ON FAST AND SLOW TIMES IN MODELS WITH DIFFUSION

M.DE ANGELIS - A.M.MONTE - P.RENNO

Facoltà di Ingegneria, Dip. di Mat. e Appl., via Claudio 21, 80125, Napoli, Italy.

The linear Kelvin-Voigt operator $L_\varepsilon$ is a typical example of wave operator $L_0$ perturbed by higher-order viscous terms as $\varepsilon u_{xxt}$. If $P_\varepsilon$ is a prefixed boundary-value problem for $L_\varepsilon$, when $\varepsilon = 0$ $L_\varepsilon$ turns into $L_0$ and $P_\varepsilon$ into a problem $P_0$ with the same initial-boundary conditions of $P_\varepsilon$. Boundary-layers are missing and the related control terms depending on the fast time are negligible. In a small time-interval, the wave behavior is a realistic approximation of $u_\varepsilon$ when $\varepsilon \to 0$. On the contrary, when $t$ is large, diffusion effects should prevail and the behavior of $u_\varepsilon$ for $\varepsilon \to 0$ and $t \to \infty$ should be analyzed. For this, a suitable functional correspondence between the Green functions $G_\varepsilon$ and $G_0$ of $P_\varepsilon$ and $P_0$ is achieved and its asymptotic behavior is rigorously examined. For this, a suitable functional correspondence between the Green functions $G_\varepsilon$ and $G_0$ of $P_\varepsilon$ and $P_0$ is derived and its asymptotic behavior is rigorously examined. As consequence, the interaction between diffusion effects and pure waves is evaluated by means of the slow time $\varepsilon t$; the main results show that in time-intervals as $(\varepsilon, 1/\varepsilon)$ pure waves are propagated nearly undisturbed, while damped oscillations predominate as from the instant $t > 1/\varepsilon$.

Green’s Function, Partial differential equations, Viscoelastic models, Singular perturbations.

1. Introduction

Consider the class of dissipative phenomena described by the following model:

$$L_\varepsilon u_\varepsilon \equiv (\varepsilon \partial_{xxt} + c^2 \partial_{xx} - \partial_{tt})u_\varepsilon = f,$$

(1.1)

where $\varepsilon, c$ are positive constants and the source term $f$ may be linear or not. In the linear case, when a prefixed boundary-initial problem $P_\varepsilon$ is stated, the knowledge of the related Green function $G_\varepsilon$ allows to solve explicitly $P_\varepsilon$. When the function $f$ is not linear, then $G_\varepsilon$ represents the explicit kernel of the integral equation to which the problem $P_\varepsilon$ can be reduced.

Two typical examples in the non-linear case are:

i) For $f = au_t + b \sin u$, the equation (1.1) is the perturbed Sine-Gordon equation which models the Josephson effect in Superconductivity [1].

ii) For $f = f(u_x, u_{xt}, u_{xx})$, the equation (1.1) is the Navier-Stokes equation for a compressible gas with small viscosity [2].

Moreover, as for the artificial viscosity methods, (1.1) represents a model of wave equations perturbed by viscous terms with a small parameter $\varepsilon$ [3], [4]. In this framework, the behavior of $u_\varepsilon$ as $\varepsilon \to 0$ must be examined and the interaction
of pure waves with the diffusion effects caused by $\varepsilon u_{xxt}$ must be estimated. This interaction is meaningful in the evolution of many dissipative models (viscoelastic liquids or solids [5] - [9], real gases with viscosity [10], magnetohydrodynamic fluids [11]).

For $\varepsilon \equiv 0$, the parabolic equation (1.1) turns into the wave equation

$$L_0 u_0 \equiv (c^2 \partial_{xx} - \partial_{tt}) u_0 = f,$$

and $\mathcal{P}_\varepsilon$ changes into a problem $\mathcal{P}_0$ for $u_0$, with the same initial-boundary conditions of $\mathcal{P}_\varepsilon$. So, boundary-layers are missing, but the approximation of $u_\varepsilon$ by $u_0$ is rough, because in large time-intervals the diffusion effects are dominant. As consequence, the asymptotic behaviour of $u_\varepsilon$ should depend on the slow time $\varepsilon t$ typical of the diffusion; on the contrary, boundary layer control terms characterized by the fast time $t/\varepsilon$, should be negligible.

Aim of the paper is to derive and analyze an appropriate functional relation between $G_\varepsilon$ and the wave Green function $G_0$ related to $\mathcal{P}_0$ (Th. 3.1). By means of this transformation, a rigorous asymptotic analysis of $G_\varepsilon$ is achieved (Th. 4.1), and $G_\varepsilon$ is approximated by solutions $v(x,t)$ of the second-order diffusion-wave equations

$$\frac{\varepsilon}{2} v_{xx} = v_t + cv_x, \quad \frac{\varepsilon}{2} \tilde{v}_{xx} = \tilde{v}_t - cv_x$$

(1.3)

which correspond to the heat equations $v_{yy} = v_\theta$, where the time-variable $\theta$ is just the slow-time ($\theta = \varepsilon t$) and the space-variables $y = x \pm ct$ are related to traveling or backward waves.

The physical meaning of the above analysis is clarified by the explicit solution related to the case $f = 0$ (n.5). Except errors of order $O(\varepsilon/t)$, this relationship is

$$u_\varepsilon(x,t) = \frac{c}{\sqrt{2\pi\varepsilon t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2\varepsilon t}} u_0(x,\tau) \, d\tau,$$

(1.4)

and it clearly shows the interaction between diffusion and wave propagation. In the time-interval $(\varepsilon, \frac{1}{\varepsilon})$, when $\varepsilon t < 1$, pure waves propagate almost without perturbation; while as from the instant $t > 1/\varepsilon$, damped oscillations predominate.

Let us remark, generally, asymptotic theories are developed for models with first and second order operators as documented in [12] and [13]. For these operators, in fact, numerous maximum theorems for the rigorous estimate of the remainder are well known.

As for the third-order model (1.1), the error of the approximation has been examined by means of the functional relation (3.1).

2. Statement of the problem

If $u_\varepsilon(x,t)$ is a function defined in the strip

$$\Omega = \{(x,t) : 0 \leq x \leq l, \ t > 0\},$$
let $\mathcal{P}_\varepsilon$ the initial-boundary value problem related to the Eq. (1.1) with conditions

$$
\begin{align*}
\begin{cases}
    u_\varepsilon(x,0) = f_0(x), & \partial_t u_\varepsilon(x,0) = f_1(x), & x \in [0,l], \\
    u_\varepsilon(0,t) = \varphi(t), & u_\varepsilon(l,t) = \psi(t), & t > 0,
\end{cases}
\end{align*}
$$

(2.1)

where $f_0, f_1, \varphi, \psi$ are arbitrary given data.

Boundary conditions (2.1) represent only an example of the analysis we are going to apply; *flux-boundary* conditions or *mixed boundary* conditions could be considered too. Further, as $f(x,t), f_0(x), f_1(x)$ are quite arbitrary, it is not restrictive assuming $\varphi = 0$, $\psi = 0$; otherwise, it suffices to put

$$
\bar{u}_\varepsilon = u_\varepsilon - \frac{x}{l}\psi - \frac{l-x}{l}\varphi; \quad \bar{f} = f + \frac{x}{l}\dot{\psi} + \frac{l-x}{l}\ddot{\varphi}
$$

(2.2)

and to modify $f_0(x), f_1(x)$ consequently.

Let $\hat{z}(s)$ denote the Laplace transform of the function $z(t)$ and let

$$
\hat{F}(x,s) = \bar{f}(x,s) - sf_0(x) - f_1(x). 
$$

(2.3)

Further, if $\sigma(\varepsilon) = s(\varepsilon s + c^2)^{-1/2}$, let

$$
\hat{g}(y,s) = \frac{\cosh((l-y)\sigma)}{2(\varepsilon s + c^2)\sigma \text{sech}(l\sigma)}
$$

(2.4)

and

$$
\hat{G}_\varepsilon(x,\xi, s) = \hat{g}(|x - \xi|, s) - \hat{g}(|x + \xi|, s).
$$

(2.5)

Then, the Laplace transform $\hat{u}_\varepsilon$ of the solution of the problem $\mathcal{P}_\varepsilon$ is given by

$$
\hat{u}_\varepsilon(x,s) = \int_0^l \left[ \hat{F}(\xi,s) + \varepsilon f_0''(\xi) \right] \hat{G}_\varepsilon(x,\xi, s) \, d\xi.
$$

(2.6)

When $\varepsilon = 0$ one has $\sigma(0) = s/c$ and (2.4),(2.5) turn into

$$
\hat{g}_0(y,s) = \frac{\cosh((l-y)s/c)}{2 (c s \text{sech}(ts/c)},
$$

(2.7)

$$
\hat{G}_0(x,\xi, s) = \hat{g}_0(|x - \xi|, s) - \hat{g}_0(|x + \xi|, s).
$$

(2.8)

Problem $\mathcal{P}_0$ is defined by (1.2), (2.1) and the Laplace transform $\hat{u}_0$ of it’s solution is:

$$
\hat{u}_0(x,s) = \int_0^l \hat{F}(\xi,s) \hat{G}_0(\xi, x, s) \, d\xi.
$$

(2.9)

By comparing (2.4),(2.5) with (2.7), (2.8), the following relationship between $\hat{G}_\varepsilon$ and $\hat{G}_0$ is pointed out:

$$
\hat{G}_\varepsilon(s) = \frac{c^2}{\varepsilon s + c^2} \hat{G}_0\left(\frac{cs}{\sqrt{\varepsilon s + c^2}}\right).
$$

(2.10)
In order to estimate $G_\varepsilon(t)$ in terms of $G_0(t)$ and to achieve the rigorous asymptotic behaviour of $G_\varepsilon(t)$ for $\varepsilon \to 0$, the correspondence (2.10) must be analyzed and its inverse Laplace transform must be established.

3. Functional dependence between $G_\varepsilon$ and $G_0$

If $I_0$ denotes the modified Bessel function of first kind and order zero, the inverse Laplace - transform of (2.10) is fully given by the following theorem:

**Theorem 3.1** - The relationship between the Green functions $G_\varepsilon(t)$ and $G_0(t)$ is:

$$G_\varepsilon(t) = \frac{c^3 e^{-c^2 t/\varepsilon}}{\varepsilon \sqrt{\pi c^2 t}} \int_0^\infty e^{-\frac{c^2 v^2}{4\varepsilon t}} dv \int_0^v G_0(u) I_0\left(\frac{2c^2}{\varepsilon} \sqrt{u(v-u)}\right) du \quad (3.1)$$

and their $L$- transforms satisfy (2.10) in the half- plane $\text{Re}(s) > -c^2/\varepsilon$.

**Proof.** For $\text{Re}(\varepsilon s + c^2) > 0$, one has:

$$\mathcal{L}(e^{-\frac{c^2 t}{\varepsilon \sqrt{\pi c^2 t}}} e^{-\frac{c^2 v^2}{4\varepsilon t}}) = \frac{e^{-\frac{c^2 \sqrt{\varepsilon s + c^2} v}{\varepsilon s + c^2}}}{\sqrt{\varepsilon s + c^2}},$$

(3.2)

therefore the Laplace tranform of (3.1) is:

$$\hat{G}_\varepsilon(s) = \frac{c^3}{\varepsilon \sqrt{\varepsilon s + c^2}} \int_0^\infty e^{-\frac{c^2 \sqrt{\varepsilon s + c^2} v}{\varepsilon s + c^2}} dv \int_0^v G_0(u) I_0\left(\frac{2c^2}{\varepsilon} \sqrt{u(v-u)}\right) du \quad (3.3)$$

$$= \frac{c^3}{\varepsilon \sqrt{\varepsilon s + c^2}} \int_0^\infty G_0(u) du \int_0^\infty e^{-\frac{c^2 (u+v)}{\varepsilon s + c^2}} I_0\left(\frac{2c^2}{\varepsilon} \sqrt{uv}\right) dv. \quad (3.4)$$

Considering that

$$\int_0^\infty e^{-\frac{c^2 \sqrt{\varepsilon s + c^2} v}{\varepsilon s + c^2}} I_0\left(\frac{2c^2}{\varepsilon} \sqrt{uv}\right) dv = \frac{\varepsilon}{c^3} e^{\frac{c^3 u}{\sqrt{\varepsilon s + c^2}}} \quad (3.4)$$

see ([14]), one has

$$\hat{G}_\varepsilon(s) = \frac{c^2}{\varepsilon s + c^2} \int_0^\infty e^{-\frac{c^2 \sqrt{\varepsilon s + c^2} u}{\varepsilon s + c^2}} G_0(u) du \quad (3.5)$$

therefore the above formula coincides with (2.10).
Consider now a first application of Theorem 3.1. By means of the transformation (3.1), it’s possible to deduce explicit expressions for $G_\varepsilon$ from known formulae of $G_0$. So, for instance, if $G_0$ is represented by its Fourier series

$$G_0 = \frac{2}{c^2 \pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{\pi}{l}ct\right) \sin\left(\frac{\pi}{l}nx\right) \sin\left(\frac{\pi}{l}n\xi\right), \quad (3.6)$$

then (3.1) allows to make Fourier series of $G_\varepsilon$ explicit. It suffices to evaluate the transform (3.1) of the time-component $G_{0,n}$ of $G_0$:

$$G_{0,n}(t) = \sin\left(\frac{\pi}{l}cn\right). \quad (3.7)$$

Owing to:

$$I = \int_0^{2v} \sin(ay)I_0(b\sqrt{y(2v-y)})dy = 2\sin(av) \frac{\sin(\sqrt{a^2-b^2})}{\sqrt{a^2-b^2}}, \quad (3.8)$$

if one puts:

$$a = \pi cn/l, \quad b = 2c^2/\varepsilon, \quad k = 2c/(\pi \varepsilon), \quad \omega_n = \sqrt{1-(n/k)^2}, \quad (3.9)$$

the integral transform (3.1) of $G_{0,n}$ gives

$$G_{\varepsilon,n}(t) = e^{-\frac{\pi^2 \varepsilon^2}{2l^2}} \sin(\omega_n t) \omega_n. \quad (3.10)$$

So, by (3.1), (3.6), (3.10), one obtains

$$G_\varepsilon = \frac{2}{c^2 \varepsilon} \sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2 \varepsilon t}{2l^2}} \sin\left(\frac{c\pi n}{l} \omega_n t\right) \frac{1}{n \omega_n} \sin\left(\frac{\pi}{l}nx\right) \sin\left(\frac{\pi}{l}n\xi\right). \quad (3.11)$$

and this formula represents the Fourier series of $G_\varepsilon$ previously established in [16] by Fourier method.

### 4. Asymptotic behavior of $G_\varepsilon$

An additional consequence of (3.1) is the asymptotic analysis of $G_\varepsilon$ when $\varepsilon$ tends to zero. Referring to (3.1), let $\tau = t/\varepsilon$ and let

$$\Gamma(u, v, \tau) = \frac{(c\sqrt{\tau})^3}{\sqrt{\pi}} G_0(tu) I_0(2c^2 \tau \sqrt{uv})e^{-c^2 \tau [1+(u+v)^2]/4}. \quad (4.1)$$

Then, the function $G_\varepsilon$ can be given the form

$$G_\varepsilon = \int \int_Q \Gamma(u, v, \tau) du \ dv, \quad (4.2)$$

where $Q \equiv \{(u, v) \in [0, \infty] \times [0, \infty]\}$.
So, $G_\varepsilon$ depends on $\varepsilon$ only through the fast time $\tau$. When $\varepsilon$ is vanishing and $t > \varepsilon$, the parameter $\tau$ goes to infinity and the asymptotic behaviour of integral (4.2) can be rigorously obtained by Laplace method [15]

To establish precise estimates of the remainder terms, let

$$\chi(\tau) = \chi_0 \tau^{-1/3}, \quad \sigma(\tau) = \sigma_0 \tau^{-1/3},$$

where $\chi_0$ and $\sigma_0$ are arbitrary real positive constants such that $\chi$ and $\sigma$ are less then one for all $\tau > 1$. Further, let

$$Q_0 \equiv \{(u, v) \in [1 - \chi, 1 + \chi] \times [1 - \sigma, 1 + \sigma]\} \subset Q.$$ 

The following Lemma holds:

**Lemma 4.1 - For all $\tau > 1$, it results:**

$$|G_\varepsilon - \int\int_{Q_0} \Gamma(u, v, \tau) du dv| < \mu e^{-\lambda^2 \tau^{1/3}},$$ \hspace{1cm} (4.5)

where the constants $\lambda^2, \mu$ depend only on $c, \chi_0, \sigma_0$.

**Proof.** For all real and positive $z$ one has $I_o(z) < e^z$, so that by (4.1) one obtains

$$|\Gamma| \leq \frac{|G_0|}{\sqrt{\pi}} (c\sqrt{\tau})^3 e^{-c^2 \tau} h(u, v)$$ \hspace{1cm} (4.6)

with

$$h(u, v) = 1 + (1/4)(u + v)^2 - 2\sqrt{uv}.$$ \hspace{1cm} (4.7)

Further, for all $(u, v) \in Q$, one has

$$h(u, v) = \frac{1}{4}[(u - 1)^2 + (v - 1)^2 + (\sqrt{u} - 1)^2(\sqrt{v} + 1)^2$$

$$+ (\sqrt{v} - 1)^2(\sqrt{u} + 1)^2] \geq \frac{1}{4}[(u - 1)^2 + (v - 1)^2].$$

Therefore, indicating by $C$ a constant independent by $\tau$, one has:

$$|\int\int_{Q - Q_0} \Gamma du dv| \leq C \int\int_{Q - Q_0} e^{-\frac{4}{\tau^2}[(u - 1)^2 + (v - 1)^2]} du dv.$$ \hspace{1cm} (4.9)

Hence, by standard computations, the estimate (4.5) is obtained.

The estimates of Lemma 4.1 allow us to define the following function

$$H(x, t, \varepsilon) = \frac{c}{\sqrt{2\pi \varepsilon t}} \int_{-\infty}^{\infty} e^{-\frac{c^2(x - \varepsilon)^2}{2\varepsilon t}} G_0(x, \tau) \, d\tau.$$ \hspace{1cm} (4.10)
and to obtain the following result:

**Theorem 4.1** - For all \((x,t) \in \Omega, \text{ and } t > \varepsilon\), one has:

\[
G_\varepsilon(x,t) = H(x,t,\varepsilon) \ (1 + \rho_1) + \rho_2, \tag{4.11}
\]

where the remainder terms \(\rho_1, \rho_2\) are decreasing functions of the fast time \(\tau = t/\varepsilon\) such that

\[
|\rho_1| \leq k_1 \tau^{-1}; \quad \rho_2 \leq k_2 e^{-\lambda^2 \tau^{1/3}}, \tag{4.12}
\]

with the constants \(k_1, k_2, \lambda^2\) depending only on \(c, \chi_0, \sigma_0\).

**Proof.** By asymptotic formulae of Bessel functions, for real, positive and large \(z\) one has ([17]):

\[
I_0(z) = \frac{e^z}{\sqrt{2\pi z}} \ (1 + r_0) \quad (|r_0| < c_0 \ z^{-1}) \tag{4.13}
\]

and so the integral of \(\Gamma\) on \(Q_0\) (see (4.1), (4.2)) can be given the form

\[
\int_{Q_0} \Gamma dudv = \frac{ct}{2\pi} \int_{1-\chi}^{1+\chi} G_0(tu) \, du \int_{1-\sigma}^{1+\sigma} \frac{1 + r_1}{(uv)^{1/4}} e^{-c^2 \tau \ h(u,v)} \, dv, \tag{4.14}
\]

where \(h(u,v)\) is defined in (4.7) and the remainder term \(r_1\) is such that \(|r_1| \leq c_1 (\tau \sqrt{uv})^{-1}\), according to (4.13).

The function \(h(u,v)\) has a point of absolute minimum in \((u_0, v_0) = (1, 1)\), where it vanishes. Therefore, it results: \(h = h_0 + h_*\) with

\[
h_0 = \frac{1}{2}((u - 1)^2 + (v - 1)^2); \quad |h_*| \leq c_* |u - 1| + |v - 1|)^3. \tag{4.15}
\]

According to the Laplace method, [15], for large \(\tau\), the dominant contribution is due to a neighbourhood of \((u_0, v_0)\), so that it’s enough to expand \(\Gamma\) near to this point. Further, if \(u\) and \(v\) are substituted by

\[
u - 1 = \frac{1}{\sqrt{\tau}} u_1, \quad v - 1 = \frac{1}{\sqrt{\tau}} v_1, \tag{4.16}
\]

the terms of third order like \(h_*\) are vanishing with order of \(\tau^{-3/2}\); besides, the limits of integration become \((-\infty, \infty)\) as: \(\chi \sqrt{\tau} = \chi_0 \tau^{1/6}\) and \(\sigma \sqrt{\tau} = \sigma_0 \tau^{1/6}\).

Then, for \(\tau \to \infty\), one has:

\[
\int_{Q_0} \Gamma dudv = \frac{c^2}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{2}{\tau^2} u^2} G_0(t + \sqrt{\varepsilon t} \ u) du \int_{-\infty}^{\infty} e^{-\frac{2}{\tau^2} v^2} dv + O(\frac{1}{\tau}). \tag{4.17}
\]

The above formula, together with the results of Lemma 4.1 (see(4.5),(4.10)), implies (4.1) and the estimate (4.12)_2 for \(\rho_2\). Finally, by means of routine calculations, the order of the error \(\rho_1\) can be rigorously specified according to (4.12)_1. ■
5. Diffusion and waves

To remark the physical meaning of the results, let consider the simple case: 
\( f = 0, f_0 = 0, f_1 \neq 0 \). By (2.6) and (2.9) one has:

\[
    u_\varepsilon = -\int_0^l f_1(\xi)G_\varepsilon(\xi, x, t)d\xi; \quad u_0 = -\int_0^l f_1(\xi)G_0(\xi, x, t)d\xi
\]  

(5.1)

Consequently, except errors of order \( O(\varepsilon/t) \), Theorem 4.1 allows to obtain (see (4.10),(4.11)) the following estimate:

\[
    u_\varepsilon(x, t) = c\sqrt{\frac{2}{\pi\varepsilon t}} \int_{-\infty}^{\infty} e^{-\frac{c^2(\tau-t)^2}{2\varepsilon t}} u_0(x, \tau) d\tau.
\]  

(5.2)

The above explicit correspondence between \( u_0 \) and \( u_\varepsilon \) clearly shows the interaction between diffusion and wave propagation.

For instance, when \( f_1 = \frac{c\pi}{l} \sin\left(\frac{\pi x}{l}\right) \), one has the pure wave

\[
    u_0(x, t) = \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi}{c} ct\right),
\]  

(5.3)

while \( u_\varepsilon \), owing to (5.2), is given by

\[
    u_\varepsilon(x, t) = e^{-\frac{c^2}{2\pi\varepsilon t}} u_0(x, t).
\]  

(5.4)

This means that the dissipation caused by \( \varepsilon u_x x t \) is significant by slow times \( \varepsilon t \) only; the terms depending on the fast time \( t/\varepsilon \) are fully negligible for all \( t > \varepsilon \).

As consequence, in the time interval \( (\varepsilon, \frac{1}{\varepsilon}) \) where \( \varepsilon t < 1 \), the pure wave (5.3) is propagated nearly undisturbed; when \( t > \frac{1}{\varepsilon} \), damped oscillations predominate.

These aspects are generally valid, whatever the initial data may be, and for appropriate \( f \). In fact, if one transforms by (4.10) the time-components \( G_{0,n}(t) = \sin\left(\frac{\pi n x}{l}\right) \) of \( G_0 \), one has [14]:

\[
    \frac{c}{\sqrt{2\pi c t t}} \int_{-\infty}^{\infty} e^{-\frac{c^2(z-\varepsilon t)^2}{2\varepsilon t}} \sin\left(\frac{\pi n c t}{l}\right) d\tau = e^{-\frac{c^2}{2\pi\varepsilon t}} \sin\left(\frac{\pi}{c} c t n\right).
\]  

(5.5)

Therefore, the Fourier series of the function \( H \) which approximates \( G_\varepsilon \) is given by

\[
    H = \frac{2}{c\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{1}{4}(\frac{\pi n}{l})^2 t} \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi n x}{l}\right).
\]  

(5.6)

This formula, together with (2.6), (4.11), makes explicit the asymptotic behavior of the solution \( u_\varepsilon \) of the problem \( P_\varepsilon \) related to arbitrary data \( f_0, f_1 \) and \( f \). As (5.6) shows, this behavior is like (5.4), with damped waves which become vanishing as from the instants \( t > 1/\varepsilon \).

It must be remarked that the function \( H \) can be given the form:

\[
    H = H^- - H^+,
\]  

with
\[ H^\pm = \frac{2}{e\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{1}{4} (\pi n)^2 ct} \sin \left( \frac{\pi n}{l} \xi \right) \cos \left( \frac{\pi n}{l} (x \pm ct) \right). \]  

(5.7)

Then, the basic variables which are typical of the diffusion and wave propagation are identified by the following

**Property 5.1** - The function \( H^- \) (or \( H^+ \)) is solution of the diffusion-wave equation

\[ \frac{\varepsilon}{2} v_{xx} = v_t + cv_x \quad \text{(or: } \frac{\varepsilon}{2} v_{xx} = v_t - cv_x \text{).} \]

(5.8)

Moreover, if one puts: \( y = x \pm ct \), \( \theta = \frac{\varepsilon}{\varepsilon^2} t \), Eq. (5.8) turns into the heat equation \( v_{yy} = \theta \), with the time-variable \( \theta \) given just by the slow time and the space-variable \( y \) related to traveling (or retrograde) waves.

Finally, we observe that \( H^\pm_x \) is given by Jacobi Theta function \( \theta_3(y, \theta) \); in fact, for \( y = x + ct \) or \( y = x - ct \), one has:

\[ H^\pm_x = \frac{1}{4cl} [\theta_3 \left( \frac{y - \xi}{2l}, \frac{\theta}{2l^2} \right) - \theta_3 \left( \frac{y + \xi}{2l}, \frac{\theta}{2l^2} \right)], \]

(5.9)

according to well-known formulae related to the heat equation.

**REFERENCES**

1. A. Barone, G. Paternò, *Physics and Application of the Josephson Effect*, Wiley, N. Y. 530 (1982).
2. V.P. Maslov, P. P. Mosolov, *Non linear wave equations perturbed by viscous terms*, Walter deGruyher Berlin N. Y. 329 (2000).
3. A.I. Kozhanov, N. A. Lar'kin, *Wave equation with nonlinear dissipation in noncylindrical Domains*, Dokl. Math 62, 2, 17-19 (2000).
4. Ali Nayfeh, *A comparison of perturbation methods for nonlinear hyperbolic waves* in *Proc. Adv. Sem. Wisconsin* 45, 223-276 (1980).
5. R. I. Tanner, *Note on the Rayleigh Problem for a Visco-Elastic Fluid*, ZAMP vol XIII, 575-576 (1962).
6. D.D Joseph, M. Renardy and J. C. Saut, *Hyperbolicity and change of type in the flow of viscoelastic fluids*, Arch Rational Mech. Analysis, 87 213-251, (1985).
7. J. A. Morrison, *Wave propagations in rods of Voigt material and visco-elastic materials with three-parameters models*, Quart. Appl. Math. 14 153-173, (1956).
8. P. Renno, *On some viscoelastic models*, Atti Acc. Lincei Rend. fis. 75 (6) 1-10, (1983).
9. Y. Shibata, *On the rate of decay of solutions to linear viscoelastic Equation*, Math. Meth. Appl. Sci.,23 203-226 (2000)
10. H. Lamb, *Hydrodynamics*, Dover Publ. Inc., (1932)
11. R. Nardini, *Soluzione di un problema al contorno della magneto idrodinamica*, Ann. Mat. Pura Appl.,35 269 (1953)
12. L.L. Bonilla and J.S. Solen, *High field limit of the Vlasov Poisson Fokker Plank system: A comparison of different perturbation methods*, Math. Models Meth. Appl. Sci. 11, 1457-1468, (2001)

13. P. Degond, *An infinite system of diffusion equation arising in trasport theory: the coupled spherical armonics expansion model*, Math. Models Meth. Appl. Sci. 11, 903-932, (2001)

14. Erdelyi, Magnus, Oberhettinger, Tricomi, *Tables of integral transforms* vol. I, Mac Graw-Hill Book (1956)

15. De Bruijn, *Asymptotic Methods in Analysis* North- Holland Publishin (1958)

16. M. De Angelis, *Asymptotic analysis for the strip problem related to a parabolic third-order operator*, Appl. Math. Letters 14 (4), 425-430 (2001)

17. G. N. Watson, *Theory of Bessel Function* Cambridge p.804 (1944)

18. P. Renno, *On a Wave Theory for the Operator ε∂t(∂t^2 - c_1^2 Δ_n) + ∂t^2 - c_0^2 Δ_n*, Ann. Mat. pura e Appl., 136(4) 355-389 (1984).