Trade-off between Performance and Reversibility of Entanglement Concentration for Pure Entangled State

Wataru Kumagai1,2 Masahito Hayashi2,3
1Graduate School of Information Sciences, Tohoku University, Japan
2Graduate School of Mathematics, Nagoya University, Japan,
3Centre for Quantum Technologies, National University of Singapore, Singapore

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It is thought that the entanglement concentration for a bipartite pure state is asymptotically reversible because the distillable entanglement and the entanglement cost coincide with each other. In order to examine this argument, we give precise formulation about the reversibility of the entanglement concentration, and show a trade-off relation between the accuracy and the reversibility of the concentration, which implies the irreversibility of the entanglement concentration. Then, we regard the entanglement concentration as entangled state compression into entanglement storage with lower dimension. Because of the irreversibility of entanglement concentration, an initial state can not be completely recovered after the compression process and loss inevitably arise in the process. We numerically calculate the loss and also derive the asymptotic formula. Then we see that the approximation of the asymptotic formula is quite accurate.

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Introduction: Entanglement is an important resource in several quantum information processes. Hence, several types of conversion for entangled state have been studied in quantum information. One of the most typical conversions is the entanglement concentration, which approximately converts multiple copies of the given pure state $\psi$ to those of EPR state $\Phi$ by using local operations and classical communications (LOCC), which can be realized without use of entanglement resource. Another fundamental topic is the entanglement dilution, which converts multiple copies of EPR state to those of the given pure state $\psi$. The optimal rate of the entanglement concentration is the von Neumann entropy $S_\psi$ of the partial density matrix of $\psi$ [7, 8], and equals the optimal rate of the entanglement dilution. Therefore, the entanglement concentration for a bipartite pure state seems to be asymptotically reversible as is pointed out in some papers [1–3]. In order to examine the reversibility, this paper focuses on the following process of concentration and its recovery operation, as is illustrated in Fig. 1. Firstly, we apply a concentration operation by LOCC that converts $n$ copies of a bipartite pure state $\psi$ to a state close to $M_n$ copies of EPR state in a quantum storage system. Secondly, we apply a recovery operation by LOCC to recover the original $n$ copies of $\psi$ from the converted state in a quantum storage system.

We evaluate errors of a concentration operation $C$ and a recovery operation $D$ by using the square fidelities as $e_n^C(M, C|\psi) := 1 - F(C(\psi^\otimes n), \Phi^\otimes M)^2$ and $e_n^R(M, C, D|\psi) := 1 - F(\psi^\otimes n, D \circ C(\psi^\otimes n))^2$, respectively. Both errors can attain 0 independently. However, in order to examine the compatibility of both operations, this paper addresses their sum

$$\delta_n(M, C, D|\psi) := e_n^C(M, C|\psi) + e_n^R(M, C, D|\psi) \quad (1)$$

where the sum is obviously between 0 and 2. If the entanglement concentration is perfectly reversible in the asymptotic limit, the minimal concentration-recovery error (MCRE) $\delta_n(\psi) := \min_{D,C,M} \delta_n(M, C, D|\psi)$ converges 0. However, it is shown that $\delta_n(\psi)$ does not approach to zero by results of Hayden and Winter [15] and Harrow and Lo [14]. Furthermore, as one of main results, we obtain the following theorem.

Theorem 1. \(\lim_{n \to \infty} \delta_n(\psi) = 1\) holds for an arbitrary bipartite pure state $\psi$ except maximally entangled states.

Theorem 1 shows a trade-off between the concentra-
tion error and the recovery error, that is, the lower the
error of the one side is, the higher that of the other is
under the constraint that the sum of two errors is 1. In
particular, when the concentration error asymptotically
goes to 0, then the recovery error goes to 1. It means
that the state which is recovered by LOCC after the con-
centration is inevitably orthogonal to the initial state in
the asymptotic situation. This fact especially implies the
irreversibility of the entanglement concentration.

When we regard the concentration as entangled state
compression, an initial state itself can not be completely
recovered after the concentration because of the irre-
versibility. Then we analyze how many copies N within
an initial state can be recovered after the concentration.

Here, the error of a recovery operation $D$ is evaluated as
$$ e^R_n(M, C, D, N | \psi) := 1 - F(\psi^{\otimes N}, D \circ C(\psi^{\otimes n}))^2, $$
and the quantity in (1) is generalized to
$$ \delta_n(M, C, D, N | \psi) := e^C_n(M, C | \psi) + e^R_n(M, C, D, N | \psi). $$
Then, the MCRE can be generalized to $\delta_n(N | \psi) := \min_{D, C, M} \delta_n(M, C, D, N | \psi)$, and
$$ N_n(\epsilon | \psi) := \max \{ N | \delta_n(N | \psi) \leq \epsilon \} \tag{2} $$
represents the maximum number of recoverable copies
under a permissible error $\epsilon$ for MCRE $\delta_n(N | \psi)$. Then,
the following theorem holds.

**Theorem 2.** $N_n(\epsilon | \psi)$ has the asymptotic expansion un-
der a permissible error $0 < \epsilon < 1$ up to the smaller order
than $\sqrt{n}$ as follows.
$$ N_n(\epsilon | \psi) = n - \frac{2}{S_{\psi}} V_{\psi}^{-1} \left( 1 - \frac{\epsilon}{2} \right) \sqrt{n} + o(\sqrt{n}), \tag{3} $$
where $V_{\psi} := \text{Tr} \left\{ (\text{Tr} B \psi)(-\log(\text{Tr} B \psi) - S_{\psi})^2 \right\}$ and
$$ G(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx. $$

Theorem 2 tells us the performance of the optimal
compression operation by the entanglement con-
centration under a error constraint. In particular,
when an initial state has $n$-copies, the number of loss
of the copies is asymptotically approximated by
$\left( 2 V_{\psi} / S_{\psi} \right) G^{-1}(1 - \frac{\epsilon}{2}) \sqrt{n}$ in the optimal process.

Relation between Entanglement Concentration and Its
Recovery Operation: In order to treat the trade-off re-
lation between the concentration error $e^C_n(M, C | \psi)$
and the recovery error $e^R_n(M, C, D | \psi)$, we introduce the minimal
transition error concerning LOCC transformations
between states $\rho$ and $\sigma$ as follows.
$$ d(\rho \to \sigma) := \min_{E \in \text{LOCC}} (1 - F(E(\rho), \sigma))^2. $$
Then, as shown in Appendix, MCRE $\delta_n(\psi)$ can be
represented as follows.

$$ \delta_n(\psi) = \min_{M_n \in \mathbb{N}} d(\psi^{\otimes n} \to \Phi^{\otimes M_n}) + d(\Phi^{\otimes M_n} \to \psi^{\otimes n}). \tag{4} $$

**Proposition 3.** $\delta_n(\psi) = \min_{M_n \in \mathbb{N}} d(\psi^{\otimes n} \to \Phi^{\otimes M_n}) + d(\Phi^{\otimes M_n} \to \psi^{\otimes n})$. \tag{4}

Proposition 3 enables us to numerically calculate the
MCRE $\delta_n(\psi)$. The behavior of the MCRE $\delta_n(\psi)$ with
respect to $n$ is shown in Fig. 2 and we can find that $\delta_n(\psi)$
converges to 1 as $n$ goes to $\infty$ as was stated in Theorem 2.
To evaluate the MCRE, we only have to analyze two
terms in the right side of (4). In the following, we eval-
uate the asymptotic behaviors of the minimal transition
errors of both directions between $\psi^{\otimes n}$ and $\Phi^{\otimes M_n}$.

It is known that $d(\psi^{\otimes n} \to \Phi^{\otimes M_n})$ goes to 0 if the rate
$\lim_{M_n \to \infty} M_n / n$ is strictly less than $S_{\psi}$ by Bennett et al. \cite{7},
and does to 1 if the rate is strictly greater than $S_{\psi}$
by Hayashi et al. \cite{6}. Similarly, $d(\Phi^{\otimes M_n} \to \psi^{\otimes n})$ goes to 0 if the rate
$\lim_{M_n \to \infty} M_n / n$ is strictly greater than $S_{\psi}$ by Bennett et al. \cite{7}.
Therefore, the asymptotic behaviors of both minimal transition
errors have been analyzed unless the rate is not $S_{\psi}$. On the other hand, when the rate strictly equals $S_{\psi}$, Hayden and Winter \cite{10} and Harrow and Lo \cite{11} independently pointed out that the minimal transition
errors depend on the coefficient of the order $\sqrt{n}$. To
investigate those errors, we treat the case when $M_n$ can
be expanded as $M_n = an + b\sqrt{n} + o(\sqrt{n})$, and focus on the coefficients $a$ and $b$ called the first and the second
order rates in information theory, respectively. Then, we
obtain the following propositions similar to results of in-
trinsic randomness \cite{12} and resolvability \cite{13} in classical
information theory.

**Proposition 4.** The following equation holds for non-
maximally bipartite pure state $\psi$:
$$ \lim_{n \to \infty} d(\psi^{\otimes n} \to \Phi^{\otimes an+b\sqrt{n}}) = \begin{cases} 0 & \text{if } a < S_{\psi} \\ G \left( \frac{b}{\sqrt{S_{\psi}}} \right) & \text{if } a = S_{\psi} \\ 1 & \text{if } a > S_{\psi}. \end{cases} \tag{5} $$

![FIG. 2: When $\psi$ is a pure entangled state $\sqrt{0.1 \ket{00} + 0.9 \ket{11}}$ on two qubit systems, the MCRE $\delta_n(\psi)$ approaches to 1 as $n$ goes to $\infty$ as above. Note that the horizontal axis represents not $n$, but $\log_2 n$.](image-url)
Proposition 5. The following equation holds for non-maximally bipartite pure state \( \psi \):
\[
\lim_{n \to \infty} \left( \Phi^{\otimes n + b\sqrt{n}} \to \psi^{\otimes n + b\sqrt{n}} \right) = \begin{cases} 
1 & \text{if } a < S_{\psi} \\
1 - G \left( \frac{b - S_{\psi}}{\sqrt{V_{\psi}}} \right) & \text{if } a = S_{\psi} \\
0 & \text{if } a > S_{\psi}.
\end{cases}
\]  
(6)

We give the proofs of Propositions \( \text{(4)} \) and \( \text{(5)} \) in Appendix. Propositions \( \text{(4)} \) and \( \text{(5)} \) describe the asymptotic behavior of the minimal transition errors for entanglement concentration and dilution, respectively. As you can see from the proofs, even if the number of copies of \( \Phi \) has lower order term as \( M_n = an + b\sqrt{n} + o(\sqrt{n}) \) (e.g. \( o(\sqrt{n}) = \log n \)), the order does not affect the above errors, and thus, Propositions \( \text{(4)} \) and \( \text{(5)} \) hold as the same form. Hence, when we would like to analyze the minimal transition errors of the entanglement concentration and dilution in the asymptotic setting, we only have to treat the first and second order rates, and can ignore the lower order term.

We can easily show Theorem \( \text{(1)} \) by using Propositions \( \text{(4)} \) and \( \text{(5)} \) as follows. Suppose that the number \( M_n \in \mathbb{N} \) attains the minimum in \( \text{(4)} \) as
\[
\delta_n(\psi) = d(\psi^{\otimes n} \to \Phi^{\otimes M_n}) + d(\Phi^{\otimes M_n} \to \psi^{\otimes n})
\]  
(7)
and has the asymptotic expansion \( M_n = an + b\sqrt{n} + o(\sqrt{n}) \). When \( a \) is not \( S_{\psi} \), either the first or the second term in the right side of \( \text{(7)} \) goes to 1 and the other does to 0 as \( n \) goes to \( \infty \) by Propositions \( \text{(4)} \) and \( \text{(5)} \) and hence the MCRE \( \delta_n(\psi) \) goes to 1. When \( a \) is \( S_{\psi} \), the first and the second term in the right side of \( \text{(7)} \) goes to \( G(b/\sqrt{V_{\psi}}) \) and \( 1 - G(b/\sqrt{V_{\psi}}) \), respectively, by Propositions \( \text{(4)} \) and \( \text{(5)} \). Those values are the integrated values of the standard normal distribution on the left and the right interval from \( b/\sqrt{V_{\psi}} \) as is shown in Fig. \( \text{5} \) and those sum is always 1 for any second order rate \( b \). Thus, Theorem \( \text{(1)} \) was verified.

Due to Theorem \( \text{(1)} \) both the concentration and its recovery operation can not be accurately performed. That is, it turned out that there does not exist the process of the concentration and its recovery operation satisfying both
\[
\lim_{n \to \infty} e_{\psi}^{C}(M_n, C_n | \psi) = 0,
\]  
(8)
\[
\lim_{n \to \infty} e_{\psi}^{R}(M_n, C_n, D_n | \psi) = 0,
\]  
(9)
although there exist the concentration \( (M_n, C_n) \) satisfying \( \text{(8)} \) and the process \( (M'_n, C'_n, D'_n) \) of the concentration and its recovery operation satisfying \( \text{(9)} \) with the common first order rates \( \lim M_n/n = \lim M'_n/n = S_{\psi} \). The fact may look strange, however, can be comprehended by the argument of the second order rates. That is, those \( M_n \) and \( M'_n \) actually have different second order rates.

Loss Evaluation in Compression Process: When we regard a pair of the concentration and its recovery as compression process into entanglement storage, an initial state itself can not be completely recovered after the process because of the irreversibility of the entanglement concentration. Here, we evaluate the maximal number \( N_n(\epsilon | \psi) \) which was defined in \( \text{(2)} \), of recoverable copies within the initial state under a permissible error \( \epsilon \) after the compression process. In order to analyze \( N_n(\epsilon | \psi) \), we focus on \( \delta_n(N | \psi) \) which can be regarded as the inverse function of \( N_n(\epsilon | \psi) \). The following proposition is a generalization of Proposition \( \text{(3)} \).

Proposition 6. When \( N \leq n \), the following holds.
\[
\delta_n(N | \psi) = \min_{M_n \in \mathbb{N}} d(\psi^{\otimes n} \to \Phi^{\otimes M_n}) + d(\Phi^{\otimes M_n} \to \psi^{\otimes N}).
\]

By using Proposition \( \text{(6)} \) we can numerically calculate \( N_n(\epsilon | \psi) \) as is dotted in Fig. \( \text{4} \) and in particular, see the minimal loss \( n - N_n(\epsilon | \psi) \) after the compression process reaches 10% of the number \( n \) of copies of the initial state even when the permissible error \( \epsilon \) is 0.1. Since the rate of the loss is not so small and can not be ignored, we have to evaluate the loss after the compression process. Fig. \( \text{4} \) also contains the line of \( n - 2\sqrt{V_{\psi}}S_{\psi}G^{-1}(1 - \frac{\epsilon}{2}) \sqrt{n} \) and we can see that the line gives quite good approximation of \( N_n(\epsilon | \psi) \). In the following, we verify that the good approximation of \( N_n(\epsilon | \psi) \) holds for any \( 0 < \epsilon < 1 \), that is, we prove Theorem \( \text{(2)} \).

We expand \( N_n(\epsilon | \psi) \) as \( a_{\psi, \epsilon}n + b_{\psi, \epsilon}\sqrt{n} + o(\sqrt{n}) \). Then, in order to obtain Theorem \( \text{(2)} \) it is enough to determine the coefficients \( a_{\psi, \epsilon} \) and \( b_{\psi, \epsilon} \). In general, the limit of \( \delta_n(\psi, an + b\sqrt{n}) \) for constants \( a \) and \( b \in \mathbb{R} \) is equal to 1 for \( a > 1 \) and is equal to 0 for \( a < 1 \) by Propositions \( \text{(4)} \) and \( \text{(5)} \). Thus, when a permissible error \( \epsilon \) is between 0 and 1, the first order rate \( a_{\psi, \epsilon} \) is 1. Moreover, we obtain the following by the definition of \( N_n(\epsilon | \psi) \) and Propositions
Conclusion: In the paper, we have treated the process of the entanglement concentration and its recovery operation for a bipartite pure state. Then we introduced the MCRE to simultaneously evaluate two kinds of errors in the process and showed the trade-off relation between the errors in the asymptotic setting. In particular, when the concentration error goes to 0, the recovery error inevitably goes to the maximal value 1, that is, the concentration is not reversible even in the asymptotic situation. Next, we analyzed compression process which consist of the entanglement concentration and its recovery operation. We introduced the maximal number $N_n(\epsilon|\psi)$ of recoverable copies after compression process, and derived the asymptotic expansion of $N_n(\epsilon|\psi)$ in Theorem 2. We also derived Propositions 3 and 6 which enable us to numerically calculate exact values for the MCRE $\delta_n(\psi)$ and the maximal number $N_n(\epsilon|\psi)$ of recoverable copies, respectively. By calculating the exact values, we verified that Theorems 1 and 2 give quite good approximation.

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\[ \epsilon = \lim_{n} \delta_n(\psi, n + b_{\psi,\epsilon} \sqrt{n}) \]
\[ = \min_{b \in \mathbb{R}} \lim_{n} \left\{ d(\psi^{\otimes n} \rightarrow \Phi^{\otimes S_{\psi} n + b\sqrt{n}}) + d(\Phi^{\otimes S_{\psi} n + b\sqrt{n}} \rightarrow \psi^{\otimes n + b_{\psi,\epsilon} \sqrt{n}}) \right\} \]
\[ = \min_{b \in \mathbb{R}} \left( G \left( \frac{b}{\sqrt{S_{\psi}}} \right) + 1 - G \left( \frac{b - S_{\psi} b_{\psi,\epsilon}}{\sqrt{S_{\psi}}} \right) \right) \]
\[ = 2G \left( \frac{S_{\psi} b_{\psi,\epsilon}}{2\sqrt{S_{\psi}}} \right), \]  

(10)

where we omitted the lower order $o(\sqrt{n})$ of $N_n(\epsilon|\psi)$. Hence, $b_{\psi,\epsilon}$ is $-2\sqrt{S_{\psi} S_{\psi}^{-1} G^{-1} (1 - \epsilon/2)}$ and Theorem 2 was verified.

Therefore, the minimal loss $n - N_n(\epsilon|\psi)$ after the compression process is asymptotically evaluated as

\[ n - N_n(\epsilon|\psi) \approx 2\sqrt{S_{\psi} S_{\psi}^{-1} G^{-1} (1 - \epsilon/2)} \sqrt{n}. \]

(11)

The coefficient of $\sqrt{n}$ in (11) rapidly increase as $\epsilon$ gets smaller as is shown in Fig. 5 and in particular, diverges to $\infty$ when $\epsilon$ is 0. Therefore, unlike the case $0 < \epsilon < 1$, the loss $n - N_n(0|\psi)$ increases faster than the order $\sqrt{n}$.

Hence, the minimal loss after the compression process may be ignored if $\epsilon$ is quite small. However, the order of the minimal loss is $\sqrt{n}$, and the ratio of the loss in comparison to the number $n$ of copies of an initial state gets smaller as $n$ gets larger. Therefore, if an error $\epsilon$ is allowed in the compression operation, the entanglement concentration works as the compression operation for an entangled pure state with the slight loss which is described in (11) for large enough $n$.

FIG. 4: When $\psi = \sqrt{1/2}(00) + \sqrt{1/3}(11)$ and $n = 3000$, values of $N_n(\epsilon|\psi)$ are dotted at some $\epsilon$. The curved line is the graph of $n - 2\sqrt{V_{\psi} S_{\psi}^{-1} G^{-1} (1 - \epsilon/2)} \sqrt{n}$ with respect to $\epsilon$. The asymptotic expansion in Theorem 2 provides a good approximation for $N_n(\epsilon|\psi)$ if $n$ is large enough.

FIG. 5: The behavior of the coefficient of $\sqrt{n}$ in (11) with respect to $\epsilon$ when $2\sqrt{V_{\psi} S_{\psi}^{-1} G^{-1} (1 - \epsilon/2)}$. The value dramatically increase as $\epsilon$ gets smaller. In particular, the value approaches to $\infty$ and 0 as a permissible error $\epsilon$ goes to 0 and 1, respectively.

[1] D. Jonathan, M. B. Plenio, Phys. Rev. Lett. 83, 1455 (1999).
[2] C. H. Bennett et al., Phys. Rev. A 63, 012307 (2001).
[3] G. Vidal, J. I. Cirac, Phys. Rev. Lett. 86, 5803 (2001).
[4] D. Yang et al., Phys. Rev. Lett. 95, 190501 (2005).
[5] A. Acin, G. Vidal, and J.I. Cirac, Quant. Inf. Comp. 3, 55 (2003).
[6] M. Hayashi et al., J. Phys. A: Math. Gen. 36, 527 (2003).
[7] C. H. Bennett et al., Phys. Rev. A, 53, 2046, (1996).
[8] P. Hayden et al., J. Phys. A 34, 6891 (2001).
Appendix A: Proofs of Propositions

Proofs of Propositions 3 and 6: Because Proposition 3 coincide with Proposition 6 when \( N = n \), it is enough to show Proposition 6. For an arbitrary pure state \( \psi \in \mathcal{H}_{AB} \), we denote the Schmidt coefficients of \( \psi \) by \( p_\psi = (p_\psi,1, \cdots, p_\psi,M) \). Let \( \Phi_L = \sum_{i=1}^{L} \sqrt{1/L} |i\rangle |i\rangle \) be a maximally entangled state with the size \( L \) on \( \mathcal{H}_L := \mathbb{C}^L \otimes \mathbb{C}^L \). When \( p^* \) shows the probability distribution which is sorted in decreasing order for the components of \( p \), we define the pure state \( \eta_{\psi,L} \) in \( \mathcal{H}_L \) as

\[
\eta_{\psi,L} = \sum_{i=1}^{J_{\psi,L} - 1} \sqrt{p^*_{\psi,i}} |i\rangle |i\rangle + \sqrt{\sum_{j=J_{\psi,L}}^{L} p^*_{\psi,j}} \sum_{i=J_{\psi,L}}^{L} |i\rangle |i\rangle \]

by using

\[
J_{\psi,L} := \max\{1\} \cup \left\{ 2 \leq j \leq L \mid \sum_{i=1}^{j-1} p^*_{\psi,i} \leq \frac{1}{L+1-j} < \sum_{i=1}^{j} p^*_{\psi,i} \right\}.
\]

Then, there exists a suitable LOCC map to transform \( \psi \) to \( \eta_{\psi,L} \) \([10]\), and we can obtain the following equation:

\[
\max_{C: \text{LOCC}} F(C(\psi), \Phi_L) = F(\eta_{\psi,L}, \Phi_L). \tag{A1}
\]

Similarly, when we define the pure state \( \zeta_{\psi,L} \) in \( \mathcal{H}_{AB} \) as

\[
\zeta_{\psi,L} = \sqrt{\sum_{i=1}^{L} p^*_{\psi,i}} \sum_{i=1}^{L} |p^*_{\psi,i} \rangle |i\rangle,
\]

there exists a suitable LOCC map to transform \( \Phi_L \) to \( \zeta_{\psi,L} \), and the following equation holds as shown in \([11]\):

\[
\max_{D: \text{LOCC}} F(\psi, D(\Phi_L)) = F(\psi, \zeta_{\psi,L}) = \sum_{i=1}^{L} p^*_{\psi,i}. \tag{A2}
\]

Moreover, the following equation holds for \( N \leq n \)

\[
\max_{C, D: \text{LOCC}} F(\psi \otimes ^{\otimes n}, D \circ C(\psi \otimes ^{\otimes n})) = \max_{D: \text{LOCC}} F(\psi \otimes ^{\otimes n}, D(\Phi \otimes ^{\otimes n})). \tag{A3}
\]

where \( C : S(\mathcal{H}_{AB}^{\otimes n}) \to S(\mathcal{H}_{AB}^{M_n}) \) and \( D : S(\mathcal{H}_A^{\otimes n}) \to S(\mathcal{H}_B^{\otimes n}) \) run over LOCC maps. The equation (A3) is obtained as follows. At first, the right side is greater than the left side in (A3) since \( C(\psi \otimes ^{\otimes n}) \) is transformed from \( \Phi \otimes ^{\otimes n} \) by a suitable LOCC. On the other hand, the right side in (A3) equals \( F(\psi \otimes ^{\otimes n}, \zeta \otimes ^{\otimes 2M_n}) \) by (A2), and \( \zeta \otimes ^{\otimes 2M_n} \) can be transformed from \( \psi \otimes ^{\otimes n} \) via \( \eta \otimes ^{\otimes 2M_n} \) if \( N \leq n \).

Due to (A3), the following inequality holds.

\[
\delta_n(\psi) \geq \min_{\Delta_n \in \mathbb{N}} d(\psi \otimes ^{\otimes n} \to \Phi \otimes ^{\otimes n} \} + d(\Phi \otimes ^{\otimes n} \to \Phi \otimes ^{\otimes n}). \tag{A4}
\]

Next, we prove the converse inequality. Let us fix an arbitrary \( M_n \in \mathbb{N} \). Since there exists a suitable LOCC map from \( \eta \otimes ^{\otimes 2M_n} \) to \( \zeta \otimes ^{\otimes 2M_n} \), we obtain

\[
\delta_n(\psi) \leq d(\eta \otimes ^{\otimes 2M_n}, \Phi) + d(\Phi, \zeta) \leq d(\psi \otimes ^{\otimes n}, \Phi) + d(\Phi \otimes ^{\otimes n}, \psi). \tag{A4}
\]

We used (A1) and (A2) to show the equality (A4). ■

Proofs of Propositions 4 and 5: Since the cases \( a \neq S_\psi \) in both propositions can be shown by the theorems of the direct part \([7]\) and the strong converse part \([6, 10]\) of entanglement concentration and dilution, we show the cases \( a = S_\psi \) in both propositions. In order to show them, we employ a function, which is similar to that used in \([12]\). For this purpose, we introduce a notation for projections. For a Hermitian matrix \( A \) and a real number \( c \), we define the projection \( \{ A \leq c \} \) as \( \sum_{a \leq c} P_a \), where the spectral decomposition of \( A \) is given as \( A = \sum a P_a \).

Then, we introduce the following function for a sequence \( \{ \rho_n \}_{n=1}^\infty \) of general quantum states.

\[
K(b, b' \mid \psi) := \lim_{n \to \infty} \text{Tr} \rho_n \{ -\log \rho_n \leq S_\psi + b \sqrt{n} \} \tag{A5}
\]

where the base of the logarithm is 2. A similar function was introduced in the context of the first order asymptotics for the analysis of the asymptotic performance of the entanglement concentration \([12]\). Then, substituting the sequence \( \{ \rho_n \}_{n=1}^\infty \) into \( \{ a \}_{n=1}^\infty \), we define

\[
K(b, b' \mid \psi) := \lim_{n \to \infty} \text{Tr} \rho_n \{ -\log \rho_n \leq S_\psi + b \sqrt{n} \} \tag{A6}
\]

In the following, \( K(b, b' \mid \psi) \) is simplified to \( K(b' \mid \psi) \).

As is pointed out by Hayden and Winter [15] and Harrow and Lo [14] in the case of \( K(b \mid \psi) \), the central limit theorem guarantees that

\[
K(b, b' \mid \psi) = G \left( \frac{b - S_\psi b'}{\sqrt{V_\psi}} \right). \tag{A7}
\]

Using the relation (A7) in the case of \( K(b \mid \psi) \), Hayden and Winter [15] and Harrow and Lo [14] roughly estimated the accuracy of entanglement concentration and dilution based on the fidelity. In order to show Propositions 4
and 5, we need the tight relation between $K(b|\psi)$ and the limit of the minimal transition errors. For this purpose, we show the following four inequalities for an arbitrary $\gamma > 0$

$$1 - K(b + \gamma|\psi) \leq \lim_{n \to \infty} d(\Phi^{S_\psi n + b\sqrt{n}} \to \psi \otimes n), \quad (A8)$$

$$\lim_{n \to \infty} d(\Phi^{S_\psi n + b\sqrt{n}} \to \psi \otimes n) \leq 1 - K(b|\psi), \quad (A9)$$

$$K(b - \gamma|\psi) \leq \lim_{n \to \infty} d(\psi \otimes n \to \Phi^{S_\psi n + b\sqrt{n}}), \quad (A10)$$

$$\lim_{n \to \infty} d(\psi \otimes n \to \Phi^{S_\psi n + b\sqrt{n}}) \leq K(b + \gamma|\psi). \quad (A11)$$

Since Proposition 4 follows from (A10), (A11) and (A7), and Proposition 5 follows from (A8), (A9) and (A7) with changing the variable as $m = S_\psi n + b\sqrt{n}$ and $b = -S_\psi^2 b'$, it is enough to show the inequalities (A8)-(A11). At first, we prove (A8). By (A2), for an arbitrary state $\psi \otimes n$, arbitrary positive integers $M_n, M'_n$, and an arbitrary LOCC $D_n$, the inequality

$$F(\psi \otimes n, D_n(\Phi^{S_\psi n + b\sqrt{n}}))^2 \leq \text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq \frac{1}{2M_n} \} + \frac{2M_n}{2M'_n}$$

holds. When $M_n = S_\psi n + b\sqrt{n}$, $M'_n = S_\psi n + (b + \gamma)\sqrt{n}$ in the inequality, we obtain (A8) by taking $\lim_{n \to \infty}$. Next, we prove (A9). By (A2), for an arbitrary state $\psi \otimes n$ and an arbitrary positive integer $M_n$, there is a LOCC $D_n$ which satisfies the inequality

$$\text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq \frac{1}{2M_n} \} \leq F(\psi \otimes n, D_n(\Phi^{S_\psi n + b\sqrt{n}}))^2. \quad (A12)$$

When $M_n = S_\psi n + b\sqrt{n}$ in the inequality, we obtain (A9) by taking the limit $n \to \infty$. Next, we prove (A10). By Lemmas 4 and 5 in [12], for an arbitrary state $\psi \otimes n$, arbitrary positive integers $M_n \geq M'_n$, and an arbitrary LOCC $C_n$, the inequality

$$F(C_n(\psi \otimes n), \Phi^{S_\psi n + b\sqrt{n}})^2 \leq \frac{1}{2M_n} \left( \sqrt{\text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq 1/2M'_n \}} \right) \sqrt{\text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq 1/2M_n \}}$$

$$+ \sqrt{2M_n - \text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq 1/2M_n \}} \sqrt{1 - \text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq 1/2M_n \}}$$

holds. Substituting $S_\psi n + b\sqrt{n}$ and $S_\psi n + (b - \gamma)\sqrt{n}$ into $M_n$ and $M'_n$ in the above inequality and taking the limit $n \to \infty$, we obtain (A10).

Finally, we prove (A11). It is enough to prove

$$\lim_{n \to \infty} d(\psi \otimes n \to \Phi^{S_\psi n + b\sqrt{n}}) \leq K(b + \gamma|\psi) \quad (A13)$$

for an arbitrary positive real number $\gamma$. When $K(b + \gamma|\psi) = 1$, the inequality is obvious. Thus, we assume $K(b + \gamma|\psi) < 1$. For an arbitrary positive integer $M_n$, we define the real number $x_{M_n}$ as a real number satisfying that $\frac{1}{2M_n} \left( 1 - h_n(x_{M_n}) \right) = 2M_n$, where $h_n(x) := \text{Tr}(\rho^{\otimes n}_\psi \to x) \{ \rho^{\otimes n}_\psi \to x \geq 0 \}$. By Lemma 9 and (1) in [12], for an arbitrary positive integer $M_n$, there is an LOCC $C_n$ which satisfies the inequality

$$1 - \text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq x_{2M_n} \} \leq F(C_n(\psi \otimes n), \Phi^{S_\psi n + b\sqrt{n}})^2 \leq 1 - d(\psi \otimes n \to \Phi^{S_\psi n + b\sqrt{n}}).$$

When we choose $M_n$ as $M_n := S_\psi n + (b + \gamma)\sqrt{n} \log(1 - h_n(2^{-S_\psi n - (b + \gamma)\sqrt{n}}))$, we can take $x_{M_n} = 2^{-S_\psi n - (b + \gamma)\sqrt{n}}$. Since

$$\lim_{n \to \infty} h_n(2^{-S_\psi n - (b + \gamma)\sqrt{n}}) \leq K(b + \gamma|\psi) \leq 1,$$

the inequality $S_\psi n + b\sqrt{n} < M_n$ holds for enough large integer $n$. Therefore,

$$\text{Tr}\rho^{\otimes n}_\psi \{ \rho^{\otimes n}_\psi \geq 2^{-S_\psi n - (b + \gamma)\sqrt{n}} \} \geq d(\psi \otimes n \to \Phi^{S_\psi n + b}\sqrt{n}) \geq d(\psi \otimes n \to \Phi^{S_\psi n \to x_{2M_n}})$$

By taking the limit $n \to \infty$ in the inequality (A13), we obtain (A11).