The Response of Optical Fibres to Gravitational Waves

Thomas B. Mieling

University of Vienna, Faculty of Physics, Vienna, Austria,
TURIS Research Platform, University of Vienna, Austria
June 22, 2021

Abstract
The response of optical fibre modes to plane gravitational waves of low frequency is computed. By solving perturbatively the Maxwell equations for step-index optical fibres in a gravitational wave background and implementing appropriate boundary conditions to describe single-mode fibres, explicit formulae for the perturbations of the phase and the polarisation of the fibre modes are obtained.

Contents
1 Introduction and Summary .......................... 2
2 Statement of the Problem .......................... 4
3 Decomposition of the Maxwell Equations ........... 5
4 The Unperturbed Modes ............................. 6
4.1 Adapted Basis .................................. 6
4.2 Unperturbed Bessel Modes ......................... 7
5 The Perturbed Modes ............................... 8
5.1 The Radial Equations ............................. 12
5.2 The Transverse Components ....................... 14
5.3 Continuity Conditions ............................ 15
6 Perturbation of the Phase .......................... 22
7 Perturbation of the Polarisation ..................... 23
8 Numerical Examples and Angular Dependence ........ 26
9 Comparison with Light Propagation in Vacuum ....... 28
10 Conclusion ....................................... 28
A Derivation of the Wave Equation .................... 30
B Continuity Conditions ............................. 30
B.1 General Structure of the Continuity Matrices ....... 31
B.2 Continuity Matrix of the Unperturbed Problem .... 32
References ........................................... 33

*ORCID: 0000-0002-6905-0183, email: thomas.mieling@univie.ac.at
1 Introduction and Summary

While free space laser interferometric detectors of gravitational waves (GW’s) such as LIGO and VIRGO are mainly analysed in the low-frequency regime $\omega_g \ll \omega$ where the GW frequency is much smaller than the laser frequency (using either the geodesic deviation equation [1–3], geometrical optics [4–9] or Maxwell’s equations directly [10–12]), GW detectors employing electromagnetic waveguides are typically discussed in the high-frequency regime [11; 13–16]. In this work, we bring together aspects of both of these kinds of analyses by studying an electromagnetic waveguide in the low-frequency regime.

One of our main motivations for such calculations is to study the effect of GW’s on light in regimes which are beyond a description in terms of geometrical optics. While such descriptions suffice to describe light propagation in vacuum or in dielectrics of large extensions (generally, whenever the wavelength is small compared to all other relevant length scales), light propagating in waveguides is only consistently described by wave optics, since the diameter of optical fibres is comparable to the optical wavelength. As a consequence, polarisation-dependent effects are not necessarily as strongly suppressed as in geometrical optics, where the transport equation for the polarisation arises only at second order in the $1/\lambda$-expansion, see e.g. the general discussion in Ref. [17, Chapter III].

Further motivation for such investigations is provided by recent analyses of the perturbation of fibre-modes by Earth’s gravitational field [18] and by Earth’s rotation [19]. These calculations assume time-independent metrics, and the study of the influence of GW’s illustrates how optical fibres react to time-dependent perturbations of the space-time metric.

Of course, there are still many technical challenges still to be overcome before fibre-optic interferometers could be used in experimental gravitational wave detection, see e.g. the discussion in Ref. [20, Appendix B]. Apart from the limited power handling capacity of optical fibres due to intensity damage and the emergence of non-linear effects at high power levels, the main obstructions seem to be shot noise (which is much more pronounced than in free-space interferometers due to fibre attenuation, cf. Refs. [21, Sect. 5.1; 22, Sect. 2.2 and 4.2]), as well as thermal and mechanical noises inside the fibre (both of which are absent in free-space interferometers). As these noise sources are more pronounced for long interferometer arms, we expect the detection of GW’s in fibre-optic interferometers to be viable only for signals of higher frequencies than those measured in LIGO and Virgo (which are in the frequency range of roughly 30 to 250 Hz, see e.g. Refs. [23–26]). Consequently, it is to be expected that fibre-optic interferometers would be most suitable for searching for continuous signals (recent searches for continuous GW’s in LIGO data can be found e.g. in Refs. [27–29]). Moreover, since such interferometers can be made much smaller than the hitherto constructed free-space interferometers, it is conceivable to construct fibre-optic interferometers of multiple arms, providing more detailed position information on the GW source than the standard two-arm Michelson interferometers. We plan to assess the experimental feasibility of such fibre-optic interferometers in the near future.

To summarise our findings: we consider a gravitational wave in transverse-traceless gauge of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon A_{\mu\nu} \cos(\omega_g(t - \kappa_i x^i)),$$  \hspace{1cm} (1.1)
propagating in an arbitrary direction \( \hat{\kappa} \). (Here, we restrict the discussion to monochromatic plane waves. For arbitrary waveforms, one may use the Fourier transform and apply the analysis provided here to every Fourier component separately.) Within this GW background, we solve perturbatively the Maxwell equations for (single-mode) step-index optical fibres and obtain the following result. Choosing the \( z \)-axis to be the symmetry axis of the waveguide, we find the perturbed optical phase for fibres much shorter than the GW wavelength (\( \omega_g z \ll 1 \)) to be

\[
\psi = \omega t - \beta z - m\theta - \frac{1}{2}\varepsilon c_1 A_{zz} \omega z \cos(\omega_g t),
\]

where \( \omega \) is the angular frequency of the mode, \( \beta \) the propagation constant and \( m \) the azimuthal mode index. The Jones matrix, describing the perturbation of the polarisation (see e.g. Refs. [30, Chap. 3; 31, Sect. 1.4] for an introduction to the Jones formalism), is found to be

\[
M = 1 - i\varepsilon c_2 \Delta^2 \omega z \cos(\omega_g t) \left( \frac{1}{2}(A_{xx} - A_{yy}) A_{xy} \right)
\]

where \( \Delta = (n_1 - n_2)/n_1 \) is the relative difference of the refractive indices in the core (\( n_1 \)) and the cladding (\( n_2 \)). The coefficients \( c_1 \) and \( c_2 \) arising here are computed numerically for typical single-mode fibres. We find that \( c_1 \approx \bar{n} \), where \( \bar{n} \) is the (frequency dependent) effective refractive index of the fibre, and \( c_2 \approx 0.3 \). The diagonal terms (of opposite sign) describe birefringence, while the off-diagonal terms (of equal sign) describe a deformation of linear polarisation to elliptic polarisation.

![Dependence of three terms](image)

**Fig. 1:** Dependence of three terms \( A_{zz}, \frac{1}{2}(A_{xx} - A_{yy}) \) and \( A_{xy} \) (in absolute value) on the angle \( \vartheta \) formed by the optical and the gravitational wave-vectors. The blue curve (“phase”) describes the angular dependence of the overall phase shift, the dashed orange line (“birefringence”) describes the difference in phase shift between the two polarisation states of light, and the dotted red curve (“deformation”) quantifies the mixing of the polarisation states.
Figure 1 shows how these three effects (phase shift, birefringence and polarisation deformation) depend on the angle $\vartheta$ between the direction of light propagation and the direction of the gravitational wave. The precise dependence on this angle and on the polarisation of the gravitational wave is discussed in detail in Section 8.

It is seen that the phase shift is maximal when the gravitational wave propagates orthogonally to the fibre axis, and vanishes when it propagates collinearly with the light ray (i.e. parallel or antiparallel). For collinear propagation, the polarisation-dependent effects are maximal: the birefringence effect is then sensitive to the $+\pmb{p}$ polarisation of the gravitational wave, while the deformation effect is sensitive to the $\times\pmb{p}$ polarisation.

2 Statement of the Problem

Consider the gravitational wave (gw) metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon A_{\mu\nu} \cos(\kappa \cdot x),$$

where $\kappa \cdot x = \kappa_\mu x^\mu$, $\kappa = \omega_g (dt - \hat{k}_i dx^i)$ with $\hat{k}_i = \hat{k}^i$ being a “spatial” unit vector (normalised with respect to the unperturbed metric) determining the direction of wave propagation, $\varepsilon$ is the amplitude of the gravitational wave (we assume $\varepsilon \ll 1$), and $A$ is a symmetric matrix of constant entries which is transverse ($A_{\mu0} = A_{\mu i} \hat{k}^i = 0$) and traceless ($\eta^{\mu\nu} A_{\mu\nu} = 0$), so we are using TT coordinates.

We wish to describe the propagation of light in a cylindrical waveguide at rest in this coordinate system. More precisely, we consider a cylindrical step-index fibre consisting of a linear dielectric with constant refractive index $n_1$ in the core and $n_2$ in the cladding (with $n_1 > n_2$). Such waveguides are typically non-magnetic (i.e. of permeability $\mu = 1$) and hence the permittivity is $\varepsilon = \sqrt{n}$.

We will neglect boundary effects from the ends of the waveguide and the outer boundary of the cladding (the waveguide is thus modelled as infinitely long and the cladding as infinitely thick), and we shall neglect deformations of the fibre, so that in cylindrical coordinates $r, \theta, z$ (related to the above TT coordinates as in flat space) the core-cladding interface is located at $r = \rho$. A more accurate model would have to take into account the perturbation of the spatial metric when formulating the boundary conditions, but at such a level of accuracy we estimate that one would also have to consider elastic deformations of the waveguide due to the gravitational wave, which is beyond the scope of this article.

Being concerned with weak gravitational waves of low frequency, we work in a perturbative setting with two expansion parameters

$$\varepsilon \ll 1, \quad \Omega := \omega_g / \omega \ll 1,$$

where $\varepsilon$ is the GW amplitude, $\omega_g$ its frequency, and $\omega$ is the laser frequency.
3 Decomposition of the Maxwell Equations

In this section, we write the Maxwell equations in the GW background (2.1) in a 3+1 form, following Ref. [18] with \( c = 1 \). To this end, set

\[ g_{ij} = g_{\tilde{ij}} = \delta_{ij} + \varepsilon A_{ij} \cos(\kappa x), \tag{3.1} \]

which is the time-dependent “spatial” metric induced on the hypersurfaces of constant time \( t \), and denote by \( \nabla \) the associated (spatial) Levi-Civita derivative.

Since \( g_{00} = -1 + O(\varepsilon^2) \) and \( \partial_0 (\det g) = O(\varepsilon^2) \), the space-time split of the Faraday two-form \( F \) and the displacement bivector \( \bar{F} \) given in Ref. [18] yields

\[ D^i = \bar{F}^{0i}, \quad H^i = \frac{1}{2} \epsilon^{ijk} \bar{F}^{jk}, \quad E_i = F_{i0}, \quad B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}, \tag{3.2} \]

where \( \epsilon \) is the spatial volume form (not to be confused with the gravitational wave amplitude \( \varepsilon \)). Here and henceforward, we neglect terms of second and higher order in \( \varepsilon \) and, for brevity, we will not write the error term \( O(\varepsilon^2) \) explicitly. The field equations then take the standard form

\[ \dot{B}^i + \epsilon^{ijk} \nabla_j E_k = 0, \quad \dot{D}^i - \epsilon^{ijk} \nabla_j H_k = 0, \quad \nabla_i B^i = 0, \quad \nabla_i D^i = 0, \tag{3.3} \]

where an overset dot indicates a time derivative. Following Ref. [32], we define the optical metric

\[ \gamma^{\mu\nu} = g^{\mu\nu} + (1 - n^2) u^\mu u^\nu, \tag{3.4} \]

where \( u \) is the four-velocity of the dielectric and \( n \) its refractive index. This allows to write the constitutive equation in the form

\[ \mu \bar{F}^{\alpha\beta} = \gamma^{\alpha\rho} \gamma^{\beta\sigma} F_{\rho\sigma}. \tag{3.5} \]

In the considered case, we have \( u^\mu = \delta^\mu_0 \) and

\[ g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon A^{\mu\nu} \cos(\kappa x), \tag{3.6} \]

where the indices of \( A \) were raised with the unperturbed metric \( \eta \). The constitutive equations then reduce to

\[ D^i = \epsilon g^{ij} E_j, \quad B^i = \mu g^{ij} H_j. \tag{3.7} \]

As the spatial metric is perturbed by the gravitational wave, this is the only part where the field equations differ from those of flat space.

To further simplify the equations, we define the Riemann-Silberstein vector

\[ Z^i := \mu D^i + jn B^i, \tag{3.8} \]
where \( j \) is an imaginary unit \( (j^2 = -1) \), such that the Maxwell equations take the form

\[
\begin{align*}
n\dot{Z}^i + j\epsilon^{ijk}\nabla_j (g_{kl}Z^l) &= 0, \quad \text{(3.9)} \\
\nabla_i Z^i &= 0, \quad \text{(3.10)}
\end{align*}
\]

wherever \( n \) is locally constant.

In the following calculations, we will use two independent (commuting) imaginary units \( i \) and \( j \), where \( i \) is reserved for the usual complex description of waves (i.e. the physical fields are the \( i \)-real parts) and \( j \) is used for this compact formulation of the field equations.

As a side remark: the algebraic structure here is that of the complex vector space \( \mathbb{C}^2 \) with \( 1 \) identified with \( (1,0) \) and \( j \) identified with \( (0,1) \), which is not equal to \(-1\), here identified with \((-1,0)\).

Using only a single complex unit, it would not be possible to use this complex description of waves while simultaneously encoding the entire electromagnetic field in a single vector field. Indeed, if \( D \) and \( B \) are complex, then a field of the form \( Z' = \mu D + iB \) would only provide incomplete information about \( D \) and \( B \), as it mixes the real part of \( D \) with the imaginary part of \( B \), and vice versa. To recover those two fields entirely, one would have to use a second complex field \( Z'' = \mu D - iB \). In contrast, the field \( Z \) defined in (3.8) allows obtaining the \( i \)-complex fields \( D \) and \( B \) simply by separating the \( j \)-real and \( j \)-imaginary parts.

4 The Unperturbed Modes

In this section, we briefly review the unperturbed solution to the problem in a notation which is convenient for the following calculations.

4.1 Adapted Basis

To describe the modes in a cylindrical waveguide, it is useful to use the complex frame

\[
e_\parallel \equiv e_z, \quad e_+ = \frac{1}{\sqrt{2}} (e_r - ie_\theta), \quad e_- = \frac{1}{\sqrt{2}} (e_r + ie_\theta),
\]

where \( (e_r, e_\theta, e_z) \) is the standard orthonormal frame adapted to cylindrical coordinates \( (r, \theta, z) \). The associated coframe is given by

\[
e^\parallel = dz, \quad e^+ = \frac{1}{\sqrt{2}} (dr + i r d\theta), \quad e^- = \frac{1}{\sqrt{2}} (dr - i r d\theta).
\]

The metric of flat space can then be written as

\[
\delta = e^+ \otimes e^- + e^- \otimes e^+ + e^\parallel \otimes e^\parallel,
\]

so raising and lowering of indices interchanges \( + \) and \( - \), e.g. for a vector \( v \) one has \( v^\pm = v_\mp \) but \( v^\parallel = v^\parallel \). The only non-vanishing components of the spatial Levi-Civita connection one-form are

\[
\omega^+_- = +id\theta = \frac{1}{\sqrt{2}r} \left( e^+ - e^- \right), \quad \omega^-_+ = -id\theta = \frac{1}{\sqrt{2}r} \left( e^- - e^+ \right).
\]
We find the cross-products to be
\[ \mathbf{e}_\parallel \times \mathbf{e}_\pm = \pm i \mathbf{e}_\pm, \quad \text{and} \quad \mathbf{e}_+ \times \mathbf{e}_- = i \mathbf{e}_\parallel, \] (4.5)
and consequently
\[ \mathbf{e}_+ \cdot (\mathbf{e}_- \times \mathbf{e}_\parallel) = +i, \] (4.6)
so that for the Levi-Civita tensor (the volume form) we obtain
\[ \epsilon_{+\parallel} := \epsilon(\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_\parallel) = +i, \quad \text{and} \quad \epsilon^{+\parallel} = -i. \] (4.7)

4.2 Unperturbed Bessel Modes

The computation of the electromagnetic modes in a cylindrical step-index waveguide is a standard calculation which is discussed in many textbooks, e.g. Refs. [33, Chapter 8; 34, Section 3.1; 35, Section 16.10]. Here, we merely state the result, expressed in terms of the \( j \)-complex field \( Z \) and referred to the \( i \)-complex basis introduced above.

The unperturbed Bessel modes take the concise form
\[ Z^{(0)} = f_0(\alpha; r)e^{i(\omega t - \beta z + m\phi)}, \] (4.8)
\[ i\zeta Z^{(0)} = \frac{1}{\sqrt{2}} \rho^2 (\beta \pm inj\omega) c_m^\pm Z^{(0)}, \] (4.9)
where
\[ \zeta = \rho^2 (n^2\omega^2 - \beta^2), \] (4.10)
and
\[ c_m^\pm = \partial_r \mp \frac{m}{r}. \] (4.11)

Here, \( \omega \) is the angular frequency of the wave, \( \beta \) the propagation constant, \( m \) is the azimuthal mode index, and \( \rho \) is the radius of the fibre core. The radial function \( f_0 \) is given by
\[ f_0(\alpha_1, \alpha_2; r) = \begin{cases} \alpha_1 J_m(Ur/\rho), & r < \rho, \\ \alpha_2 K_m(Wr/\rho), & r > \rho, \end{cases} \] (4.12)
where \( \alpha_1, \alpha_2 \) are \( j \)-complex constants (related by appropriate continuity conditions discussed below), and
\[ U = \sqrt{\rho^2 (n_1^2\omega^2 - \beta^2)}, \quad \text{and} \quad W = \sqrt{\rho^2 (\beta^2 - n_2^2\omega^2)}. \] (4.13)

The function \( f_0 \) satisfies the Bessel equation
\[ B_0 f_0(\alpha; r) = \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\zeta}{\rho^2} \right] f_0(\alpha; r) = 0, \] (4.14)
both in the core and the cladding of the fibre.

In passing, let us note that the \( c_m^\pm \) act as “ladder operators”, i.e. for any real number \( \lambda \) one
has the identities
\[ c_m^\pm J_m(\lambda r) = \mp \lambda J_{m\pm1}(\lambda r), \quad \text{and} \quad c_m^\pm K_m(\lambda r) = -\lambda K_{m\pm1}(\lambda r), \] (4.15)

see e.g. Ref. [36, Eq. 9.1.27 on p. 361 and Eq. 9.6.26 on p. 376], so that we may write the \( \pm \)-components of the field as
\[ i\zeta Z^{(0)}_\pm = \frac{1}{\sqrt{2}} \rho (\beta \pm ij\omega) f_{\pm}(\alpha; r)e^{i(\omega t - \beta z + m\theta)}, \] (4.16)

where
\[ f_{\pm}(\alpha_1, \alpha_2; r) = \rho c_m^\pm f_0(\alpha_1, \alpha_2; r) = \begin{cases} \mp \alpha_1 U J_{m\pm1}(Ur/\rho), & r < \rho, \\ -\alpha_2 W K_{m\pm1}(Wr/\rho), & r > \rho. \end{cases} \] (4.17)

The dispersion relation is then obtained by imposing continuity conditions at the core-cladding interface. As we will discuss these conditions extensively for the perturbed problem, we merely outline the basic reasoning here.

1. Maxwell’s equations imply that the field components \( D_r, E_\theta, E_z, B_r, H_\theta, H_z \) are continuous at the core-cladding interface (at \( r = \rho \)).

2. Only four of these six continuity conditions are linearly independent.

3. Since the fields have two \( j \)-complex, and thus four \( j \)-real parameters, there is a \( j \)-real equation of the form (jumps) = \( \Pi_0 \) (parameters), where \( \Pi_0 \) is a \( 4 \times 4 \) matrix.

4. The continuity conditions are thus only solvable if \( \det \Pi_0 = 0 \), which constitutes the dispersion relation for \( \beta \) and \( \omega \). This equation is transcendental (as it contains various Bessel functions) and must thus be solved numerically.

5 The Perturbed Modes

As we shall see in this section, the effect of weak GW’s on light in waveguides is twofold: (i) the phase is modulated by the GW, and (ii) due to the tensorial nature of the GW amplitude, optical sidebands with an angular dependence of the form \( e^{i(m+k)\theta} \), with \( k \neq 0 \), arise. We find the dominant effects to be (a) the correction of the phase \( (k = 0) \) and (b) the perturbation of the polarisation, which arises when \( m + k = -m \).

Outline Let us first give an outline of the calculation. We start by deriving and solving the first order correction to the wave equation for the longitudinal field component. From this, we derive the transverse components of the electromagnetic field, finding structures similar to the ones of the unperturbed case. Schematically, the correction to the central mode can be written as
\[ Z_0^{(1)} = [f^+(\alpha) + g^+(\sigma^+)]e^{i(\omega t - \beta z + m\theta + \kappa x)} + [f^-(\alpha) + g^-(\sigma^-)]e^{i(\omega t - \beta z + m\theta - \kappa x)}, \] (5.1)
where the coefficients \( \alpha \) describe the unperturbed field (they act as source terms in the wave equation for \( Z^{(1)} \)) and the coefficients \( \sigma^{\pm} \) parametrise homogeneous solutions of the wave equation. The continuity condition then takes the schematic form

\[
\Pi_0 \alpha + \epsilon [P \alpha + \Sigma^+ \sigma^+] e^{+i\kappa \cdot x} + \epsilon [P \alpha + \Sigma^- \sigma^-] e^{-i\kappa \cdot x} = 0,
\]

(5.2)

where \( \alpha, \sigma^\pm \) are column vectors, each comprising four \( j \)-real numbers which parametrise \( \alpha \) and \( \sigma^\pm \), and \( \Pi_0, P, \Sigma^\pm \) are \( j \)-real \( 4 \times 4 \)-matrices.

The first term is the same as in the unperturbed problem, so the dispersion relation \( \det \Pi_0 = 0 \) and the coefficients \( \alpha \) remain unchanged. The coefficients \( \sigma^\pm \) are then determined from

\[
\Sigma^\pm \sigma^\pm + P \alpha = 0.
\]

(5.3)

For the central mode, we find that \( \Sigma^\pm \) is close to the singular matrix \( \Pi_0 \) and its inverse has the form \((\Sigma^\pm)^{-1} = \pm \Omega^{-1} \Sigma' + O(\Omega^0)\), so that the parameters \( \sigma \) are given by

\[
\sigma^\pm = \mp \Omega^{-1} \Sigma' P \alpha + O(\Omega^0).
\]

(5.4)

In Section 6, we show that this correction of order \( 1/\Omega \) amounts to a perturbation to the phase of the electromagnetic wave.

For the sidebands, we obtain similar equations. In almost all cases \( (m + k \neq -m) \), the coefficient matrices are not close to being singular, and hence no terms of order \( 1/\Omega \) arise there. However, in the exceptional case \( m + k = -m \) we have \( \det \Pi_k = \det \Pi_0 = 0 \), so that we again obtain terms of order \( 1/\Omega \) there. We show in Section 7 that these terms describe the perturbation of the polarisation of the electromagnetic wave.

**The Polarisation of the Gravitational Wave** We start by writing the polarisation tensor \( A \) of the gw, as defined in (2.1), in the complex frame introduced above. Using

\[
dx = \frac{1}{\sqrt{2}} [e^+ e^{+i\theta} + e^- e^{-i\theta}], \quad dy = -\frac{i}{\sqrt{2}} [e^+ e^{+i\theta} - e^- e^{-i\theta}],
\]

(5.5)

one finds the components of

\[
A = A_{\mu\nu} dx^\mu \otimes dx^{\nu} = A_{ij} e^i \otimes e^j
\]

(5.6)

in the basis \( e_+, e_-, e_\| \) to be

\[
A_{\|\|} = A_{zz},
\]

(5.7)

\[
A_{+-} = A_{-+} = \frac{1}{2} (A_{xx} + A_{yy}),
\]

(5.8)

\[
A_{\|\|} = A_{\|\|} = \frac{1}{\sqrt{2}} (A_{xx} + i A_{yx}) e^{\pm i\theta},
\]

(5.9)

\[
A_{++} = \frac{1}{2} (A_{xx} - A_{yy} - 2 i A_{xy}) e^{2i\theta},
\]

(5.10)

\[
A_{--} = \frac{1}{2} (A_{xx} - A_{yy} + 2 i A_{xy}) e^{-2i\theta}.
\]

(5.11)
The condition that $A$ is traceless now translates to

$$2A_{+-} + A_{||} = 0,$$  \hfill (5.12)

while the transversality condition takes the form

$$A_{++} + \hat{k}_- + A_{-+} + A_{||} = 0,$$  \hfill (5.13)

where the frame components of $\hat{k}$ are

$$\hat{k}_\pm = \frac{1}{\sqrt{2}}(\hat{k}_x \mp i\hat{k}_y)e^{\pm i\theta}, \quad \text{and} \quad \hat{k}_|| = \hat{k}_z.$$  \hfill (5.14)

To separate the amplitudes of $A_{ij}$ from their $\theta$-dependence, we write

$$A_{||} = a_0, \quad A_{||\pm} = a_{\pm 1}e^{\pm i\theta}, \quad A_{\pm\pm} = a_{\pm 2}e^{\pm 2i\theta},$$  \hfill (5.15)

and due to the trace condition we have $A_{+-} = A_{-+} = -\frac{1}{2}a_0$.

**The Wave Equation**  As shown in Appendix A, in regions where $n$ is constant, the field equations (3.9) and (3.10) imply the wave equation

$$n^2\ddot{Z}^i - (\Delta Z)^i + R_j^i Z^j + jn\epsilon^{ijk}\nabla_j(\hat{g}_{kl}Z^l) = 0,$$  \hfill (5.16)

where $\Delta$ is the perturbed spatial vector Laplacian (i.e. associated to the perturbed spatial metric $g$) and $R_j^i$ is the spatial Ricci tensor. As the exterior derivative is independent of the connection (provided it is torsion-free), it suffices to use the unperturbed Levi-Civita connection here: $\nabla \equiv \nabla^{(0)}$.

Let us now expand the electromagnetic field $Z = Z^{(0)} + \varepsilon Z^{(1)}$ as well as the Laplacian $\Delta = \Delta^{(0)} + \varepsilon \Delta^{(1)}$. Since $Z^{(0)}$ satisfies the unperturbed equation $n^2\dot{Z}^{(0)} - \Delta^{(0)} Z^{(0)} = 0$, only terms of order $\varepsilon$ remain, so we obtain at first order

$$\Box Z^{(1)i} - (\Delta^{(1)} Z^{(0)})^i + R_j^{(1)i} Z^{(0)j} + jn\epsilon^{ijk}\nabla_j(\hat{g}_{kl}^{(1)}Z^{(0)l}) = 0,$$  \hfill (5.17)

where we have set

$$\Box^i := n^2\ddot{Z}^i - (\Delta^{(0)} Z)^i.$$  \hfill (5.18)

The derivatives of the unperturbed field are of order $\omega$, which is much larger than the terms arising from derivatives of the metric or the spatial Ricci tensor, so we neglect the last two terms in (5.17), which are suppressed by $\Omega = \omega g / \omega$ or more. At this level of accuracy, we also have

$$\Delta^{(1)} Z^{(0)i} \approx g^{(1)jk}\nabla_j \nabla_k Z^{(0)i} = -\cos(\kappa.x)A^{jk}\nabla_j \nabla_k Z^{(0)i}.$$  \hfill (5.19)

With these approximations, we arrive at the wave equation

$$\Box Z^{(1)i} + \cos(\kappa.x)A^{jk}\nabla_j \nabla_k Z^{(0)i} = 0.$$  \hfill (5.20)
We shall use one further natural approximation:

$$\cos(\kappa.x) = \cos(\omega_g(t - \hat{\kappa}_z z - r(\hat{\kappa}_x \cos \theta + \hat{\kappa}_y \sin \theta))) \approx \cos(\omega_g(t - \hat{\kappa}_z z)), \quad (5.21)$$

which is admissible since the diameter of the waveguide is negligible compared to the GW wavelength ($\rho \omega_g \ll 1$). As an illustration: for typical optical fibres, $\rho$ measures a couple of micrometers and GW wavelengths detected by LIGO measure hundreds of kilometres: $\rho \omega_g \sim 10^{-10}$, so the error terms here will be much larger than those arising from quadratic terms in the GW amplitude $\varepsilon \sim 10^{-21}$. At this level of accuracy, we may thus replace $\kappa.x$ by $\kappa.x\parallel := \omega_g(t - \hat{\kappa}_z z)$ to obtain

$$\Box Z^{(1)i} + \cos(\kappa.x\parallel) A^{jk} \nabla_j \nabla_k Z^{(0)i} = 0. \quad (5.22)$$

Using the fact that the connection one-form satisfies $\omega_{\parallel ij} = 0$, the second covariant derivative in this equation evaluates to

$$\nabla_j \nabla_k Z || = \epsilon_j(\epsilon_k(Z ||)) - \omega_{\parallel kj} \epsilon_{l}(Z ||). \quad (5.23)$$

Note that the unperturbed field $Z^{(0)}$ depends on the angle $\theta$ only as $e^{i m \theta}$. Since $A$ has components which vary with $\theta$ as $e^{\pm i \theta}$ and $e^{\pm 2i \theta}$, the first order correction will have sidebands with $m$ shifted by one and two in either direction. Moreover, we find it convenient to separate terms which oscillate like the metric perturbation (i.e. trigonometric functions of $\kappa.x\parallel$) into terms with complex exponentials and thus decompose the perturbation of the electromagnetic field (and other quantities if needed) as

$$Z^{(1)a} = \sum_{k=-2}^{2} \sum_{l=\pm 1} Z^a_{kl}(r) e^{i(\omega t - \beta z + (m+k)\theta + l \kappa.x\parallel)} . \quad (5.24)$$

Thus, in $Z^a_{kl}$ the first index refers to the vector component (typically taken with respect to the complex frame $\epsilon_{\parallel}, \epsilon_{\pm}$), the first lower index determines the angular mode number, and the second lower index determines the sign in the complex exponential $\exp(\pm ik.x\parallel)$.

For the wave equation, we discuss the central mode ($k = 0$) and the sidebands ($k \neq 0$) separately.

**Central Mode** For the central mode ($k = 0$), the only contributions come from $A^{\parallel}$ and $A^{+-} = A^{-+}$, so we have

$$(A^{jk} \nabla_j \nabla_k Z^{(0)i})_0 = A^{\parallel} \partial_z \partial_z Z^{(0)i} + A^{+-} [\epsilon_+ \epsilon_- + \epsilon_- \epsilon_+] + \frac{1}{\sqrt{2r}} (\epsilon_+ + \epsilon_-) Z^{(0)i}, \quad (5.25)$$

where the subscript indicates $k = 0$. Now, $\epsilon_{\pm}$ acts on $Z^{(0)i}$ as $\frac{1}{\sqrt{2r}} \epsilon^0_{\pm}$, as defined in (4.11), so

$$[\epsilon_+ \epsilon_- + \epsilon_- \epsilon_+] Z^{(0)i} = (\partial_r^2 + r^{-1} \partial_r - m^2 r^{-2}) Z^{(0)i}. \quad (5.26)$$
Using the unperturbed radial equation (4.14) as well as the trace condition (5.12), we thus obtain

\[(A^k \nabla_j \nabla_k Z(0)^\parallel)_0 = \frac{1}{2} a_0 (\zeta/\rho^2 - 2\beta^2) Z(0)^\parallel.\] (5.27)

**Sidebands**  For the sidebands with \(m\) shifted by \(\pm 1\), the only relevant terms are those involving \(A^\parallel = A_\pm = a_\pm e^{\pm i\theta}\), so we find

\[(A^k \nabla_j \nabla_k Z(0)^\parallel)_{\pm 1} = -\sqrt{2} i \beta a_\pm c_m Z(0)^\parallel,\] (5.28)

and for the sidebands with \(m\) shifted by \(\pm 2\), the relevant terms come from \(A^\pm = A_\pm = a_\pm e^{\pm 2i\theta}\), so

\[(A^k \nabla_j \nabla_k Z(0)^\parallel)_{\pm 2} = \frac{1}{2} a_\pm (c_m^\pm - \frac{1}{r_m} c_m^\pm) Z(0)^\parallel = \frac{1}{2} a_\pm c_m^\pm Z(0)^\parallel.\] (5.29)

### 5.1 The Radial Equations

Using separation of variables, the wave equations just derived are reduced to radial equations. We give explicit solutions Bessel equations arising this way.

**Central Mode**  Using the decomposition (5.24), the inhomogeneous wave equation (5.22) yields

\[B_0^\pm Z_{0\pm}^\parallel = \frac{1}{2} a_0 (\zeta/\rho^2 - 2\beta^2) f_0(\alpha, r),\] (5.30)

where

\[B_0^\pm = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\zeta_\pm}{\rho^2} - \frac{m^2}{r^2},\] (5.31)

\[\zeta_\pm = \rho^2 (n_\pm^2 \omega^2_\pm - \beta^2_\pm),\] (5.32)

\[\omega_\pm = \omega_\pm \omega_g,\] (5.33)

\[\beta_\pm = \beta_\pm \hat{\kappa}_g \omega_g.\] (5.34)

The homogeneous solutions to this equation which are regular at the origin and vanish at infinity are evidently

\[f_0^\pm(\sigma, r) = \begin{cases} 
\sigma_1 J_m(U_\pm r/\rho), & r < \rho, \\
\sigma_2 K_m(W_\pm r/\rho), & r > \rho,
\end{cases}\] (5.35)

where \(\sigma_1, \sigma_2\) are constants (continuity conditions relating them are discussed in a later section), and where \(U_\pm, W_\pm\) are defined in analogy to \(U, W\) in (4.13):

\[U_\pm = \sqrt{\rho^2 (n_\pm^2 \omega^2_\pm - \beta^2_\pm)} ,\] and \[W_\pm = \sqrt{\rho^2 (\beta^2_\pm - n_\pm^2 \omega^2_\pm)}.\] (5.36)
To find particular solutions, we consider the more general problem

$$\rho^2 B_\nu^\pm \rho_\nu = \begin{cases} \alpha_1 J_{m+\nu}(Ur/\rho), & r < \rho, \\ \alpha_2 K_{m+\nu}(Wr/\rho), & r > \rho, \end{cases}$$

(5.37)

where \(\alpha_1, \alpha_2\) are arbitrary constants and \(B_\nu^\pm\) is a Bessel operator of arbitrary order \(\nu:\)

$$B_\nu^\pm = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\zeta_\pm}{\rho^2} - \frac{(m + \nu)^2}{r^2}. \quad (5.38)$$

Particular solutions to this equation are given by

$$p_\nu^\pm(\alpha, r) = \begin{cases} \alpha_1 [Y_{m+\nu}(U_\pm r/\rho)\Gamma_\nu^\pm(r/\rho) + J_{m+\nu}(U_\pm r/\rho)\tilde{\Gamma}_\nu^\pm(r/\rho)], & r < \rho, \\ \alpha_2 [I_{m+\nu}(W_\pm r/\rho)\Delta_\nu^\pm(r/\rho) + K_{m+\nu}(W_\pm r/\rho)\tilde{\Delta}_\nu^\pm(r/\rho)], & r > \rho, \end{cases} \quad (5.39)$$

where

\[
\Gamma_\nu^\pm(z) = \frac{\pi}{2} \int_0^z J_\nu(U_\pm z') J_\nu(U z') z' \, dz',
\]

(5.40)

\[
\tilde{\Gamma}_\nu^\pm(z) = \frac{\pi}{2} \int_z^1 Y_\nu(U_\pm z') J_\nu(U z') z' \, dz',
\]

(5.41)

\[
\Delta_\nu^\pm(z) = -\int_{\infty}^{z} K_\nu(W_\pm z') K_\nu(W z') z' \, dz',
\]

(5.42)

\[
\tilde{\Delta}_\nu^\pm(z) = -\int_1^{z} I_\nu(W_\pm z') K_\nu(W z') z' \, dz'.
\]

(5.43)

Equation (5.37) is readily verified using the Wronskians

\[
\begin{vmatrix} J_m(r) & J'_m(r) \\ Y_m(r) & Y'_m(r) \end{vmatrix} = \frac{2}{\pi r}, \quad \text{and} \quad \begin{vmatrix} K_m(r) & K'_m(r) \\ I_m(r) & I'_m(r) \end{vmatrix} = \frac{1}{r}. \quad (5.44)
\]

cf. e.g. Ref. [36, eqns. 9.1.15 on p. 360 and 9.6.15 on p. 375]. The integral ranges were chosen as follows. The function \(\Gamma_\nu^\pm\) multiplies \(Y_\nu\) which diverges at the origin, so we chose the region of integration such that \(\Gamma_\nu^\pm\) vanishes there. The region of integration for \(\Delta_\nu^\pm\) was chosen for a similar reason, as \(I_\nu\) diverges at large distances. Finally, the ranges for \(\tilde{\Gamma}_\nu^\pm\) and \(\tilde{\Delta}_\nu^\pm\) were chosen such that these functions vanish at the core-cladding interface \(r = \rho\), where certain components of the electromagnetic field are required to be continuous.

Using the function \(p_0\), one obtain the general solution to (5.30) which is regular at the origin and vanishes at infinity:

$$Z^\parallel_{0\pm} = a_0 \left[\frac{1}{4}(\zeta - 2\rho^2 \beta) p_0^\pm(\alpha) + f_0^\pm(\sigma_{0\pm})\right], \quad (5.45)$$

where \(a_0\) was factored out for convenience.
Sidebands  Using the decomposition (5.24) for the sidebands also, the inhomogeneous wave equation (5.22) reduces to

\[ B_{\pm 1}^\|_{\pm 1} = B_{\pm 1}^\|_{\pm 1} = \frac{i}{\sqrt{2}} a_{\pm 1} \beta^{-1} f_{\pm 1}(\alpha), \]
\[ B_{\pm 2}^\|_{\pm 2} = B_{\pm 2}^\|_{\pm 2} = \frac{i}{4} a_{\pm 2} \rho^{-2} f_{\pm 2}(\alpha), \]

where \( B_{\pm}^\| \) is as in (5.38). Proceeding in the same way as for the central mode, one obtains

\[ Z_{\pm 1}^\|_{\pm 1} = a_{\pm 1} \left( \frac{i}{\sqrt{2}} \rho \beta_{\pm 1} (\mp U \alpha_1, -W \alpha_2, r) + f_{\pm 1}(\sigma_{\pm 1, \pm}, r) \right), \]
\[ Z_{\pm 1}^\|_{\pm 1} = a_{\pm 1} \left( \frac{i}{\sqrt{2}} \rho \beta_{\pm 1} (\mp U \alpha_1, -W \alpha_2, r) + f_{\pm 1}(\sigma_{\pm 1, \pm}, r) \right), \]
\[ Z_{\pm 2}^\|_{\pm 2} = a_{\pm 2} \left( \frac{i}{4} p_{\pm 2} U^2 \alpha_1, W^2 \alpha_2, r) + f_{\pm 2}(\sigma_{\pm 2, \pm}, r) \right), \]
\[ Z_{\pm 2}^\|_{\pm 2} = a_{\pm 2} \left( \frac{i}{4} p_{\pm 2} U^2 \alpha_1, W^2 \alpha_2, r) + f_{\pm 2}(\sigma_{\pm 2, \pm}, r) \right), \]

where the various \( \sigma \)'s are parameters to be determined from continuity conditions, and the functions \( f_k^\pm \) are derived from \( f_0^\pm \), defined in (5.35), by successive application of the ladder operators \( c_m^\pm \) defined in (4.11):

\[ f_{\pm 1}^+ = \rho c_{m 0} f_0^+, \quad f_{\pm 1}^- = \rho c_{m 0} f_0^-, \]
\[ f_{\pm 2}^+ = \rho^2 c_{m 1} f_0^+, \quad f_{\pm 2}^- = \rho^2 c_{m 1} f_0^- . \]

Having found the longitudinal components of the electromagnetic field, one can now compute the remaining (transverse) components.

5.2 The Transverse Components

To obtain the transverse components of \( Z \), consider the \( \pm \) components of (3.9). Expanding \( g = g^{(0)} + \varepsilon g^{(1)} \), \( Z = Z^{(0)} + \varepsilon Z^{(1)} \), and using the unperturbed equation, one obtains at first order

\[ n \partial_0 Z^{(1)l} + j \epsilon^{ijk} \nabla_j (g^{(0)} l Z^{(1)l}) + j \epsilon^{ijk} \nabla_j (g^{(1)} l Z^{(0)l}) = 0 . \]

In the last term, we may neglect derivatives of the metric perturbation (of order \( \omega_g \)) compared to derivatives of the unperturbed electromagnetic field (of order \( \omega \)) to arrive at

\[ (n \partial_0 \pm ij \partial_\nu) Z^{(1)\pm} = \pm ij [\varepsilon_{\pm} (Z^{(1)\|}) + \chi^\pm \cos(\kappa x_\|)], \]

where we have used (4.7) and have set

\[ \chi^\pm = A_{\|l} \nabla_{\|} Z^{(0)l} + i \beta A_{\|l} Z^{(0)l}. \]
Using the notation of (5.24), a direct calculation shows that

\[
\begin{align*}
\chi^+_{m+2} &= i\beta a_{-2} Z^{(0)\pm}, \\
\chi^+_{m-1} &= \frac{i}{2} a_{-1} (3\beta - i\omega) Z^{(0)\|}, \\
\chi^+_0 &= -i a_{0} (\frac{3}{2}\beta - i\omega) Z^{(0)\pm}, \\
\chi^+_1 &= \frac{1}{\sqrt{2}} a_{1} c_{m+1} Z^{(0)\pm}, \\
\chi^+_2 &= 0, \\
\chi^-_{m+2} &= i\beta a_{+2} Z^{(0)\pm}, \\
\chi^-_{m+1} &= \frac{i}{2} a_{+1} (3\beta + i\omega) Z^{(0)\|}, \\
\chi^-_0 &= -i a_{0} (\frac{3}{2}\beta + i\omega) Z^{(0)-}, \\
\chi^-_1 &= \frac{1}{\sqrt{2}} a_{-1} c_{m-1} Z^{(0)\pm}, \\
\chi^-_2 &= 0.
\end{align*}
\]

(5.57)

To arrive at the stated formula for \(\chi^\pm_0\), we have used \(c_\mp(Z^{(0)\|}) = -i(\beta \mp i\omega) Z^{(0)\pm}\) which follows from (4.9), and the trace condition (5.12). For \(\chi^\pm_{m-1}\) and \(\chi^\pm_{m+1}\) we have used

\[
\rho^2 c_{m+1} c_m f_0 = \rho^2 c_{m-1} c_m f_0 = -\zeta f_0,
\]

(5.62)

which can be shown either using the recursion relations (4.15) or by noting that \(c^-_{m+1} c^+_m = c^+_{m-1} c^-_m = B_0 - \zeta/\rho^2\) and using the Bessel equation \(B_0 f_0 = 0\). These formulae have the general structure that the radial dependence of \(\chi^+_0\) is proportional to \(a_b f_{a+b}\).

Using the notation (5.24), as well as the abbreviations \(\omega_{\pm}\) and \(\beta_{\pm}\) as defined in (5.33) and (5.34), equation (5.55) reduces to

\[
\begin{align*}
i(n\omega_+ \mp ij\beta_{\pm}) Z^{\pm}_{k+} &= \pm ij \left[ \frac{1}{\sqrt{2}} c_{m+k} Z^{\pm}_{k+} + \frac{1}{2} \chi^\pm_k \right], \\
i(n\omega_- \mp ij\beta_{\pm}) Z^{\pm}_{k-} &= \pm ij \left[ \frac{1}{\sqrt{2}} c_{m+k} Z^{\pm}_{k-} + \frac{1}{2} \chi^\pm_k \right].
\end{align*}
\]

(5.63)

(5.64)

Multiplying the first equation by \(n\omega_+ \pm ij\beta_{\pm}\) and the second one by \(n\omega_- \pm ij\beta_{\pm}\), and using the definition of \(\zeta_{\pm}\) given in (5.32), one obtains

\[
\begin{align*}
i\zeta^+ Z^{\pm}_{k+} &= \frac{1}{\sqrt{2}} \rho^2 [\beta_{\pm} \pm i\omega_{\pm}] (c_{m+k} Z^{\pm}_{k+} + \frac{1}{2} \chi^\pm_k), \\
i\zeta^- Z^{\pm}_{k-} &= \frac{1}{\sqrt{2}} \rho^2 [\beta_{\pm} \pm i\omega_{\pm}] (c_{m+k} Z^{\pm}_{k-} + \frac{1}{2} \chi^\pm_k),
\end{align*}
\]

(5.65)

(5.66)

which is structurally similar to the unperturbed equation (4.9).

Now that the field equations are solved in the core and cladding separately, we consider how these solutions match at the core-cladding interface of the waveguide.

### 5.3 Continuity Conditions

From (3.3), one finds that the field components \(D^r, E^\theta, E_z, B^r, H^\theta, H_z\) must be continuous at the core-cladding interface \(r = \rho\). As is well-known, these six continuity conditions are linearly dependent, and it suffices to impose continuity of the four components

\[
\begin{align*}
D^r &= \Re Z^r, \\
B^r &= n^{-1} \Im Z^r, \\
H^\parallel &= n^{-1} g_{\parallel i} \Im Z^i, \\
E^\parallel &= n^{-2} g_{\parallel i} \Re Z^i.
\end{align*}
\]

(5.67)
where \( \Re \) and \( \Im \) refer to the \( j \)-real and \( j \)-imaginary parts, respectively. To assess the jumps of the fields, we introduce the notation

\[
[f] := \lim_{r \nearrow \rho} f(r) - \lim_{r \searrow \rho} f(r),
\]

and write symbolically

\[
[Z] = \begin{pmatrix} \| E' \| & \| B' \| & \| H' \| \end{pmatrix}^T,
\]

which conflates the jumps of all relevant fields in one column vector.

Since the jumps arising in (5.67) are determined by the jumps of the two components

\[
Z^r = \frac{1}{\sqrt{2}}(Z^+ + Z^-),
\]

\[
g_{\| Z^i \|} = Z^{(0)} + \varepsilon Z^{(1)} + \varepsilon A_{\| Z^{(0)i} \|} \cos(\kappa \cdot x_{\|}),
\]

for the perturbed problem it suffices to consider the jumps of

\[
Z^r_{kl} = \frac{1}{\sqrt{2}}(Z^+_{kl} + Z^-_{kl}),
\]

\[
Z_{\| kl} = Z^\|_{kl} + \frac{1}{2} \chi_{\| k},
\]

where the \( Z^a_{kl} \) are the components from the decomposition (5.24), and where

\[
\chi_{\| 0} = a_0 Z^{(0)}_{\|}, \quad \chi_{\| \pm 1} = a_{\pm 1} Z^{(0)\pm}_{\|}, \quad \chi_{\| \pm 2} = 0.
\]

Similar to the decomposition (5.24), we write

\[
[Z] = \begin{pmatrix} \| Z \|^{(0)} + \varepsilon \sum_{k=-2}^{+2} \sum_{l=\pm 1} \| Z^{(1)\| k \| l} \| e^{i(k\theta + l\kappa \cdot x_{\|})} \end{pmatrix} e^{i(\omega t - \beta z + m\theta)} = 0,
\]

where the first term does not contain any \( \varepsilon \) corrections because all components of \( Z^{(1)} \) oscillate with \( \exp(\pm i\kappa \cdot x_{\|}) \).

The four \( j \)-real components of \( \| Z \|_{kl}^{(1)} \) are parametrised by \( \alpha \) (coming from the particular solutions) and \( \sigma_{kl} \) (homogeneous solutions), each having four \( j \)-real parameters. Hence, one can write

\[
\| Z \|_{kl}^{(1)} = a_k (M_{kl} \alpha + \Sigma_{kl} \sigma_{kl}) \quad \text{(no summation implied)},
\]

where \( \alpha \) and \( \sigma_{kl} \) are \( j \)-real vectors:

\[
\alpha = (\Re \alpha_1, \Re \alpha_2, \Im \alpha_1, \Im \alpha_2)^T, \quad \sigma_{kl} \text{ similar},
\]

and \( M_{kl} \) and \( \Sigma_{kl} \) are \( j \)-real \( 4 \times 4 \) matrices. Note that the jumps are not simply linear in the complex coefficients \( \alpha, \sigma_{kl} \) since (5.67) discriminates between the real and complex parts.

In Appendix B.1, it is shown that because the electric and magnetic fields are derived from a single complex field, these matrices are fully determined by their first two rows, which motivates
the following abbreviating notation:

\[
\begin{bmatrix}
  a & b & c & d \\
  e & f & g & h \\
  -c/n_1 & -d/n_2 & a/n_1 & b/n_2 \\
  -g/n_1 & -h/n_2 & e/n_1 & f/n_2
\end{bmatrix} := 
\begin{bmatrix}
  a & b & c & d \\
  e & f & g & h \\
  -c/n_1 & -d/n_2 & a/n_1 & b/n_2 \\
  -g/n_1 & -h/n_2 & e/n_1 & f/n_2
\end{bmatrix} ,
\]

As many matrices considered here have similar structures, we define the following abbreviation

\[
\begin{bmatrix}
  a_1 & a_2 & a_3 & a_4 \\
  \beta & \omega & U & W \\
\end{bmatrix}_\nu := 
\begin{bmatrix}
  \frac{i}{n_1} \nu J_\nu(U) & \frac{i}{n_2} \nu K_\nu(U) \\
  -i \nu J_\nu(U) & -i \nu K_\nu(U) & a_1 n_1 \nu \rho \omega J_\nu(U) & a_4 n_2 \nu \rho \omega K_\nu(U) \\
\end{bmatrix} ,
\]

where the parameters \(a_1, \ldots, a_4\) and \(\beta, \omega\) determine the numerical coefficients in the first row of the matrix, and the last two parameters \(U, W\) determine the arguments of the Bessel functions, whose order is given by the integer \(\nu\) which is given as a general subscript.

Within this class of matrices, we identify the following families, which are useful for the problem considered here:

\[
\Pi_k = \begin{bmatrix}
  1/U & 1/W & 1/U^2 & 1/W^2 \\
  \beta & \omega & U & W \\
\end{bmatrix}_{m+k} ,
\]

\[
\Sigma_{\pm}^k = \begin{bmatrix}
  1/U_{\pm} & 1/W_{\pm} & 1/U_{\pm}^2 & 1/W_{\pm}^2 \\
  \beta_{\pm} & \omega_{\pm} & U_{\pm} & W_{\pm} \\
\end{bmatrix}_{m+k} ,
\]

where \(\beta_{\pm}, \omega_{\pm}\) are defined in (5.33) and (5.34), and \(U_{\pm}, W_{\pm}\) are defined in (5.36). The \(\Pi\) family is used to describe the central mode and to formulate the dispersion relation, and the \(\Sigma\) matrices arise when considering the jumps of the homogeneous solutions to the radial equations.

Similar to the family (5.79), we define

\[
\begin{bmatrix}
  a_1 & a_2 & a_3 & a_4 \\
  \beta & \omega & U & W \\
\end{bmatrix}_\nu := 
\begin{bmatrix}
  \frac{i}{n_1} \nu J_\nu(U) & \frac{i}{n_2} \nu K_\nu(U) \\
  -i \nu J_\nu(U) & -i \nu K_\nu(U) & a_1 n_1 \nu \rho \omega J_\nu(U) & a_4 n_2 \nu \rho \omega K_\nu(U) \\
\end{bmatrix} ,
\]

where the Bessel functions \(J_\nu\) and \(K_\nu\) are replaced by \(Y_\nu\) and \(I_\nu\). Using this notation, let

\[
P_{\pm}^k = \begin{bmatrix}
  U/U_{\pm}^2 & W/W_{\pm}^2 \\
  \beta_{\pm} & \omega_{\pm} & U & W \\
\end{bmatrix}_{m+k} ,
\]

where

\[
\Lambda_{\pm}^k = \text{diag} \ (\Gamma_{\nu}^\pm(1), \Delta_{\nu}^\pm(1), \Gamma_{\nu}^\pm(1), \Delta_{\nu}^\pm(1)) .
\]

This is used to describe the jumps of the particular solutions \(p_{\pm}^k\) defined in (5.39). The functions \(\Gamma_{\nu}^\pm, \Delta_{\nu}^\pm\) appearing here are defined in (5.40) and (5.42), while the functions \(\Gamma_{\nu}^\pm, \Delta_{\nu}^\pm\) defined in
(5.41) and (5.43) do not contribute, since they vanish at the core-cladding interface $r = \rho$.

Having prepared the notation, we now analyse (5.75). By linear independence of the exponential functions, all terms $[Z]^{(0)}_l$ and $[Z]^{(1)}_l$ must vanish separately.

**Dispersion Relation** Because all corrections coming from the gravitational wave oscillate as $\exp(\pm i\kappa \cdot x)$, the first term in (5.75) is the same as for the unperturbed problem. As shown explicitly in Appendix B.2, the continuity condition for this part is given by

$$\Pi_0 \alpha = 0,$$

(5.85)

where $\Pi_0$ is given by (5.80). To obtain a non-trivial solution, the coefficient matrix must be singular:

$$\det \Pi_0 = 0,$$

(5.86)

which is equivalent to

$$(J_m + \mathcal{H}_m) \left( n_1^2 J_m + n_2^2 \mathcal{H}_m \right) = (m\beta/\omega)^2 \left( U^{-2} + W^{-2} \right)^2,$$

(5.87)

where we have used the abbreviations

$$J_m = \frac{J_m'(U)}{U J_m(U)}, \quad \mathcal{H}_m = \frac{K_m'(W)}{W K_m(W)},$$

(5.88)

cf. e.g. Ref. [34, Eq. 3.27 on p. 124] and Ref. [35, Eq. 16.166 on p. 511] (where $\epsilon_2/\epsilon_2$ should read $\epsilon_1/\epsilon_2$). Because $\beta$ and $\omega$ enter this equation through $U$ and $W$ as arguments of Bessel functions, this is a transcendental equation which must be solved numerically. More precisely, by solving $\det \Pi_0 = 0$ for a prescribed vacuum wavelength $\lambda = 2\pi/\omega$ and given waveguide parameters (i.e. core radius $\rho$ and refractive indices $n_1, n_2$), one can determine the effective refractive index $\bar{n} := \beta/\omega$ as a function of $\lambda$: $\bar{n} = \bar{n}(\lambda)$, which constitutes the waveguide dispersion relation. Subsequently, the coefficients $\alpha$ can be determined from (5.85) up to an overall factor which describes the amplitude of the electromagnetic wave.

In general, there are multiple solutions to this equation for given waveguide parameters and given values of $m$ and $\omega$. Here, we focus on single-mode fibres (i.e. fibres with sufficiently thin cores, operated at suitable frequencies) where there is only one single solution for $m = +1$ and there are no solutions for higher values of $m$.

**Central Mode** For the central mode ($k = 0$), the continuity condition can be written as

$$\Sigma_0^+ \sigma_{0 \pm} + P_{0}^0 \Xi \alpha + \frac{1}{2} \Psi_0 \alpha = 0.$$

(5.89)

The first term describes the jumps arising from the homogeneous solution in (5.45). In the second term, which comes from the particular solution, the matrix $\Xi$ describes the discontinuity of the factor $\frac{1}{4}(\zeta - 2\rho^2\beta^2)$ in (5.45):

$$\Xi = -\frac{1}{4} \text{diag}(2\rho^2\beta^2 - U^2, 2\rho^2\beta^2 + W^2, 2\rho^2\beta^2 - U^2, 2\rho^2\beta^2 + W^2).$$

(5.90)
Finally, the last term contains all contributions from the $\chi$-terms in (5.65), (5.66) and (5.73):

$$\Psi_0 = \left\{ \frac{1}{2} U_2^2 / U^3, \frac{1}{2} W_3^2 / W^3, \frac{1}{2} U_4^2 / U W, \frac{1}{2} W_2^2 / W^4 \right\}_m,$$

where

$$U_2^2 = +\rho^2 (n_1^2 \omega^2 - 2 \beta^2), \quad U_4^2 = +\rho^2 (n_2^2 \omega^2 - 3 \beta^2),$$

$$W_2^2 = -\rho^2 (n_1^2 \omega^2 - 2 \beta^2), \quad W_3^2 = -\rho^2 (n_2^2 \omega^2 - 3 \beta^2).$$

At the considered level of accuracy, we may neglect terms of order $\Omega$ in $P_0^{\pm}$. This matrix contains the functions $\Gamma^{\pm}_\nu$ and $\Delta^{\pm}_\nu$ (via the matrix $\Lambda^{\pm}_\nu$). In (5.40) and (5.42) we may thus replace $U^{\pm}$ and $W^{\pm}$ by $U$ and $W$ which yields the integrals

$$\Gamma_\nu(z) = \frac{\pi}{2} \int_0^z J_\nu(Uz')^2 z' dz',$$

$$\Delta_\nu(z) = -\int_z^\infty K_\nu(Wz')^2 z' dz',$$

which also arise in similar calculations describing rotating waveguides. They evaluate to

$$\Gamma_\nu(z) = \frac{\pi}{2} \frac{z^2}{2} \left( J_\nu(Uz)^2 - J_{\nu-1}(Uz)J_{\nu+1}(Uz) \right),$$

$$\Delta_\nu(z) = \frac{z^2}{2} \left( K_\nu(Wz)^2 - K_{\nu-1}(Wz)K_{\nu+1}(Wz) \right).$$

Expanding $\Sigma_{0,\pm} = \Pi_0 \pm \Omega \Sigma^{(1)} + O(\Omega^2)$, we obtain the inverse

$$(\Sigma_0^{\pm})^{-1} = \pm \Omega^{-1} \Sigma' + O(\Omega^0),$$

where

$$\Sigma' = \frac{1}{\text{tr}(\Sigma^{(1)} \text{adj} \Pi_0)} \text{adj} \Pi_0.$$

Here, $\text{adj} \Pi_0$ denotes the adjugate of $\Pi_0$, which is the transpose of the cofactor matrix. This leads to

$$\sigma_{0,\pm} = \mp \Omega^{-1} \Sigma'(P_0 \Xi + \frac{1}{2} \Psi) \alpha + O(\Omega).$$

Let us analyse this result in a bit more detail. First, $\Sigma'$ can be re-expressed in a more transparent way by considering the denominator $\text{tr}(\Sigma^{(1)} \text{adj} \Pi_0) = \partial_\omega \det \Sigma_0^\pm$. Since $\Sigma_0^+$ is obtained from $\Pi_0$ by the substitution $\omega \rightarrow \omega + \omega_\theta$ and $\beta \rightarrow \beta + \hat{\kappa} || \omega_\theta$, one has

$$\text{tr}(\Sigma^{(1)} \text{adj} \Pi_0) = \partial_\omega \det \Pi_0 + \hat{\kappa} || \partial_\beta \det \Pi_0.$$

The ratio of these partial derivatives has a physical interpretation: differentiating the defining equation for the dispersion relation $\det \Pi_0(\beta(\omega), \omega) = 0$ with respect to $\omega$ yields

$$\beta' = \frac{\partial \beta}{\partial \omega} = -\frac{\partial_\omega \det \Pi_0}{\partial_\beta \det \Pi_0},$$

19
so the ratio describes how the propagation constant $\beta$ changes with varying frequency $\omega$ (i.e. the reciprocal group velocity). Equivalently, this can be expressed in terms of the effective refractive index $\bar{n} = \bar{n}(\omega)$ as

$$\beta' = \bar{n} + \omega \frac{\partial \bar{n}}{\partial \omega}.$$  \hfill (5.103)

Hence, one obtains the alternative formula

$$\Sigma' = -\frac{\partial \beta}{\beta' - \bar{k}_\parallel} \text{det} \Pi_0 - \bar{k}_\parallel \text{adj} \Pi_0.$$  \hfill (5.104)

Next, using the definition of the adjugate and the fact that $\Pi_0$ is singular, we have $\Pi_0 \Sigma' = 0$, so $\text{im} \Sigma' \subset \ker \Pi_0$. But since the kernel of $\Pi_0$ is one-dimensional and spanned by $\alpha$, it is seen that $\sigma_{0,\pm}$ is again proportional to $\alpha$. This leads to the general formula

$$\sigma_{0,\pm} = \mp \frac{1}{4\Omega} \frac{c_1}{\beta' - \bar{k}_\parallel} \alpha,$$  \hfill (5.105)

where $c_1$ is a coefficient depending on the waveguide parameters, which we compute numerically. The overall factor $\mp 1/4$ was chosen in analogy to similar results in vacuum, where $c_1 = \beta' = 1$. This equation essentially determines the phase induced by the gravitational wave, as is shown in detail in Section 6. There, we also show that appropriate emission conditions ensure that no phase shift arises if the GW is collinear with the symmetry axis of the waveguide.

**Sidebands** For the sidebands, we find the continuity conditions to be of the form

$$\Sigma_{\pm 1} C_{\pm} \sigma_{\pm 1,\pm} - i \sqrt{2} P_{\pm 1} \sigma_{\pm 1,\pm} + \frac{1}{2} \Psi_{\pm 1} \alpha = 0,$$  \hfill (5.106)

$$\Sigma_{\pm 1} C_{\mp} \sigma_{\mp 1,\mp} - i \sqrt{2} P_{\mp 1} \sigma_{\mp 1,\mp} + \frac{1}{2} \Psi_{\mp 1} \alpha = 0,$$  \hfill (5.107)

$$\Sigma_{\pm 2} (C_{\mp})^2 \sigma_{\pm 2,\pm} + \frac{1}{4} \Psi_{\pm 2} \sigma_{\pm 2,\pm} + \frac{1}{2} \Psi_{\pm 2} \alpha = 0,$$  \hfill (5.108)

$$\Sigma_{\pm 2} (C_{\mp})^2 \sigma_{\mp 2,\mp} + \frac{1}{4} \Psi_{\mp 2} \sigma_{\mp 2,\mp} + \frac{1}{2} \Psi_{\mp 2} \alpha = 0,$$  \hfill (5.109)

where the matrices $C_k$ account for the coefficients $\mp U$ and $-W$ in (4.15)

$$C_{\pm 1} = \text{diag} (\mp U, -W, \mp U, -W) \ , \ \ \ \ C_{\pm 2} = \text{diag} \left( U^2, W^2, U^2, W^2 \right) ,$$  \hfill (5.110)

and the $\Psi$-matrices are given by

$$\Psi_{\pm 1} = \frac{1}{\sqrt{2}} \left( 2\rho \beta \Psi_{\pm 1} \right),$$  \hfill (5.111)

where

$$\Psi_{\pm 1}^{(a)} = \begin{pmatrix} \rho \beta J_{m \pm 1} & \rho \beta K_{m \pm 1} & i n_1 \frac{m \pm 1}{U} \rho \beta J_{m \pm 1} & -i n_2 \frac{m \pm 1}{W} \rho \beta K_{m \pm 1} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  \hfill (5.112)

$$\Psi_{\pm 1}^{(b)} = \begin{pmatrix} -m \pm 1 & \mp m \pm 1 & 0 & 0 \\ \pm i \rho \beta J_{m \pm 1} & i \rho \beta K_{m \pm 1} & 0 & 0 \\ \pm i \rho \beta J_{m \pm 1} & i \rho \beta K_{m \pm 1} & 0 & 0 \\ \pm i \rho \beta J_{m \pm 1} & i \rho \beta K_{m \pm 1} & 0 & 0 \end{pmatrix},$$  \hfill (5.113)
and
\[ \Psi_{\pm 2} = \frac{1}{2} \begin{vmatrix} i \frac{\rho^3}{\nu} J_{m \pm 1} & \mp i \frac{\rho^3}{\nu} K_{m \pm 1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \] (5.114)

Here, the arguments of the Bessel functions have been suppressed for brevity: \( J_\nu = J_\nu(U) \) and \( K_\nu = K_\nu(W) \).

Since we are concerned with single-mode fibres, barring one exception which is discussed below, all coefficient matrices \( \Sigma_k^\pm \) with \( k \neq 0 \) are invertible. This is because for vanishing \( \Omega \) they reduce to the non-singular matrices \( \Pi_k \) (with \( k \neq 0 \)), and invertibility is maintained for sufficiently small \( \Omega \) by continuity. Thus, as a first approximation one can replace \( \Sigma_k^\pm \) by \( \Pi_k \) to obtain

for \( k = \pm 1 \):
\[ \sigma_{k, \pm} = -C_k^{-1}\Pi_k^{-1} \left( \frac{1}{2} \Psi_k - \frac{i}{\sqrt{2}} \rho \beta \rho \nu \right) \alpha + O(\Omega), \] (5.115)

for \( k = \pm 2 \):
\[ \sigma_{k, \pm} = -C_k^{-1}\Pi_k^{-1} \left( \frac{1}{2} \Psi_k + \frac{1}{4} E_k \right) \alpha + O(\Omega), \] (5.116)

which shows explicitly that the sideband amplitudes are of order \( \varepsilon \) only, and thus much smaller than the corrections of the central mode of order \( \varepsilon \omega/\omega_g \).

The exception to the above case is the following. Since
\[ J_{-\nu}(z) = (-1)^\nu J_\nu(z), \quad K_{-\nu}(z) = +K_\nu(z), \] (5.117)

see e.g. Ref. [36, eqns. 9.1.5 on p. 358 and 9.6.6 on p. 375], the dispersion relation (5.87) is invariant under \( m \to -m \), which implies that if \( m + k = -m \), i.e. \( k = -2m \), then \( \Pi_k \) is singular again. In our case, since \( k \) ranges from \(-2\) to \(+2\), this arises only if \( m = \pm 1 \), so this effect does occur in single-mode fibres. Here, we restrict the discussion to the case \( m = +1 \), as the alternative \( m = -1 \) is completely analogous. For this special case \( m = +1 \) and \( k = -2 \), the matrices \( \Sigma_{-2}^\pm \) are close to being singular:
\[ \Sigma_{-2}^\pm = \Pi_{-2} \pm \Omega \delta \Sigma_{-2} + O(\Omega^2), \] (5.118)

which has the inverse
\[ (\Sigma_{-2}^\pm)^{-1} = \pm \Omega^{-1}\Sigma'' + O(\Omega^0), \] (5.119)

where
\[ \Sigma'' = \frac{1}{\text{tr}[\delta(\Pi_{-2}) \text{adj}(\Pi_{-2})]} \text{adj}(\Pi_{-2}), \] (5.120)

so that instead of (5.116) we obtain for \( k = -2 \)
\[ C_{-2} \sigma_{-2, \pm} = \mp \Omega^{-1}\Sigma'' \left( \frac{1}{2} \Psi_{-2} + \frac{1}{4} P_{-2} \right) \alpha + O(\Omega^0). \] (5.121)

Similar to \( \Pi_0 \) whose kernel is spanned by \( \alpha \), the kernel of \( \Pi_{-2} \) is also one-dimensional and spanned by
\[ \alpha^* = (\alpha_1, -\alpha_2, -\alpha_3, +\alpha_4). \] (5.122)
Since the range of $\Sigma''$ is the kernel of $\Pi_2$, the two vectors $C_2\sigma_{-2,\pm}$ are proportional to $\alpha^*$, and we find, similar to the central mode

$$\begin{align*}
C_2\sigma_{-2,\pm} &= \pm a_2 \frac{\Delta^2}{2\Omega} \frac{c_2}{\beta' - \kappa_{||}} \alpha^*,
\end{align*}$$

(5.123)

where $\beta'$ is the same constant as in the phase correction (5.105), and the coefficient $c_2$ is a new parameter, which we determine numerically. Here, the relative index difference $\Delta$ is defined as

$$\Delta = 1 - \frac{n_2}{n_1},$$

(5.124)

and we have factored out $\Delta^2/2$ such that $c_2$ is typically of order unity (see the numerical examples below).

As shown explicitly in Section 7, this formula essentially describes the perturbation of the polarisation by the gravitational wave.

### 6  Perturbation of the Phase

In this section, we implement boundary conditions which model a setup where monochromatic laser light of definite polarisation is injected into the fibre. As we will see, contrary to the unperturbed case, this will require adding further terms to the field computed so far. This leads to an expression for the phase correction which vanishes at the point where the light enters the fibre and grows with increasing distance. A similar construction for the eikonal in a non-dispersive regime can be found in Ref. [4, Sect. 2].

Neglecting boundary effects from the ends of the waveguide, the light sent into the waveguide (or coming out of it) is simply obtained by restricting the field to a definite value of $z$, where we consider $z = 0$ for simplicity. Also, we restrict the discussion to the “large corrections”, i.e. to $k = 0$ and $k = -2m$ where the perturbations were found to be of order $1/\Omega$. The remaining “small components” are expected to be experimentally irrelevant.

Recall that the coefficients of the central mode ($k = 0$) were given by

$$\sigma_{0,\pm} = \pm \frac{1}{4\Omega} \frac{c_1}{\beta' - \kappa_{||}} \alpha.$$  

(6.1)

Since this is proportional to the unperturbed coefficients $\alpha$, we may write the central mode ($k = 0$) of the overall field in the form

$$Z(0)_0^a + \epsilon Z(1)_0^a = \tilde{3}_a \tilde{\Phi},$$

(6.2)

where

$$\begin{align*}
\tilde{3}^|| &= f_0(\alpha; r), \\
i\zeta \tilde{3}_\pm &= \frac{1}{\sqrt{2}} \rho(\beta \pm ijn\omega) f_\pm(\alpha; r), \\
\tilde{\Phi} &= e^{i\psi(0)} \left[ 1 - \frac{\epsilon}{4\Omega} \frac{a_0 c_1}{\beta' - \kappa_{||}} \left( e^{+i\kappa_{||}x} - e^{-i\kappa_{||}x} \right) \right], \\
\psi(0) &= \omega t - \beta z + m\theta.
\end{align*}$$

(6.3-6.6)
Evaluating the perturbation terms at \( z = 0 \), we find that this describes a situation where the incoming phase is modulated in a highly specific way by the gravitational wave. This is not the situation which we want to model so that correction terms are needed: instead of the phase modulation behaviour given by \( \tilde{\Phi} \), we are concerned with the scenario where a polarised monochromatic plane wave (as emitted by a laser source) is injected into the fibre at \( z = 0 \).

Hence, we require the injected field to have a definite phase of \( \psi = \omega t + m\theta \) (and definite polarisation, see the next section) at \( z = 0 \).

Such boundary conditions can be implemented by adding appropriate solutions of the unperturbed equations of slightly shifted frequencies. Specifically, we add terms of the form

\[
3^n a = 3^n \exp(i(\omega_\pm t - (\beta \pm \omega_g\beta')z + m\theta)),
\]

where the term \( \beta' = \partial\beta/\partial\omega \) ensures that the unperturbed dispersion relation is satisfied. (The shift in the frequency also causes a slight perturbation in the amplitude, but this is of order \( \Omega \) and thus negligible here.) The corrected overall field is thus

\[
Z^a_0 = 3^n \tilde{\Phi} + \frac{\varepsilon a_0 c_1}{4\Omega \beta' - \tilde{K}} (3^n a^+ - 3^n a^-) = 3^n \Phi,
\]

where

\[
\Phi = e^{i\psi(0)} \left[ 1 - \frac{\varepsilon a_0 c_1}{4\Omega \beta' - \tilde{K}} \left( e^{i\omega_g t} (e^{-i\omega_g\beta'z} - e^{-i\omega_g\beta'z}) - e^{-i\omega_g t} (e^{i\omega_g\beta'z} - e^{i\omega_g\beta'z}) \right) \right].
\]

Assuming the fibre to be much shorter than the GW wavelength, i.e. \( \omega_g z \ll 1 \), one finally obtains

\[
\Phi \approx e^{i\psi(0)} \left[ 1 - \frac{\varepsilon a_0 c_1 \cos(\omega_g t)\omega z}{2} \right] \approx e^{i\psi(0)} \exp(-\frac{\varepsilon a_0 c_1 \cos(\omega_g t)\omega z}{2}),
\]

which can be written as a perturbation of the phase:

\[
\psi = \omega t - \beta z + m\theta - \frac{1}{2} \varepsilon a_0 c_1 \omega z \cos(\omega_g t).
\]

This shows that the phase perturbation grows linearly with the distance from the emission point and oscillates in time exactly as the gravitational wave does.

### 7 Perturbation of the Polarisation

Having identified the perturbation of the central mode \( k = 0 \) as a correction to the phase, we now show that the other “large term” with \( m + k = -m \) corresponds to a perturbation of the polarisation.

To see this, we follow Ref. [34], where it is shown that modes with real field patterns, i.e. an angular dependence of the form \( \cos \theta \) or \( \sin \theta \), correspond to linear polarisation. To describe such fields, one must take linear combinations of the solutions with \( m = +1 \) and \( m = -1 \), and we shall make the standard assumption of weak guidance \( \Delta \equiv (n_1 - n_2)/n_1 \ll 1 \). It should be noted that non-zero values of \( \Delta \) cause the modes not to be perfectly linearly polarised, but
these deviations are independent of $t$ and $z$ and can thus not be mistaken for gravitational wave signals.

Let us briefly review the construction in the unperturbed case. For brevity, we restrict the discussion to the field in the core, as the analysis of the cladding is completely analogous. Assuming weak guidance, the longitudinal field components are

\[
\begin{align*}
\text{for } m = +1 : & \quad Z^i = (1 - ij)J_{+1}(Ur/\rho)e^{i(\omega t - \beta z + \theta)}, \\
\text{for } m = -1 : & \quad Z^i = (1 + ij)J_{-1}(Ur/\rho)e^{i(\omega t - \beta z - \theta)},
\end{align*}
\]

(7.1)

cf. Ref. [34, Section 3.2]. The overall normalisation is arbitrary and of no concern for the discussion here. At the level of the amplitude vector $\alpha$, this means

\[
\begin{align*}
\text{for } m = +1 : & \quad \alpha_1 + j\alpha_2 = 1 - ij, \\
\text{for } m = -1 : & \quad \alpha^*_1 + j\alpha^*_2 = 1 + ij.
\end{align*}
\]

(7.2)

Plugging this into the expressions for the unperturbed field (4.9), setting $\beta = n\omega$ with $n \approx n_1$ and computing the Cartesian components via (5.5), one obtains the so-called HE-modes

\[
\begin{align*}
\text{HE}_{1,\pm 1} : \quad \begin{pmatrix} Z^x & Z^y \end{pmatrix} &= \mathcal{A}(r) \begin{pmatrix} \mp i - j & 1 \mp ij \end{pmatrix},
\end{align*}
\]

(7.3)

where $\mathcal{A}(r) = \frac{n\omega}{U}J_0(Ur/\rho)$. The second index distinguishes the cases $m = +1$ and $m = -1$. The so-called LP-modes are then given by the linear combinations

\[
\begin{align*}
\text{LP}_{0,1,x} &= \frac{1}{2}(\text{HE}_{1,+1} - \text{HE}_{1,-1}), \\
\text{LP}_{0,1,y} &= \frac{1}{2}(\text{HE}_{1,+1} + \text{HE}_{1,-1}),
\end{align*}
\]

(7.4)

whose $x$- and $y$-components are given by

\[
\begin{align*}
\text{LP}_{0,1,x} : \quad \begin{pmatrix} Z^x & Z^y \end{pmatrix} &= \mathcal{A}(r) \begin{pmatrix} +1 & +j \end{pmatrix}, \\
\text{LP}_{0,1,y} : \quad \begin{pmatrix} Z^x & Z^y \end{pmatrix} &= \mathcal{A}(r) \begin{pmatrix} -j & +1 \end{pmatrix}.
\end{align*}
\]

(7.5)

These modes are thus linearly polarised in the $x$- and $y$-directions, respectively.

This construction can be carried over directly to the perturbed problem. For this, we determine the coefficients of the “large sidebands” to be

\[
\begin{align*}
\text{for } m = +1 : & \quad C_{-2}\sigma_{-2,\pm} = \pm a_{-2}\frac{\Delta^2}{2\Omega}\frac{c_2}{\beta' - \hat{k}_i}\alpha^*, \\
\text{for } m = -1 : & \quad C_{+2}\sigma_{+2,\pm} = \pm a_{+2}\frac{\Delta^2}{2\Omega}\frac{c_2}{\beta' - \hat{k}_i}\alpha.
\end{align*}
\]

(7.6)

As for the central mode, we impose boundary conditions such that these sidebands do not contribute at $z = 0$. We may thus repeat the argument from the previous section to find that these sidebands grow (for $\omega_gz \ll 1$) linearly with the distance $z$ and oscillate in time as $\cos(\omega_g t)$,
which is described by the function
\[ \tilde{f}(t, z) := c_2 \Delta^2 \cos(\omega_g t) \omega z. \] (7.8)

With this, one finds (up to the overall phase correction from before) that the perturbation of the longitudinal component of the field is obtained from the unperturbed expression by the substitution
\[ (1 - ij)J_{+1} e^{+i\theta} \mapsto (1 - ij)J_{+1} e^{+i\theta} + i\epsilon a_{-2} f(t, z)(1 + ij)J_{+1} e^{-i\theta}, \] (7.9)
\[ (1 + ij)J_{-1} e^{-i\theta} \mapsto (1 + ij)J_{-1} e^{+i\theta} + i\epsilon a_{+2} f(t, z)(1 + ij)J_{-1} e^{+i\theta}, \] (7.10)

where the arguments of the Bessel functions have been suppressed for brevity: they are \( J_{\pm 1} = J_{\pm 1}(Ur/\rho) \).

Using (5.10) and (5.11), the factor arising here evaluates to
\[ ia_{\mp 2} = \frac{1}{2}(A_{xx} - A_{yy}) \mp A_{xy}. \] (7.11)

Defining the abbreviations
\[ \xi = \frac{1}{2}(A_{xx} - A_{yy}) \tilde{f}(t, z), \quad \eta = A_{xy} \tilde{f}(t, z), \] (7.12)

and forming the same linear combinations as in the unperturbed case, we find the perturbed LP-modes
\[ \tilde{L}P_{0,1,x} : \left( \begin{array}{c} Z^x \\ Z^y \end{array} \right) = \mathcal{A}(r) \left( \begin{array}{c} 1 + i\epsilon \xi + ij \epsilon \eta \\ j(1 - i\epsilon \xi) - i\epsilon \eta \end{array} \right), \] \[ \tilde{L}P_{0,1,y} : \left( \begin{array}{c} Z^x \\ Z^y \end{array} \right) = \mathcal{A}(r) \left( \begin{array}{c} 1 + i\epsilon \xi - i\epsilon \eta \\ -(1 + i\epsilon \xi) - i\epsilon \eta \end{array} \right). \] (7.13)

Comparing with the unperturbed LP-modes (7.5), we find that the effect of the gravitational wave is the following transformation of the polarisation directions:
\[ \begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} \mapsto \begin{pmatrix} 1 - i\epsilon \xi & -i\epsilon \eta \\ -i\epsilon \eta & 1 + i\epsilon \xi \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix}. \] (7.14)

The modes of different polarisation \( \epsilon_x \) and \( \epsilon_y \) thus obtain slightly different phase shifts coming from \( 1 \mp i\epsilon \xi \approx \exp(\mp i\epsilon \xi) \), which arise whenever \( A_{xx} \neq A_{yy} \). This can be interpreted as weak birefringence.

In contrast, the \( \eta \)-terms, which correspond to \( A_{xy} = A_{yx} \), cannot be absorbed into the phase, as they describe weak deformations of linear polarisation into elliptic polarisation of alternating direction.

To obtain a definite interpretation in terms of polarisation states as measured by a physical polarisation filter or polarising beam splitter, one would have to model such devices in the perturbed metric. Accordingly, one might think that the above stated transformation could be absorbed in a redefinition of the polarisation directions. However, such redefinitions of the basis cannot alter the result significantly, as the perturbed basis is necessarily related to the unperturbed one by a real transformation, which cannot absorb the complex transformation coefficients found here. Hence, both the birefringence term \( \xi \) and the deformation term \( \eta \) are
genuine physical effects and not merely coordinate artifacts. For completeness, we also give the equivalent transformation in the complex basis \( f_\pm = \frac{1}{\sqrt{2}} (e_x \pm i e_y) \), which describes circular polarisation:

\[
\begin{pmatrix}
    f_+ \\
    f_-
\end{pmatrix} \mapsto \begin{pmatrix} 1 & -\varepsilon (i \xi - \eta) \\ -\varepsilon (i \xi + \eta) & 1 \end{pmatrix} \begin{pmatrix} f_+ \\
    f_-
\end{pmatrix}.
\]

(7.15)

Also in this basis, the effect of the gravitational wave is to introduce a weak mixing of the two polarisation states which grows with the distance from \( z = 0 \) and oscillates in time.

8 Numerical Examples and Angular Dependence

Having found the first order effect of weak gravitational waves on the guided modes, let us give some numerical examples. Table 1 lists two sets of parameters for typical single-mode fibres.

| Quantity          | Fibre 1   | Fibre 2   |
|-------------------|-----------|-----------|
| \( n_1 \)         | 1.4712    | 1.4715    |
| \( n_2 \)         | 1.4659    | 1.4648    |
| \( \bar{n} \)     | 1.4682    | 1.4682    |
| \( \rho \)        | 4.1 \( \mu \)m | 4.1 \( \mu \)m |
| \( \Delta \) Relative Index Difference | 0.36% | 0.45% |
| NA Numerical Aperture | 0.125 | 0.140 |

Tab. 1: Parameters of typical single-mode fibres at a vacuum wavelength \( \lambda = 1500 \) nm. (with \( m = \pm 1 \)) operated at a vacuum wavelength of \( \lambda = 1500 \) nm. The first parameter set is the same as in Refs. [18] and [19], where the influence of Earth’s gravitational field and its rotation on waveguides was analysed. Both fibres have the same core radius and effective refractive index \( \bar{n} \), but their constituents have different refractive indices \( n_1, n_2 \). Hence, their values for the relative index difference \( \Delta = (n_1 - n_2)/n_1 \) and the numerical aperture \( NA = \sqrt{n_1^2 - n_2^2} \) also differ.

| Quantity          | Fibre 1   | Fibre 2   |
|-------------------|-----------|-----------|
| \( \beta' \) Inverse Group Velocity | 1.4716 | 1.4723 |
| \( c_1 \) Phase Parameter       | 1.4665 | 1.4662 |
| \( c_2 \) Polarisation Parameter | 0.3482 | 0.3219 |

Tab. 2: Numerical results for the coefficients \( c_1, c_2 \) and \( \beta' \).

We have implemented our calculations in Wolfram Mathematica and computed the quantities \( c_1, c_2 \) and \( \beta' \) which arise in (5.105) and (5.123). The results are given in Table 2. The inverse group velocity \( \beta' = \partial \beta / \partial \omega \) is larger than the effective refractive index \( \bar{n} \), which shows that the waveguide exhibits normal dispersion \( \partial \bar{n} / \partial \omega > 0 \Leftrightarrow \partial \bar{n} / \partial \lambda < 0 \).

The “phase parameter” \( c_1 \) determines the perturbation of the phase via (6.11). As can be seen from Table 1, this quantity is comparable to the effective refractive index \( \bar{n} \) (or equally \( n_1 \) or \( n_2 \)): the relative differences are 0.12% and 0.14%, respectively, which is even smaller than the relative index difference \( \Delta \). For most applications, the approximation \( c_1 \approx \bar{n} \) will thus be sufficient.
The “polarisation parameter” $c_2$ determines the perturbation of the polarisation according to (7.14) and (7.15) via the function $\tilde{f}$ defined in (7.8). Here, we see the reason for factoring $\Delta^2$ in (5.123) and (7.8): for the two considered fibres, the values of $\Delta^2$ differ by a factor of approximately 1.6, but the values of $c_2$ are almost the same.

Together, the two parameters $c_1, c_2$ fully characterise the influence of the gravitational wave on the fibre modes derived here. To describe the angular dependence of these effects, we choose coordinates $x, y, z$ where $z$ is the distance along the symmetry axis of the waveguide and the wave-vector of the gravitational wave lies in the $xz$-plane. The unit vector $\hat{\kappa}_\parallel$ in $\kappa = \omega_g (dt - \hat{\kappa}_i dx^i)$ and the GW amplitude $A_{ij}$ can then be parameterised by

$$
(\hat{\kappa}_i) = (\sin \vartheta, 0, \cos \vartheta), \quad (A_{ij}) = \begin{pmatrix} h_+ \cos^2 \vartheta & h_x \cos \vartheta & -h_+ \cos \vartheta \sin \vartheta \\ h_x \cos \vartheta & -h_+ & -h_x \sin \vartheta \\ -h_+ \cos \vartheta \sin \vartheta & -h_x \sin \vartheta & h_+ \sin^2 \vartheta \end{pmatrix},
$$

where $h_+$ and $h_x$ are real constants describing the two possible polarisation states of the gravitational wave. Here, $\vartheta$ denotes the incidence angle of the gravitational wave, which is not to be confused with the angular coordinate of the cylindrical coordinate system ($r, \theta, z$). The terms relevant for our purposes are

$$
A_{zz} = h_+ \sin^2 \vartheta, \quad \frac{1}{2}(A_{xx} - A_{yy}) = h_+ \frac{1}{2}(1 + \cos^2 \vartheta), \quad A_{xy} = h_x \cos \vartheta,
$$

as they determine the phase shift, the birefringence effect, and the deformation of the electromagnetic polarisation, respectively. Their angular dependence is depicted in Figure 1, where we plot only absolute values in order to obtain positive radii.

As expected, the phase shift vanishes whenever the gravitational wave propagates in the same or the opposite direction as the electromagnetic wave and is maximal when the two waves propagate in orthogonal directions. The polarisation-dependent effects are maximal in the collinear case and are minimal for orthogonal propagation. Note that the deformation of linear polarisation to elliptic polarisation vanishes for orthogonal propagation, but the birefringence effect persists in this case. Moreover, the phase and birefringence terms have a definite sign (for fixed values of $h_+$ and $h_x$) but the polarisation term $A_{xy}$ can have arbitrary sign. For example, if $h_x$ is positive, then the “right lobe” in Figure 1 corresponds to positive $A_{xy}$, while the “left lobe” corresponds to negative values.

The order-of-magnitude of the overall effect caused by continuous GW’s emitted by a rotating neutron star can be estimated by

$$
\varepsilon \approx 10^{-24} \left( \frac{e}{10^{-6}} \right) \left( \frac{I}{10^{38} \text{kg m}^2} \right) \left( \frac{f}{\text{kHz}} \right)^2 \left( \frac{d}{\text{kpc}} \right)^{-1},
$$

where $I$ is the moment of inertia about the rotation axis, $e$ the eccentricity parameter, $f = \omega_g / 2\pi$ the emitted frequency, and $d$ is the distance to the source [37; 38, Sect. 4.2]. (As no continuous GW’s have been detected so far, no precise values for these quantities are known, but the normalisations indicate plausible values.) Using $c_1 \approx \bar{n} \approx 1$, the amplitude of the phase perturbation
is found to be approximately

$$\delta \psi \approx 2 \times 10^{-6} \text{ mrad} \times \left( \frac{e}{10^{-6}} \right) \left( \frac{I}{10^{38} \text{ kg m}^2} \right) \left( \frac{f}{\text{kHz}} \right)^2 \left( \frac{d}{\text{kpc}} \right)^{-1} \left( \frac{\ell}{\text{km}} \right),$$

where $\ell$ is the length of the optical fibre. Relative to this phase perturbation, the amplitude perturbation effects are suppressed by an additional factor of $\Delta^2 \approx (0.5\%)^2 = 2.5 \times 10^{-5}$, so that we expect only the phase perturbation to be of experimental relevance. We leave an experimental feasibility assessment for future work, but to the best of our knowledge, this is beyond experimental reach in the foreseeable future.

9 Comparison with Light Propagation in Vacuum

Finally, we comment on the relation to standard results for light propagation in vacuum. First, consider the phase (1.2) for $m = 0$. In vacuum, the dispersion relation reduces to $\beta = \omega$ and the phase parameter is $c_1 = 1$, so that the overall phase takes the form

$$\psi_{\text{vac}} = k_\mu x^\mu \frac{1}{2} \varepsilon A_{zz} \omega z \cos(\omega g t), \quad (9.1)$$

where $k = \omega (dt - dz)$. This is in agreement with the standard formula

$$\psi^{(1)} = \frac{k_\mu k_\nu}{2\omega^2} \int_{\omega z}^0 h^{\mu\nu}(x^\rho + sk^\rho/\omega^2) ds, \quad (9.2)$$

where $s \mapsto x^\rho + sk^\rho/\omega^2$ is the light ray emitted from the injection point ($z = 0$) at $s = -\omega z$ and reaching the observation point at $s = 0$, cf. Ref. [6, Eq. (3.28a)]. For the particular case considered here, this integral evaluates to

$$\psi^{(1)} = -\frac{\omega A_{zz}}{\omega g(1 - \hat{k}_z)} [\sin(\kappa.x) - \sin(\kappa.x - \omega g z(1 - \hat{k}_z))], \quad (9.3)$$

and expanding to leading order in $\omega g z \ll 1$ reproduces (9.1). (As already observed in Ref. [12, Sect. 5], the second term is absent in Ref. [11, Eq. (2.20)] because of different boundary conditions used. We note that without this term the result is ill-behaved in the limit $\hat{k}_z \to 1$.)

Secondly, we note that $\Delta = (n_1 - n_2)/n_1$ in (1.3) vanishes in vacuum, so that no polarisation perturbations arise at the order considered. This is in agreement with the geometrical optics setting, where such effects arise only at next-to-leading order in $\omega g/\omega$, which is negligible in the considered setup.

Thus, the formulae presented here indeed reproduce the standard results for vacuum in the limit $n_1 \to 1$ and $n_2 \to 1$.

10 Conclusion

We have solved perturbatively the Maxwell equations for a step-index waveguide in the presence of a weak gravitational wave (GW) of low frequency. The correction of the central mode was found to describe a perturbation of the phase as expressed by (6.11). The parameter $c_1$ arising
here was found to be well approximated by the effective refractive index $\bar{n}$: the relative error is smaller than the relative index difference $\Delta = (n_1 - n_2)/n_1$.

Additionally, the “large sideband” (where the azimuthal mode index $m = \pm 1$ is flipped) was identified as a perturbation of the electromagnetic wave polarisation. This perturbation for linearly polarised fields is given explicitly in (7.14), and for circular polarisation in (7.15). Compared to the phase shift, this polarisation-dependent effect (describing birefringence and coupling of polarisation states) is suppressed by a factor of $\Delta^2$. However, contrary to the phase shift, this effect is also present when the gravitational wave propagates along the symmetry axis of the waveguide, as it is sensitive to the components $\frac{1}{2}(A_{xx} - A_{yy})$ and $A_{xy}$ of the gravitational wave amplitude. Similar results for birefringence induced by gravitational waves were obtained for plane waves in infinitely extended media [39; 40], and also in vacuum for exact plane gravitational wave backgrounds [41].

Further sidebands, where the azimuthal mode index is shifted by $\pm 1$ or $\pm 2$, were also computed (equations (5.115) and (5.116)) but found to be negligibly small compared to the two main effects just mentioned: they are suppressed by the frequency ratio $\Omega = \omega_g/\omega$.

Comparing with light propagation in vacuum, we find that the waveguide dispersion enters in intermediate steps of the calculation, e.g. in (5.105) (which is structurally similar to equation (2.20) in Ref. [11]) via the term $\beta' = \partial \beta / \partial \omega$. However, having implemented physically plausible boundary conditions modelling incoming radiation, we find that the dispersion term $\beta'$ cancelled in the final results for the phase correction (6.11) and for the perturbation of the polarisation (7.14).

Considering the calculations presented here in the context of previous calculations which determined the influence of Earth’s gravitational field and its rotation on fibre-modes, we conclude that the phase shifts in all three problems are well described using semi-heuristic arguments based on the eikonal equation in a “fictitious optical metric” where the discontinuous function $n$ in (3.4) is replaced by the constant value of the effective refractive index $\bar{n}$ (computed from the unperturbed problem). In all cases, the difference between such geometrical-optics estimates and the result obtained by solving the true field equations was essentially determined by the relative index difference $\Delta \ll 1$. In contrast, the perturbation of the polarisation found here goes beyond such simplistic models, and thus required a thorough analysis of Maxwell’s equations.

Apart from neglecting terms of order $\omega_g/\omega$ (low-frequency regime) and $\rho \omega_g$ (which is consistent since $\rho \omega \sim 1$), we have neglected the elastic response of the fibre to the gravitational wave, which influences light propagation therein due to the induced displacement of the core-cladding interface (where continuity conditions had to be imposed) and due to stress-induced changes of the optical properties of the materials. Effects of the second kind were previously studied for transparent solid bars (elasto-optical antennas), see e.g. Refs. [42–45], but such calculations have not been carried over to optical fibres yet.

Acknowledgements

I thank Piotr Chruściel, Christopher Hilweg and Stefan Palenta for helpful discussions and gratefully acknowledge funding via a fellowship of the Vienna Doctoral School in Physics (VDSP), as well as support from the Austrian Science Fund (FWF) in the course of the project P34274.
A Derivation of the Wave Equation

Here, we derive the wave equation for the electromagnetic field in the GW metric considered above, neglecting terms of order $\varepsilon^2$. Throughout this section, we raise and lower indices with the full spatial metric $g_{ij}$. Equation (3.9) can then be written as

$$n \dot{Z}^i + j \epsilon^{ijk} \nabla_j Z_k = 0, \quad \text{(A.1)}$$

where $\nabla$ denotes the spatial covariant derivative. Lowering the index, keeping in mind that the spatial metric is time dependent, one obtains the equivalent form

$$n \dot{Z}_i - n \dot{g}_{ij} Z^k + j \epsilon_{ijk} \nabla_j Z^k = 0. \quad \text{(A.2)}$$

Applying $n \partial_0$ to the first equation and using this last equation, we find

$$n^2 \ddot{Z}^i + j \epsilon^{ijk} \nabla_j (n \dot{g}_{kl} Z^l - j \epsilon_{klm} \nabla_l Z^m) = 0. \quad \text{(A.3)}$$

Here, no time derivatives of the volume form occur as the trace of the metric perturbation is time-independent, and no derivatives of the Christoffel symbols occur as the exterior derivative is independent of the connection. Using $\nabla_j Z^j = 0$ and the Ricci identity, we have

$$\epsilon^{ijk} \epsilon_{klm} \nabla_j \nabla^l Z^m = \nabla_j \nabla^l Z^i - \nabla_j \nabla^i Z^l = R^i_j Z^j - \Delta Z^i, \quad \text{(A.4)}$$

where $R^i_j$ is the spatial Ricci tensor. Inserting this into the previous equation then yields

$$n^2 \ddot{Z}^i - \Delta Z^i + R^i_j Z^j + j \epsilon^{ijk} \nabla_j (n \dot{g}_{kl} Z^l) = 0. \quad \text{(A.5)}$$

B Continuity Conditions

The jumps of the various field components are linear combinations of the various parameters $\alpha$ and $\sigma$. For example, if $w$ denotes any field component

$$[w] = a^1 \alpha_1 + a^2 \alpha_2 + s^1 \sigma_1 + s^2 \sigma_2, \quad \text{(B.1)}$$

with some coefficients $a^1, a^2, s^1, s^2$, for which we shall simply write

$$[w] = (a^1, a^2)_{\alpha} + (s^1, s^2)_{\sigma}. \quad \text{(B.2)}$$

To analyse the jumps of the real fields $E, D, B, H$, we separate the $j$-real and $j$-imaginary parts. Accordingly, we write

$$(a + jb, c + jd)_{\alpha} = [a + jb, c + jd, -b + ja, -d + jc]_{\alpha} \quad \text{(B.3)}$$
to state explicitly which coefficients multiply the $j$-real parameters $\Re \alpha_1, \Re \alpha_2, \Im \alpha_1, \Im \alpha_2$. The real and imaginary parts are then given by

\[ \Re (a + jb, c + jd)_\alpha = [a, c, -b, -d]_\alpha, \]  
\[ \Im (a + jb, c + jd)_\alpha = [b, d, a, c]_\alpha. \]  

(B.4)  

(B.5)

Collecting the above mentioned real parameters of $\alpha$ into one column vector

\[ \alpha = \begin{pmatrix} \Re \alpha_1 & \Re \alpha_2 & \Im \alpha_1 & \Im \alpha_2 \end{pmatrix}^T, \]  
and similarly for $\sigma$, we may express the dependence of $[Z]$, as defined in (5.69), in matrix form

\[ [Z] = M \alpha + \Sigma \sigma, \]  

(B.7)

where $M$ and $\Sigma$ are $j$-real matrices of dimension $4 \times 4$. In the perturbed case, we may similarly collect the real parameters of the $\sigma$’s into additional column vectors and add their contributions with additional matrices.

These “continuity matrices” fully describe the jumps of the fields at the core-cladding interface and are hence useful to implement continuity conditions. We discuss their general form and compute the matrix for the unperturbed problem explicitly.

### B.1 General Structure of the Continuity Matrices

Since the electric and magnetic fields are derived from the single complex field $Z$ defined in (3.8), the continuity matrices have only eight independent components. This can be seen from the equations (B.4) and (B.5): if $w$ denotes any field component, then $[\Re w]$ completely determines $[\Im w]$. More precisely, if the jumps of the real fields $\Re Z_r$ and $\Re Z_\parallel$ are given

\[ \begin{bmatrix} \Re Z_r \end{bmatrix} = [a, b, c, d]_\alpha, \quad \begin{bmatrix} \Re Z_\parallel \end{bmatrix} = [e, f, g, h]_\alpha, \]  

(B.8)

then the vector $[Z]$, as defined in (5.69), is fully determined:

\[ \begin{bmatrix} [D^r] \\ [E_\parallel] \\ [B^r] \\ [H_\parallel] \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e/n_1^2 & f/n_2^2 & g/n_1^2 & h/n_2^2 \\ -c/n_1 & -d/n_2 & a/n_1 & b/n_2 \\ -g/n_1 & -h/n_2 & e/n_1 & f/n_2 \end{bmatrix} \begin{bmatrix} \Re \alpha_1 \\ \Re \alpha_2 \\ \Im \alpha_1 \\ \Im \alpha_2 \end{bmatrix}. \]  

(B.9)

Thus, the first two rows (which describe $[D^r] = \Re_j [Z^r]$ and $[E_\parallel] = \Im_j [n^{-2} Z_\parallel]$) determine the entire matrix $M$ in $[Z] = M \alpha$. The form given here is in agreement with the abbreviating notation (5.78) up to redefinition of the parameters.
B.2 Continuity Matrix of the Unperturbed Problem

As an example, we calculate the continuity matrix for the unperturbed problem explicitly. From (4.9) we have

$$i\zeta Z^{(0)\parallel} = \rho^2 \beta \partial_r Z^{(0)\parallel} - i j \rho^2 \omega mn r^{-1} Z^{(0)\parallel}. \quad (B.10)$$

Using (4.8), the jumps of $Z^{(0)\parallel}$ and its radial derivative are readily found to be

$$\begin{bmatrix} Z^{(0)\parallel} \end{bmatrix} = (J_m(U), -K_m(W)), \quad (B.11)$$

$$\begin{bmatrix} \rho \partial_r Z^{(0)\parallel} \end{bmatrix} = (U J_m'(U), -W K_m'(W)). \quad (B.12)$$

Here, we omit the subscript $\alpha$, as the parameters $\sigma$ do not enter the unperturbed problem. From this, we find

$$\begin{bmatrix} i\zeta Z^{(0)\parallel} \end{bmatrix} = \rho \beta (U J_m'(U), -W K_m'(W)) - i j \rho \omega m (n_1 J_m(U), -n_2 K_m(W)). \quad (B.13)$$

Since $\zeta$ equals $+U^2$ in the core and $-W^2$ in the cladding, we deduce

$$\begin{bmatrix} Z^{(0)\parallel} \end{bmatrix} = -i \rho \beta \left( \frac{1}{U} J_m'(U), \frac{1}{W} K_m'(W) \right) - j \rho \omega m \left( \frac{n_1}{U^2} J_m(U), \frac{n_2}{W^2} K_m(W) \right), \quad (B.14)$$

and taking the $j$-real part using (B.4), we obtain

$$\begin{bmatrix} D^{(0)\parallel} \end{bmatrix} = \left[ -i \frac{\rho \beta}{U} J_m'(U), -i \frac{\rho \beta}{W} K_m'(W), mn_1 \frac{\rho \omega}{U^2} J_m(U), mn_2 \frac{\rho \omega}{W^2} K_m(W) \right]. \quad (B.15)$$

The jump of $E_{||} = n^{-2} \Re Z_{||}$ is obtained directly from (B.11):

$$\begin{bmatrix} E^{(0)\parallel} \end{bmatrix} = [n_1^{-2} J_m(U), n_2^{-2} K_m(W), 0, 0]. \quad (B.16)$$

The last two equations determine the unperturbed discontinuity matrix and the result is in agreement with $\Pi_0$ given by equation (5.80).
References

[1] R. L. Forward. “Wideband laser-interferometer gravitational-radiation experiment”. In: *Physical Review D* 17.2 (Jan. 1978), pp. 379–390. DOI: 10.1103/physrevd.17.379 (cit. on p. 2).

[2] B. F. Schutz and M. Tinto. “Antenna patterns of interferometric detectors of gravitational waves - I. Linearly polarized waves”. In: *Monthly Notices of the Royal Astronomical Society* 224.1 (Jan. 1987), pp. 131–154. DOI: 10.1093/mnras/224.1.131 (cit. on p. 2).

[3] V. Faraoni. “A common misconception about LIGO detectors of gravitational waves”. In: *General Relativity and Gravitation* 39.5 (Mar. 2007), pp. 677–684. DOI: 10.1007/s10714-007-0415-5 (cit. on p. 2).

[4] M. V. Sazhin and S. N. Markova. “Optical resonator in the field of gravitational waves”. In: *Physics Letters A* 233.1-2 (Aug. 1997), pp. 43–48. DOI: 10.1016/s0375-9601(97)00436-2 (cit. on pp. 2, 22).

[5] V. N. Rudenko and M. V. Sazhin. “Laser interferometer as a gravitational wave detector”. In: *Soviet Journal of Quantum Electronics* 10.11 (Nov. 1980), pp. 1366–1373. DOI: 10.1070/qe1980v010n11abeh010312 (cit. on p. 2).

[6] L. S. Finn. “Response of interferometric gravitational wave detectors”. In: *Physical Review D* 79.2 (Jan. 2009), p. 022002. ISSN: 1550-7998. DOI: 10.1103/PhysRevD.79.022002. arXiv: 0810.4529 (cit. on pp. 2, 28).

[7] M. Rakhmanov. “On the round-trip time for a photon propagating in the field of a plane gravitational wave”. In: *Classical and Quantum Gravity* 26.15 (Aug. 2009), p. 155010. ISSN: 0264-9381. DOI: 10.1088/0264-9381/26/15/155010 (cit. on p. 2).

[8] Neil J. Cornish. “Alternative derivation of the response of interferometric gravitational wave detectors”. In: *Physical Review D* 80.8 (Oct. 2009). DOI: 10.1103/physrevd.80.087101 (cit. on p. 2).

[9] M. J. Koop and L. S. Finn. “Physical response of light-time gravitational wave detectors”. In: *Physical Review D* 90.6 (Sept. 2014). DOI: 10.1103/physrevd.90.062002 (cit. on p. 2).

[10] F. I. Cooperstock. “The interaction between electromagnetic and gravitational waves”. In: *Annals of Physics* 47.1 (Mar. 1968), pp. 173–181. ISSN: 00034916. DOI: 10.1016/0003-4916(68)90233-9 (cit. on p. 2).

[11] J. A. Lobo. “Effect of a weak plane GW on a light beam”. In: *Classical and Quantum Gravity* 9.5 (May 1992), pp. 1385–1394. ISSN: 0264-9381. DOI: 10.1088/0264-9381/9/5/019 (cit. on pp. 2, 28, 29).

[12] F. I. Cooperstock and V. Faraoni. “Laser-interferometric detectors of gravitational waves”. In: *Classical and Quantum Gravity* 10.6 (1993), pp. 1189–1199. ISSN: 02649381. DOI: 10.1088/0264-9381/10/6/016. arXiv: 9303018 [astro-ph] (cit. on pp. 2, 28).
[13] V. B. Braginsky and M. B. Mensky. “Gravitational-electromagnetic resonance”. In: General Relativity and Gravitation 3.4 (Dec. 1972), pp. 401–402. ISSN: 0001-7701. DOI: 10.1007/BF00759177 (cit. on p. 2).

[14] A. M. Cruise. “An electromagnetic detector for very-high-frequency gravitational waves”. In: Classical and Quantum Gravity 17.13 (July 2000), pp. 2525–2530. ISSN: 0264-9381. DOI: 10.1088/0264-9381/17/13/305 (cit. on p. 2).

[15] M. Tang and F. Li. “Circular waveguide: a possible gravitational wave antenna”. In: Classical and Quantum Gravity 17.12 (June 2000), pp. 2449–2453. ISSN: 0264-9381. DOI: 10.1088/0264-9381/17/12/316 (cit. on p. 2).

[16] M. B. Mensky and V. N. Rudenko. “High-frequency gravitational wave detector with electromagnetic-gravitational resonance”. In: Gravitation and Cosmology 15.2 (Apr. 2009), pp. 167–170. ISSN: 0202-2893. DOI: 10.1134/S0202289309020133 (cit. on p. 2).

[17] M. Born and E. Wolf. Principles of Optics. 60th Anniversary Edition. Cambridge University Press, 2019. DOI: 10.1017/9781108769914 (cit. on p. 2).

[18] R. Beig et al. “Weakly gravitating isotropic waveguides”. In: Classical and Quantum Gravity 35.24 (Nov. 2018), p. 244001. DOI: 10.1088/1361-6382/aae873 (cit. on pp. 2, 5, 26).

[19] T. B. Mieling. “On the influence of Earth’s rotation on light propagation in waveguides”. In: Classical and Quantum Gravity 37.22 (Oct. 2020), p. 225001. DOI: 10.1088/1361-6382/ababb2 (cit. on pp. 2, 26).

[20] P. Linsay et al. A Study of a Long Baseline Gravitational Wave Antenna System. Tech. rep. Prepared for the National Science Foundation, 1983 (cit. on p. 2).

[21] P. R. Saulson. Fundamentals of Interferometric Gravitational Wave Detectors. World Scientific, 1994. ISBN: 981-02-1820-6. DOI: 10.1142/2410 (cit. on p. 2).

[22] R. Adhikari. “Sensitivity and noise analysis of 4 km laser interferometric gravitational wave antennae”. PhD thesis. Massachusetts Institute of Technology, July 2004 (cit. on p. 2).

[23] B. P. Abbott et al. “Observation of Gravitational Waves from a Binary Black Hole Merger”. In: Physical Review Letters 116.6 (Feb. 2016). DOI: 10.1103/physrevlett.116.061102 (cit. on p. 2).

[24] B. P. Abbott et al. “GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence”. In: Physical Review Letters 119.14 (Oct. 2017). DOI: 10.1103/physrevlett.119.141101 (cit. on p. 2).

[25] B. P. Abbott et al. “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral”. In: Physical Review Letters 119.16 (Oct. 2017). DOI: 10.1103/physrevlett.119.161101 (cit. on p. 2).

[26] R. Abbott et al. “GW190412: Observation of a binary-black-hole coalescence with asymmetric masses”. In: Physical Review D 102.4 (Aug. 2020). DOI: 10.1103/physrevd.102.043015 (cit. on p. 2).
[27] B. P. Abbott et al. “All-sky search for continuous gravitational waves from isolated neutron stars using Advanced LIGO O2 data”. In: Physical Review D 100.2 (July 2019), p. 024004. DOI: 10.1103/physrevd.100.024004 (cit. on p. 2).

[28] V. Dergachev and M. A. Papa. “Results from an extended Falcon all-sky survey for continuous gravitational waves”. In: Physical Review D 101.2 (Jan. 2020), p. 022001. DOI: 10.1103/physrevd.101.022001 (cit. on p. 2).

[29] V. Dergachev and M. A. Papa. “Results from the First All-Sky Search for Continuous Gravitational Waves from Small-Ellipticity Sources”. In: Physical Review Letters 125.17 (Oct. 2020), p. 171101. DOI: 10.1103/physrevlett.125.171101 (cit. on p. 2).

[30] N. Hodgson and H. Weber. Laser Resonators and Beam Propagation. Second Edition. Springer Series in Optical Sciences 108. Springer, New York, 2005. ISBN: 978-0-387-40078-5. DOI: 10.1007/b106789 (cit. on p. 3).

[31] J. N. Damask. Polarization Optics in Telecommunications. Springer Series in Optical Sciences 101. Springer, New York, 2004. ISBN: 978-0-387-22493-0. DOI: 10.1007/b137386 (cit. on p. 3).

[32] W. Gordon. “Zur Lichtfortpflanzung nach der Relativitätstheorie”. In: Annalen der Physik 377.22 (1923), pp. 421–456. DOI: 10.1002/andp.19233772202 (cit. on p. 5).

[33] J. D. Jackson. Classical Electrodynamics. Third Edition. John Wiley & Sons, Inc., 1998. 832 pp. ISBN: 978-0-471-30932-1 (cit. on p. 7).

[34] J.-M. Liu. Photonic Devices. Cambridge University Press, 2005. DOI: 10.1017/cbo9780511614255 (cit. on pp. 7, 18, 23, 24).

[35] C. C. Davis. Lasers and Electro-optics. 2nd edition. Cambridge University Press, 2009. DOI: 10.1017/cbo9781139016629 (cit. on pp. 7, 18).

[36] F. W. J. Olver. “Bessel Functions of Integer Order”. In: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Ed. by M. Abramowitz and I. A. Stegun. Tenth Printing, December 1972, with corrections. National Bureau of Standards Applied Mathematics Series 55. Washington, DC: U.S. Department of Commerce, 1964, pp. 355–433 (cit. on pp. 8, 13, 21).

[37] P. Jaranowski, A. Królak, and B. F. Schutz. “Data analysis of gravitational-wave signals from spinning neutron stars: The signal and its detection”. In: Physical Review D 58.6 (Aug. 1998), p. 063001. DOI: 10.1103/physrevd.58.063001 (cit. on p. 27).

[38] M. Maggiore. Gravitational Waves. Volume 1. Oxford University Press, Oct. 2007. ISBN: 0198570740 (cit. on p. 27).

[39] E. Iacopini et al. “Birefringence induced by gravitational waves: A suggestion for a new detector”. In: Physics Letters A 73.2 (Sept. 1979), pp. 140–142. DOI: 10.1016/0375-9601(79)90460-2 (cit. on p. 29).

[40] F. Pegoraro and L. A. Radicati. “Dielectric tensor and magnetic permeability in the weak field approximation of general relativity”. In: Journal of Physics A: Mathematical and General 13.7 (July 1980), pp. 2411–2421. DOI: 10.1088/0305-4470/13/7/024 (cit. on p. 29).
[41] D. Bini, P. Fortini, and A. Haney. “Electromagnetic waves in gravitational wave spacetimes”. In: Classical and Quantum Gravity 28.23 (Nov. 2011), p. 235007. DOI: 10.1088/0264-9381/28/23/235007 (cit. on p. 29).

[42] J.-Y. Vinet. “Elasto-optical detection of gravitational waves”. In: Annales de l’I.H.P. Physique théorique 30.3 (1979), pp. 251–262 (cit. on p. 29).

[43] G. R. Boyer et al. “Elastooptical antenna for detection of gravitational radiation”. In: Applied Optics 19.3 (Feb. 1980), p. 382. DOI: 10.1364/ao.19.000382 (cit. on p. 29).

[44] J.-Y. Vinet. “Elasto optical antennas”. In: Annales de Physique 10.3 (1985), pp. 253–261. DOI: 10.1051/anphys:01985001003025300 (cit. on p. 29).

[45] F. T. du Cros. “Electromagnetic and elasto-optical systems for the reception or generation of gravitational radiation”. In: Annales de Physique 10.3 (1985), pp. 263–286. DOI: 10.1051/anphys:01985001003026300 (cit. on p. 29).