SHARP EXTENSIONS FOR CONVOLUTED SOLUTIONS OF WAVE EQUATIONS

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Abstract. In this paper we give sharp extensions of convoluted solutions of wave equations in abstract Banach spaces. The main technique is to use the algebraic structure, for convolution products \( \ast \) and \( \ast_c \), of these solutions which are defined by a version of the Duhamel’s formula. We define algebra homomorphisms, for the convolution product \( \ast_c \), from a certain set of test-functions and apply our results to concrete examples of abstract wave equations.

1. Introduction

We consider the following evolution problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u, \quad (t,x) \in \mathbb{R}^+ \times V \\
u(0,x) &= u_0(x), \quad x \in V \\
u(t,\cdot)|_{\partial V} &= g(t,\cdot), \quad t > 0.
\end{aligned}
\]

(1.1)

with \( V \) is an open set in \( \mathbb{R}^n \). In [6], Duhamel proposed the following formula to express the solution of problem (1.1).

\[
u(t,x) = \int_0^t \frac{\partial}{\partial t} u(\lambda, t - \lambda, x) d\lambda, \quad t > 0,
\]

where \( u(\lambda, t, x) \) is a solution of problem (1.1) for a particular function \( g(\cdot, \lambda_0) \) fixed \( \lambda_0 \). Note that this formula reduces the Cauchy problem for an in-homogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation. This formula (known as Duhamel’s principle) is used in partial differential equations and have been studied in a large number of papers. We only mention here the paper [7] where an abstract point of view is treated to give solutions of in-homogeneous differential equations in a Banach space \( X \). Recently a fractional Duhamel’s principle have been introduced to study the Cauchy problem for inhomogeneous fractional differential equations in [30].

Let \( k : (0,T) \to \mathbb{C} \) a local integrable function, \( X \) be a Banach space and \( x \in X \). The Cauchy problem

\[
\begin{aligned}
v'(t) &= Av(t) + \int_0^t k(s)xds, \quad 0 < t < T, \\
v(0) &= 0,
\end{aligned}
\]

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is usual called $K$-convoluted Cauchy problem where $K(t) = \int_0^t k(s)x$ for $0 < t < T$. If there exists a solution of the abstract Cauchy problem $u'(t) = Au(t)$ for $0 < t < T$, $u(0) = x$ then, as usual for a nonhomogeneous equation, then we have $v = u \ast K$ with $\ast$ is the usual convolution in $\mathbb{R}^+$. The theory of local $k$-convoluted semigroup was introduced in [4, 5] and extends the classical theory of $C_0$-semigroups, see a complete treatment in [24, Section 1.3.1], [18, Chapter 2] and other details [13, 15, 20] and reference in. One of the interesting consequence of this theory is the extension property of the solution of $K_1$-convoluted problem, using a version of Duhamel’s formula, which increases the regularity of the problem, see [5, 18].

Now we consider the $K$-convoluted second order Cauchy problem (or $K$-convoluted wave equation),

\[
\begin{align*}
    v''(t) &= Av(t) + \int_0^t k(s)xds + \int_0^t (t - s)k(s)yds, & 0 < t < T, \\
    v(0) &= 0, & v'(0) = 0,
\end{align*}
\]

for $x, y \in X$. The existence of a unique solution of problem (1.2) is closely connected with the existence of a unique $k$-convoluted mild solution of the wave equation $u''(t) = Au(t)$ for $0 < t < T$, $u(0) = x$ and $u'(0) = y$ (see for example [18, Theorem 2.1.3.9, Corollary 2.1.3.16]).

The main objective of this paper is to illustrate the algebraic structure of this family of operators (above mentioned as local $k$-convoluted mild solution) and also known as local $k$-convoluted cosine functions. One condition in the definition of local $k$-convoluted cosine functions followed in this paper (Definition 3.1 (ii)) may be interpreted as a Duhamel formula for the wave problem. Other equivalent definitions of local $K$-convoluted cosine functions, using the composition property of local $k$-convoluted cosine functions ([17, Theorem 2.4]) or the Laplace transform ([15, Definition 2.1]) show this algebraic aspect in a straightforward way.

The usual convolution $\ast$ on $\mathbb{R}^+$, see (2.1), and the cosine convolution $\ast_c$ see (2.2), are the starting technical tools to show some of our results. In Section 2, we prove some identities (in particular Lemma 2.2) for these convolution products which we apply in later sections.

The extension formula in Theorem 3.3 which improves some previous results, [17, Theorem 3.1] and [18, Theorem 2.1.30 (v)], shows clearly how the algebraic properties fits perfectly with the inner structure of $k$-convoluted cosine function. This theorem and the composition property of $k$-convoluted cosine function allow to define algebra homomorphism from a particular set of test-functions in Theorem 4.1. In the global case these algebra homomorphisms were considered in [13, Theorem 6.5].

The special case of $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ with $\alpha > 0$ defines the $\alpha$-times integrated cosine function. Originally they were the first example of $k$-convoluted cosine function. Note that in this case the algebra homomorphism (for $\ast_c$) are known as distribution cosine function and were introduced in [16] and studied in [19, 25]. Algebra homomorphisms from cosine convolution algebras and $\alpha$-times integrated cosine functions were treated deeply in [27].

From other point of view, some particular connections between $k$-convoluted cosine function and vector valued homomorphisms (in particular ultradistributions and hyperfunction sines) have been studied in [21, Section 5] and [18, Section 3.6.3]. As these
authors comment, it is not clear how to obtain the corresponding results (similar to distribution cosine functions) in the case of ultradistribution and hyperfunction sines, see [21, Remark 14]. Our results in Section 4 show a path to define algebra homomorphisms using k-convoluted cosine functions.

Similar results as Theorem 3.3 and Theorem 4.1 may be proved for local k-convoluted C-cosine functions where C is an injective operator (see [17, Definition 1.1]); for \( C = I_X \), this definition corresponds to local k-convoluted C-cosine families, see Remark 3.4.

In the last section, we apply our results to several particular examples of global and local k-convoluted cosine functions which have appeared in the context of this literature. Similar results for k-convoluted semigroups, distribution semigroups and algebra homomorphisms (for \( * \)) hold and may be found in [14]. Remind that the abstract Weierstrass formula gives (global) \( \tilde{k} \)-convoluted semigroups subordinated to (global) \( k \)-convoluted cosine functions, where

\[
\tilde{k}(t) = \int_0^\infty \frac{se^{-s^2}}{2\sqrt{\pi t^{3/2}}}k(s)ds, \quad t \geq 0,
\]

see [21, Theorem 11].

2. Identities for convolutions products and k-Test function spaces

Let \( L^1_{loc}(\mathbb{R}^+) \) the set of locally integrable functions on \( \mathbb{R}^+ \). For \( f, g \in L^1_{loc}(\mathbb{R}^+) \), we consider the usual convolution given by

\[
(f \ast g)(t) = \int_0^t f(t - s)g(s)ds, \quad t \geq 0.
\]

For \( t \geq 0 \), we denote by \( \chi \) the constant function equals to 1, i.e., \( \chi(t) := 1 \), by \( I(t) := (\chi \ast \chi)(t) = t \). Consequently,

\[
(\chi \ast f)(t) = \int_0^t f(s)ds, \quad (I \ast f)(t) = \int_0^t (t - s)f(s)ds, \quad t \geq 0.
\]

We write \( f^{*1} = f, \; f^{*2} = f \ast f \) and \( f^{*n} = f \ast f^{*(n-1)} \) for \( n \in \mathbb{N} \). We also follow the notation \( \circ \) to denote the dual convolution product of \( * \) given by

\[
(f \circ g)(t) = \int_t^\infty f(s - t)g(s)ds, \quad t \geq 0,
\]

where \( f, g \in L^1(\mathbb{R}^+) \). Note that,

\[
\int_0^\infty f(t) (g \ast h)(t)dt = \int_0^\infty h(t) (g \circ f)(t)dt, \quad f, g, h \in L^1(\mathbb{R}^+).
\]

There are interesting equalities between both convolution products, for example

\[
k \circ (f \circ g) = (k \ast f) \circ g = (f \ast k) \circ g = f \circ (k \circ g), \quad f, g, k \in L^1(\mathbb{R}^+).
\]

The cosine convolution product \( f \ast_c g \) is defined by

\[
f \ast_c g := \frac{1}{2}(f \ast g + f \circ g + g \circ f), \quad f, g \in L^1(\mathbb{R}^+).
\]
For the show of the previous statement and more properties of this convolution see [13, 26, 27]. We denote by \( \hat{f} \) the usual Laplace transform of a function \( f \in L_{loc}^1(\mathbb{R}^+) \) given by

\[
\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt,
\]

for \( \lambda \in \mathbb{C} \) such that this integral converges.

We denote by \( D \) the set of \( C^{(\infty)} \) functions with compact support on \( \mathbb{R} \). We write by \( D_+ \) the set of functions defined by \( \phi_+ : [0, \infty) \to \mathbb{C} \), given by \( \phi_+(t) := \phi(t) \) for \( t \geq 0 \) and \( \phi \in D \). Note that if \( \psi \) is a \( C^{(\infty)} \) functions on \([0, \infty)\) and compact support then \( \psi \in D_+, \) see [29]. All these spaces are topological vector spaces equipped with the topology of uniform convergence on bounded subsets.

**Proposition 2.1.** Let \( \phi, \psi \in D_+ \) and \( t \geq 0 \). Then

(i) \((\phi' \ast \psi)(t) = (\phi \ast \psi')(t) + \psi(0) \phi(t) - \phi(0) \psi(t)\).

(ii) \((\phi \ast \psi')(t) = -\phi(0) \psi(t) - (\phi \ast \psi')(t)\).

(iii) \((\phi \ast \psi')(t) = -\phi(0) \psi(t) - (\phi \ast \psi')(t)\).

(iv) \((\phi' \ast \phi)(t) = \frac{1}{2} \left[ \phi \ast \psi' - \phi' \ast \psi' - \psi' \circ \phi \right](t) - \phi(0) \psi(t)\).

(v) \((\phi'' \ast \phi)(t) = (\phi \ast \phi)(t) + \psi'(0) \phi(t) - \phi'(0) \psi(t)\).

**Proof.** The part (i) appears in [31, Proposition 3.1 (iii)]. Integrating by parts we obtain

\[
(\phi' \ast \psi)(t) = \int_t^\infty \phi'(s-t) \psi(s) ds = -\phi(0) \psi(t) - (\phi \ast \psi')(t), \quad t \geq 0,
\]

hence (ii) and (iii) are shown. The item (iv) is straightforward from the definition of \(*_c\) and parts (i), (ii) and (iii). Now to show (v), we apply parts (i)-(iv) to get that

\[
(\phi'' \ast \phi)(t) = \frac{1}{2} \left[ \phi' \ast \psi' - \phi' \circ \psi' - \psi' \circ \phi' \right](t) - \phi(0) \psi(t)
\]

\[
= \frac{1}{2} \left[ \phi \ast \phi'' + \phi \ast \psi'' + \phi'' \circ \phi' \right](t) + \psi'(0) \phi(t) - \phi'(0) \psi(t).
\]

\[\square\]

**Lemma 2.2.** Let \( 0 \leq t \leq s \) and \( f, g \in L_{loc}^1(\mathbb{R}^+) \). Then

(a) \( (\chi \ast g)(t)(\chi \ast f)(s) = \int_s^{t+s} g(t + r - s) (\chi \ast f)(r) dr - \int_0^t f(t + r - s) (\chi \ast g)(r) dr \)

(b) \( (\chi \ast g)(t)(\chi \ast f)(s) = \int_{s-t}^s g(t + r - s) (\chi \ast f)(r) dr + \int_0^t f(r + s - t) (\chi \ast g)(r) dr \).

**Proof.** First, we show (a). Integrating by parts and using change of variable we have
\[
\int_0^t f(t + s - r)(\chi * g)(r)dr
\]
\[
= - \int_0^{t+s} f(t + s - r)dr \int_0^t g(u)du + \int_0^t g(r)\int_r^{t+s} f(t + s - u)du dr
\]
\[
= - \int_0^s f(r)dr \int_0^t g(u)du + \int_s^{t+s} g(t + s - r)\int_r^{t+s} f(t + s - u)du dr
\]
\[
= - \int_0^s f(r)dr \int_0^t g(u)du + \int_s^{t+s} g(t + s - r)\int_r^s f(u)du dr
\]
\[
= -(\chi * f)(s)(\chi * g)(t) + \int_s^{t+s} g(t + s - r)(\chi * f)(r)dr.
\]

Now, we prove (b), using the Fubini theorem and change the variable we obtain
\[
\int_0^t f(t + s - r)(\chi * g)(r)dr = \int_0^t g(u)\int_u^t f(t + s - r)dr du
\]
\[
= \int_{s-t}^s g(u + t - s)\int_u^t f(t + s - r)dr du = \int_{s-t}^s g(u + t - s)f(r)dr du
\]
\[
= \int_{s-t}^s g(u + t - s)\int_0^{2s-u} f(r)dr du - \int_{s-t}^s g(u + t - s)\int_0^u f(r)dr du
\]
\[
= \int_{s-t}^s g(u + t - s)\int_u^s f(r)dr du
\]
\[
= \int_s^{t+s} g(t + s - u)(\chi * f)(u)du - \int_{s-t}^s g(u + t - s)(\chi * f)(u)du
\]
\[
= \int_0^t f(r + s - t)(\chi * g)(r)dr.
\]

Hence,
\[
\int_s^{t+s} g(t + s - r)(\chi * f)(r)dr - \int_0^t f(t + s - r)(\chi * g)(r)dr =
\]
\[
\int_{s-t}^s g(r + t - s)(\chi * f)(r)dr + \int_0^t f(r + s - t)(\chi * g)(r)dr,
\]
from (a) follows (b).

\[\square\]

\textbf{Remark 2.3.} If in Lemma 2.2 we taking \( f = g \) in (a) then
\[
(\chi * f)(t)(\chi * f)(s) = \int_s^{t+s} f(t + s - r)(\chi * f)(r)dr - \int_0^t f(t + s - r)(\chi * f)(r)dr.
\]
If $s = t$ in (b) then
\[
(\chi \ast g)(t)(\chi \ast f)(t) = \int_0^t g(r) (\chi \ast f)(r) dr + \int_0^t f(r) (\chi \ast g)(r) dr.
\]
We conclude that
\[
[(\chi \ast f)(t)]^2 = \int_t^{2t} f(2t - r)(\chi \ast f)(r) dr - \int_0^t f(2t - r)(\chi \ast f)(r) dr = 2 \int_0^t f(r) (\chi \ast f)(r) dr.
\]

We write by supp($h$) the usual support of a function $h$ defined in $\mathbb{R}$. The operator $T_k^r : \mathcal{D}_+ \to \mathcal{D}_+$ is given by $f \mapsto T_k^r(f) := k \circ f$. In the case that $0 \in \text{supp}(k)$, from [13] Theorem 2.5, we have that $T_k^r$ is an injective, linear and continuous homomorphism such that
\[
T_k^r(f \circ g) = f \circ T_k^r(g), \quad f, g \in \mathcal{D}_+.
\]

According to [13] Definition 2.7, the space $\mathcal{D}_k$ is given by $\mathcal{D}_k := T_k^r(\mathcal{D}_+) \subset \mathcal{D}_+$ and the right inverse map $T_k^r$ is $W_k : \mathcal{D}_k \to \mathcal{D}_+$ defined by
\[
f(t) = T_k^r(W_k(f))(t) = \int_t^{\infty} k(s - t) W_k f(s) ds, \quad f \in \mathcal{D}_k, \quad t \geq 0.
\]
It is clear that the subspace $\mathcal{D}_k$ is also a topological vector space.

The following result are show in [13] Section 3].

**Proposition 2.4.** Take $k \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $0 \in \text{supp}(k)$. Then

(i) For $a > 0$, supp$(f) \subset [0, a]$ if and only if supp$(W_k f) \subset [0, a]$ for $f \in \mathcal{D}_k$.

(ii) $\mathcal{D}_{k \ast n} \subset \mathcal{D}_{k \ast m}$ and $W_{k \ast n} f = k^{n-m} \circ W_{k \ast n} f = W_{k \ast n}(k^{n-m} \circ f)$ for $f \in \mathcal{D}_{k \ast n}$, and $n \geq m \geq 1$.

(iii) The space $\mathcal{D}_{k \ast \infty} := \bigcap_{n=1}^{\infty} \mathcal{D}_{k \ast n}$ is a topological vector space, $W_{k \ast n} \in \mathcal{L}(\mathcal{D}_{k \ast \infty})$ and $k^{n} \circ W_{k \ast \infty} f = f$ for $f \in \mathcal{D}_{k \ast \infty}$ and $n \in \mathbb{N}$.

**Example 2.5.** (i) Let $\alpha > 0$ and $j_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. The map $W_{j_\alpha}$ is the Weyl fractional derivative of order $\alpha$ (usually written by $W^\alpha$), and $\mathcal{D}_{j_\alpha} = \mathcal{D}_+$; in the case $\alpha \in \mathbb{N}$, we have $W^\alpha = (-1)^\alpha \frac{d^\alpha}{dt^\alpha}$ (see [9] 2.3).

(ii) Let $\chi_{(0,1)}$ the characteristic function on the interval $(0, 1)$. Then the operator $T'_{\chi_{(0,1)}}$ is given by
\[
T'_{\chi_{(0,1)}}(f)(t) = \int_t^{t+1} f(s) ds, \quad f \in \mathcal{D}_+, \quad t \geq 0,
\]
and $\mathcal{D}_{\chi_{(0,1)}} = \mathcal{D}_+$ and
\[
W_{\chi_{(0,1)}} f(t) = - \sum_{n=0}^{\infty} f'(t + n), \quad f \in \mathcal{D}_+, \quad t \geq 0,
\]
see [13] Section 2].

(iii) Let $0 < \delta < 1$ and $r > 0$. The functions $K_\delta$ given by
\[
K_\delta(t) := \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{\lambda t - \lambda^\delta} d\lambda, \quad t \geq 0,
\]
verify that \( \hat{K}_h(\lambda) = e^{-\lambda^x} \) for \( \lambda \in \mathbb{C}^+ \). It is known that \( 0 \in \text{supp}(K_h) \) (see [2, p. 107]) and we may define \( W_{K_h} \) and \( D_{K_h} \). This function was considered in [3, Section 5], [20] Example 6.1] and [21] Example 6.1].

(iv) Let
\[
\mathcal{K}(\lambda) := \frac{1}{\lambda^2} \prod_{n=0}^{\infty} \frac{n^2 - \lambda}{n^2 + \lambda}, \quad \Re \lambda > 0.
\]
Then there exists a continuous and exponential bounded function \( \kappa \) in \([0, \infty)\) with \( \hat{\kappa} = \mathcal{K} \), \( 0 \in \text{supp}(\kappa) \) and \( D_\kappa \subset \mathcal{D}_+ \), see [14] Section 3]. The following example was presented in [3, Section 5] and appeared later in other references in connection to convoluted semigroups (see [20] Example 6.1] and [21] Example 6.1]).

3. Local convoluted cosine functions

The notion of \( k \)-convoluted cosine functions appears implicitly introduced in [23], as a generalization of the concept of \( n \)-times integrated cosine functions given in [1] Section 6]. We will consider the following definition of local \( k \)-convoluted cosine as appear in [15, 21]. As we commented in the Introduction the condition (ii) in the next definition may be considered as a Duhamel’s formula for the abstract wave equation.

**Definition 3.1.** Let \( A \) a closed operator, \( k \in L^1_{\text{loc}}([0, \tau)) \) and \( 0 < \tau \leq \infty \). A strongly continuous operator family \( (C_k(t))_{t \in [0, \tau)} \) is said a local \( k \)-convoluted cosine function generated by \( A \) if

(i) \( C_k(t)A \subset AC_k(t), \quad t \in [0, \tau) \),

(ii) for all \( x \in X \) and \( t \in [0, \tau) \) : 
\[
\int_0^t (t-s)C_k(s)x ds \in D(A) \text{ and } A \int_0^t (t-s)C_k(s)x ds = C_k(t)x - (\chi \ast k)(t)x.
\]

If \( A \) generates a local \( k \)-convoluted cosine function \( (C_k(t))_{t \in [0, \tau)} \) then \( C_k(0) = 0 \). Taking \( \tau = \infty \) we have that \( (C_k(t))_{t \in [0, \infty)} \) is an exponentially bounded, \( k \)-convoluted cosine function with generator \( A \), for view its properties, see [13, 15, 21], among others. For exponentially bounded functions the \( k \)-convoluted cosine functions are given in terms of the Laplace transform (see [15, Definition 2.1]).

Plugging \( k = j_\alpha \) (\( \alpha > 0 \)) in above definition, we obtain the well-known classes of \( \alpha \)-times integrated cosine function. A characterization of exponentially bounded \( \alpha \)-times integrated cosine function in terms of its Laplace transform was obtained in [22]. The relationship between \( \alpha \)-times integrated cosine function and the operator Bessel function are studied in [10].

The following proposition is a direct consequence of [15, Lemma 4.4].

**Lemma 3.2.** Let \( A \) be a generator of a local \( k \)-convoluted cosine function \( (C_k(t))_{t \in [0, \tau)}, \) and let \( h \in L^1_{\text{loc}}([0, \tau)) \) such that \( k \ast h \neq 0 \) in \( L^1_{\text{loc}}([0, \tau)) \). Then \( A \) is a generator of an \((h \ast k)\)-convoluted cosine function \( ((h \ast C_k)(t))_{t \in [0, \tau)} \).

The next theorem is the main result in this paper. Note that we may extend the support of the solution from \([0, \tau)\) to \([0, n\tau)\) for any \( n \in \mathbb{N} \). This technique, based in results given in...
the previous section, improves the result of Kostić, see [17] Theorem 3.1 and [18] Theorem 2.1.1.14 in the case \( C = I_X \).

**Theorem 3.3.** Let \( n \in \mathbb{N}, 0 < \tau \leq \infty, k \in L^1([0,(n+1)\tau)) \) and \((C_k(t))_{t \in [0,\tau)} \) is a local \( k \)–convoluted cosine function generated by \( A \). The family of operators \((C_{k^{*}(n+1)}(t))_{t \in [0,(n+1)\nu]} \) defined for \( t \in [0,\nu] \) by

\[
C_{k^{*}(n+1)}(t)x = \int_0^tk(t-r)C_{k^{*}n}(r)x \, dr,
\]

and for \( t \in [\nu, (n+1)\nu] \) as

\[
C_{k^{*}(n+1)}(t)x = 2C_{k^{*}n}(\nu)C_k(t-\nu)x + \int_0^{\nu} k(t-r)C_{k^{*}n}(r)x \, dr
\]

\[
+ \int_0^{t-\nu} k^{*n}(t-r)C_k(r)x \, dr - \int_{2\nu-1}^{\nu} k(r+t-2\nu)C_{k^{*}n}(r)x \, dr
\]

\[
- \int_0^{t-\nu} k^{*n}(r+2\nu)C_k(r)x \, dr,
\]

is a local \( k^{*}(n+1) \)–convoluted cosine function with generator \( A \) for any \( \nu < \tau \) and \( x \in X \).

**Proof.** Is simple verify that \( k^{*}(n+1) \in L^1_{\text{loc}}([0,(n+1)\tau)) \) and that \( k^{*}(n+1) \) is not identical to zero. \((C_{k^{*}(n+1)}(t))_{t \in [0,(n+1)\nu]} \) is a strongly continuous operator family which commutes with \( A \). By Proposition 3.2 one gets that \((k \ast C_{k^{*}n}(t))_{t \in [0,\nu]} \) is a local \( k^{*}(n+1) \)–convoluted cosine function with generator \( A \) and consequently (ii) of Definition 3.1 holds for all \( t \in [0,\nu] \) and \( x \in X \). It remains to be show that this condition is true for every \( t \in [\nu, (n+1)\nu] \) and \( x \in X \). Write

\[
\int_0^t (t-s)C_{k^{*}(n+1)}(s)x \, ds = \int_0^\nu (t-s)C_{k^{*}(n+1)}(s)x \, ds + \int_t^\nu (t-s)C_{k^{*}(n+1)}(s)x \, ds := J + I,
\]

but, \( J = \int_0^\nu (\nu-s)C_{k^{*}(n+1)}(s)x \, ds + (t-\nu) \int_0^\nu C_{k^{*}(n+1)}(s)x \, ds \).

Let \( I = I_1 + I_2 + I_3 - I_4 - I_5 \) where

\[
I_1 = 2C_{k^{*}n}(\nu) \int_0^t (t-s)C_k(s-\nu)x \, ds, \quad I_2 = \int_0^t (t-s) \int_0^{\nu} k(s-r)C_{k^{*}n}(r)x \, dr \, ds,
\]

\[
I_3 = \int_0^t (t-s) \int_0^{s-\nu} k^{*n}(s-r)C_k(r)x \, dr \, ds,
\]

\[
I_4 = \int_0^t (t-s) \int_{2\nu-s}^{\nu} k(r+s-2\nu)C_{k^{*}n}(r)x \, dr \, ds,
\]

\[
I_5 = \int_0^t (t-s) \int_0^{s-\nu} k^{*n}(r+s-2\nu)C_k(r)x \, dr \, ds.
\]
We have $I_1 = 2C_{k^n}(nu)\int_0^{t-nu} (t-nu-s)C_k(s)ds$. For $I_2$ note that

$$I_2 = \int_0^{nu} C_{k^n}(r)x \int_0^{t} (t-s)k(s-r)ds dr.$$ 

From the simple equality

$$\int_0^{t} (t-s)k(s-r)ds = (I * k)(t-r) - (t-nu)(\chi * k)(nu-r) - (I * k)(nu-r),$$

it follows that

$$I_2 = \int_0^{nu} (I * k)(t-r)C_{k^n}(r)xdr - \int_0^{nu} (t-nu)(\chi * k)(nu-r)C_{k^n}(r) xdr$$

$$- \int_0^{nu} (I * k)(nu-r)C_{k^n}(r) xdr = I_{21} - I_{22} - I_{23}.$$ 

First, we compute $I_{21}$.

$$I_{21} = \int_0^{nu} C_{k^n}(r)x \int_0^{t-r} (t-r-s)k(s)ds dr$$

$$= \int_0^{t-nu} k(s) \int_0^{nu} (t-r-s)C_{k^n}(r)xdr ds + \int_0^{t-nu} k(s) \int_0^{t-s} (t-r-s)C_{k^n}(r)xdr ds,$$

with change of variable $y = t-s$ we have

$$\int_0^{t-nu} k(s) \int_0^{nu} (t-r-s)C_{k^n}(r)xdr ds = \int_0^{nu} k(t-y) \int_0^{y} (y-r)C_{k^n}(r)xdr dy,$$

on the other hand,

$$\int_0^{t-nu} k(s) \int_0^{nu} (t-r-s)C_{k^n}(r)xdr ds = \int_0^{nu} C_{k^n}(r)x \int_0^{t-nu} (t-r-s)k(s)ds dr$$

$$= (I * k)(t-nu) \int_0^{nu} C_{k^n}(r)xdr + (\chi * k)(t-nu) \int_0^{nu} (nu-r)C_{k^n}(r)xdr.$$ 

Hence,

$$I_{21} = \int_0^{nu} k(t-y)(I * C_{k^n})(y)x dy + (I * k)(t-nu)(\chi * C_{k^n})(nu)$$

$$+ (\chi * k)(t-nu)(I * C_{k^n})(nu)x.$$ 

Now, for $I_{22}$ from [3.4] observe that

$$I_{22} = (t-nu)(\chi * C_{k^n})(nu)x = (t-nu)(\chi * k * C_{k^n})(nu)x = (t-nu)(\chi * C_{k(n+1)})(nu)x.$$ 

We obtain that

$$I_2 = \int_0^{nu} k(t-y)(I * C_{k^n})(y)x dy + (I * k)(t-nu)(\chi * C_{k^n})(nu)x$$

$$+ (\chi * k)(t-nu)(I * C_{k^n})(nu)x - (t-nu)(\chi * C_{k(n+1)})(nu)x - (I * k * C_{k^n})(nu)x.$$
With the change of variable $s$, we integrate twice by parts to obtain that

\[
I_3 = \int_0^{t-n\nu} C_k(r)x \int_{r+s}^t (t-s)k^{*n}(s-r) \, ds \, dr = \int_0^{t-n\nu} C_k(r)x \int_{r}^{t-r} (t-s-r)k^{*n}(s) \, ds \, dr
\]

\[
= \int_0^{t-n\nu} [(I \ast k^{*n})(t-r) - (I \ast k^{*n})(n\nu)]C_k(r)x dr - (\chi \ast k^{*n})(n\nu) (I \ast C_k)(t - n\nu)
\]

\[
= \int_0^{t-n\nu} \int_0^r C_k(s)ds (\chi \ast k^{*n})(t-r)dr - (\chi \ast k^{*n})(n\nu) (I \ast C_k)(t - n\nu)
\]

\[
= (\chi \ast k^{*n})(n\nu) (I \ast C_k)(t - n\nu) + \int_0^{t-n\nu} (I \ast C_k)(r) k^{*n}(t-r)dr
\]

\[- (\chi \ast k^{*n})(n\nu) (I \ast C_k)(t - n\nu).
\]

Hence $I_3 = \int_0^{t-n\nu} (I \ast C_k)(r) k^{*n}(t-r)dr$.

For $I_4$, using Fubini theorem and change the variable, we have

\[
I_4 = \int_{2n\nu-t}^{n\nu} C_{k^{*n}}(r)x \int_{2n\nu-r}^{t} (t-s)k^{*n}(r+s-2n\nu) \, ds \, dr = \int_{2n\nu-t}^{n\nu} (I \ast k^{*n})(t+r-2n\nu)C_{k^{*n}}(r)xdr.
\]

We integrate twice by parts to obtain that

\[
I_4 = (I \ast k)(t - n\nu) \int_0^{n\nu} C_{k^{*n}}(r)xdr - \int_{2n\nu-t}^{n\nu} (\chi \ast k)(t + r - 2n\nu) \int_0^r C_{k^{*n}}(\tau)x\tau \, d\tau \, d\tau
\]

\[
+ \int_{2n\nu-t}^{n\nu} k(t + r - 2n\nu) \int_0^r (r - \tau)C_{k^{*n}}(\tau)x\tau \, d\tau \, d\tau.
\]

Applying the same argumentation, for $I_5$, again by Fubini

\[
I_5 = \int_0^{t-n\nu} C_k(r)x \int_{n\nu+u}^{t} (t-s)k^{*n}(u+2n\nu-s) \, ds \, dr.
\]

With the change of variable $s = u + 2n\nu - s$ and integration by parts, we have

\[
I_5 = (\chi \ast k^{*n})(n\nu) \int_0^{t-n\nu} (t-n\nu-r)C_k(r)xdr - (I \ast k^{*n})(n\nu) \int_0^{t-n\nu} C_k(r)xdr
\]

\[
+ \int_0^{t-n\nu} (I \ast k^{*n})(r+2n\nu-t)C_k(r)xdr
\]

\[
= (\chi \ast k^{*n})(n\nu) \int_0^{t-n\nu} (t-n\nu-r)C_k(r)xdr - (I \ast k^{*n})(n\nu) \int_0^{t-n\nu} C_k(r)xdr
\]

\[
+ (I \ast k^{*n})(n\nu) \int_0^{t-n\nu} C_k(r)xdr - \int_0^{t-n\nu} (\chi \ast k^{*n})(r+2n\nu-t) \int_0^r C_k(s)ds \, dr
\]

\[
+ (\chi \ast k^{*n})(n\nu) \int_0^{t-n\nu} (t-n\nu-r)C_k(r)xdr - (\chi \ast k^{*n})(n\nu) \int_0^{t-n\nu} C_k(s)xds
\]

\[
+ \int_0^{t-n\nu} k^{*n}(r+2n\nu-t) \int_0^r (r-s)C_k(s)ds \, dr.
\]
Hence $I_5 = \int_0^{t-n\nu} k^{*n}(r + 2n\nu - t) (I * C_k)(r) \, dr$. Adding $I_1 + I_2 + I_3 - I_4 - I_5$, it follows that

$$I = 2C_k(n\nu)(I * C_k)(t - n\nu)x + 2(\chi * k)(t - n\nu)(I * C_k^{*n})(n\nu)x$$

$$- (t - n\nu)(\chi * C_k^{*n+1})(n\nu) + \int_0^{n\nu} [k(t - y) - k(n\nu - y)](I * C_k^{*n})(y)x \, dy$$

$$+ \int_0^{t-n\nu} (I * C_k)(r) k^{*n}(t - r) \, dr - \int_{2n\nu-t}^{n\nu} k(t + r - 2n\nu)(I * C_k^{*n})(r)x \, dr$$

$$- \int_0^{t-n\nu} k^{*n}(r + 2n\nu - t) (I * C_k)(r) \, dr.$$  

From (3.3), for every $t \in [n\nu, (n+1)\nu]$ and $x \in X$, we obtain

$$\int_0^t (t-s)C_k^{*n+1}(s)x \, ds = \int_0^{n\nu} (n\nu - s)C_k^{*n+1}(s)x \, ds + 2C_k(n\nu)(I * C_k)(t - n\nu)x$$

$$+ 2(\chi * k)(t - n\nu)(I * C_k^{*n})(n\nu)x + \int_0^{t-n\nu} (I * C_k)(r) k^{*n}(t - r) \, dr$$

$$+ \int_0^{n\nu} [k(t - y) - k(n\nu - y)](I * C_k^{*n})(y)x \, dy - \int_{2n\nu-t}^{n\nu} k(t + r - 2n\nu)(I * C_k^{*n})(r)x \, dr$$

$$- \int_0^{t-n\nu} k^{*n}(r + 2n\nu - t) (I * C_k)(r) \, dr.$$  

The last equality implies $\int_0^t (t-s)C_k^{*n+1}(s)x \, ds \in D(A)$ and

$$A \int_0^t (t-s)C_k^{*n+1}(s)x \, ds = C_k^{*n+1}(t)x - 2(\chi * k)(t - n\nu) (\chi * k^{*n})(n\nu)$$

$$- \int_0^{t-n\nu} k^{*n}(t - r)(\chi * k)(r) \, dr - \int_0^{n\nu} k(t - r)(\chi * k^{*n})(r) \, dr$$

$$+ \int_{2n\nu-t}^{n\nu} k(r + 2n\nu - t) (\chi * k^{*n})(r)x \, dr$$

$$+ \int_0^{t-n\nu} k^{*n}(r + 2n\nu - t) (\chi * k)(r) \, dr.$$  

From Lemma 2.2 we obtain that $A \int_0^t (t-s)C_k^{*n+1}(s)x \, ds = C_k^{*n+1}(t)x - (\chi * k^{*n+1})(t)x$ for $t \geq 0$ and $x \in X$.  

**Remark 3.4.** Local $k$-convoluted $C$-cosine functions were introduced and studied by Kostić in [15, 17, 18] with $C$ an injective operator. The main idea is to compose a local $k$-convoluted cosine function (possibly a family of unbounded operators) with an injective
operator $C$ to get a family of bounded operators, i.e., we have to replace the condition (ii) in Definition 3.1 by

$$A \int_0^t (t-s)C_k(s)x ds = C_k(t)x - (\chi * k)(t)Cx, \quad x \in X, \quad t \in [0, \tau),$$

and to add a commutative condition, $C_k(t)C = CC_k(t)$, for $t \in [0, \tau)$, see [17, Definition 1.1]. Note that for $C = I_X$, we obtain the local $k$-convoluted cosine function.

A similar result to Theorem 3.3 for $k$-convoluted $C$-cosine function holds and the proof is left to the reader. The main point is to express the extension formula given in (3.5) and in this case,

$$C_{k^{(n+1)}}(t)x = 2C_{k^{(n)}}(\nu \nu)C_k(t - \nu \nu)x + \int_0^{\nu \nu} k(t-r)C_{k^{(n)}}(r)Cx dr$$

$$+ \int_0^{t-\nu \nu} k^{(n)}(t-r)C_k(r)Cx dr - \int_{\nu \nu - t}^{\nu \nu} k(r + t - 2\nu \nu)C_{k^{(n)}}(r)Cx dr$$

$$- \int_0^{t-\nu \nu} k^{(n)}(r - t + 2\nu \nu)C_k(r)Cx dr.$$

Note that we recover the extension theorem given [17, Theorem 3.1] and [18, Theorem 2.1.1.14] for $n = 1$. Moreover, if we iterate [17, Theorem 3.1], then we obtain only the extension for $n = 2^m$ and $m \in \mathbb{N}$. By uniqueness of solution of problem (1.2) both expressions coincide.

Note that the extension problem for $C$-cosine functions may be also found in [32, Theorem 3.6, Corollary 3.8].

In the particular case of $k = j_{\alpha}$ we obtain the next result for $\alpha$-times integrated cosine functions.

**Corollary 3.5.** Let $n \in \mathbb{N}, 0 < \tau \leq \infty$ and $(C_{\alpha}(t))_{t \in [0, \tau)}$ is a local $\alpha$-times integrated cosine function generated by $A$. The family of operators $(C_{\alpha^{(n+1)}}(t))_{t \in [0, (n+1)\nu]}$ defined for $t \in [0, \nu \nu]$ by

$$C_{\alpha^{(n+1)}}(t)x = \int_0^t \frac{(t-r)^{\alpha - 1}}{\Gamma(\alpha)} C_{\alpha}(r)x dr,$$

and for $t \in [\nu \nu, (n+1)\nu]$ as

$$C_{\alpha^{(n+1)}}(t)x = 2C_{\alpha}(\nu \nu)C_{\alpha}(t - \nu \nu)x + \int_0^{\nu \nu} \frac{(t-r)^{\alpha - 1}}{\Gamma(\alpha)} C_{\alpha}(r)x dr$$

$$+ \int_0^{t-\nu \nu} \frac{(t-r)^{\alpha - 1}}{\Gamma(\alpha)} C_{\alpha}(r)x dr - \int_{\nu \nu - t}^{\nu \nu} \frac{(r + t - 2\nu \nu)^{\alpha - 1}}{\Gamma(\alpha)} C_{\alpha}(r)x dr$$

$$- \int_0^{t-\nu \nu} \frac{(r - t + 2\nu \nu)^{\alpha - 1}}{\Gamma(\alpha)} C_{\alpha}(r)x dr,$$

is a local $\alpha(n + 1)$-times integrated cosine function with generator $A$ for any $\nu < \tau$ and $x \in X$. 
4. Algebra homomorphisms defined by convoluted cosine functions

In this section, we show that local $k$-convoluted cosine functions are kernels to define algebra homomorphism from a certain set of test functions $D_{k^{\infty}}$ (Theorem 4.1). Note that the extension theorem (Theorem 3.3) is necessary to define the algebra homomorphisms from functions defined on $\mathbb{R}^+$. The set $D_{k^{\infty}}$ was introduced in the context of local $k$-convoluted semigroups and $k$-distribution semigroups in $D_{k^{\infty}}$. However the cosine convolution $*_{c}$ (instead of $*$ convolution in semigroup setting) is the product which we must consider in the cosine setting, see Remarks 4.2 (iii).

**Theorem 4.1.** Let $k \in L^{1}_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, and $(C_{k}(t))_{t \in [0,\tau]}$ a non-degenerate local $k$-convoluted cosine function generated by $A$. We define the map $C_{k}: D_{k^{\infty}} \to \mathcal{B}(X)$ by

$$C_{k}(f)x := \int_{0}^{\tau} W_{k^{n}}f(t)C_{k^{n}}(t)x dt, \quad x \in X, supp(f) \subset [0,\tau], n \in \mathbb{N},$$

where $(C_{k^{n}}(t))_{t \in [0,\tau]}$ is defined in Theorem 3.3. Then the following properties hold.

(i) The map $C_{k}$ is well defined, linear and bounded.

(ii) For $\phi, \psi \in D_{k^{\infty}}$, we get that

$$C_{k}(\phi *_{c} \psi) = C_{k}(\phi)C_{k}(\psi).$$

(iii) $C_{k}(f)x \in D(A)$ and $AC_{k}(f)x = C_{k}(f^{n})x + f'(0)x$ for any $f \in D_{k^{\infty}}$ and $x \in X$.

**Proof.** First we prove (i). Take $f \in D_{k^{\infty}}$ and $supp(f) \subset [0,\tau]$ for some $n \in \mathbb{N}$. Let $m \geq n$, $k^{*m} = k^{*n} * k^{*(m-n)}$, and $k^{*(m-n)} \circ W_{k^{m}}f = W_{k^{n}}f$ and $supp(W_{k^{m}}f) \subset [0,\tau]$ by Proposition 2.4 (ii) and (i) respectively. By Lemma 3.2 and the Fubini theorem, we get that

$$\int_{0}^{\tau} W_{k^{m}}f(t)C_{k^{m}}(t)x dt = \int_{0}^{\tau} W_{k^{m}}f(t)(k^{*(m-n)} \circ C_{k^{n}})(t)x dt$$

$$= \int_{0}^{\tau} (k^{*(m-n)} \circ W_{k^{m}}f)(t)C_{k^{n}}(t)x dt = \int_{0}^{\tau} W_{k^{n}}f(t)C_{k^{n}}(t)x dt$$

for $x \in X$ and we conclude that $C_{k}$ is well defined. It is direct to check that $C_{k}$ is linear and bounded.

Take $f, g \in D_{k^{\infty}}$. By [13, Theorem 2.10] we have that $f *_{c} g \in D_{k^{\infty}}$ for $n \geq 1$ and then $f *_{c} g \in D_{k^{\infty}}$. Next we show that $C_{k}(f *_{c} g) = C_{k}(f)C_{k}(g)$. Take $n \in \mathbb{N}$ such that $supp(f), supp(g) \subset [0,\tau]$ and by Proposition 2.4 (i), $supp(W_{k^{2n}}f), supp(W_{k^{2n}}g) \subset [0,\tau]$. Then $supp(f * g) \subset [0,2\tau]$ and $supp(W_{k^{2n}}(f * g)) \subset [0,2\tau]$. By the definition of the cosine product $*_{c}$ (2.2) and [13, Theorem 2.10] we have that

$$C_{k}(f *_{c} g)x = \frac{1}{2} \int_{0}^{2\tau} (W_{k^{2n}}(f * g) + f \circ W_{k^{2n}}g + g \circ W_{k^{2n}}f)(t)C_{k^{2n}}(t)x dt,$$

for $x \in X$. Again by [13, Theorem 2.10], we obtain that
\[
\int_0^{2n\tau} W_{k^{2n}}(f \ast g)(t) C_{k^{2n}}(t)x dt \\
= \int_0^{2n\tau} C_{k^{2n}}(t)x \int_0^t W_{k^{2n}} g(r) \int_{t-r}^t k^{2n}(s+r-t)W_{k^{2n}} f(s) dr dt \\
- \int_0^{2n\tau} C_{k^{2n}}(t)x \int_0^\infty W_{k^{2n}} g(r) \int_t^\infty k^{2n}(s+r-t)W_{k^{2n}} f(s) dr dt.
\]

We apply Fubini theorem to get

\[
\int_0^{2n\tau} W_{k^{2n}}(f \ast g)(t) C_{k^{2n}}(t)x dt \\
= \int_0^{2n\tau} W_{k^{2n}} g(r) \int_0^r W_{k^{2n}} f(s) \left( \int_s^{s+r} - \int_0^s k^{2n}(s+r-t)C_{k^{2n}}(t)x dt \right) dr \\
+ \int_0^{2n\tau} W_{k^{2n}} g(r) \int_0^{2n\tau-r} W_{k^{2n}} f(s) \left( \int_s^{s+r} - \int_0^r k^{2n}(s+r-t)C_{k^{2n}}(t)x dt \right) ds dr \\
+ \int_0^{2n\tau} W_{k^{2n}} g(r) \int_0^{2n\tau} W_{k^{2n}} f(s) \int_s^{2n\tau} k^{2n}(s+r-t)C_{k^{2n}}(t)x dt ds dr.
\]

For the last integral in the above equality, if \( r \leq n\tau \) then \( 2n\tau - r \geq n\tau \), and

\[
\int_0^n W_{k^{2n}} g(r) \int_0^{2n\tau-r} W_{k^{2n}} f(s) \int_s^{2n\tau} k^{2n}(s+r-t)C_{k^{2n}}(t)x dt ds dr = 0
\]

since \( \text{supp}(W_{k^{2n}}) \subset [0, n\tau] \). Analogously, if \( r > n\tau \) then

\[
\int_{n\tau}^{2n\tau} W_{k^{2n}} g(r) \int_0^{2n\tau-r} W_{k^{2n}} f(s) \int_s^{2n\tau} k^{2n}(s+r-t)C_{k^{2n}}(t)x dt ds dr = 0
\]

since \( \text{supp}(W_{k^{2n}}) \subset [0, n\tau] \). Hence

\[
\int_0^{2n\tau} W_{k^{2n}}(f \ast g)(t) C_{k^{2n}}(t)x dt \\
= \int_0^n W_{k^{2n}} g(r) \int_0^r W_{k^{2n}} f(s) \left( \int_s^{s+r} - \int_0^s k^{2n}(s+r-t)C_{k^{2n}}(t)x dt \right) dr \\
+ \int_0^n W_{k^{2n}} g(r) \int_r^{n\tau} W_{k^{2n}} f(s) \left( \int_s^{s+r} - \int_0^r k^{2n}(s+r-t)C_{k^{2n}}(t)x dt \right) ds dr.
\]

On the other hand, by the Fubini theorem we get that

\[
\int_0^{2n\tau} (f \circ W_{k^{2n}})(t) C_{k^{2n}}(t)x dt = \int_0^{2n\tau} W_{k^{2n}} g(r) \int_0^r f(r-t)C_{k^{2n}}(t)x dt dr
\]

since \( f(r-t) = (k^{2n} \circ W_{k^{2n}})(r-t) \), we have that
\[ \int_0^{2n\tau} (f \circ W_{k,2n})(t)C_{k,2n}(t) x dt \]

\[ = \int_0^{2n\tau} W_{k,2n} g(r) \int_0^r W_{k,2n}(t)x \int_{r-t}^{2n\tau} k^{s2n}(s-r+t) W_{k,2n} f(s) ds dt dr \]

\[ = \int_0^{2n\tau} W_{k,2n} g(r) \int_0^r W_{k,2n} f(s) \int_{r-s}^r k^{s2n}(s-r+t) C_{k,2n}(t) x dt ds dr \]

\[ + \int_0^{2n\tau} W_{k,2n} g(r) \int_r^{2n\tau} W_{k,2n} f(s) \int_0^r k^{s2n}(t+s-r) C_{k,2n}(t) x dt ds dr. \]

In similar way, we also get that

\[ \int_0^{2n\tau} (g \circ W_{k,2n})(t)C_{k,2n}(t) x dt \]

\[ = \int_0^{2n\tau} W_{k,2n} f(r) \int_0^r W_{k,2n} g(s) \int_{r-s}^r k^{s2n}(s-r+t) C_{k,2n}(t) x dt ds dr \]

\[ + \int_0^{2n\tau} W_{k,2n} f(r) \int_r^{2n\tau} W_{k,2n} g(s) \int_0^r k^{s2n}(s-r+t) C_{k,2n}(t) x dt ds dr. \]

We apply Fubini theorem, change the variable and use that \( \text{supp}(W_{k,2n} f), \text{supp}(W_{k,2n} g) \subset [0, n\tau] \) to obtain that

\[ \int_0^{2n\tau} (g \circ W_{k,2n})(t)C_{k,2n}(t) x dt \]

\[ = \int_0^{n\tau} W_{k,2n} g(r) \int_r^{n\tau} W_{k,2n} f(s) \int_{s-r}^s k^{s2n}(r-s+t) C_{k,2n}(t) x dt ds dr \]

\[ + \int_0^{n\tau} W_{k,2n} g(r) \int_r^{n\tau} W_{k,2n} f(s) \int_0^s k^{s2n}(t+s-r) C_{k,2n}(t) x dt ds dr. \]

We join these six summands and obtain

\[ 2C_k(f \ast_c g)x = \int_0^{n\tau} W_{k,2n} g(r) \int_r^{n\tau} W_{k,2n} f(s) \left( \int_s^{s+r} k^{s2n}(s-r+t) C_{k,2n}(t) x dt - \int_0^s k^{s2n}(s-r+t) C_{k,2n}(t) x dt \right) ds dr \]

\[ + \int_r^{n\tau} W_{k,2n} g(r) \int_r^{n\tau} W_{k,2n} f(s) \left( \int_s^{s+r} k^{s2n}(s+r-t) C_{k,2n}(t) x dt - \int_0^s k^{s2n}(s+r-t) C_{k,2n}(t) x dt \right) ds dr \]

\[ + \int_0^{n\tau} W_{k,2n} g(r) \int_r^{n\tau} W_{k,2n} f(s) \left( \int_s^{s+r} k^{s2n}(s+r-t) C_{k,2n}(t) x dt - \int_0^s k^{s2n}(s+r-t) C_{k,2n}(t) x dt \right) ds dr \]

\[ = 2 \int_0^{n\tau} W_{k,2n} g(r) \int_0^{n\tau} W_{k,2n} f(s) C_{k,2n}(r) C_{k,2n}(s) x ds dr = 2C_k(f)C_k(g)x. \]
we apply the composition property of convoluted cosine functions, \([17, \text{Theorem 2.4}]\).

To finish the proof consider \(f \in \mathcal{D}_{k^{\kappa}}\), \(\text{supp}(f) \subset [0,n\tau]\) and \(x \in X\). By \([13, \text{Lemma 2.8 (i)}]\) we have \(W_{k^{\kappa}}(f''') = (W_{k^{\kappa}})''\) and according to Definition 3.1 (ii) we have

\[
AC_k(f)x = A \int_0^{n\tau} (W_{k^{\kappa}}f)'(t) \int_0^t (t-s)C_{k^{\kappa}}(s)xdsdt \\
= \int_0^{n\tau} W_{k^{\kappa}}f'''(t) \left( C_{k^{\kappa}}(t)x - \int_0^t k^{\kappa}(s)dx \right) dt \\
= C_{k^{\kappa}}(f''\,x + \int_0^{n\tau} W_{k^{\kappa}}f'(t)k^{\kappa}(t)dt\,x) = C_{k^{\kappa}}(f'\,x + f'(0)x),
\]

for \(x \in X\) and we conclude the proof. \(\square\)

\textbf{Remark 4.2.} (i) In the conditions of Theorem 4.1, now take \(l \in L^1_{loc}(\mathbb{R}^+)\) with \(0 \in \text{supp}(l)\). By \([13, \text{Lemma 4.4}]\), the family \(((l * C_k)(t))_{t \in [0,\tau]}\) is a \(k * l\)-convoluted cosine function. Then one may prove that

\[
C_{k*l}(f) = C_k(f), \quad f \in \mathcal{D}_{(k*l)^{\kappa}}.
\]

(ii) When the operator \(A\) generates a global convoluted cosine function \((C(t))_{t \geq 0}\), the homomorphism \(C_k\) is defined from \(\mathcal{D}_k\) to \(\mathcal{B}(X)\) by

\[
C_k(f)x = \int_0^\infty W_kf(t)C_k(t)x dt, \quad x \in X, \quad f \in \mathcal{D}_k,
\]

see \([13, \text{Theorem 6.5}]\). If the family \((C(t))_{t \geq 0}\) is exponentially bounded, then the homomorphism \(C_k\) is extended to a bounded Banach algebra homomorphism, see \([13, \text{Theorem 6.7}]\).

(iii) Note that in fact the Theorem 4.1 motivates the following class of distribution cosine functions. Let \(k \in L^1_{loc}(\mathbb{R}^+)\) such that \(0 \in \text{supp}(k)\). A linear and continuous map \(C_k : \mathcal{D}_{k^{\kappa}} \rightarrow \mathcal{B}(X)\) is said a \(k\)-distribution cosine function, in short \(k\)-(DCF), if satisfies the following conditions.

\begin{itemize}
  \item[(k1)] \(C_k(\hat{\phi} * \psi) = C_k(\hat{\phi})C_k(\psi)\) for \(\phi, \psi \in \mathcal{D}_{k^{\kappa}}\).
  \item[(k2)] \(\cap \{\ker(C_k(\theta)) \mid \theta \in \mathcal{D}_{k^{\kappa}}\} = \{0\}\).
\end{itemize}

For a given \(k\)-(DCF) \(C_k\), the generator \((A, D(A))\) of \(C_k\) is defined by

\[
D(A) : = \{x \in X \mid \text{exists } y \in X \text{ such that } C_k(\theta)y = C_k(\theta')x + \theta'(0)x \text{ for any } \theta \in \mathcal{D}_{k^{\kappa}}\}; \quad A(x) = y, \quad x \in D(A).
\]

The operator \((A, D(A))\) is well-defined, closed, linear and \(C_k(\mathcal{D}_{k^{\kappa}})X \subseteq D(A)\) with \(AC_k(\phi)x = C_k(\phi''\,x + \phi'(0)x)\) for any \(\phi \in \mathcal{D}_{k^{\kappa}}\) and \(x \in X\) (we use Proposition 2.4 (v)).

By Theorem 4.1 a local convoluted cosine function defines a \(k\)-distribution cosine function.

A very important case is when \(\mathcal{D}_k = \mathcal{D}_+\). In this condition we recover the definition of almost-distribution cosine function given in \([25, \text{Definition 5}]\). In fact the equivalence between almost-distribution cosine functions, distribution cosine functions (introduced by Kostić in \([16]\)) and \(n\)-times integrated cosine functions (and other spectral conditions) were proven in \([19, \text{Theorem 3.5}]\).
(iv) Some connections between $k$-convoluted cosine function and ultradistributions and hyperfunction sines have been treated in [21 Section 5] and [18 Section 3.6.3]. In particular if $A$ is the generator of an exponentially bounded $k$-cosine function for certain $k$, then there is a ultradistribution fundamental solution of $\ast$-class for $\pm iA$ ([21 Theorem 20]). Note that the relationship between this result and Theorem 4.1 may be a matter of further investigations, see [21 Remark 14].

5. Examples and final comments

In this section we consider different examples of convoluted cosine functions which have appeared in the literature. Our results are applied in these examples to illustrate its importance.

5.1. The Laplacian on $L^p(\mathbb{R}^N)$. [8, 11, 12] It is known that the Laplacian $\Delta$ is the generator of an $\alpha$-times integrated semigroups in $L^p(\mathbb{R}^N)$ with $1 < p < \infty$ and $N \geq 1$, $(C_\alpha(t))_{t \geq 0} \subset \mathcal{B}(L^p(\mathbb{R}^N))$, if and only if $\alpha \geq (N - 1)\left(\frac{1}{p} - \frac{1}{2}\right)$, and

$$\|C_\alpha(t)\| \leq Ct^\alpha, \quad t \geq 0,$$

see [11 Theorem 4.4], [8 Proposition 3.2] and [12 Section 5]. In this case the map $C_\alpha : D_+ \to \mathcal{B}(L^p(\mathbb{R}^N))$ (Remark 4.2 (ii)) extends to a Banach algebra homomorphism $C_\alpha : \mathcal{T}_\alpha(t^\alpha) \to \mathcal{B}(L^p(\mathbb{R}^N))$ where $\mathcal{T}_\alpha(t^\alpha)$ is the completion of $D_+$ in the norm

$$\|f\|_\alpha := \int_0^\infty |W_\alpha f(t)|t^\alpha dt, \quad f \in D_+,$$

where $W_\alpha$ is the Weyl derivation of order $\alpha$, see [25 Theorem 4]. Other examples of global $\alpha$-times integrated cosine functions (generated by translation invariant operators) may be found in [8 Theorem 5.2, Theorem 5.4], [12 Theorem 3.1] and [11 Theorem 4.2].

5.2. The Laplacian on $L^2[0, \pi]$ with Dirichlet boundary conditions. [21 Example 1], [18 Example 2.1.8.1] The operator $-\Delta$ on $L^2[0, \pi]$ with Dirichlet boundary conditions generates an exponentially bounded $\kappa$-convoluted cosine function $(C_\kappa(t))_{t \geq 0}$ where $\kappa$ is given in Example 2.5 (v). By Remark 4.2 (ii), there exists an algebra homomorphism $C_\kappa : D_+ \to \mathcal{B}(L^2[0, \pi])$; in fact it extends to a Banach algebra homomorphism $C_\kappa : \mathcal{T}_\kappa(t^\beta) \to \mathcal{B}(L^2[0, \pi])$ where $\mathcal{T}_\kappa(t^\beta)$ is the completion of $D_+$ in the norm

$$\|f\|_{\kappa,e,-\beta} := \int_0^\infty |W_\kappa f(t)|e^{\beta t} dt, \quad f \in D_+,$$

for some $\beta > 0$, see [13 Corollary 6.6]. Moreover, the fractional power $\Delta^{2\kappa}$ generates an exponentially bounded $\kappa_{n+1}$-convoluted cosine function for suitable kernel $\kappa_{n+1}$ ([18, 21]). Similar results for $n \in \mathbb{N}$ are obtained following previous discursion.

5.3. Multiplication operator in $L^p(\mathbb{R})$. [21 Section 6, Example 3] Take $X = L^p(\mathbb{R})$ with $1 \leq p \leq \infty$ and we consider the multiplication operator $A$ with the maximal domain in $L^p(\mathbb{R})$,

$$Af(x) = (x + ix^2)^2 f(x), \quad x \in \mathbb{R}, f \in L^p(\mathbb{R}).$$
The operator $A$ is not the generator of any (local) integrated cosine function in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. However $A$ is the generator of a local $K_\delta$-convoluted cosine function $(C_{K_\delta}(t))_{t \in [0, \tau)}$ given by

$$C_{K_\delta}(t)f(x) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\lambda e^{\lambda t} - \lambda^x}{\lambda^2 - (x + ix^2)^2} d\lambda f(x), \quad f \in L^p(\mathbb{R}), \ x \in \mathbb{R}, \ t \in [0, \tau),$$

where $K_\delta$ is given in Example 2.5 (iii) for suitable $\delta \in (0, 1)$, a complex path $\Gamma$ and for any $\tau \in (0, \infty)$. Then $A$ is the generator of a global $K_\delta$-convoluted cosine function $(C_{K_\delta}(t))_{t \in [0, \infty)}$ and we apply the Remark 4.2 (ii) to define an algebra homomorphism $C_{K_\delta} : \mathcal{D}_{K_\delta} \to \mathcal{B}(X)$. Note that $\mathcal{D}_{K_\delta} \subsetneq \mathcal{D}_+$ (in other case $A$ generates a local integrated cosine function, see Remark 4.2 (iii)).

5.4. Multiplication operator in $L^p(\mathbb{R}^+)$. [16] Example 6.2] Take $X = L^p(\mathbb{R}^+)$ with $1 \leq p \leq \infty$ and we consider the multiplication operator $A$ with the maximal domain in $L^p(\mathbb{R})$,

$$Af(x) = (x + ie^x)^2 f(x), \quad x > 0, \ f \in L^p(\mathbb{R}).$$

The operator $A$ is the generator of (local) 1-integrated cosine function $(C_1(t))_{t \in [0, 1]}$ in $L^p(\mathbb{R}^+)$. By Theorem 3.3, the operator $A$ generates an $n$-times integrated cosine function $(C_n(t))_{t \in [0, n]}$ in $L^p(\mathbb{R}^+)$ for $1 \leq p \leq \infty$ and a distribution cosine function $C_\alpha : \mathcal{D}_+ \to \mathcal{B}(X)$.

5.5. Multiplication operator in $\ell^2$. [20] Example 1], [24] Example 1.2.6] Let $\ell^2$ be the Hilbert space of all sequences $x = (x_m)_{m \in \mathbb{N}}$ such that $\sum_{m=1}^{\infty} |x_m|^2 < \infty$, with the usual norm $\|x\| := \left(\sum_{m=1}^{\infty} |x_m|^2\right)^{\frac{1}{2}}$. Take $T > 0$ and define

$$a_m = \frac{m}{T} + i \left(\frac{e_m}{m} - \left(\frac{m}{T}\right)^2\right)^{\frac{1}{2}}, \quad m \in \mathbb{N},$$

where $i^2 = -1$. For any $\alpha > 0$ let $(C_\alpha(t))_{t > 0}$ be defined by

$$C_\alpha(t)x = \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cosh(a_ms)x_{m} ds\right)_{m=1},$$

for $x \in \ell^2$. Then $(C_\alpha(t))_{t \in [0, \alpha T)}$ is a local $\alpha$-times integrated cosine function on $\ell^2$ such that $(C_\alpha(t))_{t \in [0, \alpha T)}$ cannot be extended to $t \geq \alpha T$, in fact

$$C_\alpha(t) = \frac{U_\alpha(t) + U_\alpha(-t)}{2}, \quad t \in [0, \alpha T),$$

where $(U_\alpha(t))_{t \in [0, \alpha T)}$ are local $\alpha$-integrated semigroups, see [26] Example 1]. By Theorem 5.3, $(C_{\alpha n}(t))$ may be defined to $t < n\alpha T$. 


SHARP EXTENSIONS FOR SOLUTIONS OF WAVE EQUATIONS

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