Non-Hermitian Heisenberg representation

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Abstract

In a way paralleling the recently accepted non-Hermitian version of quantum mechanics in its Schrödinger representation (working often with the innovative and heuristically productive concept of $\mathcal{PT}$-symmetry), it is demonstrated that it is also possible to construct an analogous non-Hermitian version of quantum mechanics in its Heisenberg representation.

Keywords

quantum mechanics; non-Hermitian representation of observables; generalized Heisenberg equations;
1 Introduction

One of the paradoxes connected with the birth of quantum theory \[1\] is that it was first formulated in its less economical, matrix version called Heisenberg representation (HREP, \[2\]). Naturally, almost immediately there emerged its amended, equivalent but more economical vectorial version carrying the name of Schrödinger representation (SREP, \[3\]).

An explanation of the apparent paradox may be found in the much more friendly correspondence of the quantum HREP picture with its classical limit. As a consequence, an additional feature of the paradox is that the HREP formulation of quantum theory survived in the competition. During cca ninety years of its existence the HREP language found important specific and unique applications, typically, in the models using creation and annihilation operators \[4\] and, in particular, in the context of relativistic quantum field theory \[5\].

We feel inspired by a certain loss of ambition of the HREP theoreticians and users in the newly emergent field of the study of quantum observables in their so called non-Hermitian representations \[6\]. This loss of ambition resulted in a loss of the historical leadership because the innovated quantum theory seems currently available just in its SREP non-Hermitian version (SRNH, see, e.g., reviews \[7, 8\]). Thus, we believe that it is time to describe and discuss also a parallel upgrade of the theory in its alternative, non-Hermitian HREP-based (NHH) version.

The first steps in this direction are to be made in what follows.

2 Quantum theory using time-independent Hilbert spaces

From the historical point of view the SREP-based transition to non-Hermitian Hamiltonians and/or other observables was introduced, almost 60 years ago, by Freeman Dyson \[9\]. Unfortunately, it took several decades before the idea found another and, incidentally, most successful series of applications in nuclear physics (cf. review plus basic SRNH theory in Ref. \[10\]). The reason of delay lied, obviously, in a perceivable increase of mathematical complexity during the SREP $\rightarrow$ SRNH change of the perspective. In contrast, the main source of the persuasive delayed success of the innovated non-Hermitian theory was truly unexpected. After the SREP $\rightarrow$ SRNH upgrade of the algorithms people reported a perceivable simplification of numerical computations \[10\].

The latter observation returned attention to various unitary time-evolution SREP prescriptions with difficult (i.e., e.g., manybody) Hamiltonians $\hat{h}_{(SREP)} = \hat{h}_{(SREP)}^\dagger$,

$$i\partial_t |\varphi^{(SREP)}(t)\rangle = \hat{h}_{(SREP)} |\varphi^{(SREP)}(t)\rangle, \quad |\varphi^{(SREP)}(t)\rangle \in \mathcal{H}^{(P)}. \quad (1)$$

According to conventional textbooks the Hamiltonians may be complicated but they must be self-adjoint in the preselected Hilbert space of states in which the superscript \((P)\) stands
for “physical”. In the subsequent step the Dyson’s idea has been recalled. Its essence lies in an inclusion of a part of our a priori knowledge of the structure of $|\varphi^{(\text{SREP})}(t)\rangle$ into a non-unitarily pre-conditioned ansatz

$$|\varphi^{(\text{SREP})}(t)\rangle = \Omega|\psi^{(\text{SRNH})}(t)\rangle, \quad |\psi^{(\text{SRNH})}(t)\rangle \in \mathcal{H}^{(F)}$$

(2)

with a nontrivial operator $\Omega^\dagger \Omega = \Theta \neq I$ called metric.

Under some reasonable assumptions about the linear and time-independent operators $\Omega$ and $\Theta$ the latter ansatz converts Eq. (1) into its formally equivalent SRNH alternative

$$i\partial_t |\psi^{(\text{SRNH})}(t)\rangle = H^{(\text{SRNH})} |\psi^{(\text{SRNH})}(t)\rangle, \quad |\psi^{(\text{SRNH})}(t)\rangle \in \mathcal{H}^{(F)}.$$ (3)

Here, one works in another, auxiliary Hilbert space which is, by construction, unphysical. Its symbol is characterized by superscripted $(F)$ which abbreviates “friendly” as well as “false” $[11]$.

Although it gave the name to the theory, it is not too relevant that the transformed Hamiltonian is manifestly non-Hermitian, i.e., that we have $H^{(\text{SRNH})} \neq H^{(\text{SRNH})\dagger}$ in the $F-$superscripted Hilbert space. The reason is that the latter Hilbert space is purely auxiliary and that it is not intended to carry any quantum theoretical meaning by itself. What is only important is that in the light of the underlying theory the necessary physical meaning may easily be reinstalled if one redefines the inner product $[7,8]$. In mathematical language this means that one has to replace the unphysical $F-$superscripted Hilbert space by its “standard” physical amendment $\mathcal{H}^{(S)}$. The latter space differs from its predecessor $\mathcal{H}^{(F)}$ strictly and solely by its upgraded, metric-dependent inner product between any pair of its shared ket-vector elements,

$$\langle \psi_1 | \psi_2 \rangle^{(S)} \equiv \langle \psi_1 | \Theta | \psi_2 \rangle^{(F)}, \quad |\psi_{1,2}\rangle \in \mathcal{H}^{(F,S)}.$$ (4)

The mathematical core of the whole SRNH scheme of Refs. $[7,8,10]$ just follows from the obvious identity

$$\langle \varphi_1 | \varphi_2 \rangle = \langle \psi_1 | \Theta | \psi_2 \rangle = \langle \tilde{\psi}_1 | \psi_2 \rangle, \quad |\varphi_{1,2}\rangle \in \mathcal{H}^{(P)}, \quad |\psi_{1,2}\rangle, |\tilde{\psi}_1\rangle \in \mathcal{H}^{(F)}.$$ (5)

These relations may be read as a proof of the strict unitary equivalence $\mathcal{H}^{(S)} \equiv \mathcal{H}^{(P)}$ between the two alternative physical Hilbert spaces. As an immediate consequence one deduces the indistinguishability between the $P-$based and $S-$based physical probabilistic predictions.

One of the important advantages of the SRNH formalism is that in the light of relations $[5]$ one can completely suppress the reference to our ultimate physical $S-$superscripted SRNH Hilbert space once one decides to use the nontrivial metric, whenever needed, in explicit manner. Such a convention is also being accepted in the majority of the newer SRNH applications as reviewed in Refs. $[7,8]$ or $[12]$.

In the more pragmatic and phenomenological physical setting it makes sense to recall, once more, the instructive nuclear-physics illustration of Ref. $[10]$ in which the initial space
\( \mathcal{H}^{(P)} \) had the form of a Pauli-antisymmetrized Fock space of the half-spin fermions while the (shared) ket-vector elements of Dyson-partner spaces \( \mathcal{H}^{(F,S)} \) were the states which obeyed the perceivably simpler statistics of effective bosons. Another important class of implementations of the whole SRNH pattern are the systems in which the non-Hermitian Hamiltonians in Schrödinger Eq. (3) are pseudo-Hermitian alias \( \mathcal{PT} \) symmetric (cf. reviews \[7, 8, 12\] for more details).

3 Time-dependent maps \( \Omega(t) \)

In the conventional, Hermitian quantum models the operators of observables \( a \) are usually assumed, in the SREP setting, time-independent, \( a^{(SREP)}(t) = a^{(SREP)}(0) \). Less often, the time-dependent influence, say, of a variable external field is reflected by an induced time-dependence in \( a^{(SREP)}(t) \). In such a case, it must be treated as an independent dynamical input information. Hence, it cannot follow from any equations and must be prescribed in full extent.

In contrast, what is assumed time-independent in the conventional Hermitian HREP picture are wave functions,

\[
|\varphi^{(HREP)}(t)\rangle = |\varphi^{(HREP)}(0)\rangle .
\]

The necessary unitary transformation which mediates the transition between the SREP and HREP pictures is elementary. Its form may be found in any textbook \[4\].

Our present study is inspired by the fact that the conventional unitary SREP \( \rightarrow \) HREP transition may be given the generic form of Eq. (2). Unfortunately, this seems to be in conflict with the current literature on non-Hermitian models. Indeed, up to a few inconclusive exceptions \[13\] there emerged virtually no studies using time-dependent non-unitary maps \( \Omega(t) \) and/or nontrivial metrics \( \Theta(t) \). In \[8\], moreover, it has been shown that the stationarity of metrics is, up to an overall phase, unavoidable whenever one requires that the time-evolution generator \( H \) entering SRNH Eq. (3) represents an observable quantity.

In a way explained in Ref. \[14\] we know that the requirement of the observability of the time-evolution generator is rather artificial and that its removal leaves the whole SRNH theory fully mathematically consistent. Simultaneously, we are fully aware of the fact that many practical merits of the SRNH recipes (and, first of all, the possibility of a simplification of Eq.(1)) get definitely lost for nontrivial \( \Omega(t) \neq \Omega(0) \) \[15\]. In the context of such a conflict of our own opinions the core of our present message is twofold. Firstly, we shall outline the NHH formalism in a concrete form and, secondly, we shall demonstrate that its technical aspects are, especially in comparison with the conventional HREP, quite user-friendly and not untractable.
3.1 General case

In review [11] we started from the same Schrödinger Eq. (1) as above. The change of the perspective was that the $F \rightarrow P$ mapping was admitted arbitrarily time-dependent, $\Omega = \Omega(t)$. This enabled us to upgrade definition (2),

$$|\varphi^{(SREP)}(t)\rangle = \Omega(t) |\psi(t)\rangle = \Omega^\dagger(t) |\tilde{\psi}(t)\rangle \in \mathcal{H}^{(P)} , \quad |\psi(t)\rangle, |\tilde{\psi}(t)\rangle \in \mathcal{H}^{(F)} .$$  

(7)

Next, still without any direct reference to the prospective NHH context we set

$$\hbar_{(SREP)} = \hbar_{(SREP)}(t) = \Omega(t) H(t) \Omega^{-1}(t)$$  

(8)

and inserted all of these ansatzs in Eq. (1). Elementary differentiation led to the doublet of $F$–space Schrödinger equations

$$i \partial_t |\psi(t)\rangle = G(t) |\psi(t)\rangle , \quad |\psi(t)\rangle \in \mathcal{H}^{(F)} ,$$  

(9)

$$i \partial_t |\tilde{\psi}(t)\rangle = G^\dagger(t) |\tilde{\psi}(t)\rangle , \quad |\tilde{\psi}(t)\rangle \in \mathcal{H}^{(F)} .$$  

(10)

We emphasized that the generator $G(t) = H(t) - \Sigma(t)$ of the time-evolution of the wave functions differed from the non-Hermitian energy-operator of Eq. (8) by the easily evaluated additional Coriolis-force term of a purely kinematical origin [15],

$$\Sigma(t) = i \Omega^{-1}(t) [\partial_t \Omega(t)] = i \Omega^{-1}(t) \dot{\Omega}(t) .$$  

(11)

Now we may add that the emergence of the Coriolis force changes the time-dependence of any operator $a$ of an observable. Its $P$–space representation $a(t) = a^\dagger(t)$ as well as its $F$–space image $A(t) = \Omega^{-1}(t) a(t) \Omega(t)$ may be differentiated yielding formula

$$i \partial_t A(t) = A(t) \Sigma(t) - \Sigma(t) A(t) + \Omega^{-1}(t) [i \dot{a}(t)] \Omega(t) .$$  

(12)

This formula will be a starting point of our forthcoming considerations. It is important to notice that in its general form it is still too formal since it intermingles the information encoded in the $P$–space operator $a$ with the one encoded in its $P$–space transform $A$ and with the one encoded in the Dyson’s operator $\Omega$ itself.

3.2 Heisenberg representations

The characteristic HREP postulate (13) that the wave functions of a state remain constant in time means, in the general three-Hilbert-space context of Eqs. (9) and (10), that in the prospective NHH setting we have to select, first of all, the trivial generator, $G^{(NHH)}(t) \equiv 0$. Such a constraint implies the necessity of the coincidence between the Coriolis NHH force and the NHH observable-energy operator,

$$\Sigma_{(NHH)}(t) = H_{(NHH)}(t) .$$  

(13)
Thus, given the energy operator $H_{(NHH)}(t)$ as a dynamical input information we may reconstruct the specific Dyson’s NHH operator $\Omega(t) = \Omega_{(NHH)}(t)$ via Eq. (11). The NHH version of this operator differential equation reads

$$i\partial_t \Omega_{(NHH)}(t) = \Omega_{(NHH)}(t) H_{(NHH)}(t).$$

Thus, a time-dependent NHH Dyson map is defined, in principle at least. As an operator Cauchy problem the construction must be initiated by a suitable preselected non-unitary operator $\Omega^{(NHH)}(t_0)$. The latter initial value must be such that the NHH Hamiltonian observability remains guaranteed,

$$H_{(NHH)}^\dagger(t_0) \Theta^{(NHH)}(t_0) = \Theta^{(NHH)}(t_0) H_{(NHH)}(t_0).$$

We may conclude that the main difference between the Hermitian and non-Hermitian HREP constructions is that the operator solution $\Omega^{(NHH)}(t)$ of Eq. (14) is not unitary. Thus, in the NHH setting it is now time to add another simplifying assumption:

**Definition 1** The special stationary subfamily of the general NHH Hamiltonians of Eq. (13) will be specified by requirement $H_{(NHH)}(t) = H_{(NHH)}(0)$ and marked by a dedicated subscript, $H_{(NHH)}(0) = H_{(spec)}$.

It is worth noticing that even in the conventional Hermitian HREP recipe the use of stationary Hamiltonians simplifies a number of technicalities. The same holds true in the NHH setting. First of all, it is easy to prove the following

**Lemma 2** In the stationary special case of Definition 1 our operator differential Eq. (14) may be solved in closed form,

$$\Omega_{(spec)}(t) = \Omega_{(spec)}(t_0)e^{-i(t-t_0)H_{(spec)}}.$$  

The initial value must be compatible with the factorization of metric in Eq. (15),

$$\Theta^{(spec)}(t_0) = \Omega_{(spec)}^\dagger(t_0)\Theta^{(spec)}(t_0).$$

The following result is slightly less expectable.

**Proposition 3** Although the NHH Dyson’s map of Eq. (16) varies with time it still leads to the metric which is time-independent, $\Theta^{(spec)}(t) = \Theta^{(spec)}(t_0)$.

Proof. In the definition

$$\Theta^{(spec)}(t) = e^{i(t-t_0)H_{(spec)}} \Theta^{(spec)}(t_0)e^{-i(t-t_0)H_{(spec)}}$$

(18)
we expand one of the exponentials in the operator Taylor series. Next, we recall the Hamiltonian-observability initial condition
\[ H_{(\text{spec})}^\dagger \Theta^{(\text{spec})}(t_0) = \Theta^{(\text{spec})}(t_0) H_{(\text{spec})} \]
and intertwine all of the powers of the time-independent \( H_{(\text{spec})} \) with the central operator of the initial-value metric. Finally, we re-convert the modified Taylor series into the operator exponential and multiply the two time-dependent factors yielding the unit operator. □

We see that our stationary operator of metric defines the physical inner product in Hilbert space \( \mathcal{H}^{(S)} = \mathcal{H}^{(\text{spec})} \). In the light of the construction the product and, hence, also the space remain time-independent. This has further nontrivial consequences in \( \mathcal{H}^{(P)} \).

**Proposition 4** *In the stationary NHH scenario also the self-adjoint isospectral partner Hamiltonian operator remains stationary.*

**Proof.** In order to prove the time-independence of
\[ \mathfrak{h}_{(\text{spec})}(t) = \Omega_{(\text{spec})}(t) H_{(\text{spec})} \left[ \Omega_{(\text{spec})}(t) \right]^{-1} = \mathfrak{h}_{(\text{spec})}^\dagger(t) , \]it is sufficient to proceed in analogy with the proof of Prop. 3. The situation is now simpler because the time-dependent exponentials commute with the Hamiltonian. Hence, they immediately cancel and no operator-intertwining is needed. □

### 3.3 Heisenberg equations of motion

In the special NHH stationary case of Definition 1 the knowledge of the mapping \( \Omega_{(\text{spec})}(t) \) of Eq. (16) together with the knowledge of an operator \( a(t) \) of any observable in \( \mathcal{H}^{(P)} \) enables us to reconstruct the action \( A(t) \) of the same observable in the auxiliary, unphysical space \( \mathcal{H}^{(F)} \). The latter space merely differs from the correct and physical Hilbert space \( \mathcal{H}^{(S)} = \mathcal{H}^{(\text{spec})} \) by the “false” metric \( \Theta^{(F)} = I \) so that the same formula
\[ A^{(\text{spec})}(t) = e^{i\left(t-t_0\right) H_{(\text{spec})}} \Omega_{(\text{spec})}^{-1}(t_0) a(t) \Omega_{(\text{spec})}(t_0) e^{-i\left(t-t_0\right) H_{(\text{spec})}} \]
offers the explicit definition. No equation of motion is needed. Naturally, whenever asked for, we may still recall the universal rule (12) and write down, after appropriate substitutions, the equation of motion
\[ i\partial_t A_{(\text{spec})}(t) = A_{(\text{spec})}(t) H_{(\text{spec})} - H_{(\text{spec})} A_{(\text{spec})}(t) + K(t) . \]Here, an additional and independent dynamical input must be provided via operator
\[ K(t) = \Omega_{(\text{spec})}^{-1}(t)[ia(t)]\Omega_{(\text{spec})}(t) . \]
Unless we select dynamical scenario in which \( \partial_t a(t) = 0 \), recipe (21) remains highly formal. The knowledge of the time-derivative of observable \( a \) in \( \mathcal{H}^{(P)} \) (and of its initial value at
any \( t_{ini} \) could have been much more easily complemented by an exhaustive reconstruction of operator \( a(t) \) and by its insertion, at all times, in definition (20) of \( A^{(spec)}(t) \).

In the sense of this comment the more general non-stationary cases which do not obey Definition 1 but which still do not need any additional information are much more interesting.

**Definition 5** The observables exhibiting the manifest-time-independence property (i.e., satisfying relation \( \partial_t a_{(indep)}(t) = 0 \) in \( \mathcal{H}^{(P)} \) ) will be marked by subscript \( (indep) \).

**Theorem 6** Under the constraint imposed by Definition 5, the observable \( A_{(indep)}(t) \) which is defined as acting in the physical NHH Hilbert space \( \mathcal{H}^{(S)} = \mathcal{H}_{(NHH)} \) may be determined from the pair of operator differential evolution equations

\[
i \partial_t A_{(indep)}(t) = A_{(indep)}(t)H_{(NHH)}(t) - H_{(NHH)}(t)A_{(indep)}(t) \quad (22)
\]

and

\[
i \partial_t A^\dagger_{(indep)}(t) = A^\dagger_{(indep)}(t)H^\dagger_{(NHH)}(t) - H^\dagger_{(NHH)}(t)A^\dagger_{(indep)}(t) \quad (23)
\]

together with the obligatory coupling

\[
A^\dagger_{(indep)}(t_0)\Theta_{(NHH)}(t_0) = \Theta_{(NHH)}(t_0)A_{(indep)}(t_0) \quad (24)
\]

between the respective initial operator values.

**Proof.** The assumption of the \( P \)–space stationarity \( \partial_t a_{(indep)}(t) = 0 \) enables us to omit the last term from Eq. (12) living in friendly space \( \mathcal{H}^{(F)} \). In combination with Eq. (13) this leads to the desired pair of generalized non-Hermitian Heisenberg equations. The Hermitian-conjugation duplicity is nontrivial because we must guarantee the observability of \( A_{(indep)}(t) \) at all times, i.e., the validity of relation

\[
A^\dagger_{(indep)}(t)\Theta_{(NHH)}(t) = \Theta_{(NHH)}(t)A_{(indep)}(t) \quad (25)
\]

This relation reflects the \( P \rightarrow F \)-inherited Hermiticity *alias* crypto-Hermiticity of our observable. Once we differentiate this relation with respect to time we get a new relation which is identically satisfied. This is immediately proved when we recall (or quickly derive) an insert the well known [13] elementary identity

\[
i \partial_t \Theta(t) = \Theta(t)\Sigma(t) - \Sigma^\dagger(t)\Theta(t) \quad (26)
\]

Thus, it is necessary and sufficient to postulate Eq. (25) at \( t = t_0 \). \( \square \)
4 Summary

The use of non-unitary operators $\Omega$ in the time-independent and time-dependent regimes is not too dissimilar. One simply works with the $P - F - S$ triplet of Hilbert spaces in a way which is summarized by the following diagram,

\[ \text{P-space} \]
\( \text{textbook level quantum theory} \)
\( \text{selfadjoint observables } a \)
\( \text{selfadjoint Hamiltonian } h \)
\( \text{calculations = prohibitively complicated} \)

\[ \text{F-space} \]
\( \text{friendlier } H = \Omega^{-1} h \Omega \)
\( H \neq H^\dagger \) (space = false)
\( \text{feasible calculations} \)

\[ \text{S-space} \]
\( \text{inner product - metric } \Theta = \Omega^\dagger \Omega \)
\( \text{selfadjoint } H = H^\dagger = \Theta^{-1} H^\dagger \Theta \)
\( \text{standard interpretation} \)

In this framework we described a few basic features of the innovated, non-Hermitian Heisenberg-representation approach to quantum theory. In a restricted format of the letter we managed to demonstrate not only its mathematical consistency but also its sufficiently friendly and feasible nature.
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