Abstract

In this early preliminary report on an ongoing project, we present – to the best of our knowledge – the first study of completeness thresholds for memory safety proofs. Specifically we consider heap-manipulating programs that iterate over arrays without allocating or freeing memory. We present the first notion of completeness thresholds for program verification which reduce unbounded memory safety proofs to bounded ones. Moreover, we present some preliminary ideas on how completeness thresholds can be computed for concrete programs.
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1 Introduction

In this early preliminary report on an ongoing project, we present – to the best of our knowledge – the first study of completeness thresholds for memory safety proofs. Specifically we consider heap-manipulating programs that iterate over arrays without allocating or freeing memory. We present the first notion of completeness thresholds for program verification which reduce unbounded memory safety proofs to bounded ones. Moreover, we present some preliminary ideas on how completeness thresholds can be computed for concrete programs.

Unbounded vs Bounded Proofs Memory safety is a very basic property we want to hold for every critical program, regardless of its nature or purpose. Yet, it remains hard to prove and in general requires us to write tedious, inductive proofs. One way to automate the verification process is to settle on bounded proofs and accept bounded guarantees.

Consider a program $c$ that searches through an array of size $s$. An unbounded memory safety proof for $c$ would yield that the program is safe for any possible input, in particular for any array size, i.e., $\forall s. \text{safe}(c)$. A bounded proof that only considers input sizes smaller than 10 would only guarantees that the program is safe for any such bounded array, i.e., $\forall s < 10. \text{safe}(c)$.

Completeness Thresholds Approximating unbounded proofs by bounded ones is a technique often used in model checking. Hence, the relationship between bounded and unbounded proofs about finite state transition systems has been studied extensively [3, 5, 7, 11, 6, 2, 8]. For a finite transition system $T$ and a property of interest $\phi$, a completeness threshold is any number $k$ such that we can prove $\phi$ by only examining path prefixes of length $k$ in $T$, i.e., $T \models_k \phi \Rightarrow T \models \phi$ [5]¹.

¹Note that the term completeness threshold is used inconsistently in literature. Some papers such as [5] use the definition above, according to which completeness thresholds are not unique. Others such as [7] define them as the minimal number $k$ such that $T \models_k \phi \Rightarrow T \models \phi$, which makes them unique.
over-approximations of least completeness thresholds for different types of properties $\phi$. These over-approximations are typically described in terms of key attributes of the transition system $T$, such as the recurrence diameter $[7]$.

Heap-manipulating programs are essentially infinite state transition systems. Hence, in general, these key attributes are infinite. This vast structural difference between the programs we are interested in and the transition systems for which completeness thresholds have been studied prevents us from reusing any of the existing definitions or results.

In §2 we start by presenting basic definitions and notations that are used throughout this work. In §3 and §4 we present the syntax and semantics of the programming language we consider. In §5 and §6 we formalise our assertion language as well as the notion of memory safety we consider. Our study of completeness thresholds relies on verification conditions, which we define in §7. In §8 we present an intuitive treatment of completeness thresholds before we study them formally in §9. Finally, we describe our envisioned roadmap for this work in progress in §10.

## 2 General Notation and Basic Definitions

The following definitions and notations will be used throughout this work.

**Definition 2.1 (Tuples).** For any set $X$ we denote the set of tuples over $X$ as

$$X^* := \bigcup_{n \in \mathbb{N}} \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in X\}.$$  

We denote tuples by overlining the variable name, i.e., $\bar{x} \in X^*$, except if it is clear from the context.

For any set $X$ and for any tuple $\bar{t} = (x_1, \ldots, x_n) \in X^*$, we denote the tuple’s length by $\operatorname{len}(\bar{t}) = n$.

We define the following notation for appending a single element $e \in X$ to a tuple:

$$\bar{t} \cdot e := (x_1, \ldots, x_n, e).$$

**Definition 2.2 (Non-Empty Tuples).** For any set $X$ we denote the set of non-empty tuples over $X$ as

$$X^+ := \{t \in X^* \mid \operatorname{len}(t) > 0\}.$$  

**Definition 2.3 (Disjoint Union).** Let $A, B$ be sets. We define their disjoint union as

$$A \sqcup B := A \cup B$$

if $A \cap B = \emptyset$ and leave it undefined otherwise.
Definition 2.4 (Equivalence Classes). Let $R \subseteq X \times X$ be an equivalence relation. For any $x \in X$, we define the notation

$$\langle x \rangle_R := \{x' \in X \mid R(x, x')\} \in X/R.$$ 

We omit $R$ whenever it is clear from the context and write $\langle x \rangle$ instead of $\langle x \rangle_R$.

Notation 2.5 (Homomorphic & Isomorphic). Let $A, B$ be algebraic structures. We define the following notations:

- $A \sim B$ expresses that $A$ and $B$ are homomorphic.
- $A \simeq B$ expresses that $A$ and $B$ are isomorphic.

Remark 2.6 (Canonical Homomorphism from Tuples to Sets). Let $X$ be a set. Then, $X^*$ with concatenation and $\mathcal{P}(X)$ with union are both monoids and the canonical homomorphism from $X^*$ to $\mathcal{P}(X)$ is $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$.

Definition 2.7 (Congruence Relation between Tuples and Sets). Let $X$ be a set and let $h_X : X^* \to \mathcal{P}(X)$ be the canonical homomorphism. We define the congruence relation $\cong_X \subseteq (X^* \times \mathcal{P}(X)) \cup (\mathcal{P}(X) \times X^*)$ such that the following holds for all $t \in X^*$ and all $S \in \mathcal{P}(X)$:

$$S \cong_X t \iff t \cong_X S \iff h_X(t) = S.$$ 

Whenever the base set $X$ is clear from the context, we write $\cong$ instead of $\cong_X$. 
3 Syntax

In this section we define the syntax of the programming language that we use in the rest of this work.

Definition 3.1 (Variables). We define $\mathcal{X}$ to be an infinite set of variable symbols.

Our language allows for simple pointer arithmetic of the form $l +_{\mathcal{L}\mathbb{Z}} z$ where $l$ is a heap location and $z$ is an offset.

Definition 3.2 (Heap Locations). We define the set of heap locations $\mathcal{L}$ to be an infinite set.

We define $+_{\mathcal{L}\mathbb{Z}} : \mathcal{L} \times \mathbb{Z} \to \mathcal{L}$ to have the following properties:

- $+_{\mathcal{L}\mathbb{Z}}$ is right-associative.
- $l +_{\mathcal{L}\mathbb{Z}} 0 = l$ holds for all $l \in \mathcal{L}$.
- $l +_{\mathcal{L}\mathbb{Z}} i \neq l$ holds for all $l \in \mathcal{L}$, $i \in \mathbb{Z} \setminus \{0\}$.

When the meaning is clear from the context, we write $+$ instead of $+_{\mathcal{L}\mathbb{Z}}$.

Definition 3.3 (Unit Type). We define the unit type as $\text{Unit} := \{()\}$.

Definition 3.4 (Types). We define the set of types $\text{Types}$ syntactically as follows:

$$ T \in \text{Types} := \mathcal{L} | \mathbb{Z} | \mathbb{B} | \text{Unit} $$

Operations are pure functions that map inputs to output and cannot access the heap.

Definition 3.5 (Operations). We define $\text{Ops}$ to be a set of operations with $\text{Ops} \subseteq \{ f : D \to C \mid D \in \text{Types}^*, C \in \text{Types} \}$ and with $+_{\mathcal{L}\mathbb{Z}}$, $\prec$, $+ \in \text{Ops}$.

For each $f : D \to C \in \text{Ops}$, we use the following notation $\text{dom}(f) = D$ and $\text{codom}(f) = C$.

While this work currently only deals with arrays, our plan is to investigate completeness thresholds for programs that deal with arbitrary tree-like inductive data structures. In order to keep the semantics of our language modular and to allow for easy extensions, we introduce an unspecified set of heap commands that captures the APIs of the data structures we are interested in.

Definition 3.6 (Heap Commands). We define $\text{HCmds}$ to be a set of symbols. Every $hc \in \text{HCmds}$ represents a command that accesses the heap.

Definition 3.7 (Program syntax). We define the set of commands $\text{Cmds}$, expressions $\text{Exps}$ and values $\mathcal{V}$ syntactically by the grammar presented in Fig. 1.
\( l \in \mathcal{L} \)  
\( z \in \mathcal{Z} \)  
\( b \in \mathbb{B} \)  
\( x \in \mathcal{X} \)  
\( \text{Heap locations} \)

\( \text{Variables} \)

\( op \in Ops \quad \ni \quad +_{\mathcal{L}\mathcal{Z}} \)  
\( hc \in HCmds \)  
\( \text{Primitive Operations} \)

\( \text{Heap Commands} \)

\( v \in \mathcal{V} \quad ::= \quad l \mid z \mid b \mid () \)  
\( e \in \text{Exps} \quad ::= \quad v \mid x \mid op(\overline{e}) \)  
\( \text{Values} \)

\( \text{Expressions} \)

\( c \in \text{Cmds} \quad ::= \quad e \mid \text{let } x := c \text{ in } c \mid \)  
\( \text{if } e \text{ then } c \text{ else } c \mid \)  
\( \text{while } !e \text{ do } c \mid \)  
\( \text{for } x \text{ in } (e \text{ to } e) \text{ do } c \mid \)  
\( !e \mid !e := e \mid hc(\overline{e}) \)  
\( \text{Heap access} \)

(a) Values, expressions and commands.

\( c; c' \quad ::= \quad \text{let } x := c \text{ in } c' \quad \text{where } x \text{ is not free in } c' \)  
\( \text{if } e \text{ then } c \quad ::= \quad \text{if } e \text{ then } c \text{ else } () \)

(b) Syntactic sugar.

Figure 1: Program syntax.
4 Dynamic Semantics

In order to keep things simple, our language uses a heap but no store. Hence, variables are actually constants that can be bound to values via let commands. As a consequence, the entire evaluation state of a program is represented by the heap and the program itself. Further, expressions are pure, hence their evaluation does not depend on the heap.

**Definition 4.1** (Evaluation of Closed Expressions). We define a partial evaluation function $\llbracket \cdot \rrbracket : \text{Exps} \rightarrow \mathcal{V}$ on expressions by recursion on the structure of expressions as follows:

\[
\llbracket v \rrbracket := v \quad \text{if} \quad v \in \mathcal{V},
\]

\[
\llbracket \text{op}(e_1, \ldots, e_n) \rrbracket := \text{op}(\llbracket e_1 \rrbracket, \ldots, \llbracket e_n \rrbracket) \quad \text{if} \quad \bot \notin \{\llbracket e_1 \rrbracket, \ldots, \llbracket e_n \rrbracket\}
\]

\[
\quad \text{and} \quad (\llbracket e_1 \rrbracket, \ldots, \llbracket e_n \rrbracket) \in \text{dom}(\text{op}),
\]

\[
\llbracket e \rrbracket := \bot \quad \text{otherwise}.
\]

We identify closed expressions $e$ with their ascribed value $\llbracket e \rrbracket$.

**Definition 4.2** (Evaluation Context). We define the set of evaluation contexts $\text{EvalCtxs}$ syntactically as follows:

\[
E \in \text{EvalCtxs} ::= \text{let } x := \square \text{ in } c
\]

For any $c \in \text{Cmds}$ and $E \in \text{EvalCtxs}$, we define $E[c] := E[\square \mapsto c]$.

**Definition 4.3** (Free Variables (Commands)). We define free variables in the usual way. For any command $c$ we denote the set of variables that occur freely in $c$ by $\text{freeVars}(c)$.

**Definition 4.4** (Substitution (Commands)). We define substitution in the usual way. For any command $c$, variable $x$ and expression $e$, we denote the result of substituting every free occurrence of $x$ in $c$ with $e$ by $c[x \mapsto e]$. Further, we extend substitutions to tuples of variables and expressions in the canonical way.

We explicitly model memory errors in our operational semantics. This way we know that (i) any execution which ends in a value does not involve memory errors and (ii) any execution that does involve memory errors ends in the dedicated error state $\text{error}$.

**Definition 4.5** (Memory Errors). We denote the memory error state by $\text{error}$ and the set of potentially erroneous commands by $\text{Cmds}^+ := \text{Cmds} \cup \{\text{error}\}$. We denote potentially erroneous commands by $c^+$.
Heaps are finite collections of resources that can be manipulated by commands.

**Definition 4.6** (Physical Resources). We define the set of physical resources $\mathcal{R}^{\text{phys}}$ syntactically as follows:

$$r^p \in \mathcal{R}^{\text{phys}} ::= l \mapsto v$$

$$l \in \mathcal{L} \quad v \in \mathcal{V}$$

**Definition 4.7** (Physical Heaps). We define the set of physical heaps as

$$\text{Heaps}^{\text{phys}} := \mathcal{P}_{\text{fin}}(\mathcal{R}^{\text{phys}})$$

and the function $\text{locs}_{p\text{Res}} : \text{Heaps}^{\text{phys}} \to \mathcal{P}_{\text{fin}}(\mathcal{L})$ mapping physical heaps to the sets of allocated heap locations as

$$\text{locs}_{p\text{Res}}(h) := \{l \in \mathcal{L} \mid \exists v \in \mathcal{V}. l \mapsto v \in h\}.$$  

We denote physical heaps by $h$.

**Definition 4.8** (Basic Commands). We define the set of basic commands $\text{BCmds}$ syntactically as follows:

$$bc \in \text{BCmds} ::= e \mid \text{let } x := bc \text{ in } bc \mid \text{if } e \text{ then } bc \text{ else } bc \mid \text{while } !e \text{ do } bc \mid \text{for } x \text{ in } (e \text{ to } e) \text{ do } bc \mid !e \mid !e := e$$

The set of basic commands is the subset of $\text{Cmds}$ that consists exactly of those commands that do not involve any heap command call $hc(\overline{e})$. Remember that the set of heap commands captures the APIs of data structures. We do not want to change our operational semantics each time we want to consider a new data structure. Hence, we assume that there exists an interpretation for each heap command that describes its behaviour in terms of basic commands.

**Assumption 4.9** (Heap Command Interpretation). We assume that there exists a function $\mathcal{I}_{\text{hcmds}} : \text{HCmds} \to (\text{BCmds} \times X^*)$ that maps each heap command to a basic command and a vector of variables. Further, for every mapping of the form $\mathcal{I}_{\text{hcmds}}(hc) = (bc, (x_1, \ldots, x_n))$ the following two properties hold:
\[ \bigwedge_{i \neq j} x_i \neq x_j \]

- freeVars(bc) \subseteq \{x_1, \ldots, x_n\}.

**Definition 4.10** (Command Reduction Relation). We define a command reduction relation \( \sim_{\text{cmd}} \) according to the rules presented in Fig. 2. A reduction step has the form

\[ h, c \sim_{\text{cmd}} h', c'. \]

We define \( \sim^{\ast}_{\text{cmd}} \) as the reflexive transitive closure of \( \sim_{\text{cmd}} \).

5 Assertion Language

In the previous sections we introduced heap commands that capture the APIs of data structures. In a similar way, we introduce heap predicates that describe their memory layout.

**Assumption 5.1** (Heap Predicates). We assume that there is a set of symbols \( P \). Every \( p \in P \) represents a predicate characterising the heap. Further, we assume that there is a function \( I_{\text{hpreds}} : P \to \mathcal{P}(\text{Heaps}^{\text{phys}} \times \mathcal{V}^*) \) that maps each heap predicate symbol to a predicate over heaps and value tuples.

**Definition 5.2** (Assertions). We define the set of assertions \( \mathcal{A} \) according to the syntax presented in Figure 3.

We omit the index set \( I \) in quantifications when its choice becomes clear from the context and write \( \exists i. a(i) \) and \( \forall i. a(i) \) instead of \( \exists i \in I. a(i) \) and \( \forall i \in I. a(i) \),

**Definition 5.3.** Asserttions Model Relation We define the assertion model relation \( \models_{\mathcal{A}} \subseteq \text{Heaps}^{\text{phys}} \times \mathcal{A} \) by recursion over the structure of assertions according to the rules presented in Fig. 4.

**Definition 5.4** (Free Variables (Assertions)). We define the notion of free variables for assertions analogously to that of commands (cf. 4.3).

**Definition 5.5** (Substitution (Assertions)). We define substitution for assertions analogously to substitution for commands (cf. 4.4).

**Notation 5.6.** Free Variables of Tuples For convenience we define the following notation for any tuple of commands and assertions \( (y_1, \ldots, y_n) \in (\text{Cmds} \cup \mathcal{A})^* \):

\[
\text{freeVars}(y_1, \ldots, y_n) := \text{freeVars}(y_1) \cup \cdots \cup \text{freeVars}(y_n).
\]
| Rule Name                  | Description                                                                 |
|---------------------------|-----------------------------------------------------------------------------|
| `CmdRed-EvalCtxt`          | \[ h, c \leadsto_{cmd} h', c' \]                                           |
| `CmdRed-EvalCtxt-Fail`     | \[ h, c \leadsto_{cmd} h', \text{error} \]                                 |
| `CmdRed-IfTrue`            | \[ h, \text{if True then } c_t \text{ else } c_f \leadsto_{cmd} h, c_t \] |
| `CmdRed-IfFalse`           | \[ h, \text{if False then } c_t \text{ else } c_f \leadsto_{cmd} h, c_f \] |
| `CmdRed-While`             | x \not\in \text{freeVars}(c)                                               |
| `CmdRed-For`               | \[ h, \text{for } x \text{ in } (n \text{ to } n') \text{ do } c \leadsto_{cmd} h, c \] |
| `CmdRed-Let`               | \[ h, \text{let } x := v \text{ in } c \leadsto_{cmd} h, c[x \mapsto v] \] |
| `CmdRed-Desugar-HeapCmdCall` | \[ I_{\text{cmds}}(hc) = (bc, (x_1, \ldots, x_n)) \] |

Figure 2: Command reduction rules.
\[ e \in \text{Exps} \]
\[ p \in P \]
\[ A \subseteq A \]
index set \( I \subseteq \mathbb{Z} \)

\[
a \in A := \text{True} \mid \text{False} \mid e \mid \neg a \mid a \land a \mid a \lor a \mid a \ast a \mid e \mapsto e \mid p(e) \mid \bigvee A
\]

(a) Assertion syntax.

\[
a_1 \rightarrow a_2 := \neg a_1 \lor a_2
\]
\[
a_1 \leftrightarrow a_2 := (a_1 \rightarrow a_2) \land (a_2 \rightarrow a_1)
\]
\[
\exists i \in I. a(i) := \bigvee \{a(i) \mid i \in I\}
\]
\[
\forall i \in I. a(i) := \neg \exists i \in I. \neg a(i)
\]

(b) Syntactic sugar.

Figure 3: Assertions.

\[
\begin{align*}
h \models_A \text{True} \\
h \not\models_A \text{False} \\
h \models_A e & \iff \emptyset \models_A [e] \\
h \models_A \neg a & \iff h \not\models_A a \\
h \models A a_1 \land a_2 & \iff h \models A a_1 \land h \models A a_2 \\
h \models A a_1 \lor a_2 & \iff h \models A a_1 \lor h \models A a_2 \\
h \models A a_1 \ast a_2 & \iff \exists h_1, h_2. h = h_1 \sqcup h_2 \\& h_1 \models A a_1 \land h_2 \models A a_2 \\
h \models A l \mapsto v & \iff l \mapsto v \in h \\
h \models_A p(e_1, \ldots, e_n) & \iff \mathcal{I}_{\text{preds}}(p)(h, [e_1], \ldots, [e_n]) \\
h \models A \bigvee A & \iff \exists a \in A. h \models_A a
\end{align*}
\]

Figure 4: Assertion model relation. We write \( h \not\models_A a \) if \( h \models A a \) does not hold.
Definition 5.7 (Validity). Let $a \in A$ be an assertion with $\text{freeVars}(a) \equiv \overline{x} = (x_i)_i$. For each $i = 1, \ldots, n$, let $T_i$ be the type of variables $x_i$ and let $\overline{T} = (T_i)_i$. We call assertion $a$ valid if the following holds:

$$\forall h. \forall \overline{v} \in \overline{T}. h \models_A a[\overline{x} \mapsto \overline{v}]$$

We denote validity of $a$ by writing $\models_A a$.

6 Memory Safety

Definition 6.1 (Memory Safety of Commands and Heaps). We define the safety relation for commands $\text{safe} \subseteq \text{Heaps}^{\text{phys}} \times \text{Cmds}$ as follows:

Let $\overline{x} = (x_i)_i \equiv \text{freeVars}(c)$ be the variables occurring freely in $c$. For each $i$, let $T_i$ be the type of variable $x_i$ and let $\overline{T} = (T_i)_i$. Then,

$$\text{safe}(h, c) \iff \forall \overline{v} \in \overline{T}. \neg \exists h'. h, c[\overline{x} \mapsto \overline{v}], \rightsquigarrow_{\text{cmd}} h', \text{error}$$

We say that a command $c$ is safe under a physical heap $h$ if $\text{safe}(h, c)$ holds.

We consider a command safe under a heap if its execution does not lead to a memory error. Note that a command’s execution can get stuck without any memory error occurring. Such cases arise for not-well-typed commands such as if 13 then . . . . For this work, we only consider well-typed programs. Hence, we do not care about cases in which a program gets stuck as long as no memory error occurs.

Definition 6.2 (Memory Safety of Commands and Assertions). We define the safety relation for commands $\text{safe}_A \subseteq A \times \text{Cmds}$ as follows:

Let $\overline{x} = (x_i)_i \equiv \text{freeVars}(a, c)$ be the variables occurring freely in $a$ and $c$. For each $i$, let $T_i$ be the type of variable $x_i$ and let $\overline{T} = (T_i)_i$. Then,

$$\text{safe}_A(a, c) \iff \forall \overline{v} \in \overline{T}. \forall h. (h \models_A a[\overline{x} \mapsto \overline{v}] \Rightarrow \text{safe}(h, c[\overline{x} \mapsto \overline{v}]))$$

We say that a command $c$ is safe under an assertion $a$ if $\text{safe}_A(a, c)$ holds.

Notation 6.3. We denote preconditions by $M$. Further, we aggregate preconditions $M$ and programs $c$ into tuples $\{M\} c$.

7 Verification Conditions

A common approach in program verification is to derive a verification condition $vc$ from the program $c$ and correctness property $\phi$ in question. Instead
of verifying the program directly, we prove the verification condition. In

general, \( vc \) describes an over-approximation of all possible program behaviours.
The process is sound iff truth of the verification condition indeed implies that
our program is correct, i.e., \( \models vc \Rightarrow c \models \phi \). We proceed analogously during
our study of completeness thresholds.

**Definition 7.1** (Verification Condition). We call an assertion \( vc \in A \) a
verification condition for \( \{M\} c \) if the following holds:

\[
\models_A vc \Rightarrow \text{safe}_A(M, c)
\]

We denote verification conditions by \( vc \).

**Definition 7.2** (Precise Verification Conditions). Let \( vc \) be a verification
condition for \( \{M\} c \) and let \( x \in \text{freeVars}(M, c) \) be a free variable of type \( T \).
We call \( vc \) precise in \( x \) for \( \{M\} c \) if the following holds for every value \( v \in T \):

\[
\text{safe}_A(M[x \mapsto v], c[x \mapsto v]) \Rightarrow \models_A vc[x \mapsto v]
\]

**Assumption 7.3** (Array Predicate). We assume that there exists a predicate
symbol \( \text{array} \in P \). We further assume that \( I_{\text{hpreds}}(\text{array}) \subseteq \mathcal{L} \times \mathbb{N} \) is the
minimal relation for which the following holds:

\[
I_{\text{hpreds}}(\text{array})(h, a, s) \iff h \models_A \bigwedge_{0 \leq i < s} (a + \lfloor i \rfloor) \mapsto -
\]

We identify the assertion \( \text{array}(a, s) \) with the assertion \( \bigwedge_{0 \leq i < s} (a + \lfloor i \rfloor) \mapsto - \).

8 Intuition behind Completeness Thresholds

Before we dive into the formal definitions and lemmas about completeness
thresholds let’s look at a concrete example. The following listing shows a
program family \( p_z \) where each instance iterates through an array \( a \) of size \( s \).
The program sums up all array elements and writes the result to a heap
location \( r \). Whether or not a memory error occurs depends on the offset \( z \).

The example shows that we can reduce the original verification problem
to a simpler one by studying the program family’s verification condition. In
particular, we analyse the formula and extract a completeness threshold that
bounds the domain we have to consider in a memory safety proof for \( p_z \).
Example 8.1 (Summing up array elements). Consider the precondition \( M := \) \( \text{array}(a, s) \ast r \mapsto \_ \) the constant \( z \in \mathbb{Z} \) and the program (family) \( p_z \):

\[
\text{for } i \text{ in } (0 \text{ to } s) \text{ do (}
\begin{align*}
\text{let } e & := a[i + z] \text{ in} \\
\text{let } r' & := !r \text{ in} \\
!r & := r' + e
\end{align*}
\)
\]

Intuitively, the verification condition for memory safety of \( \{ M \} p_z \) should be equivalent to

\[
\forall a, s, r. M \to (\forall i. 0 \leq i < s \to vcc(a[i + z]) \land vcc(!r))
\]

\[
\equiv \forall a, s, r. \text{array}(a, s) \ast r \mapsto \_ \\
\to (\forall i. 0 \leq i < s \to (\exists s'. \text{array}(a, s') \land 0 \leq i + z < s') \land r \mapsto \_)
\]

\[
=: \forall s. V_0(s)
\]

By making the formula stronger, we can simplify it to the following:

\[
\forall a, s, r. \text{array}(a, s) \ast r \mapsto \_ \\
\to (\forall i. 0 \leq i < s \to (\text{array}(a, s) \land 0 \leq i + z < s) \land r \mapsto \_)
\]

\[
\equiv \forall s. \forall i. 0 \leq i < s \to 0 \leq i + z < s
\]

\[
=: \forall s. V_1(s)
\]

It holds for \( n \in \{0, 1\} \)

\[
\vdash \forall s. V_n(s) \Rightarrow \text{safe}_A(M, p_z).
\]

A completeness threshold for \( s \) is any non-empty restriction \( \emptyset \neq R \subseteq \mathbb{Z} \) of the domain of \( s \) with the following property:

\[
\vdash \forall s \in R. V_n(s) \Rightarrow \text{safe}_A(M, p_z).
\]

Intuitively, a completeness threshold \( R \) for \( s \) means that we can prove memory safety of \( p \) by only considering the array sizes in \( R \) instead of all possible sizes.

Under the assumption \( s > 0 \), we simplify the verification condition further:

\[
\forall s > 0. V_1(s) \equiv \forall s > 0. \forall i. 0 \leq i < s \to 0 \leq i + z < s \equiv z = 0
\]

We see that as long as \( s > 0 \), memory safety does not depend on the concrete value of \( s \). Hence, we are free to choose any strictly positive size as representative for this case. Let’s choose 1, by which we get the completeness threshold \( \{ y \mid s \leq 1 \} \) for \( s \). It holds

\[
\vdash \forall s \leq 1. V_1(s) \Rightarrow \text{safe}_A(M, p_z).
\]
9 Completeness Thresholds

Now that we have an intuition for what a completeness threshold should be and for how we want to use it, let’s formalise this intuition.

**Definition 9.1** (Completeness Thresholds for Quantified Assertions). Let $Q \in \{\forall, \exists\}$ be a quantifier and let $Qx \in T. a$ be a quantified assertion. Further, let $R \subseteq T$ be a restriction of the domain of $x$. We call $R$ a completeness threshold for $a$ if the following holds:

$$
\models A Qx \in R. a \quad \Rightarrow \quad \models A Qx \in T. a.
$$

**Definition 9.2** (Completeness Thresholds for Programs). Let $T$ be the type of $x$ in $\{M\} c$ and let $R \subseteq T$ be a restriction of this type. We call $R$ a completeness threshold for $x$ in $\{M\} c$ if the following holds:

$$
\forall v \in R. \text{safe}_A (M[x \mapsto v], c[x \mapsto v]) \quad \Rightarrow \quad \forall v \in T. \text{safe}_A (M[x \mapsto v], c[x \mapsto v])
$$

**Lemma 9.3** (Precision). Let $\forall x \in T. vc$ be a verification condition for $\{M\} c$ and let $R \subseteq T$ be a completeness threshold for $vc$. Further, let $vc$ be precise in $x$ for $\{M\} c$. Then $R$ is a completeness threshold for $\{M\} c$.

**Proof.** Assume that $\forall v \in R. \text{safe}_A (M[x \mapsto v], c[x \mapsto v])$ holds. We have to prove that the program is safe for the unrestricted domain, i.e., $\text{safe}_A(M, c)$. Together with the precision of $vc$ in $x$, this assumption implies $\forall v \in R. \models A vc[x \mapsto v]$ (cf. Def. 7.2), which is equivalent to $\models A \forall x \in R. vc$. $R$ is a completeness threshold for $vc$. According to Def. 9.1 this means $\models A \forall x \in T. vc$. Since the latter is a verification condition for $\{M\} c$, proposition $\text{safe}_A(M, c)$ holds by Def. 7.1. Hence, $R$ is a completeness threshold for $\{M\} c$. \qed

Finding completeness threshold for a non-precise verification condition $\forall x. vc$ does not allow us to conclude that we found a completeness threshold for the actual program $\{M\} c$. However, as long as our ultimate goal is to verify the program by proving an assertion $A$ that is at least as strong as our verification condition, i.e., $A \Rightarrow \forall x. vc$, we can leverage the completeness threshold. Hence, it makes sense to first concentrate on completeness thresholds for verification conditions. Later, we can try to relate our results to completeness thresholds for programs.

**Lemma 9.4.** Let $\forall x \in T. a$ be an assertion. Let $R \subseteq T$ be such that

$$
\forall v_1, v_2 \in R. \ (\models_A a[x \mapsto v_1] \quad \Leftrightarrow \quad \models_A a[x \mapsto v_2]).
$$

Then, for every $r \in R$ it holds that

$$
\models_A \forall x \in (T \setminus R) \cup \{r\}. a \quad \Leftrightarrow \quad \models_A \forall x \in T. a.
$$
Proof. Let \( r \in R \). With the assumption from the lemma, we get
\[
\forall v \in R. \ (\models_A a[x \mapsto r] \iff \models_A a[x \mapsto v])
\]
and hence
\[
\models_A a[x \mapsto r] \iff \models_A \forall x \in R. a
\]
Further, \( \forall x \in T. a \) is valid iff both \( \forall x \in (T \setminus R). a \) and \( \forall x \in R. a \) are valid. As we can reduce validity of the latter to validity of \( a[x \mapsto r] \), we get
\[
(\models_A \forall x \in (T \setminus R). a) \land (\models_A a[x \mapsto r]) \iff (\models_A \forall x \in (T \setminus R). a) \land (\models_A \forall x \in R. a) \iff \models_A \forall x \in T. a.
\]

\( \square \)

Corollary 9.5. Let \( \forall x \in T. a \) be an assertion. Let \( R \subseteq T \) be such that
\[
\forall v_1, v_2 \in R. \ (\models_A a[x \mapsto v_1] \iff \models_A a[x \mapsto v_2]).
\]

Then, for every \( r \in R \), the set \((T \setminus R) \cup \{r\}\) is a completeness threshold for assertion \( a \).

Proof. Follows from Lem. 9.4 and Def. 9.1. \( \square \)

Consider a verification condition \( \forall x \in T. vc \) and suppose we are interested in a completeness threshold for \( x \). By definition, the threshold is a restriction of \( x \)'s domain, i.e., \( R \subseteq T \). The lemmas and corollary above show us that one way forward is to identify a validity-preserving subset of \( T \). That is, we need to look for a subset \( R \subseteq T \) of the domain within which the concrete choice for \( x \) does not affect the validity of the verification condition. Once we got this, we can collapse \( R \) to any representative \( r \in R \) and we found our completeness threshold \((T \setminus R) \cup \{r\}\).

Notice that validity preservation of domain restrictions is a transitive property. Hence, we can easily turn the search for completeness thresholds for a fixed variable into an iterative approach.

Lemma 9.6 (Transitivity of Completeness Thresholds for Fixed Variable). Let \( R_0, R_1, R_2 \) be sets with \( R_2 \subseteq R_1 \subseteq R_0 \). Let \( a_i = \forall x \in R_i. a \) be assertions. Let \( R_1 \) and \( R_2 \) be completeness thresholds for \( x \) in \( a_0 \) and \( a_1 \), respectively. Then, \( R_2 \) is also a completeness threshold for \( x \) in \( a_0 \).

Proof. Since \( R_1 \) is a completeness threshold for \( x \) in \( a_0 = \forall x \in R_0. a \), we get
\[
\models_A \forall x \in R_1. a \Rightarrow \models_A \forall x \in R_0. a.
\]
Since $R_2$ is a completeness threshold for $x$ in $a_1 = \forall x \in R_1. a$, we get
\[ \models_A \forall x \in R_2. a \implies \models_A \forall x \in R_1. a. \]
That is,
\[ \models_A \forall x \in R_2. a \implies \models_A \forall x \in R_1. a \implies \models_A \forall x \in R_0. a \]
and hence $R_2 \subseteq R_0$ is a completeness threshold for $x$ in $a_0 = \forall x \in R_0. a$. □

**Corollary 9.7.** Let $T$ be a type and let $(R_i)_i$ be a family of sets with $R_0 = T$ and $R_{i+1} \subseteq R_i$. Let $(a_i)_i = (\forall x \in R_i. a)_i$ be a family of assertions such that each $R_{i+1}$ is a completeness threshold for $x$ in $a_i$. Then, each $R_i$ is a completeness threshold for $x$ in $a_0 = \forall x \in T. a$.

**Proof.** Follows from Lem. 9.6 by induction. □

Consider a program that traverses an array $a$ of size $s_a$ and an array $b$ of size $s_b$. When we analyse the verification condition of this program we find that it contains distinct parts that describe memory safety of the accesses to array $a$ and distinct parts for the accesses to $b$. Since both arrays describe separate parts of the heap, we can bring the verification condition into a form that reflects this. Thereby, we get a formula of the form $vc \equiv vc_a \ast vc_b$ where $vc_a$ and $vc_b$ describe memory safety in respect to $a$ and $b$, respectively.

Suppose, we want to find a completeness threshold for $s_a$. In some cases, the manipulation of both arrays is entangled which means that $s_a$ potentially affects the validity of $vc_b$. In such a case, we have no choice but to analyse the entire formula to find our completeness threshold. However, often that’s not the case and $s_a$ only shows up in the subformula $vc_a$ that actually concerns array $a$. In this case, it is sufficient to analyse $vc_a$ in order to find a completeness threshold for $s_a$.

**Lemma 9.8** (Elimination). Let $a, a_x, a'_x$ be assertions with $\forall x \in T. a \equiv \forall x \in T. a_x \ast a'_x$. Suppose the choice of $x$ does not affect the validity of $a'_x$, i.e.,
\[ \forall v \in T. \ (\models_A a'_x \iff \models_A a'_x[x \mapsto v]). \]
Let $R \subseteq T$ be a completeness threshold for $x$ in $\forall x \in T. a_x$. Then, $R$ is also a completeness threshold for $x$ in $\forall x \in T. a$.

**Proof.** Since $R$ is a completeness threshold for $x$ in $\forall x \in R. a_x$, we get
\[
\begin{align*}
\models_A \forall x \in R. a_x & \ast a'_x \\
\implies \models_A (\forall x \in R. a_x) \ast (\forall x \in R. a'_x) \\
\implies \models_A (\forall x \in T. a_x) \ast (\forall x \in R. a'_x).
\end{align*}
\]

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By using the assumption that the choice of \( x \) does not affect the validity of \( a' \) we can conclude
\[
\models_A (\forall x \in T. a_x) \ast (\forall x \in R. a') \\
\Rightarrow \models_A (\forall x \in T. a_x) \ast (\forall x \in T. a') \\
\Rightarrow \models_A \forall x \in T. a_x \ast a'.
\]

**Corollary 9.9.** Let \( a, a_x, a' \) be assertions with \( \forall x \in T. a \equiv \forall x \in T. a_x \ast a' \). Suppose \( x \) is not free in \( a' \). Let \( R \subseteq T \) be a completeness threshold for \( x \) in \( \forall x \in T. a_x \). Then, \( R \) is also a completeness threshold for \( x \) in \( \forall x \in T. a \).

**Proof.** Follows from Lem. 9.8.

### 9.1 Iteratively Extracting Completeness Thresholds

What we saw so far, gives us the tools to define an iterative process to extract completeness thresholds.

**Workflow 9.10.** Let \( \{M\} \) be a program with variables \( x_1, \ldots, x_n \) for which we would like to extract completeness thresholds.

1. Compute a verification condition for \( \{M\} \), e.g., by using weakest preconditions. The result has the form \( \forall \tau \in \overline{T}. vc \).

2. Iteratively extract completeness thresholds for each \( x_i \). For all \( i \in \{1, \ldots, n\} \):

   (a) Let \( \overline{R} \subseteq \overline{T} \) be the completeness thresholds extracted so far. (Initially \( \overline{R} = \overline{T} \).)

   (b) Bring the verification condition into the form
        \[
        \forall \tau \in \overline{R}. vc \equiv \forall \tau \in \overline{R}. \bigwedge_{0 \leq j \leq m} vc_j.
        \]

   (c) Identify a subformula \( vc' \) whose validity is not affected by the choice of \( x_i \). Bring the verification condition into the form
        \[
        \forall \tau \in \overline{R}. vc \equiv \forall \tau \in \overline{R}. vc_i \ast vc'.
        \]

   In the remaining steps it suffices to analyse \( \forall \tau \in \overline{R}. vc_i \).

   (d) Examine \( vc_i \) and extract a completeness threshold. This can either be done purely manually or by identifying patterns for which we previously proved that we can extract completeness thresholds. This step yields a completeness threshold \( R'_i \) which, in the worst case, does not yield an improvement, i.e., \( R'_i = R_i \).

   (e) Repeat this process iteratively until the extracted completeness threshold does not improve in respect to the last iteration.
9.2 Iterating over Arrays

In the following, we study patterns encountered in verification conditions of programs that iterate over arrays. The goal of this section is to formulate reusable lemmas that allow us to automate the extraction of completeness thresholds.

Lemma 9.11 (VC-CT for Bounded, Unconditional Array Access). Let $z, a, b \in \mathbb{Z}$ be constants and $vc(s) = \forall i \in \mathbb{Z}. a \leq i < s + b \rightarrow 0 \leq i + z < s$. Then, for every $r \in \mathbb{Z}$ with $r > a - b$ it holds that

$$\models_A \forall s \in \mathbb{Z}. vc(s) \iff \models_A vc(r)$$

That is, $\{r\}$ is a completeness threshold for $vc(s)$.

Proof. For $s \leq a - b$ it holds $a \leq i < s + b \equiv \text{False}$ and hence $vc(s) \equiv \text{True}$. For $s > a - b$ we get:

$$vc(s) \equiv \forall i. (a \leq i < s + b \rightarrow 0 \leq i + z < s)$$

$$\equiv \forall i. (a \leq i \rightarrow 0 \leq i + z) \land (i < s + b \rightarrow i + z < s)$$

$$\equiv \forall i. (a \leq i \rightarrow 0 \leq i + z) \land \forall i. (i < s + b \rightarrow i + z < s)$$

$$\equiv \forall i. (a \leq i \rightarrow 0 \leq i + z) \land \forall i. (i < b \rightarrow i + z < 0)$$

$$=: vc^+$$

$s$ does not occur in $vc^+$ and, hence, the concrete value of $s$ does not impact the validity of $vc^+$. By Lem. 9.5 for any choice of $r \in \mathbb{Z}$ with $r > l - r$, the set $\{r\}$ is a completeness threshold for $vc(s)$. □

10 Roadmap

This report describes a work in progress. We are currently studying patterns encountered in programs that traverse arrays. Once sufficiently many patterns have been identified, we are going to prove that we can extract completeness thresholds for a relevant class of array-traversing programs. Afterwards, we are going to generalize our findings to arbitrary tree-like inductive data structures.

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