OV-bearing modes with $\ell = 2$ of twisted anisotropic optical fibres

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Abstract. We investigated the high-order mode structure of a weakly guiding twisted anisotropic optical fibre. An analytical solution of the vector wave equation for this case is presented. We obtained analytical expressions for the higher-order modes with an azimuthal number $\ell = 2$ and their propagation constants of such a fibre, considering the mutual effect of linear anisotropy of fibre's material, twisting, including torsional mechanical stress, and spin-orbital interaction. We showed that optical vortex beams with topological charge $\pm 2$ are the modes of the fibres considered.

1. Introduction

During the last decades of the development of optics have unveiled the extreme relevance of optical vortices (OVs) [1] to the problem of increasing the data-carrying capacity of a communication channel. Nowadays, optical vortices, as beams bearing orbital angular momentum (OAM) [2], are commonly recognized as highly perspective carries of information encoded in the orbital degrees of freedom of light [3] in both free space [4, 5, 6, 7, 8, 9] and optical fibres [10, 11, 12, 13, 14, 15, 16]. Indeed, OAM of an OV, defined as $\ell \hbar$, where topological charge $\ell = 0, \pm 1, \pm 2, \ldots$, has theoretically unlimited range of values. The possibility to encode information in this extra dimension makes OVs very promising for communications. Moreover, the technique of OAM-multiplexing allows one to reach a novel level of information protection against eavesdropping [17, 18].

Since free space OV-based communication suffers strict restrictions caused by the destructive effect in the atmosphere [19], optical waveguides seem to be a promising medium for delivering the data stored in the OAM values. Much effort has been put to study the propagation of the OAM-bearing beams in optical fibres. An enduring interest was paid to the investigation of the fundamental and high-order mode behaviour in spun and twisted anisotropic fibres (TAFs) [20, 16]. In particular, TAFs were suggested for the robust transmission of information encoded in the light’s orbital degrees of freedom [16]. The main group of high-order modes with the unity azimuthal number, $|\ell| = 1$, of TAFs were obtained and discussed. At the same time, the analytical expressions describing the $|\ell| > 1$ modes in TAFs has not to our knowledge been reported. Obviously, the study of the propagation of high-order modes with $|\ell| > 1$ is of interest for developing the possibilities of OAM-based application of fibres. Thus, the goal of
this work is to obtain analytical expressions for modes with an azimuthal number $|\ell| = 2$ and their propagation constants.

2. Model and basic equation

As a model we consider a round weakly guiding twisted optical fibre with constant one-axis material anisotropy, whose principal axis lies in the plane of the cross-section of the fibre and uniformly rotates with the fibre’s axis. The permittivity of a TAF in the basis of linear polarizations $\mathbf{E} = (E_x, E_y, E_z)^T$, where $\mathbf{E}$ is the electric field and $T$ stands for ”transposed”, is given by:

$$\tilde{\varepsilon}(r, \varphi, z) = \varepsilon_{\text{IF}}(r)\hat{1} + \tilde{\varepsilon}_{\text{An}}(z) + \tilde{\varepsilon}_{\text{TMS}}(r, \varphi)$$  \hspace{1cm} (1)

Here cylindrical polar coordinates $(r, \varphi, z)$ are implied and the $z$ axis is the fibre’s axis. The first term $\varepsilon_{\text{IF}}(r) = \varepsilon_{\text{co}}[1 - 2\Delta f(r)]$ is the scalar permittivity of an ideal fibre (IF) with a core of radius $r_0$ and an infinite cladding [21], $\Delta = (\varepsilon_{\text{co}} - \varepsilon_{\text{cl}})/2\varepsilon_{\text{co}}$ is the normalized index difference, $\varepsilon_{\text{co}}$ and $\varepsilon_{\text{cl}}$ are the core and cladding values of the permittivity, respectively, and $f(r)$ is the profile function. We consider conventional step-index fibres with the profile function $f(r) = \Theta(r/r_0 - 1)$, where $\Theta$ is the unit step function, in the weak guidance (paraxial) regime, $\Delta \ll 1$. The second term in Eq. (1) describes the effect of linear anisotropy of fibre’s material (material birefringence), whose principal axis rotation angle is $\theta = qz \equiv \frac{2\pi}{\pi}z$, where $H$ is the pitch of the twist. The effect of anisotropy with a fixed orientation of its axis is taken into account by introducing material birefringence in the local base that rotates along with the fibre through an angle $\theta$. Finally, one makes the transition to the laboratory frame that has the following effect on a tensor: $\hat{H} \rightarrow \hat{R}(qz)\hat{H}\hat{R}^{-1}(qz)$, where $\hat{R}(qz)$ is the rotation matrix, yielding $\hat{\varepsilon}_{\text{An}}(z) = \delta \varepsilon_{\text{An}} \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The third term in Eq. (1), $\tilde{\varepsilon}_{\text{TMS}}(r, \varphi) = \delta \varepsilon_{\text{TMS}} R \begin{pmatrix} 0 & 0 & \sin \varphi \\ 0 & 0 & -\cos \varphi \\ \sin \varphi & -\cos \varphi & 0 \end{pmatrix}$, appears due to the influence of the twisted mechanical stress (TMS) on the optical properties of the fibre’s medium through the photoelastic effect. Here $\delta \varepsilon_{\text{TMS}} = p_{44}\varepsilon_{\text{co}}^2 r_0 q$, where $p_{44}$ is the corresponding element of the photoelastic tensor in the standard contracted notation, $R = r/r_0$.

The electric field distribution in TAFs can be obtained by solving the vector wave equation [21]:

$$\left(\nabla^2 + k^2\tilde{\varepsilon}\right)\mathbf{E} = \nabla(\text{div}\mathbf{E})$$  \hspace{1cm} (2)

where $k = 2\pi/\lambda$, $\lambda$ being the wavelength in vacuum, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. In Eq. (2) the right-hand side, which is usually referred as the gradient term, is responsible for the spin-orbital interaction (SOI) of light beams propagating in an inhomogeneous medium [22]. The Eq. (2) allows one to consider a mutual influence of polarization and spatial degrees of freedom on the beam propagation. It should be noted, however, that the significant feature of the Eq. (2) with the permittivity (1) is the violation of the translation invariance in the propagation direction, because of the $z$ -dependence of the permittivity (1). As has been shown in [23], to restore such symmetry one has to rewrite Eq. (2) into the local frame that rotates along with the anisotropy axis. The transition to the local basis implies two following steps: 1) introducing the new fields: $\mathbf{E} = \text{diag}(e^{iqx}, e^{-iqy}, 1)\mathbf{E}$, where the vector $\mathbf{E} = \text{col}(E_+, E_-, E_z)$ is written in the basis of circular polarizations with $E_\pm = (1/\sqrt{2})(E_x \mp iE_y)$; 2) passing to the new variables: $\tilde{r} = r$, $\tilde{\varphi} = \varphi - qz$, $\tilde{z} = z$. The obtained equation turns out to be translational invariant with respect to $\tilde{z}$, which
enables one to make the standard substitution: \( \hat{E}(r, \hat{\varphi}, z) = \hat{e}(r, \hat{\varphi}) \exp(i \beta z) \), where \( \beta \) is the propagation constant. The resulting equation in \( |\tilde{\Psi}\rangle = (\tilde{e}_+, \tilde{e}_-, e_z)^T \) reads as:

\[
[\hat{H}_{IF} - (\beta \hat{1} - q \hat{J}_z)^2 + \hat{V}_{TMS} + \hat{V}_{An}] |\tilde{\Psi}\rangle = 0,
\]

where the operator \( \hat{H}_{IF} \) describes the light propagation in IFs [24, 21], \( \hat{J}_z = -i \nabla_2 \hat{1} + \text{diag}(1,-1,0) \) is the total angular momentum operator, \( \hat{V}_{TMS} = (i/\sqrt{2})\delta \varepsilon_{TMS} R \begin{pmatrix} 0 & 0 & e^{i \varphi} \\ 0 & 0 & -e^{i \varphi} \end{pmatrix} \) is the operator of the TMS induced in IFs [25], and \( \hat{V}_{An} = k^2 \delta \varepsilon_{An} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \).

The obtained modified wave equation (3) can further be solved by applying the perturbation theory considering the ratio: \( \delta \varepsilon_{An}, \delta \varepsilon_{TMS} \ll 1 \).

3. OV-bearing modes with \( |\ell| = 2 \) and their propagation constants

Let the fibre supports modes with the azimuthal number \( |\ell| < 3 \) (at that, for example, fibre’s parameters can be following: \( r_0 = 6A_{He-Nc}, \Delta = 0.004 \), waveguide parameter \( V = kr_0 \sqrt{2 \Delta \varepsilon_{co}} = 4.99 \). In particular, we are interested in the high-order OV-bearing modes with the azimuthal number \( |\ell| = 2 \). According to the perturbation theory approach, the fibre modes and their propagation constants are found as a solution to the following equation:

\[
\hat{H} x_k = 0.
\]

Here the vectors \( x_k \) are four-component vectors, which components \( a_k^\ell \) are coefficients in the expansion of the desired modes in basis OVs:

\[
\{1, 2, 1, -1, 2, 1, -2, 1, -2, 1, -2\},
\]

where OV \( |\sigma, \ell\rangle = (1/\sqrt{2}) \hat{e}^{i \varphi} \left( F_\ell(r), i \sigma F_\ell(r), 1/r \tilde{\beta}_\ell \right) (r F_\ell'(r) - \sigma \sigma F_\ell) e^{i \sigma \varphi} \right)^T \), index \( \sigma = \pm 1 \) describes the handedness of circular polarization, the vortex topological charge \( \ell = \pm 2 \), \( \tilde{\beta}_\ell \) is the scalar propagation constant, and the radial function \( F_\ell(r) \) is expressed through the Bessel functions \( J_\ell(r) \) and \( K_\ell(r) \) in the core and the cladding, respectively [21].

The perturbation matrix \( \hat{H} \) in Eq. (4) is built in the standard manner by averaging operators from the Eq. (3) over the basis (5), \( \langle \sigma, \ell | \hat{H}_{IF} - (\beta \hat{1} - q \hat{J}_z)^2 + \hat{V}_{TMS} + \hat{V}_{An} | \sigma' \ell' \rangle \), and it can be brought to the form:

\[
\hat{H} = \begin{pmatrix}
D_+ + A & E & 0 & 0 \\
E & G_+ + B & 0 & 0 \\
0 & 0 & G_- + B & E \\
0 & 0 & E & D_- + A
\end{pmatrix},
\]

where terms \( D_\pm = \pm 2 \beta \left( 3q \pm \tilde{\beta}_2 \right) + 2 \tilde{\beta}_2^3 + 6gq \tilde{\beta}_2 - 9q^2 \) and \( G_\pm = \pm 2 \beta \left( q \pm \tilde{\beta}_2 \right) + 2 \tilde{\beta}_2^3 + 2gq \tilde{\beta}_2 - q^2 \) include the effect of twisting, \( g = 0.5 \varepsilon_{co} |p_{44}| \) (for silica \( g \approx 0.08 \)), matrix elements \( E = k^2 \delta \varepsilon_{An} \) are associated with the effect of linear material anisotropy. Parameters \( A = \frac{\Delta}{\sigma q} (F_2'(1) - 2), \)

\( B = \frac{\Delta}{\sigma q} (F_2'(1) + 2) \) describe the SOI, prime stands for the derivative with respect to \( R \), \( Q = \int_0^R RF_2^2(R) dR \).
Obtained modes in the local frame are given by:

\[ |\Psi_1\rangle = (\cos \alpha_+|1, 2\rangle + \sin \alpha_+|1, 2\rangle) e^{i\beta_1z}, \]
\[ |\Psi_2\rangle = (\sin \alpha_+|1, 2\rangle - \cos \alpha_+|1, 2\rangle) e^{i\beta_2z}, \]
\[ |\Psi_3\rangle = (\cos \alpha_-|1, -2\rangle + \sin \alpha_-|1, -2\rangle) e^{i\beta_3z}, \]
\[ |\Psi_4\rangle = (\sin \alpha_-|1, -2\rangle - \cos \alpha_-|1, -2\rangle) e^{i\beta_4z}, \]  \hspace{1cm} (7)

where \( \tan 2\alpha_\pm = \frac{E}{2g_\pm(1-g)|^2(A-B)/2}, \) \( 0 < \alpha_\pm \leq \pi/4. \) As is seen, the modes (7) are composed of two orthogonal OVs with the same topological charge. The energy distribution between partial fields within the modes is controlled by the competition of linear material anisotropy, twisting and the SOI via parameter \( \alpha. \) In the laboratory frame expressions (7) can be brought to the form:

\[ |\Psi_1\rangle = F_2(R) (\cos \alpha_+e_+e^{-iqz} + \sin \alpha_+e_-e^{iqz}) e^{2i(\varphi-qz)e^{i\beta_1z}}, \]
\[ |\Psi_2\rangle = F_2(R) (\sin \alpha_+e_+e^{-iqz} - \cos \alpha_+e_-e^{iqz}) e^{2i(\varphi-qz)e^{i\beta_2z}}, \]
\[ |\Psi_3\rangle = F_2(R) (\cos \alpha_-e_+e^{-iqz} + \sin \alpha_-e_-e^{iqz}) e^{-2i(\varphi-qz)e^{i\beta_3z}}, \]
\[ |\Psi_4\rangle = F_2(R) (\sin \alpha_-e_+e^{-iqz} - \cos \alpha_-e_-e^{iqz}) e^{-2i(\varphi-qz)e^{i\beta_4z}}. \]  \hspace{1cm} (8)

where \( e_\pm = (1/\sqrt{2})(1, \pm i)^T \) is the basis vectors of circular polarization. As is evident, modes (8) consist of partial waves propagating with different phase velocities. This enables us to recognize the modes of a TAFs as Bloch waves. The elliptical polarization of obtained modes results from the interplay between linear material anisotropy on the one hand side and the spin part \([16]\) of twist-induced circular birefringence and the SOI on the other. At that, the vortex structure of modes is formed by the action of the orbital part of twist-induced circular birefringence and the SOI.

The corresponding propagation constants are found to be:

\[ \beta_{1,2} = \tilde{\beta}_2 + 2q(1-g) + \frac{A+B}{4\beta_2} \pm \left( q(1-g) + \frac{A-B}{4\beta_2} \right)^2 + \left( \frac{E}{2\beta_2} \right)^2 \right)^{1/2}, \]
\[ \beta_{3,4} = \tilde{\beta}_2 - 2q(1-g) + \frac{A+B}{4\beta_2} \pm \left( q(1-g) + \frac{B-A}{4\beta_2} \right)^2 + \left( \frac{E}{2\beta_2} \right)^2 \right)^{1/2}. \]  \hspace{1cm} (9)

The propagation constants are splitting due to the different corrections caused by the anisotropy of material, twisting and the SOI. That fact allows one to expect the stability of OV-bearing modes with \( \ell = 2 \) in such a fibre.

It should be noted, that obtained results in the limiting case of untwisted fibre \( (q = 0) \) coincide with known expressions for modes and their propagation constants of anisotropic optical fibre \([24, 26]\).

4. Conclusion
The above expressions in Eqs. (8, 9) give insight about the structure of modes with the azimuthal number \( \ell = 2 \) and their corresponding propagation constants of TAFs at the arbitrary ratio of linear material anisotropy, twisting, and the SOI. The next step in this work implies the studying of practically important limiting cases of the ratio between fibre’s parameters when their modes become linearly and circularly polarized OVs. It is the subject of our further research.
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