Research Article

Equivalent Characterization on Besov Space

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In this paper, we give an equivalent characterization of the Besov space. This reveals the equivalent relation between the mixed derivative norm and single-variable norm. Fourier multiplier, real interpolation, and Littlewood-Paley decomposition are applied.

1. Introduction

In Sobolev spaces, it is known that \(\|f\|_{H^s(\mathbb{R}^n)} \sim \|f\|_{L^2(\mathbb{R}^n)} + \sum_{i=1}^{n} \|\partial_x^i f\|_{L^2(\mathbb{R}^n)}\), where \(\|f\|_{H^s(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} + \|\partial_x^i f\|_{L^2(\mathbb{R}^n)}\). Note that on the right hand side of the definition \(\|f\|_{H^s(\mathbb{R}^n)}\), it contains the mixed derivative norm \(\|\partial_x^i \partial_y^j f\|_{L^2(\mathbb{R}^n)}\). This mixed derivative norm would make the calculation more complicated or even infeasible to estimate partial differential equations with some anisotropy property, like Vlasov-Poisson equation \([1, 2]\), in fractional Sobolev space \([3]\). So, separating variables becomes necessary and meaningful.

In this paper, we aim to prove \(\|f\|_{B^s_p(\mathbb{R}^n)} \sim \sum_{j=1}^{n} \|f\|_{B^s_{p,j}(\mathbb{R}^n)}\) which realizes the separation, i.e., the right hand side does not contain the “mixed derivative” term, it only contains fractional derivative with respect to a single variable for each term. Thus, when it comes to estimate \(\|f\|_{B^s_p(\mathbb{R}^n)}\) in solving partial differential equations, it is equivalent to estimate \(\|f\|_{B^s_{p,j}(\mathbb{R}^n)}\) individually. For the other equivalent characterizations for Besov spaces, refer to \([4–7]\) and the references therein.

2. Preliminaries

We first recall definitions on Besov spaces, see \([8]\). Given \(f \in \mathcal{S}\) which is the Schwartz function, its Fourier transform \(\hat{\mathcal{F}} f = \hat{f}\) is defined by

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx,
\]

and its inverse Fourier transform is defined by \(\mathcal{F}^{-1} f(x) = \hat{f}(-x)\).

We consider \(\varphi \in \mathcal{S}\) satisfying \(\text{supp } \varphi \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}\). Setting \(\varphi_j(\xi) = \varphi(2^{-j}\xi)\) with \(j = \{1, 2, \cdots\}\), we can adjust the normalization constant in front of \(\varphi\) and choose \(\varphi_0 \in \mathcal{S}\) satisfying \(\text{supp } \varphi_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2\}\), such that

\[
\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \forall \xi \in \mathbb{R}^d.
\]

We observe

\[
\text{supp } \varphi_j \cap \text{supp } \varphi_{j'} = \emptyset \quad \text{if } |j - j'| \geq 2.
\]

Given \(f \in \mathcal{S}'\), we denote \(\Delta f = \mathcal{F}^{-1} \varphi \mathcal{F} f\). For \((s, p, r) \in \mathbb{R} \times [1,\infty] \times [1,\infty]\), then we define the inhomogeneous Besov space by

\[
B^s_{p,r} = \left\{ f \in \mathcal{S}' : \|f\|_{B^s_{p,r}} = \left\| \sum_{j=0}^{\infty} 2^{js}\|\Delta f\|_{L_p^r} \right\| < \infty \right\},
\]
with the usual interpretation for \( p = \infty \) or \( r = \infty \). Throughout this paper, all the function spaces are defined on Euclidean space \( \mathbb{R}^n \); we will omit it whenever there is no confusion.

Next, we would like to present some known results which will be used later. The first one is the unit decomposition.

**Lemma 1** (see [8], page 145). Assume that \( n \geq 2 \), and take \( \varphi \) as in the definition of Besov space. Then, there exist functions \( \chi_j \in \mathcal{S}(\mathbb{R}^n) (j = 1, \ldots, n) \), such that

\[
\sum_{j=1}^{n} \chi_j = 1 \text{ on } \text{supp } \varphi = \{ \xi : 1/2 \leq |\xi| \leq 2 \},
\]

where \( \text{supp } \chi_j \subseteq \{ \xi \in \mathbb{R}^n : |\xi_j| \geq (3/\sqrt{n})^{-1} \} (j = 1, \ldots, n) \).

Next, we recall the real interpolation characterization for Besov spaces.

**Lemma 2** (see [8], page 142). Suppose \( 1 \leq p, q \leq \infty, 0 < \theta < 1, s = (1 - \theta)s_0 + \theta s_1 \), where \( s_0 \neq s_1 \). We have

\[
\left( H_p^{s_0}, H_p^{s_1} \right)_{\theta,q} = B_p^{s_0,s_1}. \tag{6}
\]

**Remark 3.** We also have

\[
\left( H_p^{s_0}, H_p^{s_1} \right)_{\theta,q} = B_p^{s_0,s_1}. \tag{7}
\]

Its proof can be repeated the process of Lemma 2 completely.

### 3. Equivalent Characterization

Now, we are in the position to state and prove our theorems. Firstly, we apply the Fourier multiplier [9] to prove that \( H_p^s (\mathbb{R}^n) = \bigcap_{n=1}^\infty H_p^{s_n} (\mathbb{R}^n) \) directly; \( H_p^s \) space has an advantage that the factor \( (1 + |\xi|^2)^{s/2} \) is positive everywhere, which is fundamentally important when applying the Fourier multiplier theorem.

For the sake of brevity, we denote

\[
\langle \xi \rangle = \left( 1 + |\xi|^2 \right)^{1/2}. \tag{8}
\]

We have the following equivalent norm theorem in Sobolev spaces.

**Theorem 4.** Suppose \( 1 < p < \infty, s > 0 \). We have

\[
H_p^s = \bigcap_{j=1}^n H_p^{s_j}, \tag{9}
\]

where

\[
\|f\|_{H_p^s} = \| F^{-1} \langle \xi \rangle^{s} \hat{f} \|_{L_p},
\]

\[
\|f\|_{H_p^{s_j}} = \| F^{-1} \langle \xi \rangle^{s_j} \hat{f} \|_{L_p}. \tag{10}
\]

**Proof.** On the one hand, if \( f \in H_p^s \), i.e., \( \| F^{-1} \langle \xi \rangle^{s} \hat{f} \|_p < \infty \) where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \). Note that, for any \( j = 1, \ldots, n \), we have

\[
\| F^{-1} \langle \xi \rangle^{s_j} \hat{f} \|_p = \| F^{-1} \langle \xi \rangle^{s_j} \hat{f} \|_p. \tag{11}
\]

Next, we just need to show that \( m_1 (\xi) = \langle \xi \rangle^s \hat{f} \) is an \( L^p \) multiplier. To prove the assertion, we introduce an auxiliary function on \( \mathbb{R}^{n+1} \) defined by

\[
m_1 (\xi,t) = \left( t^2 + |\xi|^2 \right)^{s/2}. \tag{12}
\]

It is easy to verify that \( m_1 \) is homogeneous of degree 0 and smooth on \( \mathbb{R}^{n+1} \setminus \{0\} \). The derivatives \( \partial^\beta \hat{m}_1 \) are homogeneous of degree \( -|\beta| \) and satisfy

\[
|\partial^\beta \hat{m}_1 (\xi,t)| \leq C_{\beta} |(\xi,t)|^{-|\beta|}, \quad \text{with } C_{\beta} = \sup_{\theta \in \mathbb{R}} \left| \partial^\beta \hat{m}_1 (\theta) \right|, \tag{13}
\]

whenever \( (\xi,t) \neq 0 \) and \( \beta \) is a multiindex of \( n + 1 \) variables. In particular, taking \( \beta = (\alpha,0) \), we obtain

\[
|\partial^\alpha \hat{m}_1 (\xi,t)| \leq C_{\alpha} \left( t^2 + |\xi|^2 \right)^{-|\alpha|/2}, \tag{14}
\]

and setting \( t = 1 \), we deduce that \( |\partial^\alpha m_1 (\xi)| \leq C_{\alpha} \left( 1 + |\xi|^2 \right)^{-|\alpha|/2} \leq C_{2} |\xi|^{-|\alpha|} \), which implies that \( m_1 (\xi) \) is an \( L^p \) Fourier multiplier by the Mihlin-Hömander theorem [9] (page 446).

On the other hand, assume \( f \in \bigcap_{j=1}^n H_p^{s_j} \), that is, \( \| F^{-1} \langle \xi \rangle^{s_j} \hat{f} \|_p < \infty \), for any \( j = 1, \ldots, n \). Note that

\[
\| F^{-1} \langle \xi \rangle^{s} \hat{f} \|_p = \| F^{-1} \sum_{j=1}^n \langle \xi \rangle^{s_j} \hat{f} \|_p. \tag{15}
\]

Similarly, we can verify that \( m_2 (\xi) = \langle \xi \rangle^s / \sum_{j=1}^n \langle \xi \rangle^{s_j} \) is an \( L^p \) Fourier multiplier which finishes the proof of Theorem 4.

We return to prove the equivalent characterization on Besov spaces. However, we cannot do the same trick as in \( H_p^s \) space since \( \phi_j (\xi) \) is not positive everywhere as \( \langle \xi \rangle \). Fortunately, we have \( \left( H_p^{s_0}, H_p^{s_1} \right)_{\theta,q} = B_p^{s_0,s_1} \), see Lemma 2. This observation is favourable to prove the equivalent relation in one direction; however, for the other direction, we need a more delicate technique, in fact, we establish an identity by
applying the Littlewood-Paley decomposition [10], which is very important in our proof. In what follows, $A \leq B$ means there exists a constant $c$ independent of the main parameters such that $A \leq c BA \sim B$ means $A \leq B$ and $B \leq A$.

**Theorem 5.** Suppose $1 < p < \infty$, $1 \leq q \leq \infty$, $s > 0$. We have

$$B^p_{p,q} = \bigcap_{j=1}^{n} B^p_{p,q,s_j},$$

(16)

where $\|f\|_{B^p_{p,q}} \leq \left( \sum_{k=0}^{\infty} (1 + \epsilon)^{k \delta_s} \|f_k^j \|^p_{F^p_{p,q}} \right)^{1/p}$ and $q_k^j$ is the dyadic block of the unit decomposition for the $j$th variable as in the definition of Besov spaces.

**Proof.** We split the proof into the following two steps:

Step I. To prove

$$B^p_{p,q} \subseteq \bigcap_{j=1}^{n} B^p_{p,q,s_j}$$

(17)

Assume $f \in B^p_{p,q}$, by the real interpolation Lemma 2, we have

$$\|
\|f\|
\|
_{B^p_{p,q}} \leq \left( \sum_{k=0}^{\infty} (1 + \epsilon)^{k \delta_s} \|f_k^j \|^p_{F^p_{p,q}} \right)^{1/p}$$

(18)

where $0 < \theta < 1$, $s = (1 - \theta) s_0 + \theta s_1$, and we applied the equivalent norm for the interpolation space $(H^1_p, H^s_p)_{\theta q}$, see [8] ((3) page 39 and (5) page 40).

By Remark 3, we obtain, for any $j = 1, \ldots, n$,

$$\|
\|f\|
\|
_{B^p_{p,q,s_j}} \leq \left( \sum_{k=0}^{\infty} (1 + \epsilon)^{k \delta_s} \|f_k^j \|^p_{F^p_{p,q}} \right)^{1/p}$$

(19)

combining (18) and (19), it follows that

$$\|
\|f\|
\|
_{B^p_{p,q,s_j}} \leq \|
\|f\|
\|
_{B^p_{p,q}}$$

(20)

the arbitrariness of $j$ implies that (17) holds.

Step II. To prove

$$\bigcap_{j=1}^{n} B^p_{p,q,s_j} \subseteq B^p_{p,q}$$

(21)

For $n = 1$, it is trivial.

For $n \geq 2$, we need the following key claim.

**Claim.** There exists a positive integer $m$ depending on $n$ only such that

$$\sum_{|l-k| \leq m} \varphi_k \tilde{\chi}^j_k \varphi_l^j = \varphi_k \tilde{\chi}^j_k.$$  

(22)

where

$$\varphi_k^j(\xi) = \varphi(\frac{1}{3} \xi), \quad \tilde{\chi}^j_k(\xi) = \tilde{\chi}_j(\frac{2^k \xi}),$$

(23)

which is the dyadic block for $j$th variable, $\varphi_k$ is the usual dyadic block as in the definition of Besov spaces, and $\chi_j$ is the same as in Lemma 1.

**Proof of Claim.** By Lemma 1, we have $\varphi_k = \sum_{j=1}^{n} \varphi_k \tilde{\chi}^j_k$.

Note

$$\sum_{j=1}^{n} \varphi_k \tilde{\chi}^j_k \varphi_l^j = \varphi_k \tilde{\chi}^j_k.$$  

(24)

In order to get $\varphi_k \tilde{\chi}^j_k \varphi_l^j \neq 0$, for any chosen $j$ and $k$, we must have

$$\left\{ \begin{array}{l}
2^{k-1} \leq |\xi| \leq 2^{k+1}, \\
2^{l-1} \leq |\xi| \leq 2^{l+1},
\end{array} \right.$$

(25)

which implies that $|l - k| \leq m$ with $m = \lceil \log_2 3 \sqrt{n} \rceil + 1$, ending the proof of the claim. With this claim in mind, we get

$$2 \|f\|_{F^{-1}_{p,q}} \leq 2 \|f\|_{F^p_{p,q}} = 2 \sum_{j=1}^{n} \varphi_k \tilde{\chi}^j_k \varphi_l^j \|f\|_{F^p_{p,q}}$$

(26)

where we used the fact that $\varphi_k \tilde{\chi}^j_k$ is the Fourier multiplier.
With Young’s inequality [11], taking the \( l^q \) norm on both sides of (26) yields that

\[
\| f \|_{B^s_{p,q}} \leq C(n, s, q) \sum_{j=1}^{n} \| f \|_{B^s_{p,q}, x_j},
\]

which implies (21) holds; thus, we complete the proof of our main theorem.

**Remark 6.** The methods could be adapted to the weighted Sobolev spaces and weighted Besov space, or even in the anisotropic function space.

**Data Availability**

The data in this paper is available on request. Please contact Jingchun Chen at jingchun.chen@utoledo.edu.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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