Schrödinger representation of quantum mechanics, Berry connection, and superconductivity

Hiroyasu Koizumi

Division of Quantum Condensed Matter Physics,
Center for Computational Sciences,
University of Tsukuba, Tsukuba, Ibaraki 305-8577, Japan

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Abstract

The standard quantum mechanical electronic state calculations for molecules and solids uses the Schrödinger representation where the momentum conjugate to the coordinate \( q_r \) is given by 
\[-i\hbar \frac{\partial}{\partial q_r}.\]
This formalism contains an extra \( U(1) \) phase degree-of-freedom. We show that it can be regarded as a Berry phase arising from many-electron interaction, and when it is non-trivial, it gives rise to a current carrying ground state identified as the superconducting ground state. The connection between this superconducting state and the BCS one is presented.

In the Schrödinger’s representation explained by Dirac [1], the momenta \( p_r \)’s conjugate to canonical coordinates \( q_r \)’s are given by
\[ p_r = -i\hbar \frac{\partial}{\partial q_r}, \quad r = 1, \cdots, n \] (1)
This representation assumes the existence of eigenket for the coordinates
\[ |q_1, \cdots, q_n\rangle \] (2)

The wave function \( \psi(q_1, \cdots, q_n) \) is given using the above ket as
\[ \psi(q_1, \cdots, q_n) = \langle q_1, \cdots, q_n | \psi \rangle \] (3)
where \( |\psi\rangle \) is the ket for the physical state. The wave function must be a single-valued function of the coordinates \( q_1, \cdots, q_n \) since the coordinates here are eigenvalues of the ket \( |q_1, \cdots, q_n\rangle \), thus, must be uniquely specified.

From the viewpoint of the Heisenberg formulation of quantum theory [2], commutation relations
\[ [q_r, q_s] = 0, \quad [p_r, p_s] = 0, \quad [q_r, p_s] = i\hbar \delta_{rs} \] (4)
are more fundamental than Eq. (1). The following \( p_r \)’s are also legitimate
\[ p_r = -i\hbar \frac{\partial}{\partial q_r} + \frac{\partial F}{\partial q_r}, \quad r = 1, \cdots, n \] (5)
since they satisfy the same commutation relations.

Dirac claims that we can always use Eq. (1). The reason is that the extra term, \( \frac{\partial F}{\partial q_r} \), can be removed by the change of the wave function
\[ \psi(q_1, \cdots, q_n) \rightarrow e^{i\gamma} \psi(q_1, \cdots, q_n) \] (6)
where $\gamma$ is related to $F$ by

$$F = \hbar \gamma + \text{constant} \quad (7)$$

However, this claim has not been carefully examined so far.

In the present work, we examine the phase factor $e^{i\gamma}$ for the ground state. The standard electronic structure calculation obtains the ground state wave function $e^{i\gamma} \psi(q_1, \cdots, q_n)$ as a whole by employing a finite number of basis functions. In this procedure, $e^{i\gamma}$ is not considered explicitly.

However, there are some cases where the explicit consideration of $e^{i\gamma}$ is necessary due to the fact that the coordinates in the wave function are continuous eigenvalues for continuously parameterized ketvector $|q_1, \cdots, q_n\rangle$. Representing a continuous eigenvalue state by a finite number of basis functions may miss something. Such a phenomenon is known in quantum field theory in the context of “anomaly” [3]. We consider the case where the explicit consideration of $e^{i\gamma}$ is necessary in the following.

Generally, the ground state many-electron wave function for a $N$ electron system can be cast in the following form,

$$\Psi(x_1, \cdots, x_N) = \exp \left( i \sum_{j=1}^{N} \int_0^{r_j} A_{\Psi}^{MB}(r') \cdot dr' \right) \Psi_0(x_1, \cdots, x_N) \quad (8)$$

where $x_i$ collectively denotes the coordinate $r_i$ and the spin $\sigma_i$ of the $i$th electron, $\Psi_0(x_1, \cdots, x_N)$ is the currentless wave function that is obtained by the energy minimization; the coordinates $q_r$, ($r = 1, \cdots, n$) in Eq. (1) correspond to $r_i$, ($i = 1, \cdots, N$) with $n = 3N$ and from $\Psi$ [2, 6]. The factor $\exp \left( i \sum_{j=1}^{N} \int_0^{r_j} A_{\Psi}^{MB}(r') \cdot dr' \right)$ is the one arising from the Berry connection $A_{\Psi}^{MB}$ [4], which is defined using $\Psi(x_1, \cdots, x_N)$ by

$$A_{\Psi}^{MB}(r) = -i \int d\sigma_1 d\xi_2 \cdots d\xi_N \frac{\Psi^*(r, \sigma_1, x_2, \cdots, x_N) \nabla_r \Psi(r, \sigma_1, x_2, \cdots, x_N)}{\rho(r)^{\frac{3}{2}}} \frac{\rho(r)^{\frac{3}{2}}}{\rho(r)^{\frac{3}{2}}}

= \frac{1}{\hbar \rho(r)^{\frac{3}{2}}} \text{Re} \left( \int d\sigma_1 d\xi_2 \cdots d\xi_N \Psi^*(r, \sigma_1, \cdots, x_N) p_r \Psi(r, \sigma_1, \cdots, x_N) \right) \quad (9)$$

where $\rho$ is the electron density obtained

$$\rho(r) = \int d\sigma_1 d\xi_2 \cdots d\xi_N \Psi^*(r, \sigma_1, x_2, \cdots, x_N) \Psi(r, \sigma_1, x_2, \cdots, x_N) \quad (10)$$

The comparison of Eqs. (6) and (8) reads

$$\gamma = \sum_{j=1}^{N} \int_0^{r_j} A_{\Psi}^{MB}(r') \cdot dr' \quad (11)$$
This indicates that a non-trivial $e^{i\gamma}$ arises from a non-trivial $A^{\text{MB}}_\Psi$.

The so-called “Bloch’s theorem” states that the ground state is currentless \cite{7}. If this theorem is valid, $\Psi_0$ is the ground state wave function, and $A^{\text{MB}}_\Psi$ is trivial, giving a constant
\[ e^{i\sum_{j=1}^N \int_0^r A^{\text{MB}}_\Psi(r') \cdot dr'} \]
This situation corresponds to a normal ground state. However, if $A^{\text{MB}}_\Psi$ is non-trivial, a current carrying ground state may arise. We may identify such a ground state as a superconducting one.

Since $A^{\text{MB}}_\Psi$ is a self-referencing quantity obtained from the wave function $\Psi$ itself as seen in Eq. \cite{9}, we employ general requirements to derive it from $\Psi_0$ obtained by the usual procedure.

The requirements are

1. The single-valuedness of $\Psi(x_1, \ldots, x_N)$ as a function of $r_1, \ldots, r_N$.

2. The local charge conservation.

The first requirement is necessary since we use Eq. \cite{11} which assumes the uniqueness of $q_i$’s.

The second requirement is a usual requirement for electron systems. Actually, it is equivalent to the energy minimization requirement with respect to the variation of $\gamma$.

Our previous investigations show that current carrying ground states can be obtained using the above requirements \cite{6, 8}.

Now, we treat $\gamma$ as a field. For this purpose, it is more convenient to use $\chi$ given by
\[ \chi(r) = -2 \int_0^r A^{\text{MB}}_\Psi(r') \cdot dr' \quad (12) \]

The presence of $\chi$ makes the total energy as a functional of $\chi$, which we denote as $E[\chi]$. This $\chi$ enters only in the kinetic energy part through $\nabla \chi$ by the replacement
\[ -i\hbar \nabla \rightarrow -i\hbar \nabla - \frac{\hbar}{2} \nabla \chi \quad (13) \]

This indicates that $\nabla \chi$ gives rise to a vector potential for a “fictitious magnetic field”
\[ A^{\text{fc}} = \frac{c\hbar}{2(-e)} \nabla \chi \quad (14) \]
where $-e$ is the electron charge.

Then, using a general formula for the current from the energy functional, the current density is
\[ j = -e \frac{\delta E}{\delta A^{\text{fc}}} = -\frac{2e}{\hbar} \frac{\delta E}{\delta \nabla \chi} \quad (15) \]
The fact that \( E[\chi] \) only depends on \( \nabla \chi \) yields the following relation from the minimization requirement of the total energy with respect to \( \chi \)

\[
0 = \frac{\delta E}{\delta \chi} = -\nabla \cdot \frac{\delta E}{\delta \nabla \chi} = \frac{\hbar}{2e} \nabla \cdot j
\]  

(16)

which proves the energy minimization is equivalent to the local charge conservation as mentioned before.

Let us consider the quantization of the field \( \chi \). The canonical conjugate momentum of \( \chi \), denoted by \( \pi_\chi \), is obtained from the Lagrangian

\[
L = \langle \Psi | i\hbar \frac{\partial}{\partial t} - H | \Psi \rangle = \int d^3r \hbar \frac{\dot{\chi}}{2} \rho + i\hbar \langle \Psi_0 | \frac{\partial}{\partial t} | \Psi_0 \rangle - \langle \Psi | H | \Psi \rangle
\]  

as

\[
\pi_\chi = \frac{\delta L}{\delta \dot{\chi}} = \frac{\hbar}{2} \rho
\]  

(17)

(18)

The canonical quantization condition reads

\[
[\chi(\mathbf{r}), \pi_\chi(\mathbf{r}')] = i\hbar \delta(\mathbf{r} - \mathbf{r}')
\]  

(19)

or

\[
[\chi(\mathbf{r}), \rho(\mathbf{r}')] = 2i\delta(\mathbf{r} - \mathbf{r}')
\]  

(20)

Then, we can construct the following boson field operators

\[
\psi_\chi(\mathbf{r}) = \sqrt{\rho(\mathbf{r})} e^{i\chi(\mathbf{r})}, \quad \psi_\chi^\dagger(\mathbf{r}) = e^{-\frac{i\chi}{2}(\mathbf{r})} \sqrt{\rho(\mathbf{r})}
\]  

(21)

that satisfy the commutation relation \([\psi_\chi(\mathbf{r}), \psi_\chi^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}').\]

Integrating \( \psi_\chi^\dagger(\mathbf{r}) \) and \( \psi_\chi(\mathbf{r}) \) over the space, we obtain boson operators \( B_\chi^\dagger \) and \( B_\chi \) given by

\[
B_\chi^\dagger = \int d^3r \psi_\chi^\dagger(\mathbf{r}), \quad B_\chi = \int d^3r \psi_\chi(\mathbf{r})
\]  

(22)

They satisfy the following commutation relation

\[
[B_\chi, B_\chi^\dagger] = 1
\]  

(23)

We define the number operator

\[
\hat{N}_\chi = B_\chi^\dagger B_\chi
\]  

(24)
and introduce the eigenket $|N\chi\rangle$,

$$\hat{N}\chi|N\chi\rangle = N\chi|N\chi\rangle \quad (25)$$

The number $N\chi$ can be considered as the number of electrons participating in the collective mode described by $\chi$.

A phase operator $\hat{X}$ that is conjugate to $\hat{N}\chi$ is defined through the relations

$$B_\chi^\dagger = \sqrt{N\chi} e^{i\frac{1}{2}\hat{X}}, \quad B_\chi = e^{-i\frac{1}{2}\hat{X}} \sqrt{N\chi} \quad (26)$$

The phase and number operators satisfy the following commutation relations

$$[\hat{N}\chi, e^{\pm i\frac{1}{2}\hat{X}}] = \pm e^{\pm i\frac{1}{2}\hat{X}} \quad (27)$$

We can derive the following relations

$$e^{\pm i\frac{1}{2}\hat{X}}|N\chi\rangle \propto |N\chi \pm 1\rangle \quad (28)$$

Thus, $e^{-i\frac{1}{2}\hat{X}}$ is the number changing operator that decreases the number of electrons participating in the collective mode by one, $e^{i\frac{1}{2}\hat{X}}$ increases by one.

We now reformulate the standard theory of superconductivity, the BCS theory, using the number changing operators $e^{\pm i\frac{1}{2}\hat{X}}$. According to this theory, the origin of superconductivity is the energy gap formation due to electron-pairing [9]. The rigidity of the wave function against external perturbations envisaged by London [10] is realized by this energy gap, and one of the hallmarks of superconductivity, the exclusion of a magnetic field from the superconductor (the Meissner effect), is explained by this rigidity.

The BCS used the following variational state vector,

$$|\text{BCS}\rangle = \prod_k (u_k e^{i\theta} v_k c_{k\uparrow}^\dagger c_{-k\downarrow}) |\text{vac}\rangle \quad (29)$$

to take into account the electron pairing effect, where $|\text{vac}\rangle$ is the vacuum state, $c_{k\sigma}^\dagger$ is the creation operator for the conduction electron of effective mass $m^*$ with the wave vector $k$ and spin $\sigma$, and $u_k$ and $v_k$ are variational parameters. The obtained energy gap explains many experimental results, and a method to calculate the superconducting transition temperatures is provided as the method to calculate the energy gap formation temperature [9].

A salient feature of the BCS state vector in Eq. (29) is that it does not satisfy the conservation of the particle number. This is odd since superconductivity occurs in an isolated
superconductor where the number of electrons is fixed \[11\]. However, this non-conservation property is a crucial ingredient; it makes the phase factor \(e^{i\theta}\) physically meaningful. This breaks the global \(U(1)\) gauge symmetry, and this gauge symmetry breaking is needed to explain the Meissner effect in superconductors \[12-14\].

Using the number changing operator \(e^{iX}\), a state vector similar to the one in Eq. (29) is constructed

\[
|\text{Gnd}\rangle = \prod_k (u_k + v_k c^+_k e^{-iX}) |\text{Cnd}\rangle
\]

where the state vector \(|\text{Cnd}\rangle\) corresponds to the state given by the wave function \(\Psi\). This state corresponds to the state in which all the electrons participate in the collective mode

\[
|\text{Cnd}\rangle = |N_\chi = N\rangle
\]

The operator \(c^+_k e^{-iX}\) acting on \(|\text{Cnd}\rangle\) decreases the number of electrons participating in the collective mode by two, and increases the number of electrons in single-particle motion. Thus, the particle number is fixed in Eq. (30).

Let us consider an example using Eq. (30). It is a two dimensional system with the following single particle Hamiltonian

\[
h = -\frac{\hbar^2}{2m_e} (\partial^2_x + \partial^2_y) + U(r)
\]

For simplicity we assume that the potential \(U\) depends only on \(r\) with \(x = r \cos \phi\) and \(y = r \sin \phi\).

The coordinate part of the wave function is a product of an angular function and a radial function given by

\[
\varphi_{nm}(r, \phi) = \frac{1}{\sqrt{2\pi}} \frac{e^{im\phi}}{n!} R_{n|m|}(r)
\]

where \(m\) is an integer, \(n\) is a natural number that denotes the number of nodes of the radial wave function, \(R_{n|m|}(r)\); \(\varphi_{nm}(r, \phi)\) is the eigenfunction of \(h\) with energy \(E_{nm}\),

\[
h \varphi_{nm}(r, \phi) = E_{nm} \varphi_{nm}(r, \phi)
\]

Usually, the wave functions \(\psi_{nm\uparrow} = \varphi_{nm}(r, \phi)|\uparrow\rangle\) and \(\psi_{nm\downarrow} = \varphi_{nm}(r, \phi)|\downarrow\rangle\) are used by adopting the coordinate independent spin functions \(|\uparrow\rangle\) and \(|\downarrow\rangle\). However, we consider the
following spin functions

\[
|\Sigma_a\rangle = \frac{1}{\sqrt{2}} (e^{-\frac{i}{2}f(\phi) \sin \zeta_0} |\uparrow\rangle + e^{\frac{i}{2}f(\phi) \cos \zeta_0} |\downarrow\rangle),
\]

\[
|\Sigma_b\rangle = \frac{1}{\sqrt{2}} (-e^{-\frac{i}{2}f(\phi) \sin \zeta_0} |\uparrow\rangle + e^{\frac{i}{2}f(\phi) \cos \zeta_0} |\downarrow\rangle)
\]  

(35)

where \(f(\phi)\) is a function of \(\phi\), and \(\zeta_0\) is a constant, and use the following wave functions,

\[
\tilde{\psi}_{nma} = \varphi_{nm}(r,\phi)|\Sigma_a\rangle, \quad \tilde{\psi}_{nmb} = \varphi_{nm}(r,\phi)|\Sigma_b\rangle
\]  

(36)

Expectation values of the components of spin \(s = (s_x, s_y, s_z)\) for \(|\Sigma_a\rangle\) are given by

\[
\langle \Sigma_a | s_x | \Sigma_a \rangle = \frac{\hbar}{2} \cos f(\phi) \sin \zeta_0,
\]

\[
\langle \Sigma_a | s_y | \Sigma_a \rangle = \frac{\hbar}{2} \sin f(\phi) \sin \zeta_0,
\]

\[
\langle \Sigma_a | s_z | \Sigma_a \rangle = \frac{\hbar}{2} \cos \zeta_0
\]  

(37)

and those for \(|\Sigma_b\rangle\) are \(\langle \Sigma_b | s | \Sigma_b \rangle = -\langle \Sigma_a | s | \Sigma_a \rangle\).

We consider the case where spin-twisting occurs around the \(z\)-axis. For simplicity, we consider the following case

\[
f(\phi) = \phi
\]  

(38)

Then, the spin functions in Eq. (35) become multi-valued as follows

\[
\phi \rightarrow \phi + 2\pi; \quad \Sigma_a \rightarrow -\Sigma_a, \quad \Sigma_b \rightarrow -\Sigma_b
\]  

(39)

Note that \(\phi\) and \(\phi + 2\pi\) describe the same coordinate, however, two values of \(\Sigma_a\) and \(\Sigma_b\) arise. The wave functions in Eq. (36) also show the same multi-valuedness.

Now we take into account the phase factor \(e^{i\gamma}\). The wave functions including the phase factor \(e^{i\gamma}\) are given by

\[
\tilde{\psi}_{nma} = \tilde{\psi}_{nma} e^{-i\frac{\chi}{2}}, \quad \tilde{\psi}_{nmb} = \tilde{\psi}_{nmb} e^{-i\frac{\chi}{2}}
\]  

(40)

The above new wave functions are single-valued if the phase factor \(e^{i\frac{\chi}{2}}\) compensates the sign change of \(\tilde{\psi}_{nma}\) and \(\tilde{\psi}_{nmb}\). Since the multi-valuedness of \(\tilde{\psi}_{nma}\) and \(\tilde{\psi}_{nmb}\) arise from their \(\phi\) dependence, we may treat \(\chi\) as a function of \(\phi\).

The total energy \(E[\chi]\) depends on \(\chi\) through \(\frac{d\chi}{d\phi}\). Thus, the condition for an optimal \(\chi\) that minimizes the total energy yields \(\chi = A\phi + B\), where \(A\) and \(B\) are constants. We can
put \( B = 0 \) since it merely gives rise to a constant phase factor on the wave function. The constant \( A \) must be so chosen that \( e^{\frac{i}{2}(A \pm 1)\phi} \) is a single-valued function of the coordinate. Then, \( A \) is an odd integer, and the minimal energy one will be \( A = 1 \) or \( A = -1 \).

The total energy becomes the sum of the kinetic energy from \( e^{i\gamma} \) and the total energy from \( \psi(q_1, \cdots, q_n) \) due to the fact that the current is zero for \( \psi(q_1, \cdots, q_n) \). Thus, we have

\[
E[\chi] = \int drd\phi \frac{\rho(r) \hbar^2}{2m_e} \left( \frac{1}{2} \frac{d\chi}{d\phi} \right)^2 + \sum_{E_{nm} \leq 0} 2\tilde{E}_{nm}
\]

where \( \tilde{E}_{nm} = E_{nm} - E_F \) and \( E_F \) is the Fermi energy. The factor 2 in the second term appears due to the fact that both \( \psi_{nma} \) and \( \psi_{nmb} \) are occupied. This current carrying state is energetically higher than the currentless state.

Let us introduce the pairing interaction given by

\[
H_{\text{pair}} = \sum_{n,m,n',m'} V_{nm;m'm'} c_{nm\uparrow}^\dagger c_{n-m'\downarrow} c_{n'-m\uparrow} c_{n'm\uparrow}
\]

\[
= \sum_{n,m,n',m'} V_{nm;m'm'} c_{nm\uparrow}^\dagger c_{n-m\downarrow} e^{-i\hat{X}} e^{i\hat{X}} c_{n'-m'\downarrow} c_{n'm\uparrow}
\]

where \( c_{nm\sigma}^\dagger \) and \( c_{nm\sigma} \) are creation and annihilation operators for \( \varphi_{nm}(r, \phi)|\sigma\rangle \), and the identity \( 1 = e^{-i\hat{X}} e^{i\hat{X}} \) is used.

The total Hamiltonian is the sum of \( H_{\text{pair}} \) and the single-particle Hamiltonian given by

\[
H_0 = \sum_{n,m} \tilde{E}_{nm}(c_{nm\uparrow}^\dagger c_{nm\uparrow} + c_{nm\downarrow}^\dagger c_{nm\downarrow})
\]

We employ a BCS type variational ground state

\[
|\text{Gnd}\rangle = \prod_{n,m} (u_{nm} + v_{nm} c_{nm\uparrow}^\dagger c_{n-m\downarrow} e^{-i\hat{X}}) |\text{Gnd}\rangle
\]

where \( u_{nm} \) and \( v_{nm} \) are variational parameters that satisfy \( u_{nm}^2 + v_{nm}^2 = 1 \).

The pairing energy gap is defined as

\[
\Delta_{nm} = - \sum_{n,m,n',m'} V_{nm;n'm'} \langle \text{Gnd}| e^{i\hat{X}} c_{n'-m\downarrow} c_{n'm\uparrow} |\text{Gnd}\rangle
\]

\[
= - \sum_{n,m,n',m'} V_{nm;n'm'} u_{n'm'} v_{n'm'}
\]

where \( e^{i\hat{X}} c_{n'-m\downarrow} c_{n'm\uparrow} \) annihilates two electrons in the single-particle mode and creates two electrons in the collective mode with conserving the particle number.
As in the BCS theory, we assume \( V_{nm,n'm'} = -V \) if \( |\tilde{E}_{nm}|, |\tilde{E}_{n'm'}| \leq \hbar \omega_c \) and zero otherwise, where \( \hbar \omega_c \) is a cut-off energy \([9]\). Then, we obtain

\[
 u_{nm}^2 = \frac{1}{2} \left( 1 + \frac{\tilde{E}_{nm}}{\sqrt{E_{nm}^2 + \Delta^2}} \right), \quad v_{nm}^2 = \frac{1}{2} \left( 1 - \frac{\tilde{E}_{nm}}{\sqrt{E_{nm}^2 + \Delta^2}} \right)
\]

(46)

where the pairing energy gap is given by

\[
 \Delta \approx 2\hbar \omega_c e^{-\frac{1}{N(0)V}}
\]

(47)

with \( N(0) \) being the density of states at the Fermi energy.

The total energy becomes

\[
 E_{\text{tot}} = \int r dr d\phi \rho(r) \frac{\hbar^2}{2m_e} \left( \frac{1}{2} \frac{d\chi}{d\phi} \right)^2 + 2 \sum_{mn} \tilde{E}_{nm} v_{nm}^2 - \frac{\Delta^2}{V}
\]

\[
 = \int r dr d\phi \rho(r) \frac{\hbar^2}{2m_e} \left( \frac{1}{2} \frac{d\chi}{d\phi} \right)^2 - \frac{1}{2} N(0) V \Delta^2
\]

(48)

where the number of electrons in the collective mode is calculated as

\[
 \int r dr d\phi \rho(r) = \sum_{\tilde{E}_{nm} \leq 0} u_{nm}^2
\]

\[
 = N(0) \left( \hbar \omega_c + \sqrt{\Delta^2 - \hbar^2 \omega_c^2 + \Delta^2} \right)
\]

\[
 \approx N(0) \left( \Delta - \frac{1}{2\hbar \omega_c} \Delta^2 + \frac{1}{8\hbar^3 \omega_c^3} \Delta^4 \right)
\]

(49)

If the energy gap formation makes the current carrying state lower in energy than the currentless state, the superconducting state is realized. This example only consider one centers of spin-twisting; in reality, multi-spin-twisting centers are more energetically favorable. If we consider a more general setting by including the potential energy from the underlying ion lattice and effective field from other electrons, the spin-twisting itinerant motion occurs as the circular motion around a section of the Fermi surface of the metal \([15]\). This will correspond to a system with multi-spin-twisting-centers in the coordinate space.

When a magnetic field exists, the vector potential from magnetic field \( A_{\text{em}} \) appears in addition. Then, the kinetic energy of the collective mode is given by

\[
 E_\chi = \int d^3r \frac{\hbar^2}{2m_e} \left( \frac{1}{2} \nabla \chi - \frac{e}{c \hbar} A_{\text{em}} \right)^2
\]

(50)

Then, the supercurrent density is given by

\[
 \mathbf{j} = -e \frac{\partial E_\chi}{\partial A_{\text{em}}} = -\frac{e^2}{m_e c} \left( \mathbf{A}_{\text{em}} - \frac{c \hbar}{2e} \nabla \chi \right)
\]

(51)
This is a diamagnetic current explains the Meissner effect. Note that the ambiguity in the gauge of $A^\text{em}$ is absorbed during the optimization of $\nabla \chi$, thus, the current is gauge invariant. The period $2\pi$ of the angular variable $\chi$ yields the flux quantum $\frac{\hbar}{2e}$.

The velocity field associated with the above supercurrent is

$$v_s = \frac{e}{m_e c} \left( A^\text{em} - \frac{\hbar}{2e} \nabla \chi \right)$$

(52)

The velocity field generated inside the superconductor by rotating it with an angular velocity $\omega$ is given by $v_{\text{rot}} = \omega \times r$. Since supercurrent electrons move with the body to shield the electric field from the ion core, the condition $v_s = v_{\text{rot}}$ is satisfied. Substituting this in Eq. (52) yields the magnetic field

$$B^\text{em} = \frac{2m_e c}{e} \omega$$

(53)

inside the superconductor. This is known as the London moment phenomenon. The above formula has the free electron mass in accordance with the experimental results. Note that the effective mass $m^\ast$ is used for the supercurrent carrier in the BCS theory; thus, the mass in Eq. (53) becomes $m^\ast$ instead of $m_e$ in disagreement with the experiment. This point has been discussed by some researchers [16, 17].

The present work indicates the superconductivity is a phenomenon where the non-trivial $A_\Psi^\text{MB}$ appears. The electron pairing stabilizes it, and the BCS theory takes into account this stabilization effect using the variational wave function that breaks the global $U(1)$ gauge symmetry. However, the global $U(1)$ gauge symmetry breaking is not necessary for superconductivity but the non-trivial $e^{i\gamma}$ is.

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