Two Proofs of the Fisher Information Inequality via Data Processing Arguments

Tie Liu and Pramod Viswanath

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Abstract

Two new proofs of the Fisher information inequality (FII) using data processing inequalities for mutual information and conditional variance are presented.

1 Introduction

In parameter estimation problems, the Fisher information matrix of a measurement \( X \) relative to a vector parameter \( \theta \) is defined as

\[
J(X; \theta) \overset{\text{def}}{=} \text{COV}\left\{ \frac{\partial}{\partial \theta} \ln f_{\theta}(X) \right\}
\]

(1)

where \( \{f_{\theta}(x)\} \) is a family of probability density functions of \( X \) parameterized by \( \theta \), and \( \text{COV}\{\cdot\} \) denotes the covariance matrix calculated with respect to \( f_{\theta}(x) \). A special form of the Fisher information matrix that shows up regularly in information theory [1] and physics [2] is the Fisher information matrix of a random vector with respect to a translation parameter:

\[
J(X) \overset{\text{def}}{=} J(\theta + X; \theta) = \text{COV}\{\rho_X(X)\}
\]

(2)

where the score function \( \rho_X \) is defined as

\[
\rho_X(x) \overset{\text{def}}{=} \frac{\partial}{\partial x} \ln f(x),
\]

(3)

and \( f(x) \) is the probability density function of the random vector \( X \). Unlike in the general definition [1], this special form of the Fisher information matrix is a function of the probability density of the random vector alone, and not of its parametrization.
Let $N_1$ and $N_2$ be two independent random variables with probability density functions in the real line $\mathcal{R}$. The classical Fisher information inequality (FII) states that

$$(a + b)^2 J(N_1 + N_2) \leq a^2 J(N_1) + b^2 J(N_2), \quad \forall a, b \geq 0. \quad (4)$$

Choosing $a = 1/J(N_1)$ and $b = 1/J(N_2)$, we obtain from (4) that

$$\frac{1}{J(N_1 + N_2)} \geq \frac{1}{J(N_1)} + \frac{1}{J(N_2)}, \quad (5)$$

where the equality holds if and only if both $N_1$ and $N_2$ are Gaussian. Compared with (4), (5) is usually thought of as the canonical form of the classical FII.

The classical Stam-Blachman proof [3, 4] of the FII relies on the following conditional-mean representation of the score function for the sum of two independent random variables:

$$\rho_{N_1+N_2}(n) = \mathbb{E}[\rho_{N_i}(N_i)|N_1 + N_2 = n], \quad i = 1, 2; \quad (6)$$

and then applies the Cauchy-Schwartz inequality. Although the proof is direct and concise, it does not bring any operational meaning to the FII.

In the excellent contribution [5], Zamir showed that the FII can be proved using the following data processing inequality for Fisher information:

$$J(Y; \theta) \preceq J(X; \theta) \quad (7)$$

if $\theta \to X \to Y$ satisfies the chain relation

$$f(x, y|\theta) = f_\theta(x)f(y|x), \quad (8)$$

i.e., the conditional distribution of $Y$ given $X$ is no longer a function of the parameter $\theta$. In [5], Zamir considered the parameter estimation model:

$$\begin{cases} X_1 = a\theta + N_1 \\ X_2 = b\theta + N_2 \end{cases} \quad (9)$$

where $a$ and $b$ are two arbitrary nonnegative real numbers. Note that $\theta \to (X_1, X_2) \to X_1 + X_2$ satisfies the chain relation [5] in a trivial way. By the data processing inequality [7] for Fisher information, we have

$$J(X_1 + X_2; \theta) \leq J(X_1, X_2; \theta). \quad (10)$$

Thus, the desired inequality [11] can be obtained by substituting the parameter estimation model [9] into [10]. Moreover, it can be seen that the difference between the two sides of [11] corresponds to the loss in the Cramér-Rao bound due to the “processing” in a certain linear additive noise model for parameter estimation.

It is worthy of mentioning that an identical argument without assuming the independence between $N_1$ and $N_2$ proves a generalization of the classical FII to the dependent-variable case:

$$(a + b)^2 J(N_1 + N_2) \leq [a b] J(N_1, N_2) [a b]^t. \quad (11)$$

This result was initially proved in [6, Th. 2] using the conditional-mean representation of the score function for the sum of two dependent random variables.
2 New Proofs of the FII

Data processing is a general principle in information theory, in that any quantity under the name “information” should obey some sort of data processing inequality. In this sense, Zamir’s data processing inequality for Fisher information merely pointed out the fact that Fisher information bears the real meaning as an information quantity. Interestingly enough, at the very beginning of [5], Zamir also pointed out that the data processing principle applies to mutual information and conditional variance as well. Specifically, if random variables $W \rightarrow X \rightarrow Y$ form a Markov chain, the mutual information among them satisfies
\[ I(W; Y) \leq I(W; X), \quad \text{(12)} \]
and the conditional variances satisfy
\[ \text{VAR}[W|Y] \geq \text{VAR}[W|X] \quad \text{(13)} \]
where $\text{VAR}[W|X] \overset{\text{def}}{=} E[(W - E[W|X])^2]$. The main purpose of this note is to provide two new proofs of the FII using the more familiar data processing inequalities (12) and (13), respectively.

2.1 A Communications Proof

Consider the communication model:
\[
\begin{align*}
X_1 &= a \sqrt{t} W + N_1, \\
X_2 &= b \sqrt{t} W + N_2, \\
\end{align*}
\tag{14}
\]
where $W$ is standard Gaussian and $W$, $N_1$ and $N_2$ are pairwise independent. By the scalar De-Bruijn identity [4], we have
\[
I(W; X_1) = \frac{a^2 t}{2} J(N_1) + o(t), \tag{15}
\]
\[
I(W; X_2) = \frac{b^2 t}{2} J(N_2) + o(t), \tag{16}
\]
and
\[
I(W; X_1 + X_2) = \frac{(a + b)^2 t}{2} J(N_1 + N_2) + o(t) \tag{17}
\]
where $\frac{o(t)}{t} \to 0$ in the limit as $t \downarrow 0$. Note that $W \rightarrow (X_1, X_2) \rightarrow X_1 + X_2$ forms a trivial Markov chain. By the data processing inequality (12) for mutual information, we have
\[
I(W; X_1 + X_2) \leq I(W; X_1, X_2) \leq I(W; X_1) + I(W; X_2|X_1) \leq I(W; X_1) + I(W; X_2) \tag{18-20}
\]
where the last inequality follows from $I(W; X_2|X_1) \leq I(W; X_2)$ because of the Markov chain $X_2 \rightarrow W \rightarrow X_1$ [7, p. 33]. Substituting (15), (16) and (17) into (20), we obtain
\[
(a + b)^2 t J(N_1 + N_2) \leq a^2 t J(N_1) + b^2 t J(N_2) + o(t). \tag{21}
\]
Dividing both sides by $t$ and letting $t \downarrow 0$, we obtain the desired inequality (11). This completes the proof of the classical FII using the data processing inequality for mutual information.

**Remark.** The above proof can still go through even without assuming $W$ is Gaussian. This is because, as mentioned in the last paragraph of [8, Sec. III], the scalar De-Bruijn identity holds for any $W$ whose first four moments coincide with those of Gaussian one.

### 2.2 A Bayesian Estimation Proof

Consider the Bayesian estimation model:

\[
\begin{align*}
X_1 &= N_1 + \sqrt{a}W_1, \\
X_2 &= N_2 + \sqrt{b}W_2,
\end{align*}
\]  

(22)

where $W_1$ and $W_2$ are stand Gaussian random variables, and $N_1$, $N_2$, $W_1$ and $W_2$ are pairwise independent. The following lemma provides the needed connection between conditional variance and Fisher information.

**Lemma 1** Let $N$ and $W$ be two independent random variables. Assuming $W$ is Gaussian with zero mean and variance $\sigma^2$, we have

\[
J(N + W) = \frac{1}{\sigma^4} \left\{ \sigma^2 - \text{VAR}[N|N + W] \right\}.
\]  

(23)

**Proof:** See Appendix A. ■

**Remark.** Our proof uses the conditional-mean representation of the score function for the sum of two independent random variables, which suggests a natural connection between conditional-mean estimators and Fisher information. We mention here that Lemma 1 may also be deduced from a recent result of Guo, Shamai and Verdú [9, Th. 1], in conjunction with the scalar De-Bruijn identity. Therefore, our proof can be viewed as an alternative proof of the result of Guo, Shamai and Verdú.

Applying Lemma 1 to model (22), we obtain

\[
\begin{align*}
\text{VAR}[N_1|X_1] &= at - a^2t^2J(X_1), \\
\text{VAR}[N_2|X_2] &= bt - b^2t^2J(X_2),
\end{align*}
\]  

(24) and

\[
\text{VAR}[N_1 + N_2|X_1 + X_2] = (a + b)t - (a + b)^2t^2J(X_1 + X_2).
\]  

(25)

(26)

Note that $N_1 + N_2 \to (X_1, X_2) \to X_1 + X_2$ forms a trivial Markov chain. By the data processing inequality for conditional variance, we have

\[
\begin{align*}
\text{VAR}[N_1 + N_2|X_1 + X_2] &\geq \text{VAR}[N_1 + N_2|X_1, X_2] \\
&= \text{VAR}[N_1|X_1] + \text{VAR}[N_2|X_2]
\end{align*}
\]  

(27) and

\[
(a + b)^2J(X_1 + X_2) \leq a^2J(X_1) + b^2J(X_2).
\]  

(28)

(29)
Note that $J(X_1), J(X_2)$ and $J(X_1 + X_2)$ approach $J(N_1), J(N_2)$ and $J(N_1 + N_2)$, respectively, in the limit as $t \downarrow 0$. The desired inequality (11) thus follows from (29) by letting $t \downarrow 0$. This completes the proof of the classical FII using the data processing inequality for conditional variance.

**Remark.** As in Zamir’s proof, the necessity of the equality condition in (5) does not follow easily in either of the new proof. As a matter of fact, it becomes less apparent due to the limiting argument used in both proofs.

### A Proof of Lemma 1

Let $X = N + W$. Since score functions always have zero mean, the Fisher information of $X$ can be written as

$$J(X) = \mathbb{E}[\rho_X^2(X)].$$

(30)

By the conditional-mean representation of the score function for the sum of two independent random variables, we have

$$\rho_X(x) = \mathbb{E}[\rho_W(W)|X = x]$$

(31)

$$= \frac{1}{\sigma^2} \mathbb{E}[-W|X = x]$$

(32)

$$= \frac{1}{\sigma^2} \{\mathbb{E}[N|X = x] - x\}$$

(33)

where (32) holds because $W$ is Gaussian with zero mean and variance $\sigma^2$ so we have $\rho_W(w) = -w/\sigma^2$. It then follows from (30) and (33) that

$$J(X) = \frac{1}{\sigma^4} \mathbb{E}\left[(\mathbb{E}[N|X] - X)^2\right]$$

(34)

$$= \frac{1}{\sigma^4} \mathbb{E}\left[(W + (N - \mathbb{E}[N|X]))^2\right]$$

(35)

$$= \frac{1}{\sigma^4} \{\sigma^2 + \text{VAR}[N|X] + 2\mathbb{E}[W(N - \mathbb{E}[N|X])]\}.$$  

(36)

Further note that

$$\mathbb{E}[W(N - \mathbb{E}[N|X])] = -\mathbb{E}[(N - X)(N - \mathbb{E}[N|X])]$$

(37)

$$= -\mathbb{E}[(N - \mathbb{E}[N|X]) + (\mathbb{E}[N|X] - X)(N - \mathbb{E}[N|X])]$$

(38)

$$= -\text{VAR}[N|X]$$

(39)

because, by the orthogonality principle [10], we have $\mathbb{E}[(\mathbb{E}[N|X] - X)(N - \mathbb{E}[N|X])] = 0$. Substituting (39) into (36), we obtain the desired representation

$$J(X) = \frac{1}{\sigma^4} \{\sigma^2 - \text{VAR}[N|X]\}.$$  

(40)

This completes the proof of Lemma 1.
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