SUPPLEMENT TO THE PAPER
"ON EXISTENCE OF LOG MINIMAL MODELS II"

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Abstract. We prove a stronger version of a termination theorem appeared in the paper "On existence of log minimal models II" [2]. We essentially just get rid of the redundant assumptions so the proof is almost the same as in [2]. However, we give a detailed proof here for future reference.

1. Introduction

We work over a fixed algebraically closed field $k$ of characteristic zero. See section 2 of [2] for notation, terminology, and definition and basic properties of log minimal models. The following theorem was proved in [2, Theorem 1.5] under stronger assumptions. However, the proof works with little change under weaker assumptions.

Theorem 1.1. Let $(X/Z, B + C)$ be a lc pair of dimension $d$ such that $K_X + B + C$ is nef$/Z$, $B, C \geq 0$ and $C$ is $\mathbb{R}$-Cartier. Assume that we are given an LMMP$/Z$ on $K_X + B$ with scaling of $C$ as in Definition 1.2 with $\lambda_i$ the corresponding numbers, and $\lambda := \lim \lambda_i$. Then, the LMMP terminates in the following cases:

(i) $(X/Z, B)$ is $\mathbb{Q}$-factorial dlt, $B \geq H \geq 0$ for some ample$/Z \mathbb{R}$-divisor $H$;
(ii) $(X/Z, B)$ is $\mathbb{Q}$-factorial dlt, $C \geq H \geq 0$ for some ample$/Z \mathbb{R}$-divisor $H$, and $\lambda > 0$;
(iii) $(X/Z, B + \lambda C)$ has a log minimal model, and $\lambda \neq \lambda_j$ for any $j$.

We should remark that much of the difficulties in the proof of Theorem 1.1 are caused by the presence of non-klt singularities. It is also worth to mention that the above theorem and the corresponding Theorem 1.5 of [2] follow quite different goals. The arguments of [2] in essence do not rely on [4]. In contrast, the above theorem heavily relies on [4]. So, this paper is not intended to replace [2].

A sequence of log flips$/Z$ starting with $(X/Z, B)$ is a sequence $X_i \to X_{i+1}/Z_i$ in which $X_i \to Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$-flip$/Z$, $B_i$ is the birational transform of $B_i$ on $X_i$, and $(X_1/Z, B_1) = (X/Z, B)$. As usual, here $X_i \to Z_i$ is an extremal flipping contraction.

Definition 1.2 (LMMP with scaling) Let $(X_1/Z, B_1 + C_1)$ be a lc pair such that $K_{X_1} + B_1 + C_1$ is nef$/Z$, $B_1 \geq 0$, and $C_1 \geq 0$ is $\mathbb{R}$-Cartier. Suppose that either $K_{X_1} + B_1$ is nef$/Z$ or there is an extremal ray $R_1/Z$ such that $(K_{X_1} + B_1) \cdot R_1 < 0$.

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and \((K_{X_1} + B_1 + \lambda_1 C_1) \cdot R_1 = 0\) where 
\[
\lambda_1 := \inf \{ t \geq 0 \mid K_{X_1} + B_1 + tC_1 \text{ is nef}/Z \}
\]

When \((X_1/Z, B_1)\) is \(\mathbb{Q}\)-factorial dlt, the last sentence follows from [3, 3.1] (the same is true in general for lc pairs by the results of Ambro [1] and Fujino [5, Theorem 1.1 (6)]; however, we do not need this stronger version). If \(R_1\) defines a divisorial contraction or a log flip \(X_1 \to X_2\). We can now consider \((X_2/Z, B_2 + \lambda_1 C_2)\) where \(B_2 + \lambda_1 C_2\) is the birational transform of \(B_1 + \lambda_1 C_1\) and continue. That is, suppose that either \(K_{X_2} + B_2\) is nef/Z or there is an extremal ray \(R_2/Z\) such that \((K_{X_2} + B_2) \cdot R_2 < 0\) and \((K_{X_2} + B_2 + \lambda_2 C_2) \cdot R_2 = 0\) where 
\[
\lambda_2 := \inf \{ t \geq 0 \mid K_{X_2} + B_2 + tC_2 \text{ is nef}/Z \}
\]

By continuing this process, we obtain a sequence of numbers \(\lambda_i\) and a special kind of LMMP/Z which is called the LMMP/Z on \(K_{X_i} + B_1\) with scaling of \(C_1\); note that it is not unique. This kind of LMMP was first used by Shokurov [7]. When we refer to termination with scaling we mean termination of such an LMMP. We usually put \(\lambda = \lim \lambda_i\).

When we have a lc pair \((X/Z, B)\), we can always find an ample/Z \(\mathbb{R}\)-Cartier divisor \(C \geq 0\) such that \(K_X + B + C\) is lc and nef/Z, so we can run the LMMP/Z with scaling assuming that all the necessary ingredients exist, e.g. extremal rays, log flips.

2. Proof of the theorem

Proof of Theorem 1.1. Step 1. Proof of (i): Since \(H\) is ample/Z, we can perturb the coefficients of \(B\) hence assume that \((X/Z, B)\) is klt. If \(\lambda_i < 1\) for some \(i\), then \((X/Z, B + \lambda_i C)\) is klt. If \(\lambda_i = 1\) for every \(i\), then we can perturb \(C\) hence in any case we could assume that \((X/Z, B + C)\) is klt and then use [4].

Proof of (ii): The LMMP is also an LMMP on \(K_X + B + \frac{1}{2} C\) with scaling of \((1 - \frac{1}{2}) C\). We can replace \(B\) with \(B + \frac{1}{2} C\), \(C\) with \((1 - \frac{1}{2}) C\), and \(\lambda_i\) with \(\frac{\lambda_i - \frac{1}{2}}{1 - \frac{1}{2}}\). After this change, we can assume that \(B \geq \frac{H}{2}\). Now use (i).

Proof of (iii): Note that if

\((*) (X/Z, B)\) is \(\mathbb{Q}\)-factorial dlt, and \(C \geq H \geq 0\) for some ample/Z \(\mathbb{R}\)-divisor \(H\),

then, for each \(i\), there is a klt \(K_X + \Delta \sim_{\mathbb{R}} K_X + B + \lambda_i C/Z\). We continue the proof without assuming \((*)\) (but we will come back to \((*)\) in Step 7).

Step 2. We can replace \(B\) with \(B + \lambda C\) hence assume that \(\lambda = 0\). Moreover, we may assume that the LMMP consists of only a sequence \(X_i \to X_{i+1}/Z_i\) of log flips starting with \((X_1/Z, B_1) = (X/Z, B)\). Pick \(i\) so that \(\lambda_i > \lambda_{i+1}\). Thus, \(\text{Supp} C_{i+1}\) does not contain any lc centre of \((X_{i+1}/Z, B_{i+1} + \lambda_{i+1} C_{i+1})\) because \((X_{i+1}/Z, B_{i+1} + \lambda_i C_{i+1})\) is lc. Then, by replacing \((X/Z, B)\) with \((X_{i+1}/Z, B_{i+1})\) and \(C\) with \(\lambda_{i+1} C_{i+1}\) we may assume that no lc centre of \((X/Z, B + C)\) is inside \(\text{Supp} C\). Moreover, since there are
finitely many lc centres of \((X/Z, B)\), perhaps after truncating the sequence, we can assume that no lc centre is contracted in the sequence.

By assumptions there is a log minimal model \((Y/Z, B_Y)\) for \((X/Z, B)\). Let \(\phi: X \to Y/Z\) be the corresponding birational map. Since \(K_{X_1} + B_1 + \lambda_i C_i\) is nef/\(Z\), we may add an ample/\(Z\ \mathbb{R}\)-divisor \(G_i\) so that \(K_{X_1} + B_1 + \lambda_i C_i + G_i\) becomes ample/\(Z\), in particular, it is movable/\(Z\). We can choose the \(G_i\) so that \(\lim_{i \to \infty} G_i = 0\) in \(N^1(X_1/Z)\) where \(G_i\) is the birational transform of \(G_i\) on \(X_1 = X\). Therefore,

\[
K_X + B \equiv \lim_{i \to \infty} (K_{X_1} + B_1 + \lambda_i C_i + G_i)/Z
\]

which implies that \(K_X + B\) is a limit of movable/\(Z\ \mathbb{R}\)-divisors.

Let \(f: W \to X\) and \(g: W \to Y\) be a common log resolution of \((X/Z, B + C)\) and \((Y/Z, B_Y + C_Y)\) where \(C_Y\) is the birational transform of \(C\). By applying the negativity lemma to \(f\), we see that

\[
E := f^*(K_X + B) - g^*(K_Y + B_Y) = \sum D a(D, Y, B_Y)D - a(D, X, B)D
\]

is effective (cf. [3, Remark 2.6]) where \(D\) runs over the prime divisors on \(W\). Assume that \(D\) is a component of \(E\). If \(D\) is not exceptional/\(Y\), then it must be exceptional/\(X\) otherwise \(a(D, X, B) = a(D, Y, B_Y)\) and \(D\) cannot be a component of \(E\). On the other hand, if \(D\) is not exceptional/\(Y\) but exceptional/\(X\), then by definition of log minimal models, \(a(D, X, B) \leq a(D, Y, B_Y) = 0\) hence \(a(D, X, B) = 0\) which again shows that \(D\) cannot be a component of \(E\). Therefore, \(E\) is exceptional/\(Y\).

**Step 3.** Let \(B_W\) be the birational transform of \(B\) plus the reduced exceptional divisor of \(f\), and let \(C_W\) be the birational transform of \(C\) on \(W\). Pick a sufficiently small \(\delta \geq 0\). Take a general ample/\(Z\) divisor \(L\) so that \(K_W + B_W + \delta C_W + L\) is dlt and nef/\(Z\). Since \((X/Z, B)\) is lc,

\[
E' := K_W + B_W - f^*(K_X + B) = \sum D a(D, X, B)D \geq 0
\]

where \(D\) runs over the prime exceptional/\(X\) divisors on \(W\). So,

\[
K_W + B_W + \delta C_W = f^*(K_X + B) + E' + \delta C_W = g^*(K_Y + B_Y) + E + E' + \delta C_W
\]

Moreover, \(E'\) is also exceptional/\(Y\) because for any prime divisor \(D\) on \(Y\) which is exceptional/\(X\), \(a(D, Y, B_Y) = a(D, X, B) = 0\) hence \(D\) cannot be a component of \(E'\).

On the other hand, since \(Y\) is \(\mathbb{Q}\)-factorial, there are exceptional/\(Y\) \(\mathbb{R}\)-divisors \(F, F'\) on \(W\) such that \(C_W + F \equiv 0/\{Y\} \) and \(L + F' \equiv 0/\{Y\} \). Now run the LMMP/\(Y\) on \(K_W + B_W + \delta C_W\) with scaling of \(L\) which is the same as the LMMP/\(Y\) on \(E + E' + \delta C_W\) with scaling of \(L\). Let \(\lambda'_i\) and \(\lambda' = \lim_{i \to \infty} \lambda'_i\) be the corresponding numbers. If \(\lambda' > 0\), then by Step 1 the LMMP terminates since \(L\) is ample/\(Z\). Since \(W \to Y\) is birational, the LMMP terminates only when \(\lambda'_i = 0\) for some \(i\) which implies that \(\lambda' = 0\), a contradiction. Thus, \(\lambda' = 0\). On some model \(V\) in the process of the LMMP, the
pushdown of \(K_W + B_W + \delta C_W + \lambda'_i L\), say
\[
K_V + B_V + \delta C_V + \lambda'_i L_V
\]
\[
\equiv E_V + E'_V + \delta C_V + \lambda'_i L_V
\]
\[
\equiv E_V + E'_V - \delta F_V - \lambda'_i F'_V / Y
\]
is nef/\(Y\). Applying the negativity lemma over \(Y\) shows that \(E_V + E'_V - \delta F_V - \lambda'_i F'_V \leq 0\).
But if \(i \gg 0\), then \(E_V + E'_V \leq 0\) because \(\lambda'_i\) and \(\delta\) are sufficiently small. Therefore, \(E_V = E'_V = 0\) as \(E\) and \(E'\) are effective.

**Step 4.** We prove that \(\phi: X \to Y\) does not contract any divisors. Assume otherwise and let \(D\) be a prime divisor on \(X\) contracted by \(\phi\). Then \(D'\) the birational transform of \(D\) on \(W\) is a component of \(E\) because by definition of log minimal models \(a(D, X, B) < a(D, Y, B_Y)\). Now, in step 3 take \(\delta = 0\). The LMMP contracts \(D'\) since \(D'\) is not possible because \(K_X + B\) is a limit of movable/\(Z\) \(\mathbb{R}\)-divisors and \(D'\) is not a component of \(E'\) so the pushdown of \(K_W + B_W = f^*(K_X + B) + E'\) cannot negatively intersect a general curve on \(D' / Y\). Thus \(\phi\) does not contract divisors, in particular, any prime divisor on \(W\) which is exceptional/\(Y\) is also exceptional/\(X\). Though \(\phi\) does not contract divisors but \(\phi^{-1}\) might contract divisors. The prime divisors contracted by \(\phi^{-1}\) appear on \(W\).

**Step 5.** Now take \(\delta > 0\) in step 3 which is sufficiently small by assumptions. As mentioned, we arrive at a model \(Y' := V\) on which \(E_{Y'}, E'_{Y'} = 0\) where \(E_{Y'}\) and \(E'_{Y'}\) are the birational transforms of \(E\) and \(E'\) on \(Y'\), respectively. In view of
\[
K_{Y'} + B_{Y'} \equiv E_{Y'} + E'_{Y'} = 0 / Y
\]
we deduce that \((Y' / Z, B_{Y'})\) is a \(\mathbb{Q}\)-factorial dlt blowup of \((Y / Z, B_Y)\). Moreover, by construction \((Y' / Z, B_{Y'} + \delta C_{Y'})\) is also dlt.

**Step 6.** As in step 3,
\[
E'' := K_W + B_W + C_W - f^*(K_X + B + C) = \sum_D a(D, X, B + C) D \geq 0
\]
is exceptional/\(X\) where \(D\) runs over the prime exceptional/\(X\) divisors on \(W\). Since \(K_W + B_W + C_W \equiv E'' / X\), any LMMP/\(X\) with scaling of a suitable ample/\(X\) divisor \(L\) terminates for the same reasons as in step 3. Indeed, if \(\lambda'_i\) and \(\lambda' = \lim_{i \to \infty} \lambda'_i\) are the corresponding numbers in the LMMP, then we may assume \(\lambda' = 0\) by Step 1; on some model \(V\) in the process of the LMMP, the pushdown of \(K_W + B_W + C_W + \lambda'_i L\), say \(K_V + B_V + C_V + \lambda'_i L_V\) is nef/\(X\). But since \(W \to X\) is birational, there is some \(L' \geq 0\) such that \(L' \sim_{\mathbb{R}} -L' / X\) hence \(E''_V - \lambda'_i L'_V\) is nef/\(X\). Now if \(i \gg 0\), the negativity lemma implies that \(E''_V = 0\) hence the LMMP terminates.

So, we get a \(\mathbb{Q}\)-factorial dlt blowup \((X' / Z, B' + C')\) of \((X / Z, B + C)\) where \(K_{X'} + B'\) is the pullback of \(K_X + B\) and \(C'\) is the pullback of \(C\). In fact, \(X'\) and \(X\) are isomorphic outside the lc centres of \((X / Z, B + C)\) because the prime exceptional/\(X\) divisors on \(X'\) are exactly the pushdown of the prime exceptional/\(X\) divisors \(D\) on \(W\) with \(a(D, X, B + C) = 0\), that is, those which are not components of \(E''\). Since \(\text{Supp} C\)
does not contain any lc centre of \((X/Z, B + C)\) by step 2. \((X'/Z, B')\) is a \(\mathbb{Q}\)-factorial dlt blowup of \((X/Z, B)\) and \(C'\) is just the birational transform of \(C\). Note that the prime exceptional divisors of \(\phi^{-1}\) are not contracted/\(X'\) since their log discrepancy with respect to \((X/Z, B)\) are all 0, and so their birational transforms are not components of \(E''\).

**Step 7.** Remember that \(X_1 = X, B_1 = B,\) and \(C_1 = C\). Similarly, put \(X'_1 := X', B'_1 := B',\) and \(C'_1 := C''\). Since \(K_{X'_1} + B'_1 + \lambda_1 C'_1 \equiv 0/Z_1, K_{X'_1} + B'_1 + \lambda_1 C'_1 \equiv 0/Z_1.\) Run the LMMP\(/Z_1\) on \(K_{X'_1} + B'_1\) with scaling of some ample \(Z_1\) divisor (which is automatically also an LMMP with scaling of \(\lambda_1 C'_1\)). Assume that this LMMP terminates with a log minimal model \((X'_2/Z_1, B'_2)\). Since \((X_2/Z_1, B_2)\) is the lc model of \((X_1/Z_1, B_1)\) and of \((X'_1/Z_1, B'_1)\), \(X'_2\) maps to \(X_2\) and \(K_{X'_2} + B'_2\) is the pullback of \(K_{X_2} + B_2\). Thus, \((X'_2/Z, B'_2)\) is a \(\mathbb{Q}\)-factorial dlt blowup of \((X_2/Z, B_2)\). Since \(K_{X'_1} + B'_1 + \lambda_1 C'_1 \equiv 0/Z_1, K_{X'_2} + B'_2 + \lambda_1 C'_2 \equiv 0/Z_1\) where \(C'_2\) is the birational transform of \(C'_1\) and actually the pullback of \(C_2\). We can continue this process: that is use the fact that \(K_{X_n} + B_n + \lambda_n C_n \equiv 0/Z_n\) and \(K_{X'_n} + B'_n + \lambda_n C'_n \equiv 0/Z_n\) to run an LMMP\(/Z_2\) on \(K_{X'_2} + B'_2,\) etc. Therefore, we can lift the original sequence to a sequence in the \(\mathbb{Q}\)-factorial dlt case assuming that the following statement holds for each \(i:\)

\[
(**) \text{ some LMMP}/Z_i \text{ on } K_{X'_i} + B'_i \text{ with scaling of some ample divisor terminates with a log minimal model } (X'_{i+1}/Z_i, B'_{i+1}).
\]

In this paragraph, we show that we can assume that \((**)\) holds. First, note that if \((X/Z, B + C)\) is klt, then \(X' \to X\) is a small birational morphism and \((X'/Z, B' + C')\) is also klt hence \((**)\) holds by [1]. Now assume that \((*)\) in Step 1 holds. Then, there is a klt \(K_{X_i} + \Delta_i \sim_{\mathbb{R}} K_{X_i} + B_i + \lambda_i C_i/Z_i.\) Thus,

\[
K_{X'_i} + B'_i \sim_{\mathbb{R}} (K_{X'_i} + B'_i) + \epsilon (K_{X'_i} + \Delta'_i) =
\]

\[
(1 + \epsilon)(K_{X'_i} + \frac{1}{1 + \epsilon} B'_i + \epsilon \Delta'_i)/Z_i
\]

where \(K_{X'_i} + \Delta'_i\) is the pullback of \(K_{X_i} + \Delta_i.\) If \(\epsilon > 0\) is sufficiently small, then \((X'_i/Z, \frac{1}{1 + \epsilon} B'_i + \epsilon \Delta'_i)\) is klt hence \((**)\) again follows from [1] under \((*)\). Now note that \((**)\) itself is an LMMP under the assumption \((*)\) bearing in mind that some \(\mathbb{Q}\)-factorial dlt blowup of \((X_{i+1}/Z_i, B_{i+1})\) (which can be constructed as in Step 6) is a log minimal model of \((X'_i/Z_i, B'_i)\). So, we can assume that \((**)\) holds; otherwise we can replace the original sequence with the one in \((**)\) and repeat Steps 2-7 again.

Note that \(Y' \dashrightarrow X'\) does not contract divisors: if \(D\) is a prime divisor on \(Y'\) which is exceptional/\(X'\), then it is exceptional/\(X\) and so it is exceptional/\(Y\) by step 6; but then \(a(D,Y,B_Y) = 0 = a(D,X,B)\) and again by step 6 such divisors are not contracted/\(X',\) a contradiction. Thus, \((Y''/Z, B_{Y''})\) of step 5 is a log birational model of \((X'/Z, B')\) because \(B_{Y''}\) is the birational transform of \(B'.\) On the other hand, assume that \(D\) is a prime divisor on \(X'\) which is exceptional/\(Y''.\) Since \(X \dashrightarrow Y\) does not contract divisors by step 4, \(D\) is exceptional/\(X.\) In particular,
\(a(D, X', B') = a(D, X, B) = 0\); in this case \(a(D, Y, B_Y) = a(D, Y', B_Y') > 0\) otherwise \(D\) could not be contracted \(Y'\) by the LMMP of step 5 which started on \(W\) because the birational transform of \(D\) would not be a component of \(E + E' + \delta C_W\). So, \((Y'/Z, B_{Y'})\) is actually a log minimal model of \((X'/Z, B')\). Therefore, as in step 4, \(X' \to Y'\) does not contract divisors which implies that \(X'\) and \(Y'\) are isomorphic in codimension one. Now replace the old sequence \(X_i \to X_{i+1}/Z_i\) with the new one constructed in the last paragraph and replace \((Y/Z, B_Y)\) with \((Y'/Z, B_{Y'})\). So, from now on we can assume that \(X, X_i\) and \(Y\) are all isomorphic in codimension one, and that \((X/Z, B + C)\) is \(\mathbb{Q}\)-factorial dlt. In addition, by step 5, we can also assume that \((Y/Z, B_Y + \delta C_Y)\) is \(\mathbb{Q}\)-factorial dlt for some \(\delta > 0\).

**Step 8.** Let \(A \geq 0\) be a reduced divisor on \(W\) whose components are general ample \(Z\)-divisors such that they generate \(N^1(W/Z)\). By step 6, \((X_i/Z, B_i + C_i)\) is obtained by running a specific LMMP on \(K_W + B_W + C_W\). Every step of this LMMP is also a step of an LMMP on \(K_W + B_W + C_W + \varepsilon A\) for any sufficiently small \(\varepsilon > 0\), in particular, \((X_i/Z, B_i + C_i + \varepsilon A_1)\) is dlt where \(A_1\) is the birational transform of \(A\). For similar reasons, we can choose \(\varepsilon\) so that \((Y/Z, B_Y + \delta C_Y + \varepsilon A_Y)\) is also dlt. On the other hand, by [2 Proposition 3.2], perhaps after replacing \(\delta\) and \(\varepsilon\) with smaller positive numbers, we may assume that if \(0 \leq \delta' \leq \delta\) and \(0 \leq A' \leq \varepsilon A_Y\), then any LMMP/\(Z\) on \(K_Y + B_Y + \delta' C_Y + A_Y\) consists of only a sequence of log flips which are flops with respect to \((Y/Z, B_Y)\). Note that \(K_Y + B_Y + \delta' C_Y + A'_Y\) is a limit of movable/\(Z\) \(\mathbb{R}\)-divisors for reasons similar to those used in Step 2, so no divisor is contracted by such an LMMP.

**Step 9.** Fix some \(i \gg 0\) so that \(\lambda_i < \delta\). Then, by [2 Proposition 3.2], there is \(0 < \tau \ll \varepsilon\) such that \((X_i/Z, B_i + \lambda_i C_i + \tau A_i)\) is dlt and such that if we run the LMMP/\(Z\) on \(K_{X_i} + B_i + \lambda_i C_i + \tau A_i\) with scaling of some ample/\(Z\) divisor, then it will be a sequence of log flips which would be a sequence of flops with respect to \((X_i/Z, B_i + \lambda_i C_i)\). Moreover, since the components of \(A_i\) generate \(N^1(X_i/Z)\), we can assume that there is an ample/\(Z\) \(\mathbb{R}\)-divisor \(H \geq 0\) such that \(\tau A_i \equiv H + H'/Z\) where \(H' \geq 0\) and \((X_i/Z, B_i + \lambda_i C_i + H + H')\) is dlt. Hence the LMMP terminates by (i) and we get a model \(T\) on which both \(K_T + B_T + \lambda_i C_T\) and \(K_T + B_T + \lambda_i C_T + \tau A_T\) are nef/\(Z\). Again since the components of \(A_T\) generate \(N^1(T/Z)\), there is \(0 \leq A'_T \leq \tau A_T\) so that \(K_T + B_T + \lambda_i C_T + A'_T\) is ample/\(Z\) and \(\text{Supp } A'_T = \text{Supp } A_T\). Now run the LMMP/\(Z\) on \(K_T + B_T + \lambda_i C_T + A'_T\) with scaling of some ample/\(Z\) divisor where \(A'_T\) is the birational tranform of \(A'_T\). The LMMP terminates for reasons similar to the above and we end up with \(T\) since \(K_T + B_T + \lambda_i C_T + A'_T\) is ample/\(Z\). Moreover, the LMMP consists of only log flips which are flops with respect to \((Y/Z, B_Y)\) by Step 8 hence \(K_T + B_T\) will also be nef/\(Z\). So, by replacing \(Y\) with \(T\) we could assume that \(K_Y + B_Y + \lambda_i C_Y\) is nef/\(Z\). In particular, \(K_Y + B_Y + \lambda_j C_Y\) is nef/\(Z\) for any \(j \geq i\) since \(\lambda_j \leq \lambda_i\).

**Step 10.** Pick \(j > i\) so that \(\lambda_j < \lambda_{j-1} \leq \lambda_i\) and let \(r: U \to X_j\) and \(s: U \to Y\) be a common resolution. Then, we have
\[ r^*(K_{X_j} + B_j + \lambda_j C_j) = s^*(K_Y + B_Y + \lambda_j C_Y) \]
\[ r^*(K_{X_j} + B_j) \geq s^*(K_Y + B_Y) \]
\[ r^*C_j \leq s^*C_Y \]

where the first equality holds because both \( K_{X_j} + B_j + \lambda_j C_j \) and \( K_Y + B_Y + \lambda_j C_Y \) are nef\(/Z\) and \( X_j \) and \( Y \) are isomorphic in codimension one, the second inequality holds because \( K_Y + B_Y \) is nef\(/Z\) but \( K_{X_j} + B_j \) is not nef\(/Z\), and the third follows from the other two. Now

\[ r^*(K_{X_j} + B_j + \lambda_{j-1} C_j) \]
\[ = r^*(K_{X_j} + B_j + \lambda_j C_j) + r^*(\lambda_{j-1} - \lambda_j) C_j \]
\[ \leq s^*(K_Y + B_Y + \lambda_j C_Y) + s^*(\lambda_{j-1} - \lambda_j) C_Y \]
\[ = s^*(K_Y + B_Y + \lambda_{j-1} C_Y) \]

However, since \( K_{X_j} + B_j + \lambda_{j-1} C_j \) and \( K_Y + B_Y + \lambda_{j-1} C_Y \) are both nef\(/Z\), we have

\[ r^*(K_{X_j} + B_j + \lambda_{j-1} C_j) = s^*(K_Y + B_Y + \lambda_{j-1} C_Y) \]

This is a contradiction and the sequence of log flips terminates as claimed. \( \square \)

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