MEAN TYPE OF FUNCTIONS OF BOUNDED CHARACTERISTIC AND MARTIN FUNCTIONS IN DENJOY DOMAINS

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Abstract. Functions of bounded characteristic in simply connected domains have a classical factorization to Blaschke, outer and singular inner parts. The latter has a singular measure on the boundary assigned to it. The exponential speed of change of a function when approaching a point of a boundary (mean type) corresponds to a point mass at this point. In this paper we consider the analogous relation for functions in infinitely connected (Denjoy) domains. The factorization result holds of course with one important addition: all functions involved become multiple valued even though the initial function was single valued. The mean type now can be measured by using the Martin function of the domain. But this result does not follow from the lifting to the universal covering of the domain because of the simple (but interesting) reason that the mean types of the original and the lifted functions can be completely different.

1. Introduction

1.1. Motivation. In the theory of de Branges [dB] the main objects are various classes of meromorphic functions $f$ in $\mathbb{C}$ such that $f|\mathbb{C}_+\cup\mathbb{C}_-$ is a function of bounded characteristic in $\mathbb{C}_+$ and in $\mathbb{C}_-$. Recall that functions of bounded characteristic in, say, $\mathbb{C}_+$, are just the ratios of two bounded analytic functions in $\mathbb{C}_+$, $\varphi_1/\varphi_2$, where $\varphi_2$ is not identically zero. The most famous such class is the class of Cartwright [L].

The following number is very important in de Branges’ theory of entire function [dB]:

$$h = \limsup_{y \to \infty} \frac{\log |f(\pm iy)|}{y}.$$
This is, the so-called, mean type of the given function. This parameter plays a major part in the spectral theory of Dirac operators and 1D Schrödinger operators. Its precise role in the general theory of canonical systems was clarified by de Branges, see [dB, Section 39]. In fact, de Branges has shown the necessity of subtler grading parameter than the one given by the mean type. It is feasible that the growth in the scale of Martin function in Denjoy domains (discussed in the present article) might serve as this subtler parameter.

Indeed, in the spectral theory of differential operators, in the theory of canonical systems, and in approximation theory one routinely encounters more general classes of meromorphic or entire functions [Yu]. Namely, let $E$ be a closed unbounded subset of $\mathbb{R}$ and $\Omega = \mathbb{C} \setminus E$. Then the class $B_E$ consists of entire functions $f$ whose restriction to $\Omega$ are functions of bounded characteristic in $\Omega$. By that one should understand the following. Let $\pi : \mathbb{C}_+ \rightarrow \Omega$ be a universal covering, $f$ is called a function of bounded characteristic in $\Omega$ if its lifting $f \circ \pi$ is a function of bounded characteristic on $\mathbb{C}_+$.

Domain in $\mathbb{C}$ whose boundary belongs to $\mathbb{R}$ is called Denjoy domain, analytic properties of Denjoy domains were widely studied. Our $\Omega$ is a Denjoy domain. In Denjoy domain $\Omega$, $\infty \in \partial \Omega$, one can consider an important positive harmonic function called the Martin function at infinity.

In what follows we consider only sets $E \subset \mathbb{R}$ whose every point is Dirichlet regular.

**Definition.** Let $\Omega = \mathbb{C} \setminus E$, the symmetric Martin function $M$ at infinity is a non-zero positive harmonic function in $\Omega$ such that $M|_E = 0$ and $M(\bar{z}) = M(z)$.

The set of positive harmonic functions vanishing on $E$ form a one dimensional cone or a two dimensional cone, see [Be]. In this article we will always have the situation that this cone is one dimensional, that is any Martin function is symmetric.

If one considers the cone of all positive harmonic function in a domain, its extremal rays form the so-called Martin boundary of the domain. One can read about Martin functions and Martin boundary in [Be],[Ha], [EYu] for example.

**Definition.** By symbol $B_E(h)$ we denote the class of entire functions such that 1) they are of bounded characteristic (see above) in $\Omega = \mathbb{C}\setminus E$, that is $f \in N(\Omega)$, and

\[
2) \limsup_{y \to \pm \infty} \frac{\log |f(iy)|}{M(iy)} \leq h.
\]
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1.2. Results. To formulate a factorization theorem for \( f \in B_E(h) \) we recall several facts. Analytic functions \( F \in N(\Omega) \), that is, of bounded characteristic in \( \Omega \), have inner/outer factorization very similar to such classical factorization in the disc or half-plane [Ne], [Ga]. Outer part of \( F \) is \( F_{out} \) such that

\[
\log F_{out}(z) = u + i\tilde{u}, \quad z \in \Omega,
\]

where

\[
u(z) = \int_E \log |F(t)| d\omega(t, z),
\]

\( \omega(t, z) = \omega_\Omega(t, z) \) denotes the harmonic measure of \( \Omega \) with respect to \( z \in \Omega \), and \( \tilde{u} \) is a harmonic conjugate of \( u \) in \( \Omega \). Generically these functions \( \tilde{u} \) and \( F_{out} \) are not single valued.

Blaschke products in \( \Omega \) are defined as follows. Let \( \mathfrak{z} \) be a universal covering of \( \Omega \) by the disc \( \mathbb{D} \), let \( \Gamma \) be a Fuchsian group of this covering: \( \Omega = \mathbb{D}/\Gamma \). Blaschke factor in \( \Omega \) is defined by fixing a point \( \zeta_0 \in \mathbb{D} \), \( z_0 := \mathfrak{z}(\zeta_0) \), and considering

\[
b_{z_0}(\zeta) := \prod_{\gamma \in \Gamma} \frac{\zeta - \gamma(\zeta_0)}{1 - \gamma(\zeta_0)\zeta} \frac{\gamma(\zeta_0)}{|\gamma(\zeta_0)|}.
\]

Denoting by \( G(z, z_0) \) Green’s function of \( \Omega \), we can write

\[
G(\mathfrak{z}(\zeta), z_0) = \log \frac{1}{|b_{z_0}(\zeta)|}.
\]

We introduce now multi-valued character automorphic holomorphic function

\[
\Phi_{z_0}(z) := e^{-G(z, z_0) - i\tilde{G}(z, z_0)}.
\]

It is sometimes called complex Green’s function of \( \Omega \) with pole at \( z_0 \). Now we can write

\[
\Phi_{z_0}(\mathfrak{z}(\zeta)) = b_{z_0}(\zeta).
\]

Definition. Blaschke product in \( \Omega \) is \( B_\Omega(z) := \prod_{k=1}^{\infty} \Phi_{z_k}(z) \), where the Blaschke condition in \( \Omega \) is satisfied: \( \sum_{k=1}^{\infty} G(0, z_k) < \infty \).

The reader awaits for us to introduce the notion of singular inner function in \( \Omega \). This of course can and need be done if we want to have inner/outer factorization of functions of \( N(\Omega) \). One can introduce singular inner functions in \( \Omega \) by integrating Martin functions over Martin boundary. But let us recall the reader, that our functions will be also entire functions. This extra assumption easily implies that the singular
inner function can be only of a very simple type, it can only involve point masses in the points of Martin boundary that lie over infinity. In our case we will have \( E \subset \mathbb{R}_- \), and so ([Be]) there will be only one Martin function (up to a positive multiplicative constant) at infinity. So all possible singular inner function involved in the factorization of functions from \( B_E(h) \) can be only of the following form \( e^{a(M+i\tilde{M})} \), \( a \in \mathbb{R} \). Also, we substitute condition (2) by

\[
2') \limsup_{x \to \infty} \frac{\log |f(x)|}{M(x)} \leq h.
\]

Recall that now \( \mathbb{R}_+ \subset \Omega \).

Now we can state the following factorization theorem.

**Theorem 1** (inner/outer factorization of entire functions of bounded characteristic in \( \Omega \)). We assume that \( E \subset \mathbb{R}_- \). Let \( f \in B_E(h) \) with the best possible \( h \) for the given function, that is,

\[
\limsup_{x \to \infty} \frac{\log |f(x)|}{M(x)} = h.
\]

Then

\[
f = B_\Omega \cdot f_{\text{out}} \cdot e^{a(M+i\tilde{M})},
\]

where \( h \leq a \).

Notice that our reasoning preceding the statement of this theorem has already almost proved it, and also proved the following correspondence between the factorization of the lifted function \( f \circ \tilde{\zeta}(\zeta) \) and the factorization of the original function \( f \in B_E(h) \): if \( F := f \circ \tilde{\zeta} \), and \( F = \frac{B \cdot F_{\text{out}}}{S} \) is its factorization to a Blaschke product, an outer part, and a singular inner part, then

\[
B = B_\Omega \circ \tilde{\zeta}, \quad F_{\text{out}} = f_{\text{out}} \circ \tilde{\zeta}, \quad S = e^{-a(M+i\tilde{M})} \circ \tilde{\zeta}.
\]

But the inequality

\[
h \leq a
\]

has not been proved. This “small” claim will be one of the main results of the present paper. We also indicate cases when \( a = h \).

Let us explain the difficulty behind this seemingly “trivial” statement (3). For this purpose it is more convenient to have universal covering \( \pi : \mathbb{C}_+ \to \Omega, \pi(\infty) = \infty \). Then given \( f \in B_E(h) \) and \( F := f \circ \pi \) we can factorize \( F, \ F = \frac{B \cdot F_{\text{out}}}{S} \). Here \( S \) is a singular inner function in \( \mathbb{C}_+ \), in particular

\[
|S(z)| = e^{-u(z)}, \ u(z) = Ay + \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} d\sigma_s(t),
\]
where \( \sigma_s \) is a singular measure on \( \mathbb{R} \). The convenience of \( C^+ \) in comparison with \( D \) is explained by the special role of \( \infty \) (of course we could have had the same role assigned to point \( 1 \in \partial \mathbb{D} \)).

Recall that the symmetric Martin function \( M(z) \) is defined up to a positive multiplier.

**Question 1.** Is it possible to normalize \( M(z) \) such that \( A \) in (4) is equal to \( a \) in Theorem 1?

Clearly, measure \( A\delta_\infty + d\sigma_s \) appeared just from lifting of \( a\delta^E_\infty \). By this we mean that we have \( a\delta^E_\infty \) measure at (unique) infinity point of the Martin boundary and it defines our singular inner function \( e^{-a(M+i\tilde{M})} \) in \( \Omega \). Then we lift this singular inner function \( e^{-a(M+i\tilde{M})} \) into \( C^+ \) by \( S := e^{-a(M+i\tilde{M})} \circ \pi \), and then we consider the Poisson representation of \( \log \frac{1}{|S|} \) by singular (it is necessarily singular by the principle of correspondence of harmonic measures under lifting) measure \( A\delta_\infty + d\sigma_s \). The same can be done of course by lifting into the disc \( \mathbb{D} \) as follows \( S := e^{-a(M+i\tilde{M})} \circ \zeta \).

**Definition.** Singular measure \( A\delta_\infty + d\sigma_s \) (or corresponding singular measure on \( \partial \mathbb{D} \)) will be called the lifting of delta measure at infinity under lifting \( \pi : C^+ \to \Omega \) (correspondingly, under lifting \( \zeta : \mathbb{D} \to \Omega \), we have \( A\delta_1 + d\sigma_s \)).

Here is a related very natural question.

**Question 2.** Can the lifting of a point measure at infinity be purely continuous singular measure?

In fact, we will show that \( A \) may not have anything to do with \( a \), for example, it can be that \( A = 0, a \neq 0 \). Also we will show that the lifting of the point mass can be 1) purely point, 2) purely singular continuous measure, 3) but never the mixture of two. The answer depends on how complicated \( E \) is at infinity.

We can now explain the reason why the inequality \( h \leq a \) in Theorem 1 is far from being trivial. By definition of \( h \) we have

\[
h = \limsup_{x \to \infty} \frac{|f_{out}||B_\Omega| \log |e^{a(M+i\tilde{M})}|}{M(x)} \\
+ \limsup_{x \to \infty} \frac{\log |f_{out}(x)|}{M(x)} + \limsup_{x \to \infty} \frac{\log |B_\Omega(x)|}{M(x)} \\
\leq a + \limsup_{x \to \infty} \frac{\log |f_{out}(x)|}{M(x)} + \limsup_{x \to \infty} \frac{\log |B_\Omega(x)|}{M(x)}.
\]

To prove \( h \leq a \) one should prove the theorem:

**Theorem 2.** Let \( E \subset \mathbb{R}_- \). Then

\[
\lim_{x \to \infty} \frac{\log |f_{out}(x)|}{M(x)} = 0.
\]
Of course, the inequality
$$\limsup_{z \to \infty, z \in \Omega} \frac{\log |B_\Omega(z)|}{M(z)} \leq 0,$$
is obvious because $|B_\Omega| \leq 1$. In fact, we believe that in a special sense (but not literally)
$$\lim_{z \to \infty, z \in \Omega} \frac{\log |B_\Omega(z)|}{M(z)} = 0. \tag{6}$$
We will clarify and prove (6) in some special case.

**Theorem 3.** Let $E \subset \mathbb{R}_-$. Then
$$\lim_{x \to \infty} \frac{\log |B_\Omega(x)|}{M(x)} = 0, \tag{7}$$
if all zeros of $B_\Omega$ lie in $\mathbb{R}_- \setminus E$.

In this case we have equality $a = h$ just because
$$a = \lim_{x \to \infty} \frac{\log |e^{a(M+i\tilde{M})}|}{M(x)} \leq \limsup_{x \to \infty} \frac{|f_{\text{out}}||B_\Omega| \log |e^{a(M+i\tilde{M})}|}{M(x)}$$
$$- \liminf_{x \to \infty} \frac{\log |f_{\text{out}}(x)|}{M(x)} + \liminf_{x \to \infty} \frac{\log |B_\Omega(x)|}{M(x)} = h.$$

Now we want to look at the relationship between numbers $a$ and $A$. Number $A$ can clearly be calculated as follows using the lifting of $f$ into half-plane $\mathbb{C}_+, F(z) := f \circ \pi(z)$,
$$A = \limsup_{y \to \infty} \frac{\log |F(iy)|}{y}.$$

Let us fix a fundamental domain for the covering $\pi : \mathbb{C}_+ \to \Omega$. For a given set $E = (-\infty, a_0) \setminus \bigcup_{j \geq 1}(a_j, b_j)$ there exists a system of non intersecting half-discs $D_j := \{z \in \mathbb{C}_+ : |z - \xi_j| < \rho_j\}$, $\xi_j \in \mathbb{R}_-, \rho_j > 0$, such that $\mathbb{C}_+$ can be mapped conformally
$$\phi : \mathbb{C}_+ \to \mathcal{F}_-, \quad \mathcal{F}_- = \{\Re z > 0, \Re z < 0\} \setminus \bigcup_{j \geq 1} D_j \tag{8}$$
with the following properties
$$\phi(a_0) = 0, \quad \phi(\infty) = i\infty, \quad \phi((a_j, b_j)) = \{z : z = \xi_j + \rho_j e^{i\theta}, \ 0 < \theta < \pi\}.$$
These domain and map are defined up to a positive rescaling. In other words they are defined uniquely if we add one normalization condition, say, $\phi(1) = i$. By the symmetry principle, we extend $\phi$ to the conformal mapping from $\mathbb{C}_+ \setminus (-\infty, a_0)$ to
$$\mathcal{F} = \mathcal{F}_- \cup \{z : \bar{z} \in \mathcal{F}_-\} \cup \{z = iy : y > 0\}.$$
This domain \( \mathcal{F} \) will be fixed as an (open) fundamental domain for the covering \( \pi : \mathbb{C}_+ \to \Omega \), moreover
\[
\pi(\phi(z)) = z \in \mathbb{C} \setminus (-\infty, a_0] \subset \Omega.
\]

**Lemma 1.** Let \( E \subset \mathbb{R}_- \). Function \( M \circ \pi \) is a Poisson integral of a pure point mass measure (lifting of delta measure at infinity is pure point) if and only if
\[
\lim_{x \to +\infty} \frac{M(x)}{U(x)} > 0,
\]
where \( V(z) + iU(z) \) is the branch of \( \pi^{-1} \) given by the conformal mapping \( \phi \) (9).

**Proof.** Function \( M \circ \pi \) is a positive harmonic function in \( \mathbb{C}_+ \). By the principle of preservation of harmonic measure, its a. e. non-tangential limits are zero. So it is given by the Poisson integral of positive singular measure \( b_\delta + \mu_s \). It is well-known that \( b = \lim_{y \to +\infty} \frac{M(\pi(iy))}{y} \). We change variable in this equality: \( \xi + i\eta = \zeta = \phi(x + iy) = V(z) + iU(z) \). In particular, the right half-axis is mapped into \( \{i\eta, \eta > 0\} \). Then we see that \( b > 0 \) if and only if (10) is satisfied.

Now let us prove that if \( M \circ \pi \) has a nontrivial point mass in its measure, then this measure is pure point. If \( x \in [-\infty, \infty] \) carries a point mass than every point of the orbit \( \text{orb}(x) \) under the Fuchsian group \( \Gamma \) carries point masses of \( M \circ \pi \). Corresponding point measure is denoted by \( \nu_s \), and if we build its Poisson extension \( p(\zeta) \) it will be automatically \( \Gamma \)-invariant. This means that \( p \) is a lifting, namely, \( p(\zeta) = P \circ \pi(\zeta) \), where \( P \) is a positive harmonic function in \( \Omega \). Moreover, as \( \nu_s \) is “only a part” of singular measure of \( M \circ \pi \), we get \( P \circ \pi = P \leq M \circ \pi \). So \( P \leq M \). But \( M \) is Martin’s function, so \( P = cM \). But then \( P \circ \pi = cM \circ \pi \). This means that the measure of \( M \circ \pi \) is proportional to the measure of \( P \circ \pi \). This latter being pure point, we conclude that measure of \( M \circ \pi \) must also be pure point. \( \square \)

This simple lemma “answers” the question when \( A \neq 0 \) given that \( a \neq 0 \).

We recall that \( F \) is of bounded characteristic in \( \mathbb{C}_+ \), and for such functions this non tangential \( \limsup_{z \to \infty} \frac{\log |F(z)|}{|z|} \) can be calculated via the factorization of \( F \) into outer, Blaschke product, and singular inner functions. Namely, this \( \limsup \) is exactly the mass at infinity of the measure of the singular inner part of the factorization of \( F \).

The singular inner part of \( F \) was denoted by \( S \) and we saw that \( S = e^{-a(M+iM)} \circ \pi \). Hence, \( A \neq 0 \) if and only if \( a \cdot \limsup_{y \to \infty} \frac{M \circ \pi(iy)}{y} \neq 0 \).
Changing the variable $\zeta = \pi^{-1}(z)$, we conclude that this holds if and only if (10) holds.

1.3. **Abelian integrals and Comb domains.** Let $E$ be a union of a finite number of disjoint non-degenerated intervals,

$$E = [b_0, a_0] \setminus \cup_{j=1}^g (a_j, b_j), \quad (a_j, b_j) \subset [b_0, a_0].$$

Let us associate to these data the hyperelliptic Riemann surface

$$\mathcal{R} = \{(z, s) : s^2 = \prod_{j=0}^g (z - a_j)(z - b_j)\}.$$

One can visualize this surface as the two copies of the domain $\Omega = \mathbb{C} \setminus E$ glued in an appropriate way along the set $E$.

The Abelian integrals on this surface have a very explicit form. In particular, the Abelian integral of the third kind with logarithmic singularities at $(z_0, \pm s_0)$, $z_0 \in \mathbb{R} \setminus [b_0, a_0]$, is given by

$$H(z, z_0) = \int_{a_0}^z \frac{P_g(\lambda) s_0 d\lambda}{\lambda - z_0 s(\lambda)}$$

where $P_g(z)$ is a polynomial of degree $g$ such that $P_g(z_0) = 1$. If we fix the remaining coefficients of $P_g$ by the conditions

$$\int_{a_j}^{b_j} \frac{P_g(\lambda) s_0 d\lambda}{\lambda - z_0 s(\lambda)} = 0, \quad 1 \leq j \leq g,$$

we obtain the standard representation for the Green function in $\Omega$

$$H(z, z_0) = G(z, z_0) + i\widetilde{G}(z, z_0).$$

Note that the conditions (11) imply that $P_g$ has exactly one zero $\lambda_j$ in each gap $(a_j, b_j)$. Thus

$$H(z, z_0) = \int_{a_0}^z \frac{\prod_{j=1}^g (\lambda - \lambda_j) s(z_0) d\lambda}{\prod_{j=1}^g (z_0 - \lambda_j) \prod_{j=1}^g (\lambda - z_0) s(\lambda)}$$

represents a Schwarz-Christoffel integral. That is, $\Theta(z) := iH(z, z_0)$ maps the upper half-plane onto a special polygon, a so-called comb domain,

$$\Pi = \{u+iv : -\pi < u < 0, \ v > 0\} \setminus \{-\pi \omega_k + iv : 0 < v \leq h_k, \ 1 \leq k \leq g\},$$

where $\omega_k \neq \omega_j$ for $k \neq j$ and $h_k > 0$.

This construction one can widely generalize.
Theorem 4 ([EYu]). Let

$$\Pi = \{ u + iv : -\pi < u < 0, \ v > h(u) \},$$

where $h$ is a non-negative upper semi-continuous function bounded from above and equal 0 almost everywhere with respect to Lebesgue measure on $[-\pi, 0]$. For every such $\Pi$ and $b_0 < a_0$, $z_0 \in \mathbb{R} \setminus [b_0, a_0]$, the conformal mapping $\Theta : \mathbb{C}_+ \to \Pi$, normalized by

$$\Theta(z_0) = \infty, \Theta(a_0) = i \lim_{u \to 0} h(u), \ \Theta(b_0) = -\pi + i \lim_{u \to -\pi} h(u),$$

is related with the Green function $G(z, z_0)$ of some closed set $E$ by the formula

$$G(z, z_0) = \Im \Theta(z), \ \Im z > 0.$$

If $h(u)$ does not vanish only on a countable set $\{-\pi \omega_k\}_{k \geq 1}$ and $\lim_{k \to \infty} h(-\pi \omega_k) = 0$, then $\Theta(z)$ can be extended by continuity to $\mathbb{R}$, that is, $E$ is Dirichlet regular.

The following three possible kinds of comb domains arise from symmetric Martin functions.

Case A. $\Pi = \{ u + iv : -\infty < u < \infty, \ v > h(u) \}$.

Case B. $\Pi = \{ u + iv : u_- < u < u_+, \ v > h(u) \}$, where $u_- = -\infty$, $u_+ < \infty$ or $u_+ = \infty$, $u_- > -\infty$.

Case C. $\Pi = \{ u + iv : -\pi < u < 0, \ v > h(u) \}$ and $\sup_{u \to 0} h(u) = \infty$ and/or $\sup_{u \to -\pi} h(u) = \infty$.

As before, in all three cases $h$ is a non-negative upper semi-continuous function equal 0 almost everywhere with respect to Lebesgue measure. Also $\Theta(\infty) = \infty$ and $M(z) = \Im \Theta(z)$.

We say that $E$ is a finite gap set if the number of gaps (equivalently the number of slits in the domain $\Pi$) is finite. Note that in a finite gap case only Cases A and B are possible and the corresponding conformal mapping is related to Abelian integrals of the second type with a pole at infinity. In the case Case B the set is is semi-bounded and $E = \mathbb{R} \setminus \cup_{j=1}^g (a_j, b_j)$ in the case A.

The essential ingredient of the current paper is to deal with the results of the following sort.

Lemma 2. Let $E \subset \mathbb{R}_-$. Assume that the Martin function of $\Omega = \mathbb{C} \setminus E$ is normalized by the condition $M(P) = 1$ for some $P > 0$. Then

$$\frac{G(t, x)}{G(t, P)} \leq M(x), \ \forall x \geq P,$$

for all $t \in \mathbb{R}_- \setminus E$. 
We demonstrate two ways of proofs. One of them is based on a finite gap approximation, see Section 7. That is, for a given $E = (-\infty, a_0] \cup \bigcup_{j \geq 1} (a_j, b_j)$ we consider its approximate $E_n := (-\infty, a_0] \setminus \bigcup_{j \geq 1} (a_j, b_j)$. Respectively, $\Omega_n := \mathbb{C} \setminus E_n$. In $\Omega_n$ the relation (13) is a certain property of Abelian integrals. As soon as this is shown, we can pass to the limit in $n$, using the fact that in an appropriate normalization the Martin functions $M_n(z)$ converges to $M(z)$, as well as $\lim_{n \to \infty} G_n(z, z_0) = G(z, z_0)$ for $z, z_0 \in \Omega$. Purely potential theory approach is demonstrated in Sections 5 and 6, see Lemmas 5 and 7.

Remark. In this paper we consider Martin functions $M(z) = M(z, \infty)$, which are related to infinity in domains $\Omega = \mathbb{C} \setminus E$ with a semi-bonded $E$. It is easy to make generalization to the case of an arbitrary Denjoy domain and Martin functions related to the gap end points $a_j$ (or $b_j$). It is enough to use an evident linear fractional transform $z \mapsto \frac{1}{a_j - z}$.

However, since our construction is based essentially on inequalities of the form (13) it can not be applied directly if $M(z) = M(z, x_0)$, $x_0 \in E$, but $x_0$ is not an end point of a gap.

We conjecture that in the general case one can find a special path in $\Omega$, which terminates at the given $x_0$ and on which similar to (19) estimations are available. These paths are given by means of of combs domains related to the Abelian integrals. Namely, for a finite gap set $E$ this comb is given by an Abelian integral with pole $x_0 \in E$ of the second order. The image of the related conformal mapping is described in the following way. Consider $\mathbb{C} \setminus \mathbb{R}_+$ with a finite number of slits

$$\{\omega^+_k + iy, 0 < y \leq h^+_k\}_{k=0}^{g^+} \quad \text{and} \quad \{\omega^-_k - iy, 0 < y \leq h^-_k\}_{k=0}^{g^-},$$

where $\omega^+_0 = 0$ and $\omega^+_k > 0$, $k \geq 1$. Also $\omega^+_k \neq \omega^+_j$, $k \neq j$, and $\omega^-_k \neq \omega^-_j$, $k \neq j$. The conformal map $\Theta$ is normalized by the conditions

$$\Theta(x_0) = \infty, \Theta(a_0) = +0 - i0, \Theta(b_0) = +0 + i0.$$

For general sets $E$ this sort of combs could be defined via upper semi-continuous functions $h^\pm(u)$, $u \in \mathbb{R}_+$. Definitely, certain modifications like Cases B and C above are also required. Such $\Theta(z)$ allows us to define a special path $\{\Theta(x) \in \mathbb{C}_+ \}_{x \in \mathbb{R}_-}$ in $\Omega$, which terminates at an arbitrary point of the boundary $x_0 = \lim_{x \to -\infty} \Theta(x)$.

2. When $M \circ \pi$ has a point mass representation

Now we will give a “criterion” for (10) to happen. This criterion will be used repeatedly in this work.
Let \( \pi : \mathbb{C}_+ \to \Omega \) be a covering, \( \pi(\infty) = \infty \). We know that \( M \circ \pi \) is a positive harmonic function in \( \mathbb{C}_+ \) and by the principle of correspondence of harmonic measure it has zero non-tangential limits almost everywhere. In fact, it can be easily proved that \( \pi(x + iy) \) converges to \( E \) for almost every \( x \), hence \( M \circ \pi(x + iy) \) tends to zero for almost every \( x \).

The Poisson representation of \( M \circ \pi \) has thus the form

\[
M \circ \pi(z) = \sigma_\infty \Re z + \int \frac{\Re z}{\Im z^2 + (\Re z - x)^2} d\sigma_s(x),
\]

where \( \sigma_s \) is a singular measure on \( \mathbb{R} \).

Notice that for any positive harmonic function \( m(z) \) with a representation like that given by a non-negative measure \( \mu \) on \( \mathbb{R} \), one can find the point mass \( \mu_\infty \) as follows:

\[
\mu_\infty = \lim_{y \to \infty} \frac{m(iy)}{y}.
\]

Let \( E = (-\infty, 0] \setminus \cup_{j \geq 1} (a_j, b_j) \). Let \( G(z, t), z, t \in \Omega \) denote Green’s function of \( \Omega \). Recall that \( V(z) + iU(z) = \phi(z) \). It is originally defined in the upper half plane, see (8), and then extended by the symmetry principle to \( \mathbb{C} \setminus \mathbb{R}^- \).

Note that in the gaps

\[
U(x + i0) = U(x - i0), \quad x \in (a_j, b_j).
\]

That is, \( U(z) \) is well defined in \( \Omega \). But, by the Cauchy-Riemann equations

\[
\frac{\partial U}{\partial y}(x) = \frac{\partial V}{\partial x}(x) > 0, \quad x \in (a_j, b_j).
\]

Thus, we point out a simple thing that \( dV(x) := \frac{\partial V}{\partial x} dx \) is a positive measure on each complementary interval \( (a_j, b_j) \) of \( \mathbb{R}^- \setminus E \).

Let us fix \( P \in (0, \infty) \), say \( P = 1 \), and normalize (we recall that \( \phi(z) = V(z) + iU(z) \) was defined up to a positive rescaling)

\[
M(P) = 1, U(P) = 1.
\]

We denote

\[
\mathcal{E} := \int_{\mathbb{R}^- \setminus E} G(P, t) dV(t) > 0.
\]

We want to prove the following formula: if \( \mathcal{E} < \infty \), then there exists \( \rho > 0 \) such that

\[
\rho M(z) = U(z) + 2 \int_{\mathbb{R}^- \setminus E} G(z, t) dV(t).
\]
Theorem 5. Let $E \subset \mathbb{R}_-$, $0 \in E$. Let $U, M$ be functions introduced above and normalized at point $P$. We always suppose also that $\infty$ is the Dirichlet regular point for $\mathbb{C} \setminus E$. Then

\[(19) \quad 0 < \rho := \lim_{x \to +\infty} \frac{U(x)}{M(x)} < \infty\]

if and only if

\[(20) \quad \int_{\mathbb{R}_- \setminus E} G(P, t) dV(t) < \infty.\]

In this case, for any $z \in \Omega$ one has $\int_{\mathbb{R}_- \setminus E} G(z, t) dV(t) < \infty$, and the representation (18) holds.

Proof. Let $\int_{\mathbb{R}_- \setminus E} G(P, t) dV(t) < \infty$. Put

\[(21) \quad \rho := 1 + 2 \int_{\mathbb{R}_- \setminus E} G(P, t) dV(t),\]

and

\[\rho N(z) := U(z) + 2 \int_{\mathbb{R}_- \setminus E} G(z, t) dV(t).\]

Then $\rho \Delta N = \Delta U - dV = 2(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}) dx = 0$ by (16). Thus $N$ is a positive and harmonic function in $\Omega$. Also we assumed that all finite points of $E = \partial \Omega$ are Dirichlet regular. Hence, $N(x) = 0$ for every such $x \in E$. Therefore, $N$ is a Martin function. And by the choice of $\rho$ in (21) we have $N(P) = 1$. By the uniqueness $N = M$.

To show (19) we divide the above relation by $M$

\[\rho = \frac{U(x)}{M(x)} + 2 \int_{\mathbb{R}_- \setminus E} \frac{G(x, t)}{M(x)} dV(t).\]

Due to (13) and our integral assumption, $G(P, t)$ is an integrable majorant for $\frac{G(x, t)}{M(x)}$, $x > P$. Thus, we can pass to the limit as $x \to \infty$. Since (for a fixed $t$) $\lim_{x \to \infty} \frac{G(x, t)}{M(x)} = 0$ (because $\infty$ is the Dirichlet regular point of the boundary), we get (19). Note, that simultaneously we proved that the corresponding limit exists.

Now we assume that

\[\lim_{x \to +\infty} \frac{M(x)}{U(x)} > 0.\]

By Lemma 1, $M(\pi(z))$ is the Poisson integral of a pure point mass measure. Since one of these points of the support is infinity we have that on the universal covering

\[M(\pi(z)) - \kappa \mathbb{I} z \geq 0\]
for some positive $\kappa$. In other words $N(z) := M(z) - \kappa U(z) \geq 0$ in $\Omega$. Let $E_n = \mathbb{R}_- \setminus \bigcup_{j=1}^n (a_j, b_j)$. In the domain $\Omega_n = \mathbb{C} \setminus E_n$ this function possesses the following representation

$$N(z) = \int_{E_n \setminus E} N(t) d\omega_n(t, z) + 2\kappa \int_{\mathbb{R} \setminus E_n} G_n(z, t) dV(t) + \rho_n M_n(z),$$

where $d\omega_n(t, z)$ is the harmonic measure, and $G_n(z, t), M_n(z)$ are Green’s and Martin’s functions in $\Omega_n$ respectively. The main point is that all three terms in this representation are positive. Therefore for a normalization point $P$ we get

$$\int_{\mathbb{R}} G_n(P, t) dV(t) = \int_{\mathbb{R} \setminus E_n} G_n(P, t) dV(t) \leq \frac{1}{2\kappa} M(P) - \frac{1}{2} U(P).$$

By Fatou’s lemma

$$\int_{\mathbb{R}} G(P, t) dV(t) \leq \frac{1}{2\kappa} M(P) - \frac{1}{2} U(P) < \infty.$$

\[\square\]

3. An example of purely continuous lifting of a point mass

Now we are going to build the domain $\Omega$ for which (19) fails. In particular, the positive harmonic function $M \circ \pi$ will have a continuous singular measure, where $M$ is the Martin function at infinity of $\Omega$. We already proved that condition (19) holds if and only if (20) is satisfied for some (and then for all $P \in \Omega$).

Let us now consider $\mathbb{R}_- \setminus E := \bigcup_{n \in \mathbb{Z}} \lambda^n (a_1, b_1)$, where $\lambda > 1$ and assume that (20) holds for this set.

For the given set $G(\lambda z, \lambda t) = G(z, t)$, and the conformal map $\phi$ onto the fundamental domain and Martin’s function satisfy

$$\phi(\lambda^n z) = \rho_1^n \phi(z), \quad \text{for some } \rho_1 > 1, n \geq 1,$$

$$M(\lambda^n z) = \rho_2^n M(z), \quad \text{for some } \rho_2 > 1, n \geq 1.$$

On the other hand, due to (19), we immediately get

$$\rho_2 = \rho_1.$$

We are going to bring this to a contradiction. We recall the reader that $dV \geq 0$ in (20). Now use $G(\lambda z, \lambda t) = G(z, t)$ and $V(\lambda z) = \rho_1 V(z)$ to rewrite (20), as follows (positivity and then Fubini’s theorem)

$$\int_{a_1}^{b_1} \sum_{n \in \mathbb{Z}} \rho_1^n G(\lambda^{-n} z, t) dV(t) < \infty.$$
Lemma 3. For every $\lambda > 1$ the series
$$\sum_{k \geq 0} M(\lambda^k)G(\lambda^{-k}, t) = \infty$$
for all $t \in (a_1, b_1)$.

Proof. From the definition of $M$ we know that
$$M(\lambda^k) = \lim_{n \to \infty} \frac{G(\lambda^k, \lambda^{n+k}t)}{G(1, \lambda^{n+k}t)}, \quad t \in (a_1, b_1).$$
The general term of the series in question is, thus, precisely
$$\lim_{n \to \infty} \frac{G(1, \lambda^k t)}{G(1, \lambda^{n+k}t)}.$$
Let us assume for a moment that we can prove
\begin{equation}
G(1, \lambda^{n+k}t) \leq C(t) < \infty, \quad t \in (a_1, b_1).
\end{equation}

Then the general term above is strictly positive, hence lemma is proved. We are left to check (23). This is the same as to prove
\begin{equation}
G(\lambda^{-(n+k)}, t) \leq C G(\lambda^{-n}, t)G(\lambda^{-k}, t), \quad t \in (a_1, b_1).
\end{equation}
To prove (24) we will use the following estimate of the harmonic measure $\omega := \omega_{\Omega}$ of our domain $\Omega$, see the definition in [Ne] for example.
\begin{equation}
G(x, t) \asymp \omega([-x, 0], 1), \quad x > 0,
\end{equation}
where the constants of comparison depend only on $t$ and are positive and finite. Claim (25) is well-known, and its variants can be found in [JK], [V] for example.

Accordingly, we are left to prove that
\begin{equation}
\omega(E_{n+k}, 1) \leq C_1 \omega(E_n, 1)\omega(E_k, 1),
\end{equation}
where $E_m := E \cap [-\lambda^m, 0]$. To do that, let us consider the disc $D_k$ centred at 0 and passing through the point $\lambda^{-k}c$, where $c = \frac{1}{2}(a_1 + b_1)$. Let $T_k := \partial D_k$. Then
$$\omega(E_{n+k}, 1) = \int_{\partial(\Omega \setminus D_k)} \omega(E_{n+k}, z)d\omega_{\Omega \setminus D_k}(z, 1).$$
By Harnack’s inequality, with constant $C$ independent of $n,k$, we will then have
\begin{equation}
\omega(E_{n+k}, 1) \leq C \omega(E_{n+k}, \lambda^{-k}c)\omega_{\Omega \setminus D_k}(T_k, 1).
\end{equation}
Now let us notice several things:
1) by the self-similarity of $\Omega$ we immediately have
$$\omega(E_{n+k}, \lambda^{-k}c) = \omega(E_n, c);$$
2) by Harnack’s principle again $\omega(E_n, c) \leq \omega(E_n, 1)$;
3) the following inequality holds
$$\omega_{\Omega \setminus D_k}(T_k, 1) \leq C \omega(E_k, 1).$$

It follows from the fact: $\omega(E_k, z) \geq c_0 > 0$ uniformly for $z \in T_k$, and the constant $c_0$ does not depend on $k$. This latter claim is the combination of self-similarity (everything is like for $k = 0$) and Harnack’s principle. Now (28) follows by comparing two harmonic functions $\omega(E_k, z)$ and $\omega_{\Omega \setminus D_k}(T_k, z)$ in the same domain $\Omega \setminus D_k$. At any given point $z$ of the boundary of this domain they are both either vanish, or $\omega(E_k, z) \geq c_0 \cdot \omega_{\Omega \setminus D_k}(T_k, z)$. Therefore, $\omega(E_k, z) \geq c_0 \cdot \omega_{\Omega \setminus D_k}(T_k, z)$ holds everywhere on the boundary, and hence, everywhere inside the domain $\Omega \setminus D_k$, in particular at point $z = 1$. This is exactly (28) with $C = c_0^{-1}$.

Combining 1), 2), 3), and (27) we obtain (26), and correspondingly, (24) is proved, and Lemma is proved.

We assumed the validity of (10), and we obtained that then $M(\lambda^k) = \rho_1^k M$ with some positive constant $M$. Plugging this into the series in Lemma 3 we obtain that for all $t \in (a_1, b_1)$ the series $\sum_{k=0}^{\infty} \rho_1^k G(\lambda^{-k}, t) = \infty$. But (22) says that such series converges for almost every $t \in (a_1, b_1)$. This is a contradiction. Therefore, (10) does not hold for this example. We just proved that the lifting of the point mass can become a purely continuous singular measure.

4. LIFTING OF A POINT MASS TO A PURE POINT SPECTRUM FROM THE INFINITELY CONNECTED DOMAIN $\Omega$

Point mass lifts to a pure point measure if $\mathbb{R}_- \setminus E$ consists of finitely many open intervals (lacunas or gaps). However, we want to construct the example, when the map ($M$ is the Martin function)
$$\Theta := -\widetilde{M} + iM$$
has unbounded “teeth”. So the number of gaps should be infinite and they should substantially grow in length (we will see the details). This map is a conformal map of $\mathbb{C}_+$ onto a comb domain $\Pi := \mathbb{C}_+ \setminus \mathcal{T}$, where $\mathcal{T}$ is $\cup_{k=0}^{\infty} T_k$, and $T_k$ is a vertical segment attached to a point $\omega_k \in \mathbb{R}$. By our normalization all $\omega_k < 0$, $k \geq 1$, and we have an infinite tooth $T_0 = \{iY: Y \geq 0\}$. In fact, $\Pi$ lies in the quarter plane $Q := \{X < 0, Y > 0\}$. Let us write each tooth as follows: $T_k = \{\omega_k + it, 0 \leq t \leq h_k\}$, where $h_k = M(m_k)$ and $m_k$ is the maximum point of the function $M$ on the $k$-th gap (=complimentary interval of
(29) \[ \sup_k h_k = \sup_k M(m_k) < \infty, \]
then
\[ M(iy) \geq c\sqrt{y}, \; y \to +\infty, \; c > 0. \]
Notice that function \( M(z) \) as well as \( U(z) \) always satisfy the opposite inequality. Then (10) holds, and \( M \circ \pi \) is the Poisson integral of pure point measure, as was proved in Theorem 5.

However, we want the supremum in (29) to be infinite, and (10) to hold at the same time.

**Remark.** In fact, by constructing such an example (the supremum in (29) is infinite, but the lifting is pure point) we answer a question of Chris Bishop and Michael Sodin. They noted that if \( \sup_{k \geq 1} h_k = \infty \), then one can find on the universal covering uncountably many different rays such that \( M \circ \Theta \) goes to infinity along each of them. That is, it looks like the corresponding measure is “supported” on an uncountable set of points and “therefore” it is not pure point. In this sense our answer is slightly counterintuitive. However, the lifting of a point measure can be pure point measure even if \( \sup_K h_k = \infty \). In other words, (29) is sufficient, but not necessary for such a lifting.

The example, where supremum in (29) is infinite, and (10) holds, can be achieved in many ways. One of them is only sketched here, another is given in details.

Let us consider \( \tilde{E} := \{ x \in \mathbb{R} : -x^2 \in E \} \). One can easily see that formula
\[ \mathcal{M}(z) := M(-z^2), \; z \in \mathbb{C}_+, \; \mathcal{M}(z) := \mathcal{M}(\bar{z}), \; z \in \mathbb{C}_- \]
defines Martin function at infinity for \( \mathbb{C} \setminus \tilde{E} \). The question when
(30) \[ \mathcal{M}(z) \approx cz, \; c > 0, \]
when \( z \) approaches infinity has been thoroughly studied by Benedicks [Be]. One can reconcile his condition (this requires some work) with \( \sup_k M(m_k) = \infty \). Thus, (10) follows again from (30) and \( U(x) \leq C\sqrt{x}, \; C > 0. \)

We propose another way to build an example, where teeth may grow unboundedly, but still (10) holds with a *not necessarily maximal possible rate of grow* for \( z \) as in (30). That is, we will be reconciling (29)
and (20):
\[ \int_{\mathbb{R} - \mathbb{E}} G(P, t) dV(t) < \infty. \]

We will use the “sliding hump method” to build our domain \( \Omega \), function \( M \), and comb domain \( \Pi \) inductively. So we build the sequence of domains \( \Omega_k = \mathbb{C} \setminus E_k \), where
\[ \mathbb{R} - \mathbb{E}_k = \bigcup_{i=1}^{k} (a_i^k, b_i^k) =: \bigcup_{i=1}^{k} L_i^k. \]

We will make the lacunes \( L_k^k := (a_k^k, b_k^k) \) to tend to \(-\infty\) very fast.

Let the sequence of “teeth height” \( \{h_m\}_{m=1}^{\infty} \), such that \( \lim_{m} h_m = \infty \), be fixed.

Let \( k \)-th generation is built. Functions \( \Theta_k = -\tilde{M}_k + iM_k \) and \( \phi_k = V_k + iU_k \) are conformal maps of \( \mathbb{C}_+ \). The first one is onto the comb domain \( \Pi_k := Q \setminus \mathcal{T}_k, \mathcal{T}_k := \bigcup_{i=1}^{k} T_i \),
\[ T_j = \{ \omega_j + iY, 0 \leq Y \leq h_j \}, j = 1, \ldots, k. \]
The image of the second map is \( O_k = Q \setminus \bigcup_{j=1}^{k} D_j^k \), where \( D_j^k, j = 1, \ldots, k \) are closed disjoint discs with centers \( c_j^k, c_k^k < \cdots < c_1^k < 0 \).

We put \( \Pi_{k+1} = \Pi_k \setminus T_{k+1} \), where
\[ T_{k+1} = \{ \omega_{k+1} + iY, 0 \leq Y \leq h_{k+1} \}. \]
The point \( \omega_{k+1} \) of attachment of this tooth will be chosen very negative. If \( |\omega_{k+1}| \) is chosen very large, we can ensure that with any fixed positive \( \varepsilon \) we have \( |a_j^{k+1} - a_j^k| < \varepsilon, |b_j^{k+1} - b_j^k| < \varepsilon, j = 1, \ldots, k. \)
\[ |G_{k+1}(1, t) - G_k(1, t)| < \varepsilon, \]
and
\[ \left| \frac{dV_{k+1}}{dt} - \frac{dV_k}{dt} \right| < \varepsilon, \]
uniformly in \( t \), if \( t \) runs over a fixed compact set \( S_k \).

In particular, the integrals over all the \( (k + 1) \)-th lacunas except the last one will be easily controlled:
\[ | \sum_{j=1}^{k} \int_{L_j^{k+1}} G_{k+1}(1, t) dV_{k+1}(t) - \int_{\mathbb{R} - \mathbb{E}_k} G_k(1, t) dV_k(t) | < \varepsilon. \]

Let us write formula (18) in \( \Omega_k \) and in \( \Omega_{k+1} \) (we use also that \( U_m, M_m \) are normalized at point 1). Then we have
\[ 1 = U_k(1) = \rho_k - \frac{1}{\pi} \int_{\mathbb{R} \setminus E_k} G_k(1, t) dV_k(t), \]
\[ 1 = \rho_{k+1} - \frac{1}{\pi} \int_{\mathbb{R} \setminus (E_{k+1} \setminus L_{k+1}^{k+1})} G_{k+1}(1, t) dV_{k+1}(t) - \frac{1}{\pi} \int_{L_{k+1}^{k+1}} G_{k+1}(1, t) dV_{k+1}(t), \]

If we combine (31) with (32), (33), we immediately get

\[ |\rho_{k+1} - \rho_k - \frac{1}{\pi} \int_{L_{k+1}^{k+1}} G_{k+1}(1, t) dV_{k+1}(t)| \leq 2^{-k-1}, \]

if above we choose \( \varepsilon \) accordingly.

Now we are going to estimate \( \int_{L_{k+1}^{k+1}} G_{k+1}(1, t) dV_{k+1}(t) \). To do that we notice that \( \phi_{k+1} \) maps interval \( L_{k+1}^{k+1} \) onto the half circle \( \partial D_{k+1} \) in a one-to-one monotone fashion. In particular, the increase of \( V_{k+1} \) on this interval is exactly twice the radius of this circle, which is \( 2U_{k+1}(c_{k+1}) \).

Hence,

\[ \int_{L_{k+1}^{k+1}} dV_{k+1}(t) \leq 2U_{k+1}(c_{k+1}) \leq 2 \rho_{k+1} M_{k+1}(c_{k+1}), \]

where the last inequality follows just from (18). We got then

\[ \int_{L_{k+1}^{k+1}} dV_{k+1}(t) \leq 2 \rho_{k+1} h_{k+1}. \]

To estimate \( \int_{L_{k+1}^{k+1}} G_{k+1}(1, t) dV_{k+1}(t) \) we are now left to estimate \( G_{k+1}(1, t) \) uniformly, when \( t \in L_{k+1}^{k+1} \).

Let us fix

\[ \varepsilon_k := 2^{-(k+1)} h_{k+1}, \]

but let us move \( \omega_{k+1} \) close to \(-\infty\).

Recall that \( L_{k+1}^{k+1} = (a_{k+1}^{k+1}, b_{k+1}^{k+1}) \). We skip the indices and write \( \omega := \omega_{k+1}, L = (a, b) \). Of course, \( a, b \to -\infty \) when \( \omega \to -\infty \). Let us prove that eventually \( |a| \leq 2 |b| \). Slightly changing our normalization we can think that the conformal map \( \Theta := \Theta_{k+1} \) maps \( A_0 = i \) to \( B_0 = -1 + i \in Q \). Map \( \Theta \) maps \( \mathbb{C}_+ \) onto \( \Pi := \Pi_{k+1} \). Consider the new conformal map \( \hat{\theta} := 1/\Theta(1/z) \) mapping \( a_0 := 1/A_0 \) into \( b_0 = 1/B_0 \). Let \( \alpha \) be harmonic measure of \( \mathbb{C}_+ \) with respect to \( a_0 \), and \( \beta \) be harmonic measure of \( S := 1/\Pi \) (the image of \( \Pi \) under the map \( 1/z \)) with respect to \( b_0 \).

Suppose \( |a| > 2 |b| \) when \( \omega \) goes to \(-\infty \) over a certain sequence. Then it is immediate that

\[ \alpha((1/b, 1/a))/\alpha((1/a, 0)) \geq c > 0 \]
for all such large \( \omega \), where \( c \) does not depend of \( \omega \) and is, in fact, an absolute constant. This is just because in \( \mathbb{C}_+ \) the harmonic measure of a small interval evaluated at \( b_0 \) is comparable with the length of the interval if it lies not far from 0.

By the principle of preservation of harmonic measure we then have
\[
\beta(1/T_{k+1}) / \beta((1/\omega_{k+1}, 0)) \geq c > 0,
\]
where \( 1/T_{k+1} \) is the image of \( T_{k+1} = \{ \omega_{k+1} + it, 0 \leq t \leq h_{k+1} \} \) under the map \( 1/z \). But it is easy to see that \( \beta(1/T_{k+1}) \leq C (h_{k+1}) \) with absolute \( C \), for all large \( \omega_{k+1} \). On the other hand, it is also easy to see that \( \beta((1/\omega_{k+1}, 0)) \geq c_k (h_{k+1}) \), with positive \( c_k \), which depends on \( k \) but does not depend on \( \omega_{k+1} \). Then
\[
0 < c \leq \beta(1/T_{k+1}) / \beta((1/\omega_{k+1}, 0)) \leq C c_k (h_{k+1}).
\]
Tending \( \omega_{k+1} \) to \( -\infty \) and keeping \( h_{k+1} \) fixed we come to contradiction.

Thus, we have just proved that for all large \( \omega_{k+1} \)
\[
|a_{k+1}^1| \leq 2|b_{k+1}^1|.
\]

If we make the fractional linear transformation \( m \) sending 1 to infinity, infinity to 0 and sending \( b_{k+1}^1 \) into point 1, it will send \( \Omega_{k+1} \) to a domain we will call \( D_{k+1} \). We can see that we need to estimate \( G_{D_{k+1}}(s, \infty) \) uniformly when \( s \in I_{k+1} := (m(a_{k+1}^1), 1) \). We just proved
\[
m(a_{k+1}^1) \geq 1/2, |I_{k+1}| \leq 1/2.
\]

Hence domain \( D_{k+1} \) has boundary, which definitely includes segments \([0, 1/2]\) and \( J_{k+1} := [m(b_{k+1}^1), m(a_{k+1}^1)] = [1, m(a_{k+1}^1)] \). The latter segment has the length of the order \( h_{k+1}/a_{k+1}^1 \), which is as large as we wish.

We saw that lacune \( I_{k+1} = (m(a_{k+1}^1), 1) \) is contained in \((1/2, 1)\). By the principle of majorization of Green’s function (bigger domain, bigger function, if measured at the same points) we can see now that
\[
\max_{s \in I_{k+1}} G_{D_{k+1}}(s, \infty) \to 0, \text{ when } \omega_{k+1} \to \infty, \text{ if } |h_{k+1}| \text{ stays fixed}.
\]

In fact, we just compare \( G_{D_{k+1}}(s, \infty), s \in I_{k+1} \), with \( G_{J_{k+1}}(s, \infty) \), where the latter is Green’s function of \( \mathbb{C} \setminus ([0, 1/2] \cup J_{k+1}) \). The uniform smallness of \( G_{J_{k+1}}(s, \infty) \) on \((1/2, 1)\) is obvious because \( J_{k+1} \) becomes as long as we wish.
In particular, for large $\omega_{k+1}$ we have

$$\max_{t \in I_{k+1}} G_{k+1}(1, t) \leq \max_{s \in I_{k+1}} G_{D_{k+1}}(s, \infty) \leq 2^{-(k+1)} h_{k+1}^{-1}. \tag{35}$$

Combine this with (35) to get

$$\int_{I_{k+1}} G_{k+1}(1, t) dV_{k+1}(t) \leq 2^{-k} \rho_{k+1}. \tag{36}$$

This and (34) give us

$$\rho_{k+1} \leq \rho_k + C 2^{-k}$$

with absolute constant $C$. Then we get that $\rho_m$ will be bounded and this implies (20).

We are done. We constructed the example with infinitely growing teeth, but with (20) and (10) satisfied, which means that the point mass will be lifted to pure point measure in spite of unboundedness of the teeth.

5. Martin function dominates outer functions. The proof of Theorem 2.

A simple corollary of formula (15) is that for any outer function $F_{out}(z)$ in the unit disc $\mathbb{D}$ one can claim that

$$\lim_{r \to 1} \frac{1-r}{1+r} \log |F_{out}|(r) = 0. \tag{36}$$

Notice that Martin’s function for the disc $\mathbb{D}$ at point 1 is exactly $M(z) = \frac{1+z}{1-z}$. Therefore, (36) means

$$\lim_{r \to 1} \frac{\log |F_{out}|(r)}{M(r)} = 0. \tag{37}$$

It is natural to ask the same question in more complicated domains. At least in the case of our Denjoy type domain $\Omega = \mathbb{C} \setminus E, E \subset \mathbb{R}_-$, we will prove the validity of this dominance. This is exactly the claim of Theorem 2. Note that in this section we assume that $E$ is a compact subset in $(-\infty, 0)$. That is, $0 \in \Omega$, and it can be used as a point of normalization.

Remark. Notice that this dominance in Denjoy domain does not follow from lifting to $\mathbb{D}$ and the dominance (37) in $\mathbb{D}$. We already explained, that the lifting of $M$ in $\Omega$ may have nothing to do with Martin function in $\mathbb{D}$, because the Herglotz measure the lifting of $M$ in $\Omega$ to $\mathbb{D}$ can be even continuous.
Lemma 4. Let \( E \) be a compact set in \( \mathbb{R}_- = (\infty, 0) \). Let each point of \( E \) be Dirichlet regular. Consider \( \Omega := \mathbb{C} \setminus E \), and let \( \omega(E', z) \) denote its harmonic measure, \( E' \subset E \). Let \( \omega_0(z) := \omega((\infty, a] \cap E, z) \), for \( a \in \mathbb{R}_- \), where \( a \) is a point of accumulation of \( E \) from the left. Then for any such \( a \) the function \( x \rightarrow \omega_a(x) \) is non-decreasing on \( \mathbb{R}_+ \).

Proof. Let \( 0 < x < y, a < 0 \), and let \( \omega_a(y) < \omega_a(x) \). We want to come to a contradiction. Let \( \omega_a(y) < t < \omega_a(x) \). Consider the set \( \{ z \in \mathbb{C} \cup \infty : \omega_a(z) < t \} \). It is an open set in \( \mathbb{C} \cup \infty \) containing \( y \). It may or may not contain \( \infty \). What we care about is whether its connected component \( \mathcal{C} \) containing \( y \) is such that it contains any point of \( E \). Suppose it does not. Clearly \( \omega_a(z) \) is a harmonic function in \( \mathcal{C} \) and it is continuous in \( \bar{\mathcal{C}} \). It is identically equal to \( t \) on the boundary of \( \mathcal{C} \) (with the possible exception of point \( a \), if \( a \) happens to be in its boundary), hence inside \( \mathcal{C} \) it is also equal to \( t \). This is a contradiction with \( \omega_a(y) < t \).

Consequently, \( \mathcal{C} \) contains some point \( b \) of \( E \). Let us show that \( b \geq a \). Suppose \( b < a \). Then

\[
\omega_a(z) = 1, \text{ for all } z \in E \text{ in a small neighbourhood of } b,
\]

and this means that inside the domain \( \mathcal{C} \) there are points where value of \( \omega_a \) is as close to 1 as we wish. But these values (by the definition of \( \mathcal{C} \)) should be strictly smaller than \( t < 1 \). This is a contradiction.

Let us show that \( b \neq a \). Suppose \( b = a \). We use the assumption that \( a \) is a point of accumulation of \( E \) from the left. Then again inside the domain \( \mathcal{C} \) there are points where value of \( \omega_a \) is as close to 1 as we wish, and this is a contradiction.

Points \( b, y \) lie in the open connected set \( \mathcal{C} \), so they can be connected by a piecewise linear arc \( \gamma \) inside \( \mathcal{C} \). We can assume without the loss of generality that each straight segment of \( \gamma \) is not parallel to \( \mathbb{R} \).

On \( \gamma \) our function \( \omega_a(z) \) is strictly smaller than \( t \). By symmetry the same holds on \( \bar{\gamma} \). Lets us put \( \Gamma := \gamma \cup \bar{\gamma} \) and consider the component of infinity of \( \mathbb{C} \setminus \Gamma \), let it be \( A \), and let the bounded components be \( B_1, \ldots, B_n \). Notice three things:

1) as \( b \in E \), then \( b < 0 \),
2) \( [b, y] \subset \mathbb{R} \cap \bigcup_{i=1}^{n} \bar{B}_i \),
3) on \( \bar{B}_i \) function \( \omega_a(z) \) is strictly smaller than \( t \). In fact, this is true on the boundary of each domain \( B_i \). Every point of \( \partial B_i \) is in \( \Gamma \cup (E \cap [b, 0]) \), and \( b > a \), so \( \omega_a(z) \) is either less than \( t \) or vanishes at every boundary point, and then maximum principle works.
Combining these properties we see that
\[ \omega_a(z) < t \quad \forall \quad z \in [b, y]. \]
But \( b < 0 < x < y \), so \( \omega_a(x) < t \). This contradicts the choice of \( t \). We are done. \( \square \)

In what follows we always assume that all points of the boundary are Dirichlet regular.

**Remark.** The fact that \( a \) is a point of accumulation of \( E \) from the left was used in the proof, but it is not essential for the statement. In fact, if \( a \) has no points of \( E \) in \( (a - \varepsilon, a) \), one can just change \( a \) without changing function \( \omega_a \).

**Lemma 5.** Let \( E \subset (-\infty, 0) \). Let \( \Omega = \mathbb{C} \setminus E \). Let \( b < a, b, a \in E \). Then
\[ \frac{\omega_a(x)}{\omega_a(0)} \leq \frac{\omega_b(x)}{\omega_b(0)}, \forall x > 0. \]

**Proof.** We can first prove this for compact \( E \) and then this follows for \( E \) containing \(-\infty\) by an obvious limiting procedure.

We assume first that \( b \) is an accumulation point of \( E \) on the left. We saw that this assumption is without loss of generality. Let \( A = 1/\omega_a(0), B = 1/\omega_b(0) \). Consider function
\[ u(z) := B\omega_b(z) - A\omega_a(z) \]
and suppose that for some \( x > 0 \) we have \( u(x) < -\varepsilon < 0 \). Let \( O \) be a connected component of the set, where \( u < -\varepsilon \), containing point \( x \). The open set \( O \) must contain some point of \( E \); otherwise we come to contradiction as in the previous lemma.

Suppose \( c \in E \) and \( c < b \). Notice that \( B \geq A \) and \( \omega_b(c) = \omega_a(c) = 1 \). Then \( u(c) \geq 0 \). This is impossible. By the same reason \( c \neq b \). Otherwise, using that \( c = b \) is an accumulation point of \( E \) on the left, we get some points inside \( O \), where \( u \geq 0 \).

Hence, \( c > b \). We again connect \( c \) and \( x \) by a piecewise linear path \( \gamma \) inside the domain \( O \). We may assume that all straight segments of \( \gamma \) are not parallel to \( \mathbb{R} \). Again we consider \( \Gamma = \gamma \cup \gamma \) on which \( u < -\varepsilon \) by construction. Let \( \{B_i\}_{i=1}^n \) be bounded connected components of \( \mathbb{C} \setminus \Gamma \). Let us look at \( \partial B_i \). Its points belong either to \( \Gamma \) (where \( u < -\varepsilon \)), or to \( E \cap (b, 0) \). But at such points \( \omega_b(z) = 0 \). Therefore, at such points \( u(z) \leq 0 \).

We conclude that \( u < 0 \) at any point of \( B_i \setminus E, i = 1, \ldots, n \).

As in the previous lemma, we obviously see that \( [c, x] \subset \mathbb{R} \cap \bigcup_{i=1}^n \bar{B}_i \). Hence, for any \( z \in [c, x] \) which is not in \( E \) we conclude \( u(z) < 0 \). But
point 0 is exactly in $[c, x] \setminus E$. So $u(0) < 0$. But by definition $u(0) = 0$. We came to contradiction and proved the lemma.

\[ \square \]

**Theorem 6.** Let $E \subset (-\infty, 0)$. Let $\Omega = \mathbb{C} \setminus E$. We assume that $\infty \in \partial \Omega$. Let $M$ be its Martin function at infinity normalized as follows: $M(0) = 1$. Let $a \in E$ and $\omega_a$ be harmonic measure of $E \cap (-\infty, a]$ with respect to $\Omega$. Then

\[ \frac{\omega_a(x)}{\omega_a(0)} \leq M(x), \forall x > 0. \]

**Proof.** We use $M(z) = \lim_{b \to -\infty} \frac{\omega_b(z)}{\omega_b(0)}$. Then the claim follows from Lemma 5 immediately.

\[ \square \]

**Lemma 6.** Let $E \subset (-\infty, 0)$. Let $\Omega = \mathbb{C} \setminus E$. Let $b < a, b, a \in E$. Let $\omega_{ba}(z) := \omega_a(z) - \omega_b(z) = \omega([b, a], z)$. Then

\[ \frac{\omega_{ba}(x)}{\omega_{ba}(0)} \leq \frac{\omega_b(x)}{\omega_b(0)}, \forall x > 0. \]

Now we assume that $\infty \in \partial \Omega$. Let $M$ be the Martin function at infinity normalized as follows: $M(0) = 1$. Then

\[ \frac{\omega_{ba}(x)}{\omega_{ba}(0)} \leq M(x), \forall x > 0. \]

**Proof.** The first inequality is a direct consequence of Lemma 5. The second inequality combines the first one and Theorem 6.

\[ \square \]

**Theorem 7.** Let $\Omega = \mathbb{C} \setminus E$. We assume that $\infty \in \partial \Omega$. Let $M$ be its Martin function at infinity. Let $\psi$ be a function summable with respect to the harmonic measure $\omega(., z)$ of $\Omega$. Then

\[ \int \psi(t) \omega(dt, x) = o(M(x)), x \to +\infty. \]

**Proof.** Let $N < 0$. Let $\rho(t, x) := \frac{\omega(dt, x)}{\omega(dt, 0)}$. Then $\int \psi(t) \omega(dt, x) = \int_{-\infty}^N \psi(t) \rho(t, x) \omega(dt, 0) + \int_N^0 \psi(t) \rho(t, x) \omega(dt, 0) =: I + II$. By Lemma 6 we conclude that uniformly $\rho(t, x) \leq M(x)$. Hence

\[ I \leq M(x) \int_{-\infty}^N \psi(t) \omega(dt, 0) \leq \varepsilon M(x) \]

if $N$ is large enough by absolute value because of the fact that $\psi \in L^1(d\omega)$.  


We notice that by Lemma 6 we can write that for $t \in [N,0] \cap E$
\[ \rho(t,x) \leq \frac{\omega_N(x)}{\omega_N(0)} \leq \frac{1}{\omega_N(0)} =: C_N. \]

Then $II/M(x) \leq (C_N \int |\psi(t)|\omega(dt,0))/M(x)$, and it tends to zero when $x$ tends to infinity just because $M(x)$ goes to infinity. Theorem is proved.

\[ \square \]

Notice that we just proved Theorem 2.

6. **Martin function dominates the logarithm of Blaschke product with zeros on $\mathbb{R}_-$**

We already proved that $h \leq a$, see (3). Now we want to show some cases when we can prove the equality $h = a$.

Let $\Omega = C \setminus E$. Recall that all point of $E$ are assumed to be Dirichlet regular. We assume that $\infty \in \partial \Omega$. We can assume that

\begin{equation}
\text{(38)} \quad \text{infinity is also a Dirichlet regular point of } E.
\end{equation}

Actually for us it will be enough to have another condition (39) below. Let $M$ be its Martin function at infinity. For what follows we would need

\begin{equation}
\text{(39)} \quad \forall c \in \mathbb{R}_- \forall x \in \mathbb{R}_+, \; G(c,x) \leq C(c) < \infty.
\end{equation}

**Remark.** Notice that some condition of the type (39) is needed, because otherwise it may happen that there is no Martin function at infinity at all, or it may happen that $M(x) \sim \log x, x \to +\infty$, and $G(c,x) \sim \log x, x \to +\infty$, and then the statement below is obviously wrong. Assumption (39) is satisfied, for example, if $\infty$ is a Dirichlet regular point of the boundary $E$ of the domain $\Omega$.

**Theorem 8.** Let \( \{ c_k \}_{k=1}^\infty \) be points on $\mathbb{R}_-$, $c_1 > c_2 > \cdots > c_m > \cdots$. Let $\sum_{k=1}^\infty G(c_k,0) < \infty$. Then
\[ \sum_{k=1}^\infty G(c_k, x) = o(M(x)), \; x \to +\infty, \]
if assumption (39) is satisfied.

We need the following lemma.

**Lemma 7.** Let $a, b \in \mathbb{R}_- \cap \Omega$, $b < a$. Then
\[ \frac{G(a, x)}{G(a, 0)} \leq \frac{G(b, x)}{G(b, 0)}, \forall x \in \mathbb{R}_+. \]
Proof. Suppose that for \( x \in \mathbb{R}_+ \) we have an opposite inequality. Denote \( B := \frac{1}{G(b,0)}, A := \frac{1}{G(a,0)} \) and consider
\[
    u(z) := BG(b, z) - AG(a, z).
\]
We assumed that \( u(x) < -\varepsilon < 0 \). Consider the connected component of the open set \( \{ z \in \mathbb{C} : u(z) < -\varepsilon \} \) that contains point \( x \). Call this component \( O \).

Domain \( O \) cannot contain \( b \) inside. This is clear, because at the vicinity of \( b \) function \( G(b, z) \) is close to \( +\infty \), and so \( u(z) \) has the same property. Moreover, \( b \notin \bar{O} \) by the same reason.

Hence, \( u \) is subharmonic in \( O \). Suppose \( a \notin O \), and \( E \cap O = \emptyset \). Then \( u \) is harmonic in \( O \), and its values on \( \partial O \) are identically \( -\varepsilon \). This is obviously true except may be one point of \( \partial O \), namely, \( a \), if it so happened that \( a \in \partial O \). But \( a \) cannot belong to the boundary of \( O \), otherwise there would be a sequence of distinct points on this boundary, where \( u < -\varepsilon \), which is impossible.

Then automatically harmonic in \( O \) function \( u \) being identically \( -\varepsilon \) on \( \partial O \) will be identically \( -\varepsilon \) in \( O \). This is a contradiction with \( x \in O \).

Hence, we have to assume that either \( a \in O \) or \( E \cap O \neq \emptyset \). Suppose \( a \notin O \). Then \( E \cap O \neq \emptyset \). We can choose then a point \( z \in O \) close to \( E \cap O \), where \( G(a, z) \) and \( G(b, z) \) are as close to 0 as we wish (Dirichlet regularity assumption). Then \( u(z) \geq -\varepsilon/2 \). But on the boundary of \( O \) our subharmonic function \( u \) is \( -\varepsilon \). Maximum principle for subharmonic functions does not allow this.

There is only one possibility left: \( a \in O \). Then we choose a point \( r \in \mathbb{R}_- \cap O \) close to \( a \) at which we have
\[
    u(r) < -\varepsilon.
\]
Points \( r \) and \( x \) are in the same connected open set \( O \). Hence we can connect them by a piecewise linear path \( \gamma \) inside \( O \), consisting of finitely many straight segments. Each straight segment can be chosen not to be parallel to \( \mathbb{R} \).

As before, denote \( \Gamma := \gamma \cup \bar{\gamma} \), and notice that \( u(z) < -\varepsilon, \forall z \in \gamma \). Then, by symmetry,
\[
    u(z) < -\varepsilon, \forall z \in \Gamma.
\]

Again consider all bounded components of \( \mathbb{C} \setminus \Gamma \), let them be \( \{ B_i \}_{i=1}^n \).

Obviously \( [r, x] \subset \bigcup_{i=1}^n \bar{B}_i \). By construction, \( \gamma \) is inside \( O \), and, by symmetry, \( \Gamma \subset O \). It may happen by chance that \( b \) belongs to one of these domains \( B_i \), let us say to \( B_1 \). Then of course \( [r, x] \subset \bigcup_{i=2}^n \bar{B}_i \). In each of \( B_i \) that does not contain \( b \) function \( u \) is subharmonic, it is \( < -\varepsilon \) on \( \partial B_i \cap \Gamma \) and it is \( = 0 \) on \( \partial B_i \cap E \). Therefore, each \( s \in [r, x] \setminus E \) is either a point of intersection of \( \gamma \) and \( \mathbb{R} \) or it is in \( B_i \setminus E, i = 2, \ldots, n \).
In the first case \( u(s) < -\varepsilon \). In the second case \( u(s) < 0 \) by subharmonicity of \( u \) in \( B_i, i = 2, \ldots, n \).

So \( u(s) < 0 \) for all \( s \in [r, x] \setminus E \). But point 0 is exactly like that, so \( u(0) < 0 \). But by definition of \( u \) we have that \( u(0) = 0 \). Having a contradiction we are done with the lemma. \( \square \)

Now we can prove Theorem 8.

**Proof.** Let us write the sum (here \( x \in \mathbb{R}_+ \))

\[
\sum_{k \geq 1} G(c_k, x) = \sum_{k=1}^N G(c_k, x) + \sum_{k=N+1}^\infty \frac{G(c_k, x)}{G(c_k, 0)} G(c_k, 0) =: I + II.
\]

Recall that \( M(x) = \lim_{k \to \infty} \frac{G(c_k, x)}{G(c_k, 0)} \). By Lemma 7 it is a monotonically increasing limit. Hence,

\[
II \leq M(x) \sum_{k=N+1}^\infty G(c_k, 0) \leq \varepsilon M(x)
\]

uniformly for all \( x \in \mathbb{R}_+ \) and for as small \( \varepsilon \) as we wish, as soon as \( N \) is sufficiently large. We used here the Blaschke assumption \( \sum_{k=1}^\infty G(c_k, 0) < \infty \) of the theorem.

Having \( N \) fixed we just notice that for each \( k = 1, \ldots, N \), we have \( G(c_k, x)/M(x) \to 0 \) as \( x \to +\infty \) just by assumption (39). Theorem is proved. \( \square \)

Recall that domain \( \Omega = \mathbb{C} \setminus E, E \subset R \), is called Widom domain if

\[
\sum_{c: \nabla G(c, 0) = 0} G(c, 0) < \infty.
\]

For Widom domains function \( W(z) := \sum_{c: \nabla G(c, 0) = 0} G(c, z) \) is called Widom function. It plays an important part in Hardy space theory in such domains. Widom domains automatically have property (39).

**Corollary 1.** Let \( \Omega \) be a Widom domain, \( E \subset R_- \). Let \( W(z) := \sum_{c: \nabla G(c, 0) = 0} G(c, z) \). Then \( W(x) = o(M(x)), x \to +\infty \). Moreover, for any Blaschke product \( B_\Omega \) all of whose zeros lie in \( \Omega \cap R_- \) we have

\[
\log |B_\Omega(x)| = o(M(x)), x \to +\infty.
\]

**Corollary 2.** For any function \( f \) of bounded characteristic in domain \( \Omega = \mathbb{C} \setminus E, E \subset R_- \), such that all its zeros lie in \( R_- \) and such that (39)
is satisfied, we have \( h = a \), where \( a \) is the number in the factorization
Theorem 1, and
\[
    h = \limsup_{x \to +\infty} \frac{\log |f(x)|}{M(x)}.
\]

7. Another proof of Lemmas 7 and 2. Abelian integrals

We want to give another proof of these lemmas based on Abelian integrals, see Subsection 1.3. Again we assume first (it is enough) that \( E \) is a subset of \( \mathbb{R}_- \), say \( E \subset (-\infty, a_0] \), \( a_0 < 0 \), which consists of finitely many non-trivial intervals
\[
    E = (-\infty, a_0] \setminus \bigcup_{k=1}^g (a_k, b_k).
\]

Proof of Lemma 7. Let \( b \in (a_{k_1}, b_{k_1}) \) and \( a \in (a_{k_2}, b_{k_2}) \). By (12)
\[
    \frac{G(z, b)}{G(0, b)} - \frac{G(z, a)}{G(0, a)} = \Re H(z, a, b),
\]
where
\[
    H(z, a, b) = \int_{b_{k_1}}^z \frac{P_{g+1}^{g+1}(\lambda)}{s(\lambda)} d\lambda,
\]
and \( P_{g+1}^{g+1} \) is a certain polynomial of degree \( g + 1 \). Due to the normalization condition (11) each of \( g - 2 \) gaps, complimentary to \( (a_{k_1}, b_{k_1}) \) and \( (a_{k_2}, b_{k_2}) \), contains at least one zero \( \xi_j \) of \( P_{g+1} \). Moreover, since \( \Re H(0, a, b) = 0 \), each of intervals \( (a_0, 0) \) and \( (0, \infty) \) contains also a zero of this polynomial. Thus, we were able to localize \( g \) zeros of \( P_{g+1}(z) \). We claim that the remaining (with necessity real) zero \( \xi \) of \( P_{g+1}(z) \) belongs to the interval \( (b, a) \).

Assume that it does not. Then the standard arguments, related to integrals of Schwarz-Christoffel type, show that \( \Re H(z, a, b) > 0 \) for all \( z \in \Omega \cap (b, a) \). Indeed, by the definition \( H(z, a, b) = \Re H(z, a, b) > 0 \) in \( (b, b_{k_1}) \). Then, if \( z \) goes from \( b \) to \( a \) along the real line, its image \( w(z) = H(z, a, b) \) goes along straight lines with rotations by \( \pi/2 \) at the images of the end points \( a_j \) and \( b_j \) and by \( \pi \) at \( \xi_j \in (a_j, b_j) \). But, due to our assumption \( \xi \not\in (b, a) \), we listed all switch-argument-points in this interval. Since \( w(z) \) remains to be pure imaginary at all points \( z \in E \cap (b, a) \), the integral \( \Re w(z) \) has to be positive in all gaps, including the interval \( (a_{k_2}, a) \). That is, \( \lim_{z \to a} \Re H(z, a, b) = +\infty \) and this contradicts the definition (40), according to which \( \lim_{z \to a} \Re H(z, a, b) = -\infty \).

Thus \( \xi \in (b, a) \). Then the same arguments, related to the Schwarz-Christoffel type integrals, show that \( \Re H(z, a, b) < 0 \) in all gaps inside
(a, 0) including (a_0, 0) and $\Re H(z, a, b) > 0$ in (0, $\infty$), as well as in all gaps in (−$\infty$, b). The lemma is proved. □

Proof of Lemma 2. For $t \in \mathbb{R} \setminus E$, $P > 0$, we can repeat the argument of the previous lemma with respect to the function

$$M(z) - \frac{G(z, t)}{G(P, t)} = \Re H_P(z, t), \quad H_P(z, t) := \int_{b_k}^{z} \frac{Q_{g+1}(\lambda)}{(\lambda - t) s(\lambda)} \, d\lambda. \quad \text{(1)}$$

Now $Q_{g+1}$ must have zero at all gaps, except for that one, which contains $t \in (a_k, b_k)$. There is zero in $(a_0, P)$. Finally, the remaining real $\xi$ belongs to (−$\infty$, $t$), what follows from $\lim_{z \to t-0} \Re H_P(z, t) = -\infty$. □

Remark. Using Abelian integrals approach one can also give a proof of Lemma 5. It is essentially more simple since it does not require specification of a position of an “extra zero” $\xi$ (the number of the critical zeros just coincides with the number of the specified intervals).

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