Cohomology of modules over $H$-categories and co-$H$-categories

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Abstract

Let $H$ be a Hopf algebra. We consider $H$-equivariant modules over a Hopf module category $C$ as modules over the smash extension $C \# H$. We construct Grothendieck spectral sequences for the cohomologies as well as the $H$-locally finite cohomologies of these objects. We also introduce relative $(\mathcal{D}, H)$-Hopf modules over a Hopf comodule category $\mathcal{D}$. These generalize relative $(A, H)$-Hopf modules over an $H$-comodule algebra $A$. We construct Grothendieck spectral sequences for their cohomologies by using their rational $\text{Hom}$ objects and higher derived functors of coinvariants.

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1 Introduction

Let $H$ be a Hopf algebra over a field $K$. An $H$-category is a small $K$-linear category $\mathcal{C}$ such that the morphism space $\text{Hom}_\mathcal{C}(X,Y)$ is an $H$-module for each couple of objects $X, Y \in \text{Ob}(\mathcal{C})$ and the composition of morphisms in $\mathcal{C}$ is well-behaved with respect to the action of $H$. Similarly, a co-$H$-category is a small $K$-linear category $\mathcal{D}$ such that the morphism space $\text{Hom}_\mathcal{D}(X,Y)$ is an $H$-comodule for each couple of objects $X, Y \in \text{Ob}(\mathcal{D})$ and the composition of morphisms in $\mathcal{D}$ is well-behaved with respect to the coaction of $H$. In other words, an $H$-category is enriched over the monoidal category of $H$-modules and a co-$H$-category is enriched over the monoidal category of $H$-comodules. The purpose of this paper is to study cohomology in module categories over $H$-categories and co-$H$-categories.

The Hopf module categories that we use were first considered by Cibils and Solotar [6], where they discovered a Morita equivalence that relates Galois coverings of a category to its smash extensions via a Hopf algebra. We view these $H$-categories and the modules over them as objects of interest in their own right. We recall here that an ordinary ring may be expressed as a preadditive category with a single object. Accordingly, an arbitrary small preadditive category may be understood as a ‘ring with several objects’ (see Mitchell [20]). As such, the theories obtained by replacing rings by preadditive categories have been developed widely in the literature (see, for instance, [1], [8], [17], [18], [19], [29], [30]). In this respect, an $H$-category may be seen as an “$H$-module algebra with several objects”. Likewise, a co-$H$-category may be seen as an “$H$-comodule algebra with several objects.”

The various aspects of categorified Hopf actions and coactions on algebras have already been studied by several authors. In [13], Herscovich and Solotar obtained a Grothendieck spectral sequence for the Hochschild-Mitchell
cohomology of an $H$-comodule category appearing as an $H$-Galois extension. Hopf comodule categories were also studied in [24], where the authors introduced cleft $H$-comodule categories and extended classical results on cleft comodule algebras. More recently, Batista, Caenepeel and Vercruysse have shown in [2] that several deep theorems on Hopf modules can be extended to a categorification of Hopf algebras (see also [5]).

In this paper, we will construct a Grothendieck spectral sequence that computes the higher derived $\text{Hom}$ functors for $H$-equivariant modules over an $H$-category $\mathcal{C}$. We will also construct a spectral sequence that gives the higher derived $\text{Hom}$ functors for relative $(D, H)$-modules, where $D$ is a co-$H$-category. We will develop these cohomology theories in a manner analogous to the “$H$-finite cohomology” obtained by Guédonon [11] (see also [10]) and the cohomology of relative Hopf modules studied by Caenepeel and Guédonon in [4] respectively.

We now describe the paper in more detail. We begin in Section 2 by recalling the notion of a left $H$-category and a right co-$H$-category. For a left $H$-category $\mathcal{C}$, we have a category of $H$-invariants which will be denoted by $\mathcal{C}^H$. For a right co-$H$-category $D$, there is a corresponding category of $H$-coinvariants which will be denoted by $D^{coH}$. If $H$ is a finite dimensional Hopf algebra and $H^*$ is its linear dual, then a $K$-linear category $\mathcal{D}$ is a left $H^*$-category if and only if it is a right co-$H$-category. In that case, $D^{H^*} = D^{coH}$.

In Sections 3 and 4, we work with a left $H$-category $\mathcal{C}$. We consider right $\mathcal{C}$-modules that are equipped with an additional left $H$-equivariant structure (see Definition 3.2). This category is denoted by $(\text{Mod-}\mathcal{C})^H_H$. If $\mathcal{M}$, $\mathcal{N} \in (\text{Mod-}\mathcal{C})^H_H$, the space $\text{Hom}_{\text{Mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N})$ of right $\mathcal{C}$-module morphisms carries a left $H$-module structure whose $H$-invariants are given by $\text{Hom}_{\text{Mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N})^H = \text{Hom}_{(\text{Mod-}\mathcal{C})^H_H}(\mathcal{M}, \mathcal{N})$.

More precisely, let $(\text{Mod-}\mathcal{C})_H$ denote the category with the same objects as $(\text{Mod-}\mathcal{C})^H_H$ but whose morphisms are ordinary $\mathcal{C}$-modules morphisms. Then, we show that $(\text{Mod-}\mathcal{C})_H$ is a left $H$-category and $(\text{Mod-}\mathcal{C})^H_H$ may be recovered as the category of $H$-invariants of $(\text{Mod-}\mathcal{C})_H$. Further, we obtain that $(\text{Mod-}\mathcal{C})^H_H$ is identical to the category $\text{Mod-}(\mathcal{C}^H)$ of right modules over the smash product category $\mathcal{C}^H_H$. In particular, this shows that $(\text{Mod-}\mathcal{C})^H_H$ is a Grothendieck category. We then construct a Grothendieck spectral sequence (see Theorem 3.15)

$$R^p(-)^H(\text{Ext}^q_{\text{Mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N})) \Rightarrow (R^{p+q}\text{Hom}_{\text{Mod-}(\mathcal{C}^H)}(\mathcal{M}, -))(\mathcal{N})$$

for the higher derived $\text{Hom}$ in $\text{Mod-}(\mathcal{C}^H)$ in terms of the derived $\text{Hom}$ in $\text{Mod-}\mathcal{C}$ and the derived functor of $H$-invariants.

We proceed in Section 4 to develop the “$H$-finite cohomology” of $(\mathcal{C}^H)$-modules in a manner analogous to Guédonon [11]. If $M$ is an $H$-module, we denote by $M^{(H^*)}$ the collection of all elements $m \in M$ such that $Hm$ is a finite dimensional vector space. In particular, $M$ is said to be $H$-locally finite if $M^{(H^*)} = M$ and we let $H$-$\text{mod}$ denote the category of $H$-locally finite modules. This leads to a functor

$$L_{\text{Mod-}\mathcal{C}} : (\text{Mod-}(\mathcal{C}^H))^{op} \times \text{Mod-}(\mathcal{C}^H) \rightarrow H$-$\text{mod}$ \quad L_{\text{Mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N}) := \text{Hom}_{\text{Mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N})^{(H)}$$

We then construct a Grothendieck spectral sequence (see Theorem 4.2)

$$R^p(-)^{(H^*)}(\text{Ext}^q_{\text{Mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N})) \Rightarrow (R^{p+q}L_{\text{Mod-}\mathcal{C}}(\mathcal{M}, -))(\mathcal{N})$$

The left $H$-category $\mathcal{C}$ is said to be locally finite if every morphism space $\text{Hom}_\mathcal{C}(X, Y)$ is locally finite as an $H$-module. We denote by $\text{mod-}(\mathcal{C}^H)$ the full subcategory of $\text{Mod-}(\mathcal{C}^H)$ consisting of those left $H$-equivariant right $\mathcal{C}$-modules $\mathcal{M}$ such that $\mathcal{M}(X)$ is $H$-locally finite for each $X \in \text{Ob}(\mathcal{C})$. When $\mathcal{C}$ is left $H$-locally finite and right noetherian, we construct a spectral sequence (see Theorem 4.19)

$$R^p(-)^{(H^*)}(\text{Ext}^q_{\text{Mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N})) \Rightarrow (R^{p+q}\text{Hom}_{\text{mod-}(\mathcal{C}^H)}(\mathcal{M}, -))(\mathcal{N})$$

In Section 5, we work with a right co-$H$-category $\mathcal{D}$ and introduce the category $\mathcal{D}^{H^*}$ of relative $(D, H)$-Hopf modules (see Definition 5.1). A relative $(D, H)$-module consists of an $H$-coaction on a pair $(D, M)$, where
Let $\mathcal{D}$ be a left $\mathcal{D}$-module. In particular, $\mathcal{M}(X)$ is equipped with the structure of a right $H$-comodule for each $X \in \text{Ob}(\mathcal{D})$. We show that $\mathcal{D}_H^H$ is a Grothendieck category.

Let $\text{Comod-}H$ be the category of $H$-comodules. Thereafter, we construct a functor (see (5.3))

$$HOM_{\mathcal{D}-\text{Mod}} : (\mathcal{D}_H^H)^{op} \times \mathcal{D}_H^H \rightarrow \text{Comod-}H$$

by using the right adjoint of the functor $N \otimes (-) : \text{Comod-}H \rightarrow \mathcal{D}_H^H$ for each fixed $N \in \mathcal{D}_H^H$. In the case of an $H$-comodule algebra as considered by Caenepeel and Guédénon, the $HOM$ functor gives the collection of “rational morphisms” between relative Hopf modules (see[4, § 2]). Although the category $\mathcal{D}_H^H$ is not necessarily enriched over $\text{Comod-}H$, we see that $HOM_{\mathcal{D}-\text{Mod}}(M, N)$ behaves like a $Hom$ object. The morphisms in $Hom_{\mathcal{D}_H^H}(M, N)$ may be recovered as the $H$-coinvariants $HOM_{\mathcal{D}-\text{Mod}}(M, N)^{coH} = Hom_{\mathcal{D}_H^H}(M, N)$. We then construct a Grothendieck spectral sequence (see Theorem 5.9)

$$R^p(-)^{coH}(R^qHOM_{\mathcal{D}-\text{Mod}}(M, -))(N) \Rightarrow \left(R^{p+q}Hom_{\mathcal{D}_H^H}(M, -)\right)(N)$$

for the higher derived $Hom$ in $\mathcal{D}_H^H$.

**Notations:** Throughout the paper, $K$ is a field, $H$ is a Hopf algebra with comultiplication $\Delta$, counit $\varepsilon$ and bijective antipode $S$. We shall use Sweedler’s notation for the coproduct $\Delta(h) = \sum h_1 \otimes h_2$ and for a coaction $\rho : M \rightarrow M \otimes H$, $\rho(m) = \sum m_0 \otimes m_1$. We denote by $H^*$ the linear dual of $H$. The category of left $H$-modules will be denoted by $\text{Mod-}H$ and the category of right $H$-comodules will be denoted by $\text{Comod-}H$. For $M \in \text{Mod-}H$, we set $M^H := \{m \in M \mid hm = \varepsilon(h)m \ \forall h \in H\}$. For $M \in \text{Comod-}H$, we set $M^{\text{coH}} := \{m \in M \mid \rho(m) = m \otimes 1_H\}$.

## 2 $H$-categories and co-$H$-categories

Let $H$ be a Hopf algebra over a field $K$. Then, it is well known (see, for instance, [23, § 2.2]) that the category of $H$-modules as well as the category of $H$-comodules is monoidal. A $K$-linear category is said to be an $H$-module category (resp. an $H$-comodule category) if it is enriched over the monoidal category of $H$-modules (resp. $H$-comodules). For more on enriched categories, the reader may see, for example, [3, Chapter 6] or [16].

**Definition 2.1.** (see Cibils and Solotar [6, Definition 2.1]) Let $K$ be a field. A $K$-linear category $\mathcal{C}$ is said to be a left $H$-module category if it is enriched over the monoidal category of left $H$-modules. In other words, it satisfies the following conditions:

(i) $\text{Hom}_\mathcal{C}(X, Y)$ is a left $H$-module for all $X, Y \in \text{Ob}(\mathcal{C})$.

(ii) $h(id_X) = \varepsilon(h) \cdot id_X$ for every $X \in \text{Ob}(\mathcal{C})$ and every $h \in H$.

(iii) The composition of morphisms in $\mathcal{C}$ is $H$-equivariant, i.e., for any $h \in H$ and any pair of composable morphisms $g : X \rightarrow Y$, $f : Y \rightarrow Z$, we have

$$h(fg) = \sum h_1(f)h_2(g)$$

By a left $H$-category, we will always mean a small left $H$-module category. A right $H$-category may be defined similarly.
Definition 2.2. Let \( C \) be a left \( H \)-module category. A morphism \( f \in \text{Hom}_C(X,Y) \) is said to be \( H \)-invariant if \( h(f) = \varepsilon(h) \cdot f \) for all \( h \in H \). The subcategory whose objects are the same as those of \( C \) and whose morphisms are the \( H \)-invariant morphisms in \( C \) is denoted by \( C^H \).

Let \( A \) be a left \( H \)-module algebra. A right \( A \)-module \( M \) is said to be left \( H \)-equivariant if

(i) \( M \) is a left \( H \)-module and

(ii) the action of \( A \) on \( M \) is a morphism of \( H \)-modules, i.e., \( h(ma) = \sum h_1(m)h_2(a) \), for all \( h \in H \), \( a \in A \) and \( m \in M \).

Example 2.3. (see [15]) Let \( A \) be a left \( H \)-module algebra.

(i) Then, the category \( H M^A \) of (isomorphism classes of) all left \( H \)-equivariant finitely generated right \( A \)-modules, with right \( A \)-module morphisms between them, is an \( H \)-category. In fact, one can check that for \( X, Y \in \text{Ob}(H M^A) \), the morphism space \( \text{Hom}_A(X,Y) \) is a left \( H \)-module via

\[
h(f)(x) = \sum h_1 f(S(h_2)x) \quad \forall x \in X, \forall f \in \text{Hom}_A(X,Y)
\]

(ii) The finitely generated free right \( A \)-modules are automatically left \( H \)-equivariant. The category of (isomorphism classes of) finitely generated free right \( A \)-modules is an \( H \)-category.

We may also define the notion of a co-\( H \)-category, which replaces an \( H \)-comodule algebra (see [24]). This notion also appears implicitly in [6].

Definition 2.4. By a right co-\( H \)-category, we will mean a small \( K \)-linear category \( D \) that is enriched over the monoidal category of right \( H \)-comodules. In other words, we have:

(i) \( \text{Hom}_D(X,Y) \) is a right \( H \)-comodule for all \( X, Y \in \text{Ob}(D) \), with structure map

\[
\rho_{XY} : \text{Hom}_D(X,Y) \rightarrow \text{Hom}_D(X,Y) \otimes H, \quad \rho_{XY}(f) = \sum f_0 \otimes f_1
\]

(ii) \( \rho_{XX}(id_X) = id_X \otimes 1_H \), for any \( X \in \text{Ob}(D) \) and any \( h \in H \).

(iii) The composition of morphisms in \( D \) is \( H \)-coequivariant, i.e., for any pair of composable morphisms \( g : X \rightarrow Y, f : Y \rightarrow Z \), we have

\[
\rho_{XZ}(fg) = \sum (fg)_0 \otimes (fg)_1 = \sum f_0 g_0 \otimes f_1 g_1 = \rho_{YZ}(f) \rho_{XY}(g)
\]

A left co-\( H \)-category may be defined similarly.

A morphism \( f \in \text{Hom}_D(X,Y) \) in a right co-\( H \)-category is said to be \( H \)-coequivariant if it satisfies \( \rho_{XY}(f) = f \otimes 1_H \). The subcategory whose objects are the same as those of \( D \) and whose morphisms are \( H \)-coequivariant is denoted by \( D^{coH} \).

Proposition 2.5. Let \( H \) be a finite dimensional Hopf algebra and let \( D \) be a small \( K \)-linear category. Then, \( D \) is a right co-\( H \)-category if and only if \( D \) is a left \( H^* \)-category. Moreover, \( D^{H^*} = D^{coH} \).

Proof. Let \( \{e_1, \ldots, e_n\} \) be a basis of \( H \) and let \( \{e_1^*, \ldots, e_n^*\} \) be its dual basis. If \( D \) is a right co-\( H \)-category, then \( D \) becomes a left \( H^* \)-category with

\[
h^*(f) := \sum f_0 h^*(f_1)
\]
for all $h^* \in H^*$ and $f \in \text{Hom}_D(X,Y)$. Indeed, it is easy to check that this action makes $\text{Hom}_D(X,Y)$ a left $H^*$-module for every $X,Y \in \text{Ob}(D)$ and that

$$h^*(fg) = \sum (fg)_0 h^*((fg)_1) = \sum f_0 g_0 h^*_1(f_1 g_1) = \sum f_0 (g_0 h^*_1)(f_1) = \sum f_0 (g_0 h^*_1 f_1) = \sum h^*_1(f) h^*_2(g).$$

Conversely, if $D$ is a left $H^*$-category, then $D$ is a right co-$H$-category with

$$\rho_{XY} : \text{Hom}_D(X,Y) \to \text{Hom}_D(X,Y) \otimes H, \quad \rho_{XY} = \sum_{i=1}^n e^*_i \otimes e_i$$

It may be verified that this gives a right $H$-comodule structure on $\text{Hom}_D(X,Y)$. We need to check that the composition of morphisms in $D$ is $H$-equivariant. For any $h^* \in H^*$, $g \in \text{Hom}_D(X,Y)$ and $f \in \text{Hom}_D(Y,Z)$, we have

$$(id \otimes h^*)(\rho_{XZ}(fg)) = (id \otimes h^*)(\sum_{i=1}^n e^*_i (fg) \otimes e_i) = \sum_{i=1}^n e^*_i (fg) \otimes h^*(e_i) = \sum_{i=1}^n (h^*(e_i) e^*_i) (fg) \otimes 1_H = h^*(fg) \otimes 1_H = \sum_{i=1}^n h^*_1(f) h^*_2(g) \otimes 1_H = \sum_{i,j,s} e^*_i (f) e^*_j (g) e_i \otimes e_j$$

Since $H$ is finite dimensional, it follows that

$$\rho_{XZ}(fg) = \sum_{i,j,s} e^*_i (f) e^*_j (g) e_i \otimes e_j = \rho_{YX}(f) \rho_{XY}(g)$$

We also have

$$\text{Hom}_{D^{op}}(X,Y) = \{ f \in \text{Hom}_D(X,Y) \mid h^*(f) = e_H (h^*) f = h^*(1_H) f, \quad \forall h^* \in H^* \}$$

$$= \{ f \in \text{Hom}_D(X,Y) \mid \sum f_0 (h^* f_1) = h^*(1_H) f, \quad \forall h^* \in H^* \}$$

$$= \{ f \in \text{Hom}_D(X,Y) \mid (id \otimes h^*)(\rho_{XY}(f)) = (id \otimes h^*)(f \otimes 1_H), \quad \forall h^* \in H^* \}$$

$$= \{ f \in \text{Hom}_D(X,Y) \mid \rho_{XY}(f) = f \otimes 1_H = \text{Hom}_{D^{op}}(X,Y) \}.$$

\hfill \Box

**Remark 2.6.** Using Example 2.3 and Proposition 2.5, we can obtain several examples of co-$H$-categories. Another example of a co-$H$-category is the smash extension $C \# H$, which will be recalled in the next section.

## 3 $H$-equivariant modules and the first spectral sequence

Let $C$ be a left $H$-category. In this section, we will study the category of $H$-equivariant $C$-modules and compute their higher derived $\text{Hom}$ functors by means of a spectral sequence. We begin with the following definition (see, for instance, [21, 25]).

**Definition 3.1.** Let $C$ be a small $K$-linear category. A right module over $C$ is a $K$-linear functor $C^{op} \to \text{Vect}_K$, where $\text{Vect}_K$ denotes the category of $K$-vector spaces. Similarly, a left module over $C$ is a $K$-linear functor $C \to \text{Vect}_K$. The category of all right (resp. left) modules over $C$ will be denoted by $\text{Mod-}C$ (resp. $\text{C-Mod}$).
For each $X \in \text{Ob}(C)$, the representable functors $h_X := \text{Hom}_C(-, X)$ and $\chi h := \text{Hom}_C(X, -)$ are examples of right and left modules over $C$ respectively. Unless otherwise mentioned, by a $C$-module we will always mean a right $C$-module.

**Definition 3.2.** Let $C$ be a left $H$-category. Let $M$ be a right $C$-module with a given left $H$-module structure on $M(X)$ for each $X \in \text{Ob}(C)$. Then, $M$ is said to be a left $H$-equivariant right $C$-module if

$$h(M(f)(m)) = \sum \mathcal{M}(h \mu f)(h_1 m) \quad \forall \ h \in H, \ f \in \text{Hom}_C(X, Y), \ m \in \mathcal{M}(Y)$$

A morphism $\eta : M \rightarrow N$ of left $H$-equivariant right $C$-modules is a morphism $\eta \in \text{Hom}_{\text{Mod}-C}(M, N)$ such that $\eta(X) : M(X) \rightarrow N(X)$ is $H$-linear for each $X \in \text{Ob}(C)$. We will denote the category of left $H$-equivariant right $C$-modules by $(\text{Mod}-C)^H_C$.

By $(\text{Mod}-C)^H_H$, we will denote the category whose objects are the same as those of $(\text{Mod}-C)^H_H$, but whose morphisms are those of right $C$-modules.

**Lemma 3.3.** Let $C$ be a left $H$-category. Given $M, N \in (\text{Mod}-C)^H_H$, the $H$-module action on $\text{Hom}_{\text{Mod}-C}(M, N)$ given by

$$(h \cdot \eta)(X)(m) = \sum h_1 \eta(X)(S(h_2 m))$$

for $\eta \in \text{Hom}_{\text{Mod}-C}(M, N)$, $h \in H$, $X \in \text{Ob}(C)$, $m \in M(X)$ makes $(\text{Mod}-C)^H_H$ a left $H$-category.

**Proof.** Using the $H$-equivariance of $M$ and $N$, it may be verified that the action in (3.1) defines a left $H$-module structure on $\text{Hom}_{\text{Mod}-C}(M, N)$. We now consider $\eta \in \text{Hom}_{\text{Mod}-C}(M, N)$ and $\nu \in \text{Hom}_{\text{Mod}-C}(N, \mathcal{P})$. Then, we have

$$\sum((h_1 \nu)(X) \circ (h_2 \eta))(X)(m) = \sum((h_1 \nu)(X)(h_2 \eta(X)(S(h_2 m))))$$

This proves the result.

**Proposition 3.4.** The category $(\text{Mod}-C)^H_H$ of left $H$-equivariant right $C$-modules is identical to $((\text{Mod}-C)^H_H)$.

**Proof.** Suppose that $\eta \in \text{Hom}_{\text{Mod}-C}(M, N)^H$. We claim that $\eta(X) : M(X) \rightarrow N(X)$ is $H$-linear for each $X \in \text{Ob}(C)$. For this, we observe that

$$\eta(X)(hm) = \sum \eta(X)(\varepsilon(h_1) h_2 m) = \sum \varepsilon(h_1) \eta(X)(h_2 m) = \sum \varepsilon(h_1 \cdot \eta)(X)(h_2 m) = \sum h_1 \varepsilon(\eta)(S(h_2) h_3 m) = h \eta(X)(m)$$

for any $h \in H$ and $m \in M(X)$. Conversely, if each $\eta(X) : M(X) \rightarrow N(X)$ is $H$-linear, it is clear from the definition of the left $H$-action in (3.1) that $h \cdot \eta = \varepsilon(h) \eta$, i.e., $\eta \in \text{Hom}_{\text{Mod}-C}(M, N)^H$.

We will now study the category $(\text{Mod}-C)^H_H$ of left $H$-equivariant right $C$-modules. In particular, one may ask if $(\text{Mod}-C)^H_H$ is an abelian category. We will show that $(\text{Mod}-C)^H_H$ is in fact a Grothendieck category. For this, we will need to consider the smash product category of $C$ and $H$.

**Definition 3.5.** (see [6, §2]) Let $C$ be a left $H$-category. The smash product of $C$ and $H$, denoted by $C \# H$, is the $K$-linear category defined by

$$\text{Ob}(C \# H) := \text{Ob}(C) \quad \text{Hom}_{C \# H}(X, Y) := \text{Hom}_C(X, Y) \otimes H$$
An element of $\text{Hom}_{\mathcal{C}\#H}(X,Y)$ is a finite sum of the form $\sum g_i \# h_i$, with $g_i \in \text{Hom}_{\mathcal{C}}(X,Y)$ and $h_i \in H$. Then, the composition of morphisms in $\mathcal{C}\#H$ is determined by

$$(f \# h)(g \# h') = \sum f(h_ig) \# (h_2h')$$

for any pair of composable morphisms $g : X \rightarrow Y, f : Y \rightarrow Z$ in $\mathcal{C}$ and any $h, h' \in H$.

**Lemma 3.6.** Let $\mathcal{M} \in \text{Mod}-(\mathcal{C}\#H)$. Then, $\mathcal{M}(X)$ has a left $H$-module structure for each $X \in \text{Ob}(\mathcal{C}\#H)$ given by

$$hm := \mathcal{M}((id_X \# S(h))(m)) \quad \forall \ h \in H, \ m \in \mathcal{M}(X)$$

Further, given any morphism $\eta : \mathcal{M} \rightarrow \mathcal{N}$ in $\text{Mod}-(\mathcal{C}\#H)$, every $\eta(X) : \mathcal{M}(X) \rightarrow \mathcal{N}(X)$ is $H$-linear.

**Proof.** For $h, h' \in H$ and $m \in \mathcal{M}(X)$, we have

$$h(h'm) = (\mathcal{M}((id_X \# S(h)) \circ (id_X \# S(h')))(m)$$

$$= \mathcal{M}((id_X \# S(h'))((id_X \# S(h))(m))$$

$$= \sum \mathcal{M}(S(h_2')((id_X \# S(h_1'))S(h))(m)) \quad \text{(using Definition 2.1(ii))}$$

$$= \mathcal{M}((id_X \# S(h'h'))(m)) \quad \text{(using } \varepsilon \circ S = \varepsilon)$$

$$= (hh')(m)$$

If $\eta : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism in $\text{Mod}-(\mathcal{C}\#H)$, it may be verified easily that each $\eta(X) : \mathcal{M}(X) \rightarrow \mathcal{N}(X)$ is $H$-linear. \hfill \Box

**Proposition 3.7.** Let $\mathcal{C}$ be a left $H$-category. Then, there is a one-one correspondence between left $H$-equivariant right $\mathcal{C}$-modules and right modules over $\mathcal{C}\#H$.

**Proof.** For any $H$-equivariant $\mathcal{C}$-module $\mathcal{M}$, we have the object $\mathcal{M}'$ in $\text{Mod}-(\mathcal{C}\#H)$ defined by

$$\mathcal{M}'(X) := \mathcal{M}(X) \quad \forall X \in \text{Ob}(\mathcal{C}\#H),$$

$$\mathcal{M}'(f \# h)(m) := S^{-1}(h)\mathcal{M}(f)(m) \quad \forall f \# h \in \text{Hom}_{\mathcal{C}\#H}(Y,X), \ m \in \mathcal{M}(X).$$

(3.2)

For $f' \# h' \in \text{Hom}_{\mathcal{C}\#H}(Z,Y), f \# h \in \text{Hom}_{\mathcal{C}\#H}(Y,X)$ and $m \in \mathcal{M}(X)$, we have

$$\mathcal{M}'((f \# h) \circ (f' \# h'))(m) = \sum \mathcal{M}'((f(h_1f')) \# h'h')(m)$$

$$= \sum S^{-1}(h_2h_2')\mathcal{M}(f(h_1f'))(m)$$

$$= S^{-1}(h'h')\sum S^{-1}(h_2)\mathcal{M}(f(h_1f'))S^{-1}(h_2)(m) \quad \text{(since $M$ is $H$-equivariant)}$$

$$= S^{-1}(h'h')\mathcal{M}(S^{-1}(h_2)f)(S^{-1}(h_2)m)$$

$$= S^{-1}(h'h')\mathcal{M}(f')(S^{-1}(h_2)m)$$

$$= S^{-1}(h'h')\mathcal{M}(f')S^{-1}(h_2)\mathcal{M}(f)(m)$$

$$= (\mathcal{M}'(f' \# h') \circ \mathcal{M}'(f \# h))(m)$$

(3.2.2)

Conversely, given any $\mathcal{M}'$ in $\text{Mod}-(\mathcal{C}\#H)$, we can obtain an $H$-equivariant $\mathcal{C}$-module defined by

$$\mathcal{M}(X) := \mathcal{M}'(X) \quad \forall X \in \text{Ob}(\mathcal{C})$$

$$\mathcal{M}(f) := \mathcal{M}'(f \# 1_H) \quad \forall f \in \text{Hom}_{\mathcal{C}}(Y,X).$$
From Lemma 3.6, it follows that $M(X) = M'(X)$ has a left $H$-module structure. We now check that $M$ is indeed $H$-equivariant:

$$h(M(f)(m)) = M'(id_X \# S(h))(M'(f \# 1_H)(m))$$

$$= M'(f \# S(h))(m)$$

$$= \sum M'(((id_X \# S(h_1))(h_2 f \# 1_H))(m))$$

$$= \sum M'(h_2 f \# 1_H)(M'(id_X \# S(h_1))(m))$$

$$= \sum M(h_2 f)(h_1 m).$$

\[ \square \]

**Proposition 3.8.** Let $M$ and $N$ be right $\mathcal{C}\#H$-modules. Then, $\text{Hom}_{\text{Mod-C}}(M,N)$ is a left $H$-module and its invariants are given by $\text{Hom}_{\text{Mod-C}}(M,N)^H = \text{Hom}_{\text{Mod-C}\#H}(M,N)$.

**Proof.** We have shown in Proposition 3.7 that every right $\mathcal{C}\#H$-module is also a left $H$-equivariant right $\mathcal{C}$-module. Accordingly, we use (3.1) to give an $H$-module structure on $\text{Hom}_{\text{Mod-C}}(M,N)$ by setting

$$(h \cdot \eta)(X)(m) := \sum h_1 (\eta(X)(S(h_2)m)) \quad \forall h \in H, \ X \in \text{Ob}(\mathcal{C}), \ m \in M(X)$$

(3.3)

for any $\eta \in \text{Hom}_{\text{Mod-C}}(M,N)$. Suppose now that $\eta \in \text{Hom}_{\text{Mod-C}}(M,N)^H$. From the proof of Proposition 3.4, it follows that $\eta(X) : M(X) \to N(X)$ is $H$-linear for each $X \in \text{Ob}(\mathcal{C})$. We need to show that $\eta \in \text{Hom}_{\text{Mod-C}\#H}(M,N)$. For any $f : Y \to X$ in $\mathcal{C}$, $h \in H$ and $m \in M(X)$, we have

$$\eta(Y)(M(\eta(f)(h))(m)) = \eta(Y)(S^{-1}(h)(f)(S^{-1}(h_2)m))$$

$$= \eta(Y)(M(S^{-1}(h_1)f)(S^{-1}(h_2)m))$$

$$= \eta(N(S^{-1}(h_1)f)(S^{-1}(h_2)(\eta(X)(m))))$$

$$= \eta(f \# h)(\eta(X)(m)).$$

Conversely, let $\eta \in \text{Hom}_{\text{Mod-C}\#H}(M,N)$. Using the $H$-linearity of $\eta(X)$ from Lemma 3.6, it is clear from (3.3) that $\eta \in \text{Hom}_{\text{Mod-C}}(M,N)^H$.

\[ \square \]

**Proposition 3.9.** Let $\mathcal{C}$ be a left $H$-category. Then, the categories $\text{Mod-}(\mathcal{C}\#H)$ and $(\text{Mod-}\mathcal{C})^H$ are identical. In particular, the category $(\text{Mod-}\mathcal{C})^H$ of left $H$-equivariant right $\mathcal{C}$-modules is a Grothendieck category.

**Proof.** The fact that $\text{Mod-}(\mathcal{C}\#H)$ and $(\text{Mod-}\mathcal{C})^H$ are identical follows from Propositions 3.4, 3.7 and 3.8. Further, given any small preadditive category $\mathcal{E}$, it is well known that the category $\text{Mod-}\mathcal{E}$ is a Grothendieck category (see, for instance, [25, Example V.2.2]). Since $\mathcal{C}\#H$ is a small preadditive category, the result follows.

\[ \square \]

We denote by $M \otimes_{\mathcal{C}} (\mathcal{C}\#H)$ the extension of a right $\mathcal{C}$-module $M$ to a right $(\mathcal{C}\#H)$-module. For the general notion of extension and restriction of scalars in the case of modules over a category, see, for instance, [22, § 4]. It follows from [22, Proposition 19] that the extension of scalars is left adjoint to the restriction of scalars.

**Lemma 3.10.** Let $\mathcal{M}$ be a right $\mathcal{C}$-module. Then,

1. A right $(\mathcal{C}\#H)$-module $\mathcal{M} \otimes H$ may be obtained by setting

$$(\mathcal{M} \otimes H)(X) := \mathcal{M}(X) \otimes H$$

$$((\mathcal{M} \otimes H)(f \# h'))(m \otimes h) := \sum \mathcal{M}(h_1 f')(m) \otimes h_2 h'.$$
for any $X \in \text{Ob}(\mathcal{C} \# \mathcal{H})$, $f' \# h' \in \text{Hom}_{\mathcal{C} \# \mathcal{H}}(Y, X)$, $m \in \mathcal{M}(X)$ and $h, h' \in H$.

(2) $\mathcal{M} \otimes H$ is isomorphic to $\mathcal{M} \otimes_{\mathcal{C}} (\mathcal{C} \# \mathcal{H})$ as objects in $\text{Mod}(\mathcal{C} \# \mathcal{H})$.

Proof. (1) For any $f'' \# h'' \in \text{Hom}_{\mathcal{C} \# \mathcal{H}}(Y, X)$, $m \in \mathcal{M}(X)$ and $h, h' \in H$, we have

$$
(\mathcal{M} \otimes H)((f'' \# h'')((f'' \# h'') \otimes 1)) (m \otimes h) = \sum((\mathcal{M} \otimes H)((f'' \# h'') \otimes 1)) (m \otimes h)
$$

Further, $((\mathcal{M} \otimes H)((id_{X} \otimes 1)) (m \otimes h) = \mathcal{M}(h_{1} id_{X})(m) \otimes h_{2} = m \otimes h$. Thus, $\mathcal{M} \otimes H \in \text{Mod}(\mathcal{C} \# \mathcal{H})$.

(2) It may be easily checked that the assignment $f \mapsto \eta_{f}$, which we denote by $\phi$ by setting $M \otimes_{\mathcal{C}} H$, is surjective. This proves the result.

$$
\phi(\eta)(X)(m) = \eta(\mathcal{M}(f)(m \otimes 1) H) = \xi(\mathcal{M}(f)(m \otimes 1) H)
$$

Hence, $\phi$ is surjective. This proves the result.
Proposition 3.11. (1) The extension of scalars from $\text{Mod-}\mathcal{C}$ to $\text{Mod-}(\mathcal{C}#\mathcal{H})$ is exact.

(2) Let $I$ be an injective object in $\text{Mod-}(\mathcal{C}#\mathcal{H})$. Then, $I$ is also an injective object in $\text{Mod-}\mathcal{C}$.

Proof. Let $M, N \in \text{Mod-}\mathcal{C}$ be such that $\phi : \mathcal{C} \to N$ is a monomorphism, i.e., $\phi(X) : M(X) \to N(X)$ is a monomorphism in $\text{Vect}_K$ for each $X \in \text{Ob}(\mathcal{C})$. Applying the isomorphism in Lemma 3.10, $(\phi \otimes (\mathcal{C}#\mathcal{H}))(X) = (\phi \otimes H)(X) : M(X) \otimes H \to N(X) \otimes H$ is a monomorphism. Since extension of scalars is a left adjoint, it already preserves colimits. This proves (1). The result of (2) now follows from [27, Tag 015Y].

Lemma 3.12. Let $M \in \text{H-Mod}$ and let $N \in \text{Mod-}(\mathcal{C}#\mathcal{H})$. Then, a right $(\mathcal{C}#\mathcal{H})$-module $M \otimes N$ can be defined by setting

$$(M \otimes N)(X) := M \otimes N(X) \quad (M \otimes N)(f)(m \otimes n) := m \otimes N(f)(n)$$

for any $X \in \text{Ob}(\mathcal{C})$, $f \in \text{Hom}_\mathcal{C}(Y, X)$ and $m \otimes n \in M \otimes N(X)$.

Proof. It is clear that $M \otimes N \in \text{Mod-}\mathcal{C}$. Now for each $X \in \text{Ob}(\mathcal{C})$, the $K$-vector space $M \otimes N(X)$ has a left $H$-module structure given by

$$h(m \otimes n) := \sum h_1 m \otimes h_2 n \quad \forall \ h \in H$$

It may be easily verified that $M \otimes N$ is an $H$-equivariant right $\mathcal{C}$-module under this action. Therefore, $M \otimes N \in \text{Mod-}(\mathcal{C}#\mathcal{H})$ by Proposition 3.7.

Given any $N \in \text{Mod-}(\mathcal{C}#\mathcal{H})$, let $(-) \otimes N : \text{H-Mod} \to \text{Mod-}(\mathcal{C}#\mathcal{H})$ denote the functor which takes any $M \in \text{H-Mod}$ to $M \otimes N \in \text{Mod-}(\mathcal{C}#\mathcal{H})$.

Proposition 3.13. Let $N, P \in \text{Mod-}(\mathcal{C}#\mathcal{H})$ and let $M \in \text{H-Mod}$. Then, we have a natural isomorphism

$$\phi : \text{Hom}_{\text{Mod-}(\mathcal{C}#\mathcal{H})}(M \otimes N, P) \to \text{Hom}_{\text{H-Mod}}(M, \text{Hom}_{\text{Mod-}\mathcal{C}}(N, P))$$

given by $(\phi(\eta))(m)(X)(n) := \eta(X)(m \otimes n)$ for each $X \in \text{Ob}(\mathcal{C})$ and $m \in M, n \in N(X)$.

Proof. Let $\eta \in \text{Hom}_{\text{Mod-}(\mathcal{C}#\mathcal{H})}(M \otimes N, P)$. It may be checked that $\phi(\eta)(m) \in \text{Hom}_{\text{Mod-}\mathcal{C}}(N, P)$ for every $m \in M$. We now verify that $\phi(\eta)$ is $H$-linear, i.e., for $h \in H$:

$$\left( h(\phi(\eta)(m)) \right)(X)(n) = \sum h_1 \left( \phi(\eta)(m)(X)(S(h_2)n) \right) \quad \text{(using Proposition 3.8)}$$

$$= \sum h_1 \left( \eta(X)(m \otimes S(h_2)n) \right)$$

$$= \sum \eta(X)(h_1 m \otimes h_2 S(h_3)n)$$

$$= \eta(X)(h m \otimes n) = (\phi(\eta)(hm))(X)(n) \quad \text{(since } \eta(X) \text{ is } H\text{-linear by Lemma 3.6)}$$

Clearly, $\phi$ is injective. For $f \in \text{Hom}_{\text{H-Mod}}(M, \text{Hom}_{\text{Mod-}\mathcal{C}}(N, P))$, we consider $\nu \in \text{Hom}_{\text{Mod-}(\mathcal{C}#\mathcal{H})}(M \otimes N, P)$ determined by

$$\nu(X)(m \otimes n) := f(m)(X)(n) \quad \text{(3.4)}$$

for each $X \in \text{Ob}(\mathcal{C}), n \in N(X)$ and $m \in M$. We first check that $\nu(X) : M \otimes N(X) \to P(X)$ is $H$-linear for every $X \in \text{Ob}(\mathcal{C})$, i.e., for $h \in H$:

$$\nu(X)(h(m \otimes n)) = \sum h_1 \nu(X)(h_1 m \otimes h_2 n)$$

$$= \sum f(h_1 m)(X)(h_2 n)$$

$$= \sum h_1 f(m)(X)(S(h_2)h_3 n)$$

$$= \nu(X)(h(m \otimes n)) \quad \text{(since f is } H\text{-linear)}$$

Using the fact that $f(m) \in \text{Hom}_{\text{Mod-}\mathcal{C}}(N, P)$ for each $m \in M$, it may now be verified that $\nu \in \text{Hom}_{\text{Mod-}\mathcal{C}}(M \otimes N, P)$. From the equivalence of categories in Proposition 3.9, it follows that $\nu \in \text{Hom}_{\text{Mod-}(\mathcal{C}#\mathcal{H})}(M \otimes N, P)$. From (3.4), it is also clear that $\phi(\nu) = f$. \qed
Corollary 3.14. If \( I \) is an injective object in \( \text{Mod-}(\mathcal{C}#H) \), then \( \text{Hom}_{\text{Mod-}C}(N, I) \) is an injective object in \( H\text{-Mod} \) for any \( N \in \text{Mod-}(\mathcal{C}#H) \).

Proof. From Proposition 3.13, we know that the functor \((-) \otimes N : H\text{-Mod} \to \text{Mod-}(\mathcal{C}#H)\) is a left adjoint and therefore preserves colimits. Further, given a monomorphism \( M_1 \to M_2 \) in \( H\text{-Mod} \), it is clear from the definition in Lemma 3.12 that \( M_1 \otimes N \to M_2 \otimes N \) is a monomorphism in \( \text{Mod-}(\mathcal{C}#H) \). Hence, \((-) \otimes N : H\text{-Mod} \to \text{Mod-}(\mathcal{C}#H)\) is exact. As such, its right adjoint \( \text{Hom}_{\text{Mod-}C}(N, \_ : \text{Mod-}(\mathcal{C}#H) \to H\text{-Mod} \) preserves injectives.

We denote by \((-)^H\) the functor from \( H\text{-Mod} \) to \( \text{Vect}_K \) that takes \( M \) to \( M^H = \{ m \in M \mid hm = \varepsilon(h)m \ \forall \ h \in H \} \).

We now recall from Proposition 3.8 that we have an isomorphism
\[
\text{Hom}_{\text{Mod-}C}(M, N)^H \cong \text{Hom}_{\text{Mod-}(\mathcal{C}#H)}(M, N)
\]
for any \( M, N \in \text{Mod-}(\mathcal{C}#H) \). At the level of the derived \( \text{Hom} \) functors, this leads to the following spectral sequence.

Theorem 3.15. Let \( M, N \in \text{Mod-}(\mathcal{C}#H) \). Then, there exists a first quadrant spectral sequence:
\[
R^q\,(-)^H \, (\text{Ext}^q_{\text{Mod-}C}(M, N)) \Rightarrow \{ R^{\alpha+q} \text{Hom}_{\text{Mod-}(\mathcal{C}#H)}(M, \_ : \text{Mod-}(\mathcal{C}#H)) \}(N)
\]

Proof. We consider the functors \( \mathcal{F} : \text{Hom}_{\text{Mod-}C}(M, \_ : \text{Mod-}(\mathcal{C}#H) \to H\text{-Mod} and \( \mathcal{G} \) \) such that \( \mathcal{G} \) is a left adjoint.

By Proposition 3.8, \( \text{Hom}_{\text{Mod-}C}(N, \_ : \text{Mod-}(\mathcal{C}#H) \to \text{Vect}_K \) is given by \( \mathcal{G}(\mathcal{F})(N) = \text{Hom}_{\text{Mod-}C}(M, N)^H \). The result now follows from the Grothendieck spectral sequence for composite functors (see [9]).

4 \( H \)-locally finite modules and cohomology

We recall the definition of \( H \)-locally finite modules from [11]. For \( M \in H\text{-Mod} \) and \( m \in M \), let \( Hm \) be the \( H \)-submodule of \( M \) spanned by the elements \( hm \) for \( h \in H \). Consider
\[
M^{(H)} := \{ m \in M \mid Hm \text{ is finite dimensional as a } K\text{-vector space}\}
\]
Clearly, \( M^{(H)} \) is an \( H \)-submodule of \( M \). An \( H \)-module \( M \) is said to be \( H \)-locally finite if \( M^{(H)} = M \). The full subcategory of \( H\text{-Mod} \) whose objects are \( H \)-locally finite \( H \)-modules will be denoted by \( H\text{-mod} \).

By Proposition 3.8, \( \text{Hom}_{\text{Mod-}C}(N, P) \) is an \( H \)-module for any \( N, P \in \text{Mod-}(\mathcal{C}#H) \). We set
\[
\mathcal{L}_{\text{Mod-}C}(N, P) := \text{Hom}_{\text{Mod-}C}(N, P)^{(H)}
\]
Clearly, this defines a functor \( \mathcal{L}_{\text{Mod-}C}(N, \_ : \text{Mod-}(\mathcal{C}#H) \to H\text{-Mod} \) for every \( N \in \text{Mod-}(\mathcal{C}#H) \).

Proposition 4.1. Let \( N \in \text{Mod-}(\mathcal{C}#H) \). Then, the functor \( \mathcal{L}_{\text{Mod-}C}(N, \_ : \text{Mod-}(\mathcal{C}#H) \to H\text{-Mod} \) is right adjoint to the functor \((-) \otimes N : H\text{-mod} \to \text{Mod-}(\mathcal{C}#H) \), i.e., we have natural isomorphisms
\[
\text{Hom}_{\text{Mod-}(\mathcal{C}#H)}(M \otimes N, P) \cong \text{Hom}_{H\text{-mod}}(M, \mathcal{L}_{\text{Mod-}C}(N, P))
\]
for all \( P \in \text{Mod-}(\mathcal{C}#H) \) and \( M \in H\text{-mod} \).
Proof. Let $\phi : \text{Hom}_{\text{Mod-}(C\#H)}(M \otimes N, P) \rightarrow \text{Hom}_{H-\text{mod}}(M, \text{Hom}_{\text{Mod-C}}(N, P))$ be the isomorphism as in Proposition 3.13. Let $\eta : M \otimes N \rightarrow P$ be a morphism in $\text{Mod-}(C\#H)$. It follows that $H\phi(\eta)(m)$ is finite dimensional for each $m \in M$ by observing that $\phi(\eta)$ is $H$-linear and that $M$ is $H$-locally finite. Since $H-\text{mod}$ is a full subcategory of $H-\text{mod}$, we have $\text{Hom}_{\text{Mod-}(C\#H)}(M \otimes N, P) \cong \text{Hom}_{H-\text{mod}}(M, \text{LMod-C}(N, P)) \cong \text{Hom}_{H-\text{mod}}(M, \text{LMod-C}(N, P))$.

For any $M \in \text{Mod-}(C\#H)$, we can now consider the functor

$$\mathcal{L}_{\text{Mod-C}}(M, -) : \text{Mod-}(C\#H) \rightarrow H-\text{mod} \quad N \mapsto \mathcal{L}_{\text{Mod-C}}(M, N) \quad (4.1)$$

Since $\text{Mod-}(C\#H)$ is a Grothendieck category, we obtain derived functors $R^p\mathcal{L}_{\text{Mod-C}}(M, -) : \text{Mod-}(C\#H) \rightarrow H-\text{mod}$, $p \geq 0$. We use the boldface notation to distinguish these from the functors $R^p\mathcal{L}_{\text{Mod-C}}(M, -)$ that will appear later in the proof of Proposition 4.18 as derived functors of a restriction of $\mathcal{L}_{\text{Mod-C}}(M, -)$.

**Theorem 4.2.** Let $M, N \in \text{Mod-}(C\#H)$. We consider the functors

$$\mathcal{F} = \text{Hom}_{\text{Mod-C}}(M, -) : \text{Mod-}(C\#H) \rightarrow H-\text{Mod} \quad N \mapsto \text{Hom}_{\text{Mod-C}}(M, N)$$

Then, we have the following spectral sequence

$$R^p(\mathcal{F}(\mathcal{I}_U)) = (R^p\mathcal{L}_{\text{Mod-C}}(M, -))(N)$$

**Proof.** We have $(\mathcal{F} \circ \mathcal{G})(N) = \text{Hom}_{\text{Mod-C}}(M, N)^{(H)} = \mathcal{L}_{\text{Mod-C}}(M, N)$. By definition,

$$R^p\mathcal{F}(N) = H^p(\mathcal{F}(\mathcal{I}_U)) = H^p(\text{Hom}_{\text{Mod-C}}(M, \mathcal{I}_U))$$

where $\mathcal{I}_U$ is an injective resolution of $N$ in $\text{Mod-}(C\#H)$. By Corollary 3.11, injectives in $\text{Mod-}(C\#H)$ are also injectives in $\text{Mod-C}$. Hence, $R^p\mathcal{F}(N) = \text{Ext}_{\text{Mod-C}}^p(M, N)$. For any injective $\mathcal{I}$ in $\text{Mod-}(C\#H)$, we know that $\mathcal{F}(\mathcal{I})$ is injective in $H-\text{Mod}$ by Corollary 3.14. Since the category $H-\text{Mod}$ has enough injectives, the result now follows from Grothendieck spectral sequence for composite functors (see [9]).

**Definition 4.3.** Let $C$ be a left $H$-category.

1. $C$ is said to be $H$-locally finite if the $H$-module $\text{Hom}_C(X, Y)$ is $H$-locally finite, i.e., $\text{Hom}_C(X, Y)^{(H)} = \text{Hom}_C(X, Y)$, for all $X, Y \in \text{Ob}(C)$.

2. Let $M \in \text{Mod-}(C\#H)$. Then, $M$ is said to be $H$-locally finite if the $H$-module $M(X)$ is $H$-locally finite, i.e., $M(X)^{(H)} = M(X)$, for each $X \in \text{Ob}(C)$. The full subcategory of $\text{Mod-}(C\#H)$ whose objects are $H$-locally finite right $(C\#H)$-modules will be denoted by $\text{mod-}(C\#H)$.

If $M, M' \in H-\text{Mod}$, we know that $H$ acts diagonally on their tensor product $M \otimes M'$ over $K$, i.e., $(h \otimes m) = \sum h_1m_1 \otimes h_2m_2$ for $h \in H$, $m \in M$ and $m' \in M'$. In particular, if $M, M' \in H-\text{Mod}$, it follows that $M \otimes M' \in H-\text{mod}$. Accordingly, if $N' \in \text{mod-}(C\#H)$ and $M \in H-\text{mod}$, it is clear from the definition in Lemma 3.12 that $M \otimes N' \in \text{mod-}(C\#H)$.

**Corollary 4.4.** Let $N' \in \text{mod-}(C\#H)$. Then, the functor $\mathcal{L}_{\text{Mod-C}}(N, -) : \text{mod-}(C\#H) \rightarrow H-\text{mod}$ is right adjoint to the functor $(-) \otimes N' : H-\text{mod} \rightarrow \text{mod-}(C\#H)$, i.e., we have natural isomorphisms

$$\text{Hom}_{\text{mod-}(C\#H)}(M \otimes N', P) \cong \text{Hom}_{H-\text{mod}}(M, \mathcal{L}_{\text{Mod-C}}(N, P))$$

for all $P \in \text{mod-}(C\#H)$ and $M \in H-\text{mod}$. 

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Proof. This follows from Proposition 4.1 because \( \text{mod-}(\mathcal{C} \# H) \) is a full subcategory of \( \text{Mod-}(\mathcal{C} \# H) \).

Lemma 4.5. Let \( \mathcal{C} \) be a left \( H \)-category. Given \( X \in \text{Ob}(\mathcal{C}) \), consider the representable functor \( h_X \in \text{Mod-}\mathcal{C} \). Then, the right \( \mathcal{C} \)-module \( h_X \) is also a right \( (\mathcal{C} \# H) \)-module.

Proof. For each \( Y \in \text{Ob}(\mathcal{C}) \), we have \( h_X(Y) = \text{Hom}_\mathcal{C}(Y,X) \). Since \( \mathcal{C} \) is a left \( H \)-category, \( h_X(Y) \) has a left \( H \)-module structure. For any \( f \in \text{Hom}_\mathcal{C}(Z,Y) \), \( g \in h_X(Y) \) and \( h \in H \), we have

\[
h(h_X(f))(g) = h(gf) = \sum h_1(g)h_2(f) = \sum h_X(h_2f)(h_1g)
\]

Thus, \( h_X \) is a left \( H \)-equivariant right \( \mathcal{C} \)-module. Hence, \( h_X \in \text{Mod-}(\mathcal{C} \# H) \) by Proposition 3.9.

Lemma 4.6. (1) If \( \mathcal{I} \) is an injective in \( \text{Mod-}(\mathcal{C} \# H) \), then \( \mathcal{L}_{\text{Mod-C}}(N,\mathcal{I}) \) is an injective in \( H \)-mod for any \( N \in \text{Mod-}(\mathcal{C} \# H) \).

(2) If \( \mathcal{I} \) is an injective in \( \text{mod-}(\mathcal{C} \# H) \), then

(i) \( \mathcal{L}_{\text{Mod-C}}(N,\mathcal{I}) \) is an injective in \( H \)-mod for any \( N \in \text{mod-}(\mathcal{C} \# H) \).

(ii) Let \( \mathcal{C} \) be \( H \)-locally finite. Then, for each \( X \in \text{Ob}(\mathcal{C} \# H) \), \( \mathcal{I}(X) \) is an injective in \( H \)-mod.

Proof. (1) The functor \( \mathcal{L}_{\text{Mod-C}}(N,\mathcal{I}) : \text{Mod-}(\mathcal{C} \# H) \to H \)-mod is right adjoint to the functor \( (\cdot) \otimes N : H \)-mod \( \to \text{Mod-}(\mathcal{C} \# H) \) by Proposition 4.1. Further, the functor \( (\cdot) \otimes N \) always preserves monomorphisms (see the proof of Corollary 3.14). The result now follows from [27, Tag 015Y].

(2) The proof of (i) is exactly the same as that of (1) except that we use Corollary 4.4 in place of Proposition 4.1. To prove (ii), we consider for each \( X \in \text{Ob}(\mathcal{C} \# H) \) the representable functor \( h_X \in \text{Mod-}\mathcal{C} \). Using Lemma 4.5, we know that \( h_X \in \text{Mod-}(\mathcal{C} \# H) \). Further, since \( \mathcal{C} \) is \( H \)-locally finite, we see that \( h_X \in \text{mod-}(\mathcal{C} \# H) \). Using (i), we have \( \mathcal{L}_{\text{Mod-C}}(h_X,\mathcal{I}) \) is injective in \( H \)-mod. Finally, by Yoneda lemma, we have \( \mathcal{L}_{\text{Mod-C}}(h_X,\mathcal{I}) = \text{Hom}_{\text{Mod-C}}(h_X,\mathcal{I})(H) = \mathcal{I}(X)(H) = \mathcal{I}(X) \).

Lemma 4.7. Let \( \mathcal{C} \) be an \( H \)-locally finite category. Then, for any \( \mathcal{M} \) in \( \text{Mod-}(\mathcal{C} \# H) \), we may obtain an object \( \mathcal{M}^{(H)} \) in \( \text{mod-}(\mathcal{C} \# H) \) by setting

\[
\mathcal{M}^{(H)}(X) := \mathcal{M}(X)^{(H)} = \{ m \in \mathcal{M}(X) \mid Hm \text{ is finite dimensional} \},
\]

\[
\mathcal{M}^{(H)}(f \# h)(m) := \mathcal{M}(f \# h)(m)
\]

for any \( f \# h \in \text{Hom}_{\mathcal{C} \# H}(Y,X) \) and \( m \in \mathcal{M}^{(H)}(X) \).

Proof. We need to verify that \( \mathcal{M}(f \# h) : \mathcal{M}(X) \to \mathcal{M}(Y) \) restricts to a map \( \mathcal{M}(X)^{(H)} \to \mathcal{M}(Y)^{(H)} \). For this, we consider \( m \in \mathcal{M}(X)^{(H)} \). Since \( \mathcal{M} \in \text{Mod-}(\mathcal{C} \# H) \) may be treated as a left \( H \)-equivariant right \( \mathcal{C} \)-module as in Proposition 3.7, we obtain

\[
h'(\mathcal{M}(f \# h))(m) = h' S^{-1}(h)(\mathcal{M}(f)(m)) = \sum \mathcal{M}(h'_2 S^{-1}(h_1)f)(h'_2 S^{-1}(h_2)m)
\]

for any \( h' \in H \). Since the category \( \mathcal{C} \) is \( H \)-locally finite and \( m \in \mathcal{M}^{(H)}(X) \), it is clear from (4.2) that \( \mathcal{M}(f \# h)(m) \in \mathcal{M}(Y)^{(H)} = \mathcal{M}^{(H)}(Y) \). This proves the result.

Proposition 4.8.

(1) Let \( (-)^{(H)} : \text{H-Mod} \to \text{H-mod} \) be the functor \( N \mapsto N^{(H)} \). Then \( (-)^{(H)} \) is right adjoint to the forgetful functor from the category \( \text{H-Mod} \) to the category \( \text{H-Mod} \), i.e., we have natural isomorphisms

\[
\text{Hom}_{\text{H-Mod}}(M,N) \cong \text{Hom}_{\text{H-mod}}(M,N^{(H)})
\]

for any \( M \in \text{H-mod} \) and \( N \in \text{H-Mod} \).
(2) Let $(-)^{(H)} : \text{Mod-(C#H)} \to \text{mod-(C#H)}$ be the functor $N \to N^{(H)}$. Then $(-)^{(H)}$ is right adjoint to the forgetful functor from the category $\text{mod-(C#H)}$ to the category $\text{Mod-(C#H)}$, i.e., we have natural isomorphisms

$$\text{Hom}_{\text{mod-(C#H)}}(M,N) \cong \text{Hom}_{\text{Mod-(C#H)}}(M,N^{(H)})$$

for any $M \in \text{mod-(C#H)}$ and $N \in \text{Mod-(C#H)}$.

Proof. (1) Given any $M \in H\text{-mod}$, $N \in H\text{-Mod}$ and an $H$-module morphism $\phi : M \to N$, it is clear that $\phi(m) \in N^{(H)}$ for all $m \in M$.

(2) Let $M \in \text{mod-(C#H)}$. By Lemma 3.6, a morphism $\eta : M \to N$ in $\text{mod-(C#H)}$ induces $H$-linear morphisms $\eta(X) : M(X) \to N(X)$ for each $X \in \text{Ob}(\mathcal{C})$. Since $M(X)$ is $H$-locally finite, each $\eta(X)$ can be written as a morphism $\mathcal{M}(X) \to N(X)^{(H)}$. The result is now clear. \hfill \Box

Lemma 4.9. Let $\mathcal{C}$ be $H$-locally finite. Then,

(1) The category $\text{mod-(C#H)}$ is abelian.

(2) If $\mathcal{I}$ is an injective in $\text{Mod-(C#H)}$, then $\mathcal{I}^{(H)}$ is an injective in $\text{mod-(C#H)}$.

(3) The category $\text{mod-(C#H)}$ has enough injectives.

Proof. (1) Since $H\text{-mod}$ is closed under kernels and cokernels, it is clear that the subcategory $\text{mod-(C#H)}$ of the abelian category $\text{Mod-(C#H)}$ is closed under kernels and cokernels. Also, since products and coproducts of finitely many objects $\mathcal{M}_i$ in $\text{mod-(C#H)}$ are given by

$$(\prod \mathcal{M}_i)(X) = \left( \bigoplus \mathcal{M}_i \right)(X) = \bigoplus \mathcal{M}_i(X)$$

for $X \in \text{Ob(\mathcal{C})}$, it follows that finite products and coproducts exist and coincide in $\text{mod-(C#H)}$. Thus, the category $\text{mod-(C#H)}$ is abelian.

(2) Since the functor $(-)^{(H)}$ is right adjoint to the forgetful functor in Proposition 4.8(2) and the forgetful functor always preserves monomorphisms, this result follows from [27, Tag 015Y].

(3) Since $\text{Mod-(C#H)}$ is a Grothendieck category, it has enough injectives. The result is now clear from (2). \hfill \Box

We now recall the notions of free, finitely generated and noetherian modules over a category from [20, §3] and [21]. Given $\mathcal{M} \in \text{Mod-\mathcal{C}}$, we set $el(\mathcal{M}) := \bigcup_{X \in \text{Ob(\mathcal{C})}} \mathcal{M}(X)$ to be the collection of all elements of $\mathcal{M}$. If $m \in el(\mathcal{M})$ lies in $\mathcal{M}(X)$, we write $|m| = X$.

Definition 4.10. Let $\mathcal{C}$ be a small preadditive category and let $\mathcal{M} \in \text{Mod-\mathcal{C}}$.

(i) A family of elements $\{m_i \in el(\mathcal{M})\}_{i \in I}$ is said to generate $\mathcal{M}$ if every element $y \in el(\mathcal{M})$ can be expressed as $y = \sum_{i \in I} \mathcal{M}(f_i)(m_i)$ for some $f_i \in Hom_{\mathcal{C}}(|y|, |m_i|)$, where all but a finite number of the $f_i$ are zero. Equivalently, the family $\{m_i \in el(\mathcal{M})\}_{i \in I}$ is said to generate $\mathcal{M}$ if the induced morphism

$$\eta : \bigoplus_{i \in I} h_{|m_i|} \longrightarrow \mathcal{M}$$

which takes $(0, \ldots, 0, id_{|m_i|}, 0, \ldots, 0)$ to $m_i$ is an epimorphism. The family is said to be a basis for $\mathcal{M}$ if $\eta$ is an isomorphism. The module $\mathcal{M}$ is said to be finitely generated (resp. free) if it has a finite set of generators (resp. a basis).

(ii) The module $\mathcal{M}$ is called noetherian if it satisfies the ascending chain condition on submodules. The category $\mathcal{C}$ is said to be right noetherian if $h_X \in \text{Mod-\mathcal{C}}$ is noetherian for each $X \in \text{Ob(\mathcal{C})}$.
Proposition 4.11. Let \( C \) be a left \( \mathcal{H} \)-category. An object \( M \in \text{mod-}(C\#H) \) is finitely generated in \( \text{Mod-}(C\#H) \) if and only if there exists a finite dimensional \( V \in H\text{-Mod} \) and an epimorphism

\[
V \otimes \bigoplus_{i=1}^{n} h_{X_i} \longrightarrow M
\]

in \( \text{Mod-}(C\#H) \) for finitely many objects \( \{X_i\}_{i \in I} \) in \( C \), where each \( h_{X_i} \) is viewed as an object in \( \text{Mod-}(C\#H) \).

Proof. Let \( M \in \text{mod-}(C\#H) \) be finitely generated in \( \text{Mod-}(C\#H) \). We consider a finite generating family \( \{m_i \in \text{el}(M)\}_{i \in I} \) for \( M \). Since \( M \in \text{mod-}(C\#H) \), each \( M([m_i]) \) is \( H \)-locally finite and hence the \( H \)-module

\[
V := \bigoplus_{i \in I} Hm_i
\]

is finite dimensional. For each \( Y \in \text{Ob}(C\#H) \), we consider the morphism determined by setting

\[
\eta(Y) : V \otimes \left( \bigoplus_{i \in I} h_{[m_i]}(Y) \right) \longrightarrow M(Y) \quad \text{for } m_i \otimes (f_1, \ldots, f_n) \mapsto M(f_i \# h)(m_i)
\]

It is easy to check that \( \eta \) is a morphism in \( \text{Mod-}(C\#H) \) and \( \eta(Y) \) is an epimorphism for all \( Y \in \text{Ob}(C\#H) \). Conversely, let \( \{v_1, \ldots, v_k\} \) be a basis of a finite dimensional \( H \)-module \( V \) and \( X_1, \ldots, X_n \) be finitely many objects in \( C \) such that there is an epimorphism

\[
\eta : V \otimes \left( \bigoplus_{i=1}^{n} h_{X_i} \right) \longrightarrow M
\]

in \( \text{Mod-}(C\#H) \). It may be verified that the elements \( \{m_{ij} = \eta(X_i)(v_j \otimes \text{id}_{X_i}) \in \text{M}(X_i)\}_{i \in I, j \in J} \) give a family of generators for \( M \).

\[\square\]

Corollary 4.12. An object \( M \) in \( \text{mod-}(C\#H) \) is finitely generated in \( \text{Mod-}(C\#H) \) if and only if \( M \) is finitely generated in \( \text{Mod-}C \).

Proof. Let \( M \in \text{mod-}(C\#H) \) be finitely generated in \( \text{Mod-}C \). Then, there is a finite family \( \{m_i \in \text{el}(M)\}_{i \in I} \) of elements of \( M \) such that every \( y \in \text{el}(M) \) can be expressed as \( y = \sum_{i \in I} M(f_i)(m_i) \) for some \( f_i \in \text{Hom}_C([y], [m_i]) \). Then, \( y = \sum_{i \in I} M(f_i \# 1_{[y]})(m_i) \) and hence \( M \) is finitely generated as a \( (C\#H) \)-module.

Conversely, let \( M \in \text{mod-}(C\#H) \) be finitely generated in \( \text{Mod-}(C\#H) \) and let

\[
\eta : V \otimes \left( \bigoplus_{i=1}^{n} h_{X_i} \right) \longrightarrow M
\]

denote the epimorphism in \( \text{Mod-}(C\#H) \) as in Proposition 4.11. In particular, \( \eta \) is an epimorphism in \( \text{Mod-}C \). Then, if \( \{v_1, \ldots, v_k\} \) is a basis for \( V \), it follows from the epimorphism in (4.3) that \( \{m_{ij} = \eta(X_i)(v_j \otimes \text{id}_{X_i}) \in M(X_i)\}_{i \in I, j \in J} \) gives a finite set of generators for \( M \) as a right \( C \)-module.

\[\square\]

We remark here that if \( V \) and \( V' \) are left \( H \)-modules, then \( \text{Hom}_K(V, V') \) carries a left \( H \)-module action defined by

\[
(hf)(v) = \sum h_1 f(S(h_2)v) \quad \forall v \in V, h \in H
\]

This may be seen as the special case of the action described in Proposition 3.8 when \( C \) is the category with one object having endomorphism ring \( K \).

Lemma 4.13. Let \( V \in H\text{-Mod} \) and \( N \in \text{mod-}(C\#H) \). Then, \( \text{Hom}_K(V, N(X)) \) and \( \text{Hom}_{\text{Mod-}C}(V \otimes h_{X}, N) \) are isomorphic as objects in \( H\text{-Mod} \) for each \( X \in \text{Ob}(C) \).

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Proof. We check that the canonical isomorphism \( \phi : \text{Hom}_{\text{Mod-}\mathcal{C}}(V \otimes h_X, N) \rightarrow \text{Hom}_K(V, \text{Hom}_{\text{Mod-}\mathcal{C}}(h_X, N)) \cong \text{Hom}_K(V, N(X)) \) defined by \( \phi(\eta)(v) = \eta(X)(v \otimes \text{id}_X) \) for any morphism \( \eta \in \text{Hom}_{\text{Mod-}\mathcal{C}}(V \otimes h_X, N) \) and \( v \in V \) is \( H \)-linear:

\[
\phi(\eta)(v) = (h\eta)(X)(v \otimes \text{id}_X) \\
= \sum h_1\eta(X)(S(h_2)(v \otimes \text{id}_X)) \\
= \sum h_1\eta(X)(S(h_3)v \otimes S(h_2)\text{id}_X) \\
= \sum h_1\eta(X)(S(h_3)v \otimes \varepsilon(S(h_2))\text{id}_X) \\
= \sum h_1\eta(X)(S(h_2)v \otimes \varepsilon(h_2)) \\
= \sum h_1\eta(X)(S(h_2)v \otimes \text{id}_X) \\
= \sum h_1\eta(X)(S(h_2)v) = (h\phi(\eta))(v)
\]

\[\Box\]

**Proposition 4.14.** Let \( M \) and \( N \) be in \( \text{mod-}(\mathcal{C}\#H) \) with \( M \) finitely generated in \( \text{Mod-}(\mathcal{C}\#H) \). Then, \( \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N) \) is \( H \)-locally finite, i.e., \( \mathcal{L}_{\text{Mod-}\mathcal{C}}(M, N) = \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N)^{(H)} = \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N) \).

Proof. Since \( M \in \text{mod-}(\mathcal{C}\#H) \) is finitely generated in \( \text{Mod-}(\mathcal{C}\#H) \), there exists by Proposition 4.11 a finite dimensional \( H \)-module \( V \) and an epimorphism \( \varphi : V \otimes (\bigoplus_{i=1}^{\infty} h_X) \rightarrow M \) in \( \text{Mod-}(\mathcal{C}\#H) \) for finitely many objects \( X_1, \ldots, X_n \) in \( \mathcal{C} \). Thus we get a monomorphism

\[
\tilde{\varphi} : \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N) \rightarrow \text{Hom}_{\text{Mod-}\mathcal{C}}(V \otimes (\bigoplus_{i=1}^{\infty} h_X), N), \quad \eta \mapsto \eta \circ \varphi.
\]

(4.5)

For each \( X \in \text{Ob}(\mathcal{C}) \), \( v \in V \) and \( f \in h_{X_i}(X) \) for some chosen \( 1 \leq i \leq n \), we have

\[
\tilde{\varphi}(h\eta)(X)(v \otimes f) = (h\eta)(X)(v \otimes f) = h\eta(X)(v \otimes f) = \sum h_1\eta(X)(S(h_2)\varphi(X)(v \otimes f)) \\
= \sum h_1\eta(X)(\varphi(X)(S(h_2)(v \otimes f))) = \sum h_1\eta(X)(S(h_2)(v \otimes f)) \\
= (h\eta)(X)(v \otimes f) = (h\tilde{\varphi}(\eta))(X)(v \otimes f)
\]

This shows that \( \tilde{\varphi} \) is an \( H \)-module monomorphism. By Lemma 4.13, we know that

\[
\text{Hom}_{\text{Mod-}\mathcal{C}}(V \otimes (\bigoplus_{i=1}^{\infty} h_X), N) \cong \bigoplus_{i=1}^{\infty} \text{Hom}_K(V, N(X_i))
\]

as \( H \)-modules. Since \( V \) is finite dimensional, we know that \( \text{Hom}_K(V, N(X_i)) \cong N(X_i) \otimes V^* \) in \( \text{Vect}_K \) and it is easily seen that this is an isomorphism of \( H \)-modules. Since \( N \) is an \( H \)-locally finite \( (\mathcal{C}\#H) \)-module and \( V^* \) is \( H \)-locally finite (because \( \text{dim}_K(V^*) < \infty \)), it follows that each \( N(X_i) \otimes V^* \) is \( H \)-locally finite. The embedding in (4.5) now shows that \( \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N) \) is \( H \)-locally finite. \[\Box\]

**Lemma 4.15.** Let \( M, N \in \text{Mod-}(\mathcal{C}\#H) \). For a morphism \( \eta \in \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N) \), the following are equivalent:

1. \( \eta \in \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N)^{(H)} = \mathcal{L}_{\text{Mod-}\mathcal{C}}(M, N) \).

2. There exists a finite dimensional \( H \)-module \( V \), an element \( v \in V \) and some \( \eta \in \text{Hom}_{\text{Mod-}(\mathcal{C}\#H)}(V \otimes M, N) \) such that \( \eta(X)(v \otimes m) = \eta(X)(m) \) for each \( X \in \text{Ob}(\mathcal{C}) \) and \( m \in M(X) \).

Proof. (1) \( \Rightarrow \) (2): We put \( V = H\eta \). Since \( \eta \in \text{Hom}_{\text{Mod-}\mathcal{C}}(M, N)^{(H)} \), we see that \( V \) is finite dimensional. Let \( \{\eta_1, \ldots, \eta_k\} \) be a basis for \( V = H\eta \). Any element \( h\eta \in V \) can now be expressed as \( h\eta = \sum_{i=1}^{k} \alpha_i(h)\eta_i \).
We now define \( \hat{\eta} \in \text{Hom}_{\text{Mod}-C}(V \otimes M, N) \) by setting \( \hat{\eta}(X)(h\eta \otimes m) := (h\eta)(X)(m) \) for each \( h \in H, \ X \in \text{Ob}(C) \) and \( m \in M(X) \). It is clear that \( \hat{\eta}(\eta \otimes m) = \eta(X)(m) \).

In order to show that \( \hat{\eta} \in \text{Hom}_{\text{Mod}(C \# H)}(V \otimes M, N) \), it suffices to show that each \( \hat{\eta}(X) : V \otimes M(X) \longrightarrow N(X) \) is \( H \)-linear. For \( h' \in H \), we have

\[
\hat{\eta}(X)(h'(h \eta \otimes m)) = (h\eta)(X)(m)
\]

For any \( h \in H \), we define \( \hat{\eta}(X)(h \eta \otimes m) = \sum h_i \eta(X)(h_i \eta \otimes h_i^2 m) = \sum h_i \eta(X)(h_i \eta \otimes h_i^2 m) \).

(2) \( \Rightarrow \) (1) : We are given \( \eta \in \text{Hom}_{\text{Mod}(C \# H)}(V \otimes M, N) \). Let \( \{v_1, \ldots, v_k\} \) be a basis for \( V \) and suppose that \( hv = \sum_{i=1}^k \alpha_i(h)v_i \). For each \( 1 \leq i \leq k \), we define \( \xi_i \in \text{Hom}_{\text{Mod}-C}(M, N) \) by setting \( \xi_i(X)(m) := \eta(X)(v_i \otimes m) \) for \( X \in \text{Ob}(C) \), \( m \in M(X) \). For any \( h \in H \), we see that

\[
(h\eta)(X)(m) = \sum h_i \eta(X)(h_i S(h_2) m) = \sum h_i \eta(X)(h_i v \otimes S(h_2) m)
\]

It follows from the above that \( \hat{\eta} \) lies in the space generated by the finite collection \( \{\xi_1, \ldots, \xi_k\} \in \text{Hom}_{\text{Mod}-C}(M, N) \).

This proves the result.

Proposition 4.16. If \( I \) is an injective object in \( \text{mod}(C \# H) \), then \( \mathcal{L}_{\text{Mod}-C}(\cdot, I) \) is an exact functor from \( \text{mod}(C \# H) \) to \( \text{H-mod} \).

Proof. Let \( 0 \longrightarrow M \twoheadrightarrow N \longrightarrow P \longrightarrow 0 \) be an exact sequence in \( \text{mod}(C \# H) \) and \( I \) be an injective object in \( \text{mod}(C \# H) \). Then, \( 0 \longrightarrow \text{Hom}_{\text{Mod}(C \# H)}(P, I) \longrightarrow \text{Hom}_{\text{Mod}(C \# H)}(N, I) \longrightarrow \text{Hom}_{\text{Mod}(C \# H)}(M, I) \) is an exact sequence in \( \text{H-mod} \). Since the functor \( (\cdot)^{(H)} : \text{H-mod} \longrightarrow \text{H-mod} \) is a right adjoint by Proposition 4.8(1), it preserves monomorphisms. Thus, \( 0 \longrightarrow \mathcal{L}_{\text{Mod}(C \# H)}(P, I) \longrightarrow \mathcal{L}_{\text{Mod}(C \# H)}(N, I) \longrightarrow \mathcal{L}_{\text{Mod}(C \# H)}(M, I) \) is an exact sequence in \( \text{H-mod} \).

Let \( \eta \in \mathcal{L}_{\text{Mod}(C \# H)}(M, I) \). We set \( V := H \eta \). Then, \( V \) is a finite dimensional \( H \)-module and therefore \( V \in \text{H-mod} \). Thus, \( V \otimes M, V \otimes N \in \text{mod}(C \# H) \) and we have a monomorphism \( \text{id}_V \otimes i : V \otimes M \longrightarrow V \otimes N \) in \( \text{mod}(C \# H) \).

Now, we consider the morphism \( \zeta \in \text{Hom}_{\text{Mod}(C \# H)}(V \otimes M, I) \) defined by setting \( \zeta(X)(v \otimes m) := \eta(X)(m) \) for each \( X \in \text{Ob}(C) \), \( m \in M(X) \) and \( v \in V \). It may be verified that \( \zeta(X) \) is \( H \)-linear for each \( X \in \text{Ob}(C) \). Thus, \( \zeta \in \text{Hom}_{\text{mod}(C \# H)}(V \otimes M, I) \). Since \( I \) is injective in \( \text{mod}(C \# H) \), there exists a morphism \( \xi : V \otimes N \longrightarrow I \) in \( \text{mod}(C \# H) \) such that \( \xi(i \eta) = \zeta \). The morphism \( \xi \in \text{Hom}_{\text{mod}(C \# H)}(V \otimes N, I) \) now induces a morphism \( \xi \in \text{Hom}_{\text{Mod}(C \# H)}(V \otimes N, I) \) defined by setting \( \xi(X)(n) := \xi(X)(\eta \otimes n) \) for every \( X \in \text{Ob}(C) \) and \( n \in N(X) \). Applying Lemma 4.15, we see that \( \xi \in \mathcal{L}_{\text{Mod}(C \# H)}(N, I) \). Also, \( \xi \circ i = \eta \). This completes the proof.

Proposition 4.17. Let \( C \) be a left \( H \)-locally finite category which is right noetherian. Let \( M \in \text{mod}(C \# H) \) be finitely generated as an object in \( \text{Mod}(C \# H) \). If \( I \) is an injective object in \( \text{mod}(C \# H) \), then \( \text{Ext}_{\text{Mod}(C \# H)}^p(M, I) = 0 \) for all \( p > 0 \).

Proof. Since \( M \in \text{mod}(C \# H) \) is finitely generated in \( \text{Mod}(C \# H) \), by Proposition 4.11, there exists a finite dimensional \( H \)-module \( V_0 \) and an epimorphism

\[
\eta_0 : P_0 := V_0 \otimes \left( \bigoplus_{i=1}^n h \chi_i \right) \longrightarrow M
\]
in $\text{Mod-}(\mathcal{C}\#H)$ for finitely many objects $\{X_i\}_{1 \leq i \leq n_0}$ in $\mathcal{C}$, where $h_{X_i}$ are viewed as objects in $\text{Mod-}(\mathcal{C}\#H)$. Since $\mathcal{C}$ is $H$-locally finite, each $h_{X_i} \in \text{mod-}(\mathcal{C}\#H)$. Since $V_0$ is finite dimensional, we must have $V_0 \in H$-$\text{mod}$. Thus, $P_0 \in \text{mod-}(\mathcal{C}\#H)$. Using Proposition 4.11 and Corollary 4.12, it follows that $P_0$ is finitely generated in $\text{Mod-}\mathcal{C}$. Since $\mathcal{C}$ is right noetherian, $P_0$ is a noetherian right $\mathcal{C}$-module (see, for instance, [20, § 3]). Since the submodule of a finitely generated noetherian module is finitely generated, the $(\mathcal{C}\#H)$-submodule $K := \text{Ker}(\eta_0)$ of $P_0$ is finitely generated in $\text{Mod-}\mathcal{C}$. So, again using Proposition 4.11 and Corollary 4.12 it follows that there exists a finite dimensional $H$-module $V_1$ and an epimorphism

$$\eta_1 : P_1 := V_1 \oplus \left( \bigoplus_{j=1}^{n_1} h_{Y_j} \right) \twoheadrightarrow K,$$

in $\text{Mod-}(\mathcal{C}\#H)$ for finitely many objects $\{Y_j\}_{1 \leq j \leq n_1}$ in $\mathcal{C}$. Since $V_0$ and $V_1$ are finite dimensional $K$-vector spaces, clearly $P_0$ and $P_1$ are free right $\mathcal{C}$-modules. Moreover, $\text{Im}(\eta_1) = K = \text{Ker}(\eta_0)$. Thus, continuing in this way, we can construct a free resolution of the module $M$ in the category $\text{Mod-}\mathcal{C}$:

$$P = \cdots \to P_1 \to \cdots \to P_1 \to P_0 \to M \to 0.$$ 

Hence, we have

$$\text{Ext}^p_{\text{Mod-}\mathcal{C}}(M, I) = H^p(\text{Hom}_{\text{Mod-}\mathcal{C}}(P_*, I)), \quad p \geq 0$$

Since $\mathcal{M}$ and $\{P_i\}_{i \geq 0}$ are finitely generated in $\text{Mod-}(\mathcal{C}\#H)$, we have $L_{\text{Mod-}\mathcal{C}}(M, I) = \text{Hom}_{\text{Mod-}\mathcal{C}}(M, I)$ and $L_{\text{Mod-}\mathcal{C}}(P_1, I) = \text{Hom}_{\text{Mod-}\mathcal{C}}(P_1, I)$ for every $i \geq 0$, by Proposition 4.14. From Proposition 4.16, we know that $L_{\text{Mod-}\mathcal{C}}(\cdot, I)$ is exact and it follows that $H^p(L_{\text{Mod-}\mathcal{C}}(P_*, I)) = 0$ for all $p > 0$. This proves the result.

**Proposition 4.18.** Let $\mathcal{C}$ be a left $H$-locally finite category which is right noetherian. Let $\mathcal{M}$ and $\mathcal{N}$ be in $\text{mod-}(\mathcal{C}\#H)$ with $\mathcal{M}$ finitely generated in $\text{Mod-}(\mathcal{C}\#H)$ and let $E^*$ be an injective resolution of $\mathcal{N}$ in $\text{mod-}(\mathcal{C}\#H)$. Then,

$$\text{Ext}^p_{\text{mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N}) = H^p(\text{Hom}_{\text{Mod-}\mathcal{C}}(\mathcal{M}, E^*)), \quad p \geq 0.$$ 

**Proof.** Let $P_*$ be the free resolution of $\mathcal{M}$ in $\text{Mod-}\mathcal{C}$ constructed as in the proof of Proposition 4.17. Then, we have

$$\text{Ext}^p_{\text{mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N}) = H^p(\text{Hom}_{\text{Mod-}\mathcal{C}}(P_*, \mathcal{N})) = H^p(L_{\text{mod-}\mathcal{C}}(P_*, \mathcal{N}))$$

where the second equality follows from Proposition 4.14. Since $H$-$\text{mod}$ is an abelian category and $L_{\text{mod-}\mathcal{C}}(P_*, \mathcal{N})$ is a complex in $H$-$\text{mod}$, it follows that $H^p(L_{\text{mod-}\mathcal{C}}(P_*, \mathcal{N})) \in H$-$\text{mod}$. Hence, we may consider the family $\{L_{\text{mod-}\mathcal{C}}(\cdot, \mathcal{N})\}_{p \geq 0}$ as a $\delta$-functor from $\text{mod-}(\mathcal{C}\#H)$ to $H$-$\text{mod}$. By Proposition 4.17, $\text{Ext}^p_{\text{mod-}\mathcal{C}}(\mathcal{M}, I) = 0$, $p > 0$ for every injective object $I$ in $\text{mod-}(\mathcal{C}\#H)$. Since $\text{mod-}(\mathcal{C}\#H)$ has enough injectives, it follows that each $\text{Ext}^p_{\text{mod-}\mathcal{C}}(\cdot, \mathcal{N}) : \text{mod-}(\mathcal{C}\#H) \to H$-$\text{mod}$ is effaceable (see, for instance, [12, § III.1]).

Since $\text{mod-}(\mathcal{C}\#H)$ has enough injectives, we can consider the right derived functors

$$R^p L_{\text{mod-}\mathcal{C}}(\cdot, \mathcal{N}) : \text{mod-}(\mathcal{C}\#H) \to H$-$\text{mod}$$

For $p = 0$, we notice that $\text{Ext}^0_{\text{mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N}) = H_{\text{mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N}) = L_{\text{mod-}\mathcal{C}}(\mathcal{M}, -) = R^0 L_{\text{mod-}\mathcal{C}}(\mathcal{M}, -)$ as functors from $\text{mod-}(\mathcal{C}\#H)$ to $H$-$\text{mod}$. Since each $\text{Ext}^p_{\text{mod-}\mathcal{C}}(\mathcal{M}, -)$ is effaceable for $p > 0$, the family $\{\text{Ext}^p_{\text{mod-}\mathcal{C}}(\mathcal{M}, -)\}_{p \geq 0}$ forms a universal $\delta$-functor and it follows from [12, Corollary III.1.4] that

$$\text{Ext}^p_{\text{mod-}\mathcal{C}}(\mathcal{M}, -) = R^p L_{\text{mod-}\mathcal{C}}(\mathcal{M}, -) : \text{mod-}(\mathcal{C}\#H) \to H$-$\text{mod}$$

for every $p \geq 0$. Therefore, we have

$$\text{Ext}^p_{\text{mod-}\mathcal{C}}(\mathcal{M}, \mathcal{N}) = (R^p L_{\text{mod-}\mathcal{C}}(\mathcal{M}, -))(\mathcal{N}) = H^p(L_{\text{mod-}\mathcal{C}}(\mathcal{M}, E^*)) = H^p(\text{Hom}_{\text{Mod-}\mathcal{C}}(\mathcal{M}, E^*))$$

$\square$
Theorem 4.19. Let $C$ be a left $H$-locally finite category which is right noetherian. Fix $\mathcal{M} \in \text{mod-}(\mathcal{C}^\#H)$ with $\mathcal{M}$ finitely generated in $\text{Mod-}(\mathcal{C}^\#H)$. We consider the functors

$$\mathcal{F} = \text{Hom}_{\text{Mod-}C}(\mathcal{M}, -) : \text{mod-}(\mathcal{C}^\#H) \to \text{H-mod} \quad \mathcal{G} = (-)^H : \text{H-mod} \to \text{Vect}_K \quad \mathcal{N} \mapsto \text{Hom}_{\text{Mod-}C}(\mathcal{M}, \mathcal{N})$$

Then, we have the following spectral sequence

$$R^p(-)^H(\text{Ext}_{\text{Mod-}C}^q(M, N)) \Rightarrow \left( R^{p+q} \text{Hom}_{\text{mod-}(\mathcal{C}^\#H)}(\mathcal{M}, -) \right)(N)$$

Proof. Using Proposition 3.8 and the fact that $\text{mod-}(\mathcal{C}^\#H)$ is a full subcategory of $\text{Mod-}(\mathcal{C}^\#H)$, we have $(\mathcal{G} \circ \mathcal{F})(N) = \text{Hom}_{\text{Mod-}C}(\mathcal{M}, N)^H = \text{Hom}_{\text{mod-}(\mathcal{C}^\#H)}(\mathcal{M}, N)$. By definition,

$$R^p \mathcal{F}(N) = H^p(\mathcal{F}(\mathcal{E}^*)) = H^p(\text{Hom}_{\text{Mod-}C}(\mathcal{M}, \mathcal{E}^*))$$

where $\mathcal{E}^*$ is an injective resolution of $N$ in $\text{mod-}(\mathcal{C}^\#H)$. By Proposition 4.18, we get $R^p \mathcal{F}(N) = \text{Ext}_{\text{Mod-}C}^q(M, N)$. For any injective $\mathcal{I}$ in $\text{mod-}(\mathcal{C}^\#H)$, we know that $\mathcal{F}(\mathcal{I})$ is an injective in $\text{H-mod}$ by Proposition 4.14 and Lemma 4.6(2). Since the category $\text{H-mod}$ has enough injectives (see [11, Lemma 1.4]), the result now follows from Grothendieck spectral sequence for composite functors (see [9]).

5 Cohomology of relative $(\mathcal{D}, H)$-Hopf modules

Let $\mathcal{D}$ be a right co-$H$-category. In the notation of Definition 2.4, for any $X, Y \in \text{Ob}(\mathcal{D})$, there is an $H$-coaction on the $K$-vector space $\text{Hom}_\mathcal{D}(X, Y)$ given by $\rho_{XY}(f) := \sum f_0 \otimes f_1$. In this section, we will study the relative Hopf modules over the category $\mathcal{D}$ and describe their derived $\text{Hom}$-functors by means of spectral sequences.

We denote by $\text{Comod-}H$ the category of right $H$-comodules. If $M$ is an $H$-comodule with right $H$-coaction given by $\rho_M : M \to M \otimes H$, we set $M^{coH} := \{ m \in M \mid \rho_M(m) = m \otimes 1_H \}$ to be the coinvariants of $M$.

Definition 5.1. Let $\mathcal{D}$ be a right co-$H$-category. Let $\mathcal{M}$ be a left $\mathcal{D}$-module with a given right $H$-comodule structure $\rho_{M(X)} : M(X) \to M(X) \otimes H$, $m \mapsto \sum m_0 \otimes m_1$ on $M(X)$ for each $X$ in $\text{Ob}(\mathcal{D})$. Then, $\mathcal{M}$ is said to be a relative $(\mathcal{D}, H)$-Hopf module if the following condition holds:

$$\rho_{M(X)}(M(f)(m)) = \sum M(f_0)(m_0) \otimes f_1 m_1 \quad (5.1)$$

for any $f \in \text{Hom}_\mathcal{D}(X, Y)$ and $m \in M(X)$. We denote by $\text{D-H}_H$ the category whose objects are relative $(\mathcal{D}, H)$-Hopf modules and whose morphisms are given by

$$\text{Hom}_{\mathcal{D}-H}(M, N) := \{ \eta \in \text{Hom}_{\text{Comod-}H}(M, N) \mid \eta(X) : M(X) \to N(X) \text{ is } H\text{-colinear } \forall X \in \text{Ob}(\mathcal{D}) \}.$$  

We now recall the tensor product of $H$-comodules. Let $M, N \in \text{Comod-}H$ with $H$-coactions $\rho_M$ and $\rho_N$, respectively. Then, $M \otimes N \in \text{Comod-}H$ with $H$-coaction given by $\rho_{M \otimes N} := (\text{id} \otimes \text{id} \otimes m_{1H})((\text{id} \otimes \tau \otimes \text{id})(\rho_M \otimes \rho_N)$, where $m_{1H}$ denotes the multiplication on $H$ and $\text{id} \otimes \tau \otimes \text{id} : M \otimes (H \otimes N) \otimes H \to M \otimes (N \otimes H) \otimes H$ denotes the twist map. In other words, $\rho_{M \otimes N}(m \otimes n) = \sum m_0 \otimes n_0 \otimes m_1 n_1$ for $m \otimes n \in M \otimes N$.

Lemma 5.2. Let $M \in \text{Comod-}H$ and $N \in \text{D-H}_H$. Then, $N \otimes M$ defined by setting

$$N \otimes M(X) := N(X) \otimes M$$

$$(N \otimes M)(f)(n \otimes m) := N(f)(n) \otimes m$$

for each $X \in \text{Ob}(\mathcal{D})$, $f \in \text{Hom}_\mathcal{D}(X, Y)$ and $n \otimes m \in N(X) \otimes M$ is a relative $(\mathcal{D}, H)$-Hopf module.
Proof. Clearly, $N \otimes M$ is a left $D$-module. Since $N(X)$ is a right $H$-comodule, $N(X) \otimes M$ also carries a right $H$-comodule structure for each $X \in \text{Ob}(D)$. For any $f \in \text{Hom}_D(X,Y)$ and $n \otimes m \in N(X) \otimes M$, we have
\[
\rho((N \otimes M)(f)(n \otimes m)) = \rho(N(f)(n) \otimes m) = \sum (N(f)(n))_0 \otimes m_0 \otimes (N(f)(n))_1 m_1
\]
\[
= \sum N(f_0)(n_0) \otimes m_0 \otimes f_1 n_1 m_1
\]
\[
= \sum (N \otimes M)(f_0)(n_0 \otimes m_0) \otimes f_1 n_1 m_1
\]
\[
= \sum (N \otimes M)(f_0)(n \otimes m)_0 \otimes f_1 (n \otimes m)_1
\]
This shows that $N \otimes M$ satisfies the condition (5.1) in Definition 5.1.

From Lemma 5.2, it follows that the assignment $M \mapsto N \otimes M$ defines a functor $N \otimes (-) : \text{Comod}_H \to \mathcal{D} \mathcal{M}_H$ for each $N \in \mathcal{D} \mathcal{M}_H$.

From the definition of a co-$H$-category, it is also clear that the $D$-module $\chi h := \text{Hom}_D(X,-)$ lies in $\mathcal{D} \mathcal{M}_H$ for each $X \in \text{Ob}(D)$.

Lemma 5.3. Let $M$ be a relative $(\mathcal{D}, H)$-Hopf module and let $m \in M(X)$ for some $X \in \text{Ob}(D)$. Then, there exists a finite dimensional $H$-comodule $W_m \subseteq M(X)$ containing $m$ and a morphism $\eta_m : \chi h \otimes W_m \to M$ in $\mathcal{D} \mathcal{M}_H$ such that $\eta_m(X)(\text{id}_X \otimes m) = m$.

Proof. Using [7, Theorem 2.1.7], we know that there exists a finite dimensional $H$-subcomodule $W_m$ of $M(X)$ containing $m$. We consider the $D$-module morphism $\eta_m : \chi h \otimes W_m \to M$ defined by
\[
\eta_m(Y)(f \otimes w) := M(f)(w)
\]
for any $Y \in \text{Ob}(D)$, $f \in \chi h(Y)$ and $w \in W_m$. We now verify that $\eta_m$ is indeed a morphism in $\mathcal{D} \mathcal{M}_H$, i.e., $\eta_m(Y)$ is $H$-colinear for each $Y \in \text{Ob}(D)$:
\[
\rho(\eta_m(Y)(f \otimes w)) = \rho(M(f)(w)) = \sum M(f_0)(w_0) \otimes f_1 w_1 = \sum \eta_m(Y)(f_0 \otimes w_0) \otimes f_1 w_1 = \eta_m(Y)((f \otimes w)_0 \otimes (f \otimes w)_1)
\]

Remark 5.4. It might be tempting to view Lemma 5.3 as a Yoneda correspondence. But, we note that the finite dimensional $H$-comodules and the morphism in Lemma 5.3 determined by $m \in M(X)$ need not be unique.

Given a morphism $\eta : M \to N$ in $\mathcal{D} \mathcal{M}_H$, it may be easily verified that $\text{Ker}(\eta)$ and $\text{Coker}(\eta)$ determined by setting
\[
\text{Ker}(\eta)(X) := \text{Ker}(\eta(X)) : M(X) \to N(X)
\]
\[
\text{Coker}(\eta)(X) := \text{Coker}(\eta(X)) : M(X) \to N(X)
\]
for each $X \in \text{Ob}(D)$ are also relative $(\mathcal{D}, H)$-Hopf modules. It follows that $\eta : M \to N$ in $\mathcal{D} \mathcal{M}_H$ is a monomorphism (resp. an epimorphism) if and only if it induces monomorphisms (resp. epimorphisms) $\eta(X) : M(X) \to N(X)$ of $H$-comodules for each $X \in \text{Ob}(D)$.

Proposition 5.5. Let $D$ be a right co-$H$-category. Then, a module $M \in \mathcal{D} \mathcal{M}_H$ is finitely generated as an object in $\mathcal{D} \text{-Mod}$ if and only if there exists a finite dimensional $H$-comodule $W$ and an epimorphism
\[
\left( \bigoplus_{i \in I} X_i \right) \otimes W \to M
\]
in $\mathcal{D} \mathcal{M}_H$, for finitely many objects $\{X_i\}_{i \in I}$ in $D$. 

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Proof. Let $M \in \mathcal{D}_H$ be finitely generated as a $\mathcal{D}$-module. Then, there exists a finite collection \{\(m_i \in \text{el}(M)\)\}_{i \in I} such that every $y \in \text{el}(M)$ has the form $y = \sum_{i \in I} M(f_i)(m_i)$ for some $f_i \in \text{Hom}_\mathcal{D}(\text{el}(M), y)$. Applying Lemma 5.3, we can obtain for each $m_i$ a finite dimensional $H$-subcomodule $W_{m_i} \leq M(m_i)$ containing $m_i$ and a morphism $\eta_{m_i} : \otimes_{\text{el}(M)} h \otimes W_{m_i} \rightarrow M$ in $\mathcal{D}_H$. Setting $W := \bigoplus_{i \in I} W_{m_i}$, we have an epimorphism in $\mathcal{D}_H$ determined by

$$\eta : \left( \bigoplus_{i \in I} \otimes_{\text{el}(M)} h \right) \otimes W \rightarrow M, \quad \eta(Y)(\{f_{ij} \otimes \{w_{ij}\}_{j \in J}) := \sum_{i \in I} M(f_i)(w_i)$$

for each $Y \in \text{Ob}(\mathcal{D})$, $f_i \in \text{el}(M)$ and $w_i \in W_{m_i}$.

Conversely, let $\{w_1, \ldots, w_k\}$ be a basis of a finite dimensional $H$-comodule $W$ and $\{X_{ij} \}_{i,j \in J}$ be finitely many objects in $\mathcal{D}$ such that we have an epimorphism

$$\eta : \left( \bigoplus_{i,j} X_{ij} \right) \otimes W \rightarrow M$$

in $\mathcal{D}_H$. From the discussion above, it follows that $\eta(Y) : \left( \bigoplus_{i \in I} X_{i} \right) \otimes W \rightarrow \text{Hom}(M, Y)$ is an epimorphism in $\text{Comod}_H$ for each $Y \in \text{Ob}(\mathcal{D})$. Then, the elements $\{m_{ij} := \eta(X_{ij} \otimes w_{ij})\}_{i,j \in J}$ form a family of generators for $M$ as a $\mathcal{D}$-comodule.

We will now show that $\mathcal{D}_H$ is a Grothendieck category. This essentially follows from the fact that both $\mathcal{D} \text{-Mod}$ and $\text{Comod}_H$ are Grothendieck categories. We refer the reader, for instance, to [7, Corollary 2.2.8], for a proof of $\text{Comod}_H$ being a Grothendieck category.

**Proposition 5.6.** Let $\mathcal{D}$ be a right co-$H$-category. Then, the category $\mathcal{D}_H$ of relative $(\mathcal{D}, H)$-Hopf modules is a Grothendieck category.

**Proof.** Since the categories $\mathcal{D} \text{-Mod}$ and $\text{Comod}_H$ have kernels, cokernels and coproducts (direct sums), so does the category $\mathcal{D}_H$. The remaining properties of an abelian category are inherited by $\mathcal{D}_H$ from $\mathcal{D} \text{-Mod}$. Hence, $\mathcal{D}_H$ is a cocomplete abelian category. Directs limits are exact in $\mathcal{D}_H$ which is also a property inherited from $\mathcal{D} \text{-Mod}$. We are now left to check that $\mathcal{D}_H$ has a family of generators. For any $M$ in $\mathcal{D}_H$, it follows from Lemma 5.3 that we can find an epimorphism

$$\bigoplus_{\text{el}(M)} \otimes_{\text{el}(M)} h \otimes W_m \rightarrow M$$

in $\mathcal{D}_H$. Thus, the collection $\{X \otimes h \otimes W \}$, where $X$ ranges over all objects in $\mathcal{D}$ and $W$ ranges over all (isomorphism classes of) finite dimensional $H$-comodules, forms a generating family for $\mathcal{D}_H$ (see, for instance, the proof of [9, Proposition 1.9.1]).

For $\mathcal{N} \in \mathcal{D}_H$, we consider the functor $\mathcal{N} \otimes (-) : \text{Comod}_H \rightarrow \mathcal{D}_H$ given by $M \mapsto \mathcal{N} \otimes M$. We see that $\text{Comod}_H$ is a Grothendieck category and the functor $\mathcal{N} \otimes (-)$ preserves colimits. Therefore, by a classical result [14, Proposition 8.3.27(iii)], it has a right adjoint which we denote by $\mathcal{R}_\mathcal{N} : \mathcal{D}_H \rightarrow \text{Comod}_H$. We then define

$$HOM_{\mathcal{D}_H}(\mathcal{N}, \mathcal{P}) := \mathcal{R}_\mathcal{N}(\mathcal{P})$$

for any $\mathcal{P} \in \mathcal{D}_H$. Thus, we have a natural isomorphism

$$Hom_{\mathcal{D}_H}(\mathcal{N} \otimes M, \mathcal{P}) \cong HOM_{\text{Comod}_H}(M, \text{HOM}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{P}))$$

for $\mathcal{N}, \mathcal{P} \in \mathcal{D}_H$ and $M \in \text{Comod}_H$. In particular, when $\mathcal{D}$ is a right co-$H$-category with a single object, i.e., a right $H$-comodule algebra, then $\mathcal{N}$ and $\mathcal{P}$ are relative Hopf-modules in the classical sense of Takeuchi [26]. Then, using [4, Lemma 2.3], the definition of $HOM$ as in (5.3) recovers the standard definition of rational morphisms between relative Hopf modules as in [4, § 2] or [28]. As such, we will refer to $HOM_{\mathcal{D}_H}(-, -)$ as the “rational Hom object” in $\mathcal{D}_H$. 21
Corollary 5.7. Let $N, P \in \mathcal{D}_H$. Then, $\text{HOM}_{\mathcal{D}, \text{Mod}}(N, P)_{coH} = \text{Hom}_{\mathcal{D}_H}(N, P)$.

Proof. The result follows by choosing $M = k$ in (5.4) and the fact that $\text{Hom}_{\text{Comod-H}}(k, N) = N_{coH}$ for any $N \in \text{Comod-H}$. □

Corollary 5.8. If $\mathcal{I}$ is an injective in $\mathcal{D}_H$, then $\text{HOM}_{\mathcal{D}, \text{Mod}}(N, \mathcal{I})$ is an injective in $\text{Comod-H}$ for any $N$ in $\mathcal{D}_H$.

Proof. The fact that $\text{HOM}_{\mathcal{D}, \text{Mod}}(N, -) : \mathcal{D}_H \rightarrow \text{Comod-H}$ preserves injectives follows from the fact that its left adjoint $N \otimes (-) : \text{Comod-H} \rightarrow \mathcal{D}_H$ is an exact functor. □

At the level of higher derived functors, the result of Corollary 5.7 leads to the following spectral sequence.

Theorem 5.9. Let $M \in \mathcal{D}_H$ be a relative $(\mathcal{D}, H)$-Hopf module. We consider the functors

$$\mathcal{F} = \text{HOM}_{\mathcal{D}, \text{Mod}}(M, -) : \mathcal{D}_H \rightarrow \text{Comod-H}$$

$$\mathcal{G} = (-)_{coH} : \text{Comod-H} \rightarrow \text{Vect}_K$$

Then, we have the following spectral sequence

$$R^q(\mathcal{F} \circ \mathcal{G})(N) = (R^q \text{Hom}_{\mathcal{D}, \text{Mod}}(M, -))(N)$$

Proof. We have $(\mathcal{G} \circ \mathcal{F})(N) = \text{HOM}_{\mathcal{D}, \text{Mod}}(M, N)_{coH} = \text{Hom}_{\mathcal{D}_H}(M, N)$ by Corollary 5.7. By Corollary 5.8, the functor $\mathcal{F}$ preserves injectives. Since $\text{Comod-H}$ has enough injectives, the result now follows from Grothendieck spectral sequence for composite functors (see [9]). □

Let $M, N$ be right $H$-comodules. Let $H^*$ be the linear dual of $H$. Then, the space $\text{Hom}_K(M, N)$ carries a left $H^*$-module structure given by

$$(h^* f)(m) = \sum h^*(S^{-1}(m_1)(f(m_0)))_1 (f(m_0)_0)$$

for any $h^* \in H^*$, $f \in \text{Hom}_K(M, N)$ and $m \in M$. We now show that this $H^*$-action can be extended to relative $(\mathcal{D}, H)$-Hopf modules.

Lemma 5.10. Let $M, N \in \mathcal{D}_H$. Then, $\text{Hom}_{\mathcal{D}, \text{Mod}}(M, N)$ is a left $H^*$-module.

Proof. For $h^* \in H^*$ and $\eta \in \text{Hom}_{\mathcal{D}, \text{Mod}}(M, N)$, we set

$$(h^* \eta)(X)(m) = \sum h^*(S^{-1}(m_1)(\eta(X)(m_0)))_1 (\eta(X)(m_0)_0).$$

(5.5)

for any $X \in \text{Ob}(\mathcal{D})$ and $m \in M(X)$. We first verify that $h^* \eta$ is indeed an element in $\text{Hom}_{\mathcal{D}, \text{Mod}}(M, N)$. For any $f \in \text{Hom}_{\mathcal{D}}(X, Y)$, we have

$$ (h^* \eta)(Y)(M)(f)(m) = \sum h^*(S^{-1}((M(f)(m_1)))(\eta(Y)((M(f)(m_0))))_1 (\eta(Y)((M(f)(m_0)_0))_0$$

$$= \sum h^*(S^{-1}(f_2 m_1)(\eta(Y)((M(f)(m_0))))_1 (\eta(Y)((M(f)(m_0)_0))_0$$

(5.1)

$$= \sum h^*(S^{-1}(m_1)S^{-1}(f_1)(N(f_0)((\eta(X)(m_0))))_1 (N(f_0)((\eta(X)(m_0))))_0$$

(5.1)

$$= \sum h^*(S^{-1}(m_1)S^{-1}(f_1)(N(f_0)((\eta(X)(m_0))))_1 (N(f_0)((\eta(X)(m_0))))_0$$

(5.1)

$$= (h^* \eta)(X)(N(f)(m))$$

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Next, we verify that \((h^*g^*)\eta = h^*(g^*\eta)\) and that \(1_{H^*}\eta = \eta\), i.e., \(\varepsilon\eta = \eta\) for all \(h^*, g^* \in H^*\) and \(\eta \in \text{Hom}_{D\text{-Mod}}(M, N)\).

The latter equality follows easily and further we see that

\[
(h^*(g^*\eta))(X)(m) = \sum h^*(S^{-1}(m_1)(g^*\eta)(X)(m_0))_0 \left((g^*\eta)(X)(m_0)_0\right)
\]

\[
= \sum h^*(S^{-1}(m_1)(g^*(S^{-1}(m_0_1)\eta(X)((m_0)_0))_1)(\eta(X)((m_0)_0)_0)_0)_0
\]

\[
= \sum h^*(S^{-1}(m_2)(\eta(X)(m_0))(g^*(S^{-1}(m_1)(\eta(X)(m_0))_2)(\eta(X)(m_0))_0)
\]

\[
= \sum(h^*g^*)(S^{-1}(m_1)(\eta(X)(m_0))_1)(\eta(X)(m_0))_0
\]

\[
= ((h^*g^*)\eta)(X)(m)
\]

for all \(X \in \text{Ob}(D)\) and \(m \in M(X)\).

\[\square\]

**Lemma 5.11.** Let \(M, N \in \mathcal{D}_H^*\) and let \(\eta \in \text{Hom}_{D\text{-Mod}}(M, N)\). Then, there is a morphism \(\rho(\eta) \in \text{Hom}_{D\text{-Mod}}(M, N \otimes H)\) determined by setting

\[
\rho(\eta)(X)(m) := \sum (\eta(X)(m_0))_0 \otimes S^{-1}(m_1)(\eta(X)(m_0))_1
\]

(5.6)

for any \(X \in \text{Ob}(D)\) and \(m \in M(X)\).

**Proof.** Using (5.1) and the fact that \(\eta \in \text{Hom}_{D\text{-Mod}}(M, N)\), we have

\[
\rho(\eta)(Y)(M(f)(m)) = \sum (\eta(Y)(M(f_0)(m_0)))_0 \otimes S^{-1}(f_1m_1)(\eta(Y)(M(f_0))(m_0))_1
\]

\[
= \sum (\eta(f_0)(\eta(X)(m_0)))_0 \otimes S^{-1}(f_1m_1)(\eta(f_0)(\eta(X))(m_0))_1
\]

\[
= \sum (\eta(f_0)(\eta(X)(m_0)))_0 \otimes S^{-1}(m_1)(\eta(X)(m_0))_1
\]

\[
= \sum (\eta(f)(\eta(X)(m_0)))_0 \otimes S^{-1}(m_1)(\eta(X)(m_0))_1
\]

\[
= \rho(\eta)(Y)(X)
\]

for any \(f \in \text{Hom}_D(X, Y)\) and \(m \in M(X)\).

\[\square\]

We now recall the notion of a rational left \(H^*\)-module (see, for instance, [7]) which will be used in the next result.

Given a left \(H^*\)-module \(M\), there is a morphism \(\rho_M : M \rightarrow \text{Hom}_K(H^*, M)\) corresponding to the canonical morphism \(H^* \otimes M \rightarrow M\). There is an obvious inclusion \(M \otimes H \rightharpoonup \text{Hom}_K(H^*, M)\) given by \((m \otimes h)(h^*) = h^*(h)m\) for any \(m \in M\), \(h \in H\) and \(h^* \in H^*\).

**Definition 5.12.** (see [7, Definition 2.2.2]) A left \(H^*\)-module \(M\) is said to be rational if \(\rho_M(M) \subseteq M \otimes H\), where \(M \otimes H\) is viewed as a subspace of \(\text{Hom}_K(H^*, M)\). The full subcategory of rational \(H^*\)-modules will be denoted by \(\text{Rat}(H^*\text{-Mod})\).

If \(M\) is a right \(H\)-comodule with \(H\)-coaction \(m \mapsto \sum m_0 \otimes m_1\), then \(M\) becomes a left \(H^*\)-module via the action \(h^*m := \sum h^*(m_1)m_0\) for \(h^* \in H^*\) and \(m \in M\). This determines a functor

\[
\text{Comod}-H \rightarrow H^*\text{-Mod}
\]

It is well known (see [7, Theorem 2.2.5]) that this functor defines an equivalence of categories between \(\text{Comod}-H\) and the subcategory \(\text{Rat}(H^*\text{-Mod})\) of \(H^*\text{-Mod}\).

**Proposition 5.13.** Let \(M, N \in \mathcal{D}_H^*\) and suppose that \(M\) is finitely generated as an object in \(\mathcal{D}\text{-Mod}\). Then, \(\text{Hom}_{\mathcal{D}\text{-Mod}}(M, N)\) is a right \(H\)-comodule. In particular, \(\text{HOM}_{\mathcal{D}\text{-Mod}}(M, N) = \text{Hom}_{\mathcal{D}\text{-Mod}}(M, N)\).
Proof. Since $\mathcal{M}$ is finitely generated in $D\text{-}Mod$, by Proposition 5.5, there exists a finite dimensional $H$-comodule $W$ and an epimorphism

$$\eta : \bigoplus_{i \in I} X_i \otimes h \rightarrow W$$

in $D\text{-}H$, for finitely many objects $\{X_i\}_{i \in I}$ in $D$. From the description of epimorphisms in $D\text{-}H$ in (5.2), we know that $\eta$ is also an epimorphism in $D\text{-}Mod$. The map

$$Hom(\eta, N) : Hom_{D\text{-}Mod}(\mathcal{M}, N) \rightarrow \bigoplus_{i \in I} Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N)$$

is therefore a monomorphism for each $N \in D\text{-}H$. Using the fact that $\eta(Y)$ is $H$-colinear for each $Y \in Ob(D)$, we will now verify that the morphism $Hom(\eta, N)$ is $H^*$-linear. For any $h^* \in H^*$, $\xi \in Hom_{D\text{-}Mod}(\mathcal{M}, N)$, $Y \in Ob(D)$ and $\tilde{f} \otimes w \in (\bigoplus_{i \in I} X_i \otimes h(Y)) \otimes W$, we have

$$\begin{align*}
( Hom(\eta, N)(h^* \xi)) (Y)(\tilde{f} \otimes w) &= (h^* \xi) \otimes \eta(Y)(\tilde{f} \otimes w) \\
&= \sum h^* (S^{-1}(\eta(Y)(\tilde{f} \otimes w))) (\xi(Y)(\eta(Y)(\tilde{f} \otimes w)))_* **(\xi(Y)(\eta(Y)(\tilde{f} \otimes w)))_0 \\
&= \sum h^* (S^{-1}(\eta(Y)(\tilde{f} \otimes w))) (\xi(Y)(\eta(Y)(\tilde{f} \otimes w)))_* **(\xi(Y)(\eta(Y)(\tilde{f} \otimes w)))_0 \\
&= \sum h^* (S^{-1}(\eta(Y)(\tilde{f} \otimes w))) (\xi(Y)(\eta(Y)(\tilde{f} \otimes w)))_* **(\xi(Y)(\eta(Y)(\tilde{f} \otimes w)))_0 \\
&= (h^*(\xi \otimes \eta))(Y)(\tilde{f} \otimes w) \\
&= (h^* Hom(\eta, N)(\xi))(Y)(\tilde{f} \otimes w)
\end{align*}$$

This shows that $Hom_{D\text{-}Mod}(\mathcal{M}, N)$ is an $H^*$-submodule of $Hom_{D\text{-}Mod}(\bigoplus_{i \in I} X_i \otimes h \otimes W, N)$.

For each $i \in I$, we now prove that $\rho : Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N) \rightarrow Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N \otimes H)$, as defined in (5.6), gives an $H$-comodule structure on $Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N)$. Since $W$ is finite dimensional, we have

$$\begin{align*}
Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N \otimes H) &\cong Hom_K(W, Hom_{D\text{-}Mod}(X_i \otimes h, N \otimes H)) \\
&\cong Hom_K(W, N(X_i) \otimes H) \cong Hom_K(W, N(X_i)) \otimes H \\
&\cong Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N) \otimes H
\end{align*}$$

This gives a well defined morphism

$$\rho : Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N) \rightarrow Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N \otimes H) \cong Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N) \otimes H \quad (5.7)$$

We will verify that (5.7) gives a right $H$-coaction. For this, we do not need to show that for any $\xi \in Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N)$, we have $(\rho \otimes id)\rho(\xi) = (id \otimes \Delta)\rho(\xi)$ and $(id \otimes \rho)\rho(\xi) = \xi$. The latter equality is easy to verify. By (5.7), we know that $\rho(\xi) = \sum \xi_0 \otimes \xi_1 \in Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N) \otimes H$. Thus, for any $X \in Ob(D)$ and $u \in X \otimes h(X) \otimes W$, we have

$$\begin{align*}
((\rho \otimes id)\rho(\xi))(X)(u) &= \sum \rho(\xi_0)(X)(u) \otimes \xi_1 \\
&= \sum (\xi_0(X)(u))_0 \otimes S^{-1}(u_1)(\xi_0(X)(u))_1 \otimes \xi_1 \\
&= \sum (\xi_0(X)(u))_0 \otimes S^{-1}(u_1)(\xi_0(X)(u))_1 \otimes \sum \xi_0(X)(u) \otimes \xi_1 \otimes \xi_2.
\end{align*}$$

The third equality above follows by applying $\rho(N(X)) \otimes id_Y$ on the equality $\sum \xi_0(X)(u_0) \otimes \xi_1 = \rho(\xi)(X)(u_0)$ and the last one is obtained by applying $id_H \otimes \Delta$ on $\sum \xi_0(X)(u) \otimes \xi_1 = \sum (\xi_0(X)(u_0))_0 \otimes S^{-1}(u_1)(\xi_0(X)(u_0))_1$. Thus, we have shown that $Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N)$ is a right $H$-comodule.

Moreover, the $H^*$-action on $Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N)$ as in (5.5) is given precisely by the $H$-coaction as in (5.6). Therefore, $Hom_{D\text{-}Mod}(X_i \otimes h \otimes W, N)$ is a rational $H^*$-module. Since the category of rational $H^*$-modules

$$24$$
A morphism $N \to N'$ in $\mathcal{M}^H$ induces a morphism of functors $N \otimes (-) \to N' \otimes (-)$ and hence a morphism $\mathcal{R}_{N'} \to \mathcal{R}_N$ of their respective right adjoints. Thus, for any $\mathcal{L} \in \mathcal{M}^H$, we have a functor $\text{HOM}_{\mathcal{M},\mathcal{D}}(-, \mathcal{L}) : (\mathcal{M}^H)^{op} \to \text{Comod-H}$ which takes $N$ to $\text{HOM}_{\mathcal{M},\mathcal{D}}(N, \mathcal{L}) = \mathcal{R}_N(\mathcal{L})$.

**Proposition 5.14.** (1) For any $\mathcal{L} \in \mathcal{M}^H$, the functor $\text{HOM}_{\mathcal{M},\mathcal{D}}(-, \mathcal{L}) : (\mathcal{M}^H)^{op} \to \text{Comod-H}$ is left exact, i.e., it preserves kernels.

(2) If $\mathcal{I}$ is injective in $\mathcal{M}^H$, then $\text{HOM}_{\mathcal{M},\mathcal{D}}(-, \mathcal{I})$ is exact.

(3) If $\mathcal{I}$ is injective in $\mathcal{M}^H$, then $\text{HOM}_{\mathcal{M},\mathcal{D}}(-, \mathcal{I})$ takes every short exact sequence in $\mathcal{M}^H$ to a split short exact sequence in $\text{Comod-H}$.

**Proof.** (1) Let $\mathcal{L} : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathcal{M}^H$ and let $\mathcal{P} := \text{Coker}(\eta)$. Then, for any $T \in \text{Comod-H}$, $\text{Coker}(\eta \otimes \text{id}_T : \mathcal{M} \otimes T \to \mathcal{N} \otimes T) = \mathcal{P} \otimes T$. From the adjunction in (5.4), we now have

$$
\text{Hom}_\mathcal{M}(T, \text{HOM}_{\mathcal{M},\mathcal{D}}(\mathcal{P}, \mathcal{L})) \cong \text{Hom}_\mathcal{M}(\mathcal{P} \otimes T, \mathcal{L})
$$

$$
\cong \text{Ker}(\text{HOM}_{\mathcal{M},\mathcal{D}}(N \otimes T, \mathcal{L}) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(M \otimes T, \mathcal{L}))
$$

$$
\cong \text{Hom}_\mathcal{M}(\text{HOM}_{\mathcal{M},\mathcal{D}}(N, \mathcal{L}) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(M, \mathcal{L}))
$$

for any $T \in \text{Comod-H}$. From Yoneda Lemma, it follows that

$$
\text{HOM}_{\mathcal{M},\mathcal{D}}(\mathcal{P}, \mathcal{L}) = \text{Ker}(\text{HOM}_{\mathcal{M},\mathcal{D}}(N, \mathcal{L}) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(M, \mathcal{L}))
$$

(2) Let $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ be a short exact sequence in $\mathcal{M}^H$. From (1), we already know that

$$
0 \to \text{HOM}_{\mathcal{M},\mathcal{D}}(\mathcal{P}, T) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(N, T) \xrightarrow{q} \text{HOM}_{\mathcal{M},\mathcal{D}}(M, T)
$$

(5.8)

is exact. We need to show that $q$ is an epimorphism. For any $T \in \text{Comod-H}$, we notice that $0 \to \mathcal{M} \otimes T \to \mathcal{N} \otimes T \to \mathcal{P} \otimes T \to 0$ is still a short exact sequence in $\mathcal{M}^H$. If $\mathcal{I}$ is an injective object in $\mathcal{M}^H$, we see that

$$
0 \to \text{HOM}_{\mathcal{M},\mathcal{D}}(\mathcal{P} \otimes T, \mathcal{I}) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(N \otimes T, \mathcal{I}) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(M \otimes T, \mathcal{I}) \to 0
$$

is an exact sequence of $K$-vector spaces. Using the adjunction in (5.4), it follows that

$$
0 \to \text{Hom}_\mathcal{M}(T, \text{HOM}_{\mathcal{M},\mathcal{D}}(\mathcal{P}, \mathcal{I})) \to \text{Hom}_\mathcal{M}(T, \text{HOM}_{\mathcal{M},\mathcal{D}}(N, \mathcal{I})) \xrightarrow{\text{Hom}(T, q)} \text{Hom}_\mathcal{M}(T, \text{HOM}_{\mathcal{M},\mathcal{D}}(M, \mathcal{I})) \to 0
$$

(5.9)

is short exact in $\text{Vect}_K$. By setting $T = \text{HOM}_{\mathcal{M},\mathcal{D}}(M, \mathcal{I})$ in (5.9), we see that there exists a morphism $f : \text{HOM}_{\mathcal{M},\mathcal{D}}(M, T) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(N, T)$ of $H$-comodules such that $q \circ f$ is the identity on $\text{HOM}_{\mathcal{M},\mathcal{D}}(M, T)$. This shows that $q : \text{HOM}_{\mathcal{M},\mathcal{D}}(N, T) \to \text{HOM}_{\mathcal{M},\mathcal{D}}(M, T)$ is an epimorphism. The result of (3) is clear from the proof of (2).
Proposition 5.15. Let $D$ be a left noetherian right co-$H$-category and let $M \in \mathcal{D}^H$ be finitely generated as an object in $\mathcal{D}$-Mod. If $I$ is an injective object in $\mathcal{D}^H$, then $\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, I) = 0$ for all $p > 0$.

Proof. Since $M \in \mathcal{D}^H$ is finitely generated in $\mathcal{D}$-Mod, by Proposition 5.5, there exists a finite dimensional $H$-comodule $W_0$ and an epimorphism

$$\eta_0 : P_0 := \left( \bigoplus_{i=1}^{n_0} \chi_i \right) \otimes W_0 \twoheadrightarrow M$$

in $\mathcal{D}^H$ for finitely many objects $\{X_i\}_{i \in \mathcal{I}}$ in $\mathcal{D}$. Then, $\mathcal{K} := \text{Ker}(\eta_0)$ is a subobject of $P_0$ in $\mathcal{D}^H$. Since $D$ is left noetherian, $P_0$ is a noetherian left $D$-module (see, for instance, [20, § 3]). Thus, the submodule $\mathcal{K} := \text{Ker}(\eta_0)$ of $P_0$ is finitely generated as an object in $\mathcal{D}$-Mod. Therefore, we obtain a finite dimensional $H$-comodule $W_1$ and an epimorphism

$$\eta_1 : P_1 := \left( \bigoplus_{j=1}^{n_1} \chi_j \right) \otimes W_1 \twoheadrightarrow \mathcal{K}$$

in $\mathcal{D}^H$ for finitely many objects $\{Y_j\}_{j \in \mathcal{I}}$ in $\mathcal{D}$. Since $W_0$ and $W_1$ are finite dimensional $K$-vector spaces, clearly $P_0$ and $P_1$ are free left $D$-modules. Moreover, $\text{Im}(\eta_1) = \mathcal{K} = \text{Ker}(\eta_0)$. Continuing in this way, we can construct a free resolution of the module $M$ in the category $\mathcal{D}$-Mod:

$$P_* = \cdots \rightarrow P_1 \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

This gives us

$$\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, I) = H^p(\text{Hom}_{\mathcal{D},\text{Mod}}(P_*, I)), \quad \forall p > 0$$

Since $M$ and $\{P_i\}_{i \geq 0}$ are finitely generated in $\mathcal{D}$-Mod, it follows from Proposition 5.13 that $\text{HOM}_{\mathcal{D},\text{Mod}}(M, I) = \text{HOM}_{\mathcal{D},\text{Mod}}(M, I)$ and $\text{HOM}_{\mathcal{D},\text{Mod}}(P_1, I) = \text{HOM}_{\mathcal{D},\text{Mod}}(P_1, I)$. From Proposition 5.14, we know that the functor $\text{HOM}_{\mathcal{D},\text{Mod}}(-, I)$ is exact and it follows that $\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, I) = H^p(\text{HOM}_{\mathcal{D},\text{Mod}}(P_*, I)) = 0$ for all $p > 0$.

Proposition 5.16. Let $D$ be a left noetherian right co-$H$-category. Let $M, N \in \mathcal{D}^H$ with $M$ finitely generated as an object in $\mathcal{D}$-Mod. If $E^*$ is an injective resolution of $N$ in $\mathcal{D}^H$, then

$$\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, N) = R^pH^0(\text{Hom}_{\mathcal{D},\text{Mod}}(P_*, N)) = H^p(\text{Hom}_{\mathcal{D},\text{Mod}}(P_*, N)) \quad \forall p \geq 0.$$

Proof. Let $P_*$ be the free resolution of $M$ in $\mathcal{D}$-Mod constructed as in the proof of Proposition 5.15. Then, we have

$$\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, N) = H^p(\text{Hom}_{\mathcal{D},\text{Mod}}(P_*, N)) = H^p(\text{HOM}_{\mathcal{D},\text{Mod}}(P_*, N))$$

where the second equality follows from Proposition 5.13. Since $\text{HOM}_{\mathcal{D},\text{Mod}}(P_*, N)$ is a complex in $\text{Comod-H}$ and $\text{Comod-H}$ is an abelian category, it follows that $H^p(\text{HOM}_{\mathcal{D},\text{Mod}}(P_*, N)) \in \text{Comod-H}$. Hence, we may consider the family $\{\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, N)\}_{p \geq 0}$ as a $\delta$-functor from $\mathcal{D}^H$ to $\text{Comod-H}$.

By Proposition 5.15, $\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, I) = 0$ for every injective object $I \in \mathcal{D}^I$. Since $\mathcal{D}^H$ has enough injectives, it follows that each $\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, -) : \mathcal{D}^H \rightarrow \text{Comod-H}$ is effaceable (see, for instance, [12, § III.1.1]).

Since $\mathcal{D}^H$ has enough injectives, we can consider the right derived functors

$$R^p\text{HOM}_{\mathcal{D},\text{Mod}}(M, -) : \mathcal{D}^H \rightarrow \text{Comod-H} \quad p \geq 0$$

For $p = 0$, we notice that $\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, -) = \text{HOM}_{\mathcal{D},\text{Mod}}(M, -) = \text{HOM}_{\mathcal{D},\text{Mod}}(M, -) = R^p\text{HOM}_{\mathcal{D},\text{Mod}}(M, -)$ as functors from $\mathcal{D}^H$ to $\text{Comod-H}$. Since each $\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, -)$ is effaceable for $p > 0$, we see that the family $\{\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, -)\}_{p \geq 0}$ forms a universal $\delta$-functor and it follows from [12, Corollary III.1.1.4] that

$$\text{Ext}^p_{\mathcal{D},\text{Mod}}(M, -) = R^p\text{HOM}_{\mathcal{D},\text{Mod}}(M, -) : \mathcal{D}^H \rightarrow \text{Comod-H}$$
for every $p \geq 0$. Therefore, we have
\[ \text{Ext}^p_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{N}) = (\text{Ext}^p_{\mathcal{D}-\text{Mod}}(\mathcal{M},-)\mathcal{N}) = H^p(\text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{E}^*)) \]

Recall that by Proposition 5.13, for any $\mathcal{M} \in \mathcal{D}^{H}$ with $\mathcal{M}$ finitely generated as an object in $\mathcal{D}-\text{Mod}$, we have $\text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{N}) = \text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{N}) \in \text{Comod-}H$.

**Theorem 5.17.** Let $\mathcal{D}$ be a left noetherian right co-$H$-category. Let $\mathcal{M} \in \mathcal{D}^{H}$ with $\mathcal{M}$ finitely generated as an object in $\mathcal{D}-\text{Mod}$. We consider the functors
\[ \mathcal{F} = \text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},-) : \mathcal{D}^{H} \to \text{Comod-}H \]
\[ \mathcal{G} = (-)^{\text{co}H} : \text{Comod-}H \to \text{Vec}_{K} \]

Then, we have the following spectral sequence
\[ R^q(\mathcal{G} \circ \mathcal{F})(\mathcal{N}) = (\text{Ext}^p_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{N})) = H^q(\text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{E}^*)) \]

Proof. By Corollary 5.7, we have
\[ R^q(\mathcal{G} \circ \mathcal{F})(\mathcal{N}) = (\text{Ext}^p_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{N})) = H^q(\text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{E}^*)) \]

By definition,
\[ R^q(\mathcal{F})(\mathcal{N}) = H^q(\mathcal{F}(\mathcal{E}^*)) = H^q(\text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{E}^*)) \]

where $\mathcal{E}^*$ is an injective resolution of $\mathcal{N}$ in $\mathcal{D}^{H}$. Applying Corollary 5.16, we obtain $\text{Ext}^p_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{N}) = R^q(\mathcal{G} \circ \mathcal{F})(\mathcal{N})$. For any injective object $\mathcal{I}$ in $\mathcal{D}^{H}$, we know that $\mathcal{F}(\mathcal{I}) = \text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{I}) = \text{Hom}_{\mathcal{D}-\text{Mod}}(\mathcal{M},\mathcal{I})$ is injective in $\text{Comod-}H$ by Corollary 5.8. Since $\text{Comod-}H$ is a Grothendieck category, it has enough injectives. The result now follows from Grothendieck spectral sequence for composite functors (see [9]).

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