Compactly supported wavelets and representations of the Cuntz relations, II

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ABSTRACT

We show that compactly supported wavelets in \( L^2(\mathbb{R}) \) of scale \( N \) may be effectively parameterized with a finite set of spin vectors in \( \mathbb{C}^N \), and conversely that every set of spin vectors corresponds to a wavelet. The characterization is given in terms of irreducible representations of orthogonality relations defined from multiresolution wavelet filters.

Key words and phrases: wavelet, Cuntz algebra, representation, orthogonal expansion, quadrature mirror filter, isometry in Hilbert space

1. INTRODUCTION

Let \( L^2(\mathbb{R}) \) be the Hilbert space of all \( L^2 \)-functions on \( \mathbb{R} \). Let \( \psi \in L^2(\mathbb{R}) \), and set

\[
\psi_{n,k}(x) := 2^n \psi(2^n x - k) \quad \text{for } x \in \mathbb{R}, \text{ and } n, k \in \mathbb{Z}.
\]

(1)

We say that \( \psi \) is a wavelet (in the strict sense) if

\[
\{ \psi_{n,k} : n, k \in \mathbb{Z} \}
\]

(2)

constitutes an orthonormal basis in \( L^2(\mathbb{R}) \); and we say that \( \psi \) is a wavelet in the frame sense (tight frame) if

\[
\|f\|_{L^2(\mathbb{R})}^2 = \sum_{n,k \in \mathbb{Z}} |\langle \psi_{n,k} | f \rangle|^2
\]

(3)

holds for all \( f \in L^2(\mathbb{R}) \), where \( \langle \cdot | \cdot \rangle \) is the usual \( L^2(\mathbb{R}) \)-inner product, i.e.,

\[
\langle \psi_{n,k} | f \rangle = \int_{\mathbb{R}} \overline{\psi_{n,k}(x)} f(x) \, dx.
\]

(4)

It is known that a given wavelet \( \psi \) in the sense of frames is a (strict) wavelet if and only if \( \|\psi\|_{L^2(\mathbb{R})} = 1 \). Either way, the numbers in (4) are the wavelet coefficients.

We shall have occasion to consider scaling on \( \mathbb{R} \) other than the dyadic one, say \( x \mapsto Nx \) where \( N \in \mathbb{N}, N > 2 \). Then the analogue of (1) is

\[
\psi_{n,k}(x) := N^n \psi(N^n x - k), \quad x \in \mathbb{R}, \text{ and } n, k \in \mathbb{Z}.
\]

(5)

However, in that case, it is generically not enough to consider only one \( \psi \) in \( L^2(\mathbb{R}) \): If the wavelet is constructed from an \( N \)-subband wavelet filter as in Ref. [3], then we will be able to construct \( \psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(N-1)} \) in \( L^2(\mathbb{R}) \) such that the functions in (5) have the basis property, either in the strict sense, or in the sense of frames. In that case, the system

\[
\left\{ \psi_{n,k}^{(i)} : 1 \leq i < N, n, k \in \mathbb{Z} \right\}
\]

(6)

will constitute an orthonormal basis of \( L^2(\mathbb{R}) \), or, alternatively, a tight frame, as in (3) but with the \( \psi_{n,k}^{(i)} \) functions in place of \( \psi_{n,k} \).

In principle, there are many ways (see below) of constructing wavelets (5), but we will show in this paper that the method of quadrature mirror wavelet filters (QMF) has some features that set it apart from the alternative constructions. Several of the constructions are based on frequency subbands, and the subbands correspond to a

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family of closed subspaces in the Hilbert space $L^2(\mathbb{R})$, but we will show that this subspace structure is “optimal” for the QMF wavelets, in the sense that the subspaces cannot in a nontrivial way be refined into additional subbands. We will formulate this result in a mathematically precise fashion, which is based on a representation of the operators which define the QMF’s. In fact, we give a formula for all these QMF’s in the case of compactly supported wavelets.

The present paper is concerned with the wavelet filters which enter into the construction of the functions $\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(N-1)}$. These filters (see (5) and (6) below) are really just a finite set of numbers which relate the $\mathbb{Z}$-translates of these functions to the corresponding scalings by $x \mapsto N x$. Hence the analysis of the wavelets may be discretized via the filters, but the question arises whether or not the data which go into the wavelet filters are minimal. It turns out that representation theory is ideally suited to make the minimality question mathematically precise. (This is a QMF-multiresolution construction, and it is the minimality and efficiency of this construction which concern us here. While it is true, see, e.g., Refs. 6, 7, 8, that there are other and different possible wavelet constructions, it is not yet clear how our present techniques might adapt to the alternative constructions, although the approach in Ref. 6 is also based on operator-theoretic considerations.)

## 2. THE SCALING FUNCTION

Since the wavelets come from the multiresolution functions, it is of interest to give explicit constructions for these. We do this in the present paper, which introduces two new tools for explicit constructions of multiresolution wavelet filters, (i) a certain infinite-dimensional loop group, and (ii) a certain family of irreducible representations of orthogonality relations (the Cuntz relations). Our viewpoint makes it clear, in particular, that compactly supported wavelets may be specified effectively with a finite set of spin vectors in the $\mathbb{C}^N$ (the Cuntz relations). Hence these wavelets are given by a finite set (in arbitrary configuration) of $k$ $Q$-bits where it turns out that $k$ is half of the length of the support of the wavelet in question.

To better explain the minimality issue for multiresolution quadrature mirror (QMF) wavelet filters, we recall the scaling function $\varphi$ of a resolution in $L^2(\mathbb{R})$.

Let $g \in \mathbb{N}$, and let $a_0, a_1, \ldots, a_{2g-1}$ be given complex numbers such that

$$
\sum_{k=0}^{2g-1} a_k = 2,
$$

and

$$
\sum_{k} a_{k+2l} \bar{a}_k = \begin{cases} 
2 & \text{if } l = 0, \\
0 & \text{if } l \neq 0.
\end{cases}
$$

In the formulation of (8), and elsewhere, we adopt the convention that terms in a summation are defined to be zero when the index is not in the specified range: Hence, in (11), it is understood that $a_{k+2l} = 0$ whenever $k$ and $l$ are such that $k + 2l$ is not in $\{0, 1, \ldots, 2g - 1\}$. When $\{a_k : k = 0, 1, \ldots, 2g - 1\}$ is given subject to (7)–(8), then it is known, [9], [10] that there is an $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$, unique up to a constant multiple, such that

$$
\varphi(x) = \sum_{k=0}^{2g-1} a_k \varphi(2x - k), \quad x \in \mathbb{R},
$$

and $\varphi$ is of compact support; in fact, then

$$
\text{supp}(\varphi) \subset [0, 2g - 1].
$$

(If $H$ denotes the Hilbert transform of $L^2(\mathbb{R})$, and $\varphi$ solves (5), then $H\varphi$ does as well; but $H\varphi$ will not be of compact support if (5) holds.) In finding $\varphi$ in (5), there are methods based on iteration, on random matrix products, and on the Fourier transform, see Refs. [11, 12, 13, 14] and the various methods intertwine in the analysis of $\varphi$, i.e., in deciding when $\varphi(x)$ is continuous, or not, or if it is differentiable.

Let $\varphi$ be as in (5), and let $\mathcal{V}_0$ be the closed subspace in $\mathcal{H} := L^2(\mathbb{R})$ spanned by $\{\varphi(x - k) : k \in \mathbb{Z}\}$, i.e., by the integral translates of the scaling function $\varphi$. Let $U := U_N$ be

$$
Uf(x) := N^{-\frac{1}{2}} f\left(\frac{x}{N}\right), \quad f \in L^2(\mathbb{R}),
$$

where $\psi^{(l)}(x) = \sum_{k \in \mathbb{Z}} c_k \psi^{(l)}(2x - k)$, and $c_k$ is a complex number such that

$$
\sum_{k} c_k = 0.
$$

When $\mathcal{V}_0 \subset \mathcal{H}$ is a closed subspace, let $\mathcal{H} / \mathcal{V}_0$ denote the quotient space of $\mathcal{H}$, and let $\Pi : \mathcal{H} \to \mathcal{H} / \mathcal{V}_0$ be the quotient map. The $\Pi$-closeness of the family of scalings $\varphi(x - k)$ is shown in [13].
Table 1. Discrete vs. continuous wavelets, i.e., $\ell^2$ vs. $L^2(\mathbb{R})$

```
{0} ←⋅⋅⋅← $\mathcal{W}_2$ $\mathcal{W}_1$ $\mathcal{V}_0$   finer scales
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{0} ←⋅⋅⋅← $\mathcal{S}_0$ $\mathcal{S}_0$ $\mathcal{S}_0$ $\mathcal{S}_0$
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{0} ←⋅⋅⋅← $\mathcal{S}_0^2\mathcal{L}$ $\mathcal{S}_0\mathcal{L}$ $\mathcal{L} = S_1\ell^2$
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the unitary scaling operator in $\mathcal{H} = L^2(\mathbb{R})$. Then

$$U\mathcal{V}_0 \subset \mathcal{V}_0,$$

it is a proper subspace, and

$$\bigwedge_n U^n\mathcal{V}_0 = \{0\};$$

see Ref. 3 and Ch. 5 of Ref. 1. For $N = 2$, the situation is as in Table 1. Setting

$$\mathcal{V}_n := U^n\mathcal{V}_0,$$

and

$$\mathcal{W}_n := \mathcal{V}_{n-1} \ominus \mathcal{V}_n,$$

we arrive at the resolution

$$\mathcal{V}_0 = \sum_{n \geq 1}^{\oplus} \mathcal{W}_n,$$

and the wavelet function $\psi$ is picked in $\mathcal{W}_0$. We will set up an isomorphism between the resolution subspace $\mathcal{V}_0$ and $\ell^2(\mathbb{Z})$, and associate operators in $\ell^2(\mathbb{Z})$ with the wavelet operations in $\mathcal{V}_0 \subset L^2(\mathbb{R})$. This is of practical significance given that the operators in $\ell^2(\mathbb{Z})$ are those which are defined directly from the wavelet filters, and it is the digital filter operations which lend themselves to algorithms.

For the general case of scale $N (>2)$ the space $\mathcal{V}_0 \ominus \mathcal{V}_N\mathcal{V}_0$ splits up as a sum of orthogonal spaces $\mathcal{W}_1^{(i)}$, $i = 1, 2, \ldots, N - 1$.

3. REPRESENTATIONS OF $O_N$ AND TABLE 1
(DISCRETE VS. CONTINUOUS WAVELETS)

The practical significance of the operator system in Table 1 (scale $N = 2$) is that the operators which generate wavelets in $L^2(\mathbb{R})$ become modeled by an associated system of operators in the sequence space $\ell^2 := \ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})$. (We will do the discussion here in Section 3 just for $N = 2$, but this is merely for simplicity. It easily generalizes to arbitrary $N$.) Then the algorithms are implemented in $\ell^2$ by basic discrete operations, and only in the end are the results then “translated” back to the space $L^2(\mathbb{R})$. The space $L^2(\mathbb{R})$ is not amenable (in its own right) to discrete
computations. This is made precise by virtue of the frame operator $W: \ell^2(\mathbb{R}) \to V_0(\mathbb{R})$ which may be defined as

$$W: \ell^2 \ni (\xi_k) \mapsto \sum_{k \in \mathbb{Z}} \xi_k \varphi(x-k) \in L^2(\mathbb{R}).$$

(17)

If the scaling function $\varphi$ has then been constructed to have orthogonal translates, then $W$ will be an isometry of $\ell^2$ onto $V_0(\mathbb{R})$. Even if the functions $\{\varphi(x-k)\}_{k \in \mathbb{Z}}$ formed from $\varphi$ by $\mathbb{Z}$-translates only constitute a frame in $V_0$, then we will have the following estimates:

$$c_1 \sum_{k \in \mathbb{Z}} |\xi_k|^2 \leq \int_{-\infty}^{\infty} |(W \xi)(x)|^2 \, dx \leq c_2 \sum_{k \in \mathbb{Z}} |\xi_k|^2,$$

(18)

where $c_1$ and $c_2$ are positive constants depending only on the scaling function $\varphi$, or equivalently,

$$c_1^{1/2} \cdot \|\xi\|_{\ell^2} \leq \|W \xi\|_{L^2(\mathbb{R})} \leq c_2^{1/2} \cdot \|\xi\|_{\ell^2}.$$

(19)

**Lemma 3.1.** If the coefficients $\{a_k; k = 0, 1, \ldots, 2^g - 1\}$ from (8) satisfy the conditions in (8), then the corresponding operator $S_0: \ell^2 \to \ell^2$, given by

$$(S_0 \xi)_k = \frac{1}{\sqrt{2}} \sum_{l \in \mathbb{Z}} a_{k-2^l} \xi_l = \frac{1}{\sqrt{2}} \sum_{p \in \mathbb{Z}; p \equiv k \mod 2} a_p \xi_{k-p}, \quad k \in \mathbb{Z},$$

(20)

is isometric and satisfies the following intertwining identity:

$$WS_0 = UW,$$

(21)

where $U$ is the dyadic scaling operator in $L^2(\mathbb{R})$ which we introduced in (5). (Here we restrict attention to $N = 2$, but just for notational simplicity!) Setting

$$b_k := (-1)^k a_{2^g - 1 - k},$$

(22)

and defining a second isometric operator $S_1: \ell^2 \to \ell^2$ by formula (20) with the only modification that $(b_k)$ is used in place of $(a_k)$, we get

$$S_j^* S_k = \delta_{j,k} \mathbb{I}_{\ell^2}$$

(23)

and

$$\sum_j S_j S_j^* = \mathbb{I}_{\ell^2},$$

(24)

which are the Cuntz identities from operator theory and the operators $S_0$ and $S_1$ satisfy the identities indicated in Table 1.

**Remark 3.2.** For understanding the second line in Table 1, note that $S_0$ is a shift as an isometry, in the sense of Ref. [4], and $\mathcal{L} := S_1 \ell^2$ is a wandering subspace for $S_0$, in the sense that the spaces $\mathcal{L}$, $S_0 \mathcal{L}$, $S_0^2 \mathcal{L}$, ... are mutually orthogonal in $\ell^2$. To see this, note that (24) implies that

$$(\mathcal{L} :=) \quad S_1 \ell^2 = \ell^2 \cap S_0 \ell^2 = \ker(S_0^*).$$

(25)

As a result, we get the following:

**Corollary 3.3.** The projections onto the orthogonal subspaces in the second line of Table 1 which correspond to the $W_1, W_2, \ldots$ subspaces of the first line (see (4)) are as follows:

$$\text{proj } \mathcal{L} = S_1 S_1^* = I - S_0 S_0^*,$$

$$\vdots$$

$$\text{proj } S_0^{n-1} \mathcal{L} = S_0^{n-1} S_0^n S_0^n - S_0^n S_0^{n-1}.$$
Proof. Immediate from Lemma 3.1 and Remark 3.2.

Remark 3.4. Any system of operators \( \{ S_j \} \) satisfying (23)–(24) is said to be a representation of the \( \mathcal{C}^* \)-algebra \( \mathcal{O}_2 \), and there is a similar notion for \( \mathcal{O}_N \) when \( N > 2 \), with \( \mathcal{O}_N \) having generators \( S_0, S_1, \ldots, S_{N-1} \), but otherwise also satisfying the operator identities (23)–(24).

Definition 3.5. A representation of \( \mathcal{O}_N \) on the Hilbert space \( \ell^2 \) is said to be irreducible if there are no closed subspaces \( \{ 0 \} \subsetneq \mathcal{H}_0 \subsetneq \ell^2 \) which reduce the representation, i.e., which yield a representation of (23)–(24) on each of the two subspaces in the decomposition
\[
\ell^2 = \mathcal{H}_0 \oplus (\ell^2 \ominus \mathcal{H}_0),
\]
where
\[
\ell^2 \ominus \mathcal{H}_0 = (\mathcal{H}_0)^\perp = \{ \xi \in \ell^2 : \langle \xi | \eta \rangle = 0, \forall \eta \in \mathcal{H}_0 \}.
\]

Proof of Lemma 3.1. Most of the details of the proof are contained in Refs. 17 and 3, so we only sketch points not already covered there. The essential step (for the present applications) is the formula (21), which shows that \( W \) intertwines the isometry \( S_0 \) with the restriction of the unitary operator \( U : f \mapsto \frac{1}{\sqrt{N}} f(x/2) \) to the resolution subspace \( \mathcal{V}_0 \subset L^2(\mathbb{R}) \). We have:
\[
(UW\xi)(x) = \frac{1}{\sqrt{2}}(W\xi)\left(\frac{x}{2}\right)
\]
\[
= \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \xi_k \varphi\left(\frac{x}{2} - k\right)
\]
\[
= \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \xi_k a_l \varphi(x - 2k - l)
\]
\[
= \frac{1}{\sqrt{2}} \sum_{p \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \xi_k a_{p-2k} \right) \varphi(x - p)
\]
\[
= \sum_{p \in \mathbb{Z}} (S_0 \xi)_p \varphi(x - p)
\]
\[
= (WS_0\xi)(x)
\]
for all \( \xi \in \ell^2 \), and all \( x \in \mathbb{R} \). This proves (21).

The rest of the proof will be given in a form slightly more general than needed. For later use, we record the following table of operators on the respective Hilbert spaces \( L^2(\mathbb{T}) \cong \ell^2 \) and \( L^2(\mathbb{R}) \), and the corresponding transformation rules with respect to the operator \( W \). Let \( N \) be the scale number, and let \( (a_k)_{k=0}^{N_g-1} \) be given satisfying
\[
\sum_{k \in \mathbb{Z}} a_k + Ni\bar{a}_k = \delta_{0,i}N
\]
and set \( m_0(z) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N_g-1} a_k z^k \), \( z \in \mathbb{T} \). Then we have the following transformation rules.

| SCALING | TRANSFORMATION |
|-----------------|-----------------|
| \( L^2(\mathbb{R}) \) : | \( F \mapsto \frac{1}{\sqrt{N}} F\left(\frac{x}{N}\right) \) |
| \( \uparrow W \) | \( F(x) \mapsto F(x - 1) \) |
| \( \ell^2 \) : | \( \xi \mapsto \sum_{l} a_{k-N_l} \xi_l \) |
| \( \uparrow \text{Fourier transform} \) | \( (\xi_k) \mapsto (\xi_{k-1}) \) |
| \( L^2(\mathbb{T}) \) : | \( f \mapsto m_0(z) f(z^N) \) |
| \( f(z) \mapsto zf(z) \) |

\( T = \mathbb{R}/2\pi\mathbb{Z} \)
Now the proof may be completed by use of the following sublemma, which we state just for $N = 2$.

**Sublemma 3.6.** Let $\mathfrak{O}$ be a subspace of $L^2(\mathbb{T})$ which is invariant under multiplication by $z^2$. Then there is an $m_1 \in L^\infty(\mathbb{T})$ such that $\mathfrak{O} = \{m_1(z)f(z^2) : f \in L^2(\mathbb{T})\}$ and $\frac{1}{2} \sum_{w^2 = z} |m_1(w)|^2 = 1$, a.a. $z \in \mathbb{T}$.

**Proof.** The proof follows from the Beurling-Lax-Halmos theorem.

**Completion of proof of Lemma 3.1.** Let $m_0(z) := \frac{1}{\sqrt{2}} \sum_k a_k z^k$. Then we saw that $S_0f(z) := m_0(z)f(z^2)$ is isometric in $L^2(\mathbb{T})$, and the complementary space

$$\mathfrak{O} := L^2(\mathbb{T}) \ominus S_0L^2(\mathbb{T}) = \ker(S_0) = \left\{ f \in L^2(\mathbb{T}) : \sum_{w^2 = z} m_0(w)f(w) = 0 \right\}$$

then satisfies the condition in Sublemma 3.4. Let $m_1$ be the function determined from the sublemma. Then, after multiplication by a suitable $z^{2p}$, we will get $m_1(z) = \frac{1}{\sqrt{2}} \sum_k b_k z^k$, with the coefficients $b_k$ as in (22). Setting $S_1f(z) := m_1(z)f(z^2)$, it follows then that (23)-(24) are satisfied.

**Remark 3.7.** The significance of irreducibility (when satisfied) is that the wavelet subbands which are indicated in Table 4 are then the only subbands of the corresponding multiresolution. We will show that in fact irreducibility holds generically, but it does not hold, for example, for the Haar wavelets. In the simplest case, the Haar wavelet has $g = 2 = N$ and the numbers

$$(a_0 \ a_1 \ a_1) = (1 \ 1 \ -1). \quad (30)$$

Hence, for this representation of $\mathcal{O}_2$ on $\ell^2$, we may take $\mathcal{H}_0 = \ell^2(0,1,2,\ldots)$, and therefore $\mathcal{H}_0^+ = \ell^2(-3,-2,-1)$. Returning to the multiresolution diagram in Table 4, this means that we get additional subspaces of $L^2(\mathbb{R})$, on top of the standard ones which are listed in Table 4. Specifically, in addition to

$$\mathcal{V}_n = U^n \mathcal{V}_0 = W S_0^n \ell^2 \quad (31)$$

and

$$\mathcal{W}_n = \mathcal{V}_{n-1} \ominus \mathcal{V}_n = W S_0^{n-1} S_1 \ell^2, \quad (32)$$

we get a new system with “twice as many”, as follows: $\mathcal{V}_n^{(\pm)}$ and $\mathcal{W}_n^{(\pm)}$, where

$$\mathcal{V}_n^{(+)} = W S_0^n (\mathcal{H}_0), \quad (33)$$

$$\mathcal{W}_n^{(+)} = W S_0^n S_1 (\mathcal{H}_0); \quad (34)$$

and

$$\mathcal{V}_n^{(-)} = W S_0^n (\mathcal{H}_0^+), \quad (35)$$

$$\mathcal{W}_n^{(-)} = W S_0^n S_1 (\mathcal{H}_0^+). \quad (36)$$

For the case of the Haar wavelet, see (30),

$$\mathcal{V}_0^{(+)} \subset L^2(0,\infty), \quad \mathcal{V}_0^{(-)} \subset L^2(-\infty,0),$$

or rather, $\mathcal{V}_0$ consists of finite linear combinations of Z-translates of

$$\varphi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x \in \mathbb{R} \setminus [0,1), \end{cases} \quad (37)$$

alias the step functions of step-size one, i.e., functions in $L^2(\mathbb{R})$ which are constant between $n$ and $n + 1$ for all $n \in \mathbb{Z}$; and

$$\mathcal{V}_0^{(+)} = \mathcal{V}_0 \cap L^2(0,\infty), \quad \mathcal{V}_0^{(-)} = \mathcal{V}_0 \cap L^2(-\infty,0). \quad (38)$$
Hence we get two separate wavelets, but with translations built on \{0, 1, 2, \ldots\} and \{-3, -2, -1\}. In view of the graphics of the cascade approximation to the scaling function (see Refs. 11 and 12), it is perhaps surprising that other wavelets (different from the Haar wavelets) do not have the corresponding additional “positive vs. negative” splitting into subbands within the Hilbert space \(L^2(\mathbb{R})\).

**Remark 3.8.** There are other dyadic Haar wavelets, in addition to (37). For example, let

\[
\varphi_k(x) = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{if } 0 \leq x < 2k + 1, \\
0 & \text{if } x \in \mathbb{R} \setminus [0, 2k + 1).
\end{cases}
\]

Then it follows that there is a splitting of \(V_0\) into orthogonal subspaces which is analogous to (38), but it has many more subbands than the two, “positive vs. negative”, which are given in (38), and which are special to the standard Haar wavelet (37). For details on these other Haar wavelets, and their decompositions, we refer the reader to Proposition 8.2 of Ref. 11. The corresponding \(m\)-functions of (39) are

\[
m_0(z) = \frac{1}{\sqrt{2}} (1 + z^{2k+1}), \quad m_1(z) = \frac{1}{\sqrt{2}} (1 - z^{2k+1}), \quad z \in \mathbb{T}.
\]

Hence, after adjusting the \(O_2\)-representation \(T\) with a (rotation) \(V \in U_2(\mathbb{C})\), we have

\[
T_0 f(z) = f(z^2), \quad T_1 f(z) = z^{2k+1} f(z^2), \quad f \in L^2(\mathbb{T}) \cong \ell^2,
\]

and, of course, the two new operators \(T_0, T_1\) will satisfy the \(O_2\)-identities (23)–(24), and the corresponding representation will have the same reducing subspaces as the one defined directly from \(m_0\) and \(m_1\). The explicit decomposition of the multiresolution subspaces corresponding to (38) may be derived, via \(W\) in Table 4, from the corresponding decomposition into sums of irreducibles for the \(O_2\)-representation on \(\ell^2\) which corresponds to (38). This means that the corresponding (24) which is associated with (39) and (11) has more than two terms in its subspace configuration.

### 4. Wavelet Filters and Subbands

The power and the usefulness of the multiresolution subband filters for the analysis of wavelets and their algorithms was first demonstrated forcefully in Refs. 13 and 21; see especially p. 140 of Ref. 13 and p. 157 of Ref. 21, where the \(O_N\)-relations (23)–(24) are identified, and analyzed in the case \(N = 2\). Around the same time, A. Cohen identified and utilized the interplay between \(\ell^2\) and \(L^2(\mathbb{R})\) which, as noted in Section 3 above, is implied by the \(O_N\)-relations and their representations. But neither of those prior references takes up the construction of \(O_N\)-representations in a systematic fashion.

Of course the quadrature mirror filters (QMF’s) have a long history in electrical engineering (speech coding problems), going back to long before they were used in wavelets, but the form in which we shall use them here is well articulated, for example, in Ref. 21. Some more of the history of and literature on wavelet filters is covered well in Refs. 22 and 24.

The operators corresponding to wavelet filters may be realized on either one of the two Hilbert spaces \(\ell^2(\mathbb{Z})\) or \(L^2(\mathbb{T})\), \(\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}\), and \(L^2(\mathbb{T})\) defined from the normalized Haar measure \(\mu\) on \(\mathbb{T}\). But, of course, \(\ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})\) via the Fourier series

\[
f(z) = \sum_{k \in \mathbb{Z}} \xi_k z^k,
\]

and its inverse

\[
\xi_k = \hat{f}(k) = \int_{\mathbb{T}} z^{-k} f(z) \, d\mu(z).
\]

For a given sequence \(a_0, a_1, \ldots, a_{N_0-1}\), consider the operator \(S_0\) in \(\ell^2(\mathbb{Z})\) given by

\[
\xi \mapsto S_0 \xi;
\]
and
\[(S_0\xi)_k = \frac{1}{\sqrt{N}} \sum_l a_{k-lN} \xi_l.\] (45)

Setting
\[m_0(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k z^k,\] (46)

and
\[(\hat{S}_0 f)(z) = m_0(z) f(z^N), \quad f \in L^2(\mathbb{T}), \quad z \in \mathbb{T},\] (47)

we note that $S_0$ and $\hat{S}_0$ are really two versions of the same operator, i.e.,
\[\hat{(\hat{S}_0 f)} = S_0(\hat{f})\] (48)

when $\hat{f} = (\xi_k)$ from (43). (The first one is the discrete model, and the second, the periodic model, referring to the diagram (29).) Hence, we shall simply use the same notation $S_0$ in referring to this operator in either one of its incarnations. It is the (45) version which is used in algorithms, of course.

Let $\varphi \in L^2(\mathbb{R})$ be the compactly supported scaling function solving
\[\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(Nx - k).\] (49)

Then define the operator
\[W : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{R})\] (50)

by
\[W \xi = \sum_k \xi_k \varphi(x - k).\] (51)

The conditions on the wavelet filter $\{a_k\}$ from Section 1 and in (28), may now be restated in terms of $m_0(z)$ in (46) as follows:
\[\sum_{k=0}^{N-1} \left| m_0(z e^{i k \pi \frac{N}{N}}) \right|^2 = N,\] (52)

and
\[m_0(1) = \sqrt{N}.\] (53)

It then follows from Lemma 3.1 that $W$ in (51) maps $\ell^2(\mathbb{Z})$ onto the resolution subspace $\mathcal{V}_0(\subset L^2(\mathbb{R}))$, and that
\[U_N W = W S_0\] (54)

where
\[U_N f(x) = N^{-\frac{1}{2}} f \left( \frac{x}{N} \right), \quad f \in L^2(\mathbb{R}), \quad x \in \mathbb{R}.\] (55)

We showed in Ref. 3 that there are functions $m_1, \ldots, m_{N-1}$ such that the $N$-by-$N$ complex matrix
\[\frac{1}{\sqrt{N}} \left( m_j(e^{i k \pi \frac{N}{N}} z) \right)_{j,k=0}^{N-1}\] (56)
is unitary for all \( z \in \mathbb{T} \). If we define
\[
S_j f(z) = m_j(z)f(z^N), \quad f \in L^2(\mathbb{T}), \quad z \in \mathbb{T},
\]
then
\[
S_j^* S_k = \delta_{j,k}I_{L^2(\mathbb{T})},
\]
and
\[
\sum_{j=0}^{N-1} S_j S_j^* = I_{L^2(\mathbb{T})}.
\]
(Condition (58) is not needed for this, but only for the algorithmic operations of the Appendices in Refs. 11 and 12.) We also showed that the solutions \((m_j)_{j=0}^{N-1}\) to (50) are in 1–1 correspondence with the group of all polynomial functions
\[
A: \mathbb{T} \longrightarrow U_N(\mathbb{C})
\]
where \(U_N(\mathbb{C})\) denotes the unitary \(N \times N\) matrices. The correspondence is \(m \leftrightarrow A\) with
\[
m_j(z) = \sum_{k=0}^{N-1} A_{j,k}(z^N)z^k,
\]
and
\[
A_{j,k}(z) = \frac{1}{N} \sum_{w \in z = z} w^{-k}m_j(w).
\]
This correspondence plays a central role in the proofs in Refs. 11 and 12. We also show in Ref. 9 that if \(m_0\) is given, and if it satisfies (52), then it is possible to construct \(m_1, \ldots, m_{N-1}\) such that the extended system \(m_0, m_1, \ldots, m_{N-1}\) will satisfy (56). As a consequence, \(A\) in (52) will be a \(U_N(\mathbb{C})\)-loop, i.e., \(A: \mathbb{T} \rightarrow U_N(\mathbb{C})\), and moreover, the original \(m_0\) is then recovered from (51) for \(j = 0\).

By virtue of (58)–(59), \(L^2(\mathbb{T})\), or equivalently \(\ell^2(\mathbb{Z})\), splits up as an orthogonal sum
\[
S_j(\ell^2(\mathbb{Z})), \quad j = 0, 1, \ldots, N - 1.
\]
We saw that the wavelet transform \(W\) of (51)–(54) maps \(\ell^2(\mathbb{Z})\) onto \(\mathcal{V}_0\), and from (54) we conclude that \(W\) maps \(S_0(\ell^2(\mathbb{Z}))\) onto \(U_N(\mathcal{V}_0) =: \mathcal{V}_1\). Hence, in the \(N\)-scale wavelet case, \(W\) transforms the spaces \(S_j(\ell^2(\mathbb{Z})) \subset \ell^2(\mathbb{Z})\) onto orthogonal subspaces \(\mathcal{W}_1^{(j)}\), \(j = 1, \ldots, N - 1\) in \(L^2(\mathbb{R})\), and
\[
\mathcal{W}_1 = \mathcal{V}_0 \oplus \mathcal{V}_1 = \bigoplus_{j=1}^{N-1} \mathcal{W}_1^{(j)},
\]
where
\[
\mathcal{W}_1^{(j)} = S_j \ell^2, \quad j = 1, \ldots, N - 1.
\]
Each of the spaces \(\mathcal{V}_1\) and \(\mathcal{W}_1^{(j)}\) is split further into orthogonal subspaces corresponding to iteration of the operators \(S_0, S_1, \ldots, S_{N-1}\) of (58)–(59). It is the system \(\{S_j\}_{j=0}^{N-1}\) which is called a wavelet representation, and it follows that the wavelet decomposition may be recovered from the representation. Moreover, the variety of all wavelet representations is in 1–1 correspondence with the group of polynomial functions \(A\) as given in (50). The correspondence is fixed by (51)–(52). Operators \(\{S_j\}\) satisfying (58)–(59) are said to constitute a representation of the \(C^*\)-algebra \(O_N\), the Cuntz algebra \(\mathbb{C}_{\infty}\) and it is the irreducibility of the representations from (77) which will concern us. If a representation (77) is reducible (Definition 3.3), then there is a subspace
\[
0 \subsetneq \mathcal{H}_0 \subsetneq L^2(\mathbb{T})
\]
which is invariant under all the operators \(S_j\) and \(S_j^*\), and so the data going into the wavelet filter system \(\{m_j\}\) are then not minimal.
5. THE MAIN THEOREM

The main result will be stated in the present section, but without proof. Instead the reader is referred to Ref. [12] for the full proof, and for a detailed discussion of its implications. We noted above that the representation (57) given from a QMF system $m_j = m_j^{(A)}$ via (61)–(62) is irreducible if and only if the subbands are optimal, in that they do not admit further reduction into a refined system of closed subspaces of $L^2(\mathbb{R})$.

**Theorem 5.1.** The representation

$$S_j^{(A)} f (z) = m_j^{(A)} (z) f (z^N), \quad f \in L^2 (\mathbb{T}), \quad z \in \mathbb{T}$$

is an irreducible representation of $O_N$ on $L^2 (\mathbb{T})$ if and only if $A : \mathbb{T} \to U_N (\mathbb{C})$ does not admit a matrix corner of the form

$$V \begin{pmatrix} z^{n_0} & & & 0 \\ & \ddots & \vdots & \vdots \\ & & z^{n_{M-1}} & \end{pmatrix},$$

for some $V \in U_M (\mathbb{C})$, and where $n_0, n_1, \ldots, n_{M-1} \in \{0, 1, 2, \ldots\}$.

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