STRICTLY STRONG A∞-WEIGHTS, BESOV AND SOBOLEV CAPACITIES IN METRIC MEASURE SPACES

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ABSTRACT. This article studies strong A∞-weights in Ahlfors Q-regular and geodesic metric spaces satisfying a weak (1, s)-Poincaré inequality for some 1 < s ≤ Q ≤ ∞. It is shown that whenever max(1, Q − 1) < s ≤ Q, a function u yields a strong A∞-weight of the form w = e^{Qu} if the minimal s-weak upper gradient g_u has sufficiently small \| g_u \|_{L^{s,q}(X,\mu)} norm. Similarly, it is proved that if 1 < Q < p < ∞, then w = e^{Qu} is a strong A∞-weight whenever the Besov p-seminorm \| u \|_{B^p(X)} of u is sufficiently small.

1. Introduction

In this paper (X, d, µ) is a complete and unbounded metric measure space. In addition, we assume that it is Ahlfors Q-regular for some Q > 1. That is, there exists a constant C = c_µ such that, for each x ∈ X and all r > 0,

\[ C^{-1} r^Q ≤ \mu(B(x, r)) ≤ Cr^Q. \]

Furthermore, X is assumed to be geodesic. That is, every pair of points can be joined by a curve whose length is the distance between the points.

We will also assume that (X, d, µ) satisfies a weak (1, s)-Poincaré inequality for some s ∈ (1, Q]. That is, there exist constants C > 0 and λ ≥ 1 such that for all balls B with radius r, all measurable functions u on X and all upper gradients g of u we have

\[ \frac{1}{\mu(B)} \int_B |u - u_B| d\mu ≤ Cr \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} g^s d\mu \right)^{1/s}, \]

where \( \lambda B \) represents the ball concentric with B with radius \( \lambda \) times the radius of B whenever \( \lambda > 0 \), and \( u_E \) denotes the average of u on the measurable set \( E \subset X \) with respect to the measure µ whenever \( 0 < \mu(E) < \infty \). We recall that a nonnegative Borel function g is an upper gradient for a real-valued measurable function u on X if for all rectifiable paths \( \gamma : [0, l_\gamma] \to X \) we have

\[ |u(\gamma(0)) - u(\gamma(l_\gamma))| ≤ \int g ds. \]

Here and throughout the paper the rectifiable curve \( \gamma : [0, l_\gamma] \to X \) is assumed to be parametrized by the arc length ds, where \( l_\gamma \) is the length of \( \gamma \).

We study sufficient conditions under which we get strong A∞-weights in X. A non-trivial doubling measure \( \nu \) on X is a Radon measure for which there exists a constant \( C > 1 \) such that

\[ 0 < \nu(2B) ≤ C \nu(B) \]

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for all balls $B$.

To every doubling measure $\nu$ on $X$ we can associate a quasidistance on $X$ defined by

$$
\delta_\nu(x, y) = \nu(B_{xy})^{1/Q},
$$

where $B_{xy} := B(x, d(x, y)) \cup B(y, d(y, x))$. To say that $\delta_\nu(x, y)$ is a quasidistance means by definition that $\delta_\nu : X \times X \to [0, \infty)$ is symmetric, vanishes if and only if $x = y$, and satisfies

$$
\delta_\nu(x, z) \leq C(\delta_\nu(x, y) + \delta_\nu(y, z))
$$

for some $C \geq 1$ and all $x, y, z \in X$. If (4) was satisfied with $C = 1$, then the quasidistance $\delta_\nu$ would in fact be a distance function.

We call $\nu$ a metric doubling measure if the quasidistance $\delta_\nu$ is comparable to a distance $\delta'_\nu$; that is, there exists a distance function $\delta'_\nu$ on $X$ and a constant $C > 0$ such that

$$
C^{-1}\delta'_\nu(x, y) \leq \delta_\nu(x, y) \leq C\delta'_\nu(x, y)
$$

for all $x, y \in X$.

We say that a nonnegative function $w \in L^1_{\text{loc}}(X)$ is an $A_p$-weight with respect to the measure $\mu$ for some $1 < p < \infty$ and we write $w \in A_p(\mu)$ if there exists a constant $C \geq 1$ such that

$$
\left(\frac{1}{\mu(B)} \int_B w(x)^{-1/(p-1)} d\mu(x)\right)^{p-1} \frac{1}{\mu(B)} \int_B w(x) d\mu(x) \leq C
$$

for all balls $B \subset X$. We say that $w$ is a $A_\infty$-weight with respect to the measure $\mu$ and we write $w \in A_\infty(\mu)$ if $w$ is an $A_p$-weight with respect to $\mu$ for some $p$ in $(1, \infty)$. That is,

$$
A_\infty(\mu) = \cup_{p > 1} A_p(\mu).
$$

We define $w$ to be a strong $A_\infty$-weight if it is the density of a metric doubling measure $\nu$ and moreover, it is an $A_\infty$-weight with respect to $\mu$. That is,

$$
d\nu(x) = w(x) d\mu(x)
$$

where $w \in A_\infty(\mu)$ and $\nu$ is a metric doubling measure.

Strong $A_\infty$-weights in $\mathbb{R}^n$ were introduced in the early 90’s by David and Semmes in [DS] and [Sem] when trying to identify the subclass of $A_\infty$-weights that are comparable to the Jacobian determinants of quasiconformal mappings.

**Question 1.1.** In the Euclidean setting, metric doubling measures have densities that are $A_\infty$-weights. (See [Sem].) An open question in the metric setting is whether or not metric doubling measures necessarily have $A_\infty$-densities.

In the last few years strong $A_\infty$-weights were studied by Bonk, Heinonen, and Saksman in [BHS1] and [BHS2] and by the author in [Cos1].

In the Euclidean setting Bonk and Lang proved in [BL] that if $\nu$ is a signed Radon measure on $\mathbb{R}^2$ such that $\nu^+(\mathbb{R}^2) < 2\pi$ and $\nu^-(\mathbb{R}^2) < \infty$, then $(\mathbb{R}^2, D_\nu)$ is bi-Lipschitz equivalent to $\mathbb{R}^2$ endowed with the Euclidean metric, where

$$
D_\nu(x, y) = \inf \left\{ \int_\alpha e^u ds : \alpha \text{ analytic curve connecting } x, y \right\},
$$

$u$ is a solution of $-\Delta u = \nu$ with $|\nabla u| \in L^2(\mathbb{R}^2)$, and $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of $\nu$. In particular, it is proved that $w = e^{2u}$ is comparable to the
Jacobian of a quasiconformal mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), which implies that \( w \) is a strong \( A_\infty \)-weight.

Here we prove a result in \((X, d, \mu)\), related to [Cos1, Theorem 5.1] and to the result from [BL]. It states that \( A_\infty \)-weights of the form \( w = e^{Qu} \) are strong \( A_\infty \)-weights if \( u \) is a locally integrable function that has an upper gradient \( g \) in the Morrey space \( \mathcal{L}^{s,Q-s}(X, \mu) \) with small \( \| \cdot \|_{\mathcal{L}^{s,Q-s}(X, \mu)} \) norm for some \( s > 1 \) lying in \((Q - 1, Q]\).

We say that for \( 1 \leq s \leq Q \), the Morrey space \( \mathcal{L}^{s,Q-s}(X, \mu) \) is defined to be the linear space of locally \( \mu \)-integrable functions \( u \) on \( X \) such that

\[
\|u\|_{\mathcal{L}^{s,Q-s}(X, \mu)} = \sup_{x \in X} \sup_{r > 0} \left( r^{s-Q} \int_{B(x,r)} |u(y)|^s \, d\mu(y) \right)^{1/s}. 
\]

In particular \( \mathcal{L}^{Q,0}(X, \mu) = L^Q(X) \). We refer to [Gia, p. 65] for more information about Morrey spaces in the Euclidean setting and their use in the theory of partial differential equations.

If \((X, d, \mu)\) is an Ahlfors \( Q \)-regular metric space with \( Q > 1 \) satisfying a weak \((1, s)\)-Poincaré inequality for some \( s \in (1, Q]\), it follows from (1) that there exists a constant \( C \) depending only on \( s \) and on data of \( X \) such that

\[
[u]_{\text{BMO}(X)} \leq C \|g\|_{\mathcal{L}^{s,Q-s}(X, \mu)}
\]

whenever \( g \) is an upper gradient of \( u \). Here and throughout the paper \([u]_{\text{BMO}(X)}\) is the \textit{bounded mean oscillation} seminorm that measures the oscillation of \( u \) on balls in \( X \), given by

\[
[u]_{\text{BMO}(X)} = \sup_{a \in X} \sup_{r > 0} \frac{1}{\mu(B(a, r))} \int_{B(a,r)} |u(x) - u_{B(a,r)}| \, d\mu(x).
\]

In [BHS1, Theorem 3.1] the authors prove that if \( u \) belongs to the Bessel potential space \( L^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n), 0 < \alpha < n \), then \( w = e^{hu} \) is a strong \( A_\infty \)-weight with data depending only on \( \alpha, n \), and the \( L^{\alpha,\frac{n}{\alpha}} \)-norm of \( u \). Here we prove a result similar to [BHS1, Theorem 3.1] and [Cos1, Theorem 5.2]. This result yields strong \( A_\infty \)-weights of the form \( w = e^{Qu} \) when \( u \) has small Besov \( p \)-seminorm, \( 1 < Q < p < \infty \).

For \( 1 < Q < p < \infty \) we define

\[
B_p(X) = \{ u \in L^p(X) : \|u\|_{B_p(X)} < \infty \},
\]

where

\[
\|u\|_{B_p(X)} = \|u\|_{L^p(X)} + [u]_{B_p(X)}
\]

with

\[
[u]_{B_p(X)} = \left( \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} \, d\mu(x) \, d\mu(y) \right)^{1/p}.
\]

The expressions \( \|u\|_{B_p(X)} \) and \([u]_{B_p(X)}\) from (8) and (9) are called the \textit{Besov} \( p \)-\textit{norm} and the \textit{Besov} \( p \)-\textit{seminorm} of \( u \) respectively. If \((X, d, \mu)\) is Ahlfors \( Q \)-regular, there exists a constant \( C \) depending on \( p \) and on the data of \( X \) such that

\[
[u]_{\text{BMO}(X)} \leq C[u]_{B_p(X)}
\]

whenever \( u \in L^1_{\text{loc}}(X) \).

Besov spaces have been studied in the last decades by Jonsson and Wallin in [JW], by Fukushima and Uemura in [FU], by Xiao in [Xia], and by the author in [Cos1] and [Cos2]. Recently they have been used in the study of quasiconformal mappings in metric spaces and in geometric group theory. See [Bou] and [BP].
Capacities associated with Besov spaces were studied by Neterov in [Net1] and [Net2], by Adams and Hurri-Syrjänen in [AHS], by Adams and Xiao in [AX1] and [AX2], and by the author in [Cos1]. Bourdon in [Bou] and the author in [Cos2] studied Besov $p$-capacity in metric settings.

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2. **Preliminaries**

In this section we recall standard definitions and results. The open ball with center $x \in X$ and radius $r > 0$ is denoted $B(x, r) = \{ y \in X : d(x, y) < r \}$, the closed ball by $\overline{B}(x, r) = \{ y \in X : d(x, y) \leq r \}$, and the sphere by $S(x, r) = \{ y \in X : d(x, y) = r \}$. Throughout this paper, $C$ will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. $C(a, b, \ldots)$ is a constant that depends only on the parameters $a, b, \ldots$. Here $\Omega$ will denote a nonempty open subset of $X$. For $E \subset X$, the closure and the complement of $E$ with respect to $X$ will be denoted by $\overline{E}$ and $X \setminus E$ respectively; diam $E$ is the diameter of $E$ with respect to the metric $d$ and $E \subset\subset F$ means that $\overline{E}$ is a compact subset of $F$.

For a measurable $u : \Omega \to \mathbb{R}$, supp $u$ is the smallest closed set such that $u$ vanishes on the complement of supp $u$. We also use the spaces

\[
\text{Lip}(\Omega) = \{ \varphi : \Omega \to \mathbb{R} : \varphi \text{ is Lipschitz} \},
\]

\[
\text{Lip}_0(\Omega) = \{ \varphi : \Omega \to \mathbb{R} : \varphi \text{ is Lipschitz and supp } \varphi \subset\subset \Omega \}.
\]

2.1. **Newtonian spaces.** We introduce now some definitions and known results about Newtonian spaces to be used in this paper. Let $1 \leq s < \infty$. The $s$-modulus of a family of paths $\Gamma$ in $X$ is the number

\[
\inf \rho \int_X \rho^s \, d\mu,
\]

where the infimum is taken over all non-negative Borel measurable functions $\rho$ such that for all rectifiable paths $\gamma$ which belong to $\Gamma$ we have

\[
\int_\gamma \rho \, ds \geq 1.
\]

It is known that the $s$-modulus is an outer measure on the collection of all paths in $X$.

A property is said to hold for $s$-almost all paths, if the set of paths for which the property fails is of zero $s$-modulus. If (2) holds for $s$-almost all paths $\gamma$, then $g$ is said to be a $s$-weak upper gradient of $u$. We could have stated the definition of the weak $(1, s)$-Poincaré inequality by requiring the inequality (1) to hold for all $s$-weak upper gradients of $u$. (See [KoM].) Similarly we can define weak $(q, s)$-Poincaré inequalities for $q > 1$.

*Without further notice, we assume that $1 < s < \infty.$* We define the space $\widetilde{N}^{1,s}(X)$ to be the collection of all the functions $u$ that are $s$-integrable and have a $s$-integrable $s$-weak upper gradient $g$. This space is equipped with the norm

\[
\|u\|_{\widetilde{N}^{1,s}(X)} = \left( \|u\|_{L^s(X)}^s + \inf \|g\|_{L^s(X)}^s \right)^{1/s},
\]
where the infimum is taken over all \( s \)-weak upper gradients of \( u \). The \textit{Newtonian space} on \( X \) is the quotient space
\[
N^{1,s}(X) = \tilde{N}^{1,s}(X)/\sim
\]
with the norm \( ||u||_{N^{1,s}(X)} = ||u||_{\tilde{N}^{1,s}(X)} \), where \( u \sim v \) if and only if \( ||u - v||_{\tilde{N}^{1,s}(X)} = 0 \). For basic properties of the Newtonian spaces we refer to [Sha1]. Cheeger in [Che] gives an alternative definition which leads to the same space when \( 1 < s < \infty \). For future reference we recall some known facts (see [KiM] and [Sha2]):

(i) The functions in \( \tilde{N}^{1,s}(X) \) are defined outside a path family of \( s \)-modulus zero. This implies that the functions in \( \tilde{N}^{1,s}(X) \) cannot be changed arbitrarily on sets of measure zero.

(ii) If \( 1 < s < \infty \), every function \( u \) that has a \( s \)-integrable \( s \)-weak upper gradient has in fact a minimal \( s \)-integrable \( s \)-weak upper gradient in \( X \), denoted by \( g_u \), in the sense that if \( g \) is another \( s \)-weak upper gradient of \( u \), then \( g_u \leq g \mu \)-a.e. in \( X \).

(iii) For every \( c \in \mathbb{R} \) the minimal \( s \)-weak upper gradient satisfies \( g_u = 0 \) \( \mu \)-a.e. on the set \( \{ x \in X : u(x) = c \} \).

(iv) If \( u \in N^{1,s}(X) \) and \( v \) is a bounded Lipschitz continuous function, then \( uv \in N^{1,s}(X) \) and \( g_{uv} \leq |u|g_u + |v|g_u \mu \)-a.e.

We emphasize that these properties hold without any additional assumptions on the measure \( \mu \) and on the space \( X \).

The \( s \)-capacity of a set \( E \subset X \) is defined by (see [BBS])
\[
C_s(E) = \inf_u ||u||_{N^{1,s}(X)},
\]
where the infimum is taken over all functions \( u \in N^{1,s}(X) \) whose restriction on \( E \) is bounded below by 1. A property is said to hold \textit{s-quasieverywhere} (or \textit{s-q.e.}), if it holds everywhere except on a set of \( s \)-capacity zero. A function is \textit{s-quasicontinuous}, if there is an open set of arbitrarily small \( s \)-capacity such that the function is continuous when restricted to the complement of the set. Every function in \( \tilde{N}^{1,s}(X) \) is defined \( s \)-quasieverywhere. Moreover, if \( u, v \in N^{1,s}(X) \) and \( u = v \mu \)-a.e., then \( u \) and \( v \) belong to the same equivalence class in \( N^{1,s}(X) \).

We introduce the notion of a local Newtonian space as follows.

**Definition 2.1.** We say that \( u \) belongs to the \textit{local Newtonian space} \( N^{1,s}_{loc}(X) \) if \( u \in N^{1,s}(\Omega) \) for every open set \( \Omega \subset X \). If \( u \in N^{1,s}_{loc}(X) \) with \( 1 < s < \infty \), then \( u \) has a minimal \( s \)-weak upper gradient \( g_u \) in \( X \) in the following sense: if \( \Omega \subset X \) is an open set and \( g \) is the minimal upper gradient of \( u \) in \( \Omega \), then \( g_u = g \mu \)-a.e. in \( \Omega \).

From now on throughout the rest of the paper we assume that the measure \( \mu \) is Borel and Ahlfors \( Q \)-regular for some \( Q > 1 \). Furthermore we assume that the space supports a weak \((1, s)\)-Poincaré inequality for some \( 1 < s \leq Q \). We recall a few useful properties of Newtonian spaces that hold under these additional assumptions (see [BBS] and [KiM]):

(i) The space \( X \) is proper (that is, closed and bounded sets are compact).

(ii) Lipschitz functions are dense in \( N^{1,s}(X) \) and Lipschitz functions which vanish in the complement of an open set \( \Omega \) are dense in \( N^{1,s}_{loc}(\Omega) \), where
\[
N^{1,s}_{loc}(\Omega) = \{ u \in N^{1,s}(X) : u = 0 \text{ s-q.e. in } X \setminus \Omega \}.
\]

(iii) Every function in \( N^{1,s}(X) \) is \( s \)-quasicontinuous.
Now we introduce the relative Sobolev $s$-capacity as in [Cos3]. See also [Bjo].

**Definition 2.2.** Let $1 < s, Q < \infty$. Suppose $(X, d, \mu)$ is a proper and unbounded Ahlfors $Q$-regular metric space that satisfies a weak $(1, s)$-Poincaré inequality. Let $\Omega \subset X$ be open. For $E \subset \Omega$ we let

$$A(E, \Omega) = \{u \in N_0^{1,s}(\Omega) : u \geq 1 \text{ in a neighborhood of } E\}.$$ 

We call $A(E, \Omega)$ the set of admissible functions for the condenser $(E, \Omega)$. The relative $s$-capacity of the pair $(E, \Omega)$ is defined by

$$\text{cap}_s(E, \Omega) = \inf \left\{ \int_{\Omega} g_s \, d\mu : u \in A(E, \Omega) \right\}.$$ 

### 2.2. Besov spaces and capacities.

Now we introduce some definitions and results about Besov spaces and capacities to be used in this paper. We follow [Cos2]. See also [Cos1].

Let $1 < Q < p < \infty$ be fixed. Suppose $(X, d, \mu)$ is an Ahlfors $Q$-regular metric space. For an open set $\Omega \subset X$ we define

$$B_p(\Omega) = \{u \in B_p(X) : u = 0 \ \mu\text{-a.e. in } X \setminus \Omega\},$$

where $B_p(X)$ is defined as in (7). For a function $u \in B_p(\Omega)$ we let

$$||u||_{B_p(\Omega)} = ||u||_{B_p(X)} \text{ and } [u]_{B_p(\Omega)} = [u]_{B_p(X)}.$$

We notice that $\text{Lip}_0(\Omega) \subset B_p(\Omega)$ when $1 < Q < p < \infty$. We define $B^0_p(\Omega)$ as the closure of $\text{Lip}_0(\Omega)$ in $B_p(\Omega)$ with respect to the Besov $p$-norm. It has been proved in [Cos2] that $B_p(X)$, $B_p(\Omega)$, and $B^0_p(\Omega)$ are reflexive spaces. (See [Cos2, Lemma 3.1] and the discussion before [Cos2, Lemma 3.4].)

The Besov $p$-capacity of a set $E \subset X$ is defined by (see [Cos2])

$$\text{Cap}_{B_p}(E) = \inf\{||u||_{L^p(X)}^p + [u]_{B_p(X)}^p : u \in B^0_p(X) \},$$

where the infimum is taken over all functions $u \in B_p(X)$ that are bounded from below by 1 in an open neighborhood of $E$. A property is said to hold Besov $p$-quasieverywhere (or simply $B_p$-q.e.), if it holds everywhere except a set of Besov $p$-capacity zero. A locally integrable function $u$ is called $B_p$-quasicontinuous if there exists an open set of arbitrarily small Besov $p$-capacity such that $u$ is continuous when restricted to the complement of the set.

**Remark 2.3.** It has been shown in [Cos2] that if $u \in B_p(X)$, then there exists a $B_p$-quasicontinuous function $v$ such that $u = v \ \mu$-a.e. Such a function $v$ is called a quasicontinuous representative of $u$. In addition, we can choose $v$ to be Borel. Moreover, two such quasicontinuous representatives agree in fact $B_p$-q.e. Similar statements were proved if $u \in L^1_{\text{loc}}(X)$ with $[u]_{B_p(X)} < \infty$. (See [Cos2, Section 5].)

Suppose $\Omega \subset X$ is open. For $E \subset \Omega$ the relative Besov $p$-capacity of the condenser $(E, \Omega)$ is defined by (see [Cos2])

$$\text{cap}_{B_p}(E, \Omega) = \inf\{[u]_{B_p(\Omega)}^0 : u \in B^0_p(\Omega) \text{ and } u \geq 1 \text{ in a neighborhood of } E\}.$$
3. Main results

In this section we present the results about strong $A_\infty$-weights. We prove the following theorems.

**Theorem 3.1.** Let $1 < s \leq Q < \infty$ be fixed. We assume that $s > Q - 1$. Suppose $(X, d, \mu)$ is an Ahlfors $Q$-regular and geodesic unbounded metric space satisfying a weak $(1, s)$-Poincaré inequality. Let $u \in N^1_{loc}(X)$ be such that it has a minimal $s$-weak upper gradient $g_u$ in the Morrey space $L^{s, Q-s}(X, \mu)$. There exists a constant $\varepsilon > 0$ depending only on $s$ and on the data of $X$ such that if $\|g_u\|_{L^{s, Q-s}(X, \mu)} < \varepsilon$, then $w = e^{Qu}$ is a strong $A_\infty$-weight with data depending only on $s$ and on the data associated with $X$.

**Theorem 3.2.** Let $1 < s < Q < p < \infty$ be fixed. Suppose $(X, d, \mu)$ is an Ahlfors $Q$-regular and geodesic unbounded metric space satisfying a weak $(1, s)$-Poincaré inequality. Let $u \in L^1_{loc}(X)$ be such that $[u]_{B_p(X)} < \infty$. There exists a constant $\varepsilon > 0$ depending only on $p$ and on the data of $X$ such that if $[u]_{B_p(X)} < \varepsilon$, then $w = e^{Qu}$ is a strong $A_\infty$-weight with data depending only on $p$ and on the data associated with $X$.

For $r \in (0, \infty)$ we define the Hausdorff $r$-content of a set $E \subset X$ by

$$\Lambda_r^\infty(E) = \inf \{ \sum_i \text{diam}(G_i)^r : E \subset \bigcup_i G_i \}.$$ 

where the infimum is taken over all coverings of $E$ by open sets $G_i$.

The following lemma is a generalization of [BHS1, Lemma 3.11]. We again thank Mario Bonk for his contribution to this result.

**Lemma 3.3.** Suppose $(X, d, \mu)$ is a proper and unbounded geodesic Ahlfors $Q$-regular metric space admitting a weak $(1, Q)$-Poincaré inequality for some $1 < Q < \infty$. Suppose $0 < r \leq 1$. Let $x, y \in X$ and let $E \subset X$ be a bounded Borel set. Suppose that $B_1, \ldots, B_k$ are open balls such that $x \in B_1, y \in B_k$ and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, k - 1$. Then there exists constants $c_1$ and $C$ depending on $r$ and on the data of $X$ with the following property: if

(11) $$\Lambda_r^\infty(E) \leq c_1 d(x, y)^r,$$
then

(12) $$\sum_{i \in G_0} \text{diam}(B_i)^r > \frac{1}{(20C)^r} d(x, y)^r,$$

where

(13) $$G_0 = \left\{ i = 1, \ldots, k : \mu(E \cap B_i) \leq \frac{1}{2} \mu(B_i) \right\}.$$ 

Proof. Since $X$ is Ahlfors $Q$-regular, proper and geodesic, it follows that it is also locally linearly connected. That is, there exists a constant $C \geq 2$ such that every pair of points in $B(x, R)$ can be joined by a rectifiable path in $B(x, CR)$ and every pair
of points in $X \setminus B(x, R)$ can be joined by a continuum in $X \setminus B(x, R/C)$. (See [HeK, Section 3] and [Hei, Sections 8,9].)

We choose a family $\mathcal{I} \subset \{1, \ldots, k\}$ such that

$$CB_i \cap CB_j = \emptyset$$

whenever $i \neq j \in \mathcal{I}$ and

$$\bigcup_{i=1}^{k} CB_i \subset \bigcup_{i \in \mathcal{I}} 5CB_i,$$

where $C$ is the constant associated with the locally linear connectivity of $X$. (See [Hei, Theorem 1.16].) For every $i = 1, \ldots, k - 1$ let $x_i \in B_i \cap B_{i+1}$. We let $x_0 = x$ and $x_k = y$. Since $X$ is locally linearly connected, we have that for every $i = 1, \ldots, k$ there exists a rectifiable path $\gamma_i$ in $CB_i$ connecting $x_{i-1}$ and $x_i$. This yields a rectifiable path $\gamma \in \bigcup_{i=1}^{k} CB_i$ connecting $x$ and $y$ and therefore

$$\Lambda_{\infty}^r(\bigcup_{i \in \mathcal{I}} 5CB_i) \geq \Lambda_{\infty}^r(\bigcup_{i=1}^{k} CB_i) \geq d(x, y)^r. \quad (14)$$

We can assume without loss of generality that $\Lambda_{\infty}^r(E) > 0$. Let $(D_j)_{j \in \mathcal{J}}$ be a countable covering by open balls for $E$ such that

$$\frac{1}{5} D_i \cap \frac{1}{5} D_j = \emptyset$$

whenever $i \neq j \in \mathcal{J}$ (see [Hei, Theorem 1.16]) and such that

$$\sum_{j \in \mathcal{J}} \operatorname{diam}(D_j)^r < 2^{r+1} \Lambda_{\infty}^r(E).$$

For every $i \in \mathcal{I}$ we define

$$\mathcal{F}_i = \{j \in \mathcal{J} : D_j \cap CB_i \neq \emptyset\}.$$

We denote

$$\mathcal{G} = \{i \in \mathcal{I} : \operatorname{diam}(D_j) \leq \operatorname{diam}(CB_i) \text{ for all } j \in \mathcal{F}_i\} \text{ and } \mathcal{B} = \mathcal{I} \setminus \mathcal{G}.$$  

Suppose $i \in \mathcal{B}$. Then there exists $j = j_i \in \mathcal{F}_i$ such that

$$D_{j_i} \cap CB_i \neq \emptyset \text{ and } \operatorname{diam}(D_{j_i}) > \operatorname{diam}(CB_i).$$

We notice that

$$5CB_i \subset 14D_{j_i} \text{ and } \operatorname{diam}(5CB_i) < 14 \operatorname{diam}(D_{j_i}).$$

Therefore

$$\Lambda_{\infty}^r(\bigcup_{i \in \mathcal{B}} 5CB_i) \leq \sum_{j \in \mathcal{J}} \operatorname{diam}(14D_j)^r \leq 28^r \sum_{j \in \mathcal{J}} \operatorname{diam}(D_j)^r < 2^{r+1} 28^r \Lambda_{\infty}^r(E). \quad (15)$$

We let

$$\mathcal{G}_1 = \{i \in \mathcal{G} : \sum_{j \in \mathcal{F}_i} \operatorname{diam}(D_j)^r < c_0 \operatorname{diam}(CB_i)^r\} \text{ and } \mathcal{G}_2 = \mathcal{G} \setminus \mathcal{G}_1$$

for some $c_0$ to be chosen later. We want to evaluate

$$\sum_{i \in \mathcal{G}_2} \operatorname{diam}(5CB_i)^r.$$
Before we do that, we notice that there exists a number $M$ depending only on the data of $X$ such that every ball $D_j$ intersects at most $M$ pairwise disjoint balls $CB_i$ of bigger diameter. Therefore

\begin{equation}
\Lambda_r^\infty\left(\bigcup_{i \in \mathcal{G}_2} 5CB_i\right) \leq \sum_{i \in \mathcal{G}_2} \text{diam}(5CB_i)^r \leq 10^r \sum_{i \in \mathcal{G}_2} \text{diam}(CB_i)^r
\end{equation}

\begin{align*}
& \leq c_0^{-1} 10^r \sum_{i \in \mathcal{G}_2} \left(\sum_{j \in \mathcal{F}_i} \text{diam}(D_j)^r\right) \\
& \leq c_0^{-1} 10^r \sum_{i \in \mathcal{G}_2} \left(\sum_{j \in \mathcal{F}_i} \text{diam}(D_j)^r\right) \\
& \leq c_0^{-1} M 10^r \sum_{j \in \mathcal{F}} \text{diam}(D_j)^r < c_0^{-1} M 2^{r+1} 10^r \Lambda_r^\infty(E).
\end{align*}

We show now that if $c_0$ is taken small enough, then

\[ \mu(E \cap B_i) \leq \frac{1}{2} \mu(B_i) \text{ for every } i \in \mathcal{G}_1. \]

Indeed, for all $i \in \mathcal{G}_1$ we have

\[ \mu(E \cap B_i) \leq \mu\left(\bigcup_{j \in \mathcal{F}_i} D_j \cap CB_i\right) \leq \sum_{j \in \mathcal{F}_i} \mu(D_j) \leq \frac{c_\mu}{c_\mu} \sum_{j \in \mathcal{F}_i} \text{diam}(D_j)^Q \]

\[ \leq c_\mu \text{diam}(CB_i)^{Q-r} \sum_{j \in \mathcal{F}_i} \text{diam}(D_j)^r \leq c_0 c_\mu \text{diam}(CB_i)^Q. \]

So, if we let $c_0 = \frac{1}{2} c_\mu^{-2}(2C)^{-Q}$, we get

\[ \mu(E \cap B_i) \leq \frac{1}{2} \mu(B_i) \text{ for every } i \in \mathcal{G}_1. \]

From (14), (15), (16), the subadditivity of $\Lambda_r^\infty$, and the fact that $\mathcal{I} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{B}$, it follows that

\[ d(x, y)^r \leq \Lambda_r^\infty(\bigcup_{i \in \mathcal{I}} 5CB_i) < \sum_{i \in \mathcal{G}_1} \text{diam}(5CB_i)^r + 2^{r+1} (28^r + c_0^{-1} M 10^r) \Lambda_r^\infty(E). \]

If we choose $c_1$ such that $2^{r+1} (28^r + c_0^{-1} M 10^r) c_1 = 1 - 2^{-r}$, then we notice that

\[ (10C)^r \sum_{i \in \mathcal{G}_1} \text{diam}(B_i)^r \geq \sum_{i \in \mathcal{G}_1} \text{diam}(5CB_i)^r > 2^{-r} d(x, y)^r \]

whenever $\Lambda_r^\infty(E) < c_1 d(x, y)^r$. Since $\mathcal{G}_1 \subset \mathcal{G}_0$, this finishes the proof. \hfill \Box

**Lemma 3.4.** Suppose $1 < s, Q < \infty$. Suppose that $(X, d, \mu)$ is a complete and unbounded Ahlfors $Q$-regular metric measure space that satisfies a weak $(1, s)$-Poincaré inequality. Let $\Omega \subset X$ be open and let $E \subset \Omega$. Suppose $u \in N_0^{1,s}(\Omega)$ is compactly supported in $\Omega$. If $u \geq 1$ on $E$, then

\[ \text{cap}_s(E, \Omega) \leq \int_\Omega q_u^s d\mu. \]

**Proof.** Since $u \in N_0^{1,s}(\Omega)$ is compactly supported in $\Omega$, there exists a sequence $\varphi_j \in \text{Lip}_0(\Omega)$ converging to $u$ in $N^{1,s}(X)$. Without loss of generality we can assume that all the functions $\varphi_j$ are supported in an open set $U \subset \subset \Omega$ and that the sequence $\varphi_j$ converges to $u$ pointwise $\mu$-a.e. Since $\varphi_j$ is a Cauchy sequence in $N_0^{1,s}(\Omega)$, there is a subsequence, denoted again by $\varphi_j$, such that

\[ \|\varphi_j - \varphi_{j+1}\|_{N^{1,s}(X)} < 2^{-2j} \text{ for every } j \geq 1. \]

For the open set

\[ E_j = \{x \in X : |\varphi_j(x) - \varphi_{j+1}(x)| > 2^{-j}\} \]
we have
\[ C_s(E_j) \leq 2^{js}||\varphi_j - \varphi_{j+1}||_{N^{1,s}(X)}^s < 2^{-js}. \]
If we put
\[ G_j = \bigcup_{k=j} E_k, \]
we have from the subadditivity of the $s$-capacity that
\[ C_s(G_j)^{1/s} \leq \sum_{k=j} C_s(E_k)^{1/s} \leq \sum_{k=j} 2^{-k} = 2^{1-j}. \]
Thus the sequence $\varphi_j$ converges uniformly outside open sets of arbitrarily small $s$-capacity to a quasicontinuous function $v$ and we can assume without loss of generality that $v = 0$ on $X \setminus U$. Moreover, $v \in N_0^{1,s}(\Omega)$ because $N_0^{1,s}(\Omega)$ is a Banach space. On the other hand, $\varphi_j$ converges to $u$ $\mu$-a.e. in $X$. Thus $u$ and $v$ are two functions in $N^{1,s}(X)$ that agree $\mu$-a.e., hence they agree $s$-q.e. on $X$. We let
\[ E_0 = \{ x \in X : u(x) \neq v(x) \} \]
and $E_1 = E \setminus E_0$.
We fix $\varepsilon \in (0,1)$. We choose open sets $G \subset U$ such that $C_s(G) < \varepsilon$ and $\varphi_j \to v$ uniformly on $X \setminus G$. We let
\[ \tilde{G}_j = \{ x \in X : \varphi_j(x) > 1 - \varepsilon \}. \]
Then $\tilde{G}_j$ is open and
\[ E_1 \setminus G \subset \tilde{G}_j \text{ for } j \geq j_\varepsilon. \]
Consequently, for $j \geq j_\varepsilon$ we have via the subadditivity of the relative $s$-capacity (see [Cos3, Theorem 3.2 (vi)])
\[ \text{cap}_s(E, \Omega) = \text{cap}_s(E_1, \Omega) \leq \text{cap}_s(\tilde{G}_j, \Omega) + \text{cap}_s(G, \Omega). \]
Since $\varphi_j > 1 - \varepsilon$ on $\tilde{G}_j$, we have
\[ \text{cap}_s(\tilde{G}_j, \Omega) \leq (1 - \varepsilon)^{-s} \int_{\Omega} g_{\varphi_j}^s \, d\mu, \]
and hence by letting $j \to \infty$, we obtain
\[ \text{cap}_s(E, \Omega) \leq (1 - \varepsilon)^{-s} \int_{\Omega} g_u^s \, d\mu + \varepsilon. \]
The lemma follows by letting $\varepsilon \to 0$.

We prove Theorem 3.1 now.

Proof. Since $(X, d, \mu)$ satisfies a weak $(1, s)$-Poincaré inequality, it follows from [HaK] that $(X, d, \mu)$ satisfies in fact a weak $(s, s)$-Poincaré inequality with possibly another constant $\lambda$.

We have that $u \in N_{loc}^{1,s}(X)$ has a minimal $s$-weak upper gradient $g_u \in L^{s,Q-s}(X, \mu)$, hence we can assume without loss of generality that $u$ is a Borel $s$-quasicontinuous function. Since $g_u$ has small $L^{s,Q-s}(X, \mu)$ norm, it follows from (6) that $u$ has small BMO-seminorm. Therefore, from John-Nirenberg lemma, it follows that $w(x) = e^{Qu(x)}$ is an $A_\infty$-density with respect to $\mu$ for some doubling measure $\nu$ with data depending
on $X$. That is, (see [MP, Theorem 1.4], [MMNO, Theorem A], and [Buc, Theorem 2.2]), there exists a constant $C$ depending on $s$ and on data of $X$ such that

\begin{equation}
\frac{1}{\mu(B)} \int_B e^{Q(u(x) - u_B)} \, d\mu(x) < C \quad \text{and} \quad \int_{2B} w(x) \, d\mu(x) \leq C \int_B w(x) \, d\mu(x)
\end{equation}

for every ball $B \subset X$. We write $d\nu(x) = w(x) \, d\mu(x)$. We recall the definition of $\delta_\nu$ from (4). We shall show that there exists a constant $C \in (0, 1]$ such that

\begin{equation}
d_\nu(x_1, x_2) := \inf \sum_{i=1}^k \nu(B_i)^{1/Q} \geq C\delta_\nu(x_1, x_2)
\end{equation}

for all $x_1, x_2 \in X$, where the infimum is taken over finite chains of open balls connecting $x_1$ and $x_2$ satisfying

\begin{equation}
x_1 \in B_1, x_2 \in B_k \text{ and } B_i \cap B_{i+1} \neq \emptyset \text{ for all } i = 1, \ldots, k - 1.
\end{equation}

Indeed, (18) implies both that $d_\nu$ is a distance and that is comparable to $\delta_\nu$ as required in (5). Towards this end, fix $x_1, x_2 \in X$, $x_1 \neq x_2$. Let $\gamma$ be a geodesic segment connecting $x_1$ and $x_2$, and let $a$ be the midpoint of $\gamma$. We denote $R = d(x_1, x_2)$ and $B = B(a, R)$.

Let $\eta \in \text{Lip}_0(6B)$ be a nonnegative $1/R$-Lipschitz function such that $\eta = 1$ on $3B$. Since $u$ is $s$-quasicontinuous and Borel, it follows that $v(x) = \eta(x)|u(x) - u_{3B}|$ is a Borel $s$-quasicontinuous function in $N_{\alpha}^{1,s}(6B)$ compactly supported in $6B$.

Let $E = \{x \in 3B : |u(x) - u_{3B}| > 1\}$. We have that $E$ is a Borel set since $u$ is a Borel function. Since $v$ is an $s$-quasicontinuous function in $N_{\alpha}^{1,s}(6B)$ compactly supported in $6B$, we have from Lemma 3.4 that

\begin{align*}
\text{cap}_s(E, 6B) &\leq \int_{6B} g_u^s \, d\mu \leq \int_{6\lambda B} (\eta g_u + |u - u_{3B}| g_u)^s \, d\mu \\
&\leq C \int_{6\lambda B} g_u^s \, d\mu \leq C (6R)^{Q-s} \|g_u\|_{L^{s,Q-s}(X,\mu)}^s.
\end{align*}

This implies that

\begin{equation}
\text{cap}_s(E, 6B) \leq C \|g_u\|_{L^{s,Q-s}(X,\mu)}^s,
\end{equation}

which together with [Cos3, Theorem 4.4] yields

\begin{equation}
\frac{\Lambda_\infty(E)}{R} \leq C \frac{\text{cap}_s(E, 6B)}{(6R)^{Q-s}} \leq C_0 \|g_u\|_{L^{s,Q-s}(X,\mu)}^s.
\end{equation}

We choose $\varepsilon > 0$ such that $C_0 \varepsilon^s < c_1$ where $c_1$ is the constant from (11) and $C_0$ is the constant from the last inequality in (21).

Now let $B_1, \ldots, B_k$ be an arbitrary chain of balls connecting $x_1$ and $x_2$ as in (19). We assume first that $B_i \subset 3B$ for all $i = 1, \ldots, k$. Let $G_0$ be defined like in (13). We
have

\[ \sum_{i=1}^{k} \nu(B_i)^{1/Q} \geq \sum_{i \in G_0} \nu(B_i)^{1/Q} \geq \sum_{i \in G_0} \nu(B_i \setminus E)^{1/Q} = \sum_{i \in G_0} \left( \int_{B_i \setminus E} e^{Qu(x)} \, d\mu(x) \right)^{1/Q} \]

\[ \geq \sum_{i \in G_0} \left( \int_{B_i \setminus E} e^{Q(uB^{-1})} \, d\mu(x) \right)^{1/Q} = e^{uB^{-1}} \left( \sum_{i \in G_0} \mu(B_i \setminus E)^{1/Q} \right) \]

\[ \geq e^{uB^{-1}} \sum_{i \in G_0} \left( \frac{1}{2} \mu(B_i) \right)^{1/Q} \geq C e^{uB} \mu(3B)^{1/Q}. \]

From (17), (22), and the definition of \( \delta_\nu \) there exists \( C \) such that

\[ \sum_{i=1}^{k} \nu(B_i)^{1/Q} \geq C \left( \int_{3B} e^{Qu(x)} \, d\mu(x) \right)^{1/Q} \geq C \delta_\nu(x_1, x_2). \]

Next, if the chain \( (B_i) \) does not lie entirely in \( 3B \), then there exists a smallest number \( k' \) with \( 1 \leq k' \leq k \) such that \( B_k' \cap S(a, 2R) \neq \emptyset \). Let \( x_0 \in B_k' \cap S(a, 2R) \). Then \( B_1, \ldots, B_{k'} \) is a chain of balls connecting \( x_1 \) and \( x_0 \) and \( d(x_1, x_2) \leq d(x_1, x_0) \).

If \( B_k' \subset 3B \), then from the fact that \( x_1 \in B_1 \cap 2B \) and from the definition of \( k' \) it follows that the subchain \( B_1, \ldots, B_{k'} \) is contained in \( 3B \). Therefore we can apply the preceding argument to the chain \( B_1, \ldots, B_{k'} \) connecting the points \( x_1 \) and \( x_0 \) to conclude that (18) holds; in the opposite case, \( \text{diam } B_k' \geq R \). The doubling condition for \( \nu \) then implies \( \nu(B) \leq C \nu(B_k') \). Thus, (18) is true in all cases. This finishes the proof.

**Lemma 3.5.** Suppose \( 1 < Q < p < \infty \). Suppose \( (X, d, \mu) \) is a complete and unbounded Ahlfors \( Q \)-regular metric measure space. Let \( \Omega \subset X \) be open and let \( E \subset \Omega \). Suppose \( u \in B^0_p(\Omega) \) is a \( B_p \)-quasicontinuous function compactly supported in \( \Omega \). If \( u \geq 1 \) on \( E \), then

\[ \hbox{cap}_{B_p}(E, \Omega) \leq [u]^p_{B_p(\Omega)}. \]

**Proof.** Since \( u \in B^0_p(\Omega) \) is compactly supported in \( \Omega \), there exists a sequence \( \varphi_j \in Lip_0(\Omega) \) converging to \( u \) in \( B_p(X) \). (See [Cos2, Lemma 3.14].) Without loss of generality we can assume that all the functions \( \varphi_j \) are supported in an open set \( U \subset \subset \Omega \) and that the sequence \( \varphi_j \) converges to \( u \) pointwise \( \mu \)-a.e. Since \( \varphi_j \) is a Cauchy sequence in \( B^0_p(\Omega) \), there is a subsequence, denoted again by \( \varphi_j \), such that

\[ ||\varphi_j - \varphi_{j+1}||^p_{L_p(X)} + ||\varphi_j - \varphi_{j+1}||^p_{B_p(X)} < 2^{-j(p+1)} \]

for every \( j \geq 1 \).

For the open set

\[ E_j = \{ x \in X : |\varphi_j(x) - \varphi_{j+1}(x)| > 2^{-j} \} \]

we have

\[ \text{Cap}_{B_p}(E_j) \leq 2^{jp} \left( ||\varphi_j - \varphi_{j+1}||^p_{L_p(X)} + ||\varphi_j - \varphi_{j+1}||^p_{B_p(X)} \right) < 2^{-j}. \]

If we put

\[ G_j = \bigcup_{k=j}^\infty E_k, \]

we have via the subadditivity of the Besov \( p \)-capacity (see [Cos2, Theorem 5.3 (ii)])

\[ \text{Cap}_{B_p}(G_j) = \sum_{k=j}^\infty \text{Cap}_{B_p}(E_k) \leq \sum_{k=j}^\infty 2^{-k} = 2^{1-j}. \]
Thus the sequence $\varphi_j$ converges uniformly outside open sets of arbitrarily small Besov $p$-capacity to a $B_p$-quasicontinuous function $v$ and we can assume without loss of generality that $v = 0$ on $X \setminus U$. Moreover, $v \in B_p^0(\Omega)$ because $B_p^0(\Omega)$ is a Banach space. On the other hand, $\varphi_j$ converges to $u \mu$-a.e. in $X$. Thus $u$ and $v$ are two $B_p$-quasicontinuous functions in $B_p(X)$ that agree $\mu$-a.e., hence they agree $B_p$-q.e. (See [Kil, p.262] and [Cos2, Theorem 5.16].) We let

$$E_0 = \{x \in X : u(x) \neq v(x)\}$$

and $E_1 = E \setminus E_0$.

We fix $\varepsilon \in (0, 1)$. We choose an open set $G \subset U$ such that $\text{Cap}_{B_p}(G) < \varepsilon$ and $\varphi_j \to v$ uniformly on $X \setminus G$. We let

$$\tilde{G}_j = \{x \in X : \varphi_j(x) > 1 - \varepsilon\}.$$

Then $\tilde{G}_j$ is open and

$$E_1 \setminus G \subset \tilde{G}_j \text{ for } j \geq j_\varepsilon.$$

Consequently, for $j \geq j_\varepsilon$ we obtain via the subadditivity of the relative Besov $p$-capacity (see [Cos2, Theorem 4.2 (vi)])

$$\text{cap}_{B_p}(E, \Omega) = \text{cap}_{B_p}(E_1, \Omega) \leq \text{cap}_{B_p}(\tilde{G}_j, \Omega) + \text{cap}_{B_p}(G, \Omega).$$

Since $\varphi_j > 1 - \varepsilon$ on $\tilde{G}_j$, we have

$$\text{cap}_{B_p}(\tilde{G}_j, \Omega) \leq (1 - \varepsilon)^{-p}[\varphi_j]_{B_p(\Omega)},$$

and hence by letting $j \to \infty$, we obtain

$$\text{cap}_{B_p}(E, \Omega) \leq (1 - \varepsilon)^{-p}[u]_{B_p(\Omega)} + \varepsilon.$$

The lemma follows by letting $\varepsilon \to 0$.

Now we prove Theorem 3.2.

Proof. Since $u$ has small Besov $p$-seminorm, it follows via (10) that $u$ has small BMO-seminorm. Therefore, by John-Nirenberg lemma (see [MP, Theorem 1.4], [MMNO, Theorem A], and [Buc, Theorem 2.2]) there exists a constant $C$ depending on $p$ and on the data of $X$ such that $w = e^{\tilde{Q}u}$ is an $A_\infty$-density with respect to $\mu$ for some doubling measure $\nu$ satisfying (17) with $C$. We write $d\nu(x) = w(x) \, d\mu(x)$. We recall the definition of $\delta_\nu$ from (4). We shall show that there exists a constant $C \in (0, 1]$ such that (18) holds for all $x_1, x_2 \in X$, where the infimum is taken over finite chains of open balls connecting $x_1$ and $x_2$ satisfying (19).

Indeed, (18) implies both that $d_\nu$ is a distance and that is comparable to $\delta_\nu$ as required in (5). Towards this end, fix $x_1, x_2 \in X$. We can assume without loss of generality that $x_1 \neq x_2$. Let $\gamma$ be a geodesic segment connecting $x_1$ and $x_2$, and let $a$ be the midpoint of $\gamma$. We denote $R = d(x_1, x_2)$ and $B = B(a, R)$.

Let $\eta \in Lip_0(6B)$ be a nonnegative $1/R$-Lipschitz function such that $\eta = 1$ on $3B$. We let the function $v$ be defined by $v = \eta|u - u_{3B}|$. Since $u \in L^1_{\text{loc}}(X)$ with $[u]_{B_p(X)} < \infty$, it follows via [Cos2, Lemma 3.11] that $v \in B_p^0(6B)$. Moreover, since $v$ is compactly supported in $\Omega$, it follows via [Cos2, Theorem 5.12] that there exists a $B_p$-quasicontinuous function $\tilde{v}$ compactly supported in $\Omega$ such that $v = \tilde{v}$ $\mu$-a.e. in $\Omega$. Moreover, we can assume that $\tilde{v}$ is Borel via an argument similar to the one from Lemma 3.5. Thus we can assume without loss of generality that $v$ is $B_p$-quasicontinuous and Borel.
Let
\[ E = \{ x \in 3B : |u(x) - u_{3B}| > 1 \} = \{ x \in 3B : v(x) > 1 \}. \]
Hence \( E \) is a Borel set since \( v \) is a Borel function. From Lemma 3.5 and [Cos2, Lemma 3.11] we obtain
\[ \text{cap}_{B_p}(E, 6B) \leq [v]_{B_p(6B)} \leq C [u]_{B_p(X)} \]
where \( C \) is a constant that depends only on \( p \) and on data of \( X \). This together with [Cos2, Theorem 4.10] yields
\[ \frac{A^\infty(E)}{R} \leq C \text{cap}_{B_p}(E, 6B) \leq C [v]_{B_p(6B)} \leq C_0 [u]_{B_p(X)}, \]
where \( R \) is the radius of \( B \) and \( C_0 \) is a constant that depends only on \( p \) and on data of \( X \). We choose \( \varepsilon > 0 \) such that \( C_0 \varepsilon^p < c_1 \) where \( c_1 \) is the constant from (11) and \( C_0 \) is the constant from the last inequality in (23).

Now let \( B_1, \ldots, B_k \) be an arbitrary chain of balls connecting \( x_1 \) and \( x_2 \) as in (19). We assume first that \( B_i \subset 3B \) for all \( i = 1, \ldots, k \). Let \( \mathcal{G}_0 \) be defined like in (13). The proof now continues like in Theorem 3.1, with the only difference that the constants who depended on \( s \) and on the data of \( X \) will now depend on \( p \) and on the data of \( X \). □

Theorem 3.1 yields the following consequence:

**Theorem 3.6.** Let \( 1 < s \leq Q < \infty \) be fixed. We assume that \( s > Q - 1 \). Suppose \((X, d, \mu)\) is a metric measure space as in Theorem 3.1. Let \( u \) be a Borel function in \( N^{1,s}_{\text{loc}}(X) \) such that it has a minimal \( s \)-weak upper gradient \( g_u \) in the Morrey space \( L^{s,Q-s}(X, \mu) \). There exists a constant \( \varepsilon > 0 \) depending only on \( s \) and on the data of \( X \) such that if
\[ ||g_u||_{L^{s,Q-s}(X, \mu)} < \varepsilon, \]
then
\[ \delta_{\nu}(x_1, x_2) \leq CD_{\nu}(x_1, x_2) \] for all \( x_1, x_2 \) in \( X \),
where \( C > 0 \) is a constant depending only on \( s \) and on the data associated with \( X \) and
\[ D_{\nu}(x, y) = \inf \left\{ \int_{\gamma} e^{u} ds : \gamma \text{ a rectifiable curve connecting } x, y \right\}. \]

For a discussion about line integration see [Hei, Chapter 7].

Theorem 3.2 yields the following consequence:

**Theorem 3.7.** Let \( 1 < s < Q < p < \infty \) be fixed. Suppose \((X, d, \mu)\) is a metric measure space as in Theorem 3.2. Let \( u \) be in \( L^{1}_{\text{loc}}(X) \) such that \( [u]_{B_p(X)} < \infty \). There exists a constant \( \varepsilon > 0 \) depending only on \( p \) and on the data of \( X \) such that if \( [u]_{B_p(X)} < \varepsilon \), then
\[ \delta_{\nu}(x_1, x_2) \leq CD_{\nu}(x_1, x_2) \] for all \( x_1, x_2 \) in \( X \),
where \( C > 0 \) is a constant depending only on \( p \) and on the data associated with \( X \) and \( D_{\nu} \) is defined as in (25). Here \( \bar{u} \) is a \( B_p \)-quasicontinuous Borel representative of \( u \).

One should compare the metrics \( D_{\mu} \) in Theorems 3.6 and 3.7 to those studied in [BL], [Res] and [Cos1].
Question 3.8. Another open question is whether or not the inequality (24) can be reversed in general. The answer is yes in the Euclidean setting when \( n \geq 2 \). (See [Cos1, Theorems 5.4 and 5.5].)

We prove now Theorem 3.6.

Proof. It is easy to see that \( D_\nu \) is indeed symmetric, nonnegative and satisfies the triangle inequality. From (24) it would follow immediately that \( D_\nu \) is a distance function dominating \( \delta_\nu \). So fix \( x_1, x_2 \) in \( X \). We can assume without loss of generality that \( x_1 \neq x_2 \). Like before, let \( a \) be a point such that \( d(x_1, a) = d(a, x_2) = R/2 \), where \( R = d(x_1, x_2) \). We denote \( B = B(a, R) \). Like in the proof of Theorem 3.1, let

\[
v = \eta|u - u_{3B}| \text{ and } E = \{ x \in 3B : |u(x) - u_{3B}| > 1 \},
\]

where \( \eta \in Lip_0(6B) \) is a nonnegative \( 1/R \)-Lipschitz function such that \( \eta = 1 \) on \( 3B \).

We notice that \( E \) is a Borel set and \( v \) is a Borel and \( s \)-quasicontinuous function in \( N^1_s(6B) \) compactly supported in \( 6B \).

Let \( \gamma \) be a rectifiable curve connecting \( x_1 \) and \( x_2 \) and let \( |\gamma| \) be its image. We assume first that \( |\gamma| \subset 3B \). We obviously have

\[
\int_\gamma e^u ds \geq \int_{\gamma \cap (3B \setminus E)} e^u ds.
\]

As in the proof of Theorem 3.1, we have

\[
\frac{\Lambda^\infty_1(E)}{R} \leq C \frac{\text{cap}_s(E, 6B)}{(6R)^{2-s}} \leq C_0 \| g_u \|_{L^\infty_c(X, \mu)}^{s} < C_0 \varepsilon^s,
\]

text{hence}

\[
\Lambda^\infty_1(|\gamma| \cap (3B \setminus E)) \geq \Lambda^\infty_1(|\gamma| \cap 3B) - \Lambda^\infty_1(|\gamma| \cap E) \\
\geq R - \Lambda^\infty_1(E) \geq (1 - c_1)R
\]

if \( \varepsilon > 0 \) is small enough, where \( c_1 \) is the constant from (11).

Thus we obtain

\[
\int_\gamma e^u ds \geq \int_{\gamma \cap (3B \setminus E)} e^u ds \geq \int_{\gamma \cap (3B \setminus E)} e^{u_{3B}-1} ds \\
\geq \Lambda^\infty_1(|\gamma| \cap (3B \setminus E)) e^{u_{3B}-1} \geq CR \ e^{u_{3B}} \\
\geq C \left( \int_{3B} e^{Qu(z) d\mu} \right)^{1/Q} \geq \Lambda^\infty_1(E)^{1/Q},
\]

where the last inequality follows from (17). Therefore

\[
\int_\gamma e^u ds \geq C \left( \int_{3B} e^{Qu(z) d\mu} \right)^{1/Q} \geq C \delta_\nu(x_1, x_2) \text{ whenever } |\gamma| \subset 3B.
\]

Now we assume that \( |\gamma| \setminus 3B \neq \emptyset \). Suppose that \( \gamma \) is parametrized by its arc length parametrization. Let \( t_0 = \inf \{ t \in [0, l_\gamma] : \gamma(t) \notin B(a, 2R) \} \). Then, since \( \gamma \) is a path with \( \gamma(0), \gamma(l_\gamma) \in 2B \), it follows that

\[
0 < t_0 < l_\gamma \text{ and } \gamma([0, t_0]) \subset B(a, 2R).
\]

Let \( x_0 = \gamma(t_0) \) and let \( \tilde{\gamma} \) be the restriction of \( \gamma \) to \([0, t_0]\). Then \( x_0 \in S(a, 2R) \) and \( d(x_1, x_2) \leq d(x_1, x_0) \). Therefore

\[
\Lambda^\infty_1(|\gamma| \cap (3B \setminus E)) \geq \Lambda^\infty_1(|\tilde{\gamma}| \cap (3B \setminus E)) \geq (\Lambda^\infty_1(|\tilde{\gamma}|) - \Lambda^\infty_1(|\tilde{\gamma}| \cap E) \\
\geq d(x_1, x_0) - \Lambda^\infty_1(E) \geq (1 - c_1)R.
\]
if \( \varepsilon > 0 \) is small enough, where \( c_1 \) is the constant from (11). By repeating the argument from (27) with \( \tilde{\gamma} \) instead of \( \gamma \), we obtain

\[
\int_{\tilde{\gamma}} e^u ds \geq C \left( \int_{3B} e^{Qu(z)} d\mu(z) \right)^{1/Q}.
\]

The desired conclusion follows.

\[\square\]

Now we prove Theorem 3.7.

**Proof.** For the existence and “uniqueness” of \( B_p \)-quasicontinuous Borel representatives of \( u \) see [Cos2, Corollary 5.19]. We notice that \( D_\nu \) does not depend on the choice of the \( B_p \)-quasicontinuous Borel representative. Indeed, if \( \tilde{u} \) and \( \tilde{v} \) are two such representatives, then from [Cos2, Corollary 5.18] we have \( \tilde{u} = \tilde{v} \) \( B_p \)-q.e., which implies via [Cos2, Corollary 4.15] that

\[
\int_{\gamma} e^u ds = \int_{\gamma} e^v ds
\]

for every rectifiable curve \( \gamma \) in \( X \). So we can assume without loss of generality that \( u \) is a \( B_p \)-quasicontinuous Borel function itself.

It is easy to see that \( D_\nu \) is indeed symmetric, nonnegative and satisfies the triangle inequality. From (24) it would follow immediately that \( D_\nu \) is a distance function dominating \( \delta_\nu \). So fix \( x_1, x_2 \) in \( X \). We can assume without loss of generality that \( x_1 \neq x_2 \). Like before, let \( a \) be a point such that \( d(x_1, a) = d(a, x_2) = R/2 \), where \( R = d(x_1, x_2) \). We denote \( B = B(a, R) \).

Let \( \eta \in L^1_0(6B) \) be a nonnegative \( 1/R \)-Lipschitz function such that \( \eta = 1 \) on \( 3B \).

Like in the proof of Theorem 3.2, let

\[
v = \eta |u - u_{3B}| \quad \text{and} \quad E = \{ x \in 3B : |u(x) - u_{3B}| > 1 \} = \{ x \in 3B : v(x) > 1 \}.
\]

We notice that \( E \) is a Borel set and \( v \) is a \( B_p \)-quasicontinuous Borel function in \( B^0_p(6B) \) compactly supported in \( 6B \).

Let \( \gamma \) be a rectifiable curve connecting \( x_1 \) and \( x_2 \) and let \( |\gamma| \) be its image. We assume first that \( |\gamma| \subset 3B \). We obviously have

\[
\int_{\gamma} e^u ds \geq \int_{\gamma \cap (3B \setminus E)} e^u ds.
\]

Like in the proof of Theorem 3.2, we have

\[
\frac{\Lambda_\infty^\infty(E)}{R} \leq C \cap_{B_p}(E, 6B) \leq C_0[u]_{B_p(X)}^p < C_0 \varepsilon^p.
\]

The proof now continues like in Theorem 3.6, with the only difference that the constants who depended on \( s \) and on the data of \( X \) will now depend on \( p \) and on the data of \( X \).

\[\square\]

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