Resonant trapping of stars by merging massive black hole binaries

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ABSTRACT

A massive black hole binary might resonantly trap a star (e.g. a white dwarf) and the gas released by its tidal disruption might emit electromagnetic wave signals around the coalescence of the binary. With post-Newtonian equations of motion including gravitational radiation reaction, we numerically studied resonant trappings by black hole binaries with a mass ratio of 1/100. It is found that 2:1 (and simultaneously 4:2) mean motion resonances of the binaries would be strong and could, in principle, draw small third objects deep into relativistic regimes (e.g. ∼10 Schwarzschild radii). The inclinations of the trapped objects could increase significantly and, in some cases, retrograde orbits could be realized eventually.

Key words: gravitational waves – binaries: close.

1 INTRODUCTION

Various kinds of mean motion resonances are identified among the satellites of Jupiter and Saturn (Murray & Dermott 2000). Their origins are explained by the orbital evolution of the satellites due to their tidal interaction with the planets (Goldreich 1965). As for the orbital motions around the Sun, it is well known that Pluto and Neptune are in a 2:3 resonance (Murray & Dermott 2000). Mean motion resonances are also found in extrasolar planetary systems (Wright et al. 2011), including a large number of candidates recently provided by the Kepler mission (Lissauer et al. 2011). Here, orbital migrations in protoplanetary discs would be the underlying process to yield the resonances (Snellgrove, Papaloizou & Nelson 2001; Lee & Peale 2002; see also Raymond et al. 2008). Mean motion resonances, once established, are considered to play a fundamental role in the evolution of multiple planet and satellite systems (see e.g. Murray & Dermott 2000).

Orbits of black hole binaries are shrunk by gravitational radiation reaction, and strong gravitational waves from merging massive black hole binaries are promising targets of the Laser Interferometer Space Antenna (LISA, Bender et al. 1998). One of the related phenomena in active debates is potential electromagnetic wave counterparts associated with the mergers. Seto & Muto (2010) discussed the possibilities that stars (e.g. white dwarfs) might be resonantly trapped by massive black hole binaries and electromagnetic wave signals might be produced by the gas released by tidal disruptions of the trapped objects (see Stone & Loeb 2011 for similar arguments and also Fujii et al. 2010; Chen et al. 2011 for possible resonant effects not associated with coalescences of binary black holes). In their paper they also examined the co-orbital (1:1) resonances numerically and demonstrated that, depending on the initial conditions, a test particle can be stably trapped by a black hole binary around its Lagrangian points $L_4$ or $L_5$ until the epoch close to the coalescence of the parent binary (see also Schnittman 2010; Muto & Seto 2011).

Stabilities of the co-orbital resonances are relatively easy to study both analytically and numerically, but the critical aspect is how to originally settle the trapped objects around $L_4$ or $L_5$ under the presence of strong gravitational perturbations of the parent black hole binaries. In this paper, we analyse the standard mean motion resonances with ratios of the orbital periods $m : n$ ($m \neq n$) other than the atypical co-orbital ones (with the ratio 1:1). While there are many parameters to set up the initial conditions for a triple system composed of a black hole binary and an additional small object, here we concentrate on the primary question of whether a standard mean motion resonance can, in principle, trap a small object stably until a relativistic stage close to the merger of the parent binary. Below, by numerically integrating post-Newtonian (PN) equations of motion for triple systems, we will show preferable results for 2:1 (and simultaneously 4:2) mean motion resonances.

2 DYNAMICS OF TRIPLE SYSTEMS

2.1 Mass ratios

We study the relativistic evolution of triple systems with masses $m_0$, $m_1$ and $m_2$, including only the gravitational interaction among them. The particle $m_0(\gg m_1 + m_2)$ is the central object and the two point masses $m_1$ (inner one) and $m_2$ (outer one) are orbiting around $m_0$. Hereafter we denote $m \equiv m_0$ dropping the subscript 0 for simplicity, and put $m = 1$ in the geometrical unit with...

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\[ G = c = 1 \] In this subsection, we discuss how to choose the mass ratio \(m_1/m_2\) for our numerical simulations.

Using arguments of Hill stability for the two particles \(m_1\) and \(m_2\) orbiting around \(m\) in Newtonian dynamics, it can be analytically estimated as to how closely we can configure two (coplanar and nearly circular) stable orbits for a given mass ratio \((m_1 + m_2)/m\) (Gladman 1993; Lee, Thommes & Rasio 2009). Here we take \((m_1 + m_2)/m = 10^{-2}\) roughly corresponding to the stable limit for orbital periods of 2:1. For a black hole binary with a larger mass ratio (i.e. closer to 1), we need to increase the contrast of the two orbital periods (see also Holman & Wiegert 1999, and references therein) and the order of the involved resonance should be higher than first order. But note that, once a mean motion resonance is realized, it can maintain stability for a higher mass ratio than the simple analytical predictions. For example, the allowed mass ratios \((m_1 + m_2)/m\) for co-orbital (1:1) resonances become 0 according to Hill stability, but a linear stability analysis around the Lagrangian points \(L_3\) and \(L_4\) provides the limit \((m_1 + m_2)/m < 0.04\) (Gascheau 1843).

For the relative weight of \(m_1\) and \(m_2\), we examine the cases with \(m_1 < m_2\). This is because the orbital decay due to gravitational wave emission is faster for a larger mass (see e.g. Landau & Lifshitz 1971) and the configuration with relatively approaching orbits is a key element for yielding capture into a resonance (Sinclair 1972; Henrand & Lamaire 1983; Peale 1986). Hereafter we fix \(m_1 = 10^{-5}\) and \(m_2 = 10^{-2}\), considering a fiducial astrophysical triple system composed of two BHs \((10^{5} + 10^{3} M_{\odot})\) and a trapped \(1 M_{\odot}\) white dwarf. Numerical results for these mass ratios are presented in the next section, but we obtained qualitatively similar results for the combination \(10^{6} + 10^{1} + 1 M_{\odot}\).

### 2.2 Equations of motion

To include the general relativistic effects for triple systems, we use the 2.5PN Arnowitt–Deser–Misner (ADM) Hamiltonian which is formally expressed as follows (Schäfer 1987; Jaranowski & Schäfer 1997; Galaviz & Bruegmann 2011):

\[
H = H_N + b_1 H_{1PN} + b_2 H_{2PN} + b_3 H_{2.5PN},
\]

where \(H_N\), \(H_{1PN}\) and \(H_{2PN}\) are the 1PN and 2PN terms. These terms are conservative. \(H_{2.5PN}\) is the 2.5PN dissipative term that came to be due to gravitational radiation. In equation (1), we introduced the set of parameters \(B_{PN} \equiv (b_1, b_2, b_3)\) for convenience to compare relativistic effects, and our main results are obtained for \(B_{PN} = (0, 1, 1)\). In this paper, we ignore the spins of the black holes.

The ADM Hamiltonian \(H\) is originally given for the position variables \(x_{ia}\) and their conjugate momenta \(p_{ia}\) (\(i\) is the label for the particles and \(a\) for the spatial directions). For example, the Newtonian term is written as

\[
H_N = \frac{1}{2} \sum_{i=0}^{2} \sum_{j=\pm} \frac{p_{i}^2}{m_i} - \frac{1}{2} \sum_{i,j=\pm} \frac{m_i m_j}{d_{ij}},
\]

and the 1PN term as

\[
H_{1PN} = -\frac{1}{8} \sum_{i=0}^{2} m_i \left( \frac{p_i^2}{m_i^2} \right)^2 - \frac{1}{4} \sum_{i,j=\pm} m_i m_j \frac{d_{ij}}{d_{ij}} \left\{ 6 \frac{p_i^2}{m_i^2} - \frac{7 p_i \cdot p_j}{m_i m_j} \right. \\
\left. - \frac{(m_{ij} \cdot p_i)(m_{ij} \cdot p_j)}{m_i m_j} \right\} + \frac{1}{2} \sum_{i,j=\pm,k=\pm} \frac{m_i m_j m_k}{d_{ijk} d_{ik}},
\]

where we define \(m_{ij} = p_i \cdot p_j, d_{ij} = |x_i - x_j|\) and \(m_{ij} = (x_i - x_j)/d_{ij}\). The second and third summations in equation (3) represent interactions between different particles. Meanwhile, the 2PN term \(H_{2PN}\) is composed of many elements, and we use its lengthy expression given in Lousto & Nakano (2008) (see appendix A in their paper). The 2.5PN term \(H_{2.5PN}\) can be found e.g. in equation (41) in Jaranowski & Schäfer (1997).

In this paper, we introduce the new variables \(s_{ia} \equiv p_{ia}/m_i\) to improve accuracy at numerically handling triple systems with large mass ratios (Setho & Muto 2010; Muto & Setho 2011). This prescription is based on a technical reason, and we fully utilize the symmetrical three-body Hamiltonian without introducing approximations associated with \(m_i \ll 1\) (except for the numerical experiments discussed in Appendix A). In other words, the three particles are dealt equivalently in the post-Newtonian framework. Our equations of motion are given by appropriate partial derivatives as

\[
\frac{d x_{ia}}{d t} = \frac{1}{m_i} \frac{\partial H}{\partial x_{ia}}, \quad \frac{d s_{ia}}{d t} = -\frac{1}{m_i} \frac{\partial H}{\partial s_{ia}}.
\]

The right-hand side of these equations are well behaved even in the test particle limit \(m_i \to 0\). The 1PN contributions to the equations of motion can be derived from equation (3), but their explicit forms are also found in Lousto & Nakano (2008) (see their equations 4 and 5). For the partial derivatives of the 2PN Hamiltonian, we generate their fortran forms using Mathematica. The dissipative 2.5PN contributions for equation (4) are presented, e.g. in Galaviz & Bruegmann (2011) (their equations 12 and 13). For our triple systems, the total number of equations becomes 18. They are integrated numerically as described in the next section.

### 2.3 Orbital parameters

We denote the coordinate distances by \(d_i \equiv |x_1 - x_i|\), \(d_2 \equiv |x_2 - x_1|\) and \(d_{12} \equiv |x_1 - x_2|\) between three particles \(m_1, m_2\) and \(m(= m_0)\). The semimajor axes \(a_i\) and the eccentricities \(e_i\) of the inner particle \((i = 1)\) and the outer particle \((i = 2)\) are numerically determined from the consecutive maximum [apocentre distance \(a_1(1 + e_1)\)] and minimum [pericentre distance \(a_1(1 - e_1)\)] of \(r_i\) and the radial separation \(d_i\) between the central object \(m\) and the particle \(m_i\). The orbital angular parameters are defined in a standard manner as shown in Fig. 1 (see also Murray & Dermott 2000). The basic variables for studying resonant states are the longitude of the ascending node \(\Omega\), the argument of the pericentre \(\omega\), and the mean anomaly \(M_i = 2\pi(t - t_i)/T_i\), \(T_i\) the time of the pericentre passage, \(T_i\) is the interval between the passages). In the case of Newtonian elliptical motion, \(M_i\) is related to the true anomaly \(f_i\) (shown in Fig. 1) as

\[
M_i = f_i - e_i \sin f_i + \frac{3}{4} e_i^2 \sin 2 f_i + O(e_i^3).
\]

### 3 NUMERICAL ANALYSES

#### 3.1 Initial and termination conditions

We set up the initial conditions of our numerical calculations in the following manner. The outer particle \(m_2(10^{-2})\) is placed at \(d_2 = 300\) with \(e_2 \simeq 0\). The inner particle \(m_1(10^{-5})\) is released around the apocentre at the distance \(d_1 = 300a_1(1/10)\) with a control parameter \(q \in [0.27, 0.99]\) at 0.01 interval. We set its initial eccentricity \(e_1 \simeq 0.15\) and the relative inclination \(I_1 \simeq 0.01\) (rad). Although we would not go into the details of the earlier histories of our triple systems, we expect that such an orbital alignment would be realized, e.g. through an interaction with the disc around the central black hole. But, as we see later, an exact coplanarity is not a critical requirement for yielding a stable resonant trapping. From Kepler’s
law and the relation \( d_1 = a_1(1 - e_1) \) for the pericentre distance, the ratio of the initial orbital periods \( T_1 \) and \( T_2 \) is given as \( T_1/T_2 \sim (a_1/a_2)^{3/2} \sim 1.2g \).

We define the reference plane for the orbital angles by the initial orbital plane of the outer particle \( m_1 \) (see Fig. 1). Given \( m_1 \ll m_2 \), the total initial angular momentum of the triple is nearly normal to the reference plane (parallel to the \( Z \)-axis in Fig. 1). The two particles \( m_1 \) and \( m_2 \) are initially set nearly at conjunction with the argument of pericentre \( \Omega_1 \sim \pi/2 \).

Next we describe when we stop our runs. Given our perturbative (post-Newtonian) treatment of the relativistic effects, we conservatively terminate the integration when one of the following close encounters happens for the first time: (I) \( d_1 < 10\text{m} \), (II) \( d_{12} < 10m_2 \) and (III) \( d_2 < 10\text{m} \). We set another termination condition, (IV) \( d_1 > 30d_2 \), as a simple ejection of the lightest particle \( m_1 \) from the system. These criteria are selected primitively for particle systems. But the tidal radius of a white dwarf (mass \( 1\text{M}_\odot \) and radius \( 10^8 \text{km} \)) is given by \( \sim 10^{-3}(m_2/10^5\text{M}_\odot)^{1/3}\text{km} \) for a black hole of mass \( m_2 \), and larger than \( 10m_2 \sim 10^8(m_2/10^5\text{M}_\odot)^{1/3}\text{km} \) used for condition II. Therefore, the white dwarf would be tidally disrupted before condition II is applied. On the other hand, the central black hole \( m_3 \) can directly swallow a white dwarf.

We solve the equations of motion (4) using a fifth-order Runge–Kutta scheme with an adaptive step-size control (Press et al. 1996). Here we briefly describe two of the numerical tests done for our code. These are performed for the specific sets of parameters \((a_2, e_2) = (100, 0.1)\) and \((50,0.1)\).

(i) Binary evolution by gravitational radiation reaction. For a purely binary (tentatively putting \( m_1 = 0 \)), we compare our numerical results \( da_2/dt \) and \( de_2/dt \) with the corresponding analytical expressions obtained by Peters (1964). Since the latter results are given for \( B_{PN} = (0, 0, 1) \) in our post-Newtonian parametrization, we prepared the numerical results in the same setting \( B_{PN} = (0, 0, 1) \). We found that the differences between the analytical and numerical ones are less than 1 per cent.

(ii) Conservation of a Hamiltonian for triple systems. For several resonant triple systems with \( m_1 \neq 0 \) and \( B_{PN} = (1, 1, 0) \) (namely excluding the dissipative 2.5PN terms), we checked conservation of the total Hamiltonian \( H = H_{PN} + H_{2.5PN} + H_{3PN} \). The variation rate \( dH/dt \) is less than \( 10^{-5} \) times of the expected energy loss rate due to the 2.5PN terms.

### 3.2 Final states

Our primary interest in this paper is stable resonant evolution of relativistic triple systems. Therefore, we first provide the final semimajor axes \( a_{2,\text{fin}} \) of the outer particles \( m_2 \) at the end of our calculations (started from \( a_{2,\text{ini}} = 300 \)). In Fig. 2, the results \( a_{2,\text{fin}} \) are shown with the filled symbols. These are obtained with the parameters \( B_{PN} = (1, 1, 1) \). The squares are for case I (close encounter between the inner one \( m_1 \) and the central object \( m_2 \)), the circles are for case II (close encounter between \( m_1 \) and \( m_3 \)), and triangles for the case IV (ejection of \( m_1 \)). No run ended with condition III. Fig. 2 shows that, for \( q = 0.33 \sim 0.38 \), the inner particles \( m_1 \) were stably trapped by the outer ones \( m_2 \) down to the relativistic regime \( a_{2,\text{fin}} \sim 20 \). These are due to 2:1 mean motion resonances. In the next section, we pick one of them (\( q = 0.33 \)) and follow its evolution. Another dip around \( q = 0.55 \) was caused by 3:2 resonances that exist from the beginnings of the runs (as expected from Hill stability).

In Fig. 2 we also show the inclination \( \sin I_1 \) around \( \sim 500 \) cycles before the end of our calculations. The crosses are for retrograde orbits with \( |I_1| > \pi/2 \) and the pluses are for prograde ones with \( |I_1| < \pi/2 \). While we always had \( I_2 \sim 0 \) for the outer particle, the angle \( I_1 \) could grow significantly. In particular, for the deeply trapped ones (with small \( a_{2,\text{fin}} \)), the inner orbits could even become retrograde \( |I_1| > \pi/2 \). Many of them ended with condition I. But condition II is realized in some cases (e.g. \( q = 0.40 \) with \( a_{2,\text{fin}} \sim 60 \)), and, for the fiducial system containing a white dwarf, we might observe electromagnetic wave signals associated with the tidal disruptions. On the other hand, in our numerical samples, the ejections of \( m_1 \) were common outcomes for the 3:2 resonances.
3.3 Evolution of orbital parameters

Here we pick the full 2.5PN run for the initial parameter $q = 0.33$, and examine its time evolution. Our integration ended at $a_{2, \text{fin}} = 22$ where the retrograde inner particle $m_1$ has a close encounter with

the central one (the termination condition 1). This run is a typical example of the long-term 2:1 resonances appearing in Fig. 2 around $q = 0.35$ with the filled squares. For reference, we also show the corresponding Newtonian result (with $q = 0.33$) in Fig. 6.

Since the orbital decay is determined by gravitational radiation and we have the simple relation $m \gg m_2 \gg m_1$, the semimajor axis $a_2$ is related to the remaining time $t_r$ and the number of rotation cycles $N_1$ before the coalescence of $m$ and $m_2$ as (Peters 1964)

$$t_r \approx q \frac{256 m_1^2}{m_2^3} \left( \frac{a_2}{m} \right)^4, \quad N_1 \approx \frac{m}{64 \pi m_2} \left( \frac{a_2}{m} \right)^{5/2}$$

(6)

[or $t_r \sim 3.1 (a_2/100m)^4$ yr and $N_1 \sim 5.0 \times 10^4 (a_2/100m)^{5/2}$ for the fiducial triple with $m = 10^5 M_\odot$ and $m_2 = 10^3 M_\odot$]. Therefore we use the semimajor axis $a_2$ of the outer particle to describe the evolutionary stages of the system. The basic results are summarized in Figs 5 and 6 with time elapsing leftwards from $a_2 = 300$. The PN effects largely change the evolution of the eccentricity and the inclination of the inner particle. Below, we mainly comment on the PN calculations given in Fig. 5.

We show the semimajor axis $a_1$ of the inner orbit in Fig. 5(a). It initially oscillates until the resonant capture around $a_2 \sim 285$ (as confirmed later through a resonant argument), and then smoothly shrinks as $a_1 \sim 2^{-2/3} a_2$. In Figs 5(b) and (c), we show the eccentricities $e_1$ and $e_2$ of the two particles. They show short period oscillations. The inner one becomes $e_1 \sim 0.2$ around $a_2 = 250$, while the outer one keeps small values $e_2 < 2 \times 10^{-3}$.

The relative angle $\Omega_2 - \Omega_1$ is presented in Fig. 5(d). It soon settles to $\Omega_2 - \Omega_1 \sim \pi$. This means that the components of the angular momenta of $m_1$ and $m_2$ projected to the reference plane are nearly in the opposite directions. The oscillation at the early stage (with small $|I_2|$) would be partly due to our choice of the reference plane. For the outer particle, we initially have $I_2 = 0$ and the longitude of the ascending node $\Omega_2$ is not well defined. But other angles $\lambda_1$, $\lambda_2$, $\Omega_1$, $\sigma_1$ and $\sigma_2$ can be determined while being almost unaffected by this choice, as understood from their intrinsic geometrical meanings. Since we have $|I_2| \ll 1$ and $e_2 \ll 1$, we determine the mean longitude $\lambda_2$ using the time when $m_2$ pass the X-Z plane in Fig. 1. With $m_1 \ll m_2$, the initial total angular momentum is nearly normal to the reference plane.

In Fig. 5(e), we show the inclination angle $I_1$. The magnitude of the angle $I_1$ for the outer one stays at $I_1 \sim 0$, as expected of $m_1 \ll m_2$. In the Newtonian order, the angular momentum $J_r$ of the particle $m_1$ is related to the orbital period $T_0$ as $J_r \propto m_1 T_0^{1/2} (1 - e^2)^{1/2}$. Therefore, we define the combination

$$\Sigma \equiv \sqrt{1 - e_1^2} \sin I_1 - 1000 \times 2^{1/3} \sqrt{1 - e_2^2} \sin I_2$$

(7)

for the component of the total angular momentum projected to the reference plane, and plot it in Fig. 5(e). The result $\Sigma \approx 0$ together with the relation $\Omega_2 - \Omega_1 \sim \pi$ shows that the orientation of the total angular momentum is nearly parallel to the $Z$-axis and is almost unchanged. A similar behaviour is observed in the Newtonian results (see Fig. 6e) and also in numerical simulations of resonant planet migrations (Thommes & Lissauer 2003). These adiabatic results are reasonable, since the time-scale of the precisions $\Omega_2$ is much shorter than that of the orbital decay.

In Figs 5(f) and (g), we present the resonant arguments

$$\theta_{1,1} = 2 \lambda_2 - \lambda_1 - \sigma_1, \quad \theta_{1,1} = 4 \lambda_2 - 2 \lambda_1 - 2 \Omega_1$$

(8)

defined for a 2:1 eccentricity resonance and a 4:2 inclination resonance, respectively. Due to the symmetry with respect to the reference plane, the inclination resonances start from the second order in the standard classification of resonances (Murray & Dermott 2000).
As shown in panel Fig. 5(f), the argument $\theta_{e_1}$ initially circulates, but turns to librate around $\theta_{e_1} \sim 0$ at $a_2 \sim 285$ where the inner particle is captured into the 2:1 eccentricity resonance. The libration amplitude gradually decreases, and the inclination $I_1$ quickly increases to $\sin I_1 \sim 0.2$. On the other hand, throughout this run, the relative angle $\sigma_2 - \sigma_1$ and thus the resonant argument $\theta_{e_2} = 2\lambda_2 - \lambda_1 - \sigma_2$ circulates (see Fig. 5h). But for some other initial parameters $q$ [e.g. $q = 0.36$ with the final values $(a_{2\text{fin}}, I_1)$ similar to $q = 0.33$; see Fig. 2], we found a structured distribution for the combination $\sigma_2 - \sigma_1$. Our Newtonian run in Fig. 6(h) also shows the libration $\sigma_2 - \sigma_1 \sim \pi$ at $a_2 \lesssim 280$. At $a_2 \sim 265$, in addition to $\theta_{e_1}$, the argument $\theta_{I_1}$ is captured into libration around $\theta_{I_1/2} \sim 3\pi/2$. From Figs 5(d), (f) and (g), we can see that the other two arguments, $4\lambda_2 - 2\lambda_1 - 2\Omega_2$ and $4\lambda_2 - 2\lambda_1 - \Omega_1 - \Omega_2$, are also in libration states, and that we have $|\sigma_1 - \Omega_1| \lesssim \pi/2$ as in the case of the Kozai libration (Kozai 1962) that is caused by secular terms and known to be vulnerable to relativistic effects in the course of orbital migration. In Fig. 5(g), the distinct outlier point around $a_2 = 110$ is associated with $e_1 \ll 1$ for which the mean longitude $\lambda_1$ would not be well behaved. In the Solar system, two Saturnian satellites, Mimas and Tethys, are in the inclination resonance with the argument $4\lambda_2 - 2\lambda_1 - \Omega_1 - \Omega_2$.

Concurrently with the libration of $\theta_{I_1}$, the inclination $I_1$ starts to increase due to the accumulation of resonant effect. Similar evolution appeared in the 3:2 resonances. Growth of inclination accompanied by simultaneous (eccentricity and inclination) resonances is observed also in numerical simulations of planet migrations in Newtonian dynamics (Thommes & Lissauer 2003). But, there, relatively large eccentricities $e_1 \sim 0.6$ are required to yield the simultaneous resonances. Interestingly, our Newtonian simulation in Fig. 6 shows that we indeed need large eccentricities $e_1 \sim 0.6$ for the onset of the simultaneous resonances around $a_2 \sim 160$.

Now we briefly discuss the differences between the PN and Newtonian calculations. Here, it is important to note that the simultaneous resonances ($\theta_{e_1}$ and $\theta_{I_1}$, see equation 8) are realized with the relation $\Omega_1 \sim \sigma_1$, First we examine the Newtonian run given in Fig. 6. In the eccentricity resonance (with the argument $\theta_{e_1}$) for a nearly coplanar Newtonian system with $e_1 \ll 1$, the inner pericentre retrogrades as $0 > \sigma_1 \propto 1/e_1$ (Murray & Dermott 2000). In Fig. 6, the simultaneous resonances ($\theta_{e_1}$ and $\theta_{I_1}$) are established at
Figure 6. Same as Fig. 5 but without the 1PN and 2PN terms \([B_{\text{PN}} = (0, 0, 1)]\). This run ended with condition I around \(a_2 \sim 80\). 

Next we analyse the PN calculations given in Fig. 5. The PN effects generate a positive contribution \(\dot{\varpi}_1 > 0\) from the apsidal precession (Landau & Lifshitz 1971). Meanwhile, from close comparisons between the runs in Figs 5 and 6, we find that the nodal precession rate \(\Omega_1\) is less affected by the relativistic effects for the nearly coplanar triples. Therefore, the difference \(\varpi_1 - \Omega_1 < 0\) for the Newtonian results at \(e_1 \lesssim 0.6\) is compensated by the PN correction \((\varpi_1)_{\text{PN}} > 0\), and the condition \(\varpi_1 \sim \Omega_1\) (thereby increase of the inclination \(I_1\)) can be realized at a smaller \(e_1\), as seen in Fig. 5.

During the stable resonant evolutions of the two cases, close encounters between \(m_1\) and \(m_2\) are avoided. The inner eccentricity \(e_1\) shows a remarkable contrast between Figs 5(b) and 6(b), and this is an important parameter for condition I, as we see below. In Fig. 5, the inner particle is on a polar orbit relative to the outer one around \(a_2 \sim 60\), and then retrogrades with the final value \(\sin I_1 \sim 0.5\) \((I_1 \sim 150^\circ)\). Just before the end of this run, the eccentricity \(e_1\) goes up (not sufficiently captured in Fig. 5b), and the calculation is ended by condition I with the pericentre distance \((1 - e_1)a_1 \sim 10\). In contrast, for the Newtonian system in Fig. 6(b), the inner eccentricity \(e_1\) increases almost monotonically, and condition I is met earlier at \(a_2 \sim 75\).

4 DISCUSSIONS

We have studied the relativistic evolution of the resonant triple system with the post-Newtonian approach including terms up to the 2.5PN order. We mostly fixed masses of the system at \(m = 1\) (central object), \(m_1 = 10^{-3}\) (inner particle) and \(m_2 = 10^{-2}\) (outer particle). Here we briefly summarize our basic results. We found that an eccentricity-type 2:1 resonance (together with inclination-type 4:2 resonances) is strong and the outer particle could, in principle, trap the inner one down to the relativistic regime with a final distance \(a_2 \sim 20\). Here, the PN effects play important roles for the resonant evolution. For example, they help the onset of the simultaneous resonances at a small eccentricity \(e_1\). For the PN system, the inner eccentricity \(e_1\) can remain small, and the close encounter between \(m_1\) and \(m_2\) is delayed, compared with the corresponding Newtonian simulation in which the eccentricity \(e_1\) continues to grow. During the simultaneous resonances, we observed that a rise in the mutual inclination angle, and even retrograde orbits, could be realized.

Next we describe the astrophysical implications of our results, particularly for gravitational wave astronomy. As an example, we follow the evolution presented in Fig. 5 and introduce relevant physical units by taking \(m = 10^5\ M_\odot\). The resonant trapping of the inner one \(m_1\) continues down to \(a_2 \sim 22\), where the system becomes unstable and a close encounter between \(m_1\) and \(m\) occurs with distance \(d_1 \sim 10m\). At this point, the gravitational wave frequency...
from the main binary $m_1-m_2$ is $2/T_2 \sim 6\, \text{mHz}$. The time before its coalescence is $t_c \sim 2.7 \, \text{d}$ and we have $2N_e = 2300$ gravitational wave cycles left. Before this transition, the inspiral wave of the main binary has a phase correction of $O(m_1/m_2)$ due to the trapped particle $m_1$, compared with a purely two-body system $m_1-m_2$. It might be possible to effectively describe the waveform of the triple system in a simple manner with a small number of additional fitting parameters. Otherwise we might not make a long-term coherent integration of the gravitational wave signal before the transition.

If the trapped particle $m_1$ is a neutron star (or a stellar mass black hole) and plunges into the outer black hole (as for $q = 0.40$ in Fig. 2), the associated gravitational wave signal might be detected by future ground-based interferometers as a precursor to the merger of the parent binary that might be observed with space interferometers. But these compact objects would also have chances to fly out from the systems at very high velocities $\gtrsim 0.1c$. In the case of a white dwarf, we might observe characteristic flares of electromagnetic waves due to the gas released by its tidal disruption (Rees 1988). Detections of such electromagnetic wave signals would have significant impacts on astronomy and cosmology (e.g. identification of host galaxies).

Here we comment on the possible expansion of our study. While we have mainly investigated triple systems already captured in first-order resonances, it would be meaningful to analyse the earlier evolutionary stages. The later stages around the binary coalescence should also be studied, including a strong gravity beyond our pericentre. Spins of black holes might become important there, especially for highly inclined orbits. In addition, various astrophysical processes (e.g. interaction with a gas disc, tidal deformation of extended stars, etc.) would be worth studying.

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APPENDIX A: COHERENT RADIATION REACTION FORCE ON THE SMALL BODY

When two particles $m_1$ and $m_2$ move incoherently around $m$, the frictional forces caused by gravitational radiation can be regarded as independent. However, in the case of resonant motions, there can be coherent frictional effect between them. For an eccentric inner particle in a 2:1 resonance, we can expect this in the quadrupole order of gravitational radiation, as indicated by the coupling between the quadrupole moments of the two particles. Using the approximate formula for the radiation reaction force given in Landau & Lifshitz (1971), we made a rough estimate of the frictional force imposed on the inner particle $m_1$ by the quadrupole moment of the outer one $m_2(\gg m_1)$, and found that this work alone would not be sufficient to keep the resonant state with respect to the infalling outer one.

For reference, we also made the following numerical experiment in order to shut down the coherent frictional force caused by the main binary on the small inner particle. In equations (4), we put $B_{2,\text{SPN}} = 0$ for the lightest body ($i = 1$) but keep $B_{2,\text{SPN}} = 1$ for the other two bodies ($i = 0$ and 2). Then we obtained numerical results similar to Fig. 2.

The standard resonant transfer of energy and angular momentum from $m_1$ to $m_2$ would be the dominant and sufficient process to keep the stable resonant evolution as seen in Fig. 2. In the case of a coplanar triple with $e_1 \ll 1$, such a transfer would be realized by conjunctions after the pericentre passages of the inner particle (see e.g. Murray & Dermott 2000). The pericentre is known to be retrograde in Newtonian dynamics, and its rate is proportional to $\propto e_1^{-1}$ (Murray & Dermott 2000).

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