ASYMPTOTIC EXPANSION OF SOLUTION TO SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEM WITH CONVEX INTEGRAL QUALITY FUNCTIONAL WITH TERMINAL PART DEPENDING ON SLOW AND FAST VARIABLES

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Abstract. We consider an optimal control problem with a convex integral quality functional for a linear system with fast and slow variables in the class of piecewise continuous controls with smooth constraints on the control

\[
\begin{cases}
\dot{x}_\varepsilon = A_{11} x_\varepsilon + A_{12} y_\varepsilon + B_1 u, & t \in [0, T], \quad \|u\| \leq 1, \\
\varepsilon \dot{y}_\varepsilon = A_{22} y_\varepsilon + B_2 u, & x_\varepsilon(0) = x^0, \quad y_\varepsilon(0) = y^0, \quad \nabla \varphi_2(0) = 0, \\
J(u) := \varphi_1(x_\varepsilon(T)) + \varphi_2(y_\varepsilon(T)) + \int_0^T \|u(t)\|^2 dt \to \min,
\end{cases}
\]

where \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}^m\), \(u \in \mathbb{R}^r\); \(A_{ij}\) and \(B_i\), \(i, j = 1, 2\), are constant matrices of corresponding dimension, and the functions \(\varphi_1(\cdot)\), \(\varphi_2(\cdot)\) are continuously differentiable in \(\mathbb{R}^n\), \(\mathbb{R}^m\), strictly convex, and cofinite in the sense of the convex analysis. In the general case, for such problem, the Pontryagin maximum principle is a necessary and sufficient optimality condition and there exist unique vectors \(l_\varepsilon\) and \(\rho_\varepsilon\) determining an optimal control by the formula

\[
u_\varepsilon(T - t) := C_{1,\varepsilon}^*(t)l_\varepsilon + C_{2,\varepsilon}^*(t)\rho_\varepsilon \quad \text{s.t.} \quad S(\|C_{1,\varepsilon}^*(t)l_\varepsilon + C_{2,\varepsilon}^*(t)\rho_\varepsilon\|) = \min,
\]

where

\[
C_{1,\varepsilon}^*(t) := B_1^* e^{A_{11}t} + \varepsilon^{-1} B_2^* W_\varepsilon^*(t),
\]

\[
C_{2,\varepsilon}^*(t) := \varepsilon^{-1} B_2^* e^{A_{22}t/\varepsilon},
\]

\[
W_\varepsilon(t) := e^{A_{11}t} \int_0^t e^{-A_{11}\tau} A_{12} e^{A_{22}\tau/\varepsilon} d\tau, \quad S(\xi) := \begin{cases} 2, & 0 \leq \xi \leq 2, \\ \frac{\xi}{2}, & \xi > 2. \end{cases}
\]

The main difference of our problem from the previous papers is that the terminal part of quality functional depends on the slow and fast variables and the controlled system is a more general form. We prove that in the case of a finite number of control change points, a power asymptotic expansion can be constructed for the initial vector of dual state \(\lambda_\varepsilon = (l_\varepsilon^* \rho_\varepsilon^*)^*\), which determines the type of the optimal control.

Keywords: optimal control, singularly perturbed problems, asymptotic expansion, small parameter.

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1. Introduction

The paper is devoted to studying the asymptotics of the vector of the dual state in the problem of optimal control [1, 2, 3] of linear system with fast and slow variables, see survey [4], with an convex integral quality functional [3, Ch. 3] and smooth geometric constraints on a control.

In [5, 6], there were considered problems related with a limiting problem for problems of optimal control by a linear system with fast and slow variables. For other formulation, the asymptotics of solutions of perturbed control problem were considered in [7]–[9]. We note that a controlled system of our form but with a terminal quality functional depending on slow variables only was considered in [8].

In the present work we obtain a complete asymptotic expansion of the vector of dual system determining the optimal control. The main difference of our problem in comparison with that considered in [10] is the dependence of the terminal part of the control functional not only on slow variables but also on fast ones.

2. Formulation of problem and main relations

In the class of piece-wise continuous controls we consider the following optimal control problem:

\[
\begin{align*}
\dot{x}_\varepsilon &= A_{11}x_\varepsilon + A_{12}y_\varepsilon + B_1 u, & t \in [0, T], & \|u\| \leq 1, \\
\varepsilon \dot{y}_\varepsilon &= A_{22}y_\varepsilon + B_2 u, & x_\varepsilon(0) = x^0, & y_\varepsilon(0) = y^0, & \nabla \varphi_2(0) = 0, \\
J(u) &= \varphi_1(x_\varepsilon(T)) + \varphi_2(y_\varepsilon(T)) + \int_0^T \|u(t)\|^2 dt \to \min,
\end{align*}
\]

(1)

where \(x_\varepsilon \in \mathbb{R}^n, y_\varepsilon \in \mathbb{R}^m, u \in \mathbb{R}^r; A_{ij}, B_i, i, j = 1, 2,\) are constant matrices of an appropriate dimension and \(\varphi_1(\cdot), \varphi_2(\cdot)\) are continuously differentiable on \(\mathbb{R}^n\) and \(\mathbb{R}^m\) functions strictly convex and cofinite in the sense of the convex analysis [11, Sect. 13]. All spaces \(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^r\) are equipped with the Euclidean norm, which is everywhere denoted by the same symbol \(\| \cdot \|\).

We note that the terminal part of the quality functional depends on slow and fast variables.

For each fixed \(\varepsilon > 0\), the controlled system and the quality functional in problem (1) are of the form:

\[
\begin{align*}
\dot{z}_\varepsilon &= A_\varepsilon z_\varepsilon + B_\varepsilon u, & t \in [0, T], \\
z_\varepsilon(0) &= z^0, & \|u\| \leq 1, \\
J(u) &= \varphi(z_\varepsilon(T)) + \int_0^T \|u(t)\|^2 dt \to \min,
\end{align*}
\]

where

\[
z_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \end{pmatrix}, \quad z_\varepsilon(0) = z^0 = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}, \quad \varphi(z_\varepsilon(T)) = \varphi_1(x_\varepsilon(T)) + \varphi_2(y_\varepsilon(T)),
\]

\[
A_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ 0 & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad B_\varepsilon = \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix}.
\]

We observe that in the considered convex integral quality functional \(J\), the terminal part can be interpreted as a penalty for the error of the control at the final moment of time \(T\), while the second part reflect the energy spent for the realization of the control.

We shall say that a pair of matrices \((A, B)\) is completely controllable if the system \(\dot{x} = Ax + Bu\) is controllable.
Assumption 1. For all sufficiently small \( \varepsilon > 0 \), the pair \((A_\varepsilon, B_\varepsilon)\) is completely controllable, that is, rank \((B_\varepsilon, A_\varepsilon B_\varepsilon, \ldots, A_\varepsilon^{n+m-1} B_\varepsilon) = n + m \).

Assumption 2. All eigenvalues of the matrix \( A_{22} \) have negative real parts.

Under Assumption 1, the Pontryagin maximum principle is necessary and sufficient condition of the optimality giving the unique solution of problem (1) \([3, \text{Sect. } 3.5, \text{Thm. } 14]\).

It was shown in \([10, \text{Prop. } 1, \text{Eq. } (1.6)]\) that the function \( u_\varepsilon(t) \) is the only optimal control in problem (1), it is of the form

\[
u_\varepsilon(T - t) := \frac{B_\varepsilon^* e^{A_\varepsilon t} \lambda_\varepsilon}{S(\|B_\varepsilon e^{A_\varepsilon t} \lambda_\varepsilon\|)}, \quad S(\xi) := \begin{cases} 2, & 0 \leq \xi \leq 2, \\ \xi, & \xi > 2, \end{cases}
\]

and the vector \( \lambda_\varepsilon \) is the unique solution (in view of the cofiniteness of the function \( \varphi; [11, \text{Thm. } 26.6] \)) of the equation

\[
\nabla \varphi^*(-\lambda) = e^{A_\varepsilon T} z_0 + \int_0^T e^{A_\varepsilon \tau} B_\varepsilon \frac{B_\varepsilon^* e^{A_\varepsilon \tau} \lambda_\varepsilon}{S(\|B_\varepsilon e^{A_\varepsilon \tau} \lambda_\varepsilon\|)} d\tau.
\]

Here \( \nabla \varphi^* \) is the gradient of the function \( \varphi^* \) dual to the function \( \varphi \) in the sense of the convex analysis, see \([11, \text{Sect. } 12]\).

We note that in the considered case

\[
\varphi^*(\lambda) = \varphi_1^*(l) + \varphi_2^*(\rho) \quad \text{and} \quad \nabla \varphi_2^*(0) = 0.
\]

We shall consider the vector \( \lambda_\varepsilon \) determining the optimal control in problem (1) as \( \lambda_\varepsilon = \begin{pmatrix} l_\varepsilon \\ \rho_\varepsilon \end{pmatrix} \), where \( l_\varepsilon \in \mathbb{R}^n \), \( \rho_\varepsilon \in \mathbb{R}^m \).

Straightforward calculation of the matrix exponent of the controlled system in problem (1) gives:

\[
e^{A_\varepsilon t} := \begin{pmatrix} e^{A_{11} t} & \mathcal{W}_\varepsilon(t) \\ 0 & e^{A_{22} t} \end{pmatrix},
\]

where \( \mathcal{W}_\varepsilon(t) = A_{11} \mathcal{W}_\varepsilon(t) + A_{12} e^{A_{22} t} \) and \( \mathcal{W}_\varepsilon(0) = 0 \). This is why

\[
\mathcal{W}_\varepsilon(t) := e^{A_{11} t} \int_0^t e^{-A_{11} \tau} A_{12} e^{A_{22} \tau} d\tau.
\]

Integrating by parts in the right hand side in identity \([9]\), we obtain

\[
\mathcal{W}_\varepsilon(t) = \varepsilon \left( A_{12} e^{A_{22} t} - e^{A_{11} t} A_{12} \right) A_{22}^{-1} + \varepsilon A_{11} \mathcal{W}_\varepsilon(t) A_{22}^{-1},
\]

and by the boundedness of \( A_{12} e^{A_{22} t} - e^{A_{11} t} A_{12} \) on \([0, T]\),

\[
\mathcal{W}_\varepsilon(t) = \varepsilon \sum_{k=0}^{\infty} \varepsilon^k A_{11} \left( A_{12} e^{A_{22} t} - e^{A_{11} t} A_{12} \right) A_{22}^{-k+1}.
\]

We shall make use of the following notation:

\[
C_\varepsilon(t) = \begin{pmatrix} C_{1,\varepsilon}(t) \\ C_{2,\varepsilon}(t) \end{pmatrix} := e^{A_{11} t} B_\varepsilon = \begin{pmatrix} e^{A_{11} t} B_1 + \varepsilon^{-1} \mathcal{W}_\varepsilon(t) B_2 \\ \varepsilon^{-1} e^{A_{22} t} B_2 \end{pmatrix}.
\]
According identity (1) and notation (8), equation (3) is transformed into the system of equations
\[
\begin{aligned}
\nabla \varphi_1(-\varepsilon) &= e^{A_{11}T}x^0 + \mathcal{W}_\varepsilon(T)y^0 + \int_0^T C_{1,\varepsilon}(t)u_\varepsilon(T-t)\,dt, \\
\nabla \varphi_2(-\rho_\varepsilon) &= e^{A_{22}T/\varepsilon}y^0 + \int_0^T C_{2,\varepsilon}(t)u_\varepsilon(T-t)\,dt,
\end{aligned}
\]  

(9)

where
\[u_\varepsilon(T-t) := \frac{C_{1,\varepsilon}(t)\varepsilon + C_{2,\varepsilon}(t)\rho_\varepsilon}{S(\|C_{1,\varepsilon}(t)\varepsilon + C_{2,\varepsilon}(t)\rho_\varepsilon\|)}.
\]  

(10)

**Definition 1.** A limiting problem for problem (1) is
\[
\begin{aligned}
\dot{x}_0 &= A_0x_0 + B_0u, \quad t \in [0,T], \quad \|u\| \leq 1, \\
A_0 := A_{11}, \quad B_0 := B_1 - A_{12}A_{22}^{-1}B_2, \quad x_0(0) = x^0, \\
J_0(u) &= \varphi_1(x_0(T)) + \int_0^T \|u(t)\|^2\,dt \to \min.
\end{aligned}
\]

**Assumption 3.** The pairs of matrices \((A_0, B_0), (A_{22}, B_2)\) are completely controllable.

By [5], Assumptions 2 and 3 ensure Assumption 1 for all sufficiently small \(\varepsilon\).

Formulae (5), (7) and (8) imply
\[
C_{1,\varepsilon}(t) = C_{1,0}(t) + A_{12}A_{22}^{-1}e^{\frac{A_{22}}{\varepsilon}}B_2 + O(\varepsilon), \quad \varepsilon \to 0, \\
\frac{d}{dt}C_{1,\varepsilon}(t) = \frac{d}{dt}C_{1,0}(t) + \varepsilon^{-1}A_{12}e^{\frac{A_{22}}{\varepsilon}}B_2 + A_{11}A_{12}e^{\frac{A_{22}}{\varepsilon}}A_{22}^{-1}B_2 + O(\varepsilon), \quad \varepsilon \to 0,
\]

(11)

(12)

uniformly on the segment \([0,T]\).

We mention the known fact that under Assumption 2 there exist \(\gamma > 0\) and \(K > 0\) such that
\[\|e^{A_{22}t/\varepsilon}\| \leq Ke^{-\frac{\gamma t}{2}}.
\]  

(13)

If a vector function \(f_\varepsilon(t)\) is such that \(f_\varepsilon(t) = O(\varepsilon^\alpha)\) as \(\varepsilon \to 0\) for each \(\alpha > 0\) uniformly in \(t \in [a, b]\), we shall write \(O\) instead of \(f_\varepsilon(t)\). In particular,
\[\|e^{A_{22}t/\varepsilon}\| = O, \quad e^{-\gamma t/\varepsilon} = O \quad \text{as} \quad t \in [\varepsilon^\alpha, T], \quad p \in (0, 1),
\]

(14)

where \(\gamma > 0\).

It follows from formulae (11), (12) and estimate (13) that there exist \(K_1 > 0\) and \(\varepsilon_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0)\) and \(t \in [\sqrt{\varepsilon}, T]\), the inequalities hold
\[\|C_{1,\varepsilon}(t) - C_{1,0}(t)\| \leq K_1\varepsilon, \quad \left\| \frac{d}{dt}C_{1,\varepsilon}(t) - \frac{d}{dt}C_{1,0}(t) \right\| \leq K_1\varepsilon.
\]  

(15)

### 3. Auxiliary statements on cofinite functions

According [11] Thm. 26.6, if \(f\) is a differentiable strictly convex cofinite function on \(\mathbb{R}^n\), then \(\nabla f : \mathbb{R}^n \to \mathbb{R}^n\) is a one-to-one correspondence on \(\mathbb{R}^n\) and \(f^*\) is a differentiable strictly convex cofinite function on \(\mathbb{R}^n\).

**Lemma 1.** Let \(f\) be a differentiable strictly convex cofinite function on \(\mathbb{R}^n\), \(L\) be a non-negative linear operator in \(\mathbb{R}^n\), that is,
\[\langle \langle Ll, l \rangle \rangle \geq 0 \quad \text{for all} \quad l \in \mathbb{R}^n.
\]
Then the \( g(l) = f(l) + \frac{1}{2} \langle \|l\|, l \rangle \) is a differentiable strictly convex cofinite function on \( \mathbb{R}^n \) and \( \nabla g(l) = \nabla f(l) + \|l\| \).

**Proof.** We begin with proving that \( g(l) \) is a differentiable strictly convex cofinite function on \( \mathbb{R}^n \). We calculate the derivative of the scalar product \( \frac{1}{2} \langle \|l\|, l \rangle \) along the direction of \( \Delta l \):

\[
D \left( \frac{1}{2} \langle \|l\|, l \rangle \right) (\Delta l) = \frac{\partial}{\partial t} \left. \frac{\langle \|l\| + t\Delta l, l + t\Delta l \rangle}{2} \right|_{t=0} = \langle \|l\|, \Delta l \rangle,
\]

and we obtain that \( \nabla \left( \frac{1}{2} \langle \|l\|, l \rangle \right) = \|l\| \). According the definition [11], a convex function \( f \) is cofinite if the following relation holds:

\[
\lim_{\lambda \to +\infty} \frac{f(\lambda l)}{\lambda} = +\infty \quad \text{for all} \quad l \neq 0. \tag{16}
\]

Let us show that the function \( g(l) \) obeys this condition.

For each \( \lambda > 0 \) we have:

\[
g(\lambda l) = \lambda \frac{f(\lambda l)}{\lambda} + \frac{1}{2} \frac{\|l\| \lambda}{\lambda} = \lambda \frac{f(\lambda l)}{\lambda} + \frac{\lambda}{2} \frac{\langle \|l\|, l \rangle}{\lambda} \geq \frac{f(\lambda l)}{\lambda} \to +\infty \quad \text{as} \quad \lambda \to +\infty.
\]

\[\Box\]

**Corollary 1.** Let a function \( f \) satisfies the assumptions of Lemma 1, and \( f^* \) is a dual function for \( f \) in the sense of the convex analysis. Then the equation \( \nabla f^*(l) + \|l\| = d \) has the unique solution for each vector \( d \).

This corollary follows Lemma [1] and Theorem 26.6 in [11].

4. LIMITING VALUES OF VECTORS \( l_\varepsilon \) AND \( \rho_\varepsilon \)

**Theorem 1.** Let Assumptions 1 and 2 hold and the vector \( \lambda^*_\varepsilon = (l^*_\varepsilon, \rho^*_\varepsilon) \) is the unique solution of system [9]. Then the vectors \( l_\varepsilon, \rho_\varepsilon \) are bounded and

\[
l_\varepsilon \to l_0 \quad \text{as} \quad \varepsilon \to 0, \tag{17}
\]

where \( l_0 \) is the unique solution of the equation

\[
0 = -\nabla \varphi^*_\varepsilon(-l_\varepsilon) + e^{A^1^T}x^0 + \int_0^T C_{1,0}(t) \frac{C^*_{1,0}(t)l_\varepsilon}{S(\|C^*_{1,0}(t)l_\varepsilon\|)} dt. \tag{18}
\]

**Proof.** It is known that at the final time \( T \), the set of attainability of the controlled system in problem [9] is bounded uniformly in \( \varepsilon \in (0, \varepsilon_0] \), see, for instance, [6, Thm. 3.1]. Hence, the left hand side of equation [9] is bounded. This is why, as \( \varepsilon \to 0 \), the quantity \( \nabla \varphi^*(\lambda^*_\varepsilon) \) is bounded as well. Since the function \( \varphi^* \) is cofinite, according [11 Lm. 26.7], the vector \( \lambda^*_\varepsilon \) is bounded. Therefore, the vectors \( l_\varepsilon, \rho_\varepsilon \) are bounded.

We partition the interval of integration in the first identity [9] into two pieces: \( [0, \sqrt{\varepsilon}] \) and \( [\sqrt{\varepsilon}, T] \). Taking into consideration identity [6] and the notation [8] being representations of matrices \( W_\varepsilon(t) \) and \( C_\varepsilon(t) \) in system [9]–[10], we can write the first identity [9] as

\[
\nabla \varphi^*_1(-l_\varepsilon) = e^{A^1^T}x^0 + O(\varepsilon) + \int_{\sqrt{\varepsilon}}^T C_{1,\varepsilon}(t) \frac{C^*_{1,\varepsilon}(t)l_\varepsilon}{S(\|C^*_{1,\varepsilon}(t)l_\varepsilon\|)} dt \quad \text{as} \quad \varepsilon \to 0. \tag{19}
\]
Let $l_0$ be an arbitrary liming point of the function $l_\varepsilon$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ in identity (19), by inequalities (15) we obtain the identity

$$\nabla \varphi^*_\varepsilon(-l_0) = e^{A_{11}T}x^0 + \int_0^T C_{1,0}(t) \frac{C_{1,0}^*(t)l_0}{S\|C_{1,0}^*(t)l_0\|} dt,$$

that is, $l_0$ satisfies equation (18). This equation reads as

$$\nabla \varphi^*_\varepsilon(-l_0) + L(-l_0) = e^{A_{11}T}x^0$$

and $L \geq 0$. This is why by Corollary 1 of Lemma 1 this equation possesses the unique solution. Thus, $l_0$ is the unique limiting point for $l_\varepsilon$ and $l_\varepsilon \to l_0$ as $\varepsilon \to 0$. \hfill \Box

**Theorem 2.** Let the assumptions of Theorem 1 hold, and $B_2$ is a mapping of $\mathbb{R}^{r}$ onto $\mathbb{R}^{m}$; in particular, $r \geq m$. Then $\rho_\varepsilon \to 0$, the quantity $\{r_\varepsilon\}$ ($r_\varepsilon := \varepsilon^{-1} \rho_\varepsilon$) is bounded as $\varepsilon \to +0$ and all its limiting points $r_0$ satisfy the equation

$$0 = \int_0^{+\infty} e^{A_{22}\tau} B_2 \frac{B_0^*l_0 + B_2^*e^{A_{22}\tau}(r_0 + (A_{22}^*)^{-1}A_{12}^*l_0)}{S\left(\|B_0^*l_0 + B_2^*e^{A_{22}\tau}(r_0 + (A_{22}^*)^{-1}A_{12}^*l_0)\|\right)} d\tau. \quad (20)$$

**Proof.** We change the variable $\tau := t/\varepsilon$ in the integral in the second identity in system (9). We choose arbitrary $\delta > 0$ and taking into consideration estimate (13), we rewrite this identity as

$$\nabla \varphi^*_\delta(-\rho_\varepsilon) = 0 + \int_0^\delta e^{A_{22}\tau} B_2 \frac{\tilde{B}(\tau, \varepsilon)l_\varepsilon + B_2^*e^{A_{22}\tau}r_\varepsilon}{S\left(\|\tilde{B}(\tau, \varepsilon)l_\varepsilon + B_2^*e^{A_{22}\tau}r_\varepsilon\|\right)} d\tau + O(e^{-\gamma\delta}), \quad (21)$$

where $r_\varepsilon := \rho_\varepsilon/\varepsilon$, and

$$\tilde{B}(\tau, \varepsilon) := B_0^*e^{A_{11}\varepsilon\tau} + B_2^*e^{A_{22}\tau}(A_{22}^*)^{-1}A_{12}^*. \quad (22)$$

We note that $\tilde{B}(\tau, \varepsilon)l_\varepsilon \to \tilde{B}(\tau, 0)l_0$ as $\varepsilon \to 0$ uniformly on $[0, \delta]$ and $\tilde{B}(\tau, 0)$ is bounded on $[0, +\infty)$.

Let $\rho_0$ be an arbitrary limiting point of $\rho_\varepsilon$ as $\varepsilon \to 0$, that is, there exists $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ and $\rho_k := \rho_{\varepsilon_k} \to \rho_0$.

We assume that $r_k := r_{\varepsilon_k}$ is unbounded. Without loss of generality we suppose that

$$r_k \to \infty, \quad \frac{r_k}{\|r_k\|} \to \tau, \quad \|\tau\| = 1, \quad \rho_0 = \|\rho_0\|\tau. \quad (23)$$

Since the function $B_2^*e^{A_{22}\tau}r$ is jointly continuous in the variable $\tau$ and vector $r$, and as $r \neq 0$, by the injectivity of $B_2^*$, we have $B_2^*e^{A_{22}\tau}r \neq 0$, there exists $K_0(\delta) > 0$ such that

$$\|B_2^*e^{A_{22}\tau}r\| \geq K_0(\delta)\|r\|$$

for all $r$ and all $\tau \in [0, \delta]$. This is why, by relations (23), for all sufficiently large $k$, the inequality holds:

$$\|C_{1,\varepsilon_k}(\varepsilon\tau)l_{\varepsilon_k} + B_2^*e^{A_{22}\tau}r_k\| > 2,$$

and identity (21) becomes

$$\nabla \varphi^*_\delta(-\rho_k) = \int_0^\delta e^{A_{22}\tau} B_2 \frac{\frac{1}{\|r_k\|} \tilde{B}(\tau, \varepsilon_k)l_k + B_2^*e^{A_{22}\tau}r_k}{\|r_k\|\|\tilde{B}(\tau, \varepsilon_k)l_k + B_2^*e^{A_{22}\tau}r_k\|} d\tau + O(e^{-\gamma\delta}). \quad (24)$$
We pass to the limit in $k$ and then as $\delta \to +\infty$ in identity (24). Then in view of relations (23) we obtain the identity:

$$
\nabla \varphi^*_2(-\|\rho_0\|) = \int_0^{+\infty} e^{A_{22} \tau} B_2 \frac{B_2^* e^{A_{22} \tau}}{\|B_2^* e^{A_{22} \tau}\|} \, d\tau.
$$

We calculate the scalar product of the latter equation with $\tau$ and we obtain:

$$
\langle \nabla \varphi^*_2(-\|\rho_0\|), \tau \rangle = \int_0^{+\infty} \|B_2^* e^{A_{22} \tau}\| \, d\tau.
$$

By Assumption 3, the right hand side of the above identity is positive, while the left hand side is non-positive due to the monotonicity of $\nabla \varphi^*_2$ and the identity $\nabla \varphi^*_2(0) = 0$; this is a contradiction. Thus, $\rho_\varepsilon \to 0$.

Let the assumptions of Theorem 2 holds. Then equation (20) has the unique solution $r_0$ and $r_\varepsilon \to r_0$.

**Proof.** We introduce the notations: $l := B_0^0 l_0$, $r := r_0 + (A^*_{22})^{-1} A^*_{12} l_0$. Then equation (20) casts into the form:

$$
F(r) := \int_0^{+\infty} e^{A_{22} \tau} B_2 \frac{l + B_2^* e^{A_{22} \tau} \tau}{S(\|l + B_2^* e^{A_{22} \tau}\|)} \, d\tau = 0.
$$

If $l = 0$, we multiply identity (26) by $r$ and we obtain:

$$
\int_0^{+\infty} \frac{\|B_2^* e^{A_{22} \tau} r\|^2}{S(\|B_2^* e^{A_{22} \tau}\|)} \, d\tau = 0.
$$

Since the integrand is continuous and non-negative, we have $\|B_2^* e^{A_{22} \tau} r\| \equiv 0$ and by Assumption 3 this implies $r = 0$.

Let $l \neq 0$. Assume that there exist two different solutions $r_1 \neq r_2$ to equation (20): $F(r_1) = F(r_2) = 0$. By the Lagrange formula,

$$
0 = \langle F(r_1) - F(r_2), r_1 - r_2 \rangle = \left\langle \frac{\partial}{\partial r} F(r) \bigg|_{r = r'}, (r_1 - r_2), r_1 - r_2 \right\rangle,
$$

where $r' \in [r_1, r_2]$. Let us show that as $r_1 \neq r_2$, identity (27) is impossible.

We rewrite the integral in (26) as a sum of two integrals over two sets:

$$
E_1(r) := \{ \tau \in [0, +\infty) : \|l + B_2^* e^{A_{22} \tau}\| \leq 2 \}, \quad E_2(r) := \{ \tau \in [0, +\infty) : \|l + B_2^* e^{A_{22} \tau}\| \geq 2 \}.
$$

Then the integral in the right hand side in equation (26) is split into two integrals:

$$
F(r) = \int_{E_1(r)} e^{A_{22} \tau} B_2 \frac{l + B_2^* e^{A_{22} \tau}}{2} \, d\tau + \int_{E_2(r)} e^{A_{22} \tau} B_2 \frac{l + B_2^* e^{A_{22} \tau}}{\|l + B_2^* e^{A_{22} \tau}\|} \, d\tau.
$$

Since $B_2^* e^{A_{22} \tau} \to 0$ as $\tau \to +\infty$, the sets $E_1(r)$ and $E_2(r)$ consist of finitely many segments.
Let us find the derivative $DF(r')(\Delta r)$ of the function $F$ at the point $r'$ along the direction $\Delta r$. We employ representation (28) and the known formula:

$$D \left( \int_{\alpha(t)}^{\beta(t)} f(t,r) dt \right) (\Delta r) = \int_{\alpha(r)}^{\beta(r)} \frac{\partial f(t,r)}{\partial r} (\Delta r) dt + f(\beta(r),r) \frac{\partial \beta}{\partial r} (\Delta r) - f(\alpha(r),r) \frac{\partial \alpha}{\partial r} (\Delta r).$$

Since the integrands coincide at the common points of $E_1(r)$ and $E_2(r)$, the final formula for $DF$ involves no non-integral terms.

Since

$$\frac{\partial}{\partial r} \left( e^{A_{22}^* B_2^*} \frac{l + B_2^* e^{A_{22}^* r} r}{2} \right) (\Delta r) = C(\tau) \frac{C^*(\tau) \Delta r}{2}, \quad C(\tau) := e^{A_{22}^*} B_2,$$

and

$$\frac{\partial}{\partial r} \left( e^{A_{22}^* B_2^*} \frac{l + B_2^* e^{A_{22}^* r} r}{\|l + B_2^* e^{A_{22}^* r}\|} \right) = C(\tau) \frac{C^*(\tau) \Delta r}{\|l + C^*(\tau)r\|^2} - \frac{\langle C^*(\tau) \Delta r, l + C^*(\tau)r \rangle (l + C^*(\tau)r)}{\|l + C^*(\tau)r\|^3},$$

then

$$DF(r')(\Delta r) = DF_1(r')(\Delta r) + DF_2(r')(\Delta r),$$

$$DF_1(r')(\Delta r) = \frac{1}{2} \int_{E_1(r')} e^{A_{22}^* B_2^*} B_2^* e^{A_{22}^* \Delta r} d\tau,$$

$$DF_2(r')(\Delta r) = \int_{E_2(r')} C(\tau) \frac{C^*(\tau) \Delta r}{\|l + C^*(\tau)r\|^2} - \frac{\langle C^*(\tau) \Delta r, l + C^*(\tau)r \rangle (l + C^*(\tau)r)}{\|l + C^*(\tau)r\|^3} d\tau.$$

If $E_1(r') \neq \emptyset$, the latter identity in (29) implies $DF_1(r') > 0$. It follows from the Cauchy-Schwarz inequality and relations (29) that $DF_2(r') \geq 0$. This is why, if $E_1(r') \neq \emptyset$, then $DF(r') > 0$ and identity (27) is possible only as $\Delta r = r_1 - r_2 = 0$.

Since $\Delta r \neq 0$, it follows from identity (27) that

$$E_1(r') = \emptyset$$

and by the Cauchy-Schwarz inequality, the vector $l + B_2^* e^{A_{22}^* r}$ is parallel to the vector $B_2^* e^{A_{22}^* \Delta r}$ for all $\tau$. The identity $E_1(r') = \emptyset$ means that

$$\|l + e^{A_{22}^* r}\| \geq 2 \quad \text{for all } \tau. \quad (30)$$

By the assumptions of the theorem, $B_2^* e^{A_{22}^* \Delta r} \neq 0$. Hence, there exists a function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$l + B_2^* e^{A_{22}^* r} = \beta(\tau) B_2^* e^{A_{22}^* \Delta r} \quad \text{for all } \tau.$$

Hence, $l$ reads as $B_2 l_1$. Thus, if $l \not\in \operatorname{Im}(B_2^*)$, identity (27) is impossible.

By the injectivity of the operator $B_2^*$ we obtain that

$$\forall \tau \quad l_1 + e^{A_{22}^* r} = \beta(\tau) e^{A_{22}^* \Delta r}. \quad (31)$$

We multiply identity (31) by $e^{-A_{22}^* \tau}$ and we get:

$$e^{-A_{22}^* \tau} l_1 + r' = \beta(\tau) \Delta r. \quad (32)$$

Hence, the function $\beta(\tau)$ is infinitely differentiable. We differentiate identity (32) twice in $\tau$ and we obtain:

$$-A_{22}^* e^{-A_{22}^* \tau} l_1 = \beta'(\tau) \Delta r, \quad (A_{22}^*)^2 e^{-A_{22}^* \tau} l_1 = \beta''(\tau) \Delta r.$$
As $\tau = 0$, this gives the identities:

$$-A_{22}^2 l_1 = \beta'(0)\Delta r, \quad (A_{22}^2)^2 l_1 = \beta''(0)\Delta r. \quad (33)$$

If $\beta'(0) = 0$ or $\beta''(0) = 0$, then $l_1 = 0$ that contradicts the assumptions of the theorem.

It follows from identity (33) that

$$\beta''(0)\Delta r = (A_{22}^* l_1)^2 = -A_{22}^* \beta'(0)\Delta r,$$

that is, the vector $\Delta r$ is an eigenvector of the matrix $A_{22}^*$. Hence,

$$A_{22}^* \Delta r = -\alpha \Delta r, \quad \alpha > 0, \quad (34)$$

where $\alpha = \beta''(0)/\beta'(0)$ is an eigenvalue of the matrix $A_{22}^*$. If the matrix $A_{22}^*$ has no real eigenvalues, identity (27) is impossible.

It follows from identities (33) and (34) that the vector $l_1$ is parallel to the vector $\Delta r$. This is why by identity (32) and $r'$ is parallel to the vector $l_1$. Since $r' = r_1 - \beta_0 \Delta r$ for some $\beta_0$, it follows that the vectors $r_1, r_2$ are parallel to the vector $l_1$. Thus, in this case,

$$r_1 = \beta_i l_1, \quad r_2 = \beta_i l_2, \quad r' = \beta_3 l_1,$$

and identity (26) being valid for $r_i, i = 1, 2$ after calculating its scalar product with $l_1$, casts into the form:

$$\int_0^{+\infty} \left(1 + \beta_3 e^{-\alpha \tau}\right) e^{-\alpha \tau} \|B_2^* l_1\|^2 \frac{d\tau}{S(1 + \beta_3 e^{-\alpha \tau}) \cdot \|B_2^* l_1\|} = 0, \quad i = 1, 2. \quad (35)$$

The above identity (35) is impossible if $1 + \beta_3 e^{-\alpha \tau}$ is sign-definite on $[0, +\infty)$. Since $e^{-\alpha \tau}$ is strictly decreasing and $e^{-\alpha \tau} \to 0$ as $\tau \to +\infty$, we obtain that $\beta_i < -1, i = 1, 2$. By the relation $r' \in [r_1, r_2]$ this implies that $\beta_3 < -1$. But then there exists $\tau_0 > 0$ such that $1 + \beta_3 e^{-\alpha \tau_0} \cdot \|B_2^* l_1\| = 0$ and this contradicts inequality (30).

In what follows we suppose that

$$r = m, \quad A_{22} = -I, \quad B_2 = I. \quad (36)$$

Here $I$ stands for the identity mapping of $\mathbb{R}^m$ onto $\mathbb{R}^m$.

**Lemma 2.** Let conditions (36) and the assumptions of Theorem 1 be satisfied. Then

$$r_{\varepsilon} \to r_0 = A_{12}^* l_0 - 2B_0^* l_0 \quad \text{as} \quad \varepsilon \to 0.$$

**Proof.** Under (36), equation (20) becomes

$$\int_0^{+\infty} \frac{e^{-\tau} l + e^{-\tau} r}{S(\|l + e^{-\tau} r\|)} \, d\tau = 0, \quad (37)$$

where $l := B_0^* l_0$, $r := r_0 + (A_{22}^*)^{-1} A_{12}^* l_0$. Thanks to Theorem 3, it is sufficient to confirm that the vector $(-2l)$ is its solution. We substitute $r = -2l$ into the left hand side of equation (37), we obtain:

$$\int_0^{+\infty} \frac{(1 - 2e^{-\tau}) l}{S(1 - 2e^{-\tau} \cdot \|l\|)} \, d\tau = \left[ \xi = e^{-\tau} \right] = \int_0^{1} \frac{(1 - 2\xi)}{S(1 - 2\xi \cdot \|l\|)} \, d\xi \, l = \left[ \eta = 1 - 2\xi \right]$$

$$= \frac{1}{2} \int_{-1}^{1} \frac{\eta}{S(|\eta| \cdot \|l\|)} \, d\eta \, l = 0$$

since the integrand is odd. \qed
5. Asymptotic Expansion of Vector $\lambda_\varepsilon$ under Conditions (36)

We observe that by conditions (36) we have:

$$B_0 = B_1 + A_{12}, \quad r_0 = (A_{12}^* - 2B_0^*)l_0,$$

$$C_{1,*}(t) = B_1^*e^{A_{11}^*t} + A_{12}^*(e^{A_{11}^*t} - e^{-\frac{t}{\varepsilon}}I)\sum_{k=0}^{\infty}(-1)^k\varepsilon^k(A_{11}^*)^k.$$  (38)

It follows from identities (38) and (39) that

$$C_{1,*}(t)\lambda_{\varepsilon} = C_{1,*}(t)l_0 + C_{1,0}^*\Delta l - \varepsilon A_{12}e^{A_{11}^*t}A_{11}l_0 - 2e^{-\frac{t}{\varepsilon}}B_0^*l_0 - A_{12}^*e^{-\frac{t}{\varepsilon}}\Delta l + \varepsilon A_{12}e^{-\frac{t}{\varepsilon}}A_{11}l_0 + e^{-\frac{t}{\varepsilon}}\Delta r + \mathcal{F}_2(\varepsilon, \Delta l, \Delta r).$$  (40)

Here $\Delta l := l_\varepsilon - l_0$, $\Delta r := r_\varepsilon - r_0$, and $\mathcal{F}_2(\varepsilon, \Delta l, \Delta r)$ is a function of a second order of smallness in $\{\varepsilon, \Delta l, \Delta r\}$.

We begin with the case, when the limiting problem has a single point of the change of the type of optimal control. Suppose that for the limiting problem and the initial state of the system $x^0$ there exists the only moment of time $t = t_0 \in (0, T)$ such that

$$\|C_{1,0}(t)l_0\| < 2, \quad \|C_{1,0}(t_0)l_0\| = 2 \text{ for all } t < t_0,$n
$$\|C_{1,0}(t)l_0\| > 2 \text{ for all } t > t_0,$n
$$\frac{d}{dt}\|C_{1,0}(t)l_0\|^2 \bigg|_{t=t_0} \neq 0.$$  (41)

Lemma 3. If the condition

$$\|B_0^*l_0\| < 2$$  (42)

holds, then

$$\forall l_\varepsilon \rightarrow l_0 \forall r_\varepsilon \rightarrow (A_{12}^* - 2B_0^*)l_0 \exists \varepsilon_0 > 0 \forall \varepsilon \in (0, \varepsilon_0) \forall t \in [0, \sqrt{\varepsilon}] \|C_{1,*}(t)\lambda_{\varepsilon}\| < 2.$$  (43)

Proof. We assume the opposite; then there exits sequences $\{t_k\} \subset [0, \sqrt{\varepsilon}]$ and $\{\varepsilon_k\}$ such that $\varepsilon_k \rightarrow +0$ and

$$\|C_{1,0}(t_k)\lambda_{\varepsilon_k}\| \geq 2.$$  (44)

We let $\tau_k := t_k/\varepsilon_k$, $l_k := l_{\varepsilon_k}$, $r_k := r_{\varepsilon_k}$ and $\lambda_k := \lambda_{\varepsilon_k}$. Then by identity (40) we get:

$$C_{1,*}(t_k)\lambda_{\varepsilon_k} = C_{1,0}(l_k)l_0 - 2e^{-\tau_k}B_0^*l_0 + \mathcal{F}_1(\varepsilon_k, \Delta l_k, \Delta r_k),$$  (45)

$$\Delta l_k := l_k - l_0, \quad \Delta r_k := r_k - r_0, \quad \mathcal{F}_1(\varepsilon_k, \Delta l_k, \Delta r_k) \rightarrow 0.$$  (45)

Let $\tau_0$ be a limiting point of the sequence $\{\tau_k\}$; to shorten the notation, we suppose that $\tau_k \rightarrow \tau_0$. If $\tau_0 = +\infty$, we pass to the limit as $k \rightarrow \infty$ in identity (45) and taking into consideration that $l_k \rightarrow l_0$, $r_k \rightarrow (A_{12}^* - 2B_0^*)l_0$, we obtain: $C_{1,*}(l_k)\lambda_k \rightarrow B_0^*l_0$. But $\|B_0^*l_0\| < 2$ by assumption (41) and this contradicts condition (44).

Thus, all limiting points $\tau_0$ are finite. Then $\varepsilon_k\tau_k \rightarrow 0$ and this is why $C_{1,*}(l_k)\lambda_k \rightarrow (1 - 2e^{-\tau_0})B_0^*l_0$. But

$$\left\|(1 - 2e^{-\tau_0})B_0^*l_0\right\| = \left|1 - 2e^{-\tau_0}\right| \cdot \|B_0^*l_0\| \leq \|B_0^*l_0\| < 2,$$

and this contradicts condition (44). \qed

Theorem 4. Under condition (42), there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a single point $t_\varepsilon$ of the change of the type of optimal control in problem (4), that is,

$$\|C_{1,*}(t)\lambda_{\varepsilon}\| < 2, \quad \|C_{1,*}(t)\lambda_{\varepsilon}\| = 2 \text{ for all } t < t_\varepsilon, \quad \|C_{1,*}(t)\lambda_{\varepsilon}\| > 2 \text{ for all } t > t_\varepsilon.$$  (40)

At that, $t_\varepsilon \rightarrow t_0$ as $\varepsilon \rightarrow 0$. 

Proof. We note that by assumption (41) there exists $\delta_0 > 0$ such that

$$\frac{d}{dt}\| C^\ast_{1,0}(t)l_0 \|^2 \bigg|_{t=t_0} > 0 \quad \text{for all} \quad t \in [t_0 - \delta_0, t_0 + \delta_0].$$

By (17) and (15) and since $\| C^\ast_{1,0}(t_0 - \delta_0)l_0 \| < 2$ and $\| C^\ast_{1,0}(t_0 + \delta_0)l_0 \| > 2$, there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$ and $t \in [t_0 - \delta_0, t_0 + \delta_0]$ the inequalities hold:

$$\| C^\ast_{\varepsilon}(t_0 - \delta_0)\lambda_\varepsilon \| < 2, \quad \| C^\ast_{\varepsilon}(t_0 + \delta_0)\lambda_\varepsilon \| > 2, \quad \frac{\partial}{\partial t} \left( \| C^\ast_{\varepsilon}(t)\lambda_\varepsilon \|^2 \right) > 0.$$

This implies the existence of a single point $t_\varepsilon \in [t_0 - \delta_0, t_0 + \delta_0]$ such that for all $\varepsilon > 0$ satisfying identity $\| C^\ast_{\varepsilon}(t_\varepsilon)\lambda_\varepsilon \| = 2$.

Let us show that for all sufficiently small $\varepsilon > 0$ the equalities hold thanks to condition (43).

Then it follows from estimate (11) and condition (17) that for all sufficiently small $\varepsilon > 0$, $\varepsilon_0 \in [\sqrt{\varepsilon}, T]$ and $t \in [t_0 - \delta_0, t_0 + \delta_0]$ the inequality holds:

$$\| C^\ast_{\varepsilon}(t)\lambda_\varepsilon \| - 2 > 0.$$

Hence, $\| C^\ast_{\varepsilon}(t)\lambda_\varepsilon \| \neq 2$ for such $\varepsilon$ and $t$. On the remaining segment $[0, \sqrt{\varepsilon}]$, the relation $\| C^\ast_{\varepsilon}(t)\lambda_\varepsilon \| \neq 2$ holds thanks to condition (43).

Thus, in the considered case, the integral in (3) is also split into the sum of two integrals:

$$\int_0^T \frac{C_{\varepsilon}(t)C^\ast_{\varepsilon}(t)\lambda}{\| C^\ast_{\varepsilon}(t)\lambda \|} dt = \frac{1}{2} \int_0^{t_\varepsilon} C_{\varepsilon}(t)C^\ast_{\varepsilon}(t)\lambda dt + \int_{t_\varepsilon}^T C_{\varepsilon}(t) \frac{C^\ast_{\varepsilon}(t)\lambda}{\| C^\ast_{\varepsilon}(t)\lambda \|} dt. \quad (46)$$

Let $\Delta l_\varepsilon := l_\varepsilon - l_0$, $\Delta r_\varepsilon := r_\varepsilon - r_0$, $\Delta t_\varepsilon := t_\varepsilon - t_0$. Then

$$\lambda_\varepsilon = \left( \frac{l_0 + \Delta l_\varepsilon}{\varepsilon (r_0 + \Delta r_\varepsilon)} \right), \quad \Delta l_\varepsilon = o(1), \quad \Delta r_\varepsilon = o(1), \quad \Delta t_\varepsilon = o(1)$$

as $\varepsilon \to 0$, and by identities (2), (3), (46) and Theorem 4, the triple $\{ \Delta l_\varepsilon, \Delta r_\varepsilon, \Delta t_\varepsilon \}$ solves the following system of equations depending on the parameter $\varepsilon$:

$$\begin{align*}
0 = & F_1(\varepsilon, \Delta l, \Delta r, \Delta t) := -\nabla \varphi_1^{\ast}(-l_\varepsilon) + \nabla \varphi_1^{\ast}(-l_0) \\
& + \mathcal{W}_\varepsilon(T)y_0 + \frac{1}{2} \int_0^{t_\varepsilon} C_{1,\varepsilon}(t)C^\ast_{\varepsilon}(t)\lambda_\varepsilon dt + \int_{t_\varepsilon}^T C_{1,\varepsilon}(t) \frac{C^\ast_{\varepsilon}(t)\lambda_\varepsilon}{\| C^\ast_{\varepsilon}(t)\lambda_\varepsilon \|} dt, \\
0 = & F_2(\varepsilon, \Delta l, \Delta r, \Delta t) := -\nabla \varphi_2^{\ast}(-\varepsilon r_\varepsilon) + \nabla \varphi_2^{\ast}(0) \\
& + \frac{1}{2} \int_0^{t_\varepsilon} \varepsilon^{-1} C_{2,\varepsilon}(t)C^\ast_{\varepsilon}(t)\lambda_\varepsilon dt + \int_{t_\varepsilon}^T \varepsilon^{-1} C_{2,\varepsilon}(t) \frac{C^\ast_{\varepsilon}(t)\lambda_\varepsilon}{\| C^\ast_{\varepsilon}(t)\lambda_\varepsilon \|} dt, \\
0 = & G(\varepsilon, \Delta l, \Delta r, \Delta t) := \| C^\ast_{\varepsilon}(t + \Delta t)\lambda_\varepsilon \|^2 - \| C^\ast_{1,0}(t_0)l_0 \|^2.
\end{align*} \quad (47)$$

We note that the functions $F_1$, $F_2$ and $G$ are continuous, and $G$ is infinitely differentiable. Let us study their asymptotic expansions with respect to infinitesimals $\Delta l$, $\Delta r$ and $\Delta t$. 
By the infinite differentiability of the functions \( \varphi_1^* \) and \( \varphi_2^* \) and in view of identity \( \varphi_2^*(0) = 0 \) we obtain:

\[
-\nabla \varphi_1^*(-l_0 - \Delta l) + \nabla \varphi_1^*(-l_0) \sim D^2 \varphi_1^*(-l_0) \Delta l + \sum_{k=2}^{\infty} \Phi_{1,k}(\Delta l),
\]

\[
-\nabla \varphi_2^*(-\varepsilon \tau) + \nabla \varphi_2^*(0) \sim D^2 \varphi_2^*(0) r_0 \varepsilon + \sum_{k=2}^{\infty} \Phi_{2,k}(\varepsilon, \Delta r),
\]

where \( D^2 \varphi_1^*(-l_0) \) and \( D^2 \varphi_2^*(0) \) are second order differentials of \( \varphi_1^* \) and \( \varphi_2^* \) at the points \((-l_0)\) and \(0\), respectively, and \( \Phi_{1,k}(\Delta l) \) and \( \Phi_{2,k}(\varepsilon, \Delta r) \) are homogeneous functions of order \( k \), namely, polynomials of the components of the vector \( \Delta l \) and \( \varepsilon \).

By identity (7),

\[
W_\varepsilon(T)y_0 \sim \varepsilon e^{A_{11}T}A_{12}y_0 + \sum_{k=2}^{\infty} \varepsilon^k y_k,
\]

where \( y_k \) are known vectors.

We split each integral in the first and second identity in system of equations (47) into two parts

\[
\int_{t_0}^{t_0+\Delta t} + \int_{t_0}^{t_0+\Delta t}, \quad \int_{t_0}^{T} + \int_{t_0+\Delta t}^{T}
\]

and we denote the integrals by \( I_1(\varepsilon, \Delta \lambda), I_2(\varepsilon, \Delta \lambda), I_3(\varepsilon, \Delta \lambda) \) and \( I_4(\varepsilon, \Delta \lambda), \) respectively.

We note that by identity (7), the asymptotics of integrands in \( I_2 - I_4 \) is power in \( \varepsilon \) and the components of the vector \( \Delta \lambda \) with coefficients smoothly depending on \( t \).

To expand the integrals \( I_2 \) and \( I_3 \) in \( \Delta t \), we should additionally expand the coefficients depending on \( t \) into the Taylor series at the point \( t_0 \) and to integrate the obtained expansions over the mentioned segments.

We observe that in \( I_2 \) and \( I_3 \), the terms of the first order of smallness in \( \Delta t \) are of the form:

\[
\frac{C_{1,0}(t_0)C_{1,0}^*(t_0)l_0}{2} \Delta t, \quad -\frac{C_{1,0}(t_0)C_{1,0}^*(t_0)l_0}{\|C_{1,0}^*(t_0)l_0\|} \Delta t,
\]

respectively. Since

\[
\|C_{1,0}^*(t_0)l_0\| = 2, \quad I_2(\varepsilon, \Delta \lambda) = O(\Delta t), \quad I_2(\varepsilon, \Delta \lambda) = O(\Delta t),
\]

the expansions of the \( I_2 + I_3 \) contains no terms of the first order of smallness in \( \Delta l, \Delta r, \Delta t \) and \( \varepsilon \).

By estimate (14) and identity (39), on \([t_0, T]\) we have asymptotic identities:

\[
C_{1,\varepsilon}^*(t) = B_{11}^*(t)e^{A_{11}\varepsilon t} + A_{12e}^*e^{A_{11}\varepsilon t} \sum_{k=0}^{\infty} (-1)^k \varepsilon^k (A_{11}^*)^k, \quad C_{2,\varepsilon}^*(t) = O \quad \text{as} \quad \varepsilon \to 0. \quad (50)
\]

Hence,

\[
\frac{1}{2} \int_{t_0}^{t_0+\Delta t} \varepsilon^{-1}C_{2,\varepsilon}^*(t)\varepsilon C_{\varepsilon}^*(t)\lambda_\varepsilon dt + \int_{t_0}^{T} \varepsilon^{-1}C_{2,\varepsilon}^*(t)\frac{C_{\varepsilon}^*(t)\lambda_\varepsilon}{\|C_{\varepsilon}^*(t)\lambda_\varepsilon\|} dt = \frac{1}{2} \int_{t_0}^{t_0+\Delta t} \varepsilon^{-1}C_{2,\varepsilon}^*(t)\varepsilon C_{\varepsilon}^*(t)\lambda_\varepsilon dt + O,
\]

while the power asymptotics of the integrals \( I_i, \ i = 2, 3, 4 \) contains no \( \Delta r \).

We introduce the notation: \( \{I_i(\varepsilon, \Delta \lambda)\} \) is a linear in \( \Delta l, \Delta r, \Delta t \) and \( \varepsilon \) part of the integral \( I_i(\varepsilon, \Delta \lambda) \).
By Theorem 4, identities (50) and
\[ \int_0^{t_0} e^{-\frac{1}{2}} f(t, l, r_\epsilon) \, dt = O(\epsilon), \]
if \( f(t, l, r_\epsilon) \) is uniformly bounded on \([0, t_0]\), by simple calculations we get:

\[ (I_1(\epsilon, \Delta l))_1 = \frac{1}{2} \int_0^{t_0} C_{1,0}(t)C_{1,0}^*(t)\, dt \Delta l + \epsilon f_1 =: D_{11} \Delta l + \epsilon f_1, \]
\[ (I_3(\epsilon, \Delta l))_1 = \int_{t_0}^{T} C_{1,0}(t) \frac{\Delta l \|C_{1,0}^*(t)l_0\|^2 - (C_{1,0}^*(t)\Delta l, C_{1,0}^*(t)l_0)C_{1,0}^*(t)l_0}{\|C_{1,0}^*(t)l_0\|^3} \, dt + \epsilon f_3 =: D_{12} \Delta l + \epsilon f_3, \]
\[ (I_5(\epsilon, \Delta l))_1 = \frac{1}{4} \Delta r + \frac{1}{4} (2B_0^* - A_{12}^*) \Delta l + \epsilon f_5, \]
where \( f_1, f_3 \) and \( f_5 \) are uniquely calculated by \( l_0 \). At that, by assumption (36) and Cauchy-Schwarz inequality we have:
\[ D_{11} > 0, \quad D_{12} \geq 0. \]

By identity (50) we can find the asymptotics for the function \( G(\epsilon, \Delta l, \Delta t) \) as \( \Delta l, \Delta t \) and \( \epsilon \) tend to zero:
\[ G(\epsilon, \Delta l, \Delta t) \sim 2(C_{1,0}^*(t_0)l_0, C_{1,0}^*(t_0)\Delta l) + (C_{1,0}^*)'(t_0)l_0 \Delta t + \epsilon A_{11}^* e^{A_{11}^*t_0 l_0} \]
\[ + \sum_{k=2}^{\infty} G_k(\epsilon, \Delta l, \Delta t), \quad (C_{1,0}^*)'(t_0) := \left. \frac{d}{dt} C_{1,0}^*(t) \right|_{t=t_0}, \]
where \( G_k(\epsilon, \Delta l, \Delta t) \) are some homogeneous functions of order \( k \) in \( \epsilon \) and the components of the vectors \( \Delta l \) and \( \Delta r \).

Thus, by identities (48), (49), (51), (53) and (55), the system for the first corrector of (47) reads as
\[
\begin{align*}
\epsilon g_1 &= D^2 \varphi_1^*(-l_0) \Delta l_1 + D_{11} \Delta l_1 + D_{12} \Delta l_1 \\
\epsilon g_2 &= \frac{1}{4} \Delta r_1 + \frac{1}{4} (2B_0^* - A_{12}^*) \Delta l_1 \\
\epsilon g_3 &= 2(C_{1,0}^*(t_0)l_0, C_{1,0}^*(t_0)\Delta l_1) + (C_{1,0}^*)'(t_0)l_0, (C_{1,0}^*)'(t_0)l_0) \Delta l_1.
\end{align*}
\]

By the convexity of \( \varphi_1 \) and inequalities (54), we have
\[ D^2 \varphi_1^*(-l_0) + D_{11} + D_{12} > 0, \]
and this is why the first equation in system (56) determines uniquely \( \Delta l_1 = \epsilon l_1 \). After that by the second equation in system (56) we uniquely find \( \Delta r_1 = \epsilon r_1 \). Finally, by conditions (41), the coefficient at \( \Delta t_1 \) is non-zero and hence, by the third equation in system (56) we uniquely determine \( \Delta t_1 = \epsilon t_1 \). Thus, the linear operator of the first corrector for system (56), that is, the operator
\[
\mathcal{D} \begin{pmatrix}
\Delta l_1 \\
\Delta r_1 \\
\Delta t_1
\end{pmatrix} = \begin{pmatrix}
D^2 \varphi_1^*(-l_0) \Delta l_1 + D_{11} \Delta l_1 + D_{12} \Delta l_1 \\
\frac{1}{4} \Delta r_1 + \frac{1}{4} (2B_0^* - A_{12}^*) \Delta l_1 \\
2(C_{1,0}^*(t_0)l_0, C_{1,0}^*(t_0)\Delta l_1) + (C_{1,0}^*)'(t_0)l_0, (C_{1,0}^*)'(t_0)l_0) \Delta l_1
\end{pmatrix}
\]
is continuously invertible.
The process of determining next terms in the expansions of $\Delta l$, $\Delta r$ and $\Delta t$ is continued in a standard way. Assume that we have approximations of $\Delta l$, $\Delta r$ and $\Delta t$ up to $N$th order. Then the quantities
\[
\Delta l_{N+1} := \Delta l - \sum_{k=1}^{N} \varepsilon^k l_k, \quad \Delta r_{N+1} := \Delta r - \sum_{k=1}^{N} \varepsilon^k r_k, \quad \Delta t_{N+1} := \Delta t - \sum_{k=1}^{N} \varepsilon^k t_k
\]
satisfy the relations
\[
\mathcal{D} \begin{pmatrix}
\Delta l_{N+1} \\
\Delta r_{N+1} \\
\Delta t_{N+1}
\end{pmatrix} = O(\varepsilon^{N+1}) + O(\varepsilon \|z_{N+1}\|) + O(\|z_{N+1}\|^2), 
\]
\[
z_{N+1} := \begin{pmatrix}
\Delta l_{N+1} \\
\Delta r_{N+1} \\
\Delta t_{N+1}
\end{pmatrix}. \quad (57)
\]
By the continuous invertibility of the operator $\mathcal{D}$, by relations (57) we obtain:
\[
z_{N+1} = O(\varepsilon^{N+1}) + O(\varepsilon \|z_{N+1}\|) + O(\|z_{N+1}\|^2). \quad (58)
\]
As it was shown in [10, Stat. 2], it follows from (58) that $z_{N+1} = O(\varepsilon^{N+1})$. Thus, we have proved the following theorem.

**Theorem 5.** Let Assumptions 2 and 3 be satisfied as well as conditions (41) and (42). Then
\[
\sum_{\{t_1, t_2, \ldots, t_p\} \subset (0, T)} \sum_{\{\epsilon_1, \epsilon_2, \ldots, \epsilon_p\}} C_\epsilon^\ast(t_0) \frac{d}{dt} C_\epsilon^\ast(t_0) t_0 \neq 0,
\]
for all $t \in [0, T] \setminus \{t_i\} \cup \{0, T\}$ and condition (42) holds true.

In this case an analogue of Theorem 4 reads as follows.

**Theorem 6.** Let Assumptions (36), (42) and (59) hold true. Then there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ there exist the points $\{t_{1,\epsilon}, t_{2,\epsilon}, \ldots, t_{p,\epsilon}\} \subset (0, T)$ of the change of the type of optimal control problem (7). There are no other points of the change of the type of optimal control and $t_{i,\epsilon} \to t_i$ as $\epsilon \to 0$ for each $i = 1, \ldots, p$.

The proof of this theorem is similar to that of Theorem 4.

We note that in this case the system of equations similar to system (47) contains a set of $p$ equations $0 = G_p$ instead of one scalar equation $0 = G$; these equations correspond to the points $t_{i,\epsilon}$ and the unknowns are $\Delta l$, $\Delta r$ and $\Delta t_i$, $i = 1, \ldots, p$.

Similar to Theorem 5 we can prove the following final theorem.

**Theorem 7.** Let Assumptions 2 and 3 be satisfied as well as conditions (36), (42) and (59). Then the vectors $l_\epsilon$, $r_\epsilon$ and the moments of time $\{t_{1,\epsilon}, t_{2,\epsilon}, \ldots, t_{p,\epsilon}\}$ are expanded into power asymptotic series
\[
\begin{align*}
l_\epsilon & \approx l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k, \\
r_\epsilon & \approx (A_{12}^* - 2B_0^*)l_0 + \sum_{k=1}^{\infty} \varepsilon^k r_k, \\
t_{i,\epsilon} & \approx t_i + \sum_{k=1}^{\infty} \varepsilon^k t_{i,k}, 
\end{align*}
\]
whose coefficients can be found in a recurrent way.
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