Modelling the non-perturbative contributions to the complex heavy-quark potential

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In this paper, we propose a simple model for the complex heavy quark potential. The perturbative contributions are induced by one-gluon exchange at short distance which have been computed in the thermal field theory with hard loop approximation. The non-perturbative real and imaginary parts of the potential are described in a unified way in which one considers the long distance behavior between the quark and antiquark as an effective one-dimensional string interaction. Therefore, the non-perturbative terms in this model are assumed to be proportional to the one-dimensional Fourier transform of the resummed gluon propagator at static limit. The proportional coefficient is related to the string tension which is taken to be the same for both real and imaginary part of the potential model. The obtained real part is identical to the real-valued Karsch-Mehrz-Satz potential model which may indicate the rationality of the construction for the corresponding imaginary part. In addition, for a quantitative comparison to the current lattice simulations, we introduce an entropy contribution in the model and a reasonably good agreement between the model and lattice data is obtained.

I. INTRODUCTION

The heavy-ion experiments at RHIC and the LHC have shown very rich and interesting physics which can not be interpreted by simple extrapolation from proton-proton collisions, therefore, indicates the formation of a new form of matter – the Quark-Gluon Plasma(QGP). Heavy quarkonium dissociation has been proposed long time ago as a very sensitive probe to study the properties of the hot and dense medium\cite{1}. Bound states of heavy quarks could survive inside the plasma where the temperature \(T\) is higher than the deconfining temperature. However, color screening produced by the light quarks and gluons weakens interaction between the quark-antiquark pair and leads to the dissociation of quarkonia. Since excited states are more weakly bound than lower ones, the successive dissociations could be served as a thermometer of QGP\cite{2}.

The studies on quarkonia can be carried out in the non-relativistic limit due to their large masses, therefore, a quantum mechanical description becomes available. With some specific potential, one can solve the Schrödinger equation which determines the corresponding binding energies and decay widths of the bound states. Therefore, as the basic input in the equation of motion, heavy-quark(HQ) potential turns to be very crucial to get the correct behavior of the bound states. At zero temperature, the well-known Cornell potential can be derived from the effective field theory of QCD\cite{3}. However, the extension to finite temperature contains much more complications due to some extra \(T\)-dependent scales appearing in the effective field theory\cite{4,5}. Due to these difficulties, a phenomenological approach to study the quarkonia, i.e., the HQ potential model, has been extensively used in previous studies.

Lattice simulations on the singlet free energy and internal energy of a static quark pair provide useful information for constructing a potential model at finite temperature. Therefore, there are various lattice-based potential models on the market, see Refs.\cite{6-9} for examples. However, the calculation based on a real-valued potential model does not include any information about the width of a state. At finite temperature, the HQ potential develops an imaginary part due to Landau damping of the low-frequency gauge fields. It is an important quantity which determines the decay width of quarkonia. In fact, the calculation\cite{10} performed in the weak-coupling resummed perturbation theory leads to a complex HQ potential which, however, can only be applied to the short distance region. On the other hand, the long distance behavior of the imaginary part of the potential \(\text{Im} V\) is not clear due to the lack of the corresponding simulations on Lattice. Fortunately, progress has been made in recent years\cite{11-13}. Burnier et al. have measured the complex valued static potential by first principle simulations in quenched QCD. To reduce the finite volume artifacts in the obtained data, in a latest publication\cite{14}, they provided improved results for the complex potential where larger physical volume on the lattices is used.

The main purpose of the current paper is to construct a complex HQ potential model which is in agreement with the lattice simulations and can be used for other phenomenological studies on the in-medium properties of quarkonia. The rest of the paper is organized as follows. In Sec.\textsuperscript{II} we briefly review the calculation of the complex potential in perturbation theory which is an important component in our potential model. In Sec.\textsuperscript{III} we show a novel approach to reproduce the real-valued KMS potential model and the extension of such an approach leads to the imaginary part in the complex potential model. Without introducing any \(T\)-dependent free parameter, the extended KMS potential
model gives a fairly well prediction for the HQ potential. For a quantitatively better description of the lattice data, in Sec. V we use the lattice data of the real part of the potential to extract the Debye mass at different temperatures which are then used to predict the HQ potential. Improved results are obtained for both real and imaginary parts of the potential model. In Sec. VI we discuss the important role that the entropy contribution plays in our potential model which hints at some possible further refinements of our model. Finally, we give a short summary in Sec. VII.

II. PERTURBATIVE HEAVY QUARK POTENTIAL DUE TO ONE-GLUON EXCHANGE

At zero temperature, the Cornell potential has successfully described the experimentally observed quarkonium spectroscopy. It takes a form of a Coulomb plus a linear part,

$$V_{\text{cornell}} = -\frac{\alpha_s}{r} + \sigma r,$$  

where $\alpha_s = g^2 C_F/(4\pi)$ is the strong coupling constant, $\sigma$ is the so-called string tension which has the dimension of energy square and the separation between the quark and antiquark is denoted by $r$. At finite temperature, the Coulomb potential at short distances is replaced by the Debye-screened potential. The potential at short distances can be computed in thermal field theory with perturbation expansion. In the real time formalism, a complex potential comes out naturally. The propagator now is given by a $2 \times 2$ matrix, even at tree-level, the physical “11” component $D_{11}$ has an imaginary part which is related to the distribution function of the hot medium. In Keldysh representation, we have three independent components of the propagator named retarded ($D_R$), advanced ($D_A$) and symmetrical ($D_F$) propagators. Their relation to the physical component is given by $D_{11} = (D_R + D_A + D_F)/2$. Within hard-thermal-loop approximation, the resummed propagators which contain the medium modifications have been obtained through the Dyson-Schwinger equation. Then the perturbative HQ potential at leading order (LO) can be determined from the 3-dimensional (3D) Fourier transform of (the temporal component of) the resummed gluon propagator at static limit,

$$V_p(\hat{r}) = -g^2 C_F \int \frac{d^3p}{(2\pi)^3} (e^{ip\cdot r} - 1) \left(D^{\alpha\beta}(p_0 = 0, p)\right)_{11}.$$  

The real part of the potential comes from the Fourier transform of the retarded/advanced propagator while the imaginary part comes from the symmetric propagator in Keldysh representation. Explicitly, we have

$$\text{Re} V_p(\hat{r}) = -g^2 C_F \int \frac{d^3p}{(2\pi)^3} (e^{ip\cdot r} - 1) \frac{1}{p^2 + m_D^2} = -\alpha_s \left(m_D + \frac{e^{-\hat{r}}}{r}\right),$$

$$\text{Im} V_p(\hat{r}) = -g^2 C_F \int \frac{d^3p}{(2\pi)^3} (e^{ip\cdot r} - 1) \frac{-\pi T m_D^2}{|p|(p^2 + m_D^2)^2} = -\alpha_s T \phi(\hat{r}).$$

where $\hat{r} = rm_D$ and

$$\phi(\hat{r}) = 2 \int_0^\infty d\hat{z} \frac{z}{(\hat{z}^2 + 1)^2} \left[1 - \frac{\sin(\hat{z}\hat{r})}{\hat{z}\hat{r}}\right].$$

$m_D$ is the Debye mass which is given by $m_D^2 = (N_f + 2N_c)\frac{\alpha_s \hat{r}^2}{18}$ at leading order. Notice that for the real part, the $r$-independent term is divergent and we have subtracted a vacuum contribution $1/p^2$ in the integrand to get a finite result. The perturbation theory is not capable to deal with the medium corrections to the string contribution in the Cornell potential which will be considered by constructing phenomenological models and discussed in next section.

III. AN EXTENDED KARSCH-MEHR-SATZ HEAVY-QUARK POTENTIAL MODEL

To study the in-medium properties of the heavy bound states, such as charmonia and bottomonia, in the non-relativistic limit, a proper potential that needs to be specified in the Schrödinger equation contains typical non-perturbative physics due to the typical size of the charm and bottom quark bound states. Therefore, we can not directly use the above perturbative potential to describe the interactions.

At finite temperature, the non-perturbative contribution of the HQ potential is often obtained by constructing a potential model based on the Lattice simulations. For the real part of the potential $\text{Re} V$, we consider a very simple form in which the large $r$ interaction is modeled as a QCD string screened at the same scale as the perturbative
Coulomb contribution. It is the famous Karsch-Mehr-Satz(KMS) potential model\cite{16} and can give a qualitatively good description of the lattice data. The model reads

$$\text{Re} V_{np}(\hat{r}) = \frac{\sigma}{m_D} [1 - \exp(-\hat{r})].$$ (6)

There is a novel approach to get the above form of the non-perturbative contribution. We assume that at large distances, the quark pair exhibits an effective one-dimensional string interaction and such a contribution to the HQ potential is proportional to the 1D Fourier transform of the retarded/advanced propagator at static limit. Therefore, we have

$$\text{Re} V_{np}(\hat{r}) = -g^2 C_F k \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left( e^{ip\hat{r}} - 1 \right) \frac{1}{p^2 + m_D^2} = g^2 C_F k \frac{\sigma}{m_D} [1 - \exp(-\hat{r})] \equiv \frac{\sigma}{m_D} [1 - \exp(-\hat{r})].$$ (7)

Here, $k$ is the proportional coefficient which has the dimension of energy squared. Given the above model, we have a formal definition for the important quantity string tension, $\sigma \equiv \frac{g^2 C_F k}{m_D}$. Combining the perturbative result in Eq. (3) and the above long-distance string contribution Eq. (7), we get the following parameterization for the real part of the potential

$$\text{Re} V{I}(\hat{r}) = -\alpha_s \left( m_D + \frac{e^{-\hat{r}}}{r} \right) + \frac{\sigma}{m_D} [1 - \exp(-\hat{r})].$$ (8)

This is exactly the KMS potential model which has been considered as the HQ free energy in the previous studies\cite{17}. We use $\text{Re} V{I}$ to denote the above parameterization. In this work, we will also consider another parameterization of the HQ potential based on the internal energy which is then denoted as $\text{Re} V{II}$ to avoid any confusion in the notations.

One can easily check that in the limit of zero temperature where the Debye mass $m_D \to 0$, Eq. (8) is reduced to the vacuum Cornell potential and there is also a $T$-independent Coulombic behavior $1/r$ in the small $r$ limit as expected. The two parameters $\alpha_s$ and $\sigma$ are assumed to be unchanged in a hot medium, once determined at zero temperature. It is interesting to see if the simple parameterization Eq. (8) could reproduce the lattice data. The corresponding result is given in Fig. 1. The data from Ref. [14] are obtained in quenched QCD where $N_f = 0$ and $N_c = 3$. Due to the absence of a $T = 0$ lattice measurement, $\alpha_s = 0.272$ and $\sigma = 0.215\text{GeV}^2$ are determined\cite{14} by using the data at 113MeV. To keep the model as simple as possible, the Debye mass takes its LO perturbative form, i.e., $m_D = \sqrt{4\pi\alpha_s/C_F T} \approx 1.601T$ in the deconfined phase. In fact, a critical behavior is found by inspection the data and the deconfining temperature $T_c$ is around 290MeV. Since we will focus on the HQ potential for temperatures larger than $T_c$, we simply take the Debye mass $m_D \to 0$ once $T < T_c$. As a result, $\text{Re} V$ becomes identical to the Cornell potential in the confined phase. In fact, this consequence coincides with the lattice findings qualitatively.

From Fig. 1 we find that at small distances, $\text{Re} V{I}$ can reproduce the lattice data very well. At large distances, the predicted potential exhibits a screen behavior as suggested by the lattice data. However, quantitative agreement

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig1.png}
\caption{Comparison of $\text{Re} V$ between the lattice data (blue dots) from Ref. [14] and the extended KMS potential model discussed in Sec. III. The red solid curve denotes the model based on $\text{Re} V{II}$ while the purple dashed curve denotes the model based on $\text{Re} V{I}$.}
\end{figure}
is not observed and Re $V^I$ obviously deviates from the data. This result is actually consistent with the fact that the HQ free energy is expected to provide the lower limit of the binding energy of a quark pair. The unsucces of Eq. (8) also indicates the necessity to include the entropy contributions in the parameterization. We also mentioned that the last plot in Fig. 1 shows a nice agreement between the model and data at $T = 271$ MeV which suggests the validity of our above assumption for the confined phase, namely, the corresponding Debye mass drops to zero for $T < T_c$.

In general, the HQ potential is considered to be between the HQ free energy and internal energy $F^I$ and $F^{II}$ and the latter is considered as the higher limit of Re $V$. However, the way to add the entropy contribution is not totally clear. Lattice simulations suggest that at very small distances, the medium effect is negligible and no entropy contribution should be added to the free energy $F^I$. Instead of adding the full entropy contribution to the free energy as done in the previous studies [17], we consider adding the entropy contribution only to the non-perturbative term in Re $V$ which gives another parameterization of Re $V$ and denoted as Re $V^{II}$. The internal energy $U$ can be obtained from the free energy $F$ through the relation $U = F - T \frac{\partial F}{\partial T}$ and we have

$$\text{Re } V^{II}(\hat{r}) = -\alpha_s \left( m_D + \frac{e^{-r}}{r} \right) + \frac{2\sigma}{m_D} \left[ 1 - \exp(-\hat{r}) \right] - \frac{\sigma}{m_D} \exp(-\hat{r}).$$

The small $r$ behavior of Re $V^{II}$ is identical to that of Re $V^I$ by construction. In addition, both parameterizations reproduce the Cornell potential in the confined phase as the Debye mass is assumed to vanish. On the other hand, above $T_c$, the big difference at the large distances are observed. In the limit $r \to \infty$, Re $V^{II}$ approaches to $2\sigma/m_D - \alpha_s m_D$ which is almost two times Re $V^I(r \to \infty)$ if $T$ is not very larger than $T_c$ and the Coulombic contribution $-\alpha_s m_D$ is small. According to Fig. 1, the prediction based on Re $V^{II}$ is much better. Providing its simple form with no free parameter at finite temperature, the parameterization Eq. (9) is actually a nice model of Re $V$. For a quantitative purpose, we will also discuss its further improvement in next section.

Originally, Eq. (6) was proposed to describe the long distance behavior of Re $V$ and it has no information about the imaginary part. The description of Im $V$ at large $r$ turns to be more challenge since only few works have been done on this relative new topic [21–23]. Our goal is to construct a model that could reproduce the lattice results of Im $V$. In order to have a unified description for both real and imaginary part of the HQ potential, we adopt the same idea used above and assume Im $V$ at large distances is proportional to the 1D Fourier Transform of the symmetric gluon propagator at the static limit while the proportional coefficient $k$ is the same as that appearing in the real part. Therefore, we get

$$\text{Im } V_{np}(\hat{r}) = -g^2 C_F k \int_{-\infty}^{\infty} \frac{dp}{2\pi} (e^{ipr} - 1) \frac{-\pi T m_D^2}{|p|^2 (p^2 + m_D^2)^2} = g^2 C_F k T \frac{\sin(z\hat{r}) - 1}{2 m_D^2} \psi(\hat{r}),$$

where

$$\psi(\hat{r}) = \int_0^\infty dz \frac{2}{z(z^2 + 1)^2} \cos(z\hat{r}) - 1.$$

Accordingly, we have the following two parameterizations for the imaginary potential. One is the sum of Eqs. (10) and (11) which can be considered as an extension of the original real-valued KMS model. It reads

$$\text{Im } V^I(\hat{r}) = -\alpha_s T \phi(\hat{r}) + \frac{\sigma T}{m_D} \psi(\hat{r}).$$

The other is obtained when the “entropy” contribution is added only to the non-perturbative term in Im $V^I$. We should emphasize that the terminology “entropy” could be misleading here because the way to add such a contribution is only an ad-hoc phenomenological hypothesis. This is exactly an analogue as we did for Re $V^{II}$ and turns to be important to give a better quantitative description of the lattice data. The explicit expression is given by

$$\text{Im } V^{II}(\hat{r}) = -\alpha_s T \phi(\hat{r}) + \frac{2\sigma T}{m_D^2} [\psi(\hat{r}) + \psi'(\hat{r})],$$

with

$$\psi'(\hat{r}) = \int_0^\infty dz \frac{\hat{r} \sin(z\hat{r})}{(1 + z^2)^2}.$$

Therefore, we have two different parameterizations for the complex HQ potential, $V^I = \text{Re } V^I + i \text{Im } V^I$ and $V^{II} = \text{Re } V^{II} + i \text{Im } V^{II}$. Both of them can be considered as the extended KMS potential models. In addition,
the parameterizations for Im\(V\) contain the same parameters as the real part. Therefore, no more parameter needs to be determined and the comparisons between our models and lattice data are shown in Fig. 2.

As we can see,

\[
\text{FIG. 2: Comparison of Im}V\text{ between the lattice data (blue dots) from Ref. [14] and the extended KMS potential model discussed in Sec. III. The red solid curve denotes the model based on Im}V^{II}\text{ while the purple dashed curve denotes the model based on Im}V^{I}.
\]

Im\(V^{I}\) obviously overshoots the lattice data and the rapid increase with the quark pair separation is not an expected behavior. Surprisingly, Im\(V^{II}\) turns to be a reasonable parameterization for the imaginary potential at small distances for all the temperatures. Deviation becomes obvious when the temperature gets increased. In the confined phase, Im\(V\) is approximately zero from model prediction which is also supported by the data despite the huge uncertainty from lattice simulation. This is shown by the last plot in Fig. 2.

From the above discussion, we could conclude that adding the entropy contributions leads to a better model prediction for the real part of the potential. Interestingly, a similar term added in Im\(V\) also plays an important role to improve the results. Therefore, using the parameterization \(V^{II} = \text{Re}V^{II} + i\text{Im}V^{II}\) to model the complex HQ potential is an acceptable solution for qualitative or even semi-quantitative purpose.

Given the models above, it is also interesting to discuss the distance scales where the non-perturbative effects become important. This distance is denoted as \(r_s(T)\) which is determined through the equation \(|\text{Re}V^{p}(r_s)| = |\text{Re}V^{np}(r_s)|\) or \(|\text{Im}V^{p}(r_s)| = |\text{Im}V^{np}(r_s)|\). We first look at the real part. At zero temperature, it is easy to see that \(r_s(0) = \sqrt{\alpha_s/\sigma}\).

In order to analytically study the medium correction to \(r_s(0)\), we assume \(r_s(T) = r_s(0) + \Delta r_s(T)\) and treat \(\Delta r_s(T)\) as a perturbation by requiring \(\hat{r} < 1\). This actually corresponds to \(m_D < \sqrt{\sigma/\alpha_s}\) which is satisfied for not very high temperatures. Our calculation shows that the LO corrections can be expressed by the following equation

\[
\Delta r_s(T) = \begin{cases} 
\frac{1}{4} \sqrt{\frac{\alpha_s}{\sigma}} \hat{r}_s(0) & \text{for } \text{Re}V^{I} \\
\frac{1}{3} \sqrt{\frac{\alpha_s}{\sigma}} \hat{r}_s^2(0) & \text{for } \text{Re}V^{II}
\end{cases},
\]

where \(\hat{r}_s(0) = r_s(0)m_D\). As we can see, \(r_s(T)\) gets increasing when medium effect is considered. As compared to \(r_s(0)\), the correction is suppressed by a factor of \(\hat{r}_s(0)\) for the type I potential model while for \(\text{Re}V^{II}\), the correction appears at higher order \(\sim O(\hat{r}_s^2(0))\).

The imaginary part of the potential develops an non-zero value at finite \(T\). Both the perturbative and non-perturbative contributions vanish at \(r = 0\). When the distance starts to increase, \(|\text{Im}V^{np}|\) will exceed \(|\text{Im}V^{p}|\) at

\(^{1}\text{Notice that for the imaginary part of the potential, we actually plot its absolute values in the figures.}\)
To determine $r_s(T)$ for the imaginary part, we also consider the small $\hat{r}$ limit which allows us to expand the three functions related to $\text{Im} V$, 

$$
\phi(\hat{r}) \approx -\frac{1}{9} \hat{r}^2 (3 \ln \hat{r} - 4 + 3\gamma_E),
$$

$$
\psi(\hat{r}) \approx -\frac{1}{2} \hat{r}^2 - \frac{1}{144} \hat{r}^4 (12 \ln \hat{r} - 19 + 12\gamma_E),
$$

$$
\psi'(\hat{r}) \approx \frac{1}{2} \hat{r}^2 + \frac{1}{18} \hat{r}^4 (3 \ln \hat{r} - 4 + 3\gamma_E),
$$

where $\gamma_E$ is the Euler-Gamma constant. Then it is straightforward to get the LO result

$$
r_s(T) = \begin{cases} 
\frac{1}{m_D} e^{-\frac{\sigma}{2\alpha_s m_D} + \frac{1}{2} \gamma_E} & \text{for } \text{Im} V^I, \\
\sqrt{\frac{2\alpha_s}{\sigma}} & \text{for } \text{Im} V^{II}.
\end{cases}
$$

For the type I potential model, $r_s(T)$ is exponentially suppressed when $m_D < \sqrt{\sigma/\alpha_s}$. Interestingly, we find that for $\text{Im} V^{II}$, $r_s(T)$ is independent on the temperature$^2$. It appears at the same distance scale $\sim \sqrt{\frac{\sigma}{\alpha_s}}$ as the real part. We can also consider the $T$-dependent correction for $\text{Im} V^{II}$ at NLO which equals to $-\frac{\sqrt{\pi}}{8} r_s(0) \frac{1}{\text{Im} V^{II}(0)}$. Notice that the above discussion requires $m_D < \sqrt{\sigma/\alpha_s}$, therefore, the correction leads to a increasing $r_s(T)$.

### IV. AN IMPROVED KARSCH-MEHR-SATZ HEAVY-QUARK POTENTIAL MODEL

For the purpose of quantitatively describing the lattice data, in this section, we will discuss the improvements on the extended KMS potential model $V^{II}$ which turns to be a proper candidate for the complex HQ potential model as discussed in last section.

Providing the explicit expression in Eq. (19) and the corresponding results shown in Fig. 1 there are some possible ways to improve the model. First of all, introducing the $T$-dependence in $\alpha_s$ will ruin the good agreement at short distances which can be ruled out. In fact, the strong coupling $g$ related to $\alpha_s$ denotes the interacting strength at very small $r$ where the hot medium effect turns to be very small. In contrast, the strong coupling $g$ appearing in Eqs. (7) and (10) represents the one-dimensional interacting strength at large distances. Unlike the Coulombic behavior where no temperature effect enters, in principle, the string tension, according to its novel definition in our parameterization, could have a strong dependence on temperature. From Fig. 1 we find that $\text{Re} V^{II}$ deviates from the data at large distances where the model is dominated by $2\sigma/m_D$. A better agreement can be expected if $\sigma$ has proper $T$-dependence. A naive picture of the $T$-dependence of the string tension could be the following$^{14}$: $\sigma$ gets its maximum value in the confined phase and decreases with increasing $T$ when $T \geq T_c$. However, such a picture is not supported by the result in Fig. 1 where $\sigma$ above $T_c$ is expected to be larger than its vacuum value.

As a result, a more feasible way to improve the model is to consider the possible corrections to the Debye mass while keeping both $\alpha_s$ and $\sigma$ to be constant. Our simple assumption that $m_D$ is linear in $T$ may be not capable to capture the correct non-perturbative effects. We should point out that once the parameterizations of the HQ potential are given, the involved Debye mass should be considered as a free parameter and it is not necessary to be linear in the temperature $T$.

Therefore, we will determine the values of $m_D$ from the first principle simulations. The extraction of the imaginary part from lattice calculations gets much more challenging than the real part, so the lattice data of $\text{Re} V$ is used to extract the Debye mass. As a crosscheck, the values of $m_D$ from the fits to $\text{Re} V$ will be adopted to evaluate the imaginary part of the potential. We use the lattice data in Ref. [14] and only consider $\text{Re} V$ up to 1fm. In this region of the quark pair separations, the lattice reconstruction is most reliable and the error bars are actually very small. The optimized values$^3$ we obtain for $m_D$ at different temperatures are given in Table IV.

| $T$ [MeV] | 406 | 369 | 338 | 312 | 290 | 271 | 254 | 226 | 113 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $m_D$ [MeV] | 754 | 546 | 508 | 361 | 278 | 0 | 0 | 0 | 0 |

$^2$ Here, we also assume $1/|\ln \hat{r}| < 1$ which is a stronger requirement as compared to $\hat{r} < 1$.

$^3$ For temperatures below $T_c$, the values of $m_D$ are at the order of $10^{-6}$ or even smaller. We take them to be zero in Table IV.
TABLE I: Debye mass extracted from the model fit to the lattice result for Re $V$ in Ref. [14].

From the above table, we do find that in the deconfined phase, the Debye mass $m_D$ increases with temperature $T$ as expected. In addition, the critical behavior of $m_D$ around $T_c$ is very clear to see; it drops to zero immediately once the temperature is lower than $T_c$. According to the values of $m_D$, it doesn’t have a simple linear dependence on $T$. On the other hand, in the temperature region relevant to the quarkonium studies, we could add a non-perturbative correction to get a reasonable fit to the Debye mass. We assume the non-perturbative correction is inversely proportional to $T$ which becomes negligible at very high temperatures but important when the temperature is decreasing to $T_C$.

Namely, the extracted Debye mass can be parameterized as $m_D(T) = aT + b/T$ and the values of the parameters are given by $a = 2.687$ and $b = -0.146$. Notice that the value of $b$ is negative which indicates that the ratio $m_D/T$ decreases as $T$ approaches to $T_c$ from above. In fact, the same downward trend has been observed in a massive quasi-particle model when fitting to the equation of state[24]. However, this is actually opposite to the result from perturbation calculation where an upward trend of the ratio is found[25, 26]. As shown in Fig. 3, in the deconfined phase, the agreement between the parameterization and the values given in Table I is very good. However, our simple parameterization of $m_D$ doesn’t apply in the asymptotically high temperature limit, since the parameter $a$ is considered as a constant.

![FIG. 3: Comparison between the fitted Debye mass (red solid curve) and its values extracted from the lattice data (blue dots) in Ref. [14].](image-url)

Before we show the comparison between the improved KMS potential model\(^4\) and the lattice data, we also want to mention another parametrization of the complex HQ potential which has been discussed in Ref. [22] and is referred to as Burnier-Rothkopf(BR) model in the following. The basic idea of this model is completely different from the KMS model as discussed in this paper. BR model is constructed based on the generalized Gauss law[27, 28] and the medium effects are incorporated by an in-medium permittivity which was calculated in the Hard-thermal-loop perturbative theory[22]. The perturbative part of BR model also takes the form of the HTL Debye-screened potential as given by Eqs. 6 and 11 while the non-perturbative string parts are expressed in term of the parabolic cylinder function. Explicit forms can be found in Ref. [22].

Similar as what we did in the improved KMS model, The strong coupling constant $\alpha_s$ and the string tension $\sigma$ don’t change with the temperature in BR model, the lattice data of the real part of the potential is used to extract the free parameter $m_D$ and a very good fit is observed for Re$V$. We point out that the values of $m_D$ obtained from BR model\([14]\) differ from the results in Table I which may indicate a slightly model dependence. However, the $T$-dependent behavior of $m_D$ is very similar. Both show the same downward trend of the ratio $m_D/T$ as $T$ approaches to $T_c$ from above. The extracted values of the Debye masses at different temperatures are then used to predict the imaginary part of the HQ potential and the agreement is only satisfactory when $T$ is large and $r$ is small[14]. However, the HQ potential at temperatures close to $T_c$ is actually very important for the studies of quarkonia. Therefore, as one purpose of our work, we want to develop a model that could have a quantitatively better description of the imaginary potential.

In Fig. 4 we show the comparison between the model predictions and the lattice data for Re$V$. Besides the improved KMS model, the prediction from BR model is also plotted in this figure. Roughly speaking, both models

\(^4\) The improved KMS potential model refers to the type II parameterization Re$V^{\text{II}} + i\text{Im}V^{\text{II}}$ in which the values of the Debye mass are given in Table I.
FIG. 4: Comparison of $\text{Re}V$ between the lattice data (blue dots) from Ref. [14] and the improved KMS potential model (red solid curve) discussed in Sec. IV. The prediction from BR is also shown in this figure which is denoted by the black dashed curve with the colored error-bands from the uncertainty in the determination of $m_D$.

can well reproduce the data. In the confined phase, since the Debye mass is approximately zero, we actually have the Cornell potential and nothing changes as compared to the extended KMS model as discussed in Sec. III. Above the critical temperature, the two models behave qualitatively the same, namely, a Debye screened contribution at small distances and a screened string contribution at large distances. In addition, for temperatures slightly above $T_c$, a better agreement can be obtained by using the improved KMS model since it exhibits an upward trend at large distances which is in accordance with the data. On the other hand, at relatively high temperatures, deviations from data at intermediate and large distances appear in the improved KMS model while the BR model turns to work very well.

FIG. 5: Comparison of $\text{Im}V$ between the lattice data (blue dots) from Ref. [14] and the improved KMS potential model (red solid curve) discussed in Sec. IV. The prediction from BR is also shown in this figure which is denoted by the black dashed curve with the colored error-bands from the uncertainty in the determination of $m_D$.

Fig. 5 shows the comparisons of the imaginary part of the HQ potential. Below the critical temperature, $\text{Im} V$ gets very close to zero according to both models. In the deconfined phase, unlike the real part of the potential, these two
models have a very different behavior at large distances. The BR model turns to be saturated at large \( r \) while the improved KMS model shows a rapid increase of \( \text{Im} \, V \). Of course, the large \( r \) behavior of \( \text{Im} \, V \) needs to be further confirmed by lattice, however, the current data for \( r > 1 \text{fm} \) seems to support such a rapid increase although it contains very large error bars. It is clear that for temperatures slightly above \( T_{\text{c}} \), the improved KMS model shows a significant improvement as compared to the BR model. It reproduces the data quantitatively well at small and intermediate distances, i.e., \( r < 1 \text{fm} \). As temperatures further increase, the improved KMS model starts to deviate from the data while the BR model shows a relatively good agreement at small distances.

It still needs further work to establish an accurate model for \( \text{Im} \, V \) and the possible reasons that lead to the disagreement between the improved KMS model and lattice data at very high temperatures will be discussed in Sec. \( \text{V} \). On the other hand, for practical quarkonium studies, temperatures far above the critical temperature are not relevant since no bound states can survive at very high temperature. In addition, the typical size of the bound states is about \( 0.2 \sim 0.6 \text{fm} \). Therefore, the improved KMS model indeed can be used in Schrödinger equation to quantitatively study the properties of the bound states.

V. DISCUSSIONS ON THE ENTROPY CONTRIBUTIONS IN KMS POTENTIAL MODEL

As we already show that the parameterization based on \( \text{Re} \, V^{\text{II}} \) leads to a better fit to the lattice data as compared to \( \text{Re} \, V^{\text{I}} \), therefore, the entropy contribution plays an important role here. To make the following discussion simple and clear, in this section, we assume the Debye mass takes its LO perturbative form, namely, it is proportional to \( T \). Notice that in order to eliminate the overshooting problem at small distances as observed in Ref. \([15]\), we don’t add the perturbative entropy contribution \( -T \partial (\text{Re} \, V^{\text{I}}) / \partial T \) to \( \text{Re} \, V^{\text{II}} \), otherwise the perturbative term in type II potential model becomes \( -\alpha_s / r (1 + r) e^{-r} \) and its derivative with respect to \( T \) is \( \alpha_s e^{-r} / T > 0 \). As a result, the potential at finite \( T \) would overshoot the Cornell potential which is contradictory to the lattice data. On the other hand, such a problem is absent in our parameterizations because \( \partial (\text{Re} \, V^{\text{I}}) / \partial T = -\alpha_s m_D (1 - e^{-r}) / T \) becomes negative.\(^5\)

However, the tricky thing is that the full potential \( \text{Re} \, V \) is a sum of the perturbative and non-perturbative contributions. The perturbative entropy contribution to \( \text{Re} \, V^{\text{II}} \) at intermediate or large \( r \) can be taken into account as long as the full potential doesn’t have the overshooting problem. Obviously, such a entropy contribution \( -T \partial (\text{Re} \, V^{\text{II}}) / \partial T \) is positive and becomes important when \( T \) is large. Therefore, the deviation at high \( T \) and intermediate \( r \) as shown in the first plot in Fig. \([4]\) could be reduced if this perturbative entropy contribution is appropriately added to \( \text{Re} \, V^{\text{II}} \).

Adding a similar contribution \( -T \partial (\text{Im} \, V^{\text{I}}) / \partial T \) to the imaginary part of the potential is just an analogue of the real part and the physical motivation for doing so is not clear yet. The point is to get a proper parameterization to describe the potential and deal with the complex potential in a unified framework. Although no analytical expressions for \( \text{Im} \, V \), we can also numerically check what would happen if the perturbative contribution \( -T \partial (\text{Im} \, V^{\text{I}}) / \partial T \) is added to \( \text{Im} \, V^{\text{II}} \). The consequence is very similar as that for the real part. At very small distances, no extra contribution should be considered, otherwise the potential \( \text{Im} \, V \) could flip the sign. Notice that the combination of the negative \( \text{Im} \, V^{\text{I}} \) and \( \text{Im} \, V^{\text{II}} \) as we considered in our model leads to a negative \( \text{Im} \, V \) for all \( r \).\(^6\) However, once the positive contribution \( -T \partial (\text{Im} \, V^{\text{I}}) / \partial T \) is included, the resulting potential \( \text{Im} \, V \) is also positive at very small \( r \). On the other hand, at intermediate distances, if we include the perturbative contribution while keeping the full \( \text{Im} \, V \) negative, the absolute value of \( \text{Im} \, V \) gets suppressed. Therefore, the deviation at high \( T \) and intermediate \( r \) as shown in the first plot in Fig. \([5]\) could also be reduced.

The above discussion may provide some hints about refining the improved KMS potential model. At small distances, no extra term should be include in \( V^{\text{II}} \). At intermediate or large distances, the perturbative contribution \( -T \partial V^{\text{I}} / \partial T \) should be added or partially added to \( V^{\text{II}} \). Of course, the resulting model could be much more complicated because it may be discontinuous at the distance where such contributions set in. In addition, partially adding these contributions can be formulated by a product of a weight factor \( w(r, T) \) and the term \( -T \partial V^{\text{I}} / \partial T \). The weight factor should be in the range of \( (0, 1] \), however, its explicit form is completely arbitrary.

\(^5\) Analytically, we can also show that there is no overshooting problem for the non-perturbative contribution no matter \( \text{Re} \, V^{\text{I}}_{\text{np}} \) or \( \text{Re} \, V^{\text{II}}_{\text{np}} \) is used in the parameterization.

\(^6\) Remember that in the above figures, we actually plot the absolute values of \( \text{Im} \, V \).
VI. SUMMARY

In this paper, we proposed a model for the complex HQ potential. The model consists of two parts. The perturbative term comes from the resummed HTL perturbation theory at leading order while the non-perturbative contributions are obtained from a novel approach where the 1D Fourier transform of the resummed gluon propagator is performed. In this model, the real and imaginary part of the potential are constructed in a unified framework and share the same parameters. In fact, the construction of the imaginary part of the potential is a natural generalization of the real part where our novel approach leads to exactly the real-valued KMS potential model. In addition, we consider the “entropy” contributions to the non-perturbative terms which turns to be very important to improve the model prediction when compared with the results from lattice simulations.

Our complex HQ potential model as given by Eqs. (9) and (13) takes a relatively simple form. The real part reproduces the Cornell potential in the confined phase where $m_D \to 0$, while in the deconfined phase it gets screened for both Coulombic and linear rising string contributions. The imaginary part develops a non-vanishing contribution above $T_c$ which increases rapidly with the quark pair separation. There are three parameters appearing in the potential model where the strong coupling $\alpha_s$ and string tension $\sigma$ are assumed to be $T$-independent and can be determined from lattice simulations at zero-temperature. Therefore, there is only one free parameter $m_D$ related to the medium effect. Even in the simplest case where $m_D$ takes its LO perturbation form, the model is able to give a reasonably well description of the lattice data. To achieve a quantitatively satisfactory result, we further consider to extract Debye mass from the model fit to the in-medium Re$V$ from lattice. The obtained values of $m_D$ are then used to predict the imaginary part which turns to be a much more challenging job. The outcome suggests that in the $T - r$ region relevant to quarkonium physics, the complex potential model proposed here can offer a quantitative description of the inter-quark forces, therefore, can be used in other phenomenological studies on quarkonia.

Finally, we also discussed possible ways that could lead to further refinements of the current potential model. On the other hand, a more accurate lattice reconstruction of the HQ potential, especially for the imaginary part, is urgently needed which requires lattices with finer spacing and larger volume. It is expected to provide more information to constrain the model construction. The different behaviors predicted by different models also need to be further checked when more data from lattice becomes available.

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