Extended Diffeomorphism Algebras and Trajectories In Jet Space

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Abstract: Let the DRO (Diffeomorphism, Reparametrization, Observer) algebra $DRO(N)$ be the extension of $diff(N) \oplus diff(1)$ by its four inequivalent Virasoro-like cocycles. Here $diff(N)$ is the diffeomorphism algebra in $N$-dimensional spacetime and $diff(1)$ describes reparametrizations of trajectories in the space of tensor-valued $p$-jets. $DRO(N)$ has a Fock module for each $p$ and each representation of $gl(N)$. Analogous representations for gauge algebras (higher-dimensional Kac-Moody algebras) are also given. The reparametrization symmetry can be eliminated by a gauge fixing procedure, resulting in previously discovered modules. In this process, two $DRO(N)$ cocycles transmute into anisotropic cocycles for $diff(N)$. Thus the Fock modules of toroidal Lie algebras and their derivation algebras are geometrically explained.

1. Introduction

Consider the algebra of diffeomorphisms in $N$-dimensional spacetime, $diff(N)$. The classical representations act on tensor densities over spacetime $\mathbb{R}^N$, but this is not a good starting point for quantization. Naïvely, one would try to introduce canonical momenta and normal order, but this only works in one dimension, where this procedure gives Fock representations of the Virasoro algebra. In higher dimensions, infinities are encountered; formally, a central extension proportional to the number of time-independent functions arises. Moreover, $diff(N)$ has no central extension when $N > 1$.

diff$(N)$ acts naturally on the corresponding space of $p$-jets, $p$ finite. The infinite jet space is essentially the space of functions, insofar as functions may be identified with their Taylor series. This realization of $diff(N)$ is finite-dimensional but non-linear; diffeomorphisms act linearly on the Taylor coefficients with matrices depending non-linearly on the base point. The corresponding Fock
representation is well defined but not very interesting, because it gives us back
the original tensor densities (and derivatives thereof), and no extensions arise.

To remedy this, consider the space of trajectories in jet space. $diff(N)$ acts
naturally on this space as well, but in a highly reducible fashion; the realiza-
tion is a continuous direct sum because every point on a trajectory transforms
independently of its neighbors. This degeneracy can be lifted by adding an ex-
tra $diff(1)$ factor describing reparametrizations, and thus the total algebra is
$diff(N) \oplus diff(1)$. The DRO algebra $DRO(N)$ is the extension of this algebra
by its four independent Virasoro-like cocycles, which are non-central except in
one dimension. The canonical normal ordering with respect to reparametriza-
tions results in Fock modules for $DRO(N)$. On the group level, this corresponds
to a representation up to a local phase; only if the phase is globally constant,
the Lie algebra extension is central.

Reparametrizations are then eliminated by Hamiltonian reduction. Since they
generate first class constraints, a gauge fixing condition must be introduced; a
natural choice is to identify one coordinate with the parameter along the tra-
jectory. Poisson brackets are now replaced with Dirac brackets before normal
ordering. This yields a projective realization of $diff(N)$, which was discovered
by hand in [14] (that paper was limited to zero-jets). In particular, two of the
diff $(N) \oplus diff(1)$ cocycles transmute into the anisotropic $diff(N)$ extensions
described in that paper. By further specialization to scalar-valued jets (and
choosing a Fourier basis on the $N$-dimensional torus), we recover the results of
$\mathfrak{g}$ on the derivation algebra of toroidal Lie algebras. I thus give a complete geo-
metrical explanation of the rather surprising results in [8], and generalize in
two ways: reparametrizations are separated from diffeomorphisms, and arbitrary
tensor-valued $p$-jets are considered, not only zero-jets.

Berman and Billig [1] independently studied tensor-valued objects, but only
as modules over the “spatial” subalgebra $diff(N - 1)$. For a supersymmetric
generalization, see [13]. Proper representations were studied in [8].

It was noted by several authors [8,13] that the gauge-fixed algebra is “space-
time asymmetric” in the sense that time is a distinguished direction. In the
present work this anisotropy is isolated in the gauge fixing condition, whereas
the underlying algebraic structure is completely isotropic.

The gauge algebra $map(N, \mathfrak{g})$, i.e. the algebra of maps from $N$-dimensional
spacetime to a finite-dimensional Lie algebra $\mathfrak{g}$, has similar projective rep-
resentations. This representation theory is also developed in the present paper,
and thus the results of [8,13,14,17] on toroidal Lie algebras are geometrically
explained and generalized.

All considerations in this paper are local, but I expect that the results can be
globalized without too much difficulty. It is clear that the first de Rham homology
plays an important role, both because the basic objects are one-dimensional
trajectories and because closed one-chains appear in (2.6) below.

2. The Algebra $DRO(N)$

Let $\xi = \xi^\mu(x)\partial_\mu$, $x \in \mathbb{R}^N$, $\partial_\mu = \partial/\partial x^\mu$, be a vector field, with commutator $[\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu - \eta^\mu \partial_\mu \xi^\nu \partial_\nu$, Greek indices $\mu, \nu = 0, 1, \ldots, N - 1$ label the space-
time coordinates and the summation convention is used on all kinds of indices.
The diffeomorphism algebra (algebra of vector fields, Witt algebra) $diff(N)$ is
generated by Lie derivatives $\mathcal{L}_\xi$. In particular, we refer to diffeomorphisms on the circle as reparametrizations. They form an additional $diff(1)$ algebra with generators $L(t), t \in S^1$. $diff(N) \oplus diff(1)$ is the Lie algebra with brackets

$$
\begin{align*}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]}, \\
[L(s), \mathcal{L}_\xi] &= 0, \\
[L(s), L(t)] &= (L(s) + L(t))\delta(s - t).
\end{align*}
$$

(2.1)

Alternatively, we describe reparametrizations in terms of generators $L_f$, where $f = f(t)d/dt$ is a vector field on the circle:

$$
L_f = \int dt \ f(t)L(t).
$$

The commutator is $[f, g] = (f\dot{g} - g\dot{f})d/dt$, where a dot indicates the $t$ derivative. The assumption that $t \in S^1$ is for technical simplicity; it enables jets to be expanded in a Fourier series, but it is physically quite unjustified because it means that spacetime is periodic in the time direction. However, all we really need is that $\int dt \ F(t) = 0$ for all functions $F(t)$. Most results are unchanged if we instead take $t \in \mathbb{R}$ and replace Fourier sums with Fourier integrals everywhere.

Introduce $N$ privileged functions on the circle $q^\mu(t)$, which can be interpreted as the trajectory of an observer (or base point). Let the observer algebra $\text{Obs}(N)$ be the space of local functionals of $q^\mu(t)$, i.e. polynomial functions of $q^\mu(t)$, $\dot{q}^\mu(t)$, ... $d^kq^\mu(t)/dt^k$, $k$ finite, regarded as a commutative Lie algebra. The $\text{DRO}$ ($\text{Diffeomorphism, Reparametrization, Observer}$) algebra $\text{DRO}(N)$ is an abelian but non-central Lie algebra extension of $diff(N) \oplus diff(1)$ by $\text{Obs}(N)$:

$$
0 \rightarrow \text{Obs}(N) \rightarrow \text{DRO}(N) \rightarrow diff(N) \oplus diff(1) \rightarrow 0.
$$

(2.2)

The extension depends on the four parameters $c_j, j = 1, 2, 3, 4$, to be called abelian charges; the name is chosen in analogy with the central charge of the Virasoro algebra. The sequence (2.2) splits ($\text{DRO}(N)$ is a semi-direct product) iff all four abelian charges vanish. The brackets are given by

$$
\begin{align*}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \ \dot{\xi}^\mu(t)\left(c_1\partial_\mu\partial_\rho\xi^\mu(q(t))\partial_\sigma\eta^\sigma(q(t)) + \\
&\quad + c_2\partial_\mu\partial_\rho\xi^\mu(q(t))\partial_\sigma\eta^\sigma(q(t))\right), \\
[L_f, \mathcal{L}_\xi] &= \frac{1}{4\pi i} \int dt \ (c_3\dot{f}(t) - ia_3\dot{f}(t))\partial_\mu\xi^\mu(q(t)), \\
[L_f, L_g] &= L_{[f, g]} + \frac{c_4}{24\pi i} \int dt \ (\dot{f}(t)\dot{g}(t) - \dot{f}(t)g(t)), \\
[\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[L_f, q^\mu(t)] &= - f(t)\dot{q}^\mu(t), \\
[q^\mu(s), q^\nu(t)] &= 0.
\end{align*}
$$

(2.3)

extended to all of $\text{Obs}(N)$ by Leibniz’ rule and linearity. The parameter $a_3$ is cohomologically trivial and can be removed by the redefinition

$$
\mathcal{L}_\xi \rightarrow \mathcal{L}_\xi + \frac{ia_3}{4\pi i} \int dt \ \partial_\mu\xi^\mu(q(t)).
$$

(2.4)
The remaining four cocycles are non-trivial. We identify $c_4$ as the central charge in the Virasoro algebra generated by reparametrizations.

It is not difficult to reformulate the DRO algebra as a proper Lie algebra, by introducing a compete basis for $\text{Obs}(N)$. In fact, it suffices to consider the two linear operators $S_0(F)$ and $S_1^\rho(F_\rho)$, defined for two arbitrary functions $F(t,x)$ and $F_\rho(t,x)$, $t \in S^1$, $x \in \mathbb{R}^N$,

$$S_0(F) = \frac{1}{2\pi i} \int dt \ F(t,q(t)),$$

$$S_1^\rho(F_\rho) = \frac{1}{2\pi i} \int dt \ \dot{q}^\rho(t) F_\rho(t,q(t)). \quad (2.5)$$

$DRO(N)$ now takes the form

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]} + S_1^\rho(c_1 \partial_\rho \partial_\nu \xi^\nu \eta^\nu + c_2 \partial_\rho \partial_\mu \xi^\mu \partial_\nu \eta^\nu),$$

$$[L_f, \mathcal{L}_\xi] = \frac{1}{2} S_0((c_3 \dot{\xi} - i a_3 \dot{f}) \partial_\rho \xi^\rho),$$

$$[L_f, L_g] = L_{fg} + \frac{c_4}{12} S_0(\dot{f} \dot{g} - \dot{g} \dot{f}),$$

$$[\mathcal{L}_\xi, S_0(F)] = S_0(\xi^\mu \partial_\mu F),$$

$$[L_f, S_0(F)] = S_0(f \frac{\partial F}{\partial t} + \dot{f} F),$$

$$[\mathcal{L}_\xi, S_1^\rho(F_\rho)] = S_1^\rho(\xi^\mu \partial_\mu F_\rho + \partial_\rho \xi^\mu F_\rho),$$

$$[L_f, S_1^\rho(F_\rho)] = S_1^\rho(f \frac{\partial F_\rho}{\partial t}),$$

$$S_0(\frac{\partial F}{\partial t}) + S_1^\rho(\partial_\rho F) \equiv 0,$$

$$S_0(f) = \frac{1}{2\pi i} \int dt \ f(t), \quad \text{if } f(t) \text{ is independent of } x,$$

where $\dot{f} = df/dt$. That (2.6) defines a Lie algebra follows from the explicit realization in Theorem 1 below, but it is also straightforward to verify the Jacobi identities.

If $F_\rho(t,x)$ is independent of $t$, $S_1^\rho(F_\rho)$ is dual to closed one-forms, and as such it can be viewed as a closed one-chain. Dzhumadil’daev has given a list of $\text{diff}(N)$ extensions by modules of tensor fields; see also [4]. The cocycles $c_1$ and $c_2$ are related to his cocycles $\psi^1_3$ and $\psi^2_3$, respectively. In fact, they are equal for $S_1^\rho$ an exact one-chain, but closed one-chains are not included in Dzhumadil’daev’s list since they are not tensor modules. In one dimension, $S_1(f) = 1/(2\pi i) \int dx^0 \ f(x^0)$, so the first line in (2.6) reduces to the Virasoro algebra with central charge $c = 12(c_1 + c_2)$.

The cocycles in $DRO(N)$ have a natural origin from the diffeomorphism algebra in $(N + 1)$-dimensional space. Let its coordinates be $z^A$, where capital indices $A = -1, 0, 1, 2, ..., N$ run over $N + 1$ values and the extra direction is labelled by $-1$. $\text{diff}(N + 1)$ has an abelian extension with two Virasoro-like...
cocycles:

\[
[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} + S^C(c_1 \partial_C \partial_B X^A \partial_A Y^B + c_2 \partial_C \partial_A X^A \partial_B Y^B),
\]

\[
[\mathcal{L}_X, S^C(F_C)] = S^C(X^A \partial_A F_C + \partial_C X^A F_A),
\]

\[
S^C(\partial_C F) \equiv 0,
\]

(2.7)

where $X^A(z)\partial_A$ is an $(N+1)$-dimensional vector field. The cocycles multiplying $c_1$ and $c_2$ are simply those found by Eswara Rao and Moody [8] and myself [13], in one extra dimension. Now embed $\text{diff}(N) \oplus \text{diff}(1) \subset \text{diff}(N+1)$ in the natural manner:

$z^\mu = x^\mu, z^1 = t$, $X^A(z) = (\xi^\mu(x), f(t))$, $\mathcal{L}_X = (\mathcal{L}_\xi, L_f)$,

$S^C(F_C) = S_0(F) + S^1(F)$, where $F = F_{-1}$. Under this decomposition, (2.7) restricts to (2.6) with $c_3 = 2c_2, c_4 = 12(c_1 + c_2)$, up to a trivial cocycle. However, it is easy to see that $c_3$ and $c_4$ are in fact independent parameters, so there are four different cocycles in total. In Sect. [5] below I will show that the complicated anisotropic cocycles in [14] can be obtained from (2.3) by a gauge-fixing procedure.

We are interested in representations of $DRO(N)$ that are of lowest energy type with respect to the Hamiltonian

\[
H = L_{-i} \frac{d}{dt} = -i \int dt \mathcal{L}(t).
\]

(2.8)

Such a representation contains a cyclic state $|0\rangle$ (the vacuum), satisfying

\[
H|0\rangle = h|0\rangle,
\]

(2.9)

and $A|0\rangle = 0$ for every operator $A$ such that $[H, A] = -w_A A, w_A > 0$. The lowest energy $h$ also characterizes the representation.

3. Preliminaries

Consider the space of $V$-valued functions over spacetime, where $V$ carries an $\mathfrak{gl}(N)$ representation $\rho$. This is our configuration space which will be denoted by $Q$. A basis is given by $\phi_\alpha(x), x \in \mathbb{R}^N$, where the index $\alpha$ labels different components of tensor densities. The fields can be either bosonic or fermionic, but it is assumed that all components have the same parity. Let $m = (m_0, m_1, .., m_{N-1})$, all $m_\mu \geq 0$, be a multi-index of length $|m| = \sum_{\mu=0}^{N-1} m_\mu$, let $\mu$ be a unit vector in the $\mu^{th}$ direction, and let 0 be the multi-index of length zero. Denote by

\[
\partial_m \phi_\alpha(x) = \partial_{m_0} \partial_{m_1} .. \partial_{m_{N-1}} \partial_{N-1} \phi_\alpha(x).
\]

(3.1)
Diffeomorphisms act as follows on derivatives of tensor densities:

\[
\left[ L_\xi, \partial_\nu \phi(x) \right] = \partial_\nu \xi^\mu(x) \partial_\mu \phi(x) - \partial_\mu \xi^\mu(x) \partial_\nu \frac{\partial}{\partial x^\nu} \phi(x)
\]

\[
= -\xi^\mu(x) \delta^m_{n+\mu} \phi(x) - \sum_{|m| \leq |n|} T^m_n(\xi(x)) \partial_m \phi(x),
\]

(3.2)

\[
T^m_n(\xi) = \binom{n}{m} \partial_{n-m+\xi^\mu \phi(T^\nu_\mu)}
\]

\[
+ \binom{n}{m-\mu} (1 - \delta^m_{n-\mu}) \partial_{n-m+\xi^\mu \phi(1)},
\]

(3.3)

where the \( V \) index \((\alpha)\) was suppressed, \( m! = m_0!m_1!...m_{N-1}! \) and \( \binom{n}{m} = n!/m!(n-m)! \). Our convention is that \( \text{gl}(N) \) has basis \( T^\mu_\nu \) and brackets

\[
[T^\mu_\nu, T^\rho_\sigma] = \delta^\rho_\nu T^\mu_\sigma - \delta^\mu_\sigma T^\rho_\nu.
\]

(3.4)

\( \phi(T^\mu_\nu) \) are the matrices in the representation \( \phi \), acting on a tensor density with \( p \) upper and \( q \) lower indices and weight \( \kappa \) as follows:

\[
\phi(T^\mu_\nu) \phi_{\tau_1...\tau_q}^{\sigma_1...\sigma_p} = -\kappa \delta^\mu_\nu \phi_{\tau_1...\tau_q}^{\sigma_1...\sigma_p} + \sum_{i=1}^{p} \delta^\mu_{\nu} \phi_{\tau_1...\tau_q}^{\sigma_i...\sigma_p} - \sum_{j=1}^{q} \delta^\nu_\mu \phi_{\tau_1...\tau_q}^{\sigma_1...\sigma_{j-1}\sigma_j...\sigma_p}.
\]

(3.5)

The matrices \( T^m_n(\xi) \), with components \( T^m_n(\xi)_{\beta}^\alpha \), satisfy

\[
T^m_{n+\xi} = \partial_\nu \xi^\mu n^\mu + T^m_n(\partial_\nu \xi) + T^m_n(\xi) - \delta^m_{n-\mu} T^m_n(\xi),
\]

\[
T^m_0 = \delta_0^m \partial_\nu \xi^\mu T^\nu_\mu,
\]

\[
\partial_\nu T^m_n(\xi) = T^m_n(\partial_\nu \xi),
\]

\[
T^m_n(\xi, \eta) = \xi^\mu T^m_n(\partial_\nu \eta) - \eta^\mu T^m_n(\partial_\nu \xi)
\]

\[
+ \sum_{|m| \leq |\tau| \leq |n|} (T^m_n(\xi)T^\tau_\mu(\eta) - T^\tau_n(\eta)T^m_n(\xi)).
\]

(3.6)

In particular, \( T^m_n(\xi) = 0 \) if \(|m| > |n|\).

Let \( \text{tr} \) denote the trace in \( \text{gl}(N) \) representation \( \phi \). Define numbers \( \text{dim}(\phi) \), \( k_0(\phi) \), \( k_1(\phi) \), \( k_2(\phi) \) by

\[
\text{tr} \ 1 = \text{dim}(\phi),
\]

\[
\text{tr} T^\mu_\nu = k_0(\phi) \delta^\mu_\nu,
\]

\[
\text{tr} T^\mu_\nu T^\tau_\sigma = k_1(\phi) \delta^\mu_\nu \delta^\tau_\sigma + k_2(\phi) \delta^\mu_\nu \delta^\tau_\sigma.
\]

(3.7)

For an unconstrained tensor transforming as in (3.5),

\[
\text{dim}(\phi) = N^{p+q}, \quad k_0(\phi) = -(p-q-\kappa N)N^{p+q-1}, \quad k_1(\phi) = (p+q)N^{p+q-1}, \quad k_2(\phi) = ((p-q-\kappa N)^2 - p-q)N^{p+q-2}.
\]

(3.8)

Note that if \( \kappa = (p-q)/N \), \( \phi \) is an \( \text{sl}(N) \) representation. Let \( S_\ell \) be the symmetric representation on \( \ell \) lower indices, appropriate for multi-indices. We have
\[ \dim(S_\ell) = \sum_\mathbf{m} \delta^\mathbf{m}_\mathbf{m}, \text{ etc., where} \]

\[ \dim(S_\ell) = \binom{N - 1 + \ell}{\ell}, \quad k_0(S_\ell) = \binom{N - 1 + \ell}{\ell - 1}, \]

\[ k_1(S_\ell) = \binom{N + \ell}{\ell - 1}, \quad k_2(S_\ell) = \binom{N - 1 + \ell}{\ell - 2}. \] (3.9)

**Lemma 1.**

1. \( \sum_{|\mathbf{m}| \leq p} \delta^\mathbf{m}_\mathbf{m} \text{tr } 1 = \binom{N + p}{p} \dim(\mathfrak{g}), \)

2. \( \sum_{|\mathbf{m}| \leq p} \text{tr } T^\mathbf{m}_\mathbf{m}(\xi) = \partial_\mu \xi^\mu \binom{N + p}{p} k_0(\mathfrak{g}) + \binom{N + p + 1}{p - 1} \dim(\mathfrak{g}), \)

3. \( \sum_{|\mathbf{m}| \leq |\mathbf{n}| \leq p} \text{tr } T^\mathbf{m}_\mathbf{m}(\xi) T^\mathbf{n}_\mathbf{n}(\eta) = \partial_\mu \xi^\mu \partial_\nu \eta^\nu \binom{N + p}{p} k_1(\mathfrak{g}) + \binom{N + p + 1}{p - 1} \dim(\mathfrak{g}) + \partial_\mu \xi^\mu \partial_\nu \eta^\nu \binom{N + p}{p} k_2(\mathfrak{g}) + \binom{N + p}{p - 1} k_0(\mathfrak{g}). \)

**Proof.** If \( |\mathbf{m}| = |\mathbf{n}| = \ell, \)

\[ T^\mathbf{m}_\mathbf{m}(\xi) = \partial_\mu \xi^\mu (\mathfrak{g}(T^\mu_\nu) \delta^\mathbf{m}_\mathbf{n} + \mathfrak{g}(1) \zeta^\mathbf{m}_\mathbf{n}(T^\mu_\nu)), \] (3.10)

where \( \zeta^\mathbf{m}_\mathbf{n}(T^\mu_\nu) \) are the representation matrices in \( S_\ell, \) acting on multi-indices. Only the top values (3.10) contribute to the traces, which means that we can ignore that higher jets do not transform as \( S_\ell \)-valued zero-jets. By the definition (3.7) and (3.11),

\[ \dim(\mathfrak{g} \otimes S_\ell) = \dim(\mathfrak{g}) \cdot \dim(S_\ell), \]

\[ k_0(\mathfrak{g} \otimes S_\ell) = k_0(\mathfrak{g}) \dim(S_\ell) + \dim(\mathfrak{g}) k_0(S_\ell), \]

\[ k_1(\mathfrak{g} \otimes S_\ell) = k_1(\mathfrak{g}) \dim(S_\ell) + \dim(\mathfrak{g}) k_1(S_\ell), \]

\[ k_2(\mathfrak{g} \otimes S_\ell) = k_2(\mathfrak{g}) \dim(S_\ell) + \dim(\mathfrak{g}) k_2(S_\ell) + 2 k_0(\mathfrak{g}) k_0(S_\ell). \] (3.11)

The lemma now follows from (3.9) and the following sums:

\[ \sum_{\ell=0}^{p} \dim(S_\ell) = \binom{N + p}{p}, \quad \sum_{\ell=0}^{p} k_0(S_\ell) = \binom{N + p}{p - 1}, \]

\[ \sum_{\ell=0}^{p} k_1(S_\ell) = \binom{N + p + 1}{p - 1}, \quad \sum_{\ell=0}^{p} k_2(S_\ell) = \binom{N + p}{p - 2}. \] (3.12)
4. Jet Space Trajectories

Let \( J^p Q \) be the space of trajectories in the space of \( V \)-valued \( p \)-jets, with coordinates \((q^\mu(t), \phi_{\alpha,m}(t))\), where \(|m| \leq p \) and \( t \in S^1 \). The parameter \( t \) is referred to as time and \( q^\mu(t) \) as the observer’s trajectory in spacetime. \( DRO(N) \) acts on \( J^p Q \) as follows:

\[
[L(s), \phi_{\alpha,n}(t)] = -\sum_{|m| \leq |n|} T^m_n(\xi(q(t)))\phi_{\alpha,m}(t),
\]

\[
[L(s), \phi_{\alpha,n}(t)] = -\partial_s \xi^\alpha(q(t))\phi_{\alpha,n}(t),
\]

\[
[L(s), \phi_{\alpha,n}(t)] = -\partial_s q^\mu(t)\phi_{\alpha,n}(t),
\]

\[
[L(s), \phi_{\alpha,n}(t)] = -\partial_t q^\mu(t)\phi_{\alpha,n}(t).
\]

Clearly, there is a chain of inclusions \( J^{-1}Q \subset J^0 Q \subset J^1 Q \subset \ldots \), where \( J^{-1}Q \) consists of \( q^\mu(t) \) only. Hence \( J^p Q \) is reducible (but indecomposable) as a \( DRO(N) \) realization. This kind of reducibility is not present in the Fock modules below, because jets of all orders up to \( p \) are created from the vacuum, cf. (5.32).

We call \( \lambda \) the causal weight of \( \phi \), in contradistinction to its tensorial weight \( \kappa \). The shift parameter \( w \) can be eliminated by the redefinition

\[
\phi_{\alpha,n}(t) \rightarrow e^{-iw t} \phi_{\alpha,n}(t),
\]

so it is only defined up to an integer. The triple \((\kappa, \lambda, w)\) will collectively be referred to as the weights of \( \phi \). The observer’s trajectory \( q^\mu(t) \) has causal weight 0 but it does not transform as a zero-jet under diffeomorphisms. However, its time derivative has causal weight 1 and does transform as a (vector-valued) zero-jet,

\[
[L(s), q^\mu(t)] = \partial_s q^\mu(t)\phi_{\alpha,n}(t),
\]

\[
[L(s), q^\mu(t)] = -\partial_t q^\mu(t)\phi_{\alpha,n}(t).
\]

A point in \( J^{\infty} Q \) can be identified with a trajectory in the space of \( V \)-valued functions via generating functions: for \( x = (x^\mu) \in \mathbb{R}^N \), define

\[
\phi_\alpha(x, t) = \sum_{|m| \geq 0} \frac{1}{m!} \phi_{\alpha,m}(t)(x - q(t))^m,
\]

where

\[
(x - q(t))^m = (x^0 - q^0(t))^{m_0}(x^1 - q^1(t))^{m_1} \ldots (x^{N-1} - q^{N-1}(t))^{m_{N-1}}.
\]

\( \phi_\alpha(x, t) \) transforms as in (3.2) under diffeomorphisms and as (4.1) under reparametrizations; note that

\[
\frac{d}{dt} \phi_\alpha(x, t) = \sum_{|m| \geq 0} \frac{1}{m!} (\phi_{\alpha,m}(t)(x - q(t))^m - m_\mu q^\mu(t)\phi_{\alpha,m}(t)(x - q(t))^{m-\mu}).
\]

Moreover, \( \partial_\mu \phi_\alpha(x, t) = \phi_{\alpha,m}(x, t) \). This formula suggests that we define a map

\[
\partial_\mu : \quad J^p Q \rightarrow J^{p+1} Q,
\]

\[
\partial_\mu q^\mu(t) = \delta^\mu, \\
\partial_\mu \phi_{\alpha,n}(t) = \phi_{\alpha,n+\mu}(t).
\]
extended to the whole of \( J^r \mathcal{Q} \) by Leibniz’ rule and linearity. Further, define \( \check{\partial} \) as in (3.1). This operator satisfies

\[
\check{\partial}_\mu L_\xi = L_\xi \check{\partial}_\mu + \partial_\mu \xi^\nu \check{\partial}_\nu, \quad \check{\partial}_\mu L(s) = L(s) \check{\partial}_\mu, \tag{4.7}
\]

when acting on arbitrary functions on \( J^r \mathcal{Q} \).

5. Realization in Fock Space

Consider the symplectic space \( J^r \mathcal{P} \) obtained by adjoining to \( J^r \mathcal{Q} \) dual coordinates (jet momenta) \( (p_\mu(t), \pi^{\alpha,m}(t)) \). The graded Poisson algebra \( C^\infty(J^r \mathcal{P}) \) is the associative, graded commutative algebra on symbols \( (q^\mu(t), \phi_{\alpha,m}(t), p_\mu(t), \pi^{\alpha,m}(t)) \), equipped with a compatible graded Lie structure: the Poisson bracket. The only non-zero brackets are

\[
[p_\mu(s), q^\nu(t)] = \delta^\nu_\mu \delta(s-t), \tag{5.1}
\]
\[
[\pi^{\alpha,m}(s), \phi_{\beta,n}(t)] = \mp[\phi_{\beta,n}(t), \pi^{\alpha,m}(s)] = \delta^m_n \delta^\alpha_\beta \delta(s-t),
\]

where we here and henceforth use the convention that the upper sign refers to bosons and the lower to fermions.

All functions over \( S^1 \) can be expanded in a Fourier series; e.g.

\[
\phi_{\alpha,m}(t) = \sum_{n=-\infty}^{\infty} \hat{\phi}_{\alpha,m}(n)e^{-int} = \phi_{\alpha,m}^<(t) + \phi_{\alpha,m}^>(t), \tag{5.2}
\]

where \( \phi_{\alpha,m}^<(t) \) (\( \phi_{\alpha,m}^>(t) \)) is the sum over negative (positive) frequency modes only. \( \hat{\phi}_{\alpha,m}(0) \) will be referred to as the zero mode. Quantization amounts to replacing the Poisson brackets (5.1) by graded commutators; the Fock space \( J^r \mathcal{F} \) is the universal enveloping algebra modulo relations

\[
q_{\xi}^\mu(t) |0\rangle = p_{\xi}^\mu(t) |0\rangle = \pi_{<}^{\alpha,m}(t) |0\rangle = \phi_{\alpha,m}^<(t) |0\rangle = 0, \tag{5.3}
\]

where \( p_{\mu}(t) = \hat{p}_\mu(0) \) and \( \pi_{<}^{\alpha,m}(t) = \pi_{<}^{\alpha,m}(0) \).

Normal ordering is necessary to remove infinities and to obtain a well defined action on Fock space. Let \( f(q(t), \phi(t)) \) be a function of \( q^\mu(t), \phi(t) \), as well as its derivatives \( \phi_{,m}(t) \), but independent of the canonical momenta. Denote

\[
:f(q(t), \phi(t))p_\mu(t): = f(q(t), \phi(t))p_{\mu}^<(t) + \hat{p}_\mu(t)f(q(t), \phi(t)), \tag{5.4}
\]
\[
:f^{\beta}(q(t))\pi^{\alpha,n}(t)\phi_{\alpha,m}(t): = \pi^{\alpha,n}(t)f^{\beta}(q(t))\phi_{\alpha,m}(t) \mp \phi_{\alpha,m}(t)f^{\beta}(q(t))\pi^{\alpha,n}(t).
\]

In particular,

\[
:\pi^{n}(t) T^{m}_{n}(\xi) \phi_{\alpha,m}(t): = T^{m}_{n}(\xi)f^{\beta}(q(t))\phi_{\alpha,m}(t) \mp \phi_{\alpha,m}(t)f^{\beta}(q(t))\pi^{\alpha,n}(t).
\]

We are now ready to state the main result.
Theorem 1. The following operators provide a realization of DRO(N) on $J^pF$:

\[ L_\xi = \int dt :\xi(t)p_\mu(t): + T(\xi(t), t), \]

\[ T(\xi, t) = \mp \sum_{|m| \leq n \leq p} :\pi^m(t)T^m_n(\xi)\phi_{m}(t):, \]

\[ L(t) = -:\dot{\pi}(t)p_\mu(t): + L'(t), \]

\[ L'(t) = \pm \sum_{|m| \leq p} \left\{ -:\pi^m(t)\phi_{m}(t): + \lambda :\frac{d}{dt}(\pi^m(t)\phi_{m}(t)): \right\} + iw:\pi^m(t)\phi_{m}(t): \pm \left( \frac{N + p}{p} \right) \text{dim}(g) \frac{\lambda - \lambda^2 - w + w^2}{4\pi i}, \]

where the upper sign holds for bosons and the lower sign for fermions. The abelian charges are $c_3 = 1 + c_3'$, $a_3 = 1 + a_3'$, $c_1 = 1 + c_1'$, $c_4 = 2N + c_4'$, where

\[ c_1' = \pm \left( \frac{N + p}{p} \right) k_1(g) + \left( \frac{N + p + 1}{p - 1} \right) \text{dim}(g), \]

\[ c_2 = \pm \left( \frac{N + p}{p} \right) k_2(g) + \left( \frac{N + p}{p - 2} \right) \text{dim}(g) + 2 \left( \frac{N + p}{p - 1} \right) k_0(g), \]

\[ c_3' = \pm (2\lambda - 1) \left( \frac{N + p}{p} \right) k_0(g) + \left( \frac{N + p}{p - 1} \right) \text{dim}(g), \]

\[ a_3' = \pm (2w - 1) \left( \frac{N + p}{p} \right) k_0(g) + \left( \frac{N + p}{p - 1} \right) \text{dim}(g), \]

\[ c_4' = \pm 2(1 - 6\lambda + 6\lambda^2) \left( \frac{N + p}{p} \right) \text{dim}(g). \]

The dimensions $\text{dim}(g), k_0(g), k_1(g)$ and $k_2(g)$ were defined in (5.7) and $\lambda$ and $w$ in (4.1).

From (5.5) we read off the transformation laws for the jet momenta.

\[ [L_\xi, p_\mu(t)] = -\partial_\nu\xi^\mu(t)p_\mu(t) - T(\partial_\nu\xi(t), t), \]

\[ [L(s), p_\mu(t)] = p_\mu(s)\delta(s - t), \]

\[ [L_\xi, \pi^m(t)] = \sum_{|m| \leq n \leq p} \pi^m(t)T^m_n(\xi(t)), \]

\[ [L(s), \pi^m(t)] = -\pi^m(t)\delta(s - t) + (1 - \lambda)\pi^m(t)\delta(s - t) - iw\pi^m(t)\delta(s - t). \]

Note the range of the sum, which depends on the order of the jet. In particular, the top momentum $\pi^m(t)$, $|m| = p$, transforms as a tensor-valued zero-jet.

Without normal ordering, Theorem 1 defines a proper but highly reducible representation of $dif f_f(N)$; in fact, it is a continuous direct sum of $p$-jets, one for each value of the time parameter $t$. This degeneracy is lifted by the introduction...
of the reparametrization algebra. Using (5.3), the \( \text{diff}(N) \) generators can be written as

\[
\mathcal{L}_\xi = \int dt \ \xi^\mu(q(t))(p_\mu(t) \pm \sum_{|m| \leq p} \pi^m(t)\phi_{.m+\mu}(t)) \\
\pm \sum_{|m| \leq p} \pi^m(t)\hat{\delta}_m(\xi^\mu(q(t))\phi_{..\mu} + \partial_\nu \xi^\mu(q(t))\phi(T^\nu_\mu)\phi(t)). \quad (5.8)
\]

All formulas simplify for zero-jets. \( T(\xi, t) = \partial_\nu \xi^\mu T^\nu_\mu(t) \), where \( T^\nu_\mu(t) \) generate the Kac-Moody algebra \( gl(N) \),

\[
[T^\nu_\mu(s), T^\tau_\sigma(t)] = (\delta^\nu_\tau T^\mu_\sigma(s) - \delta^\mu_\sigma T^\nu_\tau(s))\delta(s - t) \\
\pm \frac{1}{2\pi i}(k_1(g)\partial^\tau_\nu\partial^\mu_\sigma + k_2(g)\partial^\mu_\tau\partial^\nu_\sigma)\delta(s - t), \quad (5.9)
\]

\[
[L^\nu_\mu(s), T^\sigma_\nu(t)] = T^\mu_\nu(s)\delta(s - t) \pm \frac{k_0(g)}{4\pi i}\delta^\nu_\sigma(\ddot{\delta}(s - t) + i\ddot{\delta}(s - t)).
\]

It should be stressed that the action in Theorem 1 on \( \mathcal{J}^p\mathcal{F} \) is manifestly well defined, at least for the subalgebra of polynomial vector fields. Namely, a monomial basis for \( \mathcal{J}^p\mathcal{F} \) is given by finite strings in the non-negative modes \( \hat{q}^\mu(n), \hat{p}_\mu(n), \hat{\phi}_{\alpha,m}(n), \hat{\pi}^{\alpha,m}(n), n \geq 0, |m| \leq p, \) and a generic element is a finite linear combination of such monomials. For \( \xi \) a polynomial vector field, finiteness is preserved by (5.7).

Split the delta function into positive and negative frequency parts:

\[
\delta^>(t) = \frac{1}{2\pi} \sum_{m > 0} e^{-imt}, \quad \delta^<(t) = \frac{1}{2\pi} \sum_{m \leq 0} e^{-imt}. \quad (5.10)
\]

**Lemma 2.** (5.4)

i. \( \delta^>(t)\delta^<=(-t) - \delta^>(-t)\delta^<=t) = \frac{1}{2\pi i}\delta(t), \)

ii. \( \delta^>(t)\delta^<=(-t) - \delta^>(-t)\delta^<=t) = \frac{1}{4\pi i}(\dot{\delta}(t) + i\ddot{\delta}(t)), \)

iii. \( \dot{\delta}^>(t)\delta^<=(-t) - \dot{\delta}^>(-t)\delta^<=t) = \frac{1}{12\pi i}(\ddot{\delta}(t) + \dot{\delta}(t)). \)

**Lemma 3.** Let \( \pi^1(s), \phi_B(t), s,t \in S^1 \), generate a graded Heisenberg algebra, with non-zero brackets \( [\pi^1(s), \phi_B(t)] = \delta^A_B\delta(s-t) \). Then

\[
[\pi^A\delta(s), \phi_B(t)] = \delta^A_B\delta^>(s-t), \quad [\phi_B(s), \pi^A\delta(t)] = \mp\delta^A_B\delta^>(t-s), \quad [\pi^A\delta(s), \phi_B(t)] = \delta^A_B\delta^<=s-t), \quad [\phi_B(s), \pi^A\delta(t)] = \mp\delta^A_B\delta^<=t-s).
\]
Lemma 4. Define

\[
F(t) = \mp: \pi^A(t) \dot{\phi}_A(t):, \quad E^A_B(t) = \mp: \pi^A(t) \phi_B(t):
\]  

(5.11)

where \(\pi^A(s)\) and \(\phi_B(t)\), defined as in the previous lemma, carry the same statistics (so \(E^A_B(t)\) is bosonic). Then

\[
[F(s), F(t)] = (F(s) + F(t))\delta(s-t)
\]

\[
\pm\delta^A_1 \frac{1}{4\pi i} (\dot{\delta}(s-t) + \dot{\delta}(s-t)),
\]  

(5.12)

\[
[F(s), E^A_B(t)] = E^A_B(s)\delta(s-t) \mp \delta^A_B \frac{1}{4\pi i} (\dot{\delta}(s-t) + i\dot{\delta}(s-t)),
\]  

(5.13)

\[
[E^A_B(s), E^C_D(t)] = (\delta^C_D E^A_B(s) - \delta^A_D E^C_B(s))\delta(s-t)
\]

\[
\mp \delta^A_D \delta^C_B \frac{1}{2\pi i} \delta(s-t).
\]  

(5.14)

Proof. This lemma follows by direct calculation. The technique is illustrated for (5.13) only,

\[
\begin{align*}
[F(s), E^A_B(t)] &= [\mp \pi^C_D(s) \dot{\phi}_C(s) - \dot{\phi}_C(s) \pi^C_D(s), \mp \pi^A_B(t) \phi_B(t) - \phi_B(t) \pi^A_B(t)] \\
&= \left\{ \pi^A_D(s) \frac{d}{ds} (\mp \delta^>(t-s)) \phi_B(t) \pm \pi^A_D(s) \delta^>(s-t) \dot{\phi}_B(s) \right\} \\
&\quad \pm \left\{ \delta^>(s-t) \pi^A_D(s) \phi_B(t) \mp \pi^A_D(s) \phi_B(t) \frac{d}{ds} (\mp \delta^<(t-s)) \right\} \\
&\quad \pm \left\{ \pm \phi_B(s) \pi^A_D(t) \delta^<(s-t) + \frac{d}{ds} (\mp \delta^>(t-s)) \phi_B(t) \pi^A_D(s) \right\} \\
&\quad \pm \left\{ \dot{\phi}_B(s) \delta^<(s-t) \pi^A_D(t) \pm \phi_B(t) \frac{d}{ds} (\mp \delta^<(t-s)) \pi^A_D(s) \right\} \\
&= \mp \pi^A_D(s) \phi_B(t) \frac{d}{ds} \delta(t-s) \pm \pi^A_D(t) \dot{\phi}_B(s) \delta(s-t)
\end{align*}
\]  

(5.15)

The result now follows by collecting terms and applying Lemma 3 to obtain the central extension. \(\Box\)

Proof of Theorem 1. First we note that in proving the brackets with \(q^\mu(t)\), normal ordering is irrelevant because \(L_\xi\) is linear in \(p_\mu(t)\). This part is straightforward and not given here.
We now apply Lemma 2 and integrate by parts, which yields

\[ [\mathcal{L}_\xi^0, \mathcal{L}_\eta^0] = \int ds dt \left[ \xi^\mu(s)p_\mu^\nu(s) + p_\nu^\mu(s)\xi^\nu(s) + \eta^\nu(t)p_\mu^\nu(t) + p_\nu^\mu(t)\eta^\nu(t) \right] \]

\[ = \int ds dt \left[ \xi^\mu(s)(\partial_\mu\eta^\nu(t)\delta^\xi(s-t))p_\nu^\xi(t) + \eta^\nu(t)(-\partial_\nu\xi^\mu(s)\delta^\xi(t-s))p_\mu^\xi(s) + \xi^\mu(s)p_\nu^\mu(t)(\partial_\nu\eta^\xi(t)\delta^\xi(s-t)) + (-\partial_\nu\xi^\mu(s)\delta^\xi(t-s))\eta^\xi(t)p_\mu^\nu(s) + (\partial_\nu\eta^\xi(t)\delta^\xi(s-t))\eta^\xi(t)p_\mu^\nu(s) + \eta^\nu(t)(-\partial_\nu\xi^\mu(s)\delta^\xi(t-s))\eta^\nu(t) + (\partial_\nu\eta^\xi(t)\delta^\xi(s-t))\eta^\xi(t)p_\mu^\nu(s) \right] \]

\[ = \int ds dt \left[ \xi^\mu(s)(\partial_\mu\eta^\nu(t)\delta^\xi(s-t)) + \eta^\nu(t)(-\partial_\nu\xi^\mu(s)\delta^\xi(t-s)) - \eta^\nu(t)\partial_\nu\xi^\mu(s)\delta^\xi(s-t) - \partial_\nu\eta^\xi(t)(\partial_\nu\xi^\mu(s)\delta^\xi(s-t) - \delta^\xi(t-s)\delta^\xi(s-t)) \delta^\xi(t-s) \right] \]

\[ + \partial_\nu\eta^\xi(t)(\partial_\nu\xi^\mu(s)\delta^\xi(s-t) - \delta^\xi(t-s)\delta^\xi(s-t)) \delta^\xi(t-s) \]

We now apply Lemma 2 and integrate by parts, which yields

\[ [\mathcal{L}_\xi^0, \mathcal{L}_\eta^0] = \mathcal{L}_\xi^0[\xi, \eta] + \frac{1}{2\pi i} \int ds \partial_\nu(\xi^\mu(s)\partial_\mu\eta^\nu(s)) \]

\[ = \mathcal{L}_\xi^0[\xi, \eta] + \mathcal{S}_\xi^0(\partial_\nu\partial_\mu\xi^\nu\partial_\mu\eta^\nu). \]

Let capital indices run over both tensor indices and multi-indices, e.g. \( A = (\alpha, m) \), \( \pi^A(s) = \pi^{\alpha m}(s) \), \( \phi_B(t) = \phi_{\beta, m}(t) \). Now, \( \mathcal{L}_\xi = \mathcal{L}_\xi^0 + \int dt \left\{ T(\xi(q(t)), t), T(\eta(q(t)), t) \right\} \]

where \( T(\xi, t) = T^A_B(\xi)E^A_B(t) \) in an obvious notation. \( E^A_B(t) \) is defined in (5.11), the matrices \( T^A_B(\xi) \) satisfy the relations (3.6) and Lemma 3, with

\[ \delta_A^m = \sum_{|m| \leq P} \delta_m^m, \]

\[ T^A_B(\xi) = \sum_{|m| \leq P} \text{tr} T^m_B(\xi), \]

\[ T^A_B(\xi)T^B_A(\eta) = \sum_{|m| \leq P} \text{tr} T^m_B(\xi)T^m_A(\eta). \]

It follows from (5.14) that

\[ [T(\xi(s), s), T(\eta(t), t)] \]

\[ = (T^A_B(\xi(s))T^B_A(\eta(t)) - T^B_A(\xi(s))T^A_B(\eta(t)))E^A_B(s)\delta(s-t) \]

\[ + \frac{1}{2\pi i} T^A_B(\xi(s))T^B_A(\eta(t))\delta(s-t) \]

\[ = (T(\xi(s), \eta(t)), s) - \xi^\mu(s)T(\partial_\mu\eta(t), s) + \eta^\nu(t)T(\partial_\nu\xi(s), s)\delta(s-t) \]

\[ - \frac{1}{2\pi i} (c_1^0(\partial_\mu\xi^\nu(s)\partial_\mu\eta^\nu(t) + c_2^0(\partial_\nu\xi^\mu(s)\partial_\nu\eta^\nu(t))\delta(s-t), \eta(t)). \]
where the parameters $c'_1$ and $c_2$ can now be computed from Lemma 3, with the result \( \tilde{5} \). Further,

\[
[L_0^i, T(\eta(t), t)] = \xi^\mu(t) T(\partial_\mu \eta(t), t),
\]

without extension. This concludes the proof for the \( \text{diff}(N) \) subalgebra.

Next we turn to reparametrizations. They generate a Virasoro algebra with central charge $c$, which may be written as

\[
[L(s), L(t)] = (L(s) + L(t)) \delta(s - t) + \frac{c}{24\pi i} (\delta(s - t) + \delta(t - s)). \quad (5.20)
\]

Set $L^0(s) = -:q^\mu(s)p_\mu(s):$. This is recognized as being of the same form as $F(s)$ in Lemma 3, with $N$ bosonic fields $q^\mu(s)$, and thus they generate a Virasoro algebra with central charge $2N$. Set $L(t) = L^0(t) + L'(t)$, where

\[
L'(t) = F(t) - \lambda E_A(t) - i\omega E_A(t) + \delta A \frac{\lambda - \lambda^2 - w + w^2}{4\pi i}. \quad (5.21)
\]

By Lemma 3, these operators generate a Virasoro algebra with central charge \( \pm 2(1 - 6\lambda + 6\lambda^2)\delta A \). Moreover, \([L^0(s), L'(t)] = 0\) and the parameter $c'$ in (5.6) follows from Lemma 3.

Finally, we want to prove that

\[
[L(s), L^i] = \frac{1}{4\pi i} (c_2 \partial_\mu \xi^\mu(s) + ia_3 \partial_\mu \xi^\mu(s)). \quad (5.22)
\]
\[ [L^0(s), L^0_{\xi}] = -\int dt \left[ \dot{q}^\mu(s)p^\mu_\nu(s) + p^\mu_\nu(s)\dot{q}^\mu(s) - \xi^\nu(t)p^\nu_{\Sigma}(t) + p^\nu_{\Sigma}(t)\xi^\nu(t) \right] \]

\[ = -\int dt \dot{q}^\mu(s)(\partial_\mu \xi^\nu(t)\delta^\nu(s-t))p^\nu_{\Sigma}(t) - \xi^\nu(t)\frac{d}{ds}(\delta^\nu_\nu \delta^\mu(t-s))p^\mu_\nu(s) \]

\[ + \dot{q}^\mu(s)p^\nu_{\Sigma}(t)(\partial_\mu \xi^\nu(t)\delta^\nu(s-t)) - \frac{d}{ds}(\delta^\mu_\mu \delta^\nu(t-s))\xi^\nu(t)p^\nu_{\Sigma}(s) \]

\[ + (\partial_\mu \xi^\nu(t)\delta^\nu(s-t))p^\nu_{\Sigma}(t)\dot{q}^\mu(s) - p^\nu_{\Sigma}(s)\xi^\nu(t)\frac{d}{ds}(\delta^\nu_\nu \delta^\mu(t-s)) \]

\[ + p^\nu_{\Sigma}(t)(\partial_\mu \xi^\nu(t)\delta^\nu(s-t))\dot{q}^\mu(s) - p^\nu_{\Sigma}(s)\frac{d}{ds}(\delta^\nu_\nu \delta^\mu(t-s))\xi^\nu(t) \]  \hspace{1cm} (5.23)

\[ = -\int dt \dot{q}^\mu(s)\partial_\mu \xi^\nu(t)p^\nu_{\Sigma}(t)\delta(s-t) + \partial_\mu \xi^\nu(t)\delta^\nu(s-t) \]

\[ - \xi^\nu(t)p^\nu_{\Sigma}(s)\frac{d}{ds}\delta(t-s) + p^\nu_{\Sigma}(t)\dot{q}^\mu(s)\partial_\mu \xi^\nu(t)\delta(s-t) \]

\[ - \frac{d}{ds}\delta^\nu(t-s)\partial_\mu \xi^\nu(t)\delta^\mu(s-t) - p^\nu_{\Sigma}(s)\xi^\nu(t)\frac{d}{ds}\delta(t-s) \]

\[ = \int dt - \dot{q}^\mu(s)\partial_\mu \xi^\nu(t)p^\nu_{\Sigma}(t):\delta(s-t) + :\xi^\nu(t)p^\nu_{\Sigma}(s):\delta(s-t) \]

\[ + \partial_\mu \xi^\nu(t)\delta^\nu(s-t)\delta^\mu(t-s) - \delta^\nu(t-s)\delta^\mu(s-t) \]

\[ = \frac{1}{4\pi i}(\partial_\mu \xi^\nu(s) + i\partial_\mu \xi^\nu(s)), \]

\[ [L^0(s), T(\xi(t), t)] = -\dot{q}^\mu(s)T(\partial_\mu \xi(t), t)\delta(s-t), \]  \hspace{1cm} (5.24)

\[ [L^0(s), T(\xi(t), t)] = T(\xi(t), s)\delta(s-t) \]

\[ \pm \frac{1}{4\pi i}T^A(\xi(t))((2\lambda - 1)\delta(s-t) + (2w - 1)\delta(s-t)). \]  \hspace{1cm} (5.25)

To compute \([L(s), \int dt T(\xi(t), t)],\) we note that the regular pieces from (5.24) and (5.25) cancel, whereas the extension acquires the form (5.22). The parameters \(c_3\) and \(a_3^\nu\) now follows from Lemma [1]. \(\Box\)

The Fock module described in Theorem [1] is reducible, because it can be decomposed according to the number of \(\phi^\nu\)'s, the canonical momenta counting negative. If there are several independent field species, a finer decomposition is possible. An alternative way to see this is as follows.

Let us refer to \(\phi^m(n)\) and \(\pi^m(n)\) as phase space modes of frequency \(n\). The reparametrization generators can be split as \(L(s) = L^\phi(s) + L^\pi(s)\), where the raising operators \(L^\phi(s)\) consist of Fourier modes of non-negative frequency (as measured by the Hamiltonian (2.3)), and the lowering operators \(L^\pi(s)\) consist of negative ones. Clearly, every lowering operator contains at least one negative frequency phase space mode. Because all expressions are normal ordered, lowering operators thus annihilate the vacuum. A similar decomposition should be applied to \(L^\xi = L^\phi_\xi + L^\pi_\xi\), but since \([L(s), L^\xi_\xi] = 0\) classically, there are no such lowering operators.
Define a cyclic state $|\emptyset\rangle$ to be a state annihilated by all lowering operators:

$$L^<(s)|\emptyset\rangle = 0.$$  

(5.26)

As is well known, an irreducible representation contains only one cyclic state. Since the vacuum $|0\rangle$ is cyclic, the existence of additional cyclic states signals reducibility. The following theorem describes some cyclic states and their energies, but no claim is made that the list is exhaustive.

**Theorem 2.** The lowest energy (5.24) of the Fock representation in Theorem 1 is

$$h = \mp \frac{1}{2} \left( N + \frac{p}{2} \right) \text{dim}(\rho)(w - \frac{1}{2})^2 - (\lambda - \frac{1}{2})^2).$$  

(5.27)

For a scalar bosonic zero-jet, the state $|n\rangle = (\hat{\phi}(0))^n|0\rangle$, $n \geq 0$, is cyclic with energy $h(n) = h + nw$. For fermionic tensor-valued $p$-jets, set

$$\hat{\Xi}_\ell(n) = \prod_\alpha \prod_{|m| \leq \ell} \hat{\phi}_{\alpha,m}(n),$$  

$$\hat{\Xi}_{-\ell-1}(n) = \prod_\alpha \prod_{\ell \leq |m| \leq p} \hat{\pi}_{\alpha,m}(n)$$  

(5.28)

($\ell \geq 0$), where the products run over all components. The states

$$|k,\ell\rangle = \hat{\Xi}_\ell(k-1)\ldots\hat{\Xi}_1(0)|0\rangle,$$

$$|k,-\ell-1\rangle = \hat{\Xi}_{-\ell-1}(k-1)\ldots\hat{\Xi}_{-1}(0)|0\rangle,$$

(5.29)

are cyclic, with energy

$$h(k,\ell) = h + \frac{1}{2} \left( N + \ell \right) \text{dim}(\rho)(k^2 + (2w - 1)k),$$  

(5.30)

$$h(k,-\ell-1) = h + \frac{1}{2} \left\{ \left( N + \frac{p}{2} \right) - \left( N + \ell - 1 \right) \right\} \text{dim}(\rho)(k^2 - (2w + 1)k).$$

Proof. Set $\hat{L}(m) = -i \int ds e^{ims} L(s)$. Then the Virasoro algebra takes the form

$$[\hat{L}(m), \hat{\phi}_{\alpha,m}(n)] = (n - m)\hat{\phi}_{\alpha,m}(n + m) - \frac{c}{12}(m^3 - m)\delta(m + n),$$

$$[\hat{L}(m), \hat{\phi}_{\alpha,m}(n)] = (n + (1 - \lambda)m + w)\hat{\phi}_{\alpha,m}(m + n),$$

$$[\hat{L}(m), \hat{\pi}_{\alpha,m}(n)] = (n - \lambda m - w)\hat{\pi}_{\alpha,m}(m + n),$$  

(5.31)

and the Hamiltonian $H = \hat{L}(0)$ (2.8). The action on the vacuum is (excluding the observer)

$$\hat{L}'(m)|0\rangle = \pm \sum_{n=0}^{m-1} \sum_{|m| \leq p} (n - \lambda m + w)\hat{\pi}_{\alpha,m}(m - n)\hat{\phi}_{\alpha,m}(n)|0\rangle.$$  

(5.32)
To compute parameters, note that
\[ [\hat{L}'(m), \hat{L}'(-m)][0] = (-\frac{c'}{12}m^3 + (\frac{c'}{12} - 2h)m)[0]. \]

A straightforward calculation shows that \( c' \) is given by (5.6) and \( h \) by (5.27).

The property that \( \phi(t) \) is a scalar-valued zero-jet is preserved by \( L_\xi \) and \( L(s) \). Moreover, any lowering operator gives negative-frequency phase space modes when acting on a zero mode, and hence the state is cyclic. The energy follows from \[ [\hat{L}(0), \hat{\phi}(0)] = w\hat{\phi}(0). \]

Now consider fermions and \( \ell > 0 \). When \( L_\xi \) acts on \( \hat{\Xi}_\ell(n) \), jets of order \( |m| \leq \ell \) are produced, but no higher-order jets. Also, \( L(s) \) preserves jet order. When acting on \( \Xi_\ell(n) \), a lowering operator produces a sum of terms, each containing at least one phase space mode with frequency less than \( n \), and jet order at most \( \ell \). However, the state \( |k, \ell\rangle \) is the product of all such modes, so the fermionic property makes all these terms vanish. Hence \( |k, \ell\rangle \) is cyclic. The energy \( h(k, \ell) \) follows from the following calculation and Lemma [4]:

\[ \langle k, \ell \rangle \] follows from the calculation and Lemma [4]:

\[ [\hat{L}(0), \hat{\phi}_{m}(n)] = (n + w)\phi_{m}(n), \]
\[ [\hat{L}(0), \hat{\Xi}_\ell(n)] = \sum_{|m| \leq \ell} (n + w)\hat{\Xi}_{\ell}(n) \]
\[ = (n + w)\binom{N + \ell}{\ell} \dim(g) \hat{\Xi}_{\ell}(n), \]
\[ \hat{L}(0)|k, \ell\rangle = (h + \binom{N + \ell}{\ell} \dim(g) \sum_{n=0}^{k-1} (n + w))|k, \ell\rangle. \]

The case \( \ell < 0 \) is completely analogous, except that \( L_\xi \) increases the jet order. \( \Box \)

6. Gauge Algebra

Consider the gauge algebra map\((N, g)\), i.e. maps from \( N \)-dimensional spacetime to a finite-dimensional Lie algebra \( g \), where \( g \) has basis \( J^a \) (hermitian if \( g \) is compact and semisimple), structure constants \( f^{abc} \), and Killing metric \( \delta^{ab} \). The brackets are
\[ [J^a, J^b] = if^{abc}J^c. \] (6.1)

Let \( \delta^a \propto \text{tr} J^a \) be a priviledged vector satisfying \( f^{abc} \delta^c = 0 \). Clearly, \( \delta^a = 0 \) if \( J^a \in [g, g] \), but it may be non-zero on abelian factors. The primary example is \( gl(d) \), where \( \text{tr} J^i_\mu \propto \delta^i_\mu \). Our notation is similar to [11].

Let \( X = X_\alpha(x)J^\alpha, \ x \in \mathbb{R}^N, \) be a \( g \)-valued function and define \( [X, Y] = if^{abc}X_\alpha Y_\beta J^c \). The generators of \( map(N, g) \) are denoted by \( J_X \). The DGRO (Diffeomorphism, Gauge, Reparametrization, Observer) algebra DGRO\((N, g)\) has
The expression for the matrices $J$ immediately from (2.3).

In addition to (2.3), we recognize the cocycle proportional to $a_6$ can be removed by the redefinition

$$[J_X, J_Y] = J_{[X,Y]} - (c_5 \delta^{ab} + c_6 \delta^a \delta^b) S_0^0(\partial_{\mu} X_a Y_b),$$

$$[L_f, J_X] = \frac{1}{2} \delta^a S_0 ((c_6 \tilde{f} - i a_6 \tilde{f}) X_a),$$

$$[\mathcal{L}_{\xi}, J_X] = J_{\xi^\mu \partial_\mu X} - c_7 \delta^a S_0^0(X_a \partial_\mu \partial_\mu \xi^\mu),$$

$$[J_X, S_0(F)] = [J_X, S_0'(F_\rho)] = 0,$$

in addition to (2.3). The cocycle proportional to $a_6$ can be removed by the redefinition

$$J_X \rightarrow J_X + \frac{i a_6}{2} \delta^a S_0(X_a),$$

($\delta^a [X,Y]_{a} = 0$), while the remaining terms define non-trivial extensions. In particular, we recognize the $c_5$ term as the higher-dimensional generalization of the affine Kac-Moody algebra $\tilde{g}$. The present notation has the advantage that all abelian charges $c_j, j = 1, \ldots, 8$, can be discussed collectively.

Let $M$ be a $\mathfrak{g}$ representation. We write $T^\mu_\nu = T^\mu_\nu \oplus 1, J^a = 1 \oplus J^a, 1 = 1 \oplus 1$ for elements in $gl(N) \oplus \mathfrak{g}$, and abbreviate $M^a = M(1 \oplus J^a)$. map($N, \mathfrak{g}$) acts on $J^\rho \mathcal{Q}$ and $J^\rho \mathcal{P}$ in the following fashion ($V$ indices suppressed):

$$[J_X, \phi_{m}(t)] = \sum_{|n| \leq |m|} J^m_n (X(q(t))) \phi_{m}(t),$$

$$[J_X, \pi^m(t)] = \sum_{|n| \leq |m| \leq |p|} \pi^m(t) J^m_n (X(q(t))),$$

$$J^m_n (X) \equiv \left( \begin{array}{c} n \\ m \end{array} \right) \partial_{n-m} X_a M^a,$$

$$[J_X, q^\mu(t)] = [J_X, p^\nu(t)] = 0.$$

The expression for the matrices $J^m_n (X)$, with components $J^m_n (X)^\mu_\nu$, follows immediately from

$$[J_X, \phi_{m}(t)] = \partial_n (-X_a(q(t)) M^a \phi(q(t))).$$
They satisfy the following relations:

\[
\begin{align*}
J^m_n(X) &= J^m_n(\partial_\mu X) + J^m_n(\partial_{\mu}^\sim X), \\
J^0_n(X) &= \delta^m_n X_{a} M^a, \\
\partial_\mu J^m_n(X) &= J^m_n(\partial_\mu X), \\
J^m_n([X,Y]) &= \sum_{|m| \leq |r| \leq |n|} J^r_n(X) J^m_n(Y) - J^r_n(Y) J^m_n(X), \\
J^m_n(\xi^{\mu} \partial_\mu X) &= \xi^{\mu} J^m_n(\partial_\mu X) + \sum_{|m| \leq |r| \leq |n|} T^r_n(\xi) J^m_n(X) - J^r_n(X) T^m_n(\xi).
\end{align*}
\]

(6.5)

In particular, \( J^m_n(X) = 0 \) if \(|m| > |n|\) and \( J^m_n(X) = X_{a} M^a \delta^m_n \) if \(|m| = |n|\).

Set \( \text{tr} M^a = z_M \delta^a \) and \( \text{tr} M^a M^b = y_M \delta^{ab} + w_M \delta^a \delta^b \). For \( \mathfrak{g} \) semisimple, \( w_M = z_M = 0 \) and \( y_M = \psi^2 x_M \), where \( \psi \) is the highest root of \( \mathfrak{g} \) and \( x_M \) is a positive integer (the Dynkin index of the \( \mathfrak{g} \) representation \( M \)) \cite{10}. The analog of Lemma 1 is

**Lemma 5.**

\[
\begin{align*}
i. \sum_{|m| \leq |n| \leq p} \text{tr} J^m_n(X) &= X_{a} z_M \delta^a \left( \frac{N + p}{p} \right) \dim(\mathfrak{g}), \\
ii. \sum_{|m| \leq |n| \leq p} \text{tr} J^m_n(X) J^m_n(Y) &= (y_M \delta^{ab} + w_M \delta^a \delta^b) \left( \frac{N + p}{p} \right) X_{a} Y_{b}, \\
iii. \sum_{|m| \leq |n| \leq p} \text{tr} T^m_n(\xi) J^m_n(X) &= \partial_\mu \xi^{\mu} X_{a} z_M \delta^a \left( \frac{N + p}{p} \right) k_0(\mathfrak{g}) + \left( \frac{N + p}{p} - 1 \right) \dim(\mathfrak{g}).
\end{align*}
\]

**Proof.** As in Lemma 1, only terms with \(|m| = |n|\) contribute to the sums, and we can hence think of \( J^m_n(X) \) and \( T^m_n(\xi) \) as representation matrices in \( \mathfrak{g} \otimes M \otimes S_{k} \).

Hence

\[
\begin{align*}
i. &= \sum_{\ell=0}^{p} \text{tr} M^a \dim(S_{\ell}) \dim(\mathfrak{g}), \\
ii. &= \sum_{\ell=0}^{p} \text{tr} M^a M^b X_{a} Y_{b} \dim(S_{\ell}) \dim(\mathfrak{g}), \\
iii. &= \sum_{\ell=0}^{p} X_{a} \text{tr} M^a \partial_\mu \xi^{\mu}(k_0(\mathfrak{g}) \dim(\mathfrak{g}) + \dim(\mathfrak{g}) k_0(S_{\ell})).
\end{align*}
\]

We now apply (3.12) and use the definition of \( y_M, z_M \) and \( w_M \). \( \square \)
Theorem 3. The following operators, together with the operators in Theorem 1, yield a realization of the algebra \( DGRO(N, \mathfrak{g}) \) on the Fock space \( \mathcal{F}^p \),

\[
\mathcal{J}_X = \int dt \, J(X(q(t)), t),
\]

\[
J(X, t) = \pm \sum_{|m| \leq |n| \leq p} : x^n(t) J^n_m(X) \phi_{m^*}(t). \tag{6.6}
\]

The parameters are

\[
c_5 = \mp y_M \binom{N + p}{p} \dim(\mathfrak{g}),
\]

\[
c_6 = \pm z_M \delta^a (2\lambda - 1) \binom{N + p}{p} \dim(\mathfrak{g}),
\]

\[
a_6 = \pm z_M \delta^a (2w - 1) \binom{N + p}{p} \dim(\mathfrak{g}), \tag{6.7}
\]

\[
c_7 = \mp z_M \delta^a \left( \binom{N + p}{p} k_0(\mathfrak{g}) + \binom{N + p}{p - 1} \dim(\mathfrak{g}) \right),
\]

\[
c_8 = \mp w_M \binom{N + p}{p} \dim(\mathfrak{g}).
\]

Proof. We use the same notation as in the proof of Theorem 1. In particular, capital indices \( A = (\alpha, \mathbf{m}) \) run over both internal and multi-indices, and we write \( X(s) = X(q(s)) \), etc. Equation (6.6) can be written as \( J(X, s) = J^n_A(X) E^A_B(s) \), where \( J^A_B \) satisfy relations (6.5) and \( E^A_B(s) \) is as in lemma 4. The following
formulas follow immediately from \((5.13)\) and \((5.14)\),
\[
\begin{align*}
[J(X(s), s), J(Y(t), t)] &= J_B^A(X(s))J_C^D(Y(t)) \times \\
&\times ((\delta_E^G E_B^A(s) - \delta_D^A E_B^G(s))\delta(s - t) \mp \frac{1}{2\pi i} \delta_B^A \delta_E^G \delta(s - t)) \\
&= J([X, Y](s), s)\delta(s - t) \mp \frac{1}{2\pi i} J_B^A(X(s))J_C^D(Y(t))\delta(s - t),
\end{align*}
\]
\[
\begin{align*}
[L_0^0, J(X(t), t)] &= \xi^\mu(t) J(\partial_\mu X(t), t), \\
[T(\xi(s), s), J(X(t), t)] &= T_B^A(\xi(s))J_C^D(X(t)) \times \\
&\times ((\delta_E^G E_B^A(s) - \delta_D^A E_B^G(s))\delta(s - t) \mp \frac{1}{2\pi i} \delta_B^A \delta_E^G \delta(s - t)) \\
&= (J(\xi^\mu(s)\partial_\mu X(t), s) - \xi^\mu(t) J(\partial_\mu X(t), s))\delta(s - t) \\
&\mp \frac{1}{2\pi i} T_B^A(\xi(s))J_C^D(X(t))\delta(s - t),
\end{align*}
\]
\[
\begin{align*}
[L^0(s), J(X(t), t)] &= -q^a(s) J(\partial_\mu X(t), t)\delta(s - t), \\
[L'(s), J(X(t), t)] &= J_B^A(X(t))(E_B^A(s)\dot{\delta}(s - t) \\
&\pm \frac{1}{4\pi i} \delta_B^A((2\lambda - 1)\delta(s - t) + (2w - 1)i\delta(s - t)) \\
&= J(X(t), s)\delta(s - t) \pm \frac{1}{4\pi i} J_B^A(X(t)||(2\lambda - 1)\dot{\delta}(s - t) + (2w - 1)i\dot{\delta}(s - t)).
\end{align*}
\]

We now collect terms, integrate over \(t\), and find that the regular terms give the proper algebra, while Lemma 5 give the extension parameters. \(\square\)

Since \(c_5\) must be positive in a unitary representation, the bosonic Fock space carries a non-unitary representation.

In analogy with \((5.8)\), we can write
\[
J(X, t) = \mp \sum_{|m| < p} :\pi^n m(t)\tilde{\partial}_m(X_a(q(t))M^a\phi(t)):
\]
A slight generalization is possible. The gauge connection corresponds to the jet \(A^a_{\mu, m}(t)\) with conjugate momentum \(E_a^m(t)\). \(\map(N, g)\) acts as
\[
\begin{align*}
[J_X, A^a_{\nu, n}(t)] &= -\sum_{|m| < p} J^m_n(X(q(t)))A^a_{\nu, m}(t) + \partial_{n+\nu}X^a(q(t)), \quad (6.9)
\end{align*}
\]
where the matrices $J_m^n(X)$ are taken in the adjoint representation of $g$, i.e. $(M^a)^b_c = -if^{a}_{bc}$. Thus the contribution to (6.4) is

$$J(X, t) = \sum_{|m| \leq p} \left\{ E^m_n(t) \partial_m (if^{a}_{bc} X_a A^c_{\mu, m}(t)) : + \partial_m + \partial_\mu X^a E^m_{a, \mu}(t) \right\}.$$ (6.10)

Due to the non-homogeneous term in (6.9), the Fock space does not decompose into subspaces with a fixed number of $A$'s as a map $(N, g)$ module.

7. Constraints

Representations of $DRO(N)$ can be restricted to $diff(N)$ using techniques from constrained Hamiltonian systems [3,11]. The same mechanism has appeared in mathematics under the name Drinfeld-Sokolov reduction [4].

The space $J^{P\bar{P}}$ is equipped with a natural graded symplectic structure, and it can therefore be viewed as a classical phase space. Let $P, R, ...$ label bosonic constraints $\chi_P(q, p, \phi, \pi)$. If $DRO(N)$ acts in the phase space such that all constraints are preserved, we may consider the restriction to the constraint surface $\chi_P \approx 0$. Weak equality (i.e. equality modulo constraints) is denoted by $\approx$. Constraints are classified as second or first class depending on whether the Poisson bracket matrix $C_{PR} = [\chi_P, \chi_R]$ is invertible or not. First class constraints are connected to gauge symmetries and they always generate a Lie algebra. However, it is often possible to go from first class to second class (by fixing a gauge) and back (by dropping half the constraints).

Assume that all constraints are second class, if necessary by adding gauge-fixing conditions. Then the matrix $C_{PR}$ has an inverse, denoted by $\Delta^{P\bar{R}}$. $\Delta^{P\bar{R}} C_{RS} = \delta^P_S$. The Dirac bracket

$$[A, B]^* = [A, B] - [A, \chi_P] \Delta^{P\bar{R}} [\chi_R, B]$$ (7.1)

defines a new Lie bracket which is compatible with the constraints: $[A, \chi_R]^* = 0$ for every $A \in C^\infty(J^{P\bar{P}})$.

Reparametrizations generate a Lie algebra and can hence be viewed as first class constraints. A natural gauge choice is to identify one coordinate with the time parameter. Thus, our constraints are

$$L(t) \approx 0, \quad q^0(t) - t \approx 0.$$ (7.3)

The Poisson bracket matrix $C(s, t)$ and its inverse $\Delta(s, t)$ are, on the constraint surface,

$$C(s, t) \equiv [\chi(s), \chi^T(t)] = \left[ \left( q^0(s) - s \frac{L(s)}{L(t)} \right), (q^0(t) - t L(t)) \right]$$

$$\approx \begin{pmatrix} 0 & \delta(s - t) \\ -\delta(s - t) & \frac{\epsilon^s}{2\pi} \left( \frac{\delta(s - t)}{\delta(s - t)} + \delta(s - t) \right) \end{pmatrix},$$

$$\Delta(s, t) \approx \begin{pmatrix} 0 & \frac{\epsilon^s}{2\pi} \left( \frac{\delta(s - t)}{\delta(s - t)} + \delta(s - t) \right) - \delta(s - t) \\ \frac{\epsilon^s}{2\pi} \left( \frac{\delta(s - t)}{\delta(s - t)} + \delta(s - t) \right) & 0 \end{pmatrix}.$$ (7.2)

We now solve the constraints,

$$q^0(t) = t, \quad p_0(t) = -q^i(t)p_i(t) + L'(t),$$ (7.3)
where the latin index $i = 1, 2, \ldots N - 1$ range over the remaining (“spatial”) directions. If $\mathcal{L}_\xi$ satisfy the DRO algebra (2.3) under the original bracket, the Dirac brackets become

$$
[\mathcal{L}_\xi, \mathcal{L}_\eta]^* = \mathcal{L}_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \ c_1 \partial_\eta \dot{\xi}^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) +
+ c_2 \partial_\mu \dot{\xi}^\mu(q(t)) \partial_\nu \eta^\nu(q(t)) +
+ \frac{1}{4\pi i} \int dt \ c_3 (\partial_\eta \eta^\nu(q(t)) \xi^\nu(q(t)) - \partial_\nu \xi^\nu(q(t)) \eta^\nu(q(t))) -
- ia_3 (\partial_\eta \eta^\nu(q(t)) \xi^0(q(t)) - \partial_\nu \xi^\nu(q(t)) \eta^0(q(t))) +
+ \frac{c_4}{24\pi i} \int dt \ \xi^0(q(t)) \eta^0(q(t)) - \xi^0(q(t)) \eta^0(q(t)),
$$

(7.4)

where $\dot{\xi}(q(t)) = \eta^\mu(q(t)) - \eta^\mu(q(t)) \xi^0(q(t))$, $[\eta^\mu(s), \eta^\nu(t)]^* = 0$, $[L(s), \mathcal{L}_\xi]^* = [L(s), L(t)]^* = [L(s), \eta^\mu(t)]^* = 0$, and

$$
\dot{f}(q(t)) = \eta^\mu(t) \partial_\nu f(q(t)) + \eta^\nu(t) \partial_\mu(f(q(t)) + \eta^\nu(t) \partial_\mu f(q(t)).
$$

(7.5)

Some other Dirac brackets are

$$
[p_\mu(s), p_\nu(t)]^* = (\delta^\mu_\nu p_\nu(s) + \delta^\nu_\mu p_\mu(t)) \delta(s - t),
$$

$$
[p_\mu(s), \eta^\nu(t)]^* = (\delta^\nu_\mu - \eta^\mu(t) \delta^\nu_\mu) \delta(s - t),
$$

(7.6)

$$
[p_\mu(s), T(\xi(q(t)), t)]^* = T(\partial_\mu \xi(q(s)), s) \delta(s - t) + \delta^\mu_\nu T(\xi(q(s)), s) \delta(s - t).
$$

Note that $[\mathcal{L}_\xi, \eta^0(t)]^* = 0$.

Equation (7.4) is the four-parameter extension of $diff(N)$ found in [14]; it was denoted by $diff(N; c_1, c_2, c_3, c_4)$ in that paper. The parameters $c_1$ and $c_2$ are the same as in that paper, but I have interchanged the names of the other two: $c_{10}^{old} = 12c_{10}^{new}$ and $c_4^{old} = c_4^{new}$. Note that two of the cocycles are anisotropic in the sense that they single out the $x^0$ direction. This anisotropy originates from the gauge choice $q^0(t) \approx t$. I expect other gauge choices to give rise to even more complicated cocycles. Therefore, it is natural to work with the full DRO algebra, where the cocycles are of the simple Virasoro form.

Substitution of (7.4) into (2.3) gives

$$
\mathcal{L}_\xi = \int dt \ :\xi^\mu(q(t))p_\mu(t): - :\xi^0(q(t))q^0(t)p_\mu(t):
+ \xi^0(q(t))L'(t) + T(\xi(q(t)), t),
$$

(7.7)

which is the realization found in [14]. These generators thus provide an explicit realization of the gauge-fixed algebra (7.4). In particular, the Dirac brackets agree with the original brackets since $\eta^0(t)$ and $p_0(t)$ have been eliminated.

We can recast (7.4) as a proper Lie algebra analogous to (2.6). However, this algebra acquires a very complicated form, due to the second-order derivatives in (7.4). Not only do the operators $S_0(\delta_0)$ and $S_0^\nu(F_\nu)$ enter, but two infinite families of linear operators $S_\nu^{t_1 \cdots t_n}(F_{t_1 \cdots t_n})$, $P^{t_1 \cdots t_n}(G_{t_1 \cdots t_n})$, where $F_{t_1 \cdots t_n}(t, x)$,
$G_{\rho|\nu_1..\nu_n}(t,x), t \in S^1, x \in \mathbb{R}^N,$ are arbitrary functions, totally symmetric in the indices $\nu_1..\nu_n$. They have the explicit realization

$$S_{\nu_1..\nu_n}^{\nu_1..\nu_n}(F_{\nu_1..\nu_n}) = \frac{1}{2\pi i} \int dt \dot{q}^{\nu_1}(t) \cdots \dot{q}^{\nu_n}(t) F_{\nu_1..\nu_n}(t, q(t)), \quad (7.8)$$

$$R_{\nu_1..\nu_n}^{\rho|\nu_1..\nu_n}(G_{\rho|\nu_1..\nu_n}) = \frac{1}{2\pi i} \int dt \ddot{q}^{\rho}(t) \dot{q}^{\nu_1}(t) \cdots \dot{q}^{\nu_n}(t) G_{\rho|\nu_1..\nu_n}(t, q(t)).$$

The resulting algebra was written down in [14].

The gauge algebra map$(N, g)$ is reduced along similar lines. Since $J_X$ commutes with both $L(s)$ and $q^0(t)$ (before normal ordering), the gauge-fixed realization of map$(N, g)$ is simply obtained by substituting $q^0(t) = t$ in (6.6). After normal ordering, the extension described in [14] arises, with parameters $k = c_5$, $g^a = c_6 \delta^a$ and $q^a = c_7 \delta^a$ given by (6.7); $c_8$ was not considered in that paper. Realizations of toroidal Lie algebras are obtained by further specialization to the $N$-dimensional torus.

8. Discussion

The representation theory of diffeomorphism and gauge algebras in more than one dimension has been developed. The rather obscure results in [8, 14] have been given a natural geometric explanation in terms of jet space trajectories where the reparametrization invariance has been eliminated by gauge fixing.

These manifestly well defined modules are “quantum general covariant”, in the sense that they combine a diff$(N)$ representation (general covariance) with the following quantum properties: Poisson brackets are replaced with commutators, normal ordered expressions act on a lowest-energy Fock space, and the algebra acquires an extension. Moreover, these features are obtained without the introduction of any classical background field. Therefore, these Fock modules can be viewed as natural building blocks for theories of quantum gravity.

Classically, everything could be repeated by replacing trajectories by $d$-dimensional extended objects (e.g. world sheets) in spacetime; simply reinterpret the variable $t$ in (5.5) as having $d$ components $t^i$. Reparameterization is now expressed by diff$(d)$:

$$[L_i(s), L_j(t)] = L_j(s) \partial_i \delta(s-t) + L_i(t) \partial_j \delta(s-t). \quad (8.1)$$

However, the quantum theory only exists if $d = 1$ (and trivially if $d = 0$), because otherwise normal ordering yields infinities and (8.1) has no central extension.

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