A Fermionic Hodge Star Operator

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ABSTRACT

A fermionic analogue of the Hodge star operation is shown to have an explicit operator representation in models with fermions, in spacetimes of any dimension. This operator realizes a conjugation (pairing) not used explicitly in field-theory, and induces a metric in the space of wave-function(al)s just as in exterior calculus. If made real (Hermitian), this induced metric turns out to be identical to the standard one constructed using Hermitian conjugation; the utility of the induced complex bilinear form remains unclear.

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1. Introduction, Results and Synopsis

Among models which contain fermionic degrees of freedom, we consider here the simplest general type with \( n \) spin-\( \frac{1}{2} \) variables: \( \psi^+ \). The canonically conjugate momentum of \( \psi^+ \) is again a spinor which we here denote by \( \psi^- \), and the pair satisfies the (equal time) canonical anticommutation relations \(^1\):

\[
\left\{ \psi^-, \psi^+_i \right\} = \delta^{ij}, \quad \left\{ \psi^-, \psi^-_j \right\} = 0 = \left\{ \psi^+_i, \psi^+_j \right\}.
\]

The bosonic degrees of freedom, \( \phi^a \), may be regarded as maps from the (space)time (=domain) manifold into the field space (=target) manifold. In particle physics 4-dimensional models, for example, the domain manifold is the 3+1-dimensional Minkowski spacetime and the target manifold simply \( \mathbb{C}^n \); in superstring models, the domain manifold is a Riemann surface and the target manifold is the (10-dimensional) spacetime. Supersymmetry exchanges bosonic and fermionic degrees of freedom, pairing \( \phi \leftrightarrow \psi^+ \), and has a number of very important consequences. Our main result herein however will not depend on such symmetry or even weather the number of \( \phi \)'s equals the number of \( \psi^+ \)'s.

In coordinate representation, the wave-function(al)s of course must depend on the canonical coordinates, i.e., on the \( \phi \)'s and the \( \psi^+ \)'s. The latter being nilpotent, \( (\psi^+)^2 = 0 \), any wave-function(al) can be expanded into a finite multinomial series in the \( \psi^+ \)'s, where the coefficients are function(al)s of the \( \phi \)'s. Without loss of generality then, we may consider wave-function(al)s of a fixed fermion number, i.e., fixed degree multinomials in the \( \psi^+ \)'s.

Geometrically, the bosonic variables provide local coordinates of (maps into) a general manifold \( X \). Owing to \((1.1)\), however, the fermionic variables are nilpotent \(^2\), \( (\psi^\pm)^2 = 0 \), and so span a vector space \( V \). The wave-function(al)s are then simply functions over \((X,V)\), and owing to the nilpotency of the \( \psi^+ \)'s, elements of the sheaf \( \mathcal{O}_X(\wedge^*V) \). Supersymmetry then enforces \( V = T_{\phi}(X) \); again, this is unimportant for our main result.

Explicitly, we will be concerned with super wave-function(al)s of the form

\[
|\omega; m\rangle = \sum_{i_1 \cdots i_m} \frac{1}{m!} \omega_{i_1 \cdots i_m}(\phi) \psi^{i_1} \cdots \psi^{i_m} |0\rangle,
\]

where the vacuum \(|0\rangle\) has been chosen so that

\[
\psi^+ |0\rangle \equiv 0.
\]

This explains the \( \pm \) subscript: the \( \psi^+ \)'s act as raising operators, while the \( \psi^- \)'s are the annihilation operators \(^3\).

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\(^1\) Extensions to field theory (where the canonical variables depend on spatial coordinates in addition to time) may be implemented by inserting appropriate integrations over the space-like coordinates with every summation over the variables’ indices. Alternatively, one may extend the formulae here to span over the infinitely many fermionic creation and annihilation (i.e., Fourier expansion mode) operators, relating then to the formalism of semi-infinite forms \([1]\).

\(^2\) ...except in characteristic-2; in any case, however, we assume characteristic-0 throughout.

\(^3\) We caution the Reader that our present use of subscripts \( \pm \) are unrelated, in principle, to helicity or spacetime motion—as in, e.g., right- or left-mover in 2-dimensional theories.
Owing to the formal similarity
\[ \psi_i^+ \sim dz^i, \quad \psi_i^- \sim d\lambda_i \overset{\text{def}}{=} dz^i \frac{\partial}{\partial z^i} \quad (\text{no sum on } i), \]
the super wave-function(al) \(|\omega; m\rangle\) is analogous to an \(m\)-form:
\[ \omega_{(m)} \overset{\text{def}}{=} \omega_{i_1 \cdots i_m} dz^{i_1} \wedge \cdots \wedge dz^{i_m}. \]

This formal analogy has been successfully used in the by now standard use of exterior forms and cohomology in field theory \[\text{[2,3,4]}\]. This makes it possible to reinterpret and apply field theory results in differential (and algebraic) geometry, and also the other way around.

Somewhat surprisingly, not all fermionic analogues of the general results from exterior calculus seem to have been identified so far. In particular, on any \(n\)-dimensional Riemannian manifold \(X\), there exists a Hodge star operation, \(*\), which satisfies the following axioms:

\[ *(A^m) \rightarrow A^{n-m}, \]
\[ **(\omega_{(m)}) = (-1)^{m(n-m)} \omega_{(m)}, \]
\[ *(c_1 \alpha + c_2 \beta) = c_1(*)\alpha + c_2(*)\beta, \]
\[ \alpha \wedge *\beta = \beta \wedge *\alpha, \]
\[ \alpha \wedge *\alpha = 0 \Rightarrow \alpha \equiv 0. \]

Here, \(A^m\) is the space of \(m\)-forms, \(\omega_{(m)}\) as defined in Eq. (1.5); \(c_1, c_2\) are arbitrary real constants and \(\alpha, \beta\) are any two exterior forms of the same degree. The linearity axiom (1.6a) holds for linear combinations of forms of different degrees. The symmetry axiom (1.6c), however, cannot possibly hold for forms of different degrees since the degree of the r.h.s. is \(\deg(\alpha) + n - \deg(\beta)\), whereas the degree of the l.h.s. is \(\deg(\beta) + n - \deg(\alpha)\).

Finally, note that the symmetry and the non-degeneracy axioms ensure the existence of an induced metric over the space of forms:
\[ g(\alpha, \beta) \overset{\text{def}}{=} \int_X \alpha \wedge *\beta. \]

The purpose of this note is to analyze the fermionic equivalent of the \(*\)-operation on \(\wedge^* V\) (on \(\wedge^* T_X\) when supersymmetry is present) in physics models with fermions. This turns out to be related but not equal to some known operations, such as Hermitian conjugation. In addition and unlike in exterior calculus, this fermionic \(*\)-like operation turns out to have an explicit operator representation.

Without much ado then, we proceed with a definition of this explicit \(*\)-like operator and prove that it complies with the axioms (1.6). Once this is done for the most rudimentary case of a model with a finite number of (real) fermionic degrees of freedom, we consider generalizations as appropriate for field theories and/or complex fermions.
2. The Super-Hodge Star and its Action

Begin with real theories, \textit{i.e.,} restrict to the case where all the variables and all considered functions thereof are real. We then define

\[
\ast \equiv \sum_{p=0}^{n} f_{p} \epsilon_{i_{1} \ldots i_{n}} \psi_{+}^{i_{1}} \ldots \psi_{+}^{i_{p}} \psi_{-}^{i_{p+1}} \ldots \psi_{-}^{i_{n}} ,
\]

(2.1)

where the coefficient \( f_{p} \) will be determined by requiring \( \ast \) to satisfy the axioms (1.6). No such explicit expression is known in exterior calculus.

We will need the following iterative consequences of Eqs. (1.1), (1.3) and the antisymmetry of \( \omega_{i_{1} \ldots i_{m}} \):

\[
\psi_{-}^{j_{1}} \omega_{j_{1} \ldots j_{m}} \psi_{+}^{j_{1}} \ldots \psi_{+}^{j_{m}} |0\rangle = \omega_{j_{1} \ldots j_{m}} (\delta^{ij_{1}} - \psi_{+}^{i} \psi_{+}^{j_{1}}) \psi_{+}^{j_{2}} \ldots \psi_{+}^{j_{m}} |0\rangle ,
\]

(2.2)

\[
= m \delta^{ij_{1}} \omega_{j_{1} \ldots j_{m}} \psi_{+}^{j_{2}} \ldots \psi_{+}^{j_{m}} |0\rangle .
\]

The result follows on permuting \( \psi_{-}^{i} \) all the way to the right, using the antisymmetry of \( \omega_{j_{1} \ldots j_{m}} \) to combine the \( m \) terms with \( m-1 \) \( \psi_{+}^{i} \)‘s; the final term, with \( \psi_{-}^{i} \) all the way to the right, vanishes on account of Eq. (1.3). Repeating this \( (n-p) \) times and labeling the \( \psi_{+}^{i} \)’s with some forethought, we obtain

\[
\psi_{-}^{i_{p+1}} \ldots \psi_{-}^{i_{n}} |\omega; m\rangle = \frac{1}{[m-(n-p)]!} \delta^{i_{p+1}} \ldots \delta^{i_{n-p+1}} \omega_{j_{1} \ldots j_{n}} \psi_{+}^{j_{n-p+1}} \ldots \psi_{+}^{j_{n}} |0\rangle ,
\]

(2.3)

where the right-hand side vanishes for \( m<(n-p) \) since then \( \frac{1}{[m-(n-p)]!} \equiv 0 \).

2.1. The degree axiom, Eq. (1.6a)

We then calculate:

\[
\ast |\omega; m\rangle = \sum_{p=0}^{n} f_{p} \epsilon_{i_{1} \ldots i_{n}} \psi_{+}^{i_{1}} \ldots \psi_{+}^{i_{p}} \psi_{-}^{i_{p+1}} \ldots \psi_{-}^{i_{n}} |\omega; m\rangle ,
\]

(2.4a)

\[
\left[ 2.3 \right] \sum_{p=0}^{n} \frac{f_{p}}{[m-(n-p)]!} \delta^{i_{n}j_{1}} \ldots \delta^{i_{n-p+1}j_{n-p}} \omega_{j_{1} \ldots j_{m}} \epsilon_{i_{1} \ldots i_{n}} \psi_{+}^{i_{1}} \ldots \psi_{+}^{i_{p}} \psi_{-}^{i_{p+1}} \ldots \psi_{-}^{i_{n}} |0\rangle .
\]

(2.4b)

Next, we use the identities

\[
\psi_{+}^{i_{1}} \ldots \psi_{+}^{i_{q}} \equiv \frac{1}{q!} \delta_{i_{1} \ldots i_{q}} \psi_{+}^{j_{1}} \ldots \psi_{+}^{j_{q}} ,
\]

(2.5a)

\[
\delta_{i_{1} \ldots i_{q}}^{j_{1} \ldots j_{q}} \equiv \frac{1}{(n-q)!} \epsilon_{i_{1} \ldots i_{q}j_{1} \ldots j_{q}} \epsilon_{j_{1} \ldots j_{n}} ,
\]

(2.5b)
to obtain (with \( q = 2p + m - n \))

\[
\star |\omega; m\rangle = \sum_{p=0}^{n} \frac{f_p}{[m-(n-p)]!} \delta^{i_{n-1}j_{n-1}} \delta^{i_{p+1}j_{n-p}} \omega_{j_1 \ldots j_m} \epsilon_{i_1 \ldots i_n} \\
\times \frac{1}{q!(n-q)!} e^{i_1 \ldots i_p j_{n-p+1} \ldots j_m k_{q+1} \ldots k_n} e^{\ell_1 \ldots \ell_q k_{q+1} \ldots k_n} \psi^{f_1} \ldots \psi^{f_q} |0\rangle ,
\]

(2.4c)

\[
= \sum_{p=0}^{n} \frac{f_p p!}{[m-(n-p)]!q!(n-q)!} \delta^{i_{n-1}j_{n-1}} \delta^{i_{p+1}j_{n-p}} \omega_{j_1 \ldots j_m} \\
\times \delta^{j_{n-p+1} \ldots j_m k_{q+1} \ldots k_n} \epsilon_{k_1 \ldots k_n} \psi^{k_1} \ldots \psi^{k_q} |0\rangle .
\]

(2.4d)

As \( \frac{1}{p} \equiv 0 \) for \( r < 0 \), the summands above are nonzero only for \( (m+p-n), q, (n-q) \geq 0 \). These conditions, respectively, imply

\[
p \geq n-m , \quad p \geq \frac{1}{2}(n-m) , \quad p \leq \frac{1}{2}(2n-m) .
\]

(2.6)

Since \( n \geq m \), the second inequality may be omitted, being implied by the first. The marginal case \( p = n-m \) being special, we isolate this term and find:

\[
\star |\omega; m\rangle = f_{n-m} \epsilon_{k_1 \ldots k_n} \delta^{k_{n-1}j_{n-1}} \delta^{k_{n-m+1}j_m} \frac{1}{m!} \omega_{j_1 \ldots j_m} \psi^{k_1} \ldots \psi^{k_{n-m}} |0\rangle \\
+ \sum_{p=n-m+1}^{[n-\frac{1}{2}m]} \frac{f_p p!}{[m-(n-p)]!q!(n-q)!} \\
\times \delta^{j_{n-p+1} \ldots j_m k_{q+1} \ldots k_n} \epsilon_{k_1 \ldots k_n} \psi^{k_1} \ldots \psi^{k_q} |0\rangle ,
\]

(2.7)

where \([x]\) indicates the integer part of \( x \). The indicated contractions vanish whenever \( p > n-m \), since then at least one \( \delta^{ij} \) is completely contracted with \( \omega_{k_1 \ldots k_m} \). That is, the indicated term is proportional to \( \delta^{j_1j_2} \omega_{j_1j_2 \ldots j_m} \), and this vanishes identically since \( \delta^{ij} \) is symmetric while \( \omega_{j_1 \ldots j_m} \) is totally antisymmetric.

Thus, we have obtained that

\[
\star |\omega; m\rangle = \frac{1}{(n-m)!} \omega^*_{i_1 \ldots i_{n-m}} \psi^{i_1} \ldots \psi^{i_{n-m}} |0\rangle ,
\]

(2.8)

\[
\omega^*_{i_1 \ldots i_{n-m}} = \frac{(n-m)!}{m!} f_{n-m} \epsilon_{i_1 \ldots i_n} \delta^{i_n j_1} \ldots \delta^{i_{n-m+1} j_m} \omega_{j_1 \ldots j_m} ,
\]

proving that the action of the operator \( \star \) on the wave-function(al)s \( |\omega; m\rangle \), defined as in Eq. (1.2), satisfies axiom (1.6a).

2.2. The involution axiom, Eq. (1.6b)

To fix the value of the coefficients \( f_m \) in Eq. (2.1), we calculate

\[
\star \star |\omega; m\rangle = \star \left( \frac{1}{(n-m)!} \omega^*_{i_1 \ldots i_{n-m}} \psi^{i_1} \ldots \psi^{i_{n-m}} |0\rangle \right) ,
\]

(2.9a)
\[ = f_m \epsilon_{i_1 \ldots i_n} \delta^{i_1 \ldots i_{m+1} j_{n-m}} \left[ \frac{1}{(n-m)!} \omega^*_j j_{n-m} \right] \psi_1^j \ldots \psi_m^j |0\rangle, \quad (2.9b) \]

\[ = \frac{f_m}{m!} \epsilon_{i_1 \ldots i_n} \delta^{i_1 \ldots i_{m+1} j_{n-m}} \epsilon_{j_1 \ldots j_n} \delta^{j_{n-k_1} \ldots j_{n+m+1} k_m} \psi_1^j \ldots \psi_m^j |0\rangle, \quad (2.9c) \]

\[ = \frac{f_m}{m!} \epsilon_{i_1 \ldots i_n} \left( -1 \right)^{\left( \frac{n}{2} \right)} \omega^{k_{1 \ldots k_m}} \psi_1^j \ldots \psi_m^j |0\rangle, \quad (2.9d) \]

\[ = \frac{f_m}{m!} \left( -1 \right)^{\left( \frac{n}{2} \right)} (n-m)! \psi_1^j \ldots \psi_m^j |0\rangle, \quad (2.9e) \]

\[ = \frac{f_m}{m!} \left( -1 \right)^{\left( \frac{n}{2} \right)} (n-m)! m! \omega_{i_1 \ldots i_m} \psi_1^j \ldots \psi_m^j |0\rangle. \quad (2.9f) \]

Noting that
\[ \binom{n-m}{2} = \binom{n}{2} - m(n-m) - \binom{m}{2}, \quad (2.10) \]
we set
\[ f_m = \frac{1}{m!} \left( -1 \right)^{\left( \frac{n}{2} \right)}, \quad (2.11) \]
and obtain that
\[ \star \star |\omega; m\rangle = \left( -1 \right)^{m(n-m)} |\omega; m\rangle, \quad (2.12) \]
i.e., that the \( \star \) operator satisfies the involution axiom \( (1.6d) \). Note also that in the complex case, \( n \to 2n \) and \( m \to (p+q) \), so that
\[ \left( -1 \right)^{m(n-m)} = \left( -1 \right)^{(p+q)(2n-p-q)} = \left( -1 \right)^{(p+q)2n} \left( -1 \right)^{(p+q)^2} = \left( -1 \right)^{p+q}, \quad (2.13) \]
in perfect agreement with the standard result \( 5 \).

Compliance with the linearity axiom, Eq. \( (1.6d) \), is obvious from the fact that the expression \( (2.8) \) for \( \omega^* \) is linear in the tensor coefficients \( \omega_{i_1 \ldots i_m} \).

### 2.3. The symmetry axiom, Eq. \( (1.6d) \)

To prove the analogue of Eq. \( (1.6d) \), we calculate:
\[ [\alpha_{i_1 \ldots i_m} \psi_1^{i_1} \ldots \psi_m^{i_m}] \left( \star [\beta_{j_1 \ldots j_n} \psi_1^{j_1} \ldots \psi_n^{j_n}] \right) \]
\[ = [\alpha_{i_1 \ldots i_m} \psi_1^{i_1} \ldots \psi_m^{i_m}] \times \left[ \frac{(-1)^{\binom{n}{2}}}{(n-m)!} \epsilon_{j_1 \ldots j_n} \delta^{j_{n+1} k_1} \ldots \delta^{j_{n+m+1} k_m} \beta_{k_1 \ldots k_m} \psi_1^{j_1} \ldots \psi_m^{j_m} \right], \quad (2.14a) \]
\[ = \frac{(-1)^{\binom{n}{2}}}{(n-m)!} \alpha_{i_1 \ldots i_m} \beta_{j_1 \ldots j_{n+1}} \epsilon_{j_1 \ldots j_n} \psi_1^{i_1} \ldots \psi_m^{i_m} \psi_1^{j_1} \ldots \psi_m^{j_n} \psi_1^{i_1} \ldots \psi_m^{i_m}, \quad (2.14b) \]
\[
= (-1)^{\binom{n}{2}} \frac{m(n-m)}{(n-m)!} \alpha_{i_1 \ldots i_m} \left[ (-1)^{\binom{n}{2}} \beta^{j_{n-m+1} \ldots j_n} \right]
\]
\[
\times \left[ (-1)^{m(n-m)} \epsilon_{j_{n-m+1} \ldots j_n i_1 \ldots j_{n-m}} \left[ \frac{1}{n!} \epsilon^{i_1 \ldots i_m j_1 \ldots j_{n-m}} \epsilon_{\ell_1 \ldots \ell_n} \psi^{f_1} \ldots \psi^{f_n} \right] \right],
\] (2.14c)
\[
= (-1)^{\binom{n}{2}} \frac{m! \beta^{i_1 \ldots i_m}}{(n-m)!} \delta_{j_{n-m+1} \ldots j_n} \psi^{j_1} \ldots \psi^{j_n},
\] (2.14d)
\[
= (-1)^{\binom{n}{2}} m! \beta^{i_1 \ldots i_m} \psi^{j_1} \ldots \psi^{j_n}.
\] (2.14e)

This final form is obviously symmetric with respect to the exchange \( \alpha \leftrightarrow \beta \), verifying compliance with the symmetry axiom (1.6d).

2.4. The non-degeneracy axiom, Eq. (1.6e)

Using the result (2.14d), and substituting \( \beta \rightarrow \alpha \), we obtain that
\[
\left[ \alpha_{i_1 \ldots i_m} \psi^{i_1} \ldots \psi^{i_m} \right] \star \left[ \alpha_{j_1 \ldots j_m} \psi^{j_1} \ldots \psi^{j_m} \right] = (-1)^{\binom{n}{2}} m! \alpha_{i_1 \ldots i_m} \alpha_{j_1 \ldots j_m} \psi^{i_1} \ldots \psi^{j_m}.
\] (2.15)

Since
\[
\alpha_{i_1 \ldots i_m} \alpha^{i_1 \ldots i_m} = \alpha_{i_1 \ldots i_m} \delta^{i_1 j_1} \ldots \delta^{i_m j_m} \alpha_{j_1 \ldots j_m}
\] (2.16)
is (up to the combinatorial coefficient \( \frac{1}{m!} \)) the standard norm-squared of a rank-\( m \) tensor, the expression (2.15) vanishes only if the tensor \( \alpha_{i_1 \ldots i_m} \) does. This proves non-degeneracy.

2.5. The metric

Using the symmetry and non-degeneracy of the product (2.14), we can define the bilinear form
\[
\eta(\alpha, \beta) \overset{\text{def}}{=} \int \mathcal{D}\phi \int d^n \psi_+ \left[ \frac{1}{m!} \alpha_{i_1 \ldots i_m} \psi^{i_1} \ldots \psi^{i_m} \right] \star \left[ \frac{1}{m!} \beta_{j_1 \ldots j_m} \psi^{j_1} \ldots \psi^{j_m} \right],
\] (2.17)
which may be chosen to be the metric on the space of wave-function(al)s, induced by the \( \star \)-operator.

Noting that the usual (Berezin) integration of fermions implies
\[
\int d^n \psi_+ \left[ \psi^1_+ \ldots \psi^n_+ \right] \overset{\text{def}}{=} \int d\psi^1_+ \ldots d\psi^n_+ [(-1)^{\binom{n}{2}} \psi^n_+ \ldots \psi^1_+] = (-1)^{\binom{n}{2}},
\] (2.18)
we obtain that
\[
\eta(\alpha, \beta) = \int \mathcal{D}\phi \left( \frac{1}{m!} \alpha_{i_1 \ldots i_m}(\phi) \beta^{i_1 \ldots i_m}(\phi) \right).
\] (2.19)

Setting \( \beta \rightarrow \alpha \), we see that \( \eta(\cdot, \cdot) \) is positive definite, since the standard norm (2.16) is—as long as all the quantities involved are real.
On the other hand, the standard field-theoretic metric on the space of wave-
function(al)s is calculated as
\[
\langle m; \alpha | m; \beta \rangle = \int D\phi \langle 0 | \left[ \frac{1}{m!} \psi^{i_1 \ldots i_m} \cdot \bar{\psi}^{\dagger}_{i_1 \ldots i_m} \right] \left[ \frac{1}{m!} \beta_{j_1 \ldots j_m} \psi^{\dagger}_{j_1 \ldots j_m} \cdot \bar{\psi}^{i_1 \ldots i_m} \right] | 0 \rangle, \tag{2.20a}
\]
\[
= \frac{1}{(m!)^2} \int D\phi \alpha_{i_1 \ldots i_m} \beta_{j_1 \ldots j_m} \langle 0 | \psi^{i_1 \ldots i_m} \cdot \bar{\psi}^{\dagger}_{i_1 \ldots i_m} | 0 \rangle, \tag{2.20b}
\]
\[
= \frac{1}{(m!)^2} \int D\phi \alpha_{i_1 \ldots i_m} \beta_{j_1 \ldots j_m} (m! \delta^{i_1 j_1} \ldots \delta^{i_m j_m}), \tag{2.20c}
\]
\[
= \int D\phi \left( \frac{1}{m!} \alpha_{i_1 \ldots i_m} (\phi) \beta^{i_1 \ldots i_m} (\phi) \right). \tag{2.20d}
\]
This proves that the ⋆-induced metric (2.17) is the same as the standard field-theoretic one (2.20), defined using Hermitian conjugation.

Note however, that the standard formula (2.20) based on Hermitian conjugation only involves (path-)integration over the bosonic degrees of freedom, whereas the formula (2.17) for the ⋆-induced metric involves an integration over both bosons and fermions. It would then appear that this ⋆-induced formula is better suited for manifestly supersymmetric (superfield) formulation of models which, besides simply having both bosonic and fermionic degrees of freedom, also have supersymmetry.

3. Generalizations and Summary

The immediate consequence of the existence of the ⋆-operation is the perfect pairing of wave-function(al)s: for every degree-\(m\) wave-function(al) \(| \omega; m \rangle\), there exists a degree-\((n-m)\) wave-function(al) \(\star | \omega; m \rangle\). This is recognized as Poincaré duality in the Hilbert space \(O_X(\bigwedge^* V)\). It is induced by the Poincaré duality of the target manifold (coordinatized by the \(\phi\)'s) whenever the fermions happen to span the tangent and cotangent space to the target manifold—as is the case in presence of supersymmetry. Even without supersymmetry, the dynamics of the fermions may be determined by the bosons through a \(\phi\)-dependence of the action functional for the fermions. In this more general case, the fermions span a bundle (or more generally a sheaf) over the target manifold coordinatized by the bosons, and the ⋆-operation corresponds to the Poincaré duality of this bundle. The existence of the ⋆-operation is however more general: all models with fermionic degrees of freedom have it, with or without supersymmetry.

The recondite Reader will have observed the lack of covariance in Eq. (1.1): either the \(\psi^+\)'s and the \(\psi^-\)'s should transform oppositely (so that one of them should have been covariant), or the Kronecker \(\delta\) should have been a metric. Indeed, following this second option, and writing
\[
\{ \psi^i, \psi^j_\pm \} = g^{ij}, \tag{1.1}'
\]
leads to a simple modification of Eqs. (2.8) with (2.11):
\[
\star | \omega; m \rangle = \frac{(-1)^{(n-m)}}{(n-m)! m!} \epsilon_{k_1 \ldots k_n} g^{k_1 j_1 \ldots j_m} g^{k_{n-m+1} j_{m+1} \ldots j_m} \omega_{j_1 \ldots j_m} \psi^{k_1 \ldots k_{n-m}}_\pm | 0 \rangle. \tag{2.8}'
\]
In particular, the vanishing of all but the above shown summand in (2.8) follows from the symmetry of \( g^{ij} \). Were one now to vary the metric, \( i.e. \), deform the original choice (1.1) by setting \( g^{ij} = \delta^{ij} + h^{ij} \), the action of the \(*\)-operator on \( |\omega, m\rangle \) is easily seen to be of order \( m \) in the deformation \( h^{ij} \).

As already noted, generalization to the complex case is straightforward. One realizes that \( n \) complex fermionic degrees of freedom of course amounts to \( 2n \) real ones. The sign of the square of \( \star \), given in Eq. (2.13) then agrees with the standard definition, as given for example in Ref. [3]. The remaining calculations require little or no correction. There appears however an interesting possibility in the complex case; instead of the \( \star \)-operator (2.4), we could now consider the holomorphic and the antiholomorphic “half-star” operators \(^4\):

\[
\star = \sum_{p=0}^{n} \pi_p \epsilon_{\mu_1 \cdots \mu_n} \psi_+^{\mu_1} \cdots \psi_+^{\mu_p} \psi_-^{\mu_{p+1}} \cdots \psi_-^{\mu_n}, \tag{3.1a}
\]

\[
\bar{\star} = \sum_{q=0}^{n} \bar{\pi}_q \epsilon_{\bar{\mu}_1 \cdots \bar{\mu}_n} \psi_-^{\bar{\mu}_1} \cdots \psi_-^{\bar{\mu}_q} \psi_+^{\bar{\mu}_{q+1}} \cdots \psi_+^{\bar{\mu}_n}, \tag{3.1b}
\]

with similarly determined (combinatorial) coefficients \( \pi_p, \theta_q \). The \( \bar{\psi} \)'s in (3.1d) have been reordered since the complex fermions satisfy the hermitian canonical anticommutation relations:

\[
\{ \psi_+^\mu, \psi_-^\nu \} = g^{\mu\bar{\nu}} = \{ \psi_-^\mu, \psi_+^\nu \}, \tag{3.2}
\]

where now the \( \pm \) subscripts denote (\( \pm \frac{1}{2} \)) helicity. By defining the \( (a, c) \) Clifford-Dirac vacuum as

\[
\psi_{\mu}^a |0, 0\rangle_{ac} = 0 = \psi_{\mu}^c |0, 0\rangle_{ac}, \tag{3.3}
\]

the annihilation operators are to the right in both of the operators (3.1). The preceeding arguments then also prove that they both comply with all of the axioms (1.4).

Now one can define two complex non-degenerate symmetric bilinear forms:

\[
h^+(\alpha, \beta) \overset{\text{def}}{=} \int D\phi \int d^n \bar{\psi}_- \int d^n \psi_+ \left[ \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} (\phi, \bar{\psi}_-) \psi_+^{\mu_1} \cdots \psi_+^{\mu_p} \right] \times \bar{\star} \left[ \frac{1}{p!} \beta_{\bar{\nu}_1 \cdots \bar{\nu}_p} (\bar{\phi}, \psi_+) \psi_-^{\bar{\nu}_1} \cdots \psi_-^{\bar{\nu}_p} \right], \tag{3.4}
\]

and its \( \bar{\psi}_- \leftrightarrow \psi_+ \) counterpart, \( h^- (\alpha, \beta) \). It is easy to see that neither of these is real, as would be required of a metric. In fact, the bilinear form defined in Eq. (2.17) is also complex if the quantities involved are no longer restricted to be real. The Hermitian bilinear form, which again turns out to be the same as the standard metric induced by Hermitian conjugation, may be defined as:

\[
g(\alpha, \beta) \overset{\text{def}}{=} \int D\phi \int d^n \bar{\psi}_- \int d^n \psi_+ \left[ \frac{1}{(p+q)!} \alpha_{\mu_1 \cdots \mu_p \bar{\nu}_1 \cdots \bar{\nu}_q} \psi_+^{\mu_1} \cdots \psi_+^{\mu_p} \bar{\psi}_-^{\bar{\nu}_1} \cdots \bar{\psi}_-^{\bar{\nu}_q} \right] \times \bar{\star} \left[ \frac{1}{(p+q)!} \beta_{\bar{\nu}_1 \cdots \bar{\nu}_p \bar{\mu}_1 \cdots \bar{\mu}_q} \psi_-^{\bar{\nu}_1} \cdots \psi_-^{\bar{\nu}_p} \psi_+^{\bar{\mu}_1} \cdots \psi_+^{\bar{\mu}_q} \right]. \tag{3.5}
\]

\(^4\) On Ricci-flat manifolds, these are local expressions for the covariantly constant holomorphic and antiholomorphic volume forms.
This now has the usual complex linearity properties:

\[ g(\lambda \alpha, \beta) = g(\alpha, \bar{\lambda} \beta) \]  

(3.6)

The complex conjugation entered by hand in Eq. (3.5) in fact combines the \( \star \)-operation with complex conjugation, the latter of which however performed only on the bosonic coefficients. This complexified \( \star \)-operation then pairs degree-\((p, q)\) wave-function(al)s with degree-\((-p, -q)\) complex conjugate wave-function(al)s.

This complexification of the \( \star \)-operator could have been extended over the fermions too, at the expense of inserting the appropriate sign \((-1)^{pq}\); this then would however pair degree-\((p, q)\) wave-function(al)s with degree-\((-q, -p)\) complex conjugate wave-function(al)s. Of course, these two complexified \( \star \)-operations simply represent the usual Poincaré duality with or without complex conjugation.

The “half-star” operators are however more interesting, in that the first one offers a pairing between degree-\((p, q)\) wave-function(al)s with degree-\((-p, -q)\) wave-function(al)s, while its complex conjugate pairs degree-\((p, q)\) wave-function(al)s with degree-\((-p, -q)\) wave-function(al)s. These turn out to represent the special holomorphic duality on Calabi-Yau manifolds (see for example §1.2 of Ref. [6]). Note also that the above deformation formula (2.3)’ simplifies when acting on the holomorphic volume form. If the deformation is chosen to be either of pure type, \(h^{ij}\) or mixed, \(h^{ij}\), the \( \star \)-operation is now proportional to the determinant of the deformation \(h\).

The mirror map exchanges \(\psi_+ \leftrightarrow \bar{\psi}_+\), whereupon we define the \((c, c)\) Clifford-Dirac vacuum as

\[ \psi_+^\mu |0, 0\rangle_{cc} = 0 = \psi_+^\mu |0, 0\rangle_{cc} . \]

(3.7)

Interestingly, it is again possible to write down two “twisted half-star” operators upon rewriting \(\psi_\pm^\mu \rightarrow \bar{\psi}_\pm^\mu\):

\[ \star_- \equiv \sum_{b=0}^n f_b^- \epsilon_{\mu_1 \cdots \mu_n} \bar{\psi}_{\mu_1} \cdots \bar{\psi}_{\mu_b} \psi_{\mu_{b+1}} \cdots \psi_{\mu_n} , \]  

(3.8a)

\[ \star_+ \equiv \sum_{q=0}^n f_q^+ \epsilon_{\mu_1 \cdots \mu_n} \bar{\psi}_{\mu_1} \cdots \bar{\psi}_{\mu_q} \psi_{\mu_{q+1}} \cdots \psi_{\mu_n} , \]  

(3.8b)

with similarly determined (combinatorial) coefficients \(f_b^-\), \(f_q^+\). The \(\bar{\psi}\)’s in (3.8a) have been ordered so that the annihilation operators are to the right in both of the operators (3.8). Again, they both comply with all of the axioms (1.4).

Now one can define two complex non-degenerate symmetric bilinear forms:

\[ \mu^+(\alpha, \beta) \equiv \int \mathcal{D}\phi \int d^n \bar{\psi}_- \int d^n \bar{\psi}_+ \left[ \frac{1}{|p!|} \alpha_{\mu_1 \cdots \mu_p} (\phi, \bar{\psi}_-) \bar{\psi}_{\mu_1} \cdots \bar{\psi}_{\mu_p} \right] \]

\[ \times \star \left[ \frac{1}{|q!|} \beta_{\nu_1 \cdots \nu_q} (\phi, \bar{\psi}_+) \bar{\psi}_{\nu_1} \cdots \bar{\psi}_{\nu_q} \right] \]  

(3.9a)

and its \(\bar{\psi}_- \leftrightarrow \bar{\psi}_+\) counterpart, \(\mu^- (\alpha, \beta)\). It is easy to see that neither of these is real, as would be required of a metric.

The utility of the bilinear form \(\eta(\alpha, \beta)\) and the mirror-pairs of complex bilinear forms \(h^\pm (\alpha, \beta)\), \(\mu^\pm (\alpha, \beta)\) remains unclear so far.
In conclusion, we have provided an explicit operator representation (2.1) of a fermionic analogue of Hodge star operation in all models with fermionic degrees of freedom. The Hodge star operation (duality) itself was expected since the fermions span the self-dual bundle $\wedge^* V$, the fixed degree components of which form a Gorenstein sequence. However, no analogue of the explicit operator representation of the Hodge star operation is known in exterior calculus.

This star operation induces a non-degenerate bilinear form which in the real case is identical to the standard metric on the space of wave-function(al)s, induced by Hermitian conjugation. In the complex case, the star operator (2.1) may be factorized into its the “half stars” (3.1), related to the holomorphic duality on Calabi-Yau manifolds, or the “twisted half-stars” (3.8), related to the “half-stars” by mirror symmetry. Furthermore, there exist several possible complexifications of the bilinear form induced by the star,- or the “(twisted) half star” operators, only one of which coincides with the standard Hermitian metric on the space of wave-function(al)s; the utility of the other ones remains unclear.
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