QUOTIENTS OF ONE-SIDED TRIANGULATED CATEGORIES BY RIGID SUBCATEGORIES AS MODULE CATEGORIES

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Abstract. We prove that some subquotient categories of one-sided triangulated categories are abelian. This unifies a result by Iyama-Yoshino in the case of triangulated categories and a result by Demonet-Liu in the case of exact categories.

1. Introduction

Cluster tilting theory gives a way to construct abelian categories from some triangulated categories. Let $H$ be a hereditary algebra over a field $k$, and $\mathcal{C}$ be the cluster category defined in [1] as the factor category $D^b(\text{mod } H)/\tau^{-1}\Sigma$, where $\tau$ and $\Sigma$ be the Auslander-Reiten translation and shift functor of $D^b(\text{mod } H)$ respectively. For a cluster tilting object $T$ in $\mathcal{C}$, Buan, Marsh and Reiten [2] showed that $\mathcal{C}/\text{add } \tau T \cong \text{mod } \text{End}_{\mathcal{C}}(T)^{\text{op}}$. Keller and Reiten [3] generalized this result in the case of 2-Calabi-Yau triangulated categories by showing that $\mathcal{C}/\Sigma \tau T \cong \text{mod } T$, where $T$ is a cluster tilting subcategory of $\mathcal{C}$. A general framework for cluster tilting is set up by Koenig and Zhu. They [4] showed that any quotient of a triangulated category modulo a cluster tilting subcategory carries an abelian structure. Let $\mathcal{C}$ be a triangulated category and $\mathcal{M}$ be a rigid subcategory, i.e. $\text{Hom}_\mathcal{C}(\mathcal{M}, \Sigma \mathcal{M}) = 0$. Iyama and Yoshino [5] showed that $\mathcal{M} * \Sigma \mathcal{M} / \Sigma \mathcal{M} \cong \text{mod } \mathcal{M}$. In particular, if $\mathcal{M}$ is a cluster tilting subcategory, then $\mathcal{M} * \Sigma \mathcal{M} = \mathcal{C}$, thus the work generalized some former results in [2,3,4].

Recently, Cluster tilting theory is also permitted to construct abelian categories from some exact categories. Let $\mathcal{B}$ be an exact category with enough projectives and $\mathcal{M}$ be a cluster tilting subcategory. Demonet and Liu [6] showed that $\mathcal{B}/\mathcal{M} \cong \text{mod } \mathcal{M}$, which generalized the work of Koenig and Zhu in the case of Frobenius categories.

The main aim of this article is to unify the work of Iyama-Yoshino and Demonet-Liu, and give a framework for construct abelian categories from triangulated categories and exact categories. Our setting is one-sided triangulated category, which is a natural generalization.
of triangulated category. Left and right triangulated categories were defined by Beligiannis and Marmaridis in [7]. For details and more information on one-sided triangulated categories we refer to [7-9].

The paper is organized as follows. In Section 2, we review some basic material on module categories over \( k \)-linear categories and quotient categories etc. In Section 3, we prove that some subquotient categories of right triangulated categories are module categories, which unifies the Proposition 6.2 in [4] and the Theorem 3.5 in [5]. In Section 4, we prove that some subquotient categories of left triangulated categories are module categories, which unifies the Proposition 6.2 in [4] and the Theorem 3.2 in [5]. And we will see that the case of right triangulated categories and the case of left triangulated categories are not dual.

2. Preliminaries

Throughout this paper, \( k \) denotes a field. When we say that \( C \) is a category, we always assume that \( C \) is a Hom-finite Krull-Schmidt \( k \)-linear category. For a subcategory \( M \) of category \( C \), we mean \( M \) is an additive full subcategory of \( C \) which is closed under taking direct summands. Let \( f : X \to Y \), \( g : Y \to Z \) be morphisms in \( C \), we denote by \( gf \) the composition of \( f \) and \( g \), and \( f_\ast \) the morphism \( \text{Hom}_C(M, f) : \text{Hom}_C(M, X) \to \text{Hom}_C(M, Y) \) for any \( M \in C \).

Let \( C \) be a category and \( \mathcal{X} \) be a subcategory of \( C \). A right \( \mathcal{X} \)-approximation of \( C \) in \( C \) is a map \( f : X \to C \), with \( X \in \mathcal{X} \), such that for all objects \( X' \in \mathcal{X} \), the sequence \( \text{Hom}_C(X', X) \to \text{Hom}_C(X', C) \to 0 \) is exact. If for any object \( C \in C \), there exists a right \( \mathcal{X} \)-approximation \( f : X \to C \), then \( \mathcal{X} \) is called a contravariantly finite subcategory of \( C \). Dually we have the notions of left \( \mathcal{X} \)-approximation and covariantly finite subcategory. \( \mathcal{X} \) is called functorially finite if \( \mathcal{X} \) is contravariantly finite and covariantly finite.

Let \( C \) be a category. A pseudokernel of a morphism \( v : V \to W \) in \( C \) is a morphism \( u : U \to V \) such that \( vu = 0 \) and if \( u' : U' \to V \) is a morphism such that \( vu' = 0 \), there exists \( f : U' \to U \) such that \( u' = uf \). Pseudocokernels are defined dually.

Let \( C \) be a category. A \( C \)-module is a contravariant \( k \)-linear functor \( F : C \to \text{Mod} k \). Then \( C \)-modules form an abelian category \( \text{Mod} C \). By Yoneda’s lemma, representable functors \( \text{Hom}_C(-, C) \) are projective objects in \( \text{Mod} C \). We denote by \( \text{mod} C \) the subcategory of \( \text{Mod} C \) consisting of finitely presented \( C \)-modules. One can easily check that \( \text{mod} C \) is closed under cokernels and extensions in \( \text{Mod} C \). Moreover, \( \text{mod} C \) is closed under kernels in \( \text{Mod} C \) if and only if \( C \) has pseudokernels. In this case, \( \text{mod} C \) forms an abelian category (see [10]). For example, if \( C \) is a contravariantly finite subcategory of a triangulated category, then \( \text{mod} C \) forms an abelian category.

Let \( C \) be an additive category and \( \mathcal{B} \) be a subcategory of \( C \). For any two objects \( X, Y \in C \), denote by \( \mathcal{B}(X, Y) \) the additive subgroup
of $\text{Hom}_C(X,Y)$ such that for any morphism $f \in \mathcal{B}(X,Y)$, $f$ factors through some object in $\mathcal{B}$. We denote by $\mathcal{C}/\mathcal{B}$ the quotient category whose objects are objects of $\mathcal{C}$ and whose morphisms are elements of $\text{Hom}_C(M,N)/\mathcal{B}(M,N)$. The projection functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{B}$ is an additive functor satisfying $\pi(\mathcal{B}) = 0$, and for any additive functor $F : \mathcal{C} \to \mathcal{D}$ satisfying $F(\mathcal{B}) = 0$, there exists a unique additive functor $G : \mathcal{C}/\mathcal{B} \to \mathcal{D}$ such that $F = G\pi$. We have the following two easy and useful facts.

**Lemma 2.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. If $F$ is full and dense, and there exists a subcategory $\mathcal{B}$ of $\mathcal{C}$ such that any morphism $f : X \to Y$ in $\mathcal{C}$ with $F(f) = 0$ factors through some object in $\mathcal{B}$, then $F$ induces an equivalence $\mathcal{C}/\mathcal{B} \cong \mathcal{D}$.

**Lemma 2.2.** Let $\mathcal{A}$ be an additive category, $\mathcal{B}$ and $\mathcal{C}$ be two subcategories of $\mathcal{A}$ with $\mathcal{C} \subset \mathcal{B}$. Then there exists an equivalence of categories $(\mathcal{A}/\mathcal{C})/(\mathcal{B}/\mathcal{C}) \cong \mathcal{A}/\mathcal{B}$.

**Proof.** Let $\pi_\mathcal{B} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ and $\pi_\mathcal{C} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ be the projection functors. Note that $\mathcal{C} \subset \mathcal{B}$, we have $\pi_\mathcal{B}(\mathcal{C}) = 0$, thus there exists a unique functor $F : \mathcal{A}/\mathcal{C} \to \mathcal{A}/\mathcal{B}$ such that $F\pi_\mathcal{C} = \pi_\mathcal{B}$. Since $\pi_\mathcal{B}$ is full and dense, $F$ is full and dense too.

Let $f : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(\pi_\mathcal{C}(f)) = 0$, that is $\pi_\mathcal{B}(f) = 0$. Then $f$ factors through some object in $\mathcal{B}$, thus $\pi_\mathcal{C}(f)$ factors through some object in $\mathcal{B}/\mathcal{C}$. According to Lemma 2.1, we have an equivalence of categories $(\mathcal{A}/\mathcal{C})/(\mathcal{B}/\mathcal{C}) \cong \mathcal{A}/\mathcal{B}$. \[\Box\]

3. **Subquotient categories of right triangulated categories**

Firstly, we recall some basics on right triangulated categories from [8].

**Definition 3.1.** A right triangulated category is a triple $(\mathcal{C}, \Sigma, \triangleright)$, or simply $\mathcal{C}$, where:

(a) $\mathcal{C}$ is an additive category.

(b) $\Sigma : \mathcal{C} \to \mathcal{C}$ is an additive functor, called the shift functor of $\mathcal{C}$.

(c) $\triangleright$ is a class of sequences of three morphisms of the form $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U$, called right triangles, and satisfying the following axioms:

(RTR0) If $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U$ is a right triangle, and $U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} \Sigma U'$ is a sequence of morphisms such that there exists a commutative diagram in $\mathcal{C}$

$$
\begin{array}{cccc}
U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & \Sigma U \\
\downarrow{f} & & \downarrow{g} & & \downarrow{k} & & \downarrow{\Sigma f} \\
U' & \xrightarrow{u'} & V' & \xrightarrow{v'} & W' & \xrightarrow{w'} & \Sigma U'
\end{array}
$$

$$
\begin{array}{cccc}
U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & \Sigma U \\
\downarrow{f} & & \downarrow{g} & & \downarrow{k} & & \downarrow{\Sigma f} \\
U' & \xrightarrow{u'} & V' & \xrightarrow{v'} & W' & \xrightarrow{w'} & \Sigma U'
\end{array}
$$

\]
where \(f, g, h\) are isomorphisms, then \(U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} \Sigma U'\) is also a right triangle.

(RTR1) For any \(U \in \mathcal{C}\), the sequence \(0 \to U \xrightarrow{1_U} U \to 0\) is a right triangle. And for any morphism \(u : U \to V\) in \(\mathcal{C}\), there exists a right triangle \(U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U\).

(RTR2) If \(U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U\) is a right triangle, then so is \(V \xrightarrow{v} W \xrightarrow{w} \Sigma U \xrightarrow{-u\Sigma} \Sigma V\).

(RTR3) For any two right triangles \(U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U\) and \(U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} \Sigma U'\) and any two morphisms \(f : U \to U'\), \(g : V \to V'\) such that \(gu = u'f\), there exists \(h : W \to W'\) such that the following diagram is commutative

\[
\begin{array}{c}
U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U \\
\downarrow f \quad \downarrow g \quad \downarrow h \quad \downarrow \Sigma f \\
U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} \Sigma U'.
\end{array}
\]

(RTR4) For any two right triangles \(U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U\) and \(U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} \Sigma U'\), there exists a commutative diagram

\[
\begin{array}{c}
U' \xrightarrow{u'} U \xrightarrow{v'} W \xrightarrow{w'} \Sigma U \\
\downarrow u \quad \downarrow f \quad \downarrow \Sigma f \\
U'' \xrightarrow{w'w} V \xrightarrow{p} V' \xrightarrow{q} \Sigma U'' \\
\downarrow v \quad \downarrow g \quad \downarrow \Sigma v'w \\
W \xrightarrow{w} W' \xrightarrow{\Sigma v'w} \Sigma W',
\end{array}
\]

where the second row and the third column are right triangles.

**Example 3.2.** A triangulated category \(\mathcal{C}\) is a right triangulated category, where the shift functor \(\Sigma\) is an equivalence. In this case, right triangles in \(\mathcal{C}\) are called triangles.

**Example 3.3.** (cf.[7,11]) Let \(\mathcal{B}\) be an exact category which contains enough injectives. The subcategory of injectives is denoted by \(\mathcal{I}\). Then the quotient category \(\overline{\mathcal{B}} = \mathcal{B}/\mathcal{I}\) is a right triangulated category. For any morphism \(f \in \text{Hom}_\mathcal{B}(X, Y)\), we denote its image in \(\text{Hom}_\overline{\mathcal{B}}(X, Y)\) by \(\overline{f}\). Let us recall the definitions of the shift functor \(\Sigma\) and of the distinguished right triangles. For any \(X \in \mathcal{B}\), there is a short exact sequence \(0 \to X \xrightarrow{i_X} I_X \xrightarrow{d_X} C_X \to 0\) with \(I_X \in \mathcal{I}\). For any morphism \(f : X \to Y\), we have the following commutative diagram with exact
rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \overset{ix}{\longrightarrow} & I_X & \overset{d_X}{\longrightarrow} & C_X & \longrightarrow & 0 \\
0 & \longrightarrow & Y & \overset{iy}{\longrightarrow} & I_Y & \overset{d_Y}{\longrightarrow} & C_Y & \longrightarrow & 0,
\end{array}
\]

where \( I_X, I_Y \in I \). Define \( \Sigma(X) = C_X \) and \( \Sigma f = \overline{c_f} \). We can show that the functor \( \Sigma \) is well defined. For any morphism \( f : X \rightarrow Y \), we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \overset{ix}{\longrightarrow} & I_X & \overset{d_X}{\longrightarrow} & C_X & \longrightarrow & 0 \\
0 & \longrightarrow & Y & \overset{g}{\longrightarrow} & Z & \overset{h}{\longrightarrow} & C_X & \longrightarrow & 0,
\end{array}
\]

where \( Z \) is the pushout of \( f \) and \( i_X \). Then \( X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z \xrightarrow{\overline{h}} \Sigma X \), or equivalently \( X \xrightarrow{(\overline{f}, \overline{g} - \overline{h})} Y \oplus I_X \xrightarrow{\overline{\alpha}} Z \xrightarrow{\overline{\beta}} \Sigma X \) is a distinguished right triangle. In this case, there is a short exact sequence \( 0 \rightarrow X \xrightarrow{(f, i_X)} Y \oplus I_X \xrightarrow{(g, i_f)} Z \rightarrow 0 \). And we have the following commutative diagram of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{(f, i_X)} & Y \oplus I_X & \xrightarrow{(g, i_f)} & Z & \longrightarrow & 0 \\
0 & \longrightarrow & X & \overset{i_X}{\longrightarrow} & I_X & \overset{d_X}{\longrightarrow} & \Sigma X & \longrightarrow & 0,
\end{array}
\]

So a distinguished right triangle in \( \overline{B} \) give rise to a short exact sequence in \( B \). On the other hand, Let \( 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0 \) be a short exact sequence in \( B \), then we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
0 & \longrightarrow & X & \overset{iy}{\longrightarrow} & I_Y & \overset{p}{\longrightarrow} & \Sigma X & \longrightarrow & 0,
\end{array}
\]

where \( I_Y \in I \), and \( X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z \xrightarrow{\overline{h}} \Sigma X \) is a right triangle in \( \overline{B} \) [11]. Thus, a short exact sequence in \( B \) give rise to a right triangle in \( \overline{B} \).

The following lemma can be found in [7].

**Lemma 3.4.** Let \( C \) be a right triangulated category, and \( U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma U \) be a right triangle.

(a) \( v \) is a pseudocokernel of \( u \), and \( w \) is a pseudocokernel of \( v \).
(b) If $\Sigma$ is fully faithful, then $u$ is a pseudokernel of $v$, and $v$ is a pseudokernel of $w$.

**Definition 3.5.** Let $\mathcal{C}$ be a right triangulated category. A subcategory $\mathcal{M}$ of $\mathcal{C}$ is called a rigid subcategory if $\text{Hom}_\mathcal{C}(\mathcal{M}, \Sigma \mathcal{M}) = 0$.

Let $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$. Denote by $\mathcal{M} \ast \Sigma \mathcal{M}$ the subcategory of $\mathcal{C}$ consisting of all such $X \in \mathcal{C}$ with right triangles $M_0 \to M_1 \to X \to \Sigma M_0$, where $M_0, M_1 \in \mathcal{M}$.

Now we can state the main theorem of this section.

**Theorem 3.6.** Let $\mathcal{C}$ be a right triangulated category, and $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$ satisfying:

1. $(\text{RC1})$ $\Sigma$ is fully faithful when it is restricted to $\mathcal{M}$.
2. $(\text{RC2})$ For any two objects $M_0, M_1 \in \mathcal{M}$, if $M_0 \xrightarrow{f} M_1 \xrightarrow{g} X \xrightarrow{h} \Sigma M_0$ is a right triangle in $\mathcal{C}$, then $g$ is a right $\mathcal{M}$-approximation of $X$.

Then there exists an equivalence of categories $\mathcal{M} \ast \Sigma \mathcal{M}/\Sigma \mathcal{M} \cong \text{mod}\mathcal{M}$. 

Before prove the theorem, we prove the lemma as follow.

**Lemma 3.7.** Under the same assumption as in Theorem 3.6, for any right triangle $M_0 \xrightarrow{f} M_1 \xrightarrow{g} X \xrightarrow{h} \Sigma M_0$ where $M_0, M_1 \in \mathcal{M}$, there is an exact sequence in $\text{mod}\mathcal{M}$

$$\text{Hom}_{\mathcal{M}}(-, M_0) \xrightarrow{\text{Hom}_{\mathcal{C}}(-, f)} \text{Hom}_{\mathcal{M}}(-, M_1) \xrightarrow{\text{Hom}_{\mathcal{C}}(-, g)} \text{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{M}} \to 0.$$ 

Thus, $\text{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{M}} \in \text{mod}\mathcal{M}$.

**Proof.** Let $M_0 \xrightarrow{f} M_1 \xrightarrow{g} X \xrightarrow{h} \Sigma M_0$ be a right triangle with $M_0, M_1 \in \mathcal{M}$. For any $M \in \mathcal{M}$, we claim that the following sequence is exact

$$\text{Hom}_{\mathcal{C}}(M, M_0) \xrightarrow{f} \text{Hom}_{\mathcal{C}}(M, M_1) \xrightarrow{g_*} \text{Hom}_{\mathcal{C}}(M, X) \to 0. \quad (\ast)$$

In fact, by Lemma 3.4 (a), we have $gf = 0$, hence $\text{Im} f_* \subseteq \text{Ker} g_*$. For any $t \in \text{Ker} g_*$, we have the following commutative diagram of right triangles by (RTR3)

$$
\begin{array}{ccc}
M & \xrightarrow{0} & \Sigma M \\
\downarrow t & & \downarrow \Sigma t \\
M_1 & \xrightarrow{g} & \Sigma M_0 \\
\downarrow m' & & \downarrow \Sigma f \\
\end{array}
$$

Since $\Sigma|_{\mathcal{M}}$ is full, there exists a morphism $m : M \to M_0$ such that $m' = \Sigma m$, so $\Sigma t = \Sigma(fm)$. Since $\Sigma|_{\mathcal{M}}$ is faithful, $t = fm = f_*m \in \text{Im} f_*$, then $\text{Im} f_* \supseteq \text{Ker} g_*$. Hence $\text{Im} f_* \supseteq \text{Ker} g_*$. On the other hand, by (RC2), $g_*$ is surjective. So (\ast) is exact. Since $M$ is arbitrary in $\mathcal{M}$, there exists an exact sequence

$$\text{Hom}_{\mathcal{M}}(-, M_0) \xrightarrow{\text{Hom}_{\mathcal{C}}(-, f)} \text{Hom}_{\mathcal{M}}(-, M_1) \xrightarrow{\text{Hom}_{\mathcal{C}}(-, g)} \text{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{M}} \to 0.$$ 

\qed
Proof of Theorem 3.6. By Lemma 3.7, we have an additive functor $F : \mathcal{M} \ast \Sigma \mathcal{M} \to \text{Mod}\mathcal{M}$, which is defined by $F(X) = \text{Hom}_C(-, X)|_{\mathcal{M}}$.

Firstly, we show that $F$ is dense.

For any object $G \in \text{mod}\mathcal{M}$, there exists an exact sequence

$$
\text{Hom}_\mathcal{M}(-, M') \xrightarrow{\alpha} \text{Hom}_\mathcal{M}(-, M'') \to G \to 0
$$

with $M', M'' \in \mathcal{M}$. By Yoneda’s Lemma, there exists a morphism $f : M' \to M''$ such that $\alpha = \text{Hom}_\mathcal{M}(-, f)$. Then by (RTR1), there exists a right triangle $M' \xrightarrow{f} M'' \xrightarrow{g} Z \xrightarrow{h} \Sigma M'$. By Lemma 3.7, there exists an exact sequence $\text{Hom}_\mathcal{M}(-, M') \xrightarrow{\alpha} \text{Hom}_\mathcal{M}(-, M'') \to F(Z) \to 0$, thus $G = \text{Coker} \alpha \cong F(Z)$. Hence $F$ is dense.

Secondly, we show that $F$ is full.

For any morphism $\beta : F(X) \to F(Y)$ in $\text{mod}\mathcal{M}$, because $\text{Hom}_\mathcal{M}(-, M_1)$ is a projective object in $\text{mod}\mathcal{M}$, we have the following commutative diagram with exact rows in $\text{Mod}\mathcal{M}$

$$
\text{Hom}_\mathcal{M}(-, M_0) \xrightarrow{\text{Hom}_\mathcal{M}(-, f_1)} \text{Hom}_\mathcal{M}(-, M_1) \xrightarrow{\gamma_0} F(X) \xrightarrow{\beta} 0
$$

By Yoneda’s Lemma, for $i = 0, 1$, there exists a morphism $m_i : M_i \to N_i$ such that $\gamma_i = \text{Hom}_\mathcal{M}(-, m_i)$ and $m_1 f_1 = f_2 m_0$. Hence by (RTR3) we have the following commutative diagram of right triangles

$$
\begin{array}{ccc}
M_0 & \xrightarrow{f_1} & M_1 \\
\downarrow m_0 & & \downarrow m_1 \\
N_0 & \xrightarrow{f_2} & N_1
\end{array}
\xrightarrow{i}
\begin{array}{ccc}
X & \xrightarrow{h_1} & \Sigma M_0 \\
\downarrow s & & \downarrow s \\
Y & \xrightarrow{h_2} & \Sigma N_0
\end{array}
$$

Then by Lemma 3.7, we have the following commutative diagram with exact rows in $\text{Mod}\mathcal{M}$

$$
\text{Hom}_\mathcal{M}(-, M_0) \xrightarrow{\text{Hom}_\mathcal{M}(-, f_2)} \text{Hom}_\mathcal{M}(-, M_1) \xrightarrow{\gamma_0} F(X) \xrightarrow{F(s)} 0
$$

So $\beta = F(s)$. Hence $F$ is full.

At last, in order to show $\mathcal{M} \ast \Sigma \mathcal{M}/\Sigma \mathcal{M} \cong \text{mod}\mathcal{M}$, by Lemma 2.1 we only need to prove that any morphism $t : X \to Y$ in $\mathcal{M} \ast \Sigma \mathcal{M}$ satisfying $F(t) = 0$ factors through some object in $\Sigma \mathcal{M}$.

In fact, let $M_0 \xrightarrow{f_1} M_1 \xrightarrow{g_1} X \xrightarrow{h_1} \Sigma M_0$ be a right triangle with $M_0, M_1 \in \mathcal{M}$, then $tg_1 = 0$ since $F(t) = 0$. Thus by Lemma 3.4(a), $t$ factors through $h_1$, so $t$ factors through $\Sigma M_0 \in \Sigma \mathcal{M}$. □

Applying Theorem 3.6, we can get the following two corollaries.
Corollary 3.8. ([4, Proposition 6.2]) Let $C$ be a triangulated category with the shift functor $\Sigma$ and $M$ be a rigid subcategory of $C$. Then there exists an equivalence of categories $M \prec \Sigma M/\Sigma M \cong \text{mod} M$.

Proof. Since the shift functor $\Sigma$ is an equivalence, we know that $\Sigma|_M$ is fully faithful. Let $M_0 \xrightarrow{f} M_1 \xrightarrow{g} X \xrightarrow{h} \Sigma M_0$ be a triangle in $C$, where $M_0, M_1 \in M$. Since $M$ is rigid, we know that $g$ is a right $M$-approximation of $X$ by Lemma 3.4(b). Thus, condition (RC1) and (RC2) hold. $\square$

Definition 3.9. Let $B$ be an exact category and $M$ be a full subcategory of $B$. $M$ is called rigid if $\text{Ext}^1_B(M, M) = 0$.

Corollary 3.10. ([6, Theorem 3.5]) Let $B$ be an exact category which contains enough injectives, and $M$ be a rigid subcategory of $B$ containing all injectives. Denote by $I$ the subcategory of injectives, and by $\overline{M}$ the quotient category $M/I$. Denote by $M_R$ the subcategory of objects $X$ in $B$ such that there exist short exact sequences $0 \rightarrow M_0 \rightarrow M_1 \rightarrow X \rightarrow 0$, where $M_0, M_1 \in M$. Denote by $\Sigma M$ the subcategory of objects $Y$ in $B$ such that there exist short exact sequences $0 \rightarrow M \rightarrow I \rightarrow Y \rightarrow 0$, where $M \in M$, $I \in I$. Then $M_R/\Sigma M \cong \text{mod} \overline{M}$.

Proof. According to Theorem 3.6, we prove the corollary by several steps.

(a) $\overline{M}$ is a rigid subcategory of the right triangulated category $\overline{B} = B/I$.

Let $\Sigma$ be the shift functor of $\overline{B}$, then it is easy to see that $\Sigma \overline{M} = \overline{\Sigma M}$. We claim that $\text{Hom}_{\overline{B}}(\overline{M}, \overline{\Sigma M}) = 0$. In fact, for any $f \in \text{Hom}_{\overline{B}}(M, Y)$, where $M \in \overline{M}$ and $Y \in \overline{\Sigma M}$. There is a short exact sequence $0 \rightarrow M' \xrightarrow{i} I \xrightarrow{d} Y \rightarrow 0$, where $M' \in M$, $I \in I$. Since $M$ is rigid in $B$, applying $\text{Hom}_B(M, -)$ to the short exact sequence, we have an exact sequence

$$0 \rightarrow \text{Hom}(M, M') \xrightarrow{\delta} \text{Hom}(M, I) \xrightarrow{d} \text{Hom}(M, Y) \rightarrow 0.$$ 

So $d$ is a right $M$-approximation of $Y$. Thus, $f$ factors through $I$, hence $f = 0$.

(b) $\overline{M}_R = \overline{M} \ast \Sigma \overline{M}$.

It follows from Example 3.3.

(c) $\overline{M}_R/\Sigma \overline{M} \cong \overline{M}_R/\Sigma \overline{M}$.

It follows from Lemma 2.2 since $I \subset \Sigma \overline{M} \subset \overline{M}_R$ and $\Sigma \overline{M} = \overline{\Sigma M}$.

(d) $\Sigma |_{\overline{M}}$ is fully faithful.

For any $M', M'' \in \overline{M}$, there exist two short exact sequences $0 \rightarrow M' \xrightarrow{i} I_{M'} \xrightarrow{d_{M'}} \Sigma M' \rightarrow 0$ and $0 \rightarrow M'' \xrightarrow{i_{M''}} I_{M''} \xrightarrow{d_{M''}} \Sigma M'' \rightarrow 0$, where $I_{M'}, I_{M''} \in I$, and $d_{M'}, d_{M''}$ are right $M$-approximations.

For any morphism $\alpha : \Sigma M' \rightarrow \Sigma M''$ in $B$, since $d_{M''}$ is a right $M$-approximation and $I_{M'} \in I \subset \overline{M}$, we have the following commutative
diagram with exact rows in $\mathcal{B}$

\[
\begin{array}{cccccc}
0 & \to & M' & \overset{i_{M'}}{\to} & I_{M'} & \overset{d_{M'}}{\to} & \Sigma M' & \to & 0 \\
& & \downarrow{i_{M}} & & \downarrow{j} & & \downarrow{\alpha} \\
0 & \to & M'' & \overset{i_{M''}}{\to} & I_{M''} & \overset{d_{M''}}{\to} & \Sigma M'' & \to & 0.
\end{array}
\]

Hence we have $\alpha = \Sigma \bar{m}$ by the definition of $\Sigma$, thus $\Sigma|_{\mathcal{M}}$ is full.

For any morphism $f : M' \to M''$ in $\mathcal{B}$, Since $I_{M'}$ is an injective object, we have the following commutative diagram of short exact sequences

\[
\begin{array}{cccccc}
0 & \to & M' & \overset{i_{M'}}{\to} & I_{M'} & \overset{d_{M'}}{\to} & \Sigma M' & \to & 0 \\
& & \downarrow{f} & & \downarrow{i_{f}} & & \downarrow{\Sigma f} \\
0 & \to & M'' & \overset{i_{M''}}{\to} & I_{M''} & \overset{d_{M''}}{\to} & \Sigma M'' & \to & 0.
\end{array}
\]

Suppose $\Sigma \bar{f} = 0$, then $\Sigma f$ factors through some object in $\mathcal{I}$. Because $d_{M''}$ is right $\mathcal{M}$-approximation, $\Sigma f$ factors through $I_{M''}$, i.e. there exists a morphism $a : \Sigma M' \to I_{M''}$ such that $\Sigma f = d_{M''}a$. Then $d_{M''}(i_f - ad_{M'}) = d_{M''}i_f - (\Sigma f)d_{M'} = 0$, thus there exists a morphism $b : I_{M'} \to M''$ such that $i_{M'}b = i_f - ad_{M'}$, so $i_{M''}(f - bi_{M'}) = i_{M''}f - i_{f}i_{M'} + ad_{M'}i_{M'} = 0$. Since $i_{M''}$ is a monomorphism, $f = bi_{M'}$, thus $f$ factors through $I_{M'}$. Hence $\bar{f} = 0$ and $\Sigma|_{\mathcal{M}}$ is faithful.

(e) Let $M' \xrightarrow{f} M'' \xrightarrow{g} X \xrightarrow{h} \Sigma M'$ be a right triangle in $\mathcal{B}$ with $M', M'' \in \mathcal{M}$, then $\bar{g}$ is a right $\overline{\mathcal{M}}$-approximation of $X$.

According to Example 3.3 and $\mathcal{I} \subset \mathcal{M}$, we can assume that there is a short exact sequence $0 \to M' \xrightarrow{j} M'' \xrightarrow{g} X \to 0$. Since $\mathcal{M}$ is rigid, there exists an epimorphism $\text{Hom}_{\mathcal{B}}(M, g) : \text{Hom}_{\mathcal{B}}(M, M'') \to \text{Hom}_{\mathcal{B}}(M, X)$ for any $M$ in $\mathcal{M}$. Thus we have an epimorphism $\text{Hom}_{\mathcal{B}}(M, \bar{g}) : \text{Hom}_{\mathcal{B}}(M, M'') \to \text{Hom}_{\mathcal{B}}(M, X)$, i.e. $\bar{g}$ is a right $\overline{\mathcal{M}}$-approximation of $X$. \hfill \qedsymbol

4. Subquotient categories of left triangulated categories

The definition of left triangulated category is dual to right triangulated category. For convenience, we recall the definition and some facts.

**Definition 4.1.** ([7]) A left triangulated category is a triple $(\mathcal{C}, \Omega, \preceq)$, or simply $\mathcal{C}$, where:

(a) $\mathcal{C}$ is an additive category.

(b) $\Omega : \mathcal{C} \to \mathcal{C}$ is an additive functor, called the shift functor of $\mathcal{C}$.

(c) $\preceq$ is a class of sequences of three morphisms of the form $\Omega Z \xrightarrow{\gamma} X \xrightarrow{\mu} Y \xrightarrow{\nu} Z$, called left triangles, and satisfying the following axioms:

(LTR0) If $\Omega Z \xrightarrow{\gamma} X \xrightarrow{\mu} Y \xrightarrow{\nu} Z$ is a left triangle, and $\Omega Z' \xrightarrow{\gamma'} X' \xrightarrow{\mu'} Y' \xrightarrow{\nu'} Z'$ is a sequence of morphisms such that there exists a
commutative diagram in $\mathcal{C}$

\[
\begin{array}{c}
\Omega Z \xrightarrow{x} X \xrightarrow{y} Y \xrightarrow{z} Z \\
\downarrow \Omega h \downarrow f \downarrow g \downarrow h \\
\Omega Z' \xrightarrow{x'} X' \xrightarrow{y'} Y' \xrightarrow{z'} Z',
\end{array}
\]

where $f, g, h$ are isomorphisms, then $\Omega Z' \xrightarrow{x'} X' \xrightarrow{y'} Y' \xrightarrow{z'} Z'$ is also a left triangle.

(LTR1) For any $X \in \mathcal{C}$, the sequence $0 \to X \xrightarrow{1_X} X \to 0$ is a left triangle. And for every morphism $z : Y \to Z$ in $\mathcal{C}$, there exists a left triangle $\Omega Z \xrightarrow{\Omega z} X \xrightarrow{\Omega y} Y \xrightarrow{\Omega z} Z$.

(LTR2) If $\Omega Z \xrightarrow{\Omega z} X \xrightarrow{y} Y \xrightarrow{z} Z$ is a left triangle, then so is $\Omega Y \xrightarrow{\Omega y} \Omega Z \xrightarrow{\Omega z} X \xrightarrow{y} Y \xrightarrow{z} Z$.

(LTR3) For any two left triangles $\Omega Z \xrightarrow{\Omega z} X \xrightarrow{y} Y \xrightarrow{z} Z$ and $\Omega Z' \xrightarrow{\Omega z'} X' \xrightarrow{y'} Y' \xrightarrow{z'} Z'$, and any two morphisms $g : Y \to Y', h : Z \to Z'$ such that $hz = z'g$, there exists $f : X \to X'$ making the following diagram commutative

\[
\begin{array}{c}
\Omega Z \xrightarrow{x} X \xrightarrow{y} Y \xrightarrow{z} Z \\
\downarrow \Omega h \downarrow f \downarrow g \downarrow h \\
\Omega Z' \xrightarrow{x'} X' \xrightarrow{y'} Y' \xrightarrow{z'} Z'
\end{array}
\]

(LTR4) For any two left triangles $\Omega Z \xrightarrow{\Omega z} X \xrightarrow{y} Y \xrightarrow{z} Z$ and $\Omega Z' \xrightarrow{\Omega z'} X' \xrightarrow{y'} Y \xrightarrow{z'} Z'$, there exists a commutative diagram

\[
\begin{array}{c}
\Omega Y' \xrightarrow{\Omega y'} \Omega Z \\
\downarrow x \downarrow f \downarrow g \downarrow h \\
\Omega Z' \xrightarrow{u} X' \xrightarrow{v} Y \xrightarrow{z'} Z'
\end{array}
\]

where the third row and the second column are left triangles.

**Example 4.2.** A triangulated category is a left triangulated category.

**Example 4.3.** Let $\mathcal{B}$ be an exact category with enough projectives. Denote by $\mathcal{P}$ the subcategory of $\mathcal{B}$ consisting of projectives. Then the quotient category $\mathcal{B} = \mathcal{B}/\mathcal{P}$ is a left triangulated category.

By (LTR0) and (LTR2), we have the following easy lemma.
Lemma 4.4. Let $\Omega Z \xrightarrow{\bar{x}} X \xrightarrow{\bar{y}} Y \xrightarrow{\bar{z}} Z$ be a left triangle, then so is $\Omega Y \xrightarrow{\bar{z}} \Omega Z \xrightarrow{\bar{x}} X \xrightarrow{\bar{y}} Y$.

Lemma 4.5. (cf. [8]) Let $\mathcal{C}$ be a left triangulated category. Then for any left triangle $\Omega Z \xrightarrow{\bar{z}} X \xrightarrow{\bar{y}} Y \xrightarrow{\bar{z}} Z$ and any object $U$ of $\mathcal{C}$, there exists an exact sequence

$$
\cdots \rightarrow \text{Hom}_\mathcal{C}(U, \Omega Z) \xrightarrow{\bar{z}} \text{Hom}_\mathcal{C}(U, X) \xrightarrow{\bar{y}} \text{Hom}_\mathcal{C}(U, Y) \xrightarrow{\bar{z}} \text{Hom}_\mathcal{C}(U, Z).
$$

Definition 4.6. Let $\mathcal{C}$ be a left triangulated category. A subcategory $\mathcal{M}$ of $\mathcal{C}$ is called a rigid subcategory if $\text{Hom}_\mathcal{C}(\Omega M, \Omega M) = 0$.

Let $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$. Denote by $\Omega \mathcal{M} \ast \mathcal{M}$ the subcategory of objects $X$ in $\mathcal{C}$ such that there exist left triangles $\Omega M_1 \xrightarrow{f} X \xrightarrow{g} M_0 \xrightarrow{h} M_1$ where $M_0, M_1 \in \mathcal{M}$. Now we consider the functor $H : \Omega \mathcal{M} \ast \mathcal{M} \rightarrow \text{Mod} \mathcal{M}$ defined by $H(X) = \text{Hom}_\mathcal{C}(\Omega(-), X)|_{\mathcal{M}}$.

Lemma 4.7. Let $(\mathcal{C}, \Omega, \triangleleft)$ be a left triangulated category and $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$. If $\Omega|_\mathcal{M}$ is fully faithful, then for any left triangle $\Omega M_1 \xrightarrow{f} X \xrightarrow{g} M_0 \xrightarrow{h} M_1$ where $M_0, M_1 \in \mathcal{M}$, there is an exact sequence in $\text{Mod} \mathcal{M}$

$$
\text{Hom}_\mathcal{M}(-, M_0) \xrightarrow{\text{Hom}_\mathcal{M}(-, h)} \text{Hom}_\mathcal{M}(-, M_1) \rightarrow H(X) \rightarrow 0.
$$

Thus, $H(X) \in \text{mod} \mathcal{M}$.

Proof. For any $X \in \Omega \mathcal{M} \ast \mathcal{M}$, there exists a left triangle $\Omega M_1 \xrightarrow{f} X \xrightarrow{g} M_0 \xrightarrow{h} M_1$, where $M_0, M_1 \in \mathcal{M}$. Then $\Omega M_0 \xrightarrow{\Omega h} \Omega M_1 \xrightarrow{f} X \xrightarrow{g} M_0$ is a left triangle by Lemma 4.4. Thus there exists an exact sequence by Lemma 4.5

$$
\text{Hom}_\mathcal{C}(\Omega M, \Omega M_0) \xrightarrow{(\Omega h)_*} \text{Hom}_\mathcal{C}(\Omega M, \Omega M_1) \xrightarrow{f_*} \\
\text{Hom}_\mathcal{C}(\Omega M, X) \rightarrow \text{Hom}_\mathcal{C}(\Omega M, M_0) = 0.
$$

Since $\Omega|_\mathcal{M}$ is fully faithful, we have the following commutative diagram with exact rows

$$
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(M, M_0) & \xrightarrow{h_*} & \text{Hom}_\mathcal{C}(M, M_1) \rightarrow \text{Hom}_\mathcal{C}(\Omega M, X) \rightarrow 0 \\
\text{Hom}_\mathcal{C}(\Omega M, \Omega M_0) & \xrightarrow{(\Omega h)_*} & \text{Hom}_\mathcal{C}(\Omega M, \Omega M_1) \rightarrow \text{Hom}_\mathcal{C}(\Omega M, X) \rightarrow 0,
\end{array}
$$

where $M \in \mathcal{M}$ and the vertical morphisms are isomorphisms. Thus we have an exact sequence in $\text{Mod} \mathcal{C}$

$$
\text{Hom}_\mathcal{M}(-, M_0) \xrightarrow{\text{Hom}_\mathcal{M}(-, h)} \text{Hom}_\mathcal{M}(-, M_1) \rightarrow H(X) \rightarrow 0.
$$

So $H(X) \in \text{mod} \mathcal{M}$. □
Theorem 4.8. Let $C$ be a left triangulated category, and $\mathcal{M}$ be a rigid subcategory of $C$ satisfying:

(LC1) $\Omega$ is fully faithful when it is restricted to $\mathcal{M}$.

(LC2) Let $\Omega \mathcal{M}_1 \xrightarrow{f} X \xrightarrow{g} \mathcal{M}_0 \xrightarrow{h} \mathcal{M}_1$ be a left triangle, where $\mathcal{M}_0, \mathcal{M}_1 \in \mathcal{M}$. Let $Y \in \Omega \mathcal{M} \ast \mathcal{M}$ and a morphism $t : X \to Y$ such that $tf = 0$, then $t$ factors through $g$.

Then there exists an equivalence of categories $\Omega \mathcal{M} \ast \mathcal{M} \cong \text{mod} \mathcal{M}$.

Proof. According to Lemma 4.7, we have a functor $H : \Omega \mathcal{M} \ast \mathcal{M} \to \text{mod} \mathcal{M}$.

Firstly, we show that $H$ is dense.

For any object $G \in \text{mod} \mathcal{M}$, there exists an exact sequence

$$\text{Hom}_{\mathcal{M}}(-, M') \xrightarrow{\alpha} \text{Hom}_{\mathcal{M}}(-, M'') \to G \to 0$$

with $M', M'' \in \mathcal{M}$. By Yoneda’s Lemma, there exists a morphism $h : M' \to M''$ such that $\alpha = \text{Hom}_{\mathcal{M}}(-, h)$. Then by (LTR1), there exists a left triangle $\Omega M'' \xrightarrow{f} Z \xrightarrow{g} \mathcal{M}_0 \xrightarrow{h} \mathcal{M}_1$. Hence by Lemma 4.7, there exists an exact sequence

$$\text{Hom}_{\mathcal{M}}(-, M') \xrightarrow{\alpha} \text{Hom}_{\mathcal{M}}(-, M'') \to H(Z) \to 0,$$

so $G = \text{Coker} \alpha \cong H(Z)$. Hence $H$ is dense.

Secondly, we show that $H$ is full.

For any morphism $\beta : H(X) \to H(Y)$ in $\text{mod} \mathcal{M}$. By Lemma 4.7 and because $\text{Hom}_{\mathcal{M}}(-, M_1)$ is a projective object of $\text{mod} \mathcal{M}$, we have the following commutative diagram with exact rows in $\text{Mod} \mathcal{M}$

$$\begin{array}{c}
\text{Hom}_{\mathcal{M}}(-, M_0) \xrightarrow{\text{Hom}_{\mathcal{M}}(-, h_1)} \text{Hom}_{\mathcal{M}}(-, M_1) \xrightarrow{\gamma_0} H(X) \to 0 \\
\text{Hom}_{\mathcal{M}}(-, N_0) \xrightarrow{\text{Hom}_{\mathcal{M}}(-, h_2)} \text{Hom}_{\mathcal{M}}(-, N_1) \xrightarrow{\gamma_1} H(Y) \to 0.
\end{array}$$

By Yoneda’s Lemma, for $i = 0, 1$, there exists a morphism $m_i : M_i \to N_i$ such that $\gamma_i = \text{Hom}_{\mathcal{M}}(-, m_i)$ and $m_1 h_1 = h_2 m_0$. Hence by (LTR3), we have the following commutative diagram of left triangles

$$\begin{array}{c}
\Omega M_1 \xrightarrow{f_0} X \xrightarrow{g_1} \mathcal{M}_0 \xrightarrow{h_1} \mathcal{M}_1 \\
\Omega M_1 \xrightarrow{m_0} X \xrightarrow{m_1} Y \xrightarrow{g_2} N_0 \xrightarrow{h_2} N_1.
\end{array}$$
According to the proof of Lemma 4.7, for any object $M \in \mathcal{M}$, we have the following commutative diagram with exact columns.

Thus we have the following commutative diagram with exact rows in $\text{Mod } \mathcal{M}$

So $\beta = H(s)$. Hence $H$ is full.

At last, let $X, Y$ be objects of $\Omega \mathcal{M} \ast \mathcal{M}$. We have a left triangle $\Omega M_1 \xrightarrow{f} X \xrightarrow{g} M_0 \xrightarrow{h} M_1$, where $M_0, M_1 \in \mathcal{M}$. Let $t : X \to Y$ be a morphism with $H(t) = 0$, then $tf = 0$. Thus $t$ factors through $M_0$ by (LC2). So $\Omega \mathcal{M} \ast \mathcal{M} / \mathcal{M} \cong \text{mod } \mathcal{M}$ by Lemma 2.2. $\square$

Since a triangulated category is a left triangulated category such that the shift functor is an equivalence, the conditions (LC1) and (LC2) holds automatically. Thus we have the following corollary.

**Corollary 4.9.** Let $C$ be a triangulated category with the shift functor $T$ and $\mathcal{M}$ be a rigid subcategory of $C$, then $T^{-1} \mathcal{M} \ast \mathcal{M} / \mathcal{M} \cong \text{mod } \mathcal{M}$.

**Corollary 4.10.** ([5], Theorem 3.2) Let $\mathcal{B}$ be an exact category which contains enough projectives, and $\mathcal{M}$ be a rigid subcategory of $\mathcal{B}$ containing all projectives. Denote by $\mathcal{P}$ the subcategory of projectives, and by $\mathcal{M}_L$ the quotient category $\mathcal{M} / \mathcal{P}$. Denote by $\mathcal{M}_L$ the subcategory of objects $X$ in $\mathcal{B}$ such that there exist short exact sequences $0 \to X \to M_0 \to M_1 \to 0$, where $M_0, M_1 \in \mathcal{M}$. Then $\mathcal{M}_L / \mathcal{M} \cong \text{mod } \mathcal{M}$. 
Proof. Similar to the proof of Corollary 3.10, we can prove that \( \mathcal{M} \) is a rigid subcategory of the left triangulated category \( \mathcal{B} \), and \( \mathcal{M}_L = \Omega \mathcal{M} \ast \mathcal{M} \), and \( \mathcal{M}_L/\mathcal{M} \cong \mathcal{M}_L/\mathcal{M} \), and \( \Omega |_{\mathcal{M}} \) is fully faithful. To end the proof, we only need to show that \( \mathcal{M} \) satisfies the condition (LC2).

In fact, let \( \Omega M'' \xrightarrow{f_1} X \xrightarrow{g_1} M' \xrightarrow{h_1} M'' \) be a left triangle in \( \mathcal{B} \), where \( M', M'' \in \mathcal{M} \). Since \( \mathcal{P} \subset \mathcal{M} \), we can assume that \( 0 \to X \xrightarrow{g_1} M' \xrightarrow{h_1} M'' \to 0 \) is a short exact sequence. Let \( t : X \to Y \) be a morphism satisfying \( tf_1 = 0 \), where \( Y \in \mathcal{M}_L \). Then there exists a short exact sequence \( 0 \to Y \xrightarrow{g_2} N' \xrightarrow{h_2} N'' \to 0 \), where \( N', N'' \in \mathcal{M} \). Since \( \mathcal{M} \) is rigid, it is easy to see that \( g_1 \) is a left \( \mathcal{M} \)-approximation, then we have the following commutative diagram with exact rows in \( \mathcal{B} \):

\[
\begin{array}{ccccccccc}
0 & \to & X & \xrightarrow{g_1} & M' & \xrightarrow{h_1} & M'' & \to & 0 \\
& & & \downarrow{t} & & \downarrow{m_1} & & \downarrow{m_2} \\
0 & \to & Y & \xrightarrow{g_2} & N' & \xrightarrow{h_2} & N'' & \to & 0 \\
\end{array}
\]

The lower exact sequence induces a left triangle \( \Omega N'' \xrightarrow{f_2} Y \xrightarrow{g_2} N' \xrightarrow{h_2} N'' \). We claim that \( tf_1 = f_2 \Omega m_2 \). In fact, we have the following diagram with exact rows in \( \mathcal{B} \):

\[
\begin{array}{ccccccccc}
0 & \to & \Omega M'' & \xrightarrow{i_{M''}} & P_{M''} & \xrightarrow{d_{M''}} & M'' & \to & 0 \\
0 & \to & X & \xrightarrow{g_1} & M' & \xrightarrow{h_1} & M'' & \to & 0 \\
0 & \to & \Omega N'' & \xrightarrow{i_{N''}} & P_{N''} & \xrightarrow{d_{N''}} & N'' & \to & 0 \\
0 & \to & Y & \xrightarrow{g_2} & N' & \xrightarrow{h_2} & N'' & \to & 0 \\
\end{array}
\]

where \( P_{M''}, P_{N''} \in \mathcal{P} \), and all squares are commutative except the left one and the middle one. Since \( h_2(m_1 p_M - p_{NP}) = m_2 d_{M''} - m_2 d_{M''} = 0 \), there exists a morphism \( q : P_{M''} \to Y \) such that \( g_2 q = m_1 p_M - p_{NP} \).

Then \( g_2 tf_1 - f_2 \Omega m_2 = (m_1 p_M - p_{NP}) i_{M''} - (m_1 p_M - p_{NP}) i_{M''} = 0 \). Since \( g_2 \) is a monomorphism, we get \( tf_1 = f_2 \Omega m_2 = q i_{M''} \). Thus \( tf_1 = f_2 \Omega m_2 \). Hence we have the following commutative diagram of left triangles in \( \mathcal{B} \):

\[
\begin{array}{cccc}
\Omega M'' & \xrightarrow{f_1} & X & \xrightarrow{g_1} & M' & \xrightarrow{h_1} & M'' \\
\Omega m_2 & \downarrow{f_1} & \downarrow{t} & \downarrow{m_1} & \downarrow{m_2} \\
\Omega N'' & \xrightarrow{f_2} & Y & \xrightarrow{g_2} & N' & \xrightarrow{h_2} & N'' \\
\end{array}
\]
By Lemma 4.4, we have the following commutative diagram of left triangles in $B$

\[
\begin{array}{c}
\Omega M' \xrightarrow{\Omega h_2} \Omega M'' \xrightarrow{f_1} X \xrightarrow{-g_1} M' \\
\Omega M' \xrightarrow{\Omega h_1} \Omega M'' \xrightarrow{f_1} X \xrightarrow{g_1} M' \\
\Omega N' \xrightarrow{\Omega h_2} \Omega N'' \xrightarrow{f_2} Y \xrightarrow{-g_2} N'.
\end{array}
\]

Since $f_2 \Omega m_2 = t f_1 = 0$, there exists a morphism $n' : \Omega M'' \to \Omega N'$ such that $\Omega m_2 = (\Omega h_2)n'$. Because $\Omega|_{M'}$ is fully faithful, there exists a morphism $n_1 : M'' \to N'$ such that $n'_1 = \Omega n_1$ and $m_2 = h_2 n_1$. Hence $m_2 - h_2 n_1$ factors through $P \in P$. Since $h_2$ is a epimorphism, we have the following commutative diagram in $B$:

\[
\begin{array}{c}
M'' \xrightarrow{a} P \\
\downarrow c \downarrow b \\
N' \xrightarrow{h_2} N''.
\end{array}
\]

Let $n = ca + n_1$. Then $m_2 = h_2 n_1 + ba = h_2 n_1 + h_2 ca = h_2 n$ and $n = n_1$. Since $h_2(m_1 - nh_1) = h_2 m_1 - m_2 h_1 = 0$, there exists a morphism $s : M' \to Y$ such that $g_2 s = m_1 - nh_1$. Hence $g_2(t - sg_1) = g_2 t - m_1 g_1 + nh_1 g_1 = 0$. Because $g_2$ is a monomorphism, $t = sg_1$, i.e. $t$ factors through $g_1$. Hence $f$ factors through $g_1$ in $B$. □

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