Virtual immersions and a characterization of symmetric spaces

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Abstract We define virtual immersions, as a generalization of isometric immersions in a pseudo-Riemannian vector space. We show that virtual immersions possess a second fundamental form, which is in general not symmetric. We prove that a manifold admits a virtual immersion with skew-symmetric second fundamental form, if and only if it is a symmetric space, and in this case the virtual immersion is essentially unique.

Keywords Symmetric space · Isometric immersion · Pseudo-Euclidean space

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1 Introduction

Often in Riemannian geometry, one needs to embed a Riemannian manifold into Euclidean or pseudo-Euclidean space. In this paper, we introduce a generalized and more “intrinsic” version of such embeddings and utilize them to give a new characterization of symmetric spaces.

Given a Riemannian manifold $M$ and an isometric immersion $\phi : M \rightarrow V$ into a vector space $(V, \langle , \rangle)$ endowed with a nondegenerate symmetric bilinear form (a pseudo-Euclidean vector space), then the pullback $\phi^* TV$ is a trivial vector bundle over $M$, the differential $\phi_*$ defines an immersion $\phi_* : TM \rightarrow \phi^* TV$, and the classical results on isometric immersions

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show that the canonical (flat) connection on $\phi^*TV$ induces, by projecting onto $TM$, the Levi Civita connection on $M$. We use these properties to define a virtual immersion of a Riemannian manifold $M$, as a flat bundle $M \times V$, with $V$ a pseudo-Euclidean vector space, together with an isometric embedding $TM \to M \times V$ such that the flat connection on $M \times V$ induces the Levi Civita connection on $M$ (see Definition 1 for an equivalent definition).

It turns out that, just like the usual isometric immersions, one can define a second fundamental form, but unlike the usual setting this is in general not symmetric. As a matter of fact, it can be easily shown that a virtual immersion is (locally) induced by an isometric immersion, if and only if the second fundamental form is symmetric.

In [2], we first introduced virtual immersions with $V$ Euclidean (rather than pseudo-Euclidean) in the context of verifying, for certain compact symmetric spaces, a conjecture of Marques-Neves-Schoen about the index of closed minimal hypersurfaces. In that same paper, it was proved that, when $V$ has a Euclidean metric, virtual immersions with skew-symmetric second fundamental form exist only on compact symmetric spaces (cf. [2], Theorem B).

The main result of this paper is to extend the classification of virtual immersions with skew-symmetric second fundamental form to the more general case in which the metric on $V$ is pseudo-Euclidean:

**Main Theorem** Let $(M, g)$ be a Riemannian manifold. Then, $M$ admits a virtual immersion $\Omega$ with skew-symmetric second fundamental form if and only if it is a symmetric space. In this case, $\Omega$ is essentially unique.

Virtual immersions, in other words, provide a bundle-theoretic characterization of symmetric spaces, although we expect them to have independent interest on more general spaces.

The paper is organized as follows: in Sect. 2, we define virtual immersions and their second fundamental form and establish their fundamental equations. In Sect. 3, we prove the “if” part of the Main Theorem, producing a virtual immersion with skew-symmetric second fundamental form on any symmetric space. In Sect. 4, we prove the “only if” part of the Main Theorem, showing that a virtual immersion with skew-symmetric second fundamental form forces the manifold to be a symmetric space. In this last section, we also glue the pieces together and prove the Main Theorem.

**Convention:** We will denote by $R$ the curvature tensor, and follow the sign convention $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]}Z$.

### 2 Virtual immersions

Let $(M, g)$ be a Riemannian manifold, and let $(V, \langle \cdot, \cdot \rangle)$ denote a real vector space endowed with a nondegenerate, symmetric bilinear form. We call such $(V, \langle \cdot, \cdot \rangle)$ a pseudo-Euclidean vector space. A $V$-valued virtual immersion of $M$ is, roughly speaking, an immersion of $TM$ into the trivial bundle $M \times V$, such that the natural flat connection on $M \times V$ induces the Levi-Civita connection of $M$. Such objects generalize isometric immersions of Riemannian manifolds in pseudo-Euclidean space.

Although this is the idea behind virtual immersions, we introduce such structures in a different way, more convenient for computations. See Proposition 2 for a proof that the two definitions coincide.

**Definition 1** Let $(M, g)$ be a Riemannian manifold, and $(V, \langle \cdot, \cdot \rangle)$ a finite-dimensional, pseudo-Euclidean real vector space. Let $\Omega$ be a $V$-valued one-form on $M$. We say $\Omega$ is a virtual immersion if the following two conditions are satisfied:
a) $\langle \Omega(X), \Omega(Y) \rangle = g(X, Y)$ for every $p \in M$, and every $X, Y \in T_p M$.
b) $\langle d\Omega(X, Y), \Omega(Z) \rangle = 0$ for every $p \in M$, and every $X, Y, Z \in T_p M$.

We say two virtual immersions $\Omega_i : TM \to V_i, i = 1, 2$ are equivalent if there is a linear isometry $(V_1, \langle \cdot, \cdot \rangle_1) \to (V_2, \langle \cdot, \cdot \rangle_2)$ making the obvious diagram commute.

Letting $\pi : TM \to M$ denote the foot-point projection, any virtual immersion $\Omega : TM \to V$ induces a vector bundle homomorphism $(\pi, \Omega) : TM \to M \times V$. By condition (a) in the definition, this map is an isometric immersion of (pseudo-Euclidean) vector bundles.

Fixing $p \in M$, denote by $\Omega_p : T_p M \to V$ the restriction of $\Omega$ to $T_p M$. Since $\Omega_p$ is an isometric immersion, the space $T_p M$ can be identified with its image, which we will still denote by $T_p M$. Moreover, since the metric on $T_p M$ is positive definite, its orthogonal complement $v_p M := (T_p M)^{\perp} \subset V$ is transverse to $T_p M$ and thus $V$ splits orthogonally as $V = T_p M \oplus v_p M$. This yields the orthogonal decomposition $M \times V = TM \oplus v M$. Given $(p, X) \in M \times V$, we shall write $X = X^T + X^\perp$ for the decomposition into the tangent and normal parts.

The natural flat connection $D$ on $M \times V$ induces a connection $D^T$ (respectively $D^\perp$) on $TM$ (resp. $v M$), given by $D^T_Y X = (D_X Y)^T$ (resp. $D^\perp_Y \eta = (D_X \eta)^T$). Here, $X, Y$ are vector fields on $M$, while $\eta$ is a section of the normal bundle.

**Proposition 2** Let $\Omega$ be a $V$-valued one-form on $M$ satisfying condition (a) in Definition 1. Then, $\Omega$ is a virtual immersion if and only if the flat connection $D$ on $M \times V$ satisfies $D^T = \nabla$, where $\nabla$ denotes the Levi Civita connection on $M$.

**Proof** Since $\Omega$ already satisfies condition (a), it is a virtual immersion if and only if condition (b) holds as well, that is, $d\Omega(X, Y)^T = 0$ for every point $p$ and every $X, Y \in T_p M$. Recall that

$$d\Omega(X, Y) = D_X Y - D_Y X - [X, Y]$$

so that taking the tangent part yields

$$d\Omega(X, Y)^T = D^T_X Y - D^T_Y X - [X, Y].$$

Condition (a) implies that $D^T$ is compatible with the metric $g$, and by the above formula condition (b) is equivalent to $D^T$ being torsion-free. Since these two properties characterize the Levi Civita connection, the result follows. \qed

Given a virtual immersion $\Omega : TM \to V$ and a group $\Gamma$ of isometries of $M$, we say that $\Omega$ is $\Gamma$-invariant if for every $\gamma \in \Gamma$, $\Omega \circ d\gamma = \Omega$, where $d\gamma : TM \to TM$ denotes the differential of $\gamma$. The following result is straightforward:

**Lemma 3** Let $\Omega : TM \to V$ be a virtual immersion, and let $\pi : \tilde{M} \to M$ denote a covering. Then, $\pi^* \Omega = \Omega \circ d\pi : \tilde{TM} \to V$ is a virtual immersion, which is invariant under the deck group of $\tilde{M} \to M$. Conversely, if $\Omega : TM \to V$ is invariant under a group $\Gamma$ acting freely on $M$ by isometries, and $\pi : M \to M' = M / \Gamma$ denotes the quotient, then $\Omega$ descends to a virtual immersion $\Omega' : TM' \to V$ such that $\Omega' = \pi^* \Omega'$.

Given a virtual immersion $\Omega : TM \to V$ and a linear isometric immersion $\iota : V \to W$, there is an induced virtual immersion $\iota \circ \Omega : TM \to W$. We want to rule out these trivial extensions.

**Definition 4** A virtual immersion $\Omega : TM \to V$ is called full if the image of $\Omega$ spans $V$. 
For any virtual immersion $\Omega : TM \to W$, defining the subspace $V = \text{span}(\Omega(TM))$ and letting $\iota : V \to W$ denote the inclusion, one obtains the following:

**Lemma 5** Given any virtual immersion $\Omega : TM \to W$, there exist a full immersion $\Omega' : TM \to V$ and a linear isometric immersion $\iota : V \to W$ such that $\Omega = \iota \circ \Omega'$.

Given a virtual immersion, one can define a second fundamental form and shape operator.

**Definition 6** Let $\Omega$ be a $V$-valued virtual immersion, $X, Y$ be smooth vector fields on $M$, and $\eta$ a smooth section of $\nu M$. Define the second fundamental form of $\Omega$ by

$$II : TM \times TM \to \nu M, \quad II(X, Y) = (D_X Y) \perp = D_X(\Omega(Y)) - \Omega(\nabla_X Y)$$

and the shape operator in the direction of a normal vector $\eta$ by

$$S_\eta : TM \to TM, \quad S_\eta(X) = -(D_X \eta)^T.$$

Note that the second fundamental form and the shape operator are tensors. In view of Proposition 2, we may write

$$D_X Y = \nabla_X Y + II(X, Y)$$

$$D_X \eta = -S_\eta X + D_X^\perp \eta.$$

**Example 7** Given a Riemannian manifold $M$, let $\phi : M \to V$ be an isometric immersion into a pseudo-Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$. Then, $\Omega = d\phi : TM \to V$ is a virtual immersion, with symmetric second fundamental form. On the other hand, for any virtual immersion $\Omega$, the normal part of $d\Omega(X, Y)$ equals $II(X, Y) - II(Y, X)$ and, since the tangent part of $d\Omega$ vanishes, it follows that if $II$ is symmetric, then $d\Omega = 0$, which implies that locally $\Omega = d\phi$ for some map $\phi : M \to V$. By condition (a) in the definition of virtual immersion, this map must be an isometric immersion.

**Proposition 8** Let $\Omega$ be a virtual immersion of the Riemannian manifold $(M, g)$ with values in $V$. Then, the following identities hold:

(a) Weingarten’s equation

$$\langle S_\eta(X), Y \rangle = \langle II(X, Y), \eta \rangle$$

(b) Gauss’ equation

$$R(X, Y, Z, W) = \langle II(Y, W), II(X, Z) \rangle - \langle II(X, W), II(Y, Z) \rangle$$

(c) Ricci’s equation

$$\left\langle R^\perp(X, Y)\eta, \zeta \right\rangle = -\left\langle (S_\eta^\perp S_\zeta^\perp - S_\zeta^\perp S_\eta^\perp)X, Y \right\rangle$$

(d) Codazzi’s equation

$$\langle (D_X II)(Y, Z), \eta \rangle = \langle (D_Y II)(X, Z), \eta \rangle .$$

**Proof** The proof is the same as in the classical case. For sake of completeness, we recall it here.

Fix a point $p$ and let $V = T_pM \oplus \nu_p M$ be the orthogonal splitting into tangent and normal part. Recall that this is possible even though $(V, \langle \cdot, \cdot \rangle)$ is not Euclidean, because the restriction to $T_pM$ is positive definite. Given vectors $X, Y, Z, W \in T_p M$, extend them locally to vector
fields (denoted with the same letters). Differentiating the equation $D_Y Z = \nabla_Y Z + II(Y, Z)$ with respect to $X$, one gets

$$D_X D_Y Z = D_X (\nabla_Y Z + II(Y, Z))$$

$$= \nabla_X \nabla_Y Z + II(X, \nabla_Y Z) + D_X (II(Y, Z)).$$

Since the connection $D$ is flat, its curvature vanishes, and one has

$$0 = D_{[X, Y]} Z - D_X D_Y Z + D_Y D_X Z$$

$$= \left(\nabla_{[X, Y]} Z + II([X, Y], Z)\right) - \left(\nabla_X \nabla_Y Z + II(X, \nabla_Y Z) + D_X (II(Y, Z))\right)$$

$$+ \left(\nabla_Y \nabla_X Z + II(Y, \nabla_X Z) + D_Y (II(X, Z))\right)$$

$$= R(X, Y) Z - (D_X II)(Y, Z) + (D_Y II)(X, Z).$$

(4)

Taking the product of both sides of (4) with $W \in T_p M$, one gets

$$0 = \langle R(X, Y) Z, W \rangle - \langle D_X II(Y, Z), W \rangle + \langle D_Y II(X, Z), W \rangle$$

$$= \langle R(X, Y) Z, W \rangle + \langle II(Y, Z), D_X W \rangle - \langle II(X, Z), D_Y W \rangle$$

$$= \langle R(X, Y) Z, W \rangle + \langle II(Y, Z), II(X, W) \rangle - \langle II(X, Z), II(Y, W) \rangle$$

which recovers the Gauss’ equation.

On the other hand, taking the product of equation (4) with $\eta \in \nu_p M$, one obtains

$$0 = \langle -(D_X II)(Y, Z) + (D_Y II)(X, Z), \eta \rangle$$

which is Codazzi’s Equation.

Ricci’s equation is obtained similarly, but starting with equation $D_X \eta = -S_\eta X + D_X^\perp \eta$ instead of $D_X Y = \nabla_X Y + II(X, Y)$. Weingarten’s equation is immediate. \qed

3 Virtual immersions on symmetric spaces

This section is devoted to proving the first part of the main theorem. Namely, given a symmetric space $M$, we show how to produce a virtual immersion $\Omega : TM \to V$ with skew-symmetric second fundamental form.

Since the universal cover $\tilde{M}$ of $M$ is a simply connected symmetric space, by the de Rham decomposition theorem it splits isometrically into irreducible factors, $\tilde{M} = \prod_{i=0}^k \tilde{M}_i$, where $\tilde{M}_0 = \mathbb{R}^r$ and none of the other factors is Euclidean. For each $i = 0, \ldots, k$, choose $p_i \in \tilde{M}_i$, and let $G_i$ be the subgroup of the isometry group of $M_i$, generated by transvections (i.e., products of two symmetries). Then, $G_i$ is connected and, by the standard theory of symmetric spaces, it acts transitively on $\tilde{M}_i$. Moreover, $(G_i, H_i)$ is a symmetric pair, where $H_i = (G_i)_{p_i}$. Notice that $G_0 = \mathbb{R}^r$, and $H_0 = 1$.

Let $\pi_i : G_i \to \tilde{M}_i = G_i/H_i$ denote the projection $\pi_i(g) := g \cdot p_i$. Let $g_i, h_i$ denote the Lie algebras of $G_i, H_i$ respectively, and let $m_i \subset g_i$ be a complement of $h_i$ satisfying $[m_i, m_i] \subset h_i$, $[m_i, h_i] \subset h_i$. Then, the Killing form $B_i$ on $g_i$ restricts to a negative-definite (resp. positive-definite, zero) symmetric form on $m_i$ when $\tilde{M}_i$ is of compact (resp. noncompact, Euclidean) type. Moreover, $m_i$ can be canonically identified with $T_{p_i} \tilde{M}_i$ via $(\pi_i)_*$ and, for $i > 0$, the restriction $g_{\tilde{M}_i} / \tilde{M}_i$ of the metric $g_{\tilde{M}_i}$ to $T_{p_i} \tilde{M}_i$ corresponds to $\lambda_i B_i |_{m_i}$ for some negative (resp. positive) value $\lambda_i \in \mathbb{R}$ if $\tilde{M}_i$ is of compact (resp. noncompact) type.

Letting $G = \prod_{i=0}^k G_i$ and $H = \prod_{i=0}^k H_i$, then $(G, H)$ is a symmetric pair, with $G$ acting transitively on $\tilde{M}$ and such that $H = G_{p_i}, p = (p_0, \ldots, p_k)$. In particular, $\tilde{M}$ is diffeomorphic
to $G/H$, via the map sending $[g] = [g_0, \ldots, g_k] \in G/H$ to $g \cdot p = (g_0 \cdot p_0, \ldots, g_k \cdot p_k)$. Let

$$g = \bigoplus_{i=0}^{k} g_i, \quad h = \bigoplus_{i=0}^{k} h_i, \quad m = \bigoplus_{i=0}^{k} m_i,$$

so that $g = h \oplus m$, $[m, m] \subseteq h$ and $[m, h] \subseteq m$. Define $G \times_H m$ as the quotient of $G \times m$ by the action of $H$ given by $h \cdot (g, X) = (gh^{-1}, \text{Ad}_h X)$, and denote by $[g, X]$ the image of $(g, X) \in G \times m$ under the quotient map. There is a natural $G$-action on $G \times_H m$, defined by $g' \cdot [g, X] = [g'g, X]$. Extend now the isomorphism

$$m = \bigoplus_{i=0}^{k} m_i \to \bigoplus_{i=0}^{k} T_{p_i} \tilde{M}_i = T_{\tilde{p}} \tilde{M}$$

to the $G$-equivariant bundle isomorphism $G \times_H m \to T\tilde{M}$ given by $[g, X] \mapsto dg(X)$. We can now define the virtual immersion $\tilde{\Omega}_0$ on $\tilde{M}$. Endow $g = \mathbb{R}' \oplus \bigoplus_{i=1}^{k} g_i$ with the nondegenerate symmetric bilinear form

$$\langle \cdot, \cdot \rangle = g_{\tilde{M}}|_{\mathbb{R}'} \oplus \bigoplus_{i=1}^{k} \lambda_i B_i,$$

and define

$$\tilde{\Omega}_0 : T\tilde{M} \simeq G \times_H m \longrightarrow g$$

$[g, X] \mapsto \text{Ad}_g(X)$

(5)

**Lemma 9** The $g$-valued one-form $\tilde{\Omega}_0$ defined in Equation (5) is a virtual immersion. At $[g] \in \tilde{M}$, the tangent and normal spaces are $\text{Ad}_g m$ and $\text{Ad}_g h$, respectively. The second fundamental form is skew symmetric, given by

$$II([g, X], [g, Y]) = \text{Ad}_g([X, Y]).$$

**Proof** We begin by showing that condition a) in the definition of virtual immersion holds for $\tilde{\Omega}_0$. By $G$-equivariance it is enough to show that

$$\tilde{\Omega}_0|[e] \times m : [e] \times m \to g$$

is an isometric embedding. The embedding is simply the canonical inclusion, therefore given $X, Y \in m \simeq T_{[e]} \tilde{M}$, and denoting $X_i, Y_i$ the projections of $X, Y$ onto $m_i \simeq T_{p_i} \tilde{M}_i$, one has

$$\left\langle \tilde{\Omega}_0(X), \tilde{\Omega}_0(Y) \right\rangle = \langle X, Y \rangle$$

$$= \langle X_0, Y_0 \rangle + \sum_{i=1}^{k} \langle X_i, Y_i \rangle$$

$$= g_{\tilde{M}}(X_0, Y_0) + \sum_{i=1}^{k} \lambda_i B_i(X_i, Y_i)$$

$$= g_{\tilde{M}}(X_0, Y_0) + \sum_{i=1}^{k} \tilde{g}_{\tilde{M}}(X_i, Y_i)$$

$$= g_{\tilde{M}}(X, Y).$$
It is clear from (5) that the tangent space is $\text{Ad}_g \mathfrak{m}$, thus the normal space must be $\text{Ad}_g \mathfrak{h}$.

Let $X \in \mathfrak{g}$. Under the identification of $TM$ with $G \times_H \mathfrak{m}$ that we are using, the action field $X^*$ is given by

$$X^* \mathfrak{g} = \{ g, (\text{Ad}_{g^{-1}} X)_\mathfrak{m} \}$$

Indeed, $X^* \mathfrak{g}$ is a vector of the form $\{ g, v \}$, with $v = dg^{-1}(X^* \mathfrak{g}) \in \mathfrak{m}$. One computes

$$v = dg^{-1} \left( \frac{d}{dt} \Big|_{t=0} \{ e^{tX} g \} \right) = \frac{d}{dt} \Big|_{t=0} \{ g^{-1} e^{tX} g \}$$

$$= d\pi_\mathfrak{e}(\text{Ad}_{g^{-1}} X)$$

$$= (\text{Ad}_{g^{-1}} X)_\mathfrak{m},$$

where $\pi$ denotes the map $\pi : G \to G/H$.

Given $X, Y \in \mathfrak{g}$, we then have

$$D_{X^*} \tilde{\Omega}_0(Y^*) = \frac{d}{dt} \Big|_{t=0} \tilde{\Omega}_0 \{ e^{tX} g, (\text{Ad}_{e^{tX} g})^{-1} Y \}$$

$$= \frac{d}{dt} \Big|_{t=0} \text{Ad}_{e^{tX} g} (\text{Ad}_{g^{-1} e^{-tX}} Y)_\mathfrak{m}$$

$$= \text{Ad}_g (\{ \text{Ad}_{g^{-1} X}, (\text{Ad}_{g^{-1} Y})_\mathfrak{m} \} - (\text{Ad}_{g^{-1}} [X, Y])_\mathfrak{m}).$$

By $G$-equivariance, it is enough to show that, for every $X, Y \in T_{[e]} \tilde{M} \simeq \mathfrak{m}$, we have $d\tilde{\Omega}_0(X^*, Y^*)_{[e]} = 0$ and $H(X, Y)_{[e]} = [X, Y]$. Plugging $g = e$ in the equation above, and using the fact that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, we have

$$D_{X^*} \tilde{\Omega}_0(Y^*) = [X, Y].$$

The tangent part of this is zero, so that

$$d\tilde{\Omega}_0(X^*, Y^*)_{[e]} = D_{X^*} \tilde{\Omega}_0(Y^*)_{[e]} - D_{Y^*} \tilde{\Omega}_0(X^*)_{[e]} = 0 - 0 = 0$$

which means that $\tilde{\Omega}_0$ is a virtual immersion.

Moreover, $H(X, Y)_{[e]} = D_{X^*} \tilde{\Omega}_0(Y^*)_{[e]} = [X, Y]$. \hfill $\square$

Using the lemma above, we can prove

**Lemma 10** The virtual immersion $\tilde{\Omega}_0 : T\tilde{M} \to \mathfrak{g}$ is full.

**Proof** It is enough to prove that $\tilde{\Omega}_0(T_{\tilde{\rho}} \tilde{M}) \oplus \text{span}\{H(X, Y) \mid X, Y \in T_{[e]} \tilde{M}\} = \mathfrak{g}$. By Lemma 9,

$$\tilde{\Omega}_0(T_{\tilde{\rho}} \tilde{M}) = \mathfrak{m}, \quad \text{span}\{H(X, Y) \mid X, Y \in T_{\tilde{\rho}} \tilde{M}\} = [\mathfrak{m}, \mathfrak{m}],$$

therefore this reduces to proving $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$. If not, then there exists a nonzero $H \in \mathfrak{h}$ such that $B(H, [X, Y]) = 0$ for all $X, Y$ in $\mathfrak{m}$. By Ad-invariance of the Killing form, this implies $B([H, Y], X) = 0$ for all $X, Y \in \mathfrak{m}$. Since $[H, Y] \in \bigoplus_{i=1}^s \mathfrak{g}_i$ and $B$ is nondegenerate on $\bigoplus_{i=1}^s \mathfrak{g}_i$, it follows that $[H, Y] = 0$ for every $Y \in \mathfrak{m}$. This implies that $\text{Ad}(\exp tH) \in H = G_{\tilde{\rho}}$ is the identity on $\mathfrak{m} = T_{\tilde{\rho}} \tilde{M}$, which implies $H = 0$ hence the contradiction.

Having defined the virtual immersion $\tilde{\Omega}_0$ on $\tilde{M}$, the goal is now to prove that it descends to a virtual immersion on $M$. This is equivalent to proving that $\tilde{\Omega}_0$ is invariant under the group $\Gamma$ of deck transformations of $\tilde{M} \to M$. 
Lemma 11  Let $\Gamma$ be a discrete subgroup of isometries of $\tilde{M}$ acting freely on $\tilde{M}$. Then, the virtual immersion $\tilde{\Omega}_0$ defined above is invariant under $\Gamma$ if and only if $M = \tilde{M}/\Gamma$ is a symmetric space.

Proof Suppose first that $M$ is a symmetric space, and let $\tau : \tilde{M} \to M$ denote the universal cover of $M$. Then, since the symmetry $s_{\tilde{p}}$ at any $\tilde{p} \in \tilde{M}$ is a lift of the corresponding symmetry $s_p$ at $p = \tau(\tilde{p}) \in M$, it follows that for any $\gamma \in \Gamma$, $s_{\tilde{p}} \gamma s_{\tilde{p}}$ is a lift of the identity or, in other words, $s_{\tilde{p}} \gamma s_{\tilde{p}} \in \Gamma$. Since $M = \tilde{M}/\Gamma$ is a symmetric space and in particular a homogeneous space, by the main theorem in [4] it follows that every element $\gamma \in \Gamma$ is a Clifford-Wolf translation, i.e., the displacement function $q \mapsto d(q, \gamma(q))$ is constant. In particular, for any $\tilde{p} \in \tilde{M}$ the isometry $\gamma s_{\tilde{p}} \gamma s_{\tilde{p}} = \gamma \cdot (s_{\tilde{p}} \gamma s_{\tilde{p}}) \in \Gamma$ is a Clifford-Wolf translation.

We claim that $\gamma s_{\tilde{p}} \gamma s_{\tilde{p}}$ fixes $\tilde{p}$, which implies that $\gamma s_{\tilde{p}} \gamma s_{\tilde{p}} = Id$. In fact, since $\gamma$ is a Clifford-Wolf translation, then $\gamma^{-1}(\tilde{p})$, $\tilde{p}$, $\gamma(\tilde{p})$ all lie on the same geodesic $c(t)$ (cf. [3, Theorem 1.6]). Parametrize $c(t)$ so that $c(0) = \tilde{p}$, $c(1) = \gamma(\tilde{p})$, $c(-1) = \gamma^{-1}(\tilde{p})$. Then, since $s_{\tilde{p}}(\tilde{p}) = \tilde{p}$ and $s_{\tilde{p}}(c(t)) = c(-t)$, it follows that

$$s_{\tilde{p}} \gamma s_{\tilde{p}}(\tilde{p}) = s_{\tilde{p}} \gamma(\tilde{p}) = s_{\tilde{p}}(c(1)) = c(-1) = \gamma^{-1}(\tilde{p})$$

and therefore $\gamma s_{\tilde{p}} \gamma s_{\tilde{p}}(\tilde{p}) = \tilde{p}$, thus proving the claim.

If follows that $s_{\tilde{p}} \gamma s_{\tilde{p}} = \gamma^{-1}$ and therefore, every $\gamma \in \Gamma$ commutes with every transvection. Since $G$ is generated by transvections, then $\Gamma$ commutes with $G$ and thus $\text{Ad}_{\gamma}$ acts trivially on $g$ for every $\gamma \in \Gamma$.

Given $\Omega_0 : T\tilde{M} = G \times_H m \to g$ and fixing $\gamma \in \Gamma$, the map $\Omega_0 \circ \gamma : T\tilde{M} = G \times_H m \to g$ is given by

$$(\Omega_0 \circ \gamma)(g, X) = \Omega_0([\gamma g, X]) = \text{Ad}_{\gamma}(X) = \text{Ad}_{\gamma}(\text{Ad}_g X) = \text{Ad}_g X = \Omega_0([g, X])$$

and therefore $\Omega_0$ is invariant under $\Gamma$.

On the other hand, suppose now that $\Omega_0$ is invariant under $\Gamma$. Then, for every $\gamma \in \Gamma$, $\text{Ad}_{\gamma}|_{\tilde{g}} = id$, i.e., $\Gamma$ commutes with $G$ (recall, $G$ is connected). Since $G$ acts transitively on $\tilde{M}$ it follows that every $\gamma \in \Gamma$ is a Clifford-Wolf translation: in fact, for any $\tilde{p}, \tilde{q} \in \tilde{M}$, letting $g \in G$ be such that $g \cdot \tilde{p} = \tilde{q}$, one has

$$d(\tilde{p}, \gamma \tilde{p}) = d(g \tilde{p}, \gamma(g \tilde{p})) = d(g \tilde{p}, \gamma(g \tilde{p})) = d(\tilde{q}, \gamma \tilde{q}).$$

Moreover, since $G$ is also normalized by the symmetries $s_{\tilde{p}}$ centered at any $\tilde{p} \in \tilde{M}$, it follows that $s_{\tilde{p}} \gamma s_{\tilde{p}}$, and thus $\gamma s_{\tilde{p}} \gamma s_{\tilde{p}}$, commute with $G$ for any $\gamma \in \Gamma$. In particular $\gamma s_{\tilde{p}} \gamma s_{\tilde{p}}$ is again a Clifford-Wolf translation. However, just as before it follows that $\gamma s_{\tilde{p}} \gamma s_{\tilde{p}}$ fixes $\tilde{p}$, and therefore $s_{\tilde{p}} \gamma s_{\tilde{p}} = \gamma^{-1}$. In particular, every symmetry $s_{\tilde{p}}$ satisfies $s_{\tilde{p}} \Gamma s_{\tilde{p}} = \Gamma$. Therefore, for any point $p = \tau(\tilde{p}) \in M/\Gamma$, one can define a symmetry $s_p : M \to M$ by $s_p[\tilde{q}] = [s_{\tilde{p}}(\tilde{q})]$. In particular, $M$ is a symmetric space. \hfill \Box

4 Rigidity of virtual immersions with skew-symmetric second fundamental form

In this section we prove the second half of the main theorem. Namely, given a minimal virtual immersion $\Omega : TM \to V$ with skew-symmetric second fundamental form, we prove that $M$ is a symmetric space and $\Omega$ is equivalent to the virtual immersion defined in the previous section.
Lemma 12 \ Let \((M, g)\) be a Riemannian manifold, and \(\Omega\) a \(V\)-valued virtual immersion with skew-symmetric second fundamental form \(H\). Then:

(a) \(\langle II(X, Y), II(Z, W) \rangle = \langle R(X, Y)Z, W \rangle\).
(b) \(\langle DX II(Y, Z) \rangle = -R(Y, Z)X\).
(c) \(\nabla R = 0\). In particular, \((M, g)\) is a locally symmetric space.

Proof \ (a) \ Start with Gauss’ equation (see Proposition 8(b)),

\[
\langle R(X, Y)Z, W \rangle = \langle II(Y, W), II(X, Z) \rangle - \langle II(X, W), II(Y, Z) \rangle
\]

Applying the first Bianchi identity yields

\[
0 = -2\left(\langle II(X, Y), II(Z, W) \rangle + \langle II(Y, Z), II(X, W) \rangle + \langle II(Z, X), II(Y, W) \rangle\right)
\]

so that using Gauss’ equation one more time we arrive at

\[
\langle R(X, Y)Z, W \rangle = \langle II(X, Y), II(Z, W) \rangle.
\]

(b) \ First we argue that \(\langle DX II(Y, Z) \rangle\) is tangent. Indeed, for any normal vector \(\eta\), Codazzi’s equation (Proposition 8(d)) says that

\[
\langle DX II(Y, Z), \eta \rangle = \langle DX II(X, Z), \eta \rangle.
\]

Thus the trilinear map \((X, Y, Z) \mapsto \langle DX II(Y, Z), \eta \rangle\) is symmetric in the first two entries and skew-symmetric in the last two entries, which forces it to vanish. Next we let \(W\) be any tangent vector and compute

\[
\langle DX II(Y, Z), W \rangle = \langle DX II(Y, Z), W \rangle = -\langle II(Y, Z), DX W \rangle
\]

\[
= -\langle II(Y, Z), II(X, W) \rangle = -\langle R(Y, Z)X, W \rangle
\]

where in the last equality follows we have used part (a).

(c) \ Since the natural connection \(D\) on \(M \times V\) is flat, it follows that for any vector fields \(X, Y, Z, W\), we have

\[
0 = DX(DY(II(Z, W))) - DY(DX(II(Z, W))) - D_{[X,Y]}(II(Z, W)).
\]

Fix \(p \in M\), and take vector fields such that \([X, Y] = 0\) and \(\nabla Z = \nabla W = 0\) at \(p \in M\). Then, evaluating the equation above at \(p \in M\), we have

\[
0 = DX(DY(II(Z, W))) - DY(DX(II(Z, W))) - D_{[X,Y]}(II(Z, W)).
\]

Taking the tangent part yields \(\nabla X(\nabla R)(Z, W)Y = \nabla Y(\nabla R)(Z, W)X\). Taking inner product with \(T \in T_p M\) we have

\[
\langle \nabla R\rangle(Z, W, Y, T, X) = \langle \nabla R\rangle(Z, W, X, T, Y),
\]

that is, \(\nabla R\) is symmetric in the third and fifth entries. But \(\nabla R\) is also skew-symmetric in the third and fourth entries, so that \(\nabla R = 0\). \(\Box\)
The virtual immersion $\Omega$ on $M$ lifts to a virtual immersion with skew-symmetric second fundamental form $\tilde{\Omega}$ on the universal cover $\tilde{M}$ of $M$. In the following Proposition, we prove that $\tilde{\Omega}$ is equivalent to $\Omega_0$.

**Proposition 13** Let $(\tilde{M}, g_{\tilde{M}})$ be a symmetric space, and let $\Omega_j : T\tilde{M} \to V_j$, for $j = 1, 2$ be virtual immersions with skew-symmetric second fundamental forms $\Pi_j$. Assume $V_1, V_2$ are full. Then, $\Omega_1, \Omega_2$ are equivalent.

**Proof** Define a connection $\tilde{D}$ on the vector bundle $T\tilde{M} \oplus \wedge^2 T\tilde{M}$ by

$$\tilde{D}_W(Z, \alpha) = (\nabla_W Z - R(\alpha) W, W \wedge Z + \nabla_W \alpha)$$

Here, for $\alpha = \sum_u X_u \wedge Y_u$, we define $R(\alpha) := \sum_u R(X_u, Y_u)$. Define bundle homomorphisms $\tilde{\Omega}_j : T\tilde{M} \oplus \wedge^2 T\tilde{M} \to \tilde{M} \times V_j$, for $j = 1, 2$, by

$$\tilde{\Omega}_j(Z, \alpha) = \left(\ell, \Omega_j(Z) + \Pi_j(\alpha)\right)$$

for $Z \in T_{\ell} M$, $\alpha = \sum_u X_u \wedge Y_u \in \wedge^2 T_{\ell} M$, and $\Pi(\alpha) = \sum_u \Pi(X_u, Y_u)$. By Lemma 12(b), given vector fields $Z, W$ and a section $\alpha$ of $\wedge^2 T\tilde{M}$, we have

$$(D_j)_W(\tilde{\Omega}_j(Z, \alpha)) = \tilde{\Omega}_j(\tilde{D}_W(Z, \alpha))$$  \hspace{1cm} (6)

where $D_j$ denotes the natural flat connection on $\tilde{M} \times V_j$. This implies that the image of $\tilde{\Omega}_j$ is $D_j$-parallel, and hence, by minimality of $V_j$, that $\tilde{\Omega}_j$ is onto $\tilde{M} \times V_j$. In particular, for $j = 1, 2$ the normal space in $V_j$ is spanned by $\Pi_j(X, Y)$, for $X, Y \in T_{\ell} M$.

Now, we claim that

$$\ker \tilde{\Omega}_1 = \ker \tilde{\Omega}_2 = \left\{ (0, \alpha) \mid \alpha \in \wedge^2 T_{\ell} M, R(\alpha) = 0 \right\}.$$  

Indeed, on the one hand if $R(\alpha) = 0$, then for every $\beta \in \wedge^2 T_{\ell} M$ one obtains that

$$\langle \Pi_j(\alpha), \Pi_j(\beta) \rangle = \langle R(\alpha), \beta \rangle = 0$$

by Lemma 12(a). Since the inner product on $\nu_{\ell} M \subset V_j$ is nondegenerate and the normal space in $V_j$ consists of the elements $\Pi_j(\beta)$ by the conclusion above, it follows that $\Pi(\alpha) = 0$ and thus $\tilde{\Omega}_j(0, \alpha) = 0 + \Pi_j(\alpha)$ is zero.

On the other hand, if $\tilde{\Omega}_j(Z, \alpha) = 0$, then $\Omega_j(Z) = 0$ and $\Pi_j(\alpha) = 0$, which implies $Z = 0$ and, for every $\beta \in \wedge^2 T_{\ell} M$, $0 = \langle \Pi_j(\alpha), \Pi_j(\beta) \rangle = \langle R(\alpha), \beta \rangle$ by Lemma 12(a). Since the inner product on $\wedge^2 T_{\ell} M$ is nondegenerate, it follows that $R(\alpha) = 0$ in $\wedge^2 T_{\ell} M$, and this ends the proof of the claim.

Since $\tilde{\Omega}_i, i = 1, 2$ are both surjective with the same kernel, there is a well-defined bundle isomorphism $L : M \times V_1 \to M \times V_2$ by

$$L(\tilde{\Omega}_1(Z, \alpha)) = \tilde{\Omega}_2(Z, \alpha)$$

for $Z \in T_{\ell} M, \alpha \in \wedge^2 T_{\ell} M$.

We claim that the linear map $L_p = L|_{\{p\} \times V_1} : \{p\} \times V_1 \to \{p\} \times V_2$ is independent of $p \in M$. Indeed, given two points $p, q \in M$, choose a curve $\gamma(t)$ in $M$ joining $p$ to $q$. Choose $\tilde{D}_1$-parallel vector fields $Z, X_i, Y_i$ along $\gamma(t)$ such that $\tilde{\Omega}_1(Z, \sum X_i \wedge Y_i)$ is constant equal to $v \in V_1$. Then, by (6), $\tilde{D}_j(Z, \sum X_i \wedge Y_i) \subset \ker \tilde{\Omega}_1$. But by Lemma 12(a), $\ker \tilde{\Omega}_1 = \ker \tilde{\Omega}_2$. Therefore, again by (6), we see that $L(v)$ is constant along $\gamma$, so that $L_p = L_q$. Calling this linear map $L$, we have $\tilde{\Omega}_2 = L \circ \tilde{\Omega}_1$ by construction. In particular, $\tilde{\Omega}_2 = L \circ \tilde{\Omega}_1$, finishing the proof that $\Omega_1$ and $\Omega_2$ are equivalent. \hfill $\Box$

Piecing all together, we can prove the main Theorem:
Proof of the Main Theorem  Suppose first that $M$ is a symmetric space, and let $\tilde{M}$ be its universal cover. From Lemma 9, there exists a skew-symmetric virtual immersion $\tilde{\Omega}_0 : T\tilde{M} \to V$ with $V = \mathfrak{g}$. By Lemma 11, since $M$ is symmetric, $\tilde{\Omega}_0$ is invariant under $\pi_1(M)$ and therefore $\tilde{\Omega}_0$ descends to a skew-symmetric virtual immersion $\Omega : TM \to V$. Suppose now, on the other hand, that $M$ admits a full, skew-symmetric virtual immersion $\Omega : TM \to V$. By Lemma 12, $M$ is locally symmetric, and thus the universal cover $\tilde{M}$ is a symmetric space and $\Omega$ lifts to a skew-symmetric virtual immersion $\tilde{\Omega} : T\tilde{M} \to V$ invariant under the action of $\Gamma = \pi_1(M)$. Since $\tilde{M}$ also admits the virtual immersion $\tilde{\Omega}_0$, which is full by Lemma 10, it follows from the rigidity Proposition 13 that $\tilde{\Omega} = \tilde{\Omega}_0$, and in particular $\tilde{\Omega}_0$ is invariant under the action of $\Gamma$. By Lemma 11, it follows that $M$ is a symmetric space. □

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