ON THE GROWTH OF SOLUTIONS TO THE MINIMAL SURFACE EQUATION OVER DOMAINS CONTAINING A HALFPLANE

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Abstract. We consider minimal graphs $u = u(x, y) > 0$ over unbounded domains $D$ with $u = 0$ on $\partial D$. Assuming $D$ contains a sector properly containing a halfplane, we obtain estimates on growth and provide examples illustrating a range of growth.

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1. Introduction

Let $D$ be an unbounded plane domain. In this paper we consider the boundary value problem for the minimal surface equation

$$\begin{cases}
\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad \text{and} \quad u > 0 \quad \text{in } D \\
u = 0 \quad \text{on } \partial D
\end{cases}
$$

(1.1)

We shall study the constraints on growth of nontrivial solutions to (1.1) as determined by the maximum

$$M(r) = \max u(x, y),$$

where the max is taken over the values $r = \sqrt{x^2 + y^2}$ and $(x, y) \in D$.

Perhaps the first relevant theorem in this direction was proved by Nitsche [7, p. 256] who observed that if $D$ is contained in a sector of opening strictly less than $\pi$, then $u \equiv 0$. For domains contained in a half plane, but not contained in any such sector, there are a host of solutions to (1.1) which will be discussed later. However, in this case, it has been shown [11] that if $D$ is bounded by a Jordan arc,

$$Cr \leq M(r) \leq e^{Cr} \quad (r > r_0)$$

for some positive constants $C$ and $r_0$. 


If, on the other hand, the domain $D$ contains a sector of opening $\alpha$ bigger than $\pi$, we shall show that the growth of $M(r)$ is at most linear (see Theorem 2.1 in Section 2). Regarding the bound from below, with the order $\rho$ of $u$ defined by

$$\rho = \lim_{r \to \infty} \sup \frac{\log M(r)}{\log r},$$

it follows by using the module estimates of Miklyukov [5] (see also chapter 9 in [6]) as in [10] that if $D$ omits a sector of opening $2\pi - \alpha$, ($\pi \leq \alpha \leq 2\pi$, the omitted set in the case $\alpha = 2\pi$ being a line), then the order of any nontrivial solution to (1.1) is at least $\pi/\alpha$.

The paper concludes with a list of problems and conjectures.

2. Estimates on Growth

For later convenience we shall use complex notation $z = x + iy$ for points $(x, y)$ when describing solutions to the minimal surface equation. As such, we are given a minimal graph with positive height function $u(z)$ over a domain $D$ as in (1.1).

**Theorem 2.1.** Let $D$ be a domain whose boundary is a Jordan arc, and $D$ contains a sector $S_{\lambda} := \{ z : |\arg z| \leq \lambda \}$, with $\lambda > \pi/2$. With $M(r)$ defined as above, if $u$ satisfies (1.1) in $D$ and vanishes on the boundary $\partial D$, then there exist positive constants $K$ and $R$ such that

$$M(r) \leq Kr, \quad |z| > R.$$  

Throughout, we will make use of the parametrization in isothermal coordinates by the Weierstrass functions $(x(\zeta), y(\zeta), U(\zeta))$ with $\zeta$ in the right half plane $H$, $U(\zeta) = u(x(\zeta), y(\zeta))$ and (up to additive constants)

$$\begin{cases}
  x(\zeta) = \Re \frac{1}{2} \int_{\zeta_0}^{\zeta} \omega(\zeta)(1 - G^2(\zeta))d\zeta \\
  y(\zeta) = \Re \frac{i}{2} \int_{\zeta_0}^{\zeta} \omega(\zeta)(1 + G^2(\zeta))d\zeta \\
  U(\zeta) = \Re \int_{\zeta_0}^{\zeta} \omega(\zeta) G(\zeta)d\zeta
\end{cases}$$

With this parameterization, the height function $U(\zeta)$ pulled back to the halfplane $H$ becomes a positive harmonic function in $H$ which is 0 on the imaginary axis, and thus is simply $U(\zeta) = C\Re\{\zeta\}$ for a real positive constant $C$. We may assume without loss of generality that $C = 2$.

Since $f(\zeta) := x(\zeta) + iy(\zeta)$ is harmonic in $H$, there exist analytic functions $h(\zeta)$ and $g(\zeta)$ in $H$ such that
\[ f(\zeta) = h(\zeta) + g(\zeta). \]

With this formulation, the height function then satisfies
\[ U(\zeta) = 2 \Re i \int \sqrt{h'(\zeta)g'(\zeta)} \, d\zeta, \]
and since \( U(\zeta) = \zeta \) in (2.2), it follows that
\[ g'(\zeta) = -\frac{1}{h'(\zeta)}. \]

2.1. **Proof of Theorem 2.1.** First we establish the bound (2.1) inside a sector.

**Lemma 2.2.** Let \( S_\alpha := \{ z : |\arg z| \leq \alpha < \pi/2 \} \) be a sector contained in \( H \subset D \). Then for some \( K > 0 \) the upper bound (2.1) holds in \( S_\alpha \) for all \( r \) sufficiently large:
\[ \max_{|z| = r, z \in S_\alpha} u(z) \leq Kr. \]

**Proof of Lemma.** Let \( f(\zeta), U(\zeta) \) be as above. So, \( u(f(\zeta)) = U(\zeta) = \Re \zeta \).

Let \( P := \{ \zeta : \Re f(\zeta) > 0 \} \) be the preimage of the right halfplane, and introduce a new variable \( \tilde{\zeta} \) and let \( \psi(\tilde{\zeta}) \) be a conformal map from the right half \( \tilde{\zeta} \)-plane \( H := \{ \tilde{\zeta} : \Re (\tilde{\zeta}) > 0 \} \) onto \( P \).

Define
\[
\begin{align*}
\tilde{f}(\tilde{\zeta}) &:= f(\psi(\tilde{\zeta})) \\
\tilde{g}(\tilde{\zeta}) &:= g(\psi(\tilde{\zeta})) \\
\tilde{h}(\tilde{\zeta}) &:= h(\psi(\tilde{\zeta}))
\end{align*}
\]

Then \( \tilde{f} \) is a harmonic map, and
\[ \tilde{f}(\tilde{\zeta}) = \tilde{h}(\tilde{\zeta}) + \tilde{g}(\tilde{\zeta}). \]

We wish to show that for all \( |z| > R \) in \( S_\alpha \),
\[ \frac{u(z)}{|z|} = \frac{U(\zeta)}{|f(\zeta)|} = \frac{\Re \zeta}{|f(\zeta)|} = \frac{\Re \psi(\tilde{\zeta})}{|f(\zeta)|} < K. \]

Let \( \tilde{F}(\tilde{\zeta}) = \tilde{h}(\tilde{\zeta}) + \tilde{g}(\tilde{\zeta}) \) be the analytic function with the same real part as \( \tilde{f} \). Then \( \Re \tilde{F} \) is positive in \( H \) and vanishes on \( \partial H \), and therefore, without loss of generality we may write (see [9, p. 151])
\[ \tilde{F}(\tilde{\zeta}) = \tilde{\zeta} \implies \tilde{F}'(\tilde{\zeta}) = 1. \]

The proof hinges on (2.4) along with the chain rule combined with (2.3). Now,
\[ \tilde{h}'(\tilde{\zeta}) = h'(\psi(\tilde{\zeta})) \cdot \psi'(\tilde{\zeta}). \]
and

\[(2.5) \quad \tilde{g}'(\tilde{\zeta}) = -\frac{\psi'(\tilde{\zeta})}{h'(\psi(\tilde{\zeta}))} = -\frac{\psi'(\tilde{\zeta})^2}{h'(\tilde{\zeta})}.\]

Combining this with (2.4) we have

\[1 = \tilde{F}'(\tilde{\zeta}) = \tilde{h}'(\tilde{\zeta}) - \frac{\psi'(\tilde{\zeta})^2}{h'(\tilde{\zeta})},\]

which implies

\[\tilde{h}'(\tilde{\zeta})^2 - \tilde{h}'(\tilde{\zeta}) - \psi'(\tilde{\zeta})^2 = 0.\]

Thus,

\[(2.6) \quad \tilde{h}'(\tilde{\zeta}) = \frac{1 + \sqrt{1 + 4\psi'(\tilde{\zeta})^2}}{2}.\]

Since \(\psi(\tilde{\zeta})\) is a conformal map with \(\Re\psi(\tilde{\zeta}) > 0\) in \(H\), there exists a real constant \(0 \leq c < \infty\) such that in any sector \(S_\beta := \{\tilde{\zeta} : |\arg \tilde{\zeta}| \leq \beta < \pi/2\}\) the limit \(\psi'(\tilde{\zeta}) \to c\) exists as \(\tilde{\zeta} \to \infty\) in \(S_\beta\). (see [9, p. 152])

**Case 1:** \(\psi'(\tilde{\zeta}) \to c = 0\) as \(\tilde{\zeta} \to \infty\) (with \(\tilde{\zeta}\) in \(S_\beta\)).

From (2.6) we have \(\tilde{h}'(\tilde{\zeta}) \to 1\) as \(\tilde{\zeta} \to \infty\), and using (2.5) we have \(\tilde{g}'(\tilde{\zeta}) \to 0\). Thus, \(\tilde{h}(\tilde{\zeta}) \approx \tilde{\zeta}\) and \(\tilde{g}(\tilde{\zeta}) = o(1)\), which implies that \(\tilde{f}(\tilde{\zeta}) = \tilde{h}(\tilde{\zeta}) + \tilde{g}(\tilde{\zeta}) \approx \tilde{\zeta}\).

Since \(\tilde{f} : H \to H\) is asymptotic to the identity map, given \(\alpha\), we may choose \(\beta < \pi/2\) so that \(S_\alpha \cap \{|z| > R\}\) is contained in the image of the sector \(S_\beta\) for \(R\) large enough. Thus, the estimate \(\psi'(\tilde{\zeta}) \to 0\) applies in the region \(S_\alpha\); and we have

\[\frac{u(z)}{|z|} = \frac{\Re\psi(\tilde{\zeta})}{|\tilde{f}(\tilde{\zeta})|} < \frac{|\psi(\tilde{\zeta})|}{|\tilde{f}(\tilde{\zeta})|} = o(1), \quad \text{for} \quad z \in S_\alpha \cap \{|z| > R\},\]

since \(\tilde{f}(\tilde{\zeta}) \approx \tilde{\zeta}\), and \(\psi'(\tilde{\zeta}) = o(1)\).

**Case 2:** \(\psi'(\tilde{\zeta}) \to c > 0\) as \(\tilde{\zeta} \to \infty\).

From (2.4) we have \(\Re\{\tilde{h}(\tilde{\zeta}) + \tilde{g}(\tilde{\zeta})\} = \Re\tilde{\zeta}\). Let us also estimate \(\Im \tilde{f}(\tilde{\zeta}) = \Im \tilde{h}(\tilde{\zeta}) - \Im \tilde{g}(\tilde{\zeta})\). We use (2.6) and (2.5):

\[\tilde{h}'(\tilde{\zeta}) \to \frac{1 + \sqrt{1 + 4c^2}}{2},\]
\[\tilde{g}'(\tilde{\zeta}) \to \frac{-2c^2}{1 + \sqrt{1 + 4c^2}},\]

which imply

\[\tilde{h}'(\tilde{\zeta}) - \tilde{g}'(\tilde{\zeta}) \to \frac{(1 + \sqrt{1 + 4c^2})^2 + 4c^2}{2(1 + \sqrt{1 + 4c^2})} = 1 + \frac{4c^2}{1 + \sqrt{1 + 4c^2}}.\]
Putting this together, we have
\[ \hat{h}(\zeta) + \hat{g}(\zeta) = \Re \zeta + i \left( 1 + \frac{4c^2}{1 + \sqrt{1 + 4c^2}} + o(1) \right) \Im \zeta. \]

As in the first case, given \( \alpha \), we may thus choose \( \beta < \pi/2 \) and \( R > 0 \) so that \( S_\alpha \cap \{|z| > R\} \) is contained in the image \( \hat{f}(S_\beta) \) of the sector \( S_\beta \). Then we have
\[ u(z) = \left| \frac{\Re \psi(\zeta)}{|\hat{f}(\zeta)|} < \frac{|\psi(\zeta)|}{|\hat{f}(\zeta)|} = O(1), \quad \text{for } z \in S_\alpha \cap \{|z| > R\}. \]
Indeed, \( |\hat{f}(\zeta)| = \left| \Re \zeta + i \left( 1 + \frac{4c^2}{1 + \sqrt{1 + 4c^2}} + o(1) \right) \Im \zeta \right|, \) and \( \psi'(\zeta) = O(1) \implies \psi(\zeta) = O(|\zeta|). \)

Applying Lemma 2.2 to two sectors, one rotated clockwise and the other counterclockwise, in order that their union covers \( S_\lambda \), the upper bound (2.1) is established in \( S_\lambda \). It remains to prove the estimate in the rest of \( D \).

Let \( \pi/2 < \alpha < \lambda \). We will show that the upper bound (2.1) holds in \( D \setminus \overline{S_\alpha} \).

In order to prove this, we will apply the following result from [1, Main Theorem]:

**Theorem A.** Let \( \Omega \subset \Omega_1 = \{(x,y) : x > 0, -f(x) < y < f(x)\} \), where \( f, g \in C[0,\infty), f, g \geq 0, g(0) = 0, f(t), g(t)/t \) increase as \( t \) increases, and let \( u \in C(\Omega) \cap C^2(\Omega) \). Suppose that

i) \( \text{div} \frac{\nabla u}{\sqrt{1 + |
abla u|^2}} \geq 0 \) in \( \Omega \),

ii) \( u|_{\partial \Omega \cap \{|-f(t), f(t)|\}} \leq g(x) \) for \( x \in [0, \infty) \),

iii) \( 0 < \kappa(x) := f(x)/(2g(x)) < 1 \) for all \( x \) larger than some \( x_1 > 0 \),

iv) \( \kappa(x) \) is decreasing on \( [x_1, \infty) \).

Then \( u(x, y) \leq g(x/(1 - \kappa(x))) \) for every \( (x, y) \in \Omega \) with \( x > x_1 \).

We apply this to \( \Omega = D \setminus \overline{S_\alpha} \), while taking \( \Omega_1 = \mathbb{C} \setminus \overline{S_\alpha} \). In order to relate to the setup in the theorem, reflect these domains about the y-axis, so that \( \Omega \) and \( \Omega_1 \) are in the right halfplane. Then \( \Omega_1 = \{(x, y) : x > 0, -f(x) < y < f(x)\} \), where \( f(x) = \tan(\pi - \alpha)x \). If \( C > 0 \) is sufficiently large, then \( g(x) = Cx(1 - \exp(-x)/2) \) satisfies both (iii) and (iv). We check that for \( C \) large enough, (ii) is also satisfied. Note that \( \partial \Omega \) contains points on \( \partial D \) and points on \( \partial S_\alpha \). For points on \( \partial D \), \( u = 0 \), and for points on \( \partial S_\alpha \), \( u \) has at most linear growth by Lemma 2.2. Thus, in both cases (ii) is satisfied, and
Theorem A may be applied. The result is that \( u(x, y) \leq g(x/(1 - \kappa(x))) \) for all large enough \( x \in \Omega \). Since

\[
\frac{x}{1 - \kappa(x)} = \frac{x}{1 - \tan(\pi - \alpha)/C} (1 + o(1)),
\]

and \( \tan(\pi - \alpha)/C \) is a small constant provided \( C \) is large, we have

\[ u(x, y) < Cx, \]

for all large enough \( x \in \Omega \). This completes the proof of (2.1).

2.2. A lower bound.

**Proposition 2.3.** Suppose \( D \) is a domain with \( \partial D \neq \emptyset \), and \( u(z) > 0 \) satisfies (1.1) with \( u(z) = 0 \) on \( \partial D \). Then \( u(z) \) has at least logarithmic growth.

**Proof.** Without loss of generality assume that \( 0 \in \partial D \), and consider the top half of the vertical catenoid centered at \( z = 0 \) as a “barrier” (cf. [8, p. 92]). Explicitly, let \( \cosh^{-1} \) denote the positive branch of the inverse of \( \cosh : \mathbb{R} \to \mathbb{R} \), and define

\[
G(z; r_1) := r_1 \cosh^{-1} \left( \frac{|z|}{r_1} \right), \quad |z| \geq r_1.
\]

For each \( r_1 \), \( G(z; r_1) \) satisfies (1.1).

Let \( \varepsilon > 0 \) and choose a \( \delta \)-neighborhood \( B(\delta, 0) \) of \( z = 0 \) small enough that \( u(z) < \varepsilon \) throughout \( B(\delta, 0) \cap D \).

Define \( u_\varepsilon(z) = u(z) - \varepsilon \). For \( r_1 > 0 \) small enough, \( G(|z|; r_1) > u_\varepsilon(z) \) on \( \partial B(\delta, 0) \cap D \).

For \( R > 0 \), let

\[
K_R := D \cap B(R, 0) \setminus B(\delta, 0).
\]

Fix \( R = R_0 \). Suppose \( \max_{|z|=R} |u(z)| \) grows slower than logarithmically, so it grows slower than \( G(|z|, r_1) \). Then for \( R > R_0 \) sufficiently large, \( G(|z|; r_1) > u_\varepsilon(z) \) on \( \partial K_R \). This implies the same inequality throughout \( K_{R_0} \subset K_R \). In particular, \( u_\varepsilon(z) < r_1 \cosh^{-1} \left( \frac{R_0}{r_1} \right) \) in \( K_{R_0} \). But \( r_1 > 0 \) is arbitrary, and \( r_1 \cosh^{-1} \left( \frac{R_0}{r_1} \right) \to 0 \) as \( r_1 \to 0 \). Thus, \( u_\varepsilon(z) \leq 0 \) in \( K_{R_0} \) which implies that \( u(z) \leq 0 \) since \( \varepsilon \) was arbitrary. This contradicts that \( u(z) > 0 \) in \( D \). \( \square \)

3. Examples

In this Section, we provide examples that together with the above (and previously known) results give a broad picture of the possible growth rates of minimal graphs. One notices three “regimes” illustrated in Fig. [1]. When \( D \) contains a halfplane we find nontrivial examples, but their growth rates appear to be determined by the asymptotic angle \( \pi < \beta < 2\pi \). This is reminiscent of the behavior of positive harmonic functions, hence we deem this the “Phragmén-Lindelöf regime”. However, the geometry of
$D$ plays a subtle role, since if $D$ is a true sector of opening $\beta$, even in the range $\pi < \beta < 2\pi$, then (1.1) has only the trivial solution $u \equiv 0$ [4, p.993].

When $D$ is contained in a sector $\beta < \pi$, we have a “completely rigid regime”, due to Nitsche’s theorem. At the critical angle $\beta = \pi$, an interesting phase transition occurs; there are examples with $D$ contained in a halfplane with $\beta = \pi$ exhibiting a full spectrum of possible growth rates anywhere from linear to exponential thus interpolating the known upper and lower bounds.

![Figure 1. A plot of the boundary of $D$ labeled with order $\rho$. Phragmén-Lindelöf regime: $\pi < \beta < 2\pi$, Critical regime: $\beta = \pi$, and Rigid regime: $\beta < \pi$. For the curves, from left to right the angles are $\beta = 2\pi$, $7\pi/4$, and $3\pi/2.$](image)

3.1. Examples in the “Phragmén-Lindelöf” regime $\pi < \beta < 2\pi$: In [4], there appears an example of a minimal graph with height function (pulled back to $\zeta$-plane) $U(\zeta) = 2\Re \zeta$, and harmonic map from the half plane $H := \{z = x + iy : x > 0\}$

$$z(\zeta) = \frac{(\zeta + 1)^2}{2} - \log(\overline{\zeta} + 1).$$
This example has asymptotic angle $2\pi$ and growth of order $1/2$. (See §4 for the definition of asymptotic angle.)

Let us demonstrate a whole one-parameter family of examples with asymptotic angles $\pi < \beta < 2\pi$ having growth of orders $\pi/\beta$. Let $\gamma = \beta/\pi$ (so $1 < \gamma < 2$). Then such a minimal surface is given by the harmonic map from the half plane $H$ to a region $D$

$$z(\zeta) = (\zeta + 1)^\gamma - \frac{1}{\gamma(2 - \gamma)}(\zeta + 1)^{2-\gamma}$$

together with the height function $U(\zeta) = 2\Re \zeta$.

Assuming $z(\zeta)$ is univalent, then we have growth of order $1/\gamma = \pi/\beta$ as desired, since

$$u(z) = \frac{U(\zeta)}{|z|^{1/\gamma}} = \frac{2\Re \zeta}{|z(\zeta)|^{1/\gamma}} = \frac{2\Re \zeta}{|(\zeta + 1)^\gamma - \frac{1}{\gamma(2 - \gamma)}(\zeta + 1)^{2-\gamma}|^{1/\gamma}}.$$ 

Thus, the only thing to check is that $z(\zeta)$ is univalent in $H$. Its Jacobian is

$$\gamma^2 |\zeta + 1|^{2(\gamma - 1)} - \frac{1}{\gamma^2 |\zeta + 1|^{2(\gamma - 1)}} > 0$$

since

$$\gamma^2 |\zeta + 1|^{2(\gamma - 1)} > 1.$$

Thus, global univalence can be ensured by checking the boundary behavior. We will show that the imaginary part of $z(\zeta)$ is increasing on the boundary $\zeta = it$, $-\infty < t < \infty$. The imaginary part of $z(it)$ is

$$\Im \{z(it)\} = (1 + t^2)^{\gamma/2} \sin(\gamma \tan^{-1} t) + \frac{1}{\gamma(2 - \gamma)}(1 + t^2)^{(2-\gamma)/2} \sin((2 - \gamma) \tan^{-1} t).$$

This is an odd function, so we just consider the interval $0 < t < \infty$. The second term is increasing, since it is a product of increasing functions. Indeed, $0 < 2 - \gamma < 1$ and $(1 + t^2)^{(2-\gamma)/2}$ is increasing on $0 < t < \infty$. The second term is increasing. In order to show that $(1 + t^2)^{\gamma/2} \sin(\gamma \tan^{-1} t)$ is increasing, we check that the derivative

$$\gamma(1 + t^2)^{\gamma/2 - 1} t \sin(\gamma \tan^{-1} t) + \gamma(1 + t^2)^{\gamma/2 - 1} \cos(\gamma \tan^{-1} t)$$

is positive, or equivalently that

$$t \sin(\gamma \tan^{-1} t) + \cos(\gamma \tan^{-1} t) > 0.$$

For this let $0 < \theta < \pi/2$ and take $t = \tan \theta$. Then we see that

$$\tan \theta \sin(\gamma \theta) + \cos(\gamma \theta) = \frac{\cos(\gamma - 1) \theta}{\cos \theta},$$

which is positive since $0 < \theta < \pi/2$ and $1 < \gamma < 2$. 

3.2. The critical angle $\beta = \pi$: Examples from linear growth to exponential. A plane and a horizontal catenoid sliced by a plane parallel to its axis provide two examples of minimal graphs over a domain contained in a half plane. These examples have linear and exponential growth respectively.

For each given $\rho > 1$, we provide an example contained in a halfplane (each having asymptotic angle $\beta = \pi$) with order of growth $\rho$. Let $b = 1/\rho$. Then, once again, $z(\zeta)$ has the form

$$z(\zeta) = h(\zeta) - \int h'(\zeta) d\zeta,$$

so that $U(\zeta) = 2\Re e \zeta$.

Taking $h(\zeta) = \zeta + \frac{1}{b} \zeta^b$,

$$z(\zeta) = \zeta + \frac{1}{b} \zeta^b - \bar{\zeta} + \int \frac{1}{1 + \zeta^{1-b}} d\zeta,$$

Assuming $z(\zeta)$ is univalent, $u(z)$ has order $\rho$, since

$$\frac{u(z)}{|z|^\rho} = \frac{U(\zeta)}{|z(\zeta)|^\rho} = \frac{2\Re e \zeta}{|z(\zeta)|^\rho},$$

which tends to a constant on the real axis.

It remains to check that $z(\zeta)$ is univalent in $H$. Its Jacobian is

$$|1 + \zeta^{b-1}|^2 - \frac{1}{|1 + \zeta^{b-1}|^2} > 0$$

since

$$|1 + \zeta^{b-1}|^2 > 1, \text{ for } \zeta \in H.$$

Thus, global univalence can be ensured by checking the boundary behavior. As in the previous examples we show that $\Im m \{z(\zeta)\}$ is increasing on the boundary $\zeta = it$, $-\infty < t < \infty$. This is an odd function, so we just consider the interval $0 < t < \infty$. It suffices to show that the derivative

$$\frac{d}{dt} \Im m \{z(it)\}$$

is positive. We use the identity

$$\frac{d}{dt} \Im m \{z(it)\} = \frac{d}{dt} \Im m \{h(it)\} - \frac{d}{dt} \Im m \{g(it)\} = \Re e \{h'(it)\} - \Re e \{g'(it)\},$$

to compute

$$\frac{d}{dt} \Im m \{z(it)\} = 1 + \Re e \frac{1}{(it)^{1-b}} + 1 - \Re e \frac{1}{1 + (it)^{1-b}}$$

$$> 2 - \frac{1}{1 + \Re e \{(it)^{1-b}\}} > 1.$$
We note that the domain $D$ for this example has a corner at the point $z(0)$. This can be removed by shifting the minimal graph $(x, y, u(x, y))$ in the negative $u$-direction.

4. Problems and conjectures

I. When dealing with a nonlinear equation, issues of existence and uniqueness are often complex. A survey of uniqueness results can be found in [3]. A natural question to ask here is

**Problem 1.** Is it possible for (1.1) to have more than one nontrivial solution?

II. As discussed in the introduction, for domains $D$ contained in the half plane, at least when bounded by a Jordan arc, the growth of solutions to (1.1) is at most exponential. However, it seems likely that this is true in general.

**Problem 2.** If $u$ is a solution to (1.1), then does its maximum $M(r)$ satisfy

$$M(r) \leq e^{Cr} \quad (r > r_0),$$

for some positive constants $C$ and $r_0$.

III. In the case where $D$ contains a half plane, we have been unable to ascertain a good upper bound for the maximum. However, it seems reasonable to conjecture that Theorem 2.1 holds here as well.

**Problem 3.** If $u$ is a solution to (1.1) and $D$ contains a half plane, then is it true that

$$M(r) \leq Cr \quad (r > r_0)$$

for some positive constants $C$ and $r_0$?

IV. In this paper we have shown that if $D$ contains a sector of opening $\alpha > \pi$, then any nontrivial solution has order at most 1. However, it seems likely that this might be be improved.

**Problem 4.** If $D$ contains a sector of opening $\alpha > \pi$, then is it true that the order of any nontrivial solution to (1.1) is bounded above by $\pi/(2\pi - \alpha)$? The interpretation as with the minimum bound discussed in §1 has the case $\alpha - 2\pi$ taken to mean that the omitted set is a line.

V. The results in [11] are phrased in terms of the asymptotic angle $\beta$ defined as follows. Let $\Theta(r)$ be the angular measure of the set $D \cap \{|z| = r\}$, and $\Theta^*(r) = \Theta(r)$ if $D$ does not contain the circle $|z| = r$, and $+\infty$ otherwise. Then

$$\beta = \lim_{r \to \infty} \sup_r \Theta^*(r).$$

Consideration of the case $\beta = 2\pi$ raises the following question.
**Problem 5.** If $D$ is an unbounded region bounded by a Jordan arc (taken to mean a proper curve which does not self intersect or close), then is it true that the maximum of a nontrivial solution satisfies

$$M(r) \geq C \sqrt{r} \quad (r > r_0)$$

for some positive constants $C$ and $r_0$?

VI. Returning to Nitsche’s theorem as mentioned in §1, in terms of the asymptotic angle $\beta$ it seems likely that a corresponding result should hold.

**Problem 6.** If $D$ has asymptotic angle $\beta < \pi$, and $u$ is a solution to (1.1), then must it be that $u \equiv 0$?

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