STABILITY, CONVERGENCE TO EQUILIBRIUM AND SIMULATION OF NON-LINEAR HAWKES PROCESSES WITH MEMORY KERNELS GIVEN BY THE SUM OF ERLANG KERNELS

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Non-linear Hawkes processes with memory kernels given by the sum of Erlang kernels are considered. It is shown that their stability properties can be studied in terms of an associated class of piecewise deterministic Markov processes, called Markovian cascades of successive memory terms. Explicit conditions implying the positive Harris recurrence of these processes are presented. The proof is based on integration by parts with respect to the jump times. A crucial property is the non-degeneracy of the transition semigroup which is obtained thanks to the invertibility of an associated Vandermonde matrix. For Lipschitz continuous rate functions we also show that these Markovian cascades converge to equilibrium exponentially fast with respect to the Wasserstein distance. Finally, an extension of the classical thinning algorithm is proposed to simulate such Markovian cascades.

1. Introduction. Hawkes processes have regained a lot of interest in the recent years, in particular in econometrics, as good models to account for contagion risk and clustering arrival of events. They have shown to be very useful also in neuroscience due to their capacity of reproducing the typical time dependencies observed in spike trains of neurons as well as the interaction structure of neural nets. Originally introduced by Hawkes (1971) and Hawkes and Oakes (1974) as a model for the appearances of earthquakes, their key feature is the fact that any event is able to trigger future events – for this reason, Hawkes processes are sometimes called “self-exciting point processes”. In their by now classical paper, Brémaud and Massoulié (1996) develop the stability theory of general non-linear Hawkes processes, also in a multivariate frame. Hansen, Reynaud-Bouret and Rivoirard (2015) have put the foundations for the use of Hawkes processes as models of spike trains in neuroscience, see also Chevallier et al. (2015), and recently some effort has been spent to study Hawkes processes in high dimensions, especially focusing on properties such as the propagation of chaos, see Delattre, Fournier and Hoffmann (2016) and Chevallier (2015), see also Ditlevsen and Löcherbach (2017) in a multi-class frame. Finally, we refer to Zhu (2015) for a study of the large deviation properties of non-linear Hawkes processes having Markovian intensity function.

In the present paper Hawkes processes with memory kernels given by the sum of Erlang kernels are considered. It is shown that the longtime behavior and stability properties of these processes can be studied in terms of an associated class of piecewise deterministic Markov processes (PDMPs). More precisely, let $N$ be a counting process on $\mathbb{R}_+$ character-
ized by its intensity process $$(\lambda_t)_{t \geq 0}$$ defined, for each $$t \geq 0$$, through the relation

$$\mathbb{P}(N \text{ has a jump in } [t, t + dt]|\mathcal{F}_t) = \lambda_t dt,$$

where $$\mathcal{F}_t = \sigma(N([u, s]), 0 \leq u < s \leq t)$$ and

$$(1.1) \quad \lambda_t = f \left( \delta + \int_{[0,t]} h(t - s)dN_s \right).$$

Here, $$f : \mathbb{R} \to \mathbb{R}^+$$ is the jump rate function and $$h : \mathbb{R}_+ \to \mathbb{R}$$ is the memory kernel. The parameter $$\delta \in \mathbb{R}$$ is interpreted as an initial input to the jump rate function.

The memory kernel $$h$$ is assumed to be given by the sum of Erlang kernels, that is,

$$h = \sum_{i=1}^{L} h_i,$$

where each function $$h_i$$ is of the form

$$(1.2) \quad h_i(t) = c_i e^{-\alpha_i t \frac{n_i}{n_i}}, t \geq 0,$$

where $$c_i \in \mathbb{R}, \alpha_i > 0$$ and $$n_i \in \mathbb{N}$$.

It is well known that the class of memory kernels having this form is dense in $$L^1(\mathbb{R}_+)$$, see e.g. Kammler (1976). This is the main motivation for the particular choice taken for the memory kernel $$h$$. Moreover, Erlang kernels are widely used in the modeling literature. They depend on three parameters, $$c_i, n_i$$ and $$\alpha_i$$. Here, $$n_i + 1$$ is the order of the delay of the influence of a past event on a future event. It takes its maximum absolute value at $$(n_i + 1)/\alpha_i$$ time units back in time. The mean is $$(n_i + 1)/\alpha_i$$ (if normalizing to a probability density). The higher the order of the delay, the more concentrated is the delay around its mean value, and in the limit of $$n_i \to \infty$$ while keeping $$(n_i + 1)/\alpha_i$$ fixed, the delay converges to a discrete delay. The sign of $$c_i$$ indicates if the influence of past events on future events is inhibitory or excitatory.

The use of Erlang kernels allows to relate the study of the longtime behavior of a Hawkes process having intensity (1.1) to the study of an associated system of PDMPs. More specifically, it is easily shown that the system of stochastic processes $$X_t^{(i,0)} = x_0^{(i,0)} + \int_{[0,t]} h_i(t-s)dN_s, t \geq 0$$, with $$x_0^{(i,0)} \in \mathbb{R}$$ for each $$1 \leq i \leq L$$, can be completed, by introducing $$\sum_{i=1}^{L} n_i$$ auxiliary processes, to a system of stochastic differential equations driven by a Poisson random measure, see (2.2) below. Between successive jumps of $$N$$, the evolution of each $$X_t^{(i,0)}$$, together with its auxiliary processes, is explicitly given. We shall call this class of PDMPs Markovian cascades of successive memory terms.

We prove that these Markovian cascades are recurrent in the sense of Harris under the usual sub-criticality condition

$$(1.3) \quad \|f\|_{Lip} \int_{0}^{\infty} |h(t)|dt < 1.\text{^1}$$

Under (1.3), we are able to construct a Lyapunov function implying that these processes return to a compact set infinitely often, almost surely. Under the additional condition of some minimal ellipticity, that is, some minimal jump activity, we establish, in Theorem 3, if all $$\alpha_i$$ are equal and all $$c_i$$ are of the same sign. In the case of bounded rate functions, we do not need to impose this condition.
a Doeblin like lower bound based on integration by parts with respect to the jump times. A crucial property is the non-degeneracy of the transition semigroup which is obtained thanks to the invertibility of an associated Vandermonde matrix and structure of the flow of the Markovian cascades (see (2.6) below). In the case of Lipschitz continuous rate functions we also show that the Markovian cascades converge to equilibrium exponentially fast with respect to the Wasserstein distance.

The fact that the flow governing the evolution of the Markovian cascades in between successive jump times is explicitly given enables us to introduce an efficient simulation algorithm which allows to sample from \( N \) on \([0, T]\) for any finite time horizon \( T > 0 \) and any fixed parameter \( \delta \in \mathbb{R} \). This method is straightforward to implement, and can be easily extended to multi-dimensional versions.

The paper is organized as follows. In Section 2, we present the model and provide some preliminary remarks. In Section 3, the long-time behavior of the Markovian cascades is investigated. The statement and proof for the case \( L = 1 \) of Theorem 3, establishing the Doeblin lower bound for the Markovian cascades, is also included in this section. In Section 4, a simulation algorithm to simulate simultaneously a Hawkes process with memory kernel given by the sum of Erlang kernels and its Markovian cascade is proposed. In Section 5, numerical examples are presented. Finally, in the Appendix A, we prove Theorem 3 in the general case.

2. Model definition and preliminary remarks. Throughout the article the set \( \mathbb{N} \) denotes the set of non-negative integers, \( \mathbb{N}^* \) the set of positive integers \( \{1, 2, \ldots\} \) and \( \mathcal{B}((0, \infty)) \) (resp. \( \mathcal{B}([a, b]) \), for real numbers \( 0 \leq a \leq b < \infty \) the Borel sigma-algebra on \((0, \infty)\) (resp. on \((a, b)\)).

We work on the following filtered space \((\Omega, \mathcal{F}, \mathbb{F})\). Let \( \Omega \) be the canonical path space of simple point processes, i.e.,

\[
\Omega = \{ w = (t_n)_{n \in \mathbb{N}^*: t_1 > 0, t_n \leq t_{n+1}, t_n < t_{n+1}} : t_n < +\infty, \lim_{n \to +\infty} t_n = +\infty \}.
\]

For each \( w \in \Omega \) and \( n \in \mathbb{N}^* \), we define \( T_n(w) = t_n \). For each \( w \in \Omega \), we associate the canonical point measure \( \mathcal{B}((0, \infty)) \ni A \mapsto N(w)(A) = \sum_{n \in \mathbb{N}^*} \delta_{T_n(w)}(A) \). We shall write for short \( N(A) \) rather than \( N(w)(A) \); when \( A = (0, t] \) for some \( t \geq 0 \), we simply write \( N_t \) to denote \( N((0, t]) \). Finally, we define \( \mathcal{F}_t = \sigma\{N(A) : A \in \mathcal{B}((0, t])\} \) for each \( t \geq 0 \), \( \mathcal{F} = \sigma\{N(A) : A \in \mathcal{B}((0, \infty))\} \) and \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \).

Let \( f : \mathbb{R} \to \mathbb{R}_+ \) and \( h : \mathbb{R}_+ \to \mathbb{R} \) be measurable functions.

Definition 1. A Hawkes process with parameters \((f, h)\) is a probability measure \( P \) on the filtered space \((\Omega, \mathcal{F}, \mathbb{F})\) such that the compensator of \((N_t)_{t \geq 0}\) is given by \((\int_0^t \lambda_s \, ds)_{t \geq 0}\), where \((\lambda_t)_{t \geq 0}\) is the non-negative \( \mathbb{F} \)-progressively measurable process defined for \( t \geq 0 \) by

\[
\lambda_t = f \left( \delta + \int_{[0, t]} h(t - s) \, dN_s \right)
\]

for some fixed \( \delta \in \mathbb{R} \).

The stochastic process \((\lambda_t)_{t \geq 0}\) is called intensity process. The functions \( f : \mathbb{R} \to \mathbb{R}_+ \) and \( h : \mathbb{R}_+ \to \mathbb{R} \) are called jump rate function and memory kernel respectively. The value \( \delta \) is
assumed. We shall work under the following assumptions.

**Assumption 1.** The rate function $f : \mathbb{R} \to \mathbb{R}_+$ is either bounded or Lipschitz continuous with Lipschitz constant $\|f\|_{Lip}$.

**Assumption 2.** The memory kernel $h : \mathbb{R}_+ \to \mathbb{R}$ can be written as sum of Erlang kernels, i.e, for each $t \geq 0$,

$$h(t) = \sum_{i=1}^{L} c_i e^{-\alpha_i t} t^{n_i} / n_i!,$$

where for each $1 \leq i \leq L$, $c_i \in \mathbb{R}$, $\alpha_i > 0$ and $n_i \in \mathbb{N}$.

Under Assumption 2, the intensity process (2.1) can be described by an associated PDMP. Indeed, for each $1 \leq i \leq L$ and $0 \leq k \leq n_i$, let $x_i^{(i,k)}$ be a real number and suppose that $\delta = \sum_{i=1}^{L} \sum_{k=0}^{n_i} x_i^{(i,k)}$. Writing for each $t \geq 0$,

$$X_t^{(i,k)} = x_i^{(i,k)} + \int_{[0,t]} c_i e^{-\alpha_i (t-s)} (t-s)^{(n_i-k)} / (n_i-k)! \, dN_s,$$

we have

$$\lambda_t = f \left( X_{t-}^{(1,0)} + \ldots + X_{t-}^{(L,0)} \right),$$

and one easily deduces that for $t \geq 0$ and $1 \leq i \leq L$,

$$dX_t^{(i,0)} = X_t^{(i,1)} dt - \alpha_i X_t^{(i,0)} dt$$

$$dX_t^{(i,n_i-1)} = X_t^{(i,n_i)} dt - \alpha_i X_t^{(i,n_i-1)} dt$$

$$dX_t^{(i,n_i)} = -\alpha_i X_t^{(i,n_i)} dt + c_i d\xi_t,$$

with initial condition $X_0^{(i,k)} = x_0^{(i,k)}$.

Write $\kappa = L + \sum_{i=1}^{L} n_i$. The associated PDMP is the Markov process $X = (X_t)_{t \geq 0}$ having càdlàg paths and taking values in $\mathbb{R}^\kappa$, defined, for each $t \geq 0$, by

$$X_t = \left( X_t^{(1)}, \ldots, X_t^{(L)} \right) \text{ with } X_t^{(i)} = \left( X_t^{(i,0)}, \ldots, X_t^{(i,n_i)} \right), \ 1 \leq i \leq L.$$

If $L = 1$, that is, $h$ is a pure Erlang kernel, we write for short $X_t = (X_t^{(k)}, 0 \leq k \leq n_1)$. We call the process $X$ *Markovian cascade of successive memory terms*. Its infinitesimal generator $L$ is given for any smooth test function $g : \mathbb{R}^\kappa \to \mathbb{R}$ by

$$Lg(x) = \langle F(x), \nabla g(x) \rangle + f \left( \sum_{i=1}^{L} x_i^{(i,0)} \right) \left( g(x) + \sum_{i=1}^{L} c_i e^{(i,n_i)} - g(x) \right),$$

where $x = (x^{(1)}, \ldots, x^{(L)})$ with $x^{(i)} = (x^{(i,0)}, \ldots, x^{(i,n_i)})$ and $e^{(i,n_i)} \in \mathbb{R}^\kappa$ is the unit vector having entry 1 in the coordinate $(i, n_i)$, and 0 elsewhere. Finally, $F : \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ is the vector
field associated to the system of first-order ODE’s

\[
\begin{aligned}
\frac{d}{dt} x_t^{(i,0)} &= x_t^{(i,1)} - \alpha_i x_t^{(i,0)} \\
\vdots \\
\frac{d}{dt} x_t^{(i,n_i-1)} &= x_t^{(i,n_i)} - \alpha_i x_t^{(i,n_i-1)} \\
\frac{d}{dt} x_t^{(i,n_i)} &= -\alpha_i x_t^{(i,n_i)}, \quad 1 \leq i \leq L,
\end{aligned}
\]

(2.5)

given by \( F(x) = ((F^{(1)}(x), \ldots, F^{(L)}(x))) \), where \( F^{(i)}(x) = (F^{(i,0)}(x), \ldots, F^{(i,n_i)}(x)) \) with

\[ F^{(i,k)}(x) = -\alpha_i x^{(i,k)} + x^{(i,k+1)} \quad \text{for } 0 \leq k < n_i, \quad \text{and } F^{(i,n_i)}(x) = -\alpha_i x^{(i,n_i)}. \]

Notice that jumps introduce discontinuities only in the coordinates \( X_t^{(i,n_i)} \) of \( X_t \). Figure 1 depicts a realization of the joint processes \((N_t, X_t)_{t \geq 0}\) in the case \( L = 1 \).

\[ X_t^{(0)} \]
\[ X_t^{(1)} \]
\[ X_t^{(2)} \]

\[ N_t \]

Fig 1: A finite joint realization of the Markovian cascade \( X = (X_t)_{0 \leq t \leq T} \) (upper panel) and its associated counting process \( N = (N_t)_{0 \leq t \leq T} \) (lower panel) for the choices \( L = 1, n_1 = 2, c_1 = 2, \alpha_1 = 1, T = 20 \) and \( f(x) = x/5+1 \) with initial input \( x_0 = (x_0^{(0)}, x_0^{(1)}, x_0^{(2)}) = (0, 0, 0) \).

The blue (resp. red and black) trajectory corresponds to the realization of \((X_t^{(2)})_{0 \leq t \leq T}\) (resp. \((X_t^{(1)})_{0 \leq t \leq T}\) and \((X_t^{(0)})_{0 \leq t \leq T}\)). Notice that the smaller the index \( 0 \leq k \leq n_1 \) the smoother the correspondent process \((X_0^{(k)})_{0 \leq t \leq T}\).
Hereafter, we write \( \varphi_t(x) = (\varphi_t^{(1)}(x), \ldots, \varphi_t^{(L)}(x)) \) for the unique solution, starting from \( x \in \mathbb{R}^k \), of the system (2.5). It is immediate to check that for each \( 1 \leq i \leq L \) and \( t \geq 0 \),

\[
\varphi_t^{(i)}(x) = \begin{pmatrix}
e^{-\alpha_i t(x^{(i,0)} + tx^{(i,1)} + \cdots + L^{n_i}x^{(i,n_i)})} \\
e^{-\alpha_i tx^{(i,n_i-1)} + tx^{(i,n_i)}} \\
e^{-\alpha_i tx^{(i,n_i)}}
\end{pmatrix}.
\]

(2.6)

Notice that \( \varphi_t^{(i)}(x) \) depends only on the variable \( x^{(i)} \). Given the Markovian cascade of successive memory terms (2.2)-(2.4), one recovers immediately the non-linear Hawkes processes with intensity (2.1) as shows the following proposition. In what follows, for any \( x_0 \in \mathbb{R}^k \), we write \( P_{x_0} \) for the probability measure on \( \mathbb{R}^k \) under which \( X_0 = x_0 \), and denote by \( E_{x_0} \) the expectation taken with respect to \( P_{x_0} \).

**Proposition 1.** Suppose Assumptions 1 and 2. For each \( 1 \leq i \leq L \) and \( 0 \leq k \leq n_i \), let \( x^{(i,k)}_0 \) be a real number and assume that \( \delta = \sum_{i=1}^L \sum_{k=0}^{n_i} x^{(i,k)}_0 \). Let \( X = (X_t)_{t \geq 0} \) be the Markov process having generator (2.4), starting from \( X_0 = x_0 \). Then \( X \) is non-explosive, i.e., \( X \) has \( P_{x_0} \)-almost surely a finite number of jumps on each interval \([s, t], 0 \leq s < t < \infty\).

Finally, introduce \( N_t = \sum_{s \leq t} 1_{\{\Delta X_s \neq 0\}} \) the counting process associated to the jumps of \( X \). Then \( (N_t)_{t \geq 0} \) is a non-linear Hawkes process with intensity (2.1).

**Proof.** We only have to prove the non-explosiveness of the process \( X \). In the case of bounded \( f \), nothing has to be proved. Suppose therefore that \( f \) is Lipschitz continuous and define \( g(x) = f(0)(\sum_{i=1}^L |c_i|) \) for \( x \in \mathbb{R}^k \). Let \( A = \max_{1 \leq i \leq L} \alpha_i \) and \( c = \sum_{i=1}^L |c_i| \). By plugging \( g \) in (2.4), we have

\[
Lg(x) \leq \sum_{i=1}^L \bigg( \sum_{k=0}^{n_i-1} s g(x^{(i,k)})(x^{(i,k+1)} - \alpha_i x^{(i,k)}) - \alpha_i |x^{(i,n_i)}| + |c_i| f \left( \sum_{j=1}^L x^{(j,0)} \right) \bigg)
\]

\[
\leq cf(0) + (c \| f \|_{Lip} + \alpha) \sum_{i=1}^L |x^{(i,0)}| + (1 + A) \sum_{i=1}^L \sum_{k=1}^{n_i} |x^{(i,k)}|
\]

\[
\leq Cg(x),
\]

where \( C = C(A, c, \| f \|_{Lip}) \) and \( sg(y) \) is the sign of \( y \in \mathbb{R} \). In the second inequality above we have used that \( f(y) \leq \| f \|_{Lip} |y| + f(0) \) for any \( y \in \mathbb{R} \). Thus, by applying Dynkin’s formula and then using \( Lg(x) \leq Cg(x) \), one concludes that

\[
E_{x_0}[g(X_t)] = g(x_0) + \int_0^t E_{x_0}[Lg(X_s)] ds \leq g(x_0) + C \int_0^t E_{x_0}[g(X_s)] ds.
\]

Then by Gronwall’s inequality, \( E_{x_0}[g(X_t)] \leq g(x_0)e^{Ct} \). From this last estimate we conclude the proof noticing that

\[
E_{x_0}(N([s, t])) = E_{x_0} \int_s^t f \left( \sum_{i=1}^L X^{(i,0)}_u \right) du \leq \| f \|_{Lip} \int_s^t \sum_{i=1}^L E_{x_0}[X^{(i,0)}_u] du + f(0)(t - s).
\]
Since $\sum_{i=1}^{L} |X_{u}^{(i,0)}| \leq g(X_{u})$, it follows immediately from the inequality above that

$$E_{x_{0}}(N([s, t])) \leq f \left\| \text{Lip} \int_{s}^{t} E_{x_{0}}[g(X_{u})] du + f(0)(t - s) \right\|,$$

$$\leq g(x_{0}) \frac{\|f\|_{\text{Lip}}}{C} (e^{Ct} - e^{Cs}) + f(0)(t - s) < \infty,$$

implying the assertion.

The Markovian representation of Hawkes processes having memory kernels as in Assumption 2 in terms of the PDMP (2.3)-(2.4) has two advantages. The first advantage is that stability properties and the longtime behavior of such Hawkes processes can be studied via the well-established theory of PDMPs. Since it is straightforward to simulate the PDMP (2.3)-(2.4) (see Section 4), one can also simulate Hawkes processes with memory kernels given by sum of Erlang kernels by using this representation. This is the second advantage.

In the next section we discuss stability properties of the associated PDMP (2.3)-(2.4) with random jump heights. A simulation algorithm for this PDMP will be presented in Section 4.

3. Long-time behavior of the associated Markovian cascade with random jump heights. In this section we consider the Markov process $X = (X_{t})_{t \geq 0}$ taking values in $\mathbb{R}^{\kappa}$ with (possibly) random jump heights. Its generator is given for any smooth and bounded function $g : \mathbb{R}^{\kappa} \to \mathbb{R}$ by

$$\mathcal{L}g(x) = \langle F(x), \nabla g(x) \rangle + f \left( \sum_{i=1}^{L} x_{u}^{(i,0)} \right) \int \left( g(x + \sum_{i=1}^{L} c_{i} e_{(i,n_{i})}) - g(x) \right) G(dc_{1}, \ldots, dc_{L}),$$

where $F : \mathbb{R}^{\kappa} \to \mathbb{R}^{\kappa}$ is the vector field associated to the system (2.5) and $G(dc_{1}, \ldots, dc_{L})$ is a probability measure on $\mathbb{R}^{L}$.

**Assumption 3.** The probability measure $G$ on $\mathbb{R}^{L}$ has finite first moments, i.e.,

$$\int \sum_{i=1}^{L} |c_{i}| G(dc_{1}, \ldots, dc_{L}) < \infty.$$

The above process is well-defined under Assumptions 1 and 3, as shows the following proposition.

**Proposition 2.** Assume Assumptions 1 and 3. Let $N = (N_{t})_{t \geq 0}$ be the counting process associated to the jumps of the Markov Process $X = (X_{t})_{t \geq 0}$ having generator given by (3.1), starting from $x_{0} \in \mathbb{R}^{\kappa}$. Then $N$ has $P_{x_{0}}$-almost surely a finite number of jumps on each interval $[s, t], 0 \leq s < t < \infty$.

The proof of this proposition is analogous to the proof of Proposition 1.
3.1. A Foster-Lyapunov condition. We start showing that there exists a compact set \( K \) of \( \mathbb{R}^\kappa \) such that the process \( X = (X_t)_{t \geq 0} \) possessing the generator defined in (3.1) visits \( K \) infinitely often almost surely. Let \( n = \max_{1 \leq i \leq L} n_i \) and \( \alpha = \min_{1 \leq i \leq L} \alpha_i \). In what follows, we write \( 0^\kappa \) to denote the vector in \( \mathbb{R}^\kappa \) having all coordinates equal to 0.

**Proposition 3.** Suppose Assumptions 1 and 3. Let \( \mathcal{L} \) be the generator defined in (3.1) and consider the function \( V : \mathbb{R}^\kappa \mapsto \mathbb{R}_+ \) defined by

\[
V(x) = 1 + \sum_{i=1}^{L} \sum_{k=0}^{n_i} \frac{b(k+1)}{\alpha_i^k} |x^{(i,k)}|,
\]

where \( b : \{0,1,\ldots,n+1\} \to \mathbb{R}_+ \) is a strictly increasing function. If \( f \) is not bounded but only Lipschitz continuous, we suppose moreover that

\[
\|f\|_{\text{Lip}} \left( \int \sum_{i=1}^{L} \frac{1}{\alpha_i^{n_i}} |c_i| G(dc_1,\ldots,dc_L) \right) < \alpha
\]

and choose the function \( b \) so that

\[
\frac{b(n+1)}{b(1)} \|f\|_{\text{Lip}} \left( \int \sum_{i=1}^{L} \frac{1}{\alpha_i^{n_i}} |c_i| G(dc_1,\ldots,dc_L) \right) < \alpha.
\]

Then there exist positive constants \( \lambda, \beta \) and \( R \) such that the following Foster-Lyapunov type drift condition holds

\[
\mathcal{L}V(x) \leq -\lambda V(x) + \beta 1_K(x),
\]

where \( K = \overline{B}_R(0^\kappa) \) is the (closed) ball of center \( 0^\kappa \) and radius \( R \).

**Remark 1.** If \( \alpha_i = \alpha \) for all \( 1 \leq i \leq L \) and \( \text{sg}(c_i) = \text{sg}(c_j) \) (where \( \text{sg}(u) \) is the sign of \( u \in \mathbb{R} \)) for all \( i \neq j \), then condition (3.4) is equivalent to the sub-criticality condition (1.3) required in Theorem 1 of Brémaud and Massoulié (1996). For values of \( \alpha = \min_{1 \leq i \leq L} \alpha_i \geq 1 \), we could have taken the simpler Lyapunov function \( V(x) = 1 + \sum_{i=1}^{L} \sum_{k=0}^{n_i} |x^{(i,k)}| \).

**Proof.** Indeed, one immediately verifies that

\[
\mathcal{L}V(x) = A(x) + B(x),
\]

where

\[
A(x) = \sum_{i=1}^{L} \left( \sum_{k=0}^{n_i-1} \frac{b(k+1)}{\alpha_i^k} (x^{(i,k+1)} - \alpha_i x^{(i,k)}) \text{sg}(x^{(i,k)}) - \frac{b(n_i+1)}{\alpha_i^{n_i-1}} |x^{(i,n_i)}| \right)
\]

and

\[
B(x) = f \left( \sum_{i=1}^{L} x^{(i,0)} \right) \int \sum_{i=1}^{L} \frac{b(n_i+1)}{\alpha_i^{n_i}} \left( |x^{(i,n_i)}| + c_i - |x^{(i,n_i)}| \right) G(dc_1,\ldots,dc_L).
\]
Defining $b_k = \min\{b(k+1) - b(k), 0 \leq k \leq n\}$ and $r = \alpha b_k (b(n+1))^{-1}$, it is also straightforward to check that

$$\tag{3.7} A(x) \leq -b(1) \sum_{i=1}^{L} \alpha_i |x^{(i,0)}| - \alpha \sum_{i=1}^{L} \sum_{k=1}^{n_k} \frac{1}{\alpha^k_i} (b(k+1) - b(k)) |x^{(i,k)}|$$

$$\leq -\alpha \sum_{i=1}^{L} b(1) |x^{(i,0)}| - r \sum_{i=1}^{L} \sum_{k=1}^{n_i} \frac{b(k+1)}{\alpha^k_i} |x^{(i,k)}|.$$

Suppose first that if $f$ is bounded by $f^*$. In this case, one can easily verify that

$$B(x) \leq f^* b(n+1) \int \sum_{i=1}^{L} \alpha_i^{-n_i} |c_i| G(d_{c1}, \ldots, d_{cL}).$$

Since $r < \alpha$, it follows from the above estimates that

$$\mathcal{L}V(x) \leq -rV(x) + p,$$

where $p = r + f^* b(n+1) \int \sum_{i=1}^{L} \alpha_i^{-n_i} |c_i| G(d_{c1}, \ldots, d_{cL}).$

Let $q = 1 + b(1)(1 \wedge \alpha^{-n})R$ and observe that $V(x) \geq q$, for $x \notin K = \overline{B_R}(0^e)$. Thus, taking any $R$ sufficiently large such that $p/q < r$, we deduce that

$$\mathcal{L}V(x) \leq -\left(r - \frac{p}{q}\right)V(x) + p1_K(x),$$

which proves (3.6) for bounded jump rates $f$ with $\lambda = r - \frac{p}{q} > 0$ and $\beta = p > 0$.

Assuming now that $f$ is unbounded and Lipschitz continuous, we have that

$$B(x) \leq \left(\|f\|_{Lip} \sum_{i=1}^{L} |x^{(i,0)}| + f(0)\right) b(n+1) \int \sum_{i=1}^{L} \alpha_i^{-n_i} |c_i| G(d_{c1}, \ldots, d_{cL}),$$

which, together with the first inequality in (3.7), implies that

$$\mathcal{L}V(x) \leq -dV(x) + d + f(0) b(n+1) \int \sum_{i=1}^{L} \alpha_i^{-n_i} |c_i| G(d_{c1}, \ldots, d_{cL}),$$

where $d = \left(\alpha - \|f\|_{Lip} \frac{b(n+1)}{b(1)} \int \sum_{i} \alpha_i^{-n_i} |c_i| G(d_{c1}, \ldots, d_{cL})\right) \wedge r$ is positive thanks to (3.4). Using the inequality above and proceeding as before we establish also the drift condition (3.6) for Lipschitz jump rates. 

As a corollary of Proposition 3, we obtain exponential moments for the return times to the compact set $K$ appearing in (3.6).

**Corollary 1.** Let $K = \overline{B_R}(0^e)$ and $V$ be as in Proposition 3. Write $T_K = \inf\{t > 0 : X_t \in K\}$. Then for all $\eta \leq \lambda$ and $x_0 \in \mathbb{R}^\kappa$,

$$E_{x_0}[e^{\eta T_K}] \leq V(x_0). \tag{3.8}$$

The proof of this corollary is classical, see for instance Theorem 6.1 of Down, Meyn and Tweedie (1995).
3.2. Wasserstein contraction for Lipschitz jump rates. Throughout this section we suppose that the jump rate \( f \) is Lipschitz continuous. In this case, we are able to prove the existence of a Wasserstein spectral gap of the process \( X \), under the sub-criticality condition (3.4).

More precisely, in the sequel, for any \( x \in \mathbb{R}^\kappa \), we will write \( \|x\|_1 = \sum_{i=1}^{L} \sum_{k=0}^{n_i} |x^{(i,k)}| \). Let \( \mu \) and \( \nu \) be two probability measures on \( \mathbb{R}^\kappa \). We call coupling of \( \mu \) and \( \nu \) any probability measure on \( \mathbb{R}^\kappa \times \mathbb{R}^\kappa \) whose marginals are \( \mu \) and \( \nu \), and we denote by \( \Gamma(\mu,\nu) \) the set of all such couplings. The Wasserstein distance between \( \mu \) and \( \nu \) is defined by

\[
W_1(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^\kappa} \int_{\mathbb{R}^\kappa} \|x-y\|_1^\gamma(dx,dy), \gamma \in \Gamma(\mu,\nu) \right\}.
\]

In the following, we write \( (P_t)_{t \geq 0} \) for the transition semigroup of the process \( X \) with generator (3.1). Recall that \( A = \max_{1 \leq i \leq L} \alpha_i \), \( \alpha = \min_{1 \leq i \leq L} \alpha_i \) and \( n = \max_{1 \leq i \leq L} n_i \).

**THEOREM 1.** Suppose \( f \) is Lipschitz continuous, assume condition (3.4) and choose the function \( b \) as in (3.5). Then, for any choice of probability measures \( \mu \) and \( \nu \) on \( \mathcal{B}(\mathbb{R}^\kappa) \),

\[
W_1(\mu P_t, \nu P_t) \leq \kappa e^{-dt} W_1(\mu, \nu),
\]

where

\[
\kappa = \frac{A^n \vee 1}{1 \wedge \alpha^{n+1}} \frac{b(n+1)}{b(1)},
\]

and \( d = \left( \alpha - \|f\|_{\text{Lip}} \frac{b(n+1)}{b(1)} \right) \sum \alpha_i^{-n_i} |c_i| G(d c_1, \ldots, d c_L) \right) \wedge \left( \alpha b_s (b(n+1))^{-1} \right) \), with \( b_s = \min \{ b(k+1) - b(k), 0 \leq k \leq n \} \).

**PROOF.** The assertion follows from a standard Wasserstein coupling. More precisely, denote by \( (X, \tilde{X}) \) the Markov processes taking values in \( \mathbb{R}^\kappa \times \mathbb{R}^\kappa \) having the infinitesimal generator defined for any smooth test function \( \varphi(x, y) : \mathbb{R}^\kappa \times \mathbb{R}^\kappa \to \mathbb{R} \) by

\[
\mathcal{L}_2 \varphi(x, y) = \langle F(x), \nabla_x \varphi(x, y) \rangle + \langle F(y), \nabla_y \varphi(x, y) \rangle + f \left( \sum_i x^{(i,0)} \right) \wedge f \left( \sum_i y^{(i,0)} \right) \int G(d c_1, \ldots, d c_L) \left[ \varphi(x + \sum_i c_i e_{i,n_i}, y + c_i e_{i,n_i}) - \varphi(x, y) \right]
\]

\[
+ \left( f \left( \sum_i x^{(i,0)} \right) - f \left( \sum_i y^{(i,0)} \right) \right) \int G(d c_1, \ldots, d c_L) \left[ \varphi(x + \sum_i e_{i,n_i}, y + c_i e_{i,n_i}) - \varphi(x, y) \right]
\]

\[
+ \left( f \left( \sum_i y^{(i,0)} \right) - f \left( \sum_i x^{(i,0)} \right) \right) \int G(d c_1, \ldots, d c_L) \left[ \varphi(x, y + \sum_i e_{i,n_i}) - \varphi(x, y) \right],
\]

where \( F : \mathbb{R}^\kappa \to \mathbb{R}^\kappa \) is the vector field associated to the system (2.5).

This is the usual coupling which consists of making the two processes jump together as much as possible. Define

\[
H(x, y) = \sum_{i=1}^{L} \sum_{k=0}^{n_i} \frac{b(k+1)}{\alpha_i^k} |x^{(i,k)} - y^{(i,k)}|.
\]
Then an analogous calculus as the one used in the proof of Proposition 3 yields
\[ \mathcal{L}_2 H(x, y) \leq -dH(x, y) \]
implying that
\[ E_{x,y} H(X_t, \tilde{X}_t) \leq H(x, y)e^{-dt}. \]
Observing that
\[ \|x - y\|_1 \leq A_n \lor \frac{1}{b(1)} H(x, y); \quad H(x, y) \leq \frac{b(n + 1)}{1 \land \alpha^{n+1}} \|x - y\|_1 \]
we conclude the proof.

If the process \(X\) has an invariant probability measure, then the above result states the exponential rate of convergence to equilibrium of the process with respect to the Wasserstein distance. In particular, there exists at most one invariant probability measure. In the next section we prove a stronger result, showing that the process is even recurrent in the sense of Harris, implying the existence of a unique invariant measure.

### 3.3. Harris recurrence

In this section, we use the regeneration method based on Nummelin splitting to show that \(X\) is recurrent in the sense of Harris having a unique invariant probability measure \(\pi\). We recall that

**Definition 2.** The process \((X_t)_{t \geq 0}\) is said to be recurrent in the sense of Harris with invariant measure \(\pi\) on \(B(\mathbb{R}^\kappa)\) if whenever \(\pi(A) > 0\), we have, for all \(x \in \mathbb{R}^\kappa\), \(P_x\)-almost surely

\[ \limsup_{t \to \infty} 1_A(X_t) = 1. \]

Harris recurrence of \(X\) implies in particular the uniqueness of the invariant measure \(\pi\) (up to a multiplicative constant). \(X\) is called positive Harris recurrent if \(\pi(\mathbb{R}^\kappa) < \infty\). We have the following

**Theorem 2.** Suppose that \(f\) is bounded or Lipschitz continuous satisfying (3.4). Suppose moreover that \(G(dc_1, \ldots, dc_L) = \prod_{i=1}^L G_i(dc_i)\) for probability measures \(G_i\) on \((\mathbb{R}, B(\mathbb{R}))\) satisfying \(\text{supp} (G_i) \cap \{0\}^c \neq \emptyset\), for all \(1 \leq i \leq L\). Finally, suppose that \(f\) is lower bounded. Then \((X_t)_{t \geq 0}\) is positive Harris recurrent with unique invariant measure \(\pi(dx)\).

The proof of this theorem uses the regeneration technique based on Nummelin splitting. It is well known that it is easier to implement this method in the frame of discrete time Markov chains rather then Markov processes in continuous time – although some effort has been spent to introduce regeneration times in a continuous time framework, see e.g. Löcherbach and Loukianova (2008). Therefore, we start by showing that the sampled chain \((Y_k)_{k \geq 0} = (X_{kT})_{k \geq 0}\), for some fixed \(T > 0\), is positive Harris recurrent.

We recall that the chain \((Y_k)_{k \geq 0}\) is said to be recurrent in the sense of Harris with invariant measure \(\pi\) on \(B(\mathbb{R}^\kappa)\) if whenever \(\pi(A) > 0\), we have, for all \(x \in \mathbb{R}^\kappa\), \(P_x\)-almost surely,

\[ \limsup_{k \to \infty} 1_A(Y_k) = 1. \]

Obviously, Harris recurrence of the chain \((Y_k)_{k \geq 0}\) implies
the Harris recurrence of the process \( X \), and the invariant probability measures of both processes coincide (if they exist).

The rest of this section is devoted to prove that the sampled chain \((Y_k)_{k \geq 0}\) is Harris which follows from the following Doeblin type lower bound. Recall that \((P_t)_{t \geq 0}\) denotes the transition semigroup of the process \( X \), therefore, \( P_T \) is the transition operator of the sampled chain \((Y_k)_{k \geq 0}\).

**Theorem 3.** Assume the assumptions of Theorem 2. For all \( x^* \in \mathbb{R}^n \), there exist \( R > 0 \), an open set \( I \subset \mathbb{R}^n \) and a constant \( \beta \in (0, 1) \), depending on \( I, R, L, n_1, \ldots, n_L, \alpha_1, \ldots, \alpha_L \) and \( f \), such that

\[
P_{LT}(x, dy) \geq \beta 1_C(x) \nu(dy),
\]

where \( C = B_R(x^*) \) is the (open) ball of radius \( R \) centered at \( x^* \), and where \( \nu \) is the uniform probability measure on \( I \).

**Proof.** Part I. \( L = 1 \).

We start by proving the result in the case \( L = 1, c_1 = c, \alpha_1 = \alpha \) and \( n_1 = n \), that is, \( h(t) = ce^{-\alpha t} \). The corresponding Markov process is then given by \( X_t = (X_t(0), \ldots, X_t(n)) \) taking values in \( \mathbb{R}^{n+1} \). Clearly, for all \( A \in B(\mathbb{R}^{n+1}) \),

\[
P_T(x, A) \geq E_x(1_A(X_T), N_T = n + 1).
\]

Recall the definition of the flow in (2.6). On the event \( \{N_T = n + 1\} \), starting from \( X_0 = x \), we first let the flow evolve starting from \( x \) up to some first jump time \( t_1 \). At that jump time we choose an associated jump height \( c_1 \). Then successively choose the following inter-jump waiting times \( t_2, \ldots, t_{n+1} \) under the constraint \( t_1 + \ldots + t_{n+1} < T \) and the associated jump heights \( c_2, \ldots, c_{n+1} \). Write \( s_1 = T - t_1, s_2 = T - (t_1 + t_2), \ldots, s_{n+1} = T - (t_1 + \ldots + t_{n+1}) \).

Conditionally on \( X_0 = x \), the successive choices of \( \underline{c} = (c_1, \ldots, c_{n+1}) \) and \( \underline{s} = (s_1, \ldots, s_{n+1}) \) as above, the position of \( X_T \) is given by

\[
\gamma(x, \underline{c}, \underline{s}) = \varphi_T(x) + c_1 e^{-\alpha s_1} \nu(s_1) + \ldots + c_{n+1} e^{-\alpha s_{n+1}} \nu(s_{n+1}),
\]

where for each \( 1 \leq k \leq n + 1, \)

\[
v(s_k) = \begin{pmatrix}
s_k \\
\frac{s_k}{s_{k-1}} \\
\frac{s_k}{(n-1)!} \\
\vdots \\
s_k \\
1
\end{pmatrix}.
\]

We omitted the dependence on \( T \) of the map \( \gamma(x, \underline{c}, \underline{s}) \) since we keep the value \( T > 0 \) fixed once for all and work with sequences \( \underline{s} \) satisfying the constraints \( 0 < s_{n+1} < s_n < \ldots < s_1 < T \). Finally, in what follows we shall write, for any fixed pair \((x, \underline{c})\),

\[
\gamma(x, \underline{c}) : \underline{s} \mapsto \gamma(x, \underline{c}, \underline{s}).
\]
We will use the jump noise which is created by the \( n+1 \) jumps, i.e., we will use a change of variables on the account of \( s_1, \ldots, s_{n+1} \). Therefore, in what follows we write

\[
\frac{\partial \gamma_{x,c}(s)}{\partial s} = \left[ \frac{\partial \gamma_{x,c}(s)}{\partial s_1}, \ldots, \frac{\partial \gamma_{x,c}(s)}{\partial s_{n+1}} \right]
\]

to denote the Jacobian matrix of the map \( s \mapsto \gamma_{x,c}(s) \). This matrix does not depend on the initial position \( x \) nor on the first jump height \( c_0 \). Indeed, one easily finds that

\[
\frac{\partial \gamma_{x,c}(s)}{\partial s} = \left[ C^{(1)}, \ldots, C^{(n+1)} \right],
\]

where for each \( 1 \leq k \leq n+1 \), \( C^{(k)} \) is a column vector given by

\[
C^{(k)} = c_k e^{-\alpha s_k} \left( \begin{array}{c}
\frac{s_{n-1}^k}{(n-1)!} - \alpha s_{n-1}^k \\
\frac{s_{n-2}^k}{(n-2)!} - \alpha s_{n-2}^k \\
\vdots \\
1 - \alpha s_k \\
-\alpha
\end{array} \right).
\]

As a consequence the determinant of \( \frac{\partial \gamma_{x,c}(s)}{\partial s} \) is given by

\[
\det \left( C^{(1)}, \ldots, C^{(n+1)} \right) = (-1)^{n+1} \alpha \prod_{k=1}^{n+1} c_k e^{-\alpha s_k} \det \left( \tilde{C}^{(1)}, \ldots, \tilde{C}^{(n+1)} \right),
\]

where for each \( 1 \leq k \leq n+1 \), \( \tilde{C}^{(k)} \) is a column vector given by

\[
\tilde{C}^{(k)} = \left( \begin{array}{c}
\alpha \frac{s_{n-1}^k}{(n-1)!} - s_{n-1}^k \\
\alpha \frac{s_{n-2}^k}{(n-2)!} - s_{n-2}^k \\
\vdots \\
\alpha s_k - 1 \\
1
\end{array} \right).
\]

Thus the invertibility of the matrix \( \frac{\partial \gamma_{x,c}(s)}{\partial s} \) follows from the invertibility of the matrix \( J = [\tilde{C}^{(1)}, \ldots, \tilde{C}^{(n+1)}] \). In the sequel, for each \( 1 \leq k \leq n+1 \), let \( r_k \) denote the \( k \)-th row of \( J \). By replacing successively (bottom-up) \( r_i \) by \( \alpha^{-1}(r_i + r_{i-1})(n+1-i)! \), we deduce that \( J \) is equivalent to the Vandermonde matrix

\[
\begin{pmatrix}
\frac{s_{n-1}^1}{(n-1)!} & \frac{s_{n-1}^2}{(n-2)!} & \cdots & \frac{s_{n-1}^{n+1}}{(n+1)!} \\
\frac{s_{n-2}^1}{(n-2)!} & \frac{s_{n-2}^2}{(n-1)!} & \cdots & \frac{s_{n-2}^{n+1}}{n!} \\
\vdots & \vdots & \cdots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix},
\]
which is know to be invertible if and only if $0 < s_{n+1} < s_n < \ldots < s_1$. In conclusion, we have just shown that for any $x \in \mathbb{R}^{n+1}$, any choice of $c$ having non null coordinates, the Jacobian of the map $s \mapsto \gamma_{x,c}(s)$ is invertible at any $s$ such that $0 < s_{n+1} < s_n < \ldots < s_1$.

It will be proved now that this uniform invertibility of the Jacobian matrix of the map $s \mapsto \gamma_{x,c}(s)$ implies inequality (3.11). For that sake, we shall also need the following notation. For each triple $(x,c,s)$, we write $x_0 = x$, $x_1 = \varphi_{T-s_1}(x) + c_1 e_{n+1}$ and then recursively $x_k = \varphi_{s_{k-1} - s_k}(x_{k-1}) + c_k e_{n+1}$ for $2 \leq k \leq n + 1$. The sequence $x_1, \ldots, x_{n+1}$ corresponds to the positions right after successive jumps, starting from the initial location $x \in \mathbb{R}^{n+1}$, induced by the heights $c$ and the inter-jump waiting times $T - s_1, s_1 - s_2, \ldots, s_n - s_{n+1}$ which are determined by $s$.

Introduce now for each $x \in \mathbb{R}^{n+1}$ and $t \geq 0$,

$$e(x,t) = \exp \left\{ - \int_0^t f(\varphi^{(0)}_s(x)) ds \right\}$$

and define for each triple $(x,c,s)$ (here we set $s_0 = T$),

$$q_{x,c}(s) = \left( \prod_{k=0}^{n} f(\varphi^{(0)}_{s_k - s_{k+1}}(x_k)) e(x_k, s_k - s_{k+1}) \right) e(x_{n+1}, s_{n+1}).$$

Since $f$ is bounded away from $0$ and from the definition of $e(\cdot, \cdot)$, we deduce that for any triple $(x^*, c^*, s^*)$ there are neighborhoods $W_{x^*}, U_{x^*}$ and $V_{c^*}$ of $s^*$, $x^*$ and $c^*$ respectively such that

$$\inf_{(x,c,s) \in U_{x^*} \times V_{c^*} \times W_{s^*}} q_{x,c}(s) > 0.$$ 

Let us now fix a triple $(x^*, c^*, s^*)$ such that the matrix $\frac{\partial \gamma_{x^*, c^*}(s^*)}{\partial s}$ is invertible. Recall that by (3.14), the vector $c^*$ must have all coordinates non-null. By Lemma 6.2 of Benaïm et al. (2015), there exist an open neighborhood $J_{x^*, c^*} = B_R(x^*) \times B_R(c^*)$ of the pair $(x^*, c^*)$, an open set $I \subset \mathbb{R}^{n+1}$, and for any pair $(x,c) \in J_{x^*, c^*}$, an open set $W_{x,c}$ such that

$$\hat{\gamma}_{x,c}(s) : \begin{cases} W_{x,c} \rightarrow I \\ \mathbb{R} \rightarrow \gamma_{x,c}(s), \end{cases}$$

is a diffeomorphism, with $W_{x,c} \subset W_{x,c}$ and also

$$\inf_{x,c \in J_{x^*, c^*}} \inf_{s \in W_{x,c}} \left| \det \left( \frac{\partial \gamma_{x,c}(s)}{\partial s} \right)^{-1} \right| > 0.$$ 

Reducing (if necessary) $R$, we may assume also that $J_{x^*, c^*} \subset U_{x^*} \times V_{c^*}$. Thus we have that by (3.17) and (3.18),

$$\inf_{x,c \in J_{x^*, c^*}} \inf_{s \in W_{x,c}} q_{x,c}(s) \left| \det \left( \frac{\partial \gamma_{x,c}(s)}{\partial s} \right)^{-1} \right| > 0.$$ 

\footnote{$e_{n+1}$ denotes the $n+1$-st unit vector in $\mathbb{R}^{n+1}$}
Since $\text{supp}(G) \cap \{0\} \neq \emptyset$ there exists an interval $(a, b)$ such that $0 \not\in (a, b)$ and $G((a, b)) > 0$. Thus, by taking $\xi^* = ((a + b)/2, \ldots, (a + b)/2)$, we have (reducing $R$ again if necessary) that for $1 \leq k \leq n$, $G((\xi^*^{(k)} - R, \xi^*(k) + R)) > 0$ which together with (3.19) implies

$$\beta = \left(\prod_{k=1}^{n} G((\xi^*^{(k)} - R, \xi^*(k) + R))\right) \inf_{x, \xi^* \in \mathbb{R}^{n+1}} \inf_{x, \xi^* \in \mathbb{R}^{n+1}} \inf_{x, \xi^* \in \mathbb{R}^{n+1}} q_x, (\xi^*) \det \left(\frac{\partial \gamma_x (\xi^*)}{\partial \xi^*}\right)^{-1} > 0. \quad (3.20)$$

Finally, we have for any measurable $A \in \mathcal{B}(\mathbb{R}^{n+1})$ and $x \in B_R(x^*)$, using the change of variables $y = \tilde{\gamma}_x (\xi^*)$, 

$$E_x (1_A (X_T, N_T = n + 1) \geq \int_{B_R(x^*)} G (d\xi^*) \int_{\mathbb{R}^{n+1}} q_x, (\xi^*) 1_A (\gamma_x (\xi^*)) ds_1 \ldots ds_n + 1$$

$$\geq \beta \int_{R \cap A} dy_1 \ldots dy_{n+1} = \beta \nu (A), \quad (3.21)$$

where $G (d\xi^*) = G (dc_1) \ldots G (dc_{n+1})$ and $\beta = \beta \nu (I)$, establishing the desired result in case $L = 1$.

The proof of the general case $L > 1$ follows the same strategy and is given in the Appendix. \hfill \square

We are now able to conclude the proof of Theorem 2.

PROOF OF THEOREM 2. 1) By Corollary 1, we know that $X$ comes back to $K$ infinitely often almost surely. Moreover, $\sup_{x \in K} |\varphi_t (x)| \to 0$ as $t \to \infty$, by the explicit form of the flow in (2.6). Therefore, for any $\varepsilon > 0$ there exists $t^*$ such that $\varphi_t (x) \in B_\varepsilon (0)$ for all $t \geq t^*$, for all $x \in K$. Since $f$ is bounded on $\tilde{K} := \{\varphi_t (x) : t \geq 0, x \in K\}$, we have

$$\inf_{x \in K} P_x (T_1 > t^* + 2T) > 0. \quad (4.1)$$

and therefore, using a conditional version of the Borel-Cantelli lemma, $(Y_k)_{k \in \mathbb{N}}$ visits $B_\varepsilon (0)$ infinitely often almost surely.

2) Applying the result of Theorem 3 with $x^* = 0$ and $\varepsilon = R$ and using the standard regeneration technique allows to conclude that $(Y_{\tilde{K}})_{k}$ and therefore $(X_t)_{t}$ are Harris recurrent. \hfill \square

4. Simulation Algorithm. As a consequence of Theorem 1 it follows that any Hawkes process possessing memory kernels given by the sum of Erlang kernels can be represented as the counting process associated to the jumps of its Markovian cascade. Based on this Markovian representation we propose an algorithm (hereafter Algorithm 1) for simulating such Hawkes processes.

In what follows, for any $x \in \mathbb{R}^c$, we shall write $\|x\|_\infty = \max\{\|x^{(i,k)}\|, 1 \leq i \leq L, 0 \leq k \leq n_i\}$. For a practical implementation of our algorithm the remark below will be important. Recall that $n = \max_{1 \leq i \leq L} n_i$ and $\alpha = \min_{1 \leq i \leq L} \alpha_i$. \hfill \square

**LEMMA 1.** For each $x \in \mathbb{R}^c$, let $M(x) = \max\{\|\varphi^{(i,0)}_t (x)\|, 1 \leq i \leq L, t \geq 0\}$ where $\varphi^{(i,0)}_t (x)$ is defined in (2.6). Then

$$M(x) \leq e^{\|x\|_\infty} \left(1 + \left(\frac{n}{\alpha e}\right)^n\right). \quad (4.1)$$
PROOF. Indeed, it follows from (2.6) that for each \( x \in \mathbb{R}^{n+1} \) and \( t \geq 0 \),
\[
\|\varphi_{t}^{(0)}(x)\| \leq \|x\|_{\infty}e^{-\alpha t}(1 + t + \ldots + t^{n}/n!) \leq e\|x\|_{\infty}e^{-\alpha t}(1 + t^{n}) \leq e\|x\|_{\infty}e^{-(\alpha-1)t},
\]
so that if \( \alpha > 1 \), then clearly (4.1) holds. Now, assume \( 0 < \alpha \leq 1 \). Under this assumption, from standard calculus arguments we deduce that \( \arg\max\{e^{-\alpha t}(1 + t^{n}) : t \geq 1\} = n/\alpha \). This fact and the second inequality above imply the bound in (4.1) as stated. \( \square \)

In the sequel, for any rate function \( f \) satisfying Assumption 1 we define the function \( \mathbb{R}^{\kappa} \rightarrow f^{*}(x) \) by
\[
f^{*}(x) = \begin{cases} 
    \max\{f(y) : y \in [0, LM(x)]\}, & \text{if } x \in \mathbb{R}^{\kappa}_{+} \\
    \max\{f(y) : y \in [-LM(x), 0]\}, & \text{if } x \in \mathbb{R}^{\kappa}_{-} \\
    \max\{f(y) : y \in [-LM(x), LM(x)]\}, & \text{else}
\end{cases}
\]
Here, \( L \) is the number of terms in the sum defining the memory kernel \( h \) (recall Assumption 2). It follows immediately from Lemma 1 that the function \( f^{*} \) is well-defined, that is \( f^{*}(x) \) is finite for all \( x \in \mathbb{R}^{\kappa} \). Let \( T_{0} = 0 \) and \((T_{k})_{k \geq 1}\) denote the sequence of jump times of the Markovian cascade \( X \) having generator (2.4). Observe that the non-explosiveness of \( X \) (thanks to Theorem 1) ensures that the sequence \((T_{k})_{k \geq 1}\) is well-defined. Suppose that \( X_{T_{k}} = x \) is given for some \( k \in \mathbb{N} \). Algorithm 1 works as follows. Draw an exponential random variable \( \tau \) with parameter \( f^{*}(x) \) and a uniform random variable \( U \) on \([0, 1]\). If
\[
U \leq f(\sum_{i=1}^{L} \varphi_{T_{k+i}}^{(0)}(x))/f^{*}(x),
\]
then define the next jump time \( T_{k+1} = T_{k} + \tau \). If not, repeat this procedure starting from \( X_{T_{k}+\tau} = \varphi_{\tau}(x) \). Notice that Algorithm 1 is an extension (to our framework) of the classical thinning algorithm for simulating non-homogeneous Poisson processes. Moreover, it provides an exact simulation of the Markovian cascade \( X \) (and consequently of the associated Hawkes process) in the sense that no approximation procedure is required. Its formal definition is given below as a pseudo-code.

Proposition 1 and Lemma 1 ensure that Algorithm 1 is well-defined and works properly. More precisely, we have the following result.

PROPOSITION 4. Assume Assumption 1. For any choice of \( T > 0 \), \( x_{0} \in \mathbb{R}^{\kappa} \), \( L \geq 1 \) and \( n_{i} \in \mathbb{N} \), \( c_{i} \in \mathbb{R} \), \( \alpha_{i} > 0 \) for \( 1 \leq i \leq L \), Algorithm 1 terminates almost surely within finite time. If additionally \( x_{0} \) is given as in Theorem 1, the output of Algorithm 1 follows the distribution of a Hawkes process with intensity (2.1).

It is worth noting that Algorithm 1 does not require the sub-criticality condition (1.3) for non-linear Hawkes processes. Indeed, Algorithm 1 applies for instance for the choice \( L = 1 \), \( n_{1} = 0 \), \( c_{1} = \alpha_{1} \) and \( f(x) = (\mu + x)1_{[0, \infty)}(x) \) with \( \mu \geq 0 \) for which (1.3) does not hold. The only restriction we have to impose is to work with memory kernels which are sum of Erlang kernels, which is a generalization of the approach proposed in Dassios and Zhao (2013). In the next section some numerical examples are presented both for bounded and unbounded Lipschitz jump rates \( f \).

5. Numerical Examples. In this section four numerical examples are given. Specifically, we generate first a sample of the Markovian cascade with \( L = 1 \), for a time window \( T = 100 \), order delay \( n_{1} = 3 \), jump height \( c_{1} = 1 \), decay rate \( \alpha_{1} = 1 \) and jump rate...
Algorithm 1 Simulation algorithm for the Markovian cascade $X$

1: Input: bounded or Lipschitz continuous $f$, constants $\alpha_i > 0$, $c_i \in \mathbb{R}$ and $T > 0$; and a vector of initial conditions $(X_{i}^{(k)}, 1 \leq i \leq L, 0 \leq k \leq n_i) = (x_{i}^{(k)}, 1 \leq i \leq L, 0 \leq k \leq n_i) \in \mathbb{R}^n$.
2: Output: The counting process $(N_t)_{t \in [0,T]}$
3: Initial values: $x \leftarrow (x_{0}^{(i,k)}, 1 \leq i \leq L, 0 \leq k \leq n_i)$, $D \leftarrow 0$ and $N_0 \leftarrow 0$.
4: while $D < T$ do
5:     $f^* \leftarrow f(x)$
6:     draw $\tau \sim \mathcal{E}(f^*)$
7:     if $\tau \leq T - D$ then
8:         draw $U \sim U[0,1]$  
9:         if $U \leq f(\sum_{i=1}^{n} \phi D^{(i,k)}(x))/f^*$ then
10:            $x \leftarrow x + \sum_{i=1}^{L} c_i e_i(x, n_i)$
11:            $N_t \leftarrow N_D$, for $D \leq t < D + \tau$
12:            $N_{D+\tau} \leftarrow N_D + 1$
13:        else
14:            $x \leftarrow \phi D(x)$
15:            $N_t \leftarrow N_D$, for $D \leq t \leq D + \tau$
16:        end if
17:     else
18:         $N_t \leftarrow N_D$, for $D \leq t \leq T$
19:     end if
20:     $D \leftarrow D + \tau$
21: end while
22: Return $(N_t)_{t \in [0,T]}$.

$f(x) = (\mu + x)1_{[0,\infty)}(x)$ with $\mu = 1$. We also simulate a Markovian cascade with random jump heights $c_1$ following a Normal distribution $\mathcal{N}(0,100)$ and $f(x) = (1 + (x/2)^{3/2}) \wedge 30$, keeping all others parameters as in the preceding example. The extension of Algorithm 1 for random jump heights is straightforward. Next, we simulate jointly three Markovian cascades with $L = 1$ possessing rates of decay $\alpha_1 = 0.8, \alpha_1 = 1$ and $\alpha_1 = 1.4$ respectively; in this example $T = 500, n_1 = 3$, the jump heights follow a Normal distribution $\mathcal{N}(0,100)$ and $f(x) = 1 + \sigma/(1 + e^{-\beta(x-\rho)})$ where $\sigma = 20, \beta = 1/3$ and $\rho = 10$. Finally, we simulate a Markovian cascade for the choices $L = 3, n_1 = 1, n_2 = 3, n_3 = 2, \alpha_1 = 1.3, \alpha_2 = 0.8, \alpha_3 = 1, T = 30, f(x) = (2 + \exp(x/10)) \wedge 20$, random jump heights $c_1 = c_2 = c_3$ following a Normal distribution $\mathcal{N}(0,25)$.

The results are presented in Figures 2, 3, 4 and 6 respectively. In order to test if Algorithm 1 works properly, we use the following result.

Proposition 5. Let $X$ be the Markov process whose generator is given by (2.4) with $L = 1, c_1 = 1, \alpha_1 > 0, n_1 \in \mathbb{N}$ and $f(x) = (\mu + x)1_{[0,\infty)}(x)$ with $\mu \geq 0$. For any $t \geq 0$, we write $S_t = \sum_{k=0}^{n_1} X_t^{(k)}$. Then for any $x = (x^{(0)}, \ldots, x^{(n_1)}) \in \mathbb{R}^{n_1+1}$,

\begin{equation}
E[S_t] = \sum_{k=0}^{n_1} x^{(k)} + \left\{ \begin{array}{ll}
\frac{\mu}{1-\alpha} (e^{(1-\alpha)t} - 1), & \text{if } \alpha \neq 1 \\
\mu t & \text{if } \alpha = 1.
\end{array} \right.
\end{equation}

Proof. For $\mathbb{R}^{n_1+1} \ni (y^{(0)}, \ldots, y^{(n_1)}) \mapsto g(y^{(0)}, \ldots, y^{(n_1)}) = y^{(0)} + \ldots + y^{(n_1)}$ one checks that $Lg(y^{(0)}, \ldots, y^{(n_1)}) = \mu + (1-\alpha)g(y^{(0)}, \ldots, y^{(n_1)})$. 

By Dynkin’s formula it follows that for each $t \geq 0$,

$$ E[S_t] = E[g(X_t)] = E[g(X_0)] + \int_0^t E[Lg(X_s)]ds $$

$$ = \sum_{k=0}^{n_1} x^{(k)} + \mu t + (1 - \alpha) \int_0^t E[S_s]ds, $$

from which it is easy to deduce the result by applying Gronwall’s inequality.
Fig 4: A joint realization of three Markovian cascades with $\alpha_1 = 0.8$ (upper panel), $\alpha_1 = 1$ (middle panel) and $\alpha_1 = 1.2$ (lower panel) respectively. Here, $L = 1$, $T = 500$, $n = 3$, the jump heights follow a Normal distribution $\mathcal{N}(0, 100)$ and $f(x) = 1 + \sigma/(1 + e^{-\beta(x-\rho)})$ where $\sigma = 20$, $\beta = 1/3$ and $\rho = 10$. Notice that the smaller the rate of decay $\alpha$ the larger the oscillations of the process $(X_t(0))_{0 \leq t \leq T}$ are.

Fig 5: The graph of $(E[S_t])_{0 \leq t \leq T}$, conditionally on $S_0 = 0$, for the choices $L = 1, n_1 = 3, c_1 = 1, \alpha_1 = 1.2, T = 30$ and $f(x) = (1 + x/5)1_{[0,\infty)}(x)$. The marks * corresponds to the empirical expected value $\hat{S}_t$ of $S_t$ computed at times $t \in \{0,1,\ldots,30\}$ based on 100 simulated samples of $(S_t)_{0 \leq t \leq T}$.
Fig 6: A realization of the process $X = (X_t)_{0 \leq t \leq T}$ for $L = 3$, $n_1 = 1$, $n_2 = 3$, $n_3 = 2$, $\alpha_1 = 1.3$, $\alpha_2 = 0.8$, $\alpha_3 = 1$, $T = 30$, $f(x) = (2 + \exp(x/10)) \wedge 20$, random jump heights $c_1 = c_2 = c_3$ following a Normal distribution $\mathcal{N}(0, 25)$ and initial configuration $X_0 = 0^9$.

In the upper panel it is shown separately the realization of the process $(X_t^{(1)})_{0 \leq t \leq T}$, in the middle panel (resp. lower panel) that of $(X_t^{(2)})_{0 \leq t \leq T}$ (resp. $(X_t^{(3)})_{0 \leq t \leq T}$).

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APPENDIX A: PROOF OF THEOREM 3 FOR GENERAL L

Proof. We now prove Theorem 3 in the case $L > 1$. We write $X_t = (X_t^{(1)}, \ldots, X_t^{(L)})$ and $\varphi_t(x) = (\varphi_t^{(1)}(x), \ldots, \varphi_t^{(L)}(x))$ for the flow given in (2.6). Recall that elements of $\mathbb{R}^c$ are denoted by $x = (x^{(1)}, \ldots, x^{(L)})$, where $x^{(i)} = (x^{(i,0)}, \ldots, x^{(i,n_i)})$ for each $1 \leq i \leq L$.

We prove by induction that for all $1 \leq k \leq L$, for all $x^* \in \mathbb{R}^c$ there exists a neighborhood $C$ of $x^*$ such that for all $x \in C$,

$$
(A.1) \quad P_{xT}(x, dy) \geq \beta_k \nu_k(y^{(1)}, \ldots, y^{(k)})
$$

for open sets $I_l \subset \mathbb{R}^{n_l+1}, 1 \leq l \leq k$, $\nu_k$ is the uniform density on $I_1 \times \ldots \times I_k$, and where $Q_k$ is a transition kernel from $\mathbb{R}^c \times \mathbb{R}^{n_1+\ldots+n_k+k} \to \mathbb{R}^{n_1+\ldots+n_k+L-k}$.

Proof of (A.1) for $k = 1$. We proceed as in part I and use the jump noise produced by $n_1 + 1$ jumps occurring during $[0, T]$ to produce a density for $X^{(1)}$. We impose inter-jump waiting times $t_1, \ldots, t_{n_1+1}$ under the constraint that $t_1 + \ldots + t_{n_1+1} < T$. To each jump time we associate jump heights $c_1, \ldots, c_{n_1+1}$, where each $c_i$ is an element of $\mathbb{R}^L$, that is, $c_i = (c^{(i)}_1, 1 \leq i \leq L), 1 \leq l \leq n_1 + 1$.

In what follows we shall write $\zeta = (c_1, \ldots, c_{n_1+1})$ and $\zeta^{(i)} = (c^{(i)}_1, \ldots, c^{(i)}_{n_1+1})$. Moreover, we define $\kappa = (s_1, \ldots, s_{n_1+1})$, for $s_k = T - t_1 + \ldots + t_k$. We call $\kappa$ admissible if $T > s_1 > \ldots > s_{n_1+1} > 0$.

Then, conditionally on $X_0 = x$ and on the above choices, the position of $X_T$ is given by

$$
(A.2) \quad \gamma(x, \zeta, \kappa) := \varphi_T(x) + \sum_{i=1}^L \left( c^{(i)}_1 e^{-\alpha_i s_1} v_i(s_1) + \ldots + c^{(i)}_{n_1+1} e^{-\alpha_i s_{n_1+1}} v_i(s_{n_1+1}) \right).
$$

Here, $v_i(s) \in \mathbb{R}^c, 1 \leq i \leq L$, is the vector given by $(v_i(s))^{(i,k)} = s^{n_i-k} (n_i-k)!$, for $0 \leq k \leq n_i$, and with zero entries else. We shall write shortly $\gamma^{(i)}(x, \zeta, \kappa) \in \mathbb{R}^{n_i+1}$ for the $i$—the coordinate of $\gamma(x, \zeta, \kappa)$, that is,

$$
\gamma^{(i)}(x, \zeta, \kappa) = (\gamma^{(i)}(x, \zeta, \kappa), 1 \leq i \leq L).
$$

In what follows, we will use the product form of the flow (2.6). By this we mean the fact that by the explicit form of the flow in (2.6), for each $1 \leq i \leq L$, we have that

$$
\varphi_t^{(i,k)}(x) = e^{-\alpha_i t} \sum_{m=0}^{n_i-k} \frac{t^m}{m!} x^{(i,k+m)} =: \varphi_t^{(i,k)}(x^{(i)});
$$

that is $\varphi_t^{(i)}(x) = \varphi_t^{(i)}(x^{(i)})$ does only depend on $x^{(i)}$, for any $1 \leq i \leq L$. As a consequence,

$$
\gamma^{(i)}(x, \zeta, \kappa) = \gamma^{(i)}(x^{(i)}, \zeta^{(i)}, \kappa)
$$

does also depend only on $x^{(i)}$ and on $\zeta^{(i)}$. As usual, we shall write, for any fixed pair $(x^{(i)}, \zeta^{(i)})$,

$$
\gamma^{(i)}_{(x^{(i)}, \zeta^{(i)})} : \kappa \mapsto \gamma^{(i)}(x^{(i)}, \zeta^{(i)}, \kappa).
$$
and similarly,
\[ \gamma_{(x^{(i)}, \mathbf{s})}^{(i)} : x^{(i)} \mapsto \gamma^{(i)}(x^{(i)}, \mathbf{c}^{(i)}, \mathbf{s}). \]

Fix any \( x^* \in \mathbb{R}^s \) and fix \( \mathbf{c}^* \) such that \( \mathbf{c}^{(s,i)}_i = (a_i + b_i)/2 \), for all \( 1 \leq l \leq n_1 + 1 \), where \( (a_i, b_i) \in \mathbb{R} \) such that \( 0 \notin (a_i, b_i) \) and \( G_i((a_i, b_i)) > 0 \), for all \( 1 \leq i \leq L \). Then there exists an open neighborhood \( J_i = B_R(x^{(s,1)}) \times B_R(\mathbf{c}^{(s,1)}) \) of the pair \( (x^{(s,1)}, \mathbf{c}^{(s,1)}) \) with \( R < \min_i(b_i - a_i) \), and an open set \( I \subset \mathbb{R}^{n_1+1} \) and for any pair \( (x^{(1)}, \mathbf{c}^{(1)}) \in J_1 \) an open set \( W_{x^{(1)}, \mathbf{c}^{(1)}} \subset \mathbb{R}_+^{n_1+1} \) such that
\[ \gamma_{x^{(1)}, \mathbf{c}^{(1)}}^{(1)}(\mathbf{s}) : \{ \mathbf{s} \mapsto \gamma^{(1)}(x^{(1)}, \mathbf{c}^{(1)}, \mathbf{s}) \}, \]

is a diffeomorphism. Moreover,
\[ \tilde{\beta}_1 = \inf_{x \in B_R(x^*)} \inf_{\mathbf{c} \in B_R(\mathbf{c}^*)} \inf_{\mathbf{s} \in W_{x, \mathbf{c}}^{(1)}} q_{x, \mathbf{c}}(\mathbf{s}) \left| \det \left( \frac{\partial \gamma_{x^{(1)}, \mathbf{c}^{(1)}}^{(1)}(\mathbf{s})}{\partial \mathbf{s}} \right) \right|^{-1} > 0. \]

Let now \( A_i \in \mathcal{B}(\mathbb{R}^{n_i+1}) \), for all \( 1 \leq i \leq L \). Then for all \( x \in B_R(x^*) \), using the change of variables \( y^{(1)} = \gamma_{x^{(1)}, \mathbf{c}^{(1)}}^{(1)}(\mathbf{s}) \),
\[ P_T(x, A_1 \times \ldots \times A_L) \geq \tilde{\beta}_1 \int_I 1_{A_1}(y^{(1)})dy^{(1)} \left[ \prod_{i=1}^L G_i(d\mathbf{c}^{(i)}) \right]
1_{A_2}(\gamma_{x^{(2)}, \mathbf{c}^{(2)}}^{(2)} \circ (\gamma_{x^{(1)}, \mathbf{c}^{(1)}}^{(1)})^{-1}(y^{(1)})) \ldots 1_{A_L}(\gamma_{x^{(L)}, \mathbf{c}^{(L)}}^{(L)} \circ (\gamma_{x^{(1)}, \mathbf{c}^{(1)}}^{(1)})^{-1}(y^{(1)}))
= \beta_1 \int_{\mathbb{R}^{n_1+1}} \nu^{(1)}(y^{(1)})1_{A_1}(y^{(1)})dy^{(1)} \int_{A_2 \times \ldots \times A_L} Q_1((x, y^{(1)}), dy^{(2)} \ldots dy^{(L)}), \]

where \( \nu^{(1)} \) is the uniform density on \( I \), \( \beta_1 = \tilde{\beta}_1 \nu(I) \) and \( G_i(d\mathbf{c}^{(i)}) = \prod_{i=1}^{n_i+1} G_i(d\mathbf{c}^{(i)}_i) \). This proves (A.1) for \( k = 1 \).

**Induction step**: \( k - 1 \) implies \( k \). Suppose that we have already established the result for \( k - 1 \). Let \( A_i \in \mathcal{B}(\mathbb{R}^{n_i+1}) \), for all \( 1 \leq i \leq L \). We have
\[ P_{kT}(x, A_1 \times \ldots \times A_L) = \int P_T(y, A_1 \times \ldots \times A_L)P_{(k-1)T}(x, dy)
\geq \beta_{k-1}1_C(x) \int P_T(y, A_1 \times \ldots \times A_L)\nu_{k-1}(y^{(1)}, \ldots, y^{(k-1)})
Q_{k-1}((x, y^{(1)}, \ldots, y^{(k-1)}), dy^{(k)} \ldots dy^{(L)})dy^{(1)} \ldots dy^{(k-1)}. \]

We work conditionally on the choice of \( y = (y^{(1)}, \ldots, y^{(L)}) \) and proceed as in the first part, using the jump noise (of a sufficient number of jumps) to create a density for the variable \( y^{(k)} \) and proving that the already produced density \( \nu_{k-1}(y^{(1)}, \ldots, y^{(k-1)}) \) of the first \( k - 1 \) variables is well preserved.

As in the first step, we start with
\[ P_T(y, A_1 \times \ldots \times A_L) \geq E_y \left( \prod_{i=1}^L 1_{A_i}(X_T^{(i)}), N_T = n_k + 1 \right). \]
We impose inter-jump waiting times $t_1, \ldots, t_{n_k+1}$ under the constraint that $t_1 + \ldots + t_{n_k+1} < T$ and associated jump heights $c_1, \ldots, c_{n_k+1}$, where each $c_i$ is an element of $\mathbb{R}^L$, that is, $c_l = (c_l^{(i)}, 1 \leq i \leq L)$. These inter-jump waiting times will produce a density for the $k$–th variable $X_k^{(k)}$ as in the preceding steps.

To start with, let us introduce the following notation. For all $1 \leq l < m \leq L$, we write $y_l^{(m)} := (y_l^{(1)}, \ldots, y_l^{(m)})$, $dy_l^{(m)} := dy_l^{(1)} \ldots dy_l^{(m)}$, $c_l^{(m)} := (c_l^{(1)}, \ldots, c_l^{(m)})$. For $x \in C$ we have

(A.5) \[
\int P_T(y, A_1 \times \ldots \times A_L) \nu_{k-1}(y^{1:k-1})Q_{k-1}((x, y^{1:k-1}), dy^{k:L}dy^{1:k-1}) \geq \int \nu_{k-1}(y^{1:k-1})Q_{k-1}((x, y^{1:k-1}), dy^{k:L}dy^{1:k-1}) \int \prod_{i=1}^L G_i(dy^{(i)})
\]

\[
\int ds_1 \ldots ds_{n_k+1} A_1 \times \ldots \times A_{k-1} (\gamma^{1:k-1}(y^{1:k-1}, c^{1:k-1}, s)) A_k (\gamma^k(y^k, c^k, s)) 1_{A_{k+1} \times \ldots \times A_L} (\gamma^{k+1:L}(y^{k+1:L}, c^{k+1:L}, s)) q_y c(s).
\]

Let $N_1 := \sum_{l=1}^{k-1} (n_l + 1)$ be the dimension of $y^{1:k-1}$. We introduce now for any fixed $c^{1:k}$ having all entries non zero and $y^{k} \in \mathbb{R}^{n_k+1}$,

\[
\Phi_{(c^{1:k}, y^{(k)})} : \left\{ \begin{array}{c} \mathbb{R}^{N_1} \times \mathbb{R}^{n_k+1} \rightarrow \mathbb{R}^{N_1+n_k+1} \\ (y^{1:k-1}, s) \mapsto (\gamma^{1:k-1}(y^{1:k-1}, c^{1:k-1}, s), \gamma^k(y^k, c^k, s)) \end{array} \right.
\]

We write

\[
\frac{\partial \Phi_{(c^{1:k}, y^{(k)})}(y^{1:k-1}, s)}{\partial y^{1:k-1}} = \left[ \frac{\partial \Phi_{(c^{1:k}, y^{(k)})}(y^{1:k-1}, s)}{\partial y^{(1)}} \right. \frac{\partial \Phi_{(c^{1:k}, y^{(k)})}(y^{1:k-1}, s)}{\partial y^{(2)}} \left. \ldots \frac{\partial \Phi_{(c^{1:k}, y^{(k)})}(y^{1:k-1}, s)}{\partial y^{(n_k)}} \right]
\]

to denote the Jacobian matrix of the the map $(y^{1:k-1}, s) \mapsto \Phi_{(c^{1:k}, y^{(k)})}(y^{1:k-1}, s)$. By the properties of the flow (2.6), it follows that

\[
\frac{\partial \Phi_{(c^{1:k}, y^{(k)})}(y^{1:k-1}, s)}{\partial y^{(1)}} = \begin{pmatrix}
\frac{\partial \gamma^{(1)}(y^{(1)}, c^{(1)}, s)}{\partial y^{(1)}} & \frac{\partial \gamma^{(2)}(y^{(2)}, c^{(2)}, s)}{\partial y^{(1)}} & \ldots & \frac{\partial \gamma^{(n_k)}(y^{(n_k)}, c^{(n_k)}, s)}{\partial y^{(1)}}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \ldots & 0
\end{pmatrix}
\]

where by the “cascade structure” of the flow (2.6),

\[
\frac{\partial \varphi_T(y^{(1)})}{\partial y^{(1)}} = \begin{pmatrix}
e^{-\alpha_1 T} & * & * & * \\
0 & e^{-\alpha_1 T} & * & * \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & e^{-\alpha_1 T}
\end{pmatrix}.
\]

Therefore,

\[
\frac{\partial \Phi_{(c^{1:k}, y^{(k)})}(y^{1:k-1}, s)}{\partial y^{1:k-1}} = \begin{pmatrix}
A & * \\
0 & B
\end{pmatrix},
\]

where $A$ is an $N_1 \times N_1$ upper diagonal matrix having entries of the type $e^{-\alpha_1 T}, 1 \leq i \leq k-1$, on the diagonal, where $0$ is the $(n_k + 1) \times N_1$ matrix having all zero entries, and where
\[ B = \left( \frac{\partial y^{(k)}}{\partial \mathbf{s}} \right)_{y^{(k)}}. \]  
According to Part I of the proof, \( \det B \neq 0 \) whenever \( \mathbf{c}^{(k)} \) has only non zero coordinates, for any fixed \( y^{(k)} \), for admissible \( \mathbf{s}. \) It is immediate to see that

\[
\det \left( \frac{\partial \Phi(\mathbf{c}^{(1), k}, y^{(1:k-1)})}{\partial y^{(1:k-1)} \partial \mathbf{s}} \right) = e^{-\alpha_1(n_1+1)+\ldots+\alpha_k-1(n_k+1))T} \cdot \det B \neq 0
\]

in this case. Therefore, arguing as in Part I, for any \( y^{(s,k)} \in \mathbb{R}^{n_k+1} \) and \( \mathbf{c}^{(s,1:k)} \) having non zero entries, for all \( y^{(s,1:k-1)} \) and an admissible \( \mathbf{s}^* \) there exist open neighborhoods \( J_k = B_R(y^{(s,k)}) \times B_R(\mathbf{c}^{(s,1:k)}) \) and an open set \( I \subset \mathbb{R}^{N_1+n_k+1} \) containing \( (y^{(s,1:k-1)}, \mathbf{s}^*) \) and for any pair \( (y^{(k)}, \mathbf{c}^{(1:k)}) \in J_k \) an open set \( W_{y^{(k)}}(\mathbf{c}^{(1:k)}) \) such that

\[
\Phi(\mathbf{c}^{(1), k}, y^{(1:k-1)}, \mathbf{s}) : \begin{cases} 
W_{y^{(k)}}(\mathbf{c}^{(1,k)}) & \rightarrow I \\
(y^{1:k-1}-\mathbf{s}), \mathbf{s} & \rightarrow \Phi(\mathbf{c}^{(1), k}, y^{(1:k-1)}, \mathbf{s}), \mathbf{s}
\end{cases}
\]

is a diffeomorphism.

In what follows, in order to ease notation, we shall shortly write

\[
\Phi(y^{1:k-1}, \mathbf{s}) := \Phi(\mathbf{c}^{(1), k}, y^{(1:k-1)}, \mathbf{s})
\]

and

\[
(A.6) \quad \Psi = \Phi^{-1}
\]

for the associated inverse function. \( \Psi \) taking values in some (subset of) \( \mathbb{R}^{N_1+n_k+1} \), we shall write as before \( \Psi^{(i)} \) for its coordinates and \( \Psi^{1:N_1} \) for the first \( N_1 \) of its coordinates, corresponding to \( y^{1:k-1} \), and \( \Psi^{N_1+1:N_1+n_k+1} \) for the last coordinates, corresponding to \( \mathbf{s}. \)

We choose any \( x \in C \) and \( y^* \in \text{supp} (\nu_{k-1}(y^{1:k-1})Q_{k-1}(x, y^{1:k-1}), dy^{k:L})dy^{1:k-1} \) and obtain (recall \( A.4 \) and \( A.5 \)),

\[
\int P_T(y, A_1 \times \ldots \times A_L) \nu_{k-1}(y^{1:k-1})Q_{k-1}(x, y^{1:k-1}), dy^{k:L})dy^{1:k-1} \geq \int_{B_R(y^*)} \nu_{k-1}(y^{1:k-1})Q_{k-1}(x, y^{1:k-1}), dy^{k:L})dy^{1:k-1} \int_{B_R(\mathbf{c}^{(s)})} \prod_{i=1}^{L} G_i (d\mathbf{c}^{(i)}) \int_{\mathbb{R}^{n_k+1}} W_{y^{(k)}}(\mathbf{c}^{(1,k)})(y^{1:k-1}, \mathbf{s}) q_{y, \mathbf{c}}(\mathbf{s}) 1_{A_1 \times \ldots \times A_k}(\Phi(y^{1:k-1}, \mathbf{s})) 1_{A_{k+1} \times \ldots \times A_L}(y^{k+1:L}, \mathbf{c}(\mathbf{s}))ds_1 \ldots ds_{n_k+1}.
\]

In the above formula, \( R \) is chosen sufficiently small such that \( B_R(\mathbf{c}^{(s)}) \) contains only jump heights with non zero entries. We then use the change of variables \( z^{1:k} := \Phi(y^{1:k-1}, \mathbf{s}). \)

Choose now \( R \) sufficiently small such that

\[
\beta_k := \inf_{y \in B_R(y^*)} \inf_{\mathbf{c} \in B_R(\mathbf{c}^*)} \inf_{\mathbf{s} : (y^{1:k-1}, \mathbf{s}) \in W_{y^{(k)}}(\mathbf{c}^{1:k})} q_{y, \mathbf{c}}(\mathbf{s}) \times 
\times \left| \det \left( \frac{\partial \hat{\Phi}(y^{1:k-1}, \mathbf{s})}{\partial y^{1:k-1} \partial \mathbf{s}} \right)^{-1} \right| \nu_{k-1}(y^{1:k-1}) > 0.
\]
Let $C_2 = B_R(y^{(s,k)}, \ldots, y^{(s,L)})$. Then
\[
\int P_T(y, A_1 \times \ldots \times A_L) \nu_{k-1}(y^{1,k-1}) Q_{k-1}((x, y^{1,k-1}), dy^{k:L}) dy^{1,k-1} \geq \beta_k \int \left( \prod_{j=1}^k 1_{A_j}(z^{(j)}) dz^{1:k} \right) \left[ \int_{B_R(x)} \prod_{i=1}^L G_i(d\xi^{(i)}) \right]
\int_{C_2} Q_{k-1}((x, \Psi^{1:N_1}(z^{1,k}), dy^{k:L}) 1_{A_k+1} \ldots A_L(\gamma^{k+1:L}) \circ \Psi^{N-1+1:N_1+n_k+1}(z^{1:k})
\]
\[= \beta_k \int \left( \prod_{j=1}^k 1_{A_j}(z^{(j)}) dz^{1:k} \right) \left[ \int Q_k((x, z^{1:k}), dz^{k+1:L}) 1_{A_k+1}(z^{(k+1)}) \ldots 1_{A_L}(z^{(L)}) \right].\]

Together with (A.3), this shows that (A.1) holds also for $k$, and this finishes the induction step. By taking finally $k = L$ in (A.1), this implies the assertion of the Theorem.

\[\square\]

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