THE PROJECTIVE UNITARY IRREDUCIBLE REPRESENTATIONS
OF THE POINCARÉ GROUP IN 1+2 DIMENSIONS

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ABSTRACT

We give a complete analysis of the projective unitary irreducible representations of the Poincaré group in 1+2 dimensions applying Mackey theorem and using an explicit formula for the universal covering group of the Lorentz group in 1+2 dimensions. We provide explicit formulae for all representations.

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1. Introduction

The purpose of this paper is to determine all the projective unitary irreducible representations of the Poincaré group in 1+2 dimensions. The utility of such a study becomes evident taking into account that physics in 1+2 dimensions is nowadays a subject of active research.

Our analysis is based on Mackey theorem on induced representations (see e.g. [1]). The computations are in essence straightforward, but rather tedious because of the complicated structure of the universal covering group of the Lorentz group in 1+2 dimensions (i.e. the analogue of $SL(2, C)$ from 1+3 dimensions). Cohomological arguments ensure that all the projective representations of the Poincaré group in $1 + n$ dimensions ($n > 1$) are induced by true representations of the corresponding universal covering group (see e.g. [1],[2]).

To our knowledge, the only attempt to classify all the projective unitary irreducible representations of the Poincaré group in 1+2 dimensions is contained in [3] and is based on a “sui-generis” form of Mackey theorem leading to an incomplete list of representations.

The benefit of a correct application of Mackey theorem is that it leads to explicit formulae for all the representations we are looking for.

We divide the analysis in two parts. The main part (mathematical framework, computation of the orbits, computation of the little groups, etc.) is concentrated in § 2. We defer the analysis of the projective unitary irreducible representations of the Lorentz group in 1+2 dimensions to § 3. Such an analysis has been already provided in [4] but explicit formulae are missing (the interest of the paper is centered on something else and explicit formulae are not needed but only their existence). Also we note that the same analysis is done in [5]; however one considers here the universal covering group of $SU(1, 1) \cong SL(2, R)$. By comparison, our formulae and proofs seem simpler. So we think that it is useful to provide a detailed list of these representations in an explicit way. In this way, for the benefit of the reader, all the relevant expressions concerning the projective unitary irreducible representations of the Poincaré group in $1 + 2$ dimensions will be collected together in a single paper.

The results of this paper can be used to develop the theory of invariant wave equations on the lines of [6], [7]. We note that the so-called discrete series of the covering group of the Lorentz group in 1+2 dimensions can be obtained using geometric quantization [7].
2. The Poincaré group in 1+2 dimensions

2.1 We denote by $M$ the 1+2-dimensional Minkowski space i.e. $\mathbb{R}^3$ with coordinates $(x^0, x^1, x^2)$ and with the Minkowski bilinear form:

$$\{x, y\} \equiv x^0 y^0 - x^1 y^1 - x^2 y^2. \quad (2.1)$$

The Lorentz group is:

$$L \equiv \{\Lambda \in \text{End}(M) | \{Lx, Ly\} = \{x, y\}, \forall x, y \in M\} \quad (2.2)$$

considered as a multiplicative group.

We will also consider the orthochronous Lorentz group $L^\uparrow \subset L$:

$$L^\uparrow \equiv \{\Lambda \in L | x^0 > 0 \Rightarrow (\Lambda x)^0 > 0, \forall x \in M\}, \quad (2.3)$$

the proper Lorentz group: $L_+ \subset L$:

$$L_+ \equiv \{\Lambda \in L | \text{det}(\Lambda) = 1\} \quad (2.4)$$

and the proper orthochronous Lorentz group:

$$L^\uparrow_+ \equiv L^\uparrow \cap L_+. \quad (2.5)$$

The Poincaré group is a semi-direct product:

$$P \equiv L \times_t M \quad (2.6)$$

where we are using the notations of [1]: $M$ is considered as an additive group and $t : L \rightarrow \text{Aut}(M)$ is simply:

$$t_\Lambda(x) \equiv \Lambda x.$$ 

We also have:

$$P^\uparrow \equiv L^\uparrow \times_t M, P_+ \equiv L_+ \times_t M, P^\uparrow_+ \equiv L^\uparrow_+ \times_t M.$$ 

2.2 Let us denote, generically, by $\text{Lie}(G)$ the Lie algebra of the Lie group $G$. One can prove that $H^2(\text{Lie}(P^\uparrow_+), \mathbb{R}) = 0$ (see e.g. [2]). Then as in 1+3 dimensions one proves
that the projective representations of \( \tilde{P}_+^\dagger \) are induced by true representations of the corresponding covering group \( \tilde{P}_+^\dagger \) of \( P_+^\dagger \).

To construct \( \tilde{P}_+^\dagger \) we need the universal covering group \( \tilde{L}_+^\dagger \) of \( L_+^\dagger \). It is known that \( SL(2, \mathbb{R}) \cong SU(1, 1) \) is a double covering of \( L_+^\dagger \). In [8] one can find an explicit realization for the universal covering group of \( SU(1, 1) \). From this realization one can infer an explicit realization for the universal covering group of \( SL(2, \mathbb{R}) \). We will prefer to work with \( SL(2, \mathbb{R}) \) rather than with \( SU(1, 1) \). As in [9] we define the manifold:

\[
G \equiv R \times D
\]

where:

\[
D \equiv \{ u \in C \mid |u| < 1 \}.
\]

The manifold \( G \) can be transformed into a Lie group relative to the following composition law:

\[
(x, u) \cdot (y, v) \equiv \left( x + y + \frac{1}{2i} \ln \frac{1 + e^{-2iy}uv}{1 + e^{2iy}uv}, u + e^{2iy}v \right).
\]

An explanation is needed. Let us denote:

\[
C_r \equiv \{ z \in C \mid |z| = r \} \quad (\forall r \in R_+).
\]

Then if \( z \in D \) we have \( \frac{1+z}{1+z} \in C_1 \setminus \{-1\} \) so \( \frac{1+z}{1+z} \) can be uniquely written as \( e^{2it} \) with \( t \in (-\pi/2, \pi/2) \). It is natural to put

\[
t = \frac{1}{2i} \ln \frac{1 + z}{1 + \bar{z}}
\]

and this explains (2.9). We note that the same convention can be applied for \( z \) pure imaginary.

The group \( G \) is the universal covering group of \( SL(2, \mathbb{R}) \). Indeed one can verify that the map \( \delta_1 : G \rightarrow SL(2, \mathbb{R}) \) given by:

\[
\delta_1(x, u) \equiv \frac{1}{2\sqrt{1-|u|^2}} \left( \begin{array}{cc} e^{ix}(1+u) + e^{-ix}(1+\bar{u}) & ie^{ix}(1-u) - ie^{-ix}(1-\bar{u}) \\ -ie^{ix}(1+u) + ie^{-ix}(1+\bar{u}) & ie^{ix}(1-u) + ie^{-ix}(1-\bar{u}) \end{array} \right)
\]

is well defined and it is a homomorphism.

We note that:

\[
\ker(\delta_1) = \{(2\pi n, 0) \mid n \in \mathbb{Z} \} \cong \mathbb{Z}.
\]
Next we need the covering map of $SL(2, R)$ onto $L^+_\uparrow$. To this purpose we introduce the $2 \times 2$ matrices $\tau_0, \tau_1, \tau_2$ as follows [3]:

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  (2.13)

These matrices are a basis in the linear space $H$ of $2 \times 2$ real symmetric matrices, and we have the isomorphism:

$$M \ni x \mapsto [x] \in H$$  (2.14)

where:

$$[x] \equiv x^0 \tau_0 + x^1 \tau_1 + x^2 \tau_2 = \begin{pmatrix} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}.$$  (2.15)

Then we define for any $A \in SL(2, R)$, $\delta_2(A) \in End(M)$ by:

$$[\delta_2(A)x] = A[x]A^t.$$  (2.16)

One proves that $\delta_2(A) \in L^+_\uparrow$, and $\delta_2$ is a group homomorphism with:

$$ker(\delta_2) = \{\pm 1\}.$$  (2.17)

Now it is clear that $\delta \equiv \delta_2 \circ \delta_1$ is a covering homomorphism $\delta : G \to L^+_\uparrow$ with:

$$ker(\delta) = \{(\pi n, 0) | n \in Z\} \cong Z.$$  (2.18)

Because $G$ is a simply connected Lie group it follows that it can be taken as the universal covering group of $L^+_\uparrow$.

It is clear that the universal covering group of $P^+_\uparrow$ can be taken as the inhomogeneous group

$$in(G) \equiv G \times_t M$$  (2.19)

where the homomorphism $t : G \to Aut(M)$ is:

$$t_{x, u}(a) \equiv \delta(x, u)a.$$  (2.20)

2.3 We want to classify the unitary irreducible representations of the semi-direct product $in(G)$. To fix the notations we remind the reader the content of Mackey theorem. Let $H \times_t A$ be a semi-direct product of the locally compact groups $H$ and $A$ which verify the
second axiom of countability. Suppose that \( A \) is Abelian. Here: \( t : H \to Aut(A) \) is a group homomorphism. To classify all the unitary irreducible representations of \( H \times_t A \) one goes through the following steps [1].

(a) One considers the dual \( \hat{A} \) of \( A \) and the action of \( H \) on it given by:

\[
(h \cdot \omega)(a) \equiv \omega(t_{h^{-1}}(a)). \tag{2.21}
\]

(b) One computes all the \( H \)-orbits in \( \hat{A} \). We suppose there exists a Borel cross section \( \Sigma \subset \hat{A} \) intersecting once every \( H \)-orbit.

(c) For \( \forall \omega \in Z \) one computes the “little group”: \( H_\omega \equiv \{ h \in H | h \cdot \omega = \omega \} \).

(d) We suppose that we know the complete list of unitary irreducible representations of \( H_\omega, \forall \omega \in Z \).

(e) Let \( O \subset \hat{A} \) be a \( H \)-orbit in \( \hat{A}, \omega_0 \equiv O \cap Z \) and \( \pi \) a unitary irreducible representation of \( H_\omega \) acting in the (complex) Hilbert space \( K \). As it is well known [1] one can associate to every \( \pi \) an one \((H, O, K)\)-cocy cle \( \phi^\pi \) i.e. a Borel map \( \phi^\pi : G \times O \to U(K) \) (here \( U(K) \) is the group of unitary operators in \( K \)) such that a.e. in \( G \times O \)

\[
\phi^\pi(h_1, h_2 \cdot \omega)\phi^\pi(h_2, \omega) = \phi^\pi(h_1 h_2, \omega) \tag{2.23}
\]

and \( \forall h \in H_\omega_0 \)

\[
\pi(h) = \phi^\pi(h, \omega_0). \tag{2.24}
\]

A convenient way to construct \( \phi^\pi \) is as follows. Let \( c : O \to H \) be a Borel section i.e. a Borel map such that \( \forall \omega \in O, \)

\[
c(\omega) \cdot \omega_0 = \omega. \tag{2.25}
\]

Then we can take:

\[
\phi^\pi(h, \omega) = \pi(c(h \cdot \omega)^{-1}hc(\omega)). \tag{2.26}
\]

(f) For every \( H \)-orbit \( O \) and every unitary irreducible representation \( \pi \) of \( H_\omega_0 \) in \( K \) we consider the Hilbert space \( H \equiv L^2(O, d\alpha, K) \) (where \( \alpha \) is a \( H \)-quasi-invariant measure on \( O \)) and define: \( W^{(O, \pi)}_{h, a} : H \to H \) as follows:

\[
(W^{(O, \pi)}_{h, a} f)(\omega) = \omega(a)(r_h(h^{-1} \cdot \omega))^{1/2}\phi^\pi(h, h^{-1} \cdot \omega) f(h^{-1} \cdot \omega) \tag{2.27}
\]
(where \(r_h(\cdot)\) is a version of the Radon-Nycodym derivative \(\frac{d\alpha}{da^{h-1}}\)).

Then \(W\) is a unitary irreducible representation of \(H \times_t A\).

Mackey theorem asserts that if the orbit structure is smooth (see [1]) then every unitary irreducible representation of \(H \times_t A\) is unitary equivalent to a representation of the form \(W^{(O,\pi)}\) and moreover to distinct couples \((O, \pi) \neq (O', \pi')\) correspond representations \(W^{(O,\pi)}\) and \(W^{(O',\pi)}\) which are not unitary equivalent.

In our case \(H = G = \tilde{L}_+^+\) and \(A = M\).

2.4 Steps (a) and (b) are very similar to the usual case of 1+3 dimensions. Namely we can identify \(\hat{M} \cong M\) as follows:

\[
M \ni p \mapsto \chi_p \in \hat{M}
\]

(2.28)

where:

\[
\chi_p(a) \equiv e^{i\{a,p\}}.
\]

(2.29)

Then the action of \(G\) on \(\hat{M}\) is the usual one:

\[
(x, u) \cdot p \equiv \delta(x, u)p
\]

(2.30)

and the \(G\)-orbits in \(\hat{M}\) are:

(A)

\[
X^\eta_m \equiv \{p \in \hat{M}||p||^2 = m^2, \text{sign}(p^0) = \eta\}
\]

(2.31)

(for \(m \in R_+ \cup \{0\}, \eta = \pm\))

(B)

\[
Y_m \equiv \{p \in \hat{M}||p||^2 = -m^2\}
\]

(2.32)

(for \(m \in R_+\))

(C)

\[
X_{00} \equiv \{0\}.
\]

(2.33)

Here:

\[
||p||^2 \equiv \{p, p\}.
\]

(2.34)

We take as usual as the set of representative points:

\[
\Sigma \equiv (\cup_{m \in R_+} \{\pm me_0\}) \cup \{\pm e_+\} \cup (\cup_{m \in R_+} me_2) \cup \{0\}.
\]

(2.35)
Here:

\[ e_\pm \equiv e_0 \pm e_1. \]  \hfill (2.36)

2.5 The computation of the little groups \( H_\omega \) for \( \omega \in Z \) is elementary and we provide only the final results. We have;

(A)

\[ G_{\eta_{me_0}} = \{(x, 0) | x \in R\} \cong R \]  \hfill (2.37)

(for \( m \in R_+ \)) and:

\[ G_{\eta e_+} = \left\{ \left( \frac{1}{2i} \ln \frac{1 - ib}{1 + ib} + n\pi, \frac{ib}{1 - ib} \right) | n \in Z, b \in R \right\}. \]  \hfill (2.38)

Here the expression \( \ln \frac{1 - ib}{1 + ib} \) has been defined at 2.2. We note that we have:

\[ G_{\eta e_+} \cong Z \times R \]  \hfill (2.39)

where the isomorphism is:

\[ \left( \frac{1}{2i} \ln \frac{1 - ib}{1 + ib} + n\pi, \frac{ib}{1 - ib} \right) \leftrightarrow (n, b). \]  \hfill (2.40)

(B)

\[ G_{me_2} = \left\{ \left( \pi n, \frac{a - 1}{a + 1} \right) | n \in Z, a \in R_+ \right\}. \]  \hfill (2.41)

We note that we have:

\[ G_{me_2} \cong Z \times R_+ \]

where \( R_+ \) is considered as a multiplicative group and the isomorphism is:

\[ \left( \pi n, \frac{a - 1}{a + 1} \right) \leftrightarrow (n, a). \]  \hfill (2.42)

(C) \( G_0 = G. \)

2.6 The list of all the unitary irreducible representations for the little groups above is very easy to determine.

(A) For \( G_{\eta_{me_0}} \) these representations are indexed by a number \( s \in R \). They are of the form:

\[ \pi^s(x, 0) \equiv e^{isx}. \]  \hfill (2.43)
For $G_{\eta e^+}$ one uses the group isomorphism (2.39), (2.40) and gets a list of representations indexed by a couple $(s,t)$ where $s \in R(mod\ 2)$ and $t \in R$ of the following form:

$$\pi^{s,t} \left( \frac{1}{2i} \ln \frac{1 - ib}{1 + ib} + n\pi, \frac{ib}{1 - ib} \right) \equiv e^{\pi is} e^{itb}. \quad (2.44)$$

(B) For $G_{me_2}$ we get again representations indexed by a couple $(s,t)$ where $s \in R$ (mod 2) and $t \in R$ of the following form:

$$\pi'_{s,t} \left( \pi n, \frac{a - 1}{a + 1} \right) \equiv e^{\pi is} a^t. \quad (2.45)$$

(C) The list of all unitary irreducible representations of $G$ will be given in Section 3.

2.7 If we want explicit formulae for the corresponding representations of the inhomogeneous group in $(G)$, it is necessary to determine the cocycles associated to the representations of the little group identified above. In cases (A) and (C) very simple formulae are available for a suitable choices of the Borel section $c$.

(A) For the orbit $X^\eta_m$ $(m \in R_+, \eta = \pm)$ one can show that a very simple Borel section $c : X^\eta_m \to G$ is:

$$c(p) = \left( 0, \frac{<p>}{p^0 + \eta m} \right) <p> \equiv p^1 + ip^2. \quad (2.46)$$

The corresponding cocycle $\phi^s$ is:

$$\phi^s((x, u), p) \equiv e^{isx} \left[ \frac{p^0 + \eta m + ue^{-2ix} <p>}{p^0 + \eta m + \bar{u}e^{2ix} <p>} \right]^{s/2}. \quad (2.47)$$

Here we interpret the expressions of the type $\left( \frac{a + z}{a + \bar{z}} \right)^{s/2}$ for $a \in R_+$ and $z \in C$ such that $|z| < a$ as follows. We note that we have $\frac{a + z}{a + \bar{z}} \neq -1$ so we can uniquely write $\frac{a + z}{a + \bar{z}} = e^{2it}$ with $t \in (-\pi/2, \pi/2)$. Then we put $\left( \frac{a + z}{a + \bar{z}} \right)^{s/2} \equiv e^{ist}$.

We note that (2.47) is much more simpler than the corresponding expression for the Wigner rotation obtained in [6] for the group $P^\uparrow_{\eta}$. This is another benefit of working with the covering group of $P^\uparrow_{\eta}$.

For the orbit $X^\eta_0$ a convenient cross section $c : X^\eta_0 \to G$ is:

$$c(p) = \left( x_0(p), \frac{p^0 - 1}{p^0 + 1} \right) \quad (2.48)$$
where \( x_0(p) \in (-\pi/2, \pi/2) \) is determined by:

\[
x_0(p) \equiv \text{Arg} \left( \sqrt{\frac{p^0 + p^1}{2p^0}} + i\theta(p^2) \sqrt{\frac{p^0 - p^1}{2p^0}} \right).
\] (2.49)

After some computations one can determine the corresponding cocycle, namely:

\[
\phi'^{s,t}((x, u), p) = e^{isx} \left[ \frac{p^0 + u e^{-2ix} < \bar{p} >}{p^0 + \bar{u} e^{2ix} < p >} \right]^{s/2} \times \exp \left\{ it\eta \frac{\text{Im}(ue^{-2ix} < \bar{p} >)}{p^0[(1 + |u|^2)p^0 + 2\text{Re}(ue^{-2ix} < \bar{p} >)} \right\}.
\] (2.50)

(B) A determination of a Borel section \( c : Y_m \to G \) is still possible but the expression is very complicated so we will not try to give explicit formulae in this case for the corresponding cocycle \( \phi'^{s,t} \).

(C) It is clear that if \( \pi \) is a unitary irreducible representation of \( G \), then we have:

\[
\phi^\pi((x, u), 0) = \pi(x, u).
\] (2.51)

2.8 Applying Mackey theorem we end up with the following result:

**Theorem 1**: Every unitary irreducible representation of \( in(G) \) is unitary equivalent to one of the following type:

(a) \( W^{m,\eta,s} \quad (m \in R_+, \eta = \pm, s \in R) \) acting in \( L^2(X^\eta_m, d\alpha^\eta_m) \) as follows:

\[
(W^{m,\eta,s} f)(p) = e^{i(a,p)} e^{isx} \left[ \frac{p^0 + \eta m - \bar{u} e^{2ix} < p >}{p^0 + \eta m - \bar{u} e^{-2ix} < \bar{p} >} \right]^{s/2} f((x, u)^{-1} \cdot p).
\] (2.52)

(b) \( W^{\eta,s,t} \quad (\eta = \pm, s \in R (mod 2), t \in R) \) acting in \( L^2(X^\eta_0, d\alpha^\eta_0) \) as follows:

\[
(W^{\eta,s,t} f)(p) = e^{i(a,p)} e^{isx} \left[ \frac{p^0 - \bar{u} e^{2ix} < \bar{p} >}{p^0 - \bar{u} e^{-2ix} < \bar{p} >} \right]^{s/2} \times \exp \left\{ int\eta \frac{\text{Im}(u e^{-2ix} < \bar{p} >)}{p^0[(1 + |u|^2)p^0 - 2\text{Re}(u e^{-2ix} < \bar{p} >)} \right\} \right) f((x, u)^{-1} \cdot p).
\] (2.53)

(c) \( W'^{m,s,t} \quad (m \in R_+, s \in R (mod 2), t \in R) \) acting in \( L^2(Y_m, \beta_m, K) \) according to the formula:

\[
(W'^{m,s,t} f)(p) = e^{i(a,p)} \phi'^{s,t}((x, u), (x, u)^{-1} \cdot p)f((x, u)^{-1} \cdot p).
\] (2.54)
Here $\phi_{s,t}'$ is a cocycle corresponding to the representation $\pi_{s,t}'$.

(d) $W^\pi$ acting in the Hilbert space $K(\pi)$ of the unitary irreducible representation $\pi$ of $G$ according to:

$$W^\pi_{x,u,a} = \pi(x,u).$$ (2.55)

(The measures $\alpha_{m}^\eta \ (m \in R_+ \cup 0)$, $\eta = \pm$ and $\beta_{m} \ (m \in R_+ \cup \{0\}$ are defined as in [1].)

Two different representations in the list above are not unitary equivalent.

2.9 Remarks

1) From (2.52) and (2.53) we have a sort of Jacob-Wick limit $m \to 0$:

$$\lim_{m \to 0} W^{m,\eta,s} = W^{\eta,s,0}$$

2) The infinitesimal generators are defined on a suitable Garding domain as follows:

$$(P^0 f)(p) \equiv -i \frac{d}{dt}(W_{0,0,t_{e_0}}f)(p)_{|t=0}. \quad (2.56)$$

$$(P^i f)(p) \equiv i \frac{d}{dt}(W_{0,0,t_i}f)(p)_{|t=0}. \quad (2.57)$$

$$(J f)(p) \equiv i \frac{d}{dt}(W_{t/2,0,0}f)(p)_{|t=0}. \quad (2.58)$$

$$(K^1 f)(p) \equiv i \frac{d}{dt}(W_{0,t/2,0}f)(p)_{|t=0}. \quad (2.59)$$

$$(K^2 f)(p) \equiv i \frac{d}{dt}(W_{0,0,t/2}f)(p)_{|t=0}. \quad (2.60)$$

One can compute explicitly these expressions for cases (a) and (b) in the theorem above. It is convenient to identify functions defined on $X^\eta_m$ with functions defined on $R^2$ taking into account the one-to-one correspondence:

$$R^2 \ni p \leftrightarrow \tau(p) \in X^\eta_m \quad (2.61)$$

where:

$$\tau(p) \equiv (E(p), p) \quad (2.62)$$

$$E(p) \equiv \sqrt{p^2 + m^2}. \quad (2.63)$$

We get in both cases:

$$P^\mu = \tau^\mu(p). \quad (2.64)$$
and
\[ J = i \left( p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1} \right) - s/2 \]  
(2.65)

For \( K^i \) one gets the following formulae:

- for \( W^{m,\eta,s} \):
  \[ K^i = iE(p) \frac{\partial}{\partial p^i} + s\varepsilon_{ij} \frac{p^j}{2[E(p) + \eta m]}. \]  
(2.66)

- for \( W^{\eta,s,t} \):
  \[ K^i = iE(p) \frac{\partial}{\partial p^i} + s\varepsilon_{ij} \frac{p^j}{2E(p)} + \frac{\eta t}{2} \varepsilon_{ij} \frac{p^j}{E(p)^2}. \]  
(2.67)

These formulae should be compared with the similar ones obtained in [6].

3) Let us note that by restriction to elements of the type \((x, 0, a) \in G\) (i.e. to the universal covering group of the Euclidean group in 1+2 dimensions) we get in cases (a), (b) and (c) of the theorem above a representation \( V \) acting in \( L^2(R^2, dp) \) as follows:

\[ (V_{x,a}f)(p) = e^{-i\mathbf{a} \cdot \mathbf{p}} e^{i sx} f(R(x)^{-1} \mathbf{p}). \]  
(2.68)

where:

\[ R(x) = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}. \]  
(2.69)

Indeed, for cases (a) and (b) this is immediate.

In case (c) one uses the following argument. First one has the direct integral decomposition:

\[ V = \int_{\mu \geq m} V^\mu \mu d\mu \]  
(2.70)

where \( V^\mu \) acts in \( L^2(C_\mu, d\varphi) \) as follows:

\[ (V^\mu_{x,a}f)(p) = e^{-i\mathbf{a} \cdot \mathbf{p}} \phi^{s,t}((x, 0), (x, 0)^{-1} \cdot \tau(p)) f(R(x)^{-1} \mathbf{p}). \]  
(2.71)

But the cocycle

\[ R \times C_\mu \ni (x, \mathbf{p}) \mapsto \phi^{s,t}((x, 0), \tau(p)) \in C \]  
(2.72)

corresponds to a transitive action of the group \( R \) on the Borel space \( C_\mu \) so it is determined by its restriction to the stability subgroup of, say, \( \mu e_2 \) i.e. \( \{\pi n | n \in \mathbb{Z}\} \). According to 2.6, this representation is \( \pi'_{s,0} \) (see (2.45)). Now it easily follows that the cocycle (2.72) is equivalent to the cocycle \( \phi'^{s} \) given by:

\[ \phi'^{s}(x, \mathbf{p}) = e^{isx}. \]  
(2.73)
So the representation $V^\mu$ is equivalent to the representation (denoted also by $V^\mu$), acting in the same Hilbert space as follows:

$$(V^\mu_{x,a} f)(p) = e^{-ia \cdot p} e^{isx} f(R(x)^{-1} p).$$  \hspace{1cm} (2.74)

From (2.70) and (2.74) it follows that (2.68) is valid for the case (c) of the theorem above.

Now (2.68) shows like in [1] that the corresponding systems are localizable on $R^2$. Indeed, performing a Fourier transform $F : L^2(R^2, dp) \rightarrow L^2(R^2, dx)$ one brings (2.68) to the following form: $V$ is acting in $L^2(R^2, dx)$ according to:

$$(V_{x,a} f)(x) = e^{isx} f(R(x)^{-1}(x - a)).$$  \hspace{1cm} (2.75)

Then, the corresponding projector-valued measure in $L^2(R^2, dx)$ is as usual $\beta(R^2) \ni \Delta \mapsto \chi_\Delta \in P(L^2(R^2, dx))$.

Also one can prove like in [1] that the systems corresponding to case (d) in the theorem above are not localizable on $R^2$.

4) One may wonder what is the “spin” of an elementary system of the type (a), (b) or (c). For this one must first give a canonical definition of this notion. It is interesting to note that one hardly finds a precise mathematical definition of this kind stated in the literature, although everybody has in mind the following picture [1].

Suppose one has a certain configuration space i.e. a Borel space $Q$ with a Borel action of some orthogonal group $O(Q)$ on $Q$. Next, suppose that the Hilbert space of a certain physical system is of the form $H = L^2(Q, \alpha) \otimes K$ where $\alpha$ is a quasi-invariant measure on $Q$ with respect to the action of the group $O(Q)$ and $K$ is a given Hilbert space. Finally, suppose that our system is rotationally covariant i.e. one has in $H$ a unitary representation of the universal covering group $\widetilde{O(Q)}$ of $O(Q)$. Moreover, this representation is supposed to be of the following type: $V = U \otimes W$ where

$$(U_g f)(q) = r_g(q)^{1/2} f(\delta(g)^{-1} \cdot q).$$  \hspace{1cm} (2.76)

Here $r_g(\cdot)$ is a version of the Radon-Nycodym derivative $\frac{d\alpha_g^{-1}}{d\alpha}$ and $\delta : \widetilde{O(Q)} \rightarrow O(Q)$ is the covering homomorphism.
Then we are entitled to say that the representation $U$ gives the orbital kinetic momentum of the system and the representation $W$ gives the spin of the system. In other words we can identify the infinitesimal generators of $U$ and $W$ with the orbital kinetic momentum and respectively with the spin of the system.

Now we come to our specific situation of the representations (a), (b) and (c) above. It is clear that one must make in the general framework above the following particularizations:

\[ Q = R^2, \quad O(Q) = SO(2), \quad \tilde{O}(Q) \cong R, \quad K = C. \]

The action of $SO(2)$ on $R^2$ is the usual one and the representation $W$ is simply:

\[ W_x = e^{i s x}. \quad (2.77) \]

So we can conclude that in the cases (a), (b) and (c), the spin of the system is $s$.

For a different point of view for the massless case (b), see however [10].

2.10 One may wonder now what happens for the proper orthochronous Galilei group $G_+^+$ in 1+2 dimensions. In this case the analysis is much simpler because the universal covering group for $G_+^+$ has a much more simpler expression, namely is as a manifold:

\[ \tilde{G}_+^+ = R \times R^2 \times R \times R^2 \]

with the composition law:

\[ (x, v, \eta, a) \cdot (x', v', \eta', a') = (x + x', v + R(x)v', \eta + \eta', a + R(x)a' + \eta'v), \quad (2.78) \]

where $R(x)$ has been defined at (2.69). The covering homomorphism $\delta : \tilde{G}_+^+ \rightarrow G_+^+$ is:

\[ \delta(x, v, \eta, a) = (R(x), v, \eta, a). \quad (2.79) \]

Now the analysis is straightforward and can be obtained by appropriate modifications of the similar analysis from [1] dedicated to the 1+3 dimensional case. We give only the final result:

**Theorem 2:** Every unitary irreducible representation of $\tilde{G}_+^+$ is equivalent to one of the following type:

(a) $V^{m,s}$ ($m \in \mathbb{R}^*, s \in \mathbb{R}$) acting in $L^2(R^2, dp)$ as follows:

\[
(V^{m,s}_{x,v,\eta,a}f)(p) = \exp \left\{ i \left( sx + a \cdot p + \frac{\eta p^2}{2m} + \frac{m}{2} a \cdot v \right) \right\} f(R(x)^{-1}(p + mv)). \quad (2.80)
\]

(b) $L^{r,s,t}$ ($r \in \mathbb{R}_+, s \in \mathbb{R}(mod~1), t \in \mathbb{R}$) acting in $L^2(R \times C_r, dp_0 \times d\phi)$ as follows:

\[
(L^{r,s,t}_{x,v,\eta,a}f)(p_0, p) = \exp \left\{ i \left( sx + a \cdot p + \eta p_0 + \frac{t[p, v]}{r} \right) \right\} f(p_0 + v \cdot p, R(x)^{-1}p). \quad (2.81)
\]
Here $\forall a, b \in R^2$ we have defined the antisymmetric bilinear form $[\cdot, \cdot]$ by:

$$[a, b]e_1 \wedge e_2 \equiv a \wedge b.$$  \hspace{1cm} (2.82)

(c) $L^{p_0, \rho, s}$ ($p_0 \in R, \rho \in R_+, s \in R(mod \ 1)$) acting in $L^2(C_{\rho, d\varphi})$ as follows:

$$(L^{p_0, \rho, s}_{x, \nu, \eta, a} f)(\omega) = \exp \{i (sx + \eta p_0 + \omega \cdot \nu)\} f(R(x)^{-1}\omega).$$  \hspace{1cm} (2.83)

(d) $L^{p_0, s}$ ($p_0 \in R, s \in R$) acting in $C$ as follows:

$$L^{p_0, s}_{x, \nu, \eta, a} = \exp \{i(\eta p_0 + sx)\}. \hspace{1cm} (2.84)$$

Remarks:

1) The representation $L^{p_0, s}$ induces a trivial projective representation of $G^+_\uparrow$ so it can be discarded.

2) As in [1] one can show that only $V^{m, s}$ is localizable on $R^2$.

3. The Unitary Irreducible Representation of the Universal Covering Group of $SL(2, R)$

3.1 The unitary irreducible representations of $SL(2, R) \cong SU(1, 1)$ are rather well studied in the litterature [8], [11], [12]. The corresponding problem for the universal covering group is treated in [4], [5]. However, as pointed out in the Introduction, Ref. [4] does not use the explicit construction for the covering group and the classification of the unitary irreducible representations is done in an implicit way without explicit realizations. (Also [5] is focused on $SU(1, 1)$ and the formulae are rather cumbersome). The same can be said about the analysis of the discrete series appearing in e.g. [6], [7] where only the infinitesimal generators appear.

We find it profitable for the reader to provide here all the relevant formulae for the universal covering group of $SL(2, R)$. Of course we will reproduce (in the next Subsection) a large part of the results of [4].

3.2 We identify $\text{Lie}(G) = \text{Lie}(SL(2, R))$ with the three dimensional space of $2 \times 2$ real traceless matrices. A basis in this space is $\{l_0, l_1, l_2\}$ where (see (2.13)):

$$l_0 \equiv -\frac{1}{2}\tau_3, \quad l_1 \equiv \frac{1}{2}\tau_1, \quad l_2 \equiv \frac{1}{2}\tau_2. \hspace{1cm} (3.1)$$
We have the commutations relations:

\[ [l_0, l_1] = l_2, \quad [l_0, l_2] = -l_1, \quad [l_1, l_2] = -l_0. \]  \hspace{1cm} (3.2)

Let \( T \) be a unitary irreducible representation of the group \( G \) in the Hilbert space \( H \). For any \( l \in \text{Lie}(G) \) we denote by \( H_l \) the self-adjoint operator in \( H \) determined (according to Stone-von Neumann theorem) by:

\[ e^{-itH_l} = T_{\exp(tl)} \quad (\forall t \in \mathbb{R}). \]  \hspace{1cm} (3.3)

Let us denote by \( D_l \) the domain of self-adjointness of \( H_l \). Then \( B \equiv D_{l_0} \cap D_{l_1} \cap D_{l_2} \) is a Gårding domain for \( T \), and we have in \( B \):

\[ [H_0, H_1] = iH_2, \quad [H_0, H_2] = -iH_1, \quad [H_1, H_2] = -iH_0. \]  \hspace{1cm} (3.4)

It is well known that the representation \( T \) is uniquely determined by the infinitesimal generators \( H_i (i = 0, 1, 2) \). The usual way to proceed for a non-compact group is to search for a maximal compact subgroup for which the corresponding infinitesimal generators will have a pure point spectrum. In the case of our group \( G \) the maximal compact subgroup is trivially formed by the neutral element, so apparently we cannot proceed further. However a trick of [4] shows that the spectrum of \( H_0 \) is pure point.

Next one defines on \( B \) the operator

\[ Q' \equiv (H_1)^2 + (H_2)^2 - (H_0)^2 \]  \hspace{1cm} (3.5)

and by \( Q \) the unique self-adjoint extension to \( H \). Like in [7] one proves that:

\[ Q = q I \quad (q \in \mathbb{R}). \]  \hspace{1cm} (3.6)

If we denote by \( D \subset H \) the linear subspace of finite linear combinations of eigenvectors of \( H_0 \), then one can show that \( D \) is a Gårding domain for \( T \) [8]. So it will be sufficient to determine the action of \( H_i \) \( (i = 0, 1, 2) \) on \( D \).

Let us denote:

\[ H_{\epsilon} \equiv H_1 + i\epsilon H_2 \quad (\epsilon = \pm). \]  \hspace{1cm} (3.7)

Then the final result of the infinitesimal analysis is the following. The commutation relations (3.4) are compatible with the following three cases:
I. There exists \( f \in D \) such that \( H_j^\epsilon f \neq 0 \) \( (\forall \epsilon = \pm, \forall j \in \mathbb{N}) \). In this case one can find \( \tau \in [0, 1) \) and an orthonormal base in \( H \{ f_\alpha \}_{\alpha \in \tau + Z} \) such that:

\[
H_0 f_\alpha = \alpha f_\alpha \tag{3.8}
\]

\[
H_\epsilon f_\alpha = [q + \alpha(\alpha + \epsilon)]^{1/2} f_{\alpha + \epsilon}. \tag{3.9}
\]

Moreover one must have

\[
q > \tau(1 - \tau).
\]

II. There exists \( f \in D \) such that \( f \neq 0 \) but \( H_- f = 0 \). Then one can find \( l \in R_+ \) and an orthonormal base \( \{ f_\alpha \}_{\alpha \in l + N} \) in \( H \) such that (3.8) and (3.9) stay true for the appropriate values of \( \alpha \) (we also take \( f_{l-1} \equiv 0 \)).

III. There exists \( f \in D \) such that \( f \neq 0 \) but \( H_+ f = 0 \). Then one can find \( l \in R_+ \) and an orthonormal base \( \{ f_\alpha \}_{\alpha \in -l - N} \) in \( H \) such that (3.8) and (3.9) stay true for the appropriate values of \( \alpha \) (we also take \( f_{-l+1} \equiv 0 \)).

3.3 Next one proves in [4] in a very implicit way that to every case above one has indeed a unitary irreducible representation of \( G \).

We will prove this point by some very explicit constructions. The basic result is:

**Theorem 3:** Let \( H = L^2(2\pi, d\varphi) \) and \( s \in R(\text{mod } 2), t \in R \). For any \((x, u) \in G\) let us define the linear operator \( T_{x,u}^s : H \to H \) as follows:

\[
(T_{x,u}^s f)(\varphi) = e^{is\varphi} \frac{1 - \bar{u}e^{i(\varphi - 2x)}}{1 - u\bar{e}^{i(\varphi - 2x)}} \left( \frac{1 - u\bar{e}^{i(\varphi - 2x)}}{\sqrt{1 - |u|^2}} \right)^{it-1} \times
\]

\[
f \left( \varphi - 2x + \frac{1}{i}ln \frac{1 - u\bar{e}^{i(\varphi - 2x)}}{1 - \bar{u}e^{i(\varphi - 2x)}} \right). \tag{3.10}
\]

Here \( f \) is extended to the whole real axis by periodicity with period \( 2\pi \).

Then \( T \) is a unitary representation of \( G \) in \( H \) corresponding to \( \tau = -\frac{s}{2} \in [0, 1) \) and \( q = \frac{1}{4}(t^2 + 1) \geq 1 \).

**Proof:** The fact that \( T_{x,u}^s \) is unitary follows by a convenient change of variables in the expression \( \|T_{x,u}^s f\|^2 \) and taking into account the periodicity of \( f \). For the representation property one can avoid a brute force computation rewriting a little bit (3.10).

First, we consider \( f \) as a function defined on the circle \( C_1 = \{(\zeta^1)^2 + (\zeta^2)^2 = 1\} \) taking:

\[
\zeta^1 = \cos(\varphi), \quad \zeta^2 = \sin(\varphi). \tag{3.11}
\]
Next one defines \( \forall \zeta \in C_1, \quad \tau(\zeta) \in X_0^\uparrow \) by:

\[
\tau(\zeta) \equiv (1, \zeta^1, \zeta^2). \tag{3.12}
\]

Finally, we define \( b : X_0^\uparrow \to C_1 \) as follows:

\[
b(p) \equiv \frac{p^1 + ip^2}{p^0} = \frac{<p>}{p^0}. \tag{3.13}
\]

Now one must check that (3.10) is the same as:

\[
(T_{x,u}^{s,t}f)(\zeta) = \phi^s((x,u),(x,u)^{-1} \cdot \tau(\zeta))| <(x,u) \cdot \tau(\zeta)> |^{it-1}f(b((x,u)^{-1} \cdot \tau(\zeta))b(\tau(\zeta))^{-1}). \tag{3.14}
\]

where the cocyle \( \phi^s \) has been defined at (2.47). The new realization (3.14) makes the representation property rather easy to establish.

It remains to match \( T_{x,u}^{s,t} \) with one of the cases I, II, III from the preceding Subsection. For this we need the infinitesimal generators of \( T \); they are rather easily computed:

\[
H_0 = -\left(i\frac{d}{d\varphi} + \frac{s}{2}\right) \tag{3.15}
\]

\[
H_{\epsilon} = e^{i\epsilon\varphi}\left(i\epsilon H_0 + \frac{t+i}{2}\right). \tag{3.16}
\]

Let us take for any \( n \in \mathbb{Z} \):

\[
g_n(\varphi) \equiv \frac{1}{\sqrt{2\pi}} e^{in\varphi}. \tag{3.17}
\]

Then, according to Fourier theorem, \( \{g_n\}_{n \in \mathbb{Z}} \) is an orthonormal basis in \( H \). Moreover we immediately have:

\[
H_0g_n = \left(n - \frac{s}{2}\right)g_n \tag{3.18}
\]

\[
H_{\epsilon}g_n = \left[i\epsilon \left(n - \frac{s}{2}\right) + \frac{t+i}{2}\right]g_{n+\epsilon} \tag{3.19}
\]

\[
Q = \frac{1}{4}(t^2 + 1)I. \tag{3.20}
\]

Comparing with case I it is evident that one should take \( s = -2\tau \) (with \( \tau \in [0,1) \)) and \( q = \frac{1}{4}(t^2 + 1) \). The condition \( q > \tau(1 - \tau) \) is always true. If we redefine \( g_n \to g_{n+\tau} \) we get from (3.18) and (3.19) a set of relations more familiar with I, namely:

\[
H_0g_\alpha = \alpha g_\alpha \tag{3.21}
\]
\[ H_\pm g_\alpha = \omega_\alpha \sqrt{q + \alpha(\alpha + 1)} g_{\alpha + 1} \]  
(3.22)

\[ H_- g_\alpha = \frac{1}{\omega_{\alpha - 1}} \sqrt{q + \alpha(\alpha - 1)} g_{\alpha - 1} \]  
(3.23)

Here

\[ \omega_\alpha \equiv \frac{i\alpha + \frac{t+i}{2}}{\sqrt{q + \alpha(\alpha + 1)}} \]  
(3.24)

is a complex number of modulus 1.

It is sufficient to take now \( \{ \epsilon_\alpha \}_{\alpha \in \tau + Z} \), such that \(|\epsilon_\alpha| = 1\) and \( \omega_\alpha = \frac{\epsilon_\alpha}{\epsilon_{\alpha - 1}} \) and to define the new orthonormal base \( \{ f_\alpha \}_{\alpha \in \tau + Z} \) by \( f_\alpha \equiv \epsilon_{\alpha}^{-1} g_\alpha \); then (3.21)-(3.23) go into (3.8), (3.9). Q.E.D.

We will also denote \( T_{s,t}^{s,t} \) by \( T_{\tau,q}^{\tau,q} \). According to the standard terminology the representations \( T_{\tau,q}^{\tau,q} \) \( (q \geq 1/4) \) constitute the principal series.

Let us remark that \( T_{\tau,q}^{\tau,q} \) induces a true representation of \( SL(2, \mathbb{R}) \) iff \( \tau = 0, 1/2 \) and induces a true representation of \( L_+^{\tau} \) iff \( \tau = 0 \).

### 3.4

It is clear from the preceding theorem that the case \( q < 1/4 \) can be obtained making formally \( t \to it \). Of course this modification will ruin the unitarity. The idea is to modify appropriately the expression of the scalar product [4], [12]. We define on the space \( V \) of smooth complex periodic functions with period \( 2\pi \) the representation of \( G \):

\[
(T'_{x,u} f)(\varphi) = e^{ix\varphi} \left( \frac{1 - \bar{u} e^{i(\varphi - 2x)}}{1 - ue^{-i(\varphi - 2x)}} \right)^{t-1} \times \frac{1 - ue^{-i(\varphi - 2x)}}{|1 - |u|^2|} \left( \frac{\sin(\varphi/2)}{1 - |u|^2} \right)^{t-1} e^{-i\tau \varphi}. 
\]  
(3.25)

where \( s \in R(\text{mod } 2) \) and \( t \in R_+ \).

Then, we define on \( V \) the sesquilinear form \(< \cdot, \cdot > \) by:

\[
<f, g> = \int_0^{2\pi} \int_0^{2\pi} d\varphi d\varphi' L(\varphi - \varphi') f(\varphi) g(\varphi').
\]  
(3.26)

where the kernel \( L \) is:

\[
L(\varphi) \equiv 2^t \pi e^{i\tau \varphi} B \left( \tau + \frac{1+t}{2}, -\tau + \frac{1+t}{2} \right) |\sin(\varphi/2)|^{t-1} e^{-i\tau \varphi}. \]  
(3.27)

Like above we take: \( s = -2\tau \) (with \( \tau \in [0,1) \)) and \( q = \frac{1}{4}(1 - t^2) \). The condition \( q > \tau(\tau - 1) \) gives \( t < |1 - 2\tau| \).
One proves that $<\cdot,\cdot>$ is non-degenerated as follows. We define for any $n \in \mathbb{Z}$ the functions $g_n \in V$ by:

$$g_n(\varphi) = \gamma_n e^{in\varphi} \tag{3.28}$$

where:

$$\gamma_n \equiv \left[ \frac{\Gamma(n + \theta(n)\tau + \frac{1+t}{2}) \Gamma(\theta(n)\tau + \frac{1+t}{2})}{\Gamma(n + \theta(n)\tau + \frac{1-t}{2}) \Gamma(\theta(n)\tau + \frac{1-t}{2})} \right]^{1/2}. \tag{3.29}$$

Then using [13] (§ 3.63) one can show that:

$$L(\varphi) = \sum_{n \in \mathbb{Z}} \frac{1}{\gamma_n} e^{in\varphi}. \tag{3.30}$$

It follows easily that $\{g_n\}_{n \in \mathbb{Z}}$ is an orthonormal system in $V$ and $<\cdot,\cdot>$ is positively defined. So we can obtain from $V$, by completion, a Hilbert space $H$ (in which $\{g_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis) and extend $T^{s,t}$ by continuity to $H$. It is not very hard to prove that $T^{s,t}$ is unitary with respect to the scalar product (denoted also by $<\cdot,\cdot>$) of $H$.

From (3.18)-(3.20) with $t \to i t$ we obtain ($g_n \to g_{n+\tau}$):

$$H_0 g_\alpha = \alpha g_\alpha \tag{3.31}$$

$$H_\epsilon g_\alpha = i \epsilon \eta_\alpha \sqrt{q + (\alpha + \epsilon)g_{\alpha + \epsilon}} \tag{3.32}$$

where:

$$\eta_\alpha \equiv \begin{cases} 
1 & \text{for } \alpha \in \tau + N \\
\text{sign} \left( \frac{1+t}{2} - \tau \right) & \text{for } \alpha \in \tau - N^*.
\end{cases} \tag{3.33}$$

Now let $\{\epsilon_\alpha\}_{\alpha \in \tau + N}$ such that $|\epsilon_\alpha| = 1$ and $\frac{\epsilon_{\alpha+1}}{\epsilon_\alpha} = i \eta_\alpha$. If we redefine $f_\alpha = \epsilon_\alpha^{-1}g_\alpha$ then (3.31) and (3.32) give (3.8) and (3.9).

We will also denote $T^{s,t}$ by $T^{r,q}$. The representations $T^{r,q}$ ($q < 1/4$) constitute the so-called complementary series. In this case we get true representations for $SL(2, \mathbb{R})$ and for $L^+ \uparrow$ iff $\tau = 0$.

3.5 The cases II and III from 3.2 are settled by:

**Theorem 4:** Let $F$ be the linear space of all analytic functions on the disk $D = \{z \in \mathbb{C} | |z| < 1\}$, $l \in \mathbb{R}^+$ and $\eta = \pm$. For any $(x, u) \in G$ we define $D^{l,\eta}_{x,u} : F \to F$ as follows:

$$(D^{l,+}_{x,u} f)(z) = e^{-2ilx} (1 + e^{-2ix\bar{u}z})^{-2l} (1 - |u|^2)^l f \left( \frac{z + e^{2ix}u}{e^{2ix} + \bar{u}z} \right) \tag{3.34}$$
and respectively

\[(D_{x,u}^l f)(z) = e^{2ilx}(1 + e^{2ix}uz)^{-2l}(1 - |u|^2)^l f \left( \frac{z + e^{-2ix} \bar{u}}{e^{-2ix} + uz} \right) \] (3.35)

We also define on \( F \) the Hermitean form \( \langle \cdot, \cdot \rangle \) by:

\[\langle f, g \rangle \equiv \frac{2l}{\pi} \int_D (1 - |z|^2)^{2(l-1)} f(\bar{z})g(z) d\sigma \] (3.36)

\((d\sigma \) is the surface measure on \( \mathbb{C} \)). Let \( H \) be the Hilbert space obtained from \( F \) by completion with respect to \( \langle \cdot, \cdot \rangle \), and extend \( D^{l,\eta} \) to \( H \) by continuity. Then \( T \) is a unitary representation of \( G \) in \( H \) corresponding to cases II and III for \( \eta = + \) and \( \eta = - \) respectively.

**Remark** In (3.34) and (3.35) we interpret \((1 + u)^s \) for \(|u| < 1\) taking \( \text{Arg}(1 + u) \in (-\pi, \pi) \).

**Proof:** We first remark that \( D^{l,+} \) and \( D^{l,-} \) can be obtained one from the other by complex conjugation and the substitution \( z \to \bar{z} \).

There are a number of facts which can be easily verified by direct computation namely that \( D^{l,\eta} \) is well defined, leaves \( \langle \cdot, \cdot \rangle \) invariant and have the representation property.

It remains to match these representations with the infinitesimal analysis. We do this for \( D^{l,+} \). We get immediately for the infinitesimal generators:

\[ H_0 = z \frac{d}{dz} + l. \] (3.37)

\[ H_+ = -2ilz - iz^2 \frac{d}{dz} \] (3.38)

\[ H_- = i \frac{d}{dz}. \] (3.39)

If we define \( \forall n \in \mathbb{N} \) the functions \( g_n \in F \) by:

\[ g_n(z) \equiv (-1)^n z^n \] (3.40)

then we get:

\[ H_0 g_n = (l + n)g_n \] (3.41)

\[ H_+ g_n = i(n + 2l)g_{n+1} \] (3.42)

\[ H_- g_n = -lng_{n-1}. \] (3.43)
Now we consider the functions \( f_n \in F \) given by:

\[
f_n \equiv \gamma_n g_n
\]  

(3.44)

where

\[
\gamma_n \equiv \left[ \frac{\Gamma(n + 2l)}{\Gamma(2l)\Gamma(n + 1)} \right]^{1/2}.
\]  

(3.45)

It is easy to prove that \( \{f_n\}_{n \in \mathbb{N}} \) is an orthonormal basis in \( H \) and that (3.41)-(3.43) give (3.8), (3.9) if we make \( f_n \to f_{l+n} \).

For \( D^{l-} \) one obtains the infinitesimal generators making \( H_0 \to H_0, H_\epsilon \to -(H_\epsilon)^* \). The analysis above stays true if we take \( \forall n \in -N \):

\[
g_n(z) \equiv (-1)^n z^{-n}
\]  

(3.46)

and in (3.45) make \( n \to -n \). Q.E.D.

The representations \( D^{l,\eta} \) constitute the so-called discrete series. As remarked in the Introduction they can be obtained with the help of geometric quantization [6]. Let us note that they induce true representations for \( SL(2,R) \) iff \( l \in \mathbb{N}^* \), and one easily obtains the formulae of [10]. They induces true representations for \( L^+ \) iff \( l \in N^* \).

3.6 From Subsections 2.2-2.5 we conclude that \( T^{\tau,q} \) (\( \tau \in [0,1), q > \tau(1 - \tau) \)) and \( D^{l,\eta} \) (\( l \in R_+, \eta = \pm \)) are all the unitary irreducible representations of \( G \), up to a unitary equivalence.

4. Conclusions

We have given a complete list for all the projective unitary irreducible representations of the Poincaré group in 1+2 dimensions, up to unitary equivalence. Except for the case of tachyons we have been able to produce very explicit formulae which can be of practical use in various applications.

It will be interesting to proceed further to field theory and develop the appropriate generalizations for invariant wave equations (see [6], [7]), Fock-Cook formalism for an arbitrary statistics and the basic theorems of axiomatic field theory. These subjects will be approached elsewhere.
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