CONTINUOUS AND DISCRETE FRACTIONAL OPERATORS
AND SOME FRACTIONAL FUNCTIONS

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Abstract
The classical orthogonal polynomials are usually defined by the Rodrigues’ formula. This paper refers to a fractional extension of the classical Hermite, Laguerre, Jacobi, Meixner, Krawtchouk and Hahn polynomials. By means of the Caputo operator of fractional calculus, C-Hermite, C-Laguerre, C-Legendre and the C-Jacobi functions are defined and their representation in terms of the hypergeometric functions are provided. Also, by means of the Gray and Zhang fractional difference operator, fractional Charlier, Meixner, Krawtchouk and Hahn functions are defined and their representation in terms of the hypergeometric functions are provided. Some other properties of the new defined functions are given.

1. Introduction
Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of derivatives and integrals of arbitrary (real or complex) order. It is a complex and interesting topic having interconnections with various problems of function theory, integral and differential equations, and other branches of analysis. It has been continually developed, stimulated by ideas and results in various fields of mathematical analysis. This is demonstrated by the many publications—hundreds of papers in the past years—and by the many conferences devoted to the problems of fractional calculus.

A sequence of polynomials \( \{p_n(x)\} \), where \( p_n(x) \) is of exact degree \( n \) in \( x \), is said to be orthogonal with respect to a Lebesgue-Stieltjes measure \( d\alpha(x) \) if

\[
\int_{-\infty}^{\infty} p_m(x)p_n(x)d\alpha(x) = 0, \quad m \neq n.
\]

Implicit in this definition is the assumption that the moments

\[
\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \quad n = 0, 1, 2, \ldots,
\]

are finite. If the nondecreasing, real-valued, bounded function \( \alpha(x) \) is a step-function with jumps \( \rho_j \) at \( x = x_j, \ j = 0, 1, 2, \ldots, \) then \( 1 \) and \( 2 \) take the form of a sum:

\[
\sum_{j=0}^{\infty} p_m(x_j)p_n(x_j)\rho_j = 0, \quad m \neq n
\]

and

\[
\mu_n = \sum_{j=0}^{\infty} x_j^n \rho_j, \quad n = 0, 1, 2, \ldots.
\]

A polynomial set

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(5) \( y(x) = p_n(x) = k_n x^n + \ldots \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \ k_n \neq 0) \)

is a family of classical continuous orthogonal polynomials if it is the solution of a differential equation of the type

(6) \[ \sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \]

where \( \sigma(x) = ax^2 + bx + c \) is a polynomial of at most second order and \( \tau(x) = dx + e \) is a polynomial of first order. Here, the distribution \( da(x) \) takes the form \( d\alpha(x) = \rho(x) dx, \)

where \( \rho \) is the non negative solution on \((a, b)\) of the Pearson equation \( \frac{d}{dx}(\sigma(x)\rho(x)) = \tau(x)\rho(x). \)

Up to a linear change of variable, these polynomials can be classified as (the hypergeometric representation \( p F_q \) is defined in Section 2):

(a): The Jacobi polynomials \([7, P. 216]\)

\[ P_n^{(\alpha,\beta)}(x) = 2 F_1 \left( \begin{array}{c} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{array} \middle| \frac{1 - x}{2} \right). \]

(b): The Laguerre polynomials \([7, P. 241]\)

\[ L_n^{(\alpha)}(x) = (\alpha + 1)_n F_1 \left( \begin{array}{c} -n \\ \alpha + 1 \end{array} \middle| x \right). \]

(c): The Hermite polynomials \([7, P. 250]\)

\[ H_n(x) = (2x) n F_0 \left( \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} \\ \frac{1 - x}{2} \end{array} \right). \]

Some special cases of the Jacobi polynomials are:

(a-1) The Gegenbauer / Ultraspherical polynomials \([7, P. 222]\)

They are Jacobi polynomials for \( \alpha = \beta = \lambda - \frac{1}{2} \):

\[ C_n^{(\lambda)} = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{\left(\lambda - \frac{1}{2}, \lambda + \frac{1}{2}\right)}(x) = \frac{(2\lambda)_n}{n!} F_1 \left( \begin{array}{c} -n, n + 2\lambda \\ \lambda + \frac{1}{2} \end{array} \middle| \frac{1 - x}{2} \right), \quad \lambda \neq 0. \]

(a-2) The Chebyshev polynomials \([7, P. 225]\)

The Chebyshev polynomials of the first kind can be obtained from the Jacobi polynomials by taking \( \alpha = \beta = -\frac{1}{2} \):

\[ T_n(x) = \frac{P_n^{\left(\frac{1}{2}, \frac{1}{2}\right)}(x)}{P_n^{\left(\frac{1}{2}, \frac{1}{2}\right)}(1)} = 2 F_1 \left( \begin{array}{c} -n, n \\ \frac{1}{2} \end{array} \middle| \frac{1 - x}{2} \right), \]

and the Chebyshev polynomials of the second kind can be obtained from the Jacobi polynomials by taking \( \alpha = \beta = \frac{1}{2} \):

\[ U_n(x) = (n + 1) \frac{P_n^{\left(-\frac{1}{2}, \frac{1}{2}\right)}(x)}{P_n^{\left(-\frac{1}{2}, \frac{1}{2}\right)}(1)} = (n + 1) F_1 \left( \begin{array}{c} -n, n + 2 \\ \frac{1}{2} \end{array} \middle| \frac{1 - x}{2} \right). \]
The Legendre polynomials

They are Jacobi polynomials with \( \alpha = \beta = 0:\)

\[
P_n(x) = P_n^{(0,0)}(x) = 2 \text{F}_1\left(\begin{array}{c}
-n, n + 1 \\
1
\end{array} \left| \frac{1 - x}{2} \right. \right).
\]

Also, these polynomials can be represented by a Rodrigues type formula (see [7, page 64])

\[
p_n(x) = \frac{K_n}{\rho(x)} \frac{d^n}{dx^n} \left( \rho(x) \sigma^n(x) \right).
\]

It should be noted that this representation characterizes fully the family \( p_n \) and then is sometimes used as its definition.

Also, a polynomial set \( p_n \) is a family of discrete classical orthogonal polynomials (also known as the Hahn class) if it is the solution of a difference equation of the type

\[
\sigma(x) \Delta y(x) + \tau(x) \Delta y(x) + \lambda_n y(x) = 0,
\]

Here the polynomials \( \sigma(x) \) and \( \tau(x) \) are known to satisfy a Pearson type equation

\[
\Delta(\sigma(x) \rho(x)) = \tau(x) \rho(x),
\]

where the function \( \rho(x) \) is the discrete weight function associated to the family. These polynomials can be classified as (see e.g [8]):

(d): The Hahn polynomials [7, page 204]

\[
Q_n(x; \alpha, \beta, N) = 3 \text{F}_2\left(\begin{array}{c}
-n, n + \alpha + \beta + 1 \\
\alpha + 1, -N
\end{array} \left| -x \right. \right).
\]

(e): The Krawtchouk polynomials [7, page 237]

\[
K_n(x; p, N) = 2 \text{F}_1\left(\begin{array}{c}
-n, -x \\
-N
\end{array} \left| \frac{1}{p} \right. \right).
\]

(f): The Meixner polynomials [7, page 234]

\[
M_n(x; \beta, c) = 2 \text{F}_1\left(\begin{array}{c}
-n, -x \\
\beta
\end{array} \left| 1 - \frac{1}{c} \right. \right).
\]

(g): The Charlier polynomials [7, page 247]

\[
C_n(x; \alpha) = 2 \text{F}_0\left(\begin{array}{c}
-n, -x \\
-\frac{1}{\alpha}
\end{array} \right).
\]

Also, these polynomials can be represented by a Rodrigues type formula (see [7, page 71])

\[
p_n(x) = \frac{K_n}{\rho(x)} \Delta^n \left( \rho(x-n) \prod_{k=1}^{n} \sigma(x-k-1) \right),
\]

where \( K_n \) is given by

\[
K_n = \frac{1}{\prod_{k=1}^{n} (\alpha(2n-k-1) + d)}.
\]

It should be noted that this representation characterizes fully the family \( p_n \) and then is sometimes used as its definition.

In [6], the authors defined the C-Laguerre functions from the Rodrigues representation of the Laguerre polynomials by replacing the ordinary derivative by
a fractional type derivative, then they gave several properties of the new defined functions. Following their idea, we do the same for all the classical continuous and classical discrete orthogonal polynomials listed above. It should be noted that the Caputo fractional derivative applies only when the Rodrigues representation of the polynomial set uses classical derivative as done for the classical continuous polynomials. In the case of the discrete families, we cannot apply this fractional derivative. In [1], the authors defined the fractional difference by the rather natural approach of allowing the index of differencing, in the standard expression for the nth difference, to be any real or complex number, that is

\[ \Delta^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + \alpha - k), \]

where \( \alpha \) is any real or complex number. In [2], the author employed the following definition

\[ \nabla^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - k), \]

and showed that it can be used to study long-memory time series. In [3], Gray and Zhang gave a new definition of the fractional difference which also includes the notion of the fractional sum over specific index set. One of the more important features of this definition is that the sum corresponding to the one in (11) is finite. In this paper, we make use of the definition of the fractional difference given in [3].

The paper is organised as follows:

(1) In Section 2, we present the preliminary results and definitions that are useful for a better reading of this manuscript.

(2) In Section 3, we introduce the fractional Hermite, Laguerre, Jacobi, Charlier, Meixner, Krawtchouk and Hahn functions and provide several properties of these functions such as their hypergeometric representation, the differential equations they satisfy, · · ·

2. Preliminary definitions and results

2.1. The hypergeometric series and the Gamma function

In what follows, the symbol \((a)_m\) denotes the so-called Pochhammer symbol and is defined by

\[ (a)_m = \begin{cases} 1 & \text{if } m = 0 \\ a(a + 1) \cdots (a + m - 1) & \text{if } m = 1, 2, \ldots \end{cases} \]

and the hypergeometric series is defined as

\[ {}_pF_q\left( \begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} \right) (x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!}. \]

Note that the Pochhammer fulfils the following Legendre duplication formula [12, page 22]

\[ (a)_{2n} = 2^{2n} \left( \frac{a}{2} \right)_n \left( \frac{1 + a}{2} \right)_n. \]

Proposition 2.1. [7, page 10] The following equation applies:

\[ {}_2F_1\left( \begin{array}{c} a, b \\ c \end{array} \right) (z) = (1 - z)^{-a} {}_2F_1\left( \begin{array}{c} a, c - b \\ c \end{array} \right) \left( \frac{z}{z - 1} \right). \]
Proposition 2.2. [13] page 42] The Gauss hypergeometric function $\, _2F_1\left(\frac{a, b}{c} \mid x\right)$ satisfies the differential equation

$$x(1-x)\frac{d^2}{dx^2}y(x) + [c - (a + b + 1)x]\frac{d}{dx}y(x) + aby(x) = 0.$$  

Definition 2.3. The Gamma function is defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt, \quad \forall z \in \mathbb{R}.$$  

The Gamma function satisfies the following fundamental properties:

$$\Gamma(z + 1) = z\Gamma(z), \quad (15)$$  

$$\Gamma(z + k)\Gamma(z) = (z)_k, \quad (16)$$  

By (16) the binomial coefficients can be written in terms of the $\Gamma$-function as

$$\binom{z}{k} = \frac{z(z-1)\cdots(z-k+1)}{k!} = \frac{\Gamma(z+1)}{k!\Gamma(z-k+1)},$$  

for arbitrary $z \in \mathbb{C}$, $z + 1 \neq 0, -1, \ldots$, and $z - k + 1 \neq 0, -1, \ldots$.

Note, further, the following relation between the Pochhammer symbol and the binomial coefficients,

$$\binom{z}{k} = (-1)^k\binom{k-z-1}{k} = \frac{(-1)^k}{k!}(-z)_k.$$  

Another function which is closely connected to the Gamma function is the Beta function. The later function is defined by

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p, q \in \mathbb{C}, \quad \Re(p) > 0, \ \Re(q) > 0.$$  

The Beta function is connected to the Gamma function by the formula

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad \Re(p) > 0, \ \Re(q) > 0.$$  

2.2. Riemann-Liouville and Caputo differential operators of fractional calculus

Definition 2.4. The left-sided Riemann-Liouville fractional integral of order $\nu$ of the function $f(t)$ is defined as

$$aJ_\nu f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-\tau)^{\nu-1}f(\tau)d\tau, \quad x > a.$$  

Definition 2.5. (See [10] page 68) Let $\nu > 0$ and $x > a, \nu, a, x \in \mathbb{R}$. The Riemann-Liouville differential operator of fractional calculus of order $\nu$ is defined by

$$aD_\nu f(x) := \left\{ \begin{array}{ll} \frac{1}{\Gamma(n-\nu)} \int_a^x \frac{f(\tau)}{(x-\tau)^{n-\nu+1}}d\tau, & n-1 < \nu < n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x), & \nu = n \in \mathbb{N}. \end{array} \right.$$  

Remark 2.6. If we set $D = \frac{d}{dx}$, then it is easy to see that $aD_\nu = D^n aJ_{n-\nu}$, where $n-1 \leq \nu \leq n$. 

Definition 2.7. (See [10] page 79) Let $\nu > 0$ and $x > a$, $\nu$, $a$, $x \in \mathbb{R}$. The Caputo fractional derivative or Caputo differential operator of fractional calculus of order $\nu$ is defined by

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Proof. From the definition of $a \mathcal{J}^\nu$, and the change of the variable $t = a + s(x - a)$, we have

$$a \mathcal{J}^\nu(x - a)^\beta = \int_a^x (t - a)^\beta (x - t)^{\nu - 1} dt = \frac{(x - a)^{\nu + \beta}}{\Gamma(\nu)} \int_0^1 s^\beta (1 - s)^{\nu - 1} ds = \frac{\Gamma(\beta + 1)}{\Gamma(\nu + \beta + 1)} (x - a)^{\nu + \beta}.$$

The use of Remark 2.6 and equation (28) leads to the following proposition.

**Proposition 2.10.** [10] Let $\beta > \nu - 1$ and $t > a$. The following result holds.

$$a \mathcal{D}^\nu[x^\beta] = \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - \nu)} (x - a)^{\beta - \nu},$$

$$a \mathcal{D}^*_\nu[x^\beta] = \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - \nu)} (x - a)^{\beta - \nu}.$$

Proof. The proof follows from the definition of $a \mathcal{D}^\nu$ and Proposition 2.9.

**Corollary 2.** The following result is valid.

$$0 \mathcal{D}^\nu[x^{2n}] = \frac{\Gamma(2n + 1)}{\Gamma(2n - \nu + 1)} x^{2n - \nu}.$$

Proof. Take $a = 0$ and $\beta = 2n$ in Proposition 2.10.

**Theorem 2.11.** For $x > 0$, we have:

$$0 \mathcal{D}^\nu_* e^{-x^2} = \frac{x^{-\nu}}{\Gamma(1 - \nu)} F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{1 - \nu} \\ \frac{1}{2}, \frac{1}{2 - \nu} \end{array} \bigg| -x^2 \right).$$

Proof. From the definition of $0 \mathcal{D}^\nu_*$, it follows that:

$$0 \mathcal{D}^\nu_* e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n n! \frac{\Gamma(2n + 1)}{\Gamma(2n - \nu + 1)} x^{2n - \nu} \frac{(1)^n}{n!} (x^{-\nu}) = \frac{x^{-\nu}}{\Gamma(1 - \nu)} \sum_{n=0}^{\infty} \frac{(1)^n}{n!(1 - \nu)_{2n}} (-x^2)^n = \frac{x^{-\nu}}{\Gamma(1 - \nu)} \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n n! \left( \frac{1}{2} \right)_{2n} (-x^2)^n = \frac{x^{-\nu}}{\Gamma(1 - \nu)} F_2 \left( \begin{array}{c} \frac{1}{2}, 1 \\ \frac{1}{2}, \frac{1}{2 - \nu} \end{array} \bigg| -x^2 \right).$$

2.3. Gray and Zhang fractional difference and their properties

**Definition 2.12.** [3] Let $\alpha$ and $\beta$ two complex numbers. $(\alpha)_\beta$ is defined by:

$$(\alpha)_\beta = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} & \text{when } \alpha \text{ and } \alpha + \beta \text{ are neither zero nor negative integers} \\ 1, & \text{when } \alpha = \beta = 0, \\ 0, & \text{when } \alpha = 0, \beta \text{ is not zero nor a negative integer}, \\ \text{undefined, otherwise}. \end{cases}$$
Consider the $n$-fold summation of $f$ from $a$ to $t$, that is, let:

$$\sum_{a}^{t} f(t) = \sum_{k_1=a}^{t} \sum_{k_2=a}^{k_1} \cdots \sum_{k_n=a}^{k_{n-1}} f(k_n)$$

where $t$, $k_i$ and $a$ are finite integers such that $a \leq k_i \leq k_{i-1} \leq t$. Then by repeated interchanging of summation it is easily shown that

$$\sum_{a}^{t} f(t) = \frac{1}{\Gamma(n)} \sum_{k=a}^{t} (t-k+1)_{n-1} f(k).$$

Moreover, the summation in (33) is well defined for $n = \alpha$, $\alpha$ any complex number not zero or a negative integer. The definition can be extended to negative integers by noting that for $n$ a positive integer and $\alpha$ not zero or a negative integer

$$\nabla f(t) = f(t) - f(t-1)$$

and

$$\nabla^{n} f(t) = \frac{\nabla^{n}}{\Gamma(n+\alpha)} \sum_{k=a}^{t} (t-k+1)_{n+\alpha-1} f(k) = \frac{1}{\Gamma(\alpha)} \sum_{k=a}^{t} (t-k+1)_{\alpha-1} f(k).$$

Using this, we have the definition of the $\alpha$-fold summation of $f$ from $a$ to $t$.

**Definition 2.13.** [3] For $\alpha$ any complex number, and $f$ the function defined over the integer set \{a, a+1, \cdots, t\}, the $\alpha$-order summation over \{a, a+1, \cdots, t\} is defined by:

$$\sum_{a}^{t} f(t) = \frac{\nabla^{n}}{\Gamma(n+\alpha)} \sum_{k=a}^{t} (t-k+1)_{n+\alpha-1} f(k) = \frac{1}{\Gamma(\alpha)} \sum_{k=a}^{t} (t-k+1)_{\alpha-1} f(k)$$

where $n = \max\{0, n_0\}$, $n_0$ an integer such that $0 < Re(\alpha + n_0) \leq 1$.

**Definition 2.14.** [3] For $\alpha$ any complex number, the $\alpha$th-order difference of $f(t)$ over \{a, a+1, \cdots, t\} is defined by

$$\nabla^{\alpha} f(t) = \sum_{a}^{t} f(t) = \frac{\nabla^{n}}{\Gamma(n-\alpha)} \sum_{k=a}^{t} (t-k+1)_{n-\alpha-1} f(k)$$

where $n = \max\{0, n_0\}$, $n_0$ an integer such that $0 < Re(-\alpha + n_0) \leq 1$.

Taking care that

$$\sum_{k=a}^{t} (t-k+1)_{n+\alpha-1} f(k) = \sum_{k=0}^{t-a} (k+1)_{n+\alpha-1} f(t-k),$$

the $\alpha$th-order difference of $f(t)$ over \{a, a+1, \cdots, t\} can be defined as:

$$\nabla^{\alpha} f(t) = \frac{\nabla^{n}}{\Gamma(n-\alpha)} \sum_{k=0}^{t-a} (k+1)_{n-\alpha-1} f(t-k).$$

**Proposition 2.15.** [3] For any complex number $\alpha$ and any nonnegative integer $p$ such that $p - \alpha$ is not zero or a negative integer,

$$\nabla^{\alpha} f(t) = \frac{\nabla^{p}}{\Gamma(p-\alpha)} \sum_{k=a}^{t} (t-k+1)_{p-\alpha-1} f(k).$$

**Proposition 2.16.** [3] Let $\alpha$ and $\beta$ two complex numbers. Depending on $\alpha$ and $\beta$, the following properties apply
(1) If $\alpha$ and $\beta$ are zero or positive integers, then
\[ \frac{D^\alpha D^\beta}{t^\alpha} f(t) = \frac{D^{\alpha+\beta}}{t^\alpha} f(t). \]

(2) If $\alpha$ is any complex number and $\beta$ is not a positive integer, then
\[ \frac{D^\alpha D^\beta}{t^\alpha} f(t) = \frac{D^{\alpha+\beta}}{t^\alpha} f(t). \]

(3) If $\alpha$ is not zero or a positive integer but $\beta$ is a positive integer, then
\[ \frac{D^\alpha D^\beta}{t^\alpha} f(t) = \frac{D^{\alpha+\beta}}{t^\alpha} f(t) + \frac{1}{\Gamma(-\alpha)} \sum_{l=1}^{\alpha-1} \sum_{j=a-l} (-1)^l \binom{\beta}{l} (t-l-j+1)^{-\alpha-1} f(j). \]

**Proposition 2.17.**

(1) When $\alpha$ is not a negative integer, then
\[ \frac{D^\alpha D^{-\alpha}}{t^\alpha} f(t) = f(t). \]

(2) For a constant $c$, we have
\[ \frac{D^\alpha}{t^\alpha} (cf(t) + g(t)) = c \frac{D^\alpha}{t^\alpha} f(t) + \frac{D^\alpha}{t^\alpha} g(t). \]

(3) If $m$ is a nonnegative integer,
\[ \nabla^m (f(t)g(t)) = \sum_{n=0}^{m} \binom{n}{m} \left[ \nabla^{m-n} f(t-n) \right] \nabla^n g(t). \]

(4) If $\alpha$ is not a nonnegative integer,
\[ \frac{D^\alpha}{t^\alpha} (f(t)g(t)) = \sum_{n=0}^{\alpha-1} \binom{n}{m} \left[ \frac{D^\alpha}{t^\alpha} f(t-n) \right] \nabla^n g(t). \]

**Proposition 2.18.**

If $p+1$ is not zero or a negative integer, then

(1) when $p+1 - \alpha$ is not zero or a negative integer
\[ \frac{t}{a+1} \nabla^\alpha (t-a)^p = \frac{t(t-a)^{p-\alpha}}{(p+1)^{-\alpha}}, \]

(2) when $p+1 - \alpha$ is zero or a negative integer
\[ \frac{t}{a+1} \nabla^\alpha (t-a)^p = 0. \]

3. The fractional functions

3.1. C-Hermite functions

The classical Hermite polynomials are usually defined by the following Rodrigues’ formula [7, page 251]
\[ H_n(x) = (-1)^n e^{x^2} D^n \left( e^{-x^2} \right). \]

We generalize the Hermite polynomials by taking the Caputo fractional derivative instead of the integer-order derivative in (36). We then obtain the functions we call C-Hermite functions
\[ H_\nu(x) = (-1)^n e^{x^2} 0D^\nu_* \left( e^{-x^2} \right). \]
The classical Laguerre polynomials are defined by the following Rodrigues’ formula

\[ L_n(x) = \frac{1}{n!} e^x D_n^\nu (e^{-x} x^n) . \]

The C-Laguerre functions (see [6]) are defined by

\[ L_\nu(x) = \frac{1}{\Gamma(\nu + 1)} e^x \alpha^\nu \frac{d^\nu}{dx^\nu} (e^{-x} x^{\nu + \alpha}) , \]

where \( n \in \mathbb{N}, \Re(\alpha) > 0, \ x, \nu \in \mathbb{R}, \ n - 1 < \nu < n \). The following relations are valid.

**Theorem 3.2.** [6] Let \( n \in \mathbb{N}, \ \alpha \in \mathbb{C}, \ \Re(\alpha) > 0, \ x, \nu \in \mathbb{R}, \ n - 1 < \nu < n \). Then the C-Laguerre functions can be represented by means of the confluent hypergeometric functions as

\[ L_\nu(x) = \frac{\Gamma(\nu + \alpha + 1)}{\Gamma(\nu + 1) \Gamma(\alpha + 1)} \frac{\Gamma(\nu + \alpha + 1)}{\Gamma(\nu + 1)} \left( \frac{\nu}{\alpha + 1} \right) \right| x} . \]

**Proposition 3.3.** [6] Let the conditions of Theorem 3.2 be fulfilled. Then

\[ \lim_{\nu \to \infty} L_\nu(x) = L_n(x) \]

and

\[ \frac{d}{dx} L_\nu(x) = -L_{\nu-1}(x) . \]
The classical Jacobi polynomials are usually defined by the following Rodrigues’ formula [7, page 251]

\[ (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} D^n [(1-x)^{n+\alpha} (1+x)^{n+\beta}] \]  

We generalize the Jacobi polynomials by taking the Caputo fractional derivative \([11, Page 83]\), instead of the integer-order derivative formula \([35]\) and substituting \(n\) by \(\nu\) and \(n!\) by \(\Gamma(\nu + 1)\). We obtain the functions we call C-Jacobi functions:

\[ (1-x)^\alpha (1+x)^\beta P_\nu^{(\alpha,\beta)}(x) = \frac{(-1)^\nu}{2^\nu \Gamma(\nu+1)} \frac{1}{\Gamma(\nu+\beta+1)} \frac{1}{\Gamma(\nu+\alpha+1)} \] 

For \(\nu = 1\), we write

\[ \frac{1}{\Gamma(\nu+\beta+1)} \frac{1}{\Gamma(\nu+\alpha+1)} 2F_1 \left( -\nu, \nu + \alpha + \beta + 1 ; \frac{x + 1}{2} \right) \]

**Proof.** From \([25]\), we write

\[ -1D^\nu [(1-x)^{\nu+\alpha} (1+x)^{\nu+\beta}] = \sum_{k=0}^{\infty} \frac{\nu_k}{k!} \frac{\Gamma(\nu+\beta+1)}{\Gamma(\beta+1)} (x+1)^{\beta+k} (1-x)^{\nu+\alpha-k} \]

\[ = \sum_{k=0}^{n-1} \frac{(t+1)^k - \nu}{k!(k+1-\nu)} \frac{\Gamma(\nu+\beta+1)}{\Gamma(\beta+1)} (1-x)^{\nu+\alpha} (1+x)^{\nu+\beta} \]

From the relations

\[ (1-x)^{\nu+\alpha} (1+x)^{\nu+\beta} \]

it follows that

\[ -1D^\nu [(1-x)^{\nu+\alpha} (1+x)^{\nu+\beta}] = \frac{\Gamma(\nu+\beta+1)}{\Gamma(\beta+1)} (x+1)^{\beta+k} (1-x)^{\nu+\alpha-k} \]

**Theorem 3.6.** Let \(n \in \mathbb{N}\), \(\alpha, \beta \in \mathbb{R}\), \(\alpha > -1, \beta > -1, x, \nu \in \mathbb{R}\), \(n-1 < \nu < n\). The C-Jacobi functions have the following hypergeometric representations

**Remark 3.5.** It happens that the C-Laguerre functions satisfy the Kummer differential equation \([41, Page 83]\).

**Theorem 3.4.** The C-Laguerre functions satisfy the differential equation

\[ xy'' + (\alpha + 1 - x)y' + \nu y = 0. \]
They can be defined by the Rodrigues’ formula

\[ P_{\nu}^{(\alpha, \beta)}(x) = \frac{(-1)^n \Gamma(\nu + \beta + 1)}{\Gamma(\nu + 1) \Gamma(\beta + 1)} \left( \frac{1 - x}{2} \right)^\nu \sum_{k=0}^{n} \frac{(\nu - \alpha)_{k}}{k!} \frac{(\nu + \alpha + \beta + 1)_{k}}{\beta + 1} \frac{(1 + x)^{k}}{1 - x}. \]

Next, doing the change of the variable \( x = 2z - 1 \) we have \( z = \frac{x + 1}{2} \) and then

\[ P_{\nu}^{(\alpha, \beta)}(x) = \frac{(-1)^n \Gamma(\nu + \beta + 1)}{\Gamma(\nu + 1) \Gamma(\beta + 1)} (1 - z)^\nu {}_2F_1 \left( \begin{array}{c} -\nu, -\nu - \alpha \\ \beta + 1 \end{array} \middle| \frac{1 + x}{z - 1} \right). \]

Using the property \([\text{13}]\) of the hypergeometric function \(_2F_1\), it follows that

\[ P_{\nu}^{(\alpha, \beta)}(x) = \frac{(-1)^n \Gamma(\nu + \beta + 1)}{\Gamma(\nu + 1) \Gamma(\beta + 1)} {}_2F_1 \left( \begin{array}{c} -\nu, \nu + \alpha + \beta + 1 \\ \beta + 1 \end{array} \middle| \frac{z}{x + 1} \right). \]

The proposition is then proved.

**Remark 3.7.** It can be seen that when \( \nu \) tends to a nonnegative integer \( n \), the C-Jacobi function \( P_{\nu}^{(\alpha, \beta)}(x) \) becomes the classical Jacobi polynomial \( P_n^{(\alpha, \beta)}(x) \).

**Theorem 3.8.** Let \( n \in \mathbb{N} \), \( \alpha, \beta \in \mathbb{R} \), \( \alpha > -1 \), \( \beta > 0 \), \( x, \nu \in \mathbb{R} \), \( n - 1 < \nu < n \). The C-Jacobi functions are solutions of the second order differential equation

\[ (1 - x^2) \left( P_{\nu}^{(\alpha, \beta)} \right)''(x) + (\beta - \alpha - (\alpha + \beta + 2)x) \left( P_{\nu}^{(\alpha, \beta)} \right)'(x) + \nu(\nu + \alpha + \beta + 1)P_{\nu}^{(\alpha, \beta)}(x) = 0. \]

**Proof.** Since the functions \( P_{\nu}^{(\alpha, \beta)}(x) \) have the hypergeometric representation \([\text{15}]\), the differential equation follows from \([\text{14}]\).

**Theorem 3.9.** The Jacobi functions fulfill the following relation

\[ \frac{d^k}{dx^k} P_{\nu}^{(\alpha, \beta)}(x) = \frac{(\nu + \alpha + \beta)_k}{2^k} P_{\nu-k}^{(\alpha+\beta, \beta+\beta)}(x), \quad k \in \mathbb{N}. \]

**Proof.** The proof follows from relation \([\text{13}]\) page 41

\[ \frac{d^k}{dx^k} {}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| x \right) = \frac{(a)_k (b)_k}{(c)_k} {}_2F_1 \left( \begin{array}{c} a + k, b + k \\ c + k \end{array} \middle| x \right). \]

In the following subsections, we list some particular cases of the C-Jacobi functions.

### 3.3.1. The C-Gegenbauer functions.

The classical Gegenbauer polynomials are Jacobi polynomials for \( \alpha = \beta = \lambda - \frac{1}{2} \). They can be defined by the Rodrigues’ formula

\[ (1 - x^2)^{\lambda - \frac{1}{2}} C_\lambda(x) = \frac{(-1)^n (2\lambda)}{(\lambda + \frac{1}{2})_n 2^n n!} D^\lambda \left[ (1 - x^2)^{\lambda+n-\frac{1}{2}} \right]. \]

The C-Gegenbauer functions are defined by

\[ (1 - x^2)^{\lambda - \frac{1}{2}} C_\nu^{(\lambda)}(x) = \frac{(-1)^\nu}{2^n T(\nu + 1)} \varepsilon D^\nu \left[ (1 - x^2)^{\nu+\lambda-\frac{1}{2}} \right], \]

\( n \in \mathbb{N} \), \( \lambda, x, \nu \in \mathbb{R} \), \( \lambda \geq \frac{1}{2} \), \( n - 1 < \nu < n \).
3.3.3. \textbf{The C-Chebyshev functions.} The Chebyshev polynomials of the first kind \( T_n(x) \) and \( U_n(x) \) are Jacobi polynomials for \( \alpha = \beta = -\frac{1}{2} \) and \( \alpha = \beta = \frac{1}{2} \) respectively. The have the Rodrigues’ representations \cite[page 225]{7}.

\begin{align*}
(1-x^2)^{-\frac{1}{2}} T_n(x) &= \frac{(-1)^n}{(\frac{1}{2})_n 2^n} D^n \left[(1-x^2)^n \right], \\
(1-x^2)^{\frac{1}{2}} U_n(x) &= \frac{(n+1)(-1)^n}{(\frac{3}{2})_n 2^n} D^n \left[(1-x^2)^{n+\frac{1}{2}} \right],
\end{align*}

The C-Chebyshev functions are defined by

\begin{align*}
(1-x^2)^{-\frac{1}{2}} T_n(x) &= \frac{(-1)\nu}{2^\nu \Gamma(\nu+1)} i^\nu \left[ (1-x)^\nu \frac{1}{2} \right] (1+x)^{\nu-\frac{1}{2}}, \\
(1-x^2)^{\frac{1}{2}} U_n(x) &= \frac{(-1)\nu}{2^\nu \Gamma(\nu+1)} i^\nu \left[ (1-x)^\nu \frac{1}{2} \right] (1+x)^{\nu+\frac{1}{2}}.
\end{align*}

\textbf{Theorem 3.13.} Let \( n \in \mathbb{N}, \ x, \nu \in \mathbb{R}, \ n-1 < \nu < n \). The C-Chebyshev functions have the following hypergeometric representations

\begin{align*}
T_\nu(x) &= \frac{(-1)\nu \Gamma(\nu + \frac{1}{2})}{\Gamma(\nu+1) \Gamma(\frac{3}{2})} \, _2F_1 \left( -\nu, \nu + 1; \frac{1}{2}; \frac{x+1}{2} \right), \\
U_\nu(x) &= \frac{(-1)\nu \Gamma(\nu + \frac{1}{2})}{\Gamma(\nu+1) \Gamma(\frac{3}{2})} \, _2F_1 \left( -\nu, \nu + 2; \frac{3}{2}; \frac{x+1}{2} \right).
\end{align*}

\textbf{Proof.} The proof is similar to the one of Theorem 3.6.

3.3.3. \textbf{The Legendre C-functions.} The Legendre polynomials are Jacobi polynomials for \( \alpha = \beta = 0 \). The have can be defined by the Rodrigues’ formula

\begin{equation}
P_n(x) = \frac{(-1)^n}{2^n n!} D^n \left[(1-x^2)^n \right].
\end{equation}

The C-Legendre functions are defined by

\begin{equation}
P_\nu(x) = \frac{(-1)\nu}{2^\nu \Gamma(\nu+1)} i^\nu \left[ (1-x^2)^\nu \right].
\end{equation}

\textbf{Theorem 3.14.} Let \( n \in \mathbb{N}, \ x, \nu \in \mathbb{R}, \ n-1 < \nu < n \). The C-Legendre functions have the hypergeometric representation

\begin{equation}
P_\nu(x) = (-1)\nu \, _2F_1 \left( -\nu, \nu + 1; 1; \frac{x+1}{2} \right).
\end{equation}
Remark 3.15. It can be seen that for \( \nu \) tends to a nonnegative integer \( n \), the C-Legendre functions \( P_{\nu}(x) \) become the classical Legendre polynomials \( P_n(x) \). Note that the C-Legendre functions can also be written as

\[
P_{\nu}(x) = 2F_1 \left( \begin{array}{cc}
-\nu, \nu + 1 \\
1 
\end{array} \middle| \frac{1-x}{2} \right).
\]

It is not difficult to see that

\[
P_{\nu}(x) = 2F_1 \left( \begin{array}{cc}
\nu + 1, -\nu \\
1 
\end{array} \middle| \frac{1-x}{2} \right) = P_{-\nu-1}(x).
\]

Proposition 3.16. Let \( n \in \mathbb{N} \), \( x, \nu \in \mathbb{R} \), \( n - 1 < \nu < n \). The C-Legendre functions are solutions of the second-order differential equation

\[
(1-x^2)P''_{\nu}(x) - 2xP'_{\nu}(x) + \nu(\nu + 1)P_{\nu}(x) = 0.
\]

Remark 3.17. The C-Legendre functions happen to be the Legendre functions defined in \([11, \text{page 195}]\) and so our investigation leads to a Rodrigues representation of the Legendre functions using a derivative of a fractional order.

In the sections, using the Gray and Zhang factional difference \([3]\), we define the Fractional Charlier, Fractional Meixner, the Fractional Krawtchouk and the Fractional Hahn functions and provide several properties of these functions.

3.4. Fractional Charlier functions

Definition 3.18. Let \( a \in \mathbb{C}^*, \mu \in \mathbb{C} \) and \( x \in \{0, 1, \ldots\} \). We define the fractional Charlier function \( C_\mu(x; a) \) using the Rodrigues type formula as

\[
C_\mu(x; a) = \frac{x!}{a^x} \sum_{j=0}^{\infty} \frac{x^j}{j!} \frac{1}{(-\mu)_j}.
\]

Proposition 3.19. The fractional Charlier functions \( C_\mu(x; a) \) have the following hypergeometric representation

\[
C_\mu(x; a) = 2F_0 \left( \begin{array}{cc}
-\mu, -x \\
- \frac{1}{a} 
\end{array} \middle| \frac{1}{a} \right).
\]

Proof. Applying Proposition 2.15 where \( p \) is chosen to be 0, we have:

\[
C_\mu(x; a) = \frac{x!}{a^x} \sum_{j=0}^{\infty} \frac{x^j}{j!} (-\mu)_j \frac{a^j}{(x-j)!}.
\]

If we put \( j = x - k \) then the previous relation becomes:

\[
C_\mu(x; a) = \frac{x!}{a^x} \sum_{j=0}^{\infty} \frac{(-\mu)_j}{\Gamma(-\mu)} \frac{x^j}{(x-j)!}.
\]

\[
= \frac{1}{\Gamma(-\mu)} \sum_{j=0}^{x} \frac{(-\mu)_j}{\Gamma(j+1)} \frac{x^j}{(x-j)!}.
\]

\[
= \sum_{j=0}^{x} \frac{(-\mu)_j}{\Gamma(j+1)} \frac{(-x)_j}{j!} \left( -\frac{1}{a} \right)^j.
\]
Since $x \in \mathbb{N}$ then $(-x)_j = 0$ for $j > x$ and
\[
C_\mu(x; a) = \sum_{j=0}^{\infty} \frac{(-\mu)_j (-x)_j}{j!} \left( -\frac{1}{a} \right)^j = 2F_0 \left( -\mu, -x \left| -\frac{1}{a} \right. \right).
\]

### 3.5. Fractional Meixner functions

**Definition 3.20.** Let $c \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, $\mu \in \mathbb{C}$ and $x \in \{0, 1, \ldots \}$. We define the fractional Meixner function $M_\mu(x; \beta, c)$ using the Rodrigues Type formula as
\[
M_\mu(x; \beta, c) = x! \frac{\Gamma(\beta + \mu + x)}{\Gamma(\beta + x) \Gamma(\beta + \mu)} \sum_{j=0}^{x} \frac{(-\mu)_j (-x)_j}{j! (-\beta - x + 1)_j} \left( \frac{1}{c} \right)^j.
\]

**Proposition 3.21.** The fractional Meixner function $M_\mu(x; \beta, c)$ satisfies the following sum:
\[
M_\mu(x; \beta, c) = \frac{\Gamma(\beta) \Gamma(\beta + \mu + x)}{\Gamma(\beta + x) \Gamma(\beta + \mu)} \sum_{j=0}^{x} \frac{(-\mu)_j (-x)_j}{j! (-\beta - x + 1)_j} \left( \frac{1}{c} \right)^j.
\]

**Proof.** From the definitions of the fractional difference and the fractional Meixner function, we have
\[
M_\mu(x; \beta, c) = \frac{x! \Gamma(\beta)}{\Gamma(\beta + x) c^x} \sum_{j=0}^{x} \frac{(\beta + \mu)_x c^j}{x!} \sum_{k=0}^{j} \frac{(-\mu)_j (-x)_j}{k!} \left( \frac{1}{c} \right)^j.
\]
If we put $j = x - k$ then the previous relation becomes:
\[
M_\mu(x; \beta, c) = \frac{x! \Gamma(\beta)}{\Gamma(\beta + x) c^x} \sum_{j=0}^{x} \frac{(-\mu)_j (-x)_j}{(x-j)!} \left( \frac{1}{c} \right)^j.
\]

Since $x \in \mathbb{N}$ then $(-x)_j = 0$ for $j > x$ and
\[
M_\mu(x; \beta, c) = \frac{\Gamma(\beta) \Gamma(\beta + \mu + x)}{\Gamma(\beta + x) \Gamma(\beta + \mu)} \sum_{j=0}^{\infty} \frac{(-\mu)_j (-x)_j}{j! (-\beta - x + 1)_j} \left( \frac{1}{c} \right)^j.
\]
3.6. Fractional Krawtchouk functions

**Definition 3.22.** Let \( p \in \mathbb{C}^* \), \( N \in \mathbb{N}, \mu \in \mathbb{C} \) and \( x \in \{0, 1, \cdots\} \). We define the fractional Krawtchouk function \( K_{\mu}(x; p, N) \) using the Rodrigues type formula as

\[
K_{\mu}(x; p, N) = \frac{1}{\binom{N}{x}} \sum_{j=0}^{\infty} \left( \frac{p}{1-p} \right)^j \left( \frac{N - \mu}{x} \right) \left( \frac{\mu}{N-x} \right)^x .
\]

**Proposition 3.23.** The fractional Krawtchouk function \( K_{\mu}(x; p, N) \) satisfies the following sum:

\[
K_{\mu}(x; p, N) = \Gamma(N - x + 1) \Gamma(N - \mu + 1) \sum_{j=0}^{x} \frac{(-\mu)_j (-x)_j}{j! (N - \mu - x + 1)_j} \left( \frac{1}{1-p} \right)^j .
\]

**Proof.** From the definitions of the fractional difference and the fractional Krawtchouk function, we have

\[
K_{\mu}(x; p, N) = \frac{1}{\binom{N}{x}} \sum_{j=0}^{\infty} \left( \frac{p}{1-p} \right)^j \left( \frac{N - \mu}{x} \right) \left( \frac{\mu}{N-x} \right)^x .
\]

If we put \( j = x - k \) then the previous relation becomes:

\[
K_{\mu}(x; p, N) = \frac{\Gamma(N - x + 1) \Gamma(N - \mu + 1)}{\Gamma(N + 1)} \sum_{j=0}^{x} \frac{(-\mu)_j (-x)_j}{j! (N - \mu - x + 1)_j} \left( \frac{1}{1-p} \right)^{-j} .
\]

Since \( x \in \mathbb{N} \) then \( (-x)_j = 0 \) for \( j > x \) and

\[
K_{\mu}(x; p, N) = \frac{\Gamma(N - x + 1) \Gamma(N - \mu + 1)}{\Gamma(N + 1)} \sum_{j=0}^{\infty} \frac{(-\mu)_j (-x)_j}{j! (N - \mu - x + 1)_j} \left( \frac{1}{1-p} \right)^j .
\]

3.7. Fractional Hahn functions

**Definition 3.24.** Let \( \alpha, \beta \in \mathbb{C}, N \in \mathbb{N}, \mu \in \mathbb{C} \) and \( x \in \{0, 1, \cdots\} \). We define the fractional Hahn function \( Q_{\mu}(x; \alpha, \beta, N) \) using the Rodrigues type formula as

\[
Q_{\mu}(x; \alpha, \beta, N) = \frac{(-1)^x (\beta + 1)_x}{(-\mu)_x (\alpha + x) (\beta + N - x)_x} \sum_{j=0}^{x} \left( \frac{\mu + x}{N - \mu - x} \right)^x \left( \frac{\beta + N - x}{N - \mu - x} \right)^x .
\]
PROPOSITION 3.25. The fractional Hahn function $Q_\mu(x; \alpha, \beta, N)$ satisfies the following sum:

$$Q_\mu(x; \alpha, \beta, N) = \frac{(-1)^\mu \Gamma(\beta + \mu + 1) \Gamma(-N) \Gamma(\alpha + 1) \Gamma(\alpha + x + 1)}{\Gamma(-N + \mu) \Gamma(\alpha + x + 1) \Gamma(\alpha + \mu + 1) \Gamma(-\mu + N - x) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)}$$

$$\times {}_3F_2 \left( \begin{array}{c} -\mu, -x, \beta + N - x + 1 \\ N - \mu - x, -\alpha - \mu - x \end{array} \right).$$

Proof.

$$Q_\mu(x; \alpha, \beta, N) = \frac{(-1)^\mu(\beta + \mu + 1) \Gamma(-N) \Gamma(\alpha + 1) \Gamma(\alpha + x + 1)}{\Gamma(-N + \mu) \Gamma(\alpha + x + 1) \Gamma(\alpha + \mu + 1) \Gamma(-\mu + N - x) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)}$$

$$\times \sum_{k=0}^x (x - k + 1)^{-\mu - 1} \left[ (\alpha + \mu + k) \binom{\beta + N - k}{k} \right]$$

$$= \frac{(-1)^\mu \Gamma(\beta + \mu + 1) \Gamma(-N) \Gamma(\alpha + 1) \Gamma(\alpha + x + 1)}{\Gamma(-N + \mu) \Gamma(\alpha + x + 1) \Gamma(\alpha + \mu + 1) \Gamma(-\mu + N - x) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)}$$

$$\times \sum_{k=0}^x \Gamma(x - k - \mu) \Gamma(\alpha + \mu + k + 1) \Gamma(\beta + N - k + 1) \Gamma(\beta + \mu + 1).$$

If we put $j = x - k$ then the previous relation becomes:

$$Q_\mu(x; \alpha, \beta, N) = \frac{(-1)^\mu \Gamma(\beta + \mu + 1) \Gamma(-N) \Gamma(\alpha + 1) \Gamma(\alpha + x + 1)}{\Gamma(-N + \mu) \Gamma(\alpha + x + 1) \Gamma(\alpha + \mu + 1) \Gamma(-\mu + N - x) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)}$$

$$\times \sum_{j=x}^{\infty} \Gamma(j - \mu) \Gamma(\alpha + \mu + x - j + 1) \Gamma(\beta + N - x + j + 1) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)$$

$$= \frac{(-1)^\mu \Gamma(\beta + \mu + 1) \Gamma(-N) \Gamma(\alpha + 1) \Gamma(\alpha + x + 1)}{\Gamma(-N + \mu) \Gamma(\alpha + x + 1) \Gamma(\alpha + \mu + 1) \Gamma(-\mu + N - x) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)}$$

$$\times \sum_{j=x}^{\infty} \Gamma(j - \mu) \Gamma(\alpha + \mu + x - j + 1) \Gamma(\beta + N - x + j + 1) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)$$

Since $x \in \mathbb{N}$ then $(-x)_j = 0$ for $j > x$ and

$$Q_\mu(x; \alpha, \beta, N) = \frac{(-1)^\mu \Gamma(\beta + \mu + 1) \Gamma(-N) \Gamma(\alpha + 1) \Gamma(\alpha + x + 1)}{\Gamma(-N + \mu) \Gamma(\alpha + x + 1) \Gamma(\alpha + \mu + 1) \Gamma(-\mu + N - x) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)}$$

$$\times \sum_{j=0}^{\infty} (-\mu)_j (\alpha - x)_j (\beta + x - 1)_j (-\alpha - x)_j$$

$$= \frac{(-1)^\mu \Gamma(\beta + \mu + 1) \Gamma(-N) \Gamma(\alpha + 1) \Gamma(\alpha + x + 1)}{\Gamma(-N + \mu) \Gamma(\alpha + x + 1) \Gamma(\alpha + \mu + 1) \Gamma(-\mu + N - x) \Gamma(\beta - x + 1) \Gamma(\beta - \mu + 1)}$$

$$\times {}_3F_2 \left( \begin{array}{c} -\mu, -x, \beta + N - x + 1 \\ N - \mu - x, -\alpha - \mu - x \end{array} \right).$$
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