Pesin’s entropy formula for $C^1$ non-uniformly expanding maps

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Abstract
We prove existence of equilibrium states with special properties for a class of distance expanding local homeomorphisms on compact metric spaces and continuous potentials. Moreover, we formulate a $C^1$ generalization of Pesin’s Entropy Formula: all ergodic weak-SRB-like measures satisfy Pesin’s Entropy Formula for $C^1$ non-uniformly expanding maps. We show that for weak-expanding maps such that Leb-a.e. $x$ has positive frequency of hyperbolic times, then all the necessarily existing ergodic weak-SRB-like measures satisfy Pesin’s Entropy Formula and are equilibrium states for the potential $\psi = -\log |\det Df|$. In particular, this holds for any $C^1$-expanding map and, in this case, the set of invariant probability measures that satisfy Pesin’s Entropy Formula is the weak* -closed convex hull of the ergodic weak-SRB-like measures.

Keywords Non-uniform expansion · SRB/physical-like measures · Equilibrium states · Pesin’s entropy formula · $C^1$ smooth maps · Uniform expansion

Mathematics Subject Classification Primary 37D25; Secondary 37D35 · 37D20 · 37C40

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1 Introduction

Equilibrium states, a concept originating from statistical mechanics, are special classes of probability measures on compact metric spaces $X$ that are characterized by a Variational Principle. In the classical setting, given a continuous map $T : X \to X$ on a compact metric space $X$ and a continuous function $\phi : X \to \mathbb{R}$, then a $T$-invariant probability measure $\mu$ is called $\phi$-equilibrium state (or equilibrium state for the potential $\phi$) if

$$P_{\text{top}}(T, \phi) = h_\mu(T) + \int \phi \, d\mu,$$

where $P_{\text{top}}(T, \phi) = \sup_{\lambda \in \mathcal{M}_T} \left\{ h_\lambda(T) + \int \phi \, d\lambda \right\}$,

and $\mathcal{M}_T$ is the set of all $T$-invariant probability measures. The quantity $P_{\text{top}}(T, \phi)$ is called the topological pressure and the identity on the right hand side above is a consequence of the variational principle, see e.g. [30,46] for definitions of entropy $h_\mu(T)$ and topological pressure $P_{\text{top}}(T, \phi)$.

Depending on the dynamical system, these measures can have additional properties. Sinai, Ruelle, Bowen [12,13,35,38] were the forerunners of the theory of equilibrium states of (uniformly hyperbolic) smooth dynamical systems. They established an important connection
between equilibrium states and the soon to be called physical/SRB measures. This kind of measures provides asymptotic information on a set of trajectories that one expects is large enough to be observable from the point of view of the Lebesgue measure (or some other relevant reference measure), that is, a measure $\mu$ is physical or SRB if its ergodic basin

$$B(\mu) = \left\{ x \in X : \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j x) \xrightarrow{n \to +\infty} \int \phi \, d\mu, \forall \phi \in C^0(X, \mathbb{R}) \right\}$$

has positive volume (Lebesgue measure) or other relevant distinguished measure.

Several difficulties arise when trying to extend this theory. Despite substantial progress by several authors, a global picture is still far from complete. For example, the basic strategy used by Sinai–Ruelle–Bowen was to (semi)conjugate the dynamics to a subshift of finite type, via a Markov partition. However, existence of generating Markov partitions is known only in few cases and, often, such partitions can not be finite. Moreover, there exist transformations and functions admitting no equilibrium states or SRB measure.

In many situations with some kind of expansion it is possible to ensure that equilibrium states always exist. However, equilibrium states are generally not unique and the way they are obtained does not provide additional information for the study of the dynamics. In the setting of uniformly expanding maps, equilibrium states always exist and they are unique SRB measures if the potential is Hölder continuous and the dynamics is transitive.

Some natural questions arise: when does a system have an equilibrium state? Second, what statistical properties do those probabilities exhibit with respect to the (non-invariant) reference measure? Third, when is the equilibrium state unique? The existence of equilibrium states is a relatively soft property that can often be established via compactness arguments. Statistical properties and uniqueness of equilibrium state are usually more subtle and require a better understanding of the dynamics. In our setting, we do not expect uniqueness of equilibrium states because we consider dynamical systems with low regularity (continuous or of class $C^1$ only) and only continuous potentials.

From the Thermodynamical Formalism, the answer for the first question is known to be affirmative for distance expanding maps in a compact metric space $X$ and all continuous potentials (see [30,37]). Moreover, inspired by the definition of SRB-like measure given in [19], which always exists for all continuous transformations on a compact metric space, we have an answer to the second question. That is, given a reference measure $\nu$, there always exist weak-$\nu$-SRB-like and $\nu$-SRB-like measures (generalizations of the notion of SRB measure, see the statement of the results for more details).

Our first result shows that, in our context, the $\nu$-SRB-like measures can be seen as measures that naturally arise as accumulation points of $\nu_n$-SRB measures. In addition, for topologically exact distance expanding maps, the limit measure is a $\nu$-SRB-like probability measure with full generalized basin.

Moreover, we show that if $T : X \to X$ is an open distance expanding topologically transitive map in a compact metric space $X$ and $\phi : X \to \mathbb{R}$ is continuous, then for each (necessarily existing) conformal measure $\nu_\phi$ all the (necessarily existing) $\nu_\phi$-SRB-like measures are equilibrium states for the potential $\phi$.

Let now $M$ be a compact boundaryless finite dimensional Riemannian manifold. In the 1970’s, Pesin showed how two important concepts are related: Lyapunov exponents and measure-theoretic entropy in the smooth ergodic theory of dynamical systems. In [29], Pesin showed that if $\mu$ is an $f$-invariant measure of a $C^2$ (or $C^{1+\alpha}$, $\alpha > 0$) diffeomorphism of
a compact manifold which is absolutely continuous with respect to the Lebesgue (volume) reference measure of the manifold, then

\[ h_\mu(f) = \int \Sigma^+ d\mu, \]

where \( \Sigma^+ \) denotes the sum of the positive Lyapunov exponents at a regular point, counting multiplicities.

Ledrappier and Young in [27] characterized the measures which satisfy Pesin’s Entropy Formula for \( C^2 \) diffeomorphisms. Liu, Qian and Zhu [28,31] generalized Pesin’s Entropy Formula for \( C^2 \) endomorphisms.

There are extensive results concerning Pesin’s entropy formula, but the vast majority of these results were obtained under the assumption that the dynamics is at least \( C^1+\alpha \) regular. In fact, there is still a gap between \( C^1+\alpha \) and \( C^1 \) dynamics, despite recent progress in this direction [10,14,17,18,20,32,39,40].

We formulate a \( C^1 \) generalization of Pesin’s Entropy Formula, for non-uniformly expanding local diffeomorphisms \( f \), obtaining a sufficient condition to ensure that an \( f \)-invariant probability measure satisfies Pesin’s Entropy Formula.

Moreover, we study the existence of ergodic weak-SRB-like measures and some of their properties for dynamics with some expansion: either non uniformly expanding \( C^1 \) transformations, with hyperbolic times for Lebesgue almost every point, or maps which are expanding except at a finite subset of the ambient space. It is known that not all dynamics admit ergodic SRB-like measures: see [19, Example 5.4]. Moreover, in [18] Catsigeras, Cerminara, and Enrich show that ergodic SRB-like measures do exist for \( C^1 \) Anosov diffeomorphism and, more recently in [21], Catsigeras and Troubetzkoy show that for \( C^0 \)-generic continuous interval dynamics all ergodic measures are SRB-like.

### 1.1 Statement of results

Let \( T : X \to X \) be a continuous transformation defined on a compact metric space \((X, d)\). We present first some preliminary definitions needed to state the main results.

#### 1.1.1 Topological pressure

The *dynamical ball* centered at \( x \in X \), radius \( \delta > 0 \), and length \( n \geq 1 \) is defined by

\[ B(x, n, \delta) = \{ y \in X : d(T^j x, T^j y) \leq \delta, \ 0 \leq j \leq n - 1 \}. \]

Let \( \nu \) be a Borel probability measure on \( X \). We define

\[
 h_\nu(T, x) = \lim_{\delta \to 0} \lim_{n \to +\infty} \frac{1}{n} \log \nu(B(x, n, \delta)) \text{ and } h_\nu(T, \mu) = \mu \cdot \operatorname{esssup} h_\nu(T, x).
\]

(1.1)

We note that \( \nu \) is not necessarily \( T \)-invariant. If \( \mu \) is \( T \)-invariant and ergodic, then we have \( h_\mu(T, x) = h_\mu(T) \) for \( \mu \)-a.e. \( x \in X \), where \( h_\mu(T) \) is the metric entropy of \( T \) with respect to \( \mu \).

Let \( n \) be a natural number, \( \varepsilon > 0 \) and let \( K \) be a compact subset of \( X \). A subset \( F \) of \( X \) is said to \((n, \varepsilon)\)-span \( K \) with respect to \( T \) if, for each \( x \in K \), there exists \( y \in F \) with

\[ d(T^j x, T^j y) \leq \varepsilon \text{ for all } 0 \leq j \leq n - 1, \]

that is, \( K \subset \bigcup_{x \in F} B(x, n, \varepsilon) \).
Given a compact subset $K$ of $X$, we set
\[ h(T; K) = \lim_{\epsilon \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log N(n, \epsilon, K), \]
(1.2)
where $N(n, \epsilon, K)$ denotes the smallest cardinality of any $(n, \epsilon)$-spanning set for $K$ with respect to $T$.

**Definition 1** The **topological entropy** of $T$ is $h_{\text{top}}(T) = \sup_K h(T, K)$, where the supremum is taken over the collection of all compact subset of $X$.

Let $\phi : X \to \mathbb{R}$ be a real continuous function, usually referred to as **potential**. Given an open cover $\alpha$ for $X$ we define the **pressure** $P_T(\phi, \alpha)$ of $\phi$ with respect to $\alpha$ by
\[ P_T(\phi, \alpha) := \lim_{n \to +\infty} \frac{1}{n} \log \inf_{U \subseteq U^\alpha} \left\{ \sum_{x \in E} e^{S_n\phi(U)} \right\}, \]
where the infimum is taken over all subcovers $U$ of $\alpha^n = \bigvee_{j=0}^{n-1} T^{-j}(\alpha)$ and $S_n\phi(x) := \sum_{j=0}^{n-1} \phi(T^j x)$ and $S_n\phi(U) := \sup_{x \in U} S_n\phi(x)$ in what follows.

**Definition 2** The **topological pressure** $P_{\text{top}}(T, \phi)$ of the potential $\phi$ with respect to the dynamics $T$ is defined by
\[ P_{\text{top}}(T, \phi) = \lim_{\delta \to 0} \limsup_{|\alpha| \leq \delta} \sup_{n \to +\infty} \frac{1}{n} \log \sup_{x \in E} e^{S_n\phi(x)}, \]
where the supremum is taken over all maximal $(n, \epsilon)$-separated sets $E$.

We refer the reader to [46] for more details and properties of the topological pressure.

### 1.1.2 Distance expanding maps

A continuous mapping $T : X \to X$ is said to be **distance expanding** (with respect to the metric $d$, also known as “Ruelle expanding”) if there exist constants $\lambda > 1$, $\eta > 0$ and $n \geq 1$, such that for all $x, y \in X$
\[ d(x, y) < 2\eta \quad \text{then} \quad d(T^n(x), T^n(y)) \geq \lambda d(x, y). \]
(1.3)
In the sequel we will always assume (without loss of generality, see chapter 3 in [30]) that $n = 1$, that is
\[ d(x, y) < 2\eta \quad \implies \quad d(T(x), T(y)) \geq \lambda d(x, y). \]
(1.4)
We refer the reader to [7, 22, 30] for more details and properties of distance expanding map.
1.1.3 Transfer operator

We consider the Ruelle–Perron–Fröbenius transfer operator $L_{T,\phi} = L_{\phi}$ associated to $T : X \rightarrow X$ and the continuous function (potential) $\phi : X \rightarrow \mathbb{R}$ as the linear operator defined on the space $C^0(X, \mathbb{R})$ of continuous functions $g : X \rightarrow \mathbb{R}$ by

$$L_{\phi}(g)(x) = \sum_{T(y) = x} e^{\phi(y)} g(y).$$

The dual of the Ruelle–Perron–Fröbenius transfer operator is given by

$$L_{\phi}^*: M \rightarrow M,$$

$$\eta \mapsto L_{\phi}^* \eta : C^0(X, \mathbb{R}) \rightarrow \mathbb{R},$$

$$\psi \mapsto \int L_{\phi} \psi \, d\eta,$$

where $M$ is the family of Borel probability measures in $X$.

1.1.4 SRB and weak-SRB-like probability measures

For any point $x \in X$ we define

$$\sigma_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)},$$

(1.5)

where $\delta_y$ is the Dirac delta probability measure supported on $y \in X$. The sequence (1.5) gives the empirical probabilities of the orbit of $x$. Let $M_T$ be the (non-empty) set of $T$-invariant Borel probability measures in $X$.

**Definition 4** For each point $x \in M_T$, we denote by $p\omega(x) \subset M_T$ the limit set of the empirical sequence with initial state $x$ in the weak* topology of $M$, that is,

$$p\omega(x) := \left\{ \mu \in M_T : \exists n_j \underset{j \to +\infty}{\longrightarrow} +\infty \text{ such that } \sigma_{n_j}(x) \underset{\mu}{w^*} \underset{j \to +\infty}{\longrightarrow} \mu \right\}.$$ 

**Definition 5** Fixing an underlying reference measure $\nu$ on $X$, we say that $\mu \in M_T$ is $\nu$-SRB (or $\nu$-physical) if $\nu(B(\mu)) > 0$, where

$$B(\mu) = \{x \in X : p\omega(x) = \{\mu\} \}$$

is the “ergodic basin” of $\mu$.

Let $\mu \in M_T$ and $\varepsilon > 0$ be given and consider the following measurable subsets of $X$

$$A_{\varepsilon,n}(\mu) := \{x \in X : \text{dist}(\sigma_n(x), \mu) < \varepsilon\} \quad \text{and} \quad A_{\varepsilon}(\mu) := \{x \in X : \text{dist}(p\omega(x), \mu) < \varepsilon\}.$$

(1.6)

We say that $A_{\varepsilon,n}(\mu)$ is the $\varepsilon$-pseudo basin of $\mu$ up to time $n$ and $A_{\varepsilon}(\mu)$ is the basin of $\varepsilon$-weak statistical attraction of $\mu$.

**Definition 6** Fix a reference probability measure $\nu$ for the space $X$. We say that a $T$-invariant probability measure $\mu$ is

1. $\nu$-SRB-like (or $\nu$-physical-like), if $\nu(A_{\varepsilon}(\mu)) > 0$ for all $\varepsilon > 0$;
2. $\nu$-weak-SRB-like (or $\nu$-weak-physical-like), if $\lim\sup_{n \to +\infty} \frac{1}{n} \log \nu(A_{\varepsilon,n}(\mu)) = 0$, $\forall \varepsilon > 0$. 
When $\nu = \text{Leb}$ we say that $\mu$ is simply SRB-like (or weak-SRB-like).

**Remark 1.1** It is easy to see that every $\nu$-SRB measure is also a $\nu$-SRB-like measure. Moreover, the $\nu$-SRB-like measures are a particular case of $\nu$-weak-SRB-like (see [18, Theorem 1, item B]).

**Definition 7** Given a continuous map $T : X \to X$ and a continuous function $\phi : M \to \mathbb{R}$, we say that a probability measure $\nu$ is conformal for $T$ with respect to $\phi$ (or $\phi$-conformal) if there exists $\lambda > 0$ so that $\mathcal{L}_\phi^* \nu = \lambda \nu$.

### 1.1.5 SRB-like measures as limits of SRB measures

Our first result shows that we can see the $\nu$-SRB-like measures as accumulation points of $\nu_n$-SRB measures.

**Theorem A** Let $T : X \to X$ be an open distance expanding topologically transitive map of a compact metric space $X$, $(\phi_n)_{n \geq 1}$ a sequence of Hölder continuous potentials, $(\nu_n)_{n \geq 1}$ a sequence of conformal measures associated to the pair $(T, \phi_n)$ and $(\mu_n)_{n \geq 1}$ a sequence of $\nu_n$-SRB measures. Assume that

1. $\phi_{n_j} \to \phi$ (in the topology of uniform convergence);
2. $\nu_{n_j} \wastto \nu$ (in the weak* topology);
3. $\mu_{n_j} \wastto \mu$ (in the weak* topology).

Then $\nu$ is a conformal measure for $(T, \phi)$ and $\mu$ is $\nu$-SRB-like. Moreover, $\mu$ is an equilibrium state for the potential $\phi$ and, if $T$ is topologically exact, then $\nu(X \setminus A_\varepsilon(\mu)) = 0$ for all $\varepsilon > 0$.

This is one of the motivations for the study of SRB-like measures as the natural extension of the notion of physical/SRB measure for $C^1$ maps. Additional justification is given by the results of this work.

We stress that Hölder continuous potentials are only used in this work in the assumption of Theorem A. Since all continuous potentials can be approximated by Lipschitz potentials we have, for topologically exact maps in the setting of Theorem A, that there always exists some $\phi$-conformal measure admitting a $\nu$-SRB measure with full basin of $\varepsilon$-weak statistical attraction, for all $\varepsilon > 0$.

Next result extends, in particular, the main result obtained in [20, Theorem 2.3] proved only for expanding circle maps.

**Theorem B** Let $T : X \to X$ be an open expanding topologically transitive map of a compact metric space $X$ and $\phi : X \to \mathbb{R}$ a continuous potential. For each (necessarily existing) conformal measure $\nu_\phi$ all the (necessarily existing) $\nu_\phi$-SRB-like measures are equilibrium states for the potential $\phi$.

Next we explore more properties of $\nu$-SRB-like measures.

**Definition 8** Let $T : X \to X$ be a continuous map and $\phi : X \to \mathbb{R}$ a continuous potential. Given $r > 0$ we define the subset $\mathcal{K}_r(\phi) = \{ \mu \in \mathcal{M}_T : h_\mu(T) + \int \phi d\mu \geq P_{\text{top}}(T, \phi) - r \}$. 

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Corollary C Let $T : X \to X$ be an open expanding topologically transitive map of a compact metric space $X$, $\phi : X \to \mathbb{R}$ a continuous potential and $v$ a $\phi$-conformal measure. If $\mu$ is a $\phi$-equilibrium state such that $h_v(T, \mu) < \infty$, then every ergodic component $\mu_x$ of $\mu$ is a $v$-weak-SRB-like measure. Moreover, if $\mu$ is the unique $\phi$-equilibrium state, then $\mu$ is $v$-SRB, $v(B(\mu)) = 1$ and $\mu$ satisfies the following large deviation bound: for every weak$^*$ neighborhood $\mathcal{V}$ of $\mu$ we have
\[
\limsup_{n \to +\infty} \frac{1}{n} \log v(\{x \in X : \sigma_n(x) \in M \setminus \mathcal{V}\}) \leq -I(\mathcal{V})
\]
where $I(\mathcal{V}) = \sup\{r > 0 : \mathcal{K}_r(\phi) \subset \mathcal{V}\}$.

1.1.6 Non-uniformly expanding maps

We now extend Theorem B to weaker forms of expansion.

We denote by $\| \cdot \|$ a Riemannian norm on the compact $m$-dimensional boundaryless manifold $M$, $m \geq 1$; by $d(\cdot, \cdot)$ the induced distance and by Leb a Riemannian volume form, which we call *Lebesgue measure or volume* and assume to be normalized: $\text{Leb}(M) = 1$. Note that Leb is not necessarily $f$-invariant.

Recall that a $C^1$-map $f : M \to M$ is uniformly expanding or just expanding if there is some $\lambda > 1$ such for some choice of a metric in $M$ one has
\[
\| Df(x)v \| > \lambda \| v \|, \text{ for all } x \in M \text{ and all } v \in T_x M.
\]
In the following statements we always assume $f : M \to M$ to be a $C^1$ local diffeomorphism and define
\[
\psi := -\log Jf \quad \text{and} \quad Jf = |\det Df|.
\]

The Lyapunov exponents of a $C^1$ local diffeomorphism $f$ of a compact manifold $M$ are defined by Oseledets Theorem which states that, for any $f$-invariant probability measure $\mu$, for almost all points $x \in M$ there is $\kappa(x) \geq 1$, a filtration $T_x M = F_1(x) \supset F_2(x) \supset \cdots \supset F_{\kappa(x)}(x) \supset F_{\kappa(x)+1}(x) = \{0\}$, and numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_\kappa(x)$ such $Df(x) \cdot F_1(x) = F_1(f(x))$ and
\[
\lim_{n \to +\infty} \frac{1}{n} \log \| Df^n(x)v \| = \lambda_i(x)
\]
for all $v \in F_i(x) \setminus F_{i+1}(x)$ and $0 \leq i \leq \kappa(x)$. The numbers $\lambda_i(x)$ are called Lyapunov exponents of $f$ at the point $x$. For more details on Lyapunov exponents and non-uniform hyperbolicity, see [9].

Definition 9 Let $f : M \to M$ be a $C^1$ local diffeomorphism. We say that $\mu \in \mathcal{M}_f$ satisfies Pesin’s Entropy Formula if $h_\mu(f) = \int \Sigma^+ d\mu$, where $\Sigma^+$ denotes the sum of the positive Lyapunov exponents at a regular point, counting multiplicities.

Given $0 < \sigma < 1$ we define
\[
H(\sigma) = \left\{ x \in M : \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df(f^j(x))^{-1} \| < \log \sigma \right\}.
\]  

(1.7)

Definition 10 We say that $f : M \to M$ is non-uniformly expanding if there exists $\sigma \in (0, 1)$ such that $\text{Leb}(H(\sigma)) = 1$. 
A probability measure $\nu$ (not necessarily invariant) is \textit{expanding} if there exists $\sigma \in (0, 1)$ such that $\nu(H(\sigma)) = 1$.

Next result relates non-uniform expansion and expanding measures with the Entropy Formula.

**Theorem D** Let $f : M \to M$ be non-uniformly expanding. Every expanding weak-SRB-like probability measure satisfies Pesin’s Entropy Formula. Moreover, all ergodic SRB-like probability measures are expanding.

Now we add a condition ensuring that there exists ergodic weak-SRB-like measures and that all weak-SRB-like measures are equilibrium states.

**Corollary E** Let $f : M \to M$ be non-uniformly expanding. If $P_{\text{top}}(f, \psi) = 0$, then all weak-SRB-like probability measures are $\psi$-equilibrium states and all the (necessarily existing) expanding ergodic weak-SRB-like measures satisfy Pesin’s Entropy Formula.

### 1.1.7 Weak and non-uniformly expanding maps

Now we strengthen the assumptions on non-uniform expansion to improve the properties of SRB-like measures.

**Definition 11** We say that $f$ is weak-expanding if $\|Df(x)^{-1}\| \leq 1$ for all $x \in M$.

As usual, a probability measure is \textit{atomic} if it is supported on a finite set. We denote $\text{supp}(\mu)$ the support of probability measure $\mu$ and recall that $\psi = -\log Jf = -\log |\det Df|$ in what follows.

**Corollary F** Let $f : M \to M$ be weak-expanding and non-uniformly expanding. Then,

1. all the (necessarily existing) weak-SRB-like probability measures are $\psi$-equilibrium states and, in particular, satisfy Pesin’s Entropy Formula;
2. there exists some ergodic weak-SRB-like probability measure;
3. if $\psi < 0$ (that is, $f$ is volume expanding), then there is no atomic weak-SRB-like probability measure;
4. if $D = \{x \in M : \|Df(x)^{-1}\| = 1\}$ is finite and $\psi < 0$ then almost all ergodic components of a $\psi$-equilibrium state are weak-SRB-like measures. Moreover, for all weak-SRB-like probability measures $\mu$, its ergodic components $\mu_x$ are weak-SRB-like probability measure for $\mu$-a.e. $x \in M \backslash D$.

In particular, we see that an analogous result to the existence of an atomic physical measure for a quadratic map, as obtained by Keller in [24], is not possible in the $C^1$ setting, although generically such measures must be singular with respect to any volume from.

**Corollary G** No atomic measure is an SRB measure for a $C^1$ uniformly expanding map of a compact manifold.

We now restate the previous results in the uniformly expanding setting.

**Corollary H** Let $f : M \to M$ be a $C^1$-expanding map. Then an $f$-invariant probability measure $\mu$ satisfies Pesin’s Entropy Formula if and only if its ergodic components $\mu_x$ are weak-SRB-like $\mu$-a.e. $x \in M$. Moreover, all the (necessarily existing) weak-SRB-like probability measures satisfy Pesin’s Entropy Formula, in particular, are $\psi$-equilibrium states. In addition, if $\mu$ is the unique weak-SRB-like probability measure, then $\mu$ is an ergodic SRB probability measure with $\text{Leb}(B(\mu)) = 1$, $\mu$ is the unique $\psi$-equilibrium state and satisfies a large deviation bound similar to Corollary C.
Using this we get a weak statistical stability result in the uniformly \(C^1\)-expanding setting: all weak\(^*\) accumulation points of weak-SRB-like measures are generalized convex linear combinations of ergodic weak-SRB-like measures, as follows.

**Corollary I** Let \(\{f_n : M \to M\}_{n \geq 1}\) be a sequence of \(C^1\)-expanding maps such that \(f_n \to f\) in the \(C^1\)-topology and \(f : M \to M\) be a \(C^1\)-expanding map. Let \((\mu_n)_{n \geq 1}\) be a sequence of weak-SRB-like measures associated \(f_n\). Then each accumulation point \(\mu\) of \((\mu_n)_{n \geq 1}\) is an equilibrium state for the potential \(\psi = -\log|\det Df|\) (in particular, satisfies Pesin’s Entropy Formula) and almost all ergodic components of \(\mu\) are weak-SRB-like measures.

### 1.2 Further questions

Here we list some questions that have arisen during the development of this work.

**Question 1.2** Under the same assumptions of Corollary C can we obtain a lower bound of large deviations with the same rate? (Perhaps assuming that the dynamics is topologically mixing?)

**Question 1.3** Is it possible to obtain a statistical property for the weak-SRB-like measures, as in the case of Corollary C for \(C^1\)-weak expanding and non uniformly expanding maps?

**Question 1.4** In Corollary I is it possible to show that the limit measure is weak-SRB-like?

Some examples that have naturally arisen during the development of this work motivate further questions.

The first example is attributed to Bowen and can be found in greater detail in [19, Example 5.5].

**Example 1.5** Consider a diffeomorphism \(f\) in a ball of \(\mathbb{R}^2\) with two hyperbolic saddle points \(A\) and \(B\) such that a connected component of the unstable global manifold \(W^u(A) \setminus \{A\}\) is an embedded arc that coincides with a connected component of the stable global manifold \(W^s(B) \setminus \{B\}\), and conversely, the embedded arc \(W^u(B) \setminus \{B\} = W^s(A) \setminus \{A\}\). Take \(f\) such that there exists a source \(C \subset U\) where \(U\) is the open ball with boundary \(W^u(A) \cup W^u(B)\) (Fig. 1).

If the eigenvalues of the derivative of \(f\) at \(A\) and \(B\) are adequately chosen as specified in [23,42], then the empiric sequence (1.5) for all \(x \in U \setminus \{C\}\) is not convergent. It has at least two subsequences convergent to different convex combinations of the Dirac deltas \(\delta_A\) and \(\delta_B\).

Thus, as observed in [19], the SRB-like probability measures are convex linear combinations of \(\delta_A\) and \(\delta_B\) and form a segment in the space \(M\) of probability measures. This example shows that the SRB-like measures are not necessarily ergodic.

**Fig. 1** Bowen’s eye
Moreover, the eigenvalues of $Df$ at the saddles $A$ and $B$ can be modified to obtain, instead of the result above, the convergence of the sequence (1.1) as stated in Lemma (i) of page 457 in [41]. In fact, taking conservative saddles (and $C^0$ perturbing $f$ outside small neighborhoods of the saddles $A$ and $B$ so the topological $\omega$-limit of the orbits in $U\setminus\{C\}$ still contains $A$ and $B$), one can get for all $x \in U\setminus\{C\}$ an empirical sequence (1.5) that is convergent to a single measure $\mu = \lambda\delta_A + (1 - \lambda)\delta_B$, with a fixed constant $0 < \lambda < 1$. So, $\mu$ is the unique SRB-like measure. This proves that the set of SRB-like probability measures does not depend continuously on the map.

This example motivates the following questions:

**Question 1.6** When can we say that a dynamic system $(f, \phi)$ has ergodic $v_\phi$-SRB-like measures?

From Example 1.5, we know that the set of $v_\phi$-SRB-like probability measures does not in general depend continuously on the map.

**Question 1.7** Fixing the potential $\phi$, is there continuous dependence of $v_\phi$-SRB-like probability measures as functions of the underlying map (in the $C^1$ topology)?

Recently [5] a result on this direction was obtained for Hölder and hyperbolic potentials on non-uniformly expanding maps.

**Question 1.8** Is there continuous dependence of $v_\phi$-SRB-like probability measures as functions of the potential $\phi$?

A positive response in this direction is given by Theorem A, however we still do not know in general whether the limit measure is ergodic.

Recently [16] shows that under general random perturbations only SRB measures can be empirically stochastically stable.

**Question 1.9** Can we obtain stochastic stability for SRB-like measures under certain random perturbations?

The next example is an adaptation of the ‘intermittent’ Manneville map into a local homeomorphism of the circle.

**Example 1.10** Consider $I = [-1, 1]$ and the map $\hat{f} : I \to I$ (see Fig. 2) given by

$$\hat{f}(x) = \begin{cases} 2\sqrt{x} - 1 & \text{if } x \geq 0, \\ 1 - 2\sqrt{|x|} & \text{otherwise}. \end{cases}$$

This map induces a continuous local homeomorphism $f : S^1 \to S^1$ through the identification $S^1 = I/\sim$, where $-1 \sim 1$, not differentiable at the point 0. This is a map differentiable everywhere except at a single point, having a positive frequency of hyperbolic times at Lebesgue almost every point as can be seen in [3, Sect. 5].

This example is weak expanding and non uniformly expanding and suggests that we can think of a generalization of Theorem B using the same ideas of Theorem D, replacing Lebesgue measure by an expanding $\phi$-conformal measure $\nu$ for $\phi \in C^0(X, \mathbb{R})$ and the potential $\psi = -\log Jf$ by $-\log J_\nu f$, where $J_\nu f$ is the Jacobian $f$ with respect to $\nu$. 

\[\text{Springer}\]
Question 1.11 Let $T : X \to X$ be a local homeomorphism (not necessarily distance expanding) of the compact metric space $X$ and let $\phi : X \to \mathbb{R}$ be a continuous potential. If there exists an expanding $\phi$-conformal measure $\nu$, then all the (necessarily existing) $\nu$-SRB-like measures are equilibrium states for the potential $\phi$?

It may be necessary to consider additional hypotheses about the continuous potentials, e.g. that they are hyperbolic potentials (see definition and results about Hölder hyperbolic potentials in [34]).

1.3 Organization of the paper

We present in Sect. 2 some examples of application for the main results. Section 3 contains preparatory material of the theory of $\nu$-weak-SRB-like measures and measure-theoretic entropy that will be necessary for the proofs. In Sections 4 and 5 we prove Theorems A and B. In Section 6 we use the properties of hyperbolic times and weak-SRB-like measures to prove a reformulation of Pesin’s Entropy Formula for $C^1$ non-uniformly expanding local diffeomorphism (Theorem D). In Section 7 we prove Corollary E. Finally, in Section 8 we prove the results stated in Corollaries F, C, H and I.

2 Examples of application

Here we present some examples of application. In the first subsection we describe the construction of a $C^1$ uniformly expanding map with many different SRB-like measures which, in particular, satisfies the assumptions of Theorem B and Corollary H.

In the second subsection we present a class of non uniformly expanding $C^1$ local diffeomorphisms satisfying the assumptions of Theorem D.

In the third subsection we exhibit examples of weak-expanding and non uniformly expanding transformations which are not uniformly expanding in the setting of Corollary F.
2.1 Expanding map with several absolutely continuous invariant probability measures

We present the construction of a $C^1$-expanding map $f : \mathbb{S}^1 \to \mathbb{S}^1$ which is not of class $C^{1+\varepsilon}$ for any $0 < \varepsilon < 1$ and that has several SRB-like measures. Moreover, such measures are absolutely continuous with respect to Lebesgue measure (such $f$ does not belong to the $C^1$ generic subset of $C^1(M,M)$ found in [8]).

Example 2.1 Following Bowen [11], we first adopt some notation for Cantor sets with positive Lebesgue measure. Let $I$ be a closed interval and $\alpha_n > 0$ numbers with $\sum_{n=0}^{\infty} \alpha_n < |I|$, where $|E|$ denotes the one-dimensional Lebesgue measure of the subset $E$ of $I$. Let $a = a_1 a_2 \ldots a_n$ denote a sequence of 0’s and 1’s of length $n = n(a)$; we denote the empty sequence $a = \emptyset$ with $n(\emptyset) = 0$. Define $I_\emptyset = I = [a, b]$, $I_n^* = [\frac{a+b}{2} - \frac{\alpha_0}{2}, \frac{a+b}{2} + \frac{\alpha_0}{2}]$ and $I_n^* \subset I_\emptyset$ recursively as follows.

Let $I^*_a$ and $I^*_1$ be the left and right intervals remaining when the interior of $I^*_a$ is removed from $I^*_1$; let $I^*_{\emptyset}$ be the closed interval of length $\frac{\alpha_n(a)}{3^{n(\emptyset)}}$ and having the same center as $I_{\emptyset}$ (k=0,1).

The Cantor set $K_I$ is given by $K_I = \bigcap_{m=0}^{\infty} \bigcup_{n(\emptyset)=m} I^*_a$.

This is the standard construction of the Cantor set except that we allow ourselves some flexibility in the lengths of the removed intervals. The measure of $K_I$ is $\text{Leb}(K_I) = |I| - \sum_{n=0}^{\infty} \alpha_n > 0$.

Suppose that another interval $J \supset I$ is given together with $\beta_n > 0$ such that $\sum_{n=0}^{\infty} \beta_n < |J|$. One can then construct $J^*_a$, $J^*_1$ and $K_J$ as above. Let us assume now that $\frac{\beta_n}{a_n} \to_{n \to \infty} \gamma > 0$.

Following the construction of Bowen [11], we get $g : I \to J$ a $C^1$ orientation preserving homeomorphism so that $g'(x) = \gamma$ for all $x \in K_I$ and $g'(x) > 1$ for all $x \in I$.

More precisely, let us take $J = [-1,1]$ and choose $\beta_n > 0$ with $\sum_{n=0}^{\infty} \beta_n < 2$ and $\lim_{n \to \infty} \frac{\beta_{n+1}}{\beta_n} = 1$ (e.g. $\beta_n = \frac{1}{(n+100)^2}$). Let $I = \left[\frac{\beta_0}{2},1\right]$ and $\alpha_n = \frac{\beta_{n+1}}{2}$. Then $\sum_{n=0}^{\infty} \alpha_n < 1 - \frac{\beta_0}{2}$ and $\gamma = \lim_{n \to \infty} \frac{\beta_n}{a_n} = 2$. We define a homeomorphism $G : (-I) \cup I \to J$ by $G(x) = \begin{cases} g(x) & \text{if } x \in I \\ -g(-x) & \text{if } x \in (-I) \end{cases}$, where $K_J = \bigcap_{n=0}^{\infty} G^{-n}(J)$ and $G|_{K_I} : K_I \to K_J$.

Consider now $c_1 = \left(\frac{3\beta_0}{2} - 2\right)\left(\frac{4}{\beta_0}\right)^3$, $c_2 = \left(\frac{9\beta_0}{4} - 3\right)\left(\frac{4}{\beta_0}\right)^2$; $f_1 : [-\frac{\beta_0}{2}, -\frac{\beta_0}{4}] \to [-1, -\frac{\beta_0}{4}]$ given by

$$f_1(x) = c_1 \left(x + \frac{\beta_0}{2}\right)^3 - c_2 \left(x + \frac{\beta_0}{2}\right)^2 + 2 \left(x + \frac{\beta_0}{2}\right) - 1$$

and $f_2 : [\frac{\beta_0}{4}, \frac{\beta_0}{2}] \to [\frac{\beta_0}{4}, 1]$ given by $f_2(x) = c_1 \left(x - \frac{\beta_0}{2}\right)^3 + c_2 \left(x - \frac{\beta_0}{2}\right)^2 + 2 \left(x - \frac{\beta_0}{2}\right) + 1$. Then, we have

1. $f_1\left(-\frac{\beta_0}{2}\right) = -1$, $f_1\left(-\frac{\beta_0}{4}\right) = -\frac{\beta_0}{4}$, $f_2\left(\frac{\beta_0}{4}\right) = \frac{\beta_0}{4}$ and $f_2\left(\frac{\beta_0}{2}\right) = 1$

2. $f_1^+\left(\frac{\beta_0}{2}\right) = \lim_{h \to 0^+} \frac{f_1(\frac{\beta_0}{4} + h) - f_1(\frac{\beta_0}{4})}{h} = 2$, $f_2^+\left(\frac{\beta_0}{4}\right) = 2$, $f_1^+\left(\frac{\beta_0}{2}\right) = 2$.
Consider now $J_1 = \left[-\frac{\beta_0}{4}, \frac{\beta_0}{4}\right]$, $I_1 = \left[\frac{\beta_0^2}{8}, \frac{\beta_0}{4}\right]$ and choose $\beta_n' = \frac{\alpha_n \beta_n}{4} > 0$ with $\frac{\beta_0}{2} < \beta_0' < \beta_0$ and $\beta_n' = \frac{\alpha_{n+1} \beta_{n+1}}{2}$. Let $\alpha_n' = \frac{\beta_n'}{\beta_n}$, then $\sum_{n=0}^{\infty} \alpha_n' = \sum_{n=0}^{\infty} \frac{\beta_n'}{\beta_{n+1}} < \frac{1}{2} \left(\frac{\beta_0}{2} - \beta_0'\right) = \frac{1}{2} \left(\frac{\beta_0}{2} - \frac{\beta_0^2}{8}\right)$ and $\gamma = \lim_{n \to \infty} \frac{\beta_n}{\alpha_n} = \lim_{n \to \infty} \frac{2\beta_n}{\beta_{n+1}} = 2$.

Similarly to the above construction, we obtain a homeomorphism $G_1 : (-I_1) \cup I_1 \to J_1$ given by

$$G_1(x) = \begin{cases} g_1(x) & \text{if } x \in I_1 \\ -g_1(-x) & \text{if } x \in (-I_1) \end{cases},$$

where $K_{J_1} = \bigcap_{n=0}^{\infty} G_1^{-n}(J_1)$ is a Cantor set with positive Lebesgue measure, $G_1|_{K_{J_1}} : K_{J_1} \to J_1$ and $g_1 : I_1 \to J_1$ is a $C^1$ orientation preserving homeomorphism so that $g_1'(x) = 2$ for all $x \in K_{J_1}$ and $g_1'(x) > 1$ for all $x \in I_1$.

Similarly we obtain $f_3 : [0, \frac{\beta_0^2}{8}] \to [-1, -\frac{\beta_0}{4}]$ and $f_4 : [-\frac{\beta_0}{4}, 0] \to \{\frac{\beta_0}{4}, 1\}$ such that $f_3(0) = -1$, $f_3\left(-\frac{\beta_0^2}{8}\right) = -\frac{\beta_0}{4}$, $f_4\left(-\frac{\beta_0^2}{8}\right) = 2$, $f_3'(0) = 2$, $f_4\left(-\frac{\beta_0}{4}\right) = \frac{\beta_0}{4}$, $f_4(0) = 1$, $f_4\left(-\frac{\beta_0^2}{8}\right) = 2$ and $f_4(0) = 0$.

Finally, define the function $f : J \to J$ by (see Fig. 3 for its graph)

$$f(x) = \begin{cases} G(x) & \text{if } x \in (-I) \cup I \\ f_1(x) & \text{if } x \in \left(-\frac{\beta_0}{2}, -\frac{\beta_0}{4}\right) \\ f_2(x) & \text{if } x \in \left(\frac{\beta_0}{4}, \frac{\beta_0}{2}\right) \\ f_3(x) & \text{if } x \in \left(-\frac{\beta_0^2}{8}, 0\right) \\ f_4(x) & \text{if } x \in \left(0, \frac{\beta_0^2}{8}\right) \\ f_1^{-1}(x) & \text{if } x \in \left(-\frac{\beta_0}{2}, -\frac{\beta_0}{4}\right) \cup \left(\frac{\beta_0}{4}, \frac{\beta_0}{2}\right) \cup \left(-\frac{\beta_0^2}{8}, 0\right) \cup \left(0, \frac{\beta_0^2}{8}\right) \cup (-I) \cup I \end{cases}.$$

Identifying $-1$ and 1 and making a linear change of coordinates we obtain a $C^1$ expanding map of the circle $f : S^1 \to S^1$ which is not of class $C^{1+\varepsilon}$ for any $\varepsilon > 0$; see Bowen [11].

Consider $\text{Leb}_{K_J}$ the normalized Lebesgue measure of the set $K_J$, this is, $\text{Leb}_{K_J}(A) = \text{Leb}(A \cap K_J)/\text{Leb}(K_J)$ for all measurable $A \subset J$. Denote $\Sigma_2^+ = \{0, 1\}$ and consider the homeomorphism $h : K_J \to \Sigma_2^+$ that associates each point $x \in K_J$ the sequence $a \in \Sigma_2^+$ describing its location in the set $K_J$, this is, $a$ is such that $I_{\underline{a}} \cap K_J = \{x\}$. Let $\mu$ be the Bernoulli measure in $\Sigma_2^+$ giving weight 1/2 to each digit.

**Claim 2.2** $\mu = h_* \text{Leb}_{K_J}$

We show that this relation holds for an algebra of subsets which generates the Borel $\sigma$-algebra of $\Sigma_2^+$. For $\underline{a} = a_1a_2\ldots a_n$ we have that $\mu(\{\underline{a}\}) = 1/2^n$. Moreover, $h^{-1}(\{\underline{a}\}) = I_{\underline{a}} \cap K_J$, so $m_{K_J}(h^{-1}(\{\underline{a}\})) = m_{K_J}(I_{\underline{a}} \cap K_J)$. By construction, at each step all remaining intervals in the construction of $K_J$ have the same length. Thus, the $n$-th stage contains $2^n$ intervals $I_{\underline{a}}$ (among them) all with the same Lebesgue measure. Since $\text{Leb}_{K_J}$ is a probability measure, we have

$$2^n \text{Leb}_{K_J}(I_{\underline{a}} \cap K_J) = 1 \implies \text{Leb}_{K_J}(I_{\underline{a}} \cap K_J) = \frac{1}{2^n}.$$
We conclude that $\text{Leb}_{K_J}(I_2 \cap K_J) = \text{Leb}_{K_J}(h^{-1}([a])) = \mu([a])$ for all $a = a_1a_2\ldots a_n$, $n \geq 1$ proving the Claim.

Clearly $h \circ f|_{K_J} = \sigma \circ h$ where $\sigma$ is the standard left shift $\sigma : \Sigma_2^+ \circ$. Since $\mu$ is $\sigma$-invariant, then $\text{Leb}_{K_J}$ is $f|_{K_J}$-invariant. Consequently $(f, \text{Leb}_{K_J})$ is mixing, $h_{\text{top}}(f) = \log 2 = h_{\text{Leb}_{K_J}}(f) = h_{\mu}(\sigma)$ and $\text{Leb}_{K_J} \ll \text{Leb}$.

Hence $\lim_{n \to +\infty} \sigma_n(x) = \text{Leb}_{K_J}$ for $\text{Leb}_{K_J}$-a.e. $x \in K_J$ and $|K_J| > 0$, so it follows that $\text{Leb}_{K_J}$ is a SRB-like measure. By Theorem B, $\text{Leb}_{K_J}$ is an equilibrium state for the potential $\psi = -\log |f'|$.

Similarly $\text{Leb}_{K_J}$ is also a SRB-like measure, but distinct from $\text{Leb}_{K_J}$.

We observe that this construction allows us to obtain countably many ergodic SRB-like measures by reapplying the construction to each removed subinterval of the first Cantor set.

We also note that, if we take a sequence of Hölder continuous potentials $\phi_n$ converging to $\psi = -\log |f'|$; choose $\nu_n$ a $\phi_n$-conformal measure and a $\nu_n$-SRB measure $\mu_n$ and weak* accumulation points $\nu, \mu$ as in Theorem A, since $f$ is topologically exact, then $\nu(A_\epsilon(\mu)) = 1$ for all $\epsilon > 0$. Therefore $\nu$ is not Lebesgue measure on $S^1$.

### 2.2 $C^1$ Non uniformly expanding maps

Next we present an example that is a robust ($C^1$ open) class of non-uniformly expanding $C^1$ local diffeomorphisms.

This family of maps was introduced in [4] (see also Sect. 2.1 in [1, Chapter 1]) and maps in this class exhibit non-uniform expansion Lebesgue almost everywhere but are not uniformly expanding.

**Example 2.3** Let $M$ be a compact manifold of dimension $d \geq 1$ and $f_0 : M \circ$ is a $C^1$—expanding map. Let $V \subset M$ be some small compact domain, so that the restriction
of $f_0$ to $V$ is injective. Let $f$ be any map in a sufficiently small $C^1$-neighborhood $N$ of $f_0$ so that:

1. $f$ is volume expanding everywhere: there exists $\sigma_1 > 1$ such that
   \[ |\det Df(x)| \geq \sigma_1 \text{ for every } x \in M \]

2. $f$ is expanding outside $V$: there exists $\sigma_0 < 1$ such that
   \[ \|Df(x)^{-1}\| < \sigma_0 \text{ for every } x \in M \setminus V; \]

3. $f$ is not too contracting on $V$: there is some small $\delta > 0$ such that
   \[ \|Df(x)^{-1}\| < 1 + \delta \text{ for every } x \in V. \]

Then every map $f$ in such a $C^1$-neighborhood $N$ of $f_0$ is non-uniformly expanding (see a proof in @Sect. 2.1 in [1]). By Theorem D, every expanding weak-SRB-like probability measure satisfies Pesin’s Entropy Formula, and every ergodic SRB-like measure is expanding.

Such classes of maps can be obtained through deformation of a uniformly expanding map by isotopy inside some small region.

### 2.3 Weak-expanding and non-uniformly expanding maps

Consider $\alpha > 0$ and the map $T_\alpha : [0, 1] \to [0, 1]$ defined as follows

\[
T_\alpha(x) = \begin{cases} 
  x + 2^\alpha x^{1+\alpha} & \text{if } x \in [0, 1/2) \\
  x - 2^\alpha (1 - x)^{1+\alpha} & \text{if } x \in [1/2, 1] 
\end{cases}
\]

This map defines a $C^{1+\alpha}$ map of the unit circle $S^1 := [0, 1]/\sim$ into itself, called intermittent maps. These applications are expanding, except at a neutral fixed point, the unique fixed point is 0 and $DT_\alpha(0) = 1$. The local behavior near this neutral point provides many interesting results in ergodic theory. If $\alpha \geq 1$, i.e. if the order of tangency at zero is high enough, then the Dirac mass at zero $\delta_0$ is the unique physical probability measure and so the Lyapunov exponent of Lebesgue almost all points vanishes (see [43]). This example shows that there exists systems which are weak-expanding but not non-uniformly expanding. Hence the assumption of weak expansion together with non-uniform expansion in Theorem F is not superfluous.

The following is an example of a map which is weak-expanding and non-uniformly expanding in the setting of Theorem F.

**Example 2.4** If, in the setting of the construction of the previous Example 2.3, the region $V$ of a point $p$ of a periodic orbit with period $k$, and the deformation weakens one of the eigenvalues of $(Df_0^k)_p$ in such a way that 1 becomes and eigenvalue of $Df_p^k$, then $f$ is an example of a weak expanding and non uniformly expanding $C^1$ transformation.

More precisely, consider the function $g_0(t) = \frac{t}{\log(1/t)}$, $0 < t < 1/2$, $g_0(0) = 0$. It is easy to see that

- $g_0'(t) > 0$, $0 < t < 1/2$ and so $g_0$ is strictly increasing;
- $g_0'$ is not of $\alpha$-generalized bounded variation for any $\alpha \in (0, 1)$ (see [25] for the definition of generalized bounded variation) and so $g_0'$ is not $C^\alpha$ for any $0 < \alpha < 1$.

Setting $g(t) = g_0(t)/g_0(1/2)$ we obtain $g : [0, 1/2] \to [0, 1]$ a $C^1$ strictly increasing function which is not $C^{1+\alpha}$ for any $\alpha \in (0, 1)$. Now we consider the analogous map to $T_\alpha$

\[
T(x) = \begin{cases} 
  x + xg(x) & \text{if } x \in [0, 1/2) \\
  x - (1 - x)g(1 - x) & \text{if } x \in [1/2, 1] 
\end{cases}
\]
This is now a $C^1$ map of the circle into itself which is not $C^{1+\alpha}$ for any $0 < \alpha < 1$. Moreover, letting $f_0 = T \times E$ where $E(x) = 2x \mod 1$, we see that $f_0$ satisfies items (1-3) in Example 2.3 for $V$ a small neighborhood of the fixed point $(0, 0)$.

Hence $f_0$ is a $C^1$ non-uniformly expanding map. Moreover, since $T'(0) = 1$, we have that $f_0$ is also a $C^1$ weak expanding map with $D = \{ (0, 0) \}$.

Now we extend this construction to obtain a weak expanding and non uniformly expanding $C^1$ map so that $D$ is non denumerable.

**Example 2.5** Let $K \subset I = [0, 1]$ be the middle third Cantor set and let $\beta_n(x) = d(x, K \cap [0, 3^{-n}])$, where $d$ is the Euclidean distance on $I$. Note that $\beta_n$ is Lipschitz and $g_0' \circ \beta_n$ is bounded and continuous but not of $\alpha$-generalized bounded variation for any $0 < \alpha < 1$, where $g_0$ was defined in Example 2.4. Indeed, since $(3^{-k}, 2 \cdot 3^{-k})$ is a gap of $K \cap [0, 3^{-n}]$ for all $k > n$, then

$$\frac{g_0'(\beta(3^{-k}/2)) - g_0'(\beta(3^{-k}))}{3^{-k}/2 - 3^{-k}} = 2 \cdot 3^k g_0' \left( \frac{1}{2 \cdot 3^{k-1}} \right)$$

$$= 2 \cdot 3^k \left( \frac{1}{(1-k) \log 3 - \log 2} \right)$$

is not bounded when $k \nearrow \infty$. If $h : I \to \mathbb{R}$ is given by $h(x) = x + (\int_0^x g_0' \circ \beta_n \circ g_0' \circ \beta_n)^{-1} \sin x$, where the integrals are with respect to Lebesgue measure on the real line, then $h(0) = 0$, $h(1) = 2$ and $h$ induces a $C^1$ map $h_0 : S^1 \to S^1$ whose derivative is continuous but not $C^\alpha$ for any $0 < \alpha < 1$, such that $h_0'(x) = 1$ for each $x \in K \cap [0, 3^{-n}]$ and $h_0'(x) > 1$ otherwise.

We set now $h_t(x) = x + (\int_0^x (t + g_0' \circ \beta_n) \circ g_0' \circ \beta_n)^{-1} \sin x$ for $t \in [0, 1]$, which induces a $C^1$ map of $S^1$ into itself with continuous derivative and $h_0'(x) > 1$ for $t > 0$. We then define the skew-product map $f_0(x, y) = (E(x), h_{\sin \pi x}(y))$ for $(x, y) \in S^1 \times S^1$ which is a weak expanding map with $D = \{ 0 \} \times (K \cap [0, 3^{-n}])$.

For sufficiently big $n$ it is easy to verify that $f_0$ satisfies items (1-3) in Example 2.3 for $V$ a small neighborhood of the fixed point $(0, 0)$. Hence $f_0$ is also a $C^1$ non-uniformly expanding map which is not a $C^{1+\alpha}$ map for any $0 < \alpha < 1$.

## 3 Preliminary definitions and results

In this section, we revisit the definition and some results on the theory $\nu$-SRB-like measures and also some properties of measure-theoretic entropy.

### 3.1 Invariant and $\nu$-SRB-like measures

Let $X$ be a compact metric space and $T : X \to X$ be continuous. Denote $M$ the set of all Borel probability measures on $X$ and $M_T$ the set of all $T$–invariant Borel probability measures on $X$. In $M$ fix a metric compatible with the weak* topology on $M$, e.g.

$$\text{dist}(\mu, \nu) := \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \int \phi_i d\nu \right|, \quad (3.1)$$

where $\{\phi_i\}_{i \geq 0}$ is a countable family of continuous functions that is dense in the space $C^0(X, [0, 1])$. The following will be used in the proofs of the large deviations lemma and Theorem 6.8 that are essential the proofs of Theorems B, F and D.
Lemma 3.1 For all $\varepsilon > 0$ there is $\delta > 0$ such that if $d(T^i(x), T^i(y)) < \delta$ for all $i = 0, \ldots, n - 1$ then $\text{dist}(\sigma_n(x), \sigma_n(y)) < \varepsilon$

Proof This is a simple consequence of compactness and metrizability of $M$ with the weak* topology.

Next results ensure the existence of $\nu$-SRB-like measure and consequently, by Remark 1.1, that $\nu$-weak-SRB-like measures do always exist.

Given a probability measure $\nu$ on $M$ let $W_T(\nu) \subset M_T$ be the family of weak* accumulation measures $p_\omega(x)$ for $\nu$-a.e. $x$. This is a well-defined compact subset of $M_T$ in the weak* topology as follows.

Proposition 3.2 Let $T : X \to X$ be a continuous map of a compact metric space $X$. For each reference probability measure $\nu$ there exist $W_T(\nu) \subset M_T$ the unique minimal non-empty and weak* compact set, such that $p_\omega(x) \subset W_T(\nu)$ for $\nu$-a.e. $x \in X$. When $\nu = \text{Leb}$ we denote $W_T(\text{Leb}) = W_T$.

Proof See proof of [19, Theorem 1.5], replacing Leb by $\nu$.

The subset $W_T(\nu)$ contains all the $\nu$–SRB-like measures, as follows.

Proposition 3.3 Given $\nu$ a reference probability measure, a probability measure $\mu \in M$ is $\nu$-SRB-like if and only if $\mu \in W_T(\nu)$.

Proof See proof of [20, Proposition 2.2].

For more details on SRB-like measures, see [15,18–20].

3.2 Measure-theoretic entropy

For any Borel measurable finite partition $P$ of $M$, and for any (not necessarily invariant) probability $\nu$, the entropy of $\nu$ with respect to $P$ is defined as $H(P, \nu) = - \sum_{P \in P} \nu(P) \log(\nu(P))$. Given another finite partition $\tilde{P}$ of $M$ the conditional entropy of $P$ with respect to $\tilde{P}$ is given by

$$H_\nu(P/\tilde{P}) = \sum_{P \in P} \sum_{Q \in \tilde{P}} -\nu(P \cap Q) \log \frac{\nu(P \cap Q)}{\nu(Q)}.$$

Lemma 3.4 Given $S \geq 1$ and $\varepsilon > 0$ there exists $\delta > 0$ such that, for any finite partitions $P = \{P_1, \ldots, P_S\}$ and $\tilde{P} = \{\tilde{P}_1, \ldots, \tilde{P}_S\}$ such that $\nu(P_i \triangle \tilde{P}_i) < \delta$ for all $i = 1, \ldots, S$, then $H_\nu(\tilde{P}/P) < \varepsilon$.

Proof See [45, Lemma 9.1.6].

Let we denote $P^q = \bigvee_{j=0}^{q-1} f^{-j}(P)$, where $P \lor Q = \{P \cap Q \neq \emptyset : P \in P, \ Q \in Q\}$ for any pair of finite partitions $P$ and $Q$. If $\nu \in M_f$, then

$$h(P, \nu) = \lim_{q \to +\infty} \frac{1}{q} H(P^q, \nu) = \inf_{q \geq 1} \frac{1}{q} H(P^q, \nu)$$

is the metric entropy of the partition $P$.

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Finally, the measure-theoretic entropy \( h_v(f) \) of an \( f \)-invariant measure \( v \) is defined by

\[
h_v(f) = \sup \{ h(P, v) \mid \text{where the supremum is taken over all the Borel measurable finite partitions } P \text{ of } M \}.
\]

We define the diameter \( \text{diam}(P) \) of a finite partition \( P \) as the maximum diameter of its atoms. The following results will be used in the proof of the large deviation lemma that is essential for the proof of Theorem B.

**Lemma 3.5** Let \( a_1, \ldots, a_\ell \) real numbers and let \( p_1, \ldots, p_\ell \) non-negative numbers such that

\[
\sum_{k=1}^{\ell} p_k = 1.
\]

Denote \( L = \sum_{k=1}^{\ell} e^{a_k} \). Then \( \sum_{k=1}^{\ell} p_k (a_k - \log p_k) \leq \log L \). Moreover, the equality holds if, and only if, \( p_k = \frac{e^{a_k}}{L} \) for all \( k \).

**Proof** See [45, Lemma 10.4.4]. \( \square \)

**Lemma 3.6** Let \( f : M \rightarrow M \) be a measurable function. For any sequence of not necessarily invariant probabilities \( v_n \), let \( \mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)_* v_n \) and \( \mu \) be a weak* accumulation point of \( (\mu_n) \). Let \( P \) be a finite partition of \( M \) with \( \mu(\partial P) = 0 = \mu_n(\partial P) \) for all \( n \geq 1 \). Then, for any \( \varepsilon > 0 \), there a subsequence of integers \( n_i \not\rightarrow \infty \) such that

\[
\frac{1}{n_i} H(P^{n_i}, v_{n_i}) \leq \frac{\varepsilon}{4} + h_{\mu}(f) \quad \forall i \geq 1.
\]

**Proof** Fix integers \( q \geq 1 \), and \( n \geq q \). Write \( n = aq + j \) where \( a, j \) are integer numbers such that \( 0 \leq j \leq q - 1 \). Fix a (not necessarily invariant) probability \( v \). From the properties of the entropy function \( H \) of \( v \) with respect to the partition \( P \), we obtain

\[
H(P^q, v) = H(P^{aq+j}, v) \leq H(P^{aq+q}, v) \leq H \left( \bigvee_{i=0}^{q-1} f^{-i} (P), v \right) + H \left( \bigvee_{i=1}^{a} f^{-iq} (P^l), v \right)
\]

\[
\leq \sum_{i=0}^{q-1} H(P, (f^i)_* v) + \sum_{i=1}^{a} H(P^q, (f^{-iq})_* v) \leq q \log S + \sum_{i=1}^{a} H(P^q, (f^{-iq})_* v),
\]

for all \( q \geq 1 \) and \( N \geq q \). To obtain the last inequality above recall that \( H(P, v) \leq \log S \) for all \( v \in M \) where \( S \) is the number of elements of the partition \( P \). The above inequality holds also for \( f^{-l}(P) \) instead of \( P \), for any \( l \geq 0 \), because it holds for any partition with exactly \( S \) atoms. Thus

\[
H(f^{-l}(P^q), v) \leq q \log S + \sum_{i=1}^{a} H(f^{-l}(P^q), (f^{-iq})_* v) = q \log S + \sum_{i=1}^{a} H(P^q, (f^{-iq})_* v).
\]

Adding the above inequalities for \( 0 \leq l \leq q - 1 \), we obtain on the one hand

\[
\sum_{l=0}^{q-1} H(f^{-l}(P^q), v) \leq q^2 \log S + \sum_{l=0}^{q-1} \sum_{i=1}^{a} H(P^q, (f^{-iq})_* v) = q^2 \log S + \sum_{i=1}^{a} H(P^q, (f^{-iq})_* v).
\]

On the other hand, for all \( 0 \leq l \leq q - 1 \),

\[
H(P^q, v) \leq H(P^{q+l}, v) \leq H(f^{-l}(P^q), v) + \sum_{i=0}^{l-1} H(f^{-i}(P), v) \leq H(f^{-l}(P^q), v) + q \log S.
\]
Therefore, adding the above inequalities for $0 \leq l \leq q - 1$ and joining with the inequality (3.2), we obtain $q H(P^q_1, v) \leq 2q^2 \log S + \sum_{l=0}^{aq+q-1} H(P^q_l, (f^l)_*v)$. Recalling that $n = aq + j$ with $0 \leq j \leq q - 1$, we have $aq + q \leq n + q$ and then

$$q H(P^n, v) \leq 2q^2 \log S + \sum_{l=0}^{n-1} H(P^q_l, (f^l)_*v) + \sum_{l=n}^{aq+q-1} H(P^q_l, (f^l)_*v)$$

$$\leq 3q^2 \log S + \sum_{l=0}^{n-1} H(P^q_l, (f^l)_*v),$$

where we used that the number of non-empty pieces of $P^q$ is at most $S^q$. Now we fix a sequence $n_i \to \infty$ such that $\mu_{n_i} \to \mu$ in the weak* topology; put $v = v_{n_i}$ and divide by $n_i$. Since $H$ is convex we obtain

$$\frac{1}{n_i} H(P^{n_i}, v_{n_i}) \leq \frac{3q^2 \log S}{n_i} + \frac{1}{n_i} \sum_{l=0}^{n_i-1} H(P^q_l, (f^l)_*v_{n_i}) + \frac{aq^2 \log S}{n_i} \leq \frac{3q^2 \log S}{n_i} + \frac{1}{n_i} \sum_{l=0}^{n_i-1} H(P^q_l, (f^l)_*v_{n_i}) + \frac{aq^2 \log S}{n_i} \leq \frac{3q^2 \log S}{n_i} + H(P^q, \mu_{n_i}).$$

Therefore $\frac{1}{n_i} H(P^{n_i}, v_{n_i}) \leq \frac{\varepsilon}{12} + \frac{1}{q} H(P^q, \mu_{n_i})$ for all $i \geq i_0(q) = \max\{q, 36q^2 \varepsilon^{-1} \log S\}$. Since $\mu_{n_i} \to \mu$ and $\mu_\mu(\partial P) = \mu(\partial P) = 0$ for all $n \in \mathbb{N}$, then $\lim_{i \to +\infty} H(P^q, \mu_{n_i}) = H(P^q, \mu)$. Thus there exists $i_1 > i_0(q)$ such that $\frac{1}{q} H(P^q, \mu_{n_i}) \leq \frac{1}{q} H(P^q, \mu) + \frac{\varepsilon}{12}$ for all $i \geq i_1$. Hence, we obtain $\frac{1}{n_i} H(P^{n_i}, v_{n_i}) \leq \frac{\varepsilon}{6} + \frac{1}{q} H(P^q, \mu)$ for all $i \geq i_1$.

Moreover, since by definition $\lim_{q \to +\infty} \frac{1}{q} H(P^q, \mu) = h(P, \mu)$, then there exists $q_0 \in \mathbb{N}$ such that $\frac{1}{q} H(P^q, \mu) \leq h(P, \mu) + \frac{\varepsilon}{12}$ for all $q \geq q_0$. Thus taking $i_2 := \max\{q_0, i_1\}$ we get

$$\frac{1}{n_i} H(P^{n_i}, v_{n_i}) \leq h(P, \mu) + \frac{\varepsilon}{4} \leq h_\mu(f) + \frac{\varepsilon}{4}$$

for all $i \geq i_2$. (3.3)

This completes the proof using the subsequence $(n_i)_{i \geq i_2}$ as the sequence claimed in the statement of the lemma. \qed

### 4 Continuous variation of conformal measures and SRB-like measures

Here we prove Theorem A showing that $v$-SRB-like measures can be seen as measures that naturally arise as accumulation points of $v_n$-SRB measures.

**Definition 12** A measurable function $J, T : X \to [0, +\infty)$ is the Jacobian of a map $T : X \to X$ with respect to a measure $v$ if for every Borel set $A \subset X$ on which $T$ is injective

$$v(T(A)) = \int_A J, T \, dv.$$  

Next result guarantees the existence of measures with prescribed Jacobian.

**Theorem 4.1** Let $T : X \to X$ be a local homeomorphism of a compact metric space $X$ and let $\phi : X \to \mathbb{R}$ be continuous. Then there exists a $\phi$-conformal probability measure $v = v_\phi$.
and a constant $\lambda > 0$, such that $\mathcal{L}_\phi^* v = \lambda v$. Moreover, the function $J_T = \lambda e^{-\phi}$ is the Jacobian for $T$ with respect to the measure $v$.

**Proof** See [30, Theorem 4.2.5].

We say that a continuous mapping $T : X \to X$ is open, if open sets have open images. This is equivalent to saying that if $f(x) = y$ and $y_n \to y$ then there exist $x_n \to x$ such that $f(x_n) = y_n$ for $n$ large enough.

**Definition 13** A continuous mapping $T : X \to X$ is

(a) **topologically exact** if for all non-empty open set $U \subset X$ there exists $N = N(U)$ such that $T^N U = X$.

(b) **topologically transitive** if for all non-empty open sets $U, V \subset X$ there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

Topological transitiveness ensures conformal measures give positive mass to any open subset.

**Proposition 4.2** Let $T : X \to X$ be an open distance expanding topologically transitive map and $\phi : X \to \mathbb{R}$ be continuous. Then every conformal measure $v = v_\phi$ is positive on non-empty open sets. Moreover for every $r > 0$ there exists $\alpha = \alpha(r) > 0$ such that for every $x \in X$, $v(B(x, r)) \geq \alpha$.

**Proof** See [30, Proposition 4.2.7]

For Hölder potentials it is known that there exists a unique $v$-SRB probability measure.

**Theorem 4.3** Let $T : X \to X$ an open distance expanding topologically transitive map, $v = v_\phi$ a conformal measure associated the a Hölder continuous function $\phi : X \to \mathbb{R}$. Then there exists a unique $\mu_\phi$ ergodic invariant $v$-SRB measure such that $\mu_\phi$ is an equilibrium state for $T$ and $\phi$.

**Proof** See [30, Chapter 4].

The following result shows that positively invariant sets with positive reference measure have mass uniformly bounded away from zero.

**Theorem 4.4** In the same setting of Theorem 4.3, if $G$ is a $T$-invariant set such that $v(G) > 0$, then there is a disk $\Delta$ of radius $\delta/4$ so that $v(\Delta \setminus G) = 0$.

**Proof** See the proof of [44, Lemma 5.3].

Now we are ready to prove Theorem A.

**Proof of Theorem A** Let $\lambda_{n_j} \to \lambda$, where $\mathcal{L}^*_{\phi_{n_j}} (v_{n_j}) = \lambda_{n_j} v_{n_j}$ (it is easy to see that $\lambda > 0$ since $\lambda_n = \mathcal{L}^*_{\phi_n} (v_n)(1)$ for all $n$). Then for each $\varphi \in C^0(X, \mathbb{R})$ we have

$$\int \varphi \, d(\mathcal{L}^*_{\phi_{n_j}} v_{n_j}) = \int \mathcal{L}^*_{\phi_{n_j}} (\varphi) \, d v_{n_j} \to^{j \to +\infty} \int \mathcal{L}^*_{\phi} (\varphi) \, d v = \int \varphi \, d(\mathcal{L}^*_{\phi} v)$$

thus, $\mathcal{L}^*_{\phi_{n_j}} v_{n_j} \to^{j \to +\infty} \mathcal{L}^*_{\phi} v$. Moreover, $\lambda_{n_j} v_{n_j} \to^{j \to +\infty} \lambda v$ and by uniqueness of the limit, it follows that

$$\lambda v = \lim_{j \to +\infty} \lambda_{n_j} v_{n_j} = \lim_{j \to +\infty} \mathcal{L}^*_{\phi_{n_j}} v_{n_j} = \mathcal{L}^*_{\phi} v.$$
Thus, $\nu$ is a $\phi$-conformal measure.

Let $\mu_{nj}$ be the $\nu_{nj}$-SRB measure and let $\mu = \lim_{j \to +\infty} \mu_{nj}$. For each fixed $\epsilon > 0$, let $N = N(\epsilon)$ be such that $\text{dist}((\mu_{nj},\mu_{nm}) < \frac{\epsilon}{4}$ and $\text{dist}(\mu_{nj},\mu) < \frac{\epsilon}{4}$ for all $j, m \geq N$. Thus, $A_{\epsilon/4}(\mu_{nj}) \subset A_{\epsilon/2}(\mu_{nm})$ and $A_{\epsilon/2}(\mu_{nj}) \subset A_{\epsilon}(\mu)$ for all $j, m \geq N$.

In fact, for each $x \in A_{\epsilon/4}(\mu_{nj})$ we have $\text{dist}(p\omega(x),\mu_{nj}) < \epsilon/4$. Then,

$$\text{dist}(p\omega(x),\mu_{nm}) \leq \text{dist}(p\omega(x),\mu_{nj}) + \text{dist}(\mu_{nj},\mu_{nm}) < \frac{\epsilon}{2}$$

for all $j, m \geq N$. Therefore, $A_{\epsilon/4}(\mu_{nj}) \subset A_{\epsilon/2}(\mu_{nm})$ for all $j, m \geq N$. Analogously, $A_{\epsilon/2}(\mu_{nj}) \subset A_{\epsilon}(\mu)$. Now, from Theorem 4.4, for each $j \geq 1$ there exists a disk $\Delta_{nj}$ of radius $\delta_{1}/4$ around $x_{nj}$ such that $v_{nj}(\Delta_{nj} \setminus A_{\epsilon/2}(\mu_{nj})) = 0$. Let $x = \lim_{j \to +\infty} x_{nj}$ taking a subsequence if necessary. By compactness, the sequence $(\Delta_{nj})_{j}$ accumulates on a disc $\tilde{\Delta}$ of radius $\delta_{1}/4$ around $x$. Let $\Delta$ be the disk of radius $0 < s \leq \delta_{1}/8$ around $x$. Thus, there exists $N_{0} \geq N$ such that $\Delta \subset \Delta_{nj}$ for all $j \geq N_{0}$.

Let $0 < s \leq \delta_{1}/8$ be such that $v\left(\partial(\Delta \setminus A_{\epsilon}(\mu))\right) = 0$ and note that for all $j \geq N_{0}$,

$$0 = v_{nj}\left(\Delta_{nj} \setminus A_{\epsilon/2}(\mu_{nj})\right) \geq v_{nj}\left(\Delta \setminus A_{\epsilon/2}(\mu_{nj})\right) \geq v_{nj}\left(\Delta \setminus A_{\epsilon}(\mu)\right),$$

since $\Delta \subset \Delta_{nj}$ and $A_{\epsilon/2}(\mu_{nj}) \subset A_{\epsilon}(\mu)$ for all $j \geq N_{0}$. Thus, by weak* convergence

$$v\left(\Delta \setminus A_{\epsilon}(\mu)\right) = \lim_{j \to +\infty} v_{nj}\left(\Delta \setminus A_{\epsilon}(\mu)\right) = 0. \quad (4.1)$$

Since $v$ is a conformal measure, it follows from (4.1) and Proposition 4.2 that $v(A_{\epsilon}(\mu)) > v(\Delta) > 0$. Because $\epsilon > 0$ is arbitrary, we conclude that $\mu$ is $\nu$-SRB-like.

Let $P$ be a finite Borel partition of $X$ with diameter not exceeding an expansive constant of $T$ such that $\mu(\partial P) = 0$. Then $P$ generates the Borel $\sigma$-algebra for every $T$-invariant Borel probability measure in $X$ (see [30, Lemma 2.5.5]). By Kolmogorov-Sinai Theorem, this implies that $\eta \mapsto h_{\eta}(T) = h_{\eta}(T, P)$ is upper semi-continuous at $\mu$.

Moreover, since $\int \phi_{nj} d\mu_{nj} \to \int \phi d\mu$, then $\mu_{nj}$ is an equilibrium state for $(T, \phi_{nj})$ (by Theorem 4.3) and by continuity of $\phi \mapsto P_{top}(T, \phi)$ (see [46, Theorem 9.7]) it follows that

$$h_{\mu}(T) + \int \phi d\mu \geq \lim_{j \to +\infty} \left( h_{\mu_{nj}}(T) + \int \phi_{nj} d\mu_{nj} \right) = \lim_{j \to +\infty} P_{top}(T, \phi_{nj}) = P_{top}(T, \phi).$$

This shows that $\mu$ is an equilibrium state for $T$ with respect to $\phi$.

Now we assume that $T$ is topologically exact. We know that given $\epsilon > 0$ there exists a disk $\Delta$ such that $v(\Delta \setminus A_{\epsilon}(\mu)) = 0$. Since $\Delta \subset X$ is a non-empty open set, there exists $N > 0$ such that $X = T^{N}(\Delta)$. Moreover, $A_{\epsilon}(\mu) = T(A_{\epsilon}(\mu))$, thus

$$0 = v(T^{N}(\Delta \setminus A_{\epsilon}(\mu)) \geq v(T^{N}(\Delta \setminus A_{\epsilon}(\mu)) = v(X \setminus A_{\epsilon}(\mu)).$$

Since $\epsilon > 0$ was arbitrary, this completes the proof of Theorem A.  \qed

## 5 Distance expanding maps

Here we state the main results needed to obtain the proof of Theorem B. We start by presenting some results about distance expanding maps that will be used throughout this section. We close the section with the proof of Theorem B.
5.1 Basic properties of distance expanding open maps

In what follows, $X$ is a compact metric space.

**Lemma 5.1** If $T : X \to X$ is a continuous open map, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $T(B(x, \varepsilon)) \supset B(T(x), \delta)$ for every $x \in X$.

**Proof** See [30, Lemma 3.1.2].

**Remark 5.2** If $T : X \to X$ is a distance expanding map, then by (1.3) and (1.4), for all $x \in X$, the restriction $T|_{B(x, \varepsilon)}$ is injective and therefore it has a local inverse map on $T(B(x, \varepsilon))$.

If additionally $T : X \to X$ is an open map, then, in view of Lemma 5.1, the domain of the inverse map contains the ball $B(T(x), \delta)$. So it makes sense to define the restriction of the inverse map, $T^{-1}_x : B(T(x), \delta) \to B(x, \varepsilon)$ at each $x \in X$.

**Lemma 5.3** Let $T : X \to X$ be an open distance expanding map. If $x \in X$ and $y, z \in B(T(x), \delta)$, then
\[ d(T^{-1}_x(y), T^{-1}_x(z)) \leq \lambda^{-1} d(y, z) \]
In particular $T^{-1}_x(B(T(x), \delta)) \subset B(x, \lambda^{-1}\delta)$ and
\[ B(T(x), \delta) \subset T(B(x, \lambda^{-1}\delta)) \quad (5.1) \]
for each small enough $\delta > 0$.

**Proof** The statements follow from Lemma 5.3.

**Definition 14** Let $T : X \to X$ be an open distance expanding map. For every $x \in X$, $n \geq 1$ and $j = 0, 1, \ldots, n - 1$, we write $x_j \equiv T^j(x)$. In view of Lemma 5.3 the composition
\[ T^{-1}_{x_0} \circ T^{-1}_{x_1} \circ \cdots \circ T^{-1}_{x_{n-1}} : B(T^n(x), \delta) \to X \]
is well-defined and will be denoted by $T^{-n}_x$.

**Lemma 5.4** Let $T : X \to X$ be an open distance expanding map. For every $x \in X$ we have:

1. $T^{-n}_x(A) = \bigcup_{y \in T^{-n}(x)} T^{-n}_y(A)$ for all $A \subset B(x, \delta)$;
2. $d(T^{-n}_x(y), T^{-n}_x(z)) \leq \lambda^{-n} d(y, z)$ for all $y, z \in B(T^n(x), \delta)$;
3. $T^{-n}_x(B(T^n(x), r)) \subset B(x, \min\{\varepsilon, \lambda^{-n} r\})$ for all $r \leq \delta$.

**Proof** The statements follow from Lemma 5.3.

For more details on the proofs in this subsection, see [30, Sect. 3.1].

5.2 Conformal measures

Here we relate $\phi$-conformal measures with the topological pressure of $\phi$. Next results ensure that $\phi$-conformal measures for expanding dynamics with continuous potentials are Gibbs measures; see [26, 30] for more details.

**Proposition 5.5** Let $T : X \to X$ an open distance expanding topologically transitive map, $\nu = \nu_\phi$ a conformal measure associated the a continuous function $\phi : X \to \mathbb{R}$ and $J_\nu T = \lambda > 1$ is the expansion rate; see (1.3) and (1.4).
\( \lambda e^{-\phi} \) the Jacobian for \( T \) with respect to the measure \( \nu \). Given \( \delta > 0 \), for all \( x \in X \) and all \( n \geq 1 \) there exists \( \alpha(\varepsilon) > 0 \) such that

\[
\alpha(\varepsilon)e^{-n\delta} \leq \frac{\nu(B(x, n, \varepsilon))}{\exp(S_n\phi(x) - Pn)} \leq e^{n\delta}
\]

for all \( \varepsilon > 0 \) small enough, where \( P = \log \lambda \).

**Proof** Given \( \delta > 0 \), there exist \( \gamma > 0 \) such that, for all \( x, y \in X \) with \( d(x, y) < \gamma \), we have \( |\phi(x) - \phi(y)| < \delta \). For any given fixed \( 0 < \varepsilon < \gamma \), \( x \in X \) and \( n \geq 1 \), we have

\[
\nu(B(T^n x, \varepsilon)) = \nu(T^n(B(x, n, \varepsilon))) = \int_{B(x, n, \varepsilon)} J_n T^n d\nu. \quad (5.2)
\]

Hence, since \( \nu \) attributes mass uniformly bounded away from zero to balls of fixed radius, we obtain using the uniform continuity of \( \phi \)

\[
\alpha(\varepsilon) \leq \nu(B(T^n x, \varepsilon)) = \int_{B(x, n, \varepsilon)} J_n T^n d\nu = \int_{B(x, n, \varepsilon)} \lambda^n e^{-S_n\phi} d\nu
\]

\[
\leq \lambda^n e^{-S_n\phi(x) + n\delta} \cdot \nu(B(x, n, \varepsilon)),
\]

and also, since \( \nu \) is a probability measure

\[
1 \geq \nu(B(T^n x, \varepsilon)) = \int_{B(x, n, \varepsilon)} J_n T^n d\nu \geq \lambda^n e^{-S_n\phi(x) - n\delta} \cdot \nu(B(x, n, \varepsilon)).
\]

Hence \( \alpha(\varepsilon)e^{-n\delta} \leq \nu(B(x, n, \varepsilon)) \exp(Pn - S_n\phi(x)) \leq e^{n\delta} \) where \( P = \log \lambda \).

**Lemma 5.6** Let \( T : X \to X \) an open distance expanding topologically transitive map, let \( \phi : X \to \mathbb{R} \) be continuous and \( \nu \) a probability measure such that \( \mathcal{L}_\phi^*(\nu) = \lambda \nu \). Then \( P_{\text{top}}(T, \phi) \leq P = \log \lambda \).

**Proof** Given \( \delta > 0 \) we take \( \varepsilon > 0 \) small enough as in Proposition 5.5 and \( n \geq 1 \). Let \( E_n \subset X \) be a maximal \( (n, \varepsilon) \) - separated set. Therefore \( \{B(x, n, \varepsilon) : x \in E_n\} \) covers \( X \), \( \{B(x, n, \varepsilon/2) : x \in E_n\} \) is a collection of pairwise disjoint open sets. Thus

\[
\sum_{x \in E_n} e^{S_n\phi(x)} \leq \sum_{x \in E_n} \nu(B(x, n, \varepsilon/2)) \frac{e^{n(P + \delta)}}{\alpha(\varepsilon/2)} \leq \frac{e^{n(P + \delta)}}{\alpha(\varepsilon/2)}
\]

by Proposition 5.5. Therefore \( P_{\text{top}}(T, \phi) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{x \in E_n} e^{S_n\phi(x)} \leq P + \delta \), and since \( \delta > 0 \) can be taken small enough, we conclude that \( P_{\text{top}}(T, \phi) - P \leq 0 \).

**5.3 Constructing an arbitrarily small initial partition with negligible boundary**

Let \( \nu \) be a Borel probability measure on the compact metric space \( X \). We say that \( T \) is \( \nu \)-regular map or \( \nu \) is regular for \( T \) if \( T_\ast \nu \ll \nu \), that is, if \( E \subset X \) is such that \( \nu(E) = 0 \), then \( \nu(T^{-1}(E)) = 0 \). It follows that \( T \) admits a Jacobian with respect to a \( T \)-regular measure.

We write \( \partial A \) for the topological boundary and \( \text{int}(A) \) for the interior of the subset \( A \) of the metric space \( X \). We also write \( \#A \) for the number of elements of the subset \( A \).

**Lemma 5.7** Let \( X \) be a compact metric space and \( \nu \) be a regular reference measure for \( T \), positive on non-empty open sets and let \( \delta > 0 \) be given.
(1) There exists a finite partition \( \mathcal{P} \) of \( X \) with \( \text{diam}(\mathcal{P}) < \delta \) such that every atom \( P \in \mathcal{P} \) has non-empty interior and \( \nu(\partial \mathcal{P}) = 0 \)

Let us choose one interior point \( w(P) \) in each \( P \in \mathcal{P} \) and form the set \( C_0 = \{w(P), P \in \mathcal{P}\} \).

(2) If \( (v_k)_{k \geq 1} \) is a sequence of probability measures, then there exists a partition \( \tilde{\mathcal{P}} \) of \( X \) satisfying

(a) \( \text{diam}(\tilde{\mathcal{P}}) < \delta \);
(b) \( w \in \text{int}(\partial \tilde{\mathcal{P}}(w)), \forall w \in C_0 \);
(c) \( \#\tilde{\mathcal{P}} = \#\mathcal{P} \);
(d) \( \nu(\partial \tilde{\mathcal{P}}) = 0 = v_k(\partial \tilde{\mathcal{P}}) \) for all \( k \geq 1 \); and
(e) \( \nu(\mathcal{P}(w) \Delta \tilde{\mathcal{P}}(w)) < \delta \) for all \( w \in C_0 \).

Moreover, if additionally \( T \) is a continuous open map, then \( \nu(\partial g(P)) = v_k(\partial g(P)) = 0 \) for each \( P \in \mathcal{P} \) and inverse branch \( g \) of \( T^n, k, n \geq 1 \).

Above we write \( \mathcal{P}(w) \) for the atom of \( \mathcal{P} \) containing \( w \) and similarly for \( \tilde{\mathcal{P}}(w) \), and also \( A \Delta B \) for the symmetrical difference \( A \setminus B \cup B \setminus A \) of \( A \) and \( B \). The partition \( \tilde{\mathcal{P}} \) can be seen as a small perturbation of the partition \( \mathcal{P} \).

**Proof** Let us fix \( \delta > 0 \), take \( 0 < \delta_1 \leq \delta \) and \( \mathcal{B} = \{B(\tilde{x}_l, \delta_1/8), l = 1, \ldots, q\} \) be a finite open cover of \( X \) by \( \delta_1/8 \)-balls such that \( \nu(\partial B(\tilde{x}_l, \delta_1/8)) = 0 \) for all \( l = 1, \ldots, q \). Such value of \( \delta_1 > 0 \) exists since the set of values of \( \delta_1 \) satisfying \( \nu(\partial B(\tilde{x}_l, \delta_1/8)) > 0 \) for some \( l \in \{1, \ldots, q\} \) is denumerable, because \( \nu \) is a finite measure and \( X \) is separable.

From this we define a finite partition \( \mathcal{P} \) of \( X \) as follows. We start by setting

\[
P_1 := B(\tilde{x}_1, \delta_1/8), \quad P_2 := B(\tilde{x}_2, \delta_1/8) \setminus P_1, \ldots, \quad P_k := B(\tilde{x}_k, \delta_1/8) \setminus (P_1 \cup P_2 \cup \ldots \cup P_{k-1})
\]

\[(5.3)\]

for all \( k = 2, \ldots, S \) where \( S \leq q \). We note that, if \( P_k \neq \emptyset \), then \( P_k \) has non-empty interior (since \( X \) is separable); diameter smaller than \( 2\delta_1/8 < \delta_1/4 \), and the boundary \( \partial P_k \) is a (finite) union of pieces of boundaries of balls with zero \( \nu \)-measure. We define \( \mathcal{P} \) to be the collection of elements \( P_k \) constructed above which are non-empty. This completes the proof of the first item.

Now for item (2), let \( v_k \) be an enumerable family of probability measures on \( X \), and set \( d_0 = \min\{d(w, \partial \mathcal{P}), w \in C_0\} > 0 \). Then we take \( 0 < \eta < \min\{d_0/2, \delta_1/8\} \) (note that \( d_0 \) does not depend on \( k \)) such that

\[
\nu(\partial B(\tilde{x}_j, \eta + \delta_1/8)) = v_k(\partial B(\tilde{x}_j, \eta + \delta_1/8)) \quad \text{for} \quad j = 1, \ldots, q; \quad k \geq 1,
\]

(5.4)

where the centers are the ones from the previous construction.

Again we note that such value of \( \eta \) exists since the set of values of \( \eta > 0 \) such that some of the expressions in (5.4) is positive for some \( j, k \) is at most enumerable, since the measures involved are probability measures. Thus we may take \( \eta > 0 \) satisfying (5.4) arbitrarily close to zero.

We consider now the finite open cover of \( X \) given by \( \tilde{\mathcal{B}} = \{B(\tilde{x}_j, \eta + \delta_1/8) : j = 1, \ldots, q\} \) and construct the partition \( \tilde{\mathcal{P}} \) induced by \( \tilde{\mathcal{B}} \) following the same procedure as before and the same order of construction of the atoms of \( \mathcal{P} \), as in (5.3).

We note that \( d(w, \partial B(\tilde{x}_j, \eta + \delta_1/8)) > d_0 - \eta > d_0/2 \) for all \( w \in C_0 \) and \( j = 1, \ldots, q \) by construction. Therefore, each \( w \in C_0 \) is contained in some atom \( P_w \in \tilde{\mathcal{P}} \). Moreover, there cannot be distinct \( w, w' \in C_0 \) such that \( w' \in P_w \) by the choice of \( \eta \). Thus, the number \( \#\mathcal{P} \) of atoms of \( \mathcal{P} \) is less than or equal to the number \( \#\tilde{\mathcal{P}} \) of atoms of \( \tilde{\mathcal{P}} \). On the other hand,
by construction $\#\mathcal{P} \leq S = \#\mathcal{P}$, because this is the number of elements $\#\mathcal{B} = S$ of the open cover $\mathcal{B}$. In this way we conclude that the partition $\mathcal{P}$ has the same number of atoms as $\mathcal{P}$, that is, we have proved item (c).

Moreover, it is now easy to see by the choice of $\eta$ that $\tilde{P}$ satisfies also (a), (b) and (d). In addition, every element of $\mathcal{B}$ is in a $\eta$-neighborhood of the corresponding element of $\mathcal{B}$. Since we can take $\eta > 0$ smaller without losing any of the previous conclusions, we can also obtain subitem (e). This completes the proof of item (2).

Finally, if $T$ is open and continuous, then inverse branches of $T$ are well defined by Lemma 5.1. Since $T$ is $\nu$-regular and $\partial g(P) = g(\partial P)$, then boundary of $g(P)$ still has zero $\nu$- and $v_k$-measure for every atom $P \in \mathcal{P}$ and every inverse branch $g$ of $T^n$, for each $k, n \geq 1$.

The proof of Lemma 5.7 is complete.

5.4 Expansive maps and existence of equilibrium states

A continuous transformation $T : X \to X$ of a compact metric space $X$ equipped with a metric $\rho$ is said to be (positively) expansive if and only if

$$\exists \delta > 0 \text{ such that } [\rho(T^n(x), T^n(y)) \leq \delta, \forall n \geq 0] \implies x = y$$

and the number $\delta$ above is an expansive constant for $T$.

**Theorem 5.8** If $T : X \to X$ is positively expansive, then the function $\mathcal{M}_T \ni \mu \mapsto h_\mu(T)$ is upper semi-continuous and each continuous potential $\phi : X \to \mathbb{R}$ has an equilibrium state.

**Proof** See [30, Theorem 2.5.6].

**Theorem 5.9** Every distance expanding map is positively expansive.

**Proof** See [30, Theorem 3.1.1].

**Corollary 5.10** If $T : X \to X$ is distance expanding, then each continuous potential $\phi : X \to \mathbb{R}$ has an equilibrium state. In particular, $\mathcal{K}_r(\phi) = \{\mu \in \mathcal{M}_T : h_\mu(T) + \int \phi d\mu \geq P_{top}(T, \phi) - r\} \neq \emptyset$ for all $r \geq 0$.

**Proof** The proof is immediate from Theorem 5.9 and Theorem 5.8.

5.5 Large deviations

The statement of Theorem B is a consequence of the following more abstract result, inspired in [20, Lemma 4.3] and in [26, Proposition 6.1.11].

**Proposition 5.11** Let $T : X \to X$ be an open distance expanding topologically transitive map and let $\phi : X \to \mathbb{R}$ be continuous. Fix $v = v_\phi$ a conformal measure, $r > 0$ and consider the weak* distance defined in (3.1). Then, for all $0 < \varepsilon < r$, there exists $n_0 \geq 1$ and $\kappa > 0$ such that

$$v\left(\left\{x \in X; \text{dist}(\sigma_n(x), \mathcal{K}_r(\phi)) \geq \varepsilon\right\}\right) < \kappa \exp[n(\varepsilon - r)], \forall n \geq n_0. \quad (5.5)$$

**Proof** We know by Theorem 4.1 and Proposition 4.2 that all the (necessarily existing) conformal measures $v$ are positive on non-empty open sets and $J_v T = \lambda e^{-\phi}$ is the Jacobian for $T$ with respect to the measure $v$. 

\[ \text{Springer} \]
Let \( v \) be a conformal measure, fix \( r > 0 \) and let \( 0 < \varepsilon < r \). For \( \varepsilon / 6 \), fix a constant \( \gamma > 0 \) of uniform continuity of \( \phi \), i.e., \( |\phi(x) - \phi(y)| < \varepsilon/6 \) whenever \( d(x, y) < \gamma \). Let us fix \( 0 < \xi < \gamma \) and a partition \( \mathcal{P} \) of \( X \) as in Lemma 5.7 such that \( \text{diam}(\mathcal{P}) < \xi/3 \) and choose one interior point in each atom of \( \mathcal{P} \) to construct \( C_0 = \{ w_1, \ldots, w_S \} \) where \( S = \# \mathcal{P} \).

Define \( \mathcal{A} := \{ \mu \in \mathcal{M}_T; d(\mu, \mathcal{K}_r(\phi)) \geq \varepsilon \} \) and note that \( \mathcal{A} \) is weak* compact, so it has a finite covering \( B_1, \ldots, B_\kappa \) for minimal cardinality \( \kappa \geq 1 \), with open balls \( B_i \subset \mathcal{M} \) of radius \( \xi/\kappa \). For any fixed \( n \geq 1 \) write \( C_{n,i} = \{ x \in X : \sigma_n(x) \in B_i \} \), \( C_n = \bigcup_{i=1}^\kappa C_{n,i} \), \( \tilde{C}_{n,i} = \{ x \in X : \sigma_n(x) \in \tilde{B}_i \} \) and \( \tilde{C}_n = \bigcup_{i=1}^\kappa \tilde{C}_{n,i} \), where \( \tilde{B}_i \) are open balls concentric with \( B_i \) of radius \( 2\xi/\kappa \) for \( i = 1, \ldots, \kappa \).

We note that \( C_{n,i} \subset \tilde{C}_{n,i} \) and, moreover, \( \{ x \in X : d(\sigma_n(x), \mathcal{K}_r(\phi)) \geq \varepsilon \} \subset C_n \subset \tilde{C}_n \).

**Lemma 5.12** For each \( 1 \leq i \leq \kappa \) there exists \( n_i > 0 \) such that \( v(C_{n,i}) \leq \exp \left[ n(\varepsilon - r) \right] \) for all \( n \geq n_i \).

First, let us see that it is enough to prove Lemma 5.12 to finish the proof of Proposition 5.11, and then we prove Lemma 5.12 in the following Sect. 5.5.1. In fact, if Lemma 5.12 holds, then put \( n_0 = \max_{1 \leq i \leq \kappa} n_i \) and we get

\[
v(C_n) \leq \sum_{i=1}^\kappa v(C_{n,i}) \leq \kappa \exp \left[ n(\varepsilon - r) \right]
\]

for all \( n \geq n_0 \), as needed to conclude the proof of the statement of Proposition 5.11. \( \square \)

### 5.5.1 Exponential upper bound

Now we present the proof of Lemma 5.12.

**Proof of Lemma 5.12** For \( x \in C_{n,i} \) let \( P \in \mathcal{P} \) be the atom such that \( T^n(x) \in P \) and set \( Q = T^{-n}_n(P) \). Then the family \( Q_n \) of all such sets \( Q \) is finite since both \( \mathcal{P} \) and the number of inverse branches are finite. Moreover, by the expression of \( J_rT \) in terms of \( \phi \)

\[
v(Q \cap C_{n,i}) = \int_{T^n(Q \cap C_{n,i})} J_rT^{-n}_r d\nu = \int_{T^n(Q \cap C_{n,i})} \exp \left[ \sum_{j=0}^{n-1} \phi \circ T^j - n \log \lambda \right] \circ T^{-n}_r d\nu.
\]

We observe that if \( v(C_{n,i}) = 0 \), then Lemma 5.12 becomes trivially proved. Consider the finite family of atoms \( \{ Q_1, \ldots, Q_N \} = \{ Q \in Q_n : v(Q \cap C_{n,i}) > 0 \} \) which has \( N = N(n, i) \) elements for some \( N \geq 1 \).

We note that \( v(C_{n,i}) = \sum_{k=1}^N v(Q_k \cap C_{n,i}) \). For each \( k = 1, \ldots, N \), let us take \( x_k \in Q_k \) such that \( T^n(x_k) = w_j \) for some \( j = 1, \ldots, S \). We recall that \( w_j \) are interior points of each atom of the partition \( \mathcal{P} \), so there is only one \( j = j_{k,n} \) for each \( x_k \in Q_k \) such that \( T^n(x_k) = w_j \).

Since \( \text{diam}(\mathcal{P}_n) < \text{diam}(\mathcal{P}) < \xi/3 < \xi \) for all \( n > 0 \) (recall Lemma 5.4), then \( |\phi(T^j(y))| < \xi/6 \) for all \( y \in Q_k \) and \( j = 0, \ldots, n - 1 \). Choosing \( y_k \in Q_k \cap C_{n,i} \) for each \( k \), then we get

\[
v(C_{n,i}) = \sum_{k=1}^N v(Q_k \cap C_{n,i}) \leq \sum_{k=1}^N \int_{T^n(Q_k \cap C_{n,i})} \exp \left[ \sum_{j=0}^{n-1} \left( \phi(T^j(y_k)) + \xi/6 \right) - n \log \lambda \right] d\nu.
\]
For the partition $\tilde{P}$, where $\{\tilde{Q}_k\}_{k=1}^N$ and $Q_k \cap C_{n,i} \neq \emptyset$ for $k = 1, \ldots, N(n,i)$.

Let us choose $y \in \tilde{Q}_k \cap C_{n,i}$. Then $\sigma_n(y) \in B_i$. Since $d(T^j(x_k), T^j(y)) \leq \text{diam}(\tilde{P}) < \xi$ for all $j = 0, \ldots, n-1$, then $d(\sigma_n(x_k), \sigma_n(y)) < \epsilon/3$ by Lemma 3.1. Moreover, as $B_i \subset \tilde{B}_i$ concentrically, we obtain $\sigma_{n}(x_k) \in \tilde{B}_i$.

In addition, since the ball $B_i$ is convex and $\mu_n$ is a convex combination of the measures $\sigma_n(x_k)$ (recall that $\sum \lambda_k = 1$), we deduce that $\mu_n \in B_i$.

Therefore, the weak* limit $\mu$ of any convergent subsequence of $(\mu_n)_n$ belongs to the weak* closure $\overline{B_i}$. Since the ball $\tilde{B}_i$ has radius $\frac{2\epsilon}{3}$ we get that $\mu \in M_T \setminus K_c(\phi)$. Then $\int \phi d\mu + h_\mu(T) < P_{\text{top}}(T, \phi) - r$, and hence

$$v(C_{n,i}) \leq \exp\left[n\left(\frac{\epsilon}{3} - \log \lambda_1\right)\right] \cdot L = \exp\left[n\left(\frac{\epsilon}{3} - \log \lambda\right) + \log L\right]$$

$$= \exp\left[n\left(\frac{\epsilon}{3} - \log \lambda + \int \phi d\mu_n + \frac{1}{n} H(\mathcal{P}_n, \nu_n)\right)\right].$$

Because $\int \phi d\mu_n \xrightarrow{j \to \infty} \int \phi d\mu$ (by weak* convergence), there exists $j_1 > 0$ such that $\int \phi d\mu_n \leq \int \phi d\mu + \epsilon/3$ for all $j > j_1$. By (5.7) there exists $j_0 > 0$ such that
\( \frac{1}{n_j} H(\mathcal{P}^n_{r,j}, \nu_{n_j}) \leq h_\mu(T) + \varepsilon/3, \forall j \geq j_0. \) Taking \( j_2 = \max\{j_1, j_0\} \) we have for all \( j > j_2 \)

\[

v(C_{n,j}) \leq \exp\left[ n_j \left( \frac{\varepsilon}{3} - \log\lambda + \int \phi d\mu + \frac{\varepsilon}{3} + h_\mu(T) + \frac{\varepsilon}{3} \right) \right] \\
= \exp\left[ n_j (\varepsilon - \log\lambda + \int \phi d\mu + h_\mu(T)) \right] \\
\leq \exp\left[ n_j (\varepsilon - r - \log\lambda + P_{\text{top}}(T, \phi)) \right] \leq \exp\left[ n_j (\varepsilon - r) \right],

\]

where the last inequality follows from Lemma 5.6.

Finally, by the choice of \( n_j \) satisfying (5.6), we conclude that there exist \( n_0 > 0 \) such that \( v(C_{n,j}) \leq \exp[n(\varepsilon - r)] \) for all \( n \geq n_0 \). This completes the proof of Lemma 5.12. \( \square \)

### 5.6 Proof of Theorem B

Given \( r > 0 \), consider the (non-empty) set \( \mathcal{K}_r(\phi) \subset M_T \). By the upper semicontinuity of the metric entropy (see Theorem 5.8), we have that \( \mathcal{K}_r(\phi) \) is closed, hence, weak* compact. Since \( \{\mathcal{K}_r(\phi)\}_r \) is decreasing with \( r \), we have \( \mathcal{K}_0(\phi) = \bigcap_{r>0} \mathcal{K}_r(\phi) \).

By the Variational Principle \( h_\mu(T) + \int \phi d\mu \leq P_{\top}(T, \phi) \) for all \( \mu \in M_T \). So, to prove Theorem B, we must prove that the set \( W_T(v) \) of \( v \)-SRB-like measures satisfy \( W_T(v) \subset \mathcal{K}_r(\phi) \) for all \( r > 0 \), because \( \mathcal{K}_0(\phi) = \{ \mu \in M_T : h_\mu(T) + \int \phi d\mu = P_{\top}(T, \phi) \} \). Since \( \mathcal{K}_r(\phi) \) is weak* compact, it is enough to prove the following

**Lemma 5.13** The basin of attraction of \( \mathcal{K}_r^c(\phi) \)

\[ W^s(\mathcal{K}_r^c(\phi)) := \{ x \in X : p\omega(x) \subset \mathcal{K}_r^c(\phi) \}. \]

has full \( v \)-measure: \( v(X \setminus W^s(\mathcal{K}_r^c(\phi))) = 0 \).

**Proof** By Lemma 5.11, the subset \( X_n = X_n(\varepsilon, r) = \{ x \in X : \sigma_n(x) \in \mathcal{M} \setminus \mathcal{K}_r^c(\phi) \} \)

satisfies \( v(X_n) \leq \kappa e^{n(\varepsilon - r)} \) for some \( \kappa > 0 \) any \( n > n_0 \), where \( n_0 = n_0(\varepsilon) \geq 1, \) since \( 0 < \varepsilon < r \). This implies that \( \sum_{n=1}^{+\infty} v(X_n) < \infty \). By the Borel-Cantelli Lemma, it follows that

\[

v\left( \bigcap_{n=1}^{\infty} \bigcup_{n=n_0}^{\infty} X_n \right) = 0.
\]

In other words, for \( v \)-a.e. \( x \in X \) there exists \( n_0 \geq 1 \) such that \( \sigma_n(x) \in \mathcal{K}_r^c(\phi) \) for all \( n \geq n_0 \). Hence, \( p\omega(x) \subset \mathcal{K}_r^c(\phi) \) for \( v \)-almost all the points \( x \in X \), as required. \( \square \)

The proof of Theorem B is complete.

**Remark 5.14** If the set \( \mathcal{K}_r(\phi) \) is not closed, we may substitute \( \overline{\mathcal{K}_r(\phi)} \) for \( \mathcal{K}_r(\phi) \) in the proof of Theorem B, and by the same argument we conclude that \( W_T(v) \subset \overline{\mathcal{K}_r(\phi)} \). Thus, in a more general context, where Proposition 5.11 is valid and \( \mathcal{K}_{1/n}(\phi) \neq \emptyset \) for all \( n \geq 1 \), we can say that \( v \)-SRB-like measures are “almost \( \phi \)-equilibrium states”. Indeed, given \( \mu \in W_f(v) \), then \( \mu = \lim_{n \to +\infty} h_n, \mu_n \in \mathcal{K}_{1/n}(\phi) \) for all \( n \geq 1 \). Therefore, we can find a sequence of \( T \)-invariant probability measures so that \( h_{\mu_n}(T) + \int \phi d\mu_n \geq P(T, \phi) - \frac{1}{n} \) for all \( n \geq 1 \) and \( \mu_n \to \mu \) in the weak* topology.
To obtain a \( \phi \) equilibrium state in the limit we need only assume that \( \phi \) is uniformly approximated by continuous potentials, as follows.

**Corollary 5.15** Let \( T : X \to X \) be an open distance expanding topologically transitive map of a compact metric space \( X \), \((\phi_n)_{n \geq 1}\) a sequence of continuous potentials, \((\nu_n)_{n \geq 1}\) a sequence of conformal measures associated to the \((T, \phi_n)\) and \(\mu_n\) a sequence of \(v_n\)-SRB-like measures. Assume that

1. \( \phi_{n_j} \xrightarrow{j \to +\infty} \phi \) in the topology of uniform convergence;
2. \( \mu_{n_j} \xrightarrow{w^*} \mu \) in the weak* topology.

Then \( \mu \) is an equilibrium state for the potential \( \phi \).

**Proof** Let \( \mu_{n_j} \) be a \( v_{n_j} \)-SRB-like measure and let \( \mu = \lim_{j \to +\infty} \mu_{n_j} \). Since any finite Borel partition \( \mathcal{P} \) of \( X \) with diameter not exceeding an expansive constant and satisfying \( \mu(\partial \mathcal{P}) = 0 \) generates the Borel \( \sigma \)-algebra for every Borel \( T \)-invariant probability measure in \( X \) (see [30, Lemma 2.5.5]), then Kolmogorov-Sinai Theorem implies that \( \eta \mapsto h_\eta(T) = h_\eta(T, \mathcal{P}) \) is upper semi-continuous.

Moreover, since \( \int \phi_{n_j} \, d\mu_{n_j} \to \int \phi \, d\mu \), \( \mu_{n_j} \) is an equilibrium state for \((T, \phi_{n_j})\) (by Theorem B) and by continuity of \( \varphi \mapsto P_{top}(T, \varphi) \) (see [46, Theorem 9.7]) it follows that

\[
h_{\mu}(T) + \int \phi \, d\mu \geq \lim_{j \to +\infty} \left( h_{\mu_{n_j}}(T) + \int \phi_{n_j} \, d\mu_{n_j} \right) = \lim_{j \to +\infty} P_{top}(T, \phi_{n_j}) = P_{top}(T, \phi).
\]

This shows that \( \mu \) is an equilibrium state for \( T \) with respect to \( \phi \). \( \square \)

### 6 Entropy formula

Here we state the main results needed to obtain the proof of Theorem D. Then we prove Theorem D in the last subsection.

#### 6.1 Hyperbolic times

The main technical tool used in the study of non-uniformly expanding maps is the notion of hyperbolic times, introduced in [2]. We now outline some the properties of hyperbolic times.

**Definition 15** Given \( \sigma \in (0, 1) \), we say that \( h \) is a \( \sigma \)-hyperbolic time for a point \( x \in M \) if for all \( 1 \leq k \leq h \),

\[
\prod_{j=h-k}^{h-1} \| Df(f^j(x))^{-1} \| \leq \sigma^k
\]  

(6.1)

**Remark 6.1** Throughout this section we cite results originally proved under the assumption that \( f \) is of class \( C^2 \), or \( f \in C^{1+\alpha}(M, M) \) for some \( 0 < \alpha < 1 \). But the cited results were proved without using the bounded distortion property, and therefore the proofs are easily adapted to our setting. Indeed, most of proofs are the same.
Proposition 6.2 Given $0 < \sigma < 1$, there exists $\delta_1 > 0$ such that, whenever $h$ is a $\sigma$-hyperbolic time for a point $x$, the dynamical ball $B(x, h, \delta_1)$ is mapped diffeomorphically by $f^h$ onto the ball $B(f^h(x), \delta_1)$, with
\[d(f^{h-k}(y), f^{h-k}(z)) \leq \sigma^{k/2} \cdot d(f^h(y), f^h(z))\]
for every $1 \leq k \leq h$ and $y, z \in B(x, h, \delta_1)$.

**Proof** See Lemma 5.2 in [4] \[\square\]

Remark 6.3 For an open distance expanding and topologically transitive map $T$ of a compact metric space $X$, every time $h \geq 1$ satisfies (6.2) for every $x \in X$.

Definition 16 We say that the frequency of $\sigma$-hyperbolic times for $x \in M$ is positive, if there is some $\theta > 0$ such that all sufficiently large $n \in \mathbb{N}$ there are $l \geq \theta n$ and integers $1 \leq h_1 < h_2 < \cdots < h_l \leq n$ which are $\sigma$-hyperbolic times for $x$.

The following Theorem ensures existence of infinitely many hyperbolic times Lebesgue almost every point for non-uniformly expanding maps. A complete proof can be found in [4, Section 5].

**Theorem 6.4** Let $f : M \to M$ be a $C^1$ non-uniformly expanding local diffeomorphism. Then there are $\sigma \in (0, 1)$ and there exists $\theta = \theta(\sigma) > 0$ such that $\text{Leb}$-a.e. $x \in M$ has infinitely many $\sigma$-hyperbolic times. Moreover, if we write $0 < h_1 < h_2 < h_3 < \ldots$ for the hyperbolic times of $x$, then their asymptotic frequency satisfies $\liminf_{N \to \infty} \frac{\#\{k \geq 1 : h_k \leq N\}}{N} \geq \theta$, for $\text{Leb}$-a.e. $x \in M$.

The Lemma below shows that we can translate the density of hyperbolic times into the Lebesgue measure of the set of points which have a specific (large) hyperbolic time.

**Lemma 6.5** Let $B \subset M$, $\theta > 0$ and $g : M \to M$ be a local diffeomorphisms such that $g$ has density $> 20 \theta$ of hyperbolic times for every $x \in B$. Then, given any probability measure $\nu$ on $B$ and any $n \geq 1$, there exists $h > n$ such that $\nu(\{x \in B : h is a hyperbolic time of $g$ for $x\}) > \theta/2$.

**Proof** See [6, Lemma 3.3] \[\square\]

The next result is the flexible covering lemma with hyperbolic preballs which will enable us to approximate the Lebesgue measure of a given set through the measure of families of hyperbolic preballs.

**Lemma 6.6** Let a measurable set $A \subset M$, $n \geq 1$ and $\varepsilon > 0$ be given with $\text{Leb}(A) > 0$. Let $\theta > 0$ be a lower bound for the density of hyperbolic times for Lebesgue almost every point. Then there are integers $n \leq h_1 < \cdots < h_k$ for $k = k(\varepsilon) \geq 1$ and families $E_i$ of subsets of $M$, $i = 1, \ldots, k$ such that

1. $E_1 \cup \cdots \cup E_k$ is a finite pairwise disjoint family of subsets of $M$;
2. $h_i$ is a $\sigma_{2i}$-hyperbolic time for every point in $Q$, for every element $Q \in E_i$, $i = 1, \ldots, k$;
3. every $Q \in E_i$ is the preimage of some element $P \in \mathcal{P}$ under an inverse branch of $f^{h_i}$, $i = 1, \ldots, k$;
4. there is an open set $U_1 \supset A$ containing the elements of $E_1 \cup \cdots \cup E_k$ with $\text{Leb}(U_1 \setminus A) < \varepsilon$;
5. $\text{Leb}(A \triangle \cup_i E_i) < \varepsilon$. 

\[\square\]
**Fig. 4** A sketch of $E_j$, the family of all sets $f^{-h_j}(P)$ which intersect $A_{ε,n_l}(μ)$ in points for which $h_j$ is a hyperbolic time, where $P ∈ P$. Analogously for $E_{j+1}$.

**Proof** See [6, Lemma 3.5].

**Remark 6.7** The statement of Lemma 6.6 remains valid, with the same proof, replacing $f$ by an open distance expanding and topologically transitive map $T$ of a compact metric space $X$; and Leb by a $φ$-conformal measure $ν$ for a continuous potential $φ : M → \mathbb{R}$; recall Remark 6.3.

We use this covering lemma to prove the following. Fix a reference probability measure $ν$ for the space $M$. We recall that a $f$-invariant probability measure $μ$ is $ν$-weak-SRB-like if

$$\limsup_{n → +∞} \frac{1}{n} \log ν(A_{ε,n}(μ)) = 0, \ ∀ ε > 0,$$

where $A_{ε,n}(μ)$ was defined at (1.6). We denote by $W^*_μ(ν)$ the set of $ν$-weak-SRB-like probability measures. When $ν = \text{Leb}$ we denote $W^*_μ(\text{Leb})$ by $W^*_f$.

**Proposition 6.8** Let $f : M → M$ be a non-uniformly expanding map. For each $μ ∈ W^*_f$ we have $h(μ) + \int ψ dμ ≥ 0^2$.

**Proof** Given $μ ∈ W^*_f$, let $δ_1 > 0$ be as in Proposition 6.2 and fix $τ > 0$. Since $f : M → M$ is a $C^1$ local diffeomorphism, then $f$ is a regular map. Let us take a partition $P$ of $M$ as given in Lemma 5.7 with $ν = μ$, so that $\text{diam}(P) < \xi < \frac{δ_1}{2}$ where $\xi$ is a hyperbolic time of $f$ (here $P = \{ w_1, \ldots, w_S \}$). Let $S = \#P$ (Fig. 4).

We now use Lemma 6.6 to obtain a covering of $A_{ε,n_l}(μ)$ by hyperbolic preballs. We take positive integers $l, m$ and $β_l = \frac{1}{n_l} \text{Leb}(A_{ε,n_l}(μ)) > 0$ such that $σ^2 δ_l / 4 < γ$. Then there are integers $n_l < n_l + m ≤ h_1 < h_2 < \ldots < h_k$ such that $k = k(l) ≥ 1$ (here $β_l$ takes the place of $ε$ in Lemma 6.6) and families $E_j$ of subsets of $M$, $j = 1, \ldots, k$ so that

$$\text{Leb}(A_{ε,n_l}(μ)) = \sum_{j=1}^{k} \text{Leb}(A_{ε,n_l}(μ) \cap E_j) + \sum_{j=1}^{k} \text{Leb}(A_{ε,n_l}(μ) \setminus E_j)$$

Recall that $ψ = −\log Jf = −\log |\det Df|$. 

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\[ \sum_{j=1}^{h_k} \sum_{Q \in \mathcal{E}_j} \text{Leb}(A_{\varepsilon,n_1}(\mu) \cap Q) + \beta_i, \text{ hence} \]

\[ \text{Leb}(A_{\varepsilon,n_1}(\mu)) \leq \frac{n_l}{n_l - 1} \sum_{j=1}^{h_k} \sum_{Q \in \mathcal{E}_j} \text{Leb}(A_{\varepsilon,n_1}(\mu) \cap Q), \tag{6.4} \]

where \( \mathcal{E}_j = \mathcal{E}_{h_j} \) and \( A_{\varepsilon,n_1}(\mu) \cap \mathcal{E}_j = \bigcup_{Q \in \mathcal{E}_j} A_{\varepsilon,n_1}(\mu) \cap Q \). Since \( \text{diam}(\mathcal{P}) < \frac{\delta_i}{4} \), by Lemma 6.6, \( f^{h_j}|_Q : Q \to f^{h_j}(Q) \) is a diffeomorphism for all \( Q \in \mathcal{E}_j \) and \( 1 \leq j \leq k \), where \( h_j \) is a \( \frac{\varepsilon}{2} \)-hyperbolic time for every point in \( Q \). We recall that \( Q \in \mathcal{E}_j \) is the preimage of some element \( P \in \mathcal{P} \) under an inverse branch \( \phi \) of \( f^{h_j}, j = 1, \ldots, k \). Then

\[ \text{Leb}(Q \cap A_{\varepsilon,n_1}(\mu)) = \int_{f^{h_j}(Q \cap A_{\varepsilon,n_1}(\mu))} |\det Df^{h_j}| \, d\text{Leb} = \int_{f^{h_j}(Q \cap A_{\varepsilon,n_1}(\mu))} e^{Sh_j} \circ \phi \, d\text{Leb}. \]

Note that \( S_{h_j} \psi(y) = S_{h_j} \psi(y) + S_{h_j-n_j} \psi(f^{n_j}(y)) \) and, since \( h_j \geq n_l + m \) is a hyperbolic time for all \( y \in \mathcal{E}_j \), we have \( S_{h_j-n_j} \psi(f^{n_j}(y)) = -\sum_{i=0}^{h_j-n_j-1} \log |\det Df(f^{i+n_j}(y))| \leq 0 \) and so \( S_{h_j} \psi(y) \leq S_{n_j} \psi(y) \) for all \( y \in \mathcal{E}_j \). Therefore,

\[ \text{Leb}(Q \cap A_{\varepsilon,n_1}(\mu)) \leq \int_{f^{h_j}(Q \cap A_{\varepsilon,n_1}(\mu))} e^{S_{n_j} \psi} \circ \phi \, d\text{Leb}, \quad \forall Q \in \mathcal{E}_j, \forall j = 1, \ldots, k. \]

Let us take \( y_Q \in Q \cap A_{\varepsilon,n_1}(\mu) \) for each \( Q \in \mathcal{E}_j \) such that \( \text{Leb}(Q \cap A_{\varepsilon,n_1}(\mu)) > 0 \), and let \( x_Q \in Q \) be such that \( f^{h_j}(x_Q) \in W_0 \) (recall that elements of \( W_0 \) are interior points of each atom of the partition \( \mathcal{P} \)). We write \( W_l \) for the set of all points \( x_Q \) for all \( Q \in \mathcal{E}_j \) such that \( \text{Leb}(Q \cap A_{\varepsilon,n_1}(\mu)) > 0 \) for all \( j = 1, \ldots, k \).

For each atom \( P \in \mathcal{P}^\mu \) with \( P \cap W_l \neq \emptyset \) we choose only one point of \( P \cap W_l \) to form the subset \( \hat{W}_l \). We note that for any pair \( x_Q, x'_Q \in P \cap W_l \) we have \( d(f^{i}(x_Q), f^{i}(x'_Q)) < \varepsilon, i = 0, \ldots, n_l - 1 \) so \( Q \cup Q' \subset B(x_Q, n_l, \varepsilon + \varepsilon) \) and so from (6.4) we obtain

\[ \text{Leb}(A_{\varepsilon,n_1}(\mu)) \leq \frac{n_l}{n_l - 1} \sum_{x \in \hat{W}_l} \text{Leb}(A_{\varepsilon,n_1}(\mu) \cap B(x, n_l, 2\varepsilon)) \]

and also by the choice of the initial partition and of \( \varepsilon \)

\[ \text{Leb}(B(x, n_l, 2\varepsilon)) \leq \int_{B(f^{n_l}(x), 2\varepsilon)} |\det Df^{n_l}| \, d\text{Leb} \leq e^{S_{n_l} \psi(x) + \tau/2} \text{Leb}(B(f^{n_l}(x), 2\varepsilon)). \]

So we can write \( \text{Leb}(A_{\varepsilon,n_1}(\mu)) \leq \frac{n_l}{n_l - 1} e^{\tau/2} \sum_{x \in \hat{W}_l} e^{S_{n_l} \psi(x)} \). Setting \( L(n_l) := \sum_{x \in \hat{W}_l} e^{S_{n_l} \psi(x)} \) we can rewrite

\[ \text{Leb}(A_{\varepsilon,n_1}(\mu)) \leq \frac{n_l}{n_l - 1} \exp \left[ n_l \left( \frac{\tau}{2} + \frac{1}{n_l} \log L(n_l) \right) \right]. \tag{6.5} \]

Note that since \( \sum_{x \in W_l} \lambda_x = 1 \), then \( \log L(n_l) = \sum_{x \in W_l} \lambda_x \left( S_{n_l} \psi(x) - \log \lambda_x \right) \) by Lemma 3.5.

Defining the probability measures \( \nu_{n_l} := \sum_{x \in W_l} \lambda_x \delta_x \) and \( \mu_{n_l} := \frac{1}{n_l} \sum_{i=0}^{n_l-1} (f^i)_*(\nu_{n_l}) = \sum_{x \in W_l} \lambda_x \sigma_{n_l}(x) \), we rewrite again

\[ \log L(n_l) = n_l \int \psi \, d\mu_{n_l} + H(\mathcal{P}^\mu, \nu_{n_l}). \tag{6.6} \]

\[ ^3 \text{Since the atoms of } \mathcal{P} \text{ contain at most one element from } \hat{W}_l. \]
Now we take a subsequence \( n_i \to \infty \) such that \( \mu_{n_i} \to \tilde{\mu} \) in the weak\(^*\) topology and also

\[
\lim_{i \to +\infty} \frac{1}{n_i} \log \text{Leb}(A_{E,n_i}(\mu)) = \lim_{n \to +\infty} \frac{1}{n} \log \text{Leb}(A_{E,n}(\mu)). \tag{6.7}
\]

We keep the notation \( n_i \) for simplicity in what follows.

From Proposition 6.2, we know that

\[
\max\{\text{diam}(f^l(Q)); \; Q \in E_j, \; l = 0, \ldots, n_l\} < \sigma h_j(n_l) \delta_1/4 < \sigma^m \delta_1/4 < \gamma \leq \xi,
\]

for all \( j = 1, \ldots, k \). By uniform continuity of \( \psi \) we get \(|\psi(f^l(x_Q)) - \psi(f^l(y))| < \delta_2/3\) for all \( y \in Q \) and \( l = 0, \ldots, n_l - 1 \). We observe that, for each \( x \in \hat{W}_{n_l} \) there exists \( y_Q \in Q \cap A_{E,n_l}(\mu) \) such that \( x, y_Q \in Q \). Hence \( d(f^l(x), f^l(y_Q)) < \gamma \) for all \( i = 0, \ldots, n_l - 1 \). We then have \( \text{dist}(\sigma_{n_l}(x), \sigma_{n_l}(y_Q)) < \delta_2/3 \) by Lemma 3.1 and by the triangular inequality

\[
\text{dist}(\sigma_{n_l}(x), \mu) \leq \text{dist}(\sigma_{n_l}(x), \sigma_{n_l}(y_Q)) + \text{dist}(\sigma_{n_l}(y_Q), \mu) < \delta_2/3 + \varepsilon < \delta_2,
\]

because \( y_Q \in A_{E,n_l}(\mu) \) and \( \varepsilon < \delta_2/3 \). Thus, for any \( \varphi \in C^0(M, \mathbb{R}) \),

\[
\left| \int \varphi d\mu_{n_l} - \int \varphi d\mu \right| \leq \sum_{x \in W_{n_l}} \lambda_x \left| \int \varphi d\sigma_{n_l}(x) - \int \varphi d\mu \right| < \delta_2.
\]

Hence we have \( \text{dist}(\mu, \tilde{\mu}) \leq \delta_2 \) and consequently

\[
\int \psi d\mu_{n_l} \leq \int \psi d\mu + \delta_2 \quad \text{and} \quad \int \psi d\tilde{\mu} \leq \int \psi d\mu + \delta_2. \tag{6.8}
\]

We now use again item (2) of Lemma 5.7 with \( C_0 = W_0, \; v = \tilde{\mu}, \; v_1 = \mu_{k+1}, \; k \geq 1 \) and \( \mu_1 = \mu \). We obtain a small perturbation \( \tilde{P} \) of \( P \) so that \( x \in \hat{W}_{n_l} \) still belongs to the same atom of the \( h_j \)-th refinement of \( \tilde{P} \) which intersects \( A_{E,n_l}(\mu) \), for \( Q \in E_j; \; j = 1, \ldots, k \) and \( l \geq 1 \). More precisely, \( \tilde{P} \) satisfies, besides item (2) of Lemma 5.7, the useful property:

\[
x_Q \in \tilde{Q} \cap Q \text{ for } Q \in E_j \text{ where } \tilde{Q} \in \hat{E}_j \text{ and } f^{h_j}(x_Q) = w \in P(w) \cap P_\infty(w).
\]

Now we observe that \( H(\tilde{P}^\mu, v_{n_l}) = H(\tilde{P}^\mu, v_{n_l}) \) by definition of \( v_{n_l} \) and by construction of the \( \tilde{P} \) as a perturbation of \( P \). Following the proof of Lemma 3.6 (see inequality (3.3)) there exists \( l_0 \geq 0 \) such that

\[
\frac{1}{n_l} H(\tilde{P}^\mu, v_{n_l}) = \frac{1}{n_l} H(\tilde{P}^\mu, v_{n_l}) \leq H(\tilde{P}, \tilde{\mu}) + \frac{\delta_2}{4}, \quad \forall l \geq l_0.
\]

We have \( H_{\mu}(\tilde{P}/P) < \frac{\delta_2}{4} \) by Lemma 3.4 and item (2e) from Lemma 5.7. Then, because \( h(\tilde{P}, \tilde{\mu}) \leq h(P, \mu) + H_{\mu}(\tilde{P}/P) \), we get

\[
\frac{1}{n_l} H(\tilde{P}^\mu, v_{n_l}) \leq h(P, \mu) + \frac{\delta_2}{4} \leq h(P, \mu) + \frac{\delta_2}{4} + \tau, \quad \forall l \geq l_0,
\]

where the last inequality follows by the choice of \( \delta_2 \) in (6.3). Thus,

\[
\frac{1}{n_l} H(\tilde{P}^\mu, v_{n_l}) \leq h(P, \mu) + \frac{\delta_2}{4} + \tau \leq h_{\mu}(f) + \frac{5}{4} \tau, \quad \forall l \geq l_0, \tag{6.9}
\]

and we recall that \( \delta_2 < \tau \). Combining assertions (6.5), (6.6), (6.9) and (6.8) we arrive at

\[
\text{Leb}(A_{E,n_l}(\mu)) \leq \frac{n_l}{n_l - 1} \exp \left[ n_l \left( \frac{\tau}{2} + \frac{1}{n_l} \log L(n_l) \right) \right] \tag{6.10}
\]
Lemma that clearly holds for distance expanding maps (recall Remark 6.7). Thus, we obtain potential \( \phi_0 = (\iota \iota) L \beta b y a \psi T \) distance expanding and topologically transitive map \( \mu \in H \).

Hence we obtain \( \mu \). Since \( \mu \) is a measure with non-negative \( h \)-function, the Lyapunov exponents are non-negative. Hence the sum \( h_{\mu}(f) + \int \psi d\mu + 3\tau \) for all \( \mu \in W^s_\tau(v) \). Together with Lemma 5.6, this shows that \( P_\text{top}(T, \phi) = \log \lambda \) and so all \( \nu \)-weak-SRB-like probability measures are \( \phi \)-equilibrium states.

### 6.2 Proof of Theorem D

To prove Theorem D, consider an expanding weak-SRB-like measure \( \mu \). Then there exists \( \sigma \in (0, 1) \) such that \( \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df (f^i(x))^{-1} \| \leq \log \sigma < 0 \) for \( \mu \)-a.e \( x \in M \).

Thus, the Lyapunov exponents are non-negative. Hence the sum \( \Sigma^+(x) \) of the positive Lyapunov exponents of a \( \mu \)-generic point \( x \), counting multiplicities, is such that \( \Sigma^+(x) = \lim_{n \to \infty} \frac{1}{n} \log \| \det Df^n(x) \| \) (by the Multiplicative Ergodic Theorem) and \( \int \Sigma^+ d\mu = \int \log |\det Df| d\mu = -\int \psi d\mu \) by the standard Ergodic Theorem.

For \( C^1 \)-systems, Ruelle’s Inequality [36] states that for any \( f \)-invariant probability measure \( \mu \) on the Borel \( \sigma \)-algebra of \( M \), the corresponding measure-theoretic entropy \( h_{\mu}(f) \) satisfies \( h_{\mu}(f) \leq \int \Sigma^+ d\mu \) and consequently \( h_{\mu}(f) + \int \psi d\mu \leq 0 \).

By definition, Pesin’s Entropy Formula holds if the latter difference is equal to zero. Since \( \mu \in W^s_\tau(v) \), by Proposition 6.8, we have that \( h_{\mu}(f) + \int \psi d\mu = 0 \) which proves the first statement of Theorem D.

The next result completes the proof of Theorem D.

**Proposition 6.10** Let \( f : M \to M \) be non-uniformly expanding. Then all the ergodic SRB-like probability measures are expanding probability measures.

**Proof** The assumptions on \( f \) ensure that there exists \( \sigma \in (0, 1) \) such that \( \text{Leb}(H(\sigma)) = 1 \). The proof uses the following.

**Lemma 6.11** If \( f : M \to M \) is a \( C^1 \) local diffeomorphism such that \( \text{Leb}(H(\sigma)) = 1 \) for some \( \sigma \in (0, 1) \), then each \( \mu \in W_f \) satisfies \( \int \log \|Df\|^{-1} d\mu < \log \sqrt{\sigma} \).
Proof Fix $0 < \varepsilon < -\frac{1}{2} \log \sigma$ small enough. Since that $\varphi(x) := \log \|Df(x)^{-1}\|$ is a continuous potential, from the definition of the weak* topology in space $\mathcal{M}_1$ of probability measures, we deduce that there exists $0 < \varepsilon_1 < \varepsilon$ such that if $\text{dist}(\mu, \nu) < \varepsilon_1$ then $|\int \varphi d\mu - \int \varphi d\nu| < \varepsilon$ for all $\mu, \nu \in \mathcal{M}_1$.

Let us take $\mu \in \mathcal{W}_f$, then $\text{Leb}(A_{\varepsilon_1}(\mu)) > 0$. Let $x \in A_{\varepsilon_1}(\mu) \cap H(\sigma)$ and consider $\nu_x = \rho_{\omega}(x)$ such that $\text{dist}(\mu, \nu_x) < \varepsilon_1$. Then

$$\int \log \|Df(x)^{-1}\| d\mu - \int \log \|Df(x)^{-1}\| d\nu_x < \varepsilon,$$

and therefore there exists $n_k \not\to \infty$ so that $\sigma_{n_k}(x) \wto \nu_x$. Thus

$$\int \log \|Df(x)^{-1}\| d\mu \leq \int \log \|Df(x)^{-1}\| d\nu_x = \lim_{k \to +\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \log \|Df(f^j(x))^{-1}\| + \varepsilon$$

$$\leq \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| + \varepsilon < \log \sigma + \varepsilon < \log \sqrt{\sigma}$$

as stated. \qed

Going back to the proof of the proposition, since $\mu$ is $f$-invariant and ergodic, then by the previous lemma and by the standard Ergodic Theorem

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(y))^{-1}\| = \int \log \|Df(x)^{-1}\| d\mu < \log \sqrt{\sigma} \text{ for } \mu - \text{a.e } y \in M.$$ 

Therefore $\mu$ is an expanding measure. This finishes the proof of the proposition. \qed

This completes the proof of Theorem D.

7 Ergodic weak-SRB-like measure

In this section, we prove Corollary E on the existence of ergodic weak-SRB-like measures for non-uniformly expanding local diffeomorphisms $f : M \to M$.

7.1 Ergodic expanding invariant measures

Theorem 7.1 Let $f : M \to M$ be a $C^1$ local diffeomorphism. If $\mu$ is an ergodic expanding $f$-invariant probability measure, then

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{\log(\text{Leb}(A_{\varepsilon,n}(\mu)))}{n} \geq \int \psi d\mu + h_\mu(f). \tag{7.1}$$

Proof Since $\mu$ is an ergodic probability measure, we have $\lim_{n \to +\infty} \sigma_n(x) = \mu$ for $\mu$-a.e. $x \in M$. So for $\mu$-a.e. $x \in M$, there exists $N(x) \geq 1$ such that $\text{dist}(\sigma_n(x), \mu) < \varepsilon/4$ $\forall n \geq N(x)$.

Given $\varepsilon > 0$ and any natural value of $N \geq 1$, define the set

$$B_N := \{x \in M : \text{dist}(\sigma_n(x), \mu) < \varepsilon/4 \ \forall n \geq N\}. \tag{7.2}$$
Consider $\delta_1 > 0$ such that for each $\sigma$-hyperbolic time $h \geq 1$ time for $x$, $f^h |_{B(x, h, \delta_1)}$ maps $B(x, h, \delta_1)$ diffeomorphically to the ball of radius $\delta_1$ around $f^h(x)$.

Fix $\delta > 0$ such that Lemma 3.1 holds with $\varepsilon/8$ in the place of $\varepsilon$ and fix $\xi > 0$ satisfying $|\psi(x) - \psi(y)| < \frac{\varepsilon}{8}$ if $d(x, y) < \xi$.

Consider $0 < \gamma_0 < \min\{\xi, \delta_1/2\}$ and let $N(n, \gamma, b)$ be the minimum number of points needed to $(n, \gamma)$-span a set of $\mu$-measure $b$ (see (1.2)). Choose $0 < \gamma_1 < \gamma_0$ such that

$$\liminf_{n \to +\infty} \frac{1}{n} \log N(n, 4\gamma, 1/2) \geq h_{\mu}(f) - \frac{\varepsilon}{2}, \quad \forall \gamma < \gamma_1.$$  

(7.3)

Set $A = \{x \in M : \limsup_{n \to +\infty} -\frac{1}{n} \log \text{Leb}(B(x, n, \gamma_2)) \leq h_{\text{Leb}}(f, \mu) + \frac{\varepsilon}{4}\}$ where $0 < \gamma_2 \leq \gamma_1$ is such that

$$\mu(A) > 2/3.$$  

(7.4)

This is possible by definition of $h_{\text{Leb}}(f, \mu)$ (see (1.1)). We have implicitly assumed that $h_{\text{Leb}}(f, \mu) < \infty$ here. If $h_{\text{Leb}}(f, \mu) = \infty$, then there is nothing to prove since $h_{\mu}(f) < h_{\text{top}}(f) < \infty$ because $f$ is a $C^1$ local diffeomorphism.

We note that $B_N \subset B_{N+1}$ and $\mu(\cap B_N) = 1$. So there exists $N \geq 1$ such that $\mu(B_N) \geq \frac{5}{6}$.

If $C_N := A \cap B_N$, then $\mu(C_N) \geq \frac{1}{2}$ and for all $x \in C_N$ and $n \geq N(x)$ we have:

1. $B(x, n, \gamma_2) \subset A_{\varepsilon,n}(\mu)$;
2. $\text{Leb}(B(x, n, \gamma_2)) \geq e^{-(h_{\text{Leb}}(f, \mu)+\varepsilon/2)n}$.

Note that (2) immediately follows from (7.4). Moreover, (1) holds because, given $y \in B(x, n, \gamma)$ then $d(f^j(y), f^j(x)) < \gamma$ for all $j = 0, \ldots, n - 1$. By Lemma 3.1 we have $\text{dist}(\sigma_n(y), \sigma_n(x)) < \frac{\varepsilon}{8}$. Since $x \in B_N$, by the triangular inequality

$$\text{dist}(\sigma_n(y), \mu) \leq \text{dist}(\sigma_n(y), \sigma_n(x)) + \text{dist}(\sigma_n(x), \mu) < \varepsilon.$$

Therefore, $y \in A_{\varepsilon,n}(\mu)$.

For each $n$, let $E_n = E_n(2\gamma_2)$ be a maximal $(n, 2\gamma_2)$-separated subset of points contained in $C_N$. Then $\bigcup_{n \in E_n} B(x, 4\gamma_2, n) \supset C_N$ by maximality of $E_n$ and so $\#E_n \geq N(n, 4\gamma_2, 1/2)$.

Also, given $x, y \in E_n, x \neq y$ then $B(x, n, \gamma_2) \cap B(y, n, \gamma_2) = \emptyset$. Thus,

$$\liminf_{n \to +\infty} \frac{1}{n} \log \text{Leb}(A_{\varepsilon,n}(\mu)) \geq \liminf_{n \to +\infty} \frac{1}{n} \log \sum_{x \in E_n} \text{Leb}(B(x, n, \gamma_2))$$

$$\geq \liminf_{n \to +\infty} \frac{1}{n} \log \left(\#E_n \cdot e^{-(h_{\text{Leb}}(f, \mu)+\varepsilon/2)n}\right)$$

$$\geq \liminf_{n \to +\infty} \frac{1}{n} \log N(n, 4\gamma_2, 1/2) - h_{\text{Leb}}(f, \mu) - \varepsilon/2.$$  

Thus, by (7.3) we have,

$$\liminf_{n \to +\infty} \frac{1}{n} \log \text{Leb}(A_{\varepsilon,n}(\mu)) \geq h_{\mu}(f) - h_{\text{Leb}}(f, \mu).$$  

(7.5)

Moreover, since $\mu$ is expanding, then there exist $\theta > 0$ and $\sigma \in (0, 1)$ such that $\mu$-a.e. $x \in M$ has positive frequency $\geq \theta$ of $\sigma$-hyperbolic times of $f$. Let $\tilde{M} \subset M$ be such that for all $x \in \tilde{M}$, $\lim_{n \to +\infty} \sigma_n(x) = \mu$ and $f^h |_{B(x, h, \gamma_2)}$ maps $B(x, h, \gamma_2)$ diffeomorphically to the ball of radius $\gamma_2$ around $f^h(x)$, for $h$ a $\sigma$-hyperbolic time for $x$. We observe that $|\psi(f^j(y)) - \psi(f^j(z))| < \frac{\varepsilon}{4}$ for all $z \in B(y, h, \gamma_2/2)$, since $d(f^jx, f^jy) \leq \gamma_2/2$ for all $j = 0, \ldots, h - 1$. 

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Hence, by uniform continuity of $\psi$ and because Lebesgue measure assigns mass uniformly bounded away from zero to balls of fixed radius, we obtain

$$0 < \alpha(\gamma_2) \leq \text{Leb}(B(f^h x, \gamma_2)) = \int_{B(x, h, \gamma_2)} |\det Df^h| \, d\text{Leb} \leq \int_{B(x, h, \gamma_2)} e^{-S_h \psi} \, d\text{Leb} \leq e^{-S_h \psi(x) + h \epsilon/4} \cdot \text{Leb}(B(x, h, \gamma_2)).$$

Thus

$$0 = \liminf_{h \to +\infty} \frac{1}{h} \log \alpha(\gamma_2/2) \leq \liminf_{h \to +\infty} -\frac{1}{h} S_h \psi(x) + \frac{\epsilon}{4} + \liminf_{h \to +\infty} \frac{1}{h} \log \text{Leb}(B(x, h, \gamma_2)) \leq -\limsup_{h \to +\infty} \int \psi d\sigma_h(x) + \frac{\epsilon}{4} - \limsup_{h \to +\infty} \frac{1}{h} \log \text{Leb}(B(x, h, \gamma_2)) = -\int \psi d\mu + \frac{\epsilon}{4} - h_{\text{Leb}}(f, x).$$

Therefore, $h_{\text{Leb}}(f, x) \leq -\int \psi d\mu + \frac{\epsilon}{4}$ for $\mu$-a.e $x \in M$. Hence $h_{\text{Leb}}(f, \mu) \leq -\int \psi d\mu + \frac{\epsilon}{4}$ and by (7.5)

$$\lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \text{Leb}(A_{\epsilon, n}(\mu)) \geq h_{\mu}(f) + \int \psi d\mu,$$

which gives the desired inequality. \qed

**Proposition 7.2** Let $f : M \to M$ be a $C^1$ local diffeomorphism. Every expanding ergodic $f$-invariant probability measure $\mu$ such that $\int \psi d\mu + h_{\mu}(f) \geq 0$ is a weak-SRB-like probability measure.

**Proof** Let $\mu \in \mathcal{M}_{f}$ be an expanding ergodic probability measure such that $\int \psi d\mu + h_{\mu}(f) \geq 0$. By Theorem 7.1, we have that

$$\lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{\log(\text{Leb}(A_{\epsilon, n}(\mu)))}{n} \geq h_{\mu}(f) + \int \psi d\mu \geq 0.$$  

Moreover, if $\epsilon_1 < \epsilon_2$ then $A_{\epsilon_1, n}(\mu) \subset A_{\epsilon_2, n}(\mu)$. So $\limsup_{n \to +\infty} \frac{\log(\text{Leb}(A_{\epsilon, n}(\mu)))}{n}$ is increasing with $\epsilon > 0$. Thus $\limsup_{n \to +\infty} \frac{\log(\text{Leb}(A_{\epsilon, n}(\mu)))}{n} \geq 0$ for all $\epsilon > 0$. But since $\text{Leb}$ is a probability measure, we conclude that

$$\limsup_{n \to +\infty} \frac{\log(\text{Leb}(A_{\epsilon, n}(\mu)))}{n} = 0, \quad \forall \epsilon > 0.$$

Then $\mu$ is a weak-SRB-like measure. \qed

**Remark 7.3** The statement of Theorem 7.1 still holds if we replace $f$ by an open distance expanding and topologically transitive map $T$ of a compact metric space $X$; $\text{Leb}$ by a $\phi$-conformal measure $\nu$ for some continuous potential $\phi : X \to \mathbb{R}$ with $\mathcal{L}_\phi^\mu(\nu) = \lambda \nu$ for some $\lambda > 0$. So we get

$$\lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \nu(A_{\epsilon, n}(\mu)) \geq h_{\mu}(f) + \int \psi d\mu - \log \lambda,$$

since, in the proof of Theorem 7.1, we used that $\text{Leb}$ is $\psi$-conformal together with general results from Ergodic Theory. Analogously for Proposition 7.2.
Proposition 7.4 Let \( f : M \to M \) be a \( C^1 \) local diffeomorphism. Every expanding ergodic \( f \)-invariant probability measure that satisfies Pesin’s Entropy Formula is a weak-SRB-like probability measure.

Proof Let \( \mu \in M_f \) be an expanding ergodic probability measure such that \( h_\mu(f) = \int \sum^+ d\mu \), where \( \sum^+ (x) \) is the sum of the positive Lyapunov exponents of a \( \mu \)-generic point \( x \) counting multiplicities.

Since \( \mu \) is an expanding probability measure, we deduce by the Multiplicative Ergodic that \( -\int \psi d\mu \leq \int \sum^+ d\mu = h_\mu(f) \). Therefore, \( h_\mu(f) + \int \psi d\mu \geq 0 \) and the result follows from Proposition 7.2.

\( \square \)

7.2 Ergodic expanding weak-SRB-like measures

Now we are ready to prove Corollary E.

Proof of Corollary E Since \( P_{\text{top}}(f, \psi) = 0 \), we conclude by Proposition 6.8 that all weak-SRB-like measures are equilibrium states for the potential \( \psi \), that is, \( h_\mu(f) + \int \psi d\mu = 0 \) for all \( \mu \in W^s_f \).

We know that there exists \( \sigma \in (0, 1) \) such that \( \text{Leb}(H(\sigma)) = 1 \). Given \( \mu \in W^s_f \), by Lemma 6.11 we have that \( \int \log \|Df\|^{-1} d\mu \leq \log \sqrt{\sigma} \).

By the Ergodic Decomposition Theorem, there exists \( A \subseteq M \) such that \( \mu(A) > 0 \) and for all \( y \in A, \int \log \|Df\|^{-1} d\mu_y \leq \log \sqrt{\sigma} \), where \( \mu_y \) is an ergodic component of \( \mu \).

Fix \( y_0 \in A \). By Birkhoff’s Ergodic Theorem,

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(y))^{-1}\| = \int \log \|Df\|^{-1} d\mu_{y_0} \leq \log \sqrt{\sigma},
\]

for \( \mu_{y_0} \)-a.e. \( y \in M \). Therefore \( \mu_{y_0} \) is an expanding probability measure.

Since \( P_{\text{top}}(f, \psi) = 0 \) and \( h_{\mu_y}(f) + \int \psi d\mu_y = 0 \) we conclude that \( h_{\mu_y}(f) + \int \psi d\mu_y = 0 \) for \( \mu \)-a.e \( x \in M \). In particular, by Corollary 7.2 we conclude that there exist \( y_0 \in A \) such that \( \mu_{y_0} \in W^s_f \), showing that there are ergodic expanding weak-SRB-like probability measures which satisfy Pesin’s Entropy Formula, completing the proof of Corollary E.

\( \square \)

8 Weak-expanding non-uniformly expanding maps

In this section we reformulate Pesin’s Entropy Formula for a class of weak-expanding and non-uniformly expanding maps with \( C^1 \) regularity and prove Corollary F.

8.1 Weak-SRB-like, equilibrium and expanding measures

We divide the proof of Corollary F into the next two results below.

Proposition 8.1 Let \( f : M \to M \) be weak-expanding and non-uniformly expanding. Then, all (necessarily existing) weak-SRB-like probability measures are \( \psi \)-equilibrium states and, in particular, satisfy Pesin’s Entropy Formula. Moreover, there exists some ergodic weak-SRB-like probability measure.
Proof Let $f : M \to M$ be as in statement of Corollary F. Then, for every $x \in M$ and all $v \in T_x M \setminus \{0\}$ we have $\lim_{n \to +\infty} \frac{1}{n} \log \|DF^n(x) \cdot v\| \geq 0$. Thus, the Lyapunov exponents for any given $f$ -invariant probability measure $\mu$ are non-negative. Hence $\Sigma^+(x) = \lim_{n \to +\infty} \frac{1}{n} \log |\det DF^n(x)|$ and $\int \Sigma^+ d\mu = \int \log |\det DF| d\mu = -\int \psi d\mu$.

For $C^1$-systems, Ruelle’s Inequality [36] ensures that $h_\mu(f) \leq \int \Sigma^+ d\mu$, so $h_\mu(f) + \int \psi d\mu \leq 0$. By Proposition 6.8, we have that $P_{\text{top}}(f, \psi) = 0$ and $h_\mu(f) + \int \psi d\mu = 0$ for all $\mu \in \mathcal{W}_f^\sigma$.

Therefore, all weak-SRB-like probability measures are $\psi$-equilibrium states and satisfy Pesin’s Entropy Formula. Moreover, by Corollary E, we conclude that there exist ergodic weak-SRB-like measures.

Here we obtain a sufficient condition to guarantee that all $\psi$-equilibrium states are generalized convex linear combinations of weak-SRB-like measures.

Proposition 8.2 Let $f : M \to M$ be weak-expanding and non-uniformly expanding. If $\psi < 0$, then there is no atomic weak-SRB-like probability measure. Moreover, if $\mathcal{D} = \{x \in M ; \|DF(x)^{-1}\| = 1\}$ is finite and $\psi < 0$, then almost all ergodic components of a $\psi$-equilibrium state are weak-SRB-like measures and all weak-SRB-like probability measures $\mu$ have ergodic components $\mu_x$ which are expanding weak-SRB-like probability measures for $\mu$-a.e. $x \in M \setminus \mathcal{D}$.

Proof Note that, if $\psi < 0$, then $\int \psi d\mu < 0$ for all $\mu \in \mathcal{M}_f$. On the other hand, if $\mu \in \mathcal{M}_f$ is an atomic invariant probability measure, then $h_\mu(f) = 0$. Therefore $\mu$ does not satisfy Pesin’s Entropy Formula and, by Proposition 8.1, we conclude that there is no atomic weak-SRB-like probability measure.

Since $\int \psi d\mu < 0$ and $h_\mu(f) \leq -\int \psi d\mu$ for all $\mu \in \mathcal{M}_f$, then a $\psi$-equilibrium state satisfies $h_\mu(f) + \int \psi d\mu = 0$ and so $h_{\mu_x}(f) + \int \psi d\mu_x = 0$, $\mu$-a.e. $x$ by the Ergodic Decomposition Theorem. Thus $h_{\mu_x}(f) > 0$.

Hence $\mu_x$ is non-atomic and thus expanding because $\int \log \|(DF)^{-1}\| d\mu_x < 0$ and $\mu_x$ is ergodic, for $\mu$-a.e. $x$. Such $\mu_x$ also satisfy the Entropy Formula. Therefore Proposition 7.4 ensures that $\mu_x$ is weak-SRB-like for $\mu$-a.e. $x$.

Suppose now that $\mathcal{D}$ is finite and consider $\mu \in \mathcal{W}_f^\sigma$. Then $\int \log \|(DF)^{-1}\| d\mu < 0$, for otherwise, we would have $\text{supp}(\mu) \subset \mathcal{D}$ so that $\mu$ is purely atomic, which is a contradiction with the previous conclusions. Because $\mu$ is a $\psi$-equilibrium state, then $h_{\mu_x}(f) + \int \psi d\mu_x = 0$. Moreover, $0 > \int \log \|(DF)^{-1}\| d\mu = \int_{M \setminus \mathcal{D}} \log \|(DF)^{-1}\| d\mu$ thus, by the Ergodic Decomposition Theorem, we conclude that for $\mu$-a.e $x \in M \setminus \mathcal{D}$ we have $h_{\mu_x}(f) + \int \psi d\mu_x = 0$ (remember that $P_{\text{top}}(f, \psi) = 0$ and $\int \log \|(DF)^{-1}\| d\mu_x < 0$.

Therefore $\mu$-a.e $x \in M \setminus \mathcal{D}$ has expanding ergodic components which are $\psi$-equilibrium states. By Proposition 7.2, we deduce that $\mu_x$ is a weak-SRB-like probability measure for $\mu$-a.e. $x \in M \setminus \mathcal{D}$. The proof is complete.

Putting Propositions 8.1 and 8.2 together we complete the proof of Corollary F.

8.2 Expanding case

Corollary H improves the main result of [20] and allows rewriting all the results from [20], which were only proved for $C^1$-expanding maps in circle. In this section we prove Corollaries H and I.

Proof of Corollary H The assumptions on $f$ ensure that all $f$-invariant probability measures $\mu$ are expanding. Moreover, by Proposition 6.8 and Ruelle’s Inequality, we conclude that
Proof
See [20, Corollary 2.6].

Now we are ready to prove Corollary I.

Proof of Corollary I
Let $f_n \to f$ in the $C^1$-topology, where $f_n, f : M \to M$ are $C^1$-expanding maps for all $n \geq 1$. For each $n \geq 1$ consider $\mu_n$ weak-SRB-like measures associated to $f_n$ and let $\mu$ be a weak* limit point: $\mu = \lim_{j \to +\infty} \mu_{n_j}$.

Fix $\mathcal{P}$ a generating partition for every $f_{n_j}$ for all $j \geq 1$ and such that $\mu(\partial \mathcal{P}) = 0$. This is possible, since $f_{n_j}$ is $C^1$-expanding and $f_n \to f$ in $C^1$-topology and $f$ is also $C^1$-expanding.

By the Kolmogorov-Sinai Theorem, this implies that $h_{\mu_{n_j}}(f_{n_j}) = h(\mathcal{P}, \mu_n)$ and $h_\mu(f) = h(\mathcal{P}, \mu)$, that is,

$$h_{\mu_{n_j}}(f_{n_j}) = \inf_{k \geq 1} \frac{1}{k} H(\mathcal{P}_{n_j}^k, \mu_{n_j}) \quad \text{and} \quad h_\mu(f) = \inf_{k \geq 1} \frac{1}{k} H(\mathcal{P}^k, \mu).$$

Since $\mu$ gives zero measure to the boundary of $\mathcal{P}$ then $H(\mathcal{P}_{n_j}^k, \mu_{n_j})$ converge to $H(\mathcal{P}^k, \mu)$ as $j \to \infty$. Furthermore, for every $\varepsilon > 0$ there is $n_0 \geq 1$ such that

$$h_{\mu_{n_j}}(f_{n_j}) \leq \frac{1}{n_0} H(\mathcal{P}_{n_j}^{n_0}, \mu_{n_j}) \leq \frac{1}{n_0} H(\mathcal{P}^{n_0}, \mu) + \varepsilon \leq h_\mu(f) + 2\varepsilon.$$

By Corollary H, $h_{\mu_{n_j}}(f_{n_j}) + \int \psi_{n_j} d\mu_{n_j} = 0$ for all $j \geq 0$, since $\mu_{n_j}$ is a weak-SRB-like probability measure and $\psi_{n_j} = -\log |\det Df_{n_j}|$.

Since $\psi_{n_j} \to \psi$ in the topology of uniform converge, we have that $\int \psi_{n_j} d\mu_{n_j} \to \int \psi d\mu$.

By Ruelle’s inequality, $h_\nu(f) + \int \psi d\nu \leq 0$ for any $f$-invariant probability measure $\nu$ on the Borel $\sigma$-algebra of $M$. Thus,

$$0 \geq h_\mu(f) + \int \psi d\mu \geq \lim_{n \to +\infty} \left( h_{\mu_n}(f_n) + \int \psi_n d\mu_n \right) = 0.$$
This shows that $\mu$ satisfies Pesin’s Entropy Formula, is a $\psi$-equilibrium state since $P_{\text{top}}(f, \psi) = 0$ and, by Corollary H, its ergodic components $\mu_x$ are weak-SRB-like probability measures for $\mu$-a.e $x \in M$. □

8.3 Proof of Corollary C

We are now ready to prove Corollary C.

**Proof of Corollary C** From Remarks 6.7 and 6.9, we can use Proposition 6.8, Theorem 7.1 and Proposition 7.2 in the setting of Theorem B. Let $\mu \in M_T$ be such that

$$h_\mu(T) + \int \phi \, d\mu = P_{\text{top}}(T, \phi) = \int \left( h_{\mu_x}(T) + \int \phi \, d\mu_x \right) \, d\mu(x).$$

Since we also have $h_{\mu_x}(T) + \int \phi \, d\mu_x \leq P_{\text{top}}(T, \phi)$ then $h_{\mu_x}(T) + \int \phi \, d\mu_x = P_{\text{top}}(T, \phi)$ for $\mu$-a.e $x$. Now from Theorem 7.1 (see Remark 7.3) we get

$$\lim \sup_{\epsilon \to 0^+} \lim_{n \to +\infty} \frac{1}{n} \log v(A_{\epsilon,n}(\mu_x)) \geq h_{\mu_x}(T) + \int \phi \, d\mu_x - \log \lambda,$$

and then we conclude that $\mu_x \in \mathcal{W}^u_T(\nu)$ following the same arguments in the proof of Proposition 7.2.

Assume now that $\mu$ is the unique $\phi$-equilibrium state. By Proposition 3.2 and Theorem B, we conclude that there exist a unique $\nu$-SRB-like measure $\mu$ and, by [19, Theorem 1.6], it follows that $\mu$ is $\nu$-SRB and $v(B(\mu)) = 1$.

Let now $\mathcal{V}$ be a small neighborhood of $\mu$ in $M$. Since $\kappa_r(\phi)$ is decreasing with $r$ and $\mu = \mathcal{K}_0(\phi) = \cap_{r>0} \mathcal{K}_r(\phi)$, we have that there exists $r_0 > 0$ such that $\mathcal{K}_r(\phi) \subset \mathcal{V}$ for all $0 < r < r_0$. Since $\mathcal{K}_r(\phi)$ is weak$^*$ compact (by upper semicontinuity of the metric entropy) we have $\mathcal{K}_r(\phi) = \cap_{\epsilon>0} \mathcal{K}^\epsilon_r(\phi)$, where $\mathcal{K}^\epsilon_r(\phi) = \{ \mu \in M_T : \text{dist}(\mu, \mathcal{K}_r(\phi)) \leq \epsilon \}$ with the weak$^*$ distance defined in (3.1). Let us take $0 < \epsilon < r_0$ such that $\mathcal{K}^\epsilon_r(\phi) \subset \mathcal{V}$. By Proposition 5.11, there exists $n_0 \geq 1$ and $\kappa = \kappa(\epsilon, r) > 0$ such that

$$v(\{x \in X : \sigma_n(x) \in M\setminus\mathcal{V}\}) \leq v(\{x \in X : \sigma_n(x) \in M\setminus\mathcal{K}^\epsilon_r(\phi)\}) = v(\{x \in X : \text{dist}(\sigma_n(x), \mathcal{K}_r(\phi)) \geq \epsilon\}) < \kappa e^{n(\epsilon - r)},$$

for all $n \geq n_0$. Thus $\lim sup_{n \to +\infty} \frac{1}{n} \log v(\{x \in X : \sigma_n(x) \in M\setminus\mathcal{V}\}) \geq \epsilon - r$. As $\epsilon > 0$ can be taken arbitrary small, we conclude that $\lim sup_{n \to +\infty} \frac{1}{n} \log v(\{x \in X : \sigma_n(x) \in M\setminus\mathcal{V}\}) \leq -r$ for all $0 < r < r_0$ and $r_0 = I(\mathcal{V}) := \sup \{r > 0 : \mathcal{K}_r(\phi) \subset \mathcal{V}\}$. Therefore

$$\lim sup_{n \to +\infty} \frac{1}{n} \log v(\{x \in X : \sigma_n(x) \in M\setminus\mathcal{V}\}) \leq -I(\mathcal{V}).$$

This shows that $v(\{x \in X : \sigma_n(x) \in M\setminus\mathcal{V}\})$ decreases exponentially fast with $n$ at a rate that depends on the “size” of $\mathcal{V}$.

The assumptions on $\mu$ are the same as in Corollary H with $\nu = \text{Leb}$ and $\phi = \psi$, so the upper large deviations statement of this corollary follows with the same proof. □

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References

1. Alves, J.: Statistical analysis of non-uniformly expanding dynamical systems. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2003. XXIV Colóquio Brasileiro de Matemática. [24th Brazilian Mathematics Colloquium]
2. Alves, J.F.: SRB measures for non-hyperbolic systems with multidimensional expansion. Annales scientifiques de l’Ecole normale supérieure 33(1), 1–32 (2000)
3. Alves, J.F., Araújo, V.: Hyperbolic times: frequency versus integrability. Ergod. Theory Dyn. Syst. 24, 1–18 (2004)
4. Alves, J.F., Bonatti, C., Viana, M.: SRB measures for partially hyperbolic systems whose central direction is mostly expanding. Invent. Math. 140(2), 351–398 (2000)
5. Alves, J. F., Ramos, V., Siqueira, J.: Equilibrium stability for non-uniformly hyperbolic maps. ArXiv e-prints, (July 2017)
6. Araújo, V., Pacifico, M.: Large deviations for non-uniformly expanding maps. J. Stat. Phys. 125(2), 411–453 (2006)
7. Ashley, J., Kitchens, B., Stafford, M.: Boundaries of markov partitions. Trans. Am. Math. Soc. 333(1), 177–201 (1992)
8. Avila, A., Bochi, J.: Generic expanding maps without absolutely continuous invariant σ-finite measure. Nonlinearity 19, 2717–2725 (2006)
9. Barreira, L., Pesin, Y.: Introduction to Smooth Ergodic Theory, Volume 148 of Graduate Studies in Mathematics. American Mathematical Society, Providence (2013)
10. Bessa, M., Varandas, P.: On the entropy of conservative flows. Qual. Theory Dyn. Syst. 11, 11–22 (2011)
11. Bowen, R.: A horseshoe with positive measure. Invent. Math. 29, 203–204 (1975)
12. Bowen, R.: Equilibrium states and the ergodic theory of Anosov diffeomorphisms, volume 470 of Lect. Notes in Math. Springer (1975)
13. Cao, Y., Yang, D.: On pesin’s entropy formula for dominated splittings without mixed behavior. J. Differ. Equ. 261(7), 3964–3986 (2016)
14. Catsigeras, E.: On Ilyashenko’s statistical attractors. Dyn. Syst. 29(1), 78–97 (2014)
15. Catsigeras, E.: Empiric stochastic stability of physical and pseudo-physical measures. ArXiv e-prints, (July 2017)
16. Catsigeras, E., Cerminara, M., Enrich, H.: The Pesin Entropy Formula for diffeomorphisms with dominated splitting. Ergod. Theory Dyn. Syst. 35(03), 737–761 (2015)
17. Catsigeras, E., Cerminara, M., Enrich, H.: Weak pseudo-physical measures and Pesin’s Entropy Formula for Anosov $C^1$ diffeomorphisms. Contemp. Math. 698, 69–89 (2017)
18. Hofbauer, F., Keller, G.: Quadratic maps without asymptotic measure. Commun. Math. Phys. 127, 319–337 (1990)
19. Keller, G.: Generalized bounded variation and applications to piecewise monotonic transformations. Z. Wahrsc. Verw. Gebiete 69(3), 461–478 (1985)
20. Ledrappier, F., Young, L.: The metric entropy of diffeomorphisms I. Characterization of measures satisfying Pesin’s entropy formula. Ann. Math. 122, 509–539 (1985)
21. Liu, P.D.: Pesin’s Entropy formula for endomorphisms. Nagoya Math. J. 150, 197–209 (1998)
22. Pesin, Y.B.: Characteristic Lyapunov exponents and smooth ergodic theory. Russ. Math. Surv. 324, 55–114 (1977)
23. Przytycki, F., Urbanski, M.: Conformal fractals: ergodic theory methods, volume 371 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, (2010)
11. Qian, M., Zhu, S.: SRB measures and Pesin's entropy formula for endomorphisms. Trans. Am. Math. Soc. 354(4), 1453–1471 (2002)
12. Qiu, H.: Existence and uniqueness of SRB measure on $C^1$ generic hyperbolic attractors. Commun. Math. Phys. 302(2), 345–357 (2011)
13. Quasf, A.N.: Non-ergodicity for $C^1$ expanding maps and $g$-measures. Ergod. Theory Dyn. Syst. 16(3), 531–543 (1996)
14. Ramos, V., Viana, M.: Equilibrium states for hyperbolic potentials. Nonlinearity 30(2), 825 (2017)
15. Ruelle, D.: A measure associated with Axiom A attractors. Am. J. Math. 98, 619–654 (1976)
16. Ruelle, D.: An inequality for the entropy of differentiable maps. Bol. Soc. Brasil. Mat. 9, 83–87 (1978)
17. Ruelle, D.: Thermodynamic formalism. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition. The mathematical structures of equilibrium statistical mechanics (2004)
18. Sinai, Y.: Gibbs measures in ergodic theory. Russ. Math. Surv. 27, 21–69 (1972)
19. Sun, W., Tian, X.: Dominated splitting and Pesin's Entropy Formula. Discrete Contin. Dyn. Syst. 32(4), 1421–1434 (2012)
20. Tahzibi, A.: $C^1$-generic Pesin's Entropy Formula. Comptes Rendus Mathematique 335(12), 1057–1062 (2002)
21. Takahashi, Y.: Entropy functional (free energy) for dynamical systems and their random perturbations. N. Holl. Math. Libr. Stoch. Anal. 32, 437–467 (1984)
22. Takens, F.: Heteroclinic attractors: time averages and moduli of topological conjugacy. Bull. Braz. Math. Soc. 25, 107–120 (1995)
23. Thaler, M.: Transformations on [0, 1] with infinite invariant measures. Isr. J. Math. 46(1–2), 67–96 (1983)
24. Varandas, P., Viana, M.: Existence, uniqueness and stability of equilibrium states for non-uniformly expanding maps. Ann. Inst. H. Poincaré Anal. Non Linéaire 27(2), 555–593 (2010)
25. Viana, M., Oliveira, K.: Foundations of ergodic theory, volume 151 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, (2016)
26. Walters, P.: An introduction to ergodic theory, volume 79 of Graduate Texts in Mathematics. Springer, New York (1982)

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