VERSAL DEFORMATIONS OF THREE DIMENSIONAL LIE ALGEBRAS AS \( L_\infty \) ALGEBRAS

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Abstract. We consider versal deformations of 0/3-dimensional \( L_\infty \) algebras, which correspond precisely to ordinary (non-graded) three dimensional Lie algebras. The classification of such algebras over \( \mathbb{C} \) is well known, although we shall give a derivation of this classification using an approach of treating them as \( L_\infty \) algebras. Because the symmetric algebra of a three dimensional odd vector space contains terms only of exterior degree less than or equal to three, the construction of versal deformations can be carried out completely. We give a characteristic of the moduli space of Lie algebras using deformation theory as a guide to understanding the picture.

1. Introduction

The classification of low dimensional Lie algebras has been known for a long time. For example, the classification of ordinary Lie algebras of dimension 3, the subject of this paper, appears in textbooks such as [10]. More recently, the moduli space of three dimensional Lie algebras was studied in [1, 15]. The problem of finding a versal deformation of a given object is a basic question in deformation theory because such a deformation induces all other deformations. This problem turns out to be very difficult. Versal deformation theory was first worked out for the case of Lie algebras in [4, 6] and then extended to \( L_\infty \) algebras in [7]. We apply these general results to construct versal deformations of three dimensional ordinary Lie algebras, treating them as examples of \( L_\infty \) algebras. We use the methods developed in [7, 8].

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\(L_\infty\) algebras are natural generalizations of Lie algebras and superalgebras if one considers \(\mathbb{Z}_2\)-graded vector spaces. An ordinary 3-dimensional Lie algebra is the same thing as a \(L_\infty\) algebra structure on a 0|3 (0 even and 3 odd) dimensional \(\mathbb{Z}_2\)-graded vector space. \(L_\infty\) algebras were first described in [13] and have recently been the focus of much attention [11, 2, 12, 14]. The advantage of considering Lie algebras as \(L_\infty\) algebras is that the deformation problem becomes simpler and we get a clearer insight to the moduli space of the variety of Lie algebras in a given dimension.

Even though \(L_\infty\) algebras appear in many contexts, there are only a few examples known, even in low dimensions. In [8], the authors classified \(L_\infty\) algebras of dimension less than or equal to 2. In related work [9], we studied \(L_\infty\) algebras of dimension 1|2, and in recent work with Derek Bodin [3], we studied \(L_\infty\) algebras of dimension 2|1. Our goal in this paper is to study versal deformations of 0|3 dimensional \(L_\infty\) algebras. The examples we study here are much simpler than the ones studied in [9, 3], so we are able to give a complete treatment of versal deformations.

For simplicity we will suppose that the underlying vector space is defined over \(\mathbb{C}\). Note that in the classification of Lie algebra structures, this assumption reduces the number of equivalence classes of Lie algebra structures. For example, over the reals, there are two non-equivalent classes of simple Lie algebra structures in dimension 3, while over \(\mathbb{C}\), there is only one class. However, we shall not consider the more general question of deformations over other fields.

In Section 2 we introduce \(L_\infty\) algebras and give the definition of a versal deformation — indicating a construction we will use in our computation. Section 3 treats the classification of codifferentials, giving another approach to the well-known classification of 3-dimensional Lie algebras. In Section 4 we compute versal deformations of each of these Lie algebras and also, using our deformation results, describe their moduli space.

2. Basic Definitions

2.1. \(L_\infty\) Algebras. If \(W\) is a \(\mathbb{Z}_2\)-graded vector space, then \(S(W)\) denotes the symmetric coalgebra of \(W\). If we let \(T(W)\) be the reduced tensor algebra \(T(W) = \bigoplus_{n=1}^{\infty} W^\otimes n\), then the reduced symmetric algebra \(S(W)\) is the quotient of the tensor algebra by the graded ideal generated by \(u \otimes v - (-1)^{uv} v \otimes u\) for elements \(u, v \in W\). The symmetric
algebra has a natural coalgebra structure, given by

\[ \Delta(w_1 \ldots w_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in \text{Sh}(k,n-k)} \epsilon(\sigma)w_{\sigma(1)} \ldots w_{\sigma(k)} \otimes w_{\sigma(k+1)} \ldots w_{\sigma(n)}, \]

where we denote the product in \( S(W) \) by juxtaposition, \( \text{Sh}(k,n-k) \) is the set of unshuffles of type \( (k,n-k) \), and \( \epsilon(\sigma) \) is a sign determined by \( \sigma \) (and \( w_1 \ldots w_n \)) given by

\[ w_{\sigma(1)} \ldots w_{\sigma(n)} = \epsilon(\sigma)w_1 \ldots w_n. \]

A coderivation on \( S(W) \) is a map \( \delta : S(W) \rightarrow S(W) \) satisfying

\[ \Delta \circ \delta = (\delta \otimes I + I \otimes \delta) \circ \Delta. \]

Let us suppose that the even part of \( W \) has basis \( e_1 \ldots e_m \), and the odd part has basis \( f_1 \ldots f_n \), so that \( W \) is an \( m|n \) dimensional space. Then a basis of \( S(W) \) is given by all vectors of the form \( e_1^{k_1} \ldots e_m^{k_m} f_1^{l_1} \ldots f_n^{l_n} \), where \( k_i \) is any nonnegative integer, and \( l_i \in \mathbb{Z}_2 \). An \( L_\infty \) structure on \( W \) is simply an odd codifferential on \( S(W) \), that is to say, an odd coderivation whose square is zero. The space \( \text{Coder}(W) \) can be naturally identified with \( \text{Hom}(S(W), W) \), and the Lie superalgebra structure on \( \text{Coder}(W) \) determines a Lie bracket on \( \text{Hom}(S(W), W) \) as follows. Denote \( L_m = \text{Hom}(S^m(W), W) \) so that \( L = \text{Hom}(S(W), W) \) is the direct product of the spaces \( L_i \). If \( \alpha \in L_m \) and \( \beta \in L_n \), then \([\alpha, \beta]\) is the element in \( L_{m+n-1} \) determined by

(1) \([\alpha, \beta](w_1 \ldots w_{m+n-1}) = \sum_{\sigma \in \text{Sh}(m,n-1)} \epsilon(\sigma)\alpha(\beta(w_{\sigma(1)} \ldots w_{\sigma(n-1)})w_{\sigma(m+n-1)}) - (-1)^{\alpha \beta} \sum_{\sigma \in \text{Sh}(m,n-1)} \epsilon(\sigma)\beta(\alpha(w_{\sigma(1)} \ldots w_{\sigma(m-1)})w_{\sigma(m+n-1)}). \]

Another way to express this bracket is in the form

\([\alpha, \beta] = \alpha \tilde{\beta} - (-1)^{\alpha \beta} \beta \tilde{\alpha}, \]

where for \( \varphi \in \text{Hom}(S^k(W), W) \), \( \tilde{\varphi} \) is the associated coderivation, given by

\[ \tilde{\varphi}(w_1 \ldots w_n) = \sum_{\sigma \in \text{Sh}(k,n-k)} \epsilon(\sigma)\varphi(w_{\sigma(1)} \ldots w_{\sigma(k)})w_{\sigma(k+1)} \ldots w_{\sigma(n)}. \]

If \( W \) is completely odd, and \( d \in L_2 \), then \( d \) determines an ordinary Lie algebra on \( W \), or rather on its parity reversion. This is the case we consider in the present paper. The symmetric algebra on \( W \) looks like the exterior algebra on \( W \) if we forget the grading. If we define
\[ a, b = d(ab) \] for \( a, b \in W \), then the bracket is antisymmetric because \( ba = -ab \), and moreover
\[
0 = [d, d](abc) = \frac{1}{2} \sum_{\sigma \in \text{Sh}(2, 1)} \epsilon(\sigma) d(d(\sigma(a)\sigma(b))\sigma(c))
\]
\[
= d((d(ab)c) + d(d(bc)a) - d(d(ac)b)
\]
\[
=[a, b], c + [b, c], a - [a, c], b,
\]
which is the Jacobi identity. When \( d \in L_2 \) and \( W \) has a true grading, then the same principle holds, except that one has to take into account a sign arising from the map \( S^2(W) \rightarrow \bigwedge^2(V) \), where \( V \) is the parity reversion of \( W \). Thus \( \mathbb{Z}_2 \)-graded Lie algebras are also examples of \( L_\infty \) algebras. In addition, differential graded Lie algebras, also called superalgebras, are examples of \( L_\infty \) algebras. In all these cases, the method of construction of miniversal deformations we describe here applies. One simply considers only terms that come from \( L_2 \) in the codifferentials.

Suppose that \( \tilde{g} : S(W) \rightarrow S(W') \) is a coalgebra morphism, that is a map satisfying
\[
\Delta' \circ \tilde{g} = (\tilde{g} \otimes \tilde{g}) \circ \Delta.
\]
If \( d \) and \( d' \) are \( L_\infty \) algebra structures on \( W \) and \( W' \), resp., then \( \tilde{g} \) is a homomorphism between these structures if \( \tilde{g} \circ d = d' \circ \tilde{g} \). Two \( L_\infty \) structures \( d \) and \( d' \) on \( W \) are equivalent, and we write \( d' \sim d \) when there is a coalgebra automorphism \( \tilde{g} \) of \( S(W) \) such that \( d' = \tilde{g}^* (d) = \tilde{g}^{-1} \circ d \circ \tilde{g} \). Furthermore, if \( d = d' \), then \( \tilde{g} \) is said to be an automorphism of the \( L_\infty \) algebra.

2.2. Versal Deformations. An augmented local ring \( A \) with maximal ideal \( m \) will be called an infinitesimal base if \( m^2 = 0 \), and a formal base if \( A = \varprojlim_n A/m^n \). A deformation of an \( L_\infty \) algebra structure \( d \) on \( W \) with base given by a local ring \( A \) with augmentation \( \epsilon : A \rightarrow \mathfrak{k} \), where \( \mathfrak{k} \) is the field over which \( W \) is defined, is an \( A-L_\infty \) structure \( \tilde{d} \) on \( W \otimes A \) such that the morphism of \( A-L_\infty \) algebras \( \epsilon_* = 1 \otimes \epsilon : L_A = L \otimes A \rightarrow L \otimes \mathfrak{k} = L \) satisfies \( \epsilon_*(\tilde{d}) = d \). (Here \( W \otimes A \) is an appropriate completion of \( W \otimes A \).) The deformation is called infinitesimal (formal) if \( A \) is an infinitesimal (formal) base.

In general, the cohomology \( H(D) \) of \( d \) given by the operator \( D : L \rightarrow L \) with \( D(\phi) = [\phi, d] \) may not be finite dimensional. However, \( L \) has a natural filtration \( L^n = \prod_{i=n}^{\infty} L_i \), which induces a filtration \( H^n \) on the cohomology, because \( D \) respects the filtration. Then \( H(D) \) is of finite type if \( H^n/H^{n+1} \) is finite dimensional. Since this is always true when \( W \) is finite dimensional, the examples we study here will always
be of finite type. A set $\delta_i$ will be called a basis of the cohomology, if any element $\delta$ of the cohomology can be expressed uniquely as a formal sum $\delta = \delta_i a^i$. If we identify $H(D)$ with a subspace of the space of cocycles $Z(D)$, and we choose a basis $\beta_i$ of the coboundary space $B(D)$, then any element $\zeta \in Z(D)$ can be expressed uniquely as a sum $\zeta = \delta_i a^i + \beta_i b^i$.

For each $\delta_i$, let $u^i$ be a parameter of opposite parity. Then the infinitesimal deformation $d^1 = d + \delta_i u^i$, with base $A = k[u^i]/(u^i u^j)$ is universal in the sense that if $\tilde{d}$ is any infinitesimal deformation with base $B$, then there is a unique homomorphism $f : A \to B$ such that the morphism $f_* = 1 \otimes f : L_A \to L_B$ satisfies $f_*(\tilde{d}) \sim d$.

For formal deformations, there is no universal object in the sense above. A versal deformation is a deformation $d^\infty$ with formal base $A$ such that if $\tilde{d}$ is any infinitesimal deformation with base $B$, then there is some morphism $f : A \to B$ such that $f_*(d^\infty) \sim \tilde{d}$. If $f$ is unique whenever $B$ is infinitesimal, then the versal deformation is called miniversal. In [7], we constructed a miniversal deformation for $L_\infty$ algebras with finite type cohomology.

The method of construction is as follows. Define a coboundary operator $D$ by $D(\varphi) = [\varphi, d]$. First, one constructs the universal infinitesimal deformation $d^1 = d + \delta_i u^i$, where $\delta_i$ is a graded basis of the cohomology $H(D)$ of $d$, or more correctly, a basis of a subspace of the cocycles which projects isomorphically to a basis in cohomology, and $u^i$ is a parameter whose parity is opposite to $\delta_i$. The infinitesimal assumption that the products of parameters are equal to zero gives the property that $[d^1, d^1] = 0$. Actually, we can express $[d^1, d^1] = (-1)^{\delta_j (\delta_i + 1)} [\delta_i, \delta_j] u^i u^j = \delta_k a^{ij}_k u^i u^j + \beta_k b^{ij}_k u^i u^j$, where $\beta_i$ is a basis of the coboundaries, because the bracket of $d^1$ with itself is a cocycle. Note that the right hand side is of degree 2 in the parameters, so it is zero up to order 1 in the parameters.

If we suppose that $D(\gamma_i) = -\frac{1}{2} \beta_i$, then by replacing $d^1$ with $d^2 = d^1 + \gamma_k b^{ij}_k u^i u^j$,

one obtains $[d^2, d^2] = \delta_k a^{ij}_k u^i u^j + 2[\delta_i u^i, \gamma_k b^{ij}_k u^i u^j] + [\gamma_k b^{ij}_k u^i u^j, \gamma_l b^{ij}_l u^i u^j]$.

Thus we are able to get rid of terms of degree 2 in the coboundary terms $\beta_i$, but those which involve the cohomology terms $\delta_i$ can not be eliminated. This gives rise to a set of second order relations on the parameters. One continues this process, taking the bracket of the
$n$-th order deformation $d^n$, adding some higher order terms to cancel coboundaries, obtaining higher order relations, which extend the second order relations.

Either the process continues indefinitely, in which case the miniversal deformation is expressed as a formal power series in the parameters, or after a finite number of steps, the right hand side of the bracket is zero after applying the $n$-th order relations. In this case, the miniversal deformation is simply the $n$-th order deformation. In any case, we obtain a set of relations $R_i$ on the parameters, one for each $\delta_i$, and the algebra $A = \mathbb{C}[u^i]/(R_i)$ is called the base of the miniversal deformation. Examples of the construction of miniversal deformations can be found in [5, 6, 8].

3. Classification of Lie Algebra Structures of Dimension 3

Suppose that $W = \langle f_1, f_2, f_3 \rangle$. Then $S(W)$ decomposes into three pieces.

$$S^1(W) = \langle f_1, f_2, f_3 \rangle, \quad \dim(S^1(W)) = 0|3$$

$$S^2(W) = \langle f_1f_2, f_1f_3, f_2f_3 \rangle, \quad \dim(S^2(W)) = 3|0$$

$$S^3(W) = \langle f_1f_2f_3 \rangle, \quad \dim(S^3(W)) = 0|1$$

Let $L = \text{Hom}(S(W), W)$ and $L_n = \text{Hom}(S^n(W), W)$. Then

$$L_1(W) = \{\varphi^j_I | I \in \{100, 010, 001\}, j = 1 \ldots 3\}, \quad \dim(L_1) = 9|0$$

$$L_2(W) = \{\varphi^j_I | I \in \{110, 101, 011\}, j = 1 \ldots 3\}, \quad \dim(L_2) = 0|9$$

$$L_3(W) = \{\varphi^j_{111} | j = 1 \ldots 3\}, \quad \dim(L_3) = 3|0$$

It follows that the only candidate for an odd codifferential is of the form

$$d = \varphi^{110}_1 a_1 + \varphi^{110}_2 a_2 + \varphi^{110}_3 a_3$$

$$+ \varphi^{101}_1 a_4 + \varphi^{101}_2 a_5 + \varphi^{101}_3 a_6$$

$$+ \varphi^{011}_1 a_7 + \varphi^{011}_2 a_8 + \varphi^{011}_3 a_9$$

Being a quadratic codifferential, we see that $d$ gives an $L_\infty$ structure precisely when it determines a Lie algebra structure. It is natural to consider the derived subalgebra $W' = d(S^2(W))$. Let

$$A = \begin{pmatrix}
    a_1 & a_2 & a_3 \\
    a_4 & a_5 & a_6 \\
    a_7 & a_8 & a_9 
\end{pmatrix}.$$ 

It is easy to see that the rank of $A$ is precisely equal to the dimension of the derived subalgebra. In particular, when $\text{det}(A) = 0$, the derived subalgebra has dimension less than three.
The codifferential condition \([d, d] = 0\) is equivalent to the system of three quadratic equations

\[
\begin{align*}
    a_9 a_2 - a_3 a_8 + a_6 a_1 - a_3 a_4 &= 0 \\
    -a_5 a_9 + a_8 a_6 + a_5 a_1 - a_2 a_4 &= 0 \\
    -a_4 a_9 - a_1 a_8 + a_7 a_6 + a_7 a_2 &= 0.
\end{align*}
\] (3)

Letting \(x = a_2 + a_6\), \(y = a_9 - a_1\), and \(z = -(a_4 + a_8)\), we can rewrite the above equations as homogeneous linear equations for \(x\), \(y\), \(z\):

\[
\begin{pmatrix}
    a_1 & a_2 & a_3 \\
    a_4 & a_5 & a_6 \\
    a_7 & a_8 & a_9
\end{pmatrix}
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = 0.
\]

If \(\det A \neq 0\) then the only possibility is \(x = 0\), \(y = 0\), and \(z = 0\), that is, \(a_6 = -a_2\), \(a_9 = a_1\), and \(a_8 = -a_4\). Thus

\[
A = \begin{pmatrix}
    a_1 & a_2 & a_3 \\
    a_4 & a_5 & -a_2 \\
    a_7 & -a_4 & a_1
\end{pmatrix},
\] (4)

whose determinant \(\det A = a_5 a_1^2 - 2a_1 a_2 a_4 - a_3 a_4^2 - a_7 a_2^2 - a_7 a_3 a_5\), does not vanish in general. Consequently we have only one pattern to consider for when the derived subalgebra has dimension three,

\[
d = \varphi^{110}_1 a_1 + \varphi^{110}_2 a_2 + \varphi^{110}_3 a_3 + \varphi^{101}_1 a_4 + \varphi^{101}_2 a_5 + \varphi^{101}_3 a_2 + \varphi^{011}_1 a_7 - \varphi^{011}_2 a_4 + \varphi^{011}_3 a_1.
\] (5)

When the derived subalgebra has dimension one, we can choose a basis such that the codifferential \(d\) has the simple form

\[
d = \varphi^{110}_1 a_1 + \varphi^{101}_1 a_4 + \varphi^{011}_1 a_7.
\] (6)

Moreover, it is easy to check that any coderivation of this form is a codifferential.

Finally, suppose that the derived subalgebra has dimension 2. Then we can express \(d\) in the form

\[
d = \varphi^{110}_1 a_1 + \varphi^{110}_2 a_2 + \varphi^{101}_1 a_4 + \varphi^{101}_2 a_5 + \varphi^{011}_1 a_7 + \varphi^{011}_2 a_8.
\]

However, in this case, it is easy to check that the condition \(d^2 = 0\) forces \(a_1 = 0\) and \(a_2 = 0\), or the rank of the associated matrix \(A\) must be less than 2. Thus, when the derived subalgebra has dimension 2, in an appropriate basis, we can express

\[
d = \varphi^{101}_1 a_4 + \varphi^{101}_2 a_5 + \varphi^{011}_1 a_7 + \varphi^{011}_2 a_8.
\] (7)
Each of these three cases can be reduced to a much simpler form, up to equivalence. Let us begin with the one dimensional case first. Let us suppose that \( g \) is a linear automorphism of the symmetric coalgebra of \( W \), in other words, \( g \) is generated by an invertible linear operator on \( W \). For convenience, let us denote

\[
g(1) = 1l + 2m + 3n \\
g(2) = 1r + 2s + 3t \\
g(3) = 1x + 2y + 3z
\]

Let \( G = \begin{pmatrix} l & m & n \\ r & s & t \\ x & y & z \end{pmatrix} \) be the matrix of the linear transformation, so we require \( \det(G) \neq 0 \). Let us note that on \( S^2(W) \), \( g \) is given by

\[
\begin{align*}
g(12) &= 12(ls - mr) + 13(lt - nr) + 23(mt - ns) \\
g(13) &= 12(ly - mx) + 13(lz - nx) + 23(mz - ny) \\
g(23) &= 12(ry - sx) + 13(rz - tx) + 23(sz - ty)
\end{align*}
\]

Now two codifferentials \( d' \) and \( d \) are equivalent precisely when there is some automorphism \( g \) such that \( d' = gdg^{-1} \), in other words, \( d'g = gd \). Let \( d = \varphi_{110}a_1 + \varphi_{101}a_4 + \varphi_{011}a_7 \), and \( d' = \varphi_{101} \).

Examining the conditions for these two codifferentials to be equivalent, we obtain

\[
\begin{align*}
d'g(12) &= 1(mt - ns) \\
gd(12) &= 1a_1 + 2ma_1 + 3na_1 \\
d'g(13) &= 1(mz - ny) \\
gd(13) &= 1a_4 + 2ma_4 + 3na_4 \\
d'g(23) &= 1(sz - ty) \\
gd(23) &= 1a_7 + 2ma_7 + 3na_7
\end{align*}
\]

Note that if all three coefficients \( a_i \) vanished, this would force \( \det(G) = 0 \), which is not permitted. Thus we must have \( m = n = 0 \). But then it follows that \( a_1 = a_3 = 0 \). This means the only members of the family of one dimensional codifferentials equivalent to \( d' \) are the nonzero multiples of \( d' \).

Let us now consider the case when \( d' = \varphi_{110} \). By a similar analysis to the above, it is easily shown that if either \( a_1 \) or \( a_3 \) does not vanish, then \( d' \) and \( d \) are equivalent. Thus all one dimensional solutions are equivalent to one of two codifferentials \( d = \varphi_{110} \) or \( d = \varphi_{011} \).
Next, let us consider the two dimensional solutions. Suppose that $d$ is as in equation (7), and let $B = \begin{pmatrix} a_4 & a_5 \\ a_7 & a_8 \end{pmatrix}$, so $B$ is an invertible matrix.

Let $d' = \varphi_2^{101} + \lambda \varphi_3^{011}$. Then setting $d'g = gd$ leads to the following equalities:

\[
\begin{align*}
lt - nr &= 0 \\
\lambda(mt - ns) &= 0 \\
lz - nx &= a_4l + a_5r \\
\lambda(mz - ny) &= a_4m + a_5s \\
0 &= a_4n + a_5t \\
rz - tx &= a_7l + a_8r \\
\lambda(sz - ty) &= a_7m + a_8s \\
0 &= a_7n + a_8t
\end{align*}
\]

It follows that $B \begin{bmatrix} n \\ t \end{bmatrix} = 0$, so that $n = 0$ and $t = 0$. Then we have $z(ls - mr) = \det(G) \neq 0$, and

\[
B \begin{pmatrix} l & m \\ r & s \end{pmatrix} = z \begin{pmatrix} l & \lambda m \\ r & \lambda s \end{pmatrix}
\]

Note that $z \neq 0$ and $ls - mr \neq 0$. This implies that $(l, r)$ and $(m, s)$ are linearly independent eigenvectors of $B$ with eigenvalues $z$ and $z\lambda$ respectively. Moreover, any matrix $B$ which has linearly independent eigenvectors of this form, with the ratio of eigenvalues being $\lambda$, determines a codifferential which is equivalent to $d'$. In particular, if $d_i' = \varphi_2^{101} + \lambda_i \varphi_2^{011}$, for $i = 1, 2$, then $d_i'$ is equivalent to $d'_2$ when $\lambda_2 = \lambda_1$ or $\lambda_2 = 1/\lambda_1$. Thus the nonequivalent solutions of this form parameterize the punctured unit disc in $\mathbb{C}$, with the boundary glued together. Moreover, if we consider the case $\lambda = 0$, which is a one dimensional solution, it is then easy to see that it is equivalent to the solution $d = \varphi_4^{110}$. We therefore add this solution to the family of two dimensional solutions. We shall see later, when we study versal deformations of these codifferentials, that this idea makes sense.

Now, if the matrix associated to $d$ is not diagonalizable, then there is some basis in which its matrix can be expressed in the form $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, so $d = \varphi_1^{101} + \varphi_2^{011} + \varphi_3^{011}$. Let us consider the special case $d' = \varphi_1^{101} + \varphi_2^{101} + \varphi_3^{011}$. It is evident that if we consider the diagonal automorphism $g(1) = 1, g(2) = 2\lambda, g(3) = 3\lambda$, then $d'g = gd$. 
Thus all invertible defective matrices arise from the single codifferential
\[ d = \varphi_1^{101} + \varphi_2^{101} + \varphi_2^{011}. \]

Finally, let us consider the three dimensional case, with matrix as in (4). If \( a_3 \neq 0 \), then by introducing a new basis element \( 3' = 1a_1 + 2a_2 + 3a_3 \), we can replace \( A \) with a simpler matrix of the form
\[
A = \begin{pmatrix}
0 & 0 & 1 \\
a_4 & a_5 & 0 \\
a_7 & -a_4 & 0
\end{pmatrix}.
\]
By interchanging the elements 1, 2 and 3, we can see that if any one of the elements \( a_1, a_3 \) or \( a_7 \), does not vanish, then we can make a similar transformation. On the other hand, if all of those coefficients vanish, then \( \det(A) = -2a_1a_2a_4 \), so none of the three coefficients \( a_1, a_2 \) and \( a_4 \) can vanish. Replacing the basis element 2 with \( 2' = 2 + 3 \) will result in a matrix whose \( a_3 \) term does not vanish. Thus we can assume without loss of generality that \( A \) has the simpler form above.

Now if \( a_5 \neq 0 \), then replacing 2 with \( 1 + 2 \) results in a matrix of the form
\[
A = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
a_7 & 0 & 0
\end{pmatrix},
\]
and a simple linear change of variables allows one to assume that \( a_7 = 1 \). This yields a codifferential of the form
\[ d' = \varphi_3^{110} + \varphi_2^{101} + \varphi_1^{011}. \]
Otherwise, we have \( a_5 = 0 \), and we may as well assume that \( a_7 = 0 \), otherwise, by interchanging 1 and 2, we reduce to \( d' \). Now a simple linear change of variables will allow us to assume \( a_4 = 1 \), so our second candidate for a three dimensional codifferential is
\[ d = \varphi_3^{110} + \varphi_1^{101} - \varphi_2^{011}. \]
It is easy to verify that \( G = \begin{pmatrix}
-i & 0 & 1 \\
-\frac{i}{2} & 0 & -\frac{1}{2} \\
0 & i & 0
\end{pmatrix} \)
will yield \( d'g = gd \), so these two codifferentials are equivalent.

Thus, up to equivalence, we obtain precisely the codifferentials
\[ d_1 = \varphi_1^{011}, \quad d(\lambda) = \varphi_1^{101} + \varphi_2^{011}\lambda, \quad d_2 = \varphi_1^{101} + \varphi_2^{101} + \varphi_2^{011}, \quad d_3 = \varphi_3^{110} + \varphi_2^{101} + \varphi_1^{011}, \]
where \( d(\lambda) \) is identified with \( d(\lambda^{-1}) \). This gives one family and three exceptional differentials. However, as we shall see when we study versal deformations, this classification has not yet revealed exactly how the moduli space fits together.
4. Versal Deformations of the Codifferentials

We now begin the process of constructing a miniversal deformation for each of the codifferentials we have studied. First, we compute the cohomology of the codifferential, use it to write the universal infinitesimal deformation, and then apply the bracket process above to determine a miniversal deformation and the relations on the parameters. Along the way, we will discover that the cohomology of the differentials reveals a lot of information about the moduli space of three dimensional Lie algebras. So far, what we have seen is that there is a family of codifferentials and three special cases that lie outside the family. But how do these special codifferentials fit together with the family as a moduli space?

4.1. The Codifferential \( d_3 = \varphi^{110}_3 + \varphi^{101}_2 + \varphi^{011}_1 \). This codifferential is the only one with a three dimensional derived algebra, and corresponds to the simple Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \). Accordingly, we would expect that the cohomology vanishes in all dimensions. However, keep in mind that in the usual definition of Lie algebra cohomology, there is a space \( L_0 = \text{Hom}(\mathbb{C}, W) = W \) with the bracket of a cochain and an element of \( L_0 \) given by \([\varphi, w](\alpha) = \varphi(w\alpha)\). The 1-cocycles \( Z^1 \) correspond to derivations of the Lie algebra structure determined by \( d_3 \). The 0-coboundaries are the inner derivations. For a simple Lie algebra, all derivations are inner.

Let us compute a basis of the inner derivations. We have

\[
D(1)(1) = [1, d_3](1) = -[d_3, 1](1) = d_3(11) = 0.
\]

Similarly, \( D(1)(2) = -d_3(12) = -3 \) and \( D(1)(3) = -2 \). Thus we obtain

\[
D(1) = -\varphi^{010}_3 - \varphi^{001}_2,
D(2) = \varphi^{100}_3 - \varphi^{001}_1,
D(3) = \varphi^{100}_2 + \varphi^{010}_1.
\]

From our perspective, since we do not include \( L_0 \) in our space of cochains, we expect that \( H^1 \) will be 3|0-dimensional, and all higher cohomology should vanish. This indeed turns out to be the case. We will not reproduce the calculations of the cohomology here, but we will give detailed computations later for some more interesting cases, so the method will be clear.

Nevertheless, since we have three cocycles, all even, we do have a nontrivial infinitesimal deformation. Let us adopt the convention to use the Greek letter \( \theta \) for an odd parameter, and the Roman letter
t for an even one. For an odd parameter \( \theta \), we have \( \theta^2 = 0 \) as a consequence of the graded commutativity, so we do not consider this to be a relation on our parameter algebra. Thus, we have

\[
d_3^1 = \varphi_3^{110} + \varphi_2^{101} + \varphi_1^{011} + (\varphi_3^{010} + \varphi_2^{001})\theta_1 + (\varphi_3^{100} - \varphi_1^{001})\theta_2 + (\varphi_2^{100} + \varphi_1^{010})\theta_3.
\]

In computing \([d_3^1, d_3^1]\), we note that the brackets of the cocycles with \( d_3 \) vanish, so we only need to compute the brackets of the cocycles with each other. Using the fact that the square of an odd parameter is zero, this means we only need to calculate the following brackets.

\[
\begin{align*}
[\varphi_3^{010} + \varphi_2^{001}, \varphi_3^{100} - \varphi_1^{001}] &= \varphi_3^{100} + \varphi_1^{010} \\
[\varphi_3^{010} + \varphi_2^{001}, \varphi_2^{100} + \varphi_1^{010}] &= \varphi_3^{100} - \varphi_1^{001} \\
[\varphi_3^{100} - \varphi_1^{001}, \varphi_2^{100} + \varphi_1^{010}] &= \varphi_3^{010} + \varphi_2^{011}
\end{align*}
\]

Thus we obtain

\[
[d_3^1, d_3^1] = (\varphi_2^{100} + \varphi_2^{010})\theta_1\theta_2 + (\varphi_3^{100} - \varphi_1^{001})\theta_1\theta_3 + (\varphi_3^{010} + \varphi_2^{001})\theta_2\theta_3.
\]

Of course, these are all cocycles which are not coboundaries, so we obtain the relations \( \theta_i\theta_j = 0 \) for all \( i, j \). Thus the infinitesimal deformation is minimal, and the base of the minimal deformation is given by \( \mathbb{C}[\theta_1, \theta_2, \theta_3]/(\theta_1\theta_2, \theta_1\theta_3, \theta_2\theta_3) \).

Note that the vanishing of \( H^2 \) is consistent with the observation that any small change in the codifferential \( d_3 \) will give rise to a codifferential \( d' \) which will still have a 3-dimensional derived subalgebra. Thus any small change in \( d_3 \) gives rise to the same codifferential, and we see that \( d_3 \) does not deform into any of the other codifferentials.

### 4.2. The Codifferential \( d_2 = \varphi_1^{101} + \varphi_2^{101} + \varphi_2^{011} \)

First, we give a table of all the coboundaries.

\[
\begin{align*}
D(\varphi_1^{100}) &= -\varphi_2^{101}, & D(\varphi_1^{101}) &= -\varphi_1^{111} + \varphi_2^{111} \\
D(\varphi_2^{100}) &= 0, & D(\varphi_2^{101}) &= -\varphi_2^{111} \\
D(\varphi_3^{100}) &= \varphi_2^{110} + \varphi_3^{101}, & D(\varphi_3^{101}) &= -2\varphi_3^{111} \\
D(\varphi_1^{010}) &= \varphi_1^{101} - \varphi_2^{011}, & D(\varphi_1^{101}) &= 0 \\
D(\varphi_2^{010}) &= \varphi_2^{101}, & D(\varphi_2^{101}) &= 0 \\
D(\varphi_3^{010}) &= -\varphi_1^{110} - \varphi_2^{110} + \varphi_3^{101} + \varphi_3^{011}, & D(\varphi_3^{101}) &= \varphi_2^{111} \\
D(\varphi_1^{001}) &= 0, & D(\varphi_1^{111}) &= 0 \\
D(\varphi_2^{001}) &= 0, & D(\varphi_2^{111}) &= 0 \\
D(\varphi_3^{001}) &= -\varphi_1^{101} - \varphi_2^{101} - \varphi_2^{011}, & D(\varphi_3^{111}) &= -\varphi_1^{111} - \varphi_2^{111}
\end{align*}
\]
Let $h_n$ be the dimension of $H^n$, the $n$-th cohomology group of $d$, $b_n$ be the dimension of $B^n = D(L_n)$, the space of $n$-coboundaries, and $z_n$ be the dimension of $Z^n = \ker(D : L_n \to L_{n+1})$, the space of $n$-cocycles. Note that the $n$-coboundaries are a subspace of the $n + 1$-cocycles in this notation.

Let us determine these dimensions for the codifferential $d_2$. Note that in there are three obvious 1-cocycles, $\varphi_2^{101}$, $\varphi_1^{001}$ and $\varphi_2^{001}$. In addition we also have the cocycle $\varphi_1^{100} + \varphi_2^{010}$. Thus $h_2 = z_1 = 4$, and $b_1 = 5$. Evidently, $b_2 = 3$, so we must have $z_2 = 6$. Thus $h_2 = 1$ and $h_3 = 0$. We can choose $\varphi_2^{011}$ as a basis of $H^2$. The universal infinitesimal deformation is given by

$$d_2^1 = \varphi_1^{101} + \varphi_2^{101} + (1 + t)\varphi_2^{011} + \varphi_2^{100}\theta_1 + \varphi_2^{001}\theta_2 + \varphi_2^{000}\theta_4 + (\varphi_1^{100} + \varphi_2^{010})\theta_4.$$ 

The nonzero brackets which occur in the computation of $[d_2^1, d_2^1]$ are

$$[\varphi_2^{011}, \varphi_2^{101}] = \varphi_2^{101}, \quad [\varphi_2^{001}, \varphi_1^{100} + \varphi_2^{010}] = -\varphi_2^{001}$$

$$[\varphi_1^{101}, \varphi_1^{001}] = \varphi_1^{001}, \quad [\varphi_2^{001}, \varphi_1^{100} + \varphi_2^{010}] = -\varphi_2^{001}$$

Thus we have

$$[d_2^1, d_2^1] = 2\varphi_2^{101}t\theta_1 + \varphi_2^{001}(\theta_1\theta_2 - \theta_3\theta_4) - \varphi_2^{001}\theta_2\theta_4.$$ 

Of the three cocycles appearing on the right hand side of this equation, only the first is a coboundary. Thus we obtain two second order relations, $\theta_1\theta_2 - \theta_3\theta_4 = 0$ and $\theta_2\theta_4 = 0$, and we need to add something to $d_1^1$ to obtain a second order deformation. Since $D(\varphi_1^{100}) = -\varphi_2^{101}$, we can express $d_2^2 = d_1^1 + \varphi_1^{100}t\theta_1$. We compute

$$[d_2^2, d_2^1] = \varphi_2^{001}(\theta_1\theta_2 - \theta_3\theta_4) - \varphi_1^{100}(\theta_2\theta_4 + t\theta_1\theta_2).$$ 

Since no coboundary terms occur, we obtain that $d_2^2$ is a miniversal deformation, and the base is given by

$$A = \mathbb{C}[[t, \theta_1, \theta_2, \theta_3, \theta_4]]/(\theta_1\theta_2 - \theta_3\theta_4, \theta_2\theta_4 + t\theta_1\theta_2).$$

Since there is a nontrivial deformation in the Lie algebra direction, we can explore how the deformation moves our codifferential in the moduli space. Note that $d_2^1$ is a codifferential which has two dimensional derived algebra, and it has eigenvalues 1 and $1 + t$, so it lies in the family $d(\lambda)$, and is near to $d(1)$. In fact, a punctured neighborhood of $d_2$ looks exactly like a neighborhood of $d(1)$. However, $d(1)$ is not close to $d_2$, in the sense that a small neighborhood of $d_2$ does not contain $d(1)$. When we study $d(1)$, we shall see that the opposite statement is not true.
4.3. The Family of Codifferentials \( d(\lambda) = \varphi_1^{101} + \varphi_2^{011}\lambda \). The cohomology will depend to some extent on the value of \( \lambda \). The coboundaries are given by

\[
\begin{align*}
D(\varphi_1^{100}) &= 0, & D(\varphi_1^{110}) &= -\varphi_1^{111}\lambda \\
D(\varphi_2^{100}) &= \varphi_2^{101}(1 - \lambda), & D(\varphi_2^{110}) &= -\varphi_2^{111} \\
D(\varphi_3^{100}) &= \varphi_2^{110}\lambda + \varphi_3^{101}, & D(\varphi_3^{110}) &= -\varphi_3^{111}(\lambda + 1) \\
D(\varphi_1^{010}) &= -\varphi_1^{011}(1 - \lambda), & D(\varphi_1^{111}) &= 0 \\
D(\varphi_2^{010}) &= 0, & D(\varphi_2^{111}) &= 0 \\
D(\varphi_3^{010}) &= -\varphi_1^{110} + \varphi_3^{111}\lambda, & D(\varphi_3^{101}) &= \varphi_2^{111}\lambda \\
D(\varphi_1^{001}) &= 0, & D(\varphi_1^{011}) &= 0 \\
D(\varphi_2^{001}) &= 0, & D(\varphi_2^{011}) &= 0 \\
D(\varphi_3^{001}) &= -\varphi_1^{101} - \varphi_2^{011}\lambda, & D(\varphi_3^{011}) &= -\varphi_1^{111}
\end{align*}
\]

We shall see that the only thing special about the case \( \lambda = 0 \) is that the dimension of the derived algebra drops to 1, but as far as deformations go, it will behave like a generic element of the family. The cases \( \lambda = \pm 1 \), however, are not generic in terms of their deformations. This makes sense, because in the identification of \( t d\lambda \) with \( d\lambda - 1 \), we see that every point in the unit disc has a neighborhood that is like a usual disc in \( \mathbb{C} \), with the exception of \( \pm 1 \), which are orbifold points. So it is not surprising to find some kind of exceptional behavior for these codifferentials.

4.3.1. Generic case of \( d(\lambda) \). First, we treat the generic case. Clearly, \( H^1 = \langle \varphi_1^{100}, \varphi_2^{010}, \varphi_1^{001}, \varphi_2^{001} \rangle \), so \( h_1 = 4 \) and \( b_1 = 5 \). Evidently, \( b_2 = 3 \), so \( z_2 = 6 \), \( h_2 = 1 \) and \( h_3 = 0 \). We can choose \( \varphi_2^{011} \) as a basis of \( H^2 \). Thus, the universal infinitesimal deformation is given by

\[
d(\lambda)^1 = \varphi_1^{101} + \varphi_2^{111}(\lambda + t) + \varphi_1^{100}\theta_1 + \varphi_2^{010}\theta_2 + \varphi_1^{001}\theta_3 + \varphi_2^{001}\theta_4.
\]

It is easy to verify that

\[
[d(\lambda)^1, d(\lambda)^1] = 2\varphi_1^{001}\theta_1\theta_3 + 2\varphi_2^{001}\theta_2\theta_4.
\]

Thus \( d(\lambda)^1 \) is miniversal and the base of the miniversal deformation is

\[
A = \mathbb{C}[[t, \theta_1, \theta_2, \theta_3, \theta_4]]/(\theta_1\theta_3, \theta_2\theta_4).
\]

Looking at the deformation in the Lie algebra direction, we see that the deformation simply moves along the family.
4.3.2. *The special case* $d(-1)$. Now, let us consider the special case $\lambda = -1$. Then we have an extra cocycle $\varphi_3^{110}$ in $H^2$, and correspondingly, an extra cocycle $\varphi_3^{111}$ in $H^3$. Thus $h_1 = 4$, $h_2 = 2$ and $h_3 = 1$. Thus, the universal infinitesimal deformation becomes

$$d(-1)^1 = \varphi_1^{101} + \varphi_2^{011}(-1 + t_1) + \varphi_1^{100}\theta_1 + \varphi_2^{010}\theta_2 + \varphi_1^{001}\theta_3 + \varphi_2^{001}\theta_4 + \varphi_3^{110}t_2 + \varphi_3^{111}\theta_5.$$  

Then

$$\frac{1}{2}[d(-1)^1, d(-1)^1] = \varphi_1^{001}\theta_1\theta_3 + \varphi_2^{001}\theta_2\theta_3 + \varphi_3^{110}(t_2\theta_1 + t_2\theta_2) + \varphi_3^{111}(\theta_3\theta_1 + \theta_5\theta_2 - t_1t_2) - \varphi_1^{111}\theta_5\theta_3 - \varphi_2^{111}\theta_5\theta_4 + (-\varphi_1^{110} - \varphi_3^{011})t_2\theta_3 + (-\varphi_2^{110} + \varphi_3^{101})t_2\theta_4.$$  

Note that the first four terms are cocycles, so they give rise to second order relations, while the last four terms are coboundaries, so we need to add corresponding terms to obtain the second order deformation

$$d(-1)^2 = d(-1)^1 + \varphi_3^{011}\theta_3\theta_5 + \varphi_2^{110}\theta_4\theta_5 - \varphi_3^{110}t_2\theta_3 - \varphi_3^{100}t_2\theta_4.$$  

Finally, let us compute the bracket of the second order deformation with itself. We obtain

$$\frac{1}{2}[d(-1)^2, d(-1)^2] = \varphi_1^{001}\theta_1\theta_3 + \varphi_2^{001}\theta_2\theta_3 + \varphi_3^{110}(t_2\theta_1 + t_2\theta_2) + \varphi_3^{111}(\theta_3\theta_1 + \theta_5\theta_2 - t_1t_2) + \varphi_3^{011}(\theta_3\theta_5\theta_2 - t_1t_2\theta_3) + \varphi_2^{110}(\theta_4\theta_5\theta_1 - t_1t_2\theta_4) - \varphi_3^{110}t_2\theta_3\theta_2 + \varphi_2^{110}t_2\theta_3\theta_4 - \varphi_3^{100}t_2\theta_4\theta_1 + \varphi_3^{100}t_2\theta_4\theta_3.$$  

All the terms except $\varphi_3^{011}$, $\varphi_2^{110}$, $\varphi_3^{100}$, and $\varphi_3^{100}$ are cocycles, and these exceptional terms are not even coboundaries. Thus, by the general theory, their coefficients must be zero, using the third order relations

$$\begin{align*}
\theta_1\theta_3 &= 0, & \theta_2\theta_4 &= 0, & t_2\theta_3\theta_4 &= 0 \\
t_2\theta_1 + t_2\theta_2 &= 0, & \theta_5\theta_1 + \theta_5\theta_2 - t_1t_2 &= 0
\end{align*}$$  

For example, to see that the coefficient of $\varphi_2^{110}$ vanishes, we observe that

$$\begin{align*}
\theta_4\theta_5\theta_1 - t_1t_2\theta_4 &= \theta_1\theta_5\theta_1 - (\theta_5\theta_1 + \theta_5\theta_2)\theta_4 \\
&= \theta_5\theta_1 - \theta_5\theta_1\theta_4 = 0.
\end{align*}$$  

Let us consider the induced topology on the moduli space of equivalence classes of codifferentials. This topology is not Hausdorff. If every neighborhood of a point $a$ contains the point $b$, then we note that $a$ is in the closure of $b$. In this case, we shall say that $a$ is *infinitesimally*
close to $b$. Note that $d(-1)$ is infinitesimally close to $d_3$, but not the other way around. In some sense, this explains the extra infinitesimal deformation in the Lie algebra direction.

Now at first it may seem strange that this is the first case where one of our codifferentials deforms into $d_3$. After all, generically, we would expect the matrix of a codifferential to be invertible. But look carefully at equation (5), and the form of a solution in the family, and it becomes clear that only when $\lambda = -1$ can a small change in the codifferential give a solution satisfying equation (5).

Notice also, that if we neglect odd parameters, the versal deformation has relation $t_1 t_2 = 0$. In the classical sense, this means that if we were to consider an infinitesimal deformation of the form $d' = d + t(a \varphi_2^{101} + b \varphi_3^{110})$, then the only cases where this deformation extends to a second order deformation is when either $a = 0$ or $b = 0$, in which case, the infinitesimal deformation extends trivially.

4.3.3. The special case d(1). In this case, there are two additional 1-cocycles, $\varphi_2^{100}$ and $\varphi_1^{010}$. Thus $h_1 = 6$, and $b_1 = 3$. Since $b_2 = 3$, we see that $h_2 = 3$ and $h_3 = 0$. So we pick up two extra 2-cocycles, $\varphi_2^{101}$ and $\varphi_1^{011}$. It is convenient to replace the cocycle $\varphi_2^{101}$, which we used as a basis of the cohomology in the generic case, with $\varphi_2^{101} - \varphi_2^{011}$, because it simplifies the interpretation of the bracket of the universal infinitesimal deformation with itself. Thus,

\[
d(1)^1 = \varphi_1^{101} + \varphi_2^{011} \lambda + (\varphi_1^{101} - \varphi_2^{011}) t_1 + \varphi_2^{101} t_2 + \varphi_1^{011} t_3 \\
+ \varphi_1^{100} \theta_1 + \varphi_2^{010} \theta_2 + \varphi_1^{001} \theta_3 + \varphi_2^{001} \theta_4 + \varphi_2^{100} \theta_5 + \varphi_1^{1010} \theta_6,
\]

and we compute that

\[
\frac{1}{2}[d(1)^1, d(1)^1] = (\varphi_1^{101} - \varphi_2^{011})(t_3 \theta_5 - t_2 \theta_6) - \varphi_2^{101} (2t_1 \theta_5 + t_2 (\theta_2 - \theta_1)) \\
+ \varphi_1^{011} (2t_1 \theta_6 + t_3 (\theta_2 - \theta_1)) - \varphi_1^{100} \theta_5 \theta_6 + \varphi_2^{010} \theta_5 \theta_6 \\
+ \varphi_1^{001} (\theta_1 \theta_3 - \theta_4 \theta_6) + \varphi_2^{001} (\theta_2 \theta_4 - \theta_3 \theta_5) \\
+ \varphi_2^{100} (\theta_2 \theta_5 - \theta_1 \theta_5) + \varphi_1^{1010} (\theta_1 \theta_6 - \theta_2 \theta_6),
\]

which is precisely the set of cocycles appearing in the universal infinitesimal deformation, multiplied by the second order relations. Thus the infinitesimal deformation is miniversal.

Now let us interpret how $d(1)$ fits into the moduli space. Note that there is a deformation along the family, given by the cocycle $\varphi_1^{101} - \varphi_2^{011}$, and two other directions of deformation, each of which corresponds to a deformation of $d(1)$ into the special codifferential $d_2$. In fact, if we consider the three dimensional deformation space parameterized by
(\(t_1, t_2, t_3\)), we see that precisely two curves correspond to a deformation in the \(d_2\) direction. Thus we see that \(d(1)\) is infinitesimally close to \(d_2\), although the converse is not true.

So far, we have been able to associate two of the three special codifferentials with the family in some manner.

4.4. The Codifferential \(d_1 = \varphi_{101}^0\). Now we come to the most complicated of the codifferentials. This is not surprising, because \(d_1\) gives a nilpotent Lie algebra structure, so that we expect to find a lot of deformations. The table of codifferentials is given by

\[
\begin{align*}
D(\varphi_{100}^{100}) &= \varphi_{101}^{011}, & D(\varphi_{110}^{100}) &= 0, \\
D(\varphi_{200}^{100}) &= -\varphi_{101}^{101} + \varphi_{3}^{011}, & D(\varphi_{110}^{101}) &= \varphi_{111}^{111} \\
D(\varphi_{300}^{100}) &= \varphi_{101}^{110} + \varphi_{3}^{011}, & D(\varphi_{110}^{101}) &= 0 \\
D(\varphi_{101}^{010}) &= 0, & D(\varphi_{110}^{110}) &= 0 \\
D(\varphi_{201}^{010}) &= -\varphi_{101}^{011}, & D(\varphi_{210}^{110}) &= 0 \\
D(\varphi_{301}^{010}) &= 0, & D(\varphi_{310}^{110}) &= \varphi_{111}^{111} \\
D(\varphi_{101}^{001}) &= 0, & D(\varphi_{110}^{111}) &= 0 \\
D(\varphi_{201}^{001}) &= 0, & D(\varphi_{210}^{111}) &= 0 \\
D(\varphi_{301}^{001}) &= -\varphi_{101}^{011}, & D(\varphi_{310}^{111}) &= 0
\end{align*}
\]

Clearly, \(h_1 = 6\), so \(b_1 = 3\). Since \(b_2 = 1\), we have \(h_2 = 5\) and \(h_3 = 2\). Before constructing a versal deformation, let us analyze how this codifferential sits in the moduli space. Clearly there are a lot of directions one can deform. Notice that because \(\varphi_{101}^{101} - \varphi_{201}^{011}\) and \(\varphi_{101}^{110} + \varphi_{301}^{011}\) are coboundaries, there remain three ways to deform \(d_1\) into \(d_3\), via the cocycles \(\varphi_{201}^{110}, \varphi_{301}^{101} + \varphi_{201}^{101}\) and \(\varphi_{301}^{101} - \varphi_{201}^{101}\). Generically, if we add small multiples of these cocycles, we will obtain a codifferential which is equivalent to \(d_3\). Thus \(d_1\) is infinitesimally close to \(d_3\).

Next, if we add an appropriate multiple of \(\varphi_{101}^{101} + a\varphi_{201}^{101}\), then we are constructing an element in the family, with a matrix \(B\) given by

\[
B = \begin{pmatrix} t & at \\ 1 & 0 \end{pmatrix},
\]

whose eigenvalues are given by \(\lambda_{\pm} = \frac{t + \sqrt{t^2 + 4a}}{2}\). Set \(a = ct\). Then

\[
\lambda = \frac{\lambda_-}{\lambda_+} = \frac{1 - \sqrt{1 + 4c}}{1 + \sqrt{1 + 4c}}
\]

determines the element of the family. Solving for \(c\) in terms of \(\lambda\) we obtain \(c = \frac{\lambda}{(\lambda + 1)^2}\), which gives an element of the family for any value of \(\lambda\) except \(\lambda = \pm 1\). The reason that we do not obtain an element
the cocycle $\phi$ behaves more consistently with the other members of the family than $d$.

Then there is no automorphism which takes it to $d(1)$.

Thus we conclude that $d_1$ is infinitesimally close to every codifferential except $d(1)$. From the behavior of the elements $d_2$ and $d(1)$, it is more natural to consider $d_2$ as a member of the family, because it behaves more consistently with the other members of the family than $d(1)$.

The universal infinitesimal deformation is given by

$$d = \varphi_1^{011} + \varphi_1^{110}t_1 + \varphi_1^{101}t_2 + (\varphi_2^{110} - \varphi_3^{101})t_3 + \varphi_3^{110}t_4 + \varphi_2^{101}t_5,$$

then there is no automorphism which takes it to $d(1)$.

Unfortunately, the bracket of $d_1$ with itself has terms involving almost every cocycle and coboundary.

$$[d_1, d_1] = +\varphi_1^{110}(t_1\theta_1 + t_1\theta_2 + t_2\theta_1) + \varphi_1^{011}(t_1\theta_6 - t_2\theta_2) + (\varphi_2^{110} - \varphi_3^{101})(t_3\theta_1 - t_4\theta_6 + t_5\theta_4) + 2\varphi_3^{110}(-t_3\theta_4 + t_4\theta_1 + t_4\theta_2) + 2\varphi_2^{101}(t_3\theta_6 - t_5\theta_2) - (\varphi_2^{010} - \varphi_3^{001})\theta_4\theta_6 - \varphi_1^{010}(\theta_2\theta_3 + \theta_4\theta_5)
- \varphi_2^{001}(\theta_1\theta_4 + 2t_2\theta_1) + \varphi_1^{001}(\theta_1\theta_5 + \theta_2\theta_6 + \theta_3\theta_6)
+ \varphi_2^{001}(\theta_1\theta_5 + 2t_2\theta_6)
- \varphi_1^{111}(t_1t_5 - t_2t_3 - \theta_1\theta_7 + \theta_2\theta_7 + \theta_6\theta_8)
- \varphi_2^{111}(-t_1t_3 - t_2t_4 - 2\theta_1\theta_8 - \theta_2\theta_8 + \theta_4\theta_7)
- \varphi_3^{111}(-t_1t_3 + t_2t_4 - 2\theta_1\theta_8 - \theta_2\theta_8 + \theta_4\theta_7)
+ (\varphi_1^{101} + \varphi_2^{011})(-t_3\theta_5 + t_5\theta_3) - (\varphi_1^{110} + \varphi_3^{011})(t_3\theta_3 + t_4\theta_5)
+ \varphi_1^{011}(-t_1\theta_5 + t_2\theta_3) + \varphi_1^{111}(\theta_3\theta_7 + \theta_5\theta_8).$$

If we forget about all the $\theta$ terms, then the second order relations reduce to $t_1t_5 - t_2t_3 = 0$ and $-t_1t_3 - t_2t_5 = 0$. To see why we should expect such relations, consider the matrix $A = \begin{pmatrix} t_1 & t_3 & t_4 \\ t_2 & t_5 & -t_3 \\ 1 & 0 & 0 \end{pmatrix}$ associated to the 2-cochain part of $d_1$. We know that if its determinant is nonzero, then it must be of the form $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. But then $t_1 = t_2 = 0$. Otherwise, as the relations yield, we must have dependent columns in the matrix.
Note that the last four terms are coboundaries, so we need to add something to $d_1^2$ to obtain a second order deformation. We may choose

$$d_1^2 = d_1^2 + \varphi_2^{100}(-t_3\theta_5 + t_5\theta_3) + \varphi_3^{100}(-t_3\theta_3 - t_4\theta_5) + \varphi_1^{100}(-t_1\theta_5 + t_2\theta_3) - \varphi_2^{110}(\theta_3\theta_7 + \theta_5\theta_8).$$

Let

$$b_1 = -t_3\theta_7 - \theta_5\theta_8$$
$$b_2 = -t_3\theta_5 + t_5\theta_3$$
$$b_3 = -t_3\theta_3 - t_4\theta_5$$
$$b_4 = -t_1\theta_5 + t_2\theta_3$$

Then

$$\frac{1}{2}[d_1^2, d_1^2] = +\varphi_1^{110}(t_1\theta_1 + t_1\theta_2 + t_2\theta_4 - b_1\theta_3) + \varphi_1^{101}(t_1\theta_6 - t_2\theta_2) + (\varphi_2^{110} - \varphi_3^{101})(t_3\theta_1 - t_4\theta_6 + t_5\theta_4) + \varphi_3^{110}(-2t_3\theta_4 + 2t_4\theta_1 + 2t_4\theta_2 + t_4b_4 - t_1b_3 - b_1\theta_4) + \varphi_2^{101}(2t_3\theta_6 - 2t_5\theta_2 + t_5b_1 - t_2b_2 + b_1\theta_6) - (\varphi_2^{010} - \varphi_3^{001})\theta_4\theta_6 - \varphi_3^{010}(\theta_2\theta_3 + \theta_4\theta_5 - b_4\theta_3) - \varphi_3^{010}(\theta_1\theta_4 + 2\theta_2\theta_4) + \varphi_1^{001}(\theta_1\theta_5 + \theta_2\theta_5 + \theta_3\theta_6 + b_4\theta_5) + \varphi_2^{001}(\theta_1\theta_6 + 2\theta_2\theta_6 + b_2\theta_5) + \varphi_2^{111}(t_1t_5 - t_2t_3 - \theta_1\theta_7 + \theta_2\theta_7 + \theta_6\theta_8 - b_4\theta_7 - t_2b_1) + \varphi_3^{111}(-t_1t_3 - t_2t_4 - 2\theta_1\theta_8 - \theta_2\theta_8 + \theta_4\theta_7 - b_4\theta_8) + \varphi_1^{100}(-b_3\theta_5 - b_2\theta_4) + \varphi_2^{010}b_2\theta_3 + \varphi_3^{001}b_2\theta_5 + \varphi_2^{110}(t_3b_4 - t_4b_1 + b_3b_4 + b_1\theta_1) + \varphi_2^{100}(-b_3\theta_6 + b_2b_4 - b_2\theta_2) + \varphi_3^{101}(-t_3b_4 - t_2b_3) - \varphi_2^{011}b_1\theta_5 + \varphi_3^{100}(b_3b_4 + b_3\theta_2 + b_3\theta_1 - b_2\theta_4)$$

In the end, after some work, one sees that the only remaining coboundary term is $(-\varphi_1^{101} + \varphi_2^{011})(\theta_3\theta_5\theta_7)$, so we get a third order deformation

$$d_1^3 = d_1^2 + \varphi_2^{100}\theta_3\theta_5\theta_7.$$
that the coefficients of the non-cocycle terms in the bracket of $d^2$ with itself, do vanish, using the third order relations, with the exception of some fourth order terms that cancel after adding the new fourth order terms arising in the bracket of $d^2$ with itself. Nevertheless, checking the coefficients is a useful method of avoiding misprints in the terms.

4.5. **Deformations of the Trivial Codifferential** $d_0 = 0$. There is one case which we have not touched on, the case of the trivial codifferential $d = 0$. At first it may seem as if there is little to obtain from looking at this situation, because it evidently must deform into every possible type of codifferential, so we know that it is infinitesimally close to every point in the moduli space. Moreover, since there are no coboundaries, the infinitesimal deformation is obviously miniversal.

On the other hand, we do obtain some relations, and these relations tell us something about how the moduli space is put together. In addition, all second order relations can be determined from the relations on the zero codifferential, by using appropriate values for the coefficients. Keep in mind that no information on higher order relations can be obtained in this manner.

We do not need a table of coboundaries for $d = 0$. Every cochain is a cocycle, so the universal infinitesimal deformation is given by

\[
d_0^i = \varphi_1^{10}t_1 + \varphi_2^{10}t_2 + \varphi_3^{10}t_3 + \varphi_1^{101}t_4 + \varphi_2^{101}t_5 + \varphi_3^{101}t_6 + \varphi_1^{111}t_7 + \varphi_2^{111}t_8 + \varphi_3^{111}t_9 + \varphi_1^{100}t_1 + \varphi_2^{100}t_2 + \varphi_3^{100}t_3 + \varphi_1^{010}t_4 + \varphi_2^{010}t_5 + \varphi_3^{010}t_6 + \varphi_1^{001}t_7 + \varphi_2^{001}t_8 + \varphi_3^{001}t_9 + \varphi_1^{111}t_{10} + \varphi_2^{111}t_{11} + \varphi_3^{111}t_{12}.
\]

We will not give a table of all the relations that are obtained here, but from such a table, one can construct the second order relations for any codifferential by simply substituting the $\theta$'s and $t$'s in the coefficient of a term in the bracket with the coefficients that occur in that particular infinitesimal deformation in the same places.

5. **Conclusion**

The main purpose of this paper was to illustrate how to compute versal deformations of Lie algebras. In fact, we worked in a slightly more general picture, that of $L_\infty$ algebras. The classification of Lie algebras of dimension three is well known, but we think that by studying the deformations more closely, the picture of the geometry of the moduli space of Lie algebra structures on a three dimensional vector space becomes much clearer.
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