GENERATORS OF RELATIONS FOR ANNIHILATING FIELDS

MIRKO PRIMC

Abstract. For an untwisted affine Kac-Moody Lie algebra \( \tilde{g} \), and a given positive integer level \( k \), vertex operators \( x(z) = \sum_{n \in \mathbb{Z}} x(n)z^{-n-1}, \ x \in g \), generate a vertex operator algebra \( V \). For the maximal root \( \theta \) and a root vector \( x_\theta \) of the corresponding finite-dimensional \( g \), the field \( x_\theta(z)^{k+1} \) generates all annihilating fields of level \( k \) standard \( \tilde{g} \)-modules. In this paper we study the kernel of the normal order product map \( r(z) \otimes Y(v, z) \mapsto r(z)Y(v, z) \) for \( v \in V \) and \( r(z) \) in the space of annihilating fields generated by the action of \( \frac{dz}{z} \) and \( g \) on \( x_\theta(z)^{k+1} \). We call the elements of this kernel the relations for annihilating fields, and the main result is that this kernel is generated, in certain sense, by the relation \( x_\theta(z)^{k+1} = 0 \) on \( L(\Lambda) \).

The relation (1.1) holds in general for any level \( k \) standard module of any untwisted affine Kac-Moody Lie algebra \( \tilde{g} \). Here \( \theta \) is the maximal root of the corresponding finite-dimensional Lie algebra \( g \), \( x_\theta \) is a corresponding root vector and \( x_\theta(z) = \sum_{n \in \mathbb{Z}} x_\theta(n)z^{-n-1} \) is a formal Laurent series with coefficients \( x_\theta(n) \) in the affine Lie algebra \( \tilde{g} \) (see Section 2 for notation). The relation (1.1) was studied before in [LP], but it is best understood if \( x_\theta(z)^{k+1} \) is viewed as the vertex operator \( Y(x_\theta(-1)^{k+1}z) \) associated to a singular vector \( x_\theta(-1)^{k+1}z \) in a vertex operator algebra \( V = N(k\Lambda_0) \), i.e., as an annihilating field of standard modules of level \( k \) (cf. [FLM], [DL], [Li1], [MP1]).

After a PBW spanning set is reduced to a basis, it remains to prove its linear independence. In a joint work with Arne Meurman [MP1] a sort of Gröbner base theory is used for proving linear independence. The main ingredient in the proof is

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the use of relations among annihilating fields of level \( k \) standard modules
\[
Y(r, z) = \sum_{n \in \mathbb{Z}} r_n z^{-n-1}, \quad r \in \hat{R}1 = \mathbb{C}[L_{-1}]U(\mathfrak{g})x_{\theta}(1)^{-k+1}1,
\]
“generated” by the obvious relation
\[
(1.2) \quad x_{\theta}(z) \frac{d}{dz} (x_{\theta}(z)^{k+1}) = (k+1)x_{\theta}(z)^{k+1} \frac{d}{dz} x_{\theta}(z).
\]
The relation (1.2) arises by writing the vector \( x_{\theta}(-2)x_{\theta}(-1)^{k+1}1 \) in two different ways, thus obtaining two different expressions for the same field, and the representation theory is used to generate “enough” of other relations for annihilating fields \( Y(r, z), r \in \hat{R}1 \). In \( \text{MP}2 \) a similar technique is used to construct a combinatorial basis of the basic \( \mathfrak{sl}(n, \mathbb{C}) \)-module.

By following ideas developed in \( \text{MP}1 \) and \( \text{MP}2 \), in \( \text{P2} \) a general construction of such relations for annihilating fields is given by using vertex operators \( Y(q, z) \) associated with elements \( q \) in the kernel of the map
\[
\Phi: \hat{R}1 \otimes N(k\Lambda_0) \to N(k\Lambda_0), \quad \Phi(u \otimes v) = u_{-1}v.
\]
Due to the state-field correspondence, we shall call the elements in ker \( \Phi \) relations for annihilating fields. By using these relations the problem of constructing combinatorial bases of standard modules is split into a “combinatorial part of the problem” (consisting of counting numbers \( N(n) \) of certain colored partitions of \( n \)) and a “representation theory part of the problem” (consisting of constructing certain subspaces \( Q \) in ker \( \Phi \) such that \( \dim Q(n) = N(n) \), where \( Q(n) \) is a space of coefficients of vertex operators \( Y(q, z) = \sum q(n)z^{-n-\omega^i_q} \) associated with vectors in \( Q \)). In this paper only a part of the “representation theory part of the problem” is studied, the main objective being to understand the structure of ker \( \Phi \) for general untwisted affine Kac-Moody Lie algebra \( \mathfrak{g} \).

The starting point in this paper are Theorems 3.1 and 3.2, showing that ker \( \Phi \) has certain \( (\mathcal{C}L_{-1} \ltimes \hat{\mathfrak{g}}_{\leq 0}) \)-module structure and that
\[
(1.3) \quad \ker \Phi \cong \ker \Psi
\]
as \( (\mathcal{C}L_{-1} \ltimes \hat{\mathfrak{g}}_{\leq 0}) \)-modules, where
\[
\Psi: U(\hat{\mathfrak{g}}) \otimes U(\hat{\mathfrak{g}}_{\geq 0} + \mathbb{C}z) \hat{R}1 \to N(k\Lambda_0), \quad u \otimes w \mapsto uw.
\]
This makes it possible to use the theory of induced representations of \( \hat{\mathfrak{g}} \), in particular Garland-Lepowsky’s resolution of a standard module in terms of generalized Verma modules. The main results are Theorem 5.4, saying that the \( (\mathcal{C}L_{-1} \ltimes \hat{\mathfrak{g}}_{\leq 0}) \)-module ker \( \Phi \) is generated by certain (singular) vectors, and for combinatorial arguments more convenient Theorem 6.2, saying that the \( (\mathcal{C}L_{-1} \ltimes \hat{\mathfrak{g}}) \)-module ker \( \Psi \) is generated by the vector
\[
x_{\theta}(-2) \otimes x_{\theta}(-1)^{k+1}1 - x_{\theta}(-1) \otimes x_{\theta}(-2) x_{\theta}(-1)^{k}1.
\]
Via (1.3) this vector can be seen in the vertex operator algebra \( N(k\Lambda_0) \otimes N(k\Lambda_0) \), and the vertex operator associated to this vector is
\[
(1.4) \quad x_{\theta}(z)^{k+1} \otimes \frac{d}{dz} x_{\theta}(z) - \frac{1}{k+1} \frac{d}{dz} (x_{\theta}(z)^{k+1}) \otimes x_{\theta}(z).
\]
The obvious relation (1.2) arises by taking from (1.4) normal order products. So, in a sense, the relation (1.2) generates all relations for annihilating fields of standard \( \hat{\mathfrak{g}} \)-modules of level \( k \).

Some ideas worked out in this paper come from collaboration with Arne Meurman for many years, and I thank Arne Meurman for his implicit contribution to
this work. I also thank Jim Lepowsky and Ivica Siladić for numerous stimulating discussions.

2. Vertex algebras for affine Lie algebras

Let \( \mathfrak{g} \) be a simple complex Lie algebra, \( \mathfrak{h} \) a Cartan subalgebra of \( \mathfrak{g} \) and \( \langle , \rangle \) a symmetric invariant bilinear form on \( \mathfrak{g} \). Via this form we identify \( \mathfrak{h} \) and \( \mathfrak{h}^* \) and we assume that \( \langle \theta, \theta \rangle = 2 \) for the maximal root \( \theta \) (with respect to some fixed basis of the root system). Set

\[
\hat{\mathfrak{g}} = \prod_{j \in \mathbb{Z}} \mathfrak{g} \otimes t^j + \mathbb{C}c, \quad \tilde{\mathfrak{g}} = \hat{\mathfrak{g}} + Cc.
\]

Then \( \hat{\mathfrak{g}} \) is the associated untwisted affine Lie algebra (cf. [K]) with the commutator

\[
[x(i), y(j)] = [x, y](i + j) + i\delta_{i+j,0}(x, y)c.
\]

Here, as usual, \( x(i) = x \otimes t^i \) for \( x \in \mathfrak{g} \) and \( i \in \mathbb{Z} \), \( c \) is the canonical central element, and \( [d, x(i)] = ix(i) \). Sometimes we shall denote \( \mathfrak{g} \otimes t^j \) by \( \mathfrak{g}(j) \). We identify \( \mathfrak{g} \) and \( \mathfrak{g}(0) \). Set

\[
\hat{\mathfrak{g}}_{<0} = \prod_{j < 0} \mathfrak{g} \otimes t^j, \quad \hat{\mathfrak{g}}_{\leq 0} = \prod_{j \leq 0} \mathfrak{g} \otimes t^j + \mathbb{C}d, \quad \tilde{\mathfrak{g}}_{\geq 0} = \prod_{j \geq 0} \mathfrak{g} \otimes t^j + Cc.
\]

For \( k \in \mathbb{C} \) denote by \( \mathbb{C}v_k \) the one-dimensional \( (\hat{\mathfrak{g}}_{\geq 0} + \mathbb{C}c) \)-module on which \( \tilde{\mathfrak{g}}_{\geq 0} \) acts trivially and \( c \) as the multiplication by \( k \). The untwisted affine Lie algebra \( \hat{\mathfrak{g}} \) gives rise to the vertex operator algebra (see [FZ] and [Li1]; we use the notation from [MP1])

\[
N(k\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0} + \mathbb{C}c)} \mathbb{C}v_k
\]

for level \( k \neq -g^\vee \), where \( g^\vee \) is the dual Coxeter number of \( \mathfrak{g} \); it is generated by the fields

\[
x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1}, \quad x \in \mathfrak{g},
\]

where we set \( x_n = x(n) \) for \( x \in \mathfrak{g} \). As usual, we shall write \( Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \) for the vertex operator (field) associated with a vector \( v \in N(k\Lambda_0) \). From now on we shall fix the level \( k \neq -g^\vee \), and we shall sometimes denote by \( V \) the vertex operator algebra \( N(k\Lambda_0) \).

Recall that \( N(k\Lambda_0) \otimes N(k\Lambda_0) \) is a vertex operator algebra; fields are defined by \( Y(a \otimes b, z) = Y(a, z) \otimes Y(b, z) \), the conformal vector is \( \omega \otimes 1 + 1 \otimes \omega \) (cf. [FHL]). In particular, the derivation \( D = L_{-1} \) is given by \( D \otimes 1 + 1 \otimes D \), the degree operator \( -d = L_0 \) is given by \( L_0 \otimes 1 + 1 \otimes L_0 \), and we have the action of \( \mathfrak{g} = \mathfrak{g}(0) \) given by \( x \otimes 1 + 1 \otimes x \). Set

\[
\Phi: N(k\Lambda_0) \otimes N(k\Lambda_0) \to N(k\Lambda_0), \quad \Phi(a \otimes b) = a_{-1} b.
\]

It is easy to see that \( \Phi \) intertwines the actions of \( L_{-1} \), \( L_0 \) and \( \mathfrak{g}(0) \), so that \( \ker \Phi \) is also invariant for the actions of these operators. If \( \sigma \) is an automorphism of \( V \), then clearly \( \Phi \) intertwines the actions of \( \sigma \) on \( V \otimes V \) and \( V \).
3. Tensor Products and Induced Representations

The vertex operator algebra $V = N(k\Lambda_0)$ is an induced $\tilde{g}$-module, so by restriction it is a $\tilde{g}_{<0}$-module, where $x_j = x(j)$, $x \in \mathfrak{g}$, $j < 0$, acts on

$$V = U(\hat{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0} + C_c)} \mathcal{C}v_k \cong U(\tilde{\mathfrak{g}}_{<0})$$

by the left multiplication with $x_j$. Since in general

$$(Du)_n = -nu_{n-1}$$

for $u \in V$ and $n \in \mathbb{Z}$, we have

$$(x_{-1-j} = (D^{(j)})(-1)1)_{-1} = (x_{-1-j}1)_{-1}$$

for $x \in \mathfrak{f}$ and $j \in \mathbb{Z}_{\geq 0}$, with notation $D^{(j)} = D^j/j!$.

The vertex operator algebra $V$ has a Lie algebra structure with the commutator

$$[u, v] = u_{-1}v - v_{-1}u = \sum_{n \geq 0} (-1)^n D^{(n+1)}(u_n v).$$

This Lie algebra structure is transported from the Lie algebra structure on the Lie algebra $\mathcal{L}_-(V)$ associated with the vertex Lie algebra $V$ (cf. [FF], [Li2]), via the vector space isomorphism

$$V \rightarrow \mathcal{L}_-(V), \quad u \mapsto u_{-1}$$

(cf. [DLM], or (4.14) in [P1] with $U = V$). By (3.3) the restriction of (3.5) gives a vector space isomorphism

$$\tilde{\mathfrak{g}}_{<0}1 \rightarrow \tilde{\mathfrak{g}}_{<0}, \quad u \mapsto u_{-1},$$

and the adjoint action of $\tilde{\mathfrak{g}}_{<0}1$ on $V$ with the commutator (3.4) is transported from the adjoint action of the subalgebra $\tilde{\mathfrak{g}}_{<0}$ on the Lie algebra $\mathcal{L}_-(V)$. So we have the “adjoint” action $u_{-1}: v \mapsto [u, v]$ of the Lie algebra $\tilde{\mathfrak{g}}_{<0}$ on the vertex operator algebra $V$, we shall denote it by $V_{\text{ad}}$.

Since $L_{-1}$, $L_0$ and $y_0, y \in \mathfrak{g}$, are derivations of the product $u_{-1}v$ in $V$, they are also derivations of the bracket $[u, v]$, and we can extend the adjoint action of the Lie algebra $V$ to the action of the Lie algebra $(\mathcal{C}L_{-1} + CL_0 + \mathfrak{g}(0)) \rtimes V$. Since $\tilde{\mathfrak{g}}_{<0}1$ is invariant for $L_{-1}$, $L_0$ and $y_0, y \in \mathfrak{g}$, and

$$(CL_{-1} + CL_0 + \mathfrak{g}(0)) \rtimes \tilde{\mathfrak{g}}_{<0}1 \cong CL_{-1} \rtimes \tilde{\mathfrak{g}}_{<0},$$

we can extend the “adjoint” action of the Lie algebra $\tilde{\mathfrak{g}}_{<0}$ on the vertex operator algebra $V$ to the “adjoint” action of the Lie algebra $\mathcal{C}L_{-1} \rtimes \tilde{\mathfrak{g}}_{<0}$ on $V$, we shall also denote it by $V_{\text{ad}}$.

Let $W \subseteq V$ be a $\tilde{\mathfrak{g}}_{\geq 0}$-submodule invariant for the action of $D = L_{-1}$. Then the right hand side of (3.4), and (3.2), imply that $W$ is invariant for the “adjoint” action of $\tilde{\mathfrak{g}}_{<0}$. By assumption $W$ is invariant for $L_{-1}$, $L_0$ and $\mathfrak{g}(0)$, so on $W$ we have the “adjoint” action of Lie algebra $\mathcal{C}L_{-1} \rtimes \tilde{\mathfrak{g}}_{<0}$, we shall denote it by $W_{\text{ad}}$.

Theorem 3.1. There is a unique isomorphism of $(\mathcal{C}L_{-1} \rtimes \tilde{\mathfrak{g}}_{<0})$-modules

$$\Xi_W: W_{\text{ad}} \otimes V \rightarrow U(\hat{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0} + C_c)} W$$

such that $\Xi_W(w \otimes 1) = 1 \otimes w$ for all $w \in W$. 
Proof. We shall use the identification \( (3.3) \). By definition, a linear map 
\[ \Xi: W_{ad} \otimes U(\hat{g}_{<0}) \rightarrow U(\hat{g}_{<0}) \otimes W \]
is a \( \hat{g}_{<0} \)-module homomorphism if
\[ \Xi([x, w] \otimes u) + \Xi(w \otimes x_{-1}u) = x_{-1}\Xi(w \otimes u) \]
for all \( w \in W, u \in U(\hat{g}_{<0}) \) and \( x \in \hat{g}_{<0} \mathbf{1} \subset V \). Since the universal enveloping algebra \( U(\hat{g}_{<0}) \) is a quotient of the tensor algebra \( T(\hat{g}_{<0}) \), we first define recursively a linear map
\[ \Xi: W \otimes T(\hat{g}_{<0}) \rightarrow U(\hat{g}_{<0}) \otimes W, \]
\[ \Xi^*: W \otimes T^*(\hat{g}_{<0}) \rightarrow U^*(\hat{g}_{<0}) \otimes W, \]
by setting
\[ \Xi^0(w \otimes 1) = 1 \otimes w \quad \text{for all} \quad w \in W, \]
\[ \Xi^{s+1}(w \otimes x_{-1} \otimes u) = x_{-1}\Xi^s(w \otimes u) - \Xi([x, w] \otimes u) \]
for \( s = 0, 1, 2, \ldots \). Then we see that \( \ker \Xi \supset W \otimes J \) for the ideal \( J \subset T(\hat{g}_{<0}) \) generated by elements \( x_{-1} \otimes y_{-1} - y_{-1} \otimes x_{-1} - [x_{-1}, y_{-1}] \) for \( x, y \in \hat{g}_{<0} \mathbf{1} \subset V \):
\[ \Xi^{s+1}(w \otimes (x_{-1} \otimes y_{-1} \otimes u - y_{-1} \otimes x_{-1} \otimes u)) \]
\[ - \Xi^s(w \otimes [x_{-1}, y_{-1}] \otimes u) \]
\[ = (x_{-1}y_{-1} - y_{-1}x_{-1} - [x_{-1}, y_{-1}]) \Xi^{s-1}(w \otimes u) \]
\[ + \Xi^{s-1}([([y, [x, w]] - [x, [y, w]] + [[x, y], w]) \otimes u) \]
\[ = 0. \]

Here we used the relation \([x_{-1}, y_{-1}] \mathbf{1} = [x, y] \) for the inverse of the map \( (3.6) \). By passing to the quotient we obtain a \( \hat{g}_{<0} \)-module homomorphism
\[ \Xi_W: W_{ad} \otimes U(\hat{g}_{<0}) \rightarrow U(\hat{g}_{<0}) \otimes W, \]
\[ \Xi^*_W: W_{ad} \otimes U^*(\hat{g}_{<0}) \rightarrow U^*(\hat{g}_{<0}) \otimes W. \]
From recursive relations
\[ \Xi^W_{s+1}(w \otimes x_{-1}u) = x_{-1}\Xi^W_s(w \otimes u) - \Xi^W(W([x, w] \otimes u) \]
we see by induction that \( \Xi^W \) is an isomorphism, uniquely determined by the requirement \( \Xi^W(W \otimes 1) = 1 \otimes w \). By induction \( (3.10) \) implies that the map \( \Xi^W \) intertwines the actions of \( L_{-1}, L_0 \) and \( g(0) \).

Remarks. (i) Note that, with the identifications made in \( (3.3), (3.4) \) the map \( \Xi_W \) preserves the filtration inherited from the filtration \( U_s(\hat{g}_{<0}), s \in \mathbb{Z}_{\geq 0} \), of the universal enveloping algebra \( U(\hat{g}_{<0}) \).

(ii) Let \( \sigma \) be an automorphism of \( g \). We extend \( \sigma \) to an automorphism of \( \hat{g} \) by \( \sigma(x(n)) = (\sigma x)(n) \), \( \sigma c = c \), \( \sigma d = d \), and, as well, to \( U(\hat{g}) \) and \( V \). If \( W \subset V \) is invariant for the action of \( \sigma \), then \( \sigma \) acts on \( W \otimes V \) and \( U(\hat{g}) \otimes U(\hat{g}_{>0} + C c) \) \( W \) in a natural way. Note that \( \Xi_W \) intertwines the action of \( \sigma \), as easily seen from \( (3.10) \).

Since \( W \subset V \), we have maps
\[ \Psi_W: U(\hat{g}) \otimes U(\hat{g}_{>0} + C c) \rightarrow W, \quad u \otimes w \mapsto uw, \]
\[ \Phi_W: W \otimes V \rightarrow W, \quad u \otimes w \mapsto u_{-1}w. \]
Note that the map \( \Psi_W \) is a homomorphism of \( \hat{g} \)-modules, and that \( \Psi_W \) intertwines the actions of \( L_{-1}, L_0 \) and \( \sigma \) (if \( W \) is invariant for \( \sigma \)). Hence \( \ker \Phi_W \) is a \( \hat{g} \)-module,
invariant for \( L_{-1}, L_0 \) and \( \sigma \). The following theorem relates \( \ker \Phi_W \) with induced representations of \( \tilde{\mathfrak{g}} \):

**Theorem 3.2.** The map \( \Phi_W \) is is a homomorphism of \((\mathfrak{c}L_{-1} \ltimes \tilde{\mathfrak{g}}_{<0})\)-modules and \( \Phi_W = \Psi_W \circ \Xi_W \). In particular, \( \ker \Phi_W \subset W_{ad} \otimes V \) is a \((\mathfrak{c}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})\)-module and \( \Xi_W(\ker \Phi_W) = \ker \Psi_W \).

**Remark.** Note that \( \Phi = \Phi \circ (W \otimes V) \), and likewise \( \Xi_W \) and \( \Psi_W \) are restrictions of \( \Xi_V \) and \( \Psi_V \) respectively. In what follows we shall sometimes omit the subscript \( W \), and write only \( \Phi, \Xi, \) and \( \Psi \), whenever is clear from a context which \( W \subset V \) is fixed.

**Proof.** First note that for \( w \in W, u \in V \cong U(\tilde{\mathfrak{g}}_{<0}) \) and \( x \in \tilde{\mathfrak{g}}_{<0} \) we have

\[
\Phi(w \otimes x^{-1}u) = w^{-1}x^{-1}u = x^{-1}w^{-1}u - [x^{-1}, w^{-1}]u
\]

\[
= x^{-1}w^{-1}u - \sum_{n \geq 0} \binom{-1}{n} (x_n w)_{-2-n} u
\]

\[
= x^{-1}w^{-1}u - \sum_{n \geq 0} (-1)^n (D^{(n+1)}(x_n w))_{-1} u
\]

\[
= x^{-1}w^{-1}u - ([x, w])_{-1} u = x^{-1} \Phi(w \otimes u) - \Phi([x, w] \otimes u).
\]

Also note that \( \Phi(w \otimes 1) = w = \Psi(\Xi(w \otimes 1)) \). By using the filtration \( U_s(\tilde{\mathfrak{g}}_{<0}) \), \( s \in \mathbb{Z}_{\geq 0} \), we see by induction that

\[
\Psi(\Xi(w \otimes x^{-1}u)) = x^{-1} \Psi(\Xi(w \otimes u)) - \Psi(\Xi([x, w] \otimes u))
\]

\[
= x^{-1} \Phi(w \otimes u) - \Phi([x, w] \otimes u)
\]

\[
= \Phi(w \otimes x^{-1}u),
\]

i.e., we see that \( \Phi = \Psi \circ \Xi \). \( \square \)

4. **Sugawara’s relations for annihilating fields**

From now on we fix \( k \neq -q^r \) such that \( N(k\Lambda_0) \) is a reducible \( \tilde{\mathfrak{g}} \)-module, and we denote by \( N^1(k\Lambda_0) \) its maximal \( \tilde{\mathfrak{g}} \)-submodule. Let \( R \subset N(k\Lambda_0) \) be a nonzero subspace such that:

- \( R \) is finite-dimensional,
- \( R \) is invariant for \( \tilde{\mathfrak{g}}_{\geq 0} \) and \( \tilde{\mathfrak{g}}_{>0} R = 0 \),
- \( R \subset N^1(k\Lambda_0) \).

The main example we have in mind is for \( k \in \mathbb{Z}_{>0} \) and \( R = U(\mathfrak{g})x_\theta(-1)^{k+1} \). Set

\[
\tilde{R} = \mathbb{C}\text{-span}\{r_n \mid r \in R, n \in \mathbb{Z}\},
\]

where \( r_n \) denotes a coefficient in the vertex operator \( Y(r, z) \). Then \( \tilde{R} \) is a \( \tilde{\mathfrak{g}} \)-module for the adjoint action given by the commutator formula

\[
[x_m, r_n] = \sum_{i \geq 0} \binom{m}{i} (x_i r)_{m+n-i}, \quad x \in \mathfrak{g}, \ r \in R.
\]

From now on we take

\[
W = \tilde{R} 1 \subset V
\]
and we call elements in $\ker \Psi_W$ the relations for annihilating fields (cf. [P2]). By Theorem 3.2 we may identify the relations for annihilating fields with elements of $\ker \Psi_W$, which is easier to study by using the representation theory of affine Lie algebras. The first step is to introduce Sugawara’s relations for annihilating fields as elements in

$$ N = U(\hat{\g}) \otimes U(\hat{\g}_\geq 0 + \mathbb{C}c) W \cong U(\hat{\g}_\leq 0) \otimes W. $$

Let $\{x^i\}_{i \in I}$ and $\{y^j\}_{j \in J}$ be dual bases in $\mathfrak{g}$. For $r \in R$ we define Sugawara’s relation

$$ q_r = \frac{1}{k + g^r} \sum_{i \in I} x^i(-1) \otimes y^i(0)r - 1 \otimes Dr $$

as an element of $U(\hat{\g}) \otimes U(\hat{\g}_\geq 0 + \mathbb{C}c) W$. As in the case of Casimir operator, Sugawara’s relation $q_r$ does not depend on a choice of dual bases $\{x^i\}_{i \in I}$ and $\{y^j\}_{j \in J}$. Since on $V$ we have $D = L_{-1}$, Sugawara’s construction gives (cf. [MP1], Chapter 3)

$$ L_{-1}r = \frac{1}{2(k + g^r)} \sum_{i \in I} (x^i(-1)y^i(0) + y^i(-1)x^i(0))r $$

$$ = \frac{1}{k + g^r} \sum_{i \in I} x^i(-1)y^i(0)r, $$

and we have the following proposition (cf. [MP1], Chapter 8):

**Proposition 4.1.** (i) $q_r$ is an element of $\ker \Psi_W$.

(ii) $r \mapsto q_r$ is a $\mathfrak{g}$-module homomorphism from $R$ into $\ker \Psi_W$.

(iii) $x(i)q_r = 0$ for all $x \in \mathfrak{g}$ and $i > 0$.

Let us denote the set of all Sugawara’s relations $\{q_r\}$ by

$$ Q_{\text{Sugawara}} = \{q_r \mid r \in R\} \subset \ker \Psi_W. $$

Our assumptions imply that $L_1 R = 0$, $L_0 R \subset R$ and $R \cap DR = 0$. So $\mathfrak{g}$-module $Q_{\text{Sugawara}}$ is isomorphic to $R$, and by Proposition 4.1(iii) for each maximal vector $r$ in $R$ the corresponding $q_r$ is a singular vector in the $\hat{\mathfrak{g}}$-module ker $\Psi_W$.

Our assumptions imply that $\hat{R} = \prod_{i \geq 0} D^i R$ and we set

$$ W_n = \prod_{i = 0}^n D^i R, \quad N_n = U(\hat{\g}) \otimes U(\hat{\g}_\geq 0 + \mathbb{C}c) W_n. $$

So we have a filtration of $N$ by $\hat{\mathfrak{g}}$-modules $N_n$, $n \in \mathbb{Z}_{\geq 0}$, and $N_0$ is a (sum of) generalized Verma module(s). We denote the restrictions $\Psi_W|N_n$ simply by $\Psi_n$. In particular, we have the $\hat{\mathfrak{g}}$-module homomorphism

$$ \Psi_0: U(\hat{\g}) \otimes U(\hat{\g}_\geq 0 + \mathbb{C}c) R \to N^1(k\Lambda_0), \quad u \otimes w \mapsto uw. $$

**Proposition 4.2.** As a $(\mathfrak{c}L_{-1} \ltimes \hat{\mathfrak{g}}_\leq 0)$-module ker $\Psi_W$ is generated by

$$ \ker \Psi_0 + Q_{\text{Sugawara}}. $$

**Proof.** If $v \in N_0 \cap \ker \Psi_W$, then $v \in \ker \Psi_0$. Now assume that $v \in N_n \cap \ker \Psi_W$ for $n > 0$. Then $v$ is a sum of elements of the form

$$ v = \sum_{u, r} u \otimes D^nr + w, \quad u \in U(\hat{\g}_\leq 0), \quad r \in R, \quad w \in N_{n-1}, $$

and we set

$$ \Psi_0 \Psi_v = \sum_{u, r} u \otimes D^nr + v, \quad u \in U(\hat{\g}_\leq 0), \quad r \in R, \quad w \in N_{n-1}. $$

With $\Psi_0$, the relations for the annihilating fields are given in [P2].
and to each summand $u \otimes D^n r$ add
\[ uD^{n-1}q_r = uD^{n-1}\left(\frac{1}{k + g^r} \sum_{i \in I} x^i(-1) \otimes y^i(0)r\right) - u \otimes D^n r \in \ker \Psi_W. \]

Then $v + \sum uD^{n-1}q_r \in N_{n-1} \cap \ker \Psi_W$, and in a finite number of steps we get
\[ v \in \ker \Psi_0 + \sum_{i=0}^{n-1} U(\tilde{g}_{<0})D^iQ_{\text{Sugawara}}. \]

**Remark.** By using (3.4) and (3.7) it is easy to see that for
\[ \sum_{\theta} \text{ where } \Omega = \text{ singular vector} \Psi_0 \]

\[ g_\tilde{\ell} \]

Note that for untwisted affine Lie algebras $\tilde{\ell}_i$, exactly two singular vectors of weights $r \in (1)$, the $\tilde{\ell}$-module $N_0$ is a generalized Verma module and the map $\Psi_0$ defined by (1.2) is surjective (cf. [MP1]). Hence we have the exact sequence of $\tilde{\ell}$-modules
\[ N_0 \to N(k\Lambda_0) \to L(k\Lambda_0) \to 0. \]

We shall use Garland-Lepowsky’s resolution [GL] of a standard module, in terms of generalized Verma modules, to determine generators of ker $\Psi_0$.

Denote by $W(\tilde{\ell})$ the Weyl group generated by simple reflections $r_0, r_1, \ldots, r_\ell$. Note that for $i \in \{1, \ldots, \ell\}$
\[ r_0r_i \neq r_i r_0 \text{ if and only if } \langle \alpha_0, \alpha_i^{(1)} \rangle \neq 0, \]

and that for $A_i^{(1)}$, $\ell \geq 2$, there are exactly two such $i$, for all the other untwisted affine Lie algebras $\tilde{\ell}$ there is exactly one such $i$, corresponding to $\alpha_i$ connected with $\alpha_0$ in a Dynkin diagram. With the usual notation, for $w \in W(\tilde{\ell})$ and a weight $\lambda$
\[ w \cdot \lambda = w(\lambda + \rho) - \rho. \]

**Lemma 5.1.** For $\tilde{\ell}$ of the type $A_i^{(1)}$, $\ell \geq 2$, the $\tilde{\ell}$-module ker $\Psi_0$ is generated by exactly two singular vectors of weights $r_0r_i \cdot k\Lambda_0$, $\langle \alpha_0, \alpha_i^{(1)} \rangle \neq 0$. For all the other untwisted affine Lie algebras $\tilde{\ell}$ the $\tilde{\ell}$-module ker $\Psi_0$ is generated by exactly one singular vector of weight $r_0r_i \cdot k\Lambda_0$, $\langle \alpha_0, \alpha_i^{(1)} \rangle \neq 0$.

**Proof.** A generalized Verma $\tilde{\ell}$-module $N(\Lambda)$, with $\Lambda(\alpha_i^{(1)}) \in \mathbb{Z}_{\geq 0}$ for $i \in \{1, \ldots, \ell\}$ and $\Lambda(\ell) = k$, may be defined (cf. [Le]) as an induced $\tilde{\ell}$-module $U(\tilde{\ell}) \otimes U(\tilde{\ell}_{>0} + C_0) L$, where $L = L(\Lambda|\mathfrak{h})$ is the irreducible finite-dimensional $\mathfrak{h}$-module with the highest
weight $\Lambda|\mathfrak{h}$, on which $\tilde{g}_{>0}$ acts trivially and $c$ as the multiplication by $k$. In terms of Verma modules the generalized Verma module $N(\Lambda)$ may be written as a quotient

$$
(5.1) \quad N(\Lambda) = M(\Lambda)/\left( \sum M(\varepsilon_i \cdot \Lambda) \right).
$$

By Theorem 8.7 in [GL] (cf. Theorem 5.1 in [Le]) we have Garland-Lepowsky’s resolution of $L(k\Lambda_0)$ in terms of generalized Verma modules

$$
\cdots \to E_2 \to E_1 \to E_0 \to L(k\Lambda_0) \to 0.
$$

In our setting one gets $E_0 = N(k\Lambda_0)$ and $E_1 = N(r_0 \cdot k\Lambda_0) = N_0$. In the case when $\tilde{g}$ is of the type $A_t^{(1)}$, $t \geq 2$, the $\tilde{g}$-module $E_2$ has a filtration $0 = V_0 \subset V_1 \subset V_2 = E_2$ such that $V_1/V_0$ and $V_2/V_1$ are exactly two generalized Verma modules $N(r_0 r_i \cdot k\Lambda_0)$, $\langle \alpha_0, \alpha_i' \rangle \neq 0$. Hence ker $\Psi_0$ is generated by one singular vector and one subsingular vector, their two weights being $r_0 r_i \cdot k\Lambda_0$, $\langle \alpha_0, \alpha_i' \rangle \neq 0$. Since $R$ and ker $\Psi_0$ are invariant for the action of the Dynkin diagram automorphism $\sigma$, which interchanges these two weights, both generating vectors must be singular vectors. For all the other untwisted affine Lie algebras $\tilde{g}$ one gets that $E_2 = N(r_0 r_i \cdot k\Lambda_0)$, and hence the $\tilde{g}$-module ker $\Psi_0$ is generated by exactly one singular vector of weight $r_0 r_i \cdot k\Lambda_0$, $\langle \alpha_0, \alpha_i' \rangle \neq 0$.

**Remark.** One can also prove Lemma 5.1 by using the BGG type resolution of a standard module in terms of Verma modules, due to A. Rocha-Caridi and N. R. Wallach [RW].

**Remark.** Singular vectors in Verma module $M(r_0 \cdot k\Lambda_0)$ can be computed by the Malikov-Feigin-Fuchs formula [MFF], and by passing to the quotient (5.1) we may find singular vectors in $N_0 = N(r_0 \cdot k\Lambda_0)$. In the case $A_1^{(1)}$ level $k = 1$ the singular vector of weight $\lambda = r_0 r_1 \cdot \Lambda_0$ in $M(r_0 \cdot \Lambda_0)$ is given by the formula

$$
(5.2) \quad f_0^4 f_1^1 f_0^{-2} v_{r_0 \cdot \Lambda_0},
$$

with the usual notation for generators of Kac-Moody Lie algebras. This formula gives the singular vector

$$
(\theta(-1)^2 x_{-\theta}(0) + 2x_{\theta}(-1)\theta'(-1) - 6x_{\theta}(-2)) v_{r_0 \cdot \Lambda_0}
$$

in $M(r_0 \cdot \Lambda_0)$. So we have a generator of ker $\Psi_0 \subset N_0 = N(r_0 \cdot \Lambda_0)$

$$
(5.3) \quad (2x_{\theta}(-1)\theta'(-1) - 6x_{\theta}(-2)) \otimes x_{\theta}(-1)^2 + x_{\theta}(-1)^2 \otimes x_{-\theta}(0)x_{\theta}(-1)^2 1.
$$

We shall determine singular vectors in ker $\Psi_0$ in another way.

Let us denote by $\alpha_+$ and $r_+$ the elements $\alpha_i$ and $r_i$ for which $\langle \alpha_0, \alpha_i' \rangle \neq 0$. Then

$$
(5.3) \quad r_0 r_+ \cdot k\Lambda_0 = k\Lambda_0 - \alpha_+ - (k + 1 - \langle \alpha_+, \alpha_0' \rangle) \alpha_0.
$$

Note that $\langle \alpha_+, \alpha_0' \rangle = -2$ for $A_1^{(1)}$, and $\langle \alpha_+, \alpha_0' \rangle = -1$ for all the other untwisted affine Lie algebras $\tilde{g}$. For this reason the case $A_1^{(1)}$ is somewhat different, so for now assume that $\tilde{g} \not\preceq \mathfrak{s}(2, \mathbb{C})^\circ$. Then

**Lemma 5.2.** $\theta - \alpha_+$ is a root of $\tilde{g}$ and $2\theta - \alpha_+$ is not a root of $\tilde{g}$.

**Proof.** For the type $Cl_t$, with the usual notation, $\theta = 2\varepsilon_1$ and $\alpha_+ = \alpha_1 = \varepsilon_1 - \varepsilon_2$, and the statement is clear. For all the other types $\langle \theta, \alpha_i' \rangle = 1$, so $r_+ \theta = \theta - \alpha_+$ is a positive root of $\tilde{g}$, and the statement is clear. □
Then \( x_{\theta - \alpha} = [x_{-\alpha}, x_{\theta}] \) is a root vector in \( g_{\theta - \alpha} \), and again by Lemma 5.2
\[
[x_{\theta - \alpha}, x_{\theta}] = 0.
\]

Since
\[
x_{-\alpha}(0) x_{\theta} (-1)^{k+1} 1 = (k + 1) x_{\theta - \alpha} (-1) x_{\theta} (-1)^k 1 \in R,
\]
we obviously have
\[
x_{\theta - \alpha} (-1) \otimes x_{\theta} (-1)^{k+1} 1 - x_{\theta} (-1) \otimes x_{\theta - \alpha} (-1) x_{\theta} (-1)^k 1 \in \ker \Phi_0.
\]
By (5.3) this vector is of weight \( \tau_0 r_* \cdot k \Lambda_0 \), and hence by Lemma 5.1 it must be a singular vector. So we have proved

**Proposition 5.3.** Let \( \hat{\mathfrak{g}} \not\cong sl(2, \mathbb{C})^- \) be an untwisted affine Lie algebra. Then \( \ker \Psi_0 \) is generated by the singular vector(s)
\[
x_{\theta - \alpha} (-1) \otimes x_{\theta} (-1)^{k+1} 1 - x_{\theta} (-1) \otimes x_{\theta - \alpha} (-1) x_{\theta} (-1)^k 1, \quad \langle \alpha_0, \alpha^*_1 \rangle \neq 0. 
\]

Now let \( \hat{\mathfrak{g}} = sl(2, \mathbb{C})^- \). We have the Sugawara singular vector
\[
q_{(k+1)\theta} = \frac{1}{k+1} (x_{\theta} (-1) \otimes x_{-\alpha}(0) x_{\theta} (-1)^{k+1} 1 + \frac{1}{2} \theta^\vee (-1) \otimes \theta^\vee (0) x_{\theta} (-1)^{k+1} 1) - 1 \otimes D x_{\theta} (-1)^{k+1} 1,
\]
and the obvious relation
\[
q_{(k+2)\theta} = x_{\theta} (-2) \otimes x_{\theta} (-1)^{k+1} 1 - \frac{1}{k+1} x_{\theta} (-1) \otimes D x_{\theta} (-1)^{k+1} 1
\]
(cf. [MP1]). Then obviously
\[
(k + 1) q_{(k+2)\theta} - x_{\theta} (-1) q_{(k+1)\theta} \in \ker \Psi_0
\]
is a nonzero vector of weight \( \tau_0 r_1 \cdot k \Lambda_0 = k \Lambda_0 - \alpha_1 - (k + 3) \alpha_0 \), and hence by Lemma 5.1 it must be a singular vector, proportional to (5.2) in the case \( k = 1 \).

By combining Theorem 3.2, Propositions 4.2, 5.3, and (4.3), we get:

**Theorem 5.4.** Let \( \hat{\mathfrak{g}} \not\cong sl(2, \mathbb{C})^- \) be an untwisted affine Lie algebra. Then the \((\mathcal{L}_{-1} \otimes \hat{\mathfrak{g}}_{\Psi_0})\)-module \( \ker \Phi_W \) is generated by vectors
\[
x_{\theta} (-1)^{k+1} 1 \otimes x_{\theta - \alpha} (-1) 1 - x_{\theta} (-1) x_{\theta} (-1)^k 1 \otimes x_{\theta} (-1) 1, \quad \langle \alpha_0, \alpha^*_1 \rangle \neq 0,
\]
and
\[
\frac{1}{k+1} \sum_{i \in I} g^i (0) x_{\theta} (-1)^{k+1} 1 \otimes x_i^\vee (-1) 1 + L_{-1} \left( \frac{1}{k+1} \Omega - 1 \right) x_{\theta} (-1)^{k+1} 1 \otimes 1.
\]

6. **Relation**
\[
x_{\theta} (z) \frac{d}{dz} (x_{\theta} (z)^{k+1}) = (k + 1) x_{\theta} (z)^{k+1} \frac{d}{dz} x_{\theta} (z)
\]

We have the obvious relation in \( \ker \Psi_W \)
\[
q_{(k+2)\theta} = x_{\theta} (-2) \otimes x_{\theta} (-1)^{k+1} 1 - x_{\theta} (-1) \otimes x_{\theta} (-2) x_{\theta} (-1)^k 1,
\]
and the corresponding relation in \( \ker \Phi_W \) is
\[
\Xi^{-1} (q_{(k+2)\theta}) = x_{\theta} (-1)^{k+1} 1 \otimes x_{\theta} (-2) 1 - x_{\theta} (-2) x_{\theta} (-1)^k 1 \otimes x_{\theta} (-1) 1.
\]
The vertex operator associated to the vector \( \Xi^{-1} (q_{(k+2)\theta}) \) in the vertex operator algebra \( V \otimes V \) is
\[
x_{\theta} (z)^{k+1} \otimes \frac{d}{dz} x_{\theta} (z) - \frac{1}{k+1} \frac{d}{dz} (x_{\theta} (z)^{k+1}) \otimes x_{\theta} (z),
\]
and by taking normal order products we get for the annihilating field \(x_\theta(z)^{k+1}\) the obvious relation

\[
(6.3) \quad x_\theta(z)^{k+1} \frac{d}{dz} x_\theta(z) - \frac{1}{k+1} \frac{d}{dz} (x_\theta(z)^{k+1}) x_\theta(z) = 0.
\]

The relation (6.3) plays a key role in constructions \([\text{MP1}], \text{MP2}, \text{S}\) of combinatorial bases of standard \(\mathfrak{sl}(2, \mathbb{C})^\perp\) and \(\mathfrak{sl}(3, \mathbb{C})^\perp\)-modules, and these examples showed that all necessary relations for annihilating fields of standard \(\mathfrak{sl}(2, \mathbb{C})^\perp\) and \(\mathfrak{sl}(3, \mathbb{C})^\perp\)-modules can be derived from (6.3), or (6.2), to be a bit more precise. In this section we show that for a general untwisted affine Lie algebra \(\tilde{\mathfrak{g}} \not\cong \mathfrak{sl}(2, \mathbb{C})^\perp\) we can obtain from (6.1), by using the action of \(\tilde{\mathfrak{g}}_{\geq 0}\) on \(\Psi_W\), both the singular vector(s)

\[
q_{(k+2)\theta - \alpha_*} = x_{\theta - \alpha_*}(1) \otimes x_\theta(-1)^{k+1} 1 - x_\theta(-1) \otimes x_{\theta - \alpha_*}(-1)x_\theta(-1)^{k} 1
\]

in \(\ker \Psi_0\) and the Sugawara singular vector

\[
q_{(k+1)\theta} = \frac{1}{k+2} \sum_{i \in I} x^i(-1) \otimes y^i(0)x_\theta(-1)^{k+1} 1 - 1 \otimes Dx_\theta(-1)^{k+1} 1.
\]

**Lemma 6.1.** Let \(\Omega\) be the Casimir operator for \(\mathfrak{g} \not\cong \mathfrak{sl}(2, \mathbb{C})^\perp\) and \(\lambda = (k+2)\theta - \alpha_*\). Then

\[
q_{(k+2)\theta - \alpha_*} = x_{\theta - \alpha_*}(1)q_{(k+2)\theta},
\]

\[
q_{(k+1)\theta} = \frac{k+1}{2(k+2)(k+\gamma)}(\Omega - (\lambda + 2\rho, \lambda))x_{-\theta}(1)q_{(k+2)\theta}.
\]

**Proof.** Since \(x_\theta(-1)^{k+1} 1\) is a singular vector, and \([x_{\theta - \alpha_*}, x_\theta] = 0\), the first equality is obvious. To prove the second equality, first note that

\[
(k+1)x_{-\theta}(1)q_{(k+2)\theta} = (k+2) \otimes Dx_\theta(-1)^{k+1} 1
\]

\[
-x_\theta(-1) \otimes x_{-\theta}x_\theta(-1)^{k+1} 1
\]

\[
-(k+1)\theta^\gamma(-1) \otimes x_\theta(-1)^{k+1} 1,
\]

so that

\[
q = (k+1)x_{-\theta}(1)q_{(k+2)\theta} + (k+2)q_{(k+1)\theta}
\]

is in \(\ker \Psi_0\). Since \(q\) is of degree \(-k-2\), the same as the degree of singular vector(s) in \(\ker \Psi_0\), \(q\) must be an element of a \(\mathfrak{g}\)-module generated by vector(s) \(q_{(k+2)\theta - \alpha_*}\). Hence \(\Omega q = (\lambda + 2\rho, \lambda)q\) and

\[
(k+1)(\Omega - (\lambda + 2\rho, \lambda))x_{-\theta}(1)q_{(k+2)\theta}
\]

\[
= (k+2)((\lambda + 2\rho, \lambda) - \Omega)q_{(k+1)\theta}
\]

\[
= (k+2) \cdot ((\lambda + 2\rho, \lambda) - ((k+1)\theta + 2\rho, (k+1)\theta)) \cdot q_{(k+1)\theta}
\]

\[
= (k+2) \cdot 2(k+g^\gamma) \cdot q_{(k+1)\theta}
\]

gives the second formula.

It is easy to see that for \(\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})^\perp\) we have

\[
(k+1)x_{-\theta}(1)q_{(k+2)\theta} + (k+2)q_{(k+1)\theta} = 0.
\]

Hence, by combining Proposition 4.2, Proposition 5.3 and Lemma 6.1 we get:

**Theorem 6.2.** Let \(\tilde{\mathfrak{g}}\) be an untwisted affine Lie algebra. Then the \((\mathfrak{C}L_{-1} \ltimes \tilde{\mathfrak{g}})\)-module \(\ker \Psi_W\) is generated by the vector \(q_{(k+2)\theta}\).
Remarks. (i) Although Theorem 5.4 describes generators of relations for annihilating fields, for combinatorial applications (cf. [P2], Theorem 2.12) Theorem 5.2 in conjunction with Theorem 3.2 seems to be better suited. Namely, what one needs is a description of
\[(\ker \Phi_W)_s = (V \otimes V)_s \cap \ker \Phi_W,\]
where \((V \otimes V)_s, s \in \mathbb{Z}_{\geq 0},\) is the natural filtration of \(V \otimes V\) inherited from the filtration \(U_s(\hat{g}_{\geq 0}), s \in \mathbb{Z}_{\geq 0}.\) For the corresponding filtration \((\ker \Psi_W)_s, s \in \mathbb{Z}_{\geq 0},\) of \(\ker \Psi_W\) the subspace \((\ker \Psi_W)_s\) is not preserved by the action of \(\hat{g}_{\geq 0},\) but it is preserved by the action of \(\hat{g}_{\geq 0}\). Theorem 3.2 makes one think that \(D^n q_{(k+2)\theta}\), \(n \in \mathbb{Z}_{\geq 0},\) might be generators of \((\ker \Psi_W)_{k+2}\) for the action of \(\hat{g}_{\geq 0}\)?

(ii) In the case of level \(k = 1\) standard \(\mathfrak{sl}(3, \mathbb{C})\)-modules one needs only the relations \(q_{\theta}, q_{\theta - \alpha_1}, q_{\theta - \alpha_2}, q_2 \in (\ker \Psi_W)_3\) for a construction of combinatorial bases [MP2], [S].

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Dept. of Math., Univ. of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

E-mail address: primc@math.hr