THE DEGREES OF ONESIDED RESOLVINGNESS AND THE LIMITS OF ONESIDED RESOLVING DIRECTIONS FOR ENDMORPHISMS AND AUTOMORPHISMS OF THE SHIFT

MASAKAZU NASU

ABSTRACT. We introduce the notions of the degrees of onesided resolvingness and the limits of onesided resolving directions for onto endomorphisms and automorphisms of subshifts, and develop a theory which illuminates the fundamental structure of overall dynamics of onto endomorphisms and automorphisms of subshifts. The former is not topological invariant and the latter is a topological invariant. Both of them, together with "resolving endomorphisms" of subshifts, and develop a theory which illuminates the fundamental structure of overall dynamics of onto endomorphisms and automorphisms of subshifts, and develop a theory which illuminates the fundamental structure of overall dynamics of onto endomorphisms and automorphisms of subshifts, and develop a theory which illuminates the fundamental structure of overall dynamics of onto endomorphisms and automorphisms of subshifts.

1. INTRODUCTION

This paper is a sequel of [N9]. We continue to study the dynamics of onto endomorphisms and automorphisms of the shift, in particular, the "overall dynamics" of onto endomorphisms ϕ of subshifts (X,σ), i.e. the dynamics of ϕ′σi for all i ∈ N, j ∈ Z, and the overall dynamics of onto endomorphisms of topological Markov shifts. We introduce the notions of "the degrees of onesided resolvingness" and the "limits of onesided resolving directions" for onto endomorphisms of subshifts. The former is not topological invariant and the latter is a topological invariant. Both of them, together with "resolving endomorphisms" of subshifts which were introduced and studied in [N5, N6, N9], make possible to understand the fundamental structure of the overall dynamics of onto endomorphisms and automorphisms of the shift.

Let (X,σ) be a subshift over an alphabet A. For s ≥ 1, let Ls(X) denote the set of all words a1...aj+1 that appear on some point (aj)j∈Z ∈ X with aj ∈ A. Let N ≥ 0. A local rule of neighborhood-size N on (X,σ) means a mapping f : L_{N+1}(X) → L_1(X) such that (f(a_1...a_{j+N}))_{j∈Z} ∈ X for all points (a_1)_{j∈Z} ∈ X with a_j ∈ L_1(X).

Let f : L_{N+1}(X) → L_1(X) be a local rule on (X,σ). Let J ≥ 0. We say that f is I left-redundant if for any points (a_j)_{j∈Z} and (b_j)_{j∈Z} in X with a_j, b_j ∈ L_1(X), it holds that if (a_j)_{j≥0} = (b_j)_{j≥0}, then f(a_{−1}...a_{−J+1}) = f(b_{−1}...b_{−J+1}). Symmetrically, f is said to be I right-redundant if for any points (a_j)_{j∈Z} and (b_j)_{j∈Z} in X with a_j, b_j ∈ L_1(X), it holds that if (a_j)_{j≤0} = (b_j)_{j≤0}, then f(a_{1−J}...a_1) = f(b_{1−J}...b_1). We say that f is strictly I left-redundant if it is I left-redundant but not I + 1 left-redundant. Similarly, a strictly I right-redundant local rule is defined.

Let k ≥ 0. We say that f is k right-mergeable if for any points (a_j)_{j∈Z} and (b_j)_{j∈Z} in X with a_j, b_j ∈ L_1(X), it holds that if (a_j)_{j≤0} = (b_j)_{j≤0} and f(a_{j−N}...a_j) = f(b_{j−N}...b_j) for j = 1,...,k+1, then a_1 = b_1. We say that f is k left-mergeable if for any points (a_j)_{j∈Z} and (b_j)_{j∈Z} in X with a_j, b_j ∈ L_1(X), it holds that if (a_j)_{j≥0} = (b_j)_{j≥0} and f(a_{j−N}...a_j) = f(b_{j−N}...b_j) for j = 1,...,k+1.
The following definitions can be made (Propositions 6.3 and 3.3): if $f$ is strictly $0$ right-mergible if $f$ is $0$ right-mergible. For $k \geq 1$, we say that $f$ is strictly $k$ right-mergible if $f$ is $k$ right-mergible but not $k-1$ right-mergible. Similarly, a strictly $k$ left-mergible local rule is defined for $k \geq 0$.

Let $m, n \geq 0$. Let $I, J \geq 0$. Let $\varphi$ be an endomorphism of $(m, n)$-type of a subshift $(X, \sigma)$ given by a local rule $f : L_{N+1}(X) \to L_1(X)$ with $N = m + n$ on $(X, \sigma)$, i.e. $\varphi$ is defined by

$$\varphi((a_j)_{j \in \mathbb{Z}}) = (f(a_{j-m} \ldots a_{j+n}))_{j \in \mathbb{Z}}, \ (a_j)_{j \in \mathbb{Z}} \in X, \ a_j \in L_1(X).$$

The following definitions can be made (Propositions 6.3 and 3.3): if $f$ is strictly $I$ left-redundant, then the $p$-$L$ degree $P_L(\varphi)$ of $\varphi$ is defined by

$$P_L(\varphi) = I - m;$$

if $f$ is strictly $J$ right-redundant, then the $p$-$R$ degree $P_R(\varphi)$ of $\varphi$ is defined by

$$P_R(\varphi) = J - n;$$

if $f$ is strictly $k$ right-mergible, then the $q$-$R$ degree $Q_R(\varphi)$ of $\varphi$ is defined by

$$Q_R(\varphi) = n - k;$$

if $f$ is strictly $l$ left-mergible, then the $q$-$L$ degree $Q_L(\varphi)$ of $\varphi$ is defined by

$$Q_L(\varphi) = m - l.$$ 

As is well known, $\varphi$ is right-closing (respectively, left-closing) if and only if $\varphi$ is given by a $k$ right-mergible (respectively, $k$ left-mergible) local rule for some $k \geq 0$. For convenience’s sake, if $\varphi$ is not right-closing (respectively, not left-closing), we say that $f$ is infinite right-mergible (respectively, infinite left-mergible) and we define $Q_R(\varphi) = -\infty$ (respectively, $Q_L(\varphi) = -\infty$).

Here some remarks are in order. We follow the terminology of [N9] and [N5]. The nine terms “$p$-$L$”, “$p$-$R$”, “$q$-$R$”, “$q$-$L$”, “LR”, “RL”, “RR” and “$q$-bi-resolving”, which will be called the resolving terms, have meanings only for onto endomorphisms of topological Markov shifts, and the corresponding notions for onto endomorphisms of SFTs (subshifts of finite type) can be given as “$P$ up to higher-block conjugacy between endomorphisms of subshifts” or more generally “essentially $P$”, for each one of the nine resolving terms $P$. Here the term “essentially" means “up to topological conjugacy between endomorphisms of subshifts”.

The term “weakly $P$” has a meaning generally for onto endomorphisms of subshifts for each one of the resolving terms $P$. In the remainder of this section, we describe an outline of our theory on weak resolving properties for onto endomorphisms $\varphi$ of general subshifts $(X, \sigma)$. However, all “weakly $P$” appearing in the results stated in it, where $P$ is any one of the resolving terms, can be replaced by “$P$" for the important case that $\varphi$ is an onto endomorphism of a topological Markov shift $(X, \sigma)$ with $\varphi$ one-to-one or $\sigma$ topologically transitive. Moreover, this paper treats the case in detail and contains results particular to onto endomorphisms of topological Markov shifts and to those of full shifts. (The reader who is interested in only the endomorphisms of topological Markov shifts can read only the results on them with direct proofs).

Let $\varphi$ be an onto endomorphism of a subshift $(X, \sigma)$. Then we have the following basic results (Theorems 6.12 and 5.2), which give a reason why $P_L(\varphi)$, $P_R(\varphi)$, $Q_R(\varphi)$ and $Q_L(\varphi)$ are called the degrees of onesided resolvingness of $\varphi$:
ϕ is weakly p-L if and only if \( P_L(ϕ) \geq 0 \); ϕ is weakly p-R if and only if \( P_R(ϕ) \geq 0 \); ϕ is weakly q-R if and only if \( Q_R(ϕ) \geq 0 \); ϕ is weakly q-L if and only if \( Q_L(ϕ) \geq 0 \).

Since we have

\[
P_L(ϕσ^s) = P_L(ϕ) + s, \quad P_R(ϕσ^s) = P_R(ϕ) - s, \quad Q_R(ϕσ^s) = Q_R(ϕ) + s, \quad Q_L(ϕσ^s) = Q_L(ϕ) - s,
\]
for \( s \in \mathbb{Z}, \) \( P_L(ϕ) + P_R(ϕ), Q_R(ϕ) + Q_L(ϕ), P_L(ϕ) + Q_L(ϕ) \) and \( P_R(ϕ) + Q_R(ϕ) \) are shift-invariant (Propositions 6.4 and 3.4). Except for \( Q_R(ϕ) + Q_L(ϕ), \) these are nonpositive (here and throughout the remainder of this section we assume that \( X \) has infinitely many points)(Propositions 6.13 and 7.9). Hence we have

\[
P_R(ϕ) \leq -P_L(ϕ), \quad Q_L(ϕ) \leq -P_L(ϕ), \quad P_R(ϕ) \leq -Q_R(ϕ).
\]

We have the following results for \( s \in \mathbb{Z}: \)

ϕσs is weakly p-L if and only if \( s \geq -P_L(ϕ); \) if \( s > -P_L(ϕ) \) then it is right \( \sigma \)-expansive (Proposition 6.13); ϕσs is weakly q-R if and only if \( s \geq -Q_R(ϕ); \) if \( s > -Q_R(ϕ) \) then it is left \( \sigma \)-expansive (Theorem 5.2).

ϕσs is weakly p-R if and only if \( s \leq P_R(ϕ); \) if \( s < P_R(ϕ) \) then it is left \( \sigma \)-expansive (Proposition 6.13); ϕσs is weakly q-L if and only if \( s \leq Q_L(ϕ); \) if \( s < Q_L(ϕ) \) then it is right \( \sigma \)-expansive (Theorem 5.2).

ϕσs is weakly LR up to higher block conjugacy if and only if \( s \geq C_R; \) if \( s > C_R, \) then it is expansive, where \( C_R = \max\{-P_L(ϕ), -Q_R(ϕ)\} \) (Theorem 7.8).

ϕσs is weakly RL up to higher block conjugacy if and only if \( s \leq C_L; \) if \( s < C_L, \) then it is expansive, where \( C_L = \min\{P_R(ϕ), Q_L(ϕ)\} \) (Theorem 7.8).

\( Q_R(ϕ) + Q_L(ϕ) \) can be any integer. There exists \( s \in \mathbb{Z} \) such that \( ϕσ^s \) is weakly \( q \)-biresolving up to higher block conjugacy if and only if \( Q_R(ϕ) + Q_L(ϕ) \geq 0; \) if \( Q_R(ϕ) + Q_L(ϕ) \geq 0 \), then for \( s \in \mathbb{Z}, \) \( ϕσ^s \) is weakly \( q \)-biresolving up to higher-block conjugacy if and only if \( Q_L(ϕ) \leq s \leq -Q_R(ϕ); \) if \( Q_L(ϕ) < s < -Q_R(ϕ) \), then \( ϕσ^s \) is positively expansive. (Theorem 5.3.)

We can define the real numbers \( p_L(ϕ), p_R(ϕ), q_R(ϕ) \) and \( q_L(ϕ) \) by

\[
\begin{align*}
p_L(ϕ) &= \lim_{s \to \infty} P_L(ϕ^s)/s, & p_R(ϕ) &= \lim_{s \to \infty} P_R(ϕ^s)/s, \\
q_R(ϕ) &= \lim_{s \to \infty} Q_R(ϕ^s)/s, & q_L(ϕ) &= \lim_{s \to \infty} Q_L(ϕ^s)/s.
\end{align*}
\]

which equal \( \sup_s P_L(ϕ^s)/s, \sup_s P_R(ϕ^s)/s, \sup_s Q_R(ϕ^s)/s, \sup_s Q_L(ϕ^s)/s, \) respectively (Theorem 9.2). Each of these numbers is an invariant of topological conjugacy between endomorphisms of subshifts (Theorem 9.4). We call \( -p_L(ϕ) \) the limit of \( p-L \) direction of \( ϕ, \) and call \( -q_R(ϕ), p_R(ϕ) \) and \( q_L(ϕ) \) in the same way. They are also called the limits of onesided resolving directions of \( ϕ. \)

Let \( i \in \mathbb{N} \) and let \( j \in \mathbb{Z}. \) Then we have

\[
\begin{align*}
p_L(ϕ^iσ^j) &= ip_L(ϕ) + j, & p_R(ϕ^iσ^j) &= ip_R(ϕ) - j, \\
q_R(ϕ^iσ^j) &= iq_R(ϕ) + j, & q_L(ϕ^iσ^j) &= iq_L(ϕ) - j.
\end{align*}
\]

(Proposition 9.5). Hence \( p_L(ϕ) + p_R(ϕ), q_R(ϕ) + q_L(ϕ), p_L(ϕ) + q_L(ϕ) \) and \( p_R(ϕ) + q_R(ϕ) \) are shift-invariant. Except for \( q_R(ϕ) + q_L(ϕ), \) these are nonpositive (Proposition 9.9). Therefore, we have

\[
p_R(ϕ) \leq -p_L(ϕ), \quad q_L(ϕ) \leq -p_L(ϕ), \quad p_R(ϕ) \leq -q_R(ϕ).
\]

We have the following results (Corollary 9.7):

\( ϕ^iσ^j \) is essentially weakly p-L and right \( \sigma \)-expansive if and only if \( j/i > -p_L(ϕ); \)
\(\varphi^j\sigma^i\) is essentially weakly \(q\)-R and left \(\sigma\)-expansive if and only if \(j/i > -q_R(\varphi)\); 
\(\varphi^i\sigma^j\) is essentially weakly \(p\)-R and left \(\sigma\)-expansive if and only if \(j/i < p_R(\varphi)\); 
\(\varphi^i\sigma^j\) is essentially weakly \(q\)-L and right \(\sigma\)-expansive if and only if \(j/i < q_L(\varphi)\).

Let \(c_R(\varphi) = \max\{-p_L(\varphi), -q_R(\varphi)\}\) and let \(c_L(\varphi) = \min\{p_R(\varphi), q_L(\varphi)\}\). We also have the results (Theorem 9.10) that \(\varphi^i\sigma^j\) is essentially weakly LR and expansive if and only if \(j/i > c_R(\varphi)\) and that \(\varphi^i\sigma^j\) is essentially weakly RL and expansive if and only if \(j/i < c_L(\varphi)\).

We find that \(q_R(\varphi) + q_L(\varphi)\) can be negative, zero and positive. If \(\varphi\) is an automorphism of \((X, \sigma)\), then it is nonpositive, but the converse does not hold. We obtain the following results (Theorem 9.11): \(q_R(\varphi) + q_L(\varphi) > 0\) if and only if there exist \(i \in \mathbb{N}\) and \(j \in \mathbb{Z}\) with \(\varphi^i\sigma^j\) positively expansive; if \(q_R(\varphi) + q_L(\varphi) > 0\), then \(\varphi^i\sigma^j\) is positively expansive if and only if \(-q_R(\varphi) < j/i < q_L(\varphi)\).

Mike Boyle [Bo2] proved that if \(\varphi^s\) is weakly \(p\)-L (respectively, weakly \(p\)-R) with \(s \in \mathbb{N}\), then the “dual higher block endomorphism of order \(s\)” \(\varphi^{[s]}\) is weakly \(p\)-L (respectively, weakly \(p\)-R) and hence \(\varphi\) is essentially weakly \(p\)-L (respectively, essentially weakly \(p\)-R), together with more for the case that \(\varphi\) is an automorphism (see Theorem 8.1), and moreover he suggested the possibility of proving the main results of [N5] Section 7 (on resolving endomorphisms of topological Markov shifts) without using the long theory of “resolvable textile systems” presented there. In this paper, using \(q\)-R and \(q\)-L degrees we prove that if \(\varphi^s\) is weakly \(q\)-R (respectively, weakly \(q\)-L) with \(s \in \mathbb{N}\), then \(\varphi^{[s]}\) is weakly \(q\)-R (respectively, weakly \(q\)-L). Using Boyle’s results and these we can prove that for any one of the resolving terms \(P\), if \(\varphi^s\) is weakly \(P\) for some \(s \in \mathbb{N}\), then \(\varphi\) is essentially weakly \(P\) (Section 8); they are also used to prove the results on the limits of onesided resolving directions stated above. In Section 10, using another method we prove that if \(\varphi\) is an essentially weakly \(P\) endomorphism of \((X, \sigma^s)\) for some \(s \in \mathbb{N}\), then \(\varphi\) is essentially weakly \(P\), for any one of the resolving terms \(P\). Therefore we obtain the result that for any one of the resolving terms \(P\), if \(\varphi\) is directionally essentially weakly \(P\) (i.e. \(\varphi^r\) is an essentially weakly \(P\) endomorphism of \((X, \sigma^s)\) for some \(r, s \in \mathbb{N}\)), then \(\varphi\) is essentially weakly \(P\) (Theorem 10.2). This is a generalization of the main results of [N5] Section 7 on onto endomorphisms of topological Markov shifts (satisfying the condition of the important case mentioned above) to onto endomorphisms of general subshifts, and more. (However, the theory of resolvable textile systems of [N5] Section 7 still remains to have some significant results which our theory cannot cover.)

Even for an onto endomorphism \(\varphi\) of a transitive SFT \((X, \sigma)\), it is not known how to compute the limits of onesided resolving directions \(-p_L(\varphi), -q_R(\varphi), p_R(\varphi)\) and \(q_L(\varphi)\). For \(\varphi\) in some special cases, Theorem 7.3 is useful to determine at least one of \(-p_L(\varphi)\) and \(-q_R(\varphi)\) and at least one of \(p_R(\varphi)\) and \(q_L(\varphi)\); for \(\varphi\) in some special cases, Theorem 4.4 is useful to determine \(-q_R(\varphi)\) and \(q_L(\varphi)\) (Propositions 9.15, 9.16).

We prove that if \((X, \sigma)\) is a topological Markov shift with defining matrix \(M\), and if \(\varphi\) is its endomorphism which is LR up to higher block conjugacy, then there exists a nonnegative integral matrix \(N\) with \(MN = NM\) such that for all \(i \geq 0, j \geq 1\), \((\tilde{X}, \tilde{\varphi}^i\tilde{\sigma}^j)\) is topologically conjugate to the onesided topological Markov shift with defining matrix \(N^iM^j\), where \((\tilde{X}, \tilde{\sigma})\) is the induced onesided subshift of \((X, \sigma)\) and \(\tilde{\varphi}\) is its endomorphism induced by \(\varphi\) (Theorem 7.6). We also give a construction...
method of obtaining positively-expansive endomorphisms of full shifts from right-closing and left-closing endomorphisms of the shifts (Theorem 4.8).

Finally we remark the following. Let $\varphi$ be a right or left closing endomorphism of a transitive SFT $(X, \sigma)$. Let $d_R(\varphi) = \min \{-p_L(\varphi), -q_R(\varphi)\}$ and let $d_L(\varphi) = \max \{p_R(\varphi), q_L(\varphi)\}$. Then it follows from the main result [N9 Corollary 6.6] of the preceding paper and Corollary 9.7 that for $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, if $j/i \leq d_L(\varphi)$ or $j/i \geq d_R(\varphi)$, then $\varphi^i \sigma^j$ has the pseudo-orbit tracing property whenever $\varphi^i \sigma^j$ is expansive (hence if $\varphi$ is an automorphism then $(X, \varphi^i \sigma^j)$ is conjugate to an SFT and otherwise the inverse limit system of $\varphi^i \sigma^j$ is conjugate to an SFT, whenever $\varphi^i \sigma^j$ is expansive). Note that if $d_R(\varphi) \leq d_L(\varphi)$, then for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, $\varphi^i \sigma^j$ has the pseudo-orbit tracing property whenever $\varphi^i \sigma^j$ is expansive. Typical examples of $\varphi$ with $d_R(\varphi) \leq d_L(\varphi)$ are an automorphism $\varphi$ of a transitive SFT $(X, \sigma)$ with both $\varphi$ and $\varphi^{-1}$ having memory zero and a positively-expansive endomorphism of a transitive SFT $(X, \sigma)$.

2. Preliminaries

In this section, we give preliminaries to the subsequent sections. For more information, the reader is referred to [Ki2] or [LMar] on symbolic dynamics, and [AoHi] on topological dynamics.

2.1. Some basic definitions. A commuting system $(X, \tau, \varphi)$ means an ordered pair of commuting continuous maps $\tau : X \to X$ and $\varphi : X \to X$ of a compact metric space. If $(X, \tau, \varphi)$ is a commuting system, then $\varphi$ is an endomorphism of the dynamical system $(X, \tau)$. When $\varphi$ is a homeomorphism in this definition, then $\varphi$ is an automorphism of $(X, \tau)$.

Two commuting systems $(X, \tau, \varphi)$ and $(X', \tau', \varphi')$ are said to be topologically conjugate if there exists a topological conjugacy $\theta : (X, \tau, \varphi) \to (X', \tau', \varphi')$, i.e. a homeomorphism $\theta : X \to X'$ which gives topological conjugacies $\theta : (X, \tau) \to (X, \tau')$ and $\theta : (X, \varphi) \to (X, \varphi')$ between the dynamical systems at the same time.

Let $A$ be an alphabet (i.e. a finite nonempty set of symbols). Throughout this paper, we assume that an alphabet is not a singleton.

A finite sequence $a_1 \ldots a_n$ with $a_j \in A$ and $n \geq 1$ is called a word or block of length $n$ over $A$.

Let $A^\mathbb{Z}$ be endowed with the metric $d$ such that for $x = (a_j)_{j \in \mathbb{Z}}$ and $y = (b_j)_{j \in \mathbb{Z}}$ with $a_j, b_j \in A$, $d(x, y) = 0$ if $x = y$, and otherwise $d(x, y) = 1/(1 + k)$, where $k = \min \{|j| \mid a_j \neq b_j\}$. The metric $d$ is compatible with the product topology of the discrete topology on $A$. Let $\sigma_A : A^\mathbb{Z} \to A^\mathbb{Z}$ be defined by $\sigma_A((a_j)_{j \in \mathbb{Z}}) = (a_{j+1})_{j \in \mathbb{Z}}$. The dynamical system $(A^\mathbb{Z}, \sigma_A)$ is called the full shift over $A$. For a closed subset $X$ of $A^\mathbb{Z}$ with $\sigma(X) = X$, the dynamical system $(X, \sigma)$ is called a subshift over $A$, where $\sigma$ is the restriction of $\sigma_A$ on $X$.

For a subshift $(X, \sigma)$ over an alphabet $A$ and $k \geq 1$, let $L_k(X)$ denote the set of all words $a_1 \ldots a_{j+k-1}$ that appear on some point $(a_j)_{j \in \mathbb{Z}} \in X$ with $a_j \in A$.

Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be subshifts. Let $N$ be a nonnegative integer. A mapping $f : L_{N+1}(X) \to L_1(Y)$ is called a local rule of neighborhood-size $N$ on $(X, \sigma_X)$ to $(Y, \sigma_Y)$ if $(f(a_1 \ldots a_{j+N}))_{j \in \mathbb{Z}} \in Y$ for all $(a_j)_{j \in \mathbb{Z}} \in X$ with $a_j \in L_1(X)$. If $X = Y$ in the above, then $f$ is called a local rule on $(X, \sigma_X)$.

Let $f : L_{N+1}(X) \to L_1(Y)$ be a local rule on $(X, \sigma_X)$ to $(Y, \sigma_Y)$. Let $m$ and $n$ be nonnegative integers with $m + n = N$. A mapping $\phi : X \to Y$ is called a block
map of \((m,n)\) type given by \(f\) if
\[
\phi((a_j)_{j\in\mathbb{Z}}) = (b_j)_{j\in\mathbb{Z}} \quad \text{with} \quad b_j = f(a_{j-m} \ldots a_{j+n}) \quad \text{for all} \quad j \in \mathbb{Z}.
\]
A mapping \(\phi : X \to Y\) is simply said to be of \((m,n)\)-type if there exists a local rule \(f : L_{N+1}(X) \to L_1(Y)\) with \(N = m + n\) such that \(\phi\) is a block map of \((m,n)\)-type given by \(f\). A block map of \((m,n)\) type is said to have memory \(m\) and anticipation \(n\).

For subshifts \((X,\sigma_X)\) and \((Y,\sigma_Y)\), a mapping \(\phi : X \to Y\) is a homomorphism of \((X,\sigma_X)\) into \((Y,\sigma_Y)\) (i.e. a continuous map with \(\phi \sigma_X = \sigma_Y \phi\)) if and only if \(\phi\) is a block map (Curtis-Hedlund-Lyndon Theorem \([II]\)). Hence we use the same terminology as above for homomorphisms between subshifts.

Let \(A^N = \{(a_j)_{j\in\mathbb{N}} | a_j \in A\}\) be endowed with a metric compatible with the product topology of the discrete topology on \(A\). Let \(\tilde{\sigma}_A : A^N \to A^N\) be defined by \(\tilde{\sigma}_A((a_j)_{j\in\mathbb{N}}) = (a_{j+1})_{j\in\mathbb{N}}.\) The dynamical system \((A^N, \tilde{\sigma}_A)\) is called the onesided full shift over \(A\). For a subshift \((X,\sigma)\) over \(A\), let \(\tilde{X} = \{(a_j)_{j\in\mathbb{N}} | \exists (a_j)_{j\in\mathbb{Z}} \in X\}.\) Then with the onto continuous map \(\tilde{\sigma} = \tilde{\sigma}_A|\tilde{X}\) we have a dynamical system \((\tilde{X},\tilde{\sigma})\), which is called a onesided subshift over \(A\) and is said to be induced by \((X,\sigma)\). From \((\tilde{X},\tilde{\sigma})\) we can uniquely recover \((X,\sigma)\), so that \((X,\sigma)\) is said to be induced by \((\tilde{X},\tilde{\sigma})\). Let \(s_X : X \to \tilde{X}\) be the continuous map which maps \((a_j)_{j\in\mathbb{Z}}\) to \((a_j)_{j\in\mathbb{N}}.\) If a homomorphism \(\phi\) of a subshift \((X,\sigma)\) into another \((X',\sigma')\) has memory zero, then it uniquely induces a homomorphism \(\hat{\phi}\) between the induced onesided subshifts \((\tilde{X},\tilde{\sigma})\) and \((\tilde{X}',\tilde{\sigma}')\) such that \(s_X \phi = \hat{\phi} s_X\), and conversely each homomorphism \(\phi\) of a onesided subshift \((\tilde{X},\tilde{\sigma})\) into another \((\tilde{X}',\tilde{\sigma}')\) uniquely induces a homomorphism \(\tilde{\phi}\) between the induced subshifts \((X,\sigma)\) and \((X',\sigma')\) such that \(\tilde{s_X} \tilde{\phi} = \tilde{\phi} \tilde{s_X}\).

For a subshift \((X,\sigma)\) and \(n \geq 1\), we define the higher block system of order \(n\) of \((X,\sigma)\) to be the subshift \((X^{[n]},\sigma^{[n]})\) over the alphabet \(L_n(X)\) with \(X^{[n]} = \{(a_j \ldots a_{j+n-1})_{j\in\mathbb{Z}} | (a_j)_{j\in\mathbb{Z}} \in X, a_j \in L_1(X)\}.\)

For an endomorphism \(\varphi\) of a subshift \((X,\sigma)\) and \(n \geq 1\), let \(\varphi^{[n]}\) denote the endomorphism of \((X^{[n]},\sigma^{[n]})\) such that if \(\varphi\) maps \((a_j)_{j\in\mathbb{Z}}\) to \((b_j)_{j\in\mathbb{Z}}\) with \(a_j, b_j \in L_1(X)\), then \(\varphi^{[n]}\) maps \((a_j \ldots a_{j+n-1})_{j\in\mathbb{Z}}\) to \((b_j \ldots b_{j+n-1})_{j\in\mathbb{Z}}.\) Clearly \((X,\sigma,\varphi)\) and \((X^{[n]},\sigma^{[n]},\varphi^{[n]})\) are conjugate.

Let \((X,\tau,\varphi)\) be a commuting system, where \(\tau\) is a homeomorphism. For any two \(\varphi\)-orbits \((x_i)_{i\in\mathbb{Z}}\) and \((y_i)_{i\in\mathbb{Z}}\), let
\[
\Delta((x_i)_{i\in\mathbb{Z}},(y_i)_{i\in\mathbb{Z}}) = \sup\{d_X(x_i,y_i) \mid i \in \mathbb{Z}\}.
\]
We say that \(\varphi\) is left \(\tau\)-expansive if there exists \(\delta > 0\) such that for any two \(\tau\)-orbits \((x_i)_{i\in\mathbb{Z}}\) and \((y_i)_{i\in\mathbb{Z}}\), if \(\Delta((\tau^j(x_i))_{i\in\mathbb{Z}},(\tau^j(y_i))_{i\in\mathbb{Z}}) < \delta\) for all \(j \geq 0\), then \((x_i)_{i\in\mathbb{Z}} = (y_i)_{i\in\mathbb{Z}}.\) We say that \(\tau\) is right \(\tau\)-expansive. It is clear that if \(\varphi\) is expansive, then \(\varphi\) is left \(\tau\)-expansive and right \(\tau\)-expansive.

2.2. Graph-homomorphisms. Let \(G\) be a graph. Here a graph means a directed graph which may have multiple arcs and loops. Let \(A_G\) and \(V_G\) denote the arc-set and the vertex-set, respectively, of \(G\). Let \(i_G : A_G \to V_G\) and \(t_G : A_G \to V_G\) be the mappings such that for each arc \(a \in A_G\), \(i_G(a)\) and \(t_G(a)\) are its initial and terminal vertices. Hence the graph \(G\) is represented by
\[
V_G \xleftarrow{i_G} A_G \xrightarrow{t_G} V_G.
\]
We say that \(G\) is nondegenerate if both \(i_G\) and \(t_G\) are onto.
Let $X_G$ be the set of all points $(a_j)_{j \in \mathbb{Z}}$ in $A_G^\mathbb{Z}$ such that $t_G(a_j) = i_G(a_{j+1})$ for all $j \in \mathbb{Z}$. Then we have a subshift $(X_G, \sigma_G)$ over $A_G$, which is called the topological Markov shift defined by $G$, which is called the defining graph of the shift. Throughout this paper we assume, without loss of generality, that the defining graph is always assumed to be nondegenerate. (Hence when we write $(X_G, \sigma_G)$, $G$ is defined by a topological Markov shift of any topological Markov shift is nondegenerate. (Hence when we write $(X_G, \sigma_G)$, $G$ is always assumed to be nondegenerate.)

For an alphabet $A$, let $G_A$ denote the one-vertex graph with arc-set $A$. Then the full-shift over $A$ is the topological Markov shift defined by $G_A$.

A subshift finite type, abbreviated SFT, is a subshift which is topologically conjugate to a topological Markov shift. A sofic system is a subshift which is the image of a topological Markov shift under a block map.

For graphs $\Gamma$ and $G$, a graph-homomorphism $h$ of $\Gamma$ into $G$, written by $h : \Gamma \to G$, is a pair $(h_A, h_V)$ of mappings $h_A : \Gamma \to G$ (arc-map) and $h_V : V_\Gamma \to V_G$ (vertex-map) such that the following diagram is commutative.

\[
\begin{array}{ccc}
V_\Gamma & \xleftarrow{r^\Gamma} & A_\Gamma \\
\downarrow h_V & & \downarrow h_V \\
V_G & \xleftarrow{i_G} & A_G \\
\end{array}
\]

We call a block map of $(0, 0)$ type between subshift spaces a 1-block map. For a graph-homomorphism $h : \Gamma \to G$, we define $\phi_h : X_\Gamma \to X_G$ to be the 1-block map with the local rule $h_A : A_\Gamma \to A_G$.

A graph-homomorphism $h : \Gamma \to G$ is said to be right-resolving if for each pair $(u, v)$ of $u \in V_\Gamma$ and $v \in A_\Gamma$ with $t_G(v) = h_V(u)$, there exists unique $\alpha \in A_\Gamma$ with $t_\Gamma(\alpha) = u$ and $h_A(\alpha) = v$. It is said to be left-resolving if for each pair $(u, v)$ of $u \in V_\Gamma$ and $v \in A_\Gamma$ with $i_G(v) = h_V(u)$, there exists unique $\alpha \in A_\Gamma$ with $t_\Gamma(\alpha) = u$ and $h_A(\alpha) = v$. We say that $h$ is bi-resolving if $h$ is right-resolving and left-resolving.

A graph-homomorphism $h : \Gamma \to G$ is said to be weakly right-resolving if each $\alpha \in A_\Gamma$ is uniquely determined by $i_\Gamma(\alpha)$ and $h_A(\alpha)$. We say that $h$ is weakly left-resolution if each $\alpha \in A_\Gamma$ is uniquely determined by $t_\Gamma(\alpha)$ and $h_A(\alpha)$. We say that $h$ is weakly bi-resolving if it is both right- and left-resolution.

By using the following well-known lemma, the definitions and results concerning weakly right-resolution and weakly left resolution graph-homomorphisms are interpreted as those concerning right-resolution and left-resolution graph-homomorphisms for important special cases.

**Lemma 2.1.** Let $\Gamma$ and $G$ be nondegenerate graphs. Let $h : \Gamma \to G$ be a weakly right-resolution (respectively, weakly left-resolution) graph-homomorphism.

1. If $\phi_h$ is one-to-one and onto, then $h$ is right-resolution (respectively, left-resolution).
2. If $G$ and $\Gamma$ are irreducible and have the same spectral radius, then $h$ is right-resolution (respectively, left-resolution).

Let $G$ be a graph. Let $L_0(G) = V_G$, let $L_1(G) = A_G$ and for $k \geq 2$ let $L_k(G)$ be the set of all words $a_1 \ldots a_k$ with $a_j \in A_G$ such that $t_G(a_j) = i_G(a_{j+1})$ for $j = 1, \ldots, k-1$. For $k \geq 0$, we call an element of $L_k(G)$ a path of length $k$ in $G$. Let $L(G)$ denote the set of all paths of any length in $G$. We extend $i_G : A_G \to V_G$ and $t_G : A_G \to A_G$ to $i_G : L(G) \to V_G$ and $t_G : L(G) \to V_G$, respectively, as follows: $i_G(v) = t_G(v) = v$ for $v \in L_0(G)$; for $w = a_1 \ldots a_k \in L_k(G)$, $i_G(w) = i_G(a_1)$ and
$t_G(w) = t_G(a_k)$. A path $w$ is said to go from $i_G(w)$ to $t_G(w)$, and $i_G(w)$ and $t_G(w)$ are called the initial vertex and terminal vertex of the path $w$, respectively. For a graph-homomorphism $h : \Gamma \to G$, we also extend $h_v$ and $h_A$ to $h : L(\Gamma) \to L(G)$ as follows: $h(v) = v$ for $v \in L_0(G)$; for $\mu = \alpha_1 \ldots \alpha_k \in L_k(\Gamma)$, where $k \geq 1$ and $\alpha_j \in A_\Gamma$, $h(\mu) = h(\alpha_1) \ldots h(\alpha_k)$. We say that a path $\mu$ generates $h(\mu)$ under $h$.

Let $h : \Gamma \to G$ be a graph-homomorphism between nondegenerate graphs. For $U \subset V_\Gamma$ and $w \in L(G)$, define

$$S_h^+(U, w) = \{t_\Gamma(\mu) \mid \mu \in L(\Gamma), i_\Gamma(\mu) \in U, h(\mu) = w\},$$

$$S_h^-(w, U) = \{i_\Gamma(\mu) \mid \mu \in L(\Gamma), t_\Gamma(\mu) \in U, h(\mu) = w\}.$$

and define

$$B_h^+(U, w) = \{\mu \in L(\Gamma) \mid i_\Gamma(\mu) \in U, h(\mu) = w\},$$

$$B_h^-(w, U) = \{\mu \in L(\Gamma) \mid h(\mu) = w, t_\Gamma(\mu) \in U\}.$$

We call $S_h^+(U, w)$ a $w$-successor or successor of $U$ under $h$ and call $S_h^-(w, U)$ a $w$-predecessor or predecessor of $U$ under $h$. A right-compatible set for $h$ and a left-compatible set for $h$ are defined to be a nonempty successor of a singleton of $V_\Gamma$ under $h$ and a nonempty predecessor of a singleton of $V_\Gamma$ under $h$, respectively. Let $C_h^+$ (respectively, $C_h^-$) be the set consisting of all maximal right-compatible (respectively, left-compatible) sets and their nonempty successors (respectively, predecessors). (Throughout this paper, we shall often abuse the element of a singleton to denote the singleton.) We define $h^+ : \Gamma_h^+ \to G$ to be the graph-homomorphism such that $\Gamma_h^+$ is the graph whose vertex-set is $C_h^+$ and whose arc-set is $\{(U, a) \mid U \subset C_h^+, a \in A_G, h(U) = i_G(a)\}$ and each arc $(U, a)$ goes from $U$ to $S_h^+(U, a)$ with $h^+(U, a) = a$. We also define $h^- : \Gamma_h^- \to G$ to be the graph-homomorphism such that $\Gamma_h^-$ is the graph whose vertex-set is $C_h^-$ and whose arc-set is $\{(a, U) \mid a \subset C_h^-, a \in A_G, h(U) = t_G(a)\}$, and each arc $(a, U)$ goes from $S_h^-(a, U)$ to $U$ with $h^-(a, U) = a$. We call $h^+$ (respectively, $h^-$) the induced right-resolving graph-homomorphism (respectively, induced left-resolving graph-homomorphism) of $h$.

For the case that $G = G_A$ for an alphabet $A$ in the above, the induced right-resolving and left-resolving graph-homomorphisms are the same as the induced right-resolving and left-resolving $\lambda$-graphs introduced in [N3]. (These were first introduced in [N1] for a special type of $\lambda$-graphs.) We note that generally, the induced right-resolving (respectively, left-resolving) graph-homomorphism is a weakly right-resolving (respectively, weakly left-resolving) graph-homomorphism. However we note the following:

**Lemma 2.2.**

(1) If $h$ is a graph-homomorphism between nondegenerate graphs with $\phi_h$ bijective, then $h^+$ is right-resolving and $h^-$ is left-resolving;

(2) if $h$ is a graph-homomorphism between irreducible graphs having the same spectral radius with $\phi_h$ onto, then $h^+$ is right-resolving and $h^-$ is left-resolving.

**Proof.** (1) Since $\phi_h^+$ and $\phi_h^-$ are bijections (see [N2] the proof of Lemma 5.5]), the conclusion follows by Lemma 2.1(1).

(2) By [N2] Proposition 4.1. \qed

For a graph-homomorphism $h : \Gamma \to G$ and an integer $k \geq 0$, we say that $h$ is $k$ right-mergible (respectively, $k$ left-mergible) if for any pair of paths $\mu_1 = \alpha_1 \ldots \alpha_{k+1}$
and \(\mu_2 = \beta_1 \ldots \beta_{k+1}\) in \(L_{k+1}(X_H)\) with \(\alpha_j, \beta_j \in A_H\) it holds that if \(i_{H}(\mu_1) = i_{H}(\mu_2)\) (respectively, \(t_{H}(\mu_1) = t_{H}(\mu_2)\)) then \(\alpha_1 = \beta_1\). (These notions correspond to “nonexistence of right (respectively, left) \(f\)-branch of length exceeding \(k\)” in [1].) Section 16] and appear for a special class of graph-homomorphisms in [N2] and for the general class of graph-homomorphisms with somewhat different terms in [N2] p. 400.)

For a graph \(G\) and \(n \geq 1\), we define the higher-block graph \(G^{[n]}\) of order \(n\) of \(G\) as follows: \(G^{[1]} = G\); if \(n \geq 2\), \(G^{[n]}\) is the graph such that \(A_{G^{[n]}} = L_n(G), V_{G^{[n]}} = L_{n-1}(G)\) and for \(a = a_1 \ldots a_n \in A_{G^{[n]}}\) with \(a_j \in A_G\), \(i_{G^{[n]}}(a) = a_1 \ldots a_{n-1}\) and \(t_{G^{[n]}}(a) = a_2 \ldots a_n\). For a graph-homomorphism \(h : \Gamma \rightarrow G\) and \(n \geq 1\), we define the higher-block graph-homomorphism \(h^{[n]} : \Gamma^{[n]} \rightarrow G^{[n]}\) of order \(n\) of \(h\) by \(h^{[n]}(\alpha) = h(\alpha), \alpha \in L_n(\Gamma)\).

**Remark 2.3.** Let \(h : \Gamma \rightarrow G\) be a graph-homomorphism. Let \(k \geq 0\) and let \(n \geq 1\).

1. (\(N2\) Lemma 5.6) If \(h\) is \(k\) right-mergible, then so is \(h^{[n]}\), and if \(h\) is \(k\) left-mergible, then so is \(h^{[n]}\).
2. (If \(h : \Gamma \rightarrow G\) is \(k\) right-mergible (respectively, \(k\) left-mergible), then \(\{B^+_n(h(\mu)), h(\mu)\mid \mu \in L_n(\Gamma)\} = C_{h^{[k+1]}}^+(\Gamma)\) (respectively, \(\{B^-_n(h(\mu)), t_{\mu}h(\mu)\mid \mu \in L_n(\Gamma)\} = C_{h^{[k+1]}}^-(\Gamma)\). Therefore, if \(h\) is \(k\) right-mergible (respectively, \(k\) left-mergible), then any two distinct sets in \(C^\pm_{h^{[k+1]}}\) (respectively, in \(C^\pm_{h^{[k+1]}}\)) are disjoint, and hence all sets in \(C^\pm_{h^{[k+1]}}\) (respectively, in \(C^\pm_{h^{[k+1]}}\)) are maximal right-compatible sets (respectively, maximal left-compatible sets) for \(h^{[k+1]}\).

Let \((X_G, \sigma_G)\) and \((X_H, \sigma_H)\) be topological Markov shifts. Let \(f : L_{N+1}(G) \rightarrow A_H\) be a local rule (on \((X_G, \sigma_G)\) to \((X_H, \sigma_H)\)) with \(N \geq 0\). Then we naturally define the graph-homomorphism \(q_f : G^{[N+1]} \rightarrow H\) such that \(q_f(w) = f(w)\) for \(w \in A_{G^{[N]}} = L_{N+1}(G)\).

For \(k \geq 0\), we say that \(f\) is \(k\) right-mergible (respectively, \(k\) left-mergible) if \(q_f\) is \(k\) right-mergible (respectively, \(k\) left-mergible). (These definitions are compatible with the general definition of mergibility for a local rule on a subshift to another presented in the next section.)

Suppose that \(f\) is \(k\) right-mergible with \(k \geq 0\). For \(w \in L_{N+k}(G) \cup L_{N+k+1}(G)\), we define \(D^+_f(w)\) as follows: if \(w = w_0w_1\) with \(w_0 \in L_N(G) \cup L_{N+1}(G)\) and \(w_1 \in L_k(G)\), then we define

\[
D^+_f(w) = \{ w_0w_1' \in L(G) \mid w_1' \in L_k(G), f(w_0w_1') = f(w) \}.
\]

Since \(f\) is \(k\) right-mergible, we can define a graph \(G^+_f\) and a graph-homomorphism

\[
q^{+_k} : G^+_f : H^{[k+1]}
\]

as follows: the vertex-set of \(G^+_f\) is \(\{D^+_f(w) \mid w \in L_{N+k}(G)\}\); the arc-set of \(G^+_f\) is \(\{D^+_f(w) \mid w \in L_{N+k+1}(G)\}\); each arc \(D^+_f(w)\) with \(w \in L_{N+k+1}(G)\) goes from vertex \(D^+_f(w')\) to vertex \(D^+_f(w'')\), where \(w'\) and \(w''\) are the initial and terminal subpaths, respectively, of length \(N+k\) of \(w\), and \(q^{+_k}(D^+_f(w)) = f(w)\).

Suppose that \(f\) is \(k\) left-mergible with \(k \geq 0\). For \(w \in L_{N+k}(G) \cup L_{N+k+1}(G)\), we define \(D^-_f(w)\) if \(w = w_{-1}w_0\) with \(w_{-1} \in L_k(G)\) and \(w_0 \in L_N(G) \cup L_{N+1}(G)\), then we define

\[
D^-_f(w) = \{ w_{-1}w_0 \in L(G) \mid w_{-1} \in L_k(G), f(w_{-1}w_0) = f(w) \}.
\]
Since $f$ is $k$ left-mergible, we can define a graph $G_{f,k}$ and a graph-homomorphism
\[ q_{f,k} : G_{f,k} \to G^{[k+1]} \]
in the way symmetric to the above.

Suppose that $f$ is $k$ right-mergible and $l$ left-mergible with $k, l \geq 0$. For $w \in L_{N+k+l}(G) \cup L_{N+k+l+1}(G)$, we define $D_{f,l,k}(w)$ as follows: if $w = w_{-1}w_0w_1$ with $w_{-1} \in L_l(G), w_0 \in L_N(G) \cup L_{N+1}(G)$ and $w_1 \in L_k(G)$, then we define
\[ D_{f,l,k}(w) = \{ w_{-1}w_0w_1 \in L(G) \mid w_{-1} \in L_l(G), w_0' \in L_k(G), f(w_{-1}w_0w_1') = f(w) \}. \]

These definitions are essentially due to Bruce Kitchens [Kil, K2 Section 4.3].

We easily see that $q_{f,k}^+ = (q_{f,k}^{[k+1]})^+$ up to isomorphism of graph-homomorphisms when $f$ is $k$ right-mergible, and that $q_{f,k}^- = (q_{f,k}^{[k+1]})^-$ up to isomorphism of graph-homomorphisms when $f$ is $k$ left-mergible. It is not difficult to see that $q_{f,l,k}^{+} = ((q_{f,l,k}^{[k+l+1]})^+) = ((q_{f,l,k}^{[k+l+1]})^-) +$ up to isomorphism of graph-homomorphisms when $f$ is $k$ right-mergible and $l$ left-mergible.

**Proposition 2.4** (Kitchens). Let $(X_G, \sigma_G)$ and $(X_H, \sigma_H)$ be topological Markov shifts. Let $f : L_{N+1}(G) \to A_H$ be a local rule (on $(X_G, \sigma_G)$ to $(X_H, \sigma_H)$) with $N \geq 0$. Let $k, l \geq 0$.

1. **[Kil]** When $f$ is $k$ right-mergible, $q_{f,k}^+ : G_{f,k}^+ \to H^{[k+1]}$ is weakly right-resolving. When $f$ is $l$ left-mergible, $q_{f,l}^- : G_{f,l}^- \to H^{[l+1]}$ is weakly left-resolving.

2. **[K2]** When $f$ is $k$ right-mergible and $l$ left-mergible, $q_{f,l,k}^{+} : G_{f,l,k}^{+} \to H^{[l+k+1]}$ is weakly biresolving.

### 2.3. Textile systems and textile-subsystems.

A **textile system** $T$ over a graph $G$ is an ordered pair of graph-homomorphisms $p : \Gamma \to G$ and $q : \Gamma \to G$. We write $T = (p, q : \Gamma \to G)$.

We have the following commutative diagram.

\[
\begin{array}{ccc}
V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \\
\uparrow{p_V} & & \uparrow{p_A} & & \uparrow{p_V} \\
V_T & \xleftarrow{i_T} & A_T & \xrightarrow{t_T} & V_T \\
\downarrow{q_V} & & \downarrow{q_A} & & \downarrow{q_V} \\
V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G
\end{array}
\]
If we observe this diagram vertically, then we have the ordered pair of graph-homomorphisms

\[
\begin{align*}
V_G & \xleftarrow{i_G} A_G & A_G & \xrightarrow{t_G} V_G \\
\uparrow_{p_V} & & \uparrow_{p_A} & \\
V_T & \xleftarrow{i_T} A_T & A_T & \xrightarrow{t_T} V_T.
\end{align*}
\]

This defines another textile system

\[
T^* = (p^*, q^* : \Gamma^* \to G^*)
\]
called the dual of \(T\), where \(i_{T^*} = p_A, t_{T^*} = q_A, i_G = p_V\) and \(t_G = q_V\).

Let \(T = (p, q : \Gamma \to G)\) be a textile system. Let \(\xi = \phi_p\) and let \(\eta = \phi_q\). A two-dimensional configuration \((\alpha_{ij}), i, j \in \mathbb{Z}, \alpha_{ij} \in A_T\), is called a textile woven by \(T\) if \((\alpha_{ij})_{j \in \mathbb{Z}} \in X_T\) and \(\eta((\alpha_{i-1,j})_{j \in \mathbb{Z}}) = \xi((\alpha_{ij})_{j \in \mathbb{Z}})\) for all \(i \in \mathbb{Z}\). Let \(U_T\) denote the set of all textiles woven by \(T\). Define

\[
Z_T = \{(\alpha_{0j})_{j \in \mathbb{Z}} \mid \exists (\alpha_{ij})_{i, j \in \mathbb{Z}} \in U_T\}, \quad X_T = \{\xi((\alpha_{0j})_{j \in \mathbb{Z}}) \mid \exists (\alpha_{ij})_{i, j \in \mathbb{Z}} \in U_T\}.
\]

Then we have subshifts \((Z_T, \xi_T)\) and \((X_T, \eta_T)\). We call \((X_T, \eta_T)\) the woof shift of \(T\) and \((X_T, \sigma_T)\) the warp shift of \(T\). We say that \(T\) is nondegenerate if \((X_T, \sigma_T) = (X_G, \sigma_G)\). We define onto maps \(\xi_T : Z_T \to X_T\) and \(\eta_T : Z_T \to X_T\) to be the restrictions of \(\xi\) and \(\eta\), respectively. If \(T\) is onesided 1-1, i.e., \(\xi_T\) is 1-1, then an onto endomorphism \(\varphi_T\) of \((X_T, \sigma_T)\) is defined by

\[
\varphi_T = \eta_T \xi_T^{-1}.
\]

If \(T\) is 1-1, i.e., both \(\xi_T\) and \(\eta_T\) are 1-1, then \(\varphi_T\) is an automorphism of \((X_T, \sigma_T)\). We also have the onesided subshifts \((\tilde{Z}_T, \tilde{\xi}_T)\) and \((\tilde{X}_T, \tilde{\sigma}_T)\) induced by \((Z_T, \xi_T)\) and \((X_T, \sigma_T)\), respectively.

Let \(T = (p, q : \Gamma \to G)\) be a textile system. Then \(U_T\) of is a closed subset of \(A_T^\mathbb{Z}\) equipped with the product topology of the discrete topology on \(A_T\) and is shift invariant (i.e., if \((\alpha_{i+l,j})_{l, j \in \mathbb{Z}} \in U_T\) with \(\alpha_{ij} \in A_T\), then \((\alpha_{i+l,j+l})_{l, j \in \mathbb{Z}} \in U_T\) for all \(k, l \in \mathbb{Z}\)). We define a textile subsystem of \(T\) to be a closed, shift-invariant subset of \(U_T\). Let \(U\) be a textile subsystem of \(T\). The dual \(U^*\) of \(U\) is defined by \(U^* = \{(\alpha_{ij})_{i, j \in \mathbb{Z}} \mid (\alpha_{ij})_{i, j \in \mathbb{Z}} \in U, \alpha_{ij} \in A_T\}\). Clearly \(U^*\) is a textile subsystem of \(T^*\). The woof shift \((X_U, \sigma_U)\) of \(U\) is defined by \(X_U = \{\xi_T((\alpha_{0j})_{j \in \mathbb{Z}}) \mid (\alpha_{ij})_{i, j \in \mathbb{Z}} \in U\}\) and \(\sigma_U = \sigma_T|_{X_U}\). We also define another subshift \((Z_U, \xi_U)\) by \(Z_U = \{(\alpha_{0j})_{j \in \mathbb{Z}} \mid (\alpha_{ij})_{i, j \in \mathbb{Z}} \in U\}\) and \(\xi_U = \xi_T|_{Z_U}\). We define onto maps \(\xi_U : Z_U \to X_U\) and \(\eta_U : Z_U \to X_U\) restricting \(\xi_T\) and \(\eta_T\), respectively. If \(U\) is onesided 1-1, i.e., \(\xi_U\) is one-to-one, then we have an onto endomorphism \(\varphi_U\) of \((X_U, \sigma_U)\) by \(\varphi_U = \eta_U \xi_U^{-1}\).

We say that \(U\) is 1-1 if both \(\xi_U\) and \(\eta_U\) are one-to-one. For \(n \geq 1\) we define a textile subsystem \(U[1] T[n]\) of \(T[1] T[n]\), which is called the higher block system of order \(n\) of \(U\), by \(U[n] = \{\alpha_{ij} \ldots \alpha_{i+j+n-1} j \in \mathbb{Z} \mid (\alpha_{ij})_{i, j \in \mathbb{Z}} \in U\}\).

Let \(T\) be a textile system. If \(U\) is a textile subsystem of \(T\), then for \(n \geq 1\), \(Z_{U[1] T[n]}\), is the subshift space consisting of all obis of width \(n\) (woven by \(T\)) that appear on some textile (woven by \(T\)) in \(U\), and the subshift spaces \(Z_{U[1] T[n]}\), \(n = 1, 2, \ldots\), define \(U\) (because the union of the sets \(F_n\) of “forbidden blocks” for
$Z_{((U_\gamma,\gamma_1)\alpha_1)}$ gives the set of “forbidden patterns” for $U$. Therefore if $U$ is onesided 1-1, then $Z_U$ defines $U$. For every subshift space $Z \subset Z_T$ with $\xi_T(Z) = \eta_T(Z)$ and $\xi_T|Z$ one-to-one, there exists a unique onesided 1-1 textile subsystem $U$ of $T$ with $Z_U = Z$.

A textile system $T = (p,q : \Gamma \rightarrow G)$ is said to be $p$-$L$ if $p$ is left resolving, $p$-$R$ if $p$ is right resolving, $q$-$L$ if $q$ is left resolving, and $q$-$R$ if $q$ is right resolving. An onto endomorphism $\varphi$ of a topological Markov shift $(X,\sigma)$ is called a $p$-$L$ endomorphism if there exists a onesided 1-1, nondegenerate, $p$-$L$ textile system with $(X,\sigma,\varphi) = (X_T,\sigma_T,\varphi_T)$. We similarly define $p$-$R$, $q$-$L$, and $q$-$R$ endomorphisms of topological Markov shifts.

A textile system is said to be $LR$ if it is $p$-$L$ and $q$-$R$, $RL$ if it is $p$-$R$ and $q$-$L$, $LL$ if it is $p$-$L$ and $q$-$L$, $RR$ if it is $p$-$R$ and $q$-$R$, and $q$-$biresolving$ if it is $q$-$L$ and $q$-$R$. An onto endomorphism $\varphi$ of a topological Markov shift $(X,\sigma)$ is called an $LR$ endomorphism if there exists a onesided 1-1, nondegenerate, $LR$ textile system $T$ with $(X_T,\sigma_T,\varphi_T) = (X,\sigma,\varphi)$. The definitions of $RL$, $LL$, $RR$, and $q$-$biresolving$ endomorphisms of topological Markov shifts are similarly given.

A textile system $T = (p,q : \Gamma \rightarrow G)$ is said to be weakly $p$-$L$ if $p$ is weakly left resolving. We similarly define weakly $p$-$R$, weakly $q$-$L$, and weakly $q$-$R$ textile systems. A textile system is said to be weakly $q$-$biresolving$ if $q$ is both weakly $p$-$L$ and weakly $q$-$R$. Similarly weakly $RL$ weakly $LL$ and weakly $RR$. A textile system is said to be weakly $q$-$biresolving$ if it is both weakly $q$-$R$ and weakly $q$-$L$.

An onto endomorphism $\varphi$ of a subshift $(X,\sigma)$ is said to be weakly $p$-$L$ if there exists a onesided 1-1 textile-subsystem $U$ of a weakly $p$-$L$ textile system such that $(X_U,\sigma_U,\varphi_U) = (X,\sigma,\varphi)$. We similarly define weakly $p$-$R$, weakly $q$-$L$ and weakly $q$-$R$ (onto) endomorphisms of $(X,\sigma)$. An onto endomorphism $\varphi$ of a subshift $(X,\sigma)$ is said to be weakly $LR$ if there exists a onesided 1-1 textile subsystem $U$ of a weakly $LR$ textile system such that $(X_U,\sigma_U,\varphi_U) = (X,\sigma,\varphi)$. We similarly define weakly $RL$, weakly $LL$, weakly $RR$ and weakly $q$-$biresolving$ (onto) endomorphisms of the subshift $(X,\sigma)$.

We define a textile relation-system to be an ordered pair of graph-homomorphisms $T = (p : \Gamma \rightarrow G, q : \Gamma \rightarrow H)$ (see [NH p. 191]). Let $X_T = \phi_p(X_\Gamma)$, let $Y_T = \phi_q(X_\Gamma)$ and let $\xi_T : X_T \rightarrow X_T$ and $\eta_T : X_T \rightarrow Y_T$ be the onto maps naturally induced by $\phi_p$ and $\phi_q$, respectively. $T$ is said to be onesided 1-1 if $\xi_T$ is one-to-one. For a onesided textile-relation system $T$, define $\phi_T = \eta_T \xi_T^{-1}$.

Note that a textile system is considered to be a special case of a textile relation-system. The definitions of weakly $p$-$L$, weakly $p$-$R$, weakly $q$-$L$, weakly $q$-$R$, weakly $LR$, weakly $RL$, weakly $LL$, weakly $RR$ and weakly $q$-$biresolving$ textile relation systems as well as their “non-weakly” versions should be clear.

A factor map (onto homomorphism) $\phi$ of an SFT $(X,\sigma_X)$ onto a sofic system $(Y,\sigma_Y)$ is said to be weakly $p$-$L$ if there exists a onesided 1-1, weakly $p$-$L$ textile relation system $T$ such that $\phi_T = \phi$ (with $X_T = X$ and $Y_T = Y$). The definitions of weakly $p$-$L$, weakly $q$-$R$, weakly $q$-$L$, weakly $LR$, weakly $RL$, weakly $LL$, weakly $RR$ and weakly $q$-$biresolving$ factor maps of an SFT onto a sofic system should be clear.

**Remark 2.5.** A weakly $p$-$L$ factor map of an SFT onto itself is a weakly $p$-$L$ endomorphism of the SFT, but a weakly $p$-$L$ endomorphism of an SFT is a weakly $p$-$L$ factor map of the SFT onto itself up to higher-block conjugacy. In this statement, “weakly $p$-$L$” can be replaced by any one of “weakly $p$-$R$” “weakly $q$-$R$”,...
“weakly $q$-L”, “weakly LR”, “weakly RL”, “weakly LL”, “weakly RR” and “weakly $q$-biresolving”.

Proof. Let $\varphi$ be a weakly $p$-L factor map of an SFT $(X, \sigma)$ onto itself. Then there exists a onesided 1-1, weakly $p$-L textile relation-system $T = (p : \Gamma \to G, q : \Gamma \to G)$ such that $\phi_p(X_T) = \phi_q(X_T) = X$ and $\phi_T = \varphi$. We can regard $T$ as a weakly $p$-L textile system and $U_T$ is a textile subsystem of $T$ with $\varphi_{U_T} = \varphi$.

Let $U$ be a onesided 1-1 textile-subsystem of a weakly $p$-L textile system such that $(X_U, \sigma_U)$ is an SFT. Then $(Z_U, \varsigma_U)$ is also an SFT, because $U$ is onesided 1-1. There exists $n \geq 1$ and graphs $G'$ and $\Gamma'$ such that $(X_{G'}, \sigma_{G'}) = ((X_U)^n, (\sigma_U)^n)$ and $(X_{\Gamma'}, \sigma_{\Gamma'}) = ((Z_U)^n, (\varsigma_U)^n)$. It is easily seen that we can define a weakly $p$-L textile relation-system $T' = (p' : \Gamma' \to G', q' : \Gamma' \to G')$ such that $\phi_{T'} = (\varphi_U)^n$.

The proofs of the remainder are similar.

In this paper, not only we describe results and proofs on weakly resolving (i.e. weakly $p$-L, weakly $p$-R, weakly $q$-R, weakly $q$-L, weakly LR, weakly RL, weakly LL, weakly RR, weakly $q$-biresolving) endomorphisms of general subshifts, but also we describe results and their direct proofs on resolving (i.e. $p$-L, $p$-R, $q$-R, $q$-L, $RL$, $LL$, $RR$, $q$-biresolving) endomorphisms of transitive topological Markov shifts and automorphisms of topological Markov shifts. However, by using the following remark, the latter results can be derived from the former results, though this is logically a roundabout way to obtain the latter results.

Remark 2.6. Let $\varphi$ be an onto endomorphism of an SFT $(X, \sigma)$.

1. If $\varphi$ is weakly $p$-L (respectively, weakly $p$-R), then for some $n \geq 1$, $(X^{[n]}, \sigma^{[n]})$ is a topological Markov shift and $\varphi^{[n]}$ is a $p$-L (respectively, $p$-R) endomorphism of $(X^{[n]}, \sigma^{[n]})$, and hence $\varphi$ is an essentially $p$-L (respectively, essentially $p$-R) endomorphism of $(X, \sigma)$.

2. Suppose that $\varphi$ is one-to-one or $\sigma$ is topologically transitive. If $\varphi$ is weakly $q$-R, then for some $n \geq 1$, $(X^{[n]}, \sigma^{[n]})$ is a topological Markov shift and $\varphi^{[n]}$ is a $q$-R endomorphism of $(X^{[n]}, \sigma^{[n]})$, and hence $\varphi$ is an essentially $q$-R endomorphism of $(X, \sigma)$. Moreover, in this statement “$q$-R” can be replaced by each of “$q$-L”, “LR”, “RL”, “LL” and “$q$-biresolving”.

Proof. (1) There exist a onesided 1-1 textile subsystem $U$ of a weakly $p$-L textile system $T$ such that $(X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi)$. Since $(X, \sigma) = (X_U, \sigma_U)$ is an SFT, $(Z_U, \varsigma_U)$ is an SFT because $U$ is onesided 1-1. There exists $n \geq 1$ such that both $(Z_U^{[n]}, \varsigma_U^{[n]})$ and $(X_U^{[n]}, \sigma_U^{[n]})$ are topological Markov shifts and hence there exists a onesided 1-1, nondegenerate textile system $T_0 = (p_0, q_0 : \Gamma_0 \to G_0)$ with $G_0$ and $\Gamma_0$ nondegenerate such that $U_{T_0} = U^{[n]}$. Since $T^{[n]}$ is weakly $p$-L, so is $T_0$. Since $\phi_{p_0}$ is a conjugacy, it follows from Lemma 2.1(1) that $p_0$ is left resolving. Since $(X_{T_0}, \sigma_{T_0}, \varphi_{T_0}) = (X, \sigma, \varphi)$ Hence the first version of (1) is proved. The second version is proved by symmetry.

(2) If $\varphi$ is one-to-one, then apply (1) to $\varphi^{-1}$. If $\sigma$ is transitive, then the proof is similar to that of (1), but use Lemma 2.1(2) instead of Lemma 2.1(1).

The remainder is similarly proved by using Lemma 2.1.

3. $q$-RESOLVING DEGREES

Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be subshifts. Let $f : L_{N+1}(X) \to L_1(Y)$ be a local rule on $(X, \sigma_X)$ to $(Y, \sigma_Y)$ with $N \geq 0$. Let $k$ be a nonnegative integer. We say
that \( f \) is \( k \) right-mergible if for any points \((a_j)_{j \in \mathbb{Z}}\) and \((b_j)_{j \in \mathbb{Z}}\) in \( X \) with \( a_j, b_j \in L_1(X) \), it holds that if \((a_j)_{j \leq 0} = (b_j)_{j \leq 0}\) and \( f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j) \) for \( j = 1, \ldots, k + 1 \), then \( a_1 = b_1 \). We say that \( f \) is \( k \) left-mergible if for any points \((a_j)_{j \in \mathbb{Z}}\) and \((b_j)_{j \in \mathbb{Z}}\) in \( X \) with \( a_j, b_j \in L_1(X) \), it holds that if \((a_j)_{j \geq 0} = (b_j)_{j \geq 0}\) and \( f(a_{j} \ldots a_{j+N}) = f(b_{j} \ldots b_{j+N}) \) for \( j = 1, \ldots, k + 1 \), then \( a_{-1} = b_{-1} \).

The notions defined above are compatible with the notions of being \( k \) right-mergible and being \( k \) left-mergible for graph-homomorphisms.

We say that \( f \) is strictly \( 0 \) right-mergible (respectively, strictly \( 0 \) left-mergible) if \( f \) is \( 0 \) right-mergible (respectively, \( 0 \) left-mergible). If \( k \geq 1 \), then we say that \( f \) is strictly \( k \) right-mergible (respectively, strictly \( k \) left-mergible) if \( f \) is \( k \) right-mergible but not \( k - 1 \) right-mergible (respectively, \( k \) left-mergible but not \( k - 1 \) left-mergible).

The notion of being \( k \) right-mergible (left-mergible) is a generalization of the notion of “existence of no right (respectively, left) \( f \)-branch of length exceeding \( k \)” in [H] Section 16 and the corresponding notions for a special and general classes of graph-homomorphisms and \( \Lambda \)-graphs appearing in [N1 N2 N3].

A homomorphism \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) between subshifts is said to be right-closing (respectively, left-closing) if it never collapses distinct backwardly (respectively, forwardly) \( \sigma \)-asymptotic points (Kitchens [K1]).

It is well known and easily seen that a homomorphism \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) between subshifts is right (respectively, left) closing homomorphism of \( (X, \sigma_X) \) into \( (Y, \sigma_Y) \) if and only if \( \phi \) is given by a \( k \) right-mergible (respectively, \( k \) left-mergible) local rule for some \( k \geq 0 \). For convenience’s sake, if a local rule \( f \) does not give a right-closing (respectively, left-closing) homomorphism, then we say that \( f \) is \( \infty \) right-mergible (respectively, \( \infty \) left-mergible).

**Definition 3.1.** Let \( k, m, n \geq 0 \). Suppose that a homomorphism \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) between subshifts is of \((m, n)\)-type and given by a local rule \( f : L_{N+1}(X) \to L_1(Y) \) with \( N = m + n \).

1. If \( f \) is strictly \( k \) right-mergible, then we define the \( q \)-R degree \( Q_R(\phi) \) of \( \phi \) by
   \[
   Q_R(\phi) = n - k.
   \]
2. If \( f \) is strictly \( k \) left-mergible, then we define the \( q \)-L degree \( Q_L(\phi) \) of \( \phi \) by
   \[
   Q_L(\phi) = m - k.
   \]

Furthermore, we adopt the following convention. If \( \phi \) is not right-closing, then \( Q_R(\phi) = -\infty \), and if \( \phi \) is not left-closing, then \( Q_L(\phi) = -\infty \).

We must prove that \( Q_L(\phi) \) and \( Q_R(\phi) \) are determined only by \( \phi \) for a homomorphism \( \phi \). To do this we need the following lemma.

**Lemma 3.2.** Let \( N, s, t, k, l \) be nonnegative integers. Let \( f : L_{N+1}(X) \to L_1(Y) \) be a local rule on a subshift \((X, \sigma_X)\) to another \((Y, \sigma_Y)\). Let \( g : L_{N+s+t+1}(X) \to L_1(Y) \) be the mapping such that
   \[
   g(a_{-t}a_{-t+1} \ldots a_{N+s}) = f(a_0 \ldots a_N),
   \]
   where \( a_{-t} \ldots a_{N+s} \in L_{N+s+t+1}(X) \) with \( a_j \in L_1(X) \). Then \( g \) is a local rule on \((X, \sigma_X)\) to \((Y, \sigma_Y)\). (We will call \( g \) the local rule obtained from \( f \) by adding right redundancy by \( s \) and adding left redundancy by \( t \).) If \( f \) is strictly \( k \) right-mergible, then \( g \) is strictly \( k+s \) right-mergible. If \( f \) is strictly \( l \) left-mergible, then \( g \) is strictly \( l+t \) left-mergible.
Proof. Clearly g is a local rule on \((X, \sigma_X)\) to \((Y, \sigma_Y)\).

Suppose that \(f\) is strictly \(k\) right-mergible. Suppose that \((a_j)_{j \in \mathbb{Z}}\) and \((b_j)_{j \in \mathbb{Z}}\) are two points of \((X, \sigma_X)\) with \(a_j, b_j \in L_1(X)\) such that

\[
a_j = b_j \quad \text{for all } j \leq 0, \quad \text{and} \quad g(a_{j-s-t-N} \ldots a_j) = g(b_{j-s-t-N} \ldots b_j) \quad \text{for } j = 1, \ldots, k+s+1.
\]

By definition we have

\[
f(a_{j-s-N} \ldots a_{j-s}) = f(b_{j-s-N} \ldots b_{j-s})
\]

for \(j = 1, \ldots, k+s+1\) and hence for \(j = s+1, \ldots, k+s+1\), so that we have

\[
f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j) \quad \text{for } j = 1, \ldots, k+1.
\]

Therefore, since \(f\) is \(k\) right-mergible, it follows that \(a_1 = b_1\). This implies that \(g\) is \(k+s\) right-mergible.

Since \(f\) is not \(k-1\) right-mergible with \(k \geq 1\), there exist points \((a_j)_{j \in \mathbb{Z}}\) and \((b_j)_{j \in \mathbb{Z}}\) in \(X\) with \(a_j, b_j \in L_1(X)\) such that

\[
a_j = b_j \quad \text{for all } j \leq 0, \quad a_1 \neq b_1, \quad \text{and} \quad f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j) \quad \text{for } j = 1, \ldots, k.
\]

Therefore, since \(a_{j-N} \ldots a_j = b_{j-N} \ldots b_j\) for \(j = -s+1, \ldots, 0\), it follows that

\[
f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j) \quad \text{for } j = -s+1, \ldots, k.
\]

Hence, by definition we have

\[
g(a_{j-t-N} \ldots a_{j+s}) = g(b_{j-t-N} \ldots b_{j+s})
\]

for \(j = -s+1, \ldots, k\), so that we have

\[
g(a_{j-s-t-N} \ldots a_j) = g(b_{j-s-t-N} \ldots b_j) \quad \text{for } j = 1, \ldots, k+s.
\]

Therefore, we have proved that \(g\) is not \(k+s-1\) right-mergible.

By symmetry, it follows from the result proved above that if \(f\) is strictly \(l\) mergible, then \(g\) is strictly \(l+t\) left-mergible. \(\square\)

Proposition 3.3. Let \(\phi\) be a homomorphism between subshifts. Then \(Q_R(\phi)\) and \(Q_L(\phi)\) are uniquely determined by \(\phi\).

Proof. Suppose that for \(i=1,2\), \(\phi\) is a homomorphism of \((m_i, n_i)\)-type of a subshift \((X, \sigma_X)\) into another \((Y, \sigma_Y)\) given by a local-rule \(f_i : L_{m_i+n_i+1}(X) \to L_1(Y)\).

Suppose that for \(i=1,2\), \(f_i\) is strictly \(k_i\) right-mergible. Let \(m = \max\{m_1, m_2\}\) and let \(n = \max\{n_1, n_2\}\). For \(i = 1, 2\), let \(g_i : L_{m+n}(X) \to L_1(X)\) be the local rule such that

\[
g_i(a_{-m} \ldots a_n) = f_i(a_{-m} \ldots a_n), \quad a_{-m} \ldots a_n \in L_{m+n+1}(X), \quad a_j \in L_1(X).
\]

Then, since \(\phi\) is of \((m_1, n_1)\)-type and given by \(f_1\), \(\phi\) is of \((m, n)\)-type and given by \(g_1\), for \(i = 1, 2\). Since \(f_i\) is strictly \(k_i\) right-mergible, it follows from Lemma 3.2 that \(g_i\) is strictly \(k_1+n-n_1\) right-mergible for \(i = 1, 2\). Since \(g_1 : L_{m+n+1}(X) \to L_1(Y)\) and \(g_2 : L_{m+n+1}(X) \to L_1(Y)\) give the same block-map \(\phi\) of \((m, n)\)-type, it follows that \(g_1 = g_2\). Hence \(k_1+n-n_1 = k_2+n-n_2\), so that \(n_1-k_1 = n_2-k_2\).

Similarly, it is proved that if \(f_i\) is strictly \(l_i\) left-mergible for \(i=1,2\), then \(m_1-l_1 = m_2-l_2\). \(\square\)
Proposition 3.4. Let $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ be a homomorphism between subshifts. Let $s$ be an integer. Then

$$Q_R(\phi \sigma_X^s) = Q_R(\phi) + s, \quad Q_L(\phi \sigma_X^s) = Q_L(\phi) - s$$

and hence $Q_R(\phi) + Q_L(\phi)$ is shift-invariant.

Proof. Suppose that $\phi$ is of $(m, n)$-type given by a local rule $f : L_{m+n+1}(X) \to L_1(Y)$. Increasing the redundancies of $f$ as in Lemma 3.2 if necessary, we may assume that $m, n \geq |s|$, by Proposition 3.3. Since $\phi \sigma_X^s$ is of $(m - s, n + s)$-type and given by $f$, the proposition follows. \qed

Let $N \geq 0$. For a local rule $f : L_{N+1}(X) \to L_1(Y)$, on a subshift $(X, \sigma_X)$ to another $(Y, \sigma_Y)$, the higher block local rule $f^{[s]} : L_{N+1}(X^{[s]}) \to L_1(Y^{[s]})$ of order $s$ on $(X^{[s]}, \sigma_{X^{[s]}})$ to $(Y^{[s]}, \sigma_{Y^{[s]}})$ is defined by

$$f^{[s]}((a_1 \ldots a_s) (a_2 \ldots a_{s+1}) \ldots (a_{N+1} \ldots a_{N+s}))) = f(a_1 \ldots a_{N+1}) f(a_2 \ldots a_{N+2}) \ldots f(a_s \ldots a_{s+N}), \quad a_j \in L_1(X).$$

For a homomorphism $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ between subshifts and $s \geq 1$, let the homomorphism $\phi^{[s]} : (X^{[s]}, \sigma_{X^{[s]}}) \to (Y^{[s]}, \sigma_{Y^{[s]}})$ be defined as follows: if $\phi$ maps $(a_j)_{j \in \mathbb{Z}} \in X$ to $(b_j)_{j \in \mathbb{Z}} \in Y$ with $a_j \in L_1(X)$ and $b_j \in L_1(Y)$, then $\phi^{[s]}$ maps $(a_j \ldots a_{j+s-1})_{j \in \mathbb{Z}}$ to $(b_j \ldots b_{j+s-1})_{j \in \mathbb{Z}}$.

As is easily seen, the following proposition holds:

Proposition 3.5. Let $N, m, n, k, l \geq 0$ with $N = m + n$ and let $s \geq 1$. Let $f : L_{N+1}(X) \to L_1(Y)$ be a local rule on a subshift $(X, \sigma_X)$ to another $(Y, \sigma_Y)$.

1. If $f$ is strictly $k$ right-mergible (respectively, strictly $l$ left-mergible), then the higher block local rule $f^{[s]}$ is strictly $k$ left-mergible (respectively, strictly $l$ right-mergible).

2. If a homomorphism $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ is of $(m, n)$-type and given by $f$, then $\phi^{[s]}$ is of $(m, n)$-type and given by $f^{[s]}$.

3. For a homomorphism $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$, $Q_R(\phi) = Q_R(\phi^{[s]})$ and $Q_L(\phi) = Q_L(\phi^{[s]})$.

Let $(X, \sigma_X), (Y, \sigma_Y)$, and $(Z, \sigma_Z)$ be subshifts. Let $f : L_{N+1}(X) \to L_1(Y)$ and $g : L_{N'+1}(Y) \to L_1(Z)$ be local rules on $(X, \sigma_X)$ to $(Y, \sigma_Y)$ and on $(Y, \sigma_Y)$ to $(Z, \sigma_Z)$, respectively, with $N, N' \geq 0$. We define the composition $gf : L_{N+N'+1}(X) \to L_1(Z)$ of $f$ and $g$ by

$$gf(a_1 \ldots a_{N+N'+1}) = g(f(w_1) f(w_2) \ldots f(w_{N'+1})),$$

where $w_j = a_j \ldots a_{j+N}$ for $j = 1 \ldots N' + 1$. Clearly, $gf$ is a local rule on $(X, \sigma_X)$ to $(Z, \sigma_Z)$.

Lemma 3.6. Hypotheses being the same as above, if $f$ is $k$ right-mergible (respectively, $k$ left-mergible) and $g$ is $l$ right-mergible (respectively, $l$ left-mergible), then $gf$ is $k + l$ right-mergible (respectively, $k + l$ left-mergible).

Proof. Let $(a_j)_{j \in \mathbb{Z}}$ and $(b_j)_{j \in \mathbb{Z}}$ are points of $X$ such that $(a_j)_{j \leq 0} = (b_j)_{j \leq 0}$ and $gf(a_{j-N-N'} \ldots a_{j}) = g(f(b_{j-N-N'} \ldots b_{j}))$ for $j = 1, \ldots, k+l+1$. Then $f(v_j) = f(w_j)$ for $j \leq 0$ and $g(f(v_{j-N-N'} \ldots v_{j})) = g(f(w_{j-N-N'} \ldots w_{j}))$ for $j = 1, \ldots, k+l+1$, where $v_j = a_{j-N-N'} \ldots a_{j}$ and $w_j = b_{j-N-N'} \ldots b_{j}$ for $j \leq k+l+1$. Therefore, since $g$ is $l$ right-mergible, it follows that the prefixes of length $(k+l+1) - l$ of $f(v_1) \ldots f(v_{k+l+1})$ and $f(w_1) \ldots f(w_{k+l+1})$ are the same. Hence $f(a_j-N \ldots a_j) = f(b_j-N \ldots b_j)$ for
$j = 1, \ldots, k + 1$. Therefore, $(a_j)_{j \leq 0} = (b_j)_{j \leq 0}$ and $f$ is $k$ right-mergible, we have $a_1 = b_1$. This implies that $gf$ is $k + l$ right-mergible.

The second version follows from the first one by symmetry. □

Using Lemma 3.6, we obtain:

**Proposition 3.7.** Let $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ and $\psi : (Y, \sigma_Y) \to (Z, \sigma_Z)$ be homomorphisms between subshifts. Then

$$Q_R(\psi \phi) \geq Q_R(\phi) + Q_R(\psi) \quad \text{and} \quad Q_L(\psi \phi) \geq Q_L(\phi) + Q_L(\psi).$$

For a subshift $(X, \sigma_X)$ and an integer $s \geq 1$, we define the $s$-th power system of $(X, \sigma_X)$ to be the subshift $(X^s, \sigma_X^{(s)})$ over the alphabet $L_s(X)$ with $X^s = \{(a_{j-1} \cdots a_j)_{j \in \mathbb{Z}} \mid (a_j)_{j \in \mathbb{Z}} \in X, a_j \in L_1(X)\}$.

For a homomorphism $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ between subshifts and $s \geq 1$, define $(\phi)^{(s)} : (X^s, \sigma_X^{(s)}) \to (Y^s, \sigma_Y^{(s)})$ to be the homomorphism such that if $\phi$ maps $(a_j)_{j \in \mathbb{Z}}$ to $(b_j)_{j \in \mathbb{Z}}$ with $a_j, b_j \in L_1(X)$, then $(\phi)^{(s)}$ maps $(a_{j-1} \cdots a_j)_{j \in \mathbb{Z}}$ to $(b_{j-1} \cdots b_j)_{j \in \mathbb{Z}}$ (cf. [N3, p. 12]).

**Proposition 3.8.** Let $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ be a homomorphism between subshifts. Let $s \geq 1$. Then the following holds:

1. $Q_R(\phi) \geq sQ_R((\phi)^{(s)})$;
2. $Q_L(\phi) \geq sQ_L((\phi)^{(s)})$.

**Proof.** (1) Let $m, n \geq 0$ and suppose that $\phi$ is of $(m, n)$-type and given by a local rule $f : L_{N+1}(X) \to L_1(Y)$ with $N = m + n$. It follows from Proposition 3.3 that any addition of prefix or suffix redundancies to the local rule does not change $Q_R(\phi)$. Hence we may assume that both $m$ and $n$ are divisible by $s$. Let $m' = m/s$, let $n' = n/s$ and let $N' = m' + n'$. Define $g : L_{N'+1}(X^s) \to L_1(Y^s)$ as follows: for $a_1 \cdots a_{N+s} \in L_{N+s}(X)$ with $a_j \in L_1(X)$,

$$g(c_1 \cdots c_{N'+1}) = f(a_1 \cdots a_{N+1}) f(a_2 \cdots a_{N+2}) \cdots f(a_s \cdots a_{N+s}),$$

where $c_j = a_{(j-1)s+1} \cdots a_{js}$ for $j = 1, \ldots, N' + 1$. Then $(\phi)^{(s)}$ is of $(m', n')$-type and given by the local rule $g$.

Suppose that $f$ is strictly $k$ right-mergible with $k \geq 0$. Let $k' = \lceil k/s \rceil$. Let $(a_j)_{j \in \mathbb{Z}}, (b_j)_{j \in \mathbb{Z}} \in X$ with $a_j, b_j \in L_1(X)$. For $j \in \mathbb{Z}$, let $c_j = a_{(j-1)s+1} \cdots a_{js}$ and let $d_j = b_{(j-1)s+1} \cdots b_{js}$. Then $(c_j)_{j \in \mathbb{Z}}, (d_j)_{j \in \mathbb{Z}} \in X^s$. Assume that $(c_j)_{j \geq 0} = (d_j)_{j \leq 0}$ and $g(c_j \cdots c_{N'}) = g(d_j \cdots d_{N'})$ for $j = 1, \ldots, k'+1$, then it follows that $(a_j)_{j \leq 0} = (b_j)_{j \leq 0}$ and $f(a_{j-N} \cdots a_j) = f(b_{j-N} \cdots b_j)$ for $j = 1, \ldots, (k'+1)s$. Since $f$ is $k$ right-mergible, the prefixes of length $(k'+1)s - k$ of $a_1 \cdots a_{(k'+1)s}$ and $b_1 \cdots b_{(k'+1)s}$ must be the same. Since

$$(k'+1)s - k = \lceil k/s \rceil s + s - k \geq s,$$

$c_1 = a_1 \cdots a_s = b_1 \cdots b_s = d_1$. Therefore, $g$ is $k'$ right-mergible. If $k = 0$, then $k' = 0$ and hence $g$ is strictly $k'$ right-mergible.

Assume that $k \geq 1$. Since $f$ is strictly $k$ right-mergible, it occurs that $(a_j)_{j \geq 0} = (b_j)_{j \leq 0}$, $a_1 \neq b_1$ and $f(a_{j-N} \cdots a_j) = f(b_{j-N} \cdots b_j)$ for $j = 1, \ldots, k$. Let $\tilde{c}_j = a_{(j-2)s+2} \cdots a_{(j-1)s+1}$ and let $d_j = b_{(j-2)s+2} \cdots b_{(j-1)s+1}$, for $j \in \mathbb{Z}$. Then $(\tilde{c}_j)_{j \in \mathbb{Z}}$ and $(\tilde{d}_j)_{j \in \mathbb{Z}}$ are points of $X^s$, $(\tilde{c}_j)_{j \leq 0} = (\tilde{d}_j)_{j \leq 0}$, $\tilde{c}_1 \neq \tilde{d}_1$ and $g(\tilde{c}_j \cdots \tilde{c}_{N'}) = g(\tilde{d}_j \cdots \tilde{d}_{N'})$ for

$$j = 1, 2, \ldots, 1 + \lceil (k - 1)/s \rceil = \lceil k/s \rceil = k'.$$
Therefore, it is proved that \( q \) is strictly \( k' \) right-mergible. Hence we have

\[
Q_R(\phi) = n - k \geq n - sk' = (n' - k')s = sQ_R((\phi)^s),
\]

and (1) is proved.

(2) By symmetry, (2) follows from (1). \( \square \)

4. \( q \)-resolving endomorphisms of topological Markov shifts

Proposition 4.1. Let \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) be a factor map of an SFT onto a sofic system. Let \( s \in \mathbb{Z} \).

(1) \( \phi \) is weakly \( q \)-R if and only if \( Q_R(\phi) \geq 0 \); moreover \( \phi \sigma_X^* \) is weakly \( q \)-R if and only if \( s \geq -Q_R(\phi) \).

(2) \( \phi \) is weakly \( q \)-L if and only if \( Q_L(\phi) \geq 0 \); moreover \( \phi \sigma_X^* \) is weakly \( q \)-L if and only if \( s \leq Q_L(\phi) \).

Proof. (1) Suppose \( \phi \) is weakly \( q \)-R. Then there exists a textile relation system \( T = (p : \Gamma \to G, q : \Gamma \to H) \) such that \( q \) is weakly right-resolving, \( \xi : (X_1, \sigma_{X_1}) \to (X, \sigma_X) \) is a conjugacy and \( \phi = \eta \xi_1^{-1} \). There exist \( m, n \geq 0 \) and a local rule \( g : L_{m+n+1}(X) \to L_1(\Gamma) \) such that \( \xi : X \to X_1 \) is a block-map of \((m, n)\)-type given by \( g \). Let \( f = q\phi \). Then \( \phi \) is a block-map of \((m, n)\)-type given by the local rule \( f \). Let \((a_j)_{j \in \mathbb{Z}} \) and \((b_j)_{j \in \mathbb{Z}} \) be points in \( X \) such that \((a_j)_{j \geq 0} = (b_j)_{j \geq 0} \) and \( f(a_j \ldots a_{j+n-1}) = f(b_j \ldots b_{j+n-1}) \) for \( j = 1, \ldots, n+1 \). Let \( \alpha_j = g(a_j \ldots a_{j+n}) \) and let \( \beta_j = g(b_j \ldots b_{j+n}) \) for \( j \in \mathbb{Z} \). Then it follows that \( \alpha_j = g(a_j \ldots a_n) = g(b_j \ldots b_n) = \beta_j \) and \( q(\alpha_j) = f(a_j \ldots a_n) = \beta_j \). Therefore \( a_1 = q(\alpha_1) = q(\beta_1) = b_1 \), which implies that \( f \) is \( n \) right-mergible. Therefore, if \( f \) is strictly \( k \) right-mergible, then \( k \leq n \). Thus \( Q_R(\phi) = n - k \geq 0 \).

Conversely, suppose that \( Q_R(\phi) \geq 0 \). Let \( A = L_1(X) \) and \( B = L_1(Y) \). Since \((X, \sigma_X)\) is an SFT, for all sufficiently large integer \( I \) \((X[I], \sigma_X^I)\) is the topological Markov shift whose defining graph, say \( G_1 \), is defined as follows: \( A_G = L_1(X); V_G = L_{I-1}(X); \) for \( w = a_1 \ldots a_I \in L_I(X) \) with \( a_j \in A \), \( i_G(w) = a_1 \ldots a_{I-1} \) and \( t_G(w) = a_2 \ldots a_I \). By this and Proposition 3.3, we may assume that \( \phi \) is of \((m, n)\) type and given by a local rule \( f : L_{m+n+1}(X) \to B \) with \( m, n \geq 0 \) such that \((X^{m+n+1}, \sigma_X^{m+n+1})\) is the topological Markov shift whose defining graph is \( G_{m+n+1} \). Suppose that \( f \) is strictly \( k \) right-mergible. Then \( k = n - Q_R(\phi) \). We have the graph-homomorphism \( q_f : \Gamma \to G_B \) with \( \Gamma = G_{m+n+1} \) naturally induced by \( f \). Since \( f \) is \( k \) right-mergible, so is \( q_f \).

We consider the induced right-resolving graph-homomorphism \((q_f)_+ : \Gamma^+_q \to G_B^+ \). Each arc in \( \Gamma^+_q \) is of the form \((U, b)\), where \( U \in C_q \) and \( b \) is an arc in \( G_B \) (a symbol in \( B \)). (See Section 2 for notation.) By definition, \( B^+_q(U, b) \) is the set of all arcs \( w \) in \( G_{m+n+1} \) such that \( i w_{m+n+1} = U \) and \( q_f(w) = b \). We claim that all arcs in \( B^+_q(U, b) \) are words in \( L_{m+n+1}(X) \) having the same prefix (initial subword) of length \( m + 1 + n - k \). We prove this in the following.

If \( n = 0 \), then, since \( k \leq n \), \( q_f \) is \( 0 \) right-mergible, so that every set \( U \in C_q \) is a singleton and the claim is valid. Hence we assume that \( m + n \geq 1 \). Suppose that \( w = a_1 \ldots a_{m+n+1} \) and \( \bar{w} = \bar{a}_1 \ldots \bar{a}_{m+n+1} \) are in \( B^+_q(U, b) \) with \( a_j, \bar{a}_j \in A \). Since \( a_1 \ldots a_{m+n} \) and \( \bar{a}_1 \ldots \bar{a}_{m+n} \) are in the right-compatible set \( U \), there exist
$s \geq m + n$ and paths $w_1 \ldots w_s$ and $\bar{w}_1 \ldots \bar{w}_s$ of length $s$ in $G_{m+n+1}$ with $w_j, \bar{w}_j \in A_{G_{m+n+1}} = L_{m+n+1}(X)$ starting from the same vertex and ending in $a_1 \ldots a_{m+n}$ and $\bar{a}_1 \ldots \bar{a}_{m+n}$, respectively, in $G_{m+n+1}$ and generating the same path (of length $s$) in $G_B$ (the same word over $B$) under $q_f$. Clearly $P = w_1 \ldots w_s w$ and $\bar{P} = \bar{w}_1 \ldots \bar{w}_s \bar{w}$ are paths of length $s + 1$ going from the same vertex and generating the same path. Since $q_f$ is $k$ right-mergible, it follows that the initial subpaths of length $s + 1 - k$ of $P$ and $\bar{P}$ are the same. Hence $w_{s+1-k} = \bar{w}_{s+1-k}$. Since $w_s = c_0a_1 \ldots a_{m+n}$ and $\bar{w}_s = c_0\bar{a}_1 \ldots \bar{a}_{m+n}$ with some $c_0, \bar{c_0} \in A$, it follows that $w_{s+1-k} = c_{-k+1} \ldots c_0a_1 \ldots a_{m+1+n-k}$ and $\bar{w}_{s+1-k} = \bar{c}_{-k+1} \ldots \bar{c_0}\bar{a}_1 \ldots \bar{a}_{m+1+n-k}$ with some $c_j, \bar{c}_j \in A$. Since $n \geq k$, we see that $a_1 \ldots a_{m+1+n-k} = a_1 \ldots a_{m+1+n-k}$, and the claim is proved.

Therefore we can define a onesided 1-1 textile relation system $T_0 = (\rho_0 : \Gamma_{q_f}^+ \to G_A, (q_f)^+ : \Gamma_{q_f}^+ \to G_B)$ as follows: $(\rho_0)_A((U, b))$ is the $(m + 1)$st symbol (in $A$) of any word in $G_{m+n+1}(X)$ in $B_{q_f}^+(U, b)$. Then, $T_0$ is weakly $q$-$R$, and $q_{T_0} = \phi$. Hence $\phi$ is weakly $q$-$R$.

We have proved that $\phi$ is weakly $q$-$R$ if and only if $Q_R(\phi) \geq 0$. It follows from Proposition 3.4 that for $s \in \mathbb{Z}$, $Q_R(\phi \sigma_X^s) \geq 0$ if and only if $s \geq -Q_R(\phi)$. Therefore, for $s \in \mathbb{Z}$, $\sigma_X^s$ is weakly $q$-$R$ if and only if $s \geq -Q_R(\phi)$.

(2) By symmetry, it follows from (1) that $\phi$ weakly $q$-$L$ if and only if $Q_L(\phi) \geq 0$. The remainder follows from Proposition 3.4.

\begin{theorem}
Let $\varphi$ be an onto endomorphism of a topological Markov shift $(X_G, \sigma_G)$. Let $s \in \mathbb{Z}$. If $\varphi$ is one-to-one or $\sigma_G$ is topologically transitive, then the following hold:

1. $\varphi$ is $q$-$R$ if and only if $Q_R(\varphi) \geq 0$; moreover $\varphi \sigma_X^s$ is $q$-$R$ if and only if $s \geq -Q_R(\varphi)$;

2. $\varphi$ is $q$-$L$ if and only if $Q_L(\varphi) \geq 0$; moreover $\varphi \sigma_X^s$ is $q$-$L$ if and only if $s \leq Q_L(\varphi)$.

Proof. (1) The proof that if $\varphi$ is $q$-$R$, then $Q_R(\varphi) \geq 0$, is similar to the proof of the corresponding part of Proposition 4.1.

Suppose that $Q_R(\varphi) \geq 0$. Suppose that $\varphi$ is of $(m, n)$-type and given by a local rule $f : L_{m+n+1}(G) \to A_G$ which is $k$ right-mergible with $k = n - Q_R(\varphi)$. If we modify the proof of Proposition 4.1 (the part of the proof that if $Q_R(\varphi) \geq 0$, then $\phi$ is weakly $q$-$R$) by letting $\Gamma = G^{[m+n+1]}$, replacing $G_A$ and $G_B$ by $G$ and using Lemma 2.2(2), then we directly have a onesided 1-1, nondegenerate, $q$-$R$ textile system $T_0 = (\rho_0 : (q_f)^+ : \Gamma_{q_f}^+ \to G)$ such that $q_{T_0} = \varphi$. Hence it is proved that $\varphi$ is $q$-$R$.

The remainder follows from Proposition 3.4.

(2) By symmetry, it follows from (1) that $\varphi$ is $q$-$L$ if and only if $Q_L(\varphi) \geq 0$. The remainder follows from Proposition 3.4.

\begin{proposition}
Let $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ be a factor map of an SFT onto a sofic system. Let $s \in \mathbb{Z}$.

1. If $\phi^{[t]} \sigma_X^s$ is weakly $q$-biresolving with some $t \geq 1$, then $-Q_R(\phi) \leq s \leq Q_L(\phi)$.

2. If $Q_R(\phi) + Q_L(\phi) \geq 0$, then there exists $t \geq 1$ such that $\phi^{[t]} \sigma_X^s$ is weakly $q$-biresolving for all $-Q_R(\phi) \leq s \leq Q_L(\phi)$; more precisely, we can construct
a onesided 1-1, weakly q-biresolving textile-relation system

\[ T_s = (p_s : \Gamma_0 \to G_A^{[l]}, q : \Gamma_0 \to G_B^{[l]}) \text{ with } A = L_1(X), B = L_1(Y) \]

such that \( \phi_{T_s} = \phi^{[l]}_s \sigma_{X_0}^s \) for all \( -Q_R(\phi) \leq s \leq Q_L(\phi) \).

Proof. (1) If \( \phi^{[l]}_s \sigma_{X_0}^s \) is weakly q-biresolving, then \( Q_R(\phi) + s = Q_R(\phi^{[l]}_s \sigma_{X_0}^s) \geq 0 \) and \( Q_L(\phi) - s = Q_L(\phi^{[l]}_s \sigma_{X_0}^s) \geq 0 \), by Propositions 3.4, 3.5 and 4.1, so that

\[ -Q_R(\phi) \leq s \leq Q_L(\phi). \]

(2) Suppose that \( Q_R(\phi) + Q_L(\phi) \geq 0 \). Since \((X, \sigma_X)\) is an SFT, for all sufficiently large integer \( l \), \((X^{[l]}, \sigma^{[l]}_X)\) is the topological Markov shift whose defining graph, say \( G_l \), is defined as follows: \( A_{G_l} = L_l(X); \) \( V_{G_l} = L_{l-1}(X) \); for \( w = a_1 \ldots a_l \in L_l(X) \) with \( a_j \in A \), \( i_{G_l}(w) = a_1 \ldots a_{l-1} \) and \( t_{G_l}(w) = a_2 \ldots a_l \). By this and Proposition 3.3, adding sufficiently large right and left redundancies to \( f \), we may assume that \( \phi \) is of \((m, n)\) type given by a local rule \( f : L_{m+n+1}(X) \to B \) with \( m, n \geq 0 \) such that \((X^{[m+n+1]}, \sigma^{[m+n+1]}_X)\) is the topological Markov shift whose defining graph is \( G_{m+n+1} \). Let \( k = n - Q_R(\phi) \) and let \( l = m - Q_L(\phi) \). Then \( f \) is strictly right-mergible and strictly \( l \)-left-mergible. To prove (2), it suffices to show that we can construct a onesided 1-1, weakly q-biresolving textile-relation system

\[ T_s = (p_s : \Gamma_0 \to G_A^{[l+k+1]}, q : \Gamma_0 \to G_B^{[l+k+1]}) \]

such that \( \phi_{T_s} = \phi^{[l+k+1]}_s \sigma_{X_0}^{l+k+1} \) for \( -Q_R(\phi) \leq s \leq Q_L(\phi) \).

Let \( \Gamma = G_{m+n+1} \). Let \( F : L_{1}(\Gamma) \to G_B \) be the local rule on \((X_{\Gamma}, \sigma_{\Gamma})\) to \((X_{G_B}, \sigma_{G_B})\) such that \( F(w) = f(w) \) for \( w \in A_{\Gamma} = L_{m+n+1}(X) \). Since \( f \) is \( k \)-right-mergible and \( l \)-left-mergible, so is \( F \). We use the graph-homomorphism

\[ q_{F;\Gamma,l,k}^{l+k+1} : \Gamma_{\Gamma,l,k}^{l+k+1} \to G_B^{[l+k+1]} \]

(see Subsection 2.2). Each arc \( \bar{\alpha} \) in \( \Gamma_{\Gamma,l,k}^{l+k+1} \) is written in the form

\[ \bar{\alpha} = D_{F;\Gamma,l,k}^{l+k+1}(\mu_{-1}\alpha_1), \]

where \( \mu_{-1}\alpha_1 \in L_{l+k+1}(\Gamma), \mu_{-1} \in L_1(\Gamma), \alpha \in A_{\Gamma} \) and \( \mu_1 \in L_k(\Gamma) \), and we have \( q_{F;\Gamma,l,k}^{l+k+1}(\bar{\alpha}) = F(\mu_{-1}\alpha_1) \). By definition \( \alpha \) is uniquely determined by \( \bar{\alpha} \). Since \( \Gamma = G_{m+n+1} \), the arc \( \alpha \) is a word in \( L_{m+n+1}(X) \) and hence is written

\[ \alpha = a_1 \ldots a_{m+n+1} \]

and the path \( \mu_{-1}\alpha_1 \) is written \( \mu_{-1}\alpha_1 = \alpha_{-l-1} \ldots \alpha_k \), where \( a_0 = \alpha \) and \( \alpha_j = a_{j+1} \ldots a_{j+m+n+1} \) for \( j = -l, \ldots, k \) with

\[ a_{-l-1} \ldots a_{k+m+n+1} \in L_{l+1+m+n+1}(X), \quad a_j \in A, \]

and hence we have

\[ q_{F;\Gamma,l,k}^{l+k+1}(\bar{\alpha}) = f(a_{-l-1} \ldots a_{k+m+n+1}). \]

Noting that \( -(n - k) = -Q_R(\varphi) \leq Q_L(\varphi) = m - l \), for \( -(n - k) \leq s \leq m - l \) we define a graph-homomorphism \( p_s : \Gamma_{F;\Gamma,l,k}^{l+k+1} \to G_A^{[l+k+1]} \) by

\[ p_s(\bar{\alpha}) = a_{m+1-l-s} \ldots a_{m+1+k-s}. \]

Note that the word of the right side of (4.2) is a subword of \( \alpha \) for all \( -(n - k) \leq s \leq m - l \), because its length is \( k + l + 1 \), \( p_{-(n-k)}(\bar{\alpha}) = a_{m+n+1-k-l} \ldots a_{m+n+1}, \)

\[ p_{m-l}(\bar{\alpha}) = a_1 \ldots a_{k+l+1} \text{ and } m+n+1-(k+l+1) = Q_R(\phi)+Q_L(\phi) \geq 0. \] Therefore,
Theorem 4.5

If and only if it is expansive and essentially weakly q-ϕ

Since the same spectral radius as \( q \) follows from Lemma 2.1(2), \( T_s \) is weakly q-biresolving. Clearly \( T_s \) is onesided 1-1. It follows from (4.1) and (4.2) that

\[
\varphi_{T_s} = \varphi^{[k+\ell+1]} G_{X,[k+\ell+1]}^{[k+\ell+1]} - (n - k) \leq s \leq m - l.
\]

Therefore (2) is proved.

\[ \square \]

Theorem 4.4. Suppose that \( \varphi \) is an onto endomorphism of a topological Markov shift \( (X_G, \sigma_G) \) with \( G \) irreducible and \( Q_R(\varphi) + Q_L(\varphi) \geq 0 \). Let \( s \in \mathbb{Z} \). If \( \varphi \) is of \((m,n)\)-type with \( m,n \geq 0 \) and is given by a local rule \( f : L_{m+n+1}(G) \to A_G \) which is strictly k right-mergible and strictly l left-mergible with \( k,l \geq 0 \), then for all \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\), the textile system

\[
T_s = (p_s, q_{f,l,k} : G_{f,l,k}^+ \to G^{[k+l+1]}_{f,l,k})
\]

such that

\[
p_s(D_{f,l,k}^+(w_1 a_1 \ldots a_{m+n+1} a_2)) = a_{m+1} \cdots a_{m+n+1+k-s},
\]

where \( w_1 a_1 \ldots a_{m+n+1} a_2 \in L_{m+n+1+k+1}(G) \) with \( a_j \in A_G \), \( w_1 \in L_l(G) \) and \( w_2 \in L_k(G) \), can be constructed and is onesided 1-1, nondegenerate and q-biresolving with

\[
\varphi_{T_s} = \varphi^{[k+l+1]} (\sigma_G^{[k+l+1]} s);
\]

hence \( \varphi^{[k+l+1]} (\sigma_G^{[k+l+1]} s) \) is q-biresolving for all \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\).

Proof. We have \( Q_R(\varphi) = n - k \) and \( Q_L(\varphi) = m - l \). By a proof which is similar to (but more direct than) the proof of Proposition 4.3 (take \( G \) instead of \( G_A \) and \( G_B \), and take \( G^{m+n+1} \) instead of \( G_{m+n+1} \)), we know that the textile system \( T_s \) in the theorem can be constructed and is onesided 1-1 and weakly q-biresolving with

\[
\varphi_{T_s} = \varphi^{[k+l+1]} (\sigma_G^{[k+l+1]} s)
\]

for \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\).

Since \( \varphi_{ps} \) and \( \varphi_{q_{f,l,k}} \) is onto, \( T_s \) is nondegenerate for \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\).

Since \( G \) is irreducible and \( \varphi_{ps} \) is one-to-one and onto, \( G_{f,l,k}^+ \) is irreducible and has the same spectral radius as \( G \). Therefore, since \( q_{f,l,k}^+ \) is weakly q-biresolving, it follows from Lemma 2.1(2) that \( q_{f,l,k}^+ \) is q-biresolving, so that \( T_s \) is q-biresolving for \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\). \[ \square \]

Let \( \varphi \) be an endomorphism of a subshift \((X, \sigma)\). Deciding the \( \sigma \)-expansiveness situation of \( \varphi \) means deciding which one of the following is the case for \( \varphi \):

1. \( \varphi \) is expansive;
2. \( \varphi \) is left \( \sigma \)-expansive but not right \( \sigma \)-expansive;
3. \( \varphi \) is right \( \sigma \)-expansive but not left \( \sigma \)-expansive;
4. \( \varphi \) is neither left \( \sigma \)-expansive nor right \( \sigma \)-expansive.

Here we note that \( [N9] \) the proof of Proposition 8.3(2) proves:

Theorem 4.5 \( [N9] \). An onto endomorphism of a subshift is positively expansive if and only if it is expansive and essentially weakly q-biresolving.
We also recall the following.

If \( \varphi \) is a positively expansive endomorphism of an irreducible topological Markov shift \((X, \sigma)\), then \((X, \varphi)\) is conjugate to a onesided topological Markov shift [Ku]. Hence by [N5] Theorem 3.9(1),(3), we see that an onto endomorphism of an irreducible topological Markov shift is positively expansive if and only if it is expansive and essentially \( q \)-biroyesolving.

If \( \varphi \) is a positively expansive endomorphism of an irreducible topological Markov shift \((X, \sigma)\), then \( \varphi \) is biclosing [Ku] and hence exactly \( c \)-to-one with some positive integer \( c \) [N2]: if \( \sigma \) is topologically mixing in addition, \((X, \varphi)\) is topologically-conjugate to the onesided full \( c \)-shift (see [N7] p. 4083).

**Theorem 4.6.** Let \( \varphi \) be an onto endomorphism of an irreducible topological Markov shift \((X, \sigma)\).

1. \( Q_R(\varphi) + Q_L(\varphi) \geq 0 \) if and only if there exists \( s \in \mathbb{Z} \) such that \( \varphi \sigma^s \) is \( q \)-biroyesolving up to higher block conjugacy; for \( s \in \mathbb{Z} \), \( \varphi \sigma^s \) is \( q \)-biroyesolving up to higher-block conjugacy if and only if \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\), and if \(-Q_R(\varphi) < s < Q_L(\varphi)\), then \( \varphi \sigma^s \) is positively-expansive;
2. If \( Q_R(\varphi) + Q_L(\varphi) \geq 0 \), then for \( s = -Q_R(\varphi), Q_L(\varphi)\), we can decide the \( \sigma \)-expansiveness situation of \( \varphi \sigma^s \) and hence we can decide whether \( \varphi \sigma^s \) is positively expansive or not.
3. If \( Q_R(\varphi) + Q_L(\varphi) \geq 0 \) and \( \varphi \) is \( c \)-to-one with \( c \in \mathbb{N} \), then the topological entropy \( h(\varphi \sigma^s) \) of \( \varphi \sigma^s \) is equal to \( \log c \) for all \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\), in particular we have \( h(\varphi \sigma^{-Q_R(\varphi)}) = h(\varphi \sigma^{Q_L(\varphi)}) = \log c \).

**Proof.** (1) By Proposition 3.5 and Theorems 4.2 and 4.4. \( \varphi \sigma^s \) is \( q \)-biroyesolving up to higher-block conjugacy if and only if \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\). It follows from Theorem 4.4 and [N9] Propositions 7.6(2) and 8.3(2) that if \(-Q_R(\varphi) < s < Q_L(\varphi)\), then \( \varphi \sigma^s \) is positively-expansive.

(2) Let \(-Q_R(\varphi) \leq s \leq Q_L(\varphi)\) (we are interested in only \( s = -Q_R(\varphi), Q_L(\varphi) \) here). We can construct a onesided \( L \)-1, nondegenerate, \( q \)-biroyesolving textile system \( T_s = (p_s, q_s : \varphi_s \rightarrow G^{[d]}) \) such that \( \varphi_T = (\varphi \sigma^t)^{[t]} \) for \( t \geq 1 \) which we can effectively find. Since the dual textile system \( T^*_s \) is LL, we can decide whether \( \xi \) is one-to-one or not and whether \( \eta_T \) is one-to-one or not. Hence the result follows from [N5] Theorem 2.11(1)) and [N9] Proposition 6.2.

(3) Notation being the same as above, let \( G^*_s \) be the graph such that \( T^*_s \) is defined over it. Since \( T^*_s \) is LL, we have \( \sigma_{T^*_s} = \sigma_{G^*_s} \). Therefore, using [N5] Theorem 2.13, Corollary 2.14 we have

\[
\begin{align*}
\tau(\varphi \sigma^s) &= \tau((\varphi \sigma^s)^{[t]}) = \tau(\varphi_{T^*_s}) = \tau(\sigma_{T^*_s}) = \tau(\sigma_{G^*_s}).
\end{align*}
\]

It follows from [N7] Proposition 3.1(1)] that the spectral radius of \( G^*_s \) is equal to \( c \). Therefore (3) is proved. \( \square \)

Next we shall see a construction method of positively expansive endomorphisms of full shifts. “Positively expansive endomorphisms of full-shifts” is not a very restricted subject, because Boyle and Maass [BoMa] proved that any mixing topological Markov shift having a positively expansive \( c \)-to-one endomorphism is shift equivalent to some full \( d \)-shift with \( c \) and \( d \) divisible by the same primes (they also conjectured that in this, “shift equivalent” can be replaced by topologically conjugate”) (see [N7] p. 4083).
Let $A$ be an alphabet. Let $f : A^{N+1} \to A$ and $f' : A^{N'+1} \to A$ be local rules with $N, N' \geq 0$. Let $t$ be an integer with $t \leq \min\{N, N'\}$. Let $\pi : A \times A \to A$ be a bipermutation (i.e. for any $a, a', b, b' \in A$, if $a \neq a'$ then $\pi(a, b) \neq \pi(a', b)$), and if $b \neq b'$ then $\pi(a, b) \neq \pi(a, b')$). Let $g : A^{N+N'-t+1} \to A$ be the local rule defined by

$$g(a_1 \ldots a_{N+N'-t+1}) = \pi(f'(w'), f(w)),$$

where $w'$ and $w$ are the $(N' + 1)$-prefix and the $(N + 1)$-suffix, respectively, of $a_1 \ldots a_{N+N'-t+1}$ with $a_j \in A$. We call $g$ the local rule defined by $(f', f)$, $t$ and $\pi$.

Note that this definition is possible, because $N + N' - t \geq \max\{N, N'\}$ by our hypothesis.

Let $f : A^{N+1} \to A$ be a local rule which gives an onto block-map. Let $R(f)$ (respectively, $L(f)$) be the cardinality of a maximal right (respectively, left) compatible-set for the graph-homomorphism $q_f : G_A^{\{N+1\}} \to G_A$ (see Subsection 2.2 and [11 Section 14]).

**Lemma 4.7.** Let $A$ be an alphabet. Let $f : A^{N+1} \to A$ and $f' : A^{N'+1} \to A$ be local rules with $N, N' \geq 0$. Let $g : A^{N+N'-t+1} \to A$ be the local rule defined by $(f', f)$, an integer $t$ and a bipermutation $\pi : A \times A \to A$. Let $k, l' \geq 0$.

1. If $f$ is strictly $k$ right-mergible and $t < N - k$, then $g$ is strictly $k$ right-mergible and $R(g) = R(f)$.
2. If $f'$ is strictly $l'$ left-mergible and $t < N' - l'$, then $g$ is strictly $l'$ left-mergible and $L(g) = L(f')$.

**Proof.** (1) Let $\bar{N} = N + N' - t$. Let $(a_j)_{j \in \mathbb{Z}}$ and $(b_j)_{j \in \mathbb{Z}}$ be any two points in $A^\mathbb{Z}$ with $a_j, b_j \in A$ such that $(a_j)_{j \leq 0} = (b_j)_{j \leq 0}$. Then for $j \geq 1$

$$g(a_{j-\bar{N}} \ldots a_j) = \pi(f'(a_{j-\bar{N}} \ldots a_j), f(a_{j-N} \ldots a_j)),$$

$$g(b_{j-\bar{N}} \ldots b_j) = \pi(f'(b_{j-\bar{N}} \ldots b_j), f(b_{j-N} \ldots b_j)).$$

Since $a_j = b_j$ for all $j \leq 0$ and

$$k + 1 - \bar{N} + N' = t - (N - k) + 1 \leq 0 \quad \text{by assumption},$$

we have $a_{j-\bar{N}} \ldots a_j = b_{j-\bar{N}} \ldots b_j$, for all $j = 1, \ldots, k + 1$. Therefore, for all $j = 1, \ldots, k+1$, $f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j)$ if and only if $g(a_{j-\bar{N}} \ldots a_j) = g(b_{j-\bar{N}} \ldots b_j)$. Hence if $g(a_{j-\bar{N}} \ldots a_j) = g(b_{j-\bar{N}} \ldots b_j)$ for $j = 1, \ldots, k + 1$, then $f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j)$ for $j = 1, \ldots, k + 1$, so that we have $a_1 = b_1$ by assumption. Hence $g$ is $k$ right-mergible. Also by assumption, there exist two points $(a_j)_{j \in \mathbb{Z}}, (b_j)_{j \in \mathbb{Z}} \in A^\mathbb{Z}$ such that $(a_j)_{j \leq 0} = (b_j)_{j \leq 0}, a_1 \neq b_1$ and $f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j)$ for $j = 1, \ldots, k$. Since for these points, $g(a_{j-\bar{N}} \ldots a_j) = g(b_{j-\bar{N}} \ldots b_j)$ for $j = 1, \ldots, k$, we conclude that $g$ is strictly $k$ right-mergible.

Let $a_{-\bar{N}+1} \ldots a_k$ with $a_j \in A$. Let $C = S^q_{g}((a_{-N+1} \ldots a_0, f(a_{-N+1} \ldots a_k))$ and let $D = S^q_{g}((a_{-\bar{N}+1} \ldots a_0, g(a_{-\bar{N}+1} \ldots a_k))$. Since $f$ and $g$ are $k$ right-mergible, it follows from [N2 Lemma 5.3] that $C$ is a maximal right compatible-set for $q_f$ and $D$ is a maximal right compatible-set for $q_g$. Let

$$E = \{a_{-\bar{N}+1} \ldots a_kw \mid w \in A^k, g(a_{-\bar{N}+1} \ldots a_0w) = g(a_{-\bar{N}+1} \ldots a_k)\},$$

$$F = \{a_{-\bar{N}+1} \ldots a_kw \mid w \in A^k, f(a_{-\bar{N}+1} \ldots a_0w) = f(a_{-\bar{N}+1} \ldots a_k)\}.$$

For the same reason as in the above, we see that $E = F$. Since the cardinality of $E$ is equal to that of $D$ and the cardinality of $F$ is equal to that of $C$, we have $R(g) = R(f)$.
(2) By symmetry, (2) follows from (1).

**Theorem 4.8.** Let $A$ be an alphabet with cardinality $|A|$. Let $N, N', k, l' \geq 0$. Suppose that $f : A^{N+1} \rightarrow A$ and $f' : A^{N'+1} \rightarrow A$ are local rules with $f$ strictly $k$ right-mergible and with $f'$ strictly $l'$ left-mergible. Let $t$ be an integer such that
\[ t < \min\{N - k, N' - l'\}. \]

Let $g : A^{N+N'-t+1} \rightarrow A$ be the local rule defined by $(f', f)$, $t$ and a bipermutation $\pi : A \times A \rightarrow A$. Let $\phi$ be the endomorphism of $(0, N + N' - t)$-type of the full-shift $(A^Z, \sigma_A)$ given by $g$. Then $\phi$ is strictly $k$ right-mergible and strictly $l'$ left-mergible,
\[ Q_R(\phi) + Q_L(\phi) > 0 \]
and $\phi$ is $c$-to-one with $c = |A|^{N+N'-t}/(L(f')R(f))$.

**Proof.** By Lemma 4.7, $g$ is strictly $k$ right-mergible and strictly $l'$ left-mergible. Therefore $Q_L(\phi) = 0 - l'$ and $Q_R(\phi) = N + N' - t - k$, so that
\[ Q_R(\phi) + Q_L(\phi) = N + N' - k - l' - t > 0. \]

The remainder follows from Lemma 4.7 and [H] Theorem 14.9, Section 16] (see [NT], Section 5).

Recalling Theorem 4.6(1), we know that the theorem above presents a method to obtain positively expansive endomorphisms of any full shift from any pair of a left-closing endomorphism and a right-closing endomorphism of the shift. In particular, we can obtain positively expansive endomorphism of any full shift from any biclosing endomorphism of the shift by expanding its neighborhood size sufficiently large and using a bipermutation.

5. **Weakly $q$-Resolving Endomorphisms of Subshifts**

Let $(X, \sigma)$ be a subshift. For $s \geq 1$, we define a graph $G[X, s]$ as follows.

$G[X, 1] = G_A$, where $A = L_1(X)$; if $s \geq 2$, then $A_{G[X, s]} = L_s(X)$, $V_{G[X, s]} = L_{s-1}(X)$, and for each $w = a_1 \ldots a_s \in A_{G[X, s]}$ with $a_1 \in A$, $i_G[X, s]$ and $v_G[X, s]$ map $w$ to $a_1 \ldots a_{s-1}$ and $a_2 \ldots a_s$, respectively. Let $(X^{(s)} \sigma_X^{(s)})$ be the SFT over $A$ such that $(X^{(s)})[s], (\sigma_X^{(s)})[s]) = (X_G[X, s], \sigma_G[X, s])$. That is, $(X^{(s)}, \sigma_X^{(s)})$ is the SFT with $A^s \setminus L_s(X)$ as the “set of forbidden words” that defines it. We call $(X^{(s)}, \sigma_X^{(s)})$ the approximation SFT of order $s$ of $(X, \sigma)$. If $f : L_{N+1}(X) \rightarrow A$ is a local rule (on $(X, \sigma)$) with $s \geq N + 1$, then $f$ is a local rule on $(X^{(s)}, \sigma_X^{(s)})$ to $(X^{(s-N)}, \sigma_X^{(s-N)})$. If $\phi$ is the endomorphism of $(m, n)$-type of $(X, \sigma)$ given by a local rule $f : L_{N+1}(X) \rightarrow A$ with $m, n \geq 0$, $m + n = N$, then for each $s \geq N + 1$, $f$ gives a homomorphism of $(m, n)$-type
\[ \phi_0^{(s, \phi)} : (X^{(s)}, \sigma^{(s)}) \rightarrow (X^{(s-N)}, \sigma_X^{(s-N)}), \]
which is an extension of $\phi$. Let $\phi^{(s, \phi)}$ be the factor map induced by $\phi_0^{(s, \phi)}$. Then $\phi^{(s, \phi)}$ is an extension of $\phi$, of $(m, n)$-type and given by $f$. We call $\phi^{(s, \phi)}$ the approximation factor map of order $s$ of $\phi$.

**Lemma 5.1.** Let $(X, \sigma)$ be a subshift. Let $f : L_{N+1}(X) \rightarrow L_1(X)$ be a local rule with $N \geq 0$. Let $\phi$ be an endomorphism of a subshift $(X, \sigma)$ given by $f$. 

(1) If \( f \) is strictly \( k \) right-mergible (respectively, strictly \( k \) left-mergible) with \( k \geq 0 \) as a local rule on \((X, \sigma)\), then there exists an integer \( J \geq N + 1 \) such that \( f \) is strictly \( k \) right-mergible (respectively, strictly \( k \) left-mergible) as a local rule on \((X^{(J)}, \sigma_{X^{(J)}})\) to its image under \( \phi^{(J, \varphi)} \).

(2) There exists an integer \( J \geq N + 1 \) such that \( Q_R(\phi^{(J, \varphi)}) = Q_R(\varphi) \); there exists an integer \( J \geq N + 1 \) such that \( Q_L(\phi^{(J, \varphi)}) = Q_L(\varphi) \).

Proof. (1) Since \( f \) is not \( k - 1 \) right-mergible as a local rule on \((X, \sigma)\), there exist \((a_j)_{j \in \mathbb{Z}}, (b_j)_{j \in \mathbb{Z}} \in X \) such that \((a_j)_{j \leq 0} = (b_j)_{j \leq 0}, a_1 \neq b_1, \text{ and } f(a_{j-N} \ldots a_j) = f(b_{j-N} \ldots b_j) \) for \( j = 1, \ldots, k \). Therefore, since \( X^{(s)} \supset X \), \( f \) is not \( k - 1 \) right-mergible as a local rule on \((X^{(s)}, \sigma_{X^{(s)}})\) to its image under \( \phi^{(s, \varphi)} \) for all \( s \geq N + 1 \).

Assume that there exist no \( J \geq N + 1 \) such that \( f \) is \( k \) right-mergible as a local rule on \((X^{(J)}, \sigma_{X^{(J)}})\). Then for each \( s \geq N + 1 \), there exists a pair of points \((c_j^{(s)})_{j \in \mathbb{Z}} \) and \((d_j^{(s)})_{j \in \mathbb{Z}} \) in \( X^{(s)} \) such that \((c_j^{(s)}_{j \leq 0}) = (d_j^{(s)}_{j \leq 0}), c^{(s)}_1 \neq d^{(s)}_1 \) and \( f(c_j^{(s)}_{j-N} \ldots c_j^{(s)}) = f(d_j^{(s)}_{j-N} \ldots d_j^{(s)}) \) for \( j = 1, \ldots, k + 1 \). Since \( X^{(s+1)} \subset X^{(s)} \) and \( X \equiv \cap_{s \geq N + 1} X^{(s)} \) with each \( X^{(s)} \) closed, a standard compactness argument shows that there exist points \((c_j)_{j \in \mathbb{Z}} \) and \((d_j)_{j \in \mathbb{Z}} \) in \( X \) such that \((c_j)_{j \leq 0} = (d_j)_{j \leq 0}, c_1 \neq d_1 \) and \( f(c_{j-N} \ldots c_j) = f(d_{j-N} \ldots d_j) \) for \( j = 1, \ldots, k + 1 \). This contradicts the hypothesis that \( f \) is \( k \) right mergible.

Hence the first version of (1) is proved. The second version of (1) follows from the first one by symmetry.

(2) Suppose that \( \varphi \) is of \((m, n)\)-type with \( m, n \geq 0, m + n = N \). Suppose that \( f \) is strictly \( k \) right-mergible. Then by (1), there exists \( J \geq N + 1 \) such that \( f \) is strictly \( k \) right-mergible as a local rule on \((X^{(J)}, \sigma_{X^{(J)}})\) to its image under \( \phi^{(J, \varphi)} \). Therefore, since \( \phi^{(J, \varphi)} \) is of \((m, n)\)-type and is given by \( f \), it follows that \( Q_R(\varphi) = Q_R(\phi^{(J, \varphi)}) \).

The proof for the existence of \( J \) with \( Q_L(\varphi) = Q_L(\phi^{(J, \varphi)}) \) is similar. \( \square \)

Theorem 5.2. Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). Let \( s \in \mathbb{Z} \).

(1) \( \varphi \) is weakly \( q-R \) if and only if \( Q_R(\varphi) \geq 0 \); moreover, \( \varphi^* \) is weakly \( q-R \) if and only if \( \varphi \geq -Q_R(\varphi) \); if \( \varphi \) is \( q-R \), then \( \varphi^* \) is left \( \sigma \)-expansive.

(2) \( \varphi \) is weakly \( q-L \) if and only if \( Q_L(\varphi) \geq 0 \); moreover, \( \varphi^* \) is weakly \( q-L \) if and only if \( \varphi \leq Q_L(\varphi) \); then \( \varphi^* \) is right \( \sigma \)-expansive.

Proof. (1) Suppose that \( \varphi \) is weakly \( q-R \). Then there exists a onesided 1-1 textile subsystem \( U \) of a weakly \( q-R \) textile system \( T = (p, q : \Gamma \rightarrow G) \) such that \((X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi) \). There exist nonnegative integers \( m, n \) and a local rule \( g : L_{m+n+1}(X_U) \rightarrow L_1(Z_U) \) such that \( \xi_U^{-1} : X_U \rightarrow Z_U \) is a block-map of \((m, n)\)-type given by \( g \). Let \( f = q_A g \). Then \( \varphi \) is a block-map of \((m, n)\)-type given by the local rule \( f \). By the same proof as in the proof of Proposition 4.1, we see that \( f \) is \( n \) right-mergible and hence \( Q_R(\varphi) \geq 0 \).

Suppose that \( Q_R(\varphi) \geq 0 \). By Lemma 5.1, there exists \( J \in \mathbb{N} \) such that \( Q_R(\phi^{(J, \varphi)}) = Q_R(\varphi) \). Since \( Q_R(\phi^{(J, \varphi)}) \geq 0 \), it follows from Proposition 4.1 that the factor map \( \phi^{(J, \varphi)} \) is weakly \( q-R \). Therefore, there exists a onesided 1-1 textile relation system \( T = (p : \Gamma \rightarrow G_A, q : \Gamma \rightarrow G_A) \) such that \( q \) is weakly right-resolving and \( \phi_T = \phi^{(J, \varphi)} \), where \( A = L_1(X^{(J)}) = L_1(X) \). We can regard \( T \) as a textile system \( T = (p, q : \Gamma \rightarrow G_A) \), which is weakly \( q-R \). Let \( Z = \xi_T^{-1}(X) \). Then, since \( \eta_T(Z) = \eta_T \xi_T^{-1}(X) = \phi_T(X) = \varphi(X) = X = \xi_T(Z) \) and \( \xi_T \) is one-to-one, there
exists a unique onesided 1-1 textile subsystem $U$ of $T$ with $Z_U = Z$. Since $\varphi = \varphi_U$, we conclude that $\varphi$ is weakly $q$-R.

We have proved that $\varphi$ is weakly $q$-R if and only if $Q_R(\varphi) \geq 0$. From this and Proposition 3.4 it follows that $\varphi \ast$ is weakly $q$-R if and only if $s \geq -Q_R(\varphi)$. If $s \geq -Q_R(\varphi) + 1$, then $Q_R(\varphi \ast) \geq 0$. Hence $\varphi \ast$ is weakly $q$-R, so that $\varphi \ast$ is left $\sigma$-expansive, by [N9 Proposition 7.6(2)].

(2) By symmetry, (2) follows from (1). □

**Theorem 5.3.** Let $\varphi$ be an onto endomorphism of a subshift $(X, \sigma)$. Let $s \in \mathbb{Z}$.

1. $\varphi^{[t]}$ is weakly $q$-biresolving with some $t \geq 1$ if and only if $Q_R(\varphi) \geq 0$ and $Q_L(\varphi) \geq 0$.

2. If $\varphi^{[t]}(\sigma^{[t]})^s$ is weakly $q$-biresolving with some $t \geq 1$, then

$$-Q_R(\varphi) \leq s \leq Q_L(\varphi).$$

3. If $Q_R(\varphi) + Q_L(\varphi) \geq 0$, then there exists $t \geq 1$ such that $\varphi^{[t]}(\sigma^{[t]})^s$ is weakly $q$-biresolving for all $-Q_R(\varphi) \leq s \leq Q_L(\varphi)$.

4. If $Q_R(\varphi) + Q_L(\varphi) \geq 0$, then $\varphi \ast$ is essentially weakly $q$-biresolving for $-Q_R(\varphi) \leq s \leq Q_L(\varphi)$, and positively expansive for $-Q_R(\varphi) < s < Q_L(\varphi)$.

**Proof.** It follows from Theorem 5.2 and Proposition 3.5 that if there exists $t \geq 1$ such that $\varphi^{[t]}$ is weakly $q$-biresolving, then $Q_R(\varphi) = Q_R(\varphi^{[t]}) \geq 0$ and $Q_L(\varphi) = Q_L(\varphi^{[t]}) \geq 0$. Hence the “only-if” part of (1) is proved. By this and Proposition 3.4, (2) is proved. Since the “if” part of (1) follows from (3) and (4) is proved by (3) and [N9 Propositions 7.6(2) and 8.3(2)], we prove (3) below.

Suppose that $Q_R(\varphi) + Q_L(\varphi) \geq 0$. By Lemma 5.1, there exists $J \in \mathbb{N}$ such that $Q_R(\varphi^{(J)}) = Q_R(\varphi)$ and $Q_L(\varphi^{(J)}) = Q_L(\varphi)$. Since $Q_R(\varphi^{(J)}) + Q_L(\varphi^{(J)}) \geq 0$, it follows from Proposition 4.3 that for some $t \geq 1$, we have a onesided 1-1, weakly $q$-biresolving textile-relation system

$$T_s = (p_s : \Gamma_0 \to G^{[t]}_A, q : \Gamma_0 \to G^{[t]}_A) \quad \text{with} \quad A = L_1(X)$$

such that $\phi^{(J)}(\sigma^{(J)})^t \mid_{(X^{[t]}, \sigma^{[t]})^t} \leq Q_R(\varphi^{(J)}) \leq s \leq Q_L(\varphi^{(J)})$. We can regard $T_s$ as a textile system $T_s = (p_s, q : \Gamma_0 \to G^{[t]}_A)$, which is onesided 1-1 and weakly $q$-biresolving. Let $Z_s = \xi_{T_s}^{-1}(X^{[t]})$. Then, since $\eta_{T_s}(Z_s) = \eta_{T_s}^{-1}(X^{[t]}) = \phi^{(J)}(\sigma^{(J)})^t(X^{[t]}) = X^{[t]} = \xi_{T_s}(Z_s)$ and $\xi_{T_s}$ is one-to-one, there exists a unique onesided 1-1 textile subsystem $U_s$ of $T_s$ with $Z_{U_s} = Z_s$. Since $\varphi^{[t]}(\sigma^{[t]})^s$ is weakly $q$-biresolving, we conclude that $\varphi^{[t]}(\sigma^{[t]})^s$ is weakly $q$-biresolving. □

**6. p-RESOLVING DEGREES**

Let $N \geq 0$. Let $f : L_{N+1}(X) \to L_1(Y)$ be a local rule on a subshift $(X, \sigma_X)$ to another $(Y, \sigma_Y)$. Let $I \geq 0$. We say that $f$ is $I$ left-redundant if for any points $(a_j)_{j \in \mathbb{Z}}$ and $(b_j)_{j \in \mathbb{Z}}$ in $X$ with $a_j, b_j \in L_1(X)$, it holds that if $(a_j)_{j \geq 0} = (b_j)_{j \geq 0}$, then $f(a_{-1} \ldots a_{-I+N}) = f(b_{-1} \ldots b_{-I+N}).$ Symmetrically, $f$ is said to be $I$ right-redundant if for any points $(a_j)_{j \in \mathbb{Z}}$ and $(b_j)_{j \in \mathbb{Z}}$ in $X$ with $a_j, b_j \in L_1(X)$, it holds that if $(a_j)_{j \leq 0} = (b_j)_{j \leq 0}$, then $f(a_{I-N} \ldots a_I) = f(b_{I-N} \ldots b_I).$ We say that $f$ is strictly $I$ left-redundant if it is $I$ left-redundant but not $I + 1$ left-redundant. Similarly, a strictly $I$ right-redundant local rule is defined.

We remark that if $f$ is $I$ left-redundant (respectively, $I$ right-redundant), then for all $0 \leq I' \leq I$, $f$ is $I'$ left-redundant (respectively, $I'$ right-redundant).
Definition 6.1. Let \( m, n \geq 0 \) with \( m + n = N \). Let \( I, J \geq 0 \). Let \( \phi : (X, \sigma_X) \rightarrow (Y, \sigma_Y) \) be a homomorphism between subshifts. Let \( \phi \) be of \((m, n)\)-type given by a local rule \( f : L_{N+1}(X) \rightarrow L_1(Y) \) which is strictly \( I \) left-redundant and strictly \( J \) right-redundant. We define \( p-L \) degree \( P_L(\phi) \) of \( \phi \) and the \( p-R \) degree \( P_R(\phi) \) of \( \phi \) by

\[
P_L(\phi) = I - m, \quad P_R(\phi) = J - n.
\]

The following lemma is obvious.

Lemma 6.2. Let \( N, s, t, I, J \) be nonnegative integers. Let \( f : L_{N+1}(X) \rightarrow L_1(Y) \) be a local rule on a subshift \((X, \sigma_X)\) to another \((Y, \sigma_Y)\). Let \( g : L_{N+s+t+1}(X) \rightarrow L_1(Y) \) be the mapping such that

\[
g(a_{-s}a_{-s+1} \ldots a_{N+t}) = f(a_0 \ldots a_N),
\]

where \( a_{-s} \ldots a_{N+t} \in L_{N+s+t+1}(X) \) with \( a_j \in L_1(X) \). Then \( g \) is a local rule on \((X, \sigma_X)\) to \((Y, \sigma_Y)\), if \( f \) is strictly \( I \) left-redundant, then \( g \) is strictly \( I + s \) left-redundant, and if \( f \) is strictly \( J \) right-redundant, then \( g \) is strictly \( J + t \) right-redundant.

Proposition 6.3. For a homomorphism \( \phi \) between subshifts, \( P_L(\phi) \) and \( P_R(\phi) \) are uniquely determined by \( \phi \).

Proof. Suppose that for \( i = 1, 2 \), \( \phi \) is a homomorphism of \((m_i, n_i)\)-type of a subshift \((X, \sigma_X)\) into another \((Y, \sigma_Y)\) given by a local-rule \( f_i : L_{m_i+n_i+1}(X) \rightarrow L_1(Y) \) which is strictly \( I_i \) left-redundant and strictly \( J_i \) right-redundant. Let \( m = \max\{m_1, m_2\} \) and let \( n = \max\{n_1, n_2\} \). For \( i = 1, 2 \), let \( g_i : L_{m+n}(X) \rightarrow L_1(Y) \) be the local rule such that

\[
g_i(a_{-m} \ldots a_n) = f_i(a_{-m_i} \ldots a_{n_i}), \quad a_{-m} \ldots a_n \in L_{m+n+1}(X), \quad a_j \in L_1(X).
\]

Then since \( \phi \) is of \((m_i, n_i)\)-type and given by \( f_i \), \( \phi \) is of \((m, n)\)-type and given by \( g_i \), for \( i = 1, 2 \). Since \( f_i \) is strictly \( I_i \) left-redundant and strictly \( J_i \) right-redundant, it follows from Lemma 6.2 that \( g_i \) is strictly \( I_i + (m - m_i) \) left-redundant and strictly \( J_i + (n - n_i) \) right-redundant, for \( i = 1, 2 \). Since \( g_1 : L_{m+n+1}(X) \rightarrow L_1(Y) \) and \( g_2 : L_{m+n+1}(X) \rightarrow L_1(Y) \) give the same block-map \( \phi \) of \((m, n)\)-type, it follows that \( g_1 = g_2 \). Hence \( I_1 + (m - m_1) = I_2 + (m - m_2) \) and \( J_1 + (n - n_1) = J_2 + (n - n_2) \). Thus we have \( I_1 - m_1 = I_2 - m_2 \) and \( J_1 - n_1 = J_2 - n_2 \).

Proposition 6.4. Let \( \phi \) be a homomorphism of a subshift \((X, \sigma_X)\) into another \((Y, \sigma_Y)\). Let \( s \in \mathbb{Z} \). Then

\[
P_L(\phi \sigma_X^s) = P_L(\phi) + s \quad \text{and} \quad P_R(\phi \sigma_X^s) = P_R(\phi) - s
\]

and hence \( P_R(\phi) + P_L(\phi) \) is shift-invariant.

Proof. Suppose that \( \phi \) is of \((m, n)\)-type given by a local rule \( f : L_{m+n+1}(X) \rightarrow L_1(Y) \). Increasing the redundancies of \( f \) if necessary, we may assume that \( m, n \geq |s| \), by Proposition 6.3. Since \( \phi \sigma_X^s \) is of \((m - s, n + s)\)-type and given by \( f \), the proposition follows.

As is easily seen, the following lemma holds:

Proposition 6.5. Let \((X, \sigma_X), (Y, \sigma_Y)\) be subshifts. Let \( N, I, J \geq 0 \) and let \( t \geq 1 \).
(1) If a local rule \( f : L_{N+1}(X) \to L_1(Y) \) on \((X, \sigma_X)\) to \((Y, \sigma_Y)\) is strictly I left-redundant (respectively, strictly \(J\) right-redundant), then the higher block local rule \( f^{[i]} \) is strictly I left-redundant (respectively, strictly \(J\) right-redundant).

(2) If \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) is a homomorphism, then \( P_L(\phi) = P_L(\phi^{[i]}) \) and \( P_R(\phi) = P_R(\phi^{[i]}) \).

As is easily seen, the following lemma holds:

**Lemma 6.6.** Let \((X, \sigma_X), (Y, \sigma_Y),\) and \((Z, \sigma_Z)\) be subshifts. Let \( f \) and \( g \) be local rules on \((X, \sigma_X)\) to \((Y, \sigma_Y)\) and on \((Y, \sigma_Y)\) to \((Z, \sigma_Z)\), respectively. If \( f \) is I left-redundant (\(J\) right-redundant) and \( g \) is \(J'\) left-redundant (\(J'\) right-redundant), then \(gf\) is \(I+J'\) left-redundant (\(J+J'\) right-redundant).

Using Lemma 6.6, we have:

**Proposition 6.7.** Let \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) and \( \psi : (Y, \sigma_Y) \to (Z, \sigma_Z) \) be homomorphisms between subshifts. Then

\[
P_L(\psi \phi) \geq P_L(\phi) + P_L(\psi) \quad \text{and} \quad P_R(\psi \phi) \geq P_R(\phi) + P_R(\psi).
\]

The following remark is easily seen.

**Remark 6.8.** Let \( f : L_{N+1}(X) \to L_1(Y) \) be a local rule on a topological Markov shift \((X_G, \sigma_G)\) to a sofic system \((Y, \sigma_Y)\) with \( N \geq 0 \). Let \( 0 \leq I \leq N \). Then \( f \) is I left-redundant (respectively, I right-redundant) if and only if \( f(w_1) = f(w_2) \) for any \( w_1, w_2 \in L_{N+1}(G) \) with the same terminal subpath (respectively, the same initial subpath) of length \( N + 1 - I \) in \( G \).

**Proposition 6.9.** Let \( \varphi \) be an onto endomorphism of a topological Markov shift \((X, \sigma)\). The following statements are equivalent.

1. \( P_L(\varphi) \geq 0 \) (respectively, \( P_R(\varphi) \geq 0 \)).
2. \( \varphi \) is p-L (respectively, p-R).
3. \( \varphi \) has memory zero (respectively, anticipation zero).

**Proof.** The equivalence of (2) and (3) is found as [N9] Proposition 5.1(1).

Suppose that \( \varphi \) is of \((m, n)\)-type with \( m, n \geq 0 \) and given by a local rule \( f : L_{m+n+1}(X) \to L_1(Y) \) which is strictly I left-redundant. If \( m = 0 \), then \( P_L(\varphi) = I - 0 \geq 0 \). Conversely, if \( P_L(\varphi) \geq 0 \), then \( I - m \geq 0 \) and hence we can delete the left-redundancy of \( f \) by \( m \) to obtain a local rule \( f' : L_{n+1}(X) \to L_1(Y) \) so that \( \varphi \) may be of \((0, n)\)-type and given by \( f' \). Therefore (1) and (3) are equivalent. \( \square \)

**Lemma 6.10.** Let \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) be a factor map of an SFT onto a sofic system. If \( P_L(\phi) \geq 0 \) (respectively, \( P_R(\phi) \geq 0 \)), then \( \phi \) is weakly p-L (respectively, weakly p-R).

**Proof.** Suppose that \( P_L(\varphi) \geq 0 \). Let \( A = L_1(X) \) and \( B = L_1(Y) \). Since \((X, \sigma_X)\) is an SFT, there exists \( s \geq 1 \) such that \((X^{[s]}, \sigma_X^{[s]})\) is the topological Markov shift whose defining graph, say \( G_s \), is defined as follows: \( A_{G_s} = L_s(X) \): \( V_{G_s} = L_{s-1}(X) \); for \( w = a_1 \ldots a_s \in L_s(X) \) with \( a_j \in A \), \( t_{G_s}(w) = a_1 \ldots a_{s+1} \) and \( i_{G_s}(w) = a_2 \ldots a_s \). Suppose that \( \phi \) is of \((m, n)\)-type and given by a strictly I left-redundant local rule \( f \) with \( m, n, I \geq 0 \). Then \( \phi^{[s]} : (X^{[s]}, \sigma_X^{[s]}) \to (Y^{[s]}, \sigma_Y^{[s]}) \) is of \((m, n)\)-type and given by the local rule \( f^{[s]} \), which is strictly I left-redundant, by Proposition 6.5. Since \((X^{[s]}, \sigma_X^{[s]}) = (X_{G_s}, \sigma_{G_s}) \) and \( I - m = P_L(\phi) \geq 0 \), it follows from
Remark 6.8 that we can delete the left-redundancy of \( f^{[s]} \) by \( m \) to obtain a local rule \( g : L_{n+1}(\mathcal{G}_s) \to L_1(\mathcal{G}^{[s]}_B) \) so that \( \phi^{[s]} \) may be of \((0, n)\)-type and given by \( g \).

Let \( T = (p : G_s^{[n+1]} \to G_A, q : G_s^{[n+1]} \to G_B) \) be the textile relation system such that if \( g(\alpha_0 \ldots \alpha_n) = \beta \), where \( \alpha_0 \ldots \alpha_n \in L_{n+1}(\mathcal{G}_s) \) with \( \alpha_j = a_j \ldots a_{j+s-1} \in \mathcal{A}G_s \) for \( j = 0, \ldots, n \) with each \( a_j \in \mathcal{A} \) and \( \beta = b_0 \ldots b_{s-1} \in \mathcal{B} \), then

\[
p(\alpha_0 \ldots \alpha_n) = a_0 \quad \text{and} \quad q(\alpha_0 \ldots \alpha_n) = b_0.
\]

Then it is easy to see that \( T \) is onedimensional \(1,1\) and weakly \( p\)-L and \( \phi_T = \phi \).

By symmetry, the second version follows from the first one.

**Lemma 6.11.** Let \((X, \sigma)\) be a subshift. Let \( f : L_{N+1}(X) \to L_1(X) \) be a local rule. Let \( I \geq 0 \). Let \( \varphi \) be an endomorphism of a subshift \((X, \sigma)\) given by \( f \).

1. If \( f \) is strictly \( I \)-left-redundant (respectively, strictly \( I \)-right-redundant) as a local rule on \((X, \sigma)\), then there exists an integer \( J \geq N + 1 \) such that \( f \) is strictly \( I \)-left-redundant (respectively, strictly \( I \)-right-redundant) as a local rule on \((X^{(J)}, \sigma_{X^{(J)}})\) to its image under \( \phi^{(J, \varphi)} \).

2. There exists an integer \( J \geq N + 1 \) such that \( P_L(\phi^{(J, \varphi)}) = P_L(\varphi) \); there exists an integer \( K \geq N + 1 \) such that \( P_R(\phi^{(K, \varphi)}) = P_R(\varphi) \).

**Proof.** (1) Since \( f \) is not \( I + 1 \)-left-redundant as a local rule on \((X, \sigma)\), there exist \( (a_j)_j \in \mathbb{Z} \) and \( (b_j)_j \in \mathbb{Z} \in X \) such that \( (a_j)_j \not\geq (b_j)_j \not\geq 0 \) and \( f(a_{-I} \ldots a_{-I-1-N}) \not\geq f(b_{-I} \ldots b_{-I-1-N}) \). Therefore, since \( X^{(s)} \supset X \), \( f \) is not \( I + 1 \)-left-redundant as a local rule on \((X^{(s)}, \sigma_{X^{(s)}})\) to its image under \( \phi^{(s, \varphi)} \) for all \( s \geq N + 1 \).

Assume that there exist no \( J \geq N + 1 \) such that \( f \) is \( I \)-left-redundant as a local rule on \((X^{(J)}, \sigma_{X^{(J)}})\) to its image under \( \phi^{(J, \varphi)} \). Then for each \( s \geq N + 1 \), there exists a pair of points \((c_j^{(s)})_j \in \mathbb{Z} \) and \( (d_j^{(s)})_j \in \mathbb{Z} \) in \( X^{(s)} \) such that \( (c_j^{(s)})_j \not\geq (d_j^{(s)})_j \not\geq 0 \) and \( f(c_{-I} \ldots c_{-I+N}) \not\geq f(d_{-I} \ldots d_{-I+N}) \). Since \( X^{(s+1)} \subset X^{(s)} \) and \( X = \bigcap_{s \geq N + 1} X^{(s)} \) with each \( X^{(s)} \) closed, a standard compactness argument shows that there exist points \((c_j)_j \in \mathbb{Z} \) and \( (d_j)_j \in \mathbb{Z} \) in \( X \) such that \( (c_j)_j \not\geq (d_j)_j \not\geq 0 \) and \( f(c_{-I} \ldots c_{-I+N}) \not\geq f(d_{-I} \ldots d_{-I+N}) \). This contradicts the hypothesis that \( f \) is \( I \)-left-redundant.

Hence the first version of (1) is proved. The second version of (1) follows from the first one by symmetry.

(2) Suppose that \( \varphi \) is of \((m, n)\)-type with \( m, n \geq 0, m+n = N \). Suppose that \( f \) is strictly \( I \)-left-redundant. Then by (1), there exists \( J \geq N + 1 \) such that \( f \) is strictly \( I \)-left-redundant as a local rule on \((X^{(J)}, \sigma_{X^{(J)}})\) to its image under \( \phi^{(J, \varphi)} \). Therefore, since \( \phi^{(J, \varphi)} \) is of \((m, n)\)-type and given by \( f \), it follows that \( P_L(\varphi) = P_L(\phi^{(J, \varphi)}) \).

The proof of the existence of \( K \) with \( P_R(\varphi) = P_R(\phi^{(K, \varphi)}) \) is similar.

**Theorem 6.12.** Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). The following statements are equivalent.

1. \( P_L(\varphi) \geq 0 \) (respectively, \( P_R(\varphi) \geq 0 \)).
2. \( \varphi \) is weakly \( p\)-L (respectively, weakly \( p\)-R).
3. \( \varphi \) has memory zero (respectively, anticipation zero).

**Proof.** Assume (1). By Lemma 6.11, there exists \( J \in \mathbb{N} \) such that \( P_L(\phi^{(J, \varphi)}) = P_L(\varphi) \). Since \( P_L(\phi^{(J, \varphi)}) \geq 0 \), it follows from Lemma 6.10 that the factor map \( \phi^{(J, \varphi)} \) is weakly \( p\)-L. Therefore, there exists a onedimensional \(1,1\) textile relation system \( T = (p : \Gamma \to G_A, q : \Gamma \to G_A) \) such that \( p \) is weakly left-resolving and \( \phi_T = \phi^{(J, \varphi)} \), where \( A = L_1(X^{(J)}) = L_1(X) \). We can regard \( T \) as a textile system \( T = (p, q : \)
\( \Gamma \to G_A \), which is one-sided 1-1 and weakly \( p \)-L. Let \( Z = \xi_T^{-1}(X) \). Then, since 
\[ \eta_T(Z) = \eta_T \xi_T^{-1}(X) = \phi_T(X) = \phi(X) = X = \xi_T(Z) \]
and \( \xi_T \) is one-to-one, there exists a unique one-sided 1-1 textile subsystem \( U \) of \( T \) with \( Z_U = Z \). Since \( \varphi = \varphi_U \), we conclude that \( \varphi \) is weakly \( p \)-L. Hence (2) is proved.

The equivalence of (2) and (3) is found as [N9, Proposition 7.5(1)].

Assume (3). If \( \varphi \) is \((0,n)\) type and given by a local rule which is \( I \) left-redundant, then \( P_L(\varphi) = I \geq 0 \). Hence (1) is proved.

By symmetry, the second version follows from the first one. \( \Box \)

**Proposition 6.13.** Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\).

1. \( \varphi \sigma^s \) is weakly \( p \)-L if and only if \( s \geq -P_L(\varphi) \); if \( s > -P_L(\varphi) \), then \( \varphi \sigma^s \) is right \( \sigma \)-expansive.
2. \( \varphi \sigma^s \) is weakly \( p \)-R if and only if \( s \leq P_R(\varphi) \); if \( s < P_R(\varphi) \), then \( \varphi \sigma^s \) is left \( \sigma \)-expansive.
3. If \( X \) has infinitely many points and \( \varphi \) is of \((m,n)\) type with \( m,n \geq 0 \), then 
\[ -n \leq P_R(\varphi) \leq -P_L(\varphi) \leq m. \]
4. If \( \varphi \) is an onto endomorphism of a topological Markov shift, then \( \varphi \) is weakly \( p \)-L (respectively, weakly \( p \)-R) if and only if \( \varphi \) is \( p \)-L (respectively, \( p \)-R).

**Proof.** (1) By Theorem 6.12 and Proposition 6.4, we see that \( \varphi \sigma^s \) is weakly \( p \)-L if and only if \( s \geq -P_L(\varphi) \). If \( s \geq -P_L(\varphi) + 1 \), then \( P_L(\varphi \sigma^{s-1}) \geq 0 \), so that \( \varphi \sigma^{s-1} \) is weakly \( p \)-L. Therefore \( \varphi \sigma^s \) is right \( \sigma \)-expansive, by [N9, Proposition 7.6(1)].

(2) Similar to the above.

(3) First we prove that \( P_L(\varphi) + P_R(\varphi) \leq 0 \). Assume that \( P_R(\varphi) > -P_L(\varphi) \). Then by (1) and (2), \( \varphi \sigma^{P_R(\varphi)} \) would be weakly \( p \)-L, left \( \sigma \)-expansive and weakly \( p \)-R. This contradicts [N9, Proposition 8.2(1)]. Therefore \( P_L(\varphi) + P_R(\varphi) \leq 0 \).

Since \( \varphi \sigma^m \) has memory zero, \( P_L(\varphi \sigma^m) \geq 0 \) by Theorem 6.12 and hence \( P_L(\varphi) + m \geq 0 \) by Proposition 6.4. Since \( \varphi \sigma^{-n} \) has anticipation zero, \( P_R(\varphi \sigma^{-n}) \geq 0 \) and hence \( P_R(\varphi) + n \geq 0 \).

Therefore, (3) is proved.

(4) By [N9, Proposition 5.1(1) and 7.5(1)]. \( \Box \)

### 7. Twosided Resolving Endomorphisms

**Proposition 7.1.** Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\) with infinitely many points.

1. The following statements are equivalent.
   (a) \( P_L(\varphi) = P_R(\varphi) = 0 \).
   (b) \( \varphi \) is weakly \( p \)-L and weakly \( p \)-R.
   (c) \( \varphi^t \) is a 1-block automorphism of \((X^t, \sigma^t)\) with some \( t \geq 0 \).

2. The following statements are equivalent.
   (a) \( P_L(\varphi) + P_R(\varphi) = 0 \).
   (b) \( \varphi \sigma^{P_R(\varphi)} \) is weakly \( p \)-L and weakly \( p \)-R.
   (c) \( \varphi^t(\sigma^t)^{P_R(\varphi)} \) is a 1-block automorphism of \((X^t, \sigma^t)\) with some \( t \geq 0 \).

**Proof.** (1) By Theorem 6.12, (a) implies (b).

Assume (b). Then by Theorem 6.12, \( \varphi \) is of \((0,n)\) type and of \((m,0)\) type with some \( m,n \geq 0 \). Therefore \( \varphi^{[m+n]} \) is a 1-block endomorphism and hence a 1-block automorphism of \((X^{[m+n]}, \sigma^{[m+n]})\) (because \( \varphi^{[m+n]} \) is onto). Hence (c) is proved.
Assume (c). Then \( \varphi^{[t]} \) is a weakly p-L, weakly p-R endomorphism of \( (X^{[t]}, \sigma^{[t]}) \), so that \( P_L(\varphi) = P_L(\varphi^{[t]}) \geq 0 \) and \( P_R(\varphi) = P_R(\varphi^{[t]}) \geq 0 \) by Proposition 6.5 and Theorem 6.12. It implies (a) that \( P_L(\varphi) \geq 0 \) and \( P_R(\varphi) \geq 0 \), because \( P_L(\varphi) + P_R(\varphi) \leq 0 \) by Proposition 6.13(3). Hence (a) is proved.

(2) Use (1) and Proposition 6.4.

**Proposition 7.2.** Let \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) be a factor map of an SFT onto a sofic system.

1. If \( \phi^{[t]} \) is weakly LR (respectively, weakly RL) with some \( t \geq 1 \), then \( P_L(\phi) \geq 0 \) and \( Q_R(\phi) \geq 0 \) (respectively, \( P_R(\phi) \geq 0 \) and \( Q_L(\phi) \geq 0 \).

2. If \( P_L(\phi) \geq 0 \) and \( Q_R(\phi) \geq 0 \) (respectively, \( P_R(\phi) \geq 0 \) and \( Q_L(\phi) \geq 0 \)), then there exists \( t \geq 1 \) such that we can construct a onesided 1-1, weakly LR (respectively, weakly RL) textile relation system \( T = (p : \Gamma_0 \to G_A^{[t]}, q : \Gamma_0 \to G_B^{[t]}) \) such that \( \phi_T = \phi^{[t]} \), where \( A = L_1(X), B = L_1(Y) \).

**Proof.** (1) By a straightforward modification in [N9] the proof of Proposition 5.1(1)], it is proved that any weakly p-L factor map \( \psi \) of an SFT onto a sofic system has memory zero and hence \( P_L(\psi) \geq 0 \). Therefore the first version of (1) follows from Propositions 4.1. 3.5 and 6.5. By symmetry the second version follows from the first one.

(2) Suppose that \( P_L(\phi) \geq 0 \) and \( Q_R(\phi) \geq 0 \). By the proof of Lemma 6.10, there exist \( s \geq 1 \) and a graph \( G_s \) such that \( (X^{[s]}, \sigma_{X^{[s]}}) = (X_{G_s}, \sigma_{G_s}) \), and there exist \( n \geq 0 \) and a local rule \( g : L_{n+1}(G_s) \to L_1(G_B^{[t]}) \) such that \( \phi^{[s]} \) is of \((0, n)\)-type and given by \( g \). Suppose that \( g \) is strictly \( k \) right-mergible. Then \( n - k = Q_R(\phi^{[s]}) = Q_R(\phi) \geq 0 \) (by using Proposition 3.5). Let \( G = G_s \) and let \( H = G_B^{[t]} \).

Then \( \phi^{[s]} \) is of \((0, n)\)-type given by a \( k \) right-mergible local rule \( g : L_{n+1}(G) \to L_1(H) \) with \( n \geq k \). We use the graph-homomorphism

\[
q_{g;k}^+ : G_{g;k}^+ \to H^{[k+1]}
\]

(see Subsection 2.2). Each arc in \( G_{g;k}^+ \) is written in the form

\[
D_{g;k}^+(a_1 \ldots a_{n+k+1}),
\]

where \( a_1 \ldots a_{n+k+1} \in L_{k+1}(G) \) with \( a_j \in A_G \). Arc \( D_{g;k}^+(a_1 \ldots a_{n+k+1}) \) goes from vertex \( D_{g;k}^+(a_1 \ldots a_{n+k}) \) to vertex \( D_{g;k}^+(a_2 \ldots a_{n+k+1}) \) with

\[
q_{g;k}^+(D_{g;k}^+(a_1 \ldots a_{n+k+1})) = g(a_1 \ldots a_{n+k+1}).
\]

By Proposition 2.4(1), \( q_{g;k}^+ \) is weakly right-resolving. Since \( a_1 \ldots a_{n+1} \) is uniquely determined by arc \( D_{g;k}^+(a_1 \ldots a_{n+k+1}) \), we can another graph-homomorphism \( p : G_{g;k}^+ \to G^{[k+1]} \) by

\[
p(D_{g;k}^+(a_1 \ldots a_{n+k+1})) = a_1 \ldots a_{n+k+1}.
\]

This definition is possible because \( n \geq k \). Since \( a_1 \ldots a_{k+1} \) and \( D_{g;k}^+(a_2 \ldots a_{n+k+1}) \) uniquely determine \( D_{g;k}^+(a_1 \ldots a_{n+k+1}) \), we see that \( p \) is left-resolving.

Thus we have an LR textile-relation system

\[
T = (p : G_{g;k}^+ \to G^{[k+1]}, q_{g;k}^+ : G_{g;k}^+ \to H^{[k+1]}).
\]

Clearly \( T \) is onesided 1-1 and \( \phi_T = (\phi^{[s]})^{[k+1]} = \phi^{[s+k]} \).
Therefore the first version of (2) is proved. By symmetry, the second version follows from the first one.

\textbf{Theorem 7.3.} Let \( \varphi \) be an onto endomorphism of a topological Markov shift \((X_G, \sigma_G)\). Suppose that \( \varphi \) is one-to-one or \( G \) is irreducible with \( P_L(\varphi) \geq 0 \) and \( Q_R(\varphi) \geq 0 \) (respectively, \( P_R(\varphi) \geq 0 \) and \( Q_L(\varphi) \geq 0 \)). Then there exist \( n, k \) with \( n \geq k \geq 0 \) such that \( \varphi \) is of \((0, n)\)-type (respectively, \((n, 0)\)-type) and given by a \( k \) right-mergible (respectively, \( k \) left-mergible) local rule \( f : L_{n+1}(G) \to A_G \). For any such \( n, k \), the textile system \( T = (p, q_{f;k}^+ : G^+_{f;k} \to G^{[k+1]}) \) (respectively, \( T = (p, q_{f;k}^- : G^-_{f;k} \to G^{[k+1]}) \)) such that for \( w \in L_{n+k+1}(G) \), \( p(D^+_f(w)) \) (respectively, \( p(D^-_f(w)) \)) is the initial (respectively, terminal) subpath of length \( k+1 \) of \( w \), can be constructed and is onesided 1-1 and LR (respectively, RL) with \( \varphi_T = \varphi^{[k+1]} \), and hence we can decide the expansiveness situation of \( \varphi \).

\textbf{Proof.} For the proof of the first version of Proposition 7.2(2), consider the case that \((X, \sigma_X) = (Y, \sigma_Y) = (X_G, \sigma_G)\). Then we can take \( s = 1 \) and \( H = G \) and the proof thus modified shows that the existence of required \( n, k \) and the textile system \( T \) in the theorem can be constructed and is onesided 1-1 and weakly LR with \( \varphi_T = \varphi^{[k+1]} \). Moreover, using Lemma 2.1, we see that \( T \) is LR.

Since \( T^* \) is LR, we can decide whether \( \xi_T^* \) is one-to-one or not, and whether \( \eta_T \) is one-to-one or not. Hence we can decide the expansiveness situation of \( \varphi^{[k+1]} \) and hence of \( \varphi \) by [N3] Theorem 2.5 and [N9] Proposition 6.2. Therefore the first version of the theorem is proved.

By symmetry, the second version follows from the first one.

\textbf{Corollary 7.4.} Let \( \varphi \) be an onto endomorphism of a topological Markov shift \((X, \sigma)\) such that \( \varphi \) is one-to-one or \( \sigma \) is topologically transitive. Let \( s \in \mathbb{Z} \).

\begin{enumerate}
\item Let \( C_R(\varphi) = \max\{ -P_L(\varphi), -Q_R(\varphi) \} \). Suppose that \( \varphi \) is given by a strictly \( k \) right-mergible local rule. Then there exists no \( t \geq 1 \) with \( (\varphi^*)^{[t]} \) is LR.
\begin{enumerate}
\item If \( s < C_R(\varphi) \), then there exists no \( t \geq 1 \) with \( (\varphi^*)^{[t]} \) is LR.
\item If \( C_R(\varphi) \leq s \leq C_R(\varphi) + k \), then \( (\varphi^*)^{[k+1+C_R(\varphi)-s]} \) is LR.
\item If \( s \geq k + C_R(\varphi) \) then \( \varphi^s \) is LR.
\end{enumerate}
\item We can decide the expansiveness situation of \( \varphi^{C_R(\varphi)} \).
\end{enumerate}

\begin{enumerate}
\item Let \( C_L(\varphi) = \min\{ P_R(\varphi), Q_L(\varphi) \} \). Suppose that \( \varphi \) is given by a strictly \( l \) left-mergible local rule. Then there exists no \( t \geq 1 \) with \( (\varphi^*)^{[t]} \) is LR.
\begin{enumerate}
\item If \( s > C_L(\varphi) \), then there exists no \( t \geq 1 \) with \( (\varphi^*)^{[t]} \) is LR.
\item If \( -l + C_L(\varphi) \leq s \leq C_L(\varphi) \), then \( (\varphi^*)^{[l+C_L(\varphi)+s]} \) is LR.
\item If \( s \leq -l + C_L(\varphi) \), then \( \varphi^s \) is LR.
\end{enumerate}
\item We can decide the expansiveness situation of \( \varphi^{C_L(\varphi)} \).
\end{enumerate}

\textbf{Proof.} (1) Statement (a) follows from Propositions 4.1(1), 3.5(3), 6.13(1) and 6.5(2).

To see (b), suppose \( \varphi \) is of \((m, n)\)-type and given by a local rule \( f : L_{m+n+1}(G) \to A_G \) which is strictly \( k \) right-mergible, where \( G \) is the defining graph of \((X, \sigma)\). Let \( \psi = \varphi^{C_R(\varphi)} \). Let \( g \) be the local rule obtained from \( f \) by adding left-redundancy by \( C_R(\varphi) \). Then \( g : L_{m+n+C_R(\varphi)+1}(G) \to A_G \) is strictly \( k \) right-mergible, by Lemma 3.2. Since \( \varphi \) is of \((m + C_R(\varphi), n)\)-type and given by \( g \), \( \psi \) is of \((m, n + C_R(\varphi))\)-type and given by \( g \). Since \( P_L(\psi) = P_L(\varphi) + C_R(\varphi) \geq 0 \), the left redundancy of \( g \) is not less than \( m \), so that if \( g' : L_{n+C_R(\varphi)+1}(G) \to A_G \) is the local rule obtained from \( g \) by deleting left-redundancy by \( m \), then \( g' \) is strictly \( k \) right-mergible (by Lemma 3.2) and \( \psi \) is of \((0, n + C_R(\varphi))\)-type and given by \( g' \). Therefore, since
$Q_R(\psi) = Q_R(\varphi) + C_R(\varphi) \geq 0$, it follows from Theorem 7.3 that the textile system $T = (p, q^+_{g:k} : G^+_{g:k} \to G)_{k=1}^{k+1}$ such that for each $w \in L_{a,b}(\varphi)+1(G)$, $p(D^+_g(w))$ is

the initial subpath of length $k+1$ of $w$, is onesided 1-1 and LR with $\varphi_T = \psi^{[k+1]}$. For $s = 0, \ldots, k$, we define a textile system $T^+_s = (p_s, q^+_s : G^+_{g:k} \to G^{k+1-s})$, where if $p(D^+_g(w)) = a_1 \ldots a_{k+1}$ and $q^+_s(D^+_g(w)) = b_1 \ldots b_{k+1}$ with $a_j, b_j \in A_G$ and $w \in L_{a,b}(\varphi)+1(G)$, then $p_s(D^+_g(w)) = a_1 \ldots a_{k+1-s}$ and $q^+_s(D^+_g(w)) = b_{s+1} \ldots b_{k+1}$. Then it follows by construction that $T^+_s$ is onesided 1-1 and LR and $\varphi_{T^+_s} = (\psi \sigma^+_s)^{[k+1-s]}$. Therefore, $(\varphi \sigma^{[k+1]}(\psi))^{[k+1-s]}$ is LR for $s = 0, \ldots, k$. Hence (b) is proved.

Since $\varphi^{[k+1]}(\psi)$ is LR, $\varphi^{[k+1]}(\psi)$ is LR for all $s \geq 0$, by [N5 Corollary 3.18(1)] (because $\sigma$ is LR), so that (c) is proved. By Theorem 7.3, (d) follows.

(2) By symmetry (2) follows from (1).

Statements(1)(c) and (2)(c) of Corollary 7.4 are a refinement of [N5 Proposition 6.30].

Let $G$ be a nondegenerate graph. Let $M_G = (m_{u,v})_{u,v \in V_G}$ be the adjacency matrix of $G$, i.e. the square matrix with indexing-set $V_G$ such that the $(u,v)$-component $m_{u,v}$ is the number of arcs going from vertex $u$ to vertex $v$. Let $L_G = (l_{u,a})_{u \in V_G, a \in A_G}$ denote the rectangular matrix with indexing-set $V_G \times A_G$ such that $l_{u,a} = 1$ if $i_G(a) = u$ and otherwise $l_{u,a} = 0$. Let $R_G = (r_{a,u})_{a \in A_G, u \in V_G}$, denote the rectangular matrix with indexing-set $A_G \times V_G$ such that $r_{a,v} = 1$ if $t_G(a) = v$ and otherwise $r_{a,v} = 0$. Then $M_G = L_G R_G$ and $M_G^{[2]} = R_G L_G$.

For nondegenerate graphs $G$ and $H$ with $V_G = V_H$, we let $G H$ denote the graph such that $A_{G H} = \{ab \mid t_G(a) = i_H(b), a \in A_G, b \in A_H\}$, $V_{G H} = V_G$ and $i_G(ab) = i_G(a)$ and $t_G(ab) = t_H(b)$ for $ab \in A_{G H}$, and let $\Gamma^0$ denote the graph such that $V_{\Gamma^0} = V$ and $M_{\Gamma^0}$ is the identity matrix. It is clear that $M_{G H} = M_G M_H$.

**Lemma 7.5.** Let $G$ be a nondegenerate graph. Let $t \geq 1$. Suppose that $H$ is a nondegenerate graph with $V_H = V_G^{[t]}$ such that $M_H M_G^{[t]} = M_G M_H$. Then there exists a nondegenerate graph $K$ with $V_K = V_G$ such that $L_G M_H = M_K L_G$, $M_H R_G = R_G M_K$ and hence $M_K M_G = M_G M_K$ and such that the onesided topological Markov shifts $(\tilde X_{H^{[i]}(G^{[t]})}^{t+1}, \tilde \sigma_{H^{[i]}(G^{[t]})}^{t+1})$ and $(\tilde K^{t+1}, \tilde \sigma_{K^{t+1}})$ are topologically conjugate for all $i, j \geq 0$.

**Proof.** Since $G^{[t]} = (G^{[t-1]}{[2]}$, if we show the lemma for the case that $t = 2$, then the lemma follows by induction.

Let $M_{H G^{[2]}} = (m_{a,b})_{a,b \in A_G}$. Since $M_{H G^{[2]}} = M_H M_{G^{[2]}} = M_{G^{[2]}} M_H$, we have $M_{H G^{[2]}} = (m_{a,b})_{a,b \in A_G}$. $M_H R_G L_G = R_G L_G M_H$. Since $M_{H G^{[2]}} = R_G (L_G M_H)$, it follows that for each $u \in V_G$, all $a$-rows of $M_{H G^{[2]}}$ with $a \in \tau^{-1}_G(u)$ are the same and equal to the $u$-row of $L_G M_H$. Since $M_{H G^{[2]}} = (M_H R_G) L_G$, it follows that for each $v \in V_G$, all $a$-columns of $M_{H G^{[2]}}$ with $a \in \tau^{-1}_G(v)$ are the same and equal to the $v$-column of $M_H R_G$. Hence we can define a graph $K$ by $M_K = (k_{u,v})_{u,v \in V_K}$ with $k_{u,v} = m_{a,b}$ if $a \in \tau^{-1}_G(u)$ and $b \in \tau^{-1}_G(v)$. We have $R_G M_K L_G = M_{H G^{[2]}}$. 


Therefore, since $M_{HG[2]} = R_G(L_G M_H)$, we have $M_K L_G = L_G M_H$, and since $M_{HG[2]} = (M_R G) L_G$, we have $R_G M_K = M_R R_G$. Therefore we have

$$M_K M_G = M_K L_G R_G = L_G M_H R_G = L_G R_G M_K = M_G M_K.$$ 

Since

$$M_{H(G[2])} = M_H M_{G[2]} = M_H R_G L_G (R_G L_G)^l = R_G (M_K M_R (R_G L_G)^l),$$

$$M_{K(G[2])} = M_K M_{G[2]} = (M_K L_G R_G)^l L_G R_G = (M_K L_G (R_G L_G)^l)^l R_G$$

and $R_G$ is a subdivision matrix, it follows from Williams’ theorem [Wi, Theorem G] that the onesided subshifts $(X_{H(G[2])}, \sigma_{H(G[2])})$ and $(X_{K(G[2])}, \sigma_{K(G[2])})$ are topologically conjugate.

For an onto endomorphism of a topological Markov shift $\varphi$, let $r_\varphi$ and $l_\varphi$ be the right and left multipliers of Boyle [Bo1]. (An explanation on the notions along our theory is found in [N5, pp. 52–55]. Note that $r_\varphi = 1/R(\varphi)$ and $l_\varphi = 1/L(\varphi)$ for $R(\varphi)$ and $L(\varphi)$ appearing in [Bo1] and [N5].) Let $h(\varphi)$ denote the topological entropy of $\varphi$.

We note that if $\varphi$ is an onto endomorphism of a topological Markov shift $(X, \sigma)$ and $\varphi[t]$ is an LR endomorphism of $(X[t], \sigma[t])$ for some $t \geq 1$, then $\varphi$ is a $p$, $q$-L endomorphism of $(X, \sigma)$, (by Theorems 6.12 and 4.2(1) and Propositions 6.5(2) and 3.5(3)), so that in particular, $\varphi$ has memory zero and hence the induced endomorphism $\hat{\phi}$ of the induced onesided topological Markov shift $(\hat{X}, \hat{\sigma})$ can be defined. We also note that a $p$, $q$-L endomorphism of a topological Markov shift is not necessarily LR endomorphism of the shift, as is shown in [N4].

For a subshift $(X, \sigma)$ and its endomorphism with memory zero, $(X^*, \sigma^*)$ and $\varphi^*$ denote $(\hat{X}, \hat{\sigma})$ and $\hat{\varphi}$, respectively.

**Theorem 7.6. The following statements are valid.**

1. If $\varphi$ is an onto endomorphism of a topological Markov shift $(X_G, \sigma_G)$ such that for some $t \geq 1$, $\varphi[t]$ is an LR endomorphism of $(X_G[t], \sigma_G[t])$, then $\varphi$ has memory zero and there exists a nondegenerate graph $K$ with $M_K M_G = M_G$ such that for all $i \geq 0, j \geq 1$, $(X_G, \varphi^j \sigma_G)$ is topologically conjugate to the onesided topological Markov shift $(X_{K[G]}, \sigma_{K[G]})$ and the inverse limit system of $\varphi^j \sigma_G$ is topologically conjugate to $(X_{K[G]}, \sigma_{K[G]})$.

2. If $\varphi$ is an onto endomorphism of the full $n$-shift $(X, \sigma)$ with $n \geq 1$ such that for some $t \geq 1$, $\varphi[t]$ is an LR endomorphism of $(X[t], \sigma[t])$, then for some integer $k \geq 1$,

$$h(\varphi) = h(\hat{\varphi}) = \log r_\varphi = \log k$$

and for all $i \geq 0, j \geq 1$, $(\hat{X}, \varphi^j \hat{\sigma})$ is topologically conjugate to the onesided full $k^n$-shift and the inverse limit system of $\varphi^j \sigma$ is topologically conjugate to the full $k^n$-shift.

**Proof.** (1) Let $i \geq 0$ and let $j \geq 1$. Let $\varphi[t]$ be an LR endomorphism of $(X_G[t], \sigma_G[t])$. Then there exists a nondegenerate graph $T$ over $G[t]$ such that $\varphi_T = \varphi[t]$. Hence $\varphi$ has memory zero, by Proposition 6.5(2) and Proposition 6.9. Let $G^*$ be the nondegenerate graph such that the dual $T^*$ is defined over $G^*$. Then by [N4, Proposition 6.1] that $M_{G^*[i]} M_{G^*[i]} = M_{G^*[i]} M_{G^*[i]}$. It follows from [N4, Corollary 6.7(2)] that $(X_G[t])^*, ((\varphi[t])^*)^j ((\sigma_G[t])^*)^j$ is topologically conjugate to $(\hat{X}_{G^*[i]}(G[t])), \sigma_{G^*[i]}(G[t]))$. 


By Lemma 7.5, there exists a nondegenerate graph $K$ with $M_K M_G = M_G M_K$ such that $(\hat{X}_{K^tG}, \hat{\sigma}_{K^tG})$ is topologically-conjugate to $(X_{G^t(G^t)^t}, \hat{\sigma}_{(G^t)^t(G^t)^t})$. Therefore $(\hat{X}_{G}, \hat{\psi}_{G})$ is topologically conjugate to $(X_{K^tG}, \hat{\sigma}_{K^tG})$.

The proof of the remainder of (1) is similar, but use [N5, Theorem 6.31(3)] instead of [N5 Corollary 6.7(2)].

(2) Notation being the same as in (1), we apply (1) to the case that $(X, \sigma) = (X, \sigma)$. Since the defining graph $G$ of $(X, \sigma)$ is the one-vertex graph, so is $K$, because $M_K M_G = M_G M_K$. Suppose that $K$ has $k$ arcs, then the conclusions of (2) follows from (1) except for (7.3).

Since $\varphi^{[t]}$ and $\sigma^{[t]}$ are LR endomorphisms of $(X^{[t]}, \sigma^{[t]})$, it follows from [N5 Proposition 6.1, Corollaries 3.17(1) and 3.18(1)] that there exists a onesided 1-1 LR textile system $T_{1,1}$ such that $\varphi_{T_{1,1}} = \varphi^{[t]} \sigma^{[t]}$ and the dual $T_{1,1}^t$ is defined over the graph $G^t(G)^t$ with $M_G, M_{G^{[t]}} = M_{G^{[t]}} M_G$. Hence, by [N5 Theorem 6.31(1)] and Lemma 7.5, we have

$$h(\varphi^{[t]} \sigma^{[t]}) = h(\sigma^{[t]} G^{[t]} \sigma^{[t]}) = h(\sigma_{K^tG}) = \log k n = \log k + \log n.$$ 

By Boyle [Boc], we know that for any two onto endomorphisms $\psi, \psi'$ of an irreducible topological Markov shift $r_{\psi \psi'} = r_{\psi} r_{\psi'}$ and that $r_{\psi}$ is an invariant of topological conjugacy between endomorphisms of irreducible topological Markov shifts. Using these and [N5 Theorem 6.31(2)], we see that

$$h(\varphi^{[t]} \sigma^{[t]}) = \log r_{\varphi^{[t]} \sigma^{[t]}} = \log r_{\varphi^{[t]}} + \log r_{\sigma^{[t]}} = h(\varphi^{[t]}) + h(\sigma^{[t]}) = h(\varphi) + h(\sigma) = h(\varphi) + \log n.$$ 

Therefore we have $h(\varphi) = \log k$.

Since $\varphi^{[t]}$ is LR, it follows from [N5 Theorem 6.31(2)] that $h(\varphi^{[t]}) = h((\varphi^{[t]})^{-1}) = r_{\varphi^{[t]}}$, so that $h(\varphi) = h(\varphi) = r_{\varphi}$. Therefore (7.3) is proved.

**Corollary 7.7.** Let $\varphi$ be an onto endomorphism of the full $n$-shift $(X, \sigma)$ with $n \geq 1$.

1. If $P_L(\varphi) \geq 0$ and $Q_R(\varphi) \geq 0$, then for some integer $k \geq 1$,

$$h(\varphi) = h(\varphi) = \log r_{\varphi} = \log k$$

and for all $i \geq 0$ and $j \geq 1$, the dynamical system $(\hat{X}, \hat{\varphi} J \hat{\varphi})$ is topologically conjugate to the onesided full $k^n n^j$-shift and the inverse limit system of $\varphi J \sigma^j$ is topologically conjugate to the full $k^n n^j$-shift.

2. If $P_R(\varphi) \geq 0$ and $Q_L(\varphi) \geq 0$, then for some integer $k \geq 1$,

$$h(\varphi) = h(\varphi) = \log l_{\varphi} = \log k$$

and for all $i \geq 0$ and $j \geq 1$, the inverse limit system of $\varphi J \sigma^{-j}$ is topologically conjugate to the full $k^n n^j$-shift.

**Proof.** By Theorems 7.3 and 7.6(2), (1) is proved. By symmetry, (2) follows from (1).

**Theorem 7.8.** Let $\varphi$ be an onto endomorphism of a subshift $(X, \sigma)$. Let $s \in \mathbb{Z}$.

1. Let $C_R(\varphi) = \max\{-P_L(\varphi), -Q_R(\varphi)\}$.

(a) $\varphi^{[t]}$ is weakly LR with some $t \geq 1$ if and only if $P_L(\varphi) \geq 0$ and $Q_R(\varphi) \geq 0$.

(b) If $\langle \varphi \sigma^s \rangle^{[t]}$ is weakly LR with some $t \geq 1$, then $s \geq C_R(\varphi)$; there exists $t_i \geq 1$ such that $\langle \varphi \sigma^s \rangle^{[t_i]}$ is weakly LR for all $s \geq C_R(\varphi)$.

**Proof.** By Theorems 7.3 and 7.6(2), (1) is proved. By symmetry, (2) follows from (1).
it follows from Proposition 7.2 that for some \( t \) \( 5.1 \) and 6.11, there exists \( P \) LR textile-relation system 3.5 and 6.5. .

Proof. (1)(a) The “only-if” part follows from Theorems 5.2 and 6.12 and Propositions 3.5 and 6.5.

To prove the “if” part, assume that \( P_L(\varphi) \geq 0 \) and \( Q_R(\varphi) \geq 0 \). By Lemmas 5.1 and 6.11, there exists \( J \in \mathbb{N} \) such that \( Q_R(\phi^{(J,\varphi)}) = Q_R(\varphi) \) and \( P_L(\phi^{(J,\varphi)}) = P_L(\varphi) \). Hence by the assumption, \( P_L(\phi^{(J,\varphi)}) \geq 0 \) and \( Q_R(\phi^{(J,\varphi)}) \geq 0 \). Therefore, it follows from Proposition 7.2 that for some \( t \geq 1 \) there exists onesided 1-1, weakly LR textile-relation system

\[
T = (p : \Gamma_0 \to G_A^{[t]} , q : \Gamma_0 \to G_A^{[t]} ) \quad \text{with} \quad A = L_1(X)
\]

such that \( \phi_T = (\phi^{(J,\varphi)})^{[t]} \). We can regard \( T \) as a textile system \( T = (p,q : \Gamma_0 \to G_A^{[t]} ) \) which is weakly LR. Let \( Z = \xi_T^{-1}(X^{[t]}) \). Then, since \( \eta_T(Z) = \eta_T \xi_T^{-1}(X^{[t]}) = \phi_T(X^{[t]}) = \varphi^{[t]}(X^{[t]}) = X^{[t]} = \xi_T(Z) \) and \( \xi_T \) is one-to-one, there exists a unique onesided 1-1 textile subsystem \( U \) of \( T \) with \( Z_U = Z \). Since \( \varphi^{[t]} = \varphi_U \), we conclude that \( \varphi^{[t]} \) is weakly LR.

(b) By (a) and Propositions 3.4 and 6.4, we see that if \( (\varphi^{s*})^{[t]} \) is weakly LR with some \( t \geq 1 \), then \( s \geq C_R(\varphi) \) and that there exists \( t \geq 1 \) such that \( (\varphi^{C_R(\varphi)})^{[t]} \) is weakly LR. By \([N9] \) Proposition 7.6(3), \( (\varphi^{s*})^{[t]} \) is weakly LR for all \( s \geq C_R(\varphi) \).

(c) See \([N9] \) Proposition 7.6(3).

(2) By symmetry, (2) follows from (1).

Relating to (1)(b) and (2)(b) of Theorem 7.8, we recall \([N9] \) Proposition 7.10 that a factor map \( \phi : (X, \sigma_X) \to (Y, \sigma_Y) \) is right-closing if and only if \( \varphi^{s*}_X \) is weakly LR for some (all sufficiently large) \( s \).

**Proposition 7.9.** Let \( \varphi \) be an onto endomorphism of a subshift \( (X, \sigma) \) with infinitely many points.

1. \( P_L(\varphi) + Q_L(\varphi) \) and \( P_R(\varphi) + Q_R(\varphi) \) are shift-invariant; they are nonpositive and hence

\[
Q_L(\varphi) \leq -P_L(\varphi) \quad \text{and} \quad P_R(\varphi) \leq -Q_R(\varphi).
\]

2. The following statements are equivalent:
   (a) \( \varphi \) is weakly LL (respectively, weakly RR).
   (b) \( P_L(\varphi) = Q_L(\varphi) = 0 \) (respectively, \( P_R(\varphi) = Q_R(\varphi) = 0 \)).
   (c) For some \( n \geq 0 \), \( \varphi \) is of \((0,n)\)-type (respectively, \((n,0)\)-type) and given by a zero left-mergible (respectively, zero right-mergible) local rule \( f : L_{n+1}(X) \to L_1(X) \).

3. The following statements are equivalent:
   (a) There exists \( s \in \mathbb{Z} \) with \( \varphi^{s*} \) weakly LL (respectively, weakly RR).
   (b) \( P_L(\varphi) + Q_L(\varphi) = 0 \) (respectively, \( P_R(\varphi) + Q_R(\varphi) = 0 \)).
   (c) \( \varphi^{s*} - P_L(\varphi) \) is weakly LL (respectively, \( \varphi^{s*} - Q_R(\varphi) \) is weakly RR).
(4) If $(X, \sigma)$ is a topological Markov shift and if $\varphi$ is one-to-one or $\sigma$ is transitive, then the statements in (2) and (3) with all “weakly” in them deleted hold.

Proof. (1) It follows from Propositions 3.4 and 6.4 that $P_L(\varphi) + Q_L(\varphi)$ and $P_R(\varphi) + Q_R(\varphi)$ are shift-invariant.

Assume that $P_L(\varphi) + Q_L(\varphi) > 0$. It follows from Proposition 6.13(1) that $\varphi\sigma^{-P_L(\varphi)+1}$ is weakly $p$-L and right $\sigma$-expansive. Since $-P_L(\varphi) + 1 \leq Q_L(\varphi)$ by assumption, it follows from Theorem 5.2(2) that $\varphi\sigma^{-P_L(\varphi)+1}$ would be weakly $q$-L. However this contradicts [N9, Proposition 8.2(2)] that there exists no weakly $p$-L, right $\sigma$-expansive, weakly $q$-L endomorphism of a subshift with infinitely many points. Thus $P_L(\varphi) + Q_L(\varphi) \leq 0$. By symmetry it is proved that $P_R(\varphi) + Q_R(\varphi) \leq 0$.

(2) Assume (a). Then $P_L(\varphi) \geq 0$ by Theorem 6.12, and $Q_L(\varphi) \geq 0$ by Theorem 5.2(2). Hence (b) follows by (1).

Assume (b). Since $P_L(\varphi) = 0$, it follows from Theorem 6.12 that $\varphi$ is of $(0, n)$-type and given by a local rule $f : L_{n+1}(X) \to L_1(X)$ with some $n \geq 0$. If $f$ is strictly $l$ left mergible with $l \geq 0$, then $0 - l = Q_L(\varphi) = 0$. Hence (c) is proved.

Assume (c). Let $T = (p, q : G[X, n + 1] \to G(A)$ with $A = L_1(X)$ be the textile system defined as follows: for each arc $\alpha = a_0 \ldots a_n$ in $G[X, n + 1]$ with $a_j \in A$, $p(\alpha) = a_0$ and $q(\alpha) = f(a_0 \ldots a_n)$. (Recall that $G[X, s]$ was defined in the beginning of Section 5.) Then $\xi_T$ is one-to-one. Clearly $p$ is weakly left-resolving. Since $f$ is 0 left-mergable, $q$ is weakly left-resolving. Let $Z = \xi_T^{-1}(X)$, then $\eta_T(Z) = \eta_T\xi_T^{-1}(X) = \varphi_T(X) = \varphi(X) = X = \xi_T(Z)$ and $\xi_T$ is one-to-one, there exists a unique one-sided 1-1 textile subsystem $U$ of $T$ with $Z_U = Z$. Since $\varphi = \varphi_U$, we conclude that $\varphi$ is weakly LL. Hence (a) is proved.

By symmetry, the second version is proved by the first version.

(3) By (2) and Propositions 6.4 and 3.4.

(4) Let (2’) and (3’) be the statements (2) and (3) with all “weakly” in them deleted, respectively. The proof of (2’) is the same as that of (2) except for the following.

To prove that (a) implies (b) and that (b) implies (c), use Proposition 6.9 instead of Proposition 6.12, and use Theorem 4.2(2) instead of Theorem 5.2(2).

To prove that (c) implies (a), an LL textile system $T$ is directly given as follows. If $(X, \sigma) = (X_G, \sigma_G)$, then define $T = (p, q : G^{[n+1]} \to G)$, where for $a = a_0 \ldots a_n \in A_G^{[n+1]}$ with $a_j \in A_G$, $p(\alpha) = a_0$ and $q(\alpha) = f(a_0 \ldots a_n)$. Clearly, $T$ is weakly LL. It follows from Lemma 2.1 that $T$ is LL.

The proof of (3’) is given by (2’) and Propositions 6.4 and 3.4.

□

A direct proof of (1) of Proposition 7.9 for the case that $\varphi$ is an onto endomorphism of a topological Markov shift $(X, \sigma)$ with $\varphi$ one-to-one or $\sigma$ transitive, is given by using Proposition 6.9 and [N9, Proposition 7.6(1)] instead of Proposition 6.13(1), and by using Theorem 4.2(2) and [N9, Proposition 7.6(2)] instead of Theorem 5.2(2).

For more information on LL endomorphisms of topological Markov shifts, see [N7, Section 5].

**Proposition 7.10.** Let $m, n \geq 0$. Let $\varphi$ be an onto endomorphism of $(m, n)$-type of a subshift $(X, \sigma)$ with infinitely many points. If $Q_R(\varphi) + Q_L(\varphi) \geq 0$, then

$$-n \leq P_R(\varphi) \leq -Q_R(\varphi) \leq Q_L(\varphi) \leq -P_L(\varphi) \leq m.$$
Proof. By Propositions 6.13(3) and 7.9(1).

Proposition 7.11. (1) If \( \varphi \) is an automorphism of a subshift, then
\[ Q_L(\varphi) = P_L(\varphi^{-1}) \quad \text{and} \quad Q_R(\varphi) = P_R(\varphi^{-1}). \]

(2) If \( \varphi \) is an automorphism of a subshift with infinitely many points, then
\[ Q_R(\varphi) + Q_L(\varphi) \leq 0 \quad \text{and} \quad D_L(\varphi) \leq D_R(\varphi), \]
where \( D_L(\varphi) = \max\{P_L(\varphi), Q_L(\varphi)\} \) and \( D_R(\varphi) = \min\{-P_L(\varphi), -Q_R(\varphi)\} \).

(3) If \( \varphi \) is a symbolic automorphism of a subshift with infinitely many points, then
\[ P_L(\varphi) = Q_R(\varphi) = P_R(\varphi) = Q_L(\varphi) = 0. \]

Proof. (1) Let \( s \in \mathbb{Z} \). By Proposition 6.13(1), \( s \geq -P_L(\varphi^{-1}) \) if and only \( \varphi^{-1} \sigma^s \) is weakly \( p \)-L. This holds if and only if \( \varphi \sigma^{-s} \) is weakly \( q \)-L. This holds if and only if \( -s \leq Q_L(\varphi) \), by Theorem 5.2(2). Therefore, \( s \geq -P_L(\varphi^{-1}) \) if and only if \( s \geq -Q_L(\varphi) \), so that \( P_L(\varphi^{-1}) = Q_L(\varphi) \).

The proof that \( Q_R(\varphi) = P_R(\varphi^{-1}) \) is similar.

(2) By Proposition 6.13(3), \( P_R(\varphi^{-1}) + P_L(\varphi^{-1}) \leq 0 \). Hence by the above, \( Q_R(\varphi) + Q_L(\varphi) \leq 0 \). From this, Propositions 6.13(3) and 7.9(1), it follows that \( D_L(\varphi) \leq D_R(\varphi) \).

(3) Since \( \varphi \) is a block-map of \((0,0)\) type, we have \( P_L(\varphi) = 0 \) and \( P_R(\varphi) = 0 \), by Proposition 6.13(3). Since \( \varphi^{-1} \) is a symbolic automorphism, we also have \( P_L(\varphi^{-1}) = 0 \) and \( P_R(\varphi^{-1}) = 0 \). Hence \( Q_L(\varphi) = 0 \) and \( Q_R(\varphi) = 0 \) by (1).

By Proposition 7.11(1), we see the following, which should be compared with [[N1] Theorem 7].

Remark 7.12. Let \( \varphi \) be an automorphism of a subshift. If \( Q_R(\varphi) \leq 0 \) and \( Q_L(\varphi) \leq 0 \), then \( \varphi^{-1} \) is of \((-Q_L(\varphi), -Q_R(\varphi))\)-type given by a local rule with right and left redundancies zero.

8. Endomorphisms with Resolving Powers

Let \( \varphi \) be an endomorphism of a subshift \((X, \sigma)\). Let \( s \in \mathbb{N} \). For \( x \in X \), let
\[ \rho^s_{\varphi,s}(x) = (a_{i,j})_{0 \leq i \leq s-1, j \in \mathbb{Z}}, \]
where \( a_{i,j} \in \mathbb{Z} \) if \( \varphi^i(x) \) for \( i = 0, \ldots, s-1 \) with \( a_{i,j} \in L_1(X) \). We often write
\[ \rho^s_{\varphi,s}(x) = (a_{i,j})_{0 \leq i \leq s-1, j \in \mathbb{Z}}. \]

Let
\[ X_{[s]} = \{ \rho^s_{\varphi,s}(x) \mid x \in X \}. \]
Then we have a subshift \((X_{[s]}, \sigma_{[s]})\), which is called the \( \varphi \)-dual higher-block system of order \( s \) of \((X, \sigma)\), such that
\[ L_1(X_{[s]}) \subset \{ (a_i)_{0 \leq i \leq s-1} \mid a_i \in L_1(X) \}. \]
and have a topological conjugacy \( \rho^s_{\varphi,s} : (X, \sigma) \rightarrow (X_{[s]}, \sigma_{[s]}) \), which is called the \( \varphi \)-dual higher-block conjugacy of order \( s \) or a dual higher block conjugacy. We also have an endomorphism \( \varphi^{[s]} \) of \((X_{[s]}, \sigma_{[s]})\) by
\[ \varphi^{[s]} = \rho^s_{\varphi,s} \varphi(\rho^s_{\varphi,s})^{-1}, \]
which is called the dual-higher-block endomorphism of order \( s \) of \( \varphi \). Clearly, the endomorphism \( \varphi \) of \((X, \sigma)\) is topologically conjugate to the endomorphism \( \varphi^{[s]} \) of \((X_{\{s\}}, \sigma_{\{s\}})\) through \( \rho^s \) : \((X, \sigma, \varphi) \rightarrow (X_{\{s\}}, \sigma_{\{s\}}, \varphi^{[s]})\).

Let \( N \geq 1 \). For convenience, each element \( \alpha_1 \ldots \alpha_{N+1} \) of \( L_{N+1}(X_{\{s\}}) \) such that \( \alpha_j = (a_{i,j})_{0 \leq i \leq s-1} \in L_1(X_{\{s\}}) \) with \( a_{i,j} \in L_1(X) \) will often be written

\[
(a_{i,j})_{0 \leq i \leq s-1, 1 \leq j \leq N+1}.
\]

Clearly, if \( (a_{i,j})_{0 \leq i \leq s-1, 1 \leq j \leq N+1} \in L_{N+1}(X_{\{s\}}) \), then \( a_{i,1} \ldots a_{i,N+1} \in L_{N+1}(X) \) for \( i = 0, \ldots, s-1 \).

In [N9] the proof of Proposition 8.2, it was shown that for an automorphism \( \varphi \) of a subshift \((X, \sigma)\), if \( \varphi^s \) is the identity map with \( s \geq 1 \), then \( \varphi^{[s]} \) is a symbolic automorphism (i.e., a 1-block automorphism) of \((X_{\{s\}}, \sigma_{\{s\}})\), and hence \( \varphi \) is an essentially symbolic automorphism of \((X, \sigma)\).

Mike Boyle [Bo2] proved the theorem below (cf. Remark 2.5) and suggested the possibility of proving the main results of [N5] Section 7 on resolving endomorphisms of topological Markov shifts without using the long theory of “resolvable textile systems” developed there.

**Theorem 8.1** (Boyle [Bo2]). Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). Let \( s \geq 1 \).

1. If \( \varphi^s \) is weakly p-L (respectively, weakly p-R), then so is \( \varphi^{[s]} \), and hence \( \varphi \) is essentially weakly p-L (respectively, essentially weakly p-R).
2. If \( \varphi^s \) is a weakly LR automorphism of \((X, \sigma)\), then \( \varphi \) is essentially weakly LR.
3. With the hypothesis that \((X, \sigma)\) is a topological Markov shift, it holds that if \( \varphi^s \) is p-L (respectively, p-R), then so is \( \varphi^{[s]} \) for some \( t \geq 1 \).

**Proof.** (1) Since \( \varphi^s \) is weakly p-L, it follows from [N9] Proposition 7.5(1) that \( \varphi^s \) is of \((0, n)\)-type and given by a local-rule \( f : L_{n+1}(X) \rightarrow L_1(X) \) with some \( n \geq 0 \). We define a local-rule \( F : L_{n+1}(X_{\{s\}}) \rightarrow L_1(X_{\{s\}}) \) as follows: \( F = f \) if \( s = 1 \); if \( s \geq 2 \), then for each

\[
W = (a_{i,j})_{0 \leq i \leq s-1, 1 \leq j \leq N+1} \in L_{N+1}(X_{\{s\}}), \quad a_{i,j} \in L_1(X),
\]

\[
F(W) = (a_{i,j})_{1 \leq i \leq s}, \text{ where } a_i = a_{i,1} \text{ for } i = 1, \ldots, s-1, \text{ and } a_s = f(a_{0,1} \ldots a_{0,n+1}).
\]

Then it is clear that \( \varphi^{[s]} \) is a block-map of \((0, n)\)-type given by the local-rule \( F \). Therefore, by [N9] Proposition 7.5(1), \( \varphi^{[s]} \) is weakly p-L, so that \( \varphi \) is essentially weakly p-L.

By symmetry, the second version follows from the first one.

(2) Since \( \varphi^s \) is a weakly LR automorphism of \((X, \sigma)\), it follows from (1) that \( \varphi^{[s]} \) is weakly p-L and \( \varphi^{-1} \) is weakly p-R. Hence it follows from [N9] Proposition 7.5 that for some \( t \geq 1 \), \( \varphi^{[s]} \) is weakly LR, so that \( \varphi \) is essentially weakly LR.

(3) Since \((X_{\{s\}}, \sigma_{\{s\}})\) is an SFT, for some \( t \geq 1 \), \( \varphi^{[s]} \) is an endomorphism of a topological Markov shift. Using this and using [N9] Proposition 5.1(1) instead of [N9] Proposition 7.5(1), (3) is proved. □
Moreover, (3) of the theorem gives a very short proof for \([N5]\) Theorem 7.22(1) that if \(\varphi^*\) is a \(p\)-L endomorphism of a topological Markov shift, then \(\varphi\) is an essentially \(p\)-L endomorphism of the shift, and also a similar proof to that of (2) using it together with \([N9]\) Proposition 5.1(2) instead of \([N9]\) Proposition 7.5(2) proves that if \(\varphi^*\) is an LR automorphism of a topological Markov shift, then \(\varphi\) is an essentially LR automorphism of the shift, which is the automorphism case of \([N9]\) Theorem 7.22(2).

**Proposition 8.2.** Let \(\varphi\) be an endomorphism of a subshift \((X, \sigma)\). Let \(s \geq 1\).

1. If \(P_L(\varphi^*) \geq 0\), then \(P_L(\varphi^{|s|}) \geq 0\).
2. If \(P_R(\varphi^*) \geq 0\), then \(P_R(\varphi^{|s|}) \geq 0\).

**Proof.** By Theorem 6.12 and the proof of Theorems 8.1(1). \(\square\)

**Proposition 8.3.** Let \(\varphi\) be an endomorphism of a subshift \((X, \sigma)\). Let \(s \in \mathbb{N}\).

1. If \(P_L(\varphi^{|s|}) \geq 0\), then \(P_L(\varphi^{|s+1|}) \geq 0\); if \(P_L(\varphi^{|s|}) \leq 0\), then \(P_L(\varphi^{|s+1|}) \geq P_L(\varphi^{|s|})\).
2. If \(P_R(\varphi^{|s|}) \geq 0\), then \(P_R(\varphi^{|s+1|}) \geq 0\); if \(P_R(\varphi^{|s|}) \leq 0\), then \(P_R(\varphi^{|s+1|}) \geq P_R(\varphi^{|s|})\).

**Proof.** (1) Since \(\varphi^{|s+1|} = (\varphi^{|s|})^{|2|}\), it suffices to prove the proposition for \(s = 2\).

Suppose that \(\varphi\) is of \((m, n)\)-type and given by a local rule \(f : L_{N+1}(X) \rightarrow L_1(X)\) with \(N = m + n\). Define \(g : L_{N+1}(X_{[2]}^2) \rightarrow L_1(X_{[2]}^2)\) to be the local-rule such that for each \(W = (a_{i,j})_{0 \leq i \leq 1, 1 \leq j \leq N+1} \in L_{N+1}(X_{[2]}^2)\) with \(a_{i,j} \in L_1(X)\), \(g(W) = (a_1)_{1 \leq i \leq 2}\), where

\[
\begin{align*}
    a_1 &= a_{1,m+1} \quad \text{and} \quad a_2 = f(a_{1,1} \ldots a_{1,N+1}).
\end{align*}
\]

Then it is clear that \(\varphi^{|2|}\) is of \((m, n)\)-type and given by \(g\).

Suppose that \(P_L(\varphi) \leq 0\). Then by Theorem 6.12, \(\varphi\) is of \((0, n)\)-type with some \(n \geq 0\). Then by the above, \(\varphi^{|2|}\) is of \((0, n)\)-type. Hence again by by Theorem 6.12, \(P_L(\varphi^{|2|}) \geq 0\).

Suppose that \(P_L(\varphi) \leq 0\). Suppose that \(f\) is strictly \(L\) left-redundant with \(0 \leq I \leq N\). Let \((a_{i,j})_{0 \leq i \leq 1, j \in \mathbb{Z}}\) and \((b_{i,j})_{0 \leq i \leq 1, j \in \mathbb{Z}}\) be any two points of \(X_{[2]}^2\) with \(a_{i,j}, b_{i,j} \in L_1(X)\) such that \((a_{i,j})_{0 \leq i \leq 1} = (b_{i,j})_{0 \leq i \leq 1}\). If \(g((a_{i,j})_{0 \leq i \leq 1, -I \leq j \leq -I+N}) = (a_{i,j})_{0 \leq i \leq 1}\) and \(g((b_{i,j})_{0 \leq i \leq 1, -I \leq j \leq -I+N}) = (b_{i,j})_{0 \leq i \leq 1}\), then \(a_1 = a_{1,-I+m}\) and \(b_1 = b_{1,-I+m}\). Since \(f\) is \(L\) left-redundant, it follows that \(a_2 = b_2\). Since \(P_L(\varphi^{|2|}) \geq 0\), we have \(I + m \geq 0\), so that \(a_1 = b_1\). Therefore \(g\) is \(L\) left-redundant, and hence \(P_L(\varphi^{|s|}) \geq I - m = P_L(\varphi)\).

(2) By symmetry, (2) follows from (1). \(\square\)

**Proposition 8.4.** Let \(\varphi\) be an endomorphism of a subshift \((X, \sigma)\). Let \(s \in \mathbb{N}\).

1. If \(Q_R(\varphi^*) \geq 0\), then \(Q_R(\varphi^{|s|}) \geq 0\), and if \(Q_R(\varphi^*) \leq 0\), then \(Q_R(\varphi^{|s|}) \geq Q_R(\varphi^*)\).
2. If \(Q_L(\varphi^*) \geq 0\), then \(Q_L(\varphi^{|s|}) \geq 0\), and if \(Q_L(\varphi^*) \leq 0\), then \(Q_L(\varphi^{|s|}) \geq Q_L(\varphi^*)\).

**Proof.** (1) Suppose that \(\varphi^*\) is of \((m_s, n_s)\)-type and given by a local-rule \(f_s : L_{N_s+1}(X) \rightarrow L_1(X)\) with \(m_s, n_s \geq 0\) and \(N_s = m_s + n_s\). Suppose that \(f_s\) is strictly \(k_s\) right-mergible.
We define a local-rule $F_s : L_{N_s+1}(X|_s) \to L_1(X|_s)$ as follows: $F_1 = f_1$; if $s \geq 2$, then for each
\[ W = (a_{i,j})_{0 \leq i \leq s-1, 1 \leq j \leq N_s+1} \in L_{N_s+1}(X|_s), \quad a_{i,j} \in L_1(X), \]
\[ F_s(W) = (a_{i,j})_{1 \leq i \leq s}, \where \quad a_i = a_{i,m_{s+1}} \text{ for } i = 1, \ldots, s-1, \quad \text{and} \quad a_s = f_s(a_{0,1} \ldots a_{0,N_s+1}). \]

Then it is clear that $\varphi^{|s|}$ is a block-map of $(m_s,n_s)$-type given by $F_s$.

We may assume that $s \geq 2$. Suppose that $Q_R(\varphi^*) \geq 0$. Then $k_s \leq n_s$, so that $f_s$ is $n_s$ right-mergible. To prove that $F_s$ is $n_s$ right-mergible, let
\[ (a_{i,j})_{0 \leq i \leq s, j \in \mathbb{Z}} \quad \text{and} \quad (b_{i,j})_{0 \leq i \leq s, j \in \mathbb{Z}}, \quad a_{i,j}, b_{i,j} \in L_1(X) \]
be two points in $X|_{s+1}$ such that $(a_{i,j})_{0 \leq i \leq s-1} = (b_{i,j})_{0 \leq i \leq s-1}$ for all $j \leq 0$ and
\[ F_s((a_{i,j})_{0 \leq i \leq s-1,j \in \mathbb{Z}}) = F_s((b_{i,j})_{0 \leq i \leq s-1,j \in \mathbb{Z}}) \quad \text{for } j = 1, \ldots, n_s+1. \]
Then, by the definition of $F_s$, we have
\[ (a_{i,j-n_s})_{1 \leq i \leq s-1} = (b_{i,j-n_s})_{1 \leq i \leq s-1} \quad \text{for } j = 1, \ldots, n_s + 1, \]
and
\[ f_s((a_{0,j})_{j < N_s \leq j \in \mathbb{Z}}) = f_s((b_{0,j})_{j < N_s \leq j \in \mathbb{Z}}) \quad \text{for } j = 1, \ldots, n_s + 1. \]
Since $(a_{0,j})_{j \in \mathbb{Z}}, (b_{0,j})_{j \in \mathbb{Z}} \in X$ with $(a_{0,j})_{j \leq 0} = (b_{0,j})_{j \leq 0}$ and $f_s$ is $n_s$ right-mergible, it follows from (8.2) that $a_{0,1} = b_{0,1}$. Combining this with (8.1) for the case that $j = n_s + 1$, we have $(a_{1,1})_{0 \leq i \leq s-1} = (b_{1,1})_{0 \leq i \leq s-1}$.

Therefore $F_s$ is $n_s$ right-mergible, so that $Q_R(\varphi^{|s|}) \geq 0$.

Assume that $Q_R(\varphi^*) \leq 0$. Then $k_s \geq n_s$. To prove that $F_s$ is $k_s$ right-mergible, let
\[ (a_{i,j})_{0 \leq i \leq s, j \in \mathbb{Z}} \quad \text{and} \quad (b_{i,j})_{0 \leq i \leq s, j \in \mathbb{Z}}, \quad a_{i,j}, b_{i,j} \in L_1(X) \]
be two points in $X|_{s+1}$ such that $(a_{i,j})_{0 \leq i \leq s-1} = (b_{i,j})_{0 \leq i \leq s-1}$ for all $j \leq 0$ and
\[ F_s((a_{i,j})_{0 \leq i \leq s-1,j \in \mathbb{Z}}) = F_s((b_{i,j})_{0 \leq i \leq s-1,j \in \mathbb{Z}}) \quad \text{for } j = 1, \ldots, k_s+1. \]
Then, by the definition of $F_s$, we have
\[ (a_{i,j-n_s})_{1 \leq i \leq s-1} = (b_{i,j-n_s})_{1 \leq i \leq s-1} \quad \text{for } j = 1, \ldots, k_s + 1, \]
and
\[ f_s((a_{0,j})_{0 < J_s \leq J_s}) = f_s((b_{0,j})_{0 < J_s \leq J_s}) \quad \text{for } j = 1, \ldots, k_s + 1. \]
Since $(a_{0,j})_{j \in \mathbb{Z}}, (b_{0,j})_{j \in \mathbb{Z}} \in X$ with $(a_{0,j})_{j \leq 0} = (b_{0,j})_{j \leq 0}$ and $f_s$ is $k_s$ right-mergible, it follows from (8.4) that $a_{0,1} = b_{0,1}$. Combining this with (8.3) for the case that $j = n_s + 1$, we have $(a_{1,1})_{0 \leq i \leq s-1} = (b_{1,1})_{0 \leq i \leq s-1}$.

Therefore $F_s$ is $k_s$ right-mergible, so that $Q_R(\varphi^{|s|}) \geq n_s - k_s = Q_R(\varphi^*)$.

(2) By symmetry, (2) follows from (1). \(\square\)

**Theorem 8.5.** Let $\varphi$ be an onto endomorphism of a subshift $(X, \sigma)$. Let $s \geq 1$.

1. If $\varphi^*$ is weakly $q$-$R$ (respectively, weakly $q$-$L$), then so is $\varphi^{|s|}$, and hence $\varphi$ is essentially weakly $q$-$R$ (respectively, essentially weakly $q$-$L$).

2. With the hypotheses that $(X, \sigma)$ is a topological Markov shift and that $\varphi$ is one-to-one or $\sigma$ is topologically-transitive, it holds that if $\varphi^*$ is $q$-$R$ (respectively, $q$-$L$), then so is $(\varphi^{|s|})[s]$ for some $t \geq 1$. 

**Proof.** (1) By Proposition 8.4 and Theorem 5.2.

(2) If \( \varphi^* \) is \( q-R \), then \( Q_R(\varphi^*) \geq 0 \), by Theorem 4.2. If \( Q_R(\varphi^*) \geq 0 \), then it follows from Theorem 8.4 and Proposition 3.5(3) that \( Q_R((\varphi^*[r])^{|t|}) \geq 0 \) for all \( t \geq 1 \). Since \( (X^{[*]}, \sigma[*]) \) is an SFT, for some \( t \geq 1 \), \( (\varphi^*[r])^{|t|} \) is an endomorphism of a topological Markov shift, so that by Theorem 4.2, \( (\varphi^*[r])^{|t|} \) is \( q-R \).

The proof of the second version is similar.

**Theorem 8.6.** Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). Let \( s \geq 1 \).

1. (a) If \( \varphi^* \) is weakly \( p-L \) and weakly \( q-R \) (respectively, weakly \( p-R \) and weakly \( q-L \)), then \( (\varphi^*[r])^{|t|} \) is weakly \( LR \) (respectively, weakly \( RL \)) with some \( t \geq 1 \), and hence \( \varphi \) is essentially weakly \( LR \) (respectively, essentially weakly \( RL \)).

(b) If \( \varphi^* \) is weakly \( q-R \) and weakly \( q-L \), then there exists \( t \geq 1 \) such that \( (\varphi^*[r])^{|t|} \) is weakly \( q\)-biresolving, and hence \( \varphi \) is essentially weakly \( q\)-biresolving.

(c) If \( \varphi^* \) is weakly \( p-L \) and weakly \( q-L \) (respectively, weakly \( p-R \) and weakly \( q-R \)) with \( X \) infinite, then \( \varphi^*[r] \) is weakly \( LL \) (respectively, weakly \( RR \)), and hence \( \varphi \) is essentially weakly \( LL \) (respectively, essentially weakly \( RR \)).

2. With the hypotheses that \((X, \sigma)\) is a topological Markov shift and that \( \varphi \) is one-to-one or \( \sigma \) is topologically-transitive, the following hold.

(a) If \( \varphi^* \) is \( p-L \) and \( q-R \) (respectively, \( p-R \) and \( q-L \)), then \( (\varphi^*[r])^{|t|} \) is \( LR \) (respectively, \( RL \)) with some \( t \geq 1 \).

(b) If \( \varphi^* \) is \( q-L \) and \( q-R \), then \( (\varphi^*[r])^{|t|} \) is \( q\)-biresolving with some \( t \geq 1 \).

(c) If \( \varphi^* \) is \( p-L \) and \( q-L \) (respectively, \( p-R \) and \( q-R \)) with \( X \) infinite, then \( (\varphi^*[r])^{|t|} \) is \( LL \) (respectively, \( RR \)) for some \( t \geq 1 \), and hence \( \varphi \) is essentially weakly \( LR \) (respectively, essentially weakly \( RR \)).

**Proof.** (1)(a) Since \( \varphi^* \) is weakly \( p-L \) and weakly \( q-R \), it follows from Theorems 6.12 and 5.2(1) that \( P_L(\varphi^*) \geq 0 \) and \( Q_R(\varphi^*) \geq 0 \). Hence, by Propositions 8.2(1) and 8.4(1) that \( P_L((\varphi^*[r])^{|t|}) \geq 0 \) and \( Q_R((\varphi^*[r])^{|t|}) \geq 0 \). Therefore, by Theorem 7.8(1), \( (\varphi^*[r])^{|t|} \) is weakly \( LR \) with some \( t \geq 1 \), so that \( \varphi \) is essentially weakly \( LR \).

By symmetry, the second version follows from the first one.

(1)(b) Since \( \varphi^* \) is weakly \( q-R \) and weakly \( q-L \), it follows from Theorem 5.2 that \( Q_R(\varphi^*) \geq 0 \) and \( Q_L(\varphi^*) \geq 0 \). Hence, by Proposition 8.4 that \( Q_R(\varphi^*[r]) \geq 0 \) and \( Q_L(\varphi^*[r]) \geq 0 \). Therefore, by Theorem 5.3(1), there exists \( t \geq 1 \) such that \( (\varphi^*[r])^{|t|} \) is weakly \( q\)-biresolving, and hence \( \varphi \) is essentially weakly \( q\)-biresolving.

(1)(c) Since \( \varphi^* \) is weakly \( p-L \) and weakly \( q-L \), it follows from Theorems 6.12 and 5.2(1) that \( P_L(\varphi^*) \geq 0 \) and \( Q_L(\varphi^*) \geq 0 \). Hence, by Propositions 8.2(1) and 8.4(2) that \( P_L((\varphi^*[r])^{|t|}) \geq 0 \) and \( Q_L((\varphi^*[r])^{|t|}) \geq 0 \). Therefore, by Proposition 7.9, \( (\varphi^*[r])^{|t|} \) is weakly \( LL \), and hence \( \varphi \) is essentially weakly \( LL \).

By symmetry, the second version follows from the first one.

(2)(a) Since \( \varphi^* \) is \( p-L \) and \( q-R \), it follows from Proposition 6.9 and Theorem 4.2(1) that \( P_L(\varphi^*) \geq 0 \) and \( Q_R(\varphi^*) \geq 0 \). Hence, by Propositions 8.2(1) and 8.4(1) that \( P_L((\varphi^*[r])^{|t|}) \geq 0 \) and \( Q_R((\varphi^*[r])^{|t|}) \geq 0 \). Since \((X, \sigma)\) is a topological Markov shift, there exists \( r \geq 1 \) such that \( (\varphi^*[r])^{|t|} \) is an endomorphism of a topological Markov shift. By hypothesis, it follows that \( (\varphi^*[r])^{|t|} \) is one-to-one or the shift is transitive. Since \( P_L((\varphi^*[r])^{|t|}) \geq 0 \) and \( Q_R((\varphi^*[r])^{|t|}) \geq 0 \) by Propositions 3.5 and 6.5, it follows from Theorem 7.3 that \( (\varphi^*[r])^{|t|} \) is weakly \( LR \), for some \( t \geq 1 \).
(2)(b) Since $\varphi^s$ is $q$-$L$ and $q$-$R$, it follows from Theorem 4.2 that $Q_L(\varphi^s) \geq 0$ and $Q_R(\varphi^s) \geq 0$. By similar arguments to those in (2)(a) using Theorem 4.4 instead of Theorem 7.3, we see that $(\varphi^{[t]}s)^{[t]}$ is $q$-biresolving, for some $t \geq 1$.

(2)(c) Using Proposition 6.9 and Theorem 4.2(2), we have $P_L(\varphi^s) \geq 0$ and $Q_L(\varphi^s) \geq 0$. Hence, using Propositions 8.2(1),8.4(2) and 7.9(1), we have $P_L((\varphi^s)_1) = 0$ and $Q_L((\varphi^s)_1) = 0$. There exists $t \geq 1$ such that $(\varphi^{[t]}s)^{[t]}$ is an onto endomorphism of a topological Markov shift and by Propositions 6.5 and 3.5, $P_L((\varphi^{[t]}s)^{[t]}_1) = 0$ and $Q_L((\varphi^{[t]}s)^{[t]}_1) = 0$. Hence, by Proposition 7.9(4), $(\varphi^{[t]}s)^{[t]}$ is an LL endomorphism of the shift.

We remark that the result that if $\varphi^s$ is an LL endomorphism of a topological Markov shift $(X, \sigma)$ with $\varphi$ one-to-one or $\sigma$ transitive, then $\varphi$ is essentially LL, is a result which is not given by [N9].

**Proposition 8.7.** Suppose that $\varphi$ is an onto endomorphism of a subshift $(X, \sigma)$ and is of $(m,n)$-type with $m,n \geq 0$. Let $s \geq 1$. Then

$$Q_L(\varphi^{[s]}s) \leq Q_L(\varphi^{[s+1]}s) \quad \text{and} \quad Q_R(\varphi^{[s]}s) \leq Q_R(\varphi^{[s+1]}s).$$

**Proof.** Since $\varphi^{[s+1]}s = (\varphi^{[s]}s)^{[2]}$, it suffices to prove the proposition for $s = 2$.

Suppose that $\varphi$ is given by a local rule $f : L_{N+1}(X) \to L_1(X)$ with $N = m + n$. Define $g : L_{N+1}(X^{[2]}2) \to L_1(X^{[2]}2)$ to be the local-rule such that for each $W = (a_i_j)_{0 \leq i \leq 1, 1 \leq j \leq N+1} \in L_{N+1}(X^{[2]}2)$ with $a_i_j \in L_1(X)$, $g(W) = (a_i)_{1 \leq i \leq 2}$, where

$$a_1 = a_1, m+1 \quad \text{and} \quad a_2 = f(a_1, 1 \ldots a_1, N+1).$$

Then it is clear that $\varphi^{[2]}s$ is of $(m,n)$-type and given by $g$. Noting that $a_1 = f(a_0, 1 \ldots a_0, N+1)$ in the definition of $g$, it is easy to see that if $f$ is $k$-right-mergible, then so is $g$. By symmetry we also know that if $f$ is $k$-left-mergible, then so is $g$. Since both $\varphi$ and $\varphi^{[2]}s$ are of $(m,n)$-type, we have $Q_R(\varphi) \leq Q_R(\varphi^{[2]}s)$ and $Q_L(\varphi) \leq Q_L(\varphi^{[2]}s)$. 

**Theorem 8.8.** Let $\varphi$ be an onto endomorphism of a subshift $(X, \sigma)$.

1. $\varphi$ is essentially weakly $p$-$L$ and right $\sigma$-expansive (respectively, essentially weakly $p$-$R$ and left $\sigma$-expansive) if and only if there exists $s \geq 1$ with $P_L(\varphi^s) > 0$ (respectively, $P_R(\varphi^s) > 0$).
2. $\varphi$ is essentially weakly $q$-$L$ and left $\sigma$-expansive (respectively, essentially weakly $q$-$R$ and right $\sigma$-expansive) if and only if there exists $s \geq 1$ with $Q_R(\varphi^s) > 0$ (respectively, $Q_L(\varphi^s) > 0$).
3. $\varphi$ is essentially weakly $LR$ (respectively, essentially weakly RL) and expansive if and only if there exist $s,t \geq 1$ with $P_L(\varphi^s) > 0$ and $Q_R(\varphi^t) > 0$ (respectively, $P_R(\varphi^s) > 0$ and $Q_L(\varphi^t) > 0$).
4. $\varphi$ is positively expansive if and only if there exist $s,t \geq 1$ with $Q_R(\varphi^s) > 0$ and $Q_L(\varphi^t) > 0$.
5. If $(X, \sigma)$ is a topological Markov shift and if $\varphi$ is one-to-one or $\sigma$ is topologically transitive, then statements (1), (2) and (3) with all “weakly” in them deleted hold.

**Proof.** (1) Assume that $\varphi$ is essentially weakly $p$-$L$ and left $\sigma$-expansive. Then it follows from [N9] Proposition 7.7] that there exists $s_1 \geq 1$ such that $\varphi^{s_1} \sigma^{-1}$ is essentially weakly $p$-$L$ endomorphism of $(X, \sigma)$. Hence so is $(\varphi^{s_1} \sigma^{-1})^2$ (by [N9] Remark 7.9), and $(\varphi^{s_1} \sigma^{-1})^2 \sigma$ is essentially weakly $p$-$L$ and right $\sigma$-expansive by
Proposition 7.6. Therefore, it follows from Proposition 7.11] that there exists \( s_2 \geq 1 \) such that \((\varphi^{2s_1}\sigma^{2s_2})^s\) is a weakly p-L endomorphism of \((X, \sigma)\). Hence, by Proposition 6.13(1) \( P_L(\varphi^{2s_1}) \geq s_2 > 0 \).

Conversely assume that \( P_L(\varphi^s) \geq 1 \) with \( s \geq 1 \). Then it follows from Proposition 6.13(1) that \( \varphi^s\sigma^{-1} \) is weakly p-L. Therefore \( \varphi^s \) is weakly p-L and right \( \sigma \)-expansive by Proposition 7.6. Hence, by Theorem 8.1(1), \( \varphi \) is an essentially weakly p-L, right \( \sigma \)-expansive endomorphism of \((X, \sigma)\).

By symmetry, the second version follows from the first one.

(2) By straightforward modifications in the proof above. Use Theorem 5.2(1) instead of Proposition 6.13(1), and use Theorem 8.5(1) instead of Theorem 8.1(1).

(3) The “only-if” part is proved by (1) and (2). To prove the “if” part, assume that \( P_L(\varphi^s) > 0 \) and \( Q_R(\varphi^t) > 0 \). Then by (1) and (2), \( \varphi^s \) is right \( \sigma \)-expansive and \( \varphi^t \) is left \( \sigma \)-expansive, so that \( \varphi \) is expansive (by Proposition 7.3). By Lemma 8.9 below, \( P_L(\varphi^{st}) > 0 \) and \( Q_R(\varphi^{st}) > 0 \). Hence for the same reason as in the proof of Theorem 8.6(1)(a) \( \varphi \) is essentially weakly q-biresolving. Hence by Theorem 4.5, (4) is proved.

(4) Noting Theorem 4.5, we know that the “only-if” part is proved by (2). To prove the “if” part, assume that \( Q_R(\varphi^s) > 0 \) and \( Q_L(\varphi^t) > 0 \). Then by (2) \( \varphi^s \) is right \( \sigma \)-expansive and \( \varphi^t \) is left \( \sigma \)-expansive, so that \( \varphi \) is expansive. By Lemma 8.9 below, \( Q_L(\varphi^{st}) > 0 \) and \( Q_R(\varphi^{st}) > 0 \). Therefore, for the same reason as in the proof of Theorem 8.6(1)(b), \( \varphi \) is essentially weakly q-biresolving. Hence by Theorem 4.5, (4) is proved.

(5) Let \((1'),(2')\) and \((3')\) be the statements (1) (2) and (3) with all “weakly” in them deleted, respectively.

Statement \((1')\) is proved by the proof of (1), Proposition 6.13(4) and Propositions 6.9 and 6.4, without using Proposition 6.13(1).

Statement \((2')\) is proved by a similar proof to that of (1), but use Proposition 2.6, Theorem 4.2 and Proposition 3.5 without using Proposition 6.13(1), and use Theorem 8.5(2) instead of Theorem 8.1(1).

The proof of \((3')\) is given as follows. The “only-if” part follows from \((1')\) and \((2')\). Suppose that \( P_L(\varphi^s) > 0 \) and \( Q_R(\varphi^t) > 0 \). Then by the proof of (3) we see that \( \varphi \) is expansive. By Lemma 8.9 below, \( P_L(\varphi^{st}) > 0 \) and \( Q_R(\varphi^{st}) > 0 \). Hence for the same reason as in the proof of Theorem 8.6(2)(a) \( \varphi \) is essentially LR.

A direct proof of (4) of Theorem 8.8 for the case that \( \varphi \) is an onto endomorphism of a transitive topological Markov shift \((X, \sigma)\), is given as follows. Note that \( \varphi \) is positively expansive if and only if \( \varphi \) is an expansive, essentially q-biresolving endomorphism of \((X, \sigma)\), as stated immediately after Theorem 4.5. The “only if” part follows from \((2')\) in the proof above. Suppose that \( Q_R(\varphi^s) > 0 \) and \( Q_L(\varphi^t) > 0 \). Then by the proof of (4) we see that \( \varphi \) is expansive. By Lemma 8.9 below, \( Q_R(\varphi^{st}) > 0 \) and \( Q_L(\varphi^{st}) > 0 \). Hence by a similar proof to that of Theorem 8.6(2)(b), \( \varphi \) is essentially q-biresolving.

**Lemma 8.9.** Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). Let \( s, t \geq 1 \).

1. If \( P_L(\varphi^s) > 0 \) and \( Q_R(\varphi^t) > 0 \) (respectively, \( P_R(\varphi^s) > 0 \) and \( Q_L(\varphi^t) > 0 \)), then \( P_L(\varphi^{st}) > 0 \) and \( Q_R(\varphi^{st}) > 0 \) (respectively, \( P_R(\varphi^{st}) > 0 \) and \( Q_L(\varphi^{st}) > 0 \)).

2. If \( Q_R(\varphi^s) > 0 \) and \( Q_L(\varphi^t) > 0 \), then \( Q_R(\varphi^{st}) > 0 \) and \( Q_L(\varphi^{st}) > 0 \).

**Proof.** Use Lemma 9.1 appearing in the next section. □

Here we present some results on onto endomorphisms of onesided subshifts.
Theorem 8.10. Let $\tilde{\varphi}$ be an onto endomorphism of a onesided-subshift $(\tilde{X}, \tilde{\sigma})$. Let $\varphi$ be its induced endomorphism of the induced subshift $(X, \sigma)$ of $(\tilde{X}, \tilde{\sigma})$. Then the following statements are equivalent.

1. $\tilde{\varphi}$ is positively-expansive.
2. $\varphi$ is essentially weakly $q$-R and left $\sigma$-expansive.
3. $Q_R(\varphi^s) > 0$ for some $s \geq 1$.
4. There exists a subshift $(X_0, \sigma_0)$, a left $\sigma_0$-expansive, weakly LR endomorphism $\varphi_0$ and a conjugacy $\theta : (\tilde{X}, \tilde{\sigma}) \to (X_0, \sigma_0)$ (between onesided-subshifts) such that $\varphi_0 = \tilde{\theta} \varphi \tilde{\theta}^{-1}$.

If $(\tilde{X}, \tilde{\sigma})$ is a transitive SFT, then all “weakly” can be deleted in the above statements.

Proof. Assume (1). By [N9] Remark 7.1, there exists a onesided 1-1 textile subsystem $U$ of a textile system $T$ such that $(X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi)$. Since $\varphi_U$ is positively expansive, an argument similar to that in [N9] the proof of Proposition 7.2(1) shows that there exists $m \geq 1$ such that $\xi_{(U^{[m]}),\ast}$ of $\tilde{Z}_{(U^{[m]}),\ast}$ onto $\tilde{X}_{(U^{[m]}),\ast}$ is one-to-one (see Subsection 2.3 for the notation). Therefore, $\varphi = \varphi_U$ is left $\sigma$-expansive and moreover, there exists $n \geq 0$ such that $\xi_{(U^{[m]}),\ast}$ is of $(0, n)$ type. Let $T_1 = ((T^{[m]}),\ast)^{[n+1]} = (p_1, q_1 : \Gamma_1 \to G_1)$. Then $((U^{[m]}),\ast)^{[n+1]}$ is a textile subsystem of $T_1$. Let $T_2 = (p_2, q_2 : \Gamma_2 \to G_2)$ be the textile system defined as follows: $G_2^* = G_1^*$; $\Gamma_2$ is the subgraph of $\Gamma_1$ with $A_{\Gamma_2} = L_{n+1}(Z_{(U^{[m]}),\ast})$ and $V_{\Gamma_2} = L_n(Z_{(U^{[m]}),\ast})$; $p_2^* and $q_2^*$ are the restrictions of $p_1^*$ and $q_1^*$ on $\Gamma_2$, respectively. Clearly $((U^{[m]}),\ast)^{[n+1]}$ is a textile subsystem of $T_2^*$. Since $\xi_{(U^{[m]}),\ast}$ is of $(0, n)$ type, it follows that $T_2^*$ is onesided 1-1 and weakly $p$-L. Therefore, if we let $U = (((U^{[m]}),\ast)^{[n+1]}),$ then $\tilde{U}$ is a textile subsystem of the weakly $q$-R textile system $T_2 = (T_2^*,\ast)$. Since $U$ is onesided 1-1, so is $\tilde{U}$ and $(X_U, \sigma_U, \varphi_U)$ is conjugate to $(X_0, \sigma_0, \varphi_0)$. Hence $\varphi = \varphi_U$ is an essentially weakly $q$-R endomorphism of $(X, \sigma)$. Thus (2) is proved.

Assume (2). Then, by Theorem 8.8(2), (3) follows.

Assume (3). Then, since $\varphi$ has memory zero, so is $\varphi^s$ and hence $\varphi^s$ is weakly $p$-L, by Theorem 6.12. Since $Q_R(\varphi^s) > 0$, it follows from Theorem 5.2(1) that $\varphi^s$ is weakly $q$-R and left $\sigma$-expansive. Therefore, by Theorem 8.6(1)(a), there exists $t \geq 1$ such that $(\varphi^s)^{[t]}$ is weakly LR and left $\sigma_0$-expansive, where $(X_0, \sigma_0) = (Y^t, \sigma_{Y^t})$ with $Y = X^{[s]}$. Since $\varphi$ has memory zero, $\rho_{\varphi,s}$ and its inverse have memory zero. Let $\rho_{Y,t} : (Y^t, \sigma_Y) \to (Y^t, \sigma_{Y^t})$ be the conjugacy such that for $y = (b_j)_{j \in \mathbb{Z}} \in Y$ with $b_j \in L_1(Y) \rho_{Y,t}(y) = (b_j \ldots b_{j+t})_{j \in \mathbb{Z}}$. Then $\rho_{Y,t}$ has memory zero and so does its inverse. If we let $\theta = \rho_{Y,t}(\rho_{\varphi,s}^\ast)$ and $\varphi_0 = \theta \varphi^s \theta^{-1}$, then (4) is proved.

Assume (4). There exists a onesided 1-1 textile-subsystem of $U$ of a weakly LR textile system, say $T = (p, q : \Gamma \to G)$, such that $(X_U, \sigma_U, \varphi_U) = (X_0, \sigma_0, \varphi_0)$. Since $\varphi_U$ is left $\sigma_U$-expansive, it follows from [N9] Proposition 7.2(1) that there exists $n \geq 1$ such that $\xi_{(U^{[n]}),\ast}$ is one-to-one. Since $T$ is weakly $q$-R, $(T^{[n]}),\ast$ is weakly $p$-L. Since $(U^{[n]}),\ast$ is a textile-subsystem of $(T^{[n]}),\ast$, it follows, for the same reason as in [N9] the proof of Proposition 7.5(1) that $\xi_{(U^{[n]}),\ast}$ has memory zero. This implies that for any textile $(\alpha_{i,j}),_{i,j,\mathbb{Z}} \in U$ with $\alpha_{i,j} \in A_{\Gamma}$, the “quarter textile” is determined by the sub-configuration $(\alpha_{i,j}),_{i \in \mathbb{N}, 1 \leq j \leq n}$. This implies that $\varphi_U$ is positively expansive. Therefore $\tilde{\varphi}$ is positively expansive. Thus (1) is proved.
To prove the remainder, let $(2')$ and $(4')$ be the statements obtained from (2) and (4) by deleting “weakly”, respectively. Passing through a higher-block conjugacy between endomorphisms of onesided topological Markov shifts, we may assume that $(\bar{X}, \bar{\sigma})$ is a transitive onesided topological Markov shift.

Since any positively-expansive endomorphism of a transitive onesided SFT is onto and conjugate to a onesided topological Markov shift [Ku], it follows from [N5, Theorem 2.12] and [N9, Proposition 6.2] that (4') implies (1). Clearly (4') implies (2'). By Theorem 8.8(5), (2') implies (3).

Assume (3). Then, since $\varphi$ has memory zero, so is $\varphi^s$ and hence $\varphi^s$ is p-L, by Proposition 6.9. Since $Q_R(\varphi^s) > 0$, $\varphi^s$ is q-R and left $\sigma$-expansive, by Theorem 4.2 and [N9 Proposition 7.6(2)]. By Theorem 8.6(2)(a), $(\varphi^{[s,t]}_s)^{[s,t]}$ is LR for some $t \geq 1$. Therefore in a similar way to the above, (4') is proved. \qed

9. LIMITS OF ONESIDED RESOLVING DIRECTIONS

The following lemma directly follows from Propositions 6.7 and 3.7.

Lemma 9.1. Let $\varphi$ be an onto endomorphism of a a subshift. Let $s, t \in \mathbb{N}$.

$$
P_L(\varphi^{s+t}) \geq P_L(\varphi^s) + P_L(\varphi^t), \quad P_R(\varphi^{s+t}) \geq P_R(\varphi^s) + P_R(\varphi^t), \quad Q_R(\varphi^{s+t}) \geq Q_R(\varphi^s) + Q_R(\varphi^t), \quad Q_L(\varphi^{s+t}) \geq Q_L(\varphi^s) + Q_L(\varphi^t).
$$

Therefore, it follows (from [Wa Theorem 4.9]) that the following proposition holds.

Theorem 9.2. Let $\varphi$ be an onto endomorphism of a subshift.

1. $\lim_{s \to \infty} P_L(\varphi^s)/s$ exists and equals $\sup_s P_L(\varphi^s)/s$;
2. $\lim_{s \to \infty} P_R(\varphi^s)/s$ exists and equals $\sup_s P_R(\varphi^s)/s$;
3. $\lim_{s \to \infty} Q_R(\varphi^s)/s$ exists and equals $\sup_s Q_R(\varphi^s)/s$;
4. $\lim_{s \to \infty} Q_L(\varphi^s)/s$ exists and equals $\sup_s Q_L(\varphi^s)/s$.

Definition 9.3. Let $\varphi$ be an onto endomorphism of a subshift. Define

$$
p_L(\varphi) = \lim_{s \to \infty} P_L(\varphi^s)/s, \quad p_R(\varphi) = \lim_{s \to \infty} P_R(\varphi^s)/s, \quad q_L(\varphi) = \lim_{s \to \infty} Q_L(\varphi^s)/s, \quad q_R(\varphi) = \lim_{s \to \infty} Q_R(\varphi^s)/s.
$$

We call $-p_L(\varphi)$, $p_R(\varphi)$, $q_L(\varphi)$ and $-q_R(\varphi)$ the limit of p-L direction of $\varphi$, the limit of p-R direction of $\varphi$, the limit of q-L direction of $\varphi$ and the limit of q-R direction of $\varphi$, respectively. They are also called the limits of onesided resolving directions of $\varphi$.

Each of the limits of onesided resolving directions is an invariant of topological conjugacy between endomorphisms of subshifts.

Theorem 9.4. If two onto endomorphisms $\varphi$ and $\psi$ of subshifts are topologically conjugate, then

$$
-p_L(\varphi) = -p_L(\psi), \quad -q_R(\varphi) = -q_R(\psi), \quad p_R(\varphi) = p_R(\psi), \quad q_L(\varphi) = q_L(\psi).
$$

Proof. There exists a conjugacy $\theta : (X, \sigma_X, \varphi) \to (Y, \sigma_Y, \psi)$ between commuting systems. Since $\psi = \theta \varphi \theta^{-1}$, it follows from Proposition 6.7 that $P_L(\psi^s) \geq P_L(\varphi^s) + P_L(\theta) + P_L(\theta^{-1})$ for all $s \geq 1$. Therefore $p_L(\psi^s) \geq p_L(\varphi^s)$. Similarly we have $p_L(\varphi^s) \geq p_L(\psi^s)$. Thus $-p_L(\varphi) = -p_L(\psi)$. The other equations are similarly obtained by Propositions 6.7 and 3.7. \qed
Proposition 9.5. Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). Let \( i \in \mathbb{N} \) and let \( j \in \mathbb{Z} \).

\[
p_L(\varphi^i \sigma^j) = ip_L(\varphi) + j, \quad p_R(\varphi^i \sigma^j) = ip_R(\varphi) - j, \quad q_R(\varphi^i \sigma^j) = iq_R(\varphi) + j, \quad q_L(\varphi^i \sigma^j) = iq_L(\varphi) - j,
\]

and hence each of

\[
p_L(\varphi) + p_R(\varphi), \quad p_L(\varphi) + q_L(\varphi), \quad p_R(\varphi) + q_R(\varphi), \quad q_L(\varphi) + q_R(\varphi),
\]
is shift-invariant.

Proof. Using Proposition 6.4, we have

\[
p_L(\varphi^i \sigma^j) = \lim_{s \to \infty} P_L(\varphi^{is} \sigma^{js})/s = \lim_{s \to \infty} P_L(\varphi^{is})/s + j
\]

\[
= \lim_{s \to \infty} iP_L(\varphi^{is})/(is) + j = ip_L(\varphi) + j.
\]
The remainder are similarly proved by using Propositions 6.4 and 3.4. \( \square \)

Theorem 9.6. Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). Let \( i \in \mathbb{N} \).

1. \( \varphi \) is essentially weakly \( p \)-\( L \) and right \( \sigma \)-expansive if and only if \( p_L(\varphi) > 0 \);
2. \( \varphi \) is essentially weakly \( q \)-\( R \) and left \( \sigma \)-expansive if and only if \( q_R(\varphi) > 0 \);
3. \( \varphi \) is essentially weakly \( p \)-\( R \) and left \( \sigma \)-expansive if and only if \( p_R(\varphi) > 0 \);
4. \( \varphi \) is essentially weakly \( q \)-\( L \) and right \( \sigma \)-expansive if and only if \( q_L(\varphi) > 0 \).

If \((X, \sigma)\) is an SFT and if \( \varphi \) is one-to-one or \( \sigma \) is transitive, then, then all “weakly” can be deleted in the above statements.

Proof. The theorem follows from Theorems 8.8 and 9.2. \( \square \)

Corollary 9.7. Let \( \varphi \) be an onto endomorphism of a subshift \((X, \sigma)\). Let \( i \in \mathbb{N} \) and let \( j \in \mathbb{Z} \).

1. \( \varphi^i \sigma^j \) is essentially weakly \( p \)-\( L \) and right \( \sigma \)-expansive if and only if
   \[
j/i > -p_L(\varphi);
\]
2. \( \varphi^i \sigma^j \) is essentially weakly \( q \)-\( R \) and left \( \sigma \)-expansive if and only if
   \[
j/i > -q_R(\varphi);
\]
3. \( \varphi^i \sigma^j \) is essentially weakly \( p \)-\( R \) and left \( \sigma \)-expansive if and only if
   \[
j/i < p_R(\varphi);
\]
4. \( \varphi^i \sigma^j \) is essentially weakly \( q \)-\( L \) and right \( \sigma \)-expansive if and only if
   \[
j/i < q_L(\varphi).
\]

If \((X, \sigma)\) is an SFT and if \( \varphi \) is one-to-one or \( \sigma \) is transitive, then, then all “weakly” can be deleted in the above statements.

Proof. By Proposition 9.5 and Theorem 9.6. \( \square \)

Corollary 9.8. Let \( \tilde{\varphi} \) be an onto endomorphism of a onesided subshift \((\tilde{X}, \tilde{\sigma})\). Let \( \varphi \) be its induced endomorphism of the induced subshift \((X, \sigma)\) of \((\tilde{X}, \tilde{\sigma})\). Then \( \tilde{\varphi} \) is positively expansive if and only if \( q_R(\varphi) > 0 \). Moreover, for \( i, j \in \mathbb{N} \), \( \varphi^i \tilde{\sigma}^j \) is positively expansive if and only if \( j/i > -q_R(\varphi) \).

Proof. By Theorems 8.10 and 9.6(2). \( \square \)
Proposition 9.9. Let \( \varphi \) be an onto endomorphism of a subshift with infinitely many points.

1. \( p_L(\varphi) + p_R(\varphi) \) is nonpositive, and if \( \varphi \) is of \((m,n)\)-type with \( m, n \geq 0 \), then \(-n \leq p_R(\varphi) \leq -p_L(\varphi) \leq m\).
2. \( p_L(\varphi) + q_L(\varphi) \) and \( p_R(\varphi) + q_R(\varphi) \) are nonpositive, and hence \( q_L(\varphi) \leq -p_L(\varphi) \) and \( p_R(\varphi) \leq -q_R(\varphi) \).
3. If \( q_R(\varphi) + q_L(\varphi) \geq 0 \) and \( \varphi \) is of \((m,n)\)-type with \( m, n \geq 0 \), then \(-n \leq p_R(\varphi) \leq -q_R(\varphi) \leq q_L(\varphi) \leq -p_L(\varphi) \leq m\).

Proof. (1) For \( s \geq 1 \), \( \varphi^s \) is of \((ms,ns)\)-type. Hence the result is proved by using Proposition 6.13(3) and Theorem 9.2.
(2) By Proposition 7.9(1) and Theorem 9.2.
(3) By Proposition 7.10 and Theorem 9.2. \( \square \)

For an onto endomorphism \( \varphi \) of a subshift with infinitely many points, define

\[
\begin{align*}
c_R(\varphi) &= \max\{-p_L(\varphi), -q_R(\varphi)\}, \\
c_L(\varphi) &= \min\{p_R(\varphi), q_L(\varphi)\}, \\
d_R(\varphi) &= \min\{-p_L(\varphi), -q_R(\varphi)\}, \\
d_L(\varphi) &= \max\{p_R(\varphi), q_L(\varphi)\}.
\end{align*}
\]

Then by Proposition 9.9, we have \( c_L(\varphi) \leq c_R(\varphi) \).
Clearly, \( c_L(\varphi) = c_R(\varphi) \) if and only if all limits of onesided resolving directions of \( \varphi \) coincide.

Theorem 9.10. Let \( \varphi \) be an onto endomorphism of a subshift \((X,\sigma)\). Let \( i \in \mathbb{N} \) and let \( j \in \mathbb{Z} \).

1. \( \varphi^i\sigma^j \) is essentially weakly LR and expansive if and only if \( j/i > c_R(\varphi) \).
2. \( \varphi^i\sigma^j \) is essentially weakly RL and expansive if and only if \( j/i < c_L(\varphi) \).

If \((X,\sigma)\) is an SFT and if \( \varphi \) is one-to-one or \( \sigma \) is transitive, then all "weakly" can be deleted in the above statements.

Proof. (1) If \( j/i > c_R(\varphi) \), then, by Corollary 9.7, \( \varphi^i\sigma^j \) is essentially weakly \( p\)-L, right \( \sigma \)-expansive, essentially weakly \( q\)-R and left \( \sigma \)-expansive. Therefore, using Theorem 8.8(1),(2),(3), we see that \( \varphi^i\sigma^j \) essentially weakly LR and expansive.

Conversely assume that \( \varphi^i\sigma^j \) is essentially weakly LR and expansive. Then it follows from Corollary 9.7 that \( j/i > c_R(\varphi) \).

The proof of the second version is similar.

The remainder is proved by using Theorem 8.8(5). \( \square \)

Theorem 9.11. Let \( \varphi \) be an onto endomorphism of a subshift \((X,\sigma)\). There exist \( i \in \mathbb{N} \) and \( j \in \mathbb{Z} \) with \( \varphi^i\sigma^j \) positively expansive if and only if \( q_R(\varphi) + q_L(\varphi) > 0 \).
If \( q_R(\varphi) + q_L(\varphi) > 0 \), then \( \varphi^i\sigma^j \) is positively expansive if and only if \(-q_R(\varphi) < j/i < q_L(\varphi) \).
Proof. Let \( i \in \mathbb{N} \) and let \( j \in \mathbb{Z} \). Using Theorem 8.8(1),(2),(4), we see that the following two conditions are equivalent. (a) \( \varphi^i \sigma^j \) is a right \( \sigma \)-expansive, essentially weakly \( q \)-L and left \( \sigma \)-expansive, essentially weakly \( q \)-R endomorphism of \((X, \sigma)\); (b) \( \varphi^i \sigma^j \) is positively expansive. By Corollary 9.7(2),(4), we see that (a) is equivalent to the condition (c) \(-q_R(\varphi) < j/i < q_L(\varphi)\). Hence (b) and (c) are equivalent. From this the theorem is proved. \( \square \)

A direct proof of Theorem 9.11 for the case that \( \varphi \) is an onto endomorphism of a transitive topological Markov shift \((X, \sigma)\), is given as follows. Let \( i \in \mathbb{N} \) and \( j \in \mathbb{Z} \). Let (a),(b) and (c) be the same as in the proof above. Let (a’) be the condition obtained from (a) by deleting “weakly”. Using Theorem 8.8(4),(5), we see that (a’) and (b) are equivalent (a necessary direct proof is found immediately after the proof of Theorem 8.8). By Corollary 9.7, (a’) and (c) are equivalent. Hence (b) and (c) are equivalent. From this the theorem is proved.

**Proposition 9.12.**

1. If \( \varphi \) is an automorphism of a subshift, then
   \[ q_L(\varphi) = p_L(\varphi^{-1}) \quad \text{and} \quad q_R(\varphi) = p_R(\varphi^{-1}). \]
2. If \( \varphi \) is an automorphism of a subshift with infinitely many points, then
   \[ q_R(\varphi) + q_L(\varphi) \leq 0 \quad \text{and} \quad d_L(\varphi) \leq d_R(\varphi). \]
3. If \( \varphi \) is a symbolic automorphism of a subshift with infinitely many points, then
   \[ p_L(\varphi) = q_R(\varphi) = p_R(\varphi) = q_L(\varphi) = 0. \]

**Proof.** By Proposition 7.11 and Theorem 9.2. \( \square \)

As will be proved in the next section, a “directionally essentially \( P \)” endomorphism of a subshift is the same as an “essentially \( P \)” endomorphism of the subshift, for every one of the properties “weakly \( p \)-L” “weakly \( p \)-R”, “weakly \( q \)-R”, “weakly \( q \)-L”, “weakly \( LL \)”, “weakly \( RR \)”, “weakly \( LR \)” and “weakly \( q \)-biresolving”. Therefore the following definitions and the statements of results are compatible with those of [N9].

Let \( X \) be a zero-dimensional compact metric space. Let \( S(X) \) denote the monoid of surjective continuous maps of \( X \) and let \( H(X) \) denote the group of homeomorphisms in \( S(X) \). Let \( \tau \) be in \( H(X) \) and expansive. Let \( J \) be a commutative submonoid of \( S(X) \).

Let \( PL_J(\tau) \) denote the set of all essentially weakly \( p \)-L endomorphisms of \((X, \tau)\) in \( J \). Replacing “\( p \)-L” by “\( q \)-R” “\( p \)-R” and “\( q \)-L” respectively in this definition, we respectively define \( QR_J(\tau), PR_J(\tau) \) and \( QL_J(\tau) \). Clearly \( PR_J(\tau) = PL_J(\tau^{-1}) \) and \( QL_J(\tau) = QR_J(\tau^{-1}) \). Let \( PL_J^\circ(\tau), QR_J^\circ(\tau), PR_J^\circ(\tau) \) and \( QL_J^\circ(\tau) \) be the set of all right \( \tau \)-expansive elements of \( PL_J(\tau) \), the set of all left \( \tau \)-expansive elements of \( QR_J(\tau) \), the set of all left \( \tau \)-expansive elements of \( PR_J(\tau) \) and the set of all right \( \tau \)-expansive elements of \( QL_J(\tau) \), respectively.

Let \( C_J(\tau) \) denote the set of all essentially weakly LR endomorphisms of \((X, \tau)\). Let \( C_J^\circ(\tau) \) denote the set of all expansive elements of \( C_J(\tau) \). Let

\[ D_J(\tau) = PL_J(\tau) \cup QR_J(\tau) \quad \text{and} \quad D_J^\circ(\tau) = PL_J^\circ(\tau) \cup QR_J^\circ(\tau). \]

Then

\[ D_J(\tau) \cup D_J(\tau^{-1}) = PL_J(\tau) \cup QR_J(\tau) \cup PR_J(\tau) \cup QL_J(\tau) \]
is the set of all essentially weakly resolving endomorphisms of \((X, \tau)\) in \(J\). We call
\(D_j(\tau)\) the \(\delta\)-district of \(\tau\) in \(J\), whereas we call \(C_j(\tau)\) the (essentially weakly LR) cone
of \(\tau\) in \(J\). In particular, when \((X, \tau)\) is conjugate to an SFT, we call \(C_j(\tau)\) the \(\text{ELR cone (essentially LR cone)}\) of \(\tau\) in \(J\). We will call \(\alpha_j(\tau)\) the interior of \(\alpha_j(\tau)\) when “\(\alpha\)” represents one of the symbols “PL”, “QR”, “PR”, “QL”, “C”, “D”.

Let \(i \in \mathbb{N}\) and let \(j \in \mathbb{Z}\). If \(\varphi\) is an onto endomorphism of a subshift \((X, \sigma)\) with
infinitely many points and \(J\) is the submonoid of \(S(X)\) generated by \(\{\sigma, \varphi\}\), then we have known the following:

- \(PL_j(\sigma) = \{\varphi^i \sigma^j \mid j/i > -p_L(\varphi)\}\), \(QR_j(\sigma) = \{\varphi^i \sigma^j \mid j/i > -q_R(\varphi)\}\)
- \(PR_j(\sigma^{-1}) = \{\varphi^i \sigma^j \mid j/i < p_R(\varphi)\}\), \(QL_j(\sigma^{-1}) = \{\varphi^i \sigma^j \mid j/i < q_L(\varphi)\}\)
- \(C_j(\sigma) = \{\varphi^i \sigma^j \mid j/i > c_R(\varphi)\}\), \(C_j^{-1}(\sigma) = \{\varphi^i \sigma^j \mid j/i < c_L(\varphi)\}\)
- \(D_j(\sigma) = \{\varphi^i \sigma^j \mid j/i > d_R(\varphi)\}\), \(D_j^{-1}(\sigma) = \{\varphi^i \sigma^j \mid j/i < d_L(\varphi)\}\)

\(QR_j(\sigma) \cap QL_j(\sigma) = \{\varphi^i \sigma^j \mid q_L(\varphi) < j/i < -q_R(\varphi)\}\)

is the set of all positively expansive elements of \(J\).

By the main result [N9, Corollary 6.6] of the preceding paper and Corollary 9.7, we have the following theorem.

**Theorem 9.13.** Let \(\varphi\) be a right or left-closing endomorphism of a transitive SFT \((X, \sigma)\). Let \(i \in \mathbb{N}\) and \(j \in \mathbb{Z}\). If \(j/i \leq d_L(\varphi)\) or \(j/i \geq d_R(\varphi)\), then \(\varphi^i \sigma^j\) has the pseudo-orbit tracing property whenever \(\varphi^i \sigma^j\) is expansive (hence if \(\varphi\) is one-to-one, then \((X, \varphi^i \sigma^j)\) is conjugate to an SFT, and otherwise the inverse limit system of \(\varphi^i \sigma^j\) is conjugate to an SFT, whenever \(\varphi^i \sigma^j\) is expansive).

For an onto endomorphism \(\varphi\) of \((X, \sigma)\), let us call the set
\[
\{\varphi^i \sigma^j \mid d_L(\varphi) < j/i < d_R(\varphi), i \in \mathbb{N}, j \in \mathbb{Z}\}
\]
the **genuine non-resolving zone** of \(\varphi\). Essentially weakly LL (respectively, essentially weakly RR) and essentially weakly \(q\)-biresolving endomorphisms of subshifts do not have their genuine non-resolving zones. (Recall that a positively expansive endomorphism of a subshift is the same as an expansive, essentially \(q\)-biresolving endomorphism of the shift.) There exists an automorphism \(\varphi\) of a transitive topological Markov shift \((X, \sigma)\) such that there exist \(i, j \in \mathbb{Z}\) such that \(\varphi^i \sigma^j\) is in the genuine non-resolving zone of \(\varphi\) and \((X, \varphi^i \sigma^j)\) is topologically conjugate to an SFT [N9 Section 10, Example 1]. However it still remains to be an open problem whether or not there exists an automorphism \(\varphi\) of a transitive topological Markov shift \((X, \sigma)\) such that \((X, \varphi)\) is conjugate to a subshift which is not an SFT. If such an automorphism \(\varphi\) exists, then \(\varphi\) must belong to the genuine non-resolving zone of \(\varphi\).

When \(K\) is a commutative subgroup of \(H(X)\), then the cone \(C_{\tau}(\tau)\) is the set of all essentially weakly LR automorphism of \((X, \tau)\) in \(K\), which can be called a \(\text{cone in } K\), because \(C_{\tau}(\tau) = C_{\tau'}(\tau')\) for every \(\tau' \in C_{\tau}(\tau)\). If we let \(K^\circ\) denote the set of all expansive elements in \(K\), then \(K^\circ\) is the disjoint union of distinct \(C_{\tau}(\tau)\) with \(\tau \in K^\circ\). (See [N9 Section 9] and Theorem 10.2).

Let \(\varphi\) be an onto endomorphism of a subshift \((X, \sigma)\). Each of \(-p_L(\varphi), -q_R(\varphi), p_R(\varphi)\) and \(q_L(\varphi)\) is classified into the following three types:

1. \(-p_L(\varphi)\) is said to be of type I if there exist \(i \in \mathbb{N}, j \in \mathbb{Z}\) such that \(j/i = -p_L(\varphi)\) and \(\varphi^i \sigma^j\) is essentially weakly \(p\)-L. Similarly, the definition of being type I is given for each of \(-q_R(\varphi), p_R(\varphi)\) and \(q_L(\varphi)\).
II. \(-p_L(\varphi)\) (respectively, \(-q_R(\varphi), p_R(\varphi), q_L(\varphi)\)) is said to be of type II if it is a rational number but not of type I;

III. \(-p_L(\varphi)\) (respectively, \(-q_R(\varphi), p_R(\varphi), q_L(\varphi)\)) is said to be of type III if it is an irrational number.

**Example 9.14.** Let \(r_1, r_2 \in \mathbb{R}\) with \(r_1 < 0 < r_2\). The result of Boyle and Lind [BoL] shows that there exists an automorphism \(\varphi\) of a subshift \((X, \sigma)\) (over the binary alphabet) such that the “nonexpansive directions” of \(\mathbb{Z}^2\)-action \(\alpha : \mathbb{Z}^2 \to K\) with \(\alpha^{i,j} = \varphi^i\sigma^j\) are exactly \(r_1\) and \(r_2\). Hence if \(K\) is the subgroup of \(H(X)\) generated by \(\{\sigma, \varphi\}\), then \(K = C^*_R(\sigma) \cup C^*_R(\varphi) \cup C^*_R(\sigma^{-1}) \cup C^*_R(\varphi^{-1})\) with \(C^*_R(\sigma) = \{\varphi^i\sigma^j | 1 < i < j, 1 < j/i < 2\}\) and \(C^*_R(\varphi) = \{\varphi^i\sigma^j | 1 < j/i < 2\}\), as stated in [N9, Proposition 7.6]. Hence \(c_L(\varphi) = r_1\) and \(c_R(\varphi) = r_2\).

By the above example, we know the existence of type III limits of onesided resolving directions and the existence of those of type I or II. For onto endomorphisms of SFTs, we only know examples of type I limits of onesided resolving directions. Theorems 7.3 and 4.4 are useful to determine such limits of onesided resolving directions.

**Proposition 9.15.** Let \(\varphi\) be an onto endomorphism of a subshift \((X, \sigma)\).

1. \(-p_L(\varphi) = -p_L(\varphi)\) if and only if \(\varphi^-p_L(\varphi)\) is not right \(\sigma\)-expansive.
2. \(-q_R(\varphi) = -q_R(\varphi)\) if and only if \(\varphi^-q_R(\varphi)\) is not left \(\sigma\)-expansive.
3. \(P_R(\varphi) = P_R(\varphi)\) if and only if \(\varphi^-P_R(\varphi)\) is not left \(\sigma\)-expansive.
4. \(Q_L(\varphi) = q_L(\varphi)\) if and only if \(\varphi^-Q_L(\varphi)\) is not right \(\sigma\)-expansive.

**Proof.** (1) Let \(\psi = \varphi^-P_L(\varphi)\).

Assume that \(-p_L(\varphi) \neq -p_L(\varphi)\). Since \(p_L(\varphi) = \sup_{s \in \mathbb{N}} P_L(\varphi^s)/s\), there exists \(s \geq 1\) such that \(P_L(\varphi^s)/s > P_L(\varphi)\). It follows that

\[
P_L(\psi^s) = P_L(\varphi^s \sigma^{-sP_L(\varphi)}) = P_L(\varphi^s) - sP_L(\varphi) > 0.
\]

Therefore, since \(P_L(\varphi^s\sigma^{-1}) = P_L(\varphi^s) - 1 \geq 0\), \(\psi^s\sigma^{-1}\) is a weakly \(p\)-L endomorphism of \((X, \sigma)\) (by Theorem 6.12). Hence it follows from [N9, Proposition 7.6] that \(\psi^s\) is right \(\sigma\)-expansive, so that \(\psi\) is.

Assume that \(\psi\) is right \(\sigma\)-expansive. Since \(P_L(\psi) = 0\), \(\psi\) is a weakly \(p\)-L endomorphism of \((X, \sigma)\), by Theorem 6.12. Therefore, it follows from Theorem 8.8 that there exists \(s \geq 1\) such that \(P_L(\psi^s) > 0\). Therefore \(P_L(\psi^s\sigma^{-1}) \geq 0\).

Since \(P_L(\varphi^s) - sP_L(\varphi) - 1 = P_L(\varphi^s \sigma^{-sP_L(\varphi)^{-1}}) = P_L(\psi^s\sigma^{-1}) \geq 0\), we have \(P_L(\varphi^s)/s \geq P_L(\varphi) + 1/s\). Therefore, \(P_L(\varphi) \neq \sup_{s} P_L(\varphi^s)/s = p_L(\varphi)\).

The proofs of (2),(3) and (4) are similarly given by using Theorems 6.12, 5.2 and 8.8 and [N9, Proposition 7.6].

The direct proof of the above theorem for the case that \(\varphi\) is an onto endomorphism of a topological Markov shift \((X, \sigma)\) with \(\varphi\) one-to-one or \(\sigma\) transitive, is given similarly by using Proposition 6.9, Theorems 4.2 and 8.8(5) and [N9, Proposition 7.6].

**Proposition 9.16.** Let \(\varphi\) be an onto endomorphism of a topological Markov shift \((X, \sigma)\) such that \(\varphi\) is one-to-one or \(\sigma\) is transitive.

1. If \(-P_L(\varphi) \geq -Q_R(\varphi)\) (respectively, \(-P_L(\varphi) \leq -Q_R(\varphi)\)), then we can decide whether \(\varphi^-P_L(\varphi)\) (respectively, \(\varphi^-Q_R(\varphi)\)) is right (respectively, left) \(\sigma\)-expansive or not.
(2) If $P_R(\varphi) \leq Q_L(\varphi)$ (respectively, $P_R(\varphi) \geq Q_L(\varphi)$), then we can decide whether $\varphi \sigma^{P_R(\varphi)}$ (respectively, $\varphi \sigma^{Q_L(\varphi)}$) is left (respectively, right) $\sigma$-expansive or not.

(3) If $Q_L(\varphi) \geq -Q_R(\varphi)$, then we can decide whether $\varphi \sigma^{-Q_R(\varphi)}$ (respectively, $\varphi \sigma^{-Q_L(\varphi)}$) is left (respectively, right) $\sigma$-expansive or not.

Proof. By Corollary 7.4(1)(d),(2)(d) and Theorem 4.6(2). \qed

Example 9.17. Let $A = 0, 1$. Let $f_0 : A^4 \to A$ be the local rule defined by

$$ f_0(1001) = 1, \quad f_0(1101) = 0, \quad f_0(abcd) = b \quad \text{for } a, b, c, d \in A $$

Let $f : A^4 \to A$ be the local rule such that for $w \in A^4$, $f(w) = 1$ if $f_0(w) = 0$ and $f(w) = 0$ if $f_0(w) = 1$. Let $\varphi_0$ and $\varphi$ be endomorphisms of $(1, 2)$-type of the full 2-shift $(X, \sigma) = (A^Z, \sigma_A)$ given by $f_0$ and $f$, respectively. These are the well-known automorphisms appearing in \cite{H} Section 20).

It is easy to see that $f_0$ is zero left-redundant and zero right-redundant and so is $f$. Hence $P_L(\varphi_0) = P_L(\varphi) = -1$ and $P_R(\varphi_0) = P_R(\varphi) = -2$. Since $\varphi_0^2$ is the identity, by Proposition 7.11 we have $Q_R(\varphi_0) = -2$ and $Q_L(\varphi_0) = -1$. Since the difference of $f_0$ and $f$ does not affect the $q$-R and $q$-L degrees of $\varphi_0$ and $\varphi$, we have $-Q_R(\varphi) = 2$ and $Q_L(\varphi) = -1$.

Since $\varphi_0^2$ is an essentially symbolic automorphism of $(X, \sigma)$, it is essentially symbolic automorphism of $(X, \sigma)$. (see \cite{BoLR} Proposition 2.9; cf. \cite{N} Proposition 8.2)], hence there exists an SFT $(X_0, \sigma_0)$ and a conjugacy $\theta : (X, \sigma) \to (X_0, \sigma_0)$ such that $\psi = \theta^{-1} \varphi_0 \theta$ is a symbolic automorphism. By Proposition 7.11, $P_L(\varphi) = Q_R(\varphi) = P_R(\varphi) = Q_L(\varphi) = 0$.

Thus we see that each of the degrees of onesided resolvingness is not a topological invariant, and we also see that $p_L(\psi) = Q_R(\psi) = p_R(\psi) = q_L(\psi) = 0$, by Proposition 9.12. Hence, by Theorem 9.4, $p_L(\varphi) = Q_R(\varphi) = p_R(\varphi) = q_L(\varphi) = 0$.

Consider the textile system $T_2 = (\pi_2, \gamma_2 : \Gamma_2 \to G)$ in \cite{N5} page 201]. Then $T_2$ is 1-1 and LR and $\varphi T_2 = (\varphi \sigma^2)^2$. It is easy to see that $\xi T_2$ is not one-to-one.

Therefore, $\{\varphi \sigma^2\}^2$ is not left $\sigma^{2}$-expansive. Hence $\varphi \sigma^{-Q_R(\varphi)}$ is not left $\sigma$-expansive. Therefore, it follows from Proposition 9.15(2) that $-Q_R(\varphi) = 2$.

Similar arguments shows that $\varphi \sigma^{-2}$ is not left $\sigma$-expansive, so that we know by Proposition 9.15(3) that $p_R(\varphi) = P_R(\varphi) = -2$.

As is found in \cite{N} Example 8.12(2)], $\varphi \sigma$ is not left $\sigma$-expansive (incidentally, it is not right $\sigma$-expansive either and $\varphi \sigma^{-1}$ is not right $\sigma$-expansive (incidentally, it is not left $\sigma$-expansive either). Therefore, it follows from Proposition 9.15(1),(4) that $-p_L(\varphi) = -P_L(\varphi) = 1$ and $q_L(\varphi) = Q_L(\varphi) = -1$. Hence, $\{\varphi \sigma^i \mid -1 < j/i < 1, i \in \mathbb{N}, j \in \mathbb{Z}\}$ is the genuine non-resolving zone of $\varphi$.

(Incidentally, we know, by \cite{N5} Section 10, Example 2] and \cite{N9} Proposition 7.6], that if $1 \leq j/i < 2, i \in \mathbb{N}, j \in \mathbb{Z}$, then $\varphi \sigma^j$ is right $\sigma$-expansive, but not left $\sigma$-expansive, and that if $-2 \leq j/i < -1, i \in \mathbb{N}, j \in \mathbb{Z}$, then $\varphi \sigma^j$ is right $\sigma$-expansive, but not left $\sigma$-expansive.)

Example 9.18. The reader is referred to \cite{N5} Section 10, Example 1] and \cite{N9} Example 8.12(1]). In \cite{N5} Section 10, Example 1], we find an LR automorphism $\varphi$ of a topological Markov shift $(X_M, \sigma_M)$ and 1-1, LR textile systems $T$ and $T'$ such that $(X_M, \sigma_M) = (X_T, \sigma_T) = (X_{T'}, \sigma_{T'})$, $\varphi_T = \varphi$, and $(\varphi \sigma_M)^{-1} = \varphi T$. The topological Markov shift $(X_M, \sigma_M)$ is defined by the graph $G$ with the representation matrix

$$
\begin{pmatrix}
a + b & c \\
d & e
\end{pmatrix}
$$
Let $f : L_2(G) \to A_G$ be the local rule defined by

$$
\begin{align*}
&aa \mapsto b, \ ab \mapsto c, \ ac \mapsto b, \ ba \mapsto d, \ bb \mapsto e, \ bc \mapsto d, \ ce \mapsto a, \\
&cd \mapsto a, \ ec \mapsto a, \ ed \mapsto a, \ da \mapsto b, \ db \mapsto c, \ dc \mapsto b.
\end{align*}
$$

Then $\varphi$ is of $(0,1)$-type given by $f$. It is directly seen that $f$ is strictly 0 left-redundant and strictly 0 right-redundant. It is also directly seen that $f$ is strictly 1 right- mergible and strictly 1 left-mergible. Hence

$$
-P_L(\varphi) = 0, \quad -Q_R(\varphi) = 0, \quad P_R(\varphi) = -1, \quad \text{and} \quad Q_L(\varphi) = -1.
$$

Therefore we have

$$
\varphi \sigma^{-P_L(\varphi)} = \varphi \sigma^{-Q_R(\varphi)} = \varphi_T, \quad \text{and} \quad \varphi \sigma^{P_R(\varphi)} = \varphi \sigma^{Q_L(\varphi)} = \varphi_T^{-1}.
$$

As is found in [N9, Example 8.12(1)], we easily see that $\varphi_T$ is not right $\sigma_M$-expansive and not left $\sigma_M$-expansive, and $\varphi_T^{-1}$ is not left $\sigma_M$-expansive and not right $\sigma_M$-expansive. Hence, by Proposition 9.15 we have

$$
-p_L(\varphi) = 0, \quad -q_R(\varphi) = 0, \quad p_R(\varphi) = -1, \quad \text{and} \quad q_L(\varphi) = -1.
$$

Therefore, \{\varphi \sigma^j \mid 1 < j/i < 0, i \in \mathbb{N}, j \in \mathbb{Z}\} is the genuine non-resolving zone of $\varphi$. It contains \{\varphi \sigma^j \mid -2/3 \leq j/i \leq -1/3, i \in \mathbb{N}, j \in \mathbb{Z}\}, which is an ELR cone (subtracted \{i_{X,U}\}) in the group generated by \{\varphi, \sigma_M\}, as was shown in [N9, Section 10, Example 1].

As seen above, only a little is known about the limits of onesided resolving directions even for automorphisms of transitive topological Markov shifts. Here we recall open problems presented by Boyle and Lind [BoL] ([Bo3, Questions 11.3, 11.4]) in a recast form.

Suppose that $\varphi$ is an automorphism of a subshift $(X, \sigma)$ and that all four limits of onesided resolving directions of $\varphi$ coincide. What the type of the limits can be? What it can be when $(X, \sigma)$ is an SFT?

10. DIRECTIONALLY ESSENTIALLY WEAKLY-RESOLVING ENDMORPHISMS

In this section, we prove that a “directionally essentially $P$” endomorphism of a subshift is the same as an “essentially $P$” endomorphism of the subshift, for every one of the properties “weakly $p$-$L$”, “weakly $p$-$R$”, “weakly $q$-$R$”, “weakly $q$-$L$”, “weakly $L$”, “weakly $R$”, “weakly $LR$”, “weakly $RL$” and “weakly $q$-biresolving”.

**Proposition 10.1.** Let $\varphi$ be an onto endomorphism of a subshift $(X, \sigma)$ and let $s \geq 1$. If $\varphi$ is an essentially weakly $p$-$L$ endomorphism of $(X, \sigma^s)$, then $\varphi$ is an essentially weakly $p$-$L$ endomorphism of $(X, \sigma)$. Moreover, in this statement “$p$-$L$” can be replaced by each of “$p$-$R$”, “$q$-$R$”, “$q$-$L$”, “$LR$”, “$RL$” “$LL$”, “$RR$” and “$q$-biresolving”.

**Proof.** Suppose that $\varphi$ is an essentially weakly $p$-$L$ endomorphism of $(X, \sigma^s)$. Then there exists a onesided 1-1 textile subsystem $U$ of a weakly $p$-$L$ textile system $T$ and a conjugacy $\theta : (X, \sigma^s, \varphi) \mapsto (X_U, \sigma_U, \varphi_U)$ between commuting systems. Let $\tau = \theta \theta^{-1}$. Then $\tau$ is an expansive automorphism of $(X_U, \sigma_U)$ such that $\tau^s = \sigma_U$. Since $\tau$ is expansive, there exists $k > 0$ such that if $(x_i)_{i \in \mathbb{Z}}$ is a $\tau$-orbit with $x_i = (a_{i,j})_{j \in \mathbb{Z}}$, then $(a_{i-k-1}a_{i-k} \ldots a_{i,k})_{j \in \mathbb{Z}}$ uniquely determines the orbit $(x_i)_{i \in \mathbb{Z}}$. Let $\rho_{-1} : X_U \mapsto X_{U[\omega+1]}$ be the homeomorphism which maps $(a_j)_{j \in \mathbb{Z}} \in X_U$ with $a_j \in L_1(X_U)$ to $(a_{j-1} \ldots a_{j+1})_{j \in \mathbb{Z}}$. Since $T$ is
weakly $p$-L, so is $T^{[2l+1]}$. Therefore, replacing $T^{[2l+1]}, U^{[2l+1]}$ and $p_{-l}$ by $T, U$ and $\theta$, respectively, we see that there exists a onesided 1-1 textile subsystem $U$ of a weakly $p$-L textile system $T$ and a conjugacy $\theta : (X, \sigma^*, \varphi) \to (X_U, \sigma_U, \varphi_U)$ such that $\tau$ with $\tau = \theta \eta \theta^{-1}$ is an automorphism of $(X_U, \sigma_U)$, $\tau^* = \sigma_U$ and if $(\tau^k(x))_{k \in \mathbb{Z}}$ is a $\tau$-orbit of $x \in X_U$ with $\tau^k(x) = (a_{k,j}(x))_{j \in \mathbb{Z}}, a_{i,j}(x) \in L_1(X_U)$, then $(a_{k,i}(x))_{k \in \mathbb{Z}}$ uniquely determines $x$. Let $T = (p, q : \Gamma \to G)$. For $x \in X_U$ and $k, j \in \mathbb{Z}$, let $\alpha_{k,j}(x)$ denote the arc in $\Gamma$ such that

$$\alpha_{k,j}(x))_{k,j \in \mathbb{Z}} = \xi_{U}^{-1}(\tau^k(x)).$$

We define a textile system $T = (p_0, q_0 : \Gamma_0 \to G_0)$ as follows: $G_0$ is the graph such that

$$A_{G_0} = \{(a_{k,0}(x))_{0 \leq k \leq s} \mid x \in X_U \}, \quad V_{G_0} = \{(a_{k,0}(x))_{0 \leq k \leq s-1} \mid x \in X_U \}$$

and each arc $(a_{k,0}(x))_{0 \leq k \leq s}$ goes from $(a_{k,0}(x))_{0 \leq k \leq s-1}$ to $(a_{k,0}(x))_{1 \leq k \leq s}$; $\Gamma_0$ is the graph such that

$$A_{\Gamma_0} = \{(a_{k,0}(x))_{0 \leq k \leq s} \mid x \in X_U \}, \quad V_{\Gamma_0} = \{(a_{k,0}(x))_{0 \leq k \leq s-1} \mid x \in X_U \}$$

and each arc $(a_{k,0}(x))_{0 \leq k \leq s}$ goes from $(a_{k,0}(x))_{0 \leq k \leq s-1}$ to $(a_{k,0}(x))_{1 \leq k \leq s}$; $p_0$ and $q_0$ are given as follows: for each arc $(a_{k,0}(x))_{0 \leq k \leq s}$ in $\Gamma_0$ with $x \in X_U$,

$$p_0((a_{k,0}(x))_{0 \leq k \leq s}) = (p(a_{k,0}(x)))_{0 \leq k \leq s},$$

which is equal to $(a_{k,0}(x))_{0 \leq k \leq s}$, and

$$q_0((a_{k,0}(x))_{0 \leq k \leq s}) = (q(a_{k,0}(x)))_{0 \leq k \leq s},$$

which is equal to $(a_{k,0}(\varphi_U(x)))_{0 \leq k \leq s}$.

To see that $T_0$ is weakly $p$-L, it is sufficient to show that $(a_{k,0}(x))_{0 \leq k \leq s}$ and $(a_{k,0}(x))_{1 \leq k \leq s}$ with $x \in X_U$ uniquely determine $a_{0,0}(x)$. Since $\tau^* = \sigma_U$, it follows that

$$a_{s,0}(x) = a_{0,0}(\tau^*(x)) = a_{0,0}(\sigma_U(x)) = a_{0,1}(x).$$

Since $p$ is weakly left-resolving, $a_{0,0}(x)$ and $a_{0,1}(x)$ uniquely determine $a_{0,0}(x)$. Therefore $a_{s,0}(x)$ and $a_{0,0}(x)$ uniquely determine $a_{0,0}(x)$. Hence $(a_{k,0}(x))_{0 \leq k \leq s}$ and $(a_{k,0}(x))_{1 \leq k \leq s}$ uniquely determine $a_{0,0}(x)$.

Let $Y = \{(a_{k,0}(x))_{k \in \mathbb{Z}} \mid x \in X_U \}$. Then we have a subshift $(Y, \sigma_Y)$ and a conjugacy $\chi : (X_U, \tau) \to (Y, \sigma_Y)$ by $\chi(x) = (a_{k,0}(x))_{k \in \mathbb{Z}}$. Let $\psi = \varphi_U \chi^{-1}$. Then $\chi : (X_U, \tau, \varphi_U) \to (Y, \sigma_Y, \psi)$ is a conjugacy between commuting systems. Let $Z = \{(a_{k,0}(x))_{k \in \mathbb{Z}} \mid x \in X_U \}$. Then We also have a subshift $(Z, \sigma_Z)$. It is clear that $(Z^{[s+1]}, \sigma_Z^{[s+1]})$ is a subshift of $(Z_U, \sigma_U)$, $\xi_U(Z^{[s+1]}) = \eta_U(Z^{[s+1]})$ and $\xi_{U_0} \big| Z^{[s+1]}$ is one-to-one because $(a_{k,0}(x))_{k \in \mathbb{Z}}$ uniquely determines $x \in X$. Therefore we have a textile subsystem $U_0$ of $T_0$ with $Z_{U_0} = Z^{[s+1]}$. Since $T_0$ is weakly $p$-L, $\varphi_{U_0}$ is a weakly $p$-L endomorphism of $(X_{U_0}, \sigma_{U_0})$. Passing through the conjugacies

$$(X, \sigma, \varphi) \to (X_U, \tau, \varphi_U) \to (Y, \sigma_Y, \psi) \to (Y^{[s+1]}, \sigma_Y^{[s+1]}, \psi^{[s+1]}) = (X_{U_0}, \sigma_{U_0}, \varphi_{U_0}),$$

$\varphi$ is an essentially weakly $p$-L endomorphism of $(X, \sigma)$.

To prove that “$p$-L” can be replaced by “$q$-R” in the first version of the proposition, it suffices to show that if $T$ is weakly $q$-R, then so is $T_0$. To see that $T_0$ is weakly $q$-R, it suffices to show that $(a_{k,0}(\varphi_U(x)))_{0 \leq k \leq s}$ and $(a_{k,0}(x))_{0 \leq k \leq s-1}$ with $x \in X_U$ uniquely determine $a_{s,0}(x)$. Since $\tau^* = \sigma_U$, it follows that

$$a_{s,0}(\varphi_U(x)) = a_{0,0}(\tau^*(\varphi_U(x))) = a_{0,0}(\sigma_U(\varphi_U(x))) = a_{0,1}(\varphi_U(x)).$$
Since $T$ is weakly $q$-R, $a_{0,1} (\varphi_U (x))$ and $\alpha_{0,0} (x)$ uniquely determine $\alpha_{0,1} (x)$, which is equal to $\alpha_{s,0} (x)$ as was seen above. Therefore $\alpha_{0,0} (x)$ and $a_{s,0} (\varphi_U (x))$ uniquely determine $\alpha_{s,0} (x)$. Hence $(a_{k,0} (\varphi_U (x)))_{0 \leq k \leq s}$ and $(\alpha_{k,0} (x))_{0 \leq k \leq s-1}$ uniquely determine $\alpha_{s,0} (x)$.

It is analogously proved that each of the other replacements can be made in the first statement of the proposition. $\square$

Let $\varphi$ be an endomorphism of a dynamical system $(X, \tau)$. Let $P$ be any property of endomorphisms of dynamical systems. We say that $\varphi$ is directionally $P$ if there exist $r, s \geq 1$ such that the endomorphism $\varphi^r$ of $(X, \tau^s)$ is $P$. We note that $\varphi$ is directionally essentially $P$ if and only if it is essentially directionally $P$, and hence the property of being directionally essentially $P$ is an invariant of topological conjugacy between endomorphisms of dynamical systems.

Theorem 10.2. Let $\varphi$ be an onto endomorphism of a subshift $(X, \sigma)$. If $\varphi$ is directionally essentially weakly $p$-L, then $\varphi$ is essentially weakly $p$-L. Moreover, in this statement “$p$-L” can be replaced by each of “$p$-R”, “$q$-R”, “$q$-L”, “LR”, “RL”, “LL”, “RR” and “$q$-biresolving”.

Proof. Suppose that $\varphi^r$ is an essentially weakly $p$-L endomorphism of $(X, \sigma^s)$ with $r, s \geq 1$. Then, by Proposition 10.1, $\varphi^r$ is an essentially weakly $p$-L endomorphism of $(X, \sigma)$. Therefore, by Theorem 8.1, so is $\varphi$.

The remainder is similarly proved by using Proposition 10.1 and Theorems 8.1, 8.5 and 8.6.

We cannot say that now we can cover all the results of $[N9$, Section 8$]$ that are obtained by using results of $[N5$, Section 7$]$. In fact, we cannot prove the following result $[N9$, Proposition 8.6$]$ using the methods developed in this paper. Therefore, the theory of $[N5$, Section 7$]$ is still not dispensable.

Theorem 10.3 ($[N9]$). Let $\varphi$ be an onto endomorphism of an SFT $(X, \sigma)$.

1. If $\varphi$ is essentially $p$-L and essentially $p$-R, then it is essentially a 1-block automorphism.
2. If $\varphi$ is essentially $p$-L and essentially $q$-R, then it is essentially LR.
3. If $\varphi$ is essentially $q$-L and essentially $q$-R, then it is essentially $q$-biresolving.

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19-8, 9-chôme, Takaya-Takamigaoka, Higashi-Hiroshima 739-2115, Japan
E-mail address: nasu@quartz.ocn.ne.jp