GLOBAL ATTRACTORS, EXPONENTIAL ATTRACTORS AND 
DETERMINING MODES FOR THE THREE DIMENSIONAL 
KELVIN-VOIGT FLUIDS WITH “FADEING MEMORY”

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ABSTRACT. The three-dimensional viscoelastic fluid flow equations, arising from the motion of Kelvin-Voigt (Kelvin-Voight) fluids in bounded domains is considered in this work. We investigate the long-term dynamics of such viscoelastic fluid flow equations with “fading memory” (non-autonomous). We first establish the existence of an absorbing ball in appropriate spaces for the semigroup defined for the Kelvin-Voigt fluid flow equations of order one with “fading memory” (transformed autonomous coupled system). Then, we prove that the semigroup is asymptotically compact, and hence we establish the existence of a global attractor for the semigroup. We provide estimates for the number of determining modes for both asymptotic as well as for trajectories on the global attractor. Once the differentiability of the semigroup with respect to initial data is established, we show that the global attractor has finite Hausdorff as well as fractal dimensions. We also show the existence of an exponential attractor for the semigroup associated with the transformed (equivalent) autonomous Kelvin-Voigt fluid flow equations with “fading memory”. Finally, we show that the semigroup has Ladyzhenskaya’s squeezing property and hence is quasi-stable, which also implies the existence of global as well as exponential attractor having finite fractal dimension.

1. Introduction. Viscoelastic materials are those which exhibit both viscous and elastic characteristics, when undergoing deformation and non-Newtonian fluids are those which do not follow Newton’s law of viscosity. For the past several decades, the mathematical theory of non-Newtonian and viscoelastic fluid flows have been developed by several mathematicians starting from the works of Oskolkov (see for example [2, 11, 18, 26, 30, 31, 32, 33, 34, 41], etc and the references therein). In this work, we consider a linear viscoelastic fluid with a relaxation time \( \lambda \) and retardation times \( \{\kappa_1, \kappa_2\} \). The system under our consideration is modeled through the Kelvin-Voigt fluid flow motion of order one with “fading memory” (non-autonomous) in a
bounded domain as ([33])
\[
\frac{\partial u}{\partial t} - \mu_1 \frac{\partial \Delta u}{\partial t} + (u \cdot \nabla)u - \mu_0 \Delta u - \gamma \int_0^t e^{-\delta(t-s)} \Delta u(s) ds + \nabla p = f, \ \nabla \cdot u = 0,
\]
(1)
where
\[
\mu_1 = \frac{2\kappa_2}{\lambda}, \ \mu_0 = \frac{2}{\lambda} \left( \kappa_1 - \frac{\kappa_2}{\lambda} \right), \ \gamma = \frac{2}{\lambda} \left( \nu - \frac{\kappa_1}{\lambda} + \frac{\kappa_2}{\lambda^2} \right) \quad \text{and} \quad \delta = \frac{1}{\lambda},
\]
(2)
and all quantities given in (2) are positive. Here \(\nu\) is the coefficient of kinematic viscosity. Remember that the above relation is derived under the natural assumption on stress tensor \(\sigma(t) \equiv 0\), for \(t < 0\) (see [32]). If we introduce
\[
v(t) := \int_0^t e^{-\delta(t-s)} u(s) ds,
\]
(3)
then one can rewrite the system (1) as (autonomous)
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \mu_1 \frac{\partial \Delta u}{\partial t} + (u \cdot \nabla)u - \mu_0 \Delta u - \gamma \Delta v + \nabla p &= f, \\
\frac{\partial v}{\partial t} - u + \delta v &= 0, \\
\nabla \cdot u &= 0.
\end{aligned}
\]
(4)

The system (1) with \(\gamma = 0\) is commonly known as Kelvin-Voigt equations (or Navier-Stokes-Voigt equations that governs the motion of a Kelvin-Voigt linear viscoelastic incompressible fluid or Oskolkov equations). The global solvability results of the system (1) in bounded domains are available in the literature, cf. [30, 33, 41], etc. The 2D Navier-Stokes-Voigt equation in an unbounded strip-like domain is considered in [3] and the authors established that the semigroup generated by this equation has a global attractor in weighted Sobolev spaces. The authors in [26] considered the problem of existence of a finite dimensional global attractor and established the estimates for the number of determining modes on the global attractor of Kelvin-Voigt fluids of order \(L \geq 1\). For the Kelvin-Voigt equations, the author in [33] studied the solvability of the initial boundary problem and the Cauchy problem, the existence of periodic solutions with a free term periodic in \(t\), exponential stability and attractor theory. In the work [30], the author investigated the global solvability results, asymptotic behavior and also addressed some control problems of Kelvin-Voigt fluid flow equations with “fading memory” in bounded and Poincaré domains. A local monotonicity property of the linear and nonlinear operators and a localized version of the Minty-Browder technique were used to obtain the existence of a unique weak solution. Using an \(m\)-accretive quantization of the linear and nonlinear operators, the author established the existence and uniqueness of strong solutions for the Kelvin-Voigt equations. The authors in [11] considered the Navier-Stokes-Voigt fluid model, where the instantaneous kinematic viscosity has been completely replaced by a memory term incorporating hereditary effects, in the presence of Ekman damping. They established that such a system is dissipative in the dynamical systems sense and even possesses regular global and exponential attractors of finite fractal dimension. For more details on the stability and attractors for abstract equations and viscoelastic equations with memory, interested readers are referred to see [4, 9, 10, 35], etc and the references therein.

The authors in [25] investigated the long-term dynamics of the 3D Navier-Stokes-Voigt model of viscoelastic incompressible fluid. Moreover, they showed that as
the viscosity coefficient goes to zero, the weak solutions of the Navier-Stokes-Voight equations converge to the weak solutions of the inviscid simplified Bardina model in appropriate norm. The authors in \[40\] investigated the long-term behavior of the three-dimensional Navier-Stokes-Voigt model as the regularization parameter vanishes and extended the results of \[25\]. They established the existence of global and exponential attractors of optimal regularity. At the end of the paper \[25\], the authors mentioned as a future work that the results reported in their work can be extended to Kelvin-Voigt fluid flows with “fading memory”. In this paper, we extend the results obtained in \[25\] to Kelvin-Voigt fluid flow equation with “fading memory”. The problem makes mathematically challenging as the governing equations in (1) are non-autonomous (even if \(f\) is independent of \(t\)) due to the presence of memory term and thus the corresponding solution operator does not form a one parameter family of semigroups. Hence we cannot directly use the usual methods available for the autonomous systems to study the asymptotic behavior (for example, the methods available in \[25, 40\], etc). But one can use the transformation given in (3) to transform the system (1) to an autonomous one (see (4)), whose solution operator can be represented as a one parameter family of semigroups. Then, we adopt the theory developed in \[25, 40\] to study the asymptotic behavior of the system given in (4). The asymptotic behavior of this model is studied by showing that the semigroup defined for this model is asymptotically compact, and hence the existence of a global attractor follows. We give estimates for the number of determining modes for both asymptotic as well as trajectories on the global attractor. We also show that the global attractor has finite Hausdorff and fractal dimensions. We also establish the existence of an exponential attractor for the system (4). We finally prove that the semigroup associated with the Kelvin-Voigt fluid flow equations with “fading memory” is quasi-stable, which also implies the existence of global and exponential attractor having finite fractal dimension.

The rest of the paper is structured as follows. In the next section, we describe the necessary functional settings needed to obtain the global solvability results of the system (1) in three dimensional bounded domains. Suitably defining linear and nonlinear terms, we consider an abstract formulation of the system (1). We also describe some properties of the memory kernel appearing in (1) in the same section. In section 3, we first consider the transformed system (autonomous) using the transformation given in (3), so that solution operator defines a one parameter family of semigroups. We establish the existence of an absorbing ball in \(V\) for the semigroup \(S(t), t \geq 0\) defined for the Kelvin-Voigt fluid flow equations with “fading memory” (section 3.1). Then, we show that \(S(t), t \geq 0\) is an asymptotically compact semigroup in \(X = V \times V\) (Proposition 3), and hence we prove the existence of a global attractor for the semigroup (Theorem 3.2). We also show that \(S(t)\) defined on \(X_2 = V_2 \times V_2\) has an absorbing ball in \(X_2\) in section 3.3 and hence the existence of a global attractor in \(X_2\) (Theorem 3.3). Whenever the external forcing has better regularity, these two attractors coincide (Remark 2). We give estimates for the number of determining modes for both asymptotic and trajectories on the global attractor in section 4 (Theorems 4.2 and 4.4). In section 5, we first establish the differentiability of the semigroup with respect to the initial data (Theorem 5.1). Then, we show that the global attractor for the 3D Kelvin-Voigt equations with “fading memory” has finite Hausdorff and fractal dimensions (Theorem 5.2). The existence of an exponential attractor in \(X\) for the semigroup \(S(t)\) associated with the system (21) is established in section 6. We also show that the semigroup \(S(t)\)
has Ladyzhenskaya’s squeezing property and hence is quasi-stable (Propositions 8 and 9). Furthermore, making use of Theorem 3.4.11, [5], we show the existence of global as well as exponential attractor having finite fractal dimension (Theorem 6.10). Finally, we remark that the results obtained in this work also holds true for Kelvin-Voigt fluids of order $L = 1, 2, \ldots$, with “fading memory” (see Remark 5).

2. Mathematical Formulation. In this section, we explain the necessary function spaces needed to obtain the global solvability results of the system (1) in three dimensional bounded domains. Let us consider the following three dimensional Kelvin-Voigt fluids of order one with “fading memory”:

\[
\begin{aligned}
\frac{\partial}{\partial t}[(1-\mu_1\Delta)\mathbf{u}(t,x)] - \mu_0\Delta\mathbf{u}(t,x) - (\beta * \Delta \mathbf{u})(t,x) + (\mathbf{u}(t,x) \cdot \nabla)\mathbf{u}(t,x) + \nabla p(t,x) &= \mathbf{f}(t,x) \quad \text{for } x \in \Omega, \; t > 0,
\end{aligned}
\]

with the conditions

\[
\begin{aligned}
(\nabla \cdot \mathbf{u})(t,x) &= 0, \quad \text{for } x \in \Omega, \; t > 0, \\
\mathbf{u}(t,x) &= \mathbf{0}, \quad \text{for } x \in \partial\Omega, \; t \geq 0, \\
\mathbf{u}(0,x) &= \mathbf{u}_0(x), \quad \text{for } x \in \Omega, \\
\int_{\Omega} p(t,x)dx &= 0, \quad \text{for } t \geq 0,
\end{aligned}
\]

where $\Omega$ is a bounded open domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $\mathbf{u}(t,x) \in \mathbb{R}^3$ denotes the velocity field at time $t$ and position $x$, $p(\cdot,\cdot) \in \mathbb{R}$ denotes the pressure field, $\mathbf{f}(\cdot,\cdot)$ is an external forcing, and

\[
(\beta * \Delta \mathbf{u})(t) := \int_0^t \beta(t-s)\Delta \mathbf{u}(s)ds = \gamma \int_0^t e^{-\delta(t-s)}\Delta \mathbf{u}(s)ds.
\]

The final condition in (6) is introduced for uniqueness of the pressure $p$.

2.1. Functional Setting. Let us define the space $\mathcal{V} := \{\mathbf{u} \in C_0^\infty(\Omega; \mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0\}$, where $C_0^\infty(\Omega; \mathbb{R}^3)$ is the space of all infinitely differentiable functions with compact support in $\Omega$. Let $\mathbb{H}$ and $\mathcal{V}$ be the completion of $\mathcal{V}$ in $L^2(\Omega; \mathbb{R}^3)$ and $H^1(\Omega; \mathbb{R}^3)$ norms respectively. Then under some smoothness assumptions on the boundary, we characterize the spaces $\mathbb{H}$ and $\mathcal{V}$ as $\mathbb{H} := \{\mathbf{u} \in L^2(\Omega) : \nabla \cdot \mathbf{u} = 0, \; \mathbf{u} \cdot \mathbf{n}\big|_{\partial\Omega} = 0\}$, with norm $\|\mathbf{u}\|_\mathbb{H}^2 := \int_\Omega |\nabla \mathbf{u}(x)|^2dx$, where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$, and $\mathbf{u} \cdot \mathbf{n}\big|_{\partial\Omega}$ should be understood in the sense of trace in $H^{-1/2}(\partial\Omega)$ (cf. Theorem 1.2, Chapter 1, [37]), and $\mathcal{V} := \{\mathbf{u} \in H^1_0(\Omega) : \nabla \cdot \mathbf{u} = 0\}$, with norm $\|\mathbf{u}\|_\mathcal{V}^2 := \int_\Omega |\nabla \mathbf{u}(x)|^2dx$. The inner product in the Hilbert space $\mathbb{H}$ is denoted by $(\cdot,\cdot)$ and the induced duality, for instance between the spaces $\mathcal{V}$ and its dual $\mathcal{V}'$ by $(\cdot,\cdot)$. Since $\mathcal{V}$ is densely and continuously embedded into $\mathbb{H}$ and $\mathbb{H}$ can be identified with its dual $\mathbb{H}'$ and we have the following Gelfand triple: $\mathcal{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathcal{V}'$. For a bounded domain, the embedding of $\mathcal{V} \subset \mathbb{H}$ is compact. In the sequel, we also use the notation $H^2(\Omega) := H^2(\Omega; \mathbb{R}^3)$ for the second order Sobolev spaces.

2.2. Linear operator. Let $P_\mathbb{H} : L^2(\Omega) \to \mathbb{H}$ be the Helmholtz-Hodge orthogonal projection. Then, we define

\[
\begin{aligned}
\{ Au := -P_\mathbb{H}\Delta \mathbf{u}, \; \mathbf{u} \in D(A), \\
D(A) := \mathcal{V} \cap H^2(\Omega).
\end{aligned}
\]
The operator $A$ is non-negative, self-adjoint in $\mathbb{H}$ with $V = D(A^{1/2})$ and
\[
\langle Au, u \rangle = \|u\|_V^2, \text{ for all } u \in V, \text{ so that } \|Au\|_V \leq \|u\|_V.
\] (7)

It is well known that for the bounded domain $\Omega$, the operator $A$ is invertible and its inverse $A^{-1}$ is bounded, self-adjoint and compact in $\mathbb{H}$. Hence the spectrum of $A$ consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, with $\lambda_k \to \infty$ as $k \to \infty$ of eigenvalues, and there exists an orthogonal basis $\{e_k\}_{k=1}^\infty$ of $\mathbb{H}$ consisting of eigenvectors of $A$ such that $Ae_k = \lambda_k e_k$, for all $k \in \mathbb{N}$. Thus, it is immediate that $\|\nabla u\|_V^2 \geq \lambda_1 \|u\|_V^2$, which is the Poincaré inequality. In this paper, we also need the fractional powers of $A$. For $u \in \mathbb{H}$ and $\alpha > 0$, let us define $A^{\alpha}u = \sum_{k=1}^{\infty} \lambda_k^\alpha (u, e_k)e_k$, $u \in D(A^\alpha)$, where $D(A^\alpha) = \{ u \in \mathbb{H} : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |(u, e_k)|^2 < +\infty \}$.

Here $D(A^\alpha)$ is equipped with the norm
\[
\|A^{\alpha}u\|_\mathbb{H} = \left( \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |(u, e_k)|^2 \right)^{1/2}.
\] (8)

Note that $D(A^0) = \mathbb{H}$, $D(A^{1/2}) = V$ and in general, we set $V_\alpha = D(A^{\alpha/2})$ with $\|u\|_{V_\alpha} = \|A^{\alpha/2}u\|_\mathbb{H}$. For any $s_1 < s_2$, the embedding $D(A^{s_1}) \subset D(A^{s_2})$ is also compact. Applying H"{o}lder’s inequality in the expression (8), one can obtain the following interpolation estimate:
\[
\|A^s u\|_\mathbb{H} \leq \|A^{s_1} u\|_\mathbb{H}^{\theta}\|A^{s_2} u\|_\mathbb{H}^{1-\theta},
\] (9)
for any real $s_1 < s < s_2$ and $\theta$ is given by $s = s_1 \theta + s_2(1-\theta)$.

**Remark 1.** One can show that the norms $\|u\|_V$ and $\|(I+\mu_1 A)^{1/2}u\|_\mathbb{H}$ are equivalent. It can be easily seen that
\[
\|(I+\mu_1 A)^{1/2}u\|_\mathbb{H}^2 = \|(I+\mu_1 A)u, u\| = \|u\|_V^2 + \mu_1 \|\nabla u\|_V^2 \leq \left( \frac{1}{\lambda_1} + \mu_1 \right) \|u\|_V^2,
\] (10)

using the Poincaré inequality. From (10), it is also clear that
\[
\mu_1 \|u\|_V^2 \leq \|u\|_V^2 + \mu_1 \|u\|_V^2 = \|(I+\mu_1 A)^{1/2}u\|_\mathbb{H}^2.
\] (11)

Combining (10) and (11), we get the required result.

2.3. **Nonlinear operator.** Let us define the trilinear form $b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ by
\[
b(u, v, w) = \int_{\Omega} (u(x) \cdot \nabla)v(x) \cdot w(x) dx = \sum_{i,j=1}^{3} \int_{\Omega} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx.
\]

If $u, v$ are such that the linear map $b(u, v, \cdot)$ is continuous on $V$, the corresponding element of $V'$ is denoted by $B(u, v)$. We also denote $B(u) = B(u, u) = P_H(u \cdot \nabla)u$.

An integration by parts yields
\[
\begin{cases}
\displaystyle b(u, v, v) = 0, \text{ for all } u, v \in V, \\
\displaystyle b(u, v, w) = -b(u, w, v), \text{ for all } u, v, w \in V.
\end{cases}
\] (12)

It can also be seen that $B$ maps $L^4(\Omega) \cap H$ (and so $V$) into $V'$ and
\[
\|B(u, u, v)\| = \|b(u, v, u)\| \leq \|u\|_{L^4}^2 \|\nabla v\|_H \leq 2\|u\|_{L^4}^{1/4} \|\nabla u\|_H^{3/4} \|v\|_V,
\]
for all $\mathbf{v} \in \mathcal{V}$, so that

$$
\|B(\mathbf{u})\|_{\mathcal{V}'} \leq 2\|\mathbf{u}\|_{\mathcal{H}}^{1/4}\|\nabla \mathbf{u}\|_{\mathcal{H}}^{3/4} \leq \frac{2}{\lambda_1^{1/8}}\|\mathbf{u}\|_{\mathcal{V}}^2, \quad \text{for all } \mathbf{u} \in \mathcal{V},
$$

(13)

using the Poincaré inequality. For more details on the linear and bilinear operators, interested readers are referred to see [38].

2.4. Properties of the kernel $\beta(\cdot)$. Let us discuss some properties of the kernel $\beta(t) = \gamma e^{-\delta t}$. We first define

$$(Lu)(t) := (\beta * \mathbf{u})(t) = \int_0^t \beta(t - s)\mathbf{u}(s)ds.$$  

A function $\beta(\cdot)$ is called positive kernel if the operator $L$ is positive on $L^2(0, T; \mathcal{H})$, for all $T$. That is,

$$
\int_0^T (Lu(t), \mathbf{u}(t))dt = \int_0^T \int_0^t \beta(t - s)(\mathbf{u}(s), \mathbf{u}(t))dsdt \geq 0,
$$

for all $\mathbf{u} \in \mathcal{H}$ and every $T > 0$. Using Lemma 4.1, [1], it can be easily seen that $\beta(t) = \gamma e^{-\delta t}$ is a positive kernel. Thus, we have

$$
\int_0^T ((\beta * A)\mathbf{u}(t), \mathbf{u}(t))dt = \int_0^T ((\beta * \nabla \mathbf{u})(t), \nabla \mathbf{u}(t))dt \geq 0.
$$

(14)

Applying Lemma 2.6, [30], we obtain

$$
\int_0^T \|((\beta * A)\mathbf{u})(t)\|^2_{\mathcal{V}}dt \leq \int_0^T \left(\int_0^t \beta(t - s)\|\mathbf{u}(s)\|_{\mathcal{V}}ds\right)^2 dt \leq \left(\int_0^T \beta(t)dt\right) \left(\int_0^T \|\mathbf{u}(t)\|_{\mathcal{V}}^2dt\right) \leq \gamma^{-2} \left(\int_0^T \|\mathbf{u}(t)\|_{\mathcal{V}}^2dt\right),
$$

for $\mathbf{u} \in L^2(0, T; \mathcal{V})$. Using the Cauchy-Schwarz inequality, Hölder’s inequality and (15), we can further estimate

$$
\int_0^t \langle(\beta * A)\mathbf{u}(s), \mathbf{u}(s)\rangle ds \leq \left(\int_0^t \|\beta * \nabla \mathbf{u}(s)\|^2_{\mathcal{H}}ds\right)^{1/2} \left(\int_0^t \|\nabla \mathbf{u}(s)\|^2_{\mathcal{H}}ds\right)^{1/2}
\leq \left(\int_0^t \beta(s)ds\right) \left(\int_0^t \|\nabla \mathbf{u}(s)\|^2_{\mathcal{H}}ds\right) \leq \frac{\gamma}{\delta} \int_0^t \|\mathbf{u}(s)\|_{\mathcal{V}}^2ds.
$$

(16)

2.5. Abstract formulation and weak solution. Let us take the Helmholtz-Hodge orthogonal projection $P_{\mathcal{H}}$ in (5) to write down the abstract formulation of the system (5) as:

$$
\begin{cases}
\frac{d}{dt}[(I + \mu_1 A)\mathbf{u}(t)] + \mu_0 A\mathbf{u}(t) + (\beta * A)\mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{u}(t)) = \mathbf{f}(t), & t \in (0, T), \\
\mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{V},
\end{cases}
$$

(17)
where \( f \in L^2(0, T; V') \). For notational convenience, we used \( f \) instead of \( P \Omega f \). The system (17) is also equivalent to the following system:

\[
\begin{aligned}
\frac{d\mathbf{u}(t)}{dt} + (I + \mu_A)^{-1}[\mu_0 A\mathbf{u}(t) + (\beta \ast A)\mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{u}(t))] = (I + \mu_A)^{-1}f(t), \\
\mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{V},
\end{aligned}
\]

for \( t \in (0, T) \).

Let us now provide the definition of weak solution to the system (17).

**Definition 2.1.** A function \( \mathbf{u} \in C([0, T]; \mathbb{V}) \) with \( \partial_t \mathbf{u} \in L^2(0, T; \mathbb{V}') \), \( \mathbf{u}_0 \in \mathbb{V} \) and \( \mathbf{v} \in \mathbb{V} \), \( \mathbf{u}(\cdot) \) satisfies:

\[
\begin{aligned}
\langle \partial_t[(I + \mu_A)\mathbf{u}(t)] + \mu_0 A\mathbf{u}(t) + (\beta \ast A)\mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{v}), \mathbf{v} \rangle &= \langle f(t), \mathbf{v} \rangle, \\
\lim_{t \to 0} \int_\Omega \mathbf{u}(t)\mathbf{v}dx &= \int_\Omega \mathbf{u}_0\mathbf{v}dx,
\end{aligned}
\]

and the energy equality

\[
\frac{d}{dt}(\|\mathbf{u}(t)\|_\Omega^2 + \mu_1 \|\mathbf{u}(t)\|_{\mathbb{V}}^2) + 2\mu_0 \|\mathbf{u}(t)\|_{\mathbb{V}}^2 + 2(\beta \ast \nabla \mathbf{u}(t), \nabla \mathbf{u}(t)) = 2\langle f(t), \mathbf{u}(t) \rangle.
\]

Let us now state the existence and uniqueness theorem for the system (17). A proof of the following theorem can be obtained from [33], Theorem 3.10, [30], etc.

**Theorem 2.2.** There exists a unique weak solution \( \mathbf{u}(\cdot) \) to the system (17) in the sense of Definition 2.1.

3. Global Attractor. In this section, we discuss the existence of global attractors for the three dimensional Kelvin-Voigt fluid flow equations with “fading memory”. Let us assume that \( f \in V' \) is independent of \( t \) in (17). Since the system (17) depends on memory (non-autonomous), the solution operator does not form a one parameter family of semigroups and hence we cannot use the usual procedure as in [25, 40] to obtain the global attractors. Thus, we use the transformation given in (3) to transform the system (17) as

\[
\begin{aligned}
\frac{d}{dt}[(I + \mu_A)\mathbf{u}(t)] + \mu_0 A\mathbf{u}(t) + \gamma A\mathbf{v}(t) + B(\mathbf{u}(t), \mathbf{u}(t)) &= f, \ t \in (0, T), \\
\frac{d\mathbf{v}(t)}{dt} - \mathbf{u}(t) + \delta \mathbf{v}(t) &= 0, \ t \in (0, T), \\
\mathbf{u}(0) = \mathbf{u}_0, \mathbf{v}(0) = \mathbf{v}_0 = 0,
\end{aligned}
\]

where \( \mathbf{u}_0 \in \mathbb{V} \). Note that the system (21) is autonomous and hence we can develop a similar theory available in [25, 40] for the system (21). Let us define \( U := (\mathbf{u}, \mathbf{v})^T \), so that \( U(\cdot) \) satisfies:

\[
\begin{aligned}
\mathfrak{A} \frac{dU(t)}{dt} + \mathcal{A} U(t) + \mathcal{B}(U(t)) &= \mathbf{F}, \\
U(0) = U_0,
\end{aligned}
\]

where

\[
\mathfrak{A} = \begin{pmatrix} (I + \mu_A) & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mu_0 A & \gamma A \\ -I & \delta I \end{pmatrix}, \quad \mathcal{B}(U) = \begin{pmatrix} B(\mathbf{u}) \\ 0 \end{pmatrix},
\]

\[
\mathbf{F} = \begin{pmatrix} f \\ 0 \end{pmatrix} \text{ and } U_0 = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{pmatrix}.
\]
Let us take the inner product with $A\mathbf{v}(\cdot)$ to the second equation in (21) and then integrate from $0$ to $t$ to obtain
\[
\|\mathbf{v}(t)\|^2_V + 2\delta \int_0^t \|\mathbf{v}(s)\|^2_V ds
\]
\[
= \|\mathbf{v}_0\|^2_V + 2 \int_0^t (\nabla \mathbf{u}(s), \nabla \mathbf{v}(s)) ds = \|\mathbf{v}_0\|^2_V + 2 \int_0^t \int_s^t e^{-(s-r)}(\nabla \mathbf{u}(s), \nabla \mathbf{u}(r)) dr ds
\]
\[
\leq \|\mathbf{v}_0\|^2_V + \frac{2\gamma}{\delta} \int_0^t \|\mathbf{u}(s)\|^2_V ds,
\]
where we used (16). Since the system (17) has a unique weak solution in $C([0,T];V)$, the system (21) has a unique weak solution in $C([0,T];V)$ for all initial data $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}$, then the corresponding (forward) solutions $\mathbf{x}(t)$ can be written in the form $\mathbf{x}(t) = S(t)\mathbf{x}_0$, where the semigroup $S(t)$ is uniquely determined by the dynamical system. This helps us to define a continuous semigroup $\{S(t)\}_{t \geq 0}$ in $\mathcal{X}$, where $S(t)\mathbf{u}_0 = \mathbf{U}(t)$, $t \geq 0$, where $\mathbf{U}(\cdot)$ is the unique weak solution of the system (22) with $\mathbf{U}(0) = \mathbf{U}_0 \in \mathcal{X}$, we define the space $\mathcal{X}$ with the norm $||\mathbf{U}||_{\mathcal{X}} = ||\mathbf{u}||_V^2 + \gamma ||\mathbf{v}||_V^2$.

Next, we prove the existence of a global attractor for the semigroup $S(t)$, $t \geq 0$, defined on $\mathcal{X}$ for the system (21). Our first aim is to establish the existence of an absorbing ball in $\mathcal{X}$ for $S(t), t \geq 0$ and then show that it is an asymptotically compact semigroup in $\mathcal{X}$. Finally, using Theorem 2.3.5, [5], we show the existence of a global attractor $S(t)$, $t \geq 0$ in $\mathcal{X}$. We also show that $S(t)$ defined on $\mathcal{X}_2 := \mathcal{V}_2 \times \mathcal{V}_2$ has an absorbing ball in $\mathcal{X}_2$, and hence the existence of an attractor in $\mathcal{X}_2$. Similar to the case of $\mathcal{X}$, we endow the space $\mathcal{X}_s$ with the norm $||\mathbf{U}||_{\mathcal{X}_s} = ||\mathbf{u}||_V^2 + \gamma ||\mathbf{v}||_V^2$, for $s \in \mathbb{R}$.

### 3.1. Absorbing ball in $\mathcal{X}$

Let $\mathbf{U}(t), t \geq 0$ be the unique weak solution of the system (21). Remember that $\mathbf{f}$ independent of $t$ in (21). Thanks to the existence and uniqueness of weak solution for the system (22), we can define a continuous semigroup $\{S(t)\}_{t \geq 0}$ in $\mathcal{X}$ by
\[
S(t)\mathbf{u}_0 = \mathbf{U}(t), t \geq 0,
\]
where $\mathbf{U}(\cdot)$ is the unique weak solution of (22) with $\mathbf{U}(0) = \mathbf{U}_0 \in \mathcal{X}$. Even though $\mathbf{v}_0 = \mathbf{0}$ in (21), we will be working with general $\mathbf{v}_0 \in \mathcal{V}$ in the rest of the paper. Let us first show that the map $S(t)$ is Lipschitz continuous on bounded subsets of $\mathcal{X}$.

**Lemma 3.1.** The map $S(t) : \mathcal{X} \to \mathcal{X}$, for $t \geq 0$, is Lipschitz continuous on bounded subsets of $\mathcal{X}$.

**Proof.** Let us take
\[
S(t)\mathbf{u}_0 = \mathbf{U}(t) = (\mathbf{u}_1(t), \mathbf{v}_1(t))^\top, S(t)\mathbf{v}_0 = \mathbf{V}(t) = (\mathbf{u}_2(t), \mathbf{v}_2(t))^\top,
\]
for all $t \geq 0$. Then $W(t) := U(t) - V(t) = (w(t), z(t))^\top$ satisfies:
\[
\begin{aligned}
\frac{d}{dt}[(I + \mu_1 A)w(t)] + \mu_0 Aw(t) + \gamma Az(t) + B(w(t), u_1(t)) + B(u_2(t), w(t)) &= 0, t \in (0, T), \\
\frac{dz(t)}{dt} - w(t) + \delta z(t) &= 0, t \in (0, T), \\
w(0) = w_0, z(0) = z_0.
\end{aligned}
\] (24)
\( w_0 \in V \) and \( z_0 \in V \). We take the inner product with \( w(\cdot) \) in the first equation in (24) to find
\[
\frac{1}{2} \frac{d}{dt} \left( \|w(t)\|_H^2 + \mu_1 \|\nabla w(t)\|_V^2 + \mu_0 \|\nabla w(t)\|_V^2 \right) = -\gamma (\nabla z(t), \nabla w(t)) + b(w(t), u_1(t), w(t)),
\]
where we used the fact that \( b(u_2, w, w) = 0 \). Next, we take the inner product with \( \gamma A z(\cdot) \) to the second equation in (24) to get
\[
\frac{\gamma}{2} \frac{d}{dt} \|\nabla z(t)\|_H^2 + \delta \gamma \|\nabla z(t)\|_H^2 = \gamma (\nabla w(t), \nabla z(t)).
\]
Adding (25) and (26), and integrating from 0 to \( t \), we obtain
\[
\|w(t)\|_H^2 + \mu_1 \|w(t)\|_V^2 + \gamma \|z(t)\|_V^2 + 2 \mu_0 \int_0^t \|w(s)\|_V^2 ds + 2 \delta \gamma \int_0^t \|z(s)\|_H^2 ds
\]
\[
= \|w_0\|_H^2 + \mu_1 \|w_0\|_V^2 + \gamma \|z_0\|_V^2 + 2 \int_0^t b(w(s), u_1(s), w(s)) ds.
\]
Using Hölder’s, Ladyzhenskaya’s and Poincaré’s inequalities, we estimate the term \( |b(w, u_1, w)| \) as
\[
|b(w, u_1, w)| \leq \|w\|_V^2 \|\nabla u_1\|_H \leq 2\|w\|_H^{1/2} \|\nabla w\|_H^{3/2} \|\nabla u_1\|_H \leq \frac{2}{\lambda_1^{1/4}} \|u_1\|_V \|w\|_V^2.
\]
We use (28) in (27) to deduce that
\[
\|w(t)\|_H^2 + \mu_1 \|w(t)\|_V^2 + \gamma \|z(t)\|_V^2 + 2 \mu_0 \int_0^t \|w(s)\|_V^2 ds + 2 \delta \gamma \int_0^t \|z(s)\|_H^2 ds
\]
\[
\leq \|w_0\|_H^2 + \mu_1 \|w_0\|_V^2 + \gamma \|z_0\|_V^2 + \frac{4}{\lambda_1^{1/4}} \mu_1 \int_0^t \|u_1(s)\|_V \|w(s)\|_V^2 ds.
\]
An application of Gronwall’s inequality in (29) yields
\[
\|w(t)\|_V^2 + \gamma \|z(t)\|_V^2
\]
\[
\leq \frac{\max\left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right) \right\}}{\min\{1, \mu_1\}} \left( \|w_0\|_V^2 + \gamma \|z_0\|_V^2 \right)^2 \exp\left( \frac{4}{\lambda_1^{1/4}} \mu_1 \int_0^t \|u_1(s)\|_V ds \right).
\]
Thus, we have
\[
\|S(t)U_0 - S(t)V_0\|_X
\]
\[
\leq \sqrt{\max\left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right) \right\}} \exp\left( \frac{1}{\lambda_1^{1/4}} \mu_1 \int_0^t \|u_1(s)\|_V ds \right) \|U_0 - V_0\|_X,
\]
and hence the map \( S(t) : X \to X \), for \( t \geq 0 \), is Lipschitz continuous on bounded subsets of \( X \).

**Proposition 1.** For \( 0 < \alpha \leq \min\left\{ \delta, \frac{\mu_0}{2(\alpha^2 + \mu_1)} \right\} \), the set
\[
B_1 := \left\{ V \in X : \|V\|_X \leq M_1 = \frac{1}{\sqrt{\mu_0 \min\{1, \mu_1\} \alpha}} \|f\|_V \right\},
\]
is a bounded absorbing set in $X$ for the semigroup $S(t)$. That is, given a bounded set $B \subset X$, there exists an entering time $t_B > 0$ such that $S(t)B \subset B_1$, for all $t \geq t_B$.

**Proof.** Since $\delta > 0$ and $\mu_0 > 0$, we can choose an $\alpha$ such that

$$0 < \alpha \leq \min \left\{ \delta, \frac{\mu_0}{2\left( \frac{1}{\lambda_1} + \mu_1 \right)} \right\}.$$  \hfill (33)

Let us multiply the first equation in (21) with $e^{\alpha t}$ and then take the inner product with $e^{\alpha t}u(\cdot)$ to obtain

$$\frac{1}{2} \frac{d}{dt} \left\{ e^{2\alpha t} \|u(t)\|_H^2 + \mu_1 \|\nabla u(t)\|_H^2 \right\} + \mu_0 e^{2\alpha t} \|\nabla u(t)\|_H^2$$

$$= \alpha e^{2\alpha t} \|u(t)\|_H^2 + \mu_1 \|\nabla u(t)\|_H^2 - \gamma e^{2\alpha t} \langle \nabla v(t), \nabla u(t) \rangle + e^{2\alpha t} (f, u(t)).$$  \hfill (34)

Similarly multiplying the second equation in (21) with $e^{\alpha t}$ and then taking the inner product with $e^{\alpha t}A v(\cdot)$, we find

$$\frac{\gamma}{2} \frac{d}{dt} e^{2\alpha t} \|\nabla v(t)\|_H^2 + \delta \gamma e^{2\alpha t} \|\nabla v(t)\|_H^2 = \alpha \gamma e^{2\alpha t} \|\nabla v(t)\|_H^2 + \gamma e^{2\alpha t} \langle \nabla u(t), \nabla v(t) \rangle.$$  \hfill (35)

Adding (34) and (35) and then integrating from 0 to $t$, we get

$$e^{2\alpha t} \left[ \|u(t)\|_H^2 + \mu_1 \|u(t)\|_V^2 + \gamma \|v(t)\|_V^2 \right] + 2\mu_0 \int_0^t e^{2\alpha s} \|u(s)\|_V^2 ds$$

$$+ 2\gamma (\delta - \alpha) \int_0^t e^{2\alpha s} \|v(s)\|_V^2 ds$$

$$= \|u_0\|_H^2 + \mu_1 \|u_0\|_V^2 + \gamma \|v_0\|_V^2 + 2\alpha \int_0^t e^{2\alpha s} \left[ \|u(s)\|_H^2 + \mu_1 \|u(s)\|_V^2 \right] ds$$

$$+ 2\int_0^t e^{2\alpha s} (f, u(s)) ds$$

$$\leq \|u_0\|_H^2 + \mu_1 \|u_0\|_V^2 + \gamma \|v_0\|_V^2 + 2\alpha \int_0^t e^{2\alpha s} \left[ \frac{1}{\lambda_1} + \mu_1 \right] \|u(s)\|_V^2 ds$$

$$+ 2\|f\|_V \int_0^t e^{2\alpha s} \|u(s)\|_V ds.$$  \hfill (36)

Thus from (36), we have

$$e^{2\alpha t} \left[ \|u(t)\|_H^2 + \mu_1 \|u(t)\|_V^2 + \gamma \|v(t)\|_V^2 \right] + 2\gamma (\delta - \alpha) \int_0^t e^{2\alpha s} \|v(s)\|_V^2 ds$$

$$+ \left[ \mu_0 - 2\alpha \left( \frac{1}{\lambda_1} + \mu_1 \right) \right] \int_0^t e^{2\alpha s} \|u(s)\|_V^2 ds$$

$$\leq \|u_0\|_H^2 + \mu_1 \|u_0\|_V^2 + \gamma \|v_0\|_V^2 + \frac{1}{2\mu_0 \alpha} e^{2\alpha t} \|f\|_V^2.$$  \hfill (37)

Hence, for $\alpha$ given in (33), we obtain

$$\|u(t)\|_V^2 + \gamma \|v(t)\|_V^2$$

$$\leq \max \left\{ \frac{\left( \frac{1}{\lambda_1} + \mu_1 \right)}{\min\{1, \mu_1\}} \right\} \left( \|u_0\|_H^2 + \gamma \|v_0\|_V^2 \right) e^{-2\alpha t} + \frac{1}{2\mu_0 \min\{1, \mu_1\}} \|f\|_V^2.$$  \hfill (38)
Then the linear system:

**Proposition 2.**

\[ \text{Let } G \text{ be a Galerkin approximation technique.} \]

Let \( G \) is asymptotically compact. The following proposition can be proved using a Faedo-Galerkin approximation technique.

**3.2. Asymptotic compactness.** Let us now show that the semigroup \( S(t) \), \( t \geq 0 \) is asymptotically compact. The following proposition can be proved using a Faedo-Galerkin approximation technique.

**Proposition 2.** Let \( s \in \mathbb{R} \). If \( Y_0 = (x_0, y_0)^T \in X_s, g \in L^2(0,T;V_{s-2}), s \in \mathbb{R}. \) Then the linear system:

\[
\begin{align*}
\frac{d}{dt} [(I + \mu_1 A)x(t)] + \mu_0 Ax(t) + \gamma Ay(t) &= g(t), \\
\frac{d}{dt} y(t) - x(t) + \delta y(t) &= 0, \\
x(0) &= x_0, \quad y(0) = y_0,
\end{align*}
\]

has a unique weak solution \( Y = (x, y)^T \in C([0,T]; V) \times C([0,T]; V) \). The weak solution also belongs to \( C([0,T]; V_s) \times C([0,T]; V_s) \) and the following estimate holds:

\[
\sup_{t \in [0,T]} \|Y(t)\|_{X_s} \leq C(\|Y_0\|_{X_s}) \|g\|_{L^2(0,T; V_{s-2})}.
\]

**Proposition 3.** Let \( f \in \mathbb{H} \) be time independent. Then the semigroup \( S(t), t \geq 0 \) is an asymptotically compact semigroup in \( V \).

**Proof.** Let \( f \in \mathbb{H} \) and \( (u_0, v_0)^T \in X \) be given. We write the semigroup \( S(t) \) as

\[ S(t)U_0 = R(t)U_0 + T(t)(U_0), \]

for all \( t \geq 0 \), and

\[
\limsup_{t \to \infty} [\|u(t)\|_V^2 + \gamma \|v(t)\|_V^2] \leq \frac{1}{2\mu_0 \min\{1, \mu_1\} \alpha} \|f\|_V^2. \tag{39}
\]

That is, we have

\[
\limsup_{t \to \infty} \|U(t)\|_X^2 \leq \frac{1}{2\mu_0 \min\{1, \mu_1\} \alpha} \|f\|_V^2. \tag{40}
\]

Integrating (38) from 0 to \( t \), we also obtain the following integral estimates:

\[
\int_0^t \|U(s)\|_X^2 ds \leq \frac{\max\left\{ \left( \frac{1}{\alpha} + \mu_1 \right), 1 \right\}}{2\alpha \min\{1, \mu_1\}} \left( \|u_0\|_V^2 + \gamma \|v_0\|_V^2 \right) + \frac{t \|f\|_V^2}{2\mu_0 \min\{1, \mu_1\} \alpha}, \tag{41}
\]

and

\[
\int_0^t \|U(s)\|_X ds \leq \frac{\sqrt{\max\left\{ \left( \frac{1}{\alpha} + \mu_1 \right), 1 \right\}}}{\alpha \min\{1, \mu_1\}} \|U_0\|_X + \frac{t \|f\|_V}{\sqrt{2\mu_0 \min\{1, \mu_1\} \alpha}}, \tag{42}
\]

for all \( t \geq 0 \). Moreover, it follows from (38) that the set

\[ B_1 := \left\{ V \in X : \|V\|_X \leq M_1 = \frac{1}{\sqrt{\mu_0 \min\{1, \mu_1\} \alpha}} \|f\|_V \right\}, \]

is absorbing in \( X \) for the semigroup \( S(t) \). Hence, the following uniform estimate is valid:

\[
\|U(t)\|_X \leq M_1, \quad \text{where } M_1 = \frac{1}{\sqrt{\mu_0 \min\{1, \mu_1\} \alpha}} \|f\|_V, \tag{43}
\]

for \( t \) large enough \( (t \gg 1) \) depending on the initial data. \( \square \)
where $R(t)$ is the semigroup generated by the linear problem:

\[
\begin{align*}
\frac{d}{dt} [(I + \mu_1 A)x(t)] + \mu_0 Ax(t) + \gamma Ay(t) &= 0, \\
\frac{d}{dt} y(t) - x(t) + \delta y(t) &= 0, \\
x(0) &= u_0, \quad y(0) = v_0,
\end{align*}
\]

(46)

and $Z(t) = T(t)(U_0)$ is the solution to the problem:

\[
\begin{align*}
\frac{d}{dt} [(I + \mu_1 A)w(t)] + \mu_0 Aw(t) + \gamma Az(t) &= f - B(u(t)), \\
\frac{d}{dt} z(t) - w(t) + \delta z(t) &= 0, \\
w(0) &= 0, \quad z(0) = 0.
\end{align*}
\]

(47)

The existence and uniqueness of a weak solution of (46) has been obtained in Proposition 2. Since $B(u, u) \in V'$ (see (13)), the system (47) has a unique weak solution. Let us denote $Y = (x, y)^T$ and $Z = (w, z)^T$. Then $U(t) = Y(t) + Z(t)$ is the unique weak solution of the problem (22) with the initial data $U_0$. Since $\delta > 0$ and $\mu_0 > 0$, we can choose an $\bar{\alpha}$ such that

\[
0 < \bar{\alpha} \leq \min \left\{ \delta, \frac{\mu_0}{\frac{1}{\lambda_1} + \mu_1} \right\}.
\]

(48)

Let us multiply the first equation in (46) with $e^{\bar{\alpha}t}$ and then take the inner product with $e^{\bar{\alpha}t}x(t)$ to get

\[
\frac{1}{2} \frac{d}{dt} \left[ e^{2\bar{\alpha}t} \left( \|x(t)\|_{H^1}^2 + \mu_1 \|\nabla x(t)\|_{H^1}^2 \right) \right] + \mu_0 e^{2\bar{\alpha}t} \|\nabla x(t)\|_{L^2}^2
\]

\[
= -\gamma e^{2\bar{\alpha}t} (\nabla y(t), \nabla x(t)) + \bar{\alpha} e^{2\bar{\alpha}t} \left( \|x(t)\|_{H^1}^2 + \mu_1 \|\nabla x(t)\|_{L^2}^2 \right).
\]

(49)

Now, we multiply second equation in (46) with $e^{\bar{\alpha}t}$ and then take the inner product with $\gamma e^{\bar{\alpha}t}y(t)$ to find

\[
\frac{\gamma}{2} \frac{d}{dt} \left[ e^{2\bar{\alpha}t} \|\nabla y(t)\|_{H^1}^2 \right] + \delta e^{2\bar{\alpha}t} \|\nabla y(t)\|_{L^2}^2 = \bar{\alpha} \gamma e^{2\bar{\alpha}t} \|\nabla y(t)\|_{H^1}^2 + \gamma e^{2\bar{\alpha}t} (\nabla x(t), \nabla y(t)).
\]

(50)

We add (49) and (50) and integrate from 0 to $t$ to obtain

\[
e^{2\bar{\alpha}t} \left( \|x(t)\|_{H^1}^2 + \mu_1 \|\nabla x(t)\|_{H^1}^2 + \gamma \|\nabla y(t)\|_{L^2}^2 \right) + 2\gamma (\delta - \bar{\alpha}) \int_0^t \|\nabla y(s)\|_{L^2}^2 ds
\]

\[+ 2 \left( \mu_0 - \bar{\alpha} \left( \frac{1}{\lambda_1} + \mu_1 \right) \right) \int_0^t \|\nabla x(s)\|_{H^1}^2 ds
\]

\[\leq \|u_0\|_{H^1}^2 + \mu_1 \|u_0\|_{L^2}^2 + \gamma \|v_0\|_{L^2}^2.
\]

(51)

For $\bar{\alpha}$ given in (48), from (51), we also have

\[
\|x(t)\|_{L^2}^2 + \gamma \|y(t)\|_{L^2}^2 \leq \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), \frac{1}{\min\{1, \mu_1\}} \right\} \left( \|u_0\|_{L^2}^2 + \gamma \|v_0\|_{L^2}^2 \right) e^{-2\bar{\alpha}t}.
\]

(52)
That is, we easily obtain
\[ \|Y(t)\|_{X} \leq \frac{\sqrt{\max\left\{ \frac{1}{\lambda}, \mu_1 \right\}}}{\sqrt{\min\{1, \mu_1\}}} \|Y_0\|_{X} e^{-\alpha t}, \]  
(53)
for all \( t \geq 0 \). Hence, the semigroup \( R(t) : X \to X \) is exponentially contractive.

Using Hölder’s and Gagliardo-Nirenberg’s inequalities, we estimate the term \( \|A^{-\frac{1}{2}}B(u, u)\|_{H} \) as
\[ \|A^{-\frac{1}{2}}B(u, u)\|_{H} = \sup_{\varphi \in V, \|A^{\frac{1}{2}}\varphi\|_{H}=1} b(u, u, \varphi) \leq \sup_{\varphi \in V, \|A^{\frac{1}{2}}\varphi\|_{H}=1} \|u\|_{L^\alpha} \|u\|_{V} \|\varphi\|_{L^3} \leq C \sup_{\varphi \in V, \|A^{\frac{1}{2}}\varphi\|_{H}=1} \|u\|_{L^\alpha} \|u\|_{V} \|A^{1/4}\varphi\|_{H} \leq C\|u\|_{V}^2, \]  
(54)
and hence we get \( B(u, u) \in L^\infty([0,T]; V_{\frac{1}{2}}) \). Since \( u(t) \) is the weak solution of the system (17) with \( u_0 \in V \), we know that \( u \in L^\infty(0,T; V) \). Using Proposition 2 and (54), we know that the solution \( Z(t) \) of the problem (47) belongs to \( C([0,T]; V_{\frac{1}{2}}) \times C([0,T]; V_{\frac{1}{2}}) \). Thus the operator \( T(t) \) maps \( X \) to \( X_2 = V_{\frac{1}{2}} \times V_{\frac{1}{2}} \). Note that the embedding \( X_2 \subset X \) is compact and this implies that the operator \( T(t) \) is a compact operator for each \( t > 0 \). Thus, the semigroup \( S(t) \) satisfies the conditions of Theorem 3.3, [28] (see Theorem 2.1, [25] also) and hence \( S(t), t \geq 0 \) is an asymptotically compact semigroup, which completes the proof.

Hence, we have the following theorem on the existence of a global attractor in \( X \) for the semigroup associated with the system (21).

**Theorem 3.2.** Let \( f \in H \). Then the semigroup \( S(t) : X \to X \) has an absorbing ball \( B_1 = \{ V \in X : \|V\|_X \leq M_1 \} \) and a global attractor \( A_{glob} \subset X \). Moreover, the attractor \( A_{glob} \) is compact, connected and invariant.

**Proof.** Invoking Theorem 2.3.5, [5], we get the required result. \( \Box \)

Our next aim is to establish that the global attractor \( A_{glob} \) is a bounded subset of \( X_2 = V_{\frac{1}{2}} \times V_{\frac{1}{2}} \).

**Proposition 4.** The global attractor \( A_{glob} \) is a bounded subset of \( X_2 \).

**Proof.** Note that \( U(t) = Y(t) + Z(t) \in A_{glob} \). For \( \alpha \) given in (33), let us multiply the first equation in (47) with \( e^{\alpha t} \) and then take the inner product with \( e^{\alpha t}A^{\frac{3}{2}}w(\cdot) \) to obtain
\[ \frac{1}{2} \frac{d}{dt} \left[ e^{2\alpha t} \left( \|A^{\frac{3}{2}}w(t)\|_{H}^2 + \mu_1 \|A^{\frac{3}{2}}w(t)\|_{H}^2 \right) \right] + \mu_0 e^{2\alpha t} \|A^{\frac{3}{2}}w(t)\|_{H}^2 = -\gamma e^{2\alpha t} \left( A^{\frac{3}{2}}z(t), A^{\frac{3}{2}}w(t) \right) + e^{2\alpha t} \left( f, A^{\frac{3}{2}}w(t) \right) + \alpha e^{2\alpha t} \left( \|A^{\frac{3}{2}}w(t)\|_{H}^2 + \mu_1 \|A^{\frac{3}{2}}w(t)\|_{H}^2 \right) - e^{2\alpha t} \left( B(u(t)), A^{\frac{3}{2}}w(t) \right). \]  
(55)
Now we multiply the second equation in (47) with \( e^{\alpha t} \) and then take the inner product with \( \gamma e^{\alpha t}A^{\frac{3}{2}}z(\cdot) \) to find
\[ \gamma \frac{d}{dt} \left[ e^{2\alpha t} \|A^{\frac{3}{2}}z(t)\|_{H}^2 \right] + \gamma e^{2\alpha t} \|A^{\frac{3}{2}}z(t)\|_{H}^2 \]
to inequality (53), we find that
\[
\|z(t)\|^2 + \gamma e^{2\alpha t} \langle A^\frac{3}{2} w(t), A^\frac{3}{2} z(t) \rangle.
\]
Combining the above two equations, we get
\[
\frac{1}{2} \frac{d}{dt} \left( e^{2\alpha t} \left( \|A^\frac{1}{2} w(t)\|^2 + \mu_1 \|A^\frac{3}{2} w(t)\|^2 + \gamma \|A^\frac{3}{2} z(t)\|^2 \right) \right) + \mu_0 e^{2\alpha t} \|A^\frac{3}{2} w(t)\|^2 + \delta \gamma e^{2\alpha t} \|A^\frac{3}{2} z(t)\|^2 = e^{2\alpha t} \langle f, A^\frac{3}{2} w(t) \rangle + \alpha e^{2\alpha t} \left( \|A^\frac{1}{2} w(t)\|^2 + \mu_1 \|A^\frac{3}{2} w(t)\|^2 + \|A^\frac{3}{2} z(t)\|^2 \right)
\]
\[
- e^{2\alpha t} \langle B(u(t)), A^\frac{3}{2} w(t) \rangle.
\]
The first term from the right hand side of the equality (56) can be estimated using the Cauchy-Schwarz inequality and Young's inequality as
\[
|\langle f, A^\frac{3}{2} w \rangle| = |(A^{-\frac{1}{2}} f, A^\frac{3}{2} w)| \leq \|A^{-\frac{1}{2}} f\| \|A^\frac{3}{2} w\| \leq \frac{\mu_0}{4} \|A^\frac{3}{2} w\|^2 + \frac{1}{\mu_0} \|A^{-\frac{1}{2}} f\|^2.
\]
(57)
Using (54), we estimate \(|\langle B(u), A^\frac{3}{2} w \rangle| as
\[
|\langle B(u), A^\frac{3}{2} w \rangle| = |(A^{-\frac{1}{2}} B(u), A^\frac{3}{2} w)| \leq \|A^{-\frac{1}{2}} B(u)\| \|A^\frac{3}{2} w\| \leq C \|u\| \|A^\frac{3}{2} w\|
\]
\[
\leq \frac{\mu_0}{4} \|A^\frac{3}{2} w\|^2 + \frac{C}{\mu_0} \|u\|^2.
\]
(58)
Combining (57) and (58), substituting it in (56) and then integrating from 0 to t, we find
\[
e^{2\alpha t} \left( \|A^\frac{1}{2} w(t)\|^2 + \mu_1 \|A^\frac{3}{2} w(t)\|^2 + \gamma \|A^\frac{3}{2} z(t)\|^2 \right) + \mu_0 e^{2\alpha t} \|A^\frac{3}{2} w(t)\|^2 + \delta \gamma e^{2\alpha t} \|A^\frac{3}{2} z(t)\|^2 \leq \int_0^t e^{2\alpha s} \|A^\frac{3}{2} z(s)\|^2 ds
\]
\[
+ \left[ \mu_0 - 2\alpha \left( \frac{1}{\lambda_1} + \mu_1 \right) \right] \int_0^t e^{2\alpha s} \|A^\frac{3}{2} w(s)\|^2 ds
\]
\[
\leq \int_0^t e^{2\alpha s} \left( \frac{1}{\mu_0} \|A^{-\frac{1}{2}} f\|^2 + \frac{C}{\mu_0} \|u(s)\|^2 \right) ds.
\]
(59)
For \(\alpha\) given in (33) and for large t, using (43) in (59), we get
\[
e^{2\alpha t} \left( \|A^\frac{1}{2} w(t)\|^2 + \mu_1 \|A^\frac{3}{2} w(t)\|^2 + \gamma \|A^\frac{3}{2} z(t)\|^2 \right)
\]
\[
\leq \frac{2}{\mu_0} \left( \|A^{-\frac{1}{2}} f\|^2 + CM_1^2 \right) \int_0^t e^{2\alpha s} ds.
\]
Hence, we have
\[
\|A^\frac{3}{2} w(t)\|^2 + \gamma \|A^\frac{3}{2} z(t)\|^2 \leq \frac{1}{\mu_0 \min\{\mu_1, 1\} \alpha} \left( \|A^{-\frac{1}{2}} f\|^2 + CM_1^2 \right) = L_0,
\]
that is,
\[
\|Z(t)\|_{X_2} \leq L_0, \quad \text{for } t \gg 1.
\]
Remember that the global attractor \(A_{\text{glob}}\) is invariant, \(S(t)A_{\text{glob}} = A_{\text{glob}}\) and thanks to inequality (53), we find that
\[
\|U(t) - Z(t)\|_X = \|Y(t)\|_X \leq \sqrt{\max\{\frac{1}{\lambda_1} + \mu_1, 1\}} \|Y_0\|_X e^{-\delta t},
\]
for all \( t \geq 0 \), where \( \alpha \) is defined in (48). From the above inequality, we infer that for each \( V \in A_{glob} \), there exists a sequence \( \{Z(t_k)\} \), \( t_k \to \infty \), corresponding to \( U_k(0) \in A_{glob} \) such that

\[
V = \lim_{k \to \infty} Z(t_k), \quad U_k(0) \in A_{glob}.
\] (61)

The inequality (60) tells us that the sequence \( \{Z(t_k)\} \) belongs to a ball in \( X_2 \) with radius \( L_0 \), depending only on \( M_1 \) and \( ||f||_{V \frac{1}{2}} \). Thus, the sequence \( \{Z(t_k)\} \) is precompact in \( X_2 \). Using (60) and the lowersemicontinuity property \( ||V||_{V \frac{1}{2}} \leq \liminf ||Z(t_k)||_{V \frac{1}{2}} \) easily gives that \( A_{glob} \) is bounded in \( X_2 \).

If we multiply the first equation in (47) with \( e^{\alpha t} \), take the inner product with \( e^{\alpha t} A^\frac{1}{2} w(\cdot) \), and the second equation in (47) with \( e^{\alpha t} \), take the inner product with \( \gamma e^{\alpha t} A^\frac{1}{2} z(\cdot) \), and then use similar arguments as above, one can show that \( A_{glob} \) is bounded in \( X_2 = V_2 \times V_2 \). The estimate (58) shall be replaced as

\[
||B(u), A^\frac{1}{2} w|| \leq C||A^\frac{1}{2} B(u), A^\frac{1}{2} w|| \leq C||A^\frac{1}{2} B(u)||_H ||A^\frac{1}{2} w||_H. \] (62)

We can estimate \( ||A^\frac{1}{2} B(u)||_H \) using Hölder’s and Gagliardo-Nirenberg’s inequalities as

\[
||A^\frac{1}{2} B(u)||_H = \sup_{\varphi \in V, ||A^\frac{1}{2} \varphi||_H = 1} b(u, u, \varphi) \leq \sup_{\varphi \in V, ||A^\frac{1}{2} \varphi||_H = 1} ||u||_{L^\infty} ||u||_V ||\varphi||_V \end{align*}

\[
\leq C \sup_{\varphi \in V, ||A^\frac{1}{2} \varphi||_H = 1} ||u||_{L^\infty} ||u||_V ||\varphi||_V = \sup_{\varphi \in V, ||A^\frac{1}{2} \varphi||_H = 1} ||u||_V ||u||_V ||A^\frac{1}{2} \varphi||_H = \frac{C}{\lambda_1^2} ||u||_V ||u||_V ||A^\frac{1}{2} \varphi||_H,
\]

so that from (62), we have

\[
||B(u), A^\frac{1}{2} w|| \leq \frac{C}{\lambda_1^2} ||u||_V ||u||_V ||A^\frac{1}{2} w|| \leq \frac{\mu_0}{4} ||A^\frac{1}{2} w||_H^2 + \frac{C}{\mu_0 \lambda_1^2} ||u||_V ||u||_V ||A^\frac{1}{2} \varphi||_H^2.
\]

Multiplying the first equation in (47) with \( e^{\alpha t} \), taking the inner product with \( e^{\alpha t} A^\frac{1}{2} w(\cdot) \) and the second equation in (47) with \( e^{\alpha t} \), taking the inner product with \( \gamma e^{\alpha t} A^\frac{1}{2} z(\cdot) \), and then using similar arguments as above, we can show that \( A_{glob} \) is bounded in \( X_2 = V_2 \times V_2 \) also. In this case, one has to replace (58) with

\[
||B(u), Aw|| \leq ||B(u)||_H ||Aw||_H \leq ||u||_{L^\infty} ||u||_V ||Aw||_H \leq C||u||_V ||u||_V ||Aw||_H \leq \frac{\mu_0}{4} ||Aw||_H^2 + \frac{C}{\mu_0 \lambda_1^2} ||u||_V ||u||_V ||A^\frac{1}{2} \varphi||_H^2,
\]

where we used the fact that \( V_2 \subset L^\infty(\Omega) \) (see Remark 1).

### 3.3. Absorbing ball in \( X_2 \)

Let us now show that the semigroup \( S(t) : X_2 \to X_2 \) has an absorbing ball in \( X_2 = V_2 \times V_2 = D(A) \times D(A) \).

**Proposition 5.** For \( \alpha \) given in (33), the set

\[
E_2 := \left\{ V \in X_2 : ||V||_{X_2} \leq M_2 \equiv \frac{2}{\sqrt{\mu_0 \min(1, \mu_1)} \alpha} \left( ||f||_H^2 + \frac{27CM_1}{8\mu_0 \alpha} \right)^{1/2} \right\},
\]
is a bounded absorbing set in $\mathcal{X}_2$ for the semigroup $S(t)$. That is, given a bounded set $B \subset \mathcal{X}_2$, there exists an entering time $t_B > 0$ such that $S(t)B \subset B_2$, for all $t \geq t_B$.

Proof. We multiply the first equation in (21) with $e^{\alpha t}$ and take the inner product with $e^{\alpha t}Au(\cdot)$ to obtain

$$\frac{d}{dt} e^{2\alpha t} \left[ \| A \frac{d}{dt} u(t) \|_H^2 + \mu_1 \| Au(t) \|_H^2 + \| Au(t) \|_H^2 \right] + \mu_0 e^{2\alpha t} \| Au(t) \|_H^2 = -\gamma e^{2\alpha t} \langle Av(t), Au(t) \rangle - e^{2\alpha t} \langle B(u(t)), Au(t) \rangle + e^{2\alpha t} \langle f, Au(t) \rangle + \alpha e^{2\alpha t} \left( \| A \frac{d}{dt} u(t) \|_H^2 + \mu_1 \| Au(t) \|_H^2 \right),$$

(63)

where $\alpha$ is given in (33). Let us now multiply the second equation in (21) with $e^{\alpha t}$ and take the inner product with $\gamma e^{2\alpha t}A^2v(\cdot)$ to get

$$\frac{\gamma d}{dt} e^{2\alpha t} \| Av(t) \|_H^2 + \delta \gamma e^{2\alpha t} \| Av(t) \|_H^2 = \alpha \gamma e^{2\alpha t} \| Av(t) \|_H^2 + \gamma e^{2\alpha t} \langle Au(t), Av(t) \rangle.$$  

(64)

Combining (63) and (64), we arrive at

$$\frac{d}{dt} e^{2\alpha t} \left[ \| A \frac{d}{dt} u(t) \|_H^2 + \mu_1 \| Au(t) \|_H^2 + \gamma \| Av(t) \|_H^2 \right] + \mu_0 e^{2\alpha t} \| Au(t) \|_H^2 + \delta \gamma \| Av(t) \|_H^2 = -e^{2\alpha t} \langle B(u(t)), Au(t) \rangle + e^{2\alpha t} \langle f, Au(t) \rangle + \alpha e^{2\alpha t} \left( \| A \frac{d}{dt} u(t) \|_H^2 + \mu_1 \| Au(t) \|_H^2 + \| Av(t) \|_H^2 \right).$$

(65)

We estimate $\langle (f, Au(t)) \rangle$ using the Cauchy-Schwarz inequality and Young’s inequality as

$$\langle (f, Au(t)) \rangle \leq \| f \|_V \| Au(t) \|_H \leq \frac{\mu_0}{4} \| Au(t) \|_H^2 + \frac{1}{\mu_0} \| f \|_V^2.$$

(66)

Using Hölder’s, Agmon’s and Young’s inequalities, we estimate $\| (B(u), Au) \|$ as

$$\| (B(u), Au) \| \leq \| u \|_L^2 \| u \|_V \| Au \|_H \leq C \| u \|_V^{3/2} \| Au \|_H^{3/2} \leq \frac{\mu_0}{4} \| Au \|_H^2 + \frac{27C}{4\mu_0^2} \| u \|_V^6,$$

(67)

where $C$ is the constant appearing in Agmon’s inequality. Combining (66) and (67), using it in (65), we find

$$e^{2\alpha t} \left( \| A \frac{d}{dt} u(t) \|_H^2 + \mu_1 \| Au(t) \|_H^2 + \gamma \| Av(t) \|_H^2 \right) + 2\gamma (\delta - \alpha) \int_0^t \| Av(s) \|_H^2 \, ds$$

$$+ \left[ \mu_0 - 2\alpha \left( \frac{1}{\lambda_1} + \mu_1 \right) \right] \int_0^t e^{2\alpha s} \| Au(s) \|_H^2 \, ds$$

$$\leq \left( \| A \frac{d}{dt} u_0 \|_H^2 + \mu_1 \| Au_0 \|_H^2 + \gamma \| Av_0 \|_H^2 \right)$$

$$+ \frac{2}{\mu_0} \left( \int_0^t e^{2\alpha s} \| f \|_V^2 \, ds + \frac{27C}{4\mu_0^2} \int_0^t e^{2\alpha s} \| u(s) \|_V^6 \, ds \right).$$

(68)

For $\alpha$ satisfying (33) and large enough $t$, using (43) in (68), we deduce that

$$\| Au(t) \|_H^2 + \gamma \| Av(t) \|_H^2 \leq \frac{\max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}}{\min \{ 1, \mu_1 \}} \left( \| Au_0 \|_H^2 + \gamma \| Av_0 \|_H^2 \right) e^{-2\alpha t}.$$
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\[ + \frac{1}{\min\{1, \mu_1\} \mu_0 \alpha} \left( \|f\|_H^2 + \frac{27CM_1^6}{8\mu_0^2} \right). \]

That is, we have

\[ \|A(t)U\|_X \leq \sqrt{\max\{\left\lfloor \frac{1}{\lambda_1} + \mu_1 \right\rfloor, 1\}} \|AU_0\|_X e^{-\alpha t} \]

\[ + \frac{1}{\sqrt{\min\{1, \mu_1\} \mu_0 \alpha}} \left( \|f\|_H^2 + \frac{27CM_1^6}{8\mu_0^2} \right)^{1/2}. \]

Hence, the inequality (69) assures the existence of an absorbing ball:

\[ B_2 := \left\{ V \in X_2 : \|V\|_{X_2} \leq M_2 = \frac{2}{\sqrt{\min\{\mu_1, 1\} \mu_0 \alpha}} \left( \|f\|_H^2 + \frac{27CM_1^6}{8\mu_0^2} \right)^{1/2} \right\}. \]

so that for all \( t \gg 1 \), we have \( \|A(t)U\|_X \leq M_2 \).

Thus, we have the following theorem on the existence of a global attractor in \( X_2 \) for the semigroup \( S(t) \) associated with the system (21).

**Theorem 3.3.** Let \( f \in H \). Then the semigroup \( S(t) : X_2 \to X_2 \) has a global attractor \( A^2_{\text{glob}} \subset X_2 \). The attractor \( A^2_{\text{glob}} \) is compact, connected and invariant. Furthermore, \( A^2_{\text{glob}} \) is a bounded set in \( X_3 \).

**Remark 2.** It should be noted that if we take \( f \in V_1 \) instead of \( f \in H \) in Theorem 3.2, then the attractors \( A_{\text{glob}} \) and \( A^2_{\text{glob}} \) coincide.

4. **Estimates for the Number of Determining Modes.** In this section, we provide estimates for the number of determining modes for both asymptotic (as \( t \to \infty \)) and trajectories on the attractor (for \( t \in \mathbb{R} \)) for three dimensional Kelvin-Voigt equations with “fading memory”. The concept of number of determining modes for asymptotic behavior for two dimensional Navier-Stokes equations is characterized in \([16]\). The authors in \([16]\) proved that there exists a number \( m \) such that if the first \( m \) Fourier modes of two different solutions of the 2D Navier-Stokes equations have the same asymptotic behavior, then the remaining infinitely many number of modes have the same asymptotic behavior. The number determining modes for trajectories on the global attractor for the 2D Navier-Stokes equations with Dirichlet boundary conditions is obtained in \([27]\). The author in \([27]\) established that there exists a number \( m \) such that if projections of two different trajectories on the global attractor on the \( m \) dimensional subspace of \( H \), spanned by the first \( m \) eigenfunctions of the Stokes operator, coincide for each \( t \in \mathbb{R} \), then these trajectories completely coincide for each \( t \in \mathbb{R} \). The number of determining modes estimate for both asymptotic and trajectories on the global attractor for 3D Navier-Stokes-Voigt equations is provided in \([25]\). The theory developed in \([16]\) and \([27]\) has been generalized and applied to different kinds of infinite dimensional dissipative systems including Navier-Stokes equations (see for example, \([6, 16, 17, 22, 23, 24, 28, 29, 36]\), etc and references therein).
4.1. Asymptotic determining modes. In this subsection, we provide an estimate for the number of asymptotically determining modes for the system (21). Let $P_m$ denote the $L^2$-orthogonal projection from $H$ onto the $m$-dimensional subspace $H_m = \text{span}\{e_1, e_2, \ldots, e_m\}$, where $\{e_k\}_{k=1}^\infty$ is the eigenbasis of the Stokes operator $A$. Let us also set $Q_m = I - P_m$. We take $U = (u_1, v_1)^T$ and $V = (u_2, v_2)^T$ as any two solutions of the system (22). Then, we have

$$\begin{aligned}
\frac{d}{dt} [(I + \mu_1 A)u_1(t)] + \mu_0 A u_1(t) + \gamma A v_1(t) + B(u_1(t), u_1(t)) &= f, \ t \in (0, T), \\
\frac{d}{dt} v_1(t) + \delta v_1(t) - u_1(t) &= 0, \ t \in (0, T), \\
u_1(0) &= u_0, \ v_1(0) = v_0,
\end{aligned}$$

(70)

and

$$\begin{aligned}
\frac{d}{dt} [(I + \mu_1 A)u_2(t)] + \mu_0 A u_2(t) + \gamma A v_2(t) + B(u_2(t), u_2(t)) &= f, \ t \in (0, T), \\
\frac{d}{dt} v_2(t) + \delta v_2(t) - u_2(t) &= 0, \ t \in (0, T), \\
u_2(0) &= u_0, \ v_1(0) = v_0,
\end{aligned}$$

(71)

where $(u_0^1, v_0^1)^T, (u_0^2, v_0^2)^T \in X$. Let us first give the definition of asymptotically determining modes.

**Definition 4.1.** A set of modes $\{e_1, \ldots, e_m\}$ is called asymptotically determining if

$$\lim_{t \to \infty} \|V(t) - U(t)\|_X = 0, \quad \text{whenever} \quad \lim_{t \to \infty} \|P_m(U(t) - V(t))\|_X = 0,$$

where $P_m U = (P_m u_1, P_m v_1)^T$ and $P_m V = (P_m u_2, P_m v_2)^T$.

Now we prove the following theorem on number of asymptotically determining modes for the system (21).

**Theorem 4.2.** Let us assume that

$$\lim_{t \to \infty} \|P_m(U(t) - V(t))\|_X = 0.$$

Then the first $m$ eigenfunctions of the Stokes operator are asymptotically determining for the Kelvin-Voigt equations with “fading memory” having homogeneous Dirichlet boundary conditions, provided that $m$ is large enough such that

$$\lambda_{m+1} < \frac{C||f||_{V'}}{\mu_0^2(\min\{1, \mu_1\})^2 \alpha^{-2}},$$

(72)

where $\alpha$ is given in (33).

**Proof.** Let us define $W = U - V$, where $U = (u_1, v_1)^T$ and $V = (u_2, v_2)^T$. Then $W = (w, z)^T = (u_1 - u_2, v_1 - v_2)^T$ satisfies

$$\begin{aligned}
\frac{d}{dt} [(I + \mu_1 A)w(t)] + \mu_0 A w(t) + \gamma A z(t) + B(u_1(t), w(t)) + B(w(t), u_1(t)) - B(w(t), w(t)) &= 0, \ t \in (0, T), \\
\frac{d}{dt} z(t) + \delta z(t) - w(t) &= 0, \ t \in (0, T), \\
w(0) &= w_0, \ z(0) = z_0,
\end{aligned}$$

(73)
where \( w_0 = u_1^0 - u_0^0 \) and \( z_0 = v_1^0 - v_0^0 \). For \( \alpha \) given in (33), from (39), we know that
\[
\limsup_{t \to \infty} ||U(t)||_X, \limsup_{t \to \infty} ||V(t)||_X \leq \frac{||f||_Y}{\sqrt{2\mu_0 \min\{1, \mu_1\} \alpha}}. \tag{74}
\]
Remember that for \( \varphi \in V \), we have
\[
P_m \varphi = \sum_{j=1}^{m} (\varphi, e_j)e_j, \quad A^2 P_m \varphi = \sum_{j=1}^{m} \lambda_j^{1/2} (\varphi, e_j)e_j, \\
Q_m \varphi = \sum_{j=m+1}^{\infty} (\varphi, e_j)e_j, \quad A^2 Q_m \varphi = \sum_{j=m+1}^{\infty} \lambda_j^{1/2} (\varphi, e_j)e_j,
\]
\[
||A^2 Q_m \varphi||_H^2 = \sum_{j=m+1}^{\infty} \lambda_j (\varphi, e_j)^2 \geq \lambda_{m+1} \sum_{j=m+1}^{\infty} (\varphi, e_j)^2 = \lambda_{m+1} ||Q_m \varphi||_H^2,
\]
and
\[
||A^2 P_m \varphi||_H^2 = \sum_{j=1}^{m} \lambda_j (\varphi, e_j)^2 \leq \lambda_m \sum_{j=1}^{m} (\varphi, e_j)^2 = \lambda_m ||P_m \varphi||_H^2.
\]
That is, we get
\[
||Q_m \varphi||_Y \geq \sqrt{\lambda_{m+1}} ||Q_m \varphi||_H \quad \text{and} \quad ||P_m \varphi||_Y \leq \sqrt{\lambda_m} ||P_m \varphi||_H. \tag{75}
\]
With the condition given in (72), we can choose an \( \eta \) such that
\[
0 < \eta \leq \min \left\{ \delta, \frac{1}{\lambda_{m+1} + \mu_1} \left[ \mu_0 - \frac{C}{\lambda_{m+1}^{1/4} \sqrt{\min\{1, \mu_1\} \mu_0 \alpha}} \right] \right\}. \tag{76}
\]
We define \( p(t) = P_m w(t), q(t) = Q_m w(t), s(t) = P_m z(t) \) and \( r(t) = Q_m z(t) \).
We multiply with \( e^{nt} \) to the first equation in (73), take the inner product with \( e^{nt} q(t) = e^{nt} Q_m w(t) \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( e^{2nt} \left( ||q(t)||_H^2 + \mu_1 ||q(t)||_V^2 \right) \right) + \mu_0 e^{2nt} ||q(t)||_Y^2 + e^{2nt} b(q(t), u_1(t), q(t)) \\
= -\gamma e^{2nt} \nabla q(t), \nabla q(t) - e^{2nt} b(u_1(t), p(t), q(t)) - e^{2nt} b(u_1(t), p(t), q(t)) - e^{2nt} b(p(t), q(t)) + \eta e^{2nt} ||q(t)||_H^2 \tag{77}
\]
Let us now multiply with \( e^{nt} \) to the second equation in (73), take the inner product with \( \gamma e^{nt} A q(t) = \gamma e^{nt} A Q_m z(t) \) to find
\[
\frac{\gamma}{2} \frac{d}{dt} \left( e^{2nt} ||r(t)||_V^2 \right) + \gamma \delta e^{2nt} ||r(t)||_V^2 = \eta \gamma e^{2nt} ||r(t)||_V^2 + \gamma e^{2nt} \nabla q(t), \nabla r(t) \tag{78}
\]
Adding (77) and (78), we get
\[
\frac{1}{2} \frac{d}{dt} \left( e^{2nt} \left( ||q(t)||_H^2 + \mu_1 ||q(t)||_V^2 + \gamma ||r(t)||_V^2 \right) \right) + \mu_0 e^{2nt} ||q(t)||_Y^2 + \delta e^{2nt} ||r(t)||_V^2 \\
= -e^{2nt} b(q(t), u_1(t), q(t)) - e^{2nt} b(u_1(t), p(t), q(t)) - e^{2nt} b(p(t), u_1(t), q(t)) - e^{2nt} b(p(t), p(t), q(t)) + \eta e^{2nt} ||q(t)||_H^2 + \mu_1 ||q(t)||_V^2 + \gamma ||r(t)||_V^2 \tag{79}
\]
Using Hölder’s, Ladyzhenskaya’s and Young’s inequalities, and (75), we estimate $b(q, u, q)$ as

$$
|b(q, u, q)| \leq \|q\|_2^2, |u|_1 \|v\| \leq 2\|q\|_{1/2}^1 \|q\|_{1/2}^{3/2} \|u\|_1 \|v\| \leq \frac{2}{\lambda_{m+1}^{-4}} \|q\|_2^2 \|u\|_1 \|v\|. \tag{80}
$$

Once again using Hölder’s, Gagliardo-Nirenberg’s and Young’s inequalities, and (75), we estimate $|b(u_1, p, q)|$ as

$$
|b(u_1, p, q)| \leq \|u_1\|_2 \|p\|_1 \|q\|_2 \|v\| \leq C\|u_1\|_1 \|p\|_1 \|q\|_2 \|v\| \leq C\lambda_{m+1}^{-4} \|u_1\|_1 \|p\|_1 \|q\|_2 \|v\|. \tag{81}
$$

Similarly, we have

$$
|b(p, u_1, q)| \leq \|p\|_1 \|u_1\|_1 \|q\|_2 \|v\| \leq C\lambda_{m+1}^{-4} \|u_1\|_1 \|p\|_1 \|q\|_2 \|v\|, \tag{82}
$$

$$
|b(q, p, q)| \leq \|q\|_2 \|p\|_1 \|q\|_2 \|v\| \leq C\lambda_{m+1}^{-4} \|u_1\|_1 \|p\|_1 \|q\|_2 \|v\|. \tag{83}
$$

Combining (80)-(83) and substituting it in (79), we find

$$
\frac{1}{2} \frac{d}{dt} e^{2\eta t} \left(\frac{\|q(t)\|_2^2}{\lambda_{m+1}^{-1}} + \mu_1 \|q(t)\|_2^2 + \gamma \|r(t)\|_2^2\right)
$$

$$
+ \left[\mu_0 - \left(\frac{1}{\lambda_1} + \mu_1\right)\eta - \frac{C}{\lambda_{m+1}^{-1}}\right] \left(\|u_1(t)\|_1 \|v\| + \|u_2(t)\|_1 \|v\|\right)e^{2\eta t} \|q(t)\|_2^2
$$

$$
+ \gamma (\delta - \eta) e^{2\eta t} \|r(t)\|_2^2
$$

$$
\leq \frac{C}{\lambda_{m+1}^{-4}} e^{2\eta t} \left(\|u_1(t)\|_1 \|v\| + \|u_2(t)\|_1 \|v\|\right)\|p(t)\|_2^2. \tag{84}
$$

We choose $t_1$ large enough such that $\|u_1(t)\|_2 \leq M_1$, for all $t \geq t_1$ and choose $m$ large enough so that

$$
\left[\mu_0 - \left(\frac{1}{\lambda_1} + \mu_1\right)\eta - \frac{C M_1}{\lambda_{m+1}^{-1}}\right] \leq 0, \text{ and } 0 < \eta(m) \leq \delta.
$$

Then from (84), we get

$$
\frac{1}{2} \frac{d}{dt} e^{2\eta t} \left(\frac{\|q(t)\|_2^2}{\lambda_1} + \mu_1 \|q(t)\|_2^2 + \gamma \|r(t)\|_2^2\right) \leq \frac{C M_1}{\lambda_1^{-4}} e^{2\eta t} \|p(t)\|_2^2,
$$

for a.e. $t \geq t_1$. Integrating the above inequality from 0 to $t$, we find

$$
e^{2\eta t} \left(\frac{\|q(t)\|_2^2}{\lambda_1} + \mu_1 \|q(t)\|_2^2 + \gamma \|r(t)\|_2^2\right) + 2\gamma (\delta - \eta) \int_0^t \|r(s)\|_2^2 ds
$$

$$
\leq \|q_0\|_2^2 + \mu_1 \|q_0\|_2^2 + \gamma \|r_0\|_2^2 + \frac{C M_1}{\lambda_1^{-4}} \int_0^t e^{2\eta s} \|p(s)\|_2^2 ds. \tag{85}
$$

Thus, from (85), we deduce that

$$
\|q(t)\|_2^2 + \|r(t)\|_2^2
$$

$$
\leq \frac{e^{-2\eta t}}{\min\{1, \mu_1\}} \left(\left(\frac{1}{\lambda_1} + \mu_1\right) \|q_0\|_2^2 + \gamma \|r_0\|_2^2 + \frac{C M_1}{\lambda_1^{-4}} \int_0^t e^{2\eta s} \|p(s)\|_2^2 ds\right), \tag{86}
$$
for all $t \geq t_1$. Since $\|q_0\|_{\mathcal{V}} \leq \|Q_m(u_0^1 - u_0^2)\|_{\mathcal{V}} \leq \|u_0^1 - u_0^2\|_{\mathcal{V}}$, and $\|r_0\|_{\mathcal{V}} \leq \|Q_m(v_0^1 - v_0^2)\|_{\mathcal{V}} \leq \|v_0^1 - v_0^2\|_{\mathcal{V}}$, the first term on the right hand side of the inequality (86) tends to zero as $t \to \infty$. Since $\lim_{t \to \infty} \|P_m(u_1(t) - u_2(t))\|_{\mathcal{V}} = 0$, there exists an $N > 0$ such that $\|P_m(u_1(t) - u_2(t))\|_{\mathcal{V}} \leq \varepsilon$, for all $t > N$. Thus, we have

$$\int_0^t e^{2\eta s}\|p(s)\|_{\mathcal{V}}^2 ds = \int_0^N e^{2\eta s}\|p(s)\|_{\mathcal{V}}^2 ds + \int_N^t e^{2\eta s}\|p(s)\|_{\mathcal{V}}^2 ds$$

$$\leq \int_0^N e^{2\eta s}\|p(s)\|_{\mathcal{V}}^2 ds + \frac{\varepsilon^2}{2\eta} (e^{2\eta t} - e^{2\eta N}),$$

so that

$$\lim_{t \to \infty} e^{-2\eta t} \int_0^t e^{2\eta s}\|p(s)\|_{\mathcal{V}}^2 ds \leq \lim_{t \to \infty} e^{-2\eta t} \int_0^N e^{2\eta s}\|p(s)\|_{\mathcal{V}}^2 ds + \frac{\varepsilon^2}{2\eta} \lim_{t \to \infty} [e^{2\eta t} (e^{2\eta t} - e^{2\eta N})] = \frac{\varepsilon^2}{2\eta}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{t \to \infty} e^{-2\eta t} \int_0^t e^{2\eta s}\|p(s)\|_{\mathcal{V}}^2 ds = 0.$$

Thus, we have $\lim_{t \to \infty} [\|Q_m(u_1(t) - u_2(t))\|_{\mathcal{V}}^2 + \gamma \|Q_m(v_1(t) - v_2(t))\|_{\mathcal{V}}^2] = 0$. Hence, we obtain $\lim_{t \to \infty} \|Q_m(U(t) - V(t))\|_{\mathcal{X}} = 0$, so that the first $m$ eigenfunctions of the Stokes operator are asymptotically determining for the Kelvin-Voigt equations with "fading memory" having homogeneous Dirichlet boundary conditions.

**Remark 3.** As our system is of hyperbolic type, unlike the case of 2D Navier-Stokes equations (see Lemma 4.2, [17], Proposition 4.2, [23], Theorem 4.2, [24]), in the proof of Theorem 4.2, we are not using Lemma 4.1, [17] (see Lemma 4.2, [23], Lemma 4.1, [24] also).

### 4.2. Determining modes on the attractor

Our next aim is to give an estimate of determining modes for trajectories on the attractor for the 3D Kelvin-Voigt fluid flow equations with "fading memory" (see the system (21) or (22)).

**Definition 4.3.** A set of modes $\{e_1, \ldots, e_m\}$ is called **determining on the attractor** if for each two trajectories $U(t)$ and $V(t)$ on the attractor $\mathcal{A}_{\text{glob}}$, the equality

$$\|P_m(U(t) - V(t))\|_{\mathcal{X}} = 0, \text{ for all } t \in \mathbb{R},$$

implies

$$U(t) = V(t), \text{ for all } t \in \mathbb{R}.$$ 

Now we prove the following theorem on determining modes for trajectories on the global attractor for the system (21).

**Theorem 4.4.** Let $U(\cdot)$ and $V(\cdot)$ be two solutions of the system (22) from the attractor $\mathcal{A}_{\text{glob}}$. Assume also that $P_m(U(t)) = P_m(V(t))$, for all $t \in \mathbb{R}$, where $m$ is large enough such that (72) is satisfied. Then $U(t) = V(t)$, for all $t \in \mathbb{R}$.

**Proof.** Let $U = (u_1, v_1)^T$ and $V = (u_2, v_2)^T$ be arbitrary two trajectories in the attractor $\mathcal{A}_{\text{glob}}$ of (22). Let us define $W = U - V = (w, z)^T = (u_1 - u_2, v_1 - v_2)^T$. 

Then $W(\cdot)$ satisfies:

$$
\begin{align*}
\frac{d}{dt} [(I + \mu_1 A)w(t)] + \mu_0 Aw(t) + \gamma Az(t) + B(w(t), u_1(t)) + B(u_2(t), w(t)) &= 0, \quad t \in (0, T), \\
\frac{dz(t)}{dt} + \delta z(t) - w(t) &= 0, \quad t \in (0, T), \\
w(0) = w_0, \quad z(0) = z_0,
\end{align*}
$$

(87)

where $w(0) = u_1(0) - u_2(0) = u_1^0 - u_2^0 \in V$ and $z(0) = v_1(0) - v_2(0) = v_1^0 - v_2^0 \in \mathbb{V}$. Let $Q(t) = Q_m(U(t) - V(t))$. From the condition given in (72), we can choose an $\eta > 0$ such that

$$
0 < \eta \leq \min \left\{ \delta, \frac{1}{\lambda_1^2 + \mu_1} \left[ \mu_0 - \frac{C}{\lambda_{m+1}^{1/4}} \sqrt{\min\{1, \mu_1\} \mu_0} \right] \right\}.
$$

(88)

Let us multiply the first equation in (87) with $e^{\eta t}$, take the inner product with $e^{\eta t}q = e^{\eta t}Q_m w$, and then multiply the second equation in (87) with $e^{\eta t}$, take the inner product with $\gamma e^{\eta t} A r(t) = \gamma e^{\eta t} A Q_m z(t)$, and finally add them together to find

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} & [e^{2\eta t} \left( \|q(t)\|_V^2 + \mu_1 \|q(t)\|_V^2 + \gamma \|r(t)\|_V^2 \right)] + \mu_0 e^{2\eta t} \|q(t)\|_V^2 + \delta \gamma e^{2\eta t} \|r(t)\|_V^2 \\
&= -e^{2\eta t} b(w(t), u_1(t), q(t)) - e^{2\eta t} b(u_2(t), w(t), q(t)) \\
&\quad + \eta e^{2\eta t} \left( \|q(t)\|_V^2 + \mu_1 \|q(t)\|_V^2 + \gamma \|r(t)\|_V^2 \right)
\end{align*}
$$

(89)

Let us now assume that $P_m W(t) = 0$, for all $t \in \mathbb{R}$. Then the equation (89) reduces to

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} & [e^{2\eta t} \left( \|q(t)\|_V^2 + \mu_1 \|q(t)\|_V^2 + \gamma \|r(t)\|_V^2 \right)] + \mu_0 e^{2\eta t} \|q(t)\|_V^2 + \delta \gamma e^{2\eta t} \|r(t)\|_V^2 \\
&= e^{2\eta t} b(q(t), u_1(t), q(t)) + \eta e^{2\eta t} \left( \|q(t)\|_V^2 + \mu_1 \|q(t)\|_V^2 + \gamma \|r(t)\|_V^2 \right).
\end{align*}
$$

We estimate $b(q, u_1, q)$ similarly as in (80) as

$$
|b(q, u_1, q)| \leq 2 \|q\|_V^{1/2} \|q\|_V^{3/2} \|u_1\|_V \leq \frac{2}{\lambda_{m+1}^{1/4}} \|q\|_V^2 \|u_1\|_V.
$$

(90)

Remember that on the attractor $A_{\text{glob}}$, we know that $\|u_1\|_V \leq M_1$. We use this fact, inequalities (90) and (75) in (89) to find

$$
\begin{align*}
\frac{d}{dt} & [e^{2\eta t} \left( \|q(t)\|_V^2 + \mu_1 \|q(t)\|_V^2 + \gamma \|r(t)\|_V^2 \right)] + 2 \gamma(\delta - \eta) e^{2\eta t} \|r(t)\|_V^2 \\
&\quad + 2 \left[ \mu_0 - \eta \left( \frac{1}{\lambda_1} + \mu_1 \right) - \frac{2M_1}{\lambda_{m+1}^{1/4}} \right] e^{2\eta t} \|q(t)\|_V^2 \leq 0.
\end{align*}
$$

(91)

For the condition given in (72), integrating the above inequality from $r$ to $t$, we obtain

$$
\begin{align*}
&\quad e^{2\eta t} \left( \|q(t)\|_V^2 + \mu_1 \|q(t)\|_V^2 + \gamma \|r(t)\|_V^2 \right) \\
&\leq e^{2\eta r} \left( \|q(r)\|_V^2 + \mu_1 \|q(r)\|_V^2 + \gamma \|r(r)\|_V^2 \right). \\
&\text{(92)}
\end{align*}
$$

Thus, we obtain from (92) that

$$
\|q(t)\|_V^2 + \gamma \|r(t)\|_V^2 \leq \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\} \frac{e^{-\eta(t-r)}}{\min\{1, \mu_1\}} e^{-2\eta(t-r)} \left( \|q(r)\|_V^2 + \gamma \|r(r)\|_V^2 \right).
$$

(93)
5. Estimates of Dimensions of the Global Attractor. In this section, we first show the differentiability of the semigroup with respect to the initial data. Later, we prove that the global attractor for the system (21) has finite Hausdorff and fractal dimensions.

**Theorem 5.1.** Let $U_0$ and $V_0$ be two members of $\mathcal{X}$. Then there exists a constant $K = K(||U_0||, ||V_0||)$ such that

$$||S(t)U_0 - S(t)V_0 - \Lambda(t)(U_0 - V_0)||_X \leq K||U_0 - V_0||_X,$$

where the linear operator $\Lambda(t) : \mathcal{X} \to \mathcal{X}$, for $t > 0$ is the solution operator of the problem:

$$\begin{cases}
\frac{d}{dt}((1 + \mu_1)\eta(t) + \mu_0\eta(t) + \gamma(\eta(t), u(t)) + B(\eta(t), u(t))) = 0, & t \in (0, T), \\
\frac{d\zeta(t)}{dt} + \delta\zeta(t) - \xi(t) = 0, & t \in (0, T), \\
\xi(0) = \xi_0, \zeta(0) = \zeta_0,
\end{cases} \quad (94)$$

where $\xi_0 = u_0 - u_0^2$, $\zeta_0 = v_0^2 - v_0^2$ and $U(t) = (u_1(t), v_1(t))^T = S(t)U_0$. Or in other words, for every $t > 0$, the map $S(t)U_0$, as a map $S(t) : \mathcal{X} \to \mathcal{X}$ is Fréchet differentiable with respect to the initial data, and its Fréchet derivative

$$D_S(U_0)(S(t)U_0)W_0 = \Lambda(t)W_0.$$

**Proof.** Let us define

$$\Psi(t) = \begin{pmatrix} \eta(t) \\ \varphi(t) \end{pmatrix} = \begin{pmatrix} u_1(t) - u_2(t) - \xi(t) \\ v_1(t) - v_2(t) - \zeta(t) \end{pmatrix} = S(t)\begin{pmatrix} u_1^0 - u_2^0 \\ v_1^0 - v_2^0 \end{pmatrix} - \begin{pmatrix} \xi(t) \\ \zeta(t) \end{pmatrix},$$

and $\Phi(t) = \begin{pmatrix} \xi(t) \\ \zeta(t) \end{pmatrix}$. Then $(\eta(t), \varphi(t))^T$ satisfies:

$$\begin{cases}
\frac{d}{dt}((1 + \mu_1)\eta(t) + \mu_0\eta(t) + \gamma(\eta(t), u(t)) + B(\eta(t), u(t))) = 0, & t \in (0, T), \\
B(\varphi(t), \eta(t)) = 0, & t \in (0, T), \\
\frac{d\zeta(t)}{dt} + \delta\zeta(t) - \xi(t) = 0, & t \in (0, T), \\
\xi(0) = \xi_0, \zeta(0) = \zeta_0.
\end{cases} \quad (95)$$

where $\varphi(t) = u_1(t) - u_2(t)$. Let us take the inner product with $\eta(\cdot)$ to the first equation in (95) to obtain

$$\frac{1}{2} \frac{d}{dt} \left( ||\eta(t)||_X^2 + \mu_1 ||\eta(t)||_V^2 \right) + \mu_0 ||\eta(t)||_X^2$$

$$= -\gamma(\nabla \varphi(t), \nabla \eta(t)) - B(\eta(t), u_1(t), \eta(t)) + B(\varphi(t), \eta(t), \eta(t)). \quad (96)$$
Next we take the inner product with $\gamma A\varphi(\cdot)$ to the second equation in (95) to find
\[
\gamma \frac{d}{dt} \|\varphi(t)\|_{H}^2 + \gamma \delta \|\varphi(t)\|_{V}^2 = \gamma (\nabla \eta(t), \nabla \varphi(t)).
\] (97)

Combining (96) and (97), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|\eta(t)\|_{H}^2 + \mu_1 \|\eta(t)\|_{V}^2 + \gamma \|\varphi(t)\|_{V}^2 \right) + \mu_0 \|\eta(t)\|_{V}^2 + \delta \gamma \|\varphi(t)\|_{V}^2
= -b(\eta(t), u_1(t), \eta(t)) + b(w(t), w(t), \eta(t)).
\] (98)

We estimate $|b(\eta, u_1, \eta)|$ using Hölder’s, Ladyzhenskaya’s and Young’s inequalities as
\[
|b(\eta, u_1, \eta)| \leq \|\eta\|_{V} \|\eta\|_{V}^{1/2} \|\eta\|_{V}^{3/2} \leq \frac{2}{\lambda_1^{1/4}} \|\eta\|_{V}^2.
\] (99)

Once again applying Hölder’s, Ladyzhenskaya’s, Poncaré’s and Young’s inequalities, we estimate $|b(w, w, \eta)|$ as
\[
|b(w, w, \eta)| = |b(w, \eta, w)| \leq \|\eta\|_{V} \|w\|_{V} \leq \frac{2}{\lambda_1^{1/4}} \|\eta\|_{V} \|w\|_{V}^2
\leq \frac{\mu_0}{2} \|\eta\|_{V}^2 + \frac{2}{\mu_0 \sqrt{\lambda_1}} \|w\|_{V}^4.
\] (100)

Combining (99) and (100), and using it in (98), we find
\[
\frac{d}{dt} \left( \|\eta(t)\|_{H}^2 + \mu_1 \|\eta(t)\|_{V}^2 + \gamma \|\varphi(t)\|_{V}^2 \right) + \mu_0 \|\eta(t)\|_{V}^2 + 2\delta \gamma \|\varphi(t)\|_{V}^2
\leq \frac{4}{\lambda_1^{1/4}} \|\eta(t)\|_{V}^2 + \frac{4}{\mu_0 \sqrt{\lambda_1}} \|w(t)\|_{V}^4.
\] (101)

Note that $(w(t), z(t))^T = (u_1(t) - u_2(t), v_1(t) - v_2(t))^T = (\eta U_0 - S(t)V_0)$ satisfies:
\[
\frac{d}{dt} \left( (1 + \mu_1 A)w(t) + \mu_0 Aw(t) + \gamma Az(t) + B(w(t), u_1(t)) \right)
+ B(u_1(t), w(t)) - B(w(t), w(t)) = 0, \ t \in (0, T),
\]
\[
\frac{dz(t)}{dt} + \delta z(t) - w(t) = 0, \ t \in (0, T),
\]
\[
w(0) = w_0, \ z(0) = z_0,
\]
where $w_0 = u_1^0 - u_2^0$ and $z_0 = v_1^0 - v_2^0$. Let us take the inner product with $w(\cdot)$ to the first equation in (102) to get
\[
\frac{1}{2} \frac{d}{dt} \left( \|w(t)\|_{H}^2 + \mu_1 \|w(t)\|_{V}^2 \right) + \mu_0 \|w(t)\|_{V}^2
= -\gamma (\nabla z(t), \nabla w(t)) - b(w(t), u_1(t), w(t)).
\] (103)

Now we take the inner product with $\gamma A z(\cdot)$ to the second equation in (102) to obtain
\[
\gamma \frac{d}{dt} \|z(t)\|_{V}^2 + \delta \gamma \|z(t)\|_{V}^2 = \gamma (\nabla w(t), \nabla z(t)).
\] (104)

Combining (103) and (104), we find
\[
\frac{1}{2} \frac{d}{dt} \left( \|w(t)\|_{H}^2 + \mu_1 \|w(t)\|_{V}^2 + \gamma \|z(t)\|_{V}^2 \right) + \mu_0 \|w(t)\|_{V}^2 + \delta \gamma \|z(t)\|_{V}^2
= -b(w(t), u_1(t), w(t)).
\]
A calculation similar to (99) yields
\[ |b(w, u_1, w)| \leq \frac{2}{\lambda_1^{1/4}} \|w\|_V^2. \] (105)

Using (105) in (102) and then integrating the inequality from 0 to \( t \), we obtain
\[
\|w(t)\|_V^2 + \mu_1 \|w(t)\|_V^2 + \gamma \|z(t)\|_V^2 + 2\mu_0 \int_0^t \|w(s)\|_V^2 ds + 2\gamma \delta \int_0^t \|z(s)\|_V^2 ds \\
\leq \|w_0\|_V^2 + \mu_1 \|w_0\|_V^2 + \gamma \|z_0\|_V^2 + \frac{2}{\lambda_1^{1/4}} \int_0^t \|u_1(s)\|_V \|w(s)\|_V^2 ds. \] (106)

Thus, from (106), we also get
\[
\|w(t)\|_V^2 \leq \frac{1}{\mu_1} \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right) \|w_0\|_V^2 + \gamma \|z_0\|_V^2 \right\} + \frac{2}{\mu_1 \lambda_1^{1/4}} \int_0^t \|u_1(s)\|_V \|w(s)\|_V^2 ds. \] (107)

An application of Gronwall's inequality in (107) yields
\[
\|w(t)\|_V^2 \\
\leq \frac{1}{\mu_1} \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right) \|w_0\|_V^2 + \gamma \|z_0\|_V^2 \right\} \exp \left( \frac{2}{\mu_1 \lambda_1^{1/4}} \int_0^t \|u_1(s)\|_V ds \right) \] (108)

\[
\leq \frac{1}{\mu_1} \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right) \|w_0\|_V^2 + \gamma \|z_0\|_V^2 \right\} \times \exp \left[ \frac{2}{\mu_1 \lambda_1^{1/4}} \left( \frac{\sqrt{2 \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}}}{\alpha \sqrt{\min \{1, \mu_1\}}} \|u_0\|_X + \frac{t \|f\|_V}{\sqrt{2 \mu_0 \min \{1, \mu_1\} \alpha}} \right) \right],
\]

where we used (42), for \( \alpha \) given in (33).

Using (108) in (101) and then integrating from 0 to \( t \), we find
\[
\|\eta(t)\|_V^2 + \mu_1 \|\eta(t)\|_V^2 + \gamma \|\varphi(t)\|_V^2 + \mu_0 \int_0^t \|\eta(s)\|_V^2 ds + 2\gamma \delta \int_0^t \|\varphi(s)\|_V^2 ds \\
\leq \frac{4}{\lambda_1^{1/4}} \int_0^t \|u_1(s)\|_V \|\eta(s)\|_V^2 ds + \frac{4 \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}^2}{\mu_0 \mu_1^{1/4} \sqrt{\lambda_1}} \left( \|w_0\|_V^2 + \gamma \|z_0\|_V^2 \right)^2 \times \int_0^t \exp \left[ \frac{2}{\mu_1 \lambda_1^{1/4}} \left( \frac{\sqrt{2 \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}}}{\alpha \sqrt{\min \{1, \mu_1\}}} \|u_0\|_X + \frac{s \|f\|_V}{\sqrt{2 \mu_0 \min \{1, \mu_1\} \alpha}} \right) \right] ds. \] (109)

Thus, from (109), we infer that
\[
\|\eta(t)\|_V^2 + \gamma \|\varphi(t)\|_V^2 \leq \frac{4}{\lambda_1^{1/4} \mu_1} \int_0^t \|u_1(s)\|_V (\|\eta(s)\|_V^2 + \gamma \|\varphi(s)\|_V^2) ds + \vartheta_1(t) \|W_0\|_X^4,
\] (110)
from the estimates (40)-(43), we have

\[ \theta_1(t) = \frac{4 \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}^2 \sqrt{2\mu_0 \min \{1, \mu_1\} \alpha}}{\mu_0 \mu_1^{1/2} \lambda_1^{1/4} \| f \|_{\gamma'}} \times \exp \left\{ \frac{2}{\mu_1^{1/4} \lambda_1^{1/4}} \left( \frac{\sqrt{2} \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}}{\alpha \min \{1, \mu_1\}} \| U_0 \| \chi + \frac{t \| f \|_{\gamma'}}{\sqrt{2\mu_0 \min \{1, \mu_1\} \alpha}} \right) \right\} \].

An application of Gronwall’s inequality in (110) yields

\[ \| \eta(t) \|_{\gamma'}^2 + \gamma \| g(t) \|_{\gamma'}^2 \leq \theta_1(t) \| W_0 \|_{\chi}^4 \exp \left( \frac{4}{\mu_1^{1/4} \lambda_1^{1/4}} \int_0^t \| u_1(s) \|_{\gamma'} ds \right) , \]

that is,

\[ \| \Psi(t) \|_{\chi}^2 \leq \theta_2(t) \| W_0 \|_{\chi}^4 , \]

where \( \theta_2(t) = \theta_1(t) \exp \left\{ \frac{4}{\mu_1^{1/4} \lambda_1^{1/4}} \left( \frac{\sqrt{2} \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}}{\alpha \min \{1, \mu_1\}} \| U_0 \| \chi + \frac{t \| f \|_{\gamma'}}{\sqrt{2\mu_0 \min \{1, \mu_1\} \alpha}} \right) \right\} \} .

Thus, by the definition of \( \Psi(t) \), it is immediate that

\[ \frac{\| U(t) - V(t) - \Phi(t) \| \chi}{\| U_0 - V_0 \| \chi} \leq \sqrt{\theta_2(t)} \| U_0 - V_0 \| \chi , \quad (111) \]

and hence the differentiability of \( S(t) \) with respect to the initial data follows. \( \square \)

Let \( U(\cdot) \) be the solution to (21) belonging to the global attractor \( \mathcal{A}_{\text{glob}} \). Then, from the estimates (40)-(43), we have

\[ \sup_{t \geq 0} \| U(t) \|_{\chi} \leq M_1 = \frac{1}{\sqrt{\mu_0 \min \{1, \mu_1\} \alpha}} \| f \|_{\gamma'} , \quad (112) \]

and

\[ \lim \sup_{T \to \infty} \frac{1}{T} \int_0^T \| U(t) \|_{\chi}^2 dt \leq K_1 := \frac{1}{2\mu_0 \min \{1, \mu_1\} \alpha} \| f \|_{\gamma'}^2 . \quad (113) \]

Let us set \( G^2 = (I + \mu_1 A) \), \( \tilde{u} =GU \), \( \tilde{v} = GV \) and \( \tilde{U} = (\tilde{u}, \tilde{v})^\top \). Thus, we can rewrite the system (21) as

\[ \begin{cases} \frac{d}{dt} \tilde{u}(t) = -\frac{\mu_0}{\mu_1} \tilde{u}(t) + \frac{\mu_0}{\mu_1} G^{-2} \tilde{u}(t) - \frac{\gamma}{\mu_1} \tilde{v}(t) + \frac{\gamma}{\mu_1} G^{-2} \tilde{v}(t) - G^{-1} B(G^{-1} \tilde{u}(t), G^{-1} \tilde{u}(t)) + G^{-1} f(t), & t \in (0, T), \\ \frac{d}{dt} \tilde{v}(t) = -\delta \tilde{v}(t) + \tilde{u}(t), & t \in (0, T), \\ \tilde{u}(0) = GU_0, \tilde{v}(0) = GV_0 , \end{cases} \quad (114) \]

where \( (GU_0, GV_0) \in \mathbb{H} \times \mathbb{H} \). It is important to note that the systems (114) and (21) are equivalent. But the system (114) is well posed in \( \mathbb{H} \times \mathbb{H} \) as the norms \( \| \cdot \|_{\gamma} \) and \( \| G \cdot \|_{\mathbb{H}} \) are equivalent (see Remark 1). Thus, there exists a unique weak solution \((\tilde{u}, \tilde{v})\) of (114) in \( C([0, T]; \mathbb{H}) \times C([0, T]; \mathbb{H}) \). Hence, the system (114) generates a one parameter family of strongly continuous semigroup \( S(t) \) of solution operators

\[ S(t) : \mathcal{X}_0 \to \mathcal{X}_0 , \quad U_0 \to \tilde{U}(t) = S(t)U_0 , \]
where $\tilde{U}_0 = (\tilde{u}_0, \tilde{v}_0)^\top$ and $X_0 = \mathbb{H} \times \mathbb{H}$. Since $\bar{U}(t) = G U(t)$ and $\tilde{U}_0 = G U_0$, the semigroup $\bar{S}(t)$ is connected to the original semigroup $S(t)$ through the relation
$$\bar{S}(t) = G^{-1} S(t) G.$$ 

It can be easily seen that the semigroup $\bar{S}(t)$ possesses the global attractor $\bar{A}_\text{glob}$, where

$$\bar{A}_\text{glob} = G A_\text{glob}$$

and $A_\text{glob}$ is the global attractor for $S(t)$. In the sequel, we denote $\dim_{\mathbb{F}}^X(K)$ and $\dim_{\mathbb{H}}^X(K)$ for the Hausdorff and fractal dimensions of $K \subset X$.

**Remark 4.** If we define $G^2 = (I + \mu_1 A)$, then from (10) and (11), we know that
$$\sqrt{\mu_1} \|u\|_V \leq \|G u\|_H \leq \left(\frac{1}{\chi_1} + \mu_1\right)^{1/2} \|u\|_V$$
and hence the norms $\|\cdot\|_V$ and $\|G \cdot\|_H$ are equivalent. Now, we have $\|G u - G v\|_H \leq \left(\frac{1}{\chi_1} + \mu_1\right)^{1/2} \|u - v\|_V$, so that from Proposition 3.1, Chapter VI, [39], it is immediate that $\dim_{\mathbb{F}}^X(G(K)) \leq \dim_{\mathbb{H}}^X(K)$, for any set $K \subset V$. Remember that
$$\|G^{-1} u\|^2_H = A^{1/2}(I + \mu_1 A)^{-1/2} u^2_H = \sum_{j=1}^{\infty} \frac{\lambda_j}{(I + \mu_1 A_j)} |(u, e_j)|^2 \leq \frac{1}{\mu_1} \sum_{j=1}^{\infty} |(u, e_j)|^2$$

(115)

Thus, for all $u, v \in \mathbb{H}$ (so that $G^{-1} u, G^{-1} v \in V$), we have $\|G^{-1} u - G^{-1} v\|_V \leq \frac{1}{\sqrt{\mu_1}} \|u - v\|_H$, for all $u, v \in \mathbb{H}$. Since $K \subset V$, $G(K) \subset \mathbb{H}$. Thus, using Proposition 3.1, Chapter VI, [39] once again for the map $G^{-1}$, we obtain $\dim_{\mathbb{H}}^X(G^{-1}(H)) \leq \dim_{\mathbb{F}}^X(H)$ for any subset $H$ of $\mathbb{H}$. Since $K \subset V$, we know that $G(K) \subset \mathbb{H}$ and in particular choosing $H = G(K)$, we obtain $\dim_{\mathbb{F}}^X(K) \leq \dim_{\mathbb{H}}^X(G(K))$. Hence, we have $\dim_{\mathbb{F}}^X(K) = \dim_{\mathbb{H}}^X(G(K))$. Similarly, one can also show that $\dim_{\mathbb{F}}^X(K) = \dim_{\mathbb{H}}^X(G(K))$.

We first show a bound for the fractal dimension of $\bar{A}_\text{glob}$ in $X_0$. Then, a bound for the fractal dimension of $A_\text{glob}$ in $X$ follows immediately using the following argument. From Proposition 3.1, Chapter VI, [39], we know that the fractal dimension estimates are preserved under Lipschitz maps. Moreover form Remark 4, we infer that
$$\dim_{\mathbb{F}}^X(A_\text{glob}) = \dim_{\mathbb{F}}^X(G^{-1} \bar{A}_\text{glob}) = \dim_{\mathbb{H}}^X(\bar{A}_\text{glob}).$$

In order to find a bound for the fractal dimension of $\bar{A}_\text{glob}$, we need the linear variations of the system (114). The equation of linear variations corresponding to (114) has the form
$$\frac{d}{dt} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = L(t, \bar{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix},$$
where
$$L(t, \bar{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -\frac{\mu_2}{\mu_1} w(t) + \frac{\mu_0}{\mu_1} G^{-2} w(t) - \frac{\gamma}{\mu_1} z(t) + \frac{\gamma}{\mu_1} G^{-2} z(t) \\ -G^{-1} B(G^{-1} w(t), G^{-1} \bar{u}(t)) - G^{-1} B(G^{-1} \bar{u}(t), G^{-1} w(t)) \end{pmatrix}.$$
The adjoint $L^*(t, \tilde{U})$ of $L(t, \tilde{U})$ is given by

$$
L^*(t, \tilde{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -\frac{\mu_0}{\mu_1} w(t) + \frac{\mu_0}{\mu_1} G^{-2} w(t) + z(t) - G^{-1} B(G^{-1} w(t), G^{-1} \tilde{u}(t)) \\ -G^{-1} B(G^{-1} \tilde{u}(t), G^{-1} w(t)) \\ -\frac{\gamma}{\mu_1} w(t) + \frac{\gamma}{\mu_1} G^{-2} w(t) - \delta z(t) \end{pmatrix}.
$$

Thus $\tilde{L}(t, \tilde{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = L(t, \tilde{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} + L^*(t, \tilde{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}$ can be computed as

$$
\tilde{L}(t, \tilde{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{2\mu_0}{\mu_1} w(t) + \frac{2\mu_0}{\mu_1} G^{-2} w(t) - \frac{\gamma}{\mu_1} z(t) + \frac{\gamma}{\mu_1} G^{-2} z(t) \\ -2G^{-1} B(G^{-1} w(t), G^{-1} \tilde{u}(t)) - 2G^{-1} B(G^{-1} \tilde{u}(t), G^{-1} w(t)) \\ -2\delta z(t) - \frac{\gamma}{\mu_1} w(t) + \frac{\gamma}{\mu_1} G^{-2} w(t) \end{pmatrix}.
$$

Then, we have the following important result.

**Proposition 6.** Let $W \in \mathcal{X}_0$. Then, for $\mu_0 \geq \frac{\delta \mu_1}{(1 + \frac{\mu_1}{\mu_0})}$, we have

$$
(L(t, \tilde{U})W(t), W(t)) \leq -\bar{h}_0 \|W(t)\|_{\mathcal{X}_0}^2 + \bar{h}_1(t) \|G^{-1} W(t)\|_{\mathcal{X}_0}^2,
$$

where

$$
\bar{h}_0 = \min \left\{ \frac{\mu_0}{\mu_1}, 2 \left( \delta - \frac{\mu_1}{\mu_0} \right) \right\} \quad \text{and} \quad \bar{h}_1(t) = 2 \left[ \frac{\mu_0}{\mu_1} + \frac{\gamma^2}{4\mu_0 \mu_1} + \frac{108M^2}{\mu_0^3} \|u(t)\|_{\mathcal{V}}^2 \right].
$$

**Proof.** Let us take the inner product with $\begin{pmatrix} w(t) \\ z(t) \end{pmatrix}$ in (117) to find

$$
\left( \tilde{L}(t, \tilde{U}) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}, \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \right) = -\frac{\mu_0}{\mu_1} \|w(t)\|_{\mathcal{H}}^2 + \frac{2\mu_0}{\mu_1} \|G^{-1} w(t)\|_{\mathcal{H}}^2 - 2\delta \|z(t)\|_{\mathcal{H}}^2
$$

$$
- 2 \left( \frac{\gamma}{\mu_1} - 1 \right) (z(t), w(t)) + \frac{2\gamma}{\mu_1} (G^{-1} z(t), G^{-1} w(t))
$$

$$
- 2b(G^{-1} w(t), G^{-1} \tilde{u}(t), G^{-1} w(t)).
$$

We estimate $2 \left( 1 - \frac{\gamma}{\mu_1} \right) \|z, w\|$ as

$$
2 \left( 1 - \frac{\gamma}{\mu_1} \right) \|z, w\| \leq 2 \left( 1 + \frac{\gamma}{\mu_1} \right) \|z\|_{\mathcal{H}} \|w\|_{\mathcal{H}} \leq \frac{\mu_0}{2\mu_1} \|w\|_{\mathcal{H}}^2 + 2 \left( 1 + \frac{\gamma}{\mu_1} \right)^2 \frac{\mu_1}{\mu_0} \|z\|_{\mathcal{H}}^2.
$$

Using the Cauchy-Schwarz inequality and Young’s inequality, we estimate the term $\|(G^{-1} z, G^{-1} w)\|$ as

$$
\|(G^{-1} z, G^{-1} w)\| \leq \|G^{-1} z\|_{\mathcal{H}} \|G^{-1} w\|_{\mathcal{H}} \leq \frac{\mu_0}{\gamma} \|G^{-1} w\|_{\mathcal{H}}^2 + \frac{\gamma}{4\mu_0} \|G^{-1} z\|_{\mathcal{H}}^2.
$$
From (115), we know that $\|G^{-1}u\|_{\mathcal{V}} \leq \frac{1}{\sqrt{\mu_1}} \|u\|_{\mathcal{H}}$. Using Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate $|b(G^{-1}w, G^{-1}u, G^{-1}w)|$ as

\[
|b(G^{-1}w, G^{-1}u, G^{-1}w)| \leq \|G^{-1}u\|_{\mathcal{V}}\|G^{-1}w\|_{\mathcal{H}}^2 \leq 2\|u\|_{\mathcal{V}}\|G^{-1}w\|_{\mathcal{H}}^{1/2}\|G^{-1}w\|_{\mathcal{V}}^{3/2}
\]

\[
\leq \frac{2}{\mu_1^{1/4}} \|u\|_{\mathcal{H}}\|G^{-1}w\|_{\mathcal{H}}^{1/2}\|w\|_{\mathcal{H}}^{3/2}
\]

\[
\leq \frac{\mu_0}{4\mu_1} \|w\|_{\mathcal{H}}^2 + \frac{108}{\mu_0} \|u\|_{\mathcal{V}}\|G^{-1}w\|_{\mathcal{H}}^2.
\]  

(123)

On the global attractor $\mathcal{A}_{\text{glob}}$, we estimate $\|u\|_{\mathcal{V}}^2$ as

\[
\|u\|_{\mathcal{V}}^2 = \|u\|_{\mathcal{F}}^2 \|u\|_{\mathcal{F}}^2 \leq \|u\|_{\mathcal{F}}^2 \|U\|_{\mathcal{X}}^2 \leq \|u\|_{\mathcal{F}}^2 M^2.
\]

Using this estimate in (123), we get

\[
|b(G^{-1}w, G^{-1}u, G^{-1}w)| \leq \frac{\mu_0}{4\mu_1} \|w\|_{\mathcal{H}}^2 + \frac{108M^2}{\mu_0} \|u\|_{\mathcal{F}}^2 \|G^{-1}w\|_{\mathcal{H}}^2.
\]  

(124)

Combining (122)-(124) and using it in (121), we obtain

\[
\left( \tilde{L}(t, \tilde{U}) \left( \begin{array}{c} w(t) \\ z(t) \end{array} \right), \left( \begin{array}{c} w(t) \\ z(t) \end{array} \right) \right) 
\leq -\frac{\mu_0}{\mu_1} \|w(t)\|_{\mathcal{H}}^2 - 2 \left( \delta - \left(1 + \frac{\gamma}{\mu_1} \right) \frac{2}{\mu_1} \right) \|z(t)\|_{\mathcal{H}}^2 
\]

\[
+ 2 \left[ \frac{\mu_0}{\mu_1} + \frac{\gamma^2}{4\mu_0 \mu_1} + \frac{108M^2}{\mu_0} \|u(t)\|_{\mathcal{F}}^2 \right] \|G^{-1}w(t)\|_{\mathcal{H}}^2
\]

\[
\leq -\min \left\{ \frac{\mu_0}{\mu_1}, \frac{2}{\gamma} \left( \delta - \left(1 + \frac{\gamma}{\mu_1} \right) \frac{2}{\mu_1} \right) \right\} \left( \|w(t)\|_{\mathcal{H}}^2 + \gamma \|z(t)\|_{\mathcal{H}}^2 \right) 
\]

\[
+ 2 \left[ \frac{\mu_0}{\mu_1} + \frac{\gamma^2}{4\mu_0 \mu_1} + \frac{108M^2}{\mu_0} \|u(t)\|_{\mathcal{F}}^2 \right] \|G^{-1}w(t)\|_{\mathcal{H}}^2,
\]

for $\mu_0 \geq \frac{\delta \mu_1}{(1 + \frac{\gamma}{\mu_1})^2}$. That is, we have

\[
(\tilde{L}(t, \tilde{U})W(t), W(t)) \leq -\min \left\{ \frac{\mu_0}{\mu_1}, \frac{2}{\gamma} \left( \delta - \left(1 + \frac{\gamma}{\mu_1} \right) \frac{2}{\mu_1} \right) \right\} \|W(t)\|_{\mathcal{X}_0}^2
\]  

(125)

Comparing with Theorems 4.8 and 4.9, [28] (or Theorem 2.2, [25]), we get

\[
s_0 = 0, \ s_1 = -1, \ \tilde{h}_0 = \min \left\{ \frac{\mu_0}{\mu_1}, \frac{2}{\gamma} \left( \delta - \left(1 + \frac{\gamma}{\mu_1} \right) \frac{2}{\mu_1} \right) \right\},
\]

\[
\tilde{h}_{s_1}(t) = 2 \left[ \frac{\mu_0}{\mu_1} + \frac{\gamma^2}{4\mu_0 \mu_1} + \frac{108M^2}{\mu_0} \|u(t)\|_{\mathcal{F}}^2 \right],
\]

and $\tilde{h}_{s_k}(t) = 0$, for all $k \geq 2$. \qed

**Proposition 7.** Under the assumption of Proposition 6, the global attractor $\tilde{\mathcal{A}}_{\text{glob}}$ has finite fractal dimension in $\mathcal{X}_0$ with

\[
\dim_{\mathcal{H}}(\tilde{\mathcal{A}}_{\text{glob}}) \leq \dim_{\mathcal{X}_0}(\tilde{\mathcal{A}}_{\text{glob}})
\]
\[
\begin{aligned}
\leq & \left\{ \frac{2\mu_0}{\min\left\{ \frac{\mu_0}{\mu_1}, \frac{2}{\gamma} \left( 1 + \frac{2}{\mu_1} \right) \frac{\mu_1}{\mu_0} \right\}} \tilde{C} \lambda_1 \mu_1^2 \left[ 1 + \frac{\gamma^2}{4\mu_0^2} + \frac{54\mu_1 \| f \|_V^4}{\mu_0^3 (\min\{1, \mu_1\})^2 \alpha^2} \right] \right\}^{3/2},
\end{aligned}
\]

(126)

where \( \alpha \) is defined in (33).

Proof. Let us first consider an initial orthogonal set of infinitesimal displacements \( W_{1,0}, \ldots, W_{n,0} \), for some \( n \geq 1 \). The volume of the parallelepiped spanned by \( W_{1,0}, \ldots, W_{n,0} \) is given by

\[ \mathfrak{V}_n(0) = |W_{1,0} \wedge \ldots \wedge W_{n,0}|, \]

where \( \wedge \) denotes the exterior product. Note that the evolution of such displacements obeys the following evolution equation:

\[
\begin{align*}
\frac{d}{dt} W_i(t) &= L(t, \tilde{U}) W_i(t), \\
W_i(0) &= W_{i,0},
\end{align*}
\]

(127)

for all \( i = 1, \ldots, n \). Then using Lemma 3.5, [7] (see [8] also), we know that the volume elements \( \mathfrak{V}_n(t) = |W_1(t) \wedge \ldots \wedge W_n(t)| \) satisfy

\[ \mathfrak{V}_n(t) = \mathfrak{V}_n(0) \exp \left[ \int_0^t \text{Tr}(P_n(s)L(s, \tilde{U})) ds \right], \]  

(128)

where \( P_n(s) \) is the orthogonal projection onto the linear span of \( \{W_1(t), \ldots, W_n(t)\} \) in \( X_0 \). Moreover, we know that

\[ \text{Tr}(P_n(s)L(s, \tilde{U})) = \sum_{k=1}^{n} (L(s, \tilde{U}) \varphi_k(s), \varphi_k(s)), \]

with \( n \geq 1 \) and \( \{\varphi_1(s), \ldots, \varphi_n(s)\} \), an orthonormal set spanning \( P_n(s)X_0 \). Let us define

\[ [P_nL(\tilde{U})] := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \text{Tr}(P_n(t)L(\tilde{U}, t)) dt. \]

Thus, from (128), for all \( t \geq 0 \), we obtain

\[ \mathfrak{V}_n(t) = \mathfrak{V}_n(0) \exp \left[ t \sup_{\tilde{U} \in \mathcal{A}_{\text{shock}}^+} \sup_{P_n(0)} [P_nL(\tilde{U})] \right], \]

where the supremum over \( P_n(0) \) is a supremum over all choices of initial \( n \) orthogonal set of infinitesimal displacements that have taken around \( \tilde{U} \). Now, our goal is to show that the volume elements \( \mathfrak{V}_n(t) \) decays exponentially in time whenever \( n \geq N \), with \( N > 0 \) to be determined later. The behavior of eigenvalues of the Stokes’ operator is well known in the literature (for example see Theorem 2.2, Corollary 2.2, [20]) and for all \( k \geq 1 \) is given by

\[ \lambda_k \geq \bar{C} k^{2/n}, \quad \text{where} \quad \bar{C} = \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n}, \]

(129)
\[ \omega_n = \pi^{n/2} \Gamma(1 + n/2), \quad n = 3, \] and \( |\Omega| \) is the \( n \)-dimensional Lebesgue measure of \( \Omega \). That is, \( \tilde{C} = \frac{3^{5/3} \pi^{4/3} 4^{2/3}}{5040^{2/3}} \). Using Proposition 6 and Lemma 6.2, Chapter VI, \([39]\), we estimate \( \frac{1}{T} \int_0^T \text{Tr}(P_n(t)L(\bar{U}, t)) dt \) as
\[
\frac{1}{T} \int_0^T \text{Tr}(P_n(t)L(\bar{U}, t)) dt = \frac{1}{T} \int_0^T \sum_{k=1}^n (L(t, \bar{U})\varphi_k(t), \varphi_k(t)) dt \leq \frac{1}{T} \int_0^T \sum_{k=1}^n -\tilde{h}_0 \|\varphi_j(t)\|_{X_\theta}^2 dt + \frac{1}{T} \int_0^T \tilde{h}_1(t) \sum_{k=1}^n \|G^{-1}\varphi_j(t)\|_{X_\theta}^2 dt \leq -h_0 n + \frac{1}{T} \int_0^T \tilde{h}_1(t) dt \sum_{k=1}^n -\frac{1}{(1 + \mu_1\lambda_k)} \|\varphi_j(t)\|_{X_\theta}^2 \leq -h_0 n + \frac{1}{T} \int_0^T \tilde{h}_1(t) dt \sum_{k=1}^n -\frac{1}{(1 + \tilde{C}\mu_1 k^{2/3})} \leq -h_0 n + \frac{n^{1/3}}{C\mu_1} \frac{1}{T} \int_0^T \tilde{h}_1(t) dt. \tag{130} \]

But, we can estimate \( \frac{1}{T} \int_0^T \tilde{h}_1(t) dt \) using (120), (112) and (113) as
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \tilde{h}_1(t) dt = \limsup_{T \to \infty} \frac{2}{T} \int_0^T \left[ \frac{\mu_0}{\mu_1} + \frac{\gamma^2}{4\mu_0\mu_1} + \frac{108M_1^2}{\mu_0^3} \|u(t)\|_V^2 \right] dt \leq 2 \left[ \frac{\mu_0}{\mu_1} + \frac{\gamma^2}{4\mu_0\mu_1} + \frac{108M_1^2 K_1}{\mu_0^3} \right]. \tag{131} \]

Substituting (120) and (131) in (130), we find
\[
\|P_nL(\bar{U})\| \leq -\min \left\{ \frac{\mu_0}{\mu_1} \frac{2}{\gamma} \left( \delta - \left(1 + \frac{\gamma}{\mu_1}\right)^2 \frac{\mu_1}{\mu_0} \right) \right\} n + \frac{2n^{1/3}}{C\mu_1} \left[ \frac{\mu_0}{\mu_1} + \frac{\gamma^2}{4\mu_0\mu_1} + \frac{108M_1^2 K_1}{\mu_0^3} \right]. \]

We need that the above inequality must be negative, therefore, we require
\[
n \geq N := \left\{ \frac{2\mu_0}{\min \left\{ \frac{\mu_0}{\mu_1} \frac{2}{\gamma} \left( \delta - \left(1 + \frac{\gamma}{\mu_1}\right)^2 \frac{\mu_1}{\mu_0} \right) \right\} C\mu_1^2} \left[ 1 + \frac{\gamma^2}{4\mu_0^3} + \frac{108\mu_1 M_1^2 K_1}{\mu_0^3} \right] \right\}^{3/2},
\]
where \( \tilde{C} \) is defined in (129). Using the definition of \( M_1 \) and \( K_1 \) given in (112) and (113), we obtain
\[
N = \left\{ \frac{2\mu_0}{\min \left\{ \frac{\mu_0}{\mu_1} \frac{2}{\gamma} \left( \delta - \left(1 + \frac{\gamma}{\mu_1}\right)^2 \frac{\mu_1}{\mu_0} \right) \right\} \tilde{C}\mu_1^2} \left[ 1 + \frac{\gamma^2}{4\mu_0^3} + \frac{54\mu_1 \|f\|_{V'}^2}{\mu_0^3 (\min \{1, \mu_1\})^2 \alpha^2} \right] \right\}^{3/2},
\]
which completes the proof. \( \square \)

Since \( \tilde{A}_{\text{glob}} \) has finite fractal dimension in \( X_0 \) with the bound (126), using (116), one can easily prove the following Theorem.
Theorem 5.2. Under the assumptions of Proposition 6, the global attractor $A_{\text{glob}}$ obtained in Theorem 3.2 has finite Hausdorff and fractal dimensions, which can be estimated by

$$\dim_X(A_{\text{glob}}) \leq \dim_F(A_{\text{glob}}) \leq \dim_X \left( \frac{2\mu_0}{\min \left\{ \frac{\mu_0}{\mu_1}, \frac{2}{\gamma} \left( \delta - \frac{2}{\mu_1} \right) \right\}} \right) \left[ 1 + \frac{\gamma^2}{4\mu_0^2} + \frac{54\mu_1 \|f\|_{L^6}^4}{\mu_0^2(\min\{1, \mu_1\})^2\alpha^2} \right]^{3/2},$$

where $\alpha$ is defined in (33).

6. Exponential Attractors. In this section, we establish the existence of an exponential attractor in $X$ for the semigroup $S(t)$ associated with the Kelvin-Voigt fluid flow equations with “fading memory”. We make use of Proposition 1, [13] (see Theorem 3.2, [14], Theorem A1, [41] also) to obtain the existence of an exponential attractor. As we have seen earlier, the existence and uniqueness of weak solution for the system (21), assures the existence of a continuous semigroup $\{S(t)\}$ in $X$ by $S(t)U_0 = U(t), \ t \geq 0$, where $U(\cdot)$ is the solution of (22) with $U(0) = U_0 \in X$. We have already shown the existence of a bounded absorbing set in $X$ as well as $X^2$ for the semigroup $S(t)$ associated with the system (22) (see Propositions 1 and 5 in section 3).

Definition 6.1 (Definition 3.1, [14]). A set $A_{\text{exp}}$ is an exponential attractor for $S(t)$ in a Banach space $X$ if

(i) it is compact in $X$,
(ii) it is positive invariant, that is, $S(t)A_{\text{exp}} \subset A_{\text{exp}}$, for all $t \geq 0$,
(iii) it has finite fractal dimension,
(iv) it attracts exponentially fast the bounded sets of initial data, that is, there exists a monotone function $Q$ and a constant $\alpha > 0$ such that

$$\text{dist}_H(S(t)B, A_{\text{exp}}) \leq Q(\|B\|_X) e^{-\alpha t}, \text{ for all } t \geq 0,$$

where $\text{dist}_H(X, Y)$ is the Hausdorff (non-symmetric) semidistance from a set $X$ to $Y$ in a Banach space $X$ and is defined by $\text{dist}_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_X$.

From the estimate (38) (see Proposition 1), it is immediate that

$$\sup_{t \geq 0} \sup_{U_0 \in B_1} \|S(t)U_0\|_X \leq M_1 := \left\{ \max \left\{ \frac{1}{\lambda_1} + \mu_1, \frac{1}{\mu_1} \right\} \right\}^{1/2} \frac{\|f\|_{W^1}}{\sqrt{\mu_0 \min\{1, \mu_1\} \alpha}}.$$ 

Moreover from (69)(see Proposition 5), we have the following uniform estimate:

$$\sup_{t \geq 0} \sup_{U_0 \in B_2} \|S(t)U_0\|_{X^2} \leq M_2 := \left\{ \max \left\{ \frac{1}{\lambda_1} + \mu_1, \frac{1}{\mu_1} \right\} \right\}^{1/2} \frac{1}{\sqrt{\mu_0 \min\{1, \mu_1\} \alpha}} \left( \|f\|_{B_2}^2 + \frac{27CM^6}{4\mu_0} \right)^{\frac{1}{2}}.$$ 

(132)
Let us define $D = B_1 \cap B_2$. We take $t_e > 0$ as the entering time of $D$ in the absorbing set $B_1$, and define

$$K = \bigcup_{t \geq t_e} S(t)D.$$  

We show that $K$ is invariant and compact. In order to show that $K$ is invariant, we consider

$$S(t)K = S(t)\bigcup_{\tau \geq t_e} S(\tau)D \subset \bigcup_{\tau \geq t_e} S(t+\tau)D \subset \bigcup_{\tau \geq t_e} S(\tau)D = K.$$  

Now it is left to show that $K$ is a compact set in $X$ and bounded set in $X^2$. It is enough to show the boundedness of $K$ in $X^2$, since $X^2$ is compactly embedded in $X$ and $K$ is a closed set in $X$. Let us take $W \in K$. Then, there exists a sequence $t_n \geq t_e$ and $W_n \in S(t_n)D$, such that $W_n \to W$ strongly in $X$. Thanks to the estimate (133) and hence we have

$$\|W_n\|_{X^2} \leq \tilde{M}_2.$$  

Using the weak compactness and uniqueness of limits, the Banach-Alaoglu theorem guarantees the existence of a subsequence $\{W_{n_k}\}_{k \in \mathbb{N}}$ of $\{W_n\}_{n \in \mathbb{N}}$ such that $W_{n_k} \rightharpoonup W$ weakly in $X^2$ as $k \to \infty$. Since the norm is weakly lower semicontinuous, we finally arrive at

$$\|W\|_{X^2} \leq \liminf_{k \to \infty} \|W_{n_k}\|_{X^2} \leq \tilde{M}_2,$$

and hence boundedness in $X^2$ follows. The exponentially attracting property of $K$ can be obtained similarly as in Lemma 4.6, [40].

Let us now establish our major goal, that is, to prove that the semigroup $S(t)$ associated with the system (21) admits an exponential attractor $A_{\text{exp}}$ contained and bounded in $X_2$ in the sense of Definition 6.1. Our basic idea is to use Proposition 1, [13] to get such an exponential attractor.

Let us start by decomposing the solution semigroup $S(t)$ in the following way. For $U_0 \in K$, let us split the solution of (21) as

$$S(t)U_0 = R(t)U_0 + T(t)(U_0),$$

where

$$V(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = R(t)U_0 \quad \text{and} \quad W(t) := \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = T(t)U_0,$$

solves the systems (46) and (47), respectively. Let us first prove that the semigroup $R(t)$ is an exponentially stable linear semigroup.

**Lemma 6.2.** The semigroup $R(t)$ is linear and exponentially stable.

**Proof.** Following the proof Proposition 3 (see (52)), we obtain

$$\|R(t)U_0\|_{X^2}^2 = \|V(t)\|_{X^2}^2 = \|x(t)\|_X^2 + \gamma \|y(t)\|_Y^2 \leq \max\left(\frac{1}{\gamma^2 + \mu_1}, 1\right) \min\{1, \mu_1\} \|U_0\|_X^2 e^{-\tilde{\alpha}t},$$

where $\tilde{\alpha}$ is defined in (48). Thus, we have

$$\|R(t)\|_{L(X,X)} \leq \frac{\max\left(\frac{1}{\gamma^2 + \mu_1}, 1\right)}{\sqrt{\min\{1, \mu_1\}}} e^{-\tilde{\alpha}t},$$

and hence the semigroup $R(t)$ is exponentially stable on $X$. \qed
Since the system (46) is linear, we can prove the following result in a similar fashion.

**Lemma 6.3.** Let $U_0 \in \mathcal{K}$. Then

$$\|R(t)U_0 - R(t)V_0\|_X \leq \sqrt{\max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} \frac{e^{-\tilde{\alpha}t}}{\min\{1, \mu_1\}} \|U_0 - V_0\|_X, \quad (134)$$

for all $t \geq 0$.

Let us now establish the following continuous dependence results for $S(t)$ and $T(t)$.

**Lemma 6.4.** Let $U_0, V_0 \in \mathcal{X}$. Then whenever $\|U_0\|_X, \|V_0\|_X \leq r$, we have

$$\|S(t)U_0 - S(t)V_0\|_X \leq \sqrt{\max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} \frac{e^{-\tilde{\alpha}t(r+t)}}{\min\{1, \mu_1\}} \|U_0 - V_0\|_X, \quad (135)$$

where

$$\kappa = \max \left\{ \sqrt{2 \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}}, \frac{\|f\|_{V'}}{\alpha \sqrt{\min\{1, \mu_1\}} \sqrt{2\mu_0 \min\{1, \mu_1\}} \alpha} \right\}, \quad (136)$$

and $\alpha$ is defined in (33).

**Proof.** The proof easily follows from Lemma 3.1. From (31), we have

$$\|S(t)U_0 - S(t)V_0\|_X \leq \sqrt{\max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} \frac{1}{\min\{1, \mu_1\}} \exp \left( \frac{1}{\lambda_1^{1/4} \mu_1} \int_0^t \|u_1(s)\|_Y ds \right) \|U_0 - V_0\|_X, \quad (137)$$

Using the integrability condition given in (42), we also obtain

$$\|S(t)U_0 - S(t)V_0\|_X \leq \sqrt{\max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} \frac{1}{\min\{1, \mu_1\}} \|U_0 - V_0\|_X$$

$$\times \exp \left[ \frac{1}{\lambda_1^{1/4} \mu_1} \left( \sqrt{2 \max \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} \frac{\|U_0\|_X + \frac{t\|f\|_{V'}}{\alpha \sqrt{2\mu_0 \min\{1, \mu_1\}} \alpha}}{\|U_0\|_X + \frac{t\|f\|_{V'}}{\alpha \sqrt{2\mu_0 \min\{1, \mu_1\}} \alpha}} \right) \right], \quad (138)$$

for all $t \geq 0$. Since $\|U_0\|_X \leq r$, (138) easily gives (135).

**Lemma 6.5.** The following inequality holds:

$$\|T(t)(U_0) - T(t)(V_0)\|_X \leq \kappa_1 \frac{e^{\lambda_1^{1/2} \mu_1 (r+t)}}{\min\{1, \mu_1\}} \|U_0 - V_0\|_X, \quad (139)$$
where
\[
\kappa_1 = \frac{C \sqrt{M_1 M_2 \max \left\{ \left( \frac{1}{\lambda_0} + \mu_1 \right), 1 \right\} \lambda_{1/4} \mu_1}}{\sqrt{\mu_0 \kappa} \min \{1, \mu_1\}}.
\]  

(140)

**Proof.** Let us take \( W_1(t) = T(t)(U_0) \) and \( W_2(t) = T(t)(V_0) \). Then, their difference \( W(t) = W_1(t) - W_2(t) = (w(t), z(t))^T = (w_1(t) - w_2(t), z_1(t) - z_2(t))^T \) satisfies:

\[
\begin{aligned}
\frac{d}{dt} \left[ (I + \mu_1 A) w(t) \right] + \mu_0 A w(t) + \gamma A z(t) + B(u_1(t), u(t)) \\
+ B(u(t), u_2(t)) &= 0, \quad t \in (0, T), \\
\frac{dz(t)}{dt} + \delta z(t) - w(t) &= 0, \quad t \in (0, T), \\
w(0) = 0, z(0) = 0,
\end{aligned}
\]

where \((u(t), v(t))^T = (u_1(t) - u_2(t), v_1(t) - v_2(t))^T = S(t)U_0 - S(t)V_0\). Let us now take the inner product with \( Aw(t) \) to first equation in (141) to obtain

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \|w(t)\|^2_V + \mu_1 \|Aw(t)\|^2_H + \mu_0 \|Aw(t)\|^2_H \right) \\
= -\gamma \langle Az(t), Aw(t) \rangle - \langle B(u_1(t), u(t)), Aw(t) \rangle - \langle B(u(t), u_2(t)), Aw(t) \rangle.
\end{aligned}
\]

(142)

We take the inner product with \( \gamma A^2 z(t) \) to the second equation in (141) to find

\[
\frac{\gamma}{2} \frac{d}{dt} \|Az(t)\|^2_H + \delta \gamma \|Az(t)\|^2_H = \gamma \langle Aw(t), Az(t) \rangle.
\]

(143)

Combining (142) and (143), we get

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \|w(t)\|^2_V + \mu_1 \|Aw(t)\|^2_H + \gamma \|Az(t)\|^2_H \right) + \mu_0 \|Aw(t)\|^2_H + \delta \gamma \|Az(t)\|^2_H \\
= -\langle B(u_1(t), u(t)), Aw(t) \rangle - \langle B(u(t), u_2(t)), Aw(t) \rangle.
\end{aligned}
\]

(144)

Using Hölder’s, Agmon’s and Young’s inequalities, we estimate \( \langle B(u_1, u), Aw \rangle \) as

\[
\begin{aligned}
|\langle B(u_1, u), Aw \rangle| &\leq \|u_1\|_{L^\infty} \|u\| \|Aw\|_H \\
&\leq C \|u_1\|^{1/2} \|Au_1\|^{1/2} \|u\| \|Av\|_H \\
&\leq \frac{\mu_0}{4} \|Aw\|^2_H + \frac{C}{\mu_0} \|u_1\| \|Av\| \|u\|_V^2.
\end{aligned}
\]

(145)

We estimate \( \langle B(u, u_2), Aw \rangle \) using Hölder’s, Gagliardo-Nirenberg’s, Poincaré’s and Young’s inequalities as

\[
\begin{aligned}
|\langle B(u, u_2), Aw \rangle| &\leq \|u\|_{L^6} \|\nabla u_2\|_{L^3} \|Aw\|_H \\
&\leq C \|u\| \|u_2\| \|Av\| \|u\|_V \|Aw\|_H \\
&\leq \frac{\mu_0}{4} \|Aw\|^2_H + \frac{C}{\mu_0} \|u_2\| \|Av\| \|u\|_V^2.
\end{aligned}
\]

(146)

Integrating the equality (144) from 0 to \( t \), and then using (145) and (146), we find

\[
\begin{aligned}
\|w(t)\|^2_V + \mu_1 \|Aw(t)\|^2_H + \gamma \|Az(t)\|^2_H + \mu_0 \int_0^t \|Aw(s)\|^2_H ds + \delta \gamma \int_0^t \|Az(s)\|^2_H ds \\
\leq \frac{C}{\mu_0} \int_0^t \|u_1(s)\| \|Au_1(s)\|_H + \|u_2(s)\| \|Av_2(s)\|_H \|u(s)\|_V^2 ds \\
\leq \frac{2CM_1 M_2}{\mu_0} \int_0^t \|u(s)\|^2_V ds.
\end{aligned}
\]
which completes the proof.

\[ \text{Lemma 6.6. Let } U_0 \in K. \text{ Then, we have} \]
\[ \sup_{t \geq 0} \| \tilde{U}(t) \|_X \leq \left( \frac{1}{\min\{1, \mu_1\}} \left( \frac{2}{\mu_1} \| f \|_{Y^*}^2 + \kappa \tilde{M}_1 \right) \right)^{1/2}, \]  
\[ \text{where} \]
\[ \kappa = \max \left\{ \left[ \frac{2}{\mu_1} \left( 1 + \frac{1}{4} \frac{\tilde{M}_1}{\lambda_1^{1/4}} \right) + 1 \right], \left( 1 + \frac{2\gamma^2}{\mu_0} \right) \right\}. \]

**Proof.** Let us take the inner product with \( \hat{u}(\cdot) \) to the first equation in (21) to obtain
\[ \| \tilde{u}(t) \|_H^2 + \mu_1 \| \tilde{u}(t) \|_{V^*}^2 + \gamma \| \tilde{v}(t) \|_{V^*}^2 \]
\[ = \langle f, \tilde{u}(t) \rangle - \mu_0 \langle A(u(t), \tilde{u}(t) \rangle - \langle B(u(t)), \tilde{u}(t) \rangle - \gamma \langle Av(t), \tilde{u}(t) \rangle. \]  

Next, we take the inner product with \( \gamma \tilde{v}(\cdot) \) to the second equation in (21) to find
\[ \gamma \| \tilde{v}(t) \|_{V^*}^2 = -\gamma \delta \langle v(t), A\tilde{v}(t) \rangle + \gamma \langle u(t), A\tilde{u}(t) \rangle. \]  

Adding (150) and (151), we get
\[ \| \tilde{u}(t) \|_H^2 + \mu_1 \| \tilde{u}(t) \|_{V^*}^2 + \gamma \| \tilde{v}(t) \|_{V^*}^2 \]
\[ = \langle f, \tilde{u}(t) \rangle - \mu_0 \langle A(u(t), \tilde{u}(t) \rangle - \langle B(u(t)), \tilde{u}(t) \rangle - \delta \langle v(t), A\tilde{v}(t) \rangle \]
\[ - \gamma \langle Av(t), \tilde{u}(t) \rangle + \gamma \langle u(t), A\tilde{u}(t) \rangle. \]  

Using the Cauchy-Schwarz inequality and Young’s inequality, we estimate \( |\langle f, \hat{u} \rangle| \) as
\[ |\langle f, \hat{u} \rangle| \leq \| f \|_{Y^*} \| \hat{u} \|_V \leq \frac{2}{\mu_1} \| f \|_{Y^*}^2 + \frac{\mu_1}{8} \| \hat{u} \|_V^2. \]  

Once again using the Cauchy-Schwarz inequality and Young’s inequality, we estimate \( |\langle A\hat{u}, \hat{u} \rangle| \) as
\[ |\langle A\hat{u}, \hat{u} \rangle| = |\langle \nabla u, \nabla \hat{u} \rangle| \leq \| \nabla u \|_H \| \nabla \hat{u} \|_H \leq \frac{\mu_1}{8} \| \hat{u} \|_V^2 + \frac{2}{\mu_1} \| \hat{u} \|_V^2. \]

Let us use Hölder’s, Ladyzhenskaya’s, Young’s and Poincaré’s inequalities to estimate \( |\langle B(u), \hat{u} \rangle| \) as
\[ |\langle B(u), \hat{u} \rangle| = |\langle B(u), \hat{u} \rangle, \rangle \leq \| \hat{u} \|_V \| u \|_4 \leq 2 \| \hat{u} \|_V \| u \|_V^{3/2} \| u \|_H^{1/2} \]
\[ \leq \frac{\mu_1}{8} \| \hat{u} \|_V^2 + \frac{2}{\mu_1} \| u \|_V^3 \| u \|_H \leq \frac{\mu_1}{8} \| \hat{u} \|_V^2 + \frac{2}{\mu_1 \lambda_1^{1/4}} \| u \|_V^2. \]
Using integration by parts, Hölder’s and Young’s inequalities, we estimate $\langle \mathbf{A} \mathbf{v}, \dot{\mathbf{u}} \rangle$ as

$$\langle \mathbf{A} \mathbf{v}, \dot{\mathbf{u}} \rangle = \langle \nabla \mathbf{v}, \nabla \dot{\mathbf{u}} \rangle \leq \| \nabla \mathbf{v} \|_2 \| \nabla \dot{\mathbf{u}} \|_\infty \leq \frac{\mu_1}{8\gamma} \| \dot{\mathbf{u}} \|_V^2 + \frac{2\gamma}{\mu_1} \| \mathbf{v} \|_V^2. \quad (154)$$

Similarly, we have

$$\langle \mathbf{v}, A \dot{\mathbf{v}} \rangle \leq \frac{1}{4\delta} \| \dot{\mathbf{v}} \|_V^2 + \delta \| \mathbf{v} \|_V^2, \quad (155)$$

$$\langle \mathbf{u}, A \dot{\mathbf{v}} \rangle \leq \frac{1}{4} \| \dot{\mathbf{v}} \|_V^2 + \| \mathbf{u} \|_V^2. \quad (156)$$

Combining (153)-(156) and using it in (152), we deduce that

$$\| \dot{\mathbf{u}}(t) \|_V^2 + \frac{\mu_1}{2} \| \dot{\mathbf{u}}(t) \|_V^2 + \frac{\gamma}{2} \| \mathbf{v}(t) \|_V^2$$

$$\leq \frac{2}{\mu_1} \| f \|_V^2 + \left\{ \frac{2}{\mu_1} \left( 1 + \frac{\| \mathbf{u}(t) \|_V^2}{\lambda_1^{1/4}} \right) + \gamma \right\} \| \mathbf{u}(t) \|_V^2 + \left( \delta^2 \gamma + \frac{2\gamma^2}{\mu_1} \right) \| \mathbf{v}(t) \|_V^2. \quad (157)$$

That is, we have

$$\| \dot{U}(t) \|_X^2 \leq \frac{1}{\min\{1, \mu_1\}} \times \left\{ \frac{2}{\mu_1} \| f \|_V^2 + \max \left\{ \frac{2}{\mu_1} \left( 1 + \frac{\| \mathbf{u}(t) \|_V^2}{\lambda_1^{1/4}} \right) + \gamma \right\}, \left( \delta^2 \gamma + \frac{2\gamma^2}{\mu_0} \right) \| \mathbf{u}(t) \|_X^2 \right\}. \quad (158)$$

Note that $U_0 \in \mathcal{K}$ and the estimate (132) implies the required result (148). \qed

Our next aim is to establish that the map $S(t)$ is Lipschitz continuous (see Lemma 3.1).

**Lemma 6.7.** Let $T > 0$ be arbitrary and fixed. Then, the map

$$(t, \mathbf{u}_0) \mapsto S(t)\mathbf{u}_0 : [0, T] \times \mathcal{K} \to \mathcal{K},$$

is Lipschitz continuous.

**Proof.** For $U_0, V_0 \in \mathcal{K}$ and $t_1, t_2 \in [0, T]$, we have

$$\| S(t_1)U_0 - S(t_2)V_0 \|_X \leq \| S(t_1)U_0 - S(t_1)V_0 \|_X + \| (S(t_1) - S(t_2))V_0 \|_X. \quad (159)$$

Using (135), we find

$$\| S(t_1)U_0 - S(t_1)V_0 \|_X \leq \sqrt{\max\left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} \frac{\sqrt{2}}{\sqrt{\min\{1, \mu_1\}}} \left( \tilde{M}_1 + t_1 \right) \| U_0 - V_0 \|_X. \quad (160)$$

where $\kappa$ is given in (136). Using Lemma 6.6, we also get

$$\| (S(t_1) - S(t_2))V_0 \|_X = \left\| \int_{t_1}^{t_2} \frac{d}{ds}S(s)V_0 ds \right\|_X \leq \int_{t_1}^{t_2} \| \dot{V}(s) \|_X ds$$

$$\leq \left\{ \frac{1}{\min\{1, \mu_1\}} \left( \frac{2}{\mu_1} \| f \|_V^2 + \kappa \tilde{M}_1 \right) \right\}^{1/2} |t_1 - t_2|, \quad (161)$$

where $\tilde{\kappa}$ is given in (149). Combining (160) and (161) and using it in (159), we obtain the required result. \qed

We are now ready to prove the existence of an exponential attractor for the system (21). We apply Proposition 1, [13] to obtain such an exponential attractor.
Theorem 6.8. The dynamical system $S(t)$ on $\mathcal{X}$ associated with the system (21) admits an exponential attractor $A_{exp}$ contained and bounded in $\mathcal{X}_2$ in the sense of Definition 6.1.

Proof. Remember that $\mathcal{K}$ is compact and invariant. Since the operator $S(t)$ is Lipschitz continuous (see Lemma 6.7), the first condition given in Proposition 1, [13] holds true for any $t^* \in [0,T]$. Let us fix $t^*$ according to the estimate given in Lemma 6.3 (see for example (134)), that is, we fix

$$\sqrt{\max\left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} e^{-\tilde{\alpha}t^*} = \frac{1}{8},$$

where $\tilde{\alpha}$ is defined in (48). Thus, we have

$$\|R(t)U_0 - R(t)V_0\|_{\mathcal{X}} \leq \frac{1}{8} \|U_0 - V_0\|_{\mathcal{X}},$$

for all $t \geq 0$. From Lemma 6.5, we infer that

$$\|T(t^*)U_0 - T(t^*)V_0\|_{\mathcal{X}_2} \leq C_0 \|U_0 - V_0\|_{\mathcal{X}},$$

where $C_0 = \kappa_1 e^{\frac{\kappa_1}{2\mu_1}(r+t^*)}$ and $\kappa_1$ is defined in (140). Now, one can apply Proposition 1, [13] with $L = R(t^*)$ and $K = T(t^*)$ to get an exponential attractor for the system (21).

Let us now discuss a result on the fractal dimension of exponential attractors associated with the system (21).

Theorem 6.9. The semigroup $S(t)$ admits an exponential attractor $A_{exp}$ whose fractal dimension satisfies the estimate:

$$\dim_{\mathcal{F}}^X(A_{exp}) \leq \dim_{\mathcal{F}}^X(A_{glob}) + 1,$$

(162)

where $A_{glob}$ is the global attractor for $S(t)$.

Proof. From Theorem 5.1 (see (111)), we know that $S(t)$ is differentiable. Moreover, from the Lemma 6.7, we know that (see (160) and (161))

$$\|S(t_1)U_0 - S(t_2)V_0\|_{\mathcal{X}} \leq L(t)(\|U_0 - V_0\|_{\mathcal{X}} + |t_1 - t_2|),$$

(163)

where

$$L(t) = \sqrt{\max\left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), \frac{2}{\mu_1}\|f\|_{\mathcal{Y}} + \tilde{\kappa}\tilde{M}_1 \right\}} \left( \frac{2}{\mu_1}\|f\|_{\mathcal{Y}} + \tilde{\kappa}\tilde{M}_1 \right)^{1/2}.$$

and hence $S(t)$ is Lipschitz.

Now, we need to show that the flow is an $\alpha$-contraction for all $t > 0$. In order to prove $\alpha$-contraction, we use Remark 4.6.2, [19]. Note that $S(t)U_0 = R(t)U_0 + T(t)(U_0)$, where $R(t)$ and $T(t)$ are the solution semigroups corresponding to the systems (46) and (47), respectively. From Lemma 6.3, we know that $\|R(t)\|_{\mathcal{L}(\mathcal{X};\mathcal{X})} \leq \sqrt{\max\left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right), 1 \right\}} e^{-\tilde{\alpha}t}$ and the right hand side tends to 0 as $t \to \infty$. 

Let \( \{W_n\}_{n \in \mathbb{N}} \) be a weakly convergent sequence in \( \mathcal{X} \), that is, \( W_n \to W \) weakly in \( \mathcal{X} \) as \( n \to \infty \). Since every weakly convergent sequence is bounded in \( \mathcal{X} \), \( \{W_n\}_{n \in \mathbb{N}} \) is bounded in \( \mathcal{X} \). From Lemma 6.5, we infer that (see (139))

\[
\|T(t)(W_n)\|_{\mathcal{X}_2} \leq \kappa_1 e^{-\frac{C}{\eta} (r+t)} \|W_n\|_{\mathcal{X}} < +\infty,
\]

for \( t \gg 1 \), where \( \kappa_1 \) is defined in (140). Applying the Banach-Alaoglu theorem and uniqueness of limits, we can extract a subsequence \( \{T(t)(W_{n_k})\}_{k \in \mathbb{N}} \) of \( \{T(t)(W_n)\}_{n \in \mathbb{N}} \) such that \( T(t)(W_{n_k}) \to T(t)(W) \), weakly in \( \mathcal{X}_2 \). Since the embedding of \( \mathcal{X}_2 \subset \mathcal{X} \) is compact, the sequence \( \{T(t)(W_{n_k})\}_{k \in \mathbb{N}} \) contains a strongly convergent subsequence \( \{T(t)(W_{n_j})\}_{j \in \mathbb{N}} \) in \( \mathcal{X} \), that is, \( T(t)(W_{n_j}) \to T(t)(W) \), strongly in \( \mathcal{X} \) as \( k \to \infty \). Since solution to the system (47) is unique and \( T(t) \) is the semigroup associated with the system (47), using uniqueness, one can get that the whole sequence \( \{T(t)(W_n)\}_{n \in \mathbb{N}} \) converges strongly to \( T(t)(W) \) in \( \mathcal{X} \). Hence the operator \( T(t) \) is completely continuous for sufficiently large \( t \). Using Remark 4.6.2, [19], it is immediate that \( S(t) \) is an \( \alpha \)-contraction. Invoking Theorem 2.2, [12], we assure the existence of an exponential attractor \( \mathcal{A}_{exp} \) for \( S(t) \) satisfying (162). \( \square \)

6.1. **Quasi-stability and existence of global as well as exponential attractors.** In this subsection, we show that the semigroup associated with the system (21) is quasi-stable. Moreover, making use of the Theorem 3.4.11, [5], we show the existence of global as well as exponential attractors having finite fractal dimension.

Let \( (\mathcal{X}, S(t)) \) be a dynamical system in some Banach space \( \mathcal{X} \) and \( B \subset \mathcal{X} \). Assume that there exists (a) compact seminorms (see Definition 3.1.14, [5]) \( n_1(\cdot) \) and \( n_2(\cdot) \) on the space \( \mathcal{X} \), and (b) numbers \( a_*, t_* > 0 \) and \( 0 \leq q < 1 \) such that

\[
\|S(t)y_1 - S(t)y_2\|_{\mathcal{X}} \leq a_*\|y_1 - y_2\|_{\mathcal{X}}, \quad \text{for every } y_1, y_2 \in B \text{ and } t \in [0, t_*] \quad (164)
\]

and

\[
\|P_t y_1 - P_t y_2\|_{\mathcal{X}} \leq q\|y_1 - y_2\|_{\mathcal{X}} + n_1(y_1 - y_2) + n_2(P_t y_1 - P_t y_2), \quad (165)
\]

for every \( y_1, y_2 \in B \). Then, under the conditions (164) and (165), using Proposition 3.4.10, [5], we know that the system \( (\mathcal{X}, S(t)) \) is quasi-stable on \( B \subset \mathcal{X} \).

**Proposition 8** (Ladyzhenskaya’s squeezing property). Let \( Q_m = I - P_m \). Then, for every \( 0 < q < 1, \ 0 < a \leq b < \infty \), there exists \( m_\ast = m(a, b, r, q) \) such that

\[
\|Q_m[S(t)U - S(t)U_\ast]\|_{\mathcal{X}} \leq q\|U - U_\ast\|_{\mathcal{X}}, \quad \text{for all } t \in [a, b], \quad m \geq m_\ast, \quad (166)
\]

for any \( U \) and \( U_\ast \) from the set \( \mathcal{D} \), where

\[
\mathcal{D} = \{ U \in \mathcal{X} : \|S(t)U\|_{\mathcal{X}} \leq R, \text{ for all } t \in [0, b] \}.
\]

**Proof.** Let us define \( W = U - U_\ast \), where \( U(t) = (u, v)^T \) and \( U_\ast = (u_\ast, v_\ast)^T \). We take \( U(t) = S(t)U \) and \( U_\ast(t) = S(t)U_\ast \). Then \( W = (w, z)^T = (u - u_\ast, v - v_\ast)^T \) satisfies the system (73) with \( u_1, u_2, w_0 = u_1^0 - u_2^0, z_0 = v_1^0 - v_2^0 \) replaced by \( u, u_\ast, w_0 = u - u_\ast, \) and \( z_0 = v - v_\ast \), respectively. We choose an \( m \) large enough such that such that

\[
\lambda_{m+1} > \frac{C R^4}{\mu_0^3}.
\]

With the condition given in (167), we can choose an \( \tilde{\eta} \) such that

\[
0 < \tilde{\eta} \leq \min \left\{ \delta, \frac{1}{\lambda_1^3 + \mu_1} \left[ \mu_0 - \frac{C R}{\lambda_1^{1/4} \lambda_{m+1}^{1/4}} \right] \right\}.
\]
Note that \( \|S(t)U\|_\mathcal{Y},\|S(t)U*\|_\mathcal{Y} \leq R \), for all \( U,U* \in D \). Choose \( m \) large enough so that
\[
\left[ \mu_0 - \left( \frac{1}{\lambda_1} + \mu_1 \right) \tilde{\eta} - \frac{CR}{\lambda^{1/4}_{m+1}} \right] \geq 0, \text{ and } 0 < \tilde{\eta}(m) \leq \delta.
\]

For the above condition, a calculation similar to (84) gives
\[
(\text{Quasi-stability})
\]

Proposition 9

Thus, from (168), we deduce that
\[
\|q(t)\|_Y^2 + \gamma \|r(t)\|_Y^2 \leq \left( \frac{1}{\lambda_1} + \mu_1 \right) \|q_0\|_Y^2 + \gamma \|r_0\|_Y^2 + \frac{CR}{\lambda^{1/4}_1} \int_0^t e^{2q \eta} \|p(s)\|_Y^2 ds.
\]

(168)

Thus, from (168), we deduce that
\[
\|q(t)\|_Y^2 + \gamma \|r(t)\|_Y^2 \leq \frac{1}{\min\{1, \mu_1\}} \left\{ \left( \frac{1}{\lambda_1} + \mu_1 \right) \|q_0\|_Y^2 + \gamma \|r_0\|_Y^2 + \frac{CR}{\lambda^{1/4}_1} \int_0^t e^{2q \eta} \|p(s)\|_Y^2 ds \right\}.
\]

(169)

Note that \( \|q_0\|_Y \leq \|Q_m(u-u_*)\|_Y \leq \|u-u_*\|_Y, \|r_0\|_Y \leq \|Q_m(v-v_*)\|_Y \leq \|v-v_*\|_Y \) and \( \|p(t)\|_Y \leq \|P_m(u(t)-u_*(t))\|_Y \leq \|u(t)-u_*(t)\|_Y \). Using (30), we further have
\[
\|Q_mw(t)\|_Y^2 + \|Q_mz(t)\|_Y^2 \leq \frac{\max\{1, \mu_1\}}{\min\{1, \mu_1\}} \left( \frac{CR}{\lambda^{1/4}_1 \gamma} \int_0^t e^{2q \eta} \|p(s)\|_Y^2 ds \right) \times \|u-u_*\|_Y^2 + \gamma \|v-v_*\|_Y^2,
\]

which completes the proof of (166). \( \square \)

**Proposition 9 (Quasi-stability).** For every \( 0 < q < 1, 0 < a < b < \infty \), and a forward invariant set \( \mathcal{B} \), which is a bounded set in \( \mathcal{X} \), there exists \( m = m(a,b,q,\mathcal{B}) \) such that
\[
\|S(t)U - S(t)U_*\|_\mathcal{X} \leq q\|S(r)U - S(r)U_*\|_\mathcal{X} + \|P_m[S(t)U - S(t)U_*]\|_\mathcal{X},
\]

(170)

for every \( t \in [a + r, b + r] \) and for all \( U,U_* \in \mathcal{B} \) and \( r \geq 0 \). Furthermore, the system \( (S(t), \mathcal{X}) \) is quasi-stable on \( \mathcal{B} \subset \mathcal{X} \).

**Proof.** We know that
\[
\|S(t)U - S(t)U_*\|_\mathcal{X} \leq \|Q_m[S(t)U - S(t)U_*]\|_\mathcal{X} + \|P_m[S(t)U - S(t)U_*]\|_\mathcal{X}
\]

and hence from (166), we obtain
\[
\|S(t)U - S(t)U_*\|_\mathcal{X} \leq q\|U - U_*\|_\mathcal{X} + \|P_m[S(t)U - S(t)U_*]\|_\mathcal{X},
\]

(171)

for every \( t \in [a, b] \) and for all \( U,U_* \in \mathcal{B} \). It is clear that \( \|P_m[\cdot]\|_\mathcal{X} \) is a seminorm on \( \mathcal{X} \). Thus, the condition (165) holds with \( X = \mathcal{X}, n_1 = 0 \) and \( n_2 = \|P_m[\cdot]\|_\mathcal{X} \). The Lipschitz property (164) follows from (31). Therefore, applying Proposition 3.4.10, [5], we know that the system \( (S(t), \mathcal{X}) \) is quasi-stable on \( \mathcal{B} \subset \mathcal{X} \). Using Exercise 4.3.12, [5], we infer that the relation in (171) can be written in the uniform form (170). \( \square \)
Then, based on Theorem 3.4.11, [5], we have the following result.

**Theorem 6.10.** The dynamical system \((\mathcal{X}, S(t))\) generated by the system (21) possesses a compact global attractor \(A_{\text{glob}}\) of finite fractal dimension \(\dim_X(A_{\text{glob}})\) in \(\mathcal{X}\). Moreover, the system \((\mathcal{X}, S(t))\) possesses an exponential attractor \(A_{\text{exp}}\), whose fractal dimension \(\dim_X(A_{\text{exp}})\) is finite in the phase space \(\mathcal{X}\).

**Proof.** The existence of a global attractor \(A_{\text{glob}}\) and the basic smoothness is proved Theorem 3.2. By Proposition 9, it is clear that the system satisfies the Assumption 3.4.9, [5] (see (164) and (165)) on every forward invariant set, which is bounded in \(\mathcal{X}\). Thus, we can apply Theorem 3.4.11, [5] to conclude that the global attractor has finite fractal dimension.

To prove the existence of the finite fractal dimensional exponential attractor \(A_{\text{exp}}\), we use the second part of Theorem 3.4.11, [5]. For this, we need to check the Lipschitz continuity property

\[
\|S(t_1)U - S(t_2)U\|_X \leq C_{B,T}|t_1 - t_2|, \quad t_1, t_2 \in [0, T], \ U \in B,
\]
on some forward invariant absorbing set \(B\) for \((\mathcal{X}, S(t))\), which is guaranteed by the estimates (158) and (161). The bounds for the fractal dimensions of \(A_{\text{glob}}\) and \(A_{\text{exp}}\) can be derived from the Theorems 3.4.11 and 3.2.3, [5].

**Remark 5.** The Kelvin-Voigt fluids of order \(L = 1, 2, \ldots\), with “fading memory” are described by the defining equation (see [33])

\[
\sigma(x,t) = \mu_1 \frac{\partial D(x,t)}{\partial t} + \mu_0 D(x,t) + \sum_{m=1}^{L} \beta_m \int_0^t e^{\alpha m(t-s)} D(x,s) ds,
\]

where \(\alpha_m < 0, \beta_m > 0, \sigma\) is the deviator of the stress tensor and \(D\) is the strain tensor. Then, we obtain the integro-differential equations of the motion of Kelvin-Voigt fluids with “fading memory” of order \(L = 1, 2, \ldots\), as:

\[
\frac{\partial u}{\partial t} - \mu_1 \frac{\partial \Delta u}{\partial t} + (u \cdot \nabla)u - \mu_0 \Delta u - \sum_{m=1}^{L} \beta_m \int_0^t e^{\alpha m(t-s)} \Delta u(s) ds + \nabla p = f,
\]

along with the divergence free condition. If we define \(v_m(t) := \int_0^t e^{\alpha m(t-s)} u(s) ds\), then the system (172) can be written as:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \mu_1 \frac{\partial \Delta u}{\partial t} + (u \cdot \nabla)u - \mu_0 \Delta u - \sum_{m=1}^{L} \beta_m \Delta v_m + \nabla p = f, \\
\frac{\partial v_m}{\partial t} - u - \alpha_m v_m = 0, \ m = 1, \ldots, L,
\end{cases}
\]

The results obtained in this work is also true for Kelvin-Voigt fluids of order \(L = 1, 2, \ldots\), with “fading memory” described above with some obvious modifications.

**Remark 6.** The authors in [40] proved convergence (as \(\mu_1 \to 0\)) of the (strong) global attractor of the 3D Navier-Stokes-Voigt model to the (weak) global attractor of the 3D Navier-Stokes equation. Such a convergence of the (strong) global attractor of the 3D Kelvin-Voigt fluid flow model of order one with “fading memory” to the (weak) global attractor of the 3D Oldroyd fluid flow model of order one will be discussed in a future work.
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