RENORMALIZATION GROUP FLOW AND FRAGMENTATION IN THE SELF-GRAVITATING THERMAL GAS

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Abstract

The self-gravitating thermal gas (non-relativistic particles of mass \(m\) at temperature \(T\)) is exactly equivalent to a field theory with a single scalar field \(\phi(x)\) and exponential self-interaction. We build up perturbation theory around a space dependent stationary point \(\phi_0(r)\) in a finite size domain \(\delta \leq r \leq R\), \((\delta \ll R)\), which is relevant for astrophysical applications (interstellar medium, galaxy distributions). We compute the correlations of the gravitational potential \((\phi)\) and of the density and find that they scale; the latter scales as \(r^{-2}\). A rich structure emerges in the two-point correlators from the \(\phi\) fluctuations around \(\phi_0(r)\). The \(n\)-point correlators are explicitly computed to the one-loop level. The relevant effective coupling turns out to be \(\lambda = 4\pi G m^2/(T R)\). The renormalization group equations (RGE) for the \(n\)-point correlator are derived and the RG flow for the effective coupling \(\lambda(\tau)\), \(\tau = \ln(R/\delta)\), explicitly obtained. A novel dependence on \(\tau\) emerges here. \(\lambda(\tau)\) vanishes each time \(\tau\) approaches discrete values \(\tau = \tau_n = 2\pi n/\sqrt{7} - 0, n = 0, 1, 2, \ldots\) Such RG infrared stable behaviour [\(\lambda(\tau)\) decreasing with increasing \(\tau\)] is here connected with low density self-similar fractal structures fitting one into another. For scales smaller than the points \(\tau_n\), ultraviolet unstable behaviour appears which we connect to Jeans’ unstable behaviour, growing density and fragmentation. Remarkably, we get a hierarchy of scales and Jeans lengths following the geometric progression \(R_n = R_0 e^{2\pi n/\sqrt{7}} = R_0 [10.749087\ldots]^n\). A hierarchy of this type is expected for non-spherical geometries, with a rate different from \(e^{2n/\sqrt{7}}\).
I. INTRODUCTION AND RESULTS

The statistical mechanics of the self-gravitating gas is a relevant physical problem motivated by the physics of the cold interstellar medium (ISM) and the large scale galaxy distributions [1–3].

In the grand canonical ensemble, we showed [1] that the self-gravitating gas at temperature $T$ is exactly equivalent to a field theory of a single scalar field $\phi(\vec{x})$ with exponential self-interaction through the local euclidean action,

$$S[\phi] = \frac{1}{T_{\text{eff}}} \int d^3x \left[ \frac{1}{2} (\nabla \phi(x))^2 - \mu^2 e^{\phi(x)} \right].$$

(1.1)

where

$$\mu^2 = \sqrt{\frac{2}{\pi}} z G m^2 \sqrt{T}, \quad T_{\text{eff}} = 4\pi G m^2 T.$$  

Here $m$ stands for the mass of the particles, $G$ for Newton’s constant and $z$ is the fugacity of the gas. The field $\phi$ equals the gravitational potential up to a multiplicative constant. The particle density is given by

$$\rho(\vec{r}) = -\frac{1}{T_{\text{eff}}} \nabla^2 \phi(\vec{r})$$

We analyzed this field theory perturbatively and non-perturbatively through the renormalization group approach [1]. We showed scaling behaviour (critical) for a continuous range of the temperature and of the other physical parameters. We derived in this framework the scaling relation

$$M(R) \sim R^{d_H}$$

for the mass on a region of size $R$, and

$$\Delta v \sim R^q$$

for the velocity dispersion where $q = \frac{1}{2}(d_H - 1)$. For the density-density correlations we found a power-law behaviour for large distances

$$\sim |\vec{r}_1 - \vec{r}_2|^{2d_H - 6}.$$  

The fractal dimension $d_H$ is related with the critical exponent $\nu$ of the correlation length by

$$d_H = \frac{1}{\nu}.$$  

Mean field theory yields for the scaling exponents $\nu = 1/2$, $d_H = 2$ and $q = 1/2$. Such values are compatible with the present ISM observational data: $1.4 \leq d_H \leq 2$, $0.3 \leq q \leq 0.6$.

In ref. [2] we developed a field theoretical approach to the galaxy distribution. We considered a gas of self-gravitating masses in quasi-thermal equilibrium in the Friedman-Robertson-Walker background, We derived the galaxy correlations using renormalization group methods. We found that the connected $N$-points density correlator $C(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ scales as

$$\vec{r}_1^{N(D-3)}.$$
when \( r_1 \gg r_i, \ 2 \leq i \leq N \). The theory is fully predictive and there are no free parameters in it.

Our study of the statistical mechanics of a self-gravitating system indicates that gravity provides a dynamical mechanism to produce fractal structure.

Besides the constant stationary point \( \phi_0 = -\infty \) studied in ref. [1], the field theory defined by eq.(1.1) possesses a rotationally invariant and dilatation invariant stationary point

\[
\phi_0(r) = \ln \frac{2}{\mu^2 r^2}.
\]

(1.2)

Non-constant saddle points as \( \phi_0(r) \) are clearly necessary in order to describe the physics of the self-gravitating gas.

In the present paper we investigate the perturbation theory around the stationary point given by eq.(1.2) in the finite size domain between a very large sphere of radius \( R \) and a small sphere of radius \( \delta \). Such a finite domain \( \delta \leq r \leq R \) is dictated by the physics of the problem. Both in the ISM as well as for the galaxy distributions the relevant fractal domain is bounded; beyond \( R \) and below \( \delta \) other physics (besides purely gravitational) do intervene.

The stationary point \( \phi_0(r) \) may be centered at any arbitrary point \( x_0 \) inside the domain considered. That is, we must integrate over \( x_0 \) treating it as a collective coordinate [7].

We explicitly compute the propagator of the field \( \phi \) (two points correlator of the gravitational potential) in the background \( \phi_0(r) \) to zeroth order [see eq.(3.20)]. It presents a rich structure accounting for the fluctuations around \( \phi_0(r) \).

The \( n \)-point correlation functions are explicitly computed at zero momentum to the one-loop level. The perturbative series turns out to be an expansion in the effective coupling

\[
\lambda \equiv \frac{g^2}{\mu R} = \frac{4\pi G m^2}{T R} = \frac{T_{\text{eff}}}{R}.
\]

Here,

\[
g^2 = \mu T_{\text{eff}} = (8\pi)^{3/4} \sqrt{\frac{z}{G^3/2 m^{15/4}}} \frac{m^{15/4}}{T^{3/4}}
\]

(1.3)

is the dimensionless coupling constant and \( \mu^{-1} = d_J \) is the Jeans’ length.

The renormalization group equation (RGE) for \( n \)-point correlation functions is derived and its coefficients computed to the order \( \lambda \). The RGE gives the variation of the correlation functions under a variation of scale. In the present case under variations in \( R/\delta \). We take here into account the explicit dependence on the size of the system through the position of the boundaries.

The new feature in the present model with respect to the standard cases [6] is that the propagator has a non-trivial dependence on \( R/\delta \). As a result, the coefficients in the RGE have a novel non-trivial dependence on

\[
\tau \equiv \ln \frac{R}{\delta}.
\]

We explicitly obtain the RG flow of the effective coupling \( \lambda = \lambda(\tau) \). Integrating the RGE equation from a value \( \tau = \tau_i \) where \( \lambda(\tau_i) = \lambda_i \) is small we find that \( \lambda(\tau) \) decreases until it vanishes at the first integer multiple of \( 2\pi/\sqrt{7} \). That is at
τ_n = 2πn/√7, n = 0, 1, 2, . . . . (1.4)

Just after these points τ = τ_n, the one-loop approximation ceases to be valid since the one-loop λ(τ) becomes negative.

That is, we can start to run the renormalization group at τ = τ_i with a small coupling \( \lambda_i \equiv \lambda(\tau_i) \) and keep running until τ = τ_n. At this point \( \lambda(\tau_n) = 0 \). In this interval \( \tau_i, \tau_n \), the effective coupling \( \lambda(\tau) \) decreases when the spatial scale \( \tau \) increases as usually happens in scalar field theories, which is the case here (infrared stable behaviour).

We depict in fig. 1 the running coupling constant \( \lambda(\tau) \) in the intervals \( (\tau_{n-1}, \tau_n) \) for \( n = 1, 2, 3, 4, 6, 17 \) as illustrative cases.

From eq.(1.4) we see that we have found a hierarchy of scales following the geometric progression

\[
R_0 = \delta, \\
R_1 = R_0 e^{2\pi/\sqrt{7}}, \\
\ldots \ldots \\
R_n = R_0 e^{2\pi n/\sqrt{7}} = R_0 [10.749087 \ldots]^n.
\] (1.5)

We also expect hierarchies if this type for geometries different from the spherical one. Of course, the rate \( e^{2/\sqrt{7}} \) is expected to change somewhat with the geometry. In addition, \( \lambda(\tau) \) grows when \( \tau \) decreases starting from \( \tau = \tau_i \). This suggests that one enters here a strong coupling regime. That is, an ultraviolet unstable behaviour that we connect with the instability of structures and fragmentation.

As the effective coupling \( \lambda(\tau) \) grows for decreasing \( \tau \) (ultraviolet unstable behaviour), the density fluctuations grow and the Jeans length \( d_J \) decreases, that is, smaller and smaller regions become unstable. We thus get fragmentation of the original mass structure into substructures.

Remarkably, we get a hierarchy of scales and a hierarchy of Jeans lengths following the geometric progression

\[
d_{Jn} = d_{J0} e^{2\pi n/\sqrt{7}} n = 0, 1, 2, \ldots .
\]

This paper is organized as follows: in section II we summarize the field theory approach to the gravitational gas, the properties of the theory under scale transformations and the simplest relevant stationary points. In section III we build up the perturbative series around the space dependent stationary point \( \phi_0(\mathbf{r}) \). Section IV deals with the renormalization of the theory, the derivation of the RGE, the RG flow of the effective coupling \( \lambda(\tau) \), physical results and their interpretation.

II. THE FIELD THEORY APPROACH TO THE GRAVITATIONAL GAS

As shown in ref. [1,3], the statistical mechanics of a gravitational gas can be described using a continuous field theory description.
We consider a gas of nonrelativistic point particles with mass $m$ interacting only through their mutual gravitational attraction. We ignore any relativistic or quantum effect. Moreover, we assume this gas to be in thermal equilibrium at temperature $T$. We allow the number of particles $N$ to vary through an exchange with the environment. In other words we work in the grand canonical ensemble.

This system can then be described by its grand partition function, namely;

$$Z = \sum_{N=0}^{+\infty} \frac{z^N}{N!} \int \prod_{l=1}^{N} \frac{d^3p_l \, d^3q_l}{(2\pi)^3} \, e^{-\beta H_N}, \quad (2.1)$$

where $\beta = \frac{1}{T}$ and

$$H_N = \sum_{l=1}^{N} \frac{p_l^2}{2m} - Gm^2 \sum_{1 \leq l < j \leq N} \frac{1}{|q_l - q_j|}, \quad (2.2)$$

$G$ being Newton’s constant and $z$ the fugacity of the gas.

These N-body expressions can be exactly transformed into a continuum field theoretical formula [1]. The number density field is given by

$$\rho(r) = \sum_{j=1}^{N} \delta(r - q_j), \quad (2.3)$$

and the basic expression for the partition function is:

$$Z = \int D\phi \exp \left( -\frac{1}{T_{\text{eff}}} \int d^3x \left[ \frac{1}{2} (\nabla \phi)^2 - \mu^2 e^{\phi} \right] \right), \quad (2.4)$$

where

$$\mu^2 = \sqrt{\frac{2}{\pi}} \, z \, G \, m^2 \, \sqrt{T}, \quad T_{\text{eff}} = 4\pi G m^2 / T \quad \text{and} \quad \phi(x) = 2m \sqrt{\pi G / \beta} \psi(x). \quad (2.5)$$

This expression of the partition function leads to a new insight on the self-gravitating gas. Eqs. (2.1) and eq. (2.4) are exactly equivalent. Indeed, the system is exactly described by a single scalar field $\phi(x)$ with local euclidean action,

$$S[\phi] = \frac{1}{T_{\text{eff}}} \int d^3x \left[ \frac{1}{2} (\nabla \phi(x))^2 - \mu^2 e^{\phi(x)} \right]. \quad (2.6)$$

One characteristic feature of this action is that it is unbounded from below due to the sign in front of the exponential self-interaction term. This precisely reflects the presence of an attractive force like gravitation. It is then possible that some states evolve into gravitational collapse. To avoid the divergencies linked to gravitational collapse we introduce a regularizing cutoff at short distances. A simple short distance regularization of the Newtonian force for the two-body potential is [1]

$$v_a(r) = -\frac{Gm^2}{r} \left[ 1 - \theta(a - r) \right], \quad (2.7)$$
\( \theta(x) \) being the step function. The value of the lower cutoff \( a \) depends on the physical problem under consideration.

It is useful to study the mean field equation for the stationary points of the local action. It takes the form,

\[
\nabla^2 \phi_0(x) + \mu^2 e^{\phi_0(x)} = 0. \tag{2.8}
\]

We give below an equivalence table between the \( N \)-body description and the field theoretical formulation.

\[
\begin{array}{ll}
\text{N point particles} & \text{Continuous scalar field} \\
U(r) = - Gm \sum_{1 \leq i < j \leq N} \frac{1}{|q_i - q_j|} & \rightarrow \ - \frac{T}{m} \phi(r) \\
\rho(r) = \sum_{j=1}^{N} \delta(r - q_j) & \rightarrow \ - \frac{1}{T_{\text{eff}}} \nabla^2 \phi(r) \\
4\pi \frac{Gm^2}{T} & \rightarrow \ T_{\text{eff}} \tag{2.9}
\end{array}
\]

A. Behavior under scale transformation

Let us briefly study the behaviour of the exponential field theory eq.(2.6) under scale transformations. This is a preliminary step to the investigation the renormalization group in sec. IV. Scale transformations are defined by

\[
r \xrightarrow{T_\lambda} r_\lambda = \lambda r, \tag{2.10}
\]

where \( \lambda \) is a real number. Sometimes we will consider \( \lambda = 1 + \epsilon \) close to 1. Under \( T_\lambda \) the field \( \phi \) transform as

\[
\phi(r) \xrightarrow{T_\lambda} \phi_\lambda(r) = \phi(\lambda r) + \ln \lambda^2. \tag{2.11}
\]

And the classical field equation (2.8) remains.

Since \( \phi \) is the gravitational potential the addition of the constant \( \ln \lambda^2 \) does not alter the physics.

Our field theory approach can be also generalized to a \( D \) dimensional space. The lagrangian density \( \mathcal{L} \) retain the same structure with \( D \) dimensional operators (see [1] for details),

\[
\mathcal{L}[\phi] = \frac{1}{2} (\nabla_D \phi)^2 - \mu^2 e^{\phi}, \tag{2.12}
\]
where $\nabla_D$ is the $D$ dimensional gradient.

The scale transformations $T_\lambda$ do not change and the action undergoes the following transformation under $T_\lambda$,

$$S[\phi] \xrightarrow{T_\lambda} \lambda^{2-D} S[\phi]. \quad (2.13)$$

The Noether current associated with this transformation is

$$J(r) = (r \cdot \nabla \phi + 2) \nabla \phi(r) - r \mathcal{L}. \quad (2.14)$$

Its divergence in agreement with eq. (2.13), is given by

$$\nabla J = (2 - D) \mathcal{L}. \quad (2.15)$$

We did not expect a cancellation of this divergence since the action is not invariant under $T_\lambda$ except in two space dimensions. One of the important features of this study is that, despite the presence of a characteristic length $\mu$ in the lagrangian density, the action scales under a scale transformation.

**B. Stationary points**

We consider the self-gravitating gas in a large but finite spherical box of radius $R$. In addition, we introduce a short distance cutoff at distance $r = \delta$. This cutoff $\delta \sim a$ is the same as the short-distance cutoff in the gravitational force [eq.(2.7)].

In order to extract information from the functional integral eq.(2.4) we shall search for stationary points of the action eq.(2.6).

Let $\phi_0$ be a stationary point of the action. This means that $\phi_0$ is solution of the equation

$$\nabla^2 \phi(r) + \mu^2 e^{\phi(r)} = 0 \quad (2.16)$$

It is interesting to notice that the dilated field $\phi_\lambda$ is also a stationary point.

The most simple stationary point is $\phi_0(r) = -\infty$, \quad (2.17)

which describes empty space. This is a translationally invariant stationary point. It has been studied in ref. 1.

Let us now study the non-translationally invariant but rotationally invariant stationary point

$$\phi_0(r) = \ln \frac{2}{\mu^2 r^2}. \quad (2.18)$$

This solution is also dilatation invariant under the transformation (2.11).

Less symmetrical stationary points certainly exist but their calculation would require numerical investigation of the stationary field equation.
C. Hydrostatic interpretation of the stationary points

We recall here that there is a straightforward correspondence between our stationary point equation (2.8) and the equations for the self-gravitating fluid in hydrostatic equilibrium. In terms of the gravitational potential $U(x)$ [see eq.(2.9)], eq.(2.8) takes the form

$$\nabla^2 U(r) = 4\pi G z m \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{mT}{2\pi} U(r)} .$$

This corresponds to the Poisson equation for a thermal matter distribution fulfilling the hydrostatic equilibrium of an ideal gas. The hydrostatic equilibrium condition is $\nabla P(r) = -m \rho(r) \nabla U(r)$, where $P(r)$ stands for the pressure, and the equation of state for the ideal gas $P = T \rho$ yields for the particle density $\rho(r) = \rho_0 e^{-\frac{mT}{2\pi} U(r)}$, where $\rho_0$ is a constant. Inserting this relation into the Poisson equation $\nabla^2 U(r) = 4\pi G m \rho(r)$ yields eq.(2.19) with

$$\rho_0 = z \left( \frac{mT}{2\pi} \right)^{3/2} .$$

The solution $\phi_0$ given by eq.(2.18) is singular at $r = 0$. However, such short distance singularity is inessential for the long distance properties we are investigating. There exist also spherically symmetric regular solutions known as the isothermal sphere [4].

III. PERTURBATION THEORY AROUND A NON CONSTANT STATIONARY POINT

We will build our perturbation theory around the stationary point $\phi_0$ which is invariant under rotations and dilations. That is,

$$\phi_0(x - x_0) = \ln \left( \frac{2}{\mu^2 |x - x_0|^2} \right) .$$

This stationary point is centered in an arbitrary point $x_0$.

A. Construction of the perturbation series

As usual in field theory, we will introduce an external source $\sqrt{\mu} \frac{g}{T} J(x)$ in order to define the correlation functions:

$$S[\phi] = \frac{1}{T_{eff}} \int d^3x \left[ \frac{1}{2} (\nabla_x \phi)^2 - \mu^2 e^\phi \right] - \int d^3x \sqrt{\mu} \frac{g}{T} J(x) \phi(x) .$$

$J(x)$ can be interpreted as a test-mass density. Anyway, one sets it equal to zero when computing physical quantities.

Let us now perturb around the stationary point (2.18)

$$\phi(x) = \phi_0(x - x_0) + \frac{g}{\sqrt{\mu}} \chi(x - x_0) ,$$

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where
\[ g \equiv \sqrt{\mu T_{\text{eff}}} \]  
(3.4)
is the dimensionless coupling constant and \( \chi(x) \) is, of course, the fluctuation. The action can be then split into several terms,
\[ S[\phi] = S[\phi_0] + S_2[g\chi] + S_I[g\chi] - \int d^3x \ J(x) \ \chi(x), \]  
(3.5)
where \( S_2 \) stands for the quadratic action and \( S_I \) for higher order couplings,
\[ S_2[\chi] = \frac{1}{2} \int d^3x \ (\nabla \chi(x))^2 - \mu^2 e^{\phi_0(x)} \ \chi^2(x), \]  
(3.6)
and
\[ S_I[\chi] = -\int d^3x \ \mu^2 e^{\phi_0(x)} \sum_{n=3}^{\infty} \frac{g^{n-2}}{\sqrt{\mu^{n-2}}} \frac{\chi^n(x)}{n!}. \]  
(3.7)

We have a non-polynomial interaction action, hence we shall have vertices with any number of legs in the Feynman graphs.

We compute now the field propagator in the background of the solution \( \phi_0(x) \) as the inverse of the quadratic form in \( S_2 \).

**B. Calculation of the propagator \( \mathcal{G} \) in the \( \phi_0 \) background**

We read the inverse propagator \( \mathcal{G}^{-1} \) from the quadratic action \( S_2 \),
\[ \mathcal{G}^{-1}(x - x_0, y - x_0) = \delta(x - y) \left[ -\nabla^2_{x-x_0} - \mu^2 e^{\phi_0(|x-x_0|)} \right]. \]  
(3.8)
The propagator \( \mathcal{G} \) is defined by the inversion relation:
\[ \int d^3z \ \mathcal{G}^{-1}(x - x_0, z - x_0) \mathcal{G}(z - x_0, y - x_0) = \delta^3(x - y). \]  
(3.9)
We can recast this definition as a partial differential equation,
\[ \left[ -\nabla^2_{x-x_0} - \mu^2 e^{\phi_0(|x-x_0|)} \right] \mathcal{G}(x - x_0, y - x_0) = \delta^3(x - y). \]  
(3.10)
Using the explicit form of \( \phi_0 \) and the notation \( r = x - x_0; r' = y - x_0 \), we find
\[ \left[ -\nabla^2 - \frac{2}{r^2} \right] \mathcal{G}(r, r') = \delta(r - r'). \]  
(3.11)
It is convenient to expand the correlation function in partial waves
\[ \mathcal{G}(r, r') = \sum_{l=0}^{\infty} g_l(r, r') \sum_{m=-l}^{m=l} Y_{l,m}(\theta, \phi) Y^*_{l,m}(\theta', \phi') \]  
(3.12)
where \( Y_{l,m}(\theta, \phi) \) are harmonic spherics and the function \( g_l(r, r') \) obey the ordinary differential equation.
\[
\left[-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{l(l+1) - 2}{r^2}\right] g_l(r, r') = \frac{1}{r^2} \delta(r - r') .
\] (3.13)

The general solution of the homogeneous differential equation

\[
\left[-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{l(l+1) - 2}{r^2}\right] y_l(r) = 0
\] (3.14)

can be written as

\[y_l(r) = \frac{1}{\sqrt{r}} \left( A_l \, r^{\alpha_l} + B_l \, r^{-\alpha_l} \right)\]

where \(A_l\) and \(B_l\) are arbitrary constants and

\[\alpha_l = \sqrt{(l + \frac{1}{2})^2 - 2} = \sqrt{l^2 + l - \frac{7}{4}} .\] (3.15)

As stated in the previous section we consider the gas in a finite volume of radius \(R\) with a short distance cutoff \(\delta \sim a\). Therefore, we need solutions \(g_l(r, r')\) in the interval

\[\delta \leq r \leq R, \; \delta \leq r' \leq R .\]

The radial propagator \(g_l(r, r')\) can be then expressed in terms of the solutions of the homogeneous equation (3.14) as follows:

\[g_l(r,r') = \frac{1}{r^2 \, W[y_l^<, y_l^>] } \, y_l^<(r_\leq) \, y_l^>(r_\geq) .\] (3.16)

where

\[r_\leq \equiv \min(r, r'), \; r_\geq \equiv \max(r, r') ,\]
\(y_l^<(r)\) obeys the boundary conditions appropriate for \(r = \delta\) and \(y_l^>(r)\) obeys the boundary conditions appropriate for \(r = R\). \(W[y_l^>, y_l^<]\) stands for the wronskian. Notice that \(r^2 \, W[y_l^>, y_l^<]\) is \(r\)-independent.

The simplest choice of boundary conditions is to set the fluctuations equal to zero both at \(r = \delta\) and at \(r = R\). That is,

\[y_l^<(\delta) = 0 \; , \; y_l^>(R) = 0 .\] (3.17)

The radial part of the propagator is now fully determined, it takes the form,

\[g_l(r, r') = -\frac{1}{\alpha_l \text{sh} (\alpha_l \tau)} \, \text{sh} \left[ \alpha_l \ln \left( \frac{r_\leq}{\delta} \right) \right] \, \text{sh} \left[ \alpha_l \ln \left( \frac{r_\geq}{R} \right) \right] ,\] (3.18)

where

\[\tau \equiv \ln \frac{R}{\delta} .\]

Notice that for \(l = 0\), \(\alpha_0\) being [eq.(3.13)] an imaginary number, the hyperbolic sinus turn into trigonometric sines. This is an important point since S-waves produce the leading contribution in loop calculations as we shall see in the subsequent sections.
\[ g_0(r, r') = -\frac{1}{\sqrt{\pi}} \sin \left( \frac{\sqrt{7}}{2} \tau \right) \frac{1}{\sqrt{rr'}} \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^<}{\delta} \right) \right] \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^>}{R} \right) \right], \quad (3.19) \]

We finally get for the full propagator:

\[
\mathcal{G}(r, r') = \frac{1}{4\pi\sqrt{rr'}} \left\{ -\frac{2}{\sqrt{\pi} \sin \left( \frac{\sqrt{7}}{2} \tau \right)} \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^<}{\delta} \right) \right] \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^>}{R} \right) \right] \right. \\
- \left. \sum_{l \geq 1} \frac{2l + 1}{\alpha_l \sinh(\alpha_l \tau)} \sin \left[ \alpha_l \ln \left( \frac{r^>}{R} \right) \right] \sin \left[ \alpha_l \ln \left( \frac{r^<}{\delta} \right) \right] P_l(\cos \gamma) \right\}. \quad (3.20)
\]

where we used the relation

\[
\sum_m Y_{l,m}(\theta, \phi) Y^*_{l,m}(\theta', \phi') = \frac{2l + 1}{4\pi} P_l(\cos \gamma).
\]

and \( P_l(\cos \gamma) \) stand for Legendre polynomials.

We can see that the propagator (3.25) presents a rich structure. It takes into account the behaviour of the fluctuations around the background \( \phi_0 \). Notice that the propagator (3.25) is symmetric under the exchange of \( r \) and \( r' \).

As one sees from eq.(3.25) for distances such as \( \delta \ll r \sim r' \ll R \) the two-point function of the gravitational potential scales as \( \sim 1/\sqrt{rr'} \) at the tree level.

Besides if \( \delta \ll r \ll r' \ll R \) the propagator \( \mathcal{G}(r, r') \) is dominated by the S-wave. That is,

\[
\mathcal{G}(r, r')^{r, r' \gg \delta} = \frac{1}{2\pi\sqrt{7} \sqrt{rr'}} \sin \left( \frac{\sqrt{7}}{2} \tau \right) \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^<}{\delta} \right) \right] \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^>}{R} \right) \right]
\]

C. Perturbation theory, Generating functionals and Feynman graphs

We construct now the perturbative series around the stationary point \( \phi_0 \). We use standard perturbation theory methods [1]. We shall give the main steps and the new features which appear for our model.

1. Construction of the general Green functions

We can write the generating functional of disconnected correlations as follows

\[ Z[J] = \int \frac{d^3x_0}{N} Z_{x_0}[J] \quad (3.21) \]

\[ Z_{x_0}[J] = e^{\int d^3x_0 \phi_0(x-x_0)J(x)} \frac{e^{-S_2[\chi]-S_1[\chi]+g \int d^3x J(x) \chi(x-x_0)}}{Z[\phi_0]} \quad (3.22) \]

[The name disconnected arises because topologically disconnected Feynman graphs contribute to the perturbative series of \( Z[J] \).]

Here we used that \( S[\phi_0] = 0 \) in three space dimensions. Notice first that the generating functional is now a functional of \( J(.) \). For \( J = 0 \) we recover the previous definition eq.(2.1).
As we have already mentioned, \( \phi_0(\mathbf{x} - \mathbf{x}_0) \) is a stationary point for any \( \mathbf{x}_0 \). We have therefore to integrate over \( \mathbf{x}_0 \). That is, \( \mathbf{x}_0 \) must be treated as a so called collective coordinate \( \mathbf{0} \). Such integration over \( \mathbf{x}_0 \) must be computed exactly since all points \( \mathbf{x}_0 \) contribute with similar weight due to the translational invariance within the large sphere of radius \( R \). In order to avoid double counting, the domain of functional integration for the field \( \chi(\mathbf{x}) \) must be orthogonal to the linear space of the zero modes. That is, we have to impose the constraints (3.23) implies that we must exclude the \( \ell = 1 \) term in the partial wave series. We shall denote from now on,

\[
\mathcal{N} = \left[ \frac{1}{6\pi} \int d^3 \mathbf{r} (\nabla \phi_0)^2 \right]^{\frac{3}{2}} = \left[ \frac{8}{3} (R - \delta) \right]^{\frac{3}{2}}. \tag{3.24}
\]

Since \( \phi_0(x) \) is rotationally invariant, \( \partial_x \phi_0(x) \), \( i = 1, 2, 3 \) can be expressed as linear combinations of the \( \ell = 1 \) modes of the previous section. Namely, \( y_1(r) Y_{1,\pm 1}(\theta, \phi) \) and \( y_1(r) Y_{1,0}(\theta, \phi) \). The constraint (3.23) implies that we must exclude the \( \ell = 1 \) term in the partial wave series. We shall denote from now on,

\[
\mathcal{G}(\mathbf{r}, \mathbf{r'}) = \frac{1}{4\pi\sqrt{r \cdot r'}} \left\{ \frac{2}{\sqrt{\sin (\sqrt{2} \tau)}} \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^\leq}{\delta} \right) \right] \sin \left[ \frac{\sqrt{7}}{2} \ln \left( \frac{r^\geq}{R} \right) \right] \right. \\
- \sum_{l \geq 2} \frac{2l+1}{\alpha_l \text{sh}(\alpha_l \tau)} \text{sh} \left[ \alpha_l \ln \left( \frac{r^\geq}{R} \right) \right] \text{sh} \left[ \alpha_l \ln \left( \frac{r^\leq}{\delta} \right) \right] P_l(\cos \gamma) \left\}. \tag{3.25}
\]

It is convenient to split the exponent in eq. (3.22) in two parts, separating the interaction part of the action from the gaussian part that can be easily integrated,

\[
\int D\chi \ e^{-\frac{1}{4} \int d^3 x \chi \hat{G}^{-1} \chi + \int d^3 x \ J(\mathbf{x}) \chi(\mathbf{x} - \mathbf{x}_0)} = \sqrt{\text{det} \mathcal{G}} \ e^{\frac{1}{2} \int d^3 x \ d^3 y \ J(\mathbf{x}) \mathcal{G}(\mathbf{x} - \mathbf{x}_0, \mathbf{y} - \mathbf{x}_0) \ J(\mathbf{y})}. \tag{3.26}
\]

Using this result and expanding exponentials, one gets after calculation,

\[
Z_{\mathbf{x}_0}[J] = \sqrt{\text{det} \mathcal{G}} \ e^{\frac{\mathcal{G}^T}{2} \mathcal{J}_0} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -S_l \left[ \frac{\delta}{\delta J(\mathbf{x})} \right] \right)^n \\
\times \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{2} \int \int d^3 x \ d^3 y \ J(\mathbf{x}) \mathcal{G}(\mathbf{x} - \mathbf{x}_0, \mathbf{y} - \mathbf{x}_0) \ J(\mathbf{y}) \right)^k. \tag{3.27}
\]

\( Z_{\mathbf{x}_0}[J] \) is thus expressed as a power series of \( J \). The coefficients of the \( n \)-th degree term is the disconnected \( n \) points correlation function of the field \( \phi \) (i.e the Green’s function). That is,

\[
G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{1}{Z_{\mathbf{x}_0}[0]} \left. \frac{\delta^n Z_{\mathbf{x}_0}[J]}{\delta J(\mathbf{x}_1) \ldots \delta J(\mathbf{x}_n)} \right|_{J=0}. \tag{3.28}
\]

We will compute \( G^{(2)} \) and \( G^{(1)} \) to the first orders in \( g \) in terms of Feynman diagrams.
Let us first compute the sum of “vacuum-vacuum” diagrams (without external sources $J$) $Z_{x_0}[0]$.

\[
Z_{x_0}[0] = 1 + \frac{g^2}{8\mu} \int d^3x \mu^2 e^{\phi_0(x_{x_0})} [G(x_{x_0}, x_{x_0})]^2 \\
- \frac{g^4}{32\mu^2} \int d^3x \mu^2 e^{\phi_0(x_{x_0})} [G(x_{x_0}, x_{x_0})]^3 + \mathcal{O}(g^6)
\]  

(3.29)

Here we use the notation $r_{x_0} = r - x_0$. Then, the one and two points correlation functions can be written as,

\[
G^{(1)}_{x_0}(x) = \phi_0(x_{x_0}) + \frac{1}{Z_{x_0}[0]} g^{(1)}_\chi(x_{x_0}) ,
\]

(3.30)

\[
G^{(2)}_{x_0}(x, y) = \phi_0(x_{x_0})\phi_0(y_{x_0}) + \frac{1}{Z_{x_0}[0]} \left[ \phi_0(x_{x_0})g^{(1)}_\chi(y_{x_0}) + \phi_0(y_{x_0})g^{(1)}_\chi(x_{x_0}) + g^{(2)}_\chi(x_{x_0}, y_{x_0}) \right] ,
\]

(3.31)

where,

\[
g^{(1)}_\chi(y_{x_0}) = -\frac{g}{\sqrt{\mu}} \int d^3x \mu^2 e^{\phi_0(x_{x_0})} G(x_{x_0}, x_{x_0}) G(x_{x_0}, y_{x_0})
+ \frac{g^3}{8\mu^{3/2}} \int d^3x \mu^2 e^{\phi_0(x_{x_0})} (G(x_{x_0}, x_{x_0}))^2 G(x_{x_0}, y_{x_0}) + \ldots
\]

(3.32)

\[
g^{(2)}_\chi(x_{x_0}, y_{x_0}) = G(x_{x_0}, y_{x_0}) + \frac{g^2}{2\mu} \int d^3z \mu^2 e^{\phi_0(z_{x_0})} G(x_{x_0}, z_{x_0}) G(z_{x_0}, z_{x_0}) G(z_{x_0}, y_{x_0}) + \ldots
\]

(3.33)

The $g^{(n)}_\chi$ can be straightforwardly expressed in terms of Feynman diagrams.

\[
Z_{x_0}[0] = \left\{ 1 + \bigcirc \bigcirc + \ldots \right\}
\]

\[
g^{(1)}_\chi = \left\{ \bigcirc \bigcirc \bigcirc \bigcirc + \ldots \right\}
\]
\[ g^{(2)}_\chi = \left\{ \begin{array}{c} \quad + \quad \end{array} \right\} \]

From these diagrams one can recover the previous integrals using the following Feynman rules:

- For each line: \( g(x_{x_0}, y_{x_0}) \),
- For a \( n \)-leg vertex: \( + \left( \frac{g}{\sqrt{\mu}} \right)^{n-2} \int d^3x \mu^2 e^{\phi_0(x_{x_0})} \),
- The diagram symmetry factor.

We consider now the connected correlation functions which are simpler than the disconnected ones.

### 3. Connected Green functions and vertex functions

The generating functional of connected correlations is given by

\[
W_{x_0}[J] = \ln Z_{x_0}[J].
\]  (3.34)

This is also a series in \( J \) whose coefficients are the connected Green functions,

\[
W^{(n)}_{x_0}(x_1, \ldots, x_n) = \left. \frac{\delta^n W_{x_0}[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0}.
\]  (3.35)

The name connected arises because only connected Feynman diagrams (in the topological sense) contribute to the perturbative expansion of the \( W^{(n)}_{x_0} \).

The expectation value of the field is given by the one-point function,

\[
\chi(x) = \frac{\delta W_{x_0}[J]}{\delta J(x)}.
\]  (3.36)

The generating functional of the vertices (also called 1-particle irreducible) functions is defined by the Legendre transformation,

\[
\Gamma_{x_0}[\chi] + W_{x_0}[J] = \int d^3x \chi J
\]  (3.37)

Then, it is easy to derive the inverse relation

\[
J(x) = \frac{\delta \Gamma_{x_0}[\chi]}{\delta \chi(x)}.
\]  (3.38)

The 1-particle irreducible vertex functions are given as

\[
\Gamma^{(n)}_{x_0}(x_1, \ldots, x_n) = \left. \frac{\delta^n \Gamma_{x_0}[\chi]}{\delta \chi(x_1) \ldots \delta \chi(x_n)} \right|_{J=0},
\]  (3.39)
Notice that the derivatives are taken at \( J = 0 \), that is \( \nabla = W^{(1)}_{x_0} \). So, the vertex functions are the coefficients of the development of \( \Gamma_{x_0} \) around a non-zero field.

The vertex functions are related to the connected Green functions through,

\[
\Gamma^{(1)}_{x_0}(x) = 0, \\
\Gamma^{(2)}_{x_0}(x, y) = \left[W^{(2)}_{x_0}\right]^{-1}(x, y), \\
\Gamma^{(3)}_{x_0}(x, y, z) = -\int d^3x_1 \, d^3y_1 \, d^3z_1 \, \Gamma^{(2)}_{x_0}(x, x_1) \, \Gamma^{(2)}_{x_0}(y, y_1) \, \Gamma^{(2)}_{x_0}(z, z_1) \, W^{(3)}_{x_0}(x_1, y_1, z_1).
\]

Only one-particle irreducible Feynman diagrams contribute to the vertex functions, that is, diagrams which cannot be disconnected by a single cut of an internal line. In addition, external lines are amputated to these diagrams.

The knowledge of \( \Gamma^{(2)} \) is crucial since it is the amputating operator. It can be expressed as,

\[
\Gamma^{(2)}_{x_0}(x, y) = G^{-1}(x, y) - \Sigma(x, y),
\]

where \( \Sigma(x, y) \) is the sum of one-particle irreducible diagrams with two amputated external lines. In our theory,

\[
\Sigma(x, y) = \frac{1}{2} \mu^2 e^{\phi_0(x)} G(x, x) \delta(x - y) \frac{g^2}{\mu} + \frac{1}{2} \mu^2 e^{\phi_0(x)} (G(x, x))^2 \mu^2 e^{\phi_0(y)} \frac{g^2}{\mu} + \frac{3}{2} \mu^2 e^{\phi_0(x)} (G(x, x))^2 \delta(x - y) \frac{g^4}{\mu^2} + \ldots
\]

or, with diagrams,

\[
\Sigma(x, y) = \{ \text{ } + + \text{ } \}
\]

4. Computation of \( \Gamma^{(2)} \) and \( \Gamma^{(3)} \)

We consider in this section \( \Gamma^{(2)} \) and \( \Gamma^{(3)} \). These are the relevant Green functions for renormalization.

\[
\Gamma^{(2)}_{x_0} = \{ \text{ } - \text{ } - \text{ } \}
\]

\[
\Gamma^{(3)}_{x_0} = -
\]
where the simple left arrow indicates the inverse propagator. Stumps are meant to remind of the amputated external lines. From these diagrams we can write the integrals applying our Feynman rules. We have finally to integrate over the collective coordinate $x_0$. Moreover, choosing to integrate over all but one of the vertex functions variables, yields more compact results and keep all the information about the renormalization. We display here the results for the first diagrams, where we use the notation

$$\tau = + \ln(R/\delta) ,$$

As ultraviolet regularization we use a cutoff in the angular momentum sum over $l$. That is, in each propagator we sum up to a maximum $L$. Since $l \sim kr$, we use that the momentum cutoff is $k \leq 1/\delta$ and the size cutoff is $r \leq R$. Therefore, we take

$$l < L = R/\delta = e^\tau$$

in each propagator.

$$\langle \square \rangle = C_1 \frac{g^2}{\mu} \int d^4k \int d^3x_0 \delta(x - y) \mathcal{G}(x_{0x}, x_{0x}) \mu^2 e^{\phi_0(x_{0x})}$$

$$= \frac{g^2}{\mu} \left[ \tau \sum_{l=0,l \neq 1}^{e^\tau} \frac{2l + 1}{\alpha_l} \frac{\text{ch}(\alpha_l \tau)}{\text{sh}(\alpha_l \tau)} - \sum_{l=0,l \neq 1}^{e^\tau} \frac{2l + 1}{\alpha_l^2} \right]$$

$$\langle \square \rangle = + C_2 \frac{g^2}{\mu} \int d^4k \int d^3x_0 \mathcal{G}^2(x_{0x}, y_{0x}) \mu^2 e^{\phi_0(x_{0x})} \mu^2 e^{\phi_0(y_{0x})}$$

$$= \frac{g^2}{\mu} \left[ \frac{\tau^2}{2} \sum_{l=0,l \neq 1}^{e^\tau} \frac{2l + 1}{\alpha_l^2} \frac{\text{sh}(\alpha_l \tau)}{\text{ch}(\alpha_l \tau)} + \tau \sum_{l=0,l \neq 1}^{e^\tau} \frac{(2l + 1) \text{ch}(\alpha_l \tau)}{\alpha_l^3 \text{sh}(\alpha_l \tau)} - \sum_{l=0,l \neq 1}^{e^\tau} \frac{2l + 1}{\alpha_l^4} \right]$$

$$\langle \nabla \rangle = C_5 \left( \frac{g}{\sqrt{\mu}} \right)^3 \int d^4y \int d^3x \int d^3x_0 \mathcal{G}(x_{0x}, y_{0x}) \mathcal{G}(y_{0x}, z_{0x}) \mathcal{G}(z_{0x}, x_{0x})$$

$$\times \mu^2 e^{\phi_0(x_{0x})} \mu^2 e^{\phi_0(y_{0x})} \mu^2 e^{\phi_0(z_{0x})}$$

$$= \left( \frac{g}{\sqrt{\mu}} \right)^3 \left[ \tau^3 \sum_{l=0,l \neq 1}^{e^\tau} \frac{(2l + 1) \text{ch}(\alpha_l \tau)}{\alpha_l^3 \text{sh}(\alpha_l \tau)} + \frac{3\tau^2}{2} \sum_{l=0,l \neq 1}^{e^\tau} \frac{2l + 1}{\alpha_l^2 \text{sh}(\alpha_l \tau)}$$

$$+ \frac{3\tau}{2} \sum_{l=0,l \neq 1}^{e^\tau} \frac{(2l + 1) \text{ch}(\alpha_l \tau)}{\alpha_l^3 \text{sh}(\alpha_l \tau)} - 4 \sum_{l=0,l \neq 1}^{e^\tau} \frac{2l + 1}{\alpha_l^4} \right]$$
The \( C_i \) factors are the symmetry factors of the diagrams. \( C_3 \) is the factor for the tadpole with three amputated external lines instead of two, and \( C_4 \) is the eye-like diagram with two amputated lines on one side and one on the other. Their values are,

\[
C_1 = C_2 = C_3 = \frac{1}{2}, \quad C_4 = \frac{3}{2} \quad \text{and} \quad C_5 = 1.
\]

The only averaged diagrams left to compute, at this order of approximation, are the most simple ones, the vertex and the inverse propagator. They yield

\[
\int_V d^3x_0 \int_V d^3x \ G^{-1}(x_{x_0}, y_{x_0}) = -8\pi (R - \delta), \tag{3.42}
\]

\[
\int_V d^3x_0 \int_V d^3x \int_V d^3y \ \delta(x - y)\delta(y - z) \ \frac{g}{\sqrt{\mu}} e^{\phi_0(x_{x_0})} = 8\pi (R - \delta) \frac{g}{\sqrt{\mu}}, \tag{3.43}
\]

where \( V \) is the volume of the region between the spheres with radii \( \delta \) and \( R \). The expansion of the zero-momentum vertex functions take the form,

\[
\langle \Gamma^{(2)} \rangle = -\frac{8\pi R}{N} \left\{ 1 - e^{-\tau} + \left( \frac{g^2}{\mu R} \right) \frac{1}{8\pi} \left[ +\tau^2 \sum_{l=0, l \neq 1}^{\infty} \frac{2l + 1}{2\alpha_l^2 \sinh(\alpha_l \tau)} \right] \right\}
\]

\[
\langle \Gamma^{(3)} \rangle = -\frac{8\pi R^2}{N} \left\{ 1 - e^{-\tau} + \left( \frac{g^2}{\mu R} \right) \frac{1}{8\pi} \left[ +\tau^3 \sum_{l=0, l \neq 1}^{\infty} \frac{(2l + 1) \sinh(\alpha_l \tau)}{\alpha_l^3 \sinh^3(\alpha_l \tau)} \right] \right\}
\]

\[
+ \tau^2 \sum_{l=0, l \neq 1}^{\infty} \frac{3(2l + 1)}{2\alpha_l^4 \sinh^2(\alpha_l \tau)} (1 + \alpha_l^2) + \tau \sum_{l=0, l \neq 1}^{\infty} \frac{(2l + 1) \sinh(\alpha_l \tau)}{2\alpha_l^5 \sinh(\alpha_l \tau)} (3 + 3\alpha_l^2 + \alpha_l^4)
\]

\[
- \sum_{l=0, l \neq 1}^{\infty} \frac{2l + 1}{2\alpha_l^6} (8 + 6\alpha_l^2 + \alpha_l^4) \right\} \right\} \tag{3.44}
\]

We have now all the quantities to renormalize the coupling constant at one loop order. It must be noticed that the perturbative expansion turns out to be in powers of the effective coupling

\[
\lambda \equiv \frac{g^2}{\mu R}. \tag{3.45}
\]

This fact will be very important for the behaviour of the physical quantities.

5. Zero-momentum value of the \( \Gamma^{(n)} \)

Let us investigate the \( n \) points vertex functions. We can write \( \Gamma_{x_0}^{(n)} \) at first loop order as a sum of rather simple diagrams,
\[ \Gamma^{(n)}_{x_0} = \{- \quad + \quad + \quad + \quad + \quad + \ldots \} \]

In this summation we should also include all the diagrams deduced from those drawn here by topological permutations of the vertex within a given diagram. This would create families of diagrams, with each member of a single family containing \( k_i \) \( i \)-legged vertex. However, all members of a given family have the same zero-momenta value (this can be checked on the corresponding integrals).

It is enough for renormalization to compute the diagrams at zero-momenta. Each diagram drawn above will stand for the whole family and will be assigned the sum \( C_{k_3, \ldots, k_{n+2}} \) of the symmetry coefficients of the members of the family. The averaged zero-momenta value of a representative diagram will be noted \( \langle \{ k_3, \ldots, k_{n+2} \} \rangle \). We proceed now to compute this value.

Let us call \( v = \sum_i k_i \) the number of vertex in a diagram. Then,

\[ \sum_j j k_j = n + 2v. \quad (3.46) \]

We can now recast the diagrams into integrals using Feynman rules and eq.(3.46),

\[ \langle \{ k_3, \ldots, k_{n+2} \} \rangle = C_{k_3, k_4, \ldots, k_{n+2}} \left( \frac{g}{\sqrt{\mu}} \right)^n \int d^3x_0 d^3x_1 \ldots d^3x_{v-1} G(x_1x_0, x_2x_0) \ldots G(x_vx_0, x_1x_0) \]

\[ \times \mu^2 e^{\phi(x_1x_0)} \ldots \mu^2 e^{\phi(x_v)}. \quad (3.47) \]

The sum \( C_{k_3, k_4, \ldots, k_{n+2}} \) is actually easier to compute than each single symmetry coefficient. It can be written as:

\[ C_{k_3, k_4, \ldots, k_{n+2}} = \frac{n!v!}{2v k_3! k_4! \ldots k_{n+2}} \frac{k_j}{\prod_{j=3}^{n+2} (j-2)!}. \quad (3.48) \]

Inserting the explicit propagator in eq.(3.47) and integrating over the angular variables yields:

\[ \langle \{ k_3, k_4, \ldots, k_{n+2} \} \rangle = C_{k_3, k_4, \ldots, k_{n+2}} 2^v \sum_{l=0, l \neq 1}^{v} \int dr_1 \ldots dr_v g_l(r_1, r_2) \ldots g_l(r_v, r_1) \]

\[ = C_{k_3, k_4, \ldots, k_{n+2}} 2^v \sum_{l=0, l \neq 1}^{v} \frac{\omega_v(\alpha_l \tau)}{\alpha_l^v \text{sh}^v(-\alpha_l \tau)} \quad (3.49) \]

where

\[ \omega_v(x) = v! \int_0^x dr_1 \int_0^{r_1} dr_2 \ldots \int_0^{r_{v-1}} dr_v \text{sh}^2(r_1 + x) \prod_{k=2}^{v-1} \text{sh}(r_k) \text{sh}(r_k + x) \text{sh}^2(r_v). \quad (3.50) \]
The coefficients $\omega_v$ can be directly calculated with formal calculus softwares (we have done it for $v \leq 5$). In addition, it can be established that they obey the following recurrence formula:

$$\omega_v(x) = \frac{2^v (\text{ch}(x) - 1)^v}{x^v} \left\{ (-f_1(x))^v f_v(x) + \sum_{i=1}^{v-1} (-1)^{i+1} \frac{\omega_{v-i}(x) x^{v-i}}{2^{v-i} (\text{ch}(x) - 1)^{v-i}} \right. $$

$$\left. - (-f_1(x))^{v-i} f_{v-i}(x) f_{i+1}(x) f_1^{i-1}(x) \right\} - (-1)^{n} f_{v+1}(x) f_1^{v-1}(x) \right\}$$

where,

$$f_1(x) = \frac{x}{\tanh(\frac{\pi}{2})},$$

$$f_{n+1}(x) = f_n(x) - \frac{x}{2n} f'_n(x).$$

This recurrence relation allows us to compute the $\Gamma^{(n)}$ without performing integrations.

$$\langle \Gamma^{(n)} \rangle = \sum_{\{k_i, \sum_j (j-1)k_j = v\}} C_{k_3,\ldots,k_{n+2}} 2^v \sum_{l=0, l\neq 1}^{e_r} \omega_v(\alpha_{l\tau}) \left( \frac{\omega_v(\alpha_{l\tau})}{\alpha_l^{v-1} \text{sh}(\alpha_{l\tau})} \right)$$

(3.51)

However, as far as renormalization is concerned, we need only the leading order of the $\Gamma^{(n)}$ in $\tau$. We can first establish that:

$$\langle \{k_3, \ldots, k_{n+2}\} \rangle = C_{k_3,\ldots,k_{n+2}} \left[ \tau^v \sum_{l=0, l\neq 1}^{e_r} \left( \frac{2l + 1}{\alpha_l^{v-1} \text{sh}(\alpha_{l\tau})} \right) \right. $$

$$\left. + \tau^{v-1} \frac{v}{2} \sum_{l=0, l\neq 1}^{e_r} \left( \frac{2l + 1}{\alpha_l^{v-1} \text{sh}(\alpha_{l\tau})} \right) + \mathcal{O}(\tau^{v-2}) \right] $$

(3.52)

Accordingly, only the diagrams with $n$ and $n-1$ vertex contribute to the two leading orders of $\Gamma^{(n)}$ in $\tau$. Since $C_{n,0,0,\ldots,0} = \frac{(n-1)!}{2}$ and $C_{n-2,1,0,\ldots,0} = \frac{n!}{4}$, we find the explicit formula,

$$\langle \Gamma^{(n)} \rangle = -\frac{8\pi R^2}{N} \left( \frac{g}{\sqrt{\mu R}} \right)^{n-2} \left\{ 1 - e^{-\tau} + \left( \frac{g^2}{\mu R} \right) \frac{1}{8\pi} \left[ \frac{(n-1)!}{2} \sum_{l=0, l\neq 1}^{e_r} \frac{(2l + 1) \text{ch}(\alpha_{l\tau})^{n-2}(\alpha_{l\tau})}{\alpha_l^{n-1} \text{sh}(\alpha_{l\tau})} \right] \right\}$$

$$+ \frac{n!}{4} \tau^{n-1} \left\{ \sum_{l=0, l\neq 1}^{e_r} \frac{(2l + 1) \text{ch}^{n-3}(\alpha_{l\tau})(1 + \alpha_l^2)}{\alpha_l^{n-1} \text{sh}(\alpha_{l\tau})} \right\} + \mathcal{O}(\tau^{n-2})$$

Before we turn to renormalization let us study the density correlation functions.

**D. Density correlations**

The two basic local fields here are the gravitational potential $\phi(x)$ and the particle density defined as in eq.(2.5),

$$\rho(\vec{x}) = -\frac{1}{T_{\text{eff}}} \nabla^2 \phi(\vec{x}) .$$
The correlation functions of the local density $\rho(\vec{x})$ and their long-range behaviour are particularly important.

Let us introduce a source term in the partition function in order to compute density correlations.

$$Z_{\rho}[\sigma] = \int D\phi \ e^{-S[\phi] + \int d^3x \ \rho(\vec{x}) \sigma(\vec{x})}.$$  \hspace{1cm} (3.53)

We then define density correlation functions as

$$G_{\rho}^{(n)}(x_1, \ldots, x_n) = \frac{1}{Z_{\rho}[0]} \left. \frac{\delta^n Z_{\rho}[\sigma]}{\delta \sigma(x_1) \ldots \delta \sigma(x_n)} \right|_{\sigma=0}.$$  \hspace{1cm} (3.54)

The generating functional of the connected correlation function of the density field is defined as usual as

$$W_{\rho}[\sigma] = \ln Z_{\rho}[\sigma].$$  \hspace{1cm} (3.55)

We finally define the generating functional of the vertex correlation functions of the density field:

$$\Gamma_{\rho}[\bar{\sigma}] + W_{\rho}[\sigma] = \int d^3x \ \bar{\sigma}(x) \sigma(x),$$  \hspace{1cm} (3.56)

$$\bar{\sigma}(x) = \frac{\delta W_{\rho}[\sigma]}{\delta \sigma(x)}. $$  \hspace{1cm} (3.57)

At tree level the two point density correlator takes the form

$$\langle \rho(\vec{x}) \rho(\vec{y}) \rangle = \left( \frac{1}{T_{\text{eff}}} \right)^2 \int \frac{d^3x_0}{N} \frac{\nabla^2_x \chi(\vec{x} - \vec{x}_0) \nabla^2_x \chi(\vec{y} - \vec{x}_0)}{\nabla^2_{\vec{x}} \chi(\vec{x} - \vec{y}) \nabla^2_{\vec{x}} \chi(\vec{x} - \vec{y})} \rangle$$

$$= \frac{1}{N T_{\text{eff}}} \int d^3x_0 \frac{G(\vec{x} - \vec{x}_0, \vec{y} - \vec{x}_0)}{|\vec{x} - \vec{x}_0|^2 |\vec{y} - \vec{x}_0|^2},$$  \hspace{1cm} (3.58)

where we used eq.(3.11) in the course of the calculation.

Using that $G(x, y)$ scales as $1/\sqrt{|x||y|}$, we see that this correlator scales as $1/|\vec{x} - \vec{y}|^2$.

IV. RENORMALIZATION

As usual, the renormalized vertex functions are defined as follows:

$$\Gamma_{R}^{(n)}(x_1, \ldots, x_n, g_R) \equiv Z^{n/2} \Gamma^{(n)}(x_1, \ldots, x_n, g).$$  \hspace{1cm} (4.1)

where $Z$ stands for the wave function renormalization, $g$ is the bare coupling and $g_R$ the renormalized one.

The renormalizations are defined such that the two and three points functions take their tree level values at zero external momenta. In our case, from eq.(3.44) the renormalization prescriptions take the form

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The bare vertex functions at the one-loop level are given by eq.(3.44) and can be summarized as follows,

\[
\langle \Gamma^{(2)} \rangle = -\frac{8\pi R}{N} (1 - e^{-\tau}) \left[ 1 + \left( \frac{g^2}{\mu R} \right) \frac{C(\tau)}{8\pi} \right],
\]

\[
\langle \Gamma^{(3)} \rangle = \frac{8\pi R^2}{N} (1 - e^{-\tau}) \left( \frac{g}{\sqrt{\mu R}} \right) \left[ 1 + \left( \frac{g^2}{\mu R} \right) \frac{D(\tau)}{8\pi} \right].
\]

(4.3)

where \(C(\tau)\) and \(D(\tau)\) can be read off from eqs.(3.44).

The bare coupling \(g\) and the wave function renormalization can be expressed as,

\[
g_R = g \left[ 1 + \left( \frac{g^2}{\mu R} \right) \frac{1}{8\pi} h(\tau) + \mathcal{O} \left( \frac{g^2}{\mu R} \right)^2 \right],
\]

\[
Z = 1 + \left( \frac{g^2}{\mu R} \right) \frac{1}{8\pi} B(\tau) + \mathcal{O} \left( \frac{g^2}{\mu R} \right)^2.
\]

(4.4)

\(h(\tau)\) and \(B(\tau)\) follow by inserting eqs.(4.3)-(4.4) into the renormalization prescriptions eq.(4.2) with the result,

\[
h(\tau) = D(\tau) - \frac{3}{2} C(\tau) , \quad B(\tau) = -C(\tau).
\]

(4.5)

We can thus write \(Z(\tau)\) and \(g_R\) up to the first order in \(\frac{g^2}{\mu R}\):

\[
Z(\tau) = 1 + \left( \frac{g^2}{\mu R} \right) \frac{1}{8\pi} \left[ -\tau^2 \sum_{l=0,l\neq 1}^{\epsilon^\tau} \frac{2l+1}{2 \alpha_l^3 \sh^2(\alpha_l \tau)} - \tau \sum_{l=0,l\neq 1}^{\epsilon^\tau} \frac{(2l+1) \ch(\alpha_l \tau)}{2 \alpha_l^3 \sh(\alpha_l \tau)} (1 - \alpha_l^2) \right.
\]

\[
\left. + \sum_{l=0,l\neq 1}^{\epsilon^\tau} \frac{2l+1}{\alpha_l^6} (1 + \frac{\alpha_l^2}{2}) + \mathcal{O} \left( \frac{g^2}{\mu R} \right) \right]
\]

(4.6)

\[
g_R = g \left\{ 1 + \left( \frac{g^2}{\mu R} \right) \frac{1}{8\pi} \left[ \tau^3 \sum_{l=0,l\neq 1}^{\epsilon^\tau} \frac{(2l+1) \ch(\alpha_l \tau)}{\alpha_l^3 \sh^2(\alpha_l \tau)} + \tau^2 \sum_{l=0,l\neq 1}^{\epsilon^\tau} \frac{(2l+1)}{2 \alpha_l^4 \sh^2(\alpha_l \tau)} (3 + \frac{3}{2} \alpha_l^2) \right.
\]

\[
\left. + \tau \sum_{l=0,l\neq 1}^{\epsilon^\tau} \frac{(2l+1) \ch(\alpha_l \tau)}{2 \alpha_l^5 \sh(\alpha_l \tau)} (3 + \frac{3}{2} \alpha_l^2 - \frac{1}{2} \alpha_l^4) - \sum_{l=0,l\neq 1}^{\epsilon^\tau} \frac{2l+1}{\alpha_l^6} (4 + \frac{3}{2} \alpha_l^2 - \frac{1}{4} \alpha_l^4) + \mathcal{O} \left( \frac{g^2}{\mu R} \right) \right\}
\]

(4.7)
A. The renormalization group equations

1. Derivation of the renormalization group equations

The renormalization group equations for the vertex functions can be derived as usual, by requiring the renormalized vertices to be independent of the scale $\tau$ except for the explicit dependence on the size of the system here arising from the position of the boundary.

We therefore apply on the bare correlators the differential operator

$$\frac{\partial}{\partial \tau} = \frac{1}{2} \left( R \frac{\partial}{\partial R} - \delta \frac{\partial}{\partial \delta} \right)$$

keeping the bare coupling $\lambda$ defined in eq.(3.45) fixed.

We have,

$$\frac{\partial}{\partial \tau} \left|_{\lambda} \right. \Gamma^{(n)}(x_i, \lambda, \tau) = \Delta \Gamma^{(n)}.$$

where the insertion $\Delta$ arises through the $\tau$ derivative acting on the bounds of the action:

$$\Delta \equiv -\frac{1}{2T_{\text{eff}}} \int d\Omega_u \left( R^3 \mathcal{L}(Ru) + \delta^3 \mathcal{L}(\delta u) \right).$$

Since we have chosen zero boundary conditions, only the insertion of field derivatives contribute to the correlators $\Gamma^{(n)}$. We can then set

$$\Delta = -\frac{1}{4T_{\text{eff}}} \int \left( R^3 (\nabla_x|R_u)^2 + \delta^3 (\nabla_\delta|\delta_u)^2 \right) d\Omega_u.$$

Since the boundaries are at finite distances which scale with $\tau$, their effect must be taken into account through the $\Delta$ insertion. However, on physical grounds the boundary effects through the insertion $\Delta$ are expected to be subdominant. In addition, we show in appendix B that the insertion has indeed negligible effects.

By using the relation between renormalized and bare vertex functions we get,

$$\left[ \frac{\partial}{\partial \tau} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} - \frac{n-3}{2} \gamma(\lambda_R) \right] \Gamma^{(n)}_R(x_i, \lambda_R, \tau) = \Delta \Gamma^{(n)}_R.$$

where we introduced the renormalized effective coupling

$$\lambda_R \equiv \frac{g^2_R}{\mu R}.$$

The functions $\beta$ and $\gamma$ are defined by:

$$\beta(\lambda_R) = \frac{\partial \lambda_R}{\partial \tau} \left|_{\lambda} \right.$$

$$\gamma(\lambda_R) = \frac{\partial \ln Z}{\partial \tau} \left|_{\lambda} \right.$$

We find from eq.(3.53) that

$$\gamma(\lambda_R) = 1 + \mathcal{O}(\lambda_R).$$

We now proceed to compute the RG flow of the effective coupling $\lambda(\tau)$.
2. RG flow of the effective coupling $\lambda(\tau)$

The expansion of the effective coupling constant $\lambda$ easily follows from the expansion of $g$, eq. (4.7),

$$\lambda = \lambda_R \left[ 1 - \lambda_R \frac{1}{4\pi} h(\tau) \right]. \quad (4.12)$$

Now, using eq. (4.10), we have:

$$\beta(\lambda, \tau) = -\frac{\lambda^2}{4\pi} \frac{dh(\tau)}{d\tau}$$

where $h(\tau)$ is given by eq. (4.4) and (4.7)

$$h(\tau) = \tau^3 \sum_{l=0,l\neq1}^{\infty} \frac{(2l+1) \text{ch}(\alpha_l \tau)}{\alpha_l^3 \text{sh}(\alpha_l \tau)} + \tau^2 \sum_{l=0,l\neq1}^{\infty} \frac{(2l+1)}{2\alpha_l^4 \text{sh}^2(\alpha_l \tau)} \left( 3 + \frac{3}{2} \alpha_l^2 \right)$$

$$+ \frac{\tau}{4} \sum_{l=0,l\neq1}^{\infty} \frac{(2l+1) \text{ch}(\alpha_l \tau)}{\alpha_l^5 \text{sh}(\alpha_l \tau)} \left( 6 + 3\alpha_l^2 - \alpha_l^4 \right) - \frac{1}{4} \sum_{l=0,l\neq1}^{\infty} \frac{2l+1}{\alpha_l^6} (16 + 6\alpha_l^2 - \alpha_l^4) \quad (4.13)$$

The renormalization group flow of $\lambda$ is determined by $\beta(\lambda, \tau)$ as usual by

$$\frac{d\lambda}{d\tau} = \beta(\lambda, \tau)$$

Notice that contrary to the usual cases, there is a non-trivial and explicit dependence on the scale $\tau$ in $h(\tau)$, and hence in $\beta(\lambda, \tau)$. We get,

$$\frac{d\lambda}{\lambda^2} = -\frac{1}{4\pi} h'(\tau) \, d\tau. \quad (4.14)$$

Integrating this expression from $\tau = \tau_i$ to $\tau$, we find for $\lambda(\tau)$

$$\lambda(\tau) = \frac{\lambda_i}{1 + \frac{\lambda_i}{4\pi} [h(\tau) - h(\tau_i)]}, \quad (4.15)$$

where $\lambda_i \equiv \lambda(\tau_i)$.

We proceed now to analyse the nontrivial dependence of the effective coupling constant $\lambda(\tau)$ on the scale $\tau$.

Let us define the quantities,

$$\sigma_n = \sum_{l=2}^{\infty} \frac{2l+1}{\alpha_l^{2n}} \quad .$$

We will use the $\sigma_n$ for $1 \leq n \leq 6$:

- $\sigma_1 = 2(\zeta(0) - 1) + \sum_{l=2}^{\infty} \left( \frac{2l+1}{\alpha_l} - 2 \right) = -1.874677 + O(e^{-\tau})$

- $\sigma_2 = 2\tau - 0.7871527 + 2 \, e^{-\tau} + O(e^{-2\tau})$
• \( \sigma_3 = 1.314771 + 2e^{-\tau} + O(e^{-2\tau}) \)
• \( \sigma_4 = 0.4134319 + e^{-2\tau} + O(e^{-3\tau}) \)
• \( \sigma_5 = 0.16731628 + \frac{2}{3}e^{-3\tau} + O(e^{-4\tau}) \)
• \( \sigma_6 = 0.07402368 + \frac{e^{-4\tau}}{2} + O(e^{-5\tau}) \)

Notice that only \( \sigma_2 \) grows logarithmically with the cutoffs. This enables us to reduce \( h(\tau) \) to the following form:

\[
h(\tau) = -\frac{\tau^3}{\sin^3\left(\frac{\sqrt{7}}{2}\tau\right)} \frac{8}{7\sqrt{7}} \cos\left(\frac{\sqrt{7}}{2}\tau\right) - \frac{3}{49\sin^2\left(\frac{\sqrt{7}}{2}\tau\right)} \tau^2 \\
+ \frac{37}{98\sqrt{7}} \frac{\tau}{\sin\left(\frac{\sqrt{7}}{2}\tau\right)} \cos\left(\frac{\sqrt{7}}{2}\tau\right) + \tau \left(\frac{3}{2}\sigma_5 + \frac{3}{4}\sigma_3 - \frac{1}{2}\sigma_1 + 1\right) \\
- 4\sigma_6 - \frac{3}{2}\sigma_4 + \frac{1}{2}(\sigma_2 - 2\tau) - \frac{10}{\sqrt{7}},
\]  

or with the numerical values,

\[
h(\tau) = -\frac{\tau^3}{\sin^3\left(\frac{\sqrt{7}}{2}\tau\right)} \frac{8}{7\sqrt{7}} \cos\left(\frac{\sqrt{7}}{2}\tau\right) - \frac{3}{49\sin^2\left(\frac{\sqrt{7}}{2}\tau\right)} \tau^2 \\
+ \frac{37}{98\sqrt{7}} \frac{\tau}{\sin\left(\frac{\sqrt{7}}{2}\tau\right)} \cos\left(\frac{\sqrt{7}}{2}\tau\right) \frac{37}{98\sqrt{7}} - 0.70\tau - 0.55 + O\left(\tau e^{-\tau}\right)
\]  

The effective coupling \( \lambda(\tau) \) exhibits an interesting behaviour as a function of \( \tau \).

First of all, the one-loop approximation is reliable only when \( 0 \leq \lambda(\tau) \sim \lambda_i = \lambda(\tau_i) \) or smaller. Starting at \( \tau = \tau_i \), \( \lambda(\tau) \) decreases until it vanishes at the first integer multiple of \( 2\pi/\sqrt{7} \). That is at \( \tau = 2\pi n/\sqrt{7}, \ n = 0, 1, 2, \ldots \). At these points \( h(\tau) \) becomes singular. Namely, each time

\[
\sin\left(\frac{\sqrt{7}}{2}\tau\right) = 0 \quad , \quad \tau = \tau_n = 2\pi n/\sqrt{7}
\]

we have \( \lambda(\tau_n) = 0 \). Notice that \( \lambda(\tau) \) becomes negative in the intervals

\[
\tau_n < \tau < \tau_n + \Delta_n,
\]

where \( \Delta_n \) has a specific value in each interval depending on \( \lambda_i \) and \( \tau_i \) (see fig. 1). [\( \Delta_n \) is defined by the relation: \( h(\tau_n + \Delta_n) = h(\tau_i) - 4\pi/\lambda_i \)]. This shows that the one-loop approximation ceases to be valid at \( \tau = \tau_n + 0 \).

The effective coupling constant \( \lambda(\tau) \) decreases with the scale \( \tau \) in an infinite number of disjoint intervals

\[
\tau_{n-1} + \Delta_{n-1} < \tau \leq \tau_n.
\]  

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That is, we can start to run the renormalization group at $\tau = \tau_i$, $\tau_i$ chosen such as $\tau_{n-1} < \tau_i < \tau_n$, with a small coupling $\lambda_i \equiv \lambda(\tau_i)$ and keep running until $\tau = \tau_n$. At this point $\lambda(\tau_n) = 0$. In these intervals $\tau_i, \tau_n)$, the effective coupling $\lambda(\tau)$ decreases when the space scale $\tau$ increases as usually happens in scalar field theories, which is the case here (infrared stable behaviour).

We depict in fig. 1 the running coupling constant $\lambda(\tau)$ in the intervals given by eq. (4.18) for $n = 1, 2, 3, 4, 6$ and 17 as illustrative cases.

We see that eq. (4.18) provides a hierarchy of intervals where the behaviour of the perturbative effective coupling [given by eq. (4.15)] is consistent. This hierarchy is formed by scales following the geometric progression

$$ R_0 = \delta , $$
$$ R_1 = R_0 e^{2\pi/\sqrt{7}} , $$
$$ \ldots \ldots $$
$$ R_n = R_0 e^{2\pi n/\sqrt{7}} = R_0 [10.749087 \ldots]^n . \tag{4.19} $$

Outside the intervals in $\tau$ defined by eq. (4.18), the effective coupling $\lambda(\tau)$ becomes negative, or very large or both, showing that the one-loop approximation is no more reliable.

In addition, the growth of $\lambda(\tau)$ according to eq. (4.15) when $\tau$ decreases below $\tau_i$ suggests that one enters here a strong coupling regime. That is, we find an ultraviolet unstable behaviour. We now connect this behaviour with the instability of structures and fragmentation. As already established, within a given interval, the coupling $\lambda(\tau)$ grows with decreasing $\tau$, that is with a decreasing size of the system (ultraviolet unstable behaviour). In this more and more strongly coupled regime, the density fluctuations are ever larger. This means that the Jeans length $d_J$ decreases, the instability condition (scales larger than $d_J$) becomes easily satisfied for smaller and smaller regions, thus leading to the fragmentation of the original mass structure into substructures.

In the opposite regime, $\lambda(\tau)$ decreases with increasing scale (infrared stable behaviour) and the density fluctuations decrease. Jeans’ instabilities thus become less probable as long we do not enter the next interval.

The zeroes of $\lambda(\tau)$ determine a hierarchy of structures $R_n = R_0 e^{2\pi n/\sqrt{7}}$ described above in eq. (4.19). We thus have a self-similar set of structures fitting one into each other.
Fig 1: Above is the coupling constant $\lambda(\tau)$ picture in 6 differents $\tau$ intervals; $[0, \tau_1], [\tau_1, \tau_2], [\tau_2, \tau_3], [\tau_3, \tau_4], [\tau_5, \tau_6]$ and $[\tau_{16}, \tau_{17}]$. The initial conditions are set such as $\tau_i = \tau_{n-1}$ and $\lambda_i = 0.01$. 


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APPENDIX A: COMPUTATION OF THE MOST SIMPLE FEYNMAN DIAGRAMS

In section 3.3.5 we have given the values of the 2-points and 3-points averaged vertex functions. As those values are given for a one loop order computation, only three topologically different amputated diagrams are needed. We will give some details here of the calculation of those diagrams.

1. Some useful integrals

The computation of the diagrams, as we will see later, can be brought down to that of three integrals. The calculation holds no difficulties, here are the results:

\[ \int_{0}^{-a} dx \, \text{sh}(x) \, \text{sh}(x + a) = \frac{1}{2} (a \, \text{ch}(a) - \text{sh}(a)) \]

\[ \int_{0}^{-a} dx \int_{0}^{x} dy \, \text{sh}^2(x + a) \, \text{sh}^2(y) = \frac{1}{8} \left( a^2 + a \, \text{ch}(a) \, \text{sh}(a) - 2 \, \text{sh}^2(a) \right) \]

\[ \int_{0}^{-a} dx \, \text{sh}(x + a) \, \text{sh}(x) \int_{0}^{x+a} dy \, \text{sh}^2(y) \int_{0}^{x} dz \, \text{sh}^2(z) = \]

\[ \frac{1}{16} \left( -\frac{a^3}{3} \, \text{ch}(a) - \frac{a^2}{2} \, \text{sh}(a) - \frac{a}{2} \, \text{sh}^2(a) \, \text{ch}(a) + \frac{4}{3} \, \text{sh}^3(a) \right) \]

Let us see now how these integrals appear in the diagrams.

2. One loop diagrams calculation

First is the so-called tadpole diagram, which appear in the $\phi^4$ theory for example. We compute directly its average value.

\[ \left< \varnothing \right> = C_1 \int d^3x \int d^3x_0 \, \delta(x - y) \, G(x - x_0, x_0 - x_0) \, \mu^2 \, e^{\phi_0(x-x_0)} \]

\[ = \int_\delta^{R} dr \sum_l (2l + 1) \, g_l(r, r) \]

\[ = \sum_l \frac{2l + 1}{\alpha_l \, \text{sh}(\alpha_l \tau)} \int_\delta^{R} dr \, r \, \text{sh} \left[ \alpha_l \ln \left( \frac{r}{R} \right) \right] \, \text{sh} \left[ \alpha_l \ln \left( \frac{r}{\delta} \right) \right] \]

Let us make the following change in the variables, $x = \alpha_l \ln \left( \frac{r}{\delta} \right)$,

\[ \left< \varnothing \right> = \sum_l \frac{2l + 1}{\alpha_l^2 \, \text{sh}(\alpha_l \tau)} \int_{-\alpha_l \tau}^{0} dx \, \text{sh}(x + \alpha_l \tau) \, \text{sh}(x) \]

\[ = \tau \sum_{l=0, l \neq 1}^{e^\tau} \frac{2l + 1}{2\alpha_l} \, \text{ch}(\alpha_l \tau) \, \text{sh}(\alpha_l \tau) - \sum_{l=0, l \neq 1}^{e^\tau} \frac{2l + 1}{\alpha_l^2} \]

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The next diagram, that we call eye-like, appear in the $\phi^3$ theory.

\[
\langle \bigcirc \bigcirc \rangle = -C_2 \int d^3x \int d^3x_0 \mathcal{G}(x - x_0, y - x_0) \mu^2 e^{\phi_0(x-x_0)} \mu^2 e^{\phi_0(y-x_0)}
\]

\[
= -2 \int_{\delta}^{R} dr_1 \int_{\delta}^{R} dr_2 \sum_l (2l + 1) g_l^2(r_1, r_2)
\]

We split the domain of integration into two parts where $r_1$ and $r_2$ are in a given order, either $r_1 < r_2$ or $r_1 > r_2$. For symmetry reasons the contribution of those parts to the result are the same. Beside we implement the same change in variables as in the tadpole diagram, $x_1 = \alpha_l \ln \left( \frac{r_1}{R} \right)$ and $x_2 = \alpha_l \ln \left( \frac{r_2}{R} \right)$,

\[
\langle \bigcirc \bigcirc \rangle = -4 \sum_{l=0,l\neq 1}^{\varepsilon_r} \frac{2l + 1}{\alpha_l^2 \sh^2(\alpha_l \tau)} \int_{\delta}^{R} \frac{dr_1}{r_1} \int_{\delta}^{r_1} \frac{dr_2}{r_2} \frac{\alpha_l^2 \sh^2 \alpha_l \tau}{\alpha_l^2 \sh^2 \alpha_l \tau} \sh^2 \left( \alpha_l \ln \left( \frac{r_2}{\delta} \right) \right) \sh^2 \left( \alpha_l \ln \left( \frac{r_1}{R} \right) \right)
\]

\[
= -4 \sum_{l=0,l\neq 1}^{\varepsilon_r} \frac{2l + 1}{\alpha_l^4 \sh^2(\alpha_l \tau)} \int_0^{x_1} dx_1 \int_0^{x_2} dx_2 \sh^2 (x_1 + \alpha_l \tau) \sh^2 (x_2)
\]

\[
= - \sum_{l=0,l\neq 1}^{\varepsilon_r} \frac{2l + 1}{2 \alpha_l^4 \sh^2(\alpha_l \tau)} \left[ \alpha_l^2 \tau^2 + \alpha_l \tau \sh(\alpha_l \tau) \ch(\alpha_l \tau) - 2 \sh^2(\alpha_l \tau) \right]
\]

\[
= - \frac{1}{2} \tau^2 \sum_{l=0,l\neq 1}^{\varepsilon_r} \frac{2l + 1}{\alpha_l^2 \sh^2(\alpha_l \tau)} - \frac{1}{2} \tau \sum_{l=0,l\neq 1}^{\varepsilon_r} \frac{(2l + 1) \ch(\alpha_l \tau)}{\alpha_l^3 \sh(\alpha_l \tau)} + \sum_{l=0,l\neq 1}^{\varepsilon_r} \frac{2l + 1}{\alpha_l^2}
\]

Last, the triangle diagram, that appears in the 3-points function only,

\[
\langle \bigtriangledown \rangle = -C_5 \int d^3y \int d^3x \int d^3x_0 \mathcal{G}(x - x_0, y - x_0) \mathcal{G}(x - y_0, z - x_0)
\]

\[
\times \mathcal{G}(x - z_0, x - x_0) \mu^2 e^{\phi_0(x-x_0)} \mu^2 e^{\phi_0(y-x_0)} \mu^2 e^{\phi_0(z-x_0)}
\]

\[
= -8 \sum_l (2l + 1) \int_{\delta}^{R} dr_1 \int_{\delta}^{R} dr_2 \int_{\delta}^{R} dr_3 g_l(r_1, r_2) g_l(r_2, r_3) g_l(r_3, r_1)
\]

Here again we will split the domain into six parts where the three variable are ordered. Contributions of these parts are the same. And using again the same changes in the variables:
\[ \left\langle \nabla \right\rangle = -48 \sum_{l=0, l \neq 1}^\varepsilon \frac{2l + 1}{\alpha_l^3 \sh^3(\alpha_l \tau)} \int_\delta^R dr_1 \int_\delta^{r_1} dr_2 \int_\delta^{r_2} dr_3 \sh^2 \left[ \alpha_l \ln \left( \frac{r_1}{R} \right) \right] \sh \left[ \alpha_l \ln \left( \frac{r_2}{\delta} \right) \right] \sh \left[ \alpha_l \ln \left( \frac{r_1}{\delta} \right) \right] \]

\[ = -48 \sum_{l=0, l \neq 1}^\varepsilon \frac{(2l + 1)}{\alpha_l^6 \sh^3(\alpha_l \tau)} \int_0^{-\alpha_l \tau} dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \]

\[ \sh^2 (x_1 + \alpha_l \tau) \sh (x_2) \sh (x_2 + \alpha_l \tau) \sh^2 (x_3) \]

A trick here is to integrate \( x_2 \) between 0 and \(-\alpha_0 \tau\) then the integrations on \( x_1 \) and \( x_3 \) disentangle,

\[ \left\langle \nabla \right\rangle = 48 \sum_{l=0, l \neq 1}^\varepsilon \frac{2l + 1}{\alpha_l^6 \sh^3(\alpha_l \tau)} \int_0^{-\alpha_l \tau} dx_2 \sh (x_2) \sh (x_2 + \alpha_l \tau) \int_0^{x_2 + \alpha_l \tau} dx_1 \sh^2 (x_1) \int_0^{x_2} dx_3 \sh^2 (x_3) \]

\[ = 3 \sum_{l=0, l \neq 1}^\varepsilon \frac{2l + 1}{\alpha_l^3 \sh^3(\alpha_l \tau)} \left[ -\frac{\alpha_l^3 \tau^3}{3} \ch(\alpha_l \tau) - \frac{\alpha_l^2 \tau^2}{2} \sh(\alpha_l \tau) \right. \]

\[ \left. -\frac{\alpha_l \tau}{2} \sh^2(\alpha_l \tau) \ch(\alpha_l \tau) + \frac{4}{3} \sh^3(\alpha_l \tau) \right] \]

\[ = -\tau^3 \sum_{l=0, l \neq 1}^\varepsilon \frac{(2l + 1) \ch(\alpha_l \tau)}{\alpha_l^3 \sh^3(\alpha_l \tau)} - \frac{3}{2} \tau^2 \sum_{l=0, l \neq 1}^\varepsilon \frac{2l + 1}{\alpha_l^4 \sh^2(\alpha_l \tau)} \]

\[ - \frac{3}{2} \tau \sum_{l=0, l \neq 1}^\varepsilon \frac{(2l + 1) \ch(\alpha_l \tau)}{\alpha_l^2 \sh(\alpha_l \tau)} + 4 \sum_{l=0, l \neq 1}^\varepsilon \frac{2l + 1}{\alpha_l^6} \]

It is worth to mention that the calculation of any other diagram, averaged or not, is quite longer and require the use of formal calculus programs.

**APPENDIX B: CONTRIBUTION OF BORDER TERMS IN THE RENORMALIZATION GROUP EQUATION**

To evaluate the r.h.s of the renormalization group equation, we want to compute here the action on the correlation functions of the derivative:

\[ \frac{\partial}{\partial \tau} = \frac{1}{2} \left( R \frac{\partial}{\partial R} - \delta \frac{\partial}{\partial \delta} \right) \]  

(B1)

According to our Feynmann rules, a diagram contributing to the n-points vertex function, denoted \( d_{x_0}(x_1, \ldots, x_n) \), with \( l \) loops and \( v \) vertex writes as follow;
\[ d_{x_0}(x_1, \ldots, x_n) = C \left( \frac{g}{\sqrt{\mu}} \right)^{n+2(l-1)} \int \frac{d^3 z_1}{z_1^2} \ldots \int \frac{d^3 z_n}{z_n^2} \delta(x_1 - z_1) \ldots \delta(x_n - z_n) G(z_1, \ldots, z_n), \quad (B2) \]

where \( C \) is the symmetry factor of the diagram and the specific form of the propagators product is determined by the topological structure of the diagram. We apply now the \( \tau \) derivation and make use of our boundary conditions. We find,

\[
\frac{\partial}{\partial \tau} d_{x_0}(x_1, \ldots, x_n) = \sum_{\{z_i\}} C \left( \frac{g}{\sqrt{\mu}} \right)^{n+2(l-1)} \int \frac{d^3 z_1}{z_1^2} \ldots \int \frac{d^3 z_n}{z_n^2} \delta(x_1 - z_1) \ldots \delta(x_n - z_n) G(z_1, \ldots, z_n) \frac{\partial}{\partial \tau} G(\ldots) \ldots G(\ldots, z_v),
\]

where the summation is over the derivative positions in the product. We then write:

\[
\frac{\partial}{\partial \tau} G(x, y) \equiv H(x, y)
\]

(B3)

where,

\[
H(r, r') = \frac{1}{4\pi \sqrt{rr'}} \sum_l \frac{2l + 1}{\text{sh}^2(\alpha_l \tau)} \left\{ \text{ch} \left[ \alpha_l \ln \frac{rr'}{\delta R} \right] \text{ch} (\alpha_l \tau) - \text{ch} \left[ \alpha_l \ln \frac{r'}{r} \right] \right\} P_l(\cos \gamma)
\]

(B4)

We compare now \( G \) and \( H \) for \( \tau \gg 1 \) and generic values of \( r \) and \( r' \) when \( \tau \) is not close to its critical values; \( \frac{2k\pi}{\sqrt{\tau}} \).

We can then write \( \tilde{x} = \sqrt{\frac{\xi}{\delta}}(1 + \epsilon_1) \), \( \tilde{y} = \sqrt{\frac{\xi}{\delta}}(1 + \epsilon_2) \). That is, \( x \) and \( y \) are not close to the borders. Then,

\[
g_l(x, y) = -\frac{1}{4\pi \sqrt{\delta R}} \frac{2l + 1}{2\alpha_l} \left[ 1 + (\alpha_l - 1)\epsilon_1 - (1 + \alpha_l)\epsilon_2 + O(\epsilon_1^2, \epsilon_2^2, \epsilon_1\epsilon_2) \right]
\]

(B5)

\[
h_l(x, y) \sim \frac{1}{\pi} \frac{2l + 1}{\sqrt{\delta R}} e^{-\tau \alpha_l} \left[ 1 + \alpha_l \epsilon_1 + \alpha_l \epsilon_2 + O(\epsilon_1^2, \epsilon_2^2, \epsilon_1\epsilon_2) \right]
\]

(B6)

In this second equation \( h_l \) is defined by analogy with \( g_l \). So, in this region,

\[
g_l(x, y) \gg h_l(x, y) \quad l \neq 0
\]

(B7)

and

\[
g_0(x, y) \sim h_0(x, y)
\]

(B8)

Then,

\[
G(x, y) \gg H(x, y)
\]

(B9)

This means that, in this region (i.e not close to the borders), and for non critical values of \( \tau \), the border terms are subdominant.