Ehrhart polynomials of polytopes and spectrum at infinity of Laurent polynomials

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Abstract
Gathering different results from singularity theory, geometry and combinatorics, we show that the spectrum at infinity of a tame Laurent polynomial counts (weighted) lattice points in polytopes. We deduce an effective algorithm in order to compute the Ehrhart polynomial of a simplex containing the origin as an interior point.

Keywords Toric varieties · Polytopes · Ehrhart theory · Spectrum of polytopes · Spectrum of regular functions

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1 Introduction

The spectrum at infinity of a Laurent polynomial \( f \), defined by Sabbah [13], is a sequence \( \beta_1, \ldots, \beta_\mu \) of nonnegative rational numbers which is related to various concepts in singularity theory (monodromy, Hodge and Kashiwara-Malgrange filtrations, Brieskorn lattices...) and it can be difficult to handle. Fortunately, if \( f \) is convenient and nondegenerate with respect to its Newton polytope in the sense of Kouchnirenko [10], and this is generically the case, the generating function \( \text{Spec}_f(z) := \sum_{i=1}^{\mu} z^{\beta_i} \) has a very concrete description; it is equal to the Hilbert-Poincaré series of the Jacobian ring of \( f \) graded by the Newton filtration [7]. As noticed in [5,6], it follows from Kouchnirenko’s work that

\[
\text{Spec}_f(z) = (1 - z)^n \sum_{v \in \mathbb{N}} z^{v(v)}
\]

(1)
where \( v \) is the Newton filtration of the Newton polytope \( P \) of \( f \) and \( N := \mathbb{Z}^n \). Because the right-hand side depends only on \( P \), we will also call it the Newton spectrum of \( P \) and we will denote it by \( \text{Spec}_P(z) \). With this terminology, the spectrum at infinity of a Laurent polynomial is equal to the Newton spectrum of its Newton polytope. All this is recorded in Sect. 2, where we also recall the basic definitions.

Once we have this description, it follows from the work of Stapledon [14] that the spectrum at infinity of a convenient and nondegenerate Laurent polynomial counts “weighted” lattice points in its Newton polytope \( P \) and its integer dilates. More precisely, the \( \delta \)-vector \( \delta_P(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n \) of a lattice polytope \( P \) in \( \mathbb{R}^n \) containing the origin as an interior point, hence its Ehrhart polynomial, can be obtained effortlessly from its Newton spectrum \( \text{Spec}_P(z) = \sum_{i=1}^{\mu} z^{\beta_i} \) : the coefficient \( \delta_i \) is equal to the number of \( \beta_k \)'s such that \( \beta_k \in [i-1, i] \). This is explained in Sect. 4.1. Thereby, singularity theory meets Ehrhart theory by the means of the \( \delta \)-vector; the spectrum at infinity of a convenient and nondegenerate Laurent polynomial determines the \( \delta \)-vector of its Newton polytope \( P \). And we show in Proposition 5.1 that both coincide if and only if \( P \) is reflexive.

This is particularly fruitful when \( P \) is a reduced simplex in \( \mathbb{R}^n \) in the sense of [4] (the definition of a reduced simplex and its weight is recalled in Sect. 3) because we have a very simple closed formula for its Newton spectrum, see Theorem 3.2. This formula only involves the weight of the simplex and is based on [8, Theorem 1], whose proof uses classical tools in singularity theory. It provides an effective algorithm in order to compute the \( \delta \)-vector of a reduced, not necessarily reflexive, simplex. This algorithm is explained in Sect. 4.2. For instance, let us consider the simplex

\[
\Delta := \text{conv}((1, 0, 0), (0, 2, 0), (1, 1, 1), (-3, -5, -2))
\]

in \( \mathbb{R}^3 \). By Theorem 3.2, its Newton spectrum is

\[
\text{Spec}_\Delta(z) = 1 + z + z^2 + z^3 + z^{5/4} + z^{4/3} + z^{1/2} + z^{3/2} + z^{5/3} + z^{7/4}
\]

and we get \( \delta_0 = 1, \delta_1 = 2, \delta_2 = 6, \delta_3 = 2 \). See Example 4.4.

Notice that it is also possible to get in a similar way the \( \delta \)-vectors of Newton polytopes of convenient and nondegenerate polynomials from their toric Newton spectrum, as defined in [5, Definition 3.1]. The main difference is that the Newton polytope of a polynomial does not contain the origin as an interior point and that we have to deal with a non-complete situation. This can be applied for instance to the Mordell-Pommersheim tetrahedron, that is the convex hull of \((0, 0, 0), (a, 0, 0), (0, b, 0), (0, c, 0)\) where \( a, b \) and \( c \) are positive integers, which is the Newton polytope of the Brieskorn-Pham polynomial \( f(u_1, u_2, u_3) = u_1^a + u_2^b + u_3^c \) on \( \mathbb{C}^3 \), a very familiar object in singularity theory. See Sect. 4.3.

In this context, other questions may arise: for instance, singularity theory predicts that the variance of the spectrum at infinity of a Laurent polynomial in \((\mathbb{C}^*)^n\) is bounded below by \( n/12 \) (this is a global variant of C. Hertling’s conjecture [9], see [6] and the references therein). By Proposition 5.1, this would give information about the distribution of the \( \delta \)-vector of a reflexive polytope. In the opposite direction, Ehrhart theory is useful in order to get results on the singularity side: for instance, using...
[11,12], it is readily seen that the spectrum at infinity of a tame regular function is not always unimodal. This is discussed in Sect. 5.

2 The Newton spectrum of a polytope

In this section, we define the Newton spectrum of a polytope, and we gather the results that we will use.

2.1 Polytopes (basics)

In this text, \(N\) is the lattice \(\mathbb{Z}^n\), \(M\) is the dual lattice and \(\langle , \rangle\) is the canonical pairing between \(N_\mathbb{R}\) and \(M_\mathbb{R}\). A lattice polytope is the convex hull of a finite set of \(N\). If \(P \subset N_\mathbb{R}\) is a \(n\)-dimensional lattice polytope containing the origin as an interior point there exists, for each facet \(F\) of \(P\), \(u_F \in M_\mathbb{Q}\) such that

\[
P \subset \{n \in N_\mathbb{R}, \langle u_F, n \rangle \leq 1\} \quad \text{and} \quad F = P \cap \{x \in N_\mathbb{R}, \langle u_F, x \rangle = 1\}.
\]

This provides the hyperplane presentation

\[
P = \cap_F \{x \in N_\mathbb{R}, \langle u_F, x \rangle \leq 1\}. \tag{2}
\]

The set

\[
P^\circ := \{y \in M_\mathbb{R}, \langle y, x \rangle \leq 1 \text{ for all } x \in P\} \tag{3}
\]

is the polar polytope of \(P\). It happens that \(P^\circ\) is indeed a rational polytope (the convex hull of a finite set of \(N_\mathbb{Q}\)) if \(P\) contains the origin as an interior point; the vertices of \(P^\circ\) are in one-to-one correspondence with the facets of \(P\) by

\[
\{\text{vertices of } P^\circ\} = \{u_F, F \text{ facet of } P\}. \tag{4}
\]

A lattice polytope \(P\) is reflexive if it contains the origin as an interior point and if \(P^\circ\) is a lattice polytope.

If \(P \subset N_\mathbb{R}\) is a full-dimensional lattice polytope containing the origin as an interior point, we get a complete fan \(\Sigma_P\) in \(N_\mathbb{R}\) by taking the cones over the faces of \(P\). Throughout this article, we assume that the polytope \(P\) is simplicial and we denote by \(X_{\Sigma_P}\) the complete toric variety associated with the simplicial fan \(\Sigma_P\).

The Newton function of \(P\) is the function

\[
v : N_\mathbb{R} \rightarrow \mathbb{R}
\]

which takes the value 1 at the vertices of \(P\) and which is linear on each cone of the fan \(\Sigma_P\). Alternatively, \(v(v) = \max_F \langle u_F, v \rangle\) where the maximum is taken over the facets of \(P\) and the vectors \(u_F\) are defined as in (2). The Milnor number of \(P\) is

\[
\mu_P := n! \text{vol}(P)
\]
where the volume $\text{vol}(P)$ is normalized such that the volume of the cube is equal to 1. If $P = \text{conv}(b_1, \ldots, b_r)$, the Milnor number $\mu_P$ is also the global Milnor number of the Laurent polynomial $f(u) = \sum_{i=1}^{r} c_i u^{b_i}$ for generic complex coefficients $c_i$, that is

$$\mu_P = \dim \mathbb{C}[u_1, u_1^{-1}, \ldots, u_n, u_n^{-1}] / \left( u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n} \right)$$

where $(\mathbb{C}^*)^n$ is equipped with the coordinates $u = (u_1, \ldots, u_n)$, $u^m := u_1^{m_1} \cdots u_n^{m_n}$ if $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $(u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n})$ denotes the ideal generated by the partial derivatives $u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n}$. See [10] for details.

### 2.2 The Newton spectrum of a polytope

The main object of this paper is given by Definition 2.3, which is motivated by the description of the spectrum at infinity of a Laurent polynomial in Kouchnirenko’s framework, see [5–7,13]. We first recall the construction. Let $f(u) = \sum_{m \in \mathbb{Z}^n} a_m u^m$ be a Laurent polynomial, defined on $(\mathbb{C}^*)^n$. The Newton polytope $P$ of $f$ is the convex hull of $\text{supp} f := \{ m \in \mathbb{Z}^n, a_m \neq 0 \}$ in $\mathbb{R}^n$. We assume that $P$ contains the origin as an interior point, so that $f$ is convenient in the sense of [10]. We consider the increasing filtration $N_\bullet$ on $A := \mathbb{C}[u_1, u_1^{-1}, \ldots, u_n, u_n^{-1}]$, indexed by $\mathbb{Q}$, defined by

$$N_\alpha A := \{ g \in A, \supp(g) \in \nu^{-1}([-\infty; \alpha]) \}$$

where $\nu$ is the Newton function of the Newton polytope $P$ of $f$ and $\text{supp}(g) = \{ m \in \mathbb{Z}^n, b_m \neq 0 \}$ if $g = \sum_{m \in \mathbb{Z}^n} b_m u^m \in A$. By projection, this filtration defines the Newton filtration $N_\bullet$ on $A/(u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n})$, see [5, Section 2], [7, Section 4]. On the other hand, the spectrum at infinity of $f$ is a priori a sequence $\beta_1, \ldots, \beta_{\mu_P}$ of nonnegative rational numbers (there may be repeated numbers) and we define $\text{Spec}_f(z) := \sum_{i=1}^{\mu_P} z^{\beta_i}$, see [13], [7, Section 2.e]. The following result holds if $f$ is nondegenerate with respect to $P$ in the sense of [10, Définition 1.19]:

**Theorem 2.1** Let $f$ be a convenient and nondegenerate Laurent polynomial defined on $(\mathbb{C}^*)^n$. Then

$$\text{Spec}_f(z) = \sum_{\alpha \in \mathbb{Q}} \dim \mathbb{C} \text{gr}_\alpha N A / \left( u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n} \right) z^\alpha.$$  \hspace{1cm} (5)

**Proof** By [7, Theorem 4.5], the Newton filtration and the Kashiwara-Malgrange filtration on the Brieskorn lattice of $f$ coincide if $f$ is convenient and nondegenerate. The result then follows from the description of the spectrum given in [7, Section 2.e]. \qed

For our purpose, one can take equality (5) as the definition of the spectrum at infinity of the convenient and nondegenerate Laurent polynomial $f$.  

\( \text{ Springer} \)
Corollary 2.2 Let \( f \) be a convenient and nondegenerate Laurent polynomial defined on \((\mathbb{C}^*)^n\). Then,
\[
\text{Spec}_f(z) = (1 - z)^n \sum_{v \in N} z^{\nu(v)}
\]
where \( \nu \) is the Newton function of the Newton polytope \( P \) of \( f \).

**Proof** Follows from Theorem 2.1, [10, Théorème 2.8 and Section 2.10] (where the right-hand side appears as a Hilbert-Poincaré series), [10, Théorème 4.10] and [10, Section 5.11], as in [5, Theorem 3.2].

The right-hand side of (6) depends only on \( P \). This justifies the following definition:

**Definition 2.3** Let \( P \) be a full-dimensional lattice polytope in \( \mathbb{N}^\mathbb{R} \), containing the origin as an interior point, and let \( \nu \) be its Newton function. The Newton spectrum of \( P \) is
\[
\text{Spec}_P(z) := (1 - z)^n \sum_{v \in N} z^{\nu(v)}. \tag{7}
\]

Notice that several versions of the right-hand side of (7) appear in various situations; see for instance [1, Theorem 4.3] for a relation with stringy \( E \)-functions.

We have the following geometric description of the Newton spectrum of a polytope \( P \). We define, for \( \sigma \) a cone in the fan \( \Sigma_P \) generated by the vertices \( b_1, \ldots, b_r \) of \( P \),
\[
\square(\sigma) := \left\{ \sum_{i=1}^r q_i b_i, \ q_i \in [0, 1], \ i = 1, \ldots, r \right\}
\]
and we put \( \square(\Sigma_P) := \bigcup_{\sigma \in \Sigma_P} \square(\sigma) \). Let \( \tau \) be a cone in the fan \( \Sigma_P \) and let \( N_\tau \) be the subgroup of \( N \) generated by the primitive generators of the rays contained in \( \tau \): the quotient fan \( \Sigma_P / \tau \) is the fan in \( (N / N_\tau)_{\mathbb{R}} \) with cones given by the projections of the cones in \( \Sigma_P \) containing \( \tau \).

**Proposition 2.4** Let \( P \) be a full-dimensional lattice polytope in \( \mathbb{N}^\mathbb{R} \), containing the origin as an interior point. Then,
\[
\text{Spec}_P(z) = \sum_{v \in \square(\Sigma_P) \cap N} \left[ \sum_{i=0}^{n - \dim \sigma(v)} \dim H^{2i}(X(\Sigma_P / \sigma(v)), \mathbb{Q})z^i \right] z^{\nu(v)} \tag{8}
\]
where \( X(\Sigma_P / \sigma(v)) \) is the toric variety associated with the quotient fan \( \Sigma_P / \sigma(v) \) and \( \sigma(v) \) is the smallest cone containing \( v \).

**Proof** Follows from [10, Proposition 2.6], as in [5, Proposition 3.3]. See also [14, Theorem 4.3] for another proof in a slightly different context.

**Remark 2.5** By (8), we may write \( \text{Spec}_P(z) = \sum_{i=1}^{\mu_P} z^{\beta_i} \) where \( \beta_1, \ldots, \beta_{\mu_P} \) is a sequence of nonnegative rational numbers, arranged in increasing order. We will sometimes identify the Newton spectrum of \( P \) with the sequence \( \beta_1, \ldots, \beta_{\mu_P} \).
We will make a repeated use of the following properties. In what follows, \( \text{Int } P \) denotes the interior of \( P \), \( \partial P \) denotes its boundary and, for \( I \subset \mathbb{R} \), we define \( \text{Spec}_P(z) := \sum_{\beta_i \in I} z^\beta_i \).

**Proposition 2.6** [5–7,14] Let \( P \) be a full-dimensional lattice polytope in \( \mathbb{N} \subset \mathbb{R} \) containing the origin in its interior and let \( \nu \) be its Newton function. Then,

1. \( \lim_{z \to 1} \text{Spec}_P(z) = \mu_P \),
2. \( \text{Spec}_P[0,1](z) = \sum_{v \in \text{Int } P \cap \mathbb{N}} z^{\nu(v)} \),
3. the multiplicity of 1 in \( \text{Spec}_P(z) \) is \( \text{Card}(\partial P \cap \mathbb{N}) - n \),
4. \( z^n \text{Spec}_P(z^{-1}) = \text{Spec}_P(z) \).

See [5, Corollary 3.5]. Items 1, 2 and 4 are already contained in [7]. The last property means that \( \text{Spec}_P \) is symmetric about \( \frac{n}{2} \) and is a classical specification of a singularity spectrum.

### 3 The Newton spectrum of a reduced simplex

We give in this section a closed formula for the Newton spectrum of simplices. We will say that the polytope \( \Delta := \text{conv}(v_0, \ldots, v_n) \) is a simplex if its vertices \( v_i \) belong to the lattice \( \mathbb{Z}^n \) and if it contains the origin as an interior point. The following terminology is borrowed from [4]: the weight of the simplex \( \Delta \) is the tuple \( Q(\Delta) = (q_0, \ldots, q_n) \) where

\[
q_i := |\det(v_0, \ldots, \hat{v}_i, \ldots, v_n)|
\]  

for \( i = 0, \ldots, n \) and the simplex \( \Delta \) is said to be reduced if \( \gcd(q_0, \ldots, q_n) = 1 \).

**Lemma 3.1** Let \( \Delta := \text{conv}(v_0, \ldots, v_n) \) be a simplex and let \( Q(\Delta) = (q_0, \ldots, q_n) \) be its weight. Then,

1. \( \mu_\Delta = \sum_{i=0}^n q_i \) where \( \mu_\Delta \) is the Milnor number of \( \Delta \),
2. \( \sum_{i=0}^n q_i v_i = 0 \),
3. \( (v_0, \ldots, v_n) \) generate \( N \) if and only if \( \Delta \) is reduced.

**Proof** The first two points are standard. For the third one, notice first that the module \( N_\Delta \) generated by \( (v_0, \ldots, v_n) \) is free of rank \( n \). By [4, Lemma 2.4], we have \( \det N_\Delta = \gcd(q_0, \ldots, q_n) \), and this is the index of \( N_\Delta \) in \( N \) so that \( N_\Delta = N \) if and only if \( \det N_\Delta = 1 \).

Let \( \Delta \) be a simplex and let \( Q(\Delta) = (q_0, \ldots, q_n) \) be its weight. We define

\[
F := \left\{ \frac{\ell}{q_i} | 0 \leq \ell \leq q_i - 1, \ 0 \leq i \leq n \right\}
\]

and we denote by \( f_1, \ldots, f_k \) the elements of \( F \), arranged in increasing order. We then put

\[
S_{f_i} := \{ j \mid q_j f_i \in \mathbb{Z} \} \subset \{0, \ldots, n\} \text{ and } d_i := \text{Card } S_{f_i}
\]
and we denote by $c_0, c_1, \ldots, c_{\mu} \Delta - 1$ the sequence of rational numbers $f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_k, \ldots, f_k$.

The Newton spectrum of a reduced simplex is given by the following result:

**Theorem 3.2** Assume that the simplex $\Delta$ is reduced. Then, the Newton spectrum of $\Delta$ is

$$\text{Spec}_\Delta(z) = z^{\alpha_0} + z^{\alpha_1} + \cdots + z^{\alpha_{\mu} \Delta - 1}$$

where

$$\alpha_k := k - \mu \Delta c_k$$

for $k = 0, \ldots, \mu \Delta - 1$.

**Proof** By Lemma 3.1, the vertices $v_0, \ldots, v_n$ of $\Delta$ satisfy $\sum_{i=0}^{n} q_i v_i = 0$ and they generate $\mathbb{Z}^n$ because $\Delta$ is reduced. Thus, we get the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

where the map on the right is defined by $\psi(e_i) = v_i$ for $i = 0, \ldots, n$ (we denote here by $(e_0, \ldots, e_n)$ the canonical basis of $\mathbb{Z}^{n+1}$) and the map on the left is defined by $\phi(1) = (q_0, \ldots, q_n)$. It follows that the simplex $\Delta$ is the Newton polytope of the function

$$f(u_0, \ldots, u_n) = q_0 u_0 + \cdots + q_n u_n$$

restricted to the subtorus $U \subset (\mathbb{C}^*)^{n+1}$ defined by the equation

$$u_0^{q_0} \cdots u_n^{q_n} = 1,$$

which is the convenient and nondegenerate Laurent polynomial considered in [8]. Thus, the assertion follows from Corollary 2.2 and [8, Theorem 1]. \qed

**Example 3.3** Let us consider

$$\Delta = \text{conv}((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (-1, -1, -1, -5))$$

in $\mathbb{R}^4$. We have $Q(\Delta) = (1, 1, 1, 5, 1)$ and $\mu \Delta = 9$. The sequence $c_0, c_1, \ldots, c_8$ is

$$0, 0, 0, 0, 0, 1, 2, 3, 4$$

\[ \frac{5}{5}, \frac{5}{5}, \frac{5}{5}, \frac{5}{5} \]
and the sequence $\alpha_0, \alpha_1, \ldots, \alpha_8$ is

$$0, 1, 2, 3, 4, 5 - \frac{9}{5}, 6 - \frac{18}{5}, 7 - \frac{27}{5}, 8 - \frac{36}{5}. $$

Theorem 3.2 provides $\text{Spec}_\Delta(z) = 1 + z + z^2 + z^3 + z^4 + z^{16/5} + z^{12/5} + z^{8/5} + z^{4/5}$.

**Remark 3.4** Theorem 3.2 is not true if we do not assume that $\Delta$ is reduced: for instance, let $\Delta = \text{conv}((2, 0), (0, 2), (2, 2))$ in $\mathbb{R}^2$. We have $Q(\Delta) = (4, 4, 4)$ and $\Delta$ is not reduced. By Proposition 2.6, we have

$$\text{Spec}_\Delta(z) = 1 + 3z^{1/2} + 4z + 3z^{3/2} + z^2. $$

But Theorem 3.2 would give $\text{Spec}_\Delta(z) = 4 + 4z + 4z^2$.

### 4 The spectrum as a weighted $\delta$-vector and application to the computation of (weighted) Ehrhart polynomials

In the first part of this section, we show that the spectrum at infinity of a Laurent polynomial counts weighted lattice points. In the second part, we write down an algorithm in order to compute Ehrhart polynomials of reduced simplices.

#### 4.1 The Newton spectrum of a polytope as a weighted $\delta$-vector

For the background about Ehrhart theory, we refer to the book [2]. Let $P$ be a full-dimensional lattice polytope in $\mathbb{R}^n$ and define, for a nonnegative integer $\ell$, $L_P(\ell) := \text{Card}((\ell P) \cap N)$. Then, $L_P$ is a polynomial in $\ell$ of degree $n$ (the Ehrhart polynomial of $P$), and we have

$$1 + \sum_{m \geq 1} L_P(m)z^m = \frac{\delta_0 + \delta_1 z + \cdots + \delta_n z^n}{(1 - z)^{n+1}} $$

where the $\delta_j$’s are nonnegative integers. We will write $\delta_P(z) := \delta_0 + \delta_1 z + \cdots + \delta_n z^n$. We will call $\delta_P(z)$ the $\delta$-vector of the polytope $P$. The Ehrhart polynomial of a polytope is extracted from its $\delta$-vector by means of the formula

$$L_P(z) = \delta_0 \left( \frac{z + n}{n} \right) + \delta_1 \left( \frac{z + n - 1}{n} \right) + \cdots + \delta_{n-1} \left( \frac{z + 1}{n} \right) + \delta_n \left( \frac{z}{n} \right). $$

Following [14], one defines a weighted version of these $\delta$-vectors. Let $P$ be a full-dimensional lattice polytope in $\mathbb{R}^n$, containing the origin as an interior point, and let $\nu$ be its Newton function. The weight of $v \in N$ is $\text{wt}(v) := \nu(v) - \lceil \nu(v) \rceil$ where $\lceil \rceil$ denotes the ceiling function. For $m \in \mathbb{N}$, let $L_P^\nu(m)$ be the number of lattice points in $mP$ of weight $\alpha$ and define
\[ \delta_P^\alpha(z) := (1 - z)^{n+1} \sum_{m \geq 0} L_P^m \alpha z^m. \]

By the very definition, we have \( \delta_P(z) = \sum_{\alpha \in [-1,0]} \delta_P^\alpha(z) \). The weighted \( \delta \)-vector of the polytope \( P \) is

\[ \delta_{wt}^P(z) := \sum_{\alpha \in [-1,0]} \delta_P^\alpha(z) \alpha. \]

Once the spectrum at infinity of a Laurent polynomial is identified by formula (6), the following results can be basically found in [14]. We give a proof in keeping with our framework.

**Theorem 4.1** Let \( P \) be a full-dimensional lattice polytope containing the origin as a interior point. Then,

\[ \text{Spec}_P(z) = \delta_{wt}^P(z). \]

In particular, the spectrum at infinity \( \text{Spec}_f(z) \) of a convenient and nondegenerate Laurent polynomial \( f \) is equal to the weighted \( \delta \)-vector \( \delta_{wt}^P(z) \) of its Newton polytope \( P \).

**Proof** Because \( v \in mP \cap N \) if and only if \( \nu(v) \leq m \), we get

\[
\delta_{wt}^P(z) = (1 - z)^{n+1} \sum_{m \geq 0} \left( \sum_{v \in mP} z^{\nu(v)} \right) z^m
\]
\[
= (1 - z)^{n+1} \sum_{m \geq 0} \left( \sum_{\nu(v) \leq m} z^{\nu(v)} - \lceil \nu(v) \rceil \right) z^m
\]
\[
= (1 - z)^{n+1} \sum_{m \geq 0} \left( \sum_{\nu(v) \leq m} z^{\nu(v)} \right) z^m
\]
\[
= (1 - z)^{n+1} \sum_{v \in N} \left( \sum_{\nu(v) \leq m} z^{\nu(v)} \right) z^m
\]
\[
= (1 - z)^n \sum_{v \in N} z^{\nu(v)} = \text{Spec}_P(z).
\]

The last assertion follows from Corollary 2.2. \( \Box \)

**Corollary 4.2** Let \( P \) be a full-dimensional lattice polytope containing the origin as a interior point. Let \( \delta_P(z) := \sum_{k=0}^n \delta_k z^k \) be the \( \delta \)-vector of the \( P \) and let \( \text{Spec}_P(z) := \sum_{i=1}^{\mu} z^{\beta_i} \) be its Newton spectrum. Then, for \( k = 0, \ldots, n \), the coefficient \( \delta_k \) is equal to the number of \( \beta_i \)'s such that \( \beta_i \in [k - 1, k] \).

**Proof** The result follows from Theorem 4.1 because \( \delta_P(z) = \sum_{\alpha \in [-1,0]} \delta_P^\alpha(z) \). \( \Box \)

On the singularity side, we get:
Corollary 4.3 Let $f$ be a convenient and nondegenerate Laurent polynomial and let $P$ be its Newton polytope. Let $\text{Spec}_f(z) := \sum_{i=1}^{\mu_P} z^{\beta_i}$ be the spectrum at infinity of $f$ and let $\delta_P(z)$ be the $\delta$-vector of $P$. Then $\delta_P(z) = \sum_{i=1}^{\mu_P} z^{\lceil \beta_i \rceil}$ where $\lceil \cdot \rceil$ denotes the ceiling function.

4.2 An algorithm to compute (weighted) Ehrhart polynomials and $\delta$-vectors of reduced simplices

Let $P$ be a full-dimensional lattice polytope in $\mathbb{R}^n$ containing the origin in its interior. A general recipe in order to compute the $\delta$-vector of $P$ is now clear: compute the Newton spectrum of $P$ and use Corollary 4.2. This provides an effective algorithm in order to compute the (weighted) $\delta$-vectors and the (weighted) Ehrhart polynomials of a reduced simplex $\Delta$ containing the origin as an interior point:

- compute the weight of $\Delta$ [use Eq. (9)] and check that $\Delta$ is reduced,
- compute the Newton spectrum of $\Delta$ using Theorem 3.2,
- use Theorem 4.1 in order to compute the weighted $\delta$-vector $\delta_P^{\text{wt}}(z)$,
- use Corollary 4.2 in order to get the $\delta$-vector $\delta_\Delta(z) := \delta_0 + \delta_1 z + \cdots + \delta_n z^n$ of $\Delta$ and formula (11) in order to get the Ehrhart polynomial $L_\Delta(z)$ of $\Delta$.

If $q_0 = 1$ and if $\Delta$ is moreover reflexive, another formula for the $\delta$-vector of $\Delta$ in terms of the entries of the vector $Q(\Delta)$ is given in [3, Theorem 2.2]. It should be emphasized that in our algorithm the simplex $\Delta$ needs not to be reflexive and that we may have $q_0 \neq 1$ (see example below).

Example 4.4 Let $\Delta := \text{conv}((1, 0, 0), (0, 2, 0), (1, 1, 1), (-3, -5, -2))$ in $\mathbb{R}^3$. Its weight is $Q(\Delta) = (2, 3, 4, 2)$ and $\Delta$ is reduced. We have $\mu_\Delta = 11$ and

$$\text{Spec}_\Delta(z) = 1 + z + z^2 + z^3 + z^{5/4} + z^{4/3} + z^{1/2} + z^{3/2} + z^{5/2} + z^{5/3} + z^{7/4}$$

by Theorem 3.2. Corollary 4.2 provides $\delta_0 = 1, \delta_1 = 2, \delta_2 = 6, \delta_3 = 2$, and we finally get

$$L_\Delta(z) = \frac{1}{6}(11z^3 + 6z^2 + 13z + 6).$$

Example 4.5 (Example 3.3 continued) Let us consider the simplex

$$\Delta := \text{conv}((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (-1, -1, -1, -5))$$

in $\mathbb{R}^4$. Its weight is $Q(\Delta) = (1, 1, 1, 5, 1)$ and $\Delta$ is reduced. We have $\mu_\Delta = 9$ and

$$\text{Spec}_\Delta(z) = 1 + z + z^2 + z^3 + z^4 + z^{16/5} + z^{12/5} + z^{8/5} + z^{4/5}$$

by Example 3.3. Corollary 4.2 provides $\delta_0 = 1, \delta_1 = 2, \delta_2 = 2, \delta_3 = 2, \delta_4 = 2$ and we get

$$L_\Delta(z) = \frac{1}{24}(9z^4 + 10z^3 + 75z^2 + 50z + 24).$$
We have also $\delta^0_\Delta(z) = 1 + z + z^2 + z^3 + z^4$, $\delta^{-1/5}_\Delta(z) = z$, $\delta^{-2/5}_\Delta(z) = z^2$, $\delta^{-3/5}_\Delta(z) = z^3$, $\delta^{-4/5}_\Delta(z) = z^4$ and

$$L^{-1/5}_\Delta(z) = \frac{1}{24}(z^4 + 6z^3 + 11z^2 + 6z), \quad L^{-2/5}_\Delta(z) = \frac{1}{24}(z^4 + 2z^3 - z^2 - 2z),$$

$$L^{-3/5}_\Delta(z) = \frac{1}{24}(z^4 - 2z^3 - z^2 + 2z), \quad L^{-4/5}_\Delta(z) = \frac{1}{24}(z^4 - 6z^3 + 11z^2 - 6z),$$

$$L^0_\Delta(z) = \frac{1}{24}(5z^4 + 10z^3 + 55z^2 + 50z + 24).$$

Weighted Ehrhart polynomials may have negative coefficients.

### 4.3 Ehrhart polynomials of Newton polytopes of polynomials

We end this section with a few words about Newton polytopes of polynomials. The support of a polynomial $g = \sum_{m \in \mathbb{N}^n} a_m u^m \in \mathbb{C}[u_1, \ldots, u_n]$, where $u^m := u_1^{m_1} \cdots u_n^{m_n}$ if $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, is $\text{supp}(g) := \{ m \in \mathbb{N}^n, \ a_m \neq 0 \}$ and the Newton polytope of the polynomial $g$ is the convex hull of $\{0\} \cup \text{supp}(g)$ in $\mathbb{R}^n$. The toric Newton spectrum of $g$ (or the toric Newton spectrum of its Newton polytope) is

$$\sum_{\alpha \in \mathbb{Q}} \dim_{\mathbb{C}} \text{gr}_\alpha^N\frac{\mathbb{C}[u_1, \ldots, u_n]}{(u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n})^\alpha} z^\alpha,$$

see [5, Definition 3.1]. Thanks to [5, Theorem 3.2], Theorem 4.1 and Corollary 4.2 still hold, with the same proofs, for the toric Newton spectrum of the Newton polytope of a convenient and nondegenerate polynomial. This gives a recipe in order to calculate Ehrhart polynomials of Newton polytopes of polynomials from their toric Newton spectrum. Here are two examples.

1. Let $\Delta := \text{conv}((0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c))$ where $a$, $b$ and $c$ are positive integers. It is the Newton polytope of $f(u_1, u_2, u_3) = u_1^a + u_2^b + u_3^c$. Its toric Newton spectrum is the sequence of the following rational numbers:
   - 0,
   - $\frac{i}{a}$ for $1 \leq i \leq a - 1$, $\frac{i}{b}$ for $1 \leq i \leq b - 1$, $\frac{i}{c}$ for $1 \leq i \leq c - 1$,
   - $\frac{i}{a} + \frac{j}{b}$ for $1 \leq i \leq a - 1$ and $1 \leq j \leq b - 1$, $\frac{i}{a} + \frac{j}{c}$ for $1 \leq i \leq a - 1$ and $1 \leq j \leq c - 1$,
   - $\frac{i}{a} + \frac{j}{b} + \frac{k}{c}$ for $1 \leq i \leq a - 1$, $1 \leq j \leq b - 1$ and $1 \leq k \leq c - 1$.

   Its $\delta$-vector is then given by Corollary 4.2. For instance, if $a = 2$, $b = 3$ and $c = 3$ we get $\delta_0 = 1$, $\delta_1 = 10$, $\delta_2 = 7$, $\delta_3 = 0$. If $a = 1$, $b = 1$ and $c = 1$ (this corresponds to the standard simplex in $\mathbb{R}^n$) we get of course $\delta_0 = 1$, $\delta_1 = \delta_2 = \delta_3 = 0$.

2. Let $P := \text{conv}((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, h))$ where $h$ is an integer greater or equal to 2. This is a variation of Reeve’s tetrahedron, see [2, Example 3.22]. It is the Newton polytope of $f(u_1, u_2, u_3) = u_1 + u_2 + u_3 + u_1 u_2 u_3^h$ whose
toric Newton spectrum is equal to $1 + z + z^2 + \sum_{i=1}^{h-1} z^{1+i}$. Thus, $\delta_0 = 1, \delta_1 = 1, \delta_2 = h, \delta_3 = 0$ and

$$L_P(z) = \frac{1}{6}[(h + 2)z^3 + 9z^2 + (13 - h)z + 6].$$

The coefficient of $z$ is negative if $h \geq 14$.

### 5 Reflexive polytopes and their Newton spectrum

We give in this section various characterizations of reflexive polytopes in terms of their Newton spectrum and we end with some remarks about the distribution of the $\delta$-vectors.

**Proposition 5.1** Let $P$ be a full-dimensional lattice polytope in $\mathbb{R}^n$ containing the origin in its interior and let $\delta_P(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$ be its $\delta$-vector. Then, the following are equivalent:

1. $\text{Spec}_P(z)$ is a polynomial,
2. $P$ is reflexive,
3. $\text{Spec}_P(z) = \delta_P(z) := \delta_0 + \delta_1 z + \cdots + \delta_n z^n$.

**Proof** 1 $\iff$ 2. Assume that $P$ is reflexive: if $n$ belongs to the cone $\sigma_F$ supported by the facet $F$, the Newton function of $P$ is defined by $\nu : n \mapsto \langle u_F, n \rangle$ where $u_F \in M$. By Proposition 2.4, $\text{Spec}_P(z)$ is a polynomial. Conversely, assume that $\text{Spec}_P(z)$ is a polynomial. Let $F$ be a facet of $P$. Again by Proposition 2.4, the Newton function $\nu$ of $P$ takes integral values on the fundamental domain $\square(\sigma_F)$. It follows that $\nu$ takes integral values on $\sigma_F \cap N$ and, by linearity, we get a $\mathbb{Z}$-linear map $\nu_{\sigma_F} : N_{\sigma_F} \to \mathbb{Z}$ where $N_{\sigma_F}$ is the sublattice of $N$ generated by the points of $\sigma_F \cap N$. Because we have the isomorphism $\text{Hom}_{\mathbb{Z}}(N_{\sigma_F}, \mathbb{Z}) \cong M/\sigma_F^\perp \cap M$ induced by the dual pairing between $M$ and $N$, there exists $u_F \in M$ such that $\nu(n) = \langle u_F, n \rangle$ when $n \in \sigma_F$. In particular, the equation of $F$ is $\langle u_F, x \rangle = 1$ for $u_F \in M$. Thus, $P$ is reflexive.

1 $\iff$ 3. Assume that $\text{Spec}_P(z)$ is a polynomial. Corollary 4.2 shows that $\text{Spec}_P(z) = \delta_P(z)$. Conversely, if $\text{Spec}_P(z) = \delta_P(z)$ the Newton spectrum is clearly a polynomial.

$\square$

**Remark 5.2** Recall that a polynomial $a_0 + a_1 z + \cdots + a_n z^n$ is unimodal if there exists an index $j$ such that $a_i \leq a_{i+1}$ for all $i < j$ and $a_i \geq a_{i+1}$ for all $i \geq j$. It follows from Proposition 5.1 and [11,12] that the Newton spectrum of a reflexive polytope is not always unimodal. In our setting, this can be checked as follows: let

$$\Delta := \text{conv} \left( e_1, \ldots, e_9, -\sum_{i=1}^9 a_i e_i \right)$$

where $(e_1, \ldots, e_9)$ is the standard basis of $\mathbb{R}^9$ and $(a_1, \ldots, a_9) := (1, \ldots, 1, 3)$ (this example is borrowed from [12], see also [3, Section 2]). Then, $\mu_\Delta = 12$ and...
$Q(\Delta) = (1, \ldots, 1, 3, 1)$ where 1 is counted 9-times: the simplex $\Delta$ is reduced and reflexive. From Theorem 3.2 we get

$$\text{Spec}_{\Delta}(z) = 1 + z + z^2 + 2z^3 + z^4 + z^5 + 2z^6 + z^7 + z^8 + z^9$$

and this polynomial is not unimodal. In particular, the spectrum at infinity of the Laurent polynomial

$$f(u_1, \ldots, u_9) = u_1 + \cdots + u_9 + \frac{1}{u_1 \cdots u_8 u_9}$$

is not unimodal.

**Remark 5.3** Let $P$ be a reflexive polytope. The mean of its $\delta$-vector $\delta_P(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$ is $\frac{1}{\mu_P} \sum_{i=0}^{n} \delta_i$, and is equal to $\frac{n}{2}$ because $\delta_i = \delta_{n-i}$ if $P$ is reflexive. Its variance is $\frac{1}{\mu_P} \sum_{i=0}^{n} \delta_i(i - \frac{n}{2})^2$. By Corollary 2.2 and Proposition 5.1, the inequality

$$\frac{1}{\mu_P} \sum_{i=0}^{n} \delta_i(i - \frac{n}{2})^2 \geq \frac{n}{12}$$

is expected. This is the version of Hertling’s conjecture alluded to in the introduction (see [6] for details).

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