SINGULAR PERTURBATIONS OF ORNSTEIN-UHLENBECK PROCESSES: INTEGRAL ESTIMATES AND GIRSANOV DENSITIES

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Abstract. We consider an infinite-dimensional Ornstein-Uhlenbeck operator perturbed by a singular (non-linear, non-smooth, non-autonomous) maximal monotone drift. First we prove estimates on (pseudo-weak) solutions of such equations. In addition, we obtain uniform integrability estimates of the associated Girsanov densities in terms of local estimates on the drift term and the initial point. The only assumption on the (possibly singular) drift is that it is maximal monotone and locally finite, with no assumptions made on the growth rate at infinity. Some of our results concern non-random equations as well, while probabilistic results are new even in finite-dimensional autonomous situations. In particular, consideration of a drift of non-linear growth naturally leads to a new notion of entropy.

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1. INTRODUCTION

The aim of this paper is a study of solutions to stochastic differential equations on a Hilbert space. These equations can be viewed as non-linear non-autonomous perturbations of the stochastic differential equation corresponding to the Ornstein-Uhlenbeck semigroup. The drift in the perturbed equation lacks regularity properties or even usual growth constrains (such as polynomial or exponential), and our first step is to introduce a notion of

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solutions different from the ones used previously. The main parts of our paper include a proof of existence of pseudo-weak solutions, estimates of these solutions, a proof of absolute continuity of the law of these solutions with respect to the Ornstein-Uhlenbeck process, and finally integral estimates of the corresponding Radon-Nikodym density. Our approach is a new rendition on the classical use of a Girsanov transformation to find a solution for a stochastic differential equation with a nonzero (but Lipschitz) drift. At the same time it is closely related to the Girsanov transform in infinite dimensions and its applications to prove closability of Dirichlet forms such as in [11, 43].

We would like to comment on some of the previous results both in terms of the assumptions we make and the type of results we get. We describe the setting in Section 1.1 in detail. Note that uniqueness for (1.1) is not addressed in this paper. In general one expects that Assumptions 1.1 and 1.3 would imply uniqueness (appealing to Gronwall’s lemma). But it seems that there is no one reference that would cover a general setting such as ours. For pathwise uniqueness one of the best available results is [21]. Note that the assumptions in [21] are more restrictive: A is assumed to be symmetric, and the approximations to the equation are not just Yosida, but finite-dimensional, thus introducing certain restrictions on the results. For weak (martingale solutions) uniqueness an appropriate reference is [11], where A is assumed to be symmetric in addition to more assumptions on F. For results on uniqueness for analytically weak solutions with non-Lipschitz coefficients see [11]. Finally, we observe that our assumption on A (Assumption 1.1) is satisfied for the reaction-diffusion equation as in [20, Section 9.2].

We point out that this paper does not discuss autonomous equations, in which case one can consider other questions such as uniqueness of solutions, and existence of uniqueness of the invariant measure, quasi-invariance of the semigroup and of the invariant measure, closability of the corresponding Dirichlet form and the integration by parts formula. Even though we do not give details, some of these potential applications are mentioned in Section 1.4. The literature on the subject for autonomous equations is extensive, the closest to the setting we consider are [4, 5, 14, 16, 20, 22, 23, 27, 30, 36, 45], while [43, 44] take a different approach to ergodicity in infinite dimensions.

The motivation for our study includes a better understanding of equations such as (1.1) with polynomial and other “nice” non-autonomous coefficients rather than coefficients of linear growth. In particular, we prove estimates of solutions in Section 3 and estimates of Girsanov densities in Section 4. Note that these results can be considered as an improvement of infinite-dimensional estimates by Gatarek and Goldys in [20, 41], or even of finite-dimensional estimates by Krylov [47] and [48, Chapter IV, §3] (see also [49]). The main idea behind results such as Theorem 3.10 is that we can find a suitable (finite) ϕ-moment for the solutions by looking at the behavior of the non-linearity at infinity. This might eventually lead to finding the right entropy for the system as we discuss in Remark 5.3.

1.1. Setting and assumptions. We start by introducing the setting. Our exposition in general follows [20], with the exception that the drift term $F(t,x)$ that we consider may depend on time.

Let $H$ be a real separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|\cdot|$. We denote the space of bounded linear operators equipped with the operator norm $\| \cdot \|$ by $B(H)$. The Hilbert-Schmidt norm is denoted by $\| \cdot \|_{HS}$. Let $B(H)$ be the Borel $\sigma$-algebra. Consider the following stochastic differential equation in $H$.
\[ dX_t = (AX_t + F(t, X_t)) \, dt + \sigma dW_t, \quad X_0 = x \in H, \]

where \( W_t \) is a cylindrical Wiener process in \( H \) on some filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), satisfying for all \( t \geq 0 \) the usual conditions of right continuity and \( \mathbb{P} \)-completeness. Furthermore, we suppose that the coefficients \( A, F \) and \( B \) satisfy the following assumptions.

**Assumption 1.1.** The operator \((A, D_A)\) generates a \( C_0 \)-semigroup on \( H \) denoted by \( e^{tA} \), \( t \geq 0 \). We assume that there is a \( \omega > 0 \) such that for all \( x \in D_A \)

\[ \langle Ax, x \rangle \leq -\omega |x|^2. \]

Note that Assumption 1.1 implies that \( A \) is monotone (dissipative).

**Assumption 1.2.** Both \( \sigma \) and \( \sigma^{-1} \) are in \( B(H) \) with \( \sigma \) being self-adjoint and positive. Moreover, there is an \( \alpha > 0 \) such that

\[ \int_0^\infty (1 + t^{-\alpha}) \|e^{tA}\|^2_{HS} dt < \infty. \]

**Assumption 1.3.** Let \( F(t, \cdot) : [0, \infty) \times D_F(t, \cdot) \subset H \rightarrow 2^H \) be a family of maps such that for any \( t \in [0, \infty) \) we have \( D_{F(t, \cdot)} = D_F \) is a Borel set in \( H \), and \( F(t, \cdot) \) is an \( m \)-dissipative map, that is, for any \( x_1, x_2 \in D_F \)

\[ \langle y_1 - y_2, x_1 - x_2 \rangle \leq 0, \text{ for any } y_1 \in F(t, x_1), y_2 \in F(t, x_2), t \in [0, \infty) \]

and if for any \( \alpha > 0 \) and \( t \in [0, \infty) \)

\[ \text{Range}(\alpha I - F(t, \cdot)) = H. \]

We refer to [3], Section II.3 and [4], Chapter 3 for basic facts about dissipative maps. In particular, it is known that in a Hilbert space a map is \( m \)-dissipative if and only if it is maximal dissipative, that is, it has no proper dissipative extensions. By [3, Proposition 3.5(iv), Chapter II] for any \( (t, x) \in [0, \infty) \times D_F \), the set \( F(t, x) \) is non-empty, closed and convex, and so we can consider a well-defined map

\[ F_0(t, x) := \{ y \in F(t, x) : |y| = \inf \{|z|, z \in F(t, x)\}\}, \text{ for any } x \in D_F. \]

Using the Yosida approximation to \( F \) described in Section 4 we see that the function \( F_0(t, x) \) is Borel-measurable for each \( t \in [0, \infty) \).

The next assumption is similar to the ones introduced in [22,30].

**Assumption 1.4.** We assume that there is an increasing function \( a : [0, \infty) \rightarrow [0, \infty) \) such that

\[ |F_0(t, x)| \leq a(|x|), (t, x) \in [0, \infty) \times H, \]

where we allow for \( \lim_{u \rightarrow \infty} a(u) = \infty \).

For some of the implications and possible variations of these assumptions we refer to [21, Remark 1.1].
1.2. **Pseudo-weak solutions to the main equation and their properties.** Throughout this paper we assume that Assumptions (1.1), (1.2), (1.3), (1.4) hold. We refer to [11, 54] for a discussion of how martingale solutions to (1.1) can be constructed, and for more details on such solutions. Instead we consider a different notion of solutions to (1.1) as defined below. In (1.1) we use continuous $H$-valued processes $Z_{a,t}^x$ that are the solutions of a family of regularized random ordinary differential equations

\[ dZ_{a,t} = (AZ_{a,t} + F_\alpha (t, Z_{a,t} + W_{0,A,\sigma} (t))) \, dt, \quad Z_{a,0} = x, \]

where $F_\alpha$ is the Yosida approximation to $F$ defined by (1.2) and $W_{0,A,\sigma}$ is the Ornstein-Uhlenbeck process defined by (5.4). One can use [11, Chapter 6, Theorem 1.2, page 184] to justify the existence of solutions to (1.3). We note that technically speaking [11] assumes that $F_\alpha$ is continuous in time, but it is clear that this assumption is not essential, and it is enough to assume joint mesurability in time and space, and Lipschitz continuity in space, with the Lipschitz constant uniform in time.

**Definition 1.5.** An adapted $H$-valued continuous process $X_t$ is a pseudo-weak solution to (1.1) if $X_t$ is $\mathcal{F}_t$-measurable for any $t \geq 0$ and

\[ X_t = Z_{a,t}^x + W_{0,A,\sigma} (t) \quad \mathbb{P} - \text{a.s.}, \]

where on each finite time interval $[0, T]$ the process $Z_{a,t}^x$ is an $L^2([0, T], H)$-pseudo-weak limit point of $Z_{a,t}^x$, as $\alpha \to 0$, and $Z_{a,t}^x$ is defined in (1.3) and Appendix A.

**Remark 1.6.** In Definition 1.3 one could use the notion of “locally square integrable” (uniform on compacts in probability, or ucp, see [11, Section II.4, page 58]) similarly to how it is done in the semimartingale theory. As the notion of semimartingales in infinite dimensions is delicate, we refrain from claiming or using the fact that $F_\alpha$ is continuous in time, but it is clear that this assumption is not essential, and it is enough to assume joint mesurability in time and space, and Lipschitz continuity in space, with the Lipschitz constant uniform in time.

The main results of our paper are summarized in the following theorem.

**Theorem 1.7.**

1. Under Assumptions (1.1), (1.2), (1.3), (1.4), there exists a pseudo-weak solution to the equation (1.1) as defined in Definition 1.3, which satisfies estimates

\[ |Z_{a,t}^x| \leq |x|e^{-\omega t} + \int_0^t e^{-\omega(t-s)}a(|W_{0,A,\sigma} (s)|) \, ds. \]

2. For a function $\varphi \in \mathcal{M}$ as defined in Definition 1.3, the uniform pathwise functional estimates (5.1) in Theorem 5.1 hold, which imply the $\varphi$-moments estimates

\[ \mathbb{E}^x \varphi (|X_t|^2) \leq \frac{e^{-\omega t}}{2} \varphi (4|x|^2) + \frac{1}{2} \mathbb{E}^x K_\varphi (t) + \frac{\omega t}{2} \mathbb{E}^x K_{\varphi,\omega,t} (t). \]

3. The law of the pseudo-weak solution $X_t$ to Equation (1.1) is absolutely continuous with respect to the law of $W_{x,A,\sigma}$ on any finite time interval $[0, T]$. Moreover, for any fixed positive increasing function $a(\cdot)$ in (1.2), $r > 0$, and $T > 0$, there exists an increasing positive (concave down) unbounded function $\psi_r(\cdot)$ such that the corresponding density $\rho$ satisfies

\[ \mathbb{E} \rho \psi_r (\rho) \leq 1. \]
for all initial points \( x \) with \( |x| < r \).

We prove Theorem 1.7 in Section 3 and Section 4, where we provide more detailed statements as well. These results are illustrated by Examples 3.5, 3.6, 3.7, and 4.7. Note that Theorem 1.7 addresses the absolute continuity of the laws which is a long-standing question that has been implicitly stated in a number of publications such as [51, 52].

2. Preliminaries: Yosida approximations, Gronwall’s Lemma

Notation 2.1. Throughout the paper we use the following notation.

1. \( B_b (H) \), \( C_b (H) \) denote the spaces of bounded Borel measurable, respectively continuous, real-valued functions on \( H \);
2. \( \text{Lip}_b (H) \) denotes bounded Lipschitz functions on \( H \).

One of the ingredients in [20] is the Yosida approximation (e.g. [6, 7, 12]) to \( F \) satisfying Assumption 1.3. Namely, for any \( \alpha > 0 \) define

\[
F_\alpha := \frac{1}{\alpha} (J_\alpha (x) - x), \quad x \in H,
\]

where \( J_\alpha (x) := (I - \alpha F)^{-1} (x), \quad I (x) = x. \)

Then each \( F_\alpha \) is single-valued, dissipative, Lipschitz continuous and satisfies

\[
\lim_{\alpha \to 0} F_\alpha (x) = F_0 (x), \quad x \in D (F),
\]
\[
|F_\alpha (x)| \leq |F_0 (x)|, \quad x \in D (F).
\]

It is clear from the last inequality that \( F_\alpha \) satisfy the same growth condition as \( F_0 \).

We also need some standard facts about \( C_0 \)-semigroups and their generators, most of this goes back to Hille and Yosida. We refer to [27, Chapter II] for most of the material below. Let \( \rho (A) \) be the resolvent set, then the resolvent of \( A \) is defined as

\[
R_\lambda (A) := (\lambda I - A)^{-1}, \quad \lambda \in \rho (A) \in B (H),
\]
\[
R_\lambda (A) : H \to D_A.
\]

Recall that for \( \lambda > 0 \) we have \( \| R_\lambda (A) \| \leq 1 / \lambda \). In addition,

\[
\lambda R_\lambda (A) x \underset{\lambda \to \infty}{\to} x, \quad x \in H.
\]

Finally the Yosida approximations to \( A \) are defined by

\[
A_\lambda x := \lambda AR_\lambda (A) x, \quad x \in H.
\]

The Yosida approximations to \( A \) satisfy the following properties.
\[ A_\lambda \in B(H), \]
\[ \lambda AR_\lambda(A)x = \lambda R_\lambda(A)Ax, \quad x \in D_A, \]
\[ (2.6) \]
\[ A_\lambda x = -\lambda (I - \lambda R_\lambda(A))x, \quad x \in D_A, \]
\[ \lim_{\lambda \to \infty} A_\lambda x \to Ax, \quad x \in D_A. \]

For completeness we include some of the proofs of these properties. For example, to show (2.6) we can use that for any \( x \in D_A \) we have

\[ x = (\lambda I - A)R_\lambda(A)x, \]

therefore for all \( x \in D_A \)

\[ -\lambda (I - \lambda R_\lambda(A))x = -\lambda ((\lambda I - A)R_\lambda(A)x - \lambda R_\lambda(A)x) = \]
\[ -\lambda (\lambda R_\lambda(A)x - AR_\lambda(A)x - \lambda R_\lambda(A)x) = \lambda AR_\lambda(A)x = A_\lambda x. \]

Thus for all \( x \in D_A \) by (2.6)

\[ \langle A_\lambda x, x \rangle = \langle A_\lambda x, x - \lambda R_\lambda(A)x \rangle + \langle A_\lambda x, \lambda R_\lambda(A)x \rangle = \]
\[ -\frac{1}{\lambda} \langle A_\lambda x, A_\lambda x \rangle + \langle A_\lambda x, \lambda R_\lambda(A)x \rangle = -\frac{1}{\lambda} |A_\lambda x|^2 + \lambda^2 \langle AR_\lambda(A)x, R_\lambda(A)x \rangle \leq \]
\[ \lambda^2 \langle AR_\lambda(A)x, R_\lambda(A)x \rangle \leq -\lambda^2 \omega |R_\lambda(A)x|^2. \]

Now we can use (2.4) to see that for all large enough \( \lambda \) the Yosida approximations \( A_\lambda \) satisfy Assumption 1.1.

**Remark 2.2.** We will repeatedly make use of the following elementary inequalities: for any \( a, b \geq 0 \), and \( p \geq 1 \)

\[ (a + b)^p \leq 2^{p-1}(a^p + b^p), \]
\[ e^{(a+b)^2} \leq \frac{e^{2a^2} + e^{2b^2}}{2}. \]

**Remark 2.3.** Let us recall the following well known integral form of for absolutely continuous functions which will used below. The fact that function \( u(t) \) may not be continuously differentiable will be essential in Subsection 4.1. Suppose that \( u \) is a locally bounded measurable function on the interval \([0, T]\) with \( u(0) \geq 0 \) such that there is a non-negative integrable function \( A(s) \) and a constant \( B \geq 0 \) such that for all \( t \in [0, T] \)

\[ u(t) \leq u(0) + \int_0^t A(s) - Bu(s) ds, \]

then for all \( t \in [0, T] \)

\[ u(t) \leq u(0)e^{-Bt} + \int_0^t e^{-(t-s)B}A(s) ds. \]
Indeed, if \( u^*(t) \) denotes the right hand side in (2.9), then \( u^*(t) \) is an absolutely continuous function that satisfies \( e^{-B_s} (e^{B_s} u^*(s))^\prime \leq A(s) \) for all almost all \( s \in [0, t] \), and therefore \( (e^{B_s} u^*(s))^\prime \leq e^{B_s} A(s) \) for all almost all \( s \in [0, t] \), which gives the needed inequality.

3. Almost sure \( \varphi \)-type estimates of solutions \( X_t \)

First we define the stochastic convolution

(3.1) \( W_{x,A,\sigma} (t) := e^{tA} x + \int_0^t e^{(t-s)A} \sigma dW(s), t \geq 0. \)

By Assumption 1.2 we have that \( W_{x,A,\sigma} (t) \) is well-defined and pathwise continuous (e.g. [20, Section 5.1.2]). Moreover,

\[ W_{0,A,\sigma} (t) := W_{x,A,\sigma} (t) - e^{tA} x \]

is a Gaussian random variable with values in \( H \) with the mean 0 and the covariance operator \( Q_t \) given by

\[ Q_t x = \int_0^t e^{sA} \sigma^2 e^{sA} x ds. \]

We will use the following notation for the maximum process

(3.2) \( W_{x,A,\sigma}^* (t) := \sup_{s \in [0,t]} |W_{x,A,\sigma} (s)|. \)

Later we will need the following version of the Burkholder-Davis-Gundy inequality [20, Lemma 7.7] for \( p > 1 \)

\[ \sup_{s \in [0,t]} \mathbb{E} |W_{0,A,\sigma} (s)|^{2p} = \sup_{s \in [0,t]} \mathbb{E} \left| \int_0^s e^{(s-\tau)A} \sigma dW(\tau) \right|^{2p} \leq (p (2p - 1))^p \left( \int_0^t \| e^{(t-s)A} \sigma \|_{\text{HS}}^2 ds \right)^p. \]

Hence the norms \( \sup_{s \in [0,t]} \mathbb{E} |W_{0,A,\sigma} (s)|^{2p} \) and \( \mathbb{E} \left| W_{0,A,\sigma}^* (t) \right|^{2p} \) are equivalent, and there is a constant \( C(p, A, \sigma) < \infty \) such that

(3.3) \( \sup_{s \in [0,t]} \mathbb{E} \left( W_{0,A,\sigma}^* (s) \right)^{2p} \leq C(p, A, \sigma). \)

Finally, by Fernique’s Theorem there is a (small) \( \gamma > 0 \) such that

(3.4) \( \sup_{s \geq 0} \mathbb{E} \left( e^{\gamma |W_{0,A,\sigma} (s)|^2} \right) < \infty. \)

Motivated by de la Vallée-Poussin Theorem [10, p. 19, Theorem T22] giving a criterion for uniform integrability, we define the following space of functions of one variable.

**Definition 3.1.** We denote by \( \mathcal{M} \) the space of \( C^2 \)-functions \( \varphi : (0, \infty) \to [0, \infty) \) such that

1. \( \varphi \) is an increasing convex function;
(2) the limit $\frac{u\varphi'(u)}{\varphi(u)} \to L_\varphi$ exists, and $L_\varphi \in [1, \infty]$. 

First we establish some properties of the function space $M$ depending on the value $L_\varphi$.

**Lemma 3.2** $(1 < L_\varphi < \infty)$. For any $\varphi \in M$ such that $1 < L_\varphi < \infty$, the following statements hold.

1. $\frac{\varphi(u)}{u} \to \infty$;
2. For any $c > 0$, $\omega > 0$ there are $C \geq 0$, $B > 0$ such that

$$
\varphi(u) \left[ \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \right] \leq C, \text{ for all } u \in (0, \infty).
$$

Constants $B$ and $C$ can be chosen as follows: for any $0 < B < \omega L_\varphi$

$$
C(c, \omega, B) := \max_{u \in [0, \infty)} \left( \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \varphi(u) \right) = \max_{u \in [0, u_0]} \left( \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \varphi(u) \right),
$$

where $u_0 := \max \left\{ \frac{c^2}{\omega^2}, \frac{c}{\sqrt{\omega^2 - B}} \right\}$. In particular,

$$
C \left( c, \omega, \frac{\omega}{2} \right) = \frac{\omega}{2} \varphi \left( \frac{c^2}{\omega^2} \right)
$$

**Proof.** First,

$$
\left( \frac{\varphi(u)}{u} \right)' = \left( \frac{u\varphi'(u)}{\varphi(u)} - 1 \right) \cdot \frac{\varphi(u)}{u^2} \geq \frac{1}{2} \left( L_\varphi - 1 \right) \frac{\varphi(u)}{u^2}
$$

for all large enough $u$. Then as $L_\varphi > 1$ we see that for some $K > 0$

$$
\left( \frac{\varphi(u)}{u} \right)' > \frac{K \varphi(u)}{u} > 0
$$

for all large enough $u$. Denote $H(u) := \frac{\varphi(u)}{u}$ which is positive for all positive $u$ and

$$
\frac{H'(u)}{H(u)} > \frac{K}{u}
$$

for all large enough $u$. Then for some $M > 0$

$$
H(u) = \frac{\varphi(u)}{u} > Me^{Ku} \text{ for all large enough } u,
$$

which implies that $\frac{\varphi(u)}{u} \to \infty$.

For the second statement, it is enough to check that for some $B \geq 0$

$$
\varphi(u) \left[ \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \right] \to -\infty,
$$

and so there is a $u_0 > 0$ such that

$$
\varphi'(u) \left( c\sqrt{u} - \omega u \right) + B \varphi(u) < 0 \text{ for all } u > u_0.
Then we can choose

\[ C := \max_{u \in [0, u_0]} \left( \varphi'(u) \left( c\sqrt{u} - \omega u \right) + B \varphi(u) \right). \]

Observe that

\[ \varphi'(u) \left( c\sqrt{u} - \omega u \right) = \frac{u \varphi'(u)}{\varphi(u)} \left( \frac{c}{\sqrt{u}} - \omega \right) \xrightarrow[u \to \infty]{} -\omega L \varphi, \]

and so if we choose \( B \) such that \( 0 < B < \omega L \varphi \), then

\[ \varphi(u) \left[ \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \right] \xrightarrow[u \to \infty]{} -\infty. \]

Recall that we can take \( C \) to be the maximum of the following function

\[ f(u) := \varphi'(u) \left( c\sqrt{u} - \omega u \right) + B \varphi(u) = \varphi'(u) \left( c\sqrt{u} - \omega u \right) + B \varphi(u). \]

First we take the derivative of this function

\[ f'(u) = \varphi''(u) \left( c\sqrt{u} - \omega u \right) + \varphi'(u) \left( \frac{c}{2\sqrt{u}} - \omega \right) + B \varphi'(u) = \varphi''(u) \sqrt{u} (c - \omega \sqrt{u}) + \varphi'(u) \left( \frac{c}{2\sqrt{u}} - (\omega - B) \right). \]

By assumption \( \varphi \) is an increasing convex function, and therefore \( \varphi'' \) and \( \varphi' \) are non-negative, so \( f'(u) \leq 0 \) for any \( u \geq u_0 = \max \left\{ \frac{c^2}{\omega^2}, \frac{c^2}{4(\omega - B)^2} \right\} \). Therefore we can choose

\[ C(c, \omega, B) = \max_{u \in [0, \infty)} f(u) = \max_{u \in [0, u_0]} f(u). \]

Finally, if \( B = \omega/2 \), then \( u_0 = \frac{c^2}{\omega^2} \), and

\[ C(c, \omega, \omega/2) = f(u_0) = \frac{\omega}{2} \varphi \left( \frac{c^2}{\omega^2} \right). \]

\[ \square \]

**Lemma 3.3** \((L_\varphi = 1)\). For any \( \varphi \in \mathcal{M} \) such that \( \frac{u \varphi'(u)}{\varphi(u)} \xrightarrow[u \to \infty]{} 1 \), then for any \( c > 0, \, \omega > 0 \) and any \( 0 < B < \omega \) there is a \( C \geq 0 \) such that

\[ \varphi(u) \left[ \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \right] \leq C, \text{ for all } u \in (0, \infty). \]

Constant \( C = C(c, \omega, B) \) can be chosen as in (55).

**Proof.** First,

\[ \left( \frac{\varphi(u)}{u} \right)' = \left( \frac{u \varphi'(u)}{\varphi(u)} - 1 \right) \cdot \frac{\varphi(u)}{u^2}, \]

thus we see that
for all large enough $u$. Thus $\frac{\varphi(u)}{u}$ is a positive and non-decreasing function, and therefore there is a $K > 0$ for all large enough $u$

$$\varphi(u) > Ku.$$ 

In particular, $\varphi(u) \xrightarrow[u \to \infty]{} +\infty$. For any $0 < B < \omega$

$$\left[ u\varphi'(u) \left( \frac{1}{\sqrt{u}} - \omega \right) + B \right] \xrightarrow[u \to \infty]{} B - \omega,$$

therefore

$$\varphi(u) \left[ u\varphi'(u) \left( \frac{1}{\sqrt{u}} - \omega \right) + B \right] = \varphi(u) \left[ u\varphi'(u) \left( \frac{1}{\sqrt{u}} - \omega \right) + B \right] \xrightarrow[u \to \infty]{} -\infty,$$

and so there is a $u_0 > 0$ such that

$$\varphi'(u) \left( c\sqrt{u} - \omega u \right) + B\varphi(u) < 0 \text{ for all } u > u_0.$$

Then we can choose $C$ as before.$\blacksquare$

**Lemma 3.4** $(L_\varphi = \infty)$. For any $\varphi \in \mathcal{M}$ such that $\frac{w\varphi'(u)}{\varphi(u)} \xrightarrow[u \to \infty]{} \infty$ the following statements hold.

1. $\varphi(u) \xrightarrow[u \to \infty]{} \infty$;
2. for any $c > 0$, $\omega > 0$ and any $B > 0$, there is a $C \geq 0$ such that

$$\varphi(u) \left[ \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \right] \leq C, \text{ for all } u \in (0, \infty).$$

Constant $C = C(c, \omega, B)$ can be chosen as in $(3.5)$.

**Proof.** As before

$$\left( \frac{\varphi(u)}{u} \right)' = \left( \frac{w\varphi'(u)}{\varphi(u)} - 1 \right) \cdot \frac{\varphi(u)}{u^2},$$

thus we see that

$$\left( \frac{\varphi(u)}{u} \right)' \xrightarrow[u \to \infty]{} \infty.$$

In particular, this implies that $\varphi(u) \xrightarrow[u \to \infty]{} \infty$. Similarly to the proofs of Lemma 3.2 and Lemma 3.3 we see that for any $B > 0$

$$\varphi(u) \left[ \frac{\varphi'(u)}{\varphi(u)} \left( c\sqrt{u} - \omega u \right) + B \right] \xrightarrow[u \to \infty]{} -\infty,$$
and so there is a \( u_0 > 0 \) such that
\[
\varphi' (u) \left( c\sqrt{u} - \omega u \right) + B \varphi (u) < 0, \quad \text{for all } u > u_0.
\]

Then we can choose \( C \) as before.

Note that the space \( \mathcal{M} \) is not empty, as we can see from the following examples.

**Example 3.5.** Suppose \( \varphi (u) = u^p, p \geq 1 \), then \( \varphi \in \mathcal{M} \). In this case \( L_\varphi = p \), and so for any \( 0 < B < p\omega \) we have

\[
\varphi (u) \left[ \frac{\varphi' (u)}{\varphi (u)} \left( c\sqrt{u} - \omega u \right) + B \right] \xrightarrow{u \to \infty} -\infty.
\]

To see how we can find an \( C \), observe that for any \( 0 < B < p\omega \)

\[
f (u) := \varphi (u) \left[ \frac{\varphi' (u)}{\varphi (u)} \left( c\sqrt{u} - \omega u \right) + B \right] =
\]

\[
cpu^{p-1/2} + (B - p\omega) u^p,
\]

for which

\[
f' (u) = cp \left( p - \frac{1}{2} \right) u^{p-3/2} + (B - p\omega) pu^{p-1} =
\]

\[
pu^{p-3/2} \left( c \left( p - \frac{1}{2} \right) - (p\omega - B) \sqrt{u} \right).
\]

Then the maximum of \( f \) is attained at \( u_0 = \left( \frac{c(p-\frac{1}{2})}{p\omega - B} \right)^2 \). Therefore we can choose

\[
C_p (c, \omega, B) := \frac{c}{2} \left( \frac{c (p - \frac{1}{2})}{p\omega - B} \right)^{2p-1} = \frac{c^{2p}}{2} \left( \frac{p - \frac{1}{2}}{p\omega - B} \right)^{2p-1}.
\]

In this example Lemma 3.3 holds for \( p = 1 \), while Lemma 3.2 holds for \( p > 1 \). In particular, for any \( 0 < B < \omega \) we can choose

\[
C_1 (c, \omega, B) := \frac{c^2}{4 (\omega - B)}.
\]

**Example 3.6.** Suppose \( \varphi (u) = e^u \), then \( \varphi \in \mathcal{M} \). In this case \( L_\varphi = \infty \), so we can take any positive \( B \). For example, if \( B = \omega/2 \), then for

\[
f (u) := e^u \left[ c\sqrt{u} - \omega u + B \right] = e^u \left[ c\sqrt{u} - \omega u + \frac{\omega}{2} \right]
\]

we have

\[
f' (u) = e^u \left[ c\sqrt{u} + \frac{c}{2\sqrt{u}} - \omega u - \frac{\omega}{2} \right] = e^u \left( \frac{c}{\sqrt{u} - \omega} \right) \left( u + \frac{1}{2} \right)
\]

and we can take
\[ C = f \left( \frac{c^2}{\omega^2} \right) = \frac{\omega^2}{2} e^{\frac{c^2}{\omega^2}}. \]

**Example 3.7.** Suppose \( \varphi(u) = u \ln(u + 1) \), then \( \varphi \in \mathcal{M} \). In this case \( L_{\varphi} = 1 \), so we can take any \( 0 < B < \omega \) and then \( C \) can be chosen by finding the maximum of the function

\[
 f(u) := \varphi(u) \left[ \frac{\varphi'(u)}{\varphi(u)} (c \sqrt{u} - \omega u) + B \right] = \\
 \ln(u + 1) \left( c \sqrt{u} - (\omega - B) u \right) + \frac{u}{u + 1} (c \sqrt{u} - \omega u).
\]

Note that for \( u > \left( \frac{c}{\omega - B} \right)^2 \) the function \( f(u) \) is negative. Therefore it is enough to find the maximum of \( f \) on \( \left( 0, \left( \frac{c}{\omega - B} \right)^2 \right) \). We will use a rough estimate for \( u \in \left( 0, \left( \frac{c}{\omega - B} \right)^2 \right) \)

\[
 \ln(u + 1) \left( c \sqrt{u} - (\omega - B) u \right) + \frac{u}{u + 1} (c \sqrt{u} - \omega u) \leq \\
 \frac{c^2}{4(\omega - B)} \ln(u + 1) + \frac{c^2}{4\omega} \frac{u}{u + 1} \leq \\
 \frac{c^2}{4(\omega - B)} u + \frac{c^2}{4\omega} \leq \frac{c^2}{4(\omega - B)} \left( \frac{c}{\omega - B} \right)^2 + \frac{c^2}{4\omega}.
\]

Thus we can take

\[
 C(c, \omega, B) := \frac{c^2}{4} \left( \frac{c^2}{(\omega - B)^3} + \frac{1}{\omega} \right).
\]

**Notation 3.8.** For any \( \varphi \in \mathcal{M} \) and for all \( t > 0 \) we denote the following random functions by

\[
 K_{\varphi;\omega,a(\cdot)}(t) := \varphi \left( \frac{[a(W_{0,A,\sigma}^*(t))]^2}{\omega^2} \right), \\
 K_{\varphi}(t) := \varphi \left( \frac{[W_{0,A,\sigma}^*(t)]^2}{\omega^2} \right)
\]

Note that these functions are finite a.s.

We will be using this notation for the following pairs of functions \( \varphi(x) \) and \( a(x) \): both are polynomials, or \( \varphi(x) \) is an exponential function and \( a(x) \) is a linear function (which corresponds to Fernique’s Theorem).

**Remark 3.9.** Note that often one uses an invariant measure to define an entropy of the system. As we do not want to make assumptions on such a measure, we can not introduce and study an entropy in such a general setting. Still, the introduction of \( \varphi \)-moments might eventually allow for such a notion, similarly to [11,15]. Finally, as we consider a perturbation of an Ornstein-Uhlenbeck equation, one can also incorporate a relative entropy such as considered in [28].
The variational process corresponding to (3.7) is $Z_{x,t}^\alpha$, a variational solution to the random ordinary differential equation (3.3). As $Z_{x,t}^\alpha + W_{0,A,\sigma}(t)$ is a mild solution to this equation we have a solution

$$X_{x,t}^\alpha = Z_{x,t}^\alpha + W_{0,A,\sigma}(t).$$

In what follows we abuse notation and set $Z_{x,t} = Z_{x,t}^\alpha$. Suppose $\varphi \in \mathcal{M}$ and Assumption 1.3 on $F$ and Assumption 1.4 on $A$ hold, then we can use (1.3, Chapter 6, Theorem 1.2, page 184) to claim that mild solution to (3.3) exists. In addition we have the following estimates which hold also if $A$ is replaced by $A_\alpha$, and $F$ is replaced by $F_\alpha$

$$\frac{d}{dt} \varphi \left( |Z_{0,t}|^2 \right) = 2 \varphi' \left( |Z_{0,t}|^2 \right) \langle Z_{0,t}^\alpha, Z_{0,t} \rangle = 2 \varphi' \left( |Z_{0,t}|^2 \right) (A_\alpha Z_{x,t} + F_\alpha(t, Z_{x,t} + W_{0,A,\sigma}(t)), Z_{x,t}) = 2 \varphi' \left( |Z_{0,t}|^2 \right) (A_\alpha Z_{x,t} + (F_\alpha(t, Z_{x,t} + W_{0,A,\sigma}(t)) - F_\alpha(t, W_{0,A,\sigma}(t))), Z_{x,t}) + 2 \varphi' \left( |Z_{0,t}|^2 \right) (F_\alpha(t, W_{0,A,\sigma}(t)), Z_{x,t}) \leq 2 \varphi' \left( |Z_{0,t}|^2 \right) (A_\alpha Z_{x,t}, Z_{x,t}) + 2 \varphi' \left( |Z_{0,t}|^2 \right) (F_\alpha(t, W_{0,A,\sigma}(t)), Z_{x,t}) \leq 2 \varphi' \left( |Z_{0,t}|^2 \right) (-\omega |Z_{x,t}|^2 + |Z_{x,t}||F_\alpha(t, W_{0,A,\sigma}(t))|),$$

which implies that as $\alpha \to 0$

$$\frac{d}{dt} \varphi \left( |Z_t|^2 \right) \leq 2 \varphi' \left( |Z_t|^2 \right) \left( |F(t, W_{0,A,\sigma}(t))| |Z_t| - \omega |Z_t|^2 \right).$$

This is understood in an integrated form (the derivative $\frac{d}{dt}$ is defined almost everywhere).

Using notation introduced in (3.2) and Assumption 1.3, we see that

$$\frac{d}{dt} \varphi \left( |Z_t|^2 \right) \leq 2 \varphi' \left( |Z_t|^2 \right) \left( a (W_{0,A,\sigma}^*(t)) |Z_t| - \omega |Z_t|^2 \right).$$

Depending on $\varphi$ we can use Lemma 3.2, Lemma 3.3 or Lemma 3.4 to get sharper estimates, but for the moment we use constants which might not be optimal for a particular $\varphi$, but work in all three cases. Namely, we take $B = \omega/2$ and

$$C := C \left( a (W_{0,A,\sigma}^*(t)), \omega, \frac{\omega}{2} \right) = \frac{2}{\omega} \varphi \left( \frac{\left[ a (W_{0,A,\sigma}^*(t)) \right]^2}{\omega^2} \right)$$
to see that
\[ \phi'(u) \left( a \left( W_{0,A,\sigma}^*(u) \right) \sqrt{u} - \omega u \right) \leq C - \frac{\omega}{2} \phi(u), \]
therefore
\[ \frac{d}{dt} \phi(|Z_t|^2) \leq \omega \left( \phi\left( \frac{[a \left( W_{0,A,\sigma}^*(t) \right)]^2}{\omega^2} \right) - \phi(|Z_t|^2) \right). \]

Now by Gronwall's lemma as formulated in Remark 2.3 we see that
\[ \phi(|Z_t|^2) \leq \phi(|x|^2) e^{-\omega t} + \omega \int_0^t e^{-\omega(t-s)} \phi\left( \frac{[a \left( W_{0,A,\sigma}^*(s) \right)]^2}{\omega^2} \right) ds. \]

Now we can use (2.8) and the fact that \( \phi \) is convex to see that for the solution \( X_t^x \) to (3.7) we have
\[ \phi(|X_t^x|^2) \leq \phi(2|Z_t|^2 + 2|W_{0,A,\sigma}(t)|^2) \leq \]
\[ \frac{1}{2} \phi(4|Z_t|^2) + \frac{1}{2} \phi(4|W_{0,A,\sigma}(t)|^2) \leq \]
\[ \frac{1}{2} \phi(4|x|^2) e^{-\omega t} + \frac{\omega}{2} \int_0^t e^{-\omega(t-s)} \phi\left( \frac{[a \left( W_{0,A,\sigma}^*(s) \right)]^2}{\omega^2} \right) ds \]
\[ + \frac{1}{2} \phi(4|W_{0,A,\sigma}(t)|^2) \leq \]
\[ \frac{1}{2} \phi(4|x|^2) e^{-\omega t} + \frac{1}{2} \phi(4|W_{0,A,\sigma}(t)|^2) + \]
\[ \phi\left( \frac{[a \left( W_{0,A,\sigma}^*(t) \right)]^2}{\omega^2} \right) \frac{\omega}{2} \int_0^t e^{-\omega(t-s)} ds \leq \]
\[ \frac{e^{-\omega t}}{2} \phi(4|x|^2) + \frac{1}{2} \phi(4|W_{0,A,\sigma}(t)|^2) + \frac{\omega t}{2} \phi\left( \frac{[a \left( W_{0,A,\sigma}^*(t) \right)]^2}{\omega^2} \right). \]
for all \( t \geq 0 \).

Proof. Denote

\[
Z_t^{x} := |x| e^{-\omega t} + \int_0^t e^{-\omega (t-s)} a \left( |W_{0,A,\sigma}(s)| \right) ds.
\]

We use (1.4) and [16, Chapter 6, Theorem 1.2, page 184]) to obtain that \( \mathbb{P} \)-almost surely for almost all \( t \)

\[
\frac{d}{dt} \left( |Z_{\alpha,t}|^2 \right) \leq 2 \left( -\omega |Z_{\alpha,t}|^2 + |Z_{\alpha,t}| |F_\alpha(t, W_{0,A,\sigma}(t))| \right)
\leq 2 \left( -\omega |Z_{\alpha,t}|^2 + |Z_{\alpha,t}| a \left( |W_{0,A,\sigma}(t)| \right) \right)
\]

for any \( \alpha > 0 \).

Then we use Lemma 4.1 with \( f(t) = |Z_{\alpha,t}| \), \( c(t) = 2a \left( |W_{0,A,\sigma}(t)| \right) \) to obtain that almost surely \( |Z_{\alpha,t}| \) is an absolutely continuous function satisfying

\[
\frac{d}{dt} \left( |Z_{\alpha,t}^x| \right) \leq -\omega |Z_{\alpha,t}^x| + a \left( |W_{0,A,\sigma}(t)| \right)
\]

for almost all \( t \). Therefore, we can verify assumption (2.3) in Gronwall’s Lemma in Remark 2.3 with \( B = \omega \) and \( A(s) = a \left( |W_{0,A,\sigma}(t)| \right) \). This implies that

\[
|Z_{\alpha,t}^x| \leq Z_t^{x}.\]

Note that \( Z_t^{x} \) in (4.2) does not depend on \( F \) explicitly, but only on the increasing function \( a : [0, \infty) \rightarrow [0, \infty) \) as introduced in Assumption 1 (and (2.3)). If \( Z_t^x \) is any weak limit point of the processes \( Z_{\alpha,t}^x \), as \( \alpha \rightarrow 0 \), then \( |Z_t^x| \leq Z_t^{x} \).

In particular, Proposition 4.1 shows that \( |Z_{\alpha,t}^x| \) is controlled in terms of \( a(|W_{0,A,\sigma}|) \) only, and therefore we have the following fact.

Corollary 4.2. On any finite interval \([0, T]\) we have that with probability one \( |Z_{\alpha,t}^x| \) satisfies (1.4) uniformly in \( \alpha \). In particular, this proves Theorem 1.3 (1).

Remark 4.3. We can consider the pseudo-weak solution \( Z_t^x \) of the equation

\[
dZ_t = (AZ_t + F(t, Z_t + W_{0,A,\sigma}(t))) dt, \quad Z_0 = x.
\]

Note that we can not claim that the process \( Z_t^x \) belongs to \( L^2 \left([0, T] \times \Omega; H\right) \), but we show that \( |Z_t^x| < \infty \) almost everywhere on the space \([0, T] \times \Omega\) because our construction implies that \( |Z_t^x| \leq Z_t^{x,x} := \sup_{t \in [0,T]} Z_t^{x,x} < \infty \). We observe that the Banach-Alaoglu theorem can not be applied directly in our setting because we consider random processes that are defined only a.s., and \( |Z_t^x| \) may not be integrable with respect to the probability measure \( \mathbb{P} \). However, we can avoid this difficulty because we can use the Banach-Alaoglu theorem for the family of processes defined by \( V_{\alpha,t}^x = \frac{Z_{\alpha,t}^x}{1 + |Z_{\alpha,t}^x|} \), and using Appendix [23]. The norm of these processes is uniformly bounded in \( L^2 \left([0, T] \times \Omega; H\right) \), and so there is an \( L^2 \left([0, T] \times \Omega; H\right) \)-weakly convergent subsequence. Hence, we have a sequence of positive numbers \( \alpha_k \rightarrow 0 \) such that the sequence of processes \( V_{\alpha_k,t}^x \) converges \( L^2 \left([0, T] \times \Omega; H\right) \)-weakly to a process \( V_t^x \). Note that, as a result of its construction, \( V_t^x \) is an adapted bounded process with satisfies

\[
\lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle V_{\alpha_k,t}^x, g_t \rangle dt = \mathbb{E} \int_0^T \langle V_t^x, g_t \rangle dt \quad \text{for any} \ g_t \in L^2 \left([0, T] \times \Omega; H\right).
\]

By (1.4) and (1.2) we have that almost surely for any \( t \in [0, T] \) we have \( |Z_{\alpha,t}^x| \leq Z_t^{x,x} \). Observe that \( Z_t^{x,x} \) is a real nonnegative a.s. finite random variable which does not depend neither on \( t \in [0, T] \)
nor on $\alpha > 0$. That is, $Z_{t,x}^{\ast,x}$ depends only on $T > 0, \omega > 0, x \in H, a(\cdot)$ and $W_{0,A,\sigma}$. Therefore almost everywhere on the space $[0,T] \times \Omega$ we have $|V_t^x| \leq \limsup_{k \to \infty} \frac{|Z_{t,k}^{x}|}{1+|Z_{t,k}^{x}|} \leq \frac{Z_{t,x}^{\ast,x}}{1+Z_{t,x}^{\ast,x}} < 1$ and $Z_t^x = \frac{V_t^x}{1-|V_t^x|}$ satisfies
\[
\lim_{k \to \infty} \mathbb{E} \int_0^T \frac{\langle Z_{t,k}^{x}, g_t \rangle}{1 + |Z_{t,k}^{x}|} dt = \mathbb{E} \int_0^T \frac{\langle Z_t^x, g_t \rangle}{1 + |Z_t^x|} dt.
\]

**Lemma 4.4.** Suppose that $f(t)$ is a non-negative continuous function on $\mathbb{R}$ such that $g(t) = (f(t))^2$ is an absolutely continuous function a.e. satisfying $|g'(t)| \leq c(t)f(t)$ for some measurable locally integrable function $c(t)$. Then $f(t)$ is an absolutely continuous function of $t$ a.e. satisfying $2f(t)f'(t) = g'(t)$ and $|f'(t)| \leq \frac{c(t)}{2}$.

**Proof.** For any $\varepsilon > 0$ there are countably many intervals on which $f(t) > \varepsilon$, and on any such interval the result follows from the usual chain rule for absolutely continuous functions. Then, taking the limit as $\varepsilon \to 0$, the result holds on the open set $f(t) > 0$. In particular, $f'(s) = \frac{g'(s)}{2f(s)}$ and $f(t) = f(t_0) + \int_{t_0}^t f'(s)ds$ if $f(s) > 0$ for $t_0 < s < t$. After that we write the set $f(t) > 0$ as a countable union of disjoint intervals. The continuity of $f(t)$, dominated convergence, and the standard manipulations with limits, sums, and integrals imply that $f(t) = f(0) + \int_0^t f'(s)ds \mathbb{1}_{f(s) > 0}ds$ for all $t$, which implies result. \(\square\)

### 4.2. Stopping times and Girsanov transforms.

By (1.3) we have
\[
X_t^x = Z_t^x + W_{0,A,\sigma}(t) = (Z_t^x - e^{At}x) + (W_{0,A,\sigma} + e^{At}x) := Z_t^{0,x} + W_{x,A,\sigma}(t).
\]
From this and (1.1) we have
\[
|Z_{t,x}^{0,x}| \leq |Z_{t}^{x,x}| + |e^{At}x|,
\]
where $Z_{t,x}^{x,x}$ is defined by (1.2). We define the stopping times by
\[
\tau_n^x = \inf\{t \geq 0 : Z_{t}^{x,x} + |e^{At}x| + |W_{x,A,\sigma}| \geq n\},
\]
where $W_{x,A,\sigma}$ is defined by (1.1). Note that this definition does not depend on $\alpha$.

Now we consider Girsanov transforms for the Yosida regularized equations as follows. Let
\[
\rho_\alpha(x,t) = \exp(\zeta_\alpha(x,t)),
\]
where
\[
\zeta_\alpha(x,t) = \int_0^t \langle \sigma^{-1}F_\alpha(s, W_{x,A,\sigma}(s)), dW(s) \rangle - \frac{1}{2} \int_0^t |\sigma^{-1}F_\alpha(s, W_{x,A,\sigma}(s))|^2 ds.
\]
Here $\mathbb{P}_\alpha^x$ is defined by
\[
\frac{d\mathbb{P}_\alpha^x}{d\mathbb{P}} = \rho_\alpha(x,T),
\]
which gives a mild martingale solution to (1.3). More precisely, we can define
\[
\bar{X}_x(t) := W_{x,A,\sigma}(t)
\]
and

\begin{equation}
\tilde{W}_{x, \alpha}(t) = W_t - \int_0^t \sigma^{-1} F_\alpha(s, W_{x, A, \sigma}(s)) \, ds.
\end{equation}

Then

\begin{equation}
d\tilde{X}_x(t) = dW_{x, A, \sigma}(t) = A \tilde{X}_x dt + \sigma dW_t = A \tilde{X}_x dt + F_\alpha(t, \tilde{X}_x(t)) dt + \sigma d\tilde{W}_x(t),
\end{equation}

where we used the fact that \( \tilde{W}_{x, \alpha} \) has the same probability law under \( \mathbb{P}_x^\alpha \) as the probability law of \( W \) under \( \mathbb{P} \).

**Remark 4.5** (On localization). As a side remark we would like to mention that in the infinite-dimensional case even though the processes in (4.12) are not semimartingales in general (such as in [10]), one might want to use localization to introduce

\begin{equation}
\tilde{W}_{x, \alpha}^n(t) = W_t - \int_0^{t \wedge \tau_n} \sigma^{-1} F_\alpha(s, W_{x, A, \sigma}(s)) \, ds,
\end{equation}

\begin{equation}
\rho^n_\alpha(x, t) = \exp(\zeta_\alpha(x, t \wedge \tau_n^n)).
\end{equation}

Then we can define \( \rho_\alpha(x, t) \) as a limit as \( n \to \infty \), if the limit exists. However, the localization can not be used easily for the equations with non-smooth coefficients.

### 4.3. Estimates of the Girsanov densities.

**Proof of Theorem [1,4,3].** In this proof we assume that \( r, T > 0 \) are fixed, and \( |x| < r \), \( t \in [0, T] \). Therefore we drop dependence on \( r, T \) in notation although our estimates do depend on \( r, T \).

First we apply Girsanov transformation to see that

\[ \mathbb{E} \rho_\alpha \psi(\rho_\alpha) = \mathbb{E}^x \psi(\rho_\alpha), \]

where \( \rho_\alpha \) is the density defined by (4.13). Note that the distribution of \((\tilde{W}_{x, \alpha}, \tilde{X}_x) = (\tilde{W}_{x, \alpha}, W_{x, A, \sigma})\) under the measure \( \mathbb{P}_x^\alpha \) is the same as the distribution of \((W, X)\) under the measure \( \mathbb{P} \). Then

\begin{equation}
\mathbb{E}^x \psi(\rho_\alpha) = \mathbb{E} \psi(\tilde{\rho}_\alpha),
\end{equation}

where

\begin{equation}
\tilde{\rho}_\alpha := \exp \left( \int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha(s)), dW(s) \rangle + \frac{1}{2} \int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha(s))|^2 \, ds \right).
\end{equation}
and \( \mathbb{E}_\alpha^x \) is the expectation with respect to the probability measure \( \mathbb{P}_\alpha^x \) given by (4.14). We can estimate \( \mathbb{E} |\tilde{\rho}_\alpha|^p \) as follows

\[
\mathbb{E} |\tilde{\rho}_\alpha|^p = \mathbb{E} \exp \left( p \int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha(s)), dW_s \rangle - p^2 \int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha(s))|^2 ds \right) \\
\quad \times \exp \left( \left( p^2 + \frac{p}{2} \right) \int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha(s))|^2 ds \right) \\
\quad \leq \left( \mathbb{E} \exp \left( 2p \int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha(s)), dW_s \rangle - 2p^2 \int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha(s))|^2 ds \right) \right)^{1/2} \\
\quad \times \left( \mathbb{E} \exp \left( (2p^2 + p) \int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha(s))|^2 ds \right) \right)^{1/2}.
\]

Note that the first term in the last formula is equal to the expectation of the stochastic exponential for \( 2p \int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha(s)), dW_s \rangle \), and so its expectation is 1, therefore

\[
\text{(4.17)} \quad \mathbb{E} |\tilde{\rho}_\alpha|^p \leq \left( \mathbb{E} \exp \left( (2p^2 + p) \int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha(s))|^2 ds \right) \right)^{1/2}.
\]

We can estimate, using (4.17) and Chebyshev’s inequality,

\[
\text{(4.18)} \quad \mathbb{P} (\tilde{\rho}_\alpha > M) \leq \mathbb{P} (\tilde{\rho}_\alpha^0 > M) + \mathbb{P} (\tau_n^x < t) \\
\text{(4.19)} \quad \leq \left( \exp \left( 5 \left( a(n) |\sigma^{-1} | \right)^2 T \right) \right) / M^2 + \mathbb{P} (\tau_n^x < T).
\]

Note that the coefficient 5 here appears instead of the more usual 4 because (4.16) is not a martingale. One has to compensate for the term with \( +\frac{1}{2} \) instead of the usual terms with \( -\frac{1}{2} \) that is common in stochastic exponentials.

For any given \( M \) we can find \( n_0(M) \) so that

\[
\exp \left( 5 \left( a(n_0) |\sigma^{-1} | \right)^2 T \right) < M
\]

if \( n \leq n_0 \), and then, using (4.2), define

\[
p(M, \sigma, A, a, x, T) := \min \left\{ 1, 1/M + \mathbb{P} (\tau_n^x < T) \right\}.
\]

We already assumed that \( \psi \) is increasing. Without loss of generality we can assume that \( \psi \) is differentiable and \( \psi(0) = 0 \). There is a well known formula

\[
\text{(4.20)} \quad \mathbb{E} \psi (\tilde{\rho}_\alpha) = \int_0^\infty \psi' (M) \mathbb{P} \{ \rho_\alpha > M \} dM.
\]

Finally, by the standard compactness and uniform integrability arguments, the result is true for any weak limit point of \( \rho_\alpha \) if we choose concave down \( \psi \) such that

\[
\text{(4.21)} \quad \int_0^\infty \psi' (M) p(M, \sigma, A, a, x, T) dM < 1
\]

and

\[
\text{(4.22)} \quad \lim_{M \to \infty} \psi (M) = \infty.
\]
There are infinitely many possible choices for \( \psi \). For instance
\[
\psi(M) < \frac{1}{2} (p(M, \sigma, A, a, x, T))^{-1/2}
\]
is a sufficient condition for (121). This is because it is possible to approximate \( p(M, \sigma, A, a, x, T) \) by a smooth function of \( M \) and use integration by parts.

**Corollary 4.6.** Conditions (121) and (122) imply Theorem 1.7.(3).

**Example 4.7.**
(1) If for some \( c_1, c_2 > 0 \) we have \( a(n) < c_1 n^{c_2} \) then we can choose
\[
\psi(\rho) = c_4 (\log(\rho))^{c_3}
\]
for any \( c_3 > 0 \) and some \( c_4 > 0 \), which depends on \( c_1, c_2, c_3, r, T \).
(2) If for some \( c_1, c_2 > 0 \) we have \( a(n) < c_1 e^{c_2 n} \) then we can choose
\[
\psi(\rho) = c_3 (\log(1 + (\log(1 + (\log(1 + \rho))))))
\]
for some \( c_3 > 0 \), which depends on \( c_1, c_2, r, T \).

**Proof.** This follows from estimate (1.1) in Proposition 4.4. \( \square \)

### 4.4. Applications to the autonomous equations.
Here we would like to mention several applications of our results in the case when \( F \) in (1.1) does not depend on \( t \). In this case one can use our results to prove smoothness results for an invariant measure, closability of the corresponding Dirichlet form.

Note that we do not address here under which assumptions an invariant measure exists. Suppose there is an invariant measure as described in [20], then one can use the Girsanov transform in Theorem 1.7(3) to show formally that the invariant measure is quasi-invariant under certain linear shifts. This leads to a possibility of using [3, Theorem 2.2] and [4, Theorem 1.3] to prove closability of the Dirichlet form. The main ingredient here is what is known as Hamza’s condition [35] or the lower semicontinuity of the Radon-Nikodym density as also described in [3, p.122] among other references.

Finally, there are other recent approaches to quasi-invariance of semigroups in infinite dimensions which rely on functional inequalities [8, 9, 25, 26, 31, 32, 41, 47], and these methods are not applicable to such singular perturbations as we consider in this paper.

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### Appendix A. Pseudo-weak convergence

Let \((S, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space and \((H, \langle \cdot, \cdot \rangle)\) a separable real Hilbert space. We denote by \(L^2(S; H, \mu)\) the space of \(H\)-valued square-integrable functions.

**Definition A.1.** Suppose \(\mu(S) < \infty\) and \(F, F_n : S \rightarrow H, n \in \mathbb{N}\) are \(\mathcal{F}\)-measurable. We say that \(\{F_n\}_{n=1}^\infty\) is pseudo-weakly convergent to \(F\), denoted by
\[
F_n \xrightarrow{\text{p-w}} F,
\]
if
\[
\frac{F_n}{1 + |F_n|_H} \xrightarrow{n \to \infty} \frac{F}{1 + |F|_H}
\]
in \(L^2(S; H, \mu)\). Here \(\rightharpoonup\) denotes the usual weak convergence in \(L^2(S; H, \mu)\).

**Remark A.2.** Observe that the pseudo-weak limit is unique, that is, if

\[
F_n \xrightarrow{\psi} F, \\
F_n \xrightarrow{\psi} G,
\]

then \(F = G\) \(\mu\)-a.e.

In addition, \(L^0(S; H, \mu)\)-convergence, i.e. convergence in measure

\[
\lim_{n \to \infty} \mu \{ x : |F_n - F| > \epsilon \} = 0
\]

implies pseudo-weak convergence, but these two types of convergence are not equivalent in general if \(H\) is infinite-dimensional.

We also note that the choice of the function \(r \mapsto \frac{r}{1+r}, r > 0\) used in this definition is arbitrary, and the convergence will be the same for any choice of an increasing bounded function.

**Proposition A.3.** Suppose \(F, F_n \in L^2(S; H, \mu), n \in \mathbb{N}, \mu\) is \(\sigma\)-finite and

(A.1) \[
F_n \xrightarrow{n \to \infty} F.
\]

Then

\[
|F|_H \leq \limsup_{n \to \infty} |F_n|_H \quad \mu - a.e.
\]

**Proof.** Recall that \(H\) is assumed to be separable, therefore there exist \(h_n \in H, n \in \mathbb{N}\) such that

\[
|h_n|_H = 1 \text{ for all } n \in \mathbb{N}, \\
|h|_H = \sup_n \langle h_n, h \rangle \text{ for all } h \in H.
\]

Then for all non-negative \(f \in L^\infty(S; \mathbb{R}, \mu)\) such that \(\mu(\{ f > 0 \}) < \infty\), by Fatou’s lemma and (A.1) for all \(k \in \mathbb{N}\)

\[
\int_S \langle h_k, F(y) \rangle_H f(y) \mu(dy) = \lim_{n \to \infty} \int_S \langle h_k, F_n(y) \rangle_H f(y) \mu(dy)
\]

\[
\leq \int_S \limsup_{n \to \infty} |\langle h_k, F_n(y) \rangle_H| f(y) \mu(dy) \leq \int_S \limsup_{n \to \infty} |F_n(y)|_H f(y) \mu(dy).
\]

As \(\mu\) is \(\sigma\)-finite, we have that for \(\mu\)-a.e. \(y \in S\) and all \(k \in \mathbb{N}\)

\[
\langle h_k, F(y) \rangle_H \leq \limsup_{n \to \infty} |F_n(y)|_H,
\]

which completes the proof. \(\square\)
Corollary A.4. Assume in addition that $\mu$ is finite and let $F, F_n : S \rightarrow H$, $n \in \mathbb{N}$ be $\mathcal{F}$-measurable such that

$$F_n \xrightarrow{\psi_{n \to \infty}} F.$$ 

Then

$$|F|_H \leq \limsup_{n \to \infty} |F_n|_H \quad \mu \text{-a.e.}$$

Proof. By definition of the pseudo-weak convergence and Proposition A.3 we have that on $\{\limsup_{n \to \infty} |F_n|_H < \infty\}$

$$\frac{|F|_H}{1 + |F|_H} \leq \limsup_{n \to \infty} \frac{|F_n|_H}{1 + |F_n|_H} \leq \frac{\limsup_{n \to \infty} |F_n|_H}{1 + \limsup_{n \to \infty} |F_n|_H} < 1.$$ 

Applying the non-decreasing function $r \mapsto \frac{1}{1 + r}, 0 \leq r < 1$ to both sides of this inequality proves the desired result. \hfill $\square$

Proposition A.5. Suppose $\mu$ is finite and $F_n \in L^2(S; H, \mu)$, $n \in \mathbb{N}$ are such that

$$\sup_{n \in \mathbb{N}} |F_n|_H < \infty \quad \mu \text{-a.e.}$$

Then there exists $F \in L^2(S; H, \mu)$ such that for some subsequence $\{n_k\}_{k \in \mathbb{N}}$

$$F_{n_k} \xrightarrow{\psi_{k \to \infty}} F.$$ 

Proof. Define

$$V_n := \frac{F_n}{1 + |F_n|_H}, \quad n \in \mathbb{N}.$$ 

Then $\{V_n\}_{n \in \mathbb{N}}$ is bounded in $L^2(S; H, \mu)$, therefore there exists a $V \in L^2(S; H, \mu)$ such that for some subsequence $\{n_k\}_{k \in \mathbb{N}}$

$$V_{n_k} \xrightarrow{k \to \infty} V.$$ 

By Proposition A.3 and the assumptions we have

$$|V|_H \leq \frac{\limsup_{k \to \infty} |F_{n_k}|_H}{1 + \limsup_{k \to \infty} |F_{n_k}|_H} < 1,$$

hence

$$F := \frac{V}{1 - |V|_H}$$

is well-defined and one easily checks that

$$F_{n_k} \xrightarrow{k \to \infty} F.$$ 

\hfill $\square$
References

1. S. Albeverio, M. Röckner, and T. S. Zhang, *Girsanov transform for symmetric diffusions with infinite-dimensional state space*, Ann. Probab. **21** (1993), no. 2, 961–978. MR 1217575 (94i:60087)

2. S. Albeverio and Michael Röckner, *Dirichlet forms, quantum fields and stochastic quantisation*, Tech. report, BiBoS, 1988.

3. Sergio Albeverio and Raphael Hoegh-Krohn, *Some remarks on Dirichlet forms and their applications to quantum mechanics and statistical mechanics*, Functional analysis in Markov processes (Katata/Kyoto, 1981), Lecture Notes in Math., vol. 923, Springer, Berlin-New York, 1982, pp. 120–132. MR 661620

4. Sergio Albeverio, Hiroshi Kawabi, and Michael Röckner, *Strong uniqueness for both Dirichlet operators and stochastic dynamics to Gibbs measures on a path space with exponential interactions*, J. Funct. Anal. **262** (2012), no. 2, 602–638. MR 2854715

5. Sergio Albeverio and Michael Röckner, *Classical Dirichlet forms on topological vector spaces—closability and a Cameron-Martin formula*, J. Funct. Anal. **88** (1990), no. 2, 395–436. MR 1038449

6. Viorel Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Editura Academiei Republicii Socialiste România, Bucharest, 1976. Translated from the Romanian. MR 0390843 (52 #11666)

7. R. Carmona and S. Nadtochiy, *An infinite dimensional stochastic analysis approach to local volatility dynamic models*, Commun. Stoch. Anal. **2** (2008), no. 1, 109–123. MR 2446994

8. Giuseppe Da Prato, Arnaud Debussche, and Beniamin Goldys, *Some properties of invariant measures for non symmetric dissipative stochastic systems*, Probab. Theory Related Fields **123** (2002), no. 3, 355–380. MR 1918538 (2003c:60136)

9. Giuseppe Da Prato and Jerzy Zabczyk, *Nonexplosion, boundedness, and ergodicity for stochastic semi-linear equations*, J. Differential Equations **98** (1992), no. 1, 181–195. MR 1168978
23. , Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992. MR MR1207136 (95g:60073)
24. Marzia De Donno and Maurizio Pratelli, Stochastic integration with respect to a sequence of semimartingales, In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX, Lecture Notes in Math., vol. 1874, Springer, Berlin, 2006, pp. 119–135. MR 2276892
25. Bruce K. Driver and Maria Gordina, Heat kernel analysis on infinite-dimensional Heisenberg groups, J. Funct. Anal. 255 (2008), no. 9, 2395–2461. MR 2473262
26. , Integrated Harnack inequalities on Lie groups, J. Differential Geom. 83 (2009), no. 3, 501–550. MR MR2581356
27. Klaus-Jochen Engel and Rainer Nagel, One-parameter semigroups for linear evolution equations, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR 1721989 (2000i:47075)
28. H. Föllmer, Time reversal on Wiener space, Stochastic processes—mathematics and physics (Bielefeld, 1984), Lecture Notes in Math., vol. 1158, Springer, Berlin, 1986, pp. 119–129. MR 838561 (88a:60140)
29. Dariusz Gątarek and Beniamin Goldys, On solving stochastic evolution equations by the change of drift with application to optimal control, Stochastic partial differential equations and applications (Trento, 1990), Pitman Res. Notes Math. Ser., vol. 268, Longman Sci. Tech., Harlow, 1992, pp. 180–190. MR 1222696 (94d:60096)
30. , On invariant measures for diffusions on Banach spaces, Potential Anal. 7 (1997), no. 2, 539–553. MR 1467205 (98k:60102)
31. Maria Gordina, An application of a functional inequality to quasi-invariance in infinite dimensions, pp. 251–266, Springer New York, New York, NY, 2017.
32. Maria Gordina, Michael Röckner, and Feng-Yu Wang, Dimension-independent Harnack inequalities for subordinated semigroups, Potential Anal. 34 (2011), no. 3, 293–307. MR 2782975 (2012f:60230)
33. Martin Hairer and Jonathan C. Mattingly, A theory of hypoellipticity and unique ergodicity for semilinear stochastic PDEs, Electron. J. Probab. 16 (2011), no. 23, 658–738. MR 2786645
34. , The strong feller property for singular stochastic PDEs, https://arxiv.org/abs/1610.03415, arXiv:1610.0341, 2017.
35. M.M. Hamza, Détermination des formes de dirichlet sur rn, Publications mathématiques d’Orsay, Département de Mathématique, 1975.
36. A. I. Kirillov, Infinite-dimensional analysis and quantum theory as semimartingale calculi, Uspekhi Mat. Nauk 49 (1994), no. 3(297), 43–92. MR 1289387
37. N. V. Krylov, A simple proof of the existence of a solution to the Itô equation with monotone coefficients, Teor. Veroyatnost. i Primenen. 35 (1990), no. 3, 576–580. MR 1091217
38. , Introduction to the theory of diffusion processes, Translations of Mathematical Monographs, vol. 142, American Mathematical Society, Providence, RI, 1995, Translated from the Russian manuscript by Valim Khidekel and Gennady Pasechnik. MR 1311478
39. , A simple proof of a result of A. Novikov, arXiv preprint math/0207013 (2002).
40. Carlo Marinelli and Michael Röckner, On uniqueness of mild solutions for dissipative stochastic evolution equations, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13 (2010), no. 3, 363–376. MR 2729590
41. Tai Melcher, Heat kernel analysis on semi-infinite Lie groups, J. Funct. Anal. 257 (2009), no. 11, 3552–3592. MR 2572261 (2011b:58074)
42. Michel Métivier, Semimartingales, de Gruyter Studies in Mathematics, vol. 2, Walter de Gruyter & Co., Berlin, 1982, A course on stochastic processes. MR 688144 (84i:60002)
43. Paul-A. Meyer, Probability and potentials, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1966. MR 0205288 (34 #5119)
44. A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR 710486 (85g:47061)
45. Claudia Prévôt and Michael Röckner, A concise course on stochastic partial differential equations, Lecture Notes in Mathematics, vol. 1905, Springer, Berlin, 2007. MR 2329435 (2009a:60069)
46. Philip E. Protter, Stochastic integration and differential equations, Stochastic Modelling and Applied Probability, vol. 21, Springer-Verlag, Berlin, 2005, Second edition. Version 2.1, Corrected third printing. MR 2273672
47. Michael Röckner and Feng-Yu Wang, *General extinction results for stochastic partial differential equations and applications*, J. Lond. Math. Soc. (2) **87** (2013), no. 2, 545–560. MR 3046285

48. Michael Röckner and Tu Sheng Zhang, *Uniqueness of generalized Schrödinger operators and applications*, J. Funct. Anal. **105** (1992), no. 1, 187–231. MR 1156676

49. Michael Röckner, Rongchan Zhu, and Xiangchan Zhu, *A note on stochastic semilinear equations and their associated Fokker-Planck equations*, J. Math. Anal. Appl. **415** (2014), no. 1, 83–109. MR 3173156

50. Francesco Russo and Pierre Vallois, *Elements of stochastic calculus via regularization*, Séminaire de Probabilités XL, Lecture Notes in Math., vol. 1899, Springer, Berlin, 2007, pp. 147–185. MR 2409004

51. Wilhelm Stannat, *Non-symmetric Dirichlet operators on $L^1$: existence, uniqueness and associated Markov processes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 1, 99–140. MR 1679079

52. ______, *The theory of generalized Dirichlet forms and its applications in analysis and stochastics*, Mem. Amer. Math. Soc. **142** (1999), no. 678, viii+101. MR 1632609

53. A. S. Üstünel, *Applications of integration by parts formula for infinite-dimensional semimartingales*, J. Multivariate Anal. **18** (1986), no. 2, 287–299. MR 833000

54. J. M. A. M. van Neerven, M. C. Veraar, and L. Weis, *Stochastic integration in UMD Banach spaces*, Ann. Probab. **35** (2007), no. 4, 1438–1478. MR 2330977

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