1. Introduction

In [10] and [11], it is shown that on a closed oriented surface $S$ of genus $g > 1$, there is a one-to-one correspondence between convex $\mathbb{RP}^2$ structures on $S$ and pairs $(\Sigma, U)$, where $\Sigma$ is a conformal structure on $S$ and $U$ is a holomorphic cubic differential. In this note, we compute the asymptotic values of the holonomy of the $\mathbb{RP}^2$ structure corresponding to $(\Sigma, \lambda U_0)$ as $\lambda \to \infty$ around geodesic loops of the flat metric $|U_0|^2$ which do not touch any zeros of the fixed cubic differential $U_0$. Such asymptotic holonomies are related to the compactification of the deformation space of convex $\mathbb{RP}^2$ structures on $S$ due to Inkang Kim [9] (see Section 2 below).

Theorem 1. Let $\Sigma$ be a closed Riemann surface of genus $g > 1$ and let $U_0$ be a holomorphic cubic differential on $\Sigma$. Consider a closed oriented geodesic $\mathcal{L}$ of the flat metric $|U_0|^\frac{2}{3}$ on $\Sigma$ which does not touch any of the zeros of $U_0$. In terms of the flat coordinate $z$ in which $U_0 = 2 dz^3$, represent the deck transformation corresponding to $\mathcal{L}$ as a displacement $z \mapsto z + Le^{i\theta}$ for $L > 0$. Then there is a constant $\kappa > 0$ so that the eigenvalues $\xi_1 > \xi_2 > \xi_3$ of the $\text{SL}(3, \mathbb{R})$ holonomy along $\mathcal{L}$ for the $\mathbb{RP}^2$ structure determined by the pair $(\Sigma, \lambda U_0)$ for $\lambda > 0$ satisfy

$$\kappa \xi_i > e^{\lambda \frac{2}{3}} \mu_1 L > \kappa^{-1} \xi_i$$

for $\mu_1 \geq \mu_2 \geq \mu_3$ the roots of the equation

$$\mu^3 - 3\mu - 2 \cos 3\theta = 0.$$

The techniques involved in the proof are similar to the analysis of the harmonic map equation between hyperbolic surfaces, as discussed by Mike Wolf [18] and Z.C. Han [6], and some new results on asymptotics of linear systems of ODEs.

The present paper may be thought of as something of sequel to [12], which studies the behavior of $\mathbb{RP}^2$ surfaces corresponding to $(\Sigma, U)$ as $\Sigma$ approaches the boundary of the Deligne-Mumford compactification of
the moduli space of Riemann surfaces, and $U$ degenerates to a regular cubic differential.

In future work, we hope to extend this analysis to all geodesics with respect to the singular flat metric $|U_0|^2$, including those which are singular at the zeros of $U_0$. This will allow a full description of the data Kim prescribes for the boundary of the deformation space of convex $\mathbb{RP}^2$ structures. It will also be interesting to relate the present work to harmonic maps to $\mathbb{R}$-buildings, as an extension of Wolf’s work on harmonic maps to $\mathbb{R}$-trees [19].

I would like to thank Mike Wolf, for, some years ago, pointing out the similarities between the analytic theories of convex $\mathbb{RP}^2$ structures and harmonic maps between hyperbolic surfaces. I also thank Bill Goldman for his encouragement and many fruitful discussions about $\mathbb{RP}^2$ structures, and Lee Mosher for useful discussions. The author is partially supported by NSF Grant DMS0405873.

2. THE BOUNDARY OF THE DEFORMATION SPACE OF CONVEX $\mathbb{RP}^2$ STRUCTURES

It is well known that a closed hyperbolic surface is determined by its length spectrum, which consists of the hyperbolic lengths of the unique geodesic in each free homotopy class of curves. More concretely, hyperbolic lengths of geodesic provide an embedding of Teichmüller space into $\mathbb{R}^C$, where $C$ is the set of all nontrivial conjugacy classes in $\pi_1(S)$ for a closed surface $S$ of genus $g > 1$. Then Thurston’s boundary of Teichmüller space can be recovered as the set of limit points of sequences in Teichmüller space $\subset \mathbb{R}^C$, when projected to the projective space $\mathbb{PR}^C$ [13, 11, 15].

There is an analog of this theory to convex $\mathbb{RP}^2$ surfaces due to Paulin [16], Parreau [14] and Inkang Kim [8, 9] (these authors address more general structures as well). Recall a (properly) convex $\mathbb{RP}^2$ surface $S$ is given by $S = \Omega / \Gamma$, where $\Omega \Subset \mathbb{R}^2 \subset \mathbb{RP}^2$ is a convex set and $\Gamma \subset \text{PGL}(3, \mathbb{R})$. Then each element $\gamma \in \Gamma$ may be represented as a matrix in $\text{SL}(3, \mathbb{R})$. The eigenvalues of this matrix are then analogs of the hyperbolic length (see in particular Goldman [5] for a detailed analog of the Fenchel-Nielsen theory of Teichmüller space for the case of convex $\mathbb{RP}^2$ structures). In particular, for a given $\gamma \in \Gamma$ with eigenvalues $\nu_1 > \nu_2 > \nu_3 > 0$ (the eigenvalues have this structure by [7]), the set of logarithms

$$(\ell_1, \ell_2, \ell_3) = (\log \nu_1, \log \nu_2, \log \nu_3)$$
is naturally an element of the maximal torus $t$ of the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. Kim [8] shows that a normalized version of map of logarithms of eigenvalues (into $t$) determines the $\mathbb{RP}^2$ structure. (The normalization is an analog of projectivization of $\mathbb{R}^2$ mentioned above for Teichmüller space.) Then in [16, 14, 9], the boundary of the deformation space of convex $\mathbb{RP}^2$ structures may be defined to be the boundary in $t^C$ of the image of the deformation space of all convex $\mathbb{RP}^2$ structures.

The limiting spectra in the case of both hyperbolic lengths and $\mathbb{RP}^2$ structures can be seen as naturally arising in the context of $\pi_1(S)$ actions on $\mathbb{R}$-buildings. In particular, given a background conformal structure $\Sigma_0$ on $S$, Teichmüller space can be parametrized by the unique harmonic map from $\Sigma_0$ to the target hyperbolic structure [18, Wolf]. In turn, these harmonic maps are uniquely determined by a holomorphic quadratic differential $\Psi$ on $\Sigma_0$. The key equation to solve to construct the harmonic map is

$$\Delta v + 4e^{-v}\|\Psi\|^2 - 2e^v + 2 = 0,$$

where $\Delta$ and $\| \cdot \|$ are determined by the hyperbolic metric on $\Sigma_0$. Wolf then essentially studies solutions to this equation to reproduce Thurston’s compactification of Teichmüller space as limits of hyperbolic structures for quadratic differentials $\lambda \Psi_0$ as $\lambda \to \infty$ [18], and later uses the same estimates to produce a $\pi_1(S)$ equivariant harmonic map to an appropriate $\mathbb{R}$-tree [19].

The equation we use (due to C.P. Wang [17]) to produce $\mathbb{RP}^2$ structures,

$$\Delta u + 4e^{-2u}\|U\|^2 - 2e^u - 2\kappa = 0,$$

is very similar to Wolf’s equation, with a cubic differential $U$ replacing the quadratic differential $\Psi$. There should also be an analogous theory: The limiting structures for $U = \lambda U_0$ for $\lambda \to \infty$ should be realized as an action of $\pi_1(S)$ on a $\mathbb{R}$-building, together with a $\pi_1(S)$ equivariant map. We hope to address this problem in future work.

3. Hyperbolic affine spheres and convex $\mathbb{RP}^n$ structures

Recall the standard definition of $\mathbb{RP}^n$ as the set of lines through 0 in $\mathbb{R}^{n+1}$. Consider $\pi: \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ with fiber $\mathbb{R}^*$. For a convex domain $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$ as above, then $\pi^{-1}(\Omega)$ has two connected components. Call one such component $C(\Omega)$, the cone over $\Omega$. Then any representation of a group $\Gamma$ into $\text{PGL}(n+1, \mathbb{R})$ so that $\Gamma$ acts discretely and properly discontinuously on $\Omega$ lifts to a representation
into
\[ \text{SL}^\pm(n + 1, \mathbb{R}) = \{ A \in \text{GL}(n + 1, \mathbb{R}) : \det A = \pm 1 \} \]
which acts on \( \mathcal{C}(\Omega) \). See e.g. [11].

For a properly convex \( \Omega \), there is a unique hypersurface asymptotic to the boundary of the cone \( \mathcal{C}(\Omega) \) called the hyperbolic affine sphere \([2, 3, 4]\). This hyperbolic affine sphere \( H \subset \mathcal{C}(\Omega) \) is invariant under automorphisms of \( \mathcal{C}(\Omega) \) in \( \text{SL}^\pm(n + 1, \mathbb{R}) \). The projection map \( P \) induces a diffeomorphism of \( H \) onto \( \Omega \). Affine differential geometry provides \( \text{SL}^\pm(n + 1, \mathbb{R}) \)-invariant structure on \( H \) which then descends to \( M = \Omega/\Gamma \). In particular, both the affine metric, which is a Riemannian metric conformal to the (Euclidean) second fundamental form of \( H \), and a projectively flat connection whose geodesics are the \( \mathbb{RP}^n \) geodesics on \( M \), descend to \( M \). See [11] for details. A fundamental fact about hyperbolic affine spheres is due to Cheng-Yau [4] and Calabi-Nirenberg (unpublished):

**Theorem 2.** If the affine metric on a hyperbolic affine sphere \( H \) is complete, then \( H \) is properly embedded in \( \mathbb{R}^{n+1} \) and is asymptotic to a convex cone \( C \subset \mathbb{R}^{n+1} \) which contains no line. By a volume-preserving affine change of coordinates in \( \mathbb{R}^{n+1} \), we may assume \( C = \mathcal{C}(\Omega) \) for some properly convex domain \( \Omega \) in \( \mathbb{RP}^n \).

Note that if \( S = \Omega/\Gamma \) is compact, then Cheng-Yau’s completeness condition on the affine metric is satisfied by any appropriate affine metric on \( S \).

4. **Wang’s developing map**

In dimension 2, there is a local theory due to C.P. Wang [17] which exploits the elliptic PDE nature of the problem of finding hyperbolic affine spheres to relate oriented convex \( \mathbb{RP}^2 \) surfaces to holomorphic data on Riemann surfaces. See also Labourie [10] and Loftin [12]. In particular, the affine metric of a 2-dimensional hyperbolic affine sphere induces a conformal structure on the surface, and, moreover, there is a holomorphic cubic differential \( U \) (which is essentially the difference between the Levi-Civita connection of the affine metric and the projectively flat connection of the \( \mathbb{RP}^2 \) structure) induced by the affine sphere. All this structure descends to projective quotients of the hyperbolic affine sphere. In particular, we have the following

**Theorem 3.** Given an oriented surface \( S \), the structure of a convex \( \mathbb{RP}^2 \) structure on \( S \) is equivalent to the pair of a conformal structure \( \Sigma \) and a holomorphic cubic differential \( U \) on \( S \).
Locally, the structure equations of a 2-dimensional hyperbolic affine sphere may be expressed in terms of an embedding map $f : \Omega \to \mathbb{R}^3$, where $\Omega \subset \mathbb{C}$ is a simply-connected domain. $f$ is taken to be a conformal map with respect to the affine metric $e^\psi|dz|^2$ and $U$ is a holomorphic function. Then $f$ satisfies

$$
\begin{align*}
\frac{f_{zz}}{f_z} &= \psi_z f_z + U e^{-\psi} f_z \\
\frac{f_{\bar{z}z}}{f_{\bar{z}}} &= \bar{U} e^{-\psi} f_z + \bar{\psi}_z f_{\bar{z}} \\
\frac{f_{\bar{z}z}}{f_{\bar{z}}} &= \frac{1}{2} e^\psi f
\end{align*}
$$

The conformal factor $e^\psi$ must satisfy the following integrability condition,

$$
\psi_{\bar{z}z} + |U|^2 e^{-2\psi} - \frac{1}{2} e^\psi = 0,
$$

which we call Wang’s equation. On a Riemann surface $U$ transforms as a cubic differential, and (2) becomes, with respect to a conformal background metric $h$,

$$
\Delta u + 4 e^{-2u} ||U||^2 - 2 e^u - 2\kappa = 0,
$$

where $\Delta$ is the Laplacian of $h$, $||U||^2$ is the norm-squared of $U$ with respect to the metric $h$, $\kappa$ is the Gauss curvature of $h$, and the metric $e^u h = e^\psi|dz|^2$ locally for $\psi$ given by (2).

We now study solutions to (3) for $U = \lambda U_0$ as $\lambda \to \infty$.

5. LIMITS OF THE CONFORMAL METRICS

Let $U_0$ be a holomorphic cubic differential on $\Sigma$ which is not identically zero. We study the limiting behavior of solutions to Wang’s equation (3) for solutions $u_\lambda$ as $U = \lambda U_0$ for $\lambda$ a real parameter approaching $\infty$. In his work on harmonic maps between hyperbolic surfaces and Thurston’s boundary of Teichmüller space, Mike Wolf has studied a similar equation to (3) with a holomorphic quadratic differential instead of a cubic differential [18]. The proof below is similar to the one in Han [6].

**Proposition 1.** Let $\Sigma$ be a closed Riemann surface of genus $g > 1$ equipped with a background metric $h$ and a holomorphic cubic differential $U_0$ which is not identically zero. Let $\lambda > 0$ and let $u = u_\lambda$ be the solution to (3) for $U = \lambda U_0$. Let $K$ be a compact subset of $\Sigma$ which does not contain any of the zeroes of $U_0$. Then there is a constant $C = C(\Sigma, U_0, K)$ so that

$$
\frac{1}{2} \geq ||U||^2 e^{-3u} \geq \frac{1}{2} - C\lambda^{-\frac{3}{2}}.
$$
Proof. We prove this proposition by the use of barriers. The key observation is that the singular flat conformal metric \(2\frac{1}{3}|U|^\frac{2}{3}\) provides a solution to (2) away from the zeros of \(U\).

Consider a smooth background metric \(g\) by requiring \(g = 2\frac{1}{3}|U|^\frac{2}{3}\) on \(K\) and \(\|U_0\|_g^2 \leq \frac{1}{2}\) on all \(\Sigma\). (This is possible since \(\|U_0\|_g^2 = \frac{1}{2}\) on \(K\) and \(\|U_0\|_g^2 = 0\) at the zeros of \(U_0\).)

Now for \(U = \lambda U_0\), define \(s = s_\lambda\) by
\[
ge^s = 2\frac{1}{3}|U|^\frac{2}{3} = 2\frac{1}{3}\lambda^\frac{2}{3}|U_0|^\frac{2}{3}.
\]
Note that \(s = 2\frac{2}{3}\log \lambda\) on \(K\). We may also check that \(s\) solves (3) away from the zeros of \(U\), and is equal to \(-\infty\) at the zeros of \(U\). By applying the comparison principle to (3), we find that \(u \geq s\), and so \(s\) is a subsolution of (3).

Now let \(S = S_\lambda\) be equal to \(\log r\) for \(r = r_\lambda\) the positive root of
\[
p(x) = x^3 - \sigma x^2 - \lambda^2 = 0, \quad \sigma = \max_{\Sigma}(-\kappa_g),
\]
for \(\kappa_g\) the Gauss curvature of \(g\). Then \(S\) is a supersolution of (3): At a maximum point of \(u\),
\[
0 \geq \Delta_g u = 2e^u + 2\kappa - 4e^{-2u}\|U\|_g^2,
\]
\[
\geq 2e^{-2u}(e^{3u} - \sigma e^{2u} - \lambda^2).
\]
The largest value of \(u\) for which this inequality can be true occurs when \(p(e^u) = 0\).

On \(K\) then, \(\frac{2}{3}\log \lambda \leq u \leq S\), and so
\[
\frac{1}{2} \geq \|U\|_g^2 e^{-3u} \geq \frac{1}{2} \lambda^2 e^{-3S}.
\]
Now we note that \(\bar{x} = \lambda^{-\frac{2}{3}} e^S\) solves
\[
\bar{x}^3 - \sigma \lambda^{-\frac{2}{3}} \bar{x}^2 - 1 = 0,
\]
and so for large values of \(\lambda\), \(\bar{x} = 1 + O(\lambda^{-\frac{2}{3}}).\) This proves the proposition.

Corollary 2. There is another constant \(C = C(\Sigma, U_0, K)\) so that \(|\psi_z| \leq C\lambda^{-\frac{2}{3}}\) on \(K\), where \(z\) is a local coordinate so that \(U_0 = 2\,dz^3\).

Proof. Note that in the proof and below, different uniform constants may be referred to by the same letter \(C\) depending on the context.

For \(p \in K\), choose the local coordinate \(z\) so that \(z(p) = 0\) and let consider
\[
\alpha(w) = \psi(\lambda^{-\frac{2}{3}}w) - \frac{2}{3}\log \lambda - \frac{1}{3}\log 2.
\]
Then
\[
\alpha_{w\bar{w}} = \lambda^{-\frac{2}{3}}\psi_{z\bar{z}} = 2^{-\frac{2}{3}}(e^{-2\alpha} - e^\alpha).
\]
Proposition (1) implies that there is a constant $C$ so that
$$0 \leq \alpha(\lambda^{\frac{1}{3}}z) = \psi(z) - \frac{2}{3}\log\lambda - \frac{1}{3}\log 2 \leq C\lambda^{-\frac{2}{3}}$$
for all $z$ in a neighborhood of $K$.

This implies that in any disk in the $w$-plane centered at 0, there is a constant $C$ independent of $p \in K$ and $\lambda$ large so that
$$|\alpha|, |\alpha_w| \leq C\lambda^{-\frac{2}{3}}.$$
Then the $L^p$ theory implies that on a slightly smaller disk, that $\|\alpha\|_{W^{2,p}} \leq C\lambda^{-\frac{2}{3}}$. Then, for $p > 2$, Sobolev embedding implies similar bounds for the $C^1$ norm of $\alpha$:
$$|\alpha_w| \leq C\lambda^{-\frac{2}{3}}.$$ 
Now simply compute $\psi_z = \lambda^{\frac{1}{3}}\alpha_w$. □

6. ODE estimates

Now the structure equations (1) can be recast in terms of the frame $\langle f, \lambda^{-\frac{1}{3}}f_z, \lambda^{-\frac{1}{3}}f_{\bar{z}} \rangle$ to read

$$\begin{bmatrix}
  f \\
  \lambda^{-\frac{1}{3}}f_z \\
  \lambda^{-\frac{1}{3}}f_{\bar{z}}
\end{bmatrix}_z = \begin{bmatrix}
  0 & \lambda^{\frac{1}{3}} & 0 \\
  0 & \psi_z & U e^{-\psi} \\
  \frac{1}{2} \lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0
\end{bmatrix} \begin{bmatrix}
  f \\
  \lambda^{-\frac{1}{3}}f_z \\
  \lambda^{-\frac{1}{3}}f_{\bar{z}}
\end{bmatrix}, \tag{4}
$$

$$\begin{bmatrix}
  f \\
  \lambda^{-\frac{1}{3}}f_z \\
  \lambda^{-\frac{1}{3}}f_{\bar{z}}
\end{bmatrix}_{\bar{z}} = \begin{bmatrix}
  0 & 0 & \lambda^{\frac{1}{3}} \\
  \frac{1}{2} \lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0 \\
  0 & \bar{U} e^{-\psi} & \psi_z
\end{bmatrix} \begin{bmatrix}
  f \\
  \lambda^{-\frac{1}{3}}f_z \\
  \lambda^{-\frac{1}{3}}f_{\bar{z}}
\end{bmatrix}. \tag{5}
$$

Away from the zeros of $U_0$, choose a local coordinate $z$ so that $U_0 = 2 dz^3$, and $U = \lambda U_0 = 2\lambda dz^3$. Proposition (1) and Corollary (2) then show that the matrices in the structure equations above have the form

$$\begin{bmatrix}
  0 & \lambda^{\frac{1}{3}} & 0 \\
  0 & \psi_z & U e^{-\psi} \\
  \frac{1}{2} \lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0
\end{bmatrix} = \lambda^{\frac{1}{3}} \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix} + O(\lambda^{-\frac{1}{3}}), \tag{6}
$$

$$\begin{bmatrix}
  0 & 0 & \lambda^{\frac{1}{3}} \\
  \frac{1}{2} \lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0 \\
  0 & \bar{U} e^{-\psi} & \psi_z
\end{bmatrix} = \lambda^{\frac{1}{3}} \begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{bmatrix} + O(\lambda^{-\frac{1}{3}}), \tag{7}
$$

where $O(\lambda^{-\frac{1}{3}})$ is as $\lambda \to \infty$ for all points in $K$ a compact set not containing any zero of $U_0$.

We will integrate the initial value problem along a geodesic path with respect to the metric $|U_0|^\frac{4}{3}$ which avoids the zeroes of $U_0$. These paths are simply straight lines each local complex coordinate chart with coordinate $z$ satisfying $U_0 = 2 dz^3$. In the particular case of a geodesic
loop, the system of ODEs (4-5) will compute the real projective holonomy around such a loop: along such a loop, the coordinate $z$ can be analytically continued in the universal cover, and the corresponding deck transformation corresponds to $z \mapsto z + c$ for a complex constant $c$. Therefore, the frame $\langle f, \lambda^{-\frac{1}{3}} f_z, \lambda^{-\frac{1}{3}} f_{\bar{z}} \rangle$ is the frame of a rank-3 vector bundle on the quotient whose holonomy in $GL(3, \mathbb{R})$ projects to $PGL(3, \mathbb{R})$ to compute the real projective holonomy of the geodesic loop. For more details of this argument, see e.g. Proposition 2 of [12].

Any geodesic loop of $|U_0|^\frac{2}{3}$ which avoids the zeroes of $U_0$ may be described by a starting point, at which we set the local coordinate $z$ to be 0, and a finishing point, which we set to be $z = c$ in the analytically continued $z$ coordinate. The geodesic is then the straight line segment between 0 and $c$. If $c = L e^{i\theta}$ for $L > 0$, then the holonomy with respect to the frame $\mathcal{F} = \langle f, \lambda^{-\frac{1}{3}} f_z, \lambda^{-\frac{1}{3}} f_{\bar{z}} \rangle$ is $\Phi(L)$, where $\Phi$ solves the initial value problem

$$\Phi(0) = I, \quad \frac{d\Phi}{dt} = (e^{i\theta} P + e^{-i\theta} Q)\Phi.$$  

This ODE system is equivalent to

$$\frac{d\Phi}{dt} = \left[ \lambda^\frac{1}{3} \begin{pmatrix} 0 & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & 0 & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} & 0 \end{pmatrix} + O(\lambda^{-\frac{1}{3}}) \right] \Phi.$$  

As we are primarily interested in the eigenvalues of $\Phi(L)$, we replace the matrix

$$M = \begin{pmatrix} 0 & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & 0 & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} & 0 \end{pmatrix}$$

by the conjugate diagonal matrix

$$\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

for $\mu_i$ the roots of the characteristic equation

$$\det(\mu I - M) = \mu^3 - 3\mu - 2\cos3\theta = 0.$$  

We note $M$ is diagonalizable and $\mu_i \in \mathbb{R}$. Assume $\mu_1 \geq \mu_2 \geq \mu_3$. 
Then, to compute the conjugacy class of the holonomy matrix around this geodesic loop, we compute the solution to

\[ \Phi(0) = I, \]

\[ \frac{d\Phi}{dt} = \left[ \lambda^\frac{1}{3} \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \right] \Phi, \]

where there is a constant \( C \) so that each \( b_{ij} = b_{ij}(t, \lambda) \) satisfies \( |b_{ij}| \leq C\lambda^{-\frac{4}{3}} \).

**Proposition 3.** The solution \( \Phi \) to the initial value problem (8-9) has the form

\[
\begin{pmatrix}
  e^{\lambda^\frac{1}{3} \mu_1 t} + O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_1 t}) & O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_2 t}) & O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_3 t}) \\
  O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_1 t}) & e^{\lambda^\frac{1}{3} \mu_2 t} + O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_2 t}) & O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_3 t}) \\
  O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_1 t}) & O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_2 t}) & e^{\lambda^\frac{1}{3} \mu_3 t} + O(\lambda^{-\frac{1}{3}} e^{\lambda^\frac{2}{3} \mu_3 t})
\end{pmatrix},
\]

Where the \( O \) notation denotes bounds as \( \lambda \to \infty \) that are uniform for \( t \in [0, L] \).

**Proof.** Write \( \Phi = (\phi_{ij}) \), and consider the first column \( \phi_{11}, \phi_{21}, \phi_{31} \), which satisfies the linear system

\[
\begin{align*}
  \phi_{11}(0) &= 1, & \frac{d}{dt}\phi_{11} &= (\lambda^\frac{1}{3} \mu_1 + b_{11})\phi_{11} + b_{12}\phi_{21} + b_{13}\phi_{31}, \\
  \phi_{21}(0) &= 0, & \frac{d}{dt}\phi_{21} &= b_{21}\phi_{11} + (\lambda^\frac{1}{3} \mu_2 + b_{22})\phi_{21} + b_{23}\phi_{31}, \\
  \phi_{31}(0) &= 0, & \frac{d}{dt}\phi_{31} &= b_{31}\phi_{11} + b_{32}\phi_{21} + (\lambda^\frac{1}{3} \mu_3 + b_{33})\phi_{31}.
\end{align*}
\]

Each of the above differential equations is first-order linear, and so we must have

\[
\begin{align*}
  \phi_{11} &= e^{\lambda^\frac{1}{3} \mu_1 t} \int_0^t e^{-\lambda^\frac{1}{3} \mu_1 \tau} f_{b_{11}} (b_{12}\phi_{21} + b_{13}\phi_{31}) d\tau, \\
  \phi_{21} &= e^{\lambda^\frac{1}{3} \mu_2 t} \int_0^t e^{-\lambda^\frac{1}{3} \mu_2 \tau} f_{b_{22}} (b_{21}\phi_{11} + b_{23}\phi_{31}) d\tau, \\
  \phi_{31} &= e^{\lambda^\frac{1}{3} \mu_3 t} \int_0^t e^{-\lambda^\frac{1}{3} \mu_3 \tau} f_{b_{32}} (b_{31}\phi_{11} + b_{32}\phi_{21}) d\tau.
\end{align*}
\]

The previous three equations can be seen as a map \( \mathcal{M} \) from the \( \mathbb{R}^3 \)-valued function \((\phi_{11}, \phi_{21}, \phi_{31})\) to the right-hand sides.

Now let \( N \gg 1 \) be a constant independent of \( \lambda \), and consider the Banach space \( \mathcal{B}_\lambda \) of continuous \( \mathbb{R}^3 \)-valued functions with norm

\[
\|(f_1, f_2, f_3)\|_{\mathcal{B}_\lambda} = \sup_{t \in [0, L]} \sup_i |f_i(t)| e^{-\lambda^\frac{1}{3} \mu_1 t}.
\]
Let $B_\lambda(N)$ be the closed ball of radius $N$ centered at the origin in $B_\lambda$. We claim that for $\lambda$ large enough, $\mathcal{M}$ is a contraction map from $B_\lambda(N)$ to itself, and thus the solution $(\phi_{11}, \phi_{21}, \phi_{31})$ to the ODE system, which is the fixed point of $\mathcal{M}$, must lie in $B_\lambda(N)$.

Now consider $F = (f_1, f_2, f_3)$, $G = (g_1, g_2, g_3) \in B_\lambda(N)$. Then the first component of $\mathcal{M}(F) - \mathcal{M}(G)$ is given by

$$e^{\lambda \frac{2}{3} \mu_1 t} e^{\lambda \frac{2}{3} \mu_1 \tau} b_{11} \int_0^t e^{-\lambda \frac{2}{3} \mu_1 \tau} [b_{22}(f_2 - g_2) + b_{33}(f_3 - g_3)] d\tau.$$ 

Now assume $|b_{ij}| \leq R$ and recall $t \leq L$. Then a straightforward calculation shows that the first component of $\mathcal{M}(F) - \mathcal{M}(G)$ is pointwise bounded by

$$e^{\lambda \frac{2}{3} \mu_1 t} e^{2RL} \cdot 2R \cdot L \cdot \|F - G\|_{B_\lambda},$$

and so if we choose $\lambda$ large enough so that $R \sim \lambda^{-\frac{2}{3}}$ is small enough, we may assume $e^{2RL} \cdot 2R \cdot L < 1$. Essentially the same calculation shows that $\mathcal{M} : B_\lambda(N) \to B_\lambda(N)$ for large $\lambda$, since $N \gg 1$. The two other components of $\mathcal{M}$ behave the same way. All this shows $\mathcal{M}$ is a contraction map.

Since $\mathcal{M}$ is a contraction map on the complete metric space $B_\lambda(N)$, the unique solution $(\phi_{11}, \phi_{21}, \phi_{31})$ to the ODE system is the fixed point, and so must be in $B_\lambda(N)$ for all $\lambda$ sufficiently large. Now simply apply the bounds

$$|\phi_{11}|, |\phi_{21}|, |\phi_{31}| \leq Ne^{\lambda \frac{2}{3} \mu_1 t}$$

to the fixed point equation $(\phi_{11}, \phi_{21}, \phi_{31}) = \mathcal{M}(\phi_{11}, \phi_{21}, \phi_{31})$ to show that

$$\phi_{11} = e^{\lambda \frac{2}{3} \mu_1 t} + O(\lambda^{-\frac{1}{3}} e^{\lambda \frac{2}{3} \mu_1 t}), \quad \phi_{21} = O(\lambda^{-\frac{1}{3}} e^{\lambda \frac{2}{3} \mu_1 t}), \quad \phi_{31} = O(\lambda^{-\frac{1}{3}} e^{\lambda \frac{2}{3} \mu_1 t}).$$

This justifies the first column in the matrix in Proposition 3. The argument for the other two columns is identical. □

**Theorem 4.** There is a constant $\kappa > 0$ so that the eigenvalues $\xi_1 \geq \xi_2 \geq \xi_3 > 0$ of the holonomy matrix $\Phi(L)$ satisfy

$$\kappa \xi_i > e^{\lambda \frac{2}{3} \mu_1 L} > \kappa^{-1} \xi_i$$

for $i = 1, 2, 3$.

**Proof.** Proposition 3 and the fact that $\Phi(L) \in \text{SL}(3, \mathbb{R})$ show that the characteristic polynomial of $\Phi(L)$ is

$$x^3 - (e^{\lambda \frac{2}{3} \mu_1 L} + e^{\lambda \frac{2}{3} \mu_2 L} + e^{\lambda \frac{2}{3} \mu_3 L})[1 + O(\lambda^{-\frac{1}{3}})] x^2 + (e^{\lambda \frac{2}{3} (\mu_1 + \mu_2) L} + e^{\lambda \frac{2}{3} (\mu_1 + \mu_3) L} + e^{\lambda \frac{2}{3} (\mu_2 + \mu_3) L})[1 + O(\lambda^{-\frac{1}{3}})] x - 1.$$
Kac-Vinberg showed [7] that the holonomy of any nontrivial loop in a closed oriented convex $\mathbb{RP}^2$ surface of genus $g > 1$ has positive distinct eigenvalues $\xi_1 > \xi_2 > \xi_3 > 0$. Then

$$\xi_1 + \xi_2 + \xi_3 = (e^{\lambda \frac{2}{3} \mu_1 L} + e^{\lambda \frac{2}{3} \mu_2 L} + e^{\lambda \frac{2}{3} \mu_3 L})[1 + O(\lambda^{-\frac{1}{3}})]$$

implies that there is an $\epsilon$ which goes to 0 as $\lambda \to \infty$ so that

$$(3 + \epsilon)e^{\lambda \frac{2}{3} \mu_1 L} > \xi_1 > \frac{1}{3}e^{\lambda \frac{2}{3} \mu_1 L}.$$ 

Now use the bounds on $\xi_1$ and

$$\xi_1 \xi_2 + \xi_1 \xi_3 + \xi_2 \xi_3 = (e^{\lambda \frac{2}{3} (\mu_1 + \mu_2) L} + e^{\lambda \frac{2}{3} (\mu_1 + \mu_3) L} + e^{\lambda \frac{2}{3} (\mu_2 + \mu_3) L})[1 + O(\lambda^{-\frac{1}{3}})]$$

to conclude that there is an $\epsilon' \to 0$ as $\lambda \to \infty$ so that

$$(9 + \epsilon')e^{\lambda \frac{2}{3} \mu_2 L} > \xi_2 > \frac{1}{9} - \epsilon'e^{\lambda \frac{2}{3} \mu_2 L}.$$ 

Then the theorem follows from $\mu_1 + \mu_2 + \mu_3 = 0$ and

$$\xi_1 \xi_2 \xi_3 = 1.$$

Since $\Phi(L)$ is conjugate to the holonomy matrix with respect to the frame $\langle f, \lambda^{-\frac{1}{3}} f_z, \lambda^{-\frac{1}{3}} f_{\bar{z}} \rangle$ around the loop $L$, this concludes the proof of Theorem [1]

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