PIECEWISE INTERPRETABLE HILBERT SPACES

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Abstract. We define and study piecewise interpretable Hilbert spaces in continuous logic. These are Hilbert spaces which arise as direct limits of imaginary sorts of a model \( M \) of a theory \( T \). We introduce natural examples of piecewise interpretable Hilbert spaces in a wide variety of contexts. We show that piecewise interpretable Hilbert spaces can be seen as encoding interesting model theoretic information about \( M \) or \( T \), such as properties of definable measures or Galois-theoretic information. We also show that piecewise interpretable Hilbert spaces encode unitary group representations in various ways. We carry out a systematic structural analysis of piecewise interpretable Hilbert spaces with scattered subsets. As an application of this work, we recover the classification of the unitary representations of oligomorphic groups first discovered by Tsankov (2012). Our main tool is local stability theory in continuous logic.

Contents

1. Background Material 5
  1.1. Continuous logic and type-spaces 5
  1.2. Standard facts and definitions in continuous logic 8
  1.3. Hilbert Spaces In Continuous Logic 12
2. Piecewise Interpretable Hilbert Spaces 13
  2.1. Definitions and first results 13
  2.2. Examples 14
  2.3. Prolonging Piecewise Interpretable Hilbert Spaces 20
3. Structure Theorems for Scattered Interpretable Hilbert Spaces 23
  3.1. Decomposition into \( \bigwedge \)-interpretable subspaces 26
  3.2. Strictly interpretable Hilbert spaces 28
  3.3. Some Examples and Counterexamples 32
4. Absolute Galois Groups and Associated Hilbert Spaces 34
  4.1. Interpretation of the inverse system of \( Gal(K) \) 34
  4.2. Hilbert spaces associated to \( Gal(K) \) 38
5. Unitary Representations 42
  5.1. Unitary Representations of Automorphism Groups 42
  5.2. Unitary Representations of Automorphism Groups of \( \omega \)-categorical Structures 45
  5.3. Application to Unitary Representations of Groups with Scattered Orbits 48
6. Further discussion 50
References 55
In this paper, we define piecewise interpretable Hilbert spaces and study some of their properties. These are Hilbert spaces which can be viewed as direct limits of imaginary sorts of a first-order theory $T$. While we employ continuous logic, the case where $T$ itself is a discrete first order logic theory is already of interest, and the introduction may be read with such theories in mind. We will use tools of stability theory to study the structure of such Hilbert spaces, obtaining information about the underlying theory. The stability emanates from the Hilbert space inner product formulas themselves, so no stability assumptions on the theory $T$ are required.

Definable measures, which themselves play a central role in recent model-theoretic literature, provide one rich source of examples. If $\mu$ is such a measure, the Hilbert space $L^2(\mu)$ is piecewise interpretable in $T$. The stability-theoretic viewpoint already gives useful information here; notably it was used previously to prove an independence, or 3-amalgamation theorem for definable measures (see [Hrushovski, 2015]; a more basic version was the main engine of [Hrushovski, 2012], Theorem 2.22).

A different class of piecewise interpretable Hilbert spaces arises via the absolute Galois group $G$ of a field $K$, with $T = Th(K, K^{alg})$. $G$ cannot be viewed as an imaginary sort of $T$, but it is essentially a projective limit of such sorts; this was discovered by [Cherlin et al., 1980], and forms the basis of their analysis. Here no definable measure need be present, but the Hilbert space $L^2(G, \text{Haar})$ is piecewise interpretable in $T$. This opens the way to a common treatment of some analogies between structures such as pseudo-finite fields and measurable structures, such as the independence theorem; the striking similarity was previously unexplained.

Any piecewise interpretable Hilbert space gives rise to a functor from the category of models of a theory $T$ to the category of Hilbert spaces. In particular, we obtain a homomorphism from automorphism group $G$ of any model to the unitary group of a Hilbert space, i.e. a unitary representation of $G$. A basic lemma of [Tsankov, 2012] implies in the $\omega$-categorical that conversely, all unitary representations of the automorphism group of the countable model arise in this way; giving a third and very interesting connection to a deep field.

Our initial and principal inspiration was the classification theorem for unitary representations of automorphism groups of $\omega$-categorical structures from [Tsankov, 2012]. Changing the viewpoint to that of piecewise interpretable Hilbert spaces, it becomes natural to ask whether they may admit a structure theorem under more general hypotheses than $\omega$-categoricity; and if so, what form the statement would take. The answers we found were, respectively, the notions of scatteredness and asymptotic freedom. We prove a number of structure theorems that analyze scattered representations in terms of asymptotically free ones, see notably 3.14. In particular these fully recover Tsankov’s structure theorem in the $\omega$-categorical case.

Both scatteredness and asymptotic freedom concern a complete type $p$ within a piecewise interpretable Hilbert space $H$. Scatteredness will be defined more precisely below; we mention for now a special case (referred to as ‘strict interpretability’ in the paper): whenever there are only finitely many values achieved by the inner product between elements of $p$, $p$ is scattered. This subclass already includes the $\omega$-categorical case of [Tsankov, 2012]. It also includes the Hilbert spaces arising from definable measures over pseudo-finite fields [Chatzidakis et al., 1992]; and more generally over measurable classes of [Macpherson and Steinhorn, 2008]. The Galois-theoretic examples mentioned above fall also within this fold.

$p$ is asymptotically free if for any $a \in p$, and for any $b \in p$ that is not algebraic over $a$, the elements $a, b$ are orthogonal as vectors in the Hilbert space. An extreme case occurs when $p$ forms an orthonormal set in $H$; in this case, the Hilbert space $H(p)$ generated by the realizations of $p$ does not interact with $T$ at all, and could equally be interpreted over pure equality. In general, asymptotic freedom means that the Hilbert space structure is defined using only information within the algebraic closure (bounded closure in the continuous logic...
case) of an element of $p$. In fact, the interpretation of $H(p)$ factors through a disintegrated, strongly minimal reduct of $T$ with $p$ as its universe. In the $\omega$-categorical case, this strongly minimal set can only be an equivalence relation with classes of some finite size $n$, possibly with some structure separately on each $n$-element class. In this way, the representation arises from a finite group.

Using a theorem of Howe and Moore, we show that for any algebraic group $G$ over $\mathbb{Q}$, any irreducible representation of $G(\mathbb{Q}_p)$ or $G(\mathbb{R})$ can essentially be obtained as a piecewise interpretable Hilbert space generated by an asymptotically free type (see Section 5.3 for details). As the connection is made directly to the conclusion of our theorem, we do not obtain any new implications for the irreducible representations of these classical groups; but this does show that scatteredness includes both settings for unitary representation theory, oligomorphic groups as well as algebraic groups.

Even with the above mentioned notions at hand however, the proof is not a direct generalization from the $\omega$-categorical case. It proceeds instead via a local stability analysis. The interaction of the stable, but highly non-discrete, Hilbert space formulas, with the type provided by the underlying theory $T$ turns out to imply not only finite rank properties but also a certain local modularity of forking, that forms the key to the later analysis.

The paper is structured as follows. We begin by giving a condensed exposition of some of the background model theory which will be used in this paper. This includes continuous logic for metric spaces, local stability theory, and model theory of Hilbert spaces. This material is fairly standard, so we only give basic definitions and refer the reader to the literature on these topics. Note that we use a formalism of continuous logic which is different from the formalism which has become widespread through expositions such as [Ben Yaacov et al., 2008]. The advantage of our approach is that we use the syntax of classical logic for continuous logic. Since we will need to move between classical and continuous logic, this formalism seems more appropriate for our purposes.

In the second section of this paper, we define and prove some general model theoretic facts about piecewise interpretable Hilbert spaces. We give examples of piecewise interpretable Hilbert spaces in Section 2.2. We will introduce two frameworks for working with piecewise interpretable Hilbert spaces. On one approach, a piecewise interpretable Hilbert space is a direct limit of imaginary sorts in a continuous logic structure.

The general idea is straightforward: given a structure $M$, we identify a collection of sorts of $M$ which can each be identified isometrically with subsets of a Hilbert space in such a way that the inner product maps between these subsets are definable. We then view the Hilbert space spanned by these subsets as the set of countable linear combinations of vectors in these subsets. Some care needs to be taken with these linear combinations for reasons to do with continuous logic and this leads us to a presentation of the Hilbert space as a direct limit of imaginary sorts in a continuous logic structure.

We will also introduce a useful device which we believe simplifies the study of piecewise interpretable Hilbert spaces. Given a Hilbert space $H$ piecewise interpretable in $M$, we can add new sorts to $M$ which will stand for closed Hilbert space balls containing the balls of $H$. This construction has the advantage of establishing a clear connection with the classical treatment of Hilbert spaces in continuous logic; model-theoretically it is analogous to studying a field $F$ via an embedding inside a bigger algebraically closed field, which can simplify some delicate aspects of the algebraic closure. We will show that the result of adding this new sort to $M$ loses no information about the model theory of $M$, since $M$ will be stably embedded in the new structure, with induced structure equal to the original one. This construction is more
transparent than the direct limit structure and helps make a number of statements in Section 3 simpler. In Section 3 we begin a study of the fine structure of piecewise interpretable Hilbert spaces under the assumption of scatteredness. Let $M$ be an $\omega_1$-saturated continuous logic structure and $H$ a Hilbert space piecewise interpretable in $M$. We will see a connection between the weak topology on $M$ and canonical bases for local types of elements of $H$ with respect to inner products.

A scattered subset of $H$ is a subset given by a type-definable set in $M$ whose weak closure is locally compact. Theorem 3.14 shows that a subspace of $H$ generated by a scattered subset can be expressed as an orthogonal sum of subspaces each generated by a complete type of $M$ which is asymptotically free. We then study extensions of Theorem 3.14 in order to obtain more global results, decomposing piecewise interpretable Hilbert spaces into orthogonal sums. We also see how Theorem 3.14 specialises to the context of classical logic. In Section 3.3 we give some concrete examples of the decomposition promised by Theorem 3.14.

In Section 4 we introduce a new source of piecewise interpretable Hilbert spaces. Let $M$ be a classical logic $\mathcal{L}$-structure and $K \subseteq M$ a definably closed subset. We show that $M$ interprets the projective system of finite quotients of $Gal(K)$ in the language $\mathcal{L}_P$ with an additional predicate $P$ for $K$. This generalises the classical construction of [Cherlin et al., 1980]. It follows that the $L^2$-space of $Gal(K)$ with the Haar measure is piecewise interpretable in $M$ as an $\mathcal{L}_P$-structure. We also consider the subspace of $L^2(Gal(K))$ consisting of class functions and we explain in what sense $K$ can be said to interpret this Hilbert space as an $\mathcal{L}$-structure. We will see that these Hilbert spaces have natural asymptotically free decompositions.

In Section 4 we introduce the connection between piecewise interpretable Hilbert spaces and unitary group representations and we rephrase the work of Section 3 in representation-theoretic terms. We show that representations generated by asymptotically-free complete types are induced representations and we prove a Mackey-style irreducibility criterion for these representations. In Section 5.2 we discuss representations of automorphism groups of $\omega$-categorical structures and we show that Theorem 3.14 eventually entails the classification theorem of [Tsankov, 2012]. In Section 5.3 we deduce consequences of Theorem 3.14 for arbitrary group representations with a scattered subset, even when the group is not given as the automorphism group of a continuous logic structure.

Finally, in Section 6 we clarify the model theoretic content of the proof of Theorem 3.14. This theorem is essentially proved by the application of Von Neumann’s lemma in Theorem 3.8 but we show that, in this context, Von Neumann’s lemma amounts to proving a local form of one-basedness. This may be of interest for future applications.

At the final stages of writing up this paper we became aware of Ibarlucía’s beautiful paper [Ibarlucía, 2021]. He employs the same philosophy of using stability theoretic ideas to study interpretable representations. Ibarlucía heads towards a proof of property T for the automorphism group of a continuous logic $\omega$-categorical structure, extending [Tsankov, 2012] in the discrete logic case, by a route intentionally avoiding structure theorems, which are for us the main goal. In terms of methods he is able to use stability theory over a model, whereas for us treating canonical bases that are far from models is of the essence.

The setup and main notions encountered in this paper were arrived at by joint work, while the first author was a DPhil student under the supervision of the second. The proof of the main structure theorem relating scatteredness to asymptotic freedom, including the idea of invoking Von Neumann’s lemma to prove one-basedness, are due to the first author in their entirety.

The first named author wishes to thank his PhD mate Arturo Rodriguez Fanlo for numerous enlightening discussions.
1. Background Material

In this section we present some background material used in this paper. We will sketch a presentation of continuous logic close to [Henson and Iovino, 2002], [Gomez and Pillay, 2021] or [Chang and Keisler, 1966] but the rest of the paper will not depend on it. In any version of continuous logic, one has the notion of a type-definable set. The collection of type-definable sets is closed under positive Boolean combinations and under projection. As in the discrete logic case, the projection $(\exists x) P(x,y)$ of a partial type $P(x,y)$ is the partial type $Q(y)$ such that, in a sufficiently saturated model, $Q(b)$ holds iff there exists $a$ with $Q(a,b)$. We can thus freely write formulas or sentences involving positive first order operations. These describe partial types in any formulation of continuous logic, regardless of a specific calculus. We will also recall local stability theory [Ben Yaacov and Usvyatsov, 2010] especially the notion of a Morley sequence, the basic model theory of Hilbert spaces (see [Ben Yaacov et al., 2008]), and the notion of stable embeddedness. The reader who is familiar with these may skip this section.

1.1. Continuous logic and type-spaces. Note that in this section we use some terminology which we will drop in the rest of this paper (e.g. ‘standard model’ or the type-space ‘$S_H$’). This terminology will help us highlight where continuous logic differs from classical logic.

In continuous logic, we will work with positive formulas (and their negations):

**Definition 1.1.** Let $\mathcal{L}$ be an arbitrary language for classical first-order logic. We say that an $\mathcal{L}$-formula $\phi(x)$ is positive if $\phi(x)$ is logically equivalent to a formula which uses only the logical connectives $\land$ and $\lor$ and the usual quantifiers $\forall$ and $\exists$. Write $\Pi(x)$ for the set of positive $\mathcal{L}$-formulas with free variables among the tuple $x$, and write $\Pi$ for the set of positive $\mathcal{L}$-sentences.

Note that if $\phi$ and $\psi$ are positive $\mathcal{L}$-formulas, then $(\neg \phi \rightarrow \psi)$ is positive. In general, positive formulas can have very weak expressive power but we will always work in certain languages and theories where they have strong expressive power. The restrictions we impose on $\mathcal{L}$ are the following.

We always fix a multi-sorted language $\mathcal{L}$ with sorts $\left(S_i\right)_{i \in I}$ and $\left(I_n\right)_{n \geq 1}$. We refer to the sorts $\left(S_i\right)$ as the metric sorts and to the sorts $I_n$ as the value sorts. $\mathcal{L}$ has an equality relation on every value sort but not on any metric sort. Each sort $I_n$ will be identified with the interval $[-n,n]$, so we add functions $i_{nm} : I_n \rightarrow I_m$ for $n \leq m$ which will play the role of inclusion functions. The value sorts are also equipped with the following structure of the real numbers: functions $+, -, \times$, max between the appropriate value sorts and predicates $=$ and $\leq$ (note that we choose $\leq$ and not $<$, for reasons which become clear below). In each value sort $I_n$ we also add a constant symbol for each rational number in $[-n,n]$. Each metric sort is equipped with a function $d_i : S_i \times S_i \rightarrow I_n$ for some $n$. $\mathcal{L}$ may contain more function symbols but no other relation symbols.

We say that $T$ is a continuous logic $\mathcal{L}$-theory if $T$ is a set of $\mathcal{L}$-sentences satisfying in addition the following conditions:

1. $T$ is a union of theories $T^R \cup T^+$ where $T^R$ says that the value sorts satisfy the full first-order theory of the real numbers (where $I_n$ is identified with $[-n,n]$) and $T^+$ contains only positive $\mathcal{L}$-sentences.

2. $T$ says that every $d_i$ is a pseudo-metric and there is a rational $C_i$ such that $T$ contains the sentence

   $\forall x, y \in S_i, d_i(x, y) \leq C_i$

3. $T$ says that every function symbol $f$ in $\mathcal{L}$ is a uniformly continuous function in the following way: for every rational $\epsilon > 0$ there is a rational $\delta > 0$ such that $T$ contains
the positive sentence
\[ \forall x, y (d(x, y) < \delta \rightarrow d_i(f(x), f(y)) \leq \varepsilon) \]

where \( x, y \) are finite tuples of variables appropriate for \( f \) and \( d \) is the max-metric on the sorts corresponding to the tuple \( x \) and \( d_i \) is the metric on the sort of \( f(x) \).

Given any model \( M \) of \( T \), we define \( \tilde{M} \) to be the structure obtained by quotienting every sort of \( M \) by the \( \bigwedge \)-definable equivalence relation \( E(x, y) \) which says that \( x \) and \( y \) are infinitesimally close with respect to the appropriate metric. It is easy to check that for every function symbol \( f \in \mathcal{L} \), the interpretation \( f^\tilde{M} \) of \( f \) in \( M \) induces a uniformly continuous function \( f^\tilde{M} \) on \( \tilde{M} \).

Therefore \( \tilde{M} \) is an \( \mathcal{L} \)-structure. The key property of positive formulas is the following: if \( \phi(x) \) is a positive formula and \( M \models \phi(a) \), then \( \tilde{M} \models \phi(\tilde{a}) \) where \( \tilde{a} \) is the equivalence class of \( a \) in \( M \).

Therefore, if \( M \models T \) then \( \tilde{M} \models T \). If \( M \) is \( \omega_1 \)-saturated, then the value sorts of \( \tilde{M} \) are the standard real numbers and each sort of \( \tilde{M} \) is a complete metric space.

**Definition 1.2.** We say that \( M \models T \) is a standard model if each sort of \( M \) is a complete metric space. In particular, the value sorts of \( M \) are the standard real numbers.

We say that a continuous logic \( \mathcal{L} \)-theory \( T \) is complete if there is a standard \( \mathcal{L} \)-structure \( M \) such that \( T^+ \) is the set of all positive sentences true in \( M \). In that case we write \( T = Th_{\Pi}(M) \).

Note that \( T \) may not be maximal consistent with respect to the set of all positive sentences. Nevertheless, any complete continuous logic \( \mathcal{L} \)-theory \( T \) extends to a unique continuous logic \( \mathcal{L} \)-theory which is maximal consistent with respect to positive sentences. This extension is the continuous logic \( \mathcal{L} \)-theory of any standardised nonprincipal ultrapower of \( M \) (these claims will follow from the discussion surrounding Lemma [1.3]).

Let \( M, N \) be standard \( \mathcal{L} \) structures. We say that \( N \) is an elementary extension of \( M \) (and we write \( M \prec N \)) if \( M \) is a substructure of \( N \) in the usual sense and for any tuple \( a \) in \( M \) and any positive formula \( \phi(x) \), if \( M \models \phi(a) \) then \( N \models \phi(a) \). Observe that \( Th_{\Pi}(M) \) might be a strict subset of \( Th_{\Pi}(N) \). Nevertheless, if \( M \prec N \) then \( Th_{\Pi}(M) \) and \( Th_{\Pi}(N) \) have the same extension to a continuous logic \( \mathcal{L} \)-theory maximal consistent in \( \Pi \).

Let \( M \) be a standard structure, let \( T = Th_{\Pi}(M) \) and let \( A \subseteq M \). Let \( \mathcal{L}(A) \) be the expansion of \( \mathcal{L} \) by adding constants for every element of \( A \). Write \( \Pi(x, A) \) for the set of positive \( \mathcal{L}(A) \)-formulas with free variables among the tuple \( x \). Let \( T(A) \) be the extension of \( T \) by the set of positive \( \mathcal{L}(A) \)-sentences satisfied in \( M \) where the constants in \( A \) are interpreted in the obvious way.

A partial \( \Pi(x, A) \)-type is a set \( p(x) \) of \( \Pi(x, A) \)-formulas such that \( p(x) \cup T(A) \) is consistent. A complete \( \Pi(x, A) \)-type is a partial \( \Pi(x, A) \)-type \( p(x) \) such that \( p(x) \cup T(A) \) is maximal consistent in \( \Pi(x, A) \). We write \( S_{\Pi,x}(A) \) for the set of \( \Pi(x, A) \)-types. Then \( S_{\Pi,x}(A) \) is a topological space for which \( \Pi(x, A) \) forms a basis of closed sets.

We know from classical logic that \( S_{\Pi,x}(A) \) is a compact topological space. Moreover, an easy induction on positive formulas over \( A \) shows that if \( \phi(x) \in \Pi(x, A) \), then the set of types \( p \in S_{\Pi,x}(A) \) which do not contain \( \phi(x) \) is covered by a countable union of positive formulas over \( A \). This fact depends heavily on our choice of syntax for \( \mathcal{L} \) and our background assumptions on \( T \). It follows that \( S_{\Pi,x}(A) \) is also Hausdorff (this argument can be formalised using the notion of \( \varepsilon \)-approximation discussed below).

When \( \kappa \geq \omega \), we say that a standard structure \( M \) is \( \kappa \)-saturated if for all \( A \subseteq M \) with size \( < \kappa \), \( M \) realises all complete \( \Pi(x, A) \)-types where \( x \) is a finite tuple of variables. If \( M \) is \( \kappa \)-saturated then for any \( A \subseteq M \) with \( |A| < \kappa \), any finite tuple \( a \in M \) realises a complete \( \Pi(x, A) \)-type. Moreover, for such an \( A \), \( M \) realises all \( \Pi(x, \overline{A}) \)-types, where \( \overline{A} \) is the closure of \( A \) in the appropriate metrics. We can construct \( \kappa \)-saturated standard structures in the sense
of continuous logic by taking saturated structures with respect to all $\mathcal{L}$-formulas as we usually do in classical logic and quotiening them in the way described before Definition 1.2.

If $M$ is a standard model of $T$, $A \subseteq M$ and $a \in M$, we write $tp(a/A)$ for the complete $\Pi(x, A)$-type realised by $a$ in an $|A|^+$-saturated elementary extension $N$ of $M$. Note that this is uniquely determined by the $\Pi(x, A)$-formulas satisfied by $a$ in $M$ (see the next few paragraphs for a discussion of this choice of definitions).

The readers familiar with [Henson and Iovino, 2002] will note that we have given a different definition of the type-space. This is a conscious choice but it comes with some trade-offs. One unfortunate consequence of our definitions is that it is possible to have standard structures with $T$ require any saturation in order to be realised in a standard model of $M$. Since approximate types are derived from complete $\Pi(x, A)$-types, it is more natural to work directly with $\Pi(x, A)$-types and $\tilde{S}_x(A)$, as defined in [Henson and Iovino, 2002] with an equivalent choice of definitions. The relation between $S_{\Pi,x}(A)$ and $\tilde{S}_x(A)$ is given by the following lemma:

Lemma 1.3. $S_{\Pi,x}(A)$ and $\tilde{S}_x(A)$ are homeomorphic topological spaces via the map $\alpha : p \mapsto \tilde{p}$.

Proof. $\alpha$ is continuous and surjective. To see that $\alpha$ is injective, observe that in a sufficiently saturated standard structure, for $\phi(x) \in \Pi(x, A)$, if $a \models \phi(x)$ for all $\epsilon > 0$, then $a \models \phi(x)$. Hence if $a \models \tilde{p}$ then $a \models p$ and $p$ contains a unique approximate type over $A$. Since $S_{\Pi,x}(A)$ is compact, $\alpha$ is a homeomorphism. $\Box$

The main advantage of approximate types over complete $\Pi(x)$-types is that they do not require any saturation in order to be realised in a standard model of $T$. Indeed, if $M, N$ are standard structures with $N \models T$ and $M \not\preccurlyeq N$, then $M \models \tilde{T}$ where $\tilde{T} = \{ \phi^\epsilon \mid \phi \in T, \epsilon > 0 \}$, and similarly for approximate types. This follows from 5.14 in [Henson and Iovino, 2002].

When $M \models T$ is saturated over $A$, every tuple $a \in M$ realises a complete $\Pi(x, A)$-type and Lemma 1.3 shows that there is no difference between realisations of complete $\Pi(x, A)$-types and realisations of approximate types over $A$. Since approximate types are derived from complete $\Pi(x, A)$-types, it is more natural to work directly with $S_{\Pi,x}(A)$ in order to study the structure of $M$.

In this paper we will usually work in $\omega_1$-saturated standard structures, so we can work with $S_{\Pi,x}(\emptyset)$. Whenever we need to deduce results about general standard structures, we will be careful to use Lemma 1.3 and to move from closed subsets of $S_{\Pi,x}(\emptyset)$ to closed subsets of $\tilde{S}_x(\emptyset)$ in order to translate any result about $S_{\Pi,x}(\emptyset)$ into a result about $\tilde{S}_x(\emptyset)$. It will always be clear from context how this is done.
In the remainder of this paper, unless specified otherwise, we work in continuous logic. Therefore, we say ‘model’ instead of ‘standard model’, ‘complete type’ instead of ‘complete \( \Pi \)-type’, we write \( S_x(A) \) instead of \( S_{\Pi,x}(A) \), etc.

1.2. Standard facts and definitions in continuous logic. Many basic results in classical logic go through to continuous logic unchanged. We record some definitions and results which will be used in this paper. In this section, \( T \) always denotes a complete continuous logic theory.

1.2.1. Bounded and definable closure.

**Definition 1.4.** Let \( M \models T \) and take \( A \subseteq M \). We say that a tuple \( b \in M \) is in the bounded closure of \( A \) if for every elementary extension \( N \) of \( M \), there is no infinite indiscernible sequence realising \( \text{tp}(b/A) \) in \( N \). We write \( \text{bdd}(A) \) for the bounded closure of \( A \) in \( M \).

We say that a tuple \( c \in M \) is in the definable closure of \( A \) if for every elementary extension \( N \) of \( M \), \( c \) is the only realisation of \( \text{tp}(c/A) \) in \( N \). We write \( \text{dcl}(A) \) for the definable closure of \( A \) in \( M \).

See 10.7 and 10.8 in [Ben Yaacov et al., 2008] for standard results about definable and bounded closure. Note in particular that if \( M \prec N \) and \( A \subseteq M \) then \( \text{bdd}(A) \) is the same set in \( M \) and in \( N \), and similarly for \( \text{dcl}(A) \).

1.2.2. Definable functions.

**Definition 1.5.** Let \( M \) be a \( \kappa \)-saturated model of \( T \) and take \( A \subseteq M \) of size \( < \kappa \). Let \( Y \) be a sort of \( T \), let \( X \) be a finite Cartesian products of sorts of \( T \) and let \( p \) be an \( A \)-type-definable subset of \( X \). An \( A \)-definable function \( f \) on \( p \) into \( Y \) is a function \( p(M) \to Y(M) \) such that the set \( \{(x,f(x)) \mid x \in p\} \) is an \( A \)-type-definable subset of \( p \times Y \).

When \( M \) contains \( A \) but isn’t sufficiently saturated, we say that \( f \) is \( A \)-definable if there is a sufficiently saturated elementary extension \( N \) of \( M \) and a definable function on \( N \) which restricts to \( f \).

When \( f \) is \( A \)-definable on \( p \), we often identify \( f \) with its graph in the type space \( S_x(A) \). Moreover, \( f \) is \( A \)-definable on \( p \) if and only if the function \( p \times Y \to \mathbb{R}, (x,y) \mapsto d(f(x),y) \) is \( A \)-definable on \( p \). See 9.24 in [Ben Yaacov et al., 2008].

In the special case where \( f \) is a function into a value sort, we can also view \( f \) as a continuous function \( S_x(A) \cap p \to \mathbb{R} \). This is because the type-space of a value sort can always be identified with some interval \([-n,n] \). Urysohn’s lemma then entails that \( f \) extends to a continuous function \( S_x(A) \to \mathbb{R} \) so the local definition of \( f \) on \( p \) is not usually relevant. This is a significant difference with general definable functions \( p \to Y \).

Observe also that any complete type \( q \) in \( S_x(A) \) is uniquely determined by the values \( f(q) \) where \( f \) ranges over the \( A \)-definable functions \( X \to \mathbb{R} \).

A useful technical fact is that any \( A \)-definable function \( f : X \to \mathbb{R} \) on the sort \( X \) is the uniform limit of a sequence of \( A \)-definable functions \( (f_n) \) on \( X \) such that for all \( n \), there is a finite tuple \( a_n \) in \( A \) and a \( \emptyset \)-definable function \( g_n \) such that \( f_n(x) = g_n(x,a_n) \). One way of proving this is to consider the set of \( A \)-definable functions obtained by combining basic \( L(A) \)-function symbols with the Boolean operations \( \max \) and \( \min \) and by applying operators \( \sup \) and \( \inf \). These are the functions which serve as formulas in [Ben Yaacov et al., 2008]. Note that these functions separate types in \( S_x(A) \) and form a lattice, so we can apply the Stone-Weierstrass theorem to find that they uniformly approximate all \( A \)-definable functions into \( \mathbb{R} \). One consequence of this is that \( f \) is definable over a countable subset of \( A \).

When working with definable functions, we will often write down positive formulas which contain symbols for these functions. This is a slight abuse of notation, especially when the
functions are only defined on a type-definable set $p$. In that case, a formula $f(x) = 0$ will typically correspond to an $A$-type-definable set. Nevertheless, it will always be clear how to convert such a formula into a type-definable set in $L$. Alternatively, when working with definable functions on entire sorts of $T$, we can add new symbols to the language and augment $T$ in the obvious way. This does not change the model theory of $T$.

As a final comment on definable functions, let $N$ be a model of $T$, $A \subseteq N$ and suppose $N$ is $|A|^+$-saturated. Let $f$ be $A$-definable on $p$ in $N$ and let $M \prec N$ be an elementary submodel containing $A$. Then $f$ restricts to a total function on $p(M)$, since $f(a) \in dcl(Aa)$ for all $a \in M$ in the appropriate sort and $dcl(Aa)$ does not depend on the ambient model.

1.2.3. Canonical parameters and imaginaries.

**Definition 1.6.** Let $M \models T$, let $A \subseteq M$ and let $f : X \to Y$ be an $A$-definable function. We say that an element $c$ in some sort of $M$ is a canonical parameter for $f$ if for any elementary extension $N$ of $M$, any automorphism of $N$ preserves $f$ if and only if it fixes $c$.

Note that this definition is slightly non-standard for the reason that we do not allow tuples of elements as canonical parameters. This is because canonical parameters consisting of exactly one element will play an important role in this paper. We obtain canonical parameters by adding imaginary sorts whenever we need them. Imaginary sorts in continuous logic are a special case of hyperimaginary sorts in classical logic. See [Ben Yaacov and Usvyatsov, 2010] for a detailed exposition of this technique.

**Definition 1.7.** Let $M \models T$. An imaginary sort $S$ of $M$ is a Cartesian product of at most countably many sorts $(M_n)$ of $M$ endowed with a pseudo-metric $d$ such that there is an increasing sequence $(n_k)$ in $\mathbb{N}$ and definable pseudo-metrics $d_k$ on $\prod_{i=0}^{n_k} M_i$ such that the pseudo-metrics $(d_k)$ converge uniformly to $d$ on $S$. When $d$ is not a metric, we quotient out $S$ to obtain a metric space, as usual.

If $S$ is an imaginary sort of $M$ with metric $d$, expressed as the product of the sorts $(M_n)$, we can add the sort $S$ to the language with the metric $d$ and projection maps $\pi_n : S \to M_n$.

We will use imaginary sorts in two ways. Firstly, we can use an imaginary sort to add a countable Cartesian product of metric spaces to the language: if $d_n$ is a metric on $M_n$ with diameter 1 and $S$ is the product of the sorts $(M_n)$, then we can define $d(x, y) = \sum d_n(x_i, y_i)/2^n$.

Secondly, we can use imaginary sorts to add canonical parameters for arbitrary definable functions. If $f$ is a definable function over a countable set $A \subseteq M$, we express $f$ as a uniform limit of definable functions which use only finitely many parameters, we take the Cartesian product of the sorts of $M$ to which these finite tuples of parameters belong, and we define $d$ on $S$ to be the limit of the sup-norms between the different approximations of $f$ when we vary the parameters. $d$ is usually a pseudo-metric and quotienting out by $d$ produces an element in $S$ which is a canonical parameter for $f$. See [Ben Yaacov and Usvyatsov, 2010] for a detailed exposition of this technique.

Imaginary sorts also allow us to study `definable functions’ in countably many free variables. We find it more convenient to add an imaginary sort for a countable Cartesian product and to study the definable functions (in one variable) on that sort, rather than to study definable functions with countably many variables which would require cumbersome assumptions.

1.2.4. Stability and definable types.

**Definition 1.8.** Let $M \models T$. Take $A \subseteq M$ and let $f : X \times Y \to \mathbb{R}$ be an $A$-definable function where $X$ and $Y$ are finite Cartesian products of sorts of $M$ and $f$ takes values in $\mathbb{R}$. We say $f$ is unstable if there is some elementary extension $N$ of $M$, indiscernible sequences $(a_n)$, $(b_n)$ in
$N$ and $\epsilon > 0$ such that $|f(a_n, b_m) - f(a_m, b_n)| \geq \epsilon$ for all $n \neq m$. We say $f$ is stable if $f$ is not unstable.

Note that a stable function $f(x, y)$ is really a function $f(z)$ whose variables are partitioned in such a way as to obtain stability. We will usually work with specific stable formulas and the partition will be clear from context.

**Definition 1.9.** Let $M \models T$ and let $A, B \subseteq M$. Let $p \in S_x(A)$ and let $f(x, y)$ be an $A$-definable function into $\mathbb{R}$. Let $A_y$ be the set of $y$-tuples with all entries belonging to $A$. Let $g$ be the function $A_y \to \mathbb{R}$ such that for any $a \in A_y$, $p(x)$ entails that $f(x, a) = g(a)$.

We say that $p$ is definable at $f$ over $B$ if $g$ is the restriction to $A$ of a $B$-definable function on the sort of $y$. In that case we write $g = d_p f(y)$.

We say that $p$ is definable over $B$ if $p$ is definable over $B$ at every $A$-definable function $f(x, y)$ into $\mathbb{R}$.

In this paper we will work with local stable independence. First developed in [Shelah, 1978] and [Pillay, 1986] for classical logic, local stable independence for continuous logic has roots in [Shelah, 1975] and was studied in [Ben Yaacov and Usvyatsov, 2010]. We only recall the main definition:

**Definition 1.10.** Let $M \models T$, let $A \subseteq B \subseteq M$ and $p(x) \in S(B)$. Let $\Delta$ be a set of stable functions definable over $A$ such that for every $f \in \Delta$, the variables of $f$ are partitioned into two subscripts, one of which is $x$ (so we write $f = f(x, y)$).

We say that $p(x)$ does not $\Delta$-fork over $A$ if we can add imaginary sorts to $M$ and extend $p$ to $\text{bdd}(B)$ so that $p$ is definable over $\text{bdd}(A)$ at every function $f(x, y)$ in $\Delta$. We write $A \downarrow^\Delta_B$ if $\text{tp}(A/\text{bdd}(B, C))$ does not $\Delta$-fork over $\text{bdd}(C)$.

When $\Delta$ is the set of all stable functions in the appropriate variables, we simply say that $p$ does not fork over $A$ and we write $A \downarrow^\Delta_B$.

We refer the reader to [Pillay, 1986] and [Ben Yaacov and Usvyatsov, 2010] for an exposition of the theory of stable independence. Later in this paper we will make use of Morley sequences in the context of local stability.

**Definition 1.11.** Let $M \models T$ and let $\Delta$ be a set of stable formulas. We say that a sequence $(a_n)$ in $M$ is a Morley sequence in $\Delta$ over $A$ if for all $n$ we have $a_{n+1} \downarrow^\Delta a_0 \ldots a_n$ and $\text{tp}(a_n/\text{bdd}(A)) = \text{tp}(a_0/\text{bdd}(A))$.

1.2.5. Continuous logic and classical logic. Continuous logic is a direct generalisation of classical logic, and there is a canonical way of taking a classical logic theory $T$ and viewing it as a continuous logic theory $T_{\text{cont}}$. The construction of $T_{\text{cont}}$ is as follows. Every sort of $T$ is now viewed as a metric space with the discrete metric with diameter 1. We remove the equality symbol from the sorts of $T$. Observe that there is no loss of information in doing this, since $x = y$ is equivalent to $d(x, y) = 0$ and $x \neq y$ is equivalent to $d(x, y) = 1$. We add the usual value sorts to $T$. Function symbols in the language of $T$ are unchanged. For each relation symbol $R$ in the language of $T$, we substitute in a function symbol $f_R$ which we view as the indicator function of $R$ in the corresponding continuous logic sort. It is then clear how to axiomatise a continuous logic theory $T_{\text{cont}}$ so that there is a natural correspondence between models of $T$ and models of $T_{\text{cont}}$.

In this paper, when we construct a theory $T_{\text{cont}}$ from a classical logic theory $T$, we distinguish three kinds of imaginary sorts in $T_{\text{cont}}$. Firstly we have the *classical imaginary sorts of $T_{\text{cont}}* which come from the imaginary sorts of $T$ defined in the usual way (see [Tent and Ziegler, 2019]). Secondly we have the *finitary imaginary sorts of $T_{\text{cont}}* which are imaginary sorts obtained by
of. Moreover, we can assume that

$$\Rightarrow$$

f

definable function into $\mathbb{R}$ with $y$ a finite tuple in the

sort of $D \in \mathcal{D}$ and $x$ in the sort of $a$. By (1) there is a $\bigcup \mathcal{D}(M)$-definable function $g(y)$ such that

$$g(y) = f(a, y)$$

on $D$. $g$ defines $\text{tp}(a/\bigcup \mathcal{D}(M))$ for $f$. Moreover, $g$ is definable over a countable

$$A \subseteq D(M),$$

so $\text{tp}(a/\bigcup \mathcal{D}(M))$ is definable over a subset $B$ of $\bigcup \mathcal{D}(M)$ with $|B| \leq |\mathcal{L}|$.

(2) $\Rightarrow$ (1): Let $f(x)$ be an $M$-definable function into $\mathbb{R}$ where $x$ is a finite tuple in the sort of

$D \in \mathcal{D}$. We can assume that $f(x) = g(x, b)$ where $b$ is a finite tuple in $M$ and $g$ is 0-definable.

$q = \text{tp}(b/\bigcup \mathcal{D}(M))$ is definable over some small $C \subseteq D(M)$ so we have a $C$-definable function

$$d_qg(x) = f(x).$$

(2) $\Rightarrow$ (3): Let $p(x) = \text{tp}(a/\bigcup \mathcal{D}(M)), a \in M$. Suppose that $p$ is definable over $C \subseteq \bigcup \mathcal{D}(M)$.

Let $f(x)$ be a $\bigcup \mathcal{D}(M)$-definable function into $\mathbb{R}$. If we show that the restriction of $p(x)$ to $C$
determines the value of $f(x)$, we will have shown that $p \upharpoonright C$ extends uniquely to $\bigcup \mathcal{D}(M)$.

Moreover, we can assume that $f(x) = g(x, b)$ where $b$ is a finite tuple in $\bigcup \mathcal{D}(M)$ and $g(x, y)$ is

0-definable.

By (2), $p$ is definable at $g$ over $C$ and we write $d_p g(y)$ for its definition. If the complement

of $D$ is type-definable, then the relation $y \in D$ is given by the negation of a positive formula.

Therefore $p$ contains the positive formula over $C$:

$$\forall y(y \in D \rightarrow g(x, y) = d_p g(y))$$
If $D$ is distance-definable by the function $d(y, D)$, then an easy compactness argument shows that for every $\epsilon > 0$ there is $\delta > 0$ such that $p(x)$ contains the positive formula over $C$:

$$\forall y (d(y, D) < \delta \rightarrow |g(x, y) - d_p(g(y))| \leq \epsilon)$$

In both cases, we have shown that $p \restriction C$ has a unique extension to $\bigcup D(M)$.

(3) $\Rightarrow$ (2): Let $a \in M$, write $p(x) = \text{tp}(a/\bigcup D(M))$ and let $f(x, y)$ be a definable function with $y$ a finite tuple in $D \in D$ and $x$ a tuple in the sort of $a$. By (3), there is $C \subseteq \bigcup D(M)$ such that $p(x)$ is the unique extension of $p \restriction C$ to $\bigcup D(M)$. Let $N$ be an elementary extension of $M$. Suppose there are $c, c' \in N$ both realising $p \restriction C$ and $b \in D(N)$ such that $|f(c, b) - f(c', b)| \geq \delta$ for some $\delta > 0$. Then

$$N \models \exists x, x', y (p \restriction C(x) \land p \restriction C(x') \land D(y) \land |f(x, y) - f(x', y)| \geq \delta)$$

The above is an infinite conjunction of positive formulas. Since $M \prec N$, $M$ satisfies arbitrary $\epsilon$-approximations of the above type-definable set. Since $M$ is saturated over $C$, $M$ satisfies the same type-definable set. This contradicts (3) and this proves that $p$ has a unique extension to $\bigcup D(N)$.

Now it follows from (3) that for any $M \prec N$, $p(x)$ is $C$-invariant in $N$. It follows by a standard argument that $p(x)$ is $C$-definable.

(3) $\Rightarrow$ (4): Let $\phi$ be an automorphism of $\bigcup D(M)$ and suppose that we have extended it to an automorphism $\phi : \bigcup D(M) \cup A \to \bigcup D(M) \cup B$ where $|A| < \kappa$. Let $a \in M$. There is $C \subseteq \bigcup D(M)$ with $|C| < \kappa$ such that $\text{tp}(a/A/C)$ extends uniquely to $A \cup \bigcup D(M)$. By saturation, we can find $b \in M$ such that $\phi$ extends to an automorphism $\bigcup D(M) \cup Aa \to \bigcup D(M) \cup Bb$. The result follows by a back-and-forth argument.

(4) $\Rightarrow$ (3): Suppose (3) fails for $M \models T$, where $M$ is $\kappa$-saturated with cardinality $\kappa$. Fix $a \in M$ which witnesses the failure of (3). Let $(a_i)_{i<\kappa}$ be an enumeration of the realisations of $\text{tp}(a)$ in $M$. Suppose we have constructed an isomorphism $f : C \to f(C)$ where $C$ is a subset of $\bigcup D(M)$ such that for some $\alpha < \kappa$ and all $i < \alpha$, the maps $f : aC \to a_i f(C)$ are not isomorphisms.

Suppose that $f_1 : aC \to a_i f(C)$ is an isomorphism. By the failure of (3) there is $a' \in M$ and $b \in D(M)$ such that $\text{tp}(a/C) = \text{tp}(a'/C)$ and $\text{tp}(a/bC) \neq \text{tp}(a'/bC)$. Then we can find $b'$ such that $f_1 : a' bC \to a_i b' f(C)$ is an isomorphism. Then we extend $f$ by putting $f(b) = b'$. Note that now we cannot extend $f$ to $a$ by sending $a$ to $a_a$. By enumerating $\bigcup D(M)$, we can also make sure that after $\alpha$ iterations of this procedure $f$ is defined on all of $\bigcup D(M)$. This contradicts (4).

1.3. Hilbert Spaces In Continuous Logic. We recall here basic facts about the model theory of Hilbert spaces which we will use in this paper. We refer the reader to [Ben Yaacov et al., 2008] for a more complete summary. In this paper we will usually work with Hilbert spaces over $\mathbb{R}$, but all results can be transposed to complex Hilbert spaces without modification.

On one presentation of the model theory of Hilbert spaces, the language of Hilbert spaces in continuous logic consists of countably many metric sorts, which stand for balls with radius $n$ around 0. We add appropriate inclusion maps between the metric sorts. The language consists of the usual vector space structure over $\mathbb{R}$ and a function $\langle \cdot, \cdot \rangle$ on each metric sort which stands for the inner product. The axiomatisation of the theory of infinite dimensional Hilbert spaces $T^{Hilb}$ is as expected. In this paper, we usually don’t distinguish between a Hilbert space and a model of $T^{Hilb}$. 
$T^{Hilb}$ is complete, has quantifier-elimination, is stable, and is totally categorical. The theory of Hilbert spaces does not have elimination of imaginaries, but it has weak elimination of imaginaries:

**Lemma 1.14.** Let $M \models T^{Hilb}$. Let $\alpha$ be a canonical parameter for an $M$-definable function $f$ in an arbitrary imaginary sort of $T^{Hilb}$. Then there is a closed subspace $H$ of $M$ such that each point of $H$ is in $\text{bdd}(\alpha)$ and $\alpha$ is definable over $H$.

**Proof.** See Lemma 1.2 in [Ben Yaacov and Berenstein, 2004].

**Definition 1.15.** If $H$ is a Hilbert space and $V \leq H$ is a closed subspace, write $P_V$ for the orthogonal projection onto $V$.

We will make much use of the characterisation of forking independence in Hilbert spaces. See [Ben Yaacov et al., 2008] for a proof:

**Lemma 1.16.** Let $M \models T^{Hilb}$. Let $A, B, C \subseteq M$ with $B \subseteq C$. Then $\text{bdd}(B)$ is the closed subspace of $M$ densely spanned by $B$ and $A \perp_B C$ if and only if for all $a \in A$, $P_{\text{bdd}(C)}(a) = P_{\text{bdd}(B)}(a)$.

Finally, note the two following lemmas. The first is an easy linear algebra exercise.

**Lemma 1.17.** Let $M \models T^{Hilb}$ and $A \subseteq M$ and $v \in M$. Let $(v_n)$ be an indiscernible sequence in $\text{tp}(v/A)$ in $M$. Then there is an orthogonal sequence $(w_n)$ in $M$ and $w \in M$ such that for all $i$, $v_n = w_n + w_i \perp A$, $w_n \perp w$. It follows that $(v_n)$ converges weakly to $w$ (we write $v_n \rightharpoonup w$).

Moreover, $w$ is the unique element of $M$ such that $\langle w, w \rangle = \langle v_1, v_0 \rangle = \langle w, v_n \rangle$ for all $n$.

**Lemma 1.18.** Let $M \models T^{Hilb}$. Let $(v_n)$ be a Morley sequence over $A \subseteq M$. Then $(v_n)$ converges weakly to $P_{\text{bdd}A}(v_0)$.

**Proof.** It is enough to check that $\lim_n (v_n, v_m) = \langle P_{\text{bdd}(A)}(v_0), v_m \rangle$ for all $m$. For $n > m$ we have $\langle v_n, v_m \rangle = \langle P_{\text{bdd}(A)\cup\ldots\cup v_m)}(v_n), v_m \rangle = \langle P_{\text{bdd}(A)}(v_n), v_m \rangle = \langle P_{\text{bdd}(A)}(v_0), v_m \rangle$.

2. **Piecewise Interpretable Hilbert Spaces**

In this section, we give a general exposition of piecewise interpretable Hilbert spaces. We start by going through the direct limit construction mentioned in the introduction and we show that interpretable Hilbert spaces correspond to functors on models of a theory $T$. We also clarify the role of the GNS-construction at the functorial level. In Section 2.2 we give some key examples of piecewise interpretable Hilbert spaces and in Section 2.3 we give an alternative exposition which relies on adding a new sort to the theory.

2.1. **Definitions and first results.** Let $T$ be a complete continuous logic theory in the language $\mathcal{L}$. Let $M \models T$ be a standard model. We write $S_i$ for sorts of $\mathcal{L}$ and $M_i$ for the sort in $M$ corresponding to $S_i$.

Suppose we have a partial order $(J, \leq)$ indexing a collection $(M_j)_{i \in J}$ of distinct sorts of $M$. Suppose that for any $i, j \in J$ there is $k \in J$ such that $i, j \leq k$. Finally, suppose that for any $i \leq j \in J$ there is a 0-definable map $t_{ij} : M_i \to M_j$ such that

1. for all $i \in J$, $t_{ii} = \text{id}$
2. for all $i \leq j \leq k$, $t_{ik} \circ t_{ij} = t_{ij}$

Observe that we do not require $t_{ij}$ to be injective. The direct limit $L$ of $(M_j)_{i \in J}$ is defined in the usual set-theoretic way. We refer to the sorts $(M_i)$ as the pieces of the direct limit. For each $i \in J$ we have the direct limit map $\iota : M_i \to L$. 

Definition 2.1. A piecewise interpretable Hilbert space \( H \) in \( M \) is a direct limit of imaginary sorts \( (M_i)_{i \in I} \) of \( M \) such that \( H \) is a Hilbert space and the inner product is definable between all the pieces \( M_i \). More explicitly, for all \( i,j \in I \), the map \( M_i \times M_j \to \mathbb{R}, (x,y) \mapsto \langle x,y \rangle \) is definable.

Note that we do not require that the sum and scalar multiplication operations be definable. However, it is clear that the inner product on a Hilbert space uniquely determines the sum and scalar multiplication and we will see in Lemma 2.9 that we recover these as definable maps in an appropriate sense.

The definition of piecewise interpretable Hilbert spaces is somewhat cumbersome to work with, but we will show how to give a much simpler characterisation. We begin by recalling the GNS theorem (see [Bekka et al., 2008]):

Definition 2.2. Let \( X \) be a set. A function \( f : X \times X \to \mathbb{R} \) is said to be positive-semidefinite if \( f \) is symmetric and for all \( n \geq 1 \), for all \( x_1, \ldots , x_n \in X \) and for all \( \lambda_1, \ldots , \lambda_n \in \mathbb{R} \), we have \( \sum_{i,j} \lambda_i \lambda_j f(x_i,x_j) \geq 0 \).

Theorem 2.3 (GNS Theorem). Let \( X \) be a set and let \( f : X \times X \to \mathbb{R} \) be positive-semidefinite. Then there is a Hilbert space \( H \) and a map \( F : X \to H \) such that \( F(X) \) has dense span in \( H \) and for all \( x,y \in X \), \( \langle F(x),F(y) \rangle = f(x,y) \).

We will adapt the GNS theorem to our context:

Proposition 2.4. Suppose we are given a Hilbert space \( H \), a collection of imaginary sorts \( (M_i)_{i \in I} \) of \( M \) and functions \( h_i : M_i \to H \) for all \( i \in I \) such that for all \( i,j \in I \), the map \( M_i \times M_j \to \mathbb{R}, (x,y) \mapsto \langle h_i(x),h_j(y) \rangle \) is definable.

Then there is a piecewise interpretable Hilbert space \( H(M) \) such that the sorts \( (M_i \times M_j) \) are pieces of \( H \), \( \iota(\bigcup M_i) \) has dense span in \( H(M) \), and for \( x \in M_i, y \in M_j \), \( \langle x,y \rangle_{H(M)} = \langle h_i(x),h_j(y) \rangle \).

Proof. Write \( h \) for the concatenation of all functions \( h_i \), \( (i \in I) \). By passing to a closed subspace of \( H \), we can assume without loss of generality that \( h(\bigcup M_i) \) has dense span in \( H \).

Let \( x \in H \). There is an increasing function \( \eta : \mathbb{N} \to \mathbb{N} \) and a uniformly Cauchy sequence \( (\sum_{i=\eta(n)}^{\eta(n+1)-1} \lambda_i h(x_n^i)) \) which converges to \( x \) such that \( x_n^i \in \bigcup M_{j} \). We can assume that \( \eta(n) \) is large enough so that \( \lambda_i \eta(n) \leq \eta(n) \) for all \( i \leq n \). We will decompose \( H(M) \) according to the rate of growth of \( \eta \).

Fix \( \eta : \mathbb{N} \to \mathbb{N} \) strictly increasing and fix an arbitrary countable sequence \( (i_n) \) in \( I \). We define a new imaginary sort \( M_{(i_n)}^\eta \) will be the metric completion of the infinite Cartesian product \( \prod_{n \geq 0} \prod_{k=\eta(n)}^{\eta(n+1)-1} [\eta(n),\eta(n)] \times M_{i_k} \) under the metric to be defined below. Write \( (\overline{x}_n) \) for an element of \( M_{(i_n)}^\eta \) and write \( \overline{x}_n = (\lambda_{\eta(n)},x_{\eta(n)},\ldots,\lambda_{\eta(n+1)},x_{\eta(n+1)},\ldots) \).

For all \( n \), define \( h(\overline{x}_n) = \sum_{i=\eta(n)}^{\eta(n+1)-1} \lambda_i h(x_i) \). We define inductively maps \( g_k : M_{(i_n)}^\eta \to H \). \( g_0(\overline{x}_n) \) is just \( h(\overline{x}_n) \). Given \( g_k \), define \( g_{k+1}(\overline{x}_n) = h(\overline{x}_{k+1}) \) if \( \| h(\overline{x}_{k+1}) - h(\overline{x}_n) \| \leq 2^{-n} \). Otherwise, define

\[
g_{k+1}(\overline{x}_n) = g_k(\overline{x}_n) + \frac{2^{-n}}{\| h(\overline{x}_{k+1}) - g_k(\overline{x}_n) \|} (h(\overline{x}_{k+1}) - g_k(\overline{x}_n))
\]

Then \( (g_k(\overline{x}_n))_k \) is a uniformly Cauchy sequence in \( H \). It is straightforward to check that for all \( k \), the map \( ((\overline{x}_n),\overline{g}_n) \mapsto \langle g_k(\overline{x}_n),g_k(\overline{g}_n) \rangle \) is definable. We obtain a definable map \( I((\overline{x}_n),\overline{g}_n) = \lim_k \langle g_k(\overline{x}_n),g_k(\overline{g}_n) \rangle \). \( I \) is positive-semidefinite and hence \( I \) induces

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1 This is similar to Definition 3.6 in [Ben Yaacov and Usvyatsov, 2010]
a pseudo-metric. We quotient by this pseudo-metric and take the metric completion, so that $M^n_{(i_n)}$ is identified with a subset of $H$.

We now define the direct limit structure. Choose an ordering $\leq$ of $I$. Let $J$ be the set of pairs $(\eta, (i_n))$ such that $\eta : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing and $(i_n)$ is a sequence in $I$ such that for all $n$, $(i_{\eta(n)}, \ldots, i_{\eta(n+1)-1})$ is increasing with respect to $I$. We could have chosen such an $(i_n)$ when we constructed $M^n_{(i_n)}$ above. We define a partial ordering on $J$ as follows: we say that $(\eta, (i_n)) \leq (\mu, (j_n))$ if and only if for all $n$, $\eta(n+1) - \eta(n) \leq \mu(n+1) - \mu(n)$ and $(i_{\eta(n)}, \ldots, i_{\eta(n+1)-1})$ is a subtuple of $(j_{\mu(n)}, \ldots, j_{\mu(n+1)-1})$. We define maps $M^n_{(i_n)} \rightarrow M^\mu_{(j_n)}$ for $(\eta, (i_n)) \leq (\mu, (j_n))$ by taking the obvious inclusions, adding 0 as a scalar to fill any remaining gaps. $H(M)$ is defined as the direct limit of the sorts $M^n_{(i_n)}$. We leave the remaining technical details to the reader.

Remark: Observe that, given the data of Proposition 2.4, we cannot usually find a decomposition of $H(M)$ which uses only countably many pieces. This is a departure from the usual presentation of Hilbert spaces in continuous logic.

The construction in Proposition 2.4 is somewhat arbitrary and immediately suggests the following notion of isomorphism of piecewise interpretable Hilbert spaces:

Definition 2.5. Suppose that $H, H'$ are piecewise interpretable Hilbert spaces in $M$. Say $H$ is the direct limit of $(M_j)_{j \in J}$ and $H'$ is the direct limit of $(M'_j)_{j' \in J'}$. Write $\iota : M_j \rightarrow H$ and $\iota' : M'_j \rightarrow H'$ for the inclusion maps. We say that a map $T : H \rightarrow H'$ is an embedding of piecewise interpretable Hilbert spaces if $T$ is a unitary map and for all $j \in J$ and $j' \in J'$ the set $\{(x, y) \in M_j \times M'_j \mid T\iota(x) = \iota'(y)\}$ is type-definable.

If $T$ is also an isomorphism of Hilbert spaces, we say that $T$ is an isomorphism of piecewise interpretable Hilbert spaces.

We say that $H$ and $H'$ are isomorphic if there is some $T : H \rightarrow H'$ which is an isomorphism of piecewise interpretable Hilbert spaces.

Lemma 2.6. Let $H, H'$ be piecewise interpretable Hilbert spaces in $M$. Say $H$ is the direct limit of $(M_j)_{j \in J}$ and $H'$ is the direct limit of $(M'_j)_{j' \in J'}$. Write $\iota : M_j \rightarrow H$ and $\iota' : M'_j \rightarrow H'$ for the inclusion maps. A unitary map $T : H \rightarrow H'$ is an embedding of piecewise interpretable Hilbert spaces if and only if for all $i \in J$ and $j' \in J'$ the map $M_i \times M_j \rightarrow \mathbb{R}, (x, y) \mapsto \langle T\iota(x), \iota'(y) \rangle$ is definable.

If $T$ is an isomorphism of Hilbert spaces, the same condition is equivalent to $T$ being an isomorphism of piecewise interpretable Hilbert spaces.

Proof. Suppose that $T$ is an embedding of piecewise interpretable Hilbert spaces. Take $i \in J$ and $j \in J'$ and $D$ a closed bounded subset of $\mathbb{R}$. For every $\epsilon > 0$, let $D_\epsilon$ be the closed set $\{x \in \mathbb{R} \mid \exists y \in D, |y - x| \leq \epsilon\}$. Let $M$ be an upper bound on $\{||\iota'(y)|| \mid y \in M_j\}$. By compactness, for every $\epsilon > 0$, we can find $i_\epsilon \in J$ such that for every $x \in M_i$, there is $x' \in M_{i_\epsilon}$ such that $||\iota'(x') - T\iota(x)|| < \epsilon/M$. Now observe that the set $\{(x, y) \in M_i \times M_j \mid \langle T\iota(x), \iota'(y) \rangle \in D_\epsilon\}$ is equal to the conjunction over $\epsilon > 0$ of all sets

$$\{(x, y) \in M_i \times M_j \mid \exists x' \in M_{i_\epsilon}, ||T\iota(x) - \iota'(x')|| \leq \epsilon/M \text{ and } \langle \iota'(x'), \iota'(y) \rangle \in D_\epsilon\}$$

which is type-definable, by the definition of piecewise interpretable Hilbert spaces.

Conversely, note that for $i \in J$ and $j \in J$, $T\iota(x) = \iota'(y)$ if and only if $\langle T\iota'(x), \iota(y) \rangle = \langle \iota'(y), \iota'(y) \rangle = \langle T\iota(x), T\iota(x) \rangle$.

We can now sharpen Proposition 2.4.
Lemma 2.7. As in Proposition 2.4, suppose we are given a Hilbert space $H$, a collection of distinct imaginary sorts $(M_i)_{i \in I}$ of $M$ and functions $h_i : M_i \to H$ for all $i \in I$ such that for all $i, j \in I$, the map $M_i \times M_j \to \mathbb{R}, (x, y) \mapsto \langle h_i(x), h_j(y) \rangle$ is definable.

Then there is a unique piecewise interpretable Hilbert space $H(M)$ up to isomorphism such that the sorts $(M_i)_{i \in I}$ are pieces of $H(M)$, $\iota(\bigcup M_i)$ has dense span in $H(M)$, and for $x \in M_i$, $y \in M_j$, $\langle x, y \rangle_{H(M)} = \langle h_i(x), h_j(y) \rangle_H$.

Proof. Let $H(M)$, $H'(M)$ be two piecewise interpretable Hilbert spaces satisfying the existence claim. Say $H(M)$ is the direct limit of $(M_i)_{i \in I}$ and $H'(M)$ is the direct limit of $(M_i')_{i \in I'}$. For $\lambda_1, \ldots, \lambda_n \in \mathbb{R}, x_1, \ldots, x_n \in \bigcup_{i \in I} M_i$, define $T(\sum \lambda_k t_i(x_k)) = \sum \lambda_k t'_i(x_k))$. $T$ extends to an isomorphism of Hilbert spaces $H(M) \to H'(M)$.

Pick $j \in J$, $j' \in J'$. Observe that the map which takes $(x, \lambda_1, x_1, \ldots, \lambda_n, x_n)$ where $x_k \in \bigcup_{i \in I} M_i$ and $x \in M_j$ to $(T(\lambda x), \sum \lambda_k t'_i(x_k))$ is definable. Now for every $y \in M_{j'}, t'(y)$ is the limit of a uniformly Cauchy sequence of finite linear combinations of the form $\sum \lambda_k t'_i(x_k)$ where $x_k \in \bigcup M_{i_k}$. By a standard compactness argument, we deduce that the map $M_j \times M_{j'} \to \mathbb{R}, (x, y) \mapsto \langle T(\iota(x)), t'(y) \rangle$ is definable.

Recall that we did not require direct limits of sorts of $M$ to have injective transition maps. We now show that this does not present any significant advantage.

Lemma 2.8. Let $H(M)$ be a piecewise interpretable Hilbert space in $M$. Write $H(M)$ as the direct limit of $(M_i)_{i \in I}$. Then $H(M)$ is isomorphic to a piecewise interpretable Hilbert space $H'(M)$ which is a direct limit of imaginary sorts $(M'_i)_{i \in I}$ with injective transition maps.

Proof. For every $i \in J$, let $M_i'$ be the imaginary sort of canonical parameters of the map $b_i : M_i \times M_i \to \mathbb{R}, (x, y) \mapsto \langle t_i(x), t_i(y) \rangle$. For $i \leq j \in J$, define $t_i' : M_i' \to M_j'$ as the map which takes the canonical parameter for the map $b_i(a, \ldots)$ to the canonical parameter for the map $b_j(t_i a, \ldots)$. Observe $t_i'$ is well-defined because $b_i(a, \ldots) = b_j(b, \ldots)$ if and only if $b_i(a, a) = b_j(b, b) = b_i(a, b)$, which entails $b_j(t_i(a, a)) = b_j(t_i b, \ldots)$. This also shows that for any $i, j \in J$, the map $M_i \times M_j \to \mathbb{R}, (x, y) \mapsto \langle t_i(x), t_i(y) \rangle$ factors to a definable map $M_i' \times M_j'$.

It is straightforward to prove that $H'(M)$ satisfies the lemma. □

Observe that in the above lemma, if $M_i$ is a classical (resp. finitary) imaginary sort of $M$, then $M_i'$ is also a classical (resp. finitary) imaginary sort of $H'(M)$.

As a direct application of the construction in Proposition 2.4 and Lemma 2.8, we have:

Lemma 2.9. Let $H(M)$ be a piecewise interpretable Hilbert space in $M$. Suppose $H(M)$ is the direct limit of $(M_i)_{i \in I}$. Then $H(M)$ is isomorphic to a piecewise interpretable Hilbert space $H'(M)$ in $M$ which is a direct limit of $(M'_i)_{i \in I'}$ with injective transition maps such that the Hilbert space operations on $H'(M)$ are piecewise bounded, i.e.

1. for every $i, j \in I'$, there is $k$ such that $i, j \leq k$ and $\iota(M_i) + \iota(M_j) \subseteq \iota(M_k)$ and the map $M_i \times M_j \to M_k, (x, y) \mapsto \iota^{-1}(x + y)$ is definable.
2. for every $i \in I'$ and $n \geq 0$, there is $k$ such that for $x \in M_i$ and $\lambda \in [-n, n]$, $\lambda t_i \in \iota M_k$ and the map $[-n, n] \times M_i \to M_k, (\lambda, x) \mapsto \iota^{-1}(\lambda t_i x)$ is definable.

Proof. Apply Lemma 2.8 to get $H_1(M)$ with injective transition maps and observe that the construction of Proposition 2.4 with $I = I'$ and $H = H(M)$ gives a piecewise interpretable Hilbert space with the desired properties. □

Finally, we give a category-theoretic reformulation of Proposition 2.4.

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2 Thanks to Arturo Rodríguez Fanlo for this terminology.
Lemma 2.10. Let $M \models T$. The categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ defined below are all equivalent:

Let $\mathcal{A}$ be the category of piecewise interpretable Hilbert spaces in $M$ with embeddings of piecewise interpretable Hilbert spaces.

Let $\mathcal{B}$ be the category of pairs $(H_i, (h_i)_{i \in I})$ where $H$ is a Hilbert space, $I$ indexes a set of imaginary sorts $(M_i)_{i \in I}$ of $M$ and $h_i : M_i \rightarrow H$ are maps such that

1. the set $\{h_i(M_i) \mid i \in I\}$ has dense span in $H$
2. for all $i_1, i_2 \in I$, the map $M_{i_1} \times M_{i_2} \rightarrow \mathbb{R}, (x, y) \mapsto \langle h_{i_1}(x), h_{i_2}(y) \rangle$ is definable.

The morphisms between objects $(H_i, (h_i)_{i \in I})$ and $(H_{i'}, (h_{i'})_{i' \in I'})$ of $\mathcal{B}$ are unitary maps $T : H \rightarrow H'$ such that for all $i \in I$, $i' \in I'$ the map $M_i \times M_{i'} \rightarrow \mathbb{R}, (x, y) \mapsto \langle T(h_i(x)), h_{i'}(y) \rangle$ is definable.

Let $\mathcal{C}$ be the category of pairs $((M_i)_{i \in I}, (f_{ij})_{i, j \in I})$ where $(M_i)_{i \in I}$ is a set of imaginary sorts of $M$ and the $f_{ij} : M_i \times M_j \rightarrow \mathbb{R}$ are definable functions such that their concatenation $f : \bigcup_i M_i \times \bigcup_{i,j \in I} M_i \rightarrow \mathbb{R}$ is positive semidefinite. The morphisms between objects $((M_i)_{i \in I}, (f_{ij}))$ and $((M_{i'})_{i' \in I'}, (f_{ij}'))$ of $\mathcal{C}$ are piecewise definable functions $F : \bigcup_{i' \in I'} M_{i'} \times \bigcup_{k \in I'} M_k \rightarrow \mathbb{R}$ such that

1. $F$ is positive semidefinite and $F$ extends each function $f_{ij}$ where $i, j \in I$ or $i, j \in I'$
2. writing $H_I$ and $H_{I \cup I'}$ for the Hilbert spaces induced by $\bigcup_i M_i$ and $\bigcup_{i \in I', k \in I} M_k$ respectively as in the GNS-Theorem\(^3\) the resulting Hilbert space embedding $H_{I'} \rightarrow H_{I \cup I'}$ induced from $F$ is surjective\(^4\).

Composition of morphisms in $\mathcal{C}$ is induced by the GNS theorem\(^3\).

Proof. Define $F : \mathcal{A} \rightarrow \mathcal{B}$ as follows. If the direct limit $(M_j)_{j \in J}$ is a piecewise interpretable Hilbert space with direct limit maps $\iota$, write $H$ for the direct limit of $(M_j)_{j \in J}$ with its Hilbert space structure and let $h_j = \iota : M_j \rightarrow H$. Then $F(M_j) = (H, (h_j)_{j \in J})$. It is easy to check that $F$ is natural and injective. To see that $F(\mathcal{A})$ is dense in $\mathcal{B}$, apply Proposition 2.4.

The equivalence between $\mathcal{B}$ and $\mathcal{C}$ is given by the classical GNS theorem. Note that

Closed subspaces of piecewise interpretable Hilbert spaces in $M$ are usually not piecewise interpretable Hilbert spaces. Nevertheless, note that if $H(M)$ and $H'(M)$ are piecewise interpretable in $M$ and $T : H(M) \rightarrow H'(M)$ is an embedding of interpretable Hilbert spaces, then the image of $T$ is a type-definable closed subspace of $H'(M)$.

Definition 2.11. A piecewise $\bigwedge$-interpretable Hilbert space $V(M)$ in $M$ is a type-definable closed subspace of a piecewise interpretable Hilbert space $H(M)$ in $M$. More explicitly, if $H(M)$ is the direct limit of $(M_j)_{j \in J}$ then, for every $j \in J$, $V(M) \cap M_j$ is type-definable.

All the results we have given so far have natural analogs for piecewise $\bigwedge$-interpretable Hilbert space.

Lemma 2.12. Let $H(M)$ be piecewise interpretable in $M$ and let $V(M) \leq H(M)$ be a piecewise $\bigwedge$-interpretable subspace. Suppose $H(M)$ is the direct limit of $(M_j)_{j \in J}$ and $V(M) \cap M_j := D_j$. Let $U$ be a subspace of $H(M)$ such that, for every $j$, $U \cap M_j$ is an open set in the logic topology. Then $U^\perp \cap V(M)$ is piecewise $\bigwedge$-interpretable in $H(M)$.

\(^3\) The GNS theorem gives Hilbert spaces $H_I$, $H_{I'}$ and $H_{I \cup I'}$ and maps $\phi : \bigcup_i M_i \rightarrow H_I$, $\phi' : \bigcup_{i' \in I'} M_{i'} \rightarrow H_{I'}$, $\bigcup_{k \in I'} M_k \rightarrow H_{I \cup I'}$ and Hilbert space embeddings $\phi : H_I \rightarrow H_{I \cup I'}$ and $\phi' : H_{I'} \rightarrow H_{I \cup I'}$ such that the corresponding diagram commutes. So we have $F(x, y) = \langle \phi \circ a(x), \phi' \circ b(y) \rangle$ when $x \in M_i$ and $y \in M_{i'}$.

\(^4\) It is possible to give a definition of the morphisms of $\mathcal{C}$ and their composition which does not rely explicitly on the GNS theorem by using the Gram-Schmidt orthogonalisation process and Bessel's inequality. The details are left to the interested reader.
Proof. It is enough to prove that \( U^\perp \) is piecewise \( \wedge \)-interpretable in \( H(M) \). For every \( j \in J \) write \( U \cap M_j = M_j \cap \neg \bigwedge \Sigma_j(y) \) where \( \Sigma_j \) is a set of formulas closed in \( M_j \) the logic topology. For every \( J' \in J \) write \( W_{J'} = U^\perp \cap M_{J'} \). Note that \( W_{J'}(x) \) is defined by
\[
\bigwedge_{j \in J} \{ \forall y \in M_j(\phi(y) \vee \langle x, y \rangle = 0) \mid \phi(y) \in \Sigma_j(y) \}
\]
so \( W_{J'} \) is type-definable. We only have to check that \( \iota(\bigcup J W_{J'}) \) is closed in the Hilbert space topology. If \( v \in \iota(\bigcup J W_{J'}) \) find \( j \in J \) and \( x \in M_j \) such that \( \iota(x) = v \). Since \( x \in W_{J'} \), it follows that \( v \in \bigcup_{J'} W_{J'} \).

It follows that if \( U \leq H(M) \) is a subspace as in the previous lemma, then \( H(M)/U \) is piecewise \( \wedge \)-interpretable in \( M \), if we identify it with \( U^\perp \).

Finally, we remark that one could attain an even greater level of generality by removing the assumption in Definition 2.11 that \( V(M) \) is a subspace of a piecewise interpretable Hilbert space. In other words, one could consider arbitrary direct limits of imaginary sorts of \( M \) and type-definable subsets which have a Hilbert space structure. Note that there is no trouble in generalising Proposition 2.4 to this context to obtain GNS-style constructions. However, we are not aware of any situations where such interpretable Hilbert spaces arise naturally, so we leave the details to the interested reader.

**Convention:** From now on, we only discuss piecewise interpretable Hilbert spaces in \( M \) up to isomorphism. By Lemma 2.10, we know that in order to fix such a piecewise interpretable Hilbert space, we only specify a pair \( (H, (h_i)_{i \in I}) \) where \( H \) is a Hilbert space and the maps \( h_i : M_i \rightarrow H \) as in the definition of \( \mathcal{B} \) in Lemma 2.10. We say that the interpretation is supported by the sorts \( M_i \) and we call the maps \( h_i : M_i \rightarrow H \) interpretation maps. Alternatively, it is enough to specify the sorts \( (M_i)_I \) and for every \( i, i' \in I \) a definable map \( f_{i,i'} : M_i \times M_{i'} \rightarrow \mathbb{R} \) such that the concatenation of all maps \( (f_{i,i'})_I \) is positive-semidefinite. We refer to the maps \( (f_{i,i'}) \) as the inner product maps.

**Convention:** In this paper, we will only consider ‘piecewise interpretable’ Hilbert spaces and ‘piecewise \( \wedge \)-interpretable Hilbert spaces’, so we will now refer to them simply as ‘interpretable’ or ‘\( \wedge \)-interpretable’ Hilbert spaces.

Suppose \( H(M) \) is interpretable in \( M \), supported by \( (M_i) \), with interpretation maps \( (h_i) \). Write \( f_{ij} : M_i \times M_j \rightarrow \mathbb{R}, (x, y) \mapsto \langle h_i(x), h_j(y) \rangle \) for \( i, j \in I \). Then for all \( m \geq 1 \) and for all \( \lambda_1, \ldots, \lambda_m \in \mathbb{R} \),
\[
T \models \forall x_1 \ldots \forall x_m \sum_{i \leq m} \lambda_i \lambda_j f(x_i, x_j) \geq 0
\]
where \( x_i \) is in one of the sorts \( S_j \). Hence, for every \( N \models T \), we can apply the GNS construction to \( \bigcup J N_i \). We obtain a Hilbert space \( H(N) \) supported by \( (N_i) \) with interpretation maps \( g_i : N_i \rightarrow H(N) \) such that \( f_{ij}(a, b) = \langle g_i(a), g_j(b) \rangle \) and such that \( \bigcup J g_i(N_i) \) has dense span in \( H(N) \). Therefore \( H(N) \) is piecewise interpretable in \( N \). This leads us to the notion of a Hilbert space functor:

**Definition 2.13.** A Hilbert space functor \( \mathcal{H} \) is a functor which takes every model \( M \) of \( T \) to a pair \( (H(M), (h_i)_{i \in I}) \) such that

1. \( H(M) \) is a Hilbert space interpretable in \( M \) supported by the sorts \( (M_i) \) with interpretation maps \( h_i : M_i \rightarrow H(M) \)
2. The sorts \( S_i \) corresponding to \( M_i \) and the inner product maps \( f_{ij} : S_i \times S_j \rightarrow \mathbb{R}, (x, y) \mapsto \langle h_i(x), h_j(y) \rangle \) do not depend on the choice of \( M \)
Note that a Hilbert space functor is uniquely determined by its inner product maps. The following lemma shows that a Hilbert space functor is uniquely determined by its value at a single ω-saturated model.

**Lemma 2.14.** Let \( \mathcal{H}, \mathcal{H}' \) be two Hilbert space functors. Suppose that for some ω-saturated \( M \models T, \mathcal{H}(M) \) and \( \mathcal{H}'(M) \) are isomorphic. Then for every \( N \models T, \mathcal{H}(N) \) and \( \mathcal{H}'(N) \) are isomorphic as interpretable Hilbert spaces.

**Proof.** Suppose that \( \mathcal{H}(M) \) and \( \mathcal{H}'(M) \) are the direct limits of \( (M_j)_j \) and \( (M'_{j'})_{j'} \) respectively. Write \( \iota \) and \( \iota' \) for the inclusions \( \bigcup M_j \to H(M), \bigcup M'_{j'} \to H'(M) \). Write \( T_M : H(M) \to H'(M) \) for the Hilbert space isomorphism giving the isomorphism of interpretable Hilbert spaces. Fix \( N \models T \) and take \( a \in N_j \). Find \( x \in M_j \) with \( \text{tp}(x) = \text{tp}(a) \). Suppose that \( T_M x \in \iota'M'_{j'} \).

\( T_M x \) is uniquely determined by the value \( \lambda = \langle T_M x, T_M x \rangle \) and the \( x \)-definable function \( f_x : M'_{j'} \to \mathbb{R}, y \mapsto \langle T_M x, \iota'y \rangle \). By elementarity, for every \( n \), there is \( b_n \in N_{j'} \) such that \( \langle \iota'b_n, \iota'b_n \rangle - \lambda | < 2^{-n} \) and for all \( y \in N_{j'} \), \( |f_a(y) - \langle \iota'b_n, \iota'y \rangle| < 2^{-n} \). Then \( \langle \iota'b_n \rangle \) is Cauchy and there is \( b \in N_{j'} \) such that \( \iota'b \) is the limit of \( \langle \iota'b_n \rangle \). Define \( T_N \iota a = \iota'b \). It is straightforward to check that this is well-defined and that this gives an isomorphism of interpretable Hilbert spaces. \( \square \)

### 2.2. Examples

We give some examples which illustrate a variety of sources of Hilbert space functors. We will revisit these examples in Section 5.3. See also Section 4 for another rich source of examples.

1. In classical logic, let \( T^\infty \) be the theory of an infinite set. Writing \( S \) for the main sort of \( T \), we define the inner product map \( f : S^2 \to \mathbb{R} \) by \( f(x, x) = 1 \) and \( f(x, y) = 0 \) if \( x \neq y \). This gives a Hilbert space functor \( \mathcal{H} \) such that for any \( M \models T^\infty, h(M) \) is an orthonormal set in \( H(M) \) with dense span.

Define also the inner product maps \( g(x, x) = 2 \) and \( g(x, y) = 1 \) if \( x \neq y \). Define also \( h(x, x) = 4 \) and \( h(x, y) = 3 \) if \( x \neq y \). These also give Hilbert space functors \( \mathcal{H}' \) and \( \mathcal{H}'' \) respectively. Observe that \( \mathcal{H}' \) and \( \mathcal{H}'' \) are isomorphic, but they are not isomorphic to \( \mathcal{H} \). One way of proving this is to note that for any \( M \models T, H(M) \) and \( H''(M) \) have an invariant vector under the action of \( \text{Aut}(M) \), but this is not true of \( H(M) \). This will be discussed further in Section 5.

2. Let \( T = Th(\mathbb{Z}, \leq) \). Define \( f(x, x) = 2, f(x, y) = 1 \) if \( x, y \) are consecutive, and \( f(x, y) = 0 \) otherwise. This gives a Hilbert space functor.

For a more complicated example with the same flavour, let \( V = \ell^2(\mathbb{Z}) \) and for \( n \in \mathbb{Z} \) let \( e^n = (2^{-|k+n|})_{k \in \mathbb{Z}} \). The sequence \( (e^n)_{n} \) generates \( \ell^2(\mathbb{Z}) \). Define the map \( h : \mathbb{Z} \to V, n \mapsto e^n \). Then for \( x, y \in \mathbb{Z}, \langle hx, hy \rangle \) depends only on the distance between \( x \) and \( y \), so the inner product is definable on \( \mathbb{Z} \) in \( Th(\mathbb{Z}, \leq) \). Then we are in the situation of Proposition 2.3 so \( h \) induces an interpretation of \( \ell^2(\mathbb{Z}) \) in \( \mathbb{Z} \).

For yet another example, let \( S \) be an arc of the circle \( S^1 \). \( S \) acts on \( L^2_{\mathbb{C}}(S) \) via \( f \mapsto z^n f \). Let \( V \) be the subspace of \( L^2_{\mathbb{C}}(S) \) generated by the orbit of 1 under \( S \). Then Proposition 2.3 shows that \( V \) is interpretable in \( S \).

3. Suppose \( T \) is a classical logic theory with a Keisler measure \( \mu \) on a sort \( X \) of \( T \). This means that for all \( M \models T, \mu \) is a finitely additive probability measure on the Boolean algebra \( \text{Def}_x(M) \) of \( M \)-definable subsets in the variable \( x \), where \( x \) ranges in \( X \). We view \( \text{Def}_x(M) \) as an algebra of subsets of the type space \( S_x(M) \). Suppose in addition that \( \mu \) is definable, in the sense that for any formula \( \phi(x, y) \) and any \( \lambda \geq 0 \), the set of \( a \in M \) such that \( \mu(\phi(x, a)) = \lambda \) is a definable set. The most notable case of a definable measure in model theory is perhaps the theory \( T \) of pseudofinite fields with the counting measure \( \mu \) defined in Chatzidakis et al., 1992.
Given $M \models T$, the measure $\mu$ on the algebra $\mathrm{Def}_s(M)$ extends to a countably additive probability measure on the $\sigma$-algebra $\mathcal{D}_s(M)$ generated by $\mathrm{Def}_s(M)$. We view $\mathcal{D}_s(M)$ as a $\sigma$-algebra over $S_s(M)$ but when $M$ is $\omega_1$-saturated we can also view $\mathcal{D}_s(M)$ as a $\sigma$-algebra of definable subsets of $M$ itself. Write $L^2(M,\mu)$ for the space of square-integrable functions on $S_s(M)$ with respect to $\mathcal{D}$ and $\mu$. Note that $L^2(M,\mu)$ is densely generated by functions of the form $\mathbf{1}_{\phi(x,a)}$.

For any formula $\phi(x,y)$, let $S_\phi$ be an imaginary sort of $T$ which is a Cartesian product of sorts of $T$ corresponding to the tuple $y$. If $\phi(x,y)$ and $\psi(x,y)$ are different formulas, we set the sorts $S_\phi$ and $S_\psi$ to be distinct, although they are copies of each other.

For $M \models T$ and for any formula $\phi(x,y)$, define the map $h_\phi : S_\phi \to L^2(M,\mu)$ by $h_\phi(a) = \mathbf{1}_{\phi(x,a)}$. By definability of the measure, we are in the setting of Proposition 2.4 and the system $(h_\phi, L^2(M,\mu))$ gives a Hilbert space functor $\mathcal{H}$ supported by the sorts $(S_\phi)$. Observe that $\mathcal{H}$ satisfies the following easy proposition:

**Proposition 2.15.** For any $N \models T$, there is a Hilbert space isomorphism $T : \mathcal{H}(N) \to L^2(N,\mu)$ such that for every formula $\phi(x,y)$, the map $T \circ h_\phi : S_\phi \to L^2(N,\mu)$ takes the element $b$ to the vector $\mathbf{1}_{\phi(x,b)}$.

By Proposition 2.15, it is natural to say that $T$ interprets the functor $L^2(\mu)$.

4. Given two Hilbert space functors $\mathcal{H}$ and $\mathcal{H}'$, we can form their sum $\mathcal{H} + \mathcal{H}'$ as follows. Say $\mathcal{H}$ is supported by $(S_i)_I$ with interpretation maps $(h_i)_I$ and $\mathcal{H}'$ is supported by $(S_j)_J$ with interpretation maps $(h_j)_J$. For every $M \models T$ we define $H + H'(M)$ as the direct orthogonal sum of $H(M)$ and $H'(M)$. The interpretation maps $g_k : M_k \to H + H'(M)$ are defined for $k \in I \cup J$. If $k \in I$, $g_k$ is the composition of $h_k$ with the inclusion $H(M) \to H + H'(M)$, and similarly if $k \in J$. We can also define the sum of infinitely many Hilbert space functors in the same way. In Section 3 we will show how to express certain Hilbert space functors as sums of simpler Hilbert space functors.

2.3. Prolonging Piecewise Interpretable Hilbert Spaces. Let $\mathcal{H}$ be a Hilbert space functor supported by the sorts $(S_i)$ and write $(f_{ij})$ for the inner product maps. The construction in Proposition 2.4 is rather heavy and gives a complicated decomposition of $H(M)$. We present an alternative construction which avoids these technical difficulties. We will recover the usual presentation of Hilbert spaces in continuous logic. Under this construction, the balls $B(0,n)$ in $H(M)$ become subsets of Hilbert space balls which we add to the theory $T$ as new sorts. This construction helps simplify the discussion of interpretable Hilbert spaces and easily yields basic results about model-theoretic independence in interpretable Hilbert spaces.

Recall that $T$ is in the language $\mathcal{L}$. We define an extension $T'$ of the theory $T$ in a language $\mathcal{L}'$ as follows. We add to $\mathcal{L}$ all the sorts and functions which were used in the presentation of Hilbert spaces in continuous logic from Section 1.3 and $T'$ says that these new sorts form an infinite dimensional Hilbert space. For each $i \in I$, we also add to our language a map $h_i$ from $S_i$ to one of the Hilbert space balls with radius greater than $\sup \sqrt{f_{ii}(x,x)}$. For $i,j \in I$, $T'$ contains the additional axioms

$$\forall x,y \in S_i, f_{ij}(x,y) = \langle h_i(x), h_j(y) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the ball which $h_i$ maps into. We also add axioms saying that the orthogonal complement of $\bigcup h_i(M_i)$ is infinite dimensional. We will refer to the maps $(h_i)$ as the interpretation maps as before.

Any model of $T'$ has the form $(M,H)$ where $M \models T$ and where $H$ is an infinite dimensional Hilbert space containing $H(M)$ as a subspace in the obvious way. It follows that any model
of \( T' \) is uniquely determined by its \( T \)-part and the dimension of the orthogonal complement of \( H(M) \) in the Hilbert space ball. One deduces that \( T' \) is a complete theory by applying standard saturation arguments.

We now show that adding \( H \) does not add any extra structure on \( M \). Proposition 2.16 also proves the intuitive result that the imaginary sorts which enter into the direct limit of a piecewise interpretable Hilbert space provide all the imaginaries that are needed to obtain parameters for \( T' \)-definable functions between sorts of \( T \).

**Proposition 2.16.** \( T \) is stably embedded in \( T' \). Moreover, \( T \) is fully embedded in \( T' \) in the sense that for any \((M,H) 
\models T' \) and any function \( f \) between sorts of \( T \) definable in \( L' \) over \((M,H) \) is definable in \( L \) over \( M \).

Moreover, suppose that \( H \) is the direct limit of the imaginary sorts \( (S_j) \), that the interpretation maps on each \( S_j \) are injective, and that the sorts \( (S_j) \) are real sorts of \( T \). For any \((M,H) 
\models T' \) and \( C \subseteq (M,H) \), if \( f \) is a \( C \)-definable function between sorts of \( T \) in \( L' \), then \( f \) is \( \text{dcl}(C) \cap M \)-definable in \( L \).

**Proof.** First let \((M,H) 
\models T' \) be \( \kappa \)-saturated with \(|(M,H)| = \kappa \). We show that (4) from Lemma 1.13 holds. Let \( \alpha \) be an automorphism of \( M \). By the GNS-theorem, \( \alpha \) induces an automorphism of \( H(M) \). Define \( \beta : (M,H) \to (M,H) \) as follows: on \( M, \beta = \alpha, \) on \( H(M), \) \( \beta \) is the induced Hilbert space isomorphism, and on \( H(M)^\perp, \) \( \beta \) is the identity. \( \beta \) respects all the basic relations in \( L' \) so \( \beta \) is an isomorphism and \( T \) is stably embedded in \( T' \).

To prove that \( T \) is fully embedded in \( T' \), it is enough to prove the second part of the proposition where we assume that the imaginary sorts \( (S_j) \) are part of \( T \). Let \((M,H) \) be any model of \( T' \). Let \( C \subseteq (M,H) \) and let \( f \) be a \( C \)-definable function into \( \mathbb{R} \) on a sort of \( T \). We can assume that \( f(x) = g(x,C) \) where \( g \) is \( 0 \)-definable and \( C \) is a finite tuple of \((M,H) \). We can also write \( C = ab \) where \( a \subseteq M \) and \( b \subseteq H \). Finally, we can assume that \( b = b_0b_1 \) where \( b_0 \subseteq H(M) \) and \( b_1 \subseteq H(M)^\perp \), by expressing elements of \( b \) as sums of elements of \( b_0 \) and \( b_1 \) and by noting that \( b_0b_1 \) and \( b \) are inter-definable.

Let \((N,H) \) be any elementary extension of \((M,H) \). We saw above that we can construct an automorphism of \((N,H) \) by taking any automorphism of \( N \) and extending it by any unitary automorphism of \( H(N)^\perp \). It follows that \( \text{tp}(b_1/M) \) is completely determined by the values of the inner product between elements of \( b_1 \), and in particular \( \text{tp}(b_1/M) \) is the unique extension to \( M \) of \( \text{tp}(b_1) \). It follows immediately that \( f \) is definable over \( ab_0 \).

Since we have assumed that the sorts \( (S_j) \) are part of \( T \), each element of \( b_0 \) is inter-definable with an element of \( M \). Hence we have proved that \( f \) is definable over \( \text{dcl}(C) \cap M \) in \( L' \).

In order to show that \( f \) is definable over \( \text{dcl}(C) \cap M \) in \( L \), it is enough to note that \( S_C^L(C') \cong S_{C'}^L(C') \) via the restriction map for any \( C' \subseteq M \), where \( S_C^L(C') \) is the type-space in the sort of \( f \) over \( C' \) in the language \( L \) and \( S_{C'}^L(C') \) is the corresponding type-space in \( L' \). This is easily seen by noting that in a sufficiently saturated extension \( N \) of \( M \), if \( a,b \in M \) are conjugate over \( C' \), then they are conjugate in any \((N,H) \models T' \).

Finally, we remark that so far we have only worked with functions \( f \) from sorts of \( T \) into \( \mathbb{R} \). To deduce the proposition for any definable function between the sorts of \( T \), note that the result for real functions entails that for any sorts \( X,X' \) of \( T, S_{X \times X'}^L(\text{dcl}(C) \cap M) \cong S_{X \times X'}^C(C) \). Since a function between sorts of \( T \) is defined via its graph in the appropriate type-space, the proposition follows.

**Forking Independence in Interpretable Hilbert Spaces -** This section aims to prove Proposition 2.19 which will be essential in the remainder of this paper. Proposition 2.19 says that we can use stability of the inner product maps and stable embeddedness of \( T \) in \( T' \) to obtain full forking independence in the sense of Lemma 1.16 inside interpretable Hilbert spaces.
We begin with a general lemma. Recall that, throughout this paper, ‘forking’ is always meant with respect to stable definable functions.

**Lemma 2.17.** Let $\mathcal{L}_0$ be a language and $\mathcal{L}$ an extension of $\mathcal{L}_0$, possibly with new sorts. Let $T$ be a complete $\mathcal{L}$-theory and $T_0$ the reduct of $T$ to $\mathcal{L}_0$. Suppose that $T_0$ has weak elimination of imaginaries.

Let $(M, N) \models T$ where $M \models T_0$ and $N$ denotes the new sorts in $\mathcal{L}$ added to $\mathcal{L}_0$. Take $A \subseteq B \subseteq (M, N)$ such that $A = \text{bdd}_{\mathcal{L}}(A)$. Let $p$ be a type over $B$ in a fragment of $\mathcal{L}_0$-formulas which does not fork over $A$ in the theory $T$. Then $p$ does not fork over $A \cap M$ in the theory $T_0$.

**Proof.** Let $q$ be a non-forking extension of $p$ to $M$ in $\mathcal{L}_0$. We know that $q$ is $\mathcal{L}$-definable over $A$. The usual proof of the theorem which says that stable partial types over models are definable tells us that $q$ is $\mathcal{L}_0$-definable over $M$. Let $f(x, y)$ be a stable $\mathcal{L}_0$-definable function.

Let $\alpha$ be a canonical parameter of $d_qf(y)$, viewed as an $\mathcal{L}_0$-$M$-definable function, so that $\alpha$ is an imaginary element of $M$. Working in $(M, N)^{eq}$, since $q$ is $\mathcal{L}$-definable over $A$, $\alpha$ is in the $\mathcal{L}$-definable closure of $A$. By weak elimination of imaginaries in $T_0$, $\alpha$ is in the $\mathcal{L}_0$-definable closure of some $B \subseteq \text{bdd}_{\mathcal{L}_0}(\alpha)$, so $d_qf(y)$ is $B$-definable in $\mathcal{L}_0$. Since adding imaginaries does not affect the definable or bounded closure, in $(M, N)$ we have $B \subseteq \text{bdd}_{\mathcal{L}}(A) \cap M = A \cap M$ and $q$ is $\mathcal{L}_0$-definable over $A \cap M$. \hfill $\blacksquare$

Let $\mathcal{H}$ be a Hilbert space functor supported by the sorts $(S_i)$. For $M \models T$, we write $h_i : S_i \rightarrow H(M)$ for the interpretation map. Let $(M, H) \models T'$.

**Definition 2.18.** For $A \subseteq (M, H)$, write $A \cap H$ for the union of the sets $A \cap B(0, n)$ where the $B(0, n)$ are the Hilbert sorts in $T'$. Write $A \cap H(M)$ for the union of the sets $A \cap B(0, n) \cap H(M)$ where $H(M)$ is viewed as a subspace of $H$ in the obvious way.

Write $P_A$ for the orthogonal projection onto $A \cap H$.

It is clear that if $A \subseteq (M, H)$ then $\text{bdd}(A) \cap H$ is a bdd-closed subset of $H$ viewed as a model of the theory of Hilbert spaces.

In the next proposition, we express two facts. Firstly, adding the Hilbert space balls or considering the direct limit structure of $\mathcal{H}$ provides enough imaginaries to obtain forking independence over bdd-closed sets with respect to the inner product maps. Secondly, forking independence with respect to the inner product maps in the full structures $T$ or $T'$ results in full forking independence in the Hilbert spaces in the sense of Section 1.3.

**Proposition 2.19.** Let $T$ be a theory with a Hilbert space functor $\mathcal{H}$ supported by the sorts $(S_i)$. Let $M \models T$ and $(M, H) \models T'$. Suppose $A \subseteq M$ is contained in the sorts $(S_i)$ and let $B \subseteq C \subseteq (M, H)$ be bdd-closed in the sense of $T'$. Write $h(A)$ for the result of mapping $A$ into the Hilbert space balls under the appropriate interpretation maps $h_i$.

Then $\text{tp}(A/B)$ is definable over $B$ at the functions $\langle hx, y \rangle$ where $x$ ranges in the sorts $S_i$ and $y$ is in a Hilbert space ball. If $A \downarrow_B C$ with respect to the maps $\langle hx, y \rangle$, then $h(A) \downarrow_{B \cap H} C \cap H$ and $h(A) \downarrow_{B \cap H(M)} C \cap H(M)$ in the theory of Hilbert spaces.

Suppose that $\mathcal{H}$ is the direct limit of the imaginary sorts $(S_j)_I$ and that the interpretation maps on each $S_j$ are injective. Suppose that the sorts $(S_i)$ are real sorts of $T$. Let $A \subseteq M$ as before and let $B \subseteq C \subseteq M$ be bdd-closed in the sense of $T$. Then $\text{tp}(A/B)$ is definable over $B$ with respect to the inner product maps $f_{ij}(x, y)$. If $A \downarrow_B C$ with respect to the inner product maps, then $h(A) \downarrow_{B \cap H(M)} C \cap H(M)$ in the theory of Hilbert spaces.

Let $(a_n)$ be a Morley sequence over $B$ in a sort $S_j$ with respect to the inner product maps $(f_{ij})$. Then $(ha_n)$ converges weakly to $P_B(hb_0)$.
Lemma 3.2. Let $\langle M,H \rangle \models T'$ and take $A,B,C$ as above, with $B,C$ bdd-closed in the sense of $T'$. In the theory of Hilbert spaces, we know that $\text{tp}(h(A)/B \cap H)$ is definable over the set $P_B A \subseteq B$. Therefore, in $T'$, $\text{tp}(A/B)$ is definable over $B$ at the maps $\langle h,x,y \rangle$.

Suppose that $A \nvdash_B C$ with respect to the maps $\langle h,x,y \rangle$. Then $h(A) \nvdash_B C$ with respect to the maps $\langle x,y \rangle$ because $h(A) \subseteq \text{dcl}(A)$. Hilbert spaces have weak elimination of imaginaries so Lemma 2.17 applies and we have $h(A) \nvdash_{B \cap H} C \cap H$ in the theory of Hilbert spaces. Now $h(A) \nvdash_{B \cap H(M)} C \cap H(M)$ follows by the characterisation of forking independence in Hilbert spaces and because $h(A) \subseteq H(M)$.

We now work in $T$ and we suppose that the sorts $(S_j)$ are real sorts of $T$. Let $M \models T$ and choose any extension $(M,H) \models T'$. By stable embeddedness, $\text{tp}(A/B)$ extends uniquely to $\text{bdd}_H(B)$. By the previous part of the proposition, $\text{tp}(A/\text{bdd}_H(B))$ is definable over $P_B A$ at the maps $\langle h,x,y \rangle$. Choose elements $B'$ of the sorts $S_j$ which map to the set $P_B A$, so that $B' \subseteq B \cap M$. It is clear that $\text{tp}(A/B)$ is definable over $B'$ at the inner product maps $f_{ij}$. The statement about $A \nvdash_B C$ follows in a similar manner. The statement about Morley sequences follows from Lemma 1.18. □

3. Structure Theorems for Scattered Interpretable Hilbert Spaces

In this section, we give our structure theorems for interpretable Hilbert spaces. These reduce the notion of scatteredness to the stronger notion of asymptotic freedom; see the definitions below. The main structure theorem is 3.14. In section 3.2 we show how Theorem 3.14 can be strengthened when we are working in a classical logic theory with definable functions in the sense of classical logic. This is Theorem 3.18. In Section 3.3 we give concrete examples of the decomposition promised by Theorems 3.14 and 3.18.

Fix a continuous logic theory $T$ and take $\mathcal{H}$ a Hilbert space functor with interpretation maps $(h_i)_i$ supported by the sorts $(S_i)_i$. For $i,j \in I$, write $f_{ij}$ for the inner product map $S_i \times S_j \to \mathbb{R}$, $(x,y) \mapsto \langle h_i x, h_j y \rangle$. We will work with the theory $T'$ for convenience. At no point does the orthogonal complement of $H(M)$ in $H$ become relevant in all the models we consider. For $(M,H) \models T'$ and for $A$ an arbitrary subset of $(M,H)$, we write $P_A$ for the orthogonal projection to the subspace $A \cap H(M)$. We previously wrote $P_A$ for the orthogonal projection to $A \cap H$, but a close inspection of the proofs to follow will show that $P_{A \cap H} = P_{A \cap H(M)}$ on all the domains we consider.

Definition 3.1. Let $M \models T$ and let $p$ be a partial type in some sort $S_i$, $i \in I$. We define $\mathcal{P}(p)$ in $M$ to be the closure of $h_ip$ in $H(M)$ in the weak topology.

The notation ‘$\mathcal{P}(p)$’ stands for the partial order which we will define below.

Lemma 3.2. Let $(M,H) \models T'$ and let $p$ be a partial type in some sort $S_i$. Suppose that $h_i$ maps $S_i$ to the ball with radius $n$ in $H$. Then $\mathcal{P}(p)$ is type-definable in $B(0,n)$. When $M$ is $\omega$-saturated, $\mathcal{P}(p)$ is equal to the set of weak limit points of $h_ip$ in $H(M)$.

When $M$ is $\omega_1$-saturated, $\mathcal{P}(p)$ is equal to the set $\{P_{\text{bdd}(A)}(h,b) \mid b \models p, A \subseteq M\}$ and is closed under the maps $P_{\text{bdd}(A)}$ for arbitrary $A \subseteq (M,H)$.

Proof. Recall that the weak topology has a basis of open sets of the form $\bigcap_{i=1}^n \{v \mid \langle v, v_i \rangle \in U_i\}$ where $v_i \in H$ and $U_i$ is an open interval in $\mathbb{R}$. It follows that the closure of $h_ip$ in the weak topology is the type-definable set given by the conjunctions of all formulas

$$\forall v_1, \ldots, \forall v_n \in H, \exists x \in S_i, (\phi(x) \land \bigwedge_{j=1}^n |\langle h_ix, v_j \rangle - \langle v, v_j \rangle| \leq \epsilon)$$

where $\phi$ is a formula in $p$, $n \geq 1$ and $\epsilon > 0$. 


Suppose $M$ is $\omega$-saturated and take $w$ in $\mathcal{P}(p)$. Find $a_0 \models p$ such that $\langle h, a_0, w \rangle = \langle w, w \rangle$. Given $a_0, \ldots, a_n$ find $a_{n+1} \models p$ such that $\langle h, a_{n+1}, w \rangle = \langle h, a_j, w \rangle$ and for all $j \leq n$ \( \langle h, a_{n+1}, h, a_j \rangle = \langle w, ha_i \rangle \). By Lemma 1.17 \( \langle h, a_n \rangle \rightarrow w \) so $\mathcal{P}(p)$ is the set of weak limit points of $h_p$.

Now suppose $M$ is $\omega_1$-saturated. With $w$ and $(a_i)$ as constructed above, write $A = \text{bdd}(a_n \mid n \geq 0)$. By saturation, we find a Morley sequence $(h_n)$ over $A$ in $p$ such that $(h_n h_n) \rightarrow w$. Therefore, $w = P_A^* h_n b_0 \in \mathcal{P}(p)$. The converse inclusion follows from Proposition 2.19.

Finally, take $A \subseteq M$ and $v \in \mathcal{P}(p)$. To show that $P_{\text{bdd}(A)}v \in \mathcal{P}(p)$, we can assume that $A$ is separable. Let $(a_n)$ be a sequence in $p$ such that $(h \langle a_n \rangle)$ converges weakly to $v$. Consider the following partial type in $x$ over $A$:

\[
\{\langle hx, w \rangle - \langle v, w \rangle \leq \epsilon \mid \epsilon > 0, w \in \text{bdd}(A) \cap H\}
\]

This is finitely satisfiable in $(a_n)$, so by saturation we can find a realisation $b$ in $p$. It follows that $P_A h b = P_A v$, so $P_A v \in \mathcal{P}(p)$.

**Definition 3.3.** Let $M \models T$ be $\omega_1$-saturated and let $p$ be a partial type in $S_i$. We define the partial order $\leq$ on $\mathcal{P}(p)$ as follows: we say that $v \leq w$ in $\mathcal{P}(p)$ is there is a finite sequence of $\text{bdd}$-closed subsets $A_1, \ldots, A_n$ of $M$ such that $v = P_{A_n} \ldots P_{A_1} w$.

We give a model theoretic characterisation of the set $\mathcal{P}(p)$, which will be useful when we come to study strictly interpretable Hilbert spaces. See Section 6 for a model theoretic characterisation of the partial order on $\mathcal{P}(p)$.

**Lemma 3.4.** Let $M \models T$ be $\omega_1$-saturated and let $p$ be a partial type in a sort $S_i$. Then there is a type-definable set $p^+$ in an imaginary sort of $T$ and a definable bijection $h^+: p^+ \rightarrow \mathcal{P}(p)$. $p^+$ is in fact the set of canonical bases of $f_i$-types over $M$ consistent with $p$, where $f_i(x, y) = \langle h_i x, h_i y \rangle$ on $S_i$.

**Proof.** Write $S = S_i$ and $f = f_{ii}$ and write $S_f(M)$ for the space of $f$-types over $M$. If $g_1(x), \ldots, g_n(x)$ are definable functions over $M$, we write $\text{Med}_i(g_i(x))$ for the median value of $g_1(x), \ldots, g_n(x)$. $\text{Med}_i(g_i)$ is a definable function.

Recall from elementary stability theory that for all $\epsilon > 0$ there is an integer $N_\epsilon$ such that for every $q \in S^q_f(M)$ there is an $N_\epsilon$-tuple $\overline{c}_q^\epsilon$ such that for all $y \geq n \cdot f(c_i, y) \leq \epsilon$. Write $d_q f(y, \overline{z})$ for the function $\text{Med}_{i \leq N_\epsilon} f(x_i, y)$. It follows that $d_q f$ is the uniform limit of the functions $d^{2^{-n}} f(y, \overline{c}_q^\epsilon)$ (see Lemma 7.4 in [Ben Yaacov and Usvyatsov, 2010] for more details).

Let $S^+$ be the imaginary sort of canonical parameters of the uniform limit of $(d^{2^{-n}} f(y, \overline{z}))$. Write $df(y, z)$ for this uniform limit where $z$ is in $S^+$. Let $p^+$ be the set of points in $S^+$ which are the canonical parameters of $q \in S_f(M)$ consistent with $p$. $p^+$ is type-definable, as given by the $z$-formulas:

\[
\{\forall y_1, \ldots, y_n \in S, \exists x \in S, \phi(x) \land \bigwedge_{i=1}^n f(x, y_i) = df(y_i, z) \mid n \geq 1, \phi \in p\}
\]

Let $a \models p^+$ and take $q \in S^q_f(M)$ the $f$-type defined by $a$. In a saturated elementary extension $N$ of $M$, find $b \models p \cup q$. We define $h^+ a = P_N h b$. Observe that this only depends on $a$. We show that the graph of $h^+$ is type-definable.

Observe that $q$ is definable over some small subset $A$ of $M$ and, since $M$ is $\omega_1$-saturated, we can find a Morley sequence $(c_n)$ in $(q \upharpoonright A) \cup p$ in $M$. Then $q$ is definable over $(c_n)$ and in fact for all $\epsilon > 0$, $\overline{c}_q$ can be taken to be any subset of $(c_n)$ of the size $N_\epsilon$. Then $P_N h b$ is the
weak limit of \((hc_n)\) and \((P_M hb, P_M hb) = f(c_1, c_0)\). Conversely, let \((c_n)\) be an \(f\)-indiscernible sequence in \(p\) such that for all \(\epsilon > 0\) \((c_n)\) gives the parameter for \(d^\epsilon_f\). Then for all \(d \in S\):

\[
(P_M hb, hd) = d^\epsilon_f(hd) = df(hd, (c_n)) = \lim_{\epsilon \to 0} \text{Med}_{i \leq N} f(c_i, d) = \lim_{n \to \infty} f(c_i, d)
\]

It follows that \((hc_n)\) converges weakly to \(P_M hb\) and \((P_M hb, P_M hb) = f(c_1, c_0)\). This shows that we can define the graph of \(h^+\) to be the set of pairs \((z, v)\) satisfying:

There exists an infinite indiscernible sequence \((x_n)\) in \(p\) such that for all \(n\) and all \(y \in S\),

\[
|df(y, z) - d^{2^{-n}} f(y, (x_n))| \leq 2^{-n} \quad \text{and} \quad \langle v, w \rangle = f(x_1, x_0) \quad \text{and, for all} \quad y \in S, \ \langle df(y, z) = \langle v, hy \rangle.
\]

This is a type-definable set.

We now argue that \(h^+\) is a bijection \(p^+ \to P(p)\). \(h^+\) is injective because \(S^+\) is a sort of canonical parameters of \(df\) and, for all \(a \models p^+, h^+a\) is also a canonical parameter for \(df\), as can be seen from the definition of \(h^+\). For surjectivity, take \(v \in P(p)\). Lemma 3.2 shows that \(v\) is the weak limit of some sequence \((hb_n)\) where \((b_n)\) is an \(f\)-indiscernible sequence in \(p\). Then \((b_n)\) defines a global \(f\)-type consistent with \(p\) and we can find some \(a \models p^+\) such that \(v = h^+a\). □

We will study \(P(p)\) under the assumption of scatteredness, introduced below. Under this assumption, the partial order on \(P(p)\) will be definable and \(P(p)\) will be well-founded as a partial order. This will eventually give useful structural information about \(H\).

**Definition 3.5.** Let \(p\) be a partial type in some sort \(S_i\). We say that the map \(h_i: S_i \to H\) is scattered on \(p\) if \(P(p)\) is locally compact in \(M\).

Note that this definition takes place inside an \(\omega_1\)-saturated model. While scatteredness is the most general case we will consider, the following stronger condition is of special interest in some model-theoretic situations:

**Definition 3.6.** Let \(p, q\) be partial types in the sorts \(S_i, S_j\), \(i, j \in I\). We say that the inner product map \(f_{ij}\) is strictly definable on \(p \times q\) if \(f_{ij}\) has finite range on \(p \times q\).

**Lemma 3.7.** Let \(p\) be a partial type in some sort \(S_i\). If the inner product map \(f_{ii}\) is strictly definable on \(p \times p\) then the interpretation map \(h_i\) is scattered on \(p\).

**Proof.** Let \(v, w \in P(p)\). We showed in Lemma 3.2 that \(P(p)\) is the set of weak limit points of \(h_i(p)\) so there are sequences \((a_n)\) and \((b_n)\) in \(p\) such that \(h_i a_n \to v\) and \(h_i b_n \to w\). Then \(\langle v, w \rangle = \lim_n \lim_m f_{ii}(a_n, b_m)\). By stability of the inner product map and strict definability, \(f_{ii}(a_n, a_m)\) must be eventually constant. Therefore \(\langle v, w \rangle\) is one of the finitely many values already achieved by \(f_{ii}\) on \(p \times p\). It follows that \(P(p)\) is a discrete set and hence it is locally compact. □

Finally, we note that saying that \(h_i\) is scattered on \(p\) is strictly weaker than saying that the set \(h_i\) is locally compact in \(H(M)\), for \(M\) an arbitrary model. Consider the following example. Let \(T\) be a two sorted structure \((S_1, S_2)\) where the sort \(S_1\) is an infinite set with the discrete metric and \(S_2\) is the surface of the unit ball in an infinite dimensional Hilbert space. We add the inner product map on \(S_2\), we add a function \(f: S_1 \to S_2\), and we say that \(f\) has dense image and that every fiber of \(f\) is infinite. Define the positive-definite map \(b(x, y)\) on \(S_1 \times S_1\) by saying \(b(x, x) = 2\) and \(b(x, y) = \langle f(x), f(y) \rangle\). Let \(H\) be the interpretable Hilbert space supported by \(S_1\) and write \(h\) for the interpretation map on \(S_1\).

Then for any \(\omega_1\)-saturated \(M \models T\) \(h(S_1)\) is locally compact in \(H(M)\) but \(P(p)\) contains the surface of the unit ball in \(H(M)\) and hence \(h\) is not scattered.
3.1. Decomposition into \(\wedge\)-interpretable subspaces. Until the end of Section 3.1 we make the following assumptions and notational conventions. We fix an \(\omega_1\)-saturated model \(M\) of \(T\) and we fix a type-definable set \(p\) in one of the sorts \(S_i\) supporting \(H\). We assume that \(h_i\) is scattered on \(p\). To make notation lighter, we write \(h\) instead of \(h_i\), \(S\) instead of \(S_i\) and \(f\) instead of \(f_{\alpha_i}\). Write \(H(p)\) for the \(\wedge\)-interpretable subspace of \(H\) generated by the set \(h(p)\).

The next theorem is the basic fact which shows that it is interesting to look at \(P(p)\) as a partial order. It shows that types over \(\text{bdd}\)-closed subsets of \(M\) contained in \(p\) are one-based in a restricted sense (see the discussion in Section 6).

**Theorem 3.8.** Let \(A, B\) be small subsets of \(M\) such that \(A = \text{bdd}(A)\) and \(B = \text{bdd}(B)\). Then \(A \cap H(p)\) and \(B \cap H(p)\) are orthogonal over \(A \cap B \cap H(p)\). In other words, for any \(v \in A \cap H(p)\), we have \(P_B v = P_{A \cap B} v\). Equivalently, for any \(v \in H(p)\),

\[
P_B P_A(v) = P_A P_B(v) = P_{A \cap B}(v)
\]

**Proof.** It is enough to check the lemma for \(v = hx\) for arbitrary \(x \models p\). Define \(x_0 = P_A(v)\), \(y_n = P_B(x_n)\) and \(x_{n+1} = P_A(y_n)\). It is well-known that the sequences \((x_n)\) and \((y_n)\) are convergent to \(w = P_{A \cap B} v \in P(p)\). See Theorem 13.7 in [Von Neumann, 1950] for more details.

Suppose for a contradiction that for all \(n\), \(x_n\) and \(y_n\) are distinct from \(w\). Then \(y_n \not\in A\) and \(x_n \not\in B\). By \(\omega_1\)-saturation, for every \(n\) we can find indiscernible sequences \((x^k_n)\) and \((y^k_n)\) such that \((x^k_n)\) is a sequence in \(\text{tp}(x_n/B)\) converging weakly to \(y_n\) and similarly for \((y^k_n)\). Then for any \(\epsilon > 0\) there is \(n \geq 0\) such that the sequence \((x^k_n)\) is within distance \(\epsilon\) of \(w\). Since we are assuming that \(P(p)\) is locally compact, this is a contradiction and \((x_n), (y_n)\) are eventually constant equal to \(w\).

Take \(n \geq 1\) such that \(y_n \in A \cap B\). We now show that \(x_n \in A \cap B\). Write \(x_n = y_n + \alpha\) where \(\alpha \perp B\) and \(y_{n-1} = x_n + \beta\) where \(\beta \perp A\). We have

\[
\langle \alpha, y_{n-1} \rangle = 0 = \langle \alpha, y_n + \alpha + \beta \rangle = \langle \alpha, \alpha \rangle + \langle \alpha, y_n \rangle + \langle \alpha, \beta \rangle = \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle = \langle \alpha, \alpha \rangle + \langle x_n - y_n, \beta \rangle = \langle \alpha, \alpha \rangle \text{ since } x_n - y_n \in A.
\]

So \(\alpha = 0, x_n = y_n\) and \(x_n \in A \cap B\). We use a similar calculation to show that \(y_{n-1} \in A \cap B\) when \(x_n \in A \cap B\), with \(n \geq 1\). This proves by induction that \(y_0 \in A\). Hence \(P_B P_A v = P_{A \cap B} v\) and the theorem is proved. \(\square\)

**Lemma 3.9.** For every \(v \in P(p)\), \(\{w \in P(p) \mid w \leq v\}\) is uniformly type-definable over \(v\).

**Proof.** Take \(v \in P(p)\) and define \(r(z, v)\) as the partial type which says that there is an infinite sequence \((z_n)_{n \geq 1}\) in \(P(p)\) such that

1. Setting \(z_0 = v, (z_n)_{n \geq 0}\) is an indiscernible sequence in the sense of Hilbert spaces
2. \(z_n\) converges weakly to \(z\); for \(n \geq 0\), \(\langle z, z_n \rangle = \langle z_1, z_0 \rangle = \langle z, z \rangle\).

Then \(r(z, v)\) defines \(\{w \models P(p) \mid w \leq v\}\) by Lemma \(3.2\) and Theorem \(3.8\). \(\square\)

It follows from Theorem \(3.8\) and Lemma \(3.4\) that for every \(v \in P(p)\) the set \(\{w \in P(p) \mid w \leq v\}\) is metrically compact. We can now deduce the following essential lemma:

**Lemma 3.10.** \(P(p)\) is well-founded with respect to the partial order from Definition \(3.1\).
Suppose $(x_n)$ is an infinite strictly decreasing sequence in $\mathcal{P}(p)$. By Theorem 3.8, we can write $x_n = P_{V_n}x_0$ where $V_n$ is a bdd-closed subspace of $H$ and $V_{n+1} \subseteq V_n$. Since $\{y \in \mathcal{P}(p) \mid y < x_0\}$ is metrically compact, the sequence $(x_n)$ is convergent and it follows that it converges to $z := P_Vx_0$ where $V = \bigcap_n V_n$. Then $z < x_n$ for all $n$ and $x_n \notin \text{bdd}(z)$. For every $\epsilon > 0$ we can find $n$ such that $\|x_n - z\| < \epsilon$ and we can take an infinite indiscernible sequence in $\text{tp}(x_n/z)$ which must lie in $\mathcal{P}(p)$, by Lemma 3.4. This contradicts the local compactness of $\mathcal{P}(p)$ so any decreasing sequence in $\mathcal{P}(p)$ is eventually constant. \hfill \Box

We use Lemma 3.10 to decompose $H(p)$, the Hilbert space generated by $h(p)$. Fix an enumeration $(q_\alpha)_{\alpha < \kappa}$ of the complete types in $p^+$ with the property that for any $a, b \models p^+$, if $h^+b < h^+a$ in $\mathcal{P}(p)$ then $\text{tp}(b)$ comes before $\text{tp}(a)$ in the sequence $(q_\alpha)$. For any $\alpha < \kappa$, let $V_\alpha$ be the Hilbert space spanned by the set $h^+(\bigcup_{\beta < \alpha} q_\beta)$ (set $V_0 = \{0\}$) and define $\hat{h}_\alpha : q_\alpha \to H(p)$ as $\hat{h}_\alpha(x) = P_{V_\alpha}h^+x$.

**Lemma 3.11.** Each map $\hat{h}_\alpha$ is definable on $q_\alpha$.  

Proof. We show that the map $g : q_\alpha \to H(p)$, $g(x) = P_{V_\alpha}h^+x$ is definable. Fix $x \models q_\alpha$ and write $d \geq 0$ for the distance between $h^+(x)$ and $V_\alpha$. Since $q_\alpha$ is a complete type and $V_\alpha$ is generated by a union of $\bigwedge$-interpretable Hilbert spaces, $d$ does not depend on $x$. Moreover, $g(x)$ is the unique element $v \in V_\alpha$ such that $\|h^+x - v\| = d$.

Therefore, for every $\epsilon > 0$, there is a finite $J_\epsilon \subseteq \alpha$ and $n_\epsilon \geq 0$ such that we can find $x_i \in \bigcup_{\beta \in J_\epsilon} q_\beta$ and $\lambda_i \in [-n_\epsilon, n_\epsilon]$ (i.e. $n_\epsilon$) satisfying $\|\sum_{i \leq n_\epsilon} \lambda_i h_i^+ x_i - h^+x\| \leq r + \epsilon$. $J_\epsilon$ and $n_\epsilon$ do not depend on $x$. Write $\phi_\epsilon(x, v)$ for the formula

$$\exists x_1, \ldots, x_{n_\epsilon}, \exists \lambda_1, \ldots, \lambda_{n_\epsilon}, \bigwedge_{i \leq n_\epsilon, \beta \in J_\epsilon} q_\beta(x_i)$$

$$\land \sum_{i \leq n_\epsilon} \lambda_i h_i^+ x_i - h^+x \leq r + \epsilon \land \sum_{i \leq n_\epsilon} \lambda_i h_i^+ x_i - v \leq \epsilon$$

The graph of $g$ is defined by the conjunction of all $\phi_\epsilon(x, v)$. \hfill \Box

Given the above lemma, for every $\alpha < \kappa$ we can find a complete type in an imaginary sort supporting $\mathcal{H}$ such that the interpretation map takes this type bijectively onto $\hat{h}_\alpha q_\alpha$. Renaming this type as $q_\alpha$ and this interpretation map as $\hat{h}_\alpha$, we see that we can assume the maps $\hat{h}_\alpha$ are injective. We stress that these maps $\hat{h}_\alpha$ may all be defined on complete types in different imaginary sorts.

It is easy to prove by induction that for every $\alpha \leq \kappa$ the Hilbert space generated by $\bigcup_{\beta < \alpha} \hat{h}_\beta(q_\beta)$ is equal to the Hilbert space generated by $\bigcup_{\beta < \alpha} h^+(q_\beta)$. Therefore, $H(p)$ is the orthogonal sum of the spaces generated by $\hat{h}_\alpha(q_\alpha)$. Finally, the maps $\hat{h}_\alpha$ have the important property:

**Lemma 3.12.** For every $\alpha < \kappa$, for every $x, y \in q_\alpha$, we have $\langle \hat{h}_\alpha x, \hat{h}_\alpha y \rangle = 0$ or $y \in \text{bdd}(x)$.

Proof. Suppose $x, y \in q_\alpha$ and $y \notin \text{bdd}(x)$. Let $(y_n)$ be an infinite indiscernible sequence in $\text{tp}(y/\text{bdd}(x))$. By injectivity of $\hat{h}_\alpha$, $(\hat{h}_\alpha y_n)$ is an infinite sequence in $H(p)$. Note that for all $n$

$$\langle \hat{h}_\alpha y_n, \hat{h}_\alpha x \rangle = \langle P_{V_n}h^+ y_n, P_{V_n}h^+ x \rangle = \langle h^+ y_n, P_{V_n}h^+ x \rangle = \langle w, \hat{h}_\alpha x \rangle,$$

where $w$ is the weak limit of the sequence $(h^+ y_n)$. By the choice of the enumeration $(q_\alpha)$, we know that $\langle w, \hat{h}_\alpha x \rangle = 0$ so the lemma follows. \hfill \Box

**Definition 3.13.** Suppose $q$ is a type-definable set and $g : q \to H$ is a definable map. We say that $g$ is asymptotically free if $g$ is injective and for all $x, y \in q$ we have $\langle gx, gy \rangle = 0$ or $x \in \text{bdd}(y)$. 
Note that asymptotically free maps are always scattered. We collect our results so far and repeat the assumptions we are working with:

**Theorem 3.14.** Let $\mathcal{H}$ be a Hilbert space functor for $T$ and let $p$ be a type-definable set in one of the sorts $S$ supporting $\mathcal{H}$. If $h : p \rightarrow H$ is scattered, then the $\wedge$-interpretable Hilbert space $H(p)$ generated by $h(p)$ is the orthogonal sum of $\wedge$-interpretable spaces $(H_j)_{j \in J}$ such that for all $j \in J$ there is a complete type $q_j$ in an imaginary sort of $T$ and an asymptotically free map $\tilde{h_j} : q_j \rightarrow H_j$ such that $H_j$ is generated by $\tilde{h_j}(q_j)$.

We deduce easily the following corollary when the whole Hilbert space $H$ is generated by scattered type-definable sets:

**Corollary 3.15.** Let $\mathcal{H}$ be a Hilbert space functor for $T$. Suppose that $\mathcal{H}$ is generated by $\wedge$-interpretable subspaces $(H_j)_{j \in J}$ such that each $H_j$ is generated by a scattered type-definable set. Then $H$ can be expressed as the orthogonal sum of $\wedge$-interpretable subspaces $(H_j)_{j \in J}$ such that for each $j \in J$ there is a complete type $q_j$ in an imaginary sort of $T$ and an asymptotically free map $\tilde{h_j} : q_j \rightarrow H_j$ such that $H_j$ is generated by $\tilde{h_j}(q_j)$.

**Proof.** For every $j \in J$ let $(H_j^i)_{i \in I_j}$ be a family of $\wedge$-interpretable Hilbert spaces as given by Theorem 3.14 where we write $\tilde{h_j}^i : q_j^i \rightarrow H_j^i$ for the asymptotically free map on a complete type. Let $(q_\alpha)_{\alpha < \kappa}$ be an enumeration of the types $(q_j^i)$. For $\alpha < \kappa$ define $V_\alpha$ as the Hilbert space generated by the sets $(q_\beta)_{\beta < \alpha}$ and the appropriate asymptotically free maps. For every $\alpha < \kappa$, if $q_\alpha = q_j^i$, define $h_\alpha^i x = P_{V_\alpha} h_j^i x$ for $x$ in $q_\alpha$.

As in Lemma 3.11 the maps $h_\alpha^i$ are definable on $q_\alpha$. It is clear that $H$ is the orthogonal sum of the $\wedge$-interpretable Hilbert spaces generated by each $h_\alpha^i$. Moving to another imaginary sort if necessary, we can assume that $h_\alpha^i$ is injective. We only need to check that $h_\alpha^i$ is asymptotically free.

Suppose that $q_\alpha = q_j^i$. As in Lemma 3.12 for any $x, y \in q_j^i$, $(h_\alpha^i x, h_\alpha^i y) = (h_j^i x, h_j^i y)$. Since $\tilde{h_j}^i$ is asymptotically free, it follows that $(h_\alpha^i x, h_\alpha^i y) = 0$ if $x \notin \text{bdd}(y)$ and $h_\alpha^i$ is asymptotically free. \qed

### 3.2. Strictly interpretable Hilbert spaces

In Section 3.2 we fix a Hilbert space functor $\mathcal{H}$ supported by sorts $(S_i)_{i}$. In the following results, there are no unstated assumptions on $T$ or $\mathcal{H}$. For notational simplicity, we fix a supporting sort $S$ among the $(S_i)$ with interpretation map $h$ and inner product map $f$.

In this section, we focus on the case where we are given a classical logic theory $T$, a Hilbert space functor $\mathcal{H}$ and a sort $S$ supporting $\mathcal{H}$ such that the inner product map on $S$ is strictly definable. We have already proved that strict definability entails that the interpretation map is scattered, so Theorem 3.14 applies and we have a decomposition of the interpretable Hilbert space generated by $S$ into $\wedge$-interpretable Hilbert spaces. We will show that we can sometimes gain more information about the nature of this decomposition.

First we observe that we can improve Lemma 3.4.

**Lemma 3.16.** Let $T$ be a classical logic theory and $S$ a classical sort of $T$ supporting a Hilbert space functor $\mathcal{H}$. Let $p$ be a type-definable set in $S$ and suppose that the inner product map $f$ is strictly definable on $p \times p$. Then there is a type-definable set $p^+$ in a classical imaginary sort of $T$ and a definable map $h^+ : p^+ \rightarrow H$ such that $h^+$ is bijective onto $\mathcal{P}(p)$.

**Proof.** Recall from elementary stability theory that there is a number $N$ such that for any indiscernible sequence $(x_n)$ in $p$ and $y \models p$, there is a unique $\lambda$ in the range $R$ of $f$ on $p \times p$
such that $|\{i \in \mathbb{N} \mid f(x_i, y) \neq \lambda\}| < N/2$. Then we can take $S^+$ to be $S^N/E$ where $E$ is the equivalence relation defined by

$$\forall z \in S, \text{Med}_{i \leq N} f(x_i, z) = \text{Med}_{i \leq N} f(y_i, z)$$

where $\text{Med}_{i < N} f(x_i, z)$ is the median of the set of values $\{f(x_i, z) \mid i < N\}$. We can define the graph of $h^+$ as the set of pairs $(z, v)$ in $S^+ \times H$ such that there is a sequence $(x_n)$ in $S$, possibly constant, such that

1. for all $n \geq 0$, $x_n \models p$ and $(x_n)$ is $f$-indiscernible
2. for every $k_0 < \ldots < k_N, (x_{k_0}, \ldots, x_{k_N})_E = z$
3. for all $n \geq 0$, $\langle hx_n, v \rangle = \langle h(x_1, h(x_0)) = \langle v, v \rangle$. $p^+$ is then defined as the domain of $h^+$. The verifications are left to the reader.

Now take $T$ a continuous logic theory. Observe that when the inner product map $f$ is strictly definable on a set $p$, the set of values $\{\langle v, w \rangle \mid v, w \in P(p)\}$ coincides with the set of values of $f$ on $p \times p$, so $P(p)$ is uniformly discrete. It follows from Lemma 3.9 that for any $x \models p$, the set $\{v \in P(p) \mid v \leq hx\}$ is finite. Additionally if $p$ is a complete type, then the set $p^+$ in Lemma 3.10 is a finite union of complete types.

We assume that $f$ is strictly definable on $p \times p$. For every $x \in p^+$ write $\pi(x)$ for the finite set $\{y \models p^+ \mid h^+ y < h^+ x \in \mathcal{P}(p)\}$. Write $V(x)$ for the finite dimensional subspace of $H$ spanned by $\langle h^+ y \mid y \in \pi(x) \rangle$. Define $\tilde{h}(x) = P_Y (\pi(x))$. Since $h\lambda$ is a linear combination of $\pi(x)$ and the coefficients in this linear combination only depend on the set of inner products between elements in this set, $\tilde{h}$ is strictly definable on $p^+$. By applying Lemma 2.8 we can assume that $\tilde{h}$ is injective on $p^+$.

**Lemma 3.17.** $\tilde{h}$ as defined above is asymptotically free on $p^+$.

**Proof.** Fix $x, y \in p^+$. To simplify notation, write $V = V(x)$ and $Y = \text{bdd}(y)$.

Note first that $\langle hx, \tilde{h}y \rangle = \langle P_Y hx, \tilde{h}y \rangle$, so it is enough to prove that if $x \notin Y$ then $P_Y \tilde{h}x = 0$. Since $P_Y \tilde{h}x = P_Y h^+ x - P_Y P_Y h^+ x$, it is enough to prove that $P_Y P_Y h^+ x = P_Y h^+ x$.

Write $a = h^+ x$. We show that $\|P_Y P_Y a - P_Y a\|^2 = 0$. Expanding the left hand side gives $(P_Y P_Y a, P_Y P_Y a) + (P_Y a, P_Y a) = (P_Y a, P_Y a)$.

Now we have:

1. $(P_Y P_Y a, P_Y P_Y a) = (P_Y P_Y a, a)$
2. $(a, P_Y a)$
3. $(a, P_Y a)$

The lemma follows. $\square$

We have proved:

**Theorem 3.18.** Let $T$ be a continuous logic theory, let $\mathcal{H}$ be a Hilbert space functor for $T$ and let $p$ be a type-definable set in one of the sorts $S$ supporting $\mathcal{H}$. If the inner product map $f$ is strictly definable on $p \times p$, then there is a type-definable set $p^+$ in an imaginary sort of $T$ and a definable map $\tilde{h} : p^+ \to H$ such that
(1) $\hat{h}(p^+)$ generates $H(p)$
(2) the map $\langle \hat{h}x, \hat{h}y \rangle$ is strictly definable on $p^+ \times p^+$
(3) $\hat{h}$ is asymptotically free on $p^+$.

Moreover, if $T$ is a classical theory, then $p^+$ can be taken in a classical imaginary sort of $T$.

We are now interested in the situation where $T$ is a classical logic theory and the Hilbert space functor $\mathcal{H}$ is supported by possibly more than one sort on which the inner product map is definable.

Definition 3.19. Let $T$ be a classical logic theory and $M \models T$. Suppose $H(M)$ is an interpretable Hilbert space in $M$ supported by classical imaginary sorts $S_i$ of $M$ such that the inner product map on each $S_i$ is strictly definable. Then we say that $H(M)$ is strictly interpretable in $M$.

If $H'(M)$ is $\bigwedge$-interpretable in $M$ and supported by type-definable sets in classical imaginary sorts on which the inner product map is strictly definable, we say that $H'(M)$ is strictly $\bigwedge$-interpretable.

If $\mathcal{H}$ and $\mathcal{H}'$ are the functors corresponding to $H(M)$ and $H'(M)$ as above, we say that $\mathcal{H}$ is a strict Hilbert space functor and $\mathcal{H}'$ is a $\bigwedge$-strict Hilbert space functor.

Unfortunately, the proof of Corollary 3.15 does not generalise well to the general setting of strictly interpretable Hilbert spaces and we do not know if it is possible to decompose a strict Hilbert space functor into an orthogonal sum of $\bigwedge$-strict Hilbert space functors in the style Corollary 3.15.

Nevertheless, this is possible when $T$ is a classical logic $\omega$-categorical theory. This case is of special interest. If $T$ is $\omega$-categorical and $\mathcal{H}$ is supported by classical imaginary sorts of $T$, then the inner product maps are always strictly definable and we are in the context of Theorem 3.18. Moreover, an inspection of the proof of Corollary 3.15 shows that the types $q_j$ are in classical imaginary sorts of $T$ and therefore they form definable sets. It follows that the maps $\hat{h}_j$ are strictly definable. Moreover, the $\bigwedge$-interpretable Hilbert spaces $\mathcal{H}_\alpha$ are in fact interpretable.

Therefore we have the corollary:

Corollary 3.20. Let $T$ be an $\omega$-categorical classical logic theory and let $\mathcal{H}$ be a strict Hilbert space functor. Then $\mathcal{H}$ is isomorphic to an orthogonal sum of strict Hilbert space functors $(\mathcal{H}_i)_{i \in I}$ such that each $\mathcal{H}_i$ is supported by a single complete type in a classical imaginary sort of $T$ with asymptotically free interpretation map.

We will now show that it is possible to obtain a close version of Corollary 3.15 for strictly interpretable Hilbert spaces under a relatively mild assumption on $T$. In Corollary 3.24 we will show that we can decompose strict Hilbert space functors into a sum of strict Hilbert space functors generated by asymptotically free maps. This is not as satisfactory as Corollary 3.15 or Corollary 3.20 since we do not decompose $\mathcal{H}$ as an orthogonal, or even direct, sum but we believe that it might be possible to improve further upon Corollary 3.24 to obtain richer structure theorems.

We now turn to our key assumption on $T$.

Definition 3.21. In a classical logic theory $T$, a formula $\phi(x, y)$ (possibly with parameters) has the FCP if for all $n \geq 1$ there are $a_1, \ldots, a_n$ such that $\bigwedge_{i \leq n} \phi(x, a_i)$ is inconsistent but for every $l \leq n$, $\bigwedge_{i \neq l} \phi(x, a_i)$ is consistent. If $\phi$ does not have the FCP, we say $\phi$ has the NFCP.

We say that $T$ has the weak NFCP if all stable formulas of $T$ have the NFCP.

We will use the following easy lemma about NFCP formulas, which is a weak version of Theorem II.4.6 in Shelah, 1978:
Lemma 3.22. Let $T$ be a classical theory, let $M \models T$ be $\omega$-saturated. Let $\phi(x, y)$ be a formula with the NFCP over $A \subseteq M$. There is $n \in \mathbb{N}$ such that, for all $n \leq \alpha < \omega$, any sequence $(a_i)_{i<\alpha}$ such that $\models \phi(a_i, a_j)$ for all $i \neq j$ can be extended to a sequence $(a_i)_{i<\omega}$ with the same property.

Proof. Take $n$ as given by the definition of NFCP for $\phi(x, y)$. Given $(a_i)_{i<\alpha}$, the partial type $\{\phi(x, a_i) \mid i < \alpha\}$ is $n$-consistent, so it is consistent. Take $a_\alpha$ a realisation of this partial type. \hfill \square

We now prove an analogue of Lemma 3.23 in our present context.

Lemma 3.23. Suppose that $T$ is a classical theory, $\mathcal{H}$ is a Hilbert space functor and that $S$ is a classical imaginary sort of $T$ supporting $\mathcal{H}$, with inner product map $f$ and interpretation map $h$. Let $D$ be a definable set in $S$ such that the inner product map $f$ is strictly definable on $D$. There there is a definable set $D^+$ in a classical imaginary sort of $T$ and a definable map $h^+: D^+ \to H$ such that $h^+$ is bijective onto $\mathcal{P}(D)$. Moreover, for $b \models D^+$, the set $\{a \models D^+ \mid h^+a \leq h^+b \text{ in } \mathcal{P}(D)\}$ is definable uniformly over $b$.

Proof. Throughout this proof, we write $\overline{x}$ for tuples of variables and $x$ for single variables. Write $D^+$ for the type-definable set constructed in a classical imaginary sort of $T$ in Lemma 3.16. Recall that we found an integer $N$ and a definable equivalence relation $E$ such that $D^+$ can be constructed in $S^+ := S^N/E$ and we pointed out that we can define the graph of $h^+$ as the set of pairs $(z, v)$ in $S^+ \times H$ such that there is a sequence $(x_n)$ in $S$, possibly constant, such that

\begin{enumerate}
  \item for all $n \geq 0$, $(x_n)$ is $f$-indiscernible
  \item for every $k_0 < \ldots < k_N$, $(x_{k_0}, \ldots, x_{k_N})_E = z$
  \item for all $n \geq 0$, $(hx_n, v) = (hx_1, hx_0) = (v, v)$.
\end{enumerate}

We show that $D^+$ is definable. Let $R$ be the range of $f$ on $D \times D$ and for any $\lambda \in R$, write $F_\lambda(\overline{x}, \overline{y})$ for the formula on $S^N \times S^N$ which says

\begin{enumerate}
  \item $\overline{x}$, $\overline{y}$ are in $D^N$ and $E$-equivalent
  \item for all $i, j \leq N$, $\langle hx_i, hy_j \rangle = \lambda$
  \item for all $i \neq j \leq N$, $\langle hx_i, hx_j \rangle = \langle hy_i, hy_j \rangle = \lambda$
  \item for all $i, j \leq N$, $\langle hx_i, hx_j \rangle = \langle hy_j, hy_j \rangle$
\end{enumerate}

$F_\lambda(\overline{x}, \overline{y})$ is stable so by the weak NFCP there is a number $n_\lambda$ such that for all $k \geq n_\lambda$ and $\overline{y}_1, \ldots, \overline{y}_k$, if $\{F_\lambda(\overline{x}, \overline{y}_i) \mid i \leq k\}$ is $n_\lambda$-consistent then it is consistent. Take $n_0 > n_\lambda$ for all $\lambda \in R$.

Let $D_0(\overline{x})$ be the definable set in $S^N$

$$\exists \overline{y}_1, \ldots, \overline{y}_{n_0} \bigvee_{\lambda \in R} \left( \bigwedge_{i \leq n_0} F_\lambda(\overline{x}, \overline{y}_i) \land \bigwedge_{i \neq j} F_\lambda(\overline{y}_i, \overline{y}_j) \right)$$

and let $D'$ be $D_0/E$. We check that $D'$ is in fact equal to $D^+$. $D'$ contains $p^+$ because any indiscernible sequence $(x_n)$ witnessing $p^+$ can be broken down into $n_0$ $N$-tuples which witness $D_0$. Conversely, suppose $\overline{x} \models D_0$, take $\overline{b}_1, \ldots, \overline{b}_{n_0}$ as given by $D_0$ and fix $\lambda$ such that these satisfy

$$\bigwedge_{i \leq n_0} F_\lambda(\overline{x}, \overline{b}_i) \land \bigwedge_{i \neq j} F_\lambda(\overline{b}_i, \overline{b}_j)$$

By Lemma 3.22, we can construct an infinite sequence of tuples $(\overline{b}_n)$ such that $\models F_\lambda(\overline{b}_n, \overline{b}_j) \land F_\lambda(\overline{b}_i, \overline{b}_j)$ for $i \neq j$.

Now we concatenate the tuples $\overline{b}_n$ to form an infinite sequence $(c_n)$ in $D$. By construction, $(hc_n)$ is an indiscernible sequence in the pure Hilbert space. Write $v_\overline{x}$ for the weak limit of $(hc_n)$. $v_\overline{x}$ has norm $\lambda$ and for all $x \in S$, $\langle v_\overline{x}, hx \rangle = \text{Med}_{j \leq N}(h(b_j)_n, hx) = \text{Med}_{j \leq N}(ha_j, hx)$. PIECEWISE INTERPRETABLE HILBERT SPACES 31
This entails that \( v_T \) does not depend on the choice of \((c_n)\) and that it only depends on the \(E\)-class of \( T \). Hence \((\overline{[T]}_E, v_T)\) is in the graph of \( h^+ \) and \([\overline{[T]}_E] \in D^+\). \( \square \)

We now obtain a rough analogue of Corollary 3.20 by combining Lemma 3.23 and the proof of Theorem 3.18.

**Corollary 3.24.** Assume that \( T \) is a classical logic theory with the weak NFCP. If \( \mathcal{H} \) is a strict Hilbert space functor of \( T \), then \( \mathcal{H} \) is isomorphic to a strict Hilbert space functor \( \mathcal{H}' \) supported by classical imaginary sorts \((S'_i)_i\) such that the interpretation maps \( h'_i \) are asymptotically free.

**Remark:** The proof of Corollary 3.24 also applies to Hilbert space functors \( \mathcal{H} \) such that only the inner product maps of the form \( f_{ii} \) are strictly definable. We obtain a functor \( \mathcal{H}' \) such that the inner product maps \( f_{ii}' \) are also strictly definable, but we have no information about the inner product maps \( f_{ij}' \) for \( i \neq j \).

### 3.3. Some Examples and Counterexamples

The construction of the asymptotically free interpretations maps in Theorem 3.18 or Corollary 3.24 produces imaginary sorts of \( T \) on which \( h \) is defined but it may be difficult to give a simple description of these sorts. Nevertheless, it is often easy to give directly a presentation of \( \mathcal{H} \) which satisfies Corollary 3.24 or Corollary 3.20 without going via the proofs of these theorems.

1. Let \( T \) be the theory of an infinite set with main sort \( S \) and let \( \mathcal{H}_1 \) be the Hilbert space functor supported by \( S \) with inner product map \( f(x,y) = 0 \) for \( x \neq y \) and \( f(x, x) = 1 \). Then \( \mathcal{H}_1 \) satisfies trivially the conclusion of Corollary 3.20.

Let \( \mathcal{H}_n \) be the Hilbert space functor supported by \( S^n \) with inner product map \( f(\overline{x}, \overline{y}) = k \) if \( \overline{x}, \overline{y} \) share \( k \) entries, ignoring order. Then we can take the integer \( N \) from Lemma 3.23 to be equal to \( 2n + 1 \) so the factors from Corollary 3.20 are supported on a quotient of \( S^n \). Although it is tedious to describe these explicitly, one can avoid this by noting that \( \mathcal{H}_n \) is isomorphic to the orthogonal sum of \( n \) copies of \( \mathcal{H}_1 \).

Let \( \mathcal{H}'_1 \) be the Hilbert space functor support by \( S \) with inner product map \( f(x,y) = 1 \) if \( x \neq y \) and \( f(x, x) = 2 \). Fix \( M \models T \). There is a vector \( w \) in \( H'_1(M) \) such that for any sequence \( (x_n) \) in \( M \), \( h'_1(x_n) \) converges weakly to \( w \). Note that \( \{ h'_1(x_n) - w \mid n \geq 0 \} \) is an orthogonal set.

The integer \( N \) from Lemma 3.23 is equal to 3. Let \( E \) be the equivalence relation on \( S^3 \) which identifies all triples of the form \((x, y, z)\) for \( x, y, z \) distinct. In the notation of Lemma 3.23, \( D^+ \) consists of two complete types: the type \( p \) corresponding to triples \((x, x, x)\) and the algebraic type \( q \) corresponding to the singleton \((x, y, z)\) for \( x, y, z \) distinct. For any \( M \models T \), define \( h_q : q \rightarrow H'_1(M) \) by \( h(q) = w \), and \( h_p : p \rightarrow H'_1(M) \) by \( h(x) = h'_1(x) - w \). \( h_q \) and \( h_p \) satisfy the conclusion of Corollary 3.24. We find that \( \mathcal{H}'_3 \) is isomorphic to the orthogonal sum of a copy of \( \mathcal{H}_1 \) and a one-dimensional vector space.

Note that in this case we could also follow the proof of Lemma 3.23 with \( N = 2 \).

2. For \( n \geq 1 \), let \( T_n \) be the theory of the set of unordered \( n \)-tuples over an infinite set \( X \). For \( k \leq n \), \( T_n \) has predicates \( P_k(x, y) \) to say that \( x \) and \( y \) have exactly \( k \) elements in common. Let \( \mathcal{H} \) be the Hilbert space functor supported by the main sort \( S \) with inner product map \( f \) defined by \( f(x, y) = k \) if and only if \( P_k(x, y) \). Then the structure of \( \mathcal{H} \) is similar to the structure of \( \mathcal{H}_n \) from the previous example, but imaginary sorts of \( T \) are needed to give the decomposition of corollary 3.20.

3. Let \( T = Th(\mathbb{Z}, \leq) \). All three interpretable Hilbert spaces considered in Section 2.2 Example 2 already satisfy Theorem 3.13. Note that in two of those cases, the interpretation map is scattered on the unique complete type which constitutes the domain of \( T \).
4. Let $T$ be the theory of the random graph with the usual definable measure $\mu$ and let $\mathcal{H}$ be the Hilbert space functor which takes $M \models T$ to $L^2(M, \mu)$. Let $M \models T$. For every $n \geq 1$ and every subset $\eta \subseteq \{1, \ldots, n\}$, define

$$h^n_\eta : M^n \to L^2(M, \mu)$$

$$\mathcal{I} \mapsto \sum_{i \in \eta} R(x_i, y_i) \sum_{i \in \eta} \neg R(x_i, y_i)$$

Then $\mathcal{H}$ is supported by the sorts $(S^n)$ with interpretation maps $(h^n_\eta)$. Going through the proof of Corollary 3.15 naturally leads us (with some amount of guess-work) to the maps

$$\tilde{h}^n_\eta : M^n \to L^2(M, \mu)$$

$$\mathcal{I} \mapsto (-1)^{|\{i \leq n \mid R(x_i, y_i)\}|}$$

After quotienting out by an equivalence relation to ensure these maps are injective, it is easy to check that these maps are asymptotically free and that they generate $\mathcal{H}$.

Let $(\chi_i)_{i \leq n}$ be a set of irreducible unitary characters of $G/N$ such that the set $\{\chi_i \circ \alpha \mid \alpha \in \Sigma, i \leq n\}$ is the complete list of irreducible unitary characters of $G/N$.

5. We give an example which shows the failure of Corollary 3.15 when some stable formula of $T$ has the FCP. Let $T$ be the classical logic theory of an equivalence relation $E$ such that, for each $n$, $E$ has exactly one equivalence class of cardinality $n$. Define a positive definite function by $f(x, x) = 2$ for all $x$, $f(x, y) = 1$ if $x \neq y$ and $xEy$ and $f(x, y) = 0$ if $\neg xEy$. We view $T$ as a continuous logic theory in the usual way. Write $\mathcal{H}$ for the induced Hilbert space functor. Let $M \models T$ be $\omega_1$-saturated and write $h : M \to H(M)$ for the interpretation map.

Let $p$ be the type of an element whose $E$-class is infinite. The partial order $\mathcal{P}(p)$ consists of the set $h(p)$ together with the weak limit point of each infinite $E$-class. Write $S_E$ for the quotient of $S$ by $E$ and write $p/E$ for the quotient of $p$ by $E$. Using the notation of Theorem 3.24, we can view $p \cup p/E$ as $p^\perp$ and we take $h$ as mapping $p \cup p/E$ to an orthonormal set in $H(M)$. Observe that there is no way of extending $h$ to any definable set in $S_E$ containing $p/E$ in a definable way. This strongly suggests that the conclusion of Corollary 3.15 does not hold in this case.

Furthermore, we have the following easy modification of Theorem 4.4 in [Shelah, 1978]:

**Theorem 3.25.** Let $T$ be an arbitrary classical logic theory. If $T$ does not have the weak NFCP, then there is a definable equivalence relation $E((\mathcal{F}, \mathcal{F}), (\mathcal{G}, \mathcal{G}))$ such that for all $n \geq 1$ there is a tuple $\mathcal{F}_n$ such that the formula $E((\mathcal{F}, \mathcal{F}_n), (\mathcal{G}, \mathcal{G}_n))$ is an equivalence relation with more than $n$ but only finitely many equivalence classes.

Therefore, any theory without the weak NFCP has a Hilbert space functor with the same properties as $\mathcal{H}$ defined above.

Nevertheless, with $T$ and $f$ as defined in the above example, we can define maps $g : S_E \times S_E \to \mathbb{R}$ and $g' : S_E \times S \to \mathbb{R}$ as follows: $g(x, x) = 1, g(x, y) = 0$ if $x \neq y$ and $g'(x, y) = 1$ if $[y]_E = x$ and $g'(x, y) = 0$ otherwise. Then the concatenation of the maps $f, g, g'$ is positive-definite on $S \cup S_E$ and we obtain a strictly interpretable Hilbert space functor $\mathcal{H}'$ with asymptotically free interpretation maps. For any $M \models T$, $H(M)$ is an interpretable subspace of $H'(M)$ in the obvious way. It would be interesting to know if, for arbitrary $T$ and $\mathcal{H}$, $H(M)$ is always an interpretable subspace of $H'(M)$ for some functor $\mathcal{H}'$ with asymptotically free interpretation maps.
4. Absolute Galois Groups and Associated Hilbert Spaces

In this section, we study a particular source of interpretable Hilbert spaces: the $L^2$-spaces associated to the absolute Galois group of a definably closed subset $K$ of a classical logic structure $M$.

We start by generalising the classical result of [Cherlin et al., 1980] about the co-interpretation of the absolute Galois group of a perfect field to a quite general first order setting. Assuming elimination of finite imaginaries, we show that the definable projective system associated to $\text{Gal}(K)$ is canonically interpretable in $M$ in the language of $M$ with an extra predicate $P$ for $K$. See Proposition 4.6.

We then show that $L^2(\text{Gal}(K))$ is interpretable canonically in $M$ in the language with an extra predicate $P$ for $K$ and we find a canonical asymptotically free decomposition. Using this asymptotically free decomposition, we also show that there is a sense in which $K$ can be said to interpret $L^2(\text{Gal}(K))$ in the language induced from $M$, although this interpretation is not canonical. See Proposition 4.17. Finally, we show that for arbitrary $k \geq 1$, the space of $\text{Gal}(K)$-invariant functions on $\text{Gal}(K)^k$ is canonically interpretable in $K$ in the language induced from $M$.

In this section, we work with classical logic. We fix a language $\mathcal{L}$ and a complete $\mathcal{L}$-theory $T$. We will assume that $T$ admits elimination. We assume that $T$ has elimination of finite imaginaries, which is a weakening of elimination of imaginaries:

**Definition 4.1.** $T$ admits elimination of finite imaginaries if for every finite product $S$ of sorts of $T$ and any $l \geq 1$, there is a definable set $C_{\leq l}(S)$ and a definable relation $R \subseteq C_{\leq l}(S) \times S$ such that $T$ proves that $\{R(a,y) \mid a \in C_{\leq l}(S)\}$ is precisely the set of subsets of $S$ containing at most $l$ elements.

We say that $a \in C_{\leq l}(S)$ codes the set $R(a,y)$.

We consider an enriched language $\mathcal{L}_P$ where $P$ stands for a collection of unary predicates $(P_i)$ in distinct sorts of $T$. Let $T_P$ be the $\mathcal{L}_P$-theory containing $T$ which says that $P = \text{dcl}(P)$ (i.e. if $S_1, \ldots, S_n$ and $S'$ are sorts of $T$ and $f : S_1 \times \cdots \times S_n \to S'$ is a definable function, then the image of $P_{S_1} \times \cdots \times P_{S_n}$ under $f$ is contained in $P_{S'}$).

Assume that $T$ admits elimination of finite imaginaries. Then for every finite product $S$ of sorts of $\mathcal{L}$ and $l \geq 1$, there is an $\mathcal{L}_P$-definable set $D_{\leq l}(S)$ such that for any $M \models T_P$, $D_{\leq l}(S)$ is a set of codes for the $P(M)$-definable sets contained in $S$ which contain at most $l$ elements. Namely, take $D_{\leq l}(S) = C_{\leq l}(S) \wedge P$.

Let $M \models T$ and let $K \subseteq M$. We will say that $K$ is a substructure of $M$ if $K$ is definably closed in $M$. With a choice of substructure $K$, $M$ is naturally a model of $T_P$ with $K = P(M)$. If $K$ is a substructure of $M$, we will view $K$ as an $\mathcal{L}$-structure and we will write $\text{Th}(K)$ for the complete theory of $K$ in the language $\mathcal{L}$.

When we make no special reference to $T$ or $T_P$, we will be working in $\mathcal{L}$ and viewing $K$ simply as a subset of $M$. In particular, when we write $\text{dcl}$ and $\text{acl}$, we mean the definable and algebraic closure in $M$ in the language $\mathcal{L}$.

4.1. **Interpretation of the inverse system of $\text{Gal}(K)$.** In this section, we assume that $T$ has quantifier elimination and elimination of finite imaginaries. Working with $M \models T$ and $K$ a definably closed substructure, we show that $\text{Gal}(K)$ is the inverse limit of an $\mathcal{L}_P$-definable system of finite definable groups. Although this definable inverse system is not coded in $K$, we show that $K$ codes the profinite space of $\text{Gal}(K)$-conjugacy classes of $\text{Gal}(K)^k$ for arbitrary $k \geq 1$. 
Let \( M \models T \) and let \( K \subseteq M \) be a definably closed substructure. Define \( Gal(K) \) to be the group of elementary automorphisms of \( acl(K) \) which fix \( K \) pointwise. Note that elements of \( Gal(K) \) might not extend to automorphisms \( M \). \( Gal(K) \) is a profinite group with a basis of open normal subgroups given by the family \( \text{Aut}(acl(K)/K, \phi(x, K)) \) where \( \phi(x, K) \) is the set of realisations of a complete algebraic type over \( K \). The family of groups \( \text{Aut}(\phi(x, K)/K) \) where \( \phi(x, K) \) is a complete algebraic type forms a projective system with quotient maps \( \text{Aut}(\phi(x, K)/K) \to \text{Aut}(\psi(y, K)/K) \) when \( \text{Aut}(acl(K)/K, \phi(x, K)) \) is a normal subgroup of \( \text{Aut}(acl(K)/K, \psi(y, K)) \). We say that the family of groups \( \text{Aut}(\phi(x, K)/K) \) is the inverse system of finite quotients of \( Gal(K) \).

**Remark:** This definition of \( Gal(K) \) is sensitive to the language \( L \) in several ways; in particular it concerns a quotient of the Shelah-Galois group corresponding to those sorts represented in \( L \). It yields the full Shelah-Galois group, the automorphism group of algebraic imaginary elements, when \( T \) admits full elimination of imaginaries. As there is no additional reason to assume full elimination of imaginaries, we will work with the weaker notion of elimination of finite imaginaries. For example, with \( T \) the theory of algebraically closed valued fields formulated in a single-sorted language \( L \) referring to the field sort, \( Gal(K) \) will give the field-theoretic absolute Galois group of a perfect Henselian subfield \( K \); while in a language \( L' \) with an additional sort for the residue field, a substructure \( K \) can be a Henselian subfield along with a perfect field extension of its residue field, and \( Gal(K) \) would give their combined Galois groups.

We want to show that under the assumptions on \( T \) mentioned above, \( M \) interprets in \( T_P \) the inverse system of finite quotients of \( Gal(K) \). This will generalise the classical result of \[\text{Cherlin et al., 1980}\] in the case where \( K \) is a perfect field and \( M = K_{alg} \). A detailed exposition of the construction in that setting can be found in the appendix of \[\text{Chatzidakis, 2002}\].

**Definition 4.2.** Let \( M \models T \) and let \( K \) be a definably closed substructure of \( M \). A \( K \)-definable finite structure \( c \) in \( M \) is a pair \( c = (u(c), R(c)) \) where \( u(c) \) is a finite \( K \)-definable set in \( M \) of size \( n \geq 1 \) and \( R(c) \) is a non-empty \( K \)-definable \( n \)-ary relation on \( u(c) \) such that every element of \( R(c) \) is an enumeration of \( u(c) \). Here \( u(c) \) is a subset of a finite Cartesian product of sorts of \( M \). We say that \( u(c) \) is the universe of \( c \).

We say that \( c \) is a complete \( K \)-definable finite structure if \( R(c) \) is minimal, in the sense that for any finite \( K \)-definable structure \( c' \) with the same universe as \( c \), if \( R(c') \subseteq R(c) \) then \( R(c') = R(c) \).

For \( c \) a complete finite \( K \)-definable structure, let \( Gal(c/K) \) be the finite group of elementary automorphisms of \( u(c) \cup K \) which fix \( K \).

We say that \( c \) is Galois if \( Gal(c/K) \) acts sharply transitively on \( c \).

If \( c \) is a complete finite \( K \)-definable structure in \( M \), then there is some enumeration \( u_1, \ldots, u_n \) of \( u(c) \) such that \( R(c) \) is the set of realisations of the type \( tp(u_1, \ldots, u_n/K) \). Note that \( Gal(c/K) \) acts sharply transitively on \( R(c) \). Therefore, taking a finite \( K \)-definable structure \( d \) with universe \( R(c) \) produces a Galois \( K \)-definable finite structure.

Finally, observe that \( Gal(K) \) is the projective limit of the system \( Gal(d/K) \) where \( d \) is a Galois \( K \)-definable finite structure. Therefore we will focus on Galois \( K \)-definable finite structures in what follows. This is not necessary from a technical point of view but it makes the analogy with the case of fields clearer.

We will now code uniformly the Galois structures definable over \( K \) as elements of \( M \). Their Galois groups are also coded uniformly as \( L_P \)-definable sets that we will call \( G(c) \).

**Lemma 4.3.** For every finite Cartesian product \( S \) of sorts of \( T \) and \( n \geq 1 \), there is an \( L_P \)-definable set \( C_{comp}^n(S) \) such that for any \( M \models T_P \), \( C_{comp}^n(S) \) codes the complete \( n \)-element \( K \)-definable structures contained in \( S \). \( C_{comp}^n(S) \) is contained in \( K \).
Moreover, there are \( L_P \)-definable sets \( G_n(S) \) such that for every \( M \models T_P \) and \( c \in C^\text{comp}(S) \), there is a \( K \)-definable group \( G(c) \subseteq G_n(S) \) with a \( K \)-definable action on \( u(c) \) such that \( G(c) \) is equal to \( \text{Gal}(c/K) \) as a group of permutations of \( u(c) \).

Therefore there are \( L_P \)-definable sets \( C^\text{gal}_n(S) \) such that for any \( M \models T_P \), \( C^\text{gal}_n(S) \subseteq C^\text{comp}_n(S) \) codes the Galois \( K \)-definable structures of size \( n \) in \( S \).

**Proof.** Suppose \( u(c) \) has cardinality \( n \) and is in \( S \). Then \( R(c) \) is a subset of \( S^n \) containing at most \( n! \) elements such that every element in \( R(c) \) enumerates the same \( n \)-element set. There is also a formula asserting that an element \( x \in D_{\leq n!}(S^n) \) codes a \( K \)-definable relation which is minimal in the sense of Definition 4.2. Hence all \( n \)-element complete \( K \)-definable structures in \( S \) are coded in some definable subset of \( D_{\leq n!}(S^n) \).

Consider the set of pairs \( (x, y) \in S^{2n} \) such that there is \( c \in C^\text{comp}_n(S) \) such that \( x, y \) both belong to \( R(c) \). The set of such pairs is 0-definable. Observe that any such pair \( (x, y) \) determines an element of \( \text{Gal}(c/K) \) and that a definable equivalence relation decides whether two such pairs determine the same element of \( \text{Gal}(c/K) \). By elimination of finite imaginaries, we can find a definable set \( G_n(S) \) of codes for automorphisms of \( n \)-element complete \( K \)-definable structures in \( S \).

By construction of \( G_n(S) \), we see that for every \( c \in C^\text{comp}_n(S) \) there is a \( c \)-definable group \( G(c) \) contained in \( G_n(S) \) with a \( c \)-definable action on \( u(c) \), and \( G(c) \) is canonically identified with \( \text{Gal}(c/K) \) in the obvious way. The definability of \( C^\text{gal}_n(S) \) follows. \( \square \)

Note that in Lemma 4.3, for \( M \models T_P \) and \( K = P(M) \), the definable sets \( G_n(S) \) are not usually contained in \( K \). See the discussion following Proposition 4.0. Note also that for any \( c \in C^\text{gal}_n(S) \) and \( \phi \in \text{Gal}(K) \), there is \( g_0 \in G(c) \) such that for any \( g \in G(c) \), \( \phi(g) = g_0 \), the conjugate of \( g \) by \( g_0 \).

We now show that the projective limit structure on the system of groups \( \{\text{Gal}(c/K)\} \) is definable:

**Definition 4.4.** Let \( M \models T \) and let \( K \subseteq M \) be a definably closed substructure. For \( c, c' \) finite \( K \)-definable structures, we write \( c \leq c' \) if \( u(c) \subseteq \text{dcl}(K, u(c')) \).

It is clear that for \( c \in C^\text{gal}_n(S) \) and \( c' \in C^\text{gal}_m(S') \), \( c \leq c' \) if and only if \( \text{Aut}(\text{acl}(c/K), K, c) \) is a normal subgroup of \( \text{Aut}(\text{acl}(c'/K), K, c') \).

**Lemma 4.5.** The relation \( c \leq c' \) is definable in \( T_P \) between sets \( C^\text{gal}_n(S) \) and \( C^\text{gal}_m(S') \).

**Proof.** Let \( M \models T_P \) and suppose \( c \in C^\text{gal}_n(S) \) and \( c' \in C^\text{gal}_m(S') \). Let \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_m) \) be arbitrary enumerations of \( u(c) \) and \( u(c') \). Then \( \text{tp}(ab/K) \) is algebraic so we can find \( d \in C^\text{gal}_k(S^n \times S'^m) \) such that \( u(d) \) consists of the realisations of \( \text{tp}(ab/K) \). Then \( c \leq c' \) if and only if the action of \( G(d) \) on \( u(d) \) is determined by its restriction to the coordinates in \( S'^m \). This is a definable property of \( d \). Finally, we quantify-out \( d \) to obtain a definition of \( c \leq c' \). \( \square \)

The proof of Lemma 4.3 shows that for any sets \( C^\text{gal}_n(S) \) and \( C^\text{gal}_m(S') \), there is a sort \( S'' \) and \( k \leq mm \) such that for any \( c \in C^\text{gal}_n(S) \), \( c' \in C^\text{gal}_m(S') \), there is \( c'' \in C^\text{gal}_k(S'') \) with \( c \leq c'' \) and \( c' \leq c'' \). It follows that \( \bigcup_{n,S} C^\text{gal}_n(S) \) forms a directed preorder under \( \leq \). Note that we may have \( c \neq c' \), \( c \leq c' \), and \( c' \leq c \) for \( c, c' \) in the same set \( C^\text{gal}_n(S) \) or in distinct sets.

Moreover, Lemma 4.3 shows that if \( c \in C^\text{gal}_n(S) \) and \( c' \in C^\text{gal}_m(S') \) then the relation \( c \leq c' \) determines a canonical \( K \)-definable surjective homomorphism \( G(c') \to G(c) \). It is easy to check that if we have \( c \leq c' \leq c \), then the resulting homomorphism \( G(c) \to G(c) \) is the identity. We say that the projection maps are compatible with the preorder \( \leq \) on \( \bigcup_{n,S} C^\text{gal}_n(S) \). Hence we have proved the following proposition:
Proposition 4.6. Let $M \models T$ and let $K$ be a definably closed substructure. The family of groups $G(c)$ indexed by the set

$$\bigcup\{C_n^{gal}(S) \mid n \geq 2, S \text{ finite product of sorts of } T\}$$

forms a strict $\mathcal{L}_P$-piecewise-definable projective system of finite $\mathcal{L}_P$-definable groups with the directed preorder $x \leq y$ and the induced definable homomorphisms $G(y) \to G(x)$. The inverse limit of this projective system is canonically isomorphic to $Gal(K)$.

In Proposition 4.6, we say that the family $G(c)$ is a piecewise definable projective system of finite definable groups because the underlying preorder is given by a family of definable sets. We stress that this projective system is strict because the group homomorphisms $G(y) \to G(x)$ for $x \leq y$ are surjective.

Observe that the $\mathcal{L}_P$-piecewise-definable projective system is only sensitive to the part of $M$ which contains acl$(K)$. Therefore, we can define a language $\mathcal{L}_P^*$ which contains only the quantifier-free definable relations of $\mathcal{L}$ which occur between elements of acl$(K)$ and define $T^*_p$ to be the theory of the $\mathcal{L}_P^*$-structure acl$(K)$ in the obvious way. The definable projective system of Proposition 4.6 is definable in $T^*_p$. We will not make any further references to $\mathcal{L}_P$-structure but every interpretability result about acl$(K)$ as an $\mathcal{L}_P$-structure will also holds for acl$(K)$ as an $\mathcal{L}_P^*$-structure. This stronger fact can be seen by applying those results to $T^*_p$ in place of $T$ and to the $\mathcal{L}_P^*$-structure acl$(K)$ in place of $M$ while keeping the same substructure $K$.

Finally, we remark that Proposition 4.6 entails trivially that $Gal(K)^n$ is also the inverse limit of an $\mathcal{L}_P$-definable projective system of finite groups in $M$, for any $n \geq 1$.

Let $M \models T$ and let $K$ be a substructure. Using quantifier-elimination in $T$, an inspection of the proofs of Lemmas 1.3 and 1.5 will show that the sets $C_n^{gal}(S)$ and the relation $c \leq c'$ are $\mathcal{L}$-definable subsets of $K$, which we view as an $\mathcal{L}$-structure. As a result, $Th(K)$ has partial access to the definable projective system of sets $(G(c))$. For example, there are quantifier-free $\mathcal{L}$-formulas which determine in $Th(K)$ the isomorphism type of the group $G(c)$ for $c \in C_n^{gal}(S)$.

[Dittmann, 2018] shows in the context of fields that the $\mathcal{L}_P$-definable projective system of groups $(G(c))$ is usually not interpretable in $Th(K)$. Indeed, when $c \leq c'$, $Th(K)$ does not determine a projection $G(c') \to G(c)$. This is because $Th(K)$ does not determine an embedding of the sets $u(c)$ and $u(c')$ in $M$. See [Dittmann, 2018] for proofs and a detailed account of these facts.

Nonetheless, for any $c \in C_n^{gal}(S)$ and $k \geq 1$, $K$ contains codes for each orbit on $G(c)^k$ under diagonal conjugation by $G(c)$. We say that there are the $G(c)$-conjugacy classes of $G(c)^k$. If $\phi \in Gal(c/K)$ and $g = (g_1, \ldots, g_k) \in G(c)^k$, then $\phi(g) = (g_1^\phi, \ldots, g_k^\phi)$ when we identify $\phi$ with an element of $G(c)$. Therefore, $\phi$ preserves $(g_1, \ldots, g_k)^G$. By elimination of finite imaginaries, we can construct a definable set $Conj^k_n(S) \subseteq K$ such that for every $c \in C_n^{gal}(S)$, there is a finite $K$-definable set $Conj^k_n(c) \subseteq Conj^k_n(S)$ coding the $G(c)$-conjugacy classes of $G(c)^k$.

If $c \in C_n^{gal}(S)$, $c' \in C_n^{gal}(S')$ and $c \leq c'$, then the canonical projection $\pi : G(c') \to G(c)$ induces a definable relation $\sqsubseteq$ between $Conj^k(c)$ and $Conj^k(c')$ as follows: we write $a \sqsubseteq b$ where $a \in Conj^k(c')$ and $b \in Conj^k(c)$ if $a$ is contained in $\pi^{-1}(b)$. This relation is $\mathcal{L}$-definable in $Th(K)$. Hence we have the proposition:

Proposition 4.7. Let $M \models T$ with a definably closed substructure $K$ and let $k \geq 1$. Working in $Th(K)$, the family of $\mathcal{L}$-definable finite sets $Conj^k_n(c)$ indexed by the set $\bigcup_{n,S} C_n^{gal}(S)$ forms a strict piecewise-definable projective system with the directed preorder $x \leq y$ on $\bigcup_{n,S} C_n^{gal}(S)$ and the induced relation $\sqsubseteq$. 
The inverse limit of this projective system is canonically isomorphic to the profinite space of \( \text{Gal}(K) \)-conjugacy classes of \( \text{Gal}(K)^F \).

We end this section with a comment about the opposite group to \( G(c) \). Fix \( M \models T \) and \( K \) a substructure of \( M \). For \( c \in C^\text{gal}_n(S) \), define \( G^\text{op}(c) \) the \( c \)-definable group of permutations of \( u(c) \) which commute with \( G(c) \). Since \( c \) is Galois, it is easy to show that \( G^\text{op}(c) \) is isomorphic to \( G(c) \), but not canonically: for any \( a \in u(c) \), we have an isomorphism \( G(c) \to G^\text{op}(c), g \mapsto h^{-1} \) where \( ga = ha \). Only the conjugacy classes of \( G(c) \) and \( G^\text{op}(c) \) are in a canonical bijection.

Since \( G(c) \) is equal to \( \text{Gal}(c/K) \), \( G^\text{op}(c) \) is pointwise \( K \)-invariant and hence \( G^\text{op}(c) \) is contained in \( K \). In fact, we can use the opposite groups to give a more canonical characterisation of \( K \)-definable finite Galois structures. We can define \( C^\text{gal}_n(S) \) as the set of elements \( c \) in \( D_{\leq n}(S) \) such that the set coded by \( c \) does not contain any smaller \( K \)-definable set and such that the set of permutations of \( c \) which belong to \( K \) acts sharply transitively on \( c \). This approach is more canonical than the approach through Definition 4.2, but it does not allow any strengthening of our results.

This is because the identification of \( G^\text{op}(c) \) with \( G(c) \) is not canonical and there is no canonical system of projections \( G^\text{op}(c') \to G^\text{op}(c) \) when \( c \leq c' \). If there is a \( K \)-definable surjection \( u(c') \to u(c) \), then it is true that we have a \( K \)-definable surjective homomorphism \( G^\text{op}(c') \to G^\text{op}(c) \), but different choices of surjections \( u(c') \to u(c) \) induce incompatible homomorphisms between the opposite groups. Therefore, although it is possible to work in \( T h(K) \) with an arbitrary finite collections of opposite groups, the family \( (G^\text{op}(c)) \) does not form an \( L \)-definable projective system in \( T h(K) \).

4.2. Hilbert spaces associated to \( \text{Gal}(K) \). As before, we assume that \( T \) eliminates finite imaginaries and has quantifier elimination. In this section we construct the Hilbert space functors \( \mathcal{H}_\text{gal}, \mathcal{H}^\prime_\text{gal} \) and the functors \( \mathcal{H}_\text{class,k} \) for \( k \geq 1 \). We summarise our results about these functors in the following theorem. The rest of this section is devoted to proving Theorem 4.8.

**Theorem 4.8.**

1. \( \mathcal{H}_\text{gal} \) is a \( L_{P} \)-definable Hilbert space functor such that for any \( M \models T \) and any definably closed substructure \( K \subseteq M \), \( \mathcal{H}_\text{gal}(M) \) is canonically isomorphic to \( L^2(\text{Gal}(K)) \) in \( M \).
2. \( L^2(\text{Gal}(K)) \) can be viewed as a completed orthogonal sum of finite-dimensional Hilbert spaces determined by a piecewise-interpretable family of asymptotically free \( L_P \)-definable sets in \( M \).
3. \( \mathcal{H}^\prime_\text{gal} \) is an \( L \)-definable Hilbert space functor such that for any \( M \models T \) and any definably closed substructure \( K \subseteq M \), \( \mathcal{H}^\prime_\text{gal}(K) \) is a Hilbert space interpretable in \( K \) which is abstractly isomorphic to \( L^2(\text{Gal}(K)) \) in a way that respects the orthogonal decomposition given in (2) above.
4. For every \( k \geq 1 \), \( \mathcal{H}_\text{class,k} \) is an \( L \)-definable Hilbert space functor such that for any \( M \models T \) and any definably closed substructure \( K \subseteq M \), \( \mathcal{H}_\text{class,k}(K) \) is canonically isomorphic to \( L^2(\text{Gal}(K)^k)^{\text{Gal}(K)} \) in \( K \).
5. \( L^2(\text{Gal}(K)^k)^{\text{Gal}(K)} \) can be viewed as a completed orthogonal sum of finite-dimensional Hilbert spaces given by a piecewise-interpretable family of asymptotically free \( L \)-definable sets in \( K \).

**Proof.** 1) is proved in Proposition 4.9 2) is proved in Lemma 4.10 3) is proved in Proposition 4.11 4) is proved in Proposition 4.13 and 5) is proved in the discussion following 4.13. □

We use the notation that we set up in Section 4.1. Take \( M \models T \) with a definably closed substructure \( K \) and take \( g \in G(c) \subseteq G_n(S) \) in \( M \). Then we have seen that \( G(c) \) is canonically
identified with \( \text{Gal}(K)/\text{Fix}(u(c)) \) and \( g \) is canonically identified with a coset of \( \text{Fix}(u(c)) \) in \( \text{Gal}(K) \). In order to simplify notation, we will write \( g \) for this coset.

We write \( L^2(\text{Gal}(K)) \) for the space of real or complex square-integrable functions on \( \text{Gal}(K) \) given by the Haar measure.

**Proposition 4.9.** There is a strict Hilbert space functor \( \mathcal{H}_{\text{gal}} \) on \( T_P \) supported by the definable sets \( (G_n(S)) \) with interpretation maps \( h_{S,n} \) where \( n \geq 2 \) and \( S \) ranges over finite Cartesian products of sorts of \( T \) such that for any \( M \models T_P \), \( \mathcal{H}_{\text{gal}}(M) \) can be canonically identified with \( L^2(\text{Gal}(K)) \) in the following sense:

There is a Hilbert space isomorphism \( \phi : \mathcal{H}_{\text{gal}}(M) \to L^2(\text{Gal}(K)) \) such that for any \( g \in G(c) \subseteq G_n(S) \), \( \phi \circ h_{S,n}(g) = 1_g \). \( \phi \) is necessarily unique.

**Proof.** We only need to show that for any choice of \( G_n(S) \) and \( G_m(S') \), the map \( f_{S,n,S',m} : G_n(S) \times G_m(S') \to \mathbb{R}, (g,g') \mapsto \langle 1_g, 1_{g'} \rangle \) is definable, where \( g,g' \) are identified with cosets in \( \text{Gal}(K) \) as explained above.

For any \( g \in G(c) \) and \( g' \in G(c') \), choose any \( c'' \) such that \( c,c' \leq c'' \) and write \( \pi : G(c'') \to G(c), \pi' : G(c'') \to G(c') \) for the canonical homomorphisms. Then quotienting by \( \text{Fix}(c'') \) we have

\[
\langle 1_g, 1_{g'} \rangle = \frac{|\pi^{-1}(g) \cap \pi'^{-1}(g')|}{|G(c'')|}
\]

Since \( \pi, \pi' \) are \( (c,c',c'') \)-definable, the above relation is also \( (c,c',c'') \)-definable. Quantifying over \( c'' \), we see that the inner product maps are definable and this defines \( \mathcal{H} \) as required. \( \square \)

We will say that the functor \( \mathcal{H}_{\text{gal}} \) is the interpretation of \( L^2(\text{Haar}) \) in \( T_P \). Compare with Proposition 2.15 in the case of definable measures.

We will now show how to find a natural asymptotically free decomposition of \( \mathcal{H}_{\text{gal}} \). We have seen that the equivalence relation \( (c \leq c') \wedge (c' \leq c) \) on each \( C_{n}^{\text{gal}}(S) \) induces an equivalence relation on \( G_n(S) \) and we can quotient out the sets \( C_{n}^{\text{gal}}(S) \) and \( G_n(S) \) by these equivalence relations to obtain injective interpretation maps for \( \mathcal{H}_{\text{gal}} \). We continue to write \( G_n(S) \) for the resulting imaginary sorts.

Fix \( M \models T \) and \( K \) a definably closed substructure. For every \( c \)-definable group \( G(c) \subseteq G_n(S) \), let \( H(c) \) be the finite-dimensional subspace of \( L^2(\text{Gal}(K)) \) generated by functions which are constant on cosets of \( \text{Fix}(u(c)) \). \( H(c) \) can be canonically identified with \( L^2(G(c) = L^2(\text{Gal}(c/K)) \) with the normalised counting measure. Let \( W(c) \) be the subspace of \( H(c) \) generated by the sum of the spaces \( H(c') \) for \( c' \) in arbitrary sets \( C_{n}^{\text{gal}}(S) \) such that \( c' < c \).

For \( g \in G(c) \), define \( \hat{h}_{S,n}(g) = P_{W(c)^{\perp}}(h_{S,n}(g)) \). Note that \( \hat{h}_{S,n}(g) \in H(c) \) and that \( \{ \hat{h}_{S,n}(g) \mid g \in G(c) \} \) spans \( H(c) \cap W(c)^{\perp} \).

**Lemma 4.10.** Each map \( \hat{h}_{S,n} \) defined above is definable and asymptotically free on \( G_n(S) \). In fact, \( L^2(\text{Gal}(K)) \) is the completed orthogonal sum of the finite dimensional spaces \( H(c) \cap W(c)^{\perp} \).

**Proof.** \( \hat{h}_{S,n} \) can be seen to be definable by an argument similar to the one given before Lemma 3.17. We only need to show that \( \hat{h}_{S,n}(g) \perp \hat{h}_{S,n}(g') \) when \( g,g' \) are automorphisms of finite \( K \)-definable structures which are not inter-definable.

Fix \( c,c' \in C_{n}^{\text{gal}}(S) \) such that \( c' \nless c \) and write \( N = \text{Fix}(u(c)) \) and \( N' = \text{Fix}(u(c')) \) in \( \text{Gal}(K) \). Then \( N' \) is not contained in \( N \) and \( N \cap N' \) is a proper normal subgroup of \( N \). We can find a \( K \)-definable finite Galois structure \( d \) such that \( N \cap N' = \text{Fix}(u(d)) \). Then \( c,c' \leq d \) and we can identify \( H(c) \) and \( H(c') \) with subspaces of \( H(d) \). It will be enough to show that if \( f \in H(c) \cap W(c)^{\perp} \) and \( f' \in H(c') \) then \( f \perp f' \).
Let $G_0 = NN' \leq G(d)$. Then $G(d)$ can be identified with the fiber product $G(c) \times_{G_0} G(c')$ so $L^2(G(d))$. If $f \in H(c) \cap W(c)^\perp$, then we view $f$ as a function $G(c) \to \mathbb{R}$ such that the sum of the values of $f$ over any coset of $G_0/N$ is 0. Therefore, summing over fibers of $G/G_0$, we have for any $f' \in H(c')$:

$$
\langle f, f' \rangle = \sum_{g \in G/G_0} \sum_{g_1 \in G(c)} \sum_{g_2 \in G(c') \; g_1(N'/N) = g \; g_2(N'/N) = g} f(g_1)f'(g_2)
$$

$$
= \sum_{g \in G/G_0} \left( \sum_{g_1 \in G(c) \; g_1(N'/N) = g} f(g_1) \right) \left( \sum_{g_2 \in G(c') \; g_2(N'/N) = g} f'(g_2) \right)
$$

$$
= 0.
$$

$$
\square
$$

**Remarks:**

1. We have expressed $H_{gal}$ as a strict Hilbert space functor supported by definable sets with asymptotically free interpretation maps. This matches the description of Corollary 3.2.4 although it is not clear whether the weak NFCP occurs in the present context.

2. Although each map $h_{S,n}$ is asymptotically free, we cannot guarantee that they are pairwise orthogonal because of possible identifications between $C_{gal}^n(S)$ and $C_{gal}^n(S')$. Observe also that restricting $h_{S,n}$ to different complete types in $G_n(S)$ does not give pairwise orthogonal asymptotically free maps.

3. The asymptotically free decomposition constructed above only depends on $Gal(K)$ being the inverse limit of a piecewise-definable projective system of finite groups. Therefore, the same approach gives an asymptotically free decomposition of $L^2(Gal(K)^k)$ for any $k \geq 1$.

For $M \models T$ and $K$ a definably closed substructure, it is unlikely that $K$ interprets $L^2(Gal(K))$ as an $\mathcal{L}$-structure canonically in a sense similar to Proposition 4.10. However we can use Lemma 4.10 to show that $K$ interprets a Hilbert space which is abstractly isomorphic to $L^2(Gal(K))$ in a way that respects the orthogonal decomposition of Lemma 4.10.

Indeed, with the same notation as before, $L^2(Gal(K))$ is the orthogonal sum of the finite-dimensional Hilbert spaces $H(c) \cap W(c)^\perp$. Moreover, there are quantifier-free $\mathcal{L}$-formulas contained in each $C_{gal}^n(S)$ which determine the dimension of $H(c) \cap W(c)^\perp$ for $c \in C_{gal}^n(S)$. Write $dim(x) = k$ for these formulas (where $k \leq n$).

For every set $C_{gal}^n(S) \subseteq K$ and $k \leq n$, let $D^i_k(x_i)$ ($1 \leq i \leq n$) be disjoint copies of the set $dim(x) = k$. For every $c$ satisfying $dim(c) = k$, choose an orthonormal basis $v_1(c), \ldots, v_k(c)$ of $H(c) \cap W(c)^\perp$ and define maps $h_{S,k,i} : D^i_k \to L^2(Gal(K))$ by $h_{S,k,i}(c_i) = v_i(c)$ for $i \leq k$, where $c_i$ is the copy of $c$ in $D^i_k(x_i)$. For $k < i \leq n$, define $h_{S,k,i}(c_i) = 0$. The maps $(h_{S,k,i})$ define an interpretable Hilbert space $H_{S,n}$ isomorphic to the orthogonal sum of the spaces $H(c) \cap W(c)^\perp$ as $c$ ranges over $C_{gal}^n(S)$.

Working now across different sets $C_{gal}^n(S)$ and $C_{gal}^n(S')$, if $c$ and $c'$ are inter-definable so that $H(c) = H(c')$, we can choose the same orthonormal bases $v_1(c) = v_1(c'), \ldots, v_k(c) = v_k(c')$ for $H(c) \cap W(c)^\perp = H(c') \cap W(c')^\perp$. Writing $(D^i_k(x_i))$ and $(D^i_k(y_j))$ for the copies of $dim(x) = k$ in $C_{gal}^n(S)$ and $C_{gal}^n(S')$ respectively, we set $h_{S,k,i}(c_i) = h_{S',k,i}(c_i')$. The resulting inner product maps $D^i_k \times \hat{D}^i_k \to \mathbb{R}$ are definable. Note that $c, c'$ are inter-definable if and only if $c \leq c'$ and $c' \leq c$ and this relation is definable in $Th(K)$. Therefore, the inner product maps are defined independently of the particular identifications which take place in $K$ between $C_{gal}^n(S)$ and $C_{gal}^n(S')$. 


Carrying out this construction over all sets $C_n^{gal}(S)$ and making coherent choices of bases, we obtain a strict Hilbert space functor $\mathcal{H}'_{\text{gal}}$ supported by imaginary sorts of $Th(K)$ satisfying the following proposition:

**Proposition 4.11.** For any $M \models T$ and $K$ a definably closed substructure, there is a Hilbert space isomorphism $\phi: \mathcal{H}'_{\text{gal}}(K) \to L^2(\text{Gal}(K))$ satisfying the following property: for every $c \in C_n^{gal}(S)$, there is a finite dimensional subspace $U(c)$ of $\mathcal{H}'_{\text{gal}}(K)$ such that $U(c) \subseteq \text{dcl}(c)$ and $\phi$ maps $U(c)$ isomorphically to $L^2(\text{Gal}(c/K))$.

$U(c)$ is defined at the level of the functor $\mathcal{H}'_{\text{gal}}$, in the sense $U(c)$ is determined by formulas over $c$ which do not depend on any particular choice of substructure $K$ of $M$.

**Remarks:** 1. The strict Hilbert space functor $\mathcal{H}'_{\text{gal}}$ defined above satisfies the conclusion of Corollary 3.21 by construction.

2. We can view $\mathcal{H}'_{\text{gal}}$ as a Hilbert space functor on $T_P$ since $Th(K)$ is interpretable in $T_P$. In that case, we point out that $\mathcal{H}_{\text{gal}}$ and $\mathcal{H}'_{\text{gal}}$ are not isomorphic Hilbert space functors on $T_P$.

When $G$ is a profinite group with Haar measure $\mu$ and $k \geq 1$, write $L^2(G^k)^G$ for the Hilbert space of $G$-class functions on $G^k$. These are the functions $G^k \to \mathbb{R}$ or $\mathbb{C}$ which are invariant under (diagonal) conjugation by $G$. In the following, we work with $G$-class functions on $G^k$ into $\mathbb{R}$ but all the results go through without modification to class functions into $\mathbb{C}$.

**Lemma 4.12.** Let $G$ be a profinite group with Haar measure $\mu$. Then $L^2(G^k)^G$ is generated by functions $f: G^k \to \mathbb{R}$ for which there is an open normal $N \unlhd G$ and a $G$-conjugacy class $A$ of $G^k/N^k$ such that $f = \mathbb{1}_{\pi^{-1}(A)}$ where $\pi$ is the quotient map $G^k \to G^k/N^k$.

**Proof.** Let $f$ be an arbitrary class function on $G$. Then we can find a simple function $h$ which is arbitrarily close to $f$ in the $L^2$-norm. Let $h'(x) = \int_G h(x^g)dg$. Then

$$\|h' - f\|^2 \leq \int_{G^k} \int_G (f(x^g) - h(x^g))^2dgdx$$

$$= \int_{G^k} \int_G (f(x^g) - h(x^g))^2dx dg$$

$$\leq \|f - h\|^2$$

It is easy to check that $h'$ is a simple function. Hence there is some open $N \unlhd G$ such that $h$ factors through $N^k$ and the lemma follows. \[\square\]

Now let $T$ be a theory as before, with elimination of finite imaginaries and quantifier elimination. Recall from Proposition 4.7 that there is a $\mathcal{L}$-piecewise-definable projective system such that for any $M \models T$ with $K$ a definably closed substructure, the inverse limit of this projective system in $K$ is canonically identified with the profinite space of $\text{Gal}(K)$-conjugacy classes of $\text{Gal}(K)^k$. This piecewise-definable projective system is given by the collection of definable sets $\text{Con}^k_n(S)$.

**Proposition 4.13.** For every $k \geq 1$, there is a strict Hilbert space functor $\mathcal{H}_{\text{class},k}$ for $Th(K)$ supported by the definable sets $(\text{Con}^k_n(S))$ with interpretation maps $(h_{S,n})$ such that for any $M \models T$ with $K$ a definably closed substructure, writing $G = \text{Gal}(K)$, $\mathcal{H}_{\text{class},k}(K)$ can be canonically identified with $L^2(G^k)^G$ in the following sense:

There is a Hilbert space isomorphism $\phi: \mathcal{H}_{\text{class},k}(K) \to L^2(G^k)^G$ such that for any $a \in \text{Con}^k_n(c) \subseteq \text{Con}^k_n(S)$, $\phi \circ h_{S,n}(a) = \mathbb{1}_{\pi^{-1}(a)}$ where $\pi: G^k \to \text{Gal}(c/K)^k$ is the quotient map and $a$ is canonically identified with a $\text{Gal}(c/K)$-conjugacy class of $\text{Gal}(c/K)^k$. $\phi$ is necessarily unique.
Proof. We have to show that the maps $Conj_k^c(S) \times Conj_k^c(S') \to \mathbb{R}, (a, b) \mapsto \langle \mathbb{1}_{x^{-1}(a)}, \mathbb{1}_{x^{-1}(b)} \rangle$ are definable. By Proposition 4.4 these maps are $L_P$-definable. Since quantification only takes place over $P$, these maps are $L$-definable in $K$. \hfill $\square$

We can also view $\mathcal{H}_{class,k}$ as a Hilbert space functor on $T_P$. In that case, $\mathcal{H}_{class,k}$ embeds definably into $\mathcal{H}_{gal}$ but not into $\mathcal{H}_{gal}$.

We show that the asymptotically free decomposition $L^2(Gal(K)k)$ in $M$ given by Lemma 4.10 also provides an asymptotically free decomposition of $\mathcal{H}_{class,k}$. Fix $M \models T$ and $K$ a definably closed substructure. For every $c \in C^g_{\mathcal{H}}(S)$, define $\mathcal{H}_{class,k}(c)$ the Hilbert space of $G(c)$-class functions on $G(c)^k$, viewed a subspace of $L^2(G(c)^k) := H(c)$. Recall the definition of $W(c)$ as the subspace of $H(c)$ generated by the sum of the $H(c')$ where $c' < c$. Let $U(c) = W(c) \cap \mathcal{H}_{class,k}(c)$. Since $W(c)$ is invariant under conjugation, it is clear that for any $f \in \mathcal{H}_{class,k}(c)$, $P_{U(c)}\perp(f) = P_{W(c)}\perp(f)$. Therefore we can express $L^2(Gal(K)k)Gal(K)$ as the orthogonal sum of the finite dimensional spaces $P_{U(c)}\perp(H_{class,k}(c))$. We obtain a decomposition of $\mathcal{H}_{class,k}$ in the same style as Corollary 6.24.

5. UNITARY REPRESENTATIONS

In this section, we relate interpretable Hilbert spaces to the theory of unitary group representations. We establish some general connections with asymptotic freedom in §7. In Section 5.2 we study the special case of $\omega$-categorical structures and we recover the classification theorem of Tsankov, 2012. In Section 7.6 we show that our decomposition theorems yield some results for certain unitary group representations, even when no logic is apparently present.

5.1. Unitary Representations of Automorphism Groups. In this section, we show how the theorems of Section 4 allow us to recover the key classification theorem of Tsankov, 2012 about automorphism groups of $\omega$-categorical structures and how they shed light on some of the representation theory of more general automorphism groups. In this section, unless specified otherwise, $T$ is an arbitrary continuous logic theory, $\mathcal{H}$ is a Hilbert space functor for $T$ and $M \models T$.

Definition 5.1. Let $G$ be a group and $H$ a Hilbert space, real or complex. A unitary representation $\sigma$ of $G$ on $H$ is a group action $G \times H \to H$ such that for every $g \in G$ $\sigma(g)$ is a unitary map if $H$ is a complex Hilbert space and $\sigma(g)$ is an orthogonal map if $H$ is a real Hilbert space.

When $G$ is a topological group, we say that $\sigma$ is continuous if $\sigma : G \times H \to H$ is continuous. This is equivalent to $\sigma(\cdot, v)$ being continuous for every $v \in V$.

Convention: In this paper, we only consider continuous unitary representations of topological groups, so we just say ‘representation’ instead of ‘continuous unitary representation’.

Definition 5.2. Let $G$ be a group and let $\sigma, \sigma'$ be two representations of $G$ on the Hilbert spaces $H$ and $H'$ respectively. $\sigma$ and $\sigma'$ are equivalent if there is a surjective isometry $U : H \to H'$ such that for all $g \in G$ and $v \in H$, $U(\sigma(g)v) = \sigma'(g)U(v)$. We say that $U$ intertwines $\sigma$ and $\sigma'$.

Let $T$ be a continuous logic theory and $M \models T$. Write $G = \text{Aut}(M)$. Then $G$ is a topological group with the topology of pointwise convergence. $G$ has a basis at the identity consisting of subsets of the form $\{g \in G \mid d(gA, A) < \epsilon\}$ where $A$ ranges over finite subsets of $M$ and $\epsilon > 0$. When $T$ is a classical logic theory, this is a basis of subgroups. Note we can add finitary and infinitary imaginary sorts to $M$ without changing the topology on $G$.

Suppose $\mathcal{H}$ is a Hilbert space functor for $T$ expressed as a direct limit of sorts $(S_i)_{i \in I}$. Then for any $M \models T$, we have a canonical unitary representation $\pi$ of $\text{Aut}(M)$ on $\mathcal{H}(M)$: for any
$x \in S_t$, define $\pi(g)h_ix = h_ig(x)$. By our previous comments, $\pi$ is continuous. The following lemma is an easy definition chase.

**Lemma 5.3.** Let $T$ be a continuous logic theory. If $\mathcal{H}$, $\mathcal{H}'$ are isomorphic Hilbert space functors for $T$, then for any $M \models T$, the representations of $\text{Aut}(M)$ on $H(M)$ and $H'(M)$ are equivalent.

We begin by defining the notion of induced representation in the special case where we induce from an open subgroup. See [Bekka et al., 2008] for more details. Let $G$ be a topological group and take $K$ an open subgroup of $G$. Let $\sigma$ be a representation of $K$ on the Hilbert space $V$. We suppose that $V$ is a real Hilbert space (the case of complex Hilbert spaces is completely analogous). Write $\mathcal{R}G$ for the free vector space on $G$. We define $G \otimes_\sigma V$, the $\sigma$-tensor of $G$ and $V$, to be the vector space $\mathcal{R}G \otimes V$ quotiented by a suitable subspace so that $gK \otimes_\sigma v = g \otimes \sigma(k)v$ for all $g \in G$ and $k \in K$.

We define an inner product on $G \otimes_\sigma V$ as follows. For any $g, g' \in G$ and $v, v' \in V$, if $gK \neq g'K$, then $(g \otimes_\sigma v, g' \otimes_\sigma v') = 0$. If $gK = g'K$, then find $k \in K$ such that $g' = gk$ and define $(g \otimes_\sigma v, g' \otimes_\sigma v') = (v, \sigma(k)v')$. Observe that if we choose a set of coset representatives for $G/K$, we can identify $G \otimes_\sigma V$ with the orthogonal sum of copies of $V$ indexed by $G/K$.

We will always work with the Hilbert space completion of $G \otimes_\sigma V$, so we do not introduce new notation and we also write $G \otimes_\sigma V$ for the completion. We define the induced representation of $G$ from $\sigma$, denoted $\text{Ind}^G_K(\sigma)$, as the unitary representation of $G$ on $G \otimes_\sigma V$ given by $g \cdot (g' \otimes_\sigma v) = gg' \otimes_\sigma v$. Since $K$ is open in $G$, the induced representation is continuous.

**Proposition 5.4.** Let $T$ be an arbitrary continuous logic theory with a Hilbert space functor $\mathcal{H}$ and $M \models T$. Suppose there is a complete type $p$ in a sort supporting $\mathcal{H}$ such that the interpretation map $h$ is asymptotically free on $p$. Write $H_p(M)$ for the $\mathcal{H}$-interpretable Hilbert space spanned by $h(p)$. For $x, y \models p$, write $x \sim y$ if $\text{bdd}(x) = \text{bdd}(y)$ and write $[x]$ for the equivalence class of $x$ under $\sim$.

Let $G = \text{Aut}(M)$ and suppose that for some (any) $a \models p$, the orbit of $a$ under $G$ is metrically dense in $p$. Fix $a \models p$ in $M$ and write $K$ for the open subgroup of elements of $G$ which fix $[a]$ setwise. Then the canonical representation $\pi$ of $G$ on $H_p(M)$ is equivalent to $\text{Ind}^G_K(\sigma)$ where $\sigma$ is the restriction of $\pi$ to $K$ on the Hilbert space $V$ spanned by $h([a])$.

**Proof.** Since $p$ is asymptotically free, $p$ is metrically locally compact and hence there is $\epsilon > 0$ such that for any $x, y \models p$, if $d(x, y) < \epsilon$ then $[x] = [y]$. Therefore $K$ is open in $G$. Let $A = [a]$ and write $\{A_i \mid i \in I\}$ for the orbit of $A$ under $G$ setwise (we ignore permutations of $A$). For every $i \in I$ pick $g_i \in \text{Aut}(M)$ which maps $A$ to $A_i$. Then $\{g_i \mid i \in I\}$ is a list of representatives for the left cosets of $K$ in $\text{Aut}(M)$.

Since $h$ is asymptotically free on $p$, the sets $h(A_i), i \in I$, are pairwise orthogonal in $H_p(M)$.

Write $\pi$ for the canonical representation of $\text{Aut}(M)$ on $H_p(M)$ and let $\sigma$ be the restriction of $\pi$ to $K$ on $V$, the vector space spanned by $h(A)$. Let $\iota = \text{Ind}^G_K(\sigma)$ and write $W = G \otimes_\sigma V$. We show that $\iota$ and $\pi$ are equivalent. Write also $g_i \otimes_K V$ for the subspace of $W$ given by $\{g_i \otimes v \mid v \in V\}$. Take $w \in W$ and write $w = \sum w_i$ where $w_i \in g_i \otimes v$. Let $P_i : g_i \otimes v \to V$ be the Hilbert space isomorphism taking $g_i \otimes v$ to $v$. We define

$$U(w) = \sum \pi(g_i)P_i(w_i).$$

Since $h(A_i) \perp h(A_j)$ for $i \neq j$, we have $\pi(g_i)P_i(w_i) \perp \pi(g_j)P_j(w_j)$, so $U$ is well-defined, and it is easy to check that $U$ is in fact a surjective isometry. $U$ intertwines $\iota$ and $\pi$: take $g \in G, i \in I, v \in V$. Then, writing $gg_i = g_jk$, we have

$$U(\iota(g)(g_i \otimes v)) = U(g_j \otimes_K \sigma(k)v) = \pi(g_j)(\sigma(k)v) = \pi(g_jk)v = \pi(g)(g_i \otimes v).$$

$\square$
Remark: Taking $a$ and $K$ as in Proposition 5.4, we note that $\text{Aut}(M/a)$ is a normal subgroup of $K$ contained in the kernel of $\sigma$. Write $G_1$ for the group of automorphisms of the set $[a]$. Then $\sigma$ factors through to a representation a subgroup of $G_1$. Since $[a]$ is a separable locally compact metric space, the closure of $K/\text{Aut}(M/a)$ in $G_1$ is locally compact under the metric $d(g,g') = \sup\{d(gx,g'x) \mid x \in [a]\}$. We say that the canonical representation of $G$ is obtained from the representation of $K/\text{Aut}(M/a)$ by inflation.

We recall the metric on the type spaces of a theory $T$ and the notion of $\omega$-near homogeneity, which occurs naturally in continuous logic:

Definition 5.5. Let $T$ be a continuous logic theory and $M \models T$.

Let $S(T)$ be a space of types of $T$ in a given tuple of variables. Let $M$ be a model of $T$ realising all the types in $S(T)$. Define the metric $d$ on $S(T)$ by $d(p,q) = \inf\{d(x,y) \mid x,y \in M, x \models p, y \models q\}$ where $d(x,y)$ is the metric on the relevant sort.

We say that $M$ if $\omega$-near-homogeneous is for any two finite tuples $a$ and $b$ in the same sort of $M$, for every $\epsilon > 0$ there exists $g \in \text{Aut}(M)$ such that $d(g(b),a) < d(tp(b),tp(a)) + \epsilon$.

In the above definition, the metrics on type space are defined with respect to the metrics on the relevant sorts which are always present in virtue of $T$ being a continuous logic theory of metric structures. These metrics were introduced in Section 1.1. For the purpose of this discussion, we might call these the main metrics.

Suppose that $\mathcal{H}$ is a Hilbert space functor for $T$ and let $S$ be a sort of $T$ mapping into $\mathcal{H}$ via an interpretation map $h$. Then we can define a metric on $S$ by the formula $\|hx - hy\|$. We will refer to this as the Hilbert space metric. It is worth noting that the main metric and the Hilbert space metric do not necessarily coincide. In that case, $\omega$-near-homogeneity does not necessarily entail $\omega$-near-homogeneity in the Hilbert space. A case of interest where they coincide is the case where $T$ is already given as the theory of a subset of a Hilbert space. $T$ can have additional relations but all the extra definable functions of $T$ must be uniformly continuous with respect to the Hilbert space metric.

We state a version of Mackey’s irreducibility criterion for the representations arising in Proposition 5.4. This lemma is more general than e.g. Proposition 4.1 in [Tsankov, 2012] since the subgroup $K$ below can be far from being a commensurator.

Lemma 5.6. Let $T$ be an arbitrary continuous logic theory with a Hilbert space functor $\mathcal{H}$ and $M \models T$. Suppose there is a complete type $p$ in a sort supporting $\mathcal{H}$ such that the interpretation map $h$ is asymptotically free on $p$. Write $H_p(M)$ for the $\bigwedge$-interpretable Hilbert space spanned by $h(p)$. For $x,y \models p$, write $x \sim y$ if $\text{bdd}(x) = \text{bdd}(y)$ and write $[x]$ for the equivalence class of $x$ under $\sim$.

Write $G = \text{Aut}(M)$. Fix $a \models p$ in $M$ and write $K$ for the subgroup of elements of $G$ which fix $[a]$ setwise. Let $\sigma$ be the restriction of the canonical representation $\pi$ of $G$ to $K$ on the Hilbert space $V$ spanned by $h([a])$. If $\pi$ is irreducible then $\sigma$ is irreducible.

Suppose that $N$ is an $\omega$-near-homogeneous model of $T$. Assume also that $N$ is either $\omega$-saturated or that the main metrics and the Hilbert space metrics coincide on the sorts supporting $\mathcal{H}$.

If $\sigma$ is an irreducible representation of $K$, then the canonical representation of $\text{Aut}(N)$ on $H_p(N)$ is irreducible.

Proof. The first part of the lemma is an easy fact about induced representations: if $\sigma = \sigma_1 \oplus \sigma_2$, then $\pi = \text{Ind}_{K}^{G}(\sigma_1) \oplus \text{Ind}_{K}^{G}(\sigma_2)$ so $\pi$ is reducible.

Conversely, suppose that $\sigma$ is irreducible. If we move to an $\omega_1$-strongly homogeneous elementary extension $M'$ of $M$, then the representation of the subgroup of $\text{Aut}(M')$ which fixes
[a] setwise is also irreducible on V. We now assume that M is $\omega_1$-strongly homogeneous and that $\sigma$ is irreducible. We will show that $\pi$ is irreducible and this will be enough to deduce the lemma for $\omega$-near-homogeneous models of T.

Recall that in Proposition 5.4 we expressed $H_p(M)$ as the orthogonal sum of subspaces $g_i \otimes_\sigma V$ where $(g_i)_{i \in I}$ is a set of coset representatives of $G/K$. Let 0 be an indexing element in I with $g_0 = e$ so that $V = g_0 \otimes_\sigma V$.

Suppose that we have a $G$-invariant subspace Z of $H_p(M)$ and write $P_Z$ for the orthogonal projection to Z. Then $P_Z$ commutes with G. Fix a nonzero $v \in V$. We can write $P_Z v = \sum_{j \in J} u_j$ where $J \subseteq I$ is countable and $u_j \in g_j \otimes_\sigma V$ is nonzero. Write $u_0$ for the element of the set $\{ u_j : j \in J \}$ which lies in V. Since the $u_j$ are pairwise orthogonal, $u_0 = 0$ would imply that $P_Z v = 0$. Switching if necessary to $Z^\perp$, we can assume that $P_Z v \neq 0$.

Write $J_0 = \{ j \in J \mid g_j a \notin \text{bdd}(a) \}$ and $J_1 = \{ j \in J \mid a \notin \text{bdd}(g_j a) \}$. Then $J = \{0\} \cup J_0 \cup J_1$.

We show that $J_0 = J_1 = \emptyset$. By $\omega_1$-strong homogeneity of M, we can find a sequence $(f_n)$ in $\text{Aut}(M/\text{bdd}(a))$ such for any $n \neq m$

$$\{ |f_n g_j a| \mid j \in J_0 \} \cap \{ |f_m g_j a| \mid j \in J_0 \} = \emptyset.$$  

Hence for all $n \neq m$, $\{ \pi(f_n)u_j \mid j \in J_0 \} \perp \{ \pi(f_m)u_j \mid j \in J_0 \}$. For all $n$ and $j \in J \setminus J_0$, we also have $\pi(f_n)u_j = u_j$. Now $\pi(f_n)P_Z(v) - \sum_{j \in J \setminus J_0} u_j$ and since $Z$ is G-invariant, we have $\sum_{j \in J \setminus J_0} u_j$ is at least as close to $v$ as $\sum_{j \in J} u_j$, we conclude that we have in fact $J_0 = \emptyset$.

Suppose for a contradiction that we have $j_1 \in J_1$. Let $J_2 = \{ j \in J \mid g_j a \notin \text{bdd}(g_j a) \}$. Note that $0 \in J_2$. By the same argument as above, we have $\sum_{j \in J \setminus J_2} u_j \in Z$. Therefore $\sum_{j \in J} u_j = \sum_{j \in J_2} u_j - \sum_{j \in J \setminus J_2} u_j \in Z$. Since $j_1 \notin J_2$, $\sum_{j \in J_2} u_j$ is an element of $Z$ closer to $v$ than $P_Z v$, a contradiction. Hence $J_1 = \emptyset$ and $P_Z v \in V$. Therefore, $Z \cap V$ is a nonempty $G$-invariant subspace of $V$. Since $\sigma$ is irreducible, we have $Z \cap V = V$. By irreducibility of $Z$, we have $g_i \otimes_\sigma V \subseteq Z$ for all $g_i$, and hence $Z = H_p(N)$. This proves that $\pi$ is irreducible.

Now let $N$ be an $\omega$-near-homogeneous model of $T$ and let $\pi'$ be the canonical representation of $G' = \text{Aut}(N)$ on $H_p(N)$. $\pi'$ is reducible if and only if there are nonzero $v, w \in H_p(N)$ such that for all $g \in G'$, $\pi'(g)v \perp w$. Fix nonzero $v, w \in H_p(N)$ and find elements $a, b$ in a sort of $N$ supporting $H$ such that $v = ha$ and $w = hb$ where $h$ is the interpretation map. Let $p = tp(a)$ and $q = tp(b)$. By irreducibility of the canonical representation in $\omega_1$-homogeneous models of $T$, there is $\lambda \neq 0$ such that $p(x) = \exists y(q(y) \land \langle hx, hy \rangle = \lambda)$.

If $N$ is $\omega$-saturated, we can find $c \in N$ realising $q$ and such that $\langle hc, hc \rangle = \lambda$. By $\omega$-near-homogeneity, we find $g \in G'$ such that $\langle \pi'(g)v, w \rangle$ is arbitrarily close to $\lambda$ and $\pi'$ is irreducible.

If the main metric and the Hilbert space metric coincide on the sort of $a, b$, then we have proved that $d(p, q)^2 = \langle hx, hx \rangle + \langle hy, hy \rangle - 2\lambda$ and we find $g \in G'$ such that $d(ga, b)^2$ is arbitrarily close to this value. This entails that $\langle \pi'(g)v, w \rangle \neq 0$ and $\pi'$ is irreducible.  

\section{5.2. Unitary Representations of Automorphism Groups of $\omega$-categorical Structures.}

In this section, we show that all unitary representations of automorphism groups of continuous logic $\omega$-categorical structures occur as canonical representations. This result can also be found in Ibarlucía (2021). We then show that Corollary 5.2 combined with Proposition 5.4 entails the classification theorem in Tsankov (2012). We recall some elementary facts about $\omega$-categoricity in the context of continuous logic. See Ben Yaacov et al., 2008 for details on all the background material.

**Definition 5.7.** Let $T$ be a continuous logic theory in a countable language. We say that $T$ is $\omega$-categorical if any two models of $T$ whose sorts are separable complete metric spaces $\omega$ are isomorphic.
Although the above definition is stated for theories in a countable language, we will allow ourselves to add arbitrarily many imaginary sorts to \( T \). This does not affect model theoretic properties of \( T \) and \( \omega \)-categoricity is preserved in the sense that there is a unique model of the theory such that the ‘real’ sorts form a separable metric space. In any case, we can always restrict the results which will follow so that only countably many imaginary sorts of \( T \) are mentioned, although this would make the statements slightly more awkward.

**Definition 5.8.** Let \( T \) be a continuous logic theory and \( p \) a type-definable set in a sort \( S \) of \( T \). We say that \( p \) is distance-definable if the distance function to \( p \) is a definable function on \( S \).

We say that a complete type \( p \) is principal if \( p \) is distance-definable.

Distance-definability is the continuous logic equivalent of ‘definability’ in classical logic (as distinguished from ‘type-definability’). Recall the following version of the Ryll-Nardzewski theorem in continuous logic:

**Theorem 5.9.** Let \( T \) be a continuous logic theory in a countable language. Then \( T \) is \( \omega \)-categorical if and only if every complete type in imaginary sorts of \( T \) is principal.

We record two important facts about \( \omega \)-categorical structures. Firstly, if \( M \) is the separable model of an \( \omega \)-categorical theory, then \( M \) is \( \omega \)-near homogeneous. Secondly, if \( T \) is \( \omega \)-categorical, then the metric topology on type spaces introduced in Definition \ref{def:metric-topology} and the logic topology coincide.

Next we describe a general construction which will allow us to apply our theory of interpretable Hilbert spaces to \( \omega \)-categorical theories. We note that an alternative route would be to develop a similar theory for interpretable Hilbert spaces supported by distance-definable sets. However, such a treatment is less fluid than our approach, where Hilbert spaces are supported by entire sorts of \( T \), so we prefer to introduce the construction for \( T^{\text{princ}} \).

**Definition 5.10.** Let \( T \) be a \( \omega \)-categorical continuous logic theory. Define an expansion \( T^{\text{princ}} \) of \( T \) by adding for every complete type \( p \) in finitely many variables a sort \( S_p \) and a map \( f_p : S_p \to p \). \( T^{\text{princ}} \) extends \( T \) and says that every \( f_p \) is an isometry onto \( p \).

The following is clear:

**Lemma 5.11.** Let \( T \) be an \( \omega \)-categorical continuous logic theory. Then the separable model \( M \) of \( T \) has a unique extension \( M^{\text{princ}} \) to a model of \( T^{\text{princ}} \). \( T^{\text{princ}} \) is \( \omega \)-categorical and \( \text{Aut}(M) = \text{Aut}(M^{\text{princ}}) \) in the obvious way.

Similarly, every \( \omega \)-saturated model \( M \) of \( T \) has a unique extension to a model \( M^{\text{princ}} \) of \( T \) with \( \text{Aut}(M) = \text{Aut}(M^{\text{princ}}) \). Every model of \( T^{\text{princ}} \) restricts to an \( \omega \)-saturated model of \( T \).

**Remark:** It would be of interest to write down a definition of bi-interpretability in continuous logic such that \( T^{\text{princ}} \) and \( T \) are bi-interpretable.

We will prove that all representations of groups \( \text{Aut}(M) \) where \( M \) is \( \omega \)-categorical occur as canonical representations in a way which lends itself well to the point of view of piecewise interpretable Hilbert spaces. We begin by proving a general lemma about unitary representations of automorphism groups.

**Lemma 5.12.** Let \( T \) be a continuous logic theory and let \( M \models T \). Let \( (H, \sigma) \) be a representation of \( G = \text{Aut}(M) \). Then there is a family \( V \) of nonzero vectors of \( H \) such that

1. \( H \) is the orthogonal sum of the spaces obtained by taking the closed span of \( G \cdot v \) for \( v \in V \).
2. For each \( v \in V \) there is a finite tuple \( a \) in \( M \), such that \( v \) is fixed by \( \text{Aut}(M/a) \). \( a \) does not lie in an imaginary sort of \( M \).
Proof. This is a minor modification of Lemma 3.1 in [Tsankov, 2012], which goes back to Lieberman, 1972. We repeat the argument here for convenience.

It is enough to prove that \( H \) contains a nonzero vector \( v \) satisfying (2) above. Moving to the orthogonal complement of the span of \( G \cdot v \) and applying Zorn’s lemma then gives the result.

Let \( w \in H \) be an arbitrary unit vector and fix \( \epsilon < 1 \). By continuity of the representation, there is a finite tuple \( a \in M \) and \( \delta > 0 \) such that the image of \( w \) under \( \{ \sigma(g) \mid g \in \text{Aut}(M), d(ga,a) < \delta \} \) is contained in the ball \( B(w,\epsilon) \). Let \( C \) be the closure of the convex hull of \( \text{Aut}(M/a) \cdot w \) and let \( v \) be the unique vector in \( C \) with least norm. \( v \) is nonzero. It is straightforward to check that \( C \) is invariant under \( \text{Aut}(M/a) \), so \( v \) is fixed by \( \text{Aut}(M/a) \). □

**Proposition 5.13.** Let \( T \) be an \( \omega \)-categorical continuous logic theory and let \( M \) be either the separable model of \( T \) or an arbitrary \( \omega \)-near-homogeneous \( \omega \)-saturated model of \( T \). Let \( G = \text{Aut}(M) \) and let \( \sigma \) be a representation of \( G \) on a Hilbert space \( H \). Then there is a family of Hilbert space functors \( (H_i)_{i \in I} \) on \( T^{princ} \) such that \( \sigma \) is equivalent to the canonical representation on the orthogonal sum of all \( H_i \) and each \( H_i \) is supported by a single complete type in \( T^{princ} \).

**Proof.** Let \( V \) be the family of vectors of \( H \) given by Lemma 5.12 and fix \( v \in V \). Let \( V \) be the Hilbert space spanned by \( G \cdot v \) and consider the restriction of \( \sigma \) to \( V \).

By definition of \( V \), there is a finite tuple \( a \in M \) such that \( v \) is fixed by \( \text{Aut}(M/a) \). Let \( p = \text{tp}(a) \). Since \( M \) is \( \omega \)-near-homogeneous, the orbit \( G \cdot a \) is metrically dense in \( p \). For every \( b \in G \cdot a \), define \( h(b) = \sigma(g)v \) where \( b = g(a) \). By continuity of the representation, \( h \) is continuous with respect to the metric on \( p \). Hence \( h \) induces a surjective continuous function from \( p \) to the closure of the orbit of \( v \) under \( \sigma \).

Now define \( f : p \times p \to \mathbb{R} \) by \( f(x,y) = \langle h(x), h(y) \rangle \). By \( \omega \)-near-homogeneity of \( M \), \( f \) induces a function on the space of types contained in \( p \times p \). By near-homogeneity again and by continuity of \( h \), \( f \) is continuous with respect to the metric topology on the space of types in \( p \times p \). By \( \omega \)-categoricity, \( f \) is definable. It follows that \( f \) is the inner product map of a Hilbert space functor \( H_v \) for \( T^{princ} \), supported by the sort \( S_p \).

Now we apply this construction to each \( v \in V \) and take the orthogonal sum. □

**Remark:** We emphasise that in Proposition 5.13 the Hilbert space functors can be taken to be supported by finitary imaginary sorts. This avoids situations where the imaginary elements supporting the functors could stand for enumerations of countable dense subsets of \( M \). For such an imaginary element \( a \), we would have \( \text{bdd}(a) = M \) and, as a result, the techniques from section 3 do not lead to a reduction of the representation of \( \text{Aut}(M) \). Proposition 5.13 shows that this situation can usually be avoided on a case-by-case basis.

In Proposition 5.13 if \( T \) is a classical logic theory, then the \( T^{princ} \) construction is not needed, since a complete type is a definable set \( D \) and we can define the interpretation map outside of \( D \) to be the trivial \( 0 \) map. Now the following result follows by definition, since finitary imaginary sorts in \( \omega \)-categorical theories are just classical imaginary sorts:

**Corollary 5.14.** Let \( T \) be a classical logic \( \omega \)-categorical theory and let \( M \) be an arbitrary \( \omega \)-homogeneous model of \( T \). Let \( \sigma \) be a unitary representation of \( M \) on a Hilbert space \( H \). Then there is a strict Hilbert space functor \( H \) such that \( \sigma \) is equivalent to the canonical representation of \( \text{Aut}(M) \) on \( H(M) \).

Let \( T \) be a classical logic \( \omega \)-categorical theory and let \( M \) be an \( \omega \)-homogeneous model of \( T \). Note that if \( p \) is a type in a classical imaginary sort of \( M \), then the relation \( x \in \text{acl}(y) \) is symmetric and transitive on \( p \). This is because \( \text{acl}(x) \cap p \) is a finite set with fixed cardinality. Applying Corollary 5.14, Corollary 5.20 and Proposition 5.14, we immediately see that every unitary representation of \( \text{Aut}(M) \) is an orthogonal sum of representations obtained by inflation.
from representations of groups of partial automorphisms of finite sets of the form \( \text{acl}(a) \cap \text{tp}(a) \) where \( a \) is a classical imaginary element of \( M \). This is precisely the classification theorem 5.2 in [Tsankov, 2012].

It remains to be seen if it is possible to build on the techniques developed in this paper in order to find a classification of the unitary representations of continuous logic \( \omega \)-categorical theories. We leave this an open question for future research.

5.3. Application to Unitary Representations of Groups with Scattered Orbits. In this section we show that our analysis in Section 5 gives information about all unitary representations containing a cyclic vector such that the weak closure of its orbit is locally compact. This is made possible by a general technique for constructing continuous logic structures with prescribed automorphism groups.

**Definition 5.15.** Let \( M \) be a continuous logic structure in a language \( \mathcal{L} \) with sorts \( (M_i)_{i \in I} \) and let \( G \) be a subgroup of \( \text{Aut}(M) \). We define a new structure \( M_G \) in a language \( \mathcal{L}_G \) as follows.

For every \( n \geq 1 \), for every Cartesian product \( P \) of \( n \) sorts of \( M \) and for every orbit \( O \) of \( G \) on \( P \), we add function symbols \( r_O : P \to [0, \infty) \). \( M_G \) is the structure obtained from \( M \) by interpreting each \( r_O \) as the distance in \( P \) from the metric closure of \( O \).

We refer to \( M_G \) as the \( G \)-specialisation of \( M \).

The following lemma is straightforward:

**Lemma 5.16.** Let \( M \) be a continuous logic structure, \( G \) a subgroup of \( \text{Aut}(M) \). Then \( \text{Aut}(M_G) \) is the closure of \( G \) in \( \text{Aut}(M) \), with the topology of pointwise convergence on \( M \).

Now let \( G \) be an arbitrary group and let \( H \) be a Hilbert space with a cyclic faithful representation \( \sigma \) of \( G \). Let \( v \) be a cyclic vector and let \( X \) be the closure of the orbit of \( v \) in \( H \). Let \( M \) be the continuous logic structure consisting of \( X \) with the inner product map on \( X \) induced from \( H \). Observe that the topology of pointwise convergence on \( M \) is the coarsest topology on \( G \) under which the representation \( \sigma \) is continuous. We will always work with this topology on \( G \).

The following lemma is clear:

**Lemma 5.17.** Take \( G, H \) and \( \sigma \) be as above. Then \( M_G \) is an atomic model (i.e. all types that are realised are principal) and the action of \( G \) on \( M_G \) is \( \omega \)-near-homogeneous.

\( X \) is a complete type in \( M_G \) and there is a Hilbert space functor \( \mathcal{H} \) supported by \( X \) such that the restriction to \( G \) of the canonical representation of \( \text{Aut}(M_G) \) on \( H(M_G) \) is equivalent to \( \sigma \).

We are interested in applying Theorem 5.14 to the structure \( M_G \). In Section 5 we worked with an \( \omega_1 \)-saturated structure. Knowing that \( M_G \) is atomic and \( \omega \)-near-homogeneous is sufficient to recover some information about saturated elementary extensions of \( M_G \).

**Definition 5.18.** Let \((X,d)\) be a metric space. We say that \( X \) is uniformly locally compact if there is \( \delta > 0 \) such that for all \( \epsilon > 0 \), there is \( n_\epsilon \in \mathbb{N} \) such that for all \( x_1, \ldots, x_{n_\epsilon} \in X \), if \( d(x_i, x_j) \leq \delta \) for all \( i, j \), then \( d(x_i, x_j) \leq \epsilon \) for some pair \( i \neq j \).

**Proposition 5.19.** Suppose that the weak closure of \( X \) is uniformly locally compact. Then \( \sigma \) is equivalent to an orthogonal sum of representations \( (\tau_\alpha) \) such that each \( \tau_\alpha \) has a cyclic vector \( v_\alpha \) with the property that for every \( \alpha > 0 \), the set of \( \tau_\alpha \)-conjugates \( v \) of \( v_\alpha \) such that \( |\langle v, v_\alpha \rangle| \geq \alpha \) is precompact.

**Proof.** Let \( N \) be an \( \omega_1 \)-saturated elementary extension of \( M_G \). Let \( v_1, \ldots, v_n \) be a collection of vectors in the weak closure of \( X(N) \). Find a sort \( S_i \) supporting \( \mathcal{H} \) and a complete type \( p \in S_1^n \) with a realisation \( a_1, \ldots, a_n \) such that \( \langle h_i a_k, v_k \rangle = v_k \) for all \( k \leq n \). Since the types in \( S_1^n \) that are...
realised in $M_G$ are dense in the space of types in $S^n$, we can find $q$ realised in $M_G$ such that $d(p, q) \leq \epsilon$ for arbitrarily small $\epsilon$.

Let $b_1, \ldots, b_n$ be a realisation of $q$ in $M_G$. By construction of $q$, for every $k \leq n$, $m \geq 0$ and $\delta > 0$, $b_k$ satisfies the formula

$$\exists y \in S_n, \left( d(x, y) \leq \epsilon \land \forall z_1, \ldots, z_m \in X, \exists z_{m+1} \in X, \bigwedge_{j \leq m} |\langle h z_j, h z_{m+1} \rangle - \langle h z_j, h y \rangle| \leq \delta \right)$$

It follows that in $M_G$ we can construct a sequence $(z_n)$ in $X$ such that $(h z_n)$ is weakly convergent to a vector $w_k$ with $\|w_k - h b_k\| \leq \epsilon$. Then the distances between the vectors $w_1, \ldots, w_n$ are approximately equal to the distances between $v_1, \ldots, v_n$. Hence if $P(X)$ is not locally compact in $N$, then $P(X)$ is not uniformly locally compact in $M$.

Now we can apply Theorem 3.14 to express $H(N)$ as an orthogonal sum of interpretable Hilbert spaces supported by an asymptotically free type. By Theorem 3.8, if $p$ is one of these types, then $p$ has a realisation which is in the bounded closure of some realisation of $X$. Hence $p$ is principal and this yields a decomposition of $H(M_G)$. The proposition now follows from the definition of asymptotically free types. \qed

Proposition 5.19 leads us to consider asymptotic freedom in purely group theoretic terms. If $\sigma$ is a representation of $G$ on $H$ and $v \in H$, say that the orbit of $v$ is asymptotically free if for every $\alpha > 0$, the set of conjugates $w$ of $v$ such that $|\langle w, v \rangle| \geq \alpha$ is precompact. Representations with asymptotically free orbits occur frequently. For example, if $G$ is a locally compact group and $\sigma$ is a representation of $G$ with vanishing matrix coefficients, then every orbit in $\sigma$ is asymptotically free.

More specifically, the classical result of [Howe and Moore, 1979] shows that all irreducible representations of connected algebraic groups over a local field of characteristic 0 have asymptotically free orbits. Let $G$ be such a group with an irreducible representation $\sigma$. Let $P \subseteq G$ be the preimage under $\sigma$ of the circle group in $U(H)$. Then Theorem 6.1 shows that for any $v \in H$, the map $g \mapsto |\langle \sigma(g)v, v \rangle|$ tends to 0 on $G/P$. Since the action by $P$ does not affect compactness, we see that the orbit of $v$ is asymptotically free.

In fact, the assumption of connectedness in the above is not necessary for our purposes. Indeed, if $G$ is an algebraic group as above but not necessarily connected, we can find a connected normal algebraic subgroup $G_0$ such that $G/G_0$ is finite. Then any representation $\sigma$ of $G$ on $H$ splits as a finite orthogonal sum of irreducible representations of $G_0$. For any $v \in H$, the $G$-orbit of $v$ is a finite union of $G_0$ orbits in the irreducible subrepresentations and hence the $G$-orbit of $v$ is asymptotically free.

See [Bekka and Mayer, 2000] for an overview of the Hower-Moore result and its extension to various additional cases. Although our results do not give new information about representations with the Howe-Moore property, we note that Proposition 5.4 and Lemma 6.6 apply to these representations and this raises the prospect of future applications.

If the weak closure of $X$ is only locally compact, it is still possible to replicate much of the proof of Theorem 3.14 albeit deriving a conclusion weaker than in Proposition 5.19.

**Lemma 5.20.** In $H(M_G)$, $P(X)$ is closed under the maps $P_{\text{bdd}(A)}$ for arbitrary $A \subseteq M_G$ and for any $v, w \in P(X)$, if $w = P_{\text{bdd}(A)} v$ then there is a sequence in $\text{tp}(v)$ in $M_G$ converging weakly to $w$.

**Proof.** Let $N$ be an $\omega_1$-saturated elementary extension of $M_G$. We know from Lemma 5.22 that the lemma holds in $N$, so $P(X)$ is also closed in $M_G$ under the maps $P_{\text{bdd}(A)}$. 

Now suppose \( v, w \in \mathcal{P}(X) \) and \( w = P_{\text{bdd}(A)}(v) \). Find a complete type \( p \) in an imaginary sort \( S_i \) of \( M_G \) supporting \( \mathcal{H} \) and \( x \) in \( p \) such that \( h_i x = v \). By Lemma 3.2 applied in \( N \), we know that \( w \) is in the weak closure of the set \( h_i p \). Since \( p \) is realised in \( M_G \), \( p \) is principal and there is a definable function \( d \) which defines the distance to \( p \) in the type-space containing \( p \).

Now we work entirely in \( M_G \). Suppose that we have found \( x_1, \ldots, x_n \) in \( p \) such that for all \( m < n \), \(|\langle h_i x_m, h_i x_m \rangle - \langle w, h_i x_m \rangle| \leq 1/n \). Fix \( \epsilon > 0 \) small enough, to be determined below. Since \( w \) is in the weak closure of \( h_i p \), \( w \) satisfies the formula

\[
\forall x_1, \ldots, x_n \in S_i, \exists y \in S_i, \left( d(y) \leq \epsilon \land \bigwedge_{m \leq n} \langle h_i y, h_i x_m \rangle - \langle w, h_i x_m \rangle \leq 1/2n \right)
\]

Find a realisation \( y \) of this formula over the tuple \( x_1, \ldots, x_n \) previously constructed. By \( \omega \)-near-homogeneity, there is \( x_{n+1} \) in \( p \) in \( M_G \) with \(|h_i x_{n+1} - h_i y| \leq \epsilon \). Choose \( \epsilon > 0 \) small enough so that \(|\langle h_i x_{n+1}, h_i x_m \rangle - \langle w, h_i x_m \rangle| \leq 1/(n + 1) \) for all \( m \leq n \).

In this way, we construct an infinite sequence \((x_n)\) in \( p \) such that \( (h_i x_n) \) converges weakly to \( w \) and the lemma is proved. \( \square \)

**Proposition 5.21.** Suppose that the weak closure of \( X \) in \( H \) is locally compact. Then \( \sigma \) is equivalent to an orthogonal sum of representations \((\tau_i)\) such that each \( \tau_i \) has a cyclic vector \( v_i \) satisfying the following: if \( w \) is a conjugate of \( v_i \), such that \( \langle w, v_i \rangle \neq 0 \), then the orbit of \( w \) under the stabilizer of \( v_i \) is precompact.

**Proof.** Let \( \Pi \) be the subset of \( \mathcal{P}(X) \) of elements of the form \( P_{A_1} \ldots P_{A_n} h x \) where \( A_1, \ldots, A_n \) are \( \text{bdd} \)-closed subsets of \( M_G \) and \( x \in X \). \( \Pi \) is a partial order with the order inherited from \( \mathcal{P}(X) \) in a saturated elementary extension of \( M_G \). It is straightforward to adapt the proof of Theorem 3.8 to deduce that for any \( \text{bdd} \)-closed subsets \( A_1, A_2 \) of \( M_G \), we have \( P_{A_1} P_{A_2} = P_{A_2} P_{A_1} = P_{A_1 \cap A_2} \) in \( H(M_G) \). Similarly, we can adapt the proof of Lemma 3.10 to show that \( \Pi \) is a well-founded partial order.

Let \((q_\alpha)\) be an enumeration of the complete types in \( p^+ \) which are realised in \( M_G \), contained in \( \Pi \) and such that \( h^+ q_\alpha \neq 0 \). As in the preparation for Lemma 3.11, we can assume that the enumeration \( q_\alpha \) respects the partial order on \( \Pi \). Since each \( q_\alpha \) is principal, it is straightforward to adapt the proof of Lemma 3.11 to show that for every \( \alpha \), the projection to the orthogonal complement of the subspace generated by \( \bigcup_{\beta < \alpha} q_\beta \) is a definable map on \( q_\alpha \) in \( M_G \). Write \( \hat{h}_\alpha : q_\alpha \to H(M_G) \) for each such map. By well-foundedness of \( \Pi \), \( H(M_G) \) is the orthogonal sum of the subspaces generated by \( \hat{h}_\alpha q_\alpha \). As before, we can assume that \( \hat{h}_\alpha \) is injective.

Finally, for all \( \alpha \) and realisations \( x, y \) of \( q_\alpha \) in \( M_G \), we have \( x \in \text{bdd}(y) \) or \( \langle h_i x, \hat{h}_\alpha y \rangle = 0 \). This follows by a straightforward modification of Lemma 3.12. The proposition follows. \( \square \)

6. FURTHER DISCUSSION

In this section, we discuss Definition 3.1 and Theorem 3.8 from a model theoretic point of view. At the heart of the following discussion lies the fact that Hilbert spaces are self-dual, so that we can move freely between vectors and functionals, which in model theoretic terms is the same as moving between partial types and their canonical bases.

We fix a theory \( T \) in continuous logic and a Hilbert space functor \( \mathcal{H} \) supported by \((S_i)_i \). Fix a sort \( S \) among the sorts \((S_i)_i \) with interpretation map \( h \) and inner product map \( f \). Fix also a type-definable set \( p \) in \( S \) such that for all \( x \models p \), \(|h x|\) is constant. For example, \( p \) could be a complete type. We also fix a \( \kappa \)-saturated model \( M \) of \( T \) where \( \kappa \geq \omega_1 \). Write \( H(S) \) for the subspace of \( H(M) \) spanned by the set \( h(S) \). Let \( \mathcal{P}(p) \) be as in Definition 3.1 and let \( p^+ \) be the type-definable set in an imaginary sort of \( M \) with the map \( h^+ : p^+ \to \mathcal{P}(p) \) constructed in Lemma 3.4.
Recall that $p^+$ is a set of canonical parameters of $f$-types over $M$ consistent with $p$. For $z \models p^+$, write $q^z$ for the $f$-type over $M$ consistent with $p$ defined by $z$. In the notation of Lemma 3.3, $q^z$ is defined over $M$ by the function $df(z,y)$. Note that $df(z,y)$ is stable on $S^+ \times S$, since $df(z,y) = \langle h^+z, y \rangle$. $q^z$ is uniquely determined by $z$ because the function $\|h\|$ is constant on $p$. Moreover, for any $z \models p^+$, $q^z$ induces a type $q^*_H$ over all of $H(S)$: if $a$ is any realisation of $q^z$ over $M$ and $v \in H(S)$, we must have $\langle ha, v \rangle = \langle h^+z, v \rangle$.

Now let $r(z)$ be a df-type over $M$ consistent with $p^+$. $r$ also induces a type $r_H$ over $H(S)$: for $b \models r \cup p^+$ in an elementary extension of $M$, let $r_H$ be the type of $b$ over $H(S)$ in the function $(z,v) \mapsto \langle h^+z, v \rangle$. $r_H$ is uniquely determined by $r$ since for all $x \in S$ in $M$, $\langle h^+b, hx \rangle = df(b, x)$. Observe that $r_H$ does not specify the norm of $h^+b$. However, this is not an obstacle since we are usually interested in $P_M h^+b$, and the norm of this vector is determined by the values $\langle h^+b, hx \rangle$ for $x \in M$.

**Lemma 6.1.** $p^+$ has built-in canonical bases for df-types: for any $r \in S_{df}(M)$ consistent with $p^+$, there is $b \models p^+$ in $M$ such that $r$ is definable over $b$. Moreover, for any $v \in H(S)$, $r_H(z) \models \langle h^+z, v \rangle = \lambda$ if and only if $q^*_H(x) \models \langle hx, v \rangle = \lambda$.

**Proof.** We work in the expansion $T'$ and we take the imaginary sort of $p^+$ to be a real sort of $T$, so that Proposition 2.19 applies.

We have seen that $\mathcal{P}(p)$ is a set of canonical bases for $f$-types extending $p$. Define $\mathcal{P}(p^+) = \{ P_{\text{bd}}(A) \| h^+z \mid z \models p^+, A \subseteq M \text{ small}\}$. Then $\mathcal{P}(p^+) = \mathcal{P}(p)$ by Lemma 3.3. Let $r \in S_{df}(M)$ be consistent with $p^+$. For $a \models r \cup p^+$ in an elementary extension of $M$, $r$ is definable over $P_M h^+a \in \mathcal{P}(p^+)$. Take $b$ the preimage in $p^+$ of $P_M h^+a$ under $h^+$.

Now for any $v \in H(S)$, $r_H \models \langle h^+z, v \rangle = \lambda$ if and only if $\langle h^+b, v \rangle = \lambda$ if and only if $q^*_H \models \langle hx, v \rangle = \lambda$.

From now on, we assume that the imaginary sort containing $p^+$ is a real sort of $T$. By Lemma 6.1, the results of Proposition 2.19 apply and forking independence inside $p$ or $p^+$ is equivalent to forking independence in the Hilbert space $H(S)$.

We define a partial order which gives a model-theoretic characterisation of the partial order $\leq$ on $\mathcal{P}(p)$.

**Definition 6.2.** From now on, we write $\leq_p$ for the partial order on $\mathcal{P}(p)$ as defined in Definition 3.7.

Define the relation $L_1$ on $p^+ \times p^+$ as follows: $L_1(z, z')$ if and only if $q^*_H$ extends $q^z \mid \text{bdd}(z')$. Define the relation $\leq_1$ on $p^+ \times p^+$ as the transitive closure of $L_1$.

Observe that it follows from elementary stability theory that $L_1(z, z')$ and $z \neq z'$ if and only if $q^*_H$ is a forking extension of $q^z \mid \text{bdd}(z')$.

**Lemma 6.3.** For all $z, z' \models p^+$, we have $L_1(z, z')$ if and only if there exists a bdd-closed small $A \subseteq M$ such that $h^+z' = P_A h^+z$. Hence $\leq_1$ is a partial order and is anti-isomorphic to $\leq_p$.

**Proof.** Note that $h^+z' = P_A h^+z$ if and only if $h^+z' = P_{\text{bdd}(z')} h^+z$, so we need to show that $L_1(z, z')$ if and only if $h^+z' = P_{\text{bdd}(z')} h^+z$. This is clear from the definitions of $L_1$, $q^*_H$ and $q^z$.

In the spirit of Lemma 6.1, we can also give a characterisation of $\leq_p$ in terms of df-types. Note that the same definition in terms of $f$-types gives completely analogous results in the results which will follow.

**Definition 6.4.** Define the relation $L_2$ on df-types over small bdd-closed subsets of $M$ consistent with $p^+$ as follows: if $r_1 \in S_{df}(A)$ and $r_2 \in S_{df}(B)$ and $z_1, z_2$ are the canonical parameters
Lemma 6.7. \( f \) is a one-based basis of \( M \) consistent with \( p \). Then if \( z_1, z_2 \models p^+ \) is the nonforking extension of \( (r_1)_{H} \) to \( M \).

Proof. By definition of \( (r_1)_{H} \) and \( (r_2)_{H} \), \( L_2(r_1, r_2) \) if and only if for all \( v \in \text{bdd}(z_2) \cap H, \langle h^+ z_1, v \rangle = \langle h^+ z_2, v \rangle \). This is equivalent to saying \( h^+ z_2 = P_{\text{bdd}(z_2)} h^+ z_1 \) and hence \( L_1(z_1, z_2) \).

Lemma 6.8. Let \( q \in S_{\text{df}}(A) \), and \( q \) is one-based if and only if \( d^+ \) is the nonforking extension of \( q \) over \( a \) and \( \text{bdd}(a) \).

Proof. Suppose first that \( f \)-types extending \( p \) are one-based. Take \( z, z' \in p^+ \) such that \( L_1(z, z') \). Then if \( a \) is a realisation of \( q \) over \( M \), we have both \( z \in \text{bdd}(a) \) and \( z' \in \text{bdd}(a) \). Taking independent realisations of \( q \) over \( z, z' \in \text{bdd}(z) \), we get \( z' \in \text{bdd}(z) \). Now let \( A \) be a small bdd-closed subset of \( M \) and take \( r \in S_{\text{df}}(A) \) consistent with \( p^+ \). Suppose \( a \) realises \( r \) and write \( b \) for the canonical basis of \( r \) in \( p^+ \), so that \( h^+ b = P_A h^+ a \). Let \( r' \) be the \( d^+ \)-type defined by \( b \). We have \( L_2(r', r) \) hence \( L_1(a, b) \) and hence \( b \in \text{bdd}(a) \), so \( d^+ \)-types consistent with \( p^+ \) are one-based.

The proof that if \( d^+ \)-types are one-based then \( f \)-types are one-based is similar.

Suppose now that \( f \)-types consistent with \( p \) are one-based and let \( A, B \subseteq M \) be bdd-closed. Take \( x \) in \( p^+ \) and let \( z \in p^+ \) be the canonical parameter of \( \text{tp}_f(x/B) \), so that \( h^+ z = P_B h^+ x \) and \( z \in \text{bdd}(z) \cap B \). Then \( P_A P_B h^+ x = P_A h^+ z \), which is the canonical parameter of \( \text{tp}_{\text{df}}(z/A) \). By one-basedness, this is a vector in \( \text{bdd}(z) \cap A \), so \( P_A P_B h^+ x \in A \cap B \).

Conversely, let \( q \in S_f(A) \) be consistent with \( p \) where \( A \subseteq M \) is small. Let \( a \models q \) in \( M \). The canonical parameter of \( q \) is some element \( z \models p^+ \) satisfying \( h^+ z = P_A h a \). Then \( h^+ z = P_A P_{\text{bdd}(a)} h a = P_{A \cap \text{bdd}(a)} h a \in A \cap \text{bdd}(a) \).

We now turn to a discussion of ranks.

Definition 6.6. For small bdd-closed \( A \subseteq M \), an arbitrary definable function \( g(x, y) \), and \( q \in S_{\text{df}}(A) \), we say that \( q \) is one-based if for any \( a \models q \) in \( M \), \( q \) is definable over \( A \cap \text{bdd}(a) \).

Lemma 6.7. \( f \)-types over \( M \) consistent with \( p \) are one-based if and only if \( d^+ \)-types over \( M \) consistent with \( p^+ \) are one-based if and only if \( P_A P_B = P_{A \cap B} \) on \( H(p) \).

Proof. Suppose first that \( f \)-types extending \( p \) are one-based. Take \( z, z' \in p^+ \) such that \( L_1(z, z') \). Then if \( a \) is a realisation of \( q \) over \( M \), we have both \( z \in \text{bdd}(a) \) and \( z' \in \text{bdd}(a) \). Taking independent realisations of \( q \) over \( \text{bdd}(z) \), we get \( z' \in \text{bdd}(z) \). Now let \( A \) be a small bdd-closed subset of \( M \) and take \( r \in S_{\text{df}}(A) \) consistent with \( p^+ \). Suppose \( a \) realises \( r \) and write \( b \) for the canonical basis of \( r \) in \( p^+ \), so that \( h^+ b = P_A(h^+ a) \). Let \( r' \) be the \( d^+ \)-type defined by \( b \). We have \( L_2(r', r) \) hence \( L_1(a, b) \) and hence \( b \in \text{bdd}(a) \), so \( d^+ \)-types consistent with \( p^+ \) are one-based.

The proof that if \( d^+ \)-types are one-based then \( f \)-types are one-based is similar.

Suppose now that \( f \)-types consistent with \( p \) are one-based and let \( A, B \subseteq M \) be bdd-closed. Take \( x \) in \( p^+ \) and let \( z \in p^+ \) be the canonical parameter of \( \text{tp}_f(x/B) \), so that \( h^+ z = P_B h^+ x \) and \( z \in \text{bdd}(z) \cap B \). Then \( P_A P_B h^+ x = P_A h^+ z \), which is the canonical parameter of \( \text{tp}_{\text{df}}(z/A) \). By one-basedness, this is a vector in \( \text{bdd}(z) \cap A \), so \( P_A P_B h^+ x \in A \cap B \).

Conversely, let \( q \in S_f(A) \) be consistent with \( p \) where \( A \subseteq M \) is small. Let \( a \models q \) in \( M \). The canonical parameter of \( q \) is some element \( z \models p^+ \) satisfying \( h^+ z = P_A h a \). Then \( h^+ z = P_A P_{\text{bdd}(a)} h a = P_{A \cap \text{bdd}(a)} h a \in A \cap \text{bdd}(a) \).

We now turn to a discussion of ranks.

Definition 6.8. If \( (Q, \leq) \) is an arbitrary partial order, we define the foundation rank \( F(x) \) of \( x \in Q \) as follows:

1. \( F(x) \geq 0 \) for all \( x \)
2. \( F(x) \geq \lambda \) for limit ordinal \( \lambda \) if \( F(x) \geq \alpha \) for all \( \alpha < \lambda \)
3. \( F(x) \geq \alpha + 1 \) if there is \( y \leq x \) such that \( F(y) \geq \alpha \)

\( F(x) = \infty \) and \( F(x) = \alpha \) are defined in the expected way.

The proof of the next proposition is already implicit in the proof of Theorem 3.8.

Proposition 6.9. Let \( F_P \) be the foundation rank of \( \leq_P \). If for every (any) \( x \models p \) we have \( F_P(hx) < \infty \), then \( f \)-types over \( M \) consistent with \( p \) are one-based.
Proof. This can be seen by examining the proof of Theorem 3.8. In that proof we constructed sequences \((a_n)\) and \((b_n)\) in \(P(p)\) and we showed that if they are eventually constant then Theorem 3.8 follows. Since \((a_n)\) and \((b_n)\) are decreasing sequences in \(P(p)\) and the foundation rank is ordinal-valued, \((a_n)\) and \((b_n)\) must be eventually constant. \(\square\)

In light of the anti-isomorphism between \(\leq_p\) and the partial orders \(\leq_1\) and \(\leq_2\), it is natural to define a local continuous logic version of \(U\)-rank which captures forking inside the Hilbert space:

**Definition 6.10.** For any bdd-closed subset of \(M\) and \(q\) a partial df-type in \(S^+ \times \ldots \times S^+\) over \(A\) define the relation \(V(q) \geq \alpha\) for \(\alpha\) an ordinal as follows:

1. \(V(q) \geq 0\) for all \(q\)
2. \(V(q) \geq \alpha + 1\) if there is some bdd-closed \(B \supset A\) and \(q'\) over \(B\) extending \(q\) such that \(V(q') \geq \alpha\) and \(q'\) forks over \(A\) with respect to the function \(df\).

If \(V(q) \geq \alpha\) for all \(\alpha\), we say \(V(q) = \infty\). We say \(V(q) = \alpha\) if \(V(q) \geq \alpha\) and \(V(q) < \alpha + 1\).

If \(\mathfrak{a} \in p^+ \times \ldots \times p^+\), write \(V(\mathfrak{a}/A)\) for \(V(tp(\mathfrak{a}/A))\)

**Lemma 6.11.** Let \(A \subseteq M\) be small and bdd-closed and let \(r\) \in \(S_d(A)\) be consistent with \(p^+\). Let \(z\) be the canonical parameter of \(r\) in \(p^+\). Then \(V(r)\) is equal to the foundation rank of \(z\) in \(p^+\) with the partial order \(\leq_1\).

**Proof.** If \(r \in S_d(A)\) is consistent with \(p^+\), then \(r\) has a canonical parameter \(z \in p^+\) which is contained in \(A\). We have already remarked that \(L_1(z, z')\) and \(z \neq z'\) if and only if \(q^*_H\) is a forking extension of \(q^*_H \upharpoonright \text{bdd}(z')\). The lemma follows easily. \(\square\)

Since the \(V\)-rank is a local version of the classical \(U\)-rank, we have \(V(r) \leq U(r)\) when we are in a classical logic setting. Therefore the next proposition gives an interesting criterion for obtaining one-basedness:

**Proposition 6.12.** Assume that there is some \(n \in \mathbb{N}\) such that for all 1-types \(r \in S_d(A)\) consistent with \(p^+\), \(V(r) \leq n\) where \(A \subseteq M\) is small and bdd-closed. Suppose that \(n\) is the least such integer. Then for any \(r \in S_d(A)\), if \(z\) is the canonical parameter of \(r\) in \(p^+\) then \(V(r) = n - Fp(h^+z)\). Hence \(f\)-types over \(M\) consistent with \(p\) are one-based.

**Proof.** This follows directly from Lemmas 3.8, 6.11 and Proposition 6.9. \(\square\)

When \(T\) is a classical logic theory and \(f\) is strictly definable, the \(V\)-rank coincides with the Shelah \(\omega\)-local rank defined in [Shelah, 1978]. We write down the definition here for convenience.

**Definition 6.13** ([Shelah, 1978], II.1.1). Let \(T\) be a classical logic theory, let \(\Delta(x, y)\) be a set of classical formulas and \(p(x)\) a partial type. For \(\alpha\) an ordinal, we define \(R(p) \geq \alpha\) as follows:

1. \(R(p) \geq 0\) if \(p\) is consistent
2. For \(\lambda\) a limit ordinal \(R(p) \geq \lambda\) if \(R(p) \geq \alpha\) for all \(\beta < \alpha\)
3. \(R(p) \geq \alpha + 1\) if for every finite \(p' \subseteq p\) and every \(n < \omega\) there are \(\Delta\)-types \((q_m(x))_{m < n}\) such that:
   a. for \(m \neq m' < n\), \(q_m(x) \cup q_{m'}(x)\) is inconsistent
   b. \(R(p \cup q_m) \geq \alpha\) for all \(m\).

We write \(R(p) = \alpha\) if \(R(p) \geq \alpha\) and \(R(p) \not\geq \alpha + 1\).

We leave it as an exercise to the reader to show that the \(V\)-rank and the \(\omega\)-local rank coincide when \(f\) is strictly definable in a classical logic theory. This can be proved entirely by using results of [Shelah, 1978].
Example: Let \( T \) be the classical logic theory of an infinite collection of equivalence relations \((E_n)\) such that \( E_0 \) is the trivial \( x = y \) relation and for all \( n \), each \( E_{n+1} \)-class is an infinite union of \( E_n \)-classes. Fix \( M \models T \) and for \( x \in M \) write \([x]_n\) for the \( E_n \)-class of \( x \). Viewing \( T \) as a continuous logic theory, let \( f(x, y) \) be the definable function \( f(x, y) = 1/2^n \) if \( x \in E_n y \) and \( \neg x E_{n-1} y \). It is clear that \( f \) is definable.

\( f \) defines a Hilbert space functor \( \mathcal{H} \) as follows: for every \( n \), let \( \{ e_{[x]_n} \mid x \in M \} \) be an orthonormal family of vectors in a Hilbert space \( H \). For \( n \neq m \), we take the families to be pairwise orthogonal. For every \( x \in M \) set \( h x = \sum_{k=0}^{\infty} e_{[x]_k} / 2^{k+2} \). Then \( f(x, y) = \langle hx, hy \rangle \).

Take \( M \) to be \( \omega_1 \)-saturated and let \( \mathcal{P} = \mathcal{P}(x = x) \) be the partial order associated to the set \( h(M) \). By considering indiscernible sequences in \( M \), we see that \( \mathcal{P} \) consists of the set of vectors \( \{ \sum_{k=0}^{\infty} e_{[x]_k} / 2^{k+2} \mid x \in M, n \geq 1 \} \). It is clear that \( f \)-types over \( M \) are one-based but that the partial order \( \mathcal{P} \) is not well-founded, showing that the converse of Proposition 6.9 is not true. Observe also that for every \( z \in \mathcal{P}, V(z) < \omega \), showing that the uniform bound in Proposition 6.12 is necessary.

Finally, we observe that the conclusion of Theorem 6.14 is nevertheless true for \( \mathcal{H} \): the sets \( \{ e_{[x]_n} \mid x \in M \} \) give a decomposition of \( H(M) \) into pairwise orthogonal asymptotically free families given by complete types. This suggests that it may be possible to find decomposition theorems for interpretable Hilbert spaces in settings which are more general than the scattered setting. We leave this as an open question.

We now show how asymptotically free types give strongly minimal local reducts of \( T \).

**Definition 6.14.** Let \( T \) be a continuous logic theory. We say that \( T \) is strongly minimal if for every \( M \models T \), every \( M \)-definable function \( f(x) \) in one-variable into \( \mathbb{R} \) has a unique generic value \( \alpha \), meaning that for any \( \beta \neq \alpha \), the set \( f(x) = \beta \) in \( M \) is compact.

**Lemma 6.15.** Let \( T \) be a strongly minimal continuous logic theory. Then for any \( M \models T \), \( \text{bdd} \) is a pregeometry on subsets of \( M \).

**Proof.** We only have to prove exchange: for \( A \subseteq M \) and \( a \in M \), if \( a \in \text{bdd}(Ab) \) and \( a \notin \text{bdd}(A) \), then \( b \in \text{bdd}(Aa) \). Recall that a type \( p \in S(\text{bdd}(A)) \) is uniquely determined by the values \( f(p) \) where \( f \) is any \( Ab \)-definable and into \( \mathbb{R} \). Since those definable functions all have a generic value, there is an \( Ab \)-definable function \( f \) with generic value \( \alpha \) such that \( f(a) \neq \alpha \). Moreover, since \( a \notin \text{bdd}(A) \), \( f \) is not \( A \)-definable and we can write \( f(a, b) \neq \alpha \). It follows that \( b \in \text{bdd}(Aa) \). \( \square \)

**Definition 6.16.** Let \( T \) be a strongly minimal continuous logic theory. We say that \( T \) is disintegrated if the pregeometry is trivial for any \( M \models T \), in the sense that \( \text{bdd}(A \cup B) = \text{bdd}(A) \cup \text{bdd}(B) \) for any \( A, B \subseteq M \).

Now let \( T \) be a continuous logic theory with a Hilbert space functor \( \mathcal{H} \). Let \( p \) be a complete type of \( T \) with an interpretation map \( h : p \rightarrow \mathcal{H} \) which is asymptotically free. Let \( f \) be the inner product map on \( p \). We add \( f \) as a primitive symbol to the language. Let \( M \models T \) be \( \omega_1 \)-saturated and let \( T_p \) be the theory of the set realisations of \( p \) in \( M \) in the language with \( f \) as its only symbol.

**Proposition 6.17.** \( T_p \) is a strongly minimal disintegrated continuous logic theory.

**Proof.** Write \( X \) for the set of realisations of \( p \) in \( M \) viewed as a model of \( T_p \). We consider the equivalence relation \( \sim \) on \( X \) defined to be the transitive closure of the relation \( f(x, y) \neq 0 \). For \( x \in X \), write \([x]\) for the \( \sim \)-class of \( x \). Note that \([x] \subseteq \text{bdd}(x) \) in \( M \). Since \( X \) is a complete type in \( M \), any two \( \sim \)-classes are isomorphic and the isomorphism type of these classes does not depend on the choice of model \( M \).
Let \((x_i)_I\) be a set of representatives of \(\sim\)-classes in \(X\). Then any bijection \(I \to I\) can be extended to an \(X\)-automorphism which respects \(f\). Hence in the theory \(T_p\), for \(A \subseteq X\), \(\text{bdd}(A) = \bigcup_{a \in A} [a]\) and any two \(x, y \notin \text{bdd}(A)\) are conjugate by an \(X\)-automorphism. It follows directly that \(T_p\) is strongly minimal and disintegrated. \(\square\)

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