EXPLICIT SPECIAL COVERS OF ALTERNATING LINKS

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Abstract. Given a prime, alternating link diagram, we build a special cover of the link complement whose degree is bounded by a factorial function of the crossing number. It follows that a subgroup of the link group of that index embeds into right-angled Artin and Coxeter groups. Corollaries of this result include a quantification of residual finiteness, control of the growth of Betti numbers in covers, and an explicit bound on the rank of a \( \mathbb{Z} \)-module on which the link group acts faithfully.

1. Introduction

In an influential 1982 survey article, W. Thurston \[38\] posed 24 open questions that guided the field of low-dimensional topology for the subsequent three decades. Four of these problems had to do with the virtual properties of 3–manifolds. Here, a manifold \( M \) is said to virtually satisfy a property \( P \) if some finite cover of \( M \) satisfies \( P \). Similarly, a group \( \Gamma \) is said to virtually satisfy \( P \) if a finite-index subgroup of \( \Gamma \) has \( P \). Thurston’s virtual questions were:

1. Is every aspherical 3–manifold virtually Haken, meaning, does it virtually contain an incompressible surface?
2. Does every aspherical 3–manifold \( M \) virtually have positive first Betti number? (A later strengthening asks: does \( M \) virtually have arbitrarily large Betti number?)
3. Does every hyperbolic 3–manifold (more generally, every non-positively curved 3–manifold) virtually fiber over \( S^1 \)?
4. Is every hyperbolic 3–manifold group \( \pi_1(M) \) subgroup-separable? As a special case, does an immersed, \( \pi_1 \)–injective surface in \( M \) lift to an embedding in a finite cover?

All four questions were answered positively in 2012 by Agol \[2\] and Wise \[45\], with contributions from many other mathematicians. The main breakthrough was to show that every non-positively curved 3–manifold \( M \) is virtually special, meaning that the fundamental group of \( M \) virtually embeds into a right-angled Artin group (RAAG). In turn, the subgroups of RAAGs are sufficiently well understood to permit positive answers to the above questions. Additional corollaries of virtual specialness are that hyperbolic 3–manifold groups are large (virtually surject \( F_2 \)) and linear over \( \mathbb{Z} \).

The spectacular resolution of Thurston’s virtual questions prompted mathematicians to ask for quantitative statements: what is the minimum degree of a cover with one of these desired properties? For instance, see Agol \[1, Question 11.4\].

The main result of this paper is a quantitative virtual specialness theorem for alternating link complements. An alternating link diagram is one whose crossings alternate between “under” and “over” as one traverses any component of the link. See Section 3 for a precise definition, and Figure 2 for an example.

**Theorem 1.1.** Let \( K \subset S^3 \) be a link admitting a prime, alternating diagram with \( c = c(K) \) crossings. Then \( \pi_1(S^3 \setminus K) \) has a subgroup of index at most \( 2(\lfloor \frac{c}{2} + 1 \rfloor)!^2 \) that embeds into both a right-angled Artin group and a right-angled Coxeter group.
The bound $2(\left\lceil \frac{c}{2} + 1 \right\rceil!)^2$ on the index of the special subgroup can itself be bounded by the simpler expression $12(c - 1)!$. In fact, $2(\left\lceil \frac{c}{2} + 1 \right\rceil!)^2 \leq c!$ for all $c \geq 7$. See Theorem 8.10 and Remark 8.11 for more detail.

The proof of Theorem 1.1 works with a non-positively curve square complex, called then Dehn complex $\mathcal{D}_n$, that carries the fundamental group of $S^3 \setminus K$. We will explicitly construct a finite-sheeted covering space $\tilde{X} \to \mathcal{D}_n$ whose hyperplanes are free of the pathologies studied by Haglund and Wise [21] in their work on special cube complexes. See Section 2 for a review of square complexes and specialness, and Section 3 for a review of Dehn complexes. In particular, the covering space $\tilde{X}$ that we build is both A–special and C–special according to Definition 2.10, which means that $\pi_1(\tilde{X})$ embeds into both a a right-angled Artin group and a right-angled Coxeter group (RACG).

Combining Theorem 1.1 and some ideas from its proof with standard facts from the literature produces quantitative versions of several consequences of specialness. For instance, since every RACG is linear over $\mathbb{Z}$, we obtain

**Corollary 1.2.** Let $K \subset S^3$ be a link admitting a prime, alternating diagram with $c = c(K)$ crossings. Then $\pi_1(S^3 \setminus K)$ embeds in $SL(m, \mathbb{Z})$, where $m \leq 288((c - 1)!)^2$.

We also obtain a quantitative version of largeness and the growth of Betti numbers in covers, answering the stronger form of Thurston’s question [2] for prime, alternating links:

**Corollary 1.3.** Let $K \subset S^3$ be a link admitting a prime, alternating diagram with $c = c(K) \geq 3$ crossings. Then $S^3 \setminus K$ has a cover $M$ of degree at most $12(c - 1)!$, such that $\pi_1(M) \to F_4$. Consequently, for every $n \geq 1$, there is a cover $M_n \to S^3 \setminus K$ of degree at most $12n(c - 1)!$, such that $b_1(M_n) \geq 3n + 1$.

The largeness of knot and link groups was first shown by Cooper, Long, and Reid [13, Theorem 1.3]. To make their argument effective, it would suffice to have a quantitative version of residual finiteness. Although our proof of Corollary 1.3 takes a different route, we do provide the first quantitative residual finiteness statement for prime alternating links.

**Corollary 1.4.** Let $K \subset S^3$ be a link admitting a prime, alternating diagram $\Pi(K)$ with $c = c(K)$ crossings. Let $\sigma \subset S^3 \setminus K$ be a closed curve that intersects the checkerboard surfaces of $\Pi(K)$ a total of $n$ times. Then there is a cover $M \to S^3 \setminus K$ of degree at most $(n + 1) \cdot 12(c - 1)!$ such that $\sigma$ does not lift to $M$.

Corollary 1.4 is proved by combining Theorem 1.1 with a result of Bou-Rabee, Hagen, and Patel [7, Theorem 1.1], who quantified residual finiteness in RAAGs. In subsequent work [20], Hagen and Patel also quantified subgroup separability in RAAGs. Thus we expect that Theorem 1.1 can be combined with their results to give a quantitative answer to Thurston’s question [4]. Since this application is not immediate, we leave it for future work.

1.1. *Prior work.* The study of finite covers of knot complements, and the associated study of homomorphisms from knot groups to finite groups, dates back to the work of Fox in the 1950s [16, Section 10]. Many of these investigations were directed at the conjecture that knot groups are residually finite [32, Chapter VI]. Hempel [22] proved this conjecture in the 1980s, using Thurston’s geometrization theorem for Haken manifolds. One corollary of Hempel’s theorem is that the unknot is uniquely characterized by the property that all finite quotients of $\pi_1(S^3 \setminus K)$ are cyclic. Since an epimorphism from $\pi_1(S^3 \setminus K)$ to a non-abelian finite group serves as a certificate that $K$ is non-trivial, the size of a non-abelian quotient is relevant to the complexity of unknot recognition.

For alternating knots (the main focus of this paper), Fox showed that $\pi_1(S^3 \setminus K)$ surjects a non-abelian dihedral group of order at most $2^{c(K)}$ [16, Proposition 14.10]. This provides a certificate whose size is linear in $c(K)$. Assuming the generalized Riemann hypothesis, Kuperberg showed that for every non-trivial knot $K$, some non-abelian quotient of $S^3 \setminus K$ is bounded in size by a function of the form $\exp(\text{poly}(c(K)))$, implying that unknot recognition is in the complexity class co-NP [23]. See also Broaddus [9] and Morris [30] for other constructions of non-abelian covers.
Scott’s work on surface groups in the 1970s \cite{34} led him to ask whether all 3–manifold groups (including all knot groups) are subgroup separable. That is: is every finitely generated subgroup the intersection of finite-index subgroups? This stronger form of Thurston’s question \cite{4} turns out to be false in full generality. Among the negative answers to Scott’s question, Niblo and Wise \cite{31} gave examples of non-prime, alternating knots and links whose complements are not subgroup-separable. This means these link complements cannot be virtually special.

In the early 2000s, Wise investigated subgroup separability in prime, hyperbolic, alternating link complements \cite{43}. For several prime alternating links (including the figure–8 knot, the Whitehead link, and the Borromean rings), he showed that the Dehn complex $\mathcal{D}_\Pi$ has a finite cover that embeds in a product of graphs. This implies that the Dehn complex $\mathcal{D}_\Pi$ is virtually special and its geometrically finite subgroups are separable.

Haglund and Wise conjectured that the Dehn complex of every prime alternating link is virtually special \cite{21} Remark 5.10. While this conjecture was settled by Wise’s general theorem \cite{45} Theorem 17.14, our Theorem \ref{thm:explicit-special} gives a direct and quantitative proof.

Agol, Long, and Reid generalized Scott’s work on surfaces to arithmetic hyperbolic 3–manifolds \cite{3}, showing that the fundamental group of every non-compact arithmetic 3–manifold virtually embeds into one particular RACG that acts by reflections on $\mathbb{H}^3$. This implies virtual specialness and the separability of geometrically finite subgroups. Much more recently, Chu made their methods effective \cite{11,12}, bounding the degree of a C–special cover for many arithmetic manifolds. In particular, she showed that the figure–8 knot complement has a C–special cover of degree 10.

DeBlois, Miller, and Patel extended Chu’s work to cover full commensurability classes of arithmetic hyperbolic 3–manifolds \cite{14}. Here, manifolds $M$ and $N$ are called commensurable if they are virtually isometric. The two papers \cite{11,14} combine to give the following pleasant confluence \cite{14} Corollary 1.2. Given any $\epsilon > 0$, there is a constant $C(\epsilon, [M])$ depending on $\epsilon$ and an arithmetic commensurability class, such that every non-compact arithmetic 3–manifold $M$ has a C–special cover of degree at most $120C(\epsilon, [M]) \Vol(M)^{\epsilon}$. In particular, the degree of a special cover depends mildly on the volume of $M$ but may depend sharply on the commensurability class $[M]$.

### 1.2. Lower bounds.

The following theorem provides a contrast to the arithmetic results \cite{11,14}. It shows that the degree of a special cover of the Dehn complex can be forced to go to infinity while the volume stays bounded. In particular, the degree of a special cover must depend drastically on the commensurability class of $S^3 \setminus K_n$.

**Theorem 1.5.** For every $n \geq 3$, there is a prime, alternating diagram $\Pi_n$ of a link $K_n$ with $2n + 2$ crossings, such that the following hold:

- The complement $S^3 \setminus K_n$ is hyperbolic and non-arithmetic, with $\Vol(S^3 \setminus K_n) < 6$ for all $n$.
- If $m \neq n$, the complements $S^3 \setminus K_m$ and $S^3 \setminus K_n$ are not commensurable.
- Any $A$–special or $C$–special cover of the Dehn complex of $\Pi_n$ has degree at least $n$.

One consequence of Theorem \ref{thm:explicit-special} is that the degree of a special cover of the Dehn complex must grow at least linearly in $c(K)$. This prompts the question of how fast such growth must be, or equivalently, to what extent the upper bound of Theorem \ref{thm:explicit-special} is sharp. While we do not know a complete answer to this question, we have constructed examples where special covers of the Dehn complex must have degree growing roughly factorially in the square root of the crossing number.

**Theorem 1.6.** For every sufficiently large $k$, there is a prime alternating diagram $\Pi = \Pi_k$ with $c_k \geq k^2 \log k$ crossings, such that any $A$–special or $C$–special cover of the Dehn complex $\mathcal{D}_\Pi$ has degree greater than $\exp(\sqrt{c_k \log(c_k)}/2)$.

The idea of Theorem \ref{thm:explicit-special} is to encode the first $k$ prime numbers, denoted $p_1, \ldots, p_k$, into a link diagram $\Pi_k$. See Figure 4. The crossing number of $\Pi_k$ is $c(\Pi_k) = 2 \sum p_i \geq k^2 \log k$. We will show that the degree of any special cover of the Dehn complex must be divisible by every $p_i$, hence by $\prod p_i$. The lower bound then follows from a quantitative strengthening of the prime number theorem \cite{26}.
We emphasize that the lower bounds of Theorems 1.5 and 1.6 apply to the Dehn complex \( D_\Pi \) only. It is possible that the link complement \( S^3 \setminus K \) might be homotopy equivalent to some other cube complex that has a special cover of lower degree.

1.3. Ideas in the proof of Theorem 1.1. The Dehn complex \( D_\Pi \) associated to an alternating link diagram \( \Pi(K) \) comes with a great deal of structure. To start, \( D_\Pi \) is a \( \mathcal{VH} \)-complex, meaning that its edges can be partitioned into vertical (red) and horizontal (blue) subsets. See Figure 2 for an example. After taking a certain canonical double cover (Lemma 8.4), we obtain a \( \mathcal{VH} \)-complex \( X_\Pi \) that decomposes as a graph of spaces with a single edge. This means \( X_\Pi \) can be constructed by taking the product \( \Sigma \times I \) (where the edge space \( \Sigma \) is a graph) and attaching it along \( \Sigma \times \partial I \) to two vertex spaces, each of which is a component of the vertical 1–skeleton of \( X_\Pi \). In our setting, the two vertex spaces are isomorphic bipartite graphs with two vertices, and are interchanged by a deck transformation of the cover \( X_\Pi \to D_\Pi \). See Definition 2.6 for a review of graphs of spaces.

The primary obstruction to \( X_\Pi \) being special is that the attaching maps of this graph of spaces are not embeddings. We surmount this obstruction by embedding \( X_\Pi \) in a larger graph of spaces \( Y_\Pi \) and constructing a cover of \( Y_\Pi \) where the attaching maps are embeddings.

This basic strategy tracks the approach in Wise’s work on covers of \( \mathcal{VH} \)-complexes [42,43]. To lift immersions of graphs to embeddings in finite covers, he developed the canonical completion of a graph map [42 Section 5]. This object is simplest when the target graph has a single vertex, but becomes more unwieldy in the general setting. Since our goal is to explicitly construct covers while keeping the degree as small as possible, we have found it advantageous to design a new completing object adapted to our setting. This object, called the involutive completion, takes full advantage of the situation where all the relevant graphs are bipartite and admit a partition-exchanging involution.

The graph of spaces \( Y_\Pi \) uses the involutive completion as its edge space, with the result that the attaching maps of \( Y_\Pi \) are irregular covers of the vertex spaces. The attaching maps are related by the involution, and the relationship facilitates the construction of a cover \( \tilde{Y} \to Y_\Pi \) where the attaching maps are graph isomorphisms. While \( \tilde{Y} \) was designed only with the goal of finding a graph of spaces where the attaching maps are embeddings, the symmetry provided by the involution allows us to prove that in fact \( \tilde{Y} \) is isomorphic to a product of bipartite graphs, which is special. Since \( X_\Pi \subset Y_\Pi \), its preimage \( \tilde{X} \subset \tilde{Y} \) is a special cover of \( X_\Pi \), which is also a special cover of \( D_\Pi \). The involutive completion, which plays a starring role in this construction, can be considered the main technical innovation of this paper.

1.4. Organization. Section 2 reviews standard definitions and results about \( \mathcal{VH} \)-complexes, graphs of spaces, and specialness. Section 3 reviews the construction of the Dehn complex associated to a link diagram, and some of its salient properties.

With these standard facts in hand, we prove Theorems 1.5 and 1.6 in Section 4. These proofs are self-contained apart from appeals to certain black-box facts from the literature that will not be used elsewhere in the paper. These proofs also illustrate the main difficulty that must be surmounted in Theorem 1.1 elevating the attaching map of a graph of spaces to an embedding in a finite cover.

Section 5 surveys some results about covering spaces of graphs from the point of view of permutation labelings on edges. Section 6 uses these results to construct the involutive completion, derive a number of its useful properties, and construct the space \( \tilde{Y} \).

Section 7 illustrates the construction of \( \tilde{Y} \) in the case of the \( 5_2 \) knot. The many color figures of that section give a visual preview of the proof of Theorem 1.1 in the general case.

Section 8 incorporates all of the above ideas, particularly graphs of spaces and the involutive completion, to construct a special cover of the Dehn complex \( D_\Pi \). At the end of the section, we also prove the corollaries of Theorem 1.1.

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2. $\mathcal{VH}$–complexes and specialness

All of the cube complexes that we study in this paper are 2–dimensional, and fall into the family of $\mathcal{VH}$–complexes. This restricted class of square complexes was introduced by Wise [41] to study residual finiteness of certain group amalgams. In this section, we recall some standard definitions and facts regarding $\mathcal{VH}$–complexes, as well as some background on specialness in the restricted setting of $\mathcal{VH}$–complexes.

We begin by introducing some terminology regarding graphs. A graph $G$ is a 1–complex with vertex set $V(G) = G^{(0)}$ and edge set $E(G)$ consisting of the 1–cells. When necessary, we will fix orientations on edges of a graph. We use the notation $\vec{e}$ for an oriented edge, $\overrightarrow{e}$ for the same edge with the reverse orientation, and an undecorated symbol $e$ for the underlying edge without a specified orientation. The terminus (forward endpoint) of an oriented edge $\vec{e}$ is denoted $t(\vec{e})$ and the origin (initial point) of $\vec{e}$ is denoted $o(\vec{e}) = t(\overrightarrow{e})$. For a vertex $v \in V(G)$, the incident edges are denoted $\Delta(v) = \{ e \in E(G) \mid v \in e \}.$

**Definition 2.1.** Let $G$ and $H$ be graphs. A continuous map $f : G \to H$ is combinatorial if it sends vertices to vertices and edges to edges. A combinatorial graph map $f$ is an immersion if it is locally injective on $\Delta(v)$: that is, for each $v \in V(G)$ the induced map $f_\Delta : \Delta(v) \to \Delta(f(v))$ is injective. An immersion $f$ is a covering map if and only if it is a local isomorphism, meaning that $f_\Delta$ is a bijection.

A square is the space $I \times I$ where $I = [0, 1]$, endowed with the Euclidean path-metric. A square complex $X$ is a 2–dimensional CW complex admitting a metric where (the domain of) every 1–cell is isometric to $I$ and (the domain of) every 2–cell is isometric to $I \times I$, such that the attaching maps are local isometries. The induced path-metric on $X$ with this property is called the $t^2$ metric.

A square complex is called non-positively curved (NPC) if the link of each vertex contains no loops of length less than 4. For example, the boundary of $I^2$ fails to be NPC. By a theorem of Gromov [8, Theorem II.5.20], the $t^2$ metric on a NPC complex $X$ is locally $\text{CAT}(0)$, and the lift of this metric to the universal cover of $X$ is globally $\text{CAT}(0)$.

**Definition 2.2.** A $\mathcal{VH}$–complex is a square complex $X$ with a partition of the 1–cells of $X$ into two sets, called the vertical and horizontal edges respectively, so that the attaching map of each square alternates between vertical edges and horizontal edges as the boundary is traversed cyclically.

Note that the link of any vertex in a $\mathcal{VH}$–complex is bipartite. Thus a $\mathcal{VH}$–complex is NPC if and only if the vertex links contain no two-cycles.

**Definition 2.3.** The partition of edges in a $\mathcal{VH}$–complex $X$ defines two subcomplexes of the 1–skeleton: the vertical 1–skeleton $V^{(1)}_X$ containing the vertical edges and the horizontal 1–skeleton $H^{(1)}_X$ containing the horizontal edges.

A $\mathcal{VH}$–complex $X$ has a vertical projection $\pi_V : X \to G^V_X$, where $G^V_X$ is the set of equivalence classes generated by declaring all points in a square with the same vertical coordinate to be equivalent. The horizontal projection $\pi_H : X \to G^H_X$ is defined analogously. The quotient spaces $G^V_X$ and $G^H_X$ are graphs, although the maps $\pi_V$ and $\pi_H$ may fail to be combinatorial on $X^{(1)}$. In particular, they may send midpoints of edges in $X^{(1)}$ to leaf vertices of $G^V_X$ or $G^H_X$; compare Definition 2.5.

**Definition 2.4.** A midcube of a square $[0, 1]^2$ is a segment defined by setting one coordinate to be $\frac{1}{2}$. Every midcube is either vertical or horizontal.

Given a $\mathcal{VH}$–complex $X$, we form a graph $\Sigma_V$ whose vertices are the midpoints of horizontal edges and whose edges are the vertical midcubes. Exchanging the vertical/horizontal directions produces a graph $\Sigma_H$. A vertical (resp. horizontal) hyperplane of $X$ is a connected component of $\Sigma_V$ (resp. $\Sigma_H$). We emphasize that hyperplanes are connected by definition. Observe that a vertical hyperplane is the preimage of a point under the horizontal projection $\pi_H$, and vice versa.

An edge $e$ of $X$ is dual to a hyperplane $h$ if the midpoint of $e$ is a vertex of $h$. 
Then the projection map \( \pi \) is one-to-one on the interiors of edges dual to \( h \). Equivalently, \( h \) is two-sided if and only if the edges dual to \( h \) admit a global orientation that is consistent in each square.

**Definition 2.5.** A \( VH \)-complex \( X \) is vertically (resp. horizontally) non-singular if all vertical (resp. horizontal) hyperplanes are two-sided. Observe that \( X \) is vertically non-singular precisely when the horizontal projection \( \pi_H : X \to G^H_X \) sends each vertical hyperplane to the midpoint of an edge of \( G^H_X \), which implies that \( \pi_H \) is combinatorial on \( X^{(1)} \).

**Definition 2.6.** A graph of spaces is a graph \( G \) with a vertex space \( V \) for each \( v \in V(G) \) and an edge space \( X_e \) for each \( e \in E(G) \). For each orientation \( \vec{e} \) of an edge \( e \in E(G) \), there is an attaching map \( f_{\vec{e}} : X_e \to X_{t(\vec{e})} \), where \( t(\vec{e}) \in V(G) \) is the terminal vertex of \( \vec{e} \). The total space \( X_G \) of a graph of spaces is formed by attaching \( X_e \times [0, 1] \) to \( X_{t(\vec{e})} \) and \( X_{t(\vec{e})} \) via the attaching maps for the two orientations of \( e \). The total space \( X_G \) comes with a natural quotient map \( \pi : X_G \to G \), sending \( X_e \) to \( v \) and \( X_e \times [0, 1] \) to \( e \cong [0, 1] \). See Scott and Wall [35, Section 4] for more details.

Graphs of spaces naturally arise in the context of \( VH \)-complexes.

**Proposition 2.7** (Wise [42, Proposition 4.2]). Let \( X \) be a NPC, vertically non-singular \( VH \)-complex. Then the projection map \( \pi_H : X \to G^H_X \) is the quotient map of a graph-of-spaces decomposition of \( X \), with graph \( G^H_X \), edge spaces isomorphic to vertical hyperplanes, and vertex spaces the connected components of \( V^{(1)}_X \). In this graph of spaces, all attaching maps are combinatorial immersions.

Haglund and Wise [21] defined the notion of a special cube complex in terms of the absence of certain pathologies in the hyperplanes. We recall their definition in the particular case of \( VH \)-complexes.

**Definition 2.8** ([21, Definition 3.1]). Suppose \( X \) is a \( VH \)-complex. Let \( v \in X^{(0)} \) be a vertex and \( \vec{a}, \vec{b} \) be oriented edges such that \( t(\vec{a}) = t(\vec{b}) = v \) and \( a, b \) are not consecutive in any square. Hyperplanes \( h_a, h_b \) osculate at \( (v; \vec{a}, \vec{b}) \) if \( h_a \) is dual to \( a \) and \( h_b \) is dual to \( b \). A hyperplane \( h \) self-osculates at \( (v; \vec{a}, \vec{b}) \) if \( h \) is dual to both \( a \) and \( b \). If a two-sided hyperplane \( h \) self-osculates at \( (v; \vec{a}, \vec{b}) \), we say that \( h \) directly self-osculates if some choice of global orientation on the dual edges of \( h \) induces \( \vec{a}, \vec{b} \), and indirectly self-osculates if some choice of global orientation induces \( \vec{a}, \vec{b} \). Two hyperplanes \( h, h' \) inter-osculate if they both intersect (cross transversely in some square) and osculate. See Haglund and Wise for an illustration [21, Figure 1].

**Example 2.9.** Consider the horizontal projection \( \pi_H : X \to G^H_X \) of a \( VH \)-complex \( X \) whose vertical hyperplanes are two-sided. Then a vertical hyperplane \( h \) projects to the midpoint of an edge \( e \subset G^H_X \). Furthermore, \( \pi_H^{-1}(e) \cong h \times [0, 1] \) if and only if \( h \) does not self-osculate.

There are two variants of specialness, corresponding to right-angled Artin and Coxeter groups.

**Definition 2.10** ([21, Definition 3.2]). A \( VH \)-complex \( X \) is called \( A \)-special if

- (1) every hyperplane is two-sided,
- (2) no hyperplane directly self-osculates,
- (3) no pair of hyperplanes inter-osculates.

A \( VH \)-complex \( X \) is called \( C \)-special if

- (1) \( X^{(1)} \) is bipartite,
- (2) no hyperplane self-osculates,
- (3) no pair of hyperplanes inter-osculates.

In most of the recent literature, including Wise’s survey book [44, Chapter 4] and the introduction of this paper, the term special means “\( A \)-special.” This is virtually equivalent but not equivalent to the meaning of “special” in Haglund and Wise [21].
**Definition 2.11.** Let \( \Gamma \) be a finite graph. Then we define the right-angled Artin group (RAAG)
\[
A(\Gamma) = \langle v \in V(\Gamma) : [o(\vec{e}), t(\vec{e})] = 1 \text{ for } e \in E(\Gamma) \rangle
\]
and the right-angled Coxeter group (RACG)
\[
C(\Gamma) = \langle v \in V(\Gamma) : v^2 = 1 \text{ for } v \in V(\Gamma), [o(\vec{e}), t(\vec{e})] = 1 \text{ for } e \in E(\Gamma) \rangle
\]
\[\text{Theorem 2.12 (Haglund–Wise [21, Theorem 4.2]).} \]
Let \( X \) be a VH–complex. Let \( \Gamma \) be the graph whose vertices are hyperplanes of \( X \) and whose edges correspond to intersecting hyperplanes.
If \( X \) is A–special, then \( \pi_1(X) \) embeds into the right-angled Artin group \( A(\Gamma) \). If \( X \) is C–special, then \( \pi_1(X) \) embeds into the right-angled Coxeter group \( C(\Gamma) \).

We observe that every property in Definition 2.10 passes to covers.

**Lemma 2.13 (Haglund–Wise [21, Lemma 3.7]).** Let \( \hat{X} \to X \) be a covering map of VH–complexes. If one of the following properties holds for \( X \), then the corresponding property also holds for \( \hat{X} \):
1. \( X^{(1)} \) is bipartite,
2. The hyperplanes of \( X \) are two-sided.
3. The hyperplanes of \( X \) do not self-osculate.
4. The hyperplanes of \( X \) do not inter-osculate.

The following criterion for specialness appears in Haglund and Wise [21], without being explicitly stated.

**Lemma 2.14.** Let \( A \) and \( B \) be bipartite graphs. Then any subcomplex \( X \subset A \times B \) is both A–special and C–special.

**Proof.** By [21, Example 3.3], the bipartite graphs \( A, B \) are A–special and C–special. By [21, Corollary 3.6], the product \( A \times B \) is also A–special and C–special.

By [21, Corollary 3.9], any subcomplex \( X \subset A \times B \) has the following properties: \( X^{(1)} \) is bipartite, every hyperplane of \( X \) is two-sided, and no hyperplane self-osculates. Finally, inter-osculation can be ruled out using the following argument (compare [21, Theorem 5.7]).

Suppose that a horizontal hyperplane \( \mathfrak{h} \) and a vertical hyperplane \( \mathfrak{h}' \) intersect in a square \( a \times b \subset A \times B \). Then every dual edge to \( \mathfrak{h}' \) projects to \( a \) under \( \pi_H \), and every dual edge to \( \mathfrak{h} \) projects to \( b \) under \( \pi_V \). Thus, if \( \mathfrak{h}, \mathfrak{h}' \) osculate at some vertex \( v \times w \), then \( \pi_H(v \times w) = v = t(\vec{a}) \) and \( \pi_V(v \times w) = w = t(\vec{b}) \), for appropriate orientations on \( a, b \). Hence the dual edges involved in osculation must be of the form \( v \times \vec{b} \) and \( \vec{a} \times w \). But edges of this form are consecutive in the square \( a \times b \), contradicting the definition of osculation. Thus \( \mathfrak{h} \) and \( \mathfrak{h}' \) cannot osculate, completing the proof.

### 3. The Dehn complex of a link diagram

The primary objects in this article are particular VH–complexes arising from alternating link projections. In this section, we survey some background material about link diagrams and their Dehn complexes.

**Definition 3.1.** A link \( K \subset S^3 \) is a smooth 1–manifold embedded in \( S^3 = \mathbb{R}^3 \cup \{ \infty \} \), where \( K \) is assumed by convention to be disjoint from \( \{ \infty \} \). A connected link is called a knot. Links are considered up to the equivalence relation of smooth isotopy in \( S^3 \). A regular neighborhood of \( K \) is denoted \( N(K) \).

The projection plane \( P \cong S^2 \) is the closure in \( S^3 \) of the \((x_1, x_2)\) plane of \( \mathbb{R}^3 \). Consider \( p: \mathbb{R}^3 \to \mathbb{R}^2 \subset P \), the vertical projection that forgets the third coordinate. The projection \( p \) is called regular for \( K \) if the restriction \( p|_K \) is an immersion, which deviates from being an embedding only in finitely many transverse double points. A link diagram \( \Pi = \Pi(K) \) is the image \( p(K) \) of a regular projection, where the double points (called crossings) are decorated with information indicating which preimage of the crossing has higher \( x_3 \)–coordinate. The number of crossings of a diagram is denoted \( c(\Pi) \). See Figure 2 (left) for an example.
Let $N(s) \subset P$ be a small regular neighborhood of a crossing $s \in p(K)$. The preimage $p^{-1}N(s) \cap K$ is a disjoint union of two arcs (see Figure 1, left). One of these is called the overstrand at $s$ and the other is called the understrand at $s$, depending on which has the higher $x_3$-coordinate at $p^{-1}(s)$. After a vertical isotopy (changing only the $x_3$-coordinate), we may always assume that $K$ lies in the projection plane $P$, except for a neighborhood of each crossing, where the overstrand is slightly above $P$ and the understrand is slightly below $P$.

**Definition 3.2.** A link diagram $\Pi(K)$ is called alternating if the following holds: for each component $C \subset K$, a parametrization of $C$ passes through an overstrand, then an understrand, and so on in an alternating fashion. A link $K$ is alternating if some representative of the isotopy class has an alternating diagram.

A diagram $\Pi(K)$ is called prime if, for every simple closed curve $\gamma \subset P$ intersecting $p(K)$ transversely in two non-crossing points, some component of $P \setminus \gamma$ intersects $p(K)$ in a single crossing-free arc.

**Definition 3.3.** A region of a diagram $\Pi = \Pi(K)$ is a component of $P \setminus p(K)$. It is well-known that the regions of every link diagram $\Pi$ can be two-colored, so that regions that share an arc of $p(K)$ have different colors. We will refer to the two colors as red and blue. See Figure 2 (left).

A checkerboard surface of $\Pi$ is constructed as follows. Away from a neighborhood of the crossings, the surface is the union of the red regions. In the neighborhood of a crossing $s$, the surface joins diagonally opposed red regions by a half-twisted band. The resulting red surface has boundary isotopic to $K$, hence can be taken to lie in $S^3 \setminus N(K)$. The blue surface is constructed in an analogous way, using blue regions instead of red regions.

**Remark 3.4.** To avoid some degenerate situations, we will implicitly assume that $\Pi(K)$ contains at least 2 crossings. (If $\Pi(K)$ fails this assumption, then $\pi_1(S^3 \setminus K)$ is a free group, making Theorem 1.1 and its corollaries trivial.) With this assumption, every prime diagram is connected, meaning $p(K)$ is a connected graph. In addition, every region is contractible, and the four regions adjacent to a crossing are distinct.

In the context of studying knot groups, Dehn [15, Page 157] used projection diagrams to construct a 2-dimensional presentation complex for $\pi_1(S^3 \setminus K)$. This square complex is now known as the Dehn complex.

**Definition 3.5.** Let $\Pi(K)$ be a link diagram. The Dehn complex $\mathcal{D}_\Pi$ of $\Pi$ is a square complex embedded in $S^3 \setminus K$, with the following cell structure:
• $\mathcal{D}_\Pi$ has two 0–cells, denoted $v_+$ and $v_-$, with $v_+$ above the projection plane $P$ and $v_-$ below $P$. In all our figures, $v_+$ is shown as a solid disk, while $v_-$ is shown as an open circle.

• $\mathcal{D}_\Pi$ has a 1–cell with endpoints at $v_+$ and $v_-$ corresponding to each region of $\Pi$. Edges dual to red regions are oriented from $v_-$ to $v_+$, while edges dual to blue regions are oriented from $v_+$ to $v_-$. 

• $\mathcal{D}_\Pi$ has a square 2–cell for each crossing $s$ of $\Pi$. If the four regions adjacent to $s$ are labeled $a,b,c,d$ in cyclic order, where $a$ is red and $a,b$ are separated by an under-crossing at $s$, then the attaching map of the square is $\vec{a} \cdot \vec{b} \cdot \vec{c} \cdot \vec{d}$. See Figure 1 for a visualization of how the square is positioned near $s$.

The Dehn complex $\mathcal{D}_\Pi$ is endowed with a $\mathcal{VH}$–structure by regarding the edges dual to red regions as vertical and the edges dual to blue regions as horizontal. We will use this color convention throughout the paper. See Figure 2 for an example.

**Figure 2.** Left: a diagram $\Pi$ of the 5–crossing knot $5_2$. The red (resp. blue) surface projects to the union of the red (resp. blue) regions of $S^2$. Right: a fundamental domain for the Dehn complex $\mathcal{D}_\Pi$ constructed from $\Pi$. Edges with the same label are identified.

**Proposition 3.6.** For every link diagram $\Pi(K)$, the following holds.

1. We have $\pi_1(\mathcal{D}_\Pi) \cong \pi_1(S^3 \setminus K)$.

2. $\Pi$ is prime and alternating if and only if $\mathcal{D}_\Pi$ is non-positively curved.

**Proof.** Conclusion (1) is due to Dehn [15, Page 157], and can be checked by a straightforward exercise using van Kampen’s theorem [8, Exercise II.5.42].

Conclusion (2) is due to Weinbaum [40, Theorem B], who expressed the result in terms of small cancellation theory. See Bridson and Haefliger [8, Proposition II.5.43] or Wise [43, Theorem 13.3] for direct proofs. □

**Remark 3.7.** When $\Pi$ is a prime, alternating diagram, the isomorphism of fundamental groups expressed in (1) can be promoted to a homotopy equivalence of spaces. Aitchison, Lumsden, and Rubinstein [4] showed that the link exterior $S^3 \setminus N(K)$ is homeomorphic to a 3–dimensional NPC cube complex $C$, where every 3–cube has exactly one free face. The free faces tile the tori comprising $\partial N(K)$. Retracting every 3–cube of $C$ onto the face opposite the free face induces a deformation retraction of $S^3 \setminus N(K)$ to a 2–dimensional spine, which is exactly $\mathcal{D}_\Pi$ with the embedding described in Figure 1. See Sakuma and Yokota [33, Section 2].

Our main goal is to construct a cover of $\mathcal{D}_\Pi$ that satisfies both flavors of specialness from Definition 2.10. We start with $\mathcal{D}_\Pi$ itself.
Explicit Special Covers of Alternating Links

Proposition 3.8. Let \( \Pi \) be a (connected) link diagram. Then

1. The 1-skeleton \( \mathcal{D}_\Pi^{(1)} \) is bipartite. In addition, the vertical and horizontal subgraphs of \( \mathcal{D}_\Pi^{(1)} \) are connected.
2. \( \mathcal{D}_\Pi \) has exactly two hyperplanes: one dual to the blue edges, and one dual to the red edges. Each hyperplane embeds as a spine of the corresponding checkerboard surface.
3. A hyperplane \( h \subset \mathcal{D}_\Pi \) is one-sided if and only if \( h \) contains loops of odd length, which occurs if and only if the corresponding checkerboard surface is non-orientable.
4. The hyperplanes of \( \mathcal{D}_\Pi \) do not inter-osculate.

One consequence of Proposition 3.8 is that the only hyperplane pathologies that occur in \( \mathcal{D}_\Pi \) are one-sidedness and self-osculation. As we will see (Lemma 3.9), one-sidedness can always be removed in a double cover of \( \mathcal{D}_\Pi \). Most of the effort in proving Theorem 1.1 is expended on removing self-osculation.

Proof of Proposition 3.8. Recall that \( \mathcal{D}_\Pi^{(0)} \) has exactly two vertices, \( v_+ \) and \( v_- \), and every edge joins \( v_+ \) to \( v_- \). Thus \( \mathcal{D}_\Pi^{(1)} \) is connected and bipartite. Similarly, every edge of \( V_{\Pi}^{(1)} \) and \( H_{\Pi}^{(1)} \) joins \( v_+ \) to \( v_- \), hence those subgraphs are connected and bipartite as well.

For the proofs of (2)-(3), we focus on the vertical (red) edges of \( \mathcal{D}_\Pi \), as the horizontal (blue) case is identical.

For (2), observe that the red checkerboard surface has a natural spine. This spine consists of one vertex in the interior of each red region of \( \mathcal{P} \setminus p(K) \) and one edge through each crossing, joining the two vertices of the red regions adjacent to that crossing. This spine is called the red checkerboard graph.

The red checkerboard graph embeds in a hyperplane of \( \mathcal{D}_\Pi \), as follows. Map the vertex in the center of a red region \( R \) to the midpoint of the \( \mathcal{D}_\Pi \)-edge that runs through region \( R \). Map the edge running through crossing \( s \) to the midcube connecting the two red edges of the square corresponding to \( s \). (See Figure 1.) Since the checkerboard surface is connected, this embedding crosses every red (vertical) edge of \( \mathcal{D}_\Pi \). Therefore the image of this embedding is the unique hyperplane dual to all of the vertical edges of \( \mathcal{D}_\Pi \), proving (2).

For (3), recall that we have oriented every red edge from \( v_- \) to \( v_+ \). With this convention, edges on opposite sides of a square have opposite orientations. To re-orient all red edges in a consistent direction across a hyperplane \( h \), we would have to flip every other edge crossed by \( h \), which is possible if and only if there are no loops of odd length. In a similar fashion, observe that the orientation-reversing loops in the red surface are precisely the loops that have odd length in the red checkerboard graph. (See Figure 2 where the loop \( x-y-z-x \) defines a Möbius band in the blue surface and a Möbius band in the central column of the Dehn complex \( \mathcal{D}_\Pi \).)

For (4), let \( h \) and \( h' \) be a pair of hyperplanes that intersect in \( \mathcal{D}_\Pi \). Then (up to relabeling), \( h \) is the red hyperplane and \( h' \) is the blue one. These two hyperplanes intersect in every square of \( \mathcal{D}_\Pi \), which means that they do not osculate (see Definition 3.5). Thus there is no inter-osculating in \( \mathcal{D}_\Pi \). \( \square \)

Lemma 3.9. Suppose \( \mathcal{D}_\Pi \) is the Dehn complex of an alternating link diagram \( \Pi \). Let \( \vec{a} \) be a red edge and \( \vec{b} \) a blue edge, oriented as in Definition 3.5. Then the corner of some square traverses the path \( \vec{a} \cdot \vec{b} \) at \( v_+ \), if and only if the corner of some other square traverses the path \( \vec{b} \cdot \vec{a} \) at \( v_- \).

Proof. Suppose there is a square in \( \mathcal{D}_\Pi \) whose attaching map contains the oriented path \( \vec{a} \cdot \vec{b} \) at vertex \( v_+ \). Then, according to Definition 3.5, this square corresponds to a crossing \( s \) such that the regions corresponding to \( a \) and \( \tilde{b} \) are adjacent along an arc \( \alpha \subset \Pi \) that meets \( s \) in an under-crossing. See Figure 3 top.

Let \( t \) be the crossing at the other end of \( \alpha \). Since \( \Pi \) is alternating, \( \alpha \) meets \( t \) in an over-crossing. Thus, by Definition 3.5, the attaching map for the square corresponding to \( t \) contains the oriented path \( \vec{b} \cdot \vec{a} \) at vertex \( v_- \). See Figure 3 bottom.
Figure 3. Adjacent crossings in an alternating diagram and the corresponding corners of squares in $D_\Pi$. 

The converse implication is identical, and involves following $\alpha$ from $t$ to $s$. □

Remark 3.10. The Dehn complex $D_\Pi$ associated to an alternating link diagram is the combinatorial dual of another well-studied object, namely the Menasco polyhedral decomposition [27]. This is a decomposition of $S^3 \setminus K$ into two ideal polyhedra, $P_+$ and $P_-$, with the property that there is one face for each region of $P \setminus p(K)$ and one edge at each crossing, connecting the overstrand to the understrand. Thus the polyhedra $P_+$ are dual to the vertices $v_+ \in D_\Pi$; the faces of $P_\pm$ are dual to the edges of $D_\Pi$; and the edges of $P_\pm$ are dual to the squares of $D_\Pi$. Lackenby [24, Section 5] provides a brief but very useful summary of the relationship between the polyhedral decomposition and its dual. See Lackenby’s Figure 18 in particular.

Viewed through this dual lens, Lemma 3.9 becomes the well-known statement that if two faces of $P_+$ are adjacent along an edge, then the matching faces of $P_-$ are also adjacent along an edge.

4. LOWER BOUND ON DEGREE OF A SPECIAL COVER

In this section, we prove Theorems 1.5 and 1.6, constructing examples of links whose Dehn complexes have no special covers of small degree. In both theorems, our examples come from the family of pretzel links with a particular form, shown in Figure 4. See Lickorish [25, Figure 1.7] for a general definition of pretzel link diagrams.

Lemma 4.1. Let $\ell_1, \ldots, \ell_k$ be a sequence of positive integers. Then the pretzel link diagram $\Pi = P(1, 2\ell_1-1, 1, 2\ell_2-1, \ldots, 1, 2\ell_k-1)$ shown in Figure 4 has the following properties:

1. The diagram $\Pi$ has exactly $2k$ cyclically ordered red regions, $R_1, \ldots, R_{2k}$.
2. Red region $R_{2i}$ meets one crossing to its left and $2\ell_i - 1$ crossings to its right.
3. The crossing number of $\Pi$ is $2 \sum_{i=1}^{k} \ell_i$.
4. Every hyperplane in the Dehn complex $D_\Pi$ is two-sided.

Proof. The pretzel link diagram $\Pi = P(1, 2\ell_1-1, \ldots, 1, 2\ell_k-1)$ is shown in Figure 4. This diagram has $2k$ tassels of crossings, arranged in cyclic order. Every red region borders two of these tassels (one to the left, one to the right). In particular, region $R_{2i}$ meets one crossing in the tassel to its left and $2\ell_i - 1$ crossings in the tassel to its right. Thus $R_{2i}$ meets $2\ell_i$ crossings altogether.

Observe that every crossing meets exactly one even-numbered red region $R_{2i}$. Thus the total number of crossings is $2 \sum \ell_i$. 

\( \ell_1 - 1 \quad 2 \ell_1 - 1 \quad \ldots \quad 2 \ell_k - 1 \)

Figure 4. The pretzel link diagram \( P(1, 2\ell_1 - 1, 1, 2\ell_2 - 1, \ldots, 1, 2\ell_k - 1) \). Each box represents a vertical tassel of crossings, as pictured to the right.

Finally, we check that the hyperplanes of \( D_\Pi \) are two-sided. By Proposition 3.8, a hyperplane \( \mathcal{H} \subset D_\Pi \) is one-sided if and only if \( \mathcal{H} \) contains a loop of odd length. Thus, to show \( \mathcal{H} \) is two-sided, it suffices to consider embedded loops. The red hyperplane \( \Sigma_H \) is the red checkerboard graph, and has two special vertices: \( U \) and \( L \) (corresponding to the upper and lower red region), with the property that \( \Sigma_V \setminus U \) and \( \Sigma_V \setminus L \) are both contractible. Thus any embedded loop in \( \Sigma_V \) must pass through both \( U \) and \( L \). But every path from \( U \) to \( L \) has odd length, hence every loop has even length. Thus \( \Sigma_V \) is two-sided. \( \square \)

**Proposition 4.2.** Let \( D_\Pi \) be the Dehn complex of the link diagram \( P(1, 2\ell_1 - 1, \ldots, 1, 2\ell_k - 1) \) of Figure 4. Let \( \xi: X \to D_\Pi \) be any A–special or C–special cover. Then the degree of \( \xi \) is divisible by \( \ell_i \) for every \( i \in \{1, \ldots, k\} \).

**Proof.** By Lemma 4.1, every hyperplane of \( D_\Pi \) is two-sided, hence Lemma 2.13 says that every hyperplane of \( X \) is also two-sided. In particular, if \( X \) is C–special, then \( X \) is also A–special (compare Definition 2.10). So it suffices to assume that \( X \) is an A–special cover.

Consider the horizontal projection \( \pi_H: D_\Pi \to G_H^{D_\Pi} \). Since \( D_\Pi \) has a single (two-sided) vertical hyperplane, the quotient \( G_H^{D_\Pi} \) is a single edge \( e \) with both endpoints at the same vertex. By Proposition 2.7, the projection \( \pi_H \) endows \( D_\Pi \) with the structure of a graph of spaces, with one vertex space (the red 1–skeleton of \( D_\Pi \)) and one edge space (the vertical, blue hyperplane \( \Sigma_V \)). By Definition 2.6, this means \( \Sigma_V \times e \) is attached to the vertical 1–skeleton in two ways, with attaching maps \( f_e \) and \( f_{\bar{e}} \) for the two orientations of \( e \).

Now, fix an integer \( i \in \{1, \ldots, k\} \). Let \( \gamma_i \) be the loop in the blue hyperplane \( \Sigma_V \) that starts at \( U \) and encircles red region \( R_{2i} \) going counterclockwise. Since \( \Sigma_V \) is two-sided, we choose an orientation on the blue edges as follows: orient the edge dual to \( U \) from \( v_+ \) to \( v_- \), and orient all other blue edges in a consistent transverse direction across \( \mathcal{H} \). This induces an orientation \( \hat{e} \) on the quotient edge \( e \) of the horizontal projection. We orient all red edges from \( v_- \) to \( v_+ \), with edge \( r_j \) corresponding to red region \( R_j \). With these orientations fixed, the attaching map \( f_e \) (corresponding to the terminal vertex \( t(\hat{e}) \)) sends \( \gamma_i \) to the following path of red edges:

\[ \delta_i = f_e(\gamma_i) = (\bar{r}_{2i} \cdot \bar{r}_{2i+1})^{\ell_i}, \]

meaning that the loop \( (\bar{r}_{2i} \cdot \bar{r}_{2i+1}) \) is repeated \( \ell_i \) times. See Figure 5.
Let \( n \) be the degree of the cover \( \xi: X \to D_H \). Then the vertical 1–skeleton \( V_X^{(1)} \) contains exactly \( n \) preimages of \( \vec{r}_{2i} \) and exactly \( n \) preimages of \( \vec{r}_{2i+1} \). These edges can be joined end-to-end to form some number of disjointly embedded 1–cycles in \( V_X^{(1)} \).

Let \( \vec{h} \subset X \) be a vertical hyperplane. By Proposition 2.7, there is a horizontal projection \( \pi_H: X \to G_X^H \) that induces a graph-of-spaces structure on \( X \). This projection maps \( \vec{h} \) to the midpoint of an edge \( \vec{e} \). Since \( \vec{h} \) does not directly self-osculate, each of the two attaching maps coming from the two orientations of \( \vec{e} \) is an embedding. Compare Example 2.9.

Now, consider a connected component \( \gamma_i \subset \xi^{-1}(\gamma_i) \). This is a loop in a hyperplane \( \vec{h} \subset \xi^{-1}(\Sigma_V) \), and has a length which is a multiple of \( 2\ell_i \). Let \( \vec{f} \) be the attaching map from \( \vec{h} \) to the vertex space \( t(\vec{e}) \), where \( \vec{e} \) has the orientation induced by \( \vec{e} \). By the previous paragraph, \( \vec{f} \) is an embedding, hence \( \vec{f}_i = \vec{f}(\gamma_i) \) is an embedded cycle that covers \( \delta_i \). Thus the number of preimages of \( \vec{r}_{2i} \) (resp. \( \vec{r}_{2i+1} \)) contained in \( \vec{f}_i \) is a multiple of \( \ell_i \). However, every preimage of \( \vec{r}_{2i} \) belongs to exactly one embedded cycle of \( \vec{r}_{2i} \) and \( \vec{r}_{2i+1} \) edges, and each embedded cycle is the image of exactly one connected component of \( \xi^{-1}(\gamma_i) \) under the attaching map. Therefore the number of preimages of \( \vec{r}_{2i} \), namely \( n \), is a multiple of \( \ell_i \).

The key issue that prevents a small-degree cover in Proposition 4.2 is that loops in the vertical hyperplane of \( D_H \) need to become embedded in the red (vertical) 1–skeleton of some cover of \( D_H \). Finding a cover with this property will be the primary task of Section 6 and Section 8.

For now, we proceed to the proof of Theorem 1.6. The proof uses Proposition 4.2 in combination with some facts from hyperbolic 3–manifold theory that are sourced from references. As hyperbolic geometry will not be used elsewhere in the paper, we have chosen to leave these facts as black boxes.

**Proof of Theorem 1.6** Fix an integer \( n \geq 2 \), and let \( \Pi_n = P(1, 1, 1, 2n-1) \) be a pretzel link diagram. (In Figure 4 substitute \( \ell_1 = 1 \) and \( \ell_2 = n \).) By Proposition 4.2, every A– or C–special cover of the Dehn complex of \( \Pi_n \) must have degree divisible by \( n \).

The link \( K_n \) depicted by this pretzel diagram is the two-bridge link corresponding to the continued fraction \( 1/(3 + 1/(2n-1)) \). In the terminology of Millichap and Worden [29], \( K_n \) corresponds to the word \( R^2 L^{2n-2} \). The geometry of these links and their complements is well-understood. By Menasco’s theorem [28], the complement \( S^3 \setminus K_n \) is hyperbolic for all \( n \geq 2 \). By Gehring, Maclachlan, and Martin [18], \( S^3 \setminus K_n \) is non-arithmetic for all \( n \geq 3 \). By a theorem of Millichap and Worden [29, Theorem 1.4], \( S^3 \setminus K_m \) and \( S^3 \setminus K_n \) are incommensurable when \( m \neq n \).

Finally, observe that each \( K_n \) can be obtained by \( 1/n \) Dehn filling on one component of the 3–chain link \( L = 6^1_1 \). Hyperbolic volume goes down under Dehn filling [39, Theorem 6.5.6]. Thus

\[
\text{Vol}(S^3 \setminus K_n) < \text{Vol}(S^3 \setminus L) \approx 5.333489,
\]

where the final equality is obtained by direct calculation in SnapPy.

\( \square \)
Next, we complete the proof of Theorem 1.6. We will again use Proposition 4.2 in combination with some facts from number theory [26].

**Proof of Theorem 1.6.** Let \( p_i \) denote the \( i \)th prime number, with \( p_1 = 2 \). Then, for \( k \geq 2 \), let \( \Pi_k \) denote the pretzel link diagram \( P(1, 2p_1 - 1, 1, 2p_2 - 1, \ldots, 1, 2p_k - 1) \). In particular, we substitute \( \ell_i = p_i \) in Figure 4. By Lemma 4.1, \( \Pi_k \) has crossing number

\[
c_k = 2 \sum_{i=1}^{k} p_i \geq k^2 \log k,
\]

where the lower bound is a theorem of Massias and Robin [26, Theorem C.(i)].

By Proposition 4.2, the degree \( d_k \) of any A– or C–special cover of \( D_{\Pi} \) is divisible by every \( p_i \). Since we have chosen the \( p_i \) to be distinct primes, it follows that \( d_k \geq \prod_{i=1}^{k} p_i \). It remains to show that, for sufficiently large \( k \) (specifically, \( k \geq 143 \)), we have

\[
d_k \geq \prod_{i=1}^{k} p_i > \exp(\sqrt{c_k \log(c_k)}/2).
\]

Direct calculation shows that (4.1) holds for \( 143 \leq k \leq 5106 \). Meanwhile, for \( k \geq 5107 \), we can establish the desired inequality using the work of Massias and Robin [26] on effective versions of the prime number theorem.

When \( k \geq 779 \), they prove the estimate [26, Theorem C.(vii)],

\[
c_k = 2 \sum_{i=1}^{k} p_i \leq k^2 (\log k + \log \log k - 1.463)
\]

which can be rewritten as

\[
\frac{\log c_k}{2} \leq \log k + \log \sqrt{\log k + \log \log k} - 1.463
\]

When \( k \geq 2854 \), one may easily check using Calculus that

\[
\log \sqrt{\log k + \log \log k} - 1.463 < \log \log k - 1.
\]

(To see this, compute that the derivative of the right-hand side is strictly greater than that of the left-hand side, so it suffices to substitute the value \( k = 2854 \).) Combining the last two displayed equations, we conclude that \( k \geq 2854 \) implies

\[
(4.3) \quad \frac{\log c_k}{2} < \log k + \log \log k - 1.
\]

When \( k \geq 5107 \), Massias and Robin prove the estimate [26, Theorem B.(ii)]

\[
\log d_k \geq \sum_{i=1}^{k} \log p_i \geq k (\log k + \log \log k - 1)
= k \sqrt{\log k + \log \log k - 1} \cdot \sqrt{\log k + \log \log k - 1}
> \sqrt{c_k} \cdot \sqrt{\log(c_k)}/2,
\]

where the inequality in the last line uses (4.2) and (4.3). We conclude that (4.1) holds, hence the degree of any A– or C–special cover of \( D_{\Pi} \) is \( d_k > \exp(\sqrt{c_k \log(c_k)}/2) \). \( \square \)
5. Permutation covers of graphs

In order to construct a special cover of a Dehn complex $D_{\mathcal{H}}$, we will need to construct appropriate covers of its vertical and horizontal skeleta, as well as appropriate covers of its hyperplanes. All of the above objects are graphs. To that end, this section reviews some standard material about covering spaces of graphs, viewed through the lens of permutation labelings. Our primary reference for this material is the textbook by Gross and Tucker [19, Chapter 2].

An orientation of a graph $G$ is a choice of orientation $\vec{e}$ or $\vec{e}$ for each edge $e \in E(G)$. When working with an oriented graph $G$, the oriented edge set is denoted $E^+(G)$.

Let $N$ be a finite set and $S_N$ the permutation group of $N$. We adopt the convention that $S_N$ acts on $N$ from the right. Given a symbol $p \in N$ and a permutation $\sigma \in S_N$, we denote the $\sigma$–image of $p$ by $p\sigma$. This notation is compatible with the right action: for $\sigma, \tau \in S_N$, we have $p(\sigma\tau) = (p\sigma)\tau$.

**Definition 5.1.** Let $G$ be an oriented graph, and let $S_N$ be the symmetric group on a finite set $N$. A permutation labeling on $N$ is a function $\alpha : E^+(G) \rightarrow S_N$. The function $\alpha$ can be extended to include reverse orientations by the formula $\alpha(\vec{e}) = \alpha(\vec{e})^{-1}$. It can also be extended to an oriented edge path $\gamma = \vec{e}_1\vec{e}_2\cdots \vec{e}_k$ by setting $\alpha(\gamma) = \alpha(\vec{e}_1)\alpha(\vec{e}_2)\cdots \alpha(\vec{e}_k)$. This definition is compatible with the right-action of $S_N$ on $N$.

The permutation cover derived from $\alpha$, denoted $G_{\alpha} \rightarrow G$, is a degree $|N|$ cover with vertex set $V(G) \times N$ and oriented edge set $E^+(G) \times N$, where the oriented edge $(\vec{e}, i)$ joins $(o(\vec{e}), i)$ to $(t(\vec{e}), \alpha(i))$. The covering projection is the map $(x, i) \mapsto x$ for a vertex or oriented edge $x$.

This cover $G_{\alpha}$ is typically not a regular cover. Indeed, every finite cover of a graph $G$ is a permutation cover derived from some permutation labeling.

**Theorem 5.2** ([19, Theorem 2.4.5]). If $\xi : \tilde{G} \rightarrow G$ is a finite cover of degree $n$, then for any choice of orientation on $G$, there is a permutation labeling $\alpha$ on $N = \{1, \ldots, n\}$ such that $\tilde{G} = G_{\alpha}$.

**Sketch.** Fix an orientation on $G$. For each oriented edge $\vec{e} \in E^+(G)$, the preimage $\xi^{-1}(\vec{e})$ is a matching of $\xi^{-1}(o(\vec{e}))$ to $\xi^{-1}(t(\vec{e}))$. By fixing a bijection $\xi^{-1}(v) \rightarrow \{1, \ldots, n\}$ for each $v \in V(G)$, the matching $\xi^{-1}(\vec{e})$ determines a permutation. This permutation is $\alpha(\vec{e})$.

Given a permutation labeling on $N$, $\alpha : E^+(G) \rightarrow S_N$, the lifting criterion for the permutation cover $G_{\alpha}$ can be characterized in terms of the labels.

**Lemma 5.3.** A loop $\gamma \leftrightarrow G$ based at $v \in V(G)$ lifts to $(G_{\alpha}, (v, i))$ if and only if $\alpha(\gamma) \in \text{Stab}_{S_N}(i)$.

**Proof.** By Definition 5.1, the path-lift of $\gamma$ to $(G_{\alpha}, (v, i))$ is a path $\tilde{\gamma}$ from $(v, i)$ to $(v, \alpha(i))$. This path-lift is a loop if and only if $\alpha(\gamma) = i$. □

The graph $G_{\alpha}$ is not always connected. However, the connectivity of $G_{\alpha}$ can be characterized in terms of the permutation labels. Fix a basepoint $u \in V(G)$. The local group at $u$ is the subgroup

$${\mathcal{A}}(u) = \{\alpha(\gamma) \mid \gamma \text{ is a closed edge path based at } u\}.$$

If $G$ is connected, the groups $\mathcal{A}(u)$ and $\mathcal{A}(v)$ are conjugate in $\mathcal{A}$ for any $u, v \in G$.

**Theorem 5.4** ([19, Theorem 2.5.2, Corollary 2]). Let $G$ be a connected graph and $\alpha : E^+(G) \rightarrow S_N$ a permutation labeling. Fix a base vertex $u \in V(G)$. The permutation cover $G_{\alpha}$ is connected if and only if $\mathcal{A}(u)$ is a transitive subgroup of $S_N$.

The proof of Theorem 5.4 is closely related to the observation behind Lemma 5.3.

**Definition 5.5.** Let $G$ be a graph and $\alpha : E^+(G) \rightarrow S_N$ a permutation labeling. The regular cover derived from $\alpha$ is the graph $G_{S_N}$ with vertex set $V(G) \times S_N$ and edge set $E^+(G) \times S_N$, where edge $(\vec{e}, \sigma)$ joins $(o(\vec{e}), \sigma)$ to $(t(\vec{e}), \alpha(\vec{e}), \sigma)$. The deck group $S_N$ acts from the left, with $\sigma(v, \tau) = (v, \sigma\tau)$ and $\sigma(e, \tau) = (e, \sigma\tau)$. Taking the quotient by this left action induces a covering projection $G_{S_N} \rightarrow G$, namely projection onto the first coordinate.
The lifting criterion can again be characterized in terms of permutation labels: a loop \( \gamma \mapsto G \) lifts to \( G_{S_N} \), if and only if \( \alpha(\gamma) = \text{id} \).

The local group \( A(u) \) can be used to characterize the connected components of \( G_{S_N} \), as follows.

**Definition 5.6.** Let \( G \) be a connected graph and \( \alpha : E^+(G) \to S_N \) a permutation labeling. Fix a base vertex \( u \in V(G) \) and a maximal tree \( T \subset G \). For every vertex \( v \in V(G) \), let \( \gamma_v \) be the unique directed path from \( u \) to \( v \) through \( T \). Observe that the right coset \( A(u)\alpha(\gamma_v) \) depends only on \( v \), rather than the choice of maximal tree.

Let \( G_{A(u)} \) be the connected component of \( G_{S_N} \) containing \((u,\text{id})\). The vertices of \( G_{A(u)} \) can be characterized as

\[
V(G_{A(u)}) = \{(v, \sigma) \in V(G_{S_N}) : \sigma \in A(u)\alpha(\gamma_v)\}.
\]

Then \( G_{A(u)} \) is a cover of \( G \), because the restriction of the covering projection \( G_{S_N} \to G \) to a component of \( G_{S_N} \) is again a covering map.

**Theorem 5.7 ([19] Theorem 2.5.1 and corollaries]).** Let \( G \) be a connected graph and \( \alpha : E^+(G) \to S_N \) a permutation labeling. For any basepoint \( u \in V(G) \), the connected component \( G_{A(u)} \subset G_{S_N} \) is a regular cover of \( G \), with deck group \( A(u) \), acting on the left. The vertex and edge sets of \( G_{A(u)} \) can be placed in bijective correspondence with \( V(G) \times A(u) \) and \( E(G) \times A(u) \), respectively, via \( (5.1) \).

Furthermore, every connected component of \( G_{S_N} \) is isomorphic to \( G_{A(u)} \).

We may now generalize Definition 5.6 as follows.

**Definition 5.8.** Let \( G \) be a connected graph and \( \alpha : E^+(G) \to S_N \) a permutation labeling. Let \( u \in V(G) \) be a basepoint, with local group \( A(u) \). As in Definition 5.6, fix a maximal tree \( T \) and a path \( \gamma_v \) from \( u \) to \( v \) for every \( v \in V(G) \).

For any subgroup \( B \) such that \( A(u) \subset B \subset S_N \), the regular \( B \)-cover of \( G \) derived from \( \alpha \), denoted \( G_B \), is the union of components of \( G_{S_N} \) whose vertices satisfy

\[
V(G_B) = \{(v, \sigma) \in V(G_{S_N}) : \sigma \in B\alpha(\gamma_v)\}.
\]

By Theorem 5.7, this is a regular cover of \( G \), with deck group \( B \) acting on the left.

**Lemma 5.9.** Let \( G \) be an oriented, connected graph and \( \alpha : E^+(G) \to S_N \) a permutation labeling on \( N \). For a basepoint \( u \in V(G) \), fix a subgroup \( B \) such that \( A(u) \subset B \subset S_N \).

Let \( G_o \) be the permutation cover derived from \( \alpha \) and \( G_B \) the regular \( B \)-cover derived from \( \alpha \). Assume that \( G_o \) is connected. Then, for every element \( p \in N \), the covering map \( G_B \to G \) factors through a regular cover \( \xi_p : G_B \to G_o \) with deck group \( S_p = \text{Stab}_{B}(p) \), acting on the left.

This result generalizes a construction in Gross and Tucker [19] Section 2.4.4.

**Proof.** Let \( S_p = \text{Stab}_{B}(p) \) be the subgroup of \( B \) that stabilizes \( p \). Let \( R = S_n \{B\} \) be the set of right cosets of \( S_p \). Let \( G_R \) be the graph with vertex set \( V(G) \times R \) and edge set \( E(G) \times R \), where the edge \((\vec{e}, S_p \sigma)\) joins \((o(\vec{e}), S_p \sigma)\) to \((t(\vec{e}), S_p \alpha(o(\vec{e})))\). The map \((x, \sigma) \mapsto (x, S_p \sigma)\) for \( x \in V(G) \) or \( x \in E(G) \) is readily seen to define a regular cover \( G_B \to G_R \) with deck group \( S_p \), acting on the left.

We will show that \( G_R \) is isomorphic to \( G_o \). Let \( S_p \sigma \in R \). Since \( \sigma \in B \), we have

\[
S_p \sigma = B \sigma \cap \text{Stab}_{S_N}(p) \sigma = B \cap \text{Stab}_{S_N}(p) \sigma.
\]

Next, since \( A(u) \) is transitive by Theorem 5.4 for each \( q \in N \) there is some \( \delta_{pq} \in A(u) \) such that \( pq \delta_{pq} = q \). It is an exercise in permutation groups to show that \( \text{Stab}_{S_N}(p) \cdot \delta_{pq} = \text{Stab}_{S_N}(p) \cdot (pq) \).

Since the right cosets of \( \text{Stab}_{S_N}(p) \) are the set \( \{\text{Stab}_{S_N}(p) \cdot (pq)\} \), each coset \( S_p \sigma \in R \) is equal to \( S_p \delta_{pq} \) for a unique \( q \in N \). This gives a bijection \( \varphi : R \to N \), sending \( S_p \delta_{pq} \mapsto q \).

We obtain bijections \( \varphi : V(G_R) \to V(G_o) \) and \( \varphi : E(G_R) \to E(G_o) \), where \( \varphi(x, S_p \delta_{pq}) = (x, q) \) for a vertex or edge \( x \). To see that \( \varphi \) preserves the edge relation, observe that for any \( \sigma \in S_N \) we have \( \text{Stab}_{S_N}(p) \cdot (pq) \sigma = \text{Stab}_{S_N}(p) \cdot (pq) \sigma \), where \( (pq) \sigma \) is the transposition exchanging \( p \) and \( q \sigma \). Thus \( \varphi \) is equivariant with respect to the right action of \( S_N \) on both \( N \) and \( R \). Since the edge...
relation of both $G_R$ and $G_\alpha$ is defined in terms of the right action, it follows that $\varphi$ preserves the edge relation and is a graph isomorphism.

Let $\xi_p: G_B \rightarrow G_\alpha$ denote the covering projection $(x, \sigma) \mapsto (x, S_p\sigma) \mapsto (x, p\sigma)$. The first map is the quotient by $S_p$ and the second map is $\varphi$. Then the covering projection $G_B \rightarrow G$ indeed factors through $\xi_p$.

Observe that $\xi_p(v, \text{id}) = (v, p)$, hence $\xi_p$ definitely depends on the choice of $p$. In fact, this dependence is straightforward to characterize.

**Lemma 5.10.** Suppose $G$ is an oriented graph, and $\alpha: E^+(G) \rightarrow S_N$ is a permutation labeling. Fix a basepoint $u \in V(G)$ and subgroup $B$ such that $\mathcal{A}(u) \subset B \subset S_N$. For an element $\tau \in B$ and intermediate factorization $\xi_p: G_B \rightarrow G_\alpha$ coming from Lemma 5.9, we have $\xi_p \circ \tau = \xi_p\tau$, where on the left hand side $\tau$ is acting as a deck transformation of $G_B$.

**Proof.** Two vertices or edges $(x, g)$ and $(x, h)$ are identified under $\xi_p \circ \tau$ if and only if there exists $\sigma \in S_p = \text{Stab}_B(p)$ such that $\sigma g = \tau h$, which can be rewritten as $\tau^{-1}\sigma\tau = hg^{-1}$. Such a $\sigma$ exists if and only if $hg^{-1} \in S_p\tau = S_p\tau$. Finally, $hg^{-1} \in S_p\tau$ if and only if $(x, g)$ and $(x, h)$ are identified under $\xi_p\tau$. Therefore $\xi_p \circ \tau = \xi_p\tau$. □

### 6. The involutive completion

Our construction of special covers of the Dehn complex $D_H$ is inspired and motivated by Wise’s analysis of covers of $\mathcal{VH}$–complexes \cite{Wise1, Wise2}. In the course of studying polygons of finite groups, Wise defined an object called the canonical completion (see Remark 6.4) that enables an immersion of finite graphs to factor through an embedding into a finite-sheeted cover. The idea behind the canonical completion dates back to the work of Stallings \cite{Stallings}.

In this section, we develop an analogue of the canonical completion adapted specifically for the analysis of Dehn complexes and their double covers. Our construction takes advantage of the natural bipartite structures available in this setting. It would be fair to say that the involutive completion is the main technical innovation of this paper.

We define the band graph $B_k$ to be the 2–vertex bipartite graph with $k$ edges, with vertices labeled $v_-$ and $v_+$. We fix an orientation $E^+(B_k)$, where every edge is oriented from $v_-$ toward $v_+$.

**Definition 6.1.** A symmetric bipartite graph $(G, \iota)$ is a bipartite graph $G$ with vertex partition $V(G) = V_1 \sqcup V_2$ and an involution $\iota: G \rightarrow G$ such that $\iota(V_1) = V_2$.

**Definition 6.2.** Let $(G, \iota)$ be a symmetric bipartite graph and $\lambda: G \rightarrow B_k$ a combinatorial immersion. Recall that $\lambda$ induces a set-map $\lambda\Delta: \Delta(v) \rightarrow E(B_k)$ on the set $\Delta(v)$ of edges incident to $v$.

We say that $\lambda$ is partition-symmetric if $\lambda$ is partition–preserving, and for all $v \in V(G)$, we have $\lambda\Delta(\Delta(v)) = \lambda\Delta(\Delta(v))$ as sets of unoriented edges.

**Definition 6.3.** Suppose $(G, \iota)$ is a symmetric bipartite graph and $\lambda: G \rightarrow B_k$ is a partition-symmetric immersion. The involutive completion $\mathcal{I}_\lambda(G, B_k)$ is the graph obtained from $G$ by adding edges as follows: for every orbit $\{v, \iota v\}$, and for each edge $e \in E(B_k) \setminus \lambda\Delta(\Delta(v))$, we add an edge $\hat{e}$ joining $v$ and $\iota v$. Then $\lambda$ extends to an immersion $\hat{\lambda}: \mathcal{I}_\lambda(G, B_k) \rightarrow B_k$, where $\hat{\lambda}$ coincides with $\lambda$ on $G \subseteq \mathcal{I}_\lambda(G, B_k)$ and $\hat{\lambda}$ satisfies $\hat{\lambda}(\hat{e}) = \hat{e}$ for every added edge $\hat{e}$ corresponding to $e \in E(B_k) \setminus \lambda\Delta(\Delta(v))$.

**Remark 6.4.** Given a graph immersion $\lambda: G \rightarrow R$, Wise defines a canonical completion $C(G, R)$. This object is simplest when $R = R_k$ is a bouquet of $k$ circles. In this special case, the completion is built by adding edges so that paths in the preimage of an edge $e \subset R_k$ become loops. For a general $R$, the construction of $C(G, R)$ involves a fiber-product. See Wise \cite[Definition 5.1]{Wise} for details.

In our setting, the one-vertex graph $R_k$ is replaced by the simplest possible bipartite graph with $k$ edges, namely the band graph $B_k$. This preserves the elegance of the simplest case of Wise’s construction while enabling small-degree covers.

We also note that unlike Wise’s construction, the involutive completion $\mathcal{I}_\lambda(G, B_k)$ does not necessarily retract to $G$ (compare Lemma 6.6).
Lemma 6.5. Suppose \((G, \iota)\) is a symmetric bipartite graph and \(\lambda: G \to B_k\) is a partition-symmetric immersion. The involutive completion \( \mathcal{I}_\lambda(G, B_k) \) is a well-defined, symmetric bipartite graph with an involution \( \tilde{\iota} \) that extends \( \iota \). Furthermore, \( \hat{\lambda}: \mathcal{I}_\lambda(G, B_k) \to B_k \) is a covering map that fits into the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{j} & \mathcal{I}_\lambda(G, B_k) \\
\downarrow{\lambda} & & \downarrow{\hat{\lambda}} \\
B_k & & B_k
\end{array}
\]

Proof. For every vertex \( v \in V(G) \), we have \( \lambda_\Delta(\Delta(v)) = \lambda_\Delta(\Delta(vv)) \) because \( \lambda \) is partition-symmetric. Thus

\[
E(B_k) \setminus \lambda_\Delta(\Delta(v)) = E(B_k) \setminus \lambda_\Delta(\Delta(vv)).
\]

Hence the set of edges added between vertices in an orbit \( \{v, vv\} \) does not depend on the orbit representative.

Define \( \tilde{\iota}: \mathcal{I}_\lambda(G, B_k) \to \mathcal{I}_\lambda(G, B_k) \) so that \( \tilde{\iota}(G) = \iota \), and so that \( \tilde{\iota}(\tilde{e}) = \tilde{e} \), with reversed orientation, for every added edge \( \tilde{e} \). This is a partition-reversing involution, so \( \mathcal{I}_\lambda(G, B_k) \) is symmetric bipartite. By construction, the map \( \hat{\lambda} \) is a local isomorphism of graphs, so it is a covering map.

\[ \square \]

Lemma 6.6. Suppose \((G, \iota)\) is a symmetric bipartite graph. Suppose that either \(G\) is connected or \(G\) has exactly two connected components that are exchanged by \(\iota\). If \(\lambda: G \to B_k\) is a partition-symmetric immersion but not a covering map, then \(\mathcal{I}_\lambda(G, B_k)\) is connected.

Proof. By definition, \(\mathcal{I}_\lambda(G, B_k)\) is constructed by adding edges between existing vertices of \(G\). Thus there is nothing to prove when \(G\) is connected.

Suppose now that \(G = G_0 \cup G_1\), where \(\iota(G_0) = G_1\). Since \(\lambda\) is not a covering map, there is some \(v \in V(G)\) such that \(\lambda_\Delta(\Delta(v)) \neq E(B_k)\). Let \(e \in E(B_k) \setminus \lambda_\Delta(\Delta(v))\) be an edge of \(B_k\) not covered by the neighborhood of \(v\). Without loss of generality, \(v \in G_0\), so \(vv \in G_1\). Therefore, in \(\mathcal{I}_\lambda(G, B_k)\) there is an edge \(\tilde{e}\) joining \(v\) to \(vv\). Since \(G_0\) and \(G_1\) are connected and meet every vertex of \(\mathcal{I}_\lambda(G, B_k)\), we conclude \(\mathcal{I}_\lambda(G, B_k)\) is connected.

\[ \square \]

Lemma 6.7. Suppose \((G, \iota)\) is a symmetric bipartite graph and \(\lambda: G \to B_k\) is a partition-symmetric immersion. Then \(\mathcal{I}_\lambda(G, B_k) = \mathcal{I}_{\lambda \circ \iota}(G, B_k)\) and \(\hat{\lambda} \circ \iota = \hat{\lambda} \circ \tilde{\iota}\).

Proof. Since \(\iota\) is a graph automorphism, \((\lambda \circ \iota)_\Delta(\Delta(v)) = \lambda_\Delta(\Delta(vv))\). Therefore, by equation \((6.1)\), both \(\mathcal{I}_\lambda(G, B_k)\) and \(\mathcal{I}_{\lambda \circ \iota}(G, B_k)\) are constructed from \(G\) by adding the same set of edges joining \(v\) and \(vv\) for each \(v \in V\). Further, when restricted to \(G\), we have \(\hat{\lambda} \circ \iota = \lambda \circ \iota = \hat{\lambda} \circ \tilde{\iota}\). If \(\tilde{e}\) is an added edge, \(\tilde{\lambda} \circ \iota(\tilde{e}) = e = \hat{\lambda} \circ \tilde{\iota}(\tilde{e})\) by construction.

At this point, we introduce the following simplified notation. Define \(\rho = \lambda \circ \iota\) and \(\hat{\rho} = \hat{\lambda} \circ \iota = \hat{\lambda} \circ \tilde{\iota}\). We will write \(\mathcal{I} = \mathcal{I}_\lambda(G, B_k) = \mathcal{I}_\rho(G, B_k)\) when the map \(\lambda: G \to B_k\) is clear from context. By Lemma \(6.7\), \(\mathcal{I}\) is the involutive completion of both \(\lambda\) and \(\rho\).

Example 6.8. Figure \(\square\) depicts a graph \(\Sigma\) and an immersion \(\lambda: \Sigma \to B_4\), where \(E(B_4) = \{a, b, c, d\}\). The involution \(\iota\) is given by a \(\pi\)-rotation about the center of the graph. Figure \(\square\) shows the involutive completion \(\mathcal{I} = \mathcal{I}_\lambda(\Sigma, B_4) = \mathcal{I}_\rho(\Sigma, B_4)\), where the covering maps \(\lambda: \mathcal{I} \to B_4\) and \(\hat{\rho}: \mathcal{I} \to B_4\) are now indicated by colors. The edges added in the construction of \(\mathcal{I}\) are the two edges connecting \(y\) to \(y'\). The extension \(\hat{\iota}\) sends each of these added edges to itself, with reversed orientation. We will refer to this running example several more times.

Lemma 6.9. Suppose \((G, \iota)\) is a symmetric bipartite graph and \(\lambda: G \to B_k\) is a partition-symmetric immersion. Let \(N = \{\{v, vv\} \mid v \in V(G)\}\) be the set of \(\iota\)-orbits in \(V(G)\). Then the covering maps \(\hat{\lambda}: \mathcal{I}_\lambda(G, B_k) \to B_k\) and \(\hat{\rho}: \mathcal{I}_\lambda(G, B_k) \to B_k\) induce identical permutation labels on \(E^+(B_k)\).
Proof. By Theorem 5.2, the covering map \( \hat{\lambda} : \mathcal{I}_\lambda(G, B_k) \to B_k \) defines a permutation labeling \( \alpha : E^+(B_k) \to S_N \). Similarly, the covering map \( \hat{\rho} = \hat{\lambda} \circ \iota = \hat{\lambda} \circ \hat{\iota} \) defines a permutation labeling \( \alpha' : E^+(B_k) \to S_N \). We claim that \( \alpha = \alpha' \) as functions.

Given an oriented edge \( \hat{e} \in E^+(B_k) \), the full preimage \( \hat{\lambda}^{-1}(\hat{e}) \) induces a matching from \( \hat{\lambda}^{-1}(v_-) \) to \( \hat{\lambda}^{-1}(v_+) \). This matching defines both \( \alpha \) and \( \alpha' \). If an edge in \( \hat{\lambda}^{-1}(\hat{e}) \) runs from \( v \in \hat{\lambda}^{-1}(v_-) \) to \( w \in \hat{\lambda}^{-1}(v_+) \), then, by the definition of \( \alpha \) we have

\[
\{v, w\} \alpha(\hat{e}) = \{w, \iota w\}.
\]

(Recall that the permutation label \( \alpha(\hat{e}) \in S_N \) acts on \( N \) from the right.) But then the \( \hat{\iota} \)-image of the same edge in \( \hat{\lambda}^{-1}(\hat{e}) \) runs from \( \iota w \in (\hat{\lambda} \circ \hat{\iota})^{-1}(v_-) \) to \( \iota w \in (\hat{\lambda} \circ \hat{\iota})^{-1}(v_+) \). Thus, by the definition of \( \alpha' \), we have

\[
\{v, w\} \alpha'(\hat{e}) = \{\iota w, w\} = \{v, \iota w\} \alpha(\hat{e}).
\]

Since \( \hat{e} \) and \( v \) were arbitrary, we conclude that \( \alpha = \alpha' \), as claimed. \( \square \)

Definition 6.10. Suppose \( (G, \iota) \) is a symmetric bipartite graph and \( \lambda : G \to B_k \) is a partition-symmetric immersion. Let \( \alpha : E^+(B_k) \to S_N \) be the permutation labeling constructed in Lemma 6.9. Let \( (B_k)_S \) be the regular cover of \( B_k \) derived from \( \alpha = \alpha' \).

Fix a basepoint \( v_- \in B_k \). Following Definition 5.6, let \( \tilde{\mathcal{I}}_{\lambda}(G, B_k) = (B_k)_{A(v_-)} \) be the connected component of \( (B_k)_S \) that contains \( (v_-)_{id} \). By Theorem 5.7, this component is a regular cover \( \tilde{\lambda} : \tilde{\mathcal{I}}_{\lambda}(G, B_k) \to B_k \) with deck group \( A(v_-) \). We call \( \tilde{\mathcal{I}}_{\lambda}(G, B_k) \) the regular involutive completion of \( \lambda : G \to B_k \).

Using the basepoint \( v_+ \in B_k \), let \( \tilde{\mathcal{I}}_{\rho}(G, B_k) = (B_k)_{A(v_+)} \) be the connected component of \( (B_k)_S \) that contains \( (v_+)_{id} \). This is a connected regular cover of \( B_k \) with deck group \( A(v_+) \).

When \( G \) and \( B_k \) are clear from context, we write \( \tilde{\mathcal{I}}_{\lambda} = \tilde{\mathcal{I}}_{\lambda}(G, B_k) \) and \( \tilde{\mathcal{I}}_{\rho} = \tilde{\mathcal{I}}_{\rho}(G, B_k) \). By Theorem 5.7, \( \tilde{\mathcal{I}}_{\lambda} \) is isomorphic to \( \tilde{\mathcal{I}}_{\rho} \).

For the map \( \lambda : \Sigma \to B_k \) of Example 6.8, Figure 9 shows the permutation labeling \( \alpha = \alpha' : B_k \to S_N \), where \( N = \{x, y, z\} \) is the set of \( \iota \)-orbits of vertices of \( \Sigma \). Then, Figure 10 shows the regular involutive completions \( \tilde{\mathcal{I}}_{\lambda} = \tilde{\mathcal{I}}_{\lambda}(\Sigma, B_k) \) and \( \tilde{\mathcal{I}}_{\rho} = \tilde{\mathcal{I}}_{\rho}(\Sigma, B_k) \).

Lemma 6.11. Suppose \( \mathcal{I} = \mathcal{I}_\lambda(G, B_k) \) is connected. Then the deck group \( A(v_-) \) of \( \tilde{\lambda} : \tilde{\mathcal{I}}_{\lambda} \to \mathcal{I} \) is a transitive subgroup of \( S_N \). Every choice of stabilized symbol \( q \) gives a factorization \( \xi_q^- : \tilde{\mathcal{I}}_{\lambda} \to \mathcal{I} \), so that \( \tilde{\lambda} \circ \xi_q^- = \tilde{\lambda} \).

Similarly, the deck group \( A(v_+) \) of \( \tilde{\rho} : \tilde{\mathcal{I}}_{\rho} \to B_k \) is a transitive subgroup of \( S_N \). Every choice of stabilized symbol \( q \) gives a factorization \( \xi_q^+ : \tilde{\mathcal{I}}_{\rho}(G, B_k) \to \mathcal{I}(G, B_k) \), so that \( \tilde{\rho} \circ \xi_q^+ = \tilde{\rho} \).

Proof. We focus on \( \tilde{\mathcal{I}}_{\lambda} \), as the proof for \( \tilde{\mathcal{I}}_{\rho} \) is identical.

Since \( \tilde{\mathcal{I}}_{\lambda} \) is connected by hypothesis, the local group \( A(v_-) \) must be transitive by Theorem 5.4. By Lemma 5.9, the regular cover \( \tilde{\lambda} : \tilde{\mathcal{I}} \to B_k \) factors through \( \mathcal{I}_\lambda(G, B_k) \), with the choice of factorization \( \xi_q^- : \tilde{\mathcal{I}}_{\lambda}(G, B_k) \to \mathcal{I}(G, B_k) \) determined by the stabilized element \( q \in N \). \( \square \)

Remark 6.12. While \( \mathcal{I}_\lambda \cong \mathcal{I}_\rho \), it is important to note that \( \xi_q^- \) and \( \xi_q^+ \) can be very different. In fact, in a generic situation, \( \xi_p^- \neq \xi_q^+ \) for any pair \( p, q \). See Remark 7.1 for a particular counterexample.

Our next goal is to construct a cover of \( \mathcal{I} = \mathcal{I}_\lambda(G, B_k) = \mathcal{I}_\rho(G, B_k) \) that is simultaneously compatible with the factorizations \( \xi_p^- \) and \( \xi_p^+ \). This involves permutation labels on edges of \( \mathcal{I} \).

Definition 6.13. Suppose \( \mathcal{I} = \mathcal{I}_\lambda(G, B_k) \) is connected, and endow \( \mathcal{I} \) with the orientation that makes \( \tilde{\lambda} \) orientation-preserving. Then the permutation labeling \( \alpha : E^+(B_k) \to S_N \) described in Lemma 6.9 \( \square \)
induces a labeling \((\alpha \circ \hat{\lambda}) : E^+(I) \to S_N\). For arbitrary symbols \(p, q \in N\), and an arbitrary path \(\gamma\) from \((v_-, p)\) to \((v_-, q)\), the label \((\alpha \circ \hat{\lambda})(\gamma)\) belongs to \(\mathcal{A}(v_-)\) because \(\hat{\lambda}(\gamma)\) is a loop in \(B_k\) based at \(v_-\). In particular, \(\mathcal{A}(v_-)\) contains the local group of \((\alpha \circ \hat{\lambda})\) at any vertex of \(I\) of the form \((v_-)\).

Following Definition 6.8 let \(I_{\mathcal{A}(v_-)} \subset I_{S_N}\) be the regular \(\mathcal{A}(v_-)\)-cover of \(I\) derived from \(\alpha \circ \hat{\lambda}\), with deck group \(\mathcal{A}(v_-)\). For an arbitrary \(p \in N\), this is the union of components of \(I_{S_N}\) containing every vertex of the form \(((v_-), p, \sigma)\) for \(\sigma \in \mathcal{A}(v_-)\). By the above paragraph, this subgraph of \(I_{S_N}\) does not depend on the symbol \(p\).

In an analogous fashion, the permutation labeling \(\alpha = \alpha' : E^+(B_k) \to S_N\) described in Lemma 6.9 also induces a labeling \((\alpha \circ \bar{\rho}) : E^+(I) \to S_N\). Because \(\bar{\rho}(v_-, p) = \hat{\lambda} \circ \bar{\iota}(v_-, p) = v_+\) for arbitrary \(p\), a path \(\gamma\) from \((v_-), p)\) to \((v_-, q)\), acquires a label \((\alpha \circ \bar{\rho})(\gamma)\) \(\in \mathcal{A}(v_+)\). Let \(I_{\mathcal{A}(v_+)}\) be the regular \(\mathcal{A}(v_+)\)-cover of \(I\) derived from \(\alpha \circ \bar{\rho}\), with deck group \(\mathcal{A}(v_+)\).

Lemma 6.14. Suppose \(I = I_\lambda(G, B_k)\) is connected. Let \(I_{\mathcal{A}(v_-)} \subset I_{S_N}\) be the regular \(\mathcal{A}(v_-)\)-cover of \(I\) derived from the permutation labeling \(\alpha \circ \hat{\lambda}\). Then there is a canonical isomorphism \(\varphi: I_{\mathcal{A}(v_-)} \to (I_\lambda \times N)\) of regular covers of \(I\) with deck group \(\mathcal{A}(v_-)\).

Proof. A maximal tree of \(B_k\) consists of a single edge \(\bar{e}_0\), oriented from \(v_-\) to \(v_+\). Given a vertex or edge \(x\) of \(B_k\), a vertex or edge of \(I_\lambda \times N\) lying over \(x\) has the form \((x, \sigma, p)\) where \(p \in N\), and where \(\sigma\) belongs to the appropriate right coset of \(\mathcal{A}(v_-)\). More precisely,

\[
\sigma \in \begin{cases} 
\mathcal{A}(v_-) & x = v_- \text{ or } x \in E(B_k), \\
\mathcal{A}(v_-) \alpha(\bar{e}_0) & x = v_+.
\end{cases}
\]

As in Definition 6.6, the right coset \(\mathcal{A}(v_-) \alpha(\bar{e}_0)\) is independent of the choice of \(e_0\).

We can now describe a left action of \(\mathcal{A}(v_-) \subset S_N\) on \(I_\lambda \times N\). Given a vertex or edge \((x, \sigma, p)\) and a permutation \(\tau \in \mathcal{A}(v_-)\), set

\[
\tau(x, \sigma, p) = (x, \tau \sigma, p \tau^{-1}).
\]

Observe that \(\mathcal{A}(v_-) \tau \sigma = \mathcal{A}(v_-) \sigma\), hence the image stays in \(I_\lambda \times N\). In addition,

\[
\eta \circ \tau(x, \sigma, p) = \eta(x, \tau \sigma, p \tau^{-1}) = (x, \eta \tau \sigma, p \tau^{-1} \eta^{-1}) = (\eta \tau)(x, \sigma, p),
\]

verifying that this is a left action by \(\mathcal{A}(v_-)\).

Recall from Lemma 6.11 that for every choice of symbol \(p \in N\), the stabilizer \(S_p \subset \mathcal{A}(v_-)\) acts on \(I_\lambda\) with quotient \(I\). Since \(\mathcal{A}(v_-)\) acts transitively on \(N\) by Lemma 6.11, we have (for an arbitrary \(p \in N\))

\[
I_{\mathcal{A}(v_-)} \setminus I_\lambda \times N = S_p \setminus I_\lambda \times \{p\} = I.
\]

The covering projection \((I_\lambda \times N) \to I\) is denoted \(\xi^- = \sqcup_p \xi^-\), as the restriction to each component \(I_\lambda \times \{p\}\) is exactly \(\xi^-\).

By Definition 6.13, the vertices of \(I_{\mathcal{A}(v_-)}\) are of the form \((u, \sigma)\) where \(u \in V(I)\) and \(\sigma\) belongs to the appropriate coset of \(\mathcal{A}(v_-)\). More precisely, \(\sigma \in \mathcal{A}(v_-)\) when \(u = (v_-, p)\) for some \(p\), which means that \(\sigma \in \mathcal{A}(v_-) \alpha(\bar{e}_0)\) when \(u = (v_-, q)\) for some \(q\). Since every oriented edge \(\bar{e}\) starts at a preimage of \(v_-\), edges of \(I_{\mathcal{A}(v_-)}\) are of the form \((\bar{e}', p, \sigma)\) for \(\bar{e}' \in E^+(B_k)\) and \(p \in N\) and \(\sigma \in \mathcal{A}(v_-)\).

We can use the above coordinates to define a bijection \(\varphi\) from the vertices and edges of \(I_\lambda \times N\) to those of \(I_{\mathcal{A}(v_-)}\). If \(x\) denotes a vertex or edge of \(B_k\), set

\[
\varphi(x, \sigma, p) = ((x, p \sigma), \sigma), \quad \text{where } \sigma \text{ satisfies (6.2)}.
\]

The inverse function \(\varphi^{-1}\) can be expressed as

\[
\varphi^{-1}(x, q, \sigma) = (x, \sigma, q \sigma^{-1}).
\]

Since \(\varphi\) is invertible, it must be a bijection.
Next, we check that \( \varphi \) preserves the edge relation. In \( \tilde{\mathcal{I}}_\lambda \times N \), we have
\[
o(\vec{e}, \sigma, p) = (v_-, \sigma, p), \quad t(\vec{e}, \sigma, p) = (v_+, \sigma \alpha(\vec{e}), p).
\]
In \( \mathcal{I}_A(v_-) \), we have
\[
o((\vec{e}, p\sigma), \sigma) = ((v_-, p\sigma), \sigma), \quad t((\vec{e}, p\sigma), \sigma) = ((v_+, p\sigma \alpha(\vec{e})), \sigma \alpha(\vec{e})),
\]
Thus \( \varphi \) commutes with the origin and terminus functions, preserving the edge relation. This means \( \varphi \) is a graph isomorphism.

Finally, we verify that \( \varphi \) commutes with the left action of \( \mathcal{A}(v_-) \) by deck transformations. Given \( \tau \in \mathcal{A}(v_-) \), we have
\[
\varphi \circ \tau(x, \sigma, p) = \varphi(x, \tau \sigma, p \tau^{-1}) = ((x, p \tau^{-1} \tau \sigma), \tau \sigma) = ((x, p \sigma), \tau \sigma) = \tau \circ \varphi(x, \sigma, p).
\]
Thus \( \varphi \) is an isomorphism of covers. Since the definition of \( \varphi \) in [6.4] does not involve any choices, this isomorphism is canonical.

By an identical argument, there is a canonical isomorphism \( \mathcal{I}_\mathcal{A}(v_-) \cong (\tilde{\mathcal{I}}_\rho \times N) \) of regular covers of \( \mathcal{I} \) with deck group \( \mathcal{A}(v_+) \) and quotient map \( \xi^+ = \cup \rho \xi^+_p \). See Figure [11] for a picture of one component of \( \mathcal{I}_\mathcal{A}(v_-) \) and one component of \( \mathcal{I}_\mathcal{A}(v_+) \).

We can now define the appropriate two-sided cover of \( \mathcal{I} \).

**Definition 6.15.** Suppose \( \mathcal{I} = \mathcal{I}_\lambda(G, B_k) \) is connected, and endow \( \mathcal{I} \) with the orientation that makes \( \lambda \) orientation–preserving. Let \( M = N \times \{-, +\} \), and define a permutation labeling \( \beta : E^+(\mathcal{I}) \to S_M \) via
\[
\beta(\vec{e}) = (\alpha \circ \hat{\lambda}(\vec{e}), \alpha \circ \hat{\rho}(\vec{e})).
\]
Here, the first coordinate acts by permutations on \((N, -)\) while the second coordinate acts by permutations on \((N, +)\). In particular, \( \beta \) has image in \( S_N \times S_N \subset S_M \). As in Definition 6.13, the local group of \( \beta \) at any vertex of the form \((v_-, p)\) is contained in \( \mathcal{A}(v_-) \times \mathcal{A}(v_+) \subset S_N \times S_N \).

The two-sided cover of \( \mathcal{I} \) is \( \tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\lambda(G, B_k) = \mathcal{I}_\mathcal{A}(v_-) \times \mathcal{A}(v_+) \), the regular \((\mathcal{A}(v_-) \times \mathcal{A}(v_+))\)-cover of \( \mathcal{I} \) derived from \( \beta \). See Figure [12] for one component of \( \tilde{\mathcal{I}} \).

**Lemma 6.16.** Suppose \( \mathcal{I} = \mathcal{I}_\lambda(G, B_k) \) is connected, and endow \( \mathcal{I} \) with the orientation that makes \( \lambda \) orientation–preserving. Let \( \mathcal{I}_\mathcal{A}(v_-) \) and \( \mathcal{I}_\mathcal{A}(v_+) \) be the permutation covers of \( \mathcal{I} \) constructed in Lemma 6.14, and let \( \tilde{\mathcal{I}} = \mathcal{I}_\mathcal{A}(v_-) \times \mathcal{A}(v_+) \) be the two-sided cover of \( \mathcal{I} \) with covering projection \( \xi \). There exists a cover \( \tilde{\lambda} : \tilde{\mathcal{I}}_\mathcal{A} \to \tilde{\mathcal{I}}_\lambda \) and a collapse map \( \tilde{\epsilon}_\lambda : \tilde{\mathcal{I}} \to \tilde{\mathcal{I}}_\lambda \) that fit into the following commutative diagram. In the diagram, every horizontal map is a cover of degree \( |N| \), and each of \( \tilde{\epsilon}_\lambda, \tilde{\lambda} \) is an isomorphism on each component.

\[
\begin{array}{ccc}
\tilde{\mathcal{I}}_\mathcal{A} & \xrightarrow{\pi_\lambda} & \tilde{\mathcal{I}} \\
\downarrow \tilde{\lambda} & & \downarrow \xi \mathcal{A}(v_-) \mathcal{A}(v_+) \downarrow \pi_\rho \\
\tilde{\mathcal{I}}_\lambda & \xrightarrow{\tilde{\epsilon}_\lambda} & \tilde{\mathcal{I}}_\lambda \\
\mathcal{I}(v_-) & \xrightarrow{\mathcal{A}(v_-)} & \mathcal{I} \\
B_k & \xrightarrow{\mathcal{A}(v_+)} & N = \mathcal{I}_\mathcal{A}(v_-) \\
\end{array}
\]

**Proof.** Since the deck group of \( \tilde{\mathcal{I}} \) is the direct product \( \mathcal{A}(v_-) \times \mathcal{A}(v_+) \), the actions of \{id\} \times \mathcal{A}(v_+) \) and \( \mathcal{A}(v_-) \times \{id\} \) commute. Taking the quotient by \{id\} \times \mathcal{A}(v_+) \) gives a cover \( \pi_\lambda : \tilde{\mathcal{I}} \to \mathcal{I}_\mathcal{A}(v_-) \), while taking the quotient by \( \mathcal{A}(v_-) \times \{id\} \) gives a cover \( \pi_\rho : \tilde{\mathcal{I}} \to \mathcal{I}_\mathcal{A}(v_+) \). This explains the commutative diamond on the right of the diagram.
By Lemma \ref{lem:6.14}, we have $\mathcal{I}_{A(v-)} = \tilde{\mathcal{I}}_\lambda \times N$ and $\mathcal{I}_{A(v_+)} = \tilde{\mathcal{I}}_\rho \times N$. The component $\tilde{\mathcal{I}}_\lambda \times \{p\}$ covers $\mathcal{I}$ with deck group the stabilizer $S_p \subset A(v-)$. Thus its preimage $\pi_\lambda^{-1}(\tilde{\mathcal{I}}_\lambda \times \{p\})$ covers $\mathcal{I}$ with deck group $S_p \times A(v_-)$. Similarly, the preimage $\pi_\rho^{-1}(\tilde{\mathcal{I}}_\rho \times \{q\})$ covers $\mathcal{I}$ with deck group $A(v_-) \times \mathcal{S}_q$.

Using these identifications, we define a partition

$$\tilde{\mathcal{I}} = \bigsqcup_{(p,q) \in N^2} \tilde{\mathcal{T}}^{p,q}, \quad \text{where} \quad \tilde{\mathcal{T}}^{p,q} = \pi_\lambda^{-1}(\tilde{\mathcal{I}}_\lambda \times \{p\}) \cap \pi_\rho^{-1}(\tilde{\mathcal{I}}_\rho \times \{q\}).$$

Each graph $\tilde{\mathcal{T}}^{p,q}$ is called a bin of the partition, and is a disjoint union of components of $\tilde{\mathcal{I}}$. The bins are not necessarily connected. We use the bins to define the space $\tilde{\mathcal{I}}_\lambda$ and the map $\tilde{c}_\lambda$.

Define $\tilde{c}_\lambda$: $\tilde{\mathcal{I}}_\lambda \times N \to \tilde{\mathcal{I}}_\lambda$ to be projection to the first coordinate. It follows immediately that $\tilde{c}_\lambda$ is a cover of degree $|N|$ and an isomorphism on each component.

Fix a symbol $p_0 \in N$ for the remainder of this proof. Define $\tilde{\mathcal{I}}_\lambda = \sqcup_p \tilde{\mathcal{T}}^{p,p_0}$ and the covering map $\tilde{\chi}: \tilde{\mathcal{I}}_\lambda \to \tilde{\mathcal{I}}_\lambda$ by the restriction of the composition $\tilde{c}_\lambda \circ \pi_\lambda$.

The map $\tilde{c}_\lambda$ is defined in terms of the partition of $\tilde{\mathcal{I}}$. For each symbol $q \in N$, fix a group element $\sigma_{p_0,q} \in A(v_+)$ such that $p_0 \sigma_{p_0,q} = q$. These permutations exist, since $A(v_+)$ is transitive by Lemma \ref{lem:6.11}. Moreover, $\pi_\sigma$ is equivariant with respect to the $A(v_+)$ action, hence the deck transformation $\delta^{-}_{p_0,q} = (\id, \sigma_{p_0,q}) \in A(v_-) \times A(v_+)$ satisfies $\delta^{-}_{p_0,q}(\tilde{\mathcal{T}}^{p,q}) = \tilde{\mathcal{T}}^{p_0,q}$. Indeed,

$$\delta^{-}_{p_0,q}(\tilde{\mathcal{T}}^{p,q}) = (\pi_\lambda \circ \delta^{-}_{p_0,q})^{-1}(\tilde{\mathcal{T}} \times \{p\}) \cap (\pi_\rho \circ \delta^{-}_{p_0,q})^{-1}(\tilde{\mathcal{T}} \times \{q\})$$

$$= \pi_\lambda^{-1}(\tilde{\mathcal{T}} \times \{p\}) \cap \pi_\rho^{-1}(\sigma_{p_0,q} \tilde{\mathcal{T}} \times \{q\})$$

$$= \pi_\lambda^{-1}(\tilde{\mathcal{T}} \times \{p\}) \cap \pi_\rho^{-1}(\tilde{\mathcal{T}} \times \{q\} \cap \{p_0\}) = \tilde{\mathcal{T}}^{p_0,q}.$$ 

We define $\tilde{c}_\lambda: \tilde{\mathcal{T}}^{p_0,q} \to \tilde{\mathcal{I}}_\lambda$ on the bin $\tilde{\mathcal{T}}^{p_0,q}$ to be $\delta^{-}_{p_0,q}$. Since $\delta^{-}_{p_0,q}$ is an isomorphism on each bin, which is a union of components of $\tilde{\mathcal{I}}$, we get that $\tilde{c}_\lambda$ is an isomorphism on each component. Since there are exactly $|N|$ distinct bins mapping onto each $\tilde{\mathcal{T}}^{p_0,q}$, it follows that $\tilde{c}_\lambda$ is a cover of degree $|N|$. \hfill \Box

The commutativity of the diamond on the right of the diagram follows from the definition of $\tilde{\mathcal{I}}$. Lemma \ref{lem:6.14} and Lemma \ref{lem:6.11} imply that the bottom trapezoid commutes. It remains to check that the top trapezoid commutes. We check this on each bin $\tilde{\mathcal{T}}^{p,q}$. The composition of the top horizontal arrow and the left arrow is $\tilde{c}_\lambda \circ \pi_\lambda \circ \delta^{-}_{p_0,q}$, and the composition of the downward arrow and the bottom arrow is $\tilde{c}_\lambda \circ \pi_\lambda$. Since $\delta^{-}_{p_0,q}$ is a deck transformation of the cover $\pi_\lambda: \tilde{\mathcal{I}} \to \mathcal{I}_{A(v_-)}$, we have $\pi_\lambda = \pi_\lambda \circ \delta^{-}_{p_0,q}$. Therefore $\tilde{c}_\lambda \circ \pi_\lambda \circ \delta^{-}_{p_0,q} = \tilde{c}_\lambda \circ \pi_\lambda$ as desired.

**Theorem 6.17.** Suppose $(G, \lambda)$ is a symmetric bipartite graph, $\lambda: G \to B_k$ is a partition-symmetric immersion, and $\mathcal{I} = \mathcal{I}_{A}(G, B_k)$ is connected. Then the two-sided cover $\tilde{\mathcal{I}} = \mathcal{I}_{A(v_-)} \times A(v_+)$ of $\mathcal{I}$ fits into the following commutative diagram, where every arrow is a cover. Each horizontal arrow has degree $n = |N|$, and each horizontal arrow in the top two rows is an isomorphism on each component.
Proof. Fix a choice of symbol $p_0 \in N$. The two-sided cover $\tilde{I} = \mathcal{I} \mathcal{A}(v_{-}) \times \mathcal{A}(v_{+})$ was studied in Lemma 6.16. The left half of the commutative diagram in the theorem statement is exactly the commutative diagram of Lemma 6.16. We use the fixed symbol $p_0$ in the construction of $\tilde{I}_\lambda$ and $\tilde{c}_\lambda$.

We may also apply Lemma 6.16 with $\lambda$ replaced by $\tilde{\lambda}$ and $\tilde{\lambda}$ replaced by $\tilde{\rho}$. Then the roles of $\mathcal{A}(v_{+})$ and $\mathcal{A}(v_{-})$ become interchanged, but the definition of the bin $\tilde{I}^{p,q}$ remains the same. We again use the fixed symbol $p_0$ and the transitivity of $\mathcal{A}(v_{-})$ to fix a group element $\tau_{p_0,p} \in \mathcal{A}(v_{-})$ such that $\rho_{p_0,p} = p_0$. We then use the deck transformation $\delta_{p_0,p}^+ = (\tau_{p_0,p}, id)$ to map the bin $\tilde{I}^{p,q}$ to $\tilde{I}^{p_0,q}$. Performing these modifications of Lemma 6.16 gives the right half of the commutative diagram in the theorem statement. 

The top and bottom rows of the diagram of Theorem 6.17 naturally define graphs of spaces with a one-edge defining graph, the central object as the edge space, and the left and right objects the two vertex spaces.

**Theorem 6.18.** Suppose $(G,\lambda)$ is a symmetric bipartite graph, $\lambda: G \to B_k$ is a partition-symmetric immersion, and $\mathcal{I} = \mathcal{I}_\lambda(G, B_k)$ is connected. Then the graph of spaces $\tilde{Y}$ constructed from the top row of the diagram in Theorem 6.17 is a cover of the graph of spaces $Y$ constructed from the bottom row of the diagram. The degree of the cover is at most $(n!)^2$, where $n = |V(G)|/2$.

Furthermore, $\tilde{Y}$ is isomorphic to a product $K_{n,n} \times \tilde{I}^{p_0,p_0}$.

Proof. The commutativity of the diagram of Theorem 6.17 implies that there exists a map $\zeta: \tilde{Y} \to Y$ induced by $\tilde{\lambda}$ and $\tilde{\rho}$ on the two vertex spaces and $\xi$ on the edge space. Furthermore, since $\deg(\tilde{\lambda}) = \deg(\tilde{\lambda}) = n$, it follows that $\deg(\tilde{\lambda}) = \deg(\xi)$. Similarly, $\deg(\tilde{\rho}) = \deg(\xi)$. Since $\tilde{\lambda}, \tilde{\rho}$, and $\xi$ are all covering maps of the same degree, $\zeta$ is a covering map satisfying

$$\deg(\zeta) = \deg(\tilde{\zeta}) = |\mathcal{A}(v_{-}) \times \mathcal{A}(v_{+})| \leq (n!)^2.$$ 

To show that $\tilde{Y}$ is in fact a product, we start by describing $\tilde{Y}$ as a graph of spaces with defining graph $K = K_{n,n}$, the complete bipartite graph on $n = |N|$ vertices. First, we use as vertex set $V(K) = N \times \{-, +\}$ and edge set $E(K) = N \times N$, with the edge $(p, q)$ joining $(p, -)$ to $(q, +)$.

Let $p_0$ be the choice of fixed symbol in the construction of $\tilde{I}_\lambda$ and $\tilde{I}_p$. Decompose $\tilde{Y}$ as a graph of spaces over $K$ as follows. The vertex space over $(p, -)$ is $\tilde{I}^{p_0,p_0}$ and the vertex space over $(q, +)$ is $\tilde{I}^{p_0,q}$. The edge space over $(p, q)$ is $\tilde{I}(p, q)$, with the restriction of $\tilde{c}_\lambda$ and $\tilde{c}_p$ providing the attaching maps to $(p, -)$ and $(q, +)$ respectively. This decomposition is the natural one induced by the partitions of $\tilde{I}_\lambda$, $\tilde{I}_p$ and $\tilde{I}$ into bins, as described in (6.5).

Next, we describe a graph-of-spaces morphism from this decomposition of $\tilde{Y}$ to the natural graph of spaces structure on $K \times \tilde{I}^{p_0,p_0}$. Let $\{\delta^-_{p_0,q}\}_{q \in N}$ be the maps used in the construction of $\tilde{c}_\lambda$ and $\{\delta^p_{p_0,q}\}_{p \in N}$ be the maps used in the construction of $\tilde{I}_p$. We will define $\Phi: \tilde{Y} \to K \times \tilde{I}^{p_0,p_0}$ on vertex and edge spaces. On a $(p, -)$ vertex space, set $\Phi|_{\tilde{I}^{p_0,p_0}} = \delta^p_{p_0,p} = (\tau_{p_0,p}, id)$. On a $(q, +)$ vertex space, set $\Phi|_{\tilde{I}^{q_0,p_0}} = \delta^-_{p_0,q} = (id, \sigma_{p_0,q})$. On a $(p, q)$ edge space, set

$$\Phi|_{\tilde{I}(p,q)} = \delta^-_{p_0,q} \circ \delta^p_{p_0,p} = \delta^p_{p_0,p} \circ \delta^-_{p_0,q} = (\tau_{p_0,p}, \sigma_{p_0,q}).$$

The middle equality follows because the non-trivial coordinates of $\delta^-_{p_0,q}$ and $\delta^p_{p_0,p}$ belong to commuting factors of $\mathcal{A}(v_{-}) \times \mathcal{A}(v_{+})$. For brevity, we will call the composition $\delta^p_{p_0,p} = \delta^-_{p_0,q} \circ \delta^p_{p_0,p}$.

In order to conclude that $\Phi$ is an isomorphism of graphs of spaces, it remains to check that the definition of $\Phi$ on edge spaces is compatible with the attaching maps. That is, we must verify that the following diagram commutes for each $p$ and $q$. The commutativity of the diagram is immediate from the construction.
Therefore there is a graph of spaces isomorphism $\Phi : \tilde{Y} \to K \times \tilde{I}_{p,0 \to p}$ since $K$ and $\tilde{I}_{p,0 \to p}$ are both bipartite, we conclude that as a square complex $\tilde{Y}$ is isomorphic to a product of bipartite graphs. □

7. Illustrative example: The knot $5_2$

In this section, we carry out the construction of the two-sided involutive cover (Definition 6.15) and the product graph of spaces $\tilde{Y} \cong K \times \tilde{I}_{p,0 \to p}$ using the example of a graph $\Sigma$ that forms the double cover of a hyperplane in the Dehn complex of the knot $5_2$. At the end of the construction, we find a cover of $D_\Pi$ that embeds into $\tilde{Y}$, hence a cover that is both A–special and C–special.

7.1. Double cover of the Dehn complex. The Dehn complex $D_\Pi$ of the $5_2$ knot, shown in Figure 2 has five squares. It is reproduced in Figure 6, left. As always, the $VH$ structure is chosen so that the blue edges are horizontal. The vertical hyperplane of $D_\Pi$ is one-sided, as witnessed by the orientation-reversing loop $x - y - z - x$ of length 3 in the central column.

To reach the setting where the constructions of Section 6 begin, we take a double cover $X_\Pi$ of the Dehn complex $D_\Pi$, as depicted in Figure 6, right. Call the left (orange) component of the vertical 1–skeleton $L$ and the right (red) component $R$. Both $L$ and $R$ are isomorphic to $B_4$, and are identified...
via a deck transformation \( \iota \) of the double cover. In Figure 6, \( \iota \) appears as a glide reflection along a vertical axis. Lemma 8.4 describes the construction of this cover in general.

The single vertical hyperplane \( \Sigma \) separates \( X_{\Pi} \), and in this setting it is easy to check that the attaching maps \( \lambda: \Sigma \to L \) and \( \rho: \Sigma \to R \) of the induced vertical graph of spaces satisfy \( \rho = \lambda \circ \iota \) and are partition symmetric with respect to this involution. In general, this property is verified in Lemma 8.7.

7.2. **Involutive completion of the hyperplane graph.** The first step is to build the involutive completion of the vertical hyperplane \( \Sigma \) with respect to the attaching map \( \lambda: \Sigma \to L \cong B_4 \) on the left side (the orange side in Figure 6). The restriction to \( \Sigma \) of \( \iota: X_{\Pi} \to X_{\Pi} \) can be seen in Figure 7 as the rotation by \( \pi \) of the graph diagram. The equation \( \rho = \lambda \circ \iota \) is also visible in this figure.

![Figure 7](image-url)  
**Figure 7.** The vertical hyperplane preimage \( \Sigma \). The left attaching map \( \lambda: \Sigma \to L \cong B_4 \) is indicated by orange (unprimed) edge labels, while the right attaching map \( \rho: \Sigma \to R \cong B_4 \) is indicated by red (primed) edge labels. The two components \( L \sqcup R = V^{(1)}_{X_{\Pi}} \) are pictured on the left and right.

![Figure 8](image-url)  
**Figure 8.** The involutive completion \( \Sigma_{\lambda}(\Sigma, L) \). The map \( \hat{\lambda} \) is indicated by the solid edge color, and the map \( \hat{\rho} \) is indicated by the dotted edge color.
The involutive completion $\mathcal{L} = \mathcal{L}_{\lambda}(\Sigma, B_4)$ is constructed by adding edges joining $y$ and $y'$. Figure 8 shows the completion. The solid edge color shows the image of an edge under $\hat{\lambda}$ and the dotted color the image under $\hat{\rho}$. Consistent with the definition of the involutive completion, the extension $\hat{\iota}$ of the involution of $\Sigma$ sends each added edge to itself.

Observe that since $\hat{\lambda}$ extends $\lambda$ and $\hat{\rho}$ extends $\rho$, the graph of spaces $X_{\Pi}$ embeds into a graph of spaces $Y_{\Pi}$ obtained by attaching $\mathcal{L}$ to $L$ and $R$ along the maps $\hat{\lambda}$ and $\hat{\rho}$, respectively.

7.3. Regular involutive completion. The covering map $\hat{\lambda} : \mathcal{L}_{\lambda}(\Sigma, L) \to L$ depicted in Figure 8 induces a permutation labeling $\alpha$ on the edges of $B_4$, with symbol set $N = \{x, y, z\}$. This labeling is shown in figure 9.

The regular covers $\tilde{\mathcal{L}}_{\lambda}$ and $\tilde{\mathcal{L}}_{\rho}$ derived from $\alpha$ are shown in Figure 10. Indeed, $\tilde{\mathcal{L}}_{\lambda}(\Sigma, B_4)$ and $\tilde{\mathcal{L}}_{\rho}(\Sigma, B_4)$ are isomorphic covers of the band graph $B_4$. In Figure 10 this isomorphism appears as $\pi$-rotation about the center.

Considering $\tilde{\mathcal{L}}_{\lambda}$ and $\tilde{\mathcal{L}}_{\rho}$ separately facilitates the construction of $\mathcal{L}_{A(v-)}$ and $\mathcal{L}_{A(v+)}$. Lemma 6.14 describes canonical isomorphisms $\mathcal{L}_{A(v-)} = \tilde{\mathcal{L}}_{\lambda} \times N$ and $\mathcal{L}_{A(v+)} = \tilde{\mathcal{L}}_{\rho} \times N$. Figure 11 shows a component of each of $\tilde{\mathcal{L}}_{\lambda} \times \{y\} \subset \mathcal{L}_{A(v-)}$ and a component $\tilde{\mathcal{L}}_{\rho} \times \{x\} \subset \mathcal{L}_{A(v+)}$. 
Remark 7.1. In this example, we can see that the covering maps $\xi_y^-$ and $\xi_y^+$ are not isomorphic covers. A covering isomorphism $\phi$ would need to satisfy either $\phi(x, (xy)) = (x, \sigma)$ for $\sigma = ()$ or $\sigma = (yz)$. In either case, since $\phi$ is a graph isomorphism, the $\phi$ image of the yellow edge joining $(x, (xy))$ to $(y', (yz))$ in $\tilde{I}_\lambda \times \{y\}$ must be the green edge in $\tilde{I}_\rho \times \{x\}$ with endpoint $(x, \sigma)$. However, none of the edges in $\tilde{I}$ joining $x$ to $y'$ have the edge pattern “solid yellow, dotted green”.

By Lemma 5.10, every other factorization $\xi_y^+: \tilde{I}_\rho \rightarrow \tilde{I}$ can be obtained by composing $\xi_y^+$ with a deck transformation of $\tilde{I}_\rho$. It follows that the same yellow edge forms an obstruction to an isomorphism between $\xi_y^-$ and $\xi_y^+$ for any $p \in \{x, y, z\}$.

7.4. The two-sided cover of $\tilde{I}$. Theorem 6.17 partitions the two-sided cover $\tilde{I} = \tilde{I}_{A(v_-) \times A(v_+)}$ into bins $\tilde{I}^{p,q}$ for symbols $(p, q) \in N^2$. Figure 12 depicts $\tilde{I}^{x,y}$, which covers $\tilde{I}_\lambda \times \{y\}$ and $\tilde{I}_\rho \times \{x\}$. The covering maps are indicated by vertex label, as the maps to both sides are determined by projecting a coordinate, plus the edge color: solid for the left and dotted for the right.

7.5. Assembling the graph of spaces cover. The last step of the construction of Section 9 is to assemble the graph of spaces $\tilde{Y}$. We start by fixing the symbol $x$. Next, for each symbol, we fix elements of $A(v_-)$ and $A(v_+)$ that send that symbol to $x$. In this example, $A(v_-) = A(v_+) = S_3$, so we use the identity and transpositions $(xy)$ and $(xz)$.

Once these choices are made, the covers $\tilde{L} \rightarrow L$ and $\tilde{R} \rightarrow R$ are assembled from bins of $\tilde{I}$:

$$\tilde{L} = \tilde{I}^{x,x} \cup \tilde{I}^{y,x} \cup \tilde{I}^{z,x}$$

$$\tilde{R} = \tilde{I}^{x,y} \cup \tilde{I}^{x,y} \cup \tilde{I}^{x,z}$$

The edge spaces are attached using $\delta_{x,q}^-$ and $\delta_{x,q}^+$ defined in terms of the chosen permutations. The illustrated bin $\tilde{I}^{x,y}$ is attached to $\tilde{L}$ by the identity and to $\tilde{R}$ by the isomorphism given by the action of $((xy), \text{id})$. The covering projection $\tilde{I}^{y,x} \rightarrow \tilde{L}$ is indicated by the solid color in Figure 12, and the covering projection to $\tilde{R}$ is indicated by the dashed color.

The resulting space $\tilde{Y}$ is isomorphic to $K_{3,3} \times \tilde{I}^{x,x}$. The subspace $\tilde{X} \subset \tilde{Y}$ in Theorem 8.9 that forms a special cover of $\mathcal{D}_\Pi$ can be obtained from $\tilde{Y}$ by removing the pre-images in $\tilde{I}$ of the added edges $\tilde{I} \setminus \Sigma$.
Theorem 8.1. Suppose $\Pi$ is a prime, alternating 2–braid diagram with $c = c(\Pi)$ crossings. Then the Dehn complex $D_{\Pi}$ has a cyclic cover of degree $2c$ that is isomorphic to $B_c \times O_{2c}$. In particular, this cover is $A$–special and $C$–special.

Proof. Our coloring convention $b(\Pi) = r(\Pi)$, combined with Remark 3.4 implies that $b(\Pi) = 2$ and the red regions are bigons, as in Figure 13. Thus $r(\Pi) = c$, hence the red (vertical) 1–skeleton of $D_{\Pi}$ is $V^{(1)}_{D_{\Pi}} \cong B_c$. Since there are exactly two blue regions of $\Pi$, there are exactly two blue (horizontal) edges in $D_{\Pi}$. The vertical hyperplane $\mathfrak{h}$ dual to the blue edges has exactly two vertices and runs
through \( c \) squares, hence \( \mathcal{h} \cong B_c \). In particular, all loops in \( \mathcal{h} \) have even length, hence \( \mathcal{h} \) is two-sided by Proposition 3.8.

By Proposition 3.6, \( \mathcal{D}_\Pi \) is non-positively curved. Since we have checked that \( \mathcal{D}_\Pi \) is vertically non-singular, Proposition 2.7 implies that the horizontal projection \( \pi_H: \mathcal{D}_\Pi \to G_H^\mathcal{D}_\Pi \) is a graph-of-spaces decomposition. The vertex space in this decomposition is the vertical 1-skeleton \( V_{\mathcal{D}_\Pi}^{(1)} \cong B_c \), and the edge space is the vertical hyperplane \( \mathcal{h} \cong B_c \). Still quoting Proposition 2.7, the attaching maps \( f_{\vec{e}}: \mathcal{h} \to V_{\mathcal{D}_\Pi}^{(1)} \) and \( f_{\vec{e}}: \mathcal{h} \to V_{\mathcal{D}_\Pi}^{(1)} \) are combinatorial immersions.

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Consider the monodromy \( \psi: B_c \to B_c \) of this fibration. The monodromy acts as a permutation on the edges of \( B_c \). The orbits of \( \psi \) in the edge set \( E(B_c) \) are exactly the horizontal hyperplanes of \( \mathcal{D}_\Pi \), and we know from Proposition 3.8 that \( \mathcal{D}_\Pi \) has a single horizontal hyperplane. Thus \( \psi \) acts as a \( c \)-cycle on the edges, hence \( \psi^c \) sends every edge to itself.

It might be the case that \( \psi^c \) interchanges \( v^+ \) and \( v^- \) (indeed, this will be the case when \( c \) is odd). But then \( \psi^{2c} \) sends every edge to itself with orientation preserved, hence \( \psi^{2c} = \text{id} \in \text{Aut}(B_c) \). Thus, unrolling the monodromy \( 2c \) times produces a cyclic cover of \( \mathcal{D}_\Pi \) that is isomorphic to the product \( B_c \times O_{2c} \). By Lemma 2.14, this cover is both \( A \)-special and \( C \)-special.

**Remark 8.2.** The bound of \( 2c \) obtained in Theorem 8.1 can be improved. A more fine-grained analysis of self-osculation shows that the smallest cyclic cover of \( \mathcal{D}_\Pi \) that is \( A \)- and \( C \)-special has degree \( \text{lcm}(2, p) \), where \( p \) is the smallest prime divisor of \( c \).

**Remark 8.3.** Recall from Remark 3.7 that a prime, alternating link complement \( S^3 \setminus K \) deformation retracts to \( \mathcal{D}_\Pi \). When \( \Pi = \Pi(K) \) is a 2–braid, we saw in the above proof that \( \mathcal{D}_\Pi \) fibers over \( S^1 \). By a theorem of Stallings [37], this fibration extends to a fibration of \( S^3 \setminus K \) over \( S^1 \), with finite-order monodromy. The cyclic special cover of \( \mathcal{D}_\Pi \) that we constructed in Theorem 8.1 corresponds to a cover of \( S^3 \setminus K \) with the same fiber and with trivial monodromy.

Next, we turn our attention to prime, alternating links that are not 2–braids. By a theorem of Menasco [28], the complement of such a link admits a hyperbolic metric. However, hyperbolic geometry plays no role in our argument. Instead, the main property that we will care about is that the involutive completion of a certain graph map is connected.

The next three lemmas apply to all prime alternating diagrams, including 2–braids.

**Lemma 8.4.** Let \( \Pi \) be a prime alternating link diagram with Dehn complex \( \mathcal{D}_\Pi \). There is a degree two cover \( \eta: X_\Pi \to \mathcal{D}_\Pi \) with the following properties:

1. The vertical 1-skeleton \( V_{\mathcal{D}_\Pi}^{(1)} \) lifts to \( X_\Pi \).
(2) Every square of $X_\Pi$ meets both components of $V_{X_\Pi}^{(1)} = \eta^{-1}(V^{(1)}_{D_\Pi})$.

(3) The preimage $\Sigma = \eta^{-1}(h)$ of the vertical hyperplane $h$ is two-sided, and separates the two components of $\eta^{-1}(V^{(1)}_{D_\Pi})$.

(4) The preimage $\Sigma = \eta^{-1}(h)$ is symmetric bipartite, with the non-trivial deck transformation $\iota$ exchanging the sets of the partition.

The same statements hold in $X_\Pi$ with “vertical” replaced by “horizontal”.

We call $X_\Pi$ the separating-hyperplane cover of $D_\Pi$. See Figure 14 for an example.

**Proof.** Define a homomorphism $\varphi : \pi_1(D_\Pi) \to \mathbb{Z}/2$ as follows. For any loop $\gamma$ in $D_\Pi^{(1)}$, the image $\varphi(\gamma)$ is the count (mod 2) of the vertical edges in $\gamma$. This is well-defined on $\pi_1(D_\Pi)$ because all homotopies consist of pushing $\gamma$ across a square. Observe that $\varphi(\gamma)$ is also the count (mod 2) of the horizontal edges in $\gamma$, because the total number of edges must be even. Let $\eta : X_\Pi \to D_\Pi$ be the double cover corresponding to $\ker(\varphi)$.

Since $D_\Pi^{(1)}$ is bipartite, any loop in the vertical 1–skeleton $V^{(1)}_{D_\Pi}$ must have even length, hence $V^{(1)}_{D_\Pi}$ has two disjoint lifts to $X_\Pi$. Let $V_1, V_2$ be the two components of $V^{(1)}_{X_\Pi} = \eta^{-1}(V^{(1)}_{D_\Pi})$. Similarly, the horizontal 1–skeleton $H^{(1)}_{D_\Pi}$ has two disjoint lifts to $X_\Pi$, and we call the components $H_1, H_2$.

Now, consider a square $S \subset X_\Pi$, and let $\tilde{\gamma}$ be a path in $\partial S$ connecting two opposite corners. Since $\eta(\tilde{\gamma})$ traverses one vertical and one horizontal edge, we know that $\tilde{\gamma}$ is not a closed loop. In other words, $\partial S$ meets all four vertices in $X_\Pi$: the two preimages of $v_-$ and the two preimages of $v_+$. Thus $\partial S$ meets both components of $\eta^{-1}(V^{(1)}_{D_\Pi})$ and both components of $\eta^{-1}(H^{(1)}_{D_\Pi})$. See Figure 14.

**Figure 14.** A schematic of the 1–skeleton of the separating-hyperplane cover $X_\Pi \to D_\Pi$. The vertical 1–skeleton $V^{(1)}_{X_\Pi}$ has components $V_1$ (red) and $V_2$ (cyan). The horizontal 1–skeleton $H^{(1)}_{X_\Pi}$ has components $H_1$ (blue) and $H_2$ (orange). Every square of $X_\Pi$ meets all four vertices and edges from all four of $H_1, V_1, H_2, V_2$.

Next, let $h \subset D_\Pi$ be the vertical hyperplane, and consider the preimage $\Sigma = \eta^{-1}(h)$. Since every square of $X_\Pi$ meets both $V_1$ and $V_2$, we can orient all the horizontal edges of $X_\Pi$ from $V_1$ to $V_2$. This gives a consistent orientation of all edges dual to $\Sigma$, hence $\Sigma$ two-sided. Note that $\Sigma$ separates $V_1$ from $V_2$ in each square, hence it separates them globally in $X_\Pi$.

Finally, consider the vertices of $\Sigma = \eta^{-1}(h)$. Each vertex is dual to either an edge of $H_1$ or $H_2$. Since every square of $X_\Pi$ meets both $H_1$ and $H_2$, each edge of $\Sigma$ joins a vertex dual to an edge of $H_1$ to a vertex dual to an edge of $H_2$. Thus the partition $H_1 \sqcup H_2$ induces a bipartite structure on $\Sigma$. Since the deck transformation $\iota : X_\Pi \to X_\Pi$ exchanges $H_1$ and $H_2$, it exchanges the two vertex types, and $(\Sigma, \iota)$ is symmetric bipartite. \hfill $\Box$

**Remark 8.5.** The double cover $\eta : X_\Pi \to D_\Pi$ has the following meaning in knot-theoretic language. Recall from Definition 3.5 that every red (resp. blue) edge of $D_\Pi$ is dual to a disk in the red (resp. blue) checkerboard surface. Thus, for a loop $\gamma \in \pi_1(S^3 \setminus K)$, the image $\varphi(\gamma) \in \mathbb{Z}/2$ is the count (mod 2) of the number of intersections between $\gamma$ and one of the checkerboard surfaces.

It is well known that $\pi_1(S^3 \setminus K)$ is generated by (conjugates of) meridians of the components of $K$. Each meridian intersects each checkerboard surface once. Thus $\varphi : \pi_1(S^3 \setminus K) \to \mathbb{Z}/2$ can be characterized by the property that every meridian maps to the generator of $\mathbb{Z}/2$. In particular, if $K$ is a knot, hence $H_1(S^3 \setminus K) \cong \mathbb{Z}$, it follows that $X_\Pi$ is the only double cover of $D_\Pi$.

From the point of view of images of meridians, it is straightforward to check that the $2c$–fold cyclic cover of a 2–braid constructed in Theorem 8.1 factors through the double cover $\eta$. 


Restricted to each checkerboard surface \( S \), the double cover \( \eta: \eta^{-1}(S) \to S \) is the orientation double cover. In particular, \( \eta^{-1}(S) \) is connected if and only if \( S \) is non-orientable.

The separating-hyperplane double cover \( X_\Pi \) is non-positively curved by Proposition 3.6 and vertically nonsingular by Lemma 8.4. Thus, by Proposition 2.7, the horizontal projection \( \pi_H: X_\Pi \to G^H_{X_\Pi} \) induces a graph-of-spaces decomposition. Still by Proposition 2.7, the vertex spaces are the two components \( V_1, V_2 \subset V^\Pi_{X_\Pi} \) and the edge spaces are the (one or two) vertical hyperplanes of \( X_\Pi \). Thus \( G^H_{X_\Pi} \) has two vertices and one or two edges. We introduce the following further construction and notation.

**Definition 8.6.** Let \( \eta: X_\Pi \to D_\Pi \) be the separating-hyperplane double cover, and let \( \pi_H: X_\Pi \to G^H_{X_\Pi} \) be the horizontal projection. We take a further quotient from \( G^H_{X_\Pi} \) to \( I = [0, 1] \) by identifying its (one or two) edges to a single edge. The composed map \( \pi'_{H}: X_\Pi \to I \) is called the horizontal one-edge graph of spaces for \( X_\Pi \).

In this one-edge graph of spaces, the vertex spaces \( L = \pi'_{H}^{-1}(0) \) and \( R = \pi'_{H}^{-1}(1) \) are the two connected components of the vertical 1–skeleton \( V^\Pi_{X_\Pi} \). Here, \( L \) stands for “left” and \( R \) stands for “right.” By Lemma 8.4, each of \( L \) and \( R \) is isomorphic to \( V^\Pi_{D_\Pi} \) which is isomorphic to a band graph \( B_k \). We use the covering map \( \eta \) to fix identifications of \( L \) and \( R \) with \( B_k \). The edge space \( \Sigma = \pi'_{H}^{-1}(\frac{1}{2}) = \eta^{-1}(h) \), the preimage of the vertical hyperplane \( h \subset D_\Pi \). The edge space \( \Sigma \) has one or two connected components, according to whether \( h \) is one-sided or two-sided. The attaching maps, denoted \( \lambda: \Sigma \to L \) and \( \rho: \Sigma \to R \), are combinatorial immersions by Proposition 2.7. Since the deck transformation \( \iota \) interchanges \( L \) and \( R \), and we are using the covering map as a fixed identification, we have \( \rho = \lambda \circ \iota \).

An identical construction produces the vertical one-edge graph of spaces with a projection map \( \pi'_{V}: X_\Pi \to I \), where the edge space is the preimage in \( X_\Pi \) of the horizontal hyperplane of \( D_\Pi \).

**Lemma 8.7.** Let \( \Pi \) be a prime alternating diagram, and let \( \eta: X_\Pi \to D_\Pi \) be the separating-hyperplane double cover of \( D_\Pi \). The attaching maps \( \lambda: \Sigma \to L \) and \( \rho: \Sigma \to R \) of the horizontal one-edge graph of spaces are partition symmetric with respect to the involution \( \iota \) induced by the non-trivial covering transformation of \( \eta \).

The same statement holds for the vertical one-edge graph of spaces decomposition.

**Proof.** Let \( \Sigma = \eta^{-1}(\mathcal{h}) \) be the preimage of the vertical hyperplane of \( D_\Pi \). Recall that we have used the covering map \( \eta \) to identify \( L \) and \( R \) with \( V^\Pi_{D_\Pi} \). We need to check that the attaching maps \( \lambda: \Sigma \to L \) and \( \rho: \Sigma \to R \) satisfy the two-part definition of “partition-symmetric” in Definition 6.2. By Lemma 8.4, we know that \( \Sigma \) is symmetric bipartite, with the involution \( \iota \) (coming from the deck transformation of \( \eta \)) exchanging vertices dual to edges of \( H_1 \) and vertices dual to edges of \( H_2 \). Furthermore, if horizontal edges in \( X_\Pi \) are oriented toward \( L \), then edges of \( H_1 \) point toward \( v_- \) while edges of \( H_2 \) point toward \( v_+ \) (or vice versa, exchanging \( H_1 \) with \( H_2 \)). Thus \( \lambda: \Sigma \to L \) is partition–preserving, sending vertices of one type to \( v_- \) and the other type to \( v_+ \).

The second half of Definition 6.2 requires verifying the equation \( \lambda_\Delta(\Delta(v)) = \lambda_\Delta(\Delta(\iota v)) \) for every \( \iota \)–orbit of vertices in \( \Sigma \). Let \( v \in \Sigma \) be an arbitrary vertex, and let \( \bar{e} \) be the edge of \( X_\Pi \) dual to \( v \), oriented toward \( L \). After exchanging \( v \) and \( \iota v \) if necessary, we may assume that that \( \lambda(v) = t(\eta(\bar{e})) = v_+ \).

The set \( \lambda_\Delta(\Delta(v)) \) consists of the (unoriented) lifts to \( L \) of the edges \( \bar{u} \subset V^\Pi_{D_\Pi} \) such that \( t(\eta(\bar{e})) = o(\bar{u}) \) and such that there is a square of \( D_\Pi \) whose attaching map contains \( \eta(\bar{e}) \cdot \bar{u} \). Similarly, \( \lambda_\Delta(\Delta(\iota v)) \) consists of the lifts to \( L \) of edges \( \bar{u} \subset V^\Pi_{D_\Pi} \) such that \( t(\bar{u}) = o(\iota(\bar{e})) = o(\eta(\bar{e})) \) and such that there is a square of \( D_\Pi \) whose attaching map contains the path \( \bar{u} \cdot \eta(\bar{e}) \).

Since \( t(\eta(\bar{e})) = v_+ \) and \( o(\eta(\bar{e})) = v_- \), it follows from Lemma 3.9 that

\[
\lambda_\Delta(\Delta(v)) = \lambda_\Delta(\Delta(\iota v)),
\]

hence \( \lambda \) is partition-symmetric. By an identical argument, \( \rho \) is partition-symmetric. \( \square \)
It turns out that 2–braids can be characterized in terms of properties of the attaching maps.

**Lemma 8.8.** Suppose $\Pi$ is a prime, alternating diagram. Let $\eta: X_\Pi \to D_\Pi$ be the separating-hyperplane cover. The following are equivalent:

1. One of the attaching maps of the vertical or horizontal one-edge graph of spaces for $X_\Pi$ is a covering map.
2. The diagram $\Pi$ has exactly two blue regions or exactly two red regions.
3. The diagram $\Pi$ is a closed 2–braid.

**Proof.** We begin with a useful count. Let $r = r(\Pi) = |E(V^{(1)}_{D_\Pi})|$ be the number of red faces of $\Pi$ and $b = b(\Pi) = |E(H^{(1)}_{D_\Pi})|$ be the number of blue faces of $\Pi$. The link diagram $\Pi$ induces a cell decomposition of $S^2$ (recall Remark [3.4]. This cell decomposition has $c(\Pi)$ vertices and $2c(\Pi)$ edges, because every vertex (crossing) is 4–valent. Since $\chi(S^2) = 2$, we conclude that

\[
(8.1) \quad b + r = c(\Pi) + 2.
\]

This equation holds regardless of whether $\Pi$ is a 2–braid.

$[\Pi \Rightarrow \Sigma]:$ For concreteness, we fix our attention on the left attaching map $\lambda: \Sigma \to L$ in the horizontal one-edge graph of spaces decomposition of $X_\Pi$. (The cases of $\rho$ and of the vertical decomposition are identical.) As above, we have the identifications $L = V^{(1)}_\Pi$ and $\Sigma = \eta^{-1}(h)$, the graph dual to the horizontal (blue) 1-skeleton of $X_\Pi$. For each edge $e \subset L$, the number of preimages $|\lambda^{-1}(e)|$ equals the number of squares in $D_\Pi$ adjacent to $e$, or equivalently the number of crossings adjacent to the red face of $\Pi$ corresponding to $e$. Since $\lambda: \Sigma \to L$ is a covering map by hypothesis, every red face of the diagram $\Pi$ meets the same number of crossings, namely $n = \deg(\lambda)$. Every crossing of $\Pi$ is adjacent to exactly two red faces, hence

\[
(8.2) \quad nr = 2c(\Pi).
\]

For a vertex $v \in L$, the preimage $\lambda^{-1}(v)$ consists of the vertices of $\Sigma$ dual to one component of the horizontal (blue) subgraph of $X^{(1)}_\Pi$, hence $n = \deg \lambda = |\lambda^{-1}(v)| = b$. Thus we obtain

\[
(b - 2)(r - 2) - 2 + 2r + 4 = 2c(\Pi) - 2(c(\Pi) + 2) + 4 = 0,
\]

where the second equality uses (8.1) and (8.2). Thus $b = 2$ or $r = 2$.

The equivalence $[2 \Leftrightarrow 3]$ is well-known. See, for example, Futer [17, Lemma 4]; the graph $G_A$ referenced in that lemma is the blue checkerboard graph.

We leave the final implication $[3 \Rightarrow 1]$ as an exercise, as it will not be needed below. Observe that a closely related statement was already established in the proof of Theorem [S.1].

For prime, alternating links that are not 2–braids, the next theorem provides the special cover of $D_\Pi$. The graph $K_{b,b}$ mentioned in the theorem is the complete bipartite graph with $2b$ vertices.

**Theorem 8.9.** Let $\Pi$ be a prime alternating diagram that is not a 2–braid. Let $\eta: X_\Pi \to D_\Pi$ be the separating-hyperplane double cover. Let $b = |E(H^{(1)}_{D_\Pi})|$ be the number of blue regions in $\Pi$ and $r = |E(V^{(1)}_{D_\Pi})|$ be the number of red regions in $\Pi$.

Then $X_\Pi$ embeds into a VH–complex $Y_\Pi$ that has a finite cover $\tilde{Y} \cong K_{b,b} \times A$, where $A$ is an $r$–regular bipartite graph. The covering map $\zeta: \tilde{Y} \to Y_\Pi$ restricts to a cover $\zeta: \tilde{X} \to X_\Pi$, of degree at most $(b!)^2$. Furthermore, for every midpoint $m$ of an edge of $K_{b,b}$, the intersection $(m \times A) \cap \tilde{X}$ is a regular cover of the vertical hyperplane preimage $\Sigma = \eta^{-1}(h) \subset X_\Pi$.

**Proof.** Let $\lambda: \Sigma \to L$ and $\rho: \Sigma \to R$ be the attaching maps of the one-edge horizontal graph of spaces for $X_\Pi$. As above, we use the covering map $\eta: X_\Pi \to D_\Pi$ to fix an identification of $L$ and $R$ with $V^{(1)}_{D_\Pi} \cong B_r$. 

By Lemma 8.7, λ and ρ are partition-symmetric with respect to the involution ι, where λ ◦ ι = ρ. Since Π is not a 2-braid, Lemma 8.8 implies that λ and ρ are not covering maps. Since Σ = η^{-1}(h) for the vertical hyperplane h ⊂ D_Π, it has at most two connected components. If Σ is disconnected, the covering transformation ι exchanges the components. Therefore, by Lemma 6.6, the involutive completion I_λ(Σ, B_r) is connected. By Lemma 6.7, we have I_λ(Σ, B_r) = I_ρ(Σ, B_r), so we will write I = I_λ(Σ, B_r) = I_ρ(Σ, B_r).

Let Y_Π be a νH–complex constructed as a graph of spaces with one edge, in the following fashion. The two vertex spaces are L and R (left and right). The single edge space is I, with left attaching map λ and right attaching map ρ = λ ◦ ι. Since Π embeds into I so that λ extends λ and ρ extends ρ (compare Lemmas 6.5 and 6.7), we obtain an embedding X_Π → Y_Π, preserving the νH–structures. Furthermore, the horizontal projection π_H : Y_Π → [0,1] coming from the graph-of-spaces structure is an extension of π' : X_Π → [0,1].

Let N be the set of ι–orbits of vertices in Σ. Since every vertex of Σ = η^{-1}(h) is dual to a horizontal (blue) edge of X_Π, we may also think of N as the set of blue regions of the diagram Π. In particular, |N| = b. By Lemma 6.9, the regular covers λ : I → B_r and ρ : I → B_r induce the same permutation labeling α : E^+(B_r) → S_N. Let A(v_-) and A(v_+) be the local groups of α at the two vertices of B_r.

Next, we use Definition 6.15 to construct the two-sided cover ˜I = I_{A(v_-) × A(v_+)} of I, with deck group A(v_-) × A(v_+) ⊂ S_N × S_N. By Theorem 6.17 there are corresponding covers ˜L = ˜I_λ of L = B_r and ˜R = ˜I_ρ of R = B_r, satisfying the commutative diagram shown in that theorem. By Theorem 6.18, the graph of spaces constructed using the collapse maps ˜c_λ : ˜I → ˜L and ˜c_ρ : ˜I → ˜R is a νH–complex ˜Y ≅ K_{b,b} × ˜I_{ρ0,p0}, where ˜I_{ρ0,p0} is a single bin corresponding to a symbol p_0 ∈ N. Note that ˜I_{ρ0,p0} is a regular cover of I, and in particular a bipartite r–regular graph.

Still by Theorem 6.18, the attaching maps ˜c_λ : ˜I → ˜L and ˜c_ρ : ˜I → ˜L cover λ : I → L and ρ : I → R. Thus we get a covering map ζ : ˜Y → Y_Π whose degree is |A(v_-) × A(v_+)| ≤ (b!)^2.

Theorem 6.18 does not guarantee that ˜Y is connected. However, every component of ˜Y has the form K_{b,b} × A, where A is a component of ˜I_{ρ0,p0} and a regular cover of I. We now replace ˜Y by one of its components, with a restricted covering map ζ : ˜Y → Y_Π.

Define ˜X = ζ^{-1}(X_Π) ⊂ ˜Y. Then ζ restricts to a cover ˜X → X_Π. Furthermore, for every midpoint m of an edge of K_{b,b}, the intersection (m × A) ⊂ ˜X is the preimage of Σ in a regular cover of ˜I, which is a regular cover of Σ.

**Theorem 8.10.** Let Π be a prime, alternating link diagram with c = c(Π) crossings. Then the Dehn complex D_Π has an A–special and C–special cover ˜X whose degree satisfies

\[ \deg(\tilde{X} \to D_\Pi) \leq 2(\lfloor \frac{c}{2} + 1 \rfloor)! \leq 12(c - 1)!. \]

**Proof.** Recall the standing assumption (Remark 3.3) that c ≥ 2. (If c(Π) = 0, then D_Π has no 2–cells, and is already special. If c(Π) = 1, then D_Π is a Möbius band built out of one square, and the double cover X_Π is special.)

Let b and r be the number of blue and red regions of Π, respectively. By Remark 3.4 we have b ≥ 2 and r ≥ 2. Recall that we have chosen colors so that b ≤ r.

Now, we consider two cases. If Π is a 2–braid, then by Theorem 8.1, D_Π has an A–special and C–special cover of degree at most 2c < 2(\lfloor \frac{c}{2} + 1 \rfloor)!^2.

Otherwise, Π is not a 2–braid. By Theorem 8.9, D_Π has an A–special and C–special cover of degree at most 2(b!)^2. Now, Equation (8.1) gives

\[ \frac{b}{2} + r = \frac{c + 2}{2}, \quad \text{hence} \quad b \leq \left\lfloor \frac{c}{2} + 1 \right\rfloor. \]

Thus \( \deg(\tilde{X} \to D_\Pi) \leq 2(\lfloor \frac{c}{2} + 1 \rfloor)!^2 \).
It remains to check the inequality \(2(\left\lfloor \frac{c}{2} + 1 \right\rfloor)!^2 \leq 12(c - 1)!\) for all \(c \geq 1\). This is checked by direct computation for \(c \in \{1, \ldots, 15\}\); equality holds for \(c = 4\). For larger \(c\), we use the standard estimate
\[
(8.3) \quad n \log n - (n - 1) = \int_1^n \log x \, dx \leq \log(n!) \leq \int_1^n \log(x + 1) \, dx = (n + 1) \log(n + 1) - n,
\]
to obtain
\[
\log \left( 2(\left\lfloor \frac{c}{2} + 1 \right\rfloor)!^2 \right) = \log 2 + 2 \log((\frac{c}{2} + 1)!) \\
\leq \log 2 + 2 \left( (\frac{c}{2} + 2) \log (\frac{c}{2} + 2) - (\frac{c}{2} + 1) \right) \\
= (c + 4) \log(c + 4) - (c + 3) \log 2 - (c + 2).
\]
Similarly,
\[
(8.4) \quad \log(12(c - 1)!) \geq 12(c - 1) \log(c - 1) - (c - 2).
\]
Subtracting (8.4) from (8.5) gives
\[
\log \left( \frac{12(c - 1)!}{2(\left\lfloor \frac{c}{2} + 1 \right\rfloor)!^2} \right) \geq (c - 1) \log(c - 1) - (c + 4) \log(c + 4) + (c + 3) \log 2 + \log 12 + 4.
\]
A derivative computation confirms that the function of \(c\) on the right-hand side is increasing for \(c \geq 6\) and positive for \(c \geq 16\). Thus \(2(\left\lfloor \frac{c}{2} + 1 \right\rfloor)!^2 \leq 12(c - 1)!\) for all \(c\).

Remark 8.11. Using (8.3) and (8.4), one can similarly prove that \(2(\left\lfloor \frac{c}{2} + 1 \right\rfloor)! \leq c!\) for all \(c \geq 7\). A similar computation also shows that for any fixed \(k\), we have \(2(\left\lfloor \frac{c}{2} + 1 \right\rfloor)! \leq (c - k)!\) when \(c\) is sufficiently large.

Observe that Theorem 8.10 combined with Theorem 2.12 immediately implies Theorem 1.1. We now proceed to the corollaries.

Proof of Corollary 1.2. Let \(\tilde{X}\) be the C–special cover of \(D\) constructed in Theorem 8.10. Let \(\Gamma\) be the graph whose vertices are the hyperplanes of \(\tilde{X}\) and whose edges correspond to intersecting hyperplanes. Since \(\deg(\tilde{X} \to D) \leq 12(c - 1)!\), and \(D_{\Pi}\) has exactly 2 hyperplanes, we have \(|V(\Gamma)| \leq 24(c - 1)!\).

By Theorem 2.12, \(\pi_1(\tilde{X})\) embeds into the Coxeter group \(C(\Gamma)\) with \(|V(\Gamma)|\) generators. By a theorem of Bourbaki [6, Chapitre V, §4, Section 4], a right-angled Coxeter group on \(n\) generators embeds into \(SL(n, \mathbb{Z})\). Now, since a subgroup of \(\pi_1(D_{\Pi})\) of index at most \(12(c - 1)!\) embeds into \(SL(24(c - 1)!\), \(\mathbb{Z})\), it follows that \(\pi_1(D_{\Pi})\) embeds into \(SL(m, \mathbb{Z})\), where \(m \leq 288((c - 1)!)^2\).

Proof of Corollary 1.3. Recall the hypothesis that \(c = c(\Pi) \geq 3\). Now, we consider two cases.

If \(\Pi\) is a 2–braid, then by Theorem 8.1, \(D_{\Pi}\) has a cover \(\tilde{X}\) of degree \(2c\), which is isomorphic to \(B_c \times O_{2c}\). Note that \(\chi(B_c) = 2 - c \leq -1\). Taking an additional 3–fold cover \(B \to B_c\), we obtain a cover \(\tilde{X} \to X\), where \(\tilde{X} \cong B \times O_{2c}\) and \(\chi(B) \leq -3\). Note that \(\deg(\tilde{X} \to X) = 6c < 12(c - 1)!\).

If \(\Pi\) is not a 2–braid, let \(b\) and \(r\) be the number of blue and red regions of \(\Pi\), respectively. Recall the convention that \(b \leq r\). Since \(\Pi\) is not a 2–braid, Lemma 8.8 implies \(b \geq 3\). By Theorem 8.10, \(D_{\Pi}\) has a special cover \(\tilde{X}\) of degree at most \(2(b)^2 \leq 12(c - 1)!\). Furthermore, Theorem 8.9 also says that \(\tilde{X} \cong Y \cong K_{b,b} \times A\), where the projection \(\tilde{X} \to K_{b,b}\) is onto. Observe that \(\chi(K_{b,b}) = 2b - b^2 \leq -3\).

Thus, in both cases, we have a cover \(\tilde{X}\) of degree at most \(12(c - 1)!\), and a \(\pi_1\)–surjective projection \(\tilde{X} \to B\) to a graph \(B\) with \(\chi(B) \leq -3\). Thus we get a surjection \(\varphi: \pi_1(\tilde{X}) \to F_4\).

For every \(n \geq 1\), the free group \(F_{3n+1}\) embeds in \(F_4\) as a subgroup of index \((n - 1)\). Then \(\varphi^{-1}(F_n)\) is a subgroup of \(\pi_1(D_{\Pi})\) of index at most \(12n(c - 1)!\) with a surjection to \(F_{3n+1}\). Thus the cover \(M_n \to S^3 \setminus K\) corresponding to this subgroup has Betti number \(b_1(M_n) \geq 3n + 1\).

Remark 8.12. Antolín and Minasyan [5] proved that every subgroup of a RAAG is either free abelian or surjects \(F_2\). Since the fundamental group of an alternating link complement with crossing number \(c(K) \geq 3\) is not virtually abelian, their theorem says that the A–special cover \(\tilde{X}\) has the
property that \( \pi_1(\tilde{X}) \to F_2 \). The above argument finds a surjection \( \pi_1(\tilde{X}) \to F_4 \) directly from the structure of \( \tilde{X} \) as a \( VH \)-complex.

**Proof of Corollary 1.4.** Let \( \Pi(K) \) be a prime, alternating diagram. Let \( \sigma \subset S^3 \setminus K \) be a closed curve that intersects the checkerboard surfaces of \( \Pi \) a total of \( n \) times. Thus, after a homotopy, \( \sigma \) can be taken to lie in the \( 1 \)-skeleton \( \mathcal{D}_{\Pi}^{(1)} \) as a loop of length \( n \). Compare Remark 8.5.

Let \( \tilde{X} \to \mathcal{D}_{\Pi} \) be the \( A \)-special cover constructed in Theorem 8.10 of degree at most \( 12(c-1)! \). Let \( \Gamma \) be the hyperplane graph of \( \tilde{X} \), whose vertices are hyperplanes and whose edges correspond to hyperplanes that cross.

If \( \sigma \) fails to lift to \( \tilde{X} \), we are done. Otherwise, recall that a theorem of Haglund and Wise [21, Theorem 4.2] constructs a local isometry \( \varphi: \tilde{X} \to S_F \), where \( S_F \) is the Salvetti complex of the Artin group \( A(\Gamma) \). (In the case at hand, where \( \Gamma \) has no triangles, \( S_F \) is simply the presentation complex corresponding to Definition 2.11.) Thus \( \varphi(\sigma) \) is a loop in \( S_F \) of length \( n \).

By a theorem of Bou-Rabee, Hagen, and Patel [7, Theorem 1.1], there is a degree \((n+1)\) cover of \( S_F \) where \( \varphi(\sigma) \) fails to lift. Pulling back this cover via \( \varphi \) gives a cover of \( \mathcal{D}_{\Pi} \) of degree at most \((n+1) \cdot 12(c-1)! \) where \( \sigma \) fails to lift.

\( \square \)

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