Fractional Brownian Motion: Local Modulus of Continuity with Refined Almost Sure Upper Bound and First Exit Time from One-sided Barrier

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Abstract

Based on an optimal rate wavelet series representation, we derive a local modulus of continuity result with a refined almost sure upper bound for fractional Brownian motion. The obtained upper bound of the small fractional Brownian increments is of order $O(|h|^H \sqrt{\log \log |h|^{-1}})$ as $|h| \to 0$, and an upper bound of its $p$th moment is provided, for any $p > 0$. This result fills the gap of the law of iterated logarithm for fractional Brownian motion, where the moments’ information of the random multiplier in the upper bound is missing. With this enhanced upper bound and some new results on the distribution of the maximum of fractional Brownian motion, we obtain a new and refined asymptotic estimate of the upper-tail probability for a fractional Brownian motion to first exit from a positive-valued barrier over time $T$, as $T \to +\infty$.

Keywords: Fractional Brownian motion · self-similarity · wavelets · first exit time

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1 Introduction

In this paper we first use wavelet analysis to provide a refined almost sure upper bound of fractional Brownian motion (fBm)’s increments, then apply it to study the asymptotic behavior of upper-tail probability for an fBm to first exit from a positive-valued one-sided barrier over time $T$, as $T \to +\infty$. Recall that the standard fBm $\{B^H(t)\}_{t \geq 0}$ with the Hurst parameter $H \in (0, 1)$ (see [21]) is a zero-mean continuous-time Gaussian process with $B^H(0) = 0$ a.s. and its covariance function is given by: for any $s, t \geq 0$,

$$
\text{Cov}(B^H(s), B^H(t)) = \frac{1}{2} \left\{ t^{2H} + s^{2H} - |t - s|^{2H} \right\}.
$$

(1.1)
when $H = 1/2$, $B^H$ becomes a standard Brownian motion. As a natural extension of Brownian motion, fBm nowadays plays crucial roles in a number of fields, such as signal processing, geology, biostatistics, biometrics and finance [25]. Two significant features of fBm are $H$-self-similarity and stationary increments. The former property means that: for any $a > 0$, $\{B^H(at)\}_{t \geq 0} \overset{\text{law}}{=} \{a^H B^H(t)\}_{t \geq 0}$, where $\overset{\text{law}}{=}$ denotes the equality in all finite-dimensional probability distributions. With $H$-self-similarity, the behavior of fBm over any time interval can be expressed in terms of the one over the time interval $[0, 1]$. The latter feature, stationary increments, says that for any $h > 0$, the probability distribution of the increments $\{B^H(t + h) - B^H(t)\}_{t \geq 0}$ remains invariant subject to any time shift $t \to t + s_0$, i.e., this distribution of the increment does not depend on the location $t$ but only on the time variation $h$. This feature makes the approximation of fBm’s increments particularly important. From Kolmogorov continuity theorem (see [15]) we see that, for any small value $\gamma > 0$, there are a random variable $C > 0$ having finite moments of any order and a small value $\varepsilon > 0$ (which depends on $\gamma$) such that for any $t \in [0, 1]$ and any $|h| \leq \varepsilon$,

$$|B^H(t + h) - B^H(t)| \leq C e^{H - \gamma}, \overset{\text{a.s.}}{\text{as}}. \quad (1.2)$$

Lévy’s modulus of continuity theorem for fBm (see [26]) asserts that there are a random variable $C > 0$ and a small value $\varepsilon > 0$ such that for any $t \in [0, 1]$ and any $h$ satisfying $|h| \leq \varepsilon$,

$$|B^H(t + h) - B^H(t)| \leq C e^{H} \frac{\varepsilon}{\log \varepsilon^{-1}}, \overset{\text{a.s.}}{\text{as}}. \quad (1.3)$$

By the law of iterated logarithm for fBm (see [34], Theorem 1.5), for every $t \in [0, 1]$, there exist a random variable $C(t) > 0$ and a small value $\varepsilon > 0$ such that for any $|h| \leq \varepsilon$,

$$|B^H(t + h) - B^H(t)| \leq C(t) e^{H} \frac{\varepsilon}{\log \log \varepsilon^{-1}}, \overset{\text{a.s.}}{\text{as}}. \quad (1.4)$$

In the above 3 almost sure upper bounds of small fractional Brownian increments, $e^H$ is the major term. In Lévy’s form (1.3), the deterministic multiplier of $e^H$, $\sqrt{\log \varepsilon^{-1}}$, grows more slowly than $e^{-\gamma}$ as $\varepsilon \to 0$ in Kolmogorov’s form (1.2). The Lévy’s term $\sqrt{\log \varepsilon^{-1}}$ is improved to a more slowly varying term $\sqrt{\log \log \varepsilon^{-1}}$ in the law of iterated logarithm (1.4), but this order of small increments $\Theta_{a.s.}(e^H \sqrt{\log \log \varepsilon^{-1}})$ as $\varepsilon \to 0$ can be attained only for each fixed starting point $t$ but not uniformly on $t \in [0, 1]$ (see [34], Theorems 1.4 and 1.5). Here the notation $\Theta_{a.s.}(1)$ denotes an almost surely bounded random function. More generally for 2 random functions $G$ and $H$, we say $G(\varepsilon) = \Theta_{a.s.}(H(\varepsilon))$ a.s. as $\varepsilon \to 0$, if there are a random variable $C > 0$ and a constant $\delta > 0$ such that $|G(\varepsilon)/H(\varepsilon)| \leq C$ a.s. for all $\varepsilon < \delta$. An even more serious inconvenience is: unlike Kolmogorov continuity theorem, Lévy’ form and the law of iterated logarithm give no information on the moments of the random multiplier $C$. Indeed, to show that $C$ has finite moments of every order or to evaluate the moments of $C$, we need to develop more tools or address more constraints on $\varepsilon$ than Lévy’s modulus of continuity theorem and the law of iterated logarithm for fBm. Having knowledge on the behavior of the random multiplier $C$’s moments is essential in some applications, such as evaluating the moments of the process’ supremum over a compact time interval, and studying the first exit time or statistical inferences of the processes. For examples, the upper bounds of $E(C^p)$ for all $p > 0$ are used in [1, 2] to obtain the asymptotic rates of the lower and upper bounds of the tail probabilities of the target processes’ first exit time. They are also needed
in [16, 27–29] to construct strongly consistent (a.s. convergent) estimators of the Hölder exponents of fractional and multifractional Gaussian processes. Motivated by this inconvenience, in this paper we show an enhanced almost sure upper bound \( C(t)h|H|\sqrt{\log\log|h|^{-1}} \) of \(|B^H(t+h)−B^H(t)|\) as \(|h|\) is small, where for every \( t \), the random multiplier \( C(t) \) has finite moments of every order. Moreover, an explicit upper bound of \( E[C(t)^p] \) is provided for every \( p > 0 \). Our idea of using optimal rate wavelet series representation of fBm to obtain tight upper bounds of the small increments, is originally inspired by several related works, such as [7, 18], where they used a.s. convergent series representation of multifractional or multifractal Gaussian processes to detect the law of iterated logarithm. From these works we see, not only the law of iterated logarithms can be detected, but also the moments control of the random multiplier can be made.

The new upper bound of the fractional Brownian increments obtained in this paper plays a crucial role in studying the next problem: first exit time of fBm from a positive-valued one-sided barrier. This problem has a tight relationship to Burgers equation with random initial data (see [9], Theorem 1). The problem also arises in many real world modeling problems. As an application in finance, the first exit time problem is tightly related to the first default time of a stock index or an individual asset price, as well as withdrawal or lapse behavior of policyholders in assets investments. To be more specific, in this paper we focus our attention on estimating the asymptotic probability that, starting from \( B^H(0) = 0 \) a.s., \( B^H \) does not exit from the upper barrier \( b > 0 \) over \([0, T] \) as \( T \to +\infty \). Let \( \tau_b \) denote the first exit time of \( B^H \) from the upper barrier \( b \), then we define the upper-tail probability of \( \tau_b \) as: for any \( T \geq 0 \),

\[
P(\tau_b > T) = \mathbb{P} \left( \sup_{t \in [0, T]} B^H(t) \leq b \right).
\] (1.5)

There is a rich literature on the first exit times of Gaussian processes from one-sided barrier (see e.g. [9, 20, 31]). In general, obtaining the precise asymptotic rate of \( \mathbb{P}(\tau_b > T) \) can be very challenging. Many problems remain open and only approximations of \( \mathbb{P}(\tau_b > T) \) are suggested. Studying such approximations has popularized the research on the persistence probabilities for stochastic processes, which aims at finding the persistence exponent \( \theta \) such that \( \mathbb{P}(\tau_b > T) = T^{-\theta + o(1)} \), as \( T \to +\infty \), see e.g. [3–6, 11]. Until today, the precise asymptotic rate of \( \mathbb{P}(\tau_b > T) \) is known only for Brownian motion, integrated Brownian motion and several other very special processes [1]. In the setting of fBm, the problem of estimating the asymptotic convergence rate of \( \mathbb{P}(\tau_b > T) \) in (1.5) as \( T \to -\infty \), has been heavily studied, see e.g. [1, 23, 24, 32]. Thanks to the self-similarity of fBm, the behavior of \( \mathbb{P}(\tau_b > T) \) is asymptotically equivalent to that of \( \mathbb{P}(\tau_1 > T) \) as \( T \to +\infty \). It is proved that \( \mathbb{P}(\tau_1 > T) = T^{-(1-H)+o(1)} \) as \( T \to +\infty \) for fBm, where \((1-H)\) is the so-called persistence exponent. The exact behavior of the loss factor \( T^{o(1)} \) is unknown. The most recent result is obtained by Aurzada [1], who provides so far the most precise asymptotic rate. In view of the main result Theorem 1 in [1] and its proof, we can express Aurzada’s result as follows: for any \( \gamma_1, \gamma_2 > 0 \) arbitrarily small,

\[
T^{-(1-H)(\log T)^{1/(2H)−\gamma_1}} \leq \mathbb{P}(\tau_1 > T) \leq T^{-(1-H)(\log T)^{2/H−1+\gamma_2}}, \text{ as } T \to +\infty.
\] (1.6)

(1.6) is viewed as an improvement of Molchan’s earlier result [23] given below: there is \( c > 0 \) such that

\[
T^{-(1-H)} e^{-c\sqrt{\log T}} \leq \mathbb{P}(\tau_1 > T) \leq T^{-(1-H)} e^{c\sqrt{\log T}}, \text{ as } T \to +\infty.
\] (1.7)
The proofs of (1.7) and (1.6) both involve first studying the asymptotic behavior of the quantity

\[ I(T) = \mathbb{E} \left[ \left( \int_0^T e^{B_H(t)} \, dt \right)^{-1} \right], \tag{1.8} \]

then using Tauberian theorem (see [10], Corollary 8.1.7) to build the following relationships:

\[ I(T) \approx \mathbb{E} \left( e^{-\operatorname{sup}_{t \in [0,T] |B_H(t)|}} \right) \approx \mathbb{P}(\tau_1 > T), \text{ as } T \to +\infty. \tag{1.9} \]

Although in (1.9), \( I(T) \) takes two steps to connect with \( \mathbb{P}(\tau_1 > T) \), when transforming \( I(T) \) to \( \mathbb{E} \left( e^{-\operatorname{sup}_{t \in [0,T] |B_H(t)|}} \right) \), the loss factors \( e^{\pm \epsilon \sqrt{\log T}} \) to the major term \( T^{-(1-H)} \) appear in (1.7). Later these loss factors are replaced by the more slowly varying multipliers \( \log(T)^{\pm \epsilon} \) in (1.6). Observe that, by Kolmogorov continuity theorems, the increment of fBm in the \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) space \( \sqrt{\mathbb{E}|B_H(t) - B_H(s)|^2} \) and the increment in the Euclidean space \( |B_H(t) - B_H(s)| \) differ at most by \( |t-s|^{-\gamma} \) with arbitrarily small \( \gamma > 0 \). Therefore by using an extension of Kolmogorov continuity theorem (see [30], Lemma 2.1), Aurzada [1] was able to improve the asymptotic lower bound in (1.7) to that in (1.6), where the loss factor \( e^{-c \sqrt{\log T}} \) is replaced by the more slowly varying multiplier \( \log(T)^c \). The upper bound in (1.6) is obtained through considering mainly the paths of fBm valued below 1 over the entire time interval \([0,T]\), because these paths give relatively much larger contribution on the first exit of fBm. In our framework we observe that, by the law of iterated logarithm, for every fixed \( t, \sqrt{\mathbb{E}|B_H(t+h) - B_H(t)|^2} \) and \( |B_H(t+h) - B_H(t)| \) may differ at most by \( \sqrt{\log \log |h|^{-1}} \). As the second main contribution of this paper, we step forward to improve (1.6), by taking the above advantage.

The major contribution of this paper is then bi-fold:

1. Using wavelet analysis, we provide an enhanced a.s. upper bound of the small increments of fBm, where the order is \( \mathcal{O}_{a.s.}(|h|^H \sqrt{\log \log |h|^{-1}}) \) as the time variation of the increment \( h \to 0 \), and the moments control of the random multiplier is made, see Theorem 1.1.

2. We improve the asymptotic lower bound and upper bound in (1.6), i.e., a more precise asymptotic range of \( \mathbb{P}(\tau_1 > T) \) is derived, as \( T \to +\infty \), see Theorem 1.4.

Along with the proofs of the main statements, several existing results in analysis, probability and stochastic process theory have been generalized or improved. These results have their own interests, such as Corollary 1.2, Corollary 1.3, Lemma 3.2 and Lemma 3.3.

To derive (1) we rely on an optimal rate (in the sense of Kühn and Linde [19]) wavelet series representation of fBm. Heuristically speaking, the rate optimality says that the tail \( \sum_{j=0}^{+\infty} U_j(t) c_j \) of the convergent wavelet series \( B_H(t) = \sum_{j=0}^{+\infty} U_j(t) c_j \), where \( \{c_j\}_{j \in \mathbb{Z}} \) are independent standard Gaussian variables and \( \{U_j(t)\}_{j \in \mathbb{Z}} \) are deterministic wavelet functions, is of order around \( \mathcal{O}_{a.s.}(n^{-H} \sqrt{1 + \log n}) \) as \( n \to +\infty \) (see (2.3)). This relatively fast running rate of convergence will be needed in the proof of Theorem 1.1, when upper bounding the remainder of the convergent series. To obtain (2) we will also use the fact that our upper bound of fractional Brownian increments is asymptotically smaller than the one used in (1.6). Through all the proofs the \( H \)-self-similarity and stationary increments of fBm constantly play a key role.

In what follows we state our main results.


**Theorem 1.1.** Let \( \{B^H(t)\}_{t \in [0, 1]} \) be an fBm with Hurst parameter \( H \in (0, 1) \). For every \( t \in [0, 1] \), there exists a random variable \( C(t) > 0 \) having finite moment of any positive order, such that for any \( |h| \in (0, \min\{1 - t, e^{-\epsilon}\}) \),

\[
|B^H(t + h) - B^H(t)| \leq C(t)|h|^H \sqrt{\log \log |h|^{-1}}, \text{ a.s.} \tag{1.10}
\]

Moreover, there is a constant \( c > 0 \), such that for any \( t \in [0, 1] \) and any order \( p > 0 \),

\[
E[C(t)^p] \leq c^p \sqrt{\Gamma\left(p + \frac{1}{2}\right)}, \tag{1.11}
\]

where \( \Gamma \) is the gamma function.

**Remarks.**

1. For every \( t \in [0, 1] \), the random variable \( C(t) \) depends on \( t \) and \( H \) but not on \( h \). Moreover, from the forthcoming proof of Theorem 1.1 we see that, for any \( t_1, t_2 \in [0, 1] \), \( C(t_1) \) and \( C(t_2) \) are identically distributed. This fact is consistent with the distributional stationarity of fractional Brownian increments.

2. \( h \) should satisfy the condition \( |h| \leq 1 - t \), because our series representation approach has been restricted \( t, t + h \) to belong to \([0, 1]\). However since the wavelet series representation of fBm that we have used does not depend on the choice of the time interval, Theorem 1.1 will hold for \( t, t + h \in [0, T] \) with any \( T > 0 \).

3. The condition \( |h| \leq e^{-\epsilon} \) comes from the fact that, the inequality (2.46) in the proof of Theorem 1.1 has provided the following a.s. upper bound of the small fractional Brownian increments:

\[
|B^H(t + h) - B^H(t)| \leq C(t)|h|^H \sqrt{\log \log |h|^{-1}} + |h|, \text{ a.s. as } h \to 0.
\]

And it will be proved that \( |h| \leq |h|^H \sqrt{\log \log |h|^{-1}} \) once \( |h| \leq e^{-\epsilon} \). In fact with the same proof we can show a more general result: for any \( \epsilon > 0 \) such that \( |h| \leq |h|^H \sqrt{\log \log |h|^{-1}} \) for \( |h| \leq \epsilon \), (1.10) holds for \( |h| \in (0, \min\{\epsilon, 1 - t\}) \). In this case, since the choice of \( \epsilon \) depends on the Hurst parameter \( H \), and the function \( h \to h^{1-H} (\log |\log h|)^{-1/2} \) is not necessarily monotonic on \((0, \min\{\epsilon, 1 - t\})\), we disregard discussion of this vague condition in our framework.

4. When \( H = 1/2 \), \( B^H \) becomes a standard Brownian motion. In this particular situation, our result (1.10) yields Proposition 6 in [18].

The next result is a straightforward consequence of Theorem 1.1. The improvements on the results of the first exit time problem mainly rely on it.

**Corollary 1.2.** For every \( t \in [0, 1] \), there exists a random variable \( C(t) > 0 \) satisfying (1.11), such that for any \( \delta \in (0, \min\{1 - t, e^{-\max\{W(1/2H)\}}\}] \),

\[
|B^H(t + h) - B^H(t)| \leq C(t)|h|^H \sqrt{\log \log \delta^{-1}}, \text{ a.s., for any } |h| \leq \delta, \tag{1.12}
\]

where \( W \) denotes the Lambert W function (see [13]): for every \( z > 0 \), \( W(z) \) is the unique positive-valued solution of \( xe^x = z \).

**Proof.** Thanks to the forthcoming Lemma 2.9, the function \( x \to x^H \sqrt{\log \log x^{-1}} \) is increasing over \((0, \min\{e^{-\epsilon}, e^{-\min\{W(1/2H)\}}\}]\). Now by using Theorem 1.1, for every \( t \in [0, 1] \), there is a random variable \( C(t) > 0 \) satisfying (1.11) such that, for any \( \delta \in (0, \min\{1 - t, e^{-\epsilon}, e^{-\min\{W(1/2H)\}}\}] = (0, \min\{1 - t, e^{-\max\{W(1/2H)\}}\}] \),

and any \( |h| \leq \delta, |B^H(t + h) - B^H(t)| \leq C(t)|h|^H \sqrt{\log \log |h|^{-1}} \leq C(t)|h|^H \sqrt{\log \log \delta^{-1}}. \)
A straightforward consequence of Corollary 1.2 is derivation of the following upper bounds of moments: there is a constant $c > 0$ such that for any $p > 0$, any $t \in [0,1]$ and any $\delta \in (0, \min \{(1-t, e^{-c(t)}, e^{-c|T(H)|})\}]$,
\[
E \left( \sup_{|h| \leq \delta} \left| B^H(t + h) - B^H(t) \right|^p \right) \leq c^p p^p \delta^p (\log \delta^{-1})^{p/2}.
\] (1.13)

Using a proof very similar to that of Theorem 1.1, we can also obtain the following global modulus of continuity for fBm, which provides more detail on the moments of the random multiplier in the upper bound than Lévy’ form. Its proof has been discussed in Remark 2.10.

**Corollary 1.3.** There exists a random variable $C > 0$ which does not depend on $t, h$ and satisfies (1.11), such that for any $t \in [0,1]$ and any $\delta \in (0, \min \{1-t, e^{-c|T(H)|}\})$,
\[
|B^H(t + h) - B^H(t)| \leq C\delta^H \sqrt{\log \delta^{-1}}, \text{ a.s., for any } |h| \leq \delta.
\] (1.14)

The global modulus of continuity result (1.14) directly implies: for $\delta > 0$ enough small and for any $p > 0$,
\[
E \left[ \left( \sup_{t \in [0,1]} \left| B^H(t) - B^H(s) \right| \right)^p \right] \leq c^p \Gamma (p + \frac{1}{2}) \delta^H (\log \delta^{-1})^{p/2},
\]
which extends the upper bound of Brownian increments’ moments in the inequalities (2) in [14] from Brownian motion to fBm.

Our second main result below improves (1.6), through providing more precise asymptotic bounds of $\mathbb{P}(\tau_1 > T)$. To simplify the notations, throughout this paper we denote by
\[
\log^n T = \underbrace{\log \cdots \log}_{n \text{ times}} T, \text{ for } n \geq 2 \text{ and } T > 0 \text{ sufficiently large.}
\]

**Theorem 1.4.** Let $\{B^H(t)\}_{t \geq 0}$ be an fBm with Hurst parameter $H \in (0,1)$. Let the integer $N \geq 5$, which can be arbitrarily large. For any $\gamma_1, \gamma_2 > 0$ arbitrarily small,
\[
\mathcal{L}(T, N, \gamma_1) \leq \mathbb{P}(\tau_1 > T) \leq \mathcal{U}(T, N, \gamma_2), \text{ as } T \to +\infty,
\] (1.15)

where the asymptotic lower bound
\[
\mathcal{L}(T, N, \gamma_1) = T^{-1/(1-H)} (\log T)^{-1/(2H)} (\log^2 T)^{-1/(2H)} (\log T)^{-\gamma_1};
\]
and the asymptotic upper bound
\[
\mathcal{U}(T, N, \gamma_2) = T^{-1/(1-H)} (\log T)^{1/(2H)} (\log^2 T)^{3/(2H)} (\log^3 T)^{1/H} (\log^4 T)^{1/(2H)} (\log^N T)^{\gamma_2}.
\]

Compared to the major term $T^{-1/(1-H)}$’s multipliers in (1.6), Theorem 1.4 makes them more elaborated. In addition arbitrarily slowly varying multipliers $(\log^N T)^{-c}$ and $(\log^N T)^c$ appear for the asymptotic lower and upper bounds respectively. This result narrows down the asymptotic range of $\mathbb{P}(\tau_1 > T)$ as $T \to +\infty$. For example, by our asymptotic lower bound of $\mathbb{P}(\tau_1 > T)$, there is non-existence of $c > 0$ such that $\mathbb{P}(\tau_1 > T) \leq T^{-1/(1-H)} (\log T)^{-1/(2H)} (\log^2 T)^{-1/(2H)} - c$, as $T \to +\infty$, while (1.6) gives no such information. Our asymptotic upper bound seems to improve the one in (1.6) more significantly, because it is $(\log T)^{-1/(1+O(1))}$ times of the latter one, as $T \to +\infty$. All the above statements can be verified for $H = 1/2$, because it is well-known that $\mathbb{P}(\tau_1 > T) \sim T^{-1/2}$ as $T \to +\infty$ for
Brownian motion [17]. Finally and unfortunately, whether \( P(\tau_1 > T) \sim T^{-(1-H)} \) as \( T \to +\infty \) for the fBm \( B^H \) with \( H \in (0,1) \) remains unknown.

Section 2 aims at proving Theorem 1.1. In Section 2.1 we introduce one optimal rate wavelet series representation of fBm from Meyer et al. [22]’s works. Based on this series representation we prove Theorem 1.1 in Section 2.2. Section 3 is devoted to demonstrating Theorem 1.4. The proof mainly relies on Molchan and Aurzada’s ideas but the new result also requires some non-evident findings. In Section 3.1, the asymptotic lower bound in (1.15) is obtained through evaluating the asymptotic behavior of the quantity \( I(T) \) as in (1.9). The main idea leading to the improvement of the asymptotic lower bound is the application of Corollary 1.2 and Lemma 2.6. In Section 3.2, the asymptotic upper bound in Theorem 1.4 is derived based on Aurzada’s method [1] and Corollary 1.2, with some new findings, e.g. Lemmas 3.2 and 3.3.

2 Local Modulus of Continuity of Fractional Brownian Motion with Enhanced Almost Sure Upper Bound

2.1 A Wavelet Series Representation of Fractional Brownian Motion

Meyer et al. [22] have developed several wavelet series representations of the fBm \( \{B^H(t)\}_{t \in [0,1]} \), of the form \( \sum_{j=0}^{+\infty} U_j(t) \varepsilon_j \), where \( \{U_j\}_{j \geq 0} \) are deterministic functions and \( \{\varepsilon_j\}_{j \geq 0} \) are independent identically distributed standard Gaussian variables. Later Ayache and Taqqu [8] have shown that all these wavelet series representations get their rate optimality in the sense of Kühn and Linde [19]. Therefore we pick one of these wavelet representations and prove our first main result Theorem 1.1. The goal of this section is to introduce the picked wavelet series representation of fBm. First we need to introduce some notations. For a function \( \Phi \), define the Fourier transformation and inverse Fourier transformation as follows:

\[
\hat{\Phi}(\xi) = \int_{-\infty}^{+\infty} e^{-i\xi t} \Phi(t) \, dt \quad \text{and} \quad \Phi(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\xi \xi} \hat{\Phi}(\xi) \, d\xi.
\]

**Definition 2.1.** Let \( \Psi \) be either Lemarié-Meyer wavelet with at least 2 first vanishing moments or Daubechies wavelet of class \( C^2(\mathbb{R}) \). The system \( \{2^{j/2} \Psi(2^j \cdot - k) : j, k \in \mathbb{Z}\} \) then generates an orthonormal basis in the Lebesgue space \( L^2(\mathbb{R}) \) of square integrable real-valued functions. For \( H \in (0,1) \), the function \( \Psi_H \) is defined via its Fourier transformation:

\[
\hat{\Psi}_H(\xi) = (i\xi)^{-H-1/2} \hat{\Psi}(\xi),
\]

with the convention \( \hat{\Psi}_H(0) = 0 \).

Below we introduce one of the wavelet series representations of the fBm \( \{B^H(t)\}_{t \in [0,1]} \) in [22].

**Theorem 2.2.** The fractional Brownian motion \( \{B^H(t)\}_{t \in [0,1]} \) admits the following series representation in the sense of almost sure convergence: for any \( t \in [0,1] \),

\[
B^H(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \left( \Psi_H(2^j t - k) - \Psi_H(k) \right) \varepsilon_{j,k}, \ a.s.,
\]

where \( \Psi_H \) is given in Definition 2.1 and \( \{\varepsilon_{j,k}\}_{j,k \in \mathbb{Z}} \) is a sequence of i.i.d. standard Gaussian variables.
Ayache and Taqqu [8] show that the above wavelet series representation verifies the rate optimality in the following sense [19]: there exists a set of indices $I_n \subset \mathbb{Z}^2$, such that: as $n \to +\infty$,

$$
\sqrt{\mathbb{E} \sup_{t \in [0,1]} |B^H(t) - \sum_{(j,k) \in I_n} 2^{-jH} (\Psi^H(2^j t - k) - \Psi^H(k)) \epsilon_{j,k}|^2} = \Theta(n^{-1/2} \sqrt{1 + \log n}). \tag{2.3}
$$

This result shows that the tail of the wavelet series goes to 0 as fast as $n^{-1/2} \sqrt{1 + \log n}$, which is sufficient to help to control the upper bound of the fractional Brownian increments in the proof of Theorem 1.1, see the inequalities (2.43).

### 2.2 An Enhanced Almost Sure Upper Bound of the Small Fractional Brownian Increments

In this section we prove Theorem 1.1. To this end we first list some properties of the function $\Psi^H$ and the sequence $\{\epsilon_{j,k}\}_{j,k \in \mathbb{Z}}$ which are keys to the proof.

The following lemma is Proposition 1 in [8].

**Lemma 2.3.** The function $\Psi^H$ given in Definition 2.1 satisfies the following properties: there is $c > 0$ such that for any $t \in \mathbb{R}$,

$$
|\Psi^H(t)| \leq c(2 + |t|)^{-2} \text{ and } |\Psi^H(t)| \leq c(2 + |t|)^{-3}. \tag{2.4}
$$

**Remark.** By using the mean value theorem and Lemma 2.3, there is $c > 0$ such that for any $s, t \in \mathbb{R}$,

$$
|\Psi^H(t) - \Psi^H(s)| \leq c|t - s|(2 + |t^*|)^{-3}, \tag{2.5}
$$

where $t^*$ is some value in $(\min\{s, t\}, \max\{s, t\})$.

The lemma below is Lemma 1 in [8].

**Lemma 2.4.** Let $\{\epsilon_j\}_{j \in \mathbb{Z}}$ be a sequence of standard Gaussian variables (it is not necessarily a sequence of independent variables nor a Gaussian process). There is a random variable $C > 0$ satisfying (1.11), such that for any $j \in \mathbb{Z}$, $|\epsilon_j| \leq C \sqrt{\log(2 + |j|)}$, a.s.

Note that the property (1.11) comes from the proof of Lemma 1 in [8]. Indeed, in the latter proof it is shown that we can choose

$$
C = c \max |\tilde{C}, 1|, \tag{2.6}
$$

where $c > 0$ is a deterministic constant, and the random variable $\tilde{C} > 0$ satisfies $\mathbb{E}[\tilde{C}^p] \leq c' \sqrt{\mathbb{E}|Z|^{2p}}$, with $Z \sim \mathcal{N}(0, 1)$ and $c' > 0$ being a deterministic constant independent of $p$. Observe that: for $p \geq 0$,

$$
\mathbb{E}[\max(\tilde{C}, 1)] = \mathbb{E}[\tilde{C}^p \mathbb{1} (\tilde{C} > 1)] + \mathbb{E}[\mathbb{1} (\tilde{C} \leq 1)] \leq \mathbb{E}[\tilde{C}^p] + 1 \tag{2.7}
$$

and

$$
\mathbb{E}|Z|^{2p} = \pi^{-1/2} 2^p \Gamma\left(p + \frac{1}{2}\right). \tag{2.8}
$$
It results from (2.6) - (2.8) that, for $p \geq 0$,
\[
\mathbb{E}[C^p] \leq e^{p' \pi^{-1/4} 2^{p/2}} \sqrt{\Gamma(p + 1/2)} \left( 1 + \frac{1}{c' \pi^{-1/4} 2^{p/2} \sqrt{1/(p + 1/2)}} \right)
\]
\[
\leq e^{p' \pi^{-1/4} 2^{p/2}} \sqrt{\Gamma(p + 1/2)} \left( 1 + \frac{1}{c' \pi^{-1/4} \inf_{p \geq 0} 2^{p/2} \sqrt{1/(p + 1/2)}} \right)
\]
\[
\leq e^{p_1} \sqrt{\Gamma(p + 1/2)},
\]
where the constant $c_1 > 0$ does not depend on $p$. This proves that $C$ verifies (1.11).

The following result is Lemma 2 in [8].

**Lemma 2.5.** Let $\{e_{j,k}\}_{j,k \in \mathbb{Z}}$ be a sequence of standard Gaussian variables. There is a random variable $C > 0$ satisfying (1.11), such that for any $j, k \in \mathbb{Z}$, $|e_{j,k}| \leq C \sqrt{\log(2 + |j| + |k|)}$, a.s.

The following result is a modified version of Lemma 2.5. This result is the key to the improvement of the upper bound of the increments of $\{B^H(t)\}_{t \in [0,1]}$.

**Lemma 2.6.** Fix $t \in [0,1]$. For any $j \in \mathbb{N}$, let $k_j(t)$ be the unique positive integer such that $t \in [k_j(t)2^{-j}, (k_j(t) + 1)2^{-j})$, i.e., $k_j(t) = \lfloor t \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor number. For any negative integer $j$, we set $k_j(t) = 0$. Then there is a random variable $C(t) > 0$ satisfying (1.11), such that for any $j, k \in \mathbb{Z}$, $|e_{j,k}| \leq C(t) \sqrt{\log(2 + |j| + |k - k_j(t)|)}$, a.s.

**Proof.** We denote two sets of indices by
\[
\mathcal{A} = \{(j, k) \in \mathbb{N}^2 : j \in \mathbb{N}, k \geq k_j(t)\} \quad \text{and} \quad \mathcal{B} = \{(j, k) \in \mathbb{N}^2 : j \in \mathbb{N}, k < k_j(t)\}.
\]
Recall that the following bijection from $\mathbb{N}^2$ to $\mathbb{N}$ was introduced by Cantor and it can be used to show that $\mathbb{N}^2$ is a countable set:
\[
(j, k) \mapsto \frac{(j + k - k_j(t))((j + k - k_j(t)) + 1)}{2} + j + 1.
\]
Up to a slight modification, we note that the mapping
\[
f_t(j, k) = \frac{(j + k - k_j(t))(j + k - k_j(t) + 1)}{2} + j + 1
\]
is a bijection from $\mathcal{A}$ to $\mathbb{N}$. Then replacing the indices $(j, k)$ with $f_t(j, k)$ in $\{e_{j,k}\}_{(j,k) \in \mathcal{A}}$, we can regard $\{e_{j,k}\}_{(j,k) \in \mathcal{A}}$ as a sequence indexed by $\mathbb{N}$, denoted by $\{\lambda_{f_t(j,k)}\}_{f_t(j,k) \in \mathbb{N}}$. Next by using Lemma 2.4 and the fact that $2 + f_t(j, k) \leq (2 + j + (k - k_j(t)))^2$, for $(j, k) \in \mathcal{A}$, we obtain that, there is a random variable $C_1(t) > 0$ satisfying (1.11), such that for any $(j, k) \in \mathcal{A}$,
\[
|e_{j,k}| = |\lambda_{f_t(j,k)}| \leq C_1(t) \sqrt{\log(2 + f_t(j, k))} \leq C_1(t) \sqrt{2} \sqrt{\log(2 + j + (k - k_j(t)))}, \text{ a.s.}
\]
\[
(2.10)
\]
For the second group indexed by $\mathcal{B}$, observe that the mapping
\[
g_t(j, k) = \frac{(j + k_j(t) - k)((j + k_j(t) - k) + 1)}{2} + j + 1
\]
is a bijection from $\mathcal{B}$ to $\mathbb{N}\setminus \{0\}$. Therefore $\{\epsilon_{j,k}\}_{(j,k)\in \mathcal{B}}$ can be viewed as a sequence indexed by $\mathbb{N}\setminus \{0\}$. By using analogous arguments we show that there is a random variable $C_2(t) > 0$ satisfying \((1.11)\), such that for any $(j,k) \in \mathcal{B}$,

$$|\epsilon_{j,k}| = |\lambda_{f_t(j,k)}| \leq C_2(t) \sqrt{\log(2 + j + (k_j(t) - k))}, \text{ a.s.} \quad (2.11)$$

Finally since each of the sequences $\{\epsilon_{j,-k}\}_{j,k \in \mathbb{N}}$, $\{\epsilon_{-j,k}\}_{j,k \in \mathbb{N}}$ and $\{\epsilon_{-j,-k}\}_{j,k \in \mathbb{N}}$ can be viewed as a sequence indexed by $\mathbb{N}$, the same result follows from Lemma 2 in [8]. As a summary of \((2.10)\), \((2.11)\) and the above discussion, there is a random variable $C(t) > 0$ satisfying \((1.11)\), such that for any $j, k \in \mathbb{Z}$, $|\epsilon_{j,k}| \leq C(t) \sqrt{\log(2 + j + |k - k_j(t)|)}$, a.s. The lemma is proved. \qed

**Remark.** It is worth making more efforts to explain why the random multiplier $C_1(t)$ in \((2.10)\) depends on $t$. This is because on one hand, as in the proof of Lemma 1 in [8], $C_1(t)$ depends on some stopping time of the sequence of standard Gaussian variables $\{\lambda_j\}_{j \in \mathbb{N}}$; on the other hand, given the bijection $f_t$, we can observe that $\lambda_{f_t(j,k)} = \lambda_{g(t)}$, where $g_t$ is some permutation of $(1,2,\ldots)$, which depends on $t$. Consequently, the stopping time of $\{\lambda_{g(t)}\}_{j \in \mathbb{N}}$ depends on $t$, so does $C_1(t)$.

The following statement is the inequality \((4.9)\) in [8].

**Lemma 2.7.** For any $j, k \in \mathbb{Z}$, $\log(2 + |j + |k|) \leq 4 \log(2 + |j|) \log(2 + |k|)$.

The following result will also be useful in the proof of Theorem 1.1.

**Lemma 2.8.** Let $a > 1$. There is a constant $c > 0$, such that for any integer $n \geq 0$, $\sum_{j=n}^{+\infty} a^{-j/\sqrt{\log(2+j)}} \leq c a^{-n/\sqrt{\log(2+n)}}$.

**Proof.** By using Lemma 2.7, we get

$$\sum_{j=n}^{+\infty} a^{-j/\sqrt{\log(2+j)}} \leq 2 \sum_{j=n}^{+\infty} a^{n-j/2} \sqrt{\log(2+j-n)} = 2 \sum_{j=0}^{+\infty} a^{-j/2} \sqrt{\log(2+j)} = c < +\infty.$$ 

The lemma then follows. \qed

The following lemma is an elementary result, that we will use frequently in the analysis.

**Lemma 2.9.** Let $\theta > 0$. The function $f(x) = x^\theta \sqrt{\log^2 x^{-1}}$ is increasing over the interval $(0, \min\{e^{-e}, e^{-e^{\text{W}(1/20)}}\})$, where $W$ is the Lambert W function.

**Proof.** Note that $\log^2 x^{-1} \geq 1$ for $x \leq e^{-e}$. Then

$$f'(x) = \theta x^{\theta-1} \sqrt{\log^2 x^{-1}} + x^{\theta} \left(\frac{1}{2}\right)(\log^2 x^{-1})^{-1/2} \frac{1}{\log x^{-1}} \frac{1}{x^{-1}(-1)x^{-2}} \geq 0$$

is equivalent to $\log^2(x^{-1})e^{\log^2(x^{-1})} \geq 1/(20)$. As the Lambert W function is increasing over $[0, +\infty)$ (see [13]), the above inequality is then equivalent to $x \leq \min\{e^{-e}, e^{-e^{\text{W}(1/20)}}\}$. Therefore $f$ is increasing over $(0, \min\{e^{-e}, e^{-e^{\text{W}(1/20)}}\})$. \qed

Now we are ready to prove Theorem 1.1. To avoid repetition in writing, through the rest of the paper every $C, C_1, C_2, \ldots$ denotes a positive random variable satisfying \((1.11)\), and they may be repeatedly used to mean different variables. Every $c, c', c_1, c_2, \ldots$ denotes a positive constant which does not depend on $t, h$, and they may be repeatedly used to mean different constants. If an equation or inequality contains a random component, by default it holds almost surely.
Proof. Fix \( t \in [0,1] \). Let \( h \in (0, \min\{1-t, e^{-\epsilon}\}] \). The case for \( h \in [\max\{t-1, e^{-\epsilon}\}, 0) \) can be dealt with analogously. According to (2.2), we write

\[
B^H(t+h) - B^H(t) = F_1 + F_2,
\]

where

\[
F_1 = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{+\infty} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}
\]

(2.13) and

\[
F_2 = \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}.
\]

(2.14)

Let us bound \(|F_1|\) and \(|F_2|\) respectively.

**Upper bound of \(|F_1|\):**

For any two positive integers \( J, K \), denote by

\[
\bar{F}_1^{J,K} = \sum_{j=-J}^{-1} \sum_{k=-K}^{K} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}.
\]

(2.15)

Observe that \( \lim_{J,K->\infty} \bar{F}_1^{J,K} = F_1, \) a.s. By using (2.15), the triangle inequality, (2.5) and Lemma 2.5, we obtain

\[
|\bar{F}_1^{J,K}| \leq \sum_{j=-J}^{-1} \sum_{k=-K}^{K} 2^{-jH} \left| \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right| |\epsilon_{j,k}|
\]

(2.16)

\[
\leq ch \sum_{j=-J}^{-1} \sum_{k=-K}^{K} 2^{-jH+\left(2+|2^j t_{j,k}^* - k|\right)} |\epsilon_{j,k}|
\]

\[
\leq Ch \sum_{j=-J}^{-1} \sum_{k=-K}^{K} 2^{-jH+\left(2+|2^j t_{j,k}^* - k|\right)} |\epsilon_{j,k}|
\]

(2.17)

where for \( j \in (-J, \ldots, -1) \), \( k \in (-K, \ldots, K) \), \( t_{j,k}^* \) is some value in \( (t, t+h) \), and the constant \( c > 0 \) and the random variable \( C > 0 \) do not depend on \( t, h \). Since \( j < 0 \) and \( t_{j,k}^* \in (0, 1) \), we have for all \( j \leq -1, k \in \mathbb{Z}, \)

\[
(2+|2^j t_{j,k}^* - k|)^{-3} \leq \max\{(2+|k|)^{-3}, (2+|k-1|)^{-3}\} \leq (1+|k|)^{-3}.
\]

(2.18)

It follows from (2.16), (2.17) and Lemma 2.7 that, there is a constant \( c' > 0 \) such that

\[
|\bar{F}_1^{J,K}| \leq CC'h \sum_{j=-J}^{-1} 2^{j(1-H)} \sqrt{\log(2+|j|)} \left\{ \sum_{k=-K}^{K} (1+|k|)^{-3} \sqrt{\log(2+|k|)} \right\}.
\]

(2.18)

Clearly,

\[
\sum_{j=-\infty}^{-1} 2^{j(1-H)} \sqrt{\log(2+|j|)} \left\{ \sum_{k=-\infty}^{+\infty} (1+|k|)^{-3} \sqrt{\log(2+|k|)} \right\} < +\infty.
\]

(2.19)

Finally it results from (2.18) and (2.19) that there is a random variable \( C_1 > 0 \) which does not depend on \( t, h \), such that \( |F_1| \leq C_1 h, \) a.s. Taking the limits \( J, K -> \infty \) on both hand-sides of the latter inequality yields

\[
|F_1| \leq C_1 h, \text{ a.s.}
\]

(2.20)
Upper bound of $|\mathcal{F}_2|$: 

For any two positive integers $J, K$, define

$$
\mathcal{F}_2^{J,K} = \sum_{j=0}^{J} \sum_{k=-K}^{K} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}.
$$

(2.21)

We first decompose $\mathcal{F}_2^{J,K}$ into sum of two terms with respect to the cases $k < 0$ and $k \geq 0$ below:

$$
\mathcal{F}_2^{J,K} = \mathcal{F}_2^{J,K}_{2,1} + \mathcal{F}_2^{J,K}_{2,2},
$$

(2.22)

where

$$
\mathcal{F}_2^{J,K}_{2,1} = \sum_{j=0}^{J} \sum_{k=-K}^{-1} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}
$$

(2.23)

and

$$
\mathcal{F}_2^{J,K}_{2,2} = \sum_{j=0}^{J} \sum_{k=0}^{K} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}.
$$

(2.24)

Let us first bound $|\mathcal{F}_2^{J,K}_{2,2}|$. Let $j_0$ be the unique positive integer satisfying $2^{-h} < h \leq 2^{-h+1}$, i.e., $j_0 = \lfloor \log h^{-1} / \log 2 + 1 \rfloor$. Then for any $j \in \{0, \ldots, j_0 - 1\}$, there is unique $k_j(t) \in \{0, \ldots, 2^t - 1\}$ such that $t, t + h \in [k_j(t)2^{-j}, (k_j(t) + 1)2^{-j})$, i.e., $k_j(t) = \lfloor 2^j t \rfloor = \lfloor 2^j (t + h) \rfloor$. Let $j \geq j_0$, we then write

$$
\mathcal{F}_2^{J,K}_{2,2} = \mathcal{F}_2^{J,K}_{2,2,1} + \mathcal{F}_2^{J,K}_{2,2,2},
$$

(2.25)

where

$$
\mathcal{F}_2^{J,K}_{2,2,1} = \sum_{j=j_0}^{J} \sum_{k=0}^{K} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}
$$

(2.26)

and

$$
\mathcal{F}_2^{J,K}_{2,2,2} = \sum_{j=j_0}^{J} \sum_{k=0}^{K} 2^{-jH} \left( \Psi_H(2^j(t+h) - k) - \Psi_H(2^j t - k) \right) \epsilon_{j,k}.
$$

(2.27)

To bound $|\mathcal{F}_2^{J,K}_{2,2,1}|$, we apply the triangle inequality, (2.5), Lemma 2.6 and Lemma 2.7 to obtain

$$
|\mathcal{F}_2^{J,K}_{2,2,1}| \leq C(t)h \sum_{j=0}^{j_0} \sum_{k=0}^{K} 2^{-jH} \left( 2 + |2^j t^* - k| \right)^{-3} \sqrt{\log(2 + |j|) \log(2 + |k - k_j(t)|)}
$$

(2.28)

where for $j \in \{0, \ldots, j_0 - 1\}$ and $k \in \{0, \ldots, K\}$, $t^*_{j,k}$ is some value in $(t, t + h)$ and $C(t) > 0$ is a random variable which does not depend on $h$. Since $2^h \leq 2^t$, we obtain $j_0 \leq 1 + \log(h^{-1})$. This implies that, for any $j \leq j_0 - 1$,

$$
\sqrt{\log(2 + |j|) \log(2 + \log h^{-1})} \leq \sqrt{\log(2 + \log h^{-1})}.
$$

(2.29)

Since $h \leq e^{-c}$, $\log h^{-1} \geq c$ and $\log \log h^{-1} \geq 1$, we then get

$$
\sqrt{\log(2 + \log h^{-1})} \leq \sqrt{\log 3 + \log \log h^{-1}} \leq \sqrt{\log 3 + 1} \sqrt{\log \log h^{-1}}.
$$

(2.30)

It results from (2.29) and (2.30) that, there is a constant $c_1 > 0$ such that for any $j \leq j_0 - 1$,

$$
\sqrt{\log(2 + |j|) \log(2 + \log h^{-1})} \leq c_1 \sqrt{\log \log h^{-1}}.
$$

(2.31)
Since $2^j t_{j,k}^* \in [k_j(t), k_j(t)+1)$, we have
\[ (2 + |2^j t_{j,k}^* - k|)^{-3} \leq (1 + |k - k_j(t)|)^{-3}. \] (2.32)

Next note that, for every $j = 0, \ldots, J_0 - 1$,
\[
\sum_{k=0}^{K} \sqrt{\frac{\log(2 + |k - k_j(t)|)}{(1 + |k - k_j(t)|)^3}} \leq C_2 = 2 \sum_{v=0}^{+\infty} \sqrt{\frac{\log(2 + v)}{(1 + v)^3}} < +\infty. \] (2.33)

It follows from (2.28) - (2.33) and the fact that $2^h \leq 2h^{-1}$ that
\[
|\mathcal{F}_{2,2}^{J_0}| \leq C(t) h \left\{ \sum_{j=0}^{J_0-1} \sum_{k=0}^{K} 2^{-jH} \right\} |\Psi_H(2^j(t+h) - k)| + |\Psi_H(2^j(t) - k)| \right\} |e_{j,k}|
\leq C(t) \sum_{j=0}^{J_0-1} \sum_{k=0}^{K} 2^{-jH} \left\{ 2 + 2^j(t+h) - k \right\}^2 + \left\{ 2 + 2^j(t) - k \right\}^2 \sqrt{\frac{\log(2 + |j|) \log(2 + |k - k_j(t)|)}}{v},
\] (2.35)
where $C(t) > 0$ is a random variable which does not depend on $h$. We then observe the following: similar to (2.32), since $2^j t \in [k_j(t), k_j(t)+1)$ and $2^j(t+h) \in [k'_j(t), k'_j(t)+1)$,
\[
(2 + |2^j t - k|)^{-2} \leq (1 + |k - k_j(t)|)^{-2} \quad \text{and} \quad (2 + |2^j(t+h) - k|)^{-2} \leq (1 + |k - k'_j(t)|)^{-2}. \] (2.36)

On one hand, for every $j \in (J_0, \ldots, J_0)$,
\[
\sum_{k=0}^{K} \sqrt{\frac{\log(2 + |k - k_j(t)|)}{(1 + |k - k_j(t)|)^3}} \leq C_1 = 2 \sum_{v=0}^{+\infty} \sqrt{\frac{\log(2 + v)}{(1 + v)^3}} < +\infty. \] (2.37)

With analogous argument we have
\[
\sum_{k=0}^{K} \sqrt{\frac{\log(2 + |k - k'_j(t)|)}{(1 + |k - k'_j(t)|)^3}} \leq C_1. \] (2.38)

On the other hand, by the triangle inequality and Lemma 2.7 we know
\[
\sqrt{\log(2 + |k - k'_j(t)|)} \leq \sqrt{\log(2 + |k - k'_j(t)| + |k'_j(t) - k_j(t)|)} \leq 2 \sqrt{\log(2 + |k - k'_j(t)|) \log(2 + |k'_j(t) - k_j(t)|)}. \] (2.39)

Then by Lemma 2.8 and the facts that $2^{-j_0} < h \leq 2^{-j_0+1}$ and $h \leq e^{-e}$, there are $c_2, c_3 > 0$ such that
\[
\sum_{j=J_0}^{J} 2^{-jH} \sqrt{\log(2 + |j|)} \leq \sum_{j=J_0}^{+\infty} 2^{-jH} \sqrt{\log(2 + |j|)} \leq c_2 2^{-j_0H} \sqrt{\log(2 + |J_0|)} \leq c_3 h^H \sqrt{\log h^{-1}}. \] (2.40)
Next since \( k_j(t), k'_j(t) \leq 2^j \), by a proof very similar to that of Lemma 2.8, we can show that there is a constant \( c_4 > 0 \) such that

\[
\sum_{j=0}^{J} 2^{-jH} \sqrt{\log(2 + |j|)} \sqrt{\log(2 + |k'_j(t) - k_j(t)|)} \leq c_4 2^{-j_0H} \sqrt{\log(2 + |j_0|)} \sqrt{\log(2 + |k'_0(t) - k_{j_0}(t)|)}. \tag{2.41}
\]

Observe that, since \( 2^{-j_0} < h \leq 2^{-j_0+1} \),

\[
k'_j(t) - k_j(t) = [2^h (t + h)] - [2^h t] \in \left] [2^h t + 1] - [2^h t], [2^h t + 2] - [2^h t] \right] \subset [1, 2].
\]

This together with (2.41) leads to \( \sqrt{\log(2 + |k'_j(t) - k_j(t)|)} \leq \sqrt{\log 4} \). The above fact together with (2.41) and \( 2^{-j_0} < h \leq 2^{-j_0+1} \) yields: there is a constant \( c_5 > 0 \) such that

\[
\sum_{j=0}^{J} 2^{-jH} \sqrt{\log(2 + |j|)} \sqrt{\log(2 + |k'_j(t) - k_j(t)|)} \leq c_5 h^H \sqrt{\log \log h^{-1}}. \tag{2.42}
\]

Finally it follows from (2.35) - (2.42) that

\[
|\mathcal{F}_{2,2}^J| \leq C(t) \sum_{j=0}^{J} 2^{-jH} \left\{ (1 + |k - k'_j(t)|)^{-2} + (1 + |k - k_j(t)|)^{-2} \right\} \sqrt{\log(2 + |j|)} \sqrt{\log(2 + |k - k_j(t)|)}
\]

\[
= C(t) \sum_{j=0}^{J} 2^{-jH} \sqrt{\log(2 + |j|)} \left\{ \sum_{k=0}^{K} \frac{\sqrt{\log(2 + |k - k_j(t)|)}}{\sqrt{\log(2 + |k - k'_j(t)|)}} \right\} \leq 2c_1 C(t) \sum_{j=0}^{J} 2^{-jH} \sqrt{\log(2 + |j|)} \sqrt{\log(2 + |k'_j(t) - k_j(t)|)} + c_1 C(t) \sum_{j=0}^{J} 2^{-jH} \sqrt{\log(2 + |j|)}
\]

\[
\leq C_3(t) h^H \sqrt{\log \log h^{-1}}, \quad \text{a.s.,}
\]

where \( C_3(t) > 0 \) is a random variable which does not depend on \( h \).

In order to bound \( |\mathcal{F}_{2,1}^J| \), we observe that it suffices to replace \( k \) with \( -k \) in \( \mathcal{F}_{2,2}^J \) and pursue the same proof as we have obtained (2.43). This process will result in

\[
|\mathcal{F}_{2,1}^J| \leq C_4(t) h^H \sqrt{\log \log h^{-1}}, \quad \text{a.s.,}
\tag{2.44}
\]

where \( C_4(t) > 0 \) is a random variable which does not depend on \( h \). Combining (2.22), (2.44), (2.25), (2.34), (2.43) and by taking \( J, K \to +\infty \), we derive

\[
\mathcal{F}_2 \leq C_5(t) h^H \sqrt{\log \log h^{-1}}, \quad \text{a.s.,}
\tag{2.45}
\]

where \( C_5(t) > 0 \) is a random variable which does not depend on \( h \).

Now we conclude the proof of Theorem 1.1. It results from (2.12), (2.20) and (2.45) that

\[
|B^H(t+h) - B^H(t)| \leq C(t) h^H \sqrt{\log \log h^{-1}} + h = C(t) h^H \sqrt{\log \log h^{-1}} + h = C(t) h^H \sqrt{\log \log h^{-1} (1 + h^{-1}(\log \log h^{-1})^{-1/2})}, \quad \text{a.s.,}
\tag{2.46}
\]

where \( C(t) > 0 \) is a random variable which does not depend on \( h \) and satisfies (1.11). It remains to show that the function \( f(h) = h^{1-H} (\log \log h^{-1})^{-1/2} \leq 1 \) for any \( h \leq e^{-e} \). This is true because \( f \) is increasing over \( [0, e^{-e}] \), and \( f(e^{-e}) = e^{-e(1-H)} < 1 \). (2.46) thus becomes \( |B^H(t+h) - B^H(t)| \leq 2C(t) h^H \sqrt{\log \log h^{-1}}, \) a.s., which completes the proof of the theorem for \( 0 < h \leq \min(1-t, -e^{-e}) \). Finally note that the case for \( \max(t-1, -e^{-e}) \leq h < 0 \) follows easily by switching the roles of \( t \) and \( t + h \) in the above arguments. \( \square \)
Remark 2.10. In the proof of Theorem 1.1, without introducing \( k_j(t) \)'s and \( k'_j(t) \)'s, and using Lemma 2.5, we can rewrite (2.28) as: for \( j \in [0, \ldots, J_0 - 1] \) and \( k \in [0, \ldots, K] \),

\[
|T_{2,1}^{J,K}| \leq C \sum_{j=0}^{h-1} \sum_{k=0}^{K} 2^{-J + j} (2 + 2^j t^e - k)^{-3} \sqrt{\log(2 + 1/j + |k|)} \\
\leq c_1 \sqrt{\log(2 + J_0 + 2^h)} \leq c_2 \sqrt{\log J_0} \leq c_3 \sqrt{\log h^{-1}},
\]

where \( c_1, c_2, c_3 > 0 \) and \( C > 0 \) do not depend on \( t, h \) (similar operation can be taken for \( T_{2,2}^{J,K} \)). The term \( \sqrt{\log h^{-1}} \) in the above upper bound then enlarges the overall upper bound order of \( \log h^{-1} \) from \( \Theta_{a.s.}(h^H \sqrt{\log \log h^{-1}}) \) to \( \Theta_{a.s.}(h^H \sqrt{\log h^{-1}}) \) as \( h \to 0 \). Therefore, what we can conclude is: there is a random variable \( C > 0 \) independent of \( t, h \) and satisfying (1.11) such that for any \( t \in [0, 1] \) and any \( |h| \in (1 - t, 1] \), \( \log \log \log \log h^{-1} \). Since the function \( f(h) = h^H \sqrt{\log h^{-1}} \) is increasing on \( (0, \min \{e^{-1/2H}, e^{-1}\}) \), we then have: for any \( t \in [0, 1] \) and any \( e \leq \min \{1 - t, e^{-1}, e^{-1/(2H)}\} \), \( \sup_{|h| \leq e} |B^H(t + h) - B^H(t)| \leq C e^H \sqrt{\log e^{-1}}, \) a.s.. This is nothing but Corollary 1.3.

3 Asymptotic Bounds of the Upper-tail Probability of First Exit Time

Corollary 1.2 heavily contributes to the derivation of our second main result Theorem 1.4, since the almost sure upper bound of the fractional Brownian increments, \( e^H \sqrt{\log \log e^{-1}} \), improves the one used in [1], which is \( e^{H - \gamma} \) for some \( \gamma > 0 \). This improvement will be devoted to obtaining more precise asymptotic lower and upper bounds of the upper-tail probability of fBm’s first exit time in (1.15).

Before proving Theorem 1.4, we introduce some preliminary results, which have their own interests.

The result below is a modified version of Theorem 3 in [33]. Its proof is quite similar to that of Theorem 3 in [33], so we omit it. This version can be straightforwardly applied in our proof.

Theorem 3.1. Let \( \{(Y(t))_{t \geq 0} \} \) be a continuous-time separable Gaussian process. Let \( I_1, I_2 \subseteq [0, +\infty) \) be two finite Lebesgue measure sets such that \( \sup_{s \in I_1} l_1 \leq \inf_{l_2} l_2 \) and the Lebesgue measure \( \mu(I_1 \cap I_2) = 0 \). Let \( I = I_1 \cup I_2 \). If \( \text{Cov}(Y(s), Y(t)) \geq 0 \) for all \( s, t \in I \),

\[
\text{P} \left( \inf_{l_1} Y(t) \geq 0 \right) \geq \text{P} \left( \inf_{l_2} Y(t) \geq 0 \right) \text{P} \left( \inf_{l_1} Y(t) \geq 0 \right),
\]

(3.1)

The result below is an extension of Lemma 4 in [1].

Lemma 3.2. Let the real-valued function \( g \) satisfy

(i) \( g \) is strictly increasing over \( [1, +\infty) \) and \( \lim_{t \to +\infty} g(t) = +\infty \).

(ii) \( g \) is continuously differentiable over \( [1, +\infty) \).

(iii) \( \text{The function } t \to g(t) \text{ is strictly deceasing to } 0, \text{ as } t \to [1, +\infty). \) Moreover, for any \( \gamma > 1, \)

\[
\lim_{t \to +\infty} \frac{g((\frac{\mu}{\nu} t)\gamma)}{g(t)} = 0.
\]
Then for any $a \in (0,1]$, there exists $t_0 \geq 1$ such that for any $t \geq s \geq t_0$,
\[
(g(t)^2 t^a - g(t)(t^a + 1 - (t - 1)^{a}) + 1)^{1/a} \\
\geq (g(t)^2 t^a - g(t)g(s)(t^a + s^a - (t - s)^a) + g(s)^2 s^a)^{1/a} + (g(s)^2 s^a + g(s)(s^a + 1 - (s - 1)^{a}) + 1)^{1/a}.
\]

(3.2)

Remark. Let $g$ satisfy the properties (i) - (ii) in Lemma 3.2. Define $X(t) = g(t)B^H(t) - B^H(1)$, and let $u(t) = (\mathbb{E}[X(t)^2])^{1/(2H)}$. The inequality (3.2) is then equivalent to $\mathbb{E}[X(s)X(t)] \geq \mathbb{E}[B^H(u(s))B^H(u(t))]$, for $t \geq s \geq t_0$. Therefore Lemma 3.2 serves to verify the sufficient conditions, for Theorem 3.1 (or Lemma 1 and Theorem 1 in [33]) to hold.

The result below extends Lemma 5 in [1]. Its proof is provided in the appendix.

**Lemma 3.3.** Let $H \in (0,1/2]$. Let the function $g$ satisfy $g(1) = 2$ and the properties (i) - (ii) given in Lemma 3.2. Then there exist two constants $t_0 \geq 1$ and $\kappa > 0$ such that for any $a \in \mathbb{R}$,
\[
\mathbb{P}\left(\sup_{t_0 \leq t \leq T} B^H(t) \leq -a\right) \geq \mathbb{P}\left(\sup_{0 \leq s \leq T} B^H(t) \leq 1\right) e^{-aTg(T)^2}, \text{ as } T \to +\infty.
\]

Remark. Aurzada [1] proved Lemmas 3.2 and 3.3 for a particular function $g(t) = 2(\log^2(e^t))^{1/2}$, $\lambda \in (0,1)$. As a more general result, we have proved that Lemmas 3.2 and 3.3 hold for the functions growing to $+\infty$ arbitrarily slowly, e.g., they hold for
\[
g_N(t) = \log^N(\exp(1))t, \quad N \geq 1, \text{ where } \exp^N(1) = \exp \circ \exp(1).\]

Lemma 3.3 then reveals a profound insight: let us choose $g$ to be an extremely slowly growing function, saying $g(t) = g_N(t)$ with very large value of $N$. Then by the $H$-self-similarity and Lemma 3.3, for any constant $s_0 > 0$ and any $a \in \mathbb{R}$,
\[
\mathbb{P}\left(\sup_{s_0 \leq s \leq T} B^H(t) \leq -a\right) = \mathbb{P}\left(\sup_{t_0 \leq t \leq T} B^H(t) \leq -s_0h\right) \geq c_1 \mathbb{P}\left(\sup_{0 \leq t \leq \infty} B^H(t) \leq 1\right),
\]
as $T \to +\infty$, where $\geq$ denotes “approximately greater than” and $c_1, c_2 > 0$ are some constants which do not depend on $T$. Compared to the case for $H > 1/2$ (see the forthcoming (3.30)), the above relation shows that the asymptotic behavior of the distribution of $\sup_{s_0 \leq t \leq T} B^H(t)$ with $H < 1/2$ is not significantly different from that with $H > 1/2$, as $T \to +\infty$.

Sections 3.1 - 3.2 aim at proving Theorem 1.4.

### 3.1 Asymptotic Lower Bound of $\mathbb{P}(T_1 > T)$

Let us first derive the asymptotic lower bound $\mathcal{L}(T,N,\gamma)$ in (1.15). As the paths of $B^H$ are a.s. continuous over the compact set $[0,1]$, there exists a random variable $u^* \in [0,1]$ for which $\sup_{t \in [0,1]} B^H(t) = B^H(u^*)$ a.s.. Then by using the $H$-self-similarity of $B^H$, we can write $I(T)$ in (1.8) as
\[
I(T) = \mathbb{E}\left[\left(\int_0^1 e^{B^H(tu)} T du\right)^{-1}\right] = \mathbb{E}\left[\left(\int_0^1 e^{T^H B^H(u)} T du\right)^{-1}\right] = \mathbb{E}\left[\left(\int_0^1 e^{-T^H (B^H(u^*)-B^H(u))} T du\right)^{-1}\right].
\]

(3.3)
Fix \( t = u^* \) in Corollary 1.2, then Corollary 1.2 entails that, there exists a random variable \( C = C(u^*) > 0 \) satisfying (1.11), such that for any deterministic constant \( \varepsilon_0 \in [0, e^{-\max\{W(1/2H),1\}]} \) and any \( u \in [u^* - \varepsilon_0, u^* + \varepsilon_0] \cap [0, 1] \),

\[
|B^H(u) - B^H(u^*)| \leq C\varepsilon_0^H \sqrt{\log^2 \varepsilon_0^{-1}}, \text{ a.s..} 
\]

(3.4)

The key to the improvement of the asymptotic lower bound of (1.6) then relies on selecting a proper \( \varepsilon_0 \) in (3.4), which makes the statements (1) – (3) below hold true. To this end let us first notice that, by (2.6), the random variable \( C = \max\{\bar{C}, 1\} \geq c \) a.s., for some deterministic constant \( c > 0 \). Hence we can enlarge \( c \) so that \( C \geq 1 \) a.s.. By doing so we then consider the function \( \nu(x) = x^{-H} \sqrt{\log^2 e^x} \), for \( x \geq 1 \). By definition \( \nu(x) \) is continuous and ranged in \( (0, 1] \) as \( x \in [1, +\infty) \). Since \( C^{-1} \in (0, 1] \) a.s., by the intermediate value theorem, there exists a random variable \( C_0 \geq 1 \) a.s. such that

\[
\nu(C_0) = C_0^{-H} \sqrt{\log^2 e^{C_0}} = C^{-1}, \text{ a.s..} 
\]

(3.5)

From (3.5) and the fact that \( C_0 \geq 1 \) a.s., we can easily deduce that

\[
C_0 \geq C^{1/H}(\log^2 (e^C_0))^{1/(2H)} \geq C^{1/H}, \text{ a.s..} 
\]

(3.6)

Next note that for arbitrarily small \( \eta > 0 \),

\[
\sqrt{\log^2 (e^x)} \leq \begin{cases} \sqrt{W(e/2\eta)} & \text{if } x \leq e^{W(e/2\eta)} - 1; \\ \sqrt{\eta} & \text{if } x > e^{W(e/2\eta)} - 1. 
\end{cases}
\]

This is because, by looking into its derivative (similar to Lemma 2.9), the function \( x \rightarrow x^{-\eta} \sqrt{\log^2 (e^x)} \) is increasing over \( [1, e^{W(e/2\eta)} - 1] \) and decreasing over \( (e^{W(e/2\eta)} - 1, +\infty) \). Therefore for \( \eta > 0 \) small enough, \( x^{-\eta} \sqrt{\log^2 (e^x)} \leq 1 \) for \( x \geq e^{W(e/2\eta)} - 1 \). As a result, for arbitrarily small \( \eta > 0 \),

\[
C^{-1} = C_0^{-H} \sqrt{\log^2 (e^{C_0})} \leq \begin{cases} C_0^{-H} \sqrt{W(e/2\eta)} & \text{if } C_0 \leq e^{W(e/2\eta)} - 1; \\ C_0^{-H+\eta} & \text{if } C_0 > e^{W(e/2\eta)} - 1 \text{ a.s..} 
\end{cases}
\]

(3.7)

Combining (3.6) and (3.7) we obtain, for arbitrarily small \( \eta > 0 \),

\[
C^{1/H} \leq C_0 \leq \begin{cases} C^{1/H} W(e/2\eta)^{1/(2H)} & \text{if } C_0 \leq e^{W(e/2\eta)} - 1; \\ C^{1/H-\eta} & \text{if } C_0 > e^{W(e/2\eta)} - 1 \text{ a.s..} 
\end{cases}
\]

(3.8)

Then we define \( \varepsilon_0 \) as follows:

\[
\varepsilon_0 = \min \left\{ (C_0 T(\log^2 T))^{-1}, e^{-\max\{W(1/2H),1\}} \right\}.
\]

(3.9)

By definition \( \varepsilon_0 \) becomes a random variable depending on \( C \) and \( T \). With this choice of \( \varepsilon_0 \), (3.4) holds. More importantly, the random variable \( \varepsilon_0 \) satisfies:

1. Since \( \varepsilon_0 \leq e^{-e} < 1/2 \), a.s., the Lebesgue measure

\[
\mu([u^* - \varepsilon_0, u^* + \varepsilon_0] \cap [0, 1]) \geq \varepsilon_0, \text{ a.s..} 
\]

(3.10)

2. For enough large \( T > 0 \), the terms \( T^H |B^H(u) - B^H(u^*)| \) are a.s. upper bounded by some deterministic constant \( c > 0 \). This is because: by the facts that \( x \rightarrow x^H \sqrt{\log^2 x}^{-1} \) is increasing on
(0, e^{\min(W[1/(10H), 1])})$, and that $c_0 \leq (C_0 T (\log^2 T)^{1/(2H)})^{-1}$, and also (3.5), we have

$$CT^H \sqrt{\log \log c_0^{-1}} \leq C_{C_0}^{-H} \frac{\sqrt{\log^2 (C_0T (\log^2 T)^{1/(2H)})}}{\sqrt{\log^2 T}} \leq \frac{C_{C_0}^{-H} \sqrt{\log^2 (e^\epsilon C_0)} \sqrt{\log^2 (T (\log^2 T)^{1/(2H)})}}{\sqrt{\log^2 T}} \leq \frac{\sqrt{(\log + 1) \log^2 T}}{\sqrt{\log^2 T}} = \tilde{c}_1 = \sqrt{\log^2 T + 1}, \text{ a.s.},$$

as $T \to +\infty$. In (3.11) we have made use of the following elementary inequalities: for $a \geq 1, b \geq e^\epsilon$,

$$\log^2(ab) \leq \log((\log a + 1) \log b) \leq (\log(\log a + 1) + 1) \log^2 b = \log^2(e^\epsilon a^\epsilon) \log^2 b;$$

and for $a \geq b \geq e^\epsilon$,

$$\log^2(ab) \leq \log(2 \log a) \leq (\log(2 + 1) \log^2 a.$$

(3) By definition of $c_0$, we have

$$e_0^{-1} \leq C_0 T (\log^2 T)^{1/(2H)} + \tilde{c}_2, \text{ a.s.},$$

where $\tilde{c}_2 = e^{\min(W[1/(10H), 1])}$.

Then it follows from (3.3), (3.10) - (3.12), Hölder’s inequality, Jensen’s inequality, (|x| + |y|) \leq 2^{-k-1}(|x|^k + |y|^k) for $k \geq 1$, that the term $I(T)$ can be upper bounded as:

$$I(T) \leq E \left[ \left( \int_{|\epsilon| - c_0 \epsilon + c_0 |\epsilon|} e^{-T H (B^H(u^*) - B^H(u))} T \, du \right)^{-1} e^{-T H B^H(u^*)} \right]$$

$$\leq e^{\tilde{c}_1 T^{-1}} \left( \left( C_0 T (\log^2 T)^{1/(2H)} + \tilde{c}_2 \right) e^{-T H B^H(u^*)} \right)$$

$$\leq 2 e^{\tilde{c}_1 T^{-1}} \left( \left( C_0 T (\log^2 T)^{1/(2H)} + \tilde{c}_2 \right) e^{-T H B^H(u^*)} \right)$$

where $c_1 = 4 e^{\tilde{c}_1}$ and $p, q > 1$ can be arbitrarily chosen subject to $1/p + 1/q = 1$. Next we provide upper bound of $E[C_0^p]$. By (3.8), Jensen’s inequality $(|x| + |y|) \leq 2^{-k-1}(|x|^k + |y|^k)$ for $k \geq 1$, Cauchy-Schwarz inequality and Markov’s inequality, for any $p > 1$ and arbitrarily small $\eta > 0$,

$$E[C_0^p] \leq E \left[ \left( C_1^H W(e/(2\eta))^{1/(2H)} 1 \{ C_0 \leq e^{H W(e/(2\eta)) - 1/2} \} + C_1^H (H-\eta) 1 \{ C_0 > e^{H W(e/(2\eta)) - 1/2} \} \right)^p \right]$$

$$\leq 2^{p-1} W(e/(2\eta))^{p/(2H)} E[C_0^H] + 2^{p-1} E \left[ C_0 > e^{W(e/(2\eta)) - 1/2} \right]$$

$$\leq 2^{p-1} W(e/(2\eta))^{p/(2H)} E[C_0^H] + 2^{p-1} \left( E[C^2 e^{(H-\eta) - 1/2}] \right)^{1/2}$$

$$\leq E[C] \left( e^{(H-\eta) W(e/(2\eta)) - 1/(2H)} \right)^{1/2}.$$

Since $W(e/(2\eta)) = \log(e/(2\eta)) - \log^2(e/(2\eta)) + o(1)$ as $\eta \to 0$ (see [13]), the above inequalities together with (1.11) imply that, there is $c > 0$ such that

$$E[C_0^p] \leq c^p (\log^{-1} \eta)^{p/(2H)} \sqrt{\Gamma \left( \frac{p}{H} + \frac{1}{2} \right)} + c^p \left( \Gamma \left( \frac{2p}{H - \eta} + \frac{1}{2} \right) \right)^{1/4} e^{-H e/(2\eta)}$$

$$\leq 2 c^p (\log^{-1} \eta)^{p/(2H)} \sqrt{\Gamma \left( \frac{p}{H} + \frac{1}{2} \right)}, \text{ as } \eta \to 0.$$
The last inequality in (3.14) holds, because when \( \eta < H/2 \),
\[
\sup_{p \geq 1} \frac{\Gamma((2p/(H-\eta))+1/2)}{\sqrt{\Gamma(p/H+1/2)(\log^{-1}p)(2H)}} \leq \sup_{p \geq 1} \frac{\Gamma((4p/H+1/2))}{\sqrt{\Gamma(p/H+1/2)(\log^{-1}p)(2H)}} \leq \frac{c e^{-He/(2\eta)}}{(\log^{-1}p)(2H)} \to 0,
\]
where, by Stirling's formula, \( c = \sup_{p \geq 1} \frac{\Gamma((4p/(H+1/2)))^{1/4}}{\sqrt{\Gamma(p/(H+1/2)(\log^{-1}p)(2H)}} < +\infty \). Recall the following property of Gamma function (this can be proved by applying Stirling's formula): there is \( c > 0 \) such that
\[
\Gamma(x+1) \leq c \sqrt{\pi} e^{-x} x^x, \text{ for } x \text{ large enough. Taking } x = p/H - 1/2 \text{ (we assume } p \text{ is large) in the above inequality we obtain: there is } c > 1 \text{ such that}
\]
\[
\Gamma\left(\frac{p}{H} + \frac{1}{2}\right) \leq c^p p^{p/H}.
\]
(3.15)

It follows from (3.14) and (3.15) that, there is \( c > 0 \) such that
\[
\mathbb{E}[\mathcal{C}_u^B]^{1/p} \leq c (\log^{-1}p)^{1/(2H)} p^{1/(2H)}.
\]
(3.16)

The Blumenthal's zero-one law entails that \( B^H(u^*) > 0 \) a.s. As a result, \( \mathbb{E}[e^{-qB^H(u^*)}] \leq \mathbb{E}[e^{-B^H(u^*)}] \) for \( q > 1 \). This fact together with (3.13) and (3.16) yields that, there is a constant \( c_2 > 0 \) such that
\[
\mathbb{E}[e^{-qB^H(u^*)}] \geq c_2^{-q} (\log^{-1}p)^{-q/(2H)} p^{-q/(2H)} (\log^2 T)^{-q/(2H)} (I(T))^{\delta}.
\]
(3.17)

From Molchan's result (see [23], Statement 1) we see that there is \( c_3 > 0 \) such that
\[
I(T) \geq c_3 T^{-(1-H)}, \text{ as } T \to +\infty.
\]
(3.18)

(3.17) together with (3.18) yields that, there is \( c_4 > 0 \) such that
\[
\mathbb{E}[e^{-qB^H(u^*)}] \geq c_4^{-q} (\log^{-1}p)^{-q/(2H)} p^{-q/(2H)} (\log^2 T)^{-q/(2H)} T^{-q(1-H)}.
\]
(3.19)

Let the integer \( N \geq 3 \), set \( p = \log T (\log^{N+2} T)^{-1} \) and \( \eta = (\log^{N+1} T)^{-1} \). Note that \( q = 1/(1-1/p) = 1 + \Theta(p^{-1}) \) as \( p \to +\infty \). Then the right-hand side of (3.19) can be written as
\[
\exp\left\{ -q \left[ \log(c_4) + \frac{\log^2 \eta^{-1}}{2H} + \frac{\log p}{2H} + \frac{\log T}{2H} + (1-H) \log T \right] \right\}
\]
\[
= \exp\left\{ -\left(1 + \Theta(p^{-1})\right) \left[ \log(c_4) + \frac{\log^{N+3} T}{2H} + \frac{\log^2 T}{2H} - \log^{N+3} T \right] + \frac{\log^3 T}{2H} + (1-H) \log T \right\}
\]
\[
= \exp\left\{ -\left(1-H\right) \log T + \frac{\log^2 T}{2H} + \frac{\log^3 T}{2H} + o(\log^{N+1} T) \right\}
\]
\[
= T^{-(1-H)} (\log T)^{-1/(2H)} (\log^2 T)^{-1/(2H)} (\log^3 T)^{o(1)}, \text{ as } T \to +\infty.
\]
(3.20)

It follows from (3.19) and (3.20) that for \( \gamma > 0 \) arbitrarily small,
\[
\mathbb{E}[e^{-qB^H(u^*)}] \geq T^{-(1-H)} (\log T)^{-1/(2H)} (\log^2 T)^{-1/(2H)} (\log^3 T)^{-\gamma}, \text{ as } T \to +\infty.
\]
(3.21)

Recall that Tauberian theorem (see [10], Corollary 8.1.7) says that if \( Y \) is a self-similar stationary increment process,
\[
P\left( \sup_{u \in [0,1]} Y(Tu) \leq 1 \right) \sim \mathbb{E}[\mathcal{C}_{u^*}]^{1/(2H)} Y(Tu^*), \text{ as } T \to +\infty,
\]
where \( \sim \) means the asymptotic equivalence. As a result for the fBm \( B^H \) we have: there exists \( c_5 > 0 \) such that for \( \gamma > 0 \) arbitrarily small,
\[
P\left( \sup_{u \in [0,1]} B^H(u) \leq 1 \right) = P\left( \sup_{u \in [0,1]} B^H(Tu) \leq 1 \right) \geq c_5 \mathbb{E}[e^{-B^H(Tu^*)}] = c_5 \mathbb{E}[e^{-qB^H(u^*)}] \geq T^{-(1-H)} (\log T)^{-1/(2H)} (\log^2 T)^{-1/(2H)} (\log^3 T)^{-\gamma}, \text{ as } T \to +\infty.
\]
(3.22)

The asymptotic lower bound of \( P(t_1 > T) \) in Theorem 1.4 is obtained.
3.2 Asymptotic Upper Bound of $P(\tau_1 > T)$

The main idea of obtaining the upper bound $\mathbb{E}(T, N, \gamma)$ for $P(\tau_1 > T)$ is to consider estimating the expectation in (1.8) on a special collection of paths, for which $B^H(t) \leq \varrho(t)$ for $t \in [0, T]$ (see (3.25)). These paths contribute most of the first exit of fBm, and they can be relatively easy to evaluate.

Fix a constant $\lambda \geq 1/H > 1$ and define

$$\varrho(t) = \begin{cases} 1, & t \in [0, (\lambda \log^2 T)^{1/H}]; \\ -\lambda \log^2 T, & t \in [(\lambda \log^2 T)^{1/H}, (\lambda \log T)^{1/H}]; \\ -\lambda \log T, & t \in [(\lambda \log T)^{1/H}, T]. \end{cases}$$

(3.23)

In view of (3.23), for sufficiently large $T$, we can obtain

$$\int_0^T e^{\varrho(u)} du = \int_0^{(\lambda \log^2 T)^{1/H}} e^u du + \int_{(\lambda \log^2 T)^{1/H}}^{(\lambda \log T)^{1/H}} (\log T)^{-\lambda} u + \int_{(\lambda \log T)^{1/H}}^T T^{-\lambda} du \leq 3e(\lambda \log^2 T)^{1/H}.$$  

(3.24)

It follows from (3.24) that, $I(T)$ can be lower bounded by taking the expected value restricted to the paths $B^H(t) \leq \varrho(t), t \in [0, T]$:

$$I(T) \geq E \left\{\left[1 \{t \in [0, T] : B^H(t) \leq \varrho(t)\} \left(\int_0^T e^{\varrho(u)} du\right)^{-1}\right]\right\} 
\geq E \left\{\left[1 \{t \in [0, T] : B^H(t) \leq \varrho(t)\} \left(\int_0^T e^{\varrho(u)} du\right)^{-1}\right]\right\} 
\geq (3e)^{-1} \lambda^{-1/H}P\{t \in [0, T] : B^H(t) \leq \varrho(t)\} \leq (T/\lambda)^{1/H}.$$  

(3.25)

Next we analyze the asymptotic lower bound of $P(t \in [0, T] : B^H(t) \leq \varrho(t))$. Since $B^H(s)$ and $B^H(t)$ are positively correlated for $s, t \geq 0$, we can apply Theorem 3.1 to get

$$P(t \in [0, T] : B^H(t) \leq \varrho(t)) \geq P\left(\sup_{0 \leq s \leq (\lambda \log^2 T)^{1/H}} B^H(t) \leq 1\right) 
\times P\left(\sup_{(\lambda \log^2 T)^{1/H} \leq s \leq (\lambda \log T)^{1/H}} B^H(t) \leq -\lambda \log^2 T\right) \times P\left(\sup_{(\lambda \log T)^{1/H} \leq s \leq T} B^H(t) \leq -\lambda \log T\right).$$

(3.26)

The first factor on the right-hand side of (3.26), by (3.22), can be estimated as follows: for $N \geq 5$ and for $\gamma > 0$ arbitrarily small,

$$P\left(\sup_{0 \leq s \leq (\lambda \log^2 T)^{1/H}} B^H(t) \leq 1\right) \geq (\log^2 T)^{-(1-H)/H}(\log^3 T)^{-1/(2H)}(\log^4 T)^{-1/(2H)}(\log^N T)^{-\gamma},$$

(3.27)

as $T \to +\infty$. The ways of lower bounding the second and the third factors in (3.26) are quite similar. In fact, by using the $H$-self-similarity of $B^H$ and the changes of variable $t \to t(\lambda \log^2 T)^{-1/H}$ and $t \to t(\lambda \log T)^{-1/H}$ respectively, we have

$$P\left(\sup_{(\lambda \log^2 T)^{1/H} \leq s \leq (\lambda \log T)^{1/H}} B^H(t) \leq -\lambda \log^2 T\right) = P\left(\sup_{1 \leq s \leq (\log T/\lambda \log^2 T)^{1/H}} B^H(t) \leq -1\right)$$

(3.28)

and

$$P\left(\sup_{(\lambda \log T)^{1/H} \leq s \leq T} B^H(t) \leq -\lambda \log T\right) = P\left(\sup_{1 \leq s \leq T/(\log T/\lambda \log T)^{1/H}} B^H(t) \leq -1\right) \geq P\left(\sup_{1 \leq s \leq T} B^H(t) \leq -1\right).$$

(3.29)

For bounding $P(\sup_{1 \leq s \leq T} B^H(t) \leq -1)$ from below, we deal with the cases $H \geq 1/2$ and $H < 1/2$ separately.
Case 1: \( H \geq 1/2 \).

In this case the increments of the fBm \( B^H \) are non-negatively correlated, then following the arguments in the inequalities (10) of [1], there is \( c_1 > 0 \) such that
\[
P\left( \sup_{1 \leq t \leq T} B^H(t) \leq -1 \right) \geq c_1 P\left( \sup_{0 \leq t \leq T} B^H(t) \leq 1 \right). \tag{3.30}
\]

It follows from (3.29) and (3.30) that, for \( T \) sufficiently large,
\[
P\left( \sup_{(\lambda \log T) \leq t \leq (\lambda \log T) + 1} B^H(t) \leq -\lambda \log T \right) \geq c_1 P\left( \sup_{0 \leq t \leq T} B^H(t) \leq 1 \right). \tag{3.31}
\]

By following the same arguments as in (3.29) - (3.31) and using (3.22), we can show that the second factor on the right-hand side of (3.26) verifies: there exists \( c_2 > 0 \) such that for any \( N \geq 4 \) and any \( \gamma > 0 \) arbitrarily small,
\[
P\left( \sup_{(\lambda \log T) \leq t \leq (\lambda \log T) + 1} B^H(t) \leq -\lambda \log^2 T \right) \geq c_2 P\left( \sup_{0 \leq t \leq T} B^H(t) \leq 1 \right) \tag{3.32}
\]
\[
\geq (\lambda \log T)^{-1(H+1)} (\log^2 T)^{-1(H+1)} (\log^3 T)^{-1/2H} (\log^4 T)_{(1/2H)}^{1} (\log^N T)^{-\gamma}, \text{ as } T \to +\infty.
\]

Finally combining (3.25) - (3.27), (3.31), (3.32) and using (3.18), we obtain, there exists \( c_3 > 0 \) such that for any \( N \geq 5 \) and any \( \gamma > 0 \) arbitrarily small,
\[
P\left( \sup_{0 \leq t \leq T} B^H(t) \leq 1 \right) \leq T^{-(1-H)(\log T)^{1/H-1}} (\log^2 T)^{-1/(2H)} (\log^3 T)^{1/H} (\log^4 T)_{(1/2H)}^{1} (\log^N T)^{-\gamma}, \tag{3.33}
\]
as \( T \to +\infty \).

Case 2: \( H < 1/2 \).

Let \( N \geq 4 \), we set \( g(t) = 2\sqrt{\log^{N+1}(\exp^{N+1}(1))}, \) for \( t \geq 1 \). It is easy to verify that such \( g \) satisfies \( g(1) = 2 \) and the properties (i) – (iii) in Lemma 3.2. Therefore Lemma 3.2 as well as Lemma 3.3 holds true with such \( g \). It follows from the self-similarity and Lemma 3.3 that: there are \( l_0 \geq 1 \) and \( \kappa > 0 \) such that
\[
P\left( \sup_{1 \leq t \leq T} B^H(t) \leq -1 \right) = P\left( \sup_{l_0 \leq t \leq T} B^H(t) \leq -l_0^{1/H} \right) \geq P\left( \sup_{0 \leq t \leq T(\log^{N+1}(1))^{1/2H}} B^H(t) \leq 1 \right) e^{-4l_0^{2/H} \log^{N+1}(\exp^{N+1}(1))} \tag{3.34}
\]
\[
\geq P\left( \sup_{0 \leq t \leq T} B^H(t) \leq 1 \right) (\log T)^{-\kappa},
\]
where the fact that \( \log^{N+1}(\exp^{N+1}(1)) \leq 2\log^{N+1} T \) as \( T \to +\infty \) is used. In (3.34) substituting \( T \) with \( (\log T)^{1/H}(\log^2 T)^{-1/H} \) and using (3.22) yields that, for \( \gamma > 0 \) arbitrarily small,
\[
P\left( \sup_{1 \leq t \leq (\log T)^{1/H}(\log^2 T)^{-1/H}} B^H(t) \leq -1 \right) \geq P\left( \sup_{0 \leq t \leq (\log T)^{1/H}(\log^2 T)^{-1/H}} B^H(t) \leq 1 \right) (\log T)^{1/2H} (\log^2 T)^{-1/(2H)} (\log^3 T)^{-1/2H} (\log^N T)^{-\gamma}, \tag{3.35}
\]
as \( T \to +\infty \).
Combining (3.25) - (3.29), (3.34) - (3.35) and (3.18), we obtain, for any $N \geq 5$ arbitrarily large and any $\gamma > 0$ arbitrarily small,

$$
\mathbb{P}\left( \sup_{0 \leq t \leq T} B^H(t) \leq 1 \right) \leq T^{-(1-H)\left(\log T\right)^{1/(2H)}(\log^3 T)(1/H)(\log^4 T)(1/(2H))}\left(\log^N T\right)^{\gamma},
$$

(3.36)
as $T \to +\infty$. Putting (3.33) and (3.36) together leads to the asymptotic upper bound of $\mathbb{P}(\tau_1 > T)$ for $H \in (0, 1)$. Finally the proof of Theorem 1.4 is complete.

4 Conclusions

There is a growing body of research that aims at finding the persistence exponent $\theta > 0$, such that the upper-tail probability for a stochastic process to first exit from a positive-valued barrier during the time interval $[0, T]$, can be asymptotically expressed as a polynomial form $T^{-\theta + o(1)}$, as $T \to +\infty$. The goal of this framework has been to illuminate a direction of research that aims at improving the accuracy of estimation of the loss factor $T^{o(1)}$ in the latter form. The prototypical example we have chosen here is fractional Brownian motion, but the potential applications of the methodology to more general processes extend well beyond this.

To accomplish this, first we have improved the almost sure upper bound of the small fractional Brownian increments, where the random multiplier in the upper bound has controllable moments. This result fills some gap of the law of iterated logarithm for fBm, where the moments information of the random multiplier in the upper bound is missing. In this stage, an optimal rate wavelet series representation of fBm is applied. Then we have used Molchan and Aurzada’s ideas to improve the asymptotic lower and upper bounds of the upper-tail probability for an fBm to first exit from the one-sided barrier. In order to obtain a tight upper bound of $I(T)$, the main challenge becomes to find the most suitable almost sure upper bound of fractional Brownian increments. Indeed, in the upper bound $C(t)\epsilon_0^H \sqrt{\log^2 \epsilon_0^{-1}}$, $\epsilon_0$ should be neither too large nor too small.

Progress in this field of research has been accelerating along with the study of persistence probabilities of stochastic processes. Accordingly, following the work of Aurzada [1] our main contribution here has been to demonstrate that the accuracy of the persistence probability $T^{-\theta + o(1)}$ can be further improved through a more precise evaluation of the process’ increments. This improvement of accuracy can lead to refined results on the estimation of the first exit time behavior and real world’s numerical applications will take benefits of it. It is our hope that this work has reinforced the importance of improving the accuracy of the asymptotic rate of the upper-tail probabilities for a process to first exit from a one-sided barrier.

A Proof of Lemma 3.2

Proof. The proof basically follows Aurzada’s approach for proving Lemma 4 in [1], subject to some modifications, especially on Step 2 below. First in his proof of Lemma 4 in [1], Aurzada
showed that
\[
(g(t)^2 t^a - g(t)(t^a + 1 - (t-1)^a) + 1)^{1/a} - (g(s)^2 s^a - g(s)(s^a + 1 - (s-1)^a) + 1)^{1/a} \\
- (g(t)^2 t^a - g(t)g(s)(t^a + s^a - (t-s)^a) + g(s)^2 s^a)^{1/a} \geq g(s)^{1/a} \mathcal{K},
\]
where
\[
\mathcal{K} = g(t)^{1/a} (t^a - 1)^{1/a} - g(s)^{1/a} (s^a - 1)^{1/a} - (g(t)(t-s)^a - g(t)s^a + g(s)s^a)^{1/a}.
\]
Aurzada pointed out that, in order that (A.1) holds, the following conditions are sufficient: for \( s \) being sufficiently large and \( t \geq s \geq 1 \),

1. \( 1 - g(t)(t^a - (t-1)^a) \geq 0 \).
2. \( g(s)(s^a - (s-1)^a) - g(t)(t^a - (t-1)^a) \geq 0 \).
3. \( g(t)^2 t^a - g(t)g(s)t^a \geq 0 \).
4. \( g(t)g(s) - g(t) \geq 0 \).
5. \( -g(s) + g(t) - g(t)g(s) \leq -g(s)^2 \).

All the above conditions can be easily verified by using the properties \((i) - (iii)\) of the function \( g \).

It remains to prove \( \mathcal{K} \geq 0 \) for sufficiently large \( s, t \). To this end Aurzada took 3 steps.

**Step 1:** Show that for \( s \) large enough and \( s \leq t \),
\[
(g(t)(t-s)^a - g(t)s^a + g(s)s^a)^{1/a} \leq (g(t)t^a - g(t)s^a + g(s)s^a)^{1/a} - g(s)^{1/a} s.
\]
Aurzada proved that (A.3) holds true, provided
\[
\frac{g(t)}{g(s)} \geq 1 \text{ and } \frac{t}{s} \geq \frac{g(t)}{g(s)}, \text{ as } s \text{ is large.}
\]

The first inequality in (A.4) holds since \( g \) is increasing. The second one also holds since by the properties \((i) - (iii)\) of \( g \), \( g(s)/s \) is decreasing starting from some point:
\[
\left(\frac{g(s)}{s}\right)' = g'(s) s - g(s) = a\left(\frac{1}{g(s)s^2}\right) - \frac{g(s)}{s^2} < 0, \text{ as } s \to +\infty.
\]
Therefore (A.3) holds subject to the conditions \((i) - (iii)\) on \( g \).

**Step 2:** Show that for \( s \) large enough and \( s \leq t \),
\[
g(t)^{1/a} (t^a - 1)^{1/a} - g(s)^{1/a} (s^a - 1)^{1/a} \geq (g(t)t^a - g(t)s^a + g(s)s^a)^{1/a} - g(s)^{1/a} s.
\]
In order that (A.5) holds, Aurzada pointed out that it is sufficient to prove
\[
g(t)^{1/a - 1} g'(t)(t^a - 1)^{1/a} \geq (g(t)t^a - g(t)s^a + g(s)s^a)^{1/a - 1} g'(t) (t^a - 2)
\]
and
\[
g(t)^{1/a} (t^a - 1)^{1/a - 1} a t^{-1} \geq (g(t)t^a - g(t)s^a + g(s)s^a)^{1/a - 1} (ag(t)t^{a-1} - g'(t)(s^a - 2)).
\]
First, using the fact that $g'(t) > 0$ and $g(t) \geq g(s)$ for $t \geq s$, (A.6) can be easily proved. Next we prove (A.7), which needs more efforts. Divided by $g(t)^{1/a}$, (A.7) becomes

$$(t^a - 1)^{1/a - 1} \alpha t^{a - 1} \geq (t^a - \nu)^{1/a - 1} \left(\alpha t^{a - 1} - \frac{g'(t)}{g(t)}(s^a - 2)\right), \tag{A.8}$$

where $\nu = s^a(1 - g(s)/g(t)) > 0$. In what follows two situations are considered.

**Case 1:** $s^a \geq \left(\frac{g(t)}{tg'(t)}\right)^{1+\gamma_0}$, where $\gamma_0 > 0$ is a fixed value but can be arbitrarily small.

In this case we use the fact that $\nu > 0$ and the mean value theorem to obtain

$$(1 - (t^a - \nu)^{-1/a + 1}(t^a - 1)^{1/a - 1}) \alpha t^{a - 1} \frac{g(t)}{g'(t)} \leq (1 - (t^a - 1)^{-1/a + 1}(t^a - 1)^{1/a - 1}) \alpha t^{a - 1} \frac{g(t)}{g'(t)}$$

$$= \left((t^a)^{1/a - 1} - (t^a - 1)^{1/a - 1}\right)(t^a)^{-1/a + 1} \alpha t^{a - 1} \frac{g(t)}{g'(t)}$$

$$= \left(\frac{1}{a} - 1\right) t^{a - 2} \alpha t^{a - 1} \frac{g(t)}{g'(t)} \leq ct^{a - 2} \alpha t^{a - 1} \frac{g(t)}{g'(t)} = c \frac{g(t)}{tg'(t)},$$

where $c$ is some value in $(t^a - 1, t^a)$ and the constant $c > 0$ does not depend on $s, t$. By the property (iii) of $g$, we know that $\frac{g(t)}{tg'(t)} \rightarrow +\infty$. This together with the above inequality and the fact that $s^a \geq \left(\frac{g(t)}{tg'(t)}\right)^{1+\gamma_0}$ entails $(1 - (t^a - \nu)^{-1/a + 1}(t^a - 1)^{1/a - 1}) \alpha t^{a - 1} \frac{g(t)}{g'(t)} \leq s^a - 2$, for sufficiently large $s$. This inequality is equivalent to (A.8). Hence (A.8) is proved in this case.

**Case 2:** $s^a < \left(\frac{g(t)}{tg'(t)}\right)^{1+\gamma_0}$.

In this case since $(1 + \gamma_0)/a > 1$, by using the property (iii) of $g$ we have

$$g(s) \leq g\left(\left(\frac{t}{tg'(t)}\right)^{(1+\gamma_0)/a}\right) = o(g(t)), \text{ as } t \rightarrow +\infty.$$  

As a consequence, $g(s)/g(t) \leq 1/2$ for $s$ sufficiently large. Hence for $s$ large enough,

$$\nu = s^a\left(1 - \frac{g(s)}{g(t)}\right) \geq \frac{1}{2} s^a \geq 1.$$  

This results in (A.8), because

$$(t^a - 1)^{1/a - 1} \geq (t^a - \nu)^{1/a - 1} \geq 0 \text{ and } \alpha t^{a - 1} \geq \alpha t^{a - 1} - \frac{g'(t)}{g(t)}(s^a - 2) \geq 0.$$  

(A.5) is proved hence Step 2 is complete.

Finally it follows from Steps 1 - 2 that

$$g(t)^{1/a}(t^a - 1)^{1/a} - g(s)^{1/a}(s^a - 1)^{1/a} \geq \left(g(t) t^a - g(t) s^a + g(s) s^a\right)^{1/a} - g(s)^{1/a} s$$

$$\geq \left(g(t)(t - s)^a - g(t) s^a + g(s) s^a\right).$$

This inequality is exactly $\mathcal{K} \geq 0$ (see (A.2)). The proof of Lemma 3.2 is complete. \qed
B Proof of Lemma 3.3

Proof. Through this proof we only consider \( t \geq 1 \). Let \( g \) satisfy \( g(1) = 2 \) and the properties \((i) - (iii)\) in Lemma 3.2. Next define \( X(t) = g(t)B^H(t) - B^H(1) \), for \( t \geq 1 \). Also let \( u(t) \) be the positive-valued function such that
\[
\mathbb{E}[X(t)^2] = \mathbb{E}[(g^2 B^H(g(t)))^2] = u(t)^2, \quad \text{for } t \geq 1.
\]

(B.1)

Based on the definitions of \( u(t) \) and \( g(t) \), there is a constant \( \kappa > 0 \) such that
\[
u(t) = (g(t)^2 t^{2H} - g(t)(t^{2H} + 1 - (t - 1)^2 t^{2H}) + 1)^{1/2H} \leq \kappa g(t)^{1/2}.
\]

(B.2)

The new process \( X \) satisfies the following properties:

Property 1: \( X(t) \) \( (t \geq 1) \) and \( B^H(1) \) is positively correlated. Indeed, since \( g(t) \geq 2 \) for \( t \geq 1 \), we can write
\[
\mathbb{E}[X(t)B^H(1)] = \frac{g(t)}{2} (t^{2H} + 1 - (t - 1)^2 t^{2H}) - 1 \geq \frac{g(t)}{2} - 1 \geq 0.
\]

Property 2: There is a constant \( t_0 \geq 1 \) such that \( \mathbb{E}[X(t)X(s)] \geq \mathbb{E}[(B^H(u(t))B^H(u(s)))], \) for \( t, s \geq t_0 \).

Observe that the above inequality is nothing but the inequality \((3.2)\) with \( \alpha = 2H < 1 \). As a result Property 2 holds, thanks to Lemma 3.2.

Note that the conditions of Lemma 1 and Theorem 1 in [33] are verified by Property 1 and Property 2, then it follows from Lemma 1 and Theorem 1 in [33] that
\[
P\left(\sup_{t_0 \leq t \leq T} X(t) \leq 1\right) \geq P\left(\sup_{t_0 \leq t \leq T} B^H(u(t)) \leq 1\right). \tag{B.3}
\]

Now we prove the main statement of the lemma. Since \( X(t) = g(t)B^H(t) - B^H(1) \), for any \( a \in \mathbb{R} \) we have
\[
P\left(\sup_{t_0 \leq t \leq T} B^H(t) \leq -a \right) \geq P\left(\sup_{t_0 \leq t \leq T} X(t) \leq -B^H(1) - g(T) a \right) \geq P\left(\sup_{t_0 \leq t \leq T} X(t) \leq 1, B^H(1) \leq -g(T) a - 1 \right). \tag{B.4}
\]

Since \( \text{Cov}(X(t), B^H(1)) \geq 0 \) for \( t \geq 1 \), by Theorem 3.1 we get
\[
P\left(\sup_{t_0 \leq t \leq T} X(t) \leq 1, B^H(1) \leq -g(T) a - 1 \right) \geq P\left(\sup_{t_0 \leq t \leq T} X(t) \leq 1 \right) P\left(\sup_{t_0 \leq t \leq T} B^H(1) \leq -g(T) a - 1 \right). \tag{B.5}
\]

Next recall that the standard Gaussian variable \( Z \)’s probability distribution function has the following lower bound (see e.g. [12]): for \( t > 0 \) large enough,
\[
P(Z > t) > \frac{1}{\sqrt{2\pi} t^2 + 1} \ e^{-t^2/2} \geq e^{-2t^3/3}. \tag{B.6}
\]

Since \( B^H(1) \sim \mathcal{N}(0, 1) \), it then results from \((B.6)\) that, for \( T \) large enough,
\[
P\left(B^H(1) \leq -g(T) a - 1 \right) \geq P\left(B^H(1) \geq |g(T) a + 1| \right) \geq e^{-g(T)^2 a^2}. \tag{B.7}
\]

Combining \((B.4)\) - \((B.7)\), \((B.3)\) and \((B.2)\), we get
\[
P\left(\sup_{t_0 \leq t \leq T} B^H(t) \leq -a \right) \geq P\left(\sup_{t_0 \leq t \leq T} X(t) \leq 1 \right) e^{-g(T)^2 a^2} \geq P\left(\sup_{t_0 \leq t \leq T} B^H(u(t)) \leq 1 \right) e^{-g(T)^2 a^2}
\]
\[
= P\left(\sup_{t_0 \leq t \leq T} B^H(t) \leq 1 \right) e^{-g(T)^2 a^2} \geq P\left(\sup_{0 \leq t \leq T \leq 1/H} B^H(t) \leq 1 \right) e^{-g(T)^2 a^2}.
\]

This completes the proof. \( \square \)
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