Lower bound on the relative error of mixed-state cloning and related operations

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We extend the concept of the relative error to mixed-state cloning and related physical operations, in which the ancilla contains some \textit{a priori} information about the input state. The lower bound on the relative error is obtained. It is shown that this result contributes to the stronger no-cloning theorem.

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\section{I. INTRODUCTION}

Quantum cloning is an important issue in quantum information, due to its connection to security in quantum cryptography and reflection on the nature of quantum states. It is well known that nonorthogonal pure states cannot be cloned \cite{1}. This result was generalized and extended in Ref. \cite{2}: noncommuting mixed states cannot be broadcast. In Ref. \cite{3} the stronger no-cloning theorem was established. For example, let \{\ket{s_1}, \ket{s_2}\} be any pair of nonorthogonal pure states and \{\Upsilon_1, \Upsilon_2\} be any pair of mixed states. According to the stronger no-cloning theorem, there is a physical operation \ket{s_j} \otimes \Upsilon_j \mapsto \ket{s_j} \ket{s_j}\ if and only if there is a physical operation \Upsilon_j \mapsto \ket{s_j}. In other words, the full information of the clone must be \textit{a priori} provided in the ancilla state \Upsilon_j alone \cite{3}.

The approximate quantum copying was originally considered by Bužek and Hillery \cite{4}. In addition, they examined approximate cloning machines operating on prescribed two non-orthogonal states \cite{5}. In Ref. \cite{6} such devices were called 'state-dependent cloners'. As a criterion for estimation of the state-dependent cloning, Ref. \cite{6} introduced "global fidelity" and "local fidelity." It has constructed the optimal "global" cloner that maximizes the global fidelity. The local fidelity has also been optimized. The writers of Ref. \cite{7} obtained the upper bound on the global fidelity for \(N \to L\) cloning of two states with \textit{a priori} probabilities. Ref. \cite{8} considered state-dependent \(N \to L\) cloning with respect to both the mentioned criteria.

The other category of cloners contains universal cloning machines which copy arbitrary state equally well. First such example was given by Bužek and Hillery \cite{4}. Refs. \cite{6,9} constructed the universal qubit cloner that maximizes the local fidelity. Analogous problem for multi-level quantum system was solved in Refs. \cite{10,11}. Note that the approximate cloning is interesting for several questions. The problem of security in quantum cryptography is one of obvious applications. In addition, the cloning transformations can be used to realize joint measurements of noncommuting observables \cite{12}.

Thus, the state-dependent cloning was mainly examined from the "fidelity" viewpoint. However, the state-dependent cloning is a complex subject with many facets. Important as the notions of the global fidelity and the local fidelity are, they do not cover the problem on the whole \cite{13}. An optimality criterion to widen an outlook is needed. In Ref. \cite{13} we introduced such a criterion called "relative error." We have found that minimizing the relative error is essentially different task from optimizing other quantities. The asymmetric cloner, which minimizes the relative error, was us constructed. As Ref. \cite{13} shows, the study of the relative error has allowed to complement a portrait of the state-dependent cloning.

All the above results examine the pure-state cloning. Ref. \cite{14} introduced the single qubit purification procedure that was used in extending of the input of the optimal cloners constructed in Refs. \cite{9,10} to mixed states. However, the described in Ref. \cite{14} scenario is not equivalent to the standard statement of cloning problem. The approximate copying of mixed states is interesting for various questions. For example, in some protocols Alice and Bob encode the bits 1 and 0 into two non-orthogonal pure states \cite{15}. In the reality a communication channel will inevitably suffer from noise that will have caused the bits to evolve to mixed states. Eve is then anxious for cloning of two noncommuting mixed states. Ref. \cite{16} extends the concept of state-dependent cloning to the case of mixed states. The upper bound on the global fidelity for mixed-state cloning has been established. The notion of the angle between two mixed states \cite{16} allows to give simple proof for this upper bound.

In this paper we define the relative error for mixed-state cloning and related operations in which the ancilla state contains some \textit{a priori} information about the state to be cloned. The lower bound on the relative error will be obtained. The optimization of a general unitary transformation is more difficult problem. It is not considered in the present work.
II. PRELIMINARY LEMMAS

Before definition of the relative error, we shall prove two useful statements those maintain our approach. In general, the measure of distinguishability for mixed quantum states is provided by the fidelity function. The fidelity $F(\chi, \omega)$ between two density operators $\chi$ and $\omega$ is defined by [17]

$$F(\chi, \omega) = \max |\langle X | Y \rangle|^2.$$  \hspace{1cm} (1)

Pure states $|X\rangle$ and $|Y\rangle$ are purifications of $\chi$ and $\omega$ respectively, that is $\chi = \mathrm{Tr}_E (|X\rangle\langle X|)$ and $\omega = \mathrm{Tr}_E (|Y\rangle\langle Y|)$. The quantity given by Eq. (1) is equivalent to the Uhlmann’s transition probability for mixed states [18]. Note that this usage of word “fidelity” is not unique: Refs. [2,19] define fidelity to be the square root of the present quantity. Ref. [16] parametrized the fidelity by means of the angle between mixed states, namely:

$$F(\chi, \omega) = \cos^2 \Delta(\chi, \omega),$$

$$\Delta(\chi, \omega) = \min \delta(X,Y).$$  \hspace{1cm} (3)

Here $\delta(X,Y) \in [0; \pi/2]$ is angle between vectors $|X\rangle$ and $|Y\rangle$. For any triplet $\{\chi, \omega, \rho\}$ of mixed states [16],

$$\Delta(\chi, \omega) \leq \Delta(\chi, \rho) + \Delta(\omega, \rho).$$  \hspace{1cm} (4)

This result extends the spherical triangle inequality to the case of mixed states. The first useful statements gives the upper bound on the difference between fidelities $F(\chi, \rho)$ and $F(\omega, \rho)$.

Lemma 1 For any triplet $\{\chi, \omega, \rho\}$ of mixed states,

$$|F(\chi, \rho) - F(\omega, \rho)| \leq \sin \Delta(\chi, \omega).$$  \hspace{1cm} (5)

Proof Because Eq. (4) and standard trigonometric formula $\cos^2 \alpha - \cos^2 \beta = -(\sin(\alpha + \beta) \sin(\alpha - \beta))$ [20],

$$\cos^2 \Delta_{\chi\rho} - \cos^2 \Delta_{\omega\rho} \leq \cos^2 (\Delta_{\chi\omega} - \Delta_{\omega\rho}) - \cos^2 \Delta_{\omega\rho} = \sin \Delta_{\chi\omega} \sin(2\Delta_{\omega\rho} - \Delta_{\chi\omega}) \leq \sin \Delta_{\chi\omega}.$$

We then get by a parallel argument

$$\cos^2 \Delta_{\omega\rho} - \cos^2 \Delta_{\chi\rho} \leq \sin \Delta_{\chi\omega},$$

and the two last inequalities give Eq. (5). ■

The second useful statement establishes the upper bound on the modulus of difference between probability distributions generated by two mixed states $\chi$ and $\omega$ for any measurement. Let $\{E_a\}$ be a generalized measurement (POVM). Such a measurement over the system $S$ in state $\rho$ produces outcome $a$ with probability [21,22]

$$p(a|\rho) = \mathrm{Tr}_S (E_a \rho).$$  \hspace{1cm} (6)

Lemma 2 For arbitrary measurement and any two states $\chi$ and $\omega$,

$$|p(a|\chi) - p(a|\omega)| \leq \sin \Delta_{\chi\omega}.$$

Proof Recall that POVM can be realized as an orthogonal measurement over extended system $ST$ [22] (this is insured by Neumark’s theorem). That is,

$$\mathrm{Tr}_S (E_a \rho) = \mathrm{Tr}_{ST} (\Pi_a (\rho \otimes \sigma)),$$  \hspace{1cm} (8)

where $\{\Pi_a\}$ is an orthogonal measurement. We now choose purifications $|X\rangle$ of $\chi \otimes \sigma$ and $|Y\rangle$ of $\omega \otimes \sigma$ so that $F(\chi \otimes \sigma, \omega \otimes \sigma) = |\langle X | Y \rangle|^2$. Because Eq. (8), we can write

$$p(a|\chi) = \langle X | \Pi_a \otimes \mathbf{1} | X \rangle,$$

$$p(a|\omega) = \langle Y | \Pi_a \otimes \mathbf{1} | Y \rangle.$$  \hspace{1cm} (9)

Ref. [13] proved that for arbitrary projector $\Pi$,

$$|\langle X | \Pi | X \rangle - \langle Y | \Pi | Y \rangle| \leq \sin \Delta_{XY}. $$  \hspace{1cm} (11)

Since the fidelity function is multiplicative [17], there is $F(\chi \otimes \sigma, \omega \otimes \sigma) = F(\chi, \omega)$ and therefore $\Delta_{\chi\omega} = \delta_{XY}$. Using Eqs. (9), (10) and (11), we then obtain (7). ■
III. STATEMENT OF THE PROBLEM

Let us start with a precise description of the physical operation that will be considered. A register $A$, having an $d$-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$ ($d > 1$), is initially prepared in one state from a set $\mathcal{A} = \{\rho_1, \rho_2\}$. The ancilla state $Y_j$ from a set $\mathcal{S} = \{T_1, Y_2\}$ contains some a priori (generally non-full) information about the input state of register $A$. By the ancilla we will mean a system $BE$ composed of extra register $B$, that is to receive the clone of $\rho_j$, and environment $E$. If we include an environment space then any physical operation may be expressed as a unitary evolution. Thus, the final state of two registers is described by

$$\rho_j = \text{Tr}_E \left( V(\rho_j \otimes Y_j) V^\dagger \right),$$

which is partial trace over environment space. In order to estimate a quality of cloning we shall compare $\rho_j$ with the perfect state $\rho_j \otimes \rho_j$ that would be produced by the ideal cloning. (Note that the cloning is special strong form of broadcasting [2]; the examination of approximate broadcasting is beyond the scope of the present work.)

We shall now justify the notion of the relative error for the above physical operation. Lemmas 1 and 2 insure that the sine of angle between two mixed states gives a reasonable measure of closeness for ones. We shall now use this measure to justify the notion of the relative error for discussed operations. For brevity, let us denote $\Delta_j = \Delta(\rho_j, \rho_j \otimes \rho_j)$, where $j = 1, 2$. According to Eq. (7), for any measurement

$$|p(a | \rho_j) - p(a | \rho_j \otimes \rho_j)| \leq \sin \Delta_j. \tag{13}$$

Thus, size $\sin \Delta_j$ describes upon the whole the deviation of the resulting probability distribution from the probability distribution to which it ought to tend. We define the absolute error as the sum $\sin \Delta_1 + \sin \Delta_2$. This definition extends the notion of the absolute error to the case of mixed states. However, this criterion loses sight of closeness of states $\rho_1$ and $\rho_2$. Let us take that we want distinguishing the input state of register $A$ by measurement made on the output. In order to solve the problem we compare given output $\rho_j$ to both ideal outputs $\rho_1 \otimes \rho_1$ and $\rho_2 \otimes \rho_2$. But if the ideal outputs are not sufficiently distinguishing then it is difficult. To express this in quantitative form we should use some measure of closeness for states $\rho_1 \otimes \rho_1$ and $\rho_2 \otimes \rho_2$. By Eq. (7),

$$|p(a | \rho_1 \otimes \rho_1) - p(a | \rho_2 \otimes \rho_2)| \leq \sin \Delta(\rho_1 \otimes \rho_1, \rho_2 \otimes \rho_2)$$

for any measurement. So, size $\sin \Delta(\rho_1 \otimes \rho_1, \rho_2 \otimes \rho_2)$ provides such a measure. The closeness of $\rho_1$ to $\rho_1 \otimes \rho_1$ is measured by $\sin \Delta_1$, the closeness of $\rho_2$ to $\rho_2 \otimes \rho_2$ is measured by $\sin \Delta_2$. By analogy with the case of pure states [13], the relative error is defined as follows.

**Definition** The relative error is

$$R(\mathcal{A} | \mathcal{S}) = \frac{\sin \Delta_1 + \sin \Delta_2}{\sin \Delta(\rho_1 \otimes \rho_1, \rho_2 \otimes \rho_2)}. \tag{14}$$

This definition generalizes the notion of the relative error in two significances. In the first place, it extends the mentioned notion to the case of mixed states. In the second place, it takes into account that the ancilla state can contain a priori information about the state to be cloned. We are interested in lower bound on the relative error defined above.

IV. MAIN RESULT

Let us now formulate the basic result of the present work. The desired lower bound is established by the following theorem.

**Theorem** Let $f = \sqrt{F(\rho_1, \rho_2)}$, $\phi = \sqrt{F(Y_1, Y_2)}$.

1. For $f \leq \phi \leq 1$ there holds

$$R(\mathcal{A} | \mathcal{S}) \geq f \phi - f^2 \sqrt{1 - f^2 \phi^2} / \sqrt{1 - f^2}; \tag{15}$$

2. For $0 \leq \phi \leq f$ there holds $R(\mathcal{A} | \mathcal{S}) \geq 0$.

**Proof of the theorem** (i) At first, using Eq. (4) twice, we have

$$\Delta(\rho_1 \otimes \rho_1, \rho_2 \otimes \rho_2) \leq \Delta_1 + \Delta_2 + \Delta(\tilde{\rho}_1, \tilde{\rho}_2)$$

$$\Delta_1 + \Delta_2 \geq \Delta(\rho_1 \otimes \rho_1, \rho_2 \otimes \rho_2) - \Delta(\tilde{\rho}_1, \tilde{\rho}_2). \tag{16}$$
By the multiplicativity and the unitary preservation,
\[ F(\rho_1, \rho_2) F(\Upsilon_1, \Upsilon_2) = F(\rho_1 \otimes \Upsilon_1, \rho_2 \otimes \Upsilon_2) = F\left( V(\rho_1 \otimes \Upsilon_1) V^\dagger, V(\rho_2 \otimes \Upsilon_2) V^\dagger \right). \]

Because the fidelity cannot decrease under the operation of partial trace [2],
\[ F(\rho_1, \rho_2) F(\Upsilon_1, \Upsilon_2) \leq F(\tilde{\rho}_1, \tilde{\rho}_2) \]
and
\[ \cos \Delta(\tilde{\rho}_1, \tilde{\rho}_2) \geq f. \tag{17} \]

By Eq. (17), we have
\[ -\sin \Delta(\tilde{\rho}_1, \tilde{\rho}_2) \geq -\sqrt{1 - f^2}. \tag{18} \]

According to the angle range of values,
\[ \sin \Delta_1 + \sin \Delta_2 \geq \sin(\Delta_1 + \Delta_2). \tag{19} \]

By Eqs. (19) and (16),
\[ R(\mathfrak{A} | \mathfrak{S}) \geq \cos \Delta(\tilde{\rho}_1, \tilde{\rho}_2) - \sin \Delta(\tilde{\rho}_1, \tilde{\rho}_2) \cot \Delta(\rho_1 \otimes \rho_1, \rho_2 \otimes \rho_2). \]

Using Eqs. (17) and (18), the last inequality can be rewritten as Eq. (15). Note that if \( \phi < f \) then Eq. (15) is also valid. However, it is empty, since the right-hand side of Eq. (15) becomes negative.

(ii) Suppose that \( \phi \) lies between 0 and \( f \). It suffices to show that there are states \( \Upsilon_1 \) and \( \Upsilon_2 \) such that
\[ \rho_j = \text{Tr}_E \Upsilon_j. \tag{20} \]

Then the equality \( R(\mathfrak{A} | \mathfrak{S}) = 0 \) is clearly valid. We may assume without loss of generality that states \( \Upsilon_1 \) and \( \Upsilon_2 \) are pure. If both purifications \( |Y_1\rangle \) of \( \rho_1 \) and \( |Y_2\rangle \) of \( \rho_2 \) lie in \( \mathcal{H} \otimes \mathcal{H} \) then [17]
\[ \langle Y_1 | Y_2 \rangle = |\text{Tr}(\sqrt{\rho_1} \sqrt{\rho_2} V)|, \tag{21} \]
where trace is taken over space \( \mathcal{H} \). An element \( V \) is placed in the unitary group \( \text{U}(d) \) and freely variable by choice of purifications [17]. We can use the freedom in \( V \) to make the equality \( |\langle Y_1 | Y_2 \rangle| = \phi \). To see this possibility we note that quantity \( |\langle Y_1 | Y_2 \rangle| \) ranges between 0 and \( f \). Indeed, the maximum is equal to the squared root of fidelity [17]. The minimal value is zero, because we can take orthogonal purifications. Recall that \( \text{U}(d) \) is the connected group. In fact, an arbitrary unitary matrix can be represented as \( W D W^{-1} \), where \( W \) is unitary and \( D = \text{diag}[\exp(i\theta_1), \ldots, \exp(i\theta_d)] \) (this is provided by the spectral theorem for normal matrices [23]). Replacing \( \theta_k \) by \( \theta_k \), we obtain a continuous path in \( \text{U}(d) \), that connects the given matrix \( (t = 1) \) with the identity matrix \( (t = 0) \). Since the right-hand side of Eq. (21) is continuous functional and \( \text{U}(d) \) is connected, each intermediate value is attained by some element of \( \text{U}(d) \). Thus, there are states \( \Upsilon_1 = |Y_1\rangle / |Y_1\rangle \) and \( \Upsilon_2 = |Y_2\rangle / |Y_2\rangle \), such that \( \sqrt{F(Y_1, Y_2)} = \phi \) and Eq. (20) is too valid.

At fixed \( f \), the right-hand side of Eq. (15) is increasing function of parameter \( \phi \). For \( \phi = f \) the lower bound is equal to zero and the equality \( R(\mathfrak{A} | \mathfrak{S}) = 0 \) can be reached. For example, it holds when \( \Upsilon_j = \rho_j \otimes \sigma \), i.e., the full information about the input state is a priori provided in the ancilla. Conversely, in the standard cloning there is no a priori information, i.e. \( \Upsilon_j = \Upsilon \) and \( \phi = 1 \). Then we get the bound
\[ R(\mathfrak{A}) \geq f - f^2 / \sqrt{1 + f^2}. \tag{22} \]

In general, the parameter \( \phi \) marks the top amount of information which can beforehand be contained in the ancilla. The larger \( \phi \) the less this top amount. If \( \phi = f \) then the full information of the clone can already be provided in the ancilla state. In the standard cloning, where \( \phi = 1 \), any knowledge about the input state \( \rho_j \) is a priori inaccessible. If the lower bound is seen as function of \( \phi \) then its minimum, reached at \( \phi = f \), is equal to 0 and its maximum, reached at \( \phi = 1 \), is equal to the right-hand side of Eq. (22). On the whole, these conclusions appear as plausible and contribute to the stronger no-cloning theorem.

Finally, it should be pointed out that our techniques can be applied to the \( N \rightarrow L \) operations. In this case the ancilla is composed of \( M = L - N \) extra registers and the environment. As a result, we obtain the lower bound
(i) For \( f^M \leq \phi \leq 1 \) there holds
\[ R(\mathfrak{A} | \mathfrak{S}) \geq f^N \phi - f^L \sqrt{1 - f^{2N} \phi^2} / \sqrt{1 - f^{2L}}; \tag{23} \]
(ii) For \( 0 \leq \phi \leq f^M \) there holds \( R(\mathfrak{A} | \mathfrak{S}) \geq 0 \).

At fixed \( f \), the right-hand side of Eq. (23) is increasing function of parameter \( \phi \). For \( \phi = f^M \) the lower bound is equal to zero. For example, the equality \( R(\mathfrak{A} | \mathfrak{S}) = 0 \) holds when \( \Upsilon_j = \rho_j \otimes \sigma \) and the full information is already provided in the ancilla state \( \Upsilon_j \). The right-hand side of Eq. (23) is maximal for the standard cloning in which \( \phi = 1 \).
V. CONCLUSION

We have established the lower bound on the relative error of mixed-state cloning and related physical operations, in which the ancilla state contains some a priori information about the input state. In the pure-state cloning, \( \rho_j = |s_j\rangle\langle s_j| \) for \( j = 1, 2 \) and parameter \( f = |\langle s_1|s_2\rangle| \). In this case the lower bound given by right-hand side of Eq. (22) is equivalent to the lower bound obtained in Ref. [13] for pure-state 1 \( \rightarrow \) 2 cloning. Similarly, at \( \phi = 1 \) Eq. (23) provides the extension of the lower bound deduced in Ref. [13] for pure-state \( N \rightarrow L \) cloning. For this extension the modulus of the inner product should be replaced by the square root of fidelity.

In Ref. [13] we have constructed the asymmetric cloner, minimizing the relative error. Does this cloner reach the bound in the case of mixed states? The answer is negative. The above transformation has two properties:

P1 It acts on \( \mathcal{H} \otimes \mathcal{H} \);

P2 The initial state of register \( B \) is pure.

It can be shown that two these properties do not allow to reach the lower bound given by Eq. (22) for each pair of kind \( \mathcal{A} = \{1/d, |s\rangle\langle s|\} \). We refrain from presenting the proof that is somewhat lengthy. Now we do not know whether a cloner, which reaches the established lower bound, exists. Recall that in the case of mixed states the optimal "global" cloner given in Ref. [6] does not reach the upper bound on the global fidelity [16]. Thus, there is a essential difference between pure-state and mixed-state cloning.

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