Higher discriminants of binary forms

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Abstract

We propose a method for constructing systems of polynomial equations that define submanifolds of degenerate binary forms of an arbitrary degeneracy degree.

Keywords: discriminant, degenerate form, algebraic equation

1 Introduction

This paper is devoted to studying degeneration of symmetric forms. As is known, in the manifold of all symmetric forms on \( \mathbb{C}^n \), degenerate forms constitute a submanifold that is invariant under the action of the structure group \( SL(n) \). This submanifold can be defined by a single equation \( D = 0 \), where \( D \), called the discriminant, is an \( SL(n) \)-invariant polynomial in the form coefficients.

The discriminant submanifold has an interesting internal geometry: forms with different degeneracy degrees produce a set of submanifolds in it, one embedded into another. A natural problem is to construct systems of polynomial equations that define these submanifolds. Such systems of equations allow determining not only whether a given form is degenerate but also how strongly degenerate it is.

The most general approach to solving this problem is described in [1] in terms of representations of the group \( SL(n) \). The simplest case \( n = 2 \) is related to the theory of algebraic equations in one variable. The present paper is devoted to analyzing this case in detail.

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Homogeneous symmetric forms and their degeneracy conditions often appear in physical problems. In particular, a possible application is calculating non-Gaussian generating functionals in field theory exactly (nonperturbatively). In the formalism of continuum (functional) integration, the integrals

$$Z = \int e^{S_{ij} x^i x^j} d^m x = \text{const} \cdot \frac{1}{\sqrt{\det S}},$$

called Gaussian integrals, play an important role. They are well investigated because \(S\) is a quadratic form and its discriminant \(D(S)\) is exactly equal to its determinant \(\det S\). The theory for calculating non-Gaussian integrals (with forms of an arbitrary degree) is developed worse. It is very likely related closely to the theory of discriminants and higher discriminants.

2 Binary forms and their roots

A binary \(k\)-form is a symmetric form of degree \(k\) on the space \(\mathbb{C}^2\), i.e., a homogeneous polynomial in two complex variables

$$P_k(x, y) = a_k x^k + a_{k-1} x^{k-1} y + a_{k-2} x^{k-2} y^2 + \cdots + a_0 y^k.$$

The numbers \(a_i\) are called the coefficients of the form \(P\). Because the fundamental field \(\mathbb{C}\) is algebraically closed, any such polynomial can be decomposed into a product of \(k\) linear factors:

$$P_k(x, y) = (\alpha_1 x + \beta_1 y) (\alpha_2 x + \beta_2 y) \cdots (\alpha_k x + \beta_k y).$$

Consequently, the kernel of the form \(P\) (the set of solutions of the equation \(P_k(x, y) = 0\)) consists of \(k\) one-dimensional subspaces

$$\Lambda_i = \{(x, y) \mid \alpha_i x + \beta_i y = 0\}.$$

These straight lines are called the roots of \(P\). A binary \(k\)-form in general position has exactly \(k\) nonequal roots. Degenerate forms are those with some roots equal to each other.

The roots of a binary form are one-dimensional subspaces in \(\mathbb{C}^2\), i.e., points of the projective space \(\mathbb{C}P^1\). A close relation between binary \(k\)-forms and ordinary \(k\)-th-degree algebraic equations consists in the possibility to select some chart on \(\mathbb{C}P^1\), for example, \(y \neq 0\). Because \(\dim \mathbb{C}P^1 = 1\), each root \(\Lambda_i\) is represented in this chart by one complex number \(\lambda_i\). Moreover, the equation

$$P_k(x, y) = a_k x^k + a_{k-1} x^{k-1} y + \cdots + a_0 y^k = 0$$

becomes

$$P_k(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0 = 0,$$

where \(z = x/y\) is a coordinate on a projective space in the chart \(y \neq 0\). This is a polynomial equation of degree \(k\) in one complex variable, which has exactly \(k\) roots \(\lambda_i\).
Geometrically, the space of all binary $k$-forms is a smooth manifold $S^k \equiv S^k \mathbb{C}^2 \sim \mathbb{C}^{k+1}$, in which degenerate forms constitute projective submanifolds. The main one among them is the discriminant surface $S^2_2$, the set of all $k$-forms with at least two equal roots. It is defined by one polynomial equation

$$D_k = 0,$$

where the polynomial $D_k$ is called the discriminant of $k$-forms. For example, for a quadratic form

$$P_2(x, y) = ax^2 + bxy + cy^2,$$

the discriminant is equal to

$$D_2(a, b, c) = b^2 - 4ac,$$

and for a cubic form

$$P_3(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

the discriminant is

$$D_3(a, b, c, d) = -27a^2d^2 + 18abcd - 4ac^3 - 4db^3 + b^2c^2.$$

Given the discriminant of a binary form, we can answer the question whether the form is degenerate, i.e., whether it has a pair of coincident roots. Our aim here is to construct higher discriminants, which would additionally allow determining how many roots are equal and with what multiplicities.

On the discriminant surface $S^2_2$, there are submanifolds of forms with higher degeneracy degrees. For example, $S^4_2$ is the set of all $k$-forms with three equal roots, and $S^3_4$ is the set of all $k$-forms with two pairs of equal roots.

For a given form, the degeneracy degree is determined by dividing the set of its roots into coincident roots. For any partition

$$k = k_1 + k_2 + \cdots + k_p,$$

where $k_1 \geq k_2 \geq \cdots \geq k_p$, we define $S^k_{k_1 \ldots, k_p}$ as the set of $k$-forms with such type of coincident roots. To simplify calculations, we omit the index $k_j$ in $S_{k_1 \ldots, k_n}$ if it is equal to 1. For example, the partition

$$k = 1 + 1 + \cdots + 1$$

corresponds to the space of all $k$-forms $S^k$, the partition

$$k = 2 + 1 + \cdots + 1$$

corresponds to the space of all degenerate $k$-forms $S^k_2$, and so on.

We must note that a partition $k = k_1 + k_2 + \cdots + k_p$ does not imply that the groups of coincident roots are distinct. For example, $S^4_4$ is a subset of $S^2_4$, because the coincidence of four roots is a particular case of the coincidence of
two pairs of roots. In exactly the same way, $S^k_4$ is a subset of $S^k_3$, and all these submanifolds are contained in $S^k_2$.

The submanifold $S^k_2 \subset S^k$ is defined by one polynomial equation, but submanifolds $S^k_{k_1,...,k_p}$ with $k_1 > 2$ have a lower dimension, and should therefore be defined by systems of several polynomial equations. The problem of describing such systems of equations was posed by Cayley as early as the 19th century, but it still attracts interest and is investigated using the most modern methods [2].

The group of linear transformations $\text{SL}(2, \mathbb{C})$

$$x \to G_{11}x + G_{12}y,$$

$$y \to G_{21}x + G_{22}y,$$

where $G_{11}G_{22} - G_{12}G_{21} = 1$, acts naturally on symmetric forms on $\mathbb{C}^2$. Clearly, any submanifold of degenerate forms $S^k_{k_1,...,k_p}$ just like the discriminant submanifold $S^k_2$, is invariant under this action because the coincidence of a few lines on a plane is independent of the choice of basis in this plane. Consequently, the discriminant of a binary form is an $\text{SL}(2)$-invariant polynomial of its coefficients.

The $\text{SL}$ invariance is an important property of submanifolds of degenerate forms and allows using representation theory to investigate them. This is important because the kernel of a symmetric form is a discrete set of lines only in the particular case of $\mathbb{C}^2$. For $n > 2$, an arbitrary symmetric form no longer decomposes into factors, in other words, there is no discrete set of roots. It becomes impossible to formulate degeneration in terms of coincidence of roots. A general theory of degeneration of symmetric forms should most likely be formulated in group theory terms.

But in the case of binary forms, the kernel is equal to a discrete set of roots, which can coincide with each other with different multiplicities. In this case, we can construct equations defining forms with higher degeneracy degrees using only the notion of roots, the Vieta theorem, and mathematical logic.

3 Discriminant

From now on, a binary form $P(x, y)$ is considered an inhomogeneous polynomial $P(z)$ in one complex variable $z$. The roots of the form correspond to the roots of the algebraic equation $P(z) = 0$.

We recall the main facts concerning algebraic equations of the $k$th degree in one complex variable. Such an equation has the form

$$a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0 = 0$$

and has exactly $k$ solutions $\lambda_1, \lambda_2, \ldots, \lambda_k$, which are called its roots. The roots are related to the coefficients via the Vieta formulas

$$a_{k-i} = a_k (-1)^i \cdot \sigma_i(\lambda_1, \lambda_2, \ldots, \lambda_k),$$
where \( \sigma_i \) denotes the \( i \)th elementary symmetric polynomial of roots,
\[
\sigma_1(\lambda_1, \lambda_2, \ldots, \lambda_k) = \lambda_1 + \lambda_2 + \cdots + \lambda_k,
\]
\[
\sigma_2(\lambda_1, \lambda_2, \ldots, \lambda_k) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \cdots + \lambda_{k-1} \lambda_k,
\]
\[
\vdots
\]
\[
\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_k) = \lambda_1 \lambda_2 \cdots \lambda_k.
\]

The Vieta formulas follow at once from the decomposition
\[
a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0 = a_k (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_k).
\]

Any symmetric polynomial of roots, as is known, can be represented as a polynomial of elementary symmetric polynomials \( \sigma_i \). Consequently, any symmetric polynomial of the roots is actually a function of the coefficients.

A known method for finding the discriminant of a \( k \)th degree equation is to consider the function of the roots
\[
D(\lambda_1, \lambda_2, \ldots, \lambda_k) = \prod_{i<j}(\lambda_i - \lambda_j)^2,
\]
which is a symmetric polynomial of the roots and therefore a function of the coefficients. Moreover, this polynomial is equal to zero if (and only if) some two roots coincide. These properties correspond to the definition of the discriminant. We conclude that the function \( D \) should indeed coincide with the discriminant up to a nonzero factor.

For example, for the quadratic equation \( ax^2 + bx + c = 0 \),
\[
D = (\lambda_1 - \lambda_2)^2 = \lambda_1^2 - 2\lambda_1 \lambda_2 + \lambda_2^2.
\]

Using the Vieta formulas \( b/a = -(\lambda_1 + \lambda_2) \) and \( c/a = \lambda_1 \lambda_2 \), we obtain
\[
D = \lambda_1^2 + 2\lambda_1 \lambda_2 + \lambda_2^2 - 4\lambda_1 \lambda_2 = \left(\frac{b}{a}\right)^2 - 4\frac{c}{a} = \frac{b^2 - 4ac}{a^2}.
\]

Such a method for constructing the discriminant can be generalized to an arbitrary type of root coincidence. It is necessary to write an expression in the roots that vanishes if and only if the roots coincide in the desired way (for example, three roots coincide). Here, it is convenient to use the apparatus of mathematical logic.

### 4 Higher discriminants

We consider a \( k \)-form, its roots \( \lambda_1, \lambda_2, \ldots, \lambda_k \), and the logical algebra \( \Lambda \) generated by all logical statements
\[
E_{ij} : (\lambda_i = \lambda_j)
\]
together with the logical operations AND and OR, respectively denoted by + and ×.

For any type of coincidence $S_{k_1,...,k_p}$, we let $L[S_{k_1,...,k_p}]$ denote the logical expression built from elementary expressions $E_{ij}$, which is called the definition of this type of coincidence. The definition is constructed very naturally for any type of coincidences. For example, the definition of the discriminant submanifold $S_2^k$ is that some pair of roots coincides: either the first root is equal to the second, or the first root is equal to the third, and so on. We thus obtain

$$L[S_2^k] = E_{12} \times E_{13} \times \cdots \times E_{k-1,k} = \times_{i<j} E_{ij}.$$ 

Some simple examples of definitions are

$$L[S_3^3] = E_{12} + E_{13} + E_{23},$$
$$L[S_4^2] = (E_{12} + E_{34}) \times (E_{13} + E_{24}) \times (E_{14} + E_{23}),$$
$$L[S_4^4] = E_{12} + E_{13} + E_{14} + E_{23} + E_{24} + E_{34},$$
$$L[S_4^4] = (E_{12} + E_{13} + E_{23}) \times (E_{12} + E_{14} + E_{24}) \times$$
$$\times (E_{23} + E_{24} + E_{34}) \times (E_{13} + E_{14} + E_{34}).$$

There is a direct relation between the logical expressions and equations: for each statement in $\Lambda$, there exists an equivalent system of equations imposed on roots. For example, an elementary statement $E_{ij}$ corresponds to the equation (squared for symmetry)

$$(\lambda_i - \lambda_j)^2 = 0.$$

The operation OR corresponds to multiplying the operands because a product is equal to zero if one of the factors is zero. The operation AND literally corresponds to taking a system of equations. For example,

$$E_{12} \mapsto (\lambda_1 - \lambda_2)^2 = 0,$$
$$E_{34} \mapsto (\lambda_3 - \lambda_4)^2 = 0,$$
$$E_{12} \times E_{34} \mapsto (\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2 = 0,$$
$$E_{12} + E_{34} \mapsto \begin{cases} (\lambda_1 - \lambda_2)^2 = 0, \\ (\lambda_3 - \lambda_4)^2 = 0. \end{cases}$$

We note that the discriminant case $S_2^k$ corresponds to one equation,

$$L[S_2^k] = E_{12} \times E_{13} \times \cdots \times E_{k-1,k} \mapsto \prod_{i<j}(\lambda_i - \lambda_j)^2 = 0,$$
which was already mentioned above. The coincidence of three roots for a 3-form corresponds to the system of equations

\[
L[S^3_3] = E_{12} + E_{13} + E_{23} \mapsto \begin{cases} 
(\lambda_1 - \lambda_2)^2 = 0, \\
(\lambda_1 - \lambda_3)^2 = 0, \\
(\lambda_2 - \lambda_3)^2 = 0.
\end{cases}
\]

At first glance, the logical expression

\[
L[S^4_{2,2}] = (E_{12} + E_{34}) \times (E_{13} + E_{24}) \times (E_{14} + E_{23})
\]

has no corresponding system of equations. In this and similar cases, we must expand the expression using the ordinary distributive law for \(\times\) and \(+\),

\[
a \times (b + c) = a \times b + a \times c.
\]

The definition of \(L[S^4_{2,2}]\) then becomes

\[
L[S^4_{2,2}] = E_{12} \times E_{13} \times E_{14} + E_{12} \times E_{24} \times E_{14} + E_{12} \times E_{23} \times E_{14} + E_{13} \times E_{24} \times E_{14} + E_{13} \times E_{23} \times E_{14} + E_{34} \times E_{14} + E_{24} \times E_{13} \times E_{23} + E_{34} \times E_{13} \times E_{24} + E_{34} \times E_{23} \times E_{24}.
\]

The corresponding system of equations is

\[
(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 = 0,
\]

\[
(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 = 0,
\]

\[
(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 = 0,
\]

\[
(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2 = 0,
\]

\[
(\lambda_3 - \lambda_4)^2(\lambda_1 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 = 0,
\]

\[
(\lambda_3 - \lambda_4)^2(\lambda_2 - \lambda_4)^2(\lambda_1 - \lambda_4)^2 = 0,
\]

\[
(\lambda_3 - \lambda_4)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 = 0,
\]

\[
(\lambda_3 - \lambda_4)^2(\lambda_2 - \lambda_4)^2(\lambda_2 - \lambda_3)^2 = 0.
\]

For any type of coincidences, the method described above allows constructing a system of equations imposed on the roots that is equivalent to the definition of this type of coincidence, and it consequently defines the space of forms with that type of coincidence.
The final step is to express this system of equations in terms of the coefficients of the form. We suppose that the system of equations obtained for some type of coincidence has the form

\[ P_1(\lambda_1, \lambda_2, \ldots, \lambda_k) = 0, \]
\[ \vdots \]
\[ P_m(\lambda_1, \lambda_2, \ldots, \lambda_k) = 0. \]

Generally, the polynomials \( P_i \) are not symmetric functions of the roots, and they therefore cannot be immediately expressed as functions of the coefficients. This problem is easily solved by symmetrizing this system, i.e., by passing to the equivalent system

\[ P_1 + P_2 + \cdots + P_m = 0, \]
\[ P_1P_2 + P_1P_3 + \cdots + P_mP_{m-1} = 0, \]
\[ \vdots \]
\[ P_1P_2 \cdots P_m = 0. \]

The equations in this system are symmetric in the polynomials \( P_i \). Consequently, they are symmetric in the roots and can be expressed in terms of the coefficients via the Vieta formulas. As a result, we have a solution of the problem: a system of polynomial equations for the coefficients of the form is obtained, defining a submanifold of forms with the given type of degeneracy.

5 The case \( S^3_3 \)

To illustrate the method, we consider conditions for the coincidence of three roots of a cubic binary form

\[ P_3(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \]

and the corresponding set of degenerate forms \( S^3_3 \). The definition in this case is that all three roots are equal and

\[ L[S^3_3] = E_{12} + E_{13} + E_{23}. \]

The corresponding system of equations

\[ (\lambda_1 - \lambda_2)^2 = 0, \]
\[ (\lambda_1 - \lambda_3)^2 = 0, \]
\[ (\lambda_2 - \lambda_3)^2 = 0 \]
after symmetrization becomes

\[(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2 = 0,\]

\[(\lambda_1 - \lambda_3)^2(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 + (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2 = 0,\]

\[(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 = 0\]

and can be expressed in terms of \(a, b, c,\) and \(d\) using the Vieta formulas

\[\frac{b}{a} = -(\lambda_1 + \lambda_2 + \lambda_3),\]

\[\frac{c}{a} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\]

\[\frac{d}{a} = -\lambda_1\lambda_2\lambda_3.\]

We omit the simple calculations leading to the result

\[2(b^2 - 3ac) = 0,\]

\[(b^2 - 3ac)^2 = 0,\]

\[-27a^2d^2 + 18abcd - 4ac^3 - 4db^3 + b^2c^2 = 0.\]

This system of equations is equivalent to the definition of the coincidence of three roots, and it therefore defines the submanifold \(S_3^3 \subset S^3.\) The second equation is dependent on the first and can be omitted.

We must note that this system is not \(SL(2)-\)invariant, although the submanifold \(S_3^3\) defined by this system is invariant. This means that another system, transformed under the \(SL(2)\) action, defines the same manifold. If we let \(G\) denote an arbitrary element of \(SL(2),\) then we obtain infinitely many systems

\[2Z(G) = 0,\]

\[(Z(G))^2 = 0,\]

\[-27a^2d^2 + 18abcd - 4ac^3 - 4db^3 + b^2c^2 = \text{inv} = 0,\]

where \(Z(G)\) is the orbit of \(b^2 - 3ac\) under the group action,

\[Z(G) = G_{11}^2(b^2 - 3ac) + G_{11}G_{12}(bc - 9ad) + G_{12}^2(c^2 - 3bd).\]

Any of these systems define the submanifold \(S_3^3\) of forms with three coincident roots.

An interesting fact is that the set of common zeros of the whole orbit \(Z(G),\) i.e., the set of forms for which the whole orbit \(Z(G)\) vanishes, is exactly \(S_3^3.\) This
allows regarding $Z(G)$ as a higher discriminant for the case of the coincidence of three roots out of three. The whole orbit, of course, is an $SL(2)$-invariant object. From the representation theory standpoint, this orbit realizes an irreducible representation of the $SL(2)$ group in the space of quadratic polynomials of coefficients.

6 The case $S_3^4$

As a second example, we construct a system of equations defining forms of degree four,

$$P_4(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4,$$

which have one root of multiplicity three. The definition of the considered coincidence is

$$L[S_3^4] = (E_{12} + E_{13} + E_{23}) \times (E_{12} + E_{14} + E_{24}) \times \times (E_{23} + E_{24} + E_{34}) \times (E_{13} + E_{14} + E_{34}).$$

Using standard rules of mathematical logic, we can easily verify that this definition is equivalent to the simpler logical statement

$$L' = E_{12} \times E_{34} + E_{13} \times E_{24} + E_{14} \times E_{23},$$

which corresponds to the system of equations

$$(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2 = 0,$$

$$(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_4)^2 = 0,$$

$$(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 = 0.$$

Symmetrizing this system, we obtain

$$(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2 + (\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_4)^2 + (\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 = 0,$$

$$(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_4)^2 + (\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2(\lambda_1 - \lambda_4)^2 \times \times (\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 + (\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_4)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 = 0,$$

$$(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_4)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_4)^2 = 0.$$
This system can be expressed in terms of the coefficients $a$, $b$, $c$, $d$, and $e$ using the Vieta formulas

\[
\begin{align*}
\frac{b}{a} &= -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \\
\frac{c}{a} &= \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4, \\
\frac{d}{a} &= -(\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4), \\
\frac{e}{a} &= \lambda_1 \lambda_2 \lambda_3 \lambda_4.
\end{align*}
\]

This calculation leads to the result

\[
\begin{align*}
2(c^2 - 3bd + 12ae) &= 0, \\
(c^2 - 3bd + 12ae)^2 &= 0, \\
D_4 &= 0.
\end{align*}
\]

The left-hand side of the third equation is equal to the discriminant of a fourth-degree form. As in the preceding example, the second equation is dependent on the first and can be omitted.

An interesting fact is that all equations in this system are invariant under the $SL(2)$ action. We call the invariant polynomial $c^2 - 3bd + 12ae$ the apolara. We find that the higher discriminant that is relevant for this case of the coincidence of three roots out of four is a system of two invariants: the discriminant and the apolara.

7 Conclusion

The proposed method allows obtaining systems of equations that define binary forms with an arbitrary partition of the roots into coinciding roots. The idea of the method is to select a chart on the projective space $\mathbb{C}P^1$ where roots are represented by complex numbers and the form itself is represented by a polynomial in one variable. The required system of equations is then constructed in the space of roots and finally expressed in terms of the coefficients using the symmetric Vieta formulas.

The questions of the correspondence between the orbits of the group $SL(2)$ and the conditions for the degeneracy of the forms and also of the generalization of this correspondence to the case of space dimensions $n > 2$ are still unresolved.

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