On the convergence of solutions for SPDEs under perturbation of the domain *

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Abstract

We concern the effect of domain perturbation on the behaviour of stochastic partial differential equations subject to the Dirichlet boundary condition. Under some assumptions, we get an estimate for the solutions under changes of the domain.

Keywords: Stochastic partial differential equation; Domain perturbation; Convergence of solutions;

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1 Introduction

Stochastic partial differential equations (SPDEs) have a broad spectrum of application including natural sciences and economics. The purpose of this article is to study the behavior of solution for stochastic partial differential equations with Dirichlet boundary condition under the singular domain perturbations, which means that change of variables is not possible on these domains. Under property conditions, we show how solutions of stochastic differential equations behave as a sequence of domains $\Omega_n$ converges to an open set $\Omega$ in a certain sense. The motivation to study domain perturbation comes from various sources. The main ones include shape optimization, solution structure of nonlinear problems and numerical analysis.

Domain perturbation or sometimes referred to as “perturbation of the boundary” for boundary value problems is a special topic in perturbation problems. The main characteristic is that the operators and the nonlinear term live in differential spaces which lead to the solution of differential equation live in differential spaces. Domain perturbation appears to be a simple problem if we are only interested in smooth perturbation of the domain. This is because we could perform a change of variables to consider the perturbed problems in a fixed domain and only perturb the coefficients. Hence, it turns back to a standard perturbation problem and we may apply standard techniques such as the implicit function theorem, the Liapunov-Schmidt method and the transversality theorem. Nevertheless, difficulties arrive when we perform a change of variables and standard tools are not enough (see [9]). When a change of variables is not possible, domain perturbation is even more challenging.

The fundamental question in domain perturbation is to look at how solutions behave upon varying domains. In particular we would like to know when the solutions converge and what the limit problem is. There have many papers concern on this topic, which main under the condition of Mosco convergence. For elliptic equations case see [6, 2] and references therein. In [2] the author give a sufficient condition on domains which guarantee the spectrum behaves continuously. The work of [6] prove the converge of solution for elliptic equations subject to Dirichlet boundary condition. For parabolic and evolution equation, we recommend [7, 8] and so on. In [7] the author concern domain perturbation for non-autonomous parabolic equations under the assumption of mosco converge. With such a assumption we have that the condition of mosco converge is equivalent to the strong convergence of pseudo resolvent operators for Dirichlet promble. Under the assumption of mosco converge, the author of [7] get the result of convergence of solutions for both linear and semilinear parabolic initial value problems subject to Dirichlet boundary boundary condition as well as persistence of periodic solutions under domain perturbation. There also some other papers about invariant
manifolds under the domain perturbation see [3, 14, 12], which main concern the converge of invariant manifolds under the perturbation of the domain.

For Dirichlet problems, the strong convergence of pseudo resolvent operators is equivalent to Mosco convergence (see [7], Theorem 5.2.4 or [6], Theorem 3.3). In this paper, we take the condition of strong convergence of pseudo resolvent operators relate to the domain perturbation. Compare with the Mosco condition, it is more convenient and effective for proving the convergence of solution for partial differential equations and stochastic differential equations under perturbation of the domain.

The remainder of this paper is organized as follows: In Section 2, we will review some basic properties of infinitesimal generator and its semigroups, and existence and unique of solution to stochastic partial differential equation. The result on the converge of solution for stochastic differential equation under the perturbation is described in section 3.

2 Preliminaries

Let $H$ be an infinite dimensional separable Hilbert space with norm $\| \cdot \|$. Let the sectorial operator $A : D(A) \to H$ be a self-adjoint positive linear operator with a compact resolvent. Then the spectrum of $A$ is real. We denote its spectrum by

$$\sigma(A) = \{ \lambda_n \}_{n=1}^{\infty}, \quad 0 < c \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots,$$

and an associated orthonormal family of eigenfunctions by $\{ \phi_n \}_{n=1}^{\infty}$. Since $A$ is a sectorial operator, $-A$ is the infinitesimal generator of a analytic semigroup, which is denoted by

$$e^{-At} = \frac{1}{2\pi i} \int_{\gamma} (\lambda I + A)^{-1} e^{\lambda t} d\lambda,$$

where $\gamma$ is a contour in the resolvent set of $-A$. Since $A$ is a self-adjoint operator, the formula above is equivalent to

$$e^{-At} u = \sum_{n=1}^{\infty} e^{-\lambda_n t} (u, \phi_n) \phi_n.$$

By the definition $e^{-At}$, we can easily get the following estimate

$$\| e^{-At} \|_{L(H,H)} \leq e^{-\lambda_1 t} \leq 1$$

for $t \geq 0$, which implies that $e^{-At}$ is an analytic contraction semigroup.
Consider the nonlinear stochastic partial differential equation

\[
\begin{aligned}
du + Adu &= f(u)dt + g(u)dw(t), & \text{in } & D \times (0,T) \\
\frac{\partial u}{\partial t} &= 0, & \text{on } & \partial D \times (0,T) \\
u(0) &= u_0, & \text{in } & D
\end{aligned}
\]  

(2.1)

for \( t \in [0,T] \). Here \( u \in H \), \( A \) is a sectorial operator, which will be discussed later, \( W(t) \) is the standard \( \mathbb{R} \)-valued Wiener process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

For the drift coefficients \( f(u) : H \to H \) and diffusion coefficients \( g(u) : H \to H \), we adopt the following assumptions throughout this paper.

(A.1) There exists a constant \( k_2 > 0 \) such that for any \( u, v \in H, t \in [0,T] \),

\[
\|f(u) - f(v)\|^2 + \|g(u) - g(v)\|^2 \leq k_2 \|u - v\|^2.
\]

Notice that (A.1) implies there exists a constant \( k_1 > 0 \) such that

\[
\|f(u)\|^2 + \|g(u)\|^2 \leq k_1 (1 + \|u\|^2)
\]

for any \( u \in H, t \in [0,T] \).

Now we introduce the definition of solution to Eq. (2.1) and the existence and uniqueness of solution, Both of them are taken from [13].

**Definition 2.1 (Mild solution).** An \( H \)-valued predictable process \( u(t) \) is called a mild solution of Eq. (2.1) if for any \( t \in [0,T] \)

\[
u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(u(s))ds + \int_0^t e^{-A(t-s)} g(u(s))dw(s)
\]

(2.2)

Let \( X_T \) denote the set of all continuous \( \mathcal{F}_t \)-adapted processes valued in \( H \) for \( 0 \leq t \leq T \) such that \( E \sup_{0 \leq t \leq T} \|u\|^2 < \infty \). Then \( X_T \) is a Banach space under the norm

\[
\|u\|_T = E \sup_{0 \leq t \leq T} \|u\|^2.
\]

Define an operator \( \Gamma \) in \( X_T \) as follows

\[
\Gamma u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(u(s))ds + \int_0^t e^{-A(t-s)} g(u(s))dw(s),
\]

(2.3)

for \( u \in X_T \). It is easy to prove that the operator \( \Gamma \) is well defined and Lipschitz continuous in \( X_T \). Then by the contraction mapping principle, it is easy to prove the existence and unique of mild solution for the Eq. (2.1) is the following

**Theorem 2.1.** Suppose the condition (A.1) holds true, and \( u_0 \) be a \( \mathcal{F}_0 \)-measurable random field such that \( E\|u_0\|^2 < \infty \). Then the initial-boundary value problem for the Eq. (2.1) has a unique mild solution \( u(t) \) which is a continuous adapted process in \( H \) such that \( u \in L^2(\Omega; C([0,T]; H)) \) and

\[
E \sup_{0 \leq t \leq T} \|u\|^2 \leq C(1 + E\|u_0\|^2)
\]

for some constant \( C > 0 \).
3 Solution under perturbation of the domain

In this section, we consider the following perturbation equation of Eq. (2.1)

\[
\begin{align*}
\begin{cases}
du^\epsilon + A_\epsilon u^\epsilon dt &= f^\epsilon(u^\epsilon) dt + g^\epsilon(u^\epsilon) dw(t), & \text{in } D^\epsilon \times (0, T], \\
u^\epsilon &= 0, & \text{on } \partial D^\epsilon \times (0, T], \\
u^\epsilon(0) &= u^\epsilon_0, & \text{in } D^\epsilon
\end{cases}
\end{align*}
\]

for \( \epsilon > 0 \), where \( A_\epsilon : D(A_\epsilon) \subset H^\epsilon \rightarrow H^\epsilon \) is a self-adjoint positive linear operator on a Hilbert space \( H^\epsilon \) with norm \( \| \cdot \|_\epsilon \), and \( u^\epsilon_0 \) be a \( \mathcal{F}_0 \)-measurable random field such that \( E\|u^\epsilon_0\|^2 < \infty \). We also assume that the nonlinear terms \( f^\epsilon : H^\epsilon \rightarrow H^\epsilon \) and \( g^\epsilon : H^\epsilon \rightarrow H^\epsilon \) satisfy (A.1), which guarantees the existence and unique of mild solution to the Eq. (3.1). By Theorem 2.1 for each \( \epsilon > 0 \), there is an \( H \)-valued continuous \( \mathcal{F}_t \)-adapted process \( u^\epsilon(t) \) such that

\[
u^\epsilon(t) = e^{-A_\epsilon t}u^\epsilon_0 + \int_0^t e^{-A_\epsilon(t-s)}f(u^\epsilon(s))ds + \int_0^t e^{-A_\epsilon(t-s)}g(u^\epsilon(s))dw(s)
\]

for any \( t \in [0, T] \) and \( u^\epsilon \in L^2(\Omega; C([0, T]; H^\epsilon)) \).

Note that the solutions value in different function spaces \( H^\epsilon \) for different \( \epsilon \). To deal with domain perturbation, we assume there exist bound linear operators \( P \) and \( Q \) such that

\[
P : H \rightarrow H^\epsilon, \quad Q : H^\epsilon \rightarrow H, \quad Q \circ P = I,
\]

\[
\|P\|_{\mathcal{L}(H, H^\epsilon)} \leq 2, \quad \|Q\|_{\mathcal{L}(H^\epsilon, H)} \leq 2,
\]

and

\[
\|Pu\|_{H^\epsilon} \rightarrow \|u\|_H, \quad \text{as } \epsilon \rightarrow 0
\]

for all \( u \in H \).

To derive the solution of Eq. (3.1) converges to the solution of Eq. (2.1), we also impose the following hypotheses

**H.1** For \( A \) and \( A_\epsilon \), we assume

\[
\|A_\epsilon^{-1}P - PA^{-1}\|_{\mathcal{L}(H, H^\epsilon)} = \tau(\epsilon) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.
\]

**H.2** We assume that the nonlinear terms \( g^\epsilon, f^\epsilon : H^\epsilon \rightarrow H \) for \( 0 \leq \epsilon \leq \epsilon_0 \), satisfy:

- \( f^\epsilon \) and \( g^\epsilon \) approximate \( f \) and \( g \) in the following sense,

\[
\sup_{u \in H} \|f^\epsilon(Pu) - Pf(u)\|_{H^\epsilon}^2 = \tau_1(\epsilon) \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.
\]

\[
\sup_{u \in H} \|g^\epsilon(Pu) - Pg(u)\|_{H^\epsilon}^2 = \tau_2(\epsilon) \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.
\]
• \( f \) and \( f^\epsilon \) have the uniformly bounded support, that is

\[
\text{Supp} f \subset D_R = \{ u \in H : \| u \|_H \leq R \}
\]

\[
\text{Supp} f^\epsilon \subset D_R = \{ u^\epsilon \in H^\epsilon : \| u^\epsilon \|_{H^\epsilon} \leq R \}
\]

\((H.3)\) For initial value \( u_0 \) and \( u_0^\epsilon \), we assume

\[
E \| u_0^\epsilon - \text{P} u_0 \|^2_{H^\epsilon} = \tau_0(\epsilon) \to 0, \quad \text{as } \epsilon \to 0.
\]

By the condition \((H.1)\) we have the following result, which concerns the relationship of spectrum between \( A \) and \( A^\epsilon \) (see \[3\]).

**Lemma 3.1.** If \( K_0 \) is a compact set of the complex plane with \( K_0 \subset \rho(-A) \), the resolvent set of \( A \), and hypothesis \((H.1)\) is satisfied, then there exists \( \epsilon_0(K_0) > 0 \) such that \( K_0 \subset \rho(-A^\epsilon) \) for all \( 0 < \epsilon \leq \epsilon_0(K_0) \). Moreover, we have the estimates

\[
\|(\lambda I - A^\epsilon)^{-1}\|_{L(H^\epsilon,H^\epsilon)} \leq C(K_0)
\]

for all \( \lambda \in K_0, \ 0 < \epsilon \leq \epsilon_0(K_0) \).

The result implies the upper semi-continuity of the spectrum, that is, if \( \lambda_\epsilon \in \sigma(A^\epsilon) \) and \( \lambda_\epsilon \to \lambda \) then \( \lambda \in \sigma(A) \). Also we have the resolvent operator estimate as follows (see \[3\]).

**Lemma 3.2.** Let the condition \((H.1)\) be satisfied, if \( \lambda \in \rho(-A) \) and \( \epsilon \) is small enough so that \( \lambda \in \rho(-A^\epsilon) \), we have

\[
\|(\lambda + A^\epsilon)^{-1}\text{P} - \text{P}(\lambda + A)^{-1}\|_{L(H,H^\epsilon)} \leq C(\epsilon, \lambda) \tau(\epsilon) \to 0, \quad \text{as } \epsilon \to 0.
\]

As we all known, the relationship between resolvent operator and semigroup is denoted by

\[
e^{-At} = \frac{1}{2\pi i} \int_{\gamma} (\lambda I + A)^{-1} e^{\lambda t} d\lambda \quad (3.3)
\]

where \( \gamma \) is the boundary of \( \Sigma_{-a,\phi} = \{ \lambda \in \mathbb{C} : |\text{arg}(\lambda + a)| \leq \pi - \phi \} \subset \rho(-A) \), \( \phi \in (0, \frac{\pi}{2}) \). For simply we take the \( a = 0, \phi = \frac{\pi}{4} \). Then we have

\[
\gamma = \gamma_1 \cup \gamma_2 = \{ re^{-i\frac{\pi}{4}} : 0 \leq r < \infty \} \cup \{ re^{i\frac{\pi}{4}} : 0 \leq r < \infty \}
\]

and \( C(\epsilon, \lambda) \leq 6 \) for all \( \lambda \in \Sigma_{0,\frac{\pi}{4}} \). From Lemma 3.2 we have the following estimate.
Lemma 3.3. Let (H.1) be satisfied. Then we have
\[ \|e^{-At}P - Pe^{-At}\|_{\mathcal{L}(H,H')} \leq \frac{C}{r} \tau(\epsilon) \to 0, \quad \text{as } \epsilon \to 0 \]
for any \( t \in [r, T] \), here \( r > 0 \).

Proof. By (3.3) and Lemma 3.2, we can estimate
\[
\|e^{-At}P - Pe^{-At}\|_{\mathcal{L}(H,H')} = \| \frac{1}{2\pi i} \int_{\gamma} (\lambda I + A)^{-1} P e^{\lambda t} d\lambda - \frac{1}{2\pi i} \int_{\gamma} P (\lambda I + A)^{-1} e^{\lambda t} d\lambda \|_{\mathcal{L}(H,H')}
\leq C|\int_{\gamma_1 \cup \gamma_2} \tau(\epsilon) e^{\lambda t} d\lambda|.
\]
For \( \lambda \in \gamma_1 \cup \gamma_2 \), we compute \( |e^{\lambda t}| = |e^{re^{-\frac{i\pi}{4}}}| = e^{-\frac{\sqrt{2}r}{2}t} \) with \( 0 \leq r \leq +\infty \). Then we have
\[
\|e^{-At}P - Pe^{-At}\|_{\mathcal{L}(H,H')} \leq C \tau(\epsilon) \int_0^{+\infty} e^{-\frac{\sqrt{2}r}{2}t} dr \leq \frac{C}{t} \tau(\epsilon).
\]
Hence by (H.2)
\[
\|e^{-At}P - Pe^{-At}\|_{\mathcal{L}(H,H')} \leq \frac{C}{r} \tau(\epsilon) \to 0, \quad \text{as } \epsilon \to 0
\]
for any \( t \in [r, T] \), here \( r > 0 \). \qed

Now we state and prove our main result as the following.

Theorem 3.1. Suppose the conditions (H.1) to (H.3) hold true. Then we have
\[
E \sup_{0 \leq t \leq T} \|u^\epsilon(t) - P u(t)\|^2_{H^r} \leq \frac{C(T,R)(r^2 + r + \tau_0(\epsilon) + \tau_1(\epsilon) + \tau(\epsilon))}{1 - C(T)k_2}.
\]
In particular,
\[
E \sup_{0 \leq t \leq T} \|u^\epsilon(t) - P u(t)\|^2_{H^r} \to 0,
\]
when we first let \( \epsilon \to 0 \) and then \( r \to 0 \).

Proof. From equation (2.2) and equation (3.2), we have
\[
E \sup_{0 \leq t \leq T} \|u^\epsilon(t) - P u(t)\|^2_{H^r}.
\]
\[
I_1 \leq E \sup_{0 \leq t \leq r} \|e^{-A_t}u_0^\epsilon - Pe^{-At}u_0\|^2_{L^2} + E \sup_{0 \leq t \leq r} \|e^{-A_t}u_0^\epsilon - Pe^{-At}u_0\|^2_{L^2} \\
\leq 3E \sup_{0 \leq t \leq T} \|e^{-A_t}u_0^\epsilon - Pe^{-At}u_0\|^2_{L^2} + 3E \sup_{0 \leq t \leq r} \|Pe^{-At}u_0\|^2_{L^2} + 3E \sup_{0 \leq t \leq r} \|Pe^{-At}u_0\|^2_{L^2} \\
\leq CE \sup_{0 \leq t \leq T} \|u_0^\epsilon - Pu_0\|^2_{L^2} + C\tau(\epsilon) + C\frac{\tau(\epsilon)}{r} \\
\leq CE \sup_{0 \leq t \leq T} \|e^{-A_t} - I\|^2_{L^2} + CE \|u_0 - Pu_0\|^2_{L^2} + CE \sup_{0 \leq t \leq r} \|e^{-At} - I\|^2_{L^2} \\
\leq CE \sup_{0 \leq t \leq r} \|e^{-A_t} - I\|^2_{L^2} + C\tau(\epsilon) + CE \sup_{0 \leq t \leq r} \|e^{-At} - I\|^2_{L^2} \\
\leq C\left( E \sup_{0 \leq t \leq r} \|e^{-A_t} - I\|^2_{L^2} + E \sup_{0 \leq t \leq r} \|e^{-At} - I\|^2_{L^2} \right)
\]
We obtain

\[ + C \left( \frac{\tau(\epsilon)}{r} + \tau_0(\epsilon) \right). \]

For \( I_2 \) we have

\[
I_2 \leq 2E \sup_{0 \leq t \leq T} \left\| \int_0^t e^{-A_e(t-s)} \left( f^e(u^e) - Pf(u) \right) ds \right\|_{H^\epsilon}^2 + 2E \sup_{0 \leq t \leq T} \left\| \int_0^t \left( e^{-A_e(t-s)} P - Pe^{-A(t-s)} \right) f(u) ds \right\|_{H^\epsilon}^2
\]

\[
\leq 4E \sup_{0 \leq t \leq T} \left\| \int_0^t e^{-A_e(t-s)} \left( f^e(u^e) - f^e(Pu) \right) ds \right\|_{H^\epsilon}^2 + 2E \sup_{0 \leq t \leq T} \left\| \int_0^t \left( e^{-A_e(t-s)} P - Pe^{-A(t-s)} \right) f(u) ds \right\|_{H^\epsilon}^2
\]

\[
\leq 4T^2 k_2 E \sup_{0 \leq t \leq T} \left\| u^e - Pu \right\|_{H^\epsilon}^2 + 4T^2 \left\| f^e(Pu) - Pf(u) \right\|_{H^\epsilon}^2 + 2E \sup_{0 \leq t \leq T} \left\| \int_0^t \left( e^{-A_e(t-s)} P - Pe^{-A(t-s)} \right) f(u) ds \right\|_{H^\epsilon}^2.
\]

Denote \( I_{21} = E \sup_{0 \leq t \leq T} \left\| \int_0^t \left( e^{-A_e(t-s)} P - Pe^{-A(t-s)} \right) f(u) ds \right\|_{H^\epsilon}^2. \) Then we have

\[
I_{21} = E \sup_{0 \leq t \leq T} \left\| \int_0^t e^{-A_e(t-s)} \left( f^e(Pu) - Pf(u) \right) ds \right\|_{H^\epsilon}^2
\]

\[
\leq 2E \sup_{0 \leq t \leq T} \left\| \int_0^t \left( e^{-A_e(t-s)} P - Pe^{-A(t-s)} \right) f(u) ds \right\|_{H^\epsilon}^2
\]

\[
\leq C(R, T) r^2 + C(R, T) \frac{\tau(\epsilon)^2}{r^2}.
\]

Hence we obtain

\[
I_2 \leq 4T^2 k_2 E \sup_{0 \leq t \leq T} \left\| u^e - Pu \right\|_{H^\epsilon}^2 + 4T^2 \tau_1(\epsilon) + C(R, T) r^2 + C(R, T) \frac{\tau(\epsilon)^2}{r^2}.
\]
For $I_3$ we have

\[
I_3 \leq CE \int_0^T \left\| e^{-A(t-s)}g^\epsilon(u^\epsilon) - Pe^{-A(t-s)}g(u) \right\|^2_{H^r} \, ds \\
\leq CE \int_0^T \left\| e^{-A(t-s)}(g^\epsilon(u^\epsilon) - P g(u)) \right\|^2_{H^r} \, ds \\
+ CE \int_0^T \left\| (e^{-A(t-s)}P - Pe^{-A(t-s)})g(u) \right\|^2_{H^r} \, ds \\
\leq CE \int_0^T \left\| e^{-A(t-s)}(g^\epsilon(u^\epsilon) - g^\epsilon(P u)) \right\|^2_{H^r} \, ds \\
+ CE \int_0^T \left\| e^{-A(t-s)}(g^\epsilon(P u) - P g(u)) \right\|^2_{H^r} \, ds \\
+ CE \int_0^T \left\| (e^{-A(t-s)}P - Pe^{-A(t-s)})g(u) \right\|^2_{H^r} \, ds \\
\leq CTk_2E \sup_{0 \leq t \leq T} \left\| u^\epsilon - Pu \right\|^2_{H^r} + CT\tau_2(\epsilon) + CI_{31},
\]

where

\[
I_{31} = E \int_0^T \left\| (e^{-A(t-s)}P - Pe^{-A(t-s)})g(u) \right\|^2_{H^r} \, ds.
\]

Let $l = t - s$. Note that $t \geq s$ and $0 \leq t \leq T$. Then we have

\[
I_{31} = E \int_0^t \left\| (e^{-A(t-s)}P - Pe^{-A(t-s)})g(u) \right\|^2_{H^r} \, dl \\
= E \int_0^t \left\| (e^{-A(t-s)}P - Pe^{-A(t-s)})g(u) \right\|^2_{H^r} \, dl + E \int_t^T \left\| (e^{-A(t-s)}P - Pe^{-A(t-s)})g(u) \right\|^2_{H^r} \, dl \\
\leq C(T, R)r + C(T, R) \frac{\tau(\epsilon)}{r}.
\]

A combination of $I_1$, $I_2$ and $I_3$, we finally get

\[
E \sup_{0 \leq t \leq T} \left\| u^\epsilon(t) - Pu(t) \right\|^2_{H^r} \\
\leq C(T)k_2E \sup_{0 \leq t \leq T} \left\| u^\epsilon(t) - Pu(t) \right\|^2_{H^r} \\
+ C(T, R)(r^2 + r + \tau_0(\epsilon) + \tau_1(\epsilon)) + \frac{\tau(\epsilon)^2}{r^2} + \frac{\tau(\epsilon)}{r}.
\]

We can choose a sufficiently small $T$ such that $C(T)k_2 < 1$, thus

\[
E \sup_{0 \leq t \leq T} \left\| u^\epsilon(t) - Pu(t) \right\|^2_{H^r}.
\]

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\[
\leq \frac{C(T,R)(r^2 + r + \tau_0(\epsilon) + \tau_1(\epsilon) + \frac{\tau_2(\epsilon)}{r} + \tau_3(\epsilon))}{1 - C(T)k_2}.
\]

In particular,

\[
E \sup_{0 \leq t \leq T} \| u^\epsilon(t) - P u(t) \|_{H^\epsilon}^2 \to 0
\]
as \( \epsilon \to 0 \) and then \( r \to 0 \).  

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