Negative contributions to entropy production induced by quantum coherences

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The entropy production in dissipative processes is the essence of thermodynamics and also of the arrow of time. For dissipation of quantum systems, it was recently shown that the entropy production contains indeed two contributions: a classical one and a quantum one. Here we show that for degenerate (or near-degenerate) quantum systems there are additional quantum contributions which, remarkably, can become negative. Furthermore, such negative contributions are related to significant changes in the ongoing thermodynamics. This includes phenomena such as generation of coherences between degenerate energy levels (called horizontal coherences), alteration of energy exchanges and, last but not least, reversal of the natural convergence of the populations towards the thermal equilibrium state. Going further, we establish a complementarity relation between horizontal coherences and population convergence, particularly enlightening for understanding heat flow reversals. Conservation laws of the different types of coherences are derived. Some consequences for thermal machines, fluctuation theorems and resource theory of coherence are suggested.

I. INTRODUCTION

The study of entropy production is of paramount importance due to its intimate relation with the second law of thermodynamics, the emergence of irreversibility and the arrow of time in classical and quantum systems. It is also related to other fundamental problems like reduction of performances in thermodynamic operations and thermal machines. As a recent remarkable development, for quantum systems undergoing dissipative processes (described either by Markovian evolutions or thermal operations), it was pointed out that the entropy production can be split in two contributions, an incoherent one (stemming from populations) and a coherent one (stemming from quantum coherences).

Inspired by the above insight, by intriguing questions around dramatic reductions of entropy production in presence of degenerate systems, and by the special role of “horizontal” coherences (coherences between degenerate energy levels) in heat exchanges, we uncover additional quantum contributions to the entropy production. We show that these extra quantum contributions, stemming from horizontal coherences and degenerate transitions, affect dramatically the ongoing thermodynamics. This includes the surprising possibility of reversing the natural convergence of the populations towards the equilibrium distribution. Furthermore, this phenomenon is associated to a negative contribution to the entropy production. Additionally, a second negative contribution can emerge, related to generation of horizontal coherences, which is indeed the underlying mechanism of well-known phenomena such as superradiance and bath-induced entanglement (or dissipative generation of entanglement), and affects heat exchanges. A complementary relation between horizontal coherences and reversal of the population’s convergence is established: the consumption of one fuels the other, and reciprocally. It appears to be particularly insightful for heat flow reversals.

The above results are firstly derived in the context of Markovian bath-driven dissipation. This viewpoint is extended by the end of the paper to include thermal operations (and even some a-thermal operations). Near degenerate systems are addressed in Methods. Finally, whereas coherences between energy levels of different energy are globally conserved, horizontal coherences are not. Still, a conservation law for horizontal coherences together with population convergence can be established.

These negative contributions are contrasting from the always-positive quantum and classical contributions reported in. Then, in addition to the above surprising changes in the ongoing thermodynamics, one can expect further consequences in thermal machines, fluctuation theorems, and resource theory of coherence.

II. RESULTS

Throughout this paper we consider a system $S$ of degenerate Hamiltonian $H_S = \sum_n \sum_{i=1}^{l_n} e_n \ket{n,i} \bra{n,i}$ described by eigenenergies $e_n$, eigenstates $\ket{n,i}$, and a degeneracy $l_n \geq 1$ for each energy level $n$. The system $S$ is assumed to undergo a dynamics described by a completely-positive-trace-preserving (CPTP) map $\Lambda$ such that at all time $t$ the density operator $\rho_t$ of $S$ is given by $\rho_t = \Lambda_t \rho_0$, where $\rho_0$ denotes the initial state of $S$. Furthermore, we consider that $\Lambda_t$ admits the thermal state $\rho_{th}(\beta) := Z(\beta)^{-1} e^{-\beta H_S}$ as steady state (meaning that $\Lambda_t \rho_{th}(\beta) = \rho_{th}(\beta)$), where $Z(\beta) := \text{Tr} e^{-\beta H_S}$ is the partition function and $\beta$ is an inverse temperature. In the following, $\beta$ will correspond to the underlying inverse temperature of the system in-
teracting with $S$. For now, keeping the discussion more general, we only require a CPTP map and a thermal steady state.

The rate of change of the von Neumann entropy of $S$, defined by $S(\rho_t) := -\text{Tr} \rho_t \ln \rho_t$, can be split in two terms [1–3, 43, 45, 46],

$$\frac{dS(\rho_t)}{dt} = -\text{Tr} \dot{\rho}_t \ln \rho_t = -\text{Tr} \dot{\rho}_t [\ln \rho_t - \ln \rho^h(\beta)] - \text{Tr} \dot{\rho}_t \ln \rho^h(\beta), \quad (1)$$

where $\dot{\rho}_t$ is the time derivative of $\rho_t$. The first term in \([1]\) is identified as the rate of entropy production [1–3, 43, 45, 46],

$$\Pi := -\text{Tr} \dot{\rho}_t [\ln \rho_t - \ln \rho^h(\beta)], \quad (2)$$

and is always positive (due to the contraction of the relative entropy under completely positive and trace preserving maps [43, 47], see more details in the following). The second term in \([1]\), of arbitrary sign, is the rate of entropy flow $\Phi := -\text{Tr} \dot{\rho}_t \ln \rho^h(\beta) = \beta \dot{E}_S$ and is identified as heat exchanges, with the internal energy of $S$ defined as $E_S(t) := \text{Tr} \rho_t H_S$.

The three contributions to entropy production.

We denote by $\rho_{t|\text{le}}$ the diagonal matrix obtained by canceling all non-diagonal elements of $\rho_t$ when written in the energy eigenbasis $|n,i\rangle$. Then, we defined $\rho_{t|\text{bd}}$ as the block-diagonal matrix obtained by canceling only coherences between levels of different energies. In other words, $\rho_{t|\text{bd}} := \sum_n \sum_{\ell=1}^{n} |n,\ell \rangle \rho_{t|\text{bd}} |n,\ell \rangle \langle n,\ell |$ while $\rho_{t|\text{le}} := \sum_n \sum_{\ell=1}^{n} \pi_n \rho_{t|\text{bd}} \pi_n$, where $\pi_n := \sum_{\ell=1}^{n} |n,\ell \rangle \langle n,\ell |$. In the remainder of the paper, coherences between levels of same energy is referred to as horizontal coherences. By contrast, coherences between levels of different energies is called vertical coherences. This is in reference to the respective position of the different energy levels in an energy diagram (see Fig. 1).

FIG. 1. Illustration representing the tendency of horizontal coherences (coherences between levels of same energy) to reverse the arrow of time. The entropy production can be associated to the arrow of time [4–8, 11, 12]. Therefore, as seen throughout the paper, vertical coherences (coherences between levels of different energy) always strengthen the arrow of time (represented by the blue action), whereas horizontal coherences can tend to its reversal (represented by the orange action) due to their negative contributions to the entropy production.

Note that sometimes a different terminology is used ("energetic" and "non-energetic" coherences), but here we prefer to introduce this new terminology to avoid possible confusion with the reference basis. With the above definitions we can decompose the entropy production as

$$\Pi = -\dot{C}_v - \dot{C}_h - \dot{D}_{th}. \quad (3)$$

The first term in the above identity is the time derivative of the relative entropy of vertical coherence which we defined as $C_v(t) := S(\rho_{t|\text{bd}}) - S(\rho_t)$ (equivalent to the definition in [48] for non-degenerate systems). The quantity $C_v(t)$ is a measure of vertical coherences contained in $\rho_{t|\text{bd}}$. The second term in \([3]\) is the time derivative of the relative entropy of horizontal coherence that we define as $C_h(t) := S(\rho_{t|\text{le}}) - S(\rho_{t|\text{bd}})$. In analogy with the first term, the quantity $C_h(t)$ is a measure of horizontal coherences contained in $\rho_t$. The last term $-\dot{D}_{th}$ is the time derivative of $-D_{th}(t) := -S[\rho_{t|\text{bd}} |\rho^h(\beta_B)]$ defined through the relative entropy $S(\sigma |\rho) := \text{Tr} \sigma [\ln \sigma - \ln \rho]$ which establishes a measure of distance between any two density operators $\sigma$ and $\rho$ [49]. Therefore, $S[\rho_{t|\text{bd}} |\rho^h(\beta_B)]$ measures how far the population distribution is from the thermal equilibrium distribution, and $-\dot{D}_{th}$ is the rate – or velocity – to which the population distribution converge to the thermal equilibrium distribution. Moreover, defining by $\mathcal{F}_D(t) := E_S(t) - S(\rho_{t|\text{bd}})/\beta_B$ the diagonal – or classical – free energy, we have the interesting relation $\dot{D}_{th} = \beta_B \mathcal{F}_D$.

Note that the relative entropy of vertical and horizontal coherences can be rewritten as $C_v(t) = \text{Tr} \rho_t [\ln \rho_t - \ln \rho_{t|\text{bd}}] = S(\rho_t |\rho_{t|\text{bd}})$ and $C_h(t) = \text{Tr} \rho_{t|\text{le}} [\ln \rho_{t|\text{le}} - \ln \rho_{t|\text{bd}}] = S(\rho_{t|\text{le}} |\rho_{t|\text{bd}})$.

FIG. 2. Energy level structure of $S$. Coherences between levels of same energy are called horizontal coherences whereas coherences between levels of different energy are called vertical coherences, in a direct reference to their graphical representation. The green double arrows represent degenerate transitions, playing a central role in the emergence of negative contributions to entropy production.
the following identities (see Methods),
\[
\dot{C}_v = \text{Tr} \dot{\rho}_t \left[ \ln \rho_t - \ln \rho_{t|\text{BD}} \right],
\]
\[
\dot{C}_h = \text{Tr} \dot{\rho}_t \left[ \ln \rho_{t|\text{BD}} - \ln \rho_{t|\text{BD}} \right],
\]
\[
\mathcal{D}_{\text{th}} = \text{Tr} \dot{\rho}_t \left[ \ln \rho_{t|\text{BD}} - \ln \rho^h(\beta) \right],
\]
leading to \[3\].

It is quite remarkable that the entropy production is related to the rate at which vertical coherences is consumed \[8\], corresponding to the term \(\dot{C}_v\), but additionally to the rate at which horizontal coherences is consumed, expressed by the term \(\dot{C}_h\), topped by the velocity of the population convergence to the thermal equilibrium distribution, \(\mathcal{D}_{\text{th}}\).

**Bath-driven dissipation.** In the following we study the role and behaviour of each contribution for one of the most common situation in thermodynamics and quantum dynamics: interaction with a bath. We will see that the presence of horizontal coherences has striking consequences. We consider that the system \(S\) is interacting with a stationary bath \(B\) \[20\] \[50\] \[51\], meaning that the bath state \(\rho_B\) commute with its free Hamiltonian \(H_B\), \([\rho_B, H_B] = 0\). We assume that the system-bath coupling is of the form \(V = g A_S A_B\), where \(g\) corresponds to the effective coupling strength, and \(A_S\) and \(A_B\) are observables of \(S\) and \(B\), respectively. More details regarding the physics behind of such coupling and its relation with an underlying notion of indistinguishability can be found in Section Methods ‘Collective coupling and indistinguishability’.

Under weak coupling, the Born and Markov approximations are legitimate \[43\] \[52\], so that one can derive the following master equation (in the interaction picture) for the reduced dynamics of the system
\[
\dot{\rho}_t = \mathcal{L} \rho_t := \sum_\omega \Gamma(\omega) \left[ A(\omega) \rho_t A^\dagger(\omega) - A^\dagger(\omega) A(\omega) \rho_t \right] + \text{h.c.},
\]
(5)

where \(\Gamma(\omega) = \int_0^\infty d\varepsilon e^{i\varepsilon \omega t} \text{Tr} \rho_B A_B(s) A_B,\) and \(A_B(s)\) is the bath operator \(A_B\) in the interaction picture (with respect to the free Hamiltonian \(H_B\)). The jump operators \(A(\omega)\) are defined by \[43\] \[\mathcal{A}(\omega) = \sum_{\varepsilon, \varepsilon'} \pi_{\varepsilon'} A_S \pi_{\varepsilon}.\] One equilibrium state of the dynamics \[3\] is the thermal state \(\rho^h(\beta_B)\), where \(\beta_B\) is the bath inverse temperature (or apparent temperature \[20\] \[53\] for a non-thermal stationary state).

Finally, one important characteristic of the physics described by \[5\] is the independence of the vertical coherences’ dynamics from the populations and the horizontal coherences whereas the horizontal coherences’ dynamics is coupled to the populations (see Methods “Collective coupling and indistinguishability”). This observation has deep implications as we will see in the following.

**Negative contribution of \(\dot{C}_h\).** For non-degenerate systems \(\rho_{t|\text{BD}} = \rho_{t|\text{BD}}\), so that \(\dot{C}_h = 0\) at all times. Then, \(\dot{C}_v\) and \(\mathcal{D}_{\text{th}}\) become equivalent to the coherent and diagonal contributions introduced in Eq. (12) of \[3\]. In Methods we show a simple proof of their positivity, which is similar to the following one for \(\dot{C}_v\) in a context of degenerate systems. Namely,

\[
\dot{C}_v = \frac{d}{dt} S(\rho_t|\text{BD})
\]
\[
= - \lim_{dt \to 0} \frac{1}{dt} \left[ S(\rho_{t+dt}|\text{BD}) - S(\rho_t|\text{BD}) \right]
\]
\[
= \lim_{dt \to 0} \frac{1}{dt} \left[ S(e^{dt\mathcal{L}} \rho_t|\text{BD}) - S(\rho_t|\text{BD}) \right],
\]
(6)
which is always positive since the relative entropy is contractive under completely positive and trace preserving maps \[49\] \[47\] (the map generated by \(\mathcal{L}\) defined in \[5\] being completely positive and trace preserving). The crucial step in \(6\) is

\[
\rho_{t+dt|\text{BD}} = e^{dt\mathcal{L}} \rho_{t|\text{BD}},
\]
holding since, as mentioned above, the dynamics of the populations and horizontal coherences (both contained in \(\rho_{t|\text{BD}}\)) are independent from the vertical coherences (see Methods, \[41\]). However, the dynamics of the populations and the horizontal coherences are coupled, which implies

\[
\rho_{t+dt|\text{BD}} \neq e^{dt\mathcal{L}} \rho_{t|\text{BD}}.
\]
(8)

Then,

\[
\dot{C}_h = - \lim_{dt \to 0} \frac{1}{dt} \left[ S(\rho_{t+dt|\text{BD}}|\text{BD}) - S(\rho_{t|\text{BD}}|\text{BD}) \right]
\]
\[
\neq - \lim_{dt \to 0} \frac{1}{dt} \left[ S(e^{dt\mathcal{L}} \rho_{t|\text{BD}}|\text{BD}) - S(\rho_{t|\text{BD}}|\text{BD}) \right],
\]
(9)

breaking down the guarantee of positivity of \(\dot{C}_h\). Then, the guarantee being broken, one can be sure the worst can happen: \(\dot{C}_h\) can become negative. An example of that in a quite general situation follows.

Considering the dynamics described by \(5\), we assume for instance that the system \(S\) is initially in a thermal state \(\rho_0 = \rho^h(\beta_0)\). The state of \(S\) at a later time is therefore given by (in the interaction picture)

\[
\rho_t = e^{t\mathcal{L}} \rho_0
\]
\[
= \rho_0 + t\mathcal{L} \rho_0 + \mathcal{O}(t^2),
\]
(10)
where \(\mathcal{L} := \max_\omega \Gamma(\omega)\) characterises the dissipation rate suffered by \(S\). Then, for times much smaller than \(\Gamma^{-1}\), the state of \(S\) is well approximated by the first two terms of \(10\). Using the fact that \(S\) is initially in a thermal state at inverse temperature \(\beta_0\) and the following identity
\(\mathcal{A}(\omega)\rho^{th}(\beta_0) = e^{-\omega\beta_0} \rho^{th}(\beta_0)\mathcal{A}(\omega)\), one obtains

\[
\rho_t = \rho^{th}(\beta_0) \left\{ 1 + t \sum_{\omega > 0} G(\omega) (e^{-\omega\beta_0} - e^{-\omega\beta_B}) \right\} \\
\times [\mathcal{A}(\omega)\mathcal{A}^\dagger(\omega) - e^{-\omega\beta_0}\mathcal{A}^\dagger(\omega)\mathcal{A}(\omega)] + \mathcal{O}(T^2t^2),
\]

(11)

where \(G(\omega) := \Gamma(\omega) + \Gamma^*(\omega)\) is related to the inverse bath temperature (or apparent temperature \([26, 53]\)) \(\beta_B\) through the relation \(G(-\omega)/G(\omega) = e^{-\omega\beta_B}\). Then, as soon as at least two degenerate transitions are involved in the coupling with the bath, which means the existence of two degenerate levels \(|n, i_1\rangle, |n, i_2\rangle\), and a third level \(|n', i'\rangle\) such that \(\langle n', i'|A_S|n, i_1\rangle \neq 0\) and \(\langle n', i'|A_S|n, i_2\rangle \neq 0\) (see Fig. 2), one can show that the terms \(\mathcal{A}(\omega)\mathcal{A}^\dagger(\omega)\) and \(\mathcal{A}^\dagger(\omega)\mathcal{A}(\omega)\) contain horizontal coherences. Indeed, for \(\omega = \epsilon_{n'} - \epsilon_n\),

\[
\langle n, i_1|\mathcal{A}(\omega)\mathcal{A}^\dagger(\omega)|n, i_2\rangle = \sum_{\epsilon_m - \epsilon_n = \omega} \langle n, i_1|\pi_m A_S\Pi_m A_S\pi_m|n, i_2\rangle
\]

\[
= \langle n, i_1|A_S\pi_m A_S|n, i_2\rangle \neq 0.
\]

(12)

Similarly, we have also \(\langle n, i_1|\mathcal{A}^\dagger(\omega)\mathcal{A}(\omega)|n, i_2\rangle \neq 0\). This implies \(\langle n, i_1|\rho_t|n, i_2\rangle \neq 0\) when \(\beta_B \neq \beta_0\). In other words, the presence of degenerate transitions generates horizontal coherences in \(\rho_t\). This is the underlying common mechanism of bath-induced coherences in multi-level atoms \([54, 57]\), crucial in superradiance \([31, 35]\) and bath-induced entanglement \([36, 41]\).

Therefore, since the presence of horizontal coherences implies \(\rho^{t\text{ind}} \neq \rho^{t\text{id}}\), we have

\[
-\mathcal{C}_h(t) = -S(\rho^{t\text{ind}}|\rho^{t\text{id}}) < 0,
\]

(13)

leading to the negativity of \(-\mathcal{C}_h\) (at least for small times). From the point of view of irreversibility, and the arrow of time, one can interpret the above results as a tendency of horizontal coherences to reverse the arrow of time while vertical coherences always re-enforce it, see Fig. 1.

Towards reversibility: Reduction of entropy production. Extending the conclusions of the previous paragraph, we show in the following an interesting application of horizontal coherences to reduce the irreversibility of dissipative processes. This is illustrated considering an ensemble of \(n\) spins of dimension \(s\) indistinguishable from the point of view of the bath and therefore following a dynamics described by \([3]\) (see also Methods ‘Collective coupling and indistinguishability’). The thermodynamic properties emerging from the resulting collective dissipation were studied in details in \([21]\). Using some results of \([21]\) one can show (see Methods “Reduction of irreversibility in spin ensembles”) that for an ensemble initially in a thermal state at inverse temperature \(\beta_0\), the contribution to the entropy production from the horizontal coherences is indeed negative and equal to

\[
-\Delta^\infty \mathcal{C}_h := -[\mathcal{C}_h(\infty) - \mathcal{C}_h(0)]
\]

\[
= \sum_{m=-n}^{n} e^{-\hbar \omega_B m} Z_{ns}(\beta_B) \ln I_m < 0,
\]

(14)

where \(Z_{ns}(\beta_B) := \sum_{m=-n}^{n} e^{-\hbar \omega_B m} \beta_B\) is the bath inverse temperature, and \(I_m\) is a growing function of \(n\) and \(s\) corresponding to the degeneracy of the \(m\)th excited energy level.

One could suspect that, on the other hand, such negative contribution to the entropy production would be compensated by an increase of the variation of \(-D_{th}\), \(-\Delta^\infty D_{th} := -[D_{th}(\infty) - D_{th}(0)]\). However, this is not the case. Indeed, \(-\Delta^\infty D_{th}\) is also reduced compared to independent dissipation (see Methods “Reduction of irreversibility in spin ensembles”).

Therefore, the negative contribution \(-\Delta^\infty \mathcal{C}_h\) promotes a reduction of the entropy production, reenforced by the decrease of \(-\Delta^\infty D_{th}\). This elucidates the origin of the surprising reduction of entropy production pointed out in \([21]\). As an illustration of the importance of the reduction, Fig. 3 displays for \(\hbar \omega_B \beta_0 > 1\) the plots of the ratio \(\Pi^{th}/\Pi^{col}\), where \(\Pi^{th}\) (\(\Pi^{col}\)) is the entropy production without (with) generation of horizontal coherences, which corresponds to independent (collective) dissipation \([21]\). One can see that for \(\hbar \omega_B \beta > 1\), the
entropy production of the collective dissipation $\Pi_{\text{col}}$ tends to be $n$ times smaller than $\Pi^\text{th}$ (coinciding with the analytical results derived in \cite{24}). As mentioned in the introduction, entropy production is known for degrading the performances of thermodynamic operations due for instance to energy and information dissipation \cite{13-20}. Therefore, a valuable way to reduce such losses is the introduction, entropy production is known for decreasing the entropy production of the collective dissipation $\Pi_{\text{col}}$. In fig. 4, or equivalently that the diagonal free energy $-\rho \frac{\partial}{\partial t} \rho$ implies $\Pi_{\text{col}}$, which breaks down the guarantee of positivity of $-D_{\text{th}}$ (as in \cite{9} for $\dot{\mathcal{C}}_h$). Strikingly, this means that the population distribution can go away from the thermal equilibrium distribution, as illustrated in Fig. 3, or equivalently that the diagonal free energy can increase. Furthermore, we will see that this phenomenon can indeed be related to the heat flow reversal pointed out in \cite{30}, illustrated in Fig. 5.

In the following we present a situation exhibiting such surprising properties. We consider a system $S$ following the dynamics \cite{5} and initially in a state of the form

$$\rho_0 = \rho^{\text{th}}(\beta_0) + \chi,$$  \hspace{1cm} (15)

where $\rho^{\text{th}}(\beta_0)$ is the thermal state at inverse temperature $\beta_0$ and $\chi$ is an arbitrary hermitian density matrix containing only off-diagonal terms (vertical and horizontal coherences), so that $\rho_{0|D} = \rho^{\text{th}}(\beta_0)$. Then, the velocity of the population convergence at initial times (small with respect to $\Gamma^{-1}$) is,

$$-D_{\text{th}} = -\text{Tr} \dot{\rho}_{t=0} \left( \ln \rho^{\text{th}}(\beta_0) - \ln \rho^{\text{th}}(\beta_B) \right) = (\beta_0 - \beta_B) \dot{E}_S.$$  \hspace{1cm} (16)

The heat flow $\dot{E}_S$ can be written in the following form \cite{20,53} (see also Methods “Expression of the heat flow”),

$$\dot{E}_S = \sum_{\omega>0} \omega G(\omega) \langle A(\omega)A^\dagger(\omega) \rangle_{\rho_t} (e^{-\omega \beta_B} - e^{-\omega/T(\omega)}),$$  \hspace{1cm} (17)

where $T(\omega) := \omega \left( \ln \frac{\langle A(\omega)A^\dagger(\omega) \rangle_{\rho_t}}{\langle A(\omega)A(\omega) \rangle_{\rho_t}} \right)^{-1}$ is the apparent temperature associated with the energy exchange $\omega$ \cite{20,53}. In particular, for initial states of the form \cite{15} the inverse apparent temperatures can be rewritten as

$$\frac{\omega}{T(\omega)} = \omega \beta_0 + \ln \frac{1 + e^{-c^+ \frac{1 + e^{-c^-}}{1 + e^{-c^+}}}}{1 + e^{-c^-}},$$  \hspace{1cm} (18)

where $c^- := \langle A(\omega)A^\dagger(\omega) \rangle_{\chi}/\langle A(\omega)A^\dagger(\omega) \rangle_{\rho^{\text{th}}(\beta_0)}$ and $c^+ := \langle A(\omega)A^\dagger(\omega) \rangle_{\rho^{\text{th}}(\beta_0)}/\langle A(\omega)A(\omega) \rangle_{\rho^{\text{th}}(\beta_0)}$ constitute the contribution from the horizontal coherences, highlighted in \cite{20,30,53}. When $\chi$ do not contain horizontal coherences we have $\langle A(\omega)A^\dagger(\omega) \rangle_{\chi} = \langle A^\dagger(\omega)A(\omega) \rangle_{\chi} = 0$, implying $T(\omega) = 1/\beta_0$. Consequently, in absence of horizontal coherences, the population convergence velocity becomes

$$-D_{\text{th}} = (\beta_0 - \beta_B) \sum_{\omega>0} \omega G(\omega) \langle A(\omega)A^\dagger(\omega) \rangle_{\rho_t} \times (e^{-\omega \beta_B} - e^{-\omega \beta_0}),$$  \hspace{1cm} (19)

which is always positive for any value of $\beta_0$ and $\beta_B$. However, in presence of horizontal coherences the apparent temperatures $T(\omega)$ can be risen beyond or lowered below $1/\beta_B$ \cite{30}, inverting the role of hottest and coldest system and resulting in changing the sign of the heat flow \cite{17}. Consequently, from \cite{16}, the population convergence velocity $-D_{\text{th}}$ becomes negative. It is shown in \cite{30} that such heat flow reversals are always achievable for $\beta_0$ not too far from $\beta_B$. Note that the total heat exchanged between the initial and final time (when reaching the equilibrium state) can also be inverted, leading to $(\beta_0 - \beta_B) \Delta E_S \leq 0$.

We just showed that the velocity of the population convergence $-D_{\text{th}}$ can become negative thanks to horizontal coherences, and that heat flow reversals is one of its dramatic observable consequences. In the next paragraph we go further: we show formally that heat flow reversals are powered by horizontal coherences (illustrated in Fig. 5).

**Complementarity of horizontal coherences and heat flow reversal.** We showed that both $-\dot{C}_h$ and $-D_{\text{th}}$ can become negative. However, there is a restriction: the sum of $-\dot{C}_h$ and $-D_{\text{th}}$ has to be always positive. Indeed, a variation of $-\dot{C}_h(t) - D_{\text{th}}(t)$ between instants of time $t$ and $t'$ greater than $t$ gives

$$-\Delta C_h - \Delta D_{\text{th}}$$

$$= - [S(\rho_{t'|\text{BD}}|\rho^{\text{th}}(\beta_B)) - S(\rho_{t|\text{BD}}|\rho^{\text{th}}(\beta_B))]$$

$$= - [S(e^{-t'|t} \rho_{t'|\text{BD}}|e^{-t'|t} \rho^{\text{th}}(\beta_B))]$$

$$\geq 0,$$  \hspace{1cm} (20)

where the identity \cite{7} and the contractivity of the relative entropy under completely positive and trace preserving maps \cite{13,47} were used in third and forth lines, respectively. The above inequality implies in particular that the time derivative of the sum is also always positive,

$$-\dot{C}_h - D_{\text{th}} \geq 0.$$  \hspace{1cm} (21)
The physical meaning of the inequality \((20)\) appears after re-writing the variation of \(\mathcal{D}_{\text{th}}(t)\) between the initial time \(t = 0\) and any arbitrary later time \(t\) as
\[
-\Delta \mathcal{D}_{\text{th}} = (\beta_0 - \beta_B) \Delta E_S - S(\rho_{0|\Delta}|\rho_{0|\Delta}),
\]
where \(\Delta E_S = \text{Tr} H_S(\rho_0 - \rho_0)\) is the associated variation of energy of \(S\). Note that \((22)\) is valid for populations initially thermally distributed, \(\rho_{0|\Delta} = \rho^{\text{th}}(\beta_0)\). Injecting \((22)\) in the inequality \((20)\) one obtains
\[
-\Delta \mathcal{C}_h + (\beta_0 - \beta_B) \Delta E_S - S(\rho_{0|\Delta}|\rho_{0|\Delta}) \geq 0.
\]
The quantity \((\beta_0 - \beta_B) \Delta E_S\) is always positive for initial states without horizontal coherences. However, as shown above and in \([30]\), the presence of initial horizontal coherences can reverse the heat flow \(E_S\) and the finite heat exchange \(\Delta E_S\), implying \((\beta_0 - \beta_B) E_S < 0\) and \((\beta_0 - \beta_B) \Delta E_S < 0\). From \((23)\), a reversal of finite heat exchange implies
\[
-\Delta \mathcal{C}_h \geq - (\beta_0 - \beta_B) \Delta E_S > 0,
\]
which means a strict consumption of horizontal coherences. Conversely, when horizontal coherences are not consumed, meaning that \(-\Delta \mathcal{C}_h = 0\) (or even \(-\Delta \mathcal{C}_h < 0\)), one has necessarily
\[
(\beta_0 - \beta_B) \Delta E_S \geq S(\rho_{0|\Delta}|\rho_{0|\Delta}) \geq 0,
\]
so that no reversal of heat exchange can happen. This shows explicitly that reversal of heat exchange is powered by horizontal coherences (illustration in Fig. \(5\)). Additionally, the inequality \((23)\) also implies that creation of horizontal coherences, corresponding to \(-\Delta \mathcal{C}_h < 0\), has an energetic cost
\[
(\beta_0 - \beta_B) \Delta E_S \geq \Delta \mathcal{C}_h > 0,
\]
paid in the form of heat (or “natural” heat exchange).

Finally, it is also interesting to look at the time-derivative version of \((23)\), giving at initial times \((0 \leq t < \Gamma^{-1})\),
\[
-\dot{\mathcal{C}}_h + (\beta_0 - \beta_B) \dot{E}_S \geq 0.
\]
One can obtain straightforwardly the time-derivative analogue of the above inequalities \((24)\), \((25)\), and \((26)\). The absence of the term \(S(\rho_{0|\Delta}|\rho_{0|\Delta})\) (equal to zero for \(0 \leq t \ll \Gamma^{-1}\)) in \((27)\) provides the following insight: initially, both heat flow reversal and creation of horizontal coherences do not have extra costs in horizontal coherences and energy, respectively. However, as time passes, an extra cost corresponding to \(S(\rho_{0|\Delta}|\rho_{0|\Delta}) \geq 0\) is required, as shown by \((24)\) and \((26)\). In particular, this explains why finite heat exchanges require stronger initial conditions than heat flow reversals, as pointed out and discussed in \([30]\).

**Thermal and a-thermal operations.** We now show that the above results are indeed valid in a broader context. The following considerations are inspired from \([8]\), with new results related to horizontal coherences.

Let’s consider the unitary interaction of our system of interest \(S\) with an other system \(B\) between two instant of times \(t_i\) and \(t_f\). Importantly, in the remainder of the paper \(B\) is not restricted to baths but can be of any size, even an elementary single quantum system. We denote by \(U\) the associated unitary transformation (acting on both \(S\) and \(B\)), and by \(\rho_{X,t}\) the density operator of the system \(X\) (standing for \(S\), \(B\), or \(SB\)) at an arbitrary instant of time \(t\). Note that in principle there is also no restriction on the strength of the coupling between \(S\) and \(B\) (not anymore limited to weak coupling as in \(5\)).

---

**FIG. 4. Illustration of reversal of population convergence.** Without horizontal coherences, the populations naturally converge monotonically to the thermal equilibrium distribution. The presence of horizontal coherences can have an astonishing effect: reversal of the population convergence, resulting in populations going away from the thermal equilibrium distribution. Moreover, the “fuel” enabling this reversal of the natural population convergence is the horizontal coherences themselves.

**FIG. 5. Illustration of heat flow reversal.** The system \(S\) can gain energy while interacting with a colder bath thanks to horizontal coherences acting as a “fuel” for heat flow reversal. Conversely, the system \(S\) can lose energy while interacting with a hotter bath, again thanks to horizontal coherences.
We call \textit{a-thermal}, operations satisfying the conditions of initial separability, energy conservation, and stationarity of \(\rho_{B,t}\), expressed respectively by \(\rho_{SB,t} = \rho_{S,t} \rho_{B,t}\), \([U, H_S + H_B] = 0\), and \([H_B, \rho_{B,t}] = 0\). If one asks additionally the initial state \(\rho_{B,t}\) to be thermal, the operation belongs to the well-known set of \textit{thermal operations} [42]. Quite surprisingly, the only conditions defining a-thermal operations guarantee the validity of most of the above results. More precisely, under the energy conservation condition, one can show (see Methods “Thermal operations and beyond”) that the evolution of the vertical coherences is closed (in the sense mentioned above). Thus, one can simply repeat the argument [9] (slightly adapted to finite evolution, see Methods) and show that the vertical coherences always decrease,

\[
- \Delta C_v = -|C_v(t_f) - C_v(t_i)| \geq 0.
\]  

(28)

The are two key points. First, \(\Lambda\), denoting the reduced evolution of \(S\), is a completely positive trace-preserving map (thanks to the initial separability and the unitarity of the global evolution [13]), which guarantees the contraction of the relative entropy. Second, the closed evolution of the vertical coherences, which guarantees

\[\rho_{S,t|\text{ind}} = \Lambda \rho_{S,t|\text{ind}}.\]  

(29)

By contrast, the mixing of the horizontal coherences’ dynamics with the populations’ dynamics implies

\[\rho_{S,t|\text{in}} \neq \Lambda \rho_{S,t|\text{in}},\]  

(30)

which, repeating the argument in [9], breaks down the guarantee of positivity of \(-\Delta C_h\). In Methods, some explicit conditions (relying on degenerate transitions, Fig. 2) are pinpointed in order to have effectively \(-\Delta C_h < 0\).

Assuming the existence of a thermal equilibrium state \(\rho_S(\beta_B)\), which is guaranteed for instance when \(B\) is initially in a thermal state, one can follow the definitions introduced in the beginning of the paper for the entropy production II and the measure of population distance to the equilibrium distribution \(D_{th}\). Then, for the same reason of non-closure of the populations’ dynamics, the guarantee of positivity of \(-\Delta D_{th}\) is broken. Again, in Methods, some conditions are mentioned for having divergence of the populations from the equilibrium distribution, \(-\Delta D_{th} < 0\).

Furthermore, as above, one can show with the same arguments used for vertical coherences that

\[-\Delta C_h - \Delta D_{th} \geq 0.\]  

(31)

Even though in general \(-\Delta D_{th}\) cannot be simply related to energy exchanges as in \([22]\), it is still associated to the variation of diagonal free energy, \(-\Delta D_{th} = -\beta_B \Delta F_B\), and still represents the convergence of the populations towards the thermal equilibrium distribution. Assuming for instance a situation where \(U\) generates horizontal coherences, one obtains

\[-\Delta D_{th} \geq \Delta C_h > 0,\]  

(32)

which represents the necessary cost in “population gradient” for creating horizontal coherences. In particular, \(\rho_{S,t,i|\text{in}}\) has to be far enough from \(\rho_S^B(\beta_B)\). Conversely, having \(-\Delta D_{th} < 0\), meaning a \textit{reversal} of the natural tendency of convergence of the population to the thermal equilibrium distribution, necessarily requires consumption of horizontal coherences,

\[-\Delta C_h \geq \Delta D_{th} > 0,\]  

(33)

and therefore initial states containing horizontal coherences. This result is a generalisation of [24] and of the heat flow reversal [30].

Interestingly, the above inequalities can be extended to equalities. Based on the observation that the evolution of global block diagonalisation commute with \(U\), \(U \rho_{SB,t|\text{ind}} U^\dagger = (U \rho_{SB,t} U^\dagger)|\text{ind}\) (see Methods), one can show a \textit{conservation law} of vertical coherences (already obtained in [8] in a different form),

\[\Delta C_v^{SB} = 0,\]  

(34)

which can be alternatively written as

\[-\Delta C_v^S - \Delta C_v^B = C_v^{SB}(t_f).\]  

(35)

The introduced superscripts \(S, B, \text{or } SB\) refer to the corresponding systems and \(C_{v}^{SB}(t)\) is the \textit{correlated} vertical coherences introduced in [58] as

\[C_v^{SB}(t) := C_v^{SB}(t) - C_v^S(t) - C_v^B(t) \geq 0.\]  

(36)

\(C_v^{SB}(t)\) represents the vertical coherences present in \(SB\) fruits of correlations between \(S\) and \(B\). Therefore, the conservation law (35) implies in particular that the vertical coherences consumed in \(S\) is either transferred to \(B\) or to correlated vertical coherences \(C_v^{SB}(t_f)\) between \(S\) and \(B\). Indeed, this conservation law is surprising from the perspective of decoherence theory in open quantum systems [43], which taught us that coherences are destroyed by the bath. This is because open quantum systems theory is mainly concerned about the reduced system’s dynamics and therefore changes in the bath state are mostly ignored.

By contrast, horizontal coherences are \textit{not conserved}. This is one more aspects of the fundamental difference between vertical and horizontal coherences. However, one can derive a conservation law for horizontal coherences when contributions from the population convergence is included. Namely, one can show (see Methods),

\[-\Delta C_h^B + \Delta D_{th}^B = 0,\]  

(37)

which establishes that, globally, the reversal of population convergence is exactly equal to the consumption of horizontal coherences. Conversely, generation of horizontal coherences (still globally) is exactly compensated by population convergence (which inherently carries an energetic cost). Additionally, the conservation law (37)
can be used to show that the generation of horizontal coherences affects the energy exchanges (see Methods), recovering in a straightforward way observations of [21, 23]. By contrast, when no horizontal coherences are consumed or generated, one has necessarily $\Delta D_{th}^{SB} = 0$, implying that globally, the dynamics of the populations is restricted to a region of equidistant points to the equilibrium distribution.

The conservation law [37] can alternatively be expanded as

$$-\Delta C_h^S - \Delta D_{th}^S - \Delta C_h^B - \Delta D_{th}^B = C_{SB}(t_f) + D_{SB}(t_f),$$  \hspace{1cm} (38)

where the superscripts $S$, $B$, and $SB$ refer to the corresponding systems. The quantities $C_{SB}(t_f)$ and $D_{SB}(t_f)$ are respectively the horizontal global coherences and the distance of the global populations to the thermal equilibrium distribution $\rho_{SB}^B(\beta_B)$ stemming from correlations between $S$ and $B$. Both $C_{SB}(t_f)$ and $D_{SB}(t_f)$ are defined in a similar way as in (36) and are positive. The conservation laws [37] and [38] can be seen as extensions of (31), (32), and (33) in form of equalities. Eq. (38) means that the horizontal coherences conserved in $S$ and $B$ plus the steps towards $\rho_{SB}^B(\beta_B) = \text{Tr}_{B}\rho_{SB}^B(\beta_B)$ and $\rho_{B}^B(\beta_B) = \text{Tr}_{S}\rho_{SB}^B(\beta_B)$ is recovered in the final correlated horizontal coherences and distance to $\rho_{SB}^B(\beta_B)$.

Note that the time derivative version of the results of this last part are valid only upon a divisibility property of $U$. Namely, for any $t \in [t_i, t_f]$, $U$ is divisible in two unitary evolutions $U_{t, t}^i$ and $U_{t, t}^f$ which are both energy conservatives.

### III. CONCLUDING REMARKS

The entropy production of a degenerate system can be split in three contributions. For bath-driven dissipation and thermal (or a-thermal) operations, the first contributions is always positive and is related to the consumption of vertical coherences. The second contribution stems from horizontal coherences. Strikingly different from the first one, it can be either positive or negative, which is shown to be associated respectively to consumption or generation of horizontal coherences. In particular, this explains the origin of the dramatic entropy reduction pointed out in [21]. Finally, the third contribution stems from the convergence of the populations to the thermal equilibrium distribution, and is related to the variation of the diagonal free energy. While this contribution is always positive in the absence of horizontal coherences, meaning that the populations always tends to the thermal equilibrium distribution, this natural tendency can be reversed thanks to horizontal coherences, generating an other negative contribution to the entropy production. This very surprising effect is the origin of the heat flow reversal introduced in [30]. The cost for this inversion is paid in horizontal coherences. Conversely, the generation of horizontal coherences is paid in heat or “population gradient”. Finally, a surprising global conservation law is recovered for vertical coherences, whereas horizontal coherences are shown not to be conserved. Nevertheless, the sum of the horizontal coherences plus the population convergence rate is globally conserved.

The above phenomena rely on the degeneracy of $S$ and the existence of degenerate transitions (Fig. 2). Degenerate transitions were realised experimentally in multi-level atoms for instance in [69], and also appear in diverse contexts (superradiance and bath-induced entanglement) in ensembles of subsystems collectively coupled to a common bath, with some experimental realisations for instance in [33, 34, 40, 59]. Moreover, the exact degeneracy of $S$ is indeed not necessary. We show in Methods “Dissipation of near degenerate systems” that energy gaps in the system’s spectrum of order $\delta$ are initially not resolved by the bath when $\delta$ is smaller or equal to the bath coupling $g$. Then, at least for times much smaller than $\delta^{-1}$ (which can still allow for significant dissipation of $S$), the system appears as degenerate for the bath and the phenomena described throughout the paper can take place. In particular, this includes ensembles of non-interacting or even weakly interacting subsystems with inhomogeneities as large as the bath coupling, suggesting that the above phenomena are accessible and observable.

Furthermore, motivating more research in this direction, our results seem to indicate that thermodynamics departs from classical behaviour thanks to horizontal coherences and the associated negative contributions – at least in the context of Markovian dissipations and thermal operations. Strengthening this claim, in addition to the significant effects on the ongoing thermodynamics already reported throughout the paper, one can show that negative contributions to entropy production can modify the performances of thermal machines (see Methods “Effects in thermal machines performances”). Moreover, since thermal operations can indeed generate horizontal coherences out of incoherent states, contrarily to what happens with vertical coherences [69, 61], the resource theory of vertical and horizontal coherences do not share the same set of free operations. This emphasises again the special statue of horizontal coherences and suggests the necessity of introducing a specific resource theory for horizontal coherences. Additionally, our results stress the necessity of taking into account effects from coherences, which calls for an extension or reformulation of the current formalism of fluctuation theorems [62], as also pointed out in [63, 65]. Beyond that, it would also be interesting to investigate what could be the role of horizontal coherences in work fluctuation reductions achieved through collective operations [64], violation of work fluctuation relations in processes generating coherences [67], thermodynamic uncertainty relations [13, 14, 17, 18], and dissipated work in non-adiabatic driving [9].

### IV. METHODS

Collective coupling and indistinguishability.
The most general coupling between $S$ and $B$ is of the form
\[ V = g \sum_\alpha A_{S,\alpha} A_{B,\alpha}, \]  
where $g$ corresponds to the effective coupling strength, $A_{S,\alpha}$ observables of $S$ and $A_{B,\alpha}$ observables of the bath. When two different bath observables $A_{B,\alpha}$ and $A_{B,\alpha'}$ are independent such that $\langle \rho_B A_{B,\alpha} A_{B,\alpha'} \rangle = 0$, each observable give rise to an independent dissipation channel, as if $A_{B,\alpha}$ and $A_{B,\alpha'}$ were observables of two distinct and independent baths. Contrasting with such independent dissipation, we assume in this paper a situation where the system-bath coupling give rise to a single dissipation channel only, which corresponds to a coupling of the alternative following form
\[ V = g A_S A_B. \]  
It means that all energy transitions are collectively coupled to the same bath observable $A_B$. In particular, an absorption of a bath excitation can activate any resonant transition, ending up in any corresponding excited state. Thus, the bath does not “know” which transition was activated: it cannot distinguish two (or several) different resonant transitions. This interpretation provides some insights regarding the underlying conditions for collective coupling (40), namely, the transitions should be indistinguishable from the point of view of the bath. Depending on the system, this might require some experimental arrangements. For instance for a multi-level atom, one needs parallel transition dipole moments $|B\rangle$, realised experimentally in $[69]$, or optical cavity to make the atomic transitions indistinguishable (from the point of view of the outside bath) $[54, 70]$. Conversely, if $S$ is made of an ensemble of smaller subsystems, each subsystem should be placed at spatial locations which are indistinguishable, or indiscernible, from the point of view of bath $[28, 35]$ (with an example of experimental realisations in $[34, 40]$). Alternatively, the indistinguishability from the bath’s point of view can be achieved also by adding an ancillary system between $S$ and the bath, like an optical cavity in the situations of atomic clouds $[71, 74]$, with examples of experimental realisations in $[33, 59]$.

The reduced dynamics of $S$ can be obtained using the Born and Markov approximations, valid for weak bath coupling $[43, 52]$, leading to the master equation $[\delta]$.

We now emphasise the structure imposed by the master equation $[\delta]$ on the dynamics of the populations, vertical coherences, and horizontal coherences. From $[\delta]$, the dynamics of the matrix element $\langle n, i | \rho_S | n', i' \rangle$ is
\[ \frac{d}{dt} \langle n, i | \rho_S | n', i' \rangle = \sum_\omega \Gamma(\omega) \left[ \langle n, i | A(\omega) \rho_S A(\omega) | n', i' \rangle - \langle n, i | A(\omega) | n', i' \rangle \right] + c.c. \]  
\[ (41) \]  
Then, if $n = n'$, the states $A(\omega)|n', i'\rangle$ and $A(\omega)|n, i\rangle$ lay in the eigenspace of energy $e_n + \omega = e_n + \omega$, so that the terms $\langle n, i | A(\omega) \rho_S A(\omega) | n', i' \rangle$ corresponds to horizontal coherences and populations. The state $A(\omega)|n, i\rangle$ belongs to the eigenspace of energy $e_n$, so that $\langle n, i | A(\omega) | n', i' \rangle$ also corresponds to horizontal coherences or populations (still when $n = n'$).

In other words, the dynamics of the populations and horizontal coherences are coupled. By contrast, if $n \neq n'$, one can see that only vertical coherences appear on the the right-hand side of (41), so that the dynamics of vertical coherences is not coupled neither to the populations nor to the horizontal coherences.

**Dissipation of near degenerate systems.** In Section $[\Pi]$ we assume a perfectly degenerate system $S$. In this paragraph we show that this requirement can be significantly relaxed. We consider that the Hamiltonian of $S$ is non-degenerate and of the form
\[ H_S = \sum_n \sum_{i=1}^{l_n} e_{n,i} |n, i\rangle \langle n, i|. \]  
However, we assume that for all $n$ and all $i, i'$, the energy gap $|e_{n,i} - e_{n,i'}|$ is at most of the order of magnitude $\delta$, whereas for all $n, m, i$, and $j$, $|e_{n,i} - e_{m,j}|$ is of the order of magnitude $\omega$, with $\omega \gg \delta$. In the following we refer to these properties as near degeneracy. Note that several systems, from multi-level atoms containing some close energy levels to ensembles of interacting subsystems with homogeneities, are near degenerate. Indeed, in the later situation, one can always decompose the Hamiltonian of the ensemble as $H_S = H_0 + H_{\text{inh}} + H_{\text{int}}$, where $H_0$ denotes the sum of the local free Hamiltonians of each subsystem, $H_{\text{inh}}$ corresponds to inhomogeneities representing potential small differences between the subsystems (like different energy transitions), and $H_{\text{int}}$ stands for interaction between the subsystems. Using for instance the degenerate perturbation theory $[75]$, one can easily see that $H_S$ can indeed be re-written in the form (42) satisfying the near degeneracy criteria.

For such a near degenerate system $S$, the Markovian master equation is in principle different from $[\delta]$ because the levels $|n, i\rangle$, $1 \leq i \leq l_n$ are not degenerate. Furthermore, the secular approximation $[43, 52]$ is not valid anymore if $\delta$ is of the order or smaller than $g^2 \tau_c$, where $\tau_c$ denotes the bath coherence time and $g$ is the system-bath coupling strength. In the following, we show that indeed one can recover a master equation of the form $[\delta]$ when $\delta \ll g^2 \tau_c \ll \omega$. Starting from the Born and Markov approximations (valid for $g \tau_c \ll 1$) $[43, 52]$, the reduced dynamics of $S$ before applying the secular approximation
\[ \dot{\rho}_t = \int_0^\infty du \text{Tr}_B \left[ V(t-u)\rho_t \rho_B V(t) - V(t)\rho_t \rho_B \right] + \text{h.c.} \]
\[ = \sum_{m,n,m',n',i,i',j,j'} \Gamma(e_{m,j} - e_{n,i}) \times e^{-i(e_{m,j} - e_{n,i} - e_{m',j'} + e_{n',i'})t} \rho_{n,i,m,j} A_{n,i,m,j}^* A_{n',i',m',j'} \times \left( |n,i\rangle \langle m,j| \rho_t |m',j'\rangle \langle n',i'| - |m',j'\rangle \langle n',i'| |n,i\rangle \langle m,j| \rho_t \right) + \text{h.c.} \] (43)

where \( V(t) \) is the coupling Hamiltonian in the interaction picture (with respect to \( H_S \)), the coefficients \( A_{n,i,m,j} := \langle n,i|A_S|m,j \rangle \) are the amplitudes of transitions, and the sum \( \sum_{i,i',j,j'} \) is a short notation for the 4 terms factorises in the following form

\[ e^{-i(e_{m,j} - e_{n,i} - e_{m',j'} + e_{n',i'})t} \Gamma(e_{m,j} - e_{n,i}) A_{n,i,m,j} A_{n',i',m',j'}^* \rho_{n,i,m,j} (|n,i\rangle \langle m,j| \rho_t |m',j'\rangle \langle n',i'| - |m',j'\rangle \langle n',i'| |n,i\rangle \langle m,j| \rho_t) + \text{h.c.} \]
\[ + e^{-i(e_{m,j} - e_{n,i} - e_{m',j'} + e_{n',i'})t} \Gamma(e_{m,j'} - e_{n,i'}) A_{n,i',m,j'} A_{n',i',m',j'}^* \rho_{n,i',m,j'} (|n,i'\rangle \langle m,j'| \rho_t |m,j\rangle \langle n,i| - |m,j\rangle \langle n,i| |n,i'\rangle \langle m,j'| \rho_t) + \text{h.c.} \]
\[ + \Gamma(e_{m,j} - e_{n,i}) A_{n,i,m,j} \rho_{n,i,m,j} (|n,i\rangle \langle m,j| \rho_t |m',j'\rangle \langle n',i'| - |m',j'\rangle \langle n',i'| |n,i\rangle \langle m,j| \rho_t) + \text{h.c.} \]
\[ + \Gamma(e_{m,j'} - e_{n,i'}) A_{n,i',m,j'} \rho_{n,i',m,j'} (|n,i'\rangle \langle m,j'| \rho_t |m,j\rangle \langle n,i| - |m,j\rangle \langle n,i| |n,i'\rangle \langle m,j'| \rho_t) + \text{h.c.}. \] (44)

Thus, all together we obtain \( |\Gamma(\omega + \delta) - \Gamma(\omega)| \sim g^2 \tau_c^2 \delta \ll g^2 \tau_c \sim |\Gamma(\omega)| \). Furthermore, considering times \( \tau \) such that \( \delta \tau \ll 1 \), one can approximate the above phases by 1, \( e^{-i(e_{m,j} - e_{n,i} - e_{m',j'} + e_{n',i'})t} \approx 1 \). Then, the sum (44) of the terms factorises in the following form

\[ \left| \frac{\partial \Gamma(\omega)}{\partial \omega} \right| \leq \int_0^{\tau_c} u |\text{Tr}_B A_B(u) A_B| du \]
\[ \sim \int_0^{\tau_c} u g^2 du = \frac{1}{2} g^2 \tau_c^2. \] (46)

\[ \Gamma(e_{m,j} - e_{n,i}) \left[ A_{n,i,m,j}^* |n,i\rangle \langle m,j| - A_{n,i',m',j'}^* |n,i'\rangle \langle m',j'| + A_{n,i',m,j'}^* |m',j'\rangle \langle n,i| - A_{n,i',m',j'}^* |m',j'\rangle \langle n,i| \right] \rho_t \]
\[ - \left( A_{n,i,m,j}^* |m,j\rangle \langle n,i| + A_{n,i',m',j'}^* |m',j'\rangle \langle n,i'| + A_{n,i',m',j'}^* |m',j'\rangle \langle n,i'\rangle \right) \rho_t \]
\[ + \text{h.c.} \] (47)

Repeating this procedure for all transitions \( |m,j\rangle \rightarrow |n,i\rangle \) such that \( |e_{m,j} - e_{n,i} - \omega| \leq \delta \), we can define
\[ A(\omega) = \sum_{m,n,i,j;|e_{m,j} - e_{n,i} - \omega| \leq \delta} A_{n,i,m,j} \rho_t |m,j\rangle. \] (48)
With such definition, for all times $t$ such that $(g^2 \tau_c)^{-1} \leq t \ll \delta^{-1}$, the above master equation \cite{13} can be rewritten as,

$$\dot{\rho}_t = \sum_{\omega} \Gamma(\omega)[A(\omega)\rho_t A^\dagger(\omega) - A^\dagger(\omega)A(\omega)\rho_t] + h.c.,$$

(49)

which coincides with the form of the master equation \cite{9}. Note that the condition $t \geq (g^2 \tau_c)^{-1}$ is the usual condition for the validity of the secular approximation. The new condition here, consequence of near degeneracy, is $t \ll \delta^{-1}$, which requires $\delta^{-1} \ll (g^2 \tau_c)^{-1}$ due to the previous condition. Since $g\tau_c \ll 1$ (condition for the validity of the Born and Markov approximations \cite{9}), one can see that significant inhomogeneities or interaction between subsystems of order up to $g$ still allow for a reduced dynamics of the form \cite{5}. This also means that the bath does not resolve energy differences of order $\delta$ until $\delta t \sim 1$.

**Entropy production in non-degenerate systems.**

For non-degenerate systems, $\rho_{t|D} = \rho_{t|BD}$, so that $\dot{C}_r = 0$ at all times. Moreover, $-\dot{C}_r$ and $-\dot{D}_{th}$ become equivalent to the coherent and diagonal contributions introduced in Eq. (12) of \cite{8}. One can also show that this two remaining contributions to the entropy production are always positive. Considering the time derivative of the relative entropy of coherences one obtains

$$-\dot{C}_r = -\frac{d}{dt}S(\rho_{t|BD}) = -\frac{d}{dt}S(\rho_{t|D})$$

$$= -\lim_{dt \to 0} \frac{1}{dt} \left[ S(\rho_{t+dt} \rho_{t+dt|D}) - S(\rho_{t} \rho_{t|D}) \right]$$

$$= -\lim_{dt \to 0} \frac{1}{dt} \left[ S(e^{dt\mathcal{L}} \rho_{t} e^{dt\mathcal{L}} \rho_{t|D}) - S(\rho_{t} \rho_{t|D}) \right],$$

(50)

which is always positive since the relative entropy is contractive under completely positive and trace preserving maps \cite{13, 37} (the map generated by $\mathcal{L}$ defined in \cite{5} being completely positive and trace preserving). Similarly, one can show that velocity of the population convergence to the thermal equilibrium distribution $-\dot{D}_{th}$ is, as expected, always positive, recovering the results of \cite{8}. The above result relies on the following crucial step,

$$\rho_{t+dt|D} = e^{dt\mathcal{L}} \rho_{t|D},$$

(51)

which means that the dynamics of the populations depends only on the populations themselves (remembering that we are considering non-degenerate systems). In other words, the presence of coherences does not influence the future values of the populations. This is a fundamental difference with degenerate systems, as shown in Section \cite{1}, Eq. \cite{11} is also equivalent to the Pauli equation \cite{8, 33} (which gives the dynamics of the populations in terms of themselves), and to \cite{11} applied to non-degenerate systems.

**Time derivatives of $C_r$, $C_h$, and $D_{th}$.**

While it is well-known that the time derivative of the von Neumann entropy is $-\text{Tr} \dot{\rho}_t \ln \rho_t$, it is not straightforward that similar relations hold for $C_r$, $C_h$, and $D_{th}$. In order to prove the identities \cite{11} we only need to show

$$\frac{d}{dt} \text{Tr} \rho_{t|D} \ln \rho_{t|D} = \text{Tr} \dot{\rho}_t \ln \rho_{t|D},$$

(52)

and,

$$\frac{d}{dt} \text{Tr} \rho_{t|BD} \ln \rho_{t|BD} = \text{Tr} \dot{\rho}_t \ln \rho_{t|BD}.$$ (53)

Starting with the first equality \cite{12}, from the definition of $\rho_{t|D}$ we have $\rho_{t|D} = \sum_n \sum_{l=1}^{l_n} p_{n,l} |n,i\rangle \langle n,i|$, where $p_{n,i} := \langle n,i| \rho_{t} |n,i\rangle$. Thus, $\ln \rho_{t|D} = \sum_n \sum_{l=1}^{l_n} |n,i\rangle \langle n,i| \ln p_{n,i}$ so that

$$\text{Tr} \dot{\rho}_t \ln \rho_{t|D} = \sum_n \sum_{l=1}^{l_n} p_{n,i} \ln p_{n,i}.$$ (54)

Note that the above equations shows $\text{Tr} \rho_{t|BD} \ln \rho_{t|BD} = \text{Tr} \dot{\rho}_t \ln \rho_{t|BD}$. Denoting by $\dot{p}_{n,i} := \langle n,i| \dot{\rho}_t |n,i\rangle$ the time derivative of the populations $p_{n,i}$, one obtains

$$\frac{d}{dt} \text{Tr} \rho_{t|D} \ln \rho_{t|D} = \sum_n \sum_{l=1}^{l_n} \dot{p}_{n,i} \ln p_{n,i} + \sum_n \sum_{l=1}^{l_n} \dot{p}_{n,i}$$

$$= \text{Tr} \dot{\rho}_t \ln \rho_{t|D},$$

(55)

since $\sum_n \sum_{l=1}^{l_n} \dot{p}_{n,i} = \text{Tr} \dot{\rho}_t = 0$.

For the second equality \cite{15} we can proceed in a similar way. Using the definition of $\rho_{t|BD}$ we have

$$\rho_{t|BD} = \sum_n \sum_{i,i'} \pi_n \rho_{t|BD} \pi_n$$

$$= \sum_n \sum_{i,i'} \langle n,i| \rho_{t} |n,i\rangle \langle n,i'| \rho_{t} |n,i'|.$$ (56)

which is diagonal per block (corresponding to each eigenspace). Denoting by $\{|e_{n,i}\rangle\}_{1 \leq i \leq l_n}$ a basis diagonalising $\rho_{t|BD}$ on each eigenspace we have $\rho_{t|BD} = \sum_n \sum_{i=1}^{l_n} \langle e_{n,i}| e_{n,i}\rangle e_{n,i}$, where $q_{n,i} := \langle e_{n,i}| \rho_{t} |e_{n,i}\rangle$. It is important to keep in mind that first, this basis is time-dependent, and secondly, due to the block-diagonal structure of $\rho_{t|BD}$, each $|e_{n,i}\rangle$ is a linear combination of exclusively the eigenvectors $\{|n,i\rangle\}_{1 \leq i \leq l_n}$ (spanning the eigenspace $n$). This two observations will be crucial in the following. As previously, we have

$$\text{Tr} \dot{\rho}_t \ln \rho_{t|BD} = \sum_n \sum_{i=1}^{l_n} q_{n,i} \ln q_{n,i},$$

(57)

which also shows $\text{Tr} \rho_{t|BD} \ln \rho_{t|BD} = \text{Tr} \dot{\rho}_t \ln \rho_{t|BD}$. Differently from the previous situation, the diagonalisation basis $\{|e_{n,i}\rangle\}_{1 \leq j \leq l_n}$ is time dependent. This implies

$$\dot{q}_{n,i} = \frac{d}{dt} \langle e_{n,i}| \rho_{t} |e_{n,i}\rangle + \langle e_{n,i}| \dot{\rho}_t |e_{n,i}\rangle + \langle e_{n,i}| \rho_{t} \frac{d}{dt} |e_{n,i}\rangle.$$ (58)
Then,
\[
\frac{d}{dt} \text{Tr} \rho_t \ln \rho_t \mid_{\text{BD}} = \sum_{n} \sum_{i=1}^{l_n} \langle q_{n,i} \mid \ln q_{n,i} + \dot{q}_{n,i} \rangle \tag{59}
\]
where \( \dot{q}_{n,i} \) is given by the above expression \([58]\). We have,
\[
\sum_{n} \sum_{i=1}^{l_n} \dot{q}_{n,i} = \frac{d}{dt} \sum_{n} \sum_{i=1}^{l_n} \langle e_{n,i} \mid \rho_t \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \\
= \frac{d}{dt} \text{Tr} \rho_t \nonumber \\
= 0
\tag{60}
\]
The second term is slightly more involved,
\[
\sum_{n} \sum_{i=1}^{l_n} q_{n,i} \ln q_{n,i} = \sum_{n} \sum_{i=1}^{l_n} \langle e_{n,i} \mid \rho_t \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\[
+ \sum_{n} \sum_{i=1}^{l_n} \frac{d}{dt} \langle e_{n,i} \rangle \rho_t \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\[
+ \sum_{n} \sum_{i=1}^{l_n} q_{n,i} \langle e_{n,i} \mid \rho_t \langle e_{n,i} \rangle \frac{d}{dt} \langle e_{n,i} \rangle \nonumber \]
\tag{61}
\]
Using the identity \( \langle e_{n,i} \rangle \ln q_{n,i} = \langle \rho_t \mid e_{n,i} \rangle \) the first term is equal to
\[
\sum_{n} \sum_{i=1}^{l_n} \langle e_{n,i} \mid \dot{\rho}_t \mid e_{n,i} \rangle \ln q_{n,i} = \text{Tr} \dot{\rho}_t \ln \rho_t \mid_{\text{BD}}, \tag{62}
\]
The last two terms sum up to zero. This can be shown for instance by introducing the decomposition of the identity 1 = \( \sum_n \pi_n \) on both sides of \( \rho_t \) in each term. Importantly, as mentioned previously, since each vector \( \langle e_{n,i} \rangle \) is a linear combination of exclusively the eigenvectors \( \{ \langle n, i \rangle \} \) of \( \rho_t \), its time derivative \( \frac{d}{dt} \langle e_{n,i} \rangle \) belongs to the eigenspace \( n \) (spanned by the vectors \( \{ \langle n, i \rangle \} \)). This implies in particular that \( \pi_m \frac{d}{dt} \langle e_{n,i} \rangle = 0 \) if \( n \neq m \). We obtain,
\[
\sum_{n} \sum_{i=1}^{l_n} \frac{d}{dt} \langle e_{n,i} \rangle \rho_t \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\[
= \sum_{n} \sum_{i=1}^{l_n} \frac{d}{dt} \langle e_{n,i} \rangle \sum_m \pi_m \rho_t \pi_{m'} \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\[
= \sum_{n} \sum_{i=1}^{l_n} \frac{d}{dt} \langle e_{n,i} \rangle \pi_n \rho_t \pi_n \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\[
= \sum_{n} \sum_{i=1}^{l_n} \frac{d}{dt} \langle e_{n,i} \rangle \rho_t \mid_{\text{BD}} \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\[
= \sum_{n} \sum_{i=1}^{l_n} \frac{d}{dt} \langle e_{n,i} \rangle \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\tag{63}
\]
Proceeding in a similarly for the term \( \sum_{n} \sum_{i=1}^{l_n} \langle e_{n,i} \rangle \langle e_{n,i} \rangle \ln q_{n,i} \), one finally obtains
\[
+ \sum_{n} \sum_{i=1}^{l_n} \frac{d}{dt} \langle \langle e_{n,i} \rangle \rho_t \langle e_{n,i} \rangle \ln q_{n,i} \nonumber \]
\[
+ \sum_{n} \ln q_{n,i} \langle e_{n,i} \rangle \rho_t \frac{d}{dt} \langle e_{n,i} \rangle \nonumber \]
\[
= \sum_{n} \sum_{i=1}^{l_n} \left[ \frac{d}{dt} \langle \langle e_{n,i} \rangle \rangle + \langle e_{n,i} \rangle \frac{d}{dt} \langle \langle e_{n,i} \rangle \rangle \right] \ln q_{n,i} \nonumber \]
\[
= \sum_{n} \sum_{i=1}^{l_n} q_{n,i} \ln q_{n,i} \frac{d}{dt} \langle \langle e_{n,i} \rangle \rangle \nonumber \]
\[
= 0 \tag{64}
\]
Then, all together we have shown what we announced above, equations \([52]\) and \([53]\), completing the demonstration of the identities \([6]\) in Section Results.

**Expression of the heat flow.** The heat flow between the system and the bath is defined by \( \oint \)
\[
\hat{E}_S := \text{Tr} \dot{\rho}_t H_S \tag{65}
\]
Using the dynamics described by \([\oint]\) as the time derivative of \( \rho_t \), one obtains
\[
\hat{E}_S = \sum_\omega \Gamma(\omega) \text{Tr} \left( A(\omega) \rho_t A^\dagger(\omega) - A^\dagger(\omega) A(\omega) \rho_t \right) H_S \nonumber \]
\[
+ \text{c.c.} \nonumber \]
\[
= \sum_\omega \Gamma(\omega) \text{Tr} A(\omega) \rho_t \left[ A^\dagger(\omega), H_S \right] + \text{c.c.} \nonumber \]
\[
= - \sum_\omega \omega \Gamma(\omega) \text{Tr} A(\omega) \rho_t A^\dagger(\omega) + \text{c.c.} \nonumber \]
\[
= - \sum_\omega \omega G(\omega) \langle A^\dagger(\omega) A(\omega) \rangle \rho_t \nonumber \]
\tag{66}
\]
where \( \langle O \rangle \rho_t := \text{Tr} \rho_t O \) stands for the expectation value of any operator \( O \) taken in the state \( \rho_t \). Furthermore, we used in the second line the invariance of the trace under cyclic permutations, in the third line, the commutation relation of the eigenoperators \( [A^\dagger(\omega), H_S] = -\omega A^\dagger(\omega) \), and in the fourth line, the definition \( G(\omega) := \Gamma(\omega) + \Gamma^*(\omega) \) already introduced previously. We can rewrite the above expression in an insightful way by explicitly including the negative frequencies,
\[
\hat{E}_S = - \sum_{\omega>0} \omega \langle G(\omega) \langle A^\dagger(\omega) A(\omega) \rangle \rangle \rho_t \nonumber \]
\[
- G(-\omega) \langle A(\omega) A^\dagger(\omega) \rangle \rho_t \nonumber \]
\[
= \sum_{\omega>0} \omega G(\omega) \langle A(\omega) A^\dagger(\omega) \rangle \rho_t \left( e^{-\omega \beta_B} - e^{-\omega / T(\omega)} \right) \nonumber \]
\tag{67}
\]
where \( T(\omega) := \omega \left( \frac{\ln (\mathcal{A}(\omega) \mathcal{A}^\dagger(\omega))}{\mathcal{A}(\omega) \mathcal{A}^\dagger(\omega)} \right)_\beta \) is the apparent temperature associated with the energy exchange \( \omega \) between the system and the bath, and the bath inverse temperature \( \beta_B \) can be defined through \( e^{-\beta_B \omega} := G(-\omega)/G(\omega) \). In particular, when the population of the state \( p_i \) follows a thermal distribution, as for \( p_0 \) in Eq. (15) of Section II, one can express the inverse apparent temperature as

\[
\omega = T(\omega) = \omega \beta_0 + \ln \frac{1 + e^\omega}{1 + e^{-\omega}},
\]

(68)

where \( \beta_0 \) is the inverse temperature associated with the thermal distribution of the populations, and \( e^\omega := (\mathcal{A}(\omega) \mathcal{A}(\omega))_\beta (\mathcal{A}(\omega) \mathcal{A}^\dagger(\omega))_{\rho^\beta(\beta_0)} \) and \( e^{-\omega} := (\mathcal{A}(\omega) \mathcal{A}(\omega))_\beta (\mathcal{A}(\omega) \mathcal{A}^\dagger(\omega))_{\rho^\beta(\beta_0)} \) constitute the contributions from the horizontal coherences, highlighted in \( [26, 30, 53] \). Indeed, since \( A \) is a horizontal coherence, contributions from vertical coherences do not pick up contributions from vertical coherences. In other words, when \( \chi \) do not contain horizontal coherences one has \( (\mathcal{A}(\omega) \mathcal{A}(\omega))_\chi = (\mathcal{A}(\omega) \mathcal{A}(\omega))_\chi = 0 \) implying \( T(\omega) = 1/\beta_0 \).

**Reduction of irreversibility in spin ensembles.**

In this paragraph we recall some properties of spin ensembles and theory of addition of angular momentum. We also recall the expression of the equilibrium state reached by a spin ensemble when collective interacting with a bath (as described for instance by the Eq. (3) of Section II). Considering an ensemble containing \( n \) spins of size \( s \), we denote by \( j_{z,k} \) the \( z \)-component of the angular momentum operator associated to the spin \( k \in [1;n] \), and by \( \{s, m_k \| k \}_{-s \leq m_k \leq s} \) the local eigenbasis of \( j_{z,k} \), so that \( j_{z,k} |s, m_k \rangle = \hbar m_k |s, m_k \rangle \). Then, a natural basis for the spin ensemble is

\[
|m_1, m_2, ..., m_n \rangle := \otimes_{k=1}^n |s, m_k \rangle_k,
\]

(69)

resulting from the tensor products of the local eigenbasis. One important property from the theory of addition of angular momenta is that the spin ensemble can be described alternatively by a basis obtained from the eigenvectors of the global observables \( J_z \) and \( J^2 := J_x^2 + J_y^2 + J_z^2 \), where \( J_z := \sum_{k=1}^n j_{z,k} \) (and similar definitions for the \( x \) and \( y \) components). These eigenvectors are traditionally denoted by \( |J, m_i \rangle \) in reference to their eigenvalues,

\[
\mathcal{J}^2 |J, m_i \rangle = \hbar J (J+1) |J, m_i \rangle,
\]

\[
J_z |J, m_i \rangle = \hbar m |J, m_i \rangle,
\]

(70)

with \(-J \leq m \leq J \) and \( J \in \{J_0 | n \} \), where \( J_0 = 0 \) if \( s \geq 1 \) and \( J_0 = 1/2 \) if \( s = 1/2 \) and \( n \) odd. The index \( i \) belongs to the interval \([1;l_J]\), where \( l_J \) denotes the degeneracy of the eigenspace associated to the eigenvalue \( J \) of the total spin operator \( \mathcal{J}^2 \).

The equilibrium state reached by the spin ensemble initially in a thermal state at inverse temperature \( \beta_0 \) and interacting collectively with a bath at inverse temperature \( \beta_B \) is given by \( [21] \)

\[
\rho^\infty_{\beta_B}(\beta_B) := \sum_{J=J_0}^N p_J(\beta_0) \sum_{m,J,m} \rho_{J,m}^{\beta_B}(\beta_B),
\]

(71)

where \( p_J(\beta_0) := Z_J(\beta_0)/Z(\beta_0) \), \( Z_J(\beta_0) := \sum_{m,J} e^{-J \hbar m \beta_B} Z(\beta_0) = e^{-J \hbar m \beta_B} \), and \( \rho_{J,m}^{\beta_B}(\beta_B) := Z_J(\beta_B)^{-1} \sum_{m,J} e^{-J \hbar m \beta_B} |J,m \rangle \langle J,m | \).

The details of the expression (71) are not essential here. What is important however is that \( \rho^\infty_{\beta_B}(\beta_B) \) contains horizontal coherences (in the natural basis \([m_1, m_2, ..., m_n] \)) whenever \( \beta_0 \neq \pm \beta_B \). This can be seen by the following considerations. First, note that there is a unique equilibrium state which is diagonal (in the natural basis). The reason is because any diagonal equilibrium state should satisfy the detailed balance, but there is only one diagonal state satisfying it: \( \rho^{\pm \beta}(\beta_B) \), the thermal state of inverse temperature \( \beta_B \). This thermal state is reached for \( \beta_0 = \pm \beta_B \). When \( \beta_0 \neq \pm \beta_B \), the energy of \( \rho^\infty_{\beta_B}(\beta_B) \) is different from the thermal energy of \( \rho^{\pm \beta}(\beta_B) \). Therefore, for any \( \beta_0 \neq \pm \beta_B \), the equilibrium state cannot be equal to \( \rho^{\pm \beta}(\beta_B) \) and consequently cannot be a diagonal state. Moreover, \( \rho^\infty_{\beta_B}(\beta_B) \) does not contain any vertical coherences since it is made up of statistical mixtures of collective spin states \( |J,m_i \rangle |J, m_i | \), themselves containing no vertical coherences, \( \langle m_1, ..., m_n | J, m_i | m_1', ..., m_n' \rangle = 0 \) if \( m_1 + ... + m_n = m_1' + ... + m_n' \). Consequently, \( \rho^\infty_{\beta_B}(\beta_B) \) necessarily contains horizontal coherences, implying \( \rho^\infty_{\beta_B}(\beta_B)_{|D} \neq \rho^\infty_{\beta_B}(\beta_B)_{|D} = \rho^\infty_{\beta_B}(\beta_B) \) so that

\[
-\Delta^\infty C_h = -S[\rho^\infty_{\beta_B}(\beta_B)_{|D}]_{\beta_B} < 0.
\]

Thus, since the ensemble is initially in a thermal state (with zero horizontal coherences), we have creation of horizontal coherences which corresponds to \( -\Delta^\infty C_h = -C_h(\infty) - C_h(0) = -C_h(\infty) < 0 \), promoting a reduction of the entropy production. We can even show that the value of \( \Delta^\infty C_h \) increases logarithmically with the number of degenerate levels (growing with \( n \) and \( s \)), at least in the limit \( \omega |\beta_0| \gg 1 \). One can obtain the following expression for the diagonal cut (21)

\[
-\Delta^\infty C_h = -C_h(\infty)_{|D} = -\sum_{m=-n}^n e^{-\hbar m^2 \beta_B} I_m |m_1, ..., m_n \rangle \langle m_1, ..., m_n |,
\]

(72)

where \( I_m := \sum_{J=|m|}|l_J| \) is the total number of eigenstates of \( J_z \) of eigenvalue \( \hbar m \). One can deduce from (72) the expression for the variation of horizontal coherences,
so that $\Delta^\infty C_h$ grows logarithmically with the number of degenerate levels $I_m$ (itself a growing function of $n$ and $S$).

Beyond that, the variation of $-D_{th}$, $-\Delta^\infty D_{th} := -[D_{th}(\infty) - D_{th}(0)]$, is also reduced (compared to independent dissipation). This is can be seen simply as follows. Since the equilibrium state energy is different from the thermal equilibrium energy [21], we have necessarily $\rho_{\beta,\infty}(\beta_B)_{|D} \neq \rho_{th}(\beta_B)$. Consequently, the measure of distance to the thermal distribution $\beta_{th}$ does not tend to zero but to a strictly positive value, $D_{th}(\infty)_{|col} = S[\rho_{\beta,\infty}(\beta_B)_{|D} | \rho_{th}(\beta_B)] > 0$. By comparison, for independent dissipation, the ensemble reaches the equilibrium thermal state $\rho_{th}(\beta_B)$ so that $D_{th}(\infty)_{|ind} = 0$. This makes the variation $-\Delta^\infty D_{th}$ strictly smaller for collective dissipation than for independent dissipation.

**Thermal operations and beyond.**

*Positivity of $-\Delta C_v$ and positivity break down for $C_h$ and $D_{th}$.* In this section we show that the simple conditions of energy conservation, initial separability, and initial stationarity of $B$ $\langle [H_B, \rho_{S,t}] \rangle = 0$ have rich consequences. As introduced in Section [II], we consider that the systems $S$ and $B$ interact unitarily through $U$ from $t_i$ to $t_f$. The reduced dynamics for $S$ is given by,

$$
\rho_{S,t_f} = \Lambda \rho_{S,t_i} := \text{Tr}_B U \rho_{S,t_i} U^\dagger = \sum_{\mu,\nu} M_{\mu,\nu} \rho_{S,t_i} M_{\mu,\nu}^\dagger
$$

(74)

where $M_{\mu,\nu} := \sqrt{p_{\nu}} \langle \psi_{\mu} | U | \psi_{\nu} \rangle$, $p_{\nu} := \langle \psi_{\nu} | \rho_{B,t_i} | \psi_{\nu} \rangle$, and the eigenstates and eigenenergies of $B$ are denoted respectively by $| \psi_{\nu} \rangle$ and $E_{\nu}$. We also denote by $\mathcal{H}_{S,n}$ the eigenpace of $S$ associated with the energy $E_n$. Since the initial state of $B$ is assumed to be stationary it can be written in the form $\rho_{B,t_i} = \sum_{\nu} p_{\nu} | \psi_{\nu} \rangle \langle \psi_{\nu} |$. Note that we include the possibility of $B$ being degenerate, but in order to simplify the notation we do not explicit write an extra index representing the degeneracy. It means that several $E_{\nu}$ and $p_{\nu}$ can have the same value. The system $B$ can also have an unbounded discrete spectrum or even a continuous spectrum. In this later situation one should express $H_B$ and $\rho_B$ through integrals. For simplicity again we maintain the discrete sum notations but one should bear in mind that the following results can be extended to continuous spectrum. From the above notations, the final (at time $t_f$) vertical coherences between the state $| n, i \rangle$ and $| m, j \rangle$ can be expressed as,

$$
\langle n, i | \rho_{S,t_f} | m, j \rangle = \sum_{\mu,\nu} \langle n, i | M_{\mu,\nu} \rho_{S,t_i} M_{\mu,\nu}^\dagger | m, j \rangle
$$

$$
= \sum_{\mu,\nu,q,r} \langle n, i | M_{\mu,\nu} \pi_q \rho_{S,t_i} \pi_r M_{\mu,\nu}^\dagger | m, j \rangle.
$$

(75)

“Sandwiching” the conservation energy relation $[U, H_S + H_B] = 0$ between $\langle n, i | \psi_{\mu} \rangle$ and $| q, l \rangle | \psi_{\nu} \rangle$ we obtain,

$$
\langle n, i | M_{\mu,\nu} | q, l \rangle (e_{\mu} + E_{\nu} - e_n - E_m) = 0.
$$

(76)

This implies in (75) that if $q$ is such that $e_{\mu} \neq e_n + E_m - E_{\nu}$, we must have $(n,i)|M_{\mu,\nu}|_{\pi_q} = 0$. Similarly for the term $\pi_r |M_{\mu,\nu}|_{\pi_r}$. One may conclude that the only term contributing to (75) are such that $e_{\mu} = e_n + E_m - E_{\nu}$ and $e_{\nu} = e_m + E_{\mu} - E_{\nu}$ so that if $e_n \neq e_m$, we necessarily $e_{\mu} \neq e_{\nu}$. Consequently, only vertical coherences contribute to the sum (75): the dynamics of vertical coherences depends only on vertical coherences. By contrast, if $e_n = e_m$, we have necessarily $e_{\mu} = e_{\nu}$ (and finally $q = r$), so that only horizontal coherences and populations contribute to (75). In other words, the dynamics of the populations and horizontal coherences are coupled, but both are *decoupled* from the dynamics of the vertical coherences. Consequently, the following identity holds,

$$
\rho_{S,t_f | BD} = \Lambda \rho_{S,t_i | BD},
$$

(77)

whereas

$$
\rho_{S,t_f | D} \neq \Lambda \rho_{S,t_i | D}.
$$

(78)

Then, one can repeat the argument in [6] to show that the variation of $-\Delta C_v (t)$ between $t_i$ and $t_f$ is always positive (meaning that vertical coherences are conserved). Namely,

$$
-\Delta C_v = -[C_v(t_f) - C_v(t_i)] = -[S(\rho_{S,t_f} | S_{BD}) - S(\rho_{S,t_i} | S_{BD})] = -[S(\Lambda \rho_{S,t_i} | S_{BD}) - S(\rho_{S,t_i} | S_{BD})] \geq 0.
$$

(79)

where the positivity comes from the contractivity of the relative entropy under completely positive and trace preserving maps (of which $\Lambda$ belongs since we assume $S$ and $B$ initially uncorrelated [43]). By contrast, since $\rho_{S,t_f | D} \neq \Lambda \rho_{S,t_i | D}$, the guarantee of positivity breaks down for $-\Delta C_h$ and $-\Delta D_{th}$, where $D_{th}(t)$ is defined assuming the existence of a thermal equilibrium state $\rho_{\beta,\infty} (\beta_B)$. Note that this is guaranteed at least when $B$ is initially in a thermal state at inverse temperature $\beta_B$ (thanks to energy conservation, $S$ always admits the thermal state $\rho_{\beta,\infty} (\beta_B)$ as equilibrium state). In such situation $U$ corresponds to the well-known thermal operations [42].

On the other hand, the sum of $-\Delta C_h$ and $-\Delta D_{th}$ always remains positive,

$$
-\Delta C_h - \Delta D_{th} = -[C_h(t_f) + D_{th}(t_f) - C_h(t_i) - D_{th}(t_i)]
$$

$$
= -[S(\rho_{S,t_f | BD} | S_{BD}) - S(\rho_{S,t_i | BD} | S_{BD})]
$$

$$
= -[S(\Lambda \rho_{S,t_i | BD} | S_{BD}) - S(\rho_{S,t_i | BD} | S_{BD})] \geq 0,
$$

(80)

as announced in Section [II].

Similarly to what have been done in Section [II] for the bath-driven dissipation, one can pinpoint explicit situations where $-\Delta C_h < 0$. Denoting by $| \psi_{\nu} \rangle$ and $E_{\nu}$ the (possibly degenerate) eigenstates and eigenenergies of $B$, one consequence of the energy conservation is
that the transition $\langle n, i | \psi \rangle_{U(m, j)} | \psi \rangle_{\mu}$ is equal to zero unless $e_m + E_\nu - e_m - E_\mu = 0$. In particular, one can have degenerate transitions from one state $| m, j \rangle | \psi \rangle_{\mu}$ to two degenerate states $| n, i \rangle | \psi \rangle_{\nu}$ and $| n, i' \rangle | \psi \rangle_{\nu}$, expressed by $\langle n, i | \psi \rangle_{U(m, j)} | \psi \rangle_{\mu} \neq 0$ and $\langle n, i' | \psi \rangle_{U(m, j)} | \psi \rangle_{\mu} \neq 0$. Then, if for instance $SB$ is initialised in the state $| m, j \rangle | \psi \rangle_{\mu}$, such a unitary evolution $U$ definitively generates horizontal coherences and a negative contribution to the entropy production, $-\Delta C_h < 0$. This is a mechanism analogue to the one mentioned for bath-driven dissipation in Section II, relying on degenerate transitions (illustrated in Fig. 2).

Importantly, let us consider now the same above double relaxation laws is the commutation of the global unitary evolution $U$, evolution $| \psi \rangle_{| n, i \rangle}$ and $| \psi \rangle_{| n, i' \rangle}$, the energy conservation implies the second final state of $B$ has to be changed from $| \psi \rangle_{| \psi \rangle}$ to $| \psi \rangle_{| \psi \rangle}$ such that $E_{\psi'} = e_m - e_m + E_{\mu}$. Consequently, the coherent superposition generated by $U$ disappears after tracing out $B$ (since $\langle \psi_{\psi'} | \psi_{\psi'} \rangle = 0$).

Thus, interestingly, the energy conservation intrinsically prohibits the generation of vertical coherences whereas the generation of horizontal coherences is allowed. Conservation laws. The first step to show the conservation laws is the commutation of the global unitary evolution $U$ with the operation of global block-diagonalisation. We denote by $\Delta H_{SB + H_{SB}}$ such operation. We also denote by $\{ \epsilon_k \}_{k}$ the different energy levels of the ensemble $SB$, and define

$$\Pi_k := \sum_{m, \mu; \epsilon_m + E_\mu = \epsilon_k} \pi^S_m \pi^B_\mu,$$

(81)

the projector onto the eigenspace of energy $\epsilon_k$, where $\pi^S_m$ ($\pi^B_\mu$) is itself the projector onto the eigenspace of energy $\epsilon_m$ of $S$ ($E_\mu$ of $B$). Then, the global block-diagonalising operation can be expressed as

$$\rho_{SB,t|_{\text{IND}}} = \Delta H_{SB + H_{SB}} \rho_{SB,t} = \sum_k \Pi_k \rho_{SB,t} \Pi_k.$$  

(82)

Importantly, note that $\Delta H_{SB + H_{SB}} \neq \Delta H_{SB} \Delta H_{SB}$, where $\Delta H_{SB}$ denotes the local block-diagonalising operations, $\Delta H_{SB} \rho_{SB,t} = \sum_m \pi^S_m \rho_{SB,t} \pi^S_m$ (and similarly for $B$). From these considerations, one can conclude that $U$ commute with $\Delta H_{SB + H_{SB}}$ if and only if $U$ commute with $\Pi_k$, for all $k$. Due to energy conservation, we indeed have $[U, \Pi_k] = 0$, whereas in general $[U, \pi^S_m] \neq 0$ and $[U, \pi^B_\mu] \neq 0$. Therefore, we obtain that $U$ commute with $\Delta H_{SB + H_{SB}}$ while in general this is not true for $\Delta H_{SB}$ and $\Delta H_{SB}$. This leads to the following important identity,

$$S(\rho_{SB,t|_{\text{IND}}}) = S(\rho_{SB,t|_{\text{IND}}}),$$

(83)

(where $S(\rho) := -\text{Tr} \rho \ln \rho$ denotes the von Neumann entropy). This equality implies in particular the global conservation of vertical coherences,

$$\Delta C^S_v = 0,$$

(84)

as announced in [34] of Section II. Furthermore, thanks to the initial separability of $S$ and $B$, we have,

$$\rho_{SB,t|_{\text{IND}}} = \sum_k \Pi_k \rho_{SB,t} \rho_{SB,t} \Pi_k$$

$$= \sum_k \sum_{m, \mu; \epsilon_m + E_\mu = \epsilon_k} \sum_{m', \mu'; \epsilon_m' + E_{\mu'} = \epsilon_k} \pi^S_m \rho_{SB,t} \pi^S_m \pi^B_{\mu} \rho_{SB,t} \pi^B_{\mu'}. 

(85)

$$= \sum_k \sum_{m, \mu; \epsilon_m + E_\mu = \epsilon_k} \pi^S_m \rho_{SB,t} \pi^S_m \pi^B_{\mu} \rho_{SB,t} \pi^B_{\mu}$$

$$= \sum_k \sum_{m} \sum_{\mu} \pi^S_m \rho_{SB,t} \pi^S_m \pi^B_{\mu} \rho_{SB,t} \pi^B_{\mu}$$

$$= \rho_{SB,t|_{\text{IND}}} \rho_{SB,t|_{\text{IND}}} \rho_{SB,t|_{\text{IND}}},$$

(86)

where we used $\pi^S_m \rho_{SB,t} \pi^B_\mu = \delta_{m,m} \pi^B_\mu \rho_{SB,t} \pi^B_\mu$ since we assumed that $\rho_{SB,t}$ is a stationary state, $[H_B, \rho_{SB,t}] = 0$. Without this condition, $\rho_{SB,t|_{\text{IND}}} \neq \rho_{SB,t|_{\text{IND}}} \rho_{SB,t|_{\text{IND}}}$. In other words, when any of the two system $S$ and $B$ is in a stationary state, the global and local block-diagonalising operations are identical, $\Delta H_{SB + H_B} = \Delta H_S \Delta H_B$. This leads to

$$S(\rho_{SB,t|_{\text{IND}}}) = S(\rho_{SB,t|_{\text{IND}}}) = S(\rho_{SB,t|_{\text{IND}}}) + S(\rho_{SB,t|_{\text{IND}}}) - S(\rho_{SB,t|_{\text{IND}}}).$$

(87)

Note that one can prove similarly the following identity, $\rho_{SB,t|_{\text{IND}}} \rho_{SB,t|_{\text{IND}}} = \Delta H_{SB} \rho_{SB,t} \rho_{SB,t}$, valid for any state $\rho_{SB,t}$. Together with (85) and the commutation of $U$ and $\Delta H_{SB + H_{SB}}$ one can formally prove [29] (corresponding to [20] of Section II).

The above identity (86) can be used to refine the conservation law of vertical coherences. The final relative entropy of vertical coherences of $SB$ can be rewritten as

$$C^S_v(t_f) = S(\rho_{SB,t|_{\text{IND}}}) - S(\rho_{SB,t|_{\text{IND}}})$$

$$= S(\rho_{SB,t|_{\text{IND}}}) - S(\rho_{SB,t|_{\text{IND}}}) - S(\rho_{SB,t|_{\text{IND}}})$$

$$= C^S_v(t_i) + C^B_v(t_i) - C^S_v(t_i),$$

(88)

where we used (86), the initial separability of $S$ and $B$, and the stationarity of the initial state of $B$, which implies $C^B_v(t_i) = 0$. Note that this result was already derived in [3]. It means in particular that all the vertical coherences present initially in $\rho_{SB,t|_{\text{IND}}}$ end up in $SR$. This can be made even more precise: the consumption of vertical coherences in $S$ and in $B$ is equal to the final correlated vertical coherences,

$$-\Delta C^S_v - \Delta C^B_v = C^S_v(t_f),$$

(89)

quantifies the portion of vertical coherences contained in $SB$ due to correlations between $S$ and $B$ [3]. Eq. (89) is obtained by subtracting $C^S_v(t_f)$ and $C^B_v(t_f)$ on both sides of Eq. (87).
Still based on (86), a similar conservation law can be obtained for horizontal coherences when adding contributions from population convergence. The relative entropy of horizontal coherences $C_{h}^{SB}(t_f)$ and the measure of distance $D_{th}^{SB}(t_f)$ to a global thermal equilibrium state $\rho_{SB}^{th}(\beta_B)$ are defined in the same way as for $S$. Note that due to the energy conservation, any global thermal state is a steady state of $SB$ (valid independently of the initial state of $B$). Then, we have,

$$C_{h}^{SB}(t_f) + D_{th}^{SB}(t_f)$$

$$= \text{Tr}\rho_{SB,t_f}[\ln \rho_{SB,t_f|BD} - \ln \rho_{SB}^{th}(\beta_B)]$$

$$= -S(\rho_{SB,t_f|BD}) - \text{Tr}\rho_{SB,t_f}\ln \rho_{SB}^{th}(\beta_B)$$

$$= -S(\rho_{SB,t_f|BD}) - S(\rho_{B,t_f|BD})$$

$$= -S(\rho_{SB,t_f|BD}) - S(\rho_{B,t_f|BD})$$

$$= -\text{Tr}\rho_{SB,t_f}\ln \rho_{SB}^{th}(\beta_B) - S(\rho_{B,t_f|BD})$$

$$= -S(\rho_{SB,t_f|BD}) - S(\rho_{B,t_f|BD})$$

$$= C_{h}^{SB}(t_f) + D_{th}^{SB}(t_f) + C_{h}^{B}(t_f) + D_{th}^{B}(t_f). \ (90)$$

Since $S$ and $B$ are initially uncorrelated, the identity (90) implies the following conservation law

$$\Delta C_{h}^{SB} + \Delta D_{th}^{SB} = 0. \ (91)$$

Defining concepts of correlated horizontal coherences,

$$C_{c,h}^{SB}(t) := C_{h}^{SB}(t) - C_{h}^{B}(t_f) - C_{h}^{B}(t_f) \geq 0, \ (92)$$

and correlated population distance to the thermal equilibrium state,

$$D_{c,th}^{SB}(t) := D_{th}^{SB}(t) - D_{th}^{B}(t) - D_{th}^{B}(t_f) \geq 0, \ (93)$$

in a similar way as (89), one can obtain a refined statement in the same form as (88),

$$-\Delta C_{c,h}^{SB} - \Delta D_{c,th}^{SB} - \Delta D_{th}^{B} = C_{c,h}^{SB}(t_f) + D_{c,th}^{SB}(t_f). \ (94)$$

The above identity means that the consumption of horizontal coherences in $S$ and $B$ plus the population convergence to the local equilibrium state is equal to the final correlated horizontal coherences plus correlated distance. As a corollary, horizontal coherences are not conserved, as expected. Quite curiously, (91) and (94) can be established using any global thermal state as steady state. Thus, even though $\rho_{SB}^{th}(\beta_B) = \text{Tr}_B\rho_{SB}^{th}(\beta_B)$ might not be an equilibrium state of $S$ for $\Lambda$ (since $\Lambda$ and its equilibrium states depend on the initial state of $B$), still, the conservation laws hold.

Considering a situation analogue to the paragraph “Complementarity of horizontal coherences and heat flow reversal” of Section 11, one can explicitly obtain a situation with $-\Delta D_{th}^{SB} < 0$. More precisely, one can take $B$ initially in a thermal state at inverse temperature $\beta_B$ and $S$ initially in a state $\rho_{S,0} = \rho_{SB}^{th}(\beta_B) + \chi$ composed of populations thermally distributed at the same inverse temperature $\beta_B$ and $\chi$ containing horizontal coherences (remembering that vertical coherences have no effect here). Defining the entropy production from the thermal equilibrium state $\rho_{SB}^{th}(\beta_B)$ one has initially $D_{th}^{SB}(0) = 0$ whereas $D_{th}^{SB}(t = +\infty) > 0$ since $S$ gains (loses) energy if the horizontal coherences contained in $\chi$ are such that $c^+ > c^- (c^+ c^-)$, see (18). Consequently, $-\Delta D_{th}^{SB} < 0$.

Using the above conservation law (91) (corresponding to (97) of Section 11) one can also see that the presence of initial horizontal coherences can lead to $-\Delta D_{th}^{SB} < 0$, corresponding to a global divergence of the populations from the thermal equilibrium distribution.

Finally, using (91) we can show explicitly that the generation of horizontal coherences affect the energy exchanges. This can be seen as follows. Considering $B$ in a thermal state a inverse temperature $\beta_B$ and defining the population convergence with respect to the global thermal state at inverse temperature $\beta_B$, we have the following identity, $-\Delta D_{th}^{SB} = -\beta_B F_{SB}^{B}$. From the energy conservation and using (91), one can rewrite the final diagonal entropy as

$$S(\rho_{SB,t_f|BD}) = S(\rho_{SB,t_f|BD}) + \Delta C_{h}^{SB}. \ (95)$$

Then, one can see that the final global diagonal entropy is strictly increased in a scenario where horizontal coherences are generated when compared to a situation where no horizontal coherences is generated. It means that the final populations (of $S$ or $B$, or both) are necessarily altered by the generation of horizontal coherences, implying that both the final energy and the energy exchange are altered. Thus, through the conservation laws, one recovers the observation made in [21] [25]: bath-induced coherences affects the energy exchanges.

**Effects in thermal machines performances.** In this paragraph we look at a cyclic thermal machine with a working medium $S$ containing degenerate energy levels. The working medium $S$ is successively in contact with a cold bath at temperature $T_c$ and a hot bath at temperature $T_h$. Due to the degeneracy of $S$, the coupling with each bath might involve degenerate transitions (see Fig. 2), resulting in coupled dynamics of the horizontal coherences and populations. As seen in Section 11, this might lead to negative contributions to the entropy production. Conversely, one can imagine a situation where the coupling with the baths involves no degenerate transitions (and therefore no negative contribution to the entropy production). As a result, the two kinds of dynamics might result in different entropy production (as for instance in Section 11 paragraph “Towards reversibility: Reduction of entropy production”). The aim of this paragraph is to compare the performances of the “coherent” thermal machine (when horizontal coherences and populations are coupled) to the performances of the “incoherent” thermal machine (when horizontal coherences and populations are not coupled). To simplify the discussion we consider a simple Otto cycle [77] [78] composed of the usual succession of one adiabatic stroke, one iso-
chotic stroke in contact with the cold bath, one adiabatic stroke (which can be the reverse of the first one), and finally one isochoric stroke in contact with the hot bath. For the incoherent machine we denote by $Q_c$, $Q_h$, and $\Sigma$ the heat exchanged per cycle with the cold bath (i.e., during the isochore with the cold bath), the hot bath and the entropy production per cycle, respectively. For the coherent one we denote the corresponding quantities by $Q^*_c$, $Q^*_h$, and $\Sigma^*$. During one full cycle, the entropy variation $\Delta S$ of $S$ is null and the second law can be expressed as

$$0 = \Delta S = \Sigma + \frac{Q_c}{T_c} + \frac{Q_h}{T_h}. \quad (96)$$

Similarly, for the coherent machine we have $0 = \Delta S = \Sigma^* + \frac{Q^*_c}{T_c} + \frac{Q^*_h}{T_h} = 0$. Then, assuming $\Sigma \neq \Sigma^*$ (as a result of negative contributions to the entropy production) and defining $\Delta Q_c := Q^*_c - Q_c$ and $\Delta Q_h := Q^*_h - Q_h$, we have necessarily $\Delta Q_c, \Delta Q_h \neq 0$. Considering the work extraction operating mode of the machine, we denote by $W = -Q_c - Q_h \leq 0$ the work extracted per cycle, and by $\eta = \frac{|W|}{Q_h}$ the associated efficiency (for the incoherent machine). For the coherent machine the corresponding quantities are denoted by $W^*$ and $\eta^*$.

Even though $\Delta Q_c, \Delta Q_h \neq 0$, we might have $W = W^*$. If so, one can show that the efficiency are necessarily different. More precisely, some simple manipulations give

$$\eta^* = \eta + \frac{\Delta Q_h}{Q_h} \frac{W}{Q_hQ_h^*}. \quad (97)$$

In particular, still assuming work extraction, we have $\eta^* > \eta$ if and only if $\Sigma^* < \Sigma$.

Conversely, even though $\Delta Q_c, \Delta Q_h \neq 0$, we might have $\eta = \eta^*$. Similarly, we can show that such situation implies

$$|W^*| = |W| + \left(1 + \frac{Q^*_c}{Q_hh}ight) \Delta Q_h. \quad (98)$$

In particular, one has $|W^*| > |W|$ if and only if $\Sigma^* > \Sigma$.

Then, we can draw the conclusion that any change in the entropy production per cycle inevitably affects (positively or negatively) the power or efficiency (or both) of the machine. Therefore, the alteration of the entropy production described throughout the paper can have important implication for thermal machines. As an illustration, taking for $S$ an ensemble of spins one can show that $\eta^* = \eta$ always holds and that $\Sigma^* > \Sigma$ for adequately chosen values of $T_c$ and $T_h$. This implies that $|W^*| > |W|$, recovering the results of [21]. For bad choice of $T_c$ and $T_h$, $\Sigma^* < \Sigma$ and the extracted work per cycle is degraded.

Note that this analysis fails for vertical coherences because they break the cycle: if vertical coherences are introduced at the beginning of the cycle, they are not recovered at the end of the cycle (at least when considering adiabatic strokes). By contrast, horizontal coherences do not need to be introduced, they are induced by the bath through collective coupling (or degenerate transitions). Considering more complex cycles by introducing non-adiabatic strokes, vertical coherences (in the eigenbasis of the instantaneous Hamiltonian of the working medium) can be generated by the external driving [10], so that one has the possibility to cyclically recover vertical coherences, and a similar analysis as the above one may apply.

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