The pure spinor formulation of superstrings*

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Abstract
In this paper we outline the construction of pure spinor superstrings. We consider both the open and closed pure spinor superstrings in critical and noncritical dimensions and on flat and curved target spaces with RR flux. We exhibit the integrability properties of pure spinor superstrings on curved backgrounds with RR fluxes.

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1. Introduction
There are currently three main formulations of superstrings: the Ramond–Neveu–Schwarz (RNS), the Green–Schwarz (GS) and the pure spinor (for a review see [1]). In the RNS formalism, one studies maps \((x^m, \psi^m; m = 0, \ldots, 9)\) from a two-dimensional supersymmetric worldsheet to a bosonic spacetime. The formalism lacks manifest spacetime supersymmetry and requires the introduction of a projection (GSO) in order to exhibit it. In this formalism the \((2, 2)\) worldsheet supersymmetry is related to spacetime supersymmetry.

There are various complications in the perturbative analysis in the RNS formalism, such as a requirement for summation over spin structures and a lack of a proper definition of the measure of integration on the supermoduli space. The RNS formalism is also inadequate for the quantization on backgrounds with RR fluxes, i.e. there is no simple coupling to the RR fields.

In the GS formalism we consider maps \((x^m, \theta^\alpha; m = 0, \ldots, 9; \alpha = 1, \ldots, 16)\) from a two-dimensional bosonic worldsheet to a supersymmetric spacetime. This formalism possesses a manifest spacetime supersymmetry and can be used to quantize superstrings on RR backgrounds. It is, however, difficult to analyze the GS quantum sigma model. The formalism requires a gauge fixing of a fermionic symmetry \((\kappa\text{-symmetry})\), which is known only in the light-cone gauge and hence non-covariantly. Since the equations of motion of the GS superstring do not provide a propagator for the \(\theta\)’s, the calculations in worldsheet perturbation theory are problematic.

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In the pure spinor formalism, as in the GS formalism, we consider maps \((x^m, \theta^\alpha; m = 0, \ldots, 9; \alpha = 1, \ldots, 16)\) from a two-dimensional bosonic worldsheet to a supersymmetric spacetime. We introduce additional new degrees of freedom, which are bosonic target space spinors (ghosts) \(\lambda^\alpha; \alpha = 1, \ldots, 16\). They satisfy a set of constraints that define them as pure spinors, hence the name of the superstring. The pure spinor sigma model can be quantized in a straightforward manner, since it contains additional terms that break explicitly the GS \(\kappa\) symmetry and introduce propagators for all the variables. This formalism possesses a manifest spacetime supersymmetry and can be used to quantize superstrings on RR backgrounds.

In this paper we will outline the pure spinor formulation of superstrings. The paper is organized as follows. In section 2 we will construct the pure spinor superstring in a flat ten-dimensional space. We will consider both the open and closed pure spinor superstrings and the relation to the RNS superstring. In section 3 we will consider the pure spinor superstring in curved space. After a discussion of general curved backgrounds, we will consider anti-de Sitter (AdS) backgrounds with RR flux. In section 4 we will present noncritical pure spinor superstrings in various dimensions. We will discuss various examples: the linear dilaton background, AdS_5 and AdS_4. In section 5, we will exhibit the integrability properties of pure spinor superstrings on curved backgrounds with RR fluxes. In the appendix we give some details on superalgebras and supergroups.

2. Pure spinor superstring in flat space

In this section we will construct the pure spinor superstring in a flat ten-dimensional target space [2].

2.1. Pure spinor open superstring

We will start with the construction of the open pure spinor superstring, or more precisely the holomorphic part of the closed pure spinor superstring.

2.1.1. The pure spinor superstring variables. Consider the supermanifold \((x^m, \theta^\alpha)\), where \(x^m, m = 0, \ldots, 9\) are commuting coordinates with the OPE

\[
x^m(z)x^n(0) \sim -\eta^{mn} \log|z|^2,
\]

and \(\theta^\alpha, \alpha = 1, \ldots, 16\) are worldsheet weight zero anti-commuting coordinates. \(x^m\) transform in the vector representation of the target space Lorentz group \(SO(1, 9)\), while \(\theta^\alpha\) transform in its 16 Majorana–Weyl spinor representation.

One introduces the worldsheet weight one \(p_\alpha\) as the conjugate momenta to \(\theta^\alpha\) with the OPE

\[
p_\alpha(z)\theta^\beta(0) \sim \frac{\delta^\beta_\alpha}{z}.
\]

\((p_\alpha, \theta^\beta)\) is a free fermionic \((b, c)\) system of weight \((1, 0)\). \((x^m, p_\alpha, \theta^\beta)\) are the GS variables.

Next we add a bosonic complex Weyl spinor ghost \(\lambda^\alpha; \alpha = 1, \ldots, 16\), which satisfies the pure spinor constraint

\[
\lambda^\alpha \gamma^m_{\alpha\beta} \gamma^\beta = 0 \quad m = 0, \ldots, 9.
\]

The \(\gamma^m_{\alpha\beta}\) are the symmetric 16 × 16 Pauli matrices in ten dimensions. A spinor \(\lambda^\alpha\) that satisfies the constraints (2.3) is called a pure spinor. This set of ten constraints is reducible as we will soon discuss. It reduces the number of degrees of freedom of \(\lambda^\alpha\) from 16 to 11.
Another definition of the pure spinors in even dimension \( d = 2n \), which is due to Cartan and Chevalley is

\[
\lambda^a_\alpha \gamma^{m_1 \cdots m_j}_\alpha \lambda^\beta = 0 \quad j < n,
\]

so that the pure spinor bilinear reads

\[
\lambda_\alpha \lambda_\beta = \frac{1}{n!2^n} \gamma^{m_1 \cdots m_n_} (\lambda^a_\gamma m_1 \cdots m_n_ \lambda^a_\gamma),
\]

where \( \gamma^{m_1 \cdots m_j}_\gamma \) is the antisymmetrized product of \( j \) Pauli matrices. The pure spinor that we consider in ten dimensions satisfies (2.5) with \( n = 5 \).

We note, for later use when we will discuss pure spinor superstrings in various dimensions, that this definition of the pure spinor space in \( d = 2, 4, 6 \) dimensions is trivially realized by an \( SO(d) \) Weyl spinor.

We denote the conjugate momenta to \( \lambda_\alpha \) by the worldsheet weight one bosonic target space complex \( w_\alpha \). The system \( (w_\alpha, \lambda_\alpha) \) is a curved \( (\beta, \gamma) \) system of weights \((1, 0)\). The reason that the system is not free is the set of pure spinor constraints (2.3).

The pure spinor constraints imply that \( w_\alpha \) are defined up to the gauge transformation

\[
\delta w_\alpha = \Lambda^a_\gamma m_\gamma (\gamma^m_\alpha),
\]

where \( \gamma^m_\alpha \) appears only in gauge-invariant combinations. These are the Lorentz algebra currents \( M_{mn} \), the ghost number current \( J(w, \lambda) \) which assigns ghost number 1 to \( \lambda \) and ghost number \(-1\) to \( w \)

\[
M_{mn} = w^{\gamma mn_\lambda}, \quad J(w, \lambda) = w^a_\lambda \lambda^a,
\]

and the pure spinor stress–energy tensor \( T(w, \lambda) \).

Unlike the RNS superstrings, all the variables that we use in the pure spinor superstring are of integer worldsheet spin. This is an important property of the formalism: for instance, we will have no need to sum over spin structures when computing multiloop scattering amplitudes.

### 2.1.2. The pure spinor space.

The pure spinor set of constraints (2.3) defines a curved space, which can be covered by \( 16 \) patches \( U_\alpha \) on which the \( \alpha \)th component of \( \lambda_\alpha \) is nonvanishing.

The set of constraints (2.3) is reducible. In order to solve it we rotate to Euclidean signature. The pure spinor variables \( \lambda^a_\alpha \) transform in the \( 16 \) of \( SO(10) \). Under \( SO(10) \rightarrow U(5) \simeq SU(5) \times U(1) \) we have that \( 16 \rightarrow 5 \oplus \bar{5} \). We denote the 16 components of the pure spinor in the \( U(5) \) variables by \( \lambda^a_\alpha = \lambda^a_\gamma \oplus \lambda^a_\gamma \); \( a, b = 1, \ldots, 5 \) with \( \lambda^a_\alpha = -\lambda^a_\beta \).

In these variables it is easy to solve the pure spinor set of constraints (2.3) by

\[
\lambda^+ = e^i, \quad \lambda_\alpha = u_{ab}, \quad \lambda^a = -\frac{1}{8} e^{-8} \epsilon^{abcde} u_{bc} U_{de},
\]

The pure spinor space \( \mathcal{M} \) is complex 11-dimensional, which is a cone over \( \mathcal{Q} = \frac{SO(10)}{7/5T} \).

At the origin \( \lambda^a = 0 \), both the pure spinor set of constraints (2.3) and their derivatives with respect to \( \lambda^a \) vanish. Thus, the pure spinor space has a singularity at the origin.

### 2.1.3. The pure spinor superstring action.

We will work in the worldsheet conformal gauge. The conformal gauge fixed worldsheet action of the pure spinor superstring is \( S = S_0 + S_1 \), where

\[
S_0 = \int d^2 z \left( \frac{1}{2} \dd \lambda^a \dd \lambda^a + p_a \dd \theta^a - u_a \dd \lambda^a \right)
\]

and

\[
S_1 = \int d^2 z \left( \frac{1}{4} r^{(2)} \log \Omega(\lambda) \right).
\]
The first two terms in $S_0$ correspond to the GS action written in a first-order formalism, the third term in $S_0$ is the pure spinor action and $S_1$ is a coupling of the worldsheet curvature $r^{(2)}$ to the holomorphic top form $\Omega$ of the pure spinor space $\mathcal{M}$

$$\Omega = \Omega (\lambda) \, d\lambda^1 \wedge \cdots \wedge d\lambda^{11}. \quad (2.11)$$

Note that the $(w_a, \lambda^a)$ action is holomorphic and does not depend on their complex conjugates.

The stress tensor of the $(w, \lambda)$ system reads

$$T(w,\lambda) = \frac{1}{\Omega_1(\lambda)} \frac{\partial \omega}{\partial s}, \quad (2.12)$$

and we will discuss the significance of the last term later. Note, however, that it does not contribute to the central charge of the system.

The system $(w_a, \lambda^a)$ is interacting due to the pure spinor constraints. It has the central charge $c(w,\lambda) = 22$, which is twice the complex dimension of the pure spinor space

$$T(w,\lambda)(z)T(w,\lambda)(0) \sim \frac{\text{dim}_C(\mathcal{M})}{z^4} + \cdots. \quad (2.13)$$

This can be computed, for instance, by introducing the conjugate momenta to the $U(5)$ variables (2.8) with the OPE,

$$\epsilon(z)\epsilon(0) \sim -\frac{\delta^{ab}}{z}, \quad \epsilon^{ab}u_{cd} \sim \frac{\delta^{ab}}{z}, \quad (2.14)$$

with $\delta^{ab} = \frac{1}{2}(\delta^a_1 \delta^b_2 - \delta^a_2 \delta^b_1)$. The stress–energy tensor reads

$$T(w,\lambda) = u^{ab} \partial u_{ab} + \partial \lambda \partial s + \partial \gamma^2 s, \quad (2.15)$$

giving the central charge 22.

The total central charge of the pure spinor superstring is

$$c_{\text{tot}} = c(X_m) + c(p_{\alpha}, \theta_{\alpha}) + c(w_a, \lambda^a) = 10 - 32 + 22 = 0, \quad (2.16)$$

as required by the absence of a conformal anomaly.

The ghost number anomaly reads

$$J(w,\lambda)(z)T(w,\lambda)(0) \sim \frac{8}{z^3} + \cdots = \frac{c_1(Q)}{z^3} + \cdots, \quad (2.17)$$

where $c_1(Q)$ is the first Chern class of the pure spinor cone base $Q$.

2.1.4. The BRST operator. The physical states are defined as the ghost number one cohomology of the nilpotent BRST operator

$$Q = \oint dz \, \lambda^a d_a, \quad (2.18)$$

where

$$d_a = p_a - \frac{1}{2} \gamma_{ab} \gamma^\beta \partial x_m - \frac{1}{8} \gamma_{ab} \gamma^{\alpha\beta\gamma} \partial^\alpha \partial^\beta \partial^\gamma. \quad (2.19)$$

This BRST operator is an essential ingredient of the formalism; however, it is not clear how to derive its form by a gauge fixing procedure. $Q^2 = 0$ since $d_a d_b \sim \gamma_{ab}^\beta m$ and $\lambda^\alpha \gamma_{ab}^\alpha \lambda^b = 0$.

The $d_a$ are the supersymmetric Green–Schwarz constraints. They are holomorphic and satisfy the OPE,

$$d_a(z)d_b(0) \sim -\frac{\gamma_{ab}^m \Pi_m (0)}{z}, \quad (2.20)$$

and

$$d_a(z)\Pi^m (0) \sim \frac{\gamma_{ab}^\beta \partial^\beta (0)}{z}. \quad (2.21)$$
where
\[ \Pi_m = \partial x_m + \frac{1}{2} \theta \gamma_m \partial \theta \] (2.22)
is the supersymmetric momentum. \( d_a \) acts on function on superspace \( F(x^m, \theta^a) \) as
\[ d_a(z) F(x^m(0), \theta^a(0)) \sim \frac{D_a F(x^m(0), \theta^a(0))}{z}, \] (2.23)
where
\[ D_a = \frac{\partial}{\partial \theta^a} + \frac{1}{2} \gamma_m^{ab} \partial_{a} \partial_{b} \] (2.24)
is the supersymmetric derivative in ten dimensions.

2.1.5. Massless states. Massless states are described by the ghost number one weight zero vertex operators,
\[ \mathcal{V}^{(1)} = \lambda^a A_a(x, \theta), \] (2.25)
where \( A_a(x, \theta) \) is an unconstrained spinor superfield.

The BRST cohomology conditions are
\[ Q \mathcal{V}^{(1)} = 0, \quad \mathcal{V}^{(1)} \simeq Q \Omega^{(0)}, \] (2.26)
where \( \Omega^{(0)} \) is a real scalar superfield. These imply the ten-dimensional field equations for \( A_a(x, \theta) \)
\[ \gamma_{\mu
u}^a D_a A^\mu(x, \theta) = 0, \] (2.27)
with the gauge transformation \( \delta A_a = D_a \Omega^{(0)} \), and we used relation (2.5) in ten dimensions.

These equations imply that \( A_a \) is an on-shell super Maxwell spinor superfield in ten dimensions
\[ A_a(x, \theta) = \frac{1}{2} \gamma^a \partial_\mu A^\mu(x) + \frac{i}{\sqrt{2}} \gamma^a \gamma^m \psi^\mu(x) + O(\theta^3), \] (2.28)
where \( a_m(x) \) is the gauge field and \( \psi^\mu(x) \) is the gaugino. They satisfy the super Maxwell equations
\[ \partial_\mu (\partial_\mu a_\mu - \partial_\mu a_\mu) = 0, \quad \gamma^a \partial_\mu \psi^\mu = 0, \] (2.29)
\( A_a \) is related to the gauge superfield \( A_m \) by
\[ A_m = \gamma^a \partial_a A_\mu, \] (2.30)
and \( A_m(x, \theta) = a_m(x) + O(\theta) \). Only in ten dimensions do these conditions give an on-shell vector multiplet. In lower dimensions they describe an off-shell vector multiplet.

The integrated ghost number zero vertex operator for the massless states reads
\[ \mathcal{V} = \int dz U = \int dz \left( \partial^\alpha A_\alpha + \Pi^m A_m + d_a W^a + \frac{1}{2} M_{mn} F^{mn} \right), \] (2.31)
where \( W^a \) and \( F^{mn} \) are the spinorial and bosonic field strength, respectively
\[ F_{mn} = \partial_m A_n - \partial_n A_m, \quad D_a W^\beta = \frac{1}{4} (\gamma^{mn})^\beta_\alpha F_{mn}. \] (2.32)
We have \( W^a = \psi^a + \cdots \) and \( F_{mn} = f_{mn} + \cdots \).

\( M_{mn} \) are the generators of Lorentz transformations
\[ M_{mn} = \frac{1}{2} (p \gamma_{mn} + w \gamma_{mn} \gamma). \] (2.33)
When expanded \( U \) reads
\[ U = a_m \partial x^m + \frac{1}{2} f_{mn} M^{mn} + \psi^a q_a + \cdots, \] (2.34)
where \( q_\alpha \) is the spacetime supersymmetry current,
\[
q_\alpha = p_\alpha + \frac{1}{2} \left( \partial x^\mu + \frac{1}{2} \theta \gamma^\mu \theta \right) (\gamma_\mu \theta)_\alpha.
\]

Note that, unlike the RNS, we did not need a GSO projection in order to get a supersymmetric spectrum and that the Ramond and NS sectors appear on equal footing in the pure spinor formalism.

### 2.1.6. Massive states

The analysis of the massive states proceeds in a similar way. At the first massive level \( m^2 \alpha' = 1 \), the ghost number one weight one vertex operator has the expansion
\[
U^{(1)} = \partial \lambda^\alpha A_\alpha(x, \theta) + \lambda^\alpha \partial \theta B_\alpha(x, \theta) + \partial \lambda^\alpha C_\alpha(x, \theta) + \cdots.
\]

It describes a massive spin two multiplet with 128 bosons and 128 fermions with \( g_{mn} \) traceless symmetric
\[
\eta^m g_{mn} = 0, \quad \bar{\partial}^m g_{mn} = 0,
\]
\( b_{map} \) a 3-form
\[
\bar{\partial}^m b_{map} = 0,
\]
and spin 3/2 field \( \psi_{ma} \)
\[
\bar{\partial}^m \psi_{ma} = 0, \quad \gamma^{ma} \psi_{a} = 0.
\]

This multiplet fields can be understood as the Kaluza–Klein modes of the 11-dimensional supergravity multiplet.

The procedure at the \( n \)th massive level \( m^2 = \frac{n}{\alpha'} \) is to construct a ghost number one weight \( n \) vertex operator at zero momentum by using the building blocks \( (\partial x^m, \theta^\alpha, \lambda^\alpha) \) and impose conditions \((2.26)\).

### 2.1.7. Scattering amplitudes

Consider the tree level open string scattering amplitudes. The \( n \)-point function \( A_n \) reads
\[
A_n = \langle V_1(z_1)V_2(z_2)V_3(z_3) \int dz_4 U(z_4) \cdots \int dz_n U(z_n) \rangle.
\]

\( V_i \) are dimension one, ghost number one vertex operators and \( U_i \) are dimension zero, ghost number zero vertex operators. Their construction has been discussed above.

We can use the \( SL(2, R) \) symmetry to fix three of the worldsheet coordinates, and using the free field OPEs obtain
\[
A_n = \int dz_4 \cdots \int dz_n (\lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha \beta \gamma}(z_r, k_r, \theta)),
\]

where \( k_r \) are the scattering momenta and we integrated over the nonzero modes. Thus, \( f_{\alpha \beta \gamma} \) depends only on the zero modes of \( \theta \). There were 11 bosonic zero modes of \( \lambda^\alpha \) and 16 fermionic zero modes of \( \theta^\alpha \). One expects 11 of the integration fermionic zero modes to cancel the 11 integration bosonic zero modes, leaving five fermionic \( \theta \) zero modes. A Lorentz-invariant prescription for integrating over the remaining five fermionic \( \theta \) zero modes is given by \([2]\)
\[
A_n = T_{\alpha \beta \gamma}^\mu \left( \frac{\partial}{\partial \theta} \gamma^\mu_{mn} \frac{\partial}{\partial \theta} \right) \left( \gamma_1 \frac{\partial}{\partial \theta} \right)^\rho \left( \gamma^\mu \frac{\partial}{\partial \theta} \right)^\sigma \left( \gamma^\nu \frac{\partial}{\partial \theta} \right)^\tau \int dz_4 \cdots \int dz_n f_{\alpha \beta \gamma}(z_r, k_r, \theta),
\]

where \( T_{\alpha \beta \gamma}^\mu \) is symmetrized with respect to the upper and lower indices, and \( T_{\alpha \beta \gamma}^\mu = 1, T_{\alpha \beta \gamma}^\nu T_{\alpha \beta \gamma}^\rho = T_{\alpha \beta \gamma}^\nu T_{\alpha \beta \gamma}^\rho = 0 \). For examples of explicit calculations of scattering amplitudes see, e.g., \([4]\).
2.2. Pure spinor closed superstring

The construction of the closed superstrings is straightforward. One introduces the right moving superspace variables \((\bar{p}_\alpha, \bar{\theta}_\alpha)\), the pure spinor system \((\bar{w}_\alpha, \bar{\lambda}_\alpha)\) and the nilpotent BRST operator

\[
\bar{Q} = \oint d\bar{z} \bar{\lambda}_\alpha \bar{d}_\alpha. \tag{2.43}
\]

The analysis of the spectrum proceeds by combining the left and right sectors. In our notation we will use the same spinorial indices for the right sectors.

Massless states are described by the ghost number \((1, 1)\) vertex operator

\[
V = \bar{\lambda}_\alpha \bar{\lambda}_\alpha A_{\alpha\beta}(X, \theta, \bar{\theta}), \tag{2.44}
\]

where \(A_{\alpha\beta}(X, \theta, \bar{\theta})\) is the on-shell supergravity multiplet in ten dimensions

\[
A_{\alpha\beta}(X, \theta, \bar{\theta}) = h_{mn}(\gamma_m \theta)^\alpha (\gamma_m \theta)^\beta + \psi^\gamma_n (\gamma_m \theta)^\alpha (\gamma_n \bar{\theta})^\beta + \cdots + F_{\gamma\delta} (\gamma_m \theta)^\alpha (\gamma_n \bar{\theta})^\beta (\gamma_n \bar{\theta})^\delta + \cdots, \tag{2.45}
\]

where \(h_{mn}\) is the graviton, \(\psi^\gamma_n\) is the gravitino and \(F_{\gamma\delta}\) are the RR fields,

\[
F_{\gamma\delta} = \oplus F_{k_1 \cdots k_n}^{\gamma\delta} (\gamma_{k_1} \cdots \gamma_{k_n}) \tag{2.46}
\]

We see that all the different RNS sectors (NS, NS), (R, R), (R, NS) and (NS, R) are on equal footing in this representation.

The integrated ghost number zero vertex operator for the massless states reads

\[
\mathcal{U} = \int d^2z (\partial \theta^\alpha A_{\alpha\beta} \bar{\theta}^\beta + \partial \theta^\alpha A_{\alpha\beta} \bar{\Gamma}^\beta + \cdots). \tag{2.47}
\]

2.3. Anomalies

The pure spinor system \((\lambda^\alpha, w_\alpha)\) defines a nonlinear \(\sigma\)-model due to the curved nature of the pure spinor space (2.3). There are global obstructions to define the pure spinor system on the worldsheet and on target space \([5, 6]\). They are associated with the need for holomorphic transition functions relating \((\lambda^\alpha, w_\alpha)\) on different patches of the pure spinor space, which are compatible with their OPE. They are reflected by quantum anomalies in the worldsheet and target space (pure spinor space) diffeomorphisms. The conditions for the vanishing of these anomalies are the vanishing of the integral characteristic classes

\[
\frac{1}{2} c_1(\Sigma) c_1(\mathcal{M}) = 0, \quad \frac{1}{2} p_1(\mathcal{M}) = 0, \tag{2.48}
\]

where \(c_1(\Sigma)\) is the first Chern class of the worldsheet Riemann surface, \(c_1(\mathcal{M})\) is the first Chern class of the pure spinor space \(\mathcal{M}\) and \(p_1\) is the first Pontryagin class of the pure spinor space. The vanishing of \(c_1(\mathcal{M})\) is needed for the definition of superstring perturbation theory and it implies the existence of the nowhere vanishing holomorphic top form \(\Omega(\lambda)\) on the pure spinor space \(\mathcal{M}\) that appears in the stress tensor (2.12).

The pure spinor space (2.3) has a singularity at \(\lambda^\alpha = 0\). Blowing up the singularity results in an anomalous theory. However, simply removing the origin leaves a non-anomalous theory. This means that one should consider the pure spinor variables as twistor-like variables. Indeed this is a natural interpretation of the pure spinor variables considering them from the twistor string point of view.
2.4. Mapping RNS to pure spinors

In this subsection we will construct a map from the RNS variables to the pure spinor ones [7, 8]. We will make use of a parametrization of the pure spinor components that would make the $\beta\gamma$-system structure of the pure spinor variables explicit. In this way, we will gain a new insight into the global definition of the pure spinor space and the importance of its holomorphic top form. The pure spinor stress tensor which we will obtain by the map will contain the contribution of the holomorphic top form on the pure spinor space.

In the following we will consider the holomorphic sector. The holomorphic supercharges in the $-\frac{1}{2}$ picture of the RNS superstring are given by the spin fields

$$q_+ = e^{-\phi/2 - \sum_i s_i H^i},$$

(2.49)

where the $H^i$'s are the bosons obtained from the bosonization of the RNS worldsheet matter fermions and the $s_i$'s take the values $\pm \frac{1}{2}$. These supercharges decompose into two Weyl representations.

In order to proceed with the map, one must first solve the pure spinor constraint (2.3), going to one patch of the pure spinor space. In each patch a different component of the pure spinor is nonzero. The field redefinition which we will use maps the RNS description into one patch of the pure spinor manifold. We will work on one of the patches which is described by the $SU(5) \times U(1)$ decomposition of the pure spinor $\lambda^a = (\lambda^+, \lambda^a, \lambda_{ab})$, discussed previously. The component of the pure spinor assumed to be nonzero is $\lambda^+$ corresponding to the representation $\overline{10}$ of this decomposition. In this patch one can solve for the $\lambda^a$ in terms of $\lambda^+$ and the components in the $\overline{10}$ representation $\lambda_{ab}$.

On this patch the supercharge $q_+$ which is the singlet of $SU(5)$ is raised to the $+\frac{1}{2}$ picture,

$$q_+ = b \eta e^{\beta \Phi/2 + \sum_a H^a/2} + \frac{1}{2} \sum_a \partial (\lambda^a + i x^{a+5}) e^{\phi/2 + \sum_a H^a/2 - i H^a},$$

(2.50)

while the supercharges $q_a$, corresponding to the pure spinor components $\lambda^a$ we solved for, remain in the $-\frac{1}{2}$ picture. Together they form a part of the original ten-dimensional supersymmetry algebra. One then defines the fermionic momenta

$$p_+ = b \eta e^{\beta \Phi/2 + \sum_a H^a/2}, \quad p_a = q_a$$

(2.51)

and their conjugate coordinates $\theta^+$ and $\theta^a$. Note that the OPEs of the fermionic momenta among themselves are all non-singular.

Introduce two new fields $\tilde{\phi}$ and $\tilde{\kappa}$ using

$$\eta = e^{\frac{\tilde{\phi} + \tilde{\kappa}}{2}} p_+, \quad b = e^{(\tilde{\phi} - \tilde{\kappa})/2} p_+,$$

(2.52)

yielding

$$\tilde{\phi} = -\frac{3i}{4} \sum_a H^a - \kappa - \frac{9}{4} \phi + \frac{1}{2} \chi,$$

(2.53)

$$\tilde{\kappa} = \frac{i}{4} \sum_a H^a - \kappa - \frac{3}{4} \phi - \frac{1}{2} \chi,$$

(2.54)

whose OPEs are

$$\tilde{\phi}(z)\tilde{\phi}(0) \sim -\log z, \quad \tilde{\kappa}(z)\tilde{\kappa}(0) \sim \log z.$$  

(2.55)

The reason why we choose the particular field redefinition (2.52) is that the pure spinor formalism is equivalent to the RNS formalism when we take into account all the different
pictures at the same time, which is achieved by working in the large Hilbert space, that is including the zero modes of the ghost $\xi$. But the usual cohomology of the RNS BRST charge $Q_{\text{RNS}}$ in the small Hilbert space is equivalent to the cohomology of $Q_{\text{RNS}} + \oint \eta$ in the large Hilbert space. With the redefinition (2.52), we are then mapping the $\oint \eta$ term of this extended BRST charge directly to the part $\oint \chi$ of the BRST operator (2.18).

By substituting the map into the RNS energy–momentum tensor one obtains

$$T = T_m + T_{gh} = -\frac{1}{2} \sum_m (\partial x^m)^2 - p_\alpha \partial \theta^\alpha - \sum_a p_a \theta^a - \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (\partial \kappa)^2 + \partial^2 \phi + \partial^2 \kappa.$$  

(2.56)

This can be verified to have a vanishing central charge

$$c = (10)_x + (-12)_p + (2)_{\delta \kappa} = 0.$$  

The pure spinors are reconstructed by the ordinary bosonization of a $\beta \gamma$-system [9]

$$\lambda^+ = e^{\tilde{\phi} \tilde{\kappa}}, \quad w_+ = \partial \kappa \ e^{-\tilde{\phi} - \tilde{\kappa}},$$  

(2.57)

whose OPE is

$$w_+(z) \lambda^+(0) \sim \frac{1}{z}.$$  

(2.58)

But the naive stress tensor one would expect for this $\beta \gamma$-system

$$w_+ \partial \lambda^+ = -\frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (\partial \kappa)^2 - \frac{1}{2} \partial^2 \phi - \frac{1}{2} \partial^2 \kappa,$$

does not coincide with the one we got from the map (2.56). This shows that the pure spinor stress tensor is not simply $w_+ \partial \lambda^+$ but actually

$$T_\lambda = w_+ \partial \lambda^+ - \frac{1}{2} \partial^2 \log \Omega_1(\lambda),$$  

(2.59)

where $\Omega$ is the coefficient of a top form defined on the pure spinor space [6]. By comparison we can read off the top form itself

$$\Omega = e^{-3(\tilde{\phi} + \tilde{\kappa})} = (\lambda^+)^3.$$  

(2.60)

At this point, we can map the RNS saturation rule for amplitudes on the sphere

$$\langle e^{a c d e} c \ e^{-2 \phi} \rangle = 1,$$  

(2.61)

to the pure spinor variables, obtaining

$$\langle (\lambda^+)^3 (\theta^a)^5 \rangle = 1,$$  

(2.62)

which is the prescription for the saturation of the zero modes in that we used when discussing the scattering amplitudes. Note that the third power of the pure spinor is consistent with the expression of the holomorphic top form (2.60) we just reconstructed.

In the final step in performing the map we covariantize by adding the missing coordinates and momenta. We add a BRST quartet consisting of ten $(1, 0)$ $bc$-systems $(p_{ab}, \theta^{ab})$ and ten $(1, 0)$ $\beta \gamma$-systems $(w_{ab}, \lambda^{ab})$. They have opposite central charges, so the total central charge remains unchanged. In this way we recover the full pure spinor stress tensor

$$T = -\frac{1}{2} \partial x^m \partial x_m - d_{ab} \partial \theta^a + w_{ab} \partial \lambda^a - \frac{1}{2} \partial^2 \log \Omega(\lambda).$$  

(2.63)

3. Pure spinor superstring in curved space

In this section we will consider critical pure spinor superstrings in curved ten-dimensional spaces.
3.1. General curved backgrounds

The pure spinor action in curved target space is obtained by adding to the flat target space action the integrated vertex operator for supergravity massless states and covariantizing with respect to the ten-dimensional $N = 2$ super reparametrization. Define by $Z^M = (X^m, \bar{\theta}^{\alpha}, \bar{\bar{\theta}}^{\dot{\alpha}})$ coordinates on $R^{10|32}$ superspace. The pure spinor sigma-model action takes the form

$$S = \int d^2z \left[ \frac{1}{2} (G_{MN}(Z) + B_{MN}(Z)) \partial Z^M \partial Z^N + F^{\alpha\beta} d_{\alpha} \bar{\theta}^{\beta} + \cdots \right] + S_{\lambda} + S_{\bar{\lambda}}. \quad (3.1)$$

$G_{MN}, B_{MN}, F^{\alpha\beta}, \ldots$ are background superfields

$$G_{MN} = g_{MN} + O(\theta), \quad B_{MN} = B_{MN} + O(\theta), \quad F^{\alpha\beta} = f^{\alpha\beta} + O(\theta). \quad (3.2)$$

Note that $d_{\alpha} \bar{d}_{\dot{\alpha}}$ are independent variables ($p_\alpha, \bar{p}_{\dot{\alpha}}$ do not appear explicitly).

The BRST operator reads

$$Q_B = Q + \bar{Q} = \oint d\bar{\lambda}^\alpha d_{\alpha} + \oint d\bar{\lambda}^{\dot{\alpha}} d_{\dot{\alpha}}. \quad (3.3)$$

The ten-dimensional supergravity field equations are derived by the requirement that $Q (\bar{Q})$ is holomorphic (antiholomorphic) and nilpotent [10].

3.2. Curved backgrounds with AdS symmetries

Consider pure spinor sigma models whose target space is the coset $G/H$, where $G$ is a supergroup with a $Z_4$ automorphism and the subgroup $H$ is the invariant locus of this automorphism [7]. The super Lie algebra $\mathcal{G}$ of $G$ can be decomposed into the $Z_4$ automorphism-invariant spaces

$$\mathcal{G} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \quad (3.4)$$

where the subscript keeps track of the $Z_4$ charge and in particular $\mathcal{H}_0$ is the algebra of the subgroup $H$. This decomposition satisfies the algebra ($i = 1, \ldots, 3$)

$$[\mathcal{H}_i, \mathcal{H}_j] \subset \mathcal{H}_i, \quad [\mathcal{H}_i, \mathcal{H}_j] \subset \mathcal{H}_i + \mathcal{H}_j \mod 4. \quad (3.5)$$

and the only nonvanishing supertraces are

$$\langle \mathcal{H}_i, \mathcal{H}_j \rangle \neq 0, \quad i + j = 0 \mod 4 \quad (i, j = 0, \ldots, 3). \quad (3.6)$$

We recall that the supertrace of a supermatrix $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$ is defined as $\langle M \rangle = \text{tr} A - (-1)^{\text{deg} M} \text{tr} D$, where $\text{deg} M$ is 0 for Grassmann even matrices and 1 for Grassmann odd ones.

We will denote the bosonic generators in $\mathcal{G}$ by $T_{\mu \nu} \in \mathcal{H}_0, T_a \in \mathcal{H}_2$, and the fermionic ones by $T_{\alpha \dot{\beta}} \in \mathcal{H}_1, T_{\dot{\alpha} \beta} \in \mathcal{H}_3$.

It is instructive to consider first the GS action. The worldsheet fields are the maps $g: \Sigma \rightarrow G$ and dividing by the subgroup $H$ is done by gauging the subgroup $H$ acting from the right by $g \simeq gh, h \in H$. The sigma model is further constrained by the requirement that it be invariant under the global symmetry $g \rightarrow \hat{g} g, \hat{g} \in G$. The left-invariant current is defined as

$$J = g^{-1} d g. \quad (3.7)$$

This current can be decomposed according to the $Z_4$ grading of the algebra

$$J = J_0 + J_1 + J_2 + J_3. \quad (3.8)$$
These currents are manifestly invariant under the global symmetry, which acts by left multiplication. Under the gauge transformation, which acts by right multiplication, they transform as
\[ \delta J = d \Lambda + [J, \Lambda], \quad \Lambda \in \mathcal{H}_0. \]  

Using the above properties of the algebra \( \mathcal{G} \) and the requirement of gauge invariance leads to the GS action
\[ S_{\text{GS}} = \frac{1}{4} \int \{ J_2 \wedge * J_2 + J_1 \wedge J_3 \} = \frac{1}{4} \int d^2 \sigma (\sqrt{h} m^m J_{2m} J_{2n} + \epsilon^{mn} J_{1m} J_{3n}), \]  
where \( m, n = 1, 2 \) are worldsheet indices. A \( J_0 \wedge * J_0 \) term does not appear because of gauge invariance, while the term \( J_1 \wedge J_3 \) breaks \( \kappa \)-symmetry and therefore cannot be included in the GS action. The first and second terms in the action are the kinetic and Wess–Zumino terms, respectively. The coefficient of the Wess–Zumino term is determined using \( \kappa \)-symmetry. For a particular choice of the supergroup, this GS action reproduces the GS action on \( ADSS_5 \times S^5 \) [11].

Let us turn now to the pure spinor sigma model. The worldsheet action in the pure spinor formulation of the superstring consists of a matter and a ghost sector. The worldsheet metric is in the conformal gauge and there are no reparametrization ghosts. The matter fields are formulated of the superstring consists of a matter and a ghost sector. The worldsheet metric decomposed according to the invariant spaces of the \( \mathbb{Z}_4 \) automorphism (3.8).

The Lie algebra-valued pure spinor fields and their conjugate momenta are defined as
\[ \lambda = \lambda^a T_a, \quad w = w_a \eta^{a\dot{a}} T_{\dot{a}}, \quad \bar{\lambda} = \bar{\lambda}_{\dot{a}} T_{\dot{a}}, \quad \bar{w} = \bar{w}_{\dot{a}} \eta^{a\dot{a}} T_a, \]  
where we decomposed the fermionic generators \( T \) of the super Lie algebra \( \mathcal{G} \) according to their \( \mathbb{Z}_4 \) gradings \( T_a \in \mathcal{H}_1 \) and \( T_{\dot{a}} \in \mathcal{H}_{\bar{1}} \) and used the inverse of the Cartan metric \( \eta^{a\dot{a}} \). The spinor indices here are just a reminder, the unhatted ones refer to left moving quantities, the hatted ones to right moving ones. The pure spinor currents are defined by
\[ N = -\{w, \lambda\}, \quad \bar{N} = -\{\bar{w}, \bar{\lambda}\}, \]  
which generate in the pure spinor variables the Lorentz transformations that correspond to left multiplication by elements of \( H, N, \bar{N} \in \mathcal{H}_0 \) so they indeed act on the tangent-space indices \( a \) and \( \dot{a} \) of the pure spinor variables as the Lorentz transformation. The pure spinor constraint reads
\[ \{\lambda, \lambda\} = 0, \quad \{\bar{\lambda}, \bar{\lambda}\} = 0. \]  

The sigma model should be invariant under the global transformation \( \delta g = \Sigma g, \Sigma \in \mathcal{G} \). \( J \) and \( \bar{J} \) are invariant under this global symmetry. The sigma model should also be invariant under the gauge transformation
\[ \delta \lambda J = \partial \Lambda + [J, \Lambda], \quad \delta \bar{\lambda} = \bar{\partial} \Lambda + [\bar{J}, \Lambda], \quad \delta \lambda \lambda = [\lambda, \Lambda], \quad \delta \bar{\lambda} \bar{w} = [\bar{w}, \Lambda], \quad \delta \Lambda w = [w, \Lambda], \]  
where \( \Lambda \in \mathcal{H}_0 \).

The BRST operator reads
\[ Q = \oint (d \lambda J_3 + d \bar{\lambda} \bar{J}_1), \]  
The gauge-invariant BRST-invariant sigma model takes the form
\[ S = \int d^2 \sigma \left( \frac{1}{2} J_2 J_2 + \frac{1}{4} J_1 J_3 + \frac{3}{4} J_3 J_1 + w \partial \lambda + \bar{w} \partial \bar{\lambda} + N J_0 + \bar{N} \bar{J}_0 - N \bar{N} \right). \]
This result holds for all dimensions and matches the critical $AdS_5 \times S^5$ pure spinor superstring action [12].

Let us briefly comment on the relation between the pure spinor action (3.16) and the GS action (3.10). The latter, when written in conformal gauge, reads

$$S_{GS} = \int d^2z \left( \frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 - \frac{1}{4} J_3 \bar{J}_1 \right).$$  (3.17)

To this one has to add a term which breaks $\kappa$-symmetry and adds kinetic terms for the target-space fermions and coupling to the RR-flux $F_{\alpha\beta}$,

$$S_\kappa = \int d^2z \left( d_\alpha \bar{J}_1 + \bar{d}_\alpha J_1 + F_{\alpha\beta} d_\beta \bar{d}_\beta \right) = \int d^2z \left( d \bar{J}_1 - \bar{d} J_3 + d \bar{d} \right),$$  (3.18)

where, in curved backgrounds, the $d$'s are the conjugate variables to the superspace coordinates $\theta$'s. After integrating out $d$ and $\bar{d}$ we get the complete matter part

$$S_{GS} + S_\kappa = \int d^2z \left( \frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 + \frac{3}{4} J_3 \bar{J}_1 \right).$$  (3.19)

This has to be supplemented with kinetic terms for the pure spinors and their coupling to the background

$$S_{gh} = \int d^2\bar{z} \left( w \bar{\partial} \lambda + \bar{w} \partial \bar{\lambda} + N \bar{J}_0 + \bar{N} J_0 - N \bar{N} \right)$$  (3.20)

in order to obtain the full superstring sigma model (3.16) with action $S = S_{GS} + S_\kappa + S_{gh}$.

3.3. Quantum aspects

The pure spinor action (3.16) is classically gauge invariant under the right multiplication $g \rightarrow gh$, where $h \in H$. In the following we will study the quantum properties. We will show that we can always add a local counterterm such that the quantum effective action remains gauge invariant at the quantum level. Quantum gauge invariance will then be used to prove BRST invariance.

3.3.1. Quantum gauge invariance. An anomaly in the $H$ gauge invariance would show up as a nonvanishing gauge variation of the effective action $\delta_{\Lambda} S_{\text{eff}}$ in the form of a local operator. Since there is no anomaly in the global $H$ invariance, the variation must vanish when the gauge parameter is constant and, moreover, it must have grading zero. Looking at the list of our worldsheet operators, we find that the most general form of the variation is

$$\delta S_{\text{eff}} = \int d^2z \left( c_1 N \bar{\partial} \Lambda + \bar{c}_1 \bar{N} \partial \Lambda + 2c_2 J_0 \bar{\partial} \Lambda + 2\bar{c}_2 \bar{J}_0 \partial \Lambda \right),$$  (3.21)

where $\Lambda = T_{\{ab\}}^{\alpha\beta}(z, \bar{z})$ is the local gauge parameter and $(c_1, \bar{c}_1, c_2, \bar{c}_2)$ are arbitrary coefficients. By adding the counterterm

$$S_\kappa = -\int d^2z \left( c_1 N \bar{J}_0 + \bar{c}_1 \bar{N} J_0 + (c_2 + \bar{c}_2) J_0 \bar{J}_0 \right),$$  (3.22)

we find that the total variation becomes

$$\delta_{\Lambda} (S_{\text{eff}} + S_\kappa) = (c_2 - \bar{c}_2) \int d^2\bar{z} \left( J_0 \bar{\partial} \Lambda - \bar{J}_0 \partial \Lambda \right).$$  (3.23)

On the other hand, the consistency condition on the gauge anomaly requires that

$$\delta_\Lambda S_{\text{eff}} = \delta_{\Lambda} (S_{\text{eff}} + S_\kappa),$$  (3.24)

which fixes the coefficients $c_2 = \bar{c}_2$. Therefore the action is gauge-invariant quantum mechanically.
3.3.2. Quantum BRST invariance. First, we will show that the classical BRST charge is nilpotent. We will then prove that the effective action can be made classically BRST invariant by adding a local counterterm, using triviality of a classical cohomology class. Then we will prove that order by order in perturbation theory no anomaly in the BRST invariance can appear.

As we have shown in the previous section, the action (3.16) in the pure spinor formalism is classically BRST invariant. Also, the pure spinor BRST charge is classically nilpotent on the pure spinor constraint, up to gauge invariance and the ghost equations of motion.

Consider now the quantum effective action $S_{\text{eff}}$. After the addition of a suitable counterterm, it is gauge invariant to all orders. Moreover, the classical BRST transformations commute with the gauge transformations, since the BRST charge is gauge invariant. Therefore, the anomaly in the variation of the effective action, which is a local operator, must be a gauge-invariant integrated vertex operator of ghost number one

$$\delta_{\text{BRST}} S_{\text{eff}} = \int d^2 z \Omega^{(1)}_{a \bar{a}}.$$  

(3.25)

One can show that the cohomology of such operators is empty, namely that we can add a local counterterm to cancel the BRST variation of the action [7, 12]. A crucial step in the proof is that the symmetric bispinor, constructed with the product of two pure spinors, is proportional to the middle-dimensional form (2.5).

Since there are no conserved currents of ghost number two in the cohomology, that could deform $Q^2$, the quantum modifications to the BRST charge can be chosen such that its nilpotence is preserved. In this case, we can set the anti-fields to zero and use algebraic methods to extend the BRST invariance of the effective action by induction to all orders in perturbation theory. Suppose the effective action is invariant to order $h^{n-1}$. This means that

$$\tilde{Q} S_{\text{eff}} = h^n \int d^2 z \Omega^{(1)}_{a \bar{a}} + O(h^{n+1}).$$

The quantum-modified BRST operator $\tilde{Q} = Q + Q_q$ is still nilpotent up to the equations of motion and the gauge invariance. This implies that $Q \int d^2 z \Omega^{(1)}_{a \bar{a}} = 0$. But the cohomology of ghost number one integrated vertex operators is empty, so $\Omega^{(1)}_{a \bar{a}} = Q \Sigma^{(0)}_{a \bar{a}}$, which implies

$$\tilde{Q} \left( S_{\text{eff}} - h^n \int d^2 z \Sigma^{(0)}_{a \bar{a}} \right) = O(h^{n+1}).$$

(3.26)

Therefore, order by order in perturbation theory it is possible to add a counterterm that restores BRST invariance.

3.3.3. Pure spinor beta-functions. Consider the computation of the beta-function in the pure spinor formalism in the background field method [13]. Unlike the light-cone GS formalism, one works covariantly at all stages. The contribution to the one-loop effective action coming from the pure spinor sector consists of two terms. The first term is obtained by expanding the ghost action $\frac{1}{4} \int d^2 z \langle N J_0 + \bar{N} J_0 \rangle$ to the second order in the fluctuations of the gauge current $J_0$. The trilinear couplings

$$\int d^2 z \langle \tilde{N} ([\tilde{a} X_2, X_2] + [\tilde{a} X_1, X_3] + [\tilde{a} X_3, X_1])$$

$$+ \bar{N} ([\bar{a} X_2, X_2] + [\bar{a} X_1, X_3] + [\bar{a} X_3, X_1]) \rangle,$$

(3.27)

(3.28)

generate the term $\langle \tilde{N} \bar{N} \rangle$ in the action

$$\frac{1}{8\pi} \log \mu \tilde{N}^{(ij)} \bar{N}^{(kl)} (4 R_{(ij)(kl)}(G) - 4 R_{(ij)(kl)}(H)).$$

(3.29)
There is a second contribution to the one-loop effective action in the ghost sector, coming from the operator \( \mathcal{O}(z, \bar{z}) = \langle \bar{N} N \rangle \), which couples the pure spinor Lorentz currents to the spacetime Riemann tensor. The marginal part of the OPE of \( \mathcal{O} \) with itself generates at one loop the following contribution to the effective action

\[
\frac{1}{4\pi} \int d^2z \int d^2w \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) = \frac{1}{2\pi} \log \frac{\Lambda}{\mu} R_{(i)(kl]}(H) \int d^2z \tilde{N}^{[ij]} \tilde{N}^{[kl]},
\]

which cancels the term proportional to \( R_{(i)(kl]}(H) \) in (3.29). So we are left with the following ghost contribution to the one-loop effective action in the ghost sector

\[
\frac{1}{2\pi} \log \frac{\Lambda}{\mu} \tilde{N}^{[ij]} \tilde{N}^{[kl]} R_{(i)(kl]}(G),
\]

where the explicit expression of the super Ricci tensor of the supergroup in terms of the structure constants is explained in the appendix. When the supergroup \( G \) is super Ricci flat, each coupling in the effective action vanishes by itself, all of them being separately proportional to the dual Coxeter number of the supergroup \( G \). However, for noncritical superstrings, in which the dual Coxeter number of \( G \) is nonzero the single terms do not vanish separately and to check that the total contribution vanishes.

4. Pure spinor superstrings in various dimensions

In this section we will consider noncritical pure spinor superstrings [14].

4.1. Noncritical superstrings

The critical dimension for the superstrings in flat spacetime is \( d = 10 \). In dimensions \( d < 10 \), the Liouville mode is dynamical, i.e. with \( g_{ab} = \epsilon^\phi \delta_{ab} \), the conformal (Liouville) mode \( \phi \) does not decouple and needs to be quantized as well. These superstrings are sometimes called noncritical. The Liouville mode can be interpreted as a dynamically generated dimension. Thus, if we start with superstring theory in \( d < 10 \) spacetime dimensions, we have effectively \( d + 1 \) spacetime dimensions. The total conformal anomaly vanishes for the noncritical superstrings due to the Liouville background charge. However, while this is a necessary condition for the consistency of noncritical superstrings, it is not a sufficient one.

There are various motivations to study noncritical strings. First, noncritical superstrings can provide an alternative to superstring compactifications. Second, the study of noncritical superstrings in the context of the gauge/string correspondence may provide dual descriptions of new gauge theories, and in particular QCD. In this context, one would like to study backgrounds with warped-type metrics of the form

\[
dx^2 = d\phi^2 + a^2(\phi) dx_i dx^i.
\]

The form of the warp factor \( a^2(\phi) \) determines the type of the dual gauge theory, e.g. with \( a(\phi) \sim e^\phi \) one has a conformal model, while with a warp factor vanishing at a point \( a(\phi = \phi^*) = 0 \), one has a confining one.

A complication in the study of noncritical superstrings in curved spaces is that, unlike the critical case, there is no consistent approximation where supergravity provides a valid effective description. The reason being that the \( d \)-dimensional supergravity low-energy effective action contains a cosmological constant type term of the form

\[
S \sim \int d^dx \sqrt{G} e^{-2\phi} \left( \frac{d - 10}{l_s^2} \right).
\]
which vanishes only for \( d = 10 \). This implies that the low-energy approximation \( E \ll l_s^{-1} \) is not valid when \( d \neq 10 \), and the higher order curvature terms of the form \((l_s^2 R)^n\) cannot be discarded. A manifestation of this is that solutions of the \( d \)-dimensional noncritical supergravity equations have typically curvatures of the order of the string scale \( l_s^2 R \sim O(1) \) when \( d \neq 10 \). An example is the \( AdS_d \) background with \( N \) units of RR \( d \)-form \( F_d \) flux, where one has

\[
l_s^2 R = d - 10, \quad e^{2\phi} = \frac{1}{N^2}, \quad l_s^2 F_d^2 \sim N^2. \tag{4.2}
\]

The examples that we will consider in the following are two types of backgrounds: the linear dilaton background and \( AdS_{d,d} \), for \( d = 1, 2 \).

4.2. Pure spinor spaces in various dimensions

As we noted before, the definition of the pure spinors that Cartan and Chevalley give (2.4) and (2.5) in even dimension \( d = 2n \) implies that in \( d = 2, 4, 6 \) dimensions the pure spinor is an \( SO(d) \) Weyl spinor. In some cases one needs more than just one pure spinor to construct a consistent string theory, since the pure spinor spaces are dictated by the realization of the supersymmetry algebra for the type II superstring [15].

4.2.1. Two-dimensional superstring. The left moving sector of type II superstrings in two dimensions realizes \( N = (2, 0) \) spacetime supersymmetry with two real supercharges \( Q_{\alpha} \), both of which are spacetime MW spinors of the same chirality, which are related by an \( SO(2)R \)-symmetry transformation (\( \alpha \) is not a spinor index in this case, but just enumerates supercharges of the same chirality). The corresponding superderivatives are denoted by \( D_{\alpha} \). The supersymmetry algebra reads

\[
\{D_{\alpha}, D_{\beta}\} = -\delta_{\alpha\beta} P^+, \tag{4.3}
\]

where \( P^\pm \) are the holomorphic (antiholomorphic) spacetime direction of \( AdS_2 \). The pure spinors are defined such that \( \lambda^a D_{\alpha} \) is nilpotent, so that the pure spinor condition in two dimensions reads

\[
\lambda^a \lambda^b \delta_{ab} = 0, \tag{4.4}
\]

which is solved by one Weyl spinor.

4.2.2. Four-dimensional superstring. In four dimensions, the left moving sector of the type II superstring realizes \( N = 1 \) supersymmetry, which in terms of the superderivatives \( D_{\Lambda} \) in the Dirac form reads

\[
\{D_{\Lambda}, D_{\Phi}\} = -2(\Gamma^\mu)_{m \Phi} P_m, \tag{4.5}
\]

where \( C \) is the charge conjugation matrix and \( \Lambda = 1, \ldots, 4 \). Requiring nilpotence of \( \lambda^\Lambda D_{\Lambda} \) specifies the four-dimensional pure spinor constraint

\[
\lambda^\Lambda (\Gamma^\mu)_{AB} \lambda^B = 0. \tag{4.6}
\]

If we use the Weyl notation for the spinors, under which the pure spinor is represented by a pair of Weyl and anti-Weyl spinors \( (\lambda^a, \lambda^\dot{a}) \), subject to the constraint

\[
\lambda^a \lambda^\dot{a} = 0. \tag{4.7}
\]
4.3. Linear dilaton background

The \((d + 2)\)-dimensional linear dilaton background \cite{16}

\[
\mathbb{R}^{1,d-1} \times \mathbb{R}_\phi \times U(1)_x
\]

has a flat metric in the string frame and a linear dilaton

\[
\Phi = \frac{Q}{2} \phi.
\]

The effective string coupling \(g_s = e^{\Phi/2}\) varies as we move along the \(\phi\)-direction and when considering scattering processes one needs to properly regularize the region in which the coupling diverges. We will only consider the weak coupling region \(\phi = -\infty\), where perturbative string computations are valid.

The \((d + 2)\)-dimensional RNS superstring is described in the superconformal gauge by \(2n + 1\) superfields \(X^\mu\), with \(\mu = 1, \ldots, d = 2n\), and \(X\) and by a Liouville superfield \(\Phi_1\). In components we have \(X^\mu = (x^\mu, \psi^\mu)\), \(X = (x, \psi_x)\) and \(\Phi_1 = (\phi, \psi_l)\), where the \(\psi\)'s are Majorana–Weyl fermions.

The \(d = 2n\) coordinates \(x^\mu\) parametrize the even-dimensional flat Minkowski part of the space, while the coordinate \(x\) is compactified on a circle of radius \(R = 2/Q\), whose precise value is dictated by the requirement of spacetime supersymmetry, as we will see below. The coordinate \(\phi\) parametrizes the linear dilaton direction with a background charge \(Q\). As usual, we need to add the superdiffeomorphisms ghosts \((\beta, \gamma)\) and \((b, c)\). The central charge of the system is

\[
c = \left(\frac{3}{2}\right)(2n + 1)\{X^\mu, X_\mu\} + 3\{\Phi_1, \Phi_1\} + (11)\{\phi, \phi\} - (26)\{b, c\}
\]

and the requirement that it vanishes fixes the slope of the dilaton to \(Q(n) = \sqrt{4 - n}\). For \(n = 4\), the background charge vanishes and we have eight flat coordinates plus \(\phi\) and \(x\), getting back to the flat ten-dimensional critical superstring. When \(n \neq 4\) we have noncritical superstrings.

As an example consider the pure spinor superstring in the four-dimensional linear dilaton background. The four-dimensional superstring has \(d + 1 = 3\) noncompact directions \(x^1, x^2, \phi\) and the compact \(U(1)_x\) direction \(x\) with radius \(R = 2/Q\), where \(Q = \sqrt{3}\) is the Liouville background charge. The pure spinor degrees of freedom are given by the pair of Weyl and anti-Weyl spinors \((\lambda^\alpha, \lambda^{\hat{\alpha}})\) satisfying \((4.6)\). The spacetime supersymmetry is of half the maximal, i.e. four supercharges for the closed superstring.

The BRST operator is constructed as

\[
Q = \oint \lambda^\alpha d\alpha + \oint \lambda^{\hat{\alpha}} d\hat{\alpha}
\]

in the left sector and a similar one in the right sector. Note that only half of the superderivatives in this BRST operator correspond to true supersymmetries of the linear dilaton background. This is the reason why the spectrum constructed as the BRST cohomology needs a projection in order to match the RNS one \cite{14}.

4.4. Ramond–Ramond curved backgrounds

4.4.1. AdS2. The type IIA noncritical superstring on AdS2 with RR 2-form flux is realized as the supercoset \(Osp(2|2)/SO(1, 1) \times SO(2)\). The \(Osp(2|2)\) supergroup has four bosonic generators \((E^\pm, H, \tilde{H})\) and four fermionic ones \((Q_\alpha, \tilde{Q}_\dot{\alpha})\). The index \(\alpha = \pm\) denotes the spacetime light-cone directions. The supercharges are real two-dimensional MW spinors, the index \(\alpha = 1, 2\) counts those with left spacetime chirality and the index \(\dot{\alpha} = \hat{1}, \hat{2}\) counts those
with right spacetime chirality (note that in the two-dimensional superstring $\alpha, \hat{\alpha}$ are not spinor indices but just count the multiplicity of spinors with the same chirality). To obtain $AdS_2$, we quotient by $H$ and $\tilde{H}$, which generate respectively the $SO(1, 1)$ and $SO(2)$ transformations. The $Osp(2|2)$ superalgebra and structure constants are listed in the appendix. The left invariant form $J = G^{-1} dG$ is expanded according to the grading as

$$J_0 = J^H H + J^\beta \tilde{H}, \quad J_1 = J^\alpha Q_\alpha, \quad J_2 = J^a \bar{E}_a, \quad J_3 = J^{\dot{a}} \bar{Q}_{\dot{a}}. \quad (4.9)$$

and the definition of the supertrace is

$$\langle E_a E_b \rangle = \delta_a^b + \delta_a^\beta \bar{\delta}_b^\beta, \quad \langle Q_\alpha \bar{Q}_{\dot{\alpha}} \rangle = \delta_{\alpha \dot{\alpha}}, \quad (4.10)$$

whose details are given in the appendix.

The action of the pure spinor sigma model is given by $(3.16)$, where the pure spinor $\beta \gamma$-system is defined according to $(3.11)$. The left and right moving pure spinors $\lambda^\alpha$ and $\tilde{\lambda}^{\dot{\alpha}}$ satisfy the pure spinor constraints $(4.3)$

$$\lambda^\alpha \delta_{\alpha \beta} \lambda^\beta = 0, \quad \tilde{\lambda}^{\dot{\alpha}} \delta^{\dot{\alpha} \dot{\beta}} \tilde{\lambda}^{\dot{\beta}} = 0. \quad (4.11)$$

Note that the naive central charge counting gives the correct result

$$c_{\text{tot}} = (2)_{(4)} + (-4)_{(p, \theta)} + (2)_{(w, \lambda)} = 0. \quad (4.12)$$

### 4.4.2. $AdS_4$

The noncritical type IIA superstring on $AdS_4$ with RR 4-form flux is realized as a sigma model on the $Osp(2|4)/SO(1, 3) \times SO(2)$ supersymmetry. The $Osp(2|4)$ superalgebra and structure constants are discussed in the appendix. The bosonic generators are the translations $P_\mu$, the $SO(1, 3)$ generators $J_{ab}$, for $a, b = 1, \ldots, 4$ and the $SO(2)$ generator $H$. The fermionic generators are the supercharges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$, where $\alpha, \dot{\alpha} = 1, \ldots, 4$ are four-dimensional Majorana spinor indices. We have thus $N = 2$ supersymmetry in four dimensions. The charge assignment of the generators with respect to the $\mathbb{Z}_4$ automorphism of $Osp(2|4)$ can be read from the Maurer–Cartan one forms

$$J_0 = J^{ab} J_{ab} + J^H H, \quad J_1 = J^\alpha Q_\alpha, \quad J_2 = J^a P_a, \quad J_3 = J^{\dot{a}} \bar{Q}_{\dot{a}}. \quad (4.13)$$

The pure spinor sigma model is given by $(3.16)$, where the pure spinor $\beta \gamma$-system is defined according to $(3.11)$. The left and right moving pure spinors $\lambda^\alpha$ and $\tilde{\lambda}^{\dot{\alpha}}$ are four-dimensional Dirac spinors, satisfying the pure spinor constraints $(4.5)$. Note that also here the naive central charge counting gives the correct result

$$c_{\text{tot}} = (4)_{(4)} + (-8)_{(p, \theta)} + (4)_{(w, \lambda)} = 0. \quad (4.14)$$

### 5. Integrability of pure spinor superstrings

A crucial property of sigma models on supercosets $G/H$, where the supergroup $G$ has a $\mathbb{Z}_4$ automorphism, whose invariant locus is $H$, is their classical integrability. In order to exhibit the integrability of the pure spinor sigma models, we have to construct an infinite number of BRST-invariant conserved charges.

The first step in the construction of the charges is to find a one-parameter family of currents $a(\mu)$ satisfying the flatness condition

$$da(\mu) + a(\mu) \wedge a(\mu) = 0. \quad (5.1)$$

One then constructs the Wilson line

$$U(\mu)(x, t; y, t) = \text{P exp} \left( - \int_{(y, t)}^{(x, t)} a(\mu) \right), \quad (5.2)$$

$$17$$
and obtains the infinite set of non-local charges $Q_n$ by expanding

$$U(\mu)(\infty, t; -\infty, t) = 1 + \sum_{n=1}^{\infty} \mu^n Q_n. \quad (5.3)$$

The conservation of $Q_n$ is implied by the flatness of $a(\mu)$.

The first two charges $Q_1$ and $Q_2$ generate the Yangian algebra, which is a symmetry algebra underlying the type II superstrings propagating on the AdS backgrounds with Ramond–Ramond fluxes in various dimensions. Moreover, in the pure spinor formalism one can see that this symmetry also holds at the quantum sigma-model level.

### 5.1. Classical integrability of the pure spinor sigma model

In this subsection we will demonstrate the classical integrability of the action (3.16). We have to distinguish between two cases—a non-Abelian gauge symmetry $H$ and an Abelian one, which occurs only in the two-dimensional noncritical superstrings. We will present the non-Abelian case. The construction of the flat currents in the case of an Abelian gauge group is similar.

The equations of motion of the currents $J_i$ are obtained by considering the variation $\delta g = gX$ under which $\delta J = \partial X + [J, X]$ and using the $\mathbb{Z}_4$ grading and the Maurer–Cartan equations, so that we get

\begin{align*}
\nabla \tilde{J}_3 &= -[J_1, \tilde{J}_2] - [J_2, \tilde{J}_1] + [N, \tilde{J}_3] + [\tilde{N}, J_3], \\
\nabla J_3 &= [N, J_3] + [\tilde{N}, J_3], \\
\nabla J_2 &= -[J_1, J_1] + [N, J_2] + [\tilde{N}, J_2], \\
\nabla J_1 &= [J_3, \tilde{J}_3] + [N, J_2] + [\tilde{N}, J_2], \\
\n\nabla \tilde{J}_1 &= [N, \tilde{J}_1] + [\tilde{N}, J_1], \\
\n\tilde{\nabla} J_1 &= [J_2, J_3] + [J_3, \tilde{J}_2] + [N, \tilde{J}_1] + [\tilde{N}, J_1].
\end{align*}

(5.4)\text{–}(5.9)

where $\nabla J = \partial J + [J_0, J]$ and $\tilde{\nabla} J = \tilde{\partial} J + [\tilde{J}_0, J]$ are the gauge-covariant derivatives. The equations of motion of the pure spinors and the pure spinor gauge currents are

\begin{align*}
\tilde{\nabla} \lambda &= [\tilde{N}, \lambda], \\
\nabla \tilde{\lambda} &= [N, \tilde{\lambda}], \\
\nabla N &= -[N, \tilde{N}], \\
\n\tilde{\nabla} \tilde{N} &= [N, \tilde{N}].
\end{align*}

(5.10)\text{–}(5.11)

We are looking for a one-parameter family of gauge-invariant flat currents $a(\mu)$. The left-invariant current $A = g^{-1}ag$ constructed from the flat current $a$ satisfies the equation

\begin{equation}
\nabla \tilde{A} - \tilde{\nabla} A + [A, \tilde{A}] + \sum_{i=1}^{3} ([J_i, \tilde{A}] + [A, \tilde{J}_i]) = 0.
\end{equation}

(5.12)

$A$ and $\tilde{A}$ can depend on all the currents for which there are equations of motion so

\begin{equation}
A = c_2 J_2 + c_1 J_1 + c_3 J_3 + c_N N, \quad \tilde{A} = \tilde{c}_2 J_2 + \tilde{c}_1 J_1 + \tilde{c}_3 J_3 + \tilde{c}_N \tilde{N}.
\end{equation}

(5.13)
By requiring the coefficients of the currents to satisfy (5.12) one obtains the solutions
\[ c_2 = \mu^{-1} - 1, \quad c_1 = \pm \mu^{-1/2} - 1, \quad c_3 = \pm \mu^{3/2} - 1, \quad \bar{c}_2 = \mu - 1, \]
\[ \bar{c}_1 = \pm \mu^{3/2} - 1, \quad \bar{c}_3 = \pm \mu^{1/2} - 1, \quad c_N = \mu^2 - 1, \quad \bar{c}_N = \mu^2 - 1. \]  
(5.14)

Hence, there exists a one-parameter set of flat currents.

The flat currents are given by the right-invariant versions \( a = g A g^{-1} \) and \( \bar{a} = g \bar{A} g^{-1} \) of the currents \( A \) and \( \bar{A} \) found above. The conserved charges are given by
\[ U_C = P \exp \left[ - \int_C (d\sigma a + d\bar{\sigma} \bar{a}) \right]. \]  
(5.15)

These charges are indeed BRST invariant.

The first two conserved charges can be obtained by expanding \( \mu = 1 + \epsilon \) about \( \epsilon = 0 \). To simplify the notation we will consider the right-invariant currents
\[ j_i = g J_i g^{-1}, \quad \bar{j}_i = g \bar{J}_i g^{-1}, \quad n = g \bar{N} g^{-1}, \quad \bar{n} = g \bar{N} g^{-1}. \]  
(5.16)

Using the expansion in \( \epsilon \) one gets
\[ a = - \left( \frac{1}{2} j_1 + j_2 + \frac{3}{2} j_3 + 2n \right) \epsilon + \left( \frac{1}{2} j_1 + j_2 + \frac{15}{8} j_3 + 3n \right) \epsilon^2 + O(\epsilon^3), \]
\[ \bar{a} = \left( \frac{1}{2} \bar{j}_1 + j_2 + \frac{3}{2} \bar{j}_3 + 2\bar{n} \right) \epsilon + \left( \frac{1}{2} \bar{j}_1 + j_2 + \bar{n} \right) \epsilon^2 + O(\epsilon^3), \]  
(5.17)

whence substitution in (5.15) and using \( U_C = 1 + \sum_{n=1}^\infty \epsilon^n Q_n \) yields
\begin{align*}
Q_1 &= \int_C \left[ dz \left( \frac{1}{2} j_1 + j_2 + \frac{3}{2} j_3 + 2n \right) - dz \left( \frac{3}{2} \bar{j}_1 + j_2 + \frac{1}{2} \bar{j}_3 + 2\bar{n} \right) \right], \\
Q_2 &= - \int_C \left[ dz \left( \frac{3}{8} j_1 + j_2 + \frac{15}{8} j_3 + 3n \right) + dz \left( \frac{3}{8} \bar{j}_1 + j_2 + \frac{1}{8} \bar{j}_3 + \bar{n} \right) \right] \\
&\quad + \int_C \left[ dz \left( \frac{1}{2} j_1 + j_2 + \frac{3}{2} j_3 + 2n \right) \bigg|_{(z,\bar{z})} - dz \left( \frac{3}{2} \bar{j}_1 + j_2 + \frac{1}{2} \bar{j}_3 + 2\bar{n} \right) \bigg|_{(z,\bar{z})} \right] \\
&\quad \times \int_o^{(z,\bar{z})} \left[ dz' \left( \frac{1}{2} j_1 + j_2 + \frac{3}{2} j_3 + 2n \right) \bigg|_{(z',\bar{z}')}, \\
&\quad - dz' \left( \frac{3}{2} \bar{j}_1 + j_2 + \frac{1}{2} \bar{j}_3 + 2\bar{n} \right) \bigg|_{(z',\bar{z}')} \right]. \end{align*}  
(5.19)

The first charge \( Q_1 \) is the local Noether charge. The rest of the conserved charges, which form the Yangian algebra, can be obtained by repetitive commutators of \( Q_2 \).

5.2. Quantum integrability

In this subsection, we will show that the classically conserved nonlocal currents can be made BRST-invariant quantum mechanically. In this way, we prove quantum integrability of our type II superstring theories.

Consider the charge that generates the global symmetry with respect to the supergroup \( \mathcal{G} \),
\[ q = g^\Lambda T_\Lambda = \int d\sigma \, j^\Lambda T_\Lambda, \]  
(5.21)

where \( j^\Lambda \) is the corresponding gauge-invariant current. Since this is a symmetry of the theory, the charge is BRST invariant, so we find \( \epsilon Q j = \partial_\sigma h \), where \( h = h^\Lambda T_\Lambda \) is a certain operator.
of ghost number one and weight zero. Classical nilpotence of the BRST charge implies that $Q\Omega = 0$.

Consider the operator $[h, h] :$, where $: \ldots :$ denotes a BRST-invariant normal ordering prescription. If there exists a ghost number one and weight zero operator $\Omega$, such that

$$Q\Omega = [h, h] :,$$

then there is an infinite number of nonlocal charges which are classically BRST invariant. To prove this, consider the nonlocal operator

$$k := \int^{+\infty}_{-\infty} d\sigma \int^{\sigma}_{-\infty} d\sigma' [j(\sigma), j(\sigma')] :.$$

Its BRST variation is $Qk = 2 \int^{+\infty}_{-\infty} d\sigma \rho(\sigma)$, where $\rho$ denotes a BRST-invariant normal ordering prescription. On the other hand, the BRST transformations are classically nilpotent, in fact we find $Q(2 : [j, h]) : = \partial_\sigma : [h(\sigma), h(\sigma)] :$. Since there is an operator $\Omega$ that satisfies (5.22), we have

$$Q(2 : [j, h] : - \partial_\sigma \Omega) = 0.$$

In other words, the ghost number one weight one operator $2 : [j, h] : - \partial_\sigma \Omega$ is BRST closed.

It remains to be shown that the BRST cohomology of ghost number one currents is trivial. This cohomology, in fact, is equivalent to the cohomology of ghost number two unintegrated vertex operators, by the usual descent relation

$$Q \int d\sigma \sigma^{(2)} = 0 \Rightarrow Q\sigma^{(1)} = \partial_\sigma \sigma^{(2)}.$$

At ghost number two we have only two unintegrated vertex operators that transform in the adjoint of the global supergroup $G$, namely

$$V_1 = g\lambda \bar{\lambda} g^{-1}, \quad V_2 = g\bar{\lambda} \lambda g^{-1}.$$

Their sum is BRST closed, while their difference is not. Finally, we have $V_1 + V_2 = Q\Omega^{(1)}$ where

$$\Omega^{(1)} = \frac{1}{2} g(\lambda + \bar{\lambda}) g^{-1},$$

so this classical cohomology class is empty.

Now, suppose that we have a BRST-invariant nonlocal charge $q$ at order $h^{n-1}$ in perturbation theory, namely $Qq = h^n \Omega^{(1)} + \mathcal{O}(h^{n+1})$. $\Omega^{(1)}$ must be a ghost number one local charge, since any anomaly must be proportional to a local operator. Nilpotence of the quantum BRST charge $Q = Q + Q_q$ implies that $Q\Omega^{(1)} = 0$, but the classical cohomology at ghost number one and weight one is empty, as shown above, so there exists a current $\Sigma^{(0)}(\sigma)$ such that $Q \int d\sigma \Sigma^{(0)}(\sigma) = \Omega^{(1)}$. As a result $Q(\sigma - h^n \int d\sigma \Sigma^{(0)}(\sigma)) = \mathcal{O}(h^{n+1})$. Hence, we have shown that it is possible to modify the classically BRST-invariant charges of (5.14) such that they remain BRST invariant at all orders in perturbation theory.
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Appendix. Superalgebras and supergroups

In this appendix we give some details of the superalgebras discussed in sections 3 and 4 of the paper.

A.1. Notations

The superalgebra satisfies the following commutation relations:

\[ [T_m, T_n] = f^{p}_{mn} T_p \]  \hspace{1cm} (A.1)
\[ [T_m, Q_\alpha] = f^{\beta}_{mn} Q_\beta \]  \hspace{1cm} (A.2)
\[ [Q_\alpha, Q_\beta] = A^{m}_{\alpha\beta} T_m, \]  \hspace{1cm} (A.3)

where the \( T \)'s are the bosonic (Grassman even) generators of a Lie algebra and the \( Q \)'s are the fermionic (Grassman odd) elements. The indices are \( m = 1, \ldots, d \) and \( \alpha = 1, \ldots, D \). The generators satisfy the following super-Jacobi identities:

\[ f^{p}_{mn} f^{r}_{pq} + f^{p}_{rm} f^{q}_{rp} + f^{p}_{mq} f^{q}_{rp} = 0 \]  \hspace{1cm} (A.4)
\[ F^{\gamma}_{ma} F^{\delta}_{b\gamma} - F^{\gamma}_{ma} F^{\delta}_{b\gamma} = 0 \]  \hspace{1cm} (A.5)
\[ F^{\delta}_{my} A^{a}_{\delta\beta} + F^{\delta}_{my} A^{a}_{\delta\beta} - f^{a}_{mp} A^{\beta}_{p\gamma} = 0 \]  \hspace{1cm} (A.6)
\[ A^{p}_{\beta\gamma} F^{\delta}_{pa} + A^{p}_{\gamma\alpha} F^{\delta}_{pa} + A^{p}_{\alpha\beta} F^{\delta}_{pa} = 0. \]  \hspace{1cm} (A.7)

Generally we can define a bilinear form

\[ \langle X_M, X_N \rangle = X_M X_N - (-1)^{g(X_M)g(X_N)} X_N X_M = C^P_{MN} X_P, \]  \hspace{1cm} (A.8)

where \( X \) can be either \( T \) or \( Q \) and \( P = 1, \ldots, d + D \) (say the first \( d \) are \( T \)'s and the rest \( D \) are \( Q \)'s). \( g(X_M) \) is the Grassmann grading, \( g(T) = 0 \) and \( g(Q) = 1 \) and \( C^P_{MN} \) are the structure constants. The latter satisfy the graded antisymmetry property

\[ C^P_{NM} = -(-1)^{g(X_M)g(X_P)} C^P_{MN}, \]  \hspace{1cm} (A.9)

We define the super-metric on the super-algebra as the supertrace of the generators in the fundamental representation

\[ g_{MN} = \text{Str} X_M X_N. \]  \hspace{1cm} (A.10)

We can further define raising and lowering rules when the metric acts on the structure constants

\[ C_{MNP} \equiv g_{MS} C^S_{NP} \]  \hspace{1cm} (A.11)
\[ C_{MNP} = -(-1)^{g(X_M)g(X_P)} C_{MPN} = -(-1)^{g(X_M)g(X_P)} C_{NM} \]  \hspace{1cm} (A.12)
\[ C_{MNP} = -(-1)^{g(X_M)g(X_P)g(X_R)g(X_T)g(X_U)} C_{PNM}. \]  \hspace{1cm} (A.13)
For a semi-simple super Lie algebra \(|g_{MN}| \neq 0\) and \(|h_{mn}| \neq 0\) we can define a contravariant metric tensor through the relation
\[ g_{MP} g_{PN} = \delta^N_M. \]  
(A.14)

The Killing form is defined as the supertrace of the generators in the adjoint representation
\[ K_{MN} \equiv (-1)^{\epsilon(X_M)\epsilon(X_N)} C_{PM} C_{SN} = (-1)^{\epsilon(X_M)\epsilon(X_N)} K_{NM}. \]  
(A.15)

(while on the (sub)Lie-algebra we define the metric \(K_{mn} = f^D_{mp} f^p_{nm}\)). Explicitly we have
\[ K_{mn} = h_{mn} - F^\beta_{ma} F^a_{\beta m} = K_{am}. \]  
(A.16)
\[ K_{a\beta} = F^\gamma_{ma} A^m_{\beta \gamma} - F^\gamma_{\beta m} A^m_{a \gamma} = -K_{\beta a}. \]  
(A.17)
\[ K_{ma} = K_{am} = 0. \]  
(A.18)

The Killing form is proportional to the supermetric up to the second Casimir \(C^2(G)\) of the supergroup, which is also called the dual Coxeter number
\[ K_{MN} = -C^2(G) g_{MN}. \]  
(A.19)

In section 3, we computed the one-loop beta-functions in the background field method. The sum of one-loop diagrams with fixed external lines is proportional to the Ricci tensor \(R_{MN}\) of the supergroup. The super Ricci tensor of a supergroup is defined as
\[ R_{MN}(G) = -\frac{1}{4} f^P_{MQ} f^Q_{NP} (-)^{\epsilon(X_P)}, \]  
(A.20)
and we immediately see that \(R_{MN} = -K_{MN}\); in particular, we can write it as
\[ R_{MN}(G) = \frac{C^2(G)}{4} g_{MN}. \]  
(A.21)

We considered in section 3 and 4 supergroups \(G\) with a \(\mathbb{Z}_4\) automorphism, whose zero locus is denoted by \(H\). The various RR backgrounds we discussed are realized as \(G/H\) supercosets of this kind. The bosonic submanifold is in general \(AdS_p \times S^q\), where the gauge group \(H = SO(1, p - 1) \times SO(q) \times SO(r)\), and the \(SO(r)\) factor corresponds to the non-geometric isometries. The examples we considered are

\[
\begin{array}{ccc|ccc}
G & \text{Algebra} & p & q & r & \#_{\text{any}} & C^2(G) \\
AdS_2 & \text{Osp}(1|2) & 2 & 0 & 0 & 2 & -3 \\
AdS_2 & \text{Osp}(2|2) & 2 & 0 & 2 & 4 & -2 \\
AdS_4 & \text{Osp}(2|4) & 4 & 0 & 2 & 8 & -4 \\
AdS_5 \times S^5 & PSU(2, 2|4) & A(4|4) & 5 & 5 & 0 & 32 \\
\end{array}
\]

The superspace notations will be as follows: the letters \(\{M, N, \ldots\}\) refer to elements of the supergroup \(G\), while \(\{I, J, \ldots\}\) take values in the gauge group \(H\) and finally \(\{A, B, \ldots\}\) refer to elements of the supercoset \(G/H\). The lower case letters denote the bosonic and fermionic components of the superspace indices, while \(\#_{\text{any}}\) is the number of real spacetime supercharges in the background. Then, we can rewrite the super Ricci tensor of the supergroup (A.20) making explicit the \(\mathbb{Z}_4\) grading
\[ R_{AB}(G) = -\frac{1}{4} f^C_{\tilde{A}D} f^D_{BC} (-)^C - \frac{1}{2} f^I_{\tilde{A}D} f^D_{BI} (-)^I. \]  
(A.22)

In particular, its grading two part is
\[ R_{ab}(G) = \frac{1}{4} \left( F^a_{\alpha \beta} F^\beta_{ba} + F^a_{\beta \alpha} F^\alpha_{\beta b} \right) - \frac{1}{2} f^c_{ac} f^c_{b1}. \]  
(A.23)

The Ricci tensor of the supercoset \(G/H\) is given as
\[ R_{AB}(G/H) = -\frac{1}{4} f^C_{\tilde{A}D} f^D_{BC} (-)^C - f^I_{\tilde{A}D} f^D_{BI} (-)^I. \]  
(A.24)
A.2. Osp(2|2)

The Osp(2|2) supergroup corresponds to the superalgebra $C(2)$. It has a bosonic subgroup $Sp(2) \times SO(2)$ and four real fermionic generators transforming in the $4 \oplus 4$ of $Sp(2)$. It consists of the super matrices $M$ satisfying $M^d H M = H$, where

$$
H = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
$$

The superalgebra is obtained by the commutation relations $m^d H + H m = 0$, where we parametrize

$$
m = \begin{pmatrix}
\mathfrak{sl}(2) & a & b \\
-c & d & e \\
f & h & \mathfrak{so}(2) \\
\end{pmatrix},
$$

$$
m^d = \begin{pmatrix}
\mathfrak{sl}(2)' & e & g \\
-f & -c & -d \\
-h & \mathfrak{so}(2)' \\
\end{pmatrix},
$$

so that from the condition $m^d H + H m = 0$ we find

$$
m = \begin{pmatrix}
\mathfrak{sl}(2) & a & b \\
-c & d & e \\
f & h & \mathfrak{so}(2) \\
\end{pmatrix}.
$$

The Cartan basis for the Osp(2|2) superalgebra is given by the following supermatrices. The bosonic generators are

$$
H = \begin{pmatrix}
1 & 0 \\
0 & -1 \\
\end{pmatrix},
E^+ = \begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix},
E^- = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
\end{pmatrix},
\tilde{H} = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\end{pmatrix},
$$

where $(H, E^\pm)$ are the generators of $sl(2)$ while $\tilde{H}$ is the generator of $SO(2)$. The fermionic generators $(Q_\alpha, \tilde{Q}_\alpha)$ are

$$
Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix},
Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
Q_3 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix},
$$

Finally, the Osp(2|2) superalgebra is given by

$$
[H, E^\pm] = \pm 2 E^\pm, \quad [E^+, E^-] = H, \quad [H, \tilde{H}] = 0,
[H, E^\pm] = 0, \quad [\tilde{H}, Q_\alpha] = \epsilon_{\alpha\dot{\beta}} Q_{\dot{\beta}}, \quad [\tilde{H}, \tilde{Q}_{\dot{\alpha}}] = \epsilon_{\alpha\dot{\beta}} \tilde{Q}_{\dot{\beta}}.
$$

$$
[H, Q_\alpha] = Q_\alpha, \quad [H, \tilde{Q}_{\dot{\alpha}}] = -\tilde{Q}_{\dot{\alpha}},
\{Q_\alpha, Q_\beta\} = \frac{1}{2} \delta_{\alpha\dot{\beta}} E^+, \quad \{Q_\dot{\alpha}, Q_{\dot{\beta}}\} = \frac{1}{2} \delta_{\alpha\dot{\beta}} E^-,
\{Q_\alpha, \tilde{Q}_{\dot{\alpha}}\} = \frac{1}{2} \delta_{\alpha\dot{\beta}} H + \frac{1}{2} \epsilon_{\alpha\dot{\beta}} \tilde{H},
\{E^+, Q_\alpha\} = [E^+, Q_\alpha] = 0, \quad [E^+, \tilde{Q}_{\dot{\alpha}}] = -\delta_{\alpha\dot{\beta}} Q_{\dot{\beta}},
[\tilde{E}^-, Q_\alpha] = [\tilde{E}^-, Q_\alpha] = 0, \quad [E^-, Q_{\dot{\alpha}}] = -\delta_{\alpha\dot{\beta}} \tilde{Q}_{\dot{\beta}},
[E^-, \tilde{Q}_{\dot{\alpha}}] = -\delta_{\alpha\dot{\beta}} Q_{\dot{\beta}}.
$$
We classify the generators according to their $\mathbb{Z}_4$ charge

\[
\begin{array}{c|c|c|c|c}
\mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\
\hline
\mathbf{H}, \tilde{\mathbf{H}} & \mathbf{Q}_a & \mathbf{E}^\pm & \mathbf{Q}_a
\end{array}
\]

where $m, n \in \mathbb{Z}_4$.

(A.27)

In the main text, we realize our AdS$_2$ background by quotienting with respect to the grading zero subgroup, namely $SO(1, 1) \times SO(2)$. The structure constants are

\[
f^H_{ml} = \delta^+_{m} \delta^-_{l} - \delta^-_{m} \delta^+_{l}, \quad f^H_{ml} = 0, \\
f^{cl}_{Hm} = 2(\delta^+_{m} \delta^-_{c} - \delta^-_{m} \delta^+_{c}), \quad f^{cl}_{Hm} = 0 \\
F^\alpha_{am} = \delta^+_{a} \delta^+_m, \quad F^\alpha_{am} = \delta^+_{a} \delta^+_m \\
F^\beta_{aH} = \delta^+_{a}, \quad F^\beta_{aH} = -\delta^+_{a} \\
F^\beta_{\tilde{H}a} = \epsilon_{\alpha b} \delta^\beta_{b}, \quad F^\beta_{\tilde{H}a} = \epsilon_{\alpha b} \delta^\beta_{b} \\
A^{m}_{\alpha} = \delta_{a} \delta^+_m, \quad A^{m}_{\alpha} = -\delta_{a} \delta^+_m \\
A^{H}_{\alpha \beta} = A^{H}_{\beta \alpha} = \frac{1}{2} \delta_{\alpha \beta}, \quad A^{\tilde{H}}_{\alpha \beta} = A^{\tilde{H}}_{\beta \alpha} = -\frac{1}{2} \epsilon_{\alpha \beta}.
\]

The metric on the supergroup is

\[
g_{mn} = \delta^+_{m} \delta^+_n - \delta^-_{m} \delta^-_n, \\
g_{\alpha a} = -\eta_{\alpha a} = \delta_{\alpha a}, \quad g_{ij} = 2\delta_{ij},
\]

where $m, n = \pm, i, j = H, \tilde{H}$.

The Osp(1|2) supergroup corresponds to the superalgebra $B(0|1)$. Its bosonic subgroup is $Sp(2)$ and it has two real fermionic generators transforming in the 2 of $Sp(2)$. It can be easily obtained by the one of the Osp(2|2) supergroup by simply dropping the generators $\mathbf{H}$ and $\mathbf{Q}_2, \mathbf{Q}_3$.

A.3. Osp(2|4)

The supergroup Osp(2|4) corresponds to the superalgebra $C(3)$. Its bosonic subgroup is $Sp(4) \times SO(2)$ and it has eight real fermionic generators transforming in the $4 \oplus 4$ of $Sp(4)$. We classify the generators according to their $\mathbb{Z}_4$ charge

\[
\begin{array}{c|c|c|c|c}
\mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\
\hline
\mathbf{J}_{[ab]}, \tilde{\mathbf{J}} & \mathbf{Q}_a & \mathbf{P}_a & \mathbf{Q}_a
\end{array}
\]

where $a = 0, \ldots, 3$ and $\alpha, \tilde{\alpha}$ are four-dimensional Majorana spinor indices. In the main text, we realize our AdS$_3$ background by quotienting with respect to the grading zero subgroup, namely $SO(1, 3) \times SO(2)$. The structure constants are

\[
\begin{aligned}
f^{[cd]}_{a[b]c]d]} &= \frac{1}{2} \delta_{a[b} \delta_{c]}^d, \quad f^{[cd]}_{a[b]c]d]} = -f^{[cd]}_{a[b]c]d]} \\
f^{[ef]}_{[ab][cd]} &= \frac{1}{2} \delta_{[a} \delta_{b]} \delta_{[e} \delta_{c]} \delta_{d]}, \quad f^{[ef]}_{[ab][cd]} = \frac{1}{2} \delta_{[a} \delta_{b]} \delta_{[e} \delta_{c]} \delta_{d]}, \\
F^\beta_{a[a} &= -F^\beta_{a[a} = \frac{1}{2} (Y_{\gamma})^{\alpha}_{a} \delta^\beta_{\alpha}, \quad F^\beta_{a[a} = \frac{1}{2} (Y_{\gamma})^{\alpha}_{a} \delta^\beta_{\alpha} \\
F^\beta_{a[a} &= -F^\beta_{a[a} = \frac{1}{2} (Y_{\gamma})^{\alpha}_{a} \delta^\beta_{\alpha}, \quad F^\beta_{a[a} = \frac{1}{2} (Y_{\gamma})^{\alpha}_{a} \delta^\beta_{\alpha} \\
F^\beta_{\tilde{a}[a]b]} &= \frac{1}{2} (Y_{\gamma})^{\alpha}_{a} \delta^\beta_{\alpha}, \quad F^\beta_{\tilde{a}[a]b]} = \frac{1}{2} (Y_{\gamma})^{\alpha}_{a} \delta^\beta_{\alpha} \\
A^{\alpha}_{\tilde{a}b]} &= (C\gamma^{a})_{\tilde{a}b]}, \quad A^{a}_{\tilde{a}b]} = (C\gamma^{a})_{\tilde{a}b]} \\
A^{\alpha}_{\tilde{a}b]} &= -2(\gamma^{5})_{a} \gamma^\gamma (\tilde{C})_{\gamma \beta}, \quad A^{\alpha}_{\tilde{a}b]} = 2(\gamma^{5})_{a} \gamma^\gamma (\tilde{C})_{\gamma \beta} \\
A^{a}_{\tilde{a}b]} &= -\frac{1}{2} \gamma_5 (\tilde{C})_{a \gamma} (\gamma_{ab})^\gamma_{\beta}, \quad A^{a}_{\tilde{a}b]} = -\frac{1}{2} (\tilde{C})_{a \gamma} (\gamma_{ab})^\gamma_{\beta}.
\end{aligned}
\]

(A.30)
where \( C \) is the charge conjugation matrix of \( SO(1, 3) \). The supermetric is given by

\[
\begin{align*}
\eta_{ab} &= \eta_{a\bar{b}}, \\
g_{a\bar{b}} &= 2C_{a\bar{b}}, \\
g_{[ab][c\bar{d}]} &= \eta_{a[c}\eta_{\bar{d}]b}, \\
g_{\bar{H}\bar{H}} &= 2.
\end{align*}
\]

\[\tag{A.32}\]

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