CHERN CLASSES AND SYMPLECTIC CIRCLE ACTIONS.

YUNHYUNG CHO, MIN KYU KIM, AND DONG YOUP SUH

Abstract. Let \((M, \omega)\) be a two dimensional closed symplectic manifold. Then it is a well-known fact that if \(M\) admits a symplectic circle action, then \(M\) is diffeomorphic to \(S^2\) or \(S^1 \times S^1\). For each case, any symplectic circle action on \(S^2\) is always Hamiltonian and the first Chern class \(c_1(S^2)\) is positively proportional to \([\omega]\), and any symplectic circle action on \(S^1 \times S^1\) is non-Hamiltonian and \(c_1(S^1 \times S^1) = 0\). For the case when \((M, \omega)\) is a symplectic surface with genus greater than one, \(c_1(M)\) is negatively proportional to \([\omega]\). In this paper, we extended this result in higher dimensional cases. More precisely, let \((M, \omega)\) be a closed symplectic manifold satisfying \(c_1(M) = \lambda \cdot [\omega]\) for some real number \(\lambda \in \mathbb{R}\). We prove that if \((M, \omega)\) admits a symplectic circle action, then \(\lambda \geq 0\). Also, we prove that if the action is non-Hamiltonian, then \(\lambda\) should be 0.

1. Introduction

Let \(M\) be a smooth manifold. For a Lie group \(G\), the existence or the non-existence problem of the non-trivial \(G\)-action on \(M\) has been studied for a long time. In this paper, we consider the above problem in the symplectic category in the case when \(G\) is a circle group. Let \((M, \omega)\) be a closed symplectic manifold. Then \((M, \omega)\) satisfies the following properties. (See [McS] for the details).

- There exists an almost complex structure \(J\) on \(M\) which is \(\omega\)-tame, i.e. \(\omega(J \cdot, \cdot)\) is positive definite everywhere.
- Let \(\mathfrak{J}_\omega\) be the set of \(\omega\)-tame almost complex structures on \(M\). Then \(\mathfrak{J}_\omega\) is contractible so that the tangent bundle \((TM, J)\) is unique up to complex vector bundle isomorphism. Hence \(\omega\) defines Chern classes \(c_i(M)\) for \(i = 1, 2, \ldots\).
- When a Lie group \(G\) acts on \(M\) preserving \(\omega\), then there exists a compatible almost complex structure \(J\) which is also \(G\)-invariant.

Recall that \(G\)-action on \((M, \omega)\) is called symplectic if \(\omega\) is \(G\)-invariant, or equivalently \(i_X \omega\) is closed for any fundamental vector field \(X\) on \(M\) generated by the \(G\)-action, and is called Hamiltonian if \(i_X \omega\) is exact for all \(X\). One of the most classical problem is that

- For a given symplectic Lie group \(G\)-action on \(M\), which conditions on \(M\) make the action to be Hamiltonian?

In this paper, we give an answer to the question in terms of the first Chern class of \(M\).

Theorem 1.1. Let \((M, \omega)\) be a closed symplectic manifold satisfying \(c_1(M) = \lambda \cdot [\omega]\) for some real number \(\lambda \in \mathbb{R}\).
If $\lambda > 0$, then any symplectic circle action is Hamiltonian.

If $\lambda = 0$, then there is no Hamiltonian circle action on $M$.

If $\lambda < 0$, then there is no symplectic circle action on $M$.

Remark 1.2. Note that (1) and (3) of Theorem 1.1 was proved by Ono [O], by showing the non-semipositivity of a given symplectic form. And (1) was reproved by Lupton and Oprea [LO] using the theory of Gottlieb group. In our case, we will show that if $(M, \omega)$ admits a non-Hamiltonian circle action, then there exists an $S^1$-invariant two dimensional symplectic torus $T$ whose normal bundle is trivial, which implies that $\langle c_1(M), [T] \rangle = 0$. Then it contradicts the fact that $\langle c_1(M), [T] \rangle \neq 0$ when $\lambda \neq 0$. Also, we will prove the existence of a symplectic two-sphere $S$ in any compact Hamiltonian $S^1$-space such that $\langle c_1(M), [S] \rangle > 0$.

Remark 1.3. One can analyze the theorem in terms of the topology of the group of symplectomorphism $\text{Symp}(M, \omega)$ and the group of Hamiltonian diffeomorphism $\text{Ham}(M, \omega)$ as follows. (We will not use the notion $\text{Symp}(M, \omega)$ nor $\text{Ham}(M, \omega)$ in the rest of this paper.)

$\bullet$ Let $T$ be a maximal torus of $\text{Symp}(M, \omega)$. Then if $\lambda > 0$, then $T$ is also a maximal torus of $\text{Ham}(M, \omega)$.

$\bullet$ If $\lambda = 0$, then there is no maximal torus in $\text{Ham}(M, \omega)$, which means there is no compact connected Lie group which acts on $(M, \omega)$ in Hamiltonian fashion. Therefore, any finite dimensional compact Lie subgroup of $\text{Ham}(M, \omega)$ is discrete.

$\bullet$ If $\lambda < 0$, then there is no compact connected Lie group which acts on $(M, \omega)$ preserving the symplectic structure $\omega$. Therefore, any finite dimensional compact Lie subgroup of $\text{Symp}(M, \omega)$ is discrete.

Finally, we give a brief explanation of the motivation of our paper. The following conjecture is a long-standing problem in symplectic topology

Conjecture 1.4. Let $(M, \omega)$ be a compact symplectic manifold with a symplectic circle action. If the fixed point set is non-empty and isolated, then the action is Hamiltonian.

Without the assumption “isolated”, then there is a counter-example due to D.McDuff [Mc1]. Recently, a symplectic Calabi-Yau manifold (i.e. a symplectic manifold $(M, \omega)$ with vanishing first Chern class) is studied by several people. For example, D.Panov and J.Fine constructed a family of six-dimensional compact symplectic non-Kähler Calabi-Yau manifolds in [FP]. Therefore, one can expect that there exists a symplectic non-Kähler Calabi-Yau manifold with a symplectic circle action and the non-empty isolated fixed points. If there exists, then it will be a counter-example for the conjecture [L3].

2. Proof of Theorem 1.1

Let $(M, \omega)$ be a symplectic manifold with a symplectic $S^1$-action with an integral symplectic form $\omega$, and let $J$ be an $\omega$-tame almost complex structure. For any given point
Lemma 2.2. An easy lemma which has been noted in [K] with not so detailed proof.

For each point $x_0 \in M$, the generalized moment map $\mu : M \to \mathbb{R}/\mathbb{Z} \cong S^1$ is defined by

$$\mu(x) := \int_{x_0}^x i_X \omega \mod \mathbb{Z},$$

where $X$ is the fundamental vector field on $M$ generated by the $S^1$-action. It follows immediately from definition that $\mu = i_X \omega$. Since any symplectic action is locally Hamiltonian, the generalized moment map satisfies all local properties of the usual moment map. More precisely, $\mu$ satisfies

1. $\mu$ is $S^1$-invariant, i.e. $\mu$ maps an orbit to a point in $S^1$. So any level set of $\mu$ is an $S^1$-space.
2. The set of critical points equals to $M^{S^1}$ where $M^{S^1}$ is the set of all fixed points.
3. $\mu$ is a circle-valued Morse-Bott function, and all indices are even.

When the action is non-Hamiltonian, the existence of an embedded 2-torus seems to be well-known and it was mentioned shortly in Ono’s paper [O2]. But we don’t find a good reference which includes the complete proof so that we give a complete proof below.

Proposition 2.1. Let $(M, \omega)$ be a 2n-dimensional symplectic manifold admitting symplectic $S^1$-action with an integral symplectic form $\omega$. If the action is non-Hamiltonian, then there exists an $S^1$-equivariant symplectic embedding of 2-torus $i : T^2 \hookrightarrow M$, where the circle acts freely on the left factor of $T^2 \cong S^1 \times S^1$.

Proof. As mentioned above, the symplectic circle action has a generalized moment map $\mu : M \to \mathbb{R}/\mathbb{Z}$. As noted in [At] and [Mc], the number of connected components of $\mu^{-1}(c)$ for any $c \in \mathbb{R}/\mathbb{Z}$ is constant. Call the number $m_0$, and pick a regular value $c_0 \in \mathbb{R}/\mathbb{Z}$ with respect to $\mu$. Since the action is non-Hamiltonian, any index of $\mu$ can not be equal to 0 nor $2n$. Let $M_{(1)}$ be the set of points with the trivial isotropy subgroup in $M$. Here, we prove an easy lemma which has been noted in [K] with not so detailed proof.

Lemma 2.2. There exist two points $p, q \in M_{(1)}$ in a component of $\mu^{-1}(c_0)$ which are joined by an integral curve of $-JX$.

Proof. For each point $x \in M_{(1)} \cap \mu^{-1}(c_0)$, let $\gamma_x : [0, \infty) \to M$ be the integral curve of $\text{grad } \mu = -JX$ such that $\gamma_x(0) = x$ where $M$ carries the metric $\omega(J \cdot, \cdot)$. Let $\overline{\gamma}_x$ be the reparametrization of $\gamma_x$ such that $\mu(\overline{\gamma}_x(t)) = c_0 + t$ in $S^1 \cong \mathbb{R}/\mathbb{Z}$ for $t \geq 0$. If $x$ is contained in a stable submanifold of some critical submanifold of $\mu$, then $\overline{\gamma}_x(t)$ is not defined over all $t \in [0, \infty)$. Let $N$ be the set

$$\{ x \in M_{(1)} \cap \mu^{-1}(c_0) \mid \overline{\gamma}_x \text{ is defined for } [0, m_0] \}.$$

Then $N$ is non-empty, because the complement $(M_{(1)} \cap \mu^{-1}(c_0)) - N$ is contained in an intersection of $(M_{(1)} \cap \mu^{-1}(c_0))$ and stable submanifolds of some critical submanifolds of $\mu$, which means that the codimension of $(M_{(1)} \cap \mu^{-1}(c_0)) - N$ is at most 2.

Pick a point $p_0 \in N$. Then, the curve $\overline{\gamma}_{p_0}(t)$ passes through $M_{(1)} \cap \mu^{-1}(c_0)$ at each nonnegative integer $0 \leq t \leq m_0$ by definition of $N$. Denote by $p_i \in M_{(1)} \cap \mu^{-1}(c_0)$ the point $\overline{\gamma}_{p_0}(i)$ for integer $0 \leq i \leq m_0$. Then, two points of $\{p_0, \cdots, p_{m_0}\}$ are in a component of $\mu^{-1}(c_0)$ by pigeon hole principle. Especially, pick two points $p_{i_0}, p_{j_0}$ such that
(1) \( i_0 < j_0 \),
(2) \( p_{i_0} \) and \( p_{j_0} \) are in a same connected component of \( \mu^{-1}(c_0) \),
(3) \( p_i \) and \( p_j \) are not in a same connected component of \( \mu^{-1}(c_0) \) for any \((i, j) \neq (i_0, j_0)\) such that \( i_0 \leq i < j \leq j_0 \).

Denote \( p \) and \( q \) by \( p_{i_0} \) and \( p_{j_0} \), respectively.

If dimension of \( M \) is equal to 2, then there is nothing to prove. So, we may assume that \( 2n \geq 4 \). We would show that there exists a smooth loop \( \delta : [i_0, j_0] \to M(1) \) such that

1. \( \delta(i_0) = \delta(j_0) = p \),
2. \( \mu(\delta(t)) = \mu(\overline{\gamma} p(t)) \) for any \( t \in [i_0, j_0] \),
3. \( \delta(t) \neq z \cdot \delta(t') \) for each \( z \in S^1 \) and \( t, t' \in [i_0, j_0] \) such that \( t \neq t' \).

Then, the orbit \( S^1 \cdot \delta([i_0, j_0]) \) is an equivariantly embedded 2-dimensional torus. To show that \( S^1 \cdot \delta([i_0, j_0]) \) is symplectic, we must show that \( \omega|_{S^1 \cdot \delta([i_0, j_0])} \) is non-degenerate. But it follows immediately from the fact that

\[
\omega(X, \delta'(t)) = i_X \omega(\delta'(t)) = d\mu(\delta'(t)) \neq 0,
\]

since \( \mu(\delta(t)) = \mu(\overline{\gamma} p(t)) \). Now, we construct such a \( \delta \). Pick a sufficiently small \( \epsilon > 0 \) so that \( M(1) \cap \mu^{-1}([j_0 - \epsilon, j_0]) \) is diffeomorphic to \([j_0 - \epsilon, j_0] \times (M(1) \cap \mu^{-1}(c_0)) \) through integral curves of \(-JX\). We would modify \( \overline{\gamma} p \) at \([j_0 - \epsilon, j_0] \) to obtain \( \delta \). Then \( \overline{\gamma} p(t) \) corresponds to the point \( (t, q) \) for \( t \in [j_0 - \epsilon, j_0] \) in \([j_0 - \epsilon, j_0] \times (M(1) \cap \mu^{-1}(c_0)) \), since \( \overline{\gamma} p(t) \) passes through \( q \) along \(-JX\). Here, we recall that the non-empty fixed set \( M^{2n} \) for any subgroup \( Z_n \) of \( S^1 \) is an \( S^1 \)-invariant symplectic submanifold so that \( M^{2n} \) has codimension at least two. By this, we observe that \( p \) and \( q \) are in a same connected component of \( M(1) \cap \mu^{-1}(c_0) \). So, there exists a smooth curve \( h : [j_0 - \epsilon, j_0] \to M(1) \cap \mu^{-1}([j_0 - \epsilon, j_0]) \) such that

1. \( h(t) = t \) for \( t \in [j_0 - \epsilon, j_0] \),
2. \( h|_{[j_0 - \epsilon, j_0 - \frac{\epsilon}{2}]}(t) = (t, q) \),
3. \( h|_{[j_0 - \frac{\epsilon}{2}, j_0]}(t) = (t, p) \).

By using this, define \( \delta : [i_0, j_0] \to M(1) \) as follows:

1. \( \delta = \overline{\gamma} p_0 \) on \([i_0, j_0 - \epsilon] \),
2. \( \delta = h \) on \([j_0 - \epsilon, j_0] \).

Then, we can check that \( \delta \) is a wanted loop.

Hence if the action is non-Hamiltonian, one can think of the symplectic normal bundle \( \nu(Z) \) where \( Z \) is an \( S^1 \)-invariant symplectic submanifold on \( M \) given in the Proposition 2.1. By the equivariant tubular neighborhood theorem [Br], \( \nu(Z) \) has an \( S^1 \)-equivariant vector bundle structure over \( Z \). The following lemma implies that \( \nu(Z) \) is in fact a trivial bundle.

**Lemma 2.3.** Let \( E \) be an \( S^1 \)-equivariant complex vector bundle of rank \( k \) over the 2-torus \( Z \cong S^1 \times S^1 \). If the circle action is free on \( E \), then \( E \cong Z \times \mathbb{C}^k \).
Proof. Consider the following diagram.

\[
\begin{array}{ccc}
E & \xrightarrow{\varrho} & E/S^1 \\
\downarrow{\pi} & & \downarrow{\pi'} \\
Z & \xrightarrow{\varrho} & Z/S^1 \\
\end{array}
\]

Since the action is free, \( \pi' : E/S^1 \to Z/S^1 \cong S^1 \) is a complex vector bundle of rank \( k \) over \( S^1 \) and the quotient map \( \varrho \) is a bundle morphism. Note that any complex vector bundle over \( S^1 \) is trivial, since the structure group \( U(n) \) is connected. Therefore \( E/S^1 \cong S^1 \times \mathbb{C}^k \) and hence \( E \cong \varrho^*(E/S^1) \cong Z \times \mathbb{C}^k \).

Now we are ready to prove Theorem 1.1

Proof of Theorem 1.1 Assume that the given action is non-Hamiltonian. By Proposition 2.1 there is an \( S^1 \)-equivariant symplectic embedding of 2-torus \( i : Z \hookrightarrow M \). Then we have an \( \omega \)-tame almost complex structure \( J' \) which makes \( i \) to be \( J' \)-holomorphic. (See [McS] page 3). Since an \( \omega \)-tame almost complex structure is unique up to deformation (as explained in Section 1), we have \( c_1(M) = c_1(TM, J) = c_1(TM, J') = \lambda \cdot [\omega] \). Since the normal bundle of \( i(Z) \) is trivial by Lemma 2.3 we have

\[
\langle c_1(M), [i(Z)] \rangle = 0.
\]

On the other hand, \( \langle \lambda \cdot [\omega], [i(Z)] \rangle = \lambda \cdot \text{Vol}(i(Z)) \). Hence \( \lambda = 0 \), i.e. If \( (M, \omega) \) admits a non-Hamiltonian circle action with \( c_1(M) = \lambda \cdot [\omega] \), then \( \lambda \) must be zero.

Now, assume that the action is Hamiltonian with a moment map \( \mu : M \to \mathbb{R} \). We will show that there exists an embedded symplectic sphere \( S \) such that \( \langle c_1(M), [S] \rangle > 0 \), which implies \( \lambda > 0 \). Let \( F_{\text{min}} \) (\( F_{\text{max}} \), respectively) be a fixed point which is minimal (maximal, respectively) with respect to \( \mu \). Then there is an integral curve \( \gamma \) along \(-JX\) connecting \( F_{\text{min}} \) and \( F_{\text{max}} \). Hence the orbit \( S := S^1 \cdot \gamma \) is an embedded (may not be smooth at \( F_{\text{min}} \) and \( F_{\text{max}} \)) \( S^1 \)-invariant \( J \)-holomorphic sphere in \( M \). To compute \( \langle c_1(M), [S] \rangle \), consider an \( S^1 \)-invariant normal bundle \( \nu(S) \) of \( S \). Since \( \nu(S) \) is decomposed into the sum of \( S^1 \)-equivariant line bundles and by the lemma 2.3 we have

\[
\langle c_1(M), [S] \rangle = \frac{1}{N}(p_{\text{min}} - p_{\text{max}}),
\]

where \( p_{\text{min}} \) (\( p_{\text{max}} \), respectively) is the sum of weights of the tangential \( S^1 \)-representation at \( F_{\text{min}} \) (\( F_{\text{max}} \), respectively) and \( N \) is the cardinality of the isotropy subgroup of \( S \). Since all weights are positive (negative, respectively) at \( F_{\text{min}} \) (\( F_{\text{max}} \), respectively), we have \( \langle c_1(M), [S] \rangle > 0 \). On the other hand, \( \langle [\omega], [S] \rangle > 0 \), because \( S - \{ F_{\text{min}}, F_{\text{max}} \} \) is a symplectic submanifold. Therefore, \( \langle c_1(M), [S] \rangle = \langle \lambda \cdot [\omega], [S] \rangle > 0 \) implies \( \lambda > 0 \).

Lemma 2.4. [AH] Let \( E \) be a complex line bundle over two sphere \( B \). Assume that \( B \) has a circle action with two fixed points, namely the north pole \( N \) and the south pole \( S \). If the
circle action can be lifted to $E$, then the Euler number $e$ of $E$ is given by
\[
eq -\frac{1}{k}(p_N - p_S),
\]
where $k$ is the cardinality of the stabilizer of $B$, and $p_N$ ($p_S$, respectively) is the weight of the $S^1$-representation on the fiber at $N$ ($S$, respectively).

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