On the Isomorphic Embedding of Rectangular Grids in $n$-cubes

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Abstract

All previously published work on isomorphic grid embeddings into $n$-cubes has been restricted to binary $n$-cubes. This paper describes a straightforward method for embedding a $A \times B$ grid isomorphically into a $k$-ary $n$-cube with $k > 2$.

Suppose you have meticulously assigned the tasks of an algorithm to these processors taking particular care to ensure the task assignments will require information to travel minimal distances over the network. Now, now you are asked to run this algorithm on a different parallel processing system that has a different network architecture. How can you achieve the previous performance with minimal reprogramming effort?

The answer depends on whether or not an isomorphic embedding exists between the two network architectures. Consider a graph $G = (V, E)$ with a set of vertices or nodes $V$ and a set of nondirectional edges $E$ which connect pairs of vertices. With respect to the problem above, the parallel processing system can be depicted as a graph where each processor is a vertex and the edges are interconnection links that form the communications network. The embedding of one graph (called the target graph) into another graph (called the host graph) assigns each vertex in the target graph to a vertex in the host graph. Similarly, edges in the target graph will overlay edges in the host graph.

Isomorphic embeddings do not always exist between a target and host graph, and even if they do, finding them is NP-hard. Nevertheless, isomorphic embeddings are of particular interest to the computer science field because of their strong implications for parallel processing: given two parallel machines $M_1$ and $M_2$ with network topologies $T_1$ and $T_2$ respectively,

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if \( T_2 \mapsto T_1 \), then any algorithm that runs in \( N \) steps on \( M_1 \) will likewise run in \( N \) steps on \( M_2 \). In other words, if an isomorphic embedding exists between the graphs depicting the two parallel processing systems, then little or no reprogramming effort is required. In fact, the previously attained performance on the old system will be duplicated in the new system.

![Figure 1: A \( k \)-ary \( n \)-cube with \( k = 4 \) and \( n = 3 \). Hidden nodes and edges are not shown to preserve clarity.](image)

One particularly versatile network topology is the \( n \)-cube. \( k \)-ary \( n \)-cubes are graphs with \( n \) dimensions and \( k \) nodes in each direction. (An example of a 4-ary 3-cube is shown in Figure 1.) It is well known that binary hypercubes (i.e., 2-ary \( n \)-cubes) are efficient architectures for executing parallel algorithms. Their reasonable tradeoff between number of processors and interconnectivity makes them ideally suited for solving linear algebra and graph-theoretical problems \[3\]. The performance of the more general \( k \)-ary \( n \)-cube has been analyzed by Dally \[2\].

All known previous work involving rectangular grid embeddings into \( n \)-cubes has been restricted to 2-ary \( n \)-cube host graphs. In certain restricted cases it is possible to isomorphically embed a rectangular grid into a 2-ary \( n \)-cube \[5\]—universally applicable techniques require dilation costs greater than unity \[1, 4\].

This paper describes a straightforward method for embedding a \( A \times B \) grid isomorphically into a \( k \)-ary \( n \)-cube with \( k > 2 \). The isomorphic embedding described shortly can always be accomplished with the mild restriction of \( A \leq k \). Neither \( A \) nor \( B \) are required to be an integer power of 2. Moreover, the technique does not require embedding into an intermediate graph (e.g., a binary hypercube or butterfly network).

\(^1\) “\( \mapsto \)” denotes an isomorphic embedding.
Let $b_1 b_2 \ldots b_n$ be a $n$-bit binary string where $b_i \in \{0,1\}$. There are $2^n$ unique binary patterns that can be formed with $n$-bit binary strings. A sequence of length $L$ contains $L$ $n$-bit binary strings no two of which are identical.

Let $A$ and $B$ be integers. The objective is isomorphically embed a $A \times B$ grid into a $k$-ary $n$-cube, where $A \leq k$ and $B \leq k^{(n-1)}$. For the moment, assume $A = k$ and $k$ is an integer power of two—restrictions that will be shortly removed. Each node in the $k$-ary $n$-cube is labeled with a $n \log_2 k$ bit binary label. The labeling is done in a Gray code manner such that any two nodes connected by an edge differ in only one bit position.

The binary label associated with each node in the $k$-ary $n$-cube can be partitioned into two parts

$$\{b_1 \ldots b_r \ b_{r+1} \ldots b_{n \log_2 k}\}$$

Thus any point $(x, y)$ in the 2-dimensional grid is given by using the $\lceil \log_2 A \rceil$ most significant bits to define the $x$ coordinate, and the $\lceil \log_2 B \rceil$ least significant bits to define the $y$ coordinate. Any two adjacent nodes in the cube will be adjacent in the grid—a property enforced by the Gray code labeling.

![Figure 2: An $A \times B$ grid embedding into a 4-ary 3-cube with $A = 3$ and $B = 9$. The placement of the rows of the grid are determined by the $\lceil \log_2 A \rceil$ most significant bits and the columns are labeled with the remaining bits of the binary label. The darkened nodes are used in the grid embedding.](image)

In practice, it is not necessary to have $A$ and $B$ as integer powers of 2; the use of integer ceilings in the label partitioning insures there are a sufficient number of bits. For example, consider a case where $A = 3$, $B = 9$, and the grid is to be embedded into a 4-ary 3-cube. Note that $A \leq 4$ and $B \leq 4^2 = 16$, which means the cube is large enough to contain the grid. Each node in the $n$-cube has a $3 \log_2 4 = 6$-bit binary label. Then $\lceil \log_2 A \rceil = 2$ bit positions in each binary label are reserved for grid row assignments, and $\lceil \log_2 B \rceil = 4$ bit
positions are used for grid column assignments. Incrementing the respective sets of bits in a Gray code manner will identify the ultimate assignments. Figure 2 shows a $3 \times 9$ grid embedding in a 4-ary 3-cube.

In effect, this grid embedding technique “unrolls” a high dimensional cube into a 2-dimensional grid. This unrolling does break edges in the $n$-cube and so some neighbor relationships are lost. Nevertheless, the grid embedding is isomorphic. If $k$ is not an integer power of 2, set $\tilde{k} = 2^{\lceil \log_2 k \rceil}$, and then isomorphically embed the grid into a $\tilde{k}$-ary $n$-cube.

References

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