Subsolution theorem and the Dirichlet problem for the quaternionic Monge-Ampère equation✩

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Abstract

In this paper, the author studies quaternionic Monge-Ampère equations and obtain the existence of the solutions to the Dirichlet problem for such equations in strictly pseudoconvex domains in quaternionic space. The stability and subsolution theorems are established for quaternionic Monge-Ampère equations.

Keywords: Monge-Ampère operator; plurisubharmonic function; subsolution

1. Introduction

In this paper, we are interested in solving, under possibly weak assumptions on the measure $d\mu$, the following Dirichlet problem for the quaternionic Monge-Ampère equation in a given strictly pseudoconvex domain $\Omega$ in $n$-dimensional quaternionic space $\mathbb{H}^n$:

$$\begin{cases}
    u \in PSH \cap L^\infty(\Omega) \\
    (\triangle u)^n = d\mu \\
    \lim_{\zeta \to q} u(\zeta) = \varphi(q) \quad q \in \partial \Omega, \quad \varphi \in C(\partial \Omega),
\end{cases}$$

(1.1)

where $(\triangle u)^n$ denotes the quaternionic Monge-Ampère measure of $u$. We have shown in [33] that quaternionic Monge-Ampère operator $(\triangle u)^n$ is well defined as a positive measure for locally bounded quaternionic plurisubharmonic (PSH, for short) function $u$.

The quaternionic Monge-Ampère equation, relating to the quaternionic version of Calabi-Yau conjecture, has attracted many analysts to study on it. Quaternionic analysis has important applications in the supersymmetric theory in physics. It is interesting to study the quaternionic Monge-Ampère operator

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over quaternionic manifolds, in particular to study the Dirichlet problem for the quaternionic Monge-Ampère equation \[\text{[3–6, 28, 36–38]}\].

The Dirichlet problem (1.1) in smooth case, for a smooth positive measure on the right hand side, i.e. \(d\mu = gdV\) and \(g > 0\) smooth (denote by \(dV\) the Lebesgue measure), was first considered by Alesker \[2\] on a Euclidean ball \(B \subset H^n\), then was recently solved by Zhu \[38\] on general domains in \(H^n\). For continuous data, Alesker \[2\] showed that the unique continuous solution exists for \(d\mu = gdV\), \(0 \leq g \in C(\Omega)\), and for strictly pseudoconvex domain \(\Omega \subset H^n\). The existence of solutions to the Dirichlet problem (1.1) for the quaternionic Monge-Ampère equation when the right hand side \(\mu\) is some more general positive Borel measure is still an open problem.

The existence theorems for the complex Monge-Ampère equation have been undergoing intensive research in the past few decades. In mid-seventies Bedford and Taylor \[9, 10\] solved the following Dirichlet problem for complex Monge-Ampère equation for \(d\mu = gdV, g \in C(\Omega), \) in a strictly pseudoconvex domain in \(C^n\):

\[
\begin{align*}
\{(dd^cu)^n & = d\mu \\
\lim_{\zeta \to z} u(\zeta) & = \varphi(z) \quad z \in \partial \Omega, \quad \varphi \in C(\partial \Omega).
\end{align*}
\tag{1.2}
\]

Cegrell \[13\] generalized this result to the case of bounded \(g\). Cegrell and Persson \[14\] also showed that continuous solutions exist if \(d\mu = gdV, g \in L^2(\Omega, dV)\), but for \(g \in L^1(\Omega, dV)\) this is not necessarily true \[15\]. Kołodziej \[22, 23\] proved that the above Dirichlet problem still admits a unique weak continuous solution when the right hand side \(\mu\) is a measure satisfying some sufficient condition. Since this sufficient condition is not easy to verify, Kołodziej obtained the subsolution theorem, saying that the Dirichlet problem (1.2) in a strictly pseudoconvex domain is solvable if there is a subsolution (cf. \[20, 21\]). We refer to \[25\] for a nice survey on this.

Inspired by the idea of Cegrell \[14\] we use the connection between real and quaternionic Monge-Ampère measure, which we established in \[32\], to obtain that there exists a unique solution to the Dirichlet problem (1.1) if \(d\mu = gdV, g \in L^4(\Omega, dV)\). Let \(|\cdot|_\Omega\) and \(|\cdot|_{\partial \Omega}\) denote the sup-norm on \(\Omega\) and \(\partial \Omega\).

**Theorem 1.1.** Let \(\Omega\) be a strictly pseudoconvex domain in \(H^n\). If \(0 \leq g \in L^4(\Omega), \varphi \in C(\partial \Omega)\), then the Dirichlet problem (1.1) with \(d\mu = gdV\) has a unique solution. The solution, denoted by \(U_{Q}(\varphi, g)\), is in \(C(\overline{\Omega})\) and satisfies

\[
\inf_{\partial \Omega} \varphi - C\|g\|_{L^4(\Omega)}^{\frac{1}{2}} \leq U_{Q}(\varphi, g) \leq \sup_{\partial \Omega} \varphi
\]

for some constant \(C\) depending only on \(\Omega\). And

\[
|U_{Q}(\varphi_1, g_1) - U_{Q}(\varphi_2, g_2)|_\Omega \leq |\varphi_1 - \varphi_2|_{\partial \Omega} + C\|g_1 - g_2\|_{L^4(\Omega)}^{\frac{1}{2}}
\]

for \(\varphi_1, \varphi_2 \in C(\partial \Omega)\) and \(0 \leq g_1, g_2 \in L^4(\Omega)\).
As a direct consequence of Theorem 1.1, we obtain the $L^\infty - L^4$--stability of the quaternionic Monge-Ampère equation. By Hölder’s inequality we also get the $L^p - L^q$--stability of the quaternionic Monge-Ampère equation for 

$$(p, q) \in ([1, n] \times [1, \infty]) \cup ([1, \infty] \times [4, \infty]).$$

In this paper we show that the subsolution theorem for quaternionic Monge-Ampère equation is still true by combining the above stability theorem and the well known method used by Kołodziej (cf. [20, 21, 25]).

**Theorem 1.2.** Let $\Omega$ be a strictly pseudoconvex domain in $\mathbb{H}^n$. Let $\mu$ be a finite positive Borel measure in $\Omega$ and $\varphi \in C(\partial \Omega)$. If there exists a subsolution $v$, i.e.

$$
\begin{align*}
&\begin{cases} 
v \in \text{PSH} \cap L^\infty(\Omega) \\
(\Delta v)^n \geq d\mu \\
\lim_{\zeta \to q} v(\zeta) = \varphi(q) \text{ for any } q \in \partial \Omega,
\end{cases}
\end{align*}
$$

then there exists a solution $u$ of the Dirichlet problem (1.1).

Historically, the quaternionic Monge-Ampère operator was defined by Alesker [1, 2] as the Moore determinant of the quaternionic Hessian of $u$:

$$
det(u) = det \left[ \frac{\partial^2 u}{\partial q_j \partial q_k}(q) \right]. \quad (1.3)
$$

Compared to the complex pluripotential theory, the main difficulties in the quaternionic pluripotential theory come from the noncommutativity of elements of the quaternionic Hessian and the complexity of the noncommutative Moore determinant. Alesker [3, 5] observed the relationship between the Baston operator and the quaternionic Monge-Ampère operator. The Baston operator, which is the first operator of 0-Cauchy-Fueter complex, is known explicitly [7, 35, 37]. Based on this observation, the author and Wang [33] introduced the first order differential operators $d_0$ and $d_1$ and wrote the quaternionic Monge-Ampère operator as the $n$-th exterior power of the Baston operator $\Delta = d_0d_1$. Then several results in the complex pluripotential theory were established for the quaternionic Monge-Ampère operator [30, 32, 34]. The theory of quaternionic closed positive currents established by the author and Wang [33], is an essential reason why we can still obtain the subsolution theorem for the quaternionic Monge-Ampère equation as Kołodziej did for the complex case.

2. Preliminaries

In this section, we are going to recall some basic definitions and conclusions of quaternionic Monge-Ampère operator and quaternionic closed positive currents following [33, 34].

A real valued function $f : \mathbb{H}^n \to \mathbb{R}$ is called *quaternionic plurisubharmonic* (PSH, for short) if it is upper semi-continuous and its restriction to any right
quaternionic line is subharmonic (in the usual sense). Any quaternionic PSH function is subharmonic (cf. [3] for more information about PSH functions). Denote by $PSH(\Omega)$ the class of quaternionic plurisubharmonic functions on $\Omega$.

**Proposition 2.1.** Let $\Omega$ be an open subset of $\mathbb{H}^n$.

(1). The family $PSH(\Omega)$ is a convex cone, i.e. if $\alpha, \beta$ are non-negative numbers and $u, v \in PSH(\Omega)$, then $\alpha u + \beta v \in PSH(\Omega)$; and $\max\{u, v\} \in PSH(\Omega)$.

(2). If $\Omega$ is connected and $\{u_j\} \subset PSH(\Omega)$ is a decreasing sequence, then $u = \lim_{j \to \infty} u_j \in PSH(\Omega)$ or $u \equiv -\infty$.

(3). Let $\{u_\alpha\}_{\alpha \in A} \subset PSH(\Omega)$ be such that its upper envelope $u = \sup_{\alpha \in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization $u^* \in PSH(\Omega)$.

(4). Let $\omega$ be a non-empty proper open subset of $\Omega$, $u \in PSH(\Omega), v \in PSH(\omega)$, and $\limsup_{q \to \omega} v(q) \leq u(\zeta)$ for each $\zeta \in \partial \omega \cap \Omega$, then

$$w := \begin{cases} \max\{u, v\}, & \text{in } \omega \\ u, & \text{in } \Omega \setminus \omega \end{cases} \in PSH(\Omega).$$

(5). If $u \in PSH(\Omega)$, then the standard regularization $u_\varepsilon := u * \rho_\varepsilon$ is also PSH in $\Omega_\varepsilon := \{q \in \Omega : \text{dist}(q, \partial \Omega) > \varepsilon\}$, moreover, $u_\varepsilon \searrow u$ as $\varepsilon \to 0$.

(6). If $\gamma(t)$ is a convex increasing function of a parameter $t \in \mathbb{R}$ and $u \in PSH$, then $\gamma \circ u \in PSH$.

We use the well known embedding of the quaternionic algebra $\mathbb{H}$ into $\text{End}(\mathbb{C}^2)$ defined by

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto \begin{pmatrix} x_0 + ix_1 & -x_2 - ix_3 \\ x_2 - ix_3 & x_0 - ix_1 \end{pmatrix}.$$ 

Actually we will use the conjugate embedding

$$\tau : \mathbb{H}^n \cong \mathbb{R}^{4n} \hookrightarrow \mathbb{C}^{2n \times 2},

(q_0, \ldots, q_{n-1}) \mapsto z = (z^j) \in \mathbb{C}^{2n \times 2},$$

for $q_l = x_{4l} + ix_{4l+1} + jx_{4l+2} + kx_{4l+3}$, $l = 0, 1, \ldots, n - 1$. Pulling back to the quaternionic space $\mathbb{H}^n \cong \mathbb{R}^{4n}$ by the embedding above, we define on $\mathbb{R}^{4n}$ first-order differential operators $\nabla_{j\alpha}$ as following:

$$\begin{pmatrix} \nabla_{(2)(0)} & \nabla_{(2)(1)} \\ \nabla_{(2)(0)} & \nabla_{(2)(1)} \end{pmatrix} := \begin{pmatrix} \partial_{x_{4l}} + i \partial_{x_{4l+1}} & -\partial_{x_{4l+2}} - i \partial_{x_{4l+3}} \\ \partial_{x_{4l+2}} - i \partial_{x_{4l+3}} & \partial_{x_{4l}} - i \partial_{x_{4l+1}} \end{pmatrix}. \quad (2.1)$$

$z^{k\beta}$'s can be viewed as independent variables and $\nabla_{j\alpha}$'s are derivatives with respect to these variables. The operators $\nabla_{j\alpha}$’s play very important roles in the investigating of regular functions in several quaternionic variables [19, 33].
Lemma 2.1. Let $∧ C^{2k}$ be the complex exterior algebra generated by $C^{2n}$, $0 \leq k \leq n$. Fix a basis $\{ω^0, ω^1, \ldots, ω^{2n-1}\}$ of $C^{2n}$. Let $Ω$ be a domain in $\mathbb{R}^{4n}$. Define $d_0, d_1 : C^0_0 (Ω, ∧^p C^{2n}) → C^0_0 (Ω, ∧^{p+1} C^{2n})$ by

$$
d_0 F = \sum_{k,l} \nabla_{k0} f_I \ ω^k \wedge ω^l, \quad d_1 F = \sum_{k,l} \nabla_{k1} f_I \ ω^k \wedge ω^l, \quad (2.2)
$$

for $F = \sum_I f_I ω^I \in C^∞_0 (Ω, ∧^p C^{2n})$, where the multi-index $I = (i_1, \ldots, i_p)$ and $ω^I := ω^{i_1} \wedge \ldots \wedge ω^{i_p}$. Although $d_0, d_1$ are not exterior differential, their behavior is similar to the exterior differential.

Lemma 2.1. (1) $d_0 d_1 = −d_1 d_0$. (2) $d_0^2 = d_1^2 = 0$, thus $d_0 ∆ = d_1 ∆ = 0$. (3) For $F \in C^∞_0 (Ω, ∧^p C^{2n})$, $G \in C^∞_0 (Ω, ∧^q C^{2n})$, we have

$$
da(F \wedge G) = da F \wedge G + (−1)^p F \wedge da G, \quad α = 0, 1.
$$

We say $F$ is closed if $d_0 F = d_1 F = 0$. For $u_1, \ldots, u_n \in C^2$, $∆ u_1 \wedge \ldots \wedge ∆ u_k$ is closed, $k = 1, \ldots, n$. Moreover, it follows easily from Lemma 2.1 that $∆ u_1 \wedge \ldots \wedge ∆ u_n$ satisfies the following remarkable identities:

$$
∆ u_1 \wedge ∆ u_2 \wedge \ldots \wedge ∆ u_n = d_0 (d_1 u_1 \wedge ∆ u_2 \wedge \ldots \wedge ∆ u_n) = d_0 d_1 (u_1 \wedge ∆ u_2 \wedge \ldots \wedge ∆ u_n) = ∆ (u_1 \wedge ∆ u_2 \wedge \ldots \wedge ∆ u_n). \quad (2.3)
$$

To write down the explicit expression, we define for a function $u \in C^2$,

$$
∆_{ij} u := \frac{1}{2} (\nabla_{i0} \nabla_{j1} u - \nabla_{i1} \nabla_{j0} u), \quad ∆ u = \sum_{i,j=0}^{2n-1} ∆_{ij} u \ ω^i \wedge ω^j,
$$

$$
∆ u_1 \wedge \ldots \wedge ∆ u_n = \sum_{i_1,j_1,\ldots} \Delta_{i_1,j_1} u_1 \ldots \Delta_{i_n,j_n} u_n \ ω^{i_1} \wedge ω^{j_1} \wedge \ldots \wedge ω^{i_n} \wedge ω^{j_n} = \sum_{i_1,j_1,\ldots} δ_{i_1,j_1,\ldots}^{i_n,j_n} \Delta_{i_1,j_1} u_1 \ldots \Delta_{i_n,j_n} u_n \ Ω_{2n}, \quad (2.4)
$$

for $u_1, \ldots, u_n \in C^2$, where $Ω_{2n}$ is defined as

$$
Ω_{2n} := ω^0 \wedge ω^1 \wedge \ldots \wedge ω^{2n-2} \wedge ω^{2n-1}, \quad (2.5)
$$

and $δ_{i_1,j_1,\ldots}^{i_n,j_n} :=$ the sign of the permutation from $(i_1,j_1,\ldots,i_n,j_n)$ to $(0,1,\ldots,2n−1)$, if $\{i_1,j_1,\ldots,i_n,j_n\} = \{0,1,\ldots,2n−1\}$; otherwise, $δ_{i_1,j_1,\ldots}^{i_n,j_n} = 0$.

Note that $∆ u_1 \wedge \ldots \wedge ∆ u_n$ is symmetric with respect to the permutation of $u_1, \ldots, u_n$. In particular, when $u_1 = \ldots = u_n = u$, $∆ u_1 \wedge \ldots \wedge ∆ u_n$ coincides
with \((\Delta u)^n := \wedge^n \Delta u = \Delta_n u \Omega_{2n}\). Denote by \(\Delta_n(u_1, \ldots, u_n)\) the coefficient of the form \(\Delta u_1 \wedge \ldots \wedge \Delta u_n\), i.e., \(\Delta u_1 \wedge \ldots \wedge \Delta u_n = \Delta_n(u_1, \ldots, u_n) \Omega_{2n}\). Then \(\Delta_n(u_1, \ldots, u_n)\) coincides with the mixed Monge-Ampère operator \(\det(u_1, \ldots, u_n)\) while \(\Delta_n u\) coincides with the quaternionic Monge-Ampère operator \(\det(u)\). This result was proved by Alesker (Proposition 7.1 in [3]). And in Appendix A in [33] we gave an elementary and simpler proof of the identity:

\[
\Delta u_1 \wedge \ldots \wedge \Delta u_n = n! \det(u_1, \ldots, u_n) \Omega_{2n}
\]  

(2.6)

for \(C^2\) function \(u\).

By introducing the quaternionic version of differential forms, the author and Wang defined in [33] the notions of closed positive forms and closed positive currents in the quaternionic case and our definition of closedness matches positivity well. Although a 2\(n\)-form is not an authentic differential form and we cannot integrate it, we can define

\[
\int_{\Omega} F := \int_{\Omega} \mu,
\]

(2.7)

**Lemma 2.2.** (Stokes-type formula, Lemma 3.2 in [33]) Assume that \(T = \sum_i \hat{\omega}^i\) is a smooth \((2n - 1)\)-form in \(\Omega\), where \(\hat{\omega}^i = \omega^0 \wedge \ldots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \ldots \wedge \omega^{2n-1}\). Then for smooth function \(h\), we have

\[
\int_{\Omega} h d\alpha T = -\int_{\Omega} d\alpha h \wedge T + \sum_{i=0}^{2n-1} (-1)^{i-1} \int_{\partial \Omega} h T_i \ n_{i\alpha} \ dS,
\]

where \(n_{i\alpha}, i = 0, 1, \ldots, 2n-1, \alpha = 0, 1,\) is defined by the matrix:

\[
\begin{pmatrix}
    n_{(2i)0} & n_{(2i)1} \\
    n_{(2i+1)0} & n_{(2i+1)1}
\end{pmatrix} :=
\begin{pmatrix}
    n_{4l} + i & n_{4l+1} & -n_{4l+2} - i & n_{4l+3} \\
    n_{4l+2} - i & n_{4l+3} & n_{4l} - i & n_{4l+1}
\end{pmatrix},
\]

(2.8)

\(l = 0, 1, \ldots, n - 1\). Here \(\mathbf{n} = (n_0, n_1, \ldots, n_{2n-1})\) is the unit outer normal vector to \(\partial \Omega\) and \(dS\) denotes the surface measure of \(\partial \Omega\). In particular, if \(h = 0\) on \(\partial \Omega\), we have

\[
\int_{\Omega} h d\alpha T = -\int_{\Omega} d\alpha h \wedge T, \quad \alpha = 0, 1.
\]

(2.9)

Bedford-Taylor theory [8, 10] in complex analysis can be generalized to the quaternionic case. Let \(u\) be a locally bounded PSH function and let \(T\) be a closed positive \(2k\)-current. Then \(\Delta u \wedge T\) defined by

\[
\Delta u \wedge T := \Delta(uT),
\]

(2.10)
i.e., \((\Delta u \wedge T)(\eta) := uT(\Delta \eta)\), for any test form \(\eta\), is also a closed positive current. Inductively,

\[
\Delta u_1 \wedge \ldots \wedge \Delta u_p \wedge T := \Delta(u_1 \Delta u_2 \wedge \ldots \wedge \Delta u_p \wedge T)
\]

(2.11)
is a closed positive current, when \( u_1, \ldots, u_p \in PSH \cap L^\infty_{loc}(\Omega) \). In particular, for \( u_1, \ldots, u_n \in PSH \cap L^\infty_{loc}(\Omega) \), \( \Delta u_1 \wedge \cdots \wedge \Delta u_n = \mu \Omega_{2n} \) for a well defined positive Radon measure \( \mu \).

For any strongly positive test \((2n-2p)\)-form \( \psi \) on \( \Omega \), (2.11) can be rewritten as
\[
\int_\Omega \Delta u_1 \wedge \cdots \wedge \Delta u_p \wedge \psi = \int_\Omega u_1 \Delta u_2 \wedge \cdots \wedge \Delta u_p \wedge \Delta \psi,
\]
where \( u_1, \ldots, u_p \in PSH \cap L^\infty_{loc}(\Omega) \). Since positive currents have measure coefficients, (2.12) also holds for strongly positive \( \psi \in D_0^{2n-2p}(\Omega) \) vanishing on the boundary.

The following different types of weak convergence results are powerful tools in developing pluripotential theory for the quaternionic Monge-Ampère operator. We will use these results frequently in the following.

**Lemma 2.3.** (1) (Theorem 3.1 in [24]) Let \( v^1, \ldots, v^k \in PSH \cap L^\infty_{loc}(\Omega) \). Let \( \{v^1_j\}_{j \in \mathbb{N}}, \ldots, \{v^k_j\}_{j \in \mathbb{N}} \) be decreasing sequences of PSH functions in \( \Omega \) such that \( \lim_{j \to \infty} v^j_j = v^i \) pointwisely in \( \Omega \) for each \( t \). Then the currents \( \Delta v^1_j \wedge \cdots \wedge \Delta v^k_j \) converge weakly to \( \Delta v^1 \wedge \cdots \wedge \Delta v^k \) as \( j \to \infty \).

(2) (Proposition 3.2 in [24]) Let \( v^0, \ldots, v^k \in PSH \cap L^\infty_{loc}(\Omega) \). Let \( \{v^0_j\}_{j \in \mathbb{N}}, \ldots, \{v^k_j\}_{j \in \mathbb{N}} \) be decreasing sequences of PSH functions in \( \Omega \) such that \( \lim_{j \to \infty} v^j_j = v^t \) pointwisely in \( \Omega \) for \( t = 0, \ldots, k \). Then the currents \( v^0_j \Delta v^1_j \wedge \cdots \wedge \Delta v^k_j \) converge weakly to \( v^0 \Delta v^1 \wedge \cdots \wedge \Delta v^k \) as \( j \to \infty \).

(3) (Theorem 2.1.11 in [24]) Let \( \{v_j\} \) be a sequence of continuous PSH functions in \( \Omega \). Assume that this sequence converges uniformly on compact subsets to a function \( v \). Then \( v \) is continuous PSH function. Moreover the measures \( (\Delta v_j)^n \) converge weakly to \( (\Delta v)^n \) as \( j \to \infty \).

3. Solvability and stability of quaternionic Monge-Ampère equation

In this section, we are to prove Theorem 1.1 by combining the well known results for the Dirichlet problem of real Monge-Ampère equation and the connection between real and quaternionic Monge-Ampère operator. We refer to [12, 27] for more detailed historical discussions for the real Monge-Ampère equation. Here we only mention the following two basic results.

Let \( |\cdot|_\Omega \) and \( |\cdot|_{\partial \Omega} \) denote the sup-norm on \( \Omega \) and \( \partial \Omega \). Denote by \( det_{\mathbb{R}} u \) the real Monge-Ampère measure of \( u \) in the usual sense and denote by \( det(u) \) the Moore determinant of quaternionic Hessian of \( u \) given by (1.3).

**Lemma 3.1.** (Theorem 4.1 and Lemma 3.5 in [24]) Let \( \Omega \) be bounded strictly convex in \( \mathbb{R}^m \). Let \( \varphi \in C(\partial \Omega) \) and \( 0 \leq g \in L^1(\Omega) \). Then the following Dirichlet problem has a unique solution:
\[
\begin{align*}
\begin{cases}
\text{u is convex in } \Omega \\
det_{\mathbb{R}} u &= g \text{d}V \\
\lim_{x \to \xi} u(x) &= \varphi(\xi) \text{ for all } \xi \in \partial \Omega.
\end{cases}
\end{align*}
\]
Furthermore, the solution, denoted by $U_R(\varphi, g)$, satisfies

$$\inf_{\partial\Omega} \varphi - C\|g\|_{L^1(\Omega)} \leq U_R(\varphi, g) \leq \sup_{\partial\Omega} \varphi$$

for some constant $C$ depending only on $\Omega$. And

$$|U_R(\varphi_1, g_1) - U_R(\varphi_2, g_2)|_{\Omega} \leq |\varphi_1 - \varphi_2|_{\partial\Omega} + C\|g_1 - g_2\|_{L^1(\Omega)}$$

for $\varphi_1, \varphi_2 \in C(\partial\Omega)$ and $0 \leq g_1, g_2 \in L^1(\Omega)$.

**Lemma 3.2.** (Theorem 1.1 in [12]) Let $\Omega$ be bounded strictly convex in $\mathbb{R}^m$, $\partial\Omega \in C^\infty$. If $\varphi \in C^\infty(\partial\Omega)$, $0 < g \in C^\infty(\Omega)$, then the solution $U_R(\varphi, g)$ of (3.1) exists and is in $C^\infty(\Omega)$.

Alesker [2] and Zhu [38] studied the Dirichlet problem for the quaternionic Monge-Ampère equation in terms of the original definition of the quaternionic Monge-Ampère operator as a Moore determinant $\det(u)$. As we introduced in previous sections, we [33] showed that the original definition $\det(u)$ coincides with our new definition $(\Delta u)^n$. Since we need to use the relationship between real and quaternionic Monge-Ampère operator (see Lemma 3.6 below), in this section we use the original definition of quaternionic Monge-Ampère operator as a Moore determinant $\det(u)$.

To prove Theorem 1.1, it is equivalent to prove the same conclusion for the solution $U_Q(\varphi, g)$ of the Dirichlet problem:

$$\begin{cases}
    u \in PSH \cap L^\infty(\Omega) \\
    \det(u) = gdV \\
    \lim_{\zeta \to q} u(\zeta) = \varphi(q) \quad q \in \partial\Omega, \ \varphi \in C(\partial\Omega). 
\end{cases} \quad (3.2)$$

Recall that an open bounded domain $\Omega \subset \mathbb{H}^n$ with a smooth boundary $\partial\Omega$ is called strictly pseudoconvex if for every point $q_0 \in \partial\Omega$ there exists a neighborhood $\mathcal{O}$ and a smooth strictly psh function $h$ on $\mathcal{O}$ such that $\Omega \cap \mathcal{O} = \{ h < 0 \}$, $h(q_0) = 0$, and $\nabla h(q_0) \neq 0$.

**Lemma 3.3.** (Corollary 1.3 in [38]) Let $\Omega$ be a quaternionic strictly pseudoconvex bounded domain in $\mathbb{H}^n$. If $0 < g \in C^\infty(\Omega)$, $\varphi \in C^\infty(\partial\Omega)$, then the Dirichlet problem (3.2) has a unique solution. And the solution is in $C^\infty(\Omega)$.

**Lemma 3.4.** (Theorem 1.3 in [1]) Let $\Omega$ be a quaternionic strictly pseudoconvex bounded domain in $\mathbb{H}^n$. If $0 \leq g \in C(\Omega)$, $\varphi \in C(\partial\Omega)$, then the Dirichlet problem (3.2) has a unique solution. And the solution is in $C(\Omega)$.

Here we rewrite the following comparison principle by using $\det(u)$. We [34] established these results in terms of $(\Delta u)^n$ by using the theory of quaternionic closed positive currents. We will use the comparison principle frequently in the following.
Lemma 3.5. (Comparison principle, Theorem 1.2 in [34])

(1) Let \( u, v \in \text{PSH} \cap L^\infty_{\text{loc}}(\Omega) \). If for any \( \zeta \in \partial \Omega \), \( \lim_{\zeta^{-} \rightarrow q \in \Omega} (u(q) - v(q)) \geq 0 \), then
\[
\int_{\{u < v\}} \det(v) \leq \int_{\{u < v\}} \det(u).
\]

(2) Under the assumptions of (1), the inequality \( \det(u) \leq \det(v) \) implies \( v \leq u \).

(3) If for any \( \zeta \in \partial \Omega \), \( \lim_{\zeta^{-} \rightarrow q \in \Omega} u(q) = \lim_{\zeta^{-} \rightarrow q \in \Omega} v(q) = 0 \) and \( u \leq v \) in \( \Omega \), then
\[
\int_{\Omega} \det(v) \leq \int_{\Omega} \det(u).
\]

Corollary 3.1.

\[
U_Q(\varphi_1, g_1) + U_Q(\varphi_2, g_2) \leq U_Q(\varphi_1 + \varphi_2, g_1 + g_2)
\]
\[
|U_Q(\varphi_1, g_1) - U_Q(\varphi_2, g_2)| \leq -U_Q(-|\varphi_1 - \varphi_2|, |g_1 - g_2|).
\]

Proof. By the comparison principle and superadditivity. \( \square \)

Lemma 3.6. (Proposition 4.1 in [32]) For a function \( u \in \text{PSH} \cap C^2 \), we have the inequality:
\[
\left(\frac{\det(\partial^2 u)}{\partial q_j \partial q_k}\right)^{\frac{1}{2n}} \geq 4 \left(\frac{\det(R_{\partial^2 u})}{\partial x_s \partial x_t}\right)^{\frac{1}{4n}}.
\]

Proposition 3.1. Let \( \Omega \) be a strictly convex bounded domain in \( \mathbb{H}^n \). If \( 0 \leq g \in C(\overline{\Omega}) \), \( \varphi \in C(\partial \Omega) \), then
\[
U_R(\varphi, C_0 g^4) \leq U_Q(\varphi, g), \quad C_0 = 4^{-4n}.
\]

Proof. By identifying \( \mathbb{H}^n \) with \( \mathbb{R}^{4n} \) and using Lemma 3.1, \( U_R(\varphi, C_0 g^4) \) exists and \( u := U_R(\varphi, C_0 g^4) \) is convex. By Lemma 3.2, \( U_Q(\varphi, g) \) exists. By comparison principle, it is sufficient to prove \( \det(u) \geq g \), i.e.,
\[
\int \psi \det(u) \geq \int \psi g,
\]
for any \( 0 \leq \psi \in C_0(\Omega) \).

Without loss of generality, we can assume that \( \partial \Omega \in C^\infty \). Take a sequence \( \varphi_j \in C^\infty(\partial \Omega) \) converging uniformly to \( \varphi \), and a sequence \( 0 < g_j \in C^\infty(\overline{\Omega}) \) converging uniformly to \( g \). Let \( u_j := U_R(\varphi_j, C_0 g_j^4) \). Then \( u_j \) converges uniformly to \( u \) as \( j \to \infty \) by Lemma 3.1. By weak convergence result (Lemma 2.3),
\[
\int \psi \det(u_j) \to \int \psi \det(u) \quad \text{and} \quad \int \psi g_j \to \int \psi g.
\]
Therefore, it is sufficient to prove \( 3.3 \) for \( 0 < g \in C^\infty(\overline{\Omega}) \), \( \varphi \in C^\infty(\partial \Omega) \), and \( \partial \Omega \in C^\infty \). Note that in this case \( u \in C^\infty(\overline{\Omega}) \) by Lemma 3.2. It suffices to prove that
\[
C_0(\det u)^4 \geq C_0 g^4 = \det R u
\]
for \( u \in C^\infty(\overline{\Omega}) \). Then the conclusion follows from Lemma 3.6. \( \square \)
Proof of Theorem 1.1. First assume that $\Omega$ is strictly convex. It is sufficient to prove the inequalities for $g \in C(\Omega)$. For $0 \leq g \in L^4(\Omega)$, we can take a sequence $0 \leq g_j \in C(\Omega)$ converging to $g$ in $L^4(\Omega)$. Then $U_Q(\varphi, g_j) \in C(\Omega)$ by Lemma 3.4. By the second inequality,

$$|U_Q(\varphi, g_j) - U_Q(\varphi, g_k)|_\Omega \leq C\|g_j - g_k\|_{L^4(\Omega)}^{1/4}.$$ 

Thus $u_j := U_Q(\varphi, g_j)$ converges uniformly to a continuous function $u$. And $u = U_Q(\varphi, g)$ by the Lemma 2.3 and comparison principle (Lemma 3.5).

Apply Lemma 3.1 with $m = 4n$ and Proposition 3.1 to get

$$\inf_{\partial \Omega} \varphi - C\|C\varrho \|_{L^4(\Omega)} \leq U_R(\varphi, C\varrho) \leq \sup_{\partial \Omega} \varphi,$$

i.e.,

$$\inf_{\partial \Omega} \varphi - C'\|g\|_{L^4(\Omega)}^{1/4} \leq U_Q(\varphi, g) \leq \sup_{\partial \Omega} \varphi.$$

The second inequality follows from the first inequality and Corollary 3.1.

For strictly pseudoconvex $\Omega$, we can take a bounded strictly convex domain $\Omega'$ containing $\Omega$ and extend $|g_1 - g_2|$ by zero to an $L^4$-function on $\Omega'$. Then the theorem follows.

It follows from Theorem 1.1 that for $0 \leq g \in L^4(\Omega)$,

$$\|U_Q(0, g)\|_{L^\infty(\Omega)} \leq C\|g\|_{L^4(\Omega)}^{1/4}.$$ 

We obtain the $L^\infty - L^4$-stability of the quaternionic Monge-Ampère equation. Then we get that the quaternionic Monge-Ampère equation is $L^p - L^q$-stable for $p \in [1, \infty]$ and $q \geq 4$ by Hölder’s inequality.

Remark 3.1. Here we are inspired by the article [14] written Cegrell and Persson. They proved the $L^\infty - L^2$-stability of the complex Monge-Ampère equation by using the connection between real and complex Monge-Ampère operators (due to the idea of Cheng and Yau mentioned in [3]). Blocki [11] obtained the $L^n - L^1$-stability after showing an estimate for the complex Monge-Ampère operator. For the quaternionic Monge-Ampère equation, Blocki’s estimate was generalized to the quaternionic case (cf. Lemma 4.6 in [29]), thus one can also get the $L^n - L^1$-stability for quaternionic Monge-Ampère equation by following Blocki’s method. By Hölder’s inequality we obtain the $L^p - L^q$-stability for quaternionic Monge-Ampère equation for

$$(p, q) \in ([1, n] \times [1, \infty]) \cup ([1, \infty] \times [4, \infty]).$$

We do not know the stability for other pairs $(p, q)$.

As an application of Theorem 1.1, we generalize an inequality for the mixed quaternionic Monge-Ampère measures. We prove this inequality in the smooth case in Appendix and generalize it to the nonsmooth functions in the following proposition. As for the complex Monge-Ampère measure, the nonsmooth version of this inequality has nontrivial applications [3, 16, 17, 24].
Lemma 3.7. For $u_1, u_2, \ldots, u_n \in \text{PSH} \cap C^2(\Omega)$,
\[ \det(u_1, \ldots, u_n) \geq \det(u_1)^{\frac{1}{n}} \cdots \det(u_n)^{\frac{1}{n}}. \] (3.4)

Proof. See Appendix for the proof. \[ \square \]

Proposition 3.2. Let $0 \leq g, f \in L^1(B)$, $u, v \in \text{PSH} \cap C(B)$ such that
\[ \det(u) \geq fdV, \quad \det(v) \geq gdV. \]
Then
\[ \det(u, \ldots, u, v, \ldots, v) \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} dV. \]

Proof. For smooth $u, v$, we already have
\[ \det(u, \ldots, u, v, \ldots, v) \geq \det(u)^{\frac{k}{n}} \cdot \det(v)^{\frac{n-k}{n}} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} dV. \]

Now for $g, f \in L^1(B)$, take sequences $\{g_j\}, \{f_j\}$ such that
\[ 0 < g_j \in C^\infty \to g \quad \text{in} \quad L^1(B), \]
\[ 0 < f_j \in C^\infty \to f \quad \text{in} \quad L^1(B). \]

Take $\varphi_j, \psi_j$ smooth on $\partial B$ and $\varphi_j \to u$, $\psi_j \to v$ uniformly on $\partial B$. By Lemma 3.3 there exist $u_j, v_j \in \text{PSH} \cap C^\infty(B)$ such that
\[
\begin{align*}
\{ & \det(u_j) = f_j dV \quad \text{in} \quad B \\
& u_j = \varphi_j \quad \text{on} \quad \partial B \} \\
\{ & \det(v_j) = g_j dV \quad \text{in} \quad B \\
& v_j = \psi_j \quad \text{on} \quad \partial B. \}
\end{align*}
\]

By Theorem 1.1, $u_j, v_j$ converge uniformly to $u, v$ respectively. Then it follows from Lemma 2.3 that
\[
\det(u, \ldots, u, v, \ldots, v) = \lim_{j \to \infty} \det(u_j, \ldots, u_j, v_j, \ldots, v_j) \geq \lim_{j \to \infty} (f_j)^{\frac{k}{n}} (g_j)^{\frac{n-k}{n}} dV \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} dV. \] (3.5)

Now for $0 \leq g, f \in L^1(B)$, take $g_j \in L^1(B) \nearrow g$ and $f_j \in L^1(B) \nearrow f$. By Theorem 1.1 there exist $\tilde{u}_j, \tilde{v}_j \in \text{PSH} \cap C(B)$ such that
\[ \det(\tilde{u}_j) = f_j dV \quad \text{and} \quad \det(\tilde{v}_j) = g_j dV. \]

By the comparison principle we have $\tilde{u}_j \searrow u, \tilde{v}_j \searrow v$. Then (3.5) also holds by Lemma 2.3. \[ \square \]
4. Subsolution theorem of quaternionic Monge-Ampère equation

The purpose of this section is to prove Theorem 1.2. Here we used the method from Kołodziej [20, 21, 25] for the complex Monge-Ampère equation. Nguyen [26] also use Kołodziej’s method to study the Dirichlet problem for the complex Hessian equation.

Proof of Theorem 1.2
First we can assume that \( \mu \) has compact support in \( \Omega \). This is because, for non-compactly supported measure \( \mu \), we can take a nondecreasing sequence of cut-off functions \( \chi_j \) on \( \Omega \). Then \( \chi_j \mu \) will be compact support in \( \Omega \). By Lemma 3.5, the solutions corresponding to \( \chi_j \mu \) will be bounded from below by the given subsolution and they will decrease to the solution for \( \mu \) by Lemma 2.3.

Next we can modify the subsolution \( v \) such that \( v \) is PSH in a neighborhood of \( \Omega \) and \( \lim_{\zeta \to q} v(\zeta) = 0 \) for any \( q \in \partial \Omega \). Take an open subset \( U \) such that \( \text{supp} \mu \subset U \subset \bar{\Omega} \) and define the PSH envelope \( \tilde{v} = \sup \{ w \in \text{PSH}(\Omega) : w \leq 0, w \leq v \text{ in } U \} \).

By Proposition 2.1, \( \tilde{v} \neq \tilde{v}^* \in \text{PSH}(\Omega) \). And \( v = \tilde{v} \) in \( U \), \( \lim_{\zeta \to q} \tilde{v}(\zeta) = 0 \) for any \( q \in \partial \Omega \). Since \( \text{supp} \mu \subset U \), we have \( (\Delta \tilde{v})^n \geq d\mu \). Take a defining function \( \rho \) of \( \Omega \) which is smooth and strictly PSH in a neighborhood \( \Omega_1 \) of \( \Omega \). Since \( \tilde{v} \) is bounded, we can assume \( \rho \leq \tilde{v} \) in \( \bar{U} \). Define

\[
\hat{v} = \begin{cases} 
\max\{\rho, \tilde{v}\} & \text{on } \Omega, \\
\rho & \text{on } \Omega_1 \setminus \bar{U}.
\end{cases}
\]

By Proposition 2.1, \( \hat{v} \) is a PSH function on a neighborhood of \( \Omega \) satisfying \( (\Delta \hat{v})^n \geq d\mu \). We still write \( v \) instead of \( \hat{v} \). Furthermore, we can make the support of \( d\nu := (\Delta v)^n \) compact in \( \Omega \).

Now for such subsolution \( v \) we take the standard smooth regularization \( w_j \searrow v \) in a neighborhood of \( \bar{\Omega} \) (cf. Proposition 2.1 (5)). Let \( (\Delta w_j)^n = g_j dV \). By the Radon-Nikodym theorem, \( d\mu = b d\nu \) with \( 0 \leq b \leq 1 \). Denote \( \mu_j = h g_j dV \). Since \( w_j \searrow v \), \( h(\Delta w_j)^n \) converges weakly to \( h(\Delta v)^n \) by Lemma 2.3 i.e., \( \mu_j \to \mu \) as \( j \to \infty \). As \( \mu \) has compact support, so does \( \mu_j \)'s. Note that \( h g_j \in L^p(\Omega) \) for every \( p > 0 \). Then by Theorem 1.1 we can find \( u_j \) satisfying

\[
\begin{cases}
\{ u_j \in \text{PSH}(\Omega) \cap C(\bar{\Omega}) \\
(\Delta u_j)^n = \mu_j = h g_j dV \\
u_j = \varphi & \text{on } \partial \Omega.
\end{cases}
\]

As we shall see the function \( u := (\limsup u_j)^* \) solves the equation (1.1). By passing to a subsequence we assume that \( u_j \) converges to \( u \) in \( L^1(\Omega) \). Since the smooth regularization \( \{w_j\} \) is uniformly bounded, we can choose a uniform \( C \) such that \( w_j - C < \varphi \) on \( \partial \Omega \). By the comparison principle, \( w_j - C \leq u_j \leq \sup_{\bar{\Omega}} \varphi \). It follows that \( \{u_j\} \) is uniformly bounded, thus \( u \) defined above is bounded. Now we need the following lemmas.
Lemma 4.1. (quasicontinuity theorem, Theorem 1.1 in [34]) Let $\Omega$ be an open subset of $\mathbb{H}^n$ and let $u$ be a locally bounded PSH function. Then for each $\varepsilon > 0$, there exists an open subset $\omega$ of $\Omega$ such that $\text{cap}(\omega) < \varepsilon$ and $u$ is continuous on $\Omega \setminus \omega$.

Here the quaternionic capacity of $E$ in $\Omega$ is defined in [34] as

$$\text{cap}(E) = \text{cap}(E, \Omega) = \sup \left\{ \int_E (\triangle u)^n : u \in \text{PSH}(\Omega), 0 < u < 1 \right\}.$$  \hspace{1cm} (4.1)

Lemma 4.2. If for any $a > 0$ and any compact $K \subset \Omega$ we have

$$\lim_{j \to \infty} \int_{K \cap E_j(a)} (\triangle u_j)^n = \lim_{j \to \infty} \mu_j(K \cap E_j(a)) = 0$$

with $E_j(a) := \{ u - u_j \geq a \}$, then the function $u$ defined above solves the Dirichlet problem (1.1).

Proof. By Demailly's inequality (Proposition 3.5 in [31]) we have

$$\mu_j = (\triangle u_j)^n = \chi_{E_j(a)}(\triangle u_j)^n + \chi_{\{ u - u_j < a \}}(\triangle u_j)^n \leq \chi_{E_j(a)} \mu_j + (\triangle \max\{ u - a, u_j \})^n.$$  \hspace{1cm} (4.3)

By (4.2), for any $s$ we can choose $j(s)$ such that

$$\mu_j(K \cap E_j(\frac{1}{s})) < \frac{1}{s}, \quad j \geq j(s).$$

Let $\rho_s := \max\{ u - \frac{1}{s}, u_j(s) \}$, then $\rho_s$ is PSH by Proposition 2.1. By (4.3) we have $\mu \leq \liminf_{s \to \infty} (\triangle \rho_s)^n$. By the Hartogs lemma (Theorem 4.1.9 in [18]), $\rho_s$ converges uniformly to $u$ on any compact $E$ such that $u|_E$ is continuous. Therefore by the quasicontinuity of quaternionic PSH functions (Lemma 4.1) we have $\rho_s \to u$ in capacity. Thus $(\triangle \rho_s)^n \to (\triangle u)^n$ weakly by the convergence theorem (cf. Theorem 4.1 in [30]). It follows that $\mu \leq (\triangle u)^n$.

To show the reverse inequality, we shall prove that for $\varepsilon > 0$, $\mu(\Omega) \geq \int_{\Omega_{\varepsilon}} (\triangle u)^n$, where $\Omega_{\varepsilon} = \{ q \in \Omega : \text{dist}(q, \partial \Omega) > \varepsilon \}$. Note that $\rho_s = u_j(s)$ on a neighborhood of $\partial \Omega_{\varepsilon}$ for $\varepsilon$ small enough. By (2.3), (2.4) and (2.7), we can use
the Stokes-type formula (Lemma 4.2) to get

\[
\mu(\Omega) \geq \mu(\Omega_s) \geq \liminf_{j(s) \to \infty} \mu_{j(s)}(\Omega_s) = \liminf_{j(s) \to \infty} \int_{\Omega_s} (\Delta u_{j(s)})^n
\]

\[
= \liminf_{j(s) \to \infty} \int_{\Omega_s} d_0 \left[ d_1 u_{j(s)} \wedge (\Delta u_{j(s)})^{n-1} \right]
\]

\[
= \liminf_{j(s) \to \infty} \int_{\Omega_s} \sum_{i_1} \nabla_i \left[ \sum_{j_1, j_2, \ldots} \nabla_{j_1} u_{j(s)} \Delta_{i_2 j_2} u_{j(s)} \ldots \Delta_{i_n j_n} u_{j(s)} d_{i_1 j_1 \ldots i_n j_n} \right] dV
\]

\[
= \liminf_{j(s) \to \infty} \int_{\partial \Omega_s} \sum_{i_1} \left[ \sum_{j_1, j_2, \ldots} \nabla_{j_1} u_{j(s)} \Delta_{i_2 j_2} u_{j(s)} \ldots \Delta_{i_n j_n} u_{j(s)} d_{i_1 j_1 \ldots i_n j_n} \right] n_{i_1} dS
\]

\[
= \liminf_{j(s) \to \infty} \int_{\Omega_s} (\Delta \rho_{s})^n \geq \int_{\Omega_s} (\Delta u)^n,
\]

where \( n_{i_0} \) is defined by (2.8) and \( dS \) denotes the surface measure of \( \partial \Omega \). Let \( \epsilon \to 0 \), we get \( \mu(\Omega) \geq (\Delta u)^n(\Omega) \).}

**Lemma 4.3.** (Chern-Levine-Nirenberg type estimate, Proposition 3.10 in [33]) Let \( \Omega \) be a domain in \( \mathbb{R}^n \). Let \( K,L \) be compact subsets of \( \Omega \) such that \( L \) is contained in the interior of \( K \). Then there exists a constant \( C \) depending only on \( K,L, \Omega \) such that for any \( v \in PSH(\Omega) \) and \( u_1, \ldots, u_n \in PSH \cap C^2(\Omega) \), one has

\[
\| \Delta u_1 \wedge \ldots \wedge \Delta u_k \|_L \leq C \| u_1 \|_{L^\infty(K)} \ldots \| u_k \|_{L^\infty(K)}, \quad \text{(a)}
\]

\[
\| \Delta u_1 \wedge \ldots \wedge \Delta u_k \|_L \leq C \| u_1 \|_{L^1(K)} \| u_2 \|_{L^1(K)} \ldots \| u_k \|_{L^1(K)}, \quad \text{(b)}
\]

\[
\| v \Delta u_1 \wedge \ldots \wedge \Delta u_k \|_L \leq C \| v \|_{L^1(K)} \| u_1 \|_{L^\infty(K)} \ldots \| u_k \|_{L^\infty(K)}. \quad \text{(c)}
\]

**Lemma 4.4.** Suppose that there is a subsequence \( \{u_j\} \) (we still write \( \{u_j\} \)) such that

\[
\int_{E_{j(a_0)}} (\Delta u_j)^n > A_0, \quad A_0 > 0, a_0 > 0. \quad (4.5)
\]

Then there exist \( a_p > 0, A_p > 0, k_1 > 0 \) such that

\[
\int_{E_{j(a_p)}} (\Delta v_j)^{n-p} \wedge (\Delta v_k)^p > A_p, \quad j > k > k_1, \quad \text{(4.6)}
\]

for \( v_j \)'s the solutions (by Lemma 3.4) of the Dirichlet problem

\[
\begin{align*}
&v_j \in PSH(\Omega) \cap C(\overline{\Omega}) \\
&(\Delta v_j)^n = g_j dV \\
&v_j = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Proof. We will prove it by induction over $p$. For $p = 0$, the result (4.6) follows from the hypothesis (4.5) and comparison principle. Suppose that (4.6) is true for $p < n$ and now we shall prove it for $p + 1$. Note that $\{v_j\}$ is also uniformly bounded. We can assume that $-1 < u_j, v_j < 0$. By Chern-Levine-Nirenberg estimate (Lemma 4.3) there exists $C > 0$ such that
\[
\int_{\Omega} (\triangle v_j)^q \wedge (\triangle u_j)^{n-q} \leq C,
\]
for any $q = 0, \ldots, n$. Set
\[
S := (\triangle v_j)^{n-p-1} \wedge (\triangle v_k)^p.
\]
By the induction hypothesis, there exist $a_p, A_p > 0$ and $k_1 > 0$ such that
\[
\int_{E_j(a_p)} \triangle v_j \wedge S > A_p, \quad j > k > k_1.
\]
By Lemma 4.1 for fixed $\epsilon \in (0, \frac{a_p A_p}{4(1+C)})$ (where $C$ is from (4.7)), we can choose an open set $U$ such that $\text{cap}(U, \Omega) < \frac{\epsilon}{2n+1}$, and $u, v$ are continuous off the set $U$. Then
\[
\int_{U} (\triangle (v_j + v_k))^n < 2^n \text{cap}(U, \Omega) < \frac{\epsilon}{2}, \quad \text{and} \quad \int_{U} (\triangle (u_j + v_k))^n < \frac{\epsilon}{2}. \tag{4.9}
\]
Set
\[
J_1 := \int_{\Omega} (u - u_j) \triangle v_k \wedge S, \quad J_2 := \int_{\Omega} (u - u_j) \triangle v_j \wedge S.
\]
Since $u = u_j = \varphi$ and $v_j = 0$ on the boundary $\partial \Omega$, we can use the integration by parts formula (2.12) to get
\[
J_2 - J_1 = \int_{\Omega} v_j \triangle (u - u_j) \wedge S - \int_{\Omega} v_k \triangle (u - u_j) \wedge S
\]
\[
= \int_{\Omega \setminus U} (v_j - v_k) \triangle (u - u_j) \wedge S + \int_{U} (v_j - v_k) \triangle (u - u_j) \wedge S \tag{4.10}
\]
\[
\leq \int_{\Omega \setminus U} \|v_j - v_k\| \triangle (u + u_j) \wedge S + \int_{U} \triangle (u + u_j) \wedge S,
\]
where the last inequality follows from $-1 < u_j, v_j < 0$. Note that $v_j$ converges uniformly to $v$ on $\Omega \setminus U$. Let $l > k_1$ such that $\|v_j - v_k\| < \frac{\epsilon}{2n+1}$ on $\Omega \setminus U$ for $j > k > l > k_1$, where $C$ is the constant in (4.7). By (4.7) and (4.9), each of the integral on the right hand side of (4.10) does not exceed $\frac{\epsilon}{4}$. Therefore
\[
J_2 - J_1 \leq \epsilon \leq \frac{a_p A_p}{4},
\]
for $j > k > l > k_1$.
From the upper bound of all \( u_j \) (resp. \( v_j \)) by sup \( \varphi \) (resp. 0) on the boundary, we have for \( k > k_2 > l \), in a neighborhood of \( \partial \Omega \):

\[
v_k \leq v + \epsilon, \quad \text{and} \quad u_k \leq u + \epsilon. \tag{4.11}
\]

And they still hold on \( \Omega \setminus U \) by the Hartogs lemma. By using (4.7), (4.8), (4.9) and (4.11) we get for \( j > k > k_2 \),

\[
J_2 \geq a_p \int_{\{u-u_j \geq a_p\}} \Delta v_j \wedge S + \int_{\{u-u_j < a_p\}} (u-u_j) \Delta v_j \wedge S
\]

\[
= a_p \int_{\{u-u_j \geq a_p\}} \Delta v_j \wedge S + \int_{\{u-u_j < a_p\} \cap (\Omega \setminus U)} (u-u_j) \Delta v_j \wedge S
\]

\[
+ \int_{\{u-u_j < a_p\} \cap U} (u-u_j) \Delta v_j \wedge S
\]

\[
\geq a_p \int_{\{u-u_j \geq a_p\}} \Delta v_j \wedge S - \epsilon \int_{\Omega \setminus U} \Delta v_j \wedge S - \int_{U} \Delta v_j \wedge S
\]

\[
\geq a_p A_p - \epsilon (C + 1) \geq \frac{3 a_p A_p}{4}.
\]

Fix \( d > 0 \). It follows from (4.7) that

\[
J_1 \leq \int_{\{u-u_j \geq a\}} \Delta v_k \wedge S + d \int_{\Omega} \Delta v_k \wedge S \leq \int_{\{u-u_j \geq a\}} \Delta v_k \wedge S + dC.
\]

If we take

\[
a_{p+1} = d := \frac{a_p A_p}{4C},
\]

then

\[
\int_{E_j(a_n)} \Delta v_k \wedge S \geq J_1 - dC \geq J_2 - \epsilon - dC \geq \frac{a_p A_p}{4} := A_{p+1},
\]

for \( j > k > k_2 \), which concludes the proof of the inductive step of Lemma 4.4. ☐

Now we continue to prove Theorem 1.2. It suffices to show (4.2) in Lemma 4.2. Suppose that it is not true, then by the assumption of Lemma 4.4 we have for \( p = n \) and fixed \( k > k_1 \),

\[
\int_{E_j(a_n)} \Delta v_k \wedge S > A_n, \quad j > k.
\]

Since \( \Delta v_k \wedge S \leq M_k dV \) for some \( M_k > 0 \), we have

\[
V(E_j(a_n)) \geq M_k^{-1} \int_{E_j(a_n)} \Delta v_k \wedge S > \frac{A_n}{M_k}, \quad j > k,
\]

which contradicts the fact that \( u_j \to u \) in \( L^1_{loc} \). Thus Theorem 1.2 follows. ☐

It follows from the subsolution theorem that if the Dirichlet problem (1.1) is solvable for a measure \( \nu \) and \( \mu \leq \nu \), then the Dirichlet problem (1.1) is solvable for \( \mu \) as well.
Corollary 4.1. Let $\Omega$ be a strictly pseudoconvex domain in $\mathbb{H}^n$ and $v_1, \ldots, v_n \in PSH \cap L^\infty(\Omega)$. There exists $u \in PSH \cap L^\infty(\Omega)$ satisfying any given continuous boundary data and $(\Delta u)^n = \Delta v_1 \wedge \ldots \wedge \Delta v_n$.

Proof. One can choose $v_1 + \ldots + v_n$ as a subsolution after adding a suitable PSH function. \hfill \Box

Appendix A. Proof of Lemma 3.7

Proof. For $C^2$ smooth plurisubharmonic functions $u_i$, $i = 1, \ldots, n$, their quaternionic Hessian $\left[ \frac{\partial^2 u_i}{\partial \bar{q}_j q_k} \right]$ are positive definite hyperhermitian matrices. By Theorem 1.1.15 in [1],

$$\det(u_1, \ldots, u_n) := \det \left( \left( \frac{\partial^2 u_1}{\partial \bar{q}_j q_k} \right), \ldots, \left( \frac{\partial^2 u_n}{\partial \bar{q}_j q_k} \right) \right) > 0.$$  

It follows from Aleksandrov inequality (cf. Corollary 1.1.16 in [1], or Corollary 2.16 in [38]) that

$$\det(u_1, u_2, \ldots, u_n) \geq \det(u_1, u_1, u_3, \ldots, u_n)^{\frac{1}{2}} \cdot \det(u_2, u_2, u_3, \ldots, u_n)^{\frac{1}{2}}. \quad (A.1)$$

We claim that when all functions are in $PSH \cap C^2(\Omega)$,

$$\det(u_1, u_2, \ldots, u_p, \ldots, u_p, u_1, \ldots, u_1, \ldots, u_{n-p-q}) \geq \det(u_1, u_1, \ldots, u_1, v_1, \ldots, v_{n-p-q})^{\frac{p}{p+q}} \cdot \det(u_2, u_2, \ldots, u_2, v_1, \ldots, v_{n-p-q})^{\frac{p}{p+q}}. \quad (A.2)$$

We prove this claim by induction. The case for $p = q = 1$ holds by (A.1). Assume by induction that the case for $p+q \leq m$ has already been proved. It suffices to prove it for $p+q \leq m+1$. First we need the following inequality.

$$\det(u_1, u_2, \ldots, u_p, v, \ldots) \geq \det(u_1, u_1, \ldots, u_1, v, \ldots)^{\frac{1}{p+q+1}} \cdot \det(u_2, u_2, \ldots, u_2, v, \ldots)^{\frac{p+q}{p+q+1}}. \quad (A.3)$$
By induction assumption we have

\[
\det(u_1, u_2, \ldots, u_2, v, \ldots) \\
\geq \det(u_1, \ldots, u_1, u_2, v, \ldots)^\frac{p+q-1}{p+q} \cdot \det(u_2, \ldots, u_2, v, \ldots)^\frac{p+q}{p+q+1}
\]

\[
= \det(u_1, \ldots, u_1, u_2, u_1, v, \ldots)^\frac{p+q-1}{p+q} \cdot \det(u_2, \ldots, u_2, v, \ldots)^\frac{p+q}{p+q+1}
\]

\[
\geq \left[ \det(u_1, \ldots, u_1, u_1, v, \ldots)^\frac{p+q-1}{p+q} \cdot \det(u_2, \ldots, u_2, u_1, v, \ldots)^\frac{p+q}{p+q+1} \right]^{\frac{p+q}{p+q+1}} \cdot \det(u_2, \ldots, u_2, v, \ldots)^\frac{p+q-1}{p+q+1}.
\]

Then we have

\[
\det(u_1, u_2, \ldots, u_2, v, \ldots)^\frac{(p+q)^2-1}{(p+q)^2}
\]

\[
\geq \det(u_1, \ldots, u_1, u_1, v, \ldots)^\frac{p+q-1}{p+q+1} \cdot \det(u_2, \ldots, u_2, v, \ldots)^\frac{p+q}{p+q+1}.
\]

It follows \([A.3]\). Now we complete the induction by using \([A.3]\).

\[
\det(u_1, \ldots, u_1, u_2, \ldots, u_2, v, \ldots)^\frac{p+q}{p+q+1}
\]

\[
\geq \det(u_1, \ldots, u_1, u_1, v, \ldots)^\frac{p+q-1}{p+q+1} \cdot \det(u_2, \ldots, u_2, u_1, v, \ldots)^\frac{p+q}{p+q+1}
\]

\[
\geq \det(u_1, \ldots, u_1, v, \ldots)^\frac{p+q}{p+q+1}.
\]

\[
\left[ \det(u_2, \ldots, u_2, v, \ldots)^\frac{p+q}{p+q+1} \cdot \det(u_1, \ldots, u_1, v, \ldots)^\frac{p+q}{p+q+1} \right]^{\frac{p+q}{p+q+1}} \cdot \det(u_2, \ldots, u_2, v, \ldots)^\frac{p+q}{p+q+1}.
\]

Then \([A.2]\) is proved. It follows that

\[
\det(u_1, u_2, \ldots, u_2) \geq (\det u_1)^\frac{1}{n} (\det u_2)^\frac{n-1}{n}.
\]
This is the case of (3.4) for \( u_2 = \ldots = u_n = u \). Assume that (3.4) is proved for \( u_{p+1} = \ldots = u_n = u \). We now prove it for \( u_{p+2} = \ldots = u_n = u \). By (A.2) we have

\[
det(u_1, \ldots, u_{p+1}, u, \ldots, u_{n-1}) \geq det(u_1, u_{p+1}) \cdot det(u_1, u_{p+1}, u, \ldots, u_{n-1})
\]

\[
\geq \left[ \left( \det u_1 \right)^\frac{1}{n} \cdots \left( \det u_p \right)^\frac{1}{n} \left( \det u_{p+1} \right)^\frac{1}{n} \right] \cdot \left[ \left( \det u_1 \right)^\frac{1}{n} \cdots \left( \det u_p \right)^\frac{1}{n} \left( \det u_{p+1} \right)^\frac{1}{n} \right] \cdot \left( \det u_{n-1} \right)^\frac{1}{n}.
\]

The induction is complete.

\[ \blacksquare \]

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