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NON-LOCAL POROUS MEDIA EQUATIONS WITH FRACTIONAL TIME DERIVATIVE

ESTHER S. DAUS, MARIA PIA GUALDANI, JINGJING XU, NICOLA ZAMPONI, XINYU ZHANG

ABSTRACT. In this paper we investigate existence of solutions for the system:
\[
\begin{align*}
D^\alpha_t u &= \text{div}(u \nabla p), \\
D^\alpha_t p &= -(-\Delta)^s p + u^2,
\end{align*}
\]
in $\mathbb{T}^3$ for $0 < s \leq 1$, and $0 < \alpha \leq 1$. The term $D^\alpha_t u$ denotes the Caputo derivative, which models memory effects in time. The fractional Laplacian $(-\Delta)^s$ represents the Lévy diffusion. We prove global existence of nonnegative weak solutions that satisfy a variational inequality. The proof uses several approximations steps, including an implicit Euler time discretization. We show that the proposed discrete Caputo derivative satisfies several important properties, including positivity preserving, convexity and rigorous convergence towards the continuous Caputo derivative. Most importantly, we give a strong compactness criteria for piecewise constant functions, in the spirit of Aubin-Lions theorem, based on bounds of the discrete Caputo derivative.

1. INTRODUCTION

In this manuscript we study existence of weak solutions to the following system:
\[
\begin{align*}
D^\alpha_t u &= \text{div}(u \nabla p), \\
D^\alpha_t p &= -(-\Delta)^s p + u^2,
\end{align*}
\] where the operator $D^\alpha_t$ denotes the Caputo-type time derivative
\[
D^\alpha_t f := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \, ds, \quad 0 < \alpha < 1.
\] Here $u(x, t) \geq 0$ denotes the density function and $p(x, t) \geq 0$ the pressure. The model describes the time evolution of a density function $u$ that evolves under the nonlocal continuity equation
\[
D^\alpha_t u = \text{div}(uv),
\] where the velocity is conservative, $v = \nabla p$, and $p$ is related to $u^2$ by the inverse of the fractional fully nonlocal heat operator $D^\alpha_t p + (-\Delta)^s$.

System (1.1) is a non-local-in-time version of the one recently studied in [10]. In [10] the authors proved the existence of weak solutions to
\[
\begin{align*}
\partial_t u &= \text{div}(u \nabla p), \\
\partial_t p &= -(-\Delta)^s p + u^2,
\end{align*}
\] for $x \in \mathbb{R}^2$, $\frac{1}{\beta} < s < 1$ and $\beta > 1$. The literature on (1.2) and his variants is quite large. See [6, 8, 11, 12, 13, 16, 20, 27, 28, 30] and references therein.

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The presence of $D^s_t$ makes our system quite different from (1.2). For example, techniques such as a Div-Curl Lemma do not work. The non-local structure prevents the equation from having a comparison principle. Maximum principle does not give useful insights, since at any point of maximum for $u$ we only know that $D^s_t u \leq u \Delta p$. We overcome these significant shortcomings with the introduction of ad-hoc regularization terms, together with suitable compact embeddings. We also provide a new strong compactness criterium for families of piecewise constant functions.

The authors are interested in understanding the effects and challenges that a non-local time derivative brings to a mathematical model. Time-delay memory effects are, in fact, very common in real situations. Specifically, differential equations with non-local time derivatives are used in modeling many physical and engineering processes, including particles in heat bath and soft matter with viscoelasticity ([15, 17, 18, 21, 22, 23, 24, 33]). In the past few years the study of stochastic and deterministic partial differential equations with non-local time derivatives has seen an increasing interest, see [1, 2, 3, 4, 7, 14, 21, 25, 26, 31] and the references therein.

The main result of this manuscript is summarized in the following two theorems:

**Theorem 1.1.** Let $u_{in}, p_{in} : \mathbb{T}^3 \to (0, +\infty)$ be functions such that $u_{in} \in L^2(\mathbb{T}^3)$, $p_{in} \in H^1(\mathbb{T}^3)$. For $0 < s \leq 1$ and $0 < \alpha \leq 1$, there exist functions $u, p : \mathbb{T}^3 \times [0, +\infty)$ such that for every $T > 0$

$$u \in L^\infty(0, T; L^2(\mathbb{T}^3)), \quad p \in L^2(0, T; H^{1+s}(\mathbb{T}^3)), \quad D^s_t u \in L^2(0, T; (W^{1,\infty}(\mathbb{T}^3))^\prime), \quad D^s_t p \in L^2(0, T; (L^\infty \cap H^1)^\prime(\mathbb{T}^3)),$$

which satisfy the following variational inequalities:

$$\int_0^T \int_{\mathbb{T}^3} \langle D^s_t u, \phi \rangle \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} u \nabla p \cdot \nabla \phi \, dx \, dt = 0, \quad \forall \phi \in L^2(0, T; W^{1,\infty}(\mathbb{T}^3)),$$

$$\int_0^T \int_{\mathbb{T}^3} \langle D^s_t p, \psi \rangle \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} (-\Delta)^{s/2} p (-\Delta)^{s/2} \psi \, dx \, dt$$

$$- \int_0^T \int_{\mathbb{T}^3} u^2 \psi \, dx \, dt \geq 0, \quad \forall \psi \in L^2(0, T; H^1 \cap L^\infty(\mathbb{T}^3)), \quad \lim_{t \to 0^+} u(t) = u_{in} \text{ strongly in } (W^{1,\infty}(\mathbb{T}^3))^\prime, \quad \lim_{t \to 0^+} p(t) = p_{in} \text{ strongly in } (L^\infty \cap H^1)^\prime(\mathbb{T}^3).$$

The operator $(-\Delta)^s$ is the fractional Laplacian and, on the torus, is defined via its Fourier series. More precisely, the $n$-th Fourier coefficient of $(-\Delta)^s u$ is

$$\widehat{(-\Delta)^s u}(n) = |n|^{2s} \widehat{u}(n),$$

with $\widehat{u}(n)$ the $n$-th Fourier coefficient of $u$:

$$\widehat{u}(n) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} u(x)e^{-inx} \, dx.$$

The starting point about our analysis is the observation that

$$H[u, p] := \int_{\mathbb{T}^3} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) \, dx$$

is a Lyapunov functional for (1.1) and satisfies the bound

$$D^s_t H[u, p] + \int_{\mathbb{T}^3} |(-\Delta)^{s/2} \nabla p|^2 \, dx = 0.$$
approximated by a system of elliptic problems. We introduce a novel discrete formulation for the Caputo derivative, namely

\[(D^\alpha f)_k := \Gamma_\alpha \tau^{-\alpha} \sum_{j=0}^{k-1} \lambda_{k-j} (f_{j+1} - f_j), \quad k \geq 1,\]

where the sequence \(\{\lambda_k\}\) is defined by linear recurrence:

\[\lambda_{k+1} = \sum_{j=1}^{k} ((k-j+1)^{\alpha-1} - (k-j+2)^{\alpha-1}) \lambda_j, \quad k \geq 1, \quad \lambda_1 = 1.\]

We provide several properties for the sequence \(\{\lambda_k\}\), including monotonicity, boundedness and asymptotic behavior. In particular we show that, as \(\tau \to 0\), a suitable piecewise constant function associated to \(\lambda_k\) converges, up to constants, to the function \(\frac{1}{\alpha}\). Moreover, \((D^\alpha f)_k\) satisfies some important properties, such as a discrete formulation of the fundamental theorem of calculus

\[f_n = f_0 + \frac{\tau^\alpha}{\Gamma_\alpha} \sum_{k=1}^{n} (n-k+1)^{\alpha-1} (D^\alpha f)_k,\]

and convergence towards the continuous Caputo derivative, see Proposition 5 and Proposition 3.

Once we have established the well-posedness for the discrete system, we show the existence of a sequence of solutions whose limit is a solution of the original continuous problem (with the extra viscosity term). In this step we apply strong compactness criteria. To handle our specific problem, we need strong compactness using bounds on the discrete Caputo derivative. At the best of the authors’ knowledge, there is no result of this kind in the literature. To fill this gap, we have proven the following variant of the Aubin-Lions theorem:

**Theorem 1.2.** Assume that \(X, B\) and \(Y\) are Banach spaces such that the embedding \(X \hookrightarrow B\) is compact and the embedding \(B \hookrightarrow Y\) is continuous. Let \(1 \leq p \leq \infty\), \(0 < \alpha < 1\) and \(\{f^{(\tau)}\}\) be a sequence of piecewise constant functions, satisfying

\[
\left\|f^{(\tau)}\right\|_{L^p(0,T;X)} + \left\|D^\alpha f^{(\tau)}\right\|_{L^p(0,T;Y)} \leq C_0, \tag{1.4}
\]

where \(C_0\) is a constant independent of \(\tau\) and \(D^\alpha\) is the discrete Caputo derivative operator defined in (1.3). Then \(\{f^{(\tau)}\}\) is relatively compact in \(L^p(0,T;B)\).

The proof of Theorem 1.2 uses Simon’s version of Aubin-Lions’ compactness theorem [29]. We first work with the linear interpolant functions \(\tilde{f}^{(\tau)}\) of \(f^{(\tau)}\) and show that \(\|D^\alpha f^{(\tau)}\| \geq \|D^\alpha \tilde{f}^{(\tau)}\|\); here we compare the discrete derivative of a piecewise constant function with the continuous derivative of the corresponding linear interpolant. After, we prove that the interpolants satisfy the estimate \(\|\tilde{f}^{(\tau)}(\cdot + h) - \tilde{f}^{(\tau)}\|_{L^p(0,T;Y)} \leq h^\alpha\|D^\alpha \tilde{f}^{(\tau)}\|\) and use bound (1.4) and Simon’s version of Aubin-Lions’ compactness theorem to conclude that \(\{\tilde{f}^{(\tau)}\}\) is compact in \(L^p(0,T;B)\). In the end, we show that convergence of the family of linear interpolants implies convergence of the piecewise constant functions. The limit is the same.

One could wonder if it is necessary to work with linear interpolants \(\{\tilde{f}^{(\tau)}\}\) instead with piecewise constant functions \(\{f^{(\tau)}\}\). In fact, Theorem 1 in [19] says that an estimate of the form \(\|f^{(\tau)}(\cdot + h) - f^{(\tau)}\|_{L^p(0,T;Y)} \leq Ch\) is sufficient to invoke Aubin-Lions’ theorem. Unfortunately, due to the nature of the Caputo derivative, the only estimate one can get for \(\{f^{(\tau)}\}\) is:

\[
\|f^{(\tau)}(\cdot + \tau) - f^{(\tau)}\|_{L^p(0,T;Y)} \leq C\tau^\alpha,
\]

which, alone, is not enough to guarantee compactness, see Proposition 2 in [19].
In the last step of the proof of Theorem 1.1 we remove the viscosity term $\rho \Delta u$. The major difficulty, in the approximation process, is the identification of the limit of $u^{(\rho)}$. The energy inequality provides plenty of informations for the pressure $p$, but only uniform integrability in $L^\infty(0,T,L^2(\mathbb{T}^3))$ for $u$. It is unclear, at the moment, how to use the bounds for $\nabla p$ to get useful bounds for $\nabla u$ or $u$. The authors in [10] overcome a similar problem using the Div-Curl lemma, a tool commonly employed in the study of fluid-dynamic systems. The presence of the Caputo derivative makes this method not useful. One interesting question would be if the Div-Curl lemma is still true when one considers derivatives of order strictly less than one. The authors have not explored this direction yet. Even if the lack of strong compactness prevents us to identify the limit of $u^{(\rho)}$, we are able to provide a lower bound for:

$$\int_0^T \int_{\mathbb{T}^3} (u^{(\rho)})^2 \psi \, dx \, dt.$$ 

The above integral defines a functional

$$\Psi(u) := \int_0^T \int_{\mathbb{T}^3} u^2 \psi \, dx \, dt$$

which is convex and continuous in the strong topology of $L^2(0,T,L^2(\mathbb{T}^3))$ for suitable functions $\psi$. We conclude that

$$\liminf_{\rho \to 0} \Psi(u^{(\rho)}) \geq \Psi(u),$$

where $u$ is the weak limit in the $L^2(0,T,L^2(\mathbb{T}^3))$-topology.

1.1. Outline. The rest of the paper is organized as follows. In Section 2, we prove some properties of the discrete Caputo derivatives, then prove Theorem 1.2. Sections 3 to 6 concern the proof of Theorem 1.1. In the Appendix we show a formal $L^3(0,T;L^3(\mathbb{T}^3))$ estimate for $u$, provided $\frac{1}{2} < s \leq 1$.

1.2. Notation. We list here the notations that will be used consistently throughout the paper.

- $\Gamma_\alpha$: the gamma function evaluated at $\alpha$.
- $\lceil z \rceil$: the ceiling function, namely the smallest integer greater than or equal to $z$.
- $\lfloor z \rfloor$: the floor function, namely the largest integer smaller than or equal to $z$.
- $g_+ := \max\{g,0\}$ and $g_- := \min\{g,0\}$ for measurable function $g$.

2. Preliminary and Compactness Criteria

2.1. Preliminaries. In this section we state various useful properties of the Caputo derivative. We first recall the definition [24].

**Definition 1.** If a function $f(t)$ is absolutely continuous in $(0,T)$, for $0 < \alpha < 1$ the left Caputo derivative of $f$ in variable $t \in (0,T)$ is defined as

$$D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \, ds,$$

and the right Caputo derivative is defined as

$$^*D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{f'(s)}{(s-t)^\alpha} \, ds,$$

where $\Gamma_z$ is the gamma function $\Gamma_z = \int_0^\infty e^{-x} x^{z-1} \, dx$. 
Remark 1. Alternative formulas for (2.1) and (2.2) are
\[ D_\tau^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(t) - f(0)}{t^\alpha} + \alpha \int_0^t \frac{f(t) - f(s)}{(t-s)^{1+\alpha}} ds \right), \quad (2.3) \]
and
\[ ^*D_\tau^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(T) - f(t)}{(T-t)^\alpha} + \alpha \int_t^T \frac{f(s) - f(t)}{(s-t)^{1+\alpha}} ds \right), \quad (2.4) \]
respectively.

For the purpose of our problem, we define a discrete version of (2.1) and (2.2).

Definition 2. Consider a function \( f(t) \) in \([0,T]\). Subdivide the time interval \([0,T]\) into \( N \) subintervals with uniform time step \( \tau = T/N \). Denote \( t_k := k\tau \) and \( f_k := f(t_k) \) for \( k = 0,1,\ldots,N \). The left and right discrete Caputo derivatives of \( f(t) \) of order \( \alpha \) with \( 0 < \alpha \leq 1 \) at \( t_k \) are approximated by the linear combination of values of \( f \) at \( t_0,t_1,\ldots,t_k \) as follows:
\[ (D_\tau^\alpha f)_k = \Gamma_{\alpha} \tau^{-\alpha} \sum_{j=0}^{k-1} \lambda_{k-j}(f_{j+1} - f_j), \quad k \geq 1; \]
\[ (^*D_\tau^\alpha f)_k = \Gamma_{\alpha} \tau^{-\alpha} \sum_{j=k+1}^{N} \lambda_{j-k}(f_j - f_{j-1}), \quad k \leq N-1, \]
where the sequence \( \{\lambda_k\} \) is defined by the following infinite linear recurrence:
\[ \lambda_{k+1} = \sum_{j=1}^{k} ((k-j+1)^{\alpha-1} - (k-j+2)^{\alpha-1}) \lambda_j, \quad k \geq 1, \quad \lambda_1 = 1. \quad (2.5) \]

We extend the definition for any \( t \in [0,T] \) by the backward finite difference operator:
\[ D_\tau^\alpha f(t) := \Gamma_{\alpha} \tau^{-1-\alpha} \int_0^{[t/\tau] - [s/\tau]} \lambda_{([t/\tau]-[s/\tau])}(f(s + \tau) - f(s)) \, ds, \quad (2.6) \]
with \( D_\tau^\alpha f(0) = 0 \).

When \( f(t) \) is a piecewise constant function \( f(t) := \sum_{k=1}^{\infty} f_k \chi_{(t_{k-1},t_k]}(t) \) with \( f(t) = f_m \) for \( t \leq 0 \), then (2.6) can be rewritten as
\[ D_\tau^\alpha f(\tau)(t) = \sum_{k=1}^{n}(D_\tau^\alpha f(\tau))_k \chi_{(t_{k-1},t_k]}(t). \]

For \( \alpha = 1 \) the operator \( (D_\tau^\alpha f)_k \) reduces to the classical backward finite difference operator:
\[ (D_1^\alpha f)_k = \frac{f_k - f_{k-1}}{\tau}, \quad k \geq 1, \]
since \( \lambda_j = 0 \) for all \( j \geq 2 \). For \( \alpha < 1 \) one cannot obtain an exact expression for \( \lambda_j \). We have, however, an upper bound, the asymptotic behaviour, and the proof of monotonicity, as stated in the following proposition.

Proposition 1. Let \( \{\lambda_k\}_{k \in \mathbb{N}} \) be defined as in (2.5). It holds that
\[ \lambda_{k+1} < \lambda_k \leq \frac{1}{k^\alpha}, \quad \sum_{j=1}^{k} \lambda_j \leq k^{1-\alpha}, \quad k \geq 1, \quad (2.7) \]
and as \( \tau \to 0 \)
\[ \tau^{-\alpha} \sum_{k=1}^{\infty} \lambda_k \chi_{(t_{k-1},t_k]}(t) \to \frac{t^{-\alpha}}{\Gamma_{\alpha} \Gamma_{1-\alpha}} \text{ in } L^p(0,T) \text{ with } 1 \leq p < 1/\alpha. \quad (2.8) \]
Moreover, for any sequence \( f^{(r)} \rightarrow f \) strongly convergent in \( L^q(0, T) \) for some \( q \in [1, \infty) \), we have
\[
\frac{1}{\tau^\alpha} \int_0^t \sum_{k=1}^n \lambda_k \mathcal{Y}_{(t_{k-1}, t_k]}(s) f^{(r)}(t - s) \, ds \rightarrow \frac{1}{\Gamma(1 - \alpha)} \int_0^t s^{-\alpha} f(t - s) \, ds,
\]
strongly in \( L^r(0, T) \) for every \( r < \frac{q}{1 - (1 - \alpha)q} \).

Proof. We begin by showing that the sequence \((\lambda_k)_{k \in \mathbb{N}}\) is decreasing. Due to the definition of \( \lambda_i \),
\[
\lambda_{i+1} = \sum_{j=1}^i (i - j + 1)^{\alpha-1} \lambda_j - \sum_{j=0}^{i-1} (i - j + 1)^{\alpha-1} \lambda_{j+1}, \tag{2.10}
\]
which implies
\[
\sum_{j=1}^i (i + 1 - j)^{\alpha-1} (\lambda_j - \lambda_{j+1}) = (i + 1)^{\alpha-1}, \quad i \geq 1. \tag{2.11}
\]
We prove by induction that \( \lambda_{i+1} < \lambda_i \) for \( i \geq 1 \). Since \( \lambda_2 = 1 - 2^{\alpha-1} < 1 = \lambda_1 \), the statement is true for \( i = 1 \). Let us now assume that \( \lambda_{i+1} < \lambda_i \) for \( 1 \leq i \leq k-1 \). From (2.11) evaluated at \( i = k \) it follows
\[
1 - (k + 1)^{1-\alpha} (\lambda_k - \lambda_{k+1}) = \sum_{j=1}^{k-1} \left( 1 - \frac{j}{k+1} \right)^{\alpha-1} (\lambda_j - \lambda_{j+1}).
\]
Since \( \lambda_j - \lambda_{j+1} > 0 \) for \( j = 1, \ldots, k-1 \) by inductive assumption and \( \left( 1 - \frac{j}{k+1} \right)^{\alpha-1} < \left( 1 - \frac{1}{k} \right)^{\alpha-1} \) for \( j = 1, \ldots, k-1 \), we deduce
\[
1 - (k + 1)^{1-\alpha} (\lambda_k - \lambda_{k+1}) < \sum_{j=1}^{k-1} \left( 1 - \frac{j}{k} \right)^{\alpha-1} (\lambda_j - \lambda_{j+1}).
\]
Evaluating (2.11) at \( i = k - 1 \) yields that the right-hand side of the above identity equals 1, so:
\[
1 - (k + 1)^{1-\alpha} (\lambda_k - \lambda_{k+1}) < 1,
\]
meaning that \( \lambda_k - \lambda_{k+1} > 0 \). Therefore the sequence \((\lambda_i)_{i \in \mathbb{N}}\) is decreasing.

We prove now the upper bounds for \( \lambda_k \). From (2.10) it follows
\[
\sum_{j=1}^{i+1} (i - j + 2)^{\alpha-1} \lambda_j = \sum_{j=1}^i (i - j + 1)^{\alpha-1} \lambda_{j+1} = \sum_{j=1}^i (i - j + 1)^{\alpha-1} \lambda_j, \quad i \geq 1.
\]
Thanks to the above identity, one can easily prove by induction that
\[
\sum_{j=1}^i (i - j + 1)^{\alpha-1} \lambda_j = 1, \quad i \geq 1. \tag{2.12}
\]
Thus, since \( \{\lambda_k\} \) is a decreasing sequence,
\[
1 = \sum_{j=1}^n \lambda_j (n - j + 1)^{\alpha-1} \geq \sum_{j=1}^n \lambda_j n^{\alpha-1} \geq \lambda_n n^{\alpha},
\]
which implies
\[
\lambda_n \leq n^{-\alpha}, \quad \sum_{j=1}^n \lambda_j \leq n^{1-\alpha} \quad \text{for any } n \geq 1.
\]
Define now
\[
\lambda^{(r)}(s) := \tau^{-\alpha} \sum_{j=1}^n \lambda_j \mathcal{Y}_{(t_{j-1}, t_j]}(s).
\]
Since \( \lambda^{(\tau)}(s) \leq \sum_{j=1}^{n} (j\tau)^{-\alpha}X_{(t_{j-1},t_{j})}(s) \leq s^{-\alpha} \), the sequence \( \lambda^{(\tau)} \) is uniformly bounded in \( L^{p}(0,T) \) for every \( p \in [1,1/\alpha) \) and \( \lambda^{(\tau)} \rightarrow \lambda \) in \( L^{p}(0,T) \) with \( 1 \leq p < 1/\alpha \) up to a subsequence.

Next we show (2.9). Let \( n = \lceil t/\tau \rceil \), \( t_{n} \leq t < t_{n+1} \), and \( q \leq r < \frac{q}{1 - (1/\alpha)} \). Define
\[
F_{n}^{(\tau)}(t) := \int_{0}^{t_{n}} \lambda^{(\tau)}(s)f^{(\tau)}(t-s) \, ds
\]
\[
= \int_{0}^{t} \lambda^{(\tau)}(s)f^{(\tau)}(t-s) \, ds - \int_{t_{n}}^{t} \lambda^{(\tau)}(s)f^{(\tau)}(t-s) \, ds
\]
\[
= F_{1}^{(\tau)}(t) - F_{2}^{(\tau)}(t).
\]
The term \( F_{2}^{(\tau)} \) tends to zero in \( L^{r}(0,T) \) as \( \tau \rightarrow 0 \). In fact,
\[
\left\| F_{2}^{(\tau)} \right\|_{L^{r}(0,T)} \leq \left\| \int_{0}^{t_{n}} \lambda^{(\tau)}(s)f^{(\tau)}(t-s) \, ds \right\|_{L^{r}(0,T)}
\]
\[
= \left\| \left( \lambda^{(\tau)}X_{[0,T]} \right) * (f^{(\tau)}X_{[0,\tau]}) \right\|_{L^{r}(0,T)}
\]
\[
\leq \left\| \lambda^{(\tau)} \right\|_{L^{1+r}(0,T)} \left\| f^{(\tau)} \right\|_{L^{s}(0,T)}
\]
\[
\leq C \left\| f^{(\tau)} - f \right\|_{L^{s}(0,T)} + C \left\| f \right\|_{L^{s}(0,T)},
\]
where \( 1 + \eta = (1 + 1/r - 1/q)^{-1} \in [1,1/\alpha) \), and the right-hand side of the above inequality tends to zero as \( \tau \rightarrow 0 \) since \( f^{(\tau)} \rightarrow f \) strongly in \( L^{q}(0,T) \).

On the other hand,
\[
\left\| F_{1}^{(\tau)} - \int_{0}^{t} \lambda(s)f(t-s) \, ds \right\|_{L^{q}(0,T)}
\]
\[
\leq \left\| \int_{0}^{t} \left( \lambda^{(\tau)} - \lambda \right)(s)f(t-s) \, ds \right\|_{L^{q}(0,T)} + \left\| \int_{0}^{t} \lambda^{(\tau)}(s)\left( f^{(\tau)} - f \right)(t-s) \, ds \right\|_{L^{q}(0,T)}
\]
\[
\leq \left\| \int_{0}^{t} \left( \lambda^{(\tau)} - \lambda \right)(s)f(t-s) \, ds \right\|_{L^{q}(0,T)} + \left\| \lambda^{(\tau)} \right\|_{L^{1}(0,T)} \left\| f^{(\tau)} - f \right\|_{L^{q}(0,T)}.
\]
The second term above converges to 0 as \( \tau \rightarrow 0 \). Now consider the first term. Since \( f \in L^{q}(0,T) \) and \( \left\| \lambda^{(\tau)} - \lambda \right\|_{L^{1}(0,T)} \leq C \), for \( C \) not depending on \( \tau \), by Corollary 4.28 in [9], the sequence
\[
I^{(\tau)} := \int_{0}^{t} \left( \lambda^{(\tau)} - \lambda \right)(s)f(t-s) \, ds
\]
has compact closure in \( L^{q}(0,T) \). However, for every \( \phi \in L^{q'}(0,T) \) \( (q' = q/(q-1) \) for \( q > 1 \), \( q' = \infty \) for \( q = 1) \),
\[
\int_{0}^{T} I^{(\tau)}(t)\phi(t)dt = \int_{0}^{T} \phi(t)\int_{0}^{t} \left( \lambda^{(\tau)} - \lambda \right)(s)f(t-s) \, ds \, dt
\]
\[
= \int_{0}^{T} \left( \lambda^{(\tau)} - \lambda \right)(s)\int_{s}^{T} \phi(t)f(t-s) \, dt \, ds
\]
and
\[
\left\| \int_{0}^{T} \phi(t)f(t-s) \, dt \right\| \leq \left\| \phi \right\|_{L^{q'}(s,T)} \left\| f \right\|_{L^{q}(0,T-s)} \leq \left\| \phi \right\|_{L^{q'}(0,T)} \left\| f \right\|_{L^{q}(0,T)} \quad \text{for } s \in [0,T],
\]
which means that \( s \in [0,T] \mapsto \int_{s}^{T} \phi(t)f(t-s) \, dt \in \mathbb{R} \) is in \( L^{\infty}(0,T) \). Given that \( \lambda^{(\tau)} - \lambda \rightarrow 0 \) weakly in \( L^{1}(0,T) \), we deduce that \( \int_{0}^{T} I^{(\tau)}(t)\phi(t)dt \rightarrow 0 \) for every \( \phi \in L^{q}(0,T) \), that is, \( I^{(\tau)} \rightarrow 0 \) weakly in
$L^q(0, T)$. Since we already knew that $I^{(\tau)}$ is relatively compact in $L^q(0, T)$, we conclude that $I^{(\tau)} \to 0$ strongly in $L^q(0, T)$ and therefore

$$F_1^{(\tau)} \to \int_0^t \lambda(s) f(t-s) \, ds \quad \text{strongly in } L^q(0, T).$$

However, the above convergence is also strong in $L^r(0, T)$, since $F_1^{(\tau)}$ is bounded in $L^r(0, T)$ for every $1 \leq \tilde{r} < \frac{q}{1-(1-\alpha)q}$ being the convolution of $\lambda^{(\tau)}$, which is bounded in $L^p(0, T)$ for every $1 \leq p < 1/\alpha$, and $f^{(\tau)}$, which is bounded in $L^q(0, T)$. Summarizing up we have

$$\int_0^{\tau\lfloor t/\tau \rfloor} \lambda^{(\tau)}(s) f^{(\tau)}(t-s) \, ds \to \int_0^t \lambda(s) f(t-s) \, ds \quad \text{strongly in } L^r(0, T) \quad (2.13)$$

for every $1 \leq r < \frac{q}{1-(1-\alpha)q}$, for any sequence $f^{(\tau)} \to f$ strongly convergent in $L^q(0, T)$ for some $q \in [1, \infty)$.

Our last step is finding the value of $\lambda(t)$. Since

$$\lambda_j(n-j+1)^{\alpha-1} = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} \lambda_j(n-j+1)^{\alpha-1} \, ds = \frac{1}{\tau} \int_0^{t_n} \lambda_j(n-j+1)^{\alpha-1} X_{(t_{j-1}, t_j)}(s) \, ds,$$

from (2.12) one obtains

$$1 = \sum_{j=1}^n \lambda_j(n-j+1)^{\alpha-1} = \frac{1}{\tau} \int_0^{t_n} \sum_{j=1}^n \lambda_j(n-j+1)^{\alpha-1} X_{(t_{j-1}, t_j)}(s) \, ds$$

$$= \frac{1}{\tau} \int_0^{t_n} \left( \sum_{j=1}^n \lambda_j X_{(t_{j-1}, t_j)}(s) \right) \left( \sum_{j=1}^n (n-j+1)^{\alpha-1} X_{(t_{j-1}, t_j)}(s) \right) \, ds$$

$$= \frac{1}{\tau} \int_0^{t_n} \left( \sum_{j=1}^n \lambda_j X_{(t_{j-1}, t_j)}(s) \right) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1} \, ds,$$

which means, given the definition of $\lambda^{(\tau)}$,

$$\int_0^{t_n} \lambda^{(\tau)}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1} \, ds = 1. \quad (2.14)$$

It holds

$$X_{[0, t_n]}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1} \geq X_{[0, t_\tau]}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1},$$

$$X_{[0, t_n]}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1} \leq X_{[0, t_{\tau-n}]}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1} + X_{[t_n-t_\tau, t_n]}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1},$$

which means for

$$f_1^{(\tau)}(z) \equiv (z+\tau)^{\alpha-1} X_{[\tau, T]}(z), \quad f_2^{(\tau)}(z) \equiv (z-\tau)^{\alpha-1} X_{[\tau, T]}(z)$$

that

$$f_1^{(\tau)}(t-s) \leq X_{[0, t_n]}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1} \leq f_2^{(\tau)}(t-s) + X_{[t_n-t_\tau, t_n]}(s) \left( n + 1 - \lfloor s/\tau \rfloor \right)^{\alpha-1}. \quad (2.15)$$

Estimates (2.14), (2.15) and the fact that $\lambda^{(\tau)} \equiv \tau^{-\alpha} \lambda_n = \tau^{-\alpha} \lambda_{[t/\tau]}$ on $(t_n - \tau, t_n]$ imply

$$\int_0^t \lambda^{(\tau)}(s) f^{(\tau)}(t-s) \, ds \leq \int_0^t \lambda^{(\tau)}(s) f_2^{(\tau)}(t-s) \, ds + C \lambda_{[t/\tau]}, \quad \tau \leq t \leq T, \quad (2.16)$$
with $\lambda_{[t/\tau]} \to 0$ for $\tau \to 0$. We wish to show for $\tau \to 0$ that
\[ f_i^{(\tau)} \to \cdot \alpha^{-1} \quad \text{strongly in } L^q(0,T), \quad \forall q \in [1,1/(1-\alpha)), \quad i = 1,2. \quad (2.17) \]
Clearly $f_i^{(\tau)}(z) \to |z|\alpha^{-1}$ for $\tau \to 0$ for every $z > 0$. Furthermore, if $q \in [1,1/(1-\alpha))$,
\[ \int_0^T |f_1^{(\tau)}(z)|^q \, dz \leq \int_0^T |f_2^{(\tau)}(z)|^q \, dz = \int_0^T (z-\tau)^{q(\alpha-1)} \, dz = \frac{(T-\tau)^{q(\alpha-1)+1}}{q(\alpha-1)+1} \leq \frac{T^{q(\alpha-1)+1}}{q(\alpha-1)+1}. \]
Being $q \in [1,1/(1-\alpha))$ arbitrary, we deduce via dominated convergence that (2.17) holds. From (2.16) we conclude that
\[ \int_0^t \lambda(s)(t-s)^{\alpha-1} \, ds = 1. \quad (2.18) \]
Performing the Laplace transform on both sides of (2.18) yields
\[ \int_0^{+\infty} \int_0^t e^{-kt} \lambda(s)(t-s)^{\alpha-1} \, ds \, dt = \int_0^{+\infty} e^{-kt} \, dt = \frac{1}{k}. \]
Then, interchanging the order of the integrals yields
\[ \int_0^{+\infty} e^{-ks} \lambda(s) \int_s^{+\infty} e^{-k(t-s)} (t-s)^{\alpha-1} \, dt \, ds = \frac{1}{k}, \]
which indicates
\[ \int_0^{+\infty} e^{-ks} \lambda(s) \, ds = \frac{1}{k} \left( \int_0^{+\infty} e^{-ks} s^{\alpha-1} \, ds \right)^{-1} = k^{\alpha-1} \Gamma^{-1}. \]
Performing the inverse Laplace transform on the above equation leads to
\[ \lambda(t) = \frac{t^{-\alpha}}{\Gamma \alpha \Gamma_{1-\alpha}}, \]
which finishes the proof.

Next we deal with the fractional version of fundamental theorem of calculus [25]:
\[ f(t) = f(0) + \frac{1}{\Gamma \alpha} \int_0^t (t-s)^{\alpha-1} D^\alpha_t f(s) \, ds, \quad (2.19) \]
and
\[ f(t) = f(T) - \frac{1}{\Gamma \alpha} \int_t^T (s-t)^{\alpha-1} \, ds, \quad (2.20) \]
In the following proposition we propose a discrete version of (2.19) and (2.20).

**Proposition 2** (Discrete fundamental theorem of calculus). Let $f_n := f(t_n)$ for $n = 1,\ldots,N$. We have the following identities:
\[ f_n = f_0 + \frac{\tau^\alpha}{\Gamma \alpha} \sum_{k=1}^n (n-k+1)^{\alpha-1} (D^\alpha_k f)_k, \quad n \geq 1; \quad (2.21) \]
and
\[ f_n = f_N - \frac{\tau^\alpha}{\Gamma \alpha} \sum_{k=n}^{N-1} (k-n+1)^{\alpha-1} (D^\alpha_k f)_k, \quad n \leq N-1. \quad (2.22) \]
Proof. From (2.12) one obtains
\[
\frac{\tau^{\alpha}}{\Gamma(\alpha)} \sum_{k=1}^{n} (n - k + 1)^{\alpha - 1} (D_{\tau}^{\alpha} f)_k = \sum_{k=1}^{n} (n - k + 1)^{\alpha - 1} \sum_{j=0}^{k-1} \lambda_{k-j} (f_{j+1} - f_j)
\]
\[
= \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} (n - k + 1)^{\alpha - 1} \lambda_{k-j} (f_{j+1} - f_j)
\]
\[
= \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} (n - j + 1)^{\alpha - 1} \lambda_k (f_{j+1} - f_j) = \sum_{j=0}^{n-1} (f_{j+1} - f_j)
\]
which implies (2.21). Furthermore,
\[
\frac{\tau^{\alpha}}{\Gamma(\alpha)} \sum_{k=n}^{N-1} (k - n + 1)^{\alpha - 1} (\ast D_{\tau}^{\alpha} f)_k = \sum_{k=n}^{N-1} (k - n + 1)^{\alpha - 1} \sum_{j=k+1}^{N} \lambda_{j-k} (f_j - f_{j-1})
\]
\[
= \sum_{j=n+1}^{N} \sum_{k=n}^{j-1} (k - n + 1)^{\alpha - 1} \lambda_{j-k} (f_j - f_{j-1})
\]
\[
= \sum_{j=n+1}^{N} \sum_{k=n}^{j-1} (j - n + 1)^{\alpha - 1} \lambda_k (f_j - f_{j-1}) = \sum_{j=n+1}^{N} (f_j - f_{j-1})
\]
which implies (2.22). This concludes the proof.

In the next proposition we show that the discrete Caputo derivative of an absolute continuous function converges, in a weak sense, to the continuous Caputo derivative. The proof is a straightforward consequence of Proposition 2.

Proposition 3 (Limit $\tau \to 0$). Let $X$, $Y$ be Banach space, $f : [0, T] \to X$ absolutely continuous. Assume that $F_{\tau} \equiv \sum_{k=1}^{N} (D_{\tau}^{\alpha} f)_k X_{[t_{k-1}, t_k)}$ is weakly convergent in $L^p(0, T; Y)$ to some limit $\xi$ with $p > 1/\alpha$. Then $\xi = D_{t}^{\alpha} f$.

Proof. From (2.21) it follows
\[
f(t_n) = f(0) + \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} \int_{0}^{t_n} \sum_{k=1}^{n} (n - k + 1)^{\alpha - 1} (D_{\tau}^{\alpha} f)_k X_{[t_{k-1}, t_k)} (s) \, ds
\]
\[
= f(0) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_n} \sum_{k=1}^{n} (t_n - t_k + \tau)^{\alpha - 1} (D_{\tau}^{\alpha} f)_k X_{[t_{k-1}, t_k)} (s) \, ds
\]
\[
= f(0) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_n} \left( \sum_{j=1}^{n} (t_n - t_j + \tau)^{\alpha - 1} X_{[t_{j-1}, t_j)} (s) \right) \left( \sum_{k=1}^{n} (D_{\tau}^{\alpha} f)_k X_{[t_{k-1}, t_k)} (s) \right) \, ds
\]
\[
= f(0) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_n} \left( \sum_{j=1}^{n} (t_n - t_j + \tau)^{\alpha - 1} X_{[t_{j-1}, t_j)} (s) \right) F_{\tau} (s) \, ds,
\]
since the characteristic functions have disjoint support. We now pass to the limit $\tau \to 0$ in the above identity. From (2.15), (2.17) we know that
\[
\sum_{j=1}^{n} (t_n - t_j + \tau)^{\alpha - 1} X_{[t_{j-1}, t_j)} \to (t - s)^{\alpha - 1} \quad \text{strongly in } L^{p/(p-1)}(0, t)
\]
for $p > \frac{1}{\alpha}$. Since $F_r \to \xi$ weakly in $L^p(0,T)$ by assumption, we deduce that

$$f(t) = f(0) + \frac{1}{\Gamma_0} \int_0^t (t-s)^{\alpha-1} \xi, \quad t \in [0,T],$$

which means that $\xi = D^\alpha f$ a.e. in $[0,T]$. This finishes the proof of the Proposition. \qed

We now recall three equivalent integration by parts formulas for (2.1). The first one [5] reads as

$$\int_0^T D_t^\alpha f(t) \phi(t) \, dt = -\frac{1}{\Gamma_1-\alpha} \int_0^T f(t) \frac{d}{dt} \left( \int_t^T \frac{\phi(s)}{(s-t)\alpha} \, ds \right) dt + \frac{f(0)}{\Gamma_1-\alpha} \int_0^T \frac{\phi(s)}{s\alpha} \, ds. \quad (2.23)$$

The second one can be found in [1]:

$$\int_0^T D_t^\alpha f(t) \phi(t) \, dt = -\int_0^T D_t^\alpha \phi(t) f(t) \, dt + \frac{1}{\Gamma_1-\alpha} \int_0^T \phi(t) f(t) \left[ \frac{1}{(T-t)\alpha} + \frac{1}{t\alpha} \right] dt \quad (2.24)$$

$$+ \frac{\alpha}{\Gamma_1-\alpha} \int_0^t \int_0^t \frac{(\phi(t) - \phi(s)) (f(t) - f(s))}{(s-t)^{\alpha+1}} \, ds \, dt = -\frac{1}{\Gamma_1-\alpha} \int_0^T \frac{\phi(t)f(0) + f(t)\phi(0)}{t\alpha} \, dt. \quad (2.25)$$

The last one reads as

$$\int_0^T D_t^\alpha f(t) \phi(t) \, dt = -\int_0^T f(t) \cdot D_t^\alpha \phi(t) \, dt + \phi(T) \frac{f(t)}{(T-t)^\alpha} \int_0^T \frac{f(t)}{(T-t)^\alpha} \, dt - \frac{f(0)}{\Gamma_1-\alpha} \int_0^T \frac{\phi(s)}{s\alpha} \, ds. \quad (2.26)$$

The three formulas are equivalent; each one can be derived from the others using integration by parts.

In the second part of this manuscript we will use all three. We will also use a discrete integration by parts formula, provided in the next proposition.

**Proposition 4.** Let $\phi(t)$ be a bounded function in $(0,T]$, and $f^{(r)}(t)$ be a piecewise constant function defined by $f^{(r)}(t) = f_0 \chi_{(0)}(t) + \sum_{k=1}^N f_k \chi_{(t_k-1,t_k]}(t)$. Then we have the following discrete integration by parts formula:

$$\int_0^T D_t^\alpha f^{(r)}(t) \phi(t) \, dt = -\int_0^T f^{(r)}(t) \cdot D_t^\alpha \left( D_t^\alpha \Phi \right) (t) \, dt$$

$$+ \frac{\Gamma_0}{\tau^\alpha} \Phi_N \sum_{j=1}^N \lambda_N \cdot j f_j - \frac{\Gamma_0}{\tau^\alpha} f_{0N} \chi_{(0)} \Phi_k \quad (2.26)$$

where $\Phi(t) := \int_0^t \phi(s) \, ds$ and $\Phi_k := \int_{(k-1)\tau}^{k\tau} \phi(t) \, dt$.

**Proof.** It holds

$$\int_0^T D_t^\alpha f^{(r)}(t) \phi(t) \, dt = \frac{\Gamma_0}{\tau^\alpha} \sum_{k=1}^N \left( \int_{(k-1)\tau}^{k\tau} \sum_{j=0}^{k-1} \lambda_k f_j \phi(t) \, dt \right)$$

$$= \frac{\Gamma_0}{\tau^\alpha} \sum_{k=1}^N \sum_{j=0}^{k-1} \lambda_k f_j \phi(t) \, dt.$$
We have \((D_t^1 \Phi)_k = (\Phi(k\tau) - \Phi((k-1)\tau))/\tau = \Phi_k/\tau\). Now rewrite the double summation as follows:

\[
\sum_{k=1}^{N} \sum_{j=0}^{k-1} \lambda_{k-j}(f_{j+1} - f_j) \Phi_k = \sum_{k=1}^{N} \sum_{j=1}^{k} \lambda_{k-j+1} f_j \Phi_k - \sum_{k=1}^{N} \sum_{j=0}^{k-1} \lambda_{k-j} f_j \Phi_k
\]

\[
= \sum_{k=2}^{N+1} \sum_{j=1}^{k-1} \lambda_{k-j} f_j \Phi_{k-1} - \sum_{k=1}^{N} \sum_{j=0}^{k-1} \lambda_{k-j} f_j \Phi_k
\]

\[
= N \sum_{k=2}^{N} \sum_{j=1}^{k-1} \lambda_{k-j} f_j (\Phi_{k-1} - \Phi_k) + \sum_{j=1}^{N} \lambda_{N-j+1} f_j \Phi_N - \sum_{k=1}^{N} \lambda_k f_0 \Phi_k
\]

\[
= \int_0^T D_\tau^\alpha f^{(\tau)}(t) \phi(t) \, dt = \frac{\Gamma^{\alpha}}{\tau^{\alpha}} \sum_{k=1}^{N-1} f_k \sum_{j=k+1}^{N} \lambda_{j-k} (\Phi_{j-1} - \Phi_j)
\]

\[
+ \frac{\Gamma^{\alpha}}{\tau^{\alpha}} \sum_{j=1}^{N} \lambda_{N-j} f_j - \frac{\Gamma^{\alpha}}{\tau^{\alpha}} f_0 \sum_{k=1}^{N} \lambda_k \Phi_k
\]

\[
= -\int_0^{T-\tau} f^{(\tau)}(t)^* D_\tau^\alpha (D_t^1 \Phi) (t) \, dt
\]

\[
+ \frac{\Gamma^{\alpha}}{\tau^{\alpha}} \Phi_N \sum_{j=1}^{N} \lambda_{N-j} f_j - \frac{\Gamma^{\alpha}}{\tau^{\alpha}} f_0 \sum_{k=1}^{N} \lambda_k \Phi_k.
\]

Hence

We have the following convergence theorem.

**Proposition 5.** Provided \(f^{(\tau)}(t) := \sum_{j=1}^{N} f_j \chi_{(t_{j-1}, t_j]}(t)\) is strongly convergent in \(L^1(0, T)\), for any smooth function \(\phi \in C^1(0, T)\) we have

\[
\int_0^T D_\tau^\alpha f^{(\tau)}(t) \phi(t) \, dt \rightarrow -\int_0^T f(t)^* D_\tau^\alpha \phi(t) \, dt + \frac{\phi(T)}{\Gamma^{1-\alpha}} \int_0^T f(t) \frac{(T-t)^\alpha}{(T-t)^\alpha} \, dt - \frac{f_0}{\Gamma^{1-\alpha}} \int_0^T \phi(s) \frac{s^\alpha}{s^\alpha} \, ds,
\]

as \(\tau \to 0\). Moreover, if function \(f\) is smooth enough, it holds

\[
\int_0^T D_\tau^\alpha f^{(\tau)}(t) \phi(t) \, dt \rightarrow \int_0^T D_t^\alpha f(t) \phi(t) \, dt, \quad \text{as} \ \tau \to 0.
\]

**Proof.** Let \(\Phi(t) := \int_0^t \phi(s) \, ds\) and \(\Phi_k := \int_{(k-1)\tau}^{k\tau} \phi(t) \, dt\) as in Proposition 4. Proving the above convergence is equivalent to showing that

\[-\int_0^{T-\tau} f^{(\tau)}(t)^* D_t^\alpha (D_t^1 \Phi) (t) \, dt + \frac{\Gamma^{\alpha}}{\tau^{\alpha}} \Phi_N \sum_{j=1}^{N} \lambda_{N-j} f_j - \frac{\Gamma^{\alpha}}{\tau^{\alpha}} f_0 \sum_{k=1}^{N} \lambda_k \Phi_k
\]

converges to

\[-\int_0^T f(t)^* D_t^\alpha \phi(t) \, dt + \frac{\phi(T)}{\Gamma^{1-\alpha}} \int_0^T f(t) \frac{(T-t)^\alpha}{(T-t)^\alpha} \, dt - \frac{f_0}{\Gamma^{1-\alpha}} \int_0^T \phi(s) \frac{s^\alpha}{s^\alpha} \, ds,
\]
as $\tau \to 0$. To see why, one should compare (2.25) and (2.26). Using (2.7), we have

$$\|{^*D_\tau^\alpha} (D_1^1 \Phi)\|_{L^\infty(0,T)} = \max_{0 \leq k \leq N-1} \left| \sum_{j=k+1}^N \tau^{1-\alpha} \lambda_{j-k} (D_1^2 \Phi)_j \right|$$

$$\leq \sup_{t \in [0,T]} |D_1^2 \Phi| \sum_{j=k+1}^N \lambda_{j-k}$$

$$= \sup_{t \in [0,T]} |D_1^2 \Phi| \sum_{j=1}^{N-k} \lambda_j$$

$$\leq \sup_{t \in [0,T]} |D_1^2 \Phi| \sum_{j=1}^{N-k} \lambda_j$$

$$\leq \left\| \phi \right\|_{C^1(0,T)} T^{1-\alpha}.$$

Hence $^*D_\tau^\alpha (D_1^1 \Phi) \to ^* \xi$ weakly* in $L^\infty(0,T)$. In particular $^*D_\tau^\alpha (D_1^1 \Phi) \to \xi$ weakly in $L^p(0,T)$ for every $p < \infty$, so we can employ Proposition 3 to identify the limit: $\xi = ^*D_\tau^\alpha \phi$. Since $f^{(\tau)} \to f$ strongly in $L^1(0,T)$ we deduce

$$\int_0^{T-\tau} f^{(\tau)}(t)^*D_\tau^\alpha (D_1^1 \Phi)(t) \, dt \to \int_0^T f(t)^*D_\tau^\alpha \phi(t) \, dt, \quad \text{as } \tau \to 0.$$

With $\lambda^{(\tau)}(s) := \tau^{-\alpha} \sum_{j=1}^n \lambda_j \chi_{(t_{j-1},t_j]}(s)$ we have

$$\frac{\Gamma_\alpha}{\tau^\alpha} \Phi N \sum_{j=1}^N \lambda_{N-j+1} f_j = \Gamma_\alpha \int_0^T \frac{\phi(t)}{\tau} \, dt \int_0^T \lambda^{(\tau)}(T-t) f^{(\tau)}(t) \, dt,$$

which converges to

$$\frac{\phi(T)}{\Gamma_1-\alpha} \int_0^T \frac{f(t)}{(T-t)^\alpha} \, dt$$

as $\tau \to 0$ by (2.9). Finally, using (2.8) we get

$$\frac{\Gamma_\alpha}{\tau^\alpha} \int_0^T f_0 \sum_{k=1}^N \lambda_k \Phi_k \to \frac{f_0}{\Gamma_1-\alpha} \int_0^T \frac{\phi(s)}{s^\alpha} \, ds. \quad (2.27)$$

This finishes the proof. \hfill $\square$

The following two lemmas will be useful.

**Lemma 1.** If $(D_\tau^\alpha f)_k \leq 0$ for $1 \leq k \leq n$, then $f_k \leq f_0$ for $1 \leq k \leq n$.

*Proof.* It follows directly from (2.21). \hfill $\square$

**Lemma 2.** We have

$$f_k(D_\tau^\alpha f)_k \geq \frac{1}{2} (D_\tau^\alpha f^2)_k, \quad k \geq 1.$$

*Proof.* We have to show for $k \geq 1$ that

$$\sum_{j=0}^{k-1} \lambda_{k-j} (f_{j+1} - f_j) f_k \geq \frac{1}{2} \sum_{j=0}^{k-1} \lambda_{k-j} (f_{j+1}^2 - f_j^2).$$
First, we rewrite the left hand side of the above inequality as
\[
\sum_{j=0}^{k-1} \lambda_{k-j} (f_{j+1} - f_j) f_k = -\lambda_k f_0 f_k - \sum_{j=1}^{k-1} (\lambda_{k-j+1} - \lambda_{k-j}) f_j f_k + \lambda_1 f_k^2
\]
\[
\begin{aligned}
= & \frac{1}{2} \lambda_k (f_0 - f_k)^2 + \frac{1}{2} \sum_{j=1}^{k-1} (\lambda_{k-j+1} - \lambda_{k-j}) (f_j - f_k)^2 + \lambda_1 f_k^2 \\
- & \frac{1}{2} \lambda_k f_0^2 - \frac{1}{2} \lambda_k f_k^2 - \frac{1}{2} \sum_{j=1}^{k-1} (\lambda_{k-j} - \lambda_{k-j+1}) (f_j^2 + f_k^2),
\end{aligned}
\]
which implies
\[
\sum_{j=0}^{k-1} \lambda_{k-j} (f_{j+1} - f_j) f_k \geq - \frac{1}{2} \lambda_k f_0^2 - \frac{1}{2} \lambda_k f_k^2 - \frac{1}{2} \sum_{j=1}^{k-1} (\lambda_{k-j} - \lambda_{k-j+1}) (f_j^2 + f_k^2) + \lambda_1 f_k^2
\]
\[
= \frac{1}{2} \sum_{j=0}^{k-1} \lambda_{k-j} (f_{j+1}^2 - f_j^2).
\]
Thus, the lemma is proved. \(\square\)

2.2. Compactness results. In this section we prove Theorem 1.2. For convenience we first recall the classical version of the Aubin-Lions lemma.

**Lemma 3** (Theorem 5 in [29]). Assume that \(X, B\) and \(Y\) are Banach spaces, with \(X \hookrightarrow B \hookrightarrow Y\), where \(X\) is compactly embedded in \(B\). For \(1 \leq r \leq \infty\), assume that \(F\) is bounded in \(L^r(0,T;X)\) and \(\|f(t+h,x) - f(t,x)\|_{L^r(0,T-h;Y)} \to 0\) as \(h \to 0\), uniformly for all \(f \in F\). Then \(F\) is relatively compact in \(L^r(0,T;B)\) (and in \(C([0,T];B)\) if \(r = \infty\)).

Before we start with the proof of Theorem 1.2 we need to prove three technical lemmas.

**Lemma 4.** For \(0 < \alpha < 1\), it holds that
\[
\|f(\cdot + h) - f\|_{L^p(0,T-h;Y)} \leq \frac{2h^\alpha}{1_{\alpha}} \|D_t^\alpha f\|_{L^p(0,T;Y)}, \quad h > 0,
\] (2.28)
for every \(1 \leq p \leq \infty\) and every \(f : [0,T] \to Y\) such that \(f, D_t^\alpha f \in L^p(0,T;Y)\).

**Lemma 5.** Let \(Y\) be a Banach space, and let \(f(\tau) : [0,T] \to Y\) be a piecewise constant-in-time function, namely \(f(\tau)(t) = f_0 \chi_{[t]}(t) + \sum_{n=1}^N f_n \chi_{(t_n, t_{n+1})}(t)\), with \(t_j = j\tau, f_j \in Y\) \((j = 0,\ldots,N)\), and \(T = N\tau\). The linear interpolant \(f(\tau)\) of \(f_0,\ldots,f_N\) is given by
\[
\hat{f}(\tau)(t) = \sum_{n=0}^{N-1} \left( \frac{t - t_n}{\tau} (f_{n+1} - f_n) + f_n \right) \chi_{(t_n, t_{n+1})}(t).
\] (2.29)

Then for \(0 < \alpha < 1\), a constant \(C_\alpha > 0\) independent of \(f_0,\ldots,f_N\) exists such that
\[
\|D_t^\alpha \hat{f}(\tau)\|_{L^p(0,T;Y)} \leq C_\alpha \|D_t^\alpha f(\tau)\|_{L^p(0,T;Y)}, \quad \tau > 0, \quad 1 \leq p \leq \infty.
\]

**Lemma 6.** Let \(Y\) and \(f(\tau)\) be as in Lemma 5. Then
\[
\|f(\tau)(t + \tau) - f(\tau)(t)\|_{L^p(0,T-\tau;Y)} \leq \frac{2^{1+1/p} \tau^\alpha}{\Gamma_\alpha} \|D_t^\alpha f(\tau)\|_{L^p(0,T;Y)}, \quad \tau > 0, \quad 1 \leq p \leq \infty.
\] (2.30)
Proof of Lemma 4. Since the mapping $D_t^\alpha f \mapsto f(\cdot + h) - f$ is linear, Riesz-Thorin interpolation Theorem [32, Thr. II.4.2] implies that it is enough to prove (2.28) for $p = 1$ and $p = \infty$.

We start by recalling the fundamental theorem of calculus

$$f(t) = f_0 + \frac{1}{\Gamma_\alpha} \int_0^t (t-s)^{\alpha-1} D_t^\alpha f(s)ds, \quad 0 \leq t \leq T.$$

Then

$$f(t+h) - f(t) = \frac{1}{\Gamma_\alpha} \int_0^{t+h} (t+h-s)^{\alpha-1} D_t^\alpha f(s)ds - \frac{1}{\Gamma_\alpha} \int_0^t (t-s)^{\alpha-1} D_t^\alpha f(s)ds$$

$$= \frac{1}{\Gamma_\alpha} \int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}) D_t^\alpha f(s)ds + \frac{1}{\Gamma_\alpha} \int_t^{t+h} (t+h-s)^{\alpha-1} D_t^\alpha f(s)ds$$

$$=: \mathcal{I}_1(t) + \mathcal{I}_2(t).$$

Let us estimate

$$\int_0^{T-h} \|\mathcal{I}_1(t)\|_Y dt \leq \frac{1}{\Gamma_\alpha} \int_0^{T-h} \int_0^t |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| \|D_t^\alpha f(s)\|_Y dsdt$$

$$= \frac{1}{\Gamma_\alpha} \int_0^{T-h} \int_s^{T-h} ((t-s)^{\alpha-1} - (t+h-s)^{\alpha-1}) \|D_t^\alpha f(s)\|_Y dsdt$$

$$= \frac{1}{\Gamma_\alpha} \int_0^{T-h} ((T-h-s)^{\alpha} - (T-s)^{\alpha} + h^\alpha) \|D_t^\alpha f(s)\|_Y ds.$$

We deduce that

$$\int_0^{T-h} \|\mathcal{I}_1(t)\|_Y dt \leq \frac{h^\alpha}{\Gamma_\alpha} \|D_t^\alpha f\|_{L^1(0,T;Y)}.$$

On the other hand

$$\int_0^{T-h} \|\mathcal{I}_2(t)\|_Y dt \leq \frac{1}{\Gamma_\alpha} \int_0^{T-h} \int_t^{t+h} (t+h-s)^{\alpha-1} \|D_t^\alpha f(s)\|_Y dsdt$$

$$= \frac{1}{\Gamma_\alpha} \int_0^T \int_{s-h}^s (t+h-s)^{\alpha-1} \|D_t^\alpha f(s)\|_Y dsdt$$

$$= \frac{h^\alpha}{\Gamma_\alpha} \int_0^T \|D_t^\alpha f(s)\|_Y ds = \frac{h^\alpha}{\Gamma_\alpha} \|D_t^\alpha f\|_{L^1(0,T;Y)}.$$

This means that (2.28) holds for $p = 1$. Let us now consider the case $p = \infty$:

$$\|\mathcal{I}_1(t)\|_Y \leq \|D_t^\alpha f\|_{L^\infty(0,T;Y)} \frac{1}{\Gamma_\alpha} \int_0^t ((t-s)^{\alpha-1} - (t+h-s)^{\alpha-1}) ds$$

$$= \|D_t^\alpha f\|_{L^\infty(0,T;Y)} \frac{1}{\Gamma_\alpha} (t^\alpha + h^\alpha - (t+h)^\alpha)$$

$$\leq \frac{h^\alpha}{\Gamma_\alpha} \|D_t^\alpha f\|_{L^\infty(0,T;Y)}.$$

On the other hand,

$$\|\mathcal{I}_2(t)\|_Y \leq \|D_t^\alpha f\|_{L^\infty(0,T;Y)} \frac{1}{\Gamma_\alpha} \int_t^{t+h} (t+h-s)^{\alpha-1} ds = \frac{h^\alpha}{\Gamma_\alpha} \|D_t^\alpha f\|_{L^\infty(0,T;Y)}.$$

Therefore (2.28) holds also for $p = \infty$. This finishes the proof of the Lemma. \qed
Proof of Lemma 5. Let \( t \in (0, T) \) arbitrary, \( k = \lfloor t/\tau \rfloor \). Let us begin by computing

\[
D_t^a \tilde{f}^{(\tau)}(t) = \frac{1}{\Gamma_{1-\alpha}} \int_0^t \frac{D \tilde{f}^{(\tau)}(s)}{(t-s)^a} ds
\]

\[
= \frac{1}{\Gamma_{1-\alpha}} \int_0^t (t-s)^{-\alpha} \sum_{n=0}^{N-1} \frac{f_{n+1} - f_n}{\tau} \chi_{[t_n, t_{n+1})}(s) ds
\]

\[
= \frac{1}{\Gamma_{1-\alpha}} \sum_{n=0}^{k-1} \frac{f_{n+1} - f_n}{\tau} \int_{t_n}^{t_{n+1}} (t-s)^{-\alpha} ds + \frac{1}{\Gamma_{1-\alpha}} \frac{f_{k+1} - f_k}{\tau} \int_{t_k}^t (t-s)^{-\alpha} ds
\]

\[
= \frac{1}{\Gamma_{1-\alpha}} \sum_{n=0}^{k-1} \frac{f_{n+1} - f_n}{(1-\alpha)\tau} ((t-t_n)^{1-\alpha} - (t-t_{n+1})^{1-\alpha}) + \frac{1}{\Gamma_{1-\alpha}} \frac{f_{k+1} - f_k}{(1-\alpha)\tau} (t-t_k)^{1-\alpha},
\]

and so

\[
D_t^a \tilde{f}^{(\tau)}(t) = \frac{\tau^{-\alpha}}{\Gamma_{1-\alpha}} \sum_{n=0}^{N-1} \frac{f_{n+1} - f_n}{(1-\alpha)} (g_n(t) - g_{n+1}(t)), \quad 0 \leq t \leq T,
\]

\[
g_n(t) := \tau^{-1} (t-t_n)^{1-\alpha}, \quad n \geq 0.
\]

From (2.21) one obtains

\[
f_{n+1} - f_n = \tau^{\alpha} \sum_{k=1}^{n+1} w_{n+1-k}(D_\tau^a f)_k, \quad n \geq 0,
\]

\[
w_s := \begin{cases} 
1 & s = 0 \\
(s+1)^{\alpha-1} - s^{\alpha-1} & s \geq 1.
\end{cases}
\]

Plugging the above identity into (2.31) yields

\[
D_t^a \tilde{f}^{(\tau)}(t) = \frac{1}{\Gamma_{1-\alpha} \Gamma_{\alpha}(1-\alpha)} \sum_{n=0}^{N-1} (g_n(t) - g_{n+1}(t)) \sum_{k=1}^{n+1} w_{n+1-k}(D_\tau^a f)_k
\]

\[
= \frac{1}{\Gamma_{1-\alpha} \Gamma_{\alpha}(1-\alpha)} \sum_{k=1}^{N} \sum_{n=k-1}^{N-1} (g_n(t) - g_{n+1}(t)) w_{n+1-k}(D_\tau^a f)_k,
\]

which can be rewritten as

\[
D_t^a \tilde{f}^{(\tau)}(t) = \frac{1}{\Gamma_{1-\alpha} \Gamma_{\alpha}(1-\alpha)} \sum_{k=1}^{N} \mu_k(t)(D_\tau^a f)_k,
\]

\[
\mu_k(t) := \sum_{s=0}^{N-k} w_s(g_{s+k-1}(t) - g_{s+k}(t)), \quad 1 \leq k \leq N.
\]

We aim to show that a constant \( C > 0 \) exists, depending only on \( \alpha \in (0, 1) \) and \( T > 0 \), such that \( \|\mu_k\|_{L^\infty(0,T)} \leq C \) for \( 1 \leq k \leq N, \tau > 0 \). From this fact the statement of the Lemma follows easily.
Let $\rho \equiv t/\tau \in [0, N]$. From the definitions of $w_s$ and $g_n(t)$ it follows
\[
\mu_k(t) = (\rho - k + 1)_+^{1-\alpha} - (\rho - k)_+^{1-\alpha} \\
+ \sum_{s=1}^{N-k} (s + 1)^{\alpha-1}((\rho - s - k + 1)_+^{1-\alpha} - (\rho - s - k)_+^{1-\alpha}) \\
- \sum_{s=1}^{N-k} s^{\alpha-1}((\rho - s - k + 1)_+^{1-\alpha} - (\rho - s - k)_+^{1-\alpha}) \\
= (N - k + 1)^{\alpha-1}((\rho - N + 1)_+^{1-\alpha} - (\rho - N)_+^{1-\alpha}) \\
+ \sum_{s=1}^{N-1} s^{\alpha-1}((\rho - s - k + 2)_+^{1-\alpha} - 2(\rho - s - k + 1)_+^{1-\alpha} + (\rho - s - k)_+^{1-\alpha}).
\]

Since $0 \leq \rho \leq N$ and $1 \leq k \leq N$, it holds
\[
(N - k + 1)^{\alpha-1}((\rho - N + 1)_+^{1-\alpha} - (\rho - N)_+^{1-\alpha}) \\
= (N - k + 1)^{\alpha-1}(\rho - N + 1)_+^{1-\alpha} \leq 1.
\]

It follows
\[
|\mu_k(t)| \leq 1 + \sum_{s=1}^{N-1} s^{\alpha-1} |(\rho - s - k + 2)_+^{1-\alpha} - 2(\rho - s - k + 1)_+^{1-\alpha} + (\rho - s - k)_+^{1-\alpha}|. \tag{2.33}
\]

We can clearly consider $\rho > s + k - 2$, since for $\rho \leq s + k - 2$ the sum on the right-hand side of (2.33) vanishes. Let us distinguish two cases.

**Case 1:** $s + k - 2 < \rho \leq s + k + 1$. Being $s, k$ integers, this means that $|\rho| - k \leq s \leq |\rho| - k + 2$. It follows
\[
\sum_{s=\max\{1,|\rho|-k\}}^{\lfloor |\rho| \rfloor} s^{\alpha-1} |(\rho - s - k + 2)_+^{1-\alpha} - 2(\rho - s - k + 1)_+^{1-\alpha} + (\rho - s - k)_+^{1-\alpha}| \\
\leq \sum_{s=\max\{1,|\rho|-k\}}^{\lfloor |\rho| \rfloor} s^{\alpha-1} \left|((\rho - s - k + 2)_+^{1-\alpha} - (\rho - s - k + 1)_+^{1-\alpha}) \\
+ |(\rho - s - k + 1)_+^{1-\alpha} - (\rho - s - k)_+^{1-\alpha}|\right|.
\]

Since $x \mapsto x^{1-\alpha}$ is subadditive, it holds $x^{1-\alpha} \leq (x - y)^{1-\alpha} + y^{1-\alpha}$ for $x \geq y \geq 0$, so
\[
|((\rho - s - k + 2)_+^{1-\alpha} - (\rho - s - k + 1)_+^{1-\alpha}) \\
|((\rho - s - k + 1)_+^{1-\alpha} - (\rho - s - k)_+^{1-\alpha})| \leq ((\rho - s - k + 2)_+ - (\rho - s - k + 1)_+)^{1-\alpha} \leq 1,
\]
\[
|((\rho - s - k + 1)_+^{1-\alpha} - (\rho - s - k)_+^{1-\alpha})| \leq ((\rho - s - k + 1)_+ - (\rho - s - k)_+)^{1-\alpha} \leq 1.
\]

It follows
\[
\sum_{s=1}^{N-1} s^{\alpha-1} |(\rho - s - k + 2)_+^{1-\alpha} - 2(\rho - s - k + 1)_+^{1-\alpha} + (\rho - s - k)_+^{1-\alpha}| \\
\leq 2 \sum_{s=\max\{1,|\rho|-k\}}^{\lfloor |\rho| \rfloor} s^{\alpha-1} \leq 6.
\]
Case 2: $\rho > s + k + 1$. This implies $s \leq |\rho| - k - 1$. Lagrange’s theorem yields
\[
(\rho - s - k + 2)^{1-\alpha} - 2(\rho - s - k + 1)^{1-\alpha} + (\rho - s - k)^{1-\alpha}
\]
\[= (\rho - s - k + 2)^{1-\alpha} - 2(\rho - s - k + 1)^{1-\alpha} + (\rho - s - k)^{1-\alpha}
\]
\[= -\alpha(1-\alpha)\xi^{-1-\alpha}, \quad \rho - s - k \leq \xi \leq \rho - s - k + 2,
\]
and so
\[
\sum_{s=1}^{\lfloor|\rho| - k - 1\rfloor} s^{1-\alpha} |(\rho - s - k + 2)^{1-\alpha} - 2(\rho - s - k + 1)^{1-\alpha} + (\rho - s - k)^{1-\alpha}|
\]
\[\leq \alpha(1-\alpha) \sum_{s=1}^{\lfloor|\rho| - k - 1\rfloor} s^{1-\alpha} (\rho - s - k)^{-1-\alpha}
\]
\[\leq \alpha(1-\alpha) \sum_{s=1}^{\infty} (|\rho| - s - k)^{-1-\alpha}
\]
\[\leq \alpha(1-\alpha) \sum_{r=1}^{\infty} r^{-1-\alpha} < \infty.
\]
From (2.33) the statement follows. This finishes the proof of the Lemma.

Proof of Lemma 6. From (2.21) we have
\[
f_n = f_0 + \frac{\tau}{\Gamma_\alpha} \sum_{k=1}^{n} (n-k+1)^{\alpha-1}(D^\alpha f)_k, \quad n \geq 1.
\]
For convenience, denote $a_j := j^{\alpha-1}$, then
\[
f_{n+1} - f_n = \frac{\tau}{\Gamma_\alpha} \sum_{k=1}^{n} (a_{n+2-k} - a_{n+1-k})(D^\alpha f)_k + \frac{\tau}{\Gamma_\alpha} a_1(D^\alpha f)_{n+1}.
\]
Hence
\[
\|f_{n+1} - f_n\|_Y^p = \left\| \frac{\tau}{\Gamma_\alpha} \sum_{k=1}^{n} (a_{n+2-k} - a_{n+1-k})(D^\alpha f)_k + \frac{\tau}{\Gamma_\alpha} a_1(D^\alpha f)_{n+1} \right\|_Y^p
\]
\[\leq (2/\Gamma_\alpha)^p \left[ \left( \frac{\tau}{\Gamma_\alpha} \sum_{k=1}^{n} (a_{n+1-k} - a_{n+2-k}) \|D^\alpha f_k\|_Y \right)^p + \left( \frac{\tau}{\Gamma_\alpha} a_1 \|D^\alpha f_{n+1}\|_Y \right)^p \right],
\]
thanks to the triangular and Young’s inequality. Then we have the following bounds:
\[
\left( \frac{\tau}{\Gamma_\alpha} \sum_{k=1}^{n} (a_{n+1-k} - a_{n+2-k}) \|D^\alpha f_k\|_Y \right)^p
\]
\[= \tau^{(\alpha-1)p} \left( \sum_{k=1}^{n} \tau(a_{n+1-k} - a_{n+2-k}) \|D^\alpha f_k\|_Y \right)^p
\]
\[\leq \tau^{(\alpha-1)p} \left( \sum_{k=1}^{n} \tau(a_{n+1-k} - a_{n+2-k}) \|D^\alpha f_k\|_Y \right)^{p-1} \left( \sum_{k=1}^{n} \tau(a_{n+1-k} - a_{n+2-k}) \|D^\alpha f_k\|_Y \right)
\]
\[= \tau^{(\alpha-1)p} (\tau(a_1 - a_{n+1}))^{p-1} \left( \sum_{k=1}^{n} \tau(a_{n+1-k} - a_{n+2-k}) \|D^\alpha f_k\|_Y \right)
\]
\[\leq \tau^{\alpha p} \sum_{k=1}^{n} (a_{n+1-k} - a_{n+2-k}) \|D^\alpha f_k\|_Y^p,
\]
where in the first inequality we have used a discrete version of Hölder’s inequality:
\[
\|gh\|_{L^1} \leq \|g\|_{L^\frac{p}{p-1}} \|h\|_{L^p}^{\frac{1}{p-1}}.
\]
Summarizing we get
\[
\|f_{n+1} - f_n\|^p_Y = (2/t_\alpha)^{p'_\alpha} \left[ \sum_{k=1}^n (a_{n+1-k} - a_{n+2-k}) \|(D^\alpha_{\tau} f)_k\|^p_Y + a_1^p \|(D^\alpha_{\tau} f)_{n+1}\|^p_Y \right],
\]
which yields
\[
\left\| f^{(\tau)}(t + \tau) - f^{(\tau)}(t) \right\|^p_{L^p(0,T-\tau; Y)} = \sum_{n=1}^{N-1} \tau \|f_{n+1} - f_n\|^p_Y
\]
\[
\leq (2/t_\alpha)^{p'_\alpha} \sum_{n=1}^{N-1} \tau \left( \sum_{k=1}^n (a_{n+1-k} - a_{n+2-k}) \|(D^\alpha_{\tau} f)_k\|^p_Y + a_1^p \|(D^\alpha_{\tau} f)_{n+1}\|^p_Y \right)
\]
\[
\leq (2/t_\alpha)^{p'_\alpha} \sum_{n=1}^{N-1} \sum_{k=1}^n (a_{n+1-k} - a_{n+2-k}) \|(D^\alpha_{\tau} f)_k\|^p_Y + \sum_{n=1}^{N-1} \|(D^\alpha_{\tau} f)_{n+1}\|^p_Y .
\]
Then we interchange summations and get
\[
\left\| f^{(\tau)}(t + \tau) - f^{(\tau)}(t) \right\|^p_{L^p(0,T-\tau; Y)} = (2/t_\alpha)^{p'_\alpha} \sum_{n=1}^{N-1} \sum_{k=1}^n (a_{n+1-k} - a_{n+2-k}) \|(D^\alpha_{\tau} f)_k\|^p_Y + \sum_{n=1}^{N-1} \|(D^\alpha_{\tau} f)_{n+1}\|^p_Y
\]
\[
= (2/t_\alpha)^{p'_\alpha} \sum_{n=1}^{N-1} \|(D^\alpha_{\tau} f)_k\|^p_Y (a_1 - a_{N+1-k}) + \sum_{n=1}^{N-1} \|(D^\alpha_{\tau} f)_{n+1}\|^p_Y.
\]
Since \( a_1 - a_{N+1-k} = 1 - (N + 1 - k)^{\alpha-1} \in (0, 1) \), one obtains
\[
\left\| f^{(\tau)}(t + \tau) - f^{(\tau)}(t) \right\|^p_{L^p(0,T-\tau; Y)} \leq (2/t_\alpha)^{p'_\alpha} \sum_{n=1}^{N-1} \|(D^\alpha_{\tau} f)_k\|^p_Y + \sum_{n=1}^{N-1} \|(D^\alpha_{\tau} f)_{n+1}\|^p_Y
\]
\[
= (2/t_\alpha)^{p'_\alpha} \left\| D^\alpha_{\tau} f^{(\tau)} \right\|^p_{L^p(0,T-\tau; Y)} + \left\| D^\alpha_{\tau} f^{(\tau)} \right\|^p_{L^p(0,T; Y)}
\]
\[
\leq \frac{2^{p'_\alpha}}{\Gamma_\alpha^p} \left\| D^\alpha_{\tau} f^{(\tau)} \right\|^p_{L^p(0,T; Y)}.
\]

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We denote with \( \tilde{f}^{(\tau)} \) the linear interpolant (2.29) and with \( f^{(\tau)} \) the piecewise constant function \( f^{(\tau)} = f_N x_N \tau + \sum_{n=0}^{N-1} f_n x_{(t_n,t_{n+1})} \). From (2.29), we have
\[
\left\| \tilde{f}^{(\tau)}(t) \right\|_X \leq \left\| f^{(\tau)}(t) \right\|_X + \left\| f^{(\tau)}(t + h) \right\|_X.
\]
Hence
\[
\left\| \tilde{f}^{(\tau)} \right\|_{L^p(0,T,Y)} \leq 2 \left\| f^{(\tau)} \right\|_{L^p(0,T,X)} \leq C,
\]
by the assumption of the theorem. One can apply Lemma 4 with \( f = \tilde{f}^{(\tau)} \) and use Lemma 5 to deduce
\[
\left\| \tilde{f}^{(\tau)}(t + h) - \tilde{f}^{(\tau)} \right\|_{L^p(0,T-h; Y)} \leq C_\alpha h^\alpha \left\| D^\alpha_{\tau} f^{(\tau)} \right\|_{L^p(0,T; Y)}.
\]
As a consequence, if \( \left\| D^\alpha_{\tau} f^{(\tau)} \right\|_{L^p(0,T; Y)} \leq C \), then
\[
\lim_{h \to 0} \sup_{\tau > 0} \left\| \tilde{f}^{(\tau)}(t + h) - \tilde{f}^{(\tau)} \right\|_{L^p(0,T-h; Y)} = 0.
\]
By Lemma 3, estimates (2.34) and (2.35) imply that \( \tilde{f}^{(\tau)} \) is relatively-compact in \( L^p(0,T;B) \) and therefore there exists a subsequence of \( \tilde{f}^{(\tau)} \) (still denoted with \( \tilde{f}^{(\tau)} \)) such that

\[
\tilde{f}^{(\tau)} \to f^* \text{ strongly in } L^p(0,T;B).
\]  

(2.36)

Next we show that convergence of \( \tilde{f}^{(\tau)} \) implies convergence of the corresponding piecewise constant function \( f^{(\tau)} \). Since

\[
\left\| (f^{(\tau)} - \tilde{f}^{(\tau)})(t) \right\|_{Y} = \sum_{n=0}^{N-1} \left\| \frac{\tau}{t_n - t_{n+1}} (f_{n+1} - f_n) \chi_{[t_n,t_{n+1}]}(t) \right\|_{Y} 
\leq \sum_{n=0}^{N-1} \| f_{n+1} - f_n \|_Y \chi_{[t_n,t_{n+1}]}(t),
\]

from Lemma 6 we have

\[
\left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;Y)} \leq \left\| f^{(\tau)}(t + \tau) - f^{(\tau)}(t) \right\|_{L^p(0,T - \tau;Y)} \leq C(\alpha,p) \tau^{\alpha} \left\| D^p_t f^{(\tau)} \right\|_{L^p(0,T;Y)},
\]

which implies

\[
\left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;Y)} \to 0, \text{ as } \tau \to 0,
\]  

(2.37)

by assumption of the theorem. From interpolation we know that there exists \( \theta \in (0,1) \) and \( C_\theta > 0 \) such that

\[
\left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;B)} \leq C_\theta \left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;X)}^{\theta} \left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;Y)}^{1-\theta}.
\]

Summarizing we have

\[
\left\| f^{(\tau)} - f^* \right\|_{L^p(0,T;B)} \leq \left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;B)} + \left\| \tilde{f}^{(\tau)} - f^* \right\|_{L^p(0,T;B)} 
\leq C_\theta \left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;X)}^{\theta} \left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;Y)}^{1-\theta} + \left\| \tilde{f}^{(\tau)} - f^* \right\|_{L^p(0,T;B)} 
\leq C_\theta \left\| f^{(\tau)} \right\|_{L^p(0,T;X)}^{\theta} \left\| f^{(\tau)} - \tilde{f}^{(\tau)} \right\|_{L^p(0,T;Y)}^{1-\theta} + \left\| \tilde{f}^{(\tau)} - f^* \right\|_{L^p(0,T;B)}.
\]

Therefore \( \left\| f^{(\tau)} - f^* \right\|_{L^p(0,T;B)} \to 0 \) as \( \tau \to 0 \) thanks to (2.36) and (2.37), namely \( f^{(\tau)} \) is relatively compact in \( L^p(0,T;B) \). Theorem 1.2 is proved.

\[\square\]

3. Porous Medium Equation with Caputo Time Derivative

The rest of the manuscript is devoted to the proof of Theorem 1.1. We consider the system

\[
\begin{align*}
D_\tau^\alpha u &= \text{div}(u\nabla p), \quad x \in \mathbb{T}^3, \quad t \in (0,T), \\
D_\tau^\alpha p &= -(\Delta)^\alpha p + u^2,
\end{align*}
\]  

(3.1)

with initial data \( u_{in} \) and \( p_{in} : \mathbb{T}^3 \to (0, +\infty) \) such that \( \int_{\mathbb{T}^3} u_{in}^2 + |\nabla p_{in}|^2 \, dx < +\infty \). Moreover \( 0 < s \leq 1 \) and \( 0 < \alpha \leq 1 \). The energy functional

\[
H[u, p] := \int_{\mathbb{T}^3} u^2 + \frac{1}{2} |\nabla p|^2 \, dx
\]

formally satisfies the following inequality:\footnote{In the case \( s = 1 \) we replace in (3.2) \( (\Delta)^{1/2} \nabla \) with \( \Delta \).}

\[
H[u(t), p(t)] + \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{T}^3} \frac{|(\Delta)^{\frac{\alpha}{2}} \nabla p(s)|^2}{(t-s)^{1-\alpha}} \, dx ds \leq H(u_{in}, p_{in}), \quad \text{for } t \in [0,T].
\]  

(3.2)
To see this, we first test the second equation in (3.1) by $\Delta$, and from divergence theorem we get
\[ \int_{\Omega} \nabla u^2 \cdot \nabla p \, dx = \int_{\Omega} (D_p^\alpha \nabla p) \cdot \nabla p \, dx + \int_{\Omega} |(-\Delta)^{\frac{s}{2}} \nabla p|^2 \, dx. \]
Then we test the first equation in (3.1) by $2u$ and get
\[ \int_{\Omega} 2u D_p^\alpha u \, dx = \int_{\Omega} \text{div}(u \nabla p) 2u \, dx = -\int_{\Omega} \nabla u^2 \cdot \nabla p \, dx. \]
The combination of the two yields the following equation:
\[ \int_{\Omega} 2u D_p^\alpha u \, dx + \int_{\Omega} (D_p^\alpha \nabla p) \cdot \nabla p \, dx + \int_{\Omega} |(-\Delta)^{\frac{s}{2}} \nabla p|^2 \, dx = 0. \]
Thanks to the fact that $2g D_p^\alpha g \geq D_p^\alpha g^2$ (see [25]), we get
\[ D_p^\alpha \left( \int_{\Omega} u^2 + \frac{1}{2} |\nabla p|^2 \right) + \int_{\Omega} |(-\Delta)^{\frac{s}{2}} \nabla p|^2 \, dx \leq 0. \]
The fundamental theorem of calculus (2.19) yields (3.2).

Next we divide interval $[0,T]$ into $N$ subintervals with length $\tau$ and discretize (3.1) in time. We also add extra viscosity terms as follows. For given constants $\rho, \tau, \varepsilon > 0$, functions $u_j \in H^1(\Omega)$ and $p_j \in H^{2s}(\Omega)$ such that $u_j, p_j \geq 0$, $j = 0, \ldots, k - 1$, a.e. in $\Omega$, consider the weak formulation:
\[ \int_{\Omega} (D_p^\alpha u) \phi \, dx + \int_{\Omega} \nabla p_k \cdot \nabla \phi \, dx + \int_{\Omega} \nabla u_k \cdot \nabla \phi \, dx = 0, \quad \forall \phi \in H^1(\Omega), \quad (3.3) \]
\[ \int_{\Omega} (D_p^\alpha p_k) \psi \, dx + \int_{\Omega} (-\Delta)^{s/2} p_k (-\Delta)^{s/2} \psi \, dx + \varepsilon \int_{\Omega} \nabla p_k \cdot \nabla \psi \, dx - \int_{\Omega} u_k^2 \psi \, dx = 0, \quad \forall \psi \in H^1(\Omega). \quad (3.4) \]

### 3.1. Existence of solutions for (3.3)-(3.4)

We first consider the linearized system
\[ \int_{\Omega} (D_p^\alpha u) \phi \, dx + \sigma \int_{\Omega} z^+ \nabla p_k \cdot \nabla \phi \, dx + \int_{\Omega} \nabla u_k \cdot \nabla \phi \, dx = 0, \quad \forall \phi \in H^1(\Omega), \quad (3.5) \]
\[ \int_{\Omega} (D_p^\alpha p_k) \psi \, dx + \int_{\Omega} (-\Delta)^{s/2} p_k (-\Delta)^{s/2} \psi \, dx + \varepsilon \int_{\Omega} \nabla p_k \cdot \nabla \psi \, dx - \int_{\Omega} z^2 \psi \, dx = 0, \quad \forall \psi \in H^1(\Omega), \quad (3.6) \]
with $z \in H^1(\Omega)$ and $\sigma \in [0,1]$. Define the bilinear forms
\[ B_1[u_k, \phi] := \Gamma_\alpha \int_{\Omega} u_k \phi \, dx + \int_{\Omega} \nabla u_k \cdot \nabla \phi \, dx, \]
\[ B_2[p_k, \psi] := \int_{\Omega} \phi \, dx = \Gamma_\alpha \int_{\Omega} u_k \phi \, dx + \int_{\Omega} \nabla u_k \cdot \nabla \phi \, dx, \]
and the linear functionals
\[ (F_1, \phi)_{H^1(\Omega), H^1(\Omega)'} := \Gamma_\alpha \int_{\Omega} \phi \, dx + \int_{\Omega} z^+ \nabla p_k \cdot \nabla \phi \, dx, \]
\[ (F_2, \psi)_{H^1(\Omega), H^1} := \Gamma_\alpha \int_{\Omega} \phi \, dx + \int_{\Omega} z^2 \psi \, dx. \]
Note that $B_2$ is bounded
\[ B_2[p_k, \psi] \leq C_4 \|p_k\|_{H^1} \|\psi\|_{H^1} + C_2 \|p_k\|_{H^1} \|\psi\|_{H^1} \leq C \|p_k\|_{H^1} \|\psi\|_{H^1}, \]
and coercive
\[ B_2[p_k, p_k] \geq C \|p_k\|_{H^1}^2. \]
Moreover,
\[
(F_2, \psi)_{(H^1)'\times H^1} \leq C_1 \sum_{j=0}^{k-2} \lambda_{k-j} \left[ \|p_{j+1}\|_{L^2} + \|p_j\|_{L^2} \right] \|\psi\|_{L^2} + C_2 \|z^2\|_{L^2} \|\psi\|_{L^2} \\
= C \|\psi\|_{H^1},
\]
since \( p_j \in L^2, j = 0, ..., k-1 \), and \( z \in L^{6-\delta} \), for \( \delta > 0 \). The constant \( C \) depends on \( \tau, \|z\|_{L^4}, \) and \( \|p_j\|_{L^2} \) for any \( 0 \leq j \leq k-1 \). By Lax-Milgram theorem there exists a unique solution \( p_k \in H^1(\mathbb{T}^3) \) to (3.6). In addition, the elliptic regularity theory implies that \( p_k \in H^2(\mathbb{T}^3) \). Similarly, \( B_1 \) is bounded
\[
B_1[u_k, \phi] \leq C_4 \|u_k\|_{H^1} \|\phi\|_{H^1},
\]
and coercive
\[
B_1[u_k, u_k] \geq C_2 \|u_k\|_{H^1}^2,
\]
and
\[
(F_1, \phi)_{H^1(\mathbb{T}^3)'\times H^1(\mathbb{T}^3)} \leq C_1 \sum_{j=0}^{k-2} \lambda_{k-j} \left[ \|u_{j+1}\|_{L^2} + \|u_j\|_{L^2} \right] \|\phi\|_{L^2} + C_2 \|z^2\|_{L^2} \|\nabla\phi\|_{L^2} \\
= C \|\phi\|_{H^1(\mathbb{T}^3)},
\]
since \( u_j \in L^2, j = 0, ..., k-1 \), \( z^+ \in L^4 \), \( \nabla p_k \in L^q \), \( q \geq 4 \). The constant \( C \) depends on \( \tau, \|z\|_{L^4}, \|u_j\|_{L^2}, \) and \( \|\nabla p_k\|_{L^4} \) for any \( 0 \leq j \leq k-1 \). Once more, Lax Milgram theorem yields existence and uniqueness of the solution \( u_k \in H^1(\mathbb{T}^3) \) to (3.5).

Next we use a fixed point argument to show the existence of solutions to (3.3) and (3.4). Thanks to the existence and uniqueness of solution to (3.5) and (3.6), we can define the map
\[
\mathcal{T} : (z, \sigma) \in L^{6-\delta}(\mathbb{T}^3) \times [0, 1] \rightarrow u \in L^{6-\delta}(\mathbb{T}^3).
\]

**Lemma 7.** Given any \( \sigma \in [0, 1] \), any fixed point of \( \mathcal{T}(\cdot, \sigma) \) is non-negative.

**Proof.** Choose \( \psi = p_{k-} \) as test functions and get:
\[
\int_{\mathbb{T}^3} (D^2_\tau p) p_{k-} \, dx + \int_{\mathbb{T}^3} (\Delta)^{\alpha} p_{k-} \, dx + \epsilon \int_{\mathbb{T}^3} \nabla p_{k-} \, dx + \int_{\mathbb{T}^3} u_k^2 \, dx = 0.
\]
All the terms except for the first one are non-negative. Using Definition 2 we rewrite the first term as
\[
\int_{\mathbb{T}^3} (D^2_\tau p) p_{k-} \, dx = \Gamma_\alpha \sigma^{-\alpha} \int_{\mathbb{T}^3} \lambda_k p_{in}(\nabla p_{k-}) + \sum_{j=1}^{k-1} (\lambda_{k-j} - \lambda_{k-j+1}) p_j(\nabla p_{k-}) \, dx \\
+ \Gamma_\alpha \sigma^{-\alpha} \int_{\mathbb{T}^3} p_{k-}^2 \, dx.
\]
We can see that every term is non-negative, since \( p_j \geq 0, j = 0, 1, ..., k-1 \), and
\[
\lambda_{k-1} - \lambda_k \geq 0, k = 2, 3, ...
\]
Therefore, we must have \( p_{k-} = 0 \), in other words, \( p_k \geq 0 \).

Now we show the non-negativity of \( u_k \). With \( \phi = u_{k-} \) as test function we get:
\[
\int_{\mathbb{T}^3} (D^2_\tau u_k) u_{k-} \, dx + \theta \int_{\mathbb{T}^3} \nabla u_{k-} \, dx = -\sigma \int_{\mathbb{T}^3} u_k^+ \nabla p_k \cdot \nabla u_{k-} \, dx.
\]
We have:
\[ \Gamma_\alpha \tau^{-\alpha} \int_{T^3} \lambda_k u_{in}(u_{-}) + \sum_{j=1}^{k-1} (\lambda_{k-j} - \lambda_{k-j+1}) u_j(u_{-}) \, dx \]
\[ + \Gamma_\alpha \tau^{-\alpha} \int_{T^3} u_k^2 \, dx + \varepsilon \int_{T^3} |\nabla u_{-}|^2 \, dx = 0. \]
Since every term on the left is non-negative, we must have \( u_{-} = 0 \), in other words, \( u_k \geq 0 \).

We next prove that \( \mathcal{T}(\cdot, 1) \) has a fixed point. First, \( \mathcal{T}(\cdot, 0) \) is constant since eq. (3.5) becomes independent of \( z \) when \( \sigma = 0 \). Next, we show that \( \mathcal{T} \) is continuous and compact. Taking \( \phi = u_k \) in (3.5), we have
\[ \int_{T^3} (D^\alpha u)_k u_k \, dx + \sigma \int_{T^3} z^+ \nabla p_k \cdot \nabla u_k \, dx + \varepsilon \int_{T^3} |\nabla u_k|^2 \, dx = 0, \]
which implies
\[ \Gamma_\alpha \tau^{-\alpha} \int_{T^3} u_k^2 \, dx + \varepsilon \int_{T^3} |\nabla u_k|^2 \, dx \]
\[ = -\sigma \int_{T^3} z^+ \nabla p_k \cdot \nabla u_k \, dx - \Gamma_\alpha \tau^{-\alpha} \int_{T^3} \sum_{j=0}^{k-2} \lambda_{k-j}(u_{j+1} - u_{j}) - u_{k-1} \, u_k \, dx \]
\[ \leq \frac{\sigma}{2} \int_{T^3} |\nabla u_k|^2 \, dx + 2 \varepsilon \int_{T^3} (z^+)^2 |\nabla p_k|^2 \, dx + \frac{\Gamma_\alpha \tau^{-\alpha}}{2} \int_{T^3} u_k^2 \, dx \]
\[ + 16 \Gamma_\alpha \tau^{-\alpha} \sum_{j=0}^{k-2} \lambda_{k-j}^2 \int_{T^3} u_j^2 \, dx. \]
Reorganizing the terms, we have
\[ \frac{\Gamma_\alpha \tau^{-\alpha}}{2} \int_{T^3} u_k^2 \, dx + \varepsilon \int_{T^3} |\nabla u_k|^2 \, dx \leq \Gamma_\alpha \tau^{-\alpha} \int_{T^3} \sum_{j=0}^{k-2} \lambda_{k-j}(u_{j+1} - u_{j}) - u_{k-1} \, \nabla p_k \, dx \]
\[ \leq \frac{\sigma}{2} \int_{T^3} (z^+)^4 \, dx + \frac{1}{\mu} \int_{T^3} |\nabla p_k|^4 \, dx \]
\[ + 16 \Gamma_\alpha \tau^{-\alpha} \sum_{j=0}^{k-2} \lambda_{k-j}^2 \int_{T^3} u_j^2 \, dx. \]
Now we need to estimate \( \nabla p_k \). We take \( \psi = \Delta p_k \) in (3.6), and get
\[ \int_{T^3} (D^\alpha p)_k \Delta p_k \, dx + \int_{T^3} (-\Delta)^{r/2} p_k(-\Delta)^{r/2} \Delta p_k \, dx + \varepsilon \int_{T^3} \nabla p_k \cdot \nabla \Delta p_k \, dx - \int_{T^3} (z^+)^2 \Delta p_k \, dx = 0, \]
which implies
\[ \Gamma_\alpha \tau^{-\alpha} \int_{T^3} |\nabla p_k|^2 \, dx + \varepsilon \int_{T^3} |\Delta p_k|^2 \, dx \]
\[ = -\int_{T^3} z^2 \Delta p_k \, dx - \Gamma_\alpha \tau^{-\alpha} \int_{T^3} \sum_{j=0}^{k-2} \lambda_{k-j}(\nabla p_{j+1} - \nabla p_j) - \nabla p_{k-1} \, \nabla p_k \, dx \]
\[ \leq \frac{2}{\varepsilon} \int_{T^3} (z^+)^4 \, dx + \frac{\Gamma_\alpha \tau^{-\alpha}}{2} \int_{T^3} |\nabla p_k|^2 \, dx \]
\[ + 16 \Gamma_\alpha \tau^{-\alpha} \sum_{j=0}^{k-2} \lambda_{k-j}^2 \int_{T^3} |\nabla p_j|^2 \, dx. \]
Combining the similar terms, we get
\[ -\frac{\Gamma_\alpha \tau^{-\alpha}}{2} \int_{\mathbb{T}^3} |\nabla p_k|^2 \, dx + \frac{\varepsilon}{2} \int_{\mathbb{T}^3} |\Delta p_k|^2 \, dx \leq \frac{2}{\varepsilon} \int_{\mathbb{T}^3} (z^+)^4 \, dx + 16 \Gamma_\alpha \tau^{-\alpha} \sum_{j=0}^{k-2} \lambda_{k-j}^2 \int_{\mathbb{T}^3} |\nabla p_j|^2 \, dx. \]

Given \( z^+ \in L^4 \), we have \( \nabla p_k \in H^1 \), which suggests that \( \nabla p_k \) is uniformly bounded in \( L^4 \), and therefore, \( \|u_k\|_{H^1} \leq C \) from (3.7). This shows that \( \mathcal{T} \) is bounded as an operator \( L^{6-\delta}(\mathbb{T}^3) \to H^1(\mathbb{T}^3) \). The compactness of \( \mathcal{T} \) directly follows the compact embedding \( H^1(\mathbb{T}^3) \hookrightarrow L^{6-\delta}(\mathbb{T}^3) \), while the (sequential) continuity of \( \mathcal{T} \) is proved via a standard argument.

Next, we show that fixed points of \( \mathcal{T}(\cdot, \sigma) \) are uniformly bounded in \( \sigma \) for \( \sigma \in [0, 1] \). We consider \( \phi = u_k \) and \( \psi = \Delta p_k \) as test functions respectively in (3.3) and (3.4), and take summation of the equations:
\[
\Gamma_\alpha \tau^{-\alpha} \int_{\mathbb{T}^3} \left[ \sum_{j=0}^{k-1} \lambda_{k-j}(u_{j+1} - u_j) \right] u_k \, dx + \frac{\sigma}{2} \Gamma_\alpha \tau^{-\alpha} \int_{\mathbb{T}^3} \left[ \sum_{j=0}^{k-1} \lambda_{k-j}(\nabla p_{j+1} - \nabla p_j) \right] \cdot \nabla p_k \, dx \\
+ \varphi \int_{\mathbb{T}^3} |\nabla u_k|^2 \, dx + \frac{\sigma}{2} \int_{\mathbb{T}^3} |(-\Delta)^{s/2}\nabla p_k|^2 \, dx + \frac{\sigma \varepsilon}{2} \int_{\mathbb{T}^3} (\Delta p_k)^2 \, dx = 0.
\]

From Lemma 2 we have
\[
\frac{1}{2} \Gamma_\alpha \tau^{-\alpha} \int_{\mathbb{T}^3} \sum_{j=0}^{k-1} \lambda_{k-j}(u_{j+1}^2 - u_j^2) \, dx + \frac{\sigma}{4} \Gamma_\alpha \tau^{-\alpha} \int_{\mathbb{T}^3} \sum_{j=0}^{k-1} \lambda_{k-j}(|\nabla p_{j+1}|^2 - |\nabla p_j|^2) \, dx \quad (3.8)
\]
\[ + \varphi \int_{\mathbb{T}^3} |\nabla u_k|^2 \, dx + \frac{\sigma}{2} \int_{\mathbb{T}^3} |(-\Delta)^{s/2}\nabla p_k|^2 \, dx + \frac{\sigma \varepsilon}{2} \int_{\mathbb{T}^3} (\Delta p_k)^2 \, dx \leq 0.
\]

Lemma 1 implies that \( \int_{\mathbb{T}^3} u_k^2 + \frac{\sigma}{2} |\nabla p_k|^2 \, dx \leq \int_{\mathbb{T}^3} u_{in}^2 + \frac{\sigma}{2} |\nabla p_k|^2 \, dx \) for \( \forall k = 0, \ldots, N \). Moreover, from (3.8) we have
\[
D^2 \left( \int_{\mathbb{T}^3} u_k^2 + \frac{\sigma}{2} |\nabla p_k|^2 \, dx \right) \leq -2\varphi \int_{\mathbb{T}^3} |\nabla u_k|^2 \, dx - \sigma \int_{\mathbb{T}^3} |(-\Delta)^{s/2}\nabla p_k|^2 \, dx - \sigma \varepsilon \int_{\mathbb{T}^3} (\Delta p_k)^2 \, dx,
\]
which implies, using (2.21),
\[
\int_{\mathbb{T}^3} u_k^2 + \frac{\sigma}{2} |\nabla p_k|^2 \, dx + \frac{\sigma}{2} \int_{\mathbb{T}^3} \sum_{j=1}^k (k - j + 1)^{\alpha - 1} \left( 2\varphi \int_{\mathbb{T}^3} |\nabla u_j|^2 \, dx + \sigma \int_{\mathbb{T}^3} |(-\Delta)^{s/2}\nabla p_j|^2 \, dx + \sigma \varepsilon \int_{\mathbb{T}^3} (\Delta p_j)^2 \, dx \right)
\]
\[ \leq \int_{\mathbb{T}^3} u_{in}^2 + \frac{1}{2} |\nabla p_{in}|^2 \, dx.
\]

The last estimate shows that \( u_k \) is bounded in \( H^1(\mathbb{T}^3) \) uniformly with respect to \( \sigma \).

Hence Leray-Schauder fixed point theorem yields the existence of a fixed point \( u_k \in H^1(\mathbb{T}^3) \) for \( \mathcal{T}(\cdot, 1) \), that is, a solution \( (u_k, p_k) \in H^1(\mathbb{T}^3) \times H^2(\mathbb{T}^3) \) to
\[
\int_{\mathbb{T}^3} (D^2 u_k) \phi \, dx + \int_{\mathbb{T}^3} u_k \nabla p_k \cdot \nabla \phi \, dx + \varphi \int_{\mathbb{T}^3} \nabla u_k \cdot \nabla \phi \, dx = 0,
\]
and
\[
\int_{\mathbb{T}^3} (D^2 p_k) \psi \, dx + \int_{\mathbb{T}^3} (-\Delta)^{s/2} p_k (-\Delta)^{s/2} \psi \, dx + \varepsilon \int_{\mathbb{T}^3} \nabla p_k \cdot \nabla \psi \, dx - \int_{\mathbb{T}^3} u_k^2 \psi \, dx = 0,
\]
for all \( \phi, \psi \in H^1(\mathbb{T}^3) \), such that \( u_k, p_k \geq 0 \) a.e. in \( \mathbb{T}^3 \) and
\[
H_k + \frac{\tau \alpha}{\Gamma \alpha} \sum_{i=1}^k (k - i + 1)^{\alpha - 1} \left( \varphi \int_{\mathbb{T}^3} |\nabla u_i|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^3} |(-\Delta)^{s/2}\nabla p_i|^2 \, dx + \varepsilon \int_{\mathbb{T}^3} (\Delta p_i)^2 \, dx \right) \leq H(u_{in}, p_{in}),
\]
using Proposition 2. Taking test functions \( \phi = 1 \) and \( \psi = 1 \) in (3.9) and (3.10) we get \( \int_{\Omega} (D^\alpha u)_k \, dx = 0 \), and \( \int_{\Omega} (D^\alpha p)_k \, dx - \int_{\Omega} u_k^2 \, dx = 0 \). Since \( \int_{\Omega} (D^\alpha u)_k \, dx = (D^\alpha \int_{\Omega} u \, dx)_k \), Proposition 2 implies
\[
\int u_k \, dx = \int u_{in} \, dx \quad \text{for any } k = 0, \ldots, N.
\]
Similarly, \( (D^\alpha \int_{\Omega} p \, dx)_k = \int_{\Omega} u_k^2 \, dx \leq H(u_{in}, p_{in}) \) and
\[
\int p_k \, dx \leq \int p_{in} \, dx + \frac{\tau^\alpha}{\Gamma(\alpha)} H(u_{in}, p_{in}) \sum_{i=1}^{k} (k - i + 1)^{\alpha - 1} \leq \int p_{in} \, dx + \frac{1}{\alpha \Gamma(\alpha)} H(u_{in}, p_{in}) T^\alpha,
\]
for any \( k = 0, \ldots, N \).

4. Limit \( \tau \to 0 \)

For all \( T > 0 \), let \( N = T/\tau \). Define the piecewise constant interpolant of \( \{u_k\} \) and \( \{p_k\} \), \( k = 0, \ldots, N \), respectively as
\[
u^{(\tau)}(t) = u_{in} \chi_{(0)}(t) + \sum_{k=1}^{N} u_k \chi_{(t_{k-1}, t_k]}(t),
\]
\[
\nu^{(\tau)}(t) = p_{in} \chi_{(0)}(t) + \sum_{k=1}^{N} p_k \chi_{(t_{k-1}, t_k]}(t),
\]
\[
H^{(\tau)}(t) = \int_{\Omega} \left( (\nu^{(\tau)}(t, x))^2 + \frac{1}{2} \nabla \nu^{(\tau)}(t, x)^2 \right) \, dx.
\]
By (2.6) we have
\[
(D^\alpha \nu^{(\tau)})(t) = \Gamma_\alpha \tau^{-\alpha} \sum_{k=1}^{n} \sum_{j=0}^{k-1} \lambda_{k-j} (u_{j+1} - u_j) \chi_{(t_{k-1}, t_k]}(t).
\]
So, (3.9) and (3.10) can be rewritten as
\[
\begin{aligned}
\int_{0}^{T-\tau} \int_{\Omega} \left( D^\alpha \nu^{(\tau)}(t) \right) \phi \, dx \, dt + \int_{0}^{T-\tau} \int_{\Omega} \nu^{(\tau)}(t) \nabla \nu^{(\tau)}(t) \cdot \nabla \phi \, dx \, dt \\
\quad + \varepsilon \int_{0}^{T-\tau} \int_{\Omega} \nabla \nu^{(\tau)}(t) \cdot \nabla \psi \, dx \, dt
\end{aligned}
\] (4.1)
and
\[
\begin{aligned}
\int_{0}^{T-\tau} \int_{\Omega} \left( D^\alpha \nabla \nu^{(\tau)}(t) \right) \psi \, dx \, dt + \int_{0}^{T-\tau} \int_{\Omega} (-\Delta)^{s/2} \nu^{(\tau)}(-\Delta)^{s/2} \psi \, dx \, dt \\
\quad + \varepsilon \int_{0}^{T-\tau} \int_{\Omega} \nabla \nu^{(\tau)}(t) \cdot \nabla \psi \, dx \, dt
\end{aligned}
\] (4.2)
with the energy estimates
\[
H^{(\tau)}(T) + \frac{1}{\Gamma_\alpha T^{1-\alpha}} \int_{0}^{T} \int_{\Omega} \left( g |\nabla \nu^{(\tau)}|^2 + \frac{1}{2} |(-\Delta)^{s/2} \nabla \nu^{(\tau)}|^2 + \varepsilon \frac{1}{2} |(-\Delta)^{s/2} \nu^{(\tau)}|^2 \right) \, dx \, dt \leq H(u_{in}, p_{in}).
\] (4.3)
Moreover we have the conservation of mass for \( \nu^{(\tau)} \)
\[
\int_{\Omega} \nu^{(\tau)} \, dx = \int_{\Omega} u_{in} \, dx,
\] (4.4)
and \( L^1 \)-bound for \( \nu^{(\tau)} \)
\[
\int_{\Omega} \nu^{(\tau)} \, dx \leq \int_{\Omega} p_{in} \, dx + \frac{1}{\alpha \Gamma_\alpha} H(u_{in}, p_{in}) T^\alpha.
\] (4.5)
We estimate the time derivative of the density function:
\[
\left| \int_0^T \int_{\mathbb{T}^3} \left(D_\tau^2 u^{(\tau)} \right) \phi dx dt \right| \leq \int_0^T \left\| \nabla p^{(\tau)} \right\|_{L^2(\mathbb{T}^3)} \left\| u^{(\tau)} \nabla \phi \right\|_{L^2(\mathbb{T}^2)} dt
\]
\[
+ \theta \left\| u^{(\tau)} \right\|_{L^2(0,T;H^1(\mathbb{T}^3))} \left\| \phi \right\|_{L^2(0,T;H^1(\mathbb{T}^3))}
\]
\[
\leq \left\| \nabla p^{(\tau)} \right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \left\| u^{(\tau)} \right\|_{L^2(0,T;L^6(\mathbb{T}^3))} \left\| \nabla \phi \right\|_{L^2(0,T;L^3(\mathbb{T}^3))}
\]
\[
+ \theta \left\| u^{(\tau)} \right\|_{L^2(0,T;H^1(\mathbb{T}^3))} \left\| \phi \right\|_{L^2(0,T;H^1(\mathbb{T}^3))}
\]
\[
\leq C(T, \theta) \left\| \phi \right\|_{L^2(0,T;W^{1,3}(\mathbb{T}^3))}.
\]
Therefore
\[
\left\| D_\tau^2 u^{(\tau)} \right\|_{L^2(0,T;W^{1,3}(\mathbb{T}^3))} \leq C(T, \theta). \tag{4.6}
\]

From (4.3) we have that \( u^{(\tau)} \) is uniformly bounded in \( L^2(0,T;H^1(\mathbb{T}^3)) \). Since \( H^1(\mathbb{T}^3) \) is compactly embedded in \( L^{6-\delta}(\mathbb{T}^3) \) for \( 0 < \delta < 6 \), Theorem 1.2 with \( X = H^1(\mathbb{T}^3) \), \( B = L^{6-\delta}(\mathbb{T}^3) \), \( Y = (W^{1,3})'(\mathbb{T}^3) \) yields

\[
u^{(\tau)} \to u \text{ in } L^2(0,T;L^{6-\delta}(\mathbb{T}^3)),
\]

and we also have

\[
p^{(\tau)} \rightharpoonup^\ast p \text{ in } L^\infty(0,T;L^6(\mathbb{T}^3)),
\]

and

\[
u^{(\tau)} \to u \text{ a.e. in } \mathbb{T}^3 \times [0,T].
\]

Now we start to take limit \( \tau \to 0 \). We first look at the second and third terms of (4.2). From (4.3) we know that

\[
(-\Delta)^{s/2} \nabla p^{(\tau)} \to (-\Delta)^{s/2} \nabla p, \text{ in } L^2(0,T;L^2(\mathbb{T}^3)),
\]

and

\[
\nabla p^{(\tau)} \to \nabla p, \text{ in } L^q(0,T;L^2(\mathbb{T}^3)), \forall q < \infty,
\]
as \( \tau \to 0 \), and therefore

\[
\int_0^{T-\tau} \int_{\mathbb{T}^3} \nabla p^{(\tau)} \cdot \nabla \psi dx dt \to \int_0^T \int_{\mathbb{T}^3} \nabla p \cdot \nabla \psi dx dt.
\]

Now consider the fourth term. We only need to prove that \((u^{(\tau)})^2\) converges to \( u^2 \). For all \( \psi \in L^2(0,T;L^4(\mathbb{T}^3)) \), consider

\[
\int \int [(u^{(\tau)})^2 - u^2] \psi dx dt = \int \int [(u^{(\tau)})^2 - u^{(\tau)} u + u^{(\tau)} u - u^2] \psi dx dt
\]
\[
= \int \int u^{(\tau)} \psi (u^{(\tau)} - u) dx dt + \int \int u \psi (u^{(\tau)} - u) dx dt
\]
\[
= I_1 + I_2.
\]
First, we look at $I_1$:

$$I_1 \leq \left\| u^{(\tau)} \psi \right\|_{L^2(0,T;L^{4/3}(\mathbb{T}^3))} \left\| u^{(\tau)} - u \right\|_{L^2(0,T;L^4(\mathbb{T}^3))}$$

$$\leq \left\| u^{(\tau)} \right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \left\| \psi \right\|_{L^2(0,T;L^4(\mathbb{T}^3))} \left\| u^{(\tau)} - u \right\|_{L^2(0,T;L^4(\mathbb{T}^3))}.$$  

Therefore, $I_1 \to 0$ as $\tau \to 0$. Furthermore, the convergence $I_2 \to 0$ as $\tau \to 0$ follows from the weak convergence of $u^{(\tau)}$ in $L^2(0,T;L^4(\mathbb{T}^3))$, and $u\psi \in L^2(0,T;L^{4/3}(\mathbb{T}^3))$.

By density argument and Proposition 5 we obtain

$$\int_0^T \int_{\mathbb{T}^3} \left( D^\alpha_{\tau} u^{(\tau)} \right) \phi \, dx \, dt \to \int_0^T \int_{\mathbb{T}^3} \langle D^\alpha_{\tau} u, \phi \rangle \, dx \, dt, \text{ as } \tau \to 0.$$  

Now we look at the second term in (4.1). Since $\nabla p^{(\tau)}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^3))$ from (4.3), we have the convergence

$$\nabla p^{(\tau)} \rightharpoonup \nabla p \text{ in } L^\infty(0,T;L^2(\mathbb{T}^3)).$$

Moreover, $(-\Delta)^{s/2} p^{(\tau)} \in L^2(0,T;H^1(\mathbb{T}^3))$ from (4.3), and, by Sobolev embedding, $\nabla p^{(\tau)}$ is uniformly bounded in $L^2(0,T;L^{\frac{6}{s-\delta}}(\mathbb{T}^3))$. Thus, there exists a subsequence $\nabla p^{(\tau)}$ that converges weakly in $L^2(0,T;L^{\frac{6}{s-\delta}}(\mathbb{T}^3))$. Consider

$$\int \int [u^{(\tau)} \nabla p^{(\tau)} - u \nabla p] \cdot \nabla \phi \, dx \, dt = \int \int [u^{(\tau)} \nabla p^{(\tau)} - u \nabla p^{(\tau)} + u \nabla p^{(\tau)} - u \nabla p] \cdot \nabla \phi \, dx \, dt$$

$$= \int \int u \nabla \phi \cdot (\nabla p^{(\tau)} - \nabla p) \, dx \, dt + \int \int \nabla p^{(\tau)} \cdot \nabla \phi (u^{(\tau)} - u) \, dx \, dt$$

$$= I_1 + I_2.$$  

Using $\nabla p^{(\tau)} \rightharpoonup \nabla p$ in $L^\infty(0,T;L^2(\mathbb{T}^3))$, $\nabla \phi \in L^2(0,T;L^{\frac{6}{s-\delta}}(\mathbb{T}^3))$ and $u \in L^2(0,T;L^{6-\delta}(\mathbb{T}^3))$, so that $u \nabla \phi$ is bounded in $L^1(0,T;L^2(\mathbb{T}^3))$, we have that $I_1 \to 0$ as $\tau \to 0$. We bound $I_2$ as

$$I_2 \leq \left\| \nabla p^{(\tau)} \right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \left\| \nabla \phi \right\|_{L^2(0,T;L^{\frac{6}{s-\delta}}(\mathbb{T}^3))} \left\| u^{(\tau)} - u \right\|_{L^2(0,T;L^{6-\delta}(\mathbb{T}^3))}.$$  

We have previously concluded that $u^{(\tau)} \to u$ in $L^2(0,T;L^{6-\delta}(\mathbb{T}^3))$, and $\nabla p^{(\tau)}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^3))$. Therefore, $I_2 \to 0$ as $\tau \to 0$. Hence, $\forall \phi \in L^2(0,T - \tau; \ W^{1,q}(\mathbb{T}^3))$ with $q > 3$ we have

$$\int_0^{T-\tau} \int_{\mathbb{T}^3} u^{(\tau)} \nabla p^{(\tau)} \cdot \nabla \phi \, dx \, dt \to \int_0^T \int_{\mathbb{T}^3} u \nabla p \cdot \nabla \phi \, dx \, dt.$$  

Next, we consider the third term in (4.1). From the energy estimates (4.3) we have

$$\nabla u^{(\tau)} \rightharpoonup \nabla u, \text{ in } L^2(0,T;L^2(\mathbb{T}^3)),$$

and therefore, as $\tau \to 0$

$$\theta \int_0^{T-\tau} \int_{\mathbb{T}^3} \nabla u^{(\tau)} \cdot \nabla \phi \, dx \, dt \to \theta \int_0^T \int_{\mathbb{T}^3} \nabla u \cdot \nabla \phi \, dx \, dt.$$  

Now we proceed to the first term of (4.2):

$$\int_0^T \int_{\mathbb{T}^3} \left( D^\alpha_{\tau} p^{(\tau)} \right) \psi \, dx \, dt \leq \left\| (-\Delta)^{s/2} p^{(\tau)} \right\|_{L^2(0,T;L^2(\mathbb{T}^3))} \left\| (-\Delta)^{s/2} \psi \right\|_{L^2(0,T;L^2(\mathbb{T}^3))}$$

$$+ \left\| \nabla p^{(\tau)} \right\|_{L^2(0,T;L^4(\mathbb{T}^3))} \left\| \psi \right\|_{L^2(0,T;L^{4/3}(\mathbb{T}^3))}$$

$$+ \left\| (u^{(\tau)})^2 \right\|_{L^2(0,T;L^{3/2}(\mathbb{T}^3))} \left\| \psi \right\|_{L^2(0,T;L^3(\mathbb{T}^3))}.$$
Since \((u(\tau))^2\) is uniformly bounded in \(L^\infty(0, T; L^1(\mathbb{T}^3)) \cap L^1(0, T; L^3(\mathbb{T}^3))\) by (4.3), interpolation yields
\[
\left\| (u(\tau))^2 \right\|_{L^2(0, T; L^{3/2}(\mathbb{T}^3))} \leq \left\| (u(\tau))^2 \right\|_{L^\infty(0,T; L^1(\mathbb{T}^3))} \left\| (u(\tau))^2 \right\|_{L^1(0,T; L^3(\mathbb{T}^3))} .
\]
Then \(D_\tau^p p(\tau)\) is uniformly bounded in the dual space of \(L^2(0, T; H^1(\mathbb{T}^3))\) by some constant function depending on \(T\) and on the initial data:
\[
\left\| D_\tau^p p(\tau) \right\|_{L^2(0, T; H^{-1}(\mathbb{T}^3))} \leq C(T, H_{in}). \quad (4.9)
\]
Moreover, taking into account that \(p(\tau) \in L^2(0, T; H^2(\mathbb{T}^3))\), we can use Theorem 1.2 with \(X = H^1(\mathbb{T}^3), B = L^2(\mathbb{T}^3), Y = H^{-1}(\mathbb{T}^3)\) to conclude that
\[
p(\tau) \to p \text{ in } L^2(0, T; L^2(\mathbb{T}^3)),
\]
and
\[
p(\tau) \to p \text{ a.e. in } \mathbb{T}^3 \times [0, T].
\]
By Proposition 5, we get
\[
\int_0^T \int_{\mathbb{T}^3} \left( \int_0^3 (D^\alpha_\tau p(\tau)) \psi \right) dx dt \rightarrow \int_0^T \int_{\mathbb{T}^3} \langle D^\alpha_\tau p, \psi \rangle dx dt .
\]
Summarizing, we have (after the density argument)
\[
\int_0^T \int_{\mathbb{T}^3} \langle D^\alpha_\tau p, \phi \rangle dx dt + \int_0^T \int_{\mathbb{T}^3} u \nabla p \cdot \nabla \phi dx dt + \frac{s}{2} \int_0^T \int_{\mathbb{T}^3} \nabla u \cdot \nabla \phi dx dt = 0 , \quad (4.10)
\]
\[
\int_0^T \int_{\mathbb{T}^3} \langle D^\alpha_\tau p, \psi \rangle dx dt + \int_0^T \int_{\mathbb{T}^3} \negDelta^{s/2} p \negDelta^{s/2} \psi dx dt + \frac{s}{2} \int_0^T \int_{\mathbb{T}^3} \nabla p \cdot \nabla \psi dx dt - \int_0^T \int_{\mathbb{T}^3} \nabla \psi dx dt + \int_0^T \int_{\mathbb{T}^3} u^2 dx dt = 0 , \quad (4.11)
\]
for all \(\phi \in L^2(0, T; W^{1,3}(\mathbb{T}^3))\) and \(\psi \in L^2(0, T; H^1(\mathbb{T}^3))\).

The next step is the limit \(\tau \to 0\) in the energy estimate (4.3). Taking \(\lim \inf \tau \to 0\) on both sides, by the lower weak semicontinuity of \(L^p\) norm, we get
\[
\int_{\mathbb{T}^3} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) dx + \frac{\tau^{\alpha-1}}{\Gamma_{\alpha}} \int_0^t \int_{\mathbb{T}^3} \left[ \beta |\nabla u|^2 + \frac{1}{2} \negDelta^{s/2} \nabla p |^2 + \frac{s}{2} (\negDelta p)^2 \right] dxds \quad (4.12)
\]
\[
\leq \int_{\mathbb{T}^3} \left( u_{in}^2 + \frac{1}{2} |\nabla \xi_{in}|^2 \right) dx .
\]
Furthermore, take \(\lim \inf \tau \to 0\) on both sides of (4.4) and (4.5), everywhere convergence of \(u(\tau)\) and \(p(\tau)\) yields
\[
\int_{\mathbb{T}^3} u dx \leq \int_{\mathbb{T}^3} u_{in} dx, \quad (4.13)
\]
and
\[
\int_{\mathbb{T}^3} p dx \leq \int_{\mathbb{T}^3} \xi_{in} dx + \frac{1}{\alpha \Gamma_{\alpha}} H(u_{in}, \xi_{in}) \Gamma_{\alpha} . \quad (4.14)
\]
5. Limit \( \varepsilon \to 0 \)

This section is devoted to the limit \( \varepsilon \to 0 \). For convenience we recall the statement of a compactness result proven in [25]:

**Theorem 5.1.** Let \( X, B \) and \( Y \) be Banach spaces. \( X \hookrightarrow B \) compactly and \( B \hookrightarrow Y \) continuously. Let \( 1 \leq r \leq \infty, 0 < \alpha < 1 \). Suppose \( u \in L^1_{loc}(0, T; X) \) satisfies:

\[
\|u\|_{L^r(0, T; X)} + \|D^\alpha_t u\|_{L^r(0, T; Y)} \leq C_0.
\]

Then \( u \) is relatively compact in \( L^r(0, T; B) \).

Similar to (4.6), we can conclude from (4.10) that

\[
\left\| D^\alpha_t u^{(\varepsilon)} \right\|_{L^2(0, T; (W^{1,3}(\mathbb{T}^3))))} \leq C(T, \varrho).
\]

Moreover, by (4.12) \( u^{(\varepsilon)} \) is uniformly bounded in \( L^2(0, T; H^1(\mathbb{T}^3)) \). Since \( H^1(\mathbb{T}^3) \) is compactly embedded in \( L^{6-\delta}(\mathbb{T}^3) \), Theorem 5.1 with \( X = H^1(\mathbb{T}^3), B = L^{6-\delta}(\mathbb{T}^3), Y = (W^{1,3})(\mathbb{T}^3) \) yields

\[
u^{(\varepsilon)} \rightarrow u \text{ in } L^2(0, T; L^{6-\delta}(\mathbb{T}^3)),
\]

and

\[
u^{(\varepsilon)} \rightarrow u \text{ a.e. in } \mathbb{T}^3 \times [0, T].
\]

Therefore, similarly as for (4.7), we conclude

\[
\int_0^T \int_{\mathbb{T}^3} (u^{(\varepsilon)} \psi)^2 \, dx \, dt \rightarrow \int_0^T \int_{\mathbb{T}^3} u^2 \psi \, dx \, dt, \text{ as } \varepsilon \to 0.
\]

The second term in (4.10) can be handled in the exact same way as (4.8). Summarizing up, after a density argument,

\[
\int_0^T \int_{\mathbb{T}^3} \langle D^\alpha_t u, \phi \rangle \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} u \nabla p \cdot \nabla \phi \, dx \, dt \\
+ \varrho \int_0^T \int_{\mathbb{T}^3} \nabla u \cdot \nabla \phi \, dx \, dt = 0,
\]

(5.1)

\[
\int_0^T \int_{\mathbb{T}^3} \langle D^\alpha_t p, \psi \rangle \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} (-\Delta)^{s/2} p (-\Delta)^{s/2} \psi \, dx \, dt \\
- \int_0^T \int_{\mathbb{T}^3} u^2 \psi \, dx \, dt = 0,
\]

(5.2)

for all \( \phi \in L^2(0, T; W^{1,3}(\mathbb{T}^3)) \) and \( \psi \in L^2(0, T; H^1(\mathbb{T}^3)) \).

The next step is the limit \( \varepsilon \to 0 \) in (4.12). By the lower weak semicontinuity we have

\[
\int_{\mathbb{T}^3} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) \, dx + \frac{t^{n-1}}{\Gamma_n} \left[ \varrho \int_0^t \int_{\mathbb{T}^3} |\nabla u|^2 \, dx \, ds + \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} |(-\Delta)^{s/2} \nabla p|^2 \, dx \, ds \right] \\
\leq \int_{\mathbb{T}^3} \left( u_{\varepsilon \to 0}^2 + \frac{1}{2} |\nabla p_{\varepsilon \to 0}|^2 \right) \, dx.
\]

(5.3)
Now we try to get the everywhere convergence of \( p^{(e)} \). Same as (4.9), we can bound the first term of (4.11) as
\[
\left\| D_t^\alpha p^{(e)} \right\|_{L^2(0,T;H^{-1}(T^3))} \leq \left\| (\Delta)^{\alpha/2} p^{(\tau)} \right\|_{L^2(0,T;L^2(T^3))} + \left\| \nabla p^{(\tau)} \right\|_{L^2(0,T;L^4(T^3))} \| \psi \|_{L^2(0,T;L^{4/3}(T^3))} + \left\| (u^{(\tau)})^2 \right\|_{L^2(0,T;L^{3/2}(T^3))} \| \psi \|_{L^2(0,T;L^3(T^3))}
\]
\[
\leq C(T,H_m) \| \psi \|_{L^2(0,T;H^1(T^3))}.
\]

Theorem 5.1 with \( X = H^1(T^3), B = L^2(T^3), Y = H^{-1}(T^3) \) yields
\[
p^{(e)} \to p \text{ in } L^2(0,T;L^2(T^3)),
\]
and
\[
p^{(e)} \to p \text{ a.e. in } T^3 \times [0,T].
\]

Now take \( \lim \inf_{\varepsilon \to 0} \) on (4.13) and (4.14), almost everywhere convergence of \( u^{(e)} \) and \( p^{(e)} \) yields
\[
\int_{T^3} u \, dx \leq \int_{T^3} u_{in} \, dx,
\]
and
\[
\int_{T^3} p \, dx \leq \int_{T^3} p_{in} \, dx + \frac{1}{\alpha \Gamma_\alpha} H(u_{in}, p_{in}) T^\alpha.
\]

6. LIMIT \( \varepsilon \to 0 \)

This section is devoted to the limit \( \varepsilon \to 0 \). Define
\[
\Psi(u) := \int_0^T \int_{T^3} (u)^2 \psi \, dx \, dt,
\]
with \( \psi \in L^2(0,T;H^1 \cap L^\infty(T^3)) \). We have that
\[
|\Psi(u_1) - \Psi(u_2)| = \int_0^T \int_{T^3} |u_1^2 - u_2^2| \psi \, dx \, dt
\]
\[
\leq \|u_1 - u_2\|_{L^2(0,T;L^2(T^3))} \left[ \int_0^T \int_{T^3} \psi^2(u_1 + u_2)^2 \, dx \, dt \right]^{1/2},
\]
\[
\leq \|u_1 - u_2\|_{L^2(0,T;L^2(T^3))} \|\psi\|_{L^2(0,T;L^\infty(T^3))} \|u_1 + u_2\|_{L^\infty(0,T;L^2(T^3))}.
\]

Therefore \( \Psi(\cdot) \) is continuous for the strong topology \( \|\cdot\|_{L^2(0,T;L^2(T^3))} \). Since \( \Psi \) is convex, \( u^{(e)} \to u \) in \( L^2(0,T;L^2(T^3)) \) and \( u^{(e)} \) is uniformly bounded in \( L^\infty(0,T,L^2(T^3)) \), we have that (Corollary III.8 [9])
\[
\lim \inf_{\varepsilon \to 0} \Psi(u^{(e)}) \geq \Psi(u).
\]

In other words,
\[
\int_0^T \int_{T^3} u^2 \psi \, dx \, dt \leq \lim \inf_{\varepsilon \to 0} \int_0^T \int_{T^3} (u^{(e)})^2 \psi \, dx \, dt.
\]

Next, we take a look at the second term:
\[
\int_0^T \int_{T^3} u^{(e)} \nabla p^{(e)} \cdot \nabla \phi \, dx \, dt.
\]
With $\phi \in L^2(0, T, W^{1,\infty}(\mathbb{T}^3))$, consider
\[
\int \int [u^{(e)} \nabla p^{(e)} - u \nabla p] \cdot \nabla \phi \, dx dt = \int \int [u^{(e)} \nabla p^{(e)} - u^{(e)} \nabla p + u^{(e)} \nabla p - u \nabla p] \cdot \nabla \phi \, dx dt
\]
\[
= \int \int u^{(e)} \nabla \phi \cdot (\nabla p^{(e)} - \nabla p) \, dx dt + \int \int \nabla p \cdot \nabla \phi (u^{(e)} - u) \, dx dt
\]
\[
= I_1 + I_2.
\]
We can have $I_2 \to 0$ since $u^{(e)} \rightharpoonup u$ in $L^\infty(0, T; L^2(\mathbb{T}^3))$ and $\nabla p \cdot \nabla \phi$ in $L^1(0, T; L^2(\mathbb{T}^3))$.

To handle $I_1$, we need to show strong convergence for $\nabla p^{(e)}$. For that we first bound $D_t^2 p^{(e)}$ as follows:
\[
\left| \int_0^T \int_{\mathbb{T}^3} \left< D_t^2 p^{(e)}, \psi \right> \, dx dt \right| \leq \left\| (-\Delta)^{s/2} p^{(e)} \right\|_{L^2(0,T;L^2(\mathbb{T}^3))} \left\| (-\Delta)^{s/2} \psi \right\|_{L^2(0,T;L^2(\mathbb{T}^3))}
\]
\[
+ \left\| u^{(e)} \right\|^2_{L^\infty(0,T;L^1(\mathbb{T}^3))} \left\| \psi \right\|_{L^2(0,T;L^\infty(\mathbb{T}^3))}
\]
\[
\leq C(T) \left\| \psi \right\|_{L^2(0,T;L^\infty(\mathbb{T}^3))},
\]
which implies
\[
\left\| D_t^2 p^{(e)} \right\|_{L^2(0,T;L^\infty(\mathbb{T}^3))} \leq C(T).
\]

Then Theorem 5.1 with $X = H^{s+1}(\mathbb{T}^3)$, $B = H^1(\mathbb{T}^3)$, $Y = (L^\infty \cap H^1)'(\mathbb{T}^3)$ yields $p^{(e)} \to p$ in $L^2(0,T;H^1(\mathbb{T}^3))$. We conclude that $I_1 \to 0$, since
\[
I_1 \leq \left\| u^{(e)} \right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \left\| \nabla \phi \right\|_{L^2(0,T;L^\infty(\mathbb{T}^3))} \left\| \nabla p^{(e)} - \nabla p \right\|_{L^2(0,T;L^2(\mathbb{T}^3))}.
\]

The last term
\[
\varrho \int_0^T \int_{\mathbb{T}^3} \nabla u^{(e)} \cdot \nabla \phi \, dx dt \to 0,
\]
since $\varrho \nabla u^{(e)}$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^3))$.

Furthermore, we consider the energy estimates (5.3). By taking $\lim \inf_{e \to 0}$ on both sides we have
\[
\int_{\mathbb{T}^3} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) \, dx + \frac{\alpha - 1}{1 - \alpha} \int_0^t \int_{\mathbb{T}^3} \left| (-\Delta)^{s/2} \nabla p(s, x) \right|^2 \, dx ds \leq \int_{\mathbb{T}^3} \left( u_{in}^2 + \frac{1}{2} |\nabla p_{in}|^2 \right) \, dx.
\]

Follow the same logic, with Fatou's lemma, we can pass the limits on (5.4) and (5.5):
\[
\int_{\mathbb{T}^3} u \, dx \leq \int_{\mathbb{T}^3} u_{in} \, dx,
\]
\[
\int_{\mathbb{T}^3} p \, dx \leq \int_{\mathbb{T}^3} p_{in} \, dx + \frac{1}{\alpha T} H(u_{in}, p_{in}) T^\alpha.
\]
This concludes the proof of Theorem 1.1.

7. Appendix

In the following lemma we show a formal $L^3(0,T;L^3(\mathbb{T}^3))$-estimate for $u$, provided $\frac{1}{2} < s \leq 1$. We will not use this estimate in this manuscript. We add it here for completeness, and because we believe it can be useful in the future to show compactness for $u$.

Lemma 8. If $\frac{1}{2} < s \leq 1$ the function $u$, solution to (1.1), is bounded in $L^3(0,T;L^3(\mathbb{T}^3))$ as
\[
\|u\|_{L^3(0,T;L^3(\mathbb{T}^3))} \leq C(T, H(u_{in}, p_{in})).
\]
Proof. We test the equations in (1.1) with \( p \) and \( u \) respectively. We get

\[
\int_0^T \int_{\Omega} \langle D_t^\alpha p, u \rangle \, dx \, dt + \int_0^T \int_{\Omega} u |\nabla p|^2 \, dx \, dt = 0, \tag{7.1}
\]

and

\[
\int_0^T \int_{\Omega} \langle D_t^\alpha u, p \rangle \, dx \, dt + \int_0^T \int_{\Omega} (-\Delta)^{s/2} p(-\Delta)^{s/2} u \, dx \, dt = \int_0^T \int_{\Omega} (u)^3 \, dx \, dt. \tag{7.2}
\]

Integration by parts using (2.24) yields

\[
\int_0^T \langle D_t^\alpha p(t), u(t) \rangle \, dt + \int_0^T \langle D_t^\alpha u(t), p(t) \rangle \, dt = \frac{1}{\Gamma_1 - \alpha} \int_0^T u(t)p(t) \left[ \frac{1}{(T-t)^\alpha} + \frac{1}{t^\alpha} \right] \, dt \tag{7.3}
\]

\[
+ \frac{\alpha}{\Gamma_1 - \alpha} \int_0^T \int_0^t \frac{(u(t) - u(s))(p(t) - p(s))}{(t-s)^{1+\alpha}} \, ds \, dt
\]

\[
- \frac{1}{\Gamma_1 - \alpha} \int_0^T u(t)p_{in} + p(t)u_{in} \frac{1}{t^\alpha} \, dt.
\]

Adding (7.1) to (7.2) and using (7.3), we get

\[
\int_0^T \int_{\Omega} (u)^3 \, dx \, dt = \int_0^T \int_{\Omega} u |\nabla p|^2 \, dx \, dt \tag{7.4}
\]

\[
+ \int_0^T \int_{\Omega} (-\Delta)^{s/2} p(-\Delta)^{s/2} u \, dx \, dt
\]

\[
+ \frac{1}{\Gamma_1 - \alpha} \int_0^T \int_{\Omega} u(t)p(t) \left[ \frac{1}{(T-t)^\alpha} + \frac{1}{t^\alpha} \right] \, dx \, dt
\]

\[
+ \frac{\alpha}{\Gamma_1 - \alpha} \int_0^T \int_0^t \frac{(u(t) - u(s))(p(t) - p(s))}{(t-s)^{1+\alpha}} \, ds \, dt
\]

\[
- \frac{1}{\Gamma_1 - \alpha} \int_0^T u(t)p_{in} + p(t)u_{in} \frac{1}{t^\alpha} \, dx \, dt
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5.
\]

We use Hölder’s inequality to bound \( I_1 \):

\[
I_1 = \int_0^T \int_{\Omega} u |\nabla p|^2 \, dx \, dt \leq \left[ \int_0^T \int_{\Omega} (u)^3 \, dx \, dt \right]^{\frac{2}{3}} \left[ \int_0^T \int_{\Omega} |\nabla p|^2 \, dx \, dt \right]^{\frac{1}{3}}
\]

\[
\leq \frac{1}{4} \int_0^T \int_{\Omega} (u)^3 \, dx \, dt + 2 \|\nabla p\|_{L^3(0,T;L^3(\Omega))}.
\]

Since \( s > \frac{1}{2} \), the \( \|\nabla p\|_{L^3(0,T;L^3(\Omega))} \) term can be bounded via interpolation of \( \nabla p \in L^\infty(0,T;L^2(\Omega)) \) with \( \nabla p \in L^2(0,T;L^{\frac{6}{s+1}}(\Omega)) \). We bound \( I_2 \) using (3.2) after integrating by parts. Next we analyze \( I_3 \):

\[
I_3 = \frac{1}{\Gamma_1 - \alpha} \int_0^T \int_{\Omega} u(t)p(t) \left[ \frac{1}{(T-t)^\alpha} + \frac{1}{t^\alpha} \right] \, dx \, dt
\]

\[
\leq \frac{2}{\Gamma_2 - \alpha} T^{1-\alpha} \|u\|_{L^\infty(0,T;L^2(\Omega))} \|p\|_{L^\infty(0,T;L^2(\Omega))}
\]

\[
\leq C(T, H(u_{in}, p_{in})),
\]
thanks to (3.2). Now take a close look at $I_4$:

$$I_4 = \frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_0^t \int_{\mathbb{T}^3} \frac{(u(t) - u(s))(p(t) - p(s))}{(t-s)^{1+\alpha}} \, dx \, ds \, dt$$

$$\leq \frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_0^t \int_{\mathbb{T}^3} \frac{(u(t) - u(s))^2}{(t-s)^{1+\alpha}} \, dx \, ds \, dt + \frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_0^t \int_{\mathbb{T}^3} \frac{(p(t) - p(s))^2}{(t-s)^{1+\alpha}} \, dx \, ds \, dt.$$

Integration by parts formula (2.24) gives

$$\frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_0^t \int_{\mathbb{T}^3} \frac{(u(t) - u(s))^2}{(t-s)^{1+\alpha}} \, dx \, ds \, dt = 2\alpha \int_0^T \int_{\mathbb{T}^3} u(t)D_t^\alpha u(t) \, dx \, dt + \frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} 2u(t)u_{in} \, t^\alpha \, dx \, dt - \frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} (u(t))^2 \left[ \frac{1}{(T-t)^\alpha} + \frac{1}{t^\alpha} \right] \, dx \, dt - 2\alpha \int_0^T \int_{\mathbb{T}^3} \nabla u(t) \cdot (u(t)\nabla p(t)) \, dx \, dt + \frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} 2u(t)u_{in} \, t^\alpha \, dx \, dt$$

For $A_2$ we have:

$$A_2 \leq \frac{T^{1-\alpha}}{\Gamma_{2-\alpha}} \| u(t) \|_{L^{\infty}(0,T;L^2(\mathbb{T}^3))} \| u_{in} \|_{L^{\infty}(0,T;L^2(\mathbb{T}^3))} \leq C(T, H(u_{in}, p_{in})).$$

In $A_1$, integration by parts yields

$$A_1 = -\int_0^T \int_{\mathbb{T}^3} (D_t^\alpha \nabla p) \cdot \nabla p \, dx \, dt - \int_0^T \int_{\mathbb{T}^3} \left| (-\Delta)^{s/2} \nabla p \right|^2 \, dx \, dt - \frac{1}{2} T^{1-\alpha} \int_{\mathbb{T}^3} \nabla p_{in}^2 \int_0^T \frac{1}{s^\alpha} \, ds \, dx.$$

Next we use formula (2.23) with $\phi = 1$ and get:

$$A_1 \leq \frac{1}{2\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} \frac{\| \nabla p \|^2(t) \, dt}{d} \left( \int_t^T \frac{1}{(s-t)^\alpha} \, ds \right) \, dx + \frac{1}{2\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} \nabla p_{in}^2 \int_0^T \frac{1}{s^\alpha} \, ds \, dx$$

$$= -\frac{1}{2\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} \frac{\| \nabla p \|^2(t) \, dx \, dt}{(T-t)^\alpha} + \frac{T^{1-\alpha}}{2\Gamma_{2-\alpha}} \| \nabla p_{in} \|^2_{L^2(\mathbb{T}^3)}$$

$$\leq C(T, H(u_{in}, p_{in})).$$

Summarizing

$$\frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_0^t \int_{\mathbb{T}^3} \frac{(u(t) - u(s))^2}{(t-s)^{1+\alpha}} \, dx \, ds \, dt \leq C(T, H(u_{in}, p_{in})).$$
Now we look at the second term of $I_4$:

$$\frac{\alpha}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} \frac{(p(t) - p(s))^2}{(t-s)^{1+\alpha}} \, dx \, ds \, dt$$

$$= 2 \int_0^T \int_{\mathbb{T}^3} p(t) D_t^\alpha p(t) \, dx \, dt + \frac{1}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} \frac{2p(t) p_{in}}{t^\alpha} \, dx \, dt - \frac{1}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} (p(t))^2 \left[ \frac{1}{(T-t)^\alpha} + \frac{1}{t^\alpha} \right] \, dx \, dt$$

$$\leq - \int_0^T \int_{\mathbb{T}^3} [(-\Delta) \frac{p}{2}]^2 \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} (u)^2 \, p \, dx \, dt + \frac{1}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} \frac{2p(t) p_{in}}{t^\alpha} \, dx \, dt$$

$$\leq \int_0^T \int_{\mathbb{T}^3} (u)^2 \, p \, dx \, dt + \frac{1}{\Gamma_{1-\alpha}} \int_0^T \int_{\mathbb{T}^3} \frac{2p(t) p_{in}}{t^\alpha} \, dx \, dt$$

$$= B_1 + B_2.$$

The bound for $B_2$ follows from

$$B_2 \leq \frac{T^{1-\alpha}}{\Gamma_{2-\alpha}} \|p(t)\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \|p_{in}\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq C(T,H(u_{in},p_{in})).$$

For $B_1$ we have:

$$B_1 \leq \frac{1}{2} \int_0^T \int_{\mathbb{T}^3} (u)^3 \, dx \, dt + 4 \int_0^T \int_{\mathbb{T}^3} (p)^3 \, dx \, dt$$

$$\leq \frac{1}{2} \int_0^T \int_{\mathbb{T}^3} (u)^3 \, dx \, dt + 4 \|p\|_{L^2(0,T;L^2(\mathbb{T}^3))}.$$

The second term is bounded in accordance to the interpolation of $p \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;L^6(\mathbb{T}^3))$.

Summarizing we have:

$$\frac{1}{4} \int_0^T \int_{\mathbb{T}^3} (u)^3 \, dx \, dt \leq C(T,H(u_{in},p_{in})).$$

\[ \square \]

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