Nonparametric Estimation of Conditional Expectation with Auxiliary Information and Dimension Reduction

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Abstract
Nonparametric estimation of the conditional expectation $E(Y|U)$ of an outcome $Y$ given a covariate vector $U$ is of primary importance in many statistical applications such as prediction and personalized medicine. In some problems, there is an additional auxiliary variable $Z$ in the training dataset used to construct estimators, but $Z$ is not available for future prediction or selecting patient treatment in personalized medicine. For example, in the training dataset longitudinal outcomes are observed, but only the last outcome $Y$ is concerned in the future prediction or analysis. The longitudinal outcomes other than the last point is then the variable $Z$ that is observed and related with both $Y$ and $U$. Previous work on how to make use of $Z$ in the estimation of $E(Y|U)$ mainly focused on using $Z$ in the construction of a linear function of $U$ to reduce covariate dimension for better estimation. Using $E(Y|U) = E\{E(Y|U,Z)|U\}$, we propose a two-step estimation of inner and outer expectations, respectively, with sufficient dimension reduction for kernel estimation in both steps. The information from $Z$ is utilized not only in dimension reduction, but also directly in the estimation. Because of the existence of different ways for dimension reduction, we construct two estimators that may improve the estimator without using $Z$. The improvements are shown in the convergence rate of estimators as the sample size increases to infinity as well as in the finite sample simulation performance. A real data analysis about the selection of mammography intervention is presented for illustration.

Key Words: Auxiliary information; Convergence rate; Kernel estimation; Sufficient dimension reduction; Two-step regression.
1 Introduction

In many statistical applications, a key step is to estimate the conditional expectation of $Y$ given $U = u_0$, denoted as $\psi(u_0) = E(Y|U = u_0)$ in what follows, based on a training sample dataset, where $Y$ is a response of interest, $U$ is a vector of covariates, and $u_0$ is a given specific value of $U$. Apparently, the prediction of a future $Y$ at $U = u_0$ is an example. Another example is in the area of personalized medicine in which we would like to maximize the condition expectation $E(Y|U = u_0, a)$ over several treatment options $a = 1, ..., k$ (Qian and Murphy [2011]), where $u_0$ is the vector of a future patient’s prognostic factors and demographic variables, and $Y$ is his or her future outcome. Larger (or smaller) $Y$ means better outcome. Because parametric modeling of $\psi(u_0)$ is difficult in many applications such as the personalized medicine problems, nonparametric kernel estimation of $\psi(u_0)$ (Nadaraya [1964]; Watson [1964]) has been widely considered and used. As shown in Theorem 2.2.2 of Bierens [1987], the optimal convergence rate of a kernel estimator is $n^{-m/(2m+p)}$, where $m$ is the order of kernel and $p$ is the dimension of $U$. When $p$ is not small, it is crucial to search for a matrix $B$ with the smallest possible column dimension $d_0 < p$ such that $E(Y|U) = E(Y|B^T U)$, where $B^T$ is the transpose of $B$, and hence the optimal convergence rate is improved to $n^{-m/(2m+d_0)}$. This is usually achieved by using the training data to estimate a $B$ with smallest column dimension such that $Y \perp \perp U|B^T U$, i.e., $Y$ and $U$ are independent conditional on $B^T U$, which is referred to as sufficient dimension reduction (SDR) (Li [1991]; Cook and Weisberg [1991]; Xia et al., [2002]; Li and Wang [2007]; Ma and Zhu [2012]). The linear space generated by the columns of $B$ is called the central subspace for $Y$ given $U$ and is denoted as $S(B) = S_{Y|U}$.

Besides $U$, in some problems there exists a vector $Z$ of auxiliary variables in the training sample dataset, but $Z$ is not available in the future. For example, in many clinical or observational studies, covariate $U$ and longitudinal responses $Y_1, ..., Y_T$ are observed in the training dataset, where $Y_t$ is the response at time $t$, but in the future, we may only observe $U = u_0$ to predict $Y = Y_T$ at time $T$ without the additional $Z = (Y_1, ..., Y_{T-1})$. In many situations, $Z$ may be more related with $Y$ than $U$. This raises an issue of how to make use of the available data in $Z$ in the training dataset to improve the estimation of $\psi(u_0)$.

Efforts have been made in utilizing $Z$ data in SDR. For a discrete $Z$ taking values $z_1, ..., z_L$, Chiaromonte et al. [2002] proposed to use $S_{Y|U} = S(Z_{z_1}) \oplus \cdots \oplus S(Z_{z_L})$, where $B_{z_i}$ has the smallest
column dimension such that \( Y \perp U \mid B_T^TZ = z \) and \( S_1 \oplus S_2 = \{ s_1 + s_2 : s_j \in S_j, j = 1, 2 \} \).

However, they cannot guarantee that \( S_{Y|U}^Z \) coincides with the central subspace \( S_{Y|U} = S(B) \), \( Y \perp U \mid B^TU \). Hung et al. (2015) proposed a two-stage method of searching \( B \) in a \( Z \)-envelope \( = S_{Y|U}^Z \oplus S_{Z|U} \supseteq S(B) \), where \( S_{Z|U} \) is the central subspace for \( Z \) given \( U \). Although their method utilizes \( Z \) data to produce a better \( B \) estimator, the resulting estimator of \( \psi(u_0) \) has the same convergence rate as the estimator based on an estimator of \( B \) without using \( Z \) data, because a better estimator of \( B \) does not improve the convergence rate of the estimator of \( \psi(u_0) \).

Instead, in this paper we propose an idea of using \( Z \) data in the estimation of \( \psi(u_0) \) directly, based on the following well known identity:

\[
\psi(u_0) = E(Y|U = u_0) = E \{ E(Y|Z, U = u_0)|U = u_0 \} \tag{1}
\]

We utilize the \( Z \) information in the estimation of inner expectation \( E(Y|Z, U) \) treating \( Z \) as a part of covariate as well as outer expectation \( E \{ \cdot | U = u_0 \} \) using the conditional distribution of \( Z \) given \( U \). SDR is applied in the kernel estimation of both expectations and is necessary because incorporating \( Z \) data increases the dimensions of kernels in kernel estimation.

We consider two ways of reducing dimensions, which lead to two different estimators of \( \psi(u_0) \). The first method performs SDR to find a matrix \( C_{zu} = \begin{pmatrix} C_z \\ C_u \end{pmatrix} \) so that the inner expectation is \( E(Y|Z, U) = E(Y|C_{zu}^T Z / U) = E(Y|C_z^T Z + C_u^T U) \), and then another SDR to find a matrix \( C \) with \( C_z^T Z \perp U \mid C^T U \) which implies \( E(Y|U) = E \{ E(Y|C_z^T Z + C_u^T U)|C^T U \} \). We show that the convergence rate of kernel estimator of \( \psi(u_0) \) using this method is \( n^{-m/(2m + d_1)} \) depending on \( d_1 \), the column dimension of \( C \), not the column dimension of \( C_{zu} \). Although the column dimension of \( C_{zu} \) does not affect the convergence rate, reducing \( (Z, U) \) to \( C_z^T Z + C_u^T U \) is still important for kernel estimation of the inner expectation.

However, it is not always true that \( d_1 \leq d_0 \), the dimension of the central subspace \( S_{Y|U} = S(B) \) without \( Z \). Thus, the estimator of \( \psi(u_0) \) based on the first method does not always improve the estimator without using \( Z \). To ensure obtaining an estimator with a convergence rate no slower than that of the estimator without using \( Z \) data, we propose the second method using SDR to find the following matrices: (i) a matrix \( B \) satisfying \( E(Y|U) = E(Y|B^TU) \); (ii) a matrix \( D_{zu} = \begin{pmatrix} D_z \\ D_u \end{pmatrix} \) satisfying \( E(Y|Z, B^TU) = E(Y|D_{zu}^T (B^TU) = E(Y|D_z^T Z + D_u^T B^TU) \); and (iii) a matrix \( D \) satisfying
$D_z^T Z \perp B^T U \mid D^T B^T U$. Then,

$$E(Y|U) = E(Y|B^T U) = E\{E(Y|D_z^T Z + D_u^T B^T U)|D^T B^T U}\}.$$

We show that the convergence rate of kernel estimator of $\psi(u_0)$ using this method is $n^{-m/(2m+d_2)}$ depending on $d_2$, the column dimension of $D$. By applying SDR, it is guaranteed that $d_2 \leq d_0$. In fact, $d_2 < d_0$ in many situations. For the first method, in some situations $d_1$ can be even smaller than $d_2 \leq d_0$, although it does not guarantee $d_1 \leq d_0$. See Examples 1-3 in Section 2.

Why can we improve the convergence rate in estimating $\psi(u_0)$? Without $Z$ data, the best we can do is to use the central subspace $S_{Y|U} = S(B)$ whose dimension determines the convergence rate. [Hung et al. (2015)] utilized $Z$ data to improve the estimation of $B$, but they could not improve the convergence rate. However, our approach is to use formula (1) and estimate $\psi(u_0)$ in two steps, the estimation of inner and outer expectations, with SDR in both steps. Because the convergence rate depends on the convergence rate of outer expectation estimation involving $Z$ given $U$, we may be able to make use of a space that is smaller than $S(B)$, e.g., the space generated by columns of $BD$ in our second method, which cannot be achieved without the inner expectation estimation involving $Z$ data.

Details of our proposed estimation procedures are given in Section 2 with three examples for illustration. In Section 3, we establish the asymptotic normality of proposed estimators under some regularity conditions, and obtain the optimal convergence rates and the asymptotic mean squared errors. Simulation studies under various circumstances are considered in Section 4. A real data analysis about the selection of mammography intervention methods is carried out in Section 5 to illustrate our procedure. All the technical proofs are given in the Appendix.

2 Methodology

Throughout we use $K_h$ as a generic notation for a kernel with an appropriate dimension and bandwidth $h$, i.e., $K_h$ appeared in different places may be different. Assumptions on the kernels are introduced in Section 3. Let $\{Y_i, U_i, Z_i, i = 1, ..., n\}$ be an independent and identically distributed training sample of size $n$ from $(Y, U, Z)$. Without using $Z$ data and dimension reduction, a kernel
regression estimator of \( \psi(u_0) \) defined in (1) is
\[
\hat{\psi}_p(u_0) = \frac{\sum_{i=1}^{n} Y_i K_h(U_i - u_0)}{\sum_{i=1}^{n} K_h(U_i - u_0)}
\]
where the subscript \( p \) indicates that \( \hat{\psi}_p \) uses a kernel with dimension \( p \), the dimension of \( U \).

Suppose that \( Y \perp U \mid B^T U \), where \( B \) has the smallest column dimension \( d_0 \leq p \). The estimator \( \hat{\psi}_p(u_0) \) can be improved by
\[
\hat{\psi}_{d_0}(u_0) = \frac{\sum_{i=1}^{n} Y_i K_h(\hat{B}^T U_i - \hat{B}^T u_0)}{\sum_{i=1}^{n} K_h(\hat{B}^T U_i - \hat{B}^T u_0)} \tag{2}
\]
where \( \hat{B} \) is an estimator of \( B \) by SDR.

To make use of the auxiliary information provided by \( Z \), we use identity (1) and first estimate the inner expectation \( E(Y|Z,U) \). Following the discussion in Section 1, we construct SDR estimators \( \hat{C}_z \) and \( \hat{C}_u \) of \( C_z \) and \( C_u \), respectively, with \( E(Y|Z,U) = E(Y|C_z^T Z + C_u^T U) \). Then \( E(Y|Z,U) \) can be estimated by
\[
\hat{\varphi}_1(Z,U) = \frac{\sum_{j=1}^{n} Y_j K_h(\hat{C}_z^T Z_j + \hat{C}_u^T U_j - \hat{C}_z^T Z - \hat{C}_u^T U)}{\sum_{j=1}^{n} K_h(\hat{C}_z^T Z_j + \hat{C}_u^T U_j - \hat{C}_z^T Z - \hat{C}_u^T U)} \tag{3}
\]
For the second step of estimating the outer expectation in (1), we construct an SDR estimator \( \hat{C} \) of \( C \) satisfying \( C_z^T Z \perp \perp U \mid C^T U \). Then our first proposed estimator of \( \psi(u_0) \) is
\[
\hat{\psi}_{d_1}(u_0) = \frac{\sum_{i=1}^{n} \hat{\varphi}_1(Z_i, u_0) K_h(C^T U_i - \hat{C}^T u_0)}{\sum_{i=1}^{n} K_h(C^T U_i - \hat{C}^T u_0)} \tag{4}
\]
where \( d_1 \) is the column dimension of \( C \), which is not necessarily smaller than \( d_0 \), the column dimension of \( B \). Thus, \( \hat{\psi}_{d_1}(u_0) \) is not always better than \( \hat{\psi}_{d_0}(u_0) \) in terms of convergence rate established in Section 3.

To derive an estimator having convergence rate no slower than that of \( \hat{\psi}_{d_0}(u_0) \), we build the improvement on \( (Y,Z,B^T U) \), instead of \( (Y,Z,U) \) in the derivation of \( \hat{\psi}_{d_1}(u_0) \), since \( Y \perp \perp U \mid B^T U \).

We still use (1) to do estimation in two steps. The first step is the same as the first step of constructing \( \hat{\psi}_{d_1}(u_0) \) in (4) except that \( U \) is replaced by \( B^T U \). That is, we construct SDR estimators \( \hat{D}_z \) and \( \hat{D}_u \) of \( D_z \) and \( D_u \), respectively, with \( E(Y|Z,B^T U) = E(Y|D_z^T Z + D_u^T B^T U) \), and estimate \( E(Y|Z,B^T U) \) by
\[
\hat{\varphi}_2(Z, B^T U) = \sum_{j=1}^{n} Y_j K_h(\hat{D}_z^T Z_j + \hat{D}_u^T B^T U_j - \hat{D}_z^T Z - \hat{D}_u^T B^T U)
\]
\[
\sum_{j=1}^{n} K_h(\hat{D}_j^T Z_j + \hat{D}_u^T B^T U_j - \hat{D}_j^T Z - \hat{D}_u^T B^T U)
\]

where \( \hat{B} \) is defined in \((2)\). For the second step, we construct an SDR estimator \( \hat{D} \) of \( D \) satisfying \( D_e^T Z \perp B^T U \mid D^T B^T U \). Then, our second proposed estimator of \( \psi(u_0) \) is

\[
\hat{\psi}_d(u_0) = \sum_{i=1}^{n} \hat{\varphi}_2(Z_i, \hat{B}^T u_0) K_h(\hat{D}^T \hat{B}^T U_i - \hat{D}^T \hat{B}^T u_0) / \sum_{i=1}^{n} K_h(\hat{D}^T \hat{B}^T U_i - \hat{D}^T \hat{B}^T u_0)
\]

where \( d_2 \) is the column dimension of \( D \). Note that \( d_2 \leq d_0 \), the column dimension of \( B \).

The next lemma shows the relationship among the spaces generated by \( B, C, \) and \( D \).

**Lemma 1.**

(i) If \( Y \perp U \mid B^T U, Y \perp (Z, U) \mid C^T Z + C^T U, \) and \( C^T Z \perp U \mid C^T U \), where \( B, C, C_u, \) and \( C \) all have the smallest possible column dimensions, then \( S(B) \subseteq S(C_u) \oplus S(C) \).

If, in addition, \( C^T Z \perp U | B^T U \), then \( S(C) \subseteq S(B) \).

(ii) If \( Y \perp U \mid B^T U, Y \perp (Z, B^T U) \mid D_e^T Z + D^T B^T U, \) and \( D_e^T Z \perp B^T U \mid D^T B^T U, \) where \( B, D_e, D_u, \) and \( D \) all have the smallest possible column dimensions, then \( S(B) = S(BD_u) \oplus S(BD) \).

The result in Lemma 1(i) says that \( C \) may contain column vectors that do not belong to \( S(B) \), unless \( C^T Z \perp U | B^T U \). In general, \( S(C) \) and \( S(B) \) may not have any relationship so that \( \hat{\psi}_{d_1} \) may be more or less efficient than \( \hat{\psi}_{d_0} \). On the other hand, Lemma 1(ii) indicates that \( D_u \) and \( D \) used in two steps together support the central subspace \( S(B) \), and \( \hat{\psi}_{d_2} \) is much more efficient than \( \hat{\psi}_{d_0} \) if \( S(BD) \) is truly contained in \( S(B) \), otherwise \( \hat{\psi}_{d_2} \) and \( \hat{\psi}_{d_0} \) have the same convergence rate.

Thus, \( \hat{\psi}_{d_2} \) is guaranteed to be more efficient than or as good as \( \hat{\psi}_{d_0} \) in terms of convergence rate. Regarding \( \hat{\psi}_{d_1} \) and \( \hat{\psi}_{d_2} \), there is no definite conclusion about their relative efficiency, since \( S(C) \) and \( S(D) \) do not have relationship.

Three examples are provided next for illustration on why and when the proposed estimator \( \hat{\psi}_{d_1} \) or \( \hat{\psi}_{d_2} \) is better than other estimators.

**Example 1.** Suppose that \( U \) consists of 4 components \( u_1, u_2, u_3, \) and \( u_4 \), and that random variables \( u_1, u_2, u_3, u_4, \eta_1, \eta_2, \eta_3, \) and \( \epsilon \) are mutually independent, and \( E(\epsilon) = 0 \). Assume also that \( Z \) has 3 components, \( z_1 = |u_1 - u_2| + \eta_1, z_2 = u_2 + \eta_2, \) and \( z_3 = u_1 + \eta_3, \) and that

\[
Y = (z_1 + 7u_4)(u_3^2 + 1)^{-1} + \epsilon = (|u_1 - u_2| + \eta_1 + 7u_4)(u_3^2 + 1)^{-1} + \epsilon.
\]
A straightforward calculation gives

\[
B = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad C_z = D_z = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}, \quad C_u = BD_u = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 1 \\
7 & 0
\end{pmatrix}, \quad C = BD = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

In this example, \( p = 4, d_0 = 3, \) and \( d_1 = d_2 = 1 \). When \( Z \) is not considered, \( \hat{\psi}_{d_0} \) improves \( \hat{\psi}_p \) since \( d_0 < p \). \( \hat{\psi}_{d_1} \) and \( \hat{\psi}_{d_2} \) are identical in this example, and they are more efficient than \( \hat{\psi}_{d_0} \) since \( d_1 = d_2 = 1 < d_0 = 3 \).

Here is an explanation on why our method improves \( \hat{\psi}_{d_0} \) in this particular example. In the first step of estimating the inner expectation in (1), \( Y \) is found to be related with two variables \( z_1 + 7u_4 \) and \( u_3 \); in the second step of estimating the outer expectation in (1), \( C_z^T Z = D_z^T Z = z_1 \) is found to be related with one variable \( u_1 - u_2 \). Thus, our approach “splits” the original task of estimating \( E(Y|U) \) with three variables into two tasks, estimating the inner expectation with two variables and estimating the outer expectation with one variable. It is shown in Section 3 that the convergence rate of \( \hat{\psi}_{d_1} \) or \( \hat{\psi}_{d_2} \) depends on the kernel estimation of the outer expectation and, consequently, this split produces an estimator with a faster convergence rate.

It is also interesting to notice that \( S(B) = S(C_u) \oplus S(C) \) in this example, i.e., the existence of \( Z \) splits \( S(B) \) into two orthogonal spaces and \( Z \) does not bring in any unwanted information outside of \( S(B) \).

**Example 2.** Consider the same setting as in Example 1 except that \( z_1 = -5(u_2 - \eta_1) + 0.1u_1u_3, \)
\( z_2 = 0.5|u_1| + \eta_2, \) \( z_3 = -3(|u_2| - \eta_3), \) and
\[
Y = z_1 - 0.1u_1u_3 - 3z_3 + |u_4| + 0.5\epsilon = -5(u_2 - \eta_1) + 9(|u_2| - \eta_3) + |u_4| + 0.5\epsilon.
\]

Then

\[
B = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}, \quad C_z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0
\end{pmatrix}, \quad C_u = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Without using \( Z \), \( \hat{\psi}_{d_0} \) involves \( u_2 \) and \( u_4 \) and \( d_0 = 2 \). Note that \( u_1u_3 \) is useful for \( Z \) but not for
Y. As the variable $u_1u_3$ outside the space of $S(B)$ is redundantly brought into the estimation, $\hat{\psi}_{d_1}$ uses four variables $z_1 - 3z_3$, $u_1$, $u_3$, and $u_4$ in the first step and three variables $u_1$, $u_2$, and $u_3$ in the second step. Thus, $d_1 = 3$ and $\hat{\psi}_{d_1}$ is even less efficient than $\hat{\psi}_{d_0}$.

On the other hand, the construction of $\hat{\psi}_{d_2}$ starts with $(Y, Z, B^T U)$ so that $u_1u_3$ is never in the picture, since $u_1u_3$ is independent of $(u_2, u_4)$. Note that

\[
D_z = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
-3 & 0
\end{pmatrix}, \\
D_u = \begin{pmatrix}
0 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix}, \\
D = \begin{pmatrix}
1 \\
0
\end{pmatrix}, \\
BD = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

because $D^T_z Z = z_1 - 3z_3 = -5u_2 + 0.1u_1u_3 + 9|u_2| + 5\eta_1 - 9\eta_3$ is only related to $u_2$ in $B^T U = (u_2, u_4)^T$, not $u_4$, in the second step of the estimation, and $u_1u_3$ is independent of $(u_2, u_4)$. Hence, $d_2 = 1 < d_0 = 2 < d_1 = 3$, and estimator $\hat{\psi}_{d_2}$ outperforms all other estimators in this example.

Again, $\hat{\psi}_{d_2}$ splits the task of estimating $E(Y|U)$ into two steps, with two variables $z_1 - 3z_3$ and $u_4$ in the first step and one variable $u_2$ in the second step.

**Example 3.** This is an example in which $\hat{\psi}_{d_1}$ beats $\hat{\psi}_{d_2}$ and $\hat{\psi}_{d_0}$. Consider the same setting as in Example 1 except that $z_1 = 2(-u_1 + u_4) + \eta_1$, $z_2 = (-u_1 + u_4) + \eta_2$, $z_3 = u_4 + \eta_3$, and

\[
Y = z_1 - z_2 + u_1 + u_2 + (u_1 - u_3)^2 + \epsilon = u_2 + u_4 + (u_1 - u_3)^2 + \eta_1 - \eta_2 + \epsilon
\]

In the first step of $\hat{\psi}_{d_1}$, $Y$ is related with two variables $z_1 - z_2 + u_1 + u_2$ and $u_1 - u_3$, and in the second step, $C^T_z Z = z_1 - z_2$ is a function of one variable $-u_1 + u_4$. Hence

\[
B = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & -1 \\
1 & 0
\end{pmatrix}, \\
C_z = \begin{pmatrix}
1 & 0 \\
-1 & 0 \\
0 & 0
\end{pmatrix}, \\
C_u = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & -1
\end{pmatrix}, \\
C = \begin{pmatrix}
-1 \\
0 \\
0 \\
1
\end{pmatrix}
\]

and $d_1 = 1 < d_0 = 2$.

For $\hat{\psi}_{d_2}$, we search directions in $(Z, B^T U)$ in the first step. Note that $Y$ is related to $z_1 - z_2 + u_1 + u_2$ and $u_1 - u_3$. Although $u_1 - u_3$ is exactly the second component of $B^T U$, one cannot express
$z_1 - z_2 + u_1 + u_2$ as a linear function of $Z$ and $B^T U$. Hence,

$$D_z = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad D_u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In the second step, $D_z^T Z = z_1 - z_2 = -u_1 + u_4 + \eta_1 - \eta_2$. Still, one cannot find a linear function of $B^T U$ to represent the vector $(-1 \ 0 \ 0 \ 1)^T$ related to $-u_1 + u_4$. Thus, we have to use both column vectors of $B$ despite $-u_1 + u_4$ is one dimensional. Then $D$ is the identity of order 2 and $BD = B$, i.e., in the second step we cannot reduce the dimension of $B^T U$ any more. Since $d_2 = 2 > d_1 = 1$, $\hat{\psi}_{d_1}$ turns out to be better than $\hat{\psi}_{d_2}$, and $\hat{\psi}_{d_2}$ has the same convergence rate as $\hat{\psi}_{d_0}$.

This example shows that restricting to $B^T U$ may prevent us to find the best direction in $B^T U$ for the outer expectation estimation, although it guarantees that at least we use $B^T U$ so that the resulting estimator is at least as good as $\hat{\psi}_{d_0}$.

### 3 Asymptotic Properties

This section is dedicated to the asymptotic properties of the estimators of $\psi(u_0)$ formulated in Section 2. It is clear that the asymptotic properties of $\hat{\psi}_{d_0}(u_0)$, $\hat{\psi}_{d_1}(u_0)$, and $\hat{\psi}_{d_2}(u_0)$ depend on the asymptotic convergence rates of the SDR estimators $\hat{B}$, $\hat{C}_z$, $\hat{C}_u$, $\hat{C}$, $\hat{D}_z$, $\hat{D}_u$, and $\hat{D}$ in (2)-(5).

Under reasonable conditions, it is proved in [Ma and Zhu (2012)] that SDR estimators converge at the rate $n^{-1/2}$, which is assumed throughout this paper.

In the beginning of Section 2, we introduced a generic notation $K_h$ for a kernel with bandwidth $h$. In what follows $K_h$ is chosen to be a product kernel of dimension $s$ and order $m \geq 2$ in the sense that $K_h(x) = h^{-s} \prod_{j=1}^{s} \kappa(x_j/h)$, where $x_j$ is the $j$th component of the $s$-dimensional $x$ and $\kappa(\cdot)$ is a bounded and Lipschitz continuous univariate kernel having a compact support and satisfying $\int \kappa(t)dt = 1$, $\int t^m \kappa(t)dt$ is finite and nonzero, and $\int t^l \kappa(t)dt = 0$ for all $0 < l < m$.

For $\hat{\psi}_{d_1}$, let $V = C_z^T Z + C_u^T U$, $\gamma(v)$ be the two dimensional vector whose components are $\gamma_1(v) = f_V(v)$ and $\gamma_2(v) = E(Y|V = v) f_V(v)$, and $\tilde{\gamma}(v)$ be the two dimensional vector whose components are $\tilde{\gamma}_1(v) = n^{-1} \sum_{j=1}^{n} K_h(C_z^T Z_j + C_u^T U_j - v)$ and $\tilde{\gamma}_2(v) = n^{-1} \sum_{j=1}^{n} Y_j K_h(C_z^T Z_j + C_u^T U_j - v)$. For $\hat{\psi}_{d_2}$, $V$ takes the form $V = D_z^T Z + D_u^T B^T U$ and $\tilde{\gamma}_i(v)$ is defined with $C_z^T Z_j + C_u^T U_j$ replaced by $D_z^T Z_j + D_u^T B^T U_j$. 

9
Throughout, we use $f_X(\cdot)$ to denote the probability density of a random vector $X$.

**Assumption 1.** The density $f_Y$ is bounded below from zero, i.e., there is a constant $c > 0$ such that $\inf_v f_Y(v) \geq c$.

We state the following assumptions for $\hat{\psi}_{d_1}$. For $\hat{\psi}_{d_2}$, the assumptions should be modified as in the statement of Theorem 1. To simplify expressions in assumptions and theorem, for both $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$, we use the same notation $d$, $h$ and $\bar{m}$ to denote the dimension, bandwidth, and order of the kernel used in the inner expectation estimation, and $d$, $h$ and $m$ to denote the dimension, bandwidth, and order of the kernel used in the outer expectation estimation.

**Assumption 2.** The function $\gamma(v)$ has bounded $\bar{m}$th derivative. The kernel bandwidth $h$ of the first step is of the order $n^{-\bar{s}}$ and there exists a $q > 1$ such that $\bar{s} < (1 - q^{-1})d^{-1}$ and the function $E[(1 + Y^2)^q|V = v]$ is bounded.

Assumptions 1 and 2 are similar to those in [Newey (1994)] and [Hansen (2008)], which ensure the convergence of $\hat{\gamma}(v)$ to $\gamma(v)$ uniformly in $v$. Specifically, $||\hat{\gamma}(v) - E[\gamma(v)]||_{\infty} = O_p(a_n)$ and $||E[\hat{\gamma}(v)] - \gamma(v)||_{\infty} = O_p(h^m)$, where $a_n = (\log n/nh^d)^{1/2}$ and $|| \cdot ||_{\infty}$ is the sup-norm; hence, $||\hat{\gamma}(v) - \gamma(v)||_{\infty} = O_p(a_n + h^m)$.

**Assumption 3.** Let $\hat{V} = \hat{C}_T Z + \hat{C}_T U$, and $\Omega = \{ (\hat{\gamma}, \hat{C}_z, \hat{C}) : ||\hat{\gamma}(v) - \gamma(v)||_{\infty} \leq \epsilon, ||\hat{C}_z - C_z|| \leq cn^{-1/2}, ||\hat{C} - C|| \leq cn^{-1/2} \}$ for some positive constants $c$ and $\epsilon > 0$.

(i) Uniformly on $\Omega$, the $\bar{m}$th derivative of $f_{\hat{C}}(\hat{v})$ and $E(Y|\hat{V} = \hat{v})f_{\hat{C}}(\hat{v})$ are Lipschitz-continuous functions of $\hat{v}$, the $m$th derivatives of $E[\hat{\gamma}_2(\hat{V})/\hat{\gamma}_1(\hat{V})]\hat{C}_T U = t|f_{\hat{C}_T U}(t)$ and $f_{\hat{C}_T U}(t)$ are Lipschitz-continuous functions of $t$, and $E[\hat{\gamma}_2(\hat{V})/\hat{\gamma}_1(\hat{V})]\hat{C}_T U = t$ and $E(Y^2|\hat{V} = \hat{v})$ are Lipschitz-continuous as functions of $t$ and $\hat{v}$ respectively.

(ii) Uniformly on $\Omega$, $E(Y^2|\hat{V} = \hat{C}_T Z + \hat{C}_T u)$ and $E[\hat{\gamma}_2(\hat{V})/\hat{\gamma}_1(\hat{V})]\hat{C}_T U = \hat{C}_T U$ are bounded.

(iii) Uniformly on $\Omega$, $E[\hat{\gamma}_2(\hat{V})/\hat{\gamma}_1(\hat{V})]\hat{C}_T U = \hat{C}_T U_0]f_{\hat{C}_T U}(\hat{C}_T U_0)$ and $f_{\hat{C}_T U}(\hat{C}_T U_0)$ are Lipschitz continuous functions of $\hat{C}$, and $E[|\partial E(Y|\hat{C}_T Z + \hat{C}_T U_0)/\partial \hat{C}_z u||C_T U = C_T U]$ is bounded.

(iv) $E(Y|C_T U = C_T U, C_T U = C_T U)$ is bounded.
Assumption 4. The kernel bandwidth \( h \) of the first step is of the order \( n^{-\frac{s}{d}} \) with \( s \) satisfying 
\[ m^{-1}m(2m + d)^{-1} < s < \min\{(m + d)(2m + d)^{-1}(1 + d)^{-1}, m(2m + d)^{-1}d^{-1}\}; \] 
the kernel bandwidth in the second step is \( h = \lambda^{2/(2m+d)}n^{-1/(2m+d)} \) with \( 2m > d \) and a constant \( \lambda > 0 \).

Assumptions 3 and 4 impose some constraints on the orders and bandwidths of the kernels in our two step estimation.

Assumptions 1, 3(i)-(iii) and 4 are assumed to ensure that the estimation errors of SDR estimators \( (\hat{C}_{zu}, \hat{C}) \) are asymptotically negligible \cite{Ma and Zhu 2012}. The proof can be found in Lemma 3 in the Appendix. Assumptions 4(v) and 4 are standard for the asymptotic normality of nonparametric kernel estimator \cite{Bierens 1987}.

The following result establishes the asymptotic normality as well as the convergence rates of \( \hat{\psi}_{d_1}(u_0) \) and \( \hat{\psi}_{d_2}(u_0) \) defined in (3)-(5) with a fixed \( u_0 \).

**Theorem 1.**

(i) If Assumptions 1, 4 hold with \( d = d_1 \), then there exists a function \( b(C^T u) \) such that
\[
n^{m/(2m+d_1)} \left[ \hat{\psi}_{d_1}(u_0) - \psi(u_0) \right] \xrightarrow{d} N \left( \frac{\lambda b(C^T u_0)}{\int_{C^T u}(C^T u_0)} , \frac{g(C^T u_0)}{\int_{C^T u}(C^T u_0)} \right) \int K^2(t) \, dt \]  
(6)

where \( \lambda \) is given in the bandwidth (Assumption 4), \( \xrightarrow{d} \) denotes convergence in distribution, and 
\( g(C^T u) = \text{Var} \{ E(Y|V = C^T_z Z + C^T_u u_0) | C^T U = C^T u \} \).

(ii) If Assumptions 1, 4 hold with \( d = d_2 \) and \( C_z, C_u \), and \( C \) replaced by \( D_z, BD_u, \) and \( BD \), respectively, then (6) holds with \( d_1, C_z, C_u, \) and \( C \) replaced by \( d_2, D_z, BD_u, \) and \( BD \), respectively.

The convergence rate \( n^{-m/(2m+d)} \) shown in (6) is the optimal convergence rate for \( \hat{\psi}_d(u_0) \), \( d = d_1 \) or \( d_2 \). Since \( \lambda > 0 \), the asymptotic bias of \( \hat{\psi}_d(u_0) \) has the same order as the asymptotic variance of \( \hat{\psi}_d(u_0) \) and, hence, we should consider asymptotic mean squared error. If we choose the bandwidth \( h \) to be \( o(n^{-1/(2m+d)}) \), then (6) holds with \( \lambda \) replaced by 0, but the convergence rate of the resulting estimator \( \hat{\psi}_d(u_0) \) is slower than \( n^{-m/(2m+d)} \).

Following Bierens \cite{1987}, \( \hat{\psi}_{d_0} \) defined in (2) is also asymptotically normal with convergence rate \( n^{-m/(2m+d_0)} \) if we use the same kernel order \( m \) as in the second step of \( \hat{\psi}_{d_1} \) and \( \hat{\psi}_{d_2} \). Together with Theorem 1, we conclude that the convergence rate of \( \hat{\psi}_{d_j}(u_0) \) is \( n^{-m/(2m+d_j)} \), \( j = 0, 1, 2 \), and we
can compare the three estimators by comparing $d_j$'s: the higher the dimension $d_j$ of the covariate vector used in kernel estimation of the last step, the slower the convergence rate.

By Theorem 1, the dimension $d$ in $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$ does not have a direct influence on the convergence rate. But some conditions on $d$, $h$, and $\bar{m}$ in Assumptions 2 and 4 still need to be satisfied to guarantee the asymptotic normality of the estimator. A high order $\bar{m}$ may be needed when $d$ is large. Similarly, a high order $m$ may be needed when $d_1$ or $d_2$ is large.

To end this section we provide a discussion on Assumption 1, which requires that the density $f_V(v)$ is bounded away from zero. It is a technical condition, and is sufficient but not necessary, i.e., without Assumption 1, $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$ may still perform well (see the simulation results in the next section). On the other hand, we may use the following transformation method when Assumption 1 is a concern. Note that $E(Y|V) = E(Y|\varphi(V))$ for any invertible function $\varphi$. Thus, we may use $\varphi(V)$ if the density $f_{\varphi(V)}$ is bounded away from 0. Let $\mu_V$ and $\Sigma_V$ be the mean vector and covariance matrix of $V$, respectively, and $V_S = \Sigma_V^{-1/2}(V - \mu_V)$ be the standardized $V$. We consider transformation $\varphi(V) = (\Psi_1(V_{S1}), ..., \Psi_d(V_{Sd}))^T$, where $V_{Sj}$ is the $j$th component of $V_S$ and $\Psi_j$ is a known distribution function. If $V$ is normally distributed, then a perfect choice of $\Psi_j$ is the cdf of standard normal distribution. Otherwise, we choose $\Psi_j$ to be the empirical distribution based on the $n V_{Sj}$ observations. Since $\mu_V$ and $\Sigma_V$ are unknown, they also have to be estimated using $V$ data. This method is examined in the simulation in Section 4.

4 Simulation Studies

In this section, we present simulation results to evaluate the performance of $\hat{\psi}_p$, $\hat{\psi}_{d_0}$, $\hat{\psi}_{d_1}$, and $\hat{\psi}_{d_2}$, when the sample size is $n = 200$. The estimators are evaluated at 8 different values of $E(Y|U = u_0)$ with randomly generated $u_0$'s. A second-order Epanechnikov kernel is adopted for all estimators. For the first step of $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$, bandwidth is denoted as $h$. For estimator $\hat{\psi}_p$ and $\hat{\psi}_{d_0}$ and the second step of $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$, bandwidths are of the form $h = Cn^{-1/(4+d)}$, where $d$ is $p$, $d_0$, $d_1$, or $d_2$. The values of $h$ and $C$ are selected using 10-fold cross-validation. For SDR, the semiparametric principal Hessian directions method proposed by Ma and Zhu (2012) is used.

In the first simulation study, we examine the relative performance of different estimators in the ideal situation where we know the column dimensions of matrices $B$, $C$, $D$, $C_{zu}$, and $D_{zu}$. The
situation where these dimensions are unknown is considered in the third simulation study. The following four settings are considered.

(A1) The model is given by Example 1, where four components of $U$ are uniformly distributed with lower bounds $-1, -3, -10,$ and 8, and upper bounds $7, -1, -2,$ and 18; three components of $\eta$ are uniformly distributed with lower bounds 0, 0, and 0 and upper bounds 2, 3, and 5; and $\epsilon \sim N(0, 1)$.

(A2) The model is given by Example 2, where four components of $U$ are uniformly distributed with lower bounds $3, 0, -5,$ and 8, and upper bounds $5, 9, -2,$ and 18; three components of $\eta$ are uniformly distributed with lower bounds 0, 0, and 0 and upper bounds 7, 3, and 5; and $\epsilon \sim N(0, 1)$.

(A3) The model is given by Example 3, where four components of $U$ are uniformly distributed with lower bounds $-1, -3, -10,$ and 2, and upper bounds $3, -1, -2,$ and 4; three components of $\eta$ are uniformly distributed with lower bounds 0, 0, and 0 and upper bounds 7, 3, and 5; and $\epsilon \sim N(0, 1)$.

We may replace $\hat{B}$ in $\hat{\psi}_{d_0}$ by the proposed SDR method in Hung et al. [2015] that first constructs a $Z$-envelope $\mathcal{S}_Y \supseteq \mathcal{S}_Y | U = \mathcal{S}(B)$ and then estimates $B$ in the $Z$-envelope. Let $\tilde{\psi}_{d_0}$ denote the resulting estimator of $\psi(u_0)$. Although $\tilde{\psi}_{d_0}$ does not improve $\hat{\psi}_{d_0}$ in terms of convergence rate, $\tilde{\psi}_{d_0}$ may have a better finite sample performance than $\hat{\psi}_{d_0}$ because the auxiliary $Z$ information is used in estimating $B$ through the $Z$-envelope. However, it can be shown that in settings (A1)-(A3), the $Z$-envelope is the whole space $\mathcal{R}^4$ and, hence, $\tilde{\psi}_{d_0} = \hat{\psi}_{d_0}$ as discussed by Hung et al. [2015]. To see whether $\tilde{\psi}_{d_0}$ improves $\hat{\psi}_{d_0}$, we consider another setting as follows:

(A1') The model, $Y$, and $U$ are the same as those in (A1), but $Z = z_1 = |u_1 - u_2| + \eta_1$ is one dimensional.

Under setting (A1'), the $Z$-envelope is 3-dimensional and

$$
\mathcal{S}_Y^Z | U = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}, \quad \mathcal{S}_Z | U = \begin{pmatrix}
1 \\
-1 \\
0 \\
0
\end{pmatrix}, \quad \text{Z-envelope} = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$
Simulation results with 1,000 runs under these four settings are reported in Table 1, which contains the absolute value of bias and root-mean-squared error (RMSE) of different estimators. It can be seen from Table 1 that the performance of various estimators supports our theory. In summary, the results in Table 1 indicate that, in terms of RMSE, (i) \( \hat{\psi}_{d_0} \) is better than \( \hat{\psi}_p \) due to dimension reduction from \( p \) to \( d_0 \); (ii) in setting (A1), \( d_1 < d_0 \) and \( d_2 < d_0 \), and our proposed estimators \( \hat{\psi}_{d_1} \) and \( \hat{\psi}_{d_2} \) are better than \( \hat{\psi}_{d_0} \); (iii) in setting (A2), \( d_2 < d_0 < d_1 \), and \( \hat{\psi}_{d_1} \) is better than \( \hat{\psi}_{d_0} \) but \( \hat{\psi}_{d_2} \) is worse than \( \hat{\psi}_{d_0} \); (iv) in setting (A3), \( d_1 < d_2 = d_0 \), and \( \hat{\psi}_{d_1} \) is better than \( \hat{\psi}_{d_0} \) and \( \hat{\psi}_{d_2} \) is comparable to \( \hat{\psi}_{d_0} \) except for the cases where \( E(Y \mid U = u_0) = 137.7 \) and 138.8; (v) in setting (A1'), \( \tilde{\psi}_{d_0} \) is only slightly better than \( \hat{\psi}_{d_0} \) for some cases and is still worse than \( \hat{\psi}_{d_1} \) or \( \hat{\psi}_{d_2} \).

In the second simulation study we would like to examine the effect of covariate densities not bounded away from 0 and the use of transformation discussed in the end of Section 3. We consider the following setting:

\( Y = 2z_3(z_1 + u_3) + \epsilon = 2(u_1 + \eta_3)(u_1 + u_3 + \eta_1) + \epsilon, \ z_1 = u_1 + \eta_1, \ z_2 = u_2 + \eta_2, \ z_3 = u_1 + \eta_3, \)

\( U \sim N(\mu, \Sigma), \eta \sim N(0, \Sigma_\eta), \) and \( \epsilon \sim N(0, 3) \), where \( \mu = (3, -2, -6, 3)^T \),

\[
\Sigma = \begin{pmatrix}
1 & 0 & 0.2 & 0 \\
0 & 0.3 & 0 & 0 \\
0.2 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\Sigma_\eta = \begin{pmatrix}
3 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

Under setting (B),

\[
B = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}, \quad
C_z = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}, \quad
C_u = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0
\end{pmatrix}, \quad
C = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad
C_u = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\]

\[
D_z = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}, \quad
D_u = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}, \quad
D = \begin{pmatrix}
1 \\
0
\end{pmatrix}, \quad
BD = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

Note that \( d_1 = d_2 = 1 < d_0 = 2 < p = 4 \). We still assume that the dimensions of these matrices
are known. We apply the transformation discussed in the end of Section 3 with $\Psi_j$ being either the standard normal or the empirical distribution. The resulting estimators are denoted by $\hat{\psi}_d^N$ and $\hat{\psi}_d^E$, respectively, $d = d_1$ or $d_2$.

From the results in Table 2, $\hat{\psi}_{d_2}$ has larger RMSE than $\hat{\psi}_{d_1}$, which may be true in general when $d_1 = d_2 = 1$ because when $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$ have the same convergence rate, $\hat{\psi}_{d_2}$ may have worse finite sample performance as it requires an additional application of SDR. Even this is the case, $\hat{\psi}_{d_2}$ still outperforms $\hat{\psi}_0$ and $\hat{\psi}_p$. The difference between $\hat{\psi}_d^N$ and $\hat{\psi}_d^E$ based on two transformation methods is small, indicating that the use of empirical distribution is adequate. Since $\hat{\psi}_d^N$ and $\hat{\psi}_d^E$ are comparable with $\hat{\psi}_d$, the estimator without covariate transformation, the results show that Assumption 1 is not necessary for better performance of $\hat{\psi}_d$ over $\hat{\psi}_0$ and $\hat{\psi}_p$.

So far the dimensions of matrices $B$, $C_{zu}$, $C$, $D_{zu}$, and $D$ are assumed known. In the third simulation study, we estimate these dimensions using a bootstrap procedure described by Dong and Li (2010) and recommended by Ma and Zhu (2012), with bootstrap Monte Carlo size 30. Settings (A1), (A2), and (B) are considered. Under setting (B), only results with $\hat{\psi}_d^N$ are reported since $\hat{\psi}_d$ and $\hat{\psi}_d^N$ are similar. Simulation results are shown in Table 3.

It can be seen from Table 3 that, in terms of RMSE, $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$ perform worse than those obtained using the true dimensions of matrices (Tables 1-2). But they are still better than $\hat{\psi}_0$ except for $\hat{\psi}_{d_2}$ in three cases under setting (B). Note that estimation of dimensions of subspaces is a difficult topic in the literature of SDR. More accurate estimators of the dimensions of matrices in SDR will result in better performance of our estimators $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$.

5 Data Analysis

Breast cancer has been taking a toll on the lives of women. The good news is that regular mammography screening can help to reduce mortality. Champion et al. (2014) did a Computer and Phone (CAPE) study including two tailored intervention methods, mailed DVD (abbreviated as DVD) and telephone counseling (abbreviated as TC). A CAPE randomized controlled trial was conducted to determine whether the two interventions were more efficacious than the usual care (abbreviated as UC) method at promoting mammography screening among women who are non-adherent to breast cancer screening guidelines at baseline. If the answer is yes, then we are further interested in which
of DVD, TC, and UC methods is more efficacious at promoting mammography screening for women with a particular set of demographic values. This involves estimation of \( E(Y|U = u_0, a) \) for fixed \( u_0 \) and \( a \) as well as \( \mu(a) = E(Y|a) = E\{E(Y|U, a)\} \), where \( Y \) is an outcome of interest, \( U \) is a vector of demographic variables, and \( a = DVD, TC, \) and UC corresponding to mailed DVD, telephone counseling, and usual care, respectively, which is treated as a treatment indicator.

In the CAPE dataset, there are 26 demographic variables such as age, years of education, and household income, collected at baseline of the study. The outcome we consider is perceived barriers, one of the health belief variables related with health behaviors according to the Health Belief Model. The variable of perceived barriers is the sum of grades (typically 1-5) to questions such as “I am afraid of finding out that I might have breast cancer”, “the treatment for breast cancer is worse than the cancer itself”, “having a mammogram is painful for me”, “I don’t have the time to get a mammogram”, etc. Other belief variables include perceived risk, perceived benefits, self-efficacy, breast cancer fear, and fatalism. We focus on perceived barriers for illustration.

The outcome of perceived barriers in the CAPE dataset is actually longitudinal and observed at baseline, one month after baseline, and six months after baseline. We are interested in the outcome of perceived barriers after six months from the time an individual is assigned to one of DVD, TC, and UC. Thus, \( Y = bar3tot \) is the score of perceived barriers at month six after baseline and the scores of perceived barriers at baseline (bar1tot) and at month one after baseline (bar2tot) are two components of \( Z \) that are closely related with \( Y \) but not available in the future prediction.

After eliminating units with missing data, the training dataset for our analysis contains 357, 434, and 423 sampled units for the DVD, TC, and UC methods, respectively.

Note that 26 demographic variables (covariates) are too many even for SDR. Thus, we follow the idea in [Mai and Zou (2015)]#MaiZou2015 that applies fused Kolmogorov filter to screen out some demographic variables not useful in predicting \( Y \). A fused Kolmogorov filter statistic that measures the dependence between a certain covariate \( X_j \) and the continuous response variable \( Y \) is defined as

\[
\hat{K}_j = \sum_{i=1}^{N} \hat{K}_j^{G_i}
\]

with \( \hat{K}_j^{G_i} = \max_{l,m} \sup_x |\hat{F}_j(x|H^i = l) - \hat{F}_j(x|H^i = m)| \). \( G_i \) is a uniform partition of \( Y \) with \( G_i \) slices containing the intervals bounded by the \( l/G_i \)th sample quantiles of \( Y \) for \( l = 0, ..., G_i \), and
Hi = l if Y is in the lth slice. $\hat{F}_j(x|H^i)$ is the empirical CDF of $X_j$ conditional on $H^i$. $N$ is the total number of different partitions. We pick uniform partitions $G_i$s with $G_i = 3, 4, \ldots, [\log n]$ and calculate fused Kolmogorov filter statistics for each variable $X_j$ in $bar1tot$, $bar2tot$ and all demographic variables under DVD, TC and UC methods. Results are presented in Figure 1. A higher fused Kolmogorov filter statistic indicates a stronger relationship between a covariate and $Y$. It coincides with our instinct that the two components of $Z$, $bar1tot$ and $bar2tot$, are the best predictors of $Y$ in all the three sub-datasets, as their fused Kolmogorov filter statistics are much greater than those for the demographic variables. For the sub-dataset under DVD, we keep 5 demographic variables with the biggest fused Kolmogorov filter statistics next to $bar1tot$ and $bar2tot$ and treat them as $U$, since there is a sudden decrease in fused Kolmogorov filter statistics at the 6th demographic variable SF12RP1. As suggested in Figure 1, variable $U$ under DVD contains “income3”, “educyrs”, “yearmamsum”, “SF12GH1” and “age”, which represent “household income”, “years of education”, “number of years had a mammogram in the past 2 to 5 years”, “SF12 general health scale score” and “age”, respectively. For the other two sub-datasets under TC and UC, for simplicity we just keep the 5 demographic variables next to $bar1tot$ and $bar2tot$, although these variables may be different from those under DVD. The selected $U$ variables “hcreminder”, “SF12VT1” and “SF12MH1” under TC or UC represent “whether or not received any reminders from your health care facility that it was time for you to have a mammogram”, “SF12 vitality scale score” and “SF12 mental health scale score”.

First, we would like to examine whether DVD and TC are more efficacious than UC at promoting mammography screening. Note that this can be done using $Y$ data only, i.e., the two sample t-tests based on sample means and variances. The results from the two-sample t-tests, however, show that there is no significant difference among the three methods, i.e., the p-values for rejecting $\mu(DVD) = \mu(UC)$, $\mu(TC) = \mu(UC)$, and $\mu(DVD) = \mu(TC)$ are 0.87, 0.33, and 0.43, respectively. The insufficiency results may be due to large variability in $Y$ data. If we make use of covariates, the results may be different.

Under each DVD, TC, and UC, we compute estimators $\hat{\psi}_p$, $\hat{\psi}_{d0}$, $\hat{\psi}_{d1}$, and $\hat{\psi}_{d2}$ without covariate transformation, using the procedures given in Section 2. Dimensions of matrices for using SDR estimated by the bootstrap as described in the simulation are given as follows.
Boxplots of values of $Y_i$ and $\hat{\psi}_d(U_i)$ with $d = p, d_0, d_1,$ or $d_2$ are shown in Figure 2. It can be seen that clearly $Y_i$ without using any covariate has much larger variability than $\hat{\psi}_d(U_i)$’s using covariate information, and $\hat{\psi}_p(U_i)$ has the largest variability among the four estimators using covariates. For DVD group, $\hat{\psi}_{d_2}$ has the least variability but $\hat{\psi}_{d_0}$ is not too bad. For TC, $\hat{\psi}_{d_1}$ is the best. For UC, $\hat{\psi}_{d_1}$ and $\hat{\psi}_{d_2}$ are comparable and are much less variable than $\hat{\psi}_{d_0}$.

Note that we can estimate $\mu(a) = E\{E(Y|U, a)\}$ using the average of $\hat{\psi}_d(U_i)$’s with $U_i$’s in each method group and $d = p, d_0, d_1,$ or $d_2$. Using $\hat{\psi}_d(U_i)$’s and 10,000 random permutations, we obtain p-values for testing various hypotheses based on $d = p, d_0, d_1,$ or $d_2$. The results are shown in Table 4. The reason we also consider one-sided tests is because the method with smaller $\mu(a)$ is better at promoting mammography screening.

The results in Table 4 show that TC is better than either UC or DVD with high significance when $\hat{\psi}_{d_0}$, $\hat{\psi}_{d_1}$, or $\hat{\psi}_{d_2}$ is used, but not $\hat{\psi}_p$. Thus, applying SDR is beneficial in this example. Also, all methods cannot detect any difference between DVD and UC.

The previous analysis shows some advantages of using the proposed $\hat{\psi}_{d_1}$ and/or $\hat{\psi}_{d_2}$, but the accuracy of estimators of $E(Y|U = u_0, a)$ has not been investigated. Different from the simulation study, the true value of $E(Y|U = u_0, a)$ is unknown in the real data analysis. Thus, we apply the following cross-validation to assess the accuracy for different estimation methods.

The following discussion is for a fixed $a = DVD, TC,$ or UC. We divide the dataset into 10 subsets with roughly the same sample size, say $n_c$. Let $S_c$ be one such subset, $c = 1, ..., 10$. We use data not in $S_c$ but in other subsets to obtain the estimator $\hat{\psi}_d^{(c)}$, where subscript $(c)$ indicates using data not in $S_c$ and $d = p, d_0, d_1,$ or $d_2$. Then, we assess the accuracy by $[Y_i - \hat{\psi}_d^{(c)}(U_i)]^2$ for $i \in S_c$, noting that $(Y_i, U_i), i \in S_c$, are not used in the construction of $\hat{\psi}_d^{(c)}$. After crossover all $S_c$, we estimate $E[Y - \hat{\psi}_d(U)]^2$ by

$$CV(10) = \frac{1}{10} \sum_{c=1}^{10} \frac{1}{n_c} \sum_{i \in S_c} [Y_i - \hat{\psi}_d^{(c)}(U_i)]^2.$$
However, $E[Y - \hat{\psi}_d(U)]^2$ is not the mean-squared error of $\hat{\psi}_d$. Note that
\[
E[Y - \hat{\psi}_d(U)]^2 = E[E(Y|U) - \hat{\psi}_d(U)]^2 + E[Y - E(Y|U)]^2
\]
because the cross product term
\[
E \left( [E(Y|U) - \hat{\psi}_d(U)] [Y - E(Y|U)] \right) = E \left( E \left( [E(Y|U) - \hat{\psi}_d(U)] | Y - E(Y|U) \right) \big| U \right) \\
= E \left( [E(Y|U) - \hat{\psi}_d(U)] E \left( [Y - E(Y|U)] | U \right) \right)
\]
\[
= 0.
\]
The term $E[E(Y|U) - \hat{\psi}_d(U)]^2$ is an average mean squared error (AMSE) of $\hat{\psi}_d(U)$ over all $U$ values. If we can estimate $\sigma^2 = E[Y - E(Y|U)]^2$ by $\tilde{\sigma}^2$, then we can estimate AMSE of $\hat{\psi}_d$ by $CV(10) - \tilde{\sigma}^2$.

We utilize the difference-based variance estimators proposed in [Hall et al. 1990, Hall et al. 1991] and [Munk et al. 2005] to estimate $\sigma^2$, where $U$ in $\sigma^2 = E[Y - E(Y|U)]^2$ is replaced by $\tilde{B}^T U$. Estimated values of $\sigma^2$, AMSE for $\hat{\psi}_d_0$, $\hat{\psi}_d_1$, and $\hat{\psi}_d_2$ for three groups DVD, TC, and UC are given in Table 5.

It can be seen from Table 5 that using $Z$ data helps in the estimation of $E(Y|U = u_0)$ on the average, especially when we use $\hat{\psi}_d_2$.

**Appendix**

**Proof of Lemma 1**

**Proof.** (i) Since $Y \perp (Z, U) \mid C_z^T Z + C_u^T U$, then $Y \perp (Z, U) \mid (C_z^T Z, C_u^T U)$ and further $Y \perp U \mid (C_z^T Z, C_u^T U)$. By the definition of partial central subspace in Chiaromonte et al. (2002), the partial central space $S_{Y|U}^{C_z T Z} \subseteq S(C_u)$. By Proposition 3.1 and equation (3.1) in Hung et al. (2015), it is easy to get $S(B) \subseteq S_{Y|U}^{C_z T Z} \oplus S(C) \subseteq S(C_u) \oplus S(C)$. Furthermore, if $C_z^T Z \perp U \mid B^T U$, by the definition of central subspace in Cook (1998), $S(C) \subseteq S(B)$.

(ii) For the same reason as in (i), the partial central space $S_{Y|B^T U}^{D_z T Z} \subseteq S(D_u)$, and then we can get $S(I_d) \subseteq S_{Y|B^T U}^{D_z T Z} \oplus S(D) \subseteq S(D_u) \oplus S(D)$. On the other hand, $S(D_u) \oplus S(D) \subseteq S(I_d)$. As a result, $S(D_u) \oplus S(D) = S(I_d)$, which means $S(B) = S(BD_u) \oplus S(BD)$. $\Box$

The following Lemmas 2 - 6 are all used for the proof of Theorem 1.
Lemma 2. If Assumptions 1, 3(i)(ii) and 4 hold, then

\[
\sup_{z, \Omega} \left| \frac{\tilde{\gamma}_2(\tilde{C}^T z + \tilde{C}^T u_0)}{\tilde{\gamma}_1(\tilde{C}^T z + \tilde{C}^T u_0)} - \frac{\gamma_2(C^T z + C^T u_0)}{\gamma_1(C^T z + C^T u_0)} \right| = O_p(h^m n^{-1/2} + n^{-1} h^{-(d+1)} \log n),
\]

and

\[
\sup_{\Omega} \left| \frac{1}{n} \sum_{i=1}^{n} \tilde{\gamma}_2(\tilde{C}^T Z_i + \tilde{C}^T u_0) K_h(\tilde{C}^T U_i - \tilde{C}^T u_0)
- \frac{1}{n} \sum_{i=1}^{n} \tilde{\gamma}_2(C^T Z_i + C^T u_0) K_h(C^T U_i - C^T u_0)
- E \left( \frac{\tilde{\gamma}_2(\tilde{C}^T Z + \tilde{C}^T u_0)}{\tilde{\gamma}_1(\tilde{C}^T Z + \tilde{C}^T u_0)} \tilde{C}^T U = \tilde{C}^T u_0 \right) f_{\tilde{C}^T U}(\tilde{C}^T u_0)
+ E \left( \frac{\tilde{\gamma}_2(C^T Z + C^T u_0)}{\tilde{\gamma}_1(C^T Z + C^T u_0)} C^T U = C^T u_0 \right) f_{C^T U}(C^T u_0) \right|
= O_p(h^m n^{-1/2} + n^{-1} h^{-(d+1)}).
\]

Proof. The proof is analogous to that of Lemma 3 in the supplementary materials of [Ma and Zhu (2012)]. \qed

Making use of the results in Lemma 2, we prove in the following Lemma 3 that the estimation errors of SDR are asymptotically negligible. Note that we assume that SDR estimators of \( C_{zu} \) and \( C \) converge at the rate of \( n^{-1/2} \). For the following lemmas and proofs, when necessary, we use \( \hat{\psi}_d(u_0; \hat{\gamma}, \hat{C}_{zu}, \hat{C}) \) to represent \( \hat{\psi}_d(u_0) \) based on \( \hat{\gamma} \) and \( (\hat{C}_{zu}, \hat{C}) \), and use \( \hat{\psi}_d(u_0; \tilde{\gamma}, C_{zu}, C) \) to represent \( \hat{\psi}_d(u_0) \) based on \( \tilde{\gamma} \) and \( (C_{zu}, C) \).

Lemma 3. If Assumptions 1, 3(i)-(iii) and 4 hold, then

\[
\sqrt{nh^d} \left[ \hat{\psi}_d(u_0; \hat{\gamma}, \hat{C}_{zu}, \hat{C}) - \psi(u_0) \right] = \sqrt{nh^d} \left[ \hat{\psi}_d(u_0; \tilde{\gamma}, C_{zu}, C) - \psi(u_0) \right] + o_p(1).
\]
Proof. Write
\[ \sqrt{n\bar{d}} \left[ \tilde{\psi}_d(u_0; \tilde{\gamma}, \tilde{C}_{zu}, \tilde{C}) - \psi(u_0) \right] - \sqrt{n\bar{d}} \left[ \tilde{\psi}_d(u_0; \tilde{\gamma}, C_{zu}, C) - \psi(u_0) \right] \]
\[ = \sqrt{n\bar{d}} \left[ \tilde{\psi}_d(u_0; \tilde{\gamma}, \tilde{C}_{zu}, \tilde{C}) - \tilde{\psi}_d(u_0; \tilde{\gamma}, \tilde{C}_{zu}, C) \right] + \sqrt{n\bar{d}} \left[ \tilde{\psi}_d(u_0; \tilde{\gamma}, \tilde{C}_{zu}, C) - \tilde{\psi}_d(u_0; \tilde{\gamma}, C_{zu}, C) \right] \]
\[ = R_{n1} + R_{n2} \]

For \( R_{n1} \),
\[ R_{n1} = \sqrt{n\bar{d}} \left[ \frac{n^{-1} \sum_{i=1}^{n} \tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right) \right] \times \left( \frac{n^{-1} \sum_{i=1}^{n} \tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right) \right)^{-1} \]
\[ - \sqrt{n\bar{d}} n^{-1} \sum_{i=1}^{n} \tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i \frac{K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right)}{K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right)} \]
\[ + \sqrt{n\bar{d}} n^{-1} \sum_{i=1}^{n} \tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i \frac{K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right)}{K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right)} \]
\[ = S_{n1} - S_{n2} \]

\( S_{n1} \) can be further split to
\[ S_{n1} = \sqrt{n\bar{d}} \left\{ n^{-1} \sum_{i=1}^{n} \frac{\tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right) - n^{-1} \sum_{i=1}^{n} \frac{\tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} K_h \left( \tilde{C}_U i - \tilde{C}_U u_0 \right) \right\} \]
\[ + \sqrt{n\bar{d}} \left\{ \frac{\tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} \right\} \]
\[ = \sqrt{n\bar{d}} \left\{ \frac{\tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} \right\} \]
\[ + \sqrt{n\bar{d}} \left\{ \frac{\tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} \right\} \]
\[ + \sqrt{n\bar{d}} \left\{ \frac{\tilde{\gamma}_2 \tilde{z}_i \tilde{u}_i}{\tilde{\gamma}_1 \tilde{z}_i + \tilde{C}_u u_0} \right\} \]
\[ C^T U = C^T u_0 \] \( f_{CTU}(C^T u_0) \) \[ \left[ n^{-1} \sum_{i=1}^{n} K_h (\hat{C}^T U_i - C^T u_0) \right]^{-1} \]

\[ = S_{n1} + S_{n2} \]

By Lemma 2, the numerator of \( S_{n1} \) is bounded by

\[ \sqrt{nh^d} \sup_{\Omega} \left| n^{-1} \sum_{i=1}^{n} \frac{\frac{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} K_h (\hat{C}^T U_i - C^T u_0) - n^{-1} \sum_{i=1}^{n} \frac{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} K_h (\hat{C}^T U_i - C^T u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} \right| f_{CTU}(\hat{C}^T u_0) \]

\[ + E \left[ \frac{\hat{\gamma}_2(\hat{C}_Z^T Z + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z + \hat{C}_u u_0)} C^T U = C^T u_0 \right] f_{CTU}(C^T u_0) \]

\[ = o_p \left( \sqrt{nh^d} h^{-\frac{m}{2} + n^{-1} h^{-(d+1)}} \right) \]

Also, the denominator of \( S_{n1} \) converges to \( f_{CTU}(C^T u_0) \). Hence \( S_{n1} = o_p(1) \). As to \( S_{n2} \), by Lipschitz continuity in Assumption 3(iii),

\[ \left| \frac{n^{-1} \sum_{i=1}^{n} \frac{\hat{\gamma}_2(\hat{C}_Z^T Z_i + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} K_h (C^T U_i - C^T u_0)}{n^{-1} \sum_{i=1}^{n} K_h (C^T U_i - C^T u_0)} \right| \leq \Lambda \| \hat{C} - C \| \]

for some constant \( \Lambda > 0 \), hence \( |S_{n2}| \leq o_p(\sqrt{nh^d} h^{-\frac{m}{2}}) = o_p(1) \). Then \( S_{n1} = o_p(1) \). The proof of \( S_{n2} = o_p(1) \) is similar. As to \( R_{n2} \),

\[ R_{n2} = \sqrt{nh^d} \left\{ n^{-1} \sum_{i=1}^{n} \frac{\hat{\gamma}_2(\hat{C}_Z^T Z_i + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} K_h (C^T U_i - C^T u_0) - n^{-1} \sum_{i=1}^{n} \frac{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} K_h (C^T U_i - C^T u_0) \right\} \]

\[ = \sqrt{nh^d} \left\{ n^{-1} \sum_{i=1}^{n} \left[ \frac{\hat{\gamma}_2(\hat{C}_Z^T Z_i + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} - \frac{\hat{\gamma}_2(\hat{C}_Z^T Z_i + \hat{C}_u u_0)}{\hat{\gamma}_1(\hat{C}_Z^T Z_i + \hat{C}_u u_0)} \right] K_h (C^T U_i - C^T u_0) \right\} \left[ n^{-1} \sum_{i=1}^{n} K_h (C^T U_i - C^T u_0) \right]^{-1} \]

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\[
\sqrt{nh^d} \left\{ n^{-1} \sum_{i=1}^{n} \left[ \frac{\gamma_2(\hat{C}_z^T Z_i + \hat{C}_u u_0)}{\gamma_1(\hat{C}_z^T Z_i + \hat{C}_u u_0)} - \frac{\gamma_2(C_z^T Z_i + C_u^T u_0)}{\gamma_1(C_z^T Z_i + C_u^T u_0)} \right] K_h(C^T U_i - C^T u_0) \right\} \\
= S_{n3} + S_{n4}
\]

where \(|S_{n3}| \leq O_p(\sqrt{nh^d(h^6n^{-1/2} + n^{-1}h^{-(d+1)\log n}))} = o_p(1)\) by the uniform convergence result from Lemma \ref{lemma2} and Assumption \ref{assumption4}. As to \(S_{n4}\),

\[
E \left[ \left| \frac{\gamma_2(\hat{C}_z^T Z_i + \hat{C}_u u_0)}{\gamma_1(\hat{C}_z^T Z_i + \hat{C}_u u_0)} - \frac{\gamma_2(C_z^T Z_i + C_u^T u_0)}{\gamma_1(C_z^T Z_i + C_u^T u_0)} \right| K_h(C^T U_i - C^T u_0) \right] \\
\leq \sup_{u_i, \|\hat{C}_{zu} - C_{zu}\| \leq cn^{-1/2}} \left\{ E \left[ \left| \frac{\partial}{\partial C_{zu}} E[Y|C_z^T Z + C_u^T u_0] \right|_{C_{zu} = C_{zu} + \delta(\hat{C}_{zu} - C_{zu})} \right| C^T U = C^T u \right\} \\
f_{C^T U}(C^T u) \int |K(t)| dt \|\hat{C}_{zu} - C_{zu}\| = O_p(\|\hat{C}_{zu} - C_{zu}\|)
\]

where \(\delta \in (0, 1)\). Thus \(|S_{n4}| \leq O_p(\sqrt{nh^d n^{-1/2}}) = o_p(1)\). Therefore, \(R_{n1}\) and \(R_{n2}\) are both \(o_p(1)\).

This completes the proof. \(\Box\)

By Lemma \ref{lemma3}, the estimation errors of SDR estimators \((\hat{C}_{zu}, \hat{C})\) have no effect on the asymptotic distribution of \(\hat{\psi}_d(u_0; \hat{\gamma}, \hat{C}_{zu}, \hat{C})\). Hence, in the following lemmas and proofs, we assume that \((C_{zu}, C)\) are known. The denominator of \(\hat{\psi}_d(u_0; \hat{\gamma}, C_{zu}, C)\), \(n^{-1} \sum_{i=1}^{n} K_h(C^T U_i - C^T u_0)\), converges in probability to \(f_{C^T U}(C^T u_0)\). By Slutsky’s theorem, we only need to prove that the numerator of \(\hat{\psi}_d(u_0; \hat{\gamma}, C_{zu}, C)\) is asymptotically normal. Write the numerator of \(\hat{\psi}_d(u_0; \hat{\gamma}, C_{zu}, C)\) in the form of \(n^{-1} \sum_{i=1}^{n} \psi_n(W_i, u_0; \hat{\gamma})\) with \(\psi_n(W_i, u_0; \hat{\gamma}) = \hat{\varphi}_1(Z_i, u_0)K_h(C^T U_i - C^T u_0)\) and \(W = (Z, U)\). In the following proofs, we distinguish the kernels by denoting \(\hat{K}\) and \(K\) as the kernel functions used in the first and second steps, respectively, and we use \(c, \tilde{c}\) and \(M\) as generic constants. The following Lemmas \ref{lemma4} - \ref{lemma6} together prove that

\[
\sqrt{nh^d} \frac{1}{n} \sum_{i=1}^{n} \left[ \psi_n(W_i, u_0; \hat{\gamma}) - \psi_n(W_i, u_0; \gamma) \right] = o_p(1),
\]

which means that the convergence rate of a two-step estimator is not directly affected by the kernel estimation of the inner layer, but by the kernel estimation of the outer layer.

**Lemma 4.** If Assumptions 1, 2, 3(ii) and 4 hold, then

\[
T_{n1} = \sqrt{nh^d} \frac{1}{n} \sum_{i=1}^{n} \left[ \psi_n(W_i, u_0; \hat{\gamma}) - \psi_n(W_i, u_0; \gamma) - G_n(W_i, u_0; (\hat{\gamma} - \gamma)) \right] = o_p(1),
\]
Lemma 5. By Chebyshev’s inequality, where

\[
G_n(W_i, u_0; \eta) = \frac{K_h(C^T U_i - C^T u_0)}{\gamma_1(C^T Z_i + C^T u_0)} \left[ \eta_2(C^T Z_i + C^T u_0) - \gamma_2(C^T Z_i + C^T u_0) \right] \eta_1(C^T Z_i + C^T u_0)
\]

with some functions \( \eta_j(\cdot) \), \( j = 1, 2 \).

Proof. From Assumption 1, \( \gamma_1 \) is bounded away from zero. Since \( \hat{\gamma}_1 \) converges to \( \gamma_1 \) uniformly, when \( n \) is large enough, \( \hat{\gamma}_1 \) is also bounded away from zero, i.e. both inf \( \hat{\gamma}_1 \) and inf \( \gamma_1 \geq c \). Then by Assumptions 2, 3(ii) and 4,

\[
\sqrt{n}h^d \mathbb{E} \left[ \left| \hat{\psi}_n(W_i, u_0; \hat{\gamma}) - \psi_n(W_i, u_0; \gamma) - G_n(W_i, u_0; (\hat{\gamma} - \gamma)) \right| \right]
\]

\[
= \sqrt{n}h^d \mathbb{E} \left[ |K_h(C^T U_i - C^T u_0)| \left( 1 + \left| \frac{\gamma_2}{\gamma_1} \right| \right) \| \hat{\gamma} - \gamma \|^2 \right]
\]

\[
\leq e^{-2} \sqrt{n}h^d \mathbb{E} \left[ |K_h(C^T U_i - C^T u_0)| \left( 1 + \left| \frac{\gamma_2}{\gamma_1} \right| \right) \| \hat{\gamma} - \gamma \|^2 \right]
\]

\[
= e^{-2} \frac{1}{\gamma_1} \mathbb{E} \left[ |K_h(C^T U_i - C^T u_0)| \left( 1 + |E(Y|V = C^T Z_i + C^T u_0)| \right) \right] \sqrt{n}h^d \| \hat{\gamma} - \gamma \|^2
\]

\[
\leq e^{-2} \sup_u \left\{ E \left[ 1 + |E(Y|V = C^T Z_i + C^T u_0)| \right] \| C^T U_i = C^T u \right\} \int |K(t)| \, dt
\]

\[
= o_p(1)
\]

By Chebyshev’s inequality, \( T_{n1} = o_p(1) \).

Lemma 5. Let \( G_n(W_i, u_0; \eta) \) be as defined in Lemma 4. If Assumptions 1, 2, 3(ii) (iv) and 4 hold, then

\[
T_{n2} = \sqrt{n}h^d \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; (\hat{\gamma} - \gamma)) - \int G_n(w, u_0; (\hat{\gamma} - \gamma)) \, dF \right] = o_p(1)
\]

where \( F \) is the cdf of \( W \).

Proof. Let \( \hat{\gamma}(v) = E[\hat{\gamma}(v)] \), then

\[
T_{n2} = \sqrt{n}h^d \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; (\hat{\gamma} - \gamma)) - \int G_n(w, u_0; (\hat{\gamma} - \gamma)) \, dF \right]
\]

\[
= \sqrt{n}h^d \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; (\hat{\gamma} - \gamma)) - \int G_n(w, u_0; (\hat{\gamma} - \gamma)) \, dF \right]
\]

\[
+ \sqrt{n}h^d \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; (\hat{\gamma} - \gamma)) - \int G_n(w, u_0; (\hat{\gamma} - \gamma)) \, dF \right]
\]

\[
= T_{n21} + T_{n22}
\]
We only need to prove that
\[ T_{n21} = \sqrt{nh^d} \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; (\tilde{\gamma} - \gamma)) - \int G_n(w, u_0; (\tilde{\gamma} - \gamma)) \, dF \right] \]
and
\[ T_{n22} = \sqrt{nh^d} \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; (\tilde{\gamma} - \gamma)) - \int G_n(w, u_0; (\tilde{\gamma} - \gamma)) \, dF \right] \]
are both \( o_p(1) \). As \( T_{n21} \) can be written in such way that
\[ T_{n21} = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_n(W_i, W_j) - n^{-1} \sum_{i=1}^{n} [m_{n1}(W_i) + m_{n2}(W_i)] + \mu \]
where
\[ m_n(W_i, W_j) = \frac{K_h(C^T u_i - C^T u_0)}{\gamma_1(C^T_z Z_i + C^T u_0)} \left[ Y_j - \frac{\gamma_2(C^T Z_i + C^T u_0)}{\gamma_1(C^T_z Z_i + C^T u_0)} \right] \bar{K}_h(C^T Z_i + C^T u_0 - C^T U_j) \]
\[ m_{n1}(W_i) = \int m_n(W_i, w) \, dF, \quad m_{n2}(W_i) = \int m_n(w, W_i) \, dF \]
and \( \mu = E[m_n(W_1, W_2)] \). Result from Lemma 8.4 in Newey and McFadden (1994) concerning V-statistics convergence is applied directly.

Lemma 8.4 states that if \( W_1, W_2, ..., W_n \) are i.i.d then
\[ n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_n(W_i, W_j) - n^{-1} \sum_{i=1}^{n} [m_{n1}(W_i) + m_{n2}(W_i)] + \mu = O_p \left(n^{-1} E[|m_n(W, W)|] + n^{-1} E^{1/2}[|m_n(W_1, W_2)|^2]\right) \]

In \( T_{n21} \),
\[ E[|m_n(W, W)|] \leq E \left\{ \left| K_h(C^T U - C^T u_0) \right| \frac{Y - \gamma_2(C^T Z + C^T u_0)}{\gamma_1(C^T_z Z + C^T u_0)} \bar{K}_h(0 + (C^T U - C^T u_0)) \right\} \]
\[ \leq c^{-1} E \left\{ |K_h(C^T U - C^T u_0)| K_h(C^T U - C^T u_0) \left| |Y| + \gamma_2(C^T Z + C^T u_0) \right| \bar{K}_h(0 + (C^T U - C^T u_0)) \right\} \]
\[ = c^{-1} E \left\{ |K_h(C^T U - C^T u_0)| K_h(C^T U - C^T u_0) \left| |Y| C^T U, C^T U \right| \right\} \]
\[ + E \left( |E(Y|V = C^T Z + C^T u_0)||C^T U| \right) \]
Note that \( S(C_u) \) and \( S(C) \) may overlap and it is necessary to find out the basis of \( S(C_u) \oplus S(C) \) before calculating the expectation. Split the columns of \( C_u \) and \( C \) into two parts such that \( C_u = (C_u^*, C_u^{**}) \) and \( C = (C^*, C^{**}) \), where the columns of \( C_u^* \) and \( C^* \) (with the smallest possible column dimensions) together form the basis of the space \( S(C_u) \oplus S(C) \). In this case, the columns of
$C_u^{**}$ (or $C^*$) can be written as linear combinations of the columns of $C_u^*$ and $C^*$. Let $d_u^{**}$ and $d^*$ be the column dimensions of $C_u^{**}$ and $C^*$ respectively. As the kernels are bounded from above, then there exists $M > 0$ such that $h^{-d_u^{**}}|\vec{K}((C_u^{**T}u - C_u^{**T}u_0)/h)| \leq h^{-d_u^{**}}M \leq h^{-d_m}M$ and $h^{-d^*}|K((C^*T u - C^*T u_0)/h)| \leq h^{-d^*}M \leq h^{-d_m}M$. By Assumptions 1 and 3 (ii), (iv),

$$E[\|m_n(W, W)\|] \leq E \left\{ \left| \frac{K_h(C^T U - C^T u_0)}{\gamma_1(C_z^T Z + C_u^T u_0)} \right| Y - \frac{\gamma_2(C_z^T Z + C_u^T u_0)}{\gamma_1(C_z^T Z + C_u^T u_0)} \bar{K}_h(0 + C_u^T U - C^T u_0) \right\}$$

$$\leq c^{-1} \int |\bar{K}_h(C_u^T U - C_u^T u_0)K_h(C^{**T} u - C^{**T} u_0)\bar{K}_h(C_u^{**T} U - C_u^{**T} u_0)K_h(C^{**T} u - C^{**T} u_0)|$$

$$\left[ E(|Y|C_u^T U = C_u^T u, C^T U = C^T u) + E(|Y V = C_z^T Z + C_u^T u_0)|C^T U = C^T U \right]$$

$$f_{C_u^{**T} U, C^T U}(C_u^{**T} U, C^T U) \int |\bar{K}(\bar{t})| \, d\bar{t} \int |K(t^*)| \, dt^* = O_p(h^{-d_u^{**}})$$

where $\bar{t} \in \mathbb{R}^{d - d_u^{**}}$ and $t^* \in \mathbb{R}^{d - d^*}$. As to $E[\|m_n(W_1, W_2)\|^2]$,

$$E \left[ \|m_n(W_1, W_2)\|^2 \right]$$

$$\leq E \left\{ \left| \frac{K_h(C^T U_1 - C^T u_0)}{\gamma_1(C_z^T Z_1 + C_u^T u_0)} \right| Y_2 - \frac{\gamma_2(C_z^T Z_1 + C_u^T u_0)}{\gamma_1(C_z^T Z_1 + C_u^T u_0)} \bar{K}_h(C_z^T Z_1 + C_u^T u_0 - V_2) \right|^2 \right\}$$

$$\leq I_{n1} + I_{n2}$$

where the first term

$$I_{n1} = E \left\{ \left| \frac{K_h(C^T U_1 - C^T u_0)}{\gamma_1(C_z^T Z_1 + C_u^T u_0)} \bigg( Y_2 \bar{K}_h(C_z^T Z_1 + C_u^T u_0 - V_2) \bigg) \right|^2 \right\}$$

$$\leq c^{-2} E_{W_1} \left\{ E_{W_2} \left[ K_h^2(C^T U_1 - C^T u_0)Y_2^2 \bar{K}_h^2(C_z^T Z_1 + C_u^T u_0 - V_2) \right] \right\}$$

$$= c^{-2} E_{W_1} \left\{ K_h^2(C^T U_1 - C^T u_0)E_{V_2} \left[ E(Y_2^2|V_2) \bar{K}_h^2(C_z^T Z_1 + C_u^T u_0 - V_2) \right] \right\}$$

$$\leq \bar{c} h^{-d} \sup_v \left\{ E \left( Y^2 | V = v \right) f(V(v)) \right\} \sup_u \left\{ f_{C^T U}(C^T u) \right\} \int |\bar{K}(\bar{t})| \, d\bar{t} \int |K(t^*)| \, dt^*$$

$$= O_p(h^{-d}h^{-d})$$

Similarly $I_{n2} \leq O_p(h^{-d}h^{-d})$. In all, by Assumption 4,

$$T_{n21} = \sqrt{n h^{d}} O_p \left( n^{-1} E[\|m_n(W, W)\|] + n^{-1} E[\|m_n(W_1, W_2)\|^2]^{1/2} \right)$$

$$\leq O_p \left( \sqrt{n h^{d}} h^{-d} h^{-d} \right) = o_p(1).$$

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As to $T_{n22}$, by Chebychev’s Inequality, Assumptions 1, 3 (ii) and 4, since $E(T_{n22}) = 0$,

$$P \left( \sqrt{nh^d} \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; \tilde{\gamma} - \gamma) - \int G_n(w, u_0; \tilde{\gamma} - \gamma) \, dF \right] \right) > \epsilon$$

$$= P \left( \sqrt{nh^d} \left[ \frac{1}{n} \sum_{i=1}^{n} G_n(W_i, u_0; \tilde{\gamma} - \gamma) - \int G_n(w, u_0; \tilde{\gamma} - \gamma) \, dF \right]^2 \right) > \epsilon^2$$

$$\leq \frac{nh^d}{n^2 \epsilon^2} \left\{ n(n-1) \left[ E \left[ G_n(W_i, u_0; \tilde{\gamma} - \gamma) - \int G_n(w, u_0; \tilde{\gamma} - \gamma) \, dF \right] \right]^2 \right\}$$

$$+ nE \left[ \left| G_n(W_i, u_0; \tilde{\gamma} - \gamma) - \int G_n(w, u_0; \tilde{\gamma} - \gamma) \, dF \right|^2 \right]$$

$$= h^d \epsilon^{-2} E \left[ \left| G_n(W_i, u_0; \tilde{\gamma} - \gamma) - \int G_n(w, u_0; \tilde{\gamma} - \gamma) \, dF \right|^2 \right]$$

$$\leq h^d \epsilon^{-2} E(\left| G_n(W_i, u_0; \tilde{\gamma} - \gamma) \right|^2)$$

$$\leq h^d \epsilon^{-2} E \left\{ \frac{K^2 h(C^TU_i - C^Tu_0)}{\gamma^2(C^T Z_i + C^ Tu_0)} \left[ \frac{\gamma_2(C^T Z_i + C^ Tu_0)}{\gamma_1(C^T Z_i + C^ Tu_0)}, 1 \right] \left[ \tilde{\gamma}_1 - \gamma_1, \tilde{\gamma}_2 - \gamma_2 \right] \right\}$$

$$\leq c^{-2} \epsilon^{-2} h^d E \left\{ K^2 h(C^TU_i - C^Tu_0) \left[ \left| \frac{\gamma_2(C^T Z_i + C^ Tu_0)}{\gamma_1(C^T Z_i + C^ Tu_0)} \right|^2 + 1 \right] C^TU_i \right\} \| \tilde{\gamma} - \gamma_0 \|^2_\infty$$

$$= \tilde{c} \epsilon^{-2} \sup_{u} \left\{ E \left[ \left| E[Y | V = C^T Z_i + C^Tu_0] \right|^2 + 1 \right] C^TU_i = C^Tu \right\} f_{C^TU}(C^Tu) \right\}$$

$$\int |K(t)| \, dt \| \tilde{\gamma} - \gamma_0 \|^2_\infty \to 0,$$

so $T_{n22} = o_p(1).$ \hfill \Box

For the following proof, we denote

$$s_n(V) = \int K_h(C^Tu - C^Tu_0) f_{C^TU}(V - C^Tu_0, C^Tu) \, d(C^Tu) \, f_{V}^{-1}(V)[-E(Y | V), 1]$$

and $S_n(Y, V) = s_n(V)(1, Y)^T$.

**Lemma 6.** If Assumptions 1, 2, 3 (ii) and 4 hold, then

$$T_{n3} = \sqrt{nh^d} \left[ \int G_n(w, u_0; (\tilde{\gamma} - \gamma)) \, dF - \frac{1}{n} \sum_{j=1}^{n} S_n(Y_j, V_j) \right] = o_p(1)$$

**Proof.** Since

$$\sqrt{nh^d} \int G_n(w, u_0; \tilde{\gamma} - \gamma) \, dF = \sqrt{nh^d} \int G_n(w, u_0; \tilde{\gamma}) \, dF$$

$$= \sqrt{nh^d} \frac{1}{n} \sum_{j=1}^{n} S_n(Y_j, C^T z + C^Tu_0) \tilde{K}_h(C^T z + C^Tu_0 - C^T Z_j + C^T U_j) \, d(C^T z),$$

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$T_{n3}$ can be written as
\[
T_{n3} = \sqrt{nh^d} \frac{1}{n} \sum_{j=1}^{n} \int \left[ S_n(Y_j, C_z^T z + C_u^T u_0) - S_n(Y_j, V_j) \right] \tilde{K}_h(C_z^T z + C_u^T u_0 - V_j) \, d(C_z^T z)
\]
\[
= \sqrt{nh^d} \frac{1}{n} \sum_{j=1}^{n} D_n(Y_j, V_j)
\]
by Chebyshev's inequality,
\[
P \left( \left| \sqrt{nh^d} \frac{1}{n} \sum_{j=1}^{n} D_n(Y_j, V_j) \right| > \epsilon^2 \right) \leq nh^d \left\{ n(n - 1) |E[D_n(Y_j, V_j)]|^2 + nE[D_n^2(Y_j, V_j)] \right\} / (n^2 \epsilon^2),
\]
so we only need to prove $\sqrt{nh^d}|E[D_n(Y_j, V_j)]| \to 0$ and $h^dE[D_n^2(Y_j, V_j)] \to 0$. By Assumptions 1, 2, 3 (ii) and 4,
\[
\sqrt{nh^d}|E[D_n(Y_j, V_j)]| = \sqrt{nh^d} \left| \int S_n(Y_j, C_z^T z + C_u^T u_0) \tilde{K}_h(C_z^T z + C_u^T u_0 - V_j) \, d(C_z^T z) - E[S_n(Y_j, V_j)] \right|
\]
\[
= \sqrt{nh^d} \left| \int s_n(C_z^T z + C_u^T u_0) E \left\{ \tilde{K}_h(C_z^T z + C_u^T u_0 - V_j) E[(1, Y_j)^T | V_j] \right\} \, d(C_z^T z) - E[s_n(V_j) E[(1, Y_j)^T | V_j]] \right|
\]
\[
= \sqrt{nh^d} \left| \int s_n(v) \int \tilde{K}(\bar{t}) \left\{ E[(1, Y_j)^T | V_j = v + h\bar{t}] f_V(v + h\bar{t}) - E[(1, Y_j)^T | V_j = v] f_V(v) \right\} \, d\bar{t} \right|
\]
\[
\leq \sqrt{nh^d} \left| \int s_n(v) \sum_{k=1}^{m-1} h^k [k!]^{-1} \frac{\partial^k E[(1, Y_j)^T | V_j = v] f_V(v)}{\partial v^k} \int \tilde{K}(\bar{t}) \otimes^{k} \bar{t} \, d\bar{t} \right|
\]
\[
+ \sqrt{nh^d} h^m [m!]^{-1} \int \|s_n(v)\| \, dv \left\| \frac{\partial^m E[(1, Y_j)^T | V_j = v] f_V(v)}{\partial v^m} \right\|_\infty \int \| \tilde{K}(\bar{t}) \otimes^m \bar{t} \| \, d\bar{t}
\]
\[
= O_p(\sqrt{nh^d} h^m) \to 0
\]
and by Assumptions 1 and 3 (ii),
\[
h^d E[D_n^2(Y_j, V_j)]
\]
\[
\leq h^d \left[ \left| \int S_n(Y_j, C_z^T z + C_u^T u_0) \tilde{K}_h(C_z^T z + C_u^T u_0 - V_j) \, d(C_z^T z) \right|^2 \right] + h^d E \left[ S_n^2(Y_j, V_j) \right]
\]
\[
\leq h^d \sup_{|v| < \epsilon} E[S_n^2(Y_j, V_j + v)] \left( \int |\tilde{K}(\bar{t})| \, d\bar{t} \right)^2 + h^d E \left[ S_n^2(Y_j, V_j) \right]
\]
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where its numerator can be split as

Thus, $T_{n3} = o_p(1)$ and the proof is completed. \qed

Proof of Theorem 1

Proof. The proof follows the similar ideas as in Newey and McFadden (1994) and Escanciano et al. (2014). From Lemma 3

\[
\sqrt{nh^d} \left[ \hat{\psi}_d(u_0; \gamma, \hat{C}_{zu}, \hat{C}) - \psi(u_0) \right] = \sqrt{nh^d} \left[ \hat{\psi}_d(u_0; \gamma, C_{zu}, C) - \psi(u_0) \right] + o_p(1)
\]

Moreover,

\[
\sqrt{nh^d} \left\{ \hat{\psi}_d(u_0; \gamma, C_{zu}, C) - \psi(u_0) \right\} = \sqrt{nh^d} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\gamma}_2(C_i^T Z_i + C_u^T u_0)}{\hat{\gamma}_1(C_i^T Z_i + C_u^T u_0)} K_h(C_i^T U_i - C^T u_0) \right. \\
- E[E(Y|V = C_i^T Z_i + C_u^T u_0) | C^T U = C^T u_0] \left. \right\}
\]

\[
= \sqrt{nh^d} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_h(C_i^T U_i - C^T u_0) \right. \\
- \frac{1}{n} \sum_{i=1}^{n} K_h(C_i^T U_i - C^T u_0) E[E(Y|V = C_i^T Z_i + C_u^T u_0) | C^T U = C^T u_0] \left. \right\}
\]

where its numerator can be split as

\[
\sqrt{nh^d} \frac{1}{n} \sum_{i=1}^{n} K_h(C_i^T U_i - C^T u_0) \left\{ \frac{\hat{\gamma}_2(C_i^T Z_i + C_u^T u_0)}{\hat{\gamma}_1(C_i^T Z_i + C_u^T u_0)} \right. \\
- E \left[ E(Y|V = C_i^T Z_i + C_u^T u_0) | C^T U = C^T u_0 \right] \left. \right\}
\]

\[
= T_{n1} + T_{n2} + T_{n3} + \sqrt{nh^d} \frac{1}{n} \sum_{j=1}^{n} S_n(Y_j, V_j) \\
+ \sqrt{nh^d} \frac{1}{n} \sum_{i=1}^{n} K_h(C_i^T U_i - C^T u_0) \left\{ E(Y|V = C_i^T Z_i + C_u^T u_0) \right. \\
- E \left[ E(Y|V = C_i^T Z_i + C_u^T u_0) | C^T U = C^T u_0 \right] \left. \right\}
\]

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By Lemmas 4 - 6, $T_n1$, $T_n2$ and $T_n3$ are all $O_p(1)$. It is also easy to prove
\[
\sqrt{n}h^d\frac{1}{n} \sum_{j=1}^{n} S_n(Y_j, V_j) = O_p(\sqrt{n}h^d n^{-1/2}) = o_p(1)
\]
As a result,
\[
\sqrt{n}h^d\frac{1}{n} \sum_{i=1}^{n} K_h(C^T U_i - C^T u_0) \left\{ \frac{\gamma_2(C^T_z Z_i + C^T_u u_0)}{\gamma_1(C^T_z Z_i + C^T_u u_0)} \right. \\
- E \left[ E(Y|V = C^T_z Z + C^T_u u_0)|C^T U = C^T u_0 \right] \right\}
= \sqrt{n}h^d\frac{1}{n} \sum_{i=1}^{n} K_h(C^T U_i - C^T u_0) \left\{ E[Y|V = C^T_z Z + C^T_u u_0] \right. \\
- E \left[ E(Y|V = C^T_z Z + C^T_u u_0)|C^T U = C^T u_0 \right] \right\} + o_p(1)
\]
Follow the similar proof as in Theorem 2.2.2 of Bierens (1987), when Assumptions 3 (v) and 4 are satisfied,
\[
n^{m/(2m+d)} \left[ \hat{\psi}_d(u_0; \hat{\gamma}, \hat{C}_z, \hat{C}) - \psi(u_0) \right] \implies N \left( \frac{\lambda b(C^T u_0)}{f_{C^T U}(C^T u_0)}, \frac{g(C^T u_0)}{f_{C^T U}(C^T u_0)} \int K^2(t) \, dt \right)
\]
and its optimal convergence rate is $n^{-m/(2m+d)}$.

Specifically, when $d = d_1$, $\hat{\psi}_d(u_0)$ becomes $\hat{\psi}_{d_1}(u_0)$, (6) holds and its optimal convergence rate is $n^{-m/(2m+d_1)}$; when $d = d_2$ and Assumptions 1 - 4 are satisfied with $C_z$, $C_u$, $C$ replaced by $D_z$, $BD_u$, $BD$ respectively, $\hat{\psi}_d(u_0)$ becomes $\hat{\psi}_{d_2}(u_0)$, (6) holds with $C_z$, $C_u$, $C$ replaced by $D_z$, $BD_u$, $BD$ respectively and $d = d_2$, and its optimal convergence rate is $n^{-m/(2m+d_2)}$.

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Table 1: Absolute value of bias and root mean squared error (RMSE) of different estimators at eight values of $E(Y|U = u_0)$ based on $n = 200$ and 1,000 simulations.

| setting quantity method | $E(Y|U = u_0)$ | value of $E(Y|U = u_0)$ |
|-------------------------|-----------------|---------------------------|
| (A1) | bias | $\hat{\psi}_p$ | 0.12 | 0.31 | 0.11 | 0.14 | 0.11 | 0.86 | 0.88 | 0.45 |
| | | $\hat{\psi}_{d_0}$ | 0.04 | 0.22 | 0.00 | 0.05 | 0.01 | 0.45 | 0.80 | 0.16 |
| | | $\hat{\psi}_{d_1}$ | 0.00 | 0.15 | 0.11 | 0.01 | 0.10 | 0.09 | 0.56 | 0.02 |
| | | $\hat{\psi}_{d_2}$ | 0.00 | 0.18 | 0.12 | 0.00 | 0.11 | 0.10 | 0.63 | 0.03 |
| RMSE | $\hat{\psi}_p$ | 0.36 | 0.68 | 0.44 | 0.47 | 0.41 | 1.13 | 1.84 | 1.34 |
| | $\hat{\psi}_{d_0}$ | 0.33 | 0.64 | 0.44 | 0.44 | 0.40 | 0.72 | 1.51 | 1.17 |
| | $\hat{\psi}_{d_1}$ | 0.22 | 0.33 | 0.26 | 0.25 | 0.24 | 0.30 | 0.93 | 0.54 |
| | $\hat{\psi}_{d_2}$ | 0.25 | 0.34 | 0.28 | 0.29 | 0.26 | 0.28 | 0.90 | 0.55 |
| (A1') | bias | $\hat{\psi}_p$ | 0.13 | 0.28 | 0.11 | 0.15 | 0.11 | 0.88 | 0.88 | 0.46 |
| | | $\hat{\psi}_{d_0}$ | 0.04 | 0.21 | 0.00 | 0.04 | 0.01 | 0.45 | 0.87 | 0.14 |
| | | $\hat{\psi}_{d_1}$ | 0.03 | 0.23 | 0.02 | 0.03 | 0.01 | 0.43 | 0.75 | 0.08 |
| | | $\hat{\psi}_{d_2}$ | 0.02 | 0.14 | 0.09 | 0.02 | 0.08 | 0.13 | 0.59 | 0.07 |
| RMSE | $\hat{\psi}_p$ | 0.36 | 0.66 | 0.45 | 0.46 | 0.42 | 1.14 | 1.84 | 1.34 |
| | $\hat{\psi}_{d_0}$ | 0.34 | 0.63 | 0.41 | 0.39 | 0.39 | 0.75 | 1.63 | 1.17 |
| | $\hat{\psi}_{d_1}$ | 0.34 | 0.61 | 0.39 | 0.39 | 0.39 | 0.74 | 1.54 | 1.22 |
| | $\hat{\psi}_{d_2}$ | 0.24 | 0.40 | 0.29 | 0.31 | 0.28 | 0.34 | 0.95 | 0.60 |
| (A2) | bias | $\hat{\psi}_p$ | 0.25 | 0.67 | 1.19 | 0.03 | 1.28 | 2.15 | 3.03 | 3.68 |
| | | $\hat{\psi}_{d_0}$ | 0.57 | 0.00 | 0.75 | 0.31 | 0.25 | 0.30 | 1.34 | 1.99 |
| | | $\hat{\psi}_{d_1}$ | 1.30 | 1.24 | 2.43 | 2.32 | 2.65 | 2.77 | 4.38 | 5.01 |
| | | $\hat{\psi}_{d_2}$ | 0.25 | 0.23 | 0.65 | 0.50 | 0.63 | 0.82 | 1.49 | 2.23 |
| RMSE | $\hat{\psi}_p$ | 3.67 | 4.00 | 4.12 | 4.63 | 3.57 | 3.50 | 4.81 | 5.43 |
| | $\hat{\psi}_{d_0}$ | 3.28 | 3.42 | 3.61 | 3.88 | 3.03 | 2.64 | 3.51 | 4.03 |
| | $\hat{\psi}_{d_1}$ | 3.97 | 4.03 | 4.96 | 5.30 | 4.33 | 4.08 | 5.92 | 6.53 |
| | $\hat{\psi}_{d_2}$ | 2.96 | 3.09 | 3.14 | 3.38 | 2.81 | 2.58 | 3.28 | 3.66 |
| (A3) | bias | $\hat{\psi}_p$ | 2.71 | 2.14 | 1.38 | 6.42 | 1.30 | 12.50 | 14.11 | 15.85 |
| | | $\hat{\psi}_{d_0}$ | 1.17 | 0.74 | 0.35 | 2.85 | 0.48 | 5.99 | 7.34 | 8.43 |
| | | $\hat{\psi}_{d_1}$ | 1.17 | 0.44 | 0.30 | 2.84 | 0.45 | 5.80 | 7.72 | 9.24 |
| | | $\hat{\psi}_{d_2}$ | 1.60 | 1.02 | 0.22 | 3.40 | 0.81 | 7.60 | 9.83 | 10.67 |
| RMSE | $\hat{\psi}_p$ | 4.68 | 4.81 | 5.42 | 9.75 | 7.60 | 14.97 | 15.82 | 17.83 |
| | $\hat{\psi}_{d_0}$ | 2.14 | 2.21 | 2.69 | 7.83 | 3.16 | 8.16 | 9.49 | 10.72 |
| | $\hat{\psi}_{d_1}$ | 1.83 | 1.67 | 1.82 | 6.02 | 2.67 | 7.88 | 9.59 | 12.01 |
| | $\hat{\psi}_{d_2}$ | 2.46 | 2.24 | 2.41 | 8.95 | 2.99 | 10.06 | 12.15 | 13.07 |
Table 2: Absolute value of bias and root mean squared error (RMSE) of different estimators at eight values of $E(Y|U = u_0)$ based on $n = 200$ and 1,000 simulations.

| setting | quantity | method | -23.93 | -22.70 | -18.53 | -16.99 | -14.32 | -12.35 | -10.31 | -9.09 |
|---------|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| (B)     | bias     | $\psi_p$ | 3.39   | 2.83   | 1.56   | 0.01   | 1.32   | 2.32   | 2.73   | 3.06   |
|         |          | $\psi_{d_0}$ | 1.80   | 2.44   | 2.57   | 0.35   | 0.66   | 1.57   | 1.94   | 2.04   |
|         |          | $\psi_{d_1}$ | 1.11   | 1.71   | 1.63   | 0.58   | 0.09   | 0.74   | 1.24   | 1.36   |
|         |          | $\psi_{N_1}$ | 0.79   | 1.48   | 1.55   | 0.55   | 0.22   | 0.45   | 0.91   | 0.99   |
|         |          | $\psi_{d_0}$ | 0.83   | 1.50   | 1.56   | 0.61   | 0.26   | 0.47   | 0.90   | 0.98   |
|         |          | $\psi_{d_1}$ | 1.10   | 2.15   | 2.50   | 0.71   | 0.02   | 0.78   | 1.17   | 1.23   |
|         |          | $\psi_{N_1}$ | 0.82   | 1.95   | 2.49   | 0.65   | 0.08   | 0.55   | 0.91   | 0.87   |
|         |          | $\psi_{d_0}$ | 0.81   | 1.96   | 2.49   | 0.67   | 0.11   | 0.58   | 0.90   | 0.86   |
| RMSE    |          | $\psi_p$ | 4.15   | 3.69   | 6.72   | 3.55   | 3.20   | 3.41   | 3.42   | 3.63   |
|         |          | $\psi_{d_0}$ | 3.26   | 3.54   | 4.59   | 3.27   | 3.07   | 3.03   | 2.89   | 2.91   |
|         |          | $\psi_{d_1}$ | 2.14   | 2.50   | 2.67   | 2.25   | 2.21   | 2.15   | 2.02   | 2.03   |
|         |          | $\psi_{N_1}$ | 2.03   | 2.46   | 2.69   | 2.51   | 2.47   | 2.28   | 1.95   | 1.90   |
|         |          | $\psi_{d_0}$ | 2.06   | 2.47   | 2.68   | 2.51   | 2.45   | 2.26   | 1.94   | 1.88   |
|         |          | $\psi_{d_1}$ | 2.51   | 3.05   | 4.09   | 2.91   | 2.73   | 2.47   | 2.12   | 2.09   |
|         |          | $\psi_{N_1}$ | 2.48   | 3.01   | 4.16   | 3.04   | 2.90   | 2.58   | 2.06   | 1.93   |
|         |          | $\psi_{d_0}$ | 2.48   | 3.02   | 4.25   | 3.08   | 2.96   | 2.60   | 2.06   | 1.96   |
Table 3: Absolute value of bias and root mean squared error (RMSE) of different estimators at eight values of $E(Y|U = u_0)$ based on $n = 200$ and 1,000 simulations; the dimensions of matrices in SDR are selected by bootstrap.

| setting | quantity | method | value of $E(Y|U = u_0)$ |
|---------|----------|--------|-------------------------|
| (A1) | | | 1.42 | 1.71 | 1.79 | 1.83 | 1.90 | 4.31 | 7.88 | 8.61 |
| | | $\hat{\psi}_p$ | 0.13 | 0.31 | 0.10 | 0.15 | 0.11 | 0.86 | 0.86 | 0.46 |
| | | $\hat{\psi}_{d_0}$ | 0.01 | 0.20 | 0.06 | 0.00 | 0.05 | 0.24 | 0.70 | 0.22 |
| | | $\hat{\psi}_{d_1}$ | 0.03 | 0.18 | 0.09 | 0.02 | 0.08 | 0.25 | 0.51 | 0.14 |
| | | $\hat{\psi}_{d_2}$ | 0.01 | 0.20 | 0.12 | 0.01 | 0.10 | 0.15 | 0.60 | 0.01 |
| RMSE | | $\hat{\psi}_p$ | 0.36 | 0.68 | 0.45 | 0.47 | 0.41 | 1.13 | 1.82 | 1.34 |
| | | $\hat{\psi}_{d_0}$ | 0.26 | 0.49 | 0.32 | 0.33 | 0.29 | 0.46 | 1.11 | 0.87 |
| | | $\hat{\psi}_{d_1}$ | 0.22 | 0.38 | 0.25 | 0.26 | 0.24 | 0.47 | 0.93 | 0.66 |
| | | $\hat{\psi}_{d_2}$ | 0.20 | 0.33 | 0.25 | 0.24 | 0.24 | 0.34 | 0.89 | 0.56 |

| (A2) | | | 23.57 | 26.83 | 28.40 | 29.74 | 35.00 | 36.91 | 39.46 | 41.47 |
| | | $\hat{\psi}_p$ | 0.30 | 0.59 | 1.23 | 0.09 | 1.30 | 2.06 | 3.08 | 3.56 |
| | | $\hat{\psi}_{d_0}$ | 0.53 | 0.03 | 0.77 | 0.34 | 0.41 | 0.40 | 1.58 | 2.00 |
| | | $\hat{\psi}_{d_1}$ | 0.20 | 0.10 | 0.93 | 0.86 | 1.45 | 1.64 | 2.99 | 3.51 |
| | | $\hat{\psi}_{d_2}$ | 0.14 | 0.21 | 0.67 | 0.41 | 0.75 | 0.96 | 1.77 | 2.41 |
| RMSE | | $\hat{\psi}_p$ | 3.59 | 4.00 | 4.06 | 4.68 | 3.66 | 3.45 | 4.94 | 5.38 |
| | | $\hat{\psi}_{d_0}$ | 3.29 | 3.48 | 3.58 | 4.03 | 3.08 | 2.83 | 3.79 | 4.07 |
| | | $\hat{\psi}_{d_1}$ | 3.14 | 3.45 | 3.46 | 4.08 | 3.31 | 3.23 | 4.45 | 4.88 |
| | | $\hat{\psi}_{d_2}$ | 2.88 | 3.11 | 3.12 | 3.35 | 2.83 | 2.62 | 3.38 | 3.82 |

| (B) | | | -23.93 | -22.70 | -18.53 | -16.99 | -14.32 | -12.35 | -10.31 | -9.09 |
| | | $\hat{\psi}_p$ | 3.46 | 2.87 | 1.41 | 0.01 | 1.36 | 2.34 | 2.77 | 3.11 |
| | | $\hat{\psi}_{d_0}$ | 2.56 | 2.66 | 2.50 | 0.40 | 0.85 | 1.67 | 1.72 | 1.77 |
| | | $\hat{\psi}_{d_1}$ | 2.61 | 1.99 | 2.21 | 0.86 | 0.22 | 0.61 | 0.11 | 0.22 |
| | | $\hat{\psi}_{d_2}$ | 2.20 | 2.66 | 3.13 | 0.95 | 0.01 | 0.55 | 0.19 | 0.07 |
| RMSE | | $\hat{\psi}_p$ | 4.17 | 3.66 | 6.29 | 3.42 | 3.16 | 3.36 | 3.43 | 3.64 |
| | | $\hat{\psi}_{d_0}$ | 3.73 | 3.58 | 4.37 | 2.94 | 2.88 | 2.97 | 2.79 | 2.86 |
| | | $\hat{\psi}_{d_1}$ | 3.50 | 2.84 | 3.69 | 2.75 | 2.52 | 2.27 | 2.08 | 2.18 |
| | | $\hat{\psi}_{d_2}$ | 3.49 | 3.62 | 4.67 | 3.11 | 2.86 | 2.59 | 2.16 | 2.25 |

Table 4: p-values using $\hat{\psi}_d$ and 10,000 permutations under different hypotheses
Figure 1: Fused Kolmogorov Filter Statistics for all demographic variables under DVD, TC, and UC methods.
Figure 2: Boxplots of $Y_i$ and $\hat{\psi}_d(U_i)$ in each method group, $d = p, d_0, d_1, \text{ or } d_2$.

|   | $\hat{\sigma}^2$ | AMSE($\hat{\psi}_p$) | AMSE($\hat{\psi}_{d_0}$) | AMSE($\hat{\psi}_{d_1}$) | AMSE($\hat{\psi}_{d_2}$) |
|---|------------------|------------------------|---------------------------|---------------------------|---------------------------|
| DVD | 95.372           | 101.31                 | 17.290                    | 17.700                    | 0.668                     |
| TC  | 86.803           | 105.77                 | 4.394                     | 4.944                     | 3.799                     |
| UC  | 101.35           | 157.94                 | 3.358                     | 7.453                     | 2.789                     |