We study a class of weakly identifiable location-scale mixture models for which the maximum likelihood estimates based on \( n \) i.i.d. samples are known to have lower accuracy than the classical \( n^{-\frac{1}{2}} \) error. We investigate whether the Expectation-Maximization (EM) algorithm also converges slowly for these models. We provide a rigorous characterization of EM for fitting a weakly identifiable Gaussian mixture in a univariate setting where we prove that the EM algorithm converges in order \( n^{\frac{3}{4}} \) steps and returns estimates that are at a Euclidean distance of order \( n^{-\frac{1}{8}} \) and \( n^{-\frac{1}{4}} \) from the true location and scale parameter respectively. Establishing the slow rates in the univariate setting requires a novel localization argument with two stages, with each stage involving an epoch-based argument applied to a different surrogate EM operator at the population level. We demonstrate several multivariate (\( d \geq 2 \)) examples that exhibit the same slow rates as the univariate case. We also prove slow statistical rates in higher dimensions in a special case, when the fitted covariance is constrained to be a multiple of the identity.

1 Introduction

Gaussian mixture models [20] have been used widely to model heterogeneous data in many applications arising from physical and the biological sciences. In several scenarios, the data has a large number of sub-populations and the mixture components in the data may not be well-separated. In such settings, estimating the true number of components may be difficult, so that one may end up fitting a mixture model with a number of components larger than that present in the data. Such mixture fits, referred to as over-specified mixture distributions, are commonly used by practitioners in order to deal with uncertainty in the number of components in the data [22, 11]. However, a deficiency of such models is that they are singular, meaning that their Fisher information matrices are degenerate. Given the popularity of over-specified models in practice, it is important to understand how methods for parameter estimation, including maximum likelihood and the EM algorithm, behave when applied to such models.

1.1 Background and past work

In the context of singular mixture models, an important distinction is between those that are strongly versus weakly identifiable. Chen [4] studied the class of strongly identifiable models in which, while the Fisher information matrix may be degenerate at a point, and it is not degenerate over a larger set. Studying over-specified Gaussian mixtures with known scale parameters, he showed that the accuracy of the MLE for the unknown location parameter is of the order \( n^{-\frac{1}{4}} \), which should be contrasted with the classical \( n^{-\frac{1}{2}} \) rate achieved in regular settings. A line of follow-up work has extended this type of analysis to other types of strongly identifiable mixture models; see the papers [15, 22, 19, 12] as well as the references therein for more details.

A more challenging class of mixture models are those that are only weakly identifiable, meaning that the Fisher information is degenerate over some larger set. This stronger form of singularity arises, for instance, when the scale parameter in an over-specified Gaussian mixture is also unknown [3, 5]. Ho et al. [13] characterized the behavior of MLE for a class of weakly identifiable models. They showed that the convergence rates of MLE in these models could be very slow, with the
precise rates determined by algebraic relations among the partial derivatives. However, this past work has not addressed the computational complexity of computing the MLE in a weakly identifiable model.

The focus of this paper is the intersection of statistical and computational issues associated with fitting the parameters of weakly identifiable mixture models. In particular, we study the expectation-maximization (EM) algorithm [8, 27, 21], which is the most popular algorithm for computing (approximate) MLEs in the mixture models. It is an instance of a minorization-maximization algorithm, in which at each step, a suitably chosen lower bound of the log-likelihood is maximized. There is now a lengthy line of work on the behavior of EM when applied to regular models. The classical papers [27, 23, 6] establish the asymptotic convergence of EM to a local maximum of the log-likelihood function for a general class of incomplete data models. Other papers [16, 29, 18] characterized the rate of convergence of EM for regular Gaussian mixtures. More recent years have witnessed a flurry of work on the behavior of EM for various kinds of regular mixture models [1, 26, 31, 28, 7, 30, 10, 2]; as a consequence, our understanding of EM in such cases is now relatively mature. More precisely, it is known that for Gaussian mixtures, EM converges in \(\mathcal{O}(\log(n/d))\)-steps to parameter estimates that lie within Euclidean distance \(\mathcal{O}((d/n)^{1/2})\) of the true location parameters, assuming minimal separation between the mixture components.

In our recent work [9], we studied the behavior of EM for fitting a class of non-regular mixture models, namely those in which the Fisher information is degenerate at a point, but the model remains strongly identifiable. One such class of models are Gaussian location mixtures with known scale parameters that are over-specified, meaning that the number of components in the mixture-fit exceeds the number of components in the data generating distribution. For such non-regular but strongly identifiable mixture models, they [9] showed that the EM algorithm takes \(\mathcal{O}((n/d)^{1/3})\) steps to converge to a Euclidean ball of radius \(\mathcal{O}((d/n)^{1/4})\) around the true location parameter. Recall that for such models, the MLE is known to lie at a distance \(\mathcal{O}(n^{-1/2})\) from the true parameter [4], so that even though its convergence rate as an optimization algorithm is slow; the EM algorithm nonetheless produces a solution with a statistical error of the same order as the MLE. This past work does not consider the more realistic setting in which both the location and scale parameters are unknown, and the EM algorithm is used to fit both simultaneously. Indeed, as mentioned earlier, such models may become weakly identifiable due to algebraic relations among the partial derivatives [5]. Thus, analyzing EM in the case of weakly identifiable mixtures is challenging for two reasons: the weak separation between the mixture components, and the algebraic interdependence of the partial derivatives of the log-likelihood. The main contributions of this work are (a) to highlight the dramatic differences in the convergence behavior of the EM algorithm, depending on the structure of the fitted model relative to the data-generating distribution; and (b) to analyze the EM algorithm under a few specific yet representative settings of weakly identifiable models, giving a precise analytical characterization of its convergence behavior.

1.2 Some illustrative examples

Before proceeding further, we summarize a few common aspects of the numerical experiments and the associated figures presented in the paper. Computations at the population-level were done via numerical integration on a sufficiently fine grid. For EM with finite sample size \(n\), we track its performance for several values of \(n \in \{100, 200, 400, \ldots, \}\) and report the quantity \(\hat{m}_e + 2\hat{s}_e\) on the y-axis, where \(\hat{m}_e\) and \(\hat{s}_e\), respectively, denote the mean and standard deviation across the experiments for the metric under consideration (as a function of \(n\) on the x-axis, e.g., Wasserstein error for parameter estimation in Figure 1. The stopping criteria for sample EM were: (a) the change in the iterates was small enough (< .001/n), or (b) the number of iterations was too large (greater than 100,000); criteria (a) led to convergence in most experiments. Furthermore, whenever we provide a slope, it is the slope for the least-squares fit on the log-log scale for the quantity on y-axis when fitted with the quantity reported on the x-axis. For instance, in Figure 1(a), we plot the Wasserstein error between the estimated mixture and the true mixture on the y-axis value versus the sample size n on the x-axis and also provide the slopes for the least-squares fit. In particular, in panel (a) the green dot-dashed line with the legend ‘slope= −0.09’ denotes the least-squares fit and the respective slope for the logarithmic error \(\log W_1(\mathcal{G}_e, \mathcal{G}_{\hat{n}})\) (green diamonds) with respect to the logarithmic sample size \(\log n\) when the number of components in the fitted mixture is 3. Such a result implies that the error \(W_1(\mathcal{G}_e, \mathcal{G}_{\hat{n}})\) scales as \(n^{-0.09}\) with the sample size \(n\) in our experiments.

To begin with, we consider the simplest case of over-specification with Gaussian mixture models—when the true data is generated from a zero-mean standard Gaussian distribution in \(d\) dimensions and EM is used to fit a general multi-component mixture model with different number of mixtures. (We note that fitting by one mixture model is simply a Gaussian fit.) Given the estimates for the mixture weights, location and scale parameters returned by EM, we compute the
first order Wasserstein distance\(^1\) between the true and estimated parameters. Results for \(d \in \{1, 2, 4\}\) and for various amount of over-specification are plotted in Figure 1. From these results, we notice that the decay in statistical error is \(n^{-1/2}\) when the fitted number of components is well-specified and equal to the true number of components but has a much slower rate whenever the number of fitted components is two or more. Moreover, in Section 4 (see Figure 3) we show that such a phenomenon occurs more generally in mixture models.

While a rigorous theoretical analysis of EM under over-specification in general mixture models is desirable, it remains beyond the scope of this paper. Instead, here we provide a full characterization of EM when it is used to fit the following class of models to the data drawn from standard Gaussian \(\mathcal{N}(0, I_d)\):

\[
\mathcal{G}_{\text{symm}}((\theta, \sigma^2)) = \frac{1}{2} \mathcal{N}(\theta, \sigma^2 I_d) + \frac{1}{2} \mathcal{N}(-\theta, \sigma^2 I_d). \tag{1}
\]

In particular, in this symmetric fit, we fix the mixture weights to be equal to \(\frac{1}{2}\) and require that the two components have the same scale parameter. Given the estimates \(\hat{\theta}, \hat{\sigma}\), the Wasserstein error (see equation (58) in Appendix D) in this case can be simplified as \(\|\hat{\theta}\|_2 + \sqrt{d(\sigma^2 - 1)}\). In our results to be stated later, we show that the two terms are of the same order (equations (6), (21)) and hence we primarily focus on the error \(\|\hat{\theta} - \theta\|_2\) going forward to simplify the exposition. We consider our set-up as a simple yet first step towards understanding the behavior of EM in over-specified mixtures when both location and scale parameter are unknown. In our prior work [9], we studied the slow down of EM with over-specified mixtures for estimating only the location parameter, but they assumed that the scale parameter was known and fixed. Here a more general setting is considered.

We now elaborate the choice of our class of models (1) that may appear a bit restrictive at first glance. This model turns out to be the simplest example of a weakly identifiable model in \(d = 1\). Let \(\phi\) denote the density of a Gaussian distribution with mean \(\theta\) and variance \(\sigma^2\), then we have

\[
\frac{\partial^2 \phi}{\partial \theta^2}(x; \theta, \sigma^2) = 2 \frac{\partial \phi}{\partial \sigma^2}(x; \theta, \sigma^2), \tag{2}
\]

valid for all \(x \in \mathbb{R}, \theta \in \mathbb{R}\) and \(\sigma > 0\). As alluded to earlier, models with algebraic dependence between partial derivatives lead to weak identifiability and slow statistical estimation with MLE. However, in the multivariate setting when the same parameter \(\sigma\) is shared across multiple dimensions, this algebraic relation does not hold and the model is strongly identifiable (since the Fisher information matrix is singular at \((\theta^*, \sigma^*) := (0, 1)\)). For this reason, we believe that analysis of EM for the special fit (1) may provide important insight for more general over-specified weakly identifiable models.

**Population EM:** Given \(n\) samples from a \(d\)-dimensional standard Gaussian distribution, the sample EM algorithm for location and scale parameters generates a sequence of the form \(\theta^{t+1} = M_{n,d}(\theta^t)\) and \(\sigma^{t+1}\), which is some function of \(\|\theta^{t+1}\|_2^2\); see equation (3c) for a precise definition. An abstract counterpart of the sample EM algorithm—not useful in practice but rather for theoretical understanding—is the population EM algorithm \(\overline{M}_d\), obtained in the limit of an infinite sample size (cf. equation (11b)).

In practice, running the sample EM algorithm yields an estimate \(\hat{\theta}_{n,d}\) of the unknown location parameter \(\theta^*\).
The main contribution of this paper is to provide a novel method for the operator $\theta$ based on $n$ samples. Note that the simulations indicate two distinct error scaling for $d = 1$ and $d > 1$. (b) Convergence behavior of the population-like EM sequence $\theta^{t+1} = M_d(\theta^t)$ (11b) in dimensions $d = 1$ and 2. The rate of convergence in dimension $d = 1$ is significantly slower compared to the rate in dimension $d = 2$. Overall, both the plots provide strong empirical evidence towards two distinct behaviors of the EM algorithm for dimension $d = 1$ and dimensions $d > 1$. See the Theorems 1-2, and Lemmas 1 and 3 for a theoretical justification of trends in panels (a) and (b) respectively.

Panel (a) in Figure 2 shows the scaling of the statistical estimation error $\|\hat{\theta}_{n,d} - \theta^*\|_2$ of this sample EM estimate versus the sample size $n$ on a log-log scale. The three curves correspond to dimensions $d \in \{1, 2, 16\}$, along with least-squares fits (on the log-log scale) to the data. In panel (b), we plot the Euclidean norm $\|\theta^t\|$ of the population EM iterate\(^2\) versus the iteration number $t$, with solid red line corresponding to $d = 1$ and the dash-dotted green line corresponding to $d = 2$. Observe that the algorithm converges far more slowly in the univariate case than the multivariate case. The theory to follow in this paper (see Theorems 1, 2 and Lemmas 1 and 3) provides explicit predictions for the rate at which different quantities plotted in Figure 2 should decay. We now summarize our theoretical results that are also consistent with the trends observed in Figure 2.

1.3 Our contributions

The main contribution of this paper is to provide a precise analytical characterization of the behavior of the EM algorithm for certain special cases of over-specified mixture models (1).

Univariate over-specified Gaussian mixtures: In the univariate setting ($d = 1$) of $G_{symm}$ in (1), we prove that the EM estimate has statistical estimation error of the order $n^{-\frac{1}{4}}$ and $n^{-\frac{3}{4}}$ after order $n^2$ steps for the location and scale parameters respectively. In particular, Theorem 1 provides a theoretical justification for the slow rate observed in Figure 2 (a) for $d = 1$ (red dotted line with star marks). Proving these rates requires a novel analysis, and herein lies the main technical contribution of our paper. Indeed, we show that all the analysis techniques introduced in past work on EM, including work on both the regular [1] and strongly identifiable cases [9], lead to sub-optimal rates. Our novel method is a two-stage approach that makes use of two different population level EM operators. Moreover, we also prove a matching lower bound (see Appendix B) which ensures that the upper bound of order $n^{-\frac{1}{4}}$ for the statistical error of sample EM from Theorem 1 is tight up to constant factors.

Multivariate setting with shared covariance: Given the technical challenges even in the simple univariate case, the symmetric spherical fit $G_{symm}$ in (1) serves as a special case for the multivariate setting $d \geq 2$. In this case, we establish that the sharing of scale parameter proves beneficial in the convergence of EM. Theorem 2 shows that sample EM algorithm takes $O((n/d)^{1/2})$ steps in order to converge to estimates, of the location and scale parameters respectively, that lie within distances $O((d/n)^{1/4})$ and $O(nd)^{-\frac{1}{2}}$ of the true location and scale parameters, respectively.

General multivariate setting: We want to remind the readers that we expect the Wasserstein error to scale much slowly than $n^{-\frac{1}{4}}$ (the rate mentioned in the previous paragraph) while estimating over-specified mixtures with no shared covariance. When the fitted
variance parameters are not shared across dimensions our simulations under general multi-component fits in Figure 1 demonstrate a much slower convergence of EM (for which a rigorous justification is beyond the scope of this paper).

Notation: In the paper, the expressions \(a_n \gtrless b_n\) or \(a_n \leq O(b_n)\) will be used to denote \(a_n \leq cb_n\) for some positive universal constant \(c\) that does not change with \(n\). Additionally, we write \(a_n \asymp b_n\) if both \(a_n \gtrless b_n\) and \(b_n \gtrless a_n\) hold. Furthermore, we denote \([n]\) as the set \(\{1, \ldots, n\}\) for any \(n \geq 1\). We define \(\lceil x\rceil\) as the smallest integer greater than or equal to \(x\) for any \(x \in \mathbb{R}\). The notation \(\|x\|_2^2\) stands for the \(\ell_2\) norm of vector \(x \in \mathbb{R}^d\). We use \(c, c', c_1\) etc. to denote some universal constants independent of problem parameters (which might change in value each time they appear).

1.4 EM updates for symmetric fit \(G_{\text{symm}}\)

The EM updates for Gaussian mixture models are standard, so we simply state them here. In terms of the shorthand notation \(\eta := (\theta, \sigma)\), the E-step in the EM algorithm involves computing the function

\[
Q_n(\eta'; \eta) := \frac{1}{n} \sum_{i=1}^{n} \left[ w_{\eta, \sigma}(X_i) \log \left( \phi(X_i; \theta', (\sigma')^2 I_d) \right) \right] + (1 - w_{\eta, \sigma}(X_i)) \log \left( \phi(X_i; -\theta', (\sigma')^2 I_d) \right),
\]

where the weight function is given by \(w_{\eta, \sigma}(x) = \left(1 + e^{-2\sigma^2 x^2}\right)^{-1}\). The M-step involves maximizing the \(Q_n\)-function over the pair \((\theta', \sigma')\) with \(\eta\) fixed, which yields

\[
\theta' = \frac{1}{n} \sum_{i=1}^{n} (2 w_{\eta, \sigma}(X_i) - 1) X_i, \quad \text{and} \quad (3a)
\]

\[
(\sigma')^2 = \frac{1}{d} \left( \frac{\sum_{i=1}^{n} \|X_i\|_2^2}{n} - \|\theta'\|_2^2 \right). \quad (3b)
\]

Doing some straightforward algebra, the EM updates \((\theta_n^{t+1}, \sigma_n^{t+1})\) can be succinctly defined as

\[
\theta_n^{t+1} = \frac{1}{n} \sum_{i=1}^{n} \tanh \left( \frac{X_i^T \theta_n^t}{\sum_{i=1}^{n} \|X_i\|_2^2/(nd) - \|\theta_n^t\|_2^2/d} \right) \quad =: M_{n,d}(\theta_n^t), \quad (3c)
\]

\[
\sigma_n^{t+1} = \sum_{i=1}^{n} \|X_i\|_2^2/(nd) - \|\theta_n^{t+1}\|_2^2/d.
\]

For simplicity in presentation, we refer to the operator \(M_{n,d}\) as the sample EM operator.

Organization: We present our main results in Section 2, with Section 2.1 devoted to the univariate case, Section 2.2 to the multivariate case and Section 2.3 to the simulations with more general mixtures. Our proof ideas are summarized in Section 3 and we conclude with a discussion in Section 4. The detailed proofs of all our results are deferred to the Appendices.

2 Main results

In this section, we provide our main results for the behavior of EM with the singular (symmetric) mixtures fit \(G_{\text{symm}}\) (1). Theorem 1 discusses the result for the univariate case, Theorem 2 discusses the result for multivariate case. In Section 2.3 we discuss some simulated experiments for general multivariate location-scale Gaussian mixtures.

2.1 Results for the univariate case

As discussed before, due to the relationship between the location and scale parameter, namely the updates (3c), we conclude that the EM estimate for the univariate case, Theorem 2 discusses the results for multivariate Gaussian mixtures, given \(n\) samples \(\{X_i, i \in [n]\}\), the sample EM operator is given by

\[
M_{n,1}(\theta) := \frac{1}{n} \sum_{i=1}^{n} X_i \tanh \left( \frac{X_i \theta}{\sum_{j=1}^{n} X_j^2/n - \theta^2} \right). \quad (4)
\]

We now state our first main result that characterizes the guarantees for EM under the univariate setting. Let \(I_\beta\) denote the interval \([c \exp(-\frac{2\beta}{n} + \beta), 1/10]\) where \(c\) is a positive universal constant.

Theorem 1. Fix \(\delta \in (0, 1), \beta \in (0, 1/8]\), and let \(X_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)\) for \(i = 1, \ldots, n\) such that \(n \geq \log \frac{16}{\log(1/\beta)}\). Then for any initialization \(\theta_n^0\) that satisfies \(|\theta_n^0| \in I_\beta\), the sample EM sequence \(\theta_n^{t+1} = M_{n,1}(\theta_n^t)\), satisfies

\[
|\theta_n^t - \theta^*| \leq C_0 \frac{1}{n^{1/2 - \beta}} \log^{5/4} \left(\frac{10n \log(8/\beta)}{\delta}\right), \quad (5)
\]

for all \(t \geq C_0 n^{2\beta - 6\beta - 1/2} \log n \log \frac{1}{\beta}\) with probability at least \(1 - \delta\).

See Appendix A.1 for the proof.

The bound (5) shows that with high probability after \(O(n^{3/4})\) steps the sample EM iterates converge to a ball around \(\theta^*\) whose radius is arbitrarily close to \(n^{-1/2}\). Moreover, as a direct consequence of the relation (3b), we conclude that the EM estimate for the scale parameter is of order \(n^{-\beta}\) with high probability:

\[
|\sigma_n^t - (\sigma^*)^2| = \left| \frac{\sum_{i=1}^{n} X_i^2}{n} - (\theta_n^t - \theta^*)^2 - (\sigma^*)^2 \right| \lesssim n^{-\beta} + n^{-1/2} = O(n^{-1/2}). \quad (6)
\]

where we have used the standard chi-squared concentration for the sum \(\sum_{i=1}^{n} X_i^2/n\).

Matching lower bound: In Appendix B, we prove a matching lower bound and thereby conclude that the
upper bound of order \( n^{-\frac{1}{2}} \) for the statistical error of sample EM from Theorem 1 is tight up to constant factors. In Section 2.3, we provide further evidence (cf. Figure 3) that the slow statistical rates of EM with location parameter that we derived in Theorem 1 might appear in more general settings of location-scale Gaussian mixtures as well.

### 2.2 Results for the multivariate case

Analyzing the EM updates for higher dimensions turns out to be challenging. However, for the symmetric fit in higher dimensions given by

\[
G_{symm}((\theta, \sigma^2)) = \frac{1}{2} N(\theta, \sigma^2 I_d) + \frac{1}{2} N(-\theta, \sigma^2 I_d),
\]

the sample EM operator \( M_{n,d}(\theta) \) has a closed form as already noted in the updates (3b) and (3c). Note that for the fit (7), we have assumed the same scale parameter for all dimensions. Such a fit is over-specified for data drawn from Gaussian distribution \( N(0, I_d) \). We now show that the sharing of scale parameter in the model fit across dimensions (7), leads to a faster convergence of EM in \( d \geq 2 \)—both in terms of number of steps and the final statistical accuracy. In the following result, we denote \( I_3 := [5 \left( \frac{d}{n} \right)^{1+\beta}; \frac{d}{n}] \).

**Theorem 2.** Fix \( \delta \in (0, 1) \), \( \beta \in (0, 1/4] \), and let \( X_i \sim \mathcal{N}(0, I_d) \) for \( i = 1, \ldots, n \) such that \( d \geq 2 \) and \( n \gtrsim d \log \frac{1}{\beta} \left( \log \frac{1/\beta}{\delta} \right) \). Then with any starting point \( \theta_n^0 \) such that \( \|\theta_n^0\|_2 \in I_3 \), the sample EM sequence \( \theta_n^{t+1} = M_{n,d}(\theta_n^t) \) satisfies

\[
\|\theta_n^t - \theta^*\|_2 \leq c_1 \left( \frac{d}{n} \log \frac{1/\beta}{\delta} \right)^{\frac{1}{2}-\beta},
\]

for all \( t \geq c_2 \left( \frac{d}{n} \right)^{\frac{1}{2}-2\beta} \log \frac{d}{\beta} \log \frac{1}{\beta} \) with probability at least \( 1 - \delta \).

See Appendix A.2 for the proof.

The results in Theorem 2 show that that the sample EM updates converge to a ball around \( \theta^* = 0 \) with radius arbitrarily close to \( (d/n)^{\frac{1}{2}} \) when \( d \geq 2 \). At first sight, the initialization condition \( \|\theta_n^0\|_2 \leq 1/8 \), assumed in Theorem 2, might seem pretty restrictive but Lemma 6 (in Appendix C.6) shows that for any \( \theta_n^0 \) satisfying \( \|\theta_n^0\|_2 \leq \sqrt{d} \), we have \( M_{n,d}(\theta_n^0) \leq \sqrt{2/\pi} \), with high probability. In light of this result, we may conclude that the initialization condition is Theorem 2 is not overly restrictive.

**Comparison with Theorem 1:** The scaling of order \( n^{-\frac{1}{2}} \) with \( n \) is significantly better than the univariate case \( (n^{-\frac{1}{2}}) \) stated in Theorem 1. We note that this faster statistical rate is a consequence of the sharing of the scale parameter across dimensions, and does not hold when the fit (7) has different variance parameters. Indeed, as we demonstrated in Figure 1, when the fitted components have freely varying scale parameter, the statistical rate slows down (and can be of the order \( n^{-\frac{1}{4}} \) in higher dimensions).

### 2.3 Simulations with general cases

We now present preliminary evidence that the slow statistical rates of EM with location parameter that we derived in Theorem 1 might appear in more general settings. In Figure 3, we plot the statistical error of estimates returned by sample EM when estimating all the parameters (namely weights, location and scale) simultaneously, as a function of sample size \( n \), for the following two cases:

\[
G_{d=1} = \frac{1}{6} N(-5, 1) + \frac{1}{2} N(1, 3) + \frac{1}{3} N(7, 2);
\]

\[
G_{d=2} = \frac{1}{2} N\left( \begin{bmatrix} 0 \\ I_d \end{bmatrix}, \frac{1}{2} N\left( \begin{bmatrix} 7 \\ 5 \\ 2I_d \end{bmatrix}, \frac{1}{3} N\left( \begin{bmatrix} -4 \\ -7 \end{bmatrix}, 3I \right) \right).\]

We plot the results for a \( K_{\text{fit}} \in \{3, 4, 5\} \)-mixture Gaussian model fit. When \( K_{\text{fit}} = 3 \) is true mixture the statistical rate is \( n^{-1/2} \). When it is larger, i.e., \( K_{\text{fit}} \in \{4, 5\} \), the statistical rate of EM is much larger, \( n^{-0.12} \) in panel (a) (for \( K_{\text{fit}} = 5 \)) and \( n^{-0.20} \) in panel (b) (for \( K_{\text{fit}} = 5 \)) of Figure 3. These simulations suggest that the statistical rates slower than \( n^{-\frac{1}{4}} \) and of order \( n^{-\frac{1}{4}} \) may arise in more general settings, and moreover that the rates get slower as the over-specification of the number of mixtures increases. See Section 4 for possible future work in this direction.

### 3 Analysis of EM

Deriving a sharp rate for univariate case (Theorem 1) turns out be pretty challenging and requires a thorough discussion. On the other hand, the multivariate-case considered in the paper (Theorem 2) is relatively easy due to the shared scale parameter given the techniques developed in prior works [1, ?]. See Appendix A.2 for details. We now outline the analysis of the univariate case in which we make use of several novel techniques.

#### 3.1 Proof outline

Our proof makes use of the population-to-sample analysis framework of Balakrishnan et al. [1] albeit with several new ideas. Let \( Y \sim \mathcal{N}(0, 1) \), then the population-level analog of the operator (3c) can be defined in two
The particular choice of the population-like operator \( \tilde{M}_{n,1} \) in equation (11a) was motivated by the previous works [2] with the location-scale Gaussian mixtures. We refer to this operator as the pseudo-population operator since it depends on the samples \( \{X_i, i = 1, \ldots, n\} \) and involves an expectation. Nonetheless, as we show in the sequel, analyzing \( \tilde{M}_{n,1} \) is not enough to derive sharp rates for sample EM in the over-specified setting considered in Theorem 1. A careful inspection reveals that a “better” choice of the population operator is required, which leads us to define the operator \( \overline{M}_1 \) in equation (11b). Unlike the pseudo-population operator \( \tilde{M}_{n,1} \), the operator \( \overline{M}_1 \) is indeed a population operator as it does not depend on samples \( X_1, \ldots, X_n \). Note that, this operator is obtained when we replace the sum \( \sum_{j=1}^{n} X_j^2/n \) in the definition (11a) of the operator \( \tilde{M}_{n,1} \) by its corresponding expectation \( \mathbb{E}[\|X\|^2] = 1 \). For this reason, we also refer to this operator \( \overline{M}_1 \) as the corrected population operator. In the next lemma, we state the properties of the operators defined above (here \( I_{\beta}^{\gamma} \) denotes the interval \( [cn^{-\beta} + \gamma, 1/10] \)).

**Lemma 1.** The operators \( \tilde{M}_{n,1} \) and \( \overline{M}_1 \) satisfy

\[
\begin{align*}
(1 - \frac{3\theta^6}{2}) |\theta| &\leq |\tilde{M}_{n,1}(\theta)| \leq (1 - \frac{\theta^6}{5}) |\theta|, \quad (12a) \\
(1 - \frac{\theta^6}{2}) |\theta| &\leq |\overline{M}_1(\theta)| \leq (1 - \frac{\theta^6}{5}) |\theta|, \quad (12b)
\end{align*}
\]

where bound (12a) holds for all \( |\theta| \in I_{\beta}^{\gamma} \) with high probability\(^3\) and the bound (12b) is deterministic and holds for all \( |\theta| \in [0, 3/m] \). Furthermore, for any fixed \( \delta \in (0,1) \) and any fixed \( r \geq O(n^{-\frac{1}{2}}) \), we have that

\[
\Pr \left[ \sup_{\theta \in \mathbb{B}(0,r)} |M_{n,1}(\theta) - \tilde{M}_{n,1}| \leq cr \sqrt{\frac{\log(1/\delta)}{n}} \right] \geq 1 - \delta. \quad (12c)
\]

On the other hand, for any fixed \( r \leq O(n^{-\frac{1}{2}}) \), we have

\[
\Pr \left[ \sup_{\theta \in \mathbb{B}(0,r)} |M_{n,1}(\theta) - \overline{M}_1(\theta)| \leq c_2r^3 \sqrt{\frac{\log(5n/\delta)}{n}} \right] \geq 1 - \delta. \quad (12d)
\]

See Appendix A.3 for its proof where we also numerically verify the sharpness of the results above (see Figure 4). Lemma 1 establishes that, as \( \theta \to 0 \), both the operators have similar contraction coefficient \( \gamma(\theta) \approx 1 - c\theta^6 \); thereby justifying the rates observed for \( d = 1 \) in Figure 2(b). However, their perturbation bounds are significantly different: while the error \( \sup_{\theta \in \mathbb{B}(0,r)} |M_{n,1}(\theta) - \tilde{M}_{n,1}(\theta)| \)

\(^3\)Since the operator \( \tilde{M}_{n,1} \) depends on the samples \( \{X_j, j \in [n]\} \), only a high probability bound (and not a deterministic one) is possible.
scales linearly with the radius $r$, the deviation error 
\[ \sup_{\theta \in \mathbb{B}(0,r)} |M_{n,1}(\theta) - \tilde{M}_1(\theta)| \] 
has a cubic scaling $r^3$.

**Remark:** A notable difference between the two bounds (12c) and (12d) is the range of radius $r$ over which we prove the validity of the bounds (12c) and (12d). With our tools, we establish that the perturbation bound (12c) for the operator $\tilde{M}_{n,1}$ is valid for any $r \gtrsim n^{-2}$. On the other hand, the corresponding bound (12d) for the operator $\tilde{M}_1$ is valid for any $r \lesssim n^{-3}$. We now elaborate why these different ranges of radii are helpful and make both the operators crucial to the analysis to follow.

### 3.2 A sub-optimal analysis

Using the properties of the operator $\tilde{M}_{n,1}$ from Lemma 1, we now sketch the statistical rates for the sample EM sequence, $\theta_{n+1}^t = M_{n,1}(\theta_n^t)$, that can be obtained using (a) the generic procedure outlined by Balakrishnan et al. [1] and (b) the localization argument introduced in our previous work [9]. As we show, both these arguments end up being sub-optimal as they do not provide us the rate of order $n^{-\frac{1}{2}}$ stated in Theorem 1. We use the notation:

\[ \sup_{|\theta| \leq r} \left| \tilde{M}_{n,1}(\theta) \right| / |\theta| \lesssim 1 - \epsilon^6 =: \gamma(\epsilon). \]

**Sub-optimal rate I:** The eventual radius of convergence obtained using Theorem 5(a) from the paper [1] can be determined by

\[ r / \sqrt{n} \geq \frac{1}{1 - \gamma(\epsilon)} \quad \Rightarrow \quad \epsilon \sim n^{-1/4}, \quad (13a) \]

where $r$ denotes the bound on the initialization radius $|\theta^0|$ but we have tracked dependency only on $n$. This informal computation suggests that the the sample EM iterates for location parameter are bounded by a term of order $n^{-1/4}$. This rate is clearly sub-optimal when compared to the EM rate of order $n^{-\frac{1}{2}}$ from Theorem 1.

**Sub-optimal rate II:** Next we apply the more sophisticated localization argument from the paper [9] in order to obtain a sharper rate. In contrast to the computation (13a), this argument leads to solving the equation

\[ \epsilon \cdot \frac{r / \sqrt{n}}{1 - \gamma(\epsilon)} = \epsilon \Rightarrow \frac{er / \sqrt{n}}{\epsilon^6} = \epsilon \Rightarrow \epsilon \sim n^{-\frac{1}{2}}, \quad (13b) \]

where, as before, we have only tracked dependency on $n$. This calculation allows us to conclude that the EM algorithm converges to an estimate which is at a distance of order $n^{-\frac{1}{2}}$ from the true parameter, which is again sub-optimal compared to the $n^{-\frac{1}{2}}$ rate of EM from Theorem 1.

Indeed both the conclusions above can be made rigorous (See Corollary 1 for a formal statement) to conclude that, with high probability for any $\beta \in (0, \frac{1}{12}]$

\[ |\theta_n^t - \theta^*| \leq O(n^{\frac{1}{6} + \beta}) \text{ for } t \geq O(n^{\frac{1}{2} - 6\beta}). \quad (14) \]

### 3.3 A two-staged analysis for sharp rates

In lieu of the above observations, the proof of the sharp upper bound (5) in Theorem 1 proceeds in two stages. In the first stage, invoking Corollary 1 with $\beta = \frac{1}{12}$, we conclude that with high probability the sample EM iterates converge to a ball of radius at most $r$ after $\sqrt{n}$ steps, where $r \ll n^{-1/16}$. Consequently, the sample EM iterates after $\sqrt{n}$ steps satisfy the assumptions required to invoke the perturbation bounds for the operator $\tilde{M}_1$ from Lemma 1. Thereby, in the second stage of the proof, we apply the $1 - \epsilon^6$ contraction bound (12b) of the operator $\tilde{M}_1$ in conjunction with the cubic perturbation bound (12d). Using localization argument for this stage, we establish that the EM iterates obtain a statistical error of order $n^{-1/8}$ in $O(n^{1/4})$ steps as stated in Theorem 1. See Appendix A.1 for a detailed proof.

### 4 Discussion

In this paper, we established several results characterizing the convergence behavior of EM algorithm for over-specified location-scale Gaussian mixtures. We view our analysis of EM for the symmetric singular Gaussian mixtures as the first step toward a rigorous understanding of EM for a broader class of weakly identifiable mixture models. Such a study would provide a better understanding of the singular models with weak identifiability which do arise in practice since: (a) over-specification is a common phenomenon in fitting mixture models due to weak separation between mixture components, and, (b) the parameters being estimated are often inherently dependent due to the algebraic structures of the class of kernel densities being fitted and the associated partial derivatives. We now discuss a few other directions that can serve as a natural follow-up of our work.

The slow rate of order $n^{-\frac{1}{2}}$ for EM updates with location parameter is in a sense a worst-case guarantee. In the univariate case, for the entire class of two mixture Gaussian fits, MLE exhibits the slowest known statistical rate $n^{-\frac{1}{2}}$ for the settings that we analyzed. More precisely, for certain asymmetric Gaussian mixture fits, the MLE convergence rate for the location parameter is faster than that of the symmetric equal-weighted mixture considered in this paper E.g., for the fit $1/3N(-2\theta, \sigma^2) + 2/3N(\theta, \sigma^2)$ on $N(0, 1)$ data, the MLE converges at the rate $n^{-1/6}$ and $n^{-1/3}$ respectively [14]. It is interesting to understand the effect of
such a geometric structure of the global maxima on the convergence of the EM algorithm.

Our work analyzed over-specified mixtures with a specific structure and only one extra component. As demonstrated above, the statistical rates for EM appear to be slow for general covariance fits and further appear to slow down as the number of over-specified components increases. The convergence rate of the MLE for such over-specified models is known to further deteriorate as a function of the number of extra components. It remains to understand how the EM algorithm responds to these more severe—and practically relevant—instances of over-specification.

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Supplement for “Sharp Analysis of Expectation-Maximization for Weakly Identifiable Models”

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A Proofs of main results

In this section, we present the proofs for our main results while deferring some technical results to the appendices.

A.1 Proof of Theorem 1

Our result makes use of the following corollary (proven in Appendix C.1):

**Corollary 1.** Given constants \( \delta \in (0, 1) \) and \( \beta \in (0, 1/12] \), suppose that we generate the the sample-level EM sequence \( \theta_{n}^{t+1} = M_{n,1}(\theta_{n}^{t}) \) starting from an initialization \( |\theta_{n}^{0}| \in I'_{\beta} \), and using a sample size \( n \) lower bounded as \( n \geq \log^{1/(12\beta)}(\log(1/\beta)/\delta) \). Then for all iterations \( t \geq n^{1/2-6\beta} \log(n) \log(1/\beta) \), we have

\[
|\theta_{n}^{t} - \theta^{*}| \leq c_{1} \left( \frac{1}{n} \log \frac{\log(1/\beta)}{\delta} \right)^{\frac{1}{12}-\beta},
\]  

(15)
with probability at least $1 - \delta$.

Remark: We note that the sub-optimal bound (15) obtained from Corollary 1 is not an artifact of the localization argument and arises due to the definition of the operator operator $\tilde{M}_{n,1}$ (11a). As we have alluded to earlier, indeed a finer analysis with the population EM operator $\bar{M}_1$ is required to prove the rate of $n^{-1/8}$ stated in Theorem 1. However, a key assumption in the further derivation is that the sample EM iterates $\theta_n$ can converge to a ball of radius $r \lesssim n^{-1/16}$ around $\theta^*$ in a finite number of steps, for which Corollary 1 comes in handy.

We now begin with a sketch the two stage-argument, and then provide a rigorous proof for Theorem 1.

A.1.1 Proof sketch

As mentioned earlier, the pseudo-population operator $\tilde{M}_{n,1}$ is not sufficient to achieve the sharp rate of EM iterates under the univariate symmetric Gaussian mixture fit. Therefore, we make use of corrected-population operator $\bar{M}_1$ to get a sharp statistical rate of EM. Our proof for the tight convergence rate of sample EM updates relies on a novel two-stage localization argument that we are going to sketch.

First stage argument: Plugging in $\beta = 1/84$ in Corollary 1, we obtain that for $t \gtrsim \sqrt{n} \log(n)$, with probability at least $1 - \delta$ we have that

$$|\theta_n^t - \theta^*| \leq cn^{-\frac{1}{8}} \log \frac{1}{\delta} \leq n^{-\frac{1}{16}},$$

where the second inequality follows from the large sample condition $n \geq c' \log^8 \frac{\log 84}{\delta}$. All the following claims are made conditional on the event (16).

Second stage argument: In order to keep the presentation of the proof sketch simple, we do not track constant and logarithmic factors in the arguments to follow. In epoch $\ell$, for any iteration $t$ the EM iterates satisfy $\theta_n^t \in [n^{-a_{\ell+1}}, n^{-a_{\ell}}]$ where $a_{\ell+1} > a_{\ell}$ and $a_{\ell} \leq 1/16$. Applying Lemma 1 for such iterations, we find that with high probability

$$|\bar{M}_1(\theta_n^t)| \lesssim (1 - n^{-6a_{\ell+1}}) |\theta_n^t| \quad \text{and} \quad |M_{n,1}(\theta_n^t) - \bar{M}_1(\theta_n^t)| \lesssim \frac{n^{-3a_{\ell}}}{\sqrt{n}}$$

where the first bound follows from the $1 - c\theta_n^t$ contraction bound (12b) and the second bound follows from the cubic-type Rademacher bound (12d). Invoking the basic triangle inequality $T$ times, we obtain that

$$|\theta_n^{t+T}| \overset{(i)}{=} e^{-Tn^{-6a_{\ell+1}}} n^{-a_{\ell}} + \frac{1}{1 - \gamma_{\ell}} \cdot n^{-3a_{\ell}} \overset{(ii)}{\lesssim} \frac{1}{1 - \gamma_{\ell}} \cdot \frac{n^{-3a_{\ell}}}{\sqrt{n}} = n^{6a_{\ell+1} - 3a_{\ell} - 1/2},$$

where in step (ii) we have used the fact that for large enough $T$, the first term is dominated by the second term in the RHS of step (i). To obtain a recursion for the sequence $a_{\ell}$, we set the RHS equal to $n^{-a_{\ell+1}}$. Doing so yields the recursion

$$a_{\ell+1} = \frac{3a_{\ell}}{7} + \frac{1}{14}, \quad \text{where} \quad a_0 = 1/16.$$

Solving for the limit $a_{\ell+1} = a_{\ell} = a_*$, we find that $a_* = 1/8$. Thus, we can conclude that sample EM iterates in the univariate setting converge to a ball of radius $n^{-1/8}$ as claimed in the theorem statement.

A.1.2 Formal proof of sample EM convergence rate

We now turn to providing a formal proof for the preceding arguments.
We now define the high probability event that is crucial for our proof. For any
where
we have that the event (16)
Then, for the event
In order to facilitate the proof argument later, we define the following set
ℓ
For
all the pieces together, we find that under the event
Deferring the proof of Appendix C.2, we now establish the claim (18) conditional on the event
Lemma 2.
Conditional on the event
In order to prove the claim (18), we make use of the following intermediate claim:
Formal argument:
We show that with probability at least 1 − δ the following holds:
As a consequence of this claim and the definitions (17a)-(17d) of \( a_\ell \) and \( T_\ell \), we immediately obtain that
for all number of iterates \( t \geq n^{3/4-6\beta} \log(n) \log(1/\beta) \) with probability at least 1 − δ as claimed in Theorem 1.
We now define the high probability event that is crucial for our proof. For any \( r \in \mathcal{R} \), define the event \( E_r \) as follows
Then, for the event
applying the union bound with Lemma 1 yields that \( \mathbb{P}[\mathcal{E}] \geq 1 - \delta \). All the arguments that follow are conditional on the event \( \mathcal{E} \) and hence hold with the claimed high probability.
In order to prove the claim (18), we make use of the following intermediate claim:
Lemma 2. Conditional on the event \( \mathcal{E} \), if \( \theta \leq \omega^{-a_\ell} \), then \( |M_{n,1}(\theta)| \leq \omega^{-a_\ell} \) for any \( \ell \leq \ell_\star \).
Deferring the proof of Appendix C.2, we now establish the claim (18) conditional on the event \( \mathcal{E} \) only for \( t = T_\ell \) and when \( |\theta_n^t| \in [\omega^{-a_\ell}, \omega^{-a_\ell}] \) in which we now prove using induction.
Proof of base case \( \ell = 0 \): Note that we have \( a_0 = 1/16 \) and that \( n^{-1/16} \leq \omega^{1/16} \). Also, by the definition (19) we have that the event (16) \( \subseteq \mathcal{E} \). Hence, under the event \( \mathcal{E} \) we have that \( |\theta_n^t| \leq n^{-1/16} \), for \( t \geq \sqrt{n} \log(n) \). Putting all the pieces together, we find that under the event \( \mathcal{E} \), we have \( |\theta_n^t| \leq n^{-1/16} \leq \omega^{1/16} \) and the base case follows.
Proof of inductive step: We now establish the inductive step. Note that Lemma 2 implies that we need to show the following: if $|\theta_0^n| \leq \omega^{-a_{\ell}}$ for all $t \in \{T_\ell, T_{\ell+1}, \ldots, T_{t+1} - 1\}$ for any given $\ell \leq \ell_*$, then $|\theta_n^{T_{t+1}}| \leq \omega^{-a_{\ell+1}}$. We establish this claim in two steps:

\[ \theta_n^{T_{t+1}} \leq c' \omega^{-a_{\ell+1}}, \quad \text{and,} \]

\[ \theta_n^{T_{t+1}} \leq \omega^{-a_{\ell+1}}, \quad \text{(20a)} \]

\[ \text{where } c' = (5c_2 + 1) \geq 1 \text{ is a universal constant. Note that the inductive claim follows from the bound (20b). It} \]

remains to establish the two claims (20a) and (20b) which we now do one by one.

Proof of claim (20a): Let $\Theta_\ell = \{ \theta : |\theta| \in [\omega^{-a_{\ell+1}}, \omega^{-a_{\ell}}] \}$. Now, conditional on the event $\mathcal{E}$, Lemma 1 implies that

\[ \sup_{\theta \in \Theta_\ell} |M_{n,1}(\theta) - M_1(\theta)| \leq c_2 \omega^{-3a_{\ell+1}/2}, \quad \text{and} \quad \sup_{\theta \in \Theta_\ell} |M_1(\theta)/\theta| \leq (1 - \omega^{-6a_{\ell+1}/5}) =: \gamma_\ell. \]

We can check that $\gamma_\ell \leq e^{-\omega^{a_{\ell+1}/5}}$. Unfolding the basic triangle inequality $t_{\ell}/2$ times and noting that $\theta_n^\ell \in \Theta_\ell$ for all $t \in \{T_\ell, \ldots, T_{t+1}/2\}$, we obtain that

\[ \left| \theta_n^{T_{t+1}/2} \right| \leq \gamma_\ell^{t_{\ell}/2} \left| \theta_n^{T_{t}} \right| + (1 + \gamma_\ell + \ldots + \gamma_\ell^{t_{\ell}/2-1})c_2 \omega^{-3a_{\ell+1}/2}
\]

\[ \leq e^{-t_{\ell}\omega^{-6a_{\ell+1}/10}} \cdot \omega^{-a_{\ell}} + \frac{1}{1 - \gamma_\ell} c_2 \omega^{-3a_{\ell+1}/2}
\]

\[ \leq (1 + 5c_2) \omega^{6a_{\ell+1} - 3a_{\ell+1}/2}
\]

\[ \leq (5c_2 + 1) \omega^{-a_{\ell+1}} \]

where step (i) follows from plugging in the value of $\gamma_\ell$ and invoking the definition (17c) of $t_\ell$, which leads to

\[ e^{-t_{\ell}\omega^{6a_{\ell+1}/10}} \cdot \omega^{-a_{\ell}} \leq \omega^{6a_{\ell+1} - 3a_{\ell+1}/2}. \]

Moreover, step (ii) is a direct consequence of the definition (17a) of the sequence $a_{\ell}$. Therefore, we achieve the conclusion of claim (20a).

Proof of claim (20b): The proof of this step is very similar to the previous step, except that we now use the set $\Theta_\ell' = \{ \theta : |\theta| \in [\omega^{-a_{\ell+1}}, c' \omega^{-a_{\ell+1}}] \}$ for our arguments. Applying Lemma 1, we have

\[ \sup_{\theta \in \Theta_\ell'} |M_{n,1}(\theta) - M_1(\theta)| \leq c_2 (c')^3 \omega^{-3a_{\ell+1} - 1/2}, \quad \text{and} \quad \sup_{\theta \in \Theta_\ell'} |M_1(\theta)/\theta| \leq \gamma_\ell. \]

Using the similar argument as that from the previous case, we find that

\[ \left| \theta_n^{T_{t+1} + t_{\ell}/2} \right| \leq e^{-t_{\ell}\omega^{6a_{\ell+1}/10}} c' \omega^{-a_{\ell+1}} + \frac{1}{1 - \gamma_\ell} c_2 (c')^3 \omega^{-3a_{\ell+1} - 1/2}
\]

\[ \leq (5c_2 + 1) (c')^3 \omega^{4a_{\ell+1} - 1/2} \cdot \omega^{-a_{\ell+1}}
\]

\[ \leq \omega^{-a_{\ell+1}} \]

where step (i) follows from the inequality $e^{-t_{\ell}\omega^{6a_{\ell+1}/10}} \leq \omega^{4a_{\ell+1} - 1/2}$ and the inequality

\[ \omega^{4a_{\ell+1} - 1/2} \leq \omega^{4a_{\ell+1} - 1/2} \leq \omega^{-4\beta} \leq 1/(c')^4,
\]

since $n \geq (c')^{1/\beta}c_{n,\beta}$. The claim now follows.

A.2 Proof of Theorem 2

Before proceeding further, we first derive the convergence rates for the scale parameter $\sigma_n^\ell$ using Theorem 2. Noting that $(\theta^*, \sigma^*) = (0, 1)$, we obtain the following relation

\[ \left| \left( \sigma_n^\ell \right)^2 - (\sigma^*)^2 \right| = \left| \sum_{i=1}^n \frac{||X_i||_2^2}{dn} - (\sigma^*)^2 \right| - \frac{||\theta_n^\ell - \theta^*||_2^2}{d} \]
Using standard chi-squared bounds, we obtain that
\[ \left| \frac{1}{dn} \sum_{i=1}^{n} \left\| X_i \right\|_{2}^2 - (\sigma^*)^2 \right| \lesssim (nd)^{-\frac{1}{2}}, \]
with high probability. From the bound (8), we also have \( \| \theta_n^t - \theta^* \|_{2}^2 / d \lesssim (nd)^{-\frac{1}{2}} \). Putting the pieces together, we conclude that the statistical error for the scale parameter satisfies
\[ |(\sigma_n^t)^2 - (\sigma^*)^2| \lesssim (nd)^{-\frac{1}{2}} \quad \text{for all } t \gtrsim \left( \frac{n}{d} \right)^{\frac{1}{2}}, \tag{21} \]
with high probability. Consequently, in the sequel, we focus primarily on the convergence rate for the EM estimates \( \theta_n^t \) of the location parameter, as the corresponding guarantee for the scale parameter \( \sigma_n^t \) is readily implied by it.

The proof of Theorem 2 is based on the population-to-sample analysis and follows a similar road-map as of the proofs in the paper [9]. We first analyze the population-level EM operator and then using epoch-based-localization argument derive the statistical rates (8). We make use of the following \( d \)-dimensional analog of the pseudo-population operator (cf. equation (11a)):
\[ \tilde{M}_{n,d}(\theta) := \mathbb{E}_{Y \sim \mathcal{N}(0,I_d)} \left[ Y \tanh \left( \frac{1}{\sqrt{\sum_{j=1}^{n} \| X_j \|_{2}^2 / (nd) - \| \theta \|_{2}^2 / d} \right) \right]. \tag{22} \]

In the next lemma, we establish the contraction properties and the perturbation bounds for \( \tilde{M}_{n,d} \):

**Lemma 3.** The operator \( \tilde{M}_{n,d} \) satisfies
\[ \left( 1 - \frac{3\| \theta \|_{2}^2}{4} \right) \leq \frac{\| \tilde{M}_{n,d}(\theta) \|_{2}}{\| \theta \|_{2}} \leq \left( 1 - \frac{1 - 1/d}{\delta} \right), \quad \text{for all } \| \theta \|_{2} \in I_\beta, \tag{23a} \]
with probability at least \( 1 - \delta \). Moreover, there exists a universal constant \( c_2 \) such that for any fixed \( \delta \in (0,1) \), \( \beta \in (0, \frac{1}{2}] \), and \( r \in (0, \frac{1}{5}) \) we have
\[ \mathbb{P} \left[ \sup_{\theta \in \mathbb{B}(0,r)} \| \tilde{M}_{n,d}(\theta) - \tilde{M}_{n,d}(\theta) \|_{2} \leq c_2 r \sqrt{\frac{d \log(1/\delta)}{n}} \right] \geq 1 - \delta - e^{-(nd)^{1/8}}. \tag{23b} \]

See Appendix C.3 for the proof.

Lemma 3 shows that the operator \( \tilde{M}_{n,d} \) has a faster contraction (order \( 1 - \| \theta \|_{2}^2 \)) towards zero, when compared to its univariate-version (order \( 1 - \theta^6 \) cf. (12a)). This difference between the univariate and the multivariate case had already been highlighted in Section 1.2 in Figure 2. Indeed substituting \( d = 1 \) in the bound (23a) gives us a vacuous bound for the univariate case, providing further evidence for the benefit of sharing variance among different dimensions in multivariate setting of symmetric fit (1). With Lemma 3 at hand, the proof of Theorem 2 follows by using the localization argument from the paper [9]. Mimicking the arguments similar to equation (13b), we obtain the following statistical rate:
\[ \epsilon \cdot r / \sqrt{n} = \epsilon \implies \epsilon r / \sqrt{n} = \epsilon \implies \epsilon \sim n^{-\frac{1}{2}}, \tag{24} \]

Much of the work in the proof of Theorem 2 is to establish Lemma 3. With the bounds (23a) and (23b) at hand, using the localization argument (in a manner similar to the proof of Theorem 1), easily leads to the statistical rate of order \( (d/n)^{1/4} \) as claimed in Theorem 2. The detailed proof is thereby omitted.

Moreover, similar to the arguments made in the paper [9], localization argument is necessary to derive a sharp rate. Indeed, a direct application of the framework introduced by Balakrishnan et al. [1] for our setting implies a sub-optimal rate of order \( (d/n)^{1/6} \) for the Euclidean error \( \| \theta_n^t - \theta^* \| \) (cf. (13a) and (13b)).
A.3 Proof of Lemma 1

We now prove Lemma 1 which provides the basis for the two-staged proof of Theorem 1.

The proof for the contraction property (12b) of the corrected population operator \( \tilde{M}_1 \) is similar to that of the property (12a) pseudo-population operator \( \bar{M}_{n,1} \) (albeit with a few high probability arguments replaced by deterministic arguments). Hence, while we provide a complete proof of the bound (12a) (in Section A.3.1), we only provide a proof sketch for the bound (12b) at its end. Moreover the proofs of bounds (12c) and (12d) are provided in Sections A.3.2 and A.3.3 respectively.

A.3.1 Contraction bound for population operator \( \tilde{M}_{n,1} \)

We begin by defining some notation. For \( \beta \in (0, 1/12] \) and \( \alpha \geq 1/2 - 6\beta \), we define the event \( \mathcal{E}_\alpha \) and the interval \( I_{\alpha,\beta} \) as follows

\[
\mathcal{E}_\alpha = \left\{ \frac{n}{\sum_{j=1}^{n} X_j^2/n - 1} \leq n^{-\alpha} \right\}, \quad \text{and},
\]

\[
I_{\alpha,\beta} = [3n^{-1/12+\beta}, \sqrt{9/400 - n^{-\alpha}}],
\]

where in the above notations we have omitted the dependence on \( n \), as it is clear from the context. We also use the scalars \( a \) and \( b \) to denote the following:

\[
a := 1 - n^{-\alpha} \quad \text{and} \quad b := 1 + n^{-\alpha}.
\]

With the above notation in place, observe that standard chi-squared tail bounds yield that \( \mathbb{P}[\mathcal{E}_\alpha] \geq 1 - e^{-n^{1 - 2\alpha}/8} \geq 1 - \delta \). Moreover, invoking the lower bound on \( n \) in Theorem 1, we have that \( [3n^{-1/12+\beta}, 1/10] \subseteq I_{\alpha,\beta} \). Now conditional on the high probability event \( \mathcal{E}_\alpha \), the population EM update \( \tilde{M}_{n,1}(\theta) \), in absolute value, can be upper and lower bounded as follows:

\[
|\tilde{M}_{n,1}(\theta)| \leq \mathbb{E}_Y \left[ Y \tanh \left( \frac{y |\theta|}{a - \theta^2} \right) \right] = |\theta| \mathbb{E}_Y \left[ Y \tanh \left( \frac{|\theta| X}{a - \theta^2} \right) \right], \quad \text{and},
\]

\[
|\tilde{M}_{n,1}(\theta)| \geq \mathbb{E}_Y \left[ Y \tanh \left( \frac{X |\theta|}{b - \theta^2} \right) \right] = |\theta| \mathbb{E}_Y \left[ Y \tanh \left( \frac{|\theta| Y}{b - \theta^2} \right) \right],
\]

where the last two inequalities follows directly from the definition of \( \tilde{M}_{n,1}(\theta) \) in equation (11a), and from the fact that for any fixed \( y, \theta \in \mathbb{R} \), the function \( w \rightarrow y \tanh(y |\theta| / (w - \theta^2)) \) is non-increasing in \( w \) for \( w > \theta^2 \). Consequently, in order to complete the proof, it suffices to establish the following bounds:

\[
1 - 3\theta^6/2 \leq \gamma(\theta), \quad \text{and} \quad \overline{\gamma}(\theta) \leq (1 - \theta^6/5). \tag{27}
\]

The following properties of the hyperbolic function \( x \mapsto x \tanh(x) \) are useful for our proofs:

**Lemma 4.** For any \( x \in \mathbb{R} \), the following holds

- **(Lower bound):** \( x \tanh(x) \geq x^2 - \frac{x^4}{3} + \frac{2x^6}{15} - \frac{17x^8}{315} \).
- **(Upper bound):** \( x \tanh(x) \leq x^2 - \frac{x^4}{3} + \frac{2x^6}{15} - \frac{17x^8}{315} + \frac{62x^{10}}{2835} \).

See Appendix C.4 for its proof.

Given the bounds in Lemma 4, we derive the upper and lower bounds in the inequality (27) separately.
Upper bound for $\gamma(\theta)$: Invoking the upper bound on $x\tanh(x)$ from Lemma 4, we find that
\[
\gamma(\theta) \leq \frac{a - \theta^2}{\theta^2} \left( \frac{\theta^2}{(a - \theta^2)^4} E[Y^2] - \frac{\theta^4}{3(a - \theta^2)^4} E[Y^4] + \frac{2\theta^6}{15(a - \theta^2)^6} E[Y^6] - \frac{17\theta^8}{315(a - \theta^2)^8} E[Y^8] + \frac{62\theta^{10}}{2835(a - \theta^2)^{10}} E[Y^{10}] \right).
\]
Recall that, for $Y \sim \mathcal{N}(0, 1)$, we have $E[Y^{2k}] = (2k - 1)!!$ for all $k \geq 1$. Therefore, the last inequality can be simplified to
\[
\gamma(\theta) \leq \frac{1}{a - \theta^2} - \frac{\theta^2}{(a - \theta^2)^3} + \frac{2\theta^4}{(a - \theta^2)^5} - \frac{17\theta^6}{3(a - \theta^2)^7} + \frac{62\theta^8}{3(a - \theta^2)^9}.
\tag{28}
\]
When $n^{-\alpha} + \theta^2 \leq 9/400$, we can verify that the following inequalities hold:
\[
\begin{align*}
\frac{1}{1 - n^{-\alpha} - \theta^2} &\leq 1 + (n^{-\alpha} + \theta^2) + (n^{-\alpha} + \theta^2)^2 + (n^{-\alpha} + \theta^2)^3 + 2(n^{-\alpha} + \theta^2)^4, \\
-\frac{\theta^2}{(1 - n^{-\alpha} - \theta^2)^3} &\leq -\theta^2 \left( 1 + 3(n^{-\alpha} + \theta^2) + 6(n^{-\alpha} + \theta^2)^2 + 10(n^{-\alpha} + \theta^2)^3 \right), \\
-\frac{\theta^4}{(1 - n^{-\alpha} - \theta^2)^5} &\leq -\theta^4 \left( 1 + 5(n^{-\alpha} + \theta^2) + 16(n^{-\alpha} + \theta^2)^2 \right), \\
-\frac{\theta^6}{(1 - n^{-\alpha} - \theta^2)^7} &\leq -\theta^6 \left( 1 + 7(n^{-\alpha} + \theta^2) \right), \\
-\frac{\theta^8}{(1 - n^{-\alpha} - \theta^2)^9} &\leq 5\theta^8 / 4.
\end{align*}
\]
Substituting $a = 1 - n^{-\alpha}$ into the bound (28) and doing some algebra with the above inequalities and using the fact that $\max \{\theta, n^{-\alpha}\} \leq 1$ we have that
\[
\gamma(\theta) \leq 1 - \frac{2}{3}\theta^6 + \frac{61}{6}\theta^8 + 100n^{-\alpha} \leq 1 - \frac{2}{5}\theta^6 + 100n^{-\alpha} \leq 1 - \frac{1}{5}\theta^6.
\]
The second last inequality above follows since $\theta \leq 3/20$, and the last inequality above utilizes the fact that if $\alpha \geq 1/2 - 6\beta$, then $\theta^6/5 \geq 100n^{-\alpha}$ for all $\theta \geq 3n^{-1/12+\beta}$. This completes the proof of the upper bound of $\gamma(\theta)$.

Lower bound for $\gamma(\theta)$: We start by utilizing the lower bound of $x\tanh(x)$ in the expression for $\gamma(\theta)$, which yields:
\[
\gamma(\theta) \geq \frac{1}{b - \theta^2} - \frac{\theta^2}{(b - \theta^2)^3} + \frac{2\theta^4}{(b - \theta^2)^5} - \frac{17\theta^6}{3(b - \theta^2)^7}.
\tag{29}
\]
Since $|\theta| \in [3n^{-1/12+\beta}, \sqrt{9/400} - n^{-\alpha}]$ by assumption, we have the following lower bounds:
\[
\begin{align*}
\frac{1}{1 + n^{-\alpha} - \theta^2} &\geq 1 + (\theta^2 - n^{-\alpha}) + (\theta^2 - n^{-\alpha})^2 + (\theta^2 - n^{-\alpha})^3 + (\theta^2 - n^{-\alpha})^4, \\
-\frac{\theta^2}{(1 + n^{-\alpha} - \theta^2)^3} &\geq -\theta^2 - (1 + 3(\theta^2 - n^{-\alpha}) + 6(\theta^2 - n^{-\alpha})^2 + 11(\theta^2 - n^{-\alpha})^3), \\
-\frac{\theta^4}{(1 + n^{-\alpha} - \theta^2)^5} &\geq -\theta^4 (1 + 5(\theta^2 - n^{-\alpha}) + 15(\theta^2 - n^{-\alpha})), \\
-\frac{\theta^6}{(1 + n^{-\alpha} - \theta^2)^7} &\geq -\theta^6 (1 + 8(\theta^2 - n^{-\alpha})).
\end{align*}
\]
Substituting $b = 1 + n^{-\alpha}$ into the bound (29) and doing some algebra with the above inequalities and using the fact that $\max \{\theta, n^{-\alpha}\} \leq 1$ we have that
\[
\gamma(\theta) \geq 1 - \frac{2}{3}\theta^6 - \frac{76}{3}\theta^8 - 100n^{-\alpha} \geq 1 - \frac{5}{4}\theta^6 - 100n^{-\alpha} \geq 1 - \frac{3}{2}\theta^6.
\]
The second last inequality above follows since $\theta \leq 3/20$, and the last inequality above utilizes the fact that if $\alpha \geq 1/2 - 6\beta$, then $\theta^6/4 \geq 100n^{-\alpha}$ for all $\theta \geq 3n^{-1/12+\beta}$. This completes the proof of the lower bound of $\gamma(\theta)$. 

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Proof of contraction bound for $\overline{M}_1$: Note that it suffices to repeat the arguments with $a = 1$ and $b = 1$ in the RHS of the inequalities (28) and (29) respectively. Given the other computations, the remaining steps are straightforward algebra and are thereby omitted.

A.3.2 Proof of perturbation bound for $\widetilde{M}_{n,1}$

We now prove the bound (12c) which is based on standard arguments to derive Rademacher complexity bounds. We first symmetrize with Rademacher variables, and apply the Ledoux-Talagrand contraction inequality. We then invoke results on sub-Gaussian and sub-exponential random variables, and finally perform the associated Chernoff-bound computations to obtain the desired result.

To ease the presentation, we denote $\alpha := 1/2 - 2\beta$ and $\mathcal{I} := [1 - n^{-\alpha} - 1/64, 1 - n^{-\alpha}]$. Next we fix $r \in [0, 1/8]$ and define $\tilde{r} := \frac{1}{1-n^{-\alpha}-1/64}$. For sufficiently large $n$, we have $\tilde{r} \leq 2r$. Recall the definition (25) of the event: $\mathcal{E}_\alpha = \{|\sum_{i=1}^n X_i^2/n - 1| \leq n^{-\alpha}\}$. Conditional on the event $\mathcal{E}_\alpha$, the following inequalities hold

$$
\left| M_{n,1}(\theta) - \tilde{M}_{n,1}(\theta) \right| \leq \sup_{\tilde{\theta} \in \mathbb{B}(0, r), \sigma^2 \in \mathcal{I}} \left| \frac{1}{n} \sum_{i=1}^n X_i \tanh \left( \frac{X_i \theta}{\sigma^2} \right) - \mathbb{E} \left[ Y \tanh \left( \frac{Y \theta}{\sigma^2} \right) \right] \right|
$$

$$
\leq \sup_{\tilde{\theta} \in \mathbb{B}(0, \tilde{r})} \left| \tilde{M}_n(\tilde{\theta}) - \tilde{M}(\tilde{\theta}) \right|,
$$

with all them valid for any $\theta \in \mathbb{B}(0, r)$. Here $Y$ denotes a standard normal variate $\mathcal{N}(0,1)$ whereas the operators $\tilde{M}$ and $\tilde{M}_n$ are defined as

$$
\tilde{M}(\tilde{\theta}) := \mathbb{E}[Y \tanh(Y \tilde{\theta})] \quad \text{and} \quad \tilde{M}_n(\tilde{\theta}) := \frac{1}{n} \sum_{i=1}^n X_i \tanh(X_i \tilde{\theta}).
$$

To facilitate the discussion later, we define the unconditional random variable

$$
Z := \sup_{\tilde{\theta} \in \mathbb{B}(0, \tilde{r})} \left| \tilde{M}_n(\tilde{\theta}) - \tilde{M}(\tilde{\theta}) \right|.
$$

Employing standard symmetrization argument from empirical process theory [24], we find that

$$
\mathbb{E}[\exp(\lambda Z)] \leq \mathbb{E} \left[ \exp \left( \sup_{\tilde{\theta} \in \mathbb{B}(0, r)} \frac{2\lambda}{n} \sum_{i=1}^n \varepsilon_i \tanh(X_i \tilde{\theta})X_i \right) \right],
$$

where $\varepsilon_i, i \in [n]$ are i.i.d. Rademacher random variables independent of $\{X_i, i \in [n]\}$. Noting that, the following inequality with hyperbolic function $\tanh(x)$ holds

$$
|\tanh(x \tilde{\theta}) - \tanh(x \tilde{\theta}')| \leq \left| (\theta - \tilde{\theta}) x \right| \quad \text{for all } x.
$$

Consequently for any given $x$, the function $\tilde{\theta} \mapsto \tanh(x \tilde{\theta})$ is Lipschitz. Invoking the Ledoux-Talagrand contraction result for Lipschitz functions of Rademacher processes [17] and following the proof argument from Lemma 1 in the paper [9], we obtain that

$$
Z \leq c\tilde{r} \sqrt{\frac{\log(1/\delta)}{n}}, \quad \text{with probability } \geq 1 - \delta,
$$

for some universal constant $c$. Finally, using $\tilde{r} \leq 2r$ for large $n$, we obtain that

$$
\left| M_{n,1}(\theta) - \tilde{M}_{n,1}(\theta) \right| \leq 2cr \sqrt{\frac{\log(1/\delta)}{n}}, \quad \text{with probability } \geq 1 - \delta - e^{-n^{1-2\alpha}/8},
$$

where we have also used the fact that $\mathbb{P}[\mathcal{E}_\alpha] \geq 1 - e^{-n^{1-2\alpha}/8}$ from standard chi-squared tail bounds. The bound (12c) follows and we are done.
We now prove the bound (12d). Note that it suffices to establish the following point-wise result:

$$|\overline{M}_1(\theta) - M_{n,1}(\theta)| \lesssim \frac{|\theta|^3 \log^{10}(5n/\delta)}{\sqrt{n}} \quad \text{for all} \quad |\theta| \lesssim n^{-1/6},$$

with probability at least $1 - \delta$ for any given $\delta > 0$. For the reader’s convenience, let us recall the definition of these operators

$$\overline{M}_1(\theta) = \mathbb{E} \left[ X \tanh(X\theta/(1 - \theta^2)) \right],$$

$$M_{n,1}(\theta) = \frac{1}{n} \sum_{i=1}^{n} X_i \tanh \left( X_i \theta / (a_n - \theta^2) \right),$$

where $a_n := \sum_{i=1}^{n} X_i^2 / n$. We further denote $\mu_k := \mathbb{E}_{X \sim \mathcal{N}(0,1)} [X^k]$, and $\hat{\mu}_k := \frac{1}{n} \sum_{i=1}^{n} X_i^k$. From known results on Gaussian moments, we have $\mu_{2k} = (2k - 1)!!$ for each integer $k = 1, 2, \ldots$

For any given $x$ and scalar $b$, consider the map $\theta \mapsto x \tanh(x\theta/(b - \theta^2))$. The 9-th order Taylor series for this function around $\theta = 0$ is given by

$$x \tanh(x\theta/(b - \theta^2)) = \frac{\theta x^2}{b} - \frac{\theta^3 (x^4 - 3bx^2)}{3b^3} + \theta^5 \left( \frac{2x^6}{15b^5} - \frac{x^4}{b^4} + \frac{x^2}{b^2} \right)$$

$$+ \theta^7 \left( -\frac{17x^8}{315b^7} + \frac{2x^6}{3b^5} - \frac{2x^4}{b^4} + \frac{x^2}{b^2} \right)$$

$$+ \theta^9 \left( \frac{62x^{10}}{2835b^9} - \frac{17x^8}{45b^7} + \frac{2x^6}{3b^5} - \frac{10x^4}{3b^3} + \frac{x^2}{b^2} \right) + \varepsilon,$$

(31)

where the remainder $\varepsilon$ satisfies $\varepsilon \lesssim \mathcal{O}(\theta^{11})$. Plugging in this expansion with $b = 1$ on RHS of equation (30a) and taking expectation over $X \sim \mathcal{N}(0,1)$, we obtain

$$\overline{M}_1(\theta) = \theta + \theta^3 \left( \sum_{k=1}^{2} c_{3,k} \mu_{2k} \right) + \theta^5 \left( \sum_{k=1}^{3} c_{5,k} \mu_{2k} \right) + \theta^7 \left( \sum_{k=1}^{4} c_{7,k} \mu_{2k} \right) + \theta^9 \left( \sum_{k=1}^{5} c_{9,k} \mu_{2k} \right) + \varepsilon,$$

(32a)

where we have used the notation $\mu_k := \mathbb{E}_{X \sim \mathcal{N}(0,1)} [X^k]$ and $c_{j,k}$ denote universal constants. Furthermore, plugging in the same expansion (31) with $b = a_n$ on RHS of equation (30b), we obtain the following expansion for the sample EM operator

$$M_{n,1}(\theta) = \theta + \theta^3 \left( \sum_{k=1}^{2} \frac{c_{3,k} \hat{\mu}_{2k}}{a_n \mu_{2k}} \right) + \theta^5 \left( \sum_{k=1}^{3} \frac{c_{5,k} \hat{\mu}_{2k}}{a_n^{2+k}} \right) + \theta^7 \left( \sum_{k=1}^{4} \frac{c_{7,k} \hat{\mu}_{2k}}{a_n^{3+k}} \right) + \theta^9 \left( \sum_{k=1}^{5} \frac{c_{9,k} \hat{\mu}_{2k}}{a_n^{4+k}} \right) + \varepsilon_n,$$

(32b)

where $\hat{\mu}_k$ denotes the sample mean of $X^k$, i.e., $\hat{\mu}_k := \frac{1}{n} \sum_{i=1}^{n} X_i^k$. In order to lighten the notation, we introduce the following convenient shorthand:

$$\beta_j = \sum_{k=1}^{j+1} c_{j,k} \mu_{2k} \quad \text{and} \quad \hat{\beta}_j = \sum_{k=1}^{j+1} c_{j,k} \frac{\hat{\mu}_{2k}}{a_n^{2+k}} \quad \text{for} \ j \in \{3, 5, 7, 9\} =: \mathcal{J}.$$

(33)

A careful inspection reveals that $\beta_3 = \beta_5 = 0$. With the above notations in place, we find that

$$|\overline{M}_1(\theta) - M_{n,1}(\theta)| = \left| \sum_{j \in \mathcal{J}} \theta^j (\beta_j - \hat{\beta}_j) \right| + \varepsilon$$

$$=: U_1 + U_2.$$

Therefore, it remains to establish that

$$U_1 \lesssim \frac{|\theta|^3 \log^{5}(5n/\delta)}{\sqrt{n}} \quad \text{and} \quad U_2 \lesssim \frac{|\theta|^3 \log^{5}(5n/\delta)}{\sqrt{n}},$$

(34)
Then, we have
\[ P \left( \left| \beta_j - \hat{\beta}_j \right| \geq \delta \right) \leq 1 - \delta \]
with probability at least 1 − δ for any given δ > 0. Since the remainder term is of order \( \theta^{11} \), the assumption \( |\theta| \lesssim n^{-1/16} \) ensures that the remainder term is bounded by a term of order \( \theta^3/\sqrt{n} \) and thus the bound (34) on the second term \( U_2 \) follows.

We now use concentration properties of Gaussian moments in order to prove the bound (34) on the first term \( U_1 \). Since \( |\theta| \leq 1 \), it suffices to show that
\[ \sup_{j \in J} \left| \beta_j - \hat{\beta}_j \right| \geq \frac{\log^5(5n/\delta)}{\sqrt{n}} \]  
with probability at least 1 − δ. Using the relation (33), we find that
\[ \left| \beta_j - \hat{\beta}_j \right| = \left| \sum_{k=1}^{j+1} \left( c_{j,k} \mu_{2k} - c_{j,k} \frac{\hat{\mu}_{2k}}{a_{n+k}} \right) \right| \leq \sum_{k=1}^{j+1} \frac{c_{j,k}}{a_{n+k}} \left| \mu_{2k} - \hat{\mu}_{2k} \right| + c_{j,k}(1 - a_n^{-\frac{j+1}{2}}) \mu_{2k} \]
\[ \leq C \sum_{k=1}^{j+1} \left( \left| \mu_{2k} - \hat{\mu}_{2k} \right| + \frac{\mu_{2k}}{\sqrt{n}} \right), \]  
(36)
for any \( j \in J \). Here in the last step we have used the following bounds:
\[ \max_{j \in J, k \leq j+1} c_{j,k} \leq C \quad \text{and} \quad \max_{j \in J, k \leq j+1} \left( 1 - a_n^{-\frac{j+1}{2}} \right) \leq \frac{C}{\sqrt{n}} \]
for some universal constant \( C \). Thus a lemma for the \( 1/\sqrt{n} \)-concentration\(^5\) of higher moments of Gaussian random variable is now useful:

**Lemma 5.** Let \( X_1, \ldots, X_n \) are i.i.d. samples from \( \mathcal{N}(0,1) \) and let \( \mu_{2k} := \mathbb{E}_{X \sim \mathcal{N}(0,1)}[X^{2k}] \) and \( \hat{\mu}_{2k} := \frac{1}{n} \sum_{i=1}^{n} X_i^{2k} \). Then, we have
\[ \mathbb{P} \left( \left| \hat{\mu}_{2k} - \mu_{2k} \right| \leq \frac{C_k \log^k(5n/\delta)}{\sqrt{n}} \right) \geq 1 - \delta \quad \text{for any} \quad k \geq 1, \]
where \( C_k \) denotes a universal constant depending only on \( k \).

See the Appendix C.5 for the proof.

For any \( \delta > 0 \), consider the event
\[ \mathcal{E} := \left\{ \left| \mu_{2k} - \hat{\mu}_{2k} \right| \leq \frac{C_k \log^k(5n/\delta)}{\sqrt{n}} \quad \text{for all} \quad k \in \{2, 4, \ldots, 10\} \right\}. \]
(37)
Straightforward application of union bound with Lemma 5 yields that \( \mathbb{P}[\mathcal{E}] \geq 1 - \delta \). conditional on the event \( \mathcal{E} \) inequality (35) implies that
\[ \sup_{j \in J} \left| \beta_j - \hat{\beta}_j \right| \leq C \sup_{j \in J} \sum_{k=1}^{j+1} \left( \left| \mu_{2k} - \hat{\mu}_{2k} \right| + \frac{\mu_{2k}}{\sqrt{n}} \right) \]
\[ \leq C \sup_{j \in \{3,5,7,9\}} \frac{j+1}{2} \left( \left| \mu_{j+1} - \hat{\mu}_{j+1} \right| + \frac{(j+1)!}{\sqrt{n}} \right) \]
\[ \leq \left( \begin{array}{c} j+1 \end{array} \right) \frac{(j+1)!}{\sqrt{n}} \]
\[ \leq C \left( \begin{array}{c} j \end{array} \right) \frac{(j+1)!}{\sqrt{n}} \]
\[ \leq C \frac{\log^5(5n/\delta)}{\sqrt{n}}, \]  
(38)
where step (i) follows from the definition of the event (37) and in step (ii) using the fact that \( j \leq 9 \) is bounded we absorbed all the constants into a single constant. Since the event \( \mathcal{E} \) has probability at least \( 1 - \delta \), the claim (35) now follows.

\(^5\)The bound from Lemma 5 is sub-optimal for \( k = 1 \) but is sharper than the standard tail bounds for Gaussian polynomials of degree \( 2k \) for \( k \geq 2 \). The \( 1/\sqrt{n} \) concentration of higher moments is necessary to derive the sharp rates stated in our results.
We now present the proof of the minimax bound. We introduce the shorthand $v := \sigma^2$ and $\eta := (\theta, v)$. First of all, we claim the following key upper bound of Hellinger distance between mixture densities $f_{\eta_1}$, $f_{\eta_2}$ in terms of the distances among their corresponding parameters $\eta_1$ and $\eta_2$:

$$\inf_{m, \eta_2 \in \Omega} \left( \frac{h(f_{\eta_1}, f_{\eta_2})}{\left( |\theta_1| - |\theta_2| \right)^2 + |v_1 - v_2|} \right) = 0 \quad \text{for any } r \in (1, 4).$$

(39)

Moreover, for any two densities $p$ and $q$, we denote the total variation distance between $p$ and $q$ by $V(p, q) := (1/2) \int |p(x) - q(x)| \, dx$. Similarly, the squared Hellinger distance between $p$ and $q$ is given as $h^2(p, q) = (1/2) \int \left( \sqrt{p(x)} - \sqrt{q(x)} \right)^2 \, dx$. 

A.3.4 Sharpness of bounds of Lemma 1

In Figure 4, we numerically verify the linear and cubic scaling of the bounds stated in Lemma 1.

B Minimax bound

We now show that the error of order $n^{-\frac{1}{4}}$ (up to logarithmic factors) is, in fact, tight in the standard minimax sense. Given a compact set $\Omega \subset \mathbb{R} \times (0, \infty)$, and a set of true parameters $(\theta^*, \sigma^*) \in \Omega$, suppose that we draw $n$ i.i.d. samples $\{X_i\}_{i=1}^n$ from a two-Gaussian mixture of the form $\frac{1}{2} N(\theta^*, (\sigma^*)^2) + \frac{1}{2} N(-\theta^*, (\sigma^*)^2)$. Let $(\hat{\theta}_n, \hat{\sigma}_n) \in \Omega$ denote any estimates—for the respective parameters—measurable with respect to the observed samples $X_1, \ldots, X_n \overset{i.i.d.}{\sim} f_{\theta^*, \sigma^*}$ and let $E_{(\theta^*, \sigma^*)}$ denote the corresponding expectation.

**Proposition 1.** There exists a universal constant $c_0 > 0$ (depending only on $\Omega$), such that

$$\inf_{(\theta_n, \sigma_n) \in \Omega} \sup_{(\theta^*, \sigma^*)} E_{(\theta^*, \sigma^*)} \left[ \left( |\hat{\theta}_n| - |\theta^*| \right)^2 + \left( |\hat{\sigma}_n|^2 - (\sigma^*)^2 \right) \right] \geq c_0 n^{-\frac{1}{4} - \delta} \quad \text{for any } \delta > 0.$$

See Appendix C.1 for the proof.

Based on the connection between location parameter $\theta^*_n$ and scale parameter $\sigma^*_n$ in the EM updates (cf. Equation (3c)), the minimax lower bound in Proposition 1 shows that the (non-squared) error of EM location updates $||\theta^*_n| - |\theta^*||$ is lower bounded by a term (arbitrarily close to) $n^{-\frac{1}{4}}$.

B.1 Proof of Proposition 1

We now present the proof of the minimax bound. We introduce the shorthand $v := \sigma^2$ and $\eta := (\theta, v)$. First of all, we claim the following key upper bound of Hellinger distance between mixture densities $f_{\eta_1}$, $f_{\eta_2}$ in terms of the distances among their corresponding parameters $\eta_1$ and $\eta_2$:

$$\inf_{m, \eta_2 \in \Omega} \left( \frac{h(f_{\eta_1}, f_{\eta_2})}{\left( |\theta_1| - |\theta_2| \right)^2 + |v_1 - v_2|} \right) = 0 \quad \text{for any } r \in (1, 4).$$

(39)
Taking the claim (39) as given for the moment, let us complete the proof of Proposition 1. Our proof relies on Le Cam’s lemma for establishing minimax lower bounds. In particular, for any $r \in (1, 4)$ and for any $\epsilon > 0$ sufficiently small, according to the result in equation (39), there exist $\eta_1 = (\theta_1, v_1)$ and $\eta_2 = (\theta_2, v_2)$ such that
\[ (|\theta_1| - |\theta_2|)^2 + |v_1 - v_2| = 2\epsilon \quad \text{and} \quad h(f_{\eta_1}, f_{\eta_2}) \leq c\epsilon^r \quad \text{for some universal constant} \ c. \]
From Lemma 1 from Yu [32], we obtain that
\[ \sup_{\eta \in \{\eta_1, \eta_2\}} E_\eta \left[ \left( |\hat{\theta}_n| - |\theta| \right)^2 + |(\hat{\sigma}_n)^2 - (\sigma)^2| \right] \geq \epsilon \left( 1 - V(f_{\eta_1}^n, f_{\eta_2}^n) \right), \]
where $f_{\eta}^n$ denotes the product of mixture densities $f_{\eta}$ of the data $X_1, \ldots, X_n$. A standard relation between total variation distance and Hellinger distance leads to
\[ V(f_{\eta_1}^n, f_{\eta_2}^n) \leq h(f_{\eta_1}^n, f_{\eta_2}^n) = \sqrt{1 - [1 - h^2(f_{\eta_1}, f_{\eta_2})]^n} \leq \sqrt{1 - [1 - c\epsilon^n]^n}. \]
By choosing $c\epsilon^n = 1/n$, we can verify that
\[ \sup_{\eta \in \{\eta_1, \eta_2\}} E_\eta \left[ \left( |\hat{\theta}_n| - |\theta| \right)^2 + |(\hat{\sigma}_n)^2 - (\sigma)^2| \right] \geq \epsilon \propto n^{-1/r}, \]
which establishes the claim of Proposition 1.

### B.1.1 Proof of claim (39)

In order to prove claim (39), it is sufficient to construct sequences $\eta_{1,n} = (\theta_{1,n}, v_{1,n})$ and $\eta_{2,n} = (\theta_{2,n}, v_{2,n})$ such that
\[ h(f_{\eta_{1,n}}, f_{\eta_{2,n}}) / \left( (|\theta_{1,n}| - |\theta_{2,n}|)^2 + |v_{1,n} - v_{2,n}| \right)^r \to 0 \]
as $n \to \infty$. Indeed, we construct these sequences as follows: $\theta_{2,n} = 2\theta_{1,n}$ and $v_{1,n} - v_{2,n} = 3 (\theta_{1,n})^2$ for all $n \geq 1$ while $\theta_{1,n} \to 0$ as $n \to \infty$. Direct computation leads to
\[ f_{\eta_{1,n}}(x) - f_{\eta_{2,n}}(x) = \frac{1}{2} \left( \phi(x; -\theta_{1,n}, v_{1,n}) - \phi(x; -\theta_{2,n}, v_{2,n}) \right) + \frac{1}{2} \left( \phi(x; \theta_{1,n}, v_{1,n}) - \phi(x; \theta_{2,n}, v_{2,n}) \right). \]

Invoking Taylor expansion up to the third order, we obtain that
\[ T_{1,n} = \sum_{|\alpha| \leq 3} \frac{(\theta_{2,n} - \theta_{1,n})^\alpha (v_{1,n} - v_{2,n})^{\alpha_2}}{\alpha_1! \alpha_2!} \left( \frac{\partial^{(2)} \phi}{\partial \theta^{\alpha_1} \partial v^{\alpha_2}}(x; -\theta_{2,n}, v_{2,n}) \right) + R_1(x), \]
\[ T_{2,n} = \sum_{|\alpha| \leq 3} \frac{(\theta_{1,n} - \theta_{2,n})^\alpha (v_{1,n} - v_{2,n})^{\alpha_2}}{\alpha_1! \alpha_2!} \left( \frac{\partial^{(2)} \phi}{\partial \theta^{\alpha_1} \partial v^{\alpha_2}}(x; \theta_{2,n}, v_{2,n}) \right) + R_2(x) \]
where $|\alpha| = \alpha_1 + \alpha_2$ for $\alpha = (\alpha_1, \alpha_2)$. Here, $R_1(x)$ and $R_2(x)$ are Taylor remainders that have the following explicit representations
\[ R_1(x) := 4 \sum_{|\beta| = 4} \frac{(\theta_{2,n} - \theta_{1,n})^{\beta_1} (v_{1,n} - v_{2,n})^{\beta_2}}{\beta_1! \beta_2!} \times \int_0^1 (1-t)^3 \frac{\partial^4 \phi}{\partial \theta^{\beta_1} \partial v^{\beta_2}}(x; -\theta_{2,n} + t(\theta_{2,n} - \theta_{1,n}), v_{2,n} + t(v_{1,n} - v_{2,n})) \, dt, \]
\[ R_2(x) := 4 \sum_{|\beta| = 4} \frac{(\theta_{1,n} - \theta_{2,n})^{\beta_1} (v_{1,n} - v_{2,n})^{\beta_2}}{\beta_1! \beta_2!} \times \int_0^1 (1-t)^3 \frac{\partial^4 \phi}{\partial \theta^{\beta_1} \partial v^{\beta_2}}(x; \theta_{2,n} + t(\theta_{1,n} - \theta_{2,n}), v_{2,n} + t(v_{1,n} - v_{2,n})) \, dt. \]
Recall from equation (2) that the univariate location-scale Gaussian distribution has the PDE structure of the following form
\[
\frac{\partial^2 \phi}{\partial \theta^2}(x; \theta, \sigma^2) = 2 \frac{\partial \phi}{\partial \sigma^2}(x; \theta, \sigma^2).
\]

Therefore, we can write the formulations of \(T_{1,n}\) and \(T_{2,n}\) as follows:
\[
T_{1,n} = \sum_{|\alpha| \leq 3} \frac{(\theta_{2,n} - \theta_{1,n})^\alpha_1 (v_{1,n} - v_{2,n})^\alpha_2}{\alpha_1! \alpha_2!} \frac{\partial^{\alpha_1+2\alpha_2} \phi}{\partial \theta^{\alpha_1+2\alpha_2}}(x; -\theta_{2,n}, v_{2,n}) + R_1(x),
\]
\[
T_{2,n} = \sum_{|\alpha| \leq 3} \frac{(\theta_{1,n} - \theta_{2,n})^\alpha_1 (v_{1,n} - v_{2,n})^\alpha_2}{\alpha_1! \alpha_2!} \frac{\partial^{\alpha_1+2\alpha_2} \phi}{\partial \theta^{\alpha_1+2\alpha_2}}(x; \theta_{2,n}, v_{2,n}) + R_2(x).
\]

Via a Taylor series expansion, we find that
\[
\frac{\partial^{\alpha_1+2\alpha_2} \phi}{\partial \theta^{\alpha_1+2\alpha_2}}(x; \theta_{2,n}, v_{2,n}) = \sum_{\tau=0}^{3-|\alpha|} \frac{(2\theta_{2,n})^\tau}{\tau!} \frac{\partial^{\alpha_1+2\alpha_2+\tau} \phi}{\partial \theta^{\alpha_1+2\alpha_2+\tau}}(x; -\theta_{2,n}, v_{2,n}) + R_{2,\alpha}(x)
\]
for any \(\alpha = (\alpha_1, \alpha_2)\) such that \(1 \leq |\alpha| \leq 3\). Here, \(R_{2,\alpha}\) is Taylor remainder admitting the following representation
\[
R_{2,\alpha}(x) = \sum_{\tau=4-|\alpha|}^{3} \frac{\tau (2\theta_{2,n})^\tau}{\tau!} \int_0^1 (1 - t)^{\tau-1} \frac{\partial^4 \phi}{\partial \theta^{\alpha_1+\tau} \partial \theta^{\alpha_2}}(x; -\theta_{2,n} + 2t\theta_{2,n}, v_{2,n}) dt.
\]

Governed by the above results, we can rewrite \(f_{\eta_{1,n}}(x) - f_{\eta_{2,n}}(x)\) as
\[
f_{\eta_{1,n}}(x) - f_{\eta_{2,n}}(x) = \sum_{l=1}^6 A_{l,n} \frac{\partial \phi}{\partial \theta}(x; -\theta_{2,n}, v_{2,n}) + R(x)
\]
where the explicit formulations of \(A_{l,n}\) and \(R(x)\) are given by
\[
A_{l,n} := \frac{1}{2} \sum_{\alpha_1, \alpha_2, \tau} \frac{1}{2^{\alpha_2}} \frac{(\theta_{2,n} - \theta_{1,n})^\alpha_1 (v_{1,n} - v_{2,n})^\alpha_2}{\alpha_1! \alpha_2!} + \frac{1}{2} \sum_{\alpha_1, \alpha_2, \tau} \frac{1}{2^{\alpha_2}} \frac{2^{\tau} (2\theta_{2,n})^\tau (\theta_{1,n} - \theta_{2,n})^\alpha_1 (v_{1,n} - v_{2,n})^\alpha_2}{\tau! \alpha_1! \alpha_2!},
\]
\[
R(x) := \frac{1}{2} R_1(x) + \frac{1}{2} R_2(x) + \sum_{|\alpha| \leq 2} \frac{1}{2^{\alpha_2}} \frac{(\theta_{1,n} - \theta_{2,n})^\alpha_1 (v_{1,n} - v_{2,n})^\alpha_2}{\alpha_1! \alpha_2!} R_{2,\alpha}(x)
\]
for any \(l \in [6]\) and \(x \in \mathbb{R}\). Here the ranges of \(\alpha_1, \alpha_2\) in the first sum of \(A_{l,n}\) satisfy \(\alpha_1 + 2\alpha_2 = l\) and \(1 \leq |\alpha| \leq 3\) while the ranges of \(\alpha_1, \alpha_2, \tau\) in the second sum of \(A_{l,n}\) satisfy \(\alpha_1 + 2\alpha_2 + \tau = l\), \(0 \leq \tau \leq 3 - |\alpha|\), and \(1 \leq |\alpha| \leq 3\).

From the conditions that \(\theta_{2,n} = 2\theta_{1,n}\) and \(v_{1,n} - v_{2,n} = 3(\theta_{1,n})^2\), we can check that \(A_{l,n} = 0\) for all \(1 \leq l \leq 3\). Additionally, we also have
\[
\max\{|A_{4,n}|, |A_{5,n}|, |A_{6,n}|\} \lesssim |\theta_{1,n}|^4.
\]
Given the above results, we claim that
\[
h\left(f_{\eta_{1,n}}, f_{\eta_{2,n}}\right) \lesssim |\theta_{1,n}|^8.
\]
Assume that the claim (40) is given. From the formulations of sequences \(\eta_{1,n}\) and \(\eta_{2,n}\), we can verify that
\[
\left(\left|\theta_{1,n}\right| - |\theta_{2,n}|\right)^r + |v_{1,n} - v_{2,n}|^r \lesssim |\theta_{1,n}|^{2r}.
\]
Since \(1 \leq r < 4\) and \(\theta_{1,n} \to 0\) as \(n \to \infty\), the above results lead to
\[
h\left(f_{\eta_{1,n}}, f_{\eta_{2,n}}\right) / \left(\left|\theta_{1,n}\right| - |\theta_{2,n}|\right)^r + |v_{1,n} - v_{2,n}|^r \lesssim |\theta_{1,n}|^{8-2r} \to 0.
\]
As a consequence, we achieve the conclusion of the claim (39).
B.1.2 Proof of claim (40)

The definition of Hellinger distance leads to the following equations

\[
2h^2 (f_{\eta_1,n}, f_{\eta_2,n}) = \int \frac{(f_{\eta_1,n}(x) - f_{\eta_2,n}(x))^2}{(\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2} dx
\]

\[
= \int \frac{(\sum_{l=4}^6 A_{l,n} \frac{\partial^l \phi}{\partial \theta^l}(x; \theta_{2,n}, v_{2,n}) + R(x))^2}{(\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2} dx
\]

\[
\lesssim \int \frac{\sum_{l=4}^6 \left(A_{l,n} \left(\frac{\partial^l \phi}{\partial \theta^l}(x; \theta_{2,n}, v_{2,n})\right)^2 + \beta^2\right)}{(\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2} dx \tag{41}
\]

where the last inequality is due to Cauchy-Schwarz’s inequality. According to the structure of location-scale Gaussian density, the following inequalities hold

\[
\int \left(\frac{\partial^l \phi}{\partial \theta^l}(x; \theta_{2,n}, v_{2,n})\right)^2 (\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2 dx \lesssim \int \frac{(\partial^l \phi(x; \theta_{2,n}, v_{2,n}))^2}{\phi(x; \theta_{2,n}, v_{2,n})} dx < \infty \tag{42}
\]

for \(4 \leq l \leq 6\). Note that, for any \(\beta = (\beta_1, \beta_2)\) such that \(|\beta| = 4\), we have

\[
|\theta_{2,n} - \theta_1,n|^{4+\beta_2} |v_{1,n} - v_{2,n}|^{\beta_1} \lesssim |\theta_1,n|^{4+\beta_2} \lesssim |\theta_1,n|^4.
\]

With the above bounds, an application of Cauchy-Schwarz’s inequality leads to

\[
\int \frac{R^2_1(x)}{(\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2} dx
\]

\[
\lesssim |\theta_1,n|^8 \sum_{|\beta|=4} \int \sup_{t \in [0,1]} \left(\frac{\partial^4 \phi}{\partial \theta^4 \partial v^2}(x; \theta_{2,n} + t(\theta_2,n - \theta_1,n), v_{2,n} + t(v_{1,n} - v_{2,n}))\right)^2 \phi(x; \theta_{2,n}, v_{2,n}) dx \lesssim |\theta_1,n|^8.
\]

With a similar argument, we also obtain that

\[
\int \frac{R^2_2(x)}{(\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2} dx \lesssim |\theta_1,n|^8, \quad \max_{1 \leq |\beta| \leq 4} \int \frac{R^2_{2,\alpha}(x)}{(\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2} dx \lesssim |\theta_1,n|^8.
\]

Governed by the above bounds, another application of Cauchy-Schwarz’s inequality implies that

\[
\int \frac{R^2(x)}{(\sqrt{f_{\eta_1,n}(x)} + \sqrt{f_{\eta_2,n}(x)})^2} dx \lesssim |\theta_1,n|^8. \tag{43}
\]

Combining the results from equations (41), (42), and (43), we achieve the conclusion of the claim (40).

C Proofs of auxiliary results

In this appendix, we collect the proofs of several auxiliary results stated throughout the paper.

C.1 Proof of Corollary 1

In order to ease the presentation, we only provide the proof sketch for the localization argument with this corollary. The detail proof argument for the corollary can be argued in similar fashion as that of Theorem 1. In particular,
we consider the iterations $t$ such that $\theta_n^t \in [n^{-a_\ell}, n^{-a_\ell}]$ where $a_\ell > a_r$. For all such iterations with $\theta_n^t$, invoking Lemma 1, we find that

$$|\widetilde{M}_{n,1}(\theta_n^t) - M_{n,1}(\theta_n^t)| \lesssim n^{-a_r}/\sqrt{n}.$$  

Therefore, we obtain that

$$|\theta_n^{t+T}| \leq |\widetilde{M}_{n,1}(\theta_n^{t+T-1})| + |\widetilde{M}_{n,1}(\theta_n^{t+T-1}) - M_{n,1}(\theta_n^{t+T-1})| \leq \gamma_{a_\ell} \theta_n^{t+T-1} + n^{-a_r}/\sqrt{n}.$$  

Unfolding the above inequality $T$ times, we find that

$$|\theta_n^{t+T}| \leq \gamma_{a_\ell}^2 (\theta_n^{t+T-2}) + n^{-a_r}/\sqrt{n}(1 + \gamma_m) \leq \gamma_{a_\ell}^T \theta_n^t + (1 + \gamma_{a_\ell} + \ldots + \gamma_{a_\ell}^{T-1}) n^{-a_r}/\sqrt{n} \leq e^{-Tn^{-a_\ell}} n^{-a_r} + \frac{1}{1 - \gamma_{a_\ell}} n^{-a_r}/\sqrt{n}.$$  

As $T$ is sufficiently large such that the second term is the dominant term, we find that

$$|\theta_n^{t+T}| \lesssim \frac{1}{1 - \gamma_{a_\ell}} n^{-a_r}/\sqrt{n} = n^{6a_\ell - a_r - 1/2}.$$  

Setting the RHS equal to $n^{-a_\ell}$, we obtain the recursion that

$$a_\ell = \frac{a_r}{T} + \frac{1}{14}.$$

Solving for the limit $a_\ell = a_r = a_\ast$ yields that $a_\ast = 1/12$. It suggests that we eventually have $\theta_n^t \rightarrow \mathbb{B}(0, n^{-1/2})$. As a consequence, we achieve the conclusion of the corollary.

### C.2 Proof of Lemma 2

Without loss of generality, we can assume that $|\theta| \in [\omega^{-a_{\ell+1}}, \omega^{-a_\ell}]$. Conditional on the event $\mathcal{E}$, we have that

$$|\overline{M}_1(\theta)| \leq (1 - \omega^{-6a_{\ell+1}}/5) |\theta| \quad \text{and} \quad |M_{n,1}(\theta) - \overline{M}_1(\theta)| \leq c_2 \omega^{-3a_\ell} \omega^{-1/2}.$$  

As a result, we have

$$|M_{n,1}(\theta)| \leq |M_{n,1}(\theta) - \overline{M}_1(\theta)| + |\overline{M}_1(\theta)| \leq (1 - \omega^{-6a_{\ell+1}}/5) |\theta| + c_2 \omega^{-1/2} \omega^{-3a_\ell} \leq (1 - \omega^{-6a_{\ell+1}}/5 + c_2 \omega^{-1/2} \omega^{-2a_\ell}) \omega^{-a_\ell} \leq \omega^{-a_\ell}.$$  

Here, to establish the last inequality, we have used the following observation: for $\omega = n/c_{n,\delta}$ and that $n \geq (c')^{1/\beta} c_{n,\delta}$, we have

$$c_2 \omega^{6a_{\ell+1} - 2a_\ast - 1/2} \leq 5c_2 \omega^{4a_\ast - 1/2} \leq c' \omega^{4a_\ast - 1/2} \leq c' \omega^{-4/\beta} \leq 1/(c')^3 \leq 1,$$  

which leads to $-\omega^{-6a_{\ell+1}}/5 + c_2 \omega^{-1/2} \omega^{-2a_\ell} \leq 0$. As a consequence, we achieve the conclusion of the lemma.

### C.3 Proof of Lemma 3

The proof of the perturbation bound (23b) is a standard extension of $d = 1$ case presented above in Section A.3.2, and thereby is omitted.

We now present the proof of the contraction bound (23a), which has several similarities with the proofs of bounds (12a) and (12b) from Lemma 1. In order to simplify notation, we use the shorthand $Z_{n,d} := \frac{1}{nd} \sum_{j=1}^n \|X_j\|_2^2$.

Recalling the definition (22) of operator $\widetilde{M}_{n,d}(\theta)$, we have

$$\|\widetilde{M}_{n,d}(\theta)\|_2 = \left\| E_{\theta \sim \mathcal{N}(0,1)} \left[ Y \tanh \left( \frac{Y^\top \theta}{Z_{n,d} - \|\theta\|_2^2/d} \right) \right] \right\|_2.$$  

(45)
We can find an orthonormal matrix $R$ such that $R \theta = \| \theta \|_2 e_1$, where $e_1$ is the first canonical basis in $\mathbb{R}^d$. Define the random vector $V = RY$. Since $Y \sim \mathcal{N}(0, I_d)$, we have that $V \sim \mathcal{N}(0, I_d)$. On performing the change of variables $Y = R^T V$, we find that

$$
\| \mathbb{E}_Y \left[ Y \tanh \left( \frac{V^T \theta}{Z_{n,d} - \| \theta \|_2^2/d} \right) \right] \|_2 = \| \mathbb{E}_V \left[ R^T V \tanh \left( \frac{\| \theta \|_2 V}{Z_{n,d} - \| \theta \|_2^2/d} \right) \right] \|_2
$$

where the final equality follows from the fact that

$$
\mathbb{E}[R^T V f(V_1)] = R^T \mathbb{E}[V f(V_1)] = R^T (\mathbb{E}[V f(V_1)], 0, \ldots, 0)^\top.
$$

Furthermore, the orthogonality of the matrix $R$ implies that $\| \mathbb{E}[R^T V f(V_1)] \|_2 = \| \mathbb{E}[V f(V_1)] \|_2^2$. In order to simplify the notation, we define the scalars $a, b$ and the event $\mathcal{E}_{\alpha, d}$ as follows:

$$
a := 1 - (nd)^{-\alpha}, \quad b := 1 + (nd)^{-\alpha}, \quad \text{and} \quad \mathcal{E}_{\alpha, d} = \{ |Z_{n,d} - 1| \leq (nd)^{-\alpha} \},
$$

where $\alpha$ is a suitable scalar to be specified later. Note that standard chi-squared tail bounds guarantee that

$$
\mathbb{P}(|\mathcal{E}_{\alpha, d}| \geq 1 - 2e^{-d^{2\alpha} n^{1-2\alpha}/8}.
$$

Now conditional on the event $\mathcal{E}_{\alpha, d}$, we have

$$
\| \widetilde{M}_{n, d}(\theta) \|_2 \leq \mathbb{E}_{V_1} \left[ V_1 \tanh \left( \frac{\| \theta \|_2 V}{a - \| \theta \|_2^2/d} \right) \right] = \| \mathbb{E}_{V_1} \left[ V_1 \frac{\| \theta \|_2 V}{a - \| \theta \|_2^2/d} \right] \|_2 =: \tilde{\mathbb{P}}(\theta),
$$

and

$$
\| \widetilde{M}_{n, d}(\theta) \|_2 \geq \mathbb{E}_{V_1} \left[ V_1 \tanh \left( \frac{\| \theta \|_2 V}{b - \| \theta \|_2^2/d} \right) \right] = \| \mathbb{E}_{V_1} \left[ V_1 \frac{\| \theta \|_2 V}{b - \| \theta \|_2^2/d} \right] \|_2 =: \tilde{\mathbb{P}}(\theta),
$$

where the above inequalities follow from the fact that for any fixed $y, \theta \in \mathbb{R}^d$, the function $w \mapsto y \tanh(y \| \theta \|_2 / (w - \| \theta \|_2^2/d))$ is non-increasing in $w$ for $w > \| \theta \|_2^2/d$.

Substituting $\alpha = 1/2 - 2\beta$ in the bound (48) and invoking the large sample size assumption in the theorem statement, we obtain that $\mathbb{P}[\mathcal{E}_{\alpha, d}] \geq 1 - \delta$. Putting these observations together, it remains to prove that

$$
\tilde{\mathbb{P}}(\theta) \leq \left( 1 - \frac{3|\| \theta \|_2^2/4 \right) \| \theta \|_2^2,
$$

and

$$
\tilde{\mathbb{P}}(\theta) \leq \left( 1 - \frac{1}{d} \right) \| \theta \|_2^2,
$$

for all $5(d/n)^{1/4+\beta} \leq \| \theta \|_2^2 \leq (d - 1)/(6d - 1)$ conditional on the event $\mathcal{E}_{\alpha, d}$ for $\alpha = 1/2 - 6\beta$ to obtain the conclusion of the theorem.

The proof of the claims in equation (47) relies on the following bounds on the hyperbolic function $\tanh(x)$. For any $x \in \mathbb{R}$, the following bounds hold:

$$
(\text{Upper bound}) \quad x^2 - \frac{x^4}{3} + \frac{2x^6}{15} \geq x \tanh(x) \geq x^2 - \frac{x^4}{3} \quad (\text{Upper bound}).
$$

We omit the proof of these bounds, as it is very similar to that of similar results stated and proven later in Lemma 4. We now turn to proving the bounds stated in equation (47) one-by-one.

Bounding $\tilde{\mathbb{P}}(\theta)$: Applying the upper bound (48) for $x \tanh(x)$, we obtain that

$$
\tilde{\mathbb{P}}(\theta) \leq \frac{a - \| \theta \|_2^2/d}{\| \theta \|_2^2} \left( \frac{\| \theta \|_2^2}{(a - \| \theta \|_2^2/d)^2} \mathbb{E}[V_1^2] - \frac{\| \theta \|_2^4}{3(a - \| \theta \|_2^2/d)^4} \mathbb{E}[V_1^4] + \frac{2\| \theta \|_2^6}{15(a - \| \theta \|_2^2/d)^6} \mathbb{E}[V_1^6] \right).
$$
Substituting $\mathbb{E} [V_1^{2k}] = (2k-1)!$ for $k = 1, 2, 3$ in the RHS above, we find that
\begin{equation}
\mathcal{P}(\theta) \leq \frac{1}{a - ||\theta||_2^2/d} - \frac{||\theta||_2^2}{(a - ||\theta||_2^2/d)^3} + \frac{2||\theta||_2^4}{(a - ||\theta||_2^2/d)^\alpha}.
\end{equation}

The condition $||\theta||_2^2 + (nd)^{-\alpha} \leq \frac{d-1}{6d-4} < 1/6$ implies the following bounds:
\begin{align*}
&\frac{1}{1 - (nd)^{-\alpha} - ||\theta||_2^2/d} \leq 1 + (||\theta||_2^2 + (nd)^{-\alpha}) + 3/2 \cdot (||\theta||_2^2 + (nd)^{-\alpha})^2, \\
&\frac{1}{1 - (nd)^{-\alpha} - ||\theta||_2^2/d} \geq 1 + 3 (||\theta||_2^2 + (nd)^{-\alpha}), \\
&\frac{1}{1 - (nd)^{-\alpha} - ||\theta||_2^2/d} \leq 3/2.
\end{align*}

Substituting the definitions (46a) of $a$ and $b$ and plugging the previous three bounds on the RHS of the inequality (49) yields that
\begin{align*}
\mathcal{P}(\theta) &\leq 1 + \frac{||\theta||_2^2}{d} + \frac{3||\theta||_2^4}{2d^2} - ||\theta||_2^2 \left(1 + \frac{3||\theta||_2^2}{d}\right) + 3||\theta||_2^4 + \frac{11}{2}(nd)^{-\alpha} \\
&\leq 1 - \left(1 - \frac{1}{d} \right) ||\theta||_2^2 + \left(3 - \frac{2}{d}\right) ||\theta||_2^4 + \frac{11}{2}(nd)^{-\alpha} \\
&\leq 1 - \left(1 - \frac{1}{d} \right) ||\theta||_2^2
\end{align*}
where the last step follows from the following observations that
\begin{align*}
(3 - 2/d)||\theta||_2^2 &\leq (1 - 1/d)||\theta||_2^2/2, \quad \text{for all } ||\theta||_2 \leq (d-1)/(6d-4), \\
11(nd)^{-\alpha}/2 &\leq (1 - 1/d)||\theta||_2^2/4, \quad \text{for all } ||\theta||_2 \geq (d/n)^{-1/4+\beta} \text{ when } \alpha = 1/2 - 2\beta.
\end{align*}

Therefore, the claim with an upper bound of $\mathcal{P}(\theta)$ now follows.

Bounding $\mathcal{P}(\theta)$: Using the lower bound (48) for $x \tanh(x)$, we find that
\begin{align*}
\mathcal{P}(\theta) &\geq \frac{b - ||\theta||_2^2/d}{||\theta||_2^2} \left(\frac{||\theta||_2^2}{(b - ||\theta||_2^2/d)^2} \mathbb{E} [V_1^2] - \frac{||\theta||_2^4}{3(b - ||\theta||_2^2/d)^4} \mathbb{E} [V_1^4]\right) \\
&= \frac{b - ||\theta||_2^2/d}{||\theta||_2^2} - \frac{1}{(b - ||\theta||_2^2/d)^3}.
\end{align*}

The condition $||\theta||_2 = (nd)^{-\alpha} \geq 0$ leads to
\begin{align*}
\frac{1}{1 + (nd)^{-\alpha} - ||\theta||_2^2/d} &\leq 1 + (||\theta||_2^2/d - (nd)^{-\alpha}) + (||\theta||_2^2/d - (nd)^{-\alpha})^2, \\
\frac{1}{1 + (nd)^{-\alpha} - ||\theta||_2^2/d} &\leq 1 + 4 (||\theta||_2^2/d - (nd)^{-\alpha}).
\end{align*}

Applying these inequalities to the bound (53), we obtain that
\begin{align*}
\mathcal{P}(\theta) &\geq 1 + \frac{||\theta||_2^2}{d} + \frac{||\theta||_2^4}{d^2} - ||\theta||_2^2 \left(1 + \frac{4||\theta||_2^2}{d}\right) - 2(nd)^{-\alpha} \\
&\geq 1 - ||\theta||_2^2 \left(1 - \frac{1}{d}\right) - ||\theta||_2^2 \left(\frac{4}{d} - \frac{1}{d^2}\right) - \frac{||\theta||_2^4(1 - 1/d)}{11} \\
&\geq 1 - \frac{3||\theta||_2^2}{4},
\end{align*}
where step (i) in the above inequalities follows from the observations (50)-(51) above. The lower bound (47) for $\mathcal{P}(\theta)$ now follows.
C.4 Proof of Lemma 4

The proof of this lemma relies on an evaluation of coefficients with $x^{2k}$ as $k \geq 1$. In particular, we divide the proof of the lemma into two key parts:

**Upper bound:** From the definition of hyperbolic function $\tanh(x)$, it is sufficient to demonstrate that

$$x (\exp(x) - \exp(-x)) \leq \left(x^2 - \frac{x^4}{3} + \frac{2x^6}{15} - \frac{17x^8}{315} + \frac{62x^{10}}{2835}\right) (\exp(x) + \exp(-x)).$$

Invoking the Taylor series of $\exp(x)$ and $\exp(-x)$, the above inequality is equivalent to

$$\sum_{k=0}^{\infty} \frac{2x^{2k+2}}{(2k+1)!} \leq \left(x^2 - \frac{x^4}{3} + \frac{2x^6}{15} - \frac{17x^8}{315} + \frac{62x^{10}}{2835}\right) \left(\sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!}\right).$$

Our approach to solve the above inequality is to show that the coefficients of $x^{2k}$ in the LHS is smaller than that of $x^{2k}$ in the RHS for all $k \geq 1$. In fact, when $1 \leq k \leq 3$, we can quickly check that the previous observation holds. For $k \geq 4$, it suffices to validate that

$$\frac{2}{(2k)!} - \frac{2}{3(2k-2)!} + \frac{4}{15(2k-4)!} - \frac{34}{315(2k-6)!} + \frac{124}{2835(2k-8)!} - \frac{2}{(2k+1)!} \geq 0,$$

Direct computation with the above inequality leads to

$$(k - 1)(k - 2)(k - 3)(k - 4)(496k^4 - 1736k^3 + 1430k^2 + 446k - 381) \geq 0$$

for all $k \geq 4$, which is always true. As a consequence, we achieve the conclusion with the upper bound of the lemma.

**Lower bound:** For the lower bound of the lemma, it is equivalent to prove that

$$\sum_{k=0}^{\infty} \frac{2x^{2k+2}}{(2k+1)!} \geq \left(x^2 - \frac{x^4}{3} + \frac{2x^6}{15} - \frac{17x^8}{315}\right) \left(\sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!}\right).$$

Similar to the proof technique with the upper bound, we only need to verify that

$$\frac{2}{(2k)!} - \frac{2}{3(2k-2)!} + \frac{4}{15(2k-4)!} - \frac{34}{315(2k-6)!} - \frac{2}{(2k+1)!} \leq 0$$

for any $k \geq 3$. The above inequality is identical to

$$(k - 1)(k - 2)(k - 3)(k - 4)(4352k^3 - 4352k^2 - 512k + 1472) \geq 0$$

for all $k \geq 3$, which always holds. Therefore, we obtain the conclusion with the lower bound of the lemma.

C.5 Proof of Lemma 5

The proof of this lemma is based on appropriate truncation argument. More concretely, given any positive scalar $\tau$, and the random variable $X \sim \mathcal{N}(0, 1)$, consider the pair of truncated random variables $(Y, Z)$ defined by:

$$Y := X_{2k\|X\| \leq \tau} \quad \text{and} \quad Z := X_{2k\|X\| \geq \tau}.$$  \hfill (54)

With the above notation in place, for $n$ i.i.d. samples $X_1, \ldots, X_n$ from $\mathcal{N}(0, 1)$, we have

$$\frac{1}{n} \sum_{i=1}^{n} X_i^{2k} = \frac{1}{n} \sum_{i=1}^{n} Y_i + \frac{1}{n} \sum_{i=1}^{n} Z_i := S_{Y,n} + S_{Z,n}.$$
where $S_{Y,n}$ and $S_{Z,n}$ denote the averages of the random variables $Y_i$'s and $Z_i$'s respectively. Observe that $|Y_i| \leq \tau^{2k}$ for all $i \in [n]$; consequently, by standard sub-Gaussian concentration of bounded random variables, we have

$$\mathbb{P} (|S_{Y,n} - \mathbb{E}[Y]| \geq t_1) \leq 2 \exp \left( - \frac{nt_1^2}{2\tau^{4k}} \right). \tag{55}$$

Next, applying Markov’s inequality with the non-negative random variable $S_{Z,n}$, we find that

$$\mathbb{P} (S_{Z,n} \geq t_2) \leq \frac{\mathbb{E}[S_{Z,n}]}{t_2} = \frac{\mathbb{E}[Z]}{t_2}. \tag{56}$$

By definition of the truncated random variable $Y$, we have $\mathbb{E}[Y] \leq \mathbb{E}[X^{2k}]$; moreover, an application of Holder’s inequality to $\mathbb{E}[Z]$ yields

$$\mathbb{E}[Z] = \mathbb{E} \left( X^{2k} I_{|X| \geq \tau} \right) \leq \sqrt{\mathbb{E}[X^{4k}]} \sqrt{\mathbb{P}(|X| \geq \tau)} \leq \sqrt{2\mathbb{E}[X^{4k}]} \exp(-\tau^2/4).$$

Combining the bounds on $\mathbb{E}[Y]$ and $\mathbb{E}[Z]$ with the inequalities (55) and (56) we deduce that

$$\frac{\sum_{i=1}^{n} X_{i}^{2k}}{n} \leq \mathbb{E}[Y] + t_1 + t_2 \leq \mathbb{E}[X^{2k}] + t_1 + t_2, \quad \text{and,} \tag{57a}$$

$$\frac{\sum_{i=1}^{n} X_{i}^{2k}}{n} \geq \mathbb{E}[X^{2k}] - t_1 - t_2 \sqrt{2\mathbb{E}[X^{4k}]} \exp(-\tau^2/4) \tag{57b}$$

with probability at least $1 - \exp \left( - \frac{nt_1^2}{2\tau^{4k}} \right) - \sqrt{2\mathbb{E}[X^{4k}]} \exp(-\tau^2/4)$. Finally, given any $\delta > 0$, choose the scalars $\tau, t_1, t_2$ as follows:

$$\tau = 2 \sqrt{\log \left( \frac{2\sqrt{2n\mathbb{E}[X^{4k}]}}{\delta} \right)}, \quad t_1 = \tau^2 \sqrt{\frac{1}{n} \log \left( \frac{2}{\delta} \right)} \quad \text{and} \quad t_2 = \frac{1}{\sqrt{n}}.$$

Substituting the choice of $t_1, t_2$ and $\tau$, in bounds (57a) and (57b) we conclude that with probability at least $1 - \delta$

$$\left| \frac{\sum_{i=1}^{n} X_{i}^{2k}}{n} - \mathbb{E}[X^{2k}] \right| \leq \frac{C_k \log^{k}(n/\delta)}{\sqrt{n}},$$

where $C_k$ is a universal constant that depends only on $k$. This completes the proof of Lemma 5.

## C.6 Proof of one step bound for population EM

We now describe a special one-step contraction property of the population operator.

**Lemma 6.** For any vector $\theta^0$ such that $\|\theta^0\| \leq \sqrt{d}$, we have $\|\tilde{M}_{n,d}(\theta^0)\| \leq \sqrt{2/\pi}$ with probability at least $1 - \delta$.

The proof of this lemma is a straightforward application of the proof argument in Lemma 3 in Appendix C.3. In order to simplify notations, we use the shorthand $Z_{n,d} = \sum_{j=1}^{n} ||X_j||^2/(nd)$. Recalling the definition (22) of operator $\tilde{M}_{n,d}$, we have

$$\|\tilde{M}_{n,d}(\theta)\|_2 = \mathbb{E}_{Y \sim \mathcal{N}(0,1)} \left[ Y \tanh \left( \frac{Y^\top \theta}{Z_{n,d} - ||\theta||^2/d} \right) \right]_2.$$

As demonstrated in the proof of Theorem 2, we have the equivalence

$$\|\tilde{M}_{n,d}(\theta)\|_2 = \mathbb{E} \left[ V_1 \tanh \left( \frac{||\theta||_2 V_1}{Z_{n,d} - ||\theta||^2/d} \right) \right].$$
where \( V_1 \sim \mathcal{N}(0, 1) \). Since the function \( x \tanh \left( \frac{\|\theta\|_2 x}{a - \|\theta\|_2^2 / d} \right) \) is an even function in terms of \( x \) for any given \( a \), we find that

\[
E \left[ V_1 \tanh \left( \frac{\|\theta\|_2 V_1}{Z_{n,d} - \|\theta\|_2^2 / d} \right) \right] = E \left[ |V_1| \tanh \left( \frac{\|\theta\|_2 |V_1|}{Z_{n,d} - \|\theta\|_2^2 / d} \right) \right] \\
\leq E[|V_1|] = \sqrt{\frac{2}{\pi}}
\]

where the second inequality is due to the basic inequality \( \tanh(x) \leq 1 \) for all \( x \in \mathbb{R} \). The inequality in the above display implies that regardless of the initialization \( \theta^0 \), we always have \( \|\hat{M}_{n,d}(\theta)\|_2 \leq \sqrt{\frac{2}{\pi}} \), as claimed.

**D Wasserstein Distance**

In Figures 1 and 3, we use EM to estimate all the parameters of the fitted Gaussian mixture (e.g., the parameters \( \{w_i, \mu_i, \Sigma_i, i \in [k]\} \) if the fitted mixture were \( G = \sum_{i=1}^{k} w_i \mathcal{N}(\mu_i, \Sigma_i) \)) and use first-order Wasserstein distance between the fitted model and the true model to measure the quality of the estimate. Here we briefly summarize the definition of the first-order Wasserstein distance and refer the readers to the book [25] and the paper [14] for more details. Given two Gaussian mixture distributions of the form

\[
G = \sum_{i=1}^{k} w_i \mathcal{N}(\mu_i, \Sigma_i) \quad \text{and} \quad G' = \sum_{j=1}^{k'} w_j \mathcal{N}(\mu'_j, \Sigma'_j),
\]

the first-order Wasserstein distance between the two is given by

\[
W_1(G, G') = \inf_{q \in Q} \sum_{i=1}^{k} \sum_{j=1}^{k'} q_{ij} \left( \|\theta_i - \theta'_j\|_2 + \|\Sigma_i - \Sigma'_j\|_F \right), \tag{58}
\]

where \( \|A\|_F \) denotes the Frobenius norm of the matrix \( A \) (which in turn is defined as \( \sqrt{\sum_{ij} A_{ij}^2} \)). Moreover, \( Q \) denotes the set of all couplings on \([k] \times [k']\) such that

\[
q_{ij} \in [0, 1], \quad \sum_{i=1}^{k} q_{ij} = w'_j \quad \text{and} \quad \sum_{j=1}^{k'} q_{ij} = w_i \quad \text{for all} \quad i \in [k], j \in [k'].
\]

We note that the optimization problem (58) is a linear program in the \( k \times k' \) dimensional variable \( q \) and standard linear program solvers can be used for solving it. Also, we remark that here we have abused the notation slightly since the the definition of the Wasserstein distance above is typically used for the mixing measures which only depends on the parameters of the Gaussian mixture (and not the Gaussian density). Finally, applying definition (58), we can directly conclude that for the symmetric fit (1), we have

\[
W_1 \left( \frac{1}{2} \mathcal{N}(\theta, \sigma^2 I_d) + \frac{1}{2} \mathcal{N}(-\theta, \sigma^2 I_d), \mathcal{N}(\theta_*, \sigma_*^2 I_d) \right) = \|\theta - \theta_*\|_2 + \sqrt{d/2} \sqrt{\sigma^2 - \sigma_*^2}, \tag{59}
\]

where we have assumed that \( \min \{\|\theta - \theta_*\|_2, \|\theta - \theta_*\|_2\} = \|\theta - \theta_*\|_2 \).