Quantum Hyperbolic State Sum Invariants of 3-Manifolds

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Abstract

Any triple \((W, L, \rho)\), where \(W\) is a compact closed oriented 3-manifold, \(L\) is a link in \(W\) and \(\rho\) is a flat principal \(B\)-bundle over \(W\) \((B\) is the Borel subgroup of upper triangular matrices of \(SL(2, \mathbb{C})\)), can be encoded by suitable distinguished and decorated triangulations \(T = (T, H, D)\).

For each \(T\), for each odd integer \(N \geq 3\), one defines a state sum \(K_N(T)\), based on the Faddeev-Kashaev quantum dilogarithm at \(\omega = \exp(2\pi i/N)\), such that \(K_N(W, L, \rho) = K_N(T)\) is a well-defined complex valued invariant. The purely topological, conjectural invariants \(K_N(W, L)\) proposed earlier by Kashaev correspond to the special case of the trivial flat bundle.

Moreover, we extend the definition of these invariants to the case of flat bundles on \(W \setminus L\) with not necessarily trivial holonomy along the meridians of the link’s components, and also to 3-manifolds endowed with a \(B\)-flat bundle and with arbitrary non-spherical parametrized boundary components. As a matter of fact the distinguished and decorated triangulations are strongly reminiscent of the way one represents the classical refined scissors congruence class \(\hat{\beta}(\mathcal{F})\), belonging to the extended Bloch group, of any given finite volume hyperbolic 3-manifold \(\mathcal{F}\) by using any hyperbolic ideal triangulation of \(\mathcal{F}\). We point out some remarkable specializations of the invariants; among these, the so called Seifert-type invariants, when \(W = S^3\): these seem to be good candidates in order to fully reconstruct the Jones polynomials in the main stream of quantum hyperbolic invariants. Finally, we try to set our results against the heuristic background of the Euclidean analytic continuation of \((2+1)\) quantum gravity with negative cosmological constant, regarded as a gauge theory with the non-compact group \(SO(3, 1)\) as gauge group.

Keywords: quantum dilogarithm, hyperbolic 3-manifolds, state sum invariants.

1 Introduction

In a series of papers [1, 2, 3], Kashaev proposed a conjectural infinite family \(\{K_N\}\), \(N\) being an odd positive integer, of complex valued topological invariants of links \(L\) in any oriented compact closed 3-manifold \(W\), based on the theory of quantum dilogarithms at an \(N\)’th-root of unity \(\omega = \exp(2\pi i/N)\) [4, 5]. These invariants should be computed, in a purely 3-dimensional context, as a state sum \(K_N(T)\), supported by an arbitrary distinguished and decorated triangulation of \((W, L)\), \(T = (T, H, D)\) say, if any. “Distinguished” means that \(L\) is triangulated by a Hamiltonian subcomplex \(H\) of \(T\); the rather complicated decoration \(D\) shall be specified later. The main ingredients of the state sum are the quantum-dilogarithm \(6j\)-symbols suitably associated to the tetrahedra of \(T\). On the “quantum” side, the state sum is similar to the Turaev-Viro one [6, 7]. On the “classical” side, it relates to the computation of the volume of a hyperbolic 3-manifold by the sum of the volumes of the ideal tetrahedra of any of its ideal triangulations.
Beside a somewhat neglected existence problem of such distinguished and decorated triangulations of $(W,L)$, a main question left unsettled is just the invariance of $K_N(T)$ when $T$ varies. On the other hand, Kashaev proved the invariance of $K_N(T)$ under certain “moves” on distinguished and decorated triangulations, which gives some evidence for the conjectured topological invariance.

In [2] he also defined a family $Q_N$ of topological invariants for links $L$ in $S^3$ using the solutions of the Yang-Baxter equation (i.e. the $R$-matrices) derived from the pentagon identity satisfied by the quantum-dilogarithm 6j-symbols [3]. Then he argued that these invariants coincide with the previous ones in the special case when $W = S^3$. More recently, Murakami-Murakami [3] have shown that $Q_N$ actually equals a specific coloured Jones invariant, getting, by the way, another proof that it is a well-defined invariant for links in $S^3$.

Having reformulated Kashaev’s invariants (of links in $S^3$) within the main stream of Jones polynomials has been an important achievement, but it also has the negative consequence of putting aside the original purely 3-dimensional set-up (for links in an arbitrary $W$), willingly forgetting the complicated and somewhat mysterious decorations.

On the contrary, in our opinion, Kashaev’s 3-dimensional set-up deserved to be deeper understood and developed. As a by-product, we shall see in [4] that there are good reasons to believe that it could contain a consistent part of an “exact solution” of the Euclidean analytic continuation of $(2+1)$ quantum gravity with negative cosmological constant, regarded as a gauge theory with the non-compact gauge group $SO(3,1)$ and an action of Chern-Simons type. In fact, the so called Volume Conjecture [5] about the asymptotic behaviour of $Q_N$ when $N \to \infty$ perfectly agrees, for what concerns the “real part”, with the expected “classical limit” of this theory [5, p.77]. In particular, recall that hyperbolic 3-manifolds are the pure-gravity (i.e. empty) Euclidean “classical” solutions.

The initial aim of the present paper was to work out a proof that the original 3-dimensional Kashaev set-up actually produces topological invariants. In fact, after having made it more “flexible” (as is demanded by technical and also conceptual reasons), and having better understood the decoration, we have finally obtained the following more general results that we state here in a somewhat informal way.

**Notations.** We denote by $W$ a compact closed oriented 3-manifold, $L$ is a link in $W$, $U(L)$ is a tubular neighbourhood of $L$ in $W$, and $M = W \setminus \text{Int } U(L)$. So $M$ has $n$ toral boundary components, where $n$ is the number of components of $L$; we denote them by $L_i$, $i = 1, \ldots, n$. Let $Z$ be any compact oriented 3-manifold with non-empty boundary $\partial Z = \bigcup_{i=1}^n \Sigma_i$ consisting of $n$ toral components $T_i$, and $Y$ any compact oriented 3-manifold with non-empty boundary $\Sigma$, so that each of its connected components $\Sigma_i$ has genus $g(\Sigma_i) \geq 1$.

Given $Z$, $S = \{s_i\}$ denotes a system of essential, that is non contractible, simple closed curves, one on each $T_i$.

$B = B(2,\mathbb{C})$ is the Borel subgroup of $SL(2,\mathbb{C})$ of upper triangular matrices, and $\rho$ denotes either an equivalence class of flat principal $B$-bundles on $Z$ (resp. $Y$), i.e. a bundle endowed with a flat connection, or, equivalently, an element of $\chi_B(Z) = \text{Hom}(\pi_1(Z), B)/B$ (resp. $\chi_B(Y) = \text{Hom}(\pi_1(Y), B)/B$), $B$ acting by inner automorphisms. One usually calls $\chi_B(Y)$ the character variety of $Y$ for $B$.

Finally, we denote by $\phi$ a symplectic parametrization of $\Sigma$ (see Section 2 for the precise definition). The triples $(Z, S, \rho)$ and $(Y, \phi, \rho)$ are regarded up to the natural equivalence relation induced by orientation preserving homeomorphisms.

We first consider triples $(Z, S, \rho)$.

**Theorem 1.1** A family of complex valued invariants $\{K_N(Z,S,\rho)\}$, $N$ being any odd positive integer, is defined. The triple $(Z, S, \rho)$ can be encoded by suitably decorated “distinguished” triangulations $(T,H,D)$, and $K_N(Z,S,\rho)$ can be computed as a state sum $K_N(T,H,D)$ whose value is invariant when $(T,H,D)$ varies. The original Kashaev’s purely topological invariants $K_N(W,L)$ correspond to the case when $Z = M$, each $s_i$ is a meridian of $L_i$ and $\rho$ equals the class of the trivial flat bundle, or, equivalently, equals the constant representation $\rho = 1$. 
Remark 1.2 There are two ways to look at \((Z, S, \rho)\), whence at these invariants.

(1) Let \(W\) be the closed manifold obtained by Dehn filling of each boundary component of \(Z\), along the curves of \(S\). If \(L\) denotes the link of surgery “cores”, then \(S\) becomes a family of meridians of the \(L_i\)’s, and \(\rho\) is a flat bundle on \(W \setminus L\) with, in general, a non-trivial holonomy along these meridians. In classical \((2+1)\) gravity an analogous situation arises in presence of closed lines of massive particles of a universe (see for instance [13]).

(2) We are actually considering \(Z\) with parametrized boundary. More precisely, fix a base solid torus \(H_1\) with meridian \(m\). Then each boundary component \(T_i\) of \(Z\) is parametrized by an oriented diffeomorphism \(g_i : \partial H_1 \to T_i\) such that \(g_i(m) = s_i\); \(g_i\) is well-defined up to automorphisms of \(\partial H_1\) which extend to the whole \(H_1\).

The symplectic parametrizations of surfaces cited above are defined in the spirit of Remark 1.2 (2), starting with base handlebodies of genus \(g(\Sigma_i)\) with fixed systems of meridians and so on. Then we also have:

Theorem 1.3 One can define a family of complex valued invariants \(\{K_N(Y, \phi, \rho)\}\), \(N\) being any odd positive integer, which specializes to the previous one when \(\partial Y\) is a union of tori.

Our final set-up shall be slightly different from the original Kashaev one. So we prefer to reach it step by step.

In [1, 2], we shall prove Theorem 1.1 in a particular case: \(Z = M\), each \(s_i\) is a meridian of \(L_i\) and \(\rho\) is defined on the whole of \(W\); i.e. \(\rho\) is trivial along the \(s_i\)’s; that is we actually construct invariants \(K_N(W, L, \rho)\). Note that the original Kashaev invariants already are specializations of them (with \(\rho\) trivial). To treat this case we shall need a milder modification of the original Kashaev set-up. The related existence problem of distinguished and decorated triangulations is solved in [3]. Some basic properties of these invariants are settled in [4] “projective invariance”, and “duality” (which in particular describes the behaviour of the invariants when the orientation of \(W\) changes).

Next, we treat the case when \(\rho\) does not necessarily extend to the whole of \(W\). This is done in [4] through a trick, which consists roughly in cutting up pieces of a distinguished triangulation of \((W, L)\), turning it into a decorated cell decomposition of \(Z\) with tetrahedron-like building blocks, which may be used to again define our previous state sums. Thanks to our definition of parametrized surface \((\Sigma, \phi)\), there is a natural way to recover the set-up used in the proof of Theorem 1.1 from triples \((Y, \phi, \rho)\), thus obtaining Theorem 1.3.

In [5], we consider some remarkable specializations of the invariants \(K_N(Z, S, \rho)\); among these there are the so called Seifert-type invariants of links in \(Z\)-homology spheres. For \(S^3\), they seem to be good candidates in order to fully recover the coloured Jones polynomials in terms of the quantum hyperbolic state sum invariants (generalizing [6]), and to give them a new interpretation. One finds a preliminary discussion at the end of [5] our work on this matter being in progress (cf. [12]).

We stress a qualitative contribution of the present paper in understanding the nature of the distinguished and decorated triangulations: it turns out that they are strongly reminiscent (up to a sort of reduction mod \(N\)) of the way one represents the classical refined scissors congruence class \(\tilde{\beta}(F)\) (belonging to the extended Bloch group \(\tilde{B}(\mathbb{C})\)) of any given finite volume hyperbolic 3-manifold \(\mathcal{F}\), by using any of its hyperbolic ideal triangulations [13]. It is known that the classical Rogers dilogarithm lifts, with its fundamental five-terms identity, to an analytic function \(R\) defined on \(\tilde{B}(\mathbb{C})\), and that

\[
R(\tilde{\beta}(\mathcal{F})) = i(\text{Vol}(\mathcal{F}) + iCS(\mathcal{F})),
\]

where \(CS\) denotes the Chern-Simons invariant [14]. Moreover one knows that the \(N\)-dimensional quantum dilogarithm \(\delta\)-symbols and their pentagonal relations recover respectively, in the limit \(N \to \infty\), the classical Euler dilogarithm and the Rogers five-terms identity (see [4] and, for a detailed account, [14]). All this should indicate the right conceptual framework supporting the so called volume conjecture and natural generalizations, also involving the Chern-Simons invariant; roughly speaking one expects that:

Qualitative Hyperbolic VCS-Conjecture. When \((W, L)\) admits some “hyperbolization” \(\mathcal{F}\) (for instance: \(\mathcal{F} = W \setminus L\) is a complete hyperbolic manifold or \(\mathcal{F} = W\) is hyperbolic and \(L\) is a geodesic
link, as it arises from hyperbolic Dehn surgery, then one can recover $VCS(F) = Vol(F) + iCS(F)$ by means of the asymptotic behaviour for $N \to \infty$ of the invariants $K_N(W, L, \rho)$.

Nevertheless, some substantial facts are not yet well understood, and this matter needs further investigations (see also §3.3, §6 and §7).

Finally in §8 we try to set our results against the heuristic background of the Euclidean analytic continuation of $(2+1)$ quantum gravity with negative cosmological constant, regarded as a gauge theory with the non-compact group $SO(3, 1)$ as gauge group, as defined by Witten [10]. We shall develop a few speculations about a path integral interpretation of our state sums and their relation with the absolute torsions of Farber-Turaev [17].

The aim of §3 is to provide the reader a precise statement of all the facts we use concerning our “quantum data”, allowing a substantially self-contained reading. We shall mostly insist on the geometric interpretation of the basic properties, showing how the decorations we use in this paper do arise. All the statements given without proof in the Appendix are carefully treated in [10].

2 Distinguished singular triangulations of $(W, L)$

Let us recall some facts about standard spines of 3-manifolds and their dual ideal triangulations [13, 14]. A reference is [20], but one finds also a clear discussion about this material in [1] (note that sometimes the terminologies do not agree). Using the Hauptvermutung, we will freely intermingle the differentiable, piecewise linear and topological viewpoints for 3-dimensional manifolds.

Consider a tetrahedron $\Delta$ and let $C$ be the interior of the 2-skeleton of the dual cell-decomposition. A simple polyhedron $P$ is a finite 2-dimensional polyhedron such that each point of $P$ has a neighbourhood which can be embedded into $C$. A simple polyhedron is standard (in [4] one uses the term cellular) if all the components of the natural stratification of $P$ given by singularity are open cells. Depending on the dimension, we call these components vertices, edges and regions of $P$.

Every compact 3-manifold $Y$ (which for simplicity we assume connected) with non-empty boundary $\partial Y$ has standard spines, that is standard polyhedra $P$ embedded in $\text{Int} \ Y$ such that $Y$ collapses onto $P$ (i.e. $Y$ is a regular neighbourhood of $P$). Standard spines of oriented 3-manifolds are characterized among standard polyhedra by the property of carrying an orientation, that is a suitable “screw-orientation” along the edges [21, Def. 2.1.1]. Such an oriented 3-manifold $Y$ can be reconstructed (up to orientation preserving homeomorphisms) from any of its oriented standard spines. From now on we assume that $Y$ is oriented.

A singular triangulation of a polyhedron $Q$ is a triangulation in a weak sense, namely self-adjacencies and multiple adjacencies are allowed. For any $Y$ as above, let us denote by $Q(Y)$ the polyhedron obtained by collapsing each component of $\partial Y$ to a point. An ideal triangulation of $Y$ is a singular triangulation $T$ of $Q(Y)$ such that the vertices of $T$ are precisely the points of $Q(Y)$ which correspond to the components of $\partial Y$.

For any ideal triangulation $T$ of $Y$, the 2-skeleton of the dual cell-decomposition of $Q(Y)$ is a standard spine $P(T)$ of $Y$. This procedure can be reversed, so that we can associate to each standard spine $P$ of $Y$ an ideal triangulation $T(P)$ of $Y$ such that $P(T(P)) = P$. Thus standard spines and ideal triangulations are dual equivalent viewpoints which we will freely intermingle. Note that, by removing small neighbourhoods of the vertices of $Q(Y)$, any ideal triangulation leads to a cell-decomposition of $Y$ by truncated tetrahedra which induces a singular triangulation of the boundary of $Y$.

Remark 2.1 For the following facts, a reference is [21, Ch. E]. The name “ideal triangulation” is inspired by the geometric triangulations of a non compact hyperbolic 3-manifold with finite volume (for example, the complement of a hyperbolic link in $S^3$) by ideal hyperbolic tetrahedra (possibly partially flat). It is a standard result of Epstein-Penner [22] that such triangulations exist. A geometric ideal triangulation can be regarded as a special topological ideal triangulation admitting suitable decorations by the moduli of the corresponding hyperbolic ideal tetrahedra. Then, the volume of the 3-manifold can be expressed as the sum of the volumes of the ideal
tetrahedra of any of its geometric triangulations, and it can be computed in terms of the moduli via the Bloch-Wigner dilogarithm function [23]. Of course the volume, which by Mostow’s rigidity theorem is a topological invariant, does not depend on the specific geometric ideal triangulation used to compute it. In a sense, the quantum hyperbolic state sums we are concerned with can be considered as “quantum” deformations of this “classical” hyperbolic situation, which make sense for arbitrary 3-manifolds.

Consider now our closed 3-manifold $W$. For any $r ≥ 1$ let $W'_r = W \setminus rD^3$, that is the manifold with $r$ spherical boundary components obtained by removing $r$ disjoint open balls from $W$. Clearly $Q(W'_r) = W$ and any ideal triangulation of $W'_r$ is a singular triangulation of $W'$; moreover all singular triangulations of $W$ are obtained in this way. We shall adopt the following terminology.

**Definition 2.2** A singular triangulation of $W$ is simply called a triangulation. Ordinary triangulations (where neither self-adjacencies nor multi-adjacencies are allowed) are said to be regular. An almost-regular triangulation of $W$ is a triangulation having the same vertices as a regular one. This is equivalent to saying that there exists a regular triangulation of $W$ with the same number of vertices. Given a triangulation $T$, $r_i = r_i(T)$, $i = 0, 1, 2, 3$, shall denote the number of vertices, edges, faces, tetrahedra of $T$.

The main advantage in using singular triangulations (standard spines) instead of only ordinary triangulations consists of the fact that there exists a finite set of moves which are sufficient in order to connect (by means of finite sequences of these moves) singular triangulations (standard spines) of the same manifold. On the contrary, if we pretend to keep ordinary triangulations at each steps, we are forced to consider an infinite set of moves (see [5]).

Let us recall two elementary moves on triangulations (spines) that we shall use throughout the paper; see Fig. [1].

**The 2 → 3 move.** Replace the triangulation $T$ of a portion of $Q(Y)$ made by the union of 2 tetrahedra with a common 2-face $f$ by the triangulation made by 3 tetrahedra with a new common edge which connect the two vertices opposite to $f$.

**The 1 → 4 move.** Add a new vertex in the interior of a tetrahedron $∆$ of $T$ and make from it the cone over the triangulated boundary of $∆$. The dual spine $P'$ of the triangulation $T'$ thus obtained is a spine of $Y \setminus D^3$, where $D^3$ is an open ball in the interior of $Y$.

The 2 → 3 and 3 → 2 moves can be easily reformulated in dual terms (see for instance [24, p.15], where they are called “MP-moves”; in Fig. [3] we show their branched versions). Standard spines of the same $Y$ with at least two vertices (which, of course, is a painless requirement) may always be connected by the (dual) move $2 → 3$ and its inverse. In particular, any almost-regular triangulation of $W$ can be obtained from a regular one via a finite sequence of $2 → 3$ or $3 → 2$ moves.

In order to handle singular triangulations of a closed manifold $W$, we also need a move which allows us to vary the number of vertices. Although this is not the shortest way (the so-called bubble move makes a hole in any $Y$ by introducing only two more vertices, see Proposition 8.11), the $1 → 4$ move (and its inverse) shall be convenient for our purposes. Let us describe the dual spine $P'$ of $Y \setminus D^3$. Consider the vertex $v_0$ of $P$ (which is dual to $T$) contained in $∆$. We are removing a ball around of $v_0$. The boundary $S^2(v_0)$ of this ball is a portion of $P'$ and the natural cell decomposition of $P'$ induces on it a cell decomposition which is isomorphic to the “bi-dual” tetrahedron of $v_0$; that is it is isomorphic to the cell decomposition of $∂∆$ dual to the standard triangulation.

For technical reasons we shall need a further move which we recall in Fig. [3] in terms of standard spines and that we denote by $0 → 2$ move. It is also known as lune move and is somewhat similar to the second Reidemeister move on link diagrams. Note that the inverse of the lune move is not always admissible because one could lose the standardness property when using it. However we shall need only the “positive” lune move.

The following technical result due to Makovetskii [24] shall be necessary.

**Proposition 2.3** Let $P$ and $P'$ be standard spines of $Y$. Then there exists a spine $P''$ of $Y$ such that $P''$ can be obtained from both $P$ and $P'$ via a finite sequence of $0 → 2$ and $2 → 3$ moves (we stress that they are all positive moves).
Definition 2.4 A distinguished triangulation $(T, H)$ of $(W, L)$ has the property that $H$ is Hamiltonian, that is at each vertex of $T$ there are exactly two “germs” of edges of $H$. Of course, the two germs could belong to the same edge of $H$.

It is convenient to give a slightly different description of the distinguished singular triangulations in terms of spines.

Definition 2.5 Let $Y$ be as before. Let $S$ be any finite family of $r$ disjoint simple closed curves on $\partial Y$. We say that $Q$ is a quasi-standard spine of $Y$ relative to $S$ if:

(i) $Q$ is a simple polyhedron with boundary $\partial Q$ consisting of $r$ circles. These circles bound (unilaterally) $r$ annular regions of $Q$. The other regions are cells.

(ii) $(Q, \partial Q)$ is properly embedded in $(Y, \partial Y)$ and transversely intersects $\partial Y$ at $S$.

(iii) $Q$ is a spine of $Y$.

Note that $\tilde{P} = Q \setminus \{\text{annular regions}\}$ is a simple spine of $Y$.

Lemma 2.6 Let $Y$ and $S$ be as before. Quasi-standard spines of $Y$ relative to $S$ do exist.

Proof. Let $\tilde{P}$ be any standard spine of $Y$. Consider a normal retraction $r : Y \to \tilde{P}$. Recall that $Y$ is the mapping cylinder of $r$; for each region $R$ of $\tilde{P}$, $r^{-1}(R) = R \times I$; for each edge $e$, $r^{-1}(e) = e \times \{a \text{ “tripode”}\}$; for each vertex $v$, $r^{-1}(v) = \{a \text{ “quadripode”}\}$. We can assume that $S$ is in “general position” with respect to $r$, so that the mapping cylinder of the restriction of $r$ to $S$ is a simple
spine of $Y$ relative to $S$ (with the obvious meaning of the words); possibly after doing some $0 \to 2$ moves we then obtain a quasi-standard $Q$.

**Definition 2.7** Consider $Y = M$ and $S$ formed by the union of $r_i \geq 1$ parallel copies of the meridian $m_i$ of the component $L_i$ of $L$, $i = 1, \ldots, n$. A spine of $M$ adapted to $L$ of type $(r_1, \ldots, r_n)$ is a quasi-standard spine of $M$ relative to such an $S$.

It is clear that Proposition 2.3 extends to the “adapted” setting. In particular, we have:

**Proposition 2.8** Let $P$ and $P'$ be quasi-standard spines of $Y$ relative to $S$ and of type $r = (r_1, \ldots, r_n)$. Then there exists a spine $P''$ of $Y$ relative to $S$ and of type $r$ such that $P''$ can be obtained from both $P$ and $P'$ via a finite sequence of $0 \to 2$ and $2 \to 3$ moves, and at each step we still have spines of $Y$ adapted to $S$ and of type $r$.

**Remark 2.9** If $Q$ is a spine of $M$ adapted to $L$ as before, then by filling each boundary component of $Q$ by a 2-disk, we get a standard spine $P = P(Q)$ of $W$, $r = \sum r_i$, and the dual triangulation $T(P)$ of $W$ naturally contains $L$ as a Hamiltonian subcomplex; that is we have obtained a distinguished triangulation of $(W, L)$. Vice-versa, starting from any $(T, H)$, by removing an open disk in the dual region to each edge of $H$, we pass from $P = P(T)$ to a spine $Q = Q(P)$ of $M$ adapted to $L$, of some type. So they are equivalent viewpoints.

As an immediate corollary we have:

**Corollary 2.10** For any $r \geq n$ there exist distinguished triangulations of $(W, L)$ with $r$ vertices.

Let us analyze the moves on distinguished triangulations. The $2 \to 3$ and $1 \to 4$ moves specialize to moves $(T, H) \to (T', H')$ between distinguished triangulations as follows.

Let us denote by $T \to T'$ any such a move. If $(T, H)$ is a distinguished triangulation of $(W, L)$, we want to get moves $(T, H) \to (T', H')$. In the $2 \to 3$ case there is nothing to do because $H' = H$ is still Hamiltonian. In the $1 \to 4$ case, we assume that an edge $e$ of $H$ lies in the boundary of the involved tetrahedron $\Delta$: $e$ lies in the boundary of a unique 2-face $f'$ of $T'$ containing the new vertex. Then we get the Hamiltonian $H'$ just by replacing $e$ by the other two edges of $f'$. We have also:

**Lemma 2.11** Let $(T, H)$ be a distinguished triangulation of $(W, L)$ and $T \to T'$ a $0 \to 2$ move. Then it can be completed to a move $(T, H) \to (T', H')$.

**Proof.** This is clear if we think in dual terms. If $Q$ is a spine adapted to $L$ and $Q \to Q'$ is the $0 \to 2$ move, the dual regions in $P(Q)$ to the edges in $H$ “persist” in $P(Q')$ so we find $H'$.

Finally we can solve half of the existence problem mentioned in the introduction.

**Proposition 2.12** There exist almost-regular distinguished triangulations of $(W, L)$.

**Proof.** It is enough to remark that any distinguished triangulation, which exists by Corollary 2.10, can be made almost-regular after a finite number of $1 \to 4$ moves.

### 3 Decorations

In this section we have to properly define, and possibly better understand, the decorations $D = (h, z, c)$ of distinguished triangulations of $(W, L)$. 
3.1 Branchings

Let $P$ be a standard spine of $Y$ and consider as usual the dual ideal triangulation $T = T(P)$. A branching $b$ of $T$ is a system of orientations on the edges of $T$ such that each “abstract” tetrahedron of $T$ has one source and one sink on its 1-skeleton. This is equivalent to saying that, for any 2-face $f$ of $T$, the edge-orientations do not induce an orientation of the boundary of $f$.

In dual terms, a branching is a system of orientations on the regions of $P$ such that for each edge of $P$ we have the same induced orientation only twice. In particular, note that each edge of $P$ has an induced orientation.

In the original set-up of [1] one used regular triangulations $T$ of $W$, with a given total ordering on the set of vertices; in fact the role of the ordering is just to define a branching via the natural rule: “on each edge, go from the smaller vertex towards the bigger one”. Note that one could not exclude, a priori, that after some negative moves, one eventually reaches veritable singular triangulations, for which such a total ordering no longer induces a branching.

Branchings, mostly in terms of spines, have been widely studied in [20] (see also [21]). They are rich structures: a branching of $P$ allows us to give the spine the extra structure of an embedded and oriented (hence normally oriented) branched surface in $\text{Int}(Y)$; by the way, this also justifies the name. Moreover a branched $P$ carries a suitable positively transverse combing of $Y$.

We recall here part of their combinatorial content. A branching $b$ allows to define an orientation on any cell of $T$, not only on the edges. Indeed, consider any “abstract” tetrahedron $\Delta$. For each vertex of $\Delta$ consider the number of incoming $b$-oriented edges in the 1-skeleton. This gives us an ordering $b_\Delta : \{0, 1, 2, 3\} \to V(\Delta)$ of the vertices which reproduces the branching on $\Delta$, according to the former rule. This gives us a base vertex $v_0 = b_\Delta(0)$ and an ordered triple of edges emanating from $v_0$, whence an orientation of $\Delta$. Note that this orientation may or may not agree with the orientation of $Y$: in the first case we say that $\Delta$ is of index $-1$, and it is of index 1 otherwise.

To orient 2-faces we work in a similar way on the boundary of each “abstract” 2-face $f$. We get an ordering $b_f : \{0, 1, 2\} \to V(f)$, a base vertex $v_0 = b_f(0)$, and finally an orientation of $f$. This 2-face orientation can be described in another equivalent way. Let us consider the 1-cochain $s_b$ such that $s_b(e) = 1$ for each $b$-oriented edge. Then there is a unique way to orient any 2-face $f$ such that the coboundary $\delta s_b(f) = 1$.

The corresponding dual orientation on the edges of $P$ is just the induced orientation mentioned in [3].

**Branching’s existence and transit.** This matter is carefully analyzed in [21]. In general, a given ideal triangulation of $Y$ could admit no branching, but there exist branched ideal triangulations of any $Y$. More precisely, given any system of edge-orientations $g$ on $T$ and any move $T \to T'$, a transit $(T, g) \to (T', g')$ is given by any system $g'$ of edge-orientations on $T'$ which agrees with $g$ on the “common” edges. In [21] Th. 3.4.9] one proves:

**Proposition 3.1** For any $(T, g)$ there exists a finite sequence of 2 $\to$ 3 transits such that the final $(T', g')$ is actually branched.

If $(P, b)$ is a branched spine and $P \to P'$ is either a $2 \to 3$ or a $0 \to 2$ move, then it can be completed (sometimes in a unique way, sometimes in two ways) to a branched transit $(P, b) \to (P', b')$. On the contrary, it could happen that a $3 \to 2$ or $2 \to 0$ inverse move is not “branchable” at all (see [21, Ch.3]). However we shall only use the “positive” moves. Note that otherwise it could stop any attempt to prove the invariance of the quantum hyperbolic state sums, via any argument of “move-invariance”. In Fig. 4 and Fig. 5 we show the 2 $\to$ 3 branched transits.

In fact the list is not complete, but one can see in the figures all the essentially different behaviours, and easily complete the list by applying evident symmetries. Following [20], we can distinguish two quite different kinds of branched transits: the sliding moves which actually preserve the combing, and the bumping moves which eventually change it. For the $0 \to 2$ moves there is a similar behaviour (see Fig. 4).

However, we shall not exploit this difference in the present paper. Note that the middle sliding move in Fig. 4 corresponds dually to the triangulation move shown in Fig. 4.
Given a distinguished triangulation \((T, H)\) of \((W, L)\), the first component of \(D\) is just a branching \(b\) of \(T\). If \((T, H) \rightarrow (T', H')\) is either a \(2 \rightarrow 3\), \(0 \rightarrow 2\) or \(1 \rightarrow 4\) move, we have again some branched transit \((T, H, b) \rightarrow (T', H', b')\); we have already described it in the first two cases. In the last case there are several ways to take \(b'\) which agree with \(b\) on the edges “already” present in \(T\). Anyone of these ways is a possible transit.

By combining the existence of distinguished triangulations with the above proposition we have:

**Lemma 3.2** There exist almost-regular branched distinguished triangulations \((T, H, b)\) of \((W, L)\).

**Remark 3.3** Branching’s rôle. To understand the rôle of the branching, let us go back to the “classical” hyperbolic ideal triangulations (see Remark 2.1). A hyperbolic ideal tetrahedron \(\Delta\) with ordered vertices \(v_0, \ldots, v_3\) on the Riemann sphere \(\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}\) is determined up to congruence by the cross ratio \(z = [v_0 : v_1 : v_2 : v_3] = \frac{v_2 - v_1}{v_2 - v_0} \frac{v_3 - v_0}{v_3 - v_1}\). If the tetrahedron is positively oriented \(z\) belongs to the upper half plane of \(\mathbb{C}\). Changing the ordering by even permutations produces the cyclically ordered triple \((z, 1/(1 - z), (z - 1)/z)\), which is the actual modular triple of \(\Delta\). Changing the orientation corresponds to considering the complex conjugate triple.

A branching \(b\) on the ideal triangulation allows one to choose, in a somewhat globally coherent way, for each \(\Delta\), a complex valued \(z_\Delta\) in the corresponding triple. In the usual decoration of the edges of \(\Delta\) by means of the elements of the triple, \(z_\Delta\) is associated to the first edge emanating from the base vertex \(v_0(\Delta)\), and so on, respecting the orders of the triple and of the edges.

Starting with a topological ideal triangulation of a 3-manifold \(Z\), one usually tries to make it “geometric” (proving by the way that \(Z\) is hyperbolic) by solving some suitable system of hyperbolicity equations in \(r_3\) complex indeterminates, \(r_3\) being the number of tetrahedra. Then, a
branching allows us to specify which system of equations, and this last is governed by a certain global coherence.

Such a kind of global coherence shall be important to control the behaviour of our state sums up to decorated moves. On the other hand, it is not too surprising, in the spirit of the classical situation, that the value of the state sums shall not eventually depend on the branching of the decoration.

3.2 Full cocycles

Recall that we are actually considering a triple \((W, L, \rho)\) where \(\rho\) is an equivalence class of flat principal \(B(2, \mathbb{C})\)-bundles on \(W\). Set \(B = B(2, \mathbb{C})\).

Let \((T, H, b)\) be a branched distinguished triangulation of \((W, L)\). Let \([z]\) denote the equivalence class of a cellular 1-cocycle \(z\) on \(T\) via the usual equivalence relation up to coboundaries: since a (cellular) 0-cochain \(\lambda\) is a \(B\)-valued function defined on the vertices \(V(T)\) of \(T\), \(z\) and \(z'\) are equivalent if they differ by the coboundary of some 0-cochain \(\lambda\). This means that for any \(b\)-oriented edge \(e\) with ordered end-points \(v_0, v_1\), one has \(z'(e) = \lambda(v_0)^{-1}z(e)\lambda(v_1)\). We denote the quotient set by \(H^1(T; B)\); recall that it can be identified with the set of equivalence classes of flat \(B\)-principal bundles on \(W\).

The second component \(z\) of \(D\) is a \(B\)-valued full 1-cocycle on \(T\) representing \(\rho\); we write \([z] = \rho\). For each \(b\)-oriented edge \(e\) of \(T\), \(z(e)\) is an upper triangular matrix; we denote by \(x(e) \in \mathbb{C}\) the upper-diagonal entry of this matrix. “Full” means that for each \(e\), \(x(e) \neq 0\).

**Remark 3.4** The fullness property strictly concerns the cocycle \(z\) and not its class \([z]\). It only depends on \(T\) whether a given class could be represented by full cocycles. Moreover, fullness does not depend on the branching \(b\). In fact we can define this notion by using any arbitrary system of edge-orientations.

We refer to §3 for more details and examples of \(B\)-bundles. Here we simply recall that there are two distinguished abelian subgroups of \(B\), and this fact induces some distinguished kinds of cocycles. They are:

1. the Cartan subgroup \(C = C(B)\) of diagonal matrices; it is isomorphic to the multiplicative group \(\mathbb{C}^*\). The evident isomorphism \(C \rightarrow \mathbb{C}^*\) maps \(A = (a_{ij}) \in C\) to \(a_{11}\);
2. the parabolic subgroup \(Par(B)\) of matrices with double eigenvalue 1; it is isomorphic to the additive group \(\mathbb{C}\). The evident isomorphism maps \(A = (a_{ij}) \in Par(B)\) to \(a = a_{12}\).

Denote by \(G\) any such subgroup. We get a map \(H^1(T; G) \rightarrow H^1(T; B)\), where \(H^1(T; G)\) is endowed with the natural abelian group structure. Note that \(H^1(T; Par(B)) = H^1(T; \mathbb{C})\) is isomorphic to the ordinary (singular or de Rham) 1-cohomology of \(W\).

A complex valued injective function \(u\) defined on the set of vertices of a regular distinguished triangulation \((T, H)\) of \((W, L)\) (as in the original set-up of [1]) should be regarded as a \(Par(B)\)-valued 0-cochain. Its coboundary \(z = \delta u\) is a basic example of a full cocycle representing \(0 \in H^1(T; \mathbb{C})\) (whence the trivial flat bundle in \(H^1(T; B)\)).

**Existence and transit of full cocycles.** This is a somewhat delicate matter. Let \(T\) be a triangulation of \(W\), with any edge-orientation system \(g\). Let \(z\) be a 1-cocycle on \((T, g)\). Consider any transit \((T, g) \rightarrow (T', g')\) as before. In the \(2 \rightarrow 3\) or \(0 \rightarrow 2\) cases (and their inverses), there is a unique \(z'\) on \((T', g')\) which agrees with \(z\) on the common edges. This defines a transit \((T, g, z) \rightarrow (T', g', z')\). Clearly \([z] = [z']\).

For \(1 \rightarrow 4\) moves \((T, g) \rightarrow (T', g')\), there is an infinite set of possible transits such that \([z] = [z']\) and the cocycles agree on the common edges. Moreover, given one \((T, g, z) \rightarrow (T', g', z')\) \(1 \rightarrow 4\) transit with \(z\) a full cocycle, we can always turn \(z'\) into an equivalent full \(z''\), which differs from \(z'\) by the coboundary of some 0-cochain with support consisting of the new vertex \(v\) of \(T'\). Hence for \(1 \rightarrow 4\) moves there always exists an infinite set of full transits.

Assume now that \(z\) is full, and consider \(2 \rightarrow 3\) or \(0 \rightarrow 2\) moves (or their inverses). The trouble is that, in general, \(z'\) is no longer full. If \((T, H, b) \rightarrow (T', H', b')\) is a branched distinguished transit and \(z\) is full on \((T, b)\), then we have a completion to a full transit \((T, H, b, z) \rightarrow (T', H', b', z')\) only if also the final \(z'\) is full. Otherwise we must stop.

However almost-regular triangulations have generically a good behaviour with respect to the existence and transit of full cocycles.
Proposition 3.5 Let \((T, g)\) be an almost-regular triangulation of \(W\) endowed with any edge-orientations system \(g\). Let \((T, g) = (T_1, g_1) \rightarrow \cdots \rightarrow (T_s, g_s) = (T', g')\) be any finite sequence of 2 ↔ 3 or 0 ↔ 2 orientation-transits. Then there exists a sequence \(U_i\) of open dense sets of full cocycles on \(T_i\) (in the natural subspace topology of \(B^*(T_i)\)), such that \(U_{i+1}\) is contained in the image of \(U_i\) via the elementary transit \(T_i \rightarrow T_{i+1}\), and each class \(\alpha \in H^1(T_i; B)\) can be represented by cocycles in \(U_i\).

As an immediate corollary (use the case \(s = 1\)) we make a further step towards a solution of the existence problem:

Corollary 3.6 Given a triple \((W, L, \rho)\), then a partially decorated (even almost-regular) distinguished triangulation \((T, H, b, z)\) do exists.

Proof of the proposition. As \(T\) is almost-regular, there exists a regular triangulation \(T''\) and a sequence \((T'', g'') \rightarrow \cdots \rightarrow (T, g)\) of 2 ↔ 3 or 0 ↔ 2 orientation-transits. It is enough to prove the proposition for \(T''\), so let us assume that \(T\) is regular. The conclusion of the theorem holds for \(s = 1\). In fact we can suitably perturb any \(z\) by the coboundaries of 0-cochains, see the discussion above. Now we remark that each elementary cocycle-transit \(T_i \rightarrow T_{i+1}\) can be regarded as an algebraic bijective map from the space of 1-cocycles on \(T_i\) to the space of 1-cocycles on \(T_{i+1}\). The set of full cocycles for which the full elementary transit fails are contained in a proper algebraic subvariety. So the conclusion follows, working by induction on \(s\).

Full cocycles rôle. Let \(z\) be the full cocycle of a decoration \(D\). For each edge \(e\) denote by \(t(e)\) the (11)-entry of \(z(e)\): \(x(e)\) is as before. Fix a determination \(d\) of the \(N\)'th-root holding for all entries of \(z(e)\), for all \(e\). Consider the Weil algebra \(W_N\), which is described in details in §3. Then we can associate to each \(e\) the \(N\)-dimensional standard representation \(r(e)\) of \(W_N\) characterized by the pair \((t(e)^{1/N}, x(e)^{1/N}) = (s(e), y(e))\).

This system of representations \(\{r(e)\}\) satisfies the following properties:

1. Consider any \(b\)-oriented “abstract” 2-face \(f\) of \(T\). Starting from the base vertex \(v_0(f)\), according to the orientation, we find a cycle of edges \(e_1, e_2, e_3\). The first two are positively \(b\)-oriented. The last one has the negative orientation. Then the cocycle condition on \(z\) implies that

\[ r(e_3) \text{ is, up to isomorphism, the unique irreducible summand of the representation } r(e_1) \otimes r(e_2). \]

2. Consider, in particular, the 2-face \(f\) opposite to the base vertex \(v_0(\Delta)\), in any tetrahedron \(\Delta\) of \(T\). For each \(e_i\) of \(f\), let us denote by \(e_o(i)\) the opposite edge in \(\Delta\). Then the \(y\)-components of the representations \(r(e_i), r(e_o(i))\) satisfy the following Fermat relation:

\[ y(e_3)^N y(e_o(3))^N = y(e_1)^N y(e_o(1))^N + y(e_2)^N y(e_o(2))^N. \]

3. The same conclusions hold for any other branching \(b'\) on \(T\), as they only depend on the cocycle condition.

After a full transit (once also the charge-transit shall be ruled out), we observe that:

The system of representations \(\{r(e)\}\) varies exactly in the way one needs in order to apply the fundamental algebraic identities (the pentagon, orthogonality, and bubble relations of the Appendix) satisfied by the quantum-dilogarithm c-6j-symbols.

One could ask if systems of representations \(r = \{r(e)\}\) verifying the above properties are more general than the one obtained starting from full cocycles. In fact, setting \(t(e) = s(e)^N\) and \(x(e) = y(e)^N\) one obtains a full cocycle \(z\). In a sense the system \(\{r(e)\}\) can be considered as a sort of reduction mod \(N\) of \(z\). In order to study any “classical” limit, when \(N \rightarrow \infty\), of our state sums invariants, it seems quite appropriate to consider the reductions of the same cocycle.

3.3 Charges

The “classical” source of the charges \(c\) in the decorations \(D\) clearly emerges from Neumann’s work on the Cheeger-Chern-Simons classes of hyperbolic 3-manifolds and scissors congruences of
hyperbolically polytopes. Since there is a wide literature on this subject (see the references in [13]), we shall only report a few details.

**Refined Scissors Congruence.** From the work of [23] and [24], we know that the volume of any oriented hyperbolic 3-manifold $F$ has a deep analytic relationship with another geometric invariant, the Chern-Simons invariant $CS(F)$. Recall that $CS(\cdot)$ is a $\mathbb{R}/2\pi^2\mathbb{Z}$-valued invariant defined for any oriented compact Riemannian 3-manifold, and that its definition can also be extended, with value in $\mathbb{R}/\pi^2\mathbb{Z}$, to non-compact complete and finite volume hyperbolic 3-manifolds [27]. Consider then

$$VCS(F) = Vol(F) + iCS(F).$$

It turns out that $VCS(F)$ (for any finite volume $F$), actually depends on a weaker sub-structure of the full hyperbolic structure, called the refined scissors congruence class and denoted by $\hat{\beta}(F)$ [13, 14]; it is orientation sensitive and takes values in the extended Bloch group $B(\mathbb{C})$. If $F$ is non-compact, this class may be represented with the help of the usual geometric ideal triangulations $T$; if $F$ is compact, we assume here that it is obtained by hyperbolic Dehn surgery on some non-compact one, so that we have some deformed geometric ideal triangulations $T$ of $F \setminus \mathcal{G}$, where $\mathcal{G} = \{L_i\}$ is a finite set of simple closed geodesics in $F$. Then, by abuse of notations, in both cases we shall denote these special decompositions of $F$ by $T$ and call them “ideal triangulations of $F$”.

For the sake of clarity, let us present the construction of $\hat{\beta}(F)$. It relies heavily on the functional properties of the classical Rogers dilogarithm, defined on $\mathbb{C} \setminus (]-\infty,0]\cup[1,\infty[)$ by:

$$R(z) = \frac{1}{2} \log(z) \log(1 - z) - \int_0^z \frac{\log(1 - t)}{t} dt - \pi^2/6,$$

and in particular on its five-term identity, which reads:

$$R(x) + R(y) - R(xy) = R\left(\frac{x(1 - y)}{1 - xy}\right) + R\left(\frac{y(1 - x)}{1 - xy}\right).$$

One then starts with a four components non connected covering of $\mathbb{C} \setminus \{0,1\}$:

$$\mathbb{C} = X_{00} \cup X_{10} \cup X_{01} \cup X_{11}.$$

It can be regarded as the Riemann surface for the collection of all branches of the functions $(\log(z) + p\pi i, -\log(1 - z) + q\pi i)$ on $\mathbb{C} \setminus \{0,1\}$, $(p,q) \in \mathbb{Z} \times \mathbb{Z}$. If $P$ is obtained by splitting $\mathbb{C} \setminus \{0,1\}$ along the rays $(-\infty,0)$ and $(1,\infty)$, $\mathbb{C}$ is a suitable identification space from $P \times \mathbb{Z} \times \mathbb{Z}$; namely, $X_{z_1,z_2} = \{[z,p,q] \in \mathbb{C} \mid p \equiv \epsilon_1 \text{ (mod } 2), q \equiv \epsilon_2 \text{ (mod } 2)\}$. The map

$$l([z,p,q]) = (\log(z) + p\pi i, -\log(1 - z) + q\pi i, \log(1 - z) - \log(z) - (p + q)\pi i)$$

is well-defined on $\mathbb{C}$, and it gives an identification between $\mathbb{C}$ and the set of triples of the form

$$(w_0, w_1, w_2) = (\log(z) + p\pi i, \log(z') + q\pi i, \log(z'') + r\pi i)$$

with

$$p, q, r \in \mathbb{Z} \text{ and } \sum_i w_i = 0,$$

and for some determination of the log function, where $(z, z', z'')$ is a modular triple for an ideal tetrahedron $\Delta$; in other words we adjust its dihedral angles by means of multiples of $\pi$ so that the resulting angle sum is zero. Such a triple is called a combinatorial flattening of the ideal tetrahedron. So $\mathbb{C}$ can be regarded as the set of these combinatorial flattenings, for all ideal tetrahedra of the hyperbolic space $\mathbb{H}^3$.

The Rogers dilogarithm lifts to an analytic function:

$$R : \mathbb{C} \to \mathbb{C}/\pi^2\mathbb{Z},$$

$$R([z,p,q]) = R(z) + \frac{i\pi}{2}(p\log(1 - z) + q\log(z)),$$
and it extends to the free \( \mathbb{Z} \)-module \( \mathbb{Z}[\tilde{\mathbb{C}}] \). Moreover, one may lift the classical five-term identity satisfied by the Rogers dilogarithm to the function \( R \), provided that we pass to some quotient of \( \mathbb{Z}[\mathbb{C}] \). There is such a natural maximal one \( \mathcal{P}(\mathbb{C}) \), which is called the extended pre-Bloch group. In this way, we can turn \( R : \mathcal{P}(\mathbb{C}) \to \mathbb{C}/\pi^2 \mathbb{Z} \) into a homomorphism.

Take our finite volume oriented hyperbolic 3-manifold \( F \); for any ideal triangulation \( T \) of \( F \), one can define

\[
\tilde{\beta}(T) = \sum_{\Delta \in T} \eta_\Delta[z, p, q](\Delta) \in \mathcal{P}(\mathbb{C}),
\]

where \( \eta_\Delta \in \{1, -1\} \) depends on the orientation of \( \Delta \in T \). When \( F \) is non-compact \( \tilde{\beta}(T) \) just represents the (refined) scissors congruence class \( \tilde{\beta}(F) \) of \( F \). In the compact case, how to explicitly represent \( \tilde{\beta}(F) \) is a subtler fact; anyway the above \( \tilde{\beta}(T) = \tilde{\beta}(F, \mathcal{G}) \) is adequate to represent a “classical” counterpart of our state sums, which in general depend also on the link and not only on the ambient manifold.

**Remark 3.7** Consider the map:

\[
\tilde{\delta} : \mathcal{P}(\mathbb{C}) \to \mathbb{C} \wedge \mathbb{C}
\]

\[
[z, p, q] \mapsto (\log(z) + ip\pi) \wedge (-\log(1 - z) + iq\pi).
\]

It lifts the Dehn invariant of the classical scissors congruence \([13]\). Then \( \tilde{\mathcal{B}}(\mathbb{C}) = \text{Ker} \tilde{\delta} \) is called the extended Bloch group, and one can show that \( \tilde{\beta}(F) \) or \( \tilde{\beta}(F, \mathcal{G}) \) are in \( \tilde{\mathcal{B}}(\mathbb{C}) \).

The classes \( \tilde{\beta}(F) \) or \( \tilde{\beta}(F, \mathcal{G}) \) may be seen as representatives of the fundamental class of \( F \) in the (discrete) homology group \([14, 15]\)

\[
H^1_{\text{d}}(\text{PSL}(2, \mathbb{C}); \mathbb{Z}),
\]

which itself maps surjectively onto \( \tilde{\mathcal{B}}(\mathbb{C}) \) \([14]\) (see the definition in Remark \( \tilde{\beta} \)). The relations in \( \tilde{\mathcal{P}}(\mathbb{C}) \) express the 2 \( \to \) 3 move for ideal hyperbolic tetrahedra endowed with combinatorial flattenings, and then they give the independence of \( \tilde{\beta}(T) \) from \( T \).

The relations in \( \tilde{\mathcal{P}}(\mathbb{C}) \) allow to have a global control on the values of the combinatorial flattenings, and this is made easier by the presence of a branching \( b \) of \( T \). For instance, note that a, let us say, \( X_{00} \)-flattening of a \( \Delta \) presupposes a choice of opposite edges; changing this choice turns it into a \( X_{10} \) or \( X_{01} \)-flattening.

Anyway, the point here is that part of this global coherence is expressed in terms of relations which must be satisfied by the \textit{integral} components \( (p, q) \) of the flattenings \( \{[z, p, q](\Delta)\} \), which then give, by definition, an \textit{integral charge} on \( T \). These relations are strongly reminiscent of “the” system of hyperbolicity equations (for instance the one specified by the branching) which is satisfied by the arguments of the modular triples of the ideal triangulation. The charge \( c \) of a decoration \( \mathcal{D} \) shall be, essentially, the \textit{reduction} mod \( N \) of an integral charge. In \([\tilde{\beta}]\) we shall return on these scissors congruence classes and on their relationship with the VCS-invariant.

**Integral charges.** We shall now give the formal definition of the \textit{integral charges} in our own setting. It is a straightforward adaptation of the integral charges of the previous paragraph.

Let \((T, H)\) be a distinguished triangulation of \((W, L)\). With the notations of Definitions \([23]\) and \([24]\) let us assume for simplicity that the associated \textit{simple} spine \( \tilde{P} \) of \( M \) is in fact a standard one, as in the proof of Lemma \([25]\). The following discussion could be adapted also to the case when \( \tilde{P} \) is merely simple; anyway, we could also add the \( \tilde{P} \) standardness assumption to our set-up without any substantial modification in all our arguments.

We know that the truncated tetrahedra of \( T(\tilde{P}) \) induce a triangulation \( \tau \) of \( \partial M \). Let \( s \) be an oriented simple closed curve on \( \partial M \) in general position with respect to \( \tau \). We say that \( s \) has no back-tracking with respect to \( \tau \) if it never departs a triangle of \( \tau \) across the same edge by which it entered. Thus each time \( s \) passes through a triangle, it selects the vertex between the entering and departing edges, and gives it a sign \( \in \{1, -1\} \), according as it goes past this vertex positively or negatively with respect to the boundary orientation.
Let $s$ be a simple closed curve in $M$ in general position with respect to the ideal triangulation $T(\tilde{P})$. We say that $s$ has no back-tracking with respect to $T(\tilde{P})$ if it never departs a tetrahedron of $T(\tilde{P})$ across the same 2-face by which it entered. Thus each time $s$ passes through a tetrahedron, it selects the edge between the entering and departing faces.

Fix in each component of $\partial M$ one of the meridians in the boundary of $Q$, $m_i$ say, and a simple closed curve $l_i$ in general position with respect to $\tau$, which intersects $m_i$ transversely at one point. Orient $m_i$ and $l_i$. We can assume that these curves have no back-tracking with respect to $\tau$. It is clear that any class in $H_1(\partial M; \mathbb{Z})$ may be represented in the set $\{m_i, l_i\}$ of isotopy classes of curves without back-tracking with respect to $\tau$ and generated by $m_i$ and $l_i$. In the same manner, any class in $H_1(M; \mathbb{Z}/2\mathbb{Z})$ may be represented up to isotopy by a curve in $M$ without back-tracking with respect to $T(\tilde{P})$.

Denote by $E_{\Delta}(T)$ the set of all edges of all “abstract” tetrahedra of $T$; there is a natural map $\epsilon : E_{\Delta}(T) \rightarrow E(T)$.

Definition 3.8 An integral charge on $(T, H)$ is a map $$c' : E_{\Delta}(T) \rightarrow \mathbb{Z}$$
which satisfies the following properties:

(1) For each 2-face $f$ of any abstract $\Delta$ with edges $e_1, e_2, e_3$,

$$\sum_i c'(e_i) = 1,$$

for each $e \in E(T) \setminus E(H)$,

$$\sum_{e' \in \epsilon^{-1}(e)} c'(e') = 2,$$

for each $e \in E(H)$,

$$\sum_{e' \in \epsilon^{-1}(e)} c'(e') = 0.$$

(2) Let $s$ be a curve on $\partial M$ which has no back-tracking with respect to $\tau$. Each time $s$ enters a triangle of $\tau$, $c'$ associates in a natural way an integer to the selected vertex; multiply this integer by the sign of the vertex and take the sum $\sigma(s)$ of these signed integers. Then for every $s \in \{m_i, l_i\}$,

$$\sigma(s) = 0.$$

(3) Let $s$ be any curve which has no back-tracking with respect to $T(\tilde{P})$. Each time $s$ enters a tetrahedron of $T(\tilde{P})$, $c'$ associates in a natural way an integer to the selected edge. Take the sum $\alpha(s)$ of these integers. Then, for each $s$,

$$\alpha(s) \equiv 0 \mod 2.$$

Definition 3.9 The third component $c$ of any decoration $D$, called the charge, is of the form:

$$c \equiv c'/2 \mod N,$$

where $1/2 = p + 1 \in \mathbb{Z}/N\mathbb{Z}$ (with the notations of the Appendix), and $c'$ is an integral charge on $(T, H)$.

Remark 3.10 By conditions (1), $c$ satisfies, in particular, the charge requirements of the quantum data, see Proposition 8.5. By the way, note the importance of $N$ being odd in the present definition. Conditions (2) and (3) are in fact purely homological; in particular, as already said, their validity does not depend on the particular choice of $m_i$ and $l_i$ on each component of $\partial M$. 
Charge’s existence and transit. The existence of integral charges is obtained by just rephrasing the proof of Theorem 2.4.(i) (that is of Lemma 6.1) in [28]. The only modification is in considering $T$ as an ideal triangulation of $W \setminus rD^3$ (see Remark 2.1); if we set the sum of the charges around the edges of $H$ to be equal to 0, the existence follows via the same arguments.

Also Theorem 2.4.(ii) [28] shall be important in order to prove the state sum charge-invariance. Let us first describe qualitatively this result. Let $r_0$ and $r_1$ be respectively the number of vertices and edges of $T$; an easy computation with the Euler characteristic shows that there are exactly $r_1 - r_0$ tetrahedra in $T$ (see Proposition [44]). The first condition in Definition 3.8 (1) says that there are only two independent charges on the edges of each abstract $\Delta$, and given a branching $b$ on $T$ there is a preferred such ordered pair $(c_1^\Delta, c_2^\Delta)$. Set $c_1^\Delta = w_1^\Delta, c_2^\Delta = -w_2^\Delta$ (then $c_3^\Delta = -w_1^\Delta + w_2^\Delta + 1$).

Then, there is a canonical way to write down an integral charge on $(T, b)$ as a vector in $\mathbb{Z}^{2(r_1-r_0)}$ with first components $w_1^\Delta, \Delta \in T$, and then $w_2^\Delta, \Delta \in T$. Theorem 2.4. (ii) in [28] says that integral charges lie in a specific sublattice of $\mathbb{Z}^{2(r_1-r_0)}$.

**Proposition 3.11** There exist determined $w(e) \in \mathbb{Z}^{2(r_1-r_0)}, e \in T$, such that, given any integral charge $c'$, all the other integral charges $c''$ are of the form

$$c'' = c' + \sum e \lambda_e w(e)$$

where the second addendum is an arbitrary integral linear combination of the $w(e)$.

The vectors $w(e)$ have the following form. For each tetrahedron $\Delta \in (T, b)$ glued along a specific $e$, define $r_1^\Delta(e)$ and $r_2^\Delta(e)$ as the coefficients in $c'(e^{-1}(e))$, when written in terms of $(w_1^\Delta, w_2^\Delta)$. Then $w(e)$ is the vector in $\mathbb{Z}^{2(r_1-r_0)}$ with first components $r_2^\Delta, \Delta \in T$, and then $-r_1^\Delta, \Delta \in T$. For instance, in the situation described on the right of Fig. 3 (where the ordering of the tetrahedra is induced by the ordering of the vertices), we easily see that $w(e) = (-1, 1, -1, 1, 0, 1)^t$.

![Figure 5: 2 → 3 charge transits are generated by Neumann’s vector $w(e)$](image)

Next, we describe the transit of integral charges. The transit of $D$-charges shall be obtained by reduction mod $N$. Let $(T, H) \rightarrow (T', H')$ be a $2 \rightarrow 3$ move. Let $c$ be an integral charge on $(T, H)$, and $e$ the edge that appears. Consider the two “abstract” tetrahedra $\Delta_1, \Delta_2$ of $T$ involved in the move. They determine a subset $\bar{E}(T)$ of $E_\Delta(T)$. Denote by $\bar{c}$ the restriction of $c$ to $\bar{E}(T)$. Let $c'$ be an integral charge on $(T', H')$. Consider the three “abstract” tetrahedra of $T'$ involved in the move. So we have $\bar{E}(T')$ and $\bar{c}'$ with the clear meaning of the symbols. Denote by $\bar{E}(T)$ the complement of $\bar{E}(T)$ in $E_\Delta(T)$ and $\bar{c}$ the restriction of $c$. Do similarly for $\bar{E}(T')$ and $\bar{c}'$. Clearly $\bar{E}(T)$ and $\bar{E}(T')$ can be naturally confused. The following lemma is the key point of the charge-invariance.
Lemma-Definition 3.12 We have a charge-transit \((T, H, c) \to (T', H', c')\) if:

1. For each "common edge" \(e_0 \in \epsilon_T(\tilde{E}(T)) \cap \epsilon_{T'}(\tilde{E}(T'))\) of \(T\) and \(T'\),
   \[
   \sum_{e' \in \epsilon_{T'}^{-1}(e_0)} \tilde{c}(e') = \sum_{e'' \in \epsilon_{T'}^{-1}(e_0)} \tilde{c}'(e'').
   \]

2. \(\tilde{c}\) and \(\tilde{c}'\) agree on \(\tilde{E}(T) = \tilde{E}(T')\).

Moreover, if \(c\) is an integral charge on \((T, H)\) and a function \(c'\) satisfies these two conditions, then \(c'\) is an integral charge.

Charge-transits for \(0 \to 2\) and \(1 \to 4\) moves are defined in a similar way; in particular, notice that charges are defined without any reference to branchings, and that non-branched \(0 \to 2\) moves may be obtained from \(2 \to 3\) and \(3 \to 2\) moves. Moreover, for \(1 \to 4\) moves, one must add the condition (3) that the sum of the charges is 0 around the chosen two new edges of \(H\), and 2 around the others that appear.

Proof. For this definition to make sense, we have to show that such transits actually define integral charges, and by (2) we can restrict our attention to Star\((e, T)\). Now, assuming (1), the identity \(\sum_i c(e_i) = 1\) on \(f = \Delta_1 \cap \Delta_2\) in Definition 3.8 (1) is equivalent to \(\sum_{e' \in \epsilon^{-1}(e)} c'(e') = 2\) (compare with formula (3) in §8). Hence 3.8 (1) for \(c'\) is clearly verified. One can also easily show that \(c'\) satisfy 3.8 (2)-(3), for they are homological conditions. This proves our claim.

Consider now the situation of Fig. 5; recall that the branching is only used to write down explicitly the set of charge coefficients. It is easy to verify that, assuming (1) and (2), the \(\mathbb{Z}\)-vector space of solutions \((x_1, y_1, \ldots, x_3, y_3)\) to the system of linear equations defining \(c'\) from \(c\) is generated by \(w(e) = (-1, 1, -1, 1, 0, 1)^t\). The integral charges on \(T'\) may then only differ by a \(\mathbb{Z}\)-multiple of \(w(e)\), thus concluding the proof.

3.4 Total decorations

Summing up:

1. We dispose now of a complete solution of the existence problem.

Theorem 3.13 For any \((W, L, \rho)\) there exist (even almost-regular) decorated distinguished triangulations \(T = (T, H, D = (b, z, c))\).

2. We have clearly specified what is a \(2 \to 3\), \(0 \to 2\) or \(1 \to 4\) transit \(T \to T'\) between decorated distinguished triangulations; sometimes we refer to them as decorated moves.

3. We have pointed out how the decorations are reminiscent of the refined scissors congruence classes of hyperbolic 3-manifolds, represented by means of geometric ideal triangulations.

4 The invariant \(K_N(W, L, \rho)\)

This section contains the construction of the state sums, the proof of their invariance and the proof of some of their basic properties.

4.1 State sum transit-invariance

Let \(T = (T, H, D = (b, z, c))\) be as above. The state sum is defined by:

\[
K_N(T, d) = \left( N^{-\tau_0} \prod_\Delta t^\Delta(D, \alpha, d) \prod_{e \in E(T) \setminus E(H)} x(e)^{-2p/N} \right)^N. \tag{2}
\]

In fact, denoting by \(F(T)\) the set of 2-faces of \(T\), each state \(\alpha : F(T) \to \mathbb{Z}/N\mathbb{Z}\)
together with $D$ induce a complete decoration on each tetrahedron $\Delta$, and hence an associated quantum-dilogarithm $c$-$6j$-symbol $t^h(D, \alpha, d)$ (see §3 for details). Recall that $x(e) \neq 0$ is the upper-diagonal entry of $z(e)$ and that $d$ is a determination of the $N$'th-root holding for all entries of $z(e)$ (see §2). Consider any decorated transit $T \rightarrow T'$. Assume that $d$ holds for both $T$, $T'$. Then we have:

**Proposition 4.1** The state sum is transit invariant:

$$K_N(T, d) = K_N(T', d).$$

**Proof.** In some sense this is the main achievement of [1]. The equality is a consequence of the fundamental algebraic identities satisfied by the quantum-dilogarithm $c$-$6j$-symbols: the pentagon $(2 \rightarrow 3)$, orthogonality $(0 \rightarrow 2)$, and bubble relations. The precise statements are given respectively in Propositions 8.9, 8.10 and 8.11.

### 4.2 State sum total invariance

We can finally state our first main result.

**Theorem 4.2** Let $T = (T, H, D = (b, z, c))$ be a decorated distinguished triangulation of $(W, L)$ such that $[z] = \rho$. Then, for any odd positive integer $N$, $K_N(W, L, \rho) = K_N(T, d)$ is a well-defined invariant. In particular, the original Kashaev set-up corresponds to the case of the trivial flat bundle $\rho$. These are purely topological invariants $K_N(W, L)$.

The following proposition is formally contained in the statement of the theorem.

**Proposition 4.3** (a) Let $T = (T, H, D)$ be a decorated distinguished triangulation of $(W, L)$. Let $d$ and $d'$ be two determinations of the $N$'th-root holding for $T$. Then $K_N(T, d) = K_N(T, d')$. That is, the root determination has no uninfluence, and we can omit indicating it.

(b) Let $D$ and $D'$ be two decorations on $(T, H)$, giving $T$ and $T'$. Then $K_N(T) = K_N(T')$.

**Proof of Theorem 4.2 assuming Proposition 4.3** Let us fix a model of $W$, with $L \subset W$ considered up to isotopy. We have to prove that the state sum value does not depend on the decoration $T$ used to compute it. Consider $T$ and $T'$, and assume first that the corresponding $T$ and $T'$ are almost-regular. Up to some $1 \rightarrow 4$ moves (use transit-invariance) we can assume that $T$ and $T'$ have the same vertices and that the corresponding spines of $W$ adapted to $L$ have the same type and coincide along $L$. Then, let us apply Proposition 2.8 to $T$ and $T'$; we find $T''$; as Proposition 2.8 uses only “positive” moves, the branchings transit. Moreover, we may choose the full cocycles on $T$ and $T'$ generically (use §3 and §1 a first time) so that one realizes a full cocycle-transit. Finally, the charges also transit by Lemma 3.12. Summing up, we have two decorated transits $T \rightarrow T_1$, $T' \rightarrow T_2$, where at the final step one possibly has different decorations of the same $(T'', H'')$. The transit-invariance gives us

$$K_N(T) = K_N(T_1)$$

and

$$K_N(T') = K_N(T_2).$$

Finally Proposition 4.3 gives us

$$K_N(T_1) = K_N(T_2).$$

We are done in the almost-regular setting. To finish it is enough to show that any decorated distinguished triangulation transits to an almost-regular one. This is easily achieved by means of $1 \rightarrow 4$ moves. Since one can identically pull back the decorations via (PL) homeomorphisms of the triples $(W, L, \rho)$, the proof is complete.

**Proof of Proposition 4.3** Let us fix as above a model of $W$, with $L \subset W$ considered up to isotopy.

**$N$'th-root determination invariance.** We have to prove (a) of 4.3. This is a consequence of the fact that the functions $h$ and $\omega$ in the quantum dilogarithm $c$-$6j$-symbols are homogeneous of degree $0$; see §3.
Branching-invariance. Let $\mathcal{T}$ and $\mathcal{T}'$ be two decorated distinguished triangulations of $(W,L)$ such that they only differ by the branchings $b, b'$. We have to prove that $K_N(\mathcal{T}) = K_N(\mathcal{T}')$. This is done in Lemma 8.7. It describes $K_N(\mathcal{T})$ as the $N$'th-power of a weighted trace of an operator defined on some tensor product of standard representations of $W_N$, determined by the full cocycle $z$ and the determination $d$.

Charge-invariance. Let $\mathcal{T}$ and $\mathcal{T}'$ be decorated distinguished triangulations of $(W,L)$ such that the two decorations only differ by the charges $c, c'$. We have to prove that $K_N(\mathcal{T}) = K_N(\mathcal{T}')$. We may assume that $T$ is almost-regular. We shall use the transit-invariance: so we probably have to generically change the full cocycles in order to guarantee full transits. Anyway, this is not a trouble because, by continuity, it is enough to prove the present statement for full cocycles arbitrarily close to the one of $\mathcal{T}$. Let us fix an integral charge on $(T,H)$ which we denote again $c$, of which the $D$-charge is the reduction mod $N$. Fix any edge $e$ of $T$. Consider all the charges of the form (we use the notations of Proposition 3.11)

$$c' = c + \lambda w(c), \ \lambda \in \mathbb{Z}.$$ 

Let us denote this set of charges $C(e,c,T)$. It is the set of charges which differ from $c$ only on Star$(e,T)$. Thanks to Proposition 8.11, it is enough to prove the charge-invariance when $c'$ varies in $C(e,c,T)$. In this way we have somewhat “localized” the problem. The result is an evident consequence of the following facts:

1. Let $\mathcal{T} \to \mathcal{T}''$ be any $2 \to 3$ transit such that $e$ is a common edge of $T$ and $T''$. Then the result holds for $C(e,c,T)$ iff it holds for $C(e,c'',T'')$.

2. There exists a sequence of $2 \to 3$ transits $\mathcal{T} \to \cdots \to \mathcal{T}''$ such that $e$ persists at each step, and Star$(e,T'')$ is like the final configuration of a $2 \to 3$ move with $e$ playing the role of the central common edge of the 3 tetrahedra.

3. If Star$(e,T)$ is like Star$(e,T'')$ as above, then the result holds for $C(e,c,T)$.

Property (1) is a consequence of the transit-invariance (Proposition 3.1), because $C(e,c,T)$ transits to $C(e,c'',T'')$ by the first claim of Lemma 3.12.

To prove (2) it is perhaps easier to think, for a while, in dual terms. Consider the dual region $R = R(e)$ in $P(T)$. There is a natural notion of geometric multiplicity $m(R,a)$ of $R$ at each edge $a$ of $P$, and $m(R,a) \in \{0,1,2,3\}$. We say that $R$ is embedded in $P$ iff for each $a$, $m(R,a) \in \{0,1\}$. To describe the final configuration of $e$ in $T''$ is equivalent to saying that the dual region is an embedded triangle. Each time $e$ has a proper (i.e. with two distinct vertices) edge $a$ with $m(R,a) \in \{2,3\}$, the $2 \to 3$ move at $a$ "replaces" $a$ with new edges $a'$ with $m(R,a') < m(R,a)$. If $R$ has only loops with bad multiplicity, a suitable $2 \to 3$ move at a proper edge of $P(T)$ with a common vertex with the loop "replaces" the loop with proper edges. By induction we get that, up to $2 \to 3$ moves, we can assume that $R$ is an embedded polygon. Look now at the dual situation. We possibly have more than 3 tetrahedra around $e$. It is not hard to reduce the number to 3, via some further $2 \to 3$ moves.

Property (3) is almost an immediate consequence of the $3 \to 2$ transit. In fact, given any $2 \to 3$ charge-transit $(T,c) \to (T',c')$, we know by the second claim of Lemma 3.12 that all the other charges $c''$, varying the transit $(T,c) \to (T',c'')$, exactly make $C(e,c',T')$. But there is a little subtlety: in general, the branching $b$ does not transit during a $3 \to 2$ move. Anyway, we can modify the branching $b$ on the 3 tetrahedra around $e$ in such a way that the $3 \to 2$ move becomes branchable. So we have on $T$ the original branching $b$ and another system of edge-orientations $g$. Then apply Proposition 3.1 to $(T,g)$; we find $T'$ with two branchings: $b'$ by the transit of $b$ (recall that 3.3 uses only positive moves) and the branching $b''$ over $g$. Note that, thanks to the actual proof of 3.1, $e$ persists and Star$(e,T') = Star(e,T)$. Moreover we have a charge-transit $(T,c) \to (T',c')$ with $c$ and $c'$ which agree on Star$(e,T')$. So, using the branching-invariance, we may assume that the $3 \to 2$ move is branchable, and the charge-invariance is thus proved.

Cocycle-invariance. Let $\mathcal{T}$ and $\mathcal{T}'$ be two decorated distinguished triangulations of $(W,L)$ which only differ by the full cocycles $z, z', [z] = [z']$. The two cocycles differ by a coboundary $\delta \lambda$, and it is enough to prove the result in the elementary case when the 0-cochain $\lambda$ is supported by one vertex $v_0$ of $T$. Again we have "localized" the problem. Note that the transit-invariance for $1 \to 4$ moves establishes what we need in the special case when $v_0$ is the new vertex after the move. Then we reduce the general case to this special one, by means of the transit-invariance.
We may assume that $T$ is almost-regular. As before, we probably have to generically change the full cocycles in order to guarantee full transits. Again this is not troublesome by the same continuity argument. Hence, it is enough to show that, up to decorated moves, we can modify $\text{Star}(v_0, T)$ of a given vertex of $T$ to reach the star-configuration of the special situation. But $\text{Star}(v_0, T)$ is determined by the triangulation of its boundary, that is of $\text{Link}(v_0, T)$, which is homeomorphic to $S^2$. So it is enough to control the link’s modifications. One sees that, by performing $2 \rightarrow 3$ and $1 \rightarrow 4$ moves around $v_0$ in such a way that $v_0$ persists, their trace on $\text{Link}(v_0, T)$ are $1 \rightarrow 1$ moves (2-dimensional analogues of the $2 \rightarrow 3$ moves) or $1 \rightarrow 3$ moves (2-dimensional analogues of the $1 \rightarrow 4$ moves). It is well-known that these moves are sufficient to connect any two triangulations of a given surface, so we are almost done. We only have to take into account the technical complication due to the fact that, in our situation, the $1 \rightarrow 4$ moves must be completed to moves of distinguished triangulations of $(W, L)$, that is we need to involve some edges of $H$.

The proof of Proposition $4.3$, whence of the main Theorem $4.2$ is now complete.

### 4.3 Some properties of $K_N(W, L, \rho)$

An essential question is the actual dependence of $K_N(W, L, \rho)$ on $\rho$. Here are some very partial answers. The reader is advised to consult §5 before reading this section.

**Projective invariance.** Take any $\mathcal{T}$ for $(W, L, \rho)$, having as full co-cycle $z = \{z(e)\}; \{t(e)\}$ and $\{x(e)\}$ as before. For any $\lambda \neq 0$ we can turn $z$ into $\{z_\lambda(e)\}$, where $\{t_\lambda(e)\} = \{t(e)\}$ and $\{x_\lambda(e)\} = \{x(e)\}$. In this way we get a decoration $\mathcal{T}_\lambda$ for some $(W, L, \rho_\lambda)$.

**Proposition 4.4** For each $\lambda \neq 0$,

$$K_N(W, L, \rho_\lambda) = K_N(W, L, \rho).$$

**Proof.** Since the functions $h$ and $\omega$ in the quantum dilogarithm $c$-6$j$-symbols are homogeneous of degree 0, each tetrahedron contributes (via its $c$-6$j$-symbol) by $\lambda^2p$ to the state sum $K_N(\mathcal{T}_\lambda)$. Moreover, denoting by $r_i$ the number of $i$-simplices of $T$ (remark that $r_0$ is also the number of edges of $H$), the Euler characteristic of $W$ equals:

$$\chi(W) = r_0 - r_1 + r_2 - r_3 = 0.$$ 

As $r_2 = 2r_3$ we get $r_3 = r_1 - r_0$. In the formula of $K_N(\mathcal{T}_\lambda)$, there are only $r_1 - r_0$ edge contributions coming from $T \setminus H$, each one being equal to $\lambda^{-2p}$. This concludes.

**Duality.** Let $\mathcal{T} = (T, H, \mathcal{D})$ and $z$ be as above. Let us denote by $z^*$ the complex conjugate full co-cycle, $\rho^* = [z^*], \mathcal{D}^* = (b, z^*, c)$, and $\widehat{W}$ the manifold with the opposite orientation.

**Proposition 4.5** $(K_N(W, L, \rho))^* = K_N(\widehat{W}, L, \rho^*)$. Hence, if $\rho$ is “real”, that is if it can be represented by $B(2, \mathbb{R})$-valued cocycles, then

$$(K_N(W, L, \rho))^* = K_N(\widehat{W}, L, \rho).$$

In particular

$$(K_N(W, L))^* = K_N(\widehat{W}, L).$$

**Proof.** A change of orientation of $W$ turns the quantum dilogarithm $c$-6$j$-symbol $t(\mathcal{D}, d)$ into $t(\mathcal{D}, d)$. But Proposition $5.8$ shows that

$$t^\dagger(\mathcal{D}, \alpha, d) = (t(\mathcal{D}^*, -\alpha, d))^*.$$ 

Since the state sums $K_N$ are $N$’th-powers of weighted traces (see Lemma $8.7$), they do not depend on $\alpha$. So, we get the conclusion.
5 The invariants $K_N(Z, S, \rho)$ and $K_N(Y, \phi, \rho)$

Let $(Z, S, \rho)$ be as usual, and $W$ be the closed manifold obtained by Dehn filling of each boundary component of $Z$ along $S$. If $L$ denotes the link of surgery cores, then $S$ becomes a family of meridians of the $L_i$’s and $\rho$ is a flat bundle on $W \setminus L$ with, in general, a non-trivial holonomy along these meridians. The aim of this section is to define state sum invariants for the triple $(Z, S, \rho)$.

Our construction consists in cutting up pieces of any distinguished triangulation of $(W, L)$, turning it into a decorated cell decomposition of $Z$ with tetrahedron-like building blocks, which may be used to again define our previous state sums.

Let $(T, H)$ be a distinguished triangulation of $(W, L)$ as above, with branching $b$ and charge $c$. For each edge $e \in H$ consider an open neighborhood in each tetrahedron $\Delta \in T, e \in \Delta$, with the shape of a cylinder over a triangular basis. Let $U(H)$ be the union of all such polyhedra, for all edges. If $W' = W \setminus rD^3$, then $(T, H)$ induces a natural cell-decomposition $C$ of $Z = W' \setminus \text{Int} U(H)$, with building blocks $\Delta'$ as in Fig. 6.

![Figure 6: an elementary piece of $C$](image)

Our first aim is to get a decoration of $C$. The edges of $T$, which subsist in $C$, are still $b$-oriented. In each “little” face $l$ of $C$ coming from the link of the vertices of $T$, the edges lying in the old faces of $T$ inherit their $b$-orientation. Consider now the (new) rectangular face $f$ of some $\Delta'$ at the place where an edge $e \in H$ has been removed. Orient the two parallel copies of $e$ as $e$ itself. The two other little arcs in $\partial f$ are in the boundary of some $l$: give them the orientation opposite to that induced by the other “long” edges of $l$. We thus have a canonical branching $b_C$ on $C$. Moreover, one can represent $\rho$ by a full $B_1$-cocycle $z_C$ on $C$, which follows from the same arguments used in Corollary 3.6. See again Fig. 6, where latin letters denote the values of $z_C$ on $\partial f$.

For each $\Delta'$ and each rectangular face $f \in \Delta'$, there is a unique diagonal arc in $f$ that joins the source and the sink of the branching of $f$, and it is naturally oriented. Clearly, each $\Delta'$ may then be seen as a deformation of a branched tetrahedron $\Delta$. Namely, the edges of $\Delta$ are obtained by forgetting some little edges and the parallel copies of the edges of $H$ in $\Delta'$, all pictured in dashed lines in Fig. 6. and straightening the others. There is also a full $B_1$-cocycle $z_\Delta$ on $\Delta$ induced by $z_C$. Remark that, in Fig. 7, when the bundle extends to the whole of $W$, then $h = h' = 1$ and $g' = g'' = g$. Each edge of $\Delta$ inherits the charge of the edge of $\Delta$ it comes from. Finally, we thus have constructed a set $\mathcal{T}$ of decorated branched tetrahedra $(\Delta, b^\Delta, z^\Delta, c^\Delta)$ from $(T, H, b, c)$ and $z_C$, which is sufficient to define state sums $K_N(\mathcal{T}, d)$ via formula (3) below. Denote $D^\Delta = (b^\Delta, z^\Delta, c^\Delta)$.
Note that one can glue the tetrahedra $\tilde{\Delta}$ with the pairings of faces coming from $T$. It results in a compact polyhedron $\tilde{T}$ which is simplicially equivalent to $T$, with $L$ realized as an Hamiltonian subcomplex $\tilde{H}$ inside. But $\tilde{z} = \{z^{\tilde{\Delta}}\}$, which is defined separately on each $\tilde{\Delta}$, do not have, in general, coinciding values along $\tilde{H}$.

We say that $\tilde{T}$ is a decoration of $(Z, S, \rho)$.

**Theorem 5.1** The state sum formula:

$$K_N(\tilde{T}, d) = \left( N^{-r_0} \sum_\alpha \prod_{\tilde{\Delta}} \langle D_{\tilde{\Delta}}, \alpha, d \rangle \prod_{e \in E(T) \setminus E(\tilde{H})} x(e)^{-2p/N} \right)^N. \tag{3}$$

defines a family of complex valued invariants $\{K_N(Z, S, \rho)\}$, $N$ being any odd positive integer. They recover the family $\{K_N(W, L, \rho)\}$ when $Z = M$, each $s_i$ is a meridian of $L_i$, and $\rho$ extends through $S$.

**Proof.** The only modification in the proof of Theorem 4.2 is that we have to verify that there is a well-defined state sum transit-invariance for $K_N(\tilde{T}, d)$. But this is a straightforward consequence of the fact that $2 \to 3$, $0 \to 2$ and $1 \to 4$ moves, which are purely local, apply without change on $\tilde{T}$.

Furthermore, when $Z = M$, each $s_i$ is a meridian of $L_i$ and $\rho$ extends through $S$, then $\tilde{z}$ is well-defined along $\tilde{H}$. This is the situation described in fig. 7, when $h = h' = 1$.

Next we give the definition of the invariant $K_N(Y, \phi, \rho)$. Let $Y$ be any oriented compact 3-manifold with non empty, non spherical and not necessarily toral boundary. Set $\partial Y = \Sigma_1 \cdots \Sigma_k$. The genus of $\Sigma_j$ is denoted by $g_j \geq 1$. For each $g_j$, fix a base surface $\mathcal{H}_j$ regarded as the boundary of a fixed handlebody of genus $g_j$, with a given complete system of meridians $\{m_{ji}\}_{i=1, \ldots, g_j}$.

**Definition 5.2** A symplectic parametrization $\phi_j$ of $\Sigma_j$ is the equivalence class of an homeomorphism $p_j : \mathcal{H}_j \to \Sigma_j$, considered up to self-homeomorphisms $\theta$ of $\mathcal{H}_j$ such that $\theta(m_{ji}) = m_{ji}$ (up to isotopy), and $\theta_* : H_1(\mathcal{H}_j; \mathbb{Z}) \to H_1(\mathcal{H}_j; \mathbb{Z})$ is an isometry for the standard symplectic form.
A parametrization \( \phi = (\phi_1, \ldots, \phi_k) \) of \( \Sigma \) allows one to define a triple \((Z, S, \rho_Z)\). Indeed, let \( W \) be obtained by Dehn filling of \( Y \) along the system of curves \( \phi_j(m_j) \), and \( L \) be the surgery cores specified as the symplectic duals to the \( \phi_j(m_j) \)'s. Then \( Z = W \setminus \text{Int} U(L) \), \( S = \{ \phi_j(m_j) \} \) is a system of meridians of \( L \) and \( \rho_Z \) is the restriction of \( \rho \) to \( Z \) (which embeds in \( Y \)). We thus have the following generalization of Theorem 5.1:

**Theorem 5.3** One can define a family of complex valued invariants \( \{K_N(Y, \phi, \rho)\} \), \( N \) being any odd positive integer, which may be computed as a state sum from the triple above. Clearly, it specializes to the family \( K_N(Z, S, \rho) \) when \( \partial Y \) is a union of tori.

### 6 Specializations of the invariants

Purely formal considerations allow us to specialize to closed manifolds or to framed links.

**Closed manifolds invariants.** Given \((W, \rho)\) as before, take \( L_0 \) the unknot in a 3-ball in \( W \). Clearly

\[
K_N(W, \rho) := K_N(W, L_0, \rho)
\]

is a well-defined invariant of \((W, \rho)\). Taking \( \rho \) the trivial flat bundle, we get a purely topological invariant of \( W \).

**Framed links invariants.** If \( L \) is a link in \( Y \), \( X = Y \setminus \text{Int} U(L) \) and \( S \) is made by longitudes of the components of the link \( L \) in \( Y \), then we can interpret

\[
K_N(L, S; Y, \phi, \rho) := K_N(X, (S, \phi), \rho)
\]

as an invariant of the framed link \((L, S)\) in \((Y, \phi, \rho)\).

More interesting specializations of the invariants arise by using special kinds of flat bundles \( \rho \). We have already noticed in Section 3.2 that \( B \) contains the abelian subgroups \( \text{Par}(B) \), which is isomorphic to \((\mathbb{C}, +)\), and \( C(B) \), isomorphic to the multiplicative \( \mathbb{C}^* \). Moreover \( U(1) \) naturally embeds into \( C(B) \).

**The Euler class.** Since \( B \) retracts onto \( U(1) \), any principal \( B \)-bundle (forgetting the flat structure, if any) admits a group reduction to \( U(1) \). Thus any \( B \)-bundle on \( Y \) is topologically classified by its Euler class in \( H^2(Y; \mathbb{Z}) \). Indeed, as \( \pi_2(B) = 0 \), the obstruction to define a global (continuous) section of it lies in \( H^2(Y; \pi_1(B)) \cong H^2(Y; \mathbb{Z}) \).

**Induced \( C(B) \)-bundles and abelian-type invariants.** Since we are only interested in conjugation classes of representations in \( B \), below we will often abuse of notations by writing fundamental groups of manifolds without any reference to basepoints.

Any representation \( \rho : \pi_1(Y) \to B \) induces a representation \( \rho' : \pi_1(Y) \to C(B) \), just by forgetting the upper-diagonal entries. Since \( C(B) \) is Abelian, \( \rho' \) is trivial on the kernel of the natural projection \( p : \pi_1(Y) \to H_1(Y; \mathbb{Z}) \), that is \( \rho' \) factorizes through \( H_1 \). Thus, in a sense, the group \( H^1(Y; C(B)) \cong H^1(Y; \mathbb{C}^*) \) is the “core” of our discussion, and there is a natural map

\[
\mu : H^1(Y; C(B)) \to H^1(Y; B).
\]

We have already noticed in \((3.2)\) that \( H^1(Y; \text{Par}(B)) \cong H^1(Y; \mathbb{C}) \), and that it maps via

\[
\alpha : H^1(Y; \text{Par}(B)) \to H^1(Y; B).
\]

We say that any invariant of the form \( K_N(Y, \phi, \rho = \mu(x)) \) (resp. \( K_N(Y, \phi, \rho = \alpha(x)) \)) is an invariant of multiplicative (resp. additive) abelian type.

**Abelian-type invariants and the hyperbolic \( VCS \)-conjecture.** Assume that \((W, L)\) has some “hyperbolization” \( \mathcal{F} \) in the sense we have already stipulated. In order to make more consistent the Qualitative \( VCS \)-conjecture stated in the introduction, it would be very useful to determine flat \( B \)-bundles intrinsically related to the hyperbolic manifold \( \mathcal{F} \). To make this claim precise, we are going to elaborate it in the special case of a hyperbolic knot \( L \) in \( S^3 \). So, either \( \mathcal{F} = S^3 \setminus L \), or \( \mathcal{F} \) is the (compact) result of a hyperbolic surgery along \( L \) and \( \mathcal{G} \) denotes the geodesic core of
the surgery. Using the notations of §6.2, the (refined) scissors congruence class $\hat{\beta}(F)$ and $\hat{\beta}(F,G)$ are intimately related to the affine structure induced by $F$ on the boundary torus $\partial U(L)$. This structure is Euclidean exactly in correspondence with the complete hyperbolic structure on $S^3 \setminus L$, and the linear part of its holonomy is then the identity. We interpret this fact as the main source of the role of the trivial flat bundle $\rho$ in the current volume conjecture for hyperbolic knots in $S^3$.

In the case of a hyperbolic surgery, we are considering a suitable non-complete hyperbolic structure on $S^3 \setminus L$ (close to the complete one) and $F$ is just given by its completion. The linear part of the holonomy of the affine structure is now a non-trivial representation $h : \pi_1(\partial U(L)) \to \mathbb{C}^* \cong \mathbb{C}(B)$. In particular, $h$ is non trivial on the canonical longitude of $L$ (which is homologically trivial), so it cannot be extended to the whole of $\pi_1(F)$. Nevertheless, if $m$ denotes (the copy in $F$ of) a meridian of $L$, then $h$ extends to $h : \pi_1(F \setminus m) \to C(B)$; in fact $S^3 \setminus \text{Int } U(L \cup m)$ is a $\mathbb{Z}$-homology-$\partial U(L) \times [0,1]$, and $h$ is trivial on the essential curve on $\partial U(L)$ which is filled by a 2-disc in $F$, via the surgery. This construction is completely canonical. One could interpret $m$ as a sort of cut locus for the original representation $h$. We dispose now of specific abelian-type invariants for $F \setminus \text{Int } U(m)$ (or $F \setminus \text{Int } U(m \cup G)$) which are canonically associated to $(F,G)$, and we can use them to make the VCS-conjecture precise in this case. A suitable generalization of this “cut-locus theory” for the affine holonomy on $\partial U(L)$ could also play an important role in specifying a general form of the VCS-conjecture. We shall develop these considerations in §7.

(Singular Homology)-derived invariants. Concrete examples of abelian-type invariants can be derived from the ordinary singular cohomology of $Y$.

Fix $\lambda \in \mathbb{C}$. By means of the exponential map $\exp_\lambda : \mathbb{C} \to \mathbb{C}^*$, $\exp_\lambda(z) = \exp(\lambda z)$, we get a map $\exp_\lambda : H^1(Y;\mathbb{C}) \to H^1(Y;\mathbb{C}(B))$ (which restricts to the free abelian group $H^1(Y;\mathbb{Z})/\text{Tors}$).

So the ordinary cohomology group $H^1(Y;\mathbb{C})$ maps in two ways to $H^1(Y;B)$, called respectively multiplicative (just take $\mu_\lambda = \mu \circ \exp_\lambda$), and additive (take the above $\alpha$).

For $\lambda = h = 2i\pi/k$, $k \in \mathbb{N}^*$, one also has the embedding $\exp_h : \mathbb{Z}/k\mathbb{Z} \to U(1)$, $\exp_h([m]) = \exp_\lambda(m)$, hence a map $\exp_h : H^1(Y;\mathbb{Z}/k\mathbb{Z}) \to H^1(Y;U(1))$, and finally a map also denoted by $\mu_h : H^1(Y;\mathbb{Z}/k\mathbb{Z}) \to H^1(Y;B)$. Clearly the two $\mu_h$’s “coincide” on $H^1(Y;\mathbb{Z})/\text{Tors}$ via the natural map $H^1(Y;\mathbb{Z})/\text{Tors} \to H^1(Y;\mathbb{Z}/k\mathbb{Z})$.

Remark 6.1 For $x \in H^1(Y;\mathbb{C})$, $\mu_\lambda(x)$ and $\alpha(x)$ should lead to different invariants. For example, let $L$ be a knot in $S^3$, $S$ its canonical longitude, $x$ the Poincaré dual of a generator of $H_2(Z, \partial Z; \mathbb{Z}) \cong \mathbb{Z}$, $Z = S^3 \setminus \text{Int } U(L)$. Then, for any $a \in \mathbb{C}^*$, $K_N(Y, \phi, \alpha(ax)) = K_N(Y, \phi, \alpha(x))$; this is a consequence of the “projective invariance” [La]. On the other hand, $K_N(Y, \phi, \mu_\lambda(ax))$ depends on $a$. Apparently the multiplicative approach is more sensitive.

Problem 6.2 Given $x \in H^1(Y;\mathbb{C})$, carefully analyze the dependence of $K_N(Y, \phi, \mu_\lambda(ax))$ on $(\lambda, a)$.

Seifert-type invariants. Here we specialize the last construction to the case of an oriented link $L = \prod_{i=1}^n L_i$ in a $\mathbb{Z}$-homology sphere $\mathcal{H}$. Let $F_i$ be any oriented Seifert surface for the component $L_i$ in $Z_i = \mathcal{H} \setminus \text{Int } U(L_i)$, $Z = \mathcal{H} \setminus \text{Int } U(L)$, and $l = (l_1, \ldots, l_n)$, with $l_i$ a longitude for $L_i$. As usual, $l$ represents a framing of $L$ and it can be encoded as an element of $\mathbb{Z}^n$. The surface $F_i$ represents a class in $H_2(Z_i, \partial Z_i; \mathbb{Z})$ which is the Poincaré dual of a class $f_i' \in H^1(Z_i; \mathbb{Z})$; by the embedding $j_i$ of $Z$ in $Z_i$, we get $f_i := j_i^*(f_i') \in H^1(Z;\mathbb{Z})$. Finally, for any $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$, define

$$x(c) = \sum_{i=1}^n c_i [f_i] \in H^1(Z;\mathbb{Z}).$$

Set $\lambda = 1$ or $\lambda = 2i\pi/k$, $k \in \mathbb{N}^*$. Any invariant of the form $K_N(Z, l, \mu_\lambda(x(c)))$ or $K_N(Z, l, \alpha(x(c)))$ is called a Seifert-type invariant.

We believe that Seifert-type invariants (when $\mathcal{H} = S^3$) are good candidates in order to recond the full set of coloured Jones polynomials in the main stream of the quantum hyperbolic invariants and produce new interpretations for them. Our work on this matter is in progress (see [13]). Here we limit ourselves to some considerations in that direction.
Set $\mathcal{H} = S^3$, $L$ a link in $S^3$, and $Z = S^3 \setminus \text{Int} \, U(L)$ as usual. Let $J(L,(c_1,\ldots,c_n))_{|t}$ be the non-framed version of the coloured Jones polynomial of the link $L$ in $S^3$ for $sl(2,\mathbb{C})$, evaluated in $t \in \mathbb{C}^*$ [23]. Then, each component $L_i$ is framed by its canonical longitude and coloured with the $c_i$-dimensional irreducible representation of type 1 of the simply-connected restricted integral form $U_q^{res}(sl(2,\mathbb{C}))$ [20, 29, 90]. specialized in $q = t^{\frac{1}{2}}$ a fixed square root of $t$; $J(L,(c_1,\ldots,c_n)) \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$. Moreover, $J(L,(c_1,\ldots,c_n))$ is normalized by the value on the unknot $J(O,c)_{|t} = [c]_t = \frac{t^{c/2} - t^{-c/2}}{t^{1/2} - t^{-1/2}}, t \neq 1$. Define the rational function:

$$J^o(L,(c_1,\ldots,c_n)) = \frac{J(L,(c_1,\ldots,c_n))}{[c_1]}$$

so that, in particular, $J^o(O,N)_{|\omega} = 1$.

Recall that in [3], Kashaev presented an invariant of links in $S^3$, well-defined up to $N$'th roots of unity, using an $R$-matrix derived from the canonical element of the Heisenberg double of the Weyl algebra, which is a (twisted) quantum dilogarithm [14, 16]. He also claimed that the $N$'th power of this invariant should coincide with the specialization for links in $S^3$ of his previous state-sum proposal [15]. Murakami-Murakami [6] showed that this $R$-matrix can be made into an enhanced Yang-Baxter operator equivalent to the one of the coloured Jones polynomial $J^o(L,(N,\ldots,N))$ evaluated in $\omega$. Then, combining both facts in our context would give an equality $K_N(Z,l_0,\mu_\lambda(x(0))) = (J^o(L,(N,\ldots,N))_{|\omega})^N$, where $l_0$ denotes the system of canonical longitudes of $L$ in $S^3$. But there is at present no proof of the claimed coincidence between the “$R$-matrix” invariants and the “state-sum” ones. In fact, we are able to obtain this coloured Jones polynomial with a slight modification of our setup, but in order to get an actual coincidence with the Seifert-type invariants, we believe that some computations need to be better understood. This fact deserves a careful analysis, postponed to [22], with the hope that it could reveal itself as the prototype for a full generalization. Note that $K_N(Z,l,\alpha(x(c)))$ should be sensitive to the orientation of $L$ when $i \geq 2$, since we can use Proposition 4.4 only for simultaneous changes of the orientations of all the link’s components. The evident cabling information contained in the “colours” $c_i$ of $x(c)$ makes the Seifert-type invariants sensitive to the linking matrix of $L$. Also, Seifert-type invariants $K_N(Z,l,\mu_h(x(c))), h \neq 1$, yield a “Symmetry Principle” for colours, in the style of [29, 31, §4]. The following rather informal problem contains, nevertheless, an implicit conjecture about the expected result:

**Problem 6.3** Express the framed coloured Jones polynomial $J(L,l,(c_1,\ldots,c_n))$ in terms of the Seifert-type invariants $K_N(Z,l,\mu_h(x(c_1),\ldots,c_n))$.

**Remark 6.4** Since $B$ retracts on $U(1)$, a solution of Problem 6.3 would also establish, in the case of topologically trivial $B$-bundles, a correspondence between our results and Rozansky’s work [22] on $U(1)$-reducible $SU(2)$-connections over complements of links in $S^3$ and the Jones polynomials.

## 7 Speculations on the VCS-conjecture and on (2+1) quantum gravity

### 7.1 More VCS-Conjecture

Let us adopt the notations of §1.3. Let $\mathcal{F}$ be an oriented finite volume hyperbolic 3-manifold. A fundamental result of [13, 14] and [15] relates the (refined) scissors congruence class of $\mathcal{F}$ and its VCS-invariant.

**Theorem 7.1**

$$R(\hat{\beta}(\mathcal{F})) = i (\text{Vol}(\mathcal{F}) + iCS(\mathcal{F})) .$$

If $\mathcal{F}$ is non-compact we know that $\hat{\beta}(\mathcal{F})$ can be explicitely expressed as $\hat{\beta}(\mathcal{T})$, using any geometric ideal triangulation $\mathcal{T}$ of $\mathcal{F}$. When $\mathcal{F}$ is compact it is subtler to express the scissors class by means...
of “ideal triangulations” of $\mathcal{F}$: even the meaning of such a notion is not evident a priori. However, if $\mathcal{F}$ is the result of a hyperbolic Dehn surgery with geodesic core $\mathcal{G}$, one has, in a natural way, ideal triangulations $T$ of $\mathcal{F} \setminus \mathcal{G}$ which lead to $\hat{\beta}(\mathcal{F}, \mathcal{G})$ (see §3.3). Using it and $\mathcal{F}$, we have:

**Theorem 7.2**

\[
R(\hat{\beta}(\mathcal{F}, \mathcal{G})) = i\text{VCS}(\mathcal{F}) + \frac{i\pi}{2} \sum_{i=1}^{n} \lambda(L_i) \in \mathbb{C}/\pi^2\mathbb{Z} = i\text{VCS}(\mathcal{F}, \mathcal{G}) .
\]

where $\lambda(L_i)$ denotes the length of the closed geodesic $L_i$ plus $i$ times its torsion.

Consider now a hyperbolic knot $L$ in $S^3$. Let $\mathcal{F}$ be either $S^3 \setminus L$ or the result of a hyperbolic Dehn surgery along $L$ with geodesic core $\mathcal{G}$. In §3 we have shown how to associate to $\mathcal{F}$ abelian-type invariants which we denote here $K_N(\mathcal{F})$ or $\hat{K}_N(\mathcal{F}, \mathcal{G})$ respectively. If $\mathcal{F}$ is non compact $K_N(\mathcal{F})$ is just the original Kashaev’s $K_N(S^3, L)$. In the other case $K_N(\mathcal{F}, \mathcal{G}) := K_N(\mathcal{F} \setminus \text{Int} U(m), m', \rho)$, where $m$ is (a copy in $\mathcal{F}$ of) a meridian of $L$, $m'$ is a meridian of $m$, and the flat bundle $\rho$ on $\mathcal{F} \setminus \text{Int} U(m)$ is canonically associated to the affine structure induced by the hyperbolic structure of $\mathcal{F}$ on $\partial U(L)$. Moreover, the $N$-dimensional quantum dilogarithm 6j-symbols and the pentagon identities that they satisfy may be respectively interpreted as deformations of the Euler dilogarithm and the five-term relation (1) of the classical Rogers dilogarithm. Indeed, both may be recovered from them in the limit $N \to \infty$. In the pentagon identities, this is the contribution of the powers of $\omega$ which turn the Euler dilogarithm into the Rogers one. Hence we can make the VCS-conjecture precise at least for hyperbolic knots in the 3-sphere, which looks in this case to be morally tautological.

**Conjecture 7.3 (VCS-conjecture for hyperbolic knots in $S^3$.)**

(a) When $\mathcal{F} = S^3 \setminus L$ then

\[
\text{VCS}(\mathcal{F}) = \lim_{N \to \infty} (2\pi/N) \log(K_N(\mathcal{F})) .
\]

(b) When $\mathcal{F}$ is compact

\[
\text{VCS}(\mathcal{F}, \mathcal{G}) = \lim_{N \to \infty} (2\pi/N) \log(K_N(\mathcal{F}, \mathcal{G})) .
\]

**Remark 7.4** (1) Recall that there are also non-complete hyperbolic structures on $S^3 \setminus L$ which lead, by completion, to $\mathcal{F}$ with a conical singularity along $\mathcal{G}$. The above discussion applies also to this situation.

(2) When $\mathcal{F} = S^3 \setminus L$ the conjecture fits well with Theorems 7.1 (non compact case). One would like to extend it to any $\mathcal{F} = W \setminus L$ where $L$ is a hyperbolic link in any 3-manifold $W$. But we prefer to prudently stay with a knot in $S^3$ because, in the general case, there is not a unique way to extend the trivial flat $B$-bundle on $\partial U(L)$ to $W$; in the spirit of the “path integral” ideology which we mention below, this could lead to non-trivial asymptotic contributions different from the $K_N(W, L)$ one.

(3) When $\mathcal{F}$ is compact Theorem 7.2 seems to give the most appropriate “classical” counterpart, because the quantum hyperbolic invariants should depend also on $\mathcal{G}$, not only on $\mathcal{F}$.

(4) We believe to be still far from an equally effective formulation of the VCS-conjecture for arbitrary hyperbolic links $L$ in an arbitrary $W$. In particular we should have to develop a proper general notion of “representation cut locus” (see §6).

(5) The Conjecture 7.3 is supported by some numerical computations, obtained by using a somewhat formal application of the stationary phase method on an integral formula for the coloured Jones polynomials. It is not difficult to generalize such a formula for our state sum invariants, but we stress that, in order to get an actual verification of the conjecture in this way, there are still many analytical problems related to the study of its asymptotic behaviour, even for elementary examples of knots in $S^3$. 


7.2 (2+1) quantum-gravity

Witten had already suggested a form of the VCS-Conjecture, in the context of the matter-free Euclidean (i.e. Riemannian) continuation of quantum gravity in (2+1) dimensions, with negative cosmological constant $\Lambda$ \cite{10}. This and other facts suggest that this theory is a very meaningful heuristic background for the interpretation of our invariants.

Let us briefly recall how this goes. The equation of motions of (matter-free) classical gravity with cosmological constant $\Lambda$ are obtained from the Euler-Lagrange equations associated to the Einstein-Hilbert action $I_{EH}(\Lambda)$, varying the metric of the underlying space-time manifold $W$. They say that the latter is locally homogeneous, with a scalar curvature $K$ proportional to $\Lambda$. With a view towards geometric applications, let us suppose that $W$ is oriented, closed (for simplicity) and endowed with a Riemannian structure. Then, when $\Lambda < 0$, up to a constant conformal factor, $W$ is a classical solution of 3-dimensional gravity if it is a complete hyperbolic manifold: $W \cong \mathbb{H}^3/\Gamma$, where $\Gamma$ is a discrete subgroup of $PSL(2, \mathbb{C}) \cong SO(3, 1)$.

Now, consider a principal bundle $P : SL(2, \mathbb{C}) \to E \to W$, with a connection $A \in sl(2, \mathbb{C}) \otimes \Omega^1(E)$ over it. Following Chern-Simons \cite{33}, for any global continuous section $s$ of $E$ (which exists since $\pi_1(SL(2, \mathbb{C})) = \pi_2(SL(2, \mathbb{C})) = 0$ and $SL(2, \mathbb{C})$ is connected), define the Chern-Simons 3-form for $(A, s)$ as:

$$CS(A, s) = \frac{-1}{8\pi^2} s^* \left( tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right) \in \Omega^3(W, \mathbb{C}),$$

where $tr$ denotes any non-degenerate invariant bilinear form on $sl(2, \mathbb{C})$. Then the so-called Chern-Simons action:

$$I_{CS}(A) := I_{CS}(A, s) = \frac{-1}{8\pi^2} \int_W s^* \left( tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right) \in \mathbb{C}/\mathbb{Z},$$

is well-defined mod $\mathbb{Z}$, for $I_{CS}(A, s)$ only depends on $s$ via the degree of the induced map $\tilde{s} : W \to SL(2, \mathbb{C})$. Set $\Lambda = -1$. This gives no restriction for the above classical solutions, since any discrete and faithful representation of $\pi_1(W)$ lifts to $SL(2, \mathbb{C})$. In \cite{10}, it is shown that $I_{EH} := I_{EH}(-1)$ is proportional to an action of Chern-Simons type in two essentially different ways, which we denote by $I$ and $I^\ast$. Each may be constructed by writing $A$ in terms of a frame field $e$ of $TW$ and an $SL(2, \mathbb{R})$-connection $w$ over $E$, and using respectively the Killing form and the other non-degenerate invariant form of the Lie algebra $sl(2, \mathbb{C})$.

Then, Witten’s prediction holds for the Feynman integral over the field variables, with Lagrangian $\hat{I}_h = \frac{1}{h}(I + iI)$ (where $h$ is real). More precisely, for a closed and complete hyperbolic manifold $W$, the small-$h$ limit of the partition function

$$Z(W) = \int \exp(-\hat{I}_h) \mathcal{D}e \mathcal{D}w$$

should have an exponential growth rate equal to $VCS(W)$. For an arbitrary (closed) $W$, one expects $Z(W)$ to be localized around the stationary solutions of the variation of $I$ with respect to $e$ and $w$. Moreover, by \cite{10}, these solutions should be the moduli space $\chi_{SL(2, \mathbb{C})}$ of flat $SL(2, \mathbb{C})$-connections over $W$. Then, from results of Dupont \cite{34} relating the Chern-Simons classes of flat connections to volumes of representations, this prediction is also definitely well shaped for arbitrary closed 3-manifolds.

The Turaev-Viro state sum invariants, based on the quantum group $U_q(sl(2))$ with $q = \exp(2\pi i/k)$, $k$ a positive integer, are commonly regarded as $q$-deformations of the Euclidean continuation of (2+1) quantum gravity with positive $\Lambda$ and gauge group $SU(2)$ \cite{35}, \S 11.2. They are partition functions of a well-defined TQFT; in this context, $\Lambda$ is “quantized” and it is related to $q$ by $2\pi/k = \sqrt{\Lambda}$. So, the limit for $k \to \infty$ corresponds to $\Lambda \to 0$.

Arguing similarly with respect to the action $I + iI$, one may look at the state sums invariants $K_N$, established in the present paper, as parts of $q$-deformations of the Riemannian (2+1) quantum gravity with negative $\Lambda$ and gauge group $SL(2, \mathbb{C})$. Here, the relevant quantum group is
the Borel subalgebra $W_N = BU_q(sl(2, \mathbb{C}))$, with $q = \exp(2\pi i/N)$, and it takes part only via its cyclic representation theory. Then $2\pi/N = \sqrt{-1}$, and the limit for $N \to \infty$ should correspond to $\Lambda \to 0$. From Conjecture 12 (and its hypothetical generalizations) we expect that the $K_N$ invariants capture some dominant part of the integrand of the expected full partition functions. That is we should have a dominant “rough” partition function, for which one integrates over some collection $\chi_0$ of irreducible components of $\text{Hom}(\pi_1, B)/B$, instead of the whole moduli space $\text{Hom}(\pi_1, SL(2, \mathbb{C}))/SL(2, \mathbb{C})$. Then, taking inspiration from 30, we may think for instance of partition functions for pairs $(Z, S)$, which formally look like:

$$F_N(Z, S) = \int_{\chi_0} K_N(Z, S, \rho) \, d\rho.$$ 

Considering the closed 3-manifold $W$ obtained by Dehn filling of $Z$ along $S$ with cores $L$, the link $L$ may be considered as “Wilson lines” for the integrand. In order to get complete regularizations of these partition functions for some TQFT (similarly to the Turaev-Viro state sums), one could try to restrict to some (finite) particular subsets of $\text{Hom}(\pi_1, B)/B$, and consider weighted sums of the $K_N$’s over them. A solution to Problem 6 would allow to recover in this way, for example, just the Reshetikhin-Turaev and the Turaev-Viro invariants for $sl(2, \mathbb{C})$.

By the proof of the Melvin-Morton conjecture 33, 38 and Problem 13, it could be possible that the absolute torsion $\tau(ad \circ \rho)$ (see 16) of the flat vector bundle on $W$ obtained by means of the adjoint representation of $B$ can be recovered from the $K_N$ invariants in the limit for $N$ large. This is in agreement with our proposal for $F_N$ and some speculations in 30. At this point, let us recall that

$$\tau(ad \circ \rho) = \tau(ad \circ \rho, \xi_c),$$

where $\xi_c$ is the canonical Euler structure (i.e. the Spin$^c$ structure) on $W$ with null Euler class $e(\xi_c) = 0 \in H^2(W; \mathbb{Z})$. Staying at a formal level, we would like to stress that the same technology used to define $K_N(W, L, \rho)$ can be used to treat the absolute torsions. Many basic extra-structures on 3-manifolds such as framings, spin structures and Euler structures, can be encoded in a very effective way by means of branched spines 21, 25, 39. In order to obtain also an encoding of Euler structures in terms of the canonical Euler chains carried by branched spines (as in 33), it is enough to require that the distinguished decorated triangulations $(T, H, D)$ of $(W, L)$ satisfy the further condition:

For each vertex $v_0$ of $T$ consider the “black” region $B(v_0)$ of $S^2(v_0)$, the 2-sphere around $v_0$, where the natural field on $W^r_{v_0}$, normal to the branched spine $P(T)$, points inward the ball around $v_0$. Then

$$\chi(B(v_0)) = 0.$$ 

This condition ensures that the normal field can be extended to a non-singular field on the whole of $W$, whence it determines an Euler structure $\xi$ on $W$. The Euler class $e(\xi)$ of this Euler structure can be explicitly represented by a canonical 2-co-chain on $P(T)$, and we know exactly how to modify $P(T)$ (hence $\xi$) by means of $2 \to 3$ (branched) moves in order to make $e(\xi) = 0$ (see 20 for the details).

8 Quantum data

In this Appendix we state the algebraic results needed for the construction of our invariants. Most of them were already announced by R. Kashaev in 1, 2. The proofs are often involved formal computations. Nevertheless, they do not add anything to the understanding of the rest of this paper, so we refer to 16 for the details. Recall that $\omega = \exp(2\pi i/N)$ for an odd positive integer $N$, and $N = 2p + 1$, $p \in \mathbb{N}$. Fix the determination $\omega^{1/2} = \omega^{p+1}$ for its square root. We shall henceforth denote $1/2 := p + 1 \mod N$. All other notations for manifolds, triangulations and spines are as in the rest of the paper.

Since the work of 1, we know that the quantum $6j$-symbols and, more generally, the associators of braided, spherical or balanced semisimple categories provide via the Pentagon equation a natural
framework for constructing invariants of compact 3-manifolds [3, 11]. In each of these cases, the construction heavily uses some symmetry properties of the associators for the group $S_4$ of permutations over 4 elements, which is the group of symmetries of a tetrahedron. But in our present situation, the number of irreducible cyclic representations of the Weyl algebra $W_N$, in any fixed dimension, is infinite (see Proposition 8.1), and the 6j-symbols $R(p,q,r)$ obtained in Proposition 8.3 (and then, any 6j-symbol for the regular sequences of cyclic representations of $W_N$) are not symmetric under the action of $S_4$.

One could try to overcome this difficulty by considering “branched” versions of the pentagon equation and possibly by restricting to some topologically significant subset of branched moves such as the sliding moves (see §3.3); unfortunately, this does not work. Hence we are forced to find a generalization of $R(p,q,r)$ which is covariant under $S_4$. There is a representation-theoretic interpretation of this extension, which is related to the action of $SL(2,Z)$ on $W_N$ (see [16], the discussion before Proposition 8.3 and Remark 8.6).

We first briefly recall some fundamental results concerning the Weyl algebra $W_N$. This is the unital algebra over $\mathbb{C}$ generated by elements $E, E^{-1}, D$ satisfying the commutation relation $ED = \omega DE$. It is well-known that $W_N$ can be endowed with a structure of Hopf algebra isomorphic to the simply-connected (non-restricted) integral form of the Borel subalgebra of $U_q(sl(2,C))$ [30, §9-11], specialized in $\omega$, with the following co-multiplication, co-unity and antipode maps:

$$
\Delta(E) = E \otimes E, \Delta(D) = E \otimes D + D \otimes 1,
$$

$$
\epsilon(E) = 1, \epsilon(D) = 0, S(E) = E^{-1}, S(D) = -DE^{-1}.
$$

Consider now the injective representations of $W_N$ on finite dimensional complex vector spaces, which are also called the cyclic representations of $W_N$ in the theory of quantum groups. Let us fix the dimension equal to $N$, and let $X$ and $Z$ be the $N \times N$ matrices with components:

$$
X_{ij} = \delta_{i,j+1}, \quad Z_{ij} = \omega^i \delta_{i,j},
$$

where $\delta_{i,j}$ denotes the symbol of Kronecker. Following [3], we define a standard $N$-dimensional representation $p$ of $W_N$ by:

$$
p(E) = t_p^2 Z, \quad p(D) = t_px_p X, \quad t_p, x_p \in \mathbb{C}^*,
$$

and we denote by $p^*$ (the complex conjugate representation) and by $\bar{p}$ (the inverse representation) the standard representations of $W_N$ obtained by setting

$$
t_{\bar{p}} = 1/t_p, \quad x_{\bar{p}} = -x_p
$$

and

$$
t_p^* = (t_p)^*, \quad x_p^* = (x_p^*)^*.
$$

We immediately see from the definition of the comultiplication $\Delta$ that for any pair $(p, q)$ of standard representations, we have

$$
p \otimes q(E^N) = (t_pt_q)^{2N} id \otimes id,
$$

$$
p \otimes q(D^N) = (t_pt_q)^N \left( t_p^N x_q^N + \frac{x_q^N}{t_q^N} \right) id \otimes id, \quad (4)
$$

where $id$ denotes the identity matrix in dimension $N$. In particular, for any standard representation $p$, the representation $p \otimes \bar{p}$ is no longer injective. Then we call regular an ordered sequence of cyclic representations $(p_1, \ldots, p_n)$ of $W_N$ for which any tensor product representation $p_i \otimes \ldots \otimes p_{i+j}$, $1 \leq i \leq n$, $1 \leq j \leq n - i$ is cyclic. The following proposition shows that the isomorphism classes of cyclic representations of $W_N$ may be described using only standard representations. Moreover, it implies that they are infinite in number and that their tensor product is not commutative. Hence they do not form either a semi-simple or a monoidal category.

**Proposition 8.1** i) The standard representations are irreducible and cyclic, and two standard $N$-dimensional representations $p$ and $q$ are equivalent if and only if...
\[ t_p^{2N} = t_q^{2N}, \quad t_p^N x_q^N = t_q^N x_q^N. \]

**ii)** Any irreducible cyclic representation of \( W_N \) is equivalent to a standard one.

**iii)** Fix a branch of the \( N \)’th root. If \((p, q)\) is a regular pair of \( N \)-dimensional irreducible cyclic representations of \( W_N \), the representation \( p \otimes q \) is equivalent to the direct sum of \( N \) copies of the standard representation \( pq \) defined by:

\[
t_{pq} = t_p^{f_{pq}}, \quad x_{pq} = \left( t_p^{N x_q^N} + \frac{x_p^N}{t_q^N} \right)^{1/N}.
\]

Such a representation is called a product standard representation.

Define the space of **intertwiners** for representations \( \rho \) and \( \mu \) of \( W_N \) acting respectively on the vector spaces \( V_{\rho} \) and \( V_{\mu} \) as

\[
H_{\rho, \mu} = \{ U : V_{\rho} \to V_{\mu} | U \rho(a) = \mu(a)U, \ a \in W_N \}.
\]

Then, the claims **ii)** and **iii)** of Proposition 8.3 imply that for any irreducible cyclic representations \( p, q, r \) of \( W_N \) with \((p, q)\) a regular pair, the dimension of \( H_{r, p \otimes q} \) and \( H_{p \otimes q, r} \) is equal to \( N \) if \( r \) is equivalent to \( pq \), and zero otherwise. In the first case, the intertwiners (which are injections) are called **Clebsch-Gordan operators** (CGO), and in the second case they are called **dual Clebsch-Gordan operators** (these are projectors). We shall now give an explicit basis of the CGO for a regular pair of standard representations; for the dual operators, see Proposition 8.3.

Consider the curve \( \Gamma \subset \mathbb{CP}^2 \) which is the zero set of the Fermat equation \( x^N + y^N = z^N \) (with homogeneous coordinates), and define for any positive integer \( n \) a rational function \( \omega \) (not to be confused with the root of unity \( \omega \) !) by:

\[
\omega(x, y, z|n) = \prod_{j=1}^n \frac{y}{z-x^{\omega^j}}, \quad [x, y, z] \in \Gamma \setminus \{(1,0,\omega^j) | j = 1, \ldots, n\}.
\]

Moreover, set \( \omega(x, y, z|m, n) = \omega(x, y, z|m-n)\omega^{n^2/2} \). Define also a periodic Kronecker symbol by \( \delta(n) = 1 \) if \( n \equiv 0 \mod N \), and zero otherwise. Note that the function \( \omega \) is periodic in its integer argument, with period \( N \). It satisfies very nice transformation properties for some automorphisms of \( \Gamma \), and it verifies summation formulae which are analogues in the root of unity case to well-known formulae for hypergeometric series. See [42] for a summary, and [43] and the references therein for details on their use in statistical mechanics.

**Proposition 8.2** Let \((p, q)\) be a regular pair of standard representations. For any non-zero complex number \( \nu_{p,q} \), the set \( \{ K_\alpha(p, q), \alpha = 0, \ldots, N-1 \} \) of linear operators with components

\[
K_\alpha(p, q)_{i,j}^k = \nu_{p,q} \omega^{\alpha j} \omega(t_px_q, x_p, x_q|i, \alpha)\delta(i+j-k)
\]

is a basis of \( H_{pq,p \otimes q} \).

The factor \( \nu_{p,q} \) is of course inessential here, but it is justified below.

Consider the following commutative diagram, where \((p, q, r)\) is a regular sequence of standard representations and the arrows denote injections of representations:

\[
pq \quad \rightarrow \quad p \otimes qr
\]

\[
pq \otimes r \quad \rightarrow \quad p \otimes q \otimes r
\]

This diagram is induced, for instance, by the families of compositions of CGO \( \{(id \otimes K_\delta(p, q, r)) \circ K_\gamma(p, qr)\}_{\delta, \gamma} \) for the path of injections which goes to the right, and \( \{(K_\alpha(p, q) \otimes id) \circ K_\beta(p, qr)\}_{\alpha, \beta} \).
for the path of injections which goes to the bottom. The 6\textit{j}-symbols \( R(p, q, r) \) of the above diagram are defined as the intertwiners between the two paths, i.e. it is a map:

\[
R(p, q, r) : H_{pq, p \otimes qr} \otimes H_{qr, q \otimes r} \rightarrow H_{pq, p \otimes q} \otimes H_{pq, pq \otimes r},
\]

which reads in the basis of CGO obtained in Proposition 8.2 as follows:

\[
K_\alpha(p, q)K_\beta(pq, r) = \sum_{\delta, \gamma = 0}^{N-1} R(p, q, r)\gamma^\delta_{\alpha, \beta} K_\delta(q, r)K_\gamma(p, qr).
\]

These \( 6\text{i}\)-symbols satisfy a 3-cocycle relation which is easy to obtain using (\ref{R}), and called the \textit{pentagon equation}. For a regular sequence \((p, q, r, s)\) of cyclic representations of \( \mathcal{W}_N \) it reads:

\[
R_{12}(p, q, r)R_{13}(p, qr, s)R_{23}(q, r, s) = R_{23}(pq, r, s)R_{12}(p, q, rs),
\]

where indices denote the tensor factor on which the \( 6\text{i}\)-symbols act. Next we describe the explicit form of these \( 6\text{i}\)-symbols in the basis of CGO given in Proposition 8.2.

Introduce the functions

\[
\forall x \in \mathbb{C}^*, \quad g(x) = \prod_{j=1}^{N-1} (1 - x\omega^j)^{i/N}, \quad h(x) = x^{-p} g(x)/g(1),
\]

where the branch of the \( N\text{th}-\) root is chosen by the reality condition \( g(0) = 1 \) (recall that \( N \) is odd), and \( p \) is defined by \( N = 2p + 1 \). Then fix the scalar \( \nu_{p, q} \) in Proposition 8.2 as \( \nu_{p, q} = h(x_{pq}/tpx_q) \), and set

\[
\nu_{p, q, r} = h \left( \frac{x_{pq}x_{qr}}{x_{pqr}x_{q}} \right), \quad [x] = N^{-1} \left( 1 - \frac{x^N}{1 - x} \right).
\]

Note that the functions \( \nu \) for the CGO are chosen so that, with Proposition 8.3, one can write:

\[
K_\alpha(p, q)^{k}_{ij} = R(p, q)^{l, j}_{i, k}.
\]

This is not a coincidence: one can prove that the \( 6\text{i}\)-symbols and the CGO obtained above are both representations of the canonical element of the Heisenberg double of the Weyl algebra (which is a twisted quantum dilogarithm), acting on \( H_{pq, p \otimes qr} \otimes H_{qr, q \otimes r} \). This and the relation between (\ref{R}) and (\ref{R}) are explained in (\ref{R}).

**Proposition 8.3** The \( 6\text{i}\)-symbols of the regular sequences of cyclic representations of \( \mathcal{W}_N \), in the basis of the Clebsch-Gordan operators obtained in Proposition 8.2, are described by the following components:

\[
R(p, q, r)^{\gamma, \delta}_{\alpha, \beta} = \nu_{p, q, r} \omega^{\alpha \delta} \omega(x_{pq}x_q, x_p x_r, x_{pq} x_{qr} | \gamma, \alpha) \delta(\gamma + \delta - \beta),
\]

and their inverses are given by

\[
\bar{R}(p, q, r)^{\alpha, \beta}_{\gamma, \delta} = \nu_{p, q, r}^{-1} \omega^{\alpha \delta} \frac{\delta(\gamma + \delta - \beta)}{\omega(x_{pq}x_q, x_p x_r, x_{pq} x_{qr} | \gamma, \alpha)}.\]

**Remark 8.4** Restricting attention to standard representations \( p \) where \( t_p = 1 \) would also give the above \( 6\text{i}\)-symbols (this is how they are presented in (\ref{R})). Moreover, one can continue them to nilpotent (i.e. non-cyclic) representations of \( \mathcal{W}_N \) with \( x_p = 0 \), but this is of no interest for the \( K_N \) invariants.

Let \((W, L)\) be as usual. Consider a decoration \( T = (T, H, b, z, c) \) of \((W, L)\) and fix a common determination \( d \) of the \( N\text{th}-\)root for the entries of \( \{ z(c) \} \); \( x(e) \) and \( y(e) = d(x(e)) \) are as in (\ref{R}). Each tetrahedron \( \Delta \in T \) inherits a decoration from \((T, d)\); in particular, using \( b \), the \( y(e) \)'s satisfy a specific Fermat equation, and there is defined an index for \( \Delta \), see (\ref{R}). From now on, we denote
by \( x_{ij} \) (resp. \( c_{ij} \)) the value of \( y \) (resp. the non-oriented edge) of \( \Delta \) with vertices numbered \( v_i \) and \( v_j \) via the branching. Moreover, a state \( \alpha : F(T) \to \mathbb{Z}/\mathbb{N} \) produces a number \( \alpha_k \) associated to the face of \( \Delta \) opposite to its \( k \)th-vertex.

Suppose that \( \Delta \) has index \( -1 \), and define a correspondence \( t \) as follows: send \( (\Delta, T, d, \alpha) \) to the number \( R(T, d) = R(x_{01}, x_{12}, x_{23})^{\alpha_0, \alpha_0}_{\alpha_2, \alpha_4} \). One verifies easily that under the permutation \( (v_1, v_2, v_3) \) (which preserves the index), this number turns into the component of a matrix with completely different \( x \)-parameters. Hence we are led to consider the action of elementary symmetries of \( \Delta \), considered as an abstract tetrahedron, on \( t \). The transpositions \( (v_0, v_1), (v_1, v_2), (v_2, v_3) \) are convenient, and they generate the whole group of symmetries of \( \Delta \). Since a transposition of vertices changes the index, it is natural from a topological point of view (see Proposition 8.10 below) to associate via \( t \) the component \( R(x_{01}, x_{12}, x_{23})^{\alpha_3, \alpha_1}_{\alpha_2, \alpha_0} \) to \( (\Delta, T, d, \alpha) \) when the index is 1.

Note that the change is still inessential here.

Define \( N \times N \) matrices (distinguished by the upper and lowered indices) by the components

\[
T_{m,n} = \zeta^{-1} \omega^{-m^2/2} \delta(m+n), \quad S_{m,n} = N^{-1/2} \omega^{-mn}, \quad m, n \in \mathbb{C}^*, \quad S_{m,n} = N^{-1/2} \omega^{-mn}.
\]

One may show that, for a suitable \( \zeta \), the above transpositions change the \( 6j \)-symbol associated to \( (\Delta, T, d) \) via \( t \) in two different ways. They act:

1) on the scalar factor,

2) on \( \alpha \): this action “holds” on the faces of \( \Delta \) (which permute under a permutation of the vertices), and this translates algebraically into the action of the matrices \( S \) and \( T \) on the \( 6j \)-symbols and the multiplication by a power of \( \omega \) depending affinely on \( \alpha \).

We then have to replace \( R \) and \( \tilde{R} \) by new covariant matrices. One can avoid the scalar transformation by multiplying \( R(p, q, r) \) and \( \tilde{R}(p, q, r) \) by \( (x_{pq}x_{qr})^p \). Then, note that one can act in an affine way on the \( \alpha \)-parameters of the components of \( R \) and \( \tilde{R} \) by:

- multiplying \( R \) and \( \tilde{R} \) by a power of \( \omega \) which depends affinely on \( \alpha \),

- adding a scalar in the \( \mathbb{Z}_N \) arguments of the \( \omega \) function, for instance in the first one, denoted by \( \gamma \) in Proposition 8.3, then we have to add the same scalar to \( \beta \), due to the periodic Kronecker symbol.

The following proposition describes an extension of the \( 6j \)-symbols of the regular sequences of standard representations of \( \mathcal{W}_N \) that corresponds to this description.

**Proposition 8.5** Given \( a, c \in \mathbb{Z}_N \) and a regular sequence \( (p, q, r) \) of standard representations, consider the two matrices called \( c \)-\( 6j \)-symbols and defined by

\[
R(p, q, r|a, c)^{\gamma, \delta}_{\alpha, \beta} = (x_{pq}x_{qr})^p \omega^c(\gamma - a) - ac/2 R(p, q, r)^{\gamma - a, \delta}_{\alpha, \beta - a}.
\]

\[
\tilde{R}(p, q, r|a, c)^{\alpha, \beta}_{\gamma, \delta} = (x_{pq}x_{qr})^p \omega^c(\gamma - a) + ac/2 \tilde{R}(p, q, r)^{\alpha + a, \beta + a}_{\gamma + a, \delta}.
\]

We have the following relations:

\[
\sum_{\alpha', \gamma = 0}^{N-1} R(p, q, r|a, c)^{\gamma, \delta}_{\alpha', \beta} T_{\gamma, \gamma} T_{\alpha', \alpha} = \omega^{a/4} \tilde{R}(p, q, r|a, b)^{\alpha, \delta}_{\gamma, \beta},
\]

\[
\sum_{\alpha', \delta = 0}^{N-1} R(p, q, r|a, c)^{\gamma', \delta}_{\alpha', \beta} S_{\delta, \delta} S^{\alpha, \alpha'} = \omega^{-c/4} \tilde{R}(p, q, r|b, c)^{\alpha, \gamma}_{\beta, \delta},
\]

\[
\sum_{\beta', \delta = 0}^{N-1} R(p, q, r|a, c)^{\gamma', \delta}_{\alpha, \beta'} S_{\delta, \delta} S^{\beta, \beta'} = \omega^{a/4} \tilde{R}(p, q, r|a, b)^{\gamma, \beta}_{\alpha, \delta},
\]

where \( b = 1/2 - a - c \in \mathbb{Z}_N \) and \( \zeta = (-1)^p \omega(1-N)(N-2)^2/24 \omega^{1/2} \). Note that \( \omega^{1/4} = \omega^{p^2} \) is an \( N \)th-root of 1.
It is now clear how to turn $t$ into a covariant correspondence for the action of $S_4$. The charge $c$ in $T$ clearly provides $c$-labels as in the statement. Then define:

$$\forall \Delta \in T, \ t^\Delta(T, d, \alpha) = \begin{cases} R(x_{01}, x_{12}, x_{23}|c_{01}, c_{12})^{\alpha_2, \alpha_0}_{\alpha_3, \alpha_1} & \text{if the index of } \Delta \text{ is } -1, \\ \tilde{R}(x_{01}, x_{12}, x_{23}|c_{01}, c_{12})^{\alpha_2, \alpha_0} & \text{otherwise}. \end{cases} \tag{7}$$

We say that

$$t(T, d) = \prod_\Delta t^\Delta(T, d) = \sum_\alpha \prod_\Delta t^\Delta(T, d, \alpha)$$

is the operator associated to $(T, d)$. Of course it is a complex number (since each face of $T$ appears twice in the set of faces of the abstract tetrahedra), but this definition may also be extended to simplicial complexes with boundary (see below).

**Remark 8.6** The matrices $T_{m,n}$ and $T^{m,n}$ (resp. $S_{m,n}$ and $S^{m,n}$) are inverse one to each other and

$$S^4 = \text{id}, \ S^2 = \zeta'(ST)^3, \ |\zeta'| = 1.$$ 

This uses the explicit calculation of the phase factor of $g(1)$, see [42]. Hence the matrices $S$ and $T$ define a projective $N$-dimensional representation of $SL(2, \mathbb{Z})$, which up to an $N$’th-root of unity is well-known as the modular representation on the space of characters of minimal models in Conformal Field Theory [44, 10.5].

Let us now specify the rule which determines the action of the matrices $S$ and $T$, when permuting vertices $v_i$ and $v_{i+1}$ of $\Delta$ with adjacent numberings. Each face $f \in F(\Delta)$ containing $v_i$ and $v_{i+1}$ inherits an orientation from the branching : set $\epsilon(f) = 1$ if this orientation is the one induced by $\Delta$ as a boundary, and $\epsilon(f) = -1$ otherwise. Moreover, set $\lambda(f) = 1$ if the numbering of the vertex of $f$ distinct from $v_i$ and $v_{i+1}$ is greater than $i + 1$, and $\lambda(f) = -1$ if it is lesser than $i$. Finally, define:

$$\mu_{ab}(f) = \frac{(1 + a\lambda(f))(1 + b\epsilon(f))}{4}, \ a, b = \pm.$$ 

For example, if $\Delta$ has index $-1$ and we consider the permutation $(v_0, v_1)$, the only non-zero $\mu$-number associated to the face $f_2$ is $\mu_{++}$. Then, for any permutation $(v_i, v_{i+1})$, the matrix with components

$$s_{u}(f)_{m,n} = T_{m,n}\mu_{++}(f) + T^{m,n}\mu_{--}(f) + S_{m,n}\mu_{+-}(f) + S^{m,n}\mu_{-+}(f)$$

acts on $t^\Delta(T, d)$ by multiplication through the tensor factors corresponding to the $\alpha$-labels of the faces $f$ containing both $v_i$ and $v_{i+1}$. In particular, if we change the $\epsilon$-value of such an $f$, $\mu_{++}$ turns into $\mu_{--}$ etc, and the dual matrix acts. Then, a transposition of vertices acts on the operator $t(T, d)$ only by multiplication by a power of $\omega$ (see Proposition 8.3), since for each face which is acted on and for the two tetrahedra glued on it the $\mu$-coefficients are opposite.

Set

$$Q_N(T, d) = (t(T, d))^N = \left(\sum_\alpha \prod_\Delta t^\Delta(D, \alpha, d)\right)^N,$$

and consider a maximal tree $\Gamma$ in the 1-skeleton of the dual cell decomposition of $T$. Denote by $\Gamma(T)$ the polyhedron obtained from the gluing of the abstract tetrahedra $\Delta \in T$ along the faces dual to the edges of $\Gamma$. Let $\alpha_{\Gamma}$ be a state of $\Gamma(T)$, and consider the operator

$$\Gamma_N(T, d) = \sum_{\alpha_{\Gamma}} \prod_\Delta t^\Delta(D, \alpha, d).$$

We clearly have:

$$Q_N(T, d) = (tr(\Gamma_N(T, d)))^N,$$

where $tr$ is the trace on $End(\mathbb{C}^{e_1})$ and $e$ is the number of edges of $\Gamma$. This gives an essential representation-theoretic interpretation of the state sums $K_N(T, d)$ as $N$’th-powers of weighted...
traces, the weights being given by the product of the scalars \( x(e)^{-2p/N}, e \in T \setminus H \). It makes a bridge between the “observable” topological quantities in the invariants \( K_N \) and the algebraic properties of the quantum dilogarithm operator on \( W_N \). In particular, it should be a very important tool in the study of the asymptotic behaviour of \( K_N \).

**Lemma 8.7** \( Q_N(T, d) \) (whence \( K_N(T, d) \)) does not depend on the branching \( b \) in \( T \).

**Proof.** Any change of branching of \( T \) translates on the abstract \( \Delta \)'s into a composition of transpositions of the vertices. We have seen in Proposition 8.4 that for such symmetries, the matrices \( S \) and \( T \) or their duals act on \( t^\Delta(D, d) \) through the tensor factors corresponding to the faces \( f \) containing the two permuting vertices; moreover, there is a power of \( \omega \) appearing in factor. But we saw above that the matrix action is trivial on the operator \( t(T, d) \). Then \( Q_N(T, d) \) does not depend on the branching.

Using the above terminology, this means that a change of branching turns \( \Gamma_N(T, d) \in \text{End}(\mathbb{C}^{\otimes c}) \), up to an \( N \)'th-root of unity, into a conjugate operator. Then its trace \( Q_N(T, d) \) does not depend on \( b \).

Here is a simple consequence of the definition of the c-6j-symbols:

**Proposition 8.8** Let \((p, q, r)\) be a regular sequence of standard representations of \( W_N \). We have the following identity:

\[
\widetilde{R}(p^*, q^*, r^*|a, c)_{\alpha, \delta} = \left( R(p, q, r|a, c)^{\gamma, \delta}_{-\alpha, -\beta} \right)^*,
\]

where * denotes the complex conjugation.

We now present the extension of the pentagon equation that the c-6j-symbols satisfy. It should be in the form of equation (6), with some additional conditions on the charges \( c \). Consider the branched convex polyhedron \( P_0 \) obtained by gluing two abstract tetrahedra along the face opposite to \( v_1 \) and \( v_3 \) respectively, so that \( \Delta(v_1, v_2, v_3, v_4) \) has index \(-1\) (see the left of Fig. 3). Denote by \( c^i \) the restriction of \( c \) to the tetrahedron \( \Delta^i \) which do not contain \( v_i \).

We shall now prove that the operator \( t(P_0, T, d) \) is equal to \( t(P_1, T_1, d_1) \), where \((P_1, T_1, d_1)\) is obtained via a decorated \( 2 \rightarrow 3 \) transit. This transit is dually described in the third picture of Fig. 2. The resulting equation is called the extended pentagon relation (EP relation below).

Let us first comment the behaviour of charges under decorated \( 2 \leftrightarrow 3 \) transits. It is easy to verify that there are exactly four degrees of freedom on the charges of \( (P_0, T, d) \) (e.g. \( c_{01}^3, c_{01}^4, c_{23}^3 \) and \( c_{23}^4 \)), and there is one more independant degree of freedom on the charges of \( (P_1, T_1, d_1) \) (e.g. \( c_{03}^3 \)). Some details on the geometric interpretation of the latter were given in Lemma 3.12. Then there are five degrees of freedom on the charges of the EP relation. In particular, note that we have:

\[
c_{02}^1 + c_{24}^1 + c_{40}^1 = (c_{02}^4 - c_{02}^3) + (c_{24}^0 - c_{24}^2) + (c_{04}^0 - c_{04}^3) = c_{13}^1 + c_{13}^0 + c_{13}^2 - (c_{02}^4 + c_{24}^3 + c_{40}^3),
\]

where we use the fact that opposite edges have the same charge in the second equality. Now this gives:

\[
c_{02}^1 + c_{24}^1 + c_{40}^1 \equiv c_{02}^4 + c_{24}^3 + c_{40}^3 \equiv 1/2 \quad (\text{mod } N) \iff c_{13}^1 + c_{13}^0 + c_{13}^2 \equiv 1 \quad (\text{mod } N). \tag{8}
\]

Then, we see, once again, that \( 3.8 \) (1) is the proper setting for defining the transit of charges. Let us consider the following set of independent charges for the EP relation:

\[
i = c_{01}^4, j = c_{01}^2, k = c_{12}^0, l = c_{23}^1, m = c_{12}^3.
\]

One can easily show, using Definition 3.8 (1) and the charge-transit condition, that

\[
l + m = c_{13}^2, l - i = c_{23}^2, j + k = c_{02}^4, i + j = c_{01}^3, m - k = c_{12}^4.
\]
**Proposition 8.9** Let \((p, q, r, s)\) be a regular sequence of standard representations of \(W_N\).

i) The following EP relation holds:

\[
R_{12}^4(p, q, r | i, m - k) R_{13}^2(p, qr, s | j, l - m) R_{23}^0(q, r, s | k, l - i) = x_{pq}^2 R_{23}^1(pq, r | s | j + k, l) R_{12}^3(p, q, rs | i + j, m).
\]

Setting \(x_p = x_{01}, x_q = x_{12}, x_r = x_{23}\), and \(x_s = x_{34}\), the left-hand side (resp. the right-hand side) is the operator associated via (7) to \((P_1, T_1, d_1)\) (resp. \((P_0, T, d)\)).

ii) A similar identity holds up to an \(N\)’th-root of unity for any other branchings of \(P_0\) and \(P_1\), even if one cannot blow down the branching of \(P_1\) to the one of \(P_0\).

Remark that ii) gives an alternative proof of the charge-invariance (3) in Proposition 4.3. Using the symmetry relations in Proposition 8.5, one can easily derive the whole set of EP relations, for any branching of the abstract polyhedron \(P_0\).

We give the algebraic relation that corresponds to the \(0 \rightarrow 2\) decorated transit in the case where the branching gives the simplest form of it. This branching is induced from that of \((P_1, T_1, d_1)\) above. Indeed, one can read a relation corresponding to a \(0 \rightarrow 2\) decorated transit for \(\Delta^0\) by comparing the identities obtained by applying first a \(2 \rightarrow 3\) transit on \(\Delta^0\) and then a \(3 \rightarrow 2\) transit on \(\Delta^0, \Delta^2\) and \(\Delta^4\) (this is possible, since the final configuration is branchable). Then, we have:

**Proposition 8.10** Let \((p, q, r)\) be a regular sequence of standard representations of \(W_N\). The following orthogonality relation holds:

\[
R(p, q, r | a, c) \bar{R}(p, q, r | -a, -c) = (x_{pq} x_{qr})^{2p} id \\otimes id.
\]

Finally, consider the bubble move, which consists in decomposing a face of \(T\) into three new ones with a common vertex inside and two tetrahedra glued along them. A branched relation corresponding to a bubble move is obtained by taking the partial trace over one of the indices in the above orthogonality relation:

**Proposition 8.11** Let \((p, q, r)\) be a regular sequence of standard representations of \(W_N\). The following bubble relation holds:

\[
tr_i \left( R(p, q, r | a, c) \bar{R}(p, q, r | a, c) \right) = N (x_{pq} x_{qr})^{2p} id,
\]

where \(i\) is equal to 1 or 2.

Since any \(1 \rightarrow 4\) transit may be obtained from a bubble move followed by a \(2 \rightarrow 3\) transit, Propositions 8.9 and 8.11 plus the symmetry relations give the whole set of relations corresponding to decorated \(1 \rightarrow 4\) transits. This concludes this Appendix.

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**References**

[1] R.M. Kashaev. *Quantum dilogarithm as a 6j-symbol*, Mod. Phys. Lett. A Vol. 9, No 40 (1994), 3757-3768;

[2] R.M. Kashaev. *A link invariant from quantum dilogarithm*, Mod. Phys. Lett. A Vol. 10 (1995), 1409-1418;

[3] R.M. Kashaev. *Quantum hyperbolic invariants of knots*, Discrete Integrable Geometry and Physics. A. I. Bobenko & R. Seiler éd., Oxford Science Publications (1998);
[4] L.D. Faddeev and R.M. Kashaev. Quantum dilogarithm, Mod. Phys. Lett. A Vol. 9, No 5 (1994), 427-434;

[5] R.M. Kashaev. The algebraic nature of quantum dilogarithm, Geometry and integrable models (Dubna 1994), 32-51, World Sci. Publishing, River Edge, NJ (1996);

[6] V.G. Turaev. O. Viro. State sum invariants of 3-manifolds and 6j-symbols, Topology 31 (1992), 865-904;

[7] V. Turaev. Quantum Invariants of 3-Manifolds, Studies in Math. No 18, de Gruyter, Berlin (1994);

[8] H. Murakami, J. Murakami. The colored Jones polynomials and the simplicial volume of a knot, math.GT/9905075;

[9] R.M. Kashaev. The hyperbolic volume of knots from the quantum dilogarithm, Lett. Math. Phys. 39 (1997), 269-275;

[10] E. Witten. 2+1 Dimensional gravity as an exact soluble system, Nuclear Physics B311 (1988/89), 46-78;

[11] R. Benedetti, E. Guadagnini. Geometric cone surfaces and (2+1)-gravity coupled to particles, Nuclear Physic B 588 (2000) 436-450;

[12] S. Baseilhac. The coloured Jones polynomials from the quantum hyperbolic invariants, in preparation;

[13] W.D. Neumann, Hilbert’s 3rd problem and invariants of 3-manifolds, preprint math.GT/9712226;

[14] W.D. Neumann, Extended Bloch group and the Chern-Simons class, incomplete working version, available at W. Neumann’s web page (e-mail: neumann@math.mu.oz.au);

[15] W.D. Neumann, J. Yang. Bloch invariants of hyperbolic 3-manifolds, preprint math.GT/9712224;

[16] S. Baseilhac. Dilogarithme quantique et invariants de variétés de dimension trois, Thèse, Université Paul Sabatier, Toulouse (France), to be submitted in June 2001;

[17] M. Farber and V. Turaev. Absolute torsion, preprint math.DG/9810029;

[18] S. V. Matveev. Transformations of special spines and the Zeeman conjecture, Math. USSR Izvestia No 31 (1988), 423-434;

[19] R. Piergallini. Standard moves for standard polyhedra and spines, Rend. Circ. Mat. Palermo No 37, suppl. 18 (1988), 391-414;

[20] R. Benedetti, C. Petronio. Branched Standard Spines of 3-Manifolds, Lect. Notes in Math. No 1653, Springer (1997);

[21] R. Benedetti and C. Petronio. Lectures on Hyperbolic Geometry, Springer (1992);

[22] D. B. A. Epstein, R. Penner. Euclidian decompositions of non-compact hyperbolic manifolds, J. Diff. Geom. 27 (1988), 67-80;

[23] W.D. Neumann and D. Zagier. Volumes of hyperbolic 3-manifolds, Topology 24, No 3 (1985), 307-332;

[24] A. Yu. Makovetskii. On transformations of special spines and special polyhedra, Math. Notes, Vol. 65, No 3, 1999;

[25] R. Benedetti and C. Petronio. Combed 3-manifolds with concave boundary, framed Links and pseudo-legendrian links, math.GT/0001162, to appear in J. Knot Theory and Its Ramifications;
[26] T. Yosida. The $\eta$-invariant of hyperbolic 3-manifolds, Invent. Math. 81 (1985), 473-514;

[27] R. Meyerhoff. Density of the Chern-Simons invariant for hyperbolic 3-manifolds, Low-dimensional topology and Kleinian groups (D. B. A. Epstein, ed.), Cambridge University Press (1986), 217-239;

[28] W. D. Neumann. Combinatorics of triangulations and the Chern-Simons invariant for hyperbolic 3-manifolds, Topology'90 (Columbus, OH, 1990), de Gruyter, Berlin (1992);

[29] T. T. Q. Lê, Integrality and symmetry properties of quantum link invariants, Lectures given at the summer school “Invariants de noeuds et de 3-variétés”, Inst. Fourier, Grenoble (June-July 1999);

[30] V. Chari, A. Pressley. A Guide To Quantum Groups, Cambridge University Press (1994);

[31] R. Kirby, P. Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2, \mathbb{C})$, Invent. Math. 105 (1991), 473-545;

[32] L. Rozansky. A contribution of a $U(1)$-reducible connection to quantum invariants of links I: $R$-matrix and Burau representation, math.QA/9806004;

[33] S.-S. Chern, J. Simons. Characteristic forms and geometric invariants, Ann. of Math. (2) No 99 (1974), 48-69;

[34] J. Dupont, The dilogarithm as a characteristic class for flat bundles, J. Pure and App. Algebra 44 (1987), 137-164;

[35] S. Carlip. Quantum Gravity in 2+1 Dimensions, Cambridge Monographs on Mathematical Physics, Cambridge University Press (1998);

[36] E. Witten. Topology-changing amplitudes in 2+1 dimensional gravity, Nuclear Physics B323 (1989), 113-140;

[37] P. M. Melvin, H. R. Morton. The coloured Jones function, Com. Math. Phys. No 169 (1995), 501-520;

[38] D. Bar-Natan, S. Garoufalidis. On the Melvin-Morton-Rozansky conjecture, Invent. Math. No 125 (1996), 103-133;

[39] R. Benedetti and C. Petronio. Reidemeister torsion of 3-dimensional Euler structures with simple boundary tangency and pseudo-legendrian Knots, math.GT/0002143;

[40] J. W. Barrett, B. W. Westbury. Invariants of piecewise-linear 3-manifolds, Trans. Amer. Math. Soc. Vol. 348, No 10 (1996);

[41] S. Gelfan’d, D. Kazhdan. Invariants of three-dimensional manifolds, Geom. Fun. Anal. Vol. 6, No 2 (1996);

[42] R.M. Kashaev, V.V. Mangazeev, Yu. G. Stroganov. Star-square and tetrahedron equations in the Baxter-Bazhanov model. Int. J. Mod. Phys. A Vol.8, No 8 (1993), 1399-1409;

[43] R. J. Baxter, V. V. Bazhanov, J. H. H. Perk, Functional relations for transfer matrices of the chiral Potts model, Int. J. Mod. Phys. B4, No 803 (1990);

[44] P. Di Francesco, P. Mathieu, D. Senechal. Conformal Field Theory, Graduate Texts in Contemporary Physics, Springer (1997);