Non-CMC conformal data sets which do not produce solutions of the Einstein constraint equations

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Received 16 November 2003
Published 13 January 2004
Online at stacks.iop.org/CQG/21/S233 (DOI: 10.1088/0264-9381/21/3/013)

Abstract

The conformal formulation provides a method for constructing and parametrizing solutions of the Einstein constraint equations by mapping freely chosen sets of conformal data to solutions, provided a certain set of coupled, elliptic determined PDEs (whose expression depends on the chosen conformal data) admits a unique solution. For constant mean curvature (CMC) data, it is known in almost all cases which sets of conformal data allow these PDEs to have solutions, and which do not. For non-CMC data, much less is known. Here we exhibit the first class of non-CMC data for which we can prove that no solutions exist.

PACS numbers: 04.20.Ex, 04.20.-g, 02.40.Vn

1. Introduction

While much is known about the construction and parametrization of constant mean curvature (‘CMC’) solutions of the Einstein constraint equations [1–4], the corresponding questions for non-CMC solutions have proved to be significantly more formidable. Indeed, there are virtually no results for non-CMC data without the assumption that the gradient of the mean curvature is small (‘near-CMC’ data), and even for data of this sort, what we know to date is limited to theorems prescribing certain sets of conformal data which map to near-CMC solutions [5–7]. Remarkably absent have been any results regarding sets of near-CMC conformal data which do not map to solutions of the constraints. In this work, we obtain such results for the first time.

To make things precise, we recall that an initial data set \((\Sigma^3, \gamma_{ab}, K_{cd})\) for Einstein’s theory consists of a three-dimensional manifold \(\Sigma^3\), a Riemannian metric \(\gamma_{ab}\) and a symmetric tensor field \(K_{cd}\). Such a data set generates a spacetime solution of the vacuum Einstein
equations iff it satisfies the Einstein constraint equations

\[ R - K^{cd} K_{cd} + (\text{Tr} K)^2 = 0, \tag{1} \]
\[ \nabla^m K_{mb} - \nabla_b (\text{Tr} K) = 0. \tag{2} \]

In terms of the conformal formulation, one finds solutions \((\Sigma^3, \gamma_{ab}, K_{cd})\) of the constraints (1)–(2) by (a) choosing a set of conformal data \((\Sigma^3, \lambda_{ab}, \sigma_{cd}, \tau)\), where \(\lambda_{ab}\) is a Riemannian metric, \(\sigma_{cd}\) is a divergence-free (\(\nabla_c \sigma_{cd} = 0\)) trace-free \((\lambda_{ab} \sigma_{ab} = 0)\) tensor field, and \(\tau\) is a function; (b) solving the equations

\[ \Delta \phi = \frac{1}{8} R \phi - \frac{1}{8} (\sigma_{ab} + LW_{ab}) (\sigma_{ab} + LW_{ab}) \phi^{-7} + \frac{1}{12} \tau^2 \phi^5, \tag{3} \]
\[ \nabla_a (LW)^a_b = \frac{2}{3} \phi^6 \nabla_b \tau, \tag{4} \]

for the positive definite function \(\phi\) and the vector field \(W^i\) (here \((LW)_{ab} \equiv \nabla_a W_b + \nabla_b W_a - \frac{2}{3} \lambda_{ab} \nabla_c W^c\); and (c) combining the conformal data and \((\phi, W)\) to obtain a set of initial data

\[ \gamma_{ab} = \phi^4 \lambda_{ab}, \tag{5} \]
\[ K_{ab} = \phi^{-2} (\sigma_{ab} + LW_{ab}) + \frac{1}{4} \phi^4 \lambda_{ab} \tau. \tag{6} \]

which satisfies the constraint equations (1)–(2). The goal of the conformal formulation is to determine for which sets of conformal data equations (3)–(4) can be solved uniquely, and for which sets they cannot.

For constant mean curvature data on a closed (compact without boundary) manifold, as well as for maximal (mean curvature zero) asymptotically Euclidean (AE) data, and CMC asymptotically hyperbolic (AH) data, this goal has been achieved. Classifying conformal data sets \((\Sigma, \lambda, \sigma, \tau)\) according to (i) the Yamabe class of \(\lambda_{ab}\) (i.e., whether \(\lambda\) can be conformally transformed to have scalar curvature equal to +1, 0 or −1), (ii) whether \(\sigma^{cd} \sigma_{cd}\) is identically zero or not, and (iii) whether the constant \(\tau\) is zero or not, one shows for example that any conformal data set with \(\Sigma^3\) closed, with \(\lambda \in Y^+\) (positive Yamabe class), with \(\sigma_{cd} \sigma_{cd} \neq 0\) and with \(\tau \neq 0\) does not map to a solution, while for any conformal data set with \(\Sigma^3\) closed, with \(\lambda \in Y^*\), with \(\sigma^{cd} \sigma_{cd} \neq 0\) and with \(\tau \neq 0\) does map to a solution. The complete results, and consequent parametrization for CMC data, are detailed elsewhere (see, e.g., [8]).

It is not surprising that achieving this goal of a complete parametrization is much more difficult for non-CMC solutions. In the CMC case, equations (3)–(4) decouple, and (in the vacuum case) solutions of (4) are trivial, so the analysis focuses on the single elliptic PDE (the Lichnerowicz equation)

\[ \Delta \phi = \frac{1}{8} R \phi - \frac{1}{8} \sigma^{cd} \sigma_{cd} \phi^{-7} + \frac{1}{12} \tau^2 \phi^5. \tag{7} \]

For non-CMC data, however, one has to work with the fully coupled system (3)–(4).

One of the keys to the majority of the results which have been obtained for near-CMC data, both here and prior to this work, is the pointwise estimate for \(|LW|\) which can be derived from equation (4):

\[ |LW| < C \max_{\Sigma} \phi^6 \max_{\Sigma} |\nabla \tau|. \tag{8} \]

We show in section 2 how to derive this estimate, whether or not \((\Sigma^3, \lambda_{ab})\) admits a nontrivial Killing field, and very briefly discuss there how Moncrief and the first author [5, 10] have used it and a semi-decoupled sequence argument to show that for any set of conformal data of the sort \((\Sigma^3\) closed, \(\lambda \in Y^*\), any \(\sigma\), \(\tau\) nowhere zero, small \(\max \{ |\nabla \tau| \} \) (\(\Sigma^3\) closed, \(\lambda \in Y^*\), \(\sigma\)
not identically zero, small \( \max_{\Sigma} |\nabla \tau| \), there is a unique solution to equations (1)–(2), and hence a unique corresponding near-CMC solution \((\Sigma, \gamma, K)\) of the Einstein constraints. Then in section 3 we state and prove our main result, which says that for any conformal data of the sort \((\Sigma^3, \lambda, \sigma, \tau)\) with \( R \geq 0 \), \( \sigma \equiv 0 \), \( \tau = T + \eta \) with \( T \) constant and small \( \max_{\Sigma} |\nabla \eta| \), equations (3)–(4) admit no solution. We discuss in section 4 a curious result of Alan Rendall, in which he shows that for a certain special class of non-CMC conformal data, either no solution exists to (3)–(4), or the solutions are not unique. We make some concluding remarks in section 5.

2. \(|LW|\) estimates and some existence results

To obtain the desired pointwise estimates for the quantity \(|LW|\), we start by considering the model equation

\[
\nabla^a (LX)_{ab} = J_a, \tag{9}
\]

on a smooth closed Riemannian manifold \((\Sigma^3, \lambda_{ab})\) with a specified 1-form field \(J_b\). The operator on the left-hand side of this equation, \(\nabla L\), is elliptic and self-adjoint. If in addition \((\Sigma^3, \lambda_{ab})\) admits no nontrivial conformal Killing field, then \(\nabla L\) is invertible. It then follows from standard elliptic PDE theory \([11]\) that if \(J_b\) is contained in the Sobolev space \(H^p(\Sigma^3)\) with \(p \geq 2\), then there exists a unique solution \(X\) to (9), with \(X \in H^p(\Sigma^3)\). Further, the solution \(X\) satisfies the inequality

\[
\|X\|_{H^p(\Sigma^3)} \leq c\|J\|_{H^p(\Sigma^3)}, \tag{10}
\]

where \(c\) is a constant\(^3\) depending only on \(p, k\) and the geometry of \((\Sigma^3, \lambda_{ab})\).

We want to control the absolute value of \(|LX|\) using these Sobolev norms on \(X\). To do that, we first invoke the Sobolev embedding theorem, which says (in three dimensions) that for \(p \geq 1\) and for \(k \geq 0\), if \(X \in H^p(\Sigma^3)\), then for any integer \(l\) and any number \(\alpha \in (0, 1)\) which satisfy \(l + \alpha < k - \frac{1}{p}\), one has \(X \in C^{l,\alpha}(\Sigma^3)\). It says as well that, for some constant \(C\),

\[
\|X\|_{C^{l,\alpha}} \leq c\|X\|_{H^p(\Sigma^3)}. \tag{11}
\]

If we assume that \(p > 3\) and \(J \in H^p(\Sigma^3)\), then combining (10) and (11) we have that

\[
\|X\|_{C^{0,1,\alpha}} \leq c\|J\|_{H^p(\Sigma^3)}. \tag{12}
\]

Now, it follows from the definition of the operator \(L\) and from the definition of these Hölder norms that

\[
\|LX\|_{C^0} \leq c\|\nabla X\|_{C^0} \leq c\|X\|_{C^{1,0}}. \tag{13}
\]

We also have that

\[
|LX(x)| \leq \|LX\|_{C^0}, \tag{14}
\]

for all \(x \in \Sigma^3\). Thus, setting \(k = 0\) and combining (12)–(14), we have

\[
|LX(x)| \leq c\|J\|_{L^p}. \tag{15}
\]

If we assume that \(J\) is continuous, then using the definition of the \(L^p(\Sigma^3)\) norm along with the finiteness of the measure on the closed manifold \(\Sigma^3\), we obtain finally

\[
|LX(x)| \leq c\max_{\Sigma^3} |J|. \tag{16}
\]

\(^3\) Such constants appear in many of the inequalities discussed in this section. Although these constants need not be the same, for convenience we list them all as \(‘c’\).
The pairings here can be interpreted distributionally, as duality pairings between $(\Sigma^3, \lambda_{ab})$ admitting no conformal Killing fields, and if we make sufficient a priori assumptions concerning the smoothness of $\phi$, then the argument just described allows us to derive the $[LW]$ estimate (8) from (4). What happens if $(\Sigma^3, \lambda_{ab})$ admits a nontrivial conformal Killing field? We now show, working with the model equation (9), that we still obtain this estimate.

**Lemma 1.** Let $\Sigma^3$ be a closed manifold and let $\lambda_{ab}$ be a smooth Riemannian metric on $\Sigma^3$. If the 1-form field $J_b$ is continuous on $\Sigma^3$ and satisfies the condition $\int_{\Sigma^3} V^b J_b = 0$ for any conformal Killing field $V$ of $(\Sigma^3, \lambda_{ab})$, then there exists a solution $X$ of equation (9), and every such solution satisfies the estimate (16).

**Proof.** If $(\Sigma^3, \lambda_{ab})$ admits a nontrivial conformal Killing field, then the operator $\nabla L$ is no longer invertible. However it remains elliptic and self-adjoint. Hence it follows from standard elliptic theory [11] that equation (9) admits a solution $X$ so long as $J_b$ is $L^2$ orthogonal to the kernel of $\nabla L$, and further that any such solution satisfies the elliptic estimate (for some constant $c$)

$$\|X\|_{H^3}^2 \leq c (\|J\|_{H^2}^2 + \|X\|_{H^2}^2).$$

(17)

One readily verifies that the kernel of the operator $\nabla L$ consists of the conformal Killing fields of $(\Sigma^3, \lambda_{ab})$, since by definition every conformal Killing field $V$ satisfies $LV = 0$, and since if $\nabla LZ = 0$, then $0 = (Z, \nabla LZ)_{L^2} = (LZ, LZ)_{L^2}$, which implies (using elliptic regularity) that $LZ = 0$. Thus the condition $\int_{\Sigma^3} V^b J_b = 0$ implies that equation (9) admits a solution $X$.

To obtain the desired estimate for $LX$, we need to work with the term $\|X\|_{H^2}$ in (17). We first note that the operator $\nabla L$, viewed as a map from $L^p(\Sigma^3)$ to itself, is Fredholm. It follows from Berger–Ebin splitting [9] that there is a closed subspace $Y \subset L^p(\Sigma^3)$ such that $L^p = \text{Ker}(\nabla L) \oplus Y = \text{Ker} L \oplus Y$. We now want to argue that for any vector field $V \in Y \cap H^2_0$, we have

$$\|V\|_{H^2_0} \leq c \|\nabla LV\|_{H^2_0}.$$ 

(18)

Suppose that this is not the case. Then there exists a sequence $\{V_i\} \subset Y \cap H^2_0$ such that $\|V_i\|_{H^2_0} = 1$ and $\|\nabla LV_i\|_{H^2_0} \to 0$. It follows from (17) that $\|V_i\|_{H^2_0}$ is bounded; then by the Rellich lemma, one knows that there is a subsequence of $\{V_i\}$ which converges in $L^p$. Let us call this limit $V_\infty$. Since the space $Y$ is closed, $V_\infty \in Y$. By continuity, $\|V_\infty\|_{H^2_0} = 1$. Now let $Z$ be any smooth vector field. We calculate

$$(V_\infty, \nabla LZ) = \lim_{i \to \infty} (V_i, \nabla LZ) = \lim_{i \to \infty} (\nabla LV_i, Z) = 0,$$

(19)

where the last equality is a consequence of the assumption $\|\nabla LV_i\|_{H^2_0} \to 0$. This equation shows that $V_\infty$ is a weak solution to the equation $\nabla LV_\infty = 0$. It immediately follows from elliptic regularity that $V_\infty$ is contained in the kernel of $\nabla L$ and therefore in the kernel of $L$. But since the norm of $V_\infty$ is nonzero and since by assumption $V_\infty \in Y$, we have a contradiction. This proves that the estimate (18) holds.

Now let us consider a solution $X$ of equation (9). As a consequence of the splitting $L^p = \text{Ker} L \oplus Y$, we write $X = X_Y + X_0$, with $X_0 \in \text{Ker} L$ and $X_Y \in Y$. Note that elliptic

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4 We thank Jack Lee for useful discussions concerning this proof.

5 The pairings here can be interpreted distributionally, as duality pairings between $L^p$ and $L^p$, or as ordinary integrals.
regularity guarantees that \( X, X_0 \), and therefore \( X_Y \) are all contained in \( H^p_z \). We calculate

\[
\|LX\|_{H^p_z} = \|LX_Y\|_{H^p_z} \\
\leq c\|X_Y\|_{H^p_z} \\
\leq c\left(\|\nabla LX_Y\|_{H^p_z} + \|X_Y\|_{H^p_z}\right) \\
\leq c\|\nabla LX_Y\|_{H^0_z} \\
= c\|\nabla LX\|_{H^0_z} \\
= c\|J\|_{H^p_z}.
\]

Combining this inequality with the Sobolev embedding result noted above, and using the hypothesized continuity of \( J \) on the closed manifold \( \Sigma^1 \), we obtain the pointwise estimate (8) as desired, even if \((/\Sigma^1, \lambda_{ab})\) admits nontrivial Killing fields. □

This pointwise estimate for \( |LX| \) (for \( X \) satisfying (9)) is useful in proving that equations (3)–(4) admit solutions for various classes of near-CMC conformal data, since it allows one to control the \( LW \) terms in (3) via inequalities on the conformal data, and consequently apply sub-super solution techniques to (3). Our proofs of these existence results [5, 10] rely on working with a sequence of semi-decoupled equations of the form

\[
\Delta \phi_n = \frac{1}{8} R \phi_n - \frac{1}{8} (\sigma + LW_n)(\sigma + LW_n) \phi^{-7}_n + \frac{1}{12} \tau^2 \phi^5_n, \tag{21}
\]

\[
\nabla (LW_n) = \frac{2}{3} \phi^6_{n-1} \nabla \tau, \tag{22}
\]

to be solved for the sequence of fields \((\phi_n, W_n)\). The idea is to show that, starting with a more or less arbitrary choice of \( \phi_0 \), one can solve (22) with \( n = 1 \) for \( W_1 \), solve (21) with \( n = 1 \) for \( \phi_1 \), and indeed solve the sequence of equations (22)–(21) for the sequence \((\phi_n, W_n)\). One then shows that the sequence \((\phi_n, W_n)\) converges, and that this limit is a solution of the coupled system (3)–(4).

One does not need these \(|LX|\)-type estimates to show that the sequence of solutions \((\phi_n, W_n)\) to (21)–(22) exists. They are crucial for arguing convergence of the sequence, however. In particular, deriving from (22) the estimate \( |LW_n| < c \max_{/\Sigma^1} \phi^6_{n-1} \max_{/\Sigma^1} |\nabla \tau| \), and then substituting this inequality into (21) and replacing \( n - 1 \) by \( n \), one can argue that for sufficiently small \( |\nabla \tau| \) there are upper and lower (positive) bounds for all \( \phi_n \), independent of \( n \) [5]. These uniform bounds, combined with a contraction mapping argument, lead to the proof of convergence.

We note that in our first work on the existence of solutions for near-CMC conformal data [5], we assume that the conformal metric is in the negative Yamabe class. For such metrics on closed manifolds, there are no nontrivial conformal Killing fields; thus the simpler argument for the pointwise estimate for \( |LW| \) (appearing in [5], and outlined here in the discussion prior to lemma 1) is sufficient. For our later results, in which we consider positive Yamabe and zero Yamabe conformal metrics, we need the argument of lemma 2.

3. Non-existence results

Our main result here is the following theorem:

**Theorem 2.** Let \((/\Sigma^1, \lambda_{ab}, \sigma_{cd}, \tau)\) be a set of conformal data with \( /\Sigma^1 \) closed, with the scalar curvature of \( \lambda \) non-negative, with \( \sigma^2 \) zero everywhere and with \( \tau = T + \rho \) with \( T \) a nonzero constant. For \( |\nabla \rho| / |T| \) sufficiently small, equations (3)–(4) admit no solution.
Proof. Let us assume that there is a solution \((\phi, W)\) to equations (3)–(4) for conformal data satisfying the hypothesis stated in the theorem. If we substitute the conditions \(\sigma = 0\) and \(\tau = T + \rho\) into (3)–(4), then \((\phi, W)\) must satisfy
\[
\Delta \phi = \frac{1}{2} R \phi - \frac{1}{8} (L W^{ab} W_{ab}) \phi^{-7} + \frac{1}{12} (T + \rho)^2 \phi^5,
\]
(23)
\[
\nabla_a (L W)^b_a = \frac{2}{7} \phi^6 \nabla_b \rho.
\]
(24)

If we then use (24) to derive the pointwise estimate \(|L W| < c \max_{\Sigma} \phi^6 \max_{\Sigma} |\nabla \rho|\), and substitute that into (23), we have
\[
\Delta \phi \geq \frac{1}{2} R \phi - c \left( \max_{\Sigma} |\nabla \rho| \right)^2 \left( \max_{\Sigma} \phi \right)^{12} \phi^{-7} + \frac{1}{12} (T + \rho)^2 \phi^5.
\]
(25)

Since by assumption the scalar curvature is non-negative, it follows that
\[
\Delta \phi \geq -c \left( \max_{\Sigma} |\nabla \rho| \right)^2 \left( \max_{\Sigma} \phi \right)^{12} \phi^{-7} + \frac{1}{12} (T + \rho)^2 \phi^5
\]
(26)

Now consider a point \(x_m \in \Sigma\) at which \(\phi\) achieves its global maximum. Evaluating (26) at \(x_m\), we obtain
\[
\Delta \phi(x_m) \geq \left( \max_{\Sigma} \phi \right)^5 \left[ \frac{1}{12} (T + \rho(x_m))^2 - c \max_{\Sigma} |\nabla \rho|^2 \right].
\]
(27)

For \(\max_{\Sigma} |\nabla \rho|\) sufficiently small relative to \(|T|\), the right-hand side of (27) is positive. But \(\Delta \phi(p_m)\) must be non-positive, since \(p_m\) is a global (and therefore local) maximum. Hence we have a contradiction, from which it follows that for data satisfying the conditions listed in the hypotheses, there is no solution to (3)–(4). \(\square\)

Does this same sort of non-existence result hold if, instead of requiring that \(\lambda\) has scalar curvature \(R \geq 0\), we impose the less restrictive condition that \(\lambda\) be contained in the positive or zero Yamabe class? Since the conformal method is not conformally covariant unless the conformal data are CMC [8], such a result is not an automatic consequence of theorem 1. Indeed, to date it is not clear whether or not it is true.

On the other hand, one does have a stronger result of this sort if one works with the conformal thin sandwich approach (CTSA) [12] rather than the conformal method. The procedure for constructing solutions via the CTSA is very similar to that outlined above for the conformal method: one starts by choosing a set of CTSA data \((\Sigma^3, \lambda_{ab}, U_{cd}, \tau, \eta)\) where \(\lambda_{ab}\) is a Riemannian metric, \(U_{cd}\) is a trace-free \((\lambda_{ab} \sigma_{ab} = 0)\) tensor field, \(\tau\) is a function and \(\eta\) is a function. One then seeks to solve
\[
\Delta \phi = \frac{1}{2} R \phi - \frac{1}{8} (A^{ab} A_{ab}) \phi^{-7} + \frac{1}{12} \tau^2 \phi^5,
\]
(28)
\[
\nabla_a [(2n)^{-1} (L X)^b_a] = \nabla_a [(2n)^{-1} U^b_a] + \frac{2}{7} \phi^6 \nabla_a \tau,
\]
(29)
for \(\phi\) and the vector field \(X^a\). (Here \(A_{ab} \equiv (2n)^{-1} (L X)_{ab} - U_{ab}\).) Finally, one combines the CTSA data and the solution \((\phi, X)\) to obtain a set of initial data
\[
\gamma_{ab} = \phi^4 \lambda_{ab},
\]
(30)
\[
K_{ab} = \phi^{-2} (2n)^{-1} (L X_{ab} - U_{ab}) + \frac{1}{2} \phi^4 \lambda_{ab} \tau,
\]
(31)
which satisfies the constraint equations (1)–(2) along with a choice of the lapse function \(N = \phi^0 \eta\) and the shift vector \(M^a = X^a\) which are used to generate evolution.

One of the key features of the CTSA is that, unlike the conformal method, it is conformally covariant in the sense that a solution exists for the CTSA data \((\Sigma^3, \lambda_{ab}, U_{cd}, \tau, \eta)\) if and only if it also exists for the conformally related CTSA data \((\Sigma^3, \psi^4 \lambda_{ab}, \psi^{-2} U_{cd}, \tau, \psi^6 \eta)\), where
Let us define a pair of groups, $(\Sigma^3, \lambda_{ab}, U_{cd}, \tau, \rho)$ be a set of CTSA data with $\lambda \in \mathcal{Y}^+ \cup \mathcal{Y}^0$, with $U^2$ zero everywhere and with $\tau = T + \rho$ with $T$ a nonzero constant. For $\frac{\nabla^2 \rho}{T^2}$ sufficiently small, equations (28)–(29) admit no solution.

4. Rendall’s conformal data for which there is no unique solution

While the results which we have proved in section 3 provide the first examples of sets of non-CMC conformal data and sets of non-CMC CTSA data which we know do not map to solutions of the constraint equations, earlier unpublished work of Alan Rendall describes a very special set of conformal data for which either a solution does not exist, or if it exists it is not unique\(^6\). The proof does not determine which of these possibilities holds. To stimulate further understanding of this issue, we describe Rendall’s results here.

**Theorem 3.** Let $(\Sigma^3, \lambda_{ab}, \sigma_{cd}, \sigma)$ be a set of conformal data with $\Sigma = S^2 \times S^1$, with $\lambda = \text{round sphere metric} \times \text{circle metric}$, with $\sigma^2 = 0$ and with $\sigma = f(x)$, where $x$ is the coordinate for the $S^1$ factor, and $f(-x) = -f(x)$. For such data, equations (3)–(4) either admit no solution, or admit more than one solution.

**Proof.** Let us define a pair of groups, $S$ and $Z^2$, which act on $\Sigma^3$: $S$ is the rotation group $SO(3)$ which acts on the $S^2$ component of $\Sigma^3$ in the usual way, leaving $S^1$ invariant, while $Z^2$ is the reflection group, with the element $\Psi$ reflecting $S^1$ across some central point $p_0 \in S^1$, and leaving $S^2$ invariant. We note that the data $(\lambda_{ab}, \sigma_{cd} + \lambda_{cd} \sigma)$ are invariant under the action of $S$, and the metric $\lambda_{cd}$ is invariant under the action of $\Psi$ and the quantity $\sigma_{cd} + \lambda_{cd} \sigma$ changes its sign under the $\Psi$ action. It follows that if there exists a unique solution $(\phi, W^a)$ to (3)–(4) for these conformal data, then the reconstituted data $(\gamma_{ab}, K_{cd})$ (see equations (5)–(6)) must be invariant under the $\Psi$ action, have $\gamma$ invariant under the $\Psi$ action and have $K$ change its sign under the $\Psi$ action.

So let us assume that indeed there exists a unique solution to (3)–(4), and let us label the resulting initial data set $(\gamma, K)$. We show now that this leads to a contradiction. Following Rendall [14], based on the data $(\gamma, K)$ and on its local spacetime development $g$, we define a pair of quantities (i) $\mathcal{R}(x, t)$, which is equal to the $S^2$ radius at the point $(x, t)$, where $x$ is the coordinate for $S^1$ with $x = 0$ at the point $p_0$; and (ii) $m(x, t) = \frac{1}{2} \mathcal{R}(x, t)(1 - g(\nabla \mathcal{R}(x, t), \nabla \mathcal{R}(x, t))$. As a consequence of the vacuum Einstein equations, $m$ and $\mathcal{R}$ must satisfy the equations

\begin{align*}
\nabla_\alpha \nabla_\beta \mathcal{R} &= m \frac{\mathcal{R}}{\mathcal{R}^2} \delta_{\alpha\beta}, \\
\nabla_\alpha m &= 0,
\end{align*}

where the indices $\alpha, \beta$ take on the two values $x$ and $t$. Of course follows from (33) that $m(x, t)$ is a constant; we label this constant $\dot{m}$.

We need to verify the following claim: at the point $(x, t) = (0, 0)$, $\nabla \mathcal{R}(0, 0) = 0$, and $\mathcal{R}(0, 0) = 2 \dot{m}$. The first of these claimed equations follows from the facts that (i) since $K_{cd}$ changes sign under the action of $\Psi$, we have $K_{cd}(0, 0) = 0$ and therefore $\partial_\alpha \mathcal{R} = 0$;

\[^6\] Rendall’s study of this set of data was partially motivated by Bartnik’s work [13] on sets of data (similar to this one) which evolve into spacetimes which contain no constant mean curvature Cauchy surfaces.
and (ii) since the metric is invariant under the reflection $\Psi$, $\partial_1 R(0, 0) = 0$. Thus we have $\nabla_{\partial_1} R(0, 0) = 0$. The second equation follows immediately from the first, along with the definition of $m$: we have $\hat{m} = m(0, 0) = \frac{1}{2} R(0, 0)(1 - g(\nabla R(0, 0), \nabla R(0, 0))) = \frac{1}{2} R(0, 0)$.

We next consider a global maximum point $x_m \in S^1$ of the function $R(x, 0)$. Since the data are presumed to be smooth, we have $\partial_1 R(x_m, 0) = 0$. Thus we find that

$$g(\nabla R(x_m, 0), \nabla R(x_m, 0)) = g''(\partial_1 R(x_m, 0))^2 \leq 0. \tag{34}$$

So we have, from the definition of $m$,

$$\hat{m} = m(x_m, 0) = \frac{1}{2} R(x_m, 0)(1 - g(\nabla R(x_m, 0), \nabla R(x_m, 0))) = \frac{1}{2} R(x_m, 0). \tag{35}$$

Since $x_m$ is a global maximum for $R(x, 0)$, it follows from this result that for all $x \in S^1$,

$$R(x, 0) \leq R(x_m, 0) \leq 2\hat{m}. \tag{36}$$

Now, comparing the result (36) with our earlier determination that $R(0, 0) = 2\hat{m}$, we verify that $x = 0$ is a global maximum for $R(x, 0)$. However, using the fact that $K$ vanishes at $(0, 0)$, together with the Einstein equation (32), we calculate

$$\partial_1 \partial_1 R(0, 0) = \nabla_{\partial_1} \nabla_{\partial_1} R(0, 0) = \frac{\hat{m}}{R^2} g_{\partial_1 \partial_1}(0, 0) > 0. \tag{37}$$

This contradicts our contention that $(0, 0)$ is a global maximum for $R$, completing our proof by contradiction. \hfill $\square$

5. Conclusion

In a certain sense, our main result (theorem 2) is a stability result for the non-existence of solutions to (3)-(4) for certain sets of conformal data. Specifically, we recall that for CMC conformal data of the type ($\Sigma^3$ closed, $\lambda \in \mathcal{Y}^+ \cup \mathcal{Y}^0$, $\sigma^2$ identically zero, $\tau \neq 0$), there exist no solutions [2]. Restricting this result to those special cases in which $R(\lambda) \geq 0$, we see that our new results show that if we perturb the conformal data above by allowing $\tau$ to be non-constant with small gradient, then the non-existence condition still holds. We do not expect to retain non-existence if we also perturb $\sigma^2$ away from zero, since we know that CMC conformal data of the type ($\Sigma^3$ closed, $\lambda \in \mathcal{Y}^+ \cup \mathcal{Y}^0$, $\sigma^2$ not identically zero, $\tau \neq 0$) do lead to the existence of unique solutions.

We might wish to see if other sets of CMC conformal data for which no solutions exist are stable in a similar sense. In all other such classes of CMC conformal data, the function $\tau$ is zero. (See the table in section 2 of [2].) The analysis of non-CMC conformal data with $\tau$ having zeros and also having a small gradient has thus far proved difficult. So this problem is still open.

Also wide open is the question of whether solutions exist for conformal data with no restriction on the gradient of $\tau$. The only known result concerning such data is that of Rendall, described above. New ideas are likely needed to make progress in addressing and answering this question.

Acknowledgments

This work was done at Caltech, while both of us were guests of the Caltech Numerical Relativity Visitors Program. JI is partially supported by NSF grant PHY 0099373 at the University of Oregon.

7 One could also prove this result using the fact that, as a consequence of Birkhoff’s theorem, if there were a solution to the constraints based on the choice of conformal data under discussion here, then the spacetime development of these data would necessarily be a portion of the Schwarzschild spacetime (inside the horizon).
Non-CMC conformal data sets which do not produce solutions of the Einstein constraint equations

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