Entanglement and State Preparation

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Abstract

When a subset of particles in an entangled state is measured, the state of the subset of unmeasured particles is determined by the outcome of the measurement. This first measurement may be thought of as a state preparation for the remaining particles. This type of measurement is important in quantum computing, quantum information theory and in the preparation of entangled states such as the Greenberger, Horne, and Zeilinger state.

In this paper, we examine how the duration of the first measurement affects the state of the unmeasured subsystem. We discuss the case for which the particles are photons, but the theory is sufficiently general that it can be converted to a discussion of any type of particle. The state of the unmeasured subsytem will be a pure or mixed state depending on the nature of the measurement.

In the case of quantum teleportation we show that there is an eigenvalue equation which must be satisfied for accurate teleportation. This equation provides a limitation to the states that can be accurately teleported.

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I. INTRODUCTION

The preparation of states of a system is one of the primitive notions in quantum theory \[1\]. It consists of a set of rules for preparing a physical state of a given system in the laboratory and for associating a corresponding mathematical state in the Hilbert space defined by the system. In this paper we examine how entangled states can be used for state preparation. This is of interest in quantum information theory, quantum computing and in the preparation of special states such as the Greenberger-Horne-Zeilinger (GHZ) state \[2\]. The specific question addressed is, “after a measurement is completed on a subset of particles in an entangled state, what is the state of the remaining particles?” We can formulate this as a special case of general correlation measurements in which one set of measurements must be completed before any further measurements are made. That is, the first measurement or set of measurements acts as a trigger which defines the state of the remaining particles. Alternatively, we may use the language of probability theory and say that we are studying a conditional amplitude of a subsystem, conditioned by the outcome of the measurement of a second subsystem.

An interesting example of state preparation is found in quantum teleportation \[3\]. Recall that in this case Bob and Alice share an entangled two particle state,

\[ |\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|+\rangle_A |-\rangle_B - |-\rangle_A |+\rangle_B), \] (1)

and Alice is given an arbitrary state,

\[ |\phi\rangle_C = \alpha_+|+\rangle_C + \alpha_-|--\rangle_C. \] (2)

She makes a filtering measurement on the two particle state composed of her part of the entangled state and the unknown state. This measurement yields one of the four orthogonal Bell states for the pair AC

\[ |\Psi^{(\pm)}\rangle_{AC} = \frac{1}{\sqrt{2}} (|+\rangle_A |--\rangle_C \pm |--\rangle_A |+\rangle_C), \]

\[ |\Phi^{(\pm)}\rangle_{AC} = \frac{1}{\sqrt{2}} (|+\rangle_A |+\rangle_C \pm |--\rangle_A |--\rangle_C). \] (3)
After her measurement is completed, the particle in Bob’s hands is in a definite state depending on which result Alice obtained. Therefore, if Alice knows the state she is given, she can view this procedure as the preparation of one of four definite state in Bob’s laboratory. Of course, Alice cannot predict which of the four states will be produced before she makes her measurement. That Bob ends up with this state is perfectly understandable mathematically; however, the interpretation of what has happened is controversial since it takes us into questions of the epistemology of quantum mechanics. The particle in Bob’s laboratory goes from having no state to a definite state with only local measurements being performed by Alice. This is a stark example the non-local nature of quantum theory.

In this paper, I want to discuss the mundane issues of experiments like this and ask if a part of an entangled state is measured by a detector with finite time resolution, what is the state of the “undisturbed” part system.

II. STATES OF A SYSTEM

We must be more precise in defining what it means for a system to be in a definite state. We wish to argue that a single preparation procedure produces a definite state, but to do so the procedure must be tested a number of times. For a pure state, the testing procedure means that there are measurements, which can be idealized as projection measurements, such that

\[ P|\phi\rangle = |\phi\rangle \]

for each realization of the procedure that produces the state \( |\phi\rangle \). For example, if we wish to prepare an electron with its spin up along some axis, then a Stern-Gerlach measurement along that axis is a physical realization of \( P \). If the prepared state is \( |\phi'\rangle \neq |\phi\rangle \), then \( (1 - P)|\phi'\rangle \neq 0 \).

For a mixed state, \( \rho, \rho^2 \neq \rho \), the situation is even more complicated. It is not sufficient to have a filtering measurement, idealized as a complete set of orthogonal projections \( \{P_j\} \),
\[ \sum_j P_j = 1, \quad P_i P_j = \delta_{ij} P_j. \] If we repeat the preparation many times, the result \( j \) occurs with frequency approaching \( p_j = tr \rho P_j \), but there are an infinite number of density matrices \( \rho \) with diagonal entries \( \{ p_j \} \). Therefore, the prescription for checking whether the prepared state is \( \rho \) requires a set of measurements that determine the off-diagonal elements of \( \rho \). The important point is that in principle there is a method of testing a given procedure to determine if each time it is performed it produces the state \( \rho \). Having done this, we are allowed to argue that a single such procedure will produce the state \( \rho \). Of course, in practice, we are much less rigorous, relying on theory and a few measurements to argue that a given state is prepared.

The generalization from projective measurements to positive operator valued measurements (POVM) \[ \text{I}, \text{II} \] is not difficult. In fact, the measurements that are discussed below are more closely related to general POVM’s than to projective measurements.

### III. PREPARATION OF A ONE PARTICLE STATE FROM A TWO PARTICLE ENTANGLED STATE

#### A. Idealized case

For the idealized case, we assume that idealized projection measurements can be made instantaneously. Let \( H_1 \) and \( H_2 \) be Hilbert spaces and consider the space defined by their direct product. Suppose we have a normalized bipartite state

\[ |\Psi\rangle = \sum_a c_a \phi_a \rangle_1 |\psi_a\rangle_2, \quad (5) \]

where \( \{ |\phi_a\rangle_1, a = 1, 2, \cdots \} \) is an orthonormal basis of \( H_1 \) and \( \{ |\psi_a\rangle_2, a = 1, 2, \cdots \} \) is an orthonormal basis of \( H_2 \). If the outcome of an idealized filtering measurement of the complete set of projection operators \( \{ |\phi_a\rangle_1 \langle \phi_a| \} \) gives the result \( a = r \) the state of particle 2 is instantaneously projected into the state \( |\psi_r\rangle_2 \). This is sometimes referred to as the collapse of the wave function.. The acausal behavior of quantum theory is inherent in the fact that we cannot predict, in principle, which \( r \) the measurement of 1 will yield. The non-local
nature of quantum mechanics is displayed by particle 2 going from not being in a definite state to being in a definite state even if it is far away from particle 1. In a realistic theory, such as Bohm’s theory \cite{Bohm}, for each realization of the experiment, particles 1 and 2 have definite trajectories determined in part by a non-local quantum potential acting between the particles. When we determine the trajectory on which particle 1 lies, the trajectory of particle 2 will be altered because the non-local potential acting on it changes.

It is well-known that there is no superluminal signal in this case, nothing has been transferred by the measurement of particle 1 to the neighborhood of particle 2 until a signal from the output of measuring apparatus 1 reaches 2. In other words, as soon as measurement 1 is completed the detector at 1 has acquired $-\sum_a |c_a|^2 \log_2 |c_a|^2$ bits of information. The same amount of information can be acquired by detector in the location of 2 by either measuring the state of particle 2 or receiving a signal from detector 1 containing the result of the measurement.

Now consider a less ideal case in which the measurement on 1 is a POVM, $E$. After the measurement, the state of 2 is given by the density matrix

$$\rho_2 = \frac{1}{N} \sum_{aa'} |\psi_a\rangle_2 \langle 1|\phi_a|E|\phi_{a'}\rangle_1 c_a c_{a'}^{*} \langle \psi_{a'}|,$$  \hspace{1cm} (6)

where

$$N = \sum_a \langle 1|\phi_a|E|\phi_a\rangle_1 |c_a|^2.$$  

In general, this is a mixed state. Only in the special case that $\langle 1|\phi_a|E|\phi_{a'}\rangle_1$ factors into $f_a f_{a'}^{*}$ is $\rho_2$ a pure state. This is shown in appendix1.

It is obvious that $\rho_2|\psi_a\rangle_2 = 0$ for any $a$ such that $c_a = 0$. This limits the state that can be prepared by measuring particle 1. This is important in the generalization of teleportation. In order for it to be possible to teleport a state, that state must be present in the entangle state shared by Alice and Bob.

**B. Finite time measurements**
1. Detector operators

The discussion that follows will be given in terms of the Heisenberg picture, but it is not difficult to convert to a Schrödinger picture. We shall treat the particles as photons, although the conversion to any other type of particle is not difficult. We start by specifying the measuring devices. According to Glauber [6], the detector operator for a photon linearly polarized along the \( \mathbf{e} \) direction is, the positive frequency electric field operator \( \mathbf{E} = E \mathbf{e} \) defined by

\[
E = \sum_q p(q, \mathbf{e}) e^{-iq(t-x)} a(q, \mathbf{e})
\]  

(7)

where \( a(q, \mathbf{e}) \) is the destruction operator for a photon linearly polarized in the \( \mathbf{e} \) direction with frequency \( q > 0 \). The time is measured in distance units so that the speed of light is one. We shall ignore the components of momentum in the plane of the detector surface and take \( x \) to be the coordinate normal to the detector surface. We idealize to a point detector located at \( x \) that registers a count at time \( t \).

To further understand this expression, let a photon in the state

\[
|\phi\rangle = \sum_k f(k) a(k, \mathbf{e}) |0\rangle
\]

impinge on the detector. Then, using the commutation relations

\[
[a(k, \mathbf{e}), a^\dagger(k', \mathbf{e}')] = \delta_{kk'} d(\mathbf{e}, \mathbf{e}'),
\]  

(8)

where \( d \) is the scalar product,

\[
d(\mathbf{e}, \mathbf{e}') = \mathbf{e} \cdot \mathbf{e}',
\]  

(9)

we get

\[
\langle 0 | \mathbf{E} | \phi \rangle = \sum_k f(k) p(k, \mathbf{e}) e^{-ik(t-x)} d(\mathbf{e}, \mathbf{e}').
\]

The amplitude for detection at time \( t \) is in the form of a wave packet evaluated at \( x \) the location of the detector.
The detector records a quantity proportional to the intensity or, equivalently, the counting rate,

\[ I = \frac{1}{T_m} \int_{-T_m/2}^{T+T_m/2} d\tau |\langle 0|E|\phi \rangle|^2 \]

\[ = \sum_{kk'} f(k')^* p(k',e) f(k)p(k,e) e^{i(k'-k)T} \text{sinc} \left( \frac{(k-k')T_m}{2} \right) d^2(e,e'), \]

where from here on we introduce the retarded time \( \tau = t - x \). The outcome of the measurement depends on \( f, p \) and \( T_m \). The duration of the measurement \( T_m \) determines the degree to which off-diagonal matrix elements of the state are detected. The function \( p \) determines the spectral region to which the detector is sensitive.

First, suppose that the spectral amplitude \( f(k) \) is peaked at \( k = K \) and has a width of \( \Delta k \ll K \). Also let the width of \( p \) be large compared to that of \( f \), so that \( p \) is approximately constant over the range \( \Delta k \). Under these assumptions, \( I \) depends on the parameter \( \theta = \Delta k T_m = T_m/T_k \), where \( T_k = 1/\Delta k \) is the width of the wave packet. From fig. 1 it can be seen that if \( \theta \ll 1 \), the sinc function can be replaced by 1 over the range of summation and (11) becomes

\[ I = |p(K,e)| \sum_{e} f(k') e^{-i\kappa T} d^2(e,e') |^2, \]

where

\[ k = K + \kappa. \quad (12) \]

This means that we can resolve the envelope of the wave packet by moving the detector with respect to the source. This is illustrated in the space-time diagram in fig. 2a.

If \( \theta \gg 1 \), then the sinc function restricts the integration region to \( k \approx k' \) and (11) becomes

\[ I = \pi |p(K,e)|^2 \sum_{k} |f(k)|^2 d^2(e,e'). \]

This is the usual case for single photon detectors. This is illustrated in fig. 2b.
Let us reverse the roles of $p$ and $f$, so the detector has a narrow bandwidth compared to the state. Assume that $p$ is peaked at $K_p$ with width $\Delta k_p << K_p$, such that $f$ is approximately constant over the range $\Delta k$, then we get a similar result with $p$ and $f$ interchanged. In this case, the quantity $I$ is determined by the detection function $p$ and the parameter $\theta_p = \Delta k_p T_m$. If $\theta_p >> 1$, then

$$I = \pi |f(K,e)|^2 \sum_k |p(k)|^2 d^2(e,e')$$

and the measured intensity depends on a single mode of the particle wave packet, fig. 2c. This case corresponds to placing a narrow filter in front of the detector and is often used in practice.

2. Two particle entangled states

Now suppose that a two photon entangled state is generated with one photon moving to the right and the other to the left,

$$|\Psi\rangle = \sum_{kK} f(k,K) (\xi_+ |k e_+\rangle_R |K e_\rangle_L + \xi_- |k e_\rangle_R |K e_+\rangle_L).$$

(13)

The linear polarization states are defined with respect to the orthogonal directions $e_+$ and $e_-$. The factors $\xi_\pm$ are taken to be phase factors, $|\xi_\pm| = 1$ so that $|\Psi\rangle$ is a superposition of plane wave Bell states like those defined in (3). We shall assume that $f(k,K)$, the spectral amplitude, is peaked around $k_0$ and $K_0$ with widths $\Delta k << k_0$ and $\Delta K << K_0$. This ensures that the single photon state for R, which has the spectral function $\sum_K |f(k,K)|^2$, is a quasimonochromatic wave packet, and, similarly, the single photon state for L is quasimonochromatic.

Let us now detect the right-moving photon, R, at time $t_1$ and the left-moving photon, L, at time $t_2$. The correlation function for this is given by

$$C_{12} = \langle \Psi|E_1^\dagger E_2^\dagger E_2 E_1|\Psi\rangle,$$

(14)

with the detector operators given by eq. (7).
It is unrealistic to assume that the measurements occur instantaneously, so we compute

$$\overline{C}_{12} = \int dt_2 \int dt_1 C_{12} S(t_2, t_1),$$  \quad (15)$$

where $S$ is one when the detectors are on and vanishes when they are off. In the usual coincident counting experiments, $S$ is a function of $t_2 - t_1$ that is nonvanishing over some, usually small, time interval. In this paper we are interested in the case $t_2 >> t_1$, so that we can ascribe meaning to the state of L in the time between the two measurements.

In the example we are considering, the correlation function becomes

$$C_{12} = |A_{12}|^2,$$  \quad (16)$$

where the two particle amplitude is

$$A_{12} = \langle 0 | E_2 E_1 | \Psi \rangle = \sum_K g_1(K) \langle 0 | E_2 | K \tilde{e}_1 \rangle_L,$$  \quad (17)$$

with

$$g_1(K) = \sum_k p_R(k, e_1) e^{-ikr_1} f(k, K),$$  \quad (18)$$

and the polarization state is

$$|\tilde{e}_1\rangle_L = \sum_{\sigma = \pm} |e_\sigma\rangle_L \xi_{-\sigma} d(e_1, e_{-\sigma}).$$  \quad (19)$$

If $\xi_+ = -\xi_-$, the state $|\tilde{e}_1\rangle_L$ is orthogonal to $|e_1\rangle_L$. After the measurement of R is completed, the photon L has a definite polarization state.

The first detector is a trigger which registers in a time interval $(T_1 - \frac{T_m}{2}, T_1 + \frac{T_m}{2})$. After detector one fires, the correlation function reduces to a single particle function

$$C_1 = N \sum_{K, K'} \langle 0 | E_2 | K \tilde{e}_1 \rangle_L \langle 0 | E_2 | K' \tilde{e}_1 \rangle_L^* \rho_L(K; K')$$  \quad (20)$$

where
\[
\rho_L(K; K') = \frac{1}{N} \sum_{k k'} p_R(k, e_1) p_R(k', e_1)^* f(k, K) f(k', K')^* e^{-i k T_1} \times e^{i k' T_1} T_m \sin(k - k') \frac{T_m}{2}.
\]

(21)

\(\rho_L(K; K')\) is a matrix element of the one particle density matrix for L. The normalization \(N\) is defined so that \(\text{tr} \rho_L = 1\). Finally, we have

\[
C_1 = N \text{tr} \left( \rho_L E_2^1 E_2 \right)
\]

(22)

where

\[
\rho_L = \sum_{K K'} |K, \bar{e}_1 \rangle_L \rho_L(K; K') \langle K', \bar{e}_1 |.
\]

(23)

We now investigate under what circumstances this density matrix represents a pure state. To do this we exploit the assumption that \(f\) satisfies the condition that its width \(\Delta k \ll k_0\) and use eq.(12). The sinc function is small unless \(|\kappa - \kappa'| < 2\pi / T_m\). We introduce, as we did above, \(T_k = 1 / \Delta k\), the single particle coherence time of the wave packet of particle R. The critical parameter for the following discussion is \(\theta = T_m / T_k\). As we did in section B1 above, it is simplest to consider the two extreme cases of long triggering times \(\theta \gg 1\) and short triggering times \(\theta \ll 1\). We shall see that in the first case L is, in general, in a mixed state, while in the second case, L is always in a pure state.

a. Long triggering times

For long triggering times the sinc function is non-negligible when \(|\kappa - \kappa'| < \Delta k / \theta \ll \Delta k\). In this case, as illustrated in fig. [4], we may set \(\kappa \approx \kappa'\) in \(f\) and obtain

\[
\rho_L(K; K') = \sqrt{\frac{1}{N'}} \sum_k |p_R(k, e_1)|^2 f(k, K) f(k, K')^*.
\]

(24)

In general, L is in a mixed state. We will discuss this further below.

b. Short triggering times

In this case the width of the sinc function, \(2\pi / T_m = 2\pi / (T_k \theta) \gg \Delta k\), and the sinc function may be set equal to 1 over the entire range of the summation over \(\kappa\) and \(\kappa'\). Consequently,

\[
\rho_L(K; K') = \chi(K) \chi(K')^*
\]

(25)
\[
\chi(K) = \sqrt{\frac{1}{N}} \sum_k p_R(k, e_1) f(k, K) e^{-ik\tau_1}.
\] (26)

So that \(\rho_L(K; K')\) factors. In this case, eq.(20) becomes
\[
C_1 = N|\langle 0|E_L(x_2, t_2)|\chi, \tilde{e}_1\rangle|_L|^2.
\] (27)

We interpret this as staying that upon completion of the measurement on R, L is put in the pure state
\[
|\chi, \tilde{e}_1\rangle_L = \sum_K \chi(K)|K, \tilde{e}_1\rangle_L.
\] (28)

\textit{c. The properties of the state of the left moving photon}  

The exact nature of the state of the particle moving to the left depends upon the initial entangled state and the measurement made on the right. For long triggering time, case (a) above, the explicit time of the first measurement has disappeared from the calculation. Of course, it is still present in that any measurement on L must be made after the first measurement is completed. This information is hidden by the fact that we did not include the corrections due to the width of sinc function but treated it as though it were a Dirac delta function.

Suppose that the measurement on the right was a filtering measurement in \(k\) so that \(p_R(k, e_1)\) is narrowly peaked at \(k_0\). This reduces to the short triggering time, case (b), and (25) holds with
\[
\rho_L(K; K') = \frac{1}{N'} f(k_0, K) f(k_0, K')^*.
\] (29)

so eq. (28) becomes
\[
|\chi, \tilde{e}_1\rangle_L = \sqrt{\frac{1}{N''}} \sum_K f(k_0, K)|K, \tilde{e}_1\rangle_L.
\] (30)

This is the case in which detector 1 has a narrow filter in front of it so that it projects a plane wave state of R. In general, the state of L is not a plane wave state.

A particularly interesting example of the entangled two particle state (13) is the one contemplated by Einstein, Rosen and Podolsky (EPR),
where $k_p = k_0 + K_0$. Our assumptions imply that $v(K)$ is peaked around $K_0$. Such a two-particle entangled state can approximately be realized for photons using type-II spontaneous parametric down-conversion for which $k_p$ is the pump frequency.

In case (a),

$$\rho_L(K; K') = \frac{1}{N} |p_R(k_p - K, e_1)|^2 |v(K)|^2 \delta(K - K'),$$

so that $\rho_L$ is diagonal in the basis of $|K e_1\rangle$ states. The state for case (b) becomes a plane wave state.

If the filter function $p_R$ is narrowly peaked at $k_0$, so that we have case (b) again. The state of the L photon has a spectrum determined by $f(k_0, K)$ which in turn is fixed by the original two-photon state. The result of the measurement selects the state of particle L.

A contentious issue in the interpretation of quantum mechanics is whether the uncertainty principle reflects a fundamental limitation on how well conjugate variables can be determined because of the basic quantum nature of measurement, as Heisenberg believed. This position was criticized by Popper who argued that the uncertainty principle was a statistical statement and did not imply that it was meaningless for a particle to simultaneously possess definite values of conjugate variables as they do classically. It is clear that the measurement of the uncertainty in the left moving particle is unchanged by the measurement of R if the uncertainty is computed based on all the photons moving to the left (that is, independently of whether R registers a count or not). On the other hand, if the uncertainty is measured only for those L photons whose partners are detected on the right, then the uncertainty is different. The measurement changes the uncertainty because it selects out a subset of the particles moving to the left. There is no action-at-a-distance in the sense of a force changing the uncertainty. The possible outcomes of individual experiments and the statistics of sets of experiments come from the original entangled state through $f$.

The non-local action occurs for each individual experiment, so that after the detection on the right, the photon moving to the left has gone from not having a definite state to having
a definite state. The uncertainty changes because the experiment dictates that we compute it with a conditional probability. Only those states of the particle \( L \) are considered which are associated with the triggering of the right detector and this conditioning depends on the nature of the detector through \( p_R \) and \( T_M \) [10].

The fact that measurements do not necessarily induce uncontrolled uncertainty in the sense of Heisenberg is by now well-known from the discussion of the quantum erasure [11].

IV. MEASUREMENT OF THREE PARTICLE STATES

We shall consider a three particle state that is the product of an entangled two particle state and an independent one particle state. This type of state is discussed in the original teleportation paper [3]. In that case, the measurement is performed on two of the particles and a single particle state is prepared. Alternatively, by measuring one of the particles in the entangled state, one can prepare a state of the two remaining particles. A problem related to this case has been discussed by Horne [12] in connection with measuring one particle in a four particle state to produce a GHZ state [13].

The state we will consider is

\[
|\Psi\rangle = |\Psi\rangle_{ab} |\Phi\rangle_c = \sum_{k_a,k_b} f(k_a, k_b) (|k_a e+\rangle_a |k_b e-\rangle_b + |k_a e-\rangle_a |k_b e+\rangle_b) \sum_{k_c} g(k_c) |\phi; k_c\rangle_c \tag{33}
\]

where

\[
|\phi; k_c\rangle_c = \alpha_+ |k_c e+\rangle_c + \alpha_- |k_c e-\rangle_c \tag{34}
\]

is a normalized plane wave state. The two particle entangled state is not the most general such state, but is rather a superposition of the plane wave entangled states similar to the one Bohm used in his discussion of the EPR experiment [14].

A. Measurement of the Bell states

For quantum teleportation it is necessary to perform a measurement that projects the state of the particles \( a \) and \( c \) onto the Bell states (in this section lower case \( a \) refers to Alice
and lower case b, to Bob, upper case B, to Bell). To do this it is necessary to define a Bell state detector operator. The four Bell states are defined by

\[ |B; k_1, k_2\rangle_{ab} = \sqrt{\frac{1}{2}} \sum_{\sigma=\pm} \sum_{\mu=\pm} \zeta^{(B)}_{\sigma\mu} |k_1 e_{\sigma}\rangle_a |k_2 e_{\mu}\rangle_b \]  

(35)

where the non-zero elements of the \( \zeta \) are \( \zeta_+^1 = \zeta_-^1 = 1 \), \( \zeta_+^2 = -\zeta_-^2 = 1 \), \( \zeta_+^3 = \zeta_-^3 = 1 \), and \( \zeta_+^4 = -\zeta_-^4 = 1 \), and \( e_+ \) and \( e_- \) are orthogonal polarization vectors. The Bell state detector operator is defined as

\[ E(B) = \sum_{k_1k_2} p(B)(k_1, k_2) e^{-i(k_1+k_2)\tau_B} \sum_{\sigma\mu} \zeta^{(B)}_{\sigma\mu} a(k_1, e_{\sigma}) a(k_2, e_{\mu}). \]  

(36)

The retarded time \( \tau_B = t_B - x_B \), where \( x_B \) is the coordinate normal to the detector and \( t_B \) is the time the detector registers the pair. We have chosen the form of the detector based on a model in which up-conversion is used to detect the Bell states.

Following [3], we rewrite (33) as

\[ |\Psi\rangle = \sum_{k_a k_b k_c} f(k_a, k_b) g(k_c) \sum_B |B; k_a, k_c\rangle_{ac} |\phi(B); k_b\rangle_b, \]  

(37)

where \( |\phi(B); k_b\rangle_b \) is the plane wave state associate with \( B \) in Bob’s laboratory. It is related to (34) by a spin transformation \( \Lambda_B \phi = \phi(B) \), [3].

Now suppose we measure the three particle correlation

\[ C_B = \langle \Psi | E(B)^\dagger E_b E_b E(B) | \Psi \rangle = |A_{bB}|, \]  

(38)

where the amplitude \( A_{bB} \) is given by

\[ A_{bB} = \langle 0 | E_b E(B) | \Psi \rangle = \sum_{k_a k_b k_c} U_B(k_a, k_b, k_c) e^{-i(k_a+k_c)\tau_B} \langle 0 | E_b | \phi(B); k_b \rangle_b, \]  

(39)

and

\[ U_B(k_a, k_b, k_c) = f(k_a, k_b) g(k_c) p(B)(k_a, k_c). \]  

(40)

The procedure is now the same as above, we integrate \( C_B \) over the detection time \( t_B \) of the Bell state detector in Alice’s laboratory, call the result \( \overline{C}_B \). The integration gives a sinc function that depends on the energies of the particles in Alice’s laboratory,
Finally, we express $C_B$ in terms of a density matrix for the particle in Bob’s laboratory,

$$C_B = N \sum_{k,b,k'_{b}} (0|E_{b}|\phi^{(B)}; k_{b})_{b} (0|E_{b}|\phi^{(B)}; k'_{b})_{b} \rho_{B}(k_{b}, k'_{b}),$$

with

$$\rho_{B}(k_{b}, k'_{b}) = \frac{1}{N} \sum_{k_{a},k'_{a},k_{c},k'_{c}} S_{ac} e^{-i(k_{a}+k_{c})T_{B}} U_{B}(k_{a}, k_{b}, k_{c}) e^{i(k'_{a}+k'_{c})\tau_{B}} U_{B}(k'_{a}, k'_{b}, k'_{c}),$$

where, as usual, $N$ is a normalization constant.

We are interested in accurate teleportation, so we want Bob’s state to be a pure state. To this end, we require that $S_{ac}$ in eq. (41) to be approximately equal to one over the range of integration. This entails that

$$(\Delta k_{a} + \Delta k_{c}) T_{m} << 2\pi,$$

where $\Delta k_{a}$ is the width of $f(k_{a}, k_{b})$ in the first variable and $\Delta k_{c}$ is the width of $g(k_{c})$. If (44) is satisfied, when the outcome of Alice’s Bell state measurement is $B$, (42) becomes

$$C_{B} = b\langle \chi^{(B)} | E_{b}^{\dagger} E_{b} | \chi^{(B)} \rangle_{b}.$$

The state produced in the Hilbert space of Bob’s particle is

$$|\chi^{(B)} \rangle_{b} = \sqrt{\frac{1}{N}} \sum_{k_{b}} \left[ \sum_{k_{a},k_{c}} U(k_{a}, k_{b}, k_{c}) e^{i(k_{a}+k_{c})T_{B}} \right] |\phi^{(B)}; k_{b} \rangle_{b}.$$

This is a pure state but, in general, does not have the same spectral properties of the original function, that is, it is not equal to

$$|\Phi_{B} \rangle_{b} = \sum_{k_{b}} g(k_{b}) |\Lambda_{B} \phi; k_{b} \rangle_{b}.$$

For accurate teleportation we require that these be equal up to a phase. This leads to the condition

$$\sum_{k_{c}} \left( \sum_{k_{a}} f(k_{a}, k_{b}) p^{(B)}(k_{a}, k_{c}) e^{i(k_{a}+k_{c})T_{B}} \right) g(k_{c}) = \lambda g(k_{b}).$$
This equation requires that \( g \) be an eigenvector of the operator in brackets. This operator depends on the input entangled state, \( f \), and the nature of the Bell state detector, \( p^{(B)} \). Note that the functional dependence on \( k_b \) appears in \( f \). This indicates that \( f \) limits the class of functions that can be teleported.

If there are approximations such that the operator in (47) is a constant times the identity matrix, any \( g \) consistent with (44) and these approximations can be accurately teleported. As an example of such a case, let \( f \) to be given by eq.(31). In this case (47) becomes

\[
\sum_{k_c} \left( \nu(k_b)p^{(B)}(k_p - k_b, k_c)e^{i(k_p-k_b+k_c)T_B} \right) g(k_c) = \lambda g(k_b).
\]

In addition, if

\[
p^{(B)}(k_a, k_c) = p_0^{(B)} \delta(k_a + k_c - k_0), \tag{48}
\]

then eq. (47) becomes

\[
p^{(B)} \nu(k_b) e^{ik_0 T_B} g(k_0 - k_p + k_b) = \lambda g(k_b).
\]

Finally, take \( k_0 = k_p \). We now can satisfy eq. (47) for the class of \( g(k_b) \) such that \( v(k_b) \) is approximately constant over the domain of \( k_b \) where \( g(k_b) \) is non-zero. The condition \( k_0 = k_p \) is a requirement on the detector function. For an up-conversion model of the Bell state detector, this means that the up-converted photon has the same energy as the pump photon that produced the original entangled state \( |\Psi\rangle_{ab} \) in eq. (33). In practice the approximation made here restricts the class of states that can be accurately teleported to quasimonochromatic states. On the other hand, for given \( f \) and \( p^{(B)} \) the eigenvalue equation (47) may have a richer set of solutions that permit accurate teleportation.

There is an assumption in our discussion that requires further consideration. We have assumed that the spectral functions of the entangled state and the single particle states are the same for each realization of the experiment. Usually this will not be true [15]. To see how this effects the outcome we consider a simple case. Suppose that for each experimental realization, \( f \) is the same and \( g \) has a phase factor that varies from experiment to experiment.
This might be due to the generation of the single particle state at different optical distances from the entangled pair. For the $j$th experiment suppose that

$$g^{(j)}(k) = e^{i\Theta_j(k)}g(k). \quad (49)$$

Now we must average over $j$ to compute the density matrix. This gives

$$\langle \rho_B(k_b, k_b') \rangle = \sum_j \int D\Theta_j(k) p(\Theta_j(k)) \rho^{(j)}_B(k_b, k_b'),$$

where $p(\Theta(k))$ is the probability distribution function for $\Theta(k)$, $\rho^{(j)}_B$ is given by eq.’s (43) and (44) with $g$ replaced by $g^{(j)}$. If $\Theta_j(k)$ is independent of $k$, then $\langle \rho_B \rangle = \rho_B$. On the other hand, suppose $\Theta_j(k)$ is random and that $\langle e^{i(\Theta_j(k) - \Theta_j(k'))} \rangle = \delta_{kk'}$. In this case, Bob’s state is determined by the condition

$$\langle C_B \rangle = N \text{tr} E_b^\dagger E_b \langle \rho^{(B)} \rangle$$

giving

$$\langle \rho^{(B)} \rangle = \frac{1}{N} \sum_{k_b, k_b'} |\phi^{(B)}; k_b \rangle_b \left( \sum_{k_a, k_a', k_c} e^{-i(k_a - k_a') T_B} U(k_a, k_b, k_c) U(k_a', k_b', k_c)^* \right) \langle \phi^{(B)}; k_b' \rangle.$$

In general this expression will not factor, so Bob ends up with a mixed state rather than a pure state.

To overcome this type of random phase disturbance, experimentalists usually produce the entangled state and the state to be teleported coherently [16], [17].

**B. Measurement of a single particle**

Next consider the case in which the state (33) is generated and a single particle is measured. A measurement of particle $a$ will not entangle $b$ and $c$. However, suppose that we mix $b$ and $c$ by passing the pair through a 50-50 beam splitter as illustrated in fig. 3. If $a$ is detected, the remaining pair will be partially entangled. The outgoing pair will have a density matrix of the form $\rho = \rho_{12} + \rho_{11} + \rho_{22}$ corresponding to one photon going to each detector, $\rho_{12}$, or both photons going to the same detector, $\rho_{11}$ and $\rho_{22}$. 
For a beam splitter with equal transmittance and reflectance, the field for detector 1 is

$$\Xi_1 = \frac{1}{\sqrt{2}} \sum_q p_1(q, \mathbf{e}_1) e^{-iqn_1} (ia_b(q, \mathbf{e}_1) + a_c(q, \mathbf{e}_1)) = \frac{1}{\sqrt{2}}(iE_{1b} + E_{1c}).$$

(50)

The phase factor of $i$ is associated with reflection off the beam splitter. The detector operator $E_2$ is similarly defined. The triple correlation function is composed of three non-interfering terms, one in which the particles $b$ and $c$ go to different detectors and two in which they go to the same detector.

$$C_{123} = \langle \Psi | \Xi_1^\dagger \Xi_2^\dagger E_3^\dagger E_3 \Xi_1 \Xi_2 | \Psi \rangle = |A_{12}|^2 + |A_{11}|^2 + |A_{22}|^2,$$

(51)

where

$$A_{12} = \frac{1}{2} \left( \langle 0 | E_3 E_2 | E_{1b} | 0 \rangle - \langle 0 | E_3 E_2 | E_{1c} | 0 \rangle \right)$$

(52)

$$A_{11} = \frac{i}{2} \langle 0 | E_3 E_2 | E_{1b} | \Psi \rangle$$

(53)

$$A_{22} = \frac{i}{2} \langle 0 | E_3 E_2 | E_{1c} | \Psi \rangle.$$  

(54)

The notation $E_{1b}$ means that the operator defined in (7) contains the destruction operator acting on the photon in the $b$ mode. From the point of view discussed in this paper, we must keep all these terms in order to specify the state prepared when detector 3 registers a count. In many discussions, the amplitude for both particles going to the same detector is dropped on the grounds that only the coincidences of detectors 1 and 2 are registered. It is then justified to argue that only $A_{12}$ is observed.

As shown in appendix 2,

$$A_{12} = \frac{\sqrt{N}}{2} \langle 0 | E_1 | E_2 | \chi \rangle_{12}$$

(55)

where

$$|\chi\rangle_{12} = \frac{1}{\sqrt{N}} (|\Phi'\rangle_1 |\Phi\rangle_2 + |\Phi\rangle_1 |\Phi'\rangle_2).$$

(56)

This is an entangled state composed of single particle states that are superpositions of the plane wave states (34) and (B2).
\[ |\Phi\rangle = \sum_k g(k)|\phi; k\rangle, \]
\[ |\Phi'\rangle = \sum_k \gamma(k)|\phi'; k\rangle, \]  
\[ (57) \]

where
\[ \gamma(k) = \sum_{k_3} p(k_3, e_3)e^{-ik_3\tau_3}f(k_3, k). \]

The state \(|\Phi\rangle\) is just the original input single particle state, \(|\phi, 1\rangle\).

In an identical way
\[ A_{11} = \frac{1}{2\sqrt{N}} \langle 0|E_1E_1|\chi\rangle_{11}, \]

with
\[ |\chi\rangle_{11} = \sqrt{N} \sum_{kk'} |\phi; k\rangle_1|\phi'; k'\rangle_1 g(k)\gamma(k'). \]

A similar result holds for \(A_{22}\).

We now repeat the calculation made in the first section of the paper. This case is much more complicated. We get several terms
\[ C_{123} = C_{12} + C_{11} + C_{22} \]
\[ C_{12} = \int_{T_3 - \frac{T_1}{2}}^{T_3 + \frac{T_1}{2}} d\tau_3 |A_{12}|^2 \]
\[ C_{2z} = \frac{1}{4} \int_{T_3 - \frac{T_1}{2}}^{T_3 + \frac{T_1}{2}} d\tau_3 |\langle 0|E_zE_z|\chi\rangle_{zz}|^2, \quad z = 1, 2. \]

More explicitly,
\[ C_{12} = W_1 + W_2 + 2 \Re W_3 = Ntr E_1^\dagger E_2^\dagger E_1 E_2 \rho_{12}, \]
\[ C_{mm}^{(s)} = Ntr E_m^\dagger E_m^\dagger E_mE_z \rho_{mm}, \quad m = 1, 2 \]

\(W_1\) and \(W_2\) come from the squares of the two terms in \((52)\), and \(W_3\) is the interference term between these two amplitudes. After the integration over \(\tau_3\), the parameter that determine the nature of the unmeasured pair is \(\theta = T_m/T_k\) where \(T_k = 1/\Delta k\), \(\Delta k\) is the maximum
(with respect to $k_3$) width of $p(k_3, e_3)f(k_3, k)$. If this is much less than 1, we are in the short trigger time limit and

$$\rho_{12} = |\chi\rangle_{12}\langle\chi|,$$

The overlap between the two terms is $|\langle\Omega'|\Omega\rangle|^2$. In order for this to reach a maximum it is necessary that $|\sum_k g(k)\gamma^*(k)|^2$ be a maximum or $g(k) = \lambda\gamma(k)$ for a constant $\lambda$. This places a condition on the detector function $p$ for each choice of $f$ and $g$. This case is similar to that following (47).

In this discussion, it has been assumed that the photons $b$ and $c$ are not recombined after they past the first beam splitter. If this were not the case, then the state after the beam splitter is actually a superposition of $|\chi\rangle_{12}, |\chi\rangle_{11}$, and $|\chi\rangle_{22}$. Furthermore, as in the case discussed in the previous section, if we let $g(k)$ contain a random phase, then in general the state produced after the beam splitter will be a mixed state.

**V. CONCLUSION**

If we have a set of $N$ entangled particles, the subsystems of the entangled states are not in any definite state. The effect of measuring a subset of the $M$ particles is to produce a state of $N-M$ particles. The precise nature of this state depends on the initial entangled state and the nature of the measurement. In particular, there is a time scale set by the initial entangled state and the subsystem measured such that if the duration of the measurement is long on this time scale, then the state of the $N-M$ particles prepared will be a mixed state. If the duration is short, then the state prepared is a pure state. One way to ensure that the latter case holds is to place filters in front of the measuring devices such that a definite state of the measured subsystem is projected out by the detectors. In practice, for photons, this is done using narrow spectral filters.

We have shown that accurate quantum teleportation can not be done for arbitrary states and found an integral equation that the state spectral amplitude must satisfy. This condition
shows how the teleportation of states allows local measurements in Alice’s laboratory to determine a complicated state in Bob’s laboratory. In particular it shows that the spacial information must already be present in the entangled state.

We have also seen how a measurement of one particle from an entangled pair can lead to a partially entangled state of an independent particle and the second particle form the pair. This is done by mixing the unmeasured pair on a beam splitter. However, the entangled pair that is produced is not composed of identical states in the two outputs. For this to occur, it is again necessary that the independent state and the entangled state have spectral amplitudes that are related.

VI. ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF THE FACTORIZATION OF THE POVM

For a pure state \( tr\rho_2^2 = 1 \). Using eq.( this condition becomes

\[
1 = \frac{1}{N^2} \sum_{aa'} |1\langle \phi_a | E | \phi_{a'} \rangle_1|^2 |c_a|^2 |c_{a'}|^2 \\
\leq \left( \frac{1}{N} \sum_a 1\langle \phi_a | E | \phi_{a} \rangle_1 |c_a|^2 \right)^2 = 1,
\]

since

\[
|1\langle \phi_a | E | \phi_{a'} \rangle_1|^2 \leq 1\langle \phi_a | E | \phi_{a} \rangle_1 1\langle \phi_{a'} | E | \phi_{a'} \rangle_1.
\]

The last equality is just the normalization of \( \rho_2 \). The Schwarz inequality becomes an equality if, and only if,
\[ E|\phi_{a'}\rangle_1 = z_{a'a}E|\phi_a\rangle_1 \] (A1)

for a constant \( z_{a'a} \). By taking the inner product of this equation with first with \(|\phi_a\rangle_1\) and then with \(|\phi_{a'}\rangle_1\) that for each \( a \) and \( a' \)

\[ 1\langle\phi_a|E|\phi_{a'}\rangle_1 = e^{i\theta_{aa'}}\sqrt{1\langle\phi_a|E|\phi_a\rangle_1}\sqrt{1\langle\phi_{a'}|E|\phi_{a'}\rangle_1} \]

where \( \theta_{aa} = 0 \) for all \( a \), since \( E \geq 0 \).

We now have

\[ z_{a'a} = e^{i\theta_{aa'}}\sqrt{1\langle\phi_{a'}|E|\phi_{a'}\rangle_1}/\sqrt{1\langle\phi_a|E|\phi_a\rangle_1} \]

Finally taking the inner product of \(|\phi_b\rangle_1\) with \( A_1 \) we can show that \( \theta_{ba'} = \theta_{aa'} + \theta_{ba} \) from which it follows that \( \theta_{aa'} = \xi_a - \xi_{a'} \).

APPENDIX B: CALCULATION OF OUTPUT STATE FROM BEAM SPLITTER

The Bell state given in (33) corresponds to \( B = 3 \) so the first term in \( A_{12} \) is given by

\[ \langle 0|E_3 E_2 E_1 \Psi \rangle = \sum_{k_1k_2k_3} g(k_1)f(k_3, k_2)\langle 0|E_1|\phi; k_1\rangle_1\langle 0|E_3 E_2|3; k_3, k_2\rangle_3, \] (B1)

where the index 2 refers to states after the beam splitter, the index 3 = \( a \), as shown in the fig. 3, and

\[ \langle 0|E_3 E_2|3; k_3, k_2\rangle_3 = p(k_3, e_3)e^{-ik_3\tau_3}\langle 0|E_2|\phi'; k_2\rangle, \]

with

\[ |\phi'; k_2\rangle = |k_2e_-\rangle_2d(e_3, e_+) + |k_2e_+\rangle_2d(e_3, e_-). \] (B2)

The operators \( E_1 \) and \( E_2 \) are of the form given in (3). The second term in \( A_{12} \) is

\[ \langle 0|E_3 E_2 E_1 \Psi \rangle = \sum_{k_1k_2k_3} g(k_2)f(k_3, k_1)\langle 0|E_3 E_1|3; k_3, k_1\rangle_3\langle 0|E_2|\phi; k_2\rangle_2, \] (B3)

where

\[ \langle 0|E_3 E_1|3; k_3, k_1\rangle_3 = p(k_3, e_3)e^{-ik_3\tau_3}\langle 0|E_1|\phi'; k_1\rangle_1. \]

Equation (55) now follows.
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[10] The single slit experiment analyzed by Heisenberg can be looked at in a similar fashion in which the slit makes a selection of possible outcomes. Before reaching the slit the state of the system is a plane wave $e^{i\omega z/c}$. This can be expressed as $e^{i\omega z/c} \sum_n f_{na}(y)$ where $f_{na}(y) = 1$ for $|y - na| < a/2$ and vanishes elsewhere. The slit then projects out the term $f_0(y)$ with the consequent introduction of the uncertainty in the $y$-component of momentum. I believe, Popper would argue that what we are dealing with is a statistical distribution which says nothing about the properties of individual particles. However, we
know from the study of electromagnetic theory that we can explain single slit diffraction as a scattering involving currents in the wall of the screen and the field in the slit. In this more complete theory, the near field depends on the nature of the material composing the screen, but, remarkably, the field a few wave lengths beyond the slit primarily depends on the geometry of the slit. The same must be true for the case of the diffraction of massive particles. In this regard, Heisenberg was correct, for each individual particle such physical effects are present and place physical limitations on how well we can determine the values of conjugate variables. Note that Bohm’s theory, which ascribes definite values the position and momentum of each particle, does not allow us to measure the state of a single particle any more precisely than quantum theory does. I do agree with Popper that one should not call the passage of a particle through a slit as a measurement but rather a selection.

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FIGURES

FIG. 1. Illustration of the two cases $\theta >> 1$ and $\theta << 1$.

FIG. 2. The vertical lines are the world lines for the source, S, and detector, D. The thick lines represent the limits of the signal. In c we illustrate the effect of the filter, F, in spreading the signal.

FIG. 3. One member, a, of the Bell state is detected at the detector D3. The other particle, b, goes to the beamsplitter where it is mixed with the single particle c.
$$\Delta k \approx \frac{2\pi}{T_m}$$
a. $\theta \ll 1$

b. $\theta \gg 1$

c. $T_M \gg T_p$
