Quantumness of ensemble from no-broadcasting principle

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Quantum information, though not precisely defined, is a fundamental concept of quantum information theory which predicts many fascinating phenomena and provides new physical resources. A basic problem is to recognize the features of quantum systems responsible for those phenomena. One of such important features is that non-commuting quantum states cannot be broadcast: two copies cannot be obtained out of a single copy, not even reproduced marginally on separate systems. We focus on the difference of information contents between one copy and two copies which is a basic manifestation of the gap between quantum and classical information. We show that if the chosen information measure is the Holevo quantity, the difference between the information contents of one copy and two copies is zero if and only if the states can be broadcast. We propose a new approach in defining measures of quantumness of ensembles based on the difference of information contents between the original ensemble and the ensemble of duplicated states. We comment about the permanence property of quantum states and the recently introduced superbroadcasting operation. We also provide an Appendix where we discuss the status of quantum information in quantum physics, basing on the so-called isomorphism principle.

The paper is devoted to the memory of Asher Peres.

INTRODUCTION

The main conceptual novelty of quantum information revolution was the notion of quantum information itself. Already before quantum information era there had been attempts to consider information more generally than classically [1, 2]. However those concepts did not have operational meaning. The real breakthrough turned out to be the concept of sending quantum states intact, initiated in the seminal work on teleportation [3] (for experimental realizations see [4, 5, 6]) and in Schumacher quantum noiseless coding theorem [7].

This concept was truly revolutionary, because previously, the dominating point of view was the Copenhagen interpretation. The latter is extremely epistemic: according to it, the wave function is merely a description of the measuring and preparing apparatuses. The emphasis is put on classical input (parameters of preparation) and classical output (clicks of detectors). From such a point of view, it was rather impossible to think that a message could be something else than a classical message. Indeed any process of sending message is actually some quantum experiment. If the interpretation of an experiment is dominated by the idea of having classical input and classical output, then one imagines that the sender must input a classical message and the receiver must also receive a classical message, not a quantum state.

Another obstacle against arriving at the concept of quantum information was that information was thought as closely related to knowledge: even if communicated and processed by physical means, (classical) information is something that one can get to know. However quantum information is more like a “thing” than knowledge. Knowledge can be shared, quantum information can not: if one passes some quantum information to another person, she cannot keep it at the same time. For example, in the process of teleportation the input state transferred to the receiver is destroyed at the sender site.

The above obstacles have a common denominator: in the pre-quantum-information era the emphasis was put on the subject, not on the object. This passive paradigm [8] has shaped for a long time the understanding of quantum mechanics. The quantum information revolution forces us to put emphasis on the object, because quantum information cannot be expressed in a natural way in terms of preparation and measurement procedures. Well, one can argue that everything can be earlier or later recast in terms of preparation procedure and classical outcomes. An experimenter is even forced to consider an experiment in such terms. However, if we treat such approach as the unique one, we may overlook some phenomena, in a similar way, as looking at the sky, it is rather hard to recognize the rule that governs the planetary motion.

As well known, Asher Peres was one of the authors of the paper on teleportation, which initiated the quantum information revolution. Quite paradoxically, being undoubtedly one of the fathers of the quantum information domain [9], he nevertheless maintained an epistemic point of view, considering the wave function as a mere tool for prediction of probabilities. His point of view was motivated by the fact that the attempts of ontologization of quantum mechanics
lead notoriously to paradoxes \[10\]. We hope that one can avoid this by some kind of reformulation of the notion of reality (see e.g. \[11\]).

As we have mentioned, the most distinctive feature of quantum information, which makes it a “thing” rather than “knowledge”, is that it cannot be shared (much as a (classical) physical object that can not be both here and there at the same time). For pure states, this is the content of the famous no-cloning theorem \[12, 13\] (see pioneering attempt by Wigner in \[14\]). Recently, various properties of quantum information in the context of no-cloning and dual no-deleting principle \[15\] have been analyzed \[16, 17, 18\]. In particular, in \[17\] a new angle of looking at no-cloning principle and no-deleting principles was proposed. Namely, both principles can be viewed as consequences of (i) a principle of conservation of information and (ii) the fact that in the quantum case, two copies can have more information than one copy.

In \[17\] the difference between quantum and classical information was not its conservation (which we have postulated for both), but just the relation between information of two copies and one copy. However, the no-cloning distinguishes between quantum and classical only on the level of pure states. Indeed, classical probability distribution also cannot be cloned. It is the no-broadcasting theorem \[19\] that reports full difference: it says that states can be broadcast if and only if they commute with each other. Broadcasting is a generalization of cloning that allows for correlations between copies. Thus, extending \[17\], we can say that two copies, even realized marginally on separate systems, contain more information than a single copy only if the states do not commute.

In this paper we develop a quantitative approach to this peculiar property of quantum information. To this end we start with axiomatically defined information \(I\) of an ensemble, and then study how much the information of two copies exceeds the information of one copy (where we allow the copies to be correlated). The difference denoted by \(I_q\) shows quantumness of the ensemble. For any classical ensembles (ensembles of commuting states) the difference vanishes, because such ensembles can be broadcast. A desired property of \(I\) is that it always feels that an ensemble cannot be broadcast, i.e. that for such ensemble, the difference \(I_q\) is always strictly positive. From \[19\] it follows that one can build one such function \(I\) using fidelity. However fidelity is not the best function to quantify information, as it is hard to obtain from it an extensive quantity. Therefore we turn to entropic quantities. Namely we show that also Holevo quantity

\[
\chi\left(\{p_i, \rho_i\}\right) = S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i)
\]

is good, in a sense that difference between one and two copies is always nonzero for non-broadcastable ensembles. We also discuss other possibilities.

The function \(I_q\) can also serve as an axiomatically defined quantum contents of ensemble, and can serve as a recipe to produce a new function quantifying “quantumness of ensemble” (see \[20, 21\]). By showing that Holevo quantity gives nonzero \(I_q\) for any noncommuting ensemble, we have obtained a new measure of quantumness, that it nonzero for any nonclassical ensemble.

**CLONING VERSUS BROADCASTING**

The no-cloning theorem states that there does not exist any process, which turns two distinct nonorthogonal quantum states \(\psi, \phi\) into states \(\psi \otimes \psi, \phi \otimes \phi\) respectively. In \[17\] the theorem was connected with a principle of conservation of information. Namely, two copies contain more information than one copy, and therefore it is impossible to produce two copies out of one copy. Even if the information is not conserved, it is at least monotone under operations, and still the main argument holds: cloning is in general impossible, because two copies have more information than one copy.

We actually do not know very well what information is, so it is safer to use “information monotones”, i.e. functions that do not increase under physical operations. In \[17\] we have taken entropy as a measure of informational contents of ensemble of states. Entropy is a monotone, if one restricts to pure states. (In the following, since we deal with mixed states, the Holevo function would be more appropriate.) If one is concerned with only two states, one can take a function of just two arguments, such as fidelity, and this was considered in \[22\].

Why is no-cloning a non-classical feature? The question arises, because one also cannot clone classical probabilities, by the same reason: the information contained in two samples of a probability distribution is more than that contained in single sample. One answer is the following. In the quantum world, pure states cannot be cloned, while in the classical world, pure states (i.e. probability distributions with one probability equal to 1 and all others to 0) can be cloned. However, one may not be satisfied with putting quantum and classical state on the same footing. Indeed, quantum pure states involve probabilities, so it may be more appropriate to compare quantum pure states with all classical...
states, not only pure ones. If so, then one can conclude that no-cloning principle holds both in the quantum and in the classical world [22]. However there is still a fundamental difference: in classical theory one can always broadcast information. Namely, from two classical states $\rho$ and $\sigma$ one can obtain states $\rho_{AB}$ and $\sigma_{AB}$ such that $\rho_A = \rho_B = \rho$ and $\sigma_A = \sigma_B = \sigma$. The difference between cloning and broadcasting is that in cloning one requires to obtain independent copies:

$$\rho \rightarrow \rho \otimes \rho;$$

(2)

in broadcasting correlations between the copies are allowed, i.e. we require only

$$\rho \rightarrow \rho_{AB},$$

(3)

where $\rho_A = \rho_B = \rho$.

Quantumly, broadcasting is not always possible. This can be inferred already from the no-cloning theorem: the latter says that nonorthogonal pure states cannot be cloned. However for pure states broadcasting is equivalent to cloning. The question of broadcasting quantum states was first considered in [19]. There it was shown that much more is impossible, than predicted by no-cloning theorem. Namely, the states cannot be broadcast if and only if they do not commute. Thus we see that impossibility of broadcasting is purely quantum feature, as it goes in parallel with noncommutativity - the main feature distinguishing quantum theory from classical one.

**INFORMATION CONTENTS OF ONE COPY VERSUS TWO COPIES**

In view of the above remarks, one can apply the concept of information contents to the problem of broadcasting. In quantum theory broadcasting is impossible, because, for noncommuting states, two copies (however correlated) will have more information than single copy. As we have said, we can formalize it without knowing what information really is, but rather assuming that it cannot increase under operations [23, 24]. Indeed, whatever information is, in any theory, it cannot increase under operations allowed in the theory. It follows, in particular, that for reversible operations, information is conserved.

In the context of cloning/broadcasting, we talk about sets of states. Thus information would be a monotonous function of set of states (cf. [23, 24] where we analyzed information as a function of states themselves). We require any candidate for information to satisfy the following postulates:

1. $I \geq 0,$

2. $I(S) = 0$ if and only if $S$ contains one and only one element,

3. $I(\Lambda(S)) \leq I(S)$ (monotonicity),

where $S = \{ \rho_i \}_i$ denotes a set of states, $\Lambda$ is any quantum operation, and $\Lambda(S) \equiv \{ \Lambda(\rho_i) \}_i$. An example is Holevo function of the ensemble with equal apriori probabilities. If we consider just two states, then one can also use fidelity as in [22] (actually in [19] the authors used fidelity to show that noncommuting states cannot be broadcast). In next section we will consider ensembles of states, i.e. sets of states with ascribed probabilities. The postulates are then analogous. Then the example is just Holevo function of the ensemble.

Consider an information function $I$ satisfying the above postulates. Then broadcasting of states $\sigma, \rho$ is impossible, if for any states $\rho_{AB}, \sigma_{AB}$ which have $\rho$ and $\sigma$ on both subsystems respectively the function is greater than for $\rho$ and $\sigma$:

$$I(\rho, \sigma) < I(\rho_{AB}, \sigma_{AB}).$$

(4)

Indeed, one can obtain $\rho, \sigma$ from $\rho_{AB}, \sigma_{AB}$ by partial trace. Thus for sure, $I(\rho, \sigma) \leq I(\rho_{AB}, \sigma_{AB})$. If broadcasting is possible, there exists an operation transforming $\rho$ into $\rho_{AB}$ and $\sigma$ into $\sigma_{AB}$; then we have to have also $I(\rho, \sigma) \geq I(\rho_{AB}, \sigma_{AB})$. Consequently for states that can be broadcast we have

$$I(\rho, \sigma) = I(\rho_{AB}, \sigma_{AB}).$$

(5)

In other words, broadcasting means that the operation of obtaining one copy from two copies is reversible, and thus any information monotone must be conserved. Then if there are states for which it cannot be conserved, then they cannot be broadcast.
This suggests to define a new quantity $I_q$ for any information monotone $I$. The quantity would report how much the information contents of two copies exceeds the information contents of one copy. Since two copies can be realized in many different ways, we will take the infimum over all realization of two copies. Thus, for any information monotone $I$ we define

$$I_q(\rho, \sigma) = \inf_{\rho_{AB}, \sigma_{AB}} I(\rho_{AB}, \sigma_{AB}) - I(\rho, \sigma)$$

(6)

where infimum is taken over such states $\rho_{AB}, \sigma_{AB}$ that $\rho_A = \rho_B = \rho, \sigma_A = \sigma_B = \sigma$.

Now, it would be good, if $I_q$ is nonzero if and only if the states cannot be broadcast. This would mean that the function reports presence of quantum information if only it is indeed present (i.e. if only the states cannot be broadcast). Choosing a particular information monotone, it is not easy to check that for all states giving rise to two copies, the monotone is greater than for single copies. In [19] it was shown that for fidelity it is the case. Thus if we take $I^F = 1 - F(\sigma, \rho)$, then $I_q^F$ is nonzero if and only if the states cannot be broadcast.

However fidelity cannot give rise to an extensive quantity, which we think is more appropriate to quantify information in whatever context. Moreover, fidelity can be defined just for two states, while it is convenient to extend the quantity to ensembles of more than two states. Therefore we propose to choose $I$ based on Holevo quantity $\chi$ given by

$$\chi(q_i, \rho_i) = S\left(\sum_i q_i \rho_i\right) - \sum_i q_i S(\rho_i):$$

(7)

we will denote also our information monotone by $\chi$:

$$\chi(\rho, \sigma) := \chi(\{(1/2, \rho), (1/2, \sigma)\})$$

(8)

Let us now argue that $\chi_q$ is nonzero if and only if the states cannot be broadcast.

**Proposition 1** The quantity $\chi_q(\rho, \sigma)$ is nonzero if and only if the states $\rho, \sigma$ cannot be broadcast.

**Proof.** We write the quantity $\chi_q$ as a difference of two relative entropies:

$$\chi_q = S(\rho_{ABC}|\rho_{AB} \otimes \rho_C) - S(\rho_{AC}|\rho_A \otimes \rho_C)$$

(9)

where

$$\rho_{ABC} = \frac{1}{2}|0\rangle_C\langle 0| \otimes \rho_{AB} + \frac{1}{2}|1\rangle_C\langle 1| \otimes \rho_{AB}$$

(10)

with $|i\rangle$ orthogonal states, and $\rho_{AB}, \sigma_{AB}$ are states that optimize the infimum in the definition of $\chi_q$. Now we use a theorem by Petz [23] (see also [26]) which says that if $S(\rho|\sigma) = S(\Lambda(\rho)|\Lambda(\sigma))$ where $\Lambda$ is a trace preserving completely positive (CPTP) map, then there exists another CPTP map $\Gamma$ which reverses the map $\Lambda$ on the considered states:

$$\Gamma(\Lambda(\rho)) = \rho, \quad \Gamma(\Lambda(\sigma)) = \sigma$$

(11)

In our case the map $\Lambda$ is the partial trace over system $B$. Petz gives explicit form of the map, from which it follows that in our case the map $\Gamma$ must be of the form $\Gamma_{A \rightarrow AB} \otimes \text{id}_C$. One can alternatively get it from the following two facts: the second argument of relative entropy in our formulas is a product state; the $C$ part is the same in both arguments. From the former it follows that the map must be product, from the latter that the $C$ part must be identity. Thus, if $\chi_q = 0$ then there exists a map that produces states $\rho_{AB}$ and $\sigma_{AB}$ from $\rho_A$ and $\sigma_B$, which means that the states can be broadcast. This ends the proof.

**QUANTUMNESS OF ENSEMBLE**

In [17, 24] measures of quantumness of an ensemble have been proposed. Here we propose a new way of quantifying quantumness of ensemble, by looking at the difference between information contents in one copy and information contents in two or more copies. Thus, for a given information monotone $I$ on ensembles, we define $I_q^{(n)}$ as follows

$$I_q^{(n)}(\mathcal{E}) = \inf I(\mathcal{E}_{A_1A_2...A_n}) - I(\mathcal{E})$$

(12)
where infimum is taken over all ensembles $E_{A_1 A_2 \ldots A_n}$, which, when partially traced over all subsystems but one, reproduce ensemble $E$. To obtain a particular measure of quantumness, we take $I$ to be Holevo quantity. The proposition holds also in this more general case, so that we obtain that $\chi_q^n(E)$ is nonzero if and only if there exist two states in ensemble that can not be broadcast. However, by \cite{19} we know that states can be broadcast if and only if they commute. Thus we obtain that $\chi_q^n(E)$ is zero if and only if the ensemble is entirely classical, i.e. all states commute with each other.

We now can consider the limit of $n \to \infty$. Then, for ensembles of pure states we have $I_q(\infty) = H((p_i)) - I(E)$, where $H$ is Shannon entropy. Thus here quantumness reports just how the states are indistinguishable. It would be interesting to understand what is the result for ensembles of mixed states in the limit of infinite amount of copies.

Another candidate for information monotone that should feel quantumness if only it is present is accessible information $I_{ac}$. Indeed, the original “meaning” of the Holevo quantity is that of being an upper bound to the accessible information; such a bound is achieved exactly when the states forming the ensemble commute.

That $I_{ac}$ is an information monotone comes from its very definition as the maximal mutual information between the classical input of a sender and the classical output of the receiver; if an operation could increase it, it could be used by the receiver to achieve a better mutual information.

While the Holevo quantity is in principle easily computed as a function of the ensemble alone, the evaluation of $I_{ac}$ requires to find the optimal measurement strategy to achieve the maximal mutual information. Notice that $\chi_q(E) \geq 0$ can be considered a consequence of strong subadditivity: $I_q^{ac}(E) \geq 0$, apart coming from monotonicity, can be understood as the fact that, given an optimal POVM $\{M_i\}$ for $E$, the POVM $\{M_i^{AB} = M_i \otimes 1\}$ provides a lower bound to $I_{ac}(E_{AB})$ equal to $I_{ac}(E)$. On the other hand, having at disposal two broadcast copies intuitively gives the receiver the opportunity to discern better the different states appearing in the ensemble.

However we have not been able to prove that associated $I_q^{ac}$ is nonzero if and only if the states do not commute. Such a proof is not immediate because it involves two parallel optimizations, both for $\{M_i\}$ and $\{M_i^{AB}\}$; neither the original proof of Holevo \cite{27} nor the proof by Fuchs and Caves \cite{28} can be directly applied. Let us consider the case of pure states. Then, the limiting case of quantumness based on $I_{ac}$ is simply given by $(I_{ac})_q^{\infty} = H((p_i)) - S(\rho)$. We can compare now our quantities with the quantumness proposed by Fuchs \cite{29} as $Q_F = \chi_{AC} - I_{ac}$. We thus have for ensembles of pure states

\begin{align}
\chi_q^{\infty}(E) &= H((p_i)) - S(\rho) \\
(I_{ac})_q^{\infty}(E) &= H((p_i)) - I_{ac}(E) \\
Q_F(E) &= S(\rho) - I_{ac}(E)
\end{align}

Thus we obtain that for pure state ensembles Fuchs’ measure is the difference between our two:

\begin{equation}
Q_F(E) = (I_{ac})_q^{\infty}(E) - \chi_q^{\infty}(E)
\end{equation}

It would be interesting to compare these quantities with quantumness of ensemble proposed in \cite{21}. However it is not easy to get expression of that measure.

**Permanence and superbroadcasting**

Quantumness first considered by Jozsa \cite{16}, called by him *permanence*. In particular he asked what is needed to get two copies of an ensemble of pure states, if we already possess one copy. It turns out, that one has essentially to bring in the second copy. More precisely, in \cite{16} it was proved that, given any finite set of states $\{\psi_i\}$ containing no orthogonal pairs of states and a set $\{\rho_i\}$ of (generally mixed) states indexed by the same labels, there is an operation

\begin{equation}
|\psi_i\rangle \otimes \rho_i \mapsto |\psi_i\rangle \otimes |\psi_i\rangle
\end{equation}

if and only if there is an operation $\rho_i \mapsto |\psi_i\rangle$. This indeed means that the original state $|\psi_i\rangle$ (to be cloned) does not help in the process and the information must be provided completely by means of the ancilla state $\rho_i$. It is interesting how this property can be generalized to mixed states, where the natural paradigm is broadcasting rather than cloning.
One may reformulate the problem of realizing the transformation (17) in a more general context. Suppose A and B share one unknown state out of a set of possible states \( \{ \rho_1^{AB}, \ldots, \rho_N^{AB} \} \). The task is to transform \( \rho_i^B \) into \( \rho_i^A \) for all \( i = 1, \ldots, N \), so that the reduced states are the same at both sites. B may send his (unknown) subsystem to A through a perfect quantum channel.

If the reduced states on A side are pure the shared states must be product and we go back to the original problem: if such pure states are not orthogonal, they cannot be broadcast (cloned), and the new copy must be brought in, if possible, by B by a local operation. B does everything alone, rather than help.

If instead the reduced states \( \rho_i^A \) are mixed, sending the B part to A may be useful, as the following example shows.

Consider a pair of orthogonal states

\[
|\psi_1^{AB}\rangle = |00\rangle, \quad |\psi_2^{AB}(a)\rangle = \sqrt{a}|11\rangle + \sqrt{1-a}|01\rangle,
\]

with \( 0 \leq a \leq 1/2 \). The corresponding reduced states are

\[
\rho_1^A = \rho_1^B = |0\rangle\langle 0|, \quad \rho_2^A(a) = \left( \begin{array}{cc} 1 - 2a & \sqrt{a(1-2a)} \\ \sqrt{a(1-2a)} & 2a \end{array} \right), \quad \rho_2^B(a) = \left( \begin{array}{cc} 1 - a & a \\ a & a \end{array} \right).
\]

We have

\[
[\rho_1^A, \rho_2^A(a)] = \left( \begin{array}{cc} 0 & \sqrt{a(1-2a)} \\ -\sqrt{a(1-2a)} & 0 \end{array} \right),
\]

so that for \( a \neq 0, 1/2 \) the reduced states on A side do not commute and cannot be broadcast. Moreover, for qubits it holds in general [30] that, fixed two pairs of states \( \{ \rho_1, \rho_2 \} \) and \( \{ \sigma_1, \sigma_2 \} \), there exists a CPTP map \( \Lambda \) such that \( \sigma_i = \Lambda[\rho_i] \), \( i = 1, 2 \), if and only if

\[
\|\rho_1 - t\rho_2\|_1 \geq \|\sigma_1 - t\sigma_2\|_1 \quad \forall t \in \mathbb{R}^+,
\]

with \( \|A\|_1 = \text{Tr} \sqrt{AA^\dagger} \) the trace norm. In our case the condition for

\[
\{ \rho_1^B, \rho_2^B(a) \} \mapsto \{ \rho_1^A, \rho_2^A(a) \}
\]

to be realized by an operation acting on one subsystem only can be easily checked to be satisfied only for \( a = 0 \).

Therefore for \( 0 < a < 1/2 \) the reduced states \( \rho_1^A, \rho_2^A(a) \) can neither be broadcast nor be the output of an operation performed on B subsystem only. On the other hand, since the total states are orthogonal, it is possible to implement a global transformation such that for the reduced density matrices (20) holds.

We conclude that, considering (reduced) mixed states in the broadcasting framework, global operations involving the total system whose reduced state is to be cloned may be helpful, contrary to the result (for pure states) of [18]. The question of what are the most general conditions on the set of states \( \{ \rho_i^{AB} \} \) to obtain a result similar to the latter remains open. For example, even limiting the problem to initial factorized states, i.e. \( \{ \rho_i^{AB} = \rho_i^A \otimes \rho_i^B \} \), and requiring that the \( \rho_i^A \) states do not commute pairwise, i.e. \( [\rho_i^A, \rho_j^A] \neq 0 \) for \( i \neq j \), so that they cannot be broadcast, B subsystems may help. A trivial case where this happens is the following: if \( \rho_i^A = \rho_i^{A_1} \otimes \rho_i^{A_2} \) and \( \rho_i^B = \rho_i^{B_1} \otimes \rho_i^{B_2} \), with \( \rho_i^{B_2} = \rho_i^{A_2} \), then the possibility of broadcasting the \( A_1 \) parts, i.e. \( [\rho_i^{A_1}, \rho_j^{A_1}] = 0 \) for all \( i, j \), is sufficient for broadcasting the whole states \( \rho_i^A \).

Another hint to how any concept of permanence for mixed states needs a more complex approach is given by superbroadcasting. In fact, in [31] it was proved that, given \( N \) independent copies of an arbitrary mixed qubit state \( \rho \), i.e. given \( \rho^{\otimes N} \), it is possible to broadcast \( M \) copies of it, with \( M \) arbitrarily greater than \( N \) if \( \rho \) is sufficiently mixed and \( N \geq 6 \). Such result does not contradict neither [13] nor the present work, since it is \( \rho \) which is broadcast, not \( \rho^{\otimes N} \), i.e. the task is different or, from another point of view, the initial resource is greater.

It is worth noting that if the “standard” broadcasting, from one to two copies, is possible, then it is reversible, so that the information content of the original ensemble and the broadcast ensemble is the same according to our postulates. That the same holds for superbroadcasting is not evident: after superbroadcasting we are assured to be able to get, locally, only a single copy of \( \rho \). The information content of the initial independent \( N \) copies is smeared over \( M \) qubits and, in general, may be not possible to recover it completely. We will consider this topic elsewhere.
APPENDIX: QUANTUM INFORMATION AND ISOMORPHISM PRINCIPLE

As we have dealt with characterization of quantum information it is natural to ask about its role and status in quantum physics. In particular, our motivation to discuss quantum information in the context of philosophy of physics follows in part from the fact that its impact on interpretative problems is rather little. For instance, in a recent interesting review article on interpretations of Quantum Mechanics the term “quantum information” does not occur even once [32]. Does it mean that a consistent interpretation can dispense with this notion? We think, that it is unlikely. Instead, it cannot be excluded that a permanent interpretative crisis follows from the fact that the concept of quantum information goes beyond standard axioms of quantum formalism.

In this context a basic question arises: what fundamental condition should any interpretation (philosophy) satisfy to be adequate? We believe that the “minimal” condition is that it cannot ignore the rather profound fact that Nature allows to describe itself. It seems that any philosophy (interpretation) of quantum mechanics should take it seriously. This fact suggests that Nature has an ordered structure, and this order is partially revealed in any successful mathematical description. This can be formulated as follows [8]:

Isomorphism principle: Any consistent description of Nature is a sort of isomorphism between the laws of Nature and their mathematical representation.

Such a principle could have been regarded as quite trivial before quantum mechanics was born. Indeed, in classical theories, the observer was passive, hence the structure of theory could be directly assigned to structure of Nature. Physical notions were easily associated with some realities. However, in quantum mechanics there does not exist a passive observer, and the postulate of existence of an isomorphism is nontrivial.

The isomorphism means that the theoretical structure consistent with physical phenomena, although not a real thing, is an isomorphic image of the existing reality. If accepted, it supports the view, that in quantum information era any attempts to understand quantum formalism should take into account the notion of quantum information. Indeed, one can ask: why and for what particular feature does Nature require an abstract mathematical description in terms of Hilbert space? From the point of view of the isomorphism principle, the answer is: the Hilbert space formalism reflects the structure of Nature itself, rather than being only an abstract, descriptive tool. The basic notions associated with Hilbert space formalism should consequently be also taken into account in building a consistent interpretation. One of such a basic notions is undoubtedly quantum information, and in this context, it should be treated as seriously as energy [33]. For example, basing on the close relationships between the notion of quantum information and the notion of entanglement [7, 34], in [35] principle of conservation of information was formulated in terms of entanglement: Entanglement does not change under local operations in closed system (see also [36] in this context).

According to the isomorphism principle, the quantum information, though not necessary a real thing, reflects some physical reality. In particular this allows to avoid the longstanding dilemma between Scylla ontology and Charybdis epistemology which symbolizes the two opposite (extremal) views Einstein and Bohr on the nature of physical reality in relation to quantum formalism. Indeed, the heart of the Copenhagen interpretation is a Ptolemaic paradigm, taking as its “reference frame” preparation parameters and outcomes of measurements. It involves the “surface” of reality in the sense that the wave function provides only a mathematical representation of our knowledge about the experimental setup. From this point of view “photons are clicks in photon detectors” and “there is no quantum information, there is only a quantum way of handling information” [37]. In contrast, Einstein’s ontological concept of physical reality involves the objective state of a system specified by a set of parameters independently of our knowledge of them. The Kochen-Specker theorem shows that extremal version of Einstein’s view cannot be valid any longer.

Remarkably, the isomorphism principle indicates a more suitable approach which lies between the two extremes. It involves a Copernican like active paradigm which takes as a reference frame “what is actually processed in the laboratory”. Three stages are distinguished: preparation, control and measurement. The new feature here is the introduction of the control stage as an autonomous part. The stage contains a quantum system, which may be a compound of subsystems. The latter can be localized in space, controlled individually, and communicated. In particular, the preparation part can be almost completely absorbed into the control part. For example, in quantum computation it is only necessary to prepare the standard input state. Consequently a quantum experiment can be thought of (on the conceptual level) as being mostly control followed by measurement. In accordance with the isomorphism principle the wave function is not only a tool for calculation of probabilities but it is the isomorphic image of what is actually processed in the laboratory. Basing on the information isomorphism we can claim that quantum information is carried by a quantum system and that the wave function is the image of this information. It should be emphasized here, that there is a substantial difference between the quantum information and its image: In contrast to the wave function, the quantum information itself cannot be regarded as a sequence of classical symbols.

In this context it is natural to ask how quantum information suggested by the isomorphism principle manifests itself
in reality. It seems that the simplest criterion may be related to the concept of resource. As we know, in quantum case the newly discovered resources are highly nonintuitive and much more subtle than classical ones. One may consider the following resource criterion: If a quantum property associated with quantum system can be used as resource in some nonclassical tasks, we say that it can be related to reality. In this sense, weaker than the commonly used one, such properties as quantumness of ensembles and entanglement reflect some reality. As a matter of fact, quantum information in the form of nonorthogonal quantum states and quantum information in the form of entanglement are related to each other: possibility of communication of nonorthogonal states is equivalent to possibility of sharing entanglement \cite{peres93, horodecki96, bose97}. However the full meaning of quantum information is still far from being clear \cite{wigner67}.

There is a practical reason, for which the isomorphism principle seems to be more appealing than the passive Ptolemaic paradigm. Isomorphism asks us to take seriously quantum formalism; the Copenhagen interpretation has not taken it seriously enough, and as a result the discovery that quantum states may be processed, has been done surprisingly late! A very illustrative example of a discovery that was not possible within “Ptolemaic approach” was teleportation. Namely, the measurement in Bell basis in teleportation is used as a control operation: the measurement induces a change in the state of the shared pair, and the outcomes are used to further manipulate such state.

It cannot be excluded that, not taking advantage of this lesson, we might again miss other important elements of Nature for a long time.

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