Revisiting Hartle’s model using perturbed matching theory to second order: amending the change in mass

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Abstract

Hartle’s model describes the equilibrium configuration of a rotating isolated compact body in perturbation theory up to second order in general relativity. The interior of the body is a perfect fluid with a barotropic equation of state, no convective motions and rigid rotation. That interior is matched across its surface to an asymptotically flat vacuum exterior. Perturbations are taken to second order around a static and spherically symmetric background configuration. Apart from the explicit assumptions, the perturbed configuration is constructed upon some implicit premises, in particular the continuity of the functions describing the perturbation in terms of some background radial coordinate. In this work we revisit the model within a modern general and consistent theory of perturbative matchings to second order, which is independent of the coordinates and gauges used to describe the two regions to be joined. We explore the matching conditions up to second order in full. The main particular result we present is that the radial function \( m_0 \) (in the setting of the original work) of the second order perturbation tensor, contrary to the original assumption, presents a jump at the surface of the star, which is proportional to the value of the energy density of the background configuration there. As a consequence, the change in mass \( \delta M \) needed by the perturbed configuration to keep the value of the central energy density unchanged must be amended. We also discuss some subtleties that arise when studying the deformation of the star.

Keywords: rotating compact objects, perturbation theory, matching conditions

(Some figures may appear in colour only in the online journal)
1. Introduction

Hartle’s model [14] constitutes the basis of most of the analytical studies performed to study slowly rotating stars in general relativity (GR). The formalism provides a method to construct numerical schemes in axial symmetry [28]. The model describes the axially symmetric equilibrium configuration of a rotating isolated compact body and its vacuum exterior in perturbation theory in GR. The interior of the body is a perfect fluid which satisfies a barotropic equation of state, does not have convective motions and rotates rigidly. This is matched to a stationary and axisymmetric asymptotically flat vacuum exterior region across a timelike hypersurface, and the whole model is assumed to have equatorial symmetry. By matching we mean that there is no shell of matter on the surface of the star. The approach is analytic, and makes use of a perturbative method for slow rotation around a spherically symmetric static configuration driven by a single parameter $\Omega_H^1$.

The first order perturbation, driven by a single function $\omega^H$, accounts for the rotational dragging of inertial frames. It does not change the shape of the surface of the star. The second order perturbation, in contrast, does affect the original spherical shape of the body, in agreement with the fact that this must be independent of the sense of rotation. The second order perturbation of the metric is described by three functions, $h^H$, $m^H$ and $k^H$. In addition to the deformation of the star, these functions provide the relation between the central density of the star, which is kept unperturbed, and the excess of mass $\Delta M$ between the perturbed and the static background configuration needed to keep the central density of the star unchanged, in analogy to the Newtonian approach (see [7, 8]).

Apart from the explicit assumptions made in devising the model, the construction of the perturbed configuration hides some seemingly important implicit assumptions. In this paper we focus on one of those implicit assumptions, namely the fact that the (perturbed) metric is written globally in terms of a single set of spherical-like coordinates $\{t, r, \theta, \phi\}$, that cover both the interior region (star) and exterior vacuum ($r \in (0, \infty)$), in which the function $\omega^H$ is differentiable and $h^H$, $m^H$ and $k^H$ are continuous. Explicitly,

$$\begin{aligned}
\mathrm{d}s^2 &= -e^{(\alpha)}\left(1 + 2h^H(r, \theta)\right)\mathrm{d}t^2 + e^{(\alpha)}\left(1 + 2m^H(r, \theta)/(r - 2M)\right)\mathrm{d}r^2 \\
&\quad + r^2\left(1 + 2k^H(r, \theta)\right)\left[\mathrm{d}\theta^2 + \sin^2 \theta \left(\mathrm{d}\phi - \omega^H(r, \theta)\mathrm{d}t\right)^2\right] + \mathcal{O}\left((\Omega^H)^2\right).
\end{aligned}$$

Furthermore, the radial coordinate $r$ is fixed by imposing that the function $k^H$ has no $l = 0$ term in a Legendre expansion, that is $k^H = k^H_2(r)P_2(\theta) + \ldots$. We will refer to that choice as the $k$-gauge.

In the theory of matching of spacetimes, in the exact case, the existence of (Lichnerowicz) admissible coordinates, for which the metric functions are of class $C^4$, once the matching of spacetimes is performed is known (see [2, 21]). However, how this fact translates to a perturbative scheme remains to be settled. That is, the whole background configuration (interior and exterior) can indeed be described by a metric with $C^4$ functions, but the differentiability (and even continuity) of the functions describing the perturbations in some convenient gauge is not ensured a priori. In any case, a priori explicit choice of coordinates in which the metric and its perturbations satisfy certain continuity and differentiability conditions may constitute an implicit assumption that, in principle at least, could subtract

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1 In order to ease the comparison with the original paper [14] we will use a superscript $H$ to indicate that any object $f^H$ here refers to $f$ in [14].
generality to the model. More dramatically, it could turn out to be a wrong choice, and lead to wrong outcomes.

To analyse with rigour the consequences the choices of coordinates may have in Hartle’s model, we present here the study of the matching problem by making use of the perturbed matching theory, only achieved in full generality and to second order in [19]. The global model is separated as the interior and exterior perturbed problems, matched across a perturbed matching hypersurface. The scheme we present is independent of any choice of coordinates at either side. The two problems are described in terms of two sets of functions, as described above, one for the interior and another for the exterior. To ease the reading we say a function is continuous if the values of the function at one point at one side of the matching hypersurface agree with the values of the corresponding function computed at the corresponding (same) point at the other side.

The main result we prove is that in the initial coordinates \( (1) \) used in the original work [14], the function \( \omega^H \) and its radial derivative can be taken to be continuous (see also [26]), and that in the \( k \)-gauge, the functions \( h^H \) and \( k^H \) are continuous, in agreement with [14]. However, the \( l = 0 \) sector of the function \( m^H, m^H_0 \) \((r)\), is not continuous in general, contrary to the implicit assumption in [14]. The discontinuity of \( m^H_0 \) \((r)\) turns out to be proportional to the value of the energy density of the background static and spherically symmetric configuration at the surface of the star. The consequence of this jump is that the calculation of the change in mass \( \delta M \) of the perturbed star needed to keep the central density of the star unchanged must be readdressed. The amended expression in terms of the functions used in [14] is given by equation (103).

The importance of an amended expression for \( \delta M \) lies, of course, in the existence of cases for which that correction is relevant. Indeed, for any models, i.e. equations of state, in which the energy density vanishes at the boundary of the star the expression for \( \delta M \) in [14] provides the correct value. This includes the models studied in all the subsequent works by Hartle and Thorne, in particular in [15] and [16]. As far as we know, most of the candidate models for neutron stars satisfy indeed that condition. However, some equations of state suitable to describe strange quark matter stars yield a non-zero energy density at the boundary of the star (see e.g. [10, 11, 17]). At least in that case the correcting factor in \( \delta M \) could be of (numeric) relevance. Another particular and simple case corresponds to stars with a constant energy density, as those studied in [9]. The correction to \( \delta M \) properly applies in those cases, and a numerical analysis shows that the portion of the correction over the mass computed with the missing term is not negligible by any means [25]. The fact that some relevant second order function may had a jump across the boundary has appeared previously in the literature, explicitly in e.g. [3] (see equation (46) there) and in [12], where a correct expression of \( \delta M \) is given. It is also implicit in other works such as [13]. However, the exact relationship of the functions with the original \( m^H_0 \) \((r)\) and thus the discrepancy in the computation of \( \delta M \) in [14], had not been realized at the moment. On the other hand, let us stress that the starting point of the matching procedures using perturbative schemes found so far in the literature has always relied on the prescription of the matching hypersurface defined as the set of points where the (perturbed) pressure vanishes (in a certain gauge). However, and although physically reasonable, that should be found as a consequence of the matching procedure, which ought to be constructed from first principles only.

Let us remark that the present paper is one of a series of works aimed at revisiting Hartle’s model within the modern perturbation theory, and perturbative matchings theory in particular, and thus put the model on firm grounds. In the present paper we aim specifically to the (perturbed) matching conditions. Some aspects which are shown in [14] by using the
continuity of functions, mainly the structure of the metric functions (no \( l > 2 \) sector), are still to be proven given the present state of things. That work, done in parallel, started in [26] by using the framework constructed in [18], consisting of a completely general perturbative approach to second order around static configurations of the exterior (asymptotically flat) vacuum problem of stationary and axisymmetric bodies with arbitrary matter content. All in all, given that the matching conditions may be needed in more general situations in future works, we have preferred to keep some generality in the results by focusing first on a purely geometric setting in order to impose the field equations later.

The paper is structured as follows. In section 2 we first briefly review and present the theory of perturbed matchings to second order from [19]. That is followed by the set up of the perturbed schemes needed for the (stationary and axially symmetric) geometries that are going to be used for the interior and exterior regions, together with the perturbed matching hypersurface. We present, in the form of two propositions, the necessary and sufficient conditions that the first and second order perturbations of the geometries at either side and the perturbed hypersurface must satisfy in order to match. Up to this point no Einstein’s field equations have been imposed and thus the results are purely geometric. In section 3 the interior and exterior problems at first and second order are imposed using Hartle’s model explicit assumptions: rigidly rotating perfect fluid interior with barotropic equation of state, asymptotically flat vacuum exterior, and global equatorial symmetry. The particularization of the previous propositions to Hartle’s setting is then analyzed in detail. The result concerning the interior and exterior problems at second order is finally given in the form of a theorem, in which the equations that the functions at either side must satisfy together with their corresponding matching conditions are given in full. Up to this point the whole problem has been treated within a somewhat general family of spacetime gauges, that includes the two gauges employed in [14] and most commonly used in the literature, namely the \( k \)-gauge described above and the gauge that follows the surfaces of constant energy density (or pressure). In section 4 we conclude by making explicit the link between the results shown in the preceding sections with the results as presented in [14]. The discontinuity of the \( m_0^H (r) \) function is thus shown, and the correct expression for \( \delta M \) in terms of the functions used in [14] is given.

We devote an appendix to discuss the deformation of the shape of the star, and show how the description of the perturbed hypersurface in terms of the vanishing of the pressure (in the gauge that follows surfaces of constant energy density, or pressure) holds true in the end. Let us note that the discontinuity found in the function \( m_0^H (r) \) does not affect the deformation.

In this paper we use \( G = c = 1 \), greek indices for spacetime objects and latin indices for objects relative to hypersurfaces. Spacetimes with boundary (before matching) are assumed to be \( C^1 \) (as manifolds) and oriented, with oriented boundary if any.

2. Perturbed matching to second order in brief

We take the view of modern spacetime and matching perturbation theory. A convenient starting notion in spacetime perturbation theory is a one-parameter family of spacetimes \( \{ (\mathcal{V}_\epsilon, \tilde{g}_\epsilon) \} \) with diffeomorphically related manifolds, from where we single out a background spacetime say \( (\mathcal{V}_0, g) \) with \( V_0 = V_{\epsilon=0} \) and \( g := \tilde{g}_{\epsilon=0} \). The points at each manifold of the family are identified through a diffeomorphism, say \( \psi : V_0 \rightarrow \mathcal{V}_\epsilon \). This allows us to pull back \( \tilde{g}_\epsilon \) onto the background spacetime, and thus define a family of tensors \( g_{\epsilon} := \psi^*(\tilde{g}_\epsilon) \) on \( (\mathcal{V}_0, g) \), where \( g := g_{\epsilon=0} = g_{\epsilon=0} \), a single manifold. The metric perturbation tensors are simply defined as the derivatives of \( g_\epsilon \) with respect to \( \epsilon \) evaluated on \( \epsilon = 0 \) at each order of derivation. \( K^{(1)} \)
and $K^{(2)}$ will refer to the first and second metric perturbation tensors. At this point matter fields are also introduced as a $\epsilon$-family of energy-momentum tensors $T_\epsilon$ on $(\mathcal{V}_0, g)$, and the corresponding perturbations are defined again by taking $\epsilon$-derivatives. Spacetime perturbation theory then consists of the study of the tensor fields $K^{(1)}$ and $K^{(2)}$ satisfying certain field equations on a fixed background $(\mathcal{V}_0, g)$.

Spacetime perturbation theory carries, by construction, an inherent freedom, which lies precisely on the freedom in choosing the diffeomorphism $\gamma_\epsilon$ identifying points of the manifolds. This is the so-called (spacetime) gauge freedom. Different choices of identifications lead to different, but geometrically equivalent, metric perturbation tensors. At each order in the perturbation a change of gauge is described by a vector field on the background, which measures the shift between identifications at each order. More explicitly, a change of gauge defines a $\epsilon$-parameter diffeomorphism, say $\Omega_\epsilon: \mathcal{V}_0 \to \mathcal{V}_0$. The first and second order gauge vectors, denoted as $\vec{s}_1$ and $\vec{s}_2$, can then be defined as [19]

$$\vec{s}_1 := \partial_\epsilon \Omega_\epsilon \big|_{\epsilon=0} \quad \vec{V}_2 := \partial_\epsilon \left( \partial_\epsilon \left( \Omega_{h+\epsilon} \circ \Omega_{\kappa}^{-1} \right) \big|_{\epsilon=0} \right), \quad \vec{s}_2 := \vec{V}_2 + V_\kappa \vec{s}_1.$$  \tag{2}

Indicating with a $^\#$ superscript a 'gauge transformed' quantity, the metric perturbation tensors thus transform as [5] (see [19])

$$K^{(1)^\#}_{\alpha\beta} = K^{(1)}_{\alpha\beta} + \mathcal{L}_\vec{s} g_{\alpha\beta},$$

$$K^{(2)^\#}_{\alpha\beta} = K^{(2)}_{\alpha\beta} + \mathcal{L}_\vec{s} g_{\alpha\beta} + 2\mathcal{L}_\vec{s} K^{(2)}_{\alpha\beta} - 2s_\mu^\# s_\mu^\# R_{\alpha\mu\beta\nu} + 2\nabla_\mu s_\mu^\# \nabla_\nu s_\nu^\#.$$  \tag{3}

The matching of two spacetimes with boundary, say $(\mathcal{V}^+, g^+, \Sigma^+)$ and $(\mathcal{V}^-, g^-, \Sigma^-)$, requires an identification of the boundaries, $\Sigma^+$ and $\Sigma^-$. If the boundaries are nowhere null (non-degenerate) the matching conditions (in full, so that the global Riemann tensor shows no Dirac-delta term) demand the equality of their respective first $h$ and second $\kappa$ fundamental forms. The identification of the boundaries allows the construction of an abstract manifold $\Sigma$ on which the first and second fundamental forms as coming from both sides, $h^\pm$ and $\kappa^\pm$, are pulled back so that they can be compared. The matching conditions demand the existence of one such identification for which the first and second fundamental forms agree. In particular, $\Sigma$ is endowed with the metric $h (= h^+ = h^-)$.

To study perturbation theory on a background spacetime constructed from the matching of two spacetimes one can use again the same picture. We assume two families of spacetimes with boundary $^2$ $(\mathcal{V}^\pm, \tilde{\gamma}^\pm, \Sigma^\pm)$ are matched across their respective boundaries $\tilde{\Sigma}^\pm$ for each $\epsilon$, so that there exists a corresponding family of diffeomorphically related hypersurfaces $\tilde{\Sigma}_\epsilon$ on which the first and second fundamental forms from each side are equated, $\tilde{h}_\epsilon^\pm = \tilde{\kappa}_\epsilon^\pm = \tilde{\kappa}_\epsilon^\pm$. The matching hypersurface of the background configuration is $(\Sigma_0, h)$, where $\Sigma_0 \equiv \tilde{\Sigma}_0$ and $h = h_0^+ = h_0^-$. The idea is to construct, from those tensors on $\tilde{\Sigma}_\epsilon$, corresponding families $\tilde{h}_\epsilon^\pm$ and $\tilde{\kappa}_\epsilon^\pm$ on $(\Sigma_0, h)$ containing also the information about how $\Sigma^\pm_\epsilon$ are perturbed with respect to the gauges defined at each side $w^\pm_\epsilon$, which we want to keep free. Taking $\epsilon$-derivatives on $\epsilon = 0$ one can thus construct $h^{(1)}$, $h^{(2)}$, $\kappa^{(1)}$ and $\kappa^{(2)}$ at first and second order at each side. The matching conditions to first and second order will then demand the equalities

$^2$ We refer to [19] for a proper discussion on the subtleties involved in the definition of families of spacetimes with boundary. Also, we need only to consider non-degenerate hypersurfaces $\tilde{\Sigma}_\epsilon$, without loss of generality. Their orientation will extend through $\epsilon$ by continuity.
The setting for the construction of the tensors $h^\pm$ and $\kappa^\pm$ on $(\Sigma_0, h)$ is described as follows. Take one side, say $+$, and assume each $\Sigma^{\pm}_\varepsilon$ is a submanifold with boundary $\hat{\Sigma}^{\pm}_\varepsilon$ in a larger $\hat{\Sigma}^{\pm}_\varepsilon$ with no boundary. Hence, for each $\varepsilon$, $\hat{\Sigma}^{\pm}_\varepsilon$ is an embedded hypersurface on $W^\pm_0$. Each $\Sigma^{\pm}_\varepsilon$ is projected onto $W^\pm_0$ via $\psi^\pm_\varepsilon$ in order to define an $\varepsilon$-family of hypersurfaces $\{\Sigma^{\pm}_\varepsilon\}$ on $W^\pm_0$, see the left side of figure 1. This family describes how the background $\Sigma^{\pm}_0$ changes as $\varepsilon$ varies as a set of points on $W^\pm_0$ with respect to the gauge $\psi^\pm_\varepsilon$. But this is not enough to take $\varepsilon$-derivatives. We still need to prescribe how a given point $p \in \Sigma^{\pm}_0$ is mapped onto $\Sigma^{\pm}_\varepsilon$. For that we need to prescribe first an identification $\phi^\pm_\varepsilon: \hat{\Sigma}^0_0 \rightarrow \hat{\Sigma}^\pm_\varepsilon$ for the family $\{\hat{\Sigma}^\pm_\varepsilon\}$. That comprises an additional gauge freedom, the so-called hypersurface gauge freedom [19, 24]. The diffeomorphism $\phi^\pm_\varepsilon$ infers trivially another $\phi^\pm_\varepsilon$ for the family $\{\hat{\Sigma}^\pm_\varepsilon\}$ through the embeddings on their respective $W^\pm_0$. The composition of $\phi^\pm_\varepsilon$, from $p \in \Sigma^{\pm}_0$ to $\hat{\Sigma}^\pm_\varepsilon$, and $\psi^\pm_\varepsilon$ (down to $\Sigma^{\pm}_0$) defines a path $\gamma^\pm_\varepsilon(p)$ starting at $p$ (see figure 1). The tangent vector to that path at any $p \in \Sigma^{\pm}_0$ and its acceleration define two vector fields $\vec{Z}^\pm_1$ and $\vec{Z}^\pm_2$, respectively, on $\Sigma^{\pm}_0$. The subscripts 1 and 2 refer to the fact that $\vec{Z}^\pm_1$ carries the information of the deformation of $\Sigma^{\pm}_0$ at first order, and $\vec{Z}^\pm_2$ at second order. These are the so-called perturbation vectors of $\Sigma^{\pm}_0$ [19] (see also [1, 24] for the first order). The vectors $\vec{Z}^\pm$ (we refer to both $\vec{Z}^\pm_1$ and $\vec{Z}^\pm_2$) depend on both the spacetime and the hypersurface gauges by construction. Let now $\vec{n}^\pm$ be a unit normal vector to $\Sigma^{\pm}_0$. Every $\vec{Z}^\pm$ can thus be decomposed into normal and tangent parts, i.e.

$$\vec{Z}^\pm = Q^\pm \vec{n}^\pm + \vec{T}^\pm,$$

where $\vec{T}^\pm$ is tangent to $\Sigma^{\pm}_0$. The information on how the hypersurfaces $\Sigma^{\pm}_\varepsilon$ vary as sets of points in $W^\pm_\varepsilon$ is carried only by $Q^\pm$, while $\vec{T}^\pm$ indicates how the different points within those sets are identified.

Figure 1. Family of spacetimes $(W_\varepsilon, g_\varepsilon)$, identified through the spacetime gauge $\psi_\varepsilon$, with embedded hypersurfaces $\hat{\Sigma}_\varepsilon$, identified, in turn, through the hypersurface gauge $\phi_\varepsilon$. The projections of $\hat{\Sigma}_\varepsilon$ onto the background $(W_0, g)$ via $\psi_\varepsilon$ are $\Sigma_\varepsilon$. Given $p \in \Sigma_0$ the composition $\psi_{\varepsilon}^{-1} \circ \phi_{\varepsilon}(p)$ defines the path $\gamma_{\varepsilon}(p)$ on $W_0$. 

$$h^{(1)+} = h^{(1)-}, \quad \kappa^{(1)+} = \kappa^{(1)-}, \quad h^{(2)+} = h^{(2)-}, \quad \kappa^{(2)+} = \kappa^{(2)-}$$

defined on $(\Sigma_0, h)$ at first and second order respectively.
The full calculation of the tensors \( h^{(1)}, k^{(1)}, h^{(2)} \) and \( k^{(2)} \) (let us drop the + subscripts here) in terms of the background configuration quantities plus \( K^{(1)}, Q_1, T_1^x \), and \( K^{(2)}, Q_2, T_2^x \) was performed in [19] (propositions 2 and 3), and previously in [1, 24] up to first order. We include their expressions in appendix B.

The picture discussed above makes apparent that \( T_1^x \) and \( T_2^x \) fully depend on the hypersurface gauge (as well as the spacetime gauges at either side). It is important to stress that since \( Q_1 \) and \( Q_2 \) depend on the spacetime gauge, the ‘deformation’ they describe must be understood with respect to the spacetime gauge being used. We will make use of the explicit transformations of the perturbation vectors \( \vec{Z}_1 \) and \( \vec{Z}_2 \) under spacetime gauges defined by \( \vec{s}_1 \) and \( \vec{s}_2 \). These were shown in [19] to be

\[
\vec{Z}^\xi_1 = \vec{Z}_1 - \vec{s}_1, \quad \vec{Z}^\xi_2 = \vec{Z}_2 - \vec{s}_2 - 2V_{Z_1}\vec{s}_1 + 2V_{Z_2}\vec{s}_2.
\]  

The perturbed matching conditions are shown to be (4) in terms of the background configuration quantities and \( K^{(1)}Z, Q_1Z, T_1^x \), and \( K^{(2)}Z, Q_2Z, T_2^x \) in theorem 1 in [19]. It must be stressed that the tensors \( h^{(1)}Z, h^{(2)}Z, k^{(1)}Z, k^{(2)}Z \) are spacetime gauge invariant by construction, and thus conditions (4). Moreover, although the tensors are not hypersurface gauge invariant, the matching conditions (4) are provided the background is matched [19, 24]. Let us emphasize that \( Q_1Z \) and \( T_1^x \) are a priori unknown quantities and fulfilling the matching conditions requires showing that two pairs of vectors \( \vec{Z}_1 \) and \( \vec{Z}_2 \) exist such that (4) are satisfied. The spacetime gauge freedom at either side can be exploited to fix either or both pairs \( \vec{Z}_1 \) or \( \vec{Z}_2 \) independently a priori, but this has to be carefully analyzed if additional spacetime gauge choices are made. Finally, the hypersurface gauge is common to both sides, and therefore, it can be used to fix one of the vectors \( T^+ \) or \( T^- \), but not both (at first and second order).

At either side, say +, we will call a gauge \( \psi_+ \) ‘surface-comoving’ if the hypersurfaces \( \Sigma_\psi \) do not vary, and thus agree with \( \Sigma_{\psi}^0 \) as sets of points in \( V_{\psi}^0 \). At first order that is equivalent to \( Q_1^\psi = 0 \), but at second order \( Q_2^\psi \) carries more information coming from the first order. This fact will motivate the introduction of the quantity \( Q_\psi \) in section 2.4. The gauges referred to as ‘surface gauges’ in previous works, e.g. [4, 23], require the vanishing of the whole perturbation vector \( \vec{Z} \).

### 2.1. Family of metrics

Although the original ‘perturbed’ metric in [14] is given by (1) assuming also that \( k^\mu \) has no \( l = 0 \) term, i.e. in the (spacetime) \( k \)-gauge, the determination of the matching hypersurface is made in [14] (and most other works in the literature) by resorting to another spacetime gauge, prescribed through the surfaces of constant energy density. Since we also want to examine the use of these different spacetime gauges in the literature we consider a family of metrics \( \{ g_\epsilon \} \) that can accommodate both spacetime gauges. To do that a crossed term in \( r, \theta \) is needed.

Let us thus define the following one-parameter family \( \{ g_\epsilon \} \) on \((V_0, g)\), where \( g = g_{\epsilon=0} \), taken up to order \( \epsilon^2 \)

\[
g_\epsilon = -\epsilon^2 \left( 1 + 2\epsilon^2 h(r, \theta) \right) dr^2 + \epsilon^2 \left( 1 + 2\epsilon^2 m(r, \theta) \right) d\theta^2 + 2\epsilon^2 \left( 1 + 2\epsilon^2 k(r, \theta) \right) d\phi d\theta + \epsilon^2 \left[ \left( 1 + 2\epsilon^2 k(r, \theta) \right) \right] d\theta^2 + \sin^2 \theta \left( \phi^2 - \epsilon \omega(r, \theta) dt \right) d\theta + \mathcal{O} \left( \epsilon^3 \right),
\]

(7)

where \( \epsilon \in (-\infty, \infty), r > 0, \theta \in (0, \pi) \) and \( \phi \in (0, 2\pi) \). Clearly, an arbitrary function of \( r \) can be added to \( f(r, \theta) \) with no consequences. The appearance of \( f \) differentiated is just a mere convenience. \( \{ g_\epsilon \} \) is a family of stationary and axisymmetric metrics on \((V_0, g)\). The
(unique) axial Killing vector field will be denoted by \( \vec{\eta} = \partial_r \), and we will single out the timelike Killing \( \vec{\xi} = \partial_t \). The first and second order metric perturbation tensors, \( K^{(1)} = \partial_t g |_{t=0} = \partial_t \) and \( K^{(2)} = \partial^2_t g |_{t=0} \) respectively, take thus the form

\[
K^{(1)} = -2r^2 \omega(r, \theta) \sin^2 \theta \, d\theta d\phi,
\]

\[
K^{(2)} = \left( -4e^{i\epsilon(r)} h(r, \theta) + 2r^2 \sin^2 \theta \omega^2(r, \theta) \right) \, dr^2 + 4e^{i\epsilon(r)} m(r, \theta) \, dr^2 + 4r^2 k(r, \theta) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) + 4re^{i\epsilon(r)} \partial_\theta f(r, \theta) \, d\theta d\phi,
\]

defined on the spherically symmetric and static spacetime background \((\mathcal{V}_0, g)\) with

\[
g = -e^{i\epsilon(r)} \, dr^2 + e^{i\epsilon(r)} \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right).
\]

The (spacetime) gauge transformations described by \( \vec{\xi}_1 = C \partial_\phi \), with arbitrary constant \( C \), at first order and \( \vec{\xi}_2 = 2S(r, \theta) \partial_t \), for an arbitrary \( S(r, \theta) \), are contained within the family \( g_\epsilon \).

Under the gauge \( \vec{\xi}_1 = C \partial_\phi \), the perturbation tensor \( K^{(1)} \) transforms as (3)

\[
K^{(1)}_\epsilon = -2r^2 (\omega - C) \sin^2 \theta \, d\theta d\phi,
\]

while under a change \( \vec{\xi}_2 = 2S(r, \theta) \partial_t \) (with \( \vec{\xi}_1 = C \partial_\phi \)), \( K^{(2)} \) transforms as (3)

\[
K^{(2)}_\epsilon = \left( -4e^{i\epsilon(r)} \left( h + \frac{C'}{2} \right) + 2r^2 \sin^2 \theta (\omega - C)^2 \right) \, dr^2 + 4e^{i\epsilon(r)} \left( m + e^{i\epsilon(r)} \left( S \epsilon^{(2)} \right) \right) \, dr^2 + 4r^2 \left( k + \frac{S}{r} \right) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) + 4re^{i\epsilon(r)} \partial_\theta \left( f + \frac{S}{r} \right) \, d\theta d\phi.
\]

We will refer to this class of second order gauge transformations as ‘radial’ gauges.

A (spacetime) gauge within the set of these ‘radial’ gauges will be fixed, partially or completely, whenever the functions appearing in \( K^{(1)} \), (8), and/or \( K^{(2)} \), (9), are restricted in any way. The remaining freedom would consist on the possible \( C \) and \( S(r, \theta) \) that make the changes to the components of (11) and (12) fit, component-wise, within that restriction. The \( k \)-gauge, as mentioned, consists of imposing that the function \( k(r, \theta) \) in (9) has no \( l = 0 \) part, and that \( f = 0 \). In that case, the restriction on the \( K^{(2)}_{\epsilon 0} \) component implies that \( S(r, \theta) \) cannot have \( l = 0 \) part, while the restriction on the \( K^{(2)}_{\epsilon 0} \) component needs that \( S(r, \theta) \) does not depend on \( \theta \). The only possibility is thus \( S(r, \theta) = 0 \), so that there is no freedom left. We thus say that the \( k \)-gauge fixes completely the ‘radial’ gauge.

Let us now consider a couple of background spacetimes \((\mathcal{V}_0^\pm, g^\pm)\), with corresponding coordinates \((t, \theta, \phi)\) and families of metrics \( g^\pm \) as given in (7). Greek indices will denote quantities defined on \( \mathcal{V}_0^\pm \). In what follows we present the perturbed matching over a spherically symmetric (and static) background configuration composed by the matching of \((\mathcal{V}_0^+, g^+)\) and \((\mathcal{V}_0^-, g^-)\). Let us note that by using (7) (at both sides) we will be implicitly assuming that the perturbation will be performed within the family of (spacetime) gauges for which (8) and (9) hold. We will not be using the field equations until section 3.

The structure of the original metric (1) can be clearly recovered by taking \( f = 0 \) and noting that the choice of perturbation parameter \( \epsilon \) is not relevant, since families of solutions are obtained by scaling. The physics of the model will restrict the scalability (see equation (1) in [14]). Note, however, that the relation between the radial coordinates in (1) and (7) (either \( r_\pm \)) must still be determined in order to be able to compare the functions in (1) and (7). That is the purpose of the concluding section 4.
2.2. Background configuration

The background configuration is chosen to be globally spherically symmetric and static. This translates to the fact that the matching of \((V^+_0, g^+)\) and \((V^-_0, g^-)\), through respective boundaries \(\Sigma^+_0\) and \(\Sigma^-_0\), is asked to preserve the symmetries (see [29]), both the spherical symmetry and staticity. Under that condition the hypersurfaces \(\Sigma^+_0\) and \(\Sigma^-_0\) to be matched can be finally cast as (see e.g. [20])

\[
\Sigma^+_0 = \{ t_+ = \tau, r_+ = a, \theta_+ = \theta, \varphi_+ = \phi \},
\]

\[
\Sigma^-_0 = \{ t_- = \tau, r_- = a, \theta_- = \theta, \varphi_- = \phi \},
\]

for a constant \(a > 0\), without loss of generality. The coordinates \(\{\tau, \theta, \varphi\}\) parametrize the abstract manifold \(\Sigma_0 \equiv \Sigma^+_0 = \Sigma^-_0\). Latin indices \(i, j, \ldots\) will refer to objects on \(\Sigma_0\).

The tangent vectors to \(\Sigma^+_0\) and \(\Sigma^-_0\) thus read

\[
\vec{\ell}^\pm = \partial_\tau|_{\Sigma^\pm_0}, \quad \vec{e}^\pm = \partial_\theta|_{\Sigma^\pm_0}, \quad \vec{e}_\varphi^\pm = \partial_\varphi|_{\Sigma^\pm_0},
\]

and the corresponding unit normals are

\[
\vec{n}^+ = -e^{-\frac{\lambda_+}{2}} \partial_\tau|_{\Sigma^+_0}, \quad \vec{n}^- = -e^{-\frac{\lambda_-}{2}} \partial_\tau|_{\Sigma^-_0},
\]

under the condition that \(\vec{n}^+\) points \(V^+_0\) inwards and \(\vec{n}^-\) points \(V^-_0\) outwards, so that as \(r_+\) increases one reaches \(V^-_0\), and as \(r_-\) increases one gets away of \(V^+_0\). This convention will be used in what follows in order to call \(V^+_0\) the interior and \(V^-_0\) the exterior. Clearly \(\Sigma^\pm_0\) are timelike hypersurfaces everywhere, and are (equally) oriented by construction.

The first and second fundamental forms read

\[
h^\pm dx^i dx^j = -e^{\nu \pm}(\pm 1) dr^2 + a^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),
\]

\[
\kappa^\pm dx^i dx^j = e^{-\frac{1}{2}\nu \pm}(\pm 1) \left( \frac{1}{2} e^{\nu \pm}(\pm 1) dr^2 - a \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right),
\]

where a prime denotes differentiation with respect to the corresponding radial coordinate, i.e. \(r_+\) or \(r_-\) accordingly. The matching conditions \(h^+ = h^-\) and \(\kappa^+ = \kappa^-\) are thus equivalent to

\[
[\nu] = 0, \quad [\nu'] = 0, \quad [\lambda] = 0,
\]

where we follow the usual notation \([f] = f^+|_{\Sigma^+_0} - f^-|_{\Sigma^-_0}\) for objects \(f^\pm\) defined at either side. For the sake of brevity, given a pair \(f^\pm\) satisfying \([f] = 0\), we will simply denote by \(f|_{\Sigma_0}\) either of the equivalent \(f^+|_{\Sigma^+_0}\) or \(f^-|_{\Sigma^-_0}\). The background matching hypersurface \(\Sigma_0\) is endowed with the metric \(h = -e^{\nu}(dr^2 + a^2(d\theta^2 + \sin^2 \theta d\varphi^2))\).

Once the static and spherically symmetric background configuration has been constructed we proceed to study the perturbed matching up to second order. As discussed above, the ingredients needed are the tensors that describe the perturbations at either side, i.e. the first and second metric perturbation tensors \(K^{(1)\pm}\) and \(K^{(2)\pm}\) as defined above (8) and (9), plus the two (so far unknown) perturbation vectors \(Z^\pm_1\) and \(Z^\pm_2\) given in the form (5). To ease the notation we will denote by \(Q^\pm\) and \(T^\pm = T^\pm_2(\tau, \theta, \varphi)\) the objects defined on each \(V^\pm_0\) and the corresponding pullback and pushforward quantities that live on \(\Sigma_0\). The same applies for the functions in (8) and (9), which will be denoted equivalently as functions restricted to points on \(\Sigma^\pm_0 \subset V^\pm_0\) and functions on \(\Sigma_0\) whenever that does not lead to confusion. It is not difficult to show that the fact that since the final perturbed matched spacetime is assumed to preserve the axial symmetry the functions \(Q\)
and components $T^I$ do not depend on $\phi$. Nevertheless, we will take that as an assumption. The first and second order perturbed matchings are ruled by the particularization of theorem 1 together with propositions 2 and 3 from [19] to the present setting with the above ingredients. For completeness, the explicit expressions of the first and second order first and second fundamental forms are included in appendix B.

### 2.3. First order matching

**Proposition 1.** Let $(V_0, g)$ be the static and spherically symmetric spacetime resulting from the matching of $(V^+_0, g^+)$ and $(V^-_0, g^-)$, with $g^\pm$ given by (10) with respective $\pm$ in functions and coordinates, across $\Sigma^\pm_0$, defined by (13), (14), with $a > 0$, so that the matching conditions (17) hold and the unit normals (16) are chosen following the above interior/exterior convention. Consider the metric perturbation tensors $K^{(1)}_{\pm}$ as defined in (8) at either side $V^\pm_0$, plus two unknown functions $Q_{\pm}^{(1)}$ and two unknown vectors $\vec{T}_{\pm} = \partial_{\tau} + \partial_{\vartheta}$ on $\Sigma_0$.

The necessary and sufficient conditions that $K^{(1)}_{\pm}$ must satisfy to fulfil the first order matching conditions are

$$[\omega] = b_1,$$

$$[\omega'] = 0,$$

where $b_1$ is an arbitrary constant. Regarding the perturbed matching hypersurface, if

$$2a^2 - 2 + a\omega'(a) \neq 0$$

the remaining first order matching conditions read

$$\left[\vec{T}_1\right] = b_1 r \partial_{\phi},$$

$$[Q_1] = 0, \quad Q_1[n'] = 0, \quad Q_1[\nu^\prime] = 0.$$  (22)

The proof is left to appendix B. Note that although $[Q_1] = 0$ is always a necessary condition, $(Q^{\pm}_0) = Q_1 = 0$ is not. Indeed, if the background configuration satisfies $[\nu'] = 0$ and $[\nu^\prime] = 0$, $Q_1$ can be any arbitrary function of $(\tau, \vartheta)$. Finally, let us remark that condition (20) will be satisfied in all cases we will be interested in.

### 2.4. Second order matching

Let us first define at each side $\Sigma^\pm_0$ the following quantity

$$\hat{Q}_2 := Q_2 + \kappa_{ab} T^a T^b - 2\vec{T}_1(Q_1)$$

$$= Q_2 + a e^{-\omega(a)} \left\{ \frac{\nu' \omega'}{2a^2} \left( T^\vartheta \right)^2 - \sin^2 \vartheta \left( T^\varphi \right)^2 - \left( T^\varphi \right)^2 \right\} - 2\left( T^\varphi \partial_{\vartheta} Q_1 + T^\varphi \partial_{\phi} Q_1 \right),$$

which, given the above first order matching conditions (21), leads to

$$\left[\hat{Q}_2\right] = \left[ Q_2 \right] + a e^{-\omega(a)/2} \sin^2 \vartheta b_1 \left( b_1 r - 2T^+\varphi \right) - 2\left( T^\varphi \partial_{\vartheta} [Q_1] + T^\varphi \partial_{\phi} [Q_1] \right).$$


This new $\hat{Q}_2$ will substitute the original $Q_2$ in this section. The most immediate purpose for introducing this quantity is to absorb some first order terms arising from the matching equations, and thus keep more compact expressions.³

**Proposition 2.** Let $(V_0, g)$ with $\Sigma_0$ be the static and spherically symmetric background matched spacetime as described in proposition 1, and assume that (20) is satisfied. Let it be perturbed to first order by $K^\pm_1(\tau, \theta)$ and $T^\pm_1(\tau, \theta)$ so that (18), (19), (21), (22) hold. Consider the second order metric perturbation tensor $K^{(2)\pm}$ as defined in (9) at either side, plus two unknown functions $\hat{Q}^\pm_2(\tau, \theta)$ and two unknown vectors $T^\pm_2 = T^\pm_2(\tau, \theta)\partial_\tau + T^\pm_2(\tau, \theta)\partial_\theta$ on $\Sigma_0$.

If either $[\lambda'] \neq 0$ or $[\nu'] \neq 0$, so that $(Q^\pm_2 = \hat{Q}_1 = 0$, the necessary and sufficient conditions that $K^{(2)\pm}$ must satisfy to fulfil the second order matching conditions are

$$[k] = c_1 \cos \theta + c_2 + [f]$$

$$[h] = \frac{1}{2} H_0 + \frac{1}{4} a \nu'(a) \left[ 2[k] + H_1 \cos \theta \right]$$

$$[m] = a[k'] + \frac{1}{4} e^{-\lambda'(a)/2} [\lambda'] \hat{Q}^+_2 + \frac{1}{4} \left( a \lambda'(a) + 2 \right) \left[ 2[k] + H_1 \cos \theta \right]$$

$$-[2/2] H_1 + 2c_1 \nu'(a) e^{\lambda'(a)} \cos \theta, \quad [h'] = \frac{1}{2} \nu'(a)[k'] + \frac{1}{4} e^{-\lambda'(a)/2} \left[ 2 [k'] + H_1 \cos \theta \right]$$

$$[m'] = a[k'] + \frac{1}{4} e^{-\lambda'(a)/2} \left[ 2 [k] + H_1 \cos \theta \right]$$

$$-\frac{1}{4} (H_1 + 2c_1) \nu'(a) e^{\lambda'(a)} \cos \theta, \quad (26)

for arbitrary constants $c_1, c_2, H_0$ and $H_1$ and function $\hat{Q}_2^+(\theta)$.

If $[\lambda'] = 0$ and $[\nu'] = 0$, then $[\nu'] \hat{Q}_1 = 0$ and the above equations are the same except for two changes in (26) and (27) given respectively by

$$[\lambda'] \hat{Q}_2^+ \rightarrow -e^{-\lambda'(a)/2} [\lambda'] \hat{Q}_1^2, \quad [\nu'] \hat{Q}_2^+ \rightarrow -e^{-\lambda'(a)/2} \left[ \nu' \right] \hat{Q}_1^2.$$

In all cases, the relation

$$\left[\hat{Q}_2\right] = a e^{\lambda'(a)/2} \left[ 2 [k] + H_1 \cos \theta \right]$$

must hold, hence $\left[\hat{Q}_2\right]$ cannot depend on $\tau$, and the differences $[T^\pm_2]$ satisfy

$$[T^\pm_2] = -H_0 \tau,$$

$$[T^\pm_2] = 2b_i \left( T_1^{\pm} + \tau T_1^\pm \cot \theta \right) - \frac{2}{a} e^{-\lambda'(a)/2} b_i \tau Q_1^+, \quad [T^\pm_2] = \left( b_i \tau \cos \theta \left( b_i \tau - 2T_1^+ \phi \right) + H_1 \right) \sin \theta.$$  

The proof is left to appendix B. Let us remark that in the $Q_1 \neq 0$ case, the corresponding equations for $[m]$ and $[h]$, (26) and (27) with the corresponding changes (28) (see (121) and (122) in appendix B) imply that if $[\lambda'] \neq 0$ or $[\nu'] \neq 0$ then $Q_1$ cannot depend on $\tau$. On the

³ It is not difficult to check that $\hat{Q}_2$ is hypersurface gauge invariant.
other hand, the condition \([o^x]Q_t = 0\) will be automatically satisfied in all cases of interest, once the field equations are imposed, as shown below.

### 3. Hartle’s setting for a perfect fluid interior and vacuum exterior

In this section we focus on a global configuration composed of a rigidly rotating perfect fluid ball (with no convective motions) immersed in an asymptotically flat vacuum exterior. To present the equations in this section we will drop the + and – symbols in most places if they are not necessary. Both regions can be considered to be of perfect fluid type, from which the vacuum case is recovered trivially.

Let us then impose the metrics \(\tilde{g}_a\) to satisfy the equations \(\hat{G}(\tilde{g}_a)_{\alpha\beta} = 8\pi \hat{T}_{\alpha\beta}\) for an energy momentum tensor of the form

\[
\hat{T}_a = \left( \hat{E}_a + \hat{P}_a \right) \hat{u}_a \otimes \hat{u}_a + \hat{P}_a \hat{g}_a,
\]

where \(\hat{u}_a\) is the (unit) fluid flow, and \(\hat{E}_a\) and \(\hat{P}_a\), eigenvalues of \(\hat{T}_a\), the corresponding mass-energy density and pressure. Note that the fluid vector \(\hat{u}_a\) and corresponding ‘hatted’ scalars are objects defined, still, on each \((\mathcal{V}_\varepsilon, \tilde{g}_\varepsilon)\). All these objects, in covariant form, are now pulled back through \(\psi_\varepsilon\) down onto \((\mathcal{V}_0, g)\) (see section 2). That defines the associated families of (tensoral) objects \(G_\varepsilon, T_\varepsilon, E_\varepsilon, P_\varepsilon\) and \(U_\varepsilon\) on \((\mathcal{V}_0, g)\), which therefore satisfy

\[
G(\tilde{g}_\varepsilon)_{\alpha\beta} = 8\pi T_{\alpha\beta},
\]

with

\[
T_\varepsilon = (E_\varepsilon + P_\varepsilon) U_\varepsilon \otimes U_\varepsilon + P_\varepsilon g_\varepsilon,
\]

by construction. It is worth mentioning that the (families of) objects do depend on the gauge defined by \(\psi_\varepsilon\), and thus also the right and left hand sides of (31). However, the equations (31) themselves do not depend on the gauge, in the sense that if (31) are fulfilled in one gauge, they will be satisfied in any other gauge.

On the other hand, the fluid vector in contravariant form can also be pushforwarded through \(\psi_\varepsilon^{-1}\), and thus yet obtain another family of vectors \(\vec{u}_\varepsilon\) on \((\mathcal{V}_0, g)\). Since \(\vec{u}_\varepsilon\mu_u \vec{u}_\varepsilon^\mu = -1\) at each \((\mathcal{V}_\varepsilon, \tilde{g}_\varepsilon)\), we must have \(U_{\varepsilon\mu} u^\mu = -1\) on \((\mathcal{V}_0, g)\). This can be shown to be equivalent to \(g_{\varepsilon\alpha} u^\alpha u^\beta = -1\), and corresponds to the normalization condition that \(\vec{u}_\varepsilon\) must satisfy. We can take now \(\varepsilon\)-derivatives and construct the expansion of \(\vec{u}_\varepsilon\) as \(\vec{u}_\varepsilon = \vec{u} + \varepsilon \vec{u}^{(1)} + \frac{1}{2} \varepsilon^2 \vec{u}^{(2)} + \mathcal{O}(\varepsilon^3)\), and

\[
E_\varepsilon = E + \varepsilon E^{(1)} + \frac{1}{2} \varepsilon^2 E^{(2)} + \mathcal{O}(\varepsilon^3),
\]

\[
P_\varepsilon = P + \varepsilon P^{(1)} + \frac{1}{2} \varepsilon^2 P^{(2)} + \mathcal{O}(\varepsilon^3).
\]

All functions and vector components depend on \(r\) and \(\theta\). We will consider later the existence of a barotropic equation of state for the \(\varepsilon\)-family, independent of \(\varepsilon\), so that \(P_\varepsilon\) is a function of \(E_\varepsilon\) alone. Taking \(\varepsilon\)-derivatives, such relation yields a constraint for the first and second order expansions, which must satisfy, respectively
\( p^{(1)} - \frac{\partial P}{\partial E}E^{(1)} = 0, \)
\[ (35) \]

\( p^{(2)} - \frac{\partial P}{\partial E}E^{(2)} - \frac{\partial^2 P}{\partial E^2}E^{(1)^2} = 0. \)

The absence of convective motions translates onto the condition that \( \bar{u}_e \) lies on the orbits of the group generated by \( [\vec{\eta}, \vec{\xi}] \), this is \( \bar{u}_e \propto \vec{\xi} + \kappa(\epsilon, r, \theta)\vec{\eta} \) for some function \( \kappa \). Rigid rotation demands that \( \kappa(\epsilon, r, \theta) \) does not depend on \( [r, \theta] \), so that \( \bar{u}_e \) are proportional to (timelike) Killing vector fields [27], i.e. \( \bar{u}_e = N(\epsilon)(\vec{\xi} + \kappa(\epsilon)\vec{\eta}) \) for some function \( \kappa(\epsilon) \), with \( N(\epsilon) \) fixed by the above normalization. A static background configuration forces \( \kappa(0) = 0 \), and therefore \( \kappa(\epsilon) = \epsilon(\Omega + O(\epsilon^3)) \) for some constant \( \Omega \). This constant \( \Omega \) is gauge dependent (see below, in section 3.2.1). Following [14] we assume that \( \epsilon \) drives a rotational perturbation, so that only odd powers enter \( \kappa(\epsilon) \). In components we thus demand

\[ u^\theta_\epsilon = \epsilon\Omega u^\gamma \epsilon, \quad u^\epsilon_\epsilon = u^\theta_\epsilon = 0. \]  

This (gauge-dependent) constant \( \Omega \) differs from the perturbation parameter (which we denote by \( \Omega^H \)) as defined in [14]. In the present scheme the perturbation parameter \( \epsilon \) has been defined abstractly, a priori independently of the rotation parameter \( \Omega \). The translation will be given by \( \Omega^H = \epsilon(\Omega + B) \), where \( B \) is a constant to be determined later, see section 3.2.1.

The vacuum equations are obtained by simply setting \( E = P = 0 \).

### 3.1. Background

The matter content of the interior region of the background configuration is a perfect fluid, static and spherically symmetric. Its normalized 4-velocity is \( \bar{u} = e^{\psi/2}\partial_t \). The two field equations providing \( E \) and \( P \) in terms of the metric functions are

\[ \lambda' = \frac{1}{r}\left(1 - e^{\lambda}\right) + re^{8}\pi E, \]
\[ (37) \]

\[ \nu' = \frac{1}{r}(e^{\psi} - 1) + re^{8}\pi P, \]
\[ (38) \]

while the pressure isotropy condition yields the equation

\[ 2\nu'' + \nu'(\nu' - 2) - \lambda'(2 + \nu') + \frac{4}{r}(e^{\psi} - 1) = 0, \]
\[ (39) \]

which can be also written, using (37) and (38), as

\[ P' = \frac{1}{2r}(E + P)(e^{\psi} - 1 + 8\pi r^2 e^{\psi} P). \]
\[ (40) \]

Let us now define \( M(r) \) and \( j(r) \), which will be useful for the comparison of the expressions here with those in [14], by

\[ j(r) := e^{-(\psi + \lambda)/2}, \]
\[ 1 - \frac{2M(r)}{r} = e^{-\lambda}. \]
\[ (41) \]

In the vacuum region (−) the field equations (39) imply that \( M(r_-) \) is a constant, which will be denoted by \( M \) as usual, and that
\[ e^{-\lambda_+ (r_c)} = e^{\nu_+ (r_c)} = 1 - \frac{2M}{r_c} \Rightarrow j(r_c) = 1. \] (42)

We will assume \( M > 0. \) The matching of the background (17) implies, in particular, that

\[ \nu_+ (a) = -\lambda_+ (a) = \log \left( 1 - \frac{2M}{a} \right). \] (43)

and the following expressions for the differences of the derivative of the functions of the metric in terms of the fluid variables

\[ [\nu'] = a e^{\lambda_+ (a)} 8\pi [P] = 0, \] (44)

\[ [\lambda'] = a e^{\lambda_+ (a)} 8\pi [E], \] (45)

\[ [\nu]' = \left( 1 + \frac{a\nu' (a)}{2} \right) \frac{[\lambda']}{a} = \left( 1 + \frac{a\nu' (a)}{2} \right) e^{\lambda_+ (a)} 8\pi [E]. \] (46)

Note that the difference \([E]\) corresponds to the value of the interior energy density \(E_I\) on \(\Sigma_0\), this is \([E] = E_I (a)\), for a vacuum exterior. We just prefer to keep \([E]\) in some expressions for the sake of generality, since they apply in the matching of two fluids, and the notation is, in fact, more compact.

It must be stressed that whereas the matching condition (44) implies, for a vacuum exterior, that \(P (r_c)\) must vanish on \(\Sigma_0\), the energy density \(E (a)\) stays free, \emph{a priori}. Its value will be determined, if any, by the equation of state. Let us remark finally that the condition (20) of proposition 1 is now equivalent to \(M \neq 0.\)

### 3.2. First order problem

The absence of convective motions and rigid rotation (36), together with the normalization condition, yield to first order

\[ ii^{(1)} = \Omega u^i \partial \varphi = \Omega e^{-1/2} \partial \varphi. \] (47)

The linearized energy momentum tensor is found by taking the \(\epsilon\)-derivative of (32) and evaluating on \(\epsilon = 0\), \( T_{\alpha\beta}^{(1)} = (E^{(1)} + P^{(1)}) u_\alpha u_\beta + \mathcal{P}_{\alpha\beta}^{(1)} + 2(E + P)(u_\alpha^{(1)} u_\beta) + u^\rho K_{\rho(1) \alpha\beta} + PK_{(1) \alpha\beta}. \) On the other hand, the perturbed Einstein tensor \(G_{\alpha\beta}^{(1)} = \partial (\delta_{\alpha\beta})_{\varphi=0}\) only contains \([t, \varphi]\) components. It is then straightforward to show that the first order field equations that follow from (31), i.e. \(G_{\alpha\beta}^{(1)} = 8\pi T_{\alpha\beta}^{(1)}\), imply that the first order perturbations of the pressure and density must vanish, i.e. \(E^{(1)} = P^{(1)} = 0\), and leave only one equation for \(\omega (r, \vartheta)\), that reads [14]

\[ \frac{\partial}{\partial r} \left( r^2 j \frac{\partial \omega}{\partial r} \right) + \frac{r^2 j e^{\lambda_+}}{\sin^3 \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin^3 \vartheta \frac{\partial \omega}{\partial \vartheta} \right) + 4r j' (\omega - \Omega) = 0. \] (48)

The equation for the exterior vacuum region \((-)\) is recovered by just setting \(j = 1\) in the above.

Given the regularity condition at the origin, the asymptotic behaviour at infinity and the matching conditions (18) and (19) the functions \(\omega^2 (r_\infty, \vartheta)\) can be shown to be functions of the corresponding radial coordinates only (see [26]). This is in agreement with Hartle’s argument in [14]. In particular, the exterior solution that vanishes at infinity is thus
\( \omega^r = \frac{2J}{r^3} \) \hspace{.5cm} (49)

for some constant \( J \) \cite{14}. For later use, it is easy to show that taking into account that \([J] = 0\), so that \( j(a) = 1, \) and \([j'] = -1/2[\lambda'] \) by construction, the difference of equation (48) yields

\[
[\omega'] = [\lambda'] \left( \frac{1}{2} a' \omega(a) + \frac{2}{a} (\omega^+(a) - \Omega) \right) \hspace{.5cm} (50)
\]

Regarding the perturbation of the hypersurface, let us first note that equations (45) and (46) imply that the differences \([\lambda'] \) and \([\nu'] \) are proportional to the difference \([E]\). The remark made after proposition 1 can now be stated in terms of a physical property of the interior and exterior background configuration: whenever there is a jump in the energy density at the surface, \( Q_1^- (\tau = Q_1^+) \) must vanish necessarily by (22). However, if \([E] = 0\) the function \( Q_1(\tau, \theta) \) is not determined, in principle, and enters the second order. Nevertheless, as shown in section 2.4 when analysing the determination of the surface of the rotating star at second order, \( Q_1 \) will necessarily vanish if \([E'] \neq 0\). In appendix A the whole case \([E] = 0\) is discussed.

3.2.1. On gauges at first order. We discuss next the meaning of the constant \( b_1 \), how it is related with gauges, and its role on the determination of the rotation of the perfect fluid star. Consider a spacetime gauge change in either \((V^\mu, g^x)\) defined by \( \tilde{\xi} = C \partial_x \) (we drop the \( \pm \) for clarity, the two \( C^\pm \) being independent). The rules of transformation of the first order metric perturbation tensor (3), the energy momentum tensor \( T^{(1)} \), and of the first order deformation vector (6) imply, respectively, \( \omega^\mu = \omega - C, \) \( \Omega^\mu = \Omega - C \) and \( b_1^\mu = b_1 - C \). First, note that \( \omega^r - \Omega \) is independent with respect to that gauge. This quantity is essentially the \( \tilde{\omega} \) (up to a sign) defined in \cite{14}.

As discussed, the first order matching conditions are invariant under such spacetime gauges (at either or both sides, with corresponding \( C^+ \) and \( C^- \)), that is, the first order matching conditions (18), (19), (21) and (22) transform to just the same expressions with \( g \) superscripts.

This first order gauge at either side \( \pm \) is fixed (and completely fixed) once the value of the function \( \omega^x \) is fixed at some point (or infinity). The equation for \( \omega^r \) is usually integrated in the exterior vacuum region assuming that \( \omega^r \) vanishes at \( r_\ast \to \infty \). By doing that \( \partial_r \) - is chosen to represent the \('right'\) observer at infinity. At infinity, the vector \( \partial_r \) - is thus assumed to be both unit and orthogonal, with respect to \( g_\ast \) to second order, to the axial Killing vector \( \partial_\phi \). The exterior choice of gauge thus fixes \( \omega^r \), and it is given by (49).

Nevertheless, as shown in section 2.4 when analysing the determination of the surface of the rotating star at second order, \( Q_1 \) will necessarily vanish if \([E'] \neq 0\). In appendix A the whole case \([E] = 0\) is discussed.

Regarding the interior region, the above spacetime gauge for some \( C^1 \) can then be used to get rid of one of the two constants that describe the configuration at first order, either \( b_1 \) or \( \Omega \), but clearly not both. The transformations of \( b_1 \) and \( \Omega \) suggest building a quantity defined on \( \Sigma_0 \) as

\[ \Omega_{\infty} = \Omega - b_1, \] \hspace{.5cm} (51)

invariant under the gauge \( \tilde{\eta} \). The meaning of this constant is the following. \( \Omega \) defines the rotation of the fluid flow with respect to the interior observer \( \partial_r \), and \( b_1 \) determines the tilt on \( \Sigma_0 \) between that interior observer \( \partial_r \) and the (already fixed) exterior observer \( \partial_\phi \), explicitly \( \partial_r |_{\Sigma_0} \neq \partial_\phi |_{\Sigma_0} \neq \partial_r |_{\Sigma_0} - \epsilon b_1 \partial_\phi |_{\Sigma_0} \). The difference \( \Omega_{\infty} \) thus describes the tilt of the fluid flow with respect to the continuous extension of the exterior observer to the interior, and thence, measures the rotation of the fluid with respect to the unit non-rotating observer at infinity.
The value of the ‘invariant’ quantity \( \tilde{\omega}(r) \) at the boundary can then be expressed thanks to the condition (18) as \( \omega^+(r) - \Omega = \omega^-(r) - \Omega_\infty \), i.e.

\[
\tilde{\omega}^+(a) = 2J/a^3 - \Omega_\infty.
\]

This yields the desired relation between the value of the interior \( \tilde{\omega}^+(a) \), integrated via (48) from the origin, \( J \) and the rotation of the star, thus described by \( \Omega_\infty \).

In [14] the function \( \omega \) is assumed to be ‘continuous’ by construction. In the present general setting that corresponds to a choice of gauge in the interior region for which \( b_1 = 0 \), and therefore \( \Omega(= \Omega_\infty) \) corresponds indeed to the rotation of the fluid as measured by the unit exterior observer. The relation between \( \Omega \) and \( \Omega^H \) is thus explicitly given by \( \Omega^H = \varepsilon \Omega_\infty \).

In contrast, in [3] the gauge in the interior is chosen so that the interior observer \( \partial_r \) moves with the fluid, i.e. \( \Omega = 0 \) (comoving gauge). Thereby, since the freedom one may have in the interior driven by \( \tilde{\omega} \) has been already fixed, the price to pay is a rotation in the matching hypersurface given by the constant \( b_1 \), which corresponds to the parameter \( -c_4 \Omega \) in [3], so that \( \Omega_\infty \) corresponds to ‘\( c_4 \Omega \)’ there.

### 3.3. Second order problem

We explore now the second order field equations for a perfect fluid with barotropic equation of state (the interior) and for the exterior vacuum, and particularize the second order perturbed matching conditions of proposition 2. The conditions on the fluid flow (36) together with the normalization condition now lead to \( \tilde{H}^{(2)} = \tilde{H}^{(2)}_{\partial_r} \), where

\[
\tilde{H}^{(2)}_{\partial_r} = e^{-3\varepsilon/2} \left\{ \Omega^2 g_{\theta\theta} + 2\Omega K^{(1)}_{\theta\theta} + K^{(2)}/2 \right\}.
\]

Taking the second \( \varepsilon \)-derivative of (32), evaluating on \( \varepsilon = 0 \), and using \( E^{(1)} = P^{(1)} = 0 \), the second order energy momentum tensor is found to take the form

\[
T^{(2)}_{\alpha\beta} = \left( E^{(2)} + P^{(2)} \right) u_\alpha u_\beta + P^{(2)} g_{\alpha\beta} + PK^{(2)}_{\alpha\beta} + 2(E + P) \left( u^{(2)}_{(\alpha} u_{\beta)} + u^{(1)}_{(\alpha} u^{(1)}_{\beta)} \right)
\]

\[
+ u^{(2)}_{\rho(\alpha} u^{(2)}_{\beta)} + 2u^{(1)}_{\rho(\alpha} u^{(1)}_{\beta)} + 2u^{(2)}_{\rho(\alpha} u^{(1)}_{\beta)} + u^{(2)}_{\rho(\alpha} u^{(1)}_{\beta)} + u^{(2)}_{\rho(\alpha} u^{(1)}_{\beta)}\right).
\]

As follows from (31) the second order Einstein field equations consist of equating this to the second order perturbed Einstein tensor, computed from \( g_{\varepsilon} \) as \( G^{(2)}_{\alpha\beta} := \partial_\partial G(g_{\varepsilon})_{\alpha\beta} |_{\varepsilon=0} \), that is

\[
G^{(2)}_{\alpha\beta} = 8\pi T^{(2)}_{\alpha\beta}.
\]

Given that the final purpose of the present work is to analyse the results in [14] regarding the matching problem, we assume the same angular behaviour of the functions of the second order perturbation tensor (for both the interior + and the exterior − regions). This behaviour is argued in [14] to follow from the non-dependency of the first order function \( \omega \) on any angular coordinate and equatorial symmetry. The assumption we take on the functions of \( K^{(2)}_{\pm} \) thus reads explicitly

\[
\begin{align*}
h(r, \theta) &= h_0(r) + h_2(r)P_2(\cos \theta), \\
m(r, \theta) &= m_0(r) + m_2(r)P_2(\cos \theta), \\
k(r, \theta) &= k_0(r) + k_2(r)P_2(\cos \theta), \\
f(r, \theta) &= f_2(r)P_2(\cos \theta),
\end{align*}
\]

for both the interior + and the exterior −. A straightforward calculation shows that the above angular structure assumed on the functions in \( K^{(2)} \) is inherited, via the field equations (52), by the second order energy momentum tensor, so that
Given that \( E^{(1)} = P^{(1)} = 0 \) the barotropic character of the equation of state to second order (35) translates onto the condition

\[
E^{(2)} P' - P^{(2)} E' = 0.
\]

In order to write down the second order field equations in a convenient and compact form, let us first define the following auxiliary ‘tilded’ functions

\[
\begin{align*}
\tilde{h}_0 &:= h_0 - \frac{1}{2} rv' k_0, & \tilde{m}_0 &:= m_0 - e^{i2/2}(e^{i2/2} r k_0), \\
\tilde{h}_2 &:= h_2 - \frac{1}{2} rv' f_2, & \tilde{m}_2 &:= m_2 - e^{i2/2}(e^{i2/2} r f_2), & \tilde{k}_2 &:= k_2 - f_2.
\end{align*}
\]

Clearly these quantities are invariant under the ‘radial’ gauges class of transformations (12) since e.g. both \( h - \frac{1}{2} rv' k \) and \( h - \frac{1}{2} rv' f \) are.

We introduce now the above decomposed expressions of the relevant quantities into the field equations (52). By construction the complete set of equations gets decomposed into the \( l = 0 \) and \( l = 2 \) sectors, which are independent and can thus be considered separately. Our purpose in the next two subsections is to recover and write down the field equations as closely as possible to the expressions presented in sections VII and VIII in [14]. The explicit correspondences are made in section 4.

3.3.1. The EFEs in the \( l = 0 \) sector. The \( l = 0 \) sector of the field equations (52) can be shown to provide the following expressions for the second order energy density and pressure

\[
8\pi E_0^{(2)} = \frac{4}{r^2} \left( r e^{-j} \tilde{m}_0 \right)' + \frac{8}{r j} (\omega - \Omega)^2 - \frac{1}{3} t^2 \omega'^2 + 16\pi r E' k_0, \tag{58}
\]

\[
8\pi P_0^{(2)} = \frac{4}{r^2} \left( e^{-j} r \tilde{m}_0 - \tilde{m}_0 \left( 8\pi r^2 P + 1 \right) \right) + \frac{1}{3} t^2 \omega'^2 + 16\pi r P' k_0, \tag{59}
\]

plus an equation for \( \tilde{h}_0^* \) of the form \( \tilde{h}_0^* = F_0 (\tilde{h}_0^*, \tilde{m}_0^*, \tilde{m}_0^*) \). A convenient auxiliary definition of the second order pressure is given by

\[
\tilde{P}_0 := \frac{P_0^{(2)} - 2r P' k_0}{2(E + P)} = \frac{P_0^{(2)}}{2(E + P)} + \frac{k_0}{r - 2M(r)} \left( M(r) + 4\pi r^2 P \right), \tag{60}
\]

where (40) has been used in the equality. This function is well defined at points where \( E + P = 0 \) (see below), and corresponds to the \( l = 0 \) part of the ‘pressure perturbation’ factor as defined in equation (87) in [14].

On the other hand, the \( l = 0 \) part of equation (55), i.e. \( E_0^{(2)} P' - P_0^{(2)} E' = 0 \), combined with (58), yields a direct relation between \( P_0^{(2)} \) and \( m_0' \), which written in terms of ‘tilded’ quantities reads

\[
\left( r e^{-j} \tilde{m}_0 \right)' = 4\pi r^2 \left( \frac{E'}{P'} (E + P) \tilde{P}_0 + \frac{1}{12} t^2 r^4 \omega'^2 - \frac{2}{3} t^2 \omega' \right). \tag{61}
\]

Now, the aforementioned equation for \( \tilde{h}_0^* \) can be rewritten, using (59) and (61)—and (41)—for cosmetics—as a first order ODE for \( \tilde{h}_0 \), that reads

\[ \text{These two equations correspond to (93) and (94) in [14]. Note that a global 2 factor on the right hand side here comes from the definitions (33) and (34) as compared with the definition of } \Delta G \text{ in [14], which already contains the } r^2 \text{ and } 1/2 \text{ factors.} \]
The set of functions that determines the \( l = 0 \) sector can thus be taken to be \( \{ \tilde{P}_0, \tilde{m}_0 \} \), which satisfies the system (61), (62) given regularity conditions at the origin \( r = 0 \). Equation (59) can be rewritten as

\[
\tilde{h}_0' - r e^2 \tilde{m}_0 \left( 8 \pi P + \frac{1}{r^2} \right) = 4 \pi r e^4 (E + P) \tilde{P}_0 - \frac{1}{12} e^4 r^2 \omega^2 .
\]  

(63)

It is now trivial to check that (see (90) in [14])

\[
\tilde{P}_0 + \tilde{h}_0 - \frac{1}{3} r^2 e^{-4} (\omega - \Omega)^2 = \gamma
\]  

(64)

for some constant \( \gamma \) is a first integral of (62) and (63). This relation shows, in particular, that \( \tilde{P}_0 \) is well defined in \( r, \in [0, a] \). The constant \( \gamma \) is identified in [14] as the second order to background ratio of the constant injection energy. In analogy with the Newtonian potential, \( \tilde{h}_0 \) (and thus \( h_0 \)) is determined up to an arbitrary additive constant. This constant will be determined once a condition at infinity plus some continuity across the boundary of the body are imposed. We will discuss that below. Once that is fixed, the value of \( \gamma \) still depends on one factor, that is, the conditions one may impose on \( \tilde{P}_0 \) at the origin. The latter depends on how one sets the value of the pressure (and thus of the energy density) of the rotating configuration at the origin with respect to that of the static configuration. In the present work the pressure is taken to be unchanged after the perturbation, so that \( \tilde{P}_0(0) = 0 \), as in [14] and subsequent works. Leaving \( \tilde{P}_0(0) \) as an extra parameter of the model does not introduce any remarkable effect for the purposes of our analysis. We refer to [3] (section III A) for a deeper discussion.

The fact that \( k_0 \) is ‘pure gauge’ translates onto the fact that it does not enter the set of equations, and it is therefore not determined. The quantities \( E^{(2)}_0 \) and \( P^{(2)}_0 \) are gauge dependent, and can only be computed, from (58) and (59) respectively, once \( k_0 \) is specified, i.e. by fixing the ‘radial’ gauge. The quantities independent of that choice, and thus the relevant ones, correspond to \( E^{(2)}_0 - 2E' r k_0 \) and \( P^{(2)}_0 - 2P' r k_0 \). This is the motivation for the introduction of the auxiliary function \( \tilde{P}_0 \).

The equations for \( \{ \tilde{h}_0^-, \tilde{m}_0^-, \} \) in the vacuum exterior are obtained by using (42) and the first order solution (49) in equations (58) and (59) with their left hand sides and \( P \) and \( E \) set to zero. The boundary condition is set so that \( \tilde{h}_0^- \) and \( \tilde{m}_0^- \) vanish at infinity \( r_- \to \infty \). The solutions vanishing at infinity are thus given by

\[
r_- e^{-2(r_-)} \tilde{m}_0^- (r_-) = \delta M - \frac{J^2}{r_3^-} ,
\]  

(65)

\[
\tilde{h}_0^- (r_-) = - \frac{\delta M}{r_- - 2M} + \frac{J^2}{r_3^-(r_- - 2M)} ,
\]  

(66)

where \( \delta M \) is an arbitrary constant. As mentioned above, the function \( k_0^- \) remains undetermined, under the condition that is also vanishes at infinity.

3.3.2. The EFEs in the \( l = 2 \) sector. Apart from the two field equations that provide the energy density and pressure, the \( l = 2 \) sector provides three equations. The whole set of
equations can be shown to be equivalent to the system
\[
\begin{align*}
\tilde{h}_2' + \tilde{k}_2' &= -\nu'\tilde{h}_2 + \left(\frac{1}{r} + \frac{\nu'}{2}\right)\left(-\frac{2}{3}r^3 j^j (\omega - \Omega)^2 + \frac{1}{6} j^2 r^4 \omega^2\right), \\
\tilde{h}_2' &= \left\{ -\nu' + \frac{r}{(r - 2M(r))\nu'}\left(8\pi (E + P) - \frac{4M(r)}{r^3}\right) \right\} \tilde{h}_2 - \frac{4(\tilde{h}_2 + \tilde{k}_2)}{r\nu'(r - 2M(r))} \\
&\quad + \frac{1}{6}\left(\frac{1}{2} r\nu' - \frac{1}{(r - 2M(r))\nu'}\right)r^2 j^2 (\omega - \Omega)^2 \\
&\quad - \frac{1}{3}\left(\frac{1}{2} r\nu' + \frac{1}{(r - 2M(r))\nu'}\right)r^2 (j^2)' (\omega - \Omega)^2
\end{align*}
\] (67)

plus the equation
\[
\tilde{m}_2 = \frac{1}{6} r^2 j^2 (\omega - \Omega)^2 - \frac{1}{3} r^2 (j^2)' (\omega - \Omega)^2 - \tilde{h}_2.
\] (69)

The expressions for the energy density and pressure can then be written as
\[
E_2^{(E)} - 2E'rf_2 = \frac{4E'}{3\nu'}\left(3\tilde{h}_2 + e^{\nu r^2} (\omega - \Omega)^2\right),
\] (70)
\[
P_2^{(E)} - 2P'rf_2 = -\frac{2}{3} (E + P)\left(3\tilde{h}_2 + e^{\nu r^2} (\omega - \Omega)^2\right).
\] (71)

Note that we have kept the background function \(\nu\) explicitly in order to ease the eventual comparison with the expressions in [14]. In the \(l = 2\) sector imposing the condition of a barotropic EOS does not add any further condition.

The convenient ‘pressure perturbation factor’ in this case corresponds to the following definition
\[
\tilde{P}_2 := \frac{P_2^{(E)} - 2P'rf_2}{2(E + P)},
\] (72)

so that (71) just reads
\[
\tilde{P}_2 + \tilde{h}_2 + \frac{1}{3} e^{\nu r^2} (\omega - \Omega)^2 = 0.
\] (73)

This corresponds to (91) in [14], and, together with the above (64) form the \(l = 0\) and \(l = 2\) parts of the first integral \(\gamma\), (86) in [14].

The interior region is thus determined by the solution of the pair \(\tilde{h}_2^+, \tilde{k}_2^+\) to the system (67), (68) given regularity conditions at the origin \(r_0 \to 0\), up to an arbitrary constant, say \(A'\). Then, \(\tilde{m}_2^+\) is directly obtained from (69). The function \(f_2(r)\) does not enter the equations, and thus it is, as expected, pure gauge.

In the **vacuum exterior** region only equations (67)–(69) apply. Using (42), so that in particular \(P = 0\), and (49), and given the asymptotic behaviour at \(r_+ \to \infty\), the whole set of exterior functions \(\tilde{h}_2^-, \tilde{k}_2^-, \tilde{m}_2^+\) is integrated and read
\[
\tilde{h}_2^- = A Q_2^2 \left(\frac{r_-}{M} - 1\right) + \frac{J^2_+}{r_+^2} \left(\frac{1}{M} + \frac{1}{r_-}\right).
\] (74)
\[ \tilde{k}_2^- + \tilde{h}_2^- = A \left\{ \frac{2M}{\sqrt{r_-(r_--2M)}}Q^1_2 \left( \frac{r_-}{M} - 1 \right) \right\} - \frac{J^2}{r_-^2}, \]  
(75)

\[ \tilde{m}_2^- = -AQ^2_2 \left( \frac{r_-}{M} - 1 \right) + \frac{J^2}{r_-^2} \left( \frac{1}{M} - \frac{5}{r_-} \right), \]  
(76)

where \( Q^m_\ell(x) \) stand for the associated Legendre functions of the second kind, and \( A \) is an arbitrary constant. The constants \( A' \) and \( A'' \) are to be determined once the relations between \( \{ \tilde{h}_2^-, \tilde{k}_2^- \} \) and \( \{ \tilde{h}_2^+, \tilde{k}_2^+ \} \) on the matching hypersurface \( \Sigma_0 \) are determined (see below).

### 3.3.3. The matching of the second order problem

We particularize first the matching conditions as given in proposition 2 for the particular angular expansion of the perturbation functions (53) at both sides. The field equations in the background allow us to express the differences \( [\ell'] \) and \( [\nu'] \) in terms of \([E]\) by direct use of (45) and (46). However, we will not use those relations in some places, nor the explicit form of \( \nu_\ell(r_-) \) in the exterior, to keep more compact expressions. Let us recall that condition (20) now just reads \( M \neq 0 \) given the exterior is vacuum. Clearly, for all pairs \( f^\pm(r_\pm, \theta_\pm) \) such that \( f = f_0(r) + f_2(r)P_2(\cos \theta) \) we have \([f] = [f_0] + [f_2]P_2(\cos \theta)\). Note that \([f_0]\) and \([f_2]\) are constants.

Equation (24) is thus satisfied if and only if \( c_1 = 0 \) plus

\[ [k_2] = [f_2]. \]  
(77)

The constant \( c_2 \) just corresponds to the difference \([k_0]\), i.e. \([k_0] = c_2\).

Likewise, equation (25) is satisfied if and only if \( H_1 = 0 \) plus

\[ [h_0] = \frac{H_0}{2} + \frac{1}{2}a'v(a)[k_0], \]  
(78)

\[ [h_2] = \frac{1}{2}a'v(a)[f_2]. \]  
(79)

Equation (29), since \( c_1 \) and \( H_1 \) must vanish, imposes a very particular expansion of \([\hat{Q}_2](\theta)\), explicitly

\[ \hat{Q}_2(\theta) = \left[ \hat{Q}_2^{(0)} \right] + \left[ \hat{Q}_2^{(2)} \right]P_2(\cos \theta) \]  
for some constants \([\hat{Q}_2^{(0)}]\) and \([\hat{Q}_2^{(2)}]\). Equation (29) is thus equivalent to the pair

\[ \left[ \hat{Q}_2^{(0)} \right] = 2ae^{-v(a)/2}[k_0], \]  
(80)

\[ \left[ \hat{Q}_2^{(2)} \right] = 2ae^{-v(a)/2}[f_2], \]  
(81)

where here, and in the following expressions, equation (43) is used to set \( \lambda(\alpha) = -\nu(\alpha) \). Take now the equations for the differences \([m]\) and \([h']\). In the case \([E] \neq 0 \) \((\ell' \neq 0 \) and \([\nu'] \neq 0)\), for which \( Q_1 = 0 \) necessarily, we recall we necessarily have \( \hat{Q}_2^+ = \hat{Q}_2^+(\theta) \) and therefore both \( \hat{Q}_2^\pm \) due to the above, so that

\[ \hat{Q}_2^\pm(\theta) = \hat{Q}_2^{(0)} + \hat{Q}_2^{(2)}P_2(\cos \theta), \]  
(82)
with constants \( \hat{Q}_{2,0} \) and \( \hat{Q}_{2,2} \). Thence, equation (26) holds iff
\[
[m_0] = a[\hat{k}_0] + \frac{1}{4} e^{2(a)/2} [k'] \hat{Q}_{2,0} + \frac{1}{2} (a' - a) \hat{k}_0,
\]
while equation (27) does whenever
\[
[h_0] = \frac{1}{2} \nu \hat{k}_0 + \frac{1}{4} e^{2(a)/2} [k'] \hat{Q}_{2,0} + \frac{1}{2} (a' - a) \hat{k}_0,
\]
and
\[
[m_2] = a[\hat{k}_2] + \frac{1}{4} e^{2(a)/2} [k'] \hat{Q}_{2,2} + \frac{1}{2} (a' - a) \hat{k}_2,
\]
while equation (27) does whenever
\[
[h_2] = \frac{1}{2} \nu \hat{k}_2 + \frac{1}{4} e^{2(a)/2} [k'] \hat{Q}_{2,2} + \frac{1}{2} (a' - a) \hat{k}_2.
\]

In the case \([E] = 0\), the equation \([\omega'] Q_l = 0\) provides no information, since \([\omega'] = 0\) as follows from (50) and (45). On the other hand, the equations corresponding to (26) and (27) with the changed terms (28) contain a term proportional to \(EQ\). If \([E] = 0\) we recover the above equations with \( [\hat{z}'] = [\nu'] = [E] = 0\) and therefore one only needs considering the case \([E] \neq 0\). In that case the equations imply, analogously, that \(Q_1\) does not depend on \(r\) and that it must satisfy \((Q_1)^2 = q_0 + q_2 P_2(\cos \theta)\) for some constants \(q_0\) and \(q_2\).

Some remarks are in order now, which will lead us eventually to the determination of the deformation of the matching hypersurface at second order in any ‘radial’ gauge—recall that the deformation vectors \(\vec{Z}\) are gauge dependent, and therefore the functions \(Q\) describe the deformation with respect to the gauge chosen. The appropriate quantities are constructed as follows
\[
\Xi_0 := \hat{Q}_{2,0} - 2ae^{-x/a}k_0(a), \quad \Xi_2 := \hat{Q}_{2,2} - 2ae^{-x/a}f_2(a)
\]
on \(\Sigma_0\) from either side \(+\) and \(−\). These two quantities are ‘radial’-gauge independent, since the gauge defined by \(\vec{V}_2 = 2S(r, \theta)\delta_x\) and \(\vec{S}_1 = C\delta_\phi\) induces via (6) the transformation \(\hat{Q}_2 = \hat{Q}_2 + 2ae^{x/a}2\), while \(k^x = k + S/r\) and \(f^x = f + S/r\), see (12). On the other hand, the relations (80) and (81) just read
\[
[\Xi_0] = 0, \quad [\Xi_2] = 0,
\]
meaning that the quantities coincide as computed from either side. How the actual deformation \(\Sigma^+\) out from the spherical \(\Sigma_0\) is encoded in terms of \(\Xi_0\) and \(\Xi_2\) is described in appendix A.

The above matching conditions to second order have yet to be combined with the constraints provided by the field equations at either side. We obtain the final expressions of the matching conditions to second order using the second order field equations for the perfect fluid interior and the vacuum exterior next.

Regarding the \(l = 0\) sector, the differences of the field equations do not provide any constraints to the matching conditions in the sense that the differences \([k_0]\) and \([k_0']\) remain arbitrary (constants). This, as expected, is related to the fact that \(k_0\) is pure gauge. The \(l = 0\) matching conditions (78), (83) and (85) can be written in terms of the ‘tilted’ functions (56) and the deformation functions (87) in the case \([E] \neq 0\) as follows,
\[
[h_0] = \frac{H_0}{2},
\]
\[
\left[ g_0^\prime \right] = \frac{a - M}{a(a - 2M)} [\tilde{m}_0],
\]
while if \([E] = 0\) equation (91) is replaced by

\[
\left[ \tilde{m}_0 \right] = -2\pi [E] e^{-\nu(a)/2} a \Xi_0.
\]

The background matching configuration relations (45) and (46) have been used to express the background difference functions in terms of \(E\), which is just \(E_{\tau}(a)\) (vacuum exterior), together with (42) to write

\[
a\nu'(a) = \frac{2M}{a - 2M} = e^{-\nu(a)} 2M a.
\]

The arbitrariness in shifting \(\tilde{h}_0^+ (r_i)\) corresponds here to the appearance of the free constant \(H_0\). One can always fix the shift in \(\tilde{h}_0^+ (r_i)\) in the interior simply by choosing \(H_0\). This just mirrors the fact that in Newtonian theory the potential is fixed at infinity and then taken to the interior of the body simply by imposing continuity across the boundary.

It must stressed, however, that the argument about the ‘continuity’ of \(\tilde{h}_0\) does not stand for the other function \(\tilde{m}_0\) in general. Consider first the difference of equation (59) for a vacuum exterior combined with the two matching conditions (89) and (90) at hand, which leads to the relation

\[
\left[ \tilde{m}_0 \right] = -4\pi a^3 M |E| \mathcal{P}_0(a),
\]

after using the definition (60). Note that this equation holds always, irrespective of the vanishing (or not) of \([E]\). Now, in the case \([E] \neq 0\), (91) can be finally rewritten as

\[
(2|E| \mathcal{P}_0(a)) = \mathcal{P}_0^{(2)}(a) - 2a'P^0(a)k_0^+ (a) = -\frac{M}{a^2} e^{-\nu(a)/2} |E| \Xi_0.
\]

In the \([E] = 0\) case equation (94) clearly implies \([\tilde{m}_0] = 0\) and therefore (92) yields

\[
[E] \tilde{h}_0 = 0.
\]

The implication of (94) is that the values of the functions \(\tilde{m}_0^+ (a)\) and \(\tilde{m}_0^- (a)\) coincide if and only if \([E] \mathcal{P}_0(a) = 0\). This fact turns out to be in contradiction with the assumption made in [14] stating that \(m_0^\pm\) is ‘continuous’ at the boundary, with consequences on the determination of \(\delta M\). We devote the concluding section to analyse this discrepancy and provide the correct expression for \(\delta M\).

Finally, the field equation (59) at both sides (±) can be used to replace the condition (90) by (95). To sum up, given the Einstein’s field equations hold, in the \(l = 0\) sector the set of matching conditions can be given by the two conditions (89) and either (91) or (94), plus the relation (95).

In the \(l = 2\) sector things are different, in the sense that the field equations provide, in principle, further constraints to the matching conditions. Taking the differences of the field equations (67)–(69) we obtain three equations for the differences \([\tilde{m}_2], [\tilde{k}_2], [\tilde{h}_2]\) which have to be added to the relations in (84) and (86) and the relations (77) and (79) that already determine \([\tilde{k}_2]\) and \([\tilde{h}_2]\) trivially. The number of independent equations turns out to be four plus these two trivial ones, and can be finally cast, when \([E] \neq 0\) (⇒ \(Q_l = 0\), as
\[
\begin{align*}
\left[ \hat{h}_2 \right] &= 0, \quad \left[ \tilde{h}_2 \right] = 0, \\
[E] \left\{ \hat{h}_2(a) - \frac{1}{4} \nu'(a) e^{(\nu/2)_2} \hat{\Xi}_2 + \frac{1}{3} a^2 e^{-i(\nu/2)} \left( \frac{2J}{a^3} - \Omega_{\infty} \right) \right\} = 0,
\end{align*}
\]

plus
\[
\begin{align*}
\left[ \hat{\kappa}_2 \right] &= 4\pi [E] \frac{a^2}{M} \hat{h}_2(a) + \frac{4}{3} \pi [E] \frac{a^2}{M} e^{-2i(\nu/2)} (a - M)^2 \left( \frac{2J}{a^3} - \Omega_{\infty} \right)^2, \\
\left[ \tilde{\kappa}_2 \right] &= -4\pi [E] \frac{a^2}{M} \tilde{h}_2(a) - \frac{4}{3} \pi [E] \frac{a^3}{M} (a - 2M) e^{-i(\nu/2)} \left( \frac{2J}{a^3} - \Omega_{\infty} \right)^2, \\
\left[ \tilde{m}_2 \right] &= \frac{8}{3} \pi a^4 [E] e^{-i(\nu/2)} \left( \frac{2J}{a^3} - \Omega_{\infty} \right)^2,
\end{align*}
\]

where we have used, in particular, that
\[
\nu \Omega - \nu' + - = \nu \Omega - \nu' + - = \nu \Omega - \nu' + - = \nu \Omega - \nu' + - \]
given the exterior region is vacuum. Therefore, for \([E] \neq 0\) the set of matching conditions for the \(l = 2\) sector is composed by only three equations, given by the two in (96) and (97). The three relations (98)–(100) are now a consequence of (96) and (97) and the field equations (67)–(69) at both sides.

Regarding the \([E] = 0\) case, all the above equations (96)–(100) hold except for (97), which has to be substituted by \([E] Q_1 = 0\). As a first consequence, the above matching conditions (96)–(100) always hold true, irrespective of whether or not \([E]\) vanishes. Finally, if \([E'] \neq 0\) then \(Q_1 = 0\).

Let us summarize the main results in this section so far in the form of the following theorem.

**Theorem 1.** Let \((\mathcal{V}_0, g)\) with \(\Sigma_0\) be the static and spherically symmetric background matched spacetime configuration, perturbed at either side to first order by the functions \(a_\pm (r_\pm, \theta_\pm)\) through \(K^{(1)} \pm\) as defined in (8) plus the unknowns \(\hat{Q}_1^\pm (\tau, \theta)\) and \(\hat{T}_2^\pm (\tau, \theta)\), as described in proposition 1, so that the first order matching conditions (18) and (19) plus (21) and (22) hold. Let the configuration be perturbed to second order by \(K^{(2)} \pm\) as defined in (9), plus the unknowns \(\hat{Q}_2^\pm (\tau, \theta)\) and \(\hat{T}_2^\pm (\tau, \theta)\) on \(\Sigma_0\), and assume that the interior region (+) satisfies the field equations for a perfect fluid with barotropic equation of state and that the exterior (−) region is asymptotically flat and satisfies the vacuum field equations up to second order. The energy density \(E(\tau)\) and pressure \(P(\tau)\) of the interior background configuration are given by (37) and (38) and must satisfy (40). The background exterior vacuum solution is given by (42), and we assume \(M > 0\). Consider the convenient background quantities defined in (41).

Let \(\mathbf{u}_0\) be the unit vector fluid corresponding to the interior family of metric tensors \(g_i^+ = g^+ + \varepsilon K^{(1)+} + \varepsilon^2 K^{(2)+} + \mathcal{O}(\varepsilon^3)\). Assume that \(\mathbf{u}_0\) satisfies (36) for some constant \(\Omega\). Let 1 be defined by the first order exterior solution (49).
Assume finally at both sides (±) that the first order function $\omega$ depends only on the radial coordinate, and that the second order functions are decomposed in Legendre polynomials in terms of \{$h_0$, $h_2$, $m_0$, $m_2$, $k_0$, $k_2$, $f_2$\} by (53).

Then

1. The second order pressure $P^{(2)}$ and energy density $E^{(2)}$ of the fluid inherit the same angular dependency, that is, (54) hold for some $E_0^{(2)}(r)$, $E_2^{(2)}(r)$, $P_0^{(2)}(r)$ and $P_2^{(2)}(r)$. With the help of convenient alternative ‘tilted’ counterparts, defined in (56) plus (60) and (72), the Einstein’s field equations in the interior can be expressed as the system (61), (62) and (64) for some constant $\gamma$ for the set \{$\tilde{P}_0^+$, $\tilde{m}_0^+$, $\tilde{h}_0^+$\} plus the system (67), (68), (69) for the set \{$\tilde{h}_2^+$, $\tilde{k}_2^+$, $\tilde{m}_2^+$\}. The vacuum solution at second order is given by (65), (66), (74)–(76) where $\delta M$ and $\Lambda$ are arbitrary constants.

2. Given the Einstein’s field equations of the previous point are satisfied, the necessary and sufficient conditions that the metric perturbation tensors $K^{(2)}_{±}$ must satisfy to fulfill the second order matching conditions are given by (89) and (94) for the sets \{$\tilde{P}_0^+$, $\tilde{m}_0^+$, $\tilde{h}_0^+$\}, with arbitrary constant $H_0$, and the two equations in (96) for the sets \{$\tilde{h}_2^+$, $\tilde{k}_2^+$, $\tilde{m}_2^+$\}.

Regarding the deformation of the boundary $\Sigma_0$, expressions (95) and (97) show explicitly how the quantities $\Xi_0$ and $\Xi_2$, and thus the deformation of $\Sigma_0$, are linked in a ‘radial’-gauge invariant manner to the jump in the pressure at second order across the boundary of the star through the value of the energy density of the background configuration at $\Sigma_0$. Whenever $[E] \neq 0$, equations (95) and (97) directly determine $\Xi_0$ and $\Xi_2$ in terms of arbitrary constants that are obtained by integration from the origin. Equations (95) and (97) can then be cast as

$$\Xi_0 = -\frac{2a^2}{M}e^{\psi(a)/2}P_0(a),$$

$$\Xi_2 = e^{-\psi(a)/2}\frac{2a(a - 2M)}{M}\left(h_2(a) + \frac{a^3}{3 a - 2M} \left(\frac{2J}{a} - \Omega_\infty\right)^2\right)$$

$$\Xi_2 = -\frac{2a^2}{M}e^{\psi(a)/2}\tilde{P}_2(a),$$

after using (93) and (73) in the first and second equalities in the latter, respectively.

However, if $[E] = 0$, since $\hat{Q}_2^±$ are only defined on $\Sigma_0$ we cannot determine the deformation directly from the above, in the same way $Q_1$ is undetermined in the first order problem in that case. This is to be expected. In fact, as an extreme case, when matching two vacuum regions the matching hypersurface is not determined in general. The idea is that in order to have a boundary determined by the matching, the energy density must depart from zero as one moves to the interior, so that the star indeed extends no further than, and up to, that surface. A sufficient condition is that $[E] \neq 0$. In that case it can be shown that one can make use of the gauge that follows the surfaces of constant energy density, which has been used so extensively in the literature, specially in [14]. In order to determine the deformation one can then extend $\Xi_0$ and $\Xi_2$ to the interior, say using some functions $\xi_0(r_a)$ and $\xi_2(r_a)$ in a convenient way, using that gauge, to finally obtain the deformation by continuity. This is discussed in appendix A, where it is shown, in particular, that (101) and (102) will hold also when $E(a) = 0$, under the condition that the gauge that follows the surfaces of constant energy density exists. This suggests the fact that equations (95) and (97) are expected to
appear again at higher orders, in the same way the condition \([E]Q_1 = 0\) of the first order problem appears as \([E](Q_1)^2 = 0\) at second order.

### 4. Conclusion: comparing with Hartle’s results

The gauge used in [14] at first order corresponds to setting \(b_1 = 0\) here, while at second order the starting point is the choice of gauge that corresponds here to setting \(k_0^0 = 0\) and \(f_0^0 = 0\). We refer to this choice as the \(k\)-gauge. At some point another gauge coming with the deformation is introduced. A discussion of the use of that gauge in [14] (also in [3]) can be found in appendix A.

In the \(k\)-gauge all the ‘tilded’ functions (56) and (57) correspond to the non-‘tilded’ counterparts, and in the interior region (+), the functions \(\bar{P}_0\) and \(\bar{P}_2\) are just rescalings of their respective \(P_0^{25}\) and \(P_2^{25}\), that is, \(\bar{P}_{0/2} = P_{0/2}^{25}/(2(E + P))\) = \(P_{0/2}\). To avoid having to rewrite all the previous equations without tildes we will simply use the ‘tilded’ functions in what follows.

Let us first concentrate on the \(l = 0\) sector. Regarding the interior region, the system (61)–(62) plus equation (63) for the set \([re^{-2}\tilde{m}_0^+, \bar{P}_0, h_0]\), as functions of \(r \in (0, \infty)\) coincide one by one with the coupled equations (97) and (100) plus (98) in [14] for \([m_0^H, p_0^H, h_0^H]\) as functions of \(R\), which has the same range as \(r_\pm\). To be precise, one can forget about \(r_\pm\) and \(R\) and just establish a common variable \(s\), so that the sets of equations here and in [14] hold in the range \(s \in (0, a]\). Given common conditions at \(s \to 0\) the problem for \([re^{-2}\tilde{m}_0^+, \bar{P}_0, \tilde{h}_0]\) coincides with the problem for \([m_0^H, p_0^H, h_0^H]\) and therefore \(m_0^H(s) = se^{-2s}\tilde{m}_0(s), p_0^H(s) = \bar{P}_0(s)\) and \(h_0^H(s) = \tilde{h}_0(s)\) (up to a free additive constant) necessarily for \(s \in (0, a]\), i.e. in the interior region.

In the vacuum exterior region \(m_0 = \tilde{m}_0\) and \(h_0 = \tilde{h}_0\) are given by (65) and (66) respectively. Again, these two expressions correspond to (105) and (106) in [14] for \(m_0^H\) and \(h_0^H\) respectively, in terms of a variable \(r\) in the range \(r \in [a, \infty)\).

Therefore, the matching conditions for the function \(h_0\) given by (78) and (89), and for the function \(\tilde{m}_0\) given by (91), translate directly to matching conditions on \(h_0^H\) and \(m_0^H\). As discussed previously, the free additive constant in \(\tilde{h}_0^+\) (and so in \(h_0^H\)) can be used to set \(H_0 = 0\). In an abuse of terminology, the assumption of a ‘continuous’ \(h_0^H\) is thus consistent.

The function \(m_0^H\) is also assumed to be ‘continuous’ in [14] section VII, when the value of \(m_0^H(a)\) as computed from the interior is equated to the expression of \(m_0^H(r)\) in the exterior at \(r = a\) in order to obtain the constant \(\delta M\) in (107) [14]. However, the correct matching condition is given by (94), which in the \(k\)-gauge, and since \([\lambda] = 0\), can be expressed as

\[
\left[ m_0^H \right] = -4\pi a^3/M(a - 2M)E(a)p_0^H (a)
\]

using the notation in [14]. As a result, given the value \(m_0^H(a)\) as computed from the interior, the value of the change in mass in (65) is given by

\[
\delta M = m_0^H(a) + J^2 / a^3 + 4\pi a^3/M(a - 2M)E(a)p_0^H (a).
\]

(103)

The last term corresponds to the jump of the values of \(\tilde{m}_0\) at the boundary, and it is not present in the expression for the change of mass in (107) [14] and in the subsequent works, e.g. [15, 16]. Of course, whenever the density of mass-energy vanishes at the surface of the star this term has no consequences. This will happen in many situations, as in the cases of
equations of state that imply the vanishing of the energy density at points where the pressure vanishes, polytropes for instance. In fact, in the series of papers started by [15, 16] all the equations of state considered satisfy that condition, and therefore the computation of the change of mass is not affected by the correction in (103).

However, in more general situations that is not going to be the case. As an example, models for quark stars that rely on a non-zero value of $E$ at the surface have been considered in the literature (see e.g. [10]). In particular, models of stars based on a constant background $E$ in the interior are affected by that term and the computation of the change in mass should be corrected. A future work will we devoted to revisit the models presented in [9], and find numerically the discrepancy in the values of $\delta M$.

Let us now jump to the $l = 2$ sector. In the interior region the equation (69) plus the system (67)–(68) for the set $\{r e^{a_1} \tilde{h}_2^+ + \tilde{k}_2^+ + \tilde{h}_2^*\}$ as functions of $r$, coincide one by one with equation (120) plus the coupled equations (125)–(126) in [14] for $\{m_2^H, v^H = h_2^H + k_2^H, h_2^H\}$ as functions of $R$, which has the same range as $r_i$. The same argument as in the $l = 0$ sector shows that the problems coincide and therefore we can set $m^H_2(s) = r e^{a_3(s)} \tilde{h}^+(s), h^H_2(s) = \tilde{h}^3(s)$ and $k^H_2(s) = \tilde{k}^3(s)$ for $s \in (0, a]$. In the vacuum region $h_2 = h_2'$ and $k_2 = k_2'$ are given by (74) and (75), which correspond to (139) and (140) in [14] respectively in terms of a variable $r$ in the range $r \in [a, \infty)$. The comparison of (73) with (91) in [14] implies the correspondence $p^H_2(x(s)) = \tilde{P}_2(s)$. The two matching conditions in (96) simply state that $h^H_2$ and $k^H_2$ are `continuous’ on the boundary. The assumption made in [14] regarding the $l = 2$ sector is thus consistent. This ‘continuity’ of $h^H_2$ and $k^H_2$ is finally used in order to fix the free constants $A'$ and $A$ in the interior and exterior regions respectively, thus fixing completely the global problem in the $l = 2$ sector.

We discuss finally the deformation of the boundary. In [14] the analysis of the deformation needs the introduction of a function $\xi^H(r, \theta) = \xi^H_0(r) + \xi^H_2(r) P_2(\cos \theta)$ defined in the whole interior region by imposing $P_2(R) = \tilde{P}_2(s) = P(R)$ for $R \in [0, a]$ (see also the discussion in [3]). The deformation is then determined by the values $\xi^H_0(a)$ and $\xi^H_2(a)$.

Let us recall that in the present treatment the deformation is described by $\Xi_0$ and $\Xi_2$, which are determined by equations (101) and (102) whenever $E(a) \neq 0$. In the case $E(a) = 0$ the deformation can be determined by relying on a particular gauge in order to define extensions for both $\Xi_0$ and $\Xi_2$. The correspondence of $\tilde{P}_2(r)$ and $\tilde{P}_2(\theta)$ and $\xi^H_0(r)$ and $\xi^H_2(r)$ as functions defined in the interior region with quantities in the treatment presented here rely, in fact, on the construction of those extensions. This is discussed in appendix A, where it is shown how equations (101) and (102) hold in all cases, and that the values $\xi^H_0(a)$ and $\xi^H_2(a)$ correspond to

$$\xi^H_0(a) = -\frac{1}{2} e^{a_3(a)/2} \Xi_0, \quad \xi^H_2(a) = -\frac{1}{2} e^{a_3(a)/2} \Xi_2.$$ 

(The relative minus sign comes from the orientation of the normal chosen in (16), which goes as $-\partial_\theta$.) Indeed, the former translates, via (101), to equation (117) in [14], which should in fact be corrected to $\tilde{e}^H(a) = p^H_0 a (a - 2M) / M$, whose value describes the average expansion of the shape of the star [3, 15]. The combination of the latter with (102) enters the different definitions of the ellipticity of the star found in the literature (see e.g. [3, 15]) accordingly. In particular, it provides the expression for the ellipticity as defined in [14] by $\epsilon = -\frac{1}{2 \xi^H_2(a)}$, which thus reads
\[ e = \frac{3(a - 2M)}{2M} \left( \hat{h}_2(a) + \frac{1}{3} \frac{a^3}{a - 2M} \left( \frac{2J}{a^3} - \Omega_{\infty} \right) \right), \]

in agreement with (146) in [14].

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Appendix A. Deformation of the surface and the \(E\)-gauge

We devote this appendix to discuss the deformation of the surface, and at the same time, study the relationship of the two gauges used in [14] (also in [3]).

In order to describe the deformation of the surface, motivated by the approaches taken in Newtonian theory, it has been common in the literature to focus on the surfaces of constant energy density or, equivalently, of constant pressure given a barotropic equation of state. This consists after all of a choice of gauge in which the surfaces of constant energy density (or pressure) in the interior region of the perturbed configuration are those of constant radial coordinate. This is described in [14] (see also [3]) as a change from the original coordinate \(r^H\) (the initial gauge corresponds to the \(k\)-gauge) to another \(R\) defined by (the inverse of)

\[ \{R, \theta\} \rightarrow \left\{ r^H = r^H_e(R, \theta), \theta \right\} \quad (104) \]

for some function \(r^H_e(R, \theta)\) satisfying \(r^H_0(R, \theta) = R\) and

\[ E_e \left( r^H_e(R, \theta), \theta \right) = E(R), \quad (105) \]

where \(E_\epsilon\) is the energy density corresponding to \(g_\epsilon\) (see (32)) in the \(k\)-gauge. The surfaces of constant energy density in the perturbed configuration, \(E_\epsilon\), are then those of constant \(R\), and their values correspond to the values the pressure of the background configuration \(E\) take at those \(R \in (0, a]\). In the present terminology that corresponds to moving to another gauge, to which we refer to as the \(E\)-gauge. Note that (105) is imposed for all \(\epsilon\) in some neighbourhood around 0, and therefore for all orders. To second order \(r^H_\epsilon(R, \theta)\) is specified in [14] as

\[ r^H_\epsilon(R, \theta) = R + \epsilon^2 \hat{r}^H_\epsilon(R, \theta) + O(\epsilon^3), \quad (106) \]

where for clarity we write explicitly the perturbation parameter at this point. We do not comment yet on the existence nor uniqueness of the \(E\)-gauge.

In [14] the perturbed surface is then defined as the surface of constant energy density that equals the value of the energy density at the surface of the static configuration. Explicitly, \(\Sigma\) is defined to have the form \(\Sigma_\epsilon\); \(r^H = r^H_e(a, \theta)\), which is equivalent to \(R = a\) by construction.

Let us formulate that condition in the present treatment. Indicating with a \(^{(E)}\) when a (gauge-dependent) quantity or object refers to the \(E\)-gauge, the expression (105) can be cast just as
for all orders $n \geq 1$ (note $E^{(0)(E)} = E$). At each order $n$ that condition would determine, in principle, the $E$-gauge at the corresponding order. The perturbed matching hypersurface $\Sigma_\epsilon$ would then be defined by imposing $\Sigma_\epsilon^{(E)} = \Sigma_0$ pointwise (see section 2, and note we are referring to the interior (+) region). In other words, the perturbed matching hypersurface is defined by imposing that the $E$-gauge is, at the same time, a ‘surface-comoving’ gauge.

Given a barotropic equation of state all the above can be stated in terms of the pressure. The $E$-gauge is then also determined by

$$P^{(n)(E)} = 0$$

for all $n \geq 1$. Since the interior pressure necessarily vanishes at the boundary in the background configuration, imposing that the $E$-gauge is also a ‘surface-comoving’ gauge implies that the whole perturbed pressure computed in the $E$-gauge vanishes at the perturbed boundary. This is the view taken in [3] and many other works (see e.g. [6, 22]).

Clearly, given a barotropic equation of state, the approach taken in terms of $E$ (say, approach ‘E’) and that in terms of $P$ (approach ‘P’) lead to the same conclusion. However, their justifications are of different nature, apart from the possible problems of existence.

Regarding the approach ‘E’, if $E(a) \neq 0$ the fact that the perturbed energy density attains that value $E(a)$ at the boundary may, in principle and in general, seem to constitute an assumption. Probably due to this difficulty the approach ‘P’ has seemed to be preferred in many works since the vanishing of the (perturbed) ‘pressure’ on the surface is what one would expect on physical grounds. However, that would be an erroneous statement as such, and in general, since $P_\epsilon$ is gauge dependent (see section 3). One should, at least, prove in which gauge that should happen. Indeed, the matching conditions in the exact case restrict the possible jumps of the Einstein tensor across the surface. However, it remains to be shown how this fact translates to the perturbative matching scheme in the general case. A general consistent approach should not rely, in principle, on the use of a result (the vanishing of a ‘pressure’ in a certain gauge) that has to be proven, in fact, as a consequence of the procedure.

Finally, the definition of the deformation of the star in terms of the $E$-gauge should control and take care of the existence (and uniqueness, if needed) of the gauge. For instance, in the simplest case of a constant energy density interior background $E(r) = E(a) = \text{const.}$ the $E$-gauge cannot be determined using (105), and thus, neither the deformation. Instead, the ‘P’ approach has to be used, for which the $E$-gauge can be constructed. This is implicitly done in works focused on stars of constant energy density, such as [9].

Nevertheless, the determination of $\Sigma_\epsilon$ using the $E$-gauge is well justified if $E(a) = 0$ but $E(r) \neq 0$ ($>0$ in fact) for $r \in (0, a)$, since then the perturbed star (perfect-fluid region) extends up to where $\Sigma_\epsilon$ vanishes, and no further. By the local nature of the matching, one could relax this condition to $E(a - \delta) \neq 0$ for all $\delta > 0$ in some neighbourhood of $a$. This condition (and analyticity of $E(r)$) demand that there exists $n$ such that $n$th derivative $d^nE/dr^n(a)$ at $r = a$ is non-zero. The implicit function theorem can then be applied to every differentiation of (105) with respect to $\epsilon$ evaluated at $\epsilon = 0$ in order to show that $r^\ell_\ell(a, \theta)$ can be obtained order by order from (105). The full proof is out of the scope of this appendix and will be presented elsewhere. When needed, we will simply assume that the $E$-gauge can be constructed from $r = a$ inwards.

As stressed, in the present treatment no argument about the vanishing of the pressure of the perturbed configuration $P_\epsilon$ has been made, nor any specific gauge has been used. In sections 3.2 and 3.3 it has been shown how the deformation of the boundary, described by the quantities $Q_1$ of the first order and $\Xi_0$ and $\Xi_2$ of the second order, are determined by
\( Q_1 = 0 \) when \( E(a) \neq 0 \) or \( E'(a) \neq 0 \), and (101) and (102) when \( E(a) \neq 0 \), respectively, and how that agrees with the results in [14].

In what follows we first show explicitly that the \( E \)-gauge is indeed a ‘surface gauge’ when \( E(a) \neq 0 \), at least to second order. This shows, at the same time, that the usual ‘vanishing of the pressure at the boundary’ in the exact case translates in this perturbative scenario to \( P_r^{(E)} |_{r=0} = 0 \), i.e. that the perturbed pressure in the \( E \)-gauge must vanish at the perturbed surface (at least to second order). Secondly, we use the definition of the perturbed surface when \( E(a) = 0 \) by means of the \( E \)-gauge (approach ‘\( E' \)) to show that, given the \( E \)-gauge exists (and is unique), then \( Q_1 = 0 \) and equations (101) and (102) hold even when \( E(a) = 0 \).

Not to overwhelm the notation let us drop the interior + superscripts in the following when not needed.

As shown in section 3.2, at first order we have \( E^{(1)} = P^{(1)} = 0 \), and the condition \( E(a) \neq 0 \) already implies \( Q_1 = 0 \). Therefore, the family of gauges chosen for the family (7) satisfies the \( E \)-gauge condition to first order. Since \( Q_1 = 0 \), \( \Sigma_r \) coincides at first order with \( \Sigma_0 \) as a set of points. The \( E \)-gauge is therefore a ‘surface-comoving’ gauge up to first order. A hypersurface gauge can be used to fix \( T_1^\tau = 0 \), so that the perturbed \( \Sigma \) coincides at first order with \( \Sigma_0 \) pointwise, so that the \( E \)-gauge is, moreover, a ‘surface’ gauge up to first order.

Regarding the second order, let us recall that given conditions at the origin (such that \( \bar{P}_0(0) \) vanishes) \( P_0(r) \) is fully determined by the \( l = 0 \) field equations, and \( P_2(r) \) is obtained from (73), once \( h_2(r) \) is fully determined, in turn, by the \( l = 2 \) field equations and the condition at the origin and at the boundary \( r = a \) coming from the ‘continuity’ of the functions \( h_2 \) and \( \tilde{k}_2 \). Now, the \( E \)-gauge is selected by fixing \( k_0(r) \) and \( f_2(r) \) so that \( P_0^{(2)}(E)(r) \) and \( P_2^{(2)}(E)(r) \) vanish. From (60) and (72) this is accomplished by imposing

\[
\begin{align*}
 k_0^{(E)} & = \frac{E + P}{r} \bar{P}_0, \quad f_2^{(E)} = -\frac{E + P}{r} \bar{P}_2.
\end{align*}
\]

We are ready to show that if (101) and (102) hold then \( Q_2^{(E)} = 0 \). This follows directly from the definitions (87), which in the \( E \)-gauge read

\[
\hat{Q}_{2(0)}^{(E)} = \Xi_0 + 2ae^{-\nu(a)/2}k_0^{(E)}(a), \quad \hat{Q}_{2(2)}^{(E)} = \Xi_2 + 2ae^{-\nu(a)/2}f_2^{(E)}(a).
\]

Equations (101) and (102) together with (108) evaluated on \( r = a \) readily imply \( \hat{Q}_{2(0)}^{(E)} = \hat{Q}_{2(2)}^{(E)} = 0 \). Finally, since we have chosen \( T_1^\tau = 0 \) at first order, then \( Q_2^{(E)} = Q_{2(2)}^{(E)} = 0 \) as follow from the definitions (23). It only remains, again, to choose a convenient hypersurface gauge to second order to fix \( T_2^\tau = 0 \) so that the perturbed \( \Sigma \) coincides with \( \Sigma_0 \) at second order, not only as a set of points, but pointwise. We have thus shown that the \( E \)-gauge is indeed a ‘surface gauge’ whenever \( E(a) \neq 0 \), at least to second order, as expected.

Let us consider now the case \( E(a) = 0 \) under the conditions that ensure the existence and construction of the \( E \)-gauge. The matching hypersurface \( \Sigma_x \) is then determined by the coincidence of \( \Sigma_x^{(E)} \) and \( \Sigma_0 \) pointwise (in \( V_0^+ \), mind the + superscript). This condition is equivalent, up to second order, to \( Q_1^{(E)} = Q_{2(2)}^{(E)} = 0 \) together with a hypersurface gauge choice such that \( T_1^\tau = T_2^\tau = 0 \) at each order. At first order we thus have the required result by construction. At second order, the equations defining \( \Xi_{0/2} \) (87) in the interior read then

\[
\Xi_0 = -2ae^{-\nu(a)/2}k_0^{(E)}(a), \quad \Xi_2 = -2ae^{-\nu(a)/2}f_2^{(E)}(a),
\]

which combined with (108), yield (101) and (102).

We must finally address the issue of how \( \Xi_{0/2} \), given by (101) and (102), describe the deformation of the surface. The key is to show how the deformation quantities \( \Xi_{0/2} \), defined
on $\Sigma_0$, can be extended to the interior region and how that relates to the change from the $k$-gauge to the $E$-gauge. We start by defining that change in terms of $\dot{V}_2$. Let us, for simplicity, set $\overline{s}_2 = 0$ so that $\overline{s}_2 = \dot{V}_2$. Including $\overline{s}_2 = C \dot{\phi}$ does not add anything relevant to the analysis. Recall that the $k$-gauge is defined by $k_0 = 0$ and $f_2^k = 0$. Given that the second order change $\dot{V}_2 = 2S(r, \theta) \partial_r$ induces (12) (with $C = 0$), it is immediate to check (recall the freedom in defining $f(r, \theta)$) that the change from the $k$-gauge to the $E$-gauge is accomplished by setting

$$\dot{V}_2 = 2r \left( k_0^{(E)} + f_2^{(E)} P_2(\cos \theta) \right) \partial_r = -2 \frac{E + P}{p'} \left( \tilde{P}_0 + \tilde{P}_2 \right) P_2(\cos \theta) \partial_r,$$

(109)

where the second equality follows from (108). Note that the relation $k_2^{(E)} = k_2^{(E)} - f_2^{(E)}$ holds.

On the other hand, given the definition of the second order gauge vectors in (2), the second order gauge $\dot{V}_2 = 2S(r, \theta) \partial_r$ with $\overline{s}_1 = 0$ corresponds to a diffeomorphism $\Omega_0: V_0 \rightarrow V_0$ of the form $(s, \theta) \rightarrow (R_0(s, \theta), \theta)$ for $s \in [0, a]$ defined by $R_0(s, \theta) = s + e^2 S(s, \theta)$. Given (109), we thus have

$$R_0(s, \theta) = s - e^2 E(s) + P(s) \left( \tilde{P}_0(s) + \tilde{P}_2(s) P_2(s) \right),$$

(110)

Let us recall again (see section 4) that the coordinate $R$ used in [14] ranges from 0 to $a$, and therefore (110) can be compared with the expression (106) in the form $r_0^H(s, \theta) = s + e^2 e^H(s, \theta) + O(e^3)$ to obtain

$$\xi^H = - \frac{E + P}{p'} \left( \tilde{P}_0 + \tilde{P}_2 \right) P_2(\cos \theta).$$

Now, this is in agreement with $\xi^H = \xi_0^H + \xi_2^H P_2(\cos \theta)$ for $\xi_0^H = - \frac{E + P}{p'} p_{0/2}^H$, as follows from (90) and (91) in [14] and the correspondences $p_{0/2}^H(s) = \hat{P}_{0/2}(s)$ found in section 4.

Expression (110) suggests the construction of two functions in the interior

$$\dot{\xi}_{0/2} = 2 \frac{E + P}{p'} e^{-e^2/2} \hat{P}_{0/2}.$$  

(111)

These, evaluated at $r = a$, and given that (101) and (102) hold, lead to

$$\dot{\xi}_{0/2}(a) = \Xi_{0/2}.$$

The functions $\dot{\xi}_{0/2}$ (111) are therefore extensions of $\Xi_{0/2}$, as defined in (101) and (102), to all the interior region, and are ‘radial’-gauge independent by construction. The information of the deformation of the star in the $k$-gauge is therefore encoded in the functions $\dot{\xi}_{0/2}$, whereas in the $E$-gauge that information lies in the functions $k_0^{(E)}$ and $f_2^{(E)}$.

Using the correspondence $\dot{\xi}_{0/2}(s) = -2 e^{-e^2/2} \xi_{0/2}^H s_{0/2}(s)$, so that $\Xi_{0/2} = -2 e^{-e^2/2} \xi_{0/2}^H s_{0/2}(a)$, the analysis of the deformation of the star in terms of $\Xi_{0/2}$ and $\Xi_2$ follows then from the discussions in [14] (see also [3]). Note that the minus sign in the correspondence comes from the choice of the normals as defined in (16), which point towards the origin.

**Appendix B. Proofs of propositions 1 and 2**

For the sake of completeness we include in this appendix the explicit expressions needed in order to use theorem 1 from [19], as obtained from propositions 2 and 3 from that reference, particularized for a timelike hypersurface $\Sigma_0$. Let us start by decomposing $K^{(1)}$ in its normal
and tangent parts with respect to $\tilde{n}$ as

$$K^{(1)}_{\alpha\beta} = Yn_\alpha n_\beta + n_\alpha \tau_\beta + n_\beta \tau_\alpha + K^{(1)\ell}_{\alpha\beta},$$

where the vector $\tau_\alpha$ and symmetric tensor $K^{(1)\ell}_{\alpha\beta}$ denote the projected components of $K^{(1)}_{\alpha\beta}$ on $\Sigma_0$, that is $\tau_\alpha n^\alpha = 0$ and $K^{(1)\ell}_{\alpha\beta} n^\alpha = 0$.

The first and second fundamental forms to first order, $h^{(1)}$ and $\kappa^{(1)}$, are given by the expressions [19]

$$h^{(1)}_{ij} = \mathcal{L}_{\tilde{X}}h_{ij} + 2Q_i\kappa_j + K^{(1)\ell}_{\alpha\beta} e^i_\alpha e^j_\beta,$$

$$\kappa^{(1)}_{ij} = \mathcal{L}_{\tilde{X}}\kappa_{ij} - D_i D_j Q_1 + Q_i \left( - n^\delta n^\gamma R_{\alpha\mu\beta}\epsilon^{\alpha\delta}_{\epsilon^i_\gamma} e^j_\beta + \kappa_\delta \kappa_j \right) + \frac{1}{2} Y\kappa_{ij} - n_\mu S^{(1)\mu}_{\alpha\beta} e^i_\alpha e^j_\beta, $$

(112)

where $D_i$ is the three dimensional covariant derivative of $(\Sigma_0, h)$ and

$$S^{(1)\mu}_{\beta\gamma} \equiv \frac{1}{2} \left( V_\beta K^{(1)\mu}\gamma + V_\gamma K^{(1)\mu}\beta - V^\alpha K^{(1)\mu}_{\alpha\beta} \right).$$

The first and second fundamental forms to second order, $h^{(2)}$ and $\kappa^{(2)}$, are given by the expressions [19]

$$h^{(2)}_{ij} = \mathcal{L}_{\tilde{X}}h_{ij} + 2Q_2\kappa_{ij} + K^{(2)\ell}_{\alpha\beta} e^i_\alpha e^j_\beta + 2\mathcal{L}_{\tilde{X}}h^{(1)}_{ij} - \mathcal{L}_{\tilde{X}}\mathcal{L}_{\tilde{X}}h_{ij} + \mathcal{L}_{2Q_1\tilde{X} - 2Q_i} \tilde{X} D_i \tilde{X} h_{ij} + 2T_{\tilde{X}}\kappa_{ij} + 2Q_i Y \kappa_{ij} + 2Q_j ( - n^\delta n^\gamma R_{\alpha\mu\beta}\epsilon^{\alpha\delta}_{\epsilon^i_\gamma} e^j_\beta + \kappa_\delta \kappa_j ) + 2D_i Q_1 D_j Q_1 - 4Q_i n_\mu S^{(1)\mu}_{\alpha\beta} e^i_\alpha e^j_\beta,$$

$$\kappa^{(2)}_{ij} = \mathcal{L}_{\tilde{X}}\kappa_{ij} - D_i D_j Q_2 - Q_2 n^\gamma R_{\alpha\mu\beta}\epsilon^{\alpha\gamma}_{\epsilon^i_\beta} e^j_\beta + Q_2 \kappa_\delta \kappa_i \kappa_j - n_\mu S^{(2)\mu}_{\alpha\beta} e^i_\alpha e^j_\beta$$

$$+ 2\mathcal{L}_{\tilde{X}}\kappa^{(1)}_{ij} + \kappa_\delta \left( \frac{1}{2} Y^{(2)} - \frac{1}{4} Y^2 - \left( T_\gamma + D_i Q_1 \right) \left( \tau^i \right) + D^i Q_1 \right) + 2Q_1 n^\sigma n^\gamma S^{(1)\sigma}_{\alpha\beta} e^i_\alpha e^j_\beta$$

$$+ \left( Y n_\mu + 2\tau_\mu + 2D_\mu Q_1 \right) S^{(1)\mu}_{\alpha\beta} e^i_\alpha e^j_\beta - 2Q_1 n_\mu n^\sigma$$

$$\times \left( \mathcal{L}_{\tilde{V}} S^{(1)\mu}_{\alpha\beta} \epsilon^{\alpha\mu}_{\epsilon^i_\beta} e^j_\sigma - 2n_\mu n^\rho S^{(1)\rho}_{\alpha\sigma} e^i_\alpha e^j_\sigma \right) D_j Q_1$$

$$- 2n_\mu n^\gamma n^\beta S^{(1)\gamma}_{\alpha\mu} e^i_\alpha e^j_\beta - 2Q_1 n_\mu S^{(1)\mu}_{\alpha\beta} e^i_\alpha e^j_\beta \kappa_\alpha$$

$$\times \mathcal{L}_{\tilde{V}} \left( T_\gamma (Q_1) \right) - \frac{1}{2} \mathcal{L}_{\tilde{V}} \left( T_\gamma \kappa_\alpha \right) - \frac{1}{2} Y \mathcal{L}_{\tilde{V}} \left( \tilde{X} \right) 2 n_\mu \mathcal{L}_{\tilde{V}} \left( \tilde{X} \right) h_{ij}$$

$$+ \left( 2T_\gamma (Q_1) - T_\gamma T_\gamma \kappa_\mu \kappa_\mu - Y n^\gamma R_{\alpha\mu\beta}\epsilon^{\alpha\gamma}_{\epsilon^i_\beta} e^j_\beta - \kappa_\alpha \kappa_\beta \right) + \frac{1}{2} D_i Q_1 D_j Q_1$$

$$+ D_j Q_1 D_i Q_1 - \mathcal{L}_{\tilde{X}} \mathcal{L}_{\tilde{X}} \kappa_{ij} - \mathcal{L}_{2Q_1} \left( \tilde{X} \right) D_i \tilde{X} \kappa_{ij} - 2Q_i \mathcal{L}_{\tilde{X}} \kappa_{ij}$$

$$- Q_1 \left( n^\rho n^\delta S^{(1)\gamma}_{\alpha\mu} \epsilon^{\alpha\mu}_{\epsilon^i_\beta} e^j_\sigma + 2n_\mu n^\gamma R_{\delta\beta\gamma} e^i_\alpha e^j_\beta \epsilon^\gamma_\sigma + 2n_\mu n^\gamma R_{\delta\beta\gamma} e^i_\alpha e^j_\beta \epsilon^\gamma_\sigma \right),$$

where $Y^{(2)} = K^{(2)\ell}_{\alpha\beta} n^\alpha n^\beta$,
B.1. Proof of proposition 1

Theorem 1 in [19] states that the first order matching conditions are satisfied if there exist two scalars $Q_{ij}^\pm$ and two vectors $\mathbf{T}_{ij}^\pm$ on $\Sigma_0$ such that the system of equations given by $h_{ij}^{(1)\mp} = h_{ij}^{(1)\mp}$ and $\kappa_{ij}^{(1)\mp} = \kappa_{ij}^{(1)\mp}$ admits a solution. We start by calculating $h_{ij}^{(1)}$ and $\kappa_{ij}^{(1)}$ through expressions (112). Let us recall these are objects defined on $\Sigma_0$, which is non-degenerate. The ingredients needed are the background embeddings (13), (14), with tangent basis (15) and unit normals (16), plus the first and second fundamental forms of $\Sigma_0$ (2.2), together with the first order perturbation tensors $K_{ij}^{(1)\pm}$ (8) restricted to $\Sigma_0$ at each side. The functions $Q_i (\tau, \theta)$ and vectors $\mathbf{T}_{ij}^\pm (\tau, \theta)\partial_\theta + \mathbf{T}_{ij}^{\prime\pm} (\tau, \theta)\partial_\theta$ on $\Sigma_0$ at each side are left as unknowns.

The explicit expressions of $h_{ij}^{(1)}$ and $\kappa_{ij}^{(1)}$ read

\[
h_{ij}^{(1)\pm} \, \tilde{\text{d}}x^i \, \tilde{\text{d}}x^j = e^{i(a)} \left( -2T_{ij}^{\pm\tau} + \nu'(a) e^{-\frac{1}{2}Q_{ij}^{\pm}} \right) \, \text{d}\tau^2 + 2 \left( -e^{i(a)} T_{ij}^{\pm\theta} + a^2 T_{ij}^{\pm\theta} \right) \, \text{d}\tau \, \text{d}\theta + 2a \left( T_{ij}^{\pm\theta} - \omega \right) \, \sin^2 \theta \, \text{d}\tau \, \text{d}\phi
\]

and

\[
\kappa_{ij}^{(1)\pm} \, \tilde{\text{d}}x^i \, \tilde{\text{d}}x^j = -Q_{ij}^{\pm\tau} + e^{-\frac{1}{2}Q_{ij}^{\pm}} \left( T_{ij}^{\pm\tau} + e^{\frac{1}{2}Q_{ij}^{\pm}} \left( \lambda_{ij}^{\pm}(a) \nu'(a) - 2\nu_{ij}^{\pm}(a) - 2\nu_{ij}^{\pm}(a) \right) \right) \, \text{d}\tau^2 - 2 \left( Q_{ij}^{\pm\theta} + e^{\frac{1}{2}Q_{ij}^{\pm}} \left( aT_{ij}^{\pm\theta} - \frac{1}{2} e^{i(a)} \nu'(a) T_{ij}^{\pm\theta} \right) \right) \, \text{d}\tau \, \text{d}\theta + 2ae^{-\frac{1}{2}Q_{ij}^{\pm}} \left( -T_{ij}^{\pm\phi} + \frac{1}{2} a\nu_{ij}^{\pm}(a, \theta) + \omega \right) \, \sin^2 \theta \, \text{d}\tau \, \text{d}\phi
\]

where the background matching conditions (17) have been used to set $\nu_{ij}^{\pm}(a) = \nu'(a)$, $\nu_{ij}^{\pm}(a) = \nu'(a)$ and $\lambda_{ij}^{\pm}(a) = \lambda'(a)$.

The ordered procedure used in order to obtain and integrate the difference functions is the following. First, from $[h_{ij}^{(1)\phi\theta}] = 0$ we obtain $[T_{\phi\theta}^{\pm}] = 0$. On the other hand, the derivative $[h_{ij}^{(1)\tau\phi}] = 0$ yields $[T_{ij}^{\phi}] = 0$, and therefore $[T_{ij}^{\phi}] = b_1 \tau + C_2$ for arbitrary constants $b_1$ and $C_2$. As a result, $[h_{ij}^{(1)\tau\phi}] = 0$ reads $[\omega] = b_1$.

Now, equation $[h_{ij}^{(1)\phi\theta}] \sin^2 \theta - [h_{ij}^{(1)\phi\theta}] = 0$ yields $[T_{ij}^{\phi}] \cos \theta - [T_{ij}^{\phi}] \sin \theta = 0$, which is integrated into $[T_{ij}^{\phi}] = F(\tau) \sin \theta$ for some function $F(\tau)$. Equation $[h_{ij}^{(1)\phi\theta}] = 0$ now reads $[Q_i] = e^{i(a)/2} a F \cos \theta$. On the other hand, the compatibility condition to integrate $[T_{ij}^{\phi}]$ is given by $2[h_{ij}^{(1)\tau\phi}] = [h_{ij}^{(1)\tau\phi}] = 0$, which yields $F = -F'(\tau) e^{i(a)/2} a$, and hence...
\[ [T_{t'}] = C_1 - e^{\omega(a)} a^2 \hat{F} \cos \theta \text{ for some arbitrary constant } C_1. \text{ We have so far exhausted the conditions } [h^{11}] = 0. \]

Given the above conditions, equation \([\epsilon^{(1)\phi}] = 0\) is now equivalent to \([\omega'] = 0\). The conditions on the metric perturbations have thus been obtained.

Consider the equation \([\kappa^{(1)\phi}] = 0\), which now reads \(\hat{F} a \sin a (2e^{\omega(a)} - 2 + a\omega'(a)) = 0\).

If \(2e^{\omega(a)} - 2 + a\omega'(a) \neq 0\) we then have \(\hat{F} = 0\), which due to its previous equation can only be satisfied in the trivial case \(F = 0\). From the above, in particular, \([Q_1] = 0\). Then, equations \([\kappa^{(1)\phi}] = 0\) and \([\kappa^{(1)\theta}] = 0\) just provide \(Q_1[\nu'] = 0\), from which \([\kappa^{(1)\tau}] = 0\) thus reads \(Q_1[\nu'] = 0\).

The appearance of the constants \(C_1\) and \(C_2\) is a consequence of the isometries present in the background configuration, and cannot be determined [20]. Nevertheless, they can be safely absorbed by using a isomorphic \textit{spacetime} gauge at one (any) side, say \(\vec{s}_1 = C_1 \partial_\mu + C_2 \partial_\nu\), which, by (6) leads to \(\vec{T}_1^{+} \to \vec{T}_1^{+} - \vec{s}_1^{+}\) and obviously leaves the metric perturbation tensor \(\kappa^{(1)+}\) unchanged. We can thus set \(C_1 = C_2 = 0\) without loss of generality.

\[ B.2. \text{ Proof of proposition 2} \]

The procedure is analogous to that of the previous proof. We first consider the case \([\nu'] \neq 0\) or \([\nu'] = 0\), so that \(Q_1 = 0\) necessarily. The explicit expression of \(h^{(2)\phi}\) reads

\[
h^{(2)\phi}(\theta, \phi) = \left\{ -2e^{\omega(a)} \left( T_{1,1}^{\phi} + T_{1,1}^{\phi} \right) + 2a^2 \left( T_{1,1}^{\phi} - a\omega(a, \theta) \right) \sin^2 \theta \right. \\
\left. + 2a^2 \left( T_{1,1}^{\phi} \right)^2 - 4e^{\omega(a)} h(a, \theta) + e^{-\frac{\omega(a)}{2}} e^{\omega(a)} \nu(a) \hat{Q}_2 \right\} d\theta^2 \\
+ 2 \left\{ 2a^2 T_{1,1}^{\phi} T_{1,1}^{\phi} \cos \theta \sin \theta - e^{\omega(a)} T_{1,1}^{\phi} + a^2 T_{1,1}^{\phi} T_{1,1,1}^{\phi} \right\} d\theta^2 \\
+ 2a^2 \left\{ 2 \left( T_{1,1}^{\phi} T_{1,1}^{\phi} - T_{1,1}^{\phi} - 2a^2 \omega(a, \theta) T_{1,1}^{\phi} \right) \cos \theta \\
+ \left( T_{1,1}^{\phi} - 2T_{1,1}^{\phi} \omega(a, \theta) - 2T_{1,1}^{\phi} \omega(a, \theta) \right) \sin \theta \right\} \sin \theta d\theta d\phi \\
+ 2 \left\{ a^2 \left( T_{1,1}^{\phi} \cos \theta + T_{1,1}^{\phi} \sin \theta \right)^2 \\
- a^2 \sin^2 \theta \left( T_{1,1}^{\phi} \right)^2 + a^2 \left( T_{1,1}^{\phi} \right)^2 + a^2 T_{1,1}^{\phi} - e^{\omega(a)} \left( T_{1,1}^{\phi} \right)^2 + 2a^2 k(a, \theta) - e^{-\frac{\omega(a)}{2}} a \hat{Q}_2 \right\} d\phi^2 \\
+ 2a^2 \left\{ 2T_{1,1}^{\phi} T_{1,1}^{\phi} + \left( T_{1,1}^{\phi} - 2T_{1,1}^{\phi} \omega(a, \theta) \right) \sin^2 \theta \right. \\
\left. + 2 \left( T_{1,1}^{\phi} T_{1,1}^{\phi} - T_{1,1}^{\phi} \right) \cos \theta \sin \theta \right\} d\phi d\theta \\
+ 2 \left\{ a^2 \left( T_{1,1}^{\phi} \right)^2 \cos^2 \theta \sin^2 \theta + a^2 \left( T_{1,1}^{\phi} \right)^2 \left( 1 - 2 \sin^2 \theta \right) + a^2 T_{1,1}^{\phi} \cos \theta \sin \theta \\
+ \left( 2a^2 k(a, \theta) - e^{-\frac{\omega(a)}{2}} a \hat{Q}_2 \right) \sin^2 \theta \right\} d\phi^2. \]
where we have avoided the use of ± for quantities which already coincide at both sides and we have used \( \hat{Q}_2 \), as defined in (23), instead of the original \( Q^2_2 \).

From equations \([h^{(2)}_{\phi\phi}] = 0 \) and \([h^{(2)}_{\phi\theta}] = 0 \) we obtain expressions for \( T^\phi_{L\gamma} \) and \( T^\phi_{L\phi} \) respectively. The integrability conditions are found to be automatically satisfied. The integration leads to

\[
T^\phi_{L\gamma} = 2b_1\left(T^\gamma_{L\gamma} + \tau T^\theta_{L\lambda} \cot \theta \right) + D_2
\]  

for some constant \( D_2 \). Likewise, from \([h^{(2)}_{\phi\phi}] = 0 \) and \([h^{(2)}_{\phi\theta}] \sin^2 \theta - [h^{(2)}_{\phi\phi}] = 0 \) we obtain, respectively, \( T^\phi_{L\phi} \) and \( T^\phi_{L\phi} \).

Now, this time the integrability condition provides a second order PDE for \( T^\phi_{L\phi} \), with derivatives on \( \theta \) only, which is integrated to yield

\[
T^\phi_{L\phi} = -a^2 F(\tau)e^{-\nu(\alpha)} \cos \theta + G(\tau)
\]

for some functions \( F(\tau) \), conveniently arranged, and \( G(\tau), [T^\phi_{L\phi}] \) can now be integrated in the form

\[
T^\phi_{L\phi} = \left(b_1 \tau \cos \theta \left(b_1 \tau - 2T^\theta_{L\lambda} \right) + F(\tau) + C_3 \right) \sin \theta,
\]

for some constant \( C_3 \).

Now, \([h^{(2)}_{\phi\theta}] = 0 \) provides an equation for \( \hat{Q}_2 \), explicitly

\[
\hat{Q}_2 = ae^{i(\lambda/2)} \left\{ \left(2[k] + (F(\tau) + C_3)\cos \theta \right) \right\}.
\]

The remaining equation from the equality of the second order first fundamental forms is \([h^{(2)}_{\tau\tau}] = 0 \). From its second derivative \([h^{(2)}_{\tau\tau}]_{,\tau} = 0 \) we first obtain a third order differential equation for \( F(\tau) \) which can be integrated once in order to obtain

\[
\hat{F} = e^{i(\lambda/2)}(-F + H_0 + C_3)/2a,
\]

where the constant of integration \( H_1 \) has been conveniently arranged. Using this relation back into the equation \([h^{(2)}_{\tau\tau}]_{,\tau} = 0 \) we obtain \( \hat{G} = 0 \), and therefore \( G(\tau) = -H_0 \tau + D_1 \) for some constants \( H_0 \) and \( D_1 \). Finally, \([h^{(2)}_{\tau\tau}] = 0 \) provides a relation between \( h \) and \( k \), namely

\[
[h] = \frac{1}{2} H_0 + \frac{1}{2} \nu(\alpha) \left(2[k] + H_1 \cos \theta \right).
\]

We have to impose now the equations for the perturbed second fundamental form, \([k^{(2)}_{\phi\phi}] = 0 \). The steps taken to solve the system of equations are given with enough detail in what follows so that the proof can be reproduced directly. Not to overwhelm the text we thus prefer not to include the explicit expressions of \( k^{(2)}_{\phi\phi} \) here.

Firstly, given that \( [\nu(\alpha)] = 0 \), the equations \([k^{(2)}_{\phi\phi}] = 0 \) and \([k^{(2)}_{\phi\phi}] = 0 \) are automatically satisfied. We start with the equation \([k^{(2)}_{\phi\theta}] = 0 \), which yields \( F \left(2 - 2e^{i(\lambda/2)} - \nu(\alpha) \right) = 0 \). Since \( 2 - 2e^{i(\lambda/2)} - \nu(\alpha) \neq 0 \) by assumption, we need \( \hat{F} = 0 \), and therefore, from (117) we obtain \( F + C_3 = H_1 \), which substituted on the above expressions for \( T^\phi_{L\phi}, [T^\phi_{L\phi}] \) and \( \hat{Q}_2 \) leads to

\[
T^\phi_{L\phi} = -H_0 \tau + D_1,
\]

\[
T^\phi_{L\phi} = \left(b_1 \tau \cos \theta \left(b_1 \tau - 2T^\theta_{L\lambda} \right) + H_1 \right) \sin \theta,
\]

\[
\hat{Q}_2 = ae^{i(\lambda/2)} \left\{ \left(2[k] + H_1 \cos \theta \right) \right\}.
\]

On the other hand, the combination of equations \([k^{(2)}_{\phi\phi}] \sin^2 \theta - [k^{(2)}_{\phi\phi}] = 0 \), which yields a second order PDE involving \( k \) and \( \nu(\alpha) \), with derivatives on \( \theta \) only, is integrated to obtain \( k = c_1(\tau) \cos \theta + c_2(\tau) + [f] \) for some functions \( c_1(\tau) \) and \( c_2(\tau) \). However, since \( [k]_{,\tau} = [f]_{,\tau} = 0 \), we readily have that \( c_1(\tau) = c_1 \) and \( c_2(\tau) = c_2 \) must be constants. Now, the
equation $[\kappa^{(2)}_{\theta\theta}] = 0$ provides an expression for $[m]$, which left in terms of $[k]$, in particular, can be arranged as equation (26).

The only remaining equation is given by $[\kappa^{(2)}_{\tau\tau}] = 0$. Using (26) to substitute $[m]$ in $[\kappa^{(2)}_{\tau\tau}] = 0$ we obtain a relation between $[h']$, $[k']$ and $[k]$ (and $\hat{Q}_2^+$). That relation is given explicitly by equation (27).

Furthermore, from the above expression for $[\hat{Q}_2]$ we clearly also obtain that the difference $[\hat{Q}_2]$ cannot depend on $\tau$. For the same reason, using the above equations for $[m]$ and $[h']$, and since either $[\dot{\tau}'] \neq 0$ or $[\dot{\nu}'] \neq 0$, then $\hat{Q}_2^+$ (and thus neither $\hat{Q}_2^-$) cannot depend on $\tau$.

In the case $[\dot{\tau}'] = [\dot{\nu}'] = 0$ we can have, in principle, a non-vanishing $Q_1(\tau, \theta)$. The appearance of $Q_1(\tau, \theta)$ in the expressions for $h^{(2)}_{ij}$ does not change the procedure to integrate the differences. For that reason, and due to their length, we avoid including the explicit expressions of $T^{(2)}_0$, $T^{(2)}_1$, and $T^{(2)}_0$, the integrability conditions are automatically satisfied, and the integration leads to the expression

$$
\left[ T^{(2)}_0 \right] = 2b_1(T^1_0 + \tau T^1_0 \cot \theta) - \frac{2}{a}e^{-2(\omega + i\theta)}b_1\tau Q^+_1 + D_2,
$$

for some constant $D_2$. Now, the remaining equations in the set $[h^{(2)}_{ij}] = 0$ show no terms involving $Q_1$. Therefore we obtain the same set of equations (114)–(117), $G(\tau) = -H_0 \tau + D_t$ for some constants $H_0$ and $C_1$, and $[h]$ is given by $[h] = \frac{1}{2}H_0 + \frac{1}{2}a\nu'(a)\{2[k] + H_1 \cos \theta\}$. The equation $[\kappa^{(2)}_{\theta\theta}] = 0$ reads the same as in the $Q_1 = 0$ case, and therefore the condition $\hat{F}(\tau) = 0$, assuming that $2 - 2e^{2(\omega + i\theta)} - a\nu'(a) \neq 0$, is just recovered. That again leads to $F + C_3 = H_1$. As a result $[T^{(2)}_0]$, $[T^{(2)}_1]$ and $[\hat{Q}_2]$, are also given by (118)–(120).

Likewise, the combination of equations $[\kappa^{(2)}_{\theta\theta}] \sin^2 \theta - [\kappa^{(2)}_{\theta\phi}] = 0$ does not depend on $Q_1$ either, and therefore $[k] = c_1 \cos \theta + c_2 + [f]$ all the same, for some constants $c_1$ and $c_2$. However, the equation $[\kappa^{(2)}_{\theta\phi}] = 0$ does contain a term involving $Q_1$. The expression for $[m]$ in this case is given by

$$
[m] = a[k'] - \frac{1}{4}e^{-2(\omega + i\theta)}[\dot{\lambda}'](Q)^2 + \frac{1}{4}(a\nu'(a) + 2)[2[k] + H_1 \cos \theta]
$$

$$
- \frac{1}{2}(H_1 + 2c_1)e^{2(\omega + i\theta)} \cos \theta,
$$

which used in $[\kappa^{(2)}_{\tau\tau}] = 0$ provides the following expression of $[h']$

$$
[h'] = \frac{1}{2}a\nu'(a)[k'] - \frac{1}{4}e^{-2(\omega + i\theta)}[\nu'\nu')(Q)^2 + \frac{1}{4}(a\nu'(a) + \nu'(a))[2[k] + H_1 \cos \theta]
$$

$$
- \frac{1}{4}(H_1 + 2c_1)\nu'(a)e^{2(\omega + i\theta)} \cos \theta.
$$

Finally, although equation $[\kappa^{(2)}_{\theta\phi}] = 0$ is automatically satisfied, in this case the equation $[\kappa^{(2)}_{\theta\theta}] = 0$ provides the condition $[\omega\nu']Q_1 = 0$.

As in the first order case, the constants $D_1$ and $D_2$ can be safely absorbed by using an isomorphic spacetime gauge at one (any) side, say $\hat{V}_2^+ = D_1\partial_{\tau_1} + D_2\partial_{\nu_1}$, keeping $\bar{s}_1 = 0$. Clearly $\hat{a}_2^+ = \hat{V}_2^+$ and therefore by (6) that leads to $\hat{T}_2^+ = \hat{T}_2^+ - \hat{a}_2^+$ and the second order metric perturbation tensor $K^{(2)+}$ is unchanged. We can thus set $D_1 = D_2 = 0$ without loss of generality.
References

[1] Battye R A and Carter B 2001 Generic junction conditions in brane-world scenarios Phys. Lett. B 509 331
[2] Bonnor W B and Vickers P A 1981 Junction conditions in general relativity Gen. Relativ. Gravit. 13 29–36
[3] Bradley M, Eriksson D, Fodor G and Rácz I 2007 Slowly rotating fluid balls of Petrov type D Phys. Rev. D 75 024013
[4] Brizuela D, Martín-García J M, Sperhake U and Kokkotas K D 2010 High-order perturbations of a spherical collapsing star Phys. Rev. D 82 104039
[5] Bruni M, Matarese S, Mollerach S and Sonego S 1997 Perturbations of spacetime: gauge transformations and gauge invariance at second order and beyond Class. Quantum Grav. 14 2585–606
[6] Cabezas J, Martín J, Molina A and Ruiz E 2007 An approximate global solution of Einstein’s equations for a rotating finite body Gen. Relativ. Gravit. 39 707–36
[7] Chandrasekhar S 1933 The equilibrium of distorted polytropes: I. The rotational problem Mon. Not. R. Astron. Soc. 93 390–406
[8] Chandrasekhar S and Lebovitz N R 1962 On the oscillations and the stability of rotating gaseous masses Astrophys. J. 135 248
[9] Chandrasekhar S and Miller J C 1974 On slowly rotating homogeneous masses in general relativity Mon. Not. R. Astron. Soc. 167 53–80
[10] Colpi M and Miller J C 1992 Rotational properties of strange stars Astrophys. J. 388 513–20
[11] Cuchí J E, Gil-Rivero A, Molina A and Ruiz E 2013 An approximate global solution of Einstein’s equation for a rotating compact source with linear equation of state Gen. Relativ. Gravit. 45 1457
[12] Eriksson D 2008 Perturbative methods in general relativity PhD Thesis Umeå Universitet
[13] González-Romero L M and Blázquez-Salcedo J L 2009 Core-crust transition pressure evolution in post-glitch epoch for a vela-type pulsar (arXiv:0912.0628)
[14] Hartle J B 1967 Slowly rotating relativistic stars: I. Equations of structure Astrophys. J. 150 1005–29
[15] Hartle J B and Thorne K S 1968 Slowly rotating relativistic stars: II. Models for neutron stars and supermassive stars Astrophys. J. 153 807–34
[16] Hartle J B and Thorne K S 1969 Slowly rotating relativistic stars: III. Static criterion for stability Astrophys. J. 158 719–26
[17] Lattimer J M 2012 The nuclear equation of state and neutron star masses Ann. Rev. Nucl. Part. Sci. 62 485–515
[18] MacCallum M A H, Mars M and Vera R 2007 Stationary axisymmetric exteriors for perturbations of isolated bodies in general relativity, to second order Phys. Rev. D 75 024017
[19] Mars M 2005 First- and second-order perturbations of hypersurfaces Class. Quantum Grav. 22 3325–48
[20] Mars M, Mena F C and Vera R 2007 Linear perturbations of matched spacetimes: the gauge problem and background symmetries Class. Quantum Grav. 24 3673–89
[21] Mars M and Senovilla J M M 1999 Geometry of general hypersurfaces in spacetime: junction conditions Class. Quantum Grav. 10 1865
[22] Martín J, Molina A and Ruiz E 2008 Can rigidly rotating polytropes be sources of the Kerr metric? Class. Quantum Grav. 25 105019
[23] Martín-García J and Gundlach C 2001 Gauge-invariant and coordinate-independent perturbations of stellar collapse: II. Matching to the exterior Phys. Rev. D 64 024012
[24] Mukohyama S 2000 Gauge-invariant gravitational perturbations of maximally symmetric spacetimes Phys. Rev. D 62 084015
[25] Reina B 2015 Slowly rotating homogeneous masses revisited arXiv:1503.07835
[26] Reina B and Vera R 2014 Revisiting Hartle’s model for relativistic rotating stars, progress in mathematical relativity, gravitation and cosmology (Springer Proceedings in Mathematics and Statistics, vol 60) ed A García-Parrado, F C, F Moura and E Vaz (Heideberg: Springer) pp 377–81
[27] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 Exact solutions of Einstein’s field equations Cambridge Books Online 2nd edn (Cambridge: Cambridge University Press)
[28] Stergioulas N 2003 Rotating stars in relativity Living Rev. Relativ. 6 15
[29] Vera R 2002 Symmetry-preserving matchings Class. Quantum Grav. 19 5249–64