Colloidal Dynamics on a Choreographic Time Crystal

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A choreographic time crystal is a dynamic lattice structure in which the points comprising the lattice move in a coordinated fashion. These structures were initially proposed for understanding the motion of synchronized satellite swarms. Using simulations we examine colloids interacting with a choreographic crystal consisting of traps that could be created optically. As a function of the trap strength, speed, and colloidal filling fraction, we identify a series of phases including states where the colloids organize into a dynamic chiral loop lattice as well as a frustrated induced liquid state and a choreographic lattice state. We show that transitions between these states can be understood in terms of vertex frustration effects that occur during a certain portion of the choreographic cycle. Our results can be generalized to a broader class of systems of particles coupled to choreographic structures, such as vortices, ions, cold atoms, and soft matter systems.

Crystalline states arise throughout nature and are characterized by their symmetries. Since these structures are static in time, they can be described by a single snapshot. Recently there have been proposals for dynamic crystals containing points that move in a synchronized fashion such that a single time snapshot does not reveal all the symmetries in the system. These structures are called choreographic crystals \[1\], and they are composed of a collection of points that undergo a series of repeated moves to form varying patterns that recur over time. A related idea is that in some such systems, the ground states themselves are also periodic in time, forming what are called time crystals \[2\,3\]. Generally, time crystal systems must be driven out of equilibrium and contain some form of dissipation, so they are not in a true ground state. Nevertheless, there is growing interest in creating and studying the properties of classical \[2\,6\] and quantum \[7\,9\] time and choreographic crystals in condensed matter, atomic, and even cosmological systems \[10\,11\]. Choreographic crystals represent a new type of structure and there are many open questions, including how to realize these states, what their properties are, and whether they could be coupled to other systems.

Here we examine a system of dynamic traps that form a choreographic crystal coupled to an assembly of colloidal particles. There have been many studies of colloidal trapping on static crystalline substrates \[12\,19\] or quasiperiodic lattices \[20\,21\], where melting and commensurate–incommensurate transitions were observed. Studies of the dynamics of colloids driven over such crystalline substrates have revealed locking of the motion of the colloids with a symmetry direction of the substrate lattice \[22\,26\], depinning of kinks and antikinks at incommensurate versus commensurate states \[27\,30\], and a diverse array of other dynamical phenomena \[31\,32\]. It is even possible to dynamically control and move individual traps \[33\] or to flash the traps on and off \[34\,35\], so with appropriate rules for translation, it should be feasible to create a choreographic lattice of optical traps that couple to colloidal particles. Beyond colloids, optical trapping lattices have been created for cold atom systems \[36\,37\], ions \[38\], vortices in Bose–Einstein condensates \[39\], and vortices in type-II superconductors \[40\], and therefore similar choreographic lattices could be created for these systems. Choreographic trap arrays thus represent a new type of lattice for studies of commensuration effects and dynamics.

In our simulations, we find three generic phases of colloidal dynamics depending on the strength and speed of the traps as well as the filling fraction or ratio of the number of colloids to the number of traps. In the weakly coupled regime, the colloids are temporarily trapped and organize into a dynamical chiral loop crystal. In the partially coupled regime, where a given colloid is dragged by a trap for a varied length of time before decoupling from the trap, a liquid like state appears. In the strong coupling regime, the colloids are permanently locked to the traps and themselves form a choreographic crystal. At higher filling fractions, we observe phases in which traps containing multiple colloids interact with interstitial colloids in the regions between traps. The transitions between the dynamical states are affected by the trap velocity since the colloids decouple from the traps at high trap velocities, and we map out a dynamic phase diagram as a function of trap strength and velocity. We also show that the transition into and out of the liquid phase is the result of a vertex frustration effect, similar to that found in triangular artificial spin ice \[41\,43\], which appears during the portion of the choreographic cycle when the spacing between the traps reaches its minimum value.

\textbf{Simulation}—We conduct simulations of point-like colloidal particles in a two dimensional box of size $L \times L \sqrt{3}/2$ with $L = 96.0$ where there are periodic boundary conditions in the $x$ and $y$ directions. The sample contains...
\[ N_{\text{trap}} = 576 \text{ trapping sites of radius } R_{\text{trap}} = 1.0 \] which are initially arranged in a 24 \times 24 hexagonal lattice with lattice constant \( a = 4.0 \), large enough to ensure that traps never overlap when they are translated. To create the choreographic crystal, we use the rules for motion introduced in Ref. \[1\]. The traps are divided into three subsets, as shown schematically in Fig. 1(a). Each subset of traps moves in a different direction defined by the vector \((x, y)\) which has the value \((-0.5, -\sqrt{3}/2)\) for the red traps, \((1, 0)\) for the blue traps, and \((-0.5, +\sqrt{3}/2)\) for the green traps, as indicated by the arrows. Under this motion, the traps never overlap, and they reform the original triangular lattice ordering shown in (a) every \(\tau\) time units. (b) Image of the trap positions at the point in the cycle where the spacing between the traps reaches its smallest value.

The sample contains \( N_c \) colloidal particles and we characterize the filling fraction as \( f = N_c/N_{\text{trap}} \). The dynamical evolution of the colloids is given by the following overdamped equation of motion:

\[ \frac{1}{\eta} \frac{\Delta r_{ij}}{\Delta t} = F_{pp}^i + F_{\text{trap}}^i \]

where \( \eta = 1 \) is the viscosity. The interaction between two charged colloidal particles \( i \) and \( j \) at a distance of \( r_{ij} \) is given by a screened Coulomb interaction, \( F_{pp}^{ij} = \exp(-r/r_0)F_{ij}/r^2 \), where \( r_0 = 4.0 \) is the screening length. The interaction between colloid \( i \) and trap \( k \) is given by a simple finite range harmonic spring, \( F_{\text{trap}} = (F_{\text{trap}}/R_{\text{trap}})\hat{r}_{ik} \), where \( F_{\text{trap}} \) is the maximum force at the edge of the trap and \( r_{ik} \) is the distance between the colloid and the center of the trap. The colloids are initialized at random locations with a specified minimum possible spacing between adjacent colloids. The traps are then set into motion and the system eventually settles into a steady state.

**Results**—We first consider the weak coupling regime with \( F_{\text{trap}} = 0.4 \) and \( v_{\text{trap}} = 0.5 \) at a filling of \( f = 1.0 \),...
where individual colloids can be trapped for a short time but move a distance less than a trap lattice constant. In Fig. 2(a) we illustrate the colloid and trap locations at the beginning of the simulation when the colloid positions are disordered. After several cycles of trap motion, the colloids organize into the crystalline state shown in Fig. 2(b), where the diffusion constant drops to zero and the colloids move in a nonoverlapping pattern of counterclockwise triangular loops, a state that we term a dynamic chiral lattice (DCL). The size of the colloidal orbits decreases with decreasing trap strength and falls to zero when $F_{\text{trap}} = 0$, where the colloids form a static hexagonal lattice. At $F_{\text{trap}} = 0.7$, shown in Fig. 2(c), each trap permanently captures one colloid [44]. The image of the trajectories of some of the colloids in Fig. 2(d) indicates that each colloid follows a straight line path. Here the traps are strong enough to overcome the colloid-colloid repulsive force even when the traps reach their point of closest approach, so the colloids themselves form a choreographic lattice (ChL). Although there is no net drift motion averaged over all of the colloids, individual colloids undergo ballistic motion so the diffusion constant obeys $D \propto t$ in the ChL phase. When $F_{\text{trap}}$ is increased further, we observe the same structure and dynamics.

At intermediate trapping strengths between the DCL and ChL states, the system forms a partially coupled or disordered state in which each colloid is dragged by a trap over a distance of several lattice constants before it becomes dislodged. We show a snapshot of this state in Fig. 2(e) for $F_{\text{trap}} = 0.54$, where the colloidal positions are disordered in the steady state. The corresponding trajectories of some of the colloids in Fig. 2(f) show that there is short time ballistic behavior when the colloids are dragged; however, the longer time behavior is diffusive with $D \propto \sqrt{t}$.

The mechanism that triggers the transition between the DCL and ChL state can be understood by considering the portion of the cycle in which the traps are closest together, as shown in Fig. 1(b). We can think of this structure in terms of a vertex picture, similar to that found in triangular colloidal spin ice systems [41–43]. A trap that is occupied is the equivalent of having a particle close to the vertex in one of the three arms of the triangular colloidal spin ice vertex. Vertex states are labeled by the total number of particles that are close to the vertex, giving 0-in, 1-in, 2-in, or 3-in vertices. The ice rule obeying state contains 1-in and 2-in vertices. For example, two distinct 2-in vertex configurations are illustrated in Fig. 3(c,d). The resulting frustration at intermediate trap strength prevents the colloids from reaching a repeatable ordered state, since each time the traps reach their point of closest approach, a different energy-equivalent ice-rule-obeying colloid configuration can appear. This produces a disordered structure. In a system with longer range interactions, where particles can interact over a distance spanning multiple vertices, the ice degeneracy could be lifted, causing new types of time repeated dynamical states to occur. It is also possible that other types of choreographic crystals would not have the same frustration effects during any
FIG. 4. (a) The ratio of the average colloid velocity to the trap velocity, $v_c/v_{trap}$, vs $v_{trap}$ for the system in Fig. 2 with $f = 1$ and $F_{trap} = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3$, and $1.4$, from bottom to top. The ChL phase appears when $v_c/v_{trap} = 1.0$, the DCL phase has $v_c/v_{trap} < 1/3$, and over the remaining range of $v_c/v_{trap}$ the system is in the frustrated liquid state. As the trap speed increases, the system can pass through all three states. (b) The dynamic phase diagram as a function of $F_{trap}$ versus $v_{trap}$ based on a height field plot of $v_c/v_{trap}$ with dashed lines marking the boundaries between the ChL, DCL, and intermediate frustrated liquid states. (c) The average number $n/N_{trap}$ of colloids in each trap vs $F_{trap}$ for $v_{trap} = 0.5$ and filling fractions of $f = 0.5, 0.75, 1.0, 1.25, 1.5, 2.0, 2.5, 3.0, 3.5$, and $4.0$, from bottom to top. (d) The dynamic phase diagram as a function of $F_{trap}$ versus $n/N_{trap}$ based on a height field plot of $n/(fN_{trap})$.

portion of the cycle or that choreographic crystals could exist that are frustrated during the entire cycle.

Transitions between the states can occur as a function of trap speed as well as trap strength, since the coupling between the colloids and the traps becomes weaker as the trap speed increases. In Fig. 4(a) we plot the ratio of the velocities of the colloids $v_c/v_{trap} = N_c^{-1} \sum_i^N |v_i|/v_{trap}$ versus $v_{trap}$ for the system in Fig. 2. When $v_c/v_{trap} = 1.0$, all the colloids are trapped and moving at the same velocity as the traps in the ChL state. For $v_c/v_{trap} < 1/3$, the colloids are only temporarily trapped and the DCL state appears, while for $0.3 < v_c/v_{trap} < 0.9$, the system is in the disordered state. We find a plateau near $v_c/v_{trap} = 0.33$ corresponding to a prevalence of 1-in-3 states with 1/3 of the traps occupied. Even in the DCL state, $v_c/v_{trap}$ is always larger than zero since the traps are occupied for at least a short period of time. As $F_{trap}$ increases, the onset of the ChL phase shifts to lower values of $v_{trap}$. In Fig. 4(b) we plot a dynamic phase diagram as a function of $F_{trap}$ vs $v_{trap}$ for the system in Fig. 4(a) indicating the regions where the DCL, ChL, and disordered phases occur.

We have also explored the effect of changing the filling $f$. We find the same general features described above when $f \leq 1$, while interstitial colloids begin to appear when $f > 1$. If the trap strength is large enough, however, all the colloids can eventually be trapped and the ChL phase appears with clusters of colloids at each trap. The time-averaged number of trapped colloids is given by $n$, so the average number of colloids in each trap is $n/N_{trap}$. When the traps are not strong enough to capture an average of $n/N_{trap} = f$ colloids apiece, more complex states can appear in which multiply occupied traps coexist with interstitial colloids. In general, the region over which the disordered phase appears expands as $f$ increases. In Fig. 4(c) we plot $n/N_{trap}$ versus $F_{trap}$ for varied filling. A series of plateaus occur at $n/N_{trap} = 1/f$ when the traps are strong enough to capture all the colloids with no interstitial colloids present. In Fig. 4(d) we plot a dynamic phase diagram as a function of $F_{trap}$ versus $n/N_{trap}$ highlighting the strongly coupled regime or ChL phase, the weakly coupled DCL regime, and the intermediate regime consisting of disordered states in which multiply trapped colloids can coexist with interstitial colloids.

Summary—We have examined a choreographic lattice of traps that move in a synchronized fashion without overlap. When the traps are coupled to an assembly of colloidal particles with repulsive Yukawa interactions, we observe several different dynamical regimes: a dynamically ordered chiral crystal state in which the colloids are temporarily trapped, follow loop orbits, and have zero net diffusion; a strongly coupled state in which the colloids themselves form a choreographic lattice with ballistic diffusion; and an intermediate frustrated liquid state with long time diffusive behavior. The emergence of the different states can be understood in terms of a mapping of the closest approach of the traps to a triangular colloidal spin ice vertex state. At intermediate coupling, multiple vertex states with equivalent energies are possible, resulting in frustration similar to that found in the triangular colloidal spin ice, which produces a disordered configuration. Our results could be generalized to a wide variety of different types of choreographic time crystals with dynamical substrates, and represent a new particle assembly-periodic substrate system in which commensuration effects, dynamic phases, and melting can be explored using optical traps or other methods to create a translating trap array. Beyond colloids, similar results should appear for vortices, cold atoms, and ions coupled to a choreographic lattice.

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[44] Supplementary movies are available at the following links: Dynamic chiral lattice at $F_{\text{trap}} = 0.4$, https://youtu.be/-39jaja_b4c; Choreographic lattice at $F_{\text{trap}} = 0.7$, https://youtu.be/YUWqG39CnJ0; Frustrated liquid state at $F_{\text{trap}} = 0.54$, https://youtu.be/9y_sDoJdN_E.