TOPOLOGICAL OBSTRUCTIONS TO DOMINATED SPLITTING FOR ERGODIC TRANSLATIONS ON THE HIGHER DIMENSIONAL TORUS

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ABSTRACT. Consider the space of analytic, quasi-periodic cocycles on the higher dimensional torus. We provide examples of cocycles with nontrivial Lyapunov spectrum, whose homotopy classes do not contain any cocycles satisfying the dominated splitting property. This shows that the main result in the recent work “Complex one-frequency cocycles” by A. Avila, S. Jitomirskaya and C. Sadel does not hold in the higher dimensional torus setting.

1. Introduction and statements. It is well known that the homotopy type may prevent a continuous linear cocycle over a base dynamical system from being uniformly hyperbolic. In fact, for an $\text{SL}_2(\mathbb{R})$-valued cocycle over a circle map, M. Herman remarked that the topological degrees of the base map $T: \mathbb{T} \to \mathbb{T}$ and of the matrix valued function $A: \mathbb{T} \to \text{SL}_2(\mathbb{R})$ provide topological obstructions to the uniform hyperbolicity of the cocycle. More precisely, this obstruction happens when $\text{deg}(T) - 1$ does not divide $\text{deg}(A_p)$, where for any $p \in \mathbb{P}(\mathbb{R}^2)$, $A_p: \mathbb{T} \to \mathbb{P}(\mathbb{R}^2)$ denotes the projective space induced map $A_p(x) = A(x)p$ (see [11] or [5]).

In sharp contrast with this, A. Avila, S. Jitomirskaya and C. Sadel [2] recently proved that analytic cocycles $A: \mathbb{T} \to \text{GL}_m(\mathbb{C})$ over irrational translations on the one dimensional torus $\mathbb{T}$ are always approximated by cocycles with dominated splitting (a type of uniform projective hyperbolicity), provided the Oseledets filtration is nontrivial. In particular, every homotopy class of such cocycles contains analytic cocycles with dominated splitting.

In this same direction J. Bochi [4] recently characterized those quasi-periodic cocycles which can be approximated by cocycles with dominated splitting. More precisely, Theorem A of this manuscript implies that inside each homotopy class

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of continuous quasiperiodic $GL_m(\mathbb{R})$-cocycles ($m \geq 3$), either there is no cocycle admitting a dominated splitting, or else those that admit a dominated splitting are (open and) dense inside the homotopy class.

In dynamical systems, the Bochi-Mané dichotomy refers to a generic (low regularity) dichotomy between zero Lyapunov exponents and uniform hyperbolicity, or dominated splitting in higher dimensions. This dichotomy, proved by J. Bochi [3], was first announced by R. Mañe in the context of $C^1$-area preserving diffeomorphisms of a surface. Later J. Bochi and M. Viana generalized it to $C^1$-volume preserving diffeomorphisms of any compact manifold [5]. These works [3, 5] also include versions of the dichotomy for classes of $C^0$-cocycles. Because the low regularity is essential here, it is quite surprising that the same type of dichotomy can hold in [2] for a class of analytic cocycles.

The purpose of this note is to show that the main result in the aforementioned paper [2] does not hold for cocycles over ergodic translations on the higher dimensional torus $\mathbb{T}^d$, $d \geq 2$. We obtain this by developing a simple homological obstruction to the existence of continuous invariant sections of the skew product map induced by the cocycle at the level of the Grassmannian space of a certain dimension. These examples also illustrate the dichotomy [4, Theorem A].

A somewhat related topic is that of the regularity of the Lyapunov exponents under small perturbations of the cocycle in certain topological spaces of cocycles. In [2] the authors prove continuity of the Lyapunov exponents on the space $C^\omega(T, \text{Mat}(m, \mathbb{C}))$ of analytic cocycles\(^1\) over irrational translations on the one dimensional torus. Dominated splitting plays a crucial role in their proof, more precisely, the fact that if the Oseledets filtration of the cocycle $A(x)$ is nontrivial, then for small enough $\epsilon > 0$, the complexified cocycle $A(x + iy)$ has dominated splitting for a.e. $y$ with $|y| < \epsilon$ (see [2, Lemma 4.1]). As a consequence of our main result, the analogue of this statement for ergodic translations on the higher dimensional torus does not hold (see Remark 3). However, in [6] we established by other means the continuity of the Lyapunov exponents for analytic cocycles over such translations.

We now introduce the main concepts more formally.

Let $K = \mathbb{R}$ or $K = \mathbb{C}$ refer to either the real or the complex field. Let $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ with $d \geq 2$ be the higher dimensional torus.

A continuous function $A : \mathbb{T}^d \to GL_m(K)$ and an ergodic translation $T : \mathbb{T}^d \to \mathbb{T}^d$ determine the skew-product map $F : \mathbb{T}^d \times K^m \to \mathbb{T}^d \times K^m$, $F(x, v) = (Tx, A(x)v)$.

We call the new dynamical system $F$ a linear cocycle over the base transformation $T$. Its iterates are $F^n(x, v) = (T^n x, A^{(n)}(x)v)$, where $A^{(n)}(x) := A(T^{n-1}x) \ldots A(Tx) A(x)$.

Since $T$ is usually fixed, we identify the linear cocycle $F$ with the matrix-valued function $A$, and its iterates $F^n$ with $A^{(n)}$.

The Lyapunov exponents of a linear cocycle $A$ measure the average exponential rate of growth of the iterates $A^{(n)}(x)$ along the invariant subspaces given by the Oseledets theorem.

\(^1\)We regard functions on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as 1-periodic functions on the real line. Then $C^\omega(T, \text{Mat}(m, \mathbb{C}))$ is the space of functions $A : T \to \text{Mat}(m, \mathbb{C})$ admitting a holomorphic extension to the complex strip $|\Im z| < r$, continuous up to the boundary and endowed with the uniform norm on the strip.
We say that a linear cocycle $A$ has dominated splitting with respect to $\mathbb{K}$ if there exists a continuous $F$-invariant decomposition $\mathbb{K}^m = E_1(x) \oplus \cdots \oplus E_l(x)$, where $2 \leq l \leq m$ and each $E_i$ is an $F$-invariant continuous $\mathbb{K}$ sub-bundle of the trivial bundle $\mathbb{T}^d \times \mathbb{K}^m$ such that for some $\lambda > 1$, for any $1 \leq i < j \leq l$ and for any unit vectors $v_i \in E_i(x)$, $v_j \in E_j(x)$,

$$\frac{\|A^{(n)}(x)v_i\|}{\|A^{(n)}(x)v_j\|} \geq \lambda^n$$

for all $n \in \mathbb{N}$.

In particular, as $l \geq 2$, the Oseledets decomposition of $A$ is nontrivial (its components are proper subspaces of $\mathbb{K}^m$) so the Lyapunov exponents of $A$ are not all equal.

For $\text{SL}_2(\mathbb{R})$-valued cocycles, the dominated splitting property is equivalent to uniform hyperbolicity.

Following the terminology in [2], given $1 \leq k < m$, we say that a linear cocycle $A: \mathbb{T}^d \to \text{GL}_m(\mathbb{K})$ is $k$-dominated if it admits a dominated decomposition $\mathbb{K}^m = E^+ \oplus E^-$ with $\dim E^+ = k$.

It is clear that if the linear cocycle $A$ has the dominated splitting $\mathbb{K}^m = E_1(x) \oplus \cdots \oplus E_l(x)$, then $A$ is $k$-dominated for every dimension $k = \dim(E_1) + \cdots + \dim(E_l)$ with $1 \leq i \leq l - 1$.

We are now ready to formulate the main result of this paper.

**Theorem 1.** Given integers $d \geq 2$ and $1 \leq k < m$ there exist analytic quasi-periodic cocycles $A: \mathbb{T}^d \to \text{GL}_m(\mathbb{C})$ with an invariant measurable decomposition $\mathbb{C}^m = E^+ \oplus E^-$ such that

1. $\dim E^+ = k$,
2. all Lyapunov exponents of $A|_{E^+}$ are positive,
3. all Lyapunov exponents of $A|_{E^-}$ are negative,
4. no continuous cocycle $B: \mathbb{T}^d \to \text{GL}_m(\mathbb{C})$ in the homotopy class of $A$ is $k$-dominated.

**Remark 1.** This theorem shows that the dichotomy in [2, Theorem 1.1] does not hold for analytic quasi-periodic cocycles over a torus $\mathbb{T}^d$ of dimension $d \geq 2$. In fact any sufficiently small neighborhood $V$ of $A$ is contained in the homotopy class of $A$. In this neighborhood $V$, by our continuity result [7, Theorem 6.1], assuming that the translation vector satisfies a generic Diophantine condition, the Oseledets decomposition $\mathbb{C}^m = E^+ \oplus E^-$ persists with $\dim E^+ = k$. However, in view of Theorem 1, this decomposition is never $k$-dominated.

Consider now the projective space $\mathbb{P}(\mathbb{K}^m)$ where the group $\text{GL}_m(\mathbb{K})$ acts transitively. More generally let $\text{Gr}_k(\mathbb{K}^m)$ be the Grassmannian space of all $k$-dimensional $\mathbb{K}$-linear subspaces of $\mathbb{K}^m$, which reduces to the projective space when $k = 1$.

The cocycle $F$ determines the skew-product map $\tilde{F}: \mathbb{T}^d \times \text{Gr}_k(\mathbb{K}^m) \to \mathbb{T}^d \times \text{Gr}_k(\mathbb{K}^m)$ defined by $\tilde{F}(x, V) := (Tx, A(x)V)$. Clearly the $k$-domination property implies the existence of a continuous invariant section $E^+: \mathbb{T}^d \to \text{Gr}_k(\mathbb{K}^m)$ for the bundle map $\tilde{F}$. The strategy to prove Theorem 1 is to derive topological obstructions to the existence of continuous invariant sections $\sigma: \mathbb{T}^d \to \text{Gr}_k(\mathbb{C}^m)$ of the cocycle $A$.

**Remark 2.** The statement of Theorem 1 holds also for $\text{GL}_m(\mathbb{R})$-valued cocycles over a torus $\mathbb{T}^d$ with dimension $d \geq 1$. This can be proven analogously or more simply using M. Herman’s method described in [11]. The topological obstructions
there use first homotopy groups and are applicable because the real Grassmannians $\text{Gr}_k(\mathbb{R}^m)$ are not simply connected, something which is not true about the complex Grassmannians $\text{Gr}_k(\mathbb{C}^m)$.

The paper is organized as follows. In Section 2 we provide a necessary condition for the existence of a continuous invariant section of a skew product map. In Section 3 we use the previous abstract result to provide topological obstructions to the existence of continuous invariant sections for quasi-periodic cocycles on the higher dimensional torus. This in particular implies our main theorem.

We are grateful to Christian Sadel for posing the question regarding dominated splitting for quasi-periodic cocycles on the torus of several variables, to Gustavo Granja for a valuable suggestion on using the nonexistence of homological splitting as a topological obstruction to dominated splitting, to Marcelo Viana for providing us with several references on this subject, and to the anonymous referees for pointing out the related work [4] as well as for their careful reading of the manuscript.

2. Existence of invariant sections. We call factor of linear maps any commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E \\
\pi \downarrow & & \downarrow \pi \\
F & \xrightarrow{h} & F
\end{array}
$$

(1)

where $E$, $F$ are vector spaces, $f : E \to E$, $h : F \to F$ are linear endomorphisms and $\pi : E \to F$ is a linear epimorphism. We call splitting of a factor (1) any linear map $\sigma : F \to E$ such that $\pi \circ \sigma = \text{id}_F$ and $f \circ \sigma = \sigma \circ h$. In other words $\sigma$ is an $f$-invariant section of the vector bundle $\pi : E \to F$.

Letting $K = \ker(\pi)$, by the fundamental theorem on homomorphisms, the linear epimorphism $\pi : E \to F$ induces an isomorphism $\bar{\pi} : E/K \simeq F$ through which the factor (1) can be expressed as

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E \\
\pi \downarrow & & \downarrow \pi \\
E/K & \xrightarrow{f} & E/K
\end{array}
$$

(2)

where $K$ stands for an $f$-invariant vector subspace of $E$. From these considerations it follows easily that

**Proposition 1.** The factor (1) has a splitting if and only if the vector space $E$ admits an $f$-invariant decomposition $E = G \oplus \ker(\pi)$ for some linear subspace $G$.

Let $M$ be a compact connected manifold. Consider a continuous map $T : M \to M$ and a transitive action $G \times X \to X$ of a connected Lie group $G$ on some compact connected space $X$. A continuous function $A : M \to G$ determines the skew-product map

$$
F : M \times X \to M \times X, \quad F(x, p) := (Tx, A(x) p).
$$

(3)
By definition, letting \( \pi: M \times X \to M \) stand for the canonical projection \( \pi(x, p) = x \), the following diagram commutes

\[
\begin{array}{ccc}
M \times X & \xrightarrow{F} & M \times X \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{T} & M
\end{array}
\]  \tag{4}

We call \( F \)-invariant section any continuous map \( \sigma: M \to X \) such that \( F(x, \sigma(x)) = (Tx, \sigma(Tx)) \) for all \( x \in M \).

An obvious necessary condition to the existence of an \( F \)-invariant section is the splitting property of the factor (4) at the level of homology (the reader may consult [9] for a general reference on singular homology).

**Proposition 2.** If the map (3) admits an invariant section then for each \( 0 \leq i \leq \dim M \) the homological factor

\[
\begin{array}{ccc}
H_i(M \times X, \mathbb{F}) & \xrightarrow{F_*} & H_i(M \times X, \mathbb{F}) \\
\downarrow \pi_* & & \downarrow \pi_* \\
H_i(M, \mathbb{F}) & \xrightarrow{T_*} & H_i(M, \mathbb{F})
\end{array}
\]  \tag{5}

admits a splitting.

**Proof.** By the Künneth theorem the map \( \pi_*: H_i(M \times X, \mathbb{F}) \to H_i(M, \mathbb{F}) \) is surjective. If \( \sigma: M \to X \) is an \( F \)-invariant section then its homology \( \sigma_*: H_i(M, \mathbb{F}) \to H_i(M \times X, \mathbb{F}) \) is a splitting of the homological factor (5).

The next proposition specializes the previous criterion to the case where the map \( T: M \to M \) is homotopic to the identity.

**Proposition 3.** Let \( T: M \to M \) be a continuous map homotopic to the identity. Let \( A: M \to G \) be a continuous function and 1 \( \leq k \leq \dim M \) a dimension such that:

1. \( H_k(M, \mathbb{F}) \neq \{0\} \),
2. \( H_i(M, \mathbb{F}) = \{0\} \) or \( H_{k-i}(X, \mathbb{F}) = \{0\} \), for all \( 0 < i < k \),
3. For some \( p \in X \) the map \( A_p: M \to X, A_p(x) := A(x)p \), induces a non-zero homology map in dimension \( k \), i.e., the linear map \( (A_p)_*: H_k(M, \mathbb{F}) \to H_k(X, \mathbb{F}) \) is non-zero.

Then \( F \) admits no \( F \)-invariant section.

**Proof.** By the Künneth theorem and assumptions 1-2,

\[
H_k(M \times X, \mathbb{F}) \cong H_k(M, \mathbb{F}) \otimes H_0(X, \mathbb{F}) \oplus H_0(M, \mathbb{F}) \otimes H_k(X, \mathbb{F})
\]

\[
\cong H_k(M, \mathbb{F}) \oplus H_k(X, \mathbb{F}).
\]

We are also using here that \( M \) and \( X \) are connected so that \( H_0(M, \mathbb{F}) \cong H_0(X, \mathbb{F}) \cong F \). Hence the epimorphism \( \pi_*: H_k(M \times X, \mathbb{F}) \to H_k(M, \mathbb{F}) \) has kernel

\[
\ker(\pi_*) \cong H_0(M, \mathbb{F}) \otimes H_k(X, \mathbb{F}) \cong H_k(X, \mathbb{F}).
\]

Similarly, the projection \( \pi': M \times X \to X, \pi'(x, p) = p \), induces a homology map \( \pi'_*: H_k(M \times X, \mathbb{F}) \to H_k(X, \mathbb{F}) \) with kernel

\[
\ker(\pi'_*) \cong H_k(M, \mathbb{F}) \otimes H_0(X, \mathbb{F}) \cong H_k(M, \mathbb{F}).
\]
Because $G$ is connected, each element $A(x) \in G$ induces an action $A(x) : X \to X$ which is isotopic to the identity. Therefore the homology map $F_* : H_k(M \times X, \mathbb{F}) \to H_k(M \times X, \mathbb{F})$ acts as the identity on $\ker(\pi_*)$.

Assume now, by contradiction, that $F$ admits an invariant section. By Proposition 2 there exists an $F_*$-invariant subspace $G$ such that

$$H_k(M \times X, \mathbb{F}) = \ker(\pi_*) \oplus G.$$  \hfill(6)

Since $T : M \to M$ is homotopic to $id_M$ we have $T_* = id$ on $H_k(M, \mathbb{F})$. This implies that $F_*$ is the identity map on $G$. Hence, because (6) is $F_*$-invariant, it follows that $F_*$ is the identity on $H_k(M \times X, \mathbb{F})$.

Finally, defining the inclusion map $i_p : M \to M \times X$, $i_p(x) := (x, p)$, since $A_p = \pi' \circ F \circ i_p$ we have at the homology level

$$0 \neq (A_p)_* = (\pi' \circ F \circ i_p)_* = \pi'_* \circ F_* \circ (i_p)_* = \pi'_* \circ (i_p)_* = 0.$$

We have used assumption 3 and the fact that the composition $\pi' \circ i_p$ is a constant map. This contradiction proves that $F$ admits no invariant section.

3. Consequences for quasi-periodic cocycles. Finally we show that for certain homotopy types a continuous quasi-periodic cocycle $A : \mathbb{T}^d \to \text{GL}_m(\mathbb{C})$ cannot have dominated splitting. The base dynamics is assumed to be an ergodic translation of a torus $\mathbb{T}^d$ of dimension $d \geq 2$.

Let $\text{Gr}_k(\mathbb{C}^m)$ denote the complex Grassmannian of $k$-dimensional complex subspaces of $\mathbb{C}^m$.

**Proposition 4.** Let $A : \mathbb{T}^d \to \text{GL}_m(\mathbb{C})$ be a continuous function with $d \geq 2$ and take $1 \leq k < m$. If the map $A_V : \mathbb{T}^d \to \text{Gr}_k(\mathbb{C}^m)$, $A_V(x) = A(x)V$, for some $V \in \text{Gr}_k(\mathbb{C}^m)$, induces a non-zero homology map in dimension two, i.e., $(A_V)_* : H_2(\mathbb{T}^d, \mathbb{F}) \to H_2(\text{Gr}_k(\mathbb{C}^m), \mathbb{F})$ is non zero, then the quasi-periodic cocycle $A$ has no continuous invariant section $\sigma : \mathbb{T}^d \to \text{Gr}_k(\mathbb{C}^m)$. In particular $A$ is not $k$-dominated.

**Proof.** Let us apply Proposition 3 with $M = \mathbb{T}^d$, $X = \text{Gr}_k(\mathbb{C}^m)$ and dimension $k = 2$. We have $H_0(\text{Gr}_k(\mathbb{C}^m), \mathbb{F}) = 1$ because $\text{Gr}_k(\mathbb{C}^m)$ is a connected manifold. We have $H_2(\text{Gr}_k(\mathbb{C}^m), \mathbb{F}) \geq 1$ and $\dim H_1(\text{Gr}_k(\mathbb{C}^m), \mathbb{F}) = 0$ (see [10, Section 3.2] or [8, Section 5 of Chapter 1]). We also have $H_2(\mathbb{T}^d, \mathbb{F}) \simeq \mathbb{F}(2) \neq \{0\}$ because $d \geq 2$. Therefore assumption 1 and 2 of Proposition 3 hold. On the other hand, our hypothesis implies assumption 3 of that proposition. Therefore, by Proposition 3, the map $\tilde{F} : \mathbb{T}^d \times \text{Gr}_k(\mathbb{C}^m) \to \mathbb{T}^d \times \text{Gr}_k(\mathbb{C}^m)$ does not admit any $\tilde{F}$-invariant section.

Finally, if the quasi-periodic cocycle $A$ is $k$-dominated then the $F$-invariant subbundle $E^+$ determines an $\tilde{F}$-invariant section $E^+ : \mathbb{T}^d \to \text{Gr}_k(\mathbb{C}^m)$. This contradiction proves that $A$ is not $k$-dominated.

**Corollary 1.** Consider a quasi-periodic cocycle $A : \mathbb{T}^2 \to \text{GL}_2(\mathbb{C})$. If the map $A_{\hat{v}} : \mathbb{T}^2 \to \mathbb{P}(\mathbb{C}^2)$, $A_{\hat{v}}(x) = A(x)\hat{v}$, for some $\hat{v} \in \mathbb{P}(\mathbb{C}^2)$, is not homotopic to a constant then $A$ does not have dominated splitting.

**Proof.** The projective space $\mathbb{P}(\mathbb{C}^2)$ can be identified with the Riemann sphere $\mathbb{S}^2 \equiv \mathbb{C} \cup \{\infty\}$. Since $A_{\hat{v}}$ is not homotopic to a constant, by Hopf theorem $\deg(A_{\hat{v}}) \neq 0$. Then, making the canonical identifications $H_2(\mathbb{T}^2, \mathbb{F}) \simeq \mathbb{F}$ and $H_2(\mathbb{P}(\mathbb{C}^2), \mathbb{F}) \simeq \mathbb{F}$, the homology map $(A_{\hat{v}})_* : H_2(\mathbb{T}^2, \mathbb{F}) \to H_2(\mathbb{P}(\mathbb{C}^2), \mathbb{F})$ is the multiplication by $\deg(A_{\hat{v}})$, and hence it is non zero.
Corollary 2. There are analytic functions $A : \mathbb{T}^2 \to \text{GL}_2(\mathbb{C})$ whose homotopy classes contain no quasi-periodic cocycle with dominated splitting.

Proof. Consider any analytic map $f : \mathbb{T}^2 \to \mathbb{R}^3 \setminus \{0\}$. Let $p : \mathbb{R}^3 \setminus \{0\} \to \mathbb{P}(\mathbb{C}^2)$ be the composition of the projection

$$
\mathbb{R}^3 \setminus \{0\} \ni (x, y, z) \mapsto \frac{(x, y, z)}{||(x, y, z)||} \in \mathbb{S}^2
$$

onto the unit sphere $\mathbb{S}^2$ with the stereographic projection, which maps $\mathbb{S}^2$ diffeomorphically onto the projective space $\mathbb{P}(\mathbb{C}^2) = \mathbb{C} \cup \{\infty\}$.

Assume that the parametric hypersurface $f$ has non zero winding number around 0, which implies that the composition $\phi = p \circ f : \mathbb{T}^2 \to \mathbb{P}(\mathbb{C}^2)$ has non zero degree.

Write $\phi = a/b$ as the ratio of two real analytic functions $a, b : \mathbb{T}^2 \to \mathbb{C}$, where $b$ vanishes exactly at the points $x \in \mathbb{T}^2$ where $\phi(x) = \infty$ and the pair $(a(x), b(x)) \neq (0, 0)$ for all $x \in \mathbb{T}^2$. Then the analytic function $A : \mathbb{T}^2 \to \text{GL}_2(\mathbb{C})$

$$
A(x) = \begin{pmatrix} a(x) & -b(x) \\ b(x) & a(x) \end{pmatrix}
$$

satisfies the assumption of Corollary 1 with $\hat{v} = (1, 0)$. Hence it cannot have dominated splitting.

Finally, if $B : \mathbb{T}^2 \to \text{GL}_2(\mathbb{C})$ is another continuous function homotopic to $A$ then the functions $A_\delta : \mathbb{T}^2 \to \mathbb{P}(\mathbb{C}^2), A_\delta(x) = A(x)\hat{v}$, and $B_\delta : \mathbb{T}^2 \to \mathbb{P}(\mathbb{C}^2), B_\delta(x) = B(x)\hat{v}$, are also homotopic. Hence $B_\delta$ is not homotopic to a constant and by Corollary 1 the cocycle $B$ cannot have dominated splitting either. \qed

Theorem 1 follows from the following proposition.

Proposition 5. Given integers $d \geq 2$ and $1 \leq k < m$ there exist analytic quasi-periodic cocycles $A : \mathbb{T}^d \to \text{GL}_m(\mathbb{C})$ with an invariant measurable decomposition $\mathbb{C}^m = E^+ \oplus E^-$ such that

1. $\dim E^+ = k$,
2. all Lyapunov exponents of $\hat{A}_{|E^+}$ are positive,
3. all Lyapunov exponents of $\hat{A}_{|E^-}$ are negative,
4. no continuous cocycle in the homotopy class of $\hat{A}$ admits a continuous invariant section $\sigma : \mathbb{T}^d \to \text{Gr}_k(\mathbb{C}^m)$.

Proof. Consider an analytic quasi-periodic cocycle $A : \mathbb{T}^2 \to \text{SL}_2(\mathbb{C})$ given by Corollary 2. By [1, Theorem 1], possibly perturbing it, we can assume that $A$ admits an invariant measurable decomposition $\mathbb{C}^m = E^+ \oplus E^-$ with $\dim E^- = \dim E^+ = 1$ and having non-zero Lyapunov exponents, w.r.t. an ergodic translation with frequency vector $\omega \in \mathbb{T}^2$.

Take positive numbers $\lambda > 1 > \mu$ such that $\lambda^{k-1} \mu^{m-k-1} = 1$ and let $\tilde{A} : \mathbb{T}^d \to \text{SL}_m(\mathbb{C})$ be the cocycle

$$
\tilde{A}(x) := \begin{pmatrix} \lambda I_{k-1} & 0 & 0 \\ 0 & A(\pi(x)) & 0 \\ 0 & 0 & \mu I_{m-k-1} \end{pmatrix} \in \text{SL}_m(\mathbb{C})
$$

where $I_{k-1}$ and $I_{m-k-1}$ stand for identity matrices of the specified dimensions, and $\pi : \mathbb{T}^d \to \mathbb{T}^2$ denotes the projection $\pi(x_1, \ldots, x_d) := (x_1, x_2)$. By construction the
cocycle $\hat{A}: \mathbb{T}^d \to \text{SL}_m(\mathbb{C})$ satisfies properties 1-3, w.r.t. any ergodic translation with frequency vector $\hat{\omega} \in \mathbb{T}^d$ such that $\pi(\hat{\omega}) = \omega$.

We are going to use Proposition 4 to prove item 4. For each $1 \leq i \leq m$, let $V_i \in \text{Gr}_k(\mathbb{C}^m)$ be the complex $i$-plane generated by the first $i$ vectors $e_1, \ldots, e_i$ of the canonical basis of $\mathbb{C}^m$. We claim that the map $\hat{A}_{V_i}: \mathbb{T}^d \to \text{Gr}_k(\mathbb{C}^m)$, $\hat{A}_{V_i}(x) := \hat{A}(x) V_i$, induces a non zero linear map $\omega \in \pi_2 \mathbb{T}^d$, $\pi_2(\omega) = 1$, which induces a non zero linear map at the second homology level. To relate the homologies of $\hat{A}_{V_i}$ and $A_{\epsilon_i}$ we factor the first, $\hat{A}_{V_i}$, as a composition of several maps which include the second, $A_{\epsilon_1}$.

Let $\Sigma := \{ V \in \text{Gr}_k(\mathbb{C}^m) : V_{k-1} \subset V \subset V_k \}$. This is a complex analytic submanifold of the Grassmannian space $\text{Gr}_k(\mathbb{C}^m)$, which is diffeomorphic to the complex projective line $\mathbb{P}(\mathbb{C}^2)$. The correspondence $(v_1, v_2) \mapsto V_k \supset \mathbb{C}(v_1 e_k + v_2 e_{k+1})$ determines a map $\Phi: \mathbb{P}(\mathbb{C}^2) \to \Sigma$. Then for all $x \in \mathbb{T}^d$,

$$\hat{A}_{V_i}(x) = \hat{A}(x) V_i = \Phi(A_{\epsilon_i}(\pi(x))) = (\iota \circ \Phi \circ A_{\epsilon_i} \circ \pi)(x)$$

where $\iota$ stands for the inclusion map $\iota: \Sigma \to \text{Gr}_k(\mathbb{C}^m)$.

By Künneth theorem, the linear map $\pi_*: H_2(\mathbb{T}^d, \mathbb{F}) \to H_2(\mathbb{T}^2, \mathbb{F})$ is surjective. Because $\Phi$ is a diffeomorphism, the homology map $\Phi_*$ is an isomorphism. We are left to prove that $\iota_*: H_2(\Sigma, \mathbb{F}) \to H_2(\text{Gr}_k(\mathbb{C}^m), \mathbb{F})$ is injective. This will imply that $(\hat{A}_{V_i})_*$ is non zero and, by Proposition 4, that no cocycle homotopic to $\hat{A}$ admits a continuous invariant section with values in $\text{Gr}_k(\mathbb{C}^m)$.

Let us now turn to prove the injectivity of $\iota_*$. The Grassmannian $\text{Gr}_k(\mathbb{C}^m)$ is an analytic manifold of dimension $k(m-k)$. By Schubert Calculus (see [10, Section 3.2] or [8, Section 5 of Chapter 1]), the manifold $\text{Gr}_k(\mathbb{C}^m)$ admits a class of standard cell decompositions, whose cells are referred to as Schubert cells. The closures of these cells are analytic subvarieties known as Schubert cycles. The submanifold $\Sigma$ is itself a Schubert cycle with complex dimension 1 which can be integrated in a cell decomposition

$$\{ V_k \} = \Sigma_0 \subset \Sigma = \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_N = \text{Gr}_k(\mathbb{C}^m).$$

Each space $\Sigma_i$ is an analytic subvariety obtained from $\Sigma_{i-1}$ by joining a cell with (real) even dimension and boundary contained in $\Sigma_{i-1}$.

This implies that $H_3(\Sigma_i, \Sigma_{i-1}, \mathbb{F}) = 0$ for all $i$. Hence, by the long exact sequence of the pair $(\Sigma_i, \Sigma_{i-1})$,

$$0 = H_3(\Sigma_i, \Sigma_{i-1}, \mathbb{F}) \to H_2(\Sigma_{i-1}, \mathbb{F}) \to H_2(\Sigma_i, \mathbb{F}) \to \cdots$$

is an exact sequence. Therefore, because $\iota$ can be factored as the composition of the inclusions $\Sigma_{i-1} \hookrightarrow \Sigma_i$ with $i = 2, \ldots, N$, the map $\iota$ is injective at the second homology level.

**Remark 3.** Given a cocycle $A \in C^\infty(\mathbb{T}^d, \text{GL}_m(\mathbb{R}))$ in one of the homotopy classes from Proposition 5, the cocycle $A_y: \mathbb{T}^d \to \text{GL}_m(\mathbb{C})$, $A_y(x) = A(x + iy)$, cannot have dominated splitting for any $y \in \mathbb{R}^d$.

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