ON THE PERIOD LENGTHS OF THE PARALLEL CHIP-FIRING GAME

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Abstract. The parallel chip-firing game is a periodic automaton on graphs in which vertices “fire” chips to their neighbors. In 1989, Bittar conjectured that the period of a parallel chip-firing game with \( n \) vertices is at most \( n \). Though this conjecture was disproven in 1994 by Kiwi et. al., it has been proven for particular classes of graphs, specifically trees (Bittar and Goles, 1992) and the complete graph \( K_n \) (Levine, 2008). We prove Bitar’s conjecture for complete bipartite graphs and characterize completely all possible periods for positions of the parallel chip-firing game on such graphs. Furthermore, we extend our construction of all possible periods for games on the bipartite graph to games on complete \( c \)-partite graphs, \( c > 2 \), and prove some pertinent lemmas about games on general simple connected graphs.

1. Introduction.

1.1. Definitions. The parallel chip-firing game or candy-passing game is a periodic automaton on graphs in which vertices, each of which contains some nonnegative number of chips, “fire” exactly one chip to each of their neighbors if possible. Formally, let \( G \) be an undirected graph with vertex set \( V(G) \) and edge set \( E(G) \). Define the parallel chip-firing game on \( G \) to be an automaton governed by the following rules:

- At the beginning of the game, \( \sigma(v) \) chips are placed on each vertex \( v \) in \( G \), where \( \sigma(v) \) is a nonnegative integer. Let a position of the parallel chip-firing game, denoted by \( \sigma \), be the ordered pair \( (G, \{\sigma(v), v \in G\}) \) containing the graph and the number of chips on each vertex of the graph.
- At each move or step of the game, if a vertex \( v \) has at least as many chips as it has neighbors, it will give (fire) exactly one chip to each neighbor. Such a vertex is referred to as firing; otherwise, it is non-firing. All vertices fire simultaneously (in parallel).

We employ the notation of Levine [13]. Let \( U \) denote the step operator; that is, \( U \sigma \) is the position resulting after one step is performed on \( \sigma \). Let \( U^0 \sigma = \sigma \), and \( U^m \sigma = UU^{m-1} \sigma \). We refer to \( U^m \sigma \) as the position occurring after \( m \) steps. For simplicity, we limit our discussion to connected graphs.

As the number of chips and number of vertices are both finite, there are a finite number of positions in this game. Additionally, since each position completely determines the next position, it follows that for each initial position \( \sigma \), there exist some
positive integers $p$ such that for large enough $t$, $U^t \sigma = U^{t+p} \sigma$. We refer to the minimal such $p$ as the period $p(\sigma)$ of $\sigma$, and we refer to the set $\{U^t \sigma, U^{t+1} \sigma, \ldots, U^{t+p-1} \sigma\}$ as one period of $\sigma$. Also, we call the minimal such $t$ the transient length $t_0$ of $\sigma$.

1.2. Previous Work. The parallel chip-firing game was introduced by Bitar and Goles [6] in 1992 as a special case of the general chip-firing game posited by Björner, Lovász, and Shor [7] in 1991. They [6] showed that the period of any position on a tree graph is 1 or 2. In 2008, Kominers and Kominers [11, 12] further showed that all connected graphs satisfying $\sum_{v \in G} \sigma(v) \geq 4|E(G)| - |V(G)|$ have period 1; they further established a polynomial bound for the transient length of positions on such graphs. Their result [11] that the set of all “abundant” vertices $v_i$ with $\sigma(v_i) \geq 2 \deg(v_i)$ stabilizes is particularly useful in simplifying the game.

It was conjectured by Bitar [5] that $p(\sigma) \leq |V(G)|$ for all games on all graphs $G$. However, Kiwi et. al. [10] constructed a graph on which there existed a position whose period was at least $\exp(\Omega(\sqrt{|V(G)| \log |V(G)|}))$, disproving the conjecture. Still, it is thought that excluding particular graphs constructed to force long periods, most graphs still have periods that are at most $|V(G)|$. In 2008, Levine [13] proved this for the complete graph $K_n$.

1.3. A Broader Perspective. The parallel chip-firing game is a special case of the more general chip-firing game, in which at each step, a vertex is chosen to fire. The general chip-firing game, in turn, is an example of an abelian sandpile [7], and has been shown to have deep connections in number theory, algebra, and combinatorics, ranging from elliptic curves [15] to the critical group of a graph [4] to the Tutte polynomial [14]. Bitar and Goles [6] observed that the parallel chip-firing game has “nontrivial computing capabilities,” being able to simulate the AND, NOT, and OR gates of a classical computer; later, Goles and Margenstern [9] showed that it can simulate any two-register machine, and therefore solve any theoretically solvable computational problem. Finally, the parallel chip-firing game can be used to simulate a pile of particles that falls whenever there are too many particles stacked at any point; this important problem in statistical physics is often referred to as the deterministic fixed-energy sandpile [8, 1]. The fixed-energy sandpile, in turn, is a subset of the more general study of the so-called spatially extended dynamical systems, which occur frequently in the physical sciences and even economics [2]. Such systems demonstrate the phenomenon of self-organized criticality, tending towards a “critical state” in which slight perturbations in initial position cause large, avalanche-like disturbances. Self-organized critical models such as the abelian sandpile tend to display properties of real-life systems, such as $1/f$ noise, fractal patterns, and power law distribution [2, 3]. Finally, the parallel chip-firing game is an example of a cellular automaton, the study of which have implications from biology to social science.

1.4. Our Results. In Section 2, we establish some lemmas about parallel chip-firing games on general simple connected graphs. We bound the number of chips on any single vertex in games with nontrivial period, define the notion of a complement position $\sigma_c$ of $\sigma$ and show that it has the same behavior as $\sigma$, and find a necessary and sufficient condition for a period to occur. Then, in Section 3, we find, with proof, every possible period for the complete bipartite graph $K_{a,b}$. We do so by first showing the only possible periods are of length $k$ or $2k$ for $k \leq \min(a, b)$, and then constructing games with such periods, proving our main result. Finally, in
Section 4, we construct positions on the complete $c$-partite $K_{a_1,a_2,\ldots,a_c}$ with period $p$ for all $1 \leq p \leq c \cdot \min(a_1,a_2,\ldots,a_c)$.

2. Parallel Chip-Firing on Simple Connected Graphs. Consider a simple connected graph $G$. For each vertex $v$ in $G$, let $\Phi_\sigma(v)$ denote the number of firing neighbors $w$ of $v$; that is, the number of vertices $w$ neighboring $v$ satisfying $\sigma(w) \geq \deg(w)$. A step of the parallel chip-firing game on $G$ is then defined as follows:

$$U_\sigma(v) = \begin{cases} 
\sigma(v) + \Phi_\sigma(v), & \text{if } \sigma(v) \leq \deg(v) - 1 \\
\sigma(v) + \Phi_\sigma(v) - \deg(v), & \text{if } \sigma(v) \geq \deg(v). 
\end{cases} \quad (1)$$

Define a terminating position to be a position in which no vertices fire after finitely many moves. We begin our investigation by proving some lemmas limiting the number of chips on each vertex in a game with nontrivial period (period greater than 1).

**Lemma 2.1.** For sufficiently large $t$, $U^t \sigma(v) \leq 2 \deg(v) - 1$ for all $v \in G$ in all games with nontrivial period on a connected graph $G$.

**Proof.** Kominers and Kominers [11] showed that if a vertex $v \in G$ satisfies $\sigma(v) \leq 2 \deg(v) - 1$, then $U_\sigma(v) \leq 2 \deg(v) - 1$. They then showed that if, after sufficiently many steps $t$, there still exists a vertex $v$ with $U^t \sigma(v) \geq 2 \deg(v)$, then all vertices must be firing from that step onward. Since the period of a position is 1 if and only if either all or no vertices in $G$ are firing [6], $U^t \sigma(v) \leq 2 \deg(v) - 1$ is true for any game on $G$ with nontrivial period and sufficiently large $t$. \qed

We further bound the number of chips on each vertex by generalizing a result of Levine [13]:

**Lemma 2.2.** Consider a vertex $v$ in position $\sigma$ such that $\sigma(v) \leq 2 \deg(v) - 1$. Then

$$\Phi_\sigma(v) \leq U_\sigma(v) \leq \Phi_\sigma(v) + \deg(v) - 1.$$  

**Proof.** Either $\sigma(v) < \deg(v)$ or not. We consider the cases individually.

If $0 \leq \sigma(v) \leq \deg(v) - 1$, then $U_\sigma(v) = \sigma(v) + \Phi_\sigma(v)$. So

$$\Phi_\sigma(v) \leq U_\sigma(v) \leq \Phi_\sigma(v) + \deg(v) - 1.$$  

If instead $\sigma(v) \leq 2 \deg(v) - 1$, then $U_\sigma(v) = \sigma(v) + \Phi(v) - \deg(v)$. Hence

$$\Phi(v) = \deg(v) + \Phi_\sigma(v) - \deg(v) \leq \sigma(v) + \Phi_\sigma(v) - \deg(v)$$

$$= U_\sigma(v) \leq 2 \deg(v) - 1 + \Phi_\sigma(v) - \deg(v) = \deg(v) - 1 + \Phi_\sigma(v). \quad \square$$

If a vertex $v$ satisfies $\Phi_\sigma(v) \leq \sigma(v) \leq \Phi_\sigma(v) + \deg(v) - 1$, we call it confined. Furthermore, call a position confined if all vertices in the position are confined. Note that for confined $v$, $\sigma(v) \leq \Phi_\sigma(v) + \deg(v) - 1 \leq 2 \deg(v) - 1$. Lemmas 2.1 and 2.2 imply that if $p(\sigma) > 1$, then $U^t \sigma$ is confined if $t \geq t_0$, where $t_0$ is the transient length of $\sigma$; that is, once the game reaches a position which repeats periodically, all subsequent positions are confined. We generally limit our discussion to confined positions to exclude positions with trivial periods.

Next, we define

$$F_v(t) = \begin{cases} 
1, & U^t \sigma(v) \geq \deg(v) \\
0, & U^t \sigma(v) \leq \deg(v) - 1 
\end{cases} \quad (2)$$
Lemma 2.3. Let the complement $\sigma_c$ of a confined position $\sigma$ be the position that results after replacing the $\sigma(v)$ chips on each vertex $v \in G$ with $2 \deg(v) - 1 - \sigma(v)$ chips. Then $U(\sigma_c) = (U\sigma)_c$.

Proof. We begin by noticing that since $\sigma$ is confined, each vertex $v$ has at most $2 \deg(v) - 1$ chips, so each vertex in $\sigma_c$ has a nonnegative number of chips.

Observe that a vertex $v$ fires in $\sigma_c$ exactly when it did not fire in $\sigma$. Hence, $U\sigma(v) = \sigma(v) + \Phi_\sigma(v) - F_v(0)\deg(v)$, and all but $\Phi_\sigma(v)$ neighbors will fire in $\sigma_c(v)$. So

$$U(\sigma_c(v)) = (2 \deg(v) - 1 - \sigma(v)) + (\deg(v) - \Phi_\sigma(v)) - ((1 - F_v(0))\deg(v))$$

$$= (2 \deg(v) - 1) - (\Phi_\sigma(v) + \sigma(v) - F_v(0)\deg(v)) = (U\sigma)_c(v).$$

This lemma means we may treat $\sigma$ and $\sigma_c$ as equivalent positions, as at any point during their firing, we may transform one into the other. This implies the following corollary:

Corollary 2.4. For all positions $\sigma$ on $G$, $p(\sigma) = p(\sigma_c)$.

Next, we prove a proposition that characterizes a period of the game on any connected graph $G$. For each position $\sigma$ and vertex $v \in G$, let

$$u_t(\sigma, v) = \left| \{ s \mid 0 \leq s < t, U^s\sigma(v) \geq \deg(v) \} \right|.$$

be the number of times $v$ fires in the first $t$ steps.

Proposition 2.5. The position $\sigma$ on $G$ satisfies $U^t\sigma = \sigma$ if and only if each vertex has fired the same number of times within those $t$ steps; that is, if and only if for all vertices $v, w \in G$,

$$u_t(\sigma, v) = u_t(\sigma, w) = k \geq 0. \quad (3)$$

Proof. If equation (3) holds, then by equation (1), $U^t\sigma(v) = \sigma(v) + k \cdot \deg(v) - k \cdot \deg(v) = \sigma(v)$ for all $v$, so $U^t\sigma = \sigma$. Conversely, if $U^t\sigma = \sigma$, consider the vertex $v'$ such that $u_t(\sigma, v') = k'$ is maximal. Then, since $u_t(\sigma, w) \leq k'$ for all vertices $w$ neighboring $v$,

$$U^t\sigma(v) = \sigma(v) + \sum_w u_t(\sigma, w) - k' \deg(v) \leq \sigma(v) + k' \deg(v) - k' \deg(v).$$

But as $U^t\sigma(v) = \sigma(v)$, we see that $u_t(\sigma, w) = k'$ must hold for all $w$ neighboring $v$. Since the graph is connected, we continue inductively through the entire graph to obtain equation (3).

3. Period Length of Games on $K_{a,b}$. Recall that a complete bipartite graph $G = K_{a,b}$ may be partitioned into two subsets of vertices, $L$ and $R$, such that no edges exist among vertices in the same set, but every vertex in $L$ is connected to every vertex in $R$. We refer to the sets $L, R$ as the sides of $G$. Define $a = |L|$ and $b = |R|$. As stated above, Bitar and Goles [6] showed that if no vertices or all vertices are firing, the period is 1. We consider only games whose period is greater than 1; that is, at least one vertex is firing every turn, and not all vertices fire every turn.
Let $F_L(\sigma)$ and $F_R(\sigma)$ denote the number of vertices in $L$ and $R$, respectively, that fire in $\sigma$. Then, $\Phi_v(\sigma) = F_R(\sigma)$ if $v \in L$, and $\Phi_v(\sigma) = F_L(\sigma)$ if $v \in R$. Notice that $F_R(\sigma)$ is the number of vertices in $R$ with at least $a$ chips, and $F_L(\sigma)$ is the number of vertices in $L$ with at least $b$ chips. Let $\alpha_t(L, m) = \sum_{v \in L}(u_{m+t}(\sigma, v) - u_m(\sigma, v))$ be the number of times any of the vertices in $L$ have fired in the first $t$ steps starting from, and including, $U^m\sigma$, and define $\alpha_t(R, m)$ similarly. Define $\alpha_t(L) = \alpha_t(L, 0)$ and $\alpha_t(R) = \alpha_t(R, 0)$.

Without loss of generality, we prove facts about the vertices in $L$, which also hold for vertices in $R$. In the first $t$ steps, a vertex $v$ in $L$ fires a total of $bu_t(\sigma, v)$ chips and receives $\alpha_t(R)$ chips. Hence,

$$U^t\sigma(v) - \sigma(v) = \alpha_t(R) - bu_t(\sigma, v). \quad (4)$$

Next, we prove a lemma that bounds the number of times a vertex has fired once the position is confined.

**Lemma 3.1.** Let $v, w \in L$. If $\sigma$ is confined, and $\sigma(v) \leq \sigma(w)$, then for all $t \geq 0$,

$$u_t(\sigma, v) \leq u_t(\sigma, w) \leq u_t(\sigma, v) + 1. \quad (5)$$

**Proof.** We prove this by induction on $t$. The base case, $t = 1$, is straightforward: vertices $v$ and $w$ have each fired either 0 or 1 times. If $v$ fires after step 0, then $\sigma(w) \geq \sigma(v) \geq \deg(v) = \deg(w)$ chips, and $w$ also fires. Now, assume $u_t(\sigma, v) \leq u_t(\sigma, w) \leq u_t(\sigma, v) + 1$.

If $u_t(\sigma, w) = u_t(\sigma, v)$, then by equation (4),

$$U^t\sigma(w) - \sigma(w) - (U^t\sigma(v) - \sigma(v)) = \alpha_t(R) - bu_t(\sigma, w) - (\alpha_t(R) - bu_t(\sigma, v)) \implies U^t\sigma(w) - U^t\sigma(v) = \sigma(w) - \sigma(v) \geq 0.$$

Thus, if $v$ is ready to fire after step $t$, then $w$ must be ready to fire also. It follows that

$$u_{t+1}(\sigma, v) \leq u_{t+1}(\sigma, w) \leq u_{t+1}(\sigma, v) + 1. \quad (5)$$

Otherwise, $u_t(\sigma, w) = u_t(\sigma, v) + 1$. Then, since $U^t\sigma$ is confined from Lemma 2.2, by equation (4),

$$U^t\sigma(v) - \sigma(v) - (U^t\sigma(w) - \sigma(w)) = \alpha_t(R) - bu_t(\sigma, v) - (\alpha_t(R) - bu_t(\sigma, w)) \implies U^t\sigma(v) - U^t\sigma(w) = b + \sigma(v) - \sigma(w) \geq 0$$

by Lemma 2.2, since the degrees of both $v$ and $w$ are $b$.

So, if $w$ is ready to fire after step $t$, so is $v$, and equation (5) again holds.

From the above lemma, we can deduce the following:

**Lemma 3.2.** If $\sigma$ is confined and $a | \alpha_t(L)$, then $u_t(\sigma, v) = \alpha_t(L)/a$ for all $v \in L$.

**Proof.** Let $v'$ be the vertex in $L$ with $\sigma(v')$ minimal. By Lemma 3.1, for all $v \in L$,

$$u_t(\sigma, v) \in \{m, m+1\}$$

where $m = u_t(\sigma, v')$. If $z$ is the number of vertices $w \in L$ with $u_t(\sigma, w) = m + 1$, then

$$\alpha_t(L) = \sum_{v \in L} u_t(\sigma, v) = (a-z)m + z(m+1) = am + z \equiv z \pmod{a}.$$

Since $u_t(\sigma, v') = m$, we have $z < a$. Then $z = 0$ because $z \equiv 0 \pmod{a}$; so $u_t(\sigma, v) = m$ for all $v \in L$. Since $\alpha_t(L) = \sum_{v \in L}(u_t(\sigma, v))$, this implies $u_t(\sigma, v) = \alpha_t(L)/a$ for all $v \in L$. 

Clearly, if all $a$ vertices in $L$ have fired the same number of times, then $a|\alpha_t(L)$; so we have found a necessary and sufficient condition for all vertices on the same side to fire the same number of times. But by Proposition 2.5, a period is completed when all $v \in G$ have fired the same number of times; thus, we desire a relation between the sides that forces every vertex on both sides to fire the same number of times. Our first step is the following lemma.

**Lemma 3.3.** If $\sigma$ is confined, and $\alpha_t(L) = ka$ for some positive integer $k$, then $u_{t+1}(\sigma, v) - u_1(\sigma, v) = k$ for all $v \in R$.

*Proof.* If $v \in R$ is firing, then $U^t\sigma(v) = (\sigma(v) - a) + ka - a(\alpha_t(\sigma, v) - 1)$. Since $\sigma$ is confined and $v$ is firing, $0 \leq \sigma(v) - a < a - 1$; and since $U^t\sigma$ is confined by Lemma 2.2, we have $0 \leq \sigma(v) - a + ka - a(u_t(\sigma, v) - 1) \leq 2a - 1$. These two inequalities together imply that, for firing vertices $v$, $-a < -(a - 1) \leq ka - a(u_t(\sigma, v) - 1) = ka - a(u_t(\sigma, v) - u_1(\sigma, v)) \leq 2a - 1 < 2a$.

If $v \in R$ is instead non-firing, then $U^t\sigma(v) = \sigma(v) + ka - a\alpha_t(\sigma, v)$ chips. $U^t\sigma(v)$ is confined by Lemma 2.2, so $0 \leq \sigma(v) + ka - a\alpha_t(\sigma, v) < 2a$; since $0 \leq \sigma(v) - a < a - 1$ because $v$ is non-firing, we then deduce, similarly as above, that

$$-a < -(a - 1) \leq ka - a\alpha_t(\sigma, v) = ka - a(u_t(\sigma, v) - u_1(\sigma, v)) \leq 2a - 1 < 2a$$

for non-firing vertices $v$ as well.

Therefore, for all $v \in R$, we have that

$$-a < a(k - (u_t(\sigma, v) - u_1(\sigma, v))) < 2a \implies -1 < k - (u_t(\sigma, v) - u_1(\sigma, v)) < 2,$$

so $u_t(\sigma, v) - u_1(\sigma, v) \in \{k, k - 1\}$ for all $v \in R$. If $u_t(\sigma, v) = k$, then we can compute $U^t\sigma(v) < a = \deg(v)$; hence $v$ does not fire after step $t$, and $u_{t+1}(\sigma, v) - u_1(\sigma, v) = u_t(\sigma, v) - u_1(\sigma, v) = k$. If instead $u_t(\sigma, v) = k - 1$, then $U^t\sigma(v) \geq a$, so $v$ fires after step $t$, and $u_{t+1}(\sigma, v) - u_1(\sigma, v) = u_t(\sigma, v) - u_1(\sigma, v) + 1 = k$, and we are done.\]

Note that applying this lemma to $U^m\sigma$ also means $\alpha_t(L, m) \implies \alpha_t(R, m + 1)$.

Next, recalling the definition of $F_v(t)$ in equation (2), we define $d_t(v, m) = u_{m+t}(\sigma, v) - u_m(\sigma, v) = \sum_{i=m}^{m+t-1} F_v(i)$ for nonnegative integers $m$ and positive integers $t$. Note that by definition, $\sum_{v \in L} d_t(v, m) = \alpha_t(L, m)$. Applying Lemmas 3.2 and 3.3 to the position $U^m\sigma$, we find that if $\alpha_t(L, m) = ka$, then

$$d_t(v, m) = d_t(w, m + 1) = k$$

for all vertices $v \in L$ and all vertices $w \in R$.

Now, we give a sufficient condition for a period of a position on $K_{a,b}$ to occur.

**Lemma 3.4.** If $\sigma$ is confined, and for some $m \geq 0$ and $t \geq 1$, $\alpha_t(L, m) = ka$ and $F_v(m) = F_v(m + t)$ for all $v \in L$, then $p(\sigma) = t$ if $t$ is chosen to be as small as possible.

*Proof.* By Lemma 3.2 applied to $U^m(\sigma)$, since $\alpha_t(L, m) = ka$, $d_t(v, m) = k$ for all $v \in L$. If for some $m \geq 0$, $F_v(m) = F_v(m + t)$ for all $v \in L$, then

$$k = d_t(v, m) = F_v(m) + \sum_{i=m+1}^{m+t-1} F_v(i) = F_v(m + t) + \sum_{i=m+1}^{m+t-1} F_v(i) = d_t(v, m + 1)$$

for all $v \in L$. But by equation (6), $d_t(v, m) = d_t(w, m + 1) = k$ for all $v \in L$, $w \in R$. Hence, $d_t(v, m + 1) = d_t(w, m + 1) = k$ for all vertices $v, w \in G$, by Proposition 2.5 applied to $U^{m+1} \sigma$ implies $U^{m+1+t} \sigma = U^{m+1} \sigma$, or $p(\sigma)|t$. But $t$ is taken to be as small as possible, so $p(\sigma) = t$.\]

\]
Using this fact, we limit which periods are possible for games on $K_{a,b}$.

**Proposition 3.5.** If $\sigma$ is confined and $\alpha_t(L) = ka$, then $p(\sigma) = t$ or $p(\sigma) = 2t$.

**Proof.** By iteratively applying Lemma 3.3 $m + 2$ times, along with Lemma 3.2, we find that
\[
\alpha_t(L, m) = \alpha_t(L, m + 2) = ka \implies d_t(v, m) = d_t(v, m + 2) = k
\]
for all vertices $v \in L$ and nonnegative even integers $m$.

Expanding the sums $d_t(v, m)$ and $d_t(v, m + 2)$ in terms of $F_v$, we obtain
\[
F_v(m) + F_v(m + 1) = F_v(m + t) + F_v(m + t + 1)
\]
for all nonnegative even integers $m$ and vertices $v \in L$.

If $F_v(m) = F_v(m + t)$ for all $v \in L$ and some $m \geq 0, t \geq 1$, then the period is $t$ by Lemma 3.4. Otherwise, let $X$ be the set of vertices $x \in L$ satisfying $F_x(k) \neq F_x(k + t)$ for all $k \geq 0$. The range of $F_x$ is $\{0, 1\}$, so this condition is equivalent to
\[
F_x(k) + F_x(k + t) = 1
\]
for all $k \geq 0$. We show that the period of $\sigma$ is then $2t$.

**Case 1.** $t$ is odd.

Consider some nonnegative integer $d$, and let $x \in X$. If $d$ is even, let $d' = d$; otherwise, let $d' = d + t$. If $F_x(d) = F_x(d + 1)$, then $F_x(d') = F_x(d' + 1)$ and $F_x(d' + t) = F_x(d' + t + 1)$.

But by equation (7),
\[
F_x(d') + F_x(d' + 1) = F_x(d' + t) + F_x(d' + t + 1) \implies F_x(d') = F_x(d' + t),
\]
contradicting the assumption that $F_x(k) \neq F_x(k + t)$ for all $k \geq 0$.

Hence, $F_x(d) \neq F_x(d + 1)$ for all nonnegative $d$. This implies $F_x(d + j) = F_x(d)$ for all $j \geq 0$; hence $F_x(d + 2t) = F_x(d)$ for all $x \in X$. Since $F_x(d + 2t) = F_x(d + t) = F_v(d)$ for other vertices $v \in L \setminus X$, by Lemma 3.4, $p(\sigma) = 2t$.

**Case 2.** $t$ is even.

Let $x \in X$. By equation (7),
\[
F_x(0) + F_x(1) = F_x(t) + F_x(t + 1) = F_x(2t) + F_x(2t + 1).
\]

But by equation (8),
\[
F_x(0) = 1 - F_x(t) = 1 - (1 - F_x(2t)) = F_x(2t) \text{ for all } x \in X.
\]
Then as above, $p(\sigma) = 2t$ by Lemma 3.4. $\blacksquare$

More specifically, the following corollary holds:

**Corollary 3.6.** Let $\sigma$ be a position on $K_{a,b}$. If $p(\sigma)$ is odd, $p(\sigma) \leq \min(a, b)$; and if $p(\sigma)$ is even, $p(\sigma) \leq 2 \min(a, b)$.

**Proof.** Without loss of generality, let $a \leq b$. If $p(\sigma) = 1$, $p(\sigma) \leq a$. Otherwise, $p(\sigma) > 1$. Since $p(\sigma) = p(U^t\sigma)$, we may replace $\sigma$ by $U^t\sigma$ and assume $\sigma$ is confined by Lemma 2.2.

By the Pigeonhole Principle, there must exist steps $t_1 \geq t_2 \geq t_1 - a$ with $\alpha_{t_1}(L, 0) \equiv \alpha_{t_2}(L, 0) \pmod{a}$. But then $a|\alpha_{t_1-t_2}(L, t_2)$, so by Proposition 3.5 applied to $U^t\sigma$, $p(\sigma) = k$ or $2k$, where $k = t_1 - t_2 \leq a$. Hence, if $p(\sigma)$ is odd, $p(\sigma) \leq a$, and if $p(\sigma)$ is even, $p(\sigma) \leq 2a$. $\blacksquare$

Finally, we characterize all possible periods for $\sigma$.

**Proposition 3.7.** There exist positions $\sigma$ on $G = K_{a,b}$ with period $k$ and $2k$ for all $1 \leq k \leq \min(a, b)$. 
Proof. Without loss of generality, let $a \leq b$.

Let $L_1, L_2, \ldots, L_a$ be the vertices in $L$, and $R_1, R_2, \ldots, R_b$ be the vertices in $R$. Let $k$ be a positive integer such that $2 \leq k \leq a$. We represent each position $\sigma$ on $G$ by two vectors

$$L(t) = \left( U^t\sigma(L_1), U^t\sigma(L_2), U^t\sigma(L_3), \ldots, U^t\sigma(L_a) \right),$$

$$R(t) = \left( U^t\sigma(R_1), U^t\sigma(R_2), U^t\sigma(R_3), \ldots, U^t\sigma(R_b) \right).$$

Consider the following position $\sigma_k$, which we claim has period $k$:

$$L(0) = \left( 1, 2, \ldots, k-2, k-1, \underbrace{b, b, \ldots, b}_{(a-k+1) \text{ times}} \right),$$

$$R(0) = \left( 1, 2, \ldots, k-2, k-1, \underbrace{a, a, \ldots, a}_{(b-k+1) \text{ times}} \right).$$

$L_k, L_{k+1}, \ldots, L_a$ and $R_k, R_{k+1}, \ldots, R_b$ fire, so $U\sigma_k$ is represented by

$$L(1) = \left( b - k + 2, b - k + 3, \ldots, b - 1, b, b - k + 1, b - k + 1, \ldots, b - k + 1 \right),$$

$$R(1) = \left( a - k + 2, a - k + 3, \ldots, a - 1, a, a - k + 1, a - k + 1, \ldots, a - k + 1 \right).$$

We can see that the vertices $L_i, R_j$ with $U^t\sigma_k(L_i) = b, U^t\sigma_k(R_i) = a$ satisfy $i = k-t$ for $t = 1, 2, \ldots, k-1$. So, $U^t\sigma_k = \sigma_k$ follows upon applying Proposition 2.5, noting that after $k$ steps, each vertex has fired exactly once. Hence, $\sigma_k$ has period $k$.

Next, consider the following position $\sigma_{2k}$, which we claim has period $2k$:

$$L(0) = \left( 0, 1, \ldots, k-3, k-2, k-1, k-1, \ldots, k-1 \right),$$

$$R(0) = \left( 1, 2, \ldots, k-2, k-1, \underbrace{a, a, \ldots, a}_{(b-k+1) \text{ times}} \right).$$

Note that, if at any point $\sigma(R_i) = \sigma(R_j)$, then $U\sigma(R_i) = U\sigma(R_j)$, because $R_i$ and $R_j$ have the same neighbors. So, $U\sigma_{2k}$ is represented by

$$L(1) = \left( b - k + 1, b - k + 2, \ldots, b - 2, b - 1, \underbrace{b, b, \ldots, b}_{(a-k+1) \text{ times}} \right),$$

$$R(1) = \left( 1, 2, \ldots, k-2, k-1, 0, 0, \ldots, 0 \right).$$
and $U^2\sigma_{2k}$ is represented by

$$L(1) = \left( b-k+1, b-k+2, \ldots, b-2, b-1, \underbrace{0,0,\ldots,0}_{(a-k+1) \text{ times}} \right),$$

$$R(1) = \left( a-k+2, a-k+3, \ldots, a-1, a, a-k+1, a-k+1, \ldots, a-k+1 \right).$$

We can see that for $t = 2, 3, \ldots, 2k - 1$, the vertex in $G$ that fires (has $b$ chips if it is in $L$, or $a$ chips if it is in $R$) in position $U^t\sigma_{2k}$ is

$$\begin{cases} 
R_{k-\frac{t}{2}} & \text{for } t \text{ even}, \\
L_{k-\frac{t-1}{2}} & \text{for } t \text{ odd}.
\end{cases} \quad (9)$$

So, after $2k$ steps, every vertex will have fired once, and by Proposition 2.5, $U^{2k}\sigma_{2k} = \sigma_{2k}$.

It remains to construct initial positions with period 1 or 2. The trivial game with no chips on any vertex has period 1, while the initial position where each vertex in $L$ has $b$ chips, and each vertex in $R$ has 0 chips, can be easily checked to have period 2. Thus, all periods $i, 1 \leq i \leq a$, and $2i, 1 \leq i \leq a$, are achievable.

Combining our results, we obtain our main theorem.

**Theorem 3.8.** A nonnegative integer $p$ is a possible period of a position $\sigma$ of the parallel chip-firing game on $K_{a,b}$ if and only if

$$p \in \left( \{ i \mid 1 \leq i \leq \min(a,b) \} \cup \{ 2i \mid 1 \leq i \leq \min(a,b) \} \right). \quad (10)$$

**Proof.** By Corollary 3.6, no period lengths may lie outside the sets in (10); and in Proposition 3.7, we have constructed positions with all such periods.

**4. Periods of Games on the Complete $c$-Partite Graph.** We again use the vector notation from above to represent the positions of a parallel chip-firing game on the complete $c$-partite graph $G = K_{a_1,a_2,\ldots,a_c}$ formed by joining the ant клиques $S_1, S_2, \ldots, S_c$; let the vertices in $S_i$ be $S_{i,1}, S_{i,2}, \ldots, S_{i,a_i}$ for each $1 \leq i \leq c$. Without loss of generality, we will assume $a_1 \geq a_2 \geq \ldots \geq a_c$. As above, we represent a position on $G$ by the set of vectors

$$\left\{ S_i(t) = \left( U^t\sigma(S_{i,1}), U^t\sigma(S_{i,2}), \ldots, U^t\sigma(S_{i,a_i}) \right) \right\}, \ 1 \leq i \leq c.$$

Below is a representation of a position which has period $(c-j)a_c - k + 1$ for all $0 \leq j \leq c - 1$ and $1 \leq k \leq a_c$. Note that a vertex in $S_j$ fires when it has at least $d_b = \sum_{i=0}^{c}(S_i) - S_b$ chips; here $d_b$ is the degree of any vertex in $S_b$. 


For our construction, we let

\[ S_h(0) = \begin{cases} & (h - 1), (c - 1) + (h - 1), 2(c - 1) + (h - 1), \ldots, (a_c - k - 1)(c - 1) + (h - 1) \\ & (a_c - k)(c - 1) + (h - 1), (a_c - k + 1)(c - 1) + (h - 1) - 1, \\ & \ldots, (a_c - k + (k - 2))(c - 1) + (h - 1) - (k - 2), d_h - k - \sum_{z=1}^{h-1} (a_h - a_c) \end{cases} \]

for \( 1 \leq h \leq c - j - 1 \),

\[ S_i(0) = \begin{cases} & c - 1, 2(c - 1), \ldots, (a_c - k)(c - 1), d_i, d_i, \ldots, d_i \\ & a_i - (a_c - k) \text{ times} \end{cases} \]  

for \( c - j \leq i \leq c \).

We now show that this position indeed has period \((c - j)a_c - k + 1\). Let \( F_\sigma(t) \) be the set of all firing vertices in \( U^j(\sigma) \).

It can be checked that

\[ F_\sigma(0) = \bigcup_{i=c-j}^{c} \bigcup_{m=a_c-k}^{a_c} \{ S_{i,m} \}; \]

\[ F_\sigma(t) = \bigcup_{m=a_c}^{a_c-j-t} \{ S_{c-j-t,m} \} \]

for \( 1 \leq t \leq c - j - 1 \); and

\[ F_\sigma(k_1(c - j - 1) - t_1) = \{ S_{t_1+1,c-k_1} \} \]

for \( 1 \leq k_1 \leq k, 0 \leq t_1 \leq c - j - 2 \), encompassing steps \( c - j \) through \( k(c - j - 1) \);

\[ F_\sigma(k(c - j - 1) + 1 + k_2(c - j)) = \bigcup_{i=c-j}^{c} \{ S_{c-i,a_c-(k+k_2)} \} \]

for \( 0 \leq k_2 \leq a_c - k + 1 \); and

\[ F_\sigma(k(c - j - 1) + 1 + k_2(c - j) + t_2) = \{ S_{c-j-t_2,a_c-(k+k_2)} \} \]

for \( 0 \leq k_2 \leq a_c - k - 1, 1 \leq t_2 \leq c - j - 1 \). (In fact, each vertex \( v \in G \) fires exactly when it contains \( \deg(v) \) chips.)

The latter two categories describe which vertices fire during steps \( k(c - j - 1) + 1 \) through \( k(c - j - 1) + 1 + (a_c - k - 1)(c - j) + (c - j - 1) = (k + 1)(c - j - 1) + 1 + (a_c - k - 1)(c - j) = a_c(c - j) - k \). But after this \( (a_c(c - j) - k)^{th} \) step, every vertex in \( G \) has fired exactly once; the last to fire is \( S_{t_1} \). Hence, \( U^{a_c(c - j) - k + 1} \sigma = U^0 \sigma \) by Proposition 2.5, and the period is \( a_c(c - j) - k + 1 \) as desired. This means all periods from 1 to \( c \min(a_1, a_2, \ldots, a_c) \) are achievable, as \( j \) ranges from 0 to \( c - 1 \) and \( k \) ranges from 1 to \( a_c \).
As an example, consider the following position on the graph $K_{6,5,5,4}$ with period 11:

$$S_1(0) = (0, 3, 6, 7, 7) \quad d_1 = 14$$
$$S_2(0) = (1, 4, 7, 10, 10) \quad d_2 = 15$$
$$S_3(0) = (3, 6, 15, 15) \quad d_3 = 15$$
$$S_4(0) = (3, 6, 16, 16) \quad d_4 = 16$$

For this position, $a_c = 4, c = 4, j = 1, k = 2$, and its predicted period length is $(c - j)a_c - k + 1 = 11$ as desired.

5. Discussion and Further Work. For several graphs, a proof of Bitar’s conjecture that $p(\sigma) \leq |V(G)|$ for all parallel chip-firing games on those graphs would be interesting; we proved the conjecture for the complete bipartite graph. Though we have constructed many periods of games on complete $c$-partite graphs in Section 4, there exist periods longer than those detailed. For example, take the following position on $K_{2,2,1}$, which has period 5:

$$S_1(0) = (2) \quad d_1 = 4$$
$$S_2(0) = (1, 2) \quad d_2 = 3$$
$$S_3(0) = (0, 3) \quad d_3 = 3$$

Though positions with these larger periods are more difficult to characterize generally, Bitar’s conjecture still appears to be true for complete $c$-partite graphs.

Moreover, bounding the periods of positions on vertex-regular graphs and more general bipartite graphs are directions for further research. By doubling the length of each cycle in the graph used in the counterexample by Kiwi et. al. [10], we find a counterexample on a graph containing only even cycles, that is, for the general bipartite graph.

We would also like to determine which periods less than the bound are possible. Levine [13] related period lengths of games on the complete graph to the activity, defined as $\lim_{t \to \infty} \frac{\sum_{v \in G} w_t(\sigma, v)}{v t}$. On the other hand, we believe that period lengths are related to lengths of subcycles (closed paths) of the graph $G$; in particular, we conjecture that any period length of a game on $G$ is either a divisor of the order of some subcycle of $G$, or perhaps the least common multiple of the orders of some disjoint subcycles of $G$. This agrees with known results for the tree graph [6], complete graph [13], and now the complete bipartite graph. Our numerical experiments have also verified this conjecture for cycle graphs and complete $k$-partite graphs; in fact, my correspondence with Zhai [16] has produced a proof of this conjecture for the cycle graph.

Another interesting direction to pursue is observing the implications of “reducing” the parallel chip-firing game by removing as many chips as possible from each vertex without affecting their firing pattern (without changing $F_v(t)$ for all $v \in G$ and $t \geq 0$). This reduction may simplify some games into being more approachable by induction.

Besides studying period lengths of parallel chip-firing games, an examination of the transient length of games on certain graphs would be useful in modeling real-world phenomena. Studying transient positions would also help uncover what attributes determine whether a position is within a period or not, and bounding the
transient length would make for more efficient computation of the period length of games on complex graphs.

Chip-firing games on lattices and tori have been used as cellular automaton models of the deterministic fixed-energy sandpile (see \cite{8,1}). Since most studies of sandpiles have been concerned with asymptotic measures such as the “activity,” bounding the period length of such models could serve as a measure of the fidelity of the model to the real world.

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