Billiards in confocal quadrics as a pluri-Lagrangian system

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Abstract

We illustrate the theory of one-dimensional pluri-Lagrangian systems with the example of commuting billiard maps in confocal quadrics.

1 Introduction

The aim of this note is to illustrate some of the issues of the theory of one-dimensional pluri-Lagrangian systems, developed recently in [5], with a well known example of billiards in quadrics [3, 6]. In Section 2 we recall the main positions of the theory of pluri-Lagrangian systems, including a novel explanation of the so called spectrality property introduced in [2]. Then in Section 3 we recall some basic facts about billiards in quadrics. The main new contribution is contained in Section 4 where we use the spectrality property to derive the full set of integrals of motion for commuting billiard maps in confocal quadrics.

2 Reminder on discrete 1-dimensional pluri-Lagrangian systems

Suppose we are given a 1-parameter family of pairwise commuting symplectic maps $F_\lambda : T^*M \to T^*M$, $F_\lambda(q, p) = (\tilde{q}, \tilde{p})$, possessing generating (Lagrange) functions $L(q, \tilde{q}; \lambda)$, so that

$$F_\lambda : \begin{align*}
p &= -\frac{\partial L(q, \tilde{q}; \lambda)}{\partial q}, \\
\tilde{p} &= \frac{\partial L(q, \tilde{q}; \lambda)}{\partial \tilde{q}}.
\end{align*} \quad (1)$$

When considering a second such map, say $F_\mu : T^*M \to T^*M$, we will denote its action by a hat: $F_\mu(q, p) = (\hat{q}, \hat{p})$,

$$F_\mu : \begin{align*}
p &= -\frac{\partial L(q, \hat{q}; \mu)}{\partial q}, \\
\hat{p} &= \frac{\partial L(q, \hat{q}; \mu)}{\partial \hat{q}}.
\end{align*} \quad (2)$$

The commutativity of these maps allows us to define, for any $(q_0, p_0) \in T^*M$, the function $(q, p) : \mathbb{Z}^2 \to T^*M$ by setting

$$(q(n + e_1), p(n + e_1)) = F_\lambda(q(n), p(n)), \quad (q(n + e_2), p(n + e_2)) = F_\mu(q(n), p(n)), \quad \forall n \in \mathbb{Z}^2,$$
see Figure 1 (a). Thus justifies our general short-hand notation for functions on $\mathbb{Z}^2$: if $q$ stands for $q(n)$, then $\tilde{q}$ stands for $q(n + e_1)$, while $\hat{q}$ stands for $q(n + e_2)$. We introduce a discrete 1-form $L$ on $\mathbb{Z}^2$ by setting (slightly abusing the notations) $L(n, n + e_1) = L(q, \tilde{q}; \lambda)$, respectively $L(n, n + e_2) = L(q, \hat{q}; \mu)$.

From (1), (2) we easily see that the following corner equations hold true everywhere on $\mathbb{Z}^2$:

$$\frac{\partial L(q, \tilde{q}; \lambda)}{\partial q} - \frac{\partial L(q, \hat{q}; \mu)}{\partial q} = 0,$$

$$\frac{\partial L(q, \tilde{q}; \lambda)}{\partial \tilde{q}} + \frac{\partial L(\tilde{q}, \hat{\tilde{q}}; \mu)}{\partial \tilde{q}} = 0,$$

$$\frac{\partial L(q, \hat{q}; \mu)}{\partial \hat{q}} - \frac{\partial L(\hat{q}, \hat{\tilde{q}}; \lambda)}{\partial \hat{q}} = 0.$$  \hspace{1cm} (3)

These four corner equations (E)–(E12) correspond to the four vertices of an elementary square of the lattice $\mathbb{Z}^2$, as on Figure 1 (b).

Figure 1: (a) Defining $(q, p) : \mathbb{Z}^2 \to T^* M$ for two commuting maps $F_\lambda$ and $F_\mu$. (b) Four corner equations.

Their consistency means the following. If we start with the data $q$, $\tilde{q}$, $\hat{q}$ related by the corner equation (E), and solve the corner equations (E2) and (E12) for $\hat{\tilde{q}}$, then the two values of $\hat{\tilde{q}}$ coincide identically and satisfy the corner equation (E12).

The corner equations tell us that any solution $q : \mathbb{Z}^2 \to M$ delivers a critical point to the action functional

$$S_\Gamma = \sum_{\sigma \in \Gamma} L(\sigma)$$

for any directed path $\Gamma$ in $\mathbb{Z}^2$ (under variations that fix the fields at the endpoints of the path $\Gamma$). In other words, the field $q : \mathbb{Z}^2 \to M$ solves the pluri-Lagrangian problem for the Lagrangian 1-form $L$ [5].

**Theorem 1.** The value $dL(\sigma)$ for all elementary squares $\sigma = (n, n + e_1, n + e_1 + e_2, n + e_2)$ is constant on solutions of the system of corner equations (E)–(E12):

$$dL(\sigma) := L(q, \tilde{q}; \lambda) + L(\tilde{q}, \hat{\tilde{q}}; \mu) - L(\tilde{q}, \hat{\tilde{q}}; \lambda) - L(q, \hat{q}; \mu) = c(\lambda, \mu).$$
Proof. The expression on the left-hand side of equation (3) is a function on the manifold of solutions of the system of corner equations. The manifold of solutions is of dimension $2 \dim M$, as it can be parametrized by $(q, p)$ or by $(q, \tilde{q})$. It is enough to prove that $\partial(dL(\sigma))/\partial q = 0$ and $\partial(dL(\sigma))/\partial \tilde{q} = 0$. We prove a stronger statement: if one considers $dL(\sigma)$ as a function on the manifold of dimension $4 \dim M$, parametrized by $q, \tilde{q}, \tilde{\sigma}$, and $\tilde{\gamma}$, then the gradient of this function vanishes on the submanifold of solutions of corner equations, of dimension $2 \dim M$. But this is obvious, since vanishing of the partial derivatives of $dL(\sigma)$ with respect to its 4 arguments is nothing but the corresponding corner equations.

Theorem 2. For a family $F_\lambda$ of commuting symplectic maps, the discrete pluri-Lagrangian $1$-form $L$ is closed on solutions, $dL = c(\lambda, \mu) = 0$, if and only if $\partial L(q, \tilde{q}; \lambda)/\partial \lambda$ is a common integral of motion for all $F_\mu$.

Proof. Clearly, the possible dependence of the constant $c(\lambda, \mu)$ on the parameters $\lambda, \mu$ is skew-symmetric: $c(\lambda, \mu) = -c(\mu, \lambda)$. Therefore, $c(\lambda, \mu) = 0$ is equivalent to $\partial c(\lambda, \mu)/\partial \lambda = 0$, that is, to

$$\frac{\partial L(q, \tilde{q}; \lambda)}{\partial \lambda} - \frac{\partial L(q, \tilde{\sigma}; \lambda)}{\partial \lambda} = 0 \quad (4)$$

(see equation (3); terms involving $\partial \tilde{q}/\partial \lambda$ and $\partial \tilde{\sigma}/\partial \lambda$ vanish due to the corresponding corner equations). The latter equation is equivalent to saying that $\partial L(q, \tilde{q}; \lambda)/\partial \lambda$ is an integral of motion for $F_\mu$.

The latter property is a re-formulation of the mysterious “spectrality property” discovered by Kuznetsov and Sklyanin for Bäcklund transformations [2].

3 Billiard in a quadric

We consider the billiard in an ellipsoid

$$Q = \left\{ x \in \mathbb{R}^n : \langle x, A^{-1}x \rangle = \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} = 1 \right\}. \quad (5)$$

Let $\{x_k\}_{k \in \mathbb{Z}}, x_k \in Q$, be an orbit of this billiard. Denote by

$$v_k = \frac{x_{k+1} - x_k}{|x_{k+1} - x_k|} \in S^{n-1} \quad (6)$$

the unit vector along the line $(x_k x_{k+1})$. Then the following equations define the billiard map:

$$B : \begin{cases} x_{k+1} - x_k = \mu_k v_k, \\ v_k - v_{k-1} = v_k A^{-1} x_k. \end{cases} \quad (7)$$

Here the numbers $\mu_k, v_k$ can be determined from the conditions $v_k \in S^{n-1}, x_k \in Q$, so that

$$\mu_k = |x_{k+1} - x_k|, \quad \nu_k = \langle v_k - v_{k-1}, A(v_k - v_{k-1}) \rangle^{1/2}. \quad (8)$$

One can obtain alternative expressions for $\mu_k, \nu_k$ by the following arguments. Suppose that $x_k \in Q$, and determine $\mu_k$ from the condition that $x_{k+1} = x_k + \mu_k v_k \in Q$. This gives:

$$\langle x_k + \mu_k v_k, A^{-1}(x_k + \mu_k v_k) \rangle = 1 \quad \Leftrightarrow \quad 2\mu_k \langle x_k, A^{-1} v_k \rangle + \mu_k^2 \langle v_k, A^{-1} v_k \rangle = 0,$$
so that
\[
\mu_k = -\frac{2\langle x_k, A^{-1}v_k \rangle}{\langle v_k, A^{-1}v_k \rangle} = \frac{2\langle x_{k+1}, A^{-1}v_k \rangle}{\langle v_k, A^{-1}v_k \rangle}. \tag{9}
\]
(The second expression follows in the same way by assuming that \(x_{k+1} \in \mathcal{Q}\) and requiring that \(x_k = x_{k+1} - \mu_k v_k \in \mathcal{Q}\).)

Similarly, suppose that \(v_{k-1} \in \mathcal{S}^n\) and require that \(v_k = v_{k-1} + \nu_k A^{-1}x_k \in \mathcal{S}^n\). This gives:
\[
\langle v_{k-1} + \nu_k A^{-1}x_k, v_{k-1} + \nu_k A^{-1}x_k \rangle = 1 \iff 2\nu_k \langle v_{k-1}, A^{-1}x_k \rangle + \nu_k^2 \langle A^{-1}x_k, A^{-1}x_k \rangle = 0,
\]
so that
\[
\nu_k = -\frac{2\langle v_{k-1}, A^{-1}x_k \rangle}{\langle A^{-1}x_k, A^{-1}x_k \rangle} = \frac{2\langle v_k, A^{-1}x_k \rangle}{\langle A^{-1}x_k A^{-1}x_k \rangle}. \tag{10}
\]

By the way, these alternative expressions for \(\mu_k, \nu_k\) immediately imply the following result.

**Proposition 3.** The quantity \(I = \langle x, A^{-1}v \rangle\) is an integral of motion of the billiard map.

**Proof.** Comparing the both expressions in (9), we find:
\[
\langle x_{k+1}, A^{-1}v_k \rangle = -\langle x_k, A^{-1}v_k \rangle. \tag{11}
\]

Similarly, comparing the both expressions in from (10), we find:
\[
\langle x_k, A^{-1}v_k \rangle = -\langle x_k, A^{-1}v_{k-1} \rangle. \tag{12}
\]

Combining (11) with (the shifted version of) (12), we show the desired result. \( \square \)

We are now in a position to give a Lagrangian formulation of the billiard map. Actually, there are two different such formulations. One of them is pretty well known. I do not know a reference for the second ("dual" one), however, it is based on the so called skew hodograph transformation introduced by Veselov \[7, 4\].

**First (traditional) Lagrangian formulation.** Eliminate variables \(v_k\) from (7):
\[
\frac{x_{k+1} - x_k}{\mu_k} - \frac{x_k - x_{k-1}}{\mu_{k-1}} = \nu_k A^{-1}x_k,
\]
or, according to the first expression in (8),
\[
\frac{x_{k+1} - x_k}{|x_{k+1} - x_k|} - \frac{x_k - x_{k-1}}{|x_k - x_{k-1}|} = \nu_k A^{-1}x_k. \tag{13}
\]
This can be considered as the Euler-Lagrange equation for the discrete Lagrange function
\[
L : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}, \quad L(x_k, x_{k+1}) = |x_{k+1} - x_k|. \tag{14}
\]
Here, one can interpret \(\nu_k\) as the Lagrange multiplier, which should be chosen so as to assure that \(x_{k+1} \in \mathcal{Q}\), provided \(x_{k-1} \in \mathcal{Q}\) and \(x_k \in \mathcal{Q}\).

**Second ("dual") Lagrangian formulation.** Eliminate variables \(x_k\) from (7):
\[
\frac{A(v_{k+1} - v_k)}{v_{k+1}} - \frac{A(v_k - v_{k-1})}{v_k} = \mu_k v_k,
\]
Theorem 5. 

Statement [3].

pluri-Lagrangian systems. With the help of this theory, we are going to prove the following

by

This can be considered as the Euler-Lagrange equation for the discrete Lagrange function

\[ L : S^{n-1} \times S^{n-1} \to \mathbb{R}, \quad L(v_k, v_{k+1}) = \langle v_{k+1} - v_k, A(v_{k+1} - v_k) \rangle^{1/2}. \]  

(16)

Here, one can interpret \( \mu_k \) as the Lagrange multiplier, which should be chosen so as to assure that \( v_{k+1} \in S^{n-1} \), provided \( v_{k-1} \in S^{n-1} \) and \( v_k \in S^{n-1} \).

One can consider the billiard map as the map on the space \( L \) of oriented lines in \( \mathbb{R}^n \). This space can be parametrized as follows:

\[ L \ni \ell = \{x + tv : t \in \mathbb{R}\} \iff (v, x) \in S^{n-1} \times \mathbb{R}^n. \]

Of course, in this representation one is allowed to replace \( x \in \ell \) by any other \( x' = x + t_0v \in \ell \). A canonical choice of the representative \( x' \) is the point on \( \ell \) nearest to the origin 0, that is \( x' = x - (x, v)v \). Clearly, this representative can be considered as \( x' \in T_v S^{n-1} \simeq T_v^* S^{n-1} \). Thus, one can identify \( L \) with \( T^* S^{n-1} \), and the billiard map can be considered as a map \( B : T^* S^{n-1} \to T^* S^{n-1} \), with the generating (Lagrange) function \( L : S^{n-1} \times S^{n-1} \to \mathbb{R} \). As a consequence, the map \( B : T^* S^{n-1} \to T^* S^{n-1} \) preserves the canonical 2-form on \( T^* S^{n-1} \).

4 Commuting billiard maps

We use the following classical result (see [6]).

Theorem 4. For any two quadrics \( Q_\lambda \) and \( Q_\mu \) from the confocal family

\[ Q_\lambda = \{x \in \mathbb{R}^n : Q_\lambda(x) = 1\}, \]

(17)

where

\[ Q_\lambda(x) := \langle x, (A + \lambda I)^{-1}x \rangle = \sum_{i=1}^{n} \frac{x_i^2}{a_i^2 + \lambda}, \]

(18)

the corresponding maps \( B_\lambda : T^* S^{n-1} \to T^* S^{n-1} \) and \( B_\mu : T^* S^{n-1} \to T^* S^{n-1} \) commute.

This places the billiards in confocal quadrics into the context of the theory of one-dimensional pluri-Lagrangian systems. With the help of this theory, we are going to prove the following statement [3].

Theorem 5. The maps \( B_\mu : T^* S^{n-1} \to T^* S^{n-1} \) have a set of common integrals of motion given by

\[ F_i(v, x) = v_i^2 + \sum_{j \neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i^2 - a_j^2}, \quad 1 \leq i \leq n. \]

(19)

Only \( n - 1 \) of them are functionally independent, due to \( \sum_{i=1}^{n} F_i = \langle v, v \rangle = 1 \).

\footnote{A conjecture by Tabachnikov [9] that the commutation of billiard maps characterizes confocal quadrics has been settled, under certain assumptions, in [1].}
Thus, the quantity \( x \) and the combination of the last two terms on the right-hand side is invariant under the change \( F \) is a constant (depending maybe on \( \lambda, \mu \)). The value of this constant is easily determined on a concrete billiard trajectory aligned along the big axis of either of the ellipsoids \( Q_\lambda, Q_\mu \). For such a trajectory, \( v = (1, 0, \ldots, 0) \), \( \tilde{v} = \hat{v} = -v \), and \( \hat{v} = v \). Recall that

\[
L(v, \tilde{v}; \lambda) = (\tilde{v} - v, (A + \lambda I)(\tilde{v} - v))^{1/2}.
\]

There follows immediately that \( dL(\lambda, \mu) = 0 \). Now Theorem 2 implies that the quantity \( \frac{\partial L(v, \tilde{v}; \lambda)}{\partial \lambda} \) is a common integral of motion for all \( B_\mu \). A direct computation gives:

\[
\frac{\partial L(v, \tilde{v}; \lambda)}{\partial \lambda} = \frac{\langle v - y, v - y \rangle}{\nu^2(\langle A + \lambda I \rangle(\tilde{v} - v))^{1/2}} = \frac{\nu^2}{\nu} \langle (A + \lambda I)^{-1}x, (A + \lambda I)^{-1}x \rangle
\]

(used eqs. (7), (8))

Thus, the quantity

\[
Q_\lambda(x, v) := \langle x, (A + \lambda I)^{-1}v \rangle = \sum_{i=1}^n \frac{x_i v_i}{\lambda + a_i^2}
\]

with \( v \in S^{n-1} \), \( x \in Q_\lambda \), is an integral of motion of all maps \( B_\mu : \mathcal{L} \to \mathcal{L} \) (compare with Proposition 3). However, for the map \( B_\mu \), the parametrization of the line \( \ell = \{x + tv : t \in \mathbb{R}\} \in \mathcal{L} \) by means of a point \( x \in \ell \cap Q_\lambda \) is unnatural and rather inconvenient. Actually, it would be preferable to take, for any \( B_\mu \), a representative from \( \ell \cap Q_\mu \), but a still better option would be an expression not depending on the representative at all. This is easily achieved. Observe that, as soon as \( Q_\lambda(x) = 1 \), we have

\[
Q_\lambda^2(x, v) = Q_\lambda(v) - Q_\lambda(v)Q_\lambda(x) + Q_\lambda^2(x, v),
\]

and the combination of the last two terms on the right-hand side is invariant under the change of the representative \( x \mapsto x + tv \):

\[
Q_\lambda^2(x, v) - Q_\lambda(v)Q_\lambda(x) = \sum_{i,j=1}^n \frac{x_i v_i x_j v_j - x_i^2 v_i^2}{(\lambda + a_i^2)(\lambda + a_j^2)} = \sum_{1 \leq i \neq j \leq n} \frac{x_i v_i x_j v_j - x_i^2 v_i^2}{(\lambda + a_i^2)(\lambda + a_j^2)}
\]

\[
= \sum_{1 \leq i \neq j \leq n} \frac{1}{a_i^2 - a_j^2} \left( \frac{1}{\lambda + a_i^2} - \frac{1}{\lambda + a_j^2} \right) (x_i v_i x_j v_j - x_i^2 v_i^2)
\]

\[
= \sum_{1 \leq i \neq j \leq n} \frac{1}{a_i^2 - a_j^2} \cdot \frac{1}{\lambda + a_i^2} \cdot (x_i v_i x_j v_j - x_i^2 v_i^2 + x_j v_j x_i v_i - x_j^2 v_j^2)
\]

\[
= \sum_{i=1}^n \frac{1}{\lambda + a_i^2} \sum_{j \neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i^2 - a_j^2}.
\]

As a result, we see that the maps \( B_\mu : \mathcal{L} \to \mathcal{L} \) have the following integral of motion:

\[
\sum_{i=1}^n \frac{1}{\lambda + a_i^2} \left( v_i^2 + \sum_{j \neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i^2 - a_j^2} \right) = \sum_{i=1}^n \frac{F_i}{\lambda + a_i^2}.
\]

Of course, this holds true also for each \( F_i \) individually. \( \square \)
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