In this paper, we examine the dynamical behavior of a collapsing star via fully dynamical approach. We derive and simplify the complete dynamical equation system for a collapsing spherically symmetrical star with perfect fluid, and establish some new constraints on the initial and boundary conditions. The mass density near the center can be approximately derived in the light-cone coordinate system. The results show that, the singularity can not form at the center if the equation of state satisfies some increasing and causal conditions and the initial distributions of mass-energy density and velocity are suitable. The dynamical equations can even automatically remove the weak singularity in the initial data. The analysis may give some insights into the structure of a star and the nature of the black hole.

**Keywords:** gravitational collapse, dynamical approach, black hole, singularity

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**I. INTRODUCTION**

In general relativity, we have some typical exact solutions such as the Schwarzschild, Curzon and Kerr metrics and some of their extensions to the electrovacuum solutions such as the Reissner-Nordström and Kerr-Newman metrics [1, 2, 3, 4, 5]. All these solutions have singularity and event horizon. Under some common assumptions such as positive energy condition and trapped surface for the classical matter field, Penrose and Hawking have proved that the singularity in space-time will be certainly generated from some regular states[5].

The definition of singularity in curved space-time is a subtle problem. Different from the solutions of the ordinary field equations, in general relativity we are free to use any coordinate system, the infinity of the metric may be only caused by coordinate system. Besides, the singularity of the space-time means the boundary of the manifold, where the normal description for the space-time is invalid. A way of detecting singularities within a coordinate system is to find that curvature invariants become infinite. These are scalar quantities such as $R$, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$,

*Electronic address: yqgu@fudan.edu.cn*
which measure the curvature of space-time and if they become infinite, this is a sure sign that a
region of space-time cannot be extended\[5, 6\]. A breakthrough in the understanding of space-time
singularities was the singularity theorem of Penrose\[7\], which identified general conditions under
which a space-time must be geodesic incomplete. This was then generalized to other situations
by Hawking and others. The singularity theorems are proved by contradiction. Their strength
is that the hypotheses required are very general, and their weakness is that they give very little
information about what actually happens dynamically[8].

However, as a physical description, the incomplete geodesic is rather difficult for manipulation,
and it is not a good choice to use the future properties of an evolving manifold. The singular
curvatures also have a weakness, that is, it can be easily overlooked or confused with coordinate
singularity. For example, $R \to \infty$ is easily overlooked similarly to $\Delta \frac{1}{r} = 0, (r > 0)$. We completely
do not know whether $R = 0$ or $R \to \infty$ on the horizon $r = 2m$ in the Schwarzschild space-time,
because we can not calculate it from the metric by usual means[9]. The most convenient definition
may be directly using $\rho \to \infty$. Although it is equivalent to $R \to \infty$ in mathematics, they are
different in physics. This is because, on the one hand, $\rho \to \infty$ has manifest physical meaning and
can not be overlooked. On the other hand, as calculated below, $\rho \to \infty$ occurs before the metric
itself becomes singular[10], so the space-time can be still treated as a ‘place’ when the mass-energy
density approaches infinity[6, 8].

For the static spherically symmetrical space-time generated by perfect fluid source
\[ ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{1.1} \]
we have a number of exact solutions[11, 12, 13] and generating method[14]. In these solutions
we usually expressed the mass density $\rho$ and pressure $P$ as the functions of $(A,B)$ and their
derivatives. Theoretically, such solutions form the general solution to the problem[12]. However,
in such expressions the properties of the EOS is quite obscure and concealed, and then most of the
solutions are probably unrealistic in physics.

In fact, the realistic static asymptotically flat space-time with spherical symmetry can be gen-
erally resolved by the following procedure. The dynamics for the space-time with perfect fluid can
be reduced to the following initial problem of an ordinary differential equation system[10, 15],
\[ M'(r) = 4\pi G\rho r^2, \quad M(0) = 0, \tag{1.2} \]
\[ \rho'(r) = -\frac{(\rho + P)(4\pi GPr^3 + M)}{C_s^2(r - 2M)r}, \quad \rho(0) = \rho_0, \tag{1.3} \]
in which $P = P(\rho)$ is the EOS of the fluid, $M(r)$ is the ADM mass within the ball of radial
coordinate $r$, $C_s = \sqrt{P'(\rho)}$ is the velocity of sound in the fluid. For any given $\rho_0 > 0$ we get a
unique solution. The metric components are given by

\[ A = \left( 1 - \frac{2M}{r} \right)^{-1}, \quad B = \exp \left( - \int_r^R \frac{2(4\pi GP r^3 + M)}{r(r - 2M)} \, dr \right), \quad (1.4) \]

where \( R < \infty \) is the radius of the star. \( \rho(r) = P(r) = 0 \) in the region \( r \geq R \).

For any given EOS \( P = P(\rho) \), the solution of (1.2) and (1.3) can be easily solved numerically. Similar to the case calculated in [10], we find that if the EOS satisfying the following increasing and causal conditions

\[ 0 \leq C_s \leq \frac{1}{3}, \quad C_s'(\rho) \geq 0, \quad P \rightarrow \begin{cases} P_0 \rho^\gamma, & (\gamma > 1, \rho \rightarrow 0), \\ \frac{1}{3} \rho, & (\rho \rightarrow \infty), \end{cases} \quad (1.5) \]

then all solutions of (1.2) and (1.3) are singularity-free. That is, we always have \( 0 \leq \rho \leq \rho_0 \) and \( R_s = 2M(R) < R \). The condition \( \gamma > 1 \) is necessary, which leads to the finite radius of the star \( R < \infty \) by \( C_s \rightarrow 0 \). In the case \( \gamma = 1 \), we get \( R \rightarrow \infty \), and then the space-time is no longer asymptotically flat. Or equivalently, finite such gas can not form a static star. This result reveals that, if EOS satisfies (1.5) and \( \rho_0 \) has no upper bound, for a star with any given large mass, we have regular solutions in equilibrium. The EOS of the matter is decisive for the fate of a star.

There are some skeptics who proposed special matter models which lead to global solutions or the violation of the conditions for singularity theorems[16, 17, 18, 19, 20]. The instability of naked singularities in the spherically symmetrical space-time with scalar field gravitating source was studied by D. Christodoulou[21]. There are initial data leading to the formation of naked singularities but for generic initial data this does not happen. The structure of the singularity is similar to the Schwarzschild case. The above analysis also suggests the formation of singularity in space-time should be carefully checked by realistic dynamical models.

A general dynamical analysis includes the hydrodynamics of the matter source, which is too complicated to be achieved. Noticing the fact that the asymmetry of the motion of the star, such as the rotation, is always alleviating the singularity, the collapsing of a star with spherical symmetry is probably the worst case. So if we can make an effectively discussion on this model, it will be much helpful to understand the dynamical behavior of the Einstein equation. This is the motivation of the paper.
II. DYNAMICAL EQUATIONS FOR A COLLAPSING STAR

The line element in the space-time generated by an evolving star with spherical symmetry is generally given by

\[ ds^2 = u^2 dt^2 - (v dt - w dr)^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(2.1)

Here we take the light velocity \( c = 1 \) as unit of speed. For a normal star, \((u, v, w)\) are continuous functions of \((t, r)\) with suitable smoothness. The null geodesic along the radius is described by

\[(u + v) dt - w dr = 0.\]  

(2.2)

Assume the solution is \( f(\tilde{t}, r) = C \), where \( C \) is a constant. Making light-cone coordinate transformation

\[ t = T(f(\tilde{t}, r)), \]  

(2.3)

where \( T(f) \) is any smooth function satisfying \( \partial_f T \partial_{\tilde{t}} f > 0 \), then we get the line element equivalent to (2.1) as follows.

\[ ds^2 = ab dt^2 + 2 \sqrt{b} b dt dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(2.4)

where \((a, b)\) are continuous functions of \((t, r)\) with suitable smoothness until the star becomes singular. (2.4) is similar to the “Eddington - Finkelstein coordinates”. The advantage of this coordinate system is that, the solution near the stellar center to the Einstein equation can be approximately solved under some ordinary conditions.

For the external Schwarzschild solution, we have conditions

\[ b = 1, \quad a = 1 - \frac{R_s}{r}, \quad \text{(for } r \geq R > R_s), \]  

(2.5)

where \( R = R(t) \) and \( R_s \) are respectively the stellar radius and Schwarzschild radius

\[ R_s \equiv 8\pi G \int_0^{R(t)} \rho_{\text{grav}} r^2 dr, \]  

(2.6)

in which \( \rho_{\text{grav}} \) is the total gravitating mass-energy density including influences of pressure and momentum. The detailed definition of \( \rho_{\text{grav}} \) is given below.

Denote the 4-vector speed of the fluid by

\[ U^\mu = \{ U, V, 0, 0 \}, \quad U_\mu = \{ abU + \sqrt{b} V, \sqrt{b} U, 0, 0 \}, \]  

(2.7)
which satisfies the line element equation

\[ 1 = g_{\mu\nu} U^\mu U^\nu = \left(abU + 2\sqrt{b}V\right)U.\]  

(2.8)

Solve \( U \) from it, we get

\[ U = \left(\frac{1}{\sqrt{b}\left(\sqrt{a + V^2} + V\right)}\right) = \frac{1}{a\sqrt{b}}(\sqrt{a + V^2} - V).\]  

(2.9)

So between \((U, V)\), only \( V \) is independent variable. When \(|V| \ll 1\), \( V \) is approximately equal to the usual speed.

For the perfect fluid model, the nonzero components of the energy-momentum tensor \( T_{\mu\nu} = (\rho + P)U_\mu U_\nu - Pg_{\mu\nu} \) are given by

\[ T_{tt} = b(\rho + P)\left(a\sqrt{b}U + V\right)^2 - abP, \]  

(2.10)

\[ T_{tr} = b(\rho + P)\left(a\sqrt{b}U + V\right)U - \sqrt{b}P = T_{rt}, \]  

(2.11)

\[ T_{rr} = b(\rho + P)U^2, \quad T_{\theta\theta} = Pr^2, \quad T_{\varphi\varphi} = Pr^2\sin^2\theta, \]  

(2.12)

in which \( P = P(\rho) \) is the given EOS. The reasonable EOS should satisfy the following conditions

\[ 0 < C_s = \sqrt{\frac{dP}{d\rho}} \leq \sqrt{\frac{3}{3}}, \quad (\rho > 0), \]  

(2.13)

where \( C_s \) is the velocity of sound in the fluid. For the following discussions, we only need the condition \( 0 < C_s \leq C_0 < 1 \).

The nonzero components of Einstein tensor are given by

\[ G_{tt} = \frac{1}{r}\left(\sqrt{b}\partial_t a - ab\partial_r a\right) - \frac{1}{r^2}ab(1 - a), \]  

(2.14)

\[ G_{tr} = \frac{1}{r}\sqrt{b}\partial_r a - \frac{1}{r^2}\sqrt{b}(1 - a) = G_{rt}, \quad G_{rr} = -\frac{\partial_r b}{ra}, \]  

(2.15)

\[ G_{\theta\theta} = \left(\frac{a}{r} \left(\frac{\partial_r b}{2b} + \frac{\partial_r a}{a}\right) - \frac{1 - a}{r^2} + \frac{R}{2}\right) r^2, \quad G_{\varphi\varphi} = G_{\theta\theta}\sin^2\theta, \]  

(2.16)

where the scalar curvature \( R \) depends on the second order derivatives of the metric functions \((b, a)\).

But it is not used in the following discussion, because the related equations are not independent of the used ones.

By detailed calculations, we find only the following 3 equations are independent ones in the Einstein equation \( G_{\mu\nu} = -8\pi GT_{\mu\nu} \).

\[ \partial_r b = 8\pi Gr(\rho + P)b^2U^2, \]  

(2.17)

\[ \partial_r a = 8\pi G(\rho + P)rV\sqrt{b(a + V^2)}, \]  

(2.18)

\[ \partial_r a = -4\pi Gr \left((\rho - P) + (\rho + P)abU^2\right) + \frac{1 - a}{r}. \]  

(2.19)
By (2.5), (2.6) and (2.18), we learn $R_s$ is conserved in an evolutionary star.

Among the energy-momentum conservation law $T^\mu_\nu = 0$, calculation shows that, only the equation $U_\mu T^\mu_\nu = 0$ is independent, so we get

$$U^\mu \partial_\mu \rho + (\rho + P)U^\mu_{;\mu} = 0. \tag{2.20}$$

Simplifying (2.20) and the consistent equation of (2.18) and (2.19) $\partial_r a = \partial_t a$, we get the dynamical equation for $(\rho, V)$, which is a first order hyperbolic differential equation system,

$$(1 - C_s^2)\sqrt{bU^2} \partial_t \rho + \frac{1}{2} \left[2 - (1 - C_s^2)(1 + abU^2)\right] \partial_r \rho + \frac{2(\rho + P)\sqrt{bU}}{1 + abU^2} \partial_r V = -\frac{1}{r} \left(\frac{aU^2}{1 + abU^2}(1 - a + 8\pi Gr^2 P) + 2\sqrt{bUV}\right) (\rho + P), \tag{2.21}$$

$$(1 - C_s^2)\sqrt{bU^2} \partial_t V + \frac{1}{2} \left[2 - (1 - C_s^2)(1 + abU^2)\right] \partial_r V + \frac{(1 + abU^2)C_s^2}{2(\rho + P)\sqrt{bU}} \partial_r \rho = -\frac{1}{2r} \left(\sqrt{bU}(1 - a + 8\pi Gr^2 P) + 2C_s^2(1 + abU^2)V\right). \tag{2.22}$$

The characteristic speeds are given by

$$V_1 = \frac{V}{U} + \frac{C_s}{(1 - C_s)\sqrt{bU^2}} = \frac{V + C_s\sqrt{a + V^2}}{U(1 - C_s)}, \tag{2.23}$$

$$V_2 = \frac{V}{U} - \frac{C_s}{(1 + C_s)\sqrt{bU^2}} = \frac{V - C_s\sqrt{a + V^2}}{U(1 + C_s)}. \tag{2.24}$$

The disturbance of the solution $(\rho, V)$ propagates at such speed $dr/dt = V_k$.

Among the equations (2.17)-(2.19) and (2.21)-(2.22), only 4 equations are independent, and the redundant equation can be treated as identity. For a normal star, the variables have the following range of value,

$$0 < b \leq 1, \quad 0 < a \leq 1, \quad 0 \leq \rho < \infty, \quad |V| < \infty. \tag{2.25}$$

By (2.17) and (2.19), we learn $b(\cdot, r) \in C^0([0, R])$ is a monotonic increasing function of $r$, but $a(\cdot, r) \in C^1([0, \infty))$ has a positive minimum $a_{\text{min}} > 0$. The reasonable boundary condition in the asymptotic flat space-time is given by

at $r = 0$ : $a = 1$, $\partial_r a = 0$, $V = 0$; \tag{2.26}
at $r = R$ : $b = 1$, $a = 1 - \frac{R_s}{R}$, $\rho = 0$. \tag{2.27}

The initial data of $(\rho, V)$ should satisfy (2.17), (2.19) and the above boundary condition (2.27). Besides the initial data should still satisfy a constraint similar to

$$\frac{8\pi G}{r} \int_0^r \rho(0, r)r^2 dr < 1, \tag{2.28}$$
or in ordinary function

\[ 8\pi G \rho(0,r)r^2 < 1. \]  

(2.29)

(2.28) or (2.29) means that we can not preset a black hole or singularity inside a star, because we can not use singular conditions to analyze singularity. The exact form of (2.29) will be derived below.

The Einstein equation has a scaling invariance\(^{[22]}\). Making a transformation similar to that in [10], we can get the dimensionless form of equation system (2.17)-(2.19) and (2.21)-(2.22).

Define an auxiliary function by

\[ F \equiv (\rho + P)abU^2 = (\rho + P)\frac{\sqrt{a + V^2} - V}{\sqrt{a + V^2} + V}. \]  

(2.30)

For a static star, we have \( F = \rho + P \). Substituting (2.30) into (2.19), we get the solution as

\[ a = 1 - \frac{4\pi G}{r} \int_0^r (\rho - P + F)r^2dr. \]  

(2.31)

In contrast (2.31) with (2.5), we get the total gravitational mass-energy density defined in (2.6),

\[ \rho_{\text{grav}} = \frac{1}{2}(\rho - P + F) = \frac{\rho\sqrt{a + V^2} - PV}{\sqrt{a + V^2} + V}. \]  

(2.32)

(2.32) shows the difference between \( \rho_{\text{grav}} \) and the usual mass density \( \rho \). By (2.26) and (2.27), we find that \( F \) satisfies

\[ F(t,0) = \rho(t,0) + P(t,0), \quad F(t,R) = 0. \]  

(2.33)

III. SIMPLIFICATION OF THE EQUATIONS

The above equations have a weakness, that is, the geometrical variables \((a, b)\) and mechanical variables \((\rho, V)\) couple each other in a complicated manner, which increases the difficulties for discussion. For example, we can hardly treat \( F \) as a mechanical variable or a geometrical one.

The problem can be solved by making the following transformation,

\[ U = \frac{e^{-v}}{\sqrt{ab}}, \quad V = \sqrt{a}\sinh v, \]  

(3.1)

where \( v(t,0) = 0 \), then we have

\[ U_\mu = \left( \sqrt{ab} \cosh v, \frac{e^{-v}}{\sqrt{a}}, 0, 0 \right). \]  

(3.2)
The energy function \((2.30)\) reduces to the pure mechanical quantity

\[
F = (\rho + P)e^{-2v}. \tag{3.3}
\]

The dynamical equation system \((2.17)-(2.19)\) and \((2.21)-(2.22)\) becomes

\[
\partial_t b = 8\pi Gr(\rho + P)a^{-1}be^{-2v}, \tag{3.4}
\]

\[
\partial_t a = 4\pi Gra\sqrt{b}(\rho + P) \sinh(2v), \tag{3.5}
\]

\[
\partial_r a = 8\pi Gre^{-v}(P \sinh v - \rho \cosh v) + \frac{1-a}{r}, \tag{3.6}
\]

\[
\frac{(1 - C_s^2)e^{-v}}{\sqrt{b}(\rho + P)} \partial_t \rho + a \left( \frac{C_s^2 \cosh v + \sinh v}{\rho + P} \partial_r \rho + e^v \partial_r v - \frac{1}{2r} (\cosh v - 3 \sinh v) \right)
\]

\[
= 4\pi Gr(\rho \sinh v - P \cosh v) - \frac{1}{2r} (\cosh v + \sinh v), \tag{3.7}
\]

\[
\frac{(1 - C_s^2)e^{-v}}{\sqrt{b}} \partial_t v + \frac{a}{2} \left( \frac{e^v C_s^2 \partial_r \rho}{\rho + P} + (C_s^2 \cosh v + \sinh v) \partial_r v + \frac{1}{r} (3C_s^2 \sinh v - \cosh v) \right)
\]

\[
= 4\pi Gr(C_s^2 \rho \sinh v - P \cosh v) - \frac{1}{2r} (C_s^2 \sinh v + \cosh v). \tag{3.8}
\]

The characteristic speeds become

\[
V_1 = \sqrt{\frac{abe^v}{1 - C_s}} (C_s \cosh v + \sinh v), \tag{3.9}
\]

\[
V_2 = \sqrt{\frac{abe^v}{1 + C_s}} (\sinh v - C_s \cosh v). \tag{3.10}
\]

From the above equations, we find \((a, b)\) are separated with \((\rho, P, v)\). This feature is much helpful for theoretical analysis and numeric calculation. Solving \((3.4)\) and \((3.6)\), we get

\[
a = 1 - \frac{1}{r} \int_0^r 8\pi G \rho_{\text{grav}} r^2 dr, \tag{3.11}
\]

\[
b = \exp \left( -8\pi G \int_0^R (\rho + P)a^{-1}e^{-2v}r dr \right). \tag{3.12}
\]

In which the total gravitating mass-energy density \(\rho_{\text{grav}}\) and the gravitating mass (ADM mass) \(M_{\text{ADM}}\) of the star are given by

\[
\rho_{\text{grav}} = e^{-v}(\rho \cosh v - P \sinh v), \tag{3.13}
\]

\[
M_{\text{ADM}} = \int_0^R 4\pi e^{-v}(\rho \cosh v - P \sinh v)r^2 dr. \tag{3.14}
\]

They are manifestly dependent of \((P, v)\). \(M_{\text{ADM}}\) is conserved due to \((3.5)\).

Theoretically, we can substitute \((3.11)\) and \((3.12)\) into \((3.7)\) and \((3.8)\), then we get a closed equation system for \((\rho, v)\), which includes all information of an evolving star with spherical symmetry.
Furthermore, making transformation
\[
v = \frac{1}{2} \ln \frac{1 - w}{1 + w},
\]
and substituting it into the dynamical equations, we get the coefficients of the equations in the polynomial form of \( w \). The physical meaning of \( w \) is approximately the usual velocity.

### IV. ANALYSIS OF THE SINGULARITY

In what follows, we show that, under suitable initial conditions the singularity of the space-time can not take place. When \((b > 0, a > 0)\), The space-time is normal and measurable. By (3.5), we get
\[
a(t, r) = a(0, r) \exp \left( 4\pi Gr \int_0^t \sqrt{b} \left( \rho + P \right) \sinh(2v) \, dt \right) > 0,
\]
in which \( a(0, r) \) is determined by the initial distribution \( \rho(0, r) \) via (3.11). So \( b(t, 0) \to +0 \) certainly occurs before \( a_{\text{min}} \to +0 \). According to (3.12) and \( v(t, 0) = 0, a(t, 0) = 1 \), we learn that, if \( b(t, 0) \to 0 \) then we certainly have \( \rho(t, 0) \to \infty \). That is to say, \( \rho(t, 0) \to \infty \) is a necessary condition for the singularity of space-time.

Since \( a > 0 \), by (3.12) we learn \( 0 < b \leq 1 \) is a monotonic increasing function of \( r \). By (2.13), we have \( P \propto \rho \) as \( \rho \to \infty \). So according to (3.12), \( b(t, 0) \to +0 \) only if \( \rho(t, r) \geq \rho_0(t)r^{-2}, (r \to 0) \). Even if \( \rho(t, r) \to \rho_0(t)r^{-n}, (0 < n < 2) \), we still have \( b > 0 \), and the space-time is still measurable, although the curvature is singular. The following calculation shows that, the dynamical equations (2.21) and (2.22) even can automatically remove such weak singularity. So to discuss the singularity of the space-time is transformed into to check the order of \( \rho(t, r) \to \infty \) as \( r \to 0 \). That is to say, we only need to examine the asymptotic behavior of \((\rho, V)\) near the center of the star while \( \rho \to \infty (r \to 0) \).

The disturbance of \((\rho, V)\) has only finite propagating velocity (2.23) and (2.24). Within any given finite time \( 0 \leq t \leq T_0 \), the mass density \( \rho(t, 0) \) only depends on the initial distribution of \((\rho, V)\) near the center. The worse case is that, the star has already had extremely high temperature and pressure near the center, but it is still collapsing. Here \( V < 0 \) means collapse, namely the fluid flow towards the center of the star.

In the state with extreme high temperature and pressure, the EOS of the fluid becomes simple. We have the following approximations,
\[
P \approx C_0^2 \rho, \quad C_s \approx C_0 < 1.
\]
Noticing \( V(t, 0) = 0 \), by (2.9), (3.11) and (3.12), omitting \( O(r^2) \) terms we get

\[
|V| \ll 1, \quad U = \frac{\sqrt{a} - V}{a \sqrt{b}}, \quad b = b_0(t), \quad a = 1.
\] (4.3)

Substituting (4.2)-(4.4) into (2.21) and (2.22), we have the simplified dynamical equations

\[
\frac{1 - C_0^2 \partial_r \rho}{1 + C_0^2} + \frac{C_0^2 \partial_r \rho}{1 + C_0^2} + (\partial_r V + \frac{2V}{r}) + 4\pi G C_0^2 \rho r = 0,
\] (4.5)

\[
\frac{1 - C_0^2}{C_0^2} \partial_r V + \frac{1}{1 + C_0^2} \partial_r \rho + (\partial_r V + \frac{2V}{r}) + 4\pi G \rho r = 0,
\] (4.6)

where \( d\tau \equiv \sqrt{b_0} dt \). Keep in mind that we have not made approximation for \( \rho \), so it is valid for all range value of \( \rho \).

By (4.5) and (4.6), we get a linear first order differential equation for \( \rho^{-1} \),

\[
\partial_r \rho^{-1} - \partial_r \rho^{-1} + (1 + C_0^2) \partial_r V \rho^{-1} + 4\pi G (1 + C_0^2) r = 0.
\] (4.7)

The solution of (4.7) can be exactly solved. This is the advantage of the coordinate system (2.4).

The solution is given by

\[
\rho(\tau, r) = \frac{\rho(0, \tau + r)}{f(\tau, r)} \exp \left((1 + C_0^{-2}) [V(\tau, r) - V(0, \tau + r)]\right),
\] (4.8)

in which

\[
f(\tau, r) \equiv 1 - 4\pi G (1 + C_0^2) \rho(0, \tau + r) \cdot \\
\int_0^\tau (\tau + r - x) \exp \left((1 + C_0^{-2}) [V(x, \tau + r - x) - V(0, \tau + r)]\right) \, dx.
\] (4.9)

Since \( V \) is a variable of little absolute value, so \( \rho(\tau, 0) \to \infty \) if and only if \( f(\tau, 0) \to 0 \).

For a collapsing star, we have \( V(x, r) \leq 0 \), then we have the following estimation,

\[
\exp \left((1 + C_0^{-2}) [V(x, \tau - x) - V(0, \tau)]\right) \leq \exp \left(-(1 + C_0^{-2}) V(0, \tau)\right).
\] (4.10)

Substituting it into (4.9) we get

\[
f(\tau, 0) \geq 1 - 2\pi G (1 + C_0^2) \rho(0, \tau) \tau^2 \exp \left(-(1 + C_0^{-2}) V(0, \tau)\right).
\] (4.11)

Therefore, only if the initial distribution satisfies

\[
2\pi G (1 + C_0^2) \rho(0, r) r^2 \exp \left[-(1 + C_0^{-2}) V(0, r)\right] < 1, \quad \rho(0, 0) \leq \rho_0 < \infty,
\] (4.12)

where \( \rho_0 \) is a constant, the singularity \( \rho(\tau, 0) \to \infty \) will not occur. (4.12) is the exact form of the condition (2.29) mentioned above.
By \((4.8)\), we find even the initial distribution \(\rho(0,r) \to \rho_0 r^{-n}, (0 < n < 2)\), it will become regular sonn at \(\tau > 0\). This is a negative feedback process, so \(b(t,0) \to 0\) is impossible if the initial data is suitable.

For the ultra-relativistic gas in equilibrium

\[
V = 0, \quad \partial_t \rho = 0, \quad P = \frac{1}{3} \rho, \tag{4.13}
\]

solving \((2.17)\)-(\(2.22\)) we get one solution with singular mass density\([15]\),

\[
\rho = \frac{3}{56\pi Gr^2}, \quad b = \alpha(4r + \beta), \quad a = \frac{4r + \beta}{7r}, \tag{4.14}
\]

where \((\alpha, \beta)\) are constant. Apparently, this solution is nonphysical. If take \(\rho(t,0) = \rho_0 < \infty\), by \((1.2)\) and \((1.3)\), we can check that \(R \to \infty\) and the solution is not asymptotically flat. So it is necessary for \(\gamma > 1\) in \((1.5)\) to get a finite star.

For general critical initial distribution

\[
\rho \to \frac{\rho_0}{r^2}, \quad P \to \frac{1}{3} \rho \to \frac{\rho_0}{3r^2}, \quad (r \to 0), \tag{4.15}
\]

where \(\rho_0 > 0\) is a constant. When \(r \to 0\), we have \((a \to 1, v \to 0)\). For any given \(r_0\) satisfying \(0 < r_0 - r \ll R\), by \((3.12)\), we get

\[
\begin{align*}
b(0,r) &= \exp \left( -8\pi G \left( \int_0^R + \int_0^{r_0} \right) (\rho + P)a^{-1}e^{-2v}rdr \right) \\
&\to A_1(r_0) \exp \left( -8\pi G \int_0^{r_0} \frac{4\rho_0}{3r^2}rdr \right) \\
&= A_2r^{\frac{32\pi G \rho_0}{3}} \to 0, \quad (r \to 0), \tag{4.16}
\end{align*}
\]

in which all \(0 < A_k < \infty\) are constants. \((4.16)\) means the space-time itself becomes singular, so the condition \((4.12)\) can not be modified.

V. DISCUSSION AND CONCLUSION

The above calculation shows that, the property of the EOS of matter source is decisive to the fate of the evolving star. If the EOS satisfies the increasing and causal conditions \((1.5)\), which is quite normal for ordinary particles, the singularity might be avoided.

The equation of motion for the matter source in a star is automatically derived from the energy-momentum conservation law, so we can not arbitrarily set the mass-energy distribution. Whether sufficient matter can be collected in a little ball to form a trapped surface is the key for the reality of the singularity theorems.
As analyzed in [9, 10] and shown above, we find the center of a star is not a balancing point for particles, because the gravitational potential converts into the kinetic energy of the particles. We can not treat a big star as a solid ball. The concentrated gravitating source might be unable to locate at center, it should be distributed at some balancing surface. This phenomenon may be the reason why the vacuum solution to the Einstein equation usually has not single singular point, which is different from the basic solution of Laplace equation. So for the perfect fluid matter, the cosmic censorship principle may be right, although the explanation for the black hole may be inadequate[23]. The most external singular surface may be the boundary of valid region of the exterior vacuum solution to the metric.

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