Matrix beta-integrals: an overview

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First examples of matrix beta-integrals were discovered on 1930–50s by Siegel and Hua, in 60s Gindikin obtained multi-parametric series of such integrals. We discuss beta-integrals related to symmetric spaces, their interpolation with respect to the dimension of a ground field, and adelic analogs; also we discuss beta-integrals related to flag spaces.

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1. Introduction

1.1. Euler beta-function. Recall the standard formulas for the Euler beta-function:

\[ \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{(Euler)} \]  \hfill (1.1)

\[ \int_\mathbb{R} \frac{dx}{(1+ix)^\mu(1-ix)^\nu} = \frac{2^{2-\mu-\nu}\pi \Gamma(\mu+\nu-1)}{\Gamma(\mu)\Gamma(\nu)} \quad \text{(Cauchy)} \]  \hfill (1.2)

\[ \int_0^\infty x^{\alpha-1} \frac{dx}{(1+x)^\sigma} = \frac{\Gamma(\alpha)\Gamma(\sigma-\alpha)}{\Gamma(\sigma)} \quad \text{(1.3)} \]

\[ \int_0^\pi (\sin t)^\mu e^{i\nu t} dt = \frac{\pi}{2^\mu} \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\nu)} e^{i\pi\nu/2} \quad \text{(Lobachevsky)} \]  \hfill (1.4)

The integral (1.3) is obtained from (1.1) by a substitution \( x = t/(1+t) \). Replacing the segment \([0,1]\) in (1.1) by a circle \(|x| = 1\), after simple manipulations we get (1.3). Considering a stereographic projection of the circle to a line, we come to (1.2).

1.2. Beta-integrals. 'Beta-integral' is an informal term for integrals of a type

\[ \int (\text{Product}) = \text{Product of Gamma-functions}. \]  \hfill (1.5)

There is large family of such identities (see, e.g., [2], [1]). First, we present two nice examples. The De Branges [4] – Wilson integral (1972, 1980) is given

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by

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \left| \prod_{j=1}^{4} \frac{\Gamma(a_j + ix)}{\Gamma(2ix)} \right|^2 \, dx = \prod_{1 \leq k < l \leq 4} \frac{\Gamma(a_k + a_l)}{\Gamma(a_1 + a_2 + a_3 + a_4)}.
\]

Recall that the integrand is a weight function for the Wilson orthogonal polynomials, which occupy the highest level of the Askey hierarchy [1] of hypergeometric orthogonal polynomials.

The second example is the Selberg integral, [29], 1944,

\[
\int_0^1 \ldots \int_0^1 \prod_{j=1}^{n} x_j^{\alpha - 1}(1 - t_j)^{\beta - 1} \prod_{1 \leq k < l \leq n} |t_k - t_l|^{2\gamma} \, dt_1 \ldots dt_n = \prod_{j=1}^{n} \frac{\Gamma(\alpha + (j - 1)\gamma) \Gamma(\beta + (j - 1)\gamma) \Gamma(1 + j\gamma)}{\Gamma(\alpha + \beta + (n + j - 2)\gamma) \Gamma(1 + \gamma)}.
\]

(1.6)

As the Euler beta-integral, the Selberg integral has several versions, for instance

\[
\int_0^\infty \ldots \int_0^\infty \prod_{j=1}^{n} x_j^{\alpha - 1}(1 + x_j)^{-\alpha - \beta - 2\gamma(n-1)} \prod_{1 \leq k < l \leq n} |x_k - x_l|^{2\gamma} \, dx_1 \ldots dx_n = \prod_{j=1}^{n} \frac{\Gamma(\alpha + (j - 1)\gamma) \Gamma(\beta + (j - 1)\gamma) \Gamma(1 + j\gamma)}{\Gamma(\alpha + \beta + (n + j - 2)\gamma) \Gamma(1 + \gamma)}
\]

(1.7)

\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{k=1}^{n} (1 - ix_k)^{-\alpha}(1 + ix_k)^{-\beta} \prod_{1 \leq k < l \leq n} |x_k - x_l|^{2\gamma} \, dx_1 \ldots dx_n = 2^{-(\alpha + \beta)n + \gamma n(n-1)+n} \prod_{j=1}^{n} \frac{\Gamma(\alpha + \beta - (n + j - 2)\gamma - 1) \Gamma(1 + j\gamma)}{\Gamma(\alpha - (j - 1)\gamma) \Gamma(\beta - (j - 1)\gamma) \Gamma(1 + \gamma)}
\]

(1.8)

There exists a large family of beta-integrals [14], including one-dimensional integrals (see an old overview of Askey [2]), multi-dimensional integrals, \(q\)-analogues, elliptic analogs; some occasional collection of references is [1], [11], [12], [27], [13], [32].

The topic of these notes is analogs of integrals (1.1)–(1.4), (1.6)–(1.8).

1.3. Notation.

• \(\mathbb{K}\) denote \(\mathbb{R}, \mathbb{C}\), or quaternions \(\mathbb{H}\); \(\mathfrak{d} := \text{dim}\mathbb{K}\).

• \([X]_p\) is the left upper corner of a matrix \(X\) of size \(p \times p\);

• \([X]_{pq}\) is the left upper corner of a matrix \(X\) of size \(p \times q\);

• \(X^*, X^t\) are adjoint matrix and transposed matrix;

• \(X > 0\) means that a matrix \(X\) is self-adjoint and strictly positive definite, \(X > Y\) means that \(X - Y > 0\);

• \(\|X\|\) denotes a norm of a matrix, precisely the norm of the corresponding linear operator in the standard Euclidean space. \(\|X\| = \|X^*X\|^{1/2} = \|XX^*\|^{1/2}\); for a self-adjoint matrix norm is max \(|\lambda|\) over all eigenvalues.
Spaces of matrices:
- Mat_{p,q}(\mathbb{K}) is the space of all matrices of size $p \times q$ over $\mathbb{K}$;
- Herm_n(\mathbb{K}) is the space of all Hermitian matrices ($X = X^*$) of size $n$;
- Symm_n(\mathbb{K}) is the space of all symmetric matrices ($X = X^t$) of size $n$.

The Lebesgue measure on such spaces is normalized in the most simple way. For instance, for Symm_n(\mathbb{R}) we set
\[ dX := \prod_{1 \leq k \leq l \leq n} dx_{kl}; \]
for Mat_{p,q}(\mathbb{C}), we write
\[ dZ := \prod_{1 \leq k \leq p, 1 \leq l \leq q} d \text{Re} z_{kl} d \text{Im} z_{kl}. \]

2 Hua integrals

2.1. Hua integrals. The famous book [9] 'Harmonic analysis of functions of several complex variables in classical domains' by Hua, 1958, contains calculations of a family of matrix integrals. We present two examples.

Consider the space B_{m,n} of complex $m \times n$ matrices $Z$ with $\|Z\| < 1$. The following identity holds
\[ \int_{ZZ^* < 1} \det(1 - ZZ^*)^\lambda dZ = \frac{\prod_{j=1}^n \Gamma(\lambda + j) \prod_{j=1}^m \Gamma(\lambda + j)}{\prod_{j=1}^{n+m} \Gamma(\lambda + j)} \frac{\pi^{nm}}{\pi^{nm}}. \tag{2.1} \]

Next, consider the space Symm_n(\mathbb{R}) of all real symmetric matrices of size $n$. The following identity holds
\[ \int_{\text{Symm}_n(\mathbb{R})} \frac{dT}{\det(1 + T^2)^\alpha} = \pi \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} \frac{\prod_{j=1}^{n-1} \Gamma(2\alpha - (n + j)/2)}{\prod_{j=1}^{n-1} \Gamma(2\alpha - j)}. \tag{2.2} \]

2.2. Comments: spaces and integrands. We can consider the following 10 series of matrix spaces
- $p \times q$ matrices over $\mathbb{R}$;
- symmetric $n \times n$ matrices ($X = X^t$) over $\mathbb{R}$;
- skew-symmetric $n \times n$ matrices ($X = -X^t$) over $\mathbb{R}$;
- $p \times q$ matrices over $\mathbb{C}$;
- symmetric $n \times n$ matrices over $\mathbb{C}$;
- skew-symmetric $n \times n$ matrices over $\mathbb{C}$;
- Hermitian $n \times n$ matrices ($X = X^*$) over $\mathbb{C}$;
- $p \times q$ matrices over $\mathbb{H}$;
- Hermitian $n \times n$ matrices ($X = X^*$) over $\mathbb{H}$;
- anti-Hermitian $n \times n$ matrices ($X = -X^*$) over $\mathbb{H}$.

For any space of this list, we consider a `matrix ball' $XX^* < 1$.3
For all 'matrix spaces' and all 'matrix ball', integrals

\[ \int \det(1 + XX^*)^{-\alpha}dX; \quad (2.3) \]

\[ \int_{XX^* < 1} \det(1 - XX^*)^\gamma dX \quad (2.4) \]

are long products of gamma-functions as (2.1)–(2.2). Actually, Hua evaluated 1/3 of these 20 integrals. Apparently, there is no text, where all these integrals are evaluated (and a reason, which does not excuse this, is explained in the next subsection).

The domain of integration \( B_{m,n} \subset \mathbb{C}^{nm} \) in (2.1), i.e., the matrix ball \( \| Z \| < 1 \), is a well-known object in differential geometry, representation theory, and complex analysis, since it is an Hermitian symmetric space.\(^3\)

\[ B_{p,q} = U_{p,q}/(U_p \times U_q) \]

The pseudounitary group \( U_{p,q} \) acts on this domain by linear-fractional transformations:

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : Z \mapsto U := (a + Zc)^{-1}(b + Zd). \quad (2.5) \]

The remaining 9 series of 'matrix balls' \( XX^* < 1 \) also are Riemannian symmetric spaces.\(^4\) Up to a minor inaccuracy, all Riemannian noncompact symmetric spaces admit 'matrix ball' models. The group of isometries consists of certain linear-fractional transformations (see tables of symmetric spaces in [21], Addendum D).

Meaning of the integrand \( \det(1 - ZZ^*)^\alpha \) is less obvious.\(^5\) However, any mathematician what had deal with the unit circle \( |z| < 1 \) could observe that the expression \( (1 - z^2)^\alpha \) quite often appears in formulas. The same holds for \( \det(1 - ZZ^*)^\alpha \) in the case of the matrix balls. We only point out a nice behavior of the expression under linear-fractional transformation (2.5):

\[ \det(1 - UU^*)^\alpha = \det(1 - ZZ^*)^\alpha |\det(a + zc)|^{-2\alpha}. \]

Thus integrals (2.4) are integrals of some reasonable expressions over non-compact symmetric spaces.

Integrals (2.3) are integrals over compact symmetric spaces written in co-ordinates. For instance, in (2.2) we integrate over the space \( \text{Symm}_n(\mathbb{R}) \). But \( \text{Symm}_n(\mathbb{R}) \) is a chart on the real Lagrangian Grassmannian (recall that if an operator \( T : \mathbb{R}^n \to \mathbb{R}^n \) is symmetric, then its graph is Lagrangian subspace in \( \mathbb{R}^{2n} \)).

\(^2\)Recall a definition of a determinant \( \det(X) = \det_{\mathbb{H}}(X) \) of a quaternionic matrix \( X \). Such matrix determines a transformation \( \mathbb{H}^n \to \mathbb{H}^n \) and therefore an \( \mathbb{R} \)-linear transformation \( X_{\mathbb{R}} : \mathbb{R}^{4n} \to \mathbb{R}^4 \). We set \( \det_{\mathbb{H}}(X) := \sqrt[4]{\det(X_{\mathbb{H}})}. \) In particular, \( \det(\cdot) \) is real non-negative. If entries of \( X \) are complex, then the quaternionic determinant coincides with \( |\det_{\mathbb{C}} X| \).

\(^3\)Spaces \( B_{p,q} \) also are known as Cartan domains of type I.

\(^4\)below a 'symmetric space' means a semisimple (reductive) symmetric space.

\(^5\)Hua Loo Keng evaluated volumes of Cartan domains and some compact symmetric spaces. He observed that calculations survive in a higher generality.
$\mathbb{R}^n \oplus \mathbb{R}^n$, see, e.g., [21], Sect.3.1). The Lagrangian Grassmannian is a homogeneous (symmetric) space $U_n/O_n$, see, e.g., [21], Sect. 3.3. All other 'matrix spaces' defined above are open dense charts on certain compact Riemannian symmetric spaces. Up to a minor inaccuracy, all compact symmetric spaces admit such charts (see tables of symmetric spaces in [21], Addendum D).

### 2.3. Integration over eigenvalues.

Consider the space $\text{Herm}_n(\mathbb{K})$ of all Hermitian matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H}$; equip this space with the standard Lebesgue measure. To matrix $X \in \text{Herm}_n(\mathbb{K})$, we assign the collection of its eigenvalues

$$\Lambda : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$  

(2.6)

Thus we get a map $X \mapsto \Lambda$ from $\text{Herm}_n(\mathbb{K})$ to the wedge (2.6). The distribution of eigenvalues is given by the formula

$$C_n(\mathbb{K}) \prod_{1 \leq k < l \leq n} |\lambda_k - \lambda_l|^d d\lambda_1 \ldots d\lambda_n,$$

where $C_n(\mathbb{K})$ is a certain (explicit) constant, $d = \dim K$. This can be reformulated as follows. Let $F$ be a function on $\text{Herm}_n(\mathbb{K})$ invariant with respect to the unitary group $U(n, \mathbb{K})$ $^7$

$$F(uXu^{-1}) = F(X), \quad u \in U(n, \mathbb{K}).$$

Such $F$ is a function of eigenvalues,

$$F(X) = f(\lambda_1, \ldots, \lambda_n).$$

Then we have the following integration formula holds

$$\int_{\text{Herm}_n(\mathbb{K})} F(X) \, dX =$$

$$= C_n(\mathbb{K}) \int_{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n} f(\lambda_1, \ldots, \lambda_n) \prod_{1 \leq k < l \leq n} |\lambda_k - \lambda_l|^d d\lambda_1 \ldots d\lambda_n.$$  

(2.7)

The formula is a relative of the Weyl integration formula, see derivation of several formulas of this kind in [9].

In Hua integral $\int \frac{dX}{(1 + T^2)^{-\alpha} \det(1 - iT)^\beta} \, dT$, the integrand is

$$\det(1 + T^2)^{-\alpha} = \prod_{j=1}^{n}(1 + \lambda_j^2)^{-\alpha} = \prod_{j=1}^{n}(1 + i\lambda_j)^{-\alpha}(1 - i\lambda_j)^{-\alpha}.$$

Applying the integration formula (2.7) we reduce the Hua integral (2.2) to a special case of the Selberg integral (1.8). Moreover, we get also an explicit evaluation of a more general integral

$$\int \det(1 + iT)^\alpha \det(1 - iT)^\beta \, dT.$$  

$^6$\text{Herm}_n(\mathbb{R})$ is $\text{Symm}_n(\mathbb{R})$.

$^7$U(n, \mathbb{R}) is the orthogonal group $O(n)$, $U(n, \mathbb{C})$ is the usual unitary group $U(n)$, $U(n, \mathbb{H})$ is the compact symplectic group $\text{Sp}(2n)$. 
Next, consider the space of all complex matrices of size \( m \times n \), where \( m \leq n \). To each matrix we assign a collection of its singular values \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0 \).

The distribution of singular values is given by

\[
\prod_{1 \leq k \leq n} \mu_k^{2(n-m)+1} \prod_{1 \leq k < l \leq n} (\mu_k^2 - \mu_l^2)^2 \prod_{1 \leq k \leq n} d\mu_k.
\]

The integrand in the Hua integral (2.1) is \( \prod (1 - \mu_k^2) \). After the substitution \( x_k = \mu_k^2 \), this integral also is reduced to the Selberg integral (1.8).

All 20 integrals (2.3)–(2.4) are reduced to Selberg integrals in a similar way.

### 2.4. An application of Hua calculations: projective systems of measures.

Let us return to integral (2.2). Represent a matrix \( T \) as a block matrix of size \((n - 1) + 1\),

\[
T = \begin{pmatrix} S & p \\ p^t & q \end{pmatrix}.
\]

Consider a function \( f \) on \( \text{Symm}_n(\mathbb{R}) \) depending only on \( S = [T]_{n-1} \). Then the following identity holds

\[
\int_{\text{Symm}_n(\mathbb{R})} f(S) \det \left( 1 + \begin{pmatrix} S & p \\ p^t & q \end{pmatrix}^2 \right)^{-\alpha} dS dp dq =
\]

\[
= 2^{-\alpha + 1/2} \pi^{\frac{n+1}{2}} \frac{\Gamma(2\alpha + \frac{n+1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)\Gamma(2\alpha - 1)} \int_{\text{Symm}_{n-1}(\mathbb{R})} f(S) \det(1 + S^2)^{1/2 - \alpha} dS. \tag{2.8}
\]

This formula can be extracted from the original Hua calculation (the formula also implies (2.2)).

Now fix \( \alpha > -1/2 \) and consider a measure \( \nu_{\alpha,n} \) on \( \text{Symm}_n \) given by

\[
\nu_{\alpha,n} = s_{\alpha,n} \det(1 + T^2)^{-\alpha - (n+1)/2} dT,
\]

where the normalizing constant \( s_{\alpha,n} \) is chosen to make the total measure being 1. Consider a chain of projections

\[
\ldots \leftarrow \text{Symm}_{n-1}(\mathbb{R}) \leftarrow \text{Symm}_n(\mathbb{R}) \leftarrow \ldots,
\]

where each map sends a matrix \( X \in \text{Symm}_n(\mathbb{R}) \) to its left upper corner \([X]_{n-1}(\mathbb{R})\).

According (2.8), this map sends the measure \( s_{\alpha,n} \) to the measure \( s_{\alpha,n-1} \). By the Kolmogorov consistency theorem (see, e.g., [30], §2.9) there is a measure \( \nu_{\alpha} \) on the space \( \text{Symm}_\infty(\mathbb{R}) \) of infinite symmetric matrices whose image under each map \( X \mapsto [X]_n \) is \( \nu_{\alpha,n} \).

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8Singular values of a matrix \( Z \) are eigenvalues of \( \sqrt{ZZ^*} \).

9In all these cases the parameter \( \gamma \) in Selberg integrals is 1/2, 1, 2. For some exceptional symmetric spaces distributions of invariants give \( \gamma = 4 \).
Next, consider the group of finitary orthogonal block matrices having the structure \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \). This group is isomorphic to the group \( U_\infty \) of finitary unitary matrices. It acts on \( \text{Symm}_\infty(\mathbb{R}) \) by linear-fractional transformations \((2.5)\), point out that this formula makes sense. It is easy to show that the measure \( \nu_\alpha \) is quasiinvariant with respect to such transformations, and there arises a problem of decomposition of the space \( L^2 \). We also notice that we can regard our limit space as the inverse limit of the chain of Lagrangian Grassmannians,

\[
\ldots \leftarrow U_{n-1}/O_{n-1} \leftarrow U_n/O_n \leftarrow \ldots
\]

Such construction exists for any series of compact symmetric spaces and leads to an interesting harmonic analysis on the limit objects, see [26], [16], [25], [3].

**2.5. Remarks.** 1) The construction of inverse limits does not admit an extension to non-compact symmetric spaces (i.e., to matrix balls). Of course, the chain of projections of sets

\[
\ldots \leftarrow B_{p,q} \leftarrow B_{p+1,q+1} \leftarrow B_{p+2,q+2} \leftarrow \ldots
\]

is well defined. We can consider normalized probabilistic measures

\[ s'_{\alpha,p,q,k} \det(1 - ZZ^*)^{\alpha-2k} \]

on \( B_{p+k,q+k} \). However, for sufficiently large \( k \) the integral

\[
\int_{B_{p+k,q+k}} \det(1 - ZZ^*)^{\alpha-2k} \, dZ
\]

is divergent.

2) Projective limits exist for \( p \)-adic Grassmannians, see [22].

## 3 Beta-functions of symmetric spaces

### 3.1. Gindikin beta-function of symmetric cones.

Consider the space \( \text{Pos}_n(\mathbb{K}) \) of positive definite \( n \times n \) matrices over \( \mathbb{K} \). The cone \( \text{Pos}_n(\mathbb{K}) \) is a model of the symmetric space \( \text{GL}_n(\mathbb{K})/U_n(\mathbb{K}) \); the group \( \text{GL}_n(\mathbb{K}) \) acts on \( \text{Pos}_n(\mathbb{K}) \) by transformations

\[ g : X \mapsto g^*Xg. \]

Gindikin [8], 1965, considered a matrix \( \Gamma \)-function given by

\[
\Gamma[s] := \int_{\text{Pos}_n(\mathbb{K})} e^{-tr X} \prod_{j=1}^n \det[X]^{s_j-s_{j+1}} \cdot \det X^{bn/2 - b/2 + 1} \, dX =
\]

\[
= (2\pi)^{n(n-1)b/4} \prod_{k=1}^n \Gamma \left( s_k - (k-1)\frac{b}{2} \right). \quad (3.1)
\]

\( ^{10} \)We say that a matrix \( g \) is finitary, if \( g - 1 \) has finite number of nonzero matrix elements.
Here $s_j \in \mathbb{C}$, $s_{n+1} := 0$; $[X]_p$ denotes upper left corners of size $p$ of a matrix $X$. Expressions $s_j - s_{j+1}$ are written by aesthetic reasons, we can write

$$\prod_{j=1}^{n} \det [X]_{ij}^\lambda_j$$

with arbitrary $\lambda_j$. The factor $\det X^\mathfrak{o}/2-\mathfrak{o}/2+1$ can be included to the latter product, but it is the density of the $\mathrm{GL}_n(\mathbb{K})$-invariant measure on $\mathrm{Pos}_n(\mathbb{K})$ and it is reasonable to split it from the product.

To evaluate the integral, Gindikin considers substitution $X = S^* S$, where $S$ is an upper triangular matrix with positive elements on the diagonal. After this the integral splits into a product of one-dimensional integrals.

Also the following imitation of beta-function take place

$$B[s, t] := \int_{0 < X < 1} \prod_{j=1}^{n} \left( \det [X]_j^{s_j - s_{j+1}} \cdot \det [1 - X]_j^{t_j - t_{j+1}} \right) \times \det X^\mathfrak{o}/2-\mathfrak{o}/2+1 \cdot \det (1 - X)^\mathfrak{o}/2-\mathfrak{o}/2+1 dX = \frac{\Gamma[s] \Gamma[t]}{\Gamma[s + t]} \quad (3.2)$$

A proof in [8] is an one-to-one imitation of the standard evaluation of the Euler beta-integral.

These integrals extend some results of 1920-30s (Whishart, Ingham, Siegel, see [31]).

3.2. Beta functions of Riemannian non-compact symmetric spaces.

Notice that the domain of integration $0 < X < 1$ in (3.2) is itself the symmetric space $\mathrm{GL}_n(\mathbb{K})/U_n(\mathbb{K})$. Indeed, matrix ball $ZZ^* < 1$ in the space of Hermitian matrices is a model of the symmetric space $\mathrm{GL}_n(\mathbb{K})/U_n(\mathbb{K})$. The inequality $ZZ^* < 1$ is equivalent to $-1 < Z < 1$, and we substitute $Z = -1 + 2X$.

Analogs of integrals (3.2) for 7 remaining series of Riemannian non-compact symmetric spaces were obtained in [14]. We give two well-representative examples.

In the first example we consider a symmetric space, which can be realized as a matrix wedge. Let $W_n$ be the domain (Siegel upper-half plane) of $n \times n$ complex symmetric matrices $Z$ with $\mathrm{Re} Z > 0$. This is a model of a symmetric space $\mathrm{Sp}_{2n}(\mathbb{R})/U_n$. We write $Z = T + iS$, where $T, S$ are real symmetric matrices. Then

$$\int_{T = T^* > 0, S = S^*} \prod_{j=1}^{n} \frac{\det [T]_j^{\lambda_j - \lambda_{j+1}} \det [1 + T + iS]_j^{\sigma_j - \sigma_{j+1}} \det [1 - T - iS]_j^{\tau_j - \tau_{j+1}}}{\Gamma(\sigma_k - (n - k)/2)\Gamma(\tau_k - (n - k)/2)} \times \det T^{-(n+1)} dT dS =$$

$$= \prod_{k=1}^{n} \frac{2^{2-\sigma_k-\tau_k+n-k}\pi^k \Gamma(\sigma_k - (n + k)/2)\Gamma(\sigma_k + \tau_k - \lambda_k - (n - k)/2)}{\Gamma(\tau_k - (n - k)/2)} \quad (3.3)$$

11See also [3].

12For the case of tubes $\mathrm{SO}(n, 2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$, which is slightly exceptional, see [20].
(we set $\lambda_{j+1} = \sigma_{j+1} = \tau_{j+1} = 0$).

There are also noncompact symmetric spaces, which do not admit realizations as convex matrix cones and convex matrix wedges. As an example, we consider the space $O_{p,q}/O_p \times O_q$. Let $q \geq p$. We realize this space (for details, see [14], Sect.3) as the space of real block matrices of size $(q-p) + p$ having the form

$$R = \begin{pmatrix} 1 & 0 \\ 2L & K \end{pmatrix}$$

and satisfying the dissipativity condition

$$R + R^t > 0$$

We represent $K$ as $K = M + N$, where $M$ is symmetric and $K$ is skew-symmetric. Then the dissipativity condition $R + R^t > 0$ reduces to the form

$$\begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix} > 0$$

or equivalently $M - LL^t > 0$. We have the following integrals in coordinates $L$, $M$, $K$:

$$\int_{M = M^t > 0, N = -N^t \atop M - LL^t > 0} \prod_{j=1}^p \det[M - LL^t]_j^{\lambda_j - \lambda_{j+1}} \det[1 + M + N]_j^{\sigma_j - \sigma_{j+1}} \times \det(M - LL^t)^{-(p+q)/2} \, dM \, dN \, dL = \prod_{k=1}^p \sum_{j=1}^p \sum_{j=1}^p \frac{\Gamma(\sigma_k - (q+k)/2 + 1) \Gamma(\sigma_k - (p-k)/2)}{\Gamma(\sigma_k - p+k)} (3.4)$$

3.3. Remarks. 1) Integrals (3.3)-(3.4) were written to obtain Plancherel measure for Berezin representations of classical groups, see [14], [17].

2) I do not know perfect counterparts of the integrals (3.3)-(3.4) for compact symmetric spaces. Some beta-integrals over classical groups $SO(n)$, $U(n)$, $Sp(n)$ were considered in [16], extensions to over compact symmetric spaces are more-or-less automatic. However, they depend on a smaller number of parameters.

3) On analogs of the $\Gamma$-function. To be definite, consider the space Mat$_{n,n}(\mathbb{C})$. Consider a distribution

$$\varphi(Z) = \prod_{j=1}^n |\det[Z]_j^{\lambda_j} \det[Z]_j^{p_j}|,$$

where $p_j \in \mathbb{Z}$, $\lambda_j \in \mathbb{C}$. This expression is homogeneous in the following sense: for an upper triangular matrix $A$ and a lower triangular matrix $B$,

$$\varphi(BZA) = \prod |a_{jj} b_{jj}|^{\sum_{k<j} \lambda_j (a_{jj} b_{jj})^{\sum_{k<j} p_j}} \varphi(Z).$$

The Fourier transform $\hat{\varphi}$ of $\varphi$ must be homogeneous. For $\lambda_j$ in a general position this remark allows to write $\hat{\varphi}$ up to a constant factor. This factor (it is a product
of Gamma-functions and sines) can be regarded as a matrix analog of Gamma-function. See Stein [33], 1967, Sato, Shintani [28], 1974. I do not know an exhausting text on this topic.

4 Zeta-functions of spaces of lattices

Noncompact symmetric spaces have $p$-adic counterparts, namely Bruhat–Tits buildings (see, e.g., [21], Chapter 10). Since this topic is not inside common knowledge, we will discuss an adelic variant of matrix beta-integrals.

4.1 Space of lattices. A lattice in $\mathbb{Q}^n$ is a subgroup isomorphic to $\mathbb{Z}^n$. Denote by $\text{Lat}_n$ the space of lattices in $\mathbb{Q}^n$. The group $\text{GL}_n(\mathbb{Q})$ acts on the space $\text{Lat}_n$, the stabilizer of the standard lattice $\mathbb{Z}^n$ is $\text{GL}_n(\mathbb{Z})$. Thus $\text{Lat}_n$ is a homogeneous space $\text{Lat}_n \simeq \text{GL}_n(\mathbb{Q})/\text{GL}_n(\mathbb{Z})$.

4.2 Analog of beta-integrals. We consider two coordinate flags

$0 \subset \mathbb{Z} \subset \mathbb{Z}^2 \subset \cdots \subset \mathbb{Z}^n$;

$0 \subset \mathbb{Q} \subset \mathbb{Q}^2 \subset \cdots \subset \mathbb{Q}^n$.

Consider intersections of a lattice $S$ with these flags, i.e.,

$S \cap \mathbb{Z}^k \subset S \cap \mathbb{Q}^k \subset \mathbb{R}^k$.

For a lattice $S \subset \mathbb{R}^k$ we denote by $v_k(S)$ the volume of the quotient $\mathbb{R}^k/S$. The following identity holds [19]:

$$
\sum_{S \in \text{Lat}_n(\mathbb{Q})} \prod_{j=1}^{n} v_k(S \cap \mathbb{Q}^k)^{-\beta_j+\beta_{k+1}} v_k(S \cap \mathbb{Z}^k)^{-\alpha_k+\alpha_{k+1}} = \\
= \prod_{j=1}^{n} \frac{\zeta(-\beta_j+j-1) \zeta(\alpha_j+\beta_j-n+j)}{\zeta(\alpha_j-n+j)}, \quad (4.1)
$$

where $\zeta$ is the $\zeta$-function,

$$
\zeta(s) = \sum_{k=0}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}.
$$

4.3 On Berezin kernels. It seems that holomorphic discrete series representations of semisimple Lie groups have no $p$-adic analogs. However, in [19] there were obtained analogs of the Berezin kernels and of the Berezin–Wallach set. Let us explain this on our minimal language. We define a Berezin kernel on $\text{Lat}_n$ by

$$
K_\alpha(S, T) := \frac{(v_n(R) v_n(S))^{\alpha/2}}{(v_n(R \cap S))^{\alpha}}.
$$
This kernel is positive definite if and only if
\[ \alpha = 0, 1, \ldots, n - 1, \text{ or } \alpha > n - 1. \]

Positive definiteness of the kernel means that there exists a Hilbert space \( H_\alpha \) and a total system of vectors \( \delta_S \in H_\alpha \), where \( S \) ranges in \( \text{Lat}_n \), such that
\[ \langle \delta_S, \delta_T \rangle_{H_\alpha} = K_\alpha(S, T). \]

The group \( GL_n(\mathbb{Q}_p) \) acts in the spaces \( H_\alpha \). Further picture is parallel to a theory of Berezin kernels over \( \mathbb{R} \). The formula (4.1) allows to obtain the Plancherel formula for this representation.

4.4. Remarks. 1) An analog of \( \Gamma \)-function is the Tamagawa zeta-function \[34], see also [13]. It is a sum
\[ \sum \prod_{k=1}^n v_k(S \cap \mathbb{Z}^n)^{-\alpha_k+\alpha_k+1} \]
over sublattices is \( \mathbb{Z}^n \). It can be obtained from (4.1) by a degeneration.

b) Certainly, analogs of (4.1) for symplectic and orthogonal groups must exist. As far as I know they are not yet obtained.

5 Non-radial interpolation

5.1. Rayleigh tables. Again, \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \) or quaternions \( \mathbb{H}, v = \dim \mathbb{K} \). Consider Hermitian matrices of order \( n \) over \( \mathbb{K} \).

Consider eigenvalues of \( [X]_p \) for each \( p \),
\[ \lambda_{p1} \leq \lambda_{p2} \leq \cdots \leq \lambda_{pp}. \]

We get a table \( L \)

\[
\begin{array}{ccccccc}
\lambda_{11} & & & & & \\
\lambda_{21} & \lambda_{22} & & & & \\
\lambda_{31} & \lambda_{32} & \lambda_{33} & & & \\
& \ldots & \ldots & \ldots & \ldots & \\
\lambda_{n1} & \lambda_{n2} & \lambda_{n3} & \ldots & \lambda_{n(n-2)} & \lambda_{n(n-1)} & \lambda_{nn} \\
\end{array}
\]

(5.1)

with the Rayleigh interlacing condition\[13\]
\[ \ldots \leq \lambda_{(j+1)k} \leq \lambda_{jk} \leq \lambda_{(j+1)(k+1)} \leq \ldots \]

This means that numbers \( \lambda_{kl} \) increase in 'north-east' and 'south-east' directions.

Denote by \( \mathcal{R}_n \) the space of all Rayleigh tables \[54\].

\[13\] This statement also is called the Rayleigh-Courant-Fisher theorem.
Point out that for $K = \mathbb{R}$ the number of variables $\lambda_{kl}$ coincides with $\text{dim} \text{Herm}_n(\mathbb{R})$ (but generally there are $2^n(n-1)/2$ matrices $X$ with a given $\mathcal{L}$).

Now consider the image of the Lebesgue measure on $\text{Herm}_n(\mathbb{K})$ under the map $\text{Herm}_n(\mathbb{K}) \to \mathcal{R}_n$. In other words, consider the joint distribution of eigenvalues of all $[X]_p$. It is given by the formula

$$d\rho_\mathcal{B}(\mathcal{L}) = C_n(\mathcal{B}) \prod_{2 \leq j \leq n} \prod_{1 \leq \alpha \leq j-1} |\lambda_{(j-1)\alpha} - \lambda_{j\alpha}|^{\mathcal{B}/2 - 1} \times$$

$$\times \prod_{1 \leq \rho < q \leq n} (\lambda_{nq} - \lambda_{np}) \prod_{1 \leq j \leq n} \prod_{1 \leq \alpha \leq j} d\lambda_{j\alpha}, \quad (5.2)$$

where

$$C_n(\mathcal{B}) = \frac{\pi^{n(n-1)\mathcal{B}/4}}{\Gamma(n(n-1)/2)(\mathcal{B}/2)^n}.$$ 

Notice that for $K = \mathbb{C}$ we get a total cancellation in this expression. History of this formula is not quite clear. It seems that ideologically it is contained in book [7] by Gelfand, Naimark (see evaluation of spherical functions of $\text{GL}(n, \mathbb{C})$). The measure (5.2) is used in integral representation of Jack polynomials in paper [24] by Olshanski and Okounkov. A formal proof is contained in [18], see also [6] and [10].

**5.2. Interpolation.** Now we can assume that $\mathcal{B}$ is an arbitrary complex number and interpolate matrix beta-integrals

$$\int_{\text{Herm}_n(\mathbb{K})} \prod_{k=1}^{n-1} (1 + i[X]_k)^{-\sigma_k + \sigma_{k+1} - \mathcal{B}/2} (1 - i[X]_k)^{-\tau_k + \tau_{k+1} - \mathcal{B}/2} \times$$

$$\times \det(1 + iX)^{-\sigma_n} \det(1 - iX)^{-\tau_n} dX = \prod \frac{\Gamma(\ldots)}{\Gamma(\ldots)}$$

with respect to $\mathcal{B} = \text{dim} \mathbb{K}$:

$$\int_{\mathcal{R}_n} \prod_{j=1}^{n} \prod_{\alpha=1}^{j} \prod_{p=1}^{n} (1 + i\lambda_{j\alpha})^{-\sigma_j + \sigma_{j+1} - \mathcal{B}/2} (1 - i\lambda_{j\alpha})^{-\tau_j + \tau_{j+1} - \mathcal{B}/2} \times$$

$$\times \prod_{p=1}^{n} (1 + i\lambda_{np})^{-\sigma_n} (1 - i\lambda_{np})^{-\tau_n} d\rho_\mathcal{B}(\Lambda) =$$

$$= \pi^{n(n-1)\mathcal{B}/4 + n} \prod_{j=1}^{n} \frac{\Gamma(\sigma_j + \tau_j - 1 - (j-1)\mathcal{B}/2)}{\Gamma(\sigma_j)\Gamma(\tau_j)}.$$ 

Here integration is taken over the space of all Rayleigh tables and the measure $d\rho_\mathcal{B}(\Lambda)$ is given by (5.2).
However, the proof [18] of the latter formula remains to be valid for a wider family of integrals,

\[
\int \prod_{j=1}^{n-1} \prod_{\alpha=1}^{j} (1 + i\lambda_{j\alpha})^{-\sigma_j + \sigma_{j+1} - \theta_{j\alpha}} (1 - i\lambda_{j\alpha})^{-\tau_j + \tau_{j+1} - \theta_{j\alpha}} \times \\
\times \prod_{p=1}^{n} (1 + i\lambda_{np})^{-\sigma_n} (1 - i\lambda_{np})^{-\tau_n} \times \\
\times \prod_{j=1}^{n-1} \prod_{1 \leq \alpha \leq j, 1 \leq \rho \leq j+1} \prod_{1 \leq \alpha < \beta \leq j} (\lambda_{j\alpha} - \lambda_{j(j+1)p})^{\theta_{j\alpha} - 1} \prod_{1 \leq p < q \leq n} (\lambda_{nq} - \lambda_{np}) d\Lambda = \\
= \pi^{n/2} 2^{n-\sum (\sigma_j + \tau_j)} \prod_{1 \leq \alpha \leq j \leq n-1} \Gamma(\theta_{j\alpha}) \cdot \prod_{j=1}^{n} \frac{\Gamma(\sigma_j + \tau_j - 1 - \sum_{n=1}^{j-1} \theta_{j(j-1)\alpha})}{\Gamma(\sigma_j) \Gamma(\tau_j)}. 
\]

Now the parameter \( \nu \) is replaced by \((n - 1)n/2\) parameters \( \theta_{j\alpha} \).

5.3. Remarks. The Gindikin beta-integrals admit an interpolation in the same spirit [18]. For beta-integrals (3.3)–(3.4) over wedges and more general domains an interpolation is unknown.

6 Beta-integrals over flag spaces

6.1. Beta-integrals. Now we consider upper-triangular matrices \( Z = \{z_{ij}\} \) over \( \mathbb{K} \),

\[ z_{ii} = 1, \quad z_{ij} = 0 \quad \text{for} \ i > j. \]

Denote the space of all upper-triangular matrices by \( \text{Triang}_n(\mathbb{K}) \). Recall that the space of upper-triangular matrices is chart on a flag space.

Let \([Z]_{pq}\) be left upper corners of \( Z \) of size \( p \times q \), denote

\[ s_{pq}(Z) := \det([Z]_{pq}[Z]_{pq}^*) \]

The following identity [23] holds

\[
\int_{\text{Triang}_n(\mathbb{K})} \prod_{1 \leq p < q \leq n} s_{pq}(Z)^{-\lambda_{pq}} dZ = \pi^{n(n-1)/4} \prod_{1 \leq p < q \leq n} \frac{\Gamma(\nu_{pq} - \nu/2)}{\Gamma(\nu_{pq})},
\]

where the integration is taken over the space of upper-triangular matrices, and

\[ \nu_{pq} := -\frac{1}{2}(q - p - 1)\nu + \sum_{k,m: p \leq k < q, q \leq m \leq n} \lambda_{mk}. \]

6.2. Projectivity. Consider the map \( Z \mapsto [Z]_{n-1} \) from \( \text{Triang}_n(\mathbb{K}) \) to \( \text{Triang}_{n-1}(\mathbb{K}) \). Consider a measure

\[
\prod_{p=1}^{n-1} s_{pn}(z)^{-\lambda_p} dZ^{(n)}
\]
on Triang\(_n(\mathbb{K})\). Assume
\[
\lambda_p + \lambda_{p+1} + \cdots + \lambda_{n-1} > \frac{1}{2}(n-p)d
\]
for all \(p\). Then the pushforward of this measure under the forgetting map is
\[
\pi^{\frac{(n-1)d}{2}} \prod_{1 \leq p \leq n-1} \frac{\Gamma(\lambda_p + \cdots + \lambda_n - (n-p)d/2)}{\Gamma(\lambda_p + \cdots + \lambda_n - (n-p+1)d/2)} \times \prod_{p=1}^{n-2} s_{p(n-1)}(\lfloor Z \rfloor_{n-1})^{-\lambda_p} \, d(\lfloor Z \rfloor_{n-1})
\]

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