Abstract. A smooth manifold hosts different types of submanifolds, including embedded, weakly-embedded, and immersed submanifolds. The notion of an immersed submanifold requires additional structure (namely, the choice of a topology); when this additional structure is unique, we call the subset a uniquely immersed submanifold. Diffeology provides yet another intrinsic notion of submanifold: a diffeological submanifold.

We show that from a categorical perspective, diffeology rises above the others: viewing manifolds as a concrete category over the category of sets, the initial morphisms are exactly the (diffeological) inductions, which are the diffeomorphisms with diffeological submanifolds. Moreover, if we view manifolds as a concrete category over the category of topological spaces, we recover Joris and Preissmann’s notion of pseudo-immersions.

We show that these notions are all different. In particular, a theorem of Joris from 1982 yields a diffeological submanifold whose inclusion is not an immersion, answering a question that was posed by Iglesias-Zemmour. We also characterize local inductions as those pseudo-immersions that are locally injective.

In appendices, we review a proof of Joris’ theorem, pointing at a flaw in one of the several other proofs that occur in the literature, and we illustrate how submanifolds inherit paracompactness from their ambient manifold.

1. Overview

This is a mostly-expository paper about intrinsic notions of “submanifold”.

Weakly-embedded submanifolds — for example, an irrational line in the torus — are abundant; see Example 2.6. Until some time ago, the senior author — Yael Karshon — was under the (incorrect) impression that the notion of a weakly-embedded submanifold “obviously” coincides with the diffeological notion of a submanifold (when the ambient space is a manifold). Jordan Watts pointed out that this “fact” is not obvious, and it is equivalent to an open question that Iglesias-Zemmour posed in his book [10]. After playing around with these concepts we came up with the cusp \( \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3 \} \), but we couldn’t determine if it is a diffeological submanifold. David Miyamoto then discovered Joris’ theorem [11], which implies that the cusp is a diffeological submanifold, resolving Iglesias-Zemmour’s open question. The one-page note that we planned to write about this observation evolved into the current paper.

In Section 2, we compare (embedded) submanifolds, weakly-embedded submanifolds, diffeological submanifolds, and uniquely immersed submanifolds. All of these notions are intrinsic: they are properties of a subset \( S \) of a manifold that do not require an a priori choice.
of an additional structure on $S$. This is in contrast with the notion of “immersed submanifold”, which is not intrinsic. As far as we know, the fourth of these notions — that of a uniquely immersed submanifold — is new. With the help of Joris’ theorem, we show that these four notions of submanifolds are all different.

Section 3 is about (diffeological) inductions and pseudo-immersions of manifolds. Iglesias-Zemmour’s open question was whether every induction between manifolds is an immersion; Joris’ theorem provides a negative answer. The related notion of a pseudo-immersion was defined by Joris and Preissmann [12], following a suggestion of Frölicher. Every induction between manifolds is a pseudo-immersion, but being a pseudo-immersion is a local property, so a pseudo-immersion is not necessarily an induction. Moreover, a pseudo-immersion need not even be a local induction: Joris and Preissmann found a pseudo-immersion that is not locally injective. This naturally raises the question of whether every locally injective pseudo-immersion is a local induction. We give a positive answer to this question, in Corollary 3.10. Finally, in Remark 3.12 we show that both inductions and pseudo-immersions arise naturally from a categorical point of view, as initial morphisms, when we view the category of manifolds as a concrete category over the category of sets, and, respectively, as a concrete category over the category of topological spaces.

In Appendix A we present a proof of Joris’ theorem that is based on that of Amemiya and Masuda [2]. We also point a subtle error in another author’s shorter alleged proof of this theorem. In Appendix B we review the basics of diffeology. In Appendix C we prove a characterization of inductions that we used in Remark 3.12. In Appendix D we discuss the paracompactness and second countability assumptions that appear in definitions of a manifold. Paracompactness has the advantage that this property is automatically inherited by diffeological (in particular, weakly-embedded) submanifolds. In Appendix E we compare the concepts of this paper with similar concepts in some books.

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2. Submanifolds

Submanifolds, diffeological submanifolds, and weakly-embedded submanifolds.

A smooth structure on a topological space is a maximal smooth atlas. Except where we say otherwise, a manifold is a set equipped with a topology that is Hausdorff and paracompact and with a smooth structure.

- A subset $S$ of a manifold $N$ is a submanifold (equivalently, embedded submanifold) if it has a (necessarily unique) manifold structure such that a real-valued function $f : S \to \mathbb{R}$ is smooth if and only if each point of $S$ has an open neighbourhood $V$ in $N$ and a smooth function $F : V \to \mathbb{R}$ that coincides with $f$ on $S \cap V$.

For equivalent definitions, see Remark E.2.
Some authors also refer to properly embedded submanifolds, namely, submanifolds whose inclusion map is proper. This holds if and only if the submanifold is closed in the topology of the ambient manifold.

**Example 2.1.** Open subsets of manifolds are submanifolds. Regular level sets of smooth functions are properly embedded submanifolds.

A diffeological space is a set $X$, equipped with a collection of maps (which are declared to be smooth) from open subsets of Cartesian spaces to $X$, that satisfies three axioms. Manifolds can be identified with those diffeological spaces that satisfy certain conditions. See Appendix B for details. The diffeological notion of “submanifold”, in the special case that the ambient diffeological space is a manifold, is equivalent to the first of the following two notions.

Let $N$ be a manifold, $S$ a subset of $N$, and $ι: S → N$ the inclusion map.

- The subset $S$ is a **diffeological submanifold** of $N$ if it has a (necessarily unique) manifold structure such that the following holds:

$$q: U → S \text{ is smooth if and only if } ι ∘ q: U → N \text{ is smooth.}$$

- The subset $S$ is a **weakly-embedded submanifold** of $N$ if it is a diffeological submanifold and the inclusion map $ι: S → N$ is an immersion.

Every submanifold is a weakly-embedded submanifold, but not every weakly-embedded submanifold is a submanifold;\(^{(1)}\) take, for example, the irrational line $\{ [t, \sqrt{2}t] \}$ in the torus $\mathbb{R}^2/\mathbb{Z}^2$, or the topologist’s sine curve $\{(x, y) | x = 0 \text{ or } y = \sin(1/x)\}$ in $\mathbb{R}^2$. A weakly-embedded submanifold is a submanifold if and only if its manifold topology agrees with its subset topology.

Is every diffeological submanifold weakly-embedded? No:

**Example 2.3.** The cusp $\{(x, y) ∈ \mathbb{R}^2 | x^2 = y^3\}$ is a diffeological submanifold of $\mathbb{R}^2$ that is not weakly-embedded; see Figure 2.4.

This follows from a theorem of Henri Joris from 1982 [11, Théorème 1]:

**Theorem 2.5** (Joris). *Let $(m, n)$ be a relatively prime pair of positive integers and $g: U → \mathbb{R}$ a real-valued function on a manifold $U$. Then $g$ is smooth if and only if $g^m$ and $g^n$ are smooth.*

**Details for Example 2.3.** Equip the cusp with the manifold structure that is given by the parametrization $t → (t^3, t^2)$ for $t ∈ \mathbb{R}$. Joris’ theorem implies that this manifold structure satisfies Condition (2.2). With this manifold structure, the inclusion map of the cusp into $\mathbb{R}^2$ is not an immersion. □

Weakly-embedded submanifolds are particularly useful:

\(^{(1)}\)This is an example of the red herring principle [9, Chapter 1, footnote on p. 22]: in mathematics, a red herring does not have to be either red nor a herring.
**Figure 2.4.** The cusp $x^2 = y^3$.

**Example 2.6.** On a (Hausdorff and paracompact) manifold $N$, the leaves of any (regular) foliation are weakly-embedded submanifolds [15, Theorem 19.17]. So are the leaves of any singular foliation [21, 20]. So are the orbits of any Lie group action on $N$ [5] and of any Lie algebroid action on $N$. In Lie theory, the bijection between Lie subalgebras and connected Lie subgroups is with “Lie subgroup” interpreted as weakly-embedded [15, Theorems 19.25 and 19.26].

**Remark 2.7.** A subset $S$ of a manifold $N$ is a weakly-embedded submanifold if and only if the following holds; see Remark E.3.

\[ \text{About each point of } S \text{ there is a chart } \varphi : O \to \Omega \subset \mathbb{R}^n \text{ of } N \]

\[ \text{that takes the smooth path component of the point in } S \cap O \]

\[ \text{onto the intersection of } \Omega \text{ with a linear subspace of } \mathbb{R}^n. \]

Here, the smooth path components of a subset $S$ of a manifold $N$ are the equivalence classes for the equivalence relation on $S$ that is generated by the smooth paths in $N$ that are contained in $S$. For example, let $N = \mathbb{R}^2$ and let $S$ be the union of the $y$-axis with the graph of the function $x \sin(1/x)$ over the positive $x$-axis. Then $S$ has one path component but two smooth path components (because the curve $(x, x \sin(1/x))$, for $0 \leq x \leq 1$, is not rectifiable).

**Uniquely immersed submanifolds.**

When claiming that a subset $S$ of a manifold $N$ is a manifold, what does “is” mean? This can usually be interpreted as the existence of a unique manifold structure on $N$ such that some condition holds. Being a diffeological submanifold is a reasonable condition; being an immersed or weakly-embedded submanifold are special cases. In contrast to being an immersed submanifold (see Remark E.2), these properties are intrinsic, in that they are

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(2) Under some definitions of “foliation” or “singular foliation” this is a tautology. Here is a statement that is not a tautology. A distribution on a manifold $N$ is a subbundle of $TN$; a singular distribution is a subset of $TN$ that near each point is the span of a set of (smooth) vector fields. An integral submanifold is a connected immersed submanifold (see Remark E.2) whose tangent space coincides with the singular distribution at each of its points. A singular distribution is integrable if each point is contained in an integral submanifold. The leaves — defined as the equivalence classes of the relation generated by the integral submanifolds — are then weakly-embedded integral submanifolds.
properties of a subset $S$ of a manifold $N$, not additional structures on $S$. Here is yet another intrinsic property, which we have not seen elsewhere in the literature:

- A subset $S$ of a manifold $N$ is a *uniquely immersed submanifold* if it has a unique manifold structure such that the inclusion map $S \hookrightarrow N$ is an immersion.

In contrast to embedded submanifolds, weakly-embedded submanifolds, and diffeological submanifolds, here “unique” is being assumed, not concluded.

Every weakly-embedded submanifold is uniquely immersed. The figure eight in $\mathbb{R}^2$ (cf. Remark E.2) is not uniquely immersed, because it has two distinct manifold structures with which the inclusion map is an immersion. The cusp of Example 2.3 is also not uniquely immersed, because it does not have any manifold structure with which the inclusion map is an immersion. In $\mathbb{R}^2$, the union of the $x$-axis and the open positive $y$-axis is a uniquely immersed submanifold that is not a diffeological submanifold (hence is not weakly-embedded); see Figure 2.9.

In Remark E.2 below we also discuss the (non-intrinsic!) notion of an “immersed submanifold”. Every uniquely immersed submanifold, with its induced manifold structure, is an immersed submanifold.

3. Initial maps

(Diffeological) inductions.

In diffeology, a map $f: X \rightarrow Y$ is an *induction* if it is a diffeomorphism of $X$ with a subset of $Y$, as diffeological spaces. In the special case that the spaces are manifolds, this diffeological notion becomes equivalent to the following notion.

- A map $f: M \rightarrow N$ between manifolds is an *induction* if it is a diffeomorphism of $M$ with a diffeological submanifold of $N$.

An immersion is not necessarily an induction, even if it is injective; for example, take a parametrization of the figure eight (cf. Remark E.2). Instead, if $f$ is an immersion, then $f$ is a *local induction*: each point $x \in M$ has an open neighbourhood $O$ in $M$ such that
A smooth map \( f : \mathbb{R}^n \to \mathbb{R}^m \) is an induction ([10], p. 58, Note). In his 2013 book, Iglesias-Zemmour writes “I still do not know if there exist inductions [to] \( \mathbb{R}^n \) that are not immersions” ([10], p.17, note after Exercise 14), and “Actually, it could be weird if local inductions between real domains\(^{(3)}\) were not immersions” ([10], p.58, Note). Reformulating Example 2.3, we resolve Iglesias-Zemmour’s open question:

**Example 3.1.** The smooth map \( t \mapsto (t^3, t^2) \) is an induction (by Joris’ Theorem 2.5), but it is not an immersion.

Iglesias-Zemmour writes ([10], p.58, footnote): “When J. M. Souriau wrote one of his first papers on diffeology, he named immersion what is called induction now, but after a remark from J. Pradines he changed his mind”. In an email to David Miyamoto on October 6, 2020, Iglesias Zemmour clarified that Pradines’s remark was that the figure eight is an immersion that is not an induction; Souriau’s change to “induction” was motivated by this example. Pradines made his observation while refereeing Iglesias-Zemmour’s thesis, in which Iglesias-Zemmour introduced “local inductions”, adopting Souriau’s change in nomenclature.

Every induction is smooth. For maps between manifolds, we have the following characterization of inductions.

**Lemma 3.2.** A smooth map \( f : M \to N \) between manifolds is an induction if and only if the following holds:

\[
\text{For any manifold } U \text{ and any map } p : U \to M, \\
p : U \to M \text{ is smooth if and only if } f \circ p : U \to N \text{ is smooth.}
\]

We prove Lemma 3.2 in Appendix C, but here is the key step:

**Lemma 3.4.** Let \( f : M \to N \) be a map between manifolds. If (3.3) holds, then \( f \) is one-to-one.

**Proof.** Let \( x, y \) be points of \( M \) such that \( f(x) = f(y) \). Define \( p : \mathbb{R} \to M \) by \( p(t) = x \) if \( t \leq 0 \) and \( p(t) = y \) if \( t > 0 \). Then \( f \circ p \) is constant, hence smooth. By (3.3), \( p \) is smooth, hence continuous. Because \( M \) is Hausdorff, this implies that \( x = y \). □

**Remark 3.5.** In Lemmas 3.2 and 3.4, we can take \( M \) and \( N \) to be general diffeological spaces, as long as the D-topology of \( M \) is Hausdorff. If the domain \( M \) is not Hausdorff, this does not work: take, for example, \( M := \{0, 1\} \) equipped with the coarse diffeology (all maps \( U \to M \) are plots) and \( N \) a singleton.

A weak embedding is a diffeomorphism with a weakly-embedded submanifold. Weak embeddings are exactly those maps that are both an induction and an immersion. For example, \( t \mapsto [t, \sqrt{2}t] \) defines a weak embedding from \( \mathbb{R} \) to \( \mathbb{R}^2 / \mathbb{Z}^2 \).

**Pseudo-immersions.**

Pseudo-immersions are similar to inductions, except that Condition (3.3) is only required to hold with test-maps \( p : U \to M \) that are continuous.

\(^{(3)}\)Iglesias–Zemmour calls open subsets of Cartesian spaces “real domains”
A smooth map \( f: M \to N \) between manifolds is a \textit{pseudo-immersion} if and only if the following holds:

\[
\text{For any manifold } U \text{ and any continuous map } p: U \to M, \quad p: U \to M \text{ is smooth if and only if } f \circ p: U \to N \text{ is smooth.}
\]

The restriction to continuous test-maps makes the following statements true. (The analogous statements for inductions are false.)

(i) Being a pseudo-immersion is a local property: given a map \( f: M \to N \) between manifolds, if each point \( x \in M \) has an open neighbourhood \( \mathcal{O} \) in \( M \) such that \( f|_\mathcal{O}: \mathcal{O} \to N \) is a pseudo-immersion, then \( f \) is a pseudo-immersion. And,

(ii) Every immersion is a pseudo-immersion.

Thus, the notion of a pseudo-immersion does not give rise to a notion of “submanifold” whose structure is entirely induced from the ambient manifold.

Every induction is a pseudo-immersion, but not every pseudo-immersion is an induction. For example, take the figure eight.

Every local induction is a pseudo-immersion, but not every pseudo-immersion is a local induction: the map

\[
h(x, y) := \begin{cases} 
(x^2, x^3 - xe^{\frac{1}{x^4}}, y) & \text{if } y \neq 0 \\
(x^2, x^3, 0) & \text{if } y = 0,
\end{cases}
\]

provided by Joris and Preissmann in [13], is a pseudo-immersion, but it is not injective in any neighbourhood of \((0, 0)\), because \(h(e^{-\frac{1}{t^4}}, t) = (e^{-\frac{1}{t^4}}, 0, t) = h(-e^{-\frac{1}{t^4}}, t)\). This raises the question:

\[
\text{(3.7) If a pseudo-immersion is locally injective, is it a local induction?}
\]

We answer this question in the affirmative. Recall that a map from a topological space \( A \) to a topological space \( B \) is a \textit{topological embedding} if it is a homeomorphism of \( A \) with a subset of \( B \), taken with its subspace topology.

**Exercise 3.8.** If a map \( f: A \to B \) from a locally compact space \( A \) to a Hausdorff space \( B \) is continuous and locally injective, then each point of \( A \) has a neighbourhood \( \mathcal{O} \) such that \( f|_\mathcal{O} \) is a topological embedding.

**Remark 3.9.** If a smooth map \( f: M \to N \) is a pseudo-immersion and a topological embedding, then \( f \) is an induction. Indeed, let \( p: U \to M \) be a map such that \( f \circ p: U \to N \) is smooth. Because \( f \) is a topological embedding, \( p \) is continuous. Because \( f \) is a pseudo-immersion, \( p \) is smooth.

Combining Exercise 3.8 and Remark 3.9, we obtain the answer to the Question (3.7):

**Corollary 3.10.** A smooth map between manifolds is a local induction if and only if it is a locally injective pseudo-immersion.

We summarize the various inclusion relations in Figure 3.11.
Pseudo-immersion

Local induction (= locally injective pseudo-immersion)

Injective local induction (= injective pseudo-immersion)

Induction

Immersion

Weak Embedding

\((t^2, t^2)\)

\((\sin t, \sin 2t)\)

\(0 < t < 2\pi\)

\((\sin t, \cos t)\)

\(t \in \mathbb{R}\)

Figure 3.11. The inclusion relations for pseudo-immersions, inductions, local inductions, immersions, and their injective and locally injective incarnations. The equalities follow from Corollary 3.10.

A categorical perspective.

Remark 3.12. Taking a categorical approach to smooth manifolds, the natural notion of "submanifold" is diffeological, not weakly-embedded. We now elaborate, following Definition 8.6 of "The Joy of Cats" [1].

Let \(A\) be a concrete category over a category \(X\); this means that the category \(A\) is equipped with a faithful functor \(A \to X\). We write this functor as \(A \mapsto |A|\) on objects, and we consider the set of \(A\)-morphisms \(A \to B\) as a subset of the set of \(X\)-morphisms \(|A| \to |B|\). An \(A\)-morphism \(f: A \to B\) is initial if the following holds:

For any \(A\)-object \(C\) and \(X\)-morphism \(g: |C| \to |A|\),

\(g\) is an \(A\)-morphism if and only if \(f \circ g: |C| \to |B|\) is an \(A\)-morphism.

Viewing the category of manifolds and smooth maps as a concrete category over the category of sets and set-maps, a smooth map \(f: M \to N\) is an initial morphism exactly if (3.3) holds. By Lemma 3.2, the initial morphisms are exactly the inductions.

If, instead, we view the category of manifolds and smooth maps as a concrete category over the category of topological spaces and continuous maps, a smooth map \(f: M \to N\) is an initial morphism if and only if (3.6) holds. Thus, in this context, the initial morphisms are exactly the pseudo-immersions.
Appendix A. Proofs of Joris’ Theorem

In this appendix we present a proof of Joris’ theorem that is based on that of Amemiya and Masuda [2]. Here is the strategy:

(a) Using Boman’s theorem\(^{(4)}\) [3], reduce Joris’ theorem to a statement about functions of a single variable:

\[
\text{(A.1) Let } m \text{ and } n \text{ be relatively prime positive integers, and let } g \text{ be a real-valued function of a single variable such that } g^m \text{ and } g^n \text{ are smooth. Then } g \text{ is smooth.}
\]

(b) Use the continuity of the \(m\)th or \(n\)th root to establish that \(g\) is continuous, and smooth outside its set of zeros.

(c) Prove that \(g\) is smooth outside the set \(P\) of flat zeros of \(g^m\). (The flat zeros are the points where all the derivatives vanish.)

(d) Prove that \(g\) is smooth everywhere.

We leave (a) and (b) to the reader as an exercise. Here is a proof of (c):

**Proof of (c).** Let \(t\) be a non-flat zero of \(g^m\). Then \(t\) is also a non-flat zero of \(g^n\). (Indeed, at each point, either all or none of \(g^m\), \(g^{mn}\), and \(g^n\) are flat.) By Hadamard’s lemma, we can write

\[
g^m(\tau) = (\tau - t)^M h_1(\tau) \quad \text{and} \quad g^n(\tau) = (\tau - t)^N h_2(\tau)
\]

where \(M\) and \(N\) are positive integers and where \(h_1\) and \(h_2\) are smooth functions that are non-vanishing at \(t\). Since \((g^n)^{\sigma} = (g^m)^m\),

\[
(\tau - t)^{M\sigma} h_1^\sigma(\tau) = (\tau - t)^{N\sigma} h_2^\sigma(\tau);
\]

since \(h_1\) and \(h_2\) are non-zero at \(t\), this implies that \(M\sigma = N\sigma\). So \(m\) divides \(M\sigma\). But \(\gcd(m, n) = 1\), so \(m\) must also divide \(M\). Writing \(g(\tau) = (g^m(\tau))^{1/m} = (\tau - t)^{M/m} h_1^{1/m}(\tau)\), we see that \(g\) is smooth near \(t\). \(\square\)

The proof of (d) is harder. Joris [11] proved by induction that, for each positive integer \(\sigma\),

\[
g^{(\sigma)} \text{ exists everywhere, } g^{(\sigma)} \text{ vanishes on } P, \quad \text{and } g^{(\sigma)} \text{ is flat on } P.
\]

Joris proceeded by contradiction, and invoked a combinatorial lemma that expressed \((g^m)^{(\sigma-1)}\) in terms of \((g^a)^{\sigma-1}(g^b)^{\sigma} w_{a,b}\), where \(a + b \leq m\) and \(w_{a,b}\) is continuous and flat on \(P\). This argument was quite involved. Seven years later, Amemiya and Masuda [2] presented a simpler ring-theoretic proof. Here is their key lemma, and the proof for (d).

**Lemma A.2 ([2]).** Let \(S\) be a subring of a ring \(R\), such that if \(a \in R\) and \(a^r \in S\) for every sufficiently large \(r\), then \(a \in S\). Then the ring of power series \(S[[x]]\) in \(R[[x]]\) has the same property.

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\(^{(4)}\)A real valued function \(g: U \to \mathbb{R}\) on an open subset \(U\) of a Cartesian space is smooth if the composition \(g \circ \gamma\) is smooth for any smooth curve \(\gamma: \mathbb{R} \to U\).
Proof of (d), assuming Lemma A.2.
Let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying the hypothesis of (A.1). By (b) and (c), $g$ is continuous and is smooth outside the set $P$. Set

$$R := \{h : \mathbb{R} \setminus P \to \mathbb{R} \mid h \text{ is smooth } \},$$

$$S := \{h : \mathbb{R} \setminus P \to \mathbb{R} \mid h \text{ is smooth, and } h \text{ extends to a continuous map } \mathbb{R} \to \mathbb{R} \text{ that vanishes on } P \}$$

These satisfy the conditions of Lemma A.2. Define a homomorphism $J : \mathbb{R} \to \mathbb{R}[[x]]$ by

$$J(h) := \sum_{i=0}^{\infty} x^i \frac{h^{(i)}}{i!}.$$ 

Note that $h := g|_{\mathbb{R} \setminus P}$ is in $R$.

Because $(m, n)$ is relatively prime, every sufficiently large integer $r$ can be written as non-negative integer combination of $m$ and $n$. Indeed, after possibly switching $m$ and $n$, there exist integers $a < 0 < b$ such that $am + bn = 1$. For every integer $r \geq (-a)mn$, write $r = (-a)mn + An + j$ with $A \geq 0$ and $0 \leq j < n$; then $r = (-a)(n - j)m + (A + bj)n$.

For all such $r$, since $g^m$ and $g^n$ are smooth and are flat at $P$, so is $g^r$. So $(h^r)^{(i)} = (g^r)^{(i)}|_{\mathbb{R} \setminus P}$ is in $S$ for all $i$. Therefore, $J(h^r)$ is in $S[[x]]$. Since $J$ is a homomorphism, $J(h^r) = J(h^r)$, so $J(h)^r \in S[[x]]$. By Lemma A.2, $J(h) \in S[[x]]$. This means that each $h^{(i)}$ is in $S$, or in other words that each derivative $g^{(i)} : \mathbb{R} \setminus P \to \mathbb{R}$ extends to a continuous map $\mathbb{R} \to \mathbb{R}$ that vanishes on $P$. Combined with the fact $g$ is continuous, we may conclude that $g$ is smooth (Lemma 2 in [2], a consequence of the Mean Value Theorem).

Another six years later, Myers [18] proposed a short, elementary proof of Joris’ theorem, using only Rolle’s theorem. Myres laid out the general strategy of the proof very clearly, but unfortunately, his proof that $g$ is smooth at points in $P$ lacks detail. In his induction step, Myers shows that if $g$ is of type $C^{k-1}$ and its first $k - 1$ derivatives vanish at the points of $P$ then its $k$th derivative exists and vanishes at the points of $P$. But he does not show that this $k$th derivative is continuous at the points of $P$. In fact, his argument seems to apply to any function whose square alone is smooth and flat on $P$; as shown by Glaeser [7], such a function need not be smooth.

Appendix B. Diffeology

A diffeological space is a set $X$ equipped with a collection of maps $p : U \to X$, called plots, from open subsets of Cartesian spaces to $X$, that satisfies the following three axioms:

- Constant maps are plots.
- The precomposition of a plot with a $C^\infty$ map between open subsets of Cartesian spaces is a plot.
- If the restrictions of $p : U \to X$ to elements of an open cover of $U$ are plots, then $p$ is a plot.
The subset diffeology of a subset $A$ of a diffeological space $X$ consists of those maps $p: U \to A$ whose composition with the inclusion map are plots of $X$.

A map between diffeological space is smooth if its precomposition with each plot is a plot. The map is a diffeomorphism if it is a bijection and it and its inverse are smooth.

On a diffeological space $X$, the $D$-topology is the final topology that is induced by the set of plots: a subset of $X$ is open iff its preimage under each plot is open in the domain of the plot.

Equipping each manifold $M$ with the set of $C^\infty$ maps from open subsets of Cartesian spaces to $M$, we obtain a fully faithful functor from the category of manifolds to the category of diffeological spaces. This identifies manifolds with those diffeological spaces that are locally diffeomorphic to Cartesian spaces and whose $D$-topology is Hausdorff and paracompact.

**Appendix C. Criterion for diffeological induction**

In this appendix we prove Lemma 3.2, which gives the necessary and sufficient criterion (3.3) for a smooth map $f: M \to N$ between manifolds to be a diffeological induction.

In fact, the “only if” direction of (3.3) is automatically true for all smooth $f$ (and is equivalent to the smoothness of $f$), and the criterion (3.3) is necessary and sufficient for $f: M \to N$ being an induction even if $f$ is just a set-map.

**Proof of Lemma 3.2.** Let $f: M \to N$ be a map between manifolds, let $S := \text{image } f$ be its image, let $\iota: S \to N$ be the inclusion map, and let $\hat{f}: M \to S$ denote the map $f$ viewed as a map to $S$.

First, suppose that (3.3) holds. By Lemma 3.4, $f$ is one-to-one. Equip $S$ with the manifold structure with which the bijection $\hat{f}: M \to S$ is a diffeomorphism. Then for any open subset $U$ of any Cartesian space and any map $q: U \to S$,

$$q: U \to S \text{ is smooth if and only if } p := \hat{f}^{-1} \circ q: U \to M \text{ is smooth.}$$

By (3.3),

$$p: U \to M \text{ is smooth if and only if } f \circ p: U \to N \text{ is smooth.}$$

Because $f \circ p = \iota \circ q$, this implies the condition (2.2) in the definition of diffeological submanifold. So $S$ is a diffeological submanifold of $N$. Because $\hat{f}: M \to S$ is a diffeomorphism, $f$ is an induction.

Next, suppose that $f$ is an induction. Then $S$ is a diffeological submanifold of $N$, and — with this manifold structure on $S$ — the map $\hat{f}: M \to S$ is a diffeomorphism. Because smoothness is a local property, the property (2.2) of $S$ also holds with $U$ an arbitrary manifold (and not only an open subset of a Cartesian space). Fix a manifold $U$ and a map $p: U \to M$. Because $\hat{f}: M \to S$ is a diffeomorphism,

$$p: U \to M \text{ is smooth if and only if } \hat{f} \circ p: U \to S \text{ is smooth.}$$

By the version of (2.2) for manifolds $U$,

$$\hat{f} \circ p: U \to S \text{ is smooth if and only if } \iota \circ (\hat{f} \circ p): U \to N \text{ is smooth.}$$
Because $\iota \circ \hat{f} \circ p = f \circ p$, we obtain (3.3).

APPENDIX D. PARACOMPACTNESS AND SECOND COUNTABILITY

We require manifolds to be Hausdorff and paracompact. Some authors impose the stronger requirement that manifolds be Hausdorff and second countable. With our definition, every connected component of a manifold is second countable, but there can be uncountably many connected components. See, e.g., [15, p. 30, Problem 1-5].

Remark D.1 (Topological assumptions are superfluous). In the definitions of an “embedded”, “weakly-embedded”, or “diffeological” submanifold $S$, we require a priori that its manifold topology be Hausdorff and paracompact. But since the ambient manifold $N$ is Hausdorff and paracompact and the inclusion map is continuous, these assumptions are superfluous; see Theorem D.2. Similarly, an embedded submanifold of a second countable manifold is automatically second countable. In contrast, a weakly-embedded submanifold of a second countable manifold need not be second countable. For example, the set of irrational numbers in $\mathbb{R}$, with its discrete zero-dimensional manifold structure, satisfies (2.2), so it is a weakly-embedded submanifold.

Theorem D.2. Any locally Cartesian topological space $S$ that admits an injective continuous map to a (Hausdorff and paracompact) manifold $M$ is Hausdorff and paracompact.

This result is adapted from Chapter 1, Section 11, Part 7, Theorem 1 of Bourbaki’s General Topology, [4]. Bourbaki’s result is stated with more general assumptions, and its proof is longer.

Proof of Theorem D.2. Let $S$ be a locally Cartesian topological space, and let

$$i : S \to M$$

be an injective continuous map.

Let $x, y$ be distinct points of $S$. Because $i$ is injective, $i(x) \neq i(y)$. Because $M$ is Hausdorff, $i(x)$ and $i(y)$ have disjoint neighbourhoods $U, V$ in $M$. Because $i$ is continuous, the preimages in $S$ of $U$ and $V$ are disjoint neighbourhoods of $x$ and $y$ in $S$. This shows that $S$ is Hausdorff.

Recall that a locally Cartesian Hausdorff topological space is paracompact if and only if each of its connected components is second countable. (See, e.g., [19, page 459].) Without loss of generality, we will now assume that $S$ and $M$ are connected, and we will show that $S$ is second countable.

Let $\mathcal{B}$ denote the collection of those subsets of $S$ that are homeomorphic to closed balls in Cartesian spaces. Each element of $\mathcal{B}$ is second countable. We will describe a countable open cover of $S$ that refines $\mathcal{B}$. Each element of this cover is an open subset of a second countable space, hence is second countable. Since the cover is countable, we conclude $S$ is second countable.

Fix a countable basis $\mathcal{B}_M$ of the topology of $M$.

We call a pair $(W, U)$ distinguished if $U$ is an element of $\mathcal{B}_M$ and $W$ is a component of $i^{-1}(U)$ that is contained in some element of $\mathcal{B}$. We need two facts about such pairs:
(a) For any point \(x \in S\), there is a distinguished pair \((W, U)\) with \(x \in W\):

Indeed, fix a point \(x \in S\), take \(V \in \mathcal{B}\) whose interior contains \(x\), and let
\[ F := V \cap (S \setminus V) \]
be the frontier of \(V\). Then \(F\) is compact and it does not contain \(x\). Because
\(i\) is one-to-one, \(i(F)\) does not contain \(i(x)\); because \(i\) is continuous, \(i(F)\) is compact;
because \(M\) is Hausdorff, \(i(F)\) is closed. So \(M \setminus i(F)\) is an open neighbourhood of \(i(x)\)
in \(M\). Let \(U\) be an element of \(\mathcal{B}_M\) that contains \(i(x)\) and is contained in \(M \setminus i(F)\).
Set \(W\) to be the component of \(i^{-1}(U)\) that contains \(x\). Then \(W\) is contained in (the
interior of) \(V\), and the pair \((W, U)\) is distinguished.

(b) For any distinguished pair \((W, U)\), there are at most countably many distinguished
pairs \((W', U')\) in which \(W'\) intersects \(W\):

Because \(\mathcal{B}_M\) is countable, it is enough to show that, for each \(U' \in \mathcal{B}_M\), the set
of components \(W'\) of \(i^{-1}(U')\) that intersect \(W\) is countable. Because \(S\) is locally
connected, each such \(W'\) is open in \(S\). Because (having fixed \(U'\)) the sets \(W'\) are
disjoint, and because they are open subsets of \(W\), which is separable (it is contained in
an element of \(\mathcal{B}\), which is homeomorphic to a closed ball), there can be at most
countably many such sets \(W'\).

Now, define \(x \sim x'\) if there are distinguished pairs \((W_1, U_1), \ldots, (W_n, U_n)\) with \(x \in W_1, x' \in W_n\),
and consecutive \(W_i\) intersect. This is an equivalence relation; (a) gives reflexivity.
Its equivalence classes are open; because \(S\) is connected, there is only one equivalence class.

Fix \(x\). By (a) there exists a distinguished pair \((W_1, U_1)\) with \(x \in W_1\); set \(C_1 := W_1\).
Define \(C_n\) recursively to be the union of the \(W\) from distinguished pairs \((W, U)\) such that
\(W\) intersects \(C_{n-1}\). Each \(C_n\) is a countable union by (b). Furthermore, the collection of \(C_n\)
cover \(S\): a chain \((W'_1, U'_1)\) from \(x\) to \(x'\) witnesses \(W'_i \subseteq C_{i+1}\) for each \(i\). We take our open
cover to be given by those \(W\) that occur in this construction. \(\square\)

Remark D.3. In Theorem D.2, if \(S\) is a not-necessarily-paracompact manifold and \(i\) is an
immersion, an alternative argument is to equip \(M\) with a Riemannian metric, pull it back to
a Riemannian metric on \(S\), and note that connected Riemannian manifolds are metrizable
and connected metrizable manifolds are second countable (see [19, Theorem 7, page 315]
and [19, page 459]). The advantage of Theorem D.2 is that it is purely topological, and it
applies to diffeological submanifolds that are not necessarily weakly-embedded. \(\Diamond\)

Appendix E. Remarks on the literature

Submanifolds in some textbooks.

Our notions of “(embedded) submanifold” and “weakly-embedded submanifold” are equiv-
alent to those in standard textbooks such as John Lee’s [15, Chapter 5], except that some
authors (including John Lee) require their manifolds and submanifolds to be second count-
able and not only paracompact; see Appendix D.

Remark E.1. Our definition in §2 for what it means for a subset \(S\) of a manifold \(N\) to be an
“(embedded) submanifold” is essentially an unravelling of the definition of “submanifold”
in Guillemin and Pollack [8]. A different, but equivalent, definition in the literature is that
about each point of \(S\) there is a chart \(\varphi: \mathcal{O} \to \Omega \subset \mathbb{R}^n\) of \(N\) that takes \(S \cap \mathcal{O}\) onto the
intersection of \(\Omega\) with a linear subspace of \(\mathbb{R}^n\). (Cf. Remark 2.7.)
Remark E.2. An alternative approach is through the notion of an immersed submanifold, defined as a subset \( S \), equipped with a manifold structure, such that the inclusion map is an immersion. This notion is not an intrinsic property of the subset \( S \): it requires the choice of a manifold structure on \( S \).

An (embedded) submanifold can be defined as an immersed submanifold \( S \) whose manifold topology agrees with its subset topology, and a weakly embedded submanifold can be defined as an immersed submanifold \( S \) such that Condition (2.2) holds; see e.g. [15]. In either of these cases, the manifold structure on \( S \) is unique. Our definitions of submanifold and weakly-embedded submanifold are intrinsic; they avoid an a priori choice of a manifold structure on \( S \).

Not every immersed submanifold is weakly-embedded. For example, the parametrizations of the figure eight in \( \mathbb{R}^2 \) as \((\sin t, \sin(2t))\) for \( 0 < t < 2\pi \) and for \(-\pi < t < \pi\) exhibit it as two distinct immersed submanifolds.

\[ \Diamond \]

Remark E.3. According to Kolar-Michor-Slovak [14, Def. 2.14], an initial submanifold of a manifold \( N \) is a subset \( S \) of \( N \) that satisfies the criterion (2.8). Lemma 2.15 of their book [14] can be rephrased as saying that every weakly-embedded submanifold is an initial submanifold. Lemmas 2.16 and 2.17 of their book [14] can be rephrased as saying that every initial submanifold is a weakly-embedded submanifold.

Initial maps in some textbooks.

Kolar-Michor-Slovak [14, 2.10] refer to smooth maps that satisfy the property (3.3) as initial. This term is consistent with [1]; see Remark 3.12. Kolar-Michor-Slovak do not note that initial maps are automatically injective, which is the content of our Lemma 3.4. By our Lemma 3.2, initial maps coincide with (diffeological) inductions.

While Kolar-Michor-Slovak do not work with diffeology, they note [14, Remark 2.13] that, by Joris’ theorem (Theorem 2.5), the cusp \( t \mapsto (t^m, t^n) \) is initial, but is not an immersion. This coincides with our observation in Example 3.1 that the cusp is an induction that is not an immersion. Kolar-Michor-Slovak use the cusp as evidence that “to look for all smooth [initial mappings] is too difficult” [14, Remark 2.10]. They then focus on initial injective immersions; these coincide with weak embeddings. They do not introduce a term for weak embeddings, but their notion of an “initial submanifold”, which they define using charts, is equivalent to the notion of a weakly-embedded submanifold; see Remark E.3.

Maps that satisfy the property (3.3) also appear in Jeffrey Lee’s book [16, Def. 3.10]; he calls them smoothly universal. He too does not note that such maps are necessarily injective. By our Lemma 3.2, these maps are exactly the inductions. Jeffrey Lee defines weak embeddings to be smoothly universal injective immersions [16, Def. 3.11 on p. 129]; this agrees with our usage of this term.

Henri Joris and Emmanuel Preissmann introduced the notion of a pseudo-immersion in their 1987 paper [12]. Henri Joris had studied such maps already in 1982 [11], at the suggestion of Alfred Frölicher; he called their defining property (3.6) a “universal property”. According to Joris, it was Frölicher who suggested the example \( t \mapsto (t^2, t^3) \) as a candidate for a pseudo-immersion that is not an immersion.
References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, “Abstract and Concrete Categories: The Joy of Cats”, Online Edition, http://katmat.math.uni-bremen.de/acc/.

[2] I. Amemiya and K. Masuda, “On Joris’ theorem on differentiability of functions”, Kodai Math. J. 12 (1989), 92-97.

[3] J. Boman, “Differentiability of a function and of its compositions with functions of one variable”, Math. Scand. 20 (1967), 249-268.

[4] N. Bourbaki, Elements of Mathematics: General Topology: Part 1, Addison-Wesley, Massachusetts, 1966.

[5] Domenico P.L. Castrigiano and Sandra A. Hayes, Orbits of Lie groups are weakly embedded,

[6] J. Duncan, S. G. Krantz, and H. R. Parks, Nonlinear conditions for differentiability of functions, J. d’Analyse Math. 45 (1985), 46–68.

[7] G. Glaeser, “Racine carrée d’une fonction différentiable”, Ann. Inst. Fourier (Grenoble) 13 (1963), 203-210.

[8] V. Guillemin and A. Pollack, Differential topology, Prentice-Hall, New Jersey, 1974.

[9] Morris w. Hirsch, Differential Topology, Springer-Verlag, New York Heidelberg Berlin, 1976

[10] P. Iglesias-Zemmour, Diffeology, Mathematical Surveys and Monographs, 185, American Mathematical Society, 2013.

[11] H. Joris, “Une $C^\infty$-application non-immersive qui poss'ede la propriéité universelle des immersions”, Arch. Math. 39 (1982), 269-277.

[12] H. Joris and E. Preissmann, “Pseudo-immersions”, Ann. Inst. Fourier 37 (1987), 195–221.

[13] H. Joris and E. Preissmann, “Quotients of smooth functions”, Kodai Math. J. 13 (1990), 241-264.

[14] I. Kolar, P. Michor, and J. Slovak, Natural Operations in Differential Geometry, Springer-Verlag, Berlin Heidelberg, 1993.

[15] John M. Lee, Introduction to smooth manifolds, Springer, New York Heidelberg Dordrecht London, Second Edition, 2013.

[16] Jeffrey M. Lee, Manifolds and Differential Geometry, Graduate Studies in Mathematics, 107, American Mathematical Society, 2009.

[17] P. Michor, reply to “$f^3$, $f^2$ are the cube and quadratic of $f$ respectively and both infinite differentiable on $\mathbb{R}$, how to show so is $f$.”, on MathOverflow: https://mathoverflow.net/q/127724

[18] R. Myers, “An elementary proof of Joris’ theorem”, Amer. Math. Monthly 112 (2005), 829–831.

[19] M. Spivak, “A comprehensive introduction to differential geometry”, Vol. 1, Publish or Perish, Third Edition, 2005.

[20] P. Stefan, “Accessible sets, orbits, and foliations with singularities”, Proc. London Math. Soc. (3) 29 (1974), 699–713.

[21] Héctor J. Sussman, “Orbits of families of vector fields and integrability of distributions”, Trans. Amer. Math. Soc. 180 (1973), 171–188.

[22] T. Tao, “Square, cubes, and smooth functions” (preprint): math.ucla.edu/~tao/preprints/Expository/squarecub.dvi

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