Learning with tree tensor networks: complexity estimates and model selection

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Abstract

In this paper, we propose and analyze a model selection method for tree tensor networks in an empirical risk minimization framework and analyze its performance over a wide range of smoothness classes. Tree tensor networks, or tree-based tensor formats, are prominent model classes for the approximation of high-dimensional functions in numerical analysis and data science. They correspond to sum-product neural networks with a sparse connectivity associated with a dimension partition tree $T$, widths given by a tuple $r$ of tensor ranks, and multilinear activation functions (or units). The approximation power of these model classes has been proved to be optimal (or near to optimal) for classical smoothness classes. However, in an empirical risk minimization framework with a limited number of observations, the dimension tree $T$ and ranks $r$ should be selected carefully to balance estimation and approximation errors. In this paper, we propose a complexity-based model selection strategy à la Barron, Birgé, Massart. Given a family of model classes associated with different trees, ranks, tensor product feature spaces and sparsity patterns for sparse tensor networks, a model is selected by minimizing a penalized empirical risk, with a penalty depending on the complexity of the model class. After deriving bounds of the metric entropy of tree tensor networks with bounded parameters, we deduce a form of the penalty from bounds on suprema of empirical processes. This choice of penalty yields a risk bound for the predictor associated with the selected model. In a least-squares setting, after deriving fast rates of convergence of the risk, we show that the proposed strategy is (near to) minimax adaptive to a wide range of smoothness classes including Sobolev or Besov spaces (with isotropic, anisotropic or mixed dominating smoothness) and analytic functions. We discuss the role of sparsity of the tensor network for obtaining optimal performance in several regimes. In practice, the amplitude of the penalty is calibrated with a slope heuristics method. Numerical experiments in a least-squares regression setting illustrate the performance of the strategy for the approximation of multivariate functions and univariate functions identified with tensors by tensorization (quantization).

1 Introduction

Typical tasks in statistical learning include the estimation of a regression function or of posterior probabilities for classification (supervised learning), or the estimation of the probability distribution of a random variable from samples of the distribution (unsupervised learning). These approximation tasks can be formulated as a minimization problem of a risk functional $\mathcal{R}(f)$ whose minimizer $f^*$ is the target (or oracle) function, and such that $\mathcal{R}(f) - \mathcal{R}(f^*)$ measures some discrepancy between the function $f$ and $f^*$. The risk is usually defined as

$$\mathcal{R}(f) = \mathbb{E}(\gamma(f, Z)),$$

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with \( Z = (X, Y) \) for supervised learning or \( Z = X \) for unsupervised learning, and where \( \gamma \) is a contrast function. For supervised learning, the contrast \( \gamma \) is usually chosen as \( \gamma(f, (x, y)) = \ell(y, f(x)) \) where \( \ell(y, f(x)) \) measures some discrepancy between \( y \) and the prediction \( f(x) \) for a given realization \( (x, y) \) of \((X, Y)\). In practice, given i.i.d. realizations \((Z_1, \ldots, Z_n)\) of \( Z \), an approximation \( \hat{f}_n^M \) is obtained by the minimization of an empirical risk

\[
\hat{R}_n(f) = \frac{1}{n} \sum_{i=1}^{n} \gamma(f, Z_i)
\]

over a subset of functions \( M \), also called a model class or hypothesis set. Assuming that the risk admits a minimizer \( f^M \) over \( M \), the error \( \mathcal{R}(\hat{f}_n^M) - \mathcal{R}(f^*) \) can be decomposed into two contributions: an approximation error \( \mathcal{R}(f^M) - \mathcal{R}(f^*) \) which quantifies the best we can expect from the model class \( M \), and an estimation error \( \mathcal{R}(\hat{f}_n^M) - \mathcal{R}(f^M) \) which is due to the use of a limited number of observations. For a given model class, a first problem is to understand how these errors behave under some assumptions on the target function. When considering an increasing sequence of model classes, the approximation error decreases but the estimation error usually increases. Then strategies are required for the selection of a particular model class.

In this paper, we consider the class of functions in tree-based tensor format, or tree tensor networks. These model classes are well-known approximation tools in numerical analysis and computational physics and have also been more recently considered in statistical learning. They are particular cases of feed-forward neural networks with an architecture given by a dimension partition tree and multilinear activation functions (see [31, 13]). For an overview of these tools, the reader is referred to the monograph [25] and the surveys [36, 6, 30, 11, 12]. Some results on the approximation power of tree tensor networks can be found in [38, 23, 5] for multivariate functions, or in [29, 28, 1, 2, 3] for tensorized (or quantized) functions.

A tree-based tensor format is a set of functions

\[
M^T_r(V) = \{ f \in V : \text{rank}_\alpha(f) \leq r_\alpha, \alpha \in T \},
\]

where \( T \) is a dimension partition tree over \( \{1, \ldots, d\} \), \( r = (r_\alpha) \in \mathbb{N}^{|T|} \) is a tuple of integers and \( V = V_1 \otimes \cdots \otimes V_d \) is a finite dimensional tensor space of multivariate functions (e.g., polynomials, splines), that is a tensor product feature space. A function \( f \) in \( M^T_r(V) \) is such that for each \( \alpha \in T \), the \( \alpha \)-rank \( \text{rank}_\alpha(f) \) of \( f \) is bounded by \( r_\alpha \). That means that for each \( \alpha \in T \), \( f \) admits a representation

\[
f(x) = \sum_{k=1}^{r_\alpha} g_k^\alpha(x_\alpha) h_k^\alpha(x_\alpha c)
\]
for some functions $g^\alpha_k$ and $h^\alpha_{c_k}$ of complementary groups of variables. Such a representation can be written using tensor diagram notations as

$$f(x) = g^\alpha_k h^\alpha_{c_k}$$

where $g^\alpha$ and $h^\alpha_{c}$ are order-two tensors with indices $(k, x_\alpha)$ and $(k, x_{\alpha c})$ respectively, and the edge between the two tensors has to be interpreted as a contraction of the two connected tensors.

A function $f$ in $M^T_r(V)$ admits a parametrization in terms of a collection of low-order tensors $v^\alpha = (v^\alpha_\alpha)_{\alpha \in T}$ forming a tree tensor network. For instance, for the dimension tree of Figure 1a the function $f$ admits the representation of Figure 1b using tensor diagram notations. If the tensors $v^\alpha$ are sparse, the tensor network $v$ is called a sparse tensor network. By identifying the tensors $v^\alpha$ with multilinear functions with values in $\mathbb{R}^{r_\alpha}$, the function $f$ also admits a representation as a composition of multilinear functions, that corresponds to a sum-product feed-forward neural network illustrated on Figure 2.

Model classes $M^T_r(V)$ associated with different trees (or architecture of the tensor network) are known to capture very different structures of multivariate functions. The choice of a good tree is then crucial in many applications. This requires robust strategies that select not only the ranks for a given tree but the tree and the associated ranks.

Figure 1: Dimension tree $T$ over $\{1, \ldots, 8\}$ (a) and corresponding tree tensor network $v = (v^\alpha_\alpha)_{\alpha \in T}$ (b). The vector $\phi^\nu(x_\nu) = (\phi^\nu_k(x_\nu))_{1 \leq k \leq N_\nu} \in \mathbb{R}^{N_\nu}$ represents $N_\nu$ features in the variable $x_\nu$.

The main contribution of the paper is a complexity-based strategy for the selection of a model class in an empirical risk minimization framework. Given a family $(M_m)_{m \in \mathcal{M}}$ of tensor networks (full or sparse) associated with different trees $T_m$, ranks $r_m$, feature tensor spaces $V_m$ (and different sparsity patterns for sparse tensor networks), and given the corresponding predictors $\hat{f}_m$ that minimize the empirical risk, we propose a strategy to select a particular model $\hat{m}$ with a guaranteed performance. For that purpose, we make use of the model selection approach of Barron, Birgé and Massart (see [34] for a general introduction to the topic) where $\hat{m}$ is obtained by minimizing a
penalized empirical risk
\[
\hat{R}_n(\hat{f}_m) + \text{pen}(m)
\]
with a penalty function pen(m) derived from complexity estimates of the model classes \(M_m\), of the form \(\text{pen}(m) \sim O(\sqrt{C_m/n})\) (up to logarithmic terms) in a general setting, or of the form \(\text{pen}(m) \sim O(C_m/n)\) (again up to logarithmic terms) in a bounded least-squares setting where faster convergence rates can be obtained. Here, the complexity \(C_m\) is related to the number of parameters in the tensor network (total number of entries of the tensors \(v^\alpha\)), or the number of non-zero parameters in the tensor network when exploiting sparsity of the tensors \(v^\alpha\).

In a bounded least-squares setting (for regression or density estimation), using a particular feature space based on tensorization of functions, we find that our strategy is minimax (or near to minimax) adaptive to a wide range of smoothness spaces including Sobolev or Besov spaces with isotropic, anisotropic or mixed dominating smoothness, and analytic function spaces.

In practice, the penalty is taken of the form \(\text{pen}(m) = \lambda\sqrt{C_m/n}\) (or \(\text{pen}(m) = \lambda C_m/n\) in a bounded least-squares setting), where \(\lambda\) is calibrated with the slope heuristics method proposed in [9]. The family of models can be generated by adaptive learning algorithms such as the ones proposed in [21, 20].

Note that our method is a \(\ell_0\) type approach. Convex regularization methods would be an interesting alternative route to follow. A straightforward convexification of tensor formats consists in using the sum of nuclear norms of unfoldings (see e.g. [39] for Tucker format) but this is known to be far from optimal from a statistical point of view (see [37]). A convex regularization method based on the tensor nuclear norm has been proposed for the Tucker format, or shallow tensor network, which comes with theoretical guarantees (see [42]). However, there is no straightforward extension of this approach to general tree tensor networks.

The outline of the paper is as follows. In Section 2 we describe the model class of tree tensor networks (or tree-based tensor formats) [25, 18]. In Section 3 we provide estimates of the metric and bracketing entropies in \(L^p\) spaces for tree tensor networks \(M_m\) with bounded parameters. In
Section 4 we derive bounds for the estimation error in a classical empirical risk minimization framework. These bounds are deduced from concentration inequalities for empirical processes. Then we present the complexity-based model selection approach and we derive risk bounds for particular choices of penalty in a general setting. We then introduce different collections of tensor networks (full or sparse) corresponding to different adaptive settings, where the feature space and the tree are considered either fixed or free, and we analyze the richness of these collections of models. Then in Section 5, we consider a bounded least-squares setting, for which we derive improved risk bounds adaptive to a large range of smoothness classes. Finally in Section 7 we present the practical aspects of the model selection approach, which includes the slope heuristics method for penalty calibration and the exploration strategies for the generation of a sequence of model classes and associated predictors. In Section 8 we present some numerical experiments that validate the proposed model selection strategy.

2 Tree tensor networks

We consider functions $f(x) = f(x_1, \ldots, x_d)$ defined on a product set $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_d$ and with values in $\mathbb{R}$. Typically, $\mathcal{X}_\nu$ is a subset of $\mathbb{N}$ or $\mathbb{R}$ but it could be a set of more general objects (vectors in $\mathbb{R}^{d_\nu}$, sequences, functions, graphs...).

2.1 Tensor product feature space

For each $\nu \in \{1, \ldots, d\}$, we introduce a finite-dimensional space $V_\nu$ of functions defined on $\mathcal{X}_\nu$. We let $\{\phi^\nu_{i_\nu} : i_\nu \in I^\nu\}$ be a basis of $V_\nu$, with $I^\nu = \{1, \ldots, N_\nu\}$. The functions $\phi^\nu_{i_\nu}(x_\nu)$ may be polynomials, splines, wavelets, kernel functions, or more general functions that extract features from a given input $x_\nu \in \mathcal{X}_\nu$. We let $\phi^\nu : \mathcal{X}_\nu \to \mathbb{R}^{N_\nu}$ be the associated feature map defined by $\phi^\nu(x_\nu) = (\phi^\nu_{1}(x_\nu), \ldots, \phi^\nu_{N_\nu}(x_\nu))^T \in \mathbb{R}^{N_\nu}$. The functions $\phi_i(x) = \phi^1_{i_1}(x_1) \ldots \phi^d_{i_d}(x_d)$, $i \in I = I^1 \times \ldots \times I^d$, form a basis of the tensor product space $V = V_1 \otimes \ldots \otimes V_d$. A function $f \in V$ admits a representation

$$f(x) = \sum_{i \in I} a_i \phi_i(x) = \sum_{i_1=1}^{N_1} \ldots \sum_{i_d=1}^{N_d} a_{i_1, \ldots, i_d} \phi^1_{i_1}(x_1) \ldots \phi^d_{i_d}(x_d),$$

(1)

where $a \in \mathbb{R}^I = \mathbb{R}^{N_1 \times \ldots \times N_d}$ is an algebraic tensor (or multi-dimensional array) of size $N_1 \times \ldots \times N_d$. The map $\phi$ from $\mathcal{X}$ to $\mathbb{R}^I$ which associates to $x$ the elementary tensor $\phi(x) = \phi^1(x_1) \otimes \ldots \otimes \phi^d(x_d) \in \mathbb{R}^I$ defines a tensor product feature map.

Remark 2.1. In Section 6.7, we present a particular feature space based on tensorization, that yields spaces $V_\nu$ with a tensor product structure and an identification of $f$ with a tensor of order higher than $d$.

2.2 Tree-based ranks

For any $\alpha \subset \{1, \ldots, d\} := D$, and $x \in \mathcal{X}$, we denote by $x_\alpha = (x_\nu)_{\nu \in \alpha} \in \mathcal{X}_\alpha$ the group of variables $\alpha$ that take values in $\mathcal{X}_\alpha = \times_{\nu \in \alpha} \mathcal{X}_\nu$. We let $\alpha^c = D \setminus \alpha$. 


Definition 2.2 (Ranks of multivariate functions and minimal subspaces). For a non-empty and strict subset \( \alpha \) in \( D \), the \( \alpha \)-rank of a function \( f : X \to \mathbb{R} \), denoted \( \text{rank}_\alpha(f) \), is the minimal integer \( r_\alpha \) such that

\[
f(x) = \sum_{k=1}^{r_\alpha} g_k^\alpha(x_\alpha) h_k^{\alpha^c}(x_{\alpha^c})
\]

for some functions \( g_k^\alpha : X_\alpha \to \mathbb{R} \) and \( h_k^{\alpha^c} : X_{\alpha^c} \to \mathbb{R} \). The \( r_\alpha \)-dimensional subspace spanned by the functions \( \{g_k^\alpha\}_{k=1}^{r_\alpha} \) is the \( \alpha \)-minimal subspace \( U^\text{min}_\alpha(f) \) of \( f \). For \( \alpha = \emptyset \) or \( \alpha = D \), we use the convention \( \text{rank}_{\emptyset}(f) = 1 \) and \( \text{rank}_D(f) = 1 \).

We let \( T \) be a dimension partition tree over \( D \), with root \( D \) and leaves \( \{\nu\}, 1 \leq \nu \leq d \). For a node \( \alpha \in T \), we denote by \( S(\alpha) \) the set of children of \( \alpha \). For any node \( \alpha \), we have either \( S(\alpha) = \emptyset \) (for leaf nodes) or \( S(\alpha) \geq 2 \) (for interior nodes). We denote by \( L(T) \) the set of leaves of \( T \), and by \( I(T) = \mathcal{T} \setminus \mathcal{L}(T) \) its interior nodes. For an interior node \( \alpha \in I(T) \), \( S(\alpha) \) forms a partition of \( \alpha \). The \( T \)-rank (or tree-based rank) of a function \( f \) is the tuple \( \text{rank}_T(f) = (\text{rank}_\alpha(f))_{\alpha \in T} \). The number of nodes of a dimension partition tree over \( D \) is bounded as \( |T| \leq 2d - 1 \) (with equality for a binary tree).

Remark 2.3 (Vector-valued functions). The above definition and the subsequent notions can be easily extended to the case of vector-valued functions \( f \) defined on \( X \) with values in \( \mathbb{R}^s \) \((s \in \mathbb{N})\), by identifying \( f \) with a function \( \tilde{f}(x_1, \ldots, x_d, i) = f_i(x_1, \ldots, x_d) \) of \( d + 1 \) variables. Most of the results of this paper then easily extends to this setting.

2.3 Tree tensor networks

Given a tuple \( r = (r_\alpha)_{\alpha \in T} \in \mathbb{N}^{|T|} \) we introduce the model class \( M^T_r(V) \) of functions in \( V \) with ranks bounded by \( r \),

\[
M^T_r(V) = \{ f \in V : \text{rank}_\alpha(f) \leq r_\alpha, \alpha \in T \}.
\]

The set \( M^T_r(V) \) is called a tree-based (or hierarchical) tensor format. A function \( f \in M^T_r(V) \) admits a representation \( \mathcal{R} \) for any \( \alpha \in T \), with \( \{g_k^\alpha\}_{k=1}^{r_\alpha} \) a basis of the minimal subspace \( U^\text{min}_\alpha(f) \). From the definition of minimal subspaces, \( f \) belongs to the tensor product space \( \bigotimes_{\alpha \in {S(D)}} U^\text{min}_\alpha(f) \), and therefore admits the representation (using tensor diagram notations)\(^1\)

\[
f(x) = \sum_{1 \leq k_\alpha \leq r_\alpha \text{ for } \alpha \in {S(D)}} v^D_{1,(k_\alpha)_{\alpha \in {S(D)}}} \prod_{\alpha \in {S(D)}} g_{k_\alpha}^\alpha(x_\alpha) = \begin{bmatrix} g_1^{\alpha_1} & \cdots & g_{r_{\alpha_{|S(D)|}}}^{\alpha_{|S(D)|}} \end{bmatrix}
\]

where \( v^D \) is a tensor in \( \mathbb{R}^{x_{\alpha \in S(D)}r_\alpha} \) and where \( g^\alpha(x_\alpha) = (g_k^{\alpha}(x_\alpha))_{1 \leq k \leq r_\alpha} \), with functions \( g_k^{\alpha} \in U^\text{min}_\alpha(f) \subset V_\alpha = \bigotimes_{\nu \in \alpha} V_\nu \). From the nestedness property of minimal subspaces \(^{18}\) Proposition

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\(^1\)We use tensor diagram notations where each node represents a tensor and an edge connecting two nodes represents a contraction of two tensors over one of their modes.
For any interior node $\alpha \in I(T) \setminus \{D\}$, the functions $g^\alpha_{k_\alpha} \in \bigotimes_{\beta \in S(\alpha)} U^\alpha_{\beta}(f)$ and therefore, they admit the representation

\begin{equation}
\begin{aligned}
g^\alpha_{k_\alpha}(x_\alpha) &= \sum_{1 \leq k_\beta \leq r_\beta \text{ for } \beta \in S(\alpha)} v^\alpha_{k_\alpha,(k_\beta)_{\beta \in S(\alpha)}} \prod_{\beta \in S(\alpha)} g^\beta(x_\beta) \\
&= \frac{1}{k_\alpha} \sum_{1 \leq k_\beta \leq r_\beta \text{ for } \beta \in S(\alpha)} v^\alpha_{k_\alpha,(k_\beta)_{\beta \in S(\alpha)}} \prod_{\beta \in S(\alpha)} g^\beta(x_\beta).
\end{aligned}
\end{equation}

where $v^\alpha \in \mathbb{R}^{r_\alpha \times (\times_{\beta \in S(\alpha)} r_\beta)}$. For a leaf node $\alpha \in L(T)$, the functions $g^\alpha_{k_\alpha} \in V_\alpha$ admit the representation

\begin{equation}
\begin{aligned}
g^\alpha_{k_\alpha}(x_\alpha) &= \sum_{i_\alpha \in I_\alpha} v^\alpha_{k_\alpha,i_\alpha} \phi^\alpha_{i_\alpha}(x_\alpha) \\
&= \frac{1}{k_\alpha} \sum_{i_\alpha \in I_\alpha} v^\alpha_{k_\alpha,i_\alpha} \phi^\alpha_{i_\alpha}(x_\alpha).
\end{aligned}
\end{equation}

A function $f$ in $M^T_I(V)$ therefore admits an explicit representation

\begin{equation}
\begin{aligned}
f(x) = \sum_{i_\alpha \in I_\alpha} \sum_{1 \leq k_\alpha \leq r_\alpha} \prod_{\beta \in S(\alpha)} v^\alpha_{k_\alpha,(k_\beta)_{\beta \in S(\alpha)}} \prod_{\gamma \in L(T)} v^\gamma_{k_\gamma} \prod_{\gamma \in L(T)} \phi^\gamma_{i_\gamma}(x_\gamma)
&= \frac{1}{k_\alpha} \sum_{i_\alpha \in I_\alpha} \sum_{1 \leq k_\alpha \leq r_\alpha} \prod_{\beta \in S(\alpha)} v^\alpha_{k_\alpha,(k_\beta)_{\beta \in S(\alpha)}} \prod_{\gamma \in L(T)} v^\gamma_{k_\gamma} \prod_{\gamma \in L(T)} \phi^\gamma_{i_\gamma}(x_\gamma)
\end{aligned}
\end{equation}

where the set of parameters $\mathbf{v} = (v^\alpha)_{\alpha \in T}$ form a tree tensor network (see Figure 1b for a representation using tensor diagram notations). The tensor $v^\alpha \in \mathbb{R}^{(1, \ldots , r_\alpha) \times I_\alpha} := \mathbb{R}^{K_\alpha}$, with $I_\alpha = \times_{\beta \in S(\alpha)} \{1, \ldots , r_\beta\}$ for $\alpha \in I(T)$ or $I_\alpha = \{1, \ldots , N_\alpha\} \text{ for } \alpha \in L(T)$.

**Remark 2.4** (Tree tensor networks as compositional functions). A function associated with a tree tensor network $\mathbf{v} = (v^\alpha)_{\alpha \in T}$ admits a representation as a composition of multilinear functions, by identifying a tensor $v^\alpha \in \mathbb{R}^{r_\alpha \times \ldots \times r_{\alpha N}}$ with a multilinear map from $\mathbb{R}^{r_1 \times \ldots \times r_{\alpha N}}$ to $\mathbb{R}^{r_\alpha}$. For example, for the dimension tree of Figure 1a, $f$ admits the representation

\begin{equation}
\begin{aligned}
f(x) &= v^{1,\ldots,8}(v^{1,2,3,4}(v^{1,2,3}(v^{1}(x_1)), v^{2,3}(v^{2}(x_2)), v^{3}(x_3))), v^{4}(x_4)), \\
v^{5,6,7,8}(v^{5,6}(v^{5}(x_5)), v^{6}(x_6)), v^{7}(x_7)), v^{8}(x_8)).
\end{aligned}
\end{equation}

For details, see Appendix A.

A tensor network $\mathbf{v} = (v^\alpha)_{\alpha \in T}$ is said to be a sparse tensor network if the $v^\alpha$ are sparse tensors. For $\Lambda^\alpha \subset K^\alpha$, a tensor $v^\alpha$ is said to be $\Lambda^\alpha$-sparse if $v_{i_\alpha}^\alpha = 0$ for $j \in K^\alpha \setminus \Lambda^\alpha$. For a given $\Lambda = \times_{\alpha \in T} \Lambda^\alpha$, with $K^\alpha \subset \Lambda^\alpha$, a tensor network $\mathbf{v}$ is said to be $\Lambda$-sparse if the $v^\alpha$ are $\Lambda^\alpha$-sparse for all $\alpha \in T$.  

7
2.4 Parameter space and representation map

We introduce the product space of parameters

$$\mathcal{P}_{V,T,r} := \times_{\alpha \in T} \mathcal{P}_{\alpha}, \quad \mathcal{P}_{\alpha} := \mathbb{R}^{K_{\alpha}},$$

and let $\mathcal{R}_{V,T,r}$ be the map which associates to the tensor network $\mathbf{v} \in \mathcal{P}_{V,T,r}$ the function $f = \mathcal{R}_{V,T,r}(\mathbf{v})$ defined by (6), so that

$$M^T_r(V) = \{f = \mathcal{R}_{V,T,r}(\mathbf{v}) : \mathbf{v} \in \mathcal{P}_{V,T,r}\}.$$

From the representation (6), we obtain the following

**Lemma 2.5.** The representation map $\mathcal{R}_{r,T,V}$ is a multilinear map from the product space $\mathcal{P}_{V,T,r} = \times_{\alpha \in T} \mathcal{P}_{\alpha}$ to $V$.

For $\Lambda ^{\alpha} \subset K^{\alpha}$, we denote by $\mathcal{P}_{\Lambda ^{\alpha}}^{\alpha}$ the linear subspace of $\Lambda ^{\alpha}$-sparse tensors in $\mathcal{P}_{\alpha}$. Then for $\Lambda = \times_{\alpha \in T} \Lambda ^{\alpha}$, we denote by $\mathcal{P}_{V,T,r,\Lambda} \subset \mathcal{P}_{V,T,r}$ the set of $\Lambda$-sparse tensor networks and we introduce the corresponding model class

$$M^T_{r,\Lambda}(V) = \{f = \mathcal{R}_{V,T,r}(\mathbf{v}) : \mathbf{v} \in \mathcal{P}_{V,T,r,\Lambda}\}.$$

2.5 Complexity of a tensor network

When interpreting a tensor network $\mathbf{v} \in \mathcal{P}_{V,T,r} = \times_{\alpha \in T} \mathcal{P}_{\alpha}$ as a neural network, a classical measure of complexity is the number of neurons, which is the sum of ranks $r_{\alpha}, \alpha \in T$. From an approximation or statistical perspective, a more natural measure of complexity is the number of parameters (or representation complexity), that is the dimension $\sum_{\alpha \in T} \dim(\mathcal{P}_{\alpha})$ of the corresponding parameter space $\mathcal{P}_{V,T,r}$, or the number of weights of the corresponding neural network. Then the representation complexity of $\mathbf{v}$ is

$$C(T, r, V) := \sum_{\alpha \in T} |K_{\alpha}| = \sum_{\alpha \in T} r_{\alpha} \prod_{\beta \in S(\alpha)} r_{\beta} + \sum_{\alpha \in T} r_{\alpha}N_{\alpha}. \quad (7)$$

For a sparse tensor network $\mathbf{v} \in \mathcal{P}_{V,T,r,\Lambda} = \times_{\alpha \in T} \mathcal{P}_{\Lambda ^{\alpha}}^{\alpha}$, a natural measure of complexity is given by

$$C(T, r, V, \Lambda) = \sum_{\alpha \in T} |\Lambda ^{\alpha}|, \quad (8)$$

which only counts the number of non-zero parameters (or non-zero weights in the corresponding neural network). We note that $C(T, r, V, \Lambda) \leq C(T, r, V)$. The different measures of complexity defined above lead to the definition of different approximation tools and corresponding approximation classes, see [1, 2, 3] for tensor networks, and [22] for similar results on ReLU or RePU neural networks.
3 Metric entropy of tree tensor networks

In this section, we provide an estimate of the metric entropy of the set of tree tensor networks (full or sparse) with normalized parameters. This is obtained by showing that tree tensor networks admit a Lipschitz parametrization.

We assume that the sets $X_\nu$, for all $\nu \in D = \{1, \ldots, d\}$, and the set $X$ is equipped with the product measure $\mu = \mu_1 \otimes \ldots \otimes \mu_d$. For $1 \leq p \leq \infty$, we consider the space $L^p_\mu(X)$ of real-valued measurable functions defined on $X$, with bounded norm $\| \cdot \|_{p,\mu}$ defined by

$$\|f\|_{p,\mu}^p = \int_X |f(x)|^p d\mu(x) \quad \text{for} \ 1 \leq p < \infty, \ \text{or} \ \|f\|_{\infty,\mu} = \mu\text{-ess sup}_{X} |f|.$$  

If $V_\nu \subset L^p_{\mu_\nu}(X_\nu)$ for all $\nu \in D$, then $V \subset L^p_\mu(X)$.

3.1 Normalized parametrization

A function $f \in M^T_r(V)$ admits infinitely many equivalent parametrizations. From the multilinearity of the representation map $R_{V,T,r}$ (see Lemma 2.5), it is clear that the model class $M^T_r(V)$ is a cone, i.e. $aM^T_r(V) \subset M^T_r(V)$ for any $a \in \mathbb{R}$. Given some norms $\| \cdot \|_{p,\alpha}$ on the spaces $P^\alpha = \mathbb{R}^{K^\alpha}$, $\alpha \in T$, and the corresponding product norm on $P_{V,T,r}$ defined by

$$\|(v^\alpha)_{\alpha \in T}\|_{P_{V,T,r}} = \max_{\alpha \in T} \|v^\alpha\|_{P^\alpha},$$

we have

$$M^T_r(V) = \{af : a \in \mathbb{R}, f \in M^T_r(V)_1\},$$

where $M^T_r(V)_1$ are elements of $M^T_r(V)$ with bounded parameters, defined by

$$M^T_r(V)_1 = \{f = R_{V,T,r}(v) : v \in P_{V,T,r}, \|v\|_{P_{V,T,r}} \leq 1\}. \quad (9)$$

The same normalization is used for defining the model class of sparse tensor networks $M^T_{r,\Lambda}(V)_1 = M^T_{r,\Lambda}(V) \cap M^T_r(V)_1$.

3.2 Continuity of the parametrization

We here study the continuity of the representation map $R_{V,T,r}$ as a map from $P_{V,T,r} = \times_{\alpha \in T} P^\alpha$ to $V \subset L^p_\mu(X)$. From the multilinearity of $R_{V,T,r}$ (Lemma 2.5), we easily deduce the following property.

**Lemma 3.1.** Assuming $V \subset L^p_\mu(X)$, the multilinear map $R_{V,T,r}$ from $P_{V,T,r}$ to $V \subset L^p_\mu(X)$ is continuous and such that for all $f = R_{V,T,r}((v^\alpha)_{\alpha \in T})$ in $M^T_r(V)$,

$$\|f\|_{p,\mu} \leq L_{p,\mu} \prod_{\alpha \in T} \|v^\alpha\|_{P^\alpha}$$

for some constant $L_{p,\mu} < \infty$ independent of $f$ defined by

$$L_{p,\mu} = \sup_{f = R_{V,T,r}((v^\alpha)_{\alpha \in T})} \frac{\|f\|_{p,\mu}}{\prod_{\alpha \in T} \|v^\alpha\|_{P^\alpha}}. \quad (10)$$
We denote by $B(P^\alpha)$ the unit ball of $P^\alpha$ and by $B(P_{V,T,r})$ the unit ball of $P_{V,T,r}$. The set $M_r^T(V)_1$ defined by (9) is such that

$$M_r^T(V)_1 = \mathcal{R}_{V,T,r}(B(P_{V,T,r})).$$

(11)

We then deduce that the map $\mathcal{R}_{V,T,r}$ is Lipschitz continuous on the set $M_r^T(V)_1$.

**Lemma 3.2.** Assuming $V \subset L^p_\mu(X)$, for all $f = \mathcal{R}_{V,T,r}(v)$ and $\tilde{f} = \mathcal{R}_{V,T,r}(\tilde{v})$ in $M_r^T(V)_1$,

$$\|f - \tilde{f}\|_{p,\mu} \leq L_{p,\mu} \sum_{\alpha \in T} \|v^\alpha - \tilde{v}^\alpha\|_{P^\alpha} \leq L_{p,\mu}|T|\|v - \tilde{v}\|_{P_{V,T,r}}.$$

Proof. Denoting by $\alpha_1, \ldots, \alpha_K$ the elements of $T$, we have

$$f - \tilde{f} = \sum_{k=1}^K \mathcal{R}_{V,T,r}(\tilde{v}^{\alpha_1}, \ldots, v^{\alpha_k} - \tilde{v}^{\alpha_k}, \ldots, v^{\alpha_K}).$$

Then from Lemma 3.1 we obtain

$$\|f - \tilde{f}\|_{p,\mu} \leq L_{p,\mu} \sum_{k=1}^K \|v^{\alpha_k} - \tilde{v}^{\alpha_k}\|_{P^{\alpha_k}} \prod_{i<k} \|\tilde{v}^{\alpha_i}\|_{P^{\alpha_i}} \prod_{i>k} \|v^{\alpha_i}\|_{P^{\alpha_i}},$$

(12)

and we conclude by noting that $\|v^{\alpha}\|_{P^{\alpha}} \leq 1$ and $\|\tilde{v}^{\alpha}\|_{P^{\alpha}} \leq 1$ for all $\alpha \in T$. \hfill \Box

### 3.3 Metric entropy

The metric entropy $H(\epsilon, K, \| \cdot \|_X)$ of a compact subset $K$ of a normed vector space $(X, \| \cdot \|_X)$ is defined as

$$H(\epsilon, K, \| \cdot \|_X) = \log N(\epsilon, K, \| \cdot \|_X),$$

with $N(\epsilon, K, \| \cdot \|_X)$ the covering number of $K$, which is the minimal number of balls of radius $\epsilon$ (for $\| \cdot \|_X$) necessary to cover $K$. We have the following result on the metric entropy of tensor networks with bounded parameters.

**Proposition 3.3.** Assume that $V \subset L^p_\mu(X)$, $1 \leq p \leq \infty$. The metric entropy of the model class

$$M_r^T(V)_R = \{af : a \in \mathbb{R}, |a| \leq R, f \in M_r^T(V)_1\} \quad (13)$$

in $L^p_\mu(X)$ is such that

$$H(\epsilon, M_r^T(V)_R, \| \cdot \|_{p,\mu}) \leq C(T, r, V) \log(3\epsilon^{-1}RL_{p,\mu}|T|).$$

The metric entropy in $L^p_\mu(X)$ of the model class of $\Lambda$-sparse tensors

$$M_{r,\Lambda}^T(V)_R = M_r^T(V)_R \cap M_{r,\Lambda}^T(V) \quad (14)$$

is such that

$$H(\epsilon, M_{r,\Lambda}^T(V)_R, \| \cdot \|_{p,\mu}) \leq C(T, r, V, \Lambda) \log(3\epsilon^{-1}RL_{p,\mu}|T|).$$
Proof. The covering number of the unit ball $B(\mathcal{P}^\alpha)$ of the $|K^\alpha|$-dimensional space $\mathcal{P}^\alpha$ is such that $N(\epsilon, B(\mathcal{P}^\alpha), \| \cdot \|_{\mathcal{P}^\alpha}) \leq (3\epsilon^{-1})^{|K^\alpha|}$. Then the unit ball $B(\mathcal{P}_{V,T,r})$ of the product space $\mathcal{P}_{V,T,r}$ equipped with the product topology has a covering number

$$N(\epsilon, B(\mathcal{P}_{V,T,r}), \| \cdot \|_{\mathcal{P}_{V,T,r}}) \leq \prod_{\alpha \in T} N(\epsilon, B(\mathcal{P}^\alpha), \| \cdot \|_{\mathcal{P}^\alpha}) \leq (3\epsilon^{-1})^{C(T,r,V)}$$

with $C(T,r,V) = \sum_{\alpha \in T} |K^\alpha|$. From the Lipschitz continuity of $R_{V,T,r}$ on $M^T_r(V)_1$ (Lemma 3.2), we deduce that $N(\epsilon, M^T_r(V)_1, \| \cdot \|_{p,\mu}) \leq (3\epsilon^{-1}L_{p,\mu}|T|)^{C(T,r,V)}$, from which we deduce that $N(\epsilon, M^T_r(V)_1, \| \cdot \|_{p,\mu}) \leq (3\epsilon^{-1}RL_{p,\mu}|T|)^{C(T,r,V)}$, which ends the proof of the first statement. For sparse tensors, we first note that the unit ball $B(\mathcal{P}_{\Lambda^\alpha})$ of the $|\Lambda^\alpha|$-dimensional space $\mathcal{P}_{\Lambda^\alpha}$ is such that $N(\epsilon, B(\mathcal{P}_{\Lambda^\alpha}), \| \cdot \|_{\mathcal{P}^\alpha}) \leq (3\epsilon^{-1})^{|\Lambda^\alpha|}$. Then a similar proof yields the desired upper bound with $C(T,r,V,\Lambda) = \sum_{\alpha \in T} |\Lambda^\alpha|$. □

3.4 A particular choice of norms

Assume that $V \subset L^p_\mu(\mathcal{X})$. The continuity constant $L_{p,\mu}$ of the map $R_{V,T,r}$ defined by (10) depends on $p$, $\mu$, the norms on parameter spaces $\mathcal{P}^\alpha$ and the chosen basis for $V$. We here introduce a particular choice of norms and basis functions which allows to bound the continuity constant $L_{p,\mu}$.

For any interior node $\alpha \in \mathcal{I}(T)$, we introduce a norm $\| \cdot \|_{\mathcal{P}^\alpha}$ over the space $\mathcal{P}^\alpha$ defined by

$$\|v^\alpha\|_{\mathcal{P}^\alpha} = \max_{(z_\beta)_{\beta \in S(\alpha)} \in \mathbb{R}^{|X_{V,T,r}|}} \frac{\|v^\alpha((z_\beta)_{\beta \in S(\alpha)})\|_p}{\prod_{\beta \in S(\alpha)} \|z_\beta\|_p},$$

where the tensor $v^\alpha \in \mathbb{R}^{|X_{\Lambda^\alpha}| \times (X_{V,T,r})^{|\beta|}}$ is identified with a multilinear map from $\times_{\beta \in S(\alpha)} \mathbb{R}^{|X_{V,T,r}|}$ to $\mathbb{R}^{|\Lambda^\alpha|}$, and where $\| \cdot \|_p$ refers to the vector $\ell^p$-norm (for more details, see Appendix A). For a leaf node $\alpha \in \mathcal{L}(T)$, we introduce a norm $\| \cdot \|_{\mathcal{P}^\alpha}$ over the space $\mathcal{P}^\alpha$ defined by

$$\|v^\alpha\|_{\mathcal{P}^\alpha} = \max_{z_\alpha \in \mathbb{R}^{|\Lambda^\alpha|}} \frac{\|v^\alpha(z_\alpha)\|_p}{\|z_\alpha\|_p},$$

where the order-two tensor $v^\alpha \in \mathbb{R}^{|\Lambda^\alpha| \times |\Lambda^\alpha|}$ is identified with a linear map from $\mathbb{R}^{|\Lambda^\alpha|}$ to $\mathbb{R}^{|\Lambda^\alpha|}$. This corresponds to the matrix $\ell^p$-norm of $v^\alpha$. We assume that for any $\nu \in D$, the feature map $\phi^\nu : \mathcal{X}_\nu \to \mathbb{R}^{|\Lambda^\alpha|}$ is such that $\|\phi^\nu\|_{p,\mu} = 1$. For $p = \infty$, that means that basis functions $\phi_i^\nu(x_\nu)$ have a unit norm in $L^\infty_{\mu}(\mathcal{X}_\nu)$. For $p < \infty$, that means that $\sum_{i=1}^{|\Lambda^\alpha|} \|\phi_i^\nu\|_{p,\mu} = 1$, which can be obtained by rescaling basis functions so that $\|\phi_i^\nu\|_{p,\mu} = N_{\nu}^{-1/p}$.

Proposition 3.4. Assume $V \subset L^p_\mu(\mathcal{X})$, $1 \leq p \leq \infty$. With the above choice of norms and normalization of basis functions, the continuity constant $L_{p,\mu}$ defined by (10) is such that $L_{p,\mu} \leq 1$, and for all $1 \leq q \leq p$, $L_{q,\mu} \leq \mu(\mathcal{X})^{1/q-1/p} L_{p,\mu} \leq \mu(\mathcal{X})^{1/q-1/p}$.

Proof. See Appendix [3] □

4 Risk bounds and model selection for tree tensor networks

Let $\mathcal{X}$ equipped with a finite measure $\mu$. In this section we analyze empirical risk minimization for contrasts computed over general families of functions associated to tree tensor networks built on approximation spaces in $L^\infty_\mu(\mathcal{X})$. 

11
4.1 Risk bounds for tree tensor networks

We consider a model class $M$ of tensor networks with bounded parameters (with the norms defined in Section 3.4), with $M := M_T(V)_R$ for full tensor networks or $M^{T}_r,\Lambda(V)_R$ for $\Lambda$-sparse tensor networks. We here consider as fixed the approximation space $V$, the dimension tree $T$ and the ranks $r \in \mathbb{N}^{\|T\|}$, and also the sparsity pattern $\Lambda$ for sparse tensor networks. We assume that $V \subset L^\infty_{\mu}(X)$. We denote by $C_M = C(T, r, V)$ the representation complexity of $M$ defined by (7) for full tensor networks, or $C_M = C(T, r, V, \Lambda)$ the sparse representation complexity of $M$ defined by (5). We consider a risk

$$R(f) = \mathbb{E}(\gamma(f, Z)),$$

where $Z$ is a random variable taking values in $\mathcal{Z}$ and where $\gamma : \mathbb{R}^X \times \mathcal{Z} \to \mathbb{R}$ is some contrast function. The minimizer of the risk over measurable functions defined on $\mathcal{X}$ is the target function $f^*$. For $f$ random (depending on the data), $\mathbb{E}(\gamma(f, Z))$ shall be understood as an expectation $\mathbb{E}_Z(\gamma(f, Z))$ w.r.t. $Z$ (conditional to the data). We introduce the excess risk

$$\mathcal{E}(f) = R(f) - R(f^*).$$

Given the model class $M$, we denote by $f^M$ a minimizer over $M$ of the risk $R$, and by $\hat{f}^M_n$ a minimizer over $M$ of the empirical risk

$$\hat{R}_n(f) = \frac{1}{n} \sum_{i=1}^{n} \gamma(f, Z_i),$$

which is seen as an empirical process over $M$. To obtain bounds of the estimation error, it remains to quantify the fluctuations of the centered empirical process $\hat{R}_n(f)$ defined by

$$\hat{R}_n(f) = \hat{R}_n(f) - R(f) = \frac{1}{n} \sum_{i=1}^{n} \gamma(f, Z_i) - \mathbb{E}(\gamma(f, Z)).$$

Assumption 4.1 (Bounded contrast). Assume that $\gamma$ is uniformly bounded over $M \times \mathcal{Z}$, i.e.

$$|\gamma(f, Z)| \leq B$$

holds almost surely for all $f \in M$, with $B$ a constant independent of $f$.

Assumption 4.2. Assume that $\gamma(\cdot, Z)$ is Lipschitz continuous over $M \subset L^\infty_{\mu}(X)$, i.e.

$$|\gamma(f, Z) - \gamma(g, Z)| \leq \mathcal{L}\|f - g\|_{\infty, \mu}$$

holds almost surely for all $f, g \in M$, with $\mathcal{L}$ a constant independent of $f$ and $g$.

Example 4.3 (Least-squares bounded regression). We consider a random variable $Z = (X, Y)$, with $Y$ a random variable with values in $\mathbb{R}$, $X$ a $\mathcal{X}$-valued random variable with probability law $\mu$. We consider the least-squares contrast $\gamma(f, Z) = |Y - f(X)|^2$. The excess risk $\mathcal{E}(f) = R(f) - R(f^*) = \|f - f^*\|_{2, \mu}^2$ admits $f^*(x) = \mathbb{E}[Y | X = x]$ as a minimizer. In the bounded regression setting, it is assumed that $|Y| \leq R$ almost surely. For all $f \in M$, we have $\gamma(f, Z) \leq 2(|Y|^2 + \|f\|_{2, \mu}^2)$, so that $0 \leq \gamma(f, Z) \leq B$ almost surely, with $B = 4R^2$. Also, it holds almost surely

$$|\gamma(f, Z) - \gamma(g, Z)| \leq (2|Y| + \|g\|_{\infty, \mu} + \|f\|_{\infty, \mu})\|f - g\|_{\infty, \mu}.$$
Example 4.4 \((L^2 \text{ density estimation})\). For the problem of estimating the probability distribution of a random variable \(X\), we consider \(Z = X\). We consider the estimation of the probability law \(\eta\) of \(X\). Assuming that \(\eta\) admits a density \(f^*\) with respect to the measure \(\mu\), and assuming \(f^* \in L^2_\mu(\mathcal{X})\), we consider the contrast \(\gamma(f, x) = \|f\|^2_{\mu} - 2f(x)\), so that \(\mathcal{E}(f) = \mathcal{R}(f) - \mathcal{R}(f^*) = \|f - f^*\|^2_{\mu}\) admits \(f^*\) as a minimizer. We assume that \(\mu\) is a finite measure on \(\mathcal{X}\) and that \(f^*\) is uniformly bounded by \(R\). Then \(|\gamma(f, x)| \leq B\) almost surely with \(B = R(\mu(\mathcal{X})R + 2)\). Also, for all \(f, g \in M\), we have almost surely

\[
|\gamma(f, X) - \gamma(g, X)| = \|f\|^2_{\mu} - \|g\|^2_{\mu} - 2(f(X) - g(X))| \\
\leq |\int (f - g)(f + g)d\mu| + 2\|f - g\|_{\infty, \mu} \\
\leq (\|f + g\|_{1, \mu} + 2)\|f - g\|_{\infty, \mu} \\
\leq \mathcal{L}\|f - g\|_{\infty, \mu}
\]

with \(\mathcal{L} = 2(\mu(\mathcal{X})R + 1)\).

Proposition 4.5. Under Assumptions 4.1 and 4.2, for any \(t > 0\), with probability larger than \(1 - \exp(-t)\),

\[
\mathcal{E}(\hat{f}_n^M) \leq \mathcal{E}(f^M) + 8B\sqrt{C_M}\sqrt{\frac{2\log(6\mathcal{L}B^{-1}R|T|/\sqrt{n})}{n}} + 4B\sqrt{\frac{t}{2n}}.
\]

By integrating according to \(t\), we obtain that

\[
\mathbb{E}\mathcal{E}(\hat{f}_n^M) \leq \mathcal{E}(f^M) + 8B\sqrt{C_M}\sqrt{\frac{2\log(6\mathcal{L}B^{-1}R|T|/\sqrt{n})}{n}} + 2B\sqrt{\frac{\pi}{n}}.
\]

This result is a standard application of the bounded difference inequality (see for instance Theorem 5.1 in [34]) applied to \(\sup_{f \in M} |\mathcal{R}_n(f)|\), together with a control on \(\mathbb{E}\sup_{f \in M} |\mathcal{R}_n(f)|\) with the metric entropy result of Proposition 3.3. The proof is given in Appendix C.2.

4.2 Model selection for tree tensor networks

We now consider a family of tensor networks \((M_m)_{m \in \mathcal{M}}\) indexed by a countable set \(\mathcal{M}\). For full tensor networks, a model \(M_m = M^T_{T_m}(V_m)_R\) is associated with a particular tree \(T_m\), a rank \(r_m\), an approximation space \(V_m\), and a radius \(R\). For sparse tensor networks, a model \(M_m = M^T_{r_m, \Lambda_m}(V_m)_R\) has for additional parameter a sparsity pattern \(\Lambda_m\). We denote by \(C_m\) the number of parameters of the model \(M_m\), that is \(C_m = C(T_m, r_m, V_m)\) for full tensors, or \(C_m = C(T_m, r_m, V_m, \Lambda_m)\) for sparse tensors.

For some \(m \in \mathcal{M}\), we let \(f_m\) be a minimizer of the risk over \(M_m\),

\[
f_m \in \arg\min_{f \in M_m} \mathcal{R}(f),
\]

and \(\hat{f}_m\) be a minimizer of the empirical risk over \(M_m\),

\[
\hat{f}_m \in \arg\min_{f \in M_m} \hat{\mathcal{R}}_n(f).
\]

At this stage of the procedure, we have at hand a family of predictors \(\hat{f}_m\) and our goal is to provide a strategy for selecting a good predictor in the collection. We follow a standard strategy that corresponds to the so-called Vapnik’s structural minimization of the risk method (see for instance...
Given some penalty function \( \text{pen} : \mathcal{M} \to \mathbb{R}^+ \), we define \( \hat{m} \) as the minimizer over \( \mathcal{M} \) of the criterion

\[
\text{crit}(m) := \hat{R}_n(\hat{f}_m) + \text{pen}(m),
\]

and we finally select the predictor \( \hat{f}_{\hat{m}} \) according to the criterion (16). This procedure is classical in non parametric statistics and similar model selection approaches can be found in [34, 24, 10].

For a suitable choice of penalty which takes into account both the complexity of the models and the richness of the model collection, we provide a risk bound for the selected predictor. Let

\[
N_c := N_c(\mathcal{M}) = |\{m \in \mathcal{M} : C_m = c\}|
\]

be the number of models with complexity \( c \) in the collection. The following result corresponds to the general Theorem 8.1 in [34] applied to our framework.

**Theorem 4.6.** Let \( \bar{w} > 0 \). Under Assumptions 4.1 and 4.2, if the penalty is such that

\[
\text{pen}(m) \geq \lambda_m \sqrt{C_m/n} + 2B \sqrt{\bar{w}C_m + \log(N_c)} \frac{\log(N_c)}{2n},
\]

with

\[
\lambda_m = 4B \sqrt{2 \log(6LB^{-1}R|T_m|\sqrt{n})},
\]

then the estimator \( \hat{f}_{\hat{m}} \) selected according to the criterion (16) satisfies the following risk bound

\[
\mathbb{E}(\mathcal{E}(\hat{f}_{\hat{m}})) \leq \inf_{m \in \mathcal{M}} \{\mathcal{E}(f_m) + \text{pen}(m)\} + \frac{B}{\exp(\bar{w}) - 1} \sqrt{\frac{\pi}{2n}}.
\]

**Proof.** The proof of Theorem 4.6 is given in Appendix C.3, it is a direct adaptation of the proof of Theorem 8.1 in [34].

### 4.3 Collections of models and their richness

We here present and analyze the richness of different collections of tensor networks \((M_m)_{m \in \mathcal{M}}\), where each model has a particular feature space \(V_m\), a tree \(T_m\), a tuple of ranks \(r_m\). These collections of models depend on whether the feature space and the tree are considered as fixed. More precisely, we consider the following collections of models \((M_m = M_{r_m}^{T_m}(V_m))_{m \in \mathcal{M}}\) with \(\mathcal{M}\) corresponding to one of the following collections:

- \(\mathcal{M}_{V,T}\): fixed feature space \(V_m = V\), fixed tree \(T_m = T\), variable ranks \(r_m\),
- \(\mathcal{M}_T\): variable feature space \(V_m\), fixed tree \(T\), variable ranks \(r_m\),
- \(\mathcal{M}_V\): variable feature space \(V_m\), variable tree \(T_m\), variable ranks \(r_m\).

For variable feature spaces, we classically consider that \(V_m := V_{N_m}\) with \(N_m \in \mathbb{N}^d\) and for any \(N \in \mathbb{N}^d\), \(V_N = V_{1,N_1} \otimes \ldots \otimes V_{d,N_d}\), where \((V_{\nu,N_{\nu}})_{\nu \in \mathbb{N}}\) is a sequence of subspaces of univariate functions, with \(N_{\nu} = \dim(V_{\nu,N_{\nu}})\). For variable trees, we consider trees in the family of trees with arity \(a\) (or \(a\)-ary trees), the case \(a = 2\) corresponding to (full) binary trees. The next result provides upper bounds of the complexity of the above defined families of tensor networks.
Proposition 4.7 (Collections of full tensor networks). Consider a family of full tensor networks \((M_m = M_{r_m}^T(V_m))_{m \in M}\) with \(M\) equal to \(M_{V,T}\), \(M_T\) or \(M_\star\). For any tree \(T\) and any feature space \(V\), \(N_c(M_{V,T}) \leq N_c(M_T) \leq N_c(M_\star)\), and
\[
\log(N_c(M_\star)) \leq 2a(c + d \log(c)).
\]

with \(a\) the arity of the considered trees.

Proof. See Appendix C.4.

When exploiting sparsity, we consider models \(M_m = M_{r_m}^T(V_m, \Lambda_m)\) depending on an additional sparsity pattern \(\Lambda_m\). For variable feature spaces \(V_m = V_{N_m}\), we consider models \(m\) such that \(N_m \in \mathbb{N}^d\) satisfies
\[
N_m \leq g(C_m),
\]
(19)
with \(g\) some increasing function of the complexity \(C_m\) of the model \(m\). This is a reasonable assumption from a practical point of view, where for a given complexity, we avoid the exploration of infinitely many features. We use the same notations \(M_{V,T}\), \(M_T\) and \(M_\star\) for the corresponding families of models, with \(\Lambda_m\) considered as an additional free variable. The complexities of these collections of sparse tensor networks are higher than the corresponding complexities for full tensor networks, but only up to logarithmic terms, as shown in the next result.

Proposition 4.8 (Collections of sparse tensor networks). Consider a family of sparse tensor networks \((M_m = M_{r_m}^T(V_{N_m}, \Lambda_m))_{m \in M}\) with \(M\) equal to \(M_{V,T}\), \(M_T\) or \(M_\star\), with variable sparsity patterns \(\Lambda_m\) and \(N_m \leq g(C_m)\). For any tree \(T\) and any feature space \(V\), \(N_c(M_{V,T}) \leq N_c(M_T) \leq N_c(M_\star)\), and
\[
\log(N_c(M_\star)) \leq 5ac \log(c) + 2c \log(g(c)).
\]

If we further assume that \(\log(g(c)) \leq \delta \log(c)\) for some \(\delta > 0\), then
\[
\log(N_c(M_\star)) \leq (5a + 2\delta)c \log(c).
\]

Proof. See Appendix C.5.

Together with Proposition 4.7 (or Proposition 4.8), Theorem 4.6 provides a strong justification for using a penalty proportional to \(\sqrt{C_m/n}\). However, it is known that the Vapnik’s structural minimization of the risk may lead to suboptimal rates of convergence. For instance, in the bounded regression setting, it is known that a penalty proportional to the Vapnik–Chervonenkis dimension (typically in \(O(C_m/n)\)) leads to minimax rates of convergence in various setting (see for instance Chapter 12 in [24]) whereas Vapnik’s structural minimization of the risk (typically with penalty in \(O(\sqrt{C_m/n})\)) is too pessimistic to provide fast rates of convergence.

In the case of bounded least squares contrasts, we give in Section 5 improved risk bounds. That allows us to prove that our model selection strategy is (near to) adaptive minimax in several frameworks, as shown in Section 6.

5 Oracle inequality for least squares inference with tree tensor networks

In this section, we provide an improved excess risk bound in the specific case of least squares contrasts. Our results come from Talagrand inequalities and generic chaining bounds; we follow the presentation given in the monograph [32]. The excess risk bound given below strongly relies
on the link between the excess risk and the variance of the excess loss, as explained in Chapter 5 of [32] and Chapter 8 in [34]. We then derive an improved model selection result for least squares inference by following the approach presented in Sections 8.3 and 8.4 of [34] or in Section 6.3 of [32].

Let $\gamma$ be either the least squares contrast in the bounded regression setting (as described in Example 4.3), or the least squares contrast for density estimation (as described in Example 4.4).

5.1 Improved risk bounds for least squares contrasts

We first consider as model class a tree tensor network $M = M^T_r(V)_R$ or $M = M^T_{r,\Lambda}(V)_R$ (respectively full or $\Lambda$-sparse) with bounded parameters and it assumed that the feature tensor space $V \subset L_\infty^\mu(X)$ where $\mu$ is the distribution of the random variable $X$ in the regression setting (see Example 4.3) or the reference measure for density estimation (see Example 4.4).

Proposition 5.1. Under Assumptions 4.1 and 4.2, there exists an absolute constant $A$ and a constant $\kappa$ such that for any $\epsilon \in (0, 1]$ and any $t > 0$, with probability at least $1 - A \exp(-t)$, it holds

$$\mathbb{E}(\hat{f}_n^M) \leq (1 + \epsilon)\mathbb{E}(f^M) + \frac{\kappa R^2}{n} \left[ \frac{a_T C_M}{\epsilon^2} \log^+ \left( \frac{n\epsilon^2}{a_T C_M} \right) + \frac{t}{\epsilon} \right]$$

where $a_T = 1 + \log^+ \left( \frac{|T|}{4\epsilon} \right)$, and $\kappa$ depends linearly on $\mu(X)^2$. Then by integrating according to $t$, we obtain that for any $\epsilon \in (0, 1]$,

$$\mathbb{E}(\hat{f}_n^M) \leq (1 + \epsilon)\mathbb{E}(f^M) + \frac{\kappa R^2}{n} \left[ \frac{a_T C_M}{\epsilon^2} \log^+ \left( \frac{n\epsilon^2}{a_T C_M} \right) + \frac{A}{\epsilon} \right].$$

Proof. The proof of the proposition is given in Appendix D.1.

Note that the term $a_T$ is upper bounded by a term of the order of $\log(d)$ because $|T| \leq 2d$. Thus the constants in the risk bound (20) does not explode with the dimension $d$ in regression. Note however that in density estimation, the constant $\kappa$ depends linearly on the mass $\mu(X)$ of the reference measure, which may grow exponentially with $d$.

5.2 Oracle inequality

As in Section 4.2 we now consider a family of tensor networks $(M_m)_{m \in \mathcal{M}}$ indexed by a countable set $\mathcal{M}$, with either $M_m = M^T_r(V_m)_R$ for full tensor networks, or $M_m = M^T_{r,\Lambda_m}(V_m)_R$ for sparse tensor networks. We consider features spaces $V_m \subset L_\infty^\mu(X)$ with $X$ equipped with a finite measure $\mu$. As before, $N_c(\mathcal{M})$ denotes the number of models with complexity $c$ in the collection $\mathcal{M}$ (see Section 4.3).

Theorem 5.2. Let $\bar{w} > 0$. Under Assumptions 4.1 and 4.2, there exists numerical constants $K_1$ and $K_2$ and $K_3$ such that if the penalty satisfies

$$\text{pen}(m) = K_1 R^2 \left[ \frac{b_m C_m}{\epsilon^2} \log^+ \left( \frac{n\epsilon^2}{b_m C_m} \right) + \frac{\bar{w} C_m + \log(N_{C_m})}{n\epsilon} \right]$$

2With $\mu(X) = 1$ for regression.
with \( b_m = 1 + \log^+ \left( \frac{nT_m}{4e} \right) \), then the estimator \( \hat{f}_m \) selected according to the penalized criterion (16) satisfies the following oracle inequality

\[
\mathbb{E} \mathcal{E}(\hat{f}_m) \leq \frac{1 + \varepsilon}{1 - \varepsilon} \inf_{m \in \mathcal{M}} \{ \mathcal{E}(f_m) + K_2 \text{pen}(m) \} + \frac{K_3 R^2}{\exp(\bar{w}) - 1} \frac{1 + \varepsilon}{1 - \varepsilon} \frac{1}{n}.
\]  

(21)

Proof. The proof, adapted from Theorem 6.5 in [32], is given in Appendix D.3.

For collections of models \( \mathcal{M} \) such that \( \log(N_{C_m}(\mathcal{M})) \sim C_m \log(C_m) \delta \) for some \( \delta \geq 1 \), this theorem provides an improved oracle inequality bound

\[
\mathbb{E} \mathcal{E}(\hat{f}_m) \lesssim \inf_{m \in \mathcal{M}} \mathcal{E}(f_m) + \frac{C_m}{n} \log(n) \log(C_m) \delta,
\]  

(22)

with a penalty in \( \frac{C_m}{n} \), up to logarithmic terms.

In Section 6 we will derive adaptive (near to) optimal rates of convergence for smoothness classes (in the minimax sense) from this model selection result. In Section 7.1 we explain how to calibrate the penalty in practice using the slope heuristics method.

6 Least-squares inference and minimax adaptivity for smoothness classes

Here we consider bounded least-squares inference with target functions \( f^* \) in classical smoothness spaces including Sobolev or Besov spaces (with isotropic, anisotropic or mixed dominating smoothness), or spaces of analytic functions. We consider functions defined on the hypercube \([0,1)^d\) equipped with the uniform measure \( \mu \). For clarity, we let \( L^p := L^p_\mu([0,1)^d) \).

A classical approach is to consider tensor networks with feature tensor spaces \( V_m \) that are adapted to the smoothness of the function (e.g. tensorized splines or wavelets for Besov smoothness, or tensorized polynomials for analytic functions). Here, we use an alternative and powerful approach based on tensorization of functions, which can be interpreted as a particular definition of feature space. It does not require to adapt the tool to the regularity of the function. This approach is described in Section 6.1 and Section 6.2 (for more details see [1, 3]). Then in Section 5.3 we show that our model selection strategy with this tool is minimax adaptive to a wide range of smoothness classes.

6.1 Feature space based on tensorization of functions at fixed resolution

For any integers \( b, L \in \mathbb{N} \) with \( b \geq 2 \), we introduce a uniform partition of the interval \([0,1)\) into \( b^L \) intervals of equal length \( b^{-L} \). Any \( x \in [0,1) \) can be written

\[
x = \sum_{k=1}^{L} i_k b^{-k} + b^{-L} \bar{x} := t_{b,L}(i_1, \ldots, i_L, \bar{x}),
\]

where \( (i_1, \ldots, i_L) \in \{0, \ldots, b - 1\}^L \) is the representation in base \( b \) of the integer \( i \) such that \( x \in [b^{-L}i, b^{-L}(i + 1)) \), and \( \bar{x} \in [0,1) \). The integer \( L \) is called the resolution. The map \( t_{b,L} \) is a bijection from \( \{0, \ldots, b - 1\}^d \times [0,1) \) to \([0,1)\) with inverse \( t_{b,L}^{-1}(x) = (i_1, \ldots, i_L, \bar{x}) \) such that

\[
i_k = \lfloor b^k x \rfloor \bmod b, \quad \bar{x} = b^L x - \lfloor b^L x \rfloor.
\]
A function \( f(x) \) defined on \([0,1]\) can then be linearly identified with a \((L+1)\)-variate function \( f(i_1, \ldots, i_L, \bar{x}) \) defined on \([0, \ldots, b-1]^L \times [0,1)\). The map \( \mathcal{T}_{b,L} \) which associates to a function \( f \) the multivariate function \( f \) is called the tensorization map.

For multivariate functions \( f(x_1, \ldots, x_d) \) defined on the hypercube \([0,1]^d\), we proceed in a similar way for each dimension. Each variable \( x_\nu \) is identified with a tuple \((i^\nu_1, \ldots, i^\nu_L, \bar{x}_\nu) = t_{b,L}(x_\nu)\), and \( f \) is linearly identified with a \( d(L+1) \)-variate function \( f(i^1_1, \ldots, i^1_L, \ldots, i^d_1, \ldots, i^d_L, \bar{x}_1, \ldots, \bar{x}_d) \) defined on \([0, \ldots, b-1]^Ld \times [0,1]^d\).

For any \( 1 \leq p \leq \infty \), the tensorization map \( \mathcal{T}_{b,L} \) which associates to a \( d \)-variate function \( f \) the tensor \( f \) of order \((L+1)d\) is a linear isometry from \( L_p([0,1]^d) \) to the tensor Banach space \( (\mathbb{R}^b)^{\otimes Ld} \otimes L_p([0,1]^d) = L_p^b([0, \ldots, b-1]^{Ld} \times [0,1]^d) \) equipped with the uniform measure \( \mu \) over \([0, \ldots, b-1]^{Ld} \times [0,1]^d \) [Theorem 2.2].

To define an approximation tool, we then introduce a finite-dimensional tensor space

\[
\mathbf{V}_L = (\mathbb{R}^b)^{\otimes dL} \otimes (\mathbb{P}_k)^{\otimes d}
\]

where \( \mathbb{P}_k \) is the space of univariate polynomials of degree less than \( k \). To a tensor \( f \in \mathbf{V}_L \) correspond a function \( f = \mathcal{T}_{b,L}^{-1} f \in L_\infty([0,1]^d) \) which is a spline of degree \( k \) on the uniform partition of \([0,1]^d\). This defines a feature tensor space with dimensions \( N = (N_1, \ldots, N_{(L+1)d}) \), \( N_\nu = b \) for \( 1 \leq \nu \leq Ld \) and \( N_\nu = k + 1 \) for \( \nu > Ld \), with a feature map

\[
\phi(x) = e(i^1_1) \otimes e(i^1_2) \otimes \ldots \otimes e(i^d_L) \otimes \varphi(\bar{x}_1) \otimes \ldots \otimes \varphi(\bar{x}_d)
\]

where \( e(i) \in \mathbb{R}^b \) is such that \( e(i)_j = \delta_{i,j} \) and \( \varphi(t) = (\varphi(t))_{0 \leq j \leq k} \) is a basis of \( \mathbb{P}_k \).

### 6.2 Tensor networks with variable resolution: complexity and approximation classes

Here we consider tensor networks over the tensor space \( \mathbf{V}_L \), either \( M^T_r(\mathbf{V}_L) \) for full tensor networks or \( M^{T,\Lambda}_r(\mathbf{V}_L) \) for sparse tensor networks, where \( T \) is a dimension tree over \( \{1, \ldots, d(L+1)\} \), \( r \in \mathbb{N}^{|T|} \), and \( \Lambda \) some sparsity pattern. This defines a subset of \( d \)-variate functions through the map \( \mathcal{T}_{b,L}^{-1} \). For a linear tree

\[
T = T_L := \{\{1\}, \ldots, \{d(L+1)\}, \{1,2\}, \{1,2,3\}, \ldots, \{1,\ldots, d(L+1)\}\},
\]

the tensor network corresponds to a tensor train (TT) format.

**Remark 6.1.** For the approximation of functions from classical smoothness classes, and when working with a fixed tree, this choice of tree is rather natural. Each interior node in \( T_L \) is related to a splitting of variables into a group of low-resolution variables and high-resolution variables (see discussions in [1, 3] on the impact of the tree).

#### 6.2.1 Collections of tensor networks and their richness.

We consider as an approximation tool a collection of tensor networks with variable resolutions and variable ranks with a tensor train format. More precisely, we define a collection of models \( (M_m)_{m \in M} \) in \( L_\infty([0,1]^d) \) defined by

\[
M_m = \mathcal{T}_{b,Lm}^{-1} M_{r_m}^{T_m}(\mathbf{V}_{Lm})_R \quad \text{or} \quad M_m = \mathcal{T}_{b,L}^{-1} M_{r_m,\Lambda_m}(\mathbf{V}_{Lm})_R,
\]

with variable resolutions \( L_m \in \mathbb{N} \), linear trees \( T_{Lm} \) and variable ranks \( r_m \in \mathbb{N}^{|T_{Lm}|} \).
Remark 6.2. Note that for a particular resolution $L$, we here consider a single tree $T_L$. This is sufficient for obtaining our minimax results for classical smoothness classes in Section 6.3. Working with variable trees may be relevant for highly structured functions or functions beyond classical smoothness classes. Our tree selection procedure should be able to recover a near-optimal tree, that is relevant for applications where there is no a priori for the selection of a good tree.

Note that since $T_{b,d}$ is a linear isometry from $L^p_\mu$ to $L^p_\mu$, the metric entropy $H(\epsilon, M_m, \| \cdot \|_{p,\mu})$ of $M_m$ is equal to the metric entropy of the corresponding tensor network in $L^p_\mu$.

For a model $m$ with complexity $c$, we clearly have $L_m \leq c$. Then the number $N_c(M)$ of models with complexity $c$ is such that

$$N_c(M) = \sum_{L=0}^{\infty} N_c(M_{T_L}) = \sum_{L=0}^{c} N_c(M_{T_L}),$$

with $M_{T_L}$ the collection of tensor networks with fixed tree $T_L$, fixed feature space $V_L$, and variable ranks. We deduce from Proposition 4.7 (with $d$ replaced by $(L+1)d \leq c$) and Proposition 4.8 (with a constant function $g(c) = \max\{b, k+1\}$) that

$$N_c(M) \lesssim c \log(c), \quad (23)$$

for both full and sparse tensor networks.

Given a collection of tensor networks $(M_m)_{m \in M}$ introduced above (either full or sparse), we define an approximation tool $\Phi = (\Phi_c)_{c \in \mathbb{N}}$, where the set $\Phi_c$ is the union of models with complexity less than $c$, i.e.

$$\Phi_c = \bigcup_{m \in M, C_m \leq c} M_m.$$ 

The approximation tool $\Phi$ is respectively denoted by $\Phi^F = (\Phi^F_c)_{c \in \mathbb{N}}$ and $\Phi^S = (\Phi^S_c)_{c \in \mathbb{N}}$ for full and sparse tensor networks.

6.2.2 Approximation classes

The best approximation error of $f^*$ in $L^2$ by a tensor network with complexity less than $c$ is

$$E(f^*, \Phi_c)_{L^2} = \inf_{f \in \Phi_c} E(f)^{1/2} = \inf_{f \in \Phi_c} \| f - f^* \|_{2,\mu}.$$ 

Then given a growth function $\gamma : \mathbb{N} \to \mathbb{N}$, an approximation class for tensor networks can be defined as the set of functions

$$A_\infty(\gamma, \Phi, L^2) = \{ f : \sup_{c \geq 1} \gamma(c) E(f, \Phi_c)_{L^2} < \infty \},$$

which corresponds to functions that can be approximated with tree tensor networks with an error $E(f^*, \Phi_c)$ in $O(\gamma(c)^{-1})$.

To polynomial growth functions $\gamma(c) = c^\alpha$ ($\alpha > 0$) correspond approximation classes

$$A_\infty^\alpha := A_\infty^\alpha(\Phi, L^2) = \{ f : \sup_{c \geq 1} c^\alpha E(f, \Phi_c)_{L^2} < \infty \}$$

containing functions that can be approximated by tensor networks with algebraic convergence rate in $E(f^*, \Phi_c) \lesssim c^{-\alpha}$. In [11, 12], it is proved that the sets $A_\infty^\alpha$ are quasi-Banach spaces, equipped with the quasi-norm $\| f \|_{A_\infty^\alpha} = \| f \|_{L^2} + \| f \|_{A_\infty^\alpha}$ with $\| f \|_{A_\infty^\alpha} = \sup_{c \geq 1} c^\alpha E(f, \Phi_c)_{L^2}$. A whole range
of quasi-Banach spaces $A^\beta_q$ can be defined by interpolation between $L^2$ and a space $A^\infty_\infty$, with $A^\beta_q = (L^2, A^\infty_\infty)_{\beta/\alpha,q}$, $0 < \beta < \alpha$, $0 < q \leq \infty$. The spaces $A^\alpha_\infty$ are included in $A^\infty_\infty$ and correspond to a slightly stronger convergence of approximation error.

The approximation classes associated with full and sparse tensor networks (associated with two different notions of complexity) are respectively denoted by

$$F^\alpha_q = A^\alpha_q(\Phi^F, L^2) \quad \text{and} \quad S^\alpha_q = A^\alpha_q(\Phi^S, L^2).$$

For any $0 < q \leq \infty$, we have the following continuous embeddings \[3\text{ Theorem 4.12]}

$$F^\alpha_q \hookrightarrow S^\alpha_q \hookrightarrow F^{\alpha/2}_q.$$  

That means that if full tensor networks achieve an approximation rate as $O(c^{-\alpha})$ then sparse tensor networks achieve at least the same approximation rate. However, if sparse tensor networks achieve an approximation rate as $O(c^{-\alpha})$, then full tensor networks achieve at least an approximation rate as $O(c^{-\alpha/2})$, i.e. with a possible deterioration of the rate by a factor 2.

**Remark 6.3.** We recall that the results of this section are valid for a collection of models where for a given resolution $L$, we consider a single tree $T_L$. When considering variable trees for a fixed resolution, we obtain much larger approximation classes. However, these are highly nonlinear classes and their properties have not been studied yet.

### 6.3 Rates for smoothness classes

Here we show that (near to) minimax rates can be achieved by tensor networks with our model selection strategy for a wide range of smoothness classes encompassing isotropic Besov spaces, anisotropic Besov spaces, Besov spaces with mixed dominating smoothness and spaces of analytic functions.

For that, we rely on the oracle inequality from Section 5.1, the estimates of the complexity of collections of tensor networks from Section 6.2 and approximation results from \[2, 3\].

We start by providing a useful lemma which provides of convergence of the estimator $\hat{f}_m$ for a target function in an approximation class of tensor networks. In this section, we work under the assumptions of Theorem 5.2.

**Lemma 6.4.** For any $\alpha > 0$, if $f^* \in A^\alpha_\infty(\Phi, L^2)$, the estimator $\hat{f}_m$ obtained with the model selection strategy and the approximation tool $\Phi$ (either $\Phi^F$ or $\Phi^S$) satisfies

$$\mathbb{E}\|\hat{f}_m - f^*\|_{2,\mu}^2 \lesssim n^{-2\alpha + 1} \log(n)^{4\alpha}.$$

**Proof.** Using (5.2) and the complexity estimate (23), we have

$$\mathbb{E}\mathcal{E}(\hat{f}_m) \lesssim \inf_{m \in M} \mathcal{E}(f_m) + C_m/n \log(n) \log(C_m) \lesssim \inf_{c \in \mathbb{N}} c^{-2\alpha} + c/n \log(n) \log(c).$$

Let $c$ be such that $c^{-2\alpha} = n \log(n) \log(c)$. We have

$$c = n^{1/2\alpha+1} \log(n)^{-1/2\alpha+1} \log(c)^{-1/2\alpha+1}. \quad (25)$$

For $n, c \geq 2$, we thus have $\log(c) \lesssim n^{1/2\alpha+1} \log(n)$, Together with (25), it yields $c \gtrsim n^{1/2\alpha+1} \log(n)^{-1/2\alpha+1}$ and therefore $\mathbb{E}\mathcal{E}(\hat{f}_m) \lesssim c^{-2\alpha} \lesssim n^{-2\alpha+1} \log(n)^{4\alpha}$. \[ \square \]

Next we denote by $\hat{f}^F_m$ and $\hat{f}^S_m$ the estimators obtained with our model selection strategy using full tensor networks or sparse tensor networks respectively.
6.3.1 Besov spaces with isotropic smoothness

We let $B^s_q(L^p)$ denote the Besov space of functions with regularity order $s > 0$, primary parameter $p$ and secondary parameter $q$ (see [3, 14] for a definition and characterization). The parameter $p$ is related to the norm with which the regularity is measured.

For $s < 1$ and $q = \infty$, $B^s_\infty(L^p)$ corresponds to the space Lip($s, L^p$). For $p = q$ and non-integer $s > 0$, $B^s_q(L^p)$ corresponds to the (fractional) Sobolev space $W^{s,p}$. For the special case $p = 2$, $B^s_q(L^2)$ is equal to the Sobolev space $W^{s,2}$ for any $s > 0$. For $s > d(1/\tau - 1/p)_+$, it holds that $B^s_q(L^\tau) \hookrightarrow L^p$.

It is known that the minimax rate for functions $f^* \in B^s_q(L^p)$ is lower bounded by $n^{-\frac{2s}{2s+d}}$ (see e.g. [15, 19]).

**Besov spaces $B^s_q(L^p)$ for $p \geq 2$.** We first consider Besov spaces $B^s_q(L^p)$ with smoothness measured in $L^2$ norm or stronger norm.

**Theorem 6.5** (Minimax rates for Besov spaces $B^s_q(L^p)$ for $p \geq 2$). Assume the target function $f^* \in B^s_q(L^p)$ with $s > 0$, $2 \leq p \leq \infty$ and $0 < q \leq \infty$. Then for sufficiently large $n$,

$$
\mathbb{E}\|\hat{f}_m - f^*\|^2_{2,\mu} \lesssim n^{-\frac{2s}{2s+d}} \log(n)^{\frac{4s}{2s+d}}
$$

with $\hat{s} = s$ if $k \geq s - 1/2$ or an arbitrary $\hat{s} < s$ if $k < s - 1/2$.

**Proof.** From [3, Theorem 6.6], we have the continuous embedding $B^s_q(L^p) \hookrightarrow F^\hat{s}_{q,d} \hookrightarrow F^\hat{s}_{\infty,d}$, for any $s > \hat{s} > 0$, $2 \leq p \leq \infty$ and $0 < q \leq \infty$. The result follows from Lemma 6.4. \hfill \Box

The above theorem implies that our model selection procedure with full tensor networks achieves minimax rates (up to logarithmic term) for the whole range of Besov spaces $B^s_q(L^p)$, $p \geq 2$. It is thus minimax adaptive to the regularity over these Besov spaces, i.e. it achieves minimax rates without the need to adapt the approximation tool to the regularity of the target function. Note that minimax rates for $B^s_q(L^p)$, $p \geq 2$, are also achieved with linear approximation tools such as splines, wavelets or kernel methods, but obtaining minimax adaptivity requires a suitable strategy for the selection of a particular family of splines, wavelets or kernels. Here, tensor networks are associated with spline functions of a fixed degree $k$, and minimax adaptivity is obtained for any fixed value of $k$, including $k = 0$. This is made possible by allowing models with high resolution (corresponding to deep tensor networks).

**Besov spaces $B^s_q(L^\tau)$ for $\tau < 2$.** Now we consider the case of Besov spaces $B^s_q(L^\tau)$ with a regularity measured in a weaker $L^\tau$-norm, $\tau < 2$. These are spaces of functions with ”inhomogeneous smoothness” that can be only well captured by nonlinear approximation tools. We consider spaces $B^s_q(L^\tau)$ with $1/2 < 1/\tau < s/d + 1/2$. In the usual $(1/\tau, s)$ DeVore diagram of smoothness spaces, this corresponds to Besov spaces strictly above the critical line characterized by $s = d(1/\tau - 1/2)$. Besov spaces strictly above this line ($s > d(1/\tau - 1/2)$) are compactly embedded in $L^2$, while Besov spaces strictly below this line ($s < d(1/\tau - 1/2)$) are not embedded in $L^2$.

**Theorem 6.6** (Minimax rates for Besov spaces $B^s_q(L^\tau)$). Assume the target function $f^* \in B^s_q(L^\tau)$ with $s > 0$, $1/2 < 1/\tau < s/d + 1/2$ and $0 < q \leq \tau$. Then for sufficiently large $n$, the estimators using full or sparse tensor networks respectively satisfy

$$
\mathbb{E}\|\hat{f}_m^S - f^*\|^2_{2,\mu} \lesssim n^{-\frac{2s}{2s+d}} \log(n)^{\frac{4s}{2s+d}}
$$
\[ \mathbb{E}\| \tilde{f}_{m}^{F} - f^{*} \|_{2,\mu}^{2} \lesssim n^{-\frac{2s}{2s+1} \log(n)} \]

with \( \tilde{s} = s \) if \( k \geq s - 1/2 \) or an arbitrary \( \tilde{s} < s \) if \( k < s - 1/2 \).

Proof. From \cite{3} Theorem 6.8, we have the continuous embedding \( B_{q}^{s}(L^r) \hookrightarrow S_{q}^{\tilde{s}/d} \hookrightarrow S_{\infty}^{\tilde{s}/d} \). The result then follows from Lemma 6.4 and (24).

For such spaces \( B_{q}^{s}(L^r) \) above the critical line and \( \tau < 2 \), it is known that optimal linear estimators do not achieve the optimal rate. For \( d = 1 \), optimal linear estimators achieve a rate in \( n^{-\frac{2s-2(1/\tau-1/2)}{2s+1-(1/\tau-1/2)}} \), which is larger than the minimax rate \( n^{-\frac{2s}{2s+\tau}} \). Only nonlinear methods of estimation are able to achieve the minimax rate \( \cite{10} \). The above result shows that our model selection strategy with sparse tensor networks achieves minimax rates or rates arbitrarily close to minimax (up to a logarithmic term) for the whole range of spaces \( B_{q}^{s}(L^r) \), without requiring to adapt the tool to the regularity. Note that the estimation using full tensor networks presents a slightly deteriorated rate. In this nonlinear estimation setting, exploiting sparsity of the tensor network is useful to obtain an optimal performance. Note that the chosen polynomial degree \( k \) has only a little impact on the obtained results. If this degree is adapted to the regularity \( (k \geq s-1/2) \), the minimax rate is achieved (up to logarithmic term) but any degree \( k \) (including \( k = 0 \)) allows to achieve a rate arbitrarily close to optimal.

6.3.2 Besov spaces with mixed dominating smoothness

We here consider Besov spaces \( MB_{q}^{s}(L^p) \) with mixed dominating smoothness (see \cite{3, 27, 26} for a definition and characterization). For \( p = q = 2 \), \( MB_{2}^{s}(L^2) \) corresponds to the mixed Sobolev spaces \( H^{s,mix} \) of functions \( f \) with partial derivatives \( \partial_{\alpha} f \) in \( L^2 \) for any tuple \( \alpha = (\alpha_1, \ldots, \alpha_d) \) with \( \max \alpha \alpha_\nu \leq s \).

We consider spaces \( MB_{q}^{s}(L^r) \) such that \( s > (1/\tau - 1/2)_+ \), which are embedded in \( L^2 \) and strictly above the critical embedding line (with \( \tau < 2 \) and \( s < 1/\tau - 1/2 \), \( s \) is not embedded in \( L^2 \)).

**Theorem 6.7** (Minimax rates for Besov spaces \( MB_{q}^{s}(L^r) \) with mixed dominating smoothness). Assume the target function \( f^{*} \in MB_{q}^{s}(L^r) \) with \( s > (1/\tau - 1/2)_+ \) and \( 0 < q \leq \tau \). Then for sufficiently large \( n \), the estimators using full or sparse tensor networks respectively satisfy

\[ \mathbb{E}\| \tilde{f}_{m}^{S} - f^{*} \|_{2,\mu}^{2} \lesssim n^{-\frac{2s}{2s+1} \log(n)} \]

and

\[ \mathbb{E}\| \tilde{f}_{m}^{F} - f^{*} \|_{2,\mu}^{2} \lesssim n^{-\frac{2s}{2s+2} \log(n)} \]

with \( \tilde{s} = s \) if \( k \geq s - 1/2 \) or an arbitrary \( \tilde{s} < s \) if \( k < s - 1/2 \).

Proof. From \cite{4} Theorem 6.8, we have the continuous embedding \( MB_{q}^{s}(L^r) \hookrightarrow S_{q}^{\tilde{s}/d} \hookrightarrow S_{\infty}^{\tilde{s}/d} \). The result then follows from Lemma 6.4 and (21).

For \( s > (1/p - 1/2)_+ \), it is known that the minimax rate is lower bounded by \( n^{-\frac{2s}{2s+1}} \) (up to a logarithmic term) \( \cite{40} \). Therefore, Theorem 6.7 implies that our model selection strategy using sparse tensor networks achieve a rate arbitrarily close to minimax, up to a logarithmic term. With full tensor networks, the rate is close to minimax but slightly worse. We emphasize that this result is valid for any value of \( k \), including \( k = 0 \). However, by adapting the degree \( k \) to the regularity (i.e., \( k \geq s - 1/2 \), sparse tensor networks even achieve exactly the minimax rate.

22
Note that for \( p \geq 2 \), linear estimators based on hyperbolic cross approximation \[17\] achieve minimax rates, with a suitable choice of univariate approximation tools adapted to the regularity. Let us finally mention that for \( p \geq 2 \) and full tensor networks, by using [3, Theorem 6.6], we can obtain a slightly better rate in \( n^{-\frac{2d(\alpha)}{2d+5(d)+C(d)}} \) with \( 1 < C(d) < 2 \).

### 6.3.3 Anisotropic Besov spaces

We now consider anisotropic Besov spaces \( AB_q^\alpha(L^p) \), \( \alpha = (s_1, \ldots, s_d) \in \mathbb{R}_+^d \), where \( s_\nu > 0 \) is related to the regularity order with respect to the \( \nu \)-th coordinate (see [3, 33] for a definition based on directional moduli of smoothness and the characterization of these spaces). For \( \alpha = (s_1, \ldots, s_d) \) with \( s > 0 \), \( AB_q^\alpha(L^p) \) coincides with the isotropic Besov space \( B_q^s(L^p) \). For a tuple \( \alpha \), we let \( s(\alpha) := d(s_1^{-1} + \ldots + s_d^{-1})^{-1} \) be the aggregated smoothness parameter, such that \( \min_\nu s_\nu := s \leq s(\alpha) \leq \bar{s} := \max_\nu s_\nu \).

We consider spaces \( AB_q^\alpha(L^\tau) \) with \( \alpha \) such that \( s(\alpha) > d(1/\tau - 1/2) \), which are embedded in \( L^2 \). For these spaces, the minimax rate is in \( n^{-\frac{2d(\alpha)}{2d+5(1+\alpha)}} \) [35] and this rate can be achieved by linear estimators only for \( \tau \geq 2 \).

**Theorem 6.8** (Minimax rates for anisotropic Besov spaces \( AB_q^\alpha(L^\tau) \)). Assume the target function \( f^* \in AB_q^\alpha(L^\tau) \) with \( \alpha \in \mathbb{R}_+^d \) such that \( s(\alpha) > d(1/\tau - 1/2) \) and \( 0 < q \leq \tau \). Then for sufficiently large \( n \), the estimators using full or sparse tensor networks respectively satisfy

\[
\mathbb{E}\|\hat{f}^S_m - f^*\|^2_{\mu, s} \lesssim n^{-\frac{2d}{2d+5d}} \log(n)^{\frac{4d}{2d+5d}}
\]

and

\[
\mathbb{E}\|\hat{f}^F_m - f^*\|^2_{2, \mu} \lesssim n^{-\frac{2d}{2d+2d}} \log(n)^{\frac{4d}{2d+2d}}
\]

with \( \bar{s} = s(\alpha) \) if \( k \geq \bar{s} - 1/2 \) or an arbitrary \( \bar{s} < s \) if \( k < \bar{s} - 1/2 \).

*Proof.* From [3, Theorem 6.8], we have the continuous embedding \( AB_q^\alpha(L^\tau) \hookrightarrow S_q^{\bar{s}/d} \hookrightarrow S_\infty^{\bar{s}/d} \). The result then follows from Lemma 6.4 and (24). \( \square \)

Theorem 6.4 implies that our model selection strategy using sparse tensor networks achieves a rate arbitrarily close to minimax, up to a logarithmic term. With full tensor networks, the rate is close to minimax but slightly worse. We again emphasize that this result is valid for any \( k \in \mathbb{N} \), including \( k = 0 \). However, by adapting the degree \( k \) to the highest regularity \( \bar{s} \) (i.e., \( k \geq \bar{s} - 1/2 \)), sparse tensor networks even achieve exactly the minimax rate (up to the logarithmic term).

Let us mention that for \( p \geq 2 \) and full tensor networks, by using [3, Theorem 6.6], we can obtain a slightly better rate in \( n^{-\frac{2\bar{s}}{2d+5d+C(d)d}} \) with \( 1 < C(d) < 2 \).

Note that with a sufficient anisotropy such that \( \sum_{\nu=1}^d s_\nu^{-1} \leq \beta^{-1} \) with \( \beta \) independent of \( d \), we have \( s(\alpha) \geq d\beta^{-1} \), and for an arbitrary \( \bar{\beta} < \beta \), our strategy with sparse (resp. full) tensor networks achieves a rate in \( n^{-\frac{2\bar{\beta}}{2(\bar{\beta}+1)}} \) (resp. \( n^{-\frac{\bar{\beta}}{(\bar{\beta}+1)}} \)), which is independent of the dimension \( d \).

### 6.3.4 Analytic functions.

Here, we consider the case of analytic functions on a bounded interval. We restrict the analysis to functions defined on \([0, 1]\) but the result could be easily extended to the multivariate case.

**Theorem 6.9** (Analytic functions). Assume \( f^* : [0, 1] \to \mathbb{R} \) admits an analytic extension on an open complex domain including \([0, 1]\). Then for sufficiently large \( n \),

\[
\mathbb{E}\|\hat{f}^F_m - f^*\|^2_{2, \mu} \lesssim n^{-1/2} \log(n)^{5/2}
\]
Proof. It results from [2, Main result 3.5] that the approximation error with full tensor networks converges exponentially fast as $E(f^*, \Phi^F_c)_{L^2} = O(\rho^{-c/3})$ for some $\rho > 1$ related the size of the analyticity region. That means $f^* \in \mathcal{A}_\infty(\gamma, \Phi^F_c, L^2)$ with a growth function $\gamma(c) = \rho^{c/3}$. Theorem 5.2 then implies $\mathbb{E}(\hat{f}^F_C) \lesssim \inf_{c \in \mathbb{N}} \gamma(c)^{-2} + c \log(c) \log(n)/n$, and the result is obtained by taking $c \sim (\log(n)/\log(\rho))^{3/2}$.

The rate in $n^{-1}$ (up to logarithmic terms) achieved by full tensor networks is known to be the minimax rate for analytic functions for nonparametric estimation of analytic densities [8].

6.4 Beyond smoothness classes

We have seen that the proposed strategy is (near to) minimax adaptive to a large range of classical smoothness classes. In [3, Theorem 6.9], it is proved that for any $\alpha > 0$ and any $s > 0$, it holds

$$\mathcal{F}^\alpha_q(L^2) \not\hookrightarrow \mathcal{B}^q_s(L^2),$$

that means that functions in the approximation classes of tensor networks do not need to have any smoothness in a classical sense. Tensor networks may thus achieve a good performance for functions that can not be captured by standard approximation tools such as splines or wavelets. That reveals the potential of tensor networks to achieve approximation or learning tasks for functions beyond standard smoothness classes. In particular, they have the potential to achieve a good performance in high-dimensional approximation tasks for function classes not described in terms of standard weighted or anisotropic smoothness.

Note that in [2, Proposition 5.21], it is proved that when limiting the resolution $L$ to be logarithmic in the complexity $c$ (i.e. when considering for $\Phi_c$ models for which $L = O(\log(c))$), the resulting approximation classes of tensor networks are continuously embedded in some Besov spaces. This highlights the importance of the resolution (or depth of the tensor network). Addressing learning tasks for functions beyond regularity classes requires to explore model classes with higher resolutions (i.e. with resolutions $L$ higher than $O(\log(c))$ and up to $c$).

Let us finally recall that the results of Section 6.3 have been obtained with tensor networks with variable resolution but a fixed tree at each resolution (corresponding to the tensor train format). Adaptiveness to a wide range of smoothness classes is thus achieved without tree adaptation. Much larger approximation classes are obtained by considering tensor networks with variables trees. In many high-dimensional applications, adapting the tree to the target function is necessary to achieve a good performance and circumvent the curse of dimensionality. Working with variable trees may thus be relevant to approximate highly structured functions beyond classical anisotropic smoothness spaces. Of course, this comes with a much higher computational complexity and requires in practice some exploration strategies as discussed in Section 7.

7 Practical aspects

7.1 Slope heuristics for penalty calibration

The aim of the slope heuristics method proposed by Birgé and Massart [9] is precisely to calibrate penalty function for model selection purposes. See [7] and [4] for a general presentation of the method. This method has shown very good performances and comes with mathematical guarantees in various settings. For non parametric Gaussian regression with i.i.d. error terms, see [9, 4] and references therein. The slope heuristics have several versions (see [4]).
The aim is to tune the constant $\lambda$ in a penalty of the form $\text{pen}(m) = \lambda \text{pen}_{\text{shape}}(m)$ where $\text{pen}_{\text{shape}}$ is a known penalty shape. Let $\hat{m}(\lambda)$ be the model selected by penalized criterion with constant $\lambda$:

$$\hat{m}(\lambda) \in \arg\min_{m \in M} \left\{ \hat{R}_n(\hat{f}_m) + \lambda \text{pen}_{\text{shape}}(m) \right\}.$$ 

Let $C_m$ denote the complexity of the model. The complexity jump algorithm consists of the following steps:

1. Compute the function $\lambda \mapsto \hat{m}(\lambda)$,
2. Find the constant $\hat{\lambda}^{cj} > 0$ that corresponds to the highest jump of the function $\lambda \mapsto C_{\hat{m}(\lambda)}$,
3. Select the model $\hat{m} = \hat{m}(2\hat{\lambda}^{cj})$ such that

$$\hat{m} \in \arg\min_{m \in M} \left\{ \hat{R}_n(\hat{f}_m) + 2\hat{\lambda}^{cj} \text{pen}_{\text{shape}}(m) \right\}.$$ 

### 7.2 Exploration strategy

The exploration of all possible model classes $M_T^r(V)$ (or $M_{r,\Lambda}^T(V)$) with a complexity bounded by some $c$ is intractable since the number of such models is exponential in the number of variables $d$. Therefore, strategies should be introduced to propose a set of candidate model classes $M_m, m \in M$.

In practice, a possible approach is to rely on adaptive learning algorithms from [21] (see also [20]) that generate predictors $\hat{f}_m$ (minimizing the empirical risk) in a sequence of model classes $M_T^r(V)$. Note that developing an exploration strategy for sparse tensor networks is more challenging.

#### 7.2.1 Fixed tree

For a fixed tree $T$ and fixed feature space $V$, the proposed algorithm generates a sequence of model classes $M_m = M_T^r(V_m)$ with increasing ranks $r_m, m \geq 1$, by successively increasing the $\alpha$-ranks for nodes $\alpha$ associated with the highest (estimated) truncation errors

$$\inf_{\text{rank}_\alpha(f) \leq r_m, \alpha} \mathcal{R}(f) - \mathcal{R}(f^*).$$

For the strategy described in Section 6.1 with features based on tensorization, different resolutions $L_m \in \mathbb{N}$ are explored. To each resolution $L$ corresponds a fixed tree $T = T_L$ and a fixed feature space $V_L$. For each fixed resolution, the above strategy can then be used to explore the set of possible ranks.

A more classical approach (not using the tensorization technique) is to consider variable feature spaces $V_m$ of the form $V_m := V_{N_m} = V_{1,N_{m,1}} \otimes \ldots \otimes V_{d,N_{m,d}}$, where for each dimension $\nu \in \{1, \ldots, d\}$, $(V_{\nu,k})_{k \in \mathbb{N}}$ is a given approximation tool (e.g., polynomials, wavelets). Exploring all possible tuples $N_m \in \mathbb{N}^d$ is again a combinatorial problem. The algorithm proposed in [21, 20] relies on a validation approach for the selection of a particular tuple. Note that a complexity-based model selection method could also be considered for the selection of a tuple $N_m$. 

25
7.2.2 Variable tree

Although the set of possible dimension trees over \{1, \ldots, d\} is finite, exploring this whole set of dimension trees is intractable for high and even moderate \(d\). In [21], a stochastic algorithm has been proposed for optimizing the dimension tree for the compression of a tensor. This tree optimization algorithm has been combined with the rank-adaptive strategy discussed above. The resulting algorithm generates a sequence of predictors in tree tensor networks associated with different trees. In the numerical experiments, we use this learning algorithm with tree adaptation to generate a set of candidate trees. Then the learning algorithm with rank adaptation but fixed tree is used with each of these trees. Note that this strategy provides a data-dependent collection of candidate trees. For our model selection results to remain valid, we could use a standard splitting strategy (one part of the data to identify a collection of candidate trees and the other part for the model selection strategy within this collection). Without splitting, a more advanced analysis is necessary to provide risks bounds for a model collection generated with the sample used for the model selection.

In the next section we present some numerical experiments that validate the proposed model selection method and the exploration strategy.

8 Numerical experiments

In this section, we illustrate the proposed model selection approach for supervised learning problems in a least-squares regression setting. \(Y\) is a real-valued random variable defined by

\[ Y = f^*(X) + \varepsilon \]

where \(\varepsilon\) is independent of \(X\) and has zero mean and standard deviation \(\gamma \sigma(f^*(X))\). The parameter \(\gamma\) therefore controls the noise level in relative precision.

For a given training sample, we use the learning strategies described in Section 7.2 that generate a sequence of predictors \(\hat{f}_m, m \in \mathcal{M}\), associated with a certain collection of models \(\mathcal{M}\) (which depends on the training sample). Given a set of predictors \(\hat{f}_m, m \in \mathcal{M}\), we denote by \(\hat{m}^*\) the index of the model that minimizes the risk over \(\mathcal{M}\), i.e.

\[ \hat{m}^* \in \arg \min_{m \in \mathcal{M}} \mathcal{R}(\hat{f}_m). \]

The model \(\hat{m}^*\) is the oracle model in \(\mathcal{M}\) for a given training sample.

We also denote by \(\hat{m}(\lambda)\) the model such that

\[ \hat{m}(\lambda) \in \arg \min_{m \in \mathcal{M}} \left\{ \hat{\mathcal{R}}_n(\hat{f}_m) + \lambda \text{pen}_{\text{shape}}(m) \right\}, \]

where \(\text{pen}_{\text{shape}}(m) = Cm/n\), and by \(\hat{m} = \hat{m}(2\hat{\lambda}c)\) the model selected by our model selection strategy, where \(\hat{\lambda}c\) is calibrated with the complexity jump algorithm (see Section 7.1).

We consider two different types of problems: the approximation of univariate functions defined on \((0, 1)\), identified with a multivariate function through tensorization (Section 8.1), and the approximation of multivariate functions defined on a subset of \(\mathbb{R}^d\) using classical feature tensor spaces (Section 8.2).

For a given function \(f\), the risk \(\mathcal{R}(f)\) is evaluated using a sample of size \(10^5\) independent of the training sample. Statistics of complexities and risks (such as the expected complexity \(\mathbb{E}(C_{\hat{m}})\) or the expected risk \(\mathbb{E}(\mathcal{R}(\hat{f}_m))\)) are computed using 20 different training samples.
8.1 Tensorized function

Here we consider tensor networks for the approximation of a univariate function in $L^2(0,1)$ using the tensorization approach introduced in Section 6.1 with $b = 2$, that we briefly recap. A function $f$ defined on $[0,1)$ is linearly identified with a function $f = T_L(f)$ of $L + 1$ variables defined on $\{0,1\}^L \times (0,1)$ such that

$$f(x) = T_L(f)(i_1, \ldots, i_L, \bar{x}) \quad \text{for} \quad x = 2^{-L} \left( \sum_{k=1}^{L} i_k 2^{L-k} + \bar{x} \right).$$

The map $T_L$ is the tensorization map at resolution $L$. This allows to isometrically identify the space $L^2(0,1)$ with the tensor space $\mathbb{R}^2 \otimes \cdots \otimes \mathbb{R}^2 \otimes L^2(0,1)$ of order $L + 1$. Then we consider the approximation space $V_L = \mathbb{R}^2 \otimes \cdots \otimes \mathbb{R}^2 \otimes P_0$ of $(L + 1)$-variate functions $f(i_1, \ldots, i_L, \bar{x})$ independent of the variable $\bar{x}$. The space $V_L$ is linearly identified with the space of piecewise constant functions on the uniform partition of $[0,1]$ into $2^L$ intervals. Then we consider model classes $M = T_L^{-1} M_T(V_L)$, which are piecewise constant functions $f$ whose tensorized version $T_L(f)$ is in a particular tree-based tensor format.

In the following experiments, for each $L \in \{1, \ldots, 12\}$, we consider a fixed linear binary tree $T = T_L$ (with interior nodes $\{1, \ldots, k\}$, $1 \leq k \leq L + 1$) and use the rank adaptive learning algorithm described in Section 7.2.1 to produce a sequence of 25 approximations with increasing ranks.

Three functions $f^*(x)$ are considered. The first function $f^*(x) = \sqrt{x}$ is analytic on the open interval $(0,1)$ and its derivative has a singularity at zero. The second function $f^*(x) = \frac{1}{1+x}$ is analytic on a larger interval including $[0,1]$. The third function is in the Sobolev space $H^2(0,1)$. For all functions, the proposed model selection approach shows a very good performance. It selects with high probability a model with a risk very close to the risk of the oracle $\hat{f}_{\hat{m}^*}$.

8.1.1 Tensorized function $f^*(x) = \sqrt{x}$

We consider the function $f^*(x) = \sqrt{x}$ which is analytic on the open interval $(0,1)$, with a singular derivative at zero. We observe on Figures 3 and 4 that the model selection approach selects a model close to optimal for different sample size $n$ and noise level. Tables 1 and 2 show expectations of complexities and errors for the selected estimator and illustrate the very good performance of the approach when compared to the oracle.

| $n$ | $E(C_{\hat{m}^*})$ | $E(C_{\hat{m}})$ | $E(R(\hat{f}_{\hat{m}^*}))$ | $E(R(\hat{f}_{\hat{m}}))$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 100 | 123.2           | 91.6            | 1.6e-05         | 5.0e-05         |
| 200 | 163.8           | 165.0           | 3.0e-06         | 5.1e-06         |
| 500 | 182.2           | 182.6           | 9.2e-07         | 1.2e-06         |
| 1000| 190.2           | 228.5           | 7.1e-07         | 1.4e-06         |

Table 1: Expectation of complexities and risks of the model selected by the slope heuristics, with the function $f^*(x) = \sqrt{x}$ and different values of $n$ and $\gamma = 0.001$. 

(a) Function $\lambda \mapsto C_m(\lambda)$, $\lambda^\circ j$ (red).

(b) Points $(C_m, R(\hat{f}_m), m \in \mathcal{M}$, and selected model (red).

Figure 3: Slope heuristics for the tensorized function $f^*(x) = \sqrt{x}$ with $n = 200$ and $\gamma = 0.001$.

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(a) Function $\lambda \mapsto C_m(\lambda)$, $\lambda^\circ j$ (red).

(b) Points $(C_m, R(\hat{f}_m), m \in \mathcal{M}$, and selected model (red).

Figure 4: Slope heuristics for the tensorized function $f^*(x) = \sqrt{x}$ with $n = 1000$ and $\gamma = 0.0001$.

| $\gamma$  | $\mathbb{E}(C_m^*)$ | $\mathbb{E}(C_m)$ | $\mathbb{E}(R(\hat{f}_m^*))$ | $\mathbb{E}(R(\hat{f}_m))$ |
|-----------|---------------------|-------------------|----------------------------|--------------------------|
| $10^{-3}$  | 190.2               | 228.5             | 7.1e-07                    | 1.4e-06                 |
| $10^{-4}$  | 242.8               | 251.4             | 1.5e-07                    | 2.1e-07                 |
| $10^{-5}$  | 219.8               | 267.4             | 1.3e-07                    | 2.4e-07                 |
| 0         | 218.6               | 258.6             | 1.1e-07                    | 2.1e-07                 |

Table 2: Expectation of complexities and risks of the model selected by the slope heuristics, with the function $f^*(x) = \sqrt{x}$ and different values of $\gamma$ and $n = 1000$. 
8.1.2 Tensorized function $f^*(x) = \frac{1}{1+x}$.

We consider the function $f^*(x) = \frac{1}{1+x}$ which is analytic on the interval $(-1, \infty)$ including $[0, 1]$. Figures 5 and 6 illustrate the good behaviour of the model selection approach for different sample size and noise level. Tables 3 and 4 show expectations of complexities and errors for the selected estimator and illustrate again the very good performance of the approach when compared to the oracle.

Figure 5: Slope heuristics for the tensorized function $f^*(x) = \frac{1}{1+x}$ with $n = 200$ and $\gamma = 0.001$.

Figure 6: Slope heuristics for the tensorized function $f^*(x) = \frac{1}{1+x}$ with $n = 1000$ and $\gamma = 0.0001$. 
\begin{align*}
\mathbb{E}(C_{\hat{m}}^\prime) & = \mathbb{E}(C_{\hat{m}}) \\
\mathbb{E}(R(\hat{f}_{m^\prime})) & = \mathbb{E}(R(\hat{f}_m))
\end{align*}

Table 3: Expectation of complexities and risks of the model selected by the slope heuristics, with the function $f^\star(x) = \frac{1}{1+x}$, different values of $n$ and $\gamma = 0.001$.

| $n$  | $\mathbb{E}(C_{\hat{m}}^\prime)$ | $\mathbb{E}(C_{\hat{m}})$ | $\mathbb{E}(R(\hat{f}_{m^\prime}))$ | $\mathbb{E}(R(\hat{f}_m))$ |
|------|-------------------------------|----------------------------|---------------------------------|-----------------------------|
| 100  | 88.0                          | 83.0                       | 9.3e-07                         | 1.0e-06                     |
| 200  | 97.3                          | 92.8                       | 6.4e-07                         | 6.6e-07                     |
| 500  | 92.9                          | 124.4                      | 5.8e-07                         | 6.9e-07                     |
| 1000 | 108.4                         | 107.5                      | 5.3e-07                         | 5.3e-07                     |

Table 4: Expectation of complexities and risks of the model selected by the slope heuristics, with the function $f^\star(x) = \frac{1}{1+x}$, different values of $\gamma$ and $n = 1000$.

| $\gamma$ | $\mathbb{E}(C_{\hat{m}}^\prime)$ | $\mathbb{E}(C_{\hat{m}})$ | $\mathbb{E}(R(\hat{f}_{m^\prime}))$ | $\mathbb{E}(R(\hat{f}_m))$ |
|----------|---------------------------------|----------------------------|---------------------------------|-----------------------------|
| $10^{-5}$ | 108.4                           | 107.5                      | 5.3e-07                         | 5.3e-07                     |
| $10^{-4}$ | 159.3                           | 151.1                      | 6.9e-09                         | 6.9e-09                     |
| $10^{-5}$ | 152.0                           | 182.2                      | 1.6e-09                         | 1.9e-09                     |
| 0        | 156.8                           | 155.8                      | 1.6e-09                         | 1.6e-09                     |
8.1.3 Tensorized function \( f^*(x) = g(g(x))^2 \) with \( g(x) = 1 - 2|x - \frac{1}{2}| \).

We consider the function \( f^*(x) = g(g(x))^2 \) with \( g(x) = 1 - 2|x - \frac{1}{2}| \), which is in the Sobolev space \( H^2(0, 1) \). Figures 7b and 8 illustrate again the good behaviour of the model selection approach for different sample size and noise level. And Tables 3 and 4 again illustrate again the very good performance (in expectation) for the selected estimator of the approach when compared to the oracle.

Figure 7: Slope heuristics for the tensorized function \( f^*(x) = (g(g(x)))^2 \) with \( n = 200 \) and \( \gamma = 0.001 \).

Figure 8: Slope heuristics for the tensorized function \( f^*(x) = (g(g(x)))^2 \) with \( n = 1000 \) and \( \gamma = 0.0001 \).

8.2 Multivariate functions

8.2.1 Corner peak function

We consider the function

\[
f^*(X) = \frac{1}{1 + \sum_{\nu=1}^{d} \nu^{-2} X_{\nu}}
\]

with \( d = 10 \), where the \( X_{\nu} \sim U(0, 1) \) are i.i.d. uniform random variables. The function \( f^* \) is analytic on \( [0, 1]^d \). We use the fixed balanced binary tree \( T \) of Figure 9. As univariate approximation tools,
\( n \) & \( \mathbb{E}(C_{m^*}) \) & \( \mathbb{E}(C_{\hat{m}}) \) & \( \mathbb{E}(\mathcal{R}(f_{m^*})) \) & \( \mathbb{E}(\mathcal{R}(f_{\hat{m}})) \) \\
200 & 176.4 & 181.6 & 6.3e-07 & 1.6e-06 \\
500 & 188.2 & 198.8 & 3.9e-07 & 4.1e-07 \\
1000 & 196.6 & 233.8 & 3.2e-07 & 3.5e-07 \\

Table 5: Expectation of complexities and risks for the function \( f^*(x) = (g(g(x)))^2 \), different values of \( n \) and \( \gamma = 0.001 \).

\[
\begin{array}{c|c|c|c|c}
\gamma & \mathbb{E}(C_{m^*}) & \mathbb{E}(C_{\hat{m}}) & \mathbb{E}(\mathcal{R}(f_{m^*})) & \mathbb{E}(\mathcal{R}(f_{\hat{m}})) \\
10^{-3} & 196.6 & 233.8 & 3.2e-07 & 3.5e-07 \\
10^{-4} & 195.8 & 205.8 & 1.7e-07 & 1.7e-07 \\
10^{-5} & 191.0 & 226.6 & 1.7e-07 & 1.8e-07 \\
0 & 194.0 & 232.6 & 1.7e-07 & 1.9e-07 \\
\end{array}
\]

Table 6: Expectation of complexities and risks of the model selected by the slope heuristics, with the function \( f^*(x) = (g(g(x)))^2 \), different values of \( \gamma \) and \( n = 1000 \).

we use polynomial spaces \( V_{\nu,N_{\nu}} = \mathbb{P}_{N_{\nu}-1}(X_{\nu}) \), \( \nu \in D \). Figures 10 and 11 illustrate the very good behaviour of the model selection approach for a sample size \( n = 1000 \) and noise level \( \gamma = 0.001 \), where the best model appears to be always selected. In Tables 7 and 8 we observe that the expectation of complexities and errors for the selected estimator (for different values of \( n \) and \( \gamma \)), which are of the same order as for the oracle.

Figure 9: Corner peak function. Dimension tree \( T \).
Figure 10: Slope heuristics for the Corner peak function with \( n = 1000 \) and \( \gamma = 0.001 \).

Figure 11: Slope heuristics for the Corner peak function with \( n = 1000 \) and \( \gamma = 0.001 \), superposition of 10 different samples.

| \( n \) | \( \mathbb{E}(C_{\hat{m}}^{\star}) \) | \( \mathbb{E}(C_{\hat{m}}) \) | \( \mathbb{E}(\mathcal{R}(\hat{f}_{m}^{\star})) \) | \( \mathbb{E}(\mathcal{R}(\hat{f}_{m})) \) |
|---|---|---|---|---|
| 100 | 124.1 | 73.7 | 2.1e-06 | 1.1e-05 |
| 500 | 286.7 | 291.3 | 9.8e-11 | 1.0e-10 |
| 1000 | 286.2 | 293.8 | 6.6e-11 | 6.7e-11 |

Table 7: Expectation of complexities and risks selected by the slope heuristics, with the Corner peak function, different values of \( n \) and \( \gamma = 10^{-5} \).

| \( \gamma \) | \( \mathbb{E}(C_{\hat{m}}^{\star}) \) | \( \mathbb{E}(C_{\hat{m}}) \) | \( \mathbb{E}(\mathcal{R}(\hat{f}_{m}^{\star})) \) | \( \mathbb{E}(\mathcal{R}(\hat{f}_{m})) \) |
|---|---|---|---|---|
| \( 10^{-2} \) | 95.5 | 79.8 | 5.4e-05 | 5.5e-05 |
| \( 10^{-3} \) | 143.1 | 143.1 | 5.4e-07 | 5.4e-07 |
| \( 10^{-4} \) | 223.2 | 193.7 | 5.9e-09 | 6.0e-09 |
| \( 10^{-5} \) | 286.2 | 293.8 | 6.6e-11 | 6.7e-11 |
| 0 | 598.7 | 538.4 | 2.5e-15 | 1.8e-14 |

Table 8: Expectation of complexities and risks selected by the slope heuristics, with the Corner peak function, different values of \( \gamma \) and \( n = 1000 \).
8.2.2 Borehole function

We consider the function

\[ g(U_1, \ldots, U_8) = \frac{2\pi U_3(U_4 - U_6)}{(U_2 - \log(U_1))(1 + \frac{2U_7U_3}{(U_2 - \log(U_1))U_1U_8} + U_7U_3)} \]

which models the water flow through a borehole as a function of 8 independent random variables

\[ U_1 \sim \mathcal{N}(0.1, 0.0161812), \quad U_2 \sim \mathcal{N}(7.71, 1.0056), \quad U_3 \sim U(63070, 115600), \quad U_4 \sim U(990, 1110), \quad U_5 \sim U(63.1, 116), \quad U_6 \sim U(700, 820), \quad U_7 \sim U(1120, 1680), \quad U_8 \sim U(9855, 12045). \]

Then we consider the function

\[ f^*(X_1, \ldots, X_d) = g(g_1(X_1), \ldots, g_8(X_8)), \]

where \( g_\nu \) are functions such that \( U_\nu = g_\nu(X_\nu) \), with \( X_\nu \sim \mathcal{N}(0, 1) \) for \( \nu \in \{1, 2\} \), and \( X_\nu \sim U(-1, 1) \) for \( \nu \in \{3, \ldots, 8\} \). Function \( f^* \) is thus defined on \( \mathcal{X} = \mathbb{R}^2 \times [-1, 1]^6 \). As univariate approximation tools, we use polynomial spaces \( V_{\nu,N_\nu}^\nu = P_{N_\nu-1}(X_\nu) \), \( \nu \in D \).

We use the exploration strategy described in Section 7.2.1 More precisely, we first run a learning algorithm with tree adaptation from an initial binary tree drawn randomly, with \( n = 100 \) samples. The learning algorithm visited the 9 trees plotted in Figure 12. Then for each of these trees, we start a learning algorithm with fixed tree and rank adaptation. Figures 13 to 15 illustrate the behaviour of the model selection strategy for different sample size \( n \). Table 9 shows the expectation of complexities and risks. The model selection approach shows very good performances, except for very small training size \( n = 100 \), where the approach selects a model rather far from the optimal one (in terms of expected risk and complexity).

| \( n \) | \( \mathbb{E}(C_{\hat{m}}^*) \) | \( \mathbb{E}(C_{\hat{m}}) \) | \( \mathbb{E}(\mathcal{R}(\hat{f}_{\hat{m}}^*))) \) | \( \mathbb{E}(\mathcal{R}(\hat{f}_{\hat{m}})) \) |
|---|---|---|---|---|
| 100 | 132.1 | 63.4 | 6.9e-06 | 9.3e-04 |
| 200 | 149.7 | 156.0 | 3.0e-08 | 1.1e-07 |
| 500 | 144.7 | 178.2 | 1.0e-08 | 1.8e-08 |
| 1000 | 154.1 | 194.2 | 8.3e-09 | 1.2e-08 |

Table 9: Borehole function. Expectation of complexities and risks. \( \gamma = 10^{-6} \), different \( n \).
Figure 12: Borehole function. The path of 9 trees generated by the tree adaptive learning algorithm.
Figure 13: Slope heuristics for Borehole function with $n = 100$ and $\gamma = 10^{-6}$.

Figure 14: Slope heuristics for Borehole function with $n = 200$ and $\gamma = 10^{-6}$.

Figure 15: Slope heuristics for Borehole function with $n = 1000$ and $\gamma = 10^{-6}$. 
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A Tree tensor networks as compositions of multilinear functions

A function $f = \mathcal{R}_{V,T,r}(v)$ in $M_T^r(V)$ admits a representation in terms of compositions of multilinear functions. Indeed, for any interior node $\alpha \in \mathcal{I}(T)$, the tensor $v^\alpha \in \mathbb{R}^{r_\alpha \times \beta \in S(\alpha) r_\beta}$ can be linearly identified with a multilinear map

$$v^\alpha : \bigotimes_{\beta \in S(\alpha)} \mathbb{R}^{r_\beta} \to \mathbb{R}^{r_\alpha}$$

defined by

$$v^\alpha((z_\beta)_{\beta \in S(\alpha)})_{k_\alpha} = \sum_{1 \leq k_\beta \leq r_\beta} v_{k_\alpha, (k\beta)_{\beta \in S(\alpha)}}^{\alpha} \prod_{\beta \in S(\alpha)} z_{k_\beta}^\beta$$

for $z_\beta \in \mathbb{R}^{r_\beta}$. For a given $\alpha \in T$, we let $g^\alpha(x_\alpha) = (g_{k_\alpha}^\alpha (x_\alpha))_{1 \leq k_\alpha \leq r_\alpha} \in \mathbb{R}^{r_\alpha}$. Therefore, a function $f$ in $M_T^r(V)$ admits the representation

$$f(x) = v^D((g^\alpha(x_\alpha))_{\alpha \in S(D)}).$$

where for any $\alpha \in \mathcal{I}(T) \setminus \{D\}$, $g^\alpha$ admits the representation

$$g^\alpha(x_\alpha) = v^\alpha((g_\beta^\alpha(x_\beta))_{\beta \in S(\alpha)}).$$

For a leaf node $\alpha \in \mathcal{L}(T)$, the tensor $v^\alpha$ can be linearly identified with a linear map $v^\alpha : \mathbb{R}^{n_\alpha} \to \mathbb{R}^{r_\alpha}$, and

$$g^\alpha(x_\alpha) = v^\alpha(\phi^\alpha(x_\alpha)).$$
B  Proofs of Section 3

Proof of Proposition 3.4. Let \( f = \mathcal{R}_{V:T,r}((v^\alpha)_{\alpha \in T}) \), where the tensor \( v^\alpha \) is identified with a \( \mathbb{R}^{r_\alpha} \)-valued multilinear (resp. linear) map for \( \alpha \in \mathcal{I}(T) \) (resp. \( \alpha \in \mathcal{L}(T) \)), see Appendix A for details. For \( x \in \mathcal{X} \), we first note that

\[
|f(x)| = |v^D((g^\alpha(x_\alpha))_{\alpha \in S(D)})| \leq \|v^D\|_{\mathcal{P}^\alpha} \prod_{\alpha \in S(D)} \|g(x_\alpha)\|_p,
\]

with \( \| \cdot \|_p \) the vector \( \ell^p \)-norm. Then for any interior node \( \alpha \in \mathcal{I}(T) \), we have

\[
\|g^\alpha(x_\alpha)\|_p = \|v^\alpha((g^\beta(x_\beta))_{\beta \in S(\alpha)})\|_p \leq \|v^\alpha\|_{\mathcal{P}^\alpha} \prod_{\beta \in S(\alpha)} \|g^\beta(x_\beta)\|_p,
\]

and for any leaf node \( \alpha \in \mathcal{L}(T) \),

\[
\|g^\alpha(x_\alpha)\|_p = \|v^\alpha(\phi^\alpha(x_\alpha))\|_p \leq \|v^\alpha\|_{\mathcal{P}^\alpha} \|\phi^\alpha(x_\alpha)\|_p.
\]

We deduce that

\[
|f(x)|_p \leq \prod_{\alpha \in T} \|v^\alpha\|_{\mathcal{P}^\alpha} \prod_{1 \leq \nu \leq d} \|\phi^\nu(x_\nu)\|_p,
\]

and therefore, since \( \mu \) is a product measure and from the particular normalization of functions \( \phi^\nu \), we obtain

\[
\|f\|_{p,\mu} \leq \prod_{\alpha \in T} \|v^\alpha\|_{\mathcal{P}^\alpha} \prod_{1 \leq \nu \leq d} \|\phi^\nu\|_{p,\mu} = \prod_{\alpha \in T} \|v^\alpha\|_{\mathcal{P}^\alpha},
\]

which proves that \( L_{p,\mu} \leq 1 \). Finally for \( 1 \leq q \leq p \), we note that

\[
\mu(\mathcal{X})^{1/p-1/q}\|f\|_{q,\mu} \leq \|f\|_{p,\mu} \leq \|f\|_{\infty,\mu},
\]

which yields \( L_{q,\mu} \leq \mu(\mathcal{X})^{1/q-1/p}L_{p,\mu} \).

C  Proofs of Section 4

C.1  Concentration inequalities for empirical processes

We here apply classical results to control the fluctuations of the supremum of the empirical process \( \mathcal{R}_n(f) \) over the model class \( M \).

Assumption 4.1 (Bounded contrast) yields a classical concentration inequality for the empirical process \( \mathcal{R}_n(f) \).

Lemma C.1. Under assumption 4.1 we have that for all \( \epsilon > 0 \) and all \( f \in M \)

\[
\mathbb{P}(\mathcal{R}_n(f) > \epsilon B) \lor \mathbb{P}(\mathcal{R}_n(f) < -\epsilon B) \leq e^{-n \frac{\epsilon^2}{2} B^2}.
\]  

(28)

Proof. We have \( \mathcal{R}_n(f) - \mathcal{R}(f) = \frac{1}{n} \sum_{i=1}^n A_i^f - \mathbb{E}(A_i^f) \), where the \( A_i^f = \gamma(f, Z_i) \) are i.i.d. copies of the random variable \( A^f = \gamma(f, Z) \). From Assumption 4.1 we have that \( |A_i^f| \leq B \) almost surely, so that \( A_i^f \) is subgaussian with parameter \( B^2 \) and the result simply follows from Hoeffding’s inequality.

If \( \gamma(\cdot, Z) \) is Lipschitz continuous over \( M \subset L_\mu^\infty(\mathcal{X}) \) we obtain a uniform concentration inequality for the empirical process \( \mathcal{R}_n(f) \) over \( M \):
Lemma C.2. Under Assumptions 4.1 and 4.2 we have that for all $\varepsilon > 0$ and all $f \in M$

\[ \mathbb{P}(\sup_{f \in M} \mathcal{R}_n(f) > 2\varepsilon B) \lor \mathbb{P}(\inf_{f \in M} \mathcal{R}_n(f) < -2\varepsilon B) \leq N_{\varepsilon B} e^{-\frac{n\varepsilon^2}{B^2}}, \quad (29) \]

where $N_{\varepsilon B} = N(\varepsilon B \frac{L}{2}, M, \| \cdot \|_{\infty}, \mu)$ is the covering number of $M$ at scale $\varepsilon B \frac{L}{2}$, and \[ \log N_{\varepsilon B} \leq C_M \log \left( 6LB^{-1}R|T|\varepsilon^{-1} \right). \]

Proof of Lemma C.2. Let $\gamma = \varepsilon B \frac{L}{2}$ and let $\mathcal{N}$ be a $\gamma$-net of $M$ for the $\| \cdot \|_{\infty, \mu}$-norm, with cardinal $N_{\varepsilon B}$. Using Lemma C.1 and a union bound argument, we obtain

\[ \mathbb{P}(\sup_{g \in \mathcal{N}} \mathcal{R}_n(g) > \varepsilon B) \lor \mathbb{P}(\inf_{g \in \mathcal{N}} \mathcal{R}_n(g) < -\varepsilon B) \leq N_{\varepsilon B} e^{-\frac{n\varepsilon^2}{B^2}}. \]

For any $f \in M$, there exists a $g \in \mathcal{N}$ such that $\|f - g\|_{\infty, \mu} \leq \gamma$. Noting that \[ \mathcal{R}_n(f) = \mathcal{R}_n(g) + \mathcal{R}_n(f) - \mathcal{R}_n(g) + \mathcal{R}(g) - \mathcal{R}(f), \]
we deduce from Assumption 4.2 that \[ \mathcal{R}_n(f) \leq \mathcal{R}_n(g) + 2\mathcal{L}\|f - g\|_{\infty, \mu} \leq \sup_{g \in \mathcal{N}} \mathcal{R}_n(g) + \varepsilon B, \]
and \[ \mathcal{R}_n(f) \geq \mathcal{R}_n(g) - 2\mathcal{L}\|f - g\|_{\infty, \mu} \geq \inf_{g \in \mathcal{N}} \mathcal{R}_n(g) - \varepsilon B. \]

This implies that \[ \mathbb{P}(\sup_{f \in M} \mathcal{R}_n(f) > 2\varepsilon B) \leq \mathbb{P}(\sup_{g \in \mathcal{N}} \mathcal{R}_n(g) > \varepsilon B), \]
and \[ \mathbb{P}(\inf_{f \in M} \mathcal{R}_n(f) < -2\varepsilon B) \leq \mathbb{P}(\inf_{g \in \mathcal{N}} \mathcal{R}_n(f) < -\varepsilon B), \]
which yields (29). The bound on $N_{\varepsilon B}$ directly follows from Proposition 3.3 and Proposition 3.4. \qed

Lemma C.3. Under Assumptions 4.1 and 4.2

\[ \mathbb{E}(\sup_{f \in M} |\mathcal{R}_n(f)|) \leq 4B\sqrt{C_M} \sqrt{\frac{2\log((\beta + \varepsilon)^n)}{n}}. \]

with $\beta = 6L B^{-1}R|T|$.

Proof of Lemma C.3. We have \[ \mathbb{E}(\sup_{f \in M} |\mathcal{R}_n(f)|) = \int_0^\infty \mathbb{P}(\sup_{f \in M} |\mathcal{R}_n(f)| > t) dt \]
\[ = 2B \int_0^\infty \mathbb{P}(\sup_{f \in M} |\mathcal{R}_n(f)| > 2\varepsilon B) d\varepsilon. \]
Let $\beta = 6LB^{-1}R[T]$. Then, according to Lemma C.2 for any $\delta > 0$,

$$
\mathbb{E}(\sup_{f \in M} |\tilde{R}_n(f)|) \leq 2B \left[ \delta + \int_{\delta}^{\infty} 2(\beta^{-1})e^{-n\beta^2} d\varepsilon \right],
$$

$$
= 2B \left[ \delta + 2\beta e^{\frac{CM}{2}} \int_{n\delta^2/2}^{\infty} \left( \frac{2u}{n} \right)^{-CM/2} e^{-\frac{u}{2nu}} du \right],
$$

$$
\leq 2B \left[ \delta + 2n^{-1}\beta e^{-n\delta^2/2} \right],
$$

By taking

$$
\delta = \sqrt{\frac{2CM}{n} \log((\beta \vee e)\sqrt{n})},
$$

we have

$$
n^{-1}\beta e^{-n\delta^2/2} = n^{-1}\beta e^{-n\delta^2/2} \leq \delta^{-CM-1}(\beta \vee e)^{-CM} n^{-\frac{CM}{2}}
$$

$$
\leq \delta^{-CM-1} n^{-\frac{CM}{2}-1}
$$

$$
= \delta(\delta^2 n)^{-\frac{CM}{2}-1}
$$

$$
\leq \delta
$$

where we have used the fact that $2CM \log((\beta \vee e)\sqrt{n}) \geq 1$. Then

$$
\mathbb{E}(\sup_{f \in M} |\tilde{R}_n(f)|) \leq 4B\delta,
$$

which concludes the proof.

**C.2 Proof of Proposition 4.5**

The excess risk for the estimator $\hat{f}_n^M$ satisfies

$$
\mathcal{E}(\hat{f}_n^M) = \mathcal{E}(f^M) + \mathcal{R}(\hat{f}_n^M) - \mathcal{R}(f^M),
$$

where $\mathcal{E}(f^M)$ is the best approximation error in $M$ and $\mathcal{R}(\hat{f}_n^M) - \mathcal{R}(f^M)$ is the estimation error. Using the optimality of $\hat{f}_n^M$, we obtain that the estimation error satisfies

$$
\mathcal{R}(\hat{f}_n^M) - \mathcal{R}(f^M) \leq \hat{R}_n(f^M) - \mathcal{R}(f^M) - \hat{R}_n(\hat{f}_n^M) + \mathcal{R}(f^M)
$$

$$
\leq \hat{R}_n(f^M) - \hat{R}_n(\hat{f}_n^M).
$$

Thus

$$
\mathcal{E}(\hat{f}_n^M) \leq \mathcal{E}(f^M) + 2\sup_{f \in M} |\hat{R}_n(f)|.
$$

Under Assumption 4.1, the bounded difference Inequality (see for instance Theorem 5.1 in [34]) applied to $\sup_{f \in M} |\hat{R}_n(f)|$ gives that with probability larger than $1 - \exp(-t)$,

$$
\sup_{f \in M} |\hat{R}_n(f)| \leq \mathbb{E}(\sup_{f \in M} |\hat{R}_n(f)|) + 2B\sqrt{\frac{t}{2n}}.
$$

Lemma C.3 together with Assumption 4.2 gives the risk bound.
C.3 Proof of Theorem 4.6

By definition of \( \hat{m} \), for any \( m \in M \),
\[
\mathcal{R}_n(\hat{f}_m) + \text{pen}(\hat{m}) \leq \mathcal{R}_n(f_m) + \text{pen}(m) \leq \mathcal{R}_n(f_m) + \text{pen}(m).
\]
Therefore,
\[
\mathcal{R}_n(\hat{f}_m) \leq \mathcal{R}_n(f_m) + \text{pen}(m) - \text{pen}(\hat{m})
\]
and thus
\[
\mathcal{R}(\hat{f}_m) + \mathcal{R}_n(\hat{f}_m) \leq \mathcal{R}(f_m) + \mathcal{R}_n(f_m) + \text{pen}(m) - \text{pen}(\hat{m}),
\]
where \( \mathcal{R}_n(f) \) is the centered empirical process defined in (4.1). We finally derive the following upper bound on the excess risk
\[
\mathcal{E}(\hat{f}_m) \leq \mathcal{E}(f_m) + \mathcal{R}_n(f_m) - \mathcal{R}_n(\hat{f}_m) - \text{pen}(\hat{m}) + \text{pen}(m). \tag{30}
\]
As in the proof of Proposition 4.5 by applying the bounded difference Inequality to \( \sup_{f \in M} -\mathcal{R}_n(f) \) and by Lemma C.3 it gives that for any \( t > 0 \), with probability larger than \( 1 - \exp(-t) \),
\[
\sup_{f \in M} -\mathcal{R}_n(f) \leq 4B\sqrt{C_M \frac{2 \log(6LB^{-1}R|T|\sqrt{n})}{n}} + 2B \sqrt{\frac{t}{2n}}. \tag{31}
\]
Thus, for any \( t > 0 \) and any \( m \in M \), one has with probability larger than \( 1 - \exp(-t) \),
\[
\sup_{f \in M_m} -\mathcal{R}_n(f) \leq \lambda_m \sqrt{\frac{C_m}{n}} + 2B \sqrt{\frac{t + w_m}{2n}}.
\]
Let \( w_m = \bar{w}C_m + \log(kC_m) \). Then, with probability larger than \( 1 - \sum_{m \in M} e^{-w_m - t} = 1 - \frac{1}{e^{w-m}} e^{-t} \), it holds
\[
-\mathcal{R}_n(\hat{f}_m) \leq \sup_{f \in M_m} -\mathcal{R}_n(f) \leq \lambda_m \sqrt{\frac{C_m}{n}} + 2B \sqrt{\frac{t + w_m}{2n}},
\]
which together with (30) implies that
\[
\mathcal{E}(\hat{f}_m) \leq \mathcal{E}(f_m) + \mathcal{R}_n(f_m) + \lambda_m \sqrt{\frac{C_m}{n}} + 2B \sqrt{\frac{w_m}{2n}} - \text{pen}(\hat{m}) + \text{pen}(m) + 2B \sqrt{\frac{t}{2n}}
\]
holds for all \( m \in M \). Then, with the condition (17) on the penalty function, the upper bound
\[
\mathcal{E}(\hat{f}_m) \leq \mathcal{E}(f_m) + \mathcal{R}_n(f_m) + \text{pen}(m) + 2B \sqrt{\frac{t}{2n}}
\]
holds for all \( m \in M \) simultaneously, with probability larger than \( 1 - \frac{1}{e^{w-m}} e^{-t} \). Next, integrating with respect to \( t \) gives
\[
\mathbb{E} \left[ 0 \vee \left( \mathcal{E}(\hat{f}_m) - \mathcal{E}(f_m) - \mathcal{R}_n(f_m) - \text{pen}(m) \right) \right] \leq \frac{B}{\exp(\bar{w}) - 1} \sqrt{\frac{2 \pi 1}{n/4}}.
\]
Finally, since \( \mathcal{R}_n(f_m) \) has zero mean, for any \( m \in M \),
\[
\mathbb{E}(\mathcal{E}(\hat{f}_m)) \leq \mathcal{E}(f_m) + \text{pen}(m) + \frac{B}{\exp(\bar{w}) - 1} \sqrt{\frac{\pi}{n}}
\]
and we conclude by taking the infimum over \( m \in M \).
C.4 Proof of Proposition 4.7

The collections of models have complexities
\[ N_c(M_{V,T}) = |\{ r \in \mathbb{N}^{\lvert T \rvert} : C(T, r, V_N) = c \}| \]
\[ N_c(M_T) = |\{ r \in \mathbb{N}^{\lvert T \rvert}, N \in \mathbb{N}^d : C(T, r, V_N) = c \}| \]
\[ N_c(M_\ast) = |\{ T \in T_{a,d}, r \in \mathbb{N}^{\lvert T \rvert}, N \in \mathbb{N}^d : C(T, r, V_N) = c \}| \]

where \( T_{a,d} \) denotes the collection of trees with arity \( a \) (or \( a \)-ary trees). We easily see that the above families of models have growing complexity, i.e.
\[ N_c(M_{V,T}) \leq N_c(M_T) \leq N_c(M_\ast), \]

for any \( T \) and \( V \). Let us first consider the collection \( M_T \) for a given tree \( T \). Let us recall that for a tree \( T \), a tuple \( r \in \mathbb{N}^{\lvert T \rvert} \) and a feature space \( V_N \) with \( N \in \mathbb{N}^d \), the full representation complexity is given by
\[ C(T, r, V_N) = \sum_\alpha |K_\alpha|, \]

with \( |K_\alpha| = r_\alpha \prod_{\beta \in S(\alpha)} r_\beta \) for \( \alpha \notin \mathcal{L}(T) \) and \( |K_\alpha| = r_\alpha N_\alpha \) for \( \alpha \in \mathcal{L}(T) \). Then
\[ N_c(M_T) \leq \sum_{(q_\alpha)_{\alpha \in T}} |\{ (r, N) \in \mathbb{N}^{\lvert T \rvert} \times \mathbb{N}^d : |K_\alpha| = q_\alpha, \alpha \in T \}| \]

where the sum is taken over all tuples \( (q_\alpha)_{\alpha \in T} \in \mathbb{N}^{\lvert T \rvert} \) such that \( \sum_{\alpha \in T} q_\alpha = c \). For any \( q_\alpha \in \mathbb{N} \), the number of tuples \( (r_1, \ldots, r_{a+1}) \) such that \( \prod_{k=1}^{a+1} r_k \leq q_\alpha \) is less than \( q_\alpha^{a-1} \). For \( \alpha \in \mathcal{L}(T) \), the number of pairs \( (r_\alpha, N_\alpha) \in \mathbb{N}^2 \) such that \( N_\alpha r_\alpha = q_\alpha \) is less than \( q_\alpha \). Thus for any tuple of integers \( (q_\alpha)_{\alpha \in T} \) such \( \sum_{\alpha \in T} q_\alpha = c \), the number of tuples \( (r, N) \in \mathbb{N}^{\lvert T \rvert} \times \mathbb{N}^d \) such that \( |K_\alpha| = q_\alpha \) for all \( \alpha \in T \) is less than
\[ \left( \prod_{\alpha \in T} q_\alpha \right)^{\lvert T \rvert} \leq \left( \frac{1}{\lvert T \rvert} \sum_{\alpha \in T} q_\alpha \right)^{\lvert T \rvert} \leq \left( \frac{c}{\lvert T \rvert} \right)^{\lvert T \rvert}. \]

Moreover, the number of tuple of integers \( (q_\alpha)_{\alpha \in T} \) satisfying \( \sum_{\alpha \in T} q_\alpha = c \) is bounded by \( (c+\lvert T \rvert)^c \). Thus, we deduce that
\[ N_c(M_T) \leq \binom{c + \lvert T \rvert}{c} \left( \frac{c}{\lvert T \rvert} \right)^{(a-1)|T|}. \]

Using the inequality
\[ \log \binom{k}{\ell} \leq \ell (1 + \log \frac{k}{\ell}), \]

and the fact that \( |T| \leq 2d \) and \( |T| \leq c \) for any model \( m \) with complexity \( c \), we obtain
\[ \log(N_c(M_T)) \leq c(1 + \log(\frac{c+|T|}{c})) + (a-1)|T|\log(\frac{c}{|T|}) \]
\[ \leq c(1 + \log(2)) + 2d(a-1)\log(c), \]

which yields
\[ \log(N_c(M_T)) \leq 2(a-1)(c + d\log(c)). \]
Now consider the family $\mathcal{M}_*$. We first note that
\[
\mathcal{N}_c(\mathcal{M}_*) = \sum_{T \in T_{a,d}} \mathcal{N}_c(\mathcal{M}_T),
\]
and using the bound on $\mathcal{N}_c(\mathcal{M}_T)$, we obtain
\[
\log(\mathcal{N}_c(\mathcal{M}_*)) \leq \log(|T_{a,d}|) + 2(a - 1)(c + d \log(c)),
\]
where $|T_{a,d}|$ is the number of all possible trees with arity $a$ with $d$ leaves, which is the Fuss-Catalan number
\[
C_a(d - 1) = \frac{1}{(a - 1)(d - 1) + 1} \binom{a(d - 1)}{d - 1}.
\]
Using again Inequality (32) and $d \leq c$ and $a \leq c$, we obtain
\[
\log(|T_{a,d}|) \leq (d - 1)(1 + \log(a)) \leq c + d \log(c),
\]
and finally
\[
\log(\mathcal{N}_c(\mathcal{M}_*)) \leq 2a(c + d \log(c)).
\]

C.5 Proof of Proposition 4.8

For given tree $T$, ranks $r \in \mathbb{N}^{[T]}$ and feature space $V_N$, $N \in \mathbb{N}^d$, we consider sparse tensor networks with arbitrary sparsity pattern $\Lambda \subset \mathcal{L}_{T,N,r} := \times_{\alpha \in T} K_{\alpha}$, where $K_{\alpha} = \{1, \ldots, r_{\alpha}\} \times (\times_{\beta \in S(\alpha)} \{1, \ldots, r_{\beta}\})$ for $\alpha \in \mathcal{I}(T)$, and $K_{\alpha} = \{1, \ldots, r_{\alpha}\} \times \{1, \ldots, N_{\alpha}\}$ for $\alpha \in \mathcal{L}(T)$. The sparse representation complexity of a sparse tensor network is given by $C(T, r, V_N, \Lambda) = \sum_{\alpha \in T} |\Lambda_{\alpha}|$. We recall that for models of complexity $c$, we restrict the dimensions $N$ of features spaces to be less than a certain increasing function $g(c)$ of the complexity. The collections of models of sparse tensor networks have complexities
\[
\mathcal{N}_c(\mathcal{M}_{V,N,T}) = |\{r \in \mathbb{N}^{[T]}, \Lambda \subset \mathcal{L}_{T,N,r} : C(T, r, V_N, \Lambda) = c\}|,
\]
\[
\mathcal{N}_c(\mathcal{M}_T) = |\{r \in \mathbb{N}^{[T]}, N \in \mathbb{N}^d, \Lambda \subset \mathcal{L}_{T,N,r} : C(T, r, V_N, \Lambda) = c \text{ and } N \leq g(c)\}|,
\]
\[
\mathcal{N}_c(\mathcal{M}_*) = |\{T \in T_{a,d}, r \in \mathbb{N}^{[T]}, N \in \mathbb{N}^d, \Lambda \subset \mathcal{L}_{T,N,r} : C(T, r, V_N, \Lambda) = c \text{ and } N \leq g(c)\}|,
\]
where $T_{a,d}$ denotes the collection of trees with arity $a$ (or $a$-ary trees). It is clear that $\mathcal{N}_c(\mathcal{M}_{V,T}) \leq \mathcal{N}_c(\mathcal{M}_T) \leq \mathcal{N}_c(\mathcal{M}_*)$. First, we note that
\[
\mathcal{N}_c(\mathcal{M}_T) \leq \sum_{q = (q_{\alpha})_{\alpha \in T}} |\{r \in \mathbb{N}^{[T]}, N \in \mathbb{N}^d, \Lambda \in \mathcal{L}_{T,N,r} : |\Lambda_{\alpha}| = q_{\alpha}, \alpha \in T, \text{ and } N \leq g(c)\}|,
\]
where the sum is taken over all tuples $(q_{\alpha})_{\alpha \in T} \in \mathbb{N}^{[T]}$ such that $\sum_{\alpha \in T} q_{\alpha} = c$. Then
\[
\mathcal{N}_c(\mathcal{M}_T) \leq \sum_{q = (q_{\alpha})_{\alpha \in T}} \sum_{r \in \mathbb{N}^{[T]}} \sum_{N \in \mathbb{N}^d} \mathcal{N}_{q,T,N,r}
\]
with
\[
\mathcal{N}_{q,T,N,r} = |\{\Lambda \in \mathcal{L}_{T,N,r} : |\Lambda_{\alpha}| = q_{\alpha}, \alpha \in T\}|,
\]
\[
= \prod_{\alpha \in \mathcal{I}(T)} \left( \frac{r_{\alpha} \prod_{\beta \in S(\alpha)} r_{\beta}}{q_{\alpha}} \right) \prod_{\alpha \in \mathcal{L}(T)} \left( \frac{r_{\alpha} N_{\alpha}}{q_{\alpha}} \right).
\]
For \( r \leq q \), noting that \( r_\alpha \leq |A_\alpha| = q_\alpha \) for all \( \alpha \in T \), we obtain

\[
N_{q,T,N,r} \leq \prod_{\alpha \in \mathcal{L}(T)} \left( q_\alpha \prod_{\beta \in S(\alpha)} q_\beta \right) \prod_{\alpha \in \mathcal{L}(T)} \left( q_\alpha N_\alpha \right),
\]
and then using Inequality (32),

\[
\log(N_{q,T,N,r}) \leq \sum_{\alpha \in \mathcal{L}(T)} q_\alpha \left( 1 + \log \left( \prod_{\beta \in S(\alpha)} q_\beta \right) \right) + \sum_{\alpha \in \mathcal{L}(T)} q_\alpha \left( 1 + \log (N_\alpha) \right).
\]

Then with \( N_\alpha \leq g(c) \) and \( \sum_{\alpha \in T} q_\alpha = c \), we obtain

\[
\log(N_{q,T,N,r}) \leq c(1 + a \log(c)) + c(1 + g(c)).
\]

Noting that the number of tuples \( q \) such that \( \sum_{\alpha} q_\alpha = c \) is bounded by \( \binom{c+|T|}{c} \leq e^{c(1+\log(2))} \), that the number of tuples \( r \) such that \( r \leq q \) is equal to \( \prod_{\alpha \in T} q_\alpha \leq \left( \frac{r}{|T|} \right)^{|T|} \) and that the number of tuples \( n \) less than \( g(c) \) is equal to \( g(c)^d \), we obtain

\[
\log(N_c(M_T)) \leq c(1 + \log(2)) + |T| \log(c) + d \log(g(c)) + c(1 + a \log(c)) + c(1 + g(c)).
\]

Then noting that \( |T| \leq 2d \), we obtain

\[
\log(N_c(M_T)) \leq 4c + 2d \log(c) + ac \log(c) + 2c \log(g(c)).
\]

For the collection \( M_\ast \), we follow the proof of Proposition 4.7 and deduce that

\[
\log(N_c(M_\ast)) \leq \log(|T_{a,d}|) + \max_{T \in T_{a,d}} \log(N_c(M_T))
\]

\[
\leq (d-1)(1 + \log(a)) + 4c + 2d \log(c) + ac \log(c) + 2c \log(g(c))
\]

\[
\leq 4c + 4d \log(c) + ac \log(c) + 2c \log(g(c))
\]

\[
\leq 5ac \log(c) + 2c \log(g(c)).
\]

D Proofs of Section 5

D.1 Proof of Proposition 5.1

The proof follows the presentation of [32]. The least-squares contrast \( \gamma \) corresponds either to the regression contrast or the density estimation contrast. Under the assumptions of the proposition, in both frameworks the oracle function satisfies \( \|f^*\|_{\infty,\mu} \leq R \).

- We first prove the proposition in the case where \( R = 1 \), by assuming for the moment that \( M = M_1 = M^T(V) \). For the regression framework, it is also assumed for the moment that \( |Y| \leq 1 \) almost surely. Note that we also have \( \|f^*\|_{\infty,\mu} \leq 1 \).

- For the least-squares regression contrast (see Example 4.3), we have \( \gamma(f, Z) = |Y - f(X)|^2 \). For all \( f \in M_1 \), it gives \( \gamma(f, Z) \leq 2(|Y|^2 + \|f\|_{\infty,\mu}^2) \) almost surely, so that \( 0 \leq \gamma(f, Z) \leq B \) almost surely, with \( B = 4 \). The distribution of the random variable \( X \) is denoted \( \mu \). Then, almost surely,

\[
\mathbb{E}((\gamma(f, Z) - \gamma(f^*, Z))^2) = \mathbb{E}((f^*(X) - f(X))^2 |Y - f(X) - f^*(X))^2
\]

\[
= \mathbb{E}((f^*(X) - f(X))(2Y - f(X) - f^*(X)))^2
\]

\[
= \mathbb{E}((f^*(X) - f(X))(2Y - f^*(X)))^2 + \mathbb{E}((f^*(X) - f(X))^2
\]

\[
\leq (4|Y - f^*(X)|^2 + \|f^* - f\|_{\infty,\mu}^2)\|f - f^*\|_{2,\mu}^2
\]

\[
\leq 8\|f - f^*\|_{2,\mu}^2 = 2B\|f - f^*\|_{2,\mu}^2,
\]
where the last inequality has been obtained using $|Y - f^*(X)| = |Y - \mathbb{E}(Y|X)| \leq 1$ almost surely. Let $\gamma_1 = \gamma/B$. We have $0 \leq \gamma_1 \leq 1$ and the normalized excess risk satisfies

$$\mathcal{E}_1(f) := \mathbb{E} [\gamma_1(f, Z) - \gamma_1(f^*, Z)] = \frac{1}{B} \|f - f^*\|^2_{2,\mu} = \frac{1}{B} \mathcal{E}(f)$$

and

$$\mathbb{E}((\gamma_1(f, Z) - \gamma_1(f^*, Z))^2) \leq D\|f - f^*\|^2_{2,\mu}$$

with $D = \frac{2}{B} = \frac{1}{2}$.

- We now consider the density estimation framework with $\gamma(f, x) = \|f\|^2_{2,\mu} - 2f(x)$. According to Example 4.4, $|\gamma(f, X)| \leq B = \mu(\mathcal{X}) + 2$. The excess risk satisfies $\mathcal{E}(f) = \|f^* - f\|^2_{2,\mu}$ and

$$\mathbb{E}((\gamma(f, Z) - \gamma(f^*, Z))^2) = \mathbb{E}((\|f\|^2_{2,\mu} - \|f^*\|^2_{2,\mu} + 2(f^*(X) - f(x)))^2)$$

$$\leq (\|f\|^2_{2,\mu} - \|f^*\|^2_{2,\mu} + 4\|f\|^2_{2,\mu} - \|f^*\|^2_{2,\mu})(f^* - f, f^*)_{2,\mu} + 4\|f - f^*\|^2_{2,\mu}$$

$$= (\|f\|^2_{2,\mu} - \|f^*\|^2_{2,\mu})(\|f\|^2_{2,\mu} - \|f^*\|^2_{2,\mu} + 4(f^* - f, f^*)_{2,\mu}) + 4\|f - f^*\|^2_{2,\mu}$$

$$= (f - f^*, f + f^*)_{2,\mu} \langle f - f^*, f - 3f^* \rangle_{2,\mu} + 4\|f - f^*\|^2_{2,\mu}$$

We have $\langle f - f^*, f + f^* \rangle_{2,\mu} \leq \|f\|^1_{2,\mu} \leq \mu(\mathcal{X})$, $\|f\|^1_{1,\mu} = 1 \leq \mu(\mathcal{X})^{1/2}$ and $\|f^*\|^2_{2,\mu} \leq \|f^*\|_{\infty,\mu} \|f^*\|_{1,\mu} \leq 1$. Then

$$\mathbb{E}((\gamma(f, Z) - \gamma(f^*, Z))^2) \leq (\mu(\mathcal{X}) + 2\|f + f^*\|^2_{2,\mu} + 4\|f - f^*\|^2_{2,\mu})$$

$$\leq (\mu(\mathcal{X}) + 2\mu(\mathcal{X}) + 2 + 4\|f - f^*\|^2_{2,\mu})$$

$$= 3(\mu(\mathcal{X}) + 2)\|f - f^*\|^2_{2,\mu}$$

$$= 3B\|f - f^*\|^2_{2,\mu}.$$ 

Let $\gamma_1 = \frac{1}{2B}(\gamma + B)$. Then $0 \leq \gamma_1(f, X) \leq 1$ almost surely for any $f \in M_1$. Moreover,

$$\mathcal{E}_1(f) := \mathbb{E} [\gamma_1(f, Z) - \gamma_1(f^*, Z)] = \frac{1}{2B} \mathcal{E}(f)$$

and

$$\mathbb{E}((\gamma_1(f, Z) - \gamma_1(f^*, Z))^2) \leq D\|f - f^*\|^2_{2,\mu}$$

with $D = \frac{3}{2B} \leq \frac{3}{12}$, where we have used $\mu(\mathcal{X}) \geq 1$.

- For $\delta > 0$, we introduce

$$\omega_n(\delta) = \omega_n(M_1, f^*, \delta) = \mathbb{E} \sup_{f \in M_1 \|f - f^*\|^2_{2,\mu} \leq \delta/D} \left| \frac{1}{n} \sum_{i=1}^{n} \gamma_1(f, Z_i) - \mathbb{E}[(\gamma_1(f, Z) - \mathbb{E}(\gamma_1(f, Z)))] \right|$$

Following [32] (Section 4.1 p.57), we introduce the sharp transformation $\sharp$ of the function $\omega$:

$$\omega_n^\sharp(\varepsilon) = \inf \left\{ \delta > 0 : \sup_{\sigma \geq \delta} \frac{\omega_n(\sigma)}{\sigma} \leq \varepsilon \right\}.$$
According to Proposition 4.1 in [32], there exist absolute constants \( \kappa_1 \) and \( A \) such that for any \( \varepsilon \in (0, 1] \) and any \( t > 0 \), with probability at least \( 1 - A \exp(-t) \),

\[
\mathcal{E}_1(\hat{f}_n^{M_1}) \leq (1 + \varepsilon)\mathcal{E}_1(f^{M_1}) + \frac{1}{D} \omega_n^D \left( \frac{\varepsilon}{\kappa_1 D} \right) + \frac{\kappa_1 D}{\varepsilon} t. \tag{33}
\]

The sharp transformation is monotonic: if \( \Psi_1 \leq \Psi_2 \) then \( \Psi_1^\sharp \leq \Psi_2^\sharp \) (see Appendix A.3 in [32]). Thus it remains to find an upper bound on the sharp transformation of an upper bound on \( \omega_n \).

- We use standard symmetrization and contraction arguments for Rademacher variables. The Rademacher process indexed by the class \( M_1 \) is defined by

\[
\text{Rad}_n(f) = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i)
\]

where the \( \varepsilon_i \)'s are i.i.d. Rademacher random variables (that is, \( \varepsilon_i \) takes the values +1 and −1 with probability 1/2 each) independent of the \( X_i \)'s. By the symmetrization inequality (see for instance Theorem 2.1 in [32]),

\[
\omega_n(\delta) \leq 2\mathbb{E} \sup_{f \in M_1, \|f - f^{M_1}\|_2^2 \leq \delta} \left| \text{Rad}_n \left( \gamma(f, \cdot) - \gamma(f^{M_1}, \cdot) \right) \right|.
\]

We introduce the function

\[
\Psi_n(\delta) = \mathbb{E} \sup_{f \in M_1, \|f - f^{M_1}\|_2^2 \leq \delta} \left| \text{Rad}_n(f - f^{M_1}) \right|.
\]

For bounded regression, using the contraction Lemma with Lipschitz constant equal to 2 (see for instance Theorem 2.3 in [32]),

\[
\omega_n(\delta) \leq 8\Psi_n(\delta/D).
\]

In the density estimation setting, we have \( \gamma(f, X) = \|f\|_2^2 - 2f(X) \) and since the fluctuations of a constant function are obviously zero, we obtain

\[
\omega_n(\delta) \leq 4\mathbb{E} \sup_{f \in M_1, \|f - f^{M_1}\|_2^2 \leq \delta} \left| \text{Rad}_n(f - f^{M_1}) \right| \leq 4\Psi_n(\delta/D). \tag{34}
\]

- We now introduce the subset of the \( L^2 \) ball centered at \( f^{M_1} \)

\[
M_1(\delta, f^*) = \{ f - f^{M_1} : f \in M_1, \|f - f^{M_1}\|_2^2 \leq \delta \}.
\]

In the density estimation setting, the distribution of the \( X_i \)'s is \( \eta \) and the empirical measure is denoted by \( \eta_n \). We also denote by \( \eta_n \) the empirical measure in the regression setting (by taking \( \eta = \mu \)). According to Proposition 3.3

\[
H(\varepsilon, M_1(\delta, f^*), \| \cdot \|_{2, \eta_n}) \leq H(\varepsilon, \{ f \in M_1 : \|f\|_2^2 \leq \delta \}, \| \cdot \|_{2, \eta_n}) \leq C_M \log \left( \frac{3|T|L_{2, \eta_n}}{\varepsilon} \right) \mathbb{1}_{\varepsilon \leq 2} \eta_n \text{-almost surely,}
\]

48
where $L_{2,\eta_n}$ is defined by (10) for the measure $\eta_n$ and for $p = 2$. It can be easily checked from (10) that $L_{2,\eta_n} \leq L_{\infty,\eta_n}$. Next, the inequalities $L_{\infty,\eta_n} \leq L_{\infty,\mu} \leq \eta_0 \circ \mu$ almost surely in both settings. Moreover, $L_{\infty,\mu} \leq 1$ according to Proposition 3.4. Next, the metric entropy of $M_1(\delta, f^*)$ can be upper bounded $\eta_{o} \circ \mu$-almost surely as follows:

$$H(\varepsilon, M_1(\delta, f^*), \| \cdot \|_{2, \eta_n}) \leq C_M \left[ \log \left( \frac{4e}{\varepsilon} \right) + \log^+ \left( \frac{3|T|}{4e} \right) \right] \mathbb{1}_{\varepsilon \leq 4}$$

$$\leq C_M \left[ 1 + \log^+ \left( \frac{3|T|}{4e} \right) \right] \log \left( \frac{4e}{\varepsilon} \right) \mathbb{1}_{\varepsilon \leq 4}$$

$$\leq C_M b_T h \left( \frac{2}{\varepsilon} \right)$$

with $b_T = 1 + \log^+ \left( \frac{3|T|}{4e} \right)$ and $h(u) := \log (2eu) \mathbb{1}_{u \geq \frac{1}{2}}$. We are now in position to apply Theorem D.1, which is given at the end of this section. Note that the constant function $F = 2$ is an envelope for $M_1(\delta, f^*)$ and $\|F\|_{2, \eta_n} = 2$. We can take $\sigma^2 = \delta$ in Theorem D.1 because $E_{\eta}(g(X)^2) \leq \delta$ for $g \in M_1(\delta, f^*)$. Thus, there exists an absolute constant $\kappa_2 > 0$ such that

$$\Psi_n(\delta) \leq \kappa_2 \left[ \sqrt{\frac{\delta}{n} C_M b_T h \left( \frac{2}{\sqrt{\delta n}} \right)} \vee \left( \frac{2}{n} C_M b_T h \left( \frac{2}{\sqrt{\delta n}} \right) \right) \right].$$

For regression, it can be easily checked that (see also Example 3 p.80 in [32])

$$\Psi_n^*(\varepsilon) \leq \kappa_2 \frac{C_M b_T}{\varepsilon^2 n} \log \left( \frac{16\varepsilon^2}{\kappa_2^2 C_M b_T} \right).$$

Similar calculations hold for density estimation. Together with inequalities (33) and (34), and according to the properties of the sharp transformation (see Appendix A.3 in [32]), it gives that with probability at least $1 - \mathcal{A} \exp(-t)$,

$$\mathcal{E}_1(\hat{f}_n) \leq (1 + \varepsilon) \mathcal{E}_1(f^*_{M_1}) + \frac{1}{D} \left( 8 \Psi_n \left( \frac{\cdot}{D} \right) \right) \left( \frac{\varepsilon}{\kappa_1 D} \right) + \frac{\kappa_1 D t}{\varepsilon n}$$

$$\leq (1 + \varepsilon) \mathcal{E}_1(f^*_{M_1}) + (8 \Psi_n)^\sharp \left( \frac{\varepsilon}{\kappa_1} \right) + \frac{\kappa_1 D t}{\varepsilon n}$$

$$\leq (1 + \varepsilon) \mathcal{E}_1(f^*_{M_1}) + \frac{b_T C_M}{\varepsilon^2 n} \log \left( \frac{\kappa_4 \varepsilon^2}{b_T C_M} \right) + \frac{\kappa_1 D t}{\varepsilon n},$$

where $\kappa_3$ and $\kappa_4$ are absolute constants. This completes the proof for $R = 1$, by rewriting the risk bound for the excess risk $\mathcal{E} = B \mathcal{E}_1$.

- We now consider the more general situation where $M = M^T_r(V)R$ with $R \geq 1$. We first consider regression. We assume that $|X| \leq R$ almost surely. Let $f^*, f^M$ and $\hat{f}_M$ defined as in Section 4 for the observations $Z_1, \ldots, Z_n$. We consider the least squares regression problem for the normalized data $(X_1, Y_1/R), \ldots, (X_n, Y_n/R)$ with the functional set $M_1$. For this problem the oracle $f^*_1$ satisfies $f^*_1 = f^*/R$, the best approximation $f_{M_1}^*$ on $M_1$ satisfies $f_{M_1}^* = f^*/R$ and the least squares estimator $\hat{f}_{M_1}$ also satisfies $\hat{f}_{M_1} = \hat{f}_M/R$. The risk bound (20) is valid for the normalized data (with $R = 1$) and it directly gives (20) for $R \geq 1$. The same arguments apply for proving the risk bound in the density estimation case.
D.2 An adaptation of Theorem 3.12 in [32]

We consider the same framework as in [32]. We observe \( X_1, \ldots, X_n \) according to the distribution \( \eta \) and let \( \eta_n \) be the empirical measure. Let \( F \) be a function space. Assume that the functions in \( F \) are uniformly bounded by a constant \( U \) and let \( F \leq U \) denote a measurable envelope of \( F \). We assume that \( \sigma^2 \) is a number such that

\[
\sup_{f \in F} \mathbb{E}_\eta f^2 \leq \sigma^2 \leq \| F \|_{2,\eta}.
\]

Let \( h : [0, \infty) \to [0, \infty) \) be a regularly varying function of exponent \( 0 \leq \alpha < 2 \), strictly increasing for \( u \geq 1/2 \) and such that \( h(u) = 0 \) for \( 0 \leq u < 1/2 \).

The next result is an adaptation of Theorem 3.12 in [32] which provides a better control on the constant \( \kappa_h > 0 \) when multiplying the metric entropy function by a constant. In particular in this version the constant \( \kappa_h > 0 \) depends only on \( h \) and not on \( c \).

**Theorem D.1** (Theorem 3.12 in [32]). Let \( c > 0 \). If, for all \( \varepsilon > 0 \) and \( n \geq 1 \),

\[
\log N(\varepsilon, F, \| \cdot \|_{2,\eta_n}) \leq c h \left( \frac{\| F \|_{2,\eta_n}}{\varepsilon} \right) \eta_n^{\otimes n} -\text{almost surely},
\]

then there exists a constant \( \kappa_h > 0 \) that depends only on \( h \) such that

\[
E \sup_{f \in F} |\text{Rad}_n(f)| \leq \kappa_h \left[ \frac{\sigma}{\sqrt{n}} \sqrt{c h \left( \frac{\| F \|_{2,\eta_n}}{\varepsilon} \right)} \sqrt{\frac{U}{n c h \left( \frac{\| F \|_{2,\eta_n}}{\varepsilon} \right)}} \right].
\]

**Proof.** The proof of Theorem 3.12 of [32] starts by applying Theorem 3.11 of [32]. As in [32] we assume without loss of generality that \( U = 1 \). In our context it gives

\[
E := E \sup_{f \in F} |\text{Rad}_n(f)| \leq C \sqrt{c n^{-1/2}} \mathbb{E} \int_0^{2\sigma_n} \sqrt{h \left( \frac{\| F \|_{2,\eta_n}}{\varepsilon} \right)} d\varepsilon
\]

where \( \sigma_n = \sup_{f \in F} \sum_{i=1}^n f(X_i)^2 \) and where \( C \) is an universal numerical constant. By following the lines of the proof of [32], we find that \( E \) satisfies the following inequation

\[
E \leq \sqrt{c \kappa_{h,1} n^{-1}} + \sqrt{c \kappa_{h,2} n^{-1/2} \sigma_n} \sqrt{h \left( \frac{\| F \|_{2,\eta_n}}{\sigma} \right)} + \sqrt{c \kappa_{h,3} n^{-1/2}} \sqrt{\mathbb{E} \left( \frac{\| F \|_{2,\eta_n}}{\sigma} \right)}
\]

where \( \kappa_{h,1}, \kappa_{h,2} \) and \( \kappa_{h,3} \) are positive numerical constants which only depends on the function \( h \) (see the proof of Koltchinskii for the expression of these three constants). Solving this inequation completes the proof. \( \square \)

D.3 Proof of Theorem 5.2

The proof is adapted from Theorem 6.5 in [32], which corresponds to an alternative statement of Theorem 8.5 in [34]. We follow the lines of Section 6.3 in [32] (p.107-108).

We first consider the case \( R = 1 \) and we consider the normalized contrast \( \gamma_1 \) and the normalized risk \( E_1 \) as for the proof of Proposition 5.1. We have shown that

\[
\mathbb{E} [\gamma_1(f, Z) - \gamma_1(f^*, Z)]^2 \leq D \| f - f^* \|_2^2
\]

50
where $D$ does not depend on the model $M_m$. Next, it has also been shown in the proof of Proposition 5.1 that for $\varepsilon \in (0, 1]$,

$$\omega_n^2(\varepsilon) \leq \kappa \frac{b_m C_m}{n \varepsilon^2} \log^+ \left( \frac{n \varepsilon^2}{b_m C_m} \right)$$

with $b_m = 1 + \log^+ \left( \frac{3|T_m|}{4\varepsilon} \right)$ and where $\kappa$ is an absolute constant. We consider the penalized criterion (16) with a penalty of the form

$$\text{pen}(m) = \kappa_1 \frac{b_m C_m}{n \varepsilon^2} \log^+ \frac{n \varepsilon^2}{b_m C_m} + \kappa_2 \frac{w_m}{n \varepsilon} + \kappa_3 w_m$$

where $w_m = \bar{w} C_m + \log(N C_m)$. Theorem 6.5 of [32] can be applied here with $\delta^\varepsilon_n(m) = \delta^\varepsilon_n(m) = \delta^\varepsilon_n(m) = \kappa \frac{b_m C_m}{n \varepsilon^2} \log^+ \frac{n \varepsilon^2}{b_m C_m} + K \frac{w_m + t}{n \varepsilon}$ (and thus $p_m = 0$ in the theorem) and we also note that for any $t > 0$ the penalty can be rewritten

$$\text{pen}(m) = K_1 \left[ \frac{b_m C_m}{n \varepsilon^2} \log^+ \frac{n \varepsilon^2}{b_m C_m} + \frac{w_m + t}{n \varepsilon} \right].$$

Finally, according to Theorem 6.5 in [32], there exist numerical constants $K_1$, $K_2$ and $K_3$ such that for any $t > 0$,

$$P \left( \mathcal{E}_1(\hat{f}_m) \leq \frac{1 + \varepsilon}{1 - \varepsilon} \inf_{m \in \mathcal{M}} \left\{ \mathcal{E}_1(\hat{f}_m) + K_2 \text{pen}(m) \right\} \right) \leq K_3 \sum_{m \in \mathcal{M}} \exp(-t - w_m).$$

We easily derive the oracle bound (21) by rewriting it for the contrast $\gamma$ and then by integrating this probability bound with respect to $t$. This bound generalizes to the case $R \geq 1$ as in the proof of Proposition 5.1.