Wiener-Hopf factorization indices of rational matrix functions with respect to the unit circle in terms of realization

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Abstract

As in the paper [15], our aim is to obtain explicitly the Wiener-Hopf indices of a rational $m \times m$ matrix function $R(z)$ that has no poles and no zeros on the unit circle $\mathbb{T}$ but, in contrast with [15], the function $R(z)$ is not required to be unitary on the unit circle. On the other hand, using a Douglas-Shapiro-Shields type of factorization, we show that $R(z)$ factors as $R(z) = \Xi(z)\Psi(z)$, where $\Xi(z)$ and $\Psi(z)$ are rational $m \times m$ matrix functions, $\Xi(z)$ is unitary on the unit circle and $\Psi(z)$ is an invertible outer function. Furthermore, the fact that $\Xi(z)$ is unitary on the unit circle allows us to factor as $\Xi(z) = V(z)W^*(z)$ where $V(z)$ and $W(z)$ are rational bi-inner $m \times m$ matrix functions. The latter allows us to solve the Wiener-Hopf indices problem. To derive explicit formulas for the functions $V(z)$ and $W(z)$ requires additional realization...
properties of the function $\Xi(z)$ which are given in the last two sections.

*Keywords*: Wiener-Hopf indices, bi-inner matrix functions, Douglas-Shapiro-Shields factorization, realizations.

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Dedicated to Jaap Korevaar, on the occasion of his 100-th birthday, in friendship and with gratitude.

1. Introduction

Throughout this paper $R(z)$ is an $m \times m$ rational matrix function which has no poles and no zeros on the unit circle $\mathbb{T}$. Such a function admits a right Wiener-Hopf factorization, (see, for instance, [4, 6, 7]), that is,

$$R(z) = W_-(z)D(z)W_+(z),$$

where the factors $W_-(z)$ and $W_+(z)$ are rational $m \times m$ matrix functions such that

(i) $W_-(z)$ has no poles and no zeros outside the open unit disc including infinity,

(ii) $W_+(z)$ has no poles and no zeros on the closed unit disc,

and where the middle term $D(z)$ is a diagonal matrix

$$D(z) = \text{diag} \left( z^{-\alpha_1}, \ldots, z^{-\alpha_s} \right) \oplus I_k \oplus \text{diag} \left( z^{\omega_1}, \ldots, z^{\omega_t} \right),$$

where $-\alpha_1 \leq \cdots \leq -\alpha_s < 0$ and $0 < \omega_t \leq \cdots \leq \omega_1$ are integers, and $m = s + k + t$. The numbers $\alpha_j$ and $\omega_j$ are called the right Wiener-Hopf indices of $R(z)$. Reversing the roles of $W_-(z)$ and $W_+(z)$ one obtains the definition of a left Wiener-Hopf factorization and left Wiener-Hopf indices. In what follows we will restrict to the right Wiener-Hopf indices and omit the word "right".

Wiener-Hopf factorization of (rational) matrix functions plays a role in the study of Toeplitz operators, convolution integral operators and singular integral operators, in particular, in the study of the Fredholm properties of such operators. See, e.g., [7], [12], [13], Chapters XII and XIII, and [14], Chapter XXIV. For more details, see also the introduction to [15], and the references given there.
The main goal of the present paper is to obtain explicitly the Wiener-Hopf indices of any $m \times m$ rational matrix function $R(z)$ which has no poles and no zeros on the unit circle $\mathbb{T}$. We call this the Wiener-Hopf indices problem. This problem has been solved in [15] for the case when additionally $R(z)$ is unitary for any $z \in \mathbb{T}$. In the present paper we shall solve the Wiener-Hopf indices problem for any $R(z)$ not necessarily unitary on $\mathbb{T}$. The analysis is based on the fact that the function $R(z)$ has a realization of the following type:

$$R(z) = R_0 + zC(I-zA)^{-1}B + \gamma(zI-\alpha)^{-1}\beta,$$  \hspace{1cm} (1.1)  

with the square matrices $A$ and $\alpha$ being stable, i.e., the eigenvalues of $A$ and $\alpha$ are in the open unit disc. Such a realization for $R(z)$, with $A$ and $\alpha$ stable, exists because $R(z)$ has no poles on the unit circle $\mathbb{T}$, see, e.g., [3], in particular Chapter 8.

It is the intention of this paper to solve the Wiener-Hopf indices problem by carrying out all the steps explicitly, starting from the realization (1.1). In effect the main aim is finding an algorithm for computing the Wiener-Hopf indices of $R(z)$.

The paper consists of six sections including the present introduction. In the second section, using the realization (1.1), an analogous realization is presented for the product $R^*(z)R(z)$, where $R^*(z) = R(1/\bar{z})^*$, and this realization is used to obtain in realized form an invertible outer factor $\Psi(z)$ such that

$$R^*(z)R(z) = \Psi^*(z)\Psi(z).$$  \hspace{1cm} (1.2)  

Recall that a rational matrix function $\Psi(z)$ is called an invertible outer function if $\Psi(z)$ has no poles and no zeros on the closed unit disc. Moreover, an invertible outer factor of $R(z)$ is unique up to multiplication on the left by a constant unitary matrix (see, e.g., [8], Theorems 5.2.1 and 6.1.1). The realization for $\Psi(z)$ is given in Proposition 2.3. The matrices in the realization can be constructed explicitly from the matrices in the realization of $R(z)$, the construction involves the solutions to two Stein equations and a discrete algebraic Riccati equation.

In the third section we introduce the function

$$\Xi(z) = R(z)\Psi(z)^{-1}.$$  \hspace{1cm} (1.3)  

Obviously, this function is an $m \times m$ rational matrix function. Notice that $\Xi(z)$ is uniquely determined by $\Psi(z)$, and hence is unique up to multiplication.
by a constant unitary matrix on the right. Lemma 3.1 shows that \( \Xi(z) \) is unitary for each \( z \in \mathbb{T} \) and the functions \( R(z) \) and \( \Xi(z) \) have the same right Wiener-Hopf indices. Thus in order to solve the Wiener-Hopf indices problem it suffices to solve this problem for the function \( \Xi(z) \). Since \( \Xi(z) \) is unitary for each \( z \in \mathbb{T} \), we can apply Theorem 1.1 of [15] to get these indices. To get the required explicit formulas we need to factorize \( \Xi(z) \) as 
\[ \Xi(z) = V(z)W^*(z) \]
where \( V(z) \) and \( W(z) \) are rational bi-inner \( m \times m \) matrix functions with additional results that are presented in the final two sections. Recall that a rational matrix function is called bi-inner if it has unitary values on the unit circle and no poles on the unit disc. In Section 4 we present \( \Xi(z) \) in realized form (see (4.3)) and a few related results. The realization is based on the realizations for \( R(z) \) and \( \Psi(z) \), and involves the solution to an additional Stein equation. In Section 5 we construct explicitly the bi-inner functions \( V(z) \) and \( W(z) \) appearing in the factorization \( \Xi(z) = V(z)W(z)^* \). The explicit construction of realizations for \( V(z) \) and \( W(z) \) is based on an algorithm described in Section 4.7 in [8], and involves the solution to yet one more Stein equation and a Lyapunov equation. Finally, in Section 6 we discuss some additional properties of \( V(z) \) and \( W(z) \) and their realizations.

2. The invertible outer factor of \( R^*(z)R(z) \)

Throughout \( R(z) \) is the rational \( m \times m \) matrix function given by (1.1), that is,
\[ R(z) = R_0 + zC(I - zA)^{-1}B + \gamma(zI - \alpha)^{-1}\beta, \]  
with the square matrices \( A \) and \( \alpha \) being stable, i.e., \( A \) and \( \alpha \) have all their eigenvalues located in the open unit disc. Furthermore, \( P_+ \) and \( P_- \) are the observability gramians given by
\[ P_+ - A^*P_+A = C^*C \quad \text{and} \quad P_- - \alpha^*P_-\alpha = \gamma^*\gamma. \]

Note that in general the second and third term in the realization (2.1) are not required to be minimal realizations. The next lemma provides the realization of \( R^*(z)R(z) \) starting from (2.1).

**Lemma 2.1.** The product function \( R^*(z)R(z) \) is given by
\[ R^*(z)R(z) = \tilde{D} + \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + z\tilde{B}^*(I - z\tilde{A}^*)^{-1}\tilde{C}^*, \]
Furthermore, the matrix $\tilde{A}$ is stable.

**Proof.** We have

$$R^*(z) = R(1/\bar{z})^* = R_0^* + B^*(zI - A^*)^{-1}C^* + z\beta^*(I - z\alpha^*)^{-1}\gamma^*,$$

and so

$$R^*(z)R(z) = R_0^*R_0 + zR_0^*C(I - zA)^{-1}B + R_0^*\gamma(zI - \alpha)^{-1}\beta +$$

$$+ B^*(zI - A^*)^{-1}C^*R_0 + z\beta^*(I - z\alpha^*)^{-1}\gamma^*R_0 +$$

$$+ zB^*(zI - A^*)^{-1}C^*C(I - zA)^{-1}B + z\beta^*(I - z\alpha^*)^{-1}\gamma^*\gamma(zI - \alpha)^{-1}\beta +$$

$$+ B^*(zI - A^*)^{-1}C^*\gamma(zI - \alpha)^{-1}\beta + z^2\beta^*(I - z\alpha^*)^{-1}\gamma^*C(I - zA)^{-1}B.$$ 

Let $P_+$ and $P_-$ be the observability gramians given by (2.2). Then (compare the argument following formula (2.17) in [10])

$$zB^*(zI - A^*)^{-1}C^*C(I - zA)^{-1}B =$$

$$zB^*P_+A(I - zA)^{-1}B + B^*P_+B + B^*(zI - A^*)^{-1}A^*P_+B,$$

and

$$z\beta^*(I - z\alpha^*)^{-1}\gamma^*\gamma(zI - \alpha)^{-1}\beta =$$

$$\beta^*P_\alpha(zI - \alpha)^{-1}\beta + \beta^*P_-\beta + z\beta^*(I - z\alpha^*)^{-1}\alpha^*P_-\beta.$$ 

Inserting this in the formula for $R^*(z)R(z)$ and regrouping terms a bit we obtain

$$R^*(z)R(z) = R_0^*R_0 + B^*P_+B + \beta^*P_-\beta +$$

$$+ z(R_0^*C + B^*P_+A)(I - zA)^{-1}B + B^*(zI - A^*)^{-1}(C^*R_0 + A^*P_+B)+$$

$$+ (R_0^*\gamma + \beta^*P_-\alpha)(zI - \alpha)^{-1}\beta + z\beta^*(I - z\alpha^*)^{-1}(\gamma^*R_0 + \alpha^*P_-\beta)+$$

$$+ B^*(zI - A^*)^{-1}C^*\gamma(zI - \alpha)^{-1}\beta + z^2\beta^*(I - z\alpha^*)^{-1}\gamma^*C(I - zA)^{-1}B.$$
Now note that
\[
[B^* R_0^* \gamma + \beta^* P_{-\alpha}] \left( zI - \begin{bmatrix} A^* & C^* \gamma \\ 0 & \alpha \end{bmatrix} \right)^{-1} \begin{bmatrix} C^* R_0 + A^* P_+ B \\ \beta \end{bmatrix} \\
= B^* (zI - A^*)^{-1} (C^* R_0 + A^* P_+ B) + (R_0^* \gamma + \beta^* P_{-\alpha})(zI - \alpha)^{-1} \beta + \\
+ B^* (zI - A^*)^{-1} C^* \gamma (zI - \alpha)^{-1} \beta,
\]
and
\[
z \left[ R_0^* C + B^* P_+ A \beta^* \right] \left( I - z \begin{bmatrix} A & 0 \\ \gamma^* C & \alpha^* \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \gamma^* R_0 + \alpha^* P_{-\beta} \end{bmatrix} \\
= z(R_0^* C + B^* P_+ A)(I - zA)^{-1} B + z\beta^*(I - z\alpha)^{-1}(\gamma^* R_0 + \alpha^* P_{-\beta}) + \\
+ z^2 \beta^*(I - z\alpha)^{-1} \gamma^* C(I - zA)^{-1} B.
\]
So we arrive at the following formula for \( R^*(z)R(z) \):
\[
R^*(z)R(z) = R_0^* R_0 + B^* P_+ B + \beta^* P_{-\beta} + \\
+ [B^* R_0^* \gamma + \beta^* P_{-\alpha}] \left( zI - \begin{bmatrix} A^* & C^* \gamma \\ 0 & \alpha \end{bmatrix} \right)^{-1} \begin{bmatrix} C^* R_0 + A^* P_+ B \\ \beta \end{bmatrix} + \\
+ z \left[ R_0^* C + B^* P_+ A \beta^* \right] \left( I - z \begin{bmatrix} A & 0 \\ \gamma^* C & \alpha^* \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \gamma^* R_0 + \alpha^* P_{-\beta} \end{bmatrix}. \tag{2.5} \tag{eq:RstarR}
\]
This completes the proof of the formula for \( R^*(z)R(z) \) upon using the definitions of \( \tilde{A}, \tilde{B}, \tilde{C}, \) and \( \tilde{D} \).

The fact that \( \tilde{A} \) is stable follows from the stability of \( A \) and \( \alpha \). \( \square \)

Since \( R(z) \) is an \( m \times m \) rational matrix function with no poles and no zeros on the unit circle, and since \( A \) and \( \alpha \) are stable, we can use Theorem 1.1 in [10] to prove the following lemma. In fact, the following lemma follows from the symmetric version of Theorem 1.1 in [10].

**Lemma 2.2.** Let \( \tilde{A}, \tilde{B}, \tilde{C}, \) and \( \tilde{D} \) be the matrices defined by (2.3) and (2.4). Then the algebraic Riccati equation
\[
Q - \tilde{A} Q \tilde{A}^* = \left( \tilde{B} - \tilde{A} Q \tilde{C}^* \right) \left( \tilde{D} - \tilde{C} Q \tilde{C}^* \right)^{-1} \left( \tilde{B}^* - \tilde{C} Q \tilde{A}^* \right) \tag{2.6} \tag{eq:Ricc}
\]
has a (unique) stabilizing solution \( Q \).
Given the solution $Q$ of (2.6) we define
\[ D = \left( \tilde{D} - \tilde{C}Q\tilde{C}^* \right)^{1/2}, \quad C_0 = \tilde{B}^* - \tilde{C}Q\tilde{A}^* \]
\[ B_0 = \tilde{C}^*, \quad A_0 = \tilde{A}^* - \tilde{C}^*D^{-2}C_0. \] (2.7)

Now we can apply Lemma 2.2 and the symmetric version of Theorem 1.1 in [10] to obtain the following proposition.

**Proposition 2.3.** The matrix $A_0$ is stable and $R^*(z)R(z) = \Psi^*(z)\Psi(z)$. Here $\Psi(z)$ is the outer function given by
\[ \Psi(z) = D + zD^{-1}C_0 \left( I - z\tilde{A}^* \right)^{-1} B_0, \] (2.8)
and the inverse of $\Psi(z)$ is given by
\[ \Psi(z)^{-1} = D^{-1} - zD^{-2}C_0(I - zA_0)^{-1}B_0D^{-1}. \] (2.9)

Observe that the Riccati equation can be rewritten as a Stein equation, namely
\[ Q = \begin{bmatrix} A^* & C^*\gamma \\ 0 & \alpha \end{bmatrix}QA_0 + \begin{bmatrix} C^*R_0 + A^*P_+B \\ \beta \end{bmatrix}D^{-2}C_0 \] (2.10)
\[ = \tilde{A}QA_0 + \tilde{B}D^{-2}C_0. \]

**3. The function $\Xi(z)$ and the solution of the Wiener-Hopf indices problem**

The function $\Xi(z)$ is defined by the formula
\[ \Xi(z) := R(z)\Psi(z)^{-1} \] (3.1)
where $\Psi(z)$ is the invertible outer function given by (2.8). Obviously, $\Xi(z)$ is a rational $m \times m$ matrix function.

**Lemma 3.1.** The rational function $\Xi(z)$ is unitary for each $z \in \mathbb{T}$, and the functions $R(z)$ and $\Xi(z)$ have the same right Wiener-Hopf indices.
Proof. The fact that \( \Xi(z) \) takes unitary values on the unit circle is checked by direct computation. For \( z \in \mathbb{T} \):

\[
\Xi(z)^* \Xi(z) = \Psi(z)^{-1} R(z)^* R(z) \Psi(z)^{-1} = I.
\]

Suppose that \( R(z) = W_-(z) D(z) W_+(z) \) is a right Wiener-Hopf factorization of \( R(z) \), then \( \Xi(z) = W_-(z) D(z) W_+(z) \), where \( W_+(z) = W_+(z) \Psi^{-1}(z) \). Since \( \Psi(z) \) is an invertible outer function, \( W_+ \) has no poles and no zeros on the closed unit disc. Hence the factorization \( \Xi(z) = W_-(z) D(z) W_+(z) \) is a right Wiener-Hopf factorization of \( \Xi(z) \), and as a consequence \( R(z) \) and \( \Xi(z) \) have the same right Wiener-Hopf indices. \( \square \)

**Definition 3.2.** With some ambiguity (because \( \Xi(z) \) is unique only up to multiplication on the right by a constant unitary matrix), the function \( \Xi(z) \) is said to be the left unitary factor of the rational \( m \times m \) matrix function \( R(z) \).

Our aim is to obtain the right Wiener-Hopf indices of \( R(z) \). Recall that the above lemma tells us that \( R(z) \) and \( \Xi(z) \) have the same right Wiener-Hopf indices. Thus it suffices to obtain the right Wiener-Hopf indices of the left unitary factor \( \Xi(z) \). Since \( \Xi(z) \) is unitary for each \( z \in \mathbb{T} \), we can apply Theorem 1.1 of [15] to get these indices.

As a first step (see [15, page 696]) we use the fact that \( \Xi(z) \) factors as \( \Xi(z) = V(z) W^*(z) \), where \( V(z) \) and \( W(z) \) are rational bi-inner \( m \times m \) matrix functions. Furthermore, we may assume (without loss of generality) that

\[
V(z) = D_V + z C_V (I_n - z A_V)^{-1} B_V, \tag{3.2} \label{realiV}
\]

\[
W(z) = D_W + z C_W (I_n - z A_W)^{-1} B_W, \tag{3.3} \label{realiW}
\]

with \( A_V \) and \( A_W \) being stable, and with both realizations being unitary. The latter means that

\[
\begin{bmatrix}
A_V & B_V \\
C_V & D_V
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A_W & B_W \\
C_W & D_W
\end{bmatrix}
\]

are both unitary matrices. These realizations are then minimal. In what follows we also need the unique solution \( X \) of the Stein equation

\[
X - A_V X A_W^* = B_V B_W^*, \quad \text{that is,} \quad X = \sum_{j=0}^{\infty} A_V^j B_V B_W^* (A_W^j)^* \tag{3.4} \label{Stein}
\]
(see [9] where this matrix $X$ was introduced).

Next, we need the following standard notation: for a pair $(C, A)$ we denote $\ker_k(C, A) = \cap_{j=0}^{k-1} \ker C A^j$, and for a pair $(A, B)$ we denote $\im_k(A, B) = \cup_{j=0}^{k-1} \im A^j B$. For $k = 0$ these spaces are interpreted as the full Euclidean space, respectively the zero subspace.

By a direct application of Theorem 1.1 in [15] we have the following result on the Wiener-Hopf indices of $R(z)$, as they are the same as the Wiener-Hopf indices of $\Xi(z)$.

**Theorem 3.3.** With notation as above the number $s$ of negative right Wiener-Hopf indices of the left unitary factor $\Xi(z)$ of $R(z)$ is given by

$$s = \dim \ker X - \dim \ker \begin{bmatrix} B_W^* \\ X A_W^* \end{bmatrix},$$

while the negative right Wiener-Hopf indices $-\alpha_j$, $j = 1, \ldots, s$, are given by

$$\alpha_j = \# \left\{ k \in \mathbb{N} \mid \dim \ker_{k-1} \begin{bmatrix} B_W^* \\ X A_W^* \end{bmatrix}, A_W^* \right\} - \dim \ker_k \begin{bmatrix} B_W^* \\ X A_W^* \end{bmatrix}, A_W^* \geq j \right\}.$$

Furthermore, the number $t$ of positive right Wiener-Hopf indices of $\Xi(z)$ is given by

$$t = \dim \im \begin{bmatrix} B_V & A_V X \end{bmatrix} - \dim \im X,$$

while the positive right Wiener-Hopf indices $\omega_j$, $j = 1, \ldots, t$, are given by

$$\omega_j = \# \left\{ k \in \mathbb{N} \mid \dim \im_k \begin{bmatrix} A_V, [B_V & A_V X] \end{bmatrix} - \dim \im \begin{bmatrix} A_V, [B_V & A_V X] \end{bmatrix} \geq j \right\}.$$

To make the Wiener-Hopf indices in the above theorem more explicit we need formulas for the matrices $A_V, A_W, B_V, B_W$, and $X$ in terms of the matrices occurring in the realization (1.1) of $R(z)$. This will be done in the next two sections.
4. Properties of the left unitary factor $\Xi(z)$

In this section we first produce a realization for the unitary factor function $\Xi(z)$. Recall that $\Xi(z) = R(z)\Psi(z)^{-1}$, which has unitary values on the unit circle, i.e., $\Xi(z)^{-1} = \Xi(z)^*$ for $|z| = 1$. Let $Y$ be the solution to the Stein equation

$$Y - aY A_0 = \alpha \beta D^{-2} C_0.$$  \hfill (4.1) \text{eq:SteinY}

Introduce the following notation:

$$\Xi_0 = R_0 D^{-1} - \gamma Y A_0 B_0 D^{-1} - \gamma \beta D^{-2} C_0 B_0 D^{-1},$$

$$C_1 = ([C_0] - R_0 D^{-2} C_0 - \gamma Y A_0^2 - \gamma \beta D^{-2} C_0 A_0),$$

$$\beta_1 = (\beta - Y B_0) D^{-1}. \hfill (4.2) \text{Xi0C1beta1}

Observe that $\Xi_0, C_1$ and $\beta_1$ are defined in terms of operators which we know from the preceding sections and the solution $Y$ of the Stein equation (4.1). Note also that the operators from the previous section in turn are explicitly expressed in terms of the operators in the realization of $R(z)$ and solutions to Stein equations and a Riccati equation, which can be computed. With this notation we can give an explicit formula for the unitary function $\Xi(z)$ such that $R(z) = \Xi(z)\Psi(z)$.

**Proposition 4.1.** The left unitary factor $\Xi(z)$ of the rational $m \times m$ matrix function $R(z)$ is given in realized form by the following formula:

$$\Xi(z) = \Xi_0 + \gamma (z I - \alpha)^{-1} \beta_1 + z C_1 (I - z A_0)^{-1} B_0 D^{-1}. \hfill (4.3) \text{realizationXi}

**Proof.** Compute

$$\Xi(z) = R(z)\Psi(z)^{-1} = R_0 D^{-1} + z C (I - z A)^{-1} B D^{-1} + \gamma (z I - \alpha)^{-1} \beta D^{-1}$$

$$- z R_0 D^{-2} C_0 (I - z A_0)^{-1} \left[ \gamma^* R_0 + \alpha^* P_\beta \right] D^{-1}$$

$$- z \gamma (z I - \alpha)^{-1} \beta D^{-2} C_0 (I - z A_0)^{-1} \left[ \gamma^* R_0 + \alpha^* P_\beta \right] D^{-1}$$

$$- z^2 C (I - z A)^{-1} B D^{-2} C_0 (I - z A_0)^{-1} \left[ \gamma^* R_0 + \alpha^* P_\beta \right] D^{-1}. \text{Xi0C1beta1}$$
We can combine terms as follows:

\[
\Xi(z) = R_0 D^{-1} + z \left[ C \quad -R_0 D^{-2} C_0 \right] \left( I - z \begin{bmatrix} A & -B D^{-2} C_0 \\ 0 & A_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix} D^{-1}
\]

\[
+ \gamma (z I - \alpha)^{-1} \beta D^{-1} - z \gamma (z I - \alpha)^{-1} \beta D^{-2} C_0 (I - z A_0)^{-1} B_0 D^{-1}.
\]

Now from the Riccati equation (2.10) (second row) we obtain

\[
\beta D^{-2} C_0 = [Q_{21} \quad Q_{22}] - [0 \quad \alpha] Q A_0
\]

which will be used in the last two terms of \(\Xi(z) = R(z) \Psi(z)^{-1}\). Indeed, one has

\[
z (z I - \alpha)^{-1} \beta D^{-2} C_0 (I - z A_0)^{-1} =
\]

\[
= z (z I - \alpha)^{-1} \left( [Q_{21} \quad Q_{22}] - [0 \quad \alpha] Q A_0 \right) (I - z A_0)^{-1}
\]

\[
= z (z I - \alpha)^{-1} \left( [Q_{21} \quad Q_{22}] - \alpha [Q_{21} \quad Q_{22}] A_0 \right) (I - z A_0)^{-1}
\]

\[
= (z I - \alpha)^{-1} \left( z [Q_{21} \quad Q_{22}] - \alpha [Q_{21} \quad Q_{22}] z A_0 \right) (I - z A_0)^{-1}
\]

\[
= (z I - \alpha)^{-1} \left( z [Q_{21} \quad Q_{22}] - \alpha [Q_{21} \quad Q_{22}] + \alpha [Q_{21} \quad Q_{22}] - \alpha [Q_{21} \quad Q_{22}] z A_0 \right) (I - z A_0)^{-1}
\]

\[
= (z I - \alpha)^{-1} \left( (z I - \alpha) [Q_{21} \quad Q_{22}] + \alpha [Q_{21} \quad Q_{22}] (I - z A_0) \right) (I - z A_0)^{-1}
\]

\[
= [Q_{21} \quad Q_{22}] (I - z A_0)^{-1} + (z I - \alpha)^{-1} \alpha [Q_{21} \quad Q_{22}]
\]

\[
= [Q_{21} \quad Q_{22}] (I - z A_0)^{-1} + (z I - \alpha)^{-1} [0 \quad \alpha] Q.
\]

Hence \(\Xi(z)\) is given by

\[
\Xi(z) = R_0 D^{-1} + z \left[ C \quad -R_0 D^{-2} C_0 \right] \left( I - z \begin{bmatrix} A & -B D^{-2} C_0 \\ 0 & A_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix} D^{-1}
\]

\[
+ \gamma (z I - \alpha)^{-1} \beta D^{-1} - \gamma (z I - \alpha)^{-1} [0 \quad \alpha] Q B_0 D^{-1}
\]

\[
- \gamma [Q_{21} \quad Q_{22}] (I - z A_0)^{-1} B_0 D^{-1}.
\]

Rewrite the last term as follows:

\[
\gamma [Q_{21} \quad Q_{22}] (I - z A_0)^{-1} B_0 D^{-1} =
\]

\[
= [0 \quad \gamma] Q (I - z A_0)^{-1} B_0 D^{-1}
\]

\[
= [0 \quad \gamma] Q B_0 D^{-1} + z [0 \quad \gamma] Q A_0 (I - z A_0)^{-1} B_0 D^{-1}.
\]
Then we arrive at the following formula for $\Xi(z)$:

$$
\Xi(z) = R_0 D^{-1} - \begin{bmatrix} 0 & \gamma \end{bmatrix} Q B_0 D^{-1} + \\
+ \gamma (z I - \alpha)^{-1} (\beta - \begin{bmatrix} 0 & \alpha \end{bmatrix} Q B_0) D^{-1} \\
+ z \begin{bmatrix} C & -R_0 D^{-2} C_0 \end{bmatrix} \left( I - z \begin{bmatrix} A & -B D^{-2} C_0 \\ 0 & A_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix} D^{-1} \\
- z \begin{bmatrix} 0 & \gamma \end{bmatrix} Q A_0 (I - z A_0)^{-1} B_0 D^{-1} \\
= R_0 D^{-1} - \begin{bmatrix} 0 & \gamma \end{bmatrix} Q B_0 D^{-1} \\
+ \gamma (z I - \alpha)^{-1} (\beta - \begin{bmatrix} 0 & \alpha \end{bmatrix} Q B_0) D^{-1} \\
+ z \begin{bmatrix} C & -R_0 D^{-2} C_0 - \begin{bmatrix} 0 & \gamma \end{bmatrix} Q A_0 \end{bmatrix} \times \\
\left( I - z \begin{bmatrix} A & -B D^{-2} C_0 \\ 0 & A_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix} D^{-1}.
$$

Next, consider the last term in this expression, i.e.

$$
z \begin{bmatrix} C & -R_0 D^{-2} C_0 - \begin{bmatrix} 0 & \gamma \end{bmatrix} Q A_0 \end{bmatrix} \left( I - z \begin{bmatrix} A & -B D^{-2} C_0 \\ 0 & A_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix} D^{-1}.
$$

Let $\Pi = \begin{bmatrix} I & 0 \end{bmatrix}$, and let $S = \begin{bmatrix} I & \Pi \\ 0 & I \end{bmatrix}$. Observe that $-B D^{-2} C_0 - \Pi A_0 + A \Pi = 0$. Hence

$$
S^{-1} \begin{bmatrix} A & -B D^{-2} C_0 \\ 0 & A_0 \end{bmatrix} S = \begin{bmatrix} A & 0 \\ 0 & A_0 \end{bmatrix}, \quad S^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix} = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}.
$$

Hence, applying a similarity transformation with $S$ on the realization of the last term we arrive at

$$
\Xi(z) = R_0 D^{-1} - \begin{bmatrix} 0 & \gamma \end{bmatrix} Q B_0 D^{-1} + \\
+ \gamma (z I - \alpha)^{-1} (\beta - \begin{bmatrix} 0 & \alpha \end{bmatrix} Q B_0) D^{-1} + z \begin{bmatrix} C & -R_0 D^{-2} C_0 - \begin{bmatrix} 0 & \gamma \end{bmatrix} Q A_0 \end{bmatrix} (I - z A_0)^{-1} B_0 D^{-1}.
$$

To rewrite the formula once more, consider the form \eqref{eq:2.10} of the algebraic Riccati equation and premultiply by $\begin{bmatrix} 0 & \alpha \end{bmatrix}$:

$$
\begin{bmatrix} 0 & \alpha \end{bmatrix} Q = \begin{bmatrix} 0 & \alpha^2 \end{bmatrix} Q A_0 + \alpha \beta D^{-2} C_0.
$$

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Repeating the argument, premultiply by $[0 \ \alpha^2]$ and insert the resulting expression to obtain

$$[0 \ \alpha] Q = [0 \ \alpha^2] Q A_0^2 + \alpha^2 \beta D^{-2} C_0 A_0 + \alpha \beta D^{-2} C_0.$$ 

Continuing by induction we have

$$[0 \ \alpha] Q = [0 \ \alpha^{n+1}] Q A_0^n + \sum_{j=1}^{n} \alpha^j \beta D^{-2} C_0 A_0^{j-1}.$$ 

Since $\alpha$ and $A_0$ are stable we have $\lim_{n \to \infty} \alpha^n = 0$ and $\lim_{n \to \infty} A_0^n = 0$, and so

$$[0 \ \alpha] Q = \sum_{j=1}^{\infty} \alpha^j \beta D^{-2} C_0 A_0^{j-1} = Y.$$ 

In addition, again using (2.10) yields

$$[0 \ \gamma] Q = \gamma ( [0 \ \alpha] Q A_0 + \beta D^{-2} C_0 ) = \gamma Y A_0 + \gamma \beta D^{-2} C_0.$$ 

Inserting this in the formula for $\Xi(z)$ we have

$$\Xi(z) = R_0 D^{-1} - \gamma Y A_0 B_0 D^{-1} - \gamma \beta D^{-2} C_0 B_0 D^{-1} + \gamma (z I - \alpha)^{-1} (\beta - Y B_0) D^{-1} + z \left( [C \ 0] - R_0 D^{-2} C_0 - \gamma Y A_0^2 - \gamma \beta D^{-2} C_0 A_0 \right) (I - z A_0)^{-1} B_0 D^{-1}.$$ 

Using the notations introduced in the beginning of this section, the formula for $\Xi(z)$ can be rewritten as in the statement of the proposition. □

**Remark.** Instead of (4.3) the realization of $\Xi(z)$ can also be obtained by using the matrices $A_V$ and $A_W^*$ in (3.2) and (3.3) and the matrix $X$ in (3.4) in place of the matrices $A_0$ and $\alpha$. More precisely, see, e.g., [9],

$$\Xi(z) = \Xi_0 + C_R (z I - A_W^*)^{-1} C_W^* + z C_V (I - z A_V)^{-1} B_R,$$  

(4.4) realizationXi2

where

$$B_R = B_V D_W^* + A_V X C_W^*,$$

$$C_R = D_V B_W^* + C_V X A_W^*,$$

$$\Xi_0 = D_V D_W^* + C_V X C_W^*.$$  

□
Using the fact that $\Xi(z)$ is a unitary function on $T$, we have the following extra properties for the matrices in the realization, some of which require the extra condition that $A_0^*$ and $\alpha$ have no common eigenvalues, and that, in addition, the pair $(\alpha, \beta_1)$ is controllable.

**Proposition 4.2.** Let $\Xi(z)$ be the function (4.3) taking unitary values on the unit circle. Assume that the pair $(\gamma, \alpha)$ is observable, and the pair $(A_0, B_0)$ is controllable. Then the unsymmetric Lyapunov equation

$$A_0^*P - P^*\alpha = -C_1^*\gamma,$$

has a solution. One particular solution is the zero-pole coupling matrix of $\Xi(z)$ corresponding to the pole pair $(\gamma, \alpha)$ and the zero pair $(A_0^*, C_1^*)$ corresponding to the unit disc; this solution will be denoted by $P_1$ henceforth.

Let $P_0$ be the solution to the Stein equation

$$P_0 - A_0^*P_0A_0 = C_1^*C_1.$$  \hfill (4.6)  \text{ SteinP0 }

Assume further that $A_0^*$ and $\alpha$ have no common eigenvalues. Then (4.5) has a unique solution, and the following identities hold

$$\Xi_0^*\Xi_0 + D^{-*}B_0^*P_0B_0D^{-1} + \beta_1^*P_1\beta_1 = I,$$

$$C_1^*\Xi_0 + A_0^*P_0B_0D^{-1} - P_1\beta_1 = 0.  \hfill (4.8)$$

If in addition the pair $(\alpha, \beta_1)$ is controllable, then also

$$D^*\Xi_0^*\gamma + D^*\beta_1^*P_1\alpha + B_0^*P_1 = 0.  \hfill (4.9)$$  \text{ unitarycond3 }

**Proof.** Since $(\gamma, \alpha)$ is observable and $\alpha$ and $A_0$ are stable matrices, a pole pair of $\Xi(z)$ corresponding to the unit disc is given by $(\gamma, \alpha)$. Likewise, considering $\Xi^*(z) = \Xi(\frac{1}{z})^*$, we have

$$\Xi^*(z) = \Xi_0^* + D^{-*}B_0^*(zI - A_0^*)^{-1}C_1^* + z\beta_1^*(I - z\alpha^*)^{-1}\gamma^*,$$

and since $(A_0, B_0)$ is controllable and $\Xi^*(z) = \Xi(z)^{-1}$, a zero pair of $\Xi(z)$ corresponding to the unit disc is given by $(A_0^*, C_1^*)$. Let $P_1$ be the zero-pole coupling matrix, then $P_1$ satisfies (4.5) (see, [1], Sections 4.5 and 4.4, see also [2, 3]).

Apply Lemma 2.1 to $\Xi(z)$ to obtain

$$\Xi^*(z)\Xi(z) = \Xi_0^*\Xi_0 + D^{-*}B_0^*P_0B_0D^{-1} + \beta_1^*P_1\beta_1$$

$$+ \hat{C}(zI - \hat{A})^{-1}\hat{B} + z\hat{B}^*(I - z\hat{A}^*)^{-1}\hat{C}^*, $$
where
\[\hat{C} = [D^* B^*_{0} \Xi_{0}^* \gamma + \beta_{1}^* P_{-} \alpha],\]
\[\hat{A} = \begin{bmatrix} A^*_{0} & C^*_{1} \gamma \\ 0 & \alpha \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} C_{1}^* \Xi_{0} + A^*_{0} P_{0} B_{0} D^{-1} \end{bmatrix}.\]

Since \(\Xi(z)\) is unitary on \(T\) it is now immediate that (4.7) holds, and moreover, that
\[0 = \hat{C}(zI - \hat{A})^{-1}\hat{B} = D^{-*} B^*_{0} (zI - A^*_{0})^{-1} (C_{1}^* \Xi_{0} + A^*_{0} P_{0} B_{0} D^{-1}) + (\Xi_{0}^* \gamma + \beta_{1}^* P_{-} \alpha)(zI - \alpha)^{-1} \beta_{1} + D^{-*} B^*_{0} (zI - A^*_{0})^{-1} C_{1}^* \gamma (zI - \alpha)^{-1} \beta_{1}.\]

Let \(P_{1}\) be the zero-pole coupling matrix, which is a solution to (4.5). Then
\[(zI - A^*_{0})^{-1} C_{1}^* \gamma (zI - \alpha)^{-1} = -(zI - A^*_{0})^{-1} P_{1} + P_{1} (zI - \alpha)^{-1}\]
and inserting this into the formula for \(\hat{C}(zI - \hat{A})^{-1}\hat{B}\) we obtain
\[0 = D^{-*} B^*_{0} (zI - A^*_{0})^{-1} (C_{1}^* \Xi_{0} + A^*_{0} P_{0} B_{0} D^{-1} - P_{1} \beta_{1}) + (\Xi_{0}^* \gamma + \beta_{1}^* P_{-} \alpha + D^{-*} B^*_{0} P_{1})(zI - \alpha)^{-1} \beta_{1}.\]

Now using the assumption that \(A^*_{0}\) and \(\alpha\) have no common eigenvalues each of the two terms must be zero. By controllability of \((A_{0}, B_{0})\) (4.8) follows, and under the assumption that \((\alpha, \beta_{1})\) is controllable also (4.9) follows.
\[\square\]

We conclude this section with the special case when
\[R(z) = R_{0} + zC(I - z A)^{-1} B \quad \text{with} \quad A \text{ being stable.} \quad \text{(4.10)}\]

This yields the following corollary.

**Corollary 4.3.** Assume \(R(z)\) is given by (4.10). Then the left unitary factor \(\Xi(z)\) of \(R(z)\) is a bi-inner function, and is given by
\[\Xi(z) = R_{0} D^{-1} + z(C - R_{0} D^{-2} C_{0})(I - z A_{0})^{-1} B D^{-1}.\]
Proof. In the special case where \( R(z) \) is stable, the algebraic Riccati equation (2.6) becomes

\[
Q = A^*QA + (C^*R_0 + A^*(P_+ - Q)B)(R_0^*R_0 + B^*(P_+ - Q)B)^{-1}(R_0^*C + B^*(P_+ - Q)A)
\]

and for the stabilizing solution \( Q \) we have

\[
D = (R_0^*R_0 + B^*(P_+ - Q)B)^{1/2},
C_0 = R_0^*C + B^*(P_+ - Q)A,
A_0 = A - BD^{-2}C_0.
\]

Moreover, \( B_0 = B \).

Thus the outer factor \( \Psi(z) \) and its inverse are given by

\[
\Psi(z) = D + zD^{-1}C_0(I - zA)^{-1}B,
\Psi(z)^{-1} = D^{-1} - zD^{-2}C_0(I - zA_0)^{-1}BD^{-1},
\]

while the unitary factor is given by

\[
\Xi(z) = R_0D^{-1} + z(C - R_0D^{-2}C_0)(I - zA_0)^{-1}BD^{-1}.
\]

Note that \( \Xi(z) \) is stable, so this is an inner factor, as expected for the case where \( R(z) \) is stable. In fact, because \( R(z) \) is assumed to be square, \( \Xi(z) \) is bi-inner. Compare [8] and [11]. □

5. Construction of the factorization \( \Xi(z) = V(z)W^*(z) \)

In this section \( \Xi(z) \) is the left unitary factor of \( R(z) \), and \( V(z) \) and \( W(z) \) are the bi-inner \( m \times m \) matrix functions appearing in (3.2) and (3.3). The identity \( \Xi(z) = V(z)W^*(z) \) is a Douglas-Shapiro-Shields factorization of \( \Xi(z) \) (see Section 4.7 in [4]; see also [16]). In this section we shall construct explicit formulas for the matrix functions \( V(z) \) and \( W(z) \).

DSSexplicit

Proposition 5.1. Let \( \Xi(z) \) be given by (4.3). Then the bi-inner rational matrix functions \( V(z) \) and \( W(z) \) in the factorization \( \Xi(z) = V(z)W^*(z) \) can be constructed as follows.

Let \( P_0 \) be the solution of the Stein equation

\[
P_0 - A_0^*P_0A_0 = C_1^*C_1, \text{ that is, } P_0 = \sum_{j=0}^{\infty}(A_0^*)^jC_1^*C_1A_0^j.
\]

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Then there exist matrices $B_V$ and $D_V$ such that
\[
\begin{bmatrix}
A_0 & C_1 \\
B_V^* & D_V^*
\end{bmatrix}
\begin{bmatrix}
P_0 & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_0 & B_V \\
C_1 & D_V
\end{bmatrix} =
\begin{bmatrix}
P_0 & 0 \\
0 & I
\end{bmatrix}.
\] (5.1)

Moreover, the function
\[
V(z) = D_V + zC_1(I - zA_0)^{-1}B_V
\] (5.2)
is a bi-inner factor for $\Xi(z)$ and the function $W(z)$ defined by $W^*(z) = V^*(z)\Xi(z)$ is bi-inner too.

Assume, in addition, that $A_0^*$ and $\alpha$ have no common eigenvalues and that the realization (4.3) is minimal. Let $P_1$ be the solution of (4.3). Set
\[
D_W = D_V^*\Xi_0 + B_V^*P_0B_0 D^{-1}\quad\text{and}\quad B_W^* = D_V^*\gamma + B_V^*P_1.
\]
Then
\[
W^*(z) = D_W^* + B_W^*(zI - \alpha)^{-1}\beta_1.
\] (5.3)

Note that finding $B_V$ and $D_V$ is a straightforward completion problem. Indeed, the columns of
\[
\begin{bmatrix}
P_0^{1/2}A_0 \\
C_1
\end{bmatrix}
\]
are orthonormal, and the columns of
\[
\begin{bmatrix}
B_V \\
D_V
\end{bmatrix}
\] complete the set of columns of
\[
\begin{bmatrix}
P_0^{1/2}A_0 \\
C_1
\end{bmatrix}
\] to an orthonormal basis.

**Proof.** To construct the factors in the DSS factorization explicitly from a realization of $\Xi(z)$ we follow the procedure described in Theorem 4.2.1, Remark 4.3.4 and Section 4.8.1 in [8]. That procedure leads directly to the formula (5.2) for $V(z)$. The formula may also be derived from the results in Chapter 7 of [1], assuming the pair $(C_1, A_0)$ is observable. Indeed, $V(z)$ has the pole pair $(C_1, A_0)$ corresponding to the unit disc, and has to be inner. Then $P_0$ has to be positive definite, and there are $B_V$ and $D_V$ satisfying (5.1), while $V(z)$ is given by (5.2).

It remains to compute $W(z)$. This will be done by direct computation of the product $W^*(z) = V^*(z)\Xi(z)$. Let us compute $V^*(z)\Xi(z)$ for the given $V(z)$:
\[
W^*(z) = V^*(z)\Xi(z) = (D_V^* + B_V^*(zI - A_0)^{-1}C_1^*)\times
(\Xi_0 + \gamma(zI - \alpha)^{-1}\beta_1 + zC_1(I - zA_0)^{-1}B_0 D^{-1})
= D_V^*\Xi_0 + D_V^*\gamma(zI - \alpha)^{-1}\beta_1 + zD_V^*C_1(I - zA_0)^{-1}B_0 D^{-1}
+ B_V^*(zI - A_0)^{-1}C_1^*\Xi_0 + B_V^*(zI - A_0)^{-1}C_1^*\gamma(zI - \alpha)^{-1}\beta_1
+ zB_V^*(zI - A_0)^{-1}C_1^*C_1(I - zA_0)^{-1}B_0 D^{-1}.
\]
Now the terms \( D_V^* \gamma(zI - \alpha)^{-1}\beta_1 \) and \( B_V^*(zI - A_0^*)^{-1}C_1^*\Xi_0 \) as well as \( B_V^*(zI - A_0^*)^{-1}C_1^*\gamma(zI - \alpha)^{-1}\beta_1 \) have only Fourier coefficients corresponding to negative powers of \( z \), while \( zD_V^*C_1(I - zA_0)^{-1}B_0D^{-1} \) has only Fourier coefficients corresponding to positive powers of \( z \). So the constant term of \( V^*(z)\Xi(z) \) is equal to \( D_V^*\Xi_0 \) plus the constant term of

\[
zB_V^*(zI - A_0^*)^{-1}C_1^*C_1(I - zA_0)^{-1}B_0D^{-1}.
\]

We re-express this term with the help of \( P_0 \), using

\[
(zI - A_0^*)^{-1}C_1^*C_1(I - zA_0)^{-1} = P_0A_0(I - zA_0)^{-1} + (zI - A_0^*)^{-1}P_0.
\]

this implies

\[
zB_V^*P_0A_0(I - zA_0)^{-1}B_0D^{-1} + zB_V^*(zI - A_0^*)^{-1}P_0B_0D^{-1}.
\]

The first term in the latter expression again has only Fourier coefficients corresponding to positive powers of \( z \). Furthermore,

\[
zB_V^*(zI - A_0^*)^{-1}P_0B_0D^{-1} = B_V^*(zI - A_0^* + A_0^*)(zI - A_0^*)^{-1}P_0B_0D^{-1}
\]

\[
= B_V^*P_0B_0D^{-1} + B_V^*(zI - A_0^*)^{-1}A_0^*P_0B_0D^{-1}.
\]

So the constant term of \( V^*(z)\Xi(z) \) is equal to \( D_V^*\Xi_0 + B_V^*P_0B_0D^{-1} \). It follows that

\[
D_V^* = D_V^*\Xi_0 + B_V^*P_0B_0D^{-1}.
\]

We carry the computation a bit further to compute \( W^*(z) \) directly, but it requires a bit of extra work to see that many terms cancel.

In fact, we obtain

\[
W^*(z) = D_V^*\Xi_0 + B_V^*P_0B_0D^{-1} + D_V^*\gamma(zI - \alpha)^{-1}\beta_1 +
+ zD_V^*C_1(I - zA_0)^{-1}B_0D^{-1} + B_V^*(zI - A_0^*)^{-1}C_1^*\Xi_0
+ B_V^*(zI - A_0^*)^{-1}C_1^*\gamma(zI - \alpha)^{-1}\beta_1
+ B_V^*(zI - A_0^*)^{-1}A_0^*P_0B_0D^{-1} + zB_V^*P_0A_0(I - zA_0)^{-1}B_0D^{-1}.
\]
As $B^*_V P_0 A_0 = -D^*_V C_1$ by (5.1), the two terms involving $(I - zA_0)^{-1}$ cancel, and we are left with

$$W^*(z) = D^*_V \Xi_0 + B^*_V P_0 B_0 D^{-1} + D^*_V \gamma (zI - \alpha)^{-1}\beta_1 +$$

$$+ B^*_V (zI - A_0)^{-1} C^*_1 \Xi_0 + B^*_V (zI - A_0)^{-1} C^*_1 \gamma (zI - \alpha)^{-1}\beta_1 +$$

$$+ B^*_V (zI - A_0)^{-1} A^*_0 P_0 B_0 D^{-1}.$$ 

Rewrite this by combining the fourth and sixth terms:

$$W^*(z) = D^*_V \Xi_0 + B^*_V P_0 B_0 D^{-1} + D^*_V \gamma (zI - \alpha)^{-1}\beta_1 +$$

$$+ B^*_V (zI - A_0)^{-1} (C^*_1 \Xi_0 + A^*_0 P_0 B_0 D^{-1}) +$$

$$+ B^*_V (zI - A_0)^{-1} C^*_1 \gamma (zI - \alpha)^{-1}\beta_1.$$ 

Next, we assume in addition that $(\gamma, \alpha)$ and $(A_0, B_0)$ are, respectively, observable and controllable, i.e., the realization (4.3) is minimal. Let $P_1$ be the corresponding zero-pole coupling matrix. Then $P_1$ is a solution of

$$A^*_0 P_1 - P_1 \alpha = -C^*_1 \gamma.$$ 

Hence, as in the proof of Proposition 4.2

$$W^*(z) = D^*_V \Xi_0 + B^*_V P_0 B_0 D^{-1} + (D^*_V \gamma + B^*_V P_1)(zI - \alpha)^{-1}\beta_1 +$$

$$+ B^*_V (zI - A_0)^{-1} (C^*_1 \Xi_0 + A^*_0 P_0 B_0 D^{-1} - P_1 \beta_1).$$ 

Using the assumption that $A^*_0$ and $\alpha$ have no common eigenvalues we can use (4.8), and so we have that this equals

$$W^*(z) = D^*_V \Xi_0 + B^*_V P_0 B_0 D^{-1} + (D^*_V \gamma + B^*_V P_1)(zI - \alpha)^{-1}\beta_1.$$ 

Hence we have

$$D^*_W = D^*_V \Xi_0 + B^*_V P_0 B_0 D^{-1} \quad \text{and} \quad B^*_W = D^*_V \gamma + B^*_V P_1.$$ 

This completes the proof. □

In order to apply Theorem 3.3 we need the matrix $X$ in equation (3.4), applied to the realizations (5.2) and (5.3). The following proposition gives an explicit expression for this matrix.
Proposition 5.2. Let \( V(z) \) and \( W(z) \) be given by (5.2) and (5.3), respectively. Then the corresponding matrix \( X \) solving the equation
\[
X - A_0 X \alpha = B_V B_W^* \tag{5.4}
\]
is given by
\[
X = P_0^{-1} P_1. \tag{5.5}
\]

Proof. Since \( X \) is the unique solution of (5.4), all we need to do is to check that \( P_0^{-1} P_1 \) satisfies this equation. This is done by direct checking. First we note that by taking inverses in (5.1) and re-arranging terms we have that
\[
\begin{bmatrix}
A_0 & B_V \\
C_1 & D_V \\
\end{bmatrix}
\begin{bmatrix}
P_0^{-1} & 0 \\
0 & I \\
\end{bmatrix}
\begin{bmatrix}
A_V & C_V \\
B_V & D_V \\
\end{bmatrix}
= \begin{bmatrix}
P_0^{-1} & 0 \\
0 & I \\
\end{bmatrix}
\]
In particular, \( P_0^{-1} = A_0 P_0^{-1} A_V^* \) and \( A_0 P_0^{-1} C_V^* + B_V D_V^* = 0 \).

Using (4.5) and the two equations just derived, we have
\[
\begin{align*}
P_0^{-1} P_1 - A_0 P_0^{-1} P_1 \alpha &= P_0^{-1} P_1 - A_0 P_0^{-1} (A_V^* P_1 + C_V^* \gamma) \\
&= P_0^{-1} P_1 - A_0 P_0^{-1} A_V^* P_1 - A_0 P_0^{-1} C_V^* \gamma \\
&= P_0^{-1} P_1 - (P_0^{-1} - B_V B_V^*) P_1 - A_0 P_0^{-1} C_V^* \gamma \\
&= B_V B_V^* P_1 - A_0 P_0^{-1} C_V^* \gamma = B_V B_V^* P_1 + B_V D_V^* \gamma = B_V B_W^*,
\end{align*}
\]
since \( B_W^* \) is defined by \( B_W^* = B_V^* P_1 + D_V^* \gamma \). \( \square \)

6. Appendix: some remarks on the Douglas-Shapiro-Shields factorization

In this section we discuss further properties of the DSS factorization, which first appeared in Section 3. We divide the section into three parts, Part 6.1, Part 6.2 and Part 6.3.

Part 6.1. We start with \( \Xi(z) = V(z)W^*(z) \), where \( V(z) \) and \( W(z) \) are rational bi-inner \( m \times m \) matrix functions. Furthermore, as in (3.2) and (3.3), we assume that
\[
V(z) = D_V + z C_V (I_{n_-} - z A_V)^{-1} B_V, \quad W(z) = D_W + z C_W (I_{n_+} - z A_W)^{-1} B_W
\]
are bi-inner and stable. Moreover, both realizations are unitary realizations. Thus
\[
\begin{bmatrix}
A_V & B_V \\
C_V & D_V \\
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
A_W & B_W \\
C_W & D_W \\
\end{bmatrix}
\]
are both unitary matrices. These realizations are then minimal. Also, let \( X \) be the unique solution of the Stein equation

\[
X = A_V X A_W^* + B_V B_W^*.
\]  

(6.1) \text{Steina}

Then, see [9],

\[
\Xi(z) = z C_V (I_{n_v} - z A_V)^{-1} B_R + \Xi_0 + C_R (z I_{n_v} - A_W^*)^{-1} C_W^*,
\]  

(6.2) \text{realizationXialt}

where \( B_R, C_R \) and \( \Xi_0 \) are given by

\[
B_R = B_V D_W^* + A_V X C_W^*,
\]

\[
C_R = D_V B_W^* + C_V X A_W^*,
\]

\[
\Xi_0 = D_V D_W^* + C_V X C_W^*.
\]

Minimality of the realizations of \( V(z) \) and \( W(z) \) implies that the pairs \((C_V, A_V)\) and \((C_W, A_W)\) are observable. Hence to show that the realization of \( \Xi(z) \) given by (6.2) is minimal, it suffices to prove the following proposition.

**Proposition 6.1.** The pair \((A_V, B_R)\) is controllable, and the pair \((C_R, A_W^*)\) is observable.

**Proof.** Assume the pair \((A_V, B_R)\) is not controllable. By the Hautus criterion there is an eigenvalue \( \lambda \) of \( A_V^* \) and a non-zero vector \( x \) such that

\[
A_V^* x = \lambda x \quad \text{and} \quad B_R^* x = 0.
\]

Now \( B_R^* = C_W X^* A_V^* + D_W B_V^* \). Thus

\[
C_W X^* A_V^* x + D_W B_V^* x = 0.
\]  

(6.3) \text{eq:basiscid}

We shall use that

\[
\begin{bmatrix}
A_W & B_W \\
C_W & D_W
\end{bmatrix}
\]

is unitary, i.e.,

\[
\begin{bmatrix}
A_W^* & C_W^* \\
B_W^* & D_W^*
\end{bmatrix}
\begin{bmatrix}
A_W & B_W \\
C_W & D_W
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & 1
\end{bmatrix}.
\]  

(6.4) \text{eq:Wunitary}

Firstly multiply (6.3) on the left by \( D_W^* \), to obtain that

\[
0 = D_W^* C_W X^* A_V^* x + D_W^* D_W B_V^* x.
\]

Using (6.4) and (6.1) this gives

\[
0 = -B_W^* A_W X^* A_V^* x + (I - B_W^* B_W) B_V^* x = -B_W^* (A_W X^* A_V^* + B_W B_V^*) x + B_V^* x = -B_W^* X^* x + B_V^* x.
\]

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Secondly, multiply (6.3) by $C_W^*$, and use again (6.4) and (6.1) to obtain that

$$0 = C_W^*C_WX^*A_V^*x + C_W^*D_WB_V^*x =$$

$$= (I - A_W^*A_W)X^*A_V^*x - A_W^*B_WB_V^*x =$$

$$= X^*A_V^*x - A_W^*(A_WX^*A_V^* + B_WB_V^*)x =$$

$$= X^*A_V^*x - A_W^*X^*x = (\lambda - A_W^*)X^*x.$$ 

Combining with the fact that by assumption $A_V^*x = \lambda x$ we then obtain that

$$\begin{bmatrix} B_V^* & -B_W^* \end{bmatrix} \begin{bmatrix} x \\ X^*x \end{bmatrix} = 0,$$

$$\begin{bmatrix} A_V^* & 0 \\ 0 & A_W^* \end{bmatrix} \begin{bmatrix} x \\ X^*x \end{bmatrix} = \lambda \begin{bmatrix} x \\ X^*x \end{bmatrix}.$$ 

However, the pair $\left( \begin{bmatrix} A_V & 0 \\ 0 & A_W \end{bmatrix}, \begin{bmatrix} B_V \\ -B_W \end{bmatrix} \right)$ is controllable since the realizations for $V(z)$ and $W(z)$ are minimal. So by duality and the Hautus criterion we must have $\begin{bmatrix} x \\ X^*x \end{bmatrix} = 0$, which contradicts the assumption that $x \neq 0$. This proves the first part of the claim.

The second part can be proved directly analogously to the first part. But alternatively, a proof by duality is perhaps more instructive. In fact, consider $
abla(z) = W(z)V^*(z)$, which is unitary as well as $\nabla(z)$. The corresponding Stein equation is given by

$$Y = A_WY A_V^* + B_WB_V^*.$$ 

The unique solution is then given by $Y = X^*$. Analogously to the proof of the controllability of $(A_V, B_R)$ one show that the pair $(A_W, B_{R^*})$ is controllable. Observe that

$$B_{R^*} = B_WD_V^* + A_WX^*C_V^* = C_R^*.$$ 

Hence the pair $(A_W, C_R^*)$ is controllable, and thus the dual pair $(C_R, A_W^*)$ is observable. □

**Part 6.2.** In this part we shall show that $V(z)$ and $W(z)$ are unique up to a unitary constant.

Suppose that $\nabla(z) = V(z)W^*(z) = V_1(z)W_1^*(z)$, where $V(z), W(z), V_1(z), W_1(z)$ take unitary values on the unit circle, and are all rational bi-inner.
Then, by Theorem 4.7.1 (iii) in [8], we have that $V_1(z) = V(z)U$ and $W_1(z) = W(z)U$ for some unitary constant $U$.

Next, we consider how the solution of (6.1) changes when we consider the DSS factorization $\Xi(z) = V_1(z)W_1^*(z)$ instead of $\Xi(z) = V(z)W^*(z)$. In addition to (6.2) we have

$$\Xi(z) = zC_{V_1}(I_n - zA_{V_1})^{-1}\hat{B}_R + R_0 + \hat{C}_R(zI_n - A_{W_1}^*)^{-1}C_{W_1}^*,$$  \hspace{1cm} (6.5) \hspace{1cm} \text{realizationXialt2}

where

$$\hat{B}_R = B_{V_1}D_{W_1}^* + A_{V_1}X_{W_1},$$
$$\hat{C}_R = D_{V_1}B_{W_1}^* + C_{V_1}X_{W_1},$$
$$R_0 = D_{V_1}D_{W_1} + C_{V_1}X_{W_1},$$

and $X_1$ is the solution to the equation

$$X_1 = A_{V_1}X_{W_1} + B_{V_1}B_{W_1}^*.$$ 

Because of the minimality of both realizations (6.2) and (6.5) there is an invertible matrix $S$ such that

$$A_{V_1} = S^{-1}A_VS, \quad C_{V_1} = C_VS, \quad B_R = S^{-1}\hat{B}_R$$ \hspace{1cm} (6.6) \hspace{1cm} \text{eq:avav1}

and there is an invertible matrix $T$ such that

$$A_{W_1}^* = T^{-1}A_W^*T, \quad C_{W_1}^* = T^{-1}C_W^*, \quad \hat{C}_R = C_RT.$$ \hspace{1cm} (6.7) \hspace{1cm} \text{eq:awaw1}

Note that $S$ and $T$ are the state space similarities between the realizations $(A_V, B_R, C_V)$ and $(A_{V_1}, \hat{B}_R, C_{V_1})$, and $(A_{W_1}^*, C_R, C_{W_1}^*)$ and $(A_{W_1}, \hat{C}_R, C_{W_1}^*)$, respectively.

Introduce $\Gamma_V = \text{col} (C_V A_{V_1}^j)_{j=0}^\infty$ and similarly $\Gamma_W = \text{col} (C_W A_{W_1}^j)_{j=0}^\infty$, $\Gamma_{V_1} = \text{col} (C_{V_1} A_{V_1}^j)_{j=0}^\infty$ and $\Gamma_{W_1} = \text{col} (C_{W_1} A_{W_1}^j)_{j=0}^\infty$. Denote the block Toeplitz operator with symbol $\Xi(z)$ by $T_\Xi$.

Using formulas (6.6) and (6.7), and the formula $X = \Gamma_{V_1}^*T_\Xi\Gamma_{W_1}$ (see [9]), the following proposition is immediate.

**Proposition 6.2.** We have $X = S^{-*}\Gamma_{V_1}^*T_\Xi\Gamma_{W_1}T^*$, and hence the matrices $X$ and $X_1 = \Gamma_{V_1}^*T_\Xi\Gamma_{W_1}$ are related by $X_1 = S^*XT^{-*}$. 

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Part 6.3. In this part we discuss the uniqueness of the unitary realizations of $V(z)$ and $W(z)$.

In comparison with Part 6.2 we further restrict the realizations of $V(z)$, $V_1(z)$, $W(z)$ and $W_1(z)$ to unitary realizations, that is,

$$
\begin{bmatrix}
A_V & B_V \\
C_V & D_V
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A_W & B_W \\
C_W & D_W
\end{bmatrix}
$$

as well as

$$
\begin{bmatrix}
A_{V_1} & B_{V_1} \\
C_{V_1} & D_{V_1}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A_{W_1} & B_{W_1} \\
C_{W_1} & D_{W_1}
\end{bmatrix}
$$

are unitary, then the invertible matrices $S$ and $T$ in Part 6.3 are further restricted to being unitary. Indeed,

$$
\begin{bmatrix}
A_V & B_V \\
C_V & D_V
\end{bmatrix}
= \begin{bmatrix}
S & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_V & B_V \\
C_V & D_V
\end{bmatrix}
\begin{bmatrix}
S & 0 \\
0 & I
\end{bmatrix}.
$$

Then, as $\begin{bmatrix}
A_{V_1} & B_{V_1} \\
C_{V_1} & D_{V_1}
\end{bmatrix}$ is unitary, we obtain

$$
\begin{bmatrix}
A^*_V & C^*_V \\
B^*_V & D^*_V
\end{bmatrix}
\begin{bmatrix}
(SS^*)^{-1} & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_V & B_V \\
C_V & D_V
\end{bmatrix}
= \begin{bmatrix}
(SS^*)^{-1} & 0 \\
0 & I
\end{bmatrix}.
$$

Consider the $(1,1)$-block entry in this equation:

$$
A^*_V(SS^*)^{-1}A_V + C^*_VC_V = (SS^*)^{-1}.
$$

This means that $(SS^*)^{-1}$ is a solution to the matrix equation

$$
P - A^*_VP = C^*_VC_V.
$$

However, this equation has a unique solution, since $A_V$ is stable, and since $\begin{bmatrix}
A_V & B_V \\
C_V & D_V
\end{bmatrix}$ is unitary, the solution is equal to the identity. Hence $SS^* = I$, and so $S$ is unitary. Likewise, also $T$ is unitary.

It follows that the matrix $X$, when restricting the attention to unitary realizations of $V(z)$ and $W(z)$, is unique up to unitary equivalence.

This completes Part 6.3 and hence Section 6 is completed too. $\Box$

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