Conservation Laws for SO(p,q)

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Using our previous results on the systematic construction of invariant differential operators for non-compact semisimple Lie groups we classify the conservation laws in the case of SO(p,q).

1. Introduction

In a recent paper we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study first. We decided to start with a subclass of the hermitian-symmetric algebras which share some special properties of the conformal algebra \( so(n,2) \). That is why, in view of applications to physics, we called these algebras 'conformal Lie algebras' (CLA), (or groups). Later we gave a natural way to go beyond this subclass using essentially the same results. For this we introduce the new notion of parabolic relation between two non-compact semisimple Lie algebras \( G \) and \( G' \) that have the same complexification and possess maximal parabolic subalgebras with the same complexification. Thus, for example, using results for the conformal algebra \( so(n,2) \) (for fixed \( n \)) we can obtain results for all pseudo-orthogonal algebras \( so(p,q) \) such that \( p + q = n + 2 \). In this way, in \( 3 \) (among other things) we gave the main and the reduced multiplets of indecomposable elementary representations for \( so(p,q) \) including the necessary data for all relevant invariant differential operators. We specially stressed that the classification of all invariant differential operators includes as special cases all possible conservation laws and conserved currents, unitary or not. In the present
paper we give explicitly the conservation laws in the case of $so(p,q)$.

This paper is a short sequel of\textsuperscript{3}, based on Invited talk at Group-29, Chern Institute of Mathematics, Nankai U., 20-26.8.2012. Due to the lack of space we refer to\textsuperscript{3} for motivations and extensive list of literature on the subject.

2. Preliminaries

Let $G = so(p,q)$, $p \geq q$, $p + q > 4$. We choose a maximal parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ such that:

$$\mathcal{M} = so(p-1,q-1), \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = p + q - 2.$$  

With this choice we get for the conformal algebra $so(n,2)$ the Bruhat decomposition $G = \mathcal{P} \oplus \tilde{\mathcal{N}}$ with direct physical meaning ($\tilde{\mathcal{N}} \cong \mathcal{N}$)\textsuperscript{3}.

We label the signature of the representations of $G$ as follows:

$$\chi = \{n_1, \ldots, n_h; c\},$$

$$n_j \in \mathbb{Z}/2, \quad c = d - \frac{p+q-2}{2}, \quad h = \left\lfloor \frac{p+q-2}{2} \right\rfloor,$$

$$|n_1| < n_2 < \cdots < n_h, \quad p + q \text{ even},$$

$$0 < n_1 < n_2 < \cdots < n_h, \quad p + q \text{ odd},$$

where the parameter $c$ (related to the conformal weight $d$) labels the characters of $A$, and the first $h$ entries are labels of the finite-dimensional (nonunitary for $q \neq 1$) irreps $\mu$ of $\mathcal{M}$.

We call the above induced representations $\chi = \text{Ind}_G^\mathcal{P}(\mu \otimes \nu \otimes 1)$ elementary representations of $G = SO(p,q)$. Their spaces of functions are:

$$\mathcal{C}_\chi = \{F \in C^\infty(G,V_\mu) | F(\text{gman}) = e^{-\nu(H)} \cdot D^{\mu}(m^{-1}) F(g)\}$$

where $a = \exp(H), \quad H \in \mathcal{A}, \quad m \in M = SO(p-1,q-1), \quad n \in N = \exp \mathcal{N}$.

The representation action is the left regular action:

$$(T^\chi(g)F)(g') = F(g^{-1}g'), \quad g, g' \in G.$$  

• An important ingredient in our considerations are the highest/lowest weight representations of $G^C$. These can be realized as (factor-modules of) Verma modules $V^\Lambda$ over $G^C$, where $\Lambda \in (\mathcal{H}^C)^*, \quad \mathcal{H}^C$ is a Cartan subalgebra of $G^C$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from $\chi$.

Actually, since our ERs are induced from finite-dimensional representations of $\mathcal{M}$ the Verma modules are always reducible. Thus, it is more convenient to use generalized Verma modules $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector $v_0$ is taken by the (finite-dimensional) space
For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight $d$, or the parameter $c$. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

- One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called multiplets. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines (arrows) between the vertices correspond to intertwining operators. The multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair $(\beta, m)$, where $\beta$ is a (non-compact) positive root of $G^C$, $m \in \mathbb{N}$, such that the BGG Verma module reducibility condition\(^7\) (for highest weight modules) is fulfilled:

$$ (\Lambda + \rho, \beta^\vee) = m, \quad \beta^\vee \equiv 2\beta/(\beta, \beta) \quad (4) $$

where $\rho$ is half the sum of the positive roots of $G^C$. When the above holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and $\beta$ non-compact) is embedded in the Verma module $V^\Lambda$ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector $v_s$ expressed by a polynomial $P_{m,\beta}(G^-)$ in the universal enveloping algebra $(U(G_-))v_0$, $G^-$ is the subalgebra of $G^C$ generated by the negative root generators\(^8\).

More explicitly,\(^5\) $v_{m,\beta}^s = P_{m,\beta} v_0$ (or $v_{m,\beta}^s = P_{m,\beta} V_\mu v_0$ for GVMs). Then there exists an intertwining differential operator of order $m = m_\beta$:

$$ D_{m,\beta} : \mathcal{C}_{\chi(\Lambda)} \rightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \quad (5) $$

given explicitly by:

$$ D_{m,\beta} = P_{m,\beta}(\hat{G}^-) \quad (6) $$

where $\hat{G}^-$ denotes the right action on the functions $F$.

Thus, in each such situation we have an invariant differential equation of order $m = m_\beta$:

$$ D_{m,\beta} f = f', \quad f \in \mathcal{C}_{\chi(\Lambda)}, \quad f' \in \mathcal{C}_{\chi(\Lambda - m\beta)} \quad (7) $$

In most of these situations the invariant operator $D_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $G$ is realized. Thus, studying the equations with trivial RHS:

$$ D_{m,\beta} f = 0, \quad f \in \mathcal{C}_{\chi(\Lambda)} \quad (8) $$

\(^a\)For explicit expressions for singular vectors we refer to\(^9\).
is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for \( m = m_\beta = 1 \), equations (8) are called conservation laws, and the elements \( f \in \ker D_{m,\beta} \) are called conserved currents. Below we give them explicitly for \( \mathfrak{so}(p,q) \).

3. Classification of Conservation Laws for \( \mathfrak{so}(p,q) \)

Using results of \(^4,10\)-\(^13\) we present the main multiplets (which contain the maximal number of ERs with this parabolic) with the following explicit parametrization of the ERs in the multiplets (following \(^12\)):

\[
\begin{align*}
\chi_1^\pm &= \{ \epsilon n_1, \ldots, n_h; \pm n_{h+1} \}, \quad n_h < n_{h+1}, \\
\chi_2^\pm &= \{ \epsilon n_1, \ldots, n_{h-1}, n_{h+1}; \pm n_h \} \\
\chi_3^\pm &= \{ \epsilon n_1, \ldots, n_{h-2}, n_h, n_{h+1}; \pm n_{h-1} \} \\
&\vdots \\
\chi_{h-1}^\pm &= \{ \epsilon n_1, n_2, n_4, \ldots, n_h, n_{h+1}; \pm n_3 \} \\
\chi_h^\pm &= \{ \epsilon n_1, n_3, \ldots, n_h, n_{h+1}; \pm n_2 \} \\
\chi_{h+1}^\pm &= \{ \epsilon n_2, n_3, \ldots, n_h, n_{h+1}; \pm n_1 \} \\
\epsilon &= \begin{cases} 
\pm, & p + q \text{ even} \\
1, & p + q \text{ odd} 
\end{cases}
\end{align*}
\]

(\( \epsilon = \pm \) is correlated with \( \chi^\pm \)). Clearly, the multiplets correspond 1-to-1 to the finite-dimensional irreps of \( \mathfrak{so}(p+q, \mathbb{C}) \) with signature \( \{ n_1, \ldots, n_h, n_{h+1} \} \) and we are able to use previous results due to the parabolic relation between the \( \mathfrak{so}(p,q) \) algebras for \( p + q \)-fixed. Note that the two representations in each pair \( \chi^\pm \) are called shadow fields.

Further, we denote by \( C_i^\pm \) the representation space with signature \( \chi_i^\pm \).

The ERs in the multiplet are related by intertwining integral and differential operators.

The integral operators were introduced by Knapp and Stein\(^14\). Here these operators intertwine the pairs \( C_i^\pm \) (cf. (9)):

\[
G_i^\pm : C_i^\mp \longrightarrow C_i^\pm, \quad i = 1, \ldots, 1+h .
\]

The intertwining differential operators correspond to non-compact positive roots of the root system of \( \mathfrak{so}(p+q, \mathbb{C}) \), cf.\(^5\). In the current context, compact roots of \( \mathfrak{so}(p+q, \mathbb{C}) \) are those that are roots also of the subalgebra \( \mathcal{M}^\mathbb{C} \), the rest of the roots are non-compact. We denote the differential
operators by $d_i, d'_i$. The spaces from (9) they intertwine are:

$$d_i : C^-_i \rightarrow C^-_{i+1}, \quad i = 1, \ldots, h;$$

$$d'_i : C^+_i \rightarrow C^+_{i+1}, \quad i = 1, \ldots, h - 1;$$

$$d_h : C^+_h \rightarrow C^+_h, \quad (p + q) - \text{even};$$

$$d'_h : C^-_h \rightarrow C^+_{h+1}, \quad (p + q) - \text{even};$$

$$d'_h : C^-_h \rightarrow C^+_{h+1}, \quad (p + q) - \text{odd};$$

$$d_{h+1} : C^-_{h+1} \rightarrow C^+_h, \quad (p + q) - \text{odd}. \quad (11)$$

The degrees of these intertwining differential operators are given just by the differences of the $c$ entries\[12:\]

$$\deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i}, \quad i = 1, \ldots, h, \quad (12)$$

$$\deg d'_h = n_2 + n_1, \quad (p + q) - \text{even},$$

$$\deg d_{h+1} = 2n_1, \quad (p + q) - \text{odd}.$$

where $d'_h$ is omitted from the first line for $(p + q)$ even.

Thus, we are able to list all cases of first order intertwining differential operators or conservation laws (we give only the signatures):

$$d_1 : \{ \epsilon n_1, n_2, \ldots, n_h; -n_h - 1 \} \rightarrow \{ \epsilon n_1, n_2, \ldots, n_{h-1}, n_h + 1; -n_h \};$$

$$d'_1 : \{ n_1, \ldots, n_{h-1}, n_h + 1; n_h \} \rightarrow \{ n_1, \ldots, n_h; n_h + 1 \}; \quad (13)$$

$$d_i : \{ \epsilon n_1, n_2, \ldots, n_{h+1-i}, n_{h+3-i}, \ldots, n_{h+1}; -n_{h+1-i} - 1 \} \rightarrow$$

$$\{ \epsilon n_1, n_2, \ldots, n_{h-i}, n_{h+1-i} + 1, n_{h+3-i}, \ldots, n_{h+1}; -n_{h+1-i} \},$$

$$i = 2, \ldots, h - 1;$$

$$d'_i : \{ n_1, \ldots, n_{h-i}, n_{h+1-i} + 1, n_{h+3-i}, \ldots, n_{h+1}; n_{h+1-i} \} \rightarrow$$

$$\{ n_1, \ldots, n_{h+1-i}, n_{h+3-i}, \ldots, n_{h+1}; n_{h+1-i} + 1 \},$$

$$i = 2, \ldots, h - 1;$$

$$d_h : \{ \epsilon (n_2 - 1), n_3, \ldots, n_{h+1}; -n_2 \} \rightarrow$$

$$\{ \epsilon n_2, n_3, \ldots, n_{h+1}; 1 - n_2 \}, \quad n_2 > \frac{1}{2};$$

$$d_h : \{ n_2, n_3, \ldots, n_{h+1}; n_2 - 1 \} \rightarrow \{ n_2 - 1, n_3, \ldots, n_{h+1}; n_2 \},$$

$$n_2 > \frac{1}{2}, \quad (p + q) - \text{even};$$

$$d'_h : \{ n_2 - 1, n_3, \ldots, n_{h+1}; -n_2 \} \rightarrow \{ n_2, n_3, \ldots, n_{h+1}; 1 - n_2 \},$$

$$n_2 > \frac{1}{2}, \quad (p + q) - \text{even};$$
d_h': \{ -n_2, n_3, \ldots, n_{h+1}; n_2 - 1 \} \to \{ 1 - n_2, n_3, \ldots, n_{h+1}; n_2 \},
\quad n_2 > \frac{1}{2}, \quad (p + q) - \text{even} ;

d_h': \{ n_2, n_3, \ldots, n_{h+1}; n_2 - 1 \} \to \{ n_2 - 1, n_3, \ldots, n_{h+1}; n_2 \},
\quad n_2 > 1, \quad (p + q) - \text{odd} ;

d_{h+1}': \{ -n_2, n_3, \ldots, n_{h+1}; -\frac{1}{2} \} \to \{ n_2, n_3, \ldots, n_{h+1}; \frac{1}{2} \},
\quad n_2 > 1, \quad (p + q) - \text{odd} ;
\quad \epsilon = -(1)^{p+q} .

Some of these conservation laws were given also in\(^{15}\). Although the operators are valid for arbitrary \(so(p, q)\) \((p + q \geq 5)\) the contents of the ERs is very different. This analysis was done in detail in\(^3\).

Besides the above cases there are other ERs (not related to finite-dimensional irreps of \(so(p + q, \mathbb{C})\)), and which also give rise to first order intertwining differential operators, resp. conservation laws:

\[ d_\epsilon : \tilde{\chi}_-^- \to \tilde{\chi}_+^+ , \quad (14) \]

\[ \tilde{\chi}_\pm^\epsilon = \{ \pm \epsilon \frac{1}{2}, n_3, \ldots, n_{h+1}; \pm \frac{1}{2} \} , \]

where \(\epsilon\) is defined as in (9), but here it is not correlated with \(\tilde{\chi}_\pm^\epsilon\). There is also a Knapp-Stein integral operator acting from \(\tilde{\chi}_+^\epsilon\) to \(\tilde{\chi}_-^-\). The representations \(\tilde{\chi}_-^-\) are called minimal representations.

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