Fast-Convergent Dynamics for Distributed Allocation of Resources Over Switching Sparse Networks with Quantized Communication Links

Mohammadreza Doostmohammadian, Alireza Aghasi, Mohammad Pirani, Ehsan Nekouei, Usman A. Khan, Senior Member, IEEE, Themistoklis Charalambous, Senior Member, IEEE

Abstract—This paper proposes networked dynamics to solve resource allocation problems over time-varying multi-agent networks. The state of each agent represents the amount of used resources (or produced utilities) while the total amount of resources is fixed. The idea is to optimally allocate the resources among the group of agents by minimizing the overall cost function subject to fixed sum of resources. Each agents’ information is restricted to its own state and cost function and those of its immediate in-neighbors. This is motivated by distributed applications such as mobile edge-computing, economic dispatch over smart grids, and multi-agent coverage control. This work provides a fast convergent solution (in comparison with linear dynamics), while considering relaxed network connectivity with quantized communication links. The proposed dynamics reaches optimal solution over switching (possibly disconnected) undirected networks as far as their union over some bounded non-overlapping time-intervals has a spanning-tree. We prove feasibility of the solution, uniqueness of the optimal state, and convergence to the optimal value under the proposed dynamics, where the analysis is applicable to similar 1st-order allocation dynamics with strongly sign-preserving nonlinearities, such as actuator saturation.

Index Terms—Distributed optimization, resource allocation, consensus, logarithmic quantization, spanning tree

I. INTRODUCTION

Distributed optimization has been the topic of interest in the recent machine learning, signal processing, and control literature. The literature is focused on solving the following generic optimization problem to minimize a global function which is the sum of local objective functions [1]:

$$\min_{\mathbf{x}} F(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \quad \text{subject to } C(\mathbf{x}) = 0 \quad (1)$$

The centralized solutions of (1) work under the premise that all information is available and processed at a central entity or computing node. However, in large-scale networked systems, every node/agent only has access to local information in its neighborhood. Hence, a distributed algorithm over a multi-agent system is needed, where agents cooperatively perform local computations based on their local information. Such distributed optimization solutions finds applications in multi-sensor target localization/tracking [2], edge-computing and load balancing [3], power allocation in cellular networks [4], and distributed support vector machine [5], [6]. Different constraints and solutions are considered for problem (1): unconstrained [7], [8], [9] (assuming strongly-convex/smooth objectives), inequality constraint [10], and consensus-constraint [5], [6], [11] (as $x_1 = \cdots = x_n$) aiming not only to optimize the objective cooperatively but also to drive the agents to reach consensus. In distributed resource allocation [12], [13], [14], [15], [16], [17], [18] (also referred as network resource allocation) the optimization problem is constrained with fixed summation of states, aiming to allocate a fixed amount of resources over a large-scale multi-agent network. This finds applications in economic dispatch over power networks [19], [20], [21], [22], [23], networked coverage control [24], [25], and edge-computation offloading [26], [27]. Implementing parallel dynamics at agents based on local information requires distributed algorithms to solve the problem. Some of the proposed distributed solutions include: preliminary linear solution [12], quantized solution via event-triggered communications (fixed network) [28], accelerated linear solution via adding a momentum term (heavy-ball method) [15], low communication rate protocol converging in quadratic time (via a long-term connectivity condition) [13], game-theoretic approach [18], [29], initialization-free [16], Lagragian-based solution [1] under unknown and time-varying number of resources [14] which requires one extra step of local optimization (and more computation load) at agents between every two consecutive steps of global dynamics.

Contributions: This work proposes a nonlinear dynamics for network resource allocation problem. The main purpose for considering the nonlinearity are: finite/fixed-time convergence [30], [31], quantized information [32], [33], [34], [35], and saturation/clipping [36], [33], among others. Knowing that fixed-time dynamics reaches faster convergence than linear solutions [31], [5], we propose a continuous-time state-update for distributed resource allocation with fast convergence (as compared to linear solution) while considering logarithmic-quantized information exchange among agents. We consider uniform connectivity [9], [8] for the undirected
network of agents (instead of all-time connectivity). Such sparse-connectivity only requires the union of the networks over some bounded non-overlapping time-intervals to include a spanning-tree, where, in contrast to unconstrained or consensus-constrained works [7], [8], [9], this work extends the solution to distributed resource allocation over sparsely-connected dynamic networks with quantized communications. Borrowing some ideas from convex optimization and level-set methods [37], we prove uniqueness and feasibility of optimal state under the proposed continuous-time dynamics. The convergence to this optimal value is proved via Lyapunov-type stability analysis. The convergence analysis in this paper, although used for a specific dynamics, can be easily extended to strongly sign-preserving nonlinear 1st-order continuous-time dynamics. We summarize our main contributions as follows: (i) fast-convergence (compared to linear solutions) while considering quantized data-transmissions, (ii) proving feasibility/uniqueness, and (iii) convergence to the optimal value for general strongly sign-preserving nonlinearities (e.g., fixed-time convergence, quantization, and saturation) in the continuous-time dynamics over general uniformly-connected dynamic networks.

**Paper organization:** Section II states the problem, related definitions, and introductory lemmas. Section III describes the proposed dynamics, while Section IV proves its convergence to unique optimal value. Section V provides some simulations for comparison and verifies the stability over weakly connected networks. Section VI concludes the paper.

## II. PROBLEM FORMULATION

### A. Problem Statement

Distributed resource allocation problem is formulated as,

$$\min_x F(x) = \sum_{i=1}^{n} f_i(x_i) \text{ s.t. } \sum_{i=1}^{n} x_i = K \quad (2)$$

where $x_i \in \mathbb{R}$ is the amount of resources allocated to agent $i$, $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function known by agent $i$ representing the cost as a function of resources $x_i$. This problem, also referred to as network resource allocation, aims to allocate a fixed quantity of total resources, $\sum_{i=1}^{n} x_i = K$, among a group of agents communicating over an undirected graph $G$ such that the total cost $F(x)$ is minimized. The problem is sometimes described in terms of utility functions, where the goal is to maximize the summation of utility functions. The problem might be subject to box constraints in the form $\underline{x}_i \leq x_i \leq \overline{x}_i$, for example, when the amount of resources needs to be positive and bounded for specific applications. One can eliminate these by modifying the objective function as $f'_i(x_i) = f_i(x_i) + \epsilon|x_i - \overline{x}_i|^+ + \epsilon|\underline{x}_i - x_i|^+$, with $[u]^+ = \max\{u, 0\}$. This is known as the exact penalty method described in [37]. Recall that the summation of the strictly convex $f_i(\cdot)$ and convex penalty $[\cdot]^+$ is a strictly convex function. Further, the non-smooth $[u]^+$ can be replaced by its smooth equivalent $\frac{1}{2\mu}\log(1 + \exp(\mu u))$ as in [6] or quadratic penalty $([u]^+)^2$ [38]. Applications of this setup include:

1. Economic dispatch [19], [20], [21], [23]: to allocate the electricity generation by facilities to minimize the cost while meeting the required load/demand constraints.
2. Congestion-control and load-balancing [26], [27]: to modulate traffics and data routing in telecommunication networks to gain fair allocations among the users.
3. Coverage control [24], [25]: the objective is to optimally allocate a group of networked robots/agents over a convex area in order to achieve maximum coverage.

**Remark 1:** Note the difference of (constrained) problem (2) with general (unconstrained) distributed optimization (as in [7], [8], [9]). Other than the constraint, for general distributed optimization the cost at all agents is the same (as in [7], [8], [9]). Other than the constraint, for general distributed optimization the cost at all agents is the same (as in [7], [8], [9]). Other than the constraint, for general distributed optimization the cost at all agents is the same (as in [7], [8], [9]). Other than the constraint, for general distributed optimization the cost at all agents is the same (as in [7], [8], [9]).

**Remark 2:** Note that, in case of synchronization and consensus, the 1st-order dynamics as compared to the 2nd-order dynamics (of the same linear/nonlinear type) are known to have faster convergence; see [40] page 32.

Table I compares different solutions under different constraints in the literature. Recall that the 1st-order dynamics refers to the consensus-type protocols in the form, $\dot{x}_i = \sum_{j \in N_i} f(x_j - x_i)$, while 2nd-order dynamics are in the following form, $\dot{x}_i = \sum_{j \in N_i} h(x_j - x_i)$. The convergence analysis in this paper, although used for a specific dynamics, can be easily extended to strongly sign-preserving nonlinearities (e.g., fixed-time convergence, quantization, and saturation) in the continuous-time dynamics over general uniformly-connected dynamic networks.

| Reference | Solution | Constraint $\mathcal{C}(x)$ |
|-----------|----------|-----------------------------|
| [13], [12], [15], [19] | 1st-order | $\sum_{i=1}^{n} x_i = K$ |
| [16] | 2nd-order | $\sum_{i=1}^{n} x_i = K$ |
| [7], [8], [9] | 2nd-order | $-\mathcal{C}$ |
| [10] | 2nd-order | $\sum_{i=1}^{n} g_i(x_i) \leq 0$ |
| [11], [5] | 2nd-order | $x_1 = \cdots = x_n$ |
| [39] | 1st-order | $x_1 = \cdots = x_n$ |

### B. Preliminary Definitions and Lemmas

The communication network of agents is modeled as a sequence of (possibly) time-varying undirected graphs, denoted by $G(t) = (\mathcal{V}, \mathcal{E}(t))$ with $\mathcal{V} = \{1, \ldots, n\}$. Two agents $i$ and $j$ can communicate/exchange messages at time $t$ if and only if $(i, j), (j, i) \in \mathcal{E}(t)$. Define $N_i(t) = \{j \mid (j, i) \in \mathcal{E}(t)\}$ as in-neighbors of agent $i$ at time $t$. Define $n$ by $n$ matrix $W(t)$ as the weight matrix of the graph $G(t)$, with $W_{ij} > 0$ as the weight of link $(i, j) \in \mathcal{E}(t)$ and $W_{ij} = 0$ if $(i, j) \notin \mathcal{E}(t)$.

**Definition 1:** In the undirected graph $G = (\mathcal{V}, \mathcal{E})$ define a tree as an undirected subgraph in which any two vertices in $\mathcal{V}$ are connected by exactly one path. Further, define a spanning tree as a tree which includes all of the vertices in $\mathcal{V}$ with minimum possible number of links in $\mathcal{E}$.

**Assumption 1:** There exists a sequence of non-overlapping bounded time-intervals, $[t_k, t_k + l_k]$ with $l_k \ll \infty$, where the combination of the multi-agent network across each interval $\bigcup_{t_k}^{t_k + l_k} G(t)$ has a spanning tree.
Note that the above connectivity assumption implies that there exists a path from any node $i$ to any node $j$ infinitely often. This is a weak connectivity requirement on the multi-agent network. A situation where Assumption [1] does not hold is when there are at least two nodes $i$ and $j$ in the network between which there is no connected path over time.

**Definition 2**: (37) A function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if $\nabla^2 h(x) > 0$ for $x \in \mathbb{R}^n$.

**Assumption 2**: The functions $f_i(x_i)$, $i = 1, \ldots, n$ in problem (2) are strictly convex and differentiable.

**Lemma 1** ([12], [15]): Under the Assumption 2, the resource allocation problem (2) has a unique optimal solution, say $x^*$, under the given constraint. At this touching point, say $\gamma$, due to strict convexity, only one of these level sets, say $L_\gamma$, is strictly convex if

$$F(x) = \nabla F(x) = \psi^T \mathbf{1}_n$$

where $\psi^T$ is the optimal Lagrange multiplier with the vector $\mathbf{1}_n$ as the gradient of the constraint [37].

**Definition 3**: Define the feasible set of states as $S_K = \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i = K \}$.

**Definition 4** ([37]): Given a function $g(x) : \mathbb{R}^n \to \mathbb{R}$, define the level set $L_\gamma(g)$ for a given $\gamma \in \mathbb{R}$ as the set

$$L_\gamma(g) = \{x \in \mathbb{R}^n | g(x) \leq \gamma \}.$$  

For strictly convex $g(x)$, $\mathbb{R}^n \to \mathbb{R}$, its level sets $L_\gamma(g)$ are closed and strictly convex. Assuming that problem (2) has a solution, the following lemma holds.

**Lemma 2**: Under Assumption 2 there is a unique point $x^*$ such that $\nabla F(x^*) = \alpha \mathbf{1}_n$, for every feasible set $S_K$. In other words, for every feasible set $S_K$ there is only one point $x^* \in S_K$ for which $\frac{\partial F}{\partial x_i}(x^*) = \frac{\partial F}{\partial x_j}(x^*)$, $\forall i, j \in \{1, \ldots, n\}$.

**Proof**: This lemma is a result of strict convexity of the function $F(x)$ and all its level sets (Assumption 2). Due to strict convexity, only one of these level sets, say $L_\gamma$, becomes adjacent to the constraint facet associated with $S_K$, and due to the strict convexity of the level set, the touching only happens at one point (this point in fact is the optimum point under the given constraint). At this touching point, say $x^*$, the gradient $\nabla F(x^*)$ is orthogonal to the constraint set $S_K$, where $\frac{\partial F}{\partial x_i}(x^*) = \frac{\partial F}{\partial x_j}(x^*)$ for all $i, j$. By contradiction, suppose there are two points $x^1$ and $x^2$ in $S_K$ for which $\nabla F(x^1) = \alpha \mathbf{1}_n$ and $\nabla F(x^2) = \alpha \mathbf{1}_n$, implying two points for which the level set $L_\gamma$, $\gamma = F(x^1) = F(x^2)$ is tangent to the affine constraint set $S_K$ or the two points are on two level sets $L_{\gamma_1}$, $\gamma_1 = F(x^1)$ and $L_{\gamma_2}$, $\gamma_2 = F(x^2)$ adjacent to the same affine constraint set $S_K$. Both cases contradict the strict convexity and closedness of the level sets. This proves the result by contradiction.

**Lemma 3**: Let $g_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$ be an odd mapping, i.e., $g_i(x) = -g_i(-x)$, $W \in \mathbb{R}^{n \times n}$ be a symmetric matrix, and $\varphi \in \mathbb{R}^n$. Then,

$$\sum_{i=1}^n \sum_{j=1}^n W_{ij} g_2(g_1(\varphi_j) - g_1(\varphi_i)) = -\frac{1}{2} \sum_{i,j=1}^n W_{ij} (\varphi_j - \varphi_i) g_2(g_1(\varphi_i) - g_1(\varphi_i)).$$  

**Proof**: For every $i, j$, following the symmetry of $W$ and oddness of $g_2(\cdot)$ we have,

$$\varphi_i W_{ij} g_2(g_1(\varphi_j) - g_1(\varphi_i)) + \varphi_j W_{ij} g_2(g_1(\varphi_i) - g_1(\varphi_j)) = W_{ij}(\varphi_j - \varphi_i) g_2(g_1(\varphi_j) - g_1(\varphi_i))$$

$$= W_{ij}(\varphi_j - \varphi_i) g_2(g_1(\varphi_i) - g_1(\varphi_j)).$$

and the proof follows.

**III. THE PROPOSED SOLUTION**

Typically linear dynamics is used to solve the problem (2) over undirected graphs as in [21],

$$\dot{x}_i = -\eta_1 \sum_{j \in N_i} W_{ij} \left( \frac{df_i}{dx_i} - \frac{df_j}{dx_j} \right),$$

where $\eta_1 > 0$ and symmetric stochastic weight matrix $W$ with $W_{ij} \geq 0$. By ideas from finite-time consensus [36], [41], [42], to reach faster convergence in the region $|\frac{df_i}{dx_i} - \frac{df_j}{dx_j}| < 1$, the linear dynamics (6) can be modified as follows:

$$\dot{x}_i = -\eta_1 \sum_{j \in N_i} W_{ij} \text{sgn}^v \left( \frac{df_i}{dx_i} - \frac{df_j}{dx_j} \right),$$

where $0 < v < 1$, and the function $\text{sgn}^v(x) : \mathbb{R} \to \mathbb{R}$ is,

$$\text{sgn}^v(x) = x|x|^{v-1}.$$  

The function $\text{sgn}^v(x)$ is non-Lipschitz at $x = 0$ (for $0 < v < 1$). Similar non-Lipschitz dynamics in the form $\dot{x}_i = -\eta_1 \sum_{j \in N_i} W_{ij} \text{sgn}^v(x_i - x_j)$ is proved to reach finite-time convergence [36], [43], [42] and faster than linear consensus dynamics in regions close to the consensus equilibrium (because $|\text{sgn}^v(x)| > |x|$ for $|x| < 1$). Further, due to non-Lipschitz condition on the equilibrium, consensus is reached in finite-time. However, such consensus protocols show slow convergence in regions far from the equilibrium (because $|\text{sgn}^v(x)| < |x|$ for $|x| > 1$). To overcome this, in fixed-time consensus protocols [43], typically a second term is added as $\text{sgn}^v(x)$ with $v_2 > 1$. Having $|\text{sgn}^v(x)| > |x|$ for $|x| > 1$ implies faster convergence rate than the linear case for states far from the consensus equilibrium. Therefore, the combination of the two dynamics results in faster convergence, where the convergence rate is tunable via the parameters $v_1$, $v_2$. Borrowing these ideas, this work proposes the following non-linear 1st-order dynamics to solve problem (2).

$$\dot{x}_i = -\sum_{j \in N_i} W_{ij} \left( \eta_1 \text{sgn}^{v_1} \left( \frac{df_i}{dx_i} - \frac{df_j}{dx_j} \right) + \eta_2 \text{sgn}^{v_2} \left( \frac{df_i}{dx_i} - \frac{df_j}{dx_j} \right) \right),$$

where $0 < v_1 < 1 < v_2$, $0 < \eta_2$, $\eta_1$, and $W_{ij} = W_{ji} \geq 0$. Unlike discrete-time protocols [12], [15] we do not assume $W$ to be (doubly) stochastic. Notice that, for such $v_1$, $v_2$ values, $|\text{sgn}^{v_1}(x)| + |\text{sgn}^{v_1}(x)| > |x|$; therefore, for any $|\frac{df_i}{dx_i} - \frac{df_j}{dx_j}|$, $|\dot{x}_i|$ is greater under dynamics (9) (as compared to the linear solution). Further, the convergence rate can be tuned via $v_1$, $v_2$ to be faster than similar Lipschitz/non-Lipschitz protocols.
Although $F$ is non-Lipschitz, the uniqueness of solutions to the ODE can be established as in [5].

Next, we take into account the quantization over data-transmission links by modifying the dynamics as follows,

$$
\dot{x}_i = - \sum_{j \in \mathcal{N}_i} W_{ij} \left( \eta_1 \text{sgn}^{v_1}(q \left( \frac{df_i}{dx_i} \right) - q \left( \frac{df_j}{dx_j} \right)) + \eta_2 \text{sgn}^{v_2}(q \left( \frac{df_i}{dx_i} \right) - q \left( \frac{df_j}{dx_j} \right)) \right),
$$

(10)

where function $q(z)$ represents the data-quantization (of the shared information $\frac{df_i}{dx_i}$) in the following model [33], [34],

$$
q(z) = \text{sgn}(z) \exp(q_w(\log(|z|))),
$$

(11)

where $q_w(z) = \delta \left[ \frac{z}{\delta} \right]$ is the uniform quantizer with $\delta$ as rounding to the nearest integer. The strongly sign-preserving odd function $q(\cdot)$ represents logarithmic data-quantization with level $\delta$.

For notation simplicity from this point in the paper onward we consider $\psi_i = \frac{df_i}{dx_i}$. In (10), every agent knows its own state and objective function along with information of its neighbors over $G$. At time $t$, agent $i$ shares $\psi_i = \frac{df_i}{dx_i}$ with its neighbors in $\mathcal{N}_i(t)$. The exchange of information is local and quantized, where the neighboring agent receives $q(\frac{df_i}{dx_i})$ and no agent knows the global objective and states of the other agents. In this manner, the dynamics (10) is fully distributed over the multi-agent network.

**Lemma 4:** Consider a feasible value for $x(0)$, i.e., $x(0) \in S_K$. Then, for any symmetric matrix $W$, $x(t)$ maintains its feasibility (sum-preserving) under dynamics (10) for $t > 0$.

**Proof:** $x(0) \in S_K$ implies that $\sum_{i=1}^{n} x_i(0) = K$. Then,

$$
\begin{align*}
\sum_{i=1}^{n} \dot{x}_i &= - \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} W_{ij} \left( \eta_1 \text{sgn}^{v_1}(q(\psi_i) - q(\psi_j)) + \eta_2 \text{sgn}^{v_2}(q(\psi_i) - q(\psi_j)) \right) \\
&= - \sum_{i,j=1}^{n} W_{ij} \left( \eta_1 \text{sgn}(q(\psi_i) - q(\psi_j)) + \eta_2 \text{sgn}(q(\psi_i) - q(\psi_j)) \right).
\end{align*}
$$

(12)

Recall that $\text{sgn}^{v}(q(\psi_j) - q(\psi_i)) = -\text{sgn}(q(\psi_i) - q(\psi_j))$ and $W_{ij} = W_{ji}$ (W is symmetric), therefore $\frac{d}{dt} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \dot{x}_i = 0$, implying that $\sum_{i=1}^{n} x_i(t) = \sum_{i=1}^{n} x_i(0) = K$ remains constant and feasible over time. This completes the proof. ■

This lemma implies that initializing from a feasible value, the solution under dynamics (10) remains feasible. Assuming the total resources $K$ and number of agents $n$ are known by the agents, a feasible initialization is simply $x_i(0) = \frac{K}{n}$. However, in many problems the feasibility is initially satisfied; for example, in economic dispatch the sum of initial energy production equals the demands and only needs to be optimized. Similarly, in coverage control the agents’ coverage initially includes the entire convex area and the algorithm needs to optimize the coverage [24]. Note that, unlike [11], we do not necessarily constrain the initialization such that $\sum_{i=1}^{n} \frac{df_i}{dx_i}(0) = 0$.

**IV. CONVERGENCE ANALYSIS**

We, first, characterize the unique equilibrium of the dynamics (10) and then, prove its stability.

**Theorem 1:** If Assumption 1 holds, the equilibrium point $x^*$ with $\nabla F(x^*) = \psi^* 1$ is the only equilibrium of (10).

**Proof:** For $x^*$ with $\nabla F(x^*) = \psi^* 1$, we have $x_i^* = 0$ which implies that $x^*$ is the equilibrium of (10). By contradiction suppose there exists another equilibrium $\hat{x}$ under dynamics (10) (i.e. $\hat{x} = 0$) where $\frac{df_i}{dx_i} \neq \frac{df_j}{dx_j}$ for at least two nodes $i,j$. Let $\nabla F(\hat{x}) = (\hat{\psi}_1, \ldots, \hat{\psi}_n)^T$ and consider two nodes, $a = \arg\max_{\lambda \in \{1, \ldots, n\}} \hat{\psi}_\lambda$, $b = \arg\min_{\lambda \in \{1, \ldots, n\}} \hat{\psi}_\lambda$. Clearly, $\hat{\psi}_a > \hat{\psi}_b$. From Assumption 1 and Definition 1 there is at least one path between nodes $a$ and $b$ in the graph $\bigcup_{k=t_k}^{t_{k+1}} G(t)$, including at least two nodes $\alpha$, $\beta$ such that $\hat{\psi}_a \geq \hat{\psi}_{\mathcal{N}_a}$ and $\hat{\psi}_b \leq \hat{\psi}_{\mathcal{N}_b}$ with strict inequality for at least one agent in $\mathcal{N}_a$ and $\mathcal{N}_b$. Recall that both fixed-time and quantization are sign-preserving odd functions. Thus, in a sub-domain of $[t_k, t_{k+1}]$, $\sum_{j \in \mathcal{N}_i} \text{sgn}(q(\psi_a) - q(\psi_j)) > 0$ and $\sum_{j \in \mathcal{N}_i} \text{sgn}(q(\hat{\psi}_b) - q(\psi_j)) < 0$ from dynamics (10). Therefore, $\dot{x}_a < 0$ and $\dot{x}_b > 0$ contradicting the equilibrium condition for $\hat{x}$, which proves the theorem.

To illustrate more, assume that Assumption 1 does not hold; for example, the network has two separate connected components with nodes $1, \ldots, m$ in component $G_1$ and nodes $m+1, \ldots, n$ in component $G_2$. Then, for $x^*$ as the equilibrium of (10), $\frac{df_i}{dx_i}(x^*) = \frac{df_j}{dx_j}(x^*) = \psi^* 1$ for $i,j \in \{1, \ldots, m\}$ and $\frac{df_i}{dx_i}(x^*) = \frac{df_j}{dx_j}(x^*) = \psi^* 2$ for $i,j \in \{m+1, \ldots, n\}$. Thus, such cases also prove the theorem.

**Remark 3:** In the virtue of Lemma 4, the solution $x(t)$ remains feasible under dynamics (10) i.e., $x(t) \in S_K$ for $t > 0$. Then, following Lemma 2 for initial value $x(0) \in S_K$ there is only one equilibrium $x^*$ satisfying $\nabla F(x^*) = \psi^* 1$.

**Theorem 2:** Under Assumptions 1 and 2 and starting with a feasible value $x(0) \in S_K$, the distributed dynamics (10) solves the resource allocation problem (2).

**Proof:** Following Lemma 1 let $x^*$ denote the optimal solution of problem (2) for which $\nabla F(x^*) = \psi^* 1$. We need to prove that dynamics (10) yields to the $x^*$ as its equilibrium. Recall that from Lemma 2 $x(0) \in S_K$ implies that the solution under (10) remains feasible, and therefore, $\sum_{i=1}^{n} x_i^* = K$. Let $F^* = F(x^*)$ and $\mathcal{F}(x) = F(x) - F^*$. Note that $\mathcal{F}(x) \geq 0$ can be considered as a Lyapunov function where $\mathcal{F}(x^*) = 0$. Based on Lyapunov theorem we need to show that $\dot{\mathcal{F}}(x) \leq 0$ where $\dot{\mathcal{F}}(x^*) = 0$. We have,

$$
\dot{\mathcal{F}}(x) = \nabla \mathcal{F}(x) \dot{x} = \sum_{i=1}^{n} \dot{x}_i \dot{x}_i = \sum_{i=1}^{n} \dot{x}_i \left( - \sum_{j \in \mathcal{N}_i} W_{ij} \left( \eta_1 \text{sgn}^{v_1}(q(\psi_i) - q(\psi_j)) + \eta_2 \text{sgn}^{v_2}(q(\psi_i) - q(\psi_j)) \right) \right).
$$

(13)
Then, in Lemma 3 set $g_2(\cdot)$ as $\text{sgn}(\cdot)$ and $g_1(\cdot)$ as $q(\cdot)$, 
\[
\mathcal{F}(x) = -\frac{\eta_1}{2} \sum_{i,j=1}^{n} W_{ij} |q(\psi_i) - q(\psi_j)|^{\nu_i+1} 
- \frac{\eta_2}{2} \sum_{i,j=1}^{n} W_{ij} |q(\psi_i) - q(\psi_j)|^{\nu_j+1}. \tag{13}
\]
Therefore, $\mathcal{F}(x) \leq 0$ where the accumulation set $\mathcal{I}$ includes the state values (a unique point by Theorem 1 satisfying, 
\[
\mathcal{F}(x) = 0 \iff \psi_i = \psi_j \text{ for } i, j \in \{1, \ldots, n\} \tag{14}
\]
Following Lemma 2, Remark 3 and Theorem 1 the unique point $x^*$ satisfying (14), is the unique equilibrium of dynamics (10), and, based on Lemma 4 it is the optimal solution to the problem (2). Then, following Lasalle’s invariance principle, (10) converges to $\mathcal{I} = \{x^*\}$.

The connectivity requirement in Assumption 1 gives the unique optimal state $x^*$ (with $\nabla \mathcal{F}(x^*) = \psi^* \mathbf{1}$) in Theorem 1, while Theorem 2 proves convergence to this point.

Remark 4: Eq. (13) represent the convergence rate; where,
\[
|q(\psi_i) - q(\psi_j)|^{\nu_i+1} + |q(\psi_i) - q(\psi_j)|^{\nu_j+1} 
\geq |q(\psi_i) - q(\psi_j)|^2 \tag{15}
\]
Since the RHS of the above represents the convergence rate of the linear (and linear quantized) protocols [28], [12], clearly the dynamics (10) is faster than its linear counterparts.

Remark 5: The results of this paper can be extended to take saturation effects into account [36], [33]. For example, in case of actuator saturation one may substitute $\text{sat}_\kappa(\cdot)$ instead of $\text{sgn}(\cdot)$ in dynamics (10), where $\text{sat}_\kappa(x) = x$ for $-\kappa \leq x \leq \kappa$ and $\kappa \text{sgn}(x)$ otherwise. Recall that in the proofs of the given theorems and lemmas only non-zero derivative at zero, oddness, and sign-preserving property of $\text{sgn}(\cdot)$ are used, which are also true for $\text{sat}_\kappa(\cdot)$ function. Therefore, the results on the uniqueness, feasibility, and convergence can be stated for general strongly sign-preserving nonlinearities on the agents’ dynamics and their communications.

V. NUMERICAL SIMULATIONS

For the simulations, smooth penalty $(|u|^2)^2$ [38] (with $\epsilon = 1$) for the box constraints is used to satisfy Assumption 2.

A. A Comparison Study

Consider the strictly-convex quadratic cost as [13],
\[
f_i(x_i) = b_i (x_i - a_i)^4, \tag{16}
\]
with random coefficients $b_i \in (0, 4]$, $a_i \in [-2, 4]$, and box constraints $0 \leq x_i \leq 5$. The random initial states satisfy $\sum_{i=1}^{n} x_i(0) = K = 20$ (as in Lemma 4). The multi-agent network is a cycle of $n = 10$ nodes with random stochastic link weights (this is required by [12], [15] and only for the sake of comparison). In Fig. 1 the convergence of the dynamics (10) is compared with linear [12], accelerated linear ($\beta = 0.6$) [15], quantized linear (with all-time triggered communications) [28], finite-time [19], and fixed-time [22] protocols. (with sampling time-step $\eta = 5 \times 10^{-5}$ and $v_1 = 0.1$, $v_2 = 1.6$ in dynamics (10)).

B. Simulation over Weakly Connected Sparse Networks

We evaluate the performance of the proposed solution (10) over sparse networks of $n = 100$ agents. The network switches periodically over Scale-Free networks $G_1$, $G_2$, $G_3$, and $G_4$ every 0.05 sec, where none includes a spanning tree while their union $\bigcup_{t=0}^{\infty} 10^{-2} G_r(t)$ is connected. Therefore, in Assumption $\eta_k = 0.2$ (100 time-steps with $\eta = 2 \times 10^{-3}$). The link weights are random and non-stochastic. Similar to [12], consider logarithmic strictly-convex objective function,
\[
f_i(x_i) = \frac{1}{2} a_i (x_i - c_i)^2 + \log(1 + \exp(b_i(x_i - d_i))), \tag{17}
\]
with random coefficients $a_i \in [0, 0.1]$, $b_i \in [-0.01, 0.01]$, $c_i, d_i \in [-0.5, 0.5]$, and box constraint $3 \leq x_i \leq 7$. The time-evolution of the states and absolute residual cost $\mathcal{F}(x)$ is shown in Fig. 2 with random initialization (satisfying the box constraint) $\sum_{i=1}^{100} x_i(0) = K = 500$, $\delta = 3 \times 10^{-1}$, and $v_1 = 0.3$, $v_2 = 1.6$. The residual is decreasing while, due to quantization, there remains certain bias in the steady-state.

VI. CONCLUSION

This work provides a distributed nonlinear 1st-order solution for resource allocation over dynamic undirected networks subject to quantized data transmission. The convergence is proved over sparse networks (i.e., uniform connec-
tivity over time), which allows convergence even over disconnected networks while certain weak uniform connectivity (over time) is met. The explicit discretized version of the protocol (10) (e.g., via Euler Forward method) can be used in real implementations (as shown in Section V), however, with certain lower-bound on the sampling rate.

REFERENCES

[1] M. Hong, M. Razaviyayn, Z. Luo, and J. Pang, “A unified algorithmic framework for block-structured optimization involving big data: With applications in machine learning and signal processing,” IEEE Signal Processing Magazine, vol. 33, no. 1, pp. 57–77, 2015.

[2] M. Doostmohammadian and U. A. Khan, “Topology design in network estimation: a generic approach,” in American Control Conference, Washington, DC, Jun. 2013, pp. 4140–4145.

[3] D. Dechoumiotis, N. Athanasopoulos, A. Leivadeas, N. Mitton, R. M. Jungers, and S. Papavassiliou, “Edge computing resource allocation for dynamic networks: The druid-net vision and perspective,” Sensors, vol. 20, no. 8, pp. 2191, 2020.

[4] N. Forouzan, A. M. Rabiei, M. Vehkaperä, and R. Wichman, “A distributed resource allocation scheme for self-backhauled full-duplex small cell networks,” IEEE Transactions on Vehicular Technology, vol. 70, no. 2, pp. 1461–1473, 2021.

[5] K. Garg, M. Baranwal, and D. Paccagnan, “A fixed-time convergent distributed algorithm for strongly convex functions in a time-varying network,” in IEEE Conf. on Dec. and Cont., 2020, pp. 4405–4410.

[6] M. Doostmohammadian, A. Aghasi, T. Charalambous, and U. A. Khan, “Distributed support vector machines over dynamic balanced directed networks,” IEEE Control Systems Letters, vol. 6, pp. 758–763, 2021.

[7] C. Xi, Y. S. Mai, R. Xin, E. H. Abed, and U. A. Khan, “Linear convergence in optimization over directed graphs with row-stochastic matrices,” IEEE Transactions on Automatic Control, vol. 63, no. 10, pp. 3558–3565, 2018.

[8] F. Saadati, R. Xin, and U. A. Khan, “Decentralized optimization over time-varying directed graphs with row and column-stochastic matrices,” IEEE Trans. on Automatic Control, pp. 4769–4780, 2020.

[9] A. Nedić and A. Olshevsky, “Distributed optimization over time-varying directed graphs,” IEEE Transactions on Automatic Control, vol. 60, no. 3, pp. 601–615, 2014.

[10] T. Chang, A. Nedić, and A. Scaglione, “Distributed constrained optimization by consensus-based primal-dual perturbation method,” IEEE Transactions on Automatic Control, vol. 59, no. 6, pp. 1524–1538, 2014.

[11] Z. Feng and G. Hu, “Finite-time distributed optimization with quadratic objective functions under uncertain information,” in IEEE 54th Annual Conference on Decision and Control (CDC), IEEE, 2015, pp. 208–213.

[12] G. Xiao and S. Boyd, “Optimal scaling of a gradient method for distributed resource allocation,” Journal of optimization theory and applications, vol. 129, no. 3, pp. 469–488, 2006.

[13] T. T. Doan and A. Olshevsky, “Distributed resource allocation on dynamic networks in quadratic time,” Systems & Control Letters, vol. 99, pp. 57–63, 2017.

[14] T. T. Doan and C. L. Beck, “Distributed resource allocation over dynamic networks with uncertainty,” IEEE Transactions on Automatic Control, 2020.

[15] E. Ghadimi, M. Johansson, and I. Shames, “Accelerated gradient methods for networked optimization,” in American Control Conference. IEEE, 2011, pp. 1668–1673.

[16] P. Yi, Y. Hong, and F. Liu, “Initialization-free distributed algorithms for optimal resource allocation with feasibility constraints and application to economic dispatch of power systems,” Automatica, vol. 74, pp. 259–269, 2016.

[17] W. Lin, Y. Wang, C. Li, and X. Yu, “Predefined-time optimization for distributed resource allocation,” Journal of the Franklin Institute, vol. 357, no. 16, pp. 11323–11348, 2020.

[18] D. Paccagnan and J. R. Marden, “Utility design for distributed resource allocation – part ii: Applications to submodular, covering, and supermodular problems,” IEEE Trans. on Automatic Control, 2021.

[19] G. Chen, J. Ren, and E. N. Feng, “Distributed finite-time economic dispatch of a network of energy resources,” IEEE Transactions on Smart Grid, vol. 8, no. 2, pp. 822–832, 2016.

[20] C. Li, X. Yu, T. Huang, and X. He, “Distributed optimal consensus over resource allocation network and its application to dynamical economic dispatch,” IEEE transactions on neural networks and learning systems, vol. 29, no. 6, pp. 2407–2418, 2017.

[21] A. Cherukuri and J. Cortés, “Distributed generator coordination for initialization and anytime optimization in economic dispatch,” IEEE Trans. on Control of Network Systems, vol. 2, no. 3, pp. 226–237, 2015.

[22] G. Chen and Z. Li, “A fixed-time convergent algorithm for distributed convex optimization in multi-agent systems,” Automatica, vol. 95, pp. 539–543, 2018.

[23] A. Masoumzadeh, E. Nekouei, T. Alpcan, and D. Chattopadhyay, “Impact of optimal storage allocation on price volatility in energy-only electricity markets,” IEEE Transactions on Power Systems, vol. 33, no. 2, pp. 1903–1914, 2017.

[24] H. Sayyaadi and M. Moarref, “A distributed algorithm for proportional task allocation in networks of mobile agents,” IEEE Transactions on Automatic Control, vol. 56, no. 2, pp. 405–410, Feb. 2011.

[25] M. Doostmohammadian, H. Sayyaadi, and M. Moarref, “A novel consensus protocol using facility location algorithms,” in IEEE Conf. on Control Applications & Intelligent Control, 2009, pp. 914–919.

[26] C. You, K. Huang, H. Chae, and B. Kim, “Energy-efficient resource allocation for mobile-edge computation offloading,” IEEE Transactions on Wireless Communications, vol. 16, no. 3, pp. 1877–1887, 2016.

[27] J. Guo, Z. Song, Y. Cui, Z. Liu, and Y. Ji, “Energy-efficient resource allocation for multi-user mobile edge computing,” in IEEE Global Communications Conference (GLOBECOM). IEEE, 2017, pp. 1–7.

[28] K. Li, Q. Liu, and Z. Zeng, “Quantized event-triggered communication based multi-agent system for distributed resource allocation optimization,” Information Sciences, vol. 577, pp. 336–352, 2021.

[29] E. Nekouei, T. Alpcan, and R. J. Evans, “Impact of quantized inter-agent communications on game-theoretic and distributed optimization algorithms,” in Uncertainty in Complex Networked Systems, pp. 501–532. Springer, 2018.

[30] M. Doostmohammadian and N. Meskin, “Finite-time stability under denial of service,” IEEE Systems Journal, vol. 15, no. 1, pp. 1048–1055, 2020.

[31] A. Polyakov, “Nonlinear feedback design for fixed-time stabilization of linear control systems,” IEEE Transactions on Automatic Control, vol. 57, no. 8, pp. 2106–2110, 2011.

[32] A. I. Rikos, T. Charalambous, K. H. Johansson, and C. N. Hadjicostis, “Privacy-preserving event-triggered quantized average consensus,” in IEEE Conference on Decision and Control, 2020, pp. 6246–6253.

[33] J. Wei, X. Yi, H. Sandberg, and K. H. Johansson, “Nonlinear consensus protocols with applications to quantized communication and actuation,” IEEE Transactions on Control of Network Systems, vol. 6, no. 2, pp. 598–608, 2019.

[34] M. Guo and D. V. Dimaroogonas, “Consensus with quantized relative state measurements,” Automatica, vol. 49, no. 8, pp. 2531–2537, 2013.

[35] A. I. Rikos and C. N. Hadjicostis, “Distributed average consensus under quantized communication via event-triggered mass summation,” in IEEE Conference on Decision and Control, 2018, pp. 894–899.

[36] H. Sayyaadi and M. Doostmohammadian, “Finite-time consensus in directed switching network topologies and time-delayed communications,” Scientia Iranica, vol. 18, no. 1, pp. 75–85, 2011.

[37] D Bertsekas, A Nedic, and A Ozdaglar, Convex Analysis and Optimization, Athena Scientific, Belmont, MA, 2003.

[38] Y. Nesterov, “Introductory lectures on convex programming, volume I: Basic course,” Lecture notes, vol. 3, no. 4, pp. 5, 1998.

[39] P. Lin, W. Ren, and J. A. Farrell, “Distributed continuous-time optimization: nonuniform gradient gains, finite-time convergence, and convex constraint set,” IEEE Transactions on Automatic Control, vol. 62, no. 5, pp. 2239–2253, 2017.

[40] S. Gupta, A. Campa, and S. Rufo, Statistical physics of synchronization, Springer, 2018.

[41] Z. Zuo and L. Tie, “Distributed robust finite-time nonlinear consensus protocols for multi-agent systems,” International Journal of Systems Science, vol. 47, no. 6, pp. 1366–1375, 2016.

[42] M. Doostmohammadian, “Single-bit consensus with finite-time convergence: Theory and applications,” IEEE Transactions on Aerospace and Electronic Systems, vol. 56, no. 4, pp. 3332–3338, 2020.

[43] S. E. Parsegov, A. E. Polyakov, and P. S. Shcherbakov, “Fixed-time consensus algorithm for multi-agent systems with integrator dynamics,” IFAC Proc. Volumes, vol. 46, no. 27, pp. 110–115, 2013.