Generation of pure continuous-variable entangled cluster states of four separate atomic ensembles in a ring cavity

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A practical scheme is proposed for creation of continuous variable entangled cluster states of four distinct atomic ensembles located inside a high-finesse ring cavity. The scheme does not require a set of external input squeezed fields, a network of beam splitters and measurements. It is based on nothing else than the dispersive interaction between the atomic ensembles and the cavity mode and a sequential application of laser pulses of a suitably adjusted amplitudes and phases. We show that the sequential laser pulses drive the atomic "field modes" into pure squeezed vacuum states. The state is then examined against the requirement to belong to the class of cluster states. We illustrate the method on three examples of the entangled cluster states, the so-called continuous variable linear, square and T-type cluster states.

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I. INTRODUCTION

The investigation of the continuous variable (CV) quantum information has attracted a great interest along with the development of different techniques for generation, manipulation and detection of CV multipartite entangled state [1]. The CV entangled states have many applications in various quantum information processing such as quantum teleportation [2], dense coding [3], entanglement swapping [4], and quantum teleporting [5]. Recently, Zhang and Braunstein [6] have introduced a kind of CV Gaussian multipartite entangled states, the so-called CV cluster states. The states possess the property that their entanglement is harder to destroy than that of the CV Greenberger-Horne-Zeilinger (GHZ) state. Menicucci et al. [7] have proposed a generalization of the universal quantum computation by employing the CV cluster state and an optical implementation involving squeezed-light sources, linear optics, and performing a measurement by the homodyne detection. Since then, the problem of the generation of a CV cluster state has become a very important issue. Several proposals have been put forward [8, 9, 10, 11]. Essentially all these proposals are based on linear optics schemes and measurements on the system. For example, Su et al. [10] experimentally produced the CV quadrupartite cluster state of the electromagnetic field by utilizing two amplitude-quadrature and two phase-quadrature squeezed states of light and linearly optical transformation. Three different kinds of four-mode CV cluster states, which are suitable for small-scale implementations of one-way quantum computation, have been proposed and experimentally constructed by applying squeezed light sources and a network of beam splitters [11].

Apart from the schemes based on linear optics, a numerical interest is now in the study of practical application of atomic systems to CV quantum information and computation [12, 13, 14]. This is because a large collection of identical atoms, called an atomic ensemble, can be efficiently coupled to quantum light if a collective superposition state of many atoms can be utilized for the coupling [15]. The existence of long atomic ground-state coherence lifetimes has been thus used to realize the long-lived, high-fidelity storage of quantum states that is important for long-distance quantum networking [16]. Relying on optical joint measurements of the atomic ensemble states and magnetic feedback reconstruction, a protocol to achieve high fidelity quantum state teleportation of a macroscopic atomic ensemble using a pair of quantum-correlated atomic ensembles has also been proposed by Dantan et al. [17]. The dissipation version of the Lipkin-Meshkov-Glick (LMG) model [18] and the effective Dicke model [19] have been proposed in atomic ensemble based on cavity-mediated Raman transitions. The quantum trajectories of collective atomic spin states of driven two-level atoms and coupled to a cavity field has been studied [20]. Especially, it has been proposed that the unconditional preparation of a two-mode squeezed state of effective bosonic modes realized in a pair of atomic ensembles interacting collectively with a two-mode optical cavity and laser fields [21]. However, how to prepare CV cluster states in atomic ensembles still remains an open question.

In this paper, we address this question and propose a practical scheme for creation of four-mode CV entangled cluster states of four separate atomic ensembles located inside a single-mode ring cavity. The scheme does not involve any external sources of squeezed light and networks of beam splitters used in the linear optics schemes [11]. In our scheme the atomic ensembles are driven by laser pulses and a cluster state is created by specific sequential choices of the Rabi frequencies and phases of the laser pulses. It is assumed that the atoms interact with
the laser pulses and the cavity field in a highly nonresonant dispersive manner. The dispersive interaction involves virtual states, not the excited states of the atoms. Therefore, the process of preparation of a cluster state is not affected by the atomic spontaneous emission. We illustrate the method for three types of cluster states, the so-called CV linear, square, and T-type cluster states [11] and demonstrate how these states can be deterministically prepared using the sequence of laser pulses and the cavity dissipation. In Sec. III we introduce the model and derive the effective Hamiltonian of the system. In Sec. III the wave functions of the three kinds of the four-mode CV cluster states are explicitly introduced and the concrete steps of preparation of these quadripartite cluster states are given. We summarize our results in Sec. IV.

II. THE MODEL

We consider a system consisting of four atomic ensembles located inside a high-finesse ring cavity. The cavity is composed of four mirrors that create two modes, called propagating and counter-propagating modes, to which the atomic ensembles are equally coupled. External pulse lasers that are used to drive the atomic ensembles couple to only a single propagating mode, as it is illustrated in Fig. 1. We assume that the atoms are homogeneously distributed inside the ensembles. In this case only the forward scattering occurs that allows us to neglect the coupling of the atoms to the counter-propagating mode and work in the single mode approximation [19, 21]. In other words, the cavity mode propagates with the laser fields. This condition can be easily fulfilled in trapped room-temperature atomic ensembles where fast atomic oscillations over the interaction time lead to a collectively enhanced coupling of the atoms to a single mode that is practically collinear with the laser fields [12].

The cavity is damped with the rate $\kappa$ that is assumed small to achieve a high finesse at a relatively large size of the cavity. In the current experiments with ring cavities $\kappa \approx 100 \text{ mm}$, finesses up to $F = 1.7 \times 10^9$ are achieved with the cavity length (cavity round trip) $L \approx 100 \text{ mm}$, which gives the decay rate $(\kappa/2\pi) \approx 20 \text{ kHz}$. With the relatively large length of the cavity, the mode spacing of the cavity field $\Delta \omega \approx 1 \text{ GHz}$ that is much larger than the width of the cavity modes. Thus, the one mode approximation, assumed here, appears to be practical.

The atomic ensembles contain a large number of identical four-level atoms each composed of two stable ground states, $|0_{jj'}\rangle$, $|1_{jj'}\rangle$, and two excited states $|u_{jj'}\rangle$, $|s_{jj'}\rangle$. Here, the subscript $j$ ($j = 1, 2, 3, 4$) labels the atomic ensembles, and the subscript $j'$ ($j' = 1, 2, \ldots, N_j$) labels individual atoms in a given ensemble. Such a scheme might be realized in practice, e.g., by employing alkali atoms, with $|0_{jj'}\rangle$ and $|1_{jj'}\rangle$ as different ground-state sublevels. The ground state $|0_{jj'}\rangle$ of energy $E_{0_{jj'}} = 0$ is coupled to the excited state $|s_{jj'}\rangle$ by a laser field of the Rabi frequency $\Omega_{ss_j}$ and frequency $\omega_{s_{jj'}}$ that is detuned from the laser pulses and the cavity field in a highly nonresonant dispersive manner. The dispersive interaction involves virtual states, not the excited states of the atoms. Therefore, the process of preparation of a cluster state is not affected by the atomic spontaneous emission. We illustrate the method for three types of cluster states, the so-called CV linear, square, and T-type cluster states [11] and demonstrate how these states can be deterministically prepared using the sequence of laser pulses and the cavity dissipation. In Sec. III we introduce the model and derive the effective Hamiltonian of the system. In Sec. III the wave functions of the three kinds of the four-mode CV cluster states are explicitly introduced and the concrete steps of preparation of these quadripartite cluster states are given. We summarize our results in Sec. IV.

![Fig. 1: Configuration of a ring cavity and four atomic ensembles for the preparation of entangled CV cluster states. External pulse lasers of the Rabi frequencies $\Omega_{u_j}$ and $\Omega_{s_j}$ couple to only one (propagating) mode of the cavity and interact dispersively with the atoms.](image)

![Fig. 2: Atomic level scheme. Two highly detuned laser fields of the Rabi frequencies $\Omega_{u_j}$ and $\Omega_{s_j}$ drive the atomic transitions $|1_{jj'}\rangle \rightarrow |u_{jj'}\rangle$ and $|0_{jj'}\rangle \rightarrow |s_{jj'}\rangle$, respectively. The atomic transitions $|1_{jj'}\rangle \rightarrow |s_{jj'}\rangle$ and $|0_{jj'}\rangle \rightarrow |u_{jj'}\rangle$ are coupled to the single-mode cavity with the coupling strengths $g_{uj}$ and $g_{sj}$, respectively.](image)
the atomic transition frequency by \( \Delta_{s_j} = (\omega_{s_j} - \omega_{L_{s_j}}) \), where \( \omega_{s_j} = E_{s_j'}/\hbar \) and \( E_{s_j'/j} \) is the energy of the state \( |s_j'\rangle \). Similarly, the ground state \( |1_{jj'}\rangle \) of energy \( E_{1_{jj'}} = \hbar \omega_{1_j} \) is coupled to the excited state \( |u_{jj'}\rangle \) of energy \( E_{u_{jj'}} = \hbar \omega_{u_j} \) by another laser field with the Rabi frequency \( \Omega_{1_j} \), and the angular frequency \( \omega_{L_{u_j}} \) that is detuned from the atomic transition \( |1_{jj'}\rangle \rightarrow |u_{jj'}\rangle \) by \( \Delta_{u_j} = (\omega_{u_j} - \omega_{1_j} - \omega_{L_{u_j}}) \). The atoms interact with the cavity mode of frequency \( \omega_{a} \) that simultaneously couples to the \( |u_{jj'}\rangle \leftrightarrow |0_{jj'}\rangle \) and \( |s_{jj'}\rangle \leftrightarrow |1_{jj'}\rangle \) transitions, with the coupling strengths \( g_{u_j} \) and \( g_{s_j} \), respectively. We assume that the atom-field coupling strengths are uniform through the atomic ensembles. This is consistent with current experiments involving ring cavities and large samples of trapped atoms \cite{10}. Practical sizes of the atomic samples are \( \sim 10^{10} \) nm that is smaller than the cavity mode radius \( \hbar \omega_{0} = 130 \mu \text{m} \), and also is much smaller than the practical length of a single arm of the cavity.

The system is described by the Hamiltonian, which in the interaction picture takes the form

\[
H_I = \sum_{j=1}^{N_j} \sum_{j' = 1}^{N_j} \left\{ \frac{1}{2} \Omega_{u_j} e^{i(\phi_{u_j} + \Delta_{u_j} t)} |u_{jj'}\rangle \langle 1_{jj'}| \\
+ \frac{1}{2} \Omega_{s_j} e^{i(\phi_{s_j} + \Delta_{s_j} t)} |s_{jj'}\rangle \langle 1_{jj'}| \\
+ g_{u_j} e^{i\delta_{u_j} t} a |u_{jj'}\rangle \langle 0_{jj'}| \\
+ g_{s_j} e^{i\delta_{s_j} t} a |s_{jj'}\rangle \langle 1_{jj'}| + \text{H.c.} \right\},
\]

where \( \delta_{u_j} = (\omega_{a} - \omega_{u_j}) \) and \( \delta_{s_j} = (\omega_{a} - \omega_{s_j} - \omega_{L_{s_j}}) \) are the detunings of the cavity field from the atomic resonances, and \( \phi_{s_j}, \phi_{u_j} \) are the phases of the driving fields.

In order to eliminate spontaneous scattering of photons to modes other than the privileged cavity mode, we assume that the laser fields and the cavity mode frequencies are highly detuned from the atomic resonances, i.e., the detuning \( \{ |\Delta_{u_j}|, |\Delta_{s_j}|, |\delta_{u_j}|, |\delta_{s_j}| \} \gg \{ |g_{u_j}, g_{s_j}, |\Omega_{u_j}|, |\Omega_{s_j}| \} \). This allows us to apply the adiabatic approximation \cite{18, 19, 20, 21} under which we eliminate the excited states of the atoms and obtain an effective Hamiltonian of the form \cite{10, 21}

\[
H_{\text{eff}} = \sum_{j=1}^{N_j} \left\{ \frac{\Omega_{u_j}^2}{4\Delta_{u_j}} \left( \frac{N_j}{2} + J_{z_j} \right) + \frac{\Omega_{s_j}^2}{4\Delta_{s_j}} \left( \frac{N_j}{2} - J_{z_j} \right) \right\} \\
+ \left[ \frac{g_{u_j}^2}{\Delta_{u_j}} \left( \frac{N_j}{2} - J_{z_j} \right) + \frac{g_{s_j}^2}{\Delta_{s_j}} \left( \frac{N_j}{2} + J_{z_j} \right) + \delta_{a} \right] a^\dagger a \\
+ \left[ a^\dagger \left( \beta_{u_j} J_{z_j} - \beta_{s_j} J_{z_j}^\dagger \right) + \text{H.c.} \right],
\]

where

\[
J_{z_j} = \sum_{j' = 1}^{N_j} |1_{jj'}\rangle \langle 0_{jj'}|, \quad J_{z_j} = \sum_{j' = 1}^{N_j} |0_{jj'}\rangle \langle 1_{jj'}|,
\]

are the collective operators of the atomic ensembles,

\[
\beta_{u_j} = \frac{\Omega_{u_j} g_{u_j}}{2\Delta_{u_j}} e^{i\phi_{u_j}}, \quad \beta_{s_j} = \frac{\Omega_{s_j} g_{s_j}}{2\Delta_{s_j}} e^{i\phi_{s_j}}
\]

are the effective coupling constants of the ensembles to the cavity mode, and \( \delta_{a} = \omega_{a} - (\omega_{L_{s_j}} + \omega_{L_{u_j}})/2 \) is the detuning of the cavity field from the average frequency of the laser fields. Here the frequencies of the driving lasers are also assumed to satisfy \( \omega_{L_{s_j}} - \omega_{L_{u_j}} = 2\omega_{1_j} \).

The first line of Eq. (2) represents the free energy of the ensembles. The second line represents an intensity dependent (Stark) shift of the atomic energy levels, and the third line represents the interaction between the cavity field and the atomic ensembles. The essential feature of the effective Hamiltonian (2) is that the atoms interact dispersively with the cavity mode. This means that the cavity mode will remain unpopulated during the evolution. Note the presence of nonlinear terms, \( a^\dagger J_{z_j}, a J_{z_j}^\dagger \), analogous to the counter-rotating terms, that will play the crucial role in creation of a squeezed state between the atomic ensembles.

The Hamiltonian (2) is general in the parameter values. In what follows, we will work with a simplified version of the Hamiltonian by assuming equal coupling constants of the atomic ensembles to the cavity field, \( g_{u_j} = g_{s_j} = g \), and equal detunings \( \Delta_{u_j} = \Delta_{s_j} = \Delta \). We further assume that each atomic ensemble contains the same number of atoms, i.e., \( N_j = N \), and we choose the detuning \( \delta_{a} \) such that the resonant condition with the shifted resonances, \( \delta_{a} + 4g^2 N/\Delta = 0 \), is satisfied.

A standard way to obtain a CV cluster state is to entangle a number of field (bosonic) modes. Therefore, we shall work in the field (bosonic) representation of the atomic operators and consider a procedure to entangle the corresponding number of the bosonic modes. In this approach, we make use of the Holstein-Primakoff representation \cite{25} that transforms the collective atomic operators into harmonic oscillator annihilation and creation operators \( c_j \) and \( c_j^\dagger \) of a single bosonic mode

\[
J_{z_j} = c_j^\dagger \sqrt{N - c_j^\dagger c_j}, \quad J_{z_j} = c_j c_j^\dagger - N/2.
\]

For the procedure considered here of the preparation of the cluster states, the mean number of atoms transferred to the states \( |1_{jj'}\rangle \) is expected to be much smaller than the total number of atoms in each ensemble, i.e., \( \langle c_j^\dagger c_j \rangle \ll N \). By expanding the square root in Eq. (5)
and neglecting terms of the order of $O(1/N)$, the collective atomic operators can be approximated as [21]

$$J_j^i \approx \sqrt{N} c_j^i, \quad J_{jz} \approx -N/2.$$  \hspace{1cm} (6)

Substituting these expressions into Eq. (2) and omitting the constant energy terms, we find that the effective Hamiltonian can be simplified to the form

$$H_{\text{eff}} = \frac{\sqrt{N} \kappa}{2 \Delta} \sum_{j=1}^{4} \left\{ \Omega_{u_j} \left( e^{i \phi_{u_j}} a_j^1 a_j^1 + e^{-i \phi_{u_j}} c_j^1 a_j^1 \right) + \Omega_{s_j} \left( e^{-i \phi_{s_j}} ac_j + e^{i \phi_{s_j}} c_j^1 a_j \right) \right\}.$$  \hspace{1cm} (7)

We now consider the evolution of the system under the effective Hamiltonian (7) including also a possible loss of photons due to the damping of the cavity mode. If the cavity mode is allowed to decay with a rate $\kappa$, the system then is determined by the density operator $\rho$ whose the time evolution satisfies the master equation

$$\dot{\rho} = -i [H_{\text{eff}}, \rho] + L_\alpha \rho,$$  \hspace{1cm} (8)

where

$$L_\alpha \rho = \frac{1}{2} \kappa (2a a^\dagger - a^\dagger a - \rho a^\dagger a),$$  \hspace{1cm} (9)

represents the damping of the field by the cavity decay $\kappa$. Choosing $\Omega_{u(s)}$ and $\phi_{u(s)}$ appropriately allows for the preparation of the system in a desired state that then decays to its steady-state with the rate $\kappa$.

Note that we have ignored the damping of the atoms by spontaneous emission due to large detunings assumed in the derivation of the effective Hamiltonian (7). The spontaneous emission rate due to off-resonant excitation is estimated at the rate $\gamma_{\text{eff}} = \frac{1}{3} (\gamma/2\pi) (\Omega_{u(s)}/\Delta)^2$ that with the experimental values of $\gamma = 6$ MHz for a rubidium atom and $\Omega_{u(s)}/\Delta = 0.005$ gives $\gamma_{\text{eff}} \approx 40$ Hz. The estimated value for $\gamma_{\text{eff}}$ is significantly smaller than $\kappa$ predicted for a cavity of the finesse $F = 1.7 \times 10^5$.

The master equation (8) will be use to analyze how to deterministically prepare cluster states for the four atomic ensembles by a proper choosing of the Rabi frequencies $\Omega_{u(s)}$, and the phases $\phi_{u(s)}$. We will use different sequences of the laser pulses to drive the atomic ensembles and assume that all the atoms are initially in the lowest energy state $|0\rangle_j$. As a preface, we briefly discuss how one could distinguish that a given state belongs to the class of cluster states. Simply, a given state is quantified as a cluster state if the quadrature correlations are such that in the limit of infinite squeezing, the state becomes zero eigenstate of a set of quadrature combinations [11]

$$\left( \hat{p}_a - \sum_{b \in N_a} \hat{x}_b \right) \rightarrow 0,$$  \hspace{1cm} (10)

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators (quadratures) of a mode $a$, and the modes $b$ are the nearest neighbors $N_a$ of the mode $a$. In what follows, we quantify a given state as a cluster state by evaluating the variances of linear combinations of the momentum and position operators of the involved field modes. If the variances vanish in the limit of the infinite squeezing then, according to the above definition, a given state is a cluster state.

### A. The preparation of the linear CV cluster state

Let us first concentrate on the preparation of a four-mode linear cluster state involving four separate atomic ensembles, as shown in Fig. 3(a). As we shall see, the preparation process depends crucially on the specific choice of the Rabi frequencies and phases of the driving laser fields.

In order to unconditionally prepare a linear cluster state involving four separate atomic ensembles and find its explicit form, we make an unitary transformation $d_{L_j} = T_j c_j T_j^\dagger$ (j = 1, 2, 3, 4) that transfers the $c_j$ operators into linear combinations

$$d_{L_1} = -\frac{1}{\sqrt{2}} (ic_1 + c_2),$$
$$d_{L_2} = -\frac{1}{\sqrt{10}} (ic_1 - c_2 - 2ic_3 - 2c_4),$$
$$d_{L_3} = -\frac{1}{\sqrt{2}} (c_3 + ic_4),$$
$$d_{L_4} = -\frac{1}{\sqrt{10}} (2c_1 + 2ic_2 + c_3 - ic_4).$$  \hspace{1cm} (11)

Since the operators $d_{L_j}$ commute with each other, the combined modes are orthogonal to each other. This will allow us to prepare each mode separately in a desired

III. GENERATION OF THE CV QUADRIPARTITE CLUSTER STATES

We now proceed to discuss the detailed procedure of the preparation of CV quadripartite cluster states of atomic ensembles located inside a single-mode ring cavity. In particular, we shall show how to deterministically prepare a linear cluster state, a square cluster state, and a T-shape cluster state, as shown in Fig. 3.

FIG. 3: Examples of CV quadripartite cluster states; (a) linear cluster state, (b) square cluster state, and (c) T-shape cluster state.
state. In other words, an arbitrary transformation performed on the operators of a given mode will not affect the remaining modes.

Let us now illustrate the procedure of preparing the field modes in a desired state by using laser pulses of equal length and suitably chosen magnitudes of the Rabi frequencies and phases of the driving lasers. More concretely, for a cluster state, all the modes \( d_L \) should be prepared in a squeezed vacuum state. We employ the fact that the modes can be separately prepared in the squeezed vacuum state. Afterwards, if the variances of specific combinations of the quadrature components of the field operators vanish in the limit of the infinite squeezing, the desired state is a linear CV cluster state.

In the first step of the preparation, we send a set of laser pulses driving the the atomic ensembles 1 and 2 only, the lasers driving the atomic ensembles 3 and 4 are turned off. We choose the Rabi frequencies as

\[
\Omega_{\alpha_n} = \frac{\Omega_{\alpha_n}}{r} = \sqrt{2}\Omega, \quad n = 1, 2, \\
\Omega_{\alpha_j} = \Omega_{\alpha_j} = 0, \quad j = 3, 4, \quad (12)
\]

and the phases of the driving lasers

\[
\phi_{\alpha_1} = \frac{3}{2}\pi, \quad \phi_{\alpha_2} = \frac{1}{2}\pi, \\
\phi_{\alpha_3} = \phi_{\alpha_4} = \pi. \quad (13)
\]

After the first step of the preparation, the mode \( d_L \) is left in a state that is described by the density operator \( \rho_1 = T_1\rho T_1^\dagger \), which obeys a master equation

\[
\frac{d}{dt}\rho_1 = -i\beta \left( \left( a_1^\dagger d_L + r a_1^\dagger d_L^\dagger \right) + \text{H.c.}, \rho_1 \right) + L_\alpha \rho_1, \quad (14)
\]

where \( \beta = \sqrt{N_\xi \Omega / \Delta} \) and \( r \in (0, 1) \). If we now perform the single-mode squeezing transformation for the mode \( d_L \) as \( \rho_1 = S_1(\xi)\rho_1 S_1(\xi) \) with the single-mode squeezing operator \( S_1(\xi) \) becomes

\[
S_j(\xi) = \exp \left[ \frac{\xi}{2} \left( d_{Lj}^\dagger - d_{Lj} \right)^2 \right], \quad (15)
\]

where \( \xi = \tanh^{-1}(r) \), we find that the master equation \( (14) \) becomes

\[
\frac{d}{dt}\rho_1 = \beta \sqrt{1 - r^2} \left( \left( a_{Lj}^\dagger d_{Lj} + a_{Lj}^\dagger d_{Lj}^\dagger, \rho_1 \right) + L_\alpha \rho_1. \quad (16)
\]

It can be shown from Eq. \( (16) \) that in the steady state, the cavity mode will be in the vacuum state and the mode \( d_L \) will be in a squeezed vacuum state. This is easy to understand, the master equation \( (16) \) represents two coupled modes with the cavity mode linearly damped with the rate \( \kappa \). Since the remaining modes \( d_{L2}, d_{L3}, \) and \( d_{L4} \) are decoupled from the mode \( d_L \), they remain in an undetermined state \( \rho_{d_{L2}d_{L3}d_{L4}}(\tau) \).

In order to estimate the required time to reach the steady state, we calculate eigenvalues of Eq. \( (16) \)

\[
\lambda_{\pm} = -\kappa/2 \pm \left( \frac{\kappa}{2} \right)^2 - \beta^2(1 - r^2) \right)^{1/2}, \quad (17)
\]

from which we observe that as long as \( \beta \sqrt{1 - r^2} \gg \kappa/2 \), the time for the system to reach its steady state is of order of \( \sim 2/\kappa \) \(^{27}\). Therefore, the system will definitely evolve into the steady state, provided the interaction time is sufficient long, for example, \( \tau = 4/\kappa \). The time \( \tau = 4/\kappa \) determines the time scale in our protocol for the preparation of the cluster states.

By taking the inverse unitary transformation, it follows that in the steady state, the total system is in a state determined by the density operator

\[
\rho(\tau) = S_1(\xi)|0_{\alpha_1}0_{d_{L1}}\rangle\langle 0_{\alpha_1}0_{d_{L1}}|S_1(\xi) \otimes \rho_{d_{L2}d_{L3}d_{L4}}(\tau). \quad (18)
\]

Briefly summarize what we have obtained after the first step of the preparation of a linear CV cluster state.

The application of suitably chosen laser pulses leaves the mode \( d_L \) prepared in the single-mode squeezed vacuum state, with the cavity field found in the vacuum state, and the remaining combined modes \( d_{L2}, d_{L3}, \) and \( d_{L4} \) left in the states related to their initial states.

In the second step, we turn off the first series of driving lasers and sequentially send another series of laser pulses with different parameters to preparation of the combined bosonic mode \( d_{L2} \) in the single-mode squeezed vacuum state so that a similar linearly mixing interaction between the cavity mode and another combined bosonic modes arises. As before, for the first series of pulses, a single-mode squeezed vacuum state for this combined bosonic mode can be prepared due to the cavity dissipation. The Rabi frequencies of the second series of the laser pulses, which are turned on during the time of \( t \in [\tau, 2\tau] \) for the preparation of the combined bosonic mode \( d_{L2} \) in the single-mode squeezed vacuum state \( S_3(\xi)|0_{d_{L2}}\rangle \), are

\[
\Omega_{\alpha_n} = \frac{\Omega_{\alpha_n}}{r} = \frac{2}{\sqrt{10}}\Omega, \quad n = 1, 2, \quad (19)
\]

and the phases of the driving lasers

\[
\phi_{\alpha_1} = \frac{3}{2}\pi, \quad \phi_{\alpha_2} = \frac{1}{2}\pi, \quad \phi_{\alpha_3} = \frac{3}{2}\pi, \quad \phi_{\alpha_4} = 0. \quad (20)
\]

The specific choice of the Rabi frequencies and the phases ensures the mode \( d_{L2} \) to be prepared in the state described by the density operator \( \rho_1 \) satisfying the same master as Eq. \( (14) \) but with \( d_{L1} \) replaced by \( d_{L2} \).

In the third step, which is performed during the time of \( t \in [2\tau, 3\tau] \), the combined bosonic mode \( d_{L3} \) is being prepared in the single-mode squeezed vacuum state \( S_3(\xi)|0_{d_{L3}}\rangle \). The Rabi frequencies of the third series of the laser pulses, are

\[
\Omega_{\alpha_n} = \frac{\Omega_{\alpha_n}}{r} = 0, \quad n = 1, 2, \quad (21)
\]

\[
\Omega_{\alpha_j} = \frac{\Omega_{\alpha_j}}{r} = \sqrt{2}\Omega, \quad j = 3, 4.
\]
and the phases are
\[ \begin{align*}
\phi_{u1} &= \phi_{s1} = \phi_{u2} = \phi_{s2} = 0, \\
\phi_{u3} &= \frac{3}{2}\pi, \quad \phi_{s3} = \frac{1}{2}\pi, \\
\phi_{u4} &= \phi_{s4} = \pi.
\end{align*} \tag{22} \]

The fourth series, laser pulses are turned on during the time of \( t \in (3\tau, 4\tau) \). In this final series, the combined bosonic mode \( d_{L_4} \) is prepared in the single-mode squeezed vacuum state \( S_4(\xi)|0_{d_{L_4}}\rangle \). The laser pulses required to achieve this are of the Rabi frequencies
\[ \Omega_{un} = \frac{\Omega_{sc}}{r} = \frac{4}{\sqrt{10}}\Omega, \quad n = 1, 2, \]
\[ \Omega_{uj} = \frac{\Omega_{sc}}{r} = \frac{2}{\sqrt{10}}\Omega, \quad j = 3, 4, \tag{23} \]
and phases
\[ \begin{align*}
\phi_{u_1} &= \phi_{s_1} = 0, \quad k = 1, 3, \\
\phi_{u_2} &= \frac{1}{2}\pi, \quad \phi_{s_2} = \frac{3}{2}\pi, \\
\phi_{u_4} &= \frac{3}{2}\pi, \quad \phi_{s_4} = \frac{1}{2}\pi.
\end{align*} \tag{24} \]

Therefore after enough long time \( 4\tau \), the system evolves into the state determined by the density operator
\[ \rho_1(4\tau) = |\Phi_L\rangle\langle\Phi_L| \otimes |0_a\rangle\langle 0_a|, \tag{25} \]
where
\[ \begin{align*}
|\Phi_L\rangle &= T_1|\Psi_L\rangle = S_1(\xi)|0_{d_{L_1}}\rangle \otimes S_2(\xi)|0_{d_{L_2}}\rangle \\
&\quad \otimes S_3(\xi)|0_{d_{L_3}}\rangle \otimes S_4(\xi)|0_{d_{L_4}}\rangle \tag{26} \end{align*} \]

Evidently, the transformed state is in the form of a quadripartite squeezed vacuum state \([26]\). Once all the combined modes \( d_{L_j} \) are prepared in the corresponding single-mode squeezed vacuum states, the explicit form of the state created can be obtained by reversing the unitary transformation \( T_1 \). This leads to a pure linear CV quadripartite state of the form
\[ |\psi_L\rangle = \exp\left\{ -\frac{\xi}{10} \left( c_1^2 - c_2^2 - c_3^2 + c_4^2 - 8i(c_1c_2 + c_3c_4) - 4(c_1 + ic_2)(c_3 - ic_4) \right) \right\} |\{0_c\}\rangle, \tag{27} \]
where
\[ |\{0_c\}\rangle = |0_{c_1}, 0_{c_2}, 0_{c_3}, 0_{c_4}\rangle. \tag{28} \]

We may summarize that by an appropriate driving the four atomic ensembles, the four combined bosonic modes will be ultimately prepared in four single-mode squeezed vacuum states. It can be done in four steps by appropriately choosing the laser parameters such as phases and intensities. Then, by applying an inverse unitary transformation, the pure linear CV entangled state \( |\Psi_L\rangle \) for the four atomic ensembles can be obtained.

The question remains as to whether the state \( |\psi_L\rangle \) is an example of the linear CV cluster state. To examine this, we introduce the quadrature amplitude \( q_j = (c_j + c_j^\dagger)/\sqrt{2} \) and phase \( p_j = -i(c_j - c_j^\dagger)/\sqrt{2} \) components of the four modes involved, and easily find that the variances of the linear combinations of the components, evaluated according to the definition \([10]\), are
\[ \begin{align*}
V(p_1 - q_2) &= e^{-2\xi}, \\
V(p_2 - q_1 - q_3) &= \frac{3}{2}e^{-2\xi}, \\
V(p_3 - q_2 - q_4) &= \frac{3}{2}e^{-2\xi}, \\
V(p_4 - q_3) &= e^{-2\xi}, \tag{29} \end{align*} \]
where \( V(X) = \langle X^2 \rangle - \langle X \rangle^2 \). Clearly, all the four variances tend to zero in the limit of infinite squeezing, \( \xi \to \infty \). We therefore conclude that the state \( |\Psi_L\rangle \) is a four-mode linear CV cluster state \([11]\).

\section{The preparation of the square CV cluster state}

Having discussed the procedure of creation of a linear cluster state, we now proceed to describe the realization of an entangled CV square cluster state \( |\psi_S\rangle \). The strategy is based on the same procedure used above for the preparation of the CV linear cluster state.

A square CV cluster state may be obtained by performing a unitary transformation \( d_{S_1} = T_2c_2T_1^\dagger \) to obtain superposition modes determined by the following operators
\[ \begin{align*}
d_{S_1} &= -\frac{1}{\sqrt{10}}(ic_1 + ic_2 + 2c_3 + 2c_4), \\
d_{S_2} &= -\frac{i}{\sqrt{2}}(c_1 - c_2), \\
d_{S_3} &= -\frac{1}{\sqrt{10}}(2c_1 + 2c_2 + ic_3 + ic_4), \\
d_{S_4} &= -\frac{i}{\sqrt{2}}(c_4 - c_3). \tag{30} \end{align*} \]

Similar to the preparation of the CV linear cluster state for the four separated atomic ensembles, here we can make the interaction between the cavity and the four mode \( c_j \) (\( j = 1, 2, 3, 4 \)) reduce to the coupling between the mode \( a \) and one of the combined mode \( d_{S_j} \) in the form of a combined linear mixing process by choosing proper parameters of lasers. After a long-time interaction \( \tau = 4/\kappa \), the cavity mode \( a \) and the combined mode \( d_{S_j} \) will respectively develop into the vacuum state and the squeezed vacuum state \( \exp(\xi d_{S_j}^\dagger - \text{H.c.})|0_{d_{S_j}}\rangle \) due to the cavity dissipation. Therefore through separately sending four series of driving lasers and send another series of laser pulses with different appropriate parameters such as phases and amplitudes, the four single-mode squeezed vacuum states for the combined bosonic modes \( d_{S_j} \) can be
prepared with the help of the cavity dissipation. Subsequently, by reserving the above unitary transformations, we can deterministically prepare the quadripartite square cluster state $|\psi_S\rangle$.

The four sets of the laser pulses are chosen as follows. In the time period of $t \in [0, \tau)$, we apply the first set of pulses of the parameters

$$\Omega_{u_n} = \frac{\Omega_{n}}{\tau} = \frac{2}{\sqrt{10}} \Omega, \quad n = 1, 2,$$

$$\Omega_{u_j} = \Omega_{s_j} = \frac{4}{\sqrt{10}} \Omega, \quad j = 3, 4,$$

(31)

and the phases of the driving lasers

$$\phi_{u_n} = \frac{3}{2} \pi, \quad \phi_{s_n} = \frac{1}{2} \pi, \quad n = 1, 2,$$

$$\phi_{u_j} = \phi_{s_j} = \pi, \quad j = 3, 4.$$

(32)

At the time of $t = \tau$, we turn off the first set of lasers and send the second set of laser pulses of the Rabi frequencies

$$\Omega_{u_n} = \Omega_{u_n} = \sqrt{2} \Omega, \quad n = 1, 2,$$

$$\Omega_{u_j} = \Omega_{s_j} = 0, \quad j = 3, 4,$$

(33)

and the phases

$$\phi_{u_1} = \frac{3}{2} \pi, \quad \phi_{s_1} = \frac{1}{2} \pi, \quad \phi_{u_2} = \frac{1}{2} \pi, \quad \phi_{s_2} = \frac{3}{2} \pi.$$

(34)

At the time of $t = 2\tau$, we turn off the second set of laser pulses and send the third set of the Rabi frequencies

$$\Omega_{u_n} = \Omega_{u_n} = \frac{4}{\sqrt{10}} \Omega, \quad n = 1, 2,$$

$$\Omega_{u_j} = \Omega_{s_j} = \frac{2}{\sqrt{10}} \Omega, \quad j = 3, 4,$$

(35)

and the phases

$$\phi_{u_1} = \phi_{u_2} = \phi_{u_3} = \phi_{u_4} = \frac{3}{2} \pi,$$

$$\phi_{s_1} = \phi_{s_2} = \pi, \quad \phi_{s_3} = \phi_{s_4} = \frac{3}{2} \pi.$$

(36)

At the time $t = 3\tau$, we turn off the third set of lasers and send the fourth set of laser pulses of the Rabi frequencies

$$\Omega_{u_n} = \Omega_{s_n} = 0, \quad n = 1, 2,$$

$$\Omega_{u_j} = \Omega_{s_j} = 2 \Omega, \quad j = 3, 4,$$

(37)

and the phases

$$\phi_{u_n} = \phi_{s_n} = 0, \quad n = 1, 2,$$

$$\phi_{u_3} = \frac{3}{2} \pi, \quad \phi_{s_3} = \frac{1}{2} \pi, \quad \phi_{u_4} = \frac{1}{2} \pi, \quad \phi_{s_4} = \frac{3}{2} \pi.$$

(38)

After the above sequence of the laser pulses, the system is left in a state described by the density operator

$$\rho_1(4\tau) = T_2 |\psi_S\rangle \langle \psi_S| T_2^\dagger \otimes |0_a\rangle \langle 0_a|.$$  

(39)

Inverting the transformation $T_2$, we find that the system is in the state

$$|\psi_S\rangle = \exp \left\{ -\frac{\xi}{10} [c_1^2 + c_2^2 + c_3^2 + c_4^2 - 8c_1c_2 - 8c_3c_4 -4i(c_1 + c_2)(c_3 + c_4)] - \text{H.c.} \right\} |\{0_j\} \rangle.$$  

(40)

Finally, we calculate the variances in the sum and difference operators following the general procedure for quantifying the cluster states and find that the variances for the square type state illustrated in Fig. 2(b) are given by

$$V(p_1 - q_3 - q_4) = V(p_2 - q_3 - q_4) = V(p_3 - q_1 - q_2)$$

$$= V(p_4 - q_1 - q_2) = \frac{3}{2} e^{-2 \xi}.$$  

(41)

We see that the variances tend to zero when the squeezing parameter goes to infinity, $\xi \to \infty$. Hence the state $|\psi_S\rangle$ is an example of a four-mode square cluster state $|11\rangle$. We conclude that the sequential application of the laser pulses, the CV entangled quadripartite square cluster state $|\psi_S\rangle$ is unconditionally produced.

C. The preparation of a T-shape cluster state

Finally, we apply the above formalism to describe how to prepare a T-shape entangled cluster state of the form

$$|\psi_T\rangle = \exp \left\{ \frac{\xi}{4} [c_1^2 - c_2^2 - c_3^2 - c_4^2 + 2i(c_1 + c_3 + c_4)$$

$$+ 2(c_2c_3 + c_2c_4 + c_3c_4)] - \text{H.c.} \right\} |\{0_j\} \rangle.$$  

(42)

As above, we first perform a unitary transformation $d_{T_1} = T_3 c_j T_4^\dagger$ that transfers the bosonic operators $c_j$ into superposition modes

$$d_{T_1} = \frac{\sqrt{3}}{2} \left[ ic_1 - \frac{1}{3} (c_2 + c_3 + c_4) \right],$$

$$d_{T_2} = \frac{\sqrt{6}}{3} \left[ c_2 - \frac{1}{2} (c_3 + c_4) \right],$$

$$d_{T_3} = \frac{\sqrt{2}}{2} (c_3 - c_4),$$

$$d_{T_4} = \frac{1}{2} (ic_1 + c_2 + c_3 + c_4).$$  

(43)

We will show that the transformation $T_3$ leads to the field modes that can be prepared in a T-shape cluster state. Following the similar four-step procedure as in the above two examples, we first show that by a suitable choice of the Rabi frequencies and phases of the laser fields, one can prepare the modes $d_{T_1}$ in a squeezed vacuum state.
To achieve this, we choose the following sequence of the laser pulses. In the first step, the atomic ensemble is driven by pulse lasers over the time period of $t \in [0, \tau)$, with the Rabi frequencies

$$
\begin{align*}
\Omega_{u_1} &= \frac{\Omega_1}{r} = 0, \\
\Omega_{u_n} &= \frac{\Omega_n}{r} = \frac{\sqrt{3}}{3} \Omega, \quad n = 2, 3, 4, \\
\end{align*}
$$

and the phases

$$
\begin{align*}
\phi_{u_1} &= \frac{1}{2} \pi, \quad \phi_{u_2} = \frac{3}{2} \pi, \\
\phi_{u_n} &= \phi_{s_n} = \pi, \quad j = 2, 3, 4.
\end{align*}
$$

At the time of $t = \tau$, we turn off the first set of lasers and send a second set of the pulses, with the Rabi frequencies

$$
\begin{align*}
\Omega_{u_1} &= \Omega_{s_1} = 0, \\
\Omega_{u_2} &= \Omega_{s_2} = \frac{2 \sqrt{3}}{3} \Omega, \\
\Omega_{r_n} &= \frac{\Omega_n}{r} = \frac{\sqrt{3}}{3} \Omega, \quad j = 3, 4,
\end{align*}
$$

and the phases

$$
\begin{align*}
\phi_{u_1} &= \frac{3}{2} \pi, \quad \phi_{s_1} = \frac{1}{2} \pi, \quad \phi_{u_2} = \frac{1}{2} \pi, \quad \phi_{s_2} = \frac{3}{2} \pi.
\end{align*}
$$

The following sets of pulses are: At the time of $t = 2\tau$:

$$
\begin{align*}
\Omega_{u_n} &= \frac{\Omega_n}{r} = 0, \quad n = 1, 2, \\
\Omega_{u_j} &= \frac{\Omega_j}{r} = \sqrt{2} \Omega, \quad j = 3, 4, \\
\phi_{u_1} &= \phi_{s_1} = \phi_{u_2} = \phi_{s_2} = 0, \\
\phi_{u_j} &= \phi_{s_j} = \pi, \quad j = 3, 4.
\end{align*}
$$

At the time $t = 3\tau$

$$
\begin{align*}
\Omega_{u_n} &= \frac{\Omega_n}{r} = \Omega, \quad n = 1, 2, 3, 4, \\
\phi_{u_1} &= \frac{1}{2} \pi, \quad \phi_{s_1} = \frac{3}{2} \pi, \\
\phi_{u_j} &= \phi_{s_j} = 0, \quad j = 2, 3, 4.
\end{align*}
$$

After the above sequence of the laser pulses, the system is left in a state described by the density operator

$$
\rho_1(4\tau) = T_3 |\Psi_T\rangle\langle \Psi_T| T_3^\dagger \otimes |0_1\rangle\langle 0_1|.
$$

Inverting the transformation $T_3$, we find that the system is in the pure state $|\Psi_T\rangle$.

The only thing left is to determine as to whether the state $|\Psi_T\rangle$ belongs to the class of cluster states. It is done by calculating the variances of the sum and difference operators from which we find that for the state $|\psi_T\rangle$, the variances are given by

$$
\begin{align*}
V(p_1 - q_2 - q_3 - q_4) &= 2 e^{-2\xi}, \\
V(p_2 - q_1) &= V(p_3 - q_1) = V(p_4 - q_1) = e^{-2\xi}.
\end{align*}
$$

Clearly, the variances tend to zero when $\xi \to \infty$. Consequently, we may conclude that the state $|\Psi_T\rangle$ is an example of a four-mode continuous variable T-shape cluster state.

### IV. CONCLUSIONS

We have described a practical scheme for the preparation of entangled CV cluster states of effective bosonic modes realized in four physically separated atomic ensembles interacting collectively with a single-mode optical cavity and driving laser fields. We have demonstrated robustness of the scheme on three examples of the so-called continuous variable linear, square and T-type cluster states. The basic idea of the scheme is to transfer the ensemble field modes into suitable linear combinations that can be prepared, by a sequential application of the laser pulses, in pure squeezed vacuum states. We have shown, by referring to practical parameters, that the scheme is feasible with the current experiments.

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