A solution to Ward-Takahasi-identity in QED$_3$

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Abstract

Using spectral function of photon we find the reliable results for the effects of vacuum polarization for the dressed fermion propagator in three-dimensional QED.
I. INTRODUCTION

In the previous work we studied the mass singularity of the fermion propagator in QED$_3$ in the presence of massless photon in quenched approximation[7]. We applied the method of spectral function with low-energy theorem which is known to reproduces the Bloch-Nordsieck approximation and renormalization group analysis near the mass shell in four-dimensional model[5,6,12]. In the present work we study the effect of vacuum polarization of massless
fermion on the photon propagator in the same approximation. Using the spectral function of photon we get the non-perturbative effects by integrating the quenched case with bare photon mass. However high energy behaviour of the fermion propagator does not change and we get $Z_2^{-1} = 0$ for arbitrary coupling. At sufficiently large coupling short distance singularities disappear. In this case the vacuum expectation value $\langle \bar{\psi} \psi \rangle$ becomes finite. In Section II we review the vacuum polarization of the massive fermion loop and show the structure of the photon propagator. In section III spectral function of the fermion propagator are defined and we show the way to determine it based on LSZ reduction formula and low-energy theorem. In section IV we evaluate the spectral function for quenched case with photon mass as an infrared cut-off and improve it by the photon spectral function including vacuum polarization. In Section V is devoted to analysis in momentum space of these solutions. In section VI we consider the origins of confinement in the modification of Coulomb potential by LSZ.

II. VACUUM POLARIZATION

Assuming parity invariance, we take 4-component fermion representation[6,8,9,10]. The one-loop self-energy for photon with dimensional regularization is given

$$\Pi_{\mu\nu}(k) \equiv i e^2 \int \frac{d^3 p}{(2\pi)^3} \left( \gamma_{\mu} \frac{1}{p - m} \gamma_{\nu} \frac{1}{(p - k) - m} \right)$$

$$= -e^2 T_{\mu\nu} \left[ (\sqrt{k^2} + \frac{4m^2}{\sqrt{k^2}}) \ln\left(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}}\right) - 4m \right],$$

$$T_{\mu\nu} = -(g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}), \quad \frac{d^3 p}{(2\pi)^3}. \tag{1}$$

Usual polarization function $P(k)$ is expanded in terms of $m(1/m)$

$$P(k) = -\frac{e^2}{8\pi} \left[ (\sqrt{-k^2} + \frac{4m^2}{\sqrt{-k^2}}) \ln\left(\frac{2m + \sqrt{-k^2}}{2m - \sqrt{-k^2}}\right) - 4m \right],$$

$$= +\frac{e^2}{8} \sqrt{-k^2} - 4\frac{m^2}{\sqrt{-k^2}} + \frac{64m^3}{3\pi(-k^2)} - \frac{512m^5}{15\pi(-k^2)^2} + O(m^6)(k^2 < 0),$$

$$= +\frac{e^2}{6\pi m}(-k^2) - \frac{e^2}{60\pi m^3}(-k^2)^2 + O(k^5). \tag{2}$$

If the mass $m$ is heavy expansion in terms of the inverse powers of $m$ is a weak coupling expansion and massless limit corresponds to a strong coupling limit. Here we see that the
massless limit is a correct high-energy behaviour of the vacuum polarization function $P$. The full propagator is

$$D_F(k) = -\frac{T_{\mu\nu}}{k^2 - P(k)} - d \frac{k_\mu k_\nu}{k^4}. \quad (3)$$

To see the screening effects easily, first we set $m = 0$ to neglect the threshold effects. It is equivalent to study the high-energy behaviour in momentum space. Hereafter we use the full photon propagator with $N$ fermion flavours

$$D_F(k) = -\frac{T_{\mu\nu}}{k^2 + e^2 N/8 \sqrt{k^2}} + g.f. \quad (4)$$

and do not discuss the analyticity in Minkowski space.

### III. Calculating the Spectrally Weighted Propagator

#### A. Definition of the spectral function

In this section we show how to evaluate the fermion propagator non perturbatively by the spectral representation\[5,7\]. The spectral function of the fermion is defined

$$S_F(s') = P \int ds' \frac{\gamma \cdot p \rho_1(s') + \rho_2(s')}{s' - s} + i\pi (\gamma \cdot p \rho_1(s') + \rho_2(s')), \quad (5)$$

$$\rho(p) = \frac{1}{\pi} \text{Im} S_F(p) = \gamma \cdot p \rho_1(p) + \rho_2(p)$$

$$= (2\pi)^2 \sum_n \delta(p - p_n) \langle 0|\bar{\psi}(x)|n\rangle \langle n|\overline{\psi}(0)|0\rangle. \quad (6)$$

In the quenched approximation the state $|n\rangle$ stands for a fermion and arbitrary numbers of photons,

$$|n\rangle = |r; k_1, ..., k_n\rangle, r^2 = m^2, \quad (7)$$

we have the solution which is written symbolically

$$\rho(p) = \int \frac{md^2r}{r^0} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int d^3k \theta(k^0) \delta(k^2) \sum_{\epsilon} \right) \sum_{i=1}^{n} \delta(p - r - \sum_{i=1}^{n} k_i)$$

$$\times \langle \Omega|\bar{\psi}(x)|r; k_1, ..., k_n\rangle \langle r; k_1, ..., k_n|\psi(0)|\Omega\rangle. \quad (8)$$

Here the notations

$$(f(k))_0 = 1, (f(k))_n = \prod_{i=1}^{n} f(k_i)$$
have been introduced to show the phase space of each photons. We consider the matrix element

\[ T_n = \langle \Omega | \psi | r; k_1, \ldots, k_n \rangle \]  

for \( k_n^2 \neq 0 \) by LSZ reduction formula:

\[ T_n = \epsilon_n^\mu T_{n\mu}, \]

\[ \epsilon_n^\mu T_{n\mu} = \frac{i}{\sqrt{Z_3}} \int d^3 y \exp(i k_n \cdot y) \Box_y \langle \Omega | T \psi(x) \epsilon \mu A_\mu(y) | r; k_1, \ldots, k_{n-1} \rangle \]

provided

\[ \Box_x T(\psi A_{\mu}(x) = T \psi \Box_x A_{\mu}(x) = T \psi (-j_{\mu}(x) + \frac{d - 1}{d} \partial_{\mu} (\partial \cdot A(x)), \]

\[ \partial \cdot A(+)|phys > = 0, \]

where \( d \) is a gauge fixing parameter. \( T_n \) satisfies Ward-Takahashi-identity

\[ k_{n\mu} T_{n\mu}(r; k_1, \ldots, k_n) = e \exp(i k_n \cdot x) \langle \Omega | \psi(x) | r; k_1, \ldots, k_{n-1} \rangle \],

provided

\[ \partial_{\mu} T(\psi(x)J_{\mu}(y)) = -e \psi(x) \delta(x - y). \]

Neglecting position dependence which is given by

\[ \psi(x) = \exp(-ip \cdot x)\psi(0) \exp(ip \cdot x), \]

we get the usual form

\[ k_{n\mu} T_{n\mu}(r; k_1, \ldots k_{n-1}), r^2 = m^2, \]

which implies low-energy theorems. By the low-energy theorem the fermion pole term for the external line is dominant for the infrared singularity for fermion. Let us consider the matrix element \( T_{n-1}(r; k_1, \ldots k_{n-1}) \) in which there is no photon line with external fermion. We attach \( T_{n-1} \) with one photon line with external fermion. The matrix element \( T_n \) is written as

\[ T_n^{(pole)}(r; k_1, \ldots, k_n) = T_{n-1}(r + k_n; k_1, \ldots, k_{n-1}) T_1(r; k_n), \]

where we assume \( T_{n-1}(r + k_n; k_1, \ldots, k_{n-1}) \) is regular at \( k_n^2 = 0 \). Of course only \( T_n^{(pole)} \) does not satisfy Ward-Takahashi-identity. In ref [5], the method to determine the matrix element \( T_n \)
was discussed in the following way up to $k_n^\mu$ terms which vanishes in the limit $k_n^\mu = 0$,

$$T_n^\mu = T_n^{\mu(pole)} + T_n^{\mu'} + R_n^\mu,$$

(18)

$$k_{n\mu}T_n^{\mu'} = eT_{n-1} - k_{n\mu}T_n^{\mu(pole)},$$

(19)

$$k_{n\mu}R_n^\mu = 0.$$  

(20)

The result for $T_n^{\mu(pole)}$ is

$$T_n^{\mu(pole)} = T_1(r + k_n; k_n)\Lambda_{n-1}(r + k_n; k_1, ..., k_{n-1}).$$

(21)

Here, $\Lambda_{n-1}(r; k_1, ..., k_{n-1})$ does not contain any one-fermion line with momentum $r + k_n$(off-shell) and coincides with $T_{n-1}(r; k_1, ..., k_{n-1})$ continued off the $r^2$ mass shell(by an LSZ formula for example):

$$T_{n-1}(r; k_1, ..., k_{n-1}) = \Lambda_{n-1}(r; k_1, ..., k_{n-1})|_{r^2=m^2}.$$  

(22)

For $n = 1$ case we have

$$k_\mu T_1(r + k; k) = k_\mu \frac{e^{\gamma^\mu}}{\gamma \cdot (r + k) - m} U(r) = e \langle \Omega | \psi | r \rangle = eU(r).$$

(23)

From (17),(19),(23) we see that $T_n^{\mu'}$ is sufficient to satisfy

$$k_{n\mu}T_n^{\mu'}(r; k_1, ..., k_n) = e[T_{n-1}(r; k_1, ..., k_{n-1}) - \Lambda_{n-1}(r + k_n; k_1, ..., k_{n-1})]$$

$$= -e k_n^\mu \frac{\partial}{\partial k_n^\mu} [\Lambda_{n-1}(r + k_n; k_1, ..., k_{n-1})]_{k_n = k_n^*},$$

(24)

where $k^*$ is a intermediate point for $k$. From the above it is sufficient to take

$$T_n^{\mu'} = -e \frac{\partial}{\partial k_n^\mu} \Lambda_{n-1}(r + k_n; k_1, ..., k_{n-1})|_{k_n = k_n^*} = -e \frac{\partial}{\partial k_n^\mu} \Lambda_{n-1}(r'; k_1, ..., k_{n-1})|_{r' = r + k^*}.$$  

(25)

$$T_n^{\mu(pole)} + T_n^{\mu'} = \frac{e^{\gamma^\mu}}{\gamma \cdot (r + k_n) - m} T_{n-1} + e \frac{\gamma^\mu k_n^\nu}{\gamma \cdot (r + k_n) - m} - g_\mu^{\nu'} \frac{\partial}{\partial r'} \Lambda_{n-1}|_{r' = r + k^*, r'^2 = m^2 + \epsilon}.$$  

(26)

The second term is transverse in $k_n^\mu$ and should be regular at $k_n^\mu = 0$. It is understood as the non leading term and derived by gauge invariance.

**B. Approximation to the spectral function**

If there are massless particle as photons, there exists infrared divergences near the fermion mass-shell. In these cases we cannot separate the one particle state and one particle with
multiphoton intermediate states, we must sum all intermediate states with infinite numbers of photons. The spectral representation of the propagator in position space is given.

\[ S_F(x) = \int d^3p \exp(p \cdot x) \int \frac{d\rho(s)}{p^2 + s}. \]  

(27)

First we derive the second-order spectral function and the propagator \( S_F(p) \)

\[ \overline{\rho}(x) = -e^2 \int \frac{md^2r}{(2\pi)^2r^0} \exp(ir \cdot x) \int d^3k \theta(k^0) \delta(k^2) \exp(ik \cdot x) \sum_{\lambda,s} T_1 T_1, \]  

(28)

where one-photon matrix element \( T_1 \) which is given in ref[7,10]

\[ T_1 = \langle \Omega | \psi(x) | r ; k \rangle = \left( \text{in} | T \psi_{\text{in}}(y) \gamma_\mu \psi_{\text{in}}(y) A^\mu_{\text{in}}(y) | r ; \text{in} \right) \]

\[ = ie \int d^3y d^3z S_F(x - z) \gamma_\mu \delta^{(3)}(y - z) \exp(i(k \cdot y + r \cdot z))e^\mu(k, \lambda)U(r, s) \]

\[ = -ie \frac{1}{(r + k) \cdot \gamma - m + i\epsilon} \gamma_\mu (k, \lambda) \exp(i(r + k) \cdot x)U(r, s). \]  

(29)

\[ \sum_{\lambda,s} T_1 T_1 = \frac{(r + k) \cdot \gamma + m}{(r + k)^2 - m^2} \frac{\gamma \gamma (r + m)}{2m} \frac{r + m}{(r + k)^2 - m^2} \gamma \Pi_{\mu \nu}, \]

(30)

and \( \delta(k^2) \) is a bare photon spectral function. Here \( \Pi_{\mu \nu} \) is the polarization sum

\[ \Pi_{\mu \nu} = \sum_{\lambda} \epsilon_\mu(k, \lambda) \epsilon_\nu(k, \lambda) = -g_{\mu \nu} - (d - 1) \frac{k_\mu k_\nu}{k^2}. \]  

(31)

In this case it is easy to find the explicit form of \( \rho \)

\[ \rho^{(2)}(p) = \int d^3x \exp(-ip \cdot x) \overline{\rho}(x), \]  

(32)

\[ \rho^{(2)}(p) = -e^2 \int \frac{d^2r}{(2\pi)^2r^0} \int \frac{d^2k}{(2\pi)^2k^0} \delta^{(3)}(p - k - r) \]

\[ \times \frac{1}{2} (\gamma \cdot r + m) \left[ \frac{m^2}{(r \cdot k)^2} + \frac{1}{r \cdot k} + \frac{d - 1}{k^2} \right]. \]  

(33)

At \( d = 1 \) gauge, \( \rho \) is evaluated in the center of mass system

\[ (2\pi)^2 \rho^{(2)}(p) = \frac{\pi e^2}{2 \sqrt{p^2}} \sum_{\lambda,s} T_1 T_1 \theta(p^2 - m^2) \]

\[ = \frac{\pi e^2}{2 \sqrt{p^2}} (\gamma \cdot p + m) \frac{4m^2}{(p^2 - m^2)^2} + \frac{2}{p^2 - m^2} \theta(p^2 - m^2). \]  

(34)
In this order we have the propagator in spectral form
\[
S^{(2)}(p) = \int_{m^2}^{\infty} ds \frac{\rho^{(2)}(s)}{p^2 - s + i\epsilon} = \frac{e^2}{4\pi} \int_{m^2}^{\infty} ds \frac{(\gamma \cdot p + m)(s - m^2)}{(p^2 - s + i\epsilon)^2} \left[ \frac{4m^2}{(s - m^2)^2} + \frac{2}{s - m^2} \right].
\] (35)

From the above equation we see that \(S^{(2)}\) has infrared divergences near \(p^2 = m^2\). If we sum infinite number of photons in the final state as in (8), by LSZ reduction formula the method to determine the multi-photon matrix element is given in (26). In the lowest order approximation simplest solution to the spectral function is given by exponentiation of one photon matrix element \(T_1T_1\):

\[
\langle \Omega | \psi(x) | r; k_1, \ldots, k_n \rangle \langle r; k_1, \ldots, k_n, \bar{\psi}(0) | \Omega \rangle \rightarrow \prod_{j=1}^{n} T_1(k_j)T_1(k_j),
\] (36)

\[
\overline{p}(x) = \int \frac{md^2r}{(2\pi)^2r^0} \exp(ir \cdot x) \exp(F),
\]

\[
F = \sum_{\text{one photon}} \langle \Omega | \psi(x) | r; k \rangle \langle r; k | \bar{\psi}(0) | \Omega \rangle
\]
\[
= \int d^3k \delta(k^2)\theta(k^0) \exp(ik \cdot x) \sum_{\lambda, s} T_1T_1.
\] (37)

This approximation leads to an infinite sum of ladder graphs with fixed mass. And its imaginary part is simple to evaluate by the above formula. After integration, we set \(r^2 = m^2\)

\[
S_F(x) = \int d^3p \exp(p \cdot x)(i\gamma \cdot \partial + m) \frac{\exp(-m |x|)}{2 \times 4\pi |x|}
\]
\[
\times \exp(-e^2 \int d^3k \exp(ik \cdot x)\theta(k^0)\delta(k^2)\left[ \frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{d - 1}{k^2} \right]).
\] (38)

This formula shows that the infrared divergences of photons give the radiative correction to the free position space propagator \(\exp(-m |x|)/|x|\). The radiative correction for the fermion propagator may be seen as the modification of the short distance behaviour or mass. Therefore we expand \(F\)

\[
S(x) = \frac{\exp(-m |x|)}{4\pi |x|} \exp(F(m, x))
\] (39)
in the power series of $x$ and study the correction of mass. The results are shown in the previous work. Next we use the full propagator for photon in the same way

$$D_F(x) = \int \int \mathcal{D}^3 k \exp(ik \cdot x) \frac{\rho(s) ds}{k^2 - s + i\epsilon}$$

$$= \int ds \int \mathcal{D}^3 k \exp(ik \cdot x) [\mathcal{P} \frac{\rho(s)}{k^2 - s} - i\pi\rho(s)\delta(k^2 - s)]$$

$$= \int \mathcal{D}^3 k \exp(ik \cdot x) [\text{Re} D_F(k) + i \text{Im} D_F(k)]. \quad (40)$$

Spectral functions for free and dressed photon are given by [6,11]

$$\rho^{(0)}(k) = \delta(k^2 - \mu^2),$$

$$\rho^D(k) = \frac{1}{\pi}\text{Im} D_F(k) = \frac{c\sqrt{k^2}}{k^2(k^2 + c^2)}. \quad (41)$$

One photon matrix element reads

$$2F = -e^2 \int d^3 k \exp(ik \cdot x) [i \text{Im} D_F(k) \left( \frac{m^2}{(r \cdot k)^2} + \frac{1}{k^2} - \frac{1}{k^2} \right) + \frac{d}{k^4}]. \quad (42)$$

First we perform the direct fourier transform of the propagator

$$iD_F(x) = \int \mathcal{D}^3 k \exp(ik \cdot x) \frac{1}{k^2 + c\sqrt{k^2}}$$

$$= \frac{1}{2\pi^2|x|} \int_0^\infty dk \frac{\sin(\sqrt{k^2} |x|)}{\sqrt{k^2 + c}}, \quad c = \frac{e^2 N}{8}. \quad (43)$$

On the other hand the spectral representation for photon is

$$iD_F(x) = \frac{1}{\pi} \int \mathcal{D}^3 k \exp(ik \cdot x) \int_0^\infty dp^2 \frac{\rho(p)}{k^2 + p^2}$$

$$= \int_0^\infty dp^2 \frac{\exp(-\sqrt{p^2} |x|)}{4\pi |x|} \rho(p^2). \quad (44)$$

Both ways lead the same answer;

$$iD_F(x) = \frac{1}{4\pi^2|x|} [\pi \cos(c |x|) - 2 \text{Si}(c |x|) \cos(c |x|) + 2 \text{Ci}(c |x|) \sin(c |x|)]. \quad (45)$$

These modification leads to the change of the static potential

$$V_B(x) = \int \mathcal{D}^3 k \exp(ik \cdot x) \frac{1}{k^2 + \mu^2} = K_0(\mu |x|), \quad (46)$$

$$V_R(x) = \int \mathcal{D}^3 k \exp(ik \cdot x) \frac{1}{k^2 + ck} = H_0(c |x|) - Y_0(c |x|), \quad (47)$$

$$H_0(a) = \frac{2}{\pi} \int_0^1 \frac{\sin(ax)}{\sqrt{1 - x^2}} dx, \quad Y_0(a) = -\frac{2}{\pi} \int_1^\infty \frac{\cos(ax)}{\sqrt{x^2 - 1}} dx. \quad (48)$$

$V_R(x)$ has not zero on the $|x|$ axis and decrease as $1/c|x|$. On the other hand $V_B(x)$ has zero and change its sign.
IV. APPROXIMATE SOLUTION IN POSITION SPACE

A. quenched case

To evaluate the function $F$, it is helpful to use the following parameter integral with exponential cut-off (infrared cut-off) [5,7,12]. Following the parameter trick

$$
\frac{1}{k \cdot r} = i \lim_{\epsilon \to 0} \int_{0}^{\infty} d\alpha \exp(i\alpha(k + i\epsilon) \cdot r),
$$

$$
\frac{1}{(k \cdot r)^2} = -\lim_{\epsilon \to 0} \int_{0}^{\infty} \alpha \exp(i\alpha(k + i\epsilon) \cdot r),
$$

(49)

we obtain the formula

$$F_1 = \int d^3 k \exp(ik \cdot x) D_F(k) \frac{1}{(k \cdot r)^2} = -\lim_{\mu \to 0} \int_{0}^{\infty} \alpha \exp(-\mu \alpha).$$

$$F_2 = \int d^3 k \exp(ik \cdot x) D_F(k) \frac{1}{k \cdot r} = i \lim_{\mu \to 0} \int_{0}^{\infty} \alpha \exp(-\mu \alpha).$$

(50)

(51)

Soft photon divergence corresponds to the large $\alpha$ region and $\mu$ is an infrared cut-off. For the $k_\mu k_\nu$ part there remains an infrared divergence $1/k^2$ which is independent of the $1/(r \cdot k)$ in the same gauge in (42). It is simple to evaluate this term $F_L$ by definition

$$F_L = -ie^2 \int d^3 k \exp(ik \cdot x) D_F(k) \frac{1}{k^2},$$

(52)

$$\frac{1}{4\pi^2} \int_{0}^{\infty} d\sqrt{k^2} \frac{\sin(\sqrt{k^2} |x|)}{\sqrt{k^2} |x| (k^2 + \mu^2)} = \frac{1 - \exp(-\mu |x|)}{8\pi \mu^2 |x|}.$$  

(53)

We have

$$F = ie^2 m^2 \int_{0}^{\infty} \alpha \exp(-\mu \alpha) - e^2 \int_{0}^{\infty} \alpha D_F(x + \alpha r) + ie^2 \int d^3 k \exp(ik \cdot x) D_F(k) \frac{1}{k^2}.$$  

(54)

Here we notice that the overall sign is changed by replacing the imaginary part to real part of the photon propagator. In quenced case the above formulae for the evaluation of three terms in $F$ provided the position space propagator with bare mass

$$D^{(0)}_F(x) = \frac{\exp(-\mu |x|)}{8\pi i |x|}.$$  

(55)

$$F = -\frac{e^2}{8\pi} \left( \frac{\exp(-\mu |x|)}{\mu} - |x| \text{Ei}(\mu |x|) \right) - \frac{e^2}{8\pi m} \text{Ei}(\mu |x|) + (d - 1) \frac{e^2}{8\pi \mu^2 |x|} (1 - \exp(-\mu |x|)),$$  

(56)
where
\[
\text{Ei}(z) = \int_1^\infty \frac{\exp(-zt)}{t} \, dt.
\] (57)

For the leading order in \(\mu\) we obtain
\[
F_1 = \frac{e^2}{8\pi} \left( -\frac{1}{\mu} + |x| (1 - \ln(\mu |x| - \gamma)) \right) + O(\mu),
\] (58)
\[
F_2 = \frac{e^2}{8\pi \mu} (\ln(\mu |x|) + \gamma) + O(\mu),
\] (59)
\[
F_g = \frac{e^2}{8\pi} \left( \frac{1}{\mu} - \frac{|x|}{2} \right) (d - 1) + O(\mu).
\] (60)

\[
F = \frac{e^2 (d - 2)}{8\pi \mu} + \frac{\gamma e^2}{8\pi m} + \frac{e^2}{8\pi m} \ln(\mu |x|) - \frac{e^2}{8\pi} |x| \ln(\mu |x|) - \frac{e^2}{16\pi} |x| (d - 3 + 2\gamma),
\] (61)

where \(\gamma\) is an Euler constant. In this case linear infrared divergence may cancels by higher order correction or away from threshold, at present we omit them here with constant term[6,13]. Linear term in \(|x|\) is understood as the finite mass shift from the form of the propagator in position space and \(|x|\ln(\mu |x|)\) term is position dependent mass
\[
m = \left| m_0 + \frac{e^2}{16\pi} (d - 3 + 2\gamma) \right|, \\
m(x) = m + \frac{e^2}{8\pi} \ln(\mu |x|),
\] (62)

which we will discuss in section VI. The position space propagator is written as free one multiplied by quantum correction as
\[
\frac{\exp(-m |x|)}{4\pi |x|} \exp(F) = \frac{\exp(-m |x|)}{4\pi |x|} (\mu |x|)^{D-C|x|},
\]
\[
D = \frac{e^2}{8\pi m}, C = \frac{e^2}{8\pi}.
\] (63)

From the above form we see that \(D\) acts to change the power of \(|x|\) and plays the role of anomalous dimension of the propagator[7]. If \(D \geq 1\) there is no short distance singularities as spike.

**B. unquenched case**

Here we apply the spectral function of photon to evaluate the unquenched fermion propagator. We simply integrate the function \(F(x, \mu)\) for quenched case which is given in (61), where
μ is a photon mass. Spectral function of photon is given in (41) in the Landau gauge

\[ \rho(\mu) = \frac{c}{\mu(\mu^2 + c^2)}, \]

\[ Z_3^{-1} = \int_0^\infty \rho(\mu) \mu d\mu = \frac{\pi}{2}. \tag{64} \]

An improved F is written as dispersion integral

\[ \tilde{F} = Z_3 \int_0^\infty F(\mu) \rho(\mu) \mu d\mu \]

\[ = \frac{e^2}{8c} (-2 \ln(\frac{\mu}{c}) + \frac{\gamma e^2}{8\pi m} + \frac{e^2}{8\pi m} \ln(c |x|)) \]

\[ - \frac{e^2}{8\pi} |x| \ln(c |x|) - \frac{e^2}{16\pi} |x| (3 - 2\gamma). \tag{65} \]

In this way the linear infrared divergences turn out to be a logarithmic divergence in the first term. This is the improvement by spectral function. The fermion spectral function in position space becomes

\[ \bar{\rho}(x) = \frac{\exp(-m |x|)}{4\pi |x|} \exp(\tilde{F}) \]

\[ = \frac{\exp(-m + B) |x|}{4\pi |x|} (c |x|)^{D-C|x|} \exp(\frac{\gamma e^2}{8\pi m}), \tag{66} \]

by N flavours

\[ B = \frac{c}{2N\pi} (3 - 2\gamma), \beta = \frac{-2}{N}, C = \frac{c}{N\pi}, D = \frac{c}{N\pi m}. \tag{67} \]

In this way we get a position space propagator which shows mass generation and wave renormalization in all region. We can avoid infrared divergences in the Euclid region. At large N the function damps slowly with fixed c, where mass changing effect is small for all range of |x|. For small N the function damps fast and the short distant part is dominant for mass changing effect. Short distance behaviour is determined by D. For long distance we may treat finite μ and investigate the long distance behaviour of F. In that case the F contain only ln(μ |x|) as large μ |x| and others damp faster as ln(μ |x|)/|x|. This indicate that the mass generation is a short distance effect. In the fourier tranformation

\[ \rho(p) = \int_0^\infty \frac{\sin(p |x|)}{p |x|} |x|^2 d|x| \frac{\exp(-m |x|)}{4\pi |x|} \exp(F), \tag{68} \]

first factor

\[ \frac{\sin(p |x|)}{p |x|} \]

is dominant at small |x| for both large and small p.

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V. IN MOMENTUM SPACE

A. spectrally weighted propagator

Now we turn to the fermion propagator. The momentum space propagator is given by

\[ S_F(p) = \int d^3x \exp(-ip \cdot x)S_F(x) \]

\[ = -\int d^3x \exp(-ip \cdot x)(i\gamma \cdot \partial + m)\frac{\exp(-m|\mathbf{x}|)}{4\pi|x|} \exp(F(x)). \]  (69)

where we have

\[ F(x) = (c|x|)^{D-C|x|}, D = \frac{c}{N\pi m}, C = \frac{c}{N\pi}. \]  (70)

It is known that

\[ \int d^3x \exp(-ip \cdot x)\frac{\exp(-m|\mathbf{x}|)}{4\pi|x|}(c|x|)^D \]

\[ = c^D \frac{\Gamma(D+1) \sin((D+1) \arctan(\sqrt{-p^2/m}))}{\sqrt{-p^2(p^2+m^2)^{(D+1)/2}}} \]  (71)

\[ = \frac{1}{p^2+m^2} \text{ for } D = 0, \]

\[ = \frac{2m}{(p^2+m^2)^2} \text{ for } D = 1. \]  (72)

for Euclidean momentum \( p^2 \leq 0 \). In Minkowski momentum \( p^2 \geq m^2 \) above formula is continued to

\[ \arctanh(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = -i \arctan(iz), (0 \leq z^2 \leq 1) \]  (73)

\[ \arctan h(z) = \frac{1}{2} \ln\left(\frac{1+z}{z-1}\right) \pm \frac{i\pi}{2}, \]  (74)

\[ \text{arccoth}(z) = \frac{1}{2} \ln\left(\frac{1+z}{z-1}\right), (1 \leq z^2). \]  (75)

From this definition

\[ \sinh(z) = \frac{\exp(z) - \exp(-z)}{2} = -i \sin(iz), \]

\[ \sinh((D+1) \tanh^{-1}(z)) = \frac{-i}{2} [(\frac{1+z}{z-1})^{(D+1)/2} - (\frac{z-1}{1+z})^{(D+1)/2}], \]

\[ z = \sqrt{p^2/m}. \]  (76)
The spectral function is a discontinuity in the upper half-plane of \( z \)

\[
\pi \rho(z) = -\frac{c_D D}{2} \text{Im} \frac{\Gamma(D+1)}{m|z|} \left( \frac{1}{m^2 - m^2 z^2} \right)^{(D-1)/2} \left[ (\frac{1 + z}{1 - z})^{(D+1)/2} - \left( \frac{1}{1 + z} \right)^{(D+1)/2} \right]
\]

\[
= -\frac{c_D D}{2m z m^{D+1}} \text{Im} \left[ \left( \frac{1}{1 - z} \right)^{D+1} \theta(z - 1) - \left( \frac{1}{1 + z} \right)^{D+1} \theta(-(z + 1)) \right].
\]  

(77)

There is a problem to normalize the spectral function to \( \delta(p^2 - m^2) \) in the weak coupling limit\[5\]. Here we do not evaluate the explicit spectral function. Principal part of the propagator in Minkowski space is continued to

\[
S_F(p) = (\gamma \cdot p + m)c^D \Gamma(D+1) \sinh((D+1) \text{arctanh}(\sqrt{p^2/m})) / \sqrt{p^2(m^2 - p^2)^{(D+1)/2}}, (\sqrt{p^2/m} \leq 1)
\]  

(78)

To use above formula for \( C \neq 0 \) case we use Laplace transform\[7\]

\[
F(s) = \int_0^\infty d|x| \exp(-(s - m)|x|) (\mu|x|)^{-C|x|} (s \geq 0).
\]  

(79)

This function shift the mass and we get the propagator

\[
S_F(p) = (\gamma \cdot p + m)c^D \Gamma(D+1) \int_0^\infty F(s) ds \frac{\sin((D+1) \text{arctan}(\sqrt{-p^2/(m-s)}))}{\sqrt{-p^2(p^2 + (m-s)^2)^{(D+1)/2}}},
\]  

(80)

At \( D = 0 \) and 1 we see

\[
S_F(p) = (\gamma \cdot p + m) \int_0^\infty F(s) ds \frac{1}{(p^2 + (m-s)^2)}, (D = 0).
\]  

(81)

\[
S_F(p) = (\gamma \cdot p + m) \int_0^\infty F(s) ds \frac{2|m-s|}{(p^2 + (m-s)^2)^2}, (D = 1).
\]  

(82)

VI. RENORMALIZATION CONSTANT AND ORDER PARAMETER

Hereafter we consider the renormalization constant and bare mass in our model. It is easy to evaluate the renormalization constant and bare mass by define the renormalization

\[
\bar{\psi}_0 = \sqrt{Z_2 \bar{\psi}_r}, \bar{\psi}_0 = \sqrt{Z_2 \bar{\psi}_r},
\]

\[
S_0^0 = Z_2 S_F, \frac{Z_2^{-1}}{\gamma \cdot p - m_0} = S_F(p),
\]  

(83)
\[ Z_2^{-1} = \int \rho_1(s) ds = \lim_{p \to \infty} \frac{1}{4p^2} tr(\gamma \cdot pS_F(p)) \]
\[ = \Gamma(D + 1)c^D \lim_{p \to \infty} \int_{0}^{\infty} G(s) ds \frac{\sqrt{p^2} \sin((D + 1) \arctan(\sqrt{p^2/(m - s)})}{(p^2 + (m - s)^2)^{(1+D)/2}} \to \sin(\frac{(D + 1)\pi}{2})c^D \lim_{p \to \infty} \frac{1}{p^D} = 0. \]  
(84)

\[ m_0Z_2^{-1} = m \int \rho_2(s) ds = \lim_{p \to \infty} \frac{1}{4p^2} tr(p^2S_F(p)) \to 0. \]  
(85)

There is no pole and it shows the confinement for \( D > 0 \).

Order parameter \( \langle \bar{\psi}\psi \rangle \) is given as the integral of the scalar part of the propagator in momentum space

\[ \langle \bar{\psi}\psi \rangle = -TrS_F(x), \]  
(86)

\[ \langle \bar{\psi}\psi \rangle = -2 \int_{0}^{\infty} \frac{p^2 d\sqrt{p^2} \Gamma(D + 1)c^D}{2\pi^2} \frac{2\sqrt{-p^2}}{\sqrt{((m - s)^2 + p^2)^{D+1}}}, (D = c/N\pi m). \]  
(87)

For \( D = c/N\pi m \geq 1 \) vacuum expectation value is finite. This condition is independent of the bare mass. We cannot apriori determine the value \( D \), since \( m \) is a physical mass. \( m \) and \( \Sigma(0) \) is not the same quantity but we assume they have the same order of magnitude. In numerical analysis of Dyson-Schwinger equation \( \Sigma(0) \) damps fast with \( N \) and is seen to vanish at \( N = 3 \). If we assume \( m = O(c/N) \) in the case of vanishing bare mass \( m_0 = 0 \), \( D \) becomes \( O(1) \). In our approximation if we set \( m \) equals to the second order value in the Landau gauge, we get \( D \approx 1.08 \) which is very close to 1. For \( D = 1 \) we have a similar solution of the propagator at short distance which is known by the analysis of D-S. In the analysis of Gap equation, zero momentum mass \( \Sigma(0) \) and a small critical number of flavours have been shown[1,2,3], in which same approximation was done because the vacuum polarization governs the photon propagator at low energy. The critical number of flavour \( N_c \) is a consequence of the approximation for the infrared dynamics as in quenched QED, where ultraviolet region is non-trivial and is not easy to find numerically. In the intermediate value of the coupling we solved the coupled Dyson-Schwinger equation numerically and found that the \( \Sigma(0) \) which is, \( O(c^2/4\pi) \) at \( N = 1 \), the same order of magnitude as the quenched Landau gauge[15].
VII. COULOMB-ENERGY AND SELF-ENERGY

In this section we study the origin of confinement. First we see the difference between short and long distance cases. The coefficient of \( \ln(c|x|), c|x| \ln(c|x|) \) in two cases of \( F \).

If we expand the \( D_F(x) \) in \( c|x|, 1/c|x| \), for the latter case

\[
D_F(c|x|) \simeq \frac{4e^2}{N\pi^2} \frac{1}{c^2 x^2}, \quad (c|x| \gg 1) \tag{88}
\]

we easily obtain

\[
F = \frac{8}{N\pi^2} \ln(c|x|), \quad (c|x| \gg 1). \tag{89}
\]

\[
F = \frac{(d-2)}{N} \ln\left(\frac{\mu}{c}\right) + \frac{\gamma}{N\pi m} + \frac{c}{N\pi m} \ln(c|x|) - \frac{c}{N\pi} |x| \ln(c|x|) - \frac{c}{2N\pi} |x| (d + 3 - 2\gamma)
\]

\[
, \quad (c|x| \ll 1). \tag{90}
\]

Here

\[
D = \frac{c}{N\pi m}, \quad (c|x| \ll 1),
\]

\[
= \frac{8}{N\pi^2}, \quad (c|x| \gg 1), \tag{91}
\]

\[
E = \frac{1}{N\pi}, \quad (c|x| \ll 1)
\]

\[
= 0, \quad (c|x| \gg 1). \tag{92}
\]

are both the coefficients of the Coulomb energy and self-energy which we will see below. Our approximation to the spectral function \( T_1 \overline{T_1} \) is a four particle scattering amplitude with one photon exchange. In the evaluation of \( F \) we see that \( D \) term comes from the vertex by LSZ which has an infrared singularity

\[
\frac{1}{\gamma \cdot (p+k) - m + i\epsilon} \gamma_\mu \epsilon^\mu(k, \lambda) U(p, s) \rightarrow \frac{\gamma \cdot (p+k) + m}{2p \cdot k} \gamma_\mu \epsilon^\mu(k, \lambda). \tag{93}
\]

In this way the vertex is enhanced near the on-shell as \( 1/k \) in contrast with the usual on-shell matrix element

\[
\overline{U}(p+k)\gamma_\mu U(p)\epsilon^\mu(k) \tag{94}
\]

\( E \) term comes from \( 1/(p \cdot k)^2 \) and \( 1/(p \cdot k) \). In quenched case it comes from \( 1/(p \cdot k)^2 \). Let us imagine the imaginary part of the fermion propagator. When \( p^2 \geq m^2 \) parent fermion can
decay into fermion and photon, this process gives a Coulomb energy. Namely
\[ \int_m^\infty ds \int d^3y \frac{\exp(-s|y|)}{4\pi|y|} \frac{\exp(-s|x-y|)}{4\pi|x-y|} F(x-y) = \int d^3y \rho(x-y) \rho(y) (D \ln(c|x-y|) - Cc|x-y| \ln(c|x-y|)). \] (95)

When \( p^2 \leq m^2 \) parent cannot decay, this one gives a fermion self-energy.
\[ \delta^{(3)}(0) \frac{\exp(-m|x|)}{4\pi|x|} F \simeq V(D \ln(c|x|) - Cc|x| \ln(c|x|)). \] (96)

Usually these are summed as leading infrared divergences. It may be clear if we consider the potential energy for two charged particle with modified Coulomb interaction and seek the corresponding terms in \( S_0 F \). At short distance with bare photon we have
\[ V_C(|x|) = \int \frac{d^2k}{(2\pi)^2} \exp(ik \cdot x) \frac{1}{k^2(p \cdot k)} \]
\[ = \frac{1}{4\pi} \int_0^\infty d\alpha K_0(x + ap, \mu) \propto |x| \ln(c|x|), (c|x| \ll 1). \] (97)

At long distance with dressed photon we have
\[ V_C(|x|) = \int \frac{d^2k}{(2\pi)^2} \exp(ik \cdot x) \frac{m^2}{ck(p \cdot k)} \propto \frac{1}{c|x|}(c|x| \gg 1). \] (98)

In the same way we obtain the contribution from another term which is singular as \( 1/(p \cdot k)^2 \)
\[ V_S(|x|) = \int \frac{d^2k}{(2\pi)^2} \exp(ik \cdot x) \frac{m^2}{ck(p \cdot k)^2} \]
\[ = \frac{1}{4\pi} \int d\alpha K_0(x + ap, \mu) \propto |x|^2 \ln(c|x|)(c|x| \ll 1), \]
\[ V_S(|x|) = \int \frac{d^2k}{(2\pi)^2} \exp(ik \cdot x) \frac{1}{ck(p \cdot k)^2} \propto \ln(c|x|)(c|x| \gg 1), \] (99)

which is not familiar to us. However this term drives confinement at long distance as in the quenched case. In many cases we assume the absence of mass changing effects. Of course the non-relativistic approximation is correct in that sense. But the \( m^2/(p \cdot k)^2 \) has also an infrared singular contribution to the self-energy. This one create the position dependent mass as \( M(x) = c \ln(c|x|) \). The coefficients \( D, E \) are gauge invariant provided the photon couples to conserved currents.
VIII. SUMMARY

We evaluate the fermion propagator in three dimensional QED with dressed photon by method of spectral function. Non perturbative effects are included by resummation of the infinite numbers of rainbow type diagrams with physical mass. In the evaluation of lowest order matrix element for fermion spectral function we obtain finite mass shift, Coulomb energy and position dependent mass, which is similar to the analysis of D-S equation except for the wave function renormalization. Including vacuum polarization we find the same structure. Above some coupling constant order parameter $\langle \bar{\psi} \psi \rangle$ is finite, which is independent of the symmetry which forbids finite bare mass. The arguments of confinement are given usually for the force between charged particle. In our approximation these force sets the renormalization constant $Z_2^{-1} = 0$ for arbitrary coupling. If we assume the magnitude of $m$ is generated by the second order in $e$ in the Landau gauge, our results is consistent with numerical analysis of coupled Dyson-Schwinger equation [15].

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