Local Lipschitz continuity for energy integrals with slow growth

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Abstract
We consider some energy integrals under slow growth, and we prove that the local minimizers are locally Lipschitz continuous. Many examples are given, either with subquadratic $p, q$—growth and/or anisotropic growth.

Keywords Elliptic equations · Local minimizers · Local Lipschitz continuity · $p, q$—growth · General growth

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1 Prologue

When concerned with the $W^{1,\infty}$ or $C^{1,\alpha}$ regularity of local minimizers of energy integrals of the calculus of variations of the type

$$F(u) = \int_{\Omega} f(Du(x)) \, dx$$

(1.1)

we are naturally led to require a qualified convexity condition on the energy integrand $f : \mathbb{R}^n \to \mathbb{R}$; more precisely, on the quadratic form of the $n \times n$ matrix of the second derivatives $D^2 f = (f_{\xi\xi})$ of $f$

$$g_1(|\xi|)|\lambda|^2 \leq \sum_{i,j=1}^n f_{\xi_i\xi_j}(\xi) \lambda_i \lambda_j \leq g_2(|\xi|)|\lambda|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n,$$

(1.2)

where $g_1, g_2 : [0, +\infty) \to [0, +\infty)$ are given nonnegative real functions which allow us to control the ellipticity in the minimization problem. Of course $g_1(t) \leq g_2(t)$ for all $t \in [0, +\infty)$; if $g_1$ is positive and there exists a constant $M \geq 1$ such that $g_2(t) \leq Mg_1(t)$ for all $t \in [0, +\infty)$, then we say that we are dealing with an uniformly elliptic problem. This is the case when the quadratic form $\sum_{i,j=1}^n f_{\xi_i\xi_j}(\xi) \lambda_i \lambda_j$ has a simpler equivalent behavior as $g_1(|\xi|)|\lambda|^2$ and $g_2(|\xi|)|\lambda|^2$ and the regularity process can work easier. However, the assumption $g_2 \leq Mg_1$ rules out many interesting energy integrals; in this paper, we do not assume this uniformly elliptic condition.

For instance, in the special case $f(\xi) = g(|\xi|)$ with $g : [0, +\infty) \to \mathbb{R}$, a direct computation (see for instance (6.3) in [32] and [35]) shows that

$$g_1(t) = \min \left\{ g''(t), \frac{g'(t)}{t} \right\}, \quad g_2(t) = \max \left\{ g''(t), \frac{g'(t)}{t} \right\}, \quad \forall t \in [0, +\infty),$$

where $g'$, $g''$ are the first and the second derivatives of $g$. In this context, the uniformly elliptic case corresponds to compare $g''(t)$ and $\frac{g'(t)}{t}$; i.e., to ask for the existence of two positive constants $m, M$ such that $m \frac{g'(t)}{t} \leq g''(t) \leq Mg'(t)$ for all $t \in (0, +\infty)$. The $p$–Laplacian $f(\xi) = |\xi|^p$ with $p > 1$ is a main example of uniformly elliptic energy integrand, with $g(t) = t^p$ and $\frac{g'(t)}{t} = p$. Within this uniformly elliptic context—however, nonlinearities of possibly non-polynomial type are allowed—we quote the global (i.e., up to the boundary) Lipschitz regularity results by Cianchi–Maz’ya for a class of quasilinear elliptic equations [9] and for a class of nonlinear elliptic systems [10].

Also, some energy integrands of $p, q$–growth can be uniformly elliptic; for instance, an integrand, which does not behave like a power, but which, however, is an uniformly elliptic energy integrand, is $f(\xi) = |\xi|^{a+b \sin(\log |\xi|)}$; in this case, $f(\xi)$ is a convex function for $|\xi| \geq e$ if $a, b > 0$ and $a > 1 + b \sqrt{2}$. This function $f$ satisfies the $p, q$–growth conditions with $p = a - b$ and $q = a + b$. It can be shown (see [4, 5]) that $f(\xi)$ satisfies the $\Delta_2$-condition. To notice, however, that some convex functions $f(\xi) = g(|\xi|)$ of $p, q$–growth with $p > 1$ and $q > p$ arbitrarily close to $p$ exist, they do not satisfy the $\Delta_2$-condition and the corresponding variational problem are not uniformly elliptic; see Kranso’skij-Rutickii [28, p. 28–29], Focardi-Mascolo [23, p. 342–343], Chlebicka [8, Section 2.4] and Bögelein-Duzaar-Marcellini-Scheven [5, Remark 3.3].

In this research, we are concerned with the $W^{1,\infty}$ regularity of the local minimizers of energy integrals of the calculus of variations of the type (1.1), when the quadratic form of
the second derivatives $D^2f = \left( f_{x_jx_k} \right)$ of $f$ is governed by (1.2) where $g_1, g_2 : [0, +\infty) \to [0, +\infty)$ are given nonnegative real functions, not only of polynomial type. The first local Lipschitz-continuity results under this general context have been proposed in the ’90s in [32, 33] by assuming, among other conditions, that $g_1(t), g_2(t)$ are increasing functions in $[0, +\infty)$. This, when typified by the model case $g_1(t) = t^q - t^{q-2}$, $g_2(t) = t^{r-2}$, gives $q \geq p \geq 2$. The approach to regularity, governed by (1.2) with general $g_1, g_2$ functions not necessarily monotone functions, was given by Marcellini-Papi [35]. Related regularity results, with energy-integrands $f(x, \xi) = g(x, |\xi|)$ depending on $x$ too, are due to Mascolo-Migliorini [36], Beck-Mingione [2], Di Marco-Marcellini [19], De Filippis-Mingione [17] and recently Cupini-Marcellini-Mascolo-Passarelli di Napoli [16]. See also Apushkinskaya-Bildhauer-Fuchs [1] for a local gradient bound of a priori bounded minimizers.

More precisely, in [2] Beck-Mingione consider the vector-valued case of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f(x, \xi) = g(|\xi|) + h(x)u$ with the main part $g(|\xi|)$ modulus dependent, as well as in [17, 19, 36]; they also study the scalar case $m = 1$ with the more general integrand not modulus dependent, i.e., of the form $f(x, \xi) = g(\xi) + h(x)u$, however, with a growth assumption from below of power type for some fixed exponent greater than 1 (see (1.33) in [2]).

Here, we focus our attention to slow growth integrands, for which the state-of-the-art is not so established. We give a general local $W^{1,\infty}$ regularity result for the minimizers of energy integrals of the type (1.1),(1.2) with an energy integrand $f = f(\xi)$ not necessarily depending on the modulus of $\xi$ and with $g_1, g_2 : [0, +\infty) \to [0, +\infty)$ nonnegative decreasing real functions (more precisely we require that only $g_2$ is a decreasing function), not only of polynomial type. Precise statements can be found in the next section. We treat general slow growth conditions under the ellipticity condition (1.2), where $g_2(t)$ is a decreasing (not necessarily strictly decreasing) function with respect to $t$; of course in the model case $g_2(t) = M \left( 1 + |\xi|^2 \right)^{\frac{1}{\alpha}}$ this corresponds to $q \leq 2$. As already said, in this article we do not assume uniformly elliptic conditions, nor the modulus dependence as $f(\xi) = g(|\xi|)$.

In this regularity field, specific references for slow growth are Fuchs-Mingione [24], who concentrated on the nearly linear growth, such as for instance the logarithmic case $f(\xi) = g(|\xi|) = |\xi| \log(1 + |\xi|)$; also Bildhauer, in his book [3], considered nearly-linear growth. Leonetti-Mascolo-Siepe [29] considered the subquadratic $p,q$-growth with $1 < p < q < 2$, with energy densities for instance the type $f(\xi) = g(|\xi|) = |\xi|^2 \log^a(1 + |\xi|)$.

Here, we emphasize some examples which enter in our regularity theory and which seem not to be considered in the mathematical literature on this subject. The first one is complementary to the case considered by Bousquet-Brasco [6] for exponents $p_i \geq 2$ for all $i = 1, 2, \ldots, n$; in fact, here, we can treat the model energy-integral (see Example 3.3)\[ F_1(u) = \int_{\Omega} \sum_{i=1}^{n} \left( 1 + u_i^2 \right)^{\frac{p_i}{2}} dx \] when $1 < p_i \leq 2$ for all $i = 1, 2, \ldots, n$. In Sect. 4, we propose some further examples of anisotropic energy integrands which seem to be new in the mathematical literature on this subject.

The main regularity result that we propose in this manuscript is Theorem 2.1 stated in the next section. It gives a more general regularity result than similar results that can be found in the recent mathematical literature on $p,q$-growth; see in particular the Remark 4.3 for details. Also, the integral $\int_{\Omega} |Du| \log^a (1 + |Du|) dx$, for every $a > 0$, enters in the
regularity result of Theorem 2.1. Of course, a by-product of our general Theorem 2.1 is also the \( p, q \)-growth case, when the ellipticity conditions (1.2) are satisfied with \( g_1(|\xi|) = m |\xi|^{\frac{p-2}{n}} \), \( g_2(|\xi|) = M \left( 1 + |\xi|^2 \right)^{\frac{\beta n}{2}} \), for some positive constants \( m, M \) and exponents \( 1 < p \leq q \leq 2 \) such that \( \frac{q}{p} < 1 + \frac{\beta}{n} \). As well known, this condition guarantees the Lipschitz continuity of the solutions also when \( q \geq p > 1 \) and classically this is nowadays a well-known constraint for the \( p, q \)-growth (see [31, 33, 34]).

The regularity results are stated in the next section, while in Sects. 3 and 4 some examples are considered in more details. The other sections are devoted to the proofs.

2 Introduction and statement of the main results

We assume that \( f : \mathbb{R}^n \to [0, +\infty) \) is a convex function in \( C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus B_{t_0}(0)) \) for some \( t_0 \geq 0 \), satisfying the following growth condition: there exist two continuous functions \( g_1, g_2 : [t_0, +\infty) \to (0, +\infty) \) and positive constants \( C_1, C_2, \alpha, \beta, \mu \in [0, 1] \) such that

\[
\begin{align*}
g_1(|\xi|)|\lambda|^2 &\leq \sum_{i,j=1}^n f_{\xi_i \xi_j} (\xi) \lambda_i \lambda_j \leq g_2(|\xi|)|\lambda|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n, |\xi| \geq t_0 \\
t &\mapsto t^{\alpha} g_2(t) \text{ is decreasing and } t \mapsto t g_2(t) \text{ is increasing} \\
\left( g_2(t) \right)^{\frac{n-2}{n}} &\leq C_1 t^2 |\lambda|^2, \quad \frac{1}{2} < \beta < \frac{2}{n}, \quad \forall t \geq t_0 \\
g_2(|\xi|)|\xi|^2 &\leq C_2 \left( 1 + f(\xi) \right)^\mu, \quad \alpha > 1, \quad \forall \xi \in \mathbb{R}^n, |\xi| \geq t_0 \\
f(\xi)/|\xi| &\to +\infty \quad \text{as } |\xi| \to \infty
\end{align*}
\]

where \( \frac{n-2}{n} \) in (2.1)\textsubscript{3}, in the case \( n = 2 \), must be replaced with any fixed positive number less than \( 1 - \beta \).

It is worth to highlight that we require uniform convexity and growth assumptions on \( f = f(\xi) \) only for large value of \( |\xi| \) ([7, 20–22]). We say that \( u \in W^{1,1}_0(\Omega) \) is a local minimizer of the integral functional \( F \) in (1.1) if \( f(Du) \in L^1_{\text{loc}}(\Omega) \) and

\[
\int_{\Omega'} f(Du) \, dx \leq \int_{\Omega'} f(Du + D\varphi) \, dx
\]

for every open set \( \Omega', \Omega' \subseteq \Omega \) and for every \( \varphi \in W^{1,1}_0(\Omega') \). The result for slow growth conditions, under the ellipticity condition (1.2) with \( g_1(t) \) and \( g_2(t) \) general functions, can be stated as follows.

**Theorem 2.1 (general growth)** Let us assume that \( f \) satisfies the growth assumptions in (2.1), with the parameters \( \alpha, \beta, \mu \) related by the condition

\[
2 - \mu - \alpha(n\beta - \mu) > 0.
\]

Then, any minimizer \( u \in W^{1,1}_0(\Omega) \) of (1.1) is locally Lipschitz continuous in \( \Omega \) and, for every \( 0 < \rho < R, B_R \subseteq \Omega \), there exists a positive constant \( C \) depending on \( \rho, R, C_1, C_2, \alpha, \beta, \mu, g_2(t_0) \), such that

\[
\|Du\|_{L^\infty(B_R, \mathbb{R}^n)} \leq C \left\{ \frac{1}{(R - \rho)^\mu} \int_{B_R} \left\{ 1 + f(Du) \right\} \, dx \right\}^\theta
\]
where \( \theta = \frac{(2-\mu)a}{2-\mu-a(\eta-\mu)} \).

When we specialize Theorem 2.1 to the subquadratic \( p, q \)-growth, we obtain:

**Corollary 2.2** (\( p, q \)-growth) Let \( f = f(\xi) \) be a convex function in \( C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus B_{t_0}(0)) \) for some \( t_0 \geq 0 \), satisfying the ellipticity conditions

\[
m|\xi|^{p-2} \lambda^2 \leq \sum_{i,j=1}^{n} f_{i,j}(\xi) \lambda_i \lambda_j \leq M(1+|\xi|^2)^{\frac{q-2}{2}} |\lambda|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n : |\xi| \geq t_0, \tag{2.4}
\]

for some positive constants \( m, M \) and exponents \( p, q, 1 \leq p \leq q \leq 2 \), such that

\[
\frac{q}{p} < 1 + \frac{2}{n}. \tag{2.5}
\]

Then, every local minimizer \( u \in W^{1,p}_{\text{loc}}(\Omega) \) to the energy integral in (1.1) is of class \( W^{1,\infty}_{\text{loc}}(\Omega) \) and there exists a constant \( C > 0 \), depending only on \( p, q, n, m, M \), such that, for all \( \rho, R \) with \( 0 < \rho < R \leq \rho + 1 \),

\[
||Du||_{L^\infty(B_R;\mathbb{R}^n)} \leq C \left\{ \frac{1}{(R-\rho)^n} \int_{B_R} \left[ 1 + f(Du) \right] dx \right\}^{\frac{2}{(n+2)p-nq}}. \tag{2.6}
\]

Let us briefly sketch the tools and the techniques to prove the above regularity results. A first step is an a priori estimate for smooth minimizers through the interpolation result stated in Lemma 5.1. The second step is an approximation procedure: we construct a sequence of smooth strictly convex functions \( f_k \), each of them being equal to \( f \) for large \( |\xi| \), in the same outlook in [33, 37]. More in details, if \( u \) is a local minimizer of (1.1), we consider the sequence of variational problems in a ball \( B_R, \overline{B}_R \subset \Omega \), with as integrand a suitable perturbation of \( f_k \) and boundary value data \( u_c = u \ast \varphi_k \), where \( \varphi_k \) are smooth mollifiers. Each \( u_c \) satisfies the bounded slope condition; then, by the well-know existence and Lipschitz regularity theorem by Hartman-Stampacchia [27] each problem has a unique Lipschitz continuous solution \( v_c \). By applying the a priori estimate to the sequence of the solutions, we get an uniform control in \( L^\infty \) of the gradient of \( v_c \), which allows us to transfer the Lipschitz continuity property to the original minimizer \( u \).

The plan of the paper is the following: in Sects. 3, 4, we present some examples, some of them being new in this context of general growth conditions. In Sect. 5, we give the interpolation lemma. In Sect. 6, we prove the a priori estimate for functionals with general slow growth by means of the interpolation lemma. In the last section, we prove the regularity results. As we show in the next section, the class of energy integrals that we consider is quite large, not only polynomial unbalanced \( p, q - \text{subquadratic growth} \) as in the Corollary 2.2, but also logarithmic growth (as in Examples 3.1 and 3.2) and anisotropic behavior (Example 3.3).

### 3 Examples

In this section, we present some examples of density function \( f \) for which the above assumptions hold.
Example 3.1 $f(\xi) = |\xi|^2(\log|\xi|)^{\alpha}$, $\alpha > 0$, $|\xi| \geq t_0 \geq 1$. For large $t$ (2.1) holds for $g_1(t) = \frac{2}{t(\log t)^{\alpha-1}}$ and $g_2(t) = (1 + \alpha)\frac{1}{t^{\alpha}}$. It is easy to check that (2.1) and (2.1) hold for every $\beta > \frac{1}{\alpha}$. Since $g_2(|\xi|)|\xi|^2 = (1 + \alpha)f(\xi)$, (2.1) holds for every $\alpha > 1$. Moreover, for every $\mu < \frac{1}{n}$, $\nu^\mu g_2(t)$ is decreasing in $[t_0, +\infty)$, choosing $\alpha > \frac{1}{n\beta-1}$, (2.2) follows. Therefore, Theorem 2.1 applies for every $\alpha > 0$.

Example 3.2 $f(\xi) = (|\xi| + 1)\log(|\xi|)$, $g(t) = (1 + t)L_k(t)$, $k \in \mathbb{N}$, $L_k$ defined as:

$\quad L_1(t) = \log(1 + t), \quad L_{k+1}(t) = \log(1 + L_k(t));$

therefore

$\quad L_1(t) = \frac{1}{1 + t}, \quad L_{k+1}'(t) = \frac{L_k'(t)}{1 + L_k(t)} = \frac{1}{(1 + t)(1 + L_1(t)) \cdots (1 + L_{k-1}(t))}.$

Then, we get

$\quad g'(t) = L_k(t) + \frac{1}{(1 + L_1(t)) \cdots (1 + L_{k-1}(t))} \implies g_2(t) = \frac{2}{1 + t}L_k(t);$

$\quad g''(t) = \frac{1}{(1 + t)(1 + L_1(t)) \cdots (1 + L_{k-1}(t))} \left[ 1 - \sum_{i=1}^{k-1} \frac{1}{(1 + L_1(t)) \cdots (1 + L_i(t))} \right] \implies g_1(t) = \frac{1}{2(1 + t)(1 + L_1(t)) \cdots (1 + L_{k-1}(t))}.$

Similarly to the Example 3.1, for $t$ large enough, $\mu = 1$ and $\beta = \frac{1}{\alpha}$, (2.1) and (2.1) hold. Moreover, $g_2(|\xi|)|\xi|^2 \leq 2f(\xi)$; therefore, (2.1) holds for every $\alpha > 1$. Since we can choose $\alpha$ and $\beta$ such that (2.2) holds, every local minimizer of the corresponding integral is locally Lipschitz continuous (see [24] for related results).

Example 3.3 Next, we consider the anisotropic case of the energy integral in (1.1) with

$\quad f(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n,$

where the exponents $p_i$ are greater than or equal to 2 for all $i = 1, 2, \ldots, n$. Of course, $f(\xi)$ in (3.1) is a convex function in $\mathbb{R}^n$. Note that the $n \times n$ matrix of the second derivatives $D^2f = \left(f_{\xi_i\xi_j}\right)$ of $f$ is diagonal and the corresponding quadratic form is given by

$\quad \sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j = \sum_{i=1}^{n} p_i(p_i - 1)\frac{|\xi_i|^{p_i-2}}{|\lambda_i|^2}, \quad \forall \lambda, \xi \in \mathbb{R}^n.$

This quadratic form is positive semidefinite but is not definite if (at least) one of the exponents $p_i$ is greater than 2; in fact, if for instance $p_1 > 2$, then $\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j = 0$ when $\xi = (\xi_1, 0, \ldots, 0) \neq 0$ and $\lambda = (0, \lambda_2, \ldots, \lambda_n) \neq 0$. Nevertheless, in spite of this lack of uniform convexity, without using the quadratic form in (3.2), the local $L^\infty$-bound of the minimizers has been established in [11–15, 18, 25] under some optimal conditions on the exponents $p_i > 1$. More recently, Bousquet-Brasco [6] proved that bounded minimizers of the energy integral (1.1), with $f$ as in (3.1), are locally Lipschitz continuous in $\Omega$ under the condition $p_i \geq 2$ for all $i = 1, 2, \ldots, n$.  

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In our context of slow growth, we emphasize the locally Lipschitz regularity that we deduce by Theorem 2.1 when 1 < \( p_i \leq 2 \) for all \( i = 1, 2, \ldots, n \), which should make more complete the case considered by Bousquet-Brasco [6]. More precisely, we have to change the model example \( f(\xi) \) in (3.1) since \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) there is not a function of class \( C^2 \) around \( \xi = 0 \) when \( p_i < 2 \) for some \( i \in \{1, 2, \ldots, n\} \). The corresponding not-singular model is

\[
f(\xi) = \sum_{i=1}^{n} \left( 1 + \varepsilon^2 \right)^{-\frac{p_i}{2}}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n.
\]  

(3.3)

Similarly to (3.2), we obtain the quadratic form of the \( n \times n \) matrix of the second derivatives \( D^2 f = \left( f_{\xi_i \xi_j} \right) \) of \( f \) in (3.3)

\[
\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j = \sum_{i=1}^{n} p_i \left( 1 + (p_i - 1) \varepsilon^2 \right) \left( 1 + \varepsilon^2 \right)^{-\frac{p_i}{2}} |\lambda_i|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n.
\]  

(3.4)

Since \( p_i - 2 \leq 0 \) for every \( i \in \{1, 2, \ldots, n\} \), then

\[
(1 + (p_i - 1) \varepsilon^2) \left( 1 + \varepsilon^2 \right)^{-\frac{p_i}{2}} \geq (p_i - 1) \left( 1 + \varepsilon^2 \right)^{-\frac{p_i}{2}} \geq (p_i - 1)\left( 1 + |\xi|^2 \right)^{-\frac{p_i}{2}} \geq (p_i - 1)\left( 1 + |\xi|^2 \right)^{-\frac{p_i}{2}}
\]

where \( p =: \min \{ p_i : i = 1, 2, \ldots, n \} \). We obtain

\[
\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j \geq p(p - 1) \left( 1 + |\xi|^2 \right)^{-\frac{p}{2}} |\lambda|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n.
\]  

(3.5)

Again for every \( i \in \{1, 2, \ldots, n\} \), since \( p_i - 2 \leq 0 \) we also have

\[
p_i \left( 1 + (p_i - 1) \varepsilon^2 \right) \left( 1 + \varepsilon^2 \right)^{-\frac{p_i}{2}} \leq p_i \left( 1 + \varepsilon^2 \right)^{-\frac{p_i}{2}} \leq p_i
\]

and thus, from (3.4) we deduce

\[
\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j \leq 2|\lambda|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n.
\]  

(3.6)

Therefore, by (3.5) and (3.6), we have

\[
g_1(|\xi|)|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j \leq g_2(|\xi|)|\lambda|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n,
\]  

(3.7)

where \( g_1, g_2 : [0, +\infty) \rightarrow (0, +\infty) \) are the nonnegative real functions defined by \( g_1(t) = p(p - 1)(1 + t^2)^{-\frac{p}{2}} \) and \( g_2(t) = g_2 \) constantly equal to 2. By Corollary 2.2 with \( q = 2 \), we obtain the further regularity result too.

**Corollary 3.4** (anisotropic energy integrals with slow growth) Let \( f = f(\xi) \) be the model convex function in (3.3), with \( 1 < p_i \leq 2 \) for all \( i = 1, 2, \ldots, n \). If

\[
\frac{2}{p} < 1 + \frac{2}{n} \iff p > \frac{2n}{n + 2}, \quad \text{where} \quad p =: \min_{i \in \{1, 2, \ldots, n\}} \{p_i\},
\]  

(3.8)
then every local minimizer \( u \in W^{1,p}_{\text{loc}}(\Omega) \) to the energy integral (1.1), with \( f(\xi) \) in (3.3), is of class \( W^{1,\infty}_{\text{loc}}(\Omega) \) and there exists a constant \( C > 0 \) depending only on \( p, n, m, M, \) such that, for all \( \rho, R \) with \( 0 < \rho < R \leq \rho + 1, \)

\[
\|Du(x)\|_{L^\infty(B_\rho; \mathbb{R}^n)} \leq \left( \frac{C}{(R - \rho)^n} \int_{B_R} \{1 + f(Du)\} \, dx \right)^{\frac{2}{1 + 2p - 2p}}.
\]

Note that when \( n = 2, \) the bound in (3.8) simply reduces to \( 1 < p_i \leq 2 \) for all \( i = 1, 2, \ldots, n. \) More generally, we can consider energy integrands of the form

\[
f(\xi) = \sum_{i=1}^{n} g(\xi_i), \quad \text{or} \quad f(\xi) = \sum_{i=1}^{n} g_i(\xi_i),
\]

where, for instance, \( g(t) \) or \( g_i(t) \) are one of the functions considered above in Examples 3.1 and 3.2.

\section{4 New examples of anisotropic energy functions}

We provide some applications of our Theorem 2.1, and we infer the Lipschitz continuity of the local minimizers to some class of functionals with anisotropic behavior.

\begin{example}
Consider

\[
f(\xi) = \sqrt{\sum_{i=1}^{n} \left( 1 + |\xi_i|^2 \right)^{p_i}}, \quad p_i > 1, \quad \forall \, i = 1, \ldots, n.
\]

(4.1)

With the same argument of Example 3.3, we have

\[
\frac{1}{\sqrt{n^p}} \left( 1 + |\xi|^2 \right)^{\frac{p}{2}} \leq f(\xi) \leq \sqrt{n} \left( 1 + |\xi|^2 \right)^{\frac{q}{2}}
\]

where \( p = \min_i p_i \) and \( q = \max_i p_i. \) Let us denote by \( Q(\xi, \lambda) \) the quadratic form

\[
Q(\xi, \lambda) = \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j, \quad \forall \, \lambda, \xi \in \mathbb{R}^n.
\]

(4.2)

We have

\[
f_{\xi_i \xi_i} = -p_i^2 \xi_i^2 \frac{(1 + \xi_i^2)^{2p_i - 2}}{2}\lambda_i, \quad i = 1, \ldots, n,
\]

\[
f_{\xi_i \xi_j} = -p_i p_j \xi_i \xi_j \frac{(1 + \xi_i^2)^{p_i - 1}(1 + \xi_j^2)^{p_j - 1}}{2}\lambda_i, \quad i, j = 1, \ldots, n, \quad i \neq j,
\]

and then
\[
Q(\xi, \lambda) \left( \sum_{k=1}^{n} (1 + \xi_k^2) \right)^{\frac{3}{2}}
= -(v \cdot w)^2 + \left( \sum_{k=1}^{n} (1 + \xi_k^2) \right) \sum_{i=1}^{n} p_i (1 + \xi_i^2) \lambda_i^{p-2} (1 + (2p_i - 1)\xi_i^2) \lambda_i.
\]

where \(v_i = p_i \xi_i (1 + \xi_i^2) \frac{q_i}{2} \lambda_i\) and \(w_i = (1 + \xi_i^2) \frac{q_i}{2}\). Therefore,

\[
Q(\xi, \lambda) \left( \sum_{k=1}^{n} (1 + \xi_k^2) \right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} p_i (1 + \xi_i^2) \lambda_i^{p-2} (1 + (2p_i - 1)\xi_i^2) \lambda_i
\]

\[
\leq (2q^2 - q) \sum_{i=1}^{n} \left[ (1 + \xi_i^2) \right]^{1 - \frac{1}{p}} \lambda_i^2 \leq 2q^2 \left[ \sum_{k=1}^{n} (1 + \xi_k^2) \right]^{1 - \frac{1}{q}} |\lambda|^2.
\]

For \(|\xi| \geq 1\), if \(q \leq 2\), we have

\[
Q(\xi, \lambda) \leq 2q^2 \left( \sum_{k=1}^{n} (1 + \xi_k^2) \right)^{\frac{q^2}{2q}} |\lambda|^2 \leq 2q^2 \left( 1 + \frac{1}{n} |\xi|^2 \right)^{\frac{q^2}{2} n^2} |\lambda|^2
\]

(4.3)

instead, if \(q \geq 2\) we obtain

\[
Q(\xi, \lambda) \leq 2q^2 \left( \sum_{k=1}^{n} (1 + \xi_k^2) \right)^{\frac{q^2}{2q}} |\lambda|^2 \leq 2q^2 (n(1 + |\xi|^2)^q)^{\frac{q^2}{2q}} |\lambda|^2 \leq C|\xi|^q |\lambda|^2.
\]

Moreover, since

\[
(v \cdot w)^2 \leq |v|^2 |w|^2 = \sum_{i=1}^{n} p_i^2 \xi_i^2 (1 + \xi_i^2) \lambda_i^{p-2} \sum_{k=1}^{n} (1 + \xi_k^2) \rho_k
\]

we have

\[
Q(\xi, \lambda) \left( \sum_{k=1}^{n} (1 + \xi_k^2) \right)^{\frac{1}{2}} \geq - \sum_{i=1}^{n} p_i^2 \xi_i^2 (1 + \xi_i^2) \lambda_i^{p-2} + \sum_{i=1}^{n} p_i (1 + \xi_i^2) \lambda_i^{p-2} (1 + (2p_i - 1)\xi_i^2) \lambda_i^2 = \sum_{i=1}^{n} p_i (1 + \xi_i^2) \lambda_i^{p-2} (1 + (p_i - 1)\xi_i^2) \lambda_i^2.
\]

For every \(q > 1\) and \(|\xi| \geq 1\), we deduce

\[
Q(\xi, \lambda) \geq \left( \sum_{k=1}^{n} (1 + \xi_k^2) \right)^{-\frac{1}{2}} (p^2 - p) \sum_{i=1}^{n} (1 + \xi_i^2) \lambda_i^{p-1} \lambda_i^2
\]

\[
\geq \frac{p^2 - p}{\sqrt{n}} (1 + \max_i \{|\xi_i|\}^2)^{p-1-\frac{q}{2}} |\lambda|^2 \geq c|\xi|^{2p-2-q} |\lambda|^2.
\]
We note explicitly that if $1 < p < q$, then $2p - 2 - q < p - 2$. Therefore, by denoting

$$r = 2p - q \quad \text{and} \quad s = \frac{p}{q}(q - 2) + 2$$

(4.5)

with $r \leq p \leq q \leq s \leq 2$, by (4.3) and (4.4) we obtain that $f(\xi)$ in (4.1) satisfies the assumptions $(2.1)_1$ and $(2.1)_2$ with

$$g_1(t) = c t^{r-2} \quad \text{and} \quad g_2(t) = C t^{s-2}.$$  

(4.6)

Therefore, the function $f(\xi)$ in (4.1) satisfies all assumptions in (2.1) with

$$\mu = 2 - s, \quad \beta = \frac{n-2}{2n} s - \frac{r}{2} + \frac{2}{n} \quad \text{and} \quad \alpha = \frac{s}{p},$$

when we impose the bounds

$$\alpha < \frac{2 - \mu}{n\beta - \mu} \iff s < \frac{2}{n} p + r.$$

We are in the conditions to apply Theorem 2.1. In the next Corollary 4.2, we state what we have proved by the computations above for this example, about the energy integral

$$F_2(u) = \int_{\Omega} \left( \sum_{i=1}^{n} \left( 1 + \left| u_{x_i} \right|^2 \right)^p \right)^{\frac{1}{p}}\,dx.$$  

(4.7)

**Corollary 4.2** Let $1 < p = \min_i p_i \leq q = \max_i p_i \leq 2$ and $r, s$ as in (4.5). If

$$s < \frac{2}{n} p + r \iff \frac{q}{p} < 1 + \frac{2}{n} - 2 \left( \frac{1}{p} - \frac{1}{q} \right),$$

(4.8)

then the local minimizers of the energy integral $F_2$ in (4.7) are locally Lipschitz continuous in $\Omega$.

**Remark 4.3** At a first glance, we may think that the assumption (4.8) of Corollary 4.2 is more restrictive that the similar assumption $\frac{q}{p} < 1 + \frac{2}{n}$ in Corollary 2.2, valid under the general $p, q$–growth. But, if we apply correctly Corollary 2.2 to $F_2$, on the contrary we had a more restrictive assumption than the above condition (4.8). In fact by (4.6), we have here $g_1(t) = c t^{r-2}, g_2(t) = C t^{s-2}$ and the estimate of the quadratic form $(2.1)_1$ becomes

$$c \left| \xi \right|^{r-2} |\lambda|^2 \leq \sum_{i,j=1}^{n} f_{x_i x_j}(\xi) \lambda_i \lambda_j \leq C \left| \xi \right|^{s-2} |\lambda|^2, \quad \forall \lambda, \xi \in \mathbb{R}^n : |\xi| \geq 1.$$  

(4.9)

Then, Corollary 2.2 applied to $F_2$ gives the regularity of minimizers under the bound $s < \left( 1 + \frac{2}{n} \right) r = \frac{2}{n} r + r$, which is a more restrictive condition than the above assumption (4.8) $s < \frac{2}{n} p + r$, since $r = 2p - q = p + (p - q) < p$ when $p < q$.

Therefore, the general bound of Corollary 2.2 gives a less precise result than Theorem 2.1 when applies to the energy integral (4.7). This fact also shows that Theorem 2.1 gives a more general regularity result than similar results that can be found in the recent mathematical literature on $p, q$–growth.
Example 4.4 Let

$$h(\xi) = \sqrt{\sum_{i=1}^{n} |\xi_i|^{2p_i}}, \quad p_i \geq 1, \quad (4.10)$$

$p = \min_i p_i$ and $q = \max_i p_i \leq 2$, and

$$\bar{Q}(\xi, \lambda) = \sum_{i,j=1}^{n} h_{\xi_j}(\xi) \lambda_i \lambda_j, \quad \forall \lambda, \xi \in \mathbb{R}^n. \quad (4.11)$$

We prove that the associated quadratic form to $h$ is semidefinite, i.e., $\bar{Q}(\xi, \lambda) \geq 0$. In fact,

$$\bar{Q}(\xi, \lambda) \left( \sum_{k=1}^{n} |\xi_k|^{2p_k} \right)^{\frac{3}{2}} = -\sum_{i,j=1}^{n} \text{sign}(\xi_i, \xi_j) p_i p_j |\xi_i|^{2p_i-1} |\xi_j|^{2p_j-1} \lambda_i \lambda_j + \sum_{k=1}^{n} |\xi_k|^{2p_k} \sum_{i=1}^{n} (2p_i^2 - p_i)|\xi_i|^{2p_i-2} \lambda_i^2. \quad (4.12)$$

By proceeding as above, we have

$$\bar{Q}(\xi, \lambda) \left( \sum_{k=1}^{n} |\xi_k|^{2p_k} \right)^{\frac{3}{2}} \geq \sum_{k=1}^{n} |\xi_k|^{2p_k} \sum_{i=1}^{n} (2p_i^2 - p_i)|\xi_i|^{2p_i-2} \lambda_i^2 \geq 0.$$ 

On the other hand, if $\max_i |\xi_i| \geq 1$,

$$\bar{Q}(\xi, \lambda) \left( \sum_{k=1}^{n} |\xi_k|^{2p_k} \right)^{\frac{3}{2}} \leq \sum_{i=1}^{n} (2p_i^2 - p_i)|\xi_i|^{2p_i-2} \lambda_i^2 = \sum_{i=1}^{n} (2p_i^2 - p_i)(|\xi_i|^{2p_i})^{1-\frac{1}{2}} \lambda_i^2 \leq (2q^2 - q) \sum_{i=1}^{n} \left( \sum_{k=1}^{n} |\xi_k|^{2p_k} \right)^{1-\frac{1}{2}} \lambda_i^2$$

since $\sum_{k=1}^{n} |\xi_k|^{2p_k} \geq (\max_i |\xi_i|)^{2p} \geq 1$,

$$\bar{Q}(\xi, \lambda) \left( \sum_{k=1}^{n} |\xi_k|^{2p_k} \right)^{\frac{3}{2}} \leq (2q^2 - q) \left( \sum_{k=1}^{n} |\xi_k|^{2p_k} \right)^{1-\frac{1}{2}} |\lambda|^2.$$ 

Now, again using $\max_i |\xi_i| \geq 1,$
therefore
\[
\overline{Q}(\xi, \lambda) \leq (2q^2 - q)\left(\sum_{k=1}^{n} |\xi_k|^{2p_k}\right)^{1 - \frac{1}{q}} |\lambda|^2 \leq (2q^2 - q)\left(\max_i |\xi_i|\right)^{p - 2} |\lambda|^2
\]
\[
\leq C|\xi|^{\frac{p-2}{q}}|\lambda|^2.
\]
In this case, \(\overline{Q}(\xi, \lambda) \leq C|\xi|^{q-2}|\lambda|^2\) when \(q \geq 2\). We denote by
\[
s = \frac{p}{q}(q - 2) + 2
\]
with \(1 < p = \min_i p_i \leq q = \max_i p_i \leq 2\). We consider the ellipticity conditions (2.4), with \(s\) replaced by \(q\), for the function
\[
f(\xi) = |\xi|^q + h(\xi).
\]
Since
\[
\frac{s}{p} < 1 + \frac{2}{n} \iff \frac{q}{p} < 1 + \frac{q}{n},
\]
from Corollary 2.2, we obtain the proof of a further regularity result for the following energy integral
\[
F_3(u) = \int_{\Omega} |Du|^p + \left(\sum_{i=1}^{n} |u_i|^{2p_i}\right)^{\frac{1}{2}} dx.
\]

**Corollary 4.5** \(\text{If} 1 < p = \min_i p_i \leq q = \max_i p_i \leq 2 \text{ satisfy}\)
\[
\frac{q}{p} < 1 + \frac{q}{n} \iff q < p^* = \frac{np}{n - p},
\]
then any local minimizers of \(F_3\) in (4.15) are locally Lipschitz continuous in \(\Omega\).

We can consider also different integrands related with \(h\) in (4.10). By taking in account Example 3.1 and Example 3.2, we can consider, for \(1 < q \leq 2\), \(s = 3 - \frac{2}{q} \geq q\),
\[
f(\xi) = |\xi| (\log |\xi|)^a + \sqrt{\sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^{2q}}, \quad a > 0,
\]
or
\[
f(\xi) = |\xi| L_k(|\xi|) + \sqrt{\sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^{2q}},
\]
Assumption (2.1) holds with \(g_2(t) = Ct^{s-2}\) and, respectively,
\[ g_1(t) = c \frac{(\log t)^{\alpha - 1}}{t} \quad \text{or} \quad g_1(t) = \frac{c}{(1 + t)(1 + L_1(t)) \cdots (1 + L_{k-1}(t))}, \]

\[ \mu = 2 - s, \quad \beta > \frac{n - 2}{2n} s - \frac{1}{2} + \frac{2}{n}, \quad \alpha > s. \]

Therefore, if \( s < 1 + \frac{2}{n} \), by Theorem 2.1, the corresponding local minimizers are locally Lipschitz continuous.

## 5 Interpolation lemma

As usual, we denote by \( B_R \) a generic ball of radius \( R \) compactly contained in \( \Omega \) and by \( B_\rho \) a ball of radius \( \rho < R \) concentric with \( B_R \).

**Lemma 5.1** (interpolation) Let \( v \in L_{\text{loc}}^\infty(\Omega) \) and let us assume that for some \( \theta \geq 1, c > 0 \) and for every \( \phi \) and \( R \) such that \( 0 < \rho < R \)

\[ \|v\|_{L^\theta(B_\rho)} \leq \frac{c}{(R - \rho)^n} \int_{B_R} |v| \, dx. \quad (5.1) \]

Then, for every \( \lambda \in \left( \frac{\theta - 1}{\theta}, 1 \right) \) (i.e., in particular with \( \theta(1 - \lambda) < 1 \)) there exists a constant \( c_\lambda \) such that, for every \( \phi < R \),

\[ \|v\|_{L^\lambda(B_\phi)} \leq \frac{c_\lambda}{(R - \phi)^n} \int_{B_R} |v|^\lambda \, dx. \quad (5.2) \]

**Proof** Fixed \( \lambda \in \left( \frac{\theta - 1}{\theta}, 1 \right) \), we make use of the interpolation inequality

\[ \int_{B_\phi} |v| \, dx = \int_{B_\phi} |v|^{1 - \lambda} |v|^{\lambda} \, dx \leq \|v\|_{L^\theta(B_\phi)}^{1 - \lambda} \int_{B_\phi} |v|^\lambda \, dx. \]

By the assumption (5.1), we obtain

\[ \int_{B_\phi} |v| \, dx \leq \left( \frac{c}{(R - \phi)^n} \int_{B_R} |v| \, dx \right)^{\theta(1 - \lambda)} \int_{B_\phi} |v|^\lambda \, dx. \]

We denote by \( \gamma := \theta(1 - \lambda) \), and we observe that \( 0 < \gamma < 1 \) since \( \lambda > \frac{\theta - 1}{\theta} \). Thus, the previous estimate has the equivalent form

\[ \int_{B_\phi} |v| \, dx \leq c^\gamma \int_{B_\phi} |v|^\lambda \, dx \cdot \left( \frac{1}{(R - \phi)^n} \int_{B_R} |v| \, dx \right)^\gamma. \quad (5.3) \]

Given \( \phi_0 \) and \( R_0 \), with \( 0 < \phi_0 < R_0 \leq \phi_0 + 1 \), we define a decreasing sequence \( \phi_k \) by \( \phi_k = R_0 - \frac{R_0 - \phi_0}{2^k}, \quad k = 0, 1, 2, \ldots \). In (5.3), we pose \( \phi = \phi_k \) and \( R = \phi_{k+1} \). Since \( R - \rho = \phi_{k+1} - \phi_k = \frac{R_0 - \phi_0}{2^{k+1}} \), we obtain

\[ \int_{B_{\phi_k}} |v| \, dx \leq c^\gamma \int_{B_{\phi_0}} |v|^\lambda \, dx \cdot \left( \frac{2^{n(k+1)}}{(R_0 - \phi_0)^n} \int_{B_{\phi_{k+1}}} |v| \, dx \right)^\gamma. \]
Denote \( B_k = \int_{B_{r_k}} |v| \, dx \) for \( k = 0, 1, 2, \ldots \). The last inequality becomes

\[
B_k \leq c' \int_{B_{r_0}} |v|^\lambda \, dx \cdot \frac{2^{n_f(k+1)}}{(R_0 - \rho_0)^{n_f}} B_{1}^k.
\]

We start to iterate with \( k = 0, 1, 2, \ldots \)

\[
B_0 \leq c' \int_{B_{r_0}} |v|^\lambda \, dx \cdot \frac{2^{n_f}}{(R_0 - \rho_0)^{n_f}} B_0^1
\]

\[
\leq c' \int_{B_{r_0}} |v|^\lambda \, dx \cdot \frac{2^{n_f}}{(R_0 - \rho_0)^{n_f}} \left( c' \int_{B_{r_0}} |v|^4 \, dx \cdot \frac{2^{n_f}}{(R_0 - \rho_0)^{n_f}} B_1^2 \right)^{\gamma}
\]

and for general \( k = 1, 2, 3, \ldots \) we have

\[
B_0 \leq \left( \frac{c' \int_{B_{r_0}} |v|^4 \, dx}{(R_0 - \rho_0)^{n_f}} \right)^{\sum_{i=0}^{k-1} \gamma i} \left( 2^{n_f} \sum_{i=0}^{k} (B_k)^{\gamma} \right).
\]

Since \( 0 < \gamma < 1 \), passing to the limit as \( k \to \infty \), \( \sum_{i=0}^{\infty} \gamma^i < \infty \) and \( \sum_{i=0}^{\infty} \gamma^i = \frac{1}{1 - \gamma} \). Moreover, the increasing sequence \( B_k = \int_{B_{r_k}} |v| \, dx \) is bounded by \( \int_{B_{r_0}} |v| \, dx \) for \( k = 0, 1, 2, \ldots \). Thus, \( (B_k)^{\gamma^k} = \left( \int_{B_{r_k}} |v| \, dx \right)^{\gamma^k} \leq \left( \int_{B_{r_0}} |v| \, dx \right)^{\gamma^k} \) and the right hand side converges to 1 as \( k \to \infty \). Therefore, in the limit as \( k \to \infty \), there exists a constant \( c_1 \) such that

\[
B_0 = \int_{B_{r_0}} |v| \, dx \leq c_1 \left( \frac{1}{(R_0 - \rho_0)^{n_f}} \int_{B_{r_0}} |v|^\lambda \, dx \right)^{\frac{1}{1 - \gamma}}.
\]

(5.4)

Fixed \( \rho < R \) we consider \( \overline{\rho} = \frac{R + \rho}{2} \) and, by combining the assumption (5.1) and (5.4), since \( R - \overline{\rho} = R - \rho \) and \( \gamma = \delta(1 - \lambda) \),

\[
||v||_{L^\infty(B_{\overline{\rho}})} \leq \left( \frac{c}{(R - \overline{\rho})^n} \int_{B_{\overline{\rho}}} |v| \, dx \right)^{\frac{1}{1 - \gamma}} \leq \left( \frac{c \cdot c_1}{(R - \overline{\rho})^n} \left( \frac{1}{(R - \overline{\rho})^{n_f}} \int_{B_{\overline{\rho}}} |v|^4 \, dx \right)^{\frac{1}{1 - \gamma}} \right)^{\frac{1}{1 - \gamma}} \leq c_2 \left( \frac{1}{(R - \overline{\rho})^{n_f(1 - \gamma) + n_f}} \int_{B_{\overline{\rho}}} |v|^4 \, dx \right)^{\frac{1}{1 - \gamma}} = c_3 \left( \frac{1}{(R - \overline{\rho})^n} \int_{B_{\overline{\rho}}} |v|^4 \, dx \right)^{\frac{1}{1 - \gamma}}
\]

which gives (5.2) and the proof of the Lemma is concluded. \( \square \)
6 A priori estimates

In order to simplify the notations, without loss of generality in this section we assume that \( t_0 = 1 \). First of all, we give a technical result.

**Lemma 6.1** Let us assume that \((2.1)_2\) and \((2.1)_3\) hold. Then, for every \( \gamma \geq 0 \) there exists a constant \( C_3 = C_3(\gamma, g_2(1)) > 0 \) independent of \( \gamma \), such that

\[
C_3 \left[ 1 + g_2(1 + t)^{\frac{1}{\beta}} \left( 1 + t \right)^{\frac{\gamma + 1 - \beta}{\gamma}} \right] \leq 1 + \int_0^t (1 + s)^{\frac{\gamma - 2}{2}} s \sqrt{g_1(1 + s)} \, ds
\]

for every \( t \geq 0 \), where , for \( n > 2, 2^* = \frac{2n}{n-2} \), while, for \( n = 2, 2^* \) can be any number greater than \( \frac{3}{2} \).

**Proof** If \( t \geq 0 \) then, by assumption \((2.1)_3\)

\[
1 + \int_0^t (1 + s)^{\frac{\gamma - 2}{2}} s \sqrt{g_1(1 + s)} \, ds \geq 1 + \int_0^t (1 + s)^{\frac{\gamma - 2}{2}} \frac{1}{\sqrt{C_1}} (1 + s)^{-\beta} g_2(1 + s)^{\frac{1}{\beta}} \, ds.
\]

On the other hand, since \( g_2 \) is decreasing, \( g_2(1 + s)^{\frac{1}{\beta}} \geq g_2(1 + t)^{\frac{1}{\beta}} \) and therefore

\[
1 + \frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{\beta}} \int_0^t (1 + s)^{\frac{\gamma - 2}{2}} s \, ds \leq 1 + \int_0^t (1 + s)^{\frac{\gamma - 2}{2}} \frac{1}{\sqrt{C_1}} (1 + s)^{-\beta} g_2(1 + s)^{\frac{1}{\beta}} \, ds.
\]

By Lemma 2.2 in [22], we have that (see (2.6) here): let \( a_0 > 0 \) there exists a constant \( c \) depending on \( a_0 \), but independent of \( \alpha \geq a_0 \), such that

\[
(1 + t)^{\alpha} \leq c \alpha^2 \left( 1 + \int_0^t (1 + s)^{\alpha - 2} s \, ds \right).
\]

In our case, \( \alpha := \frac{\gamma - 2}{2} - \beta + 2 = \frac{\gamma}{2} + 1 - \beta \) and \( \alpha \geq a_0 := \frac{2}{\gamma} \). Inequality (6.2) is valid for all \( t \geq 0 \) so in particular for \( t \geq 1 \) and it entails

\[
\int_0^t (1 + s)^{\alpha - 2} s \, ds \geq \frac{(1 + t)^{\alpha}}{c \alpha^2} - 1.
\]

The last inequality implies

\[
1 + \frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{\beta}} \int_0^t (1 + s)^{\frac{\gamma - 2}{2}} s \, ds
\]

\[
\geq 1 + \frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{\beta}} \left[ \frac{(1 + t)^{\frac{\gamma + 1 - \beta}{\gamma}}}{c \left( \frac{\gamma}{2} + 1 - \beta \right)^2} - 1 \right]
\]

\[
= 1 + \frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{\beta}} \frac{(1 + t)^{\frac{\gamma + 1 - \beta}{\gamma}}}{c \left( \frac{\gamma}{2} + 1 - \beta \right)^2} - \frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{\beta}}.
\]
Now, we observe that, for every $t \geq 0$, since $g_2$ is decreasing,
\[
\frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{2}} \leq \frac{1}{\sqrt{C_1}} g_2(1)^{\frac{1}{2}} =: \tilde{C}_1.
\]
Thus, summing up
\[
1 + \tilde{C}_1 + \int_0^t (1 + s)^{\frac{g_2}{2}} s \sqrt{g_1(1 + s)} \, ds \geq 1 + \frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{2}} \frac{(1 + t)^{\frac{g_2}{2} + 1 - \beta}}{c \left( \frac{g_2}{2} + 1 - \beta \right)^2}
\]
which in turn implies
\[
(1 + \tilde{C}_1) \left[ 1 + \int_0^t (1 + s)^{\frac{g_2}{2}} s \sqrt{g_1(1 + s)} \, ds \right] \geq 1 + \tilde{C}_1 + \int_0^t (1 + s)^{\frac{g_2}{2}} s \sqrt{g_1(1 + s)} \, ds
\]
\[
\geq 1 + \frac{1}{\sqrt{C_1}} g_2(1 + t)^{\frac{1}{2}} \frac{(1 + t)^{\frac{g_2}{2} + 1 - \beta}}{c \left( \frac{g_2}{2} + 1 - \beta \right)^2}
\]
therefore, by setting $\tilde{C}_2 := (1 + \tilde{C}_1) \sqrt{C_1} c$, we get (6.1) for $C_3 = \frac{1}{\tilde{C}_2}$. We note explicitly that $\tilde{C}_2$ may depend on $n$, but it is independent of $\gamma$. \hfill \Box

**Lemma 6.2** Assume that $f$ satisfies the growth assumptions (2.1)₁, (2.1)₂, (2.1)₃. In addition, assume that $f(\xi)$ is of class $C^2(\mathbb{R}^n)$ and for every $M > 0$ there exists a positive constant $\ell = \ell(M)$ such that
\[
\ell |\lambda|^2 \leq \sum_{i,j=1}^n f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j \quad \forall \lambda, \xi \in \mathbb{R}^n, |\xi| \geq M.
\] (6.3)

If $u \in W^{1,\infty}_\text{loc}(\Omega)$ is a a local minimizer of (1.1), then for every $0 < \rho < R$, $\tilde{B}_R \subset \Omega$ there exists a positive constants $c_4$ depending only on $C_1$, $\beta$, $g_2(1)$, such that
\[
\left( \|1 + (|Du| - 1)_+\|_{L^\infty(\tilde{B}_R)} \right)^{2 - n\beta} \leq \frac{c_4}{(R - \rho)^n} \int_{\tilde{B}_R} \frac{g_2(1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+)}{} \, dx.
\] (6.4)

**Proof** Since the local minimizer $u$ is in $W^{1,\infty}_\text{loc}(\Omega)$, it satisfies the Euler equation: for every open set $\Omega'$ compactly contained in $\Omega$, we have
\[
\int_{\Omega} \sum_{i=1}^n f_{\xi_i}(Du) \varphi_{x_i} \, dx = 0 \quad \forall \varphi \in W^{1,2}_0(\Omega').
\]
Moreover, by the techniques of the difference quotient (see for example [26, Ch. 8, Sect. 8.1]), $u \in W^{2,2}_\text{loc}(\Omega)$, then the second variation holds:
\[
\int_{\Omega} \sum_{i,j=1}^{n} f_{\xi,j}(Du)u_{x,j} \varphi_{x,i} \, dx = 0, \quad \forall k = 1, \ldots, n, \ \forall \varphi \in W^{1,2}_{0}(\Omega').
\]

For fixed \(k = 1, \ldots, n\), let \(\eta \in C^1(\Omega')\) be equal to 1 in \(B_\rho^c\), with support contained in \(B_R\), such that \(|D\eta| \leq \frac{2}{(8-\rho)}\), and consider \(\varphi = \eta^2 u_{x,i} \Phi((Du)_+ - 1)\) with \(\Phi\) nonnegative, increasing, locally Lipschitz continuous on \([0, +\infty)\), such that \(\Phi(0) = 0\). Here, \((a)_+\) denotes the positive part of \(a \in \mathbb{R}\); in the following, we denote \(\Phi((Du)_+ - 1)_+ = \Phi((Du)_+ - 1)_+\).

Then, a.e. in \(\Omega\)

\[
\varphi_{x,i} = 2\eta \eta^2 u_{x,i} \Phi((Du)_+ - 1)_+ + \eta^2 u_{x,j} \Phi((Du)_+ - 1)_+ + \eta^2 u_{x,i} \Phi'(1_+)(Du)_+ [((Du)_+)]_x \leq 0.
\]

Proceeding along the lines of [33], we therefore deduce that

\[
\int_{\Omega} 2\eta \Phi((Du)_+ - 1)_+ \sum_{i,j=1}^{n} f_{\xi,j}(Du)u_{x,j} \eta_{x,i} \eta_{x,k} \, dx
\]

\[
+ \int_{\Omega} \eta^2 \Phi((Du)_+ - 1)_+ \sum_{i,j=1}^{n} f_{\xi,j}(Du)u_{x,j} u_{x,k} \, dx
\]

\[
+ \int_{\Omega} \eta^2 \Phi'(1_+)(Du)_+ [((Du)_+)]_x dx = 0.
\]

We estimate the first integral in the previous equation by using the Cauchy–Schwarz inequality and the Young inequality so that

\[
\left| \int_{\Omega} 2\eta \Phi((Du)_+ - 1)_+ \sum_{i,j=1}^{n} f_{\xi,j}(Du)u_{x,j} \eta_{x,i} \eta_{x,k} \, dx \right|
\]

\[
\leq \int_{\Omega} \Phi((Du)_+ - 1)_+ \left( \eta^2 \sum_{i,j=1}^{n} f_{\xi,j}(Du)u_{x,j} u_{x,k} \right)^{\frac{1}{2}} \left( \sum_{i,j=1}^{n} f_{\xi,j}(Du)\eta_{x,i} \eta_{x,j} \eta_{x,k} \right)^{\frac{1}{2}} dx
\]

\[
\leq \frac{1}{2} \int_{\Omega} \eta^2 \Phi((Du)_+ - 1)_+ \sum_{i,j=1}^{n} f_{\xi,j}(Du)u_{x,j} u_{x,k} \, dx
\]

\[
+ 2 \int_{\Omega} \Phi((Du)_+ - 1)_+ \sum_{i,j=1}^{n} f_{\xi,j}(Du)\eta_{x,i} u_{x,j} \eta_{x,k} \, dx.
\]

Therefore, we deduce
\[ \frac{1}{2} \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ \sum_{i,j=1}^{n} f_{\xi_j}^\varepsilon(Du) u_{x_i x_j} u_{x_i x_j} \, dx \]

\[ + \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,j=1}^{n} f_{\xi_j}^\varepsilon(Du) u_{x_i} u_{x_i} [(|Du| - 1)_+]_{x_i} \, dx \]

\[ \leq 2 \int_{\Omega} \Phi(|Du| - 1)_+ \sum_{i,j=1}^{n} f_{\xi_j}^\varepsilon(Du) \eta_{x_i} u_{x_i} \eta_{x_j} u_{x_j} \, dx. \]

Since a.e. in \( \Omega \)

\[ [(|Du| - 1)_+]_{x_i} = \begin{cases} (|Du|)_{x_i} & \text{if } |Du| > 1, \\ 0 & \text{if } |Du| \leq 1, \end{cases} \]

by summing up in the previous chain of inequalities with respect to \( k = 1, \ldots, n \), we obtain

\[ \sum_{k=1}^{n} \sum_{i,j=1}^{n} f_{\xi_j}^\varepsilon(Du) u_{x_i x_k} u_{x_k} [(|Du| - 1)_+]_{x_i} \]

\[ = |Du| \sum_{i,j=1}^{n} f_{\xi_j}^\varepsilon(Du)[(|Du| - 1)_+]_{x_j} [(|Du| - 1)_+]_{x_i} \]

therefore, we deduce the estimate

\[ \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ \sum_{k,i,j=1}^{n} f_{\xi_j}^\varepsilon(Du) u_{x_i x_k} u_{x_i x_k} \, dx \]

\[ + \int_{\Omega} \eta^2 |Du| \Phi'(|Du| - 1)_+ \sum_{i,j=1}^{n} f_{\xi_j}^\varepsilon(Du) [(|Du| - 1)_+]_{x_j} [(|Du| - 1)_+]_{x_i} \, dx \]

\[ \leq 4 \int_{\Omega} \Phi(|Du| - 1)_+ \sum_{k,i,j=1}^{n} f_{\xi_j}^\varepsilon(Du) \eta_{x_i} u_{x_i} \eta_{x_j} u_{x_j} \, dx. \]

Using the inequality \(|D(|Du| - 1)_+|^2 \leq |D^2 u|^2\) and the ellipticity condition in (2.1), we obtain

\[ \int_{\Omega} \eta^2 [\Phi(|Du| - 1)_+ + |Du| \Phi'(|Du| - 1)_+]_+ g_1 (1 + (|Du| - 1)_+) |D(|Du| - 1)_+|^2 \, dx \]

\[ = \int_{\Omega} \eta^2 [\Phi(|Du| - 1)_+ + |Du| \Phi'(|Du| - 1)_+]_+ g_1 (|Du|) |D(|Du| - 1)_+|^2 \, dx \]

\[ \leq 4 \int_{\Omega} |D\eta|^2 \Phi(|Du| - 1)_+ g_2 (|Du|) |Du|^2 \, dx \]

\[ = 4 \int_{\Omega} |D\eta|^2 \Phi(|Du| - 1)_+ g_2 (1 + (|Du| - 1)_+) |Du|^2 \, dx. \]

\[ (6.5) \]

Let us define
\begin{equation}
G(t) = 1 + \int_0^t \sqrt{\Phi(s) g_1(1+s)} \, ds \quad \forall t \geq 0. \tag{6.6}
\end{equation}

By Jensen’s inequality and the monotonicity of \( \Phi \), since \( t \mapsto t g_2(t) \) is increasing,

\begin{align*}
G(t) &= 1 + \int_0^t \frac{\Phi(s)(1+s)g_1(1+s)}{1+s} \, ds \\
&\leq 1 + \int_0^t \frac{\Phi(s)(1+s)g_2(1+s)}{1+s} \, ds \\
&\leq 1 + \sqrt{\Phi(t)(1+t)g_2(1+t)} \int_0^t \frac{1}{\sqrt{1+s}} \, ds \\
&\leq 1 + 2 \sqrt{\Phi(t)(1+t)g_2(1+t) \sqrt{1+t}}.
\end{align*}

hence \( [G(t)]^2 \leq 8 \left[ 1 + \Phi(t)(1+t)^2 g_2(1+t) \right] \). On the other hand,

\begin{align*}
|D(\eta(G(|Du| - 1)_+))|^2 \\
&\leq 2 |D\eta|^2 [G((|Du| - 1)_+)]^2 + 2 \eta^2 [G'(((|Du| - 1)_+)]^2 |D((|Du| - 1)_+)|^2 \\
&\leq 16 |D\eta|^2 (1 + \Phi(|Du| - 1)_+ g_2(|Du|)|Du|^2) + 2 \eta^2 \Phi(|Du| - 1)_+ g_1(|Du|) |D(|Du|)|^2.
\end{align*}

Since \( \Phi(|Du(x)| - 1)_+ = 0 \) when \(|Du(x)| \leq 1\), by (6.5), we get

\begin{align*}
\int_{\Omega} |D(\eta G((|Du| - 1)_+))^2 |^2 \, dx \\
&\leq 24 \int_{\Omega} |D\eta|^2 (1 + \Phi(|Du| - 1)_+ g_2(|Du|)|Du|^2) \, dx \tag{6.7} \\
&= 24 \int_{\Omega} |D\eta|^2 (1 + \Phi(|Du| - 1)_+ g_2(1 + (|Du| - 1)_+)(1 + (|Du| - 1)_+)^2) \, dx.
\end{align*}

Let us assume

\begin{equation}
\Phi(t) = (1+t)^{-2} t^2 \quad \gamma \geq 0. \tag{6.8}
\end{equation}

By the Sobolev inequality, there exists a constant \( c_S \) such that

\begin{equation}
\left\{ \int_{\Omega} |\eta G((|Du| - 1)_+)|^{2^*} \, dx \right\}^{2/2^*} \leq c_S \int_{\Omega} |D(\eta(G(|Du| - 1)_+))|^2 \, dx \tag{6.9}
\end{equation}

where \( 2^* = \frac{2n}{n-2} \) if \( n > 2 \) and a number greater than \( \frac{2}{1-\beta} \) if \( n = 2 \). We apply (6.1) with the choice \( t = (|Du| - 1)_+ \)

\begin{align*}
G((|Du| - 1)_+) &= 1 + \int_{0}^{(|Du|-1)_+} (1+s)^{-\frac{n}{2}} S \sqrt{g_1(1+s)} \, ds \\
&\geq C_3 \left[ 1 + g_2(1 + (|Du| - 1)_+)^{\frac{1}{2}}(1 + (|Du| - 1)_+)^{\frac{\gamma+1-\beta}{2}} \left( \frac{\gamma}{2} + 1 - \beta \right)^2 \right]
\end{align*}

thus by (6.7) we obtain that there exists \( c = c(C_3) > 0 \) such that, for all \( \gamma \geq 0 \).
First of all, we check that 

\[ \frac{\varrho}{\varrho_1} \geq \frac{1}{c_1} \] 

for the sake of clarity we focus on the main steps. From now on, we label the constants; this will be useful in the sequel. We set \( \delta := (\gamma + 2) \), and we notice that, since \( \gamma \geq 0 \), then \( \delta \geq 2 \). Then,

\[
\left\{ \int_{B_{\rho_i}} \left[ 1 + (1 + (|Du| - 1)_+)^{\frac{\delta - 2\beta}{2}} g_2(1 + (|Du| - 1)_+) \right] dx \right\} \frac{2^*}{2^*} \leq \frac{c_1 \left( \frac{\delta^2}{R - \rho} \right)}{2^*} \int_{B_{\rho_i}} \left[ 1 + (1 + (|Du| - 1)_+)^{\delta} g_2(1 + (|Du| - 1)_+) \right] dx,
\]

(6.11)

where we used once more (6.7) and (6.9). The iteration process follows now the arguments contained in [35]; for the sake of clarity we focus on the main steps. We set \( \delta := (\gamma + 2) \), and we notice that, since \( \gamma \geq 0 \), then \( \delta \geq 2 \). Then,

\[
\left\{ \int_{\Omega} \eta^2(1 + (1 + (|Du| - 1)_+)^{\frac{\gamma + 2 - 2\beta}{2}} g_2(1 + (|Du| - 1)_+) dx \right\} \frac{2^*}{2^*} \leq 16 c \left( \frac{\gamma}{2} + 1 - \beta \right)^4 \int_{\Omega} |D\eta|^2 (1 + (1 + (|Du| - 1)_+)^{\gamma^2} g_2(1 + (|Du| - 1)_+) dx
\]

\[
\leq c (\gamma + 2)^4 \int_{\Omega} |D\eta|^2 \left[ 1 + (1 + (|Du| - 1)_+)^{\gamma^2} g_2(1 + (|Du| - 1)_+) \right] dx,
\]

(6.10)

where we used once more (6.7) and (6.9). The iteration process follows now the arguments contained in [35]; for the sake of clarity we focus on the main steps. From now on, we label the constants; this will be useful in the sequel. We set \( \delta := (\gamma + 2) \), and we notice that, since \( \gamma \geq 0 \), then \( \delta \geq 2 \). Then,

\[
\left\{ \int_{B_{\rho_i}} \left[ 1 + (1 + (|Du| - 1)_+)^{\frac{\delta - 2\beta}{2}} g_2(1 + (|Du| - 1)_+) \right] dx \right\} \frac{2^*}{2^*} \leq c_1 \left( \frac{\delta^2}{R - \rho} \right) \int_{B_{\rho_i}} \left[ 1 + (1 + (|Du| - 1)_+)^{\delta} g_2(1 + (|Du| - 1)_+) \right] dx,
\]

(6.11)

where we used once more (6.7) and (6.9). The iteration process follows now the arguments contained in [35]; for the sake of clarity we focus on the main steps. From now on, we label the constants; this will be useful in the sequel. We set \( \delta := (\gamma + 2) \), and we notice that, since \( \gamma \geq 0 \), then \( \delta \geq 2 \). Then,

\[
\left\{ \int_{B_{\rho_i}} \left[ 1 + (1 + (|Du| - 1)_+)^{\frac{\delta - 2\beta}{2}} g_2(1 + (|Du| - 1)_+) \right] dx \right\} \frac{2^*}{2^*} \leq c_1 \left( \frac{\delta^2}{R - \rho} \right) \int_{B_{\rho_i}} \left[ 1 + (1 + (|Du| - 1)_+)^{\delta} g_2(1 + (|Du| - 1)_+) \right] dx,
\]

(6.11)
where, by induction we computed
\[
\delta_{i+1} = 2 \left( \frac{2^*}{2} \right)^{i+1} - 2 \beta \sum_{k=1}^{i+1} \left( \frac{2^*}{2} \right)^{k} = 2 \left( \frac{2^*}{2} \right)^{i+1} \left[ 1 - \beta \sum_{k=0}^{i} \left( \frac{2}{2^*} \right)^{k} \right]
\]
\[
= 2 \left( \frac{2^*}{2} \right)^{i+1} \left[ 1 - \beta \frac{1 - \left( \frac{2}{2^*} \right)^{i+1}}{1 - \frac{2}{2^*}} \right] = 2 \left( \frac{2^*}{2} \right)^{i+1} \left[ 1 - \beta \frac{2^*}{2^* - 2} \right] + 2 \beta \frac{2^*}{2^* - 2}
\]
and where
\[
c_2 = \prod_{k=0}^{+\infty} \frac{c_1}{(\bar{R} - \tilde{\rho})^2} \delta_2 k^{2k+1} \left( \frac{1}{\pi} \right)^{i+1} \leq \prod_{k=0}^{+\infty} \frac{c_1}{(\bar{R} - \tilde{\rho})^2} 4 \left( \frac{2^*}{2} \right)^{2k+1} \left( \frac{1}{\pi} \right)^{i+1} = \frac{c_3}{(\bar{R} - \tilde{\rho})^{2^{2^*}/2}}.
\]

Now, by (2.1) we have that,
\[
1 + r^2 g_2(t) \leq \frac{t}{g_2(1)} g_2(1) + r^2 g_2(t) \leq \left( \frac{1}{g_2(1)} + 1 \right) r^2 g_2(t). \tag{6.13}
\]

So we can write
\[
\left[ \int_{B_\rho} (1 + (|Du| - 1)_+) \delta_1 g_2(1 + (|Du| - 1)_+)^{\tilde{\beta}^*} \right] \frac{1}{\pi}^{i+1} \leq \frac{c_4}{(\bar{R} - \tilde{\rho})^{2^{2^*}/2}} \int_{B_\rho} \left[ (1 + (|Du| - 1)_+) \delta_1 g_2(1 + (|Du| - 1)_+)^{\tilde{\beta}^*} \right] \frac{1}{\pi}^{i+1} \leq \frac{c_5}{(\bar{R} - \tilde{\rho})^{2^{2^*}/2}} \int_{B_\rho} \left[ (1 + (|Du| - 1)_+) \delta_1 g_2(1 + (|Du| - 1)_+)^{\tilde{\beta}^*} \right] \frac{1}{\pi}^{i+1},
\]
where
\[
c_4 = c_3 \left( \frac{1}{g_2(1)} + 1 \right).
\]

Finally, by (6.12), \( \delta_{i+1} \left( \frac{2}{2^*} \right)^{i+1} \rightarrow 2 - \beta \frac{2^*}{2^* - 2} \) as \( i \rightarrow +\infty \), so passing to the limit we obtain
\[
\left( \|1 + (|Du| - 1)_+\|_{L^\infty(B_\rho)} \right)^{2^{2^*/2^* - 2}} \delta_1 \beta
\]
\[
\leq \frac{c_5}{(\bar{R} - \tilde{\rho})^{2^{2^*/2^* - 2}}} \int_{B_\rho} \left[ (1 + (|Du| - 1)_+) \delta_1 g_2(1 + (|Du| - 1)_+)^{\tilde{\beta}^*} \right] \frac{1}{\pi}^{i+1}.
\]

Therefore, (6.4) is proved, in fact \( n = \frac{2^{2^*}}{2^* - 2} \) if \( n > 2 \) and \( 2 < \frac{2^{2^*}}{2^* - 2} \) if \( n = 2 \). Notice that \( 0 < 2 - n\beta < 1 \) since \( \frac{1}{n} < \beta < \frac{2}{n} \).
Lemma 6.3 Assume that $f$ satisfies the assumptions of previous lemma and (2.2) and $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ is a local minimizer of (1.1). Then, for every $0 < \rho < R$, $\bar{B}_R \subset \Omega$, there exists a positive constant $C = C(\rho, R, C_1, C_2, \alpha, \beta, \mu, g_2(\rho))$, such that

$$
\|Du\|_{L^\infty(\bar{B}_\rho)} \leq C \left\{ \frac{1}{(R - \rho)\mu} \int_{B_\rho} (1 + f(Du)) \, dx \right\}^\theta
$$

(6.14)

with $\theta = \frac{(2 - \mu)\mu}{2 - \mu - \alpha(n\beta - \mu)}$.

Proof Set

$$
V = V(x) = (1 + (|Du| - 1)_+) g_2(1 + (|Du| - 1)_+).
$$

By (2.1)$_2$, we have

$$
\|V\|_{L^\infty(\bar{B}_\rho)} \leq C_\mu \left( 1 + (|Du| - 1)_{+1} \right)^{2 - \mu}
$$

so inequality (6.4) becomes

$$
\left( \|V\|_{L^\infty(\bar{B}_\rho)} \right)^{\frac{2 - \mu}{2 - \beta}} \leq \frac{c_5}{(R - \rho)^\mu} \int_{B_\rho} V(x) \, dx.
$$

Let $\alpha > 1$ satisfies (2.2), we can apply Lemma 5.1 with $v = V$, $\theta = \frac{2 - \mu}{2 - n\beta}$ and $\lambda = \frac{1}{a}$. In fact, we have

$$
\alpha(n\beta - \mu) < 2 - \mu \iff \frac{2 - \mu}{2 - n\beta} \left( 1 - \frac{1}{\alpha} \right) < 1 \iff \theta (1 - \lambda) < 1. \quad (6.15)
$$

Therefore, we deduce the existence of a constant $c_6$ such that, for every $\rho < R$, the following estimate holds

$$
\|V\|_{L^\infty(\bar{B}_\rho)} = \|V\|_{L^\infty(\bar{B}_\rho)} \leq \frac{c_6}{(R - \rho)^\mu} \int_{B_\rho} |V|^{\frac{1}{\lambda}} \, dx.
$$

Now, by (2.1)$_4$, if $|Du| \geq 1$,

$$
|V| = (1 + (|Du| - 1)_+) g_2(1 + (|Du| - 1)_+) \leq C_2 (1 + f(Du))^{a},
$$

otherwise

$$
|V| = (1 + (|Du| - 1)_+) g_2(1 + (|Du| - 1)_+) \leq g_2(1) \leq g_2(1) (1 + f(Du))^{a}.
$$

Therefore,

$$
\|V\|_{L^\infty(\bar{B}_\rho)} \leq \left[ \frac{c_7}{(R - \rho)^\mu} \int_{B_\rho} (1 + f(Du)) \, dx \right]^{\frac{(2 - \mu)\mu}{2 - \mu - \alpha(n\beta - \mu)}} \quad (6.16)
$$

holds for $c_7 := \max\{ C_2, g_2(1) \}^{\frac{1}{\lambda}} c_4$. Finally, since by (2.1)$_2$ $V \geq g_2(1)|Du|$, (6.14) holds for $C = c_7^{\frac{(2 - \mu)\mu}{2 - \mu - \alpha(n\beta - \mu)}} / g_2(1)$ and $\theta = \frac{(2 - \mu)\mu}{2 - \mu - \alpha(n\beta - \mu)}$.  \qed
7 Proofs of the results of Sect. 2

We use the following approximation Lemma.

Lemma 7.1 Assume that $f : \mathbb{R}^n \rightarrow [0, +\infty)$ be a convex function and $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $f(Dv) \in L^1_{\mathrm{loc}}(\Omega)$. For $\Omega'$ open set compactly contained in $\Omega$ and $\varphi_\varepsilon$ be smooth mollifiers with support in $B_\varepsilon(0)$, we define $v_\varepsilon = v \ast \varphi_\varepsilon \in C^\infty(\Omega')$, i.e.,

$$v_\varepsilon(x) = \int_{B_\varepsilon(0)} \varphi_\varepsilon(y)v(x - y) \, dy, \quad x \in \Omega'. \quad (7.1)$$

Then, for every open ball $B_\rho$ compactly contained in $\Omega'$,

$$\lim_{\varepsilon \to 0} \int_{B_\rho} f(Dv_\varepsilon) \, dx = \int_{B_\rho} f(Dv) \, dx. \quad (7.2)$$

Proof By Jensen’s inequality,

$$f(Dv_\varepsilon(x)) \leq \int_{B_\varepsilon(0)} \rho_\varepsilon(y)f(Dv(x - y)) \, dy.$$ 

By integrating over $B_\rho$ for $\varepsilon$ sufficiently small, we obtain

$$\int_{B_\rho} f(Dv_\varepsilon(x)) \, dx \leq \int_{B_\varepsilon(0)} \rho_\varepsilon(y) \int_{B_\rho} f(Dv(x - y)) \, dx \, dy \leq \int_{B_\varepsilon(0)} \rho_\varepsilon(y) \int_{B_{\rho \varepsilon}} f(Dv(x)) \, dx \, dy \leq \int_{B_{\rho \varepsilon}} f(Dv(x)) \, dx$$

and then

$$\limsup_{\varepsilon \to 0} \int_{B_\rho} f(Dv_\varepsilon(x)) \, dx \leq \int_{B_\rho} f(Dv(x)) \, dx.$$

By the other hand, $Dv_\varepsilon$ converges to $Dv$ in $L^1(B_\rho, \mathbb{R}^n)$, then the lower semicontinuity of the integral yields

$$\liminf_{\varepsilon \to 0} \int_{B_\rho} f(Dv_\varepsilon(x)) \, dx \geq \int_{B_\rho} f(Dv(x)) \, dx$$

proving $(7.2)$. \qed

Proof Consider the functional $(1.1)$ with $f$ satisfying $(2.1)$. For every $k \in \mathbb{N}$, let us consider the sequence $f_k$ defined as follows (see [37]):

$$f_k(\xi) = f(\xi)(1 - \phi(\xi)) + (f \phi) \ast \eta_k(\xi), \quad (7.3)$$

where $\eta_k$ are standard mollifiers and $\phi \in C^\infty(\mathbb{R}^n)$, $0 \leq \phi(\xi) \leq 1$ for every $\xi \in \mathbb{R}^n$, $\phi(\xi) = 1$ if $|\xi| \leq t_0 + 1$ and $\phi(\xi) = 0$ if $|\xi| \geq t_0 + 2$. $f_k \in C^2(\mathbb{R}^n)$ and
\[
f_k(\xi) = \begin{cases} 
    f(\xi) & |\xi| \geq t_0 + 2 \\
    (f') * \eta_k(\xi) & |\xi| \leq t_0 + 1.
\end{cases}
\]

Therefore, the sequence \( \{f_k\}_k \) converges to \( f \) uniformly and we can suppose \( |f(\xi) - f_k(\xi)| \leq 1 \) for every \( \xi \in \mathbb{R}^n \) and \( k \in \mathbb{N} \). Moreover, for sufficiently large \( k \), \( f_k \) is a convex function and \( D^2f_k(\xi) \) is positive defined for \( |\xi| > t_0 + 1 \). Since \( f_k(\xi) = f(\xi) \) for \( |\xi| > t_0 + 2 \), then (2.1) holds with \( t_0 + 2 \) instead of \( t_0 \).

Let \( h \in C^2([0, +\infty)) \) be the positive, increasing, convex function defined by

\[
h(t) = \begin{cases} 
\frac{1}{8}(6t^2 - t^4 + 3) & t \in [0, 1) \\
\frac{1}{2}t & t \in [1, +\infty).
\end{cases}
\]

Observe that \( h \in C^2(\mathbb{R}^2) \), \( h'' \) is nonnegative and \( h'' > 0 \) in \([0, 1)\). For \( k \in \mathbb{N} \) denote

\[
\tilde{f}_k(\xi) = f_k(\xi) + \frac{1}{k} h\left(\frac{|\xi|}{t_0 + 2}\right)
\]

and define the integral functional

\[
F_k(v) = \int_{B_R} \tilde{f}_k(Dv) \, dx.
\]

Notice that \( \tilde{f}_k \in C^2(\mathbb{R}^n) \) and that it is uniformly convex on compact subsets of \( \mathbb{R}^n \). Let \( B_R \) be a ball compactly contained in \( \Omega \) and \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be a local minimizer of the functional (1.1). Since \( u \in C^2(B_R) \), then it verifies the bounded slope condition (see for example [27] and [26, Theorem 1.1 and Theorem 1.2]) and then \( F_k \) has unique minimizer \( v_{\epsilon,k} \) among Lipschitz continuous functions in \( B_R \) with boundary value \( u_\epsilon \) on \( \partial B_R \). By (2.1) and [35, (3.3)],

\[
g_1(|\xi|)|\lambda|^2 \leq \sum_{i,j} (\tilde{f}_k)_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j \leq g_2(|\xi|) + \frac{1}{k(t_0 + 2)} \frac{1}{|\xi|} |\lambda|^2,
\]

for every \( \lambda, \xi \in \mathbb{R}^n \), \( |\xi| \geq t_0 + 2 \). On the other hand, since by (2.1)_2 \( t \, g_2(t) \geq (t_0 + 2)g_2(t_0 + 2) \) for \( t \geq t_0 + 2 \),

\[
g_2(t) + \frac{1}{k(t_0 + 2)} \frac{1}{t} \leq g_2(t) + \frac{g_2(t)}{k(t_0 + 2)^2 + g_2(t_0 + 2)} \leq 2g_2(t)
\]

for \( k \) sufficiently large. Therefore, \( \tilde{f}_k \) satisfies (2.1) with \( t_0 + 2 \) instead of \( t_0 \), \( 2g_2 \) instead of \( g_2 \) and constants \( C_1 \) and \( C_2 \) independent from \( k \). Moreover, as \( k \to \infty \), \( \tilde{f}_k \to f \) uniformly on compact subsets of \( \mathbb{R}^n \) and then

\[
\lim_{k \to \infty} \int_{B_R} \tilde{f}_k(Dv) \, dx = \int_{B_R} f(Dv) \, dx \quad \text{for every } v \in W^{1,\infty}(B_R).
\]

Since \( \tilde{f}_k \) is uniformly convex on compact sets and \( v_{\epsilon,k} \) is Lipschitz continuous in \( B_R \), all the assumptions of Lemma 6.3 hold and, by the a priori estimate (6.14), for every \( \rho < R \) there exists a positive constant \( C \) depending on \( \rho, R, \alpha, \beta, \mu, C_1, C_2, g_2(t_0) \), but independent on \( k \) such that
\[ \|Dv_{\epsilon,k}\|_{L^\infty(B_{\rho})} \leq C \left\{ \frac{1}{(R-\rho)\theta} \int_{B_{\rho}} \left( 1 + \tilde{f}_k(Dv_{\epsilon,k}) \right) dx \right\} \]
\[ \leq C \left\{ \frac{1}{(R-\rho)\theta} \int_{B_{\rho}} \left( 1 + \tilde{f}_k(Du_{\epsilon}) \right) dx \right\} , \]
\[ \theta = \frac{(2-\mu)\alpha}{2-\mu-\alpha(\mu-\rho)}, \]
where the last inequality depends by the minimality of \( v_{\epsilon,k} \). Therefore, for every \( \epsilon > 0 \), we have
\[ \lim_{k \to \infty} \|Dv_{\epsilon,k}\|_{L^\infty(B_{\rho})} \leq C \left\{ \frac{1}{(R-\rho)\theta} \int_{B_{\rho}} (1 + f(Du_{\epsilon})) dx \right\} = M_{\epsilon}. \]

The sequence \( v_{\epsilon,k} \) is bounded in \( W^{1,\infty}(B_{\rho}) \) with respect to \( k \), then there exists a subsequence \( k_j \to 0 \), such that \( \{v_{\epsilon,k_j}\} \) is weakly convergent to \( \tilde{v}_{\epsilon} \) in \( W^{1,\infty}(B_{\rho}) \) and for every \( \rho < R \)

\[ \|D\tilde{v}_{\epsilon}\|_{L^\infty(B_{\rho})} \leq C \left\{ \frac{1}{(R-\rho)\theta} \int_{B_{\rho}} (1 + f(Du_{\epsilon})) dx \right\} . \] (7.4)

We prove that \( v_{\epsilon,k_j} \) converges to \( \tilde{v}_{\epsilon} \) in \( W^{1,1}(B_{R}) \) and then \( \tilde{v}_{\epsilon} \in u + W^{1,1}_0(B_{R}) \). Indeed by (2.1), and the minimality of \( v_{\epsilon,k_j} \), as \( j \to \infty \) we have
\[ \int_{B_{R}} f(Dv_{\epsilon,k_j}) dx \leq 1 + \int_{B_{R}} \tilde{f}_{k_j}(Dv_{\epsilon,k_j}) dx \leq 1 + \int_{B_{R}} \tilde{f}_{k_j}(Du_{\epsilon}) dx \to 1 + \int_{B_{R}} f(Du_{\epsilon}) dx. \]

By de la Vallée-Poussin Theorem, we can choose the sequence \( k_j \) such that \( Dv_{\epsilon,k_j} \rightharpoonup D\tilde{v}_{\epsilon} \) in \( L^1(B_{R}) \) and then \( v_{\epsilon,k_j} - u \to (\tilde{v}_{\epsilon} - u) \) in \( W^{1,1}_0(B_{R}) \). On the other hand, for every \( \delta > 0 \) and for every \( k \) sufficiently large, \( |\tilde{f}_{k}(\xi) - f(\xi)| \leq \delta \), for every \( |\xi| \leq M + 1 \), therefore by the minimality of \( v_{\epsilon,k_j} \)

\[ \int_{B_{R}} f(Dv_{\epsilon,k_j}) dx = \int_{B_{R}} (f(Dv_{\epsilon,k_j}) - \tilde{f}_{k}(Dv_{\epsilon,k_j})) + \tilde{f}_{k}(Dv_{\epsilon,k_j}) dx \]
\[ \leq \int_{B_{R}} \tilde{f}_{k}(Du_{\epsilon}) dx + \delta |B_{R}|. \]

By lower semicontinuity in \( W^{1,1}(B_{R}) \), passing to the limit for \( j \to \infty \), we get
\[ \int_{B_{R}} f(D\tilde{v}_{\epsilon}) dx \leq \liminf_{j \to \infty} \int_{B_{R}} f(Dv_{\epsilon,k_j}) dx \leq \lim_{j \to \infty} \int_{B_{R}} \tilde{f}_{k_j}(Du_{\epsilon}) dx + \delta |B_{R}| \]
\[ = \int_{B_{R}} f(Du_{\epsilon}) dx + \delta |B_{R}| \]
for every \( \delta > 0 \) and \( \rho < R \), and then for \( \rho \to R \) and \( \delta \to 0 \)
\[ \int_{B_{R}} f(D\tilde{v}_{\epsilon}) dx \leq \int_{B_{R}} f(Du_{\epsilon}) dx \]

Again by de la Vallée-Poussin Theorem and (7.4), we have that there exists a sequence \( \epsilon_j \to 0 \) such that \( v_{\epsilon_j} - u_{\epsilon_j} \to \tilde{v} - u \) in \( W^{1,1}_0(B_{R}) \) and \( \{v_{\epsilon_j}\} \) is weakly convergent to \( \tilde{v} \) in \( W^{1,\infty}(B_{\rho}) \) for every \( 0 < \rho < R \). By the lower semicontinuity of the functional
\[ \int_{B_R} f(D\tilde{v}) \, dx \leq \liminf_{j \to \infty} \int_{B_R} f(D\tilde{v}_j) \, dx \leq \lim_{j \to \infty} \int_{B_R} f(Du_j) \, dx = \int_{B_R} f(Du) \, dx. \]  

(7.5)

Then, \( \tilde{v} \) is a minimizer for (1.1) with \( \Omega = B_R \). Moreover, from (7.2) and (7.4) we have that \( \tilde{v}_\varepsilon \) converges to \( \tilde{v} \) in \( W^{1,\infty}_{\text{loc}}(\Omega) \) and

\[ \|D\tilde{v}\|_{L^\infty(B_R)} \leq \liminf_{j \to \infty} \|D\tilde{v}_j\|_{L^\infty(B_R)} \leq \lim_{j \to \infty} C \left\{ \frac{1}{(R - \rho)^n} \int_{B_R} (1 + f(Du_j)) \, dx \right\}^\theta \]

\[ = C \left\{ \frac{1}{(R - \rho)^n} \int_{B_R} (1 + f(Du)) \, dx \right\}^\theta. \]

(7.6)

Therefore, \( \tilde{v} \) and \( u \) are two different minimizers of \( F \) in \( B_R \). Since \( f(\xi) \) is strictly convex for \( |\xi| > t_0 \), by proceeding as in [20] it is possible to prove that the set

\[ E_0 := \left\{ x \in B_R : \frac{|Du(x) + D\tilde{v}(x)|}{2} > t_0 \right\}, \]

has zero measure. Therefore,

\[ \|Du\|_{L^\infty(B_R)} \leq \|Du + D\tilde{v}\|_{L^\infty(B_R)} + \|D\tilde{v}\|_{L^\infty(B_R)} \leq 2t_0 + \|D\tilde{v}\|_{L^\infty(B_R)}. \]

\[ \square \]

**Lemma 7.2** Let \( f \) be a positive function in \( C^2(\{ \xi \in \mathbb{R}^n : |\xi| \geq t_0 \}) \) satisfying ellipticity conditions (2.4). Then, there exists \( \tilde{t} \geq t_0 \) such that

\[ \frac{m}{2(p - 1)} |\xi|^p \leq f(\xi) \leq \frac{2M}{q - 1} |\xi|^q, \quad |\xi| \geq \tilde{t} \quad (7.7) \]

if \( 1 < p \leq q, \ f(\xi) \geq \frac{m}{2} |\xi| \log |\xi| \) or \( f(\xi) \leq 2M|\xi| \log |\xi| \) if \( p = 1 \) or \( q = 1 \), respectively.

**Proof** Without loss of generality, we can suppose \( t_0 = 1 \). We will prove the left hand side in (7.7), the right hand side being analogous. We consider the real function \( \varphi : [1, +\infty) \to \mathbb{R} \) defined by \( \varphi(t) = f \left( t \frac{\xi}{|\xi|} \right) \). Then, \( \varphi \in C^2[1, +\infty) \) and its first and second derivatives hold

\[ \varphi'(t) = \left( Df \left( t \frac{\xi}{|\xi|} \right), \frac{\xi}{|\xi|^2} \right); \quad \varphi''(t) = \sum_{i,j=1}^n f_{\xi_i \xi_j} \left( t \frac{\xi_i \xi_j}{|\xi|^2} \right) \frac{\xi_i \xi_j}{|\xi|^2} \]

and, by (2.4), \( \varphi''(t) \geq mt^{p-2} \). Therefore, integrating from 1 to \( t \) we obtain

\[ \varphi'(t) - \varphi'(1) \geq \begin{cases} \frac{m}{p-1} (t^{p-1} - 1) & \text{if } p > 1 \\ m \log t & \text{if } p = 1 \end{cases} \]

and again

\[ \varphi(t) - \varphi(1) \geq Df \left( \frac{\xi}{|\xi|} \right) (t - 1) + \begin{cases} \frac{m}{p-1} (t^p - t) & \text{if } p > 1 \\ m(t \log t - t) & \text{if } p = 1. \end{cases} \]

Therefore, for \( t = |\xi| \) we have
\[ f(\xi) \geq -L(|\xi| - 1) + \begin{cases} \frac{m}{p-1} (|\xi|^p - |\xi|) & \text{if } p > 1 \\ m(|\xi| \log|\xi| - |\xi|) & \text{if } p = 1 \end{cases} \]

where \( L = \max \{|Df(v)| : |v| = 1\}. \)

**Proof of Corollary 2.2** We have to prove that the assumptions of Theorem 2.1 are satisfied. First of all, notice that assumptions (2.1) hold for \( g_1(t) = mt^{p-2} \) and \( g_2(t) = Mt^{q-2} \). Moreover, for every \( t \geq 1 \),

\[ (g_2(t))^{\frac{2}{p}} = M^{2/2'} t^{(q-2)\frac{2}{p'}} = mt^{p-2} M^{2/2'} t^{(q-2)\frac{2}{p'}-(p-2)}, \]

then (2.1) holds for \( \beta = \frac{n}{2} q - \frac{p}{2} + \frac{2}{n} \). By Lemma 7.2, for \( \alpha \geq \tilde{\alpha} = q/p \) and \( |\xi| \) large enough,

\[ g_2(|\xi|)|\xi|^2 = M|\xi|^q \leq M \left( \frac{2p-2}{m} \right) \left( \frac{m}{2p-2} |\xi|^p \right)^{\alpha} \leq M \left( \frac{4}{m} \right)^{\alpha} [1 + f(\xi)]^{\alpha} \]

and (2.1) holds. Moreover, by Lemma 7.2 also (2.1) holds. Therefore, all the assumptions of Theorem 2.1 are satisfied if (2.2) holds, that is

\[ \alpha < \frac{q}{n\beta + q - 2} \quad \iff \quad \frac{q}{p} < \frac{q}{\frac{n}{2} q - \frac{n}{2} p} \quad \iff \quad \frac{q}{p} < \frac{n+2}{n} \]

since \( \mu = 2 - q \). In order to obtain the correct exponent in (2.6), notice that in this case by (6.16) we can conclude

\[ \|Du\|_{L^q(B_{\gamma})} \leq \left[ \frac{c_7}{(R - \rho)^\mu} \int_{B_{\rho_0}} (1 + f(Du)) \, dx \right]^\frac{q}{q-a(n\beta + q - 2)}. \]

Since \( \frac{q-a(n\beta + q - 2)}{(n+2)p-nq} \), we get (2.6). \( \square \)

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