Some Generalizations of Random Broken and Pick-up Stick Problems

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Abstract. We consider sticks with random lengths, which are further randomly broken into several pieces. The probability that these sticks can form a polygon is computed in this article. This Broken Pick-up Stick Problem was first asked in [1] and we give a general formula to solve this problem. Besides, we study the probability that any three sticks with independent and uniformly distributed lengths can form a triangle. Our results generalize the famous Spaghetti Problem and the Pick-up Stick Problem in probability. We present a way to transfer these continuous probability problems into discrete problems and apply combinatorial methods to address the discrete version of the problems.

1. Introduction
There is an old probability problem called the Spaghetti Problem, which is asking the probability of forming a triangle when a stick was broken into three pieces randomly. It has been already studied and extended by many mathematicians for a long time back to the early 19th century. We refer [2,3,4] for the history and background of this famous probability problem. Here, we repeat the original question as follows.

Problem 1.1 (The classical Spaghetti Problem). Randomly and independently pick two points on a spaghetti stick with length 1, and divide the stick into three parts. Calculate the probability that the resulting three pieces form a triangle.

Figure 1. Two examples on the situations of Spaghetti Problem: three sticks can form a triangle in Situation 1, but cannot in Situation 2.

As the examples shown in Figure 1, to calculate the probability for Problem 1.1, it suffices to find out the condition that the lengths of three broken sticks can form a triangle. We can either compute the probability by some integrals or use the geometry to compute the feasible region. In fact, this classical problem has many different methods to solve and the answer should be $\frac{1}{4}$. Moreover, this problem has plenty of generalizations in probability theory and has a close relation with random geometry (see [5]).
Illinois Geometry Lab has been studying a sequence of generalizations of the broken sticks problem for several years in [4, 6]. Also, there are several problems related to Problem 1.1 are raised in The American Mathematical Monthly, for instance [7, 8, 9]. Even now, there are still several open and interesting questions in this direction. Most of the extensions require new techniques in probability and geometry to address. Our current work tackle some of these open problems using discrete probability and combinatorics.

One extension is asking the probability of forming a $k$-gon if we randomly pick $k - 1$ points in a stick. It was solved by Carlos and Emiliano in [7], which states as follows.

**Theorem 1.2** (See [10]). Given any natural number $n \geq 3$. If we randomly and independently select $n - 1$ points in a stick to break the stick into $n$ pieces, then the probability that such $n$ pieces form a $n$-gon is

$$1 - \frac{n}{2^{n-1}}$$

There are other forms of related problems, for instance, asking the possibility that any three broken sticks can form a triangle when we randomly divide a stick into more than three pieces. In our work, we will consider a related but more general problem, which is called Arbitrariness Problem. Besides, instead of these broken stick problems, there is another kind of problem, called Pick-up Stick Problem. The Pick-up Stick Problem discusses the probability of forming a polygon with multiple random length sticks. In the simplest form of this question, it discusses the situation of three sticks with random lengths but without breaking. Here, we can consider these lengths are independently and identically distributed by Uniform(0, 1). The answer to this question is $\frac{1}{2}$, obviously different the simplest Spaghetti Problem. A general version of this question is stated in the following Theorem 1.3. An example of a such Pick-up Problem is shown in Figure 2.

**Theorem 1.3** (Pick-up Stick Problem [1]). Randomly and independently select $k$ sticks, which $k$ is a positive integer, with lengths chosen in a uniform distribution Uniform(0, 1). Then, the probability that these $k$ sticks can form a $k$-gon is

$$1 - \frac{1}{(k-1)!}$$

Following Theorem 1.2 and Theorem 1.3, this article will generalize these results and prove probability formulas for several problems related to broken and pick-up stick problems. We will combine the Spaghetti Problem and the Pick-up Stick Problem together to study a more general problem stated below. In terms of [1], we call this new problem as the Broken Pick-up Sticks Problem. As a result of our, we fully solve the open question raised in the last section of [1].

**Problem 1.4** (The Broken Pick-up Sticks Problem). Given any $n$ natural number $a_1, ..., a_n$, such that $k := \sum_{i=1}^{n} a_i \geq 3$. We consider $n$ sticks with independent lengths distributed by Uniform(0, 1). For $1 \leq i \leq n$, we randomly break the $i$-th stick into $a_i$ pieces. Denote the collection of all $k$ broken sticks by a set $\text{Stick}(a_1, ..., a_n)$. Then, what is the probability that these $k$ parts can form a $k$-gon?
Furthermore, we are interested in the Arbitrariness Problem in this topic. More precisely, we want to know the probability that any three broken sticks can form a triangle. This arbitrariness problem in the set-up of Theorem 1.2 has been studied in [11] by order statics and exponential random variables. In this article, we focus on the arbitrariness of Pick-up Stick Problem as below.

**Problem 1.5** (Arbitrariness of Forming Triangles in Pick-up Problem). Given \( n \) sticks with independent lengths distributed by Uniform\((0, 1)\), what is the probability that any three sticks in these \( n \) sticks can form a triangle?

1.1. **Notation**

In this work, we will always use \( \mathbb{P}(A) \) to describe the probability that the situation fit the problem (or the probability of success), which is the answer of our problem. Here, \( A \) will always be denoted as the successful event corresponding to each problem. Relatively, \( \mathbb{P}(A^c) \) as the complement describe the probability that the situation is not fit. We denote \( \Omega \) as the probability sample space. In the discrete case, the number of all possible cases is \( \#\{\Omega\} \). We use \( \mathbb{P}_{d\Omega} \) to represent the probability in discrete situations. We denote the rounding function \([x]\) as the integer part of \( x \).

2. **Main results**

In this article, we will present our main results: the solutions to Broken Pick-up Stick Problem 1.4 and Arbitrariness of forming triangles in Pick-up Problem 1.5. Before introducing the theorem, we first define some notions which we will use in the proofs.

**Definition 2.1.** Given natural numbers \( a_i \), for \( 1 \leq i \leq n \in \mathbb{N} \). We randomly and independently pick up \( n \) sticks with lengths distributed by Uniform\((0, 1)\). Denote the length of each stick by \( l_i \). Then for each \( l_i \), we randomly break it into \( a_i \) pieces. Let \( \text{Stick}(a_1, ..., a_n) \) be the collection of all broken sticks.

**Theorem 2.2** (The Broken Pick-up Sticks Problem). Given positive integers \( n \) and \( a_1, ..., a_n \). Consider the collection of sticks \( \text{Stick}(a_1, ..., a_n) \) defined above. Let \( k = \sum_{i=1}^{n} a_i \). The probability that all \( k \) parts in \( \text{Stick}(a_1, ..., a_n) \) are able to form a \( k \)-gon is

\[
\mathbb{P}(A) := \mathbb{P}(\text{Stick}(a_1, ..., a_n) \text{ can form a } k \text{-gon}) = 1 - \frac{1}{n} \cdot \sum_{i=1}^{n} \frac{a_i!}{(a_i+n-2)!} \cdot \frac{1}{2^{a_i-1}} \tag{3}
\]

**Remark 2.3.** We can consider some special cases in Theorem 2.2. If \( n \) sticks have the same number of broken parts, \( \text{Stick}(a, a, ..., a) \), then the successful probability of forming an \( an \)-gon is

\[
\mathbb{P}(A) = 1 - \frac{a!}{2^{(a-1)(a+n-2)!}} \tag{4}
\]

Hence, when \( a = 1 \), we return to Pick-up Stick Problem in Theorem 1.3. Besides, when \( n = 1 \), for \( \text{Stick}(a) \), the formula becomes

\[
\mathbb{P}(A) = 1 - \frac{a}{2^{(a-1)}} \tag{5}
\]

which is same as the original Spaghetti Problem for polygons in Theorem 1.2.

For Problem 1.5, we compute the probability that any three sticks in Pick-up Stick Problem form a triangle. The proof is finished in Section 4. The idea of the proof is analogous to the proof of Theorem 2.2 but has more difficulties in compute the limit of the probability for discrete problem.

**Theorem 2.4** (Arbitrariness of Forming Triangles in Pick-up Problem). For any integer \( n \in \mathbb{N} \), if we independently select \( n \) sticks with lengths from the distribution Uniform\((0, 1)\), then the probability that any three sticks among them can form a triangle is

\[
\frac{1}{2^{(n-2)}} \tag{6}
\]

Based on Definition 2.1, the collection of sticks with random lengths is \( \text{Stick}(1, ..., 1) \) for \( n \) ones. Like the previous question, we find a sufficient and necessary condition for any three sticks form a triangle, namely the sum of shortest two sticks are longer than the longest stick. See Lemma 4.1. Theorem 2 in [4] and Theorem 1 in [11] studied the Arbitrariness of Forming Triangles in Broke Stick...
Problem. Consider an \( n \)-piece randomly broken stick. The probability that any triple of pieces chosen from the \( n \) pieces can form a triangle is

\[
\frac{1}{\binom{2n-2}{n}}
\]  

(7)

which is much less than the probability in Theorem 2.4 for sufficiently large \( n \). This indicates that it is more difficult to guarantee any three sticks form triangle in a broken stick with lots of pieces.

3. The proof of Theorem 2.2

Our original Problem 1.4 has too many variables to study. Instead, we first introduce another problem, which is a simple and discrete version of Problem 1.4. In this case, all the lengths become integers.

**Problem 3.1** (The broken pick-up sticks problem - Simple Version). Given a positive integer \( n \), for the group of \( \text{Stick}(a_1, \ldots, a_n) \) with \( a_i \) chosen from the natural numbers, every \( \text{Stick}(a_i) \) with \( a_i \in \{a_i > 1|a_i \in \mathbb{N}^+\} \) has a fixed length 1 and every \( \text{Stick}(1) \) has the length randomly chosen from the real number range \((0, 1)\). There are totally \( \sum_{i=1}^{n} a_i = k \) parts, where the \( i \)-th stick is randomly divided into \( a_i \) parts. What is the probability that these \( h \) parts are able to form a \( k \)-gon?

**Proposition 3.2** (Necessary and Sufficient Condition of Not Forming). In Problem 3.1, the \( k \)-gon cannot be form if and only if the longest part \( L \) in \( \text{Stick}(a_1, \ldots, a_n) \) is longer than the half of the total length of all sticks \( \sum_{i=1}^{n} l_i \).

In other words,

\[
\mathbb{P}(A^c) = \mathbb{P}\left(L \geq \frac{1}{2}\sum_{i=1}^{n} l_i \right)
\]

(8)

On the basis of this, we will separate the Problem 3.1 into three situations.

**Lemma 3.3** (Two \( a_i \) more than one situation). If the \( \text{Stick}(a_1, \ldots, a_n) \) includes two or more parameters \( a_i > 1 \), for example \( \text{Stick}(3, 4, 1, 1) \), must have the probability \( \mathbb{P}(A^c) = 0 \) and \( \mathbb{P}(A) = 1 \).

**Proof.** Because that there are at least two sticks, suggest they are \( a_p \) and \( a_q \) have parameters more than one, they have the fixed-length 1, according to the problem, we have:

\[
\sum_{i=1}^{n} l_i > (l_p + l_q) = 2 \times 1
\]

(9)

Because that all the parts in \( \text{Stick}(a_i > 1) \) are less than one and \( l_j \) with \( a_j = 1 \) is less than one, it does not exist a part, which is a side of the \( h \)-gon, more than one.

\[
0 < L < 1
\]

(10)

\[
2L < (l_p + l_q) < \sum_{i=1}^{n} l_i
\]

(11)

And, with the Eq.(8),

\[
\mathbb{P}(A^c) = 0
\]

(12)

\[
\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1
\]

(13)

Secondly, because we exclude the situation with two or more \( a_i > 1 \), we discuss the situation that there is only one \( a_i > 1 \) in the group of \( \text{Stick}(a_1, \ldots, a_n) \). However, we will firstly discuss a special situation in this second part, which is \( \text{Stick}(a, 1) \), with \( a > 1 \), instead of a \( \text{Stick}(a) \) and a lot of \( \text{Stick}(1) \).

**Lemma 3.4** (Probability of \( \text{Stick}(a, 1) \)). Given a positive integer \( a \geq 1 \), for the \( \text{Stick}(a, 1) \) with a length-1-stick, which separated into \( a \) parts and a stick with random length between 0 and 1, the probability of forming a \( h \)-gon is:

\[
\mathbb{P}(A) = 1 - \frac{1}{2^{a-1}}
\]

(14)

**Proof.** In this part, we will use discrete thought. Suggest the \( \text{Stick}(a) \) has a fixed length of \( m \), which is large enough, instead of 1 and \( \text{Stick}(1) \) has an integer length randomly chosen in \((0, m)\). We will divide the \( \text{Stick}(a) \) into \( a \) parts by randomly select \((a - 1)\) points on the integer points of this \( m\)-
length stick. We suggest the Stick(1) has a length of \( k \), which satisfies \( k \in (0, m) \) and \( k \) is an integer.

To solve this situation from a discrete side, we put up a discrete version of this Problem 3.1:

**Problem 3.5** (Discrete version problem). In the group of Stick\((a, 1)\) with \( a \geq 1 \) chosen from the natural numbers, containing a stick with length \( m \), a large enough integer, divided into \( a \) parts and a stick with length \( k \), which is an integer selected from range(0, \( m \)). There are totally \((a + 1)\) parts. What’s the probability of these \((a + 1)\) parts are able to form a \((a + 1)\)-gon?

As Eq.(8), we can show the probability by Proposition 3.2:

\[
\mathbb{P}_{\text{dis}}(A^c) = \frac{\#(\{l \geq \frac{1}{2} \sum_{i=1}^{n} l_i \})}{\#(\Omega)}
\]  

(15)

According to the problem, we can get the formula of the total length:

\[
\sum_{i=1}^{n} l_i = m + k
\]  

(16)

We consider the total length \( \sum_{i=1}^{n} l_i \) as an important value in the whole proof, since that we need to determine the range of \( \mathbb{P}(A^c) \) depending on the total length. So that, we made the total length fixed by considering \( k \) a parameter, instead of a variable, as a condition.

In this assumption, we can get the universal set \( \#(\Omega) \) easily. We only need to consider the Stick\((a)\) with length \( m \), selecting \((a - 1)\) points from \((m - 1)\) points in total.

\[
\#(\text{Stick}(1) = k) = \binom{m - 1}{a - 1}
\]  

(17)

In order to satisfy the \( A^c \) situation, the longest part\((L)\):

\[
L \geq \frac{\sum_{i=1}^{n} l_i}{2} = \frac{m+k}{2}
\]  

(18)

Notice that the longest side cannot be the Stick\((1)\), since \( k < m \) and \( k < \frac{k+m}{2} \), \( L \) can only be a part of Stick\((a)\), which is divided to be very long. Suggest \( f_i \) to be the parts divided in Stick\((a)\), with \( i \in \{1, 2, \ldots, a\} \), the longest side \( L \) can be any of \( f_i \) and being any of them will not affect the final answer. So that, we can assume \( L = f_a \), but we need to multiple \( a \) to the result of this step. Thus, we get \( L \) on the rightest side of Stick\((a)\) and no points in the range that \( L \) has occupied. So, all the points chosen in Stick\((a)\) should be in the range \((0, \frac{m-k}{2})\):

\[
\mathbb{P}_{\text{dis}}\left(L \geq \frac{1}{2} \sum_{i=1}^{n} l_i \right) = \frac{\binom{m-k}{a-1}}{a-1}
\]  

(19)

Start, we use Eq.(17) times \( a \) and divides by Eq.(19) to get \( \mathbb{P}(A^c) \) under this Stick\((1) = k \) condition:

\[
\mathbb{P}_{\text{dis}}(A^c|\text{Stick}(1) = k) = \frac{a \binom{m-k}{a-1}}{\binom{m}{a-1}}
\]  

(20)

Because the conditions Stick\((1) = k, k \in \{1, \ldots, n\} \) are independent, we can add up to the sum of them and calculate the mean to eliminate this condition and find \( \mathbb{P}_{\text{dis}}(A^c) \).

\[
\mathbb{P}_{\text{dis}}(A^c) = \sum_{i=1}^{m-1} \mathbb{P}_{\text{dis}}(A^c|\text{Stick}(1) = i) \times \frac{1}{m-1} = \sum_{i=1}^{m-1} \frac{a \binom{m-i}{a-1}}{\binom{m}{a-1}} \times \frac{1}{m-1}
\]

\[
= \frac{a}{m-1} \sum_{i=1}^{m-1} \frac{\binom{m-i}{a-1}}{\binom{m}{a-1}} \times \frac{1}{m-1}
\]  

(21)

**Lemma 3.6** (Approximation of the number of combinations). For \( s \), which is big enough, and a fixed smaller \( r \), the approximation of \( \binom{s}{r} \) is equal to \( \frac{s^r}{r!} \).
As we have \( m \) big enough and a relatively small \( a \), we can approximate the formula above into the following versions:

\[
\mathbb{P}_{\text{dis}}(A^c) = \frac{a \times (a-1)!}{(m-1) \times (m-1) \times (a-1)!} \sum_{i=1}^{m-1} \frac{1}{(a-1)!} \left[ \frac{m-i}{2} \right]^{(a-1)} = \frac{a}{2^{(a-1)} \times (m-1) a} \sum_{i=1}^{m-1} (m-i)^{(a-1)}
\]

(22)

To this step, we can no longer perform discrete simplifications. For the discrete version, we got the answer above, with \( m \) the length of the fixed \( \text{Stick}(a > 1) \) and \( a \) the parameter. We are going to require \( m \) to be positive infinite instead of a very large integer and thus separate the summation part by using calculus. We take the limit of \( \mathbb{P}_{\text{dis}}(A^c) \) to get \( \mathbb{P}(A^c) \), requiring \( m \) to approach positive infinity.

\[
\mathbb{P}(A^c) = \lim_{m \to +\infty} \frac{a}{2^{(a-1)} \times (m-1) a} \sum_{i=1}^{m-1} (m-i)^{(a-1)} = \frac{a}{2^{(a-1)} \times (m-1) a} \sum_{i=1}^{m-1} \frac{1}{m-i} \times \frac{1}{a-1}
\]

(23)

So that, we can get the probability by solving this formula, keeping \( a \) as a parameter.

\[
\mathbb{P}(A^c) = \frac{1}{2^{a-1}}
\]

(24)

\[
\mathbb{P}(A) = 1 - \frac{1}{2^{a-1}}
\]

(25)

Finally, we will discuss the remaining situation, which also contain only one parameter \( a > 1 \), but many 1 in \( \text{Stick}(a, 1, 1, ..., 1) \). Notice that, the position of \( a \) parameter in \( \text{Stick}(a, 1, 1, ..., 1) \) has no affect to the result probability since they all have a same meaning. Suggest the \( \text{Stick}(a, 1, 1, ..., 1) \) with one \( a \) and \( \alpha \) 1s can be written in the form of \( \text{Stick}(a; \alpha) \).

\textbf{Lemma 3.7} (Probability of \( \text{Stick}(a; \alpha) \)). For the \( \text{Stick}(a; \alpha) \) with a length-1-stick and \( \alpha \) sticks with randomly chosen length:

\[
\mathbb{P}(A) = 1 - \frac{a!}{(a+a-1)!} \times \frac{1}{2^{a-1}}
\]

(26)

\textit{Proof.} As the previous proves, we know that the \( \text{Stick}(1) \)'s in \( \text{Stick}(a; \alpha) \) cannot be the longest part to satisfy \( L > \frac{1}{2} \sum_{i=1}^{n} l_i \), since:

\[
l_i < 1, \text{for } i > 1
\]

(27)

\[
l_i < \frac{i+1}{2}
\]

(28)

So that we only need to discuss the probability in the \( \text{Stick}(a) \), which is exactly the same in the proof above. The only limitation is that the total length should be under 2.

\[
1 > L > \frac{1}{2} \sum_{i=1}^{n} l_i
\]

(29)

\[
\sum_{i=1}^{n} l_i < 2
\]

(30)

So, \( \sum_{i=1}^{n} l_i \) for \( i > 1 \), \( \alpha \) sticks in total, should be less then 1. Still, we use the discrete thought to solve this problem, suggesting that \( \text{Stick}(a) = m \) and the \( \alpha \) \( \text{Stick}(1) \)'s have length \( \{k_1, ..., k_\alpha\} \). With the experience of the previous situation, we discuss every possible \( K = \sum_{i=1}^{\alpha} k_i \) for the \( \mathbb{P}_{\text{dis}}(A^c) \). When \( K > m \), \( \mathbb{P}(A^c) = 0 \), so that, the range of possible \( K \) is less than \( m \) and more than \( \alpha \), since every \( \text{Stick}(1) \) should at least have length 1.

\[
\mathbb{P}_{\text{dis}}(A^c) = \sum_{K=b}^{m} \mathbb{P}_{\text{dis}}(A^c | \sum_{i=1}^{\alpha} k_i = K) \cdot \mathbb{P}(\sum_{i=1}^{\alpha} k_i = K)
\]

(31)

To find \( \mathbb{P}(\sum_{i=1}^{\alpha} k_i = K) \), we can assume a \( K \)-length stick, with is the combination of all \( \text{Stick}(1) \)'s, and find the way of dividing it into \( \alpha \) parts. So that we need to choose \((\alpha - 1) \) points among \((K - 1) \) points. Then, it is easy to find the universal set of this event, which asks the total ways of forming the \( \alpha \) \( \text{Stick}(1) \)'s. That is \( m^\alpha \) since each stick has \( m \) kinds of lengths to choose.
\[ \mathbb{P}(\sum_{i=1}^{\alpha} k_i = K) = \frac{\#(\mathcal{P}(K))}{\#(\Omega)} = \frac{(K-1)^{\alpha-1}}{m^\alpha} \]  

(32)

Put this formula into \( \mathbb{P}_{\text{dis}}(A^c) \), we can get:

\[ \mathbb{P}_{\text{dis}}(A^c) = \sum_{K=a}^{m} \left( \frac{a(m-K)^{(a-1)}}{2^{(a-1)(m-1)(a-1)!}m^a} \cdot \frac{(K-1)^{(a-1)}}{(m-1)(a-1)!} \cdot \frac{1}{m} \right) \]  

(33)

By using Lemma 3.6, we can replace the combination parts into the form of powers

\[ \mathbb{P}_{\text{dis}}(A^c) = \sum_{K=a}^{m} \frac{a}{2^{(a-1)(m-1)(a-1)!}m^a} \lim_{m \to \infty} \sum_{K=a}^{m} \left( \frac{(m-K)^{(a-1)}}{m^{(a-1)}} \cdot \frac{K^{(a-1)}}{m} \cdot \frac{1}{m} \right) \]  

(34)

Noticing that, as the previous situation, we limit the length \( m \) to positive infinity to get the continuous distribution in the real number range. We can change the lower boundary to 0 since \( \alpha \) is very small comparing to \( m \).

\[ \mathbb{P}(A^c) = \lim_{m \to \infty} \sum_{K=a}^{m} \left( \frac{(m-K)^{(a-1)}}{m^{(a-1)}} \cdot \frac{K^{(a-1)}}{m} \cdot \frac{1}{m} \right) \]  

(35)

Because the defined range for the problem is \((0, 1)\), we scale \( m \) to 1 and introduce a new variable \( y = \frac{K}{m} \). Thus, \( y \in (0, 1) \) and \( \frac{1}{m} \) is dy.

\[ \mathbb{P}(A^c) = \frac{a}{2^{(a-1)(a-1)!}} \int_{0}^{1} ((1 - y)^{(a-1)} \cdot y^{(a-2)}) dy \]  

(36)

Suggest \( \int_{0}^{1} ((1 - y)^{(a-1)} \cdot y^{(a-2)}) dy = D \), we can use integration by parts to simplify \( D \).

\[ D = \frac{y^{a-1}(1-y)^{(a-1)}b}{b} \Big|_{0}^{1} - \int_{0}^{1} (a - 1)(1 - y)^{a-2} \cdot \frac{y^{a}}{a} dy = \int_{0}^{1} (a - 1)(1 - y)^{a-2} \cdot \frac{y^{a}}{a} dy \]  

(37)

By doing this time by time to eliminate the term \((1 - y)\) in the formula above, we need to do \((a - 1)\) times and then we get:

\[ D = \int_{0}^{1} (a - 1)! \cdot \frac{y^{a-2}}{(a-2)!} dy = \frac{(a-1)!}{(a-2)!} \cdot (a + 2) \]  

(38)

Then we can put \( D \) into the \( \mathbb{P}(A^c) \) formula and get:

\[ \mathbb{P}(A^c) = \frac{a!}{(a-1)!} \cdot \frac{1}{2^{a-1}} \]  

(39)

Therefore, the answer of this situation can be shown

\[ \mathbb{P}(A) = 1 - \frac{a!}{(a-1)!} \cdot \frac{1}{2^{a-1}} \]  

(40)

We finish discussing all three situations that the Probability for Stick\((a_1, ..., a_n)\) to form a \( k \)-gon in Problem 3.1. We can conclude the second situation into the third one, since the second one is situation when \( \alpha = 1 \) is the third situation. Here is the final result: There are two situations:

- More than one parameter \( a_i > 1 \): \( \mathbb{P}(A) = 1 \)
- Only one parameter \( a_i > 1 \): \( \mathbb{P}(A) = 1 - \frac{a!}{(a-1)!} \cdot \frac{1}{2^{a-1}} \)

**Proposition 3.8** (Longest side finding). Suggest \( a_{\text{great}} \) is the longest stick among the group Stick\((a_1, ..., a_n)\), the longest part \( L \) must be in the Stick\((a_{\text{great}})\), if \( L \) is longer than \( \frac{\sum_{j=1}^{n} l_j}{2} \).

**Proof.** Suggest the length of Stick\((a_{\text{great}})\) is lgreat, and \( L \) is a part of \( l_j \), with \( a_j \) is not \( a_{\text{great}} \).
Seeing that the $L$ not in the longest stick will lead to the contradiction, $L$ can only be a part of $\text{Stick}(a_{\text{great}})$. 

**Proof of Theorem 2.2.** If we assume that we know the $\text{Stick}(a_{\text{great}})$, in other words, the chosen of the longest stick, we can build the relationship between the original problem with the Problem 3.1 that we have already solved. We scale the length of all sticks in equal proportions, and make the length of $\text{Stick}(a_{\text{great}})$ with $l_{\text{great}} = 1$ as a fixed length and make all other sticks have a length taken form the range $(0, 1)$. Suppose that $l_{\text{great}} = V$, where $V$ is a fixed known parameter, and all other Sticks have length $U_1, ..., U_{n-1}$. After the scaling, $l_{\text{great}} = 1$ and others have length

$$
\frac{u_1}{V}, ..., \frac{u_{n-1}}{V} \sim \text{Uniform}(0, 1)
$$

So that, this problem turned to the Situation 2 in the Problem 3.1, the simple version. Notice that the number of sticks is $n$, including the one $\text{Stick}(a)$ and $\alpha \text{ Stick}(1)$ in the Situation 2.

$$
n = \alpha + 1
$$

So that, we can give the answer to the Broken Pick-up Stick Problem 1.4 under the assumption that $\text{Stick}(a_{\text{great}})$ is known:

$$
\mathbb{P}(A|a_{\text{great}} = a_i) = 1 - \frac{a_i!}{(a_i+n-2)!} \times \frac{1}{2^{a_i-\tau}}
$$

However, each in the group $\text{Stick}(a_1, ..., a_n)$ may be the longest one, since they are all randomly chosen. Therefore, we must find all the conditions when $a_i = a_{\text{great}}$ and take the average value, since they have the same probability of getting the longest. In this way we get the answer of this problem, presented above in Section 2.

### 4. The proof of Theorem 2.4

With the method in Section 3, we discuss the Problem 1.5, which asks if any three parts of the separating can form a triangle in $\text{Stick}(1, ..., 1)$. The arbitrariness and existence of the Spaghetti Problem was first given in [4] but without the prove. We will start with the arbitrariness of Pick-up Problem $\text{Stick}(1,1, ..., 1)$ in the current section. We will always suppose that there are $n$ sticks in this group as shown in Figure 3.

Before proving Theorem 2.4, there is an important condition to claim, which is essential for our arbitrariness problem. This lemma will help us deduce our original geometric problem to an algebraic inequality problem. The full proof of the following lemma is given in Lemma 2 of [11].

---

**Figure 3.** The example of $\text{Stick}(1, 1, 1, 1)$ that any three sticks in this collection of sticks can form a triangle.
Lemma 4.1. If any of three parts of a group of sticks can form a triangle, it has a necessary and sufficient condition that the sum of shortest two of them should be longer than the longest one.

The Lemma 4.1 is not hard to prove. Suppose that the sticks have length \(\{k_1 < \cdots < k_n\}\) in order, \(k_1 + k_2 > k_n\) is necessary for the condition. For any three sticks, \(k_a + k_b > k_c\), we know that \(k_c \leq k_n\), so \(k_a + k_b \geq k_1 + k_2\), and that \(k_a + k_b > k_c\) must be true. Thus, \(k_1 + k_2 > k_n\) is sufficient for the condition.

Proof of the Probability of Arbitrariness of Pick-up Problem. Thinking discretely, we still suggest the longest one among those \(n\) sticks have a fixed length \(M\), which is a large enough integer. Thus, all other sticks are selected with the length among the integers in the range \((0, M)\). As every other \((n-1)\) stick have \(M\) points to choose, the total number of situations is \(M^{(n-1)}\). However, considering the Lemma 4.1, we should select the shortest two sticks. We divide the total number by \((n-1)!\) to get it sorted.

\[
\#\Omega = \frac{M^{(n-1)}}{(n-1)!}
\]  

(46)

In order to get the number of situations of the sum of shortest two sticks longer than the longest, we count the situation of those two and multiple the situations of the rest sticks. Suggest the second-shortest stick is \(k_2\) and the shortest one is \(k_1\),

\[
\#\{\text{Fit for the problem}\} = \begin{cases} 
  k_1 > M - k_2 \\
  k_1 < k_2 < M \\
  \text{Others’ length are selected between} \ k_2 \ \text{and} \ M
\end{cases}
\]  

(47)

Notice that if \(k_1 + k_2 > M\), \(k_2\) should be larger than \(\frac{M}{2}\), since \(k_1\) and \(k_2\) are all smaller than \(M\). So that, for all values of \(k_2\) in the range \((\frac{M}{2}, M)\), \(k_1\) can choose from at least \((M - k_2)\) to at most \(k_2\). And the other sticks, \((n - 3)\) sticks in total, can choose from the range \((M - k_2)\), since they should be larger than \(k_2\) but less than \(M\).

Thus, we found \(k_1\) has \((M - k_2) = 2k_2 - M\) situations to choose and all other sticks have \(\binom{M - k_2}{n - 3}\) situations. We can express those situations by using summation of \(k_2\):

\[
\#\{\text{Fit for the problem}\} = \sum_{k_2 = \frac{M}{2}}^{M} (2k_2 - M) \times \binom{M - k_2}{n - 3}
\]  

(48)

Using the formula to express the probability \(P_{\text{dis}}(A)\),

\[
P_{\text{dis}}(A) = \frac{\#\{\text{Fit for the problem}\}}{\#\Omega} = \frac{(n-1)! \sum_{k_2 = \frac{M}{2}}^{M} (2k_2 - M) \times \binom{M - k_2}{n - 3}}{M^{(n-1)}}
\]  

(49)

Since \(M\) is large enough, we can use the Lemma 3.6 to simplify the formula.

\[
P_{\text{dis}}(A) = \frac{(n-1)!}{(n-3)!} \cdot \sum_{k_2 = \frac{M}{2}}^{M} \frac{(2k_2 - M) \times \binom{M - k_2}{n-3}}{M^{(n-1)}}
\]  

(50)

Similar to the previous problem, we let the \(M\) approach the positive infinity to get the \(P(A)\). By taking this limitation, the summation of \(k\), will approach the integral of \(k\).

\[
P(A) = \frac{(n-1)!}{(n-3)!} \cdot \int_{\frac{M}{2}}^{M} \frac{(2k_2 - M) \times \binom{M - k_2}{n-3}}{M^{(n-1)}} \, dk_2
\]

Suggest \(x = \frac{k}{M}\),

\[
P(A) = (n-1)(n-2) \cdot \frac{1}{2} \left(2x - 1\right) \cdot \left(1 - x\right)^{(n-3)} dx
\]  

(51)

Suppose that \(y = 1 - x\), then

\[
P(A) = (n-1)(n-2) \cdot \left(\frac{1}{2^{(n-2)}} \left(\frac{1}{n-2} + \frac{1}{n-1}\right)\right) = \frac{1}{2^{(n-2)}}
\]  

(52)
Thus, we get the final arbitrariness probability for \( n \) sticks in Pick-up Stick Problem 1.3.

5. Conclusions and Open Problems

In summary, we work out two probability problems related to broken and pick-up sticks in the literature. We first compute the probability of forming a polygon in \( \text{Stick}(a_1, \ldots, a_n) \). Here, \( \text{Stick}(a_1, \ldots, a_n) \) is collected by picking up \( n \) sticks with independent lengths in Uniform(0, 1), and then breaking each stick into \( a_i \) pieces randomly. This result solves the open problem proposed by Petersen and Tenner in [1]. We next compute the probability that arbitrary triple sticks in \( \text{Stick}(1, \ldots, 1) \) form a triangle. This can be viewed as a generalization of Theorem 2 in [4]. In the proofs, we first study the discrete version of these two problems, where all lengths of sticks are natural numbers. Then, we obtain the final answer by taking the asymptotic limit of the probability in discrete case.

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