\textbf{Abstract.} Let \( \{T_t\}_{t>0} \) be a strongly continuous semigroup of positive contractions on \( L_p(X,\mu) \) with \( 1 < p < \infty \). Let \( E \) be a UMD Banach lattice of measurable functions on another measure space \((\Omega,\nu)\). For \( f \in L_p(X;E) \) define
\[
M(f)(x,\omega) = \sup_{t>0} \frac{1}{t} \int_0^t |T_s(f(\cdot,\omega))(x)| ds, \quad (x,\omega) \in X \times \Omega.
\]
Then the following maximal ergodic inequality holds
\[
\|M(f)\|_{L_p(X;E)} \lesssim \|f\|_{L_p(X;E)}, \quad f \in L_p(X;E).
\]
If the semigroup \( \{T_t\}_{t>0} \) is additionally assumed to be analytic, then \( \{T_t\}_{t>0} \) extends to an analytic semigroup on \( L_p(X;E) \) and \( M(f) \) in the above inequality can be replaced by the following sectorial maximal function
\[
\mathcal{T}_\theta(f)(x,\omega) = \sup_{|\arg(z)| < \theta} |T_z(f(\cdot,\omega))(x)|
\]
for some \( \theta > 0 \).

Under the latter analyticity assumption and if \( E \) is a complex interpolation space between a Hilbert space and a UMD Banach space, then \( \{T_t\}_{t>0} \) extends to a analytic semigroup on \( L_p(X;E) \) and its negative generator has a bounded \( H^\infty(\Sigma_a) \) calculus for some \( \sigma < \pi/2 \).

1. Introduction

Let \((X,F,\mu)\) be a measure space and \( \{T_t\}_{t>0} \) a strongly continuous semigroup of contractions on \( L_p(X) \) for every \( 1 \leq p \leq \infty \). Consider the ergodic averages of \( \{T_t\}_{t>0} \):
\[
A_t(T) = \frac{1}{t} \int_0^t T_u du
\]
and the associated maximal operator:
\[
M(T)(f) = \sup_{t>0} |A_t(T)(f)|.
\]
The classical Dunford-Schwartz maximal ergodic inequality asserts that the maximal operator \( M(T) \) is bounded on \( L_p(X) \) for \( 1 < p \leq \infty \), and from \( L_1(X) \) to \( L_{1,\infty}(X) \).

Very recently, Charpentier and Deléaval \cite{5} proved the following vector-valued version of Dunford-Schwartz’s inequality: For any \( 1 < q < p < \infty \) and any finite sequence \( \{f_k\}_{k \geq 1} \) in \( L_p(X) \)
\[
\left\| \left( \sum_{k} (M(T)(f_k))^q \right)^{1/q} \right\|_p \lesssim \left\| \left( \sum_{k} |f_k|^q \right)^{1/q} \right\|_p.
\]
Here and in the sequel the symbol \( \lesssim \) means an inequality up to a constant depending only on the indices \( p,q \), the spaces \( E \), etc. but never on the functions in consideration.

They then asked whether \( (1) \) remains valid for \( 1 < p < q \). We answer this question by the affirmative. The proof is simply based on the transference principle that allows us to reduce \( (1) \) to the special case where \( \{T_t\}_{t>0} \) is the translation group of \( \mathbb{R} \). In the latter case, \( (1) \) is exactly Fefferman-Stein’s celebrated vector-valued maximal inequality \cite{3}. In fact, we will show more. To
state our result we need recall some definitions. An operator $T$ on $L_p(X)$ is called regular (more precisely, contractively regular) if
\[
\left\| \sup_k |T(f_k)| \right\|_p \leq \left\| \sup_k |f_k| \right\|_p
\]
for all finite sequences \( \{f_k\}_{k \geq 1} \) in $L_p(X)$. Such an operator extends to the vector-valued case. Namely, $T$ extends to a contraction on $L_p(X; E)$ for any Banach space $E$, where $L_p(X; E)$ stands for the $L_p$-space of strongly measurable functions from $X$ to $E$. For notational simplicity, this extension will be denoted still by $T$.

Obviously, positive contractions are regular. On the other hand, it is well known (and easy to check) that if $T$ is a contraction on $L_1(X)$ or $L_\infty(X)$, then $T$ is regular. Thus by interpolation, $T$ is regular on $L_p(X)$ if $T$ is a contraction on $L_p(X)$ for all $1 \leq p \leq \infty$.

We will use UMD Banach lattices. We refer to [3] for UMD spaces and [19] for Banach lattices. Recall that any $L_p$-space with $1 < p < \infty$ is a UMD space. Let $E$ be a Banach lattice of measurable functions on a measure space $(\Omega, \nu)$. The functions in $L_p(X; E)$ are viewed as functions of two variables $(x, \omega) \in X \times \Omega$.

Let $\{T_t\}_{t>0}$ be a strongly continuous semigroup of regular contractions on $L_p(X)$. So $\{T_t\}_{t>0}$ extends to a semigroup of contractions on $L_p(X; E)$ too. Define
\[
\mathcal{M}(f)(x, \omega) = M(T)(f)(x, \omega) = M(T)(f(\cdot, \omega))(x), \quad (x, \omega) \in X \times \Omega.
\]

**Theorem 1.** Let $1 < p < \infty$ and $\{T_t\}_{t>0}$ be a strongly continuous semigroup of regular contractions on $L_p(X)$. Then for any UMD Banach lattice $E$
\[
(2) \quad \left\| \mathcal{M}(f) \right\|_{L_p(X; E)} \lesssim \left\| f \right\|_{L_p(X; E)}, \quad f \in L_p(X; E).
\]

Recall that a strongly continuous semigroup $\{T_t\}_{t>0}$ on a Banach space $E$ is called analytic if there exists $\theta > 0$ such that $\{T_t\}_{t>0}$ extends to a bounded analytic function from the sector $\Sigma_\theta$ to $\mathcal{B}(E)$, where $\Sigma_\theta = \{ z \in \mathbb{C} : |\text{arg}(z)| < \theta \}$ and $\mathcal{B}(E)$ denotes the space of all bounded linear operators on $E$.

If the semigroup $\{T_t\}_{t>0}$ in Theorem 1 is further assumed to be analytic, the maximal function on the ergodic averages there can be replaced by the maximal function directly taken on the $T_t$’s. Moreover, we can also estimate the following sectorial maximal function for some $\theta > 0$
\[
\mathcal{T}_\theta(f)(x, \omega) = \sup_{z \in \Sigma_\theta} |T_z(f(\cdot, \omega))(x)|, \quad (x, \omega) \in X \times \Omega.
\]

**Theorem 2.** Let $1 < p < \infty$ and $\{T_t\}_{t>0}$ be an analytic semigroup of regular contractions on $L_p(X)$. Let $E$ be a UMD lattice on $(\Omega, \nu)$. Then there exists $\theta > 0$ such that $\{T_t\}_{t>0}$ extends to a bounded analytic function on $\Sigma_\theta$ with values in $\mathcal{B}(L_p(X; E))$ and
\[
(3) \quad \left\| \mathcal{T}_\theta(f) \right\|_{L_p(X; E)} \lesssim \left\| f \right\|_{L_p(X; E)}, \quad f \in L_p(X; E).
\]

This theorem extends [18] to the vector-valued setting. Like in [18], we can also prove its discrete analogue. Note that the ancestor of the maximal inequality [5] are Stein’s maximal ergodic inequality for symmetric diffusion semigroups, i.e., Markovian semigroups of positive contractions on $L_p(X)$ for every $1 \leq p \leq \infty$ with $T_t$ selfadjoint on $L_2(X)$ (see [23] Chapter III). Cowling [7] proved the sectorial maximal inequality for these semigroups.

**Corollary 3.** Let $E$ be a UMD lattice and $\{T_t\}_{t>0}$ be a semigroup of contractions on $L_p(X)$ for all $1 \leq p \leq \infty$. Then (2) holds for any $1 < p < \infty$.

If in addition $\{T_t\}_{t>0}$ is analytic on $L_p(X)$ for some $1 < p < \infty$, then (3) holds for any $1 < p < \infty$.

The first part of the corollary follows immediately from Theorem 1. On the other hand, the second is a consequence of Theorem 2 since it is well known that if $\{T_t\}_{t>0}$ is analytic on $L_p(X)$ for one $1 < p < \infty$, so it is for all $1 < p < \infty$. The latter fact is easily proved by complex interpolation (see [23] Chapter 3; see also the proof of Lemma 5 below). The most important case of the corollary is where every $T_t$ is a selfadjoint operator on $L_2(X)$. Then $\{T_t\}_{t>0}$ is analytic on $L_2(X)$.

The proof of Theorem 2 is based on the following result on $H^\infty$ functional calculus. We refer to [13, 16] for $H^\infty$ calculus and to [1] for complex interpolation.
Theorem 4. Let \((E_0, E_1)\) be an interpolation pair of Banach spaces and \(0 < \eta < 1\). Let \(E = (E_0, E_1)_\eta\) be the associated complex interpolation space. Assume that \(E_0\) is isomorphic to a Hilbert space and \(E_1\) is a UMD space.

(i) The extension of \(\{T_t\}_{t > 0}\) to \(L_p(X; E)\) is analytic.
(ii) \(A\) has a bounded \(H^\infty(\Sigma_\sigma)\) functional calculus for some \(\sigma < \pi/2\), where \(-A\) is the generator of \(\{T_t\}_{t > 0}\) on \(L_p(X; E)\).

In particular, if \(E\) is a UMD Banach lattice, then both assertions hold.

We will prove Theorem 1 in section 2, Theorems 2 and 3 in section 3, and conclude the paper with some further results and open problems.

2. Proof of Theorem 1

We will prove Theorem 1 in this section. This proof is a simple application of the transference principle. The argument consists in transferring ergodic inequalities like (2) to the special case where \(\{T_t\}_{t > 0}\) is the translation group of \(\mathbb{R}\). This powerful technique was invented by Calderón [4] and largely developed by Coifman and Weiss [6]. Since then it is commonly called transference principle and has been widely applied to many different situations.

Note that if \(\{T_t\}_t\) is the translation group of \(\mathbb{R}\), \(M(T)\) becomes the one-sided Hardy-Littlewood maximal function. In the latter case, \(\{T_t\}_t\) is Bourgain’s vector-valued maximal inequality [2] which extends Fefferman-Stein’s work. In fact, in this case, the lattice \(E\) does not need to be a UMD space. Following [11], \(E\) is said to have the Hardy-Littlewood property if inequality (2) holds for \(\{T_t\}_t\) equal to the translation group of \(\mathbb{R}\). Thus the transference argument presented below will show that Theorem 1 remains valid if \(E\) has the Hardy-Littlewood property.

To use transference we first need to dilate our semigroup to a group of isometries. Fendler’s principle and has been widely applied to many different situations.

We are now ready to do our transference argument. It suffices to prove (2) for \(M_a(T)(f)\) in place of \(M_a(T)(f)\) for any \(a > 0\), where

\[
M_a(T)(f)(x, \omega) = \sup_{0 < t < a} |A_t(T)(f)(x, \omega)|.
\]

Let \(A(T)(f) = \{A_t(T)(f)\}_{t > 0}\). \(A(T)(f)\) is viewed as a function of three variables \((x, \omega, t)\) on \(X \times \Omega \times (0, \infty)\). Then we can write

\[
\|M_a(T)(f)\|_{L_p(X; E)} = \|A(T)(f)\|_{L_p(X; E(L(\infty(0, a))))}.
\]

Here, given a Banach space \(B\), \(E(B)\) denotes the space of all strongly measurable functions from \(\Omega\) to \(B\) such that \(\|f(\cdot)\|_B \in E\). Its norm is defined by \(\|f(\cdot)\|_B\|_E\).

By regularity, \(P\) and \(D\) and \(S_t\) all extend to the vector-valued case. By (1), we have

\(A(T) = PA(S)D\).

So

\[
\|A(T)(f)\|_{L_p(X; E(L(\infty(0, a))))} = \|P(A(S)(D(f)))\|_{L_p(X; E(L(\infty(0, a))))}.
\]

However, the regularity of \(P\) and \(D\) implies

\[
\|P(A(S)(D(f)))\|_{L_p(X; E(L(\infty(0, a))))} \leq \|A(S)(D(f))\|_{L_p(X; E(L(\infty(0, a))))} \leq \|D(f)\|_{L_p(X; E(L(\infty(0, a)))).
\]

So we are reduced to proving (2) for \(S_t\) in place of \(T_t\). Thus we can assume that \(\{T_t\}_t\) itself extends to a group of regular isometries on \(L_p(X)\) in the rest of the proof. We will simply write \(A_t\) for...
Thus for any $b > 0$ we then deduce
\[ \left\| A(f) \right\|_{L_p(X; E(L_\infty(0, a)))} \leq \frac{1}{b} \int_0^b \left\| A(T_s(f)) \right\|_{L_p(X; E(L_\infty(0, a)))} ds. \]

Given $(x, \omega) \in X \times \Omega$ define a function $g(\cdot, x, \omega)$ on $\mathbb{R}$ by $g(s, x, \omega) = \mathbb{1}_{(0, a + b)}(s)T_s(f)(x, \omega)$. Then
\[ A_t(T_s(f))(x, \omega) = \frac{1}{t} \int_0^t T_{s+u}(f)(x, \omega) du = \frac{1}{t} \int_0^t g(s + u, x, \omega) du, \quad 0 < t < a, \quad 0 < s < b. \]

Therefore,
\[ M_a(T)(T_s(f))(x, \omega) \leq M^+(g(\cdot, x, \omega))(s) \overset{\text{def}}{=} M^+(g)(s, x, \omega), \]
where $M^+$ denotes the usual one-sided Hardy-Littlewood maximal function on $\mathbb{R}$:
\[ M^+(h)(s) = \sup_{t > 0} \frac{1}{t} \int_0^t |h(s + u)| du, \quad s \in \mathbb{R}. \]

Consequently, by [2] (see also [22])
\[
\int_0^b \left\| A(T_s(f))(x, \omega) \right\|_{L_p(X; E(L_\infty(0, a)))}^p ds = \int_0^b \left\| M_a(T)(T_s(f))(x, \omega) \right\|_{L_p(X; E(L_\infty(0, a)))}^p ds \\
\leq \int_{\mathbb{R}} \left\| M^+(g)(s, x, \omega) \right\|_{L_p(X; E(L_\infty(0, a)))}^p ds \\
\leq \int_{\mathbb{R}} \left\| g(s, x, \omega) \right\|_{L_p(X; E)}^p ds \leq \int_0^{a + b} \left\| T_s(f)(x, \omega) \right\|_{L_p(X; E)}^p ds.
\]

Taking integral over $X$ and using the regularity of $T_s$, we then get
\[
\int_0^b \left\| M_a(T)(T_s(f))(x, \omega) \right\|_{E}^p d\mu(x) ds \leq \int_0^{a + b} \int_X \left\| T_s(f)(x, \omega) \right\|_{E}^p d\mu(x) ds \leq (a + b) \left\| f \right\|_{L_p(X; E)}^p.
\]

Combining the preceding inequalities, we finally obtain
\[
\left\| M_a(T)(f) \right\|_{L_p(X; E)}^p \lesssim \frac{a + b}{b} \left\| f \right\|_{L_p(X; E)}^p.
\]
Letting $b \to \infty$ yields the desired inequality. The theorem is thus proved. \( \square \)

3. $H^\infty$ Functional Calculus

We will prove Theorems 2 and 3 in this section. Throughout the section we will fix an analytic semigroup $T = \{T_t\}_{t > 0}$ of regular contractions on $L_p(X)$ with $1 < p < \infty$. So $T : \Sigma_\theta \to B(L_p(X))$ is an analytic function for some positive angle $\theta$ and
\[
\sup_{z \in \Sigma_\theta} \left\| T_z \right\|_{B(L_p(X))} \leq C < \infty.
\]

Let $H^\infty(\Sigma_\sigma)$ be the Banach space of all bounded analytic functions on $\Sigma_\sigma$ equipped with the uniform norm $\left\| \cdot \right\|_\infty$. Recall that a sectorial operator $A$ of type $\theta$ on a Banach space $E$ is said to have a bounded $H^\infty(\Sigma_\sigma)$ calculus with $\sigma > \theta$ if there exists a constant $C$ such that for any $\varphi \in H^\infty(\Sigma_\sigma)$, $\varphi(A)$ is a well defined (unique) operator on $E$ and
\[
\left\| \varphi(A) \right\|_{B(E)} \leq C \left\| \varphi \right\|_\infty.
\]
The proof of Theorem 3 requires two lemmas.
Lemma 5. Let \((E_0, E_1)\) be an interpolation pair of Banach spaces with \(E_0\) isomorphic to a Hilbert space. Let \(E = (E_0, E_1)_\eta\) with \(0 < \eta < 1\). Then the extension of \(\{T_t\}_{t > 0}\) to \(L_p(X; E)\) is analytic. In particular, if \(E\) is a Banach lattice with nontrivial convexity and concavity, then the conclusion holds.

Proof. First observe that the assertion is obvious for \(E = \ell_p\). In this case \(\ell_p\) gives

\[
\sup_{z \in \Sigma_\theta} \left\| T_z \right\|_{B(L_p(X; \ell_p))} \leq C.
\]

We will then show the assertion for \(E = \ell_q\) with any \(1 < q < \infty\) by interpolation. Assume that \(p < q\) for the moment. Then \(\ell_q = (\ell_\infty, \ell_p)_{p/q}\). Interpolating the above inequality with the following

\[
\sup_{t > 0} \left\| T_t \right\|_{B(L_p(X; \ell_\infty))} \leq 1,
\]

we deduce

\[
\sup_{z \in \Sigma_{p/q}} \left\| T_z \right\|_{B(L_p(X; \ell_q))} \leq C^{p/q}.
\]

Thus \(T : \Sigma_{p/q} \to B(L_p(X; \ell_q))\) is a bounded analytic function. As this interpolation argument is used several times in the sequel, we give the details for the reader’s convenience. The change of variables \(z = e^{i\kappa}\) maps the sector \(\Sigma_{\theta}\) to the vertical trip \(S_{\theta} = \{\zeta = u + iv : |u| < \theta\}\). Now fix a point \(s_0 = u_0 + iv_0 \in S_{p/q} \cap \Re\) with \(u_0 \neq 0\). Choose \(u_1\) such that \(u_0 = u_1p/q\) and \(|u_1| < \theta\). Let \(f\) be in the open unit ball of \(L_p(X; \ell_q)\). Since

\[
L_p(X; \ell_q) = (L_p(X; \ell_\infty), L_p(X; \ell_p))_{p/q},
\]

there exists a continuous function \(\varphi\) on the closed strip \(\{\zeta = u + iv : 0 \leq u \leq 1\}\), analytic in the interior such that \(\varphi(p/q) = f\) and

\[
\sup_{v \in \Re} \left\| \varphi(iv) \right\|_{L_p(X; \ell_\infty)} \leq 1, \quad \sup_{v \in \Re} \left\| \varphi(1 + iv) \right\|_{L_p(X; \ell_p)} \leq 1.
\]

Now define another analytic function \(\psi\) by

\[
\psi(\zeta) = T_{e^{i\kappa} - u_0}\varphi(\zeta), \quad \zeta = u + iv, \ 0 \leq u \leq 1.
\]

Then

\[
\sup_{v \in \Re} \left\| \psi(iv) \right\|_{L_p(X; \ell_\infty)} \leq 1, \quad \sup_{v \in \Re} \left\| \psi(1 + iv) \right\|_{L_p(X; \ell_p)} \leq C.
\]

Since \(\psi(p/q) = T_{e^{i\kappa}0}(f)\), we then deduce

\[
T_{e^{i\kappa}0}(f) \in L_p(X; \ell_q) \quad \text{and} \quad \left\| T_{e^{i\kappa}0}(f) \right\|_{L_p(X; \ell_q)} \leq C^{p/q}.
\]

Taking the supremum over \(f\) in the unit ball of \(L_p(X; \ell_q)\) yields

\[
\left\| T_{e^{i\kappa}0} \right\|_{B(L_p(X; \ell_q))} \leq C^{p/q},
\]

which is the desired inequality.

The same argument applies to the case \(q < p\) with \(\ell_\infty\) replaced by \(\ell_1\).

In particular, our assertion holds for \(E = \ell_2\), so for any Hilbert space \(H\) too. Now if \(E = (E_0, E_1)_\eta\) with \(E_0\) isomorphic to a Hilbert space, then the above interpolation argument yields the analyticity of \(\{T_t\}_{t > 0}\) on \(L_p(X; E)\).

The last part of the lemma follows from Pisier’s theorem [20] which asserts that every Banach lattice \(E\) with nontrivial convexity and concavity is isomorphic to a complex interpolation space between \(L_2\) and another Banach lattice.

The following lemma might be known to experts, although it does not appear explicitly in literature. The proof presented here is explained to me by Christian Le Merdy. It is slightly simpler than my original one that uses Kalton-Weis’ square function characterization of \(H^\infty\) calculus (see [14] Theorem 12.2).

Lemma 6. Let \((E_0, E_1)\) be an interpolation pair of Banach spaces and \(E = (E_0, E_1)_\eta\) with \(0 < \eta < 1\). Assume that a sectorial operator \(A\) has a bounded \(H^\infty(\Sigma_j)\) calculus on \(E_j\) for \(j = 0, 1\). Then \(A\) has a bounded \(H^\infty(\Sigma_\sigma)\) calculus on \(E\) for any \(\sigma > \sigma_\eta = (1 - \eta)\sigma_0 + \eta\sigma_1\).
Proof. Recall that $A$ has bounded imaginary powers of angle $\theta$ on a Banach space $F$ if
$$\|A^{is}\|_{B(F)} \lesssim e^{\theta|s|}, \quad \forall s \in \mathbb{R}.$$ 

It is clear that if $A$ has a bounded $H^\infty(\Sigma_\sigma)$ calculus, then $A$ has bounded imaginary powers of angle $\sigma$. Conversely, it is proved in [8] that if $A$ has bounded imaginary powers of angle $\theta$ as well as a bounded $H^\infty(\Sigma_{\sigma'})$ calculus for some $\sigma' > \theta$, then $A$ has a bounded $H^\infty(\Sigma_\sigma)$ calculus for any $\sigma > \theta$.

Now let $\sigma' = \max(\sigma_0, \sigma_1)$. Then for any $\varphi \in H^\infty(\Sigma_{\sigma'})$ we have
$$\|\varphi(A)\|_{B(E_{\sigma'})} \lesssim \|\varphi\|_{\infty}, \quad j = 0, 1.$$ 

So by interpolation
$$\|\varphi(A)\|_{B(E)} \lesssim \|\varphi\|_{\infty}.$$ 

This means that $A$ has a bounded $H^\infty(\Sigma_{\sigma'})$ calculus on $E$. On the other hand, it is well known and easy to check that the boundedness of imaginary powers is stable under complex interpolation. Namely, if $A$ has bounded imaginary powers of angle $\theta_j$ on $E_j$ for $j = 0, 1$, then $A$ has bounded imaginary powers of angle $\theta = (1 - \eta)\theta_0 + \eta\theta_1$ on $E$. Thus the assertion follows.

Proof of Theorem 4. Part (i) is already contained in Lemma 5. It remains to prove (ii). To this end, first note that if $E$ is a UMD space, then $A$ has a bounded $H^\infty(\Sigma_{\sigma_1})$ calculus for any $\sigma_1 > \pi/2$ thanks to [12] (see also [13] Corollary 10.15). On the other hand, it is known that $A$ has a bounded $H^\infty(\Sigma_{\sigma_0})$ calculus for some $\sigma_0 < \pi/2$ on $L_p(X)$, i.e. in the scalar-valued case with $E = \mathbb{C}$ (see [13] Proposition 2.8). Thus $A$ has a bounded $H^\infty(\Sigma_{\sigma_0})$ calculus on $L_p(X; \ell_p)$ too. Now choose appropriate $q \in (1, \infty)$ and $\eta \in (0, 1)$ such that $1/2 = (1 - \eta)/p + \eta/q$. Then by Lemma 3 we deduce that $A$ has a bounded $H^\infty(\Sigma_{\sigma})$ calculus for some $\sigma < \pi/2$ on $L_p(X; \ell_2)$, so on $L_p(X; H)$ for any Hilbert space $H$ too. Finally, a second application of Lemma 3 finishes the proof of (ii).

If $E$ is a UMD Banach lattice on $\Omega$, then by [22] there exists another UMD lattice $F$ such that $E = (L_2(\Omega), F)_\eta$ with $0 < \eta < 1$, so $E$ satisfies the assumption of the theorem.

Proof of Theorem 2. Using Theorems 1 and 4 we can easily adapt the proof of [7] Theorem 7 to the present setting. Let us give the main lines for the reader’s convenience. Given $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ define
$$\Phi_\varphi(\lambda) = \exp\left(-e^{i\varphi}\lambda\right) - \int_0^1 e^{-t\lambda} dt, \quad \lambda > 0.$$ 

Let $\Psi_\varphi = \Phi_\varphi \circ \exp$. The Fourier transform of $\Psi_\varphi$ satisfies the following estimate (see [7]):
$$|\hat{\Psi}_\varphi(u)| \lesssim e^{(\pi|\varphi - \frac{\pi}{2})|u|}, \quad u \in \mathbb{R}.$$ 

By the Fourier inversion formula,
$$\Phi_\varphi(\lambda) = \frac{1}{2\pi} \int_\mathbb{R} \hat{\Psi}_\varphi(u)\lambda^{iu} du.$$ 

Now let $z = se^{i\varphi} \in \Sigma_\theta$. We then have
$$T_z = e^{-zA} = \Phi_\varphi(sA) + \int_0^1 e^{-tsA} dt$$
$$= \frac{1}{2\pi} \int_\mathbb{R} \hat{\Psi}_\varphi(u)s^{iu}A^{iu} du + \frac{1}{s} \int_0^s T_t dt.$$ 

It thus follows that
$$T_\theta(f) \lesssim \int_\mathbb{R} e^{(\theta - \frac{\pi}{2})|u||A^{iu}(f)|} du + \mathcal{M}(f).$$ 

The second term on the right hand side is estimated by Theorem 1. For the first we use Theorem 4 to conclude that $A$ has bounded imaginary powers of angle $\sigma < \pi/2$:
$$\|A^{iu}\|_{B(L_p(X, E))} \lesssim e^{\sigma|u|}.$$ 

Therefore, if $\theta < \frac{\pi}{2} - \sigma$, we get the desired estimate for the first term too.
4. More remarks and problems

The following individual convergence theorem is an easy consequence of Theorem 2.

**Proposition 7.** Keep the assumption and notation of Theorem 2. Then for any \( f \in L_p(X; E) \)
\[
\lim_{\Sigma_\theta \ni z \to 0} T_z(f) = f \quad \text{and} \quad \lim_{\Sigma_\theta \ni z \to \infty} T_z(f) = P(f) \quad \text{a.e. on} \ X \times \Omega,
\]
where \( P \) denotes the projection onto the fixed point subspace of the semigroup \( \{T_t\}_{t>0} \).

**Proof.** Let \( g \in L_p(X) \) and \( s > 0 \). Let \( \gamma \) be a circle of center \( s \) and radius \( r \) with \( r < s \sin \theta \). For any \( z \) inside \( \gamma \) by the Cauchy formula we have
\[
T_z(g) = \frac{1}{2\pi i} \int_{\gamma} T_z(g)d\zeta.
\]
Thus
\[
T_z(g) - T_s(g) = \frac{z-s}{2\pi i} \int_{\gamma} T_z(g)d\zeta/\zeta-z.
\]
Consequently,
\[
|T_z(g) - T_s(g)| \leq \frac{|z-s|}{r\pi} \int_{\gamma} |T_z(g)| d|\zeta|, \quad |z-s| < \frac{r}{2}.
\]
Note that the last integral is a function in \( L_\theta \). Therefore
\[
\lim_{z \to s} T_z(g) = T_s(g) \quad \text{a.e. on} \ X.
\]
It then follows that \( \lim_{z \to 0} T_z(T_s(g)) = T_s(g) \) a.e..

Let \( F \) be the linear span of \( \{T_s(g) \oplus h : g \in L_p(X), h \in E, s > 0\} \). Then \( F \) is dense in \( L_p(X; E) \) and by what is proved above \( \lim_{z \to 0} T_z(f) = f \) a.e. on \( X \times \Omega \) for any \( f \in F \). The assertion then follows from (3). Indeed, taking a sequence \( (f_n) \) in \( F \) such that \( f_n \to f \) in \( L_p(X; E) \), we have
\[
\lim sup_{\Sigma_\theta \ni z \to 0} |T_z(f) - f| \leq \mathcal{T}_\theta(f - f_n) + |f - f_n|.
\]

Thus by (3),
\[
\|\lim sup_{\Sigma_\theta \ni z \to 0} |T_z(f) - f|\|_{L_p(X; E)} \lesssim \|f - f_n\|_{L_p(X; E)}.
\]
Letting \( n \to \infty \), we deduce that \( \lim sup_{\Sigma_\theta \ni z \to 0} |T_z(f) - f| = 0 \) a.e..

We turn to the second limit. Let \( -A \) be the generator of \( \{T_t\}_{t>0} \). Then \( L_p(X; E) \) is decomposed into the direct sum of the null and range spaces of \( A \): \( L_p(X; E) = \mathcal{N}(A) \oplus \mathcal{R}(A) \). Moreover, \( \mathcal{N}(A) \) is the fixed point subspace of the semigroup \( \{T_t\}_{t>0} \). Thus it suffices to prove that for any \( f \in \mathcal{R}(A) \)
\[
\lim_{\Sigma_\theta \ni z \to \infty} T_z(f) = 0 \quad \text{a.e. on} \ X \times \Omega.
\]
Using (3) as in the previous part of the proof, we need only to do this for \( f \) in a dense subset of \( \mathcal{R}(A) \). It is well known that \( \{T_{t+s}(g) - T_s(g) : s > 0, t > 0, g \in L_p(X; E)\} \) is such a subset. So we are reduced to proving the above limit for \( f = T_{t+s}(g) - T_s(g) \). To this end, we will use the integral representation of \( T_z \). Let \( 0 < \sigma < \frac{\pi}{2} - \theta \) and \( \delta > 0 \) be sufficiently small. Let \( D(0, \delta) \) be the disc of center \( \delta \) and radius \( \delta \). Let \( \Gamma_\delta \) be the closed path consisting of the part of the boundary of \( \Sigma_\theta \) outside of \( D(0, \delta) \) and the part of the boundary of \( D(0, \delta) \) outside of \( \Sigma_\theta \). Then
\[
T_z = \frac{1}{2\pi i} \int_{\Gamma_\delta} e^{-z\lambda} R(\lambda, A)d\lambda, \quad z \in \Sigma_\theta,
\]
where \( R(\lambda, A) = (\lambda - A)^{-1} \). Thus for \( f = T_{t+s}(g) - T_s(g) \) as above, we have
\[
T_z(f) = \frac{1}{2\pi i} \int_{\Gamma_\delta} (e^{-(z+t+s)\lambda} - e^{-(z+s)\lambda}) R(\lambda, A)(g)d\lambda.
\]
Let \( \Gamma \) be the boundary of \( \Sigma_\theta \). By the sectoriality of \( A \), \( \|\lambda R(\lambda, A)\| \) is bounded on \( \Gamma \). So the above integral is absolutely convergent on \( \Gamma \). Thus letting \( \delta \to 0 \) we deduce
\[
T_z(f) = \frac{1}{2\pi i} \int_{\Gamma} (e^{-(z+t+s)\lambda} - e^{-(z+s)\lambda}) R(\lambda, A)(g)d\lambda.
\]
Hence for $1 < q < \infty$ with conjugate index $q'$ we have
\[ |T_2(f)| \leq \frac{t}{2\pi} \int_{\Gamma} |e^{-\lambda t} \cdot |R(\lambda, A)(g)||d\lambda| \]
\[ \leq \frac{t}{2\pi} \left( \int_{\Gamma} |e^{-q\lambda t}||d\lambda| \right)^{1/q'} \left( \int_{\Gamma} |\lambda R(\lambda, A)(g)|^q |d\lambda| \right)^{1/q} \]
\[ \lesssim \frac{1}{|z|^{1/q'}} \left( \int_{\Gamma} |e^{-q(t+s)\lambda}||\lambda R(\lambda, A)(g)|^q |d\lambda| \right)^{1/q} . \]

Since the lattice $L_p(X; E)$ has the UMD property, it is $q$-convex for some $1 < q < \infty$. Then for such a choice of $q$, the last integral represents a function in $L_p(X; E)$. Therefore,
\[ \lim_{\Sigma \geq 2z \to \infty} T_2(f) = 0 \text{ a.e. on } X \times \Omega, \]
as desired. \qed

Similarly, we have the following individual ergodic theorem corresponding to Theorem 11.

**Proposition 8.** Under the assumption of Theorem 7 we have
\[ \lim_{t \to 0} A_t(T)(f) = f \quad \text{and} \quad \lim_{t \to \infty} A_t(T)(f) = P(f) \text{ a.e. on } X \times \Omega \]
for any $f \in L_p(X; E)$.

**Proof.** Without loss of generality, we can assume that $T_t$ is positive for all $t$. Let $A_t = A_t(T)$. By virtue of the maximal inequality (2), it suffices to show the first limit for $f$ in the domain of $A$, for instance, for $f = T_s(g)$ with $s > 0$ and $g \in L_p(X; E)$. We can further assume that $f \geq 0$. Then
\[ A_t(f) - f = -\frac{1}{t} \int_0^t (t - u)T_u(A(f))du. \]
Thus
\[ |A_t(f) - f| \leq \int_0^t T_u(A(f))du \leq t A(f). \]

Since $A(f) \in L_p(X; E)$, we deduce that $\lim_{t \to 0} A_t(T)(f) = f$ a.e.

As in the proof of the previous proposition, we need only to show the second limit for $f = T_s(g) - g$ with $s > 0$ and $g \in L_p(X; E)$. For such an $f$ we have
\[ A_t(f) = \frac{1}{t} \int_t^{t+s} T_u(g)du - \frac{1}{t} \int_0^s T_u(g)du. \]
Hence for $t$ sufficiently large
\[ |A_t(f)| \leq 2 \int_t^{t+s} \frac{1}{u} |T_u(g)|du + \frac{s}{t} A(g). \]

The second term on the right hand side tends to 0 as $t \to \infty$. To treat the first one, we use again the $q$-convexity of $L_p(X; E)$. Choose $\alpha \in (0, 1)$ such that $\alpha q > 1$. Let $\beta = 1 - \alpha$. Then
\[ \int_t^{t+s} \frac{1}{u} |T_u(g)|du \leq \left( \int_t^{t+s} \frac{1}{u^\alpha} du \right)^{1/q'} \left( \int_t^{t+s} \frac{1}{u^\alpha} |T_u(g)|^q du \right)^{1/q} \]
\[ \leq \frac{1}{(1 - \beta q')^{1/q'}} \left( (t + s)^{1 - \beta q'} - t ^{1 - \beta q'} \right)^{1/q'} \left( \int_1^\infty \frac{1}{u^{\alpha q'}} |T_u(g)|^q du \right)^{1/q} . \]

The $q$-convexity of $L_p(X; E)$ implies that the last integral represents a function belonging to $L_p(X; E)$. On the other hand, the factor in the front of this integral tends to 0 as $t \to 0$. We then deduce that $\lim_{t \to 0} A_t(T)(f) = 0$ a.e. on $X \times \Omega$. \qed

**Proposition 9.** Under the assumption of Theorem 8, we have the following square function inequality
\[ \left\| \left( \int_0^\infty t^2 |\frac{\partial}{\partial t} T_t(f)|^2 dt \right)^{1/2} \right\|_{L_p(X; E)} \lesssim \| f \|_{L_p(X; E)}, \quad f \in L_p(X; E). \]
Proof. It is well known that a bounded $H^\infty(\Sigma_\sigma)$ calculus with $\sigma < \pi/2$ implies square function inequalities like \[6\]. Let us precise this. Let $\sigma'$ be another angle such that $\sigma < \sigma' < \pi/2$. Let $\phi$ be a bounded analytic function on $\Sigma_{\sigma'}$. Then by $H^\infty$ calculus $\phi(tA)$ is a well-defined bounded operator on $L_p(X; E)$ for every $t > 0$ and we have the following square function inequality
\[
\left\| \left( \int_0^\infty |\phi(tA)(f)|^2 \frac{dt}{T} \right)^{1/2} \right\|_{L_p(X; E)} \lesssim \|f\|_{L_p(X; E)}, \quad f \in L_p(X; E).
\]
This fundamental result is proved in \[8\] in the case $E = \mathbb{C}$, i.e., for the space $L_p(X)$ (see Corollary 6.7 there). As observed in \[15\] (see Lemma 5.3 there; see also \[17\]), the proof of \[8\] is valid without change for any Banach lattice with nontrivial concavity in place of $L_p(X)$. The space we are concerned here is $L_p(X; E)$ which has both nontrivial concavity and convexity. Thus \[7\] holds.

Now let $\phi$ be the function $\phi(z) = ze^{-z}$. Then
\[
\phi(tA)(f) = tAe^{-tA}(f) = -t \frac{\partial}{\partial t} T_t(f).
\]
So \[7\] becomes \[6\] for this special choice of $\phi$. \qed

We conclude this section with some open problems. The first one concerns the weak type $(1,1)$ version of inequality \[2\] under the assumption of Corollary \[3\].

**Problem 10.** Let $E$ be a UMD lattice and $\{T_t\}_{t>0}$ a semigroup of contractions on $L_p(X)$ for every $1 \leq p \leq \infty$. Does one have
\[
\|M(f)\|_{L_{1,\infty}(X; E)} \lesssim \|f\|_{L_1(X; E)}, \quad f \in L_1(X; E)?
\]
This problem is open even for $E = \ell_q$ with $1 < q < \infty$.

On the other hand, it would be interesting to determine the family of Banach spaces $E$ satisfying (i) (resp. (ii)) of Theorem \[4\]. It is easy to check that both families are closed under the passage to subspaces and quotient spaces. Pisier \[21\] proved that if $E$ is of nontrivial type, then $E$ satisfies (i) for any symmetric convolution semigroup $\{T_t\}_{t>0}$ on a locally compact abelian group.

**Problem 11.** Let $1 < p < \infty$ and $\{T_t\}_{t>0}$ be an analytic semigroup of regular contractions on $L_p(X)$.

(i) Let $E$ be a Banach space of nontrivial type. Does $\{T_t\}_{t>0}$ extend to an analytic semigroup on $L_p(X; E)$?

(ii) Let $E$ be a UMD Banach space. Does $\{T_t\}_{t>0}$ extend to an analytic semigroup on $L_p(X; E)$?

If yes, does its negative generator $A$ have a bounded $H^\infty(\Sigma_{\sigma})$ functional calculus on $L_p(X; E)$ for some $\sigma < \pi/2$?

Both parts remain open even in the most important case where $\{T_t\}_{t>0}$ is a symmetric diffusion semigroup (see \[21\] Remark 1.8). For such a semigroup the extension of $\{T_t\}_{t>0}$ to $L_p(X; E)$ is analytic for any $1 < p < \infty$ and for any UMD space $E$ (in fact, only the superreflexivity of $E$ is required; see \[21\] Remark 1.8). However, even in this special case, it is still an open problem whether $A$ has a bounded $H^\infty(\Sigma_{\sigma})$ functional calculus for some $0 < \sigma < \pi/2$. As already observed at the beginning of the proof of Theorem \[4\], $A$ has a bounded $H^\infty(\Sigma_{\sigma})$ functional calculus for some $\sigma > \pi/2$. Thus by \[13\] Theorem 5.3, the last problem is equivalent to the $R$-analyticity of $\{T_t\}_{t>0}$ on $L_p(X; E)$. Let us record this explicitly here:

**Problem 12.** Let $\{T_t\}_{t>0}$ be a symmetric diffusion semigroup. Does $\{T_t\}_{t>0}$ extend to an $R$-analytic semigroup on $L_p(X; E)$ for every $1 < p < \infty$ and every UMD space $E$?

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