ON T-NORMED INTEGRALS WITH RESPECT TO POSSIBILITY CAPACITIES ON COMPACTA

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Abstract. Riesz Theorem establishes a correspondence between the set of \( \sigma \)-additive regular Borel measures and the set of linear positively defined functionals. We consider an idempotent analogue of this correspondence between possibility capacities and functionals preserving the maximum operation and t-norm operation using t-normed integrals.

1. Introduction

Capacities (non-additive measures, fuzzy measures) were introduced by Choquet in [3] as a natural generalization of additive measures. They found numerous applications (see for example [5], [8], [24]). Capacities on compacta were considered in [14] where the important role plays the upper-semicontinuity property which connects the capacity theory with the topological structure. Categorical and topological properties of spaces of upper-semicontinuous normed capacities on compact Hausdorff spaces were investigated in [17]. In particular, there was built the capacity functor which is a functorial part of a capacity monad \( \mathbb{M} \).

In fact, the most of applications of non-additive measures to game theory, decision making theory, economics etc deal not with measures as set functions but with integrals which allow to obtain expected utility or expected pay-off. Several types of integrals with respect to non-additive measures were developed for different purposes (see for example books [11] and [4]). Such integrals are called fuzzy integrals. The most known are the Choquet integral based on the addition and the multiplication operations [3] and the Sugeno integral based on the maximum and the minimum operations [26]. If we change the minimum operation by any t-norm, we obtain the generalization of the Sugeno integral called t-normed integrals [25].

One of the important problems of the fuzzy integrals theory is characterization of integrals as functionals on some function space (see for example subchapter 4.8 in [11] devoted to characterizations of the Choquet integral and the Sugeno integral). Characterizations of t-normed integrals were discussed in [2], [18] and [23]. In fact these theorems we can consider as non-additive and non-linear analogues of well-known Riesz Theorem about a correspondence between the set of \( \sigma \)-additive regular Borel measures and the set of linear positively defined functionals.

The class of all capacities contains an important subclass of possibility capacities. By the definition a value of a possibility capacity of union of two sets is equal to maximum of values of capacities of these sets. We prove in Section 2 of this paper that the set of t-normed integrals with respect to possibility capacities is equal to...
the set of functionals which preserve the maximum and t-norms operations which are considered in [27] under name *-measures. Since the maximum operation in idempotent mathematics plays the role of addition, we can consider the maximum preserving property as an idempotent analogue of additivity. Hence we can consider possibility measures as idempotent analogue of probability measures and the above mentioned set of functionals as an idempotent analogue of the set linear positively defined functionals in Riesz Theorem.

Possibility capacities form a submonad of the capacity monad [12]. The structure of this monad is based on the maximum and minimum operations. A monad on possibility measures based on the maximum and t-norm operations was in fact considered in [16] (using a more general framework). Let us remark that not all such monads can be extended to the space of all capacities [19]. On the other hand Zarichnyi proposed to use triangular norms to construct monads on the spaces of functionals which preserve the maximum and t-norms operations [31]. We prove in Section 3 that the correspondence from Section 2 is an isomorphism of corresponding monads.

Let us remark that each monad structure leads to some abstract convexity [20]. Convexity structures are widely used to prove existence of fixed points and equilibria (see for example [10], [1], [13], [18], [21], [22]). In Section 4 we consider convexity and barycenter map generated by a t-norm.

2. Capacities: preliminaries

In what follows, all spaces are assumed to be compacta (compact Hausdorff space) except for \( \mathbb{R} \) and the spaces of continuous functions on a compactum. All maps are assumed to be continuous. By \( F(X) \) we denote the family of all closed subsets of a compactum \( X \). We shall denote the Banach space of continuous functions on a compactum \( X \) endowed with the sup-norm by \( C(X) \). For any \( c \in \mathbb{R} \) we shall denote the constant function on \( X \) taking the value \( c \) by \( c_X \). We also consider the natural lattice operations \( \vee \) and \( \wedge \) on \( C(X) \) and its sublattices \( C(X,[0, +\infty)) \) and \( C(X,[0, 1]) \).

We need the definition of capacity on a compactum \( X \). We follow a terminology of [17].

**Definition 1.** A function \( \nu : F(X) \to [0, 1] \) is called an upper-semicontinuous capacity on \( X \) if the three following properties hold for each closed subsets \( F \) and \( G \) of \( X \):

1. \( \nu(\emptyset) = 0 \),
2. if \( F \subset G \), then \( \nu(F) \leq \nu(G) \),
3. if \( \nu(F) < a \) for \( a \in [0, 1] \), then there exists an open set \( O \supset F \) such that \( \nu(O) < a \) for each compactum \( B \subset O \).

If \( F \) is a one-point set we use a simpler notation \( \nu(a) \) instead \( \nu(\{a\}) \). A capacity \( \nu \) is extended in [17] to all open subsets \( U \subset X \) by the formula

\[
\nu(U) = \sup \{ \nu(K) \mid K \text{ is a closed subset of } X \text{ such that } K \subset U \}.
\]

It was proved in [17] that the space \( MX \) of all upper-semicontinuous capacities on a compactum \( X \) is a compactum as well, if a topology on \( MX \) is defined by a subbase that consists of all sets of the form \( O_-(F, a) = \{ c \in MX \mid c(F) < a \} \), where \( F \) is a closed subset of \( X \), \( a \in [0, 1] \), and \( O_+(U, a) = \{ c \in MX \mid c(U) > a \} \), where \( U \) is an open subset of \( X \), \( a \in [0, 1] \). Since all capacities we consider here are upper-semicontinuous, in the following we call elements of \( MX \) simply capacities.

**Definition 2.** A capacity \( \nu \in MX \) for a compactum \( X \) is called a necessity (possibility) capacity if for each family \( \{A_i\}_{i \in \mathcal{T}} \) of closed subsets of \( X \) (such that \( \bigcup_{i \in \mathcal{T}} A_i \),
is a closed subset of $X$ we have
\[
\nu(\bigcap_{t \in T} A_t) = \inf_{t \in T} \nu(A_t)
\]
\[
(\nu(\bigcup_{t \in T} A_t) = \sup_{t \in T} \nu(A_t)).
\]
(See [28] for more details.)

We denote by $NX$ (PIX) a subspace of $MX$ consisting of all necessity (possibility) capacities. Since $X$ is compact and $\nu$ is upper-semicontinuous, $\nu \in NX$ iff $\nu$ satisfies the simpler requirement that $\nu(A \cap B) = \min\{\nu(A), \nu(B)\}$.

If $\nu$ is a capacity on a compactum $X$, then the function $\kappa_X(\nu)$, that is defined on the family $\mathcal{F}(X)$ by the formula $\kappa_X(\nu)(F) = 1 - \nu(X \setminus F)$, is a capacity as well. It is called the dual capacity (or conjugate capacity) to $\nu$. The mapping $\kappa_X : MX \to MX$ is a homeomorphism and an involution [17]. Moreover, $\nu$ is a necessity capacity if and only if $\kappa_X(\nu)$ is a possibility capacity. This implies in particular that $\nu \in \Pi X$ iff $\nu$ satisfies the simpler requirement that $\nu(A \cup B) = \max\{\nu(A), \nu(B)\}$. It is easy to check that $NX$ and $\Pi X$ are closed subsets of $MX$.

3. T-NORMED INTEGRALS WITH RESPECT TO POSSIBILITY CAPACITIES

Remind that a triangular norm $*$ is a binary operation on the closed unit interval $[0,1]$ which is associative, commutative, monotone and $s*1 = s$ for each $s \in [0,1]$ [15]. Let us remark that the monotonicity of $*$ implies distributivity, i.e. $(t \vee s)*l = (t*l) \vee (s*l)$ for each $t, s, l \in [0,1]$. We consider only continuous t-norms in this paper.

Integrals generated by t-norms are called t-normed integrals and were studied in [29], [30] and [25]. Denote $\varphi_t = \varphi^{-1}([t,1])$ for each $\varphi \in C(X,[0,1])$ and $t \in [0,1]$. So, for a continuous t-norm $*$, a capacity $\mu$ and a function $f \in C(X,[0,1])$ the corresponding t-normed integral is defined by the formula
\[
\int_X f d\mu = \max\{\mu(f_t) * t \mid t \in [0,1]\}.
\]

Let $X$ be a compactum. We call two functions $\varphi, \psi \in C(X,[0,1])$ comonotone (or equiordered) if $(\varphi(x_1) - \varphi(x_2)) \cdot (\psi(x_1) - \psi(x_2)) \geq 0$ for each $x_1, x_2 \in X$. Let us remark that a constant function is comonotone to any function $\psi \in C(X,[0,1])$.

Let $*$ be a continuous t-norm. We denote for a compactum $X$ by $T^*(X)$ the set of functionals $\mu : C(X,[0,1]) \to [0,1]$ which satisfy the conditions:

1. $\mu(1_X) = 1$;
2. $\mu(\psi \vee \varphi) = \mu(\psi) \vee \mu(\varphi)$ for each comonotone functions $\varphi, \psi \in C(X,[0,1])$;
3. $\mu(c \varphi * \varphi) = c * \mu(\varphi)$ for each $c \in [0,1]$ and $\varphi \in C(X,[0,1])$.

It was proved in [23] for the general case that a functional $\mu$ on $C(X,[0,1])$ belongs to $T^*(X)$ if and only if there exists a unique capacity $\nu$ such that $\mu$ is the t-normed integral with respect to $\nu$.

Following [27] we call a functional $\mu \in T^*(X)$ a $*$-measure if

1. $\mu(1_X) = 1$;
2. $\mu(\psi \vee \varphi) = \mu(\psi) \vee \mu(\varphi)$ for each functions $\varphi, \psi \in C(X,[0,1])$;
3. $\mu(c \varphi * \varphi) = c * \mu(\varphi)$ for each $c \in [0,1]$ and $\varphi \in C(X,[0,1])$.

We consider $T^*(X)$ as a subspace of the space $[0,1]^{C(X,[0,1])}$ with the product topology. We denote by $A^*(X)$ the subspace of all $*$-measures in $T^*(X)$.

**Theorem 1.** Let $\mu \in T^*(X)$. Then $\mu \in A^*(X)$ if and only if there exists a unique $\nu \in \Pi X$ such that $\mu(\varphi) = \int_X f d\nu$ for each $f \in C(X,[0,1])$. 
Proof. Necessity. We can choose any \( \nu \in M(X) \) such that \( \mu(f) = \int_X^\nu f \, dv \) for each \( f \in C(X, [0, 1]) \) by the above mentioned characterization of the t-normed integral from \([13]\). Moreover, we have

\[
\nu(A) = \inf \{ \mu(\varphi) \mid \varphi \in C(X, [0, 1]) \text{ with } \varphi \geq \chi_A \}
\]

for each closed subset \( A \) of \( X \) ([13]) (by \( \chi_A \) we denote the characteristic function of the set \( A \)). We have to show that \( \nu \in \Pi X \).

Suppose the contrary. Then there exist two closed subsets \( A \) and \( B \) of \( X \) such that \( \nu(A \cup B) > \nu(A) \lor \nu(B) \). We can choose functions \( \varphi, \psi \in C(X, [0, 1]) \) such that \( \varphi \geq \chi_A, \psi \geq \chi_B \) and \( \nu(A \cup B) > \mu(\varphi) \lor \mu(\psi) \). Since \( \mu \in A^*(X) \), we have \( \mu(\varphi) \lor \mu(\psi) = \mu(\varphi \lor \psi) \). But \( \varphi \lor \psi \geq \chi_{A \lor B} \), hence \( \mu(\varphi \lor \psi) > \nu(A \cup B) \) and we obtain a contradiction.

 Sufficiency. Let \( \nu \in \Pi X \) such that \( \mu(f) = \int_X^\nu f \, dv \) for each \( f \in C(X, [0, 1]) \). Take any functions \( \varphi, \psi \in C(X, [0, 1]) \). Evidently, we have \( (\varphi \lor \psi)_t = \varphi_t \lor \psi_t \) for each \( t \in [0, 1] \). Since \( \nu \in \Pi X \), we obtain \( \nu(\varphi \lor \psi)_t = (\nu(\varphi)_t \lor \nu(\psi)_t)_t = (\nu(\varphi)_t) \lor (\nu(\psi)_t)_t \). Since \( \ast \) is distributive, we have \( (\nu(\varphi_t) \lor \nu(\psi_t))_t \ast t = (\nu(\varphi)_t \lor \nu(\psi)_t) \ast t \leq \int_X \varphi d\nu \lor \int_X \psi d\nu \). Hence \( f_X^\nu \varphi \lor \psi d\nu \leq \int_X^\nu \varphi d\nu \lor \int_X^\nu \psi d\nu \). Inverse inequality follows from the obvious monotonicity of t-normed integral. Hence \( \mu \in A^*(X) \). \( \square \)

4. A MORPHISM OF MONADS

The main aim of this section is to show that the correspondence obtained in the previous section is a monad morphism. By \( \textbf{Comp} \) we denote the category of compact Hausdorff spaces (compacta) and continuous maps. We recall the notion of monad (or triple) in the sense of S.Eilenberg and J.Moore [6]. We define it only for the category \( \textbf{Comp} \).

A monad \( \mathcal{E} = (E, \eta, \mu) \) in the category \( \textbf{Comp} \) consists of an endofunctor \( E : \textbf{Comp} \rightarrow \textbf{Comp} \) and natural transformations \( \eta : \text{Id}_{\textbf{Comp}} \rightarrow E \) (unity), \( \mu : E^2 \rightarrow E \) (multiplication) satisfying the relations

\[
\mu \circ E\eta = \mu \circ \eta E = 1_E
\]

and

\[
\mu \circ \mu E = \mu \circ E\mu.
\]

(By \( \text{Id}_{\textbf{Comp}} \) we denote the identity functor on the category \( \textbf{Comp} \) and \( E^2 \) is the superposition \( E \circ E \) of \( E \).)

For a continuous map of compacta \( f : X \rightarrow Y \) we define the map \( \Pi f : \Pi X \rightarrow \Pi Y \) by the formula \( \Pi f(\nu)(A) = \nu(f^{-1}(A)) \) where \( \nu \in \Pi X \) and \( A \in F(Y) \). The map \( \Pi f \) is continuous. In fact, this extension of the construction \( \Pi \) defines the capacity functor \( \Pi \) in the category \( \textbf{Comp} \) (see [12] for more details).

The functor \( \Pi \) was completed to the monad \( U_* = (\Pi, \eta, \mu_*) \) (where \( * \) is a continuous t-norm) in [10], where the components of the natural transformations are defined as follows:

\[
\eta_X(x)(F) = \begin{cases} 1, & x \in F; \\ 0, & x \notin F; \end{cases}
\]

For a closed set \( F \subset X \) and for \( t \in [0, 1] \) put \( F_t = \{ c \in M X \mid c(F) \geq t \} \). Define the map \( \mu_X : \Pi^2 X \rightarrow \Pi X \) by the formula

\[
\mu_X(C)(F) = \max \{ C(F_t) \ast t \mid t \in (0, 1) \}.
\]

(Existing of max follows from Lemma 3.7 [17].)

Zarichnyi proposed to construct monads on the spaces of \( * \)-measures [31]. The components of such monad when \( * = \land \) were described and studied in detail in [17]. Let us describe it in general case.

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For a map \( \phi \in C(X, [0, 1]) \) we denote by \( \pi_\phi \) or \( \pi(\phi) \) the corresponding projection \( \pi_\phi : A^*X \to I \). For each map \( f : X \to Y \) we define the map \( A^*f : A^*X \to A^*Y \) by the formula \( \pi_\phi \circ A^*f = \pi_{\phi f} \) for \( \phi \in C(Y, [0, 1]) \). It is easy to check that the map \( A^*f \) is well defined and continuous. Hence \( A^* \) forms an endofunctor on \( \text{Comp} \) (see [27] for more details). For a compactum \( X \) we define components \( hX \) and \( mX \) of natural transformations \( h : \text{Id}_{\text{Comp}} \to A^* \), \( m : (A^*)^2 \to A^* \) by \( \pi_\phi \circ hX = \phi \) and \( \pi_\phi \circ mX = \pi(\phi) \) for all \( \phi \in C(X, [0, 1]) \). It is easy to check that the maps \( hX \) and \( mX \) are well defined, continuous for each compactum \( X \) and are components of corresponding natural transformations. Let us remark that for each \( x \in X \) the \( * \)-measure \( hX(x) \) is the Dirac measure concentrated at the point \( x \) and we denote \( hX(x) = \delta_x \).

**Proposition 1.** The triple \( \mathbb{A}^* = (A^*, h, m) \) is a monad on \( \text{Comp} \).

**Proof.** We have to check that the natural transformations \( h \) and \( m \) satisfy the monad definition.

The equality \( mX \circ hVX = mX \circ VHX = \text{id}_{VX} \) follows from the next two equalities: \( \pi_\phi \circ A^*X \circ hA^*X = \pi(\pi_\phi) \circ hA^*X = \pi_\phi \circ \pi A^*X \) and \( \pi_\phi \circ mX \circ A^*hX = \pi(\pi_\phi) \circ A^*hX = \pi_\phi \circ \pi A^*hX \).

The equality \( mX \circ A^*mX = mX \circ mA^*X \) follows from the equality \( \pi_\phi \circ mX \circ A^*mX = \pi(\pi_\phi) \circ A^*mX = \pi(\pi_\phi \circ mX) = \pi(\pi(\pi_\phi)) = \pi(\pi_\phi) \circ mA^*X = \pi_\phi \circ mX \circ mA^*X \) for each \( \phi \in C(X) \). The proposition is proved. \( \square \)

Let us remark that a partial case of the above proposition for \( * = \land \) was proved in [7].

A natural transformation \( \psi : E \to E' \) is called a morphism from a monad \( E = (E, \eta, \mu) \) into a monad \( E' = (E', \eta', \mu') \) if \( \psi \circ \eta = \eta' \) and \( \psi \circ \mu = \mu' \circ \psi E' \circ E \psi \). A monad morphism \( \psi : E \to E' \) is called an isomorphism if it has an inverse morphism of monads. It is easy to check that in \( \text{Comp} \) a monad morphism \( \psi \) is an isomorphism if each its component \( \psi X \) is a homeomorphism.

For a compactum \( X \) let us define a map \( lX : \Pi X \to A^*X \) by the formula \( lX(\nu)(\varphi) = \int_{[0, 1]}^{} \varphi d\nu \).

**Proposition 2.** The map \( lX \) is a homeomorphism.

**Proof.** Theorem 1 implies that the map \( lX \) is well defined and bijective. The continuity of \( lX \) follows from [13] Lemma 4. \( \square \)

By \( l \) we denote the natural transformation with the components \( lX \).

**Theorem 2.** The natural transformation \( l \) is an isomorphism of monads \( U_* \) and \( \mathbb{A}^* \).

**Proof.** Consider any compactum \( X \). We have to check equalities \( lX \circ \eta X = hX \) and \( lX \circ \mu X = mX \circ lA^*X \circ \Pi(lX) \).

Take any \( x \in X \) and \( f \in C(X, [0, 1]) \). Then we have

\[
\pi_f \circ lX \circ \eta X(x) = \int_{X}^{\nu \ast} f d\eta X(x) = \max\{\eta X(x)(f_t) \ast t \mid t \in [0, 1]\} = 1 \ast f(x) = \pi_f \circ hX(x).
\]

Now, consider any \( C \in \Pi^2 X \) and \( f \in C(X, [0, 1]) \). Then we have

\[
\pi_f \circ mX \circ lA^*X \circ \Pi(lX)(C) = \pi(\pi_f) \circ lA^*X(\Pi(lX)(C)) = \int_{A^*X}^{\nu \ast} \pi_f d\Pi(lX)(C) = \max\{\Pi(lX)(C)(\pi_f) \ast t \mid t \in [0, 1]\} = \max\{C(lX^{-1}(\pi_f)) \ast t \mid t \in [0, 1]\} = \max\{C(\nu \in \Pi X \mid \int_{X}^{\nu \ast} f d\nu \geq t) \ast t \mid t \in [0, 1]\} =
\]


\[= \max \{ \mathcal{C}(\{ \nu \in \Pi X | \nu(f_i) * l \geq t \}) \} * t \mid t \in [0, 1] = \]

\[= \max \{ \mathcal{C}(\{ \nu \in \Pi X | \text{there exists } l \in [0, 1] \text{ such that } \nu(f_i) * l \geq t \}) \} * t \mid t \in [0, 1] = \]

since \( \mathcal{C} \) is a possibility capacity

\[= \max \{ \max \{ \mathcal{C}(\{ \nu \in \Pi X | \nu(f_i) * l \geq t \}) \} \mid l \in [0, 1] \} * t \mid t \in [0, 1] \).

Consider any \( t, l \in [0, 1] \) and \( \nu \in \Pi X \) such that \( \nu(f_i) * l \geq t \). Then we have \( t \leq l \) by monotonicity *. Put \( b(t, l) = \inf \{ s \in [0, 1] \mid t \leq s * l \} \). It follows from continuity of * that \( l * b(t, l) = t \). Moreover, we have \( k * l \geq t \iff k \geq b(t, l) \) for each \( k \in [0, 1] \).

So, we obtain

\[\max \{ \max \{ \mathcal{C}(\{ \nu \in \Pi X | \nu(f_i) * l \geq t \}) \} \mid l \in [0, 1] \} * t \mid t \in [0, 1] = \]

\[= \max \{ \max \{ \mathcal{C}(\{ \nu \in \Pi X | \nu(f_i) \geq b(t, l) \}) \} * l * b(t, l) \mid l \in [0, 1] \} \mid t \in [0, 1] \} = \]

\[= \max \{ \max \{ \mathcal{C}(f_i)_s * s \mid s \in [0, 1] \} * l \mid l \in [0, 1] \} = \max \{ \mu \mathcal{C}(f_i)_s * l \mid l \in [0, 1] \} = \]

\[= \int_X f d\mu X(\mathcal{C}) = \pi_f \circ lX \circ \mu X(\mathcal{C}). \square \]

5. Convexity generated by a t-norm

Max-plus convex sets were introduced in [32] and found many applications (see for example [11]). Well known is also max-min convexity. We generalize this convexity changing the minimum operation by any continuous t-norm *.

Let \( T \) be a set. Given \( x, y \in [0, 1]^T \) and \( \lambda \in [0, 1] \), we denote by \( y \vee x \) the coordinatewise maximum of \( x \) and \( y \) and by \( \lambda \odot x \) the point with coordinates \( \lambda \odot x_t \), \( t \in T \). A subset \( A \) in \([0, 1]^T \) is said to be max-\(*\) convex if \( \lambda \odot a \vee b \in A \) for all \( a, b \in A \) and \( \lambda \in [0, 1] \). It is easy to check that \( A \) is max-\(*\) convex iff \( \bigvee_{i=1}^n \lambda_i \odot x_i \in A \) for all \( x_1, \ldots, x_n \in A \) and \( \lambda_1, \ldots, \lambda_n \in [0, 1] \) such that \( \bigwedge_{i=1}^n \lambda_i = 1 \). In the following we denote max-\(*\) convex compactum we mean a max-\(*\) convex compact subset of \([0, 1]^T \).

It was proved in [27] that the set \( A^*(K) \) is max-\(*\) convex compact subset of \([0, 1]^C(X, [0, 1]) \).

Let \( K \subset [0, 1]^T \) be a compact max-\(*\) convex subset. For each \( t \in T \) we put \( f_t = \pi_{t|K}^r : K \to [0, 1] \) where \( \pi_r : [0, 1]^T \to [0, 1] \) is the natural projection. Given \( \mu \in A^*(K) \), the point \( \beta_K(\mu) \in [0, 1]^T \) is defined by the conditions \( \pi_r(\beta_K(\mu)) = \mu(f_t) \) for each \( t \in T \).

**Proposition 3.** We have \( \beta_K(\mu) \in K \) for each \( \mu \in A^*(K) \) and the map \( \beta_K : A^*(K) \to K \) is continuous.

**Proof.** Let \( K \subset [0, 1]^T \) be a compact max-\(*\) convex subset. Consider the subset \( A_n^*(K) \subset A^*(K) \) defined as follows

\[A_n^*(K) = \left\{ \bigvee_{i=1}^n \lambda_i \odot x_i \mid n \in \mathbb{N}, x_1, \ldots, x_n \in A \text{ and } \lambda_1, \ldots, \lambda_n \in [0, 1] \right\} \]

such that \( \bigwedge_{i=1}^n \lambda_i = 1 \).

It is known that \( A_n^*(K) \) is dense in \( A^*(K) \) [27]. Since \( K \) is compact, it is enough to prove that \( \beta_K(\mu) \in K \) for each \( \mu = \bigvee_{i=1}^n \lambda_i \odot x_i \in A_n^*(K) \). For each \( t \in T \) we have \( \pi_r(\beta_K(\mu)) = \bigvee_{i=1}^n \lambda_i \odot \pi_r(x_i) = \bigvee_{i=1}^n \lambda_i \odot \pi_r(x_i) \). Hence \( \beta_K(\mu) = \bigvee_{i=1}^n \lambda_i \odot x_i \in K \).

Continuity of the map \( \beta_K \) follows from the definition of topology on \( A^*(K) \). \( \square \)
The map $\beta_K$ is called the $*$-barycenter map.

It was shown in [20] that each monad generates a convexity structure on its algebras. We will show in this section that convexities generated by the monad $A^*$ coincide with described above max-$*$ convexities for each continuous t-norm $*$.

We will need some categorical notions and the construction of convexities generated by a monad from [20]. Let $T = (T, \eta, \mu)$ be a monad in the category Comp.

A pair $(X, \xi)$, where $\xi : TX \to X$ is a map, is called a $T$-algebra if $\xi \circ \eta X = id_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let $(X, \xi)$, $(Y, \xi')$ be two $T$-algebras. A map $f : X \to Y$ is called a morphism of $T$-algebras if $\xi' \circ Tf = f \circ \xi$. A morphism of $T$-algebras is called an isomorphism if there exists an inverse morphism of $T$-algebras.

Let $(X, \xi)$ be an $F$-algebra for a monad $F = (F, \eta, \mu)$ and let $A$ be a closed subset of $X$. Denote by $\chi_A : X \to X/A$ (the equivalence classes are one-point sets $\{x\}$ for $x \in X \setminus A$ and the set $A$) and put $a = \chi_A(A)$.

Denote $A^+ = (F\chi_A)^{-1}(\eta(X/A)(a))$. Define the $F$-convex hull $conv_F(A)$ of $A$ as follows $conv_F(A) = \xi(A^+)$. Put additionally $conv_F(\emptyset) = \emptyset$. We define the family $C_F(X, \xi) = \{A \subset X|A$ is closed and $conv_F(A) = A\}$. The elements of the family $C_F(X, \xi)$ will be called $F$-convex.

**Proposition 4.** Let $K \subset [0, 1]^T$ be a compact max-$*$ convex subset. Then the pair $(K, \beta_K)$ is an $A^*$-algebra.

*Proof.* It is easy to see that $\beta_K \circ hK = id_K$. Consider any $A \in A^+(A^*K)$ and $t \in T$.

We have $pr_t \circ \beta_K \circ A^*(\beta_K)(\Lambda) = A^+(\beta_K)(\Lambda)(f_t) = \Lambda(f_t \circ \beta_K)$. On the other hand $pr_t \circ \beta_K \circ mK(\Lambda) = mK(\Lambda)(f_t) = \Lambda(\pi(f_t))$. But $f_t \circ \beta_K = \pi(f_t)$ by the definition of $\beta_K$, hence $\beta_K \circ A^*(\beta_K) = \beta_K \circ mK$. \hfill $\Box$

It is natural to ask whether each $A^*$-algebra has the above described form? More precisely, we have the following problem.

**Problem 1.** Let $(X, \xi)$ be an $A^*$-algebra. Is $(X, \xi)$ isomorphic to $(K, \beta_K)$ for some max-$*$ convex compactum $K \subset [0, 1]^T$?

For a max-$*$ convex compactum $K \subset [0, 1]^T$ we denote the family of $A^*$-convex subset of $K$ by $C_{A^*}(K)$ and the family of max-$*$ convex subsets by $C_{\cup^*}(K)$. The main goal of this section is to prove equality of these two families.

We will need to establish some properties of the functor $A^*$.

**Proposition 5.** Let $i : K \to X$ be a topological embedding for compacta $K$ and $X$. Then the map $A^*i : A^*K \to A^*X$ is a topological embedding too.

*Proof.* Since $A^*K$ is a compactum, it is enough to prove that $A^*i$ is injective. Consider any $\nu, \mu \in A^*K$ such that $\nu \neq \mu$. Then there exists $\varphi \in C(K, [0, 1])$ such that $\nu(\varphi) \neq \mu(\varphi)$. Take any $\varphi' \in C(X, [0, 1])$ such that $\varphi'|_K = \varphi$. Then we have $A^*i(\nu)(\varphi') = \nu(\varphi' \circ i) = \nu(\varphi) \neq \mu(\varphi) = \mu(\varphi' \circ i) = A^*i(\mu)(\varphi')$. \hfill $\Box$

Let $K$ be a closed subset of a compactum $X$. We will identify the space $A^*K$ with its image $A^*i(A^*K)$ in $A^*X$ where $i : K \hookrightarrow X$ is the identity embedding.

**Lemma 1.** Let $\mu \in A^*X$ and let $K$ be a closed subset of $X$. Then $\mu \in A^*K \subset A^*X$ iff for each $\phi_1, \phi_2 \in C(X, [0, 1])$ with $\phi_1|K = \phi_2|K$ we have $\mu(\phi_1) = \mu(\phi_2)$.

*Proof.* Let $\mu \in A^*K \subset A^*X$. Denote by $i : K \to X$ the identity embedding. Let $\phi_1, \phi_2 \in C(X, [0, 1])$ be functions with $\phi_1|K = \phi_2|K$. There exists a functional $\nu \in A^*K$ such that $A^*i(\nu) = \mu$. Then we have $\mu(\phi_1) = \nu(\phi_1|A) = \nu(\phi_2|A) = \mu(\phi_2)$.

Now let $\mu \in A^*X$ be a functional such that $\mu(\phi_1) = \mu(\phi_2)$ for each $\phi_1, \phi_2 \in C(X, [0, 1])$ with $\phi_1|A = \phi_2|A$. Then we can define a functional $\nu \in A^*K$ by the formula $\nu(\phi) = \mu(\phi')$, where $\phi \in C(X, [0, 1])$ and $\phi'$ is any extension of $\phi$ on $X$. It is easy to see that $\nu \in A^*K$ is well defined and $A^*i(\nu) = \mu$. \hfill $\Box$
Corollary 1. Let \( \mu \in A^*X \) and let \( K \) be a closed subset of \( X \). Then \( \mu \in A^*K \subset A^*X \) iff for each \( \phi \in C(X, [0, 1]) \) with \( \phi|K \equiv 0 \) we have \( \mu(\phi) = 0 \).

Lemma 2. Let \( f : X \to Y \) be a continuous map between compacta \( X \) and \( Y \) and let \( K \) be a closed subset of \( Y \). Consider any function \( \phi \in C(X, [0, 1]) \) such that \( \phi|f^{-1}(K) \equiv 0 \). Then there exists \( \psi \in C(Y, [0, 1]) \) such that \( \psi|K \equiv 0 \) and \( \phi \leq \psi \circ f \).

Proof. If \( \phi \equiv 0 \), then we put \( \psi \equiv 0 \). So, we assume that \( \max_{x \in X} \phi(x) = b > 0 \).

For \( \varepsilon > 0 \) denote
\[
B_\varepsilon = \{ y \in Y \mid \text{there exists } x \in f^{-1}(y) \text{ such that } \phi(x) \leq \varepsilon \}.
\]

Let us show that the set \( B_\varepsilon \) is closed in \( Y \). Take any \( z \notin B_\varepsilon \). Then \( f^{-1}(z) \subset U = \{ x \in X \mid \phi(x) < \varepsilon \} \). Since \( f^{-1}(z) \) is compact and \( U \) is an open subset of \( X \), we can choose an open neighborhood \( V \) of \( z \) in \( Y \) such that \( f^{-1}(V) \subset U \). Then \( z \in V \subset Y \setminus B_\varepsilon \), hence \( Y \setminus B_\varepsilon \) is open.

Let us build by the induction a sequence of closed sets \( V_i \) in \( Y \) such that \( V_i \cap K = \emptyset \), \( V_1 \subset \text{Int}V_{i+1} \) and \( V_i \supset B_{2^{-i}} \) for each \( i \in \mathbb{N} \).

Put \( V_1 = B_2 \). Suppose we have constructed \( V_1, \ldots, V_k \) for some \( k \geq 1 \). Take any open set \( V \) such that \( V_k \subset V \subset \text{Cl}V \subset Y \setminus K \). Put \( V_{k+1} = \text{Cl}V \cup B_{2^{-k}} \).

We can construct by the induction a function \( \psi \in C(Y, [0, 1]) \) such that \( \psi|K \equiv 0 \), \( \psi|V_i \equiv b \) and \( \psi(x) \in [\frac{b}{2^i}, \frac{b}{2^{i-1}}] \) for each \( x \in V_{i+1} \setminus V_i \) and \( i \in \mathbb{N} \). It is easy to see that \( \psi \) is the function we are looking for.

Lemma 3. Let \( f : X \to Y \) be a continuous map between compacta \( X \) and \( Y \) and let \( K \) be a closed subset of \( Y \). Then \( \text{we have } (A^*f)^{-1}(A^*K) \supset A^*(f^{-1}(K)). \)

Proof. It is easy to see that \( (A^*f)^{-1}(A^*K) \supset A^*(f^{-1}(K)) \). Let us prove the opposite inclusion.

Take any \( \nu \in (A^*f)^{-1}(A^*K) \). Suppose that \( \nu \notin A^*(f^{-1}(K)) \). Then there exists a function \( \phi \in C(X, [0, 1]) \) with \( \phi|f^{-1}(K) \equiv 0 \) and \( \nu(\phi) > 0 \) by Corollary 1. We can take a function \( \psi \in C(Y, [0, 1]) \) such that \( \psi|K \equiv 0 \) and \( \phi \leq \psi \circ f \) by Lemma 2. Then we have \( A^*f(\nu)(\psi) = \nu(\psi \circ f) \geq \nu(\phi) > 0 \) and we obtain a contradiction with \( A^*f(\nu) \in A^*K \).

Theorem 3. Let \( K \) be a max*-convex compactum. Then \( C_{A^*}(K, b_K) = C_{\text{vs}}(K) \).

Proof. Let \( C \) be a closed subset of \( K \). As usual we denote by \( \chi_C : K \to K/C \) the quotient map \( \chi_C : K \to K/C \) and \( c = \chi_C(C) \).

Let \( C \in C_{A^*}(K, b_K) \). Consider any \( e, d \in C \) and \( \lambda \in [0, 1] \). Put \( \nu = \lambda \ast \delta_e \lor \delta_d \in A^*C \subset A^*K \). We have \( \nu \in (A^*\chi_C)^{-1}(\delta_c) \). Since \( C \in C_{A^*}(K, b_K) \), we have \( b_K(\nu) \in C \). But \( b_K(\nu) = b_K(\lambda \ast \delta_e \lor \delta_d) = \lambda \ast e \lor d \), hence \( C \in C_{\text{vs}}(K) \).

Now, consider any \( C \in C_{\text{vs}}(K) \). Suppose the contrary \( C \notin C_{A^*}(K, b_K) \), i.e., there exists \( \nu \in (A^*\chi_C)^{-1}(\delta_c) \) such that \( b_K(\nu) \notin C \). We have by Lemma 2
\( (A^*\chi_C)^{-1}(\delta_c) = A^*(\chi_C^{-1}(c)) = A^*C \). Since \( A^*C \) is dense in \( A^*C \), we can choose \( x_1, \ldots, x_k \in C \) and \( \lambda_1, \ldots, \lambda_k \in [0, 1] \) with \( \sum_{i=1}^{k} \lambda_i = 1 \) such that \( b_K(\sum_{i=1}^{k} \lambda_i \ast \delta_{x_i}) \notin b_K^{-1}(C) \). But \( b_K(\sum_{i=1}^{k} \lambda_i \ast \delta_{x_i}) = \sum_{i=1}^{k} \lambda_i \ast x_i \in C \), because \( C \in C_{\text{vs}}(K) \). We obtain a contradiction.

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