Matrix Theory and the Six Torus

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Abstract: We review the problems associated with Matrix compactifications on $T^6$.

1 Introduction

Matrix theory claims to be a non-perturbative formulation of M-theory. However it is formulated in a background dependent way and hence compactifications pose a serious problem.

At the heart of the matrix construction lies the fact that in the special reference frame chosen several degrees of freedom decouple [1, 2]. Basically one considers a compact light-like circle [2] of radius $R$ with $N$ units of momentum $1/R$ around the circle. In the limit that $N$ and $R$ both go to infinity this goes over to the so-called infinite momentum frame (IMF) of [1], which loosely speaking is the reference frame of an observer boosted infinitely with respect to the experiment along the 11 direction.

Due to the infinite boost, in this frame $p_{11}$ of all the constituents is much bigger than any other scale in the problem. The energy of a particle becomes

$$E = \sqrt{p_{11}^2 + p_{\text{perp}}^2 + m^2} \rightarrow |p_{11}| + \frac{p_{\text{perp}}^2 + m^2}{2|p_{11}|}.$$ 

The Hamiltonian responsible for propagation in this frame is $H = E - p_{11}$. From this we see, that only modes with positive $p_{11}$ contribute to the dynamics. In this case the Hamiltonian is governed by the finite piece

$$H = \frac{p_{\text{perp}}^2 + m^2}{2|p_{11}|}.$$ 

1 Based on a talk given at the "31st International Symposium Ahrenshoop on the Theory of Elementary Particles" Buckow, September 2-6, 1997.
while all the other modes lead to a Hamiltonian which goes to infinity with growing $p_{11}$. Therefore they lead to very rapid oscillations and can be neglected. This decoupling mechanism is still valid when we compactify more spacelike directions. The main task is to identify these remaining degrees of freedom.

2 The matrix description

To analyse the dynamics of M-theory in the discrete lightcone gauge, one can use IIA/M-theory duality to map this system to a better understood IIA setup. One then identifies the degrees of freedom that survive as carriers of positive $p_{11}$ as D0 branes \[^1\]. The same philosophy suggest, that using T-duality Matrix theory on the $T^p$ is described by the dynamics of D$p$ branes. We will review this kind of construction in the following sections, especially in view of the still puzzling $T^6$ compactification. Recently it was shown by Seiberg \[^3\] that this identification can basically be derived from the assumptions of M-theory/IIA duality and Lorentz invariance of M-theory by just noting that the compact lightlike circle is Lorentz equivalent to a particular limit of a compactification on a vanishing space-like circle.

2.1 The infinite boost limit

In order to identify the surviving degrees of freedom a la \[^3\] one proceeds in two steps. One notes that the funny frame we are interested is really equivalent to a vanishing spacelike circle $R$ and one rescales all length and energy scales involved in order to keep the relevant Hamiltonian $H$ for the surviving degrees of freedom finite. These scales are the 11d Planck length $l_p$, the radii of the transverse compact directions $L_i$ and of course $R$. In terms of these quantities we can express $p_{11}, m$ and $p_{\text{perp}}$ in the expression for the energy of the relativistic excitation as

\[
\begin{align*}
  m &= \tilde{m}/l_p \\
  p_{11} &= N/R \\
  p_{\text{perp}} &= M_i/L_i
\end{align*}
\]

where $\tilde{m}, N$ and $M_i$ are fixed dimensionless numbers. The first relation just expresses the fact that we want to measure energies in terms of the 11d Planck scale, while the other two are just quantization conditions for momenta along the compact directions.

Now one wants to take the limit in such a way that for $R \to 0$

\[
H = E - p_{11} = \frac{p_{\text{perp}}^2 + m^2}{2|p_{11}|} \sim \frac{1/L_i^2 + 1/l_p^2}{1/R}
\]

is constant. Thus the limit of infinite boost parameter amounts to taking

\[
R \to 0, \quad L_i \to 0, \quad l_p \to 0 \tag{1}
\]

\[
l_p^2/R = f = \text{fixed}, \quad L_i/l_p = D_i = \text{fixed}
\]
2.2 The type IIA string theory perspective

After one identified the limit that corresponds to an infinite boost one can analyze this limit from the IIA perspective. By considering the dynamics of D0 branes (which after all are the carriers of positive $p_{11}$) in the IIA setup in this particular limit one therefore derives the matrix description for various string compactifications. The parameters of the corresponding IIA theory are

$$g_s^2 = R^3/l_p^3, \quad l_s^2 = l_p^3/R, \quad L_i = L_i.$$

We see that the limit (1) corresponds to $g_s \to 0$, $M_s = 1/l_s \to \infty$ and $L_i \to 0$. If we have no compact transverse dimensions we recover the original DLCQ version of the matrix description. The full quantum theory of the sector with $N$ units of momentum $p_{11}$ is described by the dynamics of $N$ D0 branes in the limit that the string coupling is zero, the Planck and the string scale go to infinity while the gauge coupling of the quantum mechanics on the D0 wordvolume is fixed.

Since (1) involves taking $L_i$ to zero the original IIA picture is not a good description anymore once we compactify transverse dimensions. We should perform a T-duality transformation along the transverse compact directions, thereby mapping the D0 partons into D$d$ branes, where $d$ denotes the number of compact transverse dimensions. Since the generalization of T-duality to arbitrary manifolds is problematic, we will restrict ourselves in the rest of this work to the case of tori. This way we will derive the matrix description of the DLCQ of M-theory on $T^d$. It is given by D$d$ branes wrapping a torus with radii $\Sigma_i$ in IIB (IIA) string theory for $d$ odd (even), in the following limit for the 10d string coupling $g_s$, string scale $M_s$ and 10d Planck scale $M_P$. $g_{YM}$ denotes the coupling constant of the effective SYM on the D$d$ worldvolume. The following formulas can be simply obtained by applying the usual T-duality.

$$g_s^2 = \frac{l_p^{d-3}}{R^3 V^2}, \quad M_s^2 = \frac{l_p^3}{R L_i}, \quad \Sigma_i = \frac{l_p^3}{R L_i},$$

where $V = \prod_{i=1}^d L_i$. From this we can read off how the various parameters behave in the limit (1).

$$g_s^2 \sim \begin{cases} 0 & \text{for } d < 3 \\ \text{finite} & \text{for } d = 3 \\ \infty & \text{for } d > 3 \end{cases}$$

$$M_s^2 \sim \begin{cases} \infty & \text{for } d < 7 \\ \Sigma_i & \text{finite for all } d \end{cases}$$

$$M_P^8 \sim \begin{cases} \infty & \text{for } d < 8 \\ g_{YM}^2 & \text{finite for all } d \end{cases}$$

Note that this time the sides of the torus stay finite, so we actually found a good string theory description. Also the gauge coupling of the theory on the brane is kept at a finite value, so we are really left with an interacting theory to describe the
dynamics of the M-theory setup. For \( d \leq 3 \) the string coupling is finite or vanishes, while Planck and string scale go to infinity. So we can identify the worldvolume theory on the \( Dd \) by the usual string theory techniques. This way one obviously reproduced and hence derived the usual SYM on the dual torus description.

For \( d = 4, 5, 6 \) the string coupling blows up while the Planck scale still goes to infinity. One may hope that one can use strong/weak coupling dualities to find an appropriate description in a weakly coupled theory. We will review them in the following section on a case by case basis.

At least starting from the \( T^7 \) one won’t be able to decouple the bulk gravity since the Planck scale stays finite. In the infinite boost limit several degrees of freedom decouple. The method outlined above allowed us to determine which dynamical degrees of freedom remain. It turns out that for small tori indeed all the bulk modes decouple, so that the matrix description of M-theory on \( T^d \) is described by a \( d + 1 \) dimensional theory. Even though for \( d \geq 7 \) still many bulk degrees of freedom decouple (for example all the massive string modes, since \( M_s \) is sent to infinity) at least bulk gravity remains! The matrix description of M-theory in the DLCQ on those higher tori is given by a 10 dimensional theory! The same turned out to be the case for \( d = 6 \) [4].

3 The four-, five- and sixtorus

The fourtorus:

According to our derivation above the DLCQ of M-theory on the \( T^4 \) is given by a IIA D4 brane where we take the string coupling to infinity, the string scale to infinity and the 10d Planck scale to infinity, keeping \( g_{YM}^2 = \lambda_s g_s \), the gauge coupling of the 4+1 SYM on the D4 fixed. The D4 at infinite coupling is better thought of as the M-theory 5 brane. Translating the above limit in M-theory language we find that we are interested in the worldvolume theory of the M5 in the limit that we take the radius of the 11th dimension and also the 11d Planck scale to infinity. This worldvolume theory is known to be given by the (2,0) fixed point in 6 dimensions. One thus derived the Berkooz-Rozali-Seiberg [5] description of M-theory on the \( T^4 \).

The fivetorus:

For the fivetorus we find the theory of D5 branes of IIB at infinite string coupling, sending \( M_s \) and \( M_p \) also to infinity, where as shown above the gauge coupling of the D5 worldvolume theory stays fixed. IIB string theory offers us the possibility to map this to the theory of NS5 branes at zero string coupling. The above limit maps to \( M_p \to \infty, g_s \to 0 \), keeping \( M_s \) and hence the gauge coupling of the NS5 worldvolume gauge theory fixed. This is precisely Seiberg’s realization of the 6 dimensional little string theory, which he used as the matrix description of M-theory on \( T^5 \) [6].

At this point a comment about the BPS solutions of the theory is in order. Seiberg identified the 16 BPS states preserving 1/2 of the original 32 supercharges, transforming under the \( SO(5, 5, Z) \) U-duality group of M-theory on \( T^5 \). They are bound states of the NS5 brane with D1, D3 or D5 branes. The energy of these states
is given by

$$E = \sqrt{(T_{NS5}V_{NS5})^2 + (T_DV_D)^2}$$

where $T_{brane}$ and $V_{brane}$ denote the volume and the tension of the brane respectively. This is the T-dual version of the formula $E = \sqrt{p_{11}^2 + m^2}$ for the case without transverse momentum. The $p_{11}$ momentum modes got mapped to $T_{NS5}V_{NS5}$, since this is what the D0 branes, the carriers of positive $p_{11}$, are mapped to under the ST5-duality we used. $m^2$ is the mass of the wrapped D branes. Boosting to the IMF now amounts to taking the $g_s \to 0$ limit and switching to a light cone Hamiltonian $H = E - p_{11}$. In our case this amounts to taking

$$H = \lim_{g_s \to 0} \left( \sqrt{(T_{NS5}V_{NS5})^2 + (T_DV_D)^2 - T_{NS5}V_{NS5}} \right)$$

for the light cone energy of these objects, which is the relation Seiberg used to get the excitations of the little string theory. We see again that this procedure is just equivalent to identifying the degrees of freedom that survive the infinite boost limit.

Since all the time we are working at finite $R$ we should really see the U-duality group of M-theory on $T^6$, which is the discrete version of $E_6$. Under the $SO(5,5,\mathbb{Z})$ subgroup the 27 of $E_6$ decomposes as $27 \to 10 + 16 + 1$. In addition to the 16 states identified above, there are the 10 states corresponding to longitudinal membranes and fivebranes (that is branes wrapping $R$), which are momentum and winding modes in the little string theory. They correspond to bound states at threshold, so in this case $E = p_{11} + m$ and hence their lightcone energy $H = E - p_{11} = m$ is equal to their mass in the IIB string theory. These states preserve only one quarter of the supersymmetry. This is due to the fact that in the particular reference frame we chose only half of the supercharges are linearly realized. While the transverse branes break the half that’s not visible anyway, they still appear as 1/2 SUSY bound states, while the longitudinal states break another half and we are left with only 1/4 linearly realized supercharges.

The 27th state is the fundamental carrier of $p_{11}$ itself, the wrapped NS5 brane. Its space-time mass by construction is $1/R$. Indeed the full U-duality multiplet for the $T^6$ can be found in accordance with the fact that $R$ is finite.

The sixtorus:

Applying the above procedure to M-theory on the $T^6$, one can similarly derive that the DLCQ matrix model of this theory is given by the worldvolume theory of IIA D6 branes at infinite coupling (sending $M_p$, $M_s$ to infinity, holding the gauge coupling on the D6 and hence the eleven dimensional Planck scale fixed). This is the limit proposed in [7]. There it was also found that this description yields the right moduli space and BPS states.

It was however shown by Seiberg and Sethi [4] that the worldvolume theory of the D6 does not decouple from the bulk fields in this limit. Since we sent the 10d Planck scale $M_p$ to infinity we decoupled all the 10d gravitons, but the gravitons associated with the 11th dimension of M-theory that opens up in the infinite coupling limit (the D0 branes from the IIA point of view) become massless in this limit. The coupling of these excitations to the D6 worldvolume is governed by the 11d Planck scale, which
is kept fixed in the limit that’s forced upon us by \( (1) \). We thus find that similar as in the case for \( T^{\geq 7} \) the matrix description of the DLCQ is not a d+1 dimensional theory but involves some of the bulk modes of the full type II theory.

This conclusion can also be reached by looking again at the BPS states we should see. We are still working at finite \( R \) and thus should find the discrete \( E_7 \) of \( T^7 \) compactifications. The 56 of \( E_7 \) decomposes under the \( E_6 \) as \( 56 \rightarrow 27 + \bar{27} + 1 + 1 \). In \( [\,] \) the 27 longitudinal and 27 transverse branes were again identified as bound states of the \( p_{11} \) carriers (in this case the D6) and other branes. One of the singlets is again the wrapped D6 brane itself and corresponds to the space-time state with energy \( 1/R \). The 56th state has space-time mass \( \frac{R^2 V_{l}}{l_p} \) and corresponds to the KK6 associated with the compact \( R \). But this state is precisely mapped to a D0 brane in the IIA setup describing the \( T^6 \) compactification. These thus can not decouple from the matrix theory, since they are the required missing BPS state.

By now there seem to be two lines of thought about how to live with this puzzle. On one hand one might just appeal to the magic of large \( N \). The DLCQ description of M theory on higher tori has all the problems mentioned. But once we go back to the original proposal of \( [\,] \), that is go to infinite \( N \), all these problems might go away. The D0 and the D6 repell each other. For infinite \( N \) this might be strong enough to decouple the bulk. Since we also got rid of the hidden compact dimension, we no longer would expect to see the full U-duality of the \( T^7 \) and thus the D0 branes no longer have to be there. This option was for example suggested in \( [\,] \).

The other option would be that what we’ve seen here is an indication that M-theory doesn’t want to be compactified on \( T^{\geq 6} \). This interpretation recently got some support by the observation that by replacing the \( T^6 \) by a Calabi-Yau \( [\,] \), the states that correspond to those that in our case come from the D0 do not lead to any interaction with gravity.

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