Some results on presemistar operations

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Abstract
In [O3], we introduced the notion of presemistar operation as a generalization of the notion of semistar operation and gave some characterizations of presemistar operations. In this paper we shall give some new characterizations of presemistar operations. We also introduced the notion of an s.e.a.b. presemistar operation in [O3]. One of the main results of this paper is to show that there exists an example of a proper presemistar operation which is e.a.b. but is not s.e.a.b.. We also give two methods of construction which will provide many new examples of presemistar operations.

1. Introduction
Throughout this paper, \( D \) will be an integral domain with quotient field \( K \neq D \). Let \( \mathcal{K}(D) \) be the set of all nonzero \( D \)-submodules of \( K \). Each member of \( \mathcal{K}(D) \) is called a Kaplansky fractional ideal or a \( K \)-fractional ideal of \( D \). Let \( \mathcal{F}(D) \) be the set of all nonzero fractional ideals of \( D \) and let \( \mathcal{F}_f(D) \) be the set of all nonzero finitely generated fractional ideals of \( D \). We denote the set of all nonzero integral ideals of \( D \) by \( \mathcal{I}(D) \).

First we shall recall the definition of a presemistar operation on \( D \) which was defined in [O3].

Let \( \ast \) be a self-map of \( \mathcal{K}(D) \). Then \( \ast \) is called a presemistar operation on \( D \) if the following three conditions are satisfied:

- (E) \( E \subseteq E^\ast \) for all \( E \in \mathcal{K}(D) \);
- (OP) If \( E \subseteq F \), then \( E^\ast \subseteq F^\ast \) for all \( E, F \in \mathcal{K}(D) \);
- (T) \( (aE)^\ast = aE^\ast \) for all \( a \in K \setminus \{0\} \) and all \( E \in \mathcal{K}(D) \).

As in [O3], we say that a self-map \( \ast \) of \( \mathcal{K}(D) \) has Extension Property (resp. Order Preservation Property, Transportability Property) if \( \ast \) satisfies condition (E) (resp. (OP), (T)).

A self-map \( \ast \) of \( \mathcal{K}(D) \) is called a semistar operation on \( D \) if it is a presemistar operation on \( D \) and satisfies the following condition:

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(I) $(E^*)^* = E^*$ for all $E \in \mathcal{K}(D)$.

We say that a self-map $\bullet$ of $\mathcal{K}(D)$ has Idempotence Property if $\bullet$ satisfies condition (I).

We can define a partial order $\leq$ on $\mathcal{K}(D)$ by $A \leq B$ if $A \subseteq B$ for all $A, B \in \mathcal{K}(D)$. Hence $\mathcal{K}(D)$ can be considered as a partially ordered set (for short a poset). Following [E2], a self-map $\bullet$ of $\mathcal{K}(D)$ is called a closure operation on the poset $\mathcal{K}(D)$ if $\bullet$ satisfies conditions (E), (OP) and (I). Using this terminology, a semistar operation on $D$ is a closure operation on the poset $\mathcal{K}(D)$ which satisfies condition (T) (see [E1, page 3]). It easily follows from the above result that a self-map $\bullet$ of $\mathcal{K}(D)$ is a semistar operation on $D$ if and only if $\bullet$ is a presemistar operation on $D$ and $\bullet$ is a closure operation on the poset $\mathcal{K}(D)$.

A self-map $\bullet$ of $\mathcal{F}(D)$ which satisfies conditions (E), (OP), (T) and (I) for all $E, F \in \mathcal{F}(D)$ and all $a \in K \setminus \{0\}$ is called a star operation on $D$ if $D = D^\bullet$ (see [G, Section 32]).

Here we recall some representative examples of semistar operations.

If we set $E^d = E$ (resp. $E^e = K$) for all $E \in \mathcal{K}(D)$, then the map $E \mapsto E^d$ (resp. $E \mapsto E^e$) is a semistar operation on $D$ and is called the $d$-operation (resp. the $e$-operation).

For each $E \in \mathcal{K}(D)$, we set $E^{-1} = \{x \in K \mid xE \subseteq D\}$ and $E^v = (E^{-1})^{-1}$. Then the map $E \mapsto E^v$ is a semistar operation on $D$ and is called the $v$-operation. Note that $E^v = K$ for all $E \in \mathcal{K}(D) \setminus \mathcal{F}(D)$. An element $E \in \mathcal{K}(D)$ is called a $v$-ideal of $D$ if $E = E^v$.

If we set $E^v = (E^{-1})^{-1}$ for all $E \in \mathcal{F}(D)$, then the map $E \mapsto E^v$ is a star operation on $D$ defined in [G] and is called the $v$-operation on $D$. An element $I \in \mathcal{F}(D)$ is called a $v$-ideal of $D$ or a divisorial ideal of $D$ if $I = I^v$.

In this paper, for any $E, F \in \mathcal{K}(D)$, we denote the set $\{x \in K \mid xF \subseteq E\}$ by $E : F$. In Section 2, we shall give ten new characterizations of a presemistar operation.

In Section 3, we shall give two methods by which presemistar operations of new type can be constructed. We also show fundamental properties of these presemistar operations. Furthermore we shall show that there exists an example of a proper presemistar operation which is c.e.a.b. but is not s.e.a.b.

2. Characterizations

In [O3], we have given three characterizations of a presemistar operation. In this section, we shall further give ten characterizations of a presemistar operation. We first recall the following result which will play the important role later:

**Lemma 2.1.** ([O3, Lemma 2.1]) Let $\bullet$ be a self-map of $\mathcal{K}(D)$. If $\bullet$ satisfies conditions (OP) and (T), then $\bullet$ satisfies the following condition

(P) \[ EF^* \subseteq (EF)^* \quad \text{for all} \quad E, F \in \mathcal{K}(D). \]

We say that a self-map $\bullet$ of $\mathcal{K}(D)$ has Product Property if $\bullet$ satisfies condition (P) in Lemma 2.1.

Concerning condition (P) of $\mathcal{K}(D)$ has Product Property if $\bullet$ satisfies condition (P) in Lemma 2.1.
Note 2.1. Let $\star$ be a closure operation on the poset $\mathcal{K}(D)$. Then $\star$ is a nucleus on $\mathcal{K}(D)$ if and only if $\star$ satisfies condition (P), because $\mathcal{K}(D)$ is an ordered magma (see [E2, Proposition 3.2]).

Proposition 2.1. Let $\star$ be a self-map of $\mathcal{K}(D)$. Then the following are equivalent:

1. $\star$ is a presemistar operation on $D$.
2. $\star$ satisfies conditions (E), (OP) and (P).

Proof. (1) $\Rightarrow$ (2). This follows from Lemma 2.1.

(2) $\Rightarrow$ (1). It suffices to show that condition (P) implies condition (T). Let $0 \neq a \in K \setminus \{0\}$ and let $E \in \mathcal{K}(D)$. Then $aE^* = (aD)E^* \subseteq (aDE)^* = (aE)^*$ by (P). Conversely take any $x \in (aE)^*$. then $xa^{-1} \in a^{-1}(aE)^* \subseteq (a^{-1}aE)^* = E^*$ by (P) and so $x \in aE^*$. Thus $(aE)^* \subseteq aE^*$ also holds and therefore condition (T) holds as we wanted.

Proposition 2.2. Let $\star$ be a self-map of $\mathcal{K}(D)$. Then the following are equivalent:

1. $\star$ is a presemistar operation on $D$.
2. $\star$ satisfies conditions (OP), (P) and (BE) $D \subseteq D^*$.

Proof. (1) $\Rightarrow$ (2). This follows from Proposition 2.1.

(2) $\Rightarrow$ (1). For each $E \in \mathcal{K}(D)$, we have $E = ED \subseteq ED^* \subseteq (ED)^* = E^*$ by (BE) and (P) and so condition (E) holds. Hence our assertion follows from Proposition 2.1(2) $\Rightarrow$ (1).

We say that a self-map $\star$ of $\mathcal{K}(D)$ has Basic Extension Property if $\star$ satisfies condition (BE) in Proposition 2.2.

Lemma 2.2. Let $\star$ be a self-map of $\mathcal{K}(D)$. If $\star$ satisfies conditions (OP) and (T), then $\star$ satisfies the following condition

(Q) $E : F \subseteq E^* : F^*$ for all $E, F \in \mathcal{K}(D)$.

Proof. Choose an element $x \in E : F$ with $E, F \in \mathcal{K}(D)$. Then $xF \subseteq E$ and so $xF^* = (xF)^* \subseteq E^*$ by (OP) and (T). Thus $E : F \subseteq E^* : F^*$ as desired.

We say that a self-map $\star$ of $\mathcal{K}(D)$ has Quotient Property if $\star$ satisfies condition (Q) in Lemma 2.2.

Lemma 2.3. Let $\star$ be a self-map of $\mathcal{K}(D)$. Then the following are equivalent:

1. $\star$ satisfies condition (Q).
2. $\star$ satisfies conditions (OP) and (P).
3. $\star$ satisfies the following condition

(GOP) $EF \subseteq G \Rightarrow EF^* \subseteq G^*$ for all $E, F, G \in \mathcal{K}(D)$.

4. $\star$ satisfies conditions (OP) and (T).
Proof. (1) ⇒ (2). Suppose that star satisfies condition (Q). Then \( F \subseteq EF : E \subseteq (EF)^* : E^* \) and so \( E^*F \subseteq (EF)^* \) for all \( E, F \in K(D) \). Hence star satisfies condition (P). Next, suppose that \( E \subseteq F \) with \( E, F \in K(D) \). Then \( D \subseteq F : E \subseteq F^* : E^* \) and therefore \( E^* \subseteq F^* \). Thus star satisfies condition (OP).

(2) ⇒ (3). Suppose that \( EF \subseteq G \) with \( E, F, G \in K(D) \). Then, by condition (OP), \( (EF)^* \subseteq G^* \) and by condition (P), \( EF^* \subseteq (EF)^* \). Hence \( EF^* \subseteq G^* \).

(3) ⇒ (1). Assume that star satisfies condition (GOP). For all \( E, F \in K(D) \), we have \( (E : F)F \subseteq E \) and then, by condition (GOP), \( (E : F)^* \subseteq E^* \) and so \( E^* \subseteq F^* \).

(2) ⇒ (4). This easily follows from the fact that (P) ⇒ (T) which was shown in the proof of Proposition 2.1 (2) ⇒ (1).

(4) ⇒ (2). This follows from Lemma 2.1.

We say that a self-map star of \( K(D) \) has Generalized Order Preservation Property if star satisfies condition (GOP) in Lemma 2.3.

Theorem 2.1. Let star be a self-map of \( K(D) \). Then the following are equivalent:

1. star is a presemistar operation on \( D \).
2. star satisfies conditions (E) and (Q).
3. star satisfies conditions (E) and (GOP).
4. star satisfies conditions (BE) and (Q).
5. star satisfies conditions (BE) and (GOP).

Proof. (1) ⇒ (2). This follows from Lemma 2.2.

(2) ⇒ (1). This follows from Proposition 2.1 and Lemma 2.3.

(2) ⇔ (3). This follows from Lemma 2.3.

(2) ⇒ (4). Trivial.

(4) ⇒ (5). This follows from Lemma 2.3.

(5) ⇒ (1). This follows from Proposition 2.2 and Lemma 2.3.

Lemma 2.4. Let star be a self-map of \( K(D) \). If star satisfies conditions (P) and (OP), then star satisfies the following condition:

\[ (E : F)^* \subseteq E^* : F \quad \text{for all } E, F \in K(D). \]

(SQ)

Proof. Suppose that star satisfies conditions (P) and (OP). Then \( (E : F)^* F \subseteq ((E : F)^* F)^* \subseteq E^* \) and so \( (E : F)^* \subseteq E^* : F \) for all \( E, F \in K(D) \).

We say that a self-map star of \( K(D) \) has Secondary Quotient Property if star satisfies condition (SQ) in Lemma 2.4.

Proposition 2.3. Let star be a self-map of \( K(D) \). Then

1. If star satisfies condition (SQ), then star satisfies condition (T).
2. If star satisfies condition (Q), then star satisfies condition (SQ).

Proof. (1) This follows from [O1, Lemmas 2.1 and 2.2].

(2). Suppose that star satisfies condition (Q). Then, by Lemma 2.3, star satisfies condition (GOP). Since \( (E : F)F \subseteq E \), we have \( (E : F)^* F \subseteq E^* \) and so \( (E : F)^* \subseteq E^* : F \).
Lemma 2.5. Let $\ast$ be a self-map of $K(D)$. Then the following are equivalent:

1. $\ast$ satisfies conditions (OP) and (SQ).
2. $\ast$ satisfies conditions (OP) and (P).

Proof. (1) $\Rightarrow$ (2). Suppose that $\ast$ satisfies (OP) and (SQ). Then, by Proposition 2.3, $\ast$ satisfies (OP) and (T). Hence it follows from Lemma 2.3 that $\ast$ satisfies (OP) and (P).

(2) $\Rightarrow$ (1). This easily follows from Lemma 2.4.

Remark 2.1. Let $\ast$ be a self-map of $K(D)$. Then it easily follows from Lemmas 2.3 and 2.5 that $\ast$ satisfies condition (Q) if and only if $\ast$ satisfies conditions (SQ) and (OP).

Proposition 2.4. Let $\ast$ be a self-map of $K(D)$. Then the following are equivalent:

1. $\ast$ is a presemistar operation on $D$.
2. $\ast$ satisfies conditions (E), (OP) and (SQ).

Proof. (1) $\Rightarrow$ (2). If $\ast$ is a presemistar operation on $D$, then $\ast$ satisfies (E), (OP) and (P) by Proposition 2.1. Then $\ast$ satisfies (Q) by Lemma 2.3 and hence, by Proposition 2.3, $\ast$ satisfies (SQ), that is, $\ast$ satisfies (E), (OP) and (SQ).

(2) $\Rightarrow$ (1). If $\ast$ satisfies (SQ), then $\ast$ satisfies (T) by Proposition 2.3 and so, by Lemma 2.3, $\ast$ satisfies (Q). Hence our assertion follows from Theorem 2.1.

It is easily seen that Proposition 2.3 (2) can be also derived from Lemmas 2.3 and 2.4.

Proposition 2.5. Let $\ast$ be a self-map of $K(D)$. Then

1. $\ast$ satisfies conditions (E) and (SQ) if and only if $\ast$ satisfies (ESQ) $E : F \subseteq (E : F)^* \subseteq E^* : F$ for all $E, F \in K(D)$.
2. $\ast$ satisfies conditions (E) and (P) if and only if $\ast$ satisfies (EP) $EF \subseteq EF^* \subseteq (EF)^*$ for all $E, F \in K(D)$.

Proof. The proof is straightforward.

We say that a self-map $\ast$ of $K(D)$ has Extended Secondary Quotient Property (resp. Extended Product Property) if $\ast$ satisfies condition (ESQ) (resp. (EP)) in Proposition 2.5.

Lemma 2.6. ([O2, Lemma 2.3]) Let $\ast$ be a self-map of $K(D)$. Then $\ast$ satisfies conditions (BE) and (P) if and only if $\ast$ satisfies condition (EP).

Corollary 2.1. Let $\ast$ be a self-map of $K(D)$. Then $\ast$ satisfies conditions (E) and (P) if and only if $\ast$ satisfies conditions (BE) and (P).

Proof. This follows from Lemma 2.6 and Proposition 2.5 (2).

Corollary 2.2. Let $\ast$ be a self-map of $K(D)$. Then the following are equivalent:
(1) $\ast$ is a presemistar operation on $D$.
(2) $\ast$ satisfies conditions (OP) and (ESQ).
(3) $\ast$ satisfies conditions (OP) and (EP).

Proof. (1) $\iff$ (2). This follows from Propositions 2.4 and 2.5 (1).
(1) $\iff$ (3). This follows from Propositions 2.1 and 2.5 (2).

**Proposition 2.6.** Let $\ast$ be a self-map of $K(D)$. Then the following are equivalent:

(1) $\ast$ is a presemistar operation on $D$.
(2) $\ast$ satisfies the following condition

\[(EQ) \quad E : F \subseteq E^* : F^* \subseteq E^* : F \quad \text{for all } E, F \in K(D).\]

Proof. (1) $\Rightarrow$ (2). By Theorem 2.1, $\ast$ satisfies conditions (E) and (Q). Since $F \subseteq F^*$ by (E) and so $E^* : F^* \subseteq E^* : F$ for all $E, F \in K(D)$ and hence $\ast$ satisfies condition (EQ).

(2) $\Rightarrow$ (1). If $\ast$ satisfies (EQ), then evidently $\ast$ satisfies (Q). Next, if we take $F = D$ in (EQ), then $E = E : D \subseteq E^* : D = E^*$ and so $\ast$ also satisfies (E). Hence $\ast$ is a presemistar operation on $D$ by Theorem 2.1 (2) $\Rightarrow$ (1).

We say that a self-map $\ast$ of $K(D)$ has Extended Quotient Property if $\ast$ satisfies (EQ) in Proposition 2.6.

3. Examples

Let $I$ be an integral ideal of $D$ with $I : I = D$. We set $J^{v(I)} = I : (I : J)$ for all $J \in \mathcal{F}(D)$. Then it was proved in [HHIP, Proposition 3.2] that the map $J \mapsto J^{v(I)}$ is a star operation on $D$. By definition, $J^{v(D)} = (D : J) = J^v$ for all $J \in \mathcal{F}(D)$ and so $v(D)$ is equal to the $v$-operation on $D$.

Let $I$ be an arbitrary element of $K(D)$. We set $E^{v(I)} = I : (I : E)$ for all $E \in K(D)$. Then it has been shown in [P1, Proposition 1.17 (2)] that the map $E \mapsto E^{v(I)}$ is a semistar operation on $D$. It is easy to see that $\bar{v}(D) = \bar{v}$ and $\bar{v}(K) = \bar{e}$. As noted in [P2, Example 1.8 (2)] the semistar operation $\bar{v}(I)$ with $I \in K(D)$ would be the natural semistar operation version of the star operation $v(I)$ with $I \in \mathcal{F}(D)$. For further properties of $\bar{v}(I)$, the reader is referred to [P1, Section 1.2.5].

We denote the set of all presemistar operations (resp. semistar operations) on $D$ by $PS(D)$ (resp. $S(D)$). Evidently $S(D) \subseteq PS(D)$. In $PS(D)$, we may define a partial order $\leq$ by $\ast_1 \leq \ast_2$ if and only if $E^{\ast_1} \subseteq E^{\ast_2}$ for all $E \in K(D)$. We use the notation $\ast_1 \leq \ast_2$ in case $\ast_1 \leq \ast_2$ and $\ast_1 \neq \ast_2$ hold.

An element $E \in K(D)$ is called an invertible $D$-submodule of $K$ if $EE^{-1} = D$ and an element $I \in \mathcal{F}(D)$ is called an invertible ideal of $D$ if $II^{-1} = D$. Evidently every invertible ideal of $D$ is a divisorial ideal of $D$.

First we shall define a presemistar operation $v[I]$ for each nonzero integral ideal $I$ of $D$ as follows.

**Proposition 3.1.** Let $I$ be a nonzero integral ideal of $D$. We set
Then we have the following:

1. \(v[I]\) is a presemistar operation on \(D\).
2. \(v[I] \subseteq v[I]\) for all \(I \in \mathcal{I}(D)\).
3. \(v[D] = \bar{v} \) and \(D^{v[I]} = I^{-1}\) for all \(I \in \mathcal{I}(D)\).
4. If \(v = v[I]\) holds for some \(I \in \mathcal{I}(D)\), then \(I^0 = D\).
5. If \(I\) is a divisorial ideal of \(D\), then \(\bar{v} \leq v[I]\). In particular, if \(I\) is an invertible ideal of \(D\), then \(\bar{v} \leq v[I]\).
6. If \(I \neq J\) are divisorial ideals of \(D\), then \(v[I] \neq v[J]\).
7. If \(I \subseteq J\) are nonzero integral ideals of \(D\), then \(v[J] \leq v[I]\). In particular, \(v[I] \leq v[I^2] \leq \cdots \leq v[I^n] \leq \cdots\) for each \(I \in \mathcal{I}(D)\).
8. Let \(I\) and \(J\) be nonzero ideals of \(D\). If \(v[J] \leq v[I]\), then \(I^0 \subseteq J^0\).

Proof. (1) This is evident.
2. Since \(I \subseteq J\), \(I : E \subseteq D : E\) and hence \(E^0 = D : E \subseteq D : (I : E) = E^{v[I]}\).
3. This is evident.
4. Assume that \(v = v[I]\). Then \(D^0 = D^{v[I]}\) and so \(D = D^0 = D^{v[I]} = I^{-1}\) by (3), that is, \(I^0 = D\).
5. The first statement follows from (2) and (4). The “in particular” statement is clear.
6. Conversely suppose that \(v[I] = v[J]\). Then by (3), \(I^{-1} = J^{-1}\) and therefore \(I = I^0 = J^0 = J\), a contradiction. Hence our assertion is valid.
7. If \(I \subseteq J\), then \(I : E \subseteq J : E\) and so \(E^{v[J]} = D : (I : E) \subseteq D : (I : E) = E^{v[I]}\) for all \(E \in \mathcal{K}(D)\). Hence we have \(v[J] \leq v[I]\). The “in particular” statement is clear.
8. If \(v[J] \leq v[I]\), then \(I^{-1} = D^{v[I]} \supseteq D^{v[J]} = J^{-1}\) and so \(I^0 = D : I^{-1} \subseteq D : J^{-1} = J^0\).

As in \([O3]\), a presemistar operation \(*\) on \(D\) is said to be proper if \(*\) is not a semistar operation on \(D\). Using presemistar operations of the form \(v[I]\), we can show that each integral domain \(D\) such that \(D \neq K\) has infinitely many proper presemistar operations.

**Proposition 3.2.**

1. If \(I\) is an invertible ideal of \(D\), then \(v[I]\) is a proper presemistar operation on \(D\).
2. If \(I\) is an invertible ideal of \(D\), then \(v[I^n] \neq v[I^m]\) for all \(n \neq m \in \mathbb{N}\), where \(\mathbb{N}\) is the set of all positive integers.

Proof. (1) By Proposition 3.1 (3), \(D^{v[I]} = I^{-1}\) and hence \((D^{v[I]})^{v[i]} = D : (I : I^{-1})\). But as easily seen, \(I : I^{-1} \subseteq I^2\) and so \((D^{v[I]})^{v[I]} \supseteq D : I^2 = (I^2)^{-1}\). Assume that \((D^{v[I]})^{v[I]} = D^{v[I]}\). Then \(I^{-1} \supseteq (I^2)^{-1}\) and therefore \(D = I^2(I^2)^{-1} \subseteq I^{-1}I^2 = I\), a contradiction. Hence we have \((D^{v[I]})^{v[I]} \neq D^{v[I]}\) which implies that \(v[I]\) is not a semistar operation.

(2) This follows from Proposition 3.1 (6).

**Corollary 3.1.** Let \(D\) be an integral domain such that \(D \neq K\). Then \(D\) has infinitely many proper presemistar operations.
Proof. Choose an invertible ideal \( I \) of \( D \). Then \( I^n \) is also invertible for all \( n \in \mathbb{N} \) and so, by Proposition 3.2 (1), \( v[I^n] \) is a proper presemistar operation for all \( n \in \mathbb{N} \). Hence our assertion follows from Proposition 3.2 (2).

In [FL, Definition 2.3], a semistar operation \( \star \) on \( D \) is said to be *endlich arithmetisch brauchbar* (for short *e.a.b.* if for all \( E, F, G \in \mathcal{F}_f(D) \),

\[(EAB) \quad (EF)^* \subseteq (EG)^* \Rightarrow F^* \subseteq G^*.
\]

A presemistar operation \( \star \) on \( D \) is said to be *e.a.b.* if \( \star \) satisfies the above condition (EAB).

In [G], an integral domain \( D \) is called a \( v \)-domain if the star operation \( v \) on \( D \) is *e.a.b.* In this paper an integral domain \( D \) is called a \( \bar{v} \)-domain if the semistar operation \( \bar{v} \) on \( D \) is *e.a.b.*

In [O3], we introduced the notion of *stark endlich arithmetisch brauchbar* (for short *s.e.a.b.*). A presemistar operation \( \star \) on \( D \) is said to be *s.e.a.b.* if \( \star \) satisfies the following

\[(SEAB) \quad A \subseteq (AB)^* \Rightarrow D \subseteq B^* \text{ for all } A, B \in \mathcal{F}_f(D).
\]

In [O3, Proposition 2.6], it was shown that each *s.e.a.b.* presemistar operation is *e.a.b.* and that if \( \star \) is a semistar operation on \( D \), then \( \star \) is *e.a.b.* if and only if \( \star \) is *e.a.b.*

We shall show that every \( \bar{v} \)-domain \( D \) has infinitely many *s.e.a.b.* proper presemistar operations on \( D \). To prove this, we first demonstrate the following

**Lemma 3.1.** Let \( a \neq 0 \) be a nonunit of \( D \). Then we have the following

\[ E^{v[aD]} = \frac{1}{a} E^0 \quad \text{for all } E \in \mathcal{K}(D). \]

*Proof.* For each \( E \in \mathcal{K}(D) \), we have \( E^{v[aD]} = D : (aD : E) = D : aE^{-1} = \frac{1}{a}(D : E^{-1}) = \frac{1}{a} E^0 \).

\[ \square \]

**Proposition 3.3.**

(1) Let \( D \) be a \( \bar{v} \)-domain. Then \( v[aD] \) is *s.e.a.b.* for each nonunit \( a \neq 0 \) of \( D \).

(2) If \( v[aD] \) is *s.e.a.b.* for some nonunit \( a \neq 0 \) of \( D \), then \( D \) is a \( \bar{v} \)-domain.

*Proof.*

(1) Assume that \( A \subseteq (AB)^{v[aD]} \) for a nonunit \( a \neq 0 \) of \( D \) and for all \( A, B \in \mathcal{F}_f(D) \). Then by Lemma 3.1, \( A \subseteq \frac{1}{a}(AB)^0 = (A \frac{1}{a} B)^0 \). Then, since \( \bar{v} \) is *s.e.a.b.*, we have \( D \subseteq (\frac{1}{a} B)^0 = \frac{1}{a} B^0 = B^{v[aD]} \) by Lemma 3.1. Hence \( v[aD] \) is *s.e.a.b.*.

(2) Assume that \( v[aD] \) is *s.e.a.b.* for some nonunit \( a \neq 0 \) of \( D \). Suppose that \( (AB)^0 \subseteq (AC)^0 \) with \( A, B, C \in \mathcal{F}_f(D) \). Then, by Lemma 3.1, \( (AB)^{v[aD]} = \frac{1}{a}(AB)^0 \subseteq \frac{1}{a}(AC)^0 = (AC)^{v[aD]} \). By assumption, \( v[aD] \) is *e.a.b.* and so we have \( B^{v[aD]} \subseteq C^{v[aD]} \) and then \( \frac{1}{a} B^0 \subseteq \frac{1}{a} C^0 \) by Lemma 3.1 and therefore \( B^0 \subseteq C^0 \). Thus \( (AB)^0 \subseteq (AC)^0 \) implies \( B^0 \subseteq C^0 \) for all \( A, B, C \in \mathcal{F}_f(D) \). Hence \( \bar{v} \) is *e.a.b.* and therefore \( D \) is a \( \bar{v} \)-domain.

\[ \square \]
**Corollary 3.2.** Let \( D \) be a \( \bar{v} \)-domain. Then there exist infinitely many s.e.a.b. proper presemistar operations on \( D \).

**Proof.** If we choose a nonunit \( a \neq 0 \) of \( D \), then by Proposition 3.3 (1), \( v[aD] \) is an s.e.a.b. presemistar operation on \( D \) and furthermore \( v[aD] \) is not a semistar operation by Proposition 3.2 (1). Thus \( v[a^nD] \) is an s.e.a.b. proper presemistar operation on \( D \) for all \( n \in \mathbb{N} \).

Next, we shall define a presemistar operation \( v(I) \) for each \( K \)-fractional ideal \( I \) of \( D \) such that \( D \subseteq I \) as follows.

**Proposition 3.4.** Let \( I \) be a \( K \)-fractional ideal of \( D \) such that \( D \subseteq I \). We set

\[
E^v(I) = I : (D : E) \text{ for all } E \in K(D).
\]

Then we have the following

1. \( v(I) \) is a presemistar operation on \( D \).
2. \( \bar{v} \leq v(I) \) for all \( I \in K(D) \) such that \( D \subseteq I \).
3. \( v(D) = \bar{v} \) and \( v(K) = \bar{e} \).
4. For all \( I \neq J \in K(D) \) such that \( D \subseteq I \) and \( D \subseteq J \), \( v(I) \neq v(J) \). In particular, \( \bar{v} \leq v(I) \) for all \( I \in K(D) \) such that \( D \subseteq I \).
5. If \( D \subseteq I \subseteq J \) in \( K(D) \), then \( v(I) \leq v(J) \). In particular, \( v(I) \leq v(I^2) \leq \cdots \leq v[I^n] \leq \cdots \) for all \( I \in K(D) \) such that \( D \subseteq I \).

**Proof.**

1. This is evident.
2. Since \( I \supseteq D \), \( E^0 = D : (D : E) \subseteq I : (D : E) = E^v(I) \) for all \( E \in K(D) \) and therefore \( \bar{v} \leq v(I) \).
3. This is evident.
4. By definition, \( D^v(I) = I \) for all \( I \in K(D) \) such that \( D \subseteq I \) and so our assertion is valid.
5. If \( D \subseteq I \subseteq J \in K(D) \), then \( E^v(I) = I : (D : E) \subseteq J : (D : E) = E^v(J) \) for all \( E \in K(D) \) and therefore \( v(I) \leq v(J) \). Next, if \( D \subseteq I \), then \( I^n \subseteq I^{n+1} \) for every integer \( n \geq 1 \) and so the “in particular” statement is evident.

**Proposition 3.5.** Let \( I \) be a \( K \)-fractional ideal of \( D \) such that \( D \subseteq I \). If \( I \) is not a \( v \)-ideal of \( D \), then \( v(I) \) is not a semistar operation on \( D \).

**Proof.** By definition, \( D^v(I) = I \) and therefore \( (D^v(I))^v(I) = I : (D : D^v(I)) = I : (D : I) = I : I^{-1} \supseteq D : I^{-1} = I^0 \supseteq I = D^v(I) \). Thus \( v(I) \) is not a semistar operation on \( D \).

**Proposition 3.6.** Let \( a \neq 0 \) be a nonunit of \( D \). Then \( v(\frac{1}{a}D) = v[aD] \).

**Proof.** First, note that \( D = \frac{1}{a}aD \subseteq \frac{1}{a}D \). Next, for all \( E \in K(D) \), we have \( E^{v(\frac{1}{a}D)} = \frac{1}{a}D : (D : E) = \frac{1}{a}D : E^{-1} = \frac{1}{a}(D : E^{-1}) = \frac{1}{a}E^0 = E^{v[aD]} \) by Lemma 3.1 and therefore \( v(\frac{1}{a}D) = v[aD] \).

**Lemma 3.2.** Let \( \ast \) be a presemistar operation on \( D \) and let \( I \) be an invertible ideal of \( D \). Then \( I^{-1}E^\ast = (I^{-1}E)^\ast \).
Proof. First, $I^{-1}E^* \subseteq (I^{-1}E)^*$ is clear by Lemma 2.1. Next, we also have $I(I^{-1}E)^* \subseteq (II^{-1}E)^* = E^*$ by Lemma 2.1 and so $(I^{-1}E)^* \subseteq I^{-1}E^*$ and therefore $I^{-1}E^* = (I^{-1}E)^*$. □

Lemma 3.3. Let $I$ be an invertible ideal of $D$. Then

1. $I : E = E^{-1} : I^{-1} = I^{-1}E^{-1}$ and $D : IE^{-1} = I^{-1}E^0$ for all $E \in K(D)$.
2. $v[I] = v(I^{-1})$ and furthermore $E^{v[I]} = E^{v(I^{-1})} = I^{-1}E^0$ for all $E \in K(D)$.

Proof. (1) This is straightforward.

(2) By definition, $E^{v(I^{-1})} = I^{-1} : (D : E) = I^{-1} : E^{-1}$ and $E^{v[I]} = D : (I : E) = (IE^{-1}) = (D : I) : E^{-1} = I^{-1} : E^{-1} = E^{v(I^{-1})}$ and so $v[I] = v(I^{-1})$. Furthermore, it follows from (1) that $E^{v[I]} = D : (IE^{-1}) = I^{-1}E^0$ for all $E \in K(D)$. □

Theorem 3.1. Let $D$ be a $\tilde{\nu}$-domain. Then $v[I] = v(I^{-1})$ is s.e.a.b. for each invertible ideal $I$ of $D$.

(2) If $v[I] = v(I^{-1})$ is s.e.a.b. for some invertible ideal $I$ of $D$, then $D$ is a $\tilde{\nu}$-domain.

Proof. (1) Assume that $A \subseteq (AB)^{v(I^{-1})}$ for an invertible ideal $I$ of $D$ and for all $A, B \in \mathcal{F}_f(D)$. Then by Lemma 3.3, $A \subseteq (AB)^{I^{-1}}$ and then $AI \subseteq (AB)^{I} = (AI^{-1}B)^{v}$. Then, since $\tilde{\nu}$ is s.e.a.b., we have $D \subseteq (I^{-1}B)^{v} = B^{v}I^{-1} = B^{v(I^{-1})}$ by Lemmas 3.2 and 3.3. Hence $v(I^{-1})$ is s.e.a.b.

(2) Assume that $v(I^{-1})$ is s.e.a.b. for some invertible ideal $I$ of $D$. Suppose that $(AB)^{v(I^{-1})} \subseteq (AC)^{v(I^{-1})}$ with $A, B, C \in \mathcal{F}_f(D)$. Then, $(AB)^{I^{-1}} = (AC)^{I^{-1}}$ and so by Lemma 3.3, $(AB)^{v(I^{-1})} \subseteq (AC)^{v(I^{-1})}$. Then it follows that $B^{v(I^{-1})} \subseteq C^{v(I^{-1})}$, because $v(I^{-1})$ is s.e.a.b.. Hence we obtain $B^vI^{-1} \subseteq C^vI^{-1}$ by Lemma 3.3 and so $B^v \subseteq C^v$. Thus $\tilde{\nu}$ is e.a.b. and therefore $D$ is a $\tilde{\nu}$-domain. □

Proposition 3.7. Let $R$ be an overring of $D$ such that $R : (D : R) \neq R$. Then $v(R)$ is a proper presemistar operation on $D$.

Proof. Assume that $R : (D : R) \neq R$. Then $D^{v(R)} = R : (D : R) = R$ and then $(D^{v(R)})^{v(R)} = R : (D : D^{v(R)}) = R : (D : R) \neq R = D^{v(R)}$. Hence $v(R)$ is not a semistar operation on $D$. □

Example 3.1. Let $k$ be a field and let $D = k[[X^2, X^3]]$ be the subdomain of $V = k[[X]]$ consisting of those power series with zero $X$ term. Then $v(V)$ is a proper presemistar operation on $D$. First, note that $M = X^2D + X^3D$ is the maximal ideal of $D$ and $M = D : V$. By definition, $D^{v(V)} = V$ and $(D^{v(V)})^{v(V)} = V : (D : V) = V : M$. It is easily seen that $\frac{1}{X} \notin V$. But, since $\frac{1}{X}X^2 = X \in V$ and $\frac{1}{X^3}X^2 = X^2 \in V$, we have $\frac{1}{X} \in V : M$. Then our assertion follows from Proposition 3.7.

The next example shows the existence of a proper presemistar operation which is e.a.b. but is not s.e.a.b. To prove this, we use a presemistar operation of the form $\mu(I)$ with $I \in K(D)$ such that $D \subseteq I$ defined in [O3].
Example 3.2. Let $k$ be a field and let $D = k[[X^2, X^7]]$ be the subdomain of $V = k[[X]]$ consisting of those power series with zero $X, X^3$ and $X^5$ terms. Let $I = D + \frac{1}{X}D \in K(D)$Then $XI = XD + D = V$. First, $\mu(I)$ is a proper presemistar operation on $D$ by [O3, Proposition 3.1 (5)]. Next, we shall show that $\mu(I)$ is e.a.b.. Let $(EF)\mu(I) \subseteq (EG)\mu(I)$ with $E, F, G \in F(D)$. Then $EFI \subseteq EGI$ and so $(EXI)F \subseteq (EXI)G$. But, $EXI = EV$ is a finitely generated $V$-module and therefore $EXI$ is principal, because $V$ is a DVR. Hence we have $VF \subseteq VG$, that is, $XIF \subseteq XIG$ and therefore $F\mu(I) = FI \subseteq GI = G\mu(I)$. Thus $\mu(I)$ is e.a.b.. Lastly, we shall show that $\mu(I)$ is not s.e.a.b.. Let $A = (1 + X)D + (X + X^2)D$ and $B = \frac{1}{1+X}D$. Then $BI = \frac{1}{1+X}D + \frac{1}{X+X^2}D$ and $ABI = D + XD + \frac{1}{X}D$. Since $1 + X \in D + XD$ and $X + X^2 \in D + XD$, we have $A \subseteq D + XD \subseteq ABI = (AB)\mu(I)$. But, $B\mu(I) = BI = \frac{1}{1+X}D + \frac{1}{X+X^2}D \not\in D$ and hence $\mu(I)$ is not s.e.a.b..

It would be worthwhile to remark the following:

Corollary 3.3. Let $D$ be an integral domain such that $D \neq K$. Then

(1) If $I$ is a divisorial ideal of $D$, then $\bar{v} \leq v[I] \leq \bar{e}$

(2) If $I$ is a $K$-fractional ideal of $D$ such that $D \not\subseteq I \subseteq K$, then $\bar{v} \leq v(I) \leq \bar{e}$

Proof. (1) By Proposition 3.1 (5), $\bar{v} \leq v[I]$ holds. Suppose that $v[I] = \bar{e}$. Then, since $D\bar{v}[I] = I^{-1}$, we have $I^{-1} = K$ and so $I = I^e = K^{-1} = \{0\}$, a contradiction. Hence we also have $v[I] \leq \bar{e}$.

(2) This follows from Proposition 3.4 (3) and (4), \qed

It is easy to see that Proposition 3.3 is a special case of Theorem 3.1 and Proposition 3.6 is a special case of Lemma 3.3 (2).

References

[E1] J. Elliott, Prequantales and applications to semistar operations and module systems, arXiv: 1101. 2462v1 [math. RA] 12 Jan 2011.

[E2] J. Elliott, Nuclei and applications to star, semistar, and semiprime operations, arXiv:1505. 06433v1 [math. RA] 24 May 2015.

[FH] M. Fontana and J. A. Huckaba, Localizing systems and semistar operations, in: S. Chapman and S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic Publishers, Dordrecht, 2000, 169-197.

[FL] M. Fontana and K. Alan Loper, Kronecker function rings: a general approach, in “Ideal Theoretic Methods in Commutative Algebra” (D. D. Anderson and I. J. Papick, Eds.), Marcel Dekker Lecture Notes Pure Appl. Math., 220 (2001), 189-205.

[G] R. Gilmer, Multiplicative Ideal Theory, Queen’s Papers in Pure and Applied Mathematics Vol. 90, Kingston, Ontario, Canada, 1992.

[HHP] W. J. Heinzer, J. A. Huckaba and I. J. Papick, m-canonical ideals in integral domains, Comm. Algebra, 26 (1998), 3021-3043.
[O1] A. Okabe, Note on characterizations of semistar operations and star operations on an integral domain, Math. J. Ibaraki Univ., 46 (2014), 31-36.

[O2] A. Okabe, Note on characterizations of semistar operations and star operations on an integral domain, II, Math. J. Ibaraki Univ., 47 (2015), 39-47.

[O3] A. Okabe, Presemistar operations on integral domains, Math. J. Ibaraki Univ., 48 (2016), 27-43.

[OM] A. Okabe and R. Matsuda, Semistar-operations on integral domains, Math. J. Toyama Univ. 17 (1994), 1-21.

[P1] G. Picozza, Semistar Operations and Multiplicative Ideal Theory, PhD thesis, Università degli Studi “Roma Tre”, June 2004.

[P2] G. Picozza, Star operations on overrings and semistar operations, Comm. Algebra, 33 (2005), 2051-2073.