A general method is presented for determining the maximum electric energy in a bounded region of optical fields with given time-averaged flux of electromagnetic energy. Time-harmonic fields are considered whose plane wave expansion consists of propagating plane waves only, i.e., evanescent waves are excluded. The bounded region can be quite general: it can consist of finitely many points, or be a curve, a curved surface or a bounded volume. The optimum optical field is eigenfield corresponding to the maximum eigenvalue of a compact linear integral operator which depends on the bounded region. It is explained how these optimum fields can be realized by focussing appropriate pupil fields. The special case that the region is a circular disc perpendicular to the direction of optical axis is investigated by numerical simulations.
I. Introduction

In optics it is often desirable to maximize the electric energy in a certain bounded region of space. This is for example important to optically excite certain molecules or atoms efficiently, to trap molecules or small particles using optical tweezers, to enhance scattering or absorption of light in some volume and in numerous other cases [1–3]. An important method to realize optimum concentrations of light is by shaping the pupil field of an objective lens [4]. With spatial light modulators (SLMs) not only amplitude and phase but also the polarization can be varied pixel by pixel. In this way pupil fields can be shaped to achieve optimized focused fields [5–10].

In this paper we present a general mathematical formulation for achieving optimum concentration of electric energy. With a similar method also the magnetic energy or the total electromagnetic energy, i.e., the sum of the electric and magnetic energies, could be maximized, but since at optical frequencies the main interaction with matter occurs through the electric field, it is more interesting to maximize the electric energy. To be more precise, we consider time-harmonic electromagnetic fields which propagate in a given direction, say the positive $z$-direction, and which have numerical aperture $NA$ smaller than the refractive index $n$ of the medium in which the propagation takes place. This means that the wave vectors of the plane wave expansion of the field make an angle with the positive $z$-axis which does not exceed the angle $\alpha_{\text{max}}$ where $\sin \alpha_{\text{max}} = NA/n$. The waves in the angular spectrum are thus all propagating and there are no evanescent waves.

In Section III we formulate the optimisation problem in terms of the plane wave amplitudes. The problem is to determine the amplitudes for which the electric energy in a given region is maximum for the given values of the NA and for given mean power flow. The region can be quite general: it can for example be a bounded 3D volume, a bounded...
curved surface, a bounded curve or it can consist of one or several points. Furthermore, by a slight generalisation of the formulation of the optimisation problem, we include the case of maximizing the squared modulus of only a specific electric field component, instead of the electric energy. Because our formulation is general, it includes many previously studied optimisation problems such as [6, 11] as special cases.

We remark that when evanescent waves would be allowed in the plane wave expansion, the maximum electric energy in any bounded region can, for every prescribed value of the mean flow of power, be made infinite. The reason is that the evanescent waves do not contribute to the mean power flow and therefore their amplitude is not constrained by it. The evanescent waves do however contribute to the electric energy density, therefore the energy density can be made arbitrarily large if evanescent waves would be taken into account. Excluding evanescent waves from the optimisation problem means that in this paper we study only fields that are radiated by sources which are many wavelengths from the region where the energy is maximized. We assume in particular that there are no structures and objects close to the region of interest which could generate evanescent waves by scattering.

Many different groups have contributed to the shaping and optimisation of optical fields in or near the focal point of a lens. This has led to important applications and to improved optical sensitivities and resolution. In contrast to most previous work where field enhancements are studied, we aim at determining the maximum possible energy in a given region and for a given NA and power flow. The optimisation problem has infinitely many variables (i.e., all amplitudes of the plane waves inside the given NA) and hence it is a problem in an infinitely dimensional function space. This means that the optimum fields obtained by our method are fundamental and are not only interesting from the point of view of applications but are also of theoretical interest.

In Section II and Section II A the optimisation problem is formulated mathematically and expressed in terms of the plane wave amplitudes. By applying the Lagrange multiplier rule to the optimisation problem, it is shown in Section III that the optimum plane wave amplitudes are eigenfield of a linear integral operator corresponding to the maximum eigenvalue. This linear integral operator is compact and Hermitian when the proper scalar product is from the point of view of applications but are also of theoretical interest.

In Section V we study in detail the special case of maximizing the electric energy in a disc perpendicular to the $z$-axis. By using cylindrical coordinates and applying a Fourier series to expand the functions with respect to polar angle, the 2D integral equation becomes equivalent to a set of 1D integral equations, with the radial variable as integration variable. In Section V we first discuss the case that the disc has vanishing radius, which means that the average of the electric energy in the focal point of the lens is optimized. For this case the solutions can be obtained in closed form and we retrieve previously published results. Then we consider discs with positive radius. In this case the solutions can only be obtained by numerical computations. It is found that when the radius of the disc is varied, only two types of solutions occur, namely one for which the optimum field in the pupil is predominantly linearly polarised solutions can only be obtained by numerical computations. It is found that when the radius of the disc is varied, only two types of solutions occur, namely one for which the optimum field in the pupil is predominantly linearly polarised in some direction, whereas the second type has azimuthal polarised pupil field. As the radius and the NA are varied, the numerically solutions are alternating between these two cases. For certain values of the NA and radius of the disc, both type of solutions occur, i.e., both give the same maximum electric energy.

II. THE OPTIMISATION PROBLEM

Consider a time-harmonic electromagnetic field in an unbounded homogeneous nonmagnetic lossless medium with refractive index $n$. The electromagnetic field is written as

$$\mathbf{E}(r, t) = \text{Re}[\mathbf{E}(r)e^{-i\omega t}],$$

$$\mathbf{H}(r, t) = \text{Re}[\mathbf{H}(r)e^{-i\omega t}],$$

where the frequency $\omega > 0$ and $\mathbf{E}(r)$ and $\mathbf{H}(r)$ are the complex time-independent electric and magnetic fields. We will assume that with respect to the cartesian coordinate system $(x, y, z)$ with unit vectors $\hat{x}, \hat{y}, \hat{z}$, the electromagnetic field has numerical aperture $\text{NA} \leq n$ and is propagating in the positive $z$-direction. This means that the plane wave vectors of the angular spectrum of the fields have angles with the positive $z$-axis that are smaller than $\alpha_{\text{max}},$
with $NA = n \sin \alpha_{\text{max}}$. The complex electric field can be expanded into plane waves

$$E(\mathbf{r}) = \frac{1}{4\pi^2} \int_\Omega \mathbf{A}(\mathbf{k}_\perp) e^{i\mathbf{k} \cdot \mathbf{r}} \, d\mathbf{k}_\perp,$$

where $\Omega$ is the disk in two-dimensional reciprocal space with radius $k_0NA$:

$$\Omega = \{(k_x, k_y) : \sqrt{k_x^2 + k_y^2} \leq k_0NA\},$$

where $k_0 = \omega \sqrt{\varepsilon_0} \mu_0$ is the wave number in vacuum and the vectors $\mathbf{k}$ and $\mathbf{k}_\perp$ are defined by

$$\mathbf{k}_\perp = k_z \mathbf{\hat{x}} + k_y \mathbf{\hat{y}}, \quad \text{and} \quad \mathbf{k} = \mathbf{k}_\perp + k_z \mathbf{\hat{z}},$$

where $k_z = \sqrt{k^2 - |\mathbf{k}_\perp|^2}$ and $k = k_0n$ is the wave number inside the medium. We choose the usual branch of the square root so that the cut is along the negative real axis and the square root of a positive real number is positive. Hence, the plane waves of (3) are propagating in the positive $z$-direction. Faraday’s law $\nabla \times \mathbf{E} = i\omega \mu_0 \mathbf{H}$ implies that the complex magnetic field $\mathbf{H}$ can be written as

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\omega \mu_0 4\pi^2} \int_\Omega \mathbf{k} \times \mathbf{A}(\mathbf{k}_\perp) e^{i\mathbf{k} \cdot \mathbf{r}} \, d\mathbf{k}_\perp.$$  

Apart from the fact that the fields $\mathbf{E}$ and $\mathbf{H}$ consist of a superposition of plane waves that propagate in the positive $z$-direction and whose wave vectors have angle with the $z$-axis which does not exceed $\alpha_{\text{max}}$, the field is completely general. For the time being we will not consider how such a field can be realized in practice. This issue will be addressed in Section IV where the focussing of an appropriate pupil field is described.

Let $S$ be a bounded set. $S$ can be quite general: it can for example consist of finitely many points, be a curve, a (curved) surface or a bounded volume. It will be convenient in what follows to associate with $S$ a distribution $T_S$ in $\mathbb{R}^3$ defined such that for every smooth test function $\phi(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}$:

$$\langle T_S, \phi \rangle_{\mathbb{R}^3} = \frac{1}{|S|} \int_S \phi \, dS.$$  

If $S$ is a set of finitely many points, then $|S|$ is the number of points and the integral should be interpreted as the sum of the values of $\phi$ in those points. In other words, if $S$ is a set of points, $T_S$ is a sum of delta-functions at these points, divided by the number of points in $S$. If $S$ is a curve, surface or volume, $|S|$ is the length, surface area or volume, respectively. Hence, $\langle T_S, \phi \rangle_{\mathbb{R}^3}$ is simply the average of $\phi$ over $S$. The subscript $\mathbb{R}^3$ at the bracket emphasizes that $T_S$ is a distribution on $\mathbb{R}^3$. We further elaborate on these examples in Section IIIA.

Because the electric field is free of divergence it follows from (3) that $\mathbf{A}(\mathbf{k}_\perp) \cdot \mathbf{k} = 0$, i.e., $\mathbf{A}(\mathbf{k}_\perp)$ is perpendicular to the wave vector. To incorporate this property we will write the plane wave amplitudes on the positively oriented orthonormal basis in reciprocal space defined by:

$$\mathbf{s}(\mathbf{k}_\perp) = \frac{\mathbf{\hat{k}} \times \mathbf{\hat{z}}}{|\mathbf{k} \times \mathbf{\hat{z}}|} = \frac{1}{|\mathbf{k}_\perp|} \begin{pmatrix} k_y \\ -k_x \\ 0 \end{pmatrix}, \quad \mathbf{p}(\mathbf{k}_\perp) = \frac{\mathbf{s} \times \mathbf{\hat{k}}}{|\mathbf{s} \times \mathbf{\hat{k}}|} = \frac{1}{k_\perp} \begin{pmatrix} -k_x k_z \\ -k_y k_z \\ k_\perp^2 \end{pmatrix},$$

where for a vector $\mathbf{v}$: $|\mathbf{v}| = \sqrt{|v_x|^2 + |v_y|^2 + |v_z|^2}$. Note that $\mathbf{\hat{k}} \cdot \mathbf{\hat{p}} = 0$ and $\mathbf{\hat{k}} \cdot \mathbf{s} = 0$. We write $\mathbf{A} : \Omega \to \mathbb{C}^3$ as

$$\mathbf{A}(\mathbf{k}_\perp) = a_p(\mathbf{k}_\perp) \mathbf{\hat{p}}(\mathbf{k}_\perp) + a_s(\mathbf{k}_\perp) \mathbf{s}(\mathbf{k}_\perp),$$

where $a_p$ is the component parallel to the plane through the wave vector $\mathbf{k}$ and the $z$-axis and $a_s$ is the component perpendicular to this plane. To prevent confusion with $\mathbf{A}$, which is a vector with three components, the vector field with two components: $(a_p, a_s) : \Omega \to \mathbb{C}^2$ will be denoted as $\mathbf{a}$, i.e., we write

$$\mathbf{a}(\mathbf{k}_\perp) = \begin{pmatrix} a_p(\mathbf{k}_\perp) \\ a_s(\mathbf{k}_\perp) \end{pmatrix}.$$  

The electromagnetic field written in terms of $a_p$ and $a_s$ becomes

$$E(\mathbf{r}) = \frac{1}{4\pi^2} \int_\Omega \left[ a_p(\mathbf{k}_\perp) \mathbf{\hat{p}}(\mathbf{k}_\perp) + a_s(\mathbf{k}_\perp) \mathbf{s}(\mathbf{k}_\perp) \right] e^{i\mathbf{k} \cdot \mathbf{r}} \, d\mathbf{k}_\perp,$$

(10)
\[ H(r) = n \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{1}{4\pi^2} \int_\Omega \left[ a_p(k_\perp)\hat{S}(k_\perp) - a_s(k_\perp)\hat{P}(k_\perp) \right] e^{ikr} d^2k_\perp. \] (11)

The time-averaged power flow in the positive \( z \)-direction is given by the integral over a plane \( z = \text{constant} \) of the \( z \)-component of half the real part of the complex Poynting vector \( S = \mathbf{E} \times \mathbf{H}^* \):

\[ P = \iint_{S, z} \frac{1}{2} \Re \{ S(r) \} \cdot \hat{z} \, dx \, dy. \] (12)

Note that, since there is no absorption, the integral (12) does not depend on the chosen plane \( z = \text{constant} \). Using Plancherel’s theorem together with \( \mathbf{A} \cdot \mathbf{k} = 0 \), we get as in [11, Equation 25] that the power flow (12) can be expressed in the amplitudes of the plane waves as

\[ P(\mathbf{E}) = \frac{1}{\omega \mu_0} \frac{1}{8\pi^2} \int_\Omega |a(k_\perp)|^2 k_z d^2k_\perp = \frac{1}{\omega \mu_0} \frac{1}{8\pi^2} \int_\Omega |a_p(k_\perp)|^2 + |a_s(k_\perp)|^2 k_z d^2k_\perp. \] (13)

To formulate the optimisation problem we define the functional \( G_{S,\Pi} \) as follows:

\[ G_{S,\Pi}(\mathbf{E}) = \frac{1}{|S|} \iint_S |\Pi(\mathbf{E})|^2 \, dS = \langle T_S, |\Pi(\mathbf{E})|^2 \rangle_{\mathbb{R}^3}, \] (14)

where \( \Pi : \mathbb{C}^3 \to \mathbb{C}^3 \) is a projection on some linear subspace of \( \mathbb{C}^3 \). The goal is to determine the electric field \( \mathbf{E} \) for which \( G_{S,\Pi}(\mathbf{E}) \) is maximal for given power \( P(\mathbf{E}) = P_0 \).

We give a number of examples.

1. Let \( \hat{\mathbf{v}} \) be a real unit vector and let \( \Pi(\mathbf{E}) = \mathbf{E} \cdot \hat{\mathbf{v}} \), i.e., \( \Pi \) is the projection on the direction defined by \( \hat{\mathbf{v}} \). Then

\[ G_{S,\Pi}(\mathbf{E}) = \frac{1}{|S|} \iint_S |\mathbf{E} \cdot \hat{\mathbf{v}}|^2 \, dS, \] (15)

is the average over the region \( S \) of the squared modulus of the projection of \( \mathbf{E} \) along \( \hat{\mathbf{v}} \). The optimisation problem then amounts to maximizing the average over the region \( S \) of the squared modulus of the component of \( \mathbf{E} \) along the unit vector \( \hat{\mathbf{v}} \).

2. \( \Pi = \mathcal{I} \), the identity, i.e., \( \Pi(\mathbf{E}) = \mathbf{E} \). In this case

\[ G_{S,\Pi}(\mathbf{E}) = \frac{1}{|S|} \iint_S |\mathbf{E}|^2 \, dS, \] (16)

is the averaged electric energy in the region \( S \) and the optimisation problem amounts to maximizing the electric energy averaged over the region \( S \).

3. \( \Pi(\mathbf{E}) = E_x \hat{x} + E_y \hat{y} \), i.e., \( \Pi \) is the projection on the \( z = 0 \) plane and

\[ G_{S,\Pi}(\mathbf{E}) = \frac{1}{|S|} \iint_S |E_x|^2 + |E_y|^2 \, dS. \] (17)

Hence in this case the squared modulus of the electric field perpendicular to the \( z \)-axis is maximized over the region \( S \).

A. Expression of the optimisation problem in terms of plane wave amplitudes

We will express the optimisation problem in terms of plane wave amplitudes \( \mathbf{a} \). First we express functional \( G_{S,\Pi} \) in terms of \( \mathbf{a} \). We remark that (3) implies that for every \( z \):

\[ \mathcal{F}_2(\Pi(\mathbf{E}))(k_\perp, z) = \Pi(\mathbf{A})(k_\perp) e^{ik_z z}, \] (18)

where \( \mathcal{F}_2 \) is the 2D Fourier transform defined by

\[ \mathcal{F}_2(f)(k_\perp) = \iint_{\mathbb{R}^2} f(r_\perp) e^{-ir_\perp \cdot k_\perp} \, dr_\perp. \] (19)
Its inverse is given by
\[
\mathcal{F}_2^{-1}(g)(\mathbf{r}_\perp) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} g(\mathbf{k}_\perp) e^{i \mathbf{k}_\perp \cdot \mathbf{r}_\perp} \, d\mathbf{k}_\perp,
\tag{20}
\]
where \( \mathbf{r}_\perp = (x, y) \). Furthermore, let \( \mathcal{F}_3 \) be the 3D Fourier transform defined by
\[
\mathcal{F}_3(f)(\mathbf{\xi}_\perp, \mathbf{\zeta}_z) = \iiint_{\mathbb{R}^3} f(\mathbf{r}_\perp, z) e^{-i(\mathbf{\xi}_\perp \cdot \mathbf{r}_\perp + \mathbf{\zeta}_z \cdot z)} \, d\mathbf{r}_\perp \, dz,
\tag{21}
\]
where \( \mathbf{\xi}_\perp = (\xi_x, \xi_y) \). It may seem more natural to use \( \mathbf{k}_\perp, k_z \) as Fourier variables, but in this paper the combination of \( k_z, k_x \) always implies that \( k_z = \sqrt{k_0^2 n^2 - |\mathbf{k}_\perp|^2} \) whereas in the 3D Fourier transform the three variables are independent and to prevent confusion we use therefore \( \mathbf{\xi}_\perp, \mathbf{\zeta}_z \) as variables of the 3D Fourier transform. We shall often write
\[
k_z(\mathbf{k}_\perp) = \sqrt{k_0^2 n^2 - |\mathbf{k}_\perp|^2},
\tag{22}
\]
to emphasize the dependence of \( k_z \) on \( \mathbf{k}_\perp \). The following result is derived in Appendix \ref{appendix}:
\[
G_{S,II}(E) = \frac{1}{(2\pi)^4} \int_{\Omega} \iiint_{\mathbb{R}^3} \mathcal{F}_3(T_S)(\mathbf{k}_\perp - \mathbf{k}_z', k_z(\mathbf{k}_\perp) - k_z(\mathbf{k}_z')) \Pi(\mathbf{A})(\mathbf{k}_\perp) \cdot \Pi(\mathbf{A})(\mathbf{k}_z') \, d\mathbf{k}_\perp \, d\mathbf{k}_z', \tag{23}
\]
This is the expression of \( G_{S,II} \) in terms of the plane wave amplitudes \( \mathbf{A} \). By substituting \( \mathbf{A}(\mathbf{k}_\perp) = a_p(\mathbf{k}_\perp) \hat{\mathbf{p}}(\mathbf{k}_\perp) + a_s(\mathbf{k}_\perp) \hat{\mathbf{s}}(\mathbf{k}_\perp) \) we find
\[
\Pi(\mathbf{A})(\mathbf{k}_\perp) \cdot \Pi(\mathbf{A})(\mathbf{k}_z') = M_{II}(\mathbf{k}_\perp, \mathbf{k}_z') \cdot a(\mathbf{k}_\perp) \cdot a(\mathbf{k}_z')^* ,
\tag{24}
\]
where \( M_{II} \) is the real matrix defined by
\[
M_{II}(\mathbf{k}_z', \mathbf{k}_\perp) = \begin{pmatrix} \Pi(\hat{\mathbf{p}})(\mathbf{k}_\perp) \cdot \Pi(\hat{\mathbf{p}})(\mathbf{k}_z') & \Pi(\hat{\mathbf{p}})(\mathbf{k}_\perp) \cdot \Pi(\hat{\mathbf{s}})(\mathbf{k}_z') \\ \Pi(\hat{\mathbf{s}})(\mathbf{k}_\perp) \cdot \Pi(\hat{\mathbf{p}})(\mathbf{k}_z') & \Pi(\hat{\mathbf{s}})(\mathbf{k}_\perp) \cdot \Pi(\hat{\mathbf{s}})(\mathbf{k}_z') \end{pmatrix}. \tag{25}
\]
This matrix is real because the vectors \( \hat{\mathbf{s}} \) and \( \hat{\mathbf{p}} \) are real. We remark that
\[
M_{II}(\mathbf{k}_\perp, \mathbf{k}_z') = M_{II}(\mathbf{k}_\perp, \mathbf{k}_z')^T, \tag{26}
\]
where the right-hand side is the transpose matrix. By substituting \ref{eq:24} in \ref{eq:23} we obtain the desired expression of \( G_{S,II} \) in terms of \( \mathbf{a} \):
\[
G_{S,II}(\mathbf{a}) = \frac{1}{(2\pi)^4} \int_{\Omega} \iiint_{\mathbb{R}^3} \mathcal{F}_3(T_S)(\mathbf{k}_\perp - \mathbf{k}_z', k_z(\mathbf{k}_\perp) - k_z(\mathbf{k}_z')) \cdot M_{II}(\mathbf{k}_\perp, \mathbf{k}_z') \cdot a(\mathbf{k}_\perp) \cdot a(\mathbf{k}_z')^* \, d\mathbf{k}_\perp \, d\mathbf{k}_z'. \tag{27}
\]
Remark: Because \( T_S \) is a real distribution on \( \mathbb{R}^3 \), its 3D Fourier transform satisfies:
\[
\mathcal{F}_3(T_S)(\mathbf{\xi}_\perp, \mathbf{\zeta}_z)^* = \mathcal{F}_3(T_S)(-\mathbf{\xi}_\perp, -\mathbf{\zeta}_z). \tag{28}
\]
With this property and \ref{eq:20} one can easily verify that the expression in the right-hand side of \ref{eq:27} is real, as should be.

The optimisation problem can now be formulated as a problem for the vector function \( \mathbf{a} : L^2(\Omega)^2 \rightarrow \mathbb{C}^2 \):

**Optimisation Problem 1:** max arg \( G_{S,II}(\mathbf{a}) \), for \( \mathbf{a} \in L^2(\Omega)^2 \) with \( P(\mathbf{a}) = P_0 \),

where the power is written as function of \( \mathbf{a} \) and \( P_0 \) is the total power. It is easy to see that the equality constraint on the power can be replaced by the inequality constraint \( P(\mathbf{a}) \leq P_0 \). In fact, if \( P(\mathbf{a}) < P_0 \), then \( G_{S,II}(\mathbf{a}) \) is increased by multiplying \( \mathbf{a} \) by a number larger than 1. So optimisation problem 1 is equivalent to:

**Optimisation Problem 2:** max arg \( G_{S,II}(\mathbf{a}) \), for \( \mathbf{a} \in L^2(\Omega)^2 \) with \( P(\mathbf{a}) \leq P_0 \).
III. LAGRANGE MULTIPLIER RULE FOR THE OPTIMUM PLANE WAVE AMPLITUDES

If \( a \) is a solution of Problem 2, it will satisfy the Lagrange multiplier rule \([12]\). To formulate this we need to compute the Gateaux derivatives of the functionals \( G_{S,\Pi} \) and \( P \). For the Gateaux derivative of \( G_{S,\Pi} \) (see \([27]\)) we get

\[
\delta G_{S,\Pi}(a)(b) = \lim_{t \to 0} \frac{1}{t} [G_{S,\Pi}(a + t b) - G_{S,\Pi}(a)]
= \frac{1}{8\pi^2} \text{Re} \int_{\Omega} \int_{\Omega} \int_{\Omega} \mathcal{F}_3(T_S)(k_\perp - k'_\perp, k_z(k_\perp) - k_z(k'_\perp)) \mathcal{M}_{\Pi}(k_\perp, k'_\perp) a(k_\perp) \cdot b(k_\perp)^* \, dk_\perp \, dk'_\perp,
\]

where in the last step we have used \([26]\) and \([28]\). We can similarly compute the Gateaux derivative of \( P \) (see \([13]\)):

\[
\delta P(a)(b) = \frac{1}{\omega \mu_0} \frac{1}{4\pi^2} \text{Re} \int_{\Omega} a(k_\perp) \cdot b(k_\perp)^* k_z(k_\perp) \, dk_\perp.
\]

Let \( a \) be a solution of Problem 2. According to the Lagrange multiplier rule there exists a number \( \Lambda' > 0 \) such that

\[
\delta G_{S,\Pi}(a)(b) - \Lambda' \delta P(a)(b) = 0, \text{ for all } b \in L^2(\Omega)^2.
\]

By substituting \([29]\) and \([30]\), and by choosing subsequently \( b \) real-valued and purely imaginary-valued, one can derive that

\[
\int_{\Omega} \mathcal{F}_3(T_S)(k_\perp - k'_\perp, k_z(k_\perp) - k_z(k'_\perp)) \mathcal{M}_{\Pi}(k_\perp, k'_\perp) a(k_\perp) \, dk_\perp = 2\pi^2 \Lambda' \int_{\Omega} k_z(k_\perp) a(k_\perp) = 0, \text{ for all } k_\perp \in \Omega.
\]

If we define

\[
\Lambda = \frac{2\pi^2 \Lambda'}{\omega \mu_0},
\]

and the operator \( T_{S,\Pi} : L^2(\Omega)^2 \mapsto L^2(\Omega)^2 \) by

\[
T_{S,\Pi}(a)(k_\perp) = \frac{1}{k_z(k_\perp)} \int_{\Omega} \mathcal{F}_3(T_S)(k_\perp - k'_\perp, k_z(k_\perp) - k_z(k'_\perp)) \mathcal{M}_{\Pi}(k_\perp, k'_\perp) a(k'_\perp) \, dk'_\perp,
\]

then \([32]\) implies that \( a \) is eigenvector of operator \( T_{S,\Pi} \) with eigenvalue \( \Lambda \):

\[
T_{S,\Pi}(a) - \Lambda a = 0.
\]

Note that, since \( G_{S,\Pi} \) and \( P \) are quadratic functionals

\[
\delta G_{S,\Pi}(a)(a) = 2G_{S,\Pi}(a), \text{ and } \delta P(a)(a) = 2P(a).
\]

Then \([31]\) implies for the eigenvector satisfying \( P(a) = P_0 \):

\[
G_{S,\Pi}(a) = \Lambda' P_0.
\]

We conclude that the eigenfield with the largest eigenvalue is the solution of the optimisation problem.

Summarizing, we have found that for any bounded set \( S \), (e.g., a set of finitely many points, a curve, a (curved) surface or a volume) the plane wave amplitudes of the field of which the average value of \(|\Pi(E)|^2\) over \( S \) is maximum for a given power and numerical aperture, is given by the eigenfield met maximum eigenvalue of operator \( T_{S,\Pi} \) whose kernel depends on the set \( S \) and the projection \( \Pi \). The function \( \mathcal{F}_3(T_S) \) which occurs in the kernel of \( T_{S,\Pi} \) is the 3D Fourier transform of the distribution \( T_S \) defined by \([6]\), evaluated at spatial frequencies \( k_\perp, k_z(k_\perp) \). The numerical aperture determines the domain \( \Omega \) of the space \( L^2(\Omega)^2 \) for the operator and the eigenfields.

A. Examples

We give some examples of the operator \( T_S \).
1. If $S$ consists of one point: $S = \{ (r_{10}, 0, z_0) \}$, with $r_{10} = (x_0, y_0)$, the optimisation problem amounts to maximizing $|\Pi(E)(r_{10}, 0, z_0)|^2$, i.e., the squared modulus of the projection $\Pi(E)$ in point $(r_{10}, 0, z_0)$, for the given power. In particular, if $\Pi = I$ (the identity), then the electric energy density in point $(r_{10}, 0, z_0)$ is maximized, whereas if $\Pi(E) = E \cdot \vec{\nu}$, the optimisation problem amounts to maximizing the modulus of the component of the electric field along the direction $\vec{\nu}$ in point $(r_{10}, 0, z_0)$. We have

$$T_S(r_{\perp}, z) = \delta(r_{\perp} - r_{10}, z - z_0),$$

and hence

$$\mathcal{F}_3(T_S)(\xi_{\perp}, \xi_z) = e^{-i\xi_{\perp} r_{10} - i\xi_z z_0}.$$  

Therefore operator $[\mathcal{F}_3]_{\Pi}$ becomes

$$\mathcal{T}_{S, \Pi}(\mathbf{a}(\kappa_{\perp})) = \frac{e^{-i r_{10} \cdot \kappa_{\perp} - i \xi_z k_z(\kappa_{\perp})}}{k_z(\kappa_{\perp})} \int \int e^{i r_{10} \cdot \kappa_{\perp}' - i \xi_z k_z(\kappa_{\perp}')} M_{\Pi}(\kappa_{\perp}, \kappa_{\perp}') \mathbf{a}(\kappa_{\perp}') d\kappa_{\perp}'.$$

2. Let $S$ be the part of the $z$-axis given by $-\ell/2 < z < \ell/2$. Then the optimisation problem is to maximize the average value of $|\Pi(E)|^2$ over the part of the $z$-axis given by $-\ell/2 \leq z \leq \ell/2$. We have

$$T_S(r_{\perp}, z) = \delta(r_{\perp}) \frac{1}{\ell} \mathbb{I}_{[-\ell/2, \ell/2]}(z),$$

where $\mathbb{I}_D(x) = 1$ if $x$ is in $D$ and 0 elsewhere. $T_S$ has Fourier transform,

$$\mathcal{F}_3(T_S)(\xi_{\perp}, \xi_z) = \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} e^{-i z \xi_z} \, dz = \sin(c)(\frac{\xi_z}{2}).$$

Hence $[\mathcal{F}_3]_{\Pi}$ becomes

$$\mathcal{T}_{S, \Pi}(\mathbf{a}(\kappa_{\perp})) = \frac{1}{k_z(\kappa_{\perp})} \int \int_{\Omega} \sin\left(\frac{\ell}{2} k_z(\kappa_{\perp}) - \frac{\xi_z k_z(\kappa_{\perp})}{2}\right) M_{\Pi}(\kappa_{\perp}, \kappa_{\perp}') \mathbf{a}(\kappa_{\perp}') d\kappa_{\perp}'.$$

3. If $S = B_R$ is the sphere of radius $R > 0$ and centre the origin, then the optimisation problem is to maximize for the given power the average value over this sphere of $|\Pi(E)|^2$. There holds for $r = (x, y, z)$:

$$T_S(r) = \mathbb{I}_{B_R}(r) = \frac{1}{4\pi R^3},$$

with $\mathbb{I}_{B_R}(r) = 1$ if $r < R$ and $= 0$ otherwise. We have

$$\mathcal{F}_3(T_S)(\xi_{\perp}, \xi_z) = \frac{2 J_3/2(R \sqrt{|\xi_{\perp}|^2 + \xi_z^2})}{(R \sqrt{|\xi_{\perp}|^2 + \xi_z^2})^{3/2}}.$$  

Hence,

$$\mathcal{T}_{S, \Pi}(\mathbf{a}(\kappa_{\perp})) = \frac{2}{k_z(\kappa_{\perp})} \int \int_{\Omega} \frac{J_3/2(R \sqrt{|\kappa_{\perp} - \kappa_{\perp}'|^2 + |k_z(\kappa_{\perp}) - k_z(\kappa_{\perp}')|^2})}{(R \sqrt{|\kappa_{\perp} - \kappa_{\perp}'|^2 + |k_z(\kappa_{\perp}) - k_z(\kappa_{\perp}')|^2})^{3/2}} M_{\Pi}(\kappa_{\perp}, \kappa_{\perp}') \mathbf{a}(\kappa_{\perp}') d\kappa_{\perp}'.$$

4. If $S$ is the circular disc $D_R$ of radius $R > 0$ in the plane $z = 0$ with centre the origin, then

$$T_S(r_{\perp}, z) = \mathbb{I}_{D_R}(r_{\perp}) \delta(z) = \frac{1}{\pi R^2} \mathbb{I}_{D_R}(r_{\perp}) \delta(z),$$

where $\mathbb{I}_{D_R}(r_{\perp}) = 1$ if $r_{\perp} < R$ and $= 0$ otherwise. We have

$$\mathcal{F}_3(T_S)(\xi_{\perp}, \xi_z) = 2 \frac{J_1(R |\kappa_{\perp}|)}{R |\kappa_{\perp}|}.$$  

Hence,

$$\mathcal{T}_{S, \Pi}(\mathbf{a}(\kappa_{\perp})) = \frac{2}{k_z(\kappa_{\perp})} \int \int_{\Omega} \frac{J_1(R |\kappa_{\perp} - \kappa_{\perp}'|)}{R |\kappa_{\perp} - \kappa_{\perp}'|} M_{\Pi}(\kappa_{\perp}, \kappa_{\perp}') \mathbf{a}(\kappa_{\perp}') d\kappa_{\perp}'.$$

We will study the optimisation problem for the disc in more detail in Section \ref{sec:optimisation_disc} and following sections.
B. Mathematical properties of the eigenvalue problem

We equip the space $L^2(\Omega)^2$ of square integrable vector fields $\mathbf{a} : \Omega \mapsto C^2$ (where $\Omega$ is, as before, the circle of finite numerical aperture (4)) with the scalar product:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \iint_{\Omega} \mathbf{a}(\mathbf{k}_\perp) \cdot \mathbf{b}(\mathbf{k}_\perp)^* d\mathbf{k}_\perp.$$  \hspace{1cm} (48)

This scalar product differs from the usual one by the factor $k_z(\mathbf{k}_\perp)$ in the integrand, but the corresponding norm is equivalent to the usual $L^2$-norm. Hence also with respect to this scalar product, $L^2(\Omega)^2$ is a Hilbert space. Moreover, the power $P(\mathbf{a})$ is proportional to $\langle \mathbf{a}, \mathbf{a} \rangle$. However this is not the motivation for introducing this scalar product: the reason is that with respect to this scalar product, operator $T_{S,\Pi}$ is symmetric:

$$\langle T_{S,\Pi}(\mathbf{a}), \mathbf{b} \rangle = \langle \mathbf{a}, T_{S,\Pi}(\mathbf{b}) \rangle.$$  \hspace{1cm} (49)

It is can furthermore be verified that the kernel of operator $T_{S,\Pi}$ is square integrable with respect to the measure $k_z(\mathbf{k}_\perp)k_z(\mathbf{k}_\perp')d\mathbf{k}_\perp, d\mathbf{k}_\perp'$:

$$\iint_{\Omega} \iint_{\Omega} k_z(\mathbf{k}_\perp)^2 |F_3(T_S)(\mathbf{k}_\perp - \mathbf{k}_\perp', k_z(\mathbf{k}_\perp) - k_z(\mathbf{k}_\perp'))\mathcal{M}_{\Pi}^2(\mathbf{k}_\perp, \mathbf{k}_\perp')|^2 k_z(\mathbf{k}_\perp)k_z(\mathbf{k}_\perp')d\mathbf{k}_\perp d\mathbf{k}_\perp' < \infty,$$  \hspace{1cm} (50)

for $i, j = 1, 2$. This property implies that operator $T_{S,\Pi}$ is a Hilbert-Schmidt operator, hence it is a self-adjoint compact operator $L^2(\Omega)^2 \mapsto L^2(\Omega)^2$. Therefore the spectrum of $T_{S,\Pi}$ is real and discrete with all eigenvalues having a finite number of linear independent eigenvectors. Furthermore, there exists a basis of $L^2(\Omega)^2$ of eigenvalues of $T_{S,\Pi}$ which is orthonormal with respect to the scalar product (48). The eigenvectors corresponding to the largest eigenvalue are, after being properly normalized to give the maximum allowed power, the solution of the optimisation problem. If the largest eigenvalue is not degenerate, the optimum field is unique. However in general it can happen that the largest eigenvalue is degenerate and then a finite number of linear independent solutions of the optimisation problem exist.

C. Scaling law

The optimisation problem depends on the chosen set $S$, the projection $\Pi$, the numerical aperture NA, the wavenumber $k = k_0n = 2\pi n/\lambda$ and the power $P_0$. Suppose that $S, \Pi$ and NA have been chosen and suppose that we change the size of the set $S$ by multiplying it by a number $\sigma > 0$: $S \mapsto \sigma S$. We have

$$\mathcal{F}_3(T_{\sigma S})(\xi_\perp, \xi_z) = \frac{1}{\sigma S} \iint_{\sigma S} e^{-i(\xi_\perp \cdot r + \xi_z z)} d\mathbf{r}_\perp dz$$

$$= \frac{1}{\sigma} \iint_{S} e^{-i\sigma \xi_\perp \cdot r' + \sigma \xi_z z'} d\mathbf{r}_\perp' dz'$$

$$= \mathcal{F}_3(T_S)(\sigma \xi_\perp, \sigma \xi_z).$$  \hspace{1cm} (51)

Then

$$\mathcal{F}_3(T_{\sigma S})(\mathbf{k}_\perp - \mathbf{k}_\perp', k_z(\mathbf{k}_\perp) - k_z(\mathbf{k}_\perp')) = \mathcal{F}_3(T_{\sigma S}) \left( k_\perp \frac{k_\perp'}{k}, k_z \sqrt{1 - k_\perp'^2/k^2} - k \sqrt{1 - k_\perp^2/k^2} \right)$$

$$= \mathcal{F}_3(T_{\sigma S}) \left( k_\perp \frac{k_\perp'}{k}, \sqrt{1 - k_\perp'^2/k^2} - \sqrt{1 - k_\perp^2/k^2} \right).$$  \hspace{1cm} (52)

Since $\mathcal{M}_{\Pi}$ actually is a function of $k_\perp/k, k_\perp'/k$ we write in this section

$$\mathcal{M}_{\Pi} \left( \frac{k_\perp}{k}, \frac{k_\perp'}{k} \right)$$

instead of $\mathcal{M}_{\Pi}(\mathbf{k}_\perp, \mathbf{k}_\perp')$. Substitution into (34) then gives

$$T_{\sigma S,\Pi}(\mathbf{a})(\mathbf{k}_\perp) = \frac{1}{k_z(\mathbf{k}_\perp)} \iint_{\Omega} \mathcal{F}_3(T_{\sigma S})(\mathbf{k}_\perp - \mathbf{k}_\perp', k_z(\mathbf{k}_\perp) - k_z(\mathbf{k}_\perp')) \mathcal{M}_{\Pi} \left( \frac{k_\perp}{k}, \frac{k_\perp'}{k} \right) \mathbf{a}(\mathbf{k}_\perp') d\mathbf{k}_\perp'$$

$$= \frac{k}{\sqrt{1 - k_\perp'^2/k^2}} \iint_{k_\perp'/k \leq \sin \alpha_{\text{max}}} \mathcal{F}_3(T_{\sigma S}) \left( \frac{k_\perp}{k}, \frac{k_\perp'}{k}, \sqrt{1 - k_\perp'^2/k^2} - \sqrt{1 - k_\perp^2/k^2} \right) \mathcal{M}_{\Pi} \left( \frac{k_\perp}{k}, \frac{k_\perp'}{k} \right) \mathbf{a}(\mathbf{k}_\perp') d\mathbf{k}_\perp'$$  \hspace{1cm} (53)
After dividing by $k$ this expression only depends on the product of $\sigma$ and $k$ and not on $\sigma$ and $k$ separately. By dividing eigenvalue problem (35) for $\sigma S$ by $k$ we obtain the eigenvalue problem

$$\frac{1}{k} T_{\sigma S, n}(a) - \frac{\Lambda}{k} a = 0,$$

(54)

which depends on $\sigma$ and $k$ only through the product $\sigma k$. We therefore conclude that the eigenvectors $a$ are the same if $\sigma k$ is kept constant while the eigenvalues $\Lambda$ are proportional to $k$, i.e., inversely proportional to the wavelength. Then (33) implies that

$$\Lambda' = \frac{\omega \mu_0 \Lambda}{2\pi^2} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{2\pi^2} k \Lambda \propto k^2,$$

(55)

and hence with (37) it follows that for fixed $\sigma k$ and fixed power $P_0$ the maximum value of the objective function is proportional to $k^2$.

Summarizing we conclude that if $\alpha_{\text{max}}$, $P_0$ and the product $\sigma k$ are fixed, where $\sigma$ is a scaling parameter of the set $S$ and $k$ is the wavenumber, the optimum fields are the same, while the maximum of the objective function depends quadratically on the wavenumber.

**IV. REALISATION OF THE OPTIMUM FIELDS**

An obvious way to realize the optimum field is in the focal region of a lens using spatial light modulators (SLMs) to shape the field in the entrance pupil. The numerical aperture of the lens should be at least as large as that of the optimum field. Since the plane wave amplitude of the electric field in the focal region corresponds 1-to-1 to the electric field in the entrance pupil, the desired amplitude, phase and polarization of these plane waves can be obtained by programming a number SLMs in series [5–10]. Let the electric field in the entrance pupil, the desired amplitude, phase and polarization of these plane waves can be the optimum field. Since the plane wave amplitude of the electric field in the focal region corresponds 1-to-1 to the optimum field. According to the theory of Ignatowski [13, 14], and Richards and Wolf [15] the radial and azimuthal components of the pupil field are proportional to $a_p$ and $a_s$, respectively:

$$E_p^e(\rho_e, \phi_e) = \frac{\sqrt{kk_z}}{2\pi i f} a_p \left( -k \frac{\rho_e}{f} \cos \phi_e, -k \frac{\rho_e}{f} \sin \phi_e \right),$$

(62)

$$E_s^e(\rho_e, \phi_e) = \frac{\sqrt{kk_z}}{2\pi i f} a_s \left( -k \frac{\rho_e}{f} \cos \phi_e, -k \frac{\rho_e}{f} \sin \phi_e \right).$$

(63)
where the factor $\sqrt{kk_\perp/(2\pi f)}$ is included to account for energy conservation and where

$$k_z = k\sqrt{1 - \frac{\rho^2}{f^2}}. \quad (64)$$

Hence, written on the $\{\mathbf{\hat{x}}, \mathbf{\hat{y}}\}$ basis, $\mathbf{E}^e$ becomes:

$$\mathbf{E}^e(\rho_e, \phi_e) = \sqrt{kk_\perp/(2\pi f)} \left[ a_p(-k_x, -k_y) \cos \phi_e - a_s(-k_x, -k_y) \sin \phi_e \mathbf{\hat{x}} \right.$$

$$+ \sqrt{kk_\perp/(2\pi f)} \left[ a_p(-k_x, -k_y) \sin \phi_e + a_s(-k_x, -k_y) \cos \phi_e \mathbf{\hat{y}} \right]. \quad (65)$$

The pupil field can be quite general as every point of the pupil can have its own elliptical state of polarization and the phase difference between the fields in different points of the pupil can be arbitrary.

V. OPTIMISING THE ELECTRIC ENERGY IN A DISC

In the remainder of this paper we will study the example of Section III A where the region $S$ is the disc $S = D_R = \{(r, z): r < R, z = 0\}$ and the projection is the identity: $\Pi = I$. Hence

$$G_{S,\Pi}(a) = \frac{1}{\pi R^2} \int_{D_R} |\mathbf{E}(x, y, 0)|^2 dx \, dy, \quad (66)$$

and the optimisation problem amounts to finding the field of which the electric energy averaged over the disc $D_R$ is maximum for given power $P_0$. The optimum plane wave amplitude $a$ is the eigenvector:

$$T_{S,\Pi}(a) - \Lambda a = 0. \quad (67)$$

corresponding to the largest eigenvalue $\Lambda$ of operator $T_{S,\Pi}$ defined by (47):

$$T_{S,\Pi}(a)(k_\perp) = \frac{2}{k_z(k_\perp)} \int_{\Omega} \frac{J_1(R|k_\perp - k'_\perp|)}{R|k_\perp - k'_\perp|} M_{\Pi}(k_\perp, k'_\perp) a(k'_\perp) \, dk'_\perp. \quad (68)$$

A. Expressions in terms of azimuthal and polar angles

It is convenient to change the integration variables from $k_\perp$ to azimuthal and polar angles $0 < \alpha < \alpha_{\text{max}}$ and $0 < \beta < 2\pi$, where $\alpha_{\text{max}} = \arcsin(NA/n)$. We have

$$k_x = k \sin \alpha \cos \beta, \quad k_y = k \sin \alpha \sin \beta, \quad (69)$$

so that the normalised wavevector $\mathbf{\hat{k}}$ is

$$\mathbf{\hat{k}}(k_\perp) = \mathbf{\hat{k}}(\alpha, \beta) = \begin{pmatrix} \sin \alpha \cos \beta \\ \sin \alpha \sin \beta \\ \cos \alpha \end{pmatrix}, \quad (70)$$

and $\mathbf{\hat{p}}$ and $\mathbf{\hat{s}}$ are given by

$$\mathbf{\hat{p}}(\alpha, \beta) = \begin{pmatrix} -\cos \alpha \cos \beta \\ -\cos \alpha \sin \beta \\ \sin \alpha \end{pmatrix}, \quad \mathbf{\hat{s}}(\beta) = \begin{pmatrix} \sin \beta \\ -\cos \beta \\ 0 \end{pmatrix}. \quad (71)$$

Writing

$$k'_x = k \sin \alpha' \cos \beta', \quad k'_y = k \sin \alpha' \sin \beta', \quad (72)$$

we get

$$|k_\perp - k'_\perp|^2 = k^2[\sin^2 \alpha + \sin^2 \alpha' - 2 \sin \alpha \sin \alpha' \cos(\beta - \beta')]. \quad (73)$$
and therefore
\[
\frac{2}{k_c(k_{\perp})} J_1(R|k_{\perp} - k'_{\perp}|) = \frac{2}{k \cos \alpha} \frac{J_1\left(kR\sqrt{\sin^2 \alpha + \sin^2 \alpha' - 2 \sin \alpha \sin \alpha' \cos(\beta - \beta')}\right)}{kR\sqrt{\sin^2 \alpha + \sin^2 \alpha' - 2 \sin \alpha \sin \alpha' \cos(\beta - \beta')}}. \tag{74}
\]

Furthermore, using (25) with \( \Pi = \mathcal{I} \),

\[
\mathcal{M}_\Pi(k_{\perp}, k'_{\perp}) = \mathcal{M}_\Pi(\alpha, \alpha', \beta - \beta'), \tag{75}
\]

where

\[
\mathcal{M}_\Pi(\alpha, \alpha', \beta) = \begin{pmatrix}
\cos \alpha \cos \alpha' \cos \beta + \sin \alpha \sin \alpha' \cos \beta \\
-\cos \alpha' \sin \beta \\
\cos \beta
\end{pmatrix}. \tag{76}
\]

Using

\[
dk'_{\perp} = dk_x' dk_y' = k^2 \sin \alpha' \cos \alpha' \, d\alpha' \, d\beta', \tag{77}
\]

we conclude that (68) becomes

\[
\mathcal{T}_{S,\Pi}(a)(\alpha, \beta) = \int_0^{\alpha_{\max}} \int_0^{2\pi} C_R(\alpha, \alpha', \beta - \beta') \mathcal{M}_\Pi(\alpha, \alpha', \beta - \beta') a(\alpha', \beta') \, d\alpha' \, d\beta'. \tag{78}
\]

where

\[
C_R(\alpha, \alpha', \beta) = \frac{2 k \cos \alpha' \sin \alpha' J_1(kR\sqrt{\sin^2 \alpha + \sin^2 \alpha' - 2 \sin \alpha \sin \alpha' \cos \beta})}{\cos \alpha} \frac{J_1\left(kR\sqrt{\sin^2 \alpha + \sin^2 \alpha' - 2 \sin \alpha \sin \alpha' \cos(\beta - \beta')}\right)}{kR\sqrt{\sin^2 \alpha + \sin^2 \alpha' - 2 \sin \alpha \sin \alpha' \cos(\beta - \beta')}}. \tag{79}
\]

Note that the integral with respect to \( \beta' \) is a convolution.

### B. Fourier series

We shall use a Fourier series for \( \beta \mapsto a(\alpha, \beta) \):

\[
a(\alpha, \beta) = \sum_{\ell} \hat{a}(\alpha, \ell) e^{i\ell \beta}, \tag{80}
\]

Let \( \mathcal{M}_R \) be the matrix

\[
\mathcal{M}_R(\alpha, \alpha, \beta) = C_R(\alpha, \alpha', \beta) \mathcal{M}_\Pi(\alpha, \alpha', \beta). \tag{81}
\]

Writing

\[
\mathcal{M}_R(\alpha, \alpha', \beta) = \sum_{\ell} \hat{\mathcal{M}}_R(\alpha, \alpha', \ell) e^{i\ell \beta}, \tag{82}
\]

it follows that

\[
\hat{\mathcal{M}}_R(\alpha, \alpha', \ell) = \sum_{\ell'} \hat{C}_R(\alpha, \alpha', \ell - \ell') \hat{\mathcal{M}}_\Pi(\alpha, \alpha', \ell'). \tag{83}
\]

where \( \hat{C}_R(\alpha, \alpha', \ell) \) are the Fourier coefficients of \( \beta \mapsto C_R(\alpha, \alpha', \beta) \) and

\[
\hat{\mathcal{M}}_\Pi(\alpha, \alpha', \ell) = \delta_{\ell, 0} \begin{pmatrix}
\sin \alpha \sin \alpha' & 0 \\
0 & 0
\end{pmatrix} + \delta_{\ell+1, 0} \begin{pmatrix}
\cos \alpha \cos \alpha' & \mp i \cos \alpha \\
\pm i \cos \alpha & \mp i \cos \alpha
\end{pmatrix}. \tag{84}
\]

where \( \delta_{\ell, \ell'} = 1 \) when \( \ell = \ell' \) and \( 0 \) otherwise. Hence,

\[
\hat{\mathcal{M}}_R(\alpha, \alpha', \ell) = \hat{C}_R(\alpha, \alpha', \ell + 1) \hat{\mathcal{M}}_\Pi(\alpha, \alpha', -1) + \hat{C}_R(\alpha, \alpha', \ell) \hat{\mathcal{M}}_\Pi(\alpha, \alpha', 0) + \hat{C}_R(\alpha, \alpha', \ell - 1) \hat{\mathcal{M}}_\Pi(\alpha, \alpha', 1). \tag{85}
\]
The Fourier coefficients $\hat{C}_R(\alpha, \alpha', \ell)$ are computed in Appendix B. Operator (78) can now be written as:

$$T_{S,II}(\alpha)(\alpha, \beta) = 2\pi \sum_{\ell} \int_{0}^{\alpha_{\text{max}}} \mathcal{M}_R(\alpha, \alpha', \ell) \mathbf{a}(\alpha', \ell) \, d\alpha' \, e^{i\ell\beta}.$$  \hfill (86)

By computing the Fourier coefficients of (67) it follows that the eigenvalue problem is equivalent to the following set of eigenvalue problems

$$2\pi \int_{0}^{\alpha_{\text{max}}} \mathcal{M}_R(\alpha, \alpha', \ell) \mathbf{a}(\alpha', \ell) \, d\alpha' - \Lambda \mathbf{a}(\alpha, \ell) = 0, \quad \text{for all integers } \ell,$$  \hfill (87)

(where eigenvalue $\Lambda$ depends on $\ell$). Hence we have obtained an eigenvalue problem for every Fourier component $\hat{\mathbf{a}}(\alpha, \ell)$. Because $C_R$ and $\mathcal{M}_I$ are real-valued, we have

$$\hat{C}_R(\alpha, \alpha', -\ell) = \hat{C}_R(\alpha, \alpha', \ell)^*, \quad \hat{\mathcal{M}}_I(\alpha, \alpha', -\ell) = \hat{\mathcal{M}}_I(\alpha, \alpha', \ell)^*,$$  \hfill (88)

and hence also

$$\hat{\mathcal{M}}_R(\alpha, \alpha', -\ell) = \hat{\mathcal{M}}_R(\alpha, \alpha', \ell)^*.$$  \hfill (90)

This implies that if $\hat{\mathbf{a}}(\alpha, \ell)$ is a solution of the eigenvalue problem for $\ell$, $\hat{\mathbf{a}}(\alpha, -\ell)^*$ is solution of the eigenvalue problem for $-\ell$. Furthermore the eigenvalues for $\ell$ and $-\ell$ are the same. We may therefore assume that the eigenfields $\mathbf{a}(\alpha, \beta)$ are real and harmonic in $\beta$:

$$\mathbf{a}(\alpha, \beta) = \hat{\mathbf{a}}(\alpha, \ell)^* e^{-i\ell\beta} + \hat{\mathbf{a}}(\alpha, \ell) e^{i\ell\beta} = 2 \text{Re}[\hat{\mathbf{a}}(\alpha, \ell) e^{i\ell\beta}].$$  \hfill (91)

It is clear that when $\hat{\mathbf{a}}$ is a solution for given $\ell$, so is $\hat{\mathbf{a}} e^{i\ell\psi}$, for arbitrary $\psi$. This implies that for every eigenvector $\mathbf{a}(\alpha, \beta)$, $\mathbf{a}(\alpha, \beta + \psi)$ is also eigenvector. This reflects the rotational symmetry of the problem.

The optimum field we are looking for is eigenvector for the value of $\ell$ for which the eigenvalue of (87) is largest. Because $C_R$ is an analytic function of $\beta$, we have for $\ell$ large that $\hat{C}_R(\alpha, \alpha', \beta) \to 0$ faster than any power $\ell^{-m}$, $m = 1, 2, \ldots$ and uniformly for $0 < \alpha, \alpha' < \alpha_{\text{max}}$. Hence also

$$|\hat{\mathcal{M}}_R(\alpha, \alpha', \ell)| \leq C \frac{1}{\ell^m}, \quad \text{for } \ell \to \infty$$  \hfill (92)

for some constant $C$ (depending on $m$) and uniformly in $\alpha, \alpha'$. This shows that the eigenvalues of the operator (87) become arbitrary small in the limit $\ell \to \infty$. Therefore, the maximum eigenvalue occurs for some finite $\ell$. As discussed in the section with numerical results, it can happen that the eigenvalues for different $\ell$ are the same and both maximum. In that case there are two fields with different $\ell$ which both are solutions of the optimisation problem.

Finally, we express also the power flux (13) of the solution in terms of the Fourier coefficients of the optimum plane wave amplitudes:

$$P(\mathbf{a}) = \frac{1}{\omega_0} \frac{1}{8\pi^2} \iint_{\Omega} \mathbf{a}(k_\perp)^2 k_z \, dk_\perp$$

$$= \frac{1}{\omega_0} \frac{1}{8\pi^2} \iint_{\Omega} \left[ |a_p(k_\perp)|^2 + |a_s(k_\perp)|^2 \right] k_z \, dk_\perp$$

$$= n \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{k^2}{8\pi^2} \int_{0}^{\alpha_{\text{max}}} \int_{0}^{2\pi} \left[ |a_p(\alpha, \beta)|^2 + |a_s(\alpha, \beta)|^2 \right] \cos^2 \alpha \sin \alpha \, d\beta \, d\alpha$$

$$= n \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{k^2}{4\pi} \int_{0}^{\alpha_{\text{max}}} \left[ |\tilde{a}_p(\alpha, \ell)|^2 + |\tilde{a}_s(\alpha, \ell)|^2 \right] \cos^2 \alpha \sin \alpha \, d\alpha,$$  \hfill (93)

for the optimum $\ell$.

C. Optimum pupil fields

From (60), (61) and (69) it follows that the pupil coordinates $\rho_e$, $\phi_e$ are related to $\alpha$, $\beta$ by

$$\beta = \phi_e + \pi \quad \text{and} \quad \sin \alpha = \frac{\rho_e}{r}.$$  \hfill (94)
Let
\[ a(\alpha, \beta) = 2 \text{Re}[\tilde{a}(\alpha, \ell)e^{i\ell\beta}] = 2 \text{Re}\left[ \left( \hat{a}_\rho(\alpha, \ell) \hat{a}_\alpha(\alpha, \ell) \right) e^{i\ell\beta} \right], \tag{95} \]
be a solution of eigenvalue problem \([87]\) for the value of \(\ell\) for which the eigenvalue is maximum. If we normalize \(a\) such that the power satisfies \(P(a) = P_0\), \(a\) is a solution of the optimisation problem. According to \([62]\) and \([63]\) the radial and azimuthal components of the corresponding pupil field are
\[ E^r(\rho, \phi, \varphi) = 2 \frac{k}{2\pi f} \frac{\rho^2/f^2}{2\pi f} \text{Re}[\hat{a}_\rho(\alpha, \ell)e^{i\ell\phi}], \tag{96} \]
\[ E^\varphi(\rho, \phi, \varphi) = 2 \frac{k}{2\pi f} \frac{\rho^2/f^2}{2\pi f} \text{Re}[\hat{a}_\alpha(\alpha, \ell)e^{i\ell\phi}], \tag{97} \]
where the irrelevant factor \(e^{i\ell\pi} = (-1)^\ell\) has been omitted. On the cartesian basis we have (see \([65]\)):
\[ E^\alpha(\rho, \phi) = 2 \frac{k}{2\pi f} \frac{\rho^2/f^2}{2\pi f} \left\{ \text{Re}[\hat{a}_\rho(\alpha, \ell)e^{i\ell\phi\alpha}] \cos \phi \rho - \text{Re}[\hat{a}_\alpha(\alpha, \ell)e^{i\ell\phi\alpha}] \sin \phi \rho \right\} \hat{x} \]
\[ + 2 \frac{k}{2\pi f} \frac{\rho^2/f^2}{2\pi f} \left\{ \text{Re}[\hat{a}_\rho(\alpha, \ell)e^{i\ell\phi\alpha}] \sin \phi \rho + \text{Re}[\hat{a}_\alpha(\alpha, \ell)e^{i\ell\phi\alpha}] \cos \phi \rho \right\} \hat{y}. \tag{98} \]
It is seen that the optimum pupil field is linear polarized, but that the direction of the polarisation strongly varies throughout the pupil. By multiplying \([96]\) and \([97]\) by \(e^{-i\omega t}\) and taking the real part, it follows that the azimuthal and polar components of the time dependent electric field are in phase throughout the pupil, i.e., they all have value zero at the same time during a period of the field oscillation.

D. Optimum field in the focal region

The optimum field in the focal region is the (rescaled) Fourier transform of the optimum pupil field. We rewrite the Fourier transforms in terms of integrals over polar and azimuthal angles. We have, for some \(\ell:\)
\[ a_\rho(\alpha, \beta) = 2 \text{Re}[\hat{a}_\rho(\alpha, \ell)e^{i\ell\beta}], \tag{99} \]
\[ a_\alpha(\alpha, \beta) = 2 \text{Re}[\hat{a}_\alpha(\alpha, \ell)e^{i\ell\beta}], \tag{100} \]
where \(\alpha\) and \(\beta\) are related to \(\mathbf{k}_\perp\) by \([69]\). By applying the change of integration variables \(\mathbf{k}_\perp \rightarrow (\alpha, \beta)\) to \([10]\), using \([77]\), we find that the optimum electric field in the focal region is given by
\[ E(r) = \frac{1}{4\pi^2} \int \int [a_\rho(\mathbf{k}_\perp)\tilde{\mathbf{p}}(\mathbf{k}_\perp) + a_\alpha(\mathbf{k}_\perp)\tilde{\mathbf{s}}(\mathbf{k}_\perp)] e^{ikr} d\mathbf{k}_\perp \]
\[ = \frac{k^2}{2\pi^2} \int_0^{\alpha_{\text{max}}} \int_0^{2\pi} \text{Re}[\hat{a}_\rho(\alpha, \ell)e^{i\ell\beta\rho\hat{\mathbf{p}}(\alpha, \beta)} + \hat{a}_\alpha(\alpha, \ell)e^{i\ell\beta\hat{\mathbf{s}}(\beta)}] e^{ik(x \sin \alpha \cos \beta + y \sin \alpha \sin \beta + z \cos \alpha)} \cos \alpha \sin \alpha \, d\alpha \, d\beta. \tag{101} \]
Expressed in cylindrical coordinates
\[ x = \rho \cos \phi, \quad y = \rho \sin \phi. \tag{102} \]
this becomes
\[ E(\rho, \phi, z) = \frac{k^2}{2\pi^2} \int_0^{\alpha_{\text{max}}} \int_0^{2\pi} \text{Re}[\hat{a}_\rho(\alpha, \ell)e^{i\ell\beta\rho\hat{\mathbf{p}}(\alpha, \beta)} + \hat{a}_\alpha(\alpha, \ell)e^{i\ell\beta\hat{\mathbf{s}}(\beta)}] e^{ik(\rho \sin \alpha \cos (\beta - \phi) + z \cos \alpha)} \cos \alpha \sin \alpha \, d\alpha \, d\beta. \tag{103} \]
For the magnetic field we have similarly from \([5]\):
\[ H(\rho, \phi, z) = \frac{1}{\omega \mu_0} \frac{1}{4\pi^2} \int \int [a_\rho(\mathbf{k}_\perp)\tilde{\mathbf{s}}(\mathbf{k}_\perp) - a_\alpha(\mathbf{k}_\perp)\tilde{\mathbf{p}}(\mathbf{k}_\perp)] e^{ikr} d\mathbf{k}_\perp \]
\[ = \frac{n}{\omega \mu_0} \frac{1}{4\pi^2} \int_0^{\alpha_{\text{max}}} \int_0^{2\pi} \text{Re}\left[ \hat{a}_\rho(\alpha, \ell)e^{i\ell\beta\hat{\mathbf{s}}(\beta)} - \hat{a}_\alpha(\alpha, \ell)e^{i\ell\beta\hat{\mathbf{p}}(\alpha, \beta)} \right] e^{ik(\rho \sin \alpha \cos (\beta - \phi) + z \cos \alpha)} \cos \alpha \sin \alpha \, d\alpha \, d\beta. \tag{104} \]
The integrals over $\beta$ in (103) and (104) can be computed analytically. The derivation and results are given in Appendix C.

VI. RESULTS FOR THE MAXIMUM ENERGY IN A DISC

We start with a special case for which the solution can be computed in closed form.

A. The solution for a disc with radius $R = 0$

This means that we are maximizing the electric energy density in the origin, i.e., (66) becomes

$$G_{S,\Pi}(a) = \frac{1}{\pi R^2} \iint_{D_R} |E(x, y, 0)|^2 \, dx \, dy \rightarrow |E(0)|^2.$$  \hspace{1cm} (105)

We have

$$C_{R=0}(\alpha, \alpha', \beta) = \frac{k \cos \alpha' \sin \alpha'}{\cos \alpha},$$ \hspace{1cm} (106)

so that

$$\widehat{C}_{R=0}(\alpha, \alpha', \ell) = \frac{k \cos \alpha' \sin \alpha'}{\cos \alpha} \delta_{\ell,0}. \hspace{1cm} (107)$$

Then, (84) implies:

$$\widehat{M}_{R=0}(\alpha, \alpha', 0) = \frac{k \cos \alpha' \sin \alpha'}{\cos \alpha} \left( \begin{array}{cc} \sin \alpha \sin \alpha' & 0 \\ 0 & 0 \end{array} \right),$$ \hspace{1cm} (108)

$$\widehat{M}_{R=0}(\alpha, \alpha', 1) = \frac{k \cos \alpha' \sin \alpha'}{2 \cos \alpha} \left( \begin{array}{cc} \cos \alpha \cos \alpha' & -i \cos \alpha \\ i \cos \alpha' & 1 \end{array} \right),$$ \hspace{1cm} (109)

$$\widehat{M}_{R=0}(\alpha, \alpha', \ell) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \text{ if } \ell > 1. \hspace{1cm} (110)$$

Hence the optimum solution either has $\ell = 0$ or $\ell = 1$.

We consider first $\ell = 0$. Substitution of (107) into (87) with $\ell = 0$, implies:

$$2\pi k \tan \alpha \int_0^{\alpha_{\text{max}}} \cos \alpha' \sin^2 \alpha' \widehat{a}_p(\alpha', 0) \, d\alpha' = \Lambda \widehat{a}_p(\alpha, 0), \hspace{1cm} (111)$$

and

$$\widehat{a}_s(\alpha, 0) = 0. \hspace{1cm} (112)$$

Hence, $\widehat{a}_p(\alpha, 0)$ is proportional to $\tan \alpha$ and using this fact it follows from (111) and (93):

$$\Lambda = 2\pi k \left( \frac{2}{3} - \cos \alpha_{\text{max}} + \frac{1}{3} \cos^3 \alpha_{\text{max}} \right),$$ \hspace{1cm} (113)

and

$$\widehat{a}_p(\alpha, 0) = 2\pi \left( \frac{2}{kn} \Lambda \right)^{1/2} \left( \frac{\mu_0}{\epsilon_0} \right)^{1/4} \tan \alpha. \hspace{1cm} (114)$$

Next we consider the case $\ell = 1$. By substituting (109) into (87) with $\ell = 1$, one finds

$$\int_0^{\alpha_{\text{max}}} \cos^2 \alpha' \sin \alpha' \widehat{a}_p(\alpha', 1) \, d\alpha' - i \int_0^{\alpha_{\text{max}}} \cos \alpha' \sin \alpha' \widehat{a}_s(\alpha', 1) \, d\alpha' = \frac{\Lambda}{\pi k} \widehat{a}_p(\alpha, 1),$$ \hspace{1cm} (115)

$$\frac{i}{\cos \alpha} \int_0^{\alpha_{\text{max}}} \cos^2 \alpha' \sin \alpha' \widehat{a}_p(\alpha', 1) + \frac{1}{\cos \alpha} \int_0^{\alpha_{\text{max}}} \cos \alpha' \sin \alpha' \widehat{a}_s(\alpha', 1) \, d\alpha' = \frac{\Lambda}{\pi k} \widehat{a}_s(\alpha, 1). \hspace{1cm} (116)$$
Hence, 

\[ \hat{a}_p(\alpha, 1) = C_p, \quad \hat{a}_s(\alpha, 1) = \frac{C_s}{\cos \alpha}, \quad (117) \]

where \( C_p \) and \( C_s \) are constants. Substituting (117) into (113) and (116) implies

\[ \left( \frac{1}{2}(1 - \cos^3 \alpha_{\max}) - i(1 - \cos \alpha_{\max}) \right) \begin{pmatrix} C_p \\ C_s \end{pmatrix} = \frac{\Lambda}{\pi k} \begin{pmatrix} C_p \\ C_s \end{pmatrix} \quad (118) \]

The largest eigenvalue is given by

\[ \Lambda = \pi k \left( \frac{4}{3} \cos \alpha_{\max} - \frac{1}{3} \cos^3 \alpha_{\max} \right), \quad (119) \]

with eigenvector

\[ \begin{pmatrix} C_p \\ C_s \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad (120) \]

Eigenvalue (119) is for all \( \alpha_{\max} > 0 \) strictly larger than eigenvalue (113), hence the optimum solution corresponds to \( \ell = 1 \). We have

\[ \text{Re} \left[ \hat{a}_p(\alpha, 1) e^{i\beta} \right] = C_p \cos \beta, \quad (121) \]

\[ \text{Re} \left[ \hat{a}_s(\alpha, 1) e^{i\beta} \right] = -C_p \sin \beta \cos \alpha, \quad (122) \]

The corresponding pupil field that gives the optimum field in the focal region follows from (98):

\[ E^e(\rho_e, \phi_e) = C_p \frac{k}{\pi i f} (1 - \rho_e^2 / f^2)^{1/4} \left\{ \left[ \cos(2\phi_e) + \frac{\sin^2 \phi_e}{(1 - \rho_e^2 / f^2)^{1/2}} \right] \hat{x} + \frac{1}{2} \sin(2\phi_e) \left[ 1 - \frac{1}{(1 - \rho_e^2 / f^2)^{1/2}} \right] \hat{y} \right\}. \quad (123) \]

The constant \( C_p \) can be determined by substituting \( \hat{a}_p(\alpha, 1) = C_p \) and \( \hat{a}_s(\alpha, 1) = C_p / \tan \alpha \) into (92) and requiring that the power equals \( P_0 \). We remark that the result agrees with the solution obtained by different methods in [6] and [10].

As has been mentioned after (98), the pupil field is linearly polarised in all pupil points. It follows from (123) that the pupil field is predominantly linearly polarised parallel to the \( x \)-axis with more or less constant amplitude. This is confirmed by Fig. 1 where a snapshot of the optimum pupil field is shown when \( NA = 0.95 \). As function of time the electric field vectors in all pupil points oscillate harmonically parallel to the direction of the arrows. The amplitudes of the \( x, y \), and \( z \)-components of the optimum electric field in the focal plane and the optimum electric energy density in the focal plane are shown in Figs. 2. This focal field indeed resembles that of the vectorial Airy spot, i.e., the focused field of a linearly polarised plane wave.

As was stated after (91), if \( a(\alpha, \beta) \) is eigenvector, so is \( a(\alpha, \beta + \psi) \) for arbitrary \( \psi \). The latter solution is predominantly polarised parallel to the direction which makes an angle \( \psi \) with the \( x \)-axis. Hence there is nothing special about the \( x \)-axis and it is therefore more appropriate to state that the optimum pupil fields for the case \( R = 0 \) are similar to that of a linearly polarised plane wave. When the numerical aperture is increased, the difference between the optimum pupil field and that of a linearly polarised plane wave becomes bigger.

### B. Optimum fields for general \( R \)

For general \( R > 0 \) the optimisation problem can not be solved in closed form but instead numerical computations are necessary. We explain how this can be done in Appendix D. In Fig. 3 the maximum of the electric energy density is shown as function of \( R/\lambda \) and \( NA \) for power \( P_0 = 1 \). According to the scaling law discussed in Section III C for given \( NA \), the eigenfields are the same if \( R/\lambda \) is kept constant and are independent of the power \( P_0 \). The maximum of the object functional \( G_{S_{11}} \), i.e., the maximum of the average electric energy density over the disc with radius \( R \), increases as \( 1/R^2 \) when \( R/\lambda \) is kept constant and is proportional to \( P_0 \). Hence, Fig. 3 contains information of the solutions of the optimisation problem for all \( 0.40 < NA < 0.95 \) and for the values of \( R / \lambda \) for which \( 0 < R / \lambda < 2 \).

It is seen in Fig. 3 that the maximum average electric energy density monotonically increases with \( NA \) for fixed \( R / \lambda \) and that it monotonically decreases for increasing \( R / \lambda \) when \( NA \) is fixed. Furthermore, for all optimisation problems for which we have computed the solution, we found that either \( \ell = 0 \) or \( \ell = 1 \), i.e., no value \( \ell > 1 \) was found to be
optimal. The regions in Fig. 3 for different values of \( \ell \) are separated by curves where solutions for both \( \ell = 0 \) and \( \ell = 1 \) occur. These curves seem to satisfy \( NA.R/\lambda = C \), with \( R/\lambda \geq C \) and the constant \( C \) depends on the curve.

When \( R = 0.5\lambda \) and \( NA = 0.95 \), the solution is in the large region where \( \ell = 1 \) which also contains \( R = 0 \). The optimum pupil field is shown in Fig. 4 and the corresponding focal field is shown in Figs. 5. The pupil field is similar to that of a linear polarised plane wave although the amplitude decreases towards the rim of the pupil.

In Fig. 6 a snapshot of the optimum pupil field is shown for \( R = 1.25\lambda \) and \( NA = 0.75 \), for which \( \ell = 0 \). It is found that \( \hat{a}_p(\alpha, 0) = 0 \) and in fact this property holds for all solutions where \( \ell = 0 \). Then (98) implies that the optimum pupil field is azimuthally polarised with amplitude that is rotational invariant and depends only on \( \varrho_c \). The focal field is a superposition of S-polarised plane waves and hence the \( E_z \) component of the field in the focal region vanishes.

As is seen in Figs. 7 the transverse electric field amplitudes in the focal point vanish and the electric energy density has a doughnut shape.

When the NA of the lens is increased, the optimum pupil field for the same \( R = 1.25\lambda \) becomes more concentrated at the edge of the pupil. This is confirmed by Fig. 8 where the results are shown for \( NA = 0.95 \). In this case the rotational symmetric solution: \( \ell = 0 \) applies as for \( NA = 0.75 \), but the ratios of the amplitudes in the centre to those at the edge are much smaller than in Fig. 6. The optimum electric field components in the focal region for \( R = 1.25\lambda \) and \( NA = 0.95 \) are shown in Fig. 9. They are more narrow than in Fig. 7 for \( NA = 0.75 \) (note the different scales of the figures for \( NA = 0.75 \) and \( NA = 0.95 \)).

Next we consider the optimisation problem at the border between two regions where \( \ell = 0 \) and \( \ell = 1 \). For \( R = 1.753656\lambda \) and \( NA = 0.95 \) two solutions are found. Fig. 10 and Fig. 11 show the optimum pupil field and the optimum electric field components in the focal plane for \( \ell = 0 \). We have \( \hat{a}_p(\alpha, 0) = 0 \) and hence the pupil field is azimuthally polarised. It is seen that the pupil field is strongly concentrated at the rim of the pupil similar to the case of Fig. 8. In Fig. 12 and Fig. 13 the optimum pupil field and the corresponding electric field components in the focal plane are shown for the case \( \ell = 1 \). It is seen that the pupil field amplitudes are largest at the rim. Furthermore it strongly deviates from that of a linearly polarised plane wave which is a general trend when \( R \) is increased.

To better explain the optimum pupil fields, we show in Fig. 14 the corresponding \( \hat{a}_s(\alpha, \ell = 0) \) as function of \( \alpha \).

VII. CONCLUSION

We have derived a general formulism for obtaining the electromagnetic field with given power and given numerical aperture of which the electric energy averaged over a bounded set is maximum. The set can be chosen arbitrarily: it may consist of finitely many points, it may be a curve, a (curved) surface or a three dimensional region. It has
been shown that the Lagrange multiplier rule implies that the optimum field is eigenfield with maximum eigenvalue of an integral operator whose kernel is determined by the set. This integral operator is compact and also hermitian, provided the proper scalar product is chosen. Hence its spectrum is discrete and there is a maximum eigenvalue. It was shown that when the set over which the electric energy is averaged is scaled by a parameter $\sigma > 0$, the optimum solution remains the same if the numerical aperture and the ratio of $\sigma$ and the wavelength are kept constant. We have studied in more detail the problem of maximizing the electric energy in a disc perpendicular to and symmetric with respect to the optical axis. If the radius of the disc vanishes, the energy in a single point is maximized. In this case the optimum pupil field which after focusing gives maximum electric energy density in the focal point can be computed in closed form and is similar to that of a linear polarised plane wave. For general radii, the optimum solutions must be computed numerically. It is found that when the numerical aperture is fixed and the radius of the disc is increased, the optimum pupil fields alternate between a field that resembles more or less that of a polarised plane wave with constant direction of polarisation, and an azimuthally polarized pupil field. At values of NA and the radius over the wavelength where the transitions between the two types of solutions occurs, multiple optimum fields exist.

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FIG. 3. Contour plot of the maximum energy density averaged over a disc of radius $R$, as function of NA and $R/\lambda$. The values in the left figure are normalized with respect to the maximum which occurs in this plot for $R/\lambda = 0$ and NA = 0.95. The right plot is an enlargement of the part inside the red rectangle in the left figure and is normalized to the maximum occurring in this rectangle depending on the values of $R$ and NA there holds either $\ell = 0$ or $\ell = 1$.

FIG. 4. Pupil field for $R = 0.5\lambda$ and NA = 0.95. In this case $\ell = 1$.

Appendix A: Derivation of Eq. 23

We express $G_{S,\Pi}$ in terms of the plane wave amplitudes $A$. The following derivation is formal but can be mathematically justified.

First we remark that (3) implies for every $z$:

$$F_2(\Pi(E)) (k_{\perp}, z) = \Pi(A)(k_{\perp}) e^{ik_{\perp}z},$$  \hspace{1cm} (A1)

$$F_2(\Pi(E)^*)(k_{\perp}, z) = \Pi(A)(-k_{\perp})^* e^{-ik_{\perp}z},$$  \hspace{1cm} (A2)
where \( k_z = k_z(k_\perp) \) given by (22) and \( \mathcal{F}_2 \) is the 2D Fourier transform defined by (19) and its inverse by (20). We apply Plancherel’s identity and the convolution theorem using the 3D Fourier transform:

\[
G_{S,\Pi}(E) = \langle T_S, |\Pi(E)|^2 \rangle_{\mathbb{R}^3} = \frac{1}{(2\pi)^3} \langle \mathcal{F}_3(T_S), \mathcal{F}_3[\Pi(E)\Pi(E)^*]\rangle_{\mathbb{R}^3}.
\]

(A3)

Next we write the 3D Fourier transform of \( \Pi(E) \) as the composition of the 2D and the 1D Fourier transform:

\[
\mathcal{F}_3(\Pi(E))(\xi_\perp, \xi_z) = \iint_{\mathbb{R}^3} \Pi(E)(r_\perp, z)e^{-i(\xi_\perp \cdot r_\perp + \xi_z z)} \, dr_\perp \, dz = \int_{\mathbb{R}} e^{-i\xi_z z} \mathcal{F}_2(\Pi(E))(\xi_\perp, z) \, dz.
\]

(A4)

Using (A1), we find:

\[
\mathcal{F}_3(\Pi(E))(\xi_\perp, \xi_z) = \int_{\mathbb{R}} e^{-i(\xi_z - k_z(\xi_\perp)) z} \, dz \Pi(A)(\xi_\perp) = 2\pi \delta(\xi_z - k_z(\xi_\perp)) \cdot \Pi(A)(\xi_\perp),
\]

(A5a)

\[
\mathcal{F}_3(\Pi(E)^*)(\xi_\perp, \xi_z) = 2\pi \delta(\xi_z + k_z(\xi_\perp)) \cdot \Pi(A)(-\xi_\perp)^*.
\]

(A5b)

Hence,

\[
[\mathcal{F}_3(\Pi(E))^* \star \mathcal{F}_3(\Pi(E)^*)](\xi_\perp, \xi_z) = 4\pi^2 \iint_{\mathbb{R}^2} \delta(\xi_z + k_z(\xi_\perp) - k_z(\xi_\perp - \xi'_\perp)) \Pi(A)(-\xi'_\perp)^* \cdot \Pi(A)(\xi_\perp - \xi'_\perp)^* \, d\xi'_\perp,
\]
FIG. 6. Pupil field for $R = 1.25\lambda$ and NA = 0.75. In this case $\ell = 0$.

where we have used that

$$\int_{\mathbb{R}} \delta(\xi_z - \xi_z') \delta(\xi_z + k_z(\xi_\perp - \xi_\perp')) d\xi_z' = \delta(\xi_z + k_z(\xi_\perp) - k_z(\xi_\perp - \xi_\perp')),$$

which can be verified by integrating against a test function. Substitution into (A3) yields after computing the integral over the $\delta$-function:

$$G_{S,\Pi}(E) = \frac{1}{(2\pi)^4} \int_{\Omega} \int_{\Omega} \mathcal{F}_3(T_S)(\xi_\perp, k_z(\xi_\perp - \xi_\perp') - k_z(\xi_\perp')) \Pi(A)(-\xi_\perp') \cdot \Pi(A)(\xi_\perp - \xi_\perp')^* d\xi_\perp d\xi_\perp'. \quad \text{(A6)}$$

By a change of integration variables we get:

$$G_{S,\Pi}(E) = \frac{1}{(2\pi)^4} \int_{\Omega} \int_{\Omega} \mathcal{F}_3(T_S)(k_\perp - k_\perp', k_z(k_\perp) - k_z(k_\perp')) \Pi(A)(k_\perp') \cdot \Pi(A)(k_\perp)^* dk_\perp dk_\perp'. \quad \text{(A7)}$$

This is the averaged energy density in $S$ expressed in terms of the plane wave amplitudes $A$.

**Appendix B: The Fourier coefficients of $C_R$**

In this appendix we compute the Fourier coefficients of $C_R$. As a first step, we expand the Bessel function $J_1$ in its Taylor series

$$J_1(R|k_\perp - k_\perp'|) = \frac{R|k_\perp - k_\perp'|}{R|k_\perp - k_\perp'|} \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+1)!} \left( \frac{R|k_\perp - k_\perp'|}{2} \right)^{2k}. \quad \text{(B1)}$$

Next, we recall (73) to which we apply the binomial theorem after setting $\gamma = \beta - \beta'$, $\xi = \sin \alpha$ and $\xi' = \sin \alpha'$:

$$(\xi^2 + \xi'^2 - 2\xi\xi' \cos \gamma)^k = \sum_{\ell=0}^k \binom{k}{\ell} (-1)^\ell \cos^\ell \gamma (2\xi\xi')^\ell (\xi^2 + \xi'^2)^{k-\ell}.$$
FIG. 7. Optimum focused squared electric field amplitudes and electric energy density in the focal plane for \( R = 1.25\lambda \) and NA = 0.75. The solution has \( \ell = 0 \). Top left: \(|E_x|^2\); top right: \(|E_y|^2\); bottom: \(|E|^2\). The longitudinal component \( E_z \) vanishes. The amplitude and energy density are normalised such that the maximum of the energy density is unity.

Combining with (B1):

\[
\frac{J_1(R|k_\perp - k'_\perp|)}{R|k_\perp - k'_\perp|} = \sum_{\ell=0}^{\infty} \sum_{m=\ell}^{\infty} \frac{(-1)^{m+\ell}}{(m+1)!\ell!(m-\ell)!} \left( \frac{Rk}{2} \right)^{2m} \cos^\ell \gamma (2\xi \xi')^\ell (\xi^2 + \xi'^2)^{m-\ell}.
\]

In the next step, we apply the binomial theorem to \( 2 \cos \gamma = e^{i\gamma} + e^{-i\gamma} \). Combining and rearranging gives

\[
\frac{J_1(R|k_\perp - k'_\perp|)}{R|k_\perp - k'_\perp|} = \frac{1}{4\pi^2} \sum_{\ell=-\infty}^{\infty} e^{i\ell \gamma} \sum_{s=0}^{\infty} \left( \frac{\ell}{s} \right) e^{-2is\gamma} (\xi \xi')^\ell \sum_{m=\ell}^{\infty} \frac{(\xi^2 + \xi'^2)}{(m+1)!\ell!(m-\ell)!} \left( \frac{Rk}{2} \right)^{2m} (\xi^2 + \xi'^2)^{m-\ell}.
\]

Rearranging the sums gives

\[
\frac{J_1(R|k_\perp - k'_\perp|)}{R|k_\perp - k'_\perp|} = \frac{1}{4\pi^2} \sum_{\ell=-\infty}^{\infty} e^{i\ell \gamma} \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\xi \xi')^\ell (\xi^2 + \xi'^2)^m}{s!(\ell + s)!} \left( \frac{Rk}{2} \right)^{2\ell+2s} \frac{(\xi^2 + \xi'^2)^{m-\ell}}{m!(m + \ell + 2s + 1)!} \left( \frac{Rk}{2} \right)^{2m}.
\]

The last sum over \( m \) is the expansion of a Bessel function:

\[
\sum_{m=0}^{\infty} \frac{(-1)^m \sqrt{\xi^2 + \xi'^2}^{2m}}{m!(m + \ell + 2s + 1)!} \left( \frac{Rk}{2} \right)^{2m} = 2^{\ell+2s+1} \frac{J_{\ell+2s+1}(Rk\sqrt{\xi^2 + \xi'^2})}{(Rk\sqrt{\xi^2 + \xi'^2})^{\ell+2s+1}}.
\]
FIG. 8. Pupil field for $R = 1.25\lambda$ and $\text{NA} = 0.95$. In this case $\ell = 0$.

Hence,

$$
\frac{J_1(R|k_\perp - k'_\perp|)}{R|k_\perp - k'_\perp|} = \frac{1}{4\pi^2} \sum_{\ell = -\infty}^{\infty} e^{i\ell\gamma} \sum_{s = \max(0, -\ell)}^{\infty} \frac{(\xi_{\ell'})^{\ell+2s}}{s!(\ell + s)!} \left( \frac{Rk}{2} \right)^{2\ell+4s} 2^{\ell+2s+1} J_{\ell+2s+1}\left(\frac{Rk\sqrt{\xi^2 + \xi'^2}}{(Rk\sqrt{\xi^2 + \xi'^2})^{\ell+2s+1}}\right).
$$

Since by (79)

$$
C_R(\alpha, \alpha', \beta) = \frac{2 \cos \alpha' \sin \alpha'}{\cos \alpha} J_1(kR\sqrt{\sin^2 \alpha + \sin^2 \alpha' - 2 \sin \alpha \sin \alpha' \cos \beta}) \cos \alpha \sin \alpha' \cos \beta,
$$

it follows that the Fourier coefficients of $\beta \mapsto C_R(\alpha, \alpha', \beta)$ are:

$$
\hat{C}_R(\alpha, \alpha', \ell) = \frac{1}{4\pi^2} \frac{2 \cos \alpha' \sin \alpha'}{\cos \alpha} \sum_{s = \max(0, -\ell)}^{\infty} \frac{(\sin \alpha \sin \alpha')^{\ell+2s}}{s!(\ell + s)!} \left( \frac{Rk}{\sqrt{2}} \right)^{2\ell+4s} J_{\ell+2s+1}\left(\frac{Rk\sqrt{\sin^2 \alpha + \sin^2 \alpha'}}{(Rk\sqrt{\sin^2 \alpha + \sin^2 \alpha'})^{\ell+2s+1}}\right).
$$

Recall that we have shown that $\ell$ is sufficient to consider $\ell \geq 0$. For these $\ell$ the expression can be simplified slightly. The partial sums of the series converge very fast.

**Appendix C: Analytical evaluation of the integrals with respect to polar angle of the focused field.**

We will use the following notations:

\begin{align}
N_{x,\ell}(\phi, \gamma) &= \int_0^{2\pi} e^{i\gamma \cos(\beta - \phi)} e^{i\ell\beta} \cos \beta \, d\beta, \quad \text{(C1a)} \\
N_{y,\ell}(\phi, \gamma) &= \int_0^{2\pi} e^{i\gamma \cos(\beta - \phi)} e^{i\ell\beta} \sin \beta \, d\beta, \quad \text{(C1b)} \\
N_{z,\ell}(\phi, \gamma) &= \int_0^{2\pi} e^{i\gamma \cos(\beta - \phi)} e^{i\ell\beta} \, d\beta. \quad \text{(C1c)}
\end{align}
FIG. 9. Optimum focused squared electric field amplitudes and electric energy density in the focal plane for $R = 1.25\lambda$ and $NA = 0.95$. The solution has $\ell = 0$. Top left: $|E_x|^2$, top right: $|E_y|^2$, bottom: $|E|^2$. The longitudinal component $E_z$ vanishes. The amplitude and energy density are normalised such that the maximum of the energy density is unity.

These integrals can be computed analytically, using the integral representations of the Bessel functions [10, Equation 4.7.6]:

$$N_{x,\ell}(\phi, \gamma) = -\pi i^{\ell-1} e^{i\ell\phi} [e^{i\phi} J_{\ell+1}(\gamma) - e^{-i\phi} J_{\ell-1}(\gamma)], \quad (C2a)$$

$$N_{y,\ell}(\phi, \gamma) = \pi i^\ell e^{i\ell\phi} [e^{i\phi} J_{\ell+1}(\gamma) + e^{-i\phi} J_{\ell-1}(\gamma)], \quad (C2b)$$

$$N_{z,\ell}(\phi, \gamma) = 2\pi i^\ell e^{i\ell\phi} J_{\ell}(\gamma). \quad (C2c)$$

Using these expressions it follows that

$$\int_0^{2\pi} \tilde{p}(\alpha, \beta) e^{i\ell\beta} e^{i\gamma\cos(\beta-\phi)} d\beta = \begin{pmatrix} -N_{x,\ell}(\phi, \gamma) \cos \alpha \\ -N_{y,\ell}(\phi, \gamma) \cos \alpha \\ N_{z,\ell}(\phi, \gamma) \sin \alpha \end{pmatrix}, \quad (C3)$$

and

$$\int_0^{2\pi} \tilde{s}(\beta) e^{i\ell\beta} e^{i\gamma\cos(\beta-\phi)} d\beta = \begin{pmatrix} -N_{y,\ell}(\phi, \gamma) \\ N_{x,\ell}(\phi, \gamma) \\ 0 \end{pmatrix}. \quad (C4)$$
Note that

\[ N_x, -\ell(\phi, \gamma) = (-1)^{\ell-1} N_{x, \ell}(\phi, \gamma)^*, \]
\[ N_y, -\ell(\phi, \gamma) = (-1)^{\ell-1} N_{y, \ell}(\phi, \gamma)^*, \]
\[ N_z, -\ell(\phi, \gamma) = (-1)^\ell N_{z, \ell}(\phi, \gamma)^*. \]

So that

\[
\int_0^{2\pi} \hat{p}(\alpha, \beta) e^{-i\beta} e^{i\gamma \cos(\beta - \phi)} d\beta = (-1)^\ell \begin{pmatrix}
N_{x, \ell}(\phi, \gamma)^* \cos \alpha \\
N_{y, \ell}(\phi, \gamma)^* \cos \alpha, \\
N_{z, \ell}(\phi, \gamma)^* \sin \alpha
\end{pmatrix}, \tag{C5}
\]

and

\[
\int_0^{2\pi} \hat{s}(\beta) e^{-i\beta} e^{i\gamma \cos(\beta - \phi)} d\beta = (-1)^\ell \begin{pmatrix}
N_{y, \ell}(\phi, \gamma)^* \\
-N_{x, \ell}(\phi, \gamma)^*
\end{pmatrix}. \tag{C6}
\]

Then

\[
\int_0^{2\pi} \text{Re}[\hat{a}_p(\alpha, \ell)] e^{i\ell \beta} \hat{p}(\alpha, \beta) e^{ik(\rho \sin \alpha \cos(\beta - \phi))} d\beta = \frac{1}{2} \int_0^{2\pi} \hat{a}_p(\alpha, \ell) e^{i\beta} \hat{p}(\alpha, \beta) e^{ik(\rho \sin \alpha \cos(\beta - \phi))} d\beta
\]

\[
+ \frac{1}{2} \int_0^{2\pi} \hat{a}_p(\alpha, \ell)^* e^{-i\beta} \hat{p}(\alpha, \beta) e^{ik(\rho \sin \alpha \cos(\beta - \phi))} d\beta
\]

\[
= \frac{1}{2} \begin{pmatrix}
- \text{Re}[\hat{a}_p(\alpha, \ell) N_{x, \ell}(\phi, k\rho \sin \alpha)] \cos \alpha \\
- \text{Re}[\hat{a}_p(\alpha, \ell) N_{y, \ell}(\phi, k\rho \sin \alpha)] \cos \alpha \\
\text{Im}[\hat{a}_p(\alpha, \ell) N_{z, \ell}(\phi, k\rho \sin \alpha)] \sin \alpha
\end{pmatrix}, \text{ if } \ell \text{ is odd,} \tag{C7}
\]

or

\[
= \frac{1}{2} \begin{pmatrix}
- \text{Re}[\hat{a}_p(\alpha, \ell) N_{x, \ell}(\phi, k\rho \sin \alpha)] \cos \alpha \\
- \text{Re}[\hat{a}_p(\alpha, \ell) N_{y, \ell}(\phi, k\rho \sin \alpha)] \cos \alpha \\
\text{Im}[\hat{a}_p(\alpha, \ell) N_{z, \ell}(\phi, k\rho \sin \alpha)] \sin \alpha
\end{pmatrix}, \text{ if } \ell \text{ is even}. \tag{C8}
\]
FIG. 11. Optimum focused squared electric field amplitudes and electric energy density in the focal plane or $R = 1.753656\lambda$ and NA = 0.95. The solution has $\ell = 0$. Top left: $|E_x|^2$, top right: $|E_y|^2$, bottom: $|E|^2$. The longitudinal component $E_z$ is everywhere zero. The amplitude and energy density are normalised such that the maximum of the energy density is unity.

and

$$
\int_0^{2\pi} \text{Re}[\hat{a}_s(\alpha, \ell) e^{i\ell\beta} \hat{s}(\alpha, \beta)] e^{ik(\rho \sin \alpha \cos(\beta - \phi))} d\beta = \frac{1}{2} \int_0^{2\pi} \hat{a}_s(\alpha, \ell) e^{i\ell\beta} \hat{s}(\alpha, \beta) e^{ik(\rho \sin \alpha \cos(\beta - \phi))} d\beta
$$

$$
+ \frac{1}{2} \int_0^{2\pi} \hat{a}_s(\alpha, \ell)^* e^{-i\ell\beta} \hat{s}(\alpha, \beta) e^{ik(\rho \sin \alpha \cos(\beta - \phi))} d\beta
$$

$$
= \frac{1}{2} \begin{cases} 
\text{Re}[\hat{a}_s(\alpha, \ell) N_x,\ell(\phi, k\rho \sin \alpha)] & \text{if } \ell \text{ is odd}, \\
\text{Re}[\hat{a}_s(\alpha, \ell) N_y,\ell(\phi, k\rho \sin \alpha)] & \text{if } \ell \text{ is even}.
\end{cases}
$$

(C9)

Hence, with (103):

$$
\mathbf{E}(\rho, \phi, z) = \frac{1}{2} \frac{k^2}{2\pi^2} \int_0^{\alpha_{\text{max}}} \left[ \begin{array}{cc}
\text{Re}[\hat{a}_p(\alpha, \ell) N_x,\ell(\phi, k\rho \sin \alpha)] & \cos \alpha \\
\text{Re}[\hat{a}_p(\alpha, \ell) N_y,\ell(\phi, k\rho \sin \alpha)] & \cos \alpha \\
i \text{Im}[\hat{a}_p(\alpha, \ell) N_z,\ell(\phi, k\rho \sin \alpha)] & \sin \alpha
\end{array} \right] e^{ikz \cos \alpha \cos \sin \phi d\alpha},
$$

(C11)
FIG. 12. Pupil field for $R = 1.753656\lambda$ and NA = 0.95 for which two solutions exist. In the case shown here $\ell = 1$. if $\ell$ is odd, and

$$E(\rho, \phi, z) = \frac{1}{2} \frac{k^2}{2\pi^2} \int_0^{\alpha_{\text{max}}} \left[ \begin{array}{c}
-i \text{Im}[\hat{a}_p(\alpha, \ell) N_{x,\ell}(\phi, k\rho \sin \alpha)] \cos \alpha \\
-i \text{Re}[\hat{a}_p(\alpha, \ell) N_{y,\ell}(\phi, k\rho \sin \alpha)] \sin \alpha
\end{array} \right] + \begin{array}{c}
-i \text{Im}[\hat{a}_s(\alpha, \ell) N_{x,\ell}(\phi, k\rho \sin \alpha)] \\
0
\end{array} \right] \times e^{ikz \cos \alpha \sin \alpha} d\alpha, \quad (C12)$$

if $\ell$ is even. Similarly, using (104):

$$H(\rho, \phi, z) = \frac{1}{2} \left[ \frac{\varepsilon_0}{\mu_0} \frac{k^2}{2\pi^2} \int_0^{\alpha_{\text{max}}} \begin{array}{c}
\text{Re}[\hat{a}_p(\alpha, \ell) N_{y,\ell}(\phi, k\rho \sin \alpha)] \\
\text{Re}[\hat{a}_p(\alpha, \ell) N_{x,\ell}(\phi, k\rho \sin \alpha)]
\end{array} \right] - \begin{array}{c}
-i \text{Im}[\hat{a}_s(\alpha, \ell) N_{y,\ell}(\phi, k\rho \sin \alpha)] \\
-i \text{Im}[\hat{a}_s(\alpha, \ell) N_{x,\ell}(\phi, k\rho \sin \alpha)]
\end{array} \right] \times e^{ikz \cos \alpha \sin \alpha} d\alpha, \quad (C13)$$

for $\ell$ odd, and

$$H(\rho, \phi, z) = \frac{1}{2} \left[ \frac{\varepsilon_0}{\mu_0} \frac{k^2}{2\pi^2} \int_0^{\alpha_{\text{max}}} \begin{array}{c}
-i \text{Im}[\hat{a}_p(\alpha, \ell) N_{y,\ell}(\phi, k\rho \sin \alpha)] \\
0
\end{array} \right] - \begin{array}{c}
-i \text{Im}[\hat{a}_s(\alpha, \ell) N_{x,\ell}(\phi, k\rho \sin \alpha)] \\
0
\end{array} \right] \times e^{ikz \cos \alpha \sin \alpha} d\alpha, \quad (C14)$$

for $\ell$ even. The integrals over azimuthal angle $\alpha$ have to be computed numerically.

**Appendix D: Discretization of the integral equation**

In this Appendix we will discretize (87). This means that for each $\ell \in \mathbb{Z}$ we discretize $\alpha$ and $\alpha'$ and approximate

$$\Lambda(\hat{a}_{P,\ell}(\alpha), \hat{a}_{S,\ell}(\alpha)) = \int_0^{\alpha_{\text{max}}} \hat{M}_{R,\ell}(\alpha, \alpha') \left( \frac{\hat{a}_{P,\ell}(\alpha')}{\hat{a}_{S,\ell}(\alpha')} \right) d\alpha'. \quad (D1)$$
FIG. 13. Optimum focused squared electric field amplitudes and electric energy density in the focal plane for $R = 1.753656\lambda$ and $\text{NA} = 0.95$. Top left: $|E_x|^2$, top right: $|E_y|^2$, bottom left: $|E_z|^2$ and bottom right: $|E|^2$. The amplitude and energy density are normalised such that the maximum of the energy density is unity. The solution has $\ell = 1$.

by a matrix equation. First, we substitute $s : \alpha \rightarrow (\alpha + 1)\alpha_{\text{max}}/2$ to obtain
\[
\Lambda \begin{pmatrix}
\hat{a}_{P,\ell}(s(\alpha)) \\
\hat{a}_{S,\ell}(s(\alpha))
\end{pmatrix} = \frac{\alpha_{\text{max}}}{2} \int_{-1}^{1} \hat{M}_{R,\ell}(s(\alpha), s(\alpha')) \begin{pmatrix}
\hat{a}_{P,\ell}(s(\alpha')) \\
\hat{a}_{S,\ell}(s(\alpha'))
\end{pmatrix} d\alpha'.
\]
(D2)

We discretize the integral with the Gaussian quadrature rule on the interval $-1 < s(\alpha) < 1$, which will, given the number of data points $N$ return nodal points $-1 = \alpha'_1 < \alpha'_2 < \ldots < \alpha'_N = 1$ and weights $(w_n)_{n=1}^N$ so that we can write
\[
\Lambda \begin{pmatrix}
\hat{a}_{P,\ell}(s(\alpha)) \\
\hat{a}_{S,\ell}(s(\alpha))
\end{pmatrix} = \frac{\alpha_{\text{max}}}{2} \sum_{n=1}^{N} w_n \hat{M}_{R,\ell}(s(\alpha), s(\alpha'_n)) \begin{pmatrix}
\hat{a}_{P,\ell}(s(\alpha'_n)) \\
\hat{a}_{S,\ell}(s(\alpha'_n))
\end{pmatrix}.
\]

If we discretize $\alpha$ on the integration nodal points, we get $N$ equations, that is for each $m = 1,\ldots, N$ we have
\[
\Lambda \begin{pmatrix}
\hat{a}_{P,\ell}(s(\alpha_m)) \\
\hat{a}_{S,\ell}(s(\alpha_m))
\end{pmatrix} = \frac{\alpha_{\text{max}}}{2} \sum_{n=1}^{N} w_n \hat{M}_{R,\ell}(s(\alpha_m), s(\alpha'_n)) \begin{pmatrix}
\hat{a}_{P,\ell}(s(\alpha'_n)) \\
\hat{a}_{S,\ell}(s(\alpha'_n))
\end{pmatrix},
\]
for $m = 1,\ldots, N$. We rewrite this as a matrix eigenvalue problem. Let $(\alpha_i)_{i=1}^N$ be the integration nodal points with corresponding weights $(w_i)_{i=1}^N$ and set
\[
\alpha = \begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_N
\end{pmatrix}, \quad w = \text{diag}\{w_1,\ldots, w_N\} \quad \text{and} \quad W = \text{diag}\{w, w\}.\]
Next we define the block matrix $M_D$:

$$M_D = \begin{pmatrix} M_{11}^D & M_{12}^D \\ M_{21}^D & M_{22}^D \end{pmatrix},$$

(D3)

where the matrices $M_{mn}^D$ are defined as

$$M_{mn}^D = \hat{M}_{R, mn}(s(\alpha), s(\alpha')).$$

Using this, we can write the discretized equation as an eigenvalue problem

$$\frac{2\Lambda}{\alpha_{\text{max}}} \begin{pmatrix} \hat{a}_{P, \ell}(s(\alpha)) \\ \hat{a}_{S, \ell}(s(\alpha)) \end{pmatrix} = M_D \begin{pmatrix} \hat{a}_{P, \ell}(s(\alpha)) \\ \hat{a}_{S, \ell}(s(\alpha)) \end{pmatrix},$$

(D4)

The method we have used above is the so-called Nyström method. For the discretized problem to be a good approximant to the solution of integral equation (D1), the solution of (D4) should converge to it as $N \to \infty$. 

FIG. 14. The function $\alpha \mapsto \hat{a}_s(\alpha, \ell = 0)$ for the optimum pupil fields of Figs. 6 (top left), 8 (top right), 10 (bottom).
As we have seen in Section III B the integral operator is compact. Applying [17, Theorem 3] gives us the numerical stability for problem (D4).

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