On description of spatial topologies in quantum gravity

A. A. Kirillov

Institute for Applied Mathematics and Cybernetics
10 Uljanova str., Nizhny Novgorod, 603005, Russia
e-mail: Kirillovl@focus.nnov.su

Moscow 1994
A new approach is suggested which allows to describe phenomenologically arbitrary topologies of the Universe. It consists in a generalization of third quantization. This quantization is carried out for the case of asymptotic closeness to a cosmological singularity. It is also pointed out that suggested approach leads to a modification of the ordinary quantum field theory. In order to show this modification we consider the example of a free massless scalar field.

1 Introduction

It is widely accepted that quantum fluctuations of metrics at small-scale distances can change spatial topology of the Universe [1],[2]. Effects connected with topology changing have been considered already in Refs.[3-6]. Nevertheless, an adequate mathematical scheme for description of processes of such a kind is absent yet. In this paper we suggest an approach which, as we hope, gives a possibility for, at least, a phenomenological description of arbitrary spatial topologies. To this end we use a generalization of third quantization.

Third quantization has been already used in quantum cosmology for description of ”wormholes” and ”baby universes” [3-6] (which lead as it was shown to the loss of quantum coherence) as well as for description of ”spontaneous quantum creation of a universe from nothing” [7],[8] proposed earlier in Ref.[9]. Note however that in the all considered cases a number of small closed universes (which branch off the our large Universe) is a variable quantity but it turns out that the topology of each closed universe is fixed. To describe different possible topologies one should modify the procedure of the third quantization. The simplest way to do this is to make it local. Such a possibility follows from the fact that the Wheeler-DeWitt equation which governs the evolution of a wave function of the Universe consists of an infinite set of the Klein-Gordon type equations (one local Wheeler-DeWitt equation at each point $x$ of three-space $S$). We note that this fact is in accordance with another one that in General Relativity time has only a local meaning. Therefore, one may attempt to quantize every local Wheeler-DeWitt equation independently. Such procedure we call the Local Third Quantization (LTQ).

There is one problem with LTQ procedure in general case. The fact is that all local Wheeler-DeWitt equations are strongly coupled to each other and so it is very difficult to carry out this procedure. Nevertheless there is a situation when connection between different local Wheeler-DeWitt equations disappear, at least, in main order. It is just the case if one treats a behavior of a gravitational field close to a cosmological singularity.

It was shown in Refs.[10,11] that a general inhomogeneous gravitational field at the singularity may be considered as a continuum of uncoupled homogeneous (of IX type) fields. Indeed, nearby the singularity it is always possible to choose an ”elementary” volume $\Delta V$, in which the gravitational field is homogeneous in leading order (we do not take into account the presence of matter for the sake of simplicity and because it does not change properties of the gravitational field). In the vicinity of the singularity the horizon size tends to zero ($l_h \to 0$) and different ”elementary” domains $\Delta V(x)$ of the three-space $S$, do not affect on each other and may be considered independently (for validity of that the following condition must be fulfilled: $(\Delta V)^{1/3} \ll l_h$). The LTQ procedure consists then of the assumption that quantization is carried out independently for each elementary spatial domain $\Delta V(x)$. Furthermore, one may assume that all these domains are indistinguishable. Localization of third quantization is achieved in the limit as $\Delta V \to 0$ only. Notice that in this limit every such ”elementary” domain contains only one ”physical point” of space and, therefore, under the
distributions of matter and gravitational field. At each point of \( S \) these degrees of freedom form a set. Thus, under the **Local Third Quantization** one may mean the independent third quantization of all such sets. In ordinary field theory it is more convenient to use Fourier transformation for physical degrees of freedom that is expansion in modes. So in the case of free fields local third quantization consists in third quantization of the field modes.

## 2 A Gravitational Field in the Vicinity of a Cosmological Singularity

As it was pointed out above the problem of the local third quantization of the gravitational field at the singularity is reduced to the third quantization of the homogeneous field. It was shown in Refs.\([8,11]\) (see also Ref.\([12]\)) that at the singularity the quantum states of the homogeneous field (or in our case of an "elementary" spatial domain \( \Delta V(x) \)) may be classified by some quantum number \( n(n = 0,1,...) \). When third quantization is imposed, the wave function becomes a field operator and can be expanded in the form (here for simplicity we assume that \( \Psi \) is a real scalar function)

\[
\Psi = \sum C_n U_n + C_n^+ U_n^* ,
\]

where \( \{U_n, U_n^*\} \) is an arbitrary complete orthonormal set of solutions of the Wheeler-DeWitt equation:

\[
(\Delta + V)U_n = 0
\]

here \( V \) is a potential, \( \Delta = \frac{1}{\sqrt{-g}} \partial_A \sqrt{-g} G^{AB} \partial_B \) and \( G_{AB} \) is the metric on a minisuperspace \( W \) (for more detail see Ref.\([11]\)). The operators \( C_n \) and \( C_n^+ \) satisfy the standard (anti)commutation relations

\[
[C_n, C_m^+] = \delta_{n,m}
\]

where \( \pm \) relates to the two possible statistics of the wave function.

In the case of inhomogeneous field the Wheeler-DeWitt equation is split up into the set of the uncoupled equations of the (2.2) type, each of which contains the variables describing a gravitational field at a fixed point \( x \) of the three-manifold \( S \):

\[
(\Delta(x) + V_x)\Psi = 0.
\]

The space \( H \) of solutions of the Wheeler-DeWitt equation has the form of the tensor product of the spaces \( H_x \) (written as \( H = \prod_{x \in S} H_x \)) where \( H_x \) is the space of solutions of Eq.(2.2). Then one may introduce a set of wave functions \( \Psi_x \) and secondly quantize every local Wheeler-DeWitt equation (2.4) independently. Thus the operators (2.3) acquire additional dependence of spatial coordinates. The LTQ procedure consists thus in the replacement of the relations (2.3) by the following ones

\[
[C(x, n), C^+(y, m)] = \delta_{n,m}\delta(x - y).
\]

Using the operator algebra (2.5) one can construct a set of states with an arbitrary number of domains (with an arbitrary density of points for physical continuum). In particular, the vacuum state is determined by the relations \( C(x, n) \mid 0 >= 0 \) (for arbitrary \( x \in S \)) and, therefore, this state corresponds to the absence of all points of physical space and consequently the absence of all field observables. In other words in this state there is no a physical continuum as it is. The operator \( N(x, n) = C^+(x, n)C(x, n) \) has the ordinary meaning of number of elementary domains.
Considered theory includes conventional quantum gravity as a particular case. Indeed, let us consider the set of states \( \{ | A > \} \) which have the form

\[
| n(x) > = \prod_{x \in S} C^+(x, n(x)) | 0 > .
\]  

(2.6)

These states describe the case when there is just one elementary domain at each point \( x \in S \) and, therefore, the following conditions are fulfilled:

\[
\theta(x) | A >= 0, a s x \in S
\]  

(2.7)

(i.e. the number of point of physical continuum having the coordinate \( x \) coincides with that of the \( S \)). Obviously that for these states topology of physical space is the same as that of the \( S \).

In order to illustrate a nontrivial topology of the Universe one may construct a handle on \( S \).

In our approach the existence of the handle is indexed by the fact that quantum states of the gravitational field \( U_n(x) \) are triple-valued functions of spatial coordinates (in some region \( K \in S \)). Therefore, the states describing the handle may be taken in the form

\[
| n(x); [p(x)], [q(x)]_K > = \prod_{y \in K} C^+(y, p(y))C^+(y, q(y)) \prod_{z \in S} C^+(z, n(z)) | 0 > .
\]

It is obvious that due to indistinguishability of domains one may speak about topology of physical space in a usual sense in quasi-classical limit only. Indeed, in this limit one can introduce a set of maps such that metric functions become single-valued.

Evidently, one of possible applications of LTQ is a description of effects connected with the "space-time foam"[1], [2]. In particular, it should display itself in the existence of the so-called vacuum fluctuations connected with the creation and annihilation of virtual points of physical space. It should be also noted that the numbers \( N(x) \) vary during the evolution [7], [8]. This means that the structure of the foam is not fixed and must be determined dynamically. Besides, there is an interesting possibility that at small distances the spatial continuum has "hollows" (i.e. \( N(k) \rightarrow 0 \) if \( k \rightarrow \infty \), where \( N(k) = (2\pi)^{-2/3} \int N(x) \exp(-ikx)d^3x \)). Thus, in this way, one may attempt to overcome the divergences problem in conventional quantum gravity.

## 3 On a Modification of the Ordinary Field Theory

The foamy structure of the spacetime must be reflected in a universal way on the structure of the conventional field theory. As an example we consider now a free massless scalar field \( \varphi \).

If one writes the Fourier expansion for \( \varphi \)

\[
\varphi(x, t) = (2\pi)^{-2/3} \int \frac{d^3k}{\sqrt{2k}} \left\{ A(k)e^{ikx-ikt} + A^+(k)e^{-ikx+ikt} \right\}
\]

(3.1)

(here \( k = | \mathbf{k} | \)), then the Hamiltonian of the field takes the form of the sum of independent non-interacting oscillators

\[
H = \int \frac{k}{2} \left\{ A(k)A^+(k) + A^+(k)A(k) \right\} d^3k.
\]

(3.2)

Since, as it was mentioned in sec.2, the number of spatial domains \( N(k) \) may be a variable quan-
where dependence of the operators upon the quantities \( k \) and \( n \) is connected with the classification of the states of a separate oscillator (the spectrum of the oscillator has the form \( \epsilon(k, n) = kn + \epsilon_0(k) \), where the quantity \( \epsilon_0(k) \) gives the contribution of vacuum fluctuations of the field). In the vacuum state \(|0\rangle\) (which is determined now by \( C(k, n) \mid 0 \rangle = 0 \) field oscillators (and all field observables) are absent. The operator of total energy of the field may be generalized in a natural way as

\[
E = \sum \epsilon(k, n)C^+(k, n)C(k, n).
\]

(3.4)

The connection with the standard field variables may be determined with the help of operators which increase (decrease) the energy of system on \( k \) \([\hat{E}, A^{(+)}(k)]_\epsilon = \pm kA^{(+)}(k)\)

\[
A^+(k) = \sum_{n=0}^{\infty} (n + 1)^{1/2}C^+(k, n + 1)C(k, n),
\]

(3.5)

\[
A(k) = \sum_{n=0}^{\infty} (n + 1)^{1/2}C^+(k, n)C(k, n + 1).
\]

(3.6)

It may be seen from (3.4)-(3.6) that the operators \( A \) and \( A^+ \) satisfy the commutation relations

\[
[A(k), A^+(k')]_\epsilon = N(k)\delta^3(k - k'),
\]

(3.7)

where \( N(k) = \sum_{n=0}^{\infty} C^+(k, n)C(k, n) \) is the complete number of spatial domains related to the wave number \( k \). If one restricts oneself by the states (2.6) with \( N(k) = 1 \), then the operators \( A^+(k) \) and \( A(k) \) will be surely coincided with the standard operators of the creation and annihilation of scalar particles.

As it was mentioned in sec.2, the quantities \( N(k, n) = C^+(k, n)C(k, n) \) must be determined by dynamics. However, they may be estimated under the simple considerations. It is clear that in the absence of the gravitational interaction the quantities \( N(k, n) \) remain constant. Then, for instance, under the assumption of the bounded density \( N < \infty \) of oscillators satisfying the Fermi statistics it is easy to find that the occupation numbers corresponding to the ground state are

\[
N(k, n) = \theta(\mu - \epsilon(k, n)),
\]

(3.8)

where \( \theta(x) = \{ 0 \text{ for } x < 0 \text{ and } 1 \text{ for } x > 0 \} \), and \( \mu \) is determined via the full number of oscillators \( N = \sum N(k, n) \). Using (3.8) one can found the number of oscillators corresponding to the wave vector \( k \)

\[
N(k) = \sum_{n=0}^{\infty} \theta(\mu - \epsilon(k, n)) = [1 + (\mu - \epsilon_0(k))/k],
\]

(3.9)

here \([x]\) denotes the entire part of the number \( x \). In particular, from (3.9) one can see that \( N(k) = 0 \) as \( \mu < \epsilon_0(k) \).

4 Concluding Remarks

For the excited states formed by the acting of the operators \( A^+(k) \) on the ground state (3.8) the operator \( N(k) \) is the usual function (3.9). Let us consider the excitations of the field (scalar particles) described by the thermal equilibrium state corresponding to the temperature \( T \) (one could
where \( \Phi^2(k) = k^2 N(k) \frac{1}{2} cth\left(\frac{k}{2T}\right) \). In the region of the wave numbers \( k \ll (T, \mu) \) the spectrum of fluctuations of the field is occurred to be the scale-independent: \( \Phi^2(k) \approx TkN(k) = T\mu \).

We also notice that the ground state determined by the occupation numbers (3.8) has a bounded energy density of the field which may be considered as a “dark matter”. Besides, we note that the mentioned property of spectrum to be scale-invariant on large scales for the thermal equilibrium state, actually, does not depend on the statistics of the oscillators (i.e. upon the choosing of the sign \( \pm \) in (3.3), (2.5)).

Acknowledgments

I am grateful to A.A.Starobinsky, V.D.Ivashchuk and D.V.Turaev for useful discussions and critics.

References

[1] J.A.Wheeler, in: Relativity Groups and Topology, eds B.S. and C.M. DeWitt, Gordan and Breach, New York.1964.
[2] S.W.Hawking, Nuclear Phys., B114, 349 (1978).
[3] S.W.Hawking, Phys. Rev. D37, 904 (1988).
[4] S.Gidings, A.Strominger, Nucl. Phys. B307, 854 (1988).
[5] G.V.Lavrelashvili, V.A.Rubakov, P.G.Tinyakov, Nucl. Phys. B299, 757 (1988).
[6] S.Coleman, Nucl. Phys. B310, 643 (1988).
[7] V.A.Rubakov. Phys. Lett. B214, 503 (1988).
[8] A.A.Kirillov, Sov. Phys. JETP Lett. 55, 561 (1992).
[9] L.P.Grishchuk and Ya.B.Zeldovich. Quantum Structure of Space and Time. (Cambridge Univ. Press, 1982) p.387; Ya.B.Zeldovich and A.A.Starobinsky, Pis’ma Astron. Zh. 10, 323 (1984) [Sov. Astron. Lett. 10, 135 (1984)].
[10] A.A.Kirillov, Sov. Phys. JETP 76, 355 (1993).
[11] A.A.Kirillov, Int. Jour. Mod. Phys. D3, 1 (1994) (see also A.A.Kirillov, Ph.D. Thesis, Center for Surface and Vacuum Research, Moscow, 1993, [in Russian]).
[12] C.W.Misner, Phys.Rev.186,1319 (1969).