Critical behaviour of the compactified $\lambda\phi^4$ theory

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Abstract

We investigate the critical behaviour of the $N$-component Euclidean $\lambda\phi^4$ model at leading order in $\frac{1}{N}$-expansion. We consider it in three situations: confined between two parallel planes a distance $L$ apart from one another, confined to an infinitely long cylinder having a square cross-section of area $A$ and to a cubic box of volume $V$. Taking the mass term in the form $m_0^2 = \alpha(T - T_0)$, we retrieve Ginzburg-Landau models which are supposed to describe samples of a material undergoing a phase transition, respectively in the form of a film, a wire and of a grain, whose bulk transition temperature ($T_0$) is known. We obtain equations for the critical temperature as functions of $L$ (film), $A$ (wire), $V$ (grain) and of $T_0$, and determine the limiting sizes sustaining the transition.

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1. Introduction

Models with fields confined in spatial dimensions play important roles both in field theory and in quantum mechanics. Relevant examples are the Casimir effect and superconducting films, where confinement is carried on by appropriate boundary conditions. For Euclidean field theories, imaginary time and the spatial coordinates are treated exactly on the same footing, so that an extended Matsubara formalism can be applied for dealing with the breaking of invariance along any one of the spatial directions.

Relying on this fact, in the present work we discuss the critical behaviour of the Euclidean $\lambda\phi^4$ model compactified in one, two and three spatial dimensions. We implement the spontaneous symmetry breaking by taking the bare mass coefficient in the Lagrangean parametrized as $m_0^2 = \alpha(T - T_0)$, with $\alpha > 0$ and the parameter $T$ varying in an interval containing $T_0$. With this choice, considering the system confined between two parallel planes a distance $L$ apart from one another, in an infinitely long square cylinder with cross-section area $A = L^2$, and in a cube of volume $V = L^3$, in dimension $D = 3$, we obtain Ginzburg-Landau models describing phase transitions in samples of a material in the form of a film, a wire and a grain, respectively, $T_0$ standing for the bulk transition temperature. Such descriptions apply to physical circumstances where no gauge fluctuations need to be considered.

We start presenting a recapitulation of the general procedure developed in Ref. [1] to treat the massive $(\lambda\phi^4)_D$ theory in Euclidean space, compactified in a $d$-dimensional subspace, with $d \leq D$. This permits to extend to an arbitrary subspace some results in the literature for finite temperature field theory [2] and for the behaviour of field theories in presence of spatial boundaries [1, 3, 4]. We shall consider the vector $N$-component $(\lambda\phi^4)_D$ Euclidean theory at leading order in $\frac{1}{N}$, thus allowing for non-perturbative results, the system being submitted to the constraint of compactification of a $d$-dimensional subspace. After that, besides the review of the situation $d = 1$ (already studied in Refs. [3]), we extend the investigation to the two other particularly interesting cases of $d = 2$ and $d = 3$. These three situations above mentioned correspond respectively to the system confined between parallell...

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planes (a film), confined to an infinitely long cylinder of square cross-section (a wire) and to a finite cubic box (a grain).

For these situations, in the framework of the Ginzburg-Landau model we derive equations for the critical temperature as a function of the confining dimensions. For a film, we show that the critical temperature decreases linearly with the inverse of the film thickness while, for a square cross-section wire and for a cubic grain, we obtain that the critical temperature decreases linearly with the inverse of the square root of the cross-section area $A$ and with the inverse of the cubic root of the grain volume $V$, respectively. In all cases, we are able to calculate the minimal system size (thickness, cross-section area, or volume) below which the phase transition does not take place.

2. The compactified model

In this Section we review the analytical methods of compactification of the $N$-component Euclidean $\lambda \phi^4$ model developed in Ref. [1]. We consider the model described by the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + \frac{1}{2} \tilde{m}^2 \phi_a \phi_a + \frac{\lambda}{N} (\phi_a \phi_a)^2,$$

(1)

in Euclidean $D$-dimensional space, confined to a $d$-dimensional spatial rectangular box of sides $L_j$, $j = 1, 2, ..., d$. In the above equation $\lambda$ is the renormalized coupling constant, $\tilde{m}_0^2$ is a boundary-modified mass parameter depending on \( \{L_i\} i = 1, 2...d \), in such a way that

$$\lim_{\{L_i\} \to \infty} \tilde{m}_0^2(L_1, ..., L_d) = m_0^2(T) \equiv \alpha (T - T_0),$$

(2)

$m_0^2(T)$ being the constant mass parameter present in the usual free-space Ginzburg-Landau model. In Eq. (2), $T_0$ represents the bulk transition temperature. Summation over repeated “color” indices $a$ is assumed. To simplify the notation in the following we drop out the color indices, summation over them being understood in field products. We will work in the approximation of neglecting boundary corrections to the coupling constant. A precise definition of the boundary-modified mass parameter will be given later for the situation of $D = 3$ with $d = 1$, $d = 2$ and $d = 3$, corresponding respectively to a film of thickness $L_1$, to a wire of rectangular section $L_1 \times L_2$ and to a grain of volume $L_1 \times L_2 \times L_3$.

We use Cartesian coordinates $r = (x_1, ..., x_d, z)$, where $z$ is a $(D - d)$-dimensional vector, with corresponding momentum $k = (k_1, ..., k_d, q)$, $q$ being a $(D - d)$-dimensional vector in momentum space. Then the generating functional of correlation functions has the form,

$$Z = \int \mathcal{D} \phi^\dagger \mathcal{D} \phi \exp \left( - \int_0^L d^d r \int d^{D-d} z \mathcal{H}(\phi, \nabla \phi) \right),$$

(3)

where $L = (L_1, ..., L_d)$, and we are allowed to introduce a generalized Matsubara prescription, performing the following multiple replacements (compactification of a $d$-dimensional subspace),

$$\int \frac{dk_i}{2\pi} \rightarrow \frac{1}{L_i} \sum_{n_i = -\infty}^{+\infty} ; \quad k_i \rightarrow \frac{2n_i \pi}{L_i}, \quad i = 1, 2, ..., d.$$  

(4)

A simpler situation is the system confined simultaneously between two parallel planes a distance $L_1$ apart from one another normal to the $x_1$-axis and two other parallel planes, normal to the $x_2$-axis separated by a distance $L_2$ (a “wire” of rectangular section).

We start from the well known expression for the one-loop contribution to the zero-temperature effective potential $[6]$,

$$U_1(\varphi_0) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s} \left[ 12\lambda \varphi_0^2 \right]^s \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + m^2)^s}$$

(5)

where $m$ is the physical mass and $\varphi_0$ is the normalized vacuum expectation value of the field (the classical field). In the following, to deal with dimensionless quantities in the regularization procedures, we introduce parameters

$$c = \frac{m}{2\pi \mu}, \quad b_i = \frac{1}{L_i \mu}, \quad g = \frac{\lambda}{4\pi^2 \mu^{D-2}}, \quad \phi_0^2 = \frac{\varphi_0^2}{\mu^{D-2}}.$$  

(6)
where $\mu$ is a mass scale. In terms of these parameters and performing the replacements [1], the one-loop contribution to the effective potential can be written in the form,

$$
U_1(\phi_0, b_1, ..., b_d) = \mu^D b_1 \cdots b_d \sum_{s=1}^{\infty} \left( \frac{(-1)^s}{2s} \right) \left[ 12g_{\phi_0}^2 \right]^s \sum_{n_1, ..., n_d = -\infty}^{+\infty} \int \frac{d^{D-d}q'}{(b_1^2 n_1^2 + \cdots + b_d^2 n_d^2 + c^2 + q'^2)^s},
$$

(7)

where $q' = q/2\pi \mu$ is dimensionless. Using a well-known dimensional regularization formula [2] to perform the integration over the $(D - d)$ non-compactified momentum variables, we obtain

$$
U_1(\phi_0, b_1, ..., b_d) = \mu^D b_1 \cdots b_d \sum_{s=1}^{\infty} f(D, d, s) \left[ 12g_{\phi_0}^2 \right]^s A_d^2 \left( s - \frac{D - d}{2}; b_1, ..., b_d \right),
$$

(8)

where

$$
f(D, d, s) = \pi^{(D-d)/2} \frac{(-1)^{s+1}}{2s!} \Gamma(s - \frac{D - d}{2})
$$

(9)

and

$$
A_d^2(\nu; b_1, ..., b_d) = \sum_{n_1, ..., n_d = -\infty}^{+\infty} \left( b_1^2 n_1^2 + \cdots + b_d^2 n_d^2 + c^2 \right)^{-\nu} = \frac{1}{e\nu} + 2 \sum_{i=1}^{d} \sum_{n_i = 1}^{\infty} \left( b_i^2 n_i^2 + c^2 \right)^{-\nu} + 2^d \sum_{i < j} \sum_{n_i, n_j = 1}^{\infty} \left( b_i^2 n_i^2 + b_j^2 n_j^2 + c^2 \right)^{-\nu} + \cdots + 2^d \sum_{n_1, ..., n_d = 1}^{\infty} \left( b_1^2 n_1^2 + \cdots + b_d^2 n_d^2 + c^2 \right)^{-\nu}.
$$

(10)

Next we can proceed generalizing to several dimensions the mode-sum regularization prescription described in Ref. [3]. This generalization has been done in [1] and we briefly describe here its principal steps. From the identity,

$$
\frac{1}{\Delta^\nu} = \frac{1}{\Gamma(\nu)} \int_0^{\infty} dt \, e^{\nu \cdot \text{thesis work} -1} e^{-\Delta t},
$$

(11)

and using the following representation for Bessel functions of the third kind, $K_\nu$,

$$
2(a/b)^{\frac{\nu}{2}} K_\nu(2\sqrt{ab}) = \int_0^{\infty} dx \, x^{\nu-1} e^{-(a/x) - bx},
$$

(12)

we obtain after some rather long but straightforward manipulations [1],

$$
A_d^2(\nu; b_1, ..., b_d) = \frac{\nu - \frac{d}{2} + \frac{\nu - \frac{d}{2}}{2} \Gamma \left( \nu - \frac{d}{2} \right)}{b_1 \cdots b_d \Gamma(\nu)} \left[ 2^{\nu - \frac{d}{2} - 1} \right] \left( \frac{\nu}{2} \right)^{d-\nu} + 2 \sum_{i=1}^{d} \sum_{n_i = 1}^{\infty} \left( \frac{n_i}{2\pi c b_i} \right)^{-\nu} K_{\nu - \frac{\nu}{2}} \left( \frac{2\pi c n_i}{b_i} \right) + \cdots + 2^d \sum_{n_1, ..., n_d = 1}^{\infty} \left( \frac{1}{2\pi c} \sqrt{\frac{n_1^2}{b_1^2} + \cdots + \frac{n_d^2}{b_d^2}} \right)^{-\nu} K_{\nu - \frac{\nu}{2}} \left( 2\pi c \sqrt{\frac{n_1^2}{b_1^2} + \cdots + \frac{n_d^2}{b_d^2}} \right).
$$

(13)

Taking $\nu = s - (D - d)/2$ in Eq. (13) and inserting it in Eq. (8), we obtain the one-loop correction to the effective potential in $D$ dimensions with a compactified $d$-dimensional subspace in the form (recovering the dimensionful parameters)

$$
U_1(\phi_0, L_1, ..., L_d) = \sum_{s=1}^{\infty} \left[ 12g_{\phi_0}^2 \right]^s h(D, s) \left[ 2^{s - \frac{d}{2} - 2} \Gamma(s - \frac{D}{2}) m^{D-2s} + \sum_{i=1}^{d} \sum_{n_i = 1}^{\infty} \left( \frac{m}{L_i n_i} \right)^{\frac{\nu}{2} - s} K_{\nu - \frac{\nu}{2}} \left( m \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2} \right) + \cdots + 2^{d-1} \sum_{n_1, ..., n_d = 1}^{\infty} \left( \frac{m}{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right)^{\frac{\nu}{2} - s} K_{\nu - \frac{\nu}{2}} \left( m \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right) \right],
$$

(14)
with

$$h(D, s) = \frac{1}{2^{d/2+s-1} \pi^{D/2}} \frac{(-1)^{s+1}}{s \Gamma(s)}.$$  \number{15}

Criticality is attained when the inverse squared correlation length, $\xi^{-2}(L_1, \ldots, L_d, \varphi_0)$, vanishes in the large-$N$ gap equation,

$$\xi^{-2}(L_1, \ldots, L_d, \varphi_0) = \tilde{m}_0^2(L_1, \ldots, L_d) + \frac{24 \lambda}{L_1 \cdots L_d} \sum_{n_1, \ldots, n_d = -\infty}^{\infty} \int \frac{d^{D-d} q}{(2\pi)^{D-d}} \frac{1}{q^2 + \left(\frac{2m}{L_1}\right)^2 + \cdots + \left(\frac{2m}{L_d}\right)^2 + \xi^{-2}(L_1, \ldots, L_d, \varphi_0)}.$$  \number{16}

where $\varphi_0$ is the normalized vacuum expectation value of the field (different from zero in the ordered phase). In the disordered phase, $\varphi_0$ vanishes and the inverse correlation length equals the physical mass, given below by Eq. (18).

Recalling the condition,

$$\left. \frac{\partial^2}{\partial \varphi^2} U(D, L_1, L_2) \right|_{\varphi=0} = m^2$$  \number{17}

where $U$ is the sum of the tree-level and one-loop contributions to the effective potential (remembering that at the large-$N$ limit it is enough to take the one-loop contribution to the effective potential), we obtain

$$m^2(L_1, \ldots, L_d) = \tilde{m}_0^2(L_1, \ldots, L_d) + \frac{24 \lambda}{(2\pi)^{D/2}} \left[ \sum_{i=1}^{d} \sum_{n_i=1}^{\infty} \left( \frac{m}{L_i n_i} \right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1} (m L_i n_i) \right] + 2 \sum_{i<j=1}^{d} \sum_{n_i, n_j=1}^{\infty} \left( \frac{m}{\sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}} \right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1} \left( m \sqrt{L_i^2 n_i^2 + L_j^2 n_j^2} \right) + \cdots + 2^{d-1} \sum_{n_1, \ldots, n_d=1}^{\infty} \left( \frac{m}{\sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2}} \right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1} \left( m \sqrt{L_1^2 n_1^2 + \cdots + L_d^2 n_d^2} \right).$$  \number{18}

Notice that, in writing Eq. (18), we have suppressed the parcel $2^{-\frac{D}{2}-1} \Gamma(1 - \frac{D}{2}) m^{D-2}$ from its square bracket, the parcel that emerges from the first term in the square bracket of Eq. (17). This expression, which does not depend explicitly on $L_i$, diverges for $D$ even due to the poles of the gamma function; in this case, this parcel is subtracted to get a renormalized mass equation. For $D$ odd, $\Gamma(1 - \frac{D}{2})$ is finite but we also subtract this term (corresponding to a finite renormalization) for sake of uniformity; besides, for $D \geq 3$, the factor $m^{D-2}$ does not contribute in the criticality.

The vanishing of Eq. (18) defines criticality for our compactified system. We claim that, taking $d = 1$, $d = 2$, and $d = 3$ with $D = 3$, we are able to describe respectively the critical behaviour of samples of materials in the form of films, wires and grains. Notice that the parameter $m$ in the right hand side of Eq. (18) is the boundary-modified mass $m(L_1, \ldots, L_d)$, which means that Eq. (18) is a self-consistency equation, a very complicated modified Schwinger-Dyson equation for the mass, not soluble by algebraic means. Nevertheless, as we will see in the next sections, a solution is possible at criticality, which allows us to obtain a closed formula for the boundary-dependent critical temperature.

3. Critical behaviour for films

We now consider the simplest particular case of the compactification of only one spatial dimension, the system confined between two parallel planes a distance $L$ apart from one another. This case has been already considered in Ref. [3], and we also analyze it here for completeness. Thus, from Eq. (18), taking $d = 1$, we get in the disordered phase

$$m^2(L) = \tilde{m}_0^2(L) + \frac{24 \lambda}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left( \frac{m}{n L} \right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1} (n L m),$$  \number{19}
where \( L (= L_1) \) is the separation between the planes, the film thickness. If we limit ourselves to the neighbourhood of criticality \( (m^2 \approx 0) \) and consider \( L \) finite and sufficiently small, we may use an asymptotic formula for small values of the argument of Bessel functions,

\[
K_\nu(z) \approx \frac{1}{2} \Gamma(\nu) \left( \frac{z}{2} \right)^{-\nu} \quad (z \approx 0),
\]

and Eq. (19) reduces, for \( D > 3 \), to

\[
m^2(L) \approx m_0^2(L) + \frac{6\lambda}{\pi^{D/2} L^{D-2}} \Gamma\left( \frac{D}{2} - 1 \right) \zeta(D-2)
\]

where \( \zeta(D-2) \) is the Riemann \( \zeta \)-function, defined for \( \text{Re}\{D-2\} > 1 \) by the series

\[
\zeta(D-2) = \sum_{n=1}^{\infty} \frac{1}{n^{D-2}}.
\]

(22)

It is worth mentioning that for \( D = 4 \), taking \( m^2(L) = 0 \) and making the appropriate changes \( (L \rightarrow \beta, \lambda \rightarrow \lambda/4!) \), Eq. (21) is formally identical to the high-temperature (low values of \( \beta \)) critical equation obtained in Ref. [9], thus providing a check of our calculations.

For \( D = 3 \), Eq. (28) can be made physically meaningful by a regularization procedure as follows. We consider the analytic continuation of the \( \zeta \)-function, leading to a meromorphic function having only one simple pole at \( z = 1 \), which satisfies the reflection formula

\[
\zeta(z) = \frac{1}{\Gamma(z/2)} \Gamma\left( \frac{1-z}{2} \right) \pi^{z-1/2} \zeta(1-z).
\]

(23)

Next, remembering the formula,

\[
\lim_{z \rightarrow 1} \left[ \zeta(z) - \frac{1}{z-1} \right] = \gamma,
\]

(24)

where \( \gamma \approx 0.5772 \) is the Euler-Mascheroni constant, we define the \( L \)-dependent bare mass for \( D \approx 3 \), in such a way that the pole at \( D = 3 \) in Eq. (21) is suppressed, that is we take

\[
m_0^2(L) \approx M - \frac{1}{(D-3) \pi L} \frac{6\lambda}{\pi^{D/2} L^{D-2}} \Gamma\left( \frac{D}{2} - 1 \right) \zeta(D-2)
\]

(25)

where \( M \) is independent of \( D \). To fix the finite term, we make the simplest choice satisfying

\[
M = m_0^2(T) = \alpha (T - T_0),
\]

(26)

\( T_0 \) being the bulk critical temperature. In this case, using Eq. (25) in Eq. (21) and taking the limit as \( D \rightarrow 3 \), the \( L \)-dependent renormalized mass term in the vicinity of criticality becomes

\[
m^2(L) \approx \alpha (T - T_c(L)),
\]

(27)

where the modified, \( L \)-dependent, transition temperature is given by

\[
T_c(L) = T_0 - C_1 \frac{\lambda}{\alpha L},
\]

(28)

\( L \) being the thickness of the film, with the constant \( C_1 \) given by

\[
C_1 = \frac{6\gamma}{\pi} \approx 1.1024.
\]

(29)

From this equation, we see that for \( L \) smaller than

\[
L_{\text{min}} = C_1 \frac{\lambda}{\alpha T_0},
\]

(30)
4. Critical behaviour for wires

We now focus on the situation where two spatial dimensions are compactified. From Eq. (13), taking $d = 2$, we get (in the disordered phase)

$$m^2(L_1, L_2) = m_0^2(L_1, L_2) + \frac{24\lambda}{(2\pi)^{D/2}} \left[ \sum_{n=1}^{\infty} \left( \frac{m}{nL_1} \right)^{D-1} K_{D-1}(nL_1m) + \sum_{n=1}^{\infty} \left( \frac{m}{nL_2} \right)^{D-1} K_{D-1}(nL_2m) \right] + 2 \sum_{n_1,n_2=1}^{\infty} \left( \frac{m}{\sqrt{L_1^2n_1^2 + L_2^2n_2^2}} \right)^{D-1} K_{D-1}(m\sqrt{L_1^2n_1^2 + L_2^2n_2^2}) \right].$$

(31)

If we limit ourselves to the neighborhood of criticality, $m^2 \approx 0$, and taking both $L_1$ and $L_2$ finite and sufficiently small, we may use Eq. (20) to rewrite Eq. (31) as

$$m^2(L_1, L_2) \approx m_0^2(L_1, L_2) + \frac{6\lambda}{\pi^{D/2}} \Gamma \left( \frac{D}{2} - 1 \right) \times \left[ \left( \frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \zeta(D - 2) + 2E_2 \left( \frac{D}{2} - 1 \right) \right],$$

(32)

where $E_2 \left( \frac{D-2}{2}; L_1, L_2 \right)$ is the generalized (multidimensional) Epstein zeta-function defined by

$$E_2 \left( \frac{D-2}{2}; L_1, L_2 \right) = \sum_{n_1, n_2 = 1}^{\infty} \left[ L_1^2n_1^2 + L_2^2n_2^2 \right]^{-\left( \frac{D-2}{2} \right)},$$

(33)

for $\text{Re}\{D\} > 3$.

As mentioned before, the Riemann zeta-function $\zeta(D - 2)$ has an analytical extension to the whole complex $D$-plane, having an unique simple pole (of residue 1) at $D = 3$. One can also construct analytical continuations (and recurrence relations) for the multidimensional Epstein functions which permit to write them in terms Kelvin and Riemann zeta functions. To start one considers the analytical continuation of the Epstein-Hurwitz zeta-function given by

$$\sum_{n=1}^{\infty} (n^2 + p^2)^{-\nu} = -\frac{1}{2} p^{-2\nu} + \frac{\sqrt{\pi}}{2\Gamma(\nu)} \left[ \Gamma \left( \nu - \frac{1}{2} \right) + 4 \sum_{n=1}^{\infty} (\pi pn)^{-\nu} K_{\nu - \frac{1}{2}}(2\pi pn) \right].$$

(34)

Using this relation to perform one of the sums in (33) leads immediately to the question of which sum is firstly evaluated. As it is done in Ref. 10, whatever the sum one chooses to perform firstly, the manifest $L_1 \leftrightarrow L_2$ symmetry of Eq. (33) is lost; in order to preserve this symmetry, we adopt here a symmetrized summation. Generalizing the prescription introduced in 3, we consider the multidimensional Epstein function defined as the symmetrized summation

$$E_d(\nu; L_1, \ldots, L_d) = \frac{1}{d!} \sum_{\sigma} \sum_{n_1 = 1}^{\infty} \cdots \sum_{n_d = 1}^{\infty} \left[ \sum_{i=1}^{d} \sigma_i^2 n_i^2 + \cdots + \sigma_d^2 n_d^2 \right]^{-\nu},$$

(35)

where $\sigma_i = \sigma(L_i)$, with $\sigma$ running in the set of all permutations of the parameters $L_1, \ldots, L_d$, and the summations over $n_1, \ldots, n_d$ being taken in the given order. Applying (34) to perform the sum over $n_d$, one gets

$$E_d(\nu; L_1, \ldots, L_d) = -\frac{1}{2d} \sum_{i=1}^{d} E_{d-1}(\nu, \ldots, \tilde{L}_i, \ldots) \left[ \frac{\sqrt{\pi}}{2d \Gamma(\nu)} \Gamma \left( \frac{\nu - 1}{2} \right) \sum_{i=1}^{d} \frac{1}{L_i} E_{d-1}(\nu - \frac{1}{2}, \ldots, \tilde{L}_i, \ldots) + \frac{2\sqrt{\pi}}{d \Gamma(\nu)} W_d \left( \nu - \frac{1}{2}, L_1, \ldots, L_d \right) \right].$$

(36)
where the hat over the parameter \( L_i \) in the functions \( E_{d-1} \) means that it is excluded from the set \{ \( L_1, ..., L_d \) \} (the others being the \( d-1 \) parameters of \( E_{d-1} \)), and

\[
W_d(\eta; L_1, ..., L_d) = \sum_{i=1}^{d} \frac{1}{L_i} \sum_{n_1, ..., n_d=1}^{\infty} \left( \frac{\pi n_i}{L_i \sqrt{\sum n_i^2 + \sum \cdot \cdot \cdot}} \right)^{\eta} K_{\eta} \left( \frac{2\pi n_i}{L_i} \sqrt{\sum n_i^2 + \sum \cdot \cdot \cdot} \right), \tag{37}
\]

with \((\sum \cdot \cdot \cdot \sum \cdot \cdot \cdot)\) representing the sum \( \sum_{j=1}^{d} L_j^2 n_j^2 \). In particular, noticing that \( E_{1}(\nu; L_j) = L_j^{-2
u}\zeta(2\nu) \), one finds

\[
E_2 \left( \frac{D-2}{2}; L_1^2, L_2^2 \right) = -\frac{1}{4} \left( \frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \zeta(D-2) + \frac{\sqrt{\pi} \Gamma(D-3)}{4 \Gamma(D-2)} \left( \frac{1}{L_1 L_2^{D-3}} + \frac{1}{L_1^{D-3} L_2} \right) \zeta(D-3) \tag{38}
\]

Using the above expression, the Eq. (32) can be rewritten as

\[
m^2(L_1, L_2) \approx \tilde{m}_0^2(L_1, L_2) + \frac{3\lambda}{\pi D/2} \left[ \left( \frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \Gamma \left( \frac{D-2}{2} \right) \zeta(D-2) + \sqrt{\pi} \left( \frac{1}{L_1 L_2^{D-3}} + \frac{1}{L_1^{D-3} L_2} \right) \Gamma \left( \frac{D-3}{2} \right) \zeta(D-3) + 2\sqrt{\pi} W_2 \left( \frac{D-3}{2}; L_1, L_2 \right) \right]. \tag{39}
\]

This equation presents no problems for \( 3 < D < 4 \) but, for \( D = 3 \), the first and second terms between brackets of Eq. (39) are divergent due to the \( \zeta \)-function and \( \Gamma \)-function, respectively. We can deal with divergences remembering the property in Eq. (24) and using the expansion of \( \Gamma(D/2) \) around \( D = 3 \),

\[
\Gamma \left( \frac{D-3}{2} \right) \approx \frac{2}{D-3} + \Gamma' \tag{40}
\]

\( \Gamma'(z) \) standing for the derivative of the \( \Gamma \)-function with respect to \( z \). For \( z = 1 \) it coincides with the Euler digamma-function \( \psi(1) \), which has the particular value \( \psi(1) = -\gamma \). We notice however, that differently from the case treated in the previous section, where it was necessary to apply a renormalization procedure, here the two divergent terms generated by the use of formulas (24) and (10) cancel exactly between them. No renormalization is needed. Thus, for \( D = 3 \), taking the bare mass given by \( \tilde{m}_0^2(L_1, L_2) = \alpha (T - T_0) \), we obtain the renormalized boundary-dependent mass term in the form

\[
m^2(L_1, L_2) \approx \alpha (T - T_c(L_1, L_2)) \tag{41}
\]

with the boundary-dependent critical temperature given by

\[
T_c(L_1, L_2) = T_0 - \frac{9\lambda}{2\pi \alpha} \left( \frac{1}{L_1} + \frac{1}{L_2} \right) - \frac{6\lambda}{\pi \alpha} W_2(0; L_1, L_2), \tag{42}
\]

where

\[
W_2(0; L_1, L_2) = \sum_{n_1, n_2=1}^{\infty} \left\{ \frac{1}{L_1} K_0 \left( 2\pi L_2 n_1 n_2 \right) + \frac{1}{L_2} K_0 \left( 2\pi L_1 n_1 n_2 \right) \right\}. \tag{43}
\]

The quantity \( W_2(0; L_1, L_2) \), appearing in Eq. (42), involves complicated double sums, very difficult to handle for \( L_1 \neq L_2 \); in particular, it is not possible to take limits such as \( L_i \to \infty \). For this reason we will restrict ourselves to the case \( L_1 = L_2 \). For a wire with square cross-section, we have \( L_1 = L_2 = L = \sqrt{A} \) and Eq. (42) reduces to

\[
T_c(A) = T_0 - C_2 \frac{\lambda}{\alpha \sqrt{A}}, \tag{44}
\]

where \( C_2 \) is a constant given by

\[
C_2 = \frac{9\gamma}{\pi} + \frac{12}{\pi} \sum_{n_1, n_2=1}^{\infty} K_0(2\pi n_1 n_2) \approx 1.6571. \tag{45}
\]
We see that the critical temperature of the square wire depends on the bulk critical temperature and the Ginzburg-Landau parameters $\alpha$ and $\lambda$ (which are characteristics of the material constituting the wire), and also on the area of its cross-section. Since $T_c$ decreases linearly with the inverse of the side of the square, this suggests that there is a minimal area for which $T_c(A_{\min}) = 0$,

$$A_{\min} = \left(\frac{C_2}{\alpha T_0}\right)^2; \quad (46)$$

for square wires of cross-section areas smaller than this value, in the context of our model the transition should be suppressed. On topological grounds, we expect that (apart from appropriate coefficients) our result should be independent of the cross-section shape of the wire, at least for cross-sectional regular polygons.

5. Critical behaviour for grains

We now turn our attention to the case where all three spatial dimensions are compactified, corresponding to the system confined in a box of sides $L_1, L_2, L_3$. Taking $d = 3$ in Eq. (18) and using Eq. (20), we obtain (for sufficiently small $L_1, L_2, L_3$ and in the neighbourhood of classicality, $m^2 \approx 0$)

$$m^2(L_1, L_2, L_3) \approx m_0^2(L_1, L_2, L_3) + \frac{6\lambda}{\pi^{D/2}} \Gamma \left(\frac{D-2}{2}\right) \left[\sum_{i=1}^{3} \frac{\zeta(D-2)}{L_i^{D-2}} + 2 \sum_{i<j=1}^{3} E_2 \left(\frac{D-2}{2}; L_i, L_j\right) + 4 E_3 \left(\frac{D-2}{2}; L_1, L_2, L_3\right)\right], \quad (47)$$

where $E_3(\nu; L_1, L_2, L_3) = \sum_{n_1, n_2, n_3=1}^{\infty} \left[L_1^2 n_1^2 + L_2^2 n_2^2 + L_3^2 n_3^2\right]^{-\nu}$ and the functions $E_2$ are given by Eq. (38).

The analytical structure of the function $E_3 \left(\frac{D-2}{2}; L_1, L_2, L_3\right)$ can be obtained from the general symmetrized recurrence relation given by Eqs. (46–47); explicitly, one has

$$E_3 \left(\frac{D-2}{2}; L_1, L_2, L_3\right) = -\frac{1}{6} \sum_{i<j=1}^{3} E_2 \left(\frac{D-2}{2}; L_i, L_j\right) + \frac{\sqrt{\pi} \Gamma(D-3)}{6 \Gamma(D-2)} \sum_{i,j,k=1}^{3} \frac{1 + \varepsilon_{ijk}}{L_i} E_2 \left(\frac{D-2}{2}; L_j, L_k\right)$$

$$+ \frac{2\sqrt{\pi}}{3 \Gamma(D-2)} W_3 \left(\frac{D-3}{2}; L_1, L_2, L_3\right), \quad (48)$$

where $\varepsilon_{ijk}$ is the totally antisymmetric symbol and the function $W_3$ is a particular case of Eq. (37). Using Eqs. (38) and (48), the boundary dependent mass can be written as

$$m^2(L_1, L_2, L_3) \approx m_0^2(L_1, L_2, L_3) + \frac{6\lambda}{\pi^{D/2}} \left[\frac{1}{3} \Gamma \left(\frac{D-2}{2}\right) \sum_{i=1}^{3} \frac{1}{L_i^{D-2}} \zeta(D-2)$$

$$+ \frac{\sqrt{\pi}}{6} \zeta(D-3) \sum_{i<j=1}^{3} \left(\frac{1}{L_i^{D-3} L_j} + \frac{1}{L_j^{D-3} L_i}\right) \Gamma \left(\frac{D-3}{2}\right) + \frac{4\sqrt{\pi}}{3} \sum_{i<j=1}^{3} \frac{3}{2} W_2 \left(\frac{D-3}{2}; L_i, L_j\right)$$

$$+ \frac{\pi}{6} \zeta(D-4) \Gamma \left(\frac{D-4}{2}\right) \sum_{i,j,k=1}^{3} \frac{1 + \varepsilon_{ijk}}{2} \frac{1}{L_i} \left(\frac{1}{L_j^{D-4} L_k} + \frac{1}{L_k^{D-4} L_j}\right)$$

$$+ \frac{2\pi}{3} \sum_{i,j,k=1}^{3} \frac{1 + \varepsilon_{ijk}}{2} \frac{1}{L_i} W_2 \left(\frac{D-4}{2}; L_j, L_k\right) + \frac{8\sqrt{\pi}}{3} W_3 \left(\frac{D-3}{2}; L_1, L_2, L_3\right)\right]. \quad (49)$$

The first two terms in the square bracket of Eq. (48) diverge as $D \to 3$ due to the poles of the $\Gamma$ and $\zeta$-functions. However, as it happens in the case of wires, using Eqs. (24) and (40), it can be shown that these divergences cancel
exactly one another. After some simplifications, for \( D = 3 \), the boundary dependent mass \( \tilde{m}_0^2(L_1, L_2, L_3) \) becomes
\[
\begin{align*}
m^2(L_1, L_2, L_3) & \approx \tilde{m}_0^2(L_1, L_2, L_3) + \frac{6\lambda}{\pi} \left[ \frac{\gamma}{2} \sum_{i=1}^{3} \frac{1}{T_i} + \frac{4}{3} \sum_{i<j}^{3} W_2(0; L_i, L_j) + \frac{\pi}{18} \sum_{i,j,k=1}^{3} \frac{(1 + \varepsilon_{ijk})}{2} \frac{L_i}{L_j L_k} \right] \\
& \quad + \frac{2\sqrt{\pi}}{3} \sum_{i,j,k=1}^{3} \left( 1 + \varepsilon_{ijk} \right) \frac{1}{L_i} W_2 \left( -\frac{1}{2}; L_j, L_k \right) + \frac{8}{3} W_3(0; L_1, L_2, L_3) \right].
\end{align*}
\]

As before, since no divergences need to be suppressed, we can take the bare mass given by \( \tilde{m}_0^2(L_1, L_2, L_3) = \alpha(T - T_0) \) and rewrite the renormalized mass as \( m^2(L_1, L_2, L_3) \approx \alpha(T - T_c(L_1, L_2, L_3)) \). The expression of \( T_c(L_1, L_2, L_3) \) can be easily obtained from Eq. (50), but it is a very complicated formula, involving multiple sums, which makes almost impossible a general analytical study for arbitrary parameters \( L_1, L_2, L_3 \); thus, we restrict ourselves to the situation where \( L_1 = L_2 = L_3 = L \), corresponding to a cubic box of volume \( V = L^3 \). In this case, the boundary dependent critical temperature reduces to
\[
T_c(V) = T_0 - C_3 \frac{\lambda}{\alpha V^{1/3}},
\]
where the constant \( C_3 \) is given by (using that \( K_{-\frac{3}{2}}(z) = \sqrt{\frac{2\pi}{z^2}} \)\( e^{-z} \))
\[
C_3 = 1 + \frac{9\gamma}{\pi} + \frac{12}{\pi} \sum_{n_1, n_2=1}^{\infty} \frac{e^{-2\pi n_1 n_2}}{n_1} \left( 1 + \frac{48}{\pi} \sum_{n_1, n_2=1}^{\infty} K_0(2\pi n_1 n_2) + \frac{48}{\pi} \sum_{n_1, n_2, n_3=1}^{\infty} K_0 \left( 2\pi n_1 \sqrt{n_2^2 + n_3^2} \right) \right) \approx 2.7657.
\]
One sees that the minimal volume of the cubic grain sustaining the transition is
\[
V_{\text{min}} = \left( C_3 \frac{\lambda}{\alpha T_0} \right)^3.
\]

6. Conclusions

In this paper we have discussed the spontaneous symmetry breaking of the \( \langle \lambda \phi^4 \rangle_D \) theory compactified in \( d \leq D \) Euclidean dimensions, extending some results of Ref. 1. We have parametrized the bare mass term in the form \( m_0^2(T - T_0) \), thus placing the analysis within the Ginzburg-Landau framework. We focused in the situations with \( D = 3 \) and \( d = 1, 2, 3 \), corresponding (in the context of condensed matter systems) to films, wires and grains, respectively, undergoing phase transitions which may be described by (mean-field) Ginzburg-Landau models. This generalizes to more compactified dimensions previous investigations on the superconducting transition in films, both without 2 and in the presence of a magnetic field 11. In all cases studied here, in the absence of gauge fluctuations, we found that the boundary-dependent critical temperature decreases linearly with the inverse of the linear dimension \( L \), \( T_c(L) = T_0 - C_d \lambda/\alpha L \) where \( \alpha \) and \( \lambda \) are the Ginzburg-Landau parameters, \( T_0 \) is the bulk transition temperature and \( C_d \) is a constant equal to 1.1024, 1.6571 and 2.6757 for \( d = 1 \) (film), \( d = 2 \) (square wire) and \( d = 3 \) (cubic grain), respectively. Such behaviour suggests the existence of a minimal size of the system below which the transition is suppressed.

These findings seems to be in qualitative agreement with results for the existence of a minimal thickness for disappearance of superconductivity in films 12,13,14,15. Also, experimental investigations in nanowires searching to establish whether there is a limit to how thin a superconducting wire can be, while retaining its superconducting character, have also drawn the attention of researchers; for example, in Ref. 17 the behaviour of nanowires has been studied. Similar questions have also been rised concerning the behaviour of superconducting nanograins 17,18. Nevertheless, an important point to be emphasized is that our results are obtained in a field-theoretical framework.
and do not depend on microscopic details of the material involved nor account for the influence of manufacturing aspects of the sample; in other words, our results emerge solely as a topological effect of the compactification of the Ginzburg-Landau model in a subspace. Detailed microscopic analysis is required if one attempts to account quantitatively for experimental observations which might deviate from our mean field results.

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