Abstract. We establish Arazy-Cwikel type properties for the family of couples \((\ell^p, \ell^q)\), \(0 \leq p < q \leq \infty\), and show that \((\ell^p, \ell^q)\) is a Calderón-Mityagin couple if and only if \(q \geq 1\). Moreover, we identify interpolation orbits of elements with respect to this couple for all \(p\) and \(q\) such that \(0 \leq p < q \leq \infty\) and obtain a simple positive solution of a Levitina-Sukochev-Zanin problem, clarifying its connections with whether \((\ell^p, \ell^q)\) has the Calderón-Mityagin property or not.

1. Introduction

Nowadays, the interpolation theory of operators is rather completely presented in several excellent books; see, for example, Bergh and Lofström [7], Bennett and Sharpley [6], Brudnyi and Kruglyak [8], Krein, Petunin and Semenov [33], Triebel [58]. In these monographs the reader can find not only a systematic treatment of problems within the theory itself, but also valuable applications of interpolation methods and results to various other fields of mathematics. Let us also mention several books which contain applications of interpolation theory to a variety of fields: in [24, 27] there are applications to harmonic analysis, in [36, 37, 59] to Banach space theory, and in [31, 4] to classical systems in \(L^p\) spaces and in other rearrangement invariant spaces. Furthermore, the papers [34, 35] (and the references therein) include applications to noncommutative analysis, and, finally, the survey [28] contains a very attractive account of the interaction between interpolation theory and the geometry of Banach spaces. Of course, the above list is far from being complete.

One of the reasons for there being such fruitful applicability of interpolation theory is that, for many couples \((X_0, X_1)\), we can effectively describe the class \(\text{Int}(X_0, X_1)\) of all interpolation spaces. In most of the known cases of couples \((X_0, X_1)\) for which this is possible, this description is formulated by using the Peetre \(K\)-functional, which plays an important role in the theory. For those couples the terminology \textit{Calderón couple} or \textit{Calderón-Mityagin couple} is often used. This

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is because the first example of such a couple was obtained by Calderón \[13\] and Mityagin \[40\]. They proved independently that a Banach function space \(X\) on an arbitrary underlying measure space is an interpolation space with respect to the couple \((L^1, L^\infty)\) on that measure space if and only if the following monotonicity property\[1\] holds: if \(f \in X\), \(g \in L^1 + L^\infty\) and
\[
\int_0^t g^*(s) \, ds \leq \int_0^t f^*(s) \, ds, \quad t > 0
\]
(where \(h^*\) denotes the nonincreasing left-continuous rearrangement of \(|h|\), then \(g \in X\) and \(\|g\|_X \leq \|f\|_X\). Peetre \[48, 49\] had proved (cf. also a similar result due independently to Oklander \[42\], and cf. also \[32\] pp. 158–159) that the functional \(\int_0^t f^*(s) \, ds\) is in fact the \(K\)-functional of the function \(f \in L^1 + L^\infty\) for the couple \((L^1, L^\infty)\). So the results of \[13, 40\] naturally suggested the form that analogous results for couples other than \((L^1, L^\infty)\) might take, expressed in terms of the \(K\)-functional for those couples. This led many mathematicians to search for such analogous results. Let us mention at least some of the many results of this kind which were obtained: Lorentz-Shimogaki\[2\] \[38\] \(((L^p, L^\infty), 1 < p < \infty)\), Sedaev-Semenov \[56\] (couples of weighted \(L^1\)-spaces), Dmitriev \[20\] (relative interpolation of couples \((L^1(w_0), L^1(w_1))\) and \((L^1, L^\infty)\)), Peetre \[50\] (relative interpolation of an arbitrary Banach couple with a couple of weighted \(L^\infty\)-spaces) (see also \[9\] p. 589, Theorem 4.4.16] or \[19\] p. 29, Theorem 4.1 for this result), Sparr \[57\] (couples of weighted \(L^p\)-spaces, \(1 \leq p \leq \infty\), and Kalton \[29\] (couples of rearrangement invariant spaces).

It is worth to note, that for an arbitrary Banach couple, the uniform \(K\)-monotone interpolation spaces, which are closely related to the Calderón-Mityagin property (see Definition \[2.1\] and Remark \[2.5\] below) can also be described in a more concrete way. This important fact is due to Brudnyi and Kruglyak \[8, 9\] and follows from their proof (see \[9\] pp. 503-504) of a conjecture due to S. G. Krein \[23\]. One of its consequences is that, for every Banach couple \((X_0, X_1)\) with the Calderón-Mityagin property, the family \(\text{Int}(X_0, X_1)\) of all its interpolation spaces can be parameterized by the set of so-called \(K\)-method parameters.

Moreover, using the Calderón-Mityagin property\[3\] of couples of \(L^p\)-spaces in the range \(1 \leq p \leq \infty\), Arazy and Cwikel proved, in \[1\], that for all \(1 \leq p < q \leq \infty\) and for each underlying measure space
\[
\text{Int} (L^p, L^q) = \text{Int} \left( L^1, L^q \right) \cap \text{Int} (L^p, L^\infty).
\]
Later on, Bykov and Ovchinnikov obtained a similar result for families of interpolation spaces, corresponding to weighted couples of shift-invariant ideal sequence spaces \([10]\).

On the negative side, Ovchinnikov and Dmitriev \([47]\) showed that the couple \((\ell^1(L^1), \ell^1(L^\infty))\) of vector-valued sequences is not a Calderón-Mityagin couple. Neither is the couple \((L^p, W^{1,p})\) when \(p \in (2, \infty)\). (See \([14, p. 218]\).) Later on, Ovchinnikov \([44]\) (see also \([39]\)) proved the same result for the couple \((L^1 + L^\infty, L^1 \cap L^\infty)\) on \((0, \infty)\). One can find more examples of couples of rearrangement invariant spaces of this kind in Kalton’s work \([29]\). Many of these results contain a description of interpolation orbits, which cannot be obtained by the real \(K\)-method (see e.g. \([43, 45, 46, 21, 22]\)). As shown by Theorem 2.2 of \([17, pp. 36–37]\), if \(X_0\) and \(X_1\) are both \(\sigma\)-order continuous Banach lattices of measurable functions with the Fatou property on the same underlying \(\sigma\)-finite measure space \(\Omega\) and if at least one of these spaces does not coincide to within equivalence of norm with some weighted \(L^p\) space on \(\Omega\). then there exist weight functions \(w_0\) and \(w_1\) on \(\Omega\) for which the couple of weighted lattices \((X_0,w_0, X_1,w_1)\) is not a Calderón-Mityagin couple.

We refer to the article \([16]\) for additional details about Calderón-Mityagin couples.

All of the results listed above were obtained for couples of Banach spaces. But there were also some ventures beyond Banach couples. In \([57]\) Sparr was in fact also able to treat couples of weighted \(L^p\) spaces for \(p \in (0, \infty)\) under suitable hypotheses, and then Cwikel \([15]\) considered the couple \((\ell^p, \ell^\infty)\) also for \(p\) in this extended range. New questions have recently arisen (see, for instance, \([35, 18]\)) that require analogous results for more general situations, say, for quasi-Banach couples or even for couples of quasi-normed Abelian groups. The extension of the basic concepts and constructions of interpolation theory to the latter setting was initiated long ago by Peetre and Sparr in \([52]\).

We recall that \(\ell^0\) is the linear space (sometimes considered merely as an Abelian group) of all eventually zero sequences \(x = (x_k)_{k=1}^\infty\), equipped with the “norm”\(4\)

\[\|x\|_{\ell^0} := \text{card}(\text{supp} \, x),\]

where \(\text{supp} \, x\) is the support of \(x\). This space is an analogue of the space or normed Abelian group \(L^0\), which consists of all measurable functions on \((0, \infty)\) with supports of finite measure, equipped with the quasi-norm \(\|f\|_{L^0} := m\{t > 0 : f(t) \neq 0\}\) ( \(m\) is the Lebesgue measure) and of the space of operators \(\mathcal{S}_0(A,B)\) introduced on p. 249 and p. 256 respectively of \([52]\). Comparing some simple calculations with \(L^0\) in \([52]\) with some quantities appearing implicitly in \([34, 35]\) and \([18]\) can lead one to understand that \(\ell^0\) can play a useful role in studying interpolation properties of \(\ell^p\) spaces for \(p > 0\). Note also that independently \(\ell^0\) appeared explicitly in \([3]\), where a description of orbits of elements in the couple \((\ell^0, \ell^1)\) is given.

\(4\)Although \(\| \cdot \|_{\ell^0}\) does not satisfy the condition \(\| \lambda f \|_{\ell^0} = |\lambda| \| f \|_{\ell^0}\) for scalars \(\lambda\), it is a \((1, \infty)\)-norm or \(\infty\)-norm on the Abelian group \(\ell^0\) in the terminology of \([52, p. 219]\).
The two main aims of this paper are, first of all, to completely determine for which values of $p$ and $q$ in the range $0 \leq p < q \leq \infty$, the couple $(\ell^p, \ell^q)$ has the Calderón-Mityagin property and then, secondly, to extend a property analogous to the Arazy-Cwikel property to the couples $(\ell^p, \ell^q)$, with $p$ and $q$ in the enlarged range $0 \leq p < q \leq \infty$ and with the role of $L^1$ in (1.1) now played by $\ell^0$.

Note that there are close connections between the present paper and the paper [12]. Although [12] mainly considers the couples $(L^p, L^q)$ of function spaces on $(0, \infty)$, it also deals with interpolation properties of the analogous sequence space couples $(\ell^p, \ell^q)$ for the range $0 \leq p < q \leq \infty$. However, in contrast to our paper, the authors of [12] restrict themselves to studying the Calderón-Mityagin case, i.e., for values $q \geq 1$. It seems that some of the results in [12] for this case could be used to establish some of our results, and vice versa. We shall comment more explicitly about connections with [12] at appropriate places in our text, however we have kept our approach almost self-contained.

The couple $(\ell^0, \ell^\infty)$ has some advantages over the corresponding Banach couple $(\ell^1, \ell^\infty)$. In particular, as remarked in [12], it is well-known that there exist symmetric Banach sequence spaces, which are not interpolation spaces with respect to the latter couple (see e.g. [37] Example 2.a.11, p. 128). In contrast to that, every symmetric quasi-Banach sequence space $E$ is an interpolation space with respect to the couple $(\ell^0, \ell\infty)$ (this can be obtained by obvious modifications of reasoning in the papers [26] and [2], where the analogous property is proved for the couple $(L^0, L^\infty)$ on $(0, \infty)$ and rearrangement invariant quasi-Banach function spaces).

Some other partial results for the couples $(\ell^p, \ell^q)$, in the non-Banach case, were obtained more recently in [18] [35] [12] [8] [11]. Moreover, in [12], the above Arazy-Cwikel property has been proved for the couple $(L^0, L^\infty)$ of measurable functions on the semi-axis $(0, \infty)$ with the Lebesgue measure. Observe however that there are differences in the properties of the quasi-Banach spaces $\ell^p$ and $L^p$ that are essential in our context; for instance, if $p \in (0, 1)$, then $(\ell^p)^* = \ell^1$ while $(L^p)^* = \{0\}$ (see Section 2.2).

In general, the above-mentioned Brudnyi-Kruglyak result about a description of all interpolation spaces with respect to Calderón-Mityagin couples of Banach spaces cannot be extended to the class of quasi-Banach couples. Nevertheless, whenever $p$ and $q$ are such that the couple $(\ell^p, \ell^q)$ is a Calderón-Mityagin couple (including in the non-Banach case), then every interpolation space with respect to $(\ell^p, \ell^q)$ can be described by using the real $K$-method of interpolation. Moreover, but discussion of this is deferred to a forthcoming paper [5], a similar result holds for a rather wide subclass of quasi-Banach couples (the latter paper will also deal with some other related problems).

Let us describe now the main results of the paper in more detail. In Section 2, we give preliminaries with basic definitions and results. So, we address some versions of the Holmstedt inequality and give descriptions of the $K$- and
\( \mathcal{E} \)-functionals for couples of \( \ell^p \)-spaces. Section 3 contains some auxiliary (apparently well-known) results, in particular, an extrapolation theorem for operators bounded on \( \ell^p \), \( 0 < p < 1 \).

The central result of the next section is Theorem 4.1, which extends the above-mentioned Arazy-Cwikel theorem (1.1) to the sequence space setting, showing that

\[
\text{Int} \left( \ell^p, \ell^q \right) = \text{Int} \left( \ell^s, \ell^q \right) \cap \text{Int} \left( \ell^p, \ell^r \right)
\]

for all \( 0 \leq s < p < q < r \leq \infty \). It is worth noting that (1.2) holds, in particular, in the range \( 0 \leq s < p < q < r < 1 \), i.e., when all the couples involved in (1.2) fail to have the Calderón-Mityagin property. This fact indicates that Arazy-Cwikel type properties of couples do not imply that they are Calderón-Mityagin ones. Observe that a closely related result, under the additional restriction \( q \geq 1 \), has been proved in [12, Theorem 5.6].

The main ingredient in the proof of relations (1.2) is Theorem 4.5, which has also other interesting consequences. The first of them, Corollary 4.6, states that the condition \( q \geq 1 \) ensures that \( (\ell^p, \ell^q) \) is a uniform Calderón-Mityagin couple. The second, Corollary 4.8, presents a complete description of interpolation orbits of elements of the space \( \ell^q \) with respect to the couple \( (\ell^0, \ell^q) \) both in the cases \( q \geq 1 \) and \( q < 1 \). Let us mention also the result of Theorem 4.11 which is a consequence of Theorem 1.1 combined with a self-improvement property of interpolation between the spaces \( \ell^0 \) and \( \ell^q \). Theorem 4.11 states that, if \( E \in \text{Int} \left( \ell^0, \ell^q \right) \), then there exists \( p > 0 \) such that \( E \in \text{Int} \left( \ell^p, \ell^q \right) \). Hence, interpolation of quasi-Banach spaces with respect to the couple \( (\ell^0, \ell^q) \) can be reduced, in fact, to that with respect to the couples \( (\ell^p, \ell^q) \) with \( p > 0 \). This phenomenon allows us to obtain rather simply, in the case \( q \geq 1 \), the positive answer to the Levitina-Sukochev-Zanin conjecture, which was posed in [35] and resolved in [12] (its earlier version in majorization terms may be found in the preprint [34]). Moreover, we reveal its connections with the Calderón-Mityagin property of the couple \( (\ell^p, \ell^q) \), showing that the answer to the latter conjecture is negative if \( 0 < q < 1 \).

In Section 5 we prove that \( (\ell^p, \ell^q) \) is not a Calderón-Mityagin couple whenever \( 0 < p < q < 1 \) (see Theorem 5.2). In fact, we obtain a stronger result, which reads that for every \( g \in \ell^q \setminus \ell^p \) there exists \( f \in \ell^q \) satisfying the condition

\[
\mathcal{K} \left( t, g; \ell^p, \ell^q \right) \leq \mathcal{K} \left( t, f; \ell^p, \ell^q \right), \quad t > 0,
\]

but \( g \neq Tf \) for every linear operator \( T \) bounded in \( \ell^p \) and \( \ell^q \). Combining Theorem 5.2 with Corollary 4.6, we conclude that \( (\ell^p, \ell^q) \) is a uniform Calderón-Mityagin couple if and only if \( q \geq 1 \).

Considering the above-mentioned Levitina-Sukochev-Zanin conjecture, Cwikel and Nilsson have introduced, in [18], the so-called \( S_q \)-property expressed in terms of a majorization inequality. In Section 6 we show that for every \( q \geq 1 \) a quasi-Banach sequence space \( E \) has the \( S_q \)-property if and only if \( E \in \text{Int}(\ell^0, \ell^q) \) (see Theorem 6.2 and Corollary 6.3).
In the concluding Section 7 we prove that the couple \((\ell^p, \ell^q)\), with \(0 \leq p < q < 1\), does not have the uniform Calderón-Mityagin property (see Theorem 7.1). Clearly, the latter result is weaker than Theorem 5.2 of Section 5. However, for the reader’s convenience, we provide its independent proof, which is much shorter and simpler than that of Theorem 5.2.

We dedicate this paper to the memory of Professor Jaak Peetre (1935 – 2019). His profound ideas and research have played an essential role in many mathematical topics, including those in this paper. The second and third authors have been fortunate to have him as a very respected and close friend, mentor and colleague for many years.

2. Preliminaries

2.1. Interpolation of operators and the Calderón-Mityagin property.

Let us recall some basic constructions and definitions related to the interpolation theory of operators. For more detailed information we refer to [7, 9, 6, 33, 45].

In this paper we are mainly concerned with interpolation within the class of quasi-Banach sequence spaces while linear bounded operators are considered as the corresponding morphisms. All linear spaces considered will be over the reals. But it should be possible to readily extend much of the theory that we develop also to the case of complex linear spaces.

A pair \(\vec{X} = (X_0, X_1)\) of quasi-Banach spaces is called a **quasi-Banach couple** if \(X_0\) and \(X_1\) are both linearly and continuously embedded in some Hausdorff linear topological space. In particular, every pair of arbitrary quasi-Banach sequence lattices \(E_0\) and \(E_1\) forms a quasi-Banach couple, because convergence in a quasi-Banach sequence lattice implies coordinate-wise convergence.

For each quasi-Banach couple \((X_0, X_1)\) we define the **intersection** \(X_0 \cap X_1\) and the **sum** \(X_0 + X_1\) as the quasi-Banach spaces equipped with the quasi-norms

\[
\|x\|_{X_0 \cap X_1} := \max \{\|x\|_{X_0}, \|x\|_{X_1}\},
\]

and

\[
\|x\|_{X_0 + X_1} := \inf \{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i, i = 0, 1\},
\]

respectively. A linear space \(X\) is called **intermediate** with respect to a quasi-Banach couple \(\vec{X} = (X_0, X_1)\) (or is said to be **between** \(X_0\) and \(X_1\)) if it is a quasi-Banach space and satisfies \(X_0 \cap X_1 \subset X \subset X_0 + X_1\) where both of these inclusions are continuous.

If \(\vec{X} = (X_0, X_1)\) and \(\vec{Y} = (Y_0, Y_1)\) are quasi-Banach couples, then we let \(\mathcal{L}(\vec{X}, \vec{Y})\) denote the space of all linear operators \(T : X_0 + X_1 \to Y_0 + Y_1\) that are bounded from \(X_i\) in \(Y_i\), \(i = 0, 1\), equipped with the quasi-norm

\[
\|T\|_{\mathcal{L}(\vec{X}, \vec{Y})} := \max_{i=0,1} \|T\|_{X_i \to Y_i}.
\]

In the case when \(X_i = Y_i, i = 0, 1\), we simply write \(\mathcal{L}(\vec{X})\) or \(\mathcal{L}(X_0, X_1)\).
Let \( \vec{X} = (X_0, X_1) \) be a quasi-Banach couple and let \( X \) be an intermediate space between \( X_0 \) and \( X_1 \). Then, \( X \) is called an interpolation space with respect to the couple \( \vec{X} \) (or between \( X_0 \) and \( X_1 \)) if every operator \( T \in \mathcal{L}(\vec{X}) \) is bounded on \( X \). In this case, we write: \( X \in \text{Int}(X_0, X_1) \).

Recall that, by the Aoki-Rolewicz theorem (see e.g. [7, Lemma 3.10.1]), every quasi-Banach space is a \( F \)-space (i.e., the topology in that space is generated by a complete invariant metric). In particular, this applies to the space \( \mathcal{L}(\vec{X}) \) which is obviously a quasi-Banach space with respect to the quasi-norm \( \|T\| \rightarrow \max \{\|T\|_{X_0 \rightarrow X_0}, \|T\|_{X_1 \rightarrow X_1}\} \) (cf. (2.1)), and also with respect to the larger quasi-norm \( \|T\| \rightarrow \max \{\|T\|_{X_0 \rightarrow X_0}, \|T\|_{X_1 \rightarrow X_1}, \|T\|_{X \rightarrow X}\} \) whenever the quasi-Banach space \( X \) is an interpolation space with respect to the quasi-Banach couple \( \vec{X} = (X_0, X_1) \). As is well known (see e.g. [54, Theorem 2.2.15]), the Closed Graph Theorem and the equivalent Bounded Inverse Theorem (see e.g. [54, Corollary 2.2.12]) hold for \( F \)-spaces. Therefore, by using exactly the same reasoning as required for the Banach case (see Theorem 2.4.2 of [7, p. 28]), we have the following:

if \( X \) is an interpolation quasi-Banach space with respect to a quasi-Banach couple \( \vec{X} = (X_0, X_1) \), then there exists a constant \( C > 0 \) such that \( \|T\|_{X \rightarrow X} \leq C \|T\|_{\mathcal{L}(\vec{X})} \) for every \( T \in \mathcal{L}(\vec{X}) \). The least constant \( C \), satisfying the last inequality for all such \( T \), is called the interpolation constant of \( X \) with respect to the couple \( \vec{X} \).

One of the most important ways of constructing interpolation spaces is based on use of the Peetre \( K \)-functional, which is defined for an arbitrary quasi-Banach couple \( (X_0, X_1) \), for every \( x \in X_0 + X_1 \) and each \( t > 0 \) as follows:

\[
K(t, x; X_0, X_1) := \inf\{\|x_0\|_{X_0} + t|\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i\}.
\]

For each fixed \( x \in X_0 + X_1 \) one can easily show that the function \( t \mapsto K(t, x; X_0, X_1) \) is continuous, non-decreasing, concave and non-negative on \((0, \infty)\) [7, Lemma 3.1.1]. On the other hand, for each fixed \( t > 0 \), the functional \( x \mapsto K(t, x; X_0, X_1) \) is an equivalent quasi-norm on \( X_0 + X_1 \).

As already discussed at some length in the introduction, for quite a large class of (quasi-)Banach couples, the \( K \)-functional can be used to describe all interpolation (quasi-)Banach spaces with respect to those couples. We first need the following definition:

**Definition 2.1.** Let \( X \) be an intermediate space with respect to a quasi-Banach couple \( \vec{X} = (X_0, X_1) \). Then, \( X \) is said to be a \( K \)-monotone space with respect to this couple if whenever elements \( x \in X \) and \( y \in X_0 + X_1 \) satisfy

\[
K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1), \quad \text{for all } t > 0,
\]

it follows that \( y \in X \). If it also follows that \( \|y\|_X \leq C \|x\|_X \), for a constant \( C \) which does not depend on \( x \) and \( y \), then we say that \( X \) is a uniform \( K \)-monotone space with respect to the couple \( \vec{X} \). The infimum of all constants \( C \) with this...
property is referred as the $K$-monotonicity constant of $X$. Clearly, each $K$-monotone space with respect to the couple $\vec{X}$ is an interpolation space between $X_0$ and $X_1$.

Note that every $K$-monotone Banach space with respect to a couple of Banach lattices is also a uniform $K$-monotone space with respect to this couple [17, Theorem 6.1].

**Definition 2.2.** Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be two quasi-Banach couples and let $x \in X_0 + X_1$, $x \neq 0$. The orbit $\text{Orb}_{(X_0, X_1)}(x; Y_0, Y_1)$ of $x$ with respect to the class of operators $\mathcal{L}(\vec{X}, \vec{Y})$ is the linear space

$$\left\{ T x : T \in \mathcal{L}(\vec{X}, \vec{Y}) \right\}.$$ 

This space may be equipped with the quasi-norm defined by

$$\|y\|_{\text{Orb}(x)} := \inf \left\{ \|T\|_{\mathcal{L}(\vec{X}, \vec{Y})} : y = Tx, T \in \mathcal{L}(\vec{X}, \vec{Y}) \right\}.$$ 

In the case when $(X_0, X_1) = (Y_0, Y_1)$ we will use the shortened notation $\text{Orb}(x; X_0, X_1)$.

Since any orbit $\text{Orb}(x; X_0, X_1)$ can be regarded as a quotient of the quasi-Banach space $\mathcal{L}(\vec{X})$, it is a quasi-Banach space. If for every nonzero $x \in X_0 + X_1$ there exists a linear functional $x^* \in (X_0 + X_1)^*$ with $\langle x, x^* \rangle \neq 0$ then $X_0 \cap X_1$ is contained in $\text{Orb}(x; X_0, X_1)$ continuously (see e.g. [13, Section 1.6, p. 368]). It is easy to see that then, moreover, each orbit $\text{Orb}(x; X_0, X_1)$ is an interpolation space between $X_0$ and $X_1$.

A similar concept may be defined by using the $K$-functional.

**Definition 2.3.** Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be two quasi-Banach couples. The $K$-orbit of an element $x \in X_0 + X_1$, $x \neq 0$, which we denote by $K - \text{Orb}_{(X_0, X_1)}(x; Y_0, Y_1)$ is the space of all $y \in Y_0 + Y_1$ such that the following quasi-norm

$$\|y\|_{K - \text{Orb}(x)} := \sup_{t > 0} \frac{K(t, y; Y_0, Y_1)}{K(t, x; X_0, X_1)}$$ 

is finite. If $(X_0, X_1) = (Y_0, Y_1)$, then we simplify the above notation to $K - \text{Orb}(x; X_0, X_1)$.

One can easily check that each $K$-orbit of an element $x \in X_0 + X_1$, $x \neq 0$, is an interpolation quasi-normed space between $X_0$ and $X_1$.

It is obvious that for every quasi-Banach couple $(X_0, X_1)$ and each $x \in X_0 + X_1$ we have

$$\text{Orb}(x; X_0, X_1) \subset K - \text{Orb}(x; X_0, X_1)$$

with constant 1.
Definition 2.4. A quasi-Banach couple $\vec{X} = (X_0, X_1)$ is said to be a Calderón-Mityagin couple (or to have the Calderón-Mityagin property) if for each $x \in X_0 + X_1$

$$\mathcal{K} - \text{Orb}(x; X_0, X_1) \subset \text{Orb}(x; X_0, X_1),$$
i.e., if for every $y \in \mathcal{K} - \text{Orb}(x; X_0, X_1)$ there exists an operator $T \in \mathfrak{L}(\vec{X})$ so that $\|T\|_{\mathfrak{L}(\vec{X})} \leq C \|y\|_{\mathcal{K} - \text{Orb}(x)}$, where $C$ is independent of $x$ and $y$, then $\vec{X}$ is called a uniform Calderón-Mityagin couple (or we say that $\vec{X}$ has the uniform Calderón-Mityagin property).

The name of the last property is justified by the fact that historically the first result in this direction was a theorem which describes all interpolation spaces with respect to the Banach couple $(L^1, L^\infty)$, proved independently by Calderón [13] and Mityagin [40]. In our terminology, this result is equivalent to the assertion that $(L^1, L^\infty)$ is a uniform Calderón-Mityagin couple.

Remark 2.5. The condition that $(X_0, X_1)$ is a Calderón-Mityagin couple, obviously implies that every interpolation space with respect to $(X_0, X_1)$ is also a $\mathcal{K}$-monotone space. Furthermore, if $(X_0, X_1)$ is a uniform Calderón-Mityagin couple, this clearly implies that every interpolation space $X$ with interpolation constant $C_1$ is a uniform $\mathcal{K}$-monotone space with $\mathcal{K}$-monotonicity constant not exceeding $CC_1$, where $C$ is the constant appearing in Definition 2.4.

2.2. Some quasi-Banach sequence spaces and quasi-normed groups. As was said above, we will consider, mainly, quasi-Banach spaces which consist of sequences $x = (x_k)_{k=1}^\infty$ of real numbers with the linear coordinate-wise operations. When $0 < p < \infty$, we, as usual, let $\ell^p$ denote the linear space of all sequences for which the quasi-norm

$$\|x\|_{\ell^p} := \left( \sum_{k=1}^\infty |x_k|^p \right)^{1/p}$$
is finite, and $\ell^\infty$ denotes the linear space of all bounded sequences with the usual norm

$$\|x\|_{\ell^\infty} := \sup_{k=1,2,...} |x_k|.$$For every $0 < p < \infty$ and all $x, y \in \ell^p$ we have

$$\|x + y\|_{\ell^p} \leq \max(1, 2^{(1-p)/p}) (\|x\|_{\ell^p} + \|y\|_{\ell^p})$$
(see e.g. [7, Lemma 3.10.3]).

Obviously, $(\ell^p, \ell^q)$ is a quasi-Banach couple for all $p, q$ with $0 < p, q \leq \infty$. But we also need to deal with a limiting case of such couples.

Consideration of the limit of $\|x\|_{\ell^p}^p = \sum_{k=1}^\infty |x_k|^p$ as $p$ tends to 0 provides the motivation for defining $\ell^0$ to be the set of all sequences $x = (x_k)_{k=1}^\infty$ that are eventually zero, i.e., those that satisfy

$$\|x\|_{\ell^0} := \text{card}(\text{supp } x) < \infty,$$

(2.2)
where \( \text{supp } x := \{ k \in \mathbb{N} : x_k \neq 0 \} \). Observe that \( \ell^0 \) is a linear space with respect to the usual coordinate-wise operations and hence we can consider linear operators defined on \( \ell^0 \). However, in contrast to the case of \( \ell^p \) for every \( p > 0 \), \( \ell^0 \) is not a quasi-Banach space, but rather a \textit{quasi-normed group} as defined in [52] (see also [7, § 3.10]). The functional \( \| \cdot \|_{\ell^0} \), although it is sub-additive, does not have the homogeneity property required for a quasi-norm of a linear space. Indeed, \( \ell^0 \) is an Abelian group of sequences, where the group operation is coordinate-wise addition.

\textbf{Remark 2.6.} According to the terminology introduced by Peetre and Sparr in Definitions 1.1 and 2.2 of [52, p. 219 and pp. 224-225], \( \ell^0 \), when equipped with the functional (2.2), is an example of a quasi-normed group, and, more specifically, it is a \((1, 1)\)-normed Abelian group, and also a \((1, 1 | 0)\)-normed vector space.

The extension of the basic concepts and constructions of the interpolation theory to the class of quasi-normed Abelian groups was initiated by Peetre and Sparr in the above mentioned paper [52] (see also [7, § 3.11] and [9]). In this case the role of morphisms is played, instead of bounded linear operators, by bounded homomorphisms. Recall that a mapping \( T : X \to X \) on a group \( X \) is called a homomorphism on \( X \) if \( T(x+y) = Tx + Ty \) for all \( x, y \in X \). As in [52, Definition 1.2, p. 223], a homomorphism \( T \) on \( X \) is called bounded if

\[
\|T\|_{X \to X} := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} < \infty.
\]

Note that \( \ell^0 \) is complete and is linearly and continuously embedded into the quasi-Banach space \( \ell^q \) for every \( 0 < q \leq \infty \) (the functional \( \|x\|_{\ell^0} \) generates the discrete topology on \( \ell^0 \)).

We shall adopt the following conventions related to homomorphisms which are bounded on the couple \((\ell^0, \ell^q)\).

\textbf{Definition 2.7.} (i) : For each \( q \) with \( 0 < q \leq \infty \) we let \( \mathcal{L}(\ell^0, \ell^q) \) denote the set of all bounded linear operators on \( \ell^q \) whose restrictions to \( \ell^0 \) are bounded homomorphisms.

(ii) : We let \( \text{Int} (\ell^0, \ell^q) \) denote the class of all quasi-normed Abelian groups \( E \) which satisfy the continuous inclusions \( \ell^0 \subset E \subset \ell^q \) and which are also quasi-Banach spaces with respect to their given group quasi-norms and for which \( T : E \to E \) is bounded for each \( T \in \mathcal{L}(\ell^0, \ell^q) \).

\( \mathcal{L}(\ell^0, \ell^q) \) is obviously a linear space and therefore also an Abelian group. Analogously to the usage for couples of quasi-Banach spaces we define

\[
\|T\|_{\mathcal{L}(\ell^0, \ell^q)} := \max(\|T\|_{\ell^0 \to \ell^q}, \|T\|_{\ell^q \to \ell^q})
\]

for every \( T \) in the set \( \mathcal{L}(\ell^0, \ell^q) \). Then \( T \mapsto \|T\|_{\mathcal{L}(\ell^0, \ell^q)} \) is a group quasi-norm on this set.
As shown in Remark 2.2 in [12], using the proof of Theorem 2.1 of that paper, if $E \in \text{Int} (\ell^0, \ell^q)$, then there is a constant $C$ such that $\|T\|_{E \to E} \leq C$ for every $T \in \mathcal{L}(\ell^0, \ell^q)$ with $\|T\|_{\mathcal{L}(\ell^0, \ell^q)} \leq 1$.

We adopt a variant of Definition 2.2 and define the orbit of an element $x \in \ell^q$ with respect to the couple $(\ell^0, \ell^q)$ to be the linear space $\text{Orb}(x; \ell^0, \ell^q)$ of all $y \in \ell^q$, representable in the form $y = T x$, where $T$ is a bounded linear operator in $\ell^q$ and is a bounded homomorphism in $\ell^0$. We can consider this space as a quasi-normed Abelian group by endowing it with the group quasi-norm

$$\|y\|_{\text{Orb}(x)} := \inf \{ T \|_{\mathcal{L}(\ell^0, \ell^q)} \},$$

where the infimum is taken over all $T \in \mathcal{L}(\ell^0, \ell^q)$ such that $y = Tx$.

Given any $q \in (0, \infty]$, suppose that $x = (x_n)_{n=1}^{\infty}$ is an arbitrary non-zero element of $\ell^0 + \ell^q = \ell^q$ so that $x_k \neq 0$ for at least one $k \in \mathbb{N}$. For that $k$ let $x^*$ be the obviously continuous linear functional on $\ell^q = \ell^0 + \ell^q$ defined by $\langle y, x^* \rangle = y_k$ for each element $y = (y_n)_{n=1}^{\infty} \in \ell^q$. Since $\langle x, x^* \rangle \neq 0$, we can reason in the same way as in [15, § 1.6, p. 368] (see also Section 2.1), and show that $\text{Orb}(x; \ell^0, \ell^q)$ is an interpolation quasi-normed group between $\ell^0$ and $\ell^q$.

Note that an inspection of the proofs related to a description of orbits of elements in the couples $(\ell^0, \ell^q)$, $0 < q \leq \infty$, in the papers [2] and [3] shows that these are completely consistent with the above definitions. This fact will allow us to further apply the results of these papers.

The space $\ell^p$, with $0 < p < \infty$ (resp. $\ell^0$) is an example of a symmetric quasi-Banach space (resp. group). Recall that a quasi-Banach sequence space (or group) $E$ is said to be a quasi-Banach sequence lattice if from $|y_k| \leq |x_k|$, $k = 1, 2, \ldots$, and $(x_n) \in E$ it follows that $(y_k) \in E$ and $\|(y_k)\|_E \leq \|(x_k)\|_E$. If additionally $E \subset \ell^\infty$ and the conditions $y^*_k = x^*_k$, $k = 1, 2, \ldots$, $(x_n) \in E$ imply that $(y_k) \in E$ and $\|(y_k)\|_E = \|(x_k)\|_E$, then $E$ is called symmetric. If $(u_k)_{k=1}^{\infty}$ is any bounded sequence then, in what follows, $(u^*_k)_{k=1}^{\infty}$ denotes the nonincreasing permutation of the sequence $(|u_k|)_{k=1}^{\infty}$ defined by

$$u^*_k := \inf_{\text{card } A=k-1} \sup_{i \in \mathbb{N} \setminus A} |u_i|, \quad k \in \mathbb{N}. $$

A quasi-Banach sequence lattice $E$ has the Fatou property if from $x_n \in E$, $n = 1, 2, \ldots$, $\sup_{n=1,2,\ldots} \|x_n\|_E < \infty$ and $x_n \to x$ coordinate-wise as $n \to \infty$ it follows that $x \in E$ and $\|x\|_E \leq \lim \inf_{n \to \infty} \|x_n\|_E$.

Recall that for all $0 \leq p < r < q \leq \infty$ the space $\ell^r$ is an interpolation space between $\ell^p$ and $\ell^q$ (see e.g. [7, Theorem 7.2.2 and Corollary 7.2.3]). Moreover, $\ell^r$ may be obtained by applying the classical real $K$-method to the couple $(\ell^p, \ell^q)$ [7, Theorem 7.1.7].

2.3. The Holmstedt formula and related $K$-functionals. Further, we repeatedly use the following well-known result due to Holmstedt [25], which is referred usually as the Holmstedt formula.

\[\text{The Holmstedt formula.} \]
Let $0 < p < q < \infty$. Then, there exists a positive constant $C_{p,q}$, depending only on $p$ and $q$, such that for every $f \in L^p + L^q$ on an arbitrary underlying measure space it holds

$$
\mathcal{K}(t, f; L^p, L^q) \leq \left( \int_0^t (f^*(s))^p \, ds \right)^{1/p} + t \left( \int_t^\infty (f^*(s))^q \, ds \right)^{1/q} \leq C_{p,q} \mathcal{K}(t, f; L^p, L^q), \quad t > 0,
$$

(2.3)

where $f^*$ is the nonincreasing left-continuous rearrangement of the function $|f|$ and $\alpha$ is given by the formula $1/\alpha = 1/p - 1/q$. Similarly, in the case when $q = \infty$ we have

$$
\mathcal{K}(t, f; L^p, L^\infty) \leq \left( \int_0^t (f^*(s))^p \, ds \right)^{1/p} \leq C_{p,\infty} \mathcal{K}(t, f; L^p, L^\infty), \quad t > 0.
$$

(2.4)

If the underlying measure space is the set of positive integers equipped with the counting measure, the couple $(L^p, L^q)$ can be naturally identified with the couple $(\ell^p, \ell^q)$ and so, setting $\tilde{f} := \sum_{n=1}^\infty f_n \chi_{[n-1, n]}$ for every sequence $(f_n)_{n=1}^\infty$, we have

$$
\mathcal{K}(t, (f_n); \ell^p, \ell^q) = \mathcal{K} \left( t, \tilde{f}; L^p(0, \infty), L^q(0, \infty) \right), \quad t > 0.
$$

(2.5)

Therefore, since $(\tilde{f})^* = \tilde{f}^*$, from (2.3) and (2.4) it follows that

$$
\mathcal{K}(t, (f_n); \ell^p, \ell^q) \leq \left( \int_0^t (\tilde{f}^*(s))^p \, ds \right)^{1/p} + t \left( \int_t^\infty (\tilde{f}^*(s))^q \, ds \right)^{1/q} \leq C_{p,q} \mathcal{K}(t, (f_n); \ell^p, \ell^q), \quad t > 0,
$$

(2.6)

and

$$
\mathcal{K}(t, (f_n); \ell^p, \ell^\infty) \leq \left( \int_0^t (\tilde{f}^*(s))^p \, ds \right)^{1/p} \leq C_{p,\infty} \mathcal{K}(t, (f_n); \ell^p, \ell^\infty), \quad t > 0.
$$

(2.7)

Let us define now, for every $0 < p < q < \infty$, the operators $P_p$ and $Q_q$ as follows: if $f \in L^p(0, \infty) + L^q(0, \infty)$, then

$$
P_p f(t) := \left( \int_0^t (f^*(s))^p \, ds \right)^{1/p}, \quad t > 0,
$$

and

$$
Q_q f(t) := \left( \int_t^\infty (f^*(s))^q \, ds \right)^{1/q}, \quad t > 0.
$$

By these notations, inequalities (2.6) and (2.7) can be rewritten as follows:

$$
\mathcal{K}(t, (f_n); \ell^p, \ell^q) \leq P_p \tilde{f}(t^\alpha) + tQ_q \tilde{f}(t^\alpha) \leq C_{p,q} \mathcal{K}(t, (f_n); \ell^p, \ell^q), \quad t > 0,
$$

(2.8)

and

$$
\mathcal{K}(t, (f_n); \ell^p, \ell^\infty) \leq P_p \tilde{f}(t^\alpha) \leq C_{p,\infty} \mathcal{K}(t, (f_n); \ell^p, \ell^\infty), \quad t > 0.
$$

(2.9)
In the sequence case, we define the operators \( P_p \) and \( Q_q \) by setting for every \( x = (x_k)_{k=1}^\infty \)

\[
P_p x = ((P_p x)_n), \quad (P_p x)_n := \left( \sum_{k=1}^n (x_k^*)^p \right)^{1/p}, \quad n \in \mathbb{N},
\]

\[
Q_q x = ((Q_q x)_n), \quad (Q_q x)_n := \left( \sum_{k=n}^\infty (x_k^*)^q \right)^{1/q}, \quad n \in \mathbb{N}.
\]

Clearly, for all \( x = (x_k)_{k=1}^\infty \in \ell^q \) and \( n \in \mathbb{N} \) we have

\[
(P_p x)_n = P_p \tilde{x}(n) \quad \text{and} \quad (Q_q x)_n = Q_q \tilde{x}(n).
\]

Consequently, inequalities (2.8) and (2.9) imply for all \( n \in \mathbb{N} \)

\[
(2.10) \quad \mathcal{K} (n^{1/\alpha}, x; \ell^p, \ell^q) \leq (P_p x)_n + n^{1/\alpha} (Q_q x)_n \leq C_{p,q} \mathcal{K} (n^{1/\alpha}, x; \ell^p, \ell^q)
\]

and

\[
(2.11) \quad \mathcal{K} (n^{1/p}, x; \ell^p, \ell^\infty) \leq (P_p x)_n \leq C_{p,\infty} \mathcal{K} (n^{1/p}, x; \ell^p, \ell^\infty).
\]

Let \((X_0, X_1)\) be a compatible pair of quasi-normed groups \((X_0, X_1)\). We introduce the approximation \( \mathcal{E} \)-functional by

\[
\mathcal{E}(t, x; X_0, X_1) := \inf \{ \|x - x_0\|_{X_1} : x_0 \in X_0, \|x_0\|_{X_0} \leq t \}, \quad x \in X_0 + X_1, \quad t > 0
\]

(cf. [7, Chapter 7]). Clearly, the mapping \( t \mapsto \mathcal{E}(t, x; X_0, X_1) \) is a decreasing function on \((0, \infty)\). There is the following connection between the \( \mathcal{E} \)- and \( \mathcal{K} \)-functionals (see [7, §7.1]):

\[
(2.12) \quad \mathcal{K}(t, x; X_0, X_1) = \inf_{s > 0} (s + t \mathcal{E}(s, x; X_0, X_1)), \quad t > 0.
\]

On the other hand, it is known (see e.g. [7, Lemma 7.1.3]) that for arbitrary \( x \in X_0 + X_1 \) we have

\[
\sup_{s > 0} s^{-1} (\mathcal{K}(s, x; X_0, X_1) - t) = \mathcal{E}^*(t, x; X_0, X_1), \quad t > 0,
\]

where \( \mathcal{E}^*(t, x; X_0, X_1) \) is the greatest convex minorant of \( \mathcal{E}(t, x; X_0, X_1) \), and also that for each \( \gamma \in (0, 1) \)

\[
\mathcal{E}^*(t, x; X_0, X_1) \leq \mathcal{E}(t, x; X_0, X_1) \leq (1 - \gamma)^{-1} \mathcal{E}^*(\gamma t, x; X_0, X_1), \quad t > 0.
\]

Assuming now that \( \mathcal{K}(t, y; X_0, X_1) \leq C \mathcal{K}(t, x; X_0, X_1) \) for all \( t > 0 \) and applying the last inequalities for \( \gamma = 1/2 \), we get

\[
\mathcal{E}(2t, y; X_0, X_1) \leq 2 \mathcal{E}^*(t, y; X_0, X_1) = 2 \sup_{s > 0} s^{-1} (\mathcal{K}(s, y; X_0, X_1) - t)
\]

\[
\leq 2 C \sup_{s > 0} s^{-1} (\mathcal{K}(s, x; X_0, X_1) - t/C) = 2 C \mathcal{E}^*(t/C, x; X_0, X_1)
\]

\[
\leq 2 C \mathcal{E}(t/C, x; X_0, X_1), \quad t > 0.
\]

As a result, by (2.12) and the latter inequality, we arrive at the following useful implications:

\[
(2.13) \quad \mathcal{E}(t, y) \leq \mathcal{E}(t, x), \quad t > 0 \implies \mathcal{K}(t, y) \leq \mathcal{K}(t, x), \quad t > 0,
\]
and for every $C > 0$

$$\mathcal{K}(t, y) \leq C\mathcal{K}(t, x), \quad t > 0 \quad \Rightarrow \quad \mathcal{E}(t, y) \leq 2C\mathcal{E}(t/(2C), x), \quad t > 0,$$

where $\mathcal{E}(t, z) := \mathcal{E}(t, z; X_0, X_1)$ and $\mathcal{K}(t, z) := \mathcal{K}(t, z; X_0, X_1)$, $z \in X_0 + X_1$.

Further, we will apply the above implications to the couples $(\ell_0^q, \ell_0^q)$, $0 < q \leq \infty$, and $(L^0, L^\infty)$ of (equivalence classes of) measurable functions on the semi-axis $(0, \infty)$ with the Lebesgue measure $m$. Here, $L^0 = L^0(0, \infty)$ is the group (with respect to the usual addition) of all measurable functions on $(0, \infty)$ with supports of finite measure, equipped by the quasi-norm

$$\|f\|_{L^0} := m\{t > 0 : f(t) \neq 0\}.$$ 

One can easily check that, for any $x = (x_k)_{k=1}^\infty \in \ell^q$ and all $t \geq 0$, we have

$$\mathcal{E}(t, x; \ell^q, \ell^q) = \begin{cases} (Qx)[t+1] = (\sum_{k=\lceil t \rceil+1}^{\infty} (x_k^*)^q)^{1/q} & \text{if } q < \infty \\ x^*_{\lceil t \rceil+1} & \text{if } q = \infty, \end{cases}$$

while for every $f \in L^0 + L^\infty$ and all $t > 0$

$$\mathcal{E}(t, f; L^0, L^\infty) = f^*(t).$$

We will use the standard (quasi-)Banach space notation (see e.g. [36] and [37]). In particular, throughout the paper, by $e_n$, $n \in \mathbb{N}$, we denote the vectors of the standard basis in sequence spaces, and for every sequences $x = (x_n)_{n=1}^\infty$, $y = (y_n)_{n=1}^\infty$ we set

$$\langle x, y \rangle := \sum_{n=1}^{\infty} x_n y_n \quad \text{(if the series converges)}.$$ 

By $[t]$ we denote the integer part of a number $t \in \mathbb{R}$ and by $\chi_A$ the characteristic function of a set $A \subset \mathbb{R}$. In what follows, $C$, $c$ etc. denote constants whose value may change from line to line or even within lines.

3. Auxiliary results

In this section we provide a self-contained presentation of some simple and apparently well-known facts.

3.1. An extension theorem for operators bounded on $\ell^p$-spaces, $0 < p < 1$. The purpose of this subsection is to give a detailed account related to an extension (or extrapolation) theorem for linear operators bounded on $\ell^p$, where $0 < p < 1$. This is one of the manifestations of the general principle, which allows to extend linear mappings bounded on quasi-Banach spaces to their Banach linear hulls (see, for instance, [30] and also [51], where a connection of these constructions with the interpolation theory is clarified).

**Theorem 3.1.** Let $S : \ell^q \to \ell^q$ be a linear map, where $0 \leq q < 1$. Then for every $r$, with $q < r \leq 1$, there is a linear extension $R : \ell^r \to \ell^r$ of $S$ such that

$$\|R\|_{\ell^r \to \ell^r} \leq \|S\|_{\ell^q \to \ell^q}.$$
We begin with proving an auxiliary result, where the following notation will be used. Let \( T : \ell^0 \to \ell^\infty \) be a bounded linear map. Then \( T \) can be identified with an infinite matrix \( \{t_{j,k}\}_{j,k=1}^\infty \), where \( t_{j,k} = \langle e_j, T(e_k) \rangle \). If \( x = (x_n)_{n=1}^\infty \in \ell^0 \) then \( Tx = y, \ y = (y_j)_{j=1}^\infty \) is defined by the finite sum

\[
y_j = \sum_{k=1}^\infty t_{j,k}x_k.
\]

For an arbitrary \( 0 < q \leq \infty \) let \( \Omega_q \) denote the space of all linear maps \( T : \ell^0 \to \ell^\infty \) such that the quantity

\[
\Theta_q(T) := \sup \{ \|Tx\|_{\ell^q} : x \in \ell^0, \|x\|_{\ell^q} = 1 \}
\]

is finite. Clearly, if \( T \in \Omega_q \) with the matrix \( (t_{j,k})_{j,k=1}^\infty \), then for each positive integer \( k \) the sequence \( t_k := (t_{j,k})_{j=1}^\infty = T(e_k) \) belongs to \( \ell^q \) and moreover

\[
\|t_k\|_{\ell^q} = \|T(e_k)\|_{\ell^q} \leq \Theta_q(T), \ k = 1, 2, \ldots.
\]

Hence, we see that the condition

\[
\sup_{k=1, 2, \ldots} \|t_k\|_{\ell^q} < \infty
\]

is necessary for \( T \in \Omega_q \). Furthermore, we have

**Lemma 3.2.** Let \( T : \ell^0 \to \ell^\infty \) be a linear map with the matrix \( (t_{j,k})_{j,k=1}^\infty \). Let \( t_k = (t_{j,k})_{j=1}^\infty, k \in \mathbb{N} \). Then, if \( 0 < q \leq 1 \) we have

(i) \( T \in \Omega_q \iff \sup_k \|t_k\|_{\ell^q} < \infty \) and

\[
\Theta_q(T) = \sup_{k=1, 2, \ldots} \|t_k\|_{\ell^q}.
\]

(ii) if \( T \in \Omega_q \) then there exists an extension \( \tilde{T} \) of \( T \) to \( \ell^q \) with

\[
\|\tilde{T}\|_{\ell^q \to \ell^q} = \sup_{k=1, 2, \ldots} \|t_k\|_{\ell^q}.
\]

**Proof.** (i). If \( T \in \Omega_q \) then the reasoning preceding to the lemma implies condition (3.2).

Conversely, assume that we have (3.2). Then, for each \( x = (x_n)_{n=1}^\infty \in \ell^0 \), denoting \( Tx = y, \ y = (y_j)_{j=1}^\infty \), and taking into account that \( 0 < q \leq 1 \), we have

\[
|y_j| = \sum_{k=1}^\infty t_{j,k}x_k \leq \sum_{k=1}^\infty |t_{j,k}| |x_k| \leq \left( \sum_{k=1}^\infty |t_{j,k}|^q |x_k|^q \right)^{1/q}.
\]
Hence,

\[ \|Tx\|_{\ell^q}^p = \sum_{j=1}^{\infty} |y_j|^p \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |t_{j,k}|^p |x_k|^p \]

\[ = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |t_{j,k}|^q \right) |x_k|^q = \sum_{k=1}^{\infty} \|t_k\|_{\ell^q}^q |x_k|^q \leq \sum_{k=1}^{\infty} \sup_{k=1,2,...} \|t_k\|_{\ell^q} \|x\|_{\ell^q}^q. \]  

(3.5)

Therefore, \( T \in \Omega_q \) and \( \Theta_q(T) \leq \sup_{k=1,2,...} \|t_k\|_{\ell^q} \). Moreover, combining this with inequality (3.1) and taking into account that \( \ell^0 \) is dense in \( \ell^1 \), we get (3.3).

(ii). Let \( T \in \Omega_q \). Since \( \ell^0 \) is dense in \( \ell^q \), it follows from inequality (3.5) that we can define the linear extension \( \tilde{T} : \ell^q \to \ell^q \) of \( T \) by \( \tilde{T}x = y \), where \( x = (x_j)_{j=1}^{\infty} \), \( y_j = \sum_{k=1}^{\infty} t_{j,k}x_k \). Since \( \|\tilde{T}\|_{\ell^q \to \ell^q} = \Theta_q(T) \), then in view of (3.3), formula (3.4) is also verified.

Proof of Theorem 7.1. Let \( T \) be the restriction of the given operator \( S \) to the space \( \ell^q \). Then \( T \in \Omega_q \) and \( \Theta_q(T) = \|S\|_{\ell^q \to \ell^q} \). It is obvious that the extension \( \tilde{T} \) of \( T \) to \( \ell^q \) defined in Lemma 3.2 is equal to \( S \). Now, if \((t_{j,k})_{j=1}^{\infty} \) is a matrix associated with \( T \) and \( t_k = (t_{j,k})_{j=1}^{\infty} \), the embedding \( \ell^q \subset \ell^r \), for \( q < r \), implies that \( \|t_k\|_{\ell^r} \leq \|t_k\|_{\ell^q} \), \( k = 1,2,..., \) and hence \( T \in \Omega_r \) with \( \Theta_r(T) \leq \Theta_q(T) \). Therefore, by Lemma 3.2 there exists an extension \( R \) of \( T \) defined on \( \ell^r \). Clearly, the restriction of \( R \) to \( \ell^q \) equals \( S \) and

\[ \|R\|_{\ell^r \to \ell^q} = \Theta_r(T) \leq \Theta_q(T) = \|S\|_{\ell^q \to \ell^q}, \]

which completes the proof. \( \square \)

**Corollary 3.3.** If \( 0 \leq p < q < 1 \), then we have

\[ \text{Int}(\ell^p, \ell^q) = \{X \subseteq \ell^q : X \in \text{Int}(\ell^p, \ell^1)\}. \]

**Proof.** Let \( X \in \text{Int}(\ell^p, \ell^q) \). Then, \( X \subseteq \ell^q \). Moreover, since \( \ell^q \in \text{Int}(\ell^p, \ell^1) \), then for each \( S \in \mathfrak{L}(\ell^p, \ell^1) \) we have \( S : \ell^q \to \ell^q \). Thus, \( S : X \to X \) and so \( X \in \text{Int}(\ell^p, \ell^1) \).

For the converse, assume that \( X \subseteq \ell^q \) and \( X \in \text{Int}(\ell^p, \ell^1) \). Then, \( \ell^p \subseteq X \subseteq \ell^q \). Furthermore, if \( S \in \mathfrak{L}(\ell^p, \ell^q) \), an application of Theorem 3.1 gives us a linear extension \( R : \ell^1 \to \ell^1 \) of \( S \). Since \( R : X \to X \), then \( S = R|_{\ell^q} : X \to X \). As a result, we conclude that \( X \in \text{Int}(\ell^p, \ell^q) \). \( \square \)

3.2. An interpolation property of symmetric quasi-Banach sequence spaces. It is well known that there are symmetric Banach function (resp. sequence) spaces which are not interpolation spaces with respect to the couple \((L^1(0, \infty), L^\infty(0, \infty))\) (resp. \((\ell^1, \ell^\infty))\) (see e.g. [55], [33] Theorem II.5.5, [37] Example 2.a.11). In contrast to that, all symmetric quasi-Banach function (resp. sequence) spaces are interpolation spaces with respect to the couple \((L^0(0, \infty), L^\infty(0, \infty))\) (resp. \((\ell^0, \ell^\infty))\) [2], [26]. Moreover, it turns out that each
symmetric quasi-Banach sequence space is an interpolation space between $\ell^p$ and $\ell^\infty$ for some appropriate $p > 0$.

For every $n \in \mathbb{N}$ we define the dilation operator $D_n : \ell^\infty \to \ell^\infty$ by

$$D_n : (x_k)_{k=1}^\infty \mapsto (x_{[(1+k)/n]})_{k=1}^\infty.$$  

Let $E$ be a symmetric quasi-normed sequence space. By the Aoki-Rolewicz theorem (see e.g. [7, Lemma 3.10.1]), we can define a subadditive functional on $E$, which is equivalent to the functional $\| \cdot \|_{E}^{\sigma_E}$ for some $\sigma_E > 0$ (which is called often the Aoki-Rolewicz index). Therefore, since $E$ is symmetric, by the definition of $D_n$, one can easily deduce that $\| D_n \|_{E \to E} \leq C n^{1/\sigma_E}$, $n = 1, 2, \ldots$. Hence, the upper Boyd index $q_E$ of $E$, defined by

$$q_E := \lim_{n \to \infty} \frac{\log \| D_n \|_{E \to E}}{\log n},$$

does not exceed $1/\sigma_E$ and so it is finite.

**Proposition 3.4.** For every symmetric quasi-Banach sequence space $E$ there exists $p > 0$ such that $E \in \text{Int}(\ell^p, \ell^\infty)$.

In the proof of this result we will use

**Lemma 3.5.** ([11, Theorem 1] or [12, Proposition 5.7]). Let $E$ be a symmetric quasi-Banach sequence space and let $p \in (0, \sigma_E)$. Then, the operator

$$x = (x_k)_{k=1}^\infty \mapsto \left\{ \left( \frac{1}{n} \sum_{k=1}^{n} (x_k^*)^p \right)^{1/p} \right\}_{n=1}^\infty$$

is bounded in $E$.

**Proof of Proposition 3.4.** By [13, Theorem 3], it suffices to prove that there is a constant $C >$ such that for any $x \in E$ and $y \in \ell^\infty$ satisfying

$$\sum_{k=1}^{n} (y_k^*)^p \leq \sum_{k=1}^{n} (x_k^*)^p, \quad n = 1, 2, \ldots,$$

we have $y \in E$ and $\|y\|_E \leq C \|x\|_E$.

Since

$$y_n^* \leq \left( \frac{1}{n} \sum_{k=1}^{n} (y_k^*)^p \right)^{1/p}, \quad n = 1, 2, \ldots,$$

then by the preceding lemma we have

$$\|y\|_E \leq \left\| \left( \frac{1}{n} \sum_{k=1}^{n} (y_k^*)^p \right)^{1/p} \right\|_E \leq \left\| \left( \frac{1}{n} \sum_{k=1}^{n} (x_k^*)^p \right)^{1/p} \right\|_E \leq C \|x\|_E.$$

$\square$
4. Arazy-Cwikel type properties for the scale of \( \ell^p \)-spaces, 
\[ 0 \leq p \leq \infty. \]

In [1], Arazy and Cwikel have proved that for all \( 1 \leq p < q \leq \infty \) and for each underlying measure space
\[ \text{Int} (L^p, L^q) = \text{Int} (L^1, L^q) \cap \text{Int} (L^p, L^\infty). \]

Recently, a similar result has been obtained in [12] for function quasi-Banach spaces on \((0, \infty)\) with the Lebesgue measure; namely, it was proved there that for all \( 0 < p < q < \infty \)
\[ \text{Int} (L^p, L^q) = \text{Int} (L^0, L^q) \cap \text{Int} (L^p, L^\infty). \]

In this section, we extend the above Arazy-Cwikel theorem in the sequence space setting.

**Theorem 4.1.** Let \( 0 \leq s < p < q < r \leq \infty \). Then, we have
\[ \text{Int} (\ell^p, \ell^q) = \text{Int} (\ell^s, \ell^r) \cap \text{Int} (\ell^p, \ell^q). \]

Further, in Section 5, we prove that the couple \((\ell^p, \ell^q)\) fails to have the Calderón-Mityagin property if \( 0 \leq p < q < 1 \). Thus, according to Theorem 4.1, Arazy-Cwikel type results do not imply, in general, that the corresponding couples possess necessarily the Calderón-Mityagin property.

The proof of Theorem 4.1 will be based on using a series of auxiliary results, the first of them is well-known.

**Proposition 4.2.** [15, Theorem 3] Let \( 0 < p < \infty \), and let \( x = (x_k)_{k=1}^\infty \in \ell^\infty \), \( y = (y_k)_{k=1}^\infty \in \ell^\infty \) be two sequences satisfying
\[ \sum_{k=1}^n (y_k^*)^p \leq \sum_{k=1}^n (x_k^*)^p, \quad n = 1, 2, \ldots \]

Then there exists a linear operator \( T : \ell^\infty \to \ell^\infty \) such that \( \|T\|_{\ell^p \to \ell^p} \leq 8^{1/p} \), \( \|T\|_{\ell^\infty \to \ell^\infty} \leq 2^{1/p} \) and \( Tx = y \).

**Proposition 4.3.** Let \( 1 \leq q < \infty \), \( C \geq 1 \), and let \( x = (x_k)_{k=1}^\infty \in \ell^q \), \( y = (y_k)_{k=1}^\infty \in \ell^q \) be two sequences such that
\[ \sum_{k=n}^{\infty} (y_k^*)^q \leq C \sum_{k=\lceil (n-1)/C \rceil + 1}^{\infty} (x_k^*)^q, \quad n = 1, 2, \ldots \]

Then there exists a linear operator \( S : \ell^q \to \ell^q \) such that \( \|S\|_{\ell^0 \to \ell^0} \leq 9(C + 1) \), \( \|S\|_{\ell^q \to \ell^q} \leq 6(C + 1) \) and \( Sx = y \).

**Proof.** Without loss of generality, we can assume that \( x = (x_n)_{n=1}^\infty \) and \( y = (y_n)_{n=1}^\infty \) are nonnegative and nonincreasing.

By [3, Theorem 1; see also its proof], we can find a positive linear operator \( Q : \ell^1 \to \ell^1 \) such that \( \|Q\|_{\ell^0 \to \ell^0} \leq 9(C + 1) \), \( \|Q\|_{\ell^1 \to \ell^1} \leq 6(C + 1) \) and \( Q((x_k^*)_{k=1}^\infty) = ((y_k^*)_{k=1}^\infty) \). Let us define the mapping \( Q' \) by
\[ Q'((z_k)_{k=1}^\infty) := (Q((|z_k|^q)_{k=1}^\infty))^{1/q}. \]
Then, \( Q' \) is a subadditive and positively homogeneous operator bounded on \( \ell^q \) such that \( \|Q'|_{\ell^p \to \ell^q} \leq \|Q\|_{\ell^p \to \ell^q} \), \( \|Q'|_{\ell^1 \to \ell^q} \leq \|Q\|_{\ell^1 \to \ell^1} \) and \( Q' x = y \).

Now, let us define the linear operator \( S' \) on the one-dimensional subspace \( H_x := \{ \alpha x, \alpha \in \mathbb{R} \} \) in \( \ell^q \) by \( S'(\alpha x) := \alpha y, \alpha \in \mathbb{R} \). Then, \( S' z \leq Q' z, \ z \in H_x, \) and \( S' x = y \). Hence, by the Hahn-Banach-Kantorovich theorem (see e.g. [53, p. 120]), there exists a linear extension \( S \) of \( S' \) to the whole of \( \ell^q \) such that \( S z \leq Q' z \) for all \( z \in \ell^q \). Since the operator \( S \) satisfies all the requirements, the proof is completed. \( \square \)

**Remark 4.4.** Alternatively, instead of Theorem 1 from [3], in the proof of Proposition 4.3 one can apply Lemma 5.2 from [12] together with using the dilation operators (see [36]).

**Theorem 4.5.** Suppose \( 0 < p < q < \infty \) and \( q \geq 1 \). Let \( x = (x_k)_{k=1}^\infty \in \ell^q \) and \( y = (y_k)_{k=1}^\infty \in \ell^q \) be two sequences such that

\[
\left( \sum_{k=1}^n (y_k^*)^p \right)^{1/p} + n^{1/\alpha} \left( \sum_{k=n}^\infty (y_k^*)^q \right)^{1/q} \leq \left( \sum_{k=1}^n (x_k^*)^p \right)^{1/p} + n^{1/\alpha} \left( \sum_{k=n}^\infty (x_k^*)^q \right)^{1/q}, \quad n \in \mathbb{N},
\]

where \( 1/\alpha = 1/p - 1/q \).

Then, we can find linear operators \( T : \ell^\infty \to \ell^\infty \) and \( S : \ell^q \to \ell^q \) such that \( \|T\|_{\ell^p,\ell^\infty} \leq 8^{1/p}, \|S\|_{\ell^p,\ell^q} \leq 18 \), with \( y = Tx + Sx \).

**Proof.** As above, we can (and will) assume that \( x = (x_n)_{n=1}^\infty \) and \( y = (y_n)_{n=1}^\infty \) are nonnegative and nonincreasing sequences. The proof below will be modelled on the arguments used in [1].

Let us define

\[
A(n) := \sum_{k=1}^n (x_k^* - y_k^*), \quad n \in \mathbb{N}, \quad \text{and} \quad A := \{ n \in \mathbb{N} : A(n) \geq 0 \},
\]

\[
B(n) := \sum_{k=n}^\infty (x_k^* - y_k^*), \quad n \in \mathbb{N}, \quad \text{and} \quad B := \{ n \in \mathbb{N} : B(n) \geq 0 \}.
\]

Then, from the assumption of the theorem it follows that \( A \cup B = \mathbb{N} \).

Observe that in the case when \( A = \mathbb{N} \) it follows

\[
\sum_{k=1}^n y_k^* \leq \sum_{k=1}^n x_k^*, \quad n \in \mathbb{N},
\]

and hence Proposition 4.2 implies that \( y = Tx \) for some operator \( T : \ell^\infty \to \ell^\infty \) bounded in the couple \( (\ell^p, \ell^\infty) \). Similarly, if \( B = \mathbb{N} \) then

\[
\sum_{k=n}^\infty y_k^* \leq \sum_{k=n}^\infty x_k^*, \quad n \in \mathbb{N}.
\]

Consequently, since \( q \geq 1 \), by Proposition 4.3 \( y = Sx \) for some \( S : \ell^q \to \ell^q \) bounded in the couple \( (\ell^q, \ell^q) \). So, in these cases the desired result follows.

Assume now that neither \( A = \mathbb{N} \) nor \( B = \mathbb{N} \). We represent \( A \) as the union of successive maximal pairwise disjoint intervals of positive integers, i.e., \( A = \)
$U_{i \in I_1}A_i$, where $A_i = [n_i, m_i]$, $m_i + 1 < n_{i+1}$. Let $B = \cup_{i \in I_2}B_i$ be the corresponding union for $B$. These collections of intervals may be finite or infinite.

Let $1 \in A$, i.e., $A_1 = [1, m_1]$. Then, $A(\min (l, m_1)) \geq 0$ for every $l \in \mathbb{N}$, and hence we have

$$
\sum_{j=1}^{l} \left( (\chi_{A_j}x)_{j}^{*} \right)^{p} = \sum_{j=1}^{\min(l, m_1)} x_{j}^{p} = A(\min (l, m_1)) + \sum_{j=1}^{\min(l, m_1)} y_{j}^{p} \\
\geq \sum_{j=1}^{\min(l, m_1)} y_{j}^{p} = \sum_{j=1}^{l} \left( (\chi_{A_j}y)_{j}^{*} \right)^{p}.
$$

Suppose now that $A_i = [n_i, m_i]$, where $n_i \geq 2$, be a finite interval. Then, $\min (l + n_i - 1, m_i) \in A$, $n_i - 1 \in A^c := \mathbb{N} \setminus A$ for all $l \in \mathbb{N}$. Therefore, $A(\min (l + n_i - 1, m_i)) \geq 0$, $A(n_i - 1) \leq 0$ and

$$
\sum_{j=1}^{l} \left( (\chi_{A_j}x)_{j}^{*} \right)^{p} = \sum_{j=n_i}^{l+n_i-1} (\chi_{A_j}x)_{j}^{p} = \sum_{j=n_i}^{\min(l+n_i-1, m_i)} x_{j}^{p} \\\n= A(\min (l + n_i - 1, m_i)) - A(n_i - 1) + \sum_{j=n_i}^{\min(l+n_i-1, m_i)} y_{j}^{p} \\
\geq \sum_{j=n_i}^{\min(l+n_i-1, m_i)} y_{j}^{p} = \sum_{j=1}^{l} \left( (\chi_{A_j}y)_{j}^{*} \right)^{p}.
$$

If finally $A_i = [n_i, \infty)$ then, for any $l \in \mathbb{N}$, we have $n_i + l - 1 \in A_i$ and $n_i - 1 \in A^c$. Hence, as above,

$$
\sum_{j=1}^{l} \left( (\chi_{A_j}x)_{j}^{*} \right)^{p} = \sum_{j=n_i}^{n_i+l-1} x_{j}^{p} = A(n_i + l - 1) - A(n_i - 1) + \sum_{j=n_i}^{n_i+l-1} y_{j}^{p} \\
\geq \sum_{j=n_i}^{n_i+l-1} y_{j}^{p} = \sum_{j=1}^{l} \left( (\chi_{A_j}y)_{j}^{*} \right)^{p}.
$$

Thus, by the estimates obtained and Proposition 4.13 for each $i \in I_1$, we can select a linear operator $T_i : \ell^\infty \rightarrow \ell^\infty$, $\|T_i\|_{\mathcal{L}(\ell^p, \ell^\infty)} \leq 8^{1/p}$, with $T_i(\chi_{A_j}x) = \chi_{A_j}y$. Now setting $T = \sum_{i \in I_1} \chi_{A_i}T_i \chi_{A_i}$, we see that the operator $T$ is well-defined on $\ell^\infty$,

$$
\|T\|_{\mathcal{L}(\ell^p, \ell^\infty)} \leq \sup_{i \in I_1} \|T_i\|_{\mathcal{L}(\ell^p, \ell^\infty)} \leq 8^{1/p}
$$

and $T(\chi_{A_i}x) = \chi_{A_i}y$.

Similarly, let $B_i = [n'_i, m'_i]$ be a finite interval, and let $l \in \mathbb{N}$ be such that $l \leq m'_i - n'_i + 1$. Then, $l + n'_i - 1 \in B$ and $m'_i + 1 \in B^c$. Consequently,
\[ B(l + n'_i - 1) \geq 0 \text{ and } B(m'_i + 1) \leq 0. \text{ Hence,} \]
\[
\sum_{j=l}^{\infty} \left( (\chi_{B,x})_{j}^* \right)^q = \sum_{j=l+n'_i-1}^{m'_i} x_j^q = B(l + n'_i - 1) - B(m'_i + 1) \\
+ \sum_{j=l+n'_i-1}^{m'_i} y_j^q \geq \sum_{j=l+n'_i-1}^{m'_i} y_j^q = \sum_{j=l}^{\infty} \left( (\chi_{B,y})_{j}^* \right)^q.
\]

Observe that the latter inequality holds also for \( l > m'_i - n'_i + 1 \), because in this case its both sides vanish.

In the case when \( B_i = [n'_i, \infty) \) for every \( l \in \mathbb{N} \) we get
\[
\sum_{j=l}^{\infty} \left( (\chi_{B,x})_{j}^* \right)^q = \sum_{j=n'_i+l-1}^{\infty} x_j^q = B(n'_i + l - 1) + \sum_{j=n_i+l-1}^{\infty} y_j^q \\
\geq \sum_{j=n_i+l-1}^{\infty} y_j^q = \sum_{j=l}^{\infty} \left( (\chi_{B,y})_{j}^* \right)^q,
\]
because of \( n'_i + l - 1 \in B_i \) for all \( l \in \mathbb{N} \).

As a result, by using Proposition 4.3 for every \( i \in I_2 \) we can find an operator \( S_i : \ell^q \rightarrow \ell^q, \|S_i\|_{\mathcal{L}(\ell^p, \ell^q)} \leq 18 \), with \( S_i(\chi_{B,x}) = \chi_{B,y} \). Then, the operator \( S' := \sum_{i=1}^{\infty} \chi_{B_i} S_i \chi_{B_i} \) is well-defined on \( \ell^q \),
\[
\|S'\|_{\mathcal{L}(\ell^p, \ell^q)} \leq \sup_{i \in I_2} \|S_i\|_{\mathcal{L}(\ell^p, \ell^q)} \leq 18
\]
and \( S'(\chi_{B,x}) = \chi_{B,y} \). Denoting \( S := \chi_{B\setminus A} S' \), we see that \( \|S\|_{\mathcal{L}(\ell^p, \ell^q)} \leq 18 \), and \( S x = \chi_{B\setminus A} y \). Since
\[
y = \chi_A y + \chi_{B\setminus A} y = T x + S x,
\]
the operators \( T : \ell^\infty \rightarrow \ell^\infty \) and \( S : \ell^q \rightarrow \ell^q \) satisfy all the requirements and so the proof of the theorem is completed. \(\square\)

It is a classical result of the interpolation theory that the couple \( (L^p, L^q), 1 \leq p < q \leq \infty \), has the uniform Calderón-Mityagin property with respect to the class of all interpolation Banach spaces (note that it is also a special case of the well-known Sparr theorem, see [57]). The preceding results of this section combined with the observations made in Subsection 2.2 imply the following extension of this result to the quasi-Banach case in the sequence space setting.

**Corollary 4.6.** Let \( 0 \leq p < q \leq \infty \) and \( q \geq 1 \). Then \( (\ell^p, \ell^q) \) is a uniform Calderón-Mityagin couple.

**Proof.** Let \( x = (x_k)_{k=1}^{\infty} \in \ell^q \) and \( y = (y_k)_{k=1}^{\infty} \in \ell^q \) be two nonincreasing and nonnegative sequences such that
\[
K(t, y; \ell^p, \ell^q) \leq K(t, x; \ell^p, \ell^q), \quad t > 0.
\]
We consider four cases depending on values of the numbers \( p \) and \( q \) separately.
Let first $0 < p < q = \infty$. Then, by (2.11), we have
\[
\sum_{k=1}^{n} (y_k^*)^p \leq C_{p,\infty}^p \sum_{k=1}^{n} (x_k^*)^p, \quad n = 1, 2, \ldots
\]
Hence, from Proposition 4.2 it follows the existence of a linear operator $T : \ell^\infty \to \ell^\infty$ such that $\|T\|_{\ell^p,\ell^\infty} \leq 8^{1/p}C_{p,\infty}$ and $Tx = y$. Thus, $y \in \text{Orb}(x; \ell^p, \ell^\infty)$ and $\|y\|_{\text{Orb}(x)} \leq 8^{1/p}C_{p,\infty}$.

Suppose now $p = 0$ and $q = \infty$. Then from (2.14), (2.15) and the inequality $\lfloor k/2 \rfloor + 1 \geq \lceil (k + 1)/2 \rceil$, $k = 1, 2, \ldots$, it follows that
\[
y_k \leq 2x_{\lfloor k/2 \rfloor + 1} \leq 2x_{\lceil (k + 1)/2 \rceil} = 2(D_2x)_k, \quad k = 1, 2, \ldots,
\]
where $D_2$ is the doubling operator (see (3.6)). For each $u = (u_k)_{k=1}^\infty \in \ell^\infty$ we define the multiplication operator $T$ by
\[
T u \left( u_k \cdot \frac{y_k}{2(D_2x)_k} \right)_{k=1}^\infty.
\]
 Obviously, $\|T\|_{\ell^p,\ell^\infty} \leq 1$. Therefore, if $S := 2TD_2$, then $\|S\|_{\ell^p,\ell^\infty} \leq 2$ and $Sx = y$. Thus, $y \in \text{Orb}(x; \ell^0, \ell^\infty)$ and $\|y\|_{\text{Orb}(x)} \leq 2$.

Next, if $0 = p < q < \infty$, from estimate (4.1), implication (2.14) with $X_0 = \ell^0$, $X_1 = \ell^q$ and formula (2.15) it follows that
\[
\sum_{k=n}^{\infty} (y_k^*)^q \leq 2 \sum_{k=[(n-1)/2]+1}^{\infty} (x_k^*)^q, \quad n = 1, 2, \ldots
\]
Then, by Proposition 4.3 there exists a linear operator $S : \ell^q \to \ell^q$ such that $\|S\|_{\ell^p,\ell^q} \leq 27$ and $Sx = y$. Therefore, we obtain again that $y \in \text{Orb}(x; \ell^0, \ell^q)$ and $\|y\|_{\text{Orb}} \leq 27$.

Finally, suppose $0 < p < q < \infty$. Then, combining (4.1) with the Holmstedt formula (2.10) and applying Theorem 4.5 we can find linear operators $T : \ell^\infty \to \ell^\infty$ and $S : \ell^q \to \ell^q$ such that $\|T\|_{\ell^p,\ell^\infty} \leq 8^{1/p}C_{p,q}$ and $\|S\|_{\ell^p,\ell^q} \leq 18C_{p,q}$, with $y = Tx + Sx$. Since by interpolation $T$ (resp. $S$) is bounded in $\ell^q$ (resp. $\ell^p$) and $\|T\|_{\ell^p,\ell^q} \leq C_{p,q}$ (resp. $\|S\|_{\ell^p,\ell^q} \leq C_{p,q}$), we conclude that $y \in \text{Orb}(x; \ell^p, \ell^q)$ and $\|y\|_{\text{Orb}(x)} \leq C_{p,q}$. Thus, the theorem is proved.

**Remark 4.7.** According to [12, Theorem 1.1], the result, analogous to Theorem 4.5 holds for the couple $(L^p, L^q)$ of functions defined on the semi-axis $(0, \infty)$ with the Lebesgue measure for all $0 \leq p < q \leq \infty$ without any extra conditions imposed on $q$. Consequently, arguing in the same way as in the proof of Corollary 4.6 we conclude that the couple $(L^p, L^q)$ is a uniform Calderón-Mityagin couple for all $0 \leq p < q \leq \infty$. At the same time, it is worth to note that the condition $q \geq 1$ cannot be skipped in Theorem 4.5 and Corollary 4.6 (see Corollary 5.3 below), which shows an essential difference in interpolation properties of quasinormed $L^p$-couples in function and sequence cases.

Applying Corollary 4.6 and Theorem 3.1 we obtain a complete description of orbits of elements in the couple $(\ell^p, \ell^q)$ for all nonnegative values $p$ and $q$. 

Corollary 4.8. Let $0 \leq p < q \leq \infty$. For every $x \in \ell^q$

(a) if $q \geq 1$, then $\text{Orb} (x; \ell^p, \ell^q) = K - \text{Orb} (x; \ell^p, \ell^q)$;

(b) if $q < 1$, then $\text{Orb} (x; \ell^p, \ell^q) = \text{Orb} (x; \ell^p, \ell^1) = K - \text{Orb} (x; \ell^p, \ell^1)$.

Moreover, the quasi-norms of the above spaces are equivalent with constants independent of $x \in \ell^q$.

Now, we are able to prove the following additivity property for orbits of elements with respect to the couples $(\ell^p, \ell^q)$, $0 \leq p < q \leq \infty$ (cf. [10]).

Proposition 4.9. Let $0 \leq s < p < q < r \leq \infty$. Then, for each $x \in \ell^p + \ell^q$ we have

$$\text{Orb} (x; \ell^p, \ell^q) = \text{Orb} (x; \ell^s, \ell^q) + \text{Orb} (x; \ell^p, \ell^r),$$

and the quasi-norms of these spaces are equivalent with a constant independent of $x$.

Proof. Observe that, by interpolation, with a constant independent of $x$ it follows

$$\text{Orb} (x; \ell^p, \ell^q) \supset \text{Orb} (x; \ell^s, \ell^q) + \text{Orb} (x; \ell^p, \ell^r).$$

Therefore, it remains to prove the opposite embedding.

Assume first that $q \geq 1$. Then, if $y \in \text{Orb} (x; \ell^p, \ell^q)$, from Corollary 4.8 it follows that $y \in K - \text{Orb} (x; \ell^p, \ell^q)$. Then, according to Theorem 4.5, we may write $y = T_0 x + T_1 x$, where $T_0 \in \mathcal{L}(\ell^q, \ell^q)$ and $T_1 \in \mathcal{L}(\ell^q, \ell^\infty)$. Hence, again by interpolation $T_0 : \ell^s \to \ell^q$, $T_1 : \ell^r \to \ell^q$. This yields $T_0 x \in \text{Orb} (x; \ell^q, \ell^q)$ and $T_1 x \in \text{Orb} (x; \ell^p, \ell^r)$. Thus, with a constant independent of $x$

$$\text{Orb} (x; \ell^p, \ell^q) \subset \text{Orb} (x; \ell^s, \ell^q) + \text{Orb} (x; \ell^p, \ell^r),$$

and in this case the result follows.

Let now $q < 1$. Applying successively Corollary 4.8 and the above result for the couple $(\ell^p, \ell^1)$ and the numbers $s < p < 1 < \infty$, we get

$$\text{Orb} (x; \ell^p, \ell^q) = \text{Orb} (x; \ell^p, \ell^1)$$

$$= \text{Orb} (x; \ell^s, \ell^1) + \text{Orb} (x; \ell^p, \ell^\infty)$$

$$\subset \text{Orb} (x; \ell^s, \ell^q) + \text{Orb} (x; \ell^p, \ell^r),$$

and everything is done. \(\square\)

In the proof of the next result we follow the reasoning used in the paper [10].

Proposition 4.10. Let $(X_0, X_3)$ and $(X_1, X_2)$ be two couples of quasi-Banach Abelian groups such that $X_2 \in \text{Int} (X_0, X_3)$ and $X_3 \in \text{Int} (X_1, X_2)$. Moreover, assume that

\[ X_2 \cap X_3 \subseteq (X_0 \cap X_3) + (X_1 \cap X_2), \quad (X_0 + X_3) \cap (X_1 + X_2) \subseteq X_2 + X_3, \]

and for every $x \in X_2 + X_3$

\[ \text{Orb} (x; X_2, X_3) = \text{Orb} (x; X_0, X_3) + \text{Orb} (x; X_1, X_2). \]

Then,

$$\text{Int} (X_2, X_3) = \text{Int} (X_0, X_3) \cap \text{Int} (X_1, X_2).$$
Proof. Suppose first $X \in \text{Int}(X_2, X_3)$. If $T \in \mathcal{L}(X_0, X_3)$ it follows by interpolation that $T : X_2 \to X_2$ and hence $T \in \mathcal{L}(X_2, X_3)$. This implies that $T : X \to X$, i.e., $X \in \text{Int}(X_0, X_3)$. In the same manner one can check that $X \in \text{Int}(X_1, X_2)$.

Conversely, let $X \in \text{Int}(X_0, X_3) \cap \text{Int}(X_1, X_2)$. Then, by (4.2),

$$X_2 \cap X_3 \subseteq (X_0 \cap X_3) + (X_1 \cap X_2) \subseteq X \subseteq (X_0 + X_3) \cap (X_1 + X_2) \subseteq X_2 + X_3,$$

that is, $X$ is an intermediate space between $X_2$ and $X_3$.

Moreover, for each $x \in X$ we have $\text{Orb}(x; X_0, X_3) \subseteq X$ and $\text{Orb}(x; X_1, X_2) \subseteq X$. Hence, applying assumption (4.3), we conclude that $\text{Orb}(x; X_2, X_3) \subseteq X$. Thus, $X \in \text{Int}(X_2, X_3)$, and the desired result follows. \hfill \square

Now, we are ready to prove that the full scale of $\ell^p$-spaces, $0 \leq p \leq \infty$, possesses the Arazy-Cwikel property.

Proof of Theorem 4.1. It suffices to apply Propositions 4.9 and 4.10 \hfill \square

In conclusion of this section we deduce the following result, which contains, in particular, a solution of the conjecture stated by Levitina, Sukochev and Zanin in the paper [35] (its earlier version may be found in the preprint [34]).

As above, let $\sigma_E$ denote the Aoki-Rolewicz index of the quasi-Banach sequence space $E$ (see Section 2.1).

Theorem 4.11. Let $q > 0$ and let $E$ be a quasi-Banach sequence space. The following assertions are equivalent:

(i) $E \in \text{Int}(\ell^0, \ell^q)$;

(ii) $E \in \text{Int}(\ell^p, \ell^q)$ for each $p \in (0, \sigma_E)$;

(iii) $E \in \text{Int}(\ell^0, \ell^q)$ and $E \in \text{Int}(\ell^p, \ell^\infty)$ for each $p \in (0, \sigma_E)$.

Proof. Observe that from (i) it follows that $E$ is symmetric (see e.g. [12, Lemma 1.11]). Therefore, the implication (i) $\implies$ (iii) is an immediate consequence of Proposition 3.3 (see also its proof). In turn, the equivalence (ii) $\iff$ (iii) follows from Theorem 4.1. Since the implication (iii) $\implies$ (i) is obvious, the proof is completed. \hfill \square

By the latter result, we are able to determine the exact assumptions under which the above-mentioned Levitina-Sukochev-Zanin conjecture is resolved in affirmative. To justify this claim, we consider the following conditions, assuming that $q > 0$ and $E$ is a quasi-Banach sequence space such that $E \subseteq \ell^\infty$:

(a) for any $x = (x_n)_{n=1}^\infty \in E$ and $y = (y_n)_{n=1}^\infty \in \ell^\infty$ such that

$$\sum_{n=m}^\infty (y_n^*)^q \leq \sum_{n=m}^\infty (x_n^*)^q, \ m \in \mathbb{N},$$

we have $y \in E$ and $\|y\|_E \leq C\|x\|_E$, where $C$ depends only on $E$ and $q$;

(b) there exists $p \in (0, q)$ such that $E \in \text{Int}(\ell^p, \ell^\infty)$.

In [35], the authors were asking whether the conditions (a) and (b) are equivalent, and then, in [12], the affirmative answer to this question has been given in the case when $q \geq 1$ (see also [18]).
To show a connection of the above question with Theorem 4.11, suppose that elements \( x \in E \) and \( y \in \ell^q \) satisfy the condition
\[
K(t, y; \ell^0, \ell^q) \leq K(t, x; \ell^0, \ell^q), \quad t > 0.
\]
Then, by (2.14) and (2.15), we have
\[
\sum_{n=m}^{\infty} (y_n^*)^q \leq 2^q \sum_{n=(m-1)/2+1}^{\infty} (x_n^*)^q, \quad m \in \mathbb{N}.
\]
One can easily check that
\[
\sum_{n=(m-1)/2+1}^{\infty} (x_n^*)^q \leq \sum_{n=m}^{\infty} (D_2 x^*)_n^q, \quad m \in \mathbb{N}.
\]
In consequence,
\[
\sum_{n=m}^{\infty} (y_n^*)^q \leq 2^q \sum_{n=m}^{\infty} (D_2 x^*)_n^q, \quad m \in \mathbb{N}.
\]
If the condition (a) holds, then clearly the space \( E \) is symmetric and hence from the latter inequality it follows easily that
\[
\|y\|_E \leq 2C\|D_2 x\|_E \leq 4C\|x\|_E,
\]
where \( C \) is the constant from (a). Therefore, \( E \) is a uniform \( K \)-monotone space with respect to the couple \( (\ell^0, \ell^q) \) and hence \( E \in \text{Int} (\ell^0, \ell^q) \). Thus, (a) implies condition (i) from Theorem 4.11 and so equivalence \((i) \iff (ii)\) of this theorem shows that the implication \((a) \implies (b)\) holds, in fact, for each \( q > 0 \) (including also the non-Calderón-Mityagin range \( 0 < q < 1 \)).

Regarding the converse direction, the situation in the cases \( q \geq 1 \) and \( 0 < q < 1 \) is different. If \( q \geq 1 \), then \( (\ell^0, \ell^q) \) is a uniform Calderón-Mityagin couple (see Corollary 4.6). Hence, from (b) it follows that \( E \) is a uniformly \( K \)-monotone space with respect to \( (\ell^0, \ell^q) \), and so we have (a). Thus, in this case all conditions (a), (b) and (i) are equivalent.

If \( 0 < q < 1 \), in general, the implication \((b) \implies (a)\) is not longer true. Indeed, since the couple \( (\ell^0, \ell^q) \) has not the Calderón-Mityagin property (see Theorem 5.2 in the next section), there is a non-\( K \)-monotone interpolation space \( E_0 \) between \( \ell^0 \) and \( \ell^q \). The above discussion shows then that \( E_0 \) does not satisfy the condition (a). At the same time, since from Theorem 4.11 it follows that \( E_0 \in \text{Int} (\ell^p, \ell^q) \) for each \( p \in (0, r_{E_0}) \), the condition (b) for \( E_0 \) is fulfilled. This proves the last claim.

5. A CHARACTERIZATION OF COUPLES \((\ell^p, \ell^q)\) WITH THE CALDERÓN-MITYAGIN PROPERTY.

Here, we prove one of the main results of this paper, showing that \((\ell^p, \ell^q)\) fails to be a Calderón-Mityagin couple whenever \( 0 \leq p < q < 1 \). In fact, we establish even a stronger result, which implies, in particular, that for every sequence \( g \in \ell^q \setminus \ell^p \)
there exists a sequence \( f \in \ell^q \) such that \( g \in K - \text{Orb}(f, \ell^p, \ell^q) \setminus \text{Orb}(f, \ell^p, \ell^q) \).

First, we introduce the following useful notion.

**Definition 5.1.** Let \( \vec{X} = (X_0, X_1) \) be a quasi-Banach couple (or more generally, a couple of quasi-Banach Abelian groups) and let \( y \in X_0 + X_1 \). Then, we say that \( y \) is a **Calderón-Mityagin element** (**CM-element**, in brief) with respect to \( \vec{X} \) provided if the inequality

\[
K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1), \quad t > 0,
\]

which holds for an element \( x \in X_0 + X_1 \), implies the existence of an operator \( T \in \mathcal{L}(X_0, X_1) \) with \( Tx = y \). The set of all CM-elements with respect to the couple \( \vec{X} \) we will denote by \( CM(\vec{X}) = CM(X_0, X_1) \).

Clearly, \( CM(\vec{X}) = X_0 + X_1 \) if and only if the couple \( \vec{X} \) has the Calderón-Mityagin property.

Let a quasi-Banach couple \( \vec{X} \) be such that the sum \( X_0 + X_1 \) is continuously embedded into a Banach space \( Z \). Show that then we have

\[
CM(\vec{X}) \supset X_0 \cap X_1.
\]

Indeed, let inequality (5.1) to be hold for some \( y \in X_0 \cap X_1 \) and \( x \in X_0 + X_1 \). Clearly, we may assume that \( x \neq 0 \). Then, according to Hahn-Banach Theorem, there is a linear functional \( x^* \in Z^* \), \( \|x^*\|_{Z^*} = 1/\|x\|_Z \), with \( x^*(x) = 1 \). Now, if the operator \( T \) is defined by \( Tu := x^*(u)y \), then \( T \) is bounded from \( X_0 + X_1 \) into \( X_0 \cap X_1 \) and \( Tx = y \). Moreover, for \( i = 0, 1 \) we have

\[
\|Tu\|_{X_i} \leq |x^*(u)\|\|y\|_{X_i} \leq \|x^*\|_{Z^*}\|u\|_Z\|y\|_{X_i} \leq \frac{C\|y\|_{X_i}}{\|x\|_Z}\|u\|_{X_i},
\]

where \( C \) is the embedding constant of \( X_0 + X_1 \) into \( Z \), and (5.2) is proved.

In particular, since \( \ell^p \subset \ell^1 \) continuously if \( r \in (0, 1) \), we have that \( \ell^p \subset CM(\ell^p, \ell^q) \) for all \( 0 < p < q \leq \infty \). Moreover, this result can be extended also to the extreme case \( p = 0 \). Indeed, assuming that elements \( x \in \ell^0 \) and \( y \in \ell^0 \), \( y \neq 0 \), satisfy (5.1) in the case \( \vec{X} = (\ell^0, \ell^q) \), one can readily check that the operator \( Tu := x^*(u)y \), where \( x^* \in (\ell^1)^* = \ell^{\infty} \) such that \( \|x^*\|_{\ell^{\infty}} = 1/\|x\|_{\ell^1} \) and \( x^*(x) = 1 \), is a bounded homomorphism on \( \ell^0 \) and \( \|T\|_{\ell^0 \to \ell^0} \leq \|y\|_{\ell^0} \).

The following result shows that for the couple \( (\ell^p, \ell^q) \), \( 0 \leq p < q < 1 \), embedding (5.2) turns into an equality.

**Theorem 5.2.** Let \( 0 \leq p < q < 1 \). Then, \( CM((\ell^p, \ell^q)) = \ell^p \). In particular, \( (\ell^p, \ell^q) \) is not a Calderón-Mityagin couple.

Applying Theorems 4.6 and 5.2, we get the following characterization of couples \((\ell^p, \ell^q)\) with the Calderón-Mityagin property.
Corollary 5.3. Let $0 \leq p < q \leq \infty$. Then, the following conditions are equivalent:

(i) $(\ell^p, \ell^q)$ is a Calderón-Mityagin couple;
(ii) $(\ell^p, \ell^q)$ is a uniform Calderón-Mityagin couple;
(iii) $q \geq 1$.

As we will see a little bit later, Theorem 5.2 is a straightforward consequence of the following result.

Theorem 5.4. Assume $0 \leq p < q < r \leq \infty$. Let $g = (g_n)_{n=1}^{\infty} \in \ell^q \setminus \ell^p$ be a nonnegative and nonincreasing sequence. Then, there exists a nonnegative, nonincreasing sequence $f = (f_n)_{n=1}^{\infty}$ such that

\begin{align*}
\text{(5.3)} & \quad f \in \ell^q \text{ and } \|f\|_{\ell^q} = \|g\|_{\ell^q}, \\
\text{(5.4)} & \quad K\left( t, g; \ell^0, \ell^q \right) \leq K\left( t, f; \ell^0, \ell^q \right), \quad t > 0,
\end{align*}

but

\begin{align*}
\text{(5.5)} & \quad \lim \inf_{t \to \infty} \frac{K\left( t, f; \ell^p, \ell^r \right)}{K\left( t, g; \ell^p, \ell^r \right)} = 0.
\end{align*}

Before proceeding with the proof of this theorem, we deduce some its consequences.

Corollary 5.5. Let $0 \leq p < q < r \leq \infty$, $q \geq 1$. For any $g \in \ell^q \setminus \ell^p$ there exists $f \in \ell^q$ such that

\begin{align*}
\text{g} & \in \text{Orb} (f, \ell^s, \ell^q) \text{ for each } s \in [0, q) \\
\text{and} & \\
\text{g} & \notin \text{Orb} (f, \ell^p, \ell^r).
\end{align*}

Proof. Clearly, we can assume that $f$ is nonnegative and nonincreasing. Then, by Theorem 5.4 there exists $f \in \ell^q$ satisfying conditions (5.4) and (5.5). Recall that, for every $0 < s < q$, $\ell^s$ is an interpolation space between $\ell^0$ and $\ell^q$, which can be obtained by applying the classical real $K$-method to the couple $(\ell^0, \ell^q)$ (see Section 2.2 and [7, Theorem 7.1.7]). Therefore, combining inequality (5.4) together with the well-known reiteration theorem (see e.g. [9, Theorem 3.1.2, p. 332]), we can find, for every $s \in [0, q)$, a constant $C_{q,s}$ such that

\begin{align*}
\text{(5.6)} & \quad K\left( t, g; \ell^s, \ell^0 \right) \leq C_{q,s} \cdot K\left( t, f; \ell^s, \ell^0 \right), \quad t > 0.
\end{align*}

Therefore, since $q \geq 1$, from Theorem 4.6 it follows that $g \in \text{Orb} (f, \ell^s, \ell^0)$.

On the other hand, let us assume that $g \in \text{Orb} (f; \ell^p, \ell^r)$, that is, $g = Wf$ for some operator $W \in \mathcal{L}(\ell^p, \ell^r)$. Hence,

\begin{align*}
K\left( t, g; \ell^p, \ell^r \right) = K\left( t, Wf; \ell^p, \ell^r \right) \leq ||W||_{\mathcal{L}(\ell^p, \ell^r)} K\left( t, f; \ell^p, \ell^r \right), \quad t > 0,
\end{align*}

which contradicts (5.5).
Proof of Theorem 5.2. Let $0 \leq p < q < 1$. We need to show that every $g \in \ell^q \setminus \ell^p$ is not a CM-element with respect to the couple $(\ell^p, \ell^q)$.

Given $g \in \ell^q \setminus \ell^p$ we select $f \in \ell^q$ as in Theorem 5.4 with $r = 1$. Then, reasoning in the same way as in the proof of the preceding corollary, we conclude that for each $s \in [0, q)$ inequality (5.6) holds with some constant $C_{q,s}$.

On the other hand, we claim that

$$\tag{5.7} g \notin \Orb(f; \ell^p, \ell^{q'}) \text{ for every } r' \in [q, 1).$$

Indeed, to get the contrary, assume that $g \in \Orb(f; \ell^p, \ell^{q'})$ for some $r' \in [q, 1)$. This means that there exists an operator $V \in \mathcal{L}(\ell^p, \ell^{q'})$ such that $Vf = g$. Since $r' < 1$, by Theorem 3.1 $V$ has a linear extension $\bar{V} : \ell^1 \to \ell^1$ such that $\|V\|_{\ell^1 \to \ell^1} \leq \|V\|_{\ell^{q'} \to \ell^{q'}}$. Then,

$$\mathcal{K}(t, g; \ell^p, \ell^{q'}) = \mathcal{K}(t, \bar{V}f; \ell^p, \ell^1) \leq \|V\|_{\ell^p \to \ell^1} \mathcal{K}(t, f; \ell^p, \ell^1), \quad t > 0,$$

which is a contradiction in view of (5.5) (with $r = 1$).

Now, setting $s = p$ and $r' = q$ in (5.6) and (5.7) respectively, we see that every element $g \in \ell^q \setminus \ell^p$ fails to be a CM-element with respect to the couple $(\ell^p, \ell^q)$. Therefore, by (5.2), we have $CM(\ell^p, \ell^q) = \ell^p$.

The second assertion of the theorem that $(\ell^p, \ell^q)$ is not a Calderón-Mityagin couple is now almost obvious. Indeed, take for $g$ any element from $\ell^q \setminus \ell^p$ (say, $g = (n^{-\sigma})_{n=1}^\infty$, where $\sigma \in (1/q, 1/p)$). Since $g \notin CM(\ell^p, \ell^q)$, there is $f \in \ell^q$ such that $g \in \mathcal{K} - \Orb(f; \ell^p, \ell^q) \setminus \Orb(f; \ell^p, \ell^q)$. Clearly, this yields the desired result.

Now, we proceed with the proof of Theorem 5.4.

Let $a, b$ be arbitrary positive integers. Let $T_{a,b}$ be the linear map defined on every function $h : [0, \infty) \to \mathbb{R}$ as follows:

$$T_{a,b}(h)(x) := \begin{cases} h(x) & \text{if } 0 < x < a \\ b^{-1/q}h(a + \frac{x-a}{b}) & \text{if } a \leq x < \infty. \end{cases}$$

Denote by $\Phi$ the collection of all functions $h : [0, \infty) \to \mathbb{R}$ which are constant on each interval $[n-1, n), n \in \mathbb{N}$. Then $T_{a,b} : \Phi \to \Phi$. Moreover, if $h$ is a nonnegative and nonincreasing function, then $T_{a,b}h$ is also nonnegative and nonincreasing.

Further properties of operators $T_{a,b}$ we will need are collected in the following lemma.

**Lemma 5.6.** Assume that $0 < p < q < r < \infty$ and $h \in \Phi$ is a nonnegative, nonincreasing function. Then for any $a, b \in \mathbb{N}$

$$\tag{5.8} \lim_{b \to \infty} \int_0^t (T_{a,b}h(x))^p \, dx = \int_0^a h(x)^p \, dx, \quad t > a,$$

$$\int_t^\infty (T_{a,b}h(x))^q \, dx \geq \int_t^\infty h(x)^q \, dx, \quad t > 0,$$
(5.10) \[ \int_t^\infty (T_{a,b} h(x))^q \, dx = \int_t^\infty h(x)^q \, dx, \quad 0 \leq t \leq a. \]

If additionally \( h \in L^r(0, \infty) \), then \( T_{a,b} h \in L^r(0, \infty) \), and

(5.11) \[ \lim_{b \to \infty} \int_a^\infty (T_{a,b} h(x))^r \, dx = 0. \]

If \( \int_t^\infty h(x)^r \, dx > 0 \) for all \( t > 0 \), then we have

(5.12) \[ \int_t^\infty (T_{a,b} h(x))^r \, dx \leq \int_t^\infty h(x)^r \, dx \]

whenever a positive integer \( b \) satisfies the condition

(5.13) \[ b > \left( \frac{\int_a^\infty h(y)^r \, dy}{\int_t^\infty h(y)^r \, dy} \right)^{\frac{q}{r-q}}. \]

We postpone the proof of this lemma, which is a series of elementary calculations, until the end of this section.

Proof of Theorem 5.4. As above, for some technical purposes, it will be convenient to consider, instead of sequences \( h = (h_n)_{n=1}^\infty \), the step functions \( \tilde{h}(t) \) defined for \( t > 0 \) by

\[ \tilde{h}(t) := \sum_{n=1}^\infty h_n \chi_{[n-1,n)}(t). \]

We will use further an infinite composition of suitable operators of the form \( T_{a,b} \) applied to the function \( \tilde{g} = \sum_{n=1}^\infty g_n \chi_{[n-1,n)} \) to obtain as a result the nonnegative and nonincreasing function \( \tilde{f} \in \Phi \) such that the corresponding sequence \( f = (f_n)_{n=1}^\infty \) will possess the required properties (5.3), (5.4) and (5.5).

Let \( (a_n)_{n=1}^\infty \) and \( (b_n)_{n=1}^\infty \) be two sequences of positive integers such that the sequence \( (a_n)_{n=1}^\infty \) is strictly increasing and hence \( \lim_{n \to \infty} a_n = \infty \). Next, choosing these sequences in a special way, we consider the sequence of functions \( (G_n)_{n=1}^\infty \) defined by \( G_1 = T_{a_1,b_1} \tilde{g} \), \( G_n = T_{a_n,b_n} (G_{n-1}) \) for \( n \geq 2 \). Then, by the definition of \( T_{a,b} \),

(5.14) \[ G_n(x) = G_m(x), \quad \text{for each} \ n \geq m \quad \text{and for all} \ x \in [0,a_{m+1}). \]

Since \( \bigcup_{m=1}^\infty [0,a_{m+1}) = [0, \infty) \), there exists the pointwise limit

\[ \tilde{f}(x) := \lim_{n \to \infty} G_n(x), \quad x > 0. \]

From (5.14) it follows that

(5.15) \[ \tilde{f}(x) = G_n(x), \quad \text{for each} \ n \geq m \quad \text{if} \ x \in [0,a_{m+1}). \]

It is clear also that \( \tilde{f} \in \Phi \) and it is a nonnegative and nonincreasing function. Now, the main task is to show that, if the auxiliary sequences \( (a_n)_{n=1}^\infty \) and \( (b_n)_{n=1}^\infty \) are constructed suitably, the sequence \( (f_n)_{n=1}^\infty \) of values of \( \tilde{f} \) will satisfy the conditions (5.3), (5.4) and (5.5).
At this stage we will assume that $0 < p < r < \infty$. Let us establish first some properties of the functions $G_n$ and $\tilde{f}$.

Since $g \in \ell^q$, from Lemma 5.6 (see (5.10)) it follows that $G_n \in L^q(0, \infty)$ and
\begin{equation}
\int_0^\infty G_n(x)^q \, dx = \int_0^\infty \tilde{g}(x)^q \, dx = \sum_{m=1}^{\infty} g_m^q, \quad n \in \mathbb{N}.
\end{equation}
Consequently, for every fixed positive integer $n$ we have
\[ \lim_{y \to \infty} \int_y^\infty G_n(x)^q \, dx = 0, \]
and hence the set
\[ \left\{ m \in \mathbb{N} : \int_m^\infty G_n(x)^q \, dx \leq \frac{1}{n} \right\} \]
is non-empty. Let
\begin{equation}
\gamma_n := \min \left\{ m \in \mathbb{N} : \int_m^\infty G_n(x)^q \, dx \leq \frac{1}{n} \right\}.
\end{equation}
Next, by (5.15), for each $t \geq 0$ and all $n$ satisfying $a_{n+1} \geq t$, we have
\begin{equation}
\int_t^\infty G_n(x)^q \, dx = \int_t^{a_{n+1}} G_n(x)^q \, dx + \int_{a_{n+1}}^\infty G_n(x)^q \, dx
\end{equation}
\[ = \int_t^{a_{n+1}} \tilde{f}(x)^q \, dx + \int_{a_{n+1}}^\infty G_n(x)^q \, dx. \]
Observe that for each integer $n \geq 2$ the function $G_n$ depends only on the given function $\tilde{g}$ and previously chosen $a_k, b_k$, with $1 \leq k \leq n$. Consequently, after completing the first $n$ steps, we may select $a_{n+1}$ so that
\[ a_{n+1} \geq \gamma_n, \]
where $\gamma_n$ is the positive integer defined in (5.17). Then,
\[ \int_{a_{n+1}}^\infty G_n(x)^q \, dx \leq \frac{1}{n}, \]
and hence, passing to the limit as $n$ tends to infinity in (5.18), we get
\begin{equation}
\int_t^\infty \tilde{f}(x)^q \, dx = \lim_{n \to \infty} \int_t^{a_{n+1}} G_n(x)^q \, dx \quad \text{for each } t \geq 0.
\end{equation}
In particular, setting $t = 0$, by (5.16), we have
\[ \sum_{m=1}^{\infty} f_m^q = \int_0^\infty \tilde{f}(x)^q \, dx = \int_0^\infty \tilde{g}(x)^q \, dx = \sum_{m=1}^{\infty} g_m^q < \infty. \]
Thus, the first desired condition, (5.3), is established.

Furthermore, repeated applications of (5.9) imply that
\[ \int_t^\infty G_n(x)^q \, dx \geq \int_t^\infty \tilde{g}(x)^q \, dx \quad \text{for every } n \in \mathbb{N} \text{ and all } t > 0. \]
Hence, taking limits and using (5.19) yields
\[ \int_t^\infty \tilde{f}(x)^q \, dx \geq \int_t^\infty \tilde{g}(x)^q \, dx, \quad t > 0, \]
which implies that
\[ \sum_{j=m}^\infty g_j^q \leq \sum_{j=m}^\infty f_j^q \quad \text{for all } m \in \mathbb{N}. \]

Now, recalling that for each nonnegative, nonincreasing sequence \( h = (h_n)_{n=1}^\infty \) we have
\[ \mathcal{K}(t, h; \ell^0, \ell^q) = \inf_{m=0, 1, 2, \ldots} (m + t \cdot \mathcal{E}(m, h; \ell^0, \ell^q)), \quad t > 0, \]
where
\[ \mathcal{E}(m, h; \ell^0, \ell^q) = \left( \sum_{j=m+1}^\infty h_j^q \right)^{1/q} \]
(see e.g. [7, § 7.1] or formulas (2.12) and (2.15) in Section 2.3), we conclude that the second required condition (5.4) holds as well.

Thus, it remains only to prove (imposing suitable additional hypotheses on the sequences \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\) the last required condition (5.5). Here, we will use the assumption that \( g \notin \ell^p \), and so
\[ (5.20) \quad \tilde{g} \notin L^p(0, \infty). \]
Along with the previously introduced sequence \((\gamma_n)_{n=1}^\infty\) we will need a new sequence \((\delta_n)_{n=1}^\infty\). For definiteness, we set \( a_1 = b_1 = \delta_1 = 1 \) and hence \( G_1 = T_1, \tilde{g} = \tilde{g} \). Moreover, as was specified above, \( \gamma_1 \) is the least positive integer \( m \) satisfying the inequality \( \int_m^\infty G_1(x)^q \, dx \leq 1 \).

Suppose that \( n \geq 1 \) and \( a_k, b_k, \delta_k, G_k, \gamma_k \) are determined for all \( 1 \leq k \leq n \). To pass to the next step, we first set
\[ (5.21) \quad a_{n+1} := \max (\gamma_n, \delta_n, a_n + 1). \]
Let now \( \delta_{n+1} \) be the smallest positive integer with the following properties:
\[ (5.22) \quad \delta_{n+1} > na_{n+1} \]
and
\[ \int_0^{\delta_{n+1}} \tilde{g}(x)^p \, dx \geq 2n^p \int_0^{a_{n+1}} G_n(x)^p \, dx. \]
Observe that such an integer exists because of (5.20). Therefore, by (5.8), we infer
\[ (5.23) \quad \lim_{b \to \infty} \int_0^{\delta_{n+1}} \left( T_{a_{n+1}, b} G_n(x) \right)^p \, dx = \int_0^{a_{n+1}} G_n(x)^p \, dx \leq \frac{1}{2n^p} \int_0^{\delta_{n+1}} \tilde{g}(x)^p \, dx. \]
Combining the fact that \( \tilde{g} \) is nonincreasing with (5.20), we see that \( \tilde{g}(x) > 0 \) and hence
\[ \int_0^{\delta_{n+1}} \tilde{g}(x)^p \, dx > 0. \]
Moreover, since the functions \( \tilde{g} \) and \( G_n \) belong to the intersection \( \Phi \cap L^{r} (0, \infty) \), the embedding \( \ell^{q} \subset \ell^{r} \) ensures that
\[
(5.24) \quad \tilde{g} \in L^{r} (0, \infty) \quad \text{and} \quad G_n \in L^{r} (0, \infty).
\]

Then, using (5.11) with the function \( G_n \) and the number \( a_{n+1} \) instead of \( h \) and \( a \) respectively, we conclude
\[
(5.25) \quad \lim_{b \to \infty} \int_{a_{n+1}}^{\infty} (T_{a_{n+1}, b} G_n (x))^r \, dx = 0
\]

As was noted, the function \( \tilde{g} \) is strictly positive at all points of \((0, \infty)\). Hence, \( G_n (x) > 0 \) if \( x > 0 \). Therefore, in view of (5.24), we can apply the last result of Lemma 5.6, according to that, for each \( t > 0 \), every positive integer \( b \), satisfying the condition
\[
b > \left( \frac{\int_{a_{n+1}}^{\infty} G_n (y)^r \, dy}{\int_{t}^{\infty} G_n (y)^r \, dy} \right)^{q/(r-q)},
\]

satisfies also the inequality
\[
\int_{t}^{\infty} (T_{a_{n+1}, b} G_n (x))^r \, dx \leq \int_{t}^{\infty} G_n (x)^r \, dx.
\]

Thus, from the condition
\[
b > \left( \frac{\int_{a_{n+1}}^{\infty} G_n (y)^r \, dy}{\int_{a_{n+1}}^{\infty} G_n (y)^r \, dy} \right)^{q/(r-q)}
\]

and the fact that \( a_k < a_n \) for \( k < n \) it follows that
\[
(5.26) \quad \int_{a_k}^{\infty} (T_{a_{n+1}, b} G_n (x))^r \, dx \leq \int_{a_k}^{\infty} G_n (x)^r \, dx \quad \text{for all} \quad 1 \leq k \leq n.
\]

Summarizing all, we see that, in view of (5.26), (5.23) and (5.25), it is possible to select an integer \( b_{n+1} \) such that the function \( G_{n+1} = T_{a_{n+1}, b_{n+1}} (G_n) \) satisfies the following conditions:
\[
(5.27) \quad \int_{a_k}^{\infty} G_{n+1} (x)^r \, dx \leq \int_{a_k}^{\infty} G_n (x)^r \, dx \quad \text{for all} \quad 1 \leq k \leq n,
\]
\[
(5.28) \quad \int_{0}^{\delta_{n+1}} G_{n+1} (x)^p \, dx \leq \frac{1}{n^p} \int_{0}^{\delta_{n+1}} \tilde{g} (x)^p \, dx,
\]
\[
(5.29) \quad \int_{a_{n+1}}^{\infty} G_{n+1} (x)^r \, dx \leq \frac{1}{n^r} \int_{a_{n+1}}^{\infty} \tilde{g} (x)^r \, dx.
\]

Having chosen \( a_{n+1}, \delta_{n+1}, b_{n+1} \) and so \( G_{n+1} \), we determine then the number \( \gamma_{n+1} \) by formula (5.17). This means that the step \( n+1 \) is completed. Proceeding in the same way, we obtain the sequences \( (a_n)_{n=1}^{\infty}, (\delta_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (G_n)_{n=1}^{\infty} \) and \( (\gamma_n)_{n=1}^{\infty} \) (in view of (5.20), this process does not finish after only finitely many steps). Moreover, as was observed above, there exists the pointwise limit \( \tilde{f} \) of the sequence \( \{G_n\}_{n=1}^{\infty} \) on \((0, \infty)\). Note that, thanks to the choice of \( a_{n+1} \) determined by (5.21), the reasoning from the first part of the proof shows that the sequence
of values of the function $\tilde{f}$ satisfies conditions (5.3) and (5.4). Therefore, it remains only to deduce (5.5).

By (5.28), (5.15) and the inequality $\delta_{n+1} \leq a_{n+2}$ (see (5.21)), it follows that

$$
\int_0^{\delta_{n+1}} \tilde{f}(x)^p \, dx \leq \frac{1}{n^p} \int_0^{\delta_{n+1}} \tilde{g}(x)^p \, dx.
$$

Moreover, applying (5.27), for each integer $n$ and $m \geq n + 1$ we have

$$
\int_{a_{n+1}}^{\infty} G_m(x)^r \, dx \leq \int_{a_{n+1}}^{\infty} G_{n+1}(x)^r \, dx,
$$

and hence by iteration

$$
\int_{a_{n+1}}^{\infty} G_{n+1}(x)^r \, dx \leq \int_{a_{n+1}}^{\infty} G_{n+2}(x)^r \, dx \leq \ldots \leq \int_{a_{n+1}}^{\infty} G_{n+1}(x)^r \, dx \leq 1
$$

Consequently, for each $R \geq a_{n+1}$ and all $m \geq n + 1$, in view of (5.29), we infer that

$$
\int_{a_{n+1}}^{R} G_m(x)^r \, dx \leq \int_{a_{n+1}}^{\infty} G_{n+1}(x)^r \, dx \leq \frac{1}{n^r} \int_{\delta_{n+1}}^{\infty} \tilde{g}(x)^r \, dx.
$$

Since $0 \leq G_m(x) \leq \tilde{g}(0)$ for all $x \geq 0$ and $m \in \mathbb{N}$, using dominated convergence on the interval $[a_{n+1}, R]$, we have

$$
\int_{a_{n+1}}^{R} \tilde{f}(x)^r \, dx = \lim_{m \to \infty} \int_{a_{n+1}}^{R} G_m(x)^r \, dx \leq \frac{1}{n^r} \int_{\delta_{n+1}}^{\infty} \tilde{g}(x)^r \, dx,
$$

and then, passing to the limit as $R \to \infty$, we obtain

$$
\int_{a_{n+1}}^{\infty} \tilde{f}(x)^r \, dx \leq \frac{1}{n^r} \int_{\delta_{n+1}}^{\infty} \tilde{g}(x)^r \, dx.
$$

Let $1/\alpha = 1/p - 1/r$. By the Holmstedt formula (2.3), from (5.22), (5.30) and (5.31) it follows that

$$
\mathcal{K}(\delta_{n+1}^{1/\alpha} \tilde{f}; L^p(0, \infty), L^r(0, \infty)) \\
\leq \left( \int_0^{\delta_{n+1}} \tilde{f}(x)^p \, dx \right)^{1/p} + \delta_{n+1}^{1/\alpha} \left( \int_{\delta_{n+1}}^{\infty} \tilde{f}(x)^r \, dx \right)^{1/r} \\
\leq \left( \frac{1}{n^p} \int_0^{\delta_{n+1}} \tilde{g}(x)^p \, dx \right)^{1/p} + \delta_{n+1}^{1/\alpha} \left( \frac{1}{n^r} \int_{\delta_{n+1}}^{\infty} \tilde{g}(x)^r \, dx \right)^{1/r} \\
= \frac{1}{n} \left( \left( \int_0^{\delta_{n+1}} \tilde{g}(x)^p \, dx \right)^{1/p} + \delta_{n+1}^{1/\alpha} \left( \int_{\delta_{n+1}}^{\infty} \tilde{g}(x)^r \, dx \right)^{1/r} \right) \\
\leq \frac{C_{p,r}}{n} \mathcal{K}(\delta_{n+1}^{1/\alpha} \tilde{g}; L^p(0, \infty), L^r(0, \infty)), n = 1, 2, \ldots
$$
Hence, for the corresponding sequences $f$ and $g$ it holds
\[
\lim_{n \to \infty} \frac{K\left(\delta_{n+1}^{1/\alpha}; f; \ell^p, \ell^r\right)}{K\left(\delta_{n+1}^{1/\alpha}; g; \ell^p, \ell^r\right)} = 0.
\]

Since $\delta_{n+1} > a_{n+1}$, then (5.21) ensures that $\delta_{n+1} \to 0$ as $n \to \infty$. Thus, relation (5.5) is established, and thus the proof of the theorem in the case when $0 < p < r < \infty$ is completed.

Consider now the case $0 < p < r = \infty$. The only change which should be made in the above construction of the sequences $(a_n)_{n=1}^\infty$, $(\delta_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$, $(G_n)_{n=1}^\infty$ and $(\gamma_n)_{n=1}^\infty$ is that in the step $n + 1$ the value of $b_{n+1}$ is required to satisfy only one condition, namely (5.28). Clearly, then estimate (5.30) follows in the same way. Combining it with the Holmstedt formula (2.4), we get, as above, that
\[
\lim_{n \to \infty} \frac{K\left(\delta_{n+1}^{1/p}; f; \ell^p, \ell^\infty\right)}{K\left(\delta_{n+1}^{1/p}; g; \ell^p, \ell^\infty\right)} = 0.
\]

Thus, (5.5) is proved. Since the proofs of (5.3) and (5.4) do not require any modifications in this case, the result follows.

Let now $0 = p < r < \infty$. In this case in the step $n + 1$ we choose the value of $b_{n+1}$ so that only conditions (5.27) and (5.29) to be satisfied. Then, we obtain (5.31), which being combined with inequality (5.22), gives us
\[
\int_{a_{n+1}}^\infty \tilde{f}(x)^r \, dx \leq \frac{1}{nr} \int_{na_{n+1}}^\infty \tilde{g}(x)^r \, dx.
\]
Consequently, we have
\[
\sum_{k=a_{n+1}+1}^\infty f_k^r \leq \frac{1}{nr} \sum_{k=na_{n+1}+1}^\infty g_k^r, \quad n \in \mathbb{N}.
\]

Let us assume that (5.5) does not hold, i.e., there is a constant $C > 0$ such that
\[
K\left(t, g; \ell^0, \ell^r\right) \leq CK\left(t, f; \ell^0, \ell^r\right) \quad \text{for all} \quad t > 0.
\]
Then, by implication (2.14) and formula (2.15), we have
\[
\sum_{k=m}^\infty g_k^r \leq (2C)^r \sum_{k=[(m-1)/(2C)]+1}^\infty f_k^r, \quad m \in \mathbb{N}.
\]
Substituting $m = ([2C] + 1)a_{n+1} + 1$, $n \in \mathbb{N}$, in the latter inequality, we get
\[
\sum_{k=([2C]+1)a_{n+1}+1}^\infty g_k^r \leq (2C)^r \sum_{k=a_{n+1}+1}^\infty f_k^r, \quad n \in \mathbb{N}.
\]
Combining this together with (5.32), we come to the estimate
\[
\sum_{k=([2C]+1)a_n+1}^{\infty} g_k \leq \left( \frac{2C}{r} \right)^r \sum_{k=n}^{\infty} g_k, \quad n \in \mathbb{N}.
\]
Since \( g_k > 0 \) for all \( k \in \mathbb{N} \), the latter is impossible for sufficiently large \( n \). Therefore, (5.33) does not hold, and hence we have (5.5). Noting that (5.3) and (5.4) are still satisfied, we complete the proof.

Finally, we consider the case when \( p = 0 \) and \( r = \infty \). The first part of our process may be conducted in the same way as earlier. Namely, we can find sequences of positive integers \( (a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \) and \( (\gamma_n)_{n=1}^{\infty} \) such that the sequence \( (a_n)_{n=1}^{\infty} \) is strictly increasing, \( a_{n+1} \geq \gamma_n \) and the pointwise limit \( \bar{f} \) of the sequence of functions \( (G_n)_{n=1}^{\infty} \) defined by \( G_1 = T_{a_1,b_1}(\bar{g}) = \bar{g}, \ G_n = T_{a_n,b_n}(G_{n-1}), n \geq 2, \) satisfies conditions (5.3) and (5.4). To get the remaining condition (5.5), we need somewhat to modify (in fact, to simplify) the above procedure.

In this case the sequence \( (\delta_n)_{n=1}^{\infty} \) is not needed, and we have only to arrange the choice of the sequence \( (b_n)_{n=1}^{\infty} \), which was arbitrary by now.

Since \( g \in \ell^q \setminus \ell^0 \), then \( \text{card}(\text{supp} \ g) = \infty \). Therefore, the function \( \bar{g}(x) \) is strictly positive for all \( x > 0 \). Suppose that \( n \geq 1 \) and \( a_k, b_k, G_k, \gamma_k \) are defined for all \( 1 \leq k \leq n \) (as above, \( a_1 = b_1 = \gamma_1 = 1 \)). To pass to the next step, we set
\[
(5.34) \quad a_{n+1} := \max(\gamma_n, a_n + 1) \quad \text{and} \quad b_{n+1} := \left[ \left( \frac{G_n(a_{n+1})}{\bar{g}(n(a_{n+1} + 1))} \right)^q \right] + 1.
\]
Then, by the definition of the operators \( T_{a,b} \), we have for all \( n \geq 2 \)
\[
G_n(a_{n+1}) \leq b_n^{-1/q} G_{n-1}(a_n) \leq \frac{1}{n} \bar{g}(n(a_{n+1} + 1)).
\]
Combining this together with (5.15), we get
\[
\bar{f}(a_{n+1}) \leq \frac{1}{n} \bar{g}(n(a_{n+1} + 1)), \quad n \geq 2,
\]
which yields for all \( n \geq 2 \)
\[
(5.35) \quad \lim_{x \to \infty} \inf_{x/n} \frac{\bar{f}(x/n)}{\bar{g}(x)} = 0.
\]
Let us show that (5.35) implies (5.5) in the case \( p = 0 \) and \( r = \infty \).

Assuming the contrary, for some positive \( C \) we have
\[
(5.36) \quad K(t,g;\ell^0,\ell^\infty) \leq CK(t,f;\ell^0,\ell^\infty), \quad t > 0.
\]
This implies that
\[
K(t,\bar{g};L^0(0,\infty),L^\infty(0,\infty)) \leq CK(t,\bar{f};L^0(0,\infty),L^\infty(0,\infty)), \quad t > 0.
\]
Consequently, from (2.14) and (2.16) it follows
\[
\bar{g}(x) \leq 2C \bar{f}(x/2C) \quad \text{for all} \ x > 0.
\]
Since this inequality contradicts relation (5.35), inequality (5.36) fails for any \( C \). As a result, (5.5) holds, and so the proof of Theorem 5.4 is completed. \( \square \)
Proof of Lemma 5.6. Recall that \( h \) is assumed to be a nonnegative, nonincreasing function in \( \Phi \) (i.e., constant on each interval of the form \([n-1, n)\), \( n \in \mathbb{N} \)).

We begin with proving (5.8). Fix \( a \in \mathbb{N} \) and \( t > a \). Then, for each \( b \in \mathbb{N} \), we have

\[
\int_0^t (T_{a,b} h(x))^p \, dx = \int_0^a (T_{a,b} h(x))^p \, dx + \int_a^t (T_{a,b} h(x))^p \, dx
\]

(5.37)

\[
= \int_0^a h(x)^p \, dx + \int_a^t (T_{a,b} h(x))^p \, dx.
\]

Since

\[
0 \leq \int_a^t (T_{a,b} h(x))^p \, dx = \int_a^t b^{-p/q} \left( h \left( a + \frac{x-a}{b} \right) \right)^p \, dx
\]

\[
\leq b^{-p/q} (t-a) h(0)^p,
\]

it follows that

\[
\lim_{b \to \infty} \int_a^t (T_{a,b} h(x))^p \, dx = 0.
\]

Combining this together with (5.37), we get (5.8).

To obtain (5.9) and (5.10), note first that, for \( t \geq a \), the change of variables gives us

\[
\int_t^\infty (T_{a,b} h(x))^q \, dx = \int_{a+(t-a)/b}^\infty h(x)^q \, dx.
\]

(5.38)

Since \( b \geq 1 \) we have that \( t \geq a + (t-a)/b \) whenever \( t \geq a \), which together with (5.38) implies (5.9) in the case \( t \geq a \). If \( 0 < t < a \), then using the definition of the operator \( T_{a,b} \) and (5.38) for \( t = a \), we obtain that

\[
\int_t^\infty (T_{a,b} h(x))^q \, dx = \int_a^a (T_{a,b} h(x))^q \, dx + \int_a^\infty (T_{a,b} h(x))^q \, dx
\]

\[
= \int_t^a h(x)^q \, dx + \int_a^\infty h(x)^q \, dx = \int_t^\infty h(x)^q \, dx,
\]

which establishes (5.10) and also the remaining case of inequality (5.9).

Next, for any fixed \( a \in \mathbb{N} \), by the same change of variables as above, we get

\[
\int_a^\infty (T_{a,b} h(x))^r \, dx = b^{1-r/q} \int_a^\infty h(x)^r \, dx.
\]

Hence, it follows that if \( h \in L^r(0, \infty) \) then \( T_{a,b} h \in L^r(0, \infty) \) for all \( a, b \in \mathbb{N} \). Moreover, we see that (5.11) holds if \( h \in L^r(0, \infty) \), since \( 0 < q < r \).

Finally, we have to show that a positive integer $b$ satisfies (5.12) whenever it satisfies (5.13). First, for each $t \in [0, a]$ it holds

\[
\int_t^\infty (T_{a,b}h(x))^r \, dx = \int_t^a (T_{a,b}h(x))^r \, dx + \int_a^\infty (T_{a,b}h(x))^r \, dx
\]

\[= \int_t^a h(x)^r \, dx + b^{1-r/q} \int_a^\infty h(x)^r \, dx\]

\[\leq \int_t^\infty h(x)^r \, dx,
\]

and so, for such values of $t$, (5.12) is satisfied by each positive integer $b$.

Further, we rewrite (5.13) in the form

\[
(5.39) \quad b^{1-r/q} \int_a^\infty h(x)^r \, dx \leq \int_t^\infty h(x)^r \, dx.
\]

Then, for $t > a$ we have

\[
\int_t^\infty (T_{a,b}h(x))^r \, dx \leq \int_t^\infty (T_{a,b}h(x))^r \, dx
\]

\[= b^{1-r/q} \int_a^\infty h(x)^r \, dx\]

\[\leq \int_t^\infty h(x)^r \, dx.
\]

Therefore, (5.12) is obtained for all $t > 0$, and so the proof of the lemma is completed. \qed

6. About the $S_q$-property

In [18], in connection with the conjecture stated by Levitina, Sukochev and Zanin (see Theorem 4.11 and the subsequent discussion), Cwikel and Nilsson have introduced the following notion.

**Definition 6.1.** Let $q \geq 1$ and let $E \neq \{0\}$ be a normed sequence space, $E \subseteq \ell^q$. Then, $E$ has the $S_q$-property provided that there is a constant $C$ if, whenever $x = (x_n)_{n=1}^\infty \in E$ and $y = (y_n)_{n=1}^\infty \in \ell^q$ are two sequences, which satisfy the conditions:

\[
(6.1) \quad \sum_{n=1}^\infty |x_n|^q = \sum_{n=1}^\infty |y_n|^q
\]

and

\[
(6.2) \quad \sum_{n=1}^m (x_n^*)^q \leq \sum_{n=1}^m (y_n^*)^q \text{ for all } m \in \mathbb{N},
\]

then it follows that $y \in E$ and $\|y\|_E \leq C \|x\|_E$. 
It is clear that this definition may be extended to a more general situation when \( q > 0 \) and \( E \) is a quasi-Banach sequence space.

The following result shows that the \( S_q \)-property of a quasi-Banach sequence lattice \( E \) is closely related to the fact that \( E \in \operatorname{Int}(\ell^0, \ell^q) \).

**Theorem 6.2.** Let \( 0 < q < \infty \) and \( E \) be a quasi-Banach sequence lattice. Then, the following conditions are equivalent:

(a) \( E \) has the \( S_q \)-property;

(b) \( E \) is a uniform \( K \)-monotone space with respect to the couple \((\ell^0, \ell^q)\).

Therefore, from the condition (a) it follows that \( E \in \operatorname{Int}(\ell^0, \ell^q) \). In the case when \( q \geq 1 \), the converse holds as well, i.e., \( E \) has the \( S_q \)-property if and only if \( E \in \operatorname{Int}(\ell^0, \ell^q) \).

**Proof.** (a) \( \implies \) (b). Assume that sequences \( x = (x_n)_{n=1}^\infty \in E \) and \( y = (y_n)_{n=1}^\infty \in \ell^q \) satisfy the condition

\[
\mathcal{K}(t, y; \ell^0, \ell^q) \leq \mathcal{K}(t, x; \ell^0, \ell^q), \quad t > 0.
\]

Then, in the same way as in the end of Section 4, we have

\[
\sum_{n=m}^\infty (y_n^*)^q \leq 2^q \sum_{n=m}^\infty (D_2x^*)_n^q, \quad m \in \mathbb{N},
\]

or, denoting \( u_n = 2(D_2x^*)_n, \ n = 1, 2, \ldots \),

\[
(6.3) \quad \sum_{n=m}^\infty (y_n^*)^q \leq \sum_{n=m}^\infty u_n^q, \quad \text{for all } m \in \mathbb{N}.
\]

Further, we will use a reasoning from the proof of Theorem 5.3 in [18]. Since \( E \subseteq \ell^q \), it follows that \( \lim_{n \to \infty} y_n = 0 \). Select \( n_1 \in \mathbb{N} \) with \( |y_{n_1}| = y_1^* \). Let \( z = (z_n)_{n=1}^\infty \) be a sequence such that \( z_n = y_n, \ n \neq n_1, \ z_1^* = |z_{n_1}| \geq y_1^* = |y_{n_1}| \) and \( \sum_{n=1}^\infty (z_n^*)^q = \sum_{n=1}^\infty u_n^q \). Then, by (6.3), for all \( m \in \mathbb{N} \)

\[
\sum_{n=m}^\infty (z_n^*)^q \leq \sum_{n=m}^\infty u_n^q,
\]

and hence we have

\[
\sum_{n=1}^m u_n^q = \sum_{n=1}^\infty u_n^q - \sum_{n=m+1}^\infty u_n^q
\]

\[
= \sum_{n=1}^\infty (z_n^*)^q - \sum_{n=m+1}^\infty u_n^q
\]

\[
\leq \sum_{n=1}^\infty (z_n^*)^q - \sum_{n=m+1}^\infty (z_n^*)^q = \sum_{n=1}^m (z_n^*)^q.
\]

It is clear that every quasi-Banach sequence lattice, satisfying the \( S_q \)-property, is symmetric. Therefore, \( u = (u_n) \in E \) and \( \|u\|_E = 2\|D_2x^*\|_E \leq 4\|x\|_E \). Consequently, since \( E \) has the \( S_q \)-property, combining this together with the preceding relations, we get that \( (z_n) \in E \) and \( \|z\|_E \leq 4C\|x\|_E \), where \( C \) is the \( S_q \)-property.
constant. Moreover, by the definition of \( z \), we have \(|y| \leq |z|\). Thus, since \( E \) is a lattice, \( \|y\|_E \leq \|z\|_E \leq 4C \|x\|_E \). Summarizing all, we conclude that \( E \) is a uniform \( \mathcal{K} \)-monotone space with respect to the couple \((\ell^0, \ell^q)\).

\((b) \implies (a)\). Let now \( E \) be a uniform \( \mathcal{K} \)-monotone space with respect to the couple \((\ell^0, \ell^q)\). Suppose that sequences \( x = (x_n)_{n=1}^\infty \in E \) and \( y = (y_n)_{n=1}^\infty \in \ell^q \) satisfy conditions \((6.1)\) and \((6.2)\). Then, the same argument as in the first part of the proof yields

\[
\sum_{n=m}^\infty (y_n^*)^q \leq \sum_{n=m}^\infty (x_n^*)^q, \quad m \in \mathbb{N}.
\]

This inequality, combined with formula \((2.15)\) and implication \((2.13)\), yields

\[
\mathcal{K}(t, y; \ell^0, \ell^q) \leq \mathcal{K}(t, x; \ell^0, \ell^q), \quad t > 0.
\]

Hence, by the assumption, \( y \in E \) and \( \|y\|_E \leq C \|x\|_E \), where \( C \) is the \( \mathcal{K} \)-monotonicity constant of \( E \) with respect to the couple \((\ell^0, \ell^q)\).

Since every uniform \( \mathcal{K} \)-monotone space with respect to a couple of quasi-Banach spaces is also an interpolation space with respect to this couple, from \((a)\) it follows that \( E \in Int(\ell^0, \ell^q) \).

Finally, assume that \( q \geq 1 \). Then, by Corollary \(4.6\), every interpolation space between \( \ell^0 \) and \( \ell^q \) is a uniform \( \mathcal{K} \)-monotone space. This fact, combined together with already proved implication \((b) \implies (a)\), implies the last assertion of the theorem.

From Theorems \(6.2\) and \(4.11\) we get

**Corollary 6.3.** Let \( q \geq 1 \). A quasi-Banach sequence lattice \( E \) has the \( S_q \)-property if and only if \( E \in Int(\ell^p, \ell^q) \) for some \( p > 0 \).

Recall now the following definition from [18]. For every sequence \( x = (x_n)_{n=1}^\infty \) and each \( N \in \mathbb{N} \), let \((x_n^{(N)})_{n=1}^\infty \) be the truncated sequence defined by \( x_n^{(N)} = x_n \) if \( 1 \leq n \leq N \) and \( x_n^{(N)} = 0 \) if \( n > N \). We say that a normed sequence lattice \( E \) has the *weak Fatou property* if there is a constant \( R \) such that, for every sequence \( x = (x_n)_{n=1}^\infty \) of nonnegative numbers with \((x_n^{(N)})_{n=1}^\infty \in E \) for all \( N \in \mathbb{N} \) and \( \sup_{N \in \mathbb{N}} \|x_n^{(N)}\|_E < \infty \), we have \( x \in E \) and

\[
\|x\|_E \leq R \sup_{N \in \mathbb{N}} \|x_n^{(N)}\|_E.
\]

Obviously, each normed sequence lattice with the Fatou property (see Section \(2.2\)) has the weak Fatou property.

According to the main result of [18], if \( q > 1 \), then every normed sequence lattice \( E \) with the weak Fatou property has the \( S_q \)-property if and only \( E \) is an interpolation space between \( \ell^1 \) and \( \ell^q \). Moreover, by interpolation, from the assumption \( E \in Int(\ell^1, \ell^q) \) it follows that \( E \in Int(\ell^p, \ell^q) \) for all \( 0 \leq p < 1 \). Therefore, applying Theorem \(6.2\) we get the following result, which in a sense complements Corollary \(3.3\).

**Corollary 6.4.** Let \( q > 1 \) and let \( E \) be a Banach sequence lattice with the weak Fatou property. Then, the following conditions are equivalent:
(i) \( E \in \text{Int} (\ell^p, \ell^q) \) for all \( p \in [0, 1) \);
(ii) \( E \in \text{Int} (\ell^p, \ell^q) \) for some \( p \in [0, 1) \);
(iii) \( E \in \text{Int} (\ell^1, \ell^q) \);
(iv) \( E \) has the \( S_q \)-property.

7. (\( \ell^p, \ell^q \)) is not a uniform Calderón-Mityagin couple if \( 0 \leq p < q < 1 \).

It is a long-standing problem in the interpolation theory if a quasi-Banach couple with the Calderón-Mityagin property possesses its uniform version as well (see, for instance, [28, p. 1150]). In fact, by now this question is open when being restricted to the narrower classes of Banach couples or even of couples of Banach lattices.

In Section 5, we proved that the couple (\( \ell^p, \ell^q \)) does not have the Calderón-Mityagin property whenever \( 0 \leq p < q < 1 \). As a consequence, we conclude that this couple fails to have its uniform version. For the reader’s convenience, we present here an independent proof of the latter result, which is much shorter and simpler than that of Theorem 5.2.

**Theorem 7.1.** The couple (\( \ell^p, \ell^q \)), with \( 0 \leq p < q < 1 \), does not have the uniform Calderón-Mityagin property.

Let \( P_p \) and \( Q_q \) be the operators introduced in Section 2.2. Taking into account the Holmstedt formula (2.10) if \( p > 0 \) and relations (2.14) and (2.15) if \( p = 0 \), one can easily see that Theorem 7.1 is a straightforward consequence of the following proposition.

**Proposition 7.2.** Let \( 0 \leq p < q < 1 \). Then, given arbitrarily large positive constant \( C \) there exist two nonnegative, nonincreasing sequences \( x = (x_n)_{n=1}^{\infty} \) and \( y = (y_n)_{n=1}^{\infty} \) in \( \ell^q \) satisfying the conditions

\[
(7.1) \quad (P_p y)_n + n^{1/\alpha} (Q_q y)_n \leq (P_p x)_n + n^{1/\alpha} (Q_q x)_n, \quad n = 1, 2, \ldots, \quad \text{if } p > 0,
\]

where \( 1/\alpha = 1/p - 1/q \), and

\[
(7.2) \quad (Q_q y)_n \leq (Q_q x)_n, \quad n = 1, 2, \ldots, \quad \text{if } p = 0,
\]

such that for every linear operator \( S : \ell^q \to \ell^q \) with \( S x = y \) we have

\[
(7.3) \quad \|S\|_{\ell^q \to \ell^q} \geq C.
\]

**Proof.** Taking for \( y \) the element \( e_1 \) of the unit vector basis, we consider the cases \( p > 0 \) and \( p = 0 \) separately.

(a) \( p > 0 \). Since every \( K \)-functional \( K(t, x; X_0, X_1) \) is an increasing function in \( t \), by (2.10), the sum \( (P_p z)_n + n^{1/\alpha} (Q_q z)_n \), for each \( z \in \ell^q \), is almost increasing in \( n \), i.e.,

\[
(7.4) \quad (P_p z)_n + n^{1/\alpha} (Q_q z)_n \leq C_{p,q} ((P_p z)_m + m^{1/\alpha} (Q_q z)_m) \quad \text{if } n \leq m,
\]

where \( C_{p,q} \geq 1 \) depends only on \( p \) and \( q \).

Given any constant \( C > 0 \), choose a positive integer \( N \) so that

\[
(7.5) \quad (2C_{p,q})^{-1} N^{1/q-1} > C.
\]
Next, we set $x = (x_n)_{n=1}^\infty$, where $x_n = 2C_{p,q}N^{-1/q}$ if $1 \leq n \leq N$, and $x_n = 0$ if $n > N$. Then, the first entry $(Q_q x)_1$ of the sequence $Q_q x$ is defined by

$$(Q_q x)_1 = \left( \sum_{n=1}^{N} x_n^q \right)^{1/q} = 2C_{p,q}.$$  

Therefore, by (7.3), the right-hand side of inequality (7.1) is not less than 2 for all $n \in \mathbb{N}$. On the other hand, $(P_{p,y})_n = 1$, $n \in \mathbb{N}$, and $(Q_q y)_1 = 1$, $(Q_q y)_n = 0$, $n \geq 2$. Hence, the left-hand side of (7.1) does not exceed 2, and so for the above $x$ and $y$ inequality (7.1) holds.

Let now $S$ be a linear operator such that $S : \ell^q \to \ell^q$ with $S x = y$. Clearly, $S$ is defined by a sequence of bounded linear functionals on $\ell^q$. In particular, setting $\Lambda (z) := \langle Sz, e_1 \rangle$, we have $\Lambda (x) = \langle Sz, e_1 \rangle = \langle y, e_1 \rangle = 1$ and 

$$|\Lambda (z)| \leq \|S\| \|z\|_{\ell^q}.$$  

Consequently, if $\beta_n := \Lambda (e_n)$, $n \in \mathbb{N}$, we have $|\beta_n| \leq \|S\|$. Hence, 

$$1 = |\Lambda (x)| = \left| \sum_{j=1}^{N} \beta_n x_n \right| \leq 2C_{p,q} \sum_{j=1}^{N} \beta_n N^{-1/q} \leq 2C_{p,q} \|S\| N^{1-1/q}.$$  

According to the choice of $N$ in (7.5), this implies that 

$$\|S\| \geq (2C_{p,q})^{-1} N^{1/q-1} > C,$$

and in this case the result follows.

(b) $p = 0$. Given constant $C > 0$, we take $N \in \mathbb{N}$ satisfying the inequality $N^{1/q-1} > C$. Let $x = (x_n)_{n=1}^\infty$, where $x_n = N^{-1/q}$ if $1 \leq n \leq N$, and $x_n = 0$ if $n > N$. As above, we have $(Q_q x)_1 = 1$. Therefore, since $(Q_q y)_1 = 1$ and $(Q_q y)_n = 0$ for all $n \geq 2$, inequality (7.2) holds.

If $S$ is a linear operator such that $S : \ell^q \to \ell^q$ with $S x = y$, the same reasoning as in the case (a) shows that $\|S\| \geq N^{1/q-1} > C$, and so the proof is completed.

\[
\Box
\]

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