Bounded-excess flows in cubic graphs

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Abstract

An \((r, \alpha)\)-bounded-excess flow \(((r, \alpha)\)-flow) in an orientation of a graph \(G = (V, E)\) is an assignment \(f: E \to [1, r-1]\), such that for every vertex \(x \in V\),
\[|\sum_{e \in E^+(x)} f(e) - \sum_{e \in E^-(x)} f(e)| \leq \alpha.\]
\(E^+(x)\) respectively \(E^-(x)\), is the set of edges directed from, respectively toward \(x\). Bounded-excess flows suggest a generalization of Circular nowhere-zero flows (cnzf), which can be regarded as \((r,0)\)-flows. We define \((r, \alpha)\) as Stronger or equivalent to \((s, \beta)\), if the existence of an \((r, \alpha)\)-flow in a cubic graph always implies the existence of an \((s, \beta)\)-flow in the same graph. We then study the structure of the bounded-excess flow strength poset. Among other results, we define the Trace of a point in the \(r - \alpha\) plane by
\[tr(r, \alpha) = \frac{r - 2\alpha}{1 - \alpha}\]
and prove that among points with the same trace the stronger is the one with the smaller \(\alpha\) (and larger \(r\)). For example, if a cubic graph admits a \(k\)-nzf (trace \(k\) with \(\alpha = 0\)), then it admits an \((r, \frac{k-\alpha}{k-2})\)-flow for every \(r, 2 \leq r \leq k\). A significant part of the article is devoted to proving the main result: Every cubic graph admits a \((3, \frac{1}{2})\)-flow, and there exists a graph which does not admit any stronger bounded-excess flow. Notice that \(tr(3, \frac{1}{2}) = 5\) so it can be considered a step in the direction of the 5-flow Conjecture. Our result is the best possible for all cubic graphs while the seemingly stronger 5-flow Conjecture relates only to bridgeless graphs. We also show that if the circular-flow number of a cubic graph is strictly less than 5, then it admits a \((3\frac{1}{2}, \frac{1}{3})\)-flow (trace \(4\)). We conjecture such a flow to exist in every cubic graph with a perfect matching, other than the...
Petersen graph. This conjecture is a stronger version of the Ban-Linial Conjecture [1]. Our work here strongly relies on the notion of Orientable $k$-weak bisections, a certain type of $k$-weak bisections. $k$-Weak bisections are defined and studied by L. Esperet, G. Mazzuoccolo, and M. Tarsi [4].

**KEYWORDS**
Ban-Linial Conjecture, cubic graphs, $k$-weak bisections, nowhere-zero flows

1 | INTRODUCTION

1.1 | Preliminaries

We assume familiarity with the theory of Nowhere-zero flows (nzf) (see [10] for a thorough study) and Circular nowhere-zero flows (cnzf).

**Definition 1.** Given two real numbers $r \geq 2$ and $\alpha \geq 0$, an $(r, \alpha)$-bounded-excess flow, $(r, \alpha)$-flow for short, in a directed graph $D = (V, E)$ is an assignment $f: E \to [1, r−1]$, such that $f$ is a flow in $D$, with possibly some deficiency or excess, which does not exceed $\alpha$ per vertex. That is, for every vertex $x \in V$, $|\sum_{e \in E^+(x)} f(e) − \sum_{e \in E^-(x)} f(e)| \leq \alpha$, where $E^+(x)$, respectively $E^-(x)$, is the set of edges directed from, respectively toward $x$ in $D$.

With that notation an $r$-cnzf can be referred to as an $(r, 0)$-flow. Notice that while nzf are restricted to bridgeless graphs, this is not the case for bounded-excess flows, where a bridge can carry some excess from one of its “sides” to the other.

We say that an undirected graph $G$ admits an $(r, \alpha)$-flow if there exists an orientation of $G$ which admits such a flow.

In this article we study bounded-excess flows in cubic, not necessarily simple, graphs. Notice that, given an $(r, \alpha)$-flow, a loop $e = (x, x)$ does not carry any excess. The flow value on the additional edge (just one when $d(x) = 3$) incident with $x$ entirely accumulates in $x$ and it is by definition at least 1, which implies $\alpha \geq 1$. We will later observe (Lemma 9) that such flows trivially exist, regardless of the value of $r$ in every cubic graph. Parallel edges on the other hand are essential for our results, for example, the bound in Theorem 22 is tight for multigraphs but not for simple graphs. Accordingly, in this article a graph may have parallel edges, but no loops.

For motivation and relevance to cnzf problems see Section 4.

**Definitions and Notation 2** (Rather than a stand-alone definition, we post under that title lists of related definitions and notational remarks).

- A Vertex partition of a graph $G = (V, E)$ is a partition $\Psi = (V_1, V_2)$ of its vertex set $V$ into two disjoint subsets, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$. 
Whenever it comes convenient, we use vertex partition and vertex coloring \( \Psi : V \to \{1, 2\} \) as synonyms, where \( \Psi(v) = i \Leftrightarrow v \in V_i \Rightarrow \) “the color of \( v \) is \( i \”).

Given a partition \((V_1, V_2)\) of a graph \( G \), a subgraph of \( G \) whose vertex set is contained in \( V_i \), \( i = 1, \) or \( i = 2, \) is referred to as monochromatic.

Once interpreted as a vertex 2-coloring a vertex partition of a graph naturally induces a vertex partition of every subgraph, and conversely, the union of vertex partitions of vertex disjoint subgraphs naturally provides a coloring of their graph union.

A Bisection of \( G = (V, E) \) is a vertex partition \((V_1, V_2)\) into two subsets of equal size, \( \vert V_1 \vert = \vert V_2 \vert \).

Let \((V_1, V_2)\) be a vertex partition of a graph \( G = (V, E) \) and let \( A \subseteq V \) be any set of vertices. We then use the following notation:

- The set of edges with one endvertex in \( A \) and the other one in \( V \setminus A \), known as the edge-cut induced by \( A \), is denoted here by \( E^i(A) \). Its cardinality \( \vert E^i(A) \vert \) is denoted by \( d(A) \). When more than one graph (or subgraph) is involved we can use \( d^i(A) \) to avoid ambiguity.

- Subject to a given orientation of \( G, E^1(A) \) is partitioned into \( E^+(A) \) and \( E^-(A) \), the sets of edges directed from \( A \) and into \( A \). We also denote in that case \( d^+(A) = \vert E^+(A) \vert \) and \( d^-(A) = \vert E^-(A) \vert \).

- Given a vertex partition \( \Psi = (V_1, V_2) \), \( \delta_\Psi(A) = \vert V_2 \cap A \vert - \vert V_1 \cap A \vert \) and \( \Delta_\Psi(A) = \vert \delta_\Psi(A) \vert \). When there is no confusion we omit \( \Psi \) and use \( \delta(A) \) and \( \Delta(A) \).

The following terminology is restricted to cubic graphs:

- An orientation of a cubic graph is Balanced if the outdegree of every vertex is either 1 or 2 (namely, there is no vertex of outdegree 0 or 3).

- We say that a bisection \( \Psi = (V_1, V_2) \) of a cubic graph \( G \) is Orientable if there exists an (clearly balanced) orientation of \( G \), where \( V_i, i = 1, 2, \) are the sets of vertices with outdegree \( i \), or equivalently for every vertex \( v, \Psi(v) = d^+(v) \).

The following is a simple instance of a theorem on feasible orientations of graphs. As we did not find a direct reference to that specific instance, we leave it as an exercise for the reader (a hint: use Hall’s theorem to match vertices \( \times \) outdegree into edges):

**Lemma 3.** A bisection \((V_1, V_2)\) of a cubic graph \( G = (V, E) \) is orientable if and only if \( d(A) \geq \Delta(A) \) for every set of vertices \( A \subseteq V \).

Our next result relies on the following classical variant of the “Max-flow Min-cut” theorem which deals with feasible flows in Flow-Networks with upper and lower edge capacities (see, eg, [2, p. 88], where credit for that result is given to J. Hoffman):

**Theorem 4.** Given a flow-network which consists of a directed graph \( D = (V, E) \) with upper and lower capacity functions \( u : E \to R \) and \( l : E \to R \), \( l(e) \leq u(e) \), there exists a feasible flow (no excess is allowed) \( f : E \to R \) which satisfies \( l(e) \leq f(e) \leq u(e) \) for every edge \( e \), if and only if:

For every set of vertices \( A \subseteq V \)

\[
\sum_{e \in E^+(A)} l(e) \leq \sum_{e \in E^-(A)} u(e). \tag{1}
\]
1.2  An elemental theorem

We can now state a necessary and sufficient condition for the existences of an \((r, \alpha)\)-flow in a cubic graph:

**Theorem 5.** A cubic graph \(G = (V, E)\) admits an \((r, \alpha)\)-flow with \(r \geq 2\) and \(0 \leq \alpha < 3\) if and only if there exists a bisection \((V_1, V_2)\) of \(G\) which complies with the following two conditions:

1. For every subset \(A\) of \(V\), \(d(A) \geq \Delta(A)\) and
2. For every subset \(A\) of \(V\), \(\alpha \geq \frac{2d(A) - (d(A) - \Delta(A))r}{2 |A|}\).

Condition 1 becomes redundant if \((V_1, V_2)\) is known to be orientable.

**Proof.** Given an \((r, \alpha)\)-flow in an orientation \(D\) of a cubic graph \(G = (V, E)\), \(\alpha < 3\) guarantees a balanced orientation. Accordingly, \((V_i, V_{\bar{i}})\), where \(V_i, i = 1, 2\) are the sets of vertices with outdegree \(i\), is an orientable bisection and as such satisfies Condition 1.

Transform now \(D\) into a flow-network \(\bar{G}\) by inserting one additional “excess collector” vertex \(x\) and a couple of antiparallel edges \(xv\) and \(vx\) for every vertex \(v \in V\), one directed from \(x\) toward \(v\) and the other one from \(v\) toward \(x\). Set the lower capacity of the original edges of \(G\) to 1 and their upper capacity to \(r - 1\). The edges incident with \(x\) all get lower capacity 0 and upper capacity \(\alpha\).

Equivalence between an \((r, \alpha)\)-flow in \(D\) and a feasible flow in the network \(\bar{G}\) is apparent. \(\bar{G}\) therefore complies with condition (1) of Theorem 4. We now show that Condition 2 (of Theorem 5) is satisfied by the original (undirected) graph \(G\):

When (1) is stated for a set \(A \subseteq V\), that is, a set of vertices of \(\bar{G}\) which does not include \(x\), it becomes

\[
1 \cdot d^+(A) + 0 \cdot |A| \leq (r - 1)d^+(A) + \alpha |A|.
\]

The first summand in each side reflects the capacity of the original edges of \(G\) and the second summand relates to the edges incident with \(x\). The above can be restated as

\[
\alpha \geq \frac{d^+(A) + \Delta(A) - d^-(A)r}{|A|}.
\]

Clearly \(d^+(A) + d^-(A) = d(A), d^+(A) - d^-(A) = \Delta(A),\) and hence \(d^-(A) = \frac{d(A) - \Delta(A)}{2}\), which provides

\[
\alpha \geq \frac{2d(A) - (d(A) - \Delta(A))r}{2|A|}.
\]

We should still consider sets of vertices of \(\bar{G}\) which do include the excess collector \(x\). Such a set is of the form \((V \setminus A) \cup \{x\}\), where \(A\) is a subset of \(V\). Notice that the number of edges incident with \(x\) in \(E^+(V \setminus A) \cup \{x\}\) and in \(E^-(V \setminus A) \cup \{x\}\) is \(|A|\). Inequality (1) for that set is then

\[
1 \cdot d^+(V \setminus A) + 0 \cdot |A| \leq (r - 1)d^-(V \setminus A) + \alpha |A|.
\]
Since $d^+(V \setminus A) = d^-(A)$ and $d^-(V \setminus A) = d^+(A)$, (4) is obtained from (2) when $d^+$ and $d^-$ exchange roles, which finally comes to replacing $\delta(A)$ by $-\delta(A)$ in (3):

$$\alpha \geq \frac{2d(A) - (d(A) + \delta(A))r}{2|A|}. \quad (5)$$

Depending on the sign of $\delta(A)$, either (3) or (5) is redundant and they hence combine into Condition 2 of Theorem 5.

For the proof of the “if” direction: Let $G = (V, E)$ and a bisection $(V_1, V_2)$ of $G$ comply with Conditions 1 and 2 of Theorem 5. By Lemma 3 and Condition 1 we assume an orientation $D$ of $G$, where $V_i, i = 1, 2$, are the sets of vertices with outdegree $i$. Construct now the flow-network $\tilde{G}$ as described above. Notice that the derivation of Condition 2 from (1) of Theorem 4 is fully reversible: Condition 2 clearly implies (5) and (3). Then (2) is obtained from (3) by the substitutions $d(A) = d^+(A) + d^-(A)$ and $\delta(A) = d^+(A) - d^-(A)$. Finally (2) translates into (1) for all sets $A$ which do not include $x$. Replacing (3) by (5) takes care of all sets which do include $x$. Theorem 4 confirms the existence of a feasible flow in $\tilde{G}$ and equivalently an $(r, \alpha)$-flow in the orientation $D$ of $G$. \hfill $\Box$

Theorem 5 and its proof are generalizations of the case $\alpha = 0$, stated and proved in [5,8]. When $\alpha = 0$ Condition 1 is implied by Condition 2 and therefore not explicitly stated in [5,8].

## 2. THE BOUNDED-EXCESS FLOWS POSET

### 2.1. Terminology

Whenever $r \leq s$ an $r$-cnzf is stronger than an $s$-cnzf, in the sense that the existence of the first implies that of the second. In this section we study the more complex two-dimensional hierarchy among bounded-excess flows.

**Definitions and Notation 6.**

- We use the notation $(r, \alpha) \preceq (s, \beta)$ when every cubic graph which admits an $(r, \alpha)$-flow also admits an $(s, \beta)$-flow. We say in that case that $(r, \alpha)$ is stronger or equivalent to $(s, \beta)$.
- $(r, \alpha)$ and $(s, \beta)$ are equivalent if $((r, \alpha) \preceq (s, \beta)) \land ((s, \beta) \preceq (r, \alpha))$.

“Strong is Small” may be confusing, yet we prefer consistency with nzf.

Properties of the $\preceq$ order can be visualized in the $r - \alpha$ plane (more accurately the upper-right quadrant of that plane, with $(2, 0)$ as the origin). Some specific points, lines, and regions on that plane play major roles in our analysis. To ease the formulation we adopt the following labeling (Occasionally referring to Figure 1 while reading this and the following sections are advised.)

**Remark.** Considering Theorem 22 the strength poset collapses at $(3\frac{1}{2}, 1\frac{1}{2})$ into a universal weakest equivalence class, denoted in Figure 1 by $\Omega$. Nonetheless, we chose to state our definitions, results, and proofs in a general setting (“an integer $k$” rather than $k = 4$ in some cases, or $k < 6$ in most). We found that approach preferable for better insight into
the subject without making things notably harder or more complex than treating each case separately.

**Definitions and Notation 7.** Given an integer \( k \geq 3 \), we define

- \( L_k \) is the segment of the line \( \alpha = \frac{k-r}{k-2} \) between the points \((2,1)\) (excluded) and \((k,0)\) (included).
- \( M_k \) is the upper part of \( L_k \) from \((3+\frac{k-3}{k-1}, \frac{k-3}{k-1})\) (included) to \((2,1)\).
- \( A_k = \{(r, \alpha)|2 < r < 4 \land (\frac{k-r}{k-2} \leq \alpha < \frac{k-3}{k-1}, \frac{k-3}{k-1})\} \) is the half open triangle whose vertices are \((3+\frac{k-3}{k-1}, \frac{k-3}{k-1}), (2,1)\), and \((4+\frac{k-3}{k-1}, \frac{k-3}{k-1})\), with the lower and the left \((M_k)\) edges included, but not the upper-right edge and its two endvertices.
- Let the Upper-right domain \( \text{urd} (r_0, \alpha_0) \) of a point \((r_0, \alpha_0)\) be the upper-right closed unbounded polygonal domain whose vertices are \((2, \infty), (2, 1), (r_0, \alpha_0), \) and \((\infty, \alpha_0)\).
- For a cubic graph \( G \), let the bounded-excess domain \( \text{bed} (G) \) be the set of all pairs \((r, \alpha)\) such that \( G \) admits an \((r, \alpha)\)-flow.
- If \( D \) is a (balanced) orientation of \( G \), then \( \text{bed}(D) \) is the subset of \( \text{bed}(G) \) consisting of all points \((r, \alpha)\) for which there exists an \((r, \alpha)\)-flow in \( D \).
- For a given point \((r_0, \alpha_0)\) we define \( \text{span}(r_0, \alpha_0) = \{(r, \alpha)|(r_0, \alpha_0) \in \text{bed}(G)\} \). Equivalently,

\[
\text{span}(r_0, \alpha_0) = \bigcap_{G \mid (r_0, \alpha_0) \in \text{bed}(G)} \text{bed}(G).
\]

- Although the main tool applied along this article, namely, Theorem 5 is stated in terms of orientable bisections, flows are more intuitively associated with orientations. As every balanced orientation of a cubic graph gives rise to the orientable bisection \( \Psi(v) = d^+(v) \), we refer in the sequel even without stating it explicitly to that bisection when associating Theorem 5 with a specific balanced orientation.
2.2 The Trace of a bounded-excess flow and the role of \( \text{urdf}(r_0, \alpha_0) \)

When a balanced orientation \( D \) (with the associated bisection \( (\Psi(v) = d^+(v)) \)) of \( G = (V, E) \) and a set \( A \subseteq V \) (and therefore \( d(A) \), \( \Delta(A) \), and \( |A| \)) are kept fixed, Condition 2 of Theorem 5, as a linear inequality, corresponds to an (upper-right) half-plane of the \( r - \alpha \) plane. Given a balanced orientation \( D \) of a cubic graph \( G = (V, E) \), the set \( \text{bed}(D) \) of points \( (r, \alpha) \) which complies with that inequality for every \( A \subseteq V \) is then an intersection of (finitely many) half-planes and as such it is a convex unbounded polygonal domain. \( \text{bed}(G) \) is the union of \( \text{bed}(D) \) over all balanced orientations \( D \) of \( G \). That is still an unbounded polygonal domain. Convexity is not a priori guaranteed, yet clearly:

**Lemma 8.** If an \( (r, \alpha) \)-flow and an \( (s, \beta) \)-flow both exist in the same orientation \( D \) of a cubic graph \( G \), then the line segment between \( (r, \alpha) \) and \( (s, \beta) \) is entirely contained in \( \text{bed}(D) \), and therefore also in \( \text{bed}(G) \).

In particular:

**Lemma 9.** Every balanced orientation of every cubic graph \( G \) admits a \( (2, 1) \)-flow.

**Proof.** \( f(e) = 1 \) for every edge \( e \) clearly does the job.

The points on the line defined by \( (r_0, \alpha_0) \) and \( (2, 1) \) are characterized by their:

**Definition 10.** The Trace of a point \( (r_0, \alpha_0) \), \( r_0 \geq 2 \), \( \alpha_0 < 1 \) is defined as

\[
\text{tr}(r_0, \alpha_0) = \frac{r_0 - 2\alpha_0}{1 - \alpha_0}.
\]

The line through \( (2, 1) \) and \( (r_0, \alpha_0) \) intersects with the \( r \)-axis at \( r = \text{tr}(r_0, \alpha_0) \). In particular, the trace of an \( r \)-cnzf is \( r \).

Notice that the line segment \( L_k \) consists of all the points \( p \) for which \( \text{tr}(p) = k \).
Lemma 8 implies:

**Lemma 11.** Let \( p = (r_0, \alpha_0) \) and \( q = (r_1, \alpha_1) \) be two points on the \( r - \alpha \) plane, such that \( tr(p) = tr(q) \) and \( \alpha_0 \leq \alpha_1 \) (equivalently \( r_0 \geq r_1 \)); then \( p \preceq q \).

Obviously if \( r \leq s \) and \( \alpha \leq \beta \), then \( (r, \alpha) \preceq (s, \beta) \). Combining with Lemma 11, it yields:

**Theorem 12.** \( \text{urd}(r, \alpha) \subseteq \text{span}(r, \alpha) \).

### 2.3 Bounded-excess flows and \( k \)-weak bisections

The following definition is taken from [4]:

**Definition 13.** Let \( k \geq 3 \) be an integer. A bisection \( (V_1, V_2) \) of a cubic graph \( G = (V, E) \) is a \( k \)-weak bisection if every connected monochromatic subgraph is a tree on at most \( k - 2 \) vertices.

A \( k \)-strong bisection is also defined in [4] and it is known [5] to be equivalent to a \( k \)-nzf. The following theorem reflects a similar connection between orientable \( k \)-weak bisections and bounded-excess flows:

**Theorem 14.** Let \( G \) be a cubic graph and \( k \geq 3 \) an integer. The following three statements are equivalent:

1. \( G \) admits an orientable \( k \)-weak bisection.
2. \( M_k \subseteq \text{bed}(G) \).
3. \( G \) admits an \( (r_0, \alpha_0) \)-flow with \( tr(r_0, \alpha_0) < k + 1 \).

**Proof:** 1 \( \Rightarrow \) 2: Let \( (V_1, V_2) \) be an orientable \( k \)-weak bisection of \( G = (V, E) \) and let \( A \subseteq V \) be a set of vertices of \( G \). Define \( A_1 = A \cap V_1 \) and \( A_2 = A \cap V_2 \). As \( A_2 \) induces a forest in a cubic graph, \( d(A_2) = |A_2| + 2c \), where \( c \) is the number of connected components induced by \( |A_2| \). Each such component has at most \( k - 2 \) vertices so \( c \geq \frac{|A_2|}{k-2} \) and \( d(A_2) \geq |A_2| + 2 \frac{|A_2|}{k-2} = \frac{k}{k-2} |A_2| \). As \( G \) is cubic at most \( 3|A_1| \) edges in \( E^l(A_2) \) have their second endvertex in \( A_1 \). Consequently

\[
d(A) \geq \frac{k}{k-2} |A_2| - 3|A_1|.
\]

We can assume \( |A_2| \geq |A_1| \) (otherwise exchange roles between \( A_1 \) and \( A_2 \)), which implies \( \Delta(A) = \delta(A) \) so \( |A_2| = \frac{|A| + \Delta(A)}{2} \) and \( |A_1| = \frac{|A| - \Delta(A)}{2} \). When plugged into the last inequality it yields:

\[
(k - 2)d(A) + (k - 3)|A| \geq (2k - 3)\Delta(A).
\]

We now divide by \( k - 1 \) to get, after some manipulations:

\[
\frac{k - 3}{k - 1} \geq \frac{2d(A) - (d(A) - \Delta(A))(3 + \frac{k - 3}{k - 1})}{2|A|}.
\]
which is obtained from Condition 2 of Theorem 5 with \( r = 3 + \frac{k-3}{k-1} \) and \( \alpha = \frac{k-3}{k-1} \). Theorem 5 implies the existence of a \((3 + \frac{k-3}{k-1} k-3 k-3\) )-flow and by Lemma 11, \( M_k \subseteq \text{bed}(G) \).

2 \( \Rightarrow \) 3: Points on \( M_k \) are of trace \( k < k + 1 \).

3 \( \Rightarrow \) 1: Assume an \((r_0, \alpha_0)\)-flow in an orientation of \( G = (V,E) \). The partition \((V_1, V_2)\), where \( V_i, i=1,2, \) are the sets of vertices with outdegree \( i \), is clearly an orientable bisection. If it is not a \( k \)-weak bisection, then there exists a monochromatic subgraph which is either a cycle on a set of vertices \( A \), where \( d(A) = \Delta(A) = |A| \), or a tree on a set of \( k - 1 \) vertices \( A \), where \( d(A) = k + 1 \) and \( |A| = \Delta(A) = k - 1 \). Condition 2 of Theorem 5 then either yields \( \alpha_0 \geq 1 \) which can be ignored, or

\[
\alpha_0 \geq \frac{2(k + 1) - 2r_0}{2(k - 1)}.
\]

So \((r_0, \alpha_0)\) lies on or on the right side of the line \( \alpha = \frac{k+1-r}{k-1} \), namely, on or on the right side of \( L_{k+1} \), where the trace is at least \( k + 1 \). We proved \( \neg 1 \Rightarrow \neg 3 \). \( \square \)

### 2.4 Classification of some regions of the \( r - \alpha \) plane

Properties of points in the labeled regions of the \( r - \alpha \) plane, depicted in Figure 1 can now be deduce:

**Corollary 15.** No cubic graph admits an \((r, \alpha)\)-flow with \( tr(r, \alpha) < 3 \) (ie, \( (r, \alpha) \) belongs to the region denoted by \( O \) in Figure 1).

**Proof.** Condition 2 of Theorem 5, when applied to a singleton \( A = \{x\} \), where \( d(A) = 3 \) and \( |A| = \Delta(A) = 1 \), yields \( \alpha \geq 3 - r \), that is, \( tr(r, \alpha) \geq 3 \) No \((r, \alpha)\)-flow therefore exists with \( tr(r, \alpha) < 3 \). \( \square \)

On the other extreme end:

**Lemma 16.** Every cubic graph \( G = (V,E) \) admits a \((2,1)\)-flow (no trace is defined for that point).

**Proof.** Considering Lemma 9 it suffices to prove the existence of a balanced orientation: Turn \( G \) into a 4-regular graph \( H \) by inserting \( \frac{|V|}{2} \) new edges which form a perfect matching. Select an Eulerian orientation of \( H \) and remove the extra edges. \( \square \)

**Corollary 17.** Every two points in the same triangle \( A_k \) are equivalent.

**Proof.** Let \( p \) be a point in \( A_k \). By Theorem 12, \( 3 + \frac{k-3}{k-1}, \frac{k-3}{k-1} \) \( \preceq p \). Clearly \( tr(p) < k + 1 \) so by Theorem 14, \( p \preceq (3 + \frac{k-3}{k-1}, \frac{k-3}{k-1}) \). All points of \( A_k \) are then equivalent to \( (3 + \frac{k-3}{k-1}, \frac{k-3}{k-1}) \) and consequently also to each other. \( \square \)

Equivalence of all points in \([(0,3), (0,4))\) is stated (using different notations of course) in [8] as the simplest instance of significantly more involved results which apply to regular multigraphs in general (not just cubic).
Let us remark that every point \( p \) such that \( k \leq tr(p) < k + 1 \) is at least as strong, but in general not equivalent to the points of \( A_k \). Furthermore:

**Theorem 18.** There are infinitely many nonequivalent points in the \( r - \alpha \) plane.

**Proof.** For every rational number \( r_0 \) in \([4,5]\), there exists a cubic graph \( G \) with \( \phi_c(G) = r_0 \) [6]. As a result, no two points on the line segment from \((4,0)\) to \((5,0)\) (rational, or not) are equivalent.

Another characteristic of the \( \leq \) order is that it is a proper partial order (not a full order).

**Theorem 19.** There are two points \( p \) and \( q \) such that neither \( p \leq q \) nor \( q \leq p \).

**Proof.** The Petersen graph admits a 5-nzf, that is, a \((5,0)\)-flow, but it does not admit a 4-weak bisection (see, eg, [4]) and therefore neither it admits a \((3^1_3, 1^1_3)\)-flow (Theorem 14). On the other hand take any cubic graph with a bridge which does admit an orientable 4-weak bisection and hence also a \((3^1_3, 1^1_3)\)-flow. Since the graph has a bridge it does not admit any \( r \)-cnzf, in particular not a 5-nzf. An example of such a graph is presented by the diagram at the left side of Figure 4. We believe (Conjecture 44) that any cubic graph with a bridge which admits a perfect matching will do as well.

2.5 Some noteworthy instances and corollaries

The following is an explicit formulation of Lemma 11, where \( p = (r_0, \alpha_0) \) are the points \((k,0)\), \( k \in \{3, 4, 5, 6\} \):

**Corollary 20.**

- A cubic graph is bipartite if and only if it admits a 3-nzf and therefore, an \((r, 3-r)\)-flow for every \( r, 2 \leq r \leq 3 \).
- A cubic graph is 3-edge-colorable if and only if it admits a 4-nzf and therefore, an \((r, \frac{4-r}{2})\)-flow for every \( r, 2 \leq r \leq 4 \).
- A cubic graph admits a 5-nzf (every bridgeless cubic graph if the assertion of the 5-flow conjecture [9] holds) if and only if it admits an \((r, \frac{5-r}{3})\)-flow for every \( r, 2 \leq r \leq 5 \).
- Every bridgeless cubic graph admits a 6-nzf [7] and therefore an \((r, \frac{6-r}{4})\)-flow for every \( r, 2 \leq r \leq 6 \).

The following simple instance of Theorem 14 applies to a significant family of Vizing’s class two cubic graphs, which includes all “Classical Snarks” except for the Petersen graph (see, eg, [3]).

**Corollary 21.** If the circular-flow number \( \phi_c(G) \) of a bridgeless cubic graph \( G = (V, E) \) is strictly smaller than 5, then \( G \) admits a \((3^1_3, 1^1_3)\)-flow.

**Proof.** The trace of \((r,0)\) (corresponding to an \( r \)-cnzf) is \( r \). If \( \phi_c(G) < 5 \), then \( G \) admits a \( p \)-flow with \( tr(p) < 5 \) and Theorem 14 applies.

In Section 3 we present the main result of this article.
3 EVERY CUBIC GRAPH ADMITS A \((3^{1/2}, 1^{1/2})\)-FLOW

**Theorem 22.** Every cubic graph admits a \((3^{1/2}, 1^{1/2})\)-flow.

To prove Theorem 22 we have it restated by means of Theorem 14 as

**Theorem 23.** Every cubic graph admits an orientable 5-weak bisection.

**Proof.** Theorem 11 of [4] states the existence of a 5-weak bisection for every cubic graph. The proof in [4] however does not guaranty an orientable bisection. To reach that goal we modify and significantly extend the proof of Theorem 11 of [4].

The following lemma summarizes the part of our proof where we explicitly rely on [4]:

**Lemma 24.** Every cubic graph \(G = (V, E)\) contains a spanning factor \(F\) where every connected component is either a path on at least two vertices, or a cycle, and the following conditions hold:

1. The two endvertices of an odd (number of vertices) path in \(F\) are nonadjacent.
2. An endvertex of one path in \(F\) and an endvertex of another path in \(F\) are nonadjacent.
3. A chord in an odd cycle of \(F\) (if exits) is parallel to an edge of that cycle.
4. If a vertex \(y\) of an odd cycle \(C\) of \(F\) is adjacent to a vertex \(x\) of another component, then \(x\) is an internal vertex of an odd path of \(F\).
5. An odd path of \(F\) is connected to at most one vertex of at most one odd cycle of \(F\).

**References** to these results and their proofs in Section 3 of [4] are (Claim numbers refer to that article. The factor \(F\) is denoted in [4] by \(P^*\)):

1. Definition 12 (of [4]). The proof in [4] starts with Vizing’s theorem which although it does not hold for multigraphs in general, it trivially applies to cubic multigraphs.
2. Claim 13.
3. Claim 14. In [4] a graph is assumed to be simple. If parallel edges are allowed, the proof of Claim 14 remains valid for the case where \(u\) and \(v\) are nonconsecutive vertices of \(C\).
4. Claim 15. It is proved in [4] that \(x\) lies in an even position of an odd path, but that additional fact is irrelevant to our needs here. Furthermore, if \(C\) is a cordless cycle in a simple cubic graph, as it is assumed to be in [4], then every vertex of \(C\) is adjacent to a vertex out of \(C\). We have chosen a weaker formulation because this is not the case when parallel edges are allowed and we also wish the lemma to apply to certain subgraphs of \(G\), where the degree of a vertex can be less than 3.
5. Claims 15 to 17.

**Definitions and Notation 25.**

- Until the end of the current section \(F\) is a factor of a cubic graph \(G = (V, E)\), which complies with Lemma 24. Every connected component of \(F\) is referred to as an \(F\)-component.
- A vertex \(v \in V\) is external if it is an endvertex of a path of \(F\), and also:
• Two external vertices on each odd cycle C are the two endvertices of an arbitrarily selected simple (not two parallel edges) edge of C. At least every second edge along C is simple so that selection is clearly doable.
• A vertex which is not external is Internal.
• It turns out that every vertex of an even cycle is internal.
• An edge which connects an external vertex of an odd cycle C to a vertex which does not belong to C (on an odd path by Lemma 24-4) is called a Critical edge.
• Let C be an F-component on k vertices v₁, v₂, ..., vₖ as ordered along C. Unless C is an even cycle, v₁ and vₖ are the external vertices. If C is an even cycle they are arbitrarily selected two consecutive vertices.
• An alternating coloring Ψ of C is either a Parity coloring—Ψ(vᵢ) = 1 if i is odd and Ψ(vᵢ) = 2 if i is even, or a Counter-parity coloring—Ψ(vᵢ) = 2 if i is odd and Ψ(vᵢ) = 1 if i is even.
• An alternating coloring of a subgraph of G whose vertex set is a union of F-components, is the union of alternating colorings of these components.

The following coloring rule will be obeyed in our construction of an orientable 5-weak bisection of G:

**Rule 26.** If each of the two external vertices of an odd cycle C is incident with a critical edge, then at least one of these two critical edges is bichromatic, that is, its two endvertices differ in color.

**Claim 27.** A monochromatic connected subgraph obtained by an alternating coloring which complies with Rule 26 is a tree on at most three vertices.

**Proof.** We say that a vertex is Good if it differs in color from at least two of its neighbors (a neighbor through two parallel edges counts for that matter as two) and it is otherwise Bad. Observe that no vertex shares color with all its three neighbors and hence a connected monochromatic subgraph is a tree on at most three vertices if and only if it does not contain two adjacent bad vertices of the same color. An internal vertex is clearly good. By Lemma 24-1,2,4 two external vertices of the same color are adjacent only if they are the external vertices u and v of an odd cycle C. We have to show that at least one of these two external vertices is good. Let us assume that the common color of u and v is 1. The other neighbor on C of each one of them is of color 2 (alternating coloring) so if one of them is adjacent to that other neighbor through parallel edges, then it is good. Otherwise, Rule 26 guaranties for one of u and v a second neighbor of color 2 which makes that endvertex good. □

Our goal is to describe an alternating coloring of G which complies with Rule 26 and also defines an orientable bisection.

Let us refer to an edge of G which does not belong to F as a Skeletal edge (s-edge). We now construct a spanning graph S of G which fully contains all the F-components and also a set EX of s-edges such that when every F-component is contracted into a single vertex the graph obtained from S is a tree. (For that to be possible we assume that G is connected.)

The first step in the construction of S is to include in EX every critical edge. This step is meant to gain control over the edges relevant to Rule 26. Lemma 24-5 guaranties no violation of the “tree like” structure of S.
To complete the construction of $S$ we add to $E_X$ additional $s$-edges until the required property is reached, that is, $S$ becomes a tree when every $F$-component is contracted into a single vertex. See Figure 3, where an $F$-component is represented by a horizontal line (with an arc underneath if it is a cycle) and the skeletal edges are vertical.

$S$ as a subgraph of $G$ is subcubic. We now generalize the notion of an orientable bisection for graphs which are not necessarily cubic. That definition is not associated with an actual orientation but it carries the essence of Condition 1 of Theorem 5:

**Definition 28.** A partition $\Psi = (V_1, V_2)$ of an even (number of vertices) graph $H = (V', E')$ is an orientable bisection if for every set of vertices $A \subseteq V'$:

$$d_H(A) \geq \Delta_\Psi(A).$$

The inequality above where $A = V'$ implies that $\Psi$ is indeed a bisection.

**Definitions and Notation 29.**

- In the sequel we construct a coloring $\Psi$ of $S$ which complies with Rule 26 and with the inequalities of Definition 28. Let us call such a coloring $\Psi$ a **Valid coloring**. An example of the obtained coloring is depicted in Figure 3.

- The removal of $k$ $s$-edges decomposes $S$ into $k + 1$ connected subgraphs. Each of these components is referred to as a skeletal subgraph ($s$-subgraph). An $s$-subgraph either entirely contains an $F$-component $C$, or it is disjoint from $C$.

- An $s$-subgraph $H = (V', E')$ is even or odd according to the parity of $|V'|$.

- We say that an $s$-edge $e$ of $S$ is even or odd according to the parity of each of the two $s$-subgraphs obtained by the removal of $e$ (the same parity since $S$ is even). Similarly $e$ is even or odd in an even $s$-subgraph $H$ of $S$ according to the parity of the two subgraphs of $H$ obtained by the removal of $e$.

**Claim 30.** The removal of a sequence of even $s$-edges decomposes $S$ into even disjoint $s$-subgraphs.

![Figure 3]( تصویر 3: رنگ آمیزی آمیخته از پیامدهای اضافی از $s$-برقی $S$ به سطح $G$ با تعداد کمی از $s$-برقی $S$ تا زمانی که معیار مورد نیازی را در یک رأس به هر $F$-برقی تغییر دهیم. در نمودار 3، $F$-برقی توسط یک خط عمودی (با قطعیت زیرین اگر یک مدار در دسترس باشد) و اجاق $s$-برقی طرفی شده توسط $s$-برقی.)
Proof. Assume to the contrary a minimal sequence $Q$ of even s-edges, such that the removal of the last (the order does not really matter) one $e$ decomposes one of the existing even components into two odd ones. At that stage the decomposition includes exactly two odd components and the others are all even. Reinsert the edges of $Q \setminus \{e\}$ one by one. On each such step two components are merged into one. As $e$ remains outside, the two odd components remain separated from each other, so after each step there are still exactly two odd components. Finally when $e$ is the only edge left outside, $S$ is decomposed into two odd s-subgraphs, in contradiction with the assumption that $e$ is even. □

It is worth noting that even s-subgraphs can also be generated by the removal of odd s-edges. For example, the removal of two s-edges of which one is odd provides one even s-subgraph $H$ (and two odd ones), such that an even edge of $S$ may be odd in $H$. The assertion of the last claim, therefore, cannot be taken for granted without a proof.

Definition 31. A Prime-even s-subgraph (pes-subgraph) is obtained from $S$ by repeatedly removing even s-edges until the remaining s-edges are all odd.

Claim 32. To prove the existence of a valid coloring of $S$ as well as of every even s-subgraph $H$ obtained from $S$ by the removal of any set of even s-edges, it suffices to prove the existence of such a coloring for every pes-subgraph. When applying Rule 26 to a subgraph $H$ we should consider only odd cycles with two critical edges which both belong to $H$.

Proof. If $H$ is a pes-subgraph, then we are done. Otherwise, let $e$ be an even s-edge whose removal decomposes $H$ into two even subgraphs $H_1$ and $H_2$. We can assume by induction the existence of valid colorings of $H_1$ and of $H_2$. The union $\Psi$ of these two colorings is clearly an orientable alternating bisection of $H$. As for Rule 26, by induction it is satisfied for odd cycles with two critical edges in the same subgraph $H_i$. Attention should be paid to the case where the removed edge $e$ is a critical edge of an odd cycle $C$, in one of the subgraphs. Notice that switching between the colors 1 and 2 in one of the two even subgraphs $H_1$ or $H_2$ does not compromise the orientable bisection. That way we can guaranty the two endvertices of $e$ to be of distinct colors, as required by Rule 26. See Figure 3. □

Definitions and Notation 33.

- Let $H$ be a pes-subgraph of $S$. We select an $F$-component $R_H$ to be the root of $H$.
- The removal of an s-edge $e$ decomposes $H$ into two odd s-subgraphs, the Lower side of $e$, $D(e)$ which contains $R_H$ and its Upper side $U(e)$ which does not contain $R_H$.
- Accordingly, the endvertices of $e$ are its Upper endvertex and its Lower endvertex.
- The subgraph $\bar{e}$ consists of $e$ and its two endvertices.
- The union $B$ of $\bar{e}$ and $U(e)$ is a Branch of $H$. $\bar{e}$ is the Stem of the branch $B = \bar{e} \cup U(e)$ and $U(e)$ is the Top of $B$, also denoted by $t(B)$.
- The $F$-factor $C$ in $t(B)$ which includes the upper vertex of the stem is the Base of the branch $B$.
- The stem of a branch $B$ is also referred to as the stem of the base of $B$. Every $F$-factor in $H$, other than the root $R_H$ is the base of a branch and hence has a stem.
- The lower endvertex of the stem $\bar{e}$ of a branch $B$ is the Heel of $B$, denoted by $\text{heel}(B)$, while the upper endvertex is the Heel of $\bar{e}$. The reason for that somewhat confusing terminology will be cleared soon.
• A branch can also be obtained by contracting the lower side of its stem into a single vertex (the heel). Contracting an odd subgraph into a single vertex preserves subgraph parity so, like $H$, every branch is prime-even in the sense that the removal of any s-edge provides two disjoint odd subgraphs. Accordingly we define a prime-even generalized s-subgraph (pegs-subgraph) $T$ of a pes-subgraph $H$ to be either $H$ itself or a branch in $H$.

• The base of a pes-subgraph $H$ when referred to as a pegs-subgraph is its root $R_H$.

• Let $T$ be a pegs-subgraph and let $C$ be the base of $T$. A branch $L$ whose heel belongs to $C$ is a limb of $T$. When $T$ is a branch, the stem $\bar{e}$ of $T$ also counts as one of its limbs.

• The upper endvertex of the stem $\bar{e}$ of a branch belongs to the base of that branch. For that reason, heel($\bar{e}$) is counterintuitively its upper endvertex as previously defined. The lower endvertex of the stem forms its top subgraph $t(\bar{e})$. See Figure 2, where the stem is drawn from the base $C$ upwards with its lower endvertex at the top. In Figure 3 we get a more global view where the stems are drawn from each component downwards toward the root.

Obviously:

Observation 34. All clauses of Lemma 24 apply to a pes-subgraph $H$ (as well as to any s-subgraph) of $S$, where $F$ is restricted to the $F$-components which are contained in $H$.

As for a branch $B$, Lemma 24-1,2,3,5 still similarly apply. Lemma 24-4, however, poses an issue when the base of $B$ is an odd cycle $C$ and the heel of the stem is incident with an external vertex of $C$. In that case the “out of $C$” neighbor is the lower endvertex of the stem which is a single vertex while the $F$-component to which it belongs is not contained in $B$.

We say that an odd cycle $C$ in a pegs-subgraph $T$ is bicritical in $T$ if both its external vertices are incident with critical edges which belong to $T$.

Let us restate a stronger version of Rule 26 which applies to any pegs-subgraph $T$, be it a branch or a pes-subgraph:

Rule 35. Let $C$ be a bicritical odd cycle in a pegs-subgraph $T$. At least one critical edge of $C$ which does not belong to the stem of $C$ should be bichromatic, that is, its two endvertices should differ in color.

Definition 36. A valid orientable bisection of a pegs-subgraph $T=(V',E')$ is an alternating coloring $\Psi$ of $T$ which complies with Rule 35 and with the condition $d(A)_T \geq \Delta_\Psi(A)$ for every subset of vertices $A \in V'$.

We have set the necessary tools to state and prove:

Lemma 37. Every pegs-subgraph $T$ admits a valid orientable bisection.

Proof. Let the base $C$ of $T$ be an $F$-component on $k$ vertices $v_1, v_2, ..., v_k$ as ordered along $C$. As previously stated, if $C$ is not an even cycle, then $v_1$ and $v_k$ are selected to be the external vertices of $C$. If $C$ is an even cycle, then $v_1$ and $v_k$ are two arbitrarily selected consecutive vertices along $C$.

Notice that a vertex of $C$ is not necessarily incident with an s-edge and if $C$ is a path each of $v_1$ and $v_k$ may be incident with two s-edges, accordingly, the number $m$ of limbs of $T$ can be smaller or larger or equal to $k$. Nonetheless:
Claim 38. The number $m$ of limbs of $T$ is of the same parity as the order $k$ of the base $C$.

Proof. The number of limbs $m$ is also the number of skeletal edges incident with $C$. Removal of these $s$-edges decomposes $T$ into $m + 1$ components of which $m$ are odd limbs’ tops and the last one is the base $C$. The parity of the total number of vertices in $T$ is therefore the parity of $m + k$. As $T$ is even then $m$ implies $m \equiv k$ (modulo 2). (See Figure 3. Remember to count the stem among the limbs of each branch.)

Let $L_1, L_2, ..., L_m$ be the limbs of $T$ in nondecreasing order of the indices of their heels among $v_1, v_2, ..., v_k$. The proof proceeds by induction: We assume the existence of a valid bisection $\Psi_j$ for every limb $L_j$ of $T$ (verification for the smallest pegs-subgraphs is left for the end of the proof).

Let $K_j$ be the vertex set of $t(L_j)$ the top of $L_j$ ($K_j$ includes all vertices of $L_j$ except for its heel). As $\Psi_j$ is a bisection and the heel of $L_j$ is a single vertex, $\Delta_{\Psi_j}(K_j) = 1$. We say that the color of the top set $K_j$ is 2 if $\delta_{\Psi_j}(K_j) = 1$, or it is 1 if $\delta_{\Psi_j}(K_j) = -1$.

Observe that $\Psi_j$ remains valid if the colors 1 and 2 are switched. Accordingly, we can freely chose the color of each limb top $K_j$ without violating the validity of the colorings $\Psi_j$.

We perform that choice according to the following rules:

Rule 39.

- If the base $C$ is any $F$-component other than a bicritical odd cycle, then $K_j$ gets its counter-parity color, that is, 2 if $j$ is odd and 1 if $j$ is even.
- If $C$ is a bicritical odd cycle in $T$, then assume that $v_1$ is adjacent to a vertex $x$ on an odd path in $K_1$ and that the limb $L_1$ is not the stem of $C$ (otherwise reverse the order of $v_1, ..., v_k$). For $2 \leq j \leq m - 1$ let $K_j$ get its parity color (1 if $j$ is odd and 2 if $J$ is even). Now select a color for $K_1$ such that $\Psi_1(x) = 2$. Conclude with coloring $K_m$ to make its color distinct from the color of $K_1$ (see Figure 2).
- So far we have an orientable bisection $\Psi_j$ for each limb. Their union however does not necessarily covers the entire subgraph $T$, as some vertices of $C$ may not belong to limbs. Also the sought coloring should be alternating on each $F$-component. Thus we finalize the definition of a bisection $\Psi$ of $T$ by recoloring the vertices $v_1, ..., v_k$ each with its parity color. Clearly an alternating coloring.

Following Rule 39, one can verify that in either case half of the $k + m$ elements-vertices of $C$ and limbs’ tops-are colored 1 and the other half are colored 2. Consequently, $\Psi$ is indeed a bisection. In making that observation notice the equal parity of $k$ and $m$ and the fact that when both are odd the color 1 has majority in $C$ and the color 2 gains majority among the limbs’ tops (As before, the color $\Psi(K_j)$ of a top set is the one with majority among the vertices of $K_j$. The contribution of a top set to $\delta_{\Psi}$ is the same as that of a single vertex of the same color).

Let us now verify that $\Psi$ is a valid orientable bisection of $T$:

Rule 26, in its more specific formulation as Rule 35, can be assumed (induction) to be obeyed by each of the colorings $\Psi_j$ of the limbs $L_j$. Inserting the base $C$ of $T$ may be relevant to the rule only if $C$ is either a bicritical odd cycle in $T$ or an odd path (Lemma 24-4).

Assume that $C$ is a bicritical odd cycle. We chose in that case the color of $K_1$ such that the vertex $x$ to which the external vertex $v_1$ is adjacent, is of color 2. Then $v_1$ as a vertex of $C$ is assigned with its parity color, namely, 1 (Rule 39). Consequently, Rule 35 is obeyed also by the new coloring $\Psi$ of $T$ (see Figure 2, where $K_1$ is assumed to be colored 2 for the endvertex $x$ of the critical edge to get the color 2).
We now consider the case where the base \( M_j \) of a certain limb \( L_j \) is a bicritical odd cycle, and the base \( C \) of \( T \) is an odd path. \( M_j \) in that case is connected to \( C \) by its stem, which may include an external vertex of \( M_j \). Nonetheless, Rule 35 when applied to \( L_j \) (induction) guaranties that an external vertex \( y \) of \( M_j \), which does not belong to the stem of \( M_j \) is already adjacent through a critical edge to a vertex \( x \) which differs from \( y \) in color. So we are good in that case as well.

The vertices of each \( F \)-component get their parity color so \( \Psi \) is an alternating coloring as required for a valid bisection.

It remains to show that \( \Psi \) is an orientable bisection.

**Claim 40.** For the condition \( d(A) \geq \Delta(A) \) of Definition 28 it suffices to consider sets of vertices \( A \) such that for every limb \( L_j \) of \( T \), \( A \) includes either all or none of the vertices of \( L_j \).

**Proof.** Let \( A \) be a set of vertices of \( T \) which does not include \( \text{heel}(L_j) \) but does contain a nonempty subset \( A' \) of \( K_j \). It implies that \( A \setminus A' \) is disjoint form \( L_j \) so we can assume to have proved

\[
d(A\setminus A') \geq \Delta_{\Psi}(A\setminus A').
\]

\( \Psi \) and \( \Psi_j \) are identical on \( K_j \) and therefore on \( A' \). So \( \Delta_{\Psi}(A') = \Delta_{\Psi}(A') \) (regardless of the color \( \Psi(\text{heel}(L_j)) \) which might have changed). As \( \Psi_j \) is an orientable bisection

\[
d(A') \geq \Delta_{\Psi}(A'),
\]

the relevant edge-cuts are disjoint so

\[
d(A) = d(A\setminus A') + d(A').
\]

Also \( \delta_{\Psi}(A) = \delta_{\Psi}(A\setminus A') + \delta_{\Psi}(A') \), namely,

\[
\Delta_{\Psi}(A) \leq \Delta_{\Psi}(A\setminus A') + \Delta_{\Psi}(A').
\]

It all comes to the required inequality

\[
d(A) \geq \Delta_{\Psi}(A),
\]

\( \Psi \) is a bisection and as such provides the same values of \( d \) and \( \Delta \) to a set and to its complement. Accordingly, the above also applies to sets \( A \) which do include \( \text{heel}(L_j) \). \( \square \)

The family of sets \( A \) for which \( d(A) \geq \Delta(A) \) should be proved can be further reduced:

**Claim 41.** For the proof of \( d(A) \geq \Delta(A) \), it suffices to consider sets \( A_I \), where \( I \) is an interval \([i_l, i_r] = \{i | i_l \leq i \leq i_r \}\) of consecutive integers between 1 and \( k \).

**Proof.** For an integer \( i, 1 \leq i \leq k \) let \( P_i \) be the set of vertices of the limb whose heel is \( v_i \), or the singleton \( \{v_i\} \) if there is no such limb.
According to Claim 40 a set \( A_I \), for which the inequality should be proved is defined by a set of indices \( J \in \{1, 2, ..., k\} \) as

\[
A_I = \bigcup_{j \in J} P_j.
\]

Vertices of \( C \) whose indices do not belong to \( J \) separate \( J \) into a set \( \mathcal{I} \) of disjoint intervals of consecutive integers. It is apparent that \( d \) as well as \( \delta \) can be computed separately for each \( A_I, I \in \mathcal{I} \) and then sum up to obtain \( d(A_I) \) and \( \delta(A_I) \). As \( d(A_I) \) is nonnegative and \( \Delta(A_I) = \|\delta(A_I)\| \) it suffices to confirm \( d(A_I) \geq \Delta(A_I) \) for every \( I \in \mathcal{I} \).

In the sequel \( I \) is an interval of consecutive integers in \([1, k]\). We also assume that \( I \) is not the entire interval \([1, k]\) (for which \( A_I \) is the entire set of vertices with \( d = \delta = 0 \). \( d(A_I) = 2 \) whenever \( I \) is an Internal interval, that is, if \( C \) is a cycle or \( C \) is a path where \( 1 \not\in I \) and \( k \not\in I \). If \( C \) is a path, then \( d(A_I) = 1 \) if \( I \) is a Terminal Interval which either include 1 or \( k \) (not both).

Let \( I \) be the interval \([i_l, i_r]\). For the computation of \( \delta(A_I) \) we represent the colors of the vertices of \( A_I \) by two sequences of the colors 1 and 2:

- The Base sequence \( B_I \) consisting of the colors \( \Psi(v_{i_l}), \Psi(v_{i_l+1}), ..., \Psi(v_{i_r}) \). Since \( \Psi \) is an alternating coloring this is always an alternating sequence.
- The second is the Top sequence \( W_I \) of the colors \( \Psi(K_{i_l}), ..., \Psi(K_{i_r}) \) of the top sets of the limbs which belong to \( A_I \). These colors are set according to Rule 39.

Let us consider first the case where the base \( C \) of \( T \) is not a bicritical odd cycle. In that case both the base and the top sequences are alternating (see Rule 39). The difference between the numbers of 2s and 1s in an alternating sequence is at most 1, so it sums up to at most 2 for the union of the two sequences. That comes to \( \Delta(A_I) \leq 2 \). As \( I \) is internal \( d(A_I) = 2 \) so

\[
d(A_I) \geq \Delta(A_I)
\]
as required.

If \( I \) is a terminal interval of a path, say \( I = [i_l, i_r] \), then either \( \delta(B_I) = 0 \) if \( |I| \) is even, or \( \delta(B_I) = -1 \) if \( |I| \) is odd. Rule 39 implies in that case that the first term of \( W_I \) is \( \Psi_1(K_{i_l}) = 2 \) (unless \( W_I \) is empty, which makes no exception), so either \( \delta(W_I) = 0 \) or \( \delta(W_I) = 1 \). Summing this up provides \( \Delta(A_I) \leq 1 \). As \( I \) is terminal \( d(A_I) = 1 \) so the required inequality holds. If \( I = [i_l, k] \) we rely on the equal parity of \( k \) and \( m \) so this time the last (rather than the first) terms of \( B_I \) and of \( W_I \) are distinct, which leads to the same computation and final result.

Now to the case where the base \( C \) is a bicritical odd cycle in \( T \). In that case \( v_1 \) and \( v_k \) both are colored 1 and both are heels of limbs of \( T \). Rule 39 guaranties the top of one of these two limbs to be colored 2. This limb, say \( L_1 \) (with no loss of generality), contributes a 1 to the base sequence and a 2 to the top sequence. When \( L_1 \) is removed both the base and the top sequences (for the entire subgraph \( T \)) become alternating even sequences, where \( \Delta(A_I) \leq 2 \) for every interval \( I \). That does not change when \( 1 \in I \) because \( \delta(P_1) = 0 \). Since \( C \) is a cycle there are no terminal intervals and \( d(A_I) = 2 \) for every interval \( I \). \( d(A_I) \geq \Delta(A_I) \) follows (see Figure 2).

To initialize the induction, Lemma 37 should be verified where \( T \) consists of a single \( F \)-component. \( T \) can either be an even isolated \( F \)-component, or a branch which consists of an odd base \( C \) and its stem. An isolated even \( F \)-component provides an even alternating base
sequence and an empty top sequence. An odd base \( C \) with a stem yields an odd base sequence and a top sequence consisting of a single 2. Neither of the two poses an exception to the proof schema described above.

\[ \Box \]

**Concluding the proof of Theorem 23 (and Theorem 22):**

Lemma 37 where \( T = S \) provides an alternating orientable bisection \( \Psi \) of \( S \).

Rule 26 is meant to guaranty that at least one of the two external vertices of an odd cycle is “good” in the sense that it differs in colors from at least two of its neighbors. Let us summarize the verification of that property for the original cubic graph \( G \).

In the proof of Lemma 37 we took care of odd cycles with two critical edges within a pegs-subgraph \( T \). Let us observe that it indeed suffices.

When constructing \( S \) we started with including in \( S \) every critical edge of \( G \). If an external vertex \( y \) of an odd cycle \( C \) is not incident with a critical edge within a pegs-subgraph \( T \), then either there is no such edge in \( G \) or it was removed as an even edge when decomposing \( S \) into pes-subgraphs. In the first case \( y \) connects to a neighbor \( x \) on \( C \) with two parallel edges. That makes \( y \) good because \( x \) counts as two and its color differs from the color of \( y \) (alternating coloring). In the second case \( y \) can be considered good as shown in the proof of Claim 32.

As \( S \) is a spanning subgraph of \( G \), both graphs \( G \) and \( S \) share the same vertex set \( V \). Therefore the bisection \( \Psi \) applies to \( G \) as well as it does for \( S \) with the same value of \( \Delta \Psi(A) \) for every \( A \subseteq V \). On the other hand, the edge set of \( S \) is a subset of \( E \) which implies \( d_G(A) \geq d_S(A) \). The inequalities \( d(A) \geq \Delta(A) \) therefore hold for the bisection \( \Psi \) of \( G \) and \( \Psi \) is indeed an alternating orientable bisection of \( G \), which complies with Rule 26.

Claim 27 then asserts that a connected monochromatic subgraph induced by \( \Psi \) is a tree on at most 3 vertices. By Definition 13, \( \Psi \) is an orientable 5-weak bisection of \( G \).

**Theorem 22** cannot be improved as it provides a tight result:

**Theorem 42.** There exists a cubic graph \( G \) with \( \text{bed}(G) = \text{urd}(3 \frac{1}{2}, \frac{1}{2}) \).

**Proof:** Let \( f \) be an \((r, \alpha)\)-flow in any balanced orientation of the graph \( G \) on the diagram at the right side of Figure 4. Two of the three edges incident with the vertex \( x \) are directed both into \( x \), or both from \( x \) outwards. The total flow on these two edges is at least 2 and it spreads as (positive or negative) excess among four vertices, which makes \( \alpha \geq \frac{1}{2} \). A brief case analysis shows that \( G \) does not admit a 4-weak bisection (any bisection either induces two monochromatic parallel edges or a monochromatic path on three vertices). By Theorem 14 \( tr(r, \alpha) \geq 5 \) and since \( \alpha \geq \frac{1}{2} \) it implies \((r, \alpha) \in \text{urd}(3 \frac{1}{2}, \frac{1}{2})\), so \( \text{bed}(G) \subseteq \text{urd}(3 \frac{1}{2}, \frac{1}{2}) \). Theorem 22 on the other hand yields \((3 \frac{1}{2}, \frac{1}{2}) \in \text{bed}(G)\). \( \text{bed}(G) = \text{urd}(3 \frac{1}{2}, \frac{1}{2}) \) follows. \( \Box \)

**Figure 4** An orientable 4-weak bisection with no 5-nzf (left). A graph \( G \) with \( \text{bed}(G) = \text{urd}(3 \frac{1}{2}, \frac{1}{2}) \) (right). nzf, Nowhere-zero flows.
A straightforward conclusion is:

**Corollary 43.** The intersection of \( \text{bed}(G) \) over all cubic graphs \( G \) is \( \text{span}(3\frac{1}{2}, \frac{1}{2}) = \text{urd}(3\frac{1}{2}, \frac{1}{2}) \). That region is denoted by \( \Omega \) in Figure 1.

**Remark.** If restricted to cubic graphs which admit a perfect matching (bridgeless graphs included), Theorem 22 has a much simpler and 10 times shorter proof. It is basically the second proof of Theorem 11, presented for that restricted case in [4], with some modifications.

### 4 | OPEN PROBLEMS AND CONCLUDING REMARKS

We tend to believe that the wider two-dimensional scope of bounded-excess flows can lead to new insight and better understanding of cnzf. The “Balanced Valuations” principle [5,8] behind Theorem 5 provides a simple—if and only if—condition for the existence of nzf, yet, although it is already known for more than 40 years it has rarely been used and so far failed to produce significant results ([8] may be considered an exception). This article, however, relies almost solely on Theorem 5. Our main result, Theorem 22, is proved independently of other “Nowhere-zero flow theory” results, such as the 6-flow theorem. Still, there exists a line (literally, see Figure 1) of potential progress from our result down to (literally again, on a two-dimensional framework) the assertion of the 5-flow conjecture. Also the tightness of the \( (3\frac{1}{2}, \frac{1}{2}) \) bound established in Theorem 22 may shed some light on the difficulty of settling the 5-flow conjecture.

#### 4.1 | Two conjectures

\( \phi_c(G) < 5 \) is not a necessary condition for the result in Corollary 21. Following A. Ban and N. Linial [1], the revised Ban-Linial Conjecture (Conjecture 9 in [4]) asserts that every cubic graph which admits a perfect matching, other than the Petersen graph, admits a 4-weak bisection. We hereby suggest the following stronger version:

**Conjecture 44.** Every cubic graph \( G \) which admits a perfect matching, other than the Petersen graph, admits an orientable 4-weak bisection and equivalently, a \( (3\frac{1}{2}, \frac{1}{2}) \)-flow.

A perfect matching is not necessary for a 4-weak bisection. However, infinitely many cubic graphs which admit no 4-weak bisection are presented in [4], so the scope of Conjecture 44 cannot be extended further to include all cubic graphs.

The graph in Figure 4 contains pairs of parallel edges. When analyzing the smallest example, presented in [4], of a simple cubic graph \( G \) with no 4-weak bisection we found, along similar lines, that \( \text{bed}(G) = \text{urd}(4\frac{1}{4}, \frac{1}{4}) \) (notice that \( (4\frac{1}{4}, \frac{1}{4}) \), of trace 5 lies on the lower part of \( L_5 \)). We have reasons to believe:

**Conjecture 45.** For every simple cubic graph \( G, (4\frac{1}{4}, \frac{1}{4}) \in \text{bed}(G) \).
4.2 | What does bed(G) look like?

Little do we actually know about the shape of bed(G) in general.

For every cubic graph $G$, bed(G) is a closed unbounded (to the upper-right) polygonal domain. Always among its sides are two infinite ones, a vertical side on the line $r = 2$ from $(2, 1)$ to $(2, \infty)$ and a horizontal one on $\alpha = \alpha_m$ from a certain point $(r_m, \alpha_m)$ to $(\infty, \alpha_m)$. If $G$ is bridgeless, then $(r_m, \alpha_m) = (\phi_c(G), 0)$. If there exists a bridge in $G$, then a lower bound for $\alpha$ is $\frac{1}{|V_m|}$, where $|V_m|$ is the number of vertices in the smaller side of the bridge. The actual minimum value $\alpha_m$ may be larger than that, see Theorem 42 and its proof. By Theorem 22 the point $(3\frac{1}{2}, \frac{1}{2})$ is always in bed(G).

Results in this article almost solely rely on the analysis of a certain single (balanced) orientation, rather than understanding the union of bed(D) over several orientations $D$ of a graph $G$. The proof of Theorem 14 asserts that the existence of a $p$-flow with $tr(p) < k + 1$ implies the existence of a $(3 + \frac{k-3}{k-1}, \frac{k-3}{k-1})$-flow in the same orientation $D$. By Lemma 8 the line segment between the corresponding two points is contained in bed(D) and in bed(G). Other than the above, what we can add at that stage, are mostly questions. Following is a rather arbitrary list of questions. At that point we cannot tell how hard or easy they are and how interesting the answers may be:

1. Take a bridgeless graph $G$ with $4 < \phi_c(G) < 5$, say $\phi_c(G) = 4\frac{1}{2}$ (for existence, see, eg, [6]). The line segment from $(3\frac{1}{3}, \frac{1}{3})$ to $(4\frac{1}{2}, 0)$ is contained in bed(G). Is it a side of bed(G)?
2. Are the vertices of bed(G) of the previous question $(2, \infty)$, $(2, 1)$, $(3\frac{1}{3}, \frac{1}{3})$, $(4\frac{1}{2}, 0)$, and $(\infty, 0)$?
3. If $G$ is bridgeless, does bed(G) depends solely on $\phi_c(G)$?
4. Does every cubic graph $G$ have a Dominant orientation $D$ such that bed(G) = bed(D), that is, bed(D') $\subseteq$ bed(D) for every orientation $D'$ of $G$?
5. Is bed(G) always convex (It sure is if the answer to the previous question is affirmative)?
6. Is there a constant bound to the number of sides of bed(G) (that is, a bound to the number of sets $A$ relevant to Condition 2 of Theorem 5)?
7. Considering the proof of Theorem 18: Are there two equivalent points in the quadrilateral whose four vertices are $(4, 0)$, $(5, 0)$, $(4, \frac{1}{3})$, and $(3\frac{1}{3}, \frac{1}{3})$?
8. Given a cubic graph $G$, does there always exist a point $p$ in the $r - \alpha$ plane such that bed $(G) = \text{span}(p)$? If exists, such a point represents a strongest bounded-excess flow in $G$, which is a two-dimensional generalization of the circular-flow number $\phi_c(G)$.
9. Inspired by the 5-flow Conjecture: Is there a cubic graph $G$ such that bed(G) has a finite vertex whose trace is larger than 5 (obviously true if the assertion of the 5-flow Conjecture is false)?
10. In the quest for settling the 5-flow Conjecture, can we prove the existence of a point $(r_0, \alpha_0)$ with $tr(r_0, \alpha_0) = 5$ and $\alpha_0 < \frac{1}{2}$ (equivalently $r_0 > 3\frac{1}{2}$) such that every bridgeless cubic graph admits an $(r_0, \alpha_0)$-flow?

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