gamma > 0, \alpha < 1/2 \) and \( \varepsilon \) sufficiently small.

Here and in the sequel \( t \) is allowed to behave as \( \varepsilon|\log(\varepsilon)| \).

Estimate (5.3) is obvious. Indeed if \( \chi_1 = 0 \) the first or the last collision must satisfy either \( |t - t_1| \leq 2\varepsilon/|v| \) or \( t_Q \leq 2\varepsilon/|v| \). Hence (5.3) follows easily.

A similar argument can be used to estimate \( \chi_{ov} \). Indeed if \( \chi_{ov} = 1 \) it must be \( t_i - t_{i+1} \leq 2\varepsilon/|v| \) for some \( i = 1, \ldots, (Q-1) \). Therefore proceeding as before

\[
\begin{align*}
\mathbb{E}^M_{x,v}(\chi_{ov}f_0(\xi(-t), \eta(-t))) \\
&\leq \frac{2\varepsilon}{|v|} e^{-2|v|\varepsilon^{-2\alpha t}t} \sum_{Q > 1} (Q - 1) (2|v|\varepsilon^{-2\alpha})^Q (Q - 1)! t^{Q-1} \\
&\leq 2\varepsilon \|f_0\|_{L^\infty} t (2|v|\varepsilon^{-2\alpha})^2 \leq C|v|\varepsilon^\gamma t,
\end{align*}
\]

for some \( \gamma > 0, \alpha < 1/4 \) and \( \varepsilon \) sufficiently small.

Next we pass to the control of the recollision event. We proceed similarly as in [4] and in [5]. Let \( t_i \) the first time the light particle hits the \( i \)-th scattering, \( \eta_i^- \) the incoming velocity, \( \eta_i^+ \) the outgoing velocity and \( t_i^- \) the exit time. Moreover we fix the axis in such a way that \( \eta_i^+ \) is parallel to the \( x \) axis (see figure 7). We have

\[
\chi_{rec} \leq \sum_{i=1}^{Q} \chi_{rec}^{i,j},
\]

where \( \chi_{rec}^{i,j} = 1 \) if and only if \( b_i \) (constructed via the sequence \( t_1, \rho_1, \ldots, t_i, \rho_i \)) is recollided in the time interval \( (t_j^-, t_{j+1}^-) \).

Note that, since \( |\theta_i| \leq C\varepsilon^{\alpha}, \) where \( \theta_i \) is the \( i \)-th scattering angle, in order to have a recollision it must be an intermediate velocity \( \eta_k, k = i + 1, \ldots, j - 1 \) such that

\[
|\eta_k^+ \cdot \eta_j^+| \leq C\varepsilon^\alpha |v|^2,
\]

namely \( \eta_k^+ \) is almost orthogonal to \( \eta_j^+ \) (see the figure). Then

\[
\chi_{rec} \leq \sum_{i=1}^{Q} \sum_{j=1}^{Q} \sum_{k=i+1}^{j-1} \chi_{rec}^{i,j,k},
\]
where \( \chi_{k}^{i,j,k} = 1 \) if and only if \( \chi_{i}^{j,k} = 1 \) and \( (5.6) \) is fulfilled.

Fix now all the parameters \( \rho_1, \ldots, \rho_Q, t_1, \ldots, t_Q \) but \( t_{k+1} \) and perform such a time integration. The two branches of the trajectory \( l_1, l_2 \) are rigid so that, if the recollision happen the time integration with respect to \( t_{k+1} \) is restricted to a time interval proportional to \( AB \). More precisely it is bounded by

\[
\frac{2\varepsilon}{|v| \cos \varepsilon \alpha} \leq \frac{2\varepsilon}{|v|}.
\]

Performing all the other integrations and summing over \( i, j, k \) we obtain

\[
\int_{\mathcal{B}(0,M)} f_0(\xi(-t), \eta(-t)) \chi_{\text{int}} dxdv.
\]

Figure 7.
Here $\chi_{\text{int}} = 1$ for those values of $x, v$ for which an interference takes place and $B(0, M) := \{(x, v) \in \mathbb{R}^2 \times \mathbb{R}^2; \text{s.t. } |x|^2 + |v|^2 < M\}$.

By the Liouville Theorem we can integrate over the variables $(\xi(-t), \eta(-t)) = (x_0, v_0)$ as independent variables

$$I = \int_{B_t(0, M)} f_0(x_0, v_0) \chi_{\text{rec}} \, dx_0 \, dv_0,$$

where

$$B_t(0, M) = \{(\xi(-t), \eta(-t)) \text{ s.t. } (x, v) \in B(0, M)\}.$$

Note that $\chi_{\text{int}}(x, v) = \chi_{\text{rec}}(x_0, v_0)$, since a backward interference is a forward recollision. Clearly

$$B_t(0, M) \subset B(0, M(1 + t))$$

where $B(0, M(1 + t))$ is the ball of radius $M(1 + t)$ in $\mathbb{R}^4$.

Thus

$$I \leq \int_{B(0, M(1 + t))} f_0(x_0, v_0) \chi_{\text{rec}} \, dx_0 \, dv_0.$$

Therefore, by using estimate (5.8) and (5.12)

$$\int_{B(0, M)} M^7_x (\chi_{\text{int}} f_0(\xi(-t), \eta(-t))) \, dx \, dv \leq C \varepsilon^7 M^7 (1 + t)^7.$$

This concludes the proof of Proposition 3.2.
6. Concluding Remarks

The diffusive limit analyzed in the present paper is suggested by the divergence of $\overline{B}$ for the particular choice of the potential we are considering. However the same techniques could work in presence of a smooth, radial, short-range potential $\phi$.

**Theorem 6.1.** Under the same hypothesis of Theorem 2.1, assume $\phi \in C^2([0,1])$. Scale the variables, the density and the potential according to

\[
\begin{align*}
  x & \to \varepsilon x \\
  t & \to \varepsilon^\lambda t \\
  \mu_x & = \varepsilon^{-(2\alpha+\lambda+1)} \mu \\
  \phi & \to \varepsilon^\alpha \phi.
\end{align*}
\]

Then, for $t > 0$ and $\varepsilon \to 0$, there exists $\lambda_0 = \lambda(\alpha)$ s.t. for $\lambda < \lambda(\alpha)$

\[f_\varepsilon(x,v,t) \to \rho(x,t)\]

solution of the heat equation

\[
\begin{align*}
  \partial_t \rho &= D \Delta \rho \\
  \rho(x,0) &= \langle f_0 \rangle,
\end{align*}
\]

with $D$ given by the Green-Kubo formula

\[D = \frac{1}{\mu |v|} \int_{S_1} v \cdot (- \Delta_{|S_1|}^{-1} v) \, dv.\]

The convergence is in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$.

The significance and the proof of the above theorem is clear. The kinetic regime describes the system for kinetic times $O(1)$. One can go further to diffusive times provided that $\lambda$ is not too large. Indeed the distribution function $f_\varepsilon$ “almost” solves

\[
\begin{align*}
  (\varepsilon^\lambda \partial_t + v \cdot \nabla_x) f_\varepsilon & \approx \varepsilon^{-2\alpha-\lambda} \rho \varepsilon f_\varepsilon \\
  & \approx \varepsilon^{-\lambda} \varepsilon \Delta_{|S_1|} f_\varepsilon,
\end{align*}
\]

for which the arguments of Section 3 do apply. In other words there is a scale of time for which the system diffuses. However such times are not so large to prevent the Markov property. Obviously the diffusion coefficient is computed in terms of the limiting Markov process. We can give an estimate, certainly not optimal, of the coefficient $\lambda$ appearing in (6.1). Estimating recollisions and interferences as in Section 4, setting $\gamma = 1 - 8(\alpha + \frac{1}{2})$, the condition on $\lambda$ is (see (5.13))

\[\gamma - 7\lambda > 0 \quad \text{i.e.} \quad \lambda < \frac{1 - 8\alpha}{11}.
\]

Although the scaling we are considering in Theorem 6.1 is quite particular, the aim is the same as in [6] where the same problem has been approached for the weak-coupling limit ($\alpha = \frac{1}{2}$) of a quantum system.

Recently we were aware of a result concerning the diffusion limit of a test particle of a hard-core system at thermal equilibrium [1]. Also in this case the quantitative control of the pathological trajectories allows to reach larger times in which a diffusive regime is outlined.
Acknowledgments.
We are indebted to S. Simonella and H. Spohn for illuminating discussions.

Appendix

A. Appendix (on the scattering problem associated to a circular potential barrier)

The potential energy for a finite potential barrier is given by

\[ \phi(r) = \begin{cases} \phi_0 & \text{if } r \leq 1 \\ 0 & \text{if } r > 1 \end{cases} \]

The light particle, of unitary mass, moves in a straight line with energy \( E = \frac{1}{2} v^2 > \phi_0 \). Let \( \rho \) be the impact parameter. For small impact parameters the particle will pass through the barrier, for large ones the particle will be reflected. Inside the barrier the velocity is a constant \( v = \bar{v} \) (\( \bar{v} < v \)). The complete trajectory of the light particle which passes through the barrier consists of three straight lines and is symmetrical about a radial line perpendicular to the interior path.

Let \( \alpha \) be the angle of incidence (the inside angle between the trajectory and a radial line to the point of contact with the barrier at \( r = 1 \)) and \( \beta \) the angle of refraction (the corresponding external angle). We assume that the radius of the circle is \( r = 1 \). According to the geometry of the problem \( \alpha \) and \( \beta \) are such that

\[ \sin \beta = \frac{v}{\bar{v}} \sin \alpha \]

where \( \sin \alpha = \rho \).

The angle of deflection is \( \theta = 2(\beta - \alpha) \). Thanks to the energy and angular momentum conservation the expression for the refractive index becomes

\[ n = \frac{\sin \alpha}{\sin \beta} = \frac{\bar{v}}{v} = \sqrt{1 - \frac{2\phi_0}{v^2}} \]

and so we have a scattering angle defined in the following way:

\[ \theta(\rho) = \begin{cases} 2(\arcsin \left( \frac{\rho}{n} \right) - \arcsin(\rho)) & \text{if } \rho \leq n \\ 2\arccos(\rho) & \text{if } \rho > n. \end{cases} \]

In the first case the particle passes through the barrier (for \( \rho \leq n \)), and in the second one the particle is reflected (for \( \rho > n \)). The maximum scattering angle \( \theta_{\text{max}} = 2\arccos(n) \) is the angle at which the particle scatters tangentially to the barrier. The differential scattering cross section

\[ \Psi(\theta) = \left| \frac{\partial \rho}{\partial \theta} \right| \]

is then:

\[ \Psi(\theta) = \begin{cases} \frac{n[\cos(\theta/2) - n][1-n \cos(\theta/2)]}{(1+n^2-2n \cos(\theta/2))^{3/2}} & \text{if } \theta \leq 2\arccos(n) \\ \frac{1 - \cos^2(\theta/2)}{1/2} & \text{otherwise} \end{cases} \]

Scaling now the potential as \( \phi(r) \to \epsilon^\alpha \phi(r) \), the previous formulas still hold. Thus, according to this scaling, the refractive index becomes,

\[ n_\epsilon = \sqrt{1 - \frac{2\epsilon^\alpha \phi_0}{v^2}} \]
to replace into (A.3), and (A.4).

**Appendix B. Appendix (on the diffusion coefficient)**

In this section we show that the diffusion coefficient is divergent for the circular potential barrier (A.1). At this level we assume that $\phi_0 = 1$ to simplify the following expressions.

We need to compute

\begin{equation}
\tilde{B} := \lim_{\varepsilon \to 0} \frac{\mu \varepsilon^{-2\alpha}}{2} |v| \int_{-1}^{1} \theta^2(\rho) \, d\rho.
\end{equation}

Thanks to the symmetry for the scattering problem

\begin{equation}
\varepsilon^{-2\alpha} \int_{-1}^{1} \theta^2(\rho) \, d\rho = 2 \varepsilon^{-2\alpha} \int_{0}^{1} \theta^2(\rho) \, d\rho.
\end{equation}

According to (A.3):

\begin{equation}
2 \varepsilon^{-2\alpha} \int_{0}^{1} \theta^2(\rho) \, d\rho = 2 \varepsilon^{-2\alpha} \left( \int_{0}^{n_{\varepsilon}(1-\delta)} \theta^2(\rho) \, d\rho + \int_{n_{\varepsilon}(1-\delta)}^{1} \theta^2(\rho) \, d\rho \right)
\end{equation}

Our aim is to perform a Taylor expansion of the first branch of $\theta(\rho)$ for $\rho \geq 0$, $\rho/n_{\varepsilon} < (1-\delta)$, with $\delta > 0$. We have

\[
\arcsin \left( \frac{\rho}{n_{\varepsilon}} \right) = \arcsin(\rho) + \frac{1}{\sqrt{1-\rho^2}} \left( \frac{\rho}{n_{\varepsilon}} - \rho \right) + R_1 \left( \frac{\rho}{n_{\varepsilon}} \right),
\]

where

\begin{equation}
R_1 \left( \frac{\rho}{n_{\varepsilon}} \right) = \frac{\bar{\rho}}{2(1-\rho^2)^{3/2}} \left( \frac{\rho}{n_{\varepsilon}} - \rho \right)^2 \rho < \bar{\rho} < \frac{\rho}{n_{\varepsilon}}.
\end{equation}

Then, looking at the first integral in the r.h.s of (B.3), we have to split it as

\[
\varepsilon^{-2\alpha} \int_{0}^{n_{\varepsilon}(1-\delta)} \theta^2(\rho) \, d\rho = \varepsilon^{-2\alpha} \int_{0}^{n_{\varepsilon}(1-\delta)} \theta^2(\rho) \, d\rho + \varepsilon^{-2\alpha} \int_{n_{\varepsilon}(1-\delta)}^{1} \theta^2(\rho) \, d\rho.
\]

Thus

\begin{equation}
A = \varepsilon^{-2\alpha} \int_{0}^{n_{\varepsilon}(1-\delta)} [2(\arcsin(\rho/n_{\varepsilon}) - \arcsin(\rho))]^2 \, d\rho
\end{equation}

\begin{equation}
\leq 4\varepsilon^{-2\alpha} \left[ \int_{0}^{n_{\varepsilon}(1-\delta)} \frac{1-n_{\varepsilon}}{n_{\varepsilon}^2} \left( \frac{\rho^2}{1-\rho^2} \right) \, d\rho + \int_{0}^{n_{\varepsilon}(1-\delta)} R_1 (\rho/n_{\varepsilon})^2 \, d\rho \right] +
\end{equation}

\begin{equation}
4\varepsilon^{-2\alpha} \left[ \left( \int_{0}^{n_{\varepsilon}(1-\delta)} R_1 (\rho/n_{\varepsilon}) \right)^{1/2} \left( \int_{0}^{n_{\varepsilon}(1-\delta)} \frac{1-n_{\varepsilon}}{n_{\varepsilon}^2} \frac{\rho^2}{1-\rho^2} \, d\rho \right)^{1/2} \right].
\end{equation}

It is sufficient to compute the first two integrals. Let $A_1$ and $A_2$ be the first and the second integrals respectively. We have

\begin{equation}
A_1 = \varepsilon^{-2\alpha} (1-n_{\varepsilon})^2 \int_{0}^{n_{\varepsilon}(1-\delta)} \frac{\rho^2}{1-\rho^2} \, d\rho
\end{equation}

\begin{equation}
= -\frac{\varepsilon^{-2\alpha} (1-n_{\varepsilon})^2}{2} \left[ 2n_{\varepsilon}(1-\delta) + \log(1-n_{\varepsilon}(1-\delta)) - \log(1+n_{\varepsilon}(1-\delta)) \right].
\end{equation}
Using that \( n = 1 - \frac{\varepsilon}{|v|^2} + o(\varepsilon^2) \), from (B.6) it is clear that

\[
A_1 \approx -\frac{\varepsilon^{-2\alpha}}{2} (1 - n_\varepsilon)^2 (\log(1 - n_\varepsilon (1 - \delta))) = -\frac{\varepsilon^{-2\alpha}}{2} \left( \frac{\varepsilon^{2\alpha}}{|v|^4} \right) \log(\varepsilon^\alpha (1 - \delta) + \varepsilon). 
\]

A straightforward computation shows that the right hand side of the previous expression is

\[
A_1 \approx -\frac{\varepsilon^{-2\alpha}}{2} \left( \log(\varepsilon^\alpha) + \log(1 - \delta) \right) 
= -\frac{\varepsilon^{2\alpha}}{2 |v|^4} \left( \log(\varepsilon^\alpha) + \delta \right)
= \frac{\alpha}{2 |v|^4} |\log(\varepsilon)| \left( 1 + \frac{\delta (1 - \frac{1}{\varepsilon^\alpha})}{|\log(\varepsilon^\alpha)|} \right).
\]

Choosing \( \delta = \frac{\varepsilon^\alpha}{|\log(\varepsilon)|^2} \) with \( \gamma \in (0, \alpha/2) \), it follows \( \delta/\varepsilon^\alpha \to 0 \).

In order to compute \( A_2 \), we need the following estimate for the remainder term

\[
|R_1 (\rho/n_\varepsilon)| \leq \frac{1}{2 n_\varepsilon} \frac{1}{1 - \rho^2/n_\varepsilon^2} \left( \frac{\rho}{n_\varepsilon} - \rho \right)^2.
\]

Then

\[
A_2 \leq \varepsilon^{-2\alpha} \int_0^{n_\varepsilon (1 - \delta)} \frac{\rho^2}{n_\varepsilon^2 (1 - \rho^2/n_\varepsilon^2)^2} \left( \frac{\rho}{n_\varepsilon} - \rho \right)^4 d\rho
\]

\[
\approx \varepsilon^{-2\alpha} n_\varepsilon \int_0^{1 - \delta} \frac{u^2}{2 (1 - u^2)^3} u^4 (1 - n_\varepsilon) d\rho
\approx \varepsilon^{-2\alpha} n_\varepsilon \int_0^{1 - \delta} \frac{(1 - v)^6}{v^3} (1 - n_\varepsilon) d\rho \approx \frac{\varepsilon^{-2\alpha} n_\varepsilon (1 - n_\varepsilon)^4}{2 \delta^2}.
\]

Also in this case, the only significant contribution is given by

\[
\frac{\varepsilon^{-2\alpha} (1 - n_\varepsilon)^4}{\delta^2} \sim \frac{\varepsilon^{-2\alpha \varepsilon^{4\alpha}}}{\delta^2} \varepsilon^{\alpha} \to 0
\]

again for \( \delta = \frac{\varepsilon^\alpha}{|\log(\varepsilon)|^2} \) with \( \gamma \in (0, \alpha/2) \). This shows that

\[ A = A_1 (1 + O(\varepsilon)). \]

Now we compute \( B \) in (B.3), namely

\[
B = \varepsilon^{-2\alpha} \int_{n_\varepsilon (1 - \delta)}^{n_\varepsilon} [2(\arcsin(\rho/n_\varepsilon) - \arcsin(\rho))]^2 d\rho
\]

\[
\approx \varepsilon^{-2\alpha} \int_{n_\varepsilon (1 - \delta)}^{n_\varepsilon} \left( \int_0^{\rho/n_\varepsilon} dx - \frac{1}{\sqrt{1 - x^2}} \right)^2 d\rho.
\]
Since
\[ \int_{\rho}^{1} dx \frac{1}{\sqrt{1-x^2}} = \int_{\rho}^{1} dx \frac{1}{\sqrt{(1-x)(1+x)}} \]
\[ \leq \frac{1}{1+\rho} \int_{\rho}^{1} dx \frac{1}{\sqrt{(1-x)}} \approx \int_{1-\rho}^{1} \frac{1}{\sqrt{u}} \]
\[ = \frac{1}{\sqrt{(1-\rho)}} \left( \sqrt{(1-\rho)} - \sqrt{(1-\rho/n)} \right) \]
in (B.10) we have
\[ B = \varepsilon^{-2\alpha} \int_{n \varepsilon (1-\delta)}^{n \varepsilon} \frac{1}{(1-\rho)} \left( \sqrt{(1-\rho)} - \sqrt{(1-\rho/n)} \right)^2 \, d\rho \]
\[ \leq \varepsilon^{-2\alpha} \int_{n \varepsilon (1-\delta)}^{n \varepsilon} (\rho/n - \rho) \, d\rho \]
\[ = \varepsilon^{-2\alpha} n \varepsilon (1 - n \varepsilon)(1 - (1-\delta)^2) \approx \varepsilon^{-2\alpha} \varepsilon^{2\alpha} \delta. \]
Again, with the previous choice for \( \delta \), this term vanishes in the limit for \( \varepsilon \to 0 \).

The second integral in the right hand side of (B.3) reads
\[ \varepsilon^{-2\alpha} \int_{n \varepsilon}^{1} \theta^2(\rho) \, d\rho = \varepsilon^{-2\alpha} \int_{n \varepsilon}^{1} (\pi - 2 \arcsin(\rho))^2 \, d\rho \]
\[ \approx \varepsilon^{-2\alpha} (1 - n \varepsilon)^2 \approx \varepsilon^{-2\alpha} \varepsilon^{2\alpha} \frac{1}{|v|^2} = \frac{1}{|v|^2}. \]
Therefore the only contribution in the limit is the one given by (B.7) and we obtain
\[ \tilde{B} := \lim_{\varepsilon \to 0} \frac{\mu \varepsilon^{-2\alpha}}{2 |v|} \int_{-1}^{1} \theta^2(\rho) \, d\rho = \lim_{\varepsilon \to 0} \mu \left[ \frac{2\alpha}{|v|^3} \log(\varepsilon) \right] = +\infty, \]
and finally
\[ B := \lim_{\varepsilon \to 0} \frac{\mu \varepsilon^{-2\alpha}}{2 |v| \log \varepsilon} \int_{-1}^{1} \theta^2(\rho) \, d\rho = \frac{2\alpha}{|v|^7} \mu. \]

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