A STABILIZER-FREE $C^0$ WEAK GALERKIN METHOD FOR THE BIHARMONIC EQUATIONS

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Abstract. In this article, we present and analyze a stabilizer-free $C^0$ weak Galerkin (SF-C0WG) method for solving the biharmonic problem. The SF-C0WG method is formulated in terms of cell unknowns which are $C^0$ continuous piecewise polynomials of degree $k+2$ with $k \geq 0$ and in terms of face unknowns which are discontinuous piecewise polynomials of degree $k+1$. The formulation of this SF-C0WG method is without the stabilized or penalty term and is as simple as the $C^1$ conforming finite element scheme of the biharmonic problem. Optimal order error estimates in a discrete $H^2$-like norm and the $H^1$ norm for $k \geq 0$ are established for the corresponding WG finite element solutions. Error estimates in the $L^2$ norm are also derived with an optimal order of convergence for $k > 0$ and sub-optimal order of convergence for $k = 0$. Numerical experiments are shown to confirm the theoretical results.

Key words. weak Galerkin, finite element method, weak Laplacian, biharmonic equations

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

1. Introduction. We consider the biharmonic equation of the form

\begin{align*}
\Delta^2 u &= f, \quad \text{in } \Omega, \tag{1.1a} \\
u &= g_D, \quad \text{on } \Gamma, \tag{1.1b} \\
\frac{\partial u}{\partial n} &= g_N, \quad \text{on } \Gamma, \tag{1.1c}
\end{align*}

where $\Omega$ is a bounded polytopal domain in $\mathbb{R}^2$ and $\Gamma = \partial \Omega$.

In the case of homogeneous boundary conditions $g_D = g_N = 0$, the variational form of problem (1.1a)-(1.1c) reads as: find $u \in H^2_0(\Omega)$ such that

\begin{align*}
(\Delta u, \Delta v) &= (f, v), \quad \forall v \in H^2_0(\Omega), \tag{1.2}
\end{align*}

where $H^2_0(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial \Omega$.

For the case of nonhomogeneous boundary conditions, assume that $g_D$ and $g_N$ are the Dirichlet boundary data of some function in $H^2(\Omega)$, that is, there exists $\psi \in H^2(\Omega)$ such that

\begin{align*}
\Delta^2 \psi &= 0, \quad \text{in } \Omega, \\
\psi &= g_D, \quad \text{on } \Gamma, \\
\frac{\partial \psi}{\partial n} &= g_N, \quad \text{on } \Gamma.
\end{align*}

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Then by setting $\tilde{u} = u - \psi$, we arrive at the weak form (1.2) for $\tilde{u}$. Therefore for brevity, but without loss of generality, we will assume homogeneous boundary conditions in the remainder of this paper.

It is well known that $H^2$-conforming finite element methods for problem (1.1a)-(1.1c) involve $C^1$ finite elements, which are of complex implementation and contain high order polynomials even in two dimensions. For example, Argyris and Bell finite elements have 21 and 18 degrees of freedom per triangle, respectively.

In order to avoid the use of such $C^1$ elements, nonconforming finite elements have been used to solve biharmonic problems. Morley element [13] is one of the most popular nonconforming finite elements for the biharmonic equations, which only uses quadratic piecewise polynomials on triangle elements in two dimensional domains and doesn’t need any stabilization along mesh interfaces. However, it can not be generalized to arbitrarily high order polynomials.

Discontinuous Galerkin (DG) approaches can also be applied to the biharmonic problems. The first discontinuous Galerkin method—the interior penalty method for the fourth order PDE was presented in [2], which uses fully discontinuous piecewise polynomials as basis functions. A nonsymmetric version of interior penalty method was proposed and analyzed in [11]. Although the DG methods have the advantage of using arbitrarily high-order elements, they also have some disadvantages. The weak forms are more complicated than those used for conforming and nonconforming finite element methods. The discrete linear system of the DG method is large because it has a large number of degrees of freedom. To reduce the degrees of freedom of DG methods, $C^0$ interior penalty ($C0IP$) methods have been proposed for the fourth order PDEs first in [6] and then analyzed in [4], where the simple Lagrange elements are used and the continuity of the function derivatives are weakly enforced by stabilization terms on interior edges. However, the $C0IP$ methods still have the disadvantage of complex weak form and the need for the penalty parameters.

Another approach to avoid the use of $C^1$ elements is the mixed methods [1, 7, 12], which reduces the biharmonic problem to a system of two second order elliptic problems. One of the main drawbacks of the mixed formulation is that the mixed method leads to saddle-point linear system, which causes difficulty in efficiently solving the linear algebra system.

The weak Galerkin (WG) finite element method was first introduced for the second order elliptic problems in [23]. One of its main characteristics is the use of the concept of weak functions and its weak derivatives. The classical differential operators, such as the gradient and the Laplacian, are approximated by weak differential operator defined as distributions, which are further approximated by piecewise polynomials. These weakly defined functions and differential operators make the WG methods highly flexible in choosing finite element spaces and using polytopal meshes. In recent years, the WG method has been a focus of great interest in the scientific community. Several WG methods have been developed to solve a wide variety of partial differential equations, e.g., [8–10, 17–19, 22]. Especially, there are some works [14–16, 20, 21, 25, 28] for biharmonic equations. Compared with the DG methods, there is no penalty parameters needed to tune in the formulation of WG methods. Similar to the DG methods, the WG methods also involve stabilization along mesh skeleton, which makes the implementation of DG and WG methods more complex than the ones of conforming and nonconforming finite element methods.
Most recently, a new WG method without stabilizer term was presented for the second order elliptic problems in [26], where we can remove the stabilization and pay the price in the form of using high enough degree of polynomials in the definition of the weak gradient. The resulting numerical scheme is as simple as conforming finite element scheme and it is easy to implement. The idea has been extended to the biharmonic equations in [27], where a stabilizer-free WG (SFWG) method has been proposed which uses full discontinuous piecewise polynomials of degrees \( k + 2 \), \( k + 1 \) with \( k \geq 0 \), respectively, for discretization of the unknown solution \( u \), the trace of \( u \) and the trace of normal derivative \( \frac{\partial u}{\partial n} \) on the skeleton of the mesh. For triangular mesh, the minimum degree of polynomials used for the computation of the weak Laplacian is \( k + 7 \) in theory and is \( k + 4 \) in practical computation. As it is pointed out in [27], it is a challenging task to compute weak Laplacian and its numerical integration when the degree of polynomials used in the computation of weak Laplacian is very high.

In this paper, we will present and analyze a stabilizer-free \( C^0 \) weak Galerkin method to approximate the solutions of the biharmonic problem (1.1a)-(1.1c). The method is formulated in terms of face unknowns which are discontinuous piecewise polynomials of degree \( k + 1 \) with \( k \geq 0 \) and in terms of cell unknowns which are \( C^0 \) continuous piecewise polynomials of degree \( k + 2 \). We have proved that, for triangular mesh, it is enough to take \( k + 3 \) as the degree of polynomials used in the computation of weak Laplacian. In comparison with the SFWG method [27], the SF-C0WG methods in this paper involve fewer degrees of freedom because nodal values are shared on inter-element boundaries.

The outline of this paper is as follows. In Section 2, we introduce some notations and the formulation of our SF-C0WG method and the related methods. Two energy-like norms and their equivalence and the well-posedness of the SF-C0WG method are discussed in Section 3. Then, in Section 4, we derive an error equation which plays an important role in our error estimates. The error analysis of our SF-C0WG method for \( H^2 \)-like norm and the \( L^2 \) and \( H^1 \) norm are established in Section 5 and 6, respectively. Finally, in Section 7, we report some numerical experiment results to confirm the theoretical analysis developed.

2. Weak Galerkin Finite Element Methods. Let \( T_h \) be a quasi-uniform triangulation of the domain \( \Omega \). Denote by \( E_h \) the set of all edges in \( T_h \), and let \( E_h^0 = E_h \setminus \Gamma \) be the set of all interior edges.

For convenience, we adopt the following notations,

\[
(u, w)_{T_h} = \sum_{K \in T_h} (u, w)_K = \sum_{K \in T_h} \int_K u v \, dx,
\]

\[
(v, w)_{\partial T_h} = \sum_{K \in T_h} (v, w)_{\partial K} = \sum_{K \in T_h} \int_{\partial K} u v \, ds.
\]

For any nonnegative integer \( m \), let \( P_m(D) \) denote the set of polynomials defined on \( D \) with degree no more than \( m \), where \( D \) may be an element \( K \) of \( T_h \) or an edge \( e \) of \( E_h \). In what follows, we often consider the broken polynomial spaces

\[
P_m(T_h) := \{ v \in L^2(\Omega) : v|_K \in P_m(K), \forall K \in T_h \},
\]
and
\[ \mathbb{P}_m(\mathcal{E}_h) := \{ v \in L^2(\mathcal{E}_h) : v|_e \in \mathbb{P}_m(e), \ \forall e \in \mathcal{E}_h \}. \]

First of all, we introduce a set of normal directions on \( \mathcal{E}_h \) as follows
\[ D_h = \{ n_e : n_e \text{ is unit and normal to } e, \ e \in \mathcal{E}_h \}. \]
Then, a weak Galerkin finite element space \( V_h \) for \( k \geq 0 \) is defined by
\[ V_h = \{ v = \{ v_0, v_n \} : v_0 \in S_h, v_n \in \mathbb{P}_{k+1}(\mathcal{E}_h) \}, \]
with
\[ S_h = \{ w \in H^1_0(\Omega) : w|_K \in \mathbb{P}_{k+2}(K), \ \forall K \in \mathcal{T}_h \}, \]
where \( v_n \) can be viewed as an approximation of \( \frac{\partial v_0}{\partial n} := \nabla v_0 \cdot n_e \).

Denote by \( V^0_h \) a subspace of \( V_h \) with vanishing traces,
\[ V^0_h = \{ v = \{ v_0, v_n \} \in V_h, \ v_n|_e = 0, \ e \subset \partial K \cap \Gamma \}. \]

**Definition 2.1 (Weak Laplacian).** For any function \( v = \{ v_0, v_n \} \in V_h \), its weak Laplacian \( \Delta_{w,m} v \), is piecewisely defined as the unique polynomial \( (\Delta_{w,m} v)|_K \in \mathbb{P}_m(K) \) such that
\[ (\Delta_{w,m} v, \varphi)_K = - (\nabla v_0, \nabla \varphi)_K + \langle v_n n_e \cdot n, \varphi \rangle_{\partial K}, \ \forall \varphi \in \mathbb{P}_m(K), \]
for any \( K \in \mathcal{T}_h \).

Now, we are ready to present our stabilizer-free \( C^0 \) weak Galerkin finite element method for the biharmonic problem (1.1a)-(1.1c).

**Method 1 (SF-C0WG Method).** The stabilizer-free \( C^0 \) weak Galerkin finite element scheme for solving problem (1.1a)-(1.1c) is defined as follows: find \( u_h = \{ u_0, u_n n_e \} \in V^0_h \) such that
\[ A_h(u_h, v_h) = (f, v_0), \ \forall v_h = \{ v_0, v_n n_e \} \in V^0_h, \]
where the bilinear form \( a_h(\cdot, \cdot) \) is defined by
\[ A_h(v, w) := (\Delta_{w,k+3} v, \Delta_{w,k+3} w)_{\mathcal{T}_h}, \ \forall v, w \in V_h. \]

**Remark 2.1.** Using the same WG finite element space \( V^0_h \) defined by (2.4), a \( C^0 \) weak Galerkin finite element method has been presented in [15], which is stated as follows:

**Method 2 (C0WG Method).** The \( C^0 \) weak Galerkin finite element scheme for solving problem (1.1a)-(1.1c) is defined as follows: find \( u_h = \{ u_0, u_n n_e \} \in V^0_h \) such that
\[ A_{wg}(u_h, v_h) = (f, v_0), \ \forall v_h = \{ v_0, v_n n_e \} \in V^0_h, \]
where the bilinear form \( a_h(\cdot, \cdot) \) is defined by

\[
A_{\text{wg}}(v, w) := (\Delta w, k v, \Delta w, k w)_{\mathcal{T}_h} + s_h(v, w), \quad \forall v, w \in V_h,
\]

with the stabilizer term

\[
s_h(v, w) = \sum_{K \in \mathcal{T}_h} h_K^{-1} (\frac{\partial v_0}{\partial n_e} - v_n, \frac{\partial w_0}{\partial n_e} - w_n)_{\partial K}, \quad \forall v, w \in V_h.
\]

From the formulation of the SF-C0WG method (2.6) and the C0WG method (2.7), we can see that: the SF-C0WG method is obtained by removing the stabilizer \( s_h(\cdot, \cdot) \) in the C0WG method via raising the degree of polynomials used in the definition of the weak Laplacian from \( k \) to \( k + 3 \). A comparison of numerical performance of both WG methods is discussed in Section 7.

**Remark 2.2.** Using the \( C^0 \) conforming finite element space \( S_h \) defined by (2.3), a \( C^0 \) interior penalty method has been presented in [4, 6], which is stated as follows:

**Method 3 (C0IP Method).** The \( C^0 \) interior penalty method for solving problem (1.1a)-(1.1c) is defined as follows: find \( u_h \in S_h \) such that

\[
A_{\text{dg}}(u_h, v_h) = (f, v_h), \quad \forall v_h \in S_h,
\]

where the bilinear form \( A_{\text{dg}}(\cdot, \cdot) \) is defined as follows: for any \( v, w \in S_h,

\[
A_{\text{dg}}(v, w) := (D^2 v, D^2 w)_{\mathcal{T}_h} - \langle [\nabla v], \{\frac{\partial^2 w}{\partial n_e^2}\}\rangle_{\mathcal{E}_h} - \langle [\nabla w], \{\frac{\partial^2 v}{\partial n_e^2}\}\rangle_{\mathcal{E}_h} + j_h(v, w),
\]

with the stabilizer term

\[
j_h(v, w) = \sum_{e \in \mathcal{E}_h} \eta h_e^{-1} \langle [\nabla v], [\nabla w]\rangle_e, \quad \forall v, w \in S_h.
\]

Here the penalty parameter \( \eta \) is a positive constant. For any \( v \in H^2(\mathcal{T}_h) \), the jump \([\nabla v]\) and the average \( \{\frac{\partial^2 v}{\partial n_e^2}\}\) are defined as follows.

Let \( e \in \mathcal{E}_h^b \) be the common edge of \( K_1 \) and \( K_2 \) of \( \mathcal{T}_h \) and \( n_i, i = 1, 2 \) denote by the outward unit normal vector of the boundary \( \partial K_i, i = 1, 2 \). We define on the edge \( e \)

\[
\{\frac{\partial^2 v}{\partial n_e^2}\} = \frac{1}{2} \left( \frac{\partial^2 v_1}{\partial n_1^2} + \frac{\partial^2 v_2}{\partial n_2^2} \right) \quad \text{and} \quad [\nabla v] = \nabla v_1 \cdot n_1 + \nabla v_2 \cdot n_2,
\]

where \( v_i = v|_{K_i}, i = 1, 2 \). On a boundary edge \( e \subset \partial \Omega \), we simply take \( \{\frac{\partial^2 v}{\partial n_e^2}\} = \frac{\partial^2 v}{\partial n_e^2} \) and \([\nabla v] = \nabla v \cdot n \).

Compared with the C0IP method (2.8), our SF-C0WG method (2.6) has a simple formulation without any integration term on the edges of \( \mathcal{E}_h \), which will simplify the implementation of the corresponding numerical scheme and reduce the assembling time of stiffness matrix. Although the SF-C0WG method (2.6) has more degrees of freedom than the C0IP method (2.8), numerical experiments in Section 7 indicate that its total computational time is less than that of the C0IP method (2.8).
3. Well Posedness. For simplicity of notation, from now on we shall drop the subscript \( k + 3 \) in the notation \( \Delta_{w,k+3} \) for the discrete weak Laplacian.

In order to analyze the SF-C0WG method (2.6), we introduce two \( H^2 \)-like norms \( \| \cdot \| \) and \( \| \cdot \|_{2,h} \) over \( V^0_h \) by

\[
\| v \| = \left[ \sum_{K \in T_h} \| \Delta_w v \|^2_{L^2(K)} \right]^{1/2},
\]

and

\[
\| v \|_{2,h} = \left[ \sum_{K \in T_h} \left( \| \Delta v_0 \|^2_{L^2(K)} + h_K^{-1} \| \frac{\partial v_0}{\partial n_e} - v_n \|^2_{L^2(\partial K)} \right) \right]^{1/2},
\]

for all \( v \in V^0_h \). Obviously, \( \| \cdot \|_{2,h} \) is indeed a norm on \( V^0_h \). We will show \( \| \cdot \| \) is also a norm by proving that the norms \( \| \cdot \|_{2,h} \) and \( \| \cdot \| \) are equivalent on the finite element space \( V^0_h \) in Lemma 3.2.

In what follows, the trace inequality is a frequently used analysis tool, which states as [22]: for any function \( \phi \in H^1(K) \), there holds

\[
\| \phi \|^2_{L^2(\partial K)} \leq C \left( h_K^{-1} \| \phi \|^2_{L^2(K)} + h_K \| \nabla \phi \|^2_{L^2(K)} \right).
\]

The follow lemma plays a key role in the proof of Lemma 3.2.

**Lemma 3.1.** For any \( v = \{v_0, v_n n_e\} \in V_h \) and \( K \in T_h \), there exists a polynomial \( \varphi \in P_{k+3}(K) \) such that

\[
(\Delta v_0, \varphi)_K = 0, \quad ((\nabla v_0 - v_n n_e) \cdot n, \varphi)_{\partial K} = \|(\nabla v_0 - v_n n_e) \cdot n\|^2_{L^2(\partial K)},
\]

and

\[
\| \varphi \|_{L^2(K)} \leq Ch^{1/2}_K \|(\nabla v_0 - v_n n_e) \cdot n\|_{L^2(\partial K)}.
\]

**Proof.** For any \( K \in T_h \), let \( e_i, i = 1, 2, 3 \) be the three edges of \( K \) and \( \lambda_i \)'s are the barycentric coordinates of \( K \). Then, we define a polynomial \( \varphi_i \in P_{k+3}(K) \) for \( i = 1, 2, 3 \), respectively, by requiring that

\[
\varphi_i = \prod_{j=1, j \neq i}^3 \lambda_j q,
\]

with \( q \in P_{k+1}(K) \) and such that

\[
(\varphi_i, \tau)_{e_i} = ((\nabla v_0 - v_n n_e) \cdot n, \tau)_{e_i}, \quad \forall \tau \in P_{k+1}(e_i),
\]

\[
(\varphi_i, \tau)_K = 0, \quad \forall \tau \in P_k(K).
\]

Since there are

\[
(k + 2) + \frac{1}{2}(k + 1)(k + 2) = \frac{1}{2}(k + 2)(k + 3)
\]
equations and the same number of unknowns in the linear system (3.6a)-(3.6b), the existence and uniqueness of $\varphi_i$ are equivalent.

Assume that both $\varphi_i$ and $\hat{\varphi}_i$ satisfy the linear system (3.6a)-(3.6b), we will prove their difference $d_i = \varphi_i - \hat{\varphi}_i$ vanishes on $K$. From (3.5)-(3.6b), we know that $d_i$ can be expressed as

$$d_i = \prod_{j=1, j \neq i}^3 \lambda_j \tilde{q},$$

with $\tilde{q} \in \mathbb{P}_{k+1}(K)$ and satisfies the following conditions,

$$\langle d_i, \tau \rangle_{e_i} = 0, \quad \forall \tau \in \mathbb{P}_{k+1}(e_i),$$

$$\langle d_i, \tau \rangle_K = 0, \quad \forall \tau \in \mathbb{P}_k(K).$$

It follows from (3.8a) that $d_i = 0$ on $e_i$, which together with (3.7) implies $\tilde{q}$ in (3.7) can be written as $\tilde{q} = \lambda_i \omega$ with $\omega \in \mathbb{P}_k(K)$. Therefore, we have

$$d_i = \prod_{j=1}^3 \lambda_j \omega, \quad \text{with } \omega \in \mathbb{P}_k(K),$$

which combining with (3.8b) implies $d_i = 0$ on $K$.

Hence, the linear system (3.6a)-(3.6b) has a unique solution $\varphi_i$ in the form of (3.5), which belongs to $\mathbb{P}_{k+3}(K)$.

Then, by a scaling arguments, we have

$$\|\varphi_i\|_{L^2(K)} \leq C h_{K_k}^{1/2} \|\varphi_i\|_{L^2(\partial K)}.$$

Thanks to (3.5), it is known that $\varphi_i = 0$ on $e_j$ for $j \neq i$. Then, $\|\varphi_i\|_{L^2(\partial K)} = \|\varphi_i\|_{L^2(e_i)}$. Therefore,

$$\|\varphi_i\|_{L^2(K)} \leq C h_{K_k}^{1/2} \|\varphi_i\|_{L^2(e_i)}.$$

Let $\theta_i(x) = \prod_{j=1, j \neq i}^3 \lambda_j(x)$. Using (3.5) and (3.6a), we have

$$\langle \theta_i q, \tau \rangle_{e_i} = \langle \varphi_i, \tau \rangle_{e_i} \leq \|\nabla v_0 - v_n n_e\|_{L^2(e_i)} \|\tau\|_{L^2(e_i)}.$$

Taking $\tau = q$ in the above inequality, and by the second mean value theorem of integrals, there exist a point $\varepsilon_1 \in e_i$ such that

$$\theta_i(\varepsilon_1) \|q\|_{L^2(e_i)}^2 = \langle \theta_i q, q \rangle_{e_i} \leq \|\nabla v_0 - v_n n_e\|_{L^2(e_i)} \|q\|_{L^2(e_i)}^2.$$

Then, after cancelling $\|q\|_{L^2(e_i)}$, we obtain

$$\|q\|_{L^2(e_i)}^2 \leq \theta_i^{-1}(\varepsilon_1) \|\nabla v_0 - v_n n_e\|_{L^2(e_i)}.$$

Therefore, using (3.5) and the second mean value theorem of integrals again, there exist a point $\varepsilon_2 \in e_i$ such that

$$\|\varphi_i\|_{L^2(e_i)} = \sqrt{\langle \theta_i^2, q^2 \rangle_{e_i}} = \theta_i(\varepsilon_2) \|q\|_{L^2(e_i)}.$$
which together with (3.10) leads to
\[\|\varphi_i\|_{L^2(e_i)} \leq \theta_i(\varepsilon_2)\theta_i^{-1}(\varepsilon_1)\|\nabla v_0 - v_n e\| \cdot n\|_{L^2(e_i)}.\]
Thus, from (3.9) and the above inequality, we obtain
\[\|\varphi_i\|_{L^2(K)} \leq Ch_K^{1/2}\|\nabla v_0 - v_n e\| \cdot n\|_{L^2(e_i)}.\]
Finally, choosing \(\varphi = \sum_{i=1}^{3} \varphi_i\) ends the proof. \(\square\)

**Lemma 3.2.** There exist two positive constants \(C_1\) and \(C_2\) such that for any \(v = \{v_0, v_n e\} \in V_h\), we have
\[C_1\|v\|_{2,h} \leq \|v\| \leq C_2\|v\|_{2,h}.\]

**Proof.** For any \(v = \{v_0, v_n e\} \in V_h\) and \(\varphi \in \mathbb{P}_{k+3}(K)\), it follows from the definition of weak Laplacian (2.5) and integration by parts that
\[(\Delta_w v, \varphi)_K = -\langle \nabla v_0, \nabla \varphi\rangle_K + \langle v_n e \cdot n, \varphi\rangle_{\partial K}
= (\Delta v_0, \varphi)_K + \langle (v_n e - \nabla v_0) \cdot n, \varphi\rangle_{\partial K}.\]  
(3.11)

By letting \(\varphi = \Delta_w v\) in (3.11) we arrive at
\[\|\Delta_w v\|^2_{L^2(K)} = (\Delta v_0, \Delta_w v)_K + \langle (v_n e - \nabla v_0) \cdot n, \Delta_w v\rangle_{\partial K}.\]

From the trace inequality (3.3) and the inverse inequality, we have
\[
\|\Delta_w v\|^2_{L^2(K)} \leq \|\Delta v_0\|_{L^2(K)}\|\Delta_w v\|_{L^2(K)} + \|\langle (v_n e - \nabla v_0) \cdot n\|_{L^2(\partial K)}\|\Delta_w v\|_{L^2(\partial K)}
\leq C\|\Delta v_0\|_{L^2(K)} + h_K^{-1/2}\|\langle (v_n e - \nabla v_0) \cdot n\|_{L^2(\partial K)}\|\Delta_w v\|_{L^2(\partial K)},
\]
which implies
\[\|\Delta_w v\|_{L^2(K)} \leq C\left(\|\Delta v_0\|_{L^2(K)} + h_K^{-1/2}\|\langle (v_n e - \nabla v_0) \cdot n\|_{L^2(\partial K)}\right),\]
and consequently
\[\|v\| \leq C_2\|v\|_{2,h}.\]

Next we will prove
\[\sum_{K \in T_h} h_K^{-1}\|\nabla v_0 - v_n e\| \cdot n\|_{L^2(\partial K)} \leq C\|v\|^2.\]  
(3.12)

Let \(\varphi_0\) be obtained from Lemma 3.1, taking \(\varphi = \varphi_0\) in (3.11) yields
\[\|\langle (v_n e - \nabla v_0) \cdot n\|_{L^2(\partial K)} = (\Delta_w v, \varphi_0)_K \leq \|\Delta_w v\|_{L^2(K)}\|\varphi_0\|_{L^2(K)}
\leq C h_K^{1/2}\|\Delta_w v\|_{L^2(K)}\|\langle (v_n e - \nabla v_0) \cdot n\|_{L^2(\partial K)},\]
which implies (3.12).
Finally, by letting $\varphi = \Delta v_0$ in (3.11) we arrive at
\[\|\Delta v_0\|_{L^2(K)}^2 = (\Delta v_0, \Delta w v_0)_K - \langle (v_n n_e - \nabla v_0) \cdot n, \Delta w v_0 \rangle_{\partial K}\]
Using the trace inequality (3.3), the inverse inequality, and (3.12), one has
\[\|\Delta v_0\|_{L^2(K)}^2 \leq C \|\Delta w v_0\|_{L^2(K)} \|\Delta v_0\|_{L^2(K)},\]
which gives
\[\sum_{K \in \mathcal{T}_h} \|\Delta v_0\|_{L^2(K)}^2 \leq C \|v\|_{L^2}^2,\]
which together with (3.12) yields
\[\|v\| \geq C_1 \|v\|_{L^2}^2,\]
The proof is completed. □

In the following lemma we will prove the well-posedness of the SF-C0WG method (2.6).

**Lemma 3.3.** The SF-C0WG finite element scheme (2.6) has a unique solution.

**Proof.** To show the well-posedness of (2.6) assume that $f = g_D = g_N = 0$. We will show that $u_h$ vanishes. Take $v = u_h$ in (2.6). It follows that
\[\langle \Delta w u_h, \Delta w u_h \rangle_{\mathcal{T}_h} = 0.\]
Then Lemma 3.2 implies $\|u_h\|_{L^2} = 0$. Consequently, we have $\Delta u_0 = 0$, $\nabla u_0 \cdot n_e = u_n$ on $\partial K$. Thus $u_0$ is the solution of (1.1a)-(1.1c) with $f = g_D = g_N = 0$. We have $u_0 = 0$, then $u_n = 0$, which ends the proof. □

**4. An Error Equation.** Let $Q_0 : H^2(\Omega) \to S_h$ be the Scott-Zhang interpolation operator introduced in [15], which has the following properties:

a. [15, Page 493] $Q_0$ preserves polynomial of degree up to $k + 2$, i.e., $Q_0 v = v \in P_{k+2}(T_h)$.

b. [15, Lemma 8.2] $Q_0$ preserves the face mass of order $k$, i.e.,
\[\langle v - Q_0 v, p \rangle_e = 0, \quad \forall p \in P_k(e), \; e \in \mathcal{E}_h.\]

c. [15, Theorem 8.1] For any $v \in H^\gamma(\Omega)$ with $\gamma \geq 2$, there holds
\[\left(\sum_{K \in \mathcal{T}_h} h^{2s} \|v - Q_0 v\|_{H^s(K)}^2\right)^{1/2} \leq Ch^{\min\{k+3,\gamma\}} \|v\|_{H^\gamma(\Omega)}, \quad 0 \leq s \leq 2.\]

Now for the true solution $u$ of (1.1a)-(1.1c), we introduce an interpolation operator $Q_h : H^2(\Omega) \to V_h$ such that on each element $K \in \mathcal{T}_h$,
\[Q_h u = \{Q_0 u, Q_n (\frac{\partial u}{\partial n_e}) n_e \},\]
where $Q_n$ denotes the element-wise defined $L^2$ projections from $L^2(e)$ onto $P_{k+1}(e)$ for each $e \subset \partial K$.

Define the error between the WG solution $u_h = \{u_0, u_n n_e\}$ and the projection $Q_h u = \{Q_0 u, Q_n(\frac{\partial u}{\partial n_e}) n_e\}$ of the exact solution $u$ as

$$e_h = Q_h u - u_h := \{e_0, e_n n_e\},$$

with

$$e_0 = Q_0 u - u_0, \quad e_n = Q_n(\frac{\partial u}{\partial n_e}) - u_n.$$

The aim of this section is to obtain an error equation that $e_h$ satisfies.

**Lemma 4.1.** Let $\pi_h$ be an element-wise defined $L^2$ projections onto $P_{k+3}(K)$ on each element $K \in T_h$. For any $K \in T_h$ and $w \in H^2(\Omega)$, we have

$$\left(\Delta w(Q_h w), v\right)_K = \left(\Delta Q_0 w, v\right)_K + \left(Q_n(\frac{\partial w}{\partial n_e}) n_e \cdot n, v\right)_{\partial K},$$

for any $v \in P_{k+3}(K)$.

**Proof.** From the definition (2.5) of weak Laplacian it follows that

$$\left(\Delta w(Q_h w), v\right)_K = -\left(\nabla Q_0 w, \nabla v\right)_K + \left(Q_n(\frac{\partial w}{\partial n_e}) n_e \cdot n, v\right)_{\partial K},$$

for any $v \in P_{k+3}(K)$.

Using integration by parts, we get

$$-\left(\nabla Q_0 w, \nabla v\right)_K = \left(\Delta Q_0 w, v\right)_K - \left(\nabla Q_0 w \cdot n, v\right)_{\partial K}.$$

Plugging (4.5) into (4.4), and recalling that

$$Q_n(\frac{\partial w}{\partial n_e}) n_e \cdot n = Q_n(\frac{\partial w}{\partial n})$$

yields (4.3). The proof is completed. □

**Lemma 4.2 (Error Equation).** Let $u$ and $u_h$ be the solutions of the problem (1.1a)-(1.1c) and the SF-C0WG scheme (2.6), respectively. For any $v \in V_h^0$, we have

$$A_h(e_h, v) = \ell(u, v),$$

where $\ell(u, v) := \sum_{i=1}^2 \ell_i(u, v)$, with

$$\ell_1(u, v) := \left(\Delta w(Q_h u) - \pi_h \Delta u, \Delta w(v)\right)_{T_h},$$

$$\ell_2(u, v) := \left(\Delta u - \pi_h \Delta u, (\nabla v_0 - v_n n_e) \cdot n\right)_{\partial T_h}.$$

**Proof.** For $v = \{v_0, v_n n_e\} \in V_h^0$, testing (1.1a) by $v_0$ and using the fact that

$$\sum_{K \in T_h} \left(\Delta u, v_n n_e \cdot n\right)_{\partial K} = 0$$
and integration by parts, we arrive at

\[(f, v_0) = (\Delta^2 u, v_0)_{T_h} \equiv (\Delta u, \Delta v_0)_{T_h} - (\Delta u, \nabla v_0 \cdot n)_{\partial T_h} + \langle \nabla (\Delta u) \cdot n, v_0 \rangle_{\partial T_h} \]

\[(4.8) = (\Delta u, \Delta v_0)_{T_h} - (\Delta u, (\nabla v_0 - v_n e) \cdot n)_{\partial T_h}. \]

Next we investigate the term \((\Delta u, \Delta v_0)_{T_h}\) in the above equation. Using (4.3), integration by parts and the definition of weak Laplacian, we have

\[(\Delta u, \Delta v_0)_{T_h} = (\pi_h \Delta u, \Delta v_0)_{T_h} \equiv -\langle \nabla v_0, \nabla (\pi_h \Delta u) \rangle_{T_h} + \langle \nabla v_0 \cdot n, \pi_h \Delta u \rangle_{\partial T_h} \]

\[(5.1) = (\Delta w, v_n)_{T_h} + \langle \nabla v_0 - v_n e \rangle_{T_h} \cdot n, \pi_h \Delta u \rangle_{\partial T_h} \]

which together with (4.8) yields

\[(5.2) (f, v_0) = A_h(Q_h u, v) - \ell (u, v) - \langle \nabla v_0 - v_n e \rangle \cdot n, \Delta w - \pi_h \Delta u \rangle_{\partial T_h} \]

which implies that

\[A_h(Q_h u, v) = (f, v_0) + \sum_{i=1}^2 \ell (u, v). \]

Subtracting (2.6) from the above equation ends the proof. \(\square\)

5. An Error Estimate in the \(H^2\)-like Norm. We will obtain the optimal convergence rate for the solution \(u_h\) of the SF-CSW method (2.6) in a discrete \(H^2\) norm.

**Lemma 5.1.** Assume \(w \in H^{\gamma+2}(\Omega)\) with \(\gamma > 0\). There exists a constant \(C\) such that the following estimates hold true:

\[(5.1) \left\| \sum_{K \in T_h} h_K \| \Delta w - \pi_h \Delta w \|^2_{L^2(\partial K)} \right\|^{1/2} \leq Ch^{\min\{k+4, \gamma\}} \| w \|_{H^{\gamma+2}(\Omega)}, \]

\[(5.2) \left\| \sum_{K \in T_h} h_K^{-1} \frac{\partial}{\partial n} (Q_0 w) - Q_n (\frac{\partial w}{\partial n}) \right\|^2_{L^2(\partial K)} \leq Ch^{\min\{k+1, \gamma\}} \| w \|_{H^{\gamma+2}(\Omega)}, \]

\[(5.3) \| \Delta w(Q_h w) - \pi_h \Delta w \|_{L^2(T_h)} \leq Ch^{\min\{k+1, \gamma\}} \| w \|_{H^{\gamma+2}(\Omega)}. \]

**Proof.** By the trace inequality (3.3) and the approximation property of the \(L^2\) orthogonal projection \(\pi_h\), we have

\[h_K \| \Delta w - \pi_h \Delta w \|^2_{L^2(\partial K)} \leq C \| \Delta w - \pi_h \Delta w \|_{L^2(\partial K)} + h_K^2 \| \nabla (\Delta w - \pi_h \Delta w) \|_{L^2(K)} \]

\[\leq Ch^{2\min\{k+4, \gamma\}} \| \Delta w \|_{H^{\gamma}(K)} \]

\[\leq Ch^{2\min\{k+4, \gamma\}} \| w \|_{H^{\gamma+2}(K)}. \]
Taking the summation of the above inequalities over all $K \in \mathcal{K}_h$, we completes the proof of (5.1).

Next, we turn to the estimate (5.2). It follows from the definition of $Q_0$ and $Q_n$ that
\[
\| \frac{\partial}{\partial n} (Q_0 w - Q_n \frac{\partial w}{\partial n}) \|_{L^2(\partial K)} \\
\leq \| \frac{\partial}{\partial n} (Q_0 w - w) \|_{L^2(\partial K)} + \| Q_n \frac{\partial w}{\partial n} - Q_n \frac{\partial w}{\partial n} \|_{L^2(\partial K)} \\
\leq 2 \| \frac{\partial}{\partial n} (Q_0 w - w) \|_{L^2(\partial K)}.
\]
(5.4)

Furthermore, using the trace inequality (3.3) and the approximation property (4.2) of $Q_0$, we obtain
\[
\| \frac{\partial}{\partial n} (Q_0 w - w) \|_{L^2(\partial K)}^2 \\
\leq C(h_K^{-1} \| \nabla (Q_0 w - w) \|_{L^2(K)}^2 + h_K \| \nabla^2 (Q_0 w - w) \|_{L^2(K)}^2) \\
\leq C h_K^{\min(2k+2,2\gamma+1)} |w|_{H^{\gamma+2}(\Omega)}^2,
\]
which together with (5.4) yields
\[
\sum_{K \in \mathcal{K}_h} h_K^{-1} \| \frac{\partial}{\partial n} (Q_0 w - Q_n \frac{\partial w}{\partial n}) \|_{L^2(\partial K)}^2 \leq C h_K^{\min(2k+2,2\gamma+1)} |w|_{H^{\gamma+2}(\Omega)}^2,
\]
which ends the proof of (5.2).

Now we consider the estimate (5.3). For any $v \in \mathbb{P}_{k+3}(\mathcal{T}_h)$, from (4.3) and the orthogonal property of the $L^2$ projection $\pi_h$, it follows that
\[
(\Delta w(Q_h w - \pi_h w, v)_{\mathcal{T}_h} \\
= (\Delta (Q_0 w - w), v)_{\mathcal{T}_h} + (Q_n \frac{\partial u}{\partial n} - \frac{\partial}{\partial n} (Q_0 w), v)_{\partial \mathcal{T}_h} \\
= I_1 + I_2.
\]
(5.5)

From the Cauchy-Schwarz inequality and the approximation property (4.2) of $Q_0$, one has
\[
|I_1| \leq \sum_{K \in \mathcal{K}_h} \| \Delta (Q_0 w - w) \|_{L^2(K)} \| v \|_{L^2(K)} \\
\leq \left( \sum_{K \in \mathcal{K}_h} \| Q_0 w - w \|_{H^2(K)}^2 \right)^{1/2} \| v \|_{L^2(\mathcal{T}_h)}^{1/2} \\
\leq C h_K^{\min(1+1,\gamma)} |w|_{H^{\gamma+2}(\Omega)} \| v \|_{L^2(\mathcal{T}_h)}.
\]
(5.6)

Using the Cauchy-Schwarz inequality, (5.2) and the inverse inequality, we arrive at
\[
|I_2| \leq \left( \sum_{K \in \mathcal{K}_h} h_K \| (Q_n \frac{\partial \nu}{\partial n} - \frac{\partial}{\partial n} (Q_0 w)) \|_{L^2(\partial K)}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{K}_h} h_K \| v \|_{L^2(\partial K)}^2 \right)^{1/2} \\
\leq C h_K^{\min(1+1,\gamma)} |w|_{H^{\gamma+2}(\Omega)} \| v \|_{L^2(\mathcal{T}_h)}.
\]
which together with (5.5) and (5.6) yields

$$
|\Delta w(Q_h w) - \pi_h w, v_{\mathcal{T}_h}| \leq C h^{\min\{k+1,\gamma\}} |w|_{H^{\gamma+2}(\Omega)} \|v\|_{L^2(\mathcal{T}_h)}.
$$

Taking $v = \Delta w(Q_h w) - \pi_h w$ in the above inequality ends the proof of (5.3). \(\square\)

**Lemma 5.2.** Assume $w \in H^{\gamma+2}(\Omega)$ with $\gamma > 0$. There exists a constant $C$ such that the following estimates hold true:

$$
|\ell_1(w, v)| \leq C h^{\min\{k+1,\gamma\}} |w|_{H^{\gamma+2}(\Omega)} \|v\|, \tag{5.7}
$$

$$
|\ell_2(w, v)| \leq C h^{\min\{k+4,\gamma\}} |w|_{H^{\gamma+2}(\Omega)} \|v\|, \tag{5.8}
$$

for any $v \in V_h^0$.

**Proof.** Using the Cauchy-Schwarz inequality and (5.3) of Lemma 5.1, we have

$$
|\ell_1(w, v)| = |(\Delta w(Q_h w) - \pi_h w, \Delta w v)_{\mathcal{T}_h}|
\leq \|\Delta w(Q_h w) - \pi_h w\|_{L^2(\mathcal{T}_h)} \|\Delta w v\|_{L^2(\mathcal{T}_h)}
\leq C h^{\min\{k+1,\gamma\}} |w|_{H^{\gamma+2}(\Omega)} \|v\|.
$$

It follows from the Cauchy-Schwarz inequality, (5.1), and Lemma 3.2 that

$$
|\ell_2(w, v)| = \left| \sum_{K \in \mathcal{T}_h} \langle \Delta w - \pi_h \Delta w, (\nabla v_0 - v_0 n_e) \cdot n \rangle_{\partial K} \right|
\leq \left( \sum_{K \in \mathcal{T}_h} h_K \|\Delta w - \pi_h \Delta w\|^2_{L^2(\partial K)} \right)^{1/2}
\times \left( \sum_{K \in \mathcal{T}_h} h_K^{-1} \|\nabla v_0 - v_0 n_e \cdot n\|^2_{L^2(\partial K)} \right)^{1/2}
\leq C h^{\min\{k+4,\gamma\}} |w|_{H^{\gamma+2}(\Omega)} \|v\|_{2,h}
\leq C h^{\min\{k+4,\gamma\}} |w|_{H^{\gamma+2}(\Omega)} \|v\|.
$$

We have completed the proof. \(\square\)

**Theorem 5.3.** Let $u_h \in V_h$ be the solution arising from the SF-C0WG scheme (2.6). Assume that the exact solution $u \in H^{k+3}(\Omega)$. Then, there exists a constant $C$ such that

$$
\|Q_h u - u_h\| \leq C h^{k+1} |u|_{H^{k+3}(\Omega)} \tag{5.9}
$$

**Proof.** Taking $v = e_h$ in the error equation (4.6) and using Lemma 5.2 with $\gamma = k + 1$, we arrive at

$$
\|e_h\|^2 = \ell(u, e_h) \leq C h^{k+1} |u|_{H^{k+3}(\Omega)} \|e_h\|,
$$

which completes the proof. \(\square\)
6. Error Estimates in the $L^2$ Norm and $H^1$ Norm. In this section, we will provide estimates for the $L^2$ norm and $H^1$ norm of the error between the exact solution $u$ and its corresponding WG finite element solution $u_h$.

Firstly, let us introduce the following dual problem

\begin{align}
\Delta^2 \phi &= \chi \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \Gamma, \\
\nabla \phi \cdot n &= 0 \quad \text{on } \Gamma.
\end{align}

Assume that the dual problem has the $H^{\alpha+2}$-regularity in the sense that there exists a constant $C$ such that

\begin{equation}
\|\phi\|_{H^{\alpha+2}(\Omega)} \leq C \|\chi\|_{H^{\alpha-2}(\Omega)}, \quad \text{for } \alpha = 1, 2.
\end{equation}

For $\chi \in H^{\alpha-2}(\Omega)$ with $\alpha > 0$, the $H^{\alpha+2}$-regularity has been proved for smooth domains in any dimension\cite{5}. The $H^4$-regularity has been proved by Blum and Rannacher in \cite{3} for the two dimensional convex polygonal domains with inner angles less than 126.28°.

**Lemma 6.1.** Let $\phi \in H^{\alpha+2}(\Omega)$ with $\alpha = 1, 2$. Then, there holds

\begin{equation}
|\Delta_w(Q_h \phi)|_{H^\alpha(T_h)} \leq C h^{\min\{k+1-\alpha,0\}} \|\phi\|_{H^{\alpha+2}(\Omega)}.
\end{equation}

**Proof.** The proof is given in Appendix. $\square$

**Lemma 6.2.** Assume $u \in H^{k+1}(\Omega)$ and $\phi \in H^{\alpha+2}(\Omega)$ with $\alpha = 1, 2$. Then for $k \geq 0$, there holds

\begin{align}
|\ell_1(u,Q_h \phi)| &\leq C h^{\min\{2k+2,k+1-\alpha\}} |u|_{H^{k+3}(\Omega)} \|\phi\|_{H^{\alpha+2}(\Omega)}, \\
|\ell_2(u,Q_h \phi)| &\leq C h^{\min\{2k+2,k+1-\alpha\}} |u|_{H^{k+3}(\Omega)} \|\phi\|_{H^{\alpha+2}(\Omega)}.
\end{align}

**Proof.** Let $P^{\alpha-1}_h$ be the $L^2$ orthogonal projection onto the piecewise polynomial space $P^{\alpha-1}(T_h)$. For simplicity, denote by $\phi_h = \Delta_w(Q_h \phi)$ and $\hat{\phi}_h = P^{\alpha-1}_h(\phi_h)$. Then,

\begin{align}
\ell_1(u,Q_h \phi) &= (\Delta_w(Q_h u) - \pi_h \Delta u, \phi_h)_{T_h} \\
&= (\Delta_w(Q_h u) - \pi_h \Delta u, \phi_h - \hat{\phi}_h)_{T_h} + (\Delta_w(Q_h u) - \pi_h \Delta u, \hat{\phi}_h)_{T_h} \\
&= T_1 + T_2.
\end{align}

Using the Cauchy-Schwarz inequality, (5.3) of Lemma 5.1 and (6.5), one has

\begin{align}
|T_1| &= |(\Delta_w(Q_h u) - \pi_h \Delta u, \phi_h - \hat{\phi}_h)_{T_h}| \\
&\leq \|\Delta_w(Q_h u) - \pi_h \Delta u\|_{L^2(T_h)} \|\phi_h - \hat{\phi}_h\|_{L^2(T_h)} \\
&\leq C h^{k+1} |u|_{H^{k+3}(\Omega)} \|\phi_h\|_{H^\alpha(\Omega)} \\
&\leq C h^{k+1+\alpha} |u|_{H^{k+3}(\Omega)} \|\phi_h\|_{H^\alpha(\Omega)} \\
&\leq C h^{\min\{2k+2,k+1+\alpha\}} |u|_{H^{k+3}(\Omega)} \|\phi_h\|_{H^{\alpha+2}(\Omega)}.
\end{align}
Now we turn to the estimate of the term $T_2$. Firstly, we rewrite $T_2$ as follows:

$$
T_2 = (\Delta_w(Q_hu) - \pi_h \Delta u, \hat{\phi}_h)_{\mathcal{T}_h}
$$

$$
= (\Delta(Q_0u - u), \hat{\phi}_h)_{\mathcal{T}_h} + (Q_n(\frac{\partial u}{\partial n}) - \frac{\partial u}{\partial n} (Q_0u), \hat{\phi}_h)_{\partial \mathcal{T}_h}
$$

$$
= -(\nabla(Q_0u - u), \nabla \hat{\phi}_h)_{\mathcal{T}_h} + (Q_n(\frac{\partial u}{\partial n}) - \frac{\partial u}{\partial n} \hat{\phi}_h)_{\partial \mathcal{T}_h}
$$

(6.10)

$$
= J_1 + J_2.
$$

For the first term $J_1$, we discuss it in the following two cases:

- In the case of $\alpha = 1$, $\nabla \hat{\phi}_h = 0$ since $\hat{\phi}_h = P_h^0(\phi_h) \in P_0(\mathcal{T}_h)$. Therefore, $J_1 = 0$.

- In the case of $\alpha = 2$, $\nabla \hat{\phi}_h$ is a piecewise constant vector due to $\hat{\phi}_h = P_h^1(\phi_h) \in P_1(\mathcal{T}_h)$. Then, by Green’s formula and (4.1), we get

$$
J_1 = \sum_{K \in \mathcal{T}_h} -\langle Q_0u - u, \nabla \hat{\phi}_h \cdot n \rangle_{\partial K} = 0.
$$

Thus, in both cases $\alpha = 1$ and $\alpha = 2$, we have

(6.11)

$$
J_1 = 0.
$$

As to the second term $J_2$, recalling the fact

$$
\langle Q_n(\frac{\partial u}{\partial n}) - \frac{\partial u}{\partial n} \Delta \phi \rangle_{\partial \mathcal{T}_h} = 0,
$$

we split $J_2$ into the following two terms:

$$
J_2 = \langle Q_n(\frac{\partial u}{\partial n}) - \frac{\partial u}{\partial n} \hat{\phi}_h \rangle_{\partial \mathcal{T}_h}
$$

$$
= \langle Q_n(\frac{\partial u}{\partial n}) - \frac{\partial u}{\partial n} P_h^{\alpha-1}(\Delta_w(Q_h \phi) - \Delta \phi) \rangle_{\partial \mathcal{T}_h}
$$

$$
+ \langle Q_n(\frac{\partial u}{\partial n}) - \frac{\partial u}{\partial n} P_h^{\alpha-1}(\Delta \phi) \rangle_{\partial \mathcal{T}_h}.
$$

And then, by Cauchy-Schwarz inequality and (5.2) of Lemma 5.1 with $\gamma = k + 1$, we get

$$
|J_2| \leq \left( \sum_{K \in \mathcal{T}_h} h_K^{-1} ||Q_n(\frac{\partial u}{\partial n}) - \frac{\partial u}{\partial n}||^2_{L^2(\partial K)} \right)^{1/2} (\Theta_1^{1/2} + \Theta_2^{1/2})
$$

(6.12)

$$
\leq C h^{k+1} ||u||_{H^{k+3}(\Omega)} (\Theta_1^{1/2} + \Theta_2^{1/2}),
$$

where

$$
\Theta_1 := \sum_{K \in \mathcal{T}_h} h_K ||P_h^{\alpha-1}(\Delta_w(Q_h \phi) - \Delta \phi)||^2_{L^2(\partial K)},
$$

$$
\Theta_2 := \sum_{K \in \mathcal{T}_h} h_K ||P_h^{\alpha-1}(\Delta \phi) - \Delta \phi||^2_{L^2(\partial K)}.
$$

From the trace inequality and the stability of $L^2$ projection $P_h^{\alpha-1}$, it follows that

$$
\Theta_1 \leq C \sum_{K \in \mathcal{T}_h} ||P_h^{\alpha-1}(\Delta_w(Q_h \phi) - \Delta \phi)||^2_{L^2(K)} \leq C ||\Delta_w(Q_h \phi) - \Delta \phi||^2_{L^2(\mathcal{T}_h)}.
$$
Then, by the triangle inequality and (5.3) of Lemma 5.1, we arrive at
\[
\Theta_1 \leq C(\|\Delta w(Q_h \phi) - \pi_h \Delta \phi\|^2_{L^2(T_h)} + \|\pi_h \Delta \phi - \Delta \phi\|^2_{L^2(T_h)}) \\
\leq C h^{2 \min\{k+1, \alpha\}} |\phi|^2_{H^{k+2}(\Omega)}.
\]
(6.13)

It follows from the trace inequality and the approximation property of \(L^2\) projection \(P^\alpha_{h-1}\) that
\[
\Theta_2 \leq \sum_{K \in T_h} (h_K \|\Delta \phi\|^2_{L^2(K)} + h_K \|\pi_h \Delta \phi - \Delta \phi\|^2_{H^1(\partial K)}) \\
\leq C h^{2\alpha} |\phi|^2_{H^{k+2}(\Omega)},
\]
which together with (6.13) and (6.12) leads to
\[
|J_2| \leq C h^{\min\{2k+2, k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{k+2}(\Omega)}.
\]
(6.14)

Collecting (6.10), (6.11) and (6.14) yields
\[
|T_2| \leq C h^{\min\{2k+2, k+1+\alpha\}}|u|_{H^{k+3}(\Omega)}|\phi|_{H^{k+2}(\Omega)},
\]
which combining with (6.9) and (6.8) completed the proof of (6.6).

As to the proof of (6.7), from the Cauchy-Schwarz inequality and Lemma 5.1 with \(\gamma = k + 1\) it follows that
\[
|\ell_2(u, Q_h \phi)| = \left| \sum_{T \in T_h} \frac{\partial}{\partial n} \frac{\partial}{\partial n} (Q_0 \phi) - Q_n (\frac{\partial \phi}{\partial n}) n \right| \\
\leq \left( \sum_{K \in T_h} h_K \|\Delta u - \pi_h \Delta u\|^2_{L^2(\partial K)} \right)^{1/2} \times \\
\left( \sum_{K \in T_h} h_K^{-1} \|\frac{\partial}{\partial n} (Q_0 \phi) - Q_n (\frac{\partial \phi}{\partial n})\|^2_{L^2(\partial K)} \right)^{1/2} \\
\leq C h^{k+1} |u|_{H^{k+3}(\Omega)} \cdot h^{\min\{k+1, \alpha\}} |\phi|_{H^{k+2}(\Omega)} \\
\leq C h^{\min\{2k+2, k+1+\alpha\}} |u|_{H^{k+3}(\Omega)} |\phi|_{H^{k+2}(\Omega)}.
\]

The proof is completed. \(\Box\)

**Theorem 6.3.** Let \(u_h = \{u_0, u_n, n_e\} \in V_h\) be the solution of the SF-C0WG scheme (2.6). Assume that the exact solution \(u \in H^{k+3}(\Omega)\) and the regularity assumption (6.4) holds true. Then, there exists a constant \(C\) such that
\[
\|Q_0 u - u_0\|^2_{L^2(\Omega)} \leq C h^{k+3-\delta_{k,\alpha}}|u|_{H^{k+3}(\Omega)}
\]
and
\[
\|\nabla (Q_0 u - u_0)\|_{L^2(\Omega)} \leq C h^{k+2}|u|_{H^{k+3}(\Omega)}.
\]
(6.15) (6.16)

Here \(\delta_{i, j}\) is the usual Kronecker’s delta with value 1 when \(i = j\) and value 0 otherwise.
Proof. Testing (6.1) by error function $e_0$ and then using a similar procedure as in
the proof of the equation (4.9), we obtain
\begin{equation}
(\chi, e_0) = (\Delta^2 \phi, e_0)_{\Omega} = A_h(e_h, Q_h \phi) - \ell(\phi, e_h).
\end{equation}

The error equation (4.6) gives
\begin{equation}
A_h(e_h, Q_h \phi) = \ell(u, Q_h \phi),
\end{equation}
which combining with (6.17) leads to
\begin{equation}
(\chi, e_0) = \ell(u, Q_h \phi) - \ell(\phi, e_h).
\end{equation}

In view of Lemma 6.2, we infer that
\begin{equation}
\|\ell(u, Q_h \phi)\| \leq C h^{\min\{2k+2, k+1+\alpha\}} \|u\|_{H^{k+3}(\Omega)} \|\phi\|_{H^{\alpha+2}(\Omega)}
\end{equation}

Using Lemma 5.2 with $\gamma = \alpha$ and Theorem 5.3, we have
\begin{equation}
|\ell(\phi, e_h)| \leq C h^{\min\{k+1, \alpha\}} \|\phi\|_{H^{\alpha+2}(\Omega)} \|e_h\|
\leq C h^{\min\{2k+2, k+1+\alpha\}} \|u\|_{H^{k+3}(\Omega)} \|\phi\|_{H^{\alpha+2}(\Omega)},
\end{equation}
which combining with (6.18) and (6.20) leads to
\begin{equation}
|\ell(u, Q_h \phi)| \leq C h^{\min\{2k+2, k+1+\alpha\}} \|u\|_{H^{k+3}(\Omega)} \|\phi\|_{H^{\alpha+2}(\Omega)}.
\end{equation}

For the $L^2$-norm estimate of $e_0$, taking $\chi = e_0$ in the dual problem (6.1)-(6.3),
and then using the estimate of (6.20) with the $H^4$-regularity, we find
\begin{equation}
\|e_0\|_{L^2(\Omega)}^2 \leq C h^{\min\{2k+2, k+3\}} \|u\|_{H^{k+3}(\Omega)} \|\phi\|_{H^{4}(\Omega)},
\end{equation}
which together with the assumption (6.4) with $\alpha = 2$:
\begin{equation}
\|\phi\|_{H^{4}(\Omega)} \leq C \|e_0\|_{L^2(\Omega)}
\end{equation}
completes the proof of (6.15).

Then using the estimate of (6.20) with the $H^3$-regularity yields
\begin{equation}
|(\chi, e_0)| \leq C h^{k+2} \|u\|_{H^{k+3}(\Omega)} \|\phi\|_{H^{3}(\Omega)},
\end{equation}
which together with the assumption (6.4) with $\alpha = 1$:
\begin{equation}
\|\phi\|_{H^{3}(\Omega)} \leq C \|\chi\|_{H^{-1}(\Omega)}
\end{equation}
leads to
\begin{equation}
\|\nabla e_0\|_{L^2(\Omega)} = \sup_{\chi \in H^{-1}(\Omega)} \frac{(\chi, e_0)}{\|\chi\|_{H^{-1}(\Omega)}} \leq C h^{k+2} \|u\|_{H^{k+3}(\Omega)},
\end{equation}
which ends the proof of (6.16).

By the triangle inequality, from Theorem 6.3 and (4.2), we immediately obtain
the $L^2$ norm and $H^1$ norm error estimates between the exact solution $u$ and its WG
finite element approximation $u_0$ as follows:
Corollary 6.4. Let \( u_h = \{ u_0, u_n, n_x \} \in V_h \) be the solution of the SF-C0WG scheme (2.6). Assume that the exact solution \( u \in H^{k+3}(\Omega) \) and the regularity assumption (6.4) holds true. Then, there exists a constant \( C \) such that

\[
\| u - u_0 \|_{L^2(\Omega)} \leq C h^{k+3-\delta_{0,0}} |u|_{H^{k+3}(\Omega)}
\]

and

\[
\| \nabla (u - u_0) \|_{L^2(\Omega)} \leq C h^{k+2} |u|_{H^{k+3}(\Omega)}.
\]

Here \( \delta_{i,j} \) is the usual Kronecker’s delta with value 1 when \( i = j \) and value 0 otherwise.

7. Numerical Experiments. In this section, we conduct some numerical experiments to verify the theoretical prediction on the SF-C0WG method (2.6) and also to compare its numerical performance to the C0WG method (2.7) and the C0IP method (2.8).

Example 1. Consider the model problem (1.1a)-(1.1c) with \( \Omega = (0,1)^2 \). The source data \( f \) and boundaries data \( g_D \) and \( g_N \) are chosen so that the exact solution is

\[
\text{\( u = \sin(\pi x) \sin(\pi y) \).}
\]

The initial mesh in our computation is shown in Figure 7.1, which is generated by MATLAB function `initmesh`. The next level of mesh is derived by uniformly refining the previous level of mesh. The errors and the orders of convergence for the SF-C0WG method (2.6) with \( k = 0 \) and \( k = 1 \) are reported in Tables 7.1, which confirm the theoretical prediction in Theorem 5.3 and Theorem 6.3.

Table 7.2 lists the errors and the rates of convergence for the C0WG method (2.7). The results in Table 7.1 and Table 7.2 show that both the SF-C0WG method
and the C0WG method converge with the same rates, but the accuracy reached on a given mesh with a given polynomial degree is significant different. The SF-C0WG method is more accuracy than the C0WG method.

Table 7.1: Error profiles and convergence rates of the SF-C0WG method.

| k  | level | $\|Q_h u - u_h\|$ | Rate | $\|\nabla (u - u_0)\|$ | Rate | $\|u - u_0\|$ | Rate |
|----|-------|-----------------|------|-----------------|------|-----------------|------|
| 0  | 1     | 3.34E+00        | -    | 8.06E-02        | -    | 8.15E-03        | -    |
|    | 2     | 1.66E+00        | 1.0076 | 2.03E-02        | 1.9899 | 2.00E-03        | 2.0395 |
|    | 3     | 8.25E-01        | 1.0095 | 5.20E-03        | 1.9653 | 5.09E-04        | 1.9710 |
|    | 4     | 4.11E-01        | 1.0059 | 1.32E-03        | 1.9787 | 1.29E-04        | 1.9760 |
|    | 5     | 2.05E-01        | 1.0031 | 3.32E-04        | 1.9915 | 3.26E-05        | 1.9893 |
| 1  | 1     | 3.61E-01        | -    | 5.79E-03        | -    | 2.97E-04        |ting      |
|    | 2     | 9.12E-02        | 1.9853 | 7.26E-04        | 2.9952 | 2.16E-05        | 3.7835 |
|    | 3     | 2.28E-02        | 1.9975 | 8.99E-05        | 3.0144 | 1.41E-06        | 3.9374 |
|    | 4     | 5.71E-03        | 2.0001 | 1.12E-05        | 3.9787 | 1.29E-06        | 3.9760 |
|    | 5     | 1.43E-03        | 2.0004 | 1.39E-06        | 3.0048 | 6.29E-09        | 3.8238 |

Table 7.2: Error profiles and convergence rates of the C0WG method.

| k  | level | $\|Q_h u - u_h\|$ | Rate | $\|\nabla (u - u_0)\|$ | Rate | $\|u - u_0\|$ | Rate |
|----|-------|-----------------|------|-----------------|------|-----------------|------|
| 0  | 1     | 4.73E+00        | -    | 6.35E-01        | -    | 1.36E-01        | -    |
|    | 2     | 2.33E+00        | 1.0189 | 1.52E-01        | 2.0620 | 3.35E-02        | 2.0236 |
|    | 3     | 1.16E+00        | 1.0046 | 3.77E-02        | 2.0109 | 8.35E-03        | 2.0033 |
|    | 4     | 5.81E-01        | 1.0018 | 9.41E-03        | 2.0047 | 2.08E-03        | 2.0019 |
|    | 5     | 2.90E-01        | 1.0008 | 2.35E-03        | 2.0022 | 5.21E-04        | 2.0011 |
| 1  | 1     | 7.14E-01        | -    | 7.91E-02        | -    | 4.46E-03        | -    |
|    | 2     | 1.91E-01        | 1.9043 | 1.04E-02        | 2.9287 | 3.04E-04        | 3.8743 |
|    | 3     | 4.89E-02        | 1.9629 | 1.33E-03        | 2.9627 | 1.96E-05        | 3.9522 |
|    | 4     | 1.24E-02        | 1.9825 | 1.70E-04        | 2.9743 | 1.25E-06        | 3.9737 |
|    | 5     | 3.11E-03        | 1.9914 | 2.14E-05        | 2.9849 | 7.89E-08        | 3.9852 |

Table 7.3 shows the errors and the rates of convergence for the C0IP method (2.8). The errors in the first column of Table 7.3 is measured in the following $H^2$-like norm tailored for the C0IP method:

$$
\|v\|_{dg} := \sum_{K \in T_h} |v|_{H^2(K)}^2 + \sum_{e \in E_h} h_e^{-1} \|\nabla v\|_{L^2(e)}^2 \right]^{1/2}.
$$

The results in Table 7.1 and Table 7.3 show that both the SF-C0WG method and the C0IP method converge with the same rate and the accuracies are also similar when the errors are measured in $H^1$ semi-norm and $L^2$ norm.

A comparison of the assembling time and solving time for both the C0WG method and the SF-C0WG method is displayed in Table 7.4. It can be observed that the
Table 7.3: Error profiles and convergence rates of the C0IP method.

| k | level | $\|u - u_h\|_d^g$ | Rate | $\|\nabla(u - u_h)\|$ | Rate | $\|u - u_h\|$ | Rate |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1.33E+00 | – | 8.47E-02 | – | 1.02E-02 | – |
|   | 2 | 6.40E-01 | 1.0955 | 2.24E-02 | 1.9150 | 2.91E-03 | 1.8035 |
|   | 3 | 3.20E-01 | 1.0009 | 5.92E-03 | 1.9222 | 7.99E-04 | 1.8635 |
|   | 4 | 1.60E-01 | 0.9955 | 1.52E-03 | 1.9584 | 2.10E-04 | 1.9310 |
|   | 5 | 8.03E-02 | 0.9983 | 3.86E-04 | 1.9824 | 5.35E-05 | 1.9689 |

| k | level | $\|u - u_h\|_d^g$ | Rate | $\|\nabla(u - u_h)\|$ | Rate | $\|u - u_h\|$ | Rate |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2.00E-01 | – | 6.24E-03 | – | 3.47E-04 | – |
|   | 2 | 5.15E-02 | 1.9533 | 7.83E-04 | 2.9950 | 2.58E-05 | 3.7489 |
|   | 3 | 1.31E-02 | 1.9780 | 9.62E-05 | 3.0238 | 1.70E-06 | 3.9194 |
|   | 4 | 3.30E-03 | 1.9870 | 1.19E-05 | 3.0157 | 1.09E-07 | 3.9712 |
|   | 5 | 8.29E-04 | 1.9930 | 1.48E-06 | 3.0076 | 6.53E-09 | 4.0564 |

Table 7.4: Comparison of assembling time and solving time for the C0WG method and the SF-C0WG method.

| k | level | C0WG method | SF-C0WG method |
|---|---|---|---|
|   | Assembling Time | Solving Time | Assembling Time | Solving Time |
| 0 | 1 | 0.052233 | 0.001690 | 0.065710 | 0.003116 |
|   | 2 | 0.175486 | 0.007218 | 0.157510 | 0.006127 |
|   | 3 | 0.734831 | 0.039118 | 0.546378 | 0.030866 |
|   | 4 | 2.602549 | 0.171534 | 2.240196 | 0.133192 |
|   | 5 | 10.67890 | 0.874419 | 8.766250 | 0.628217 |
| 1 | 1 | 0.347160 | 0.027630 | 0.057160 | 0.003116 |
|   | 2 | 0.184210 | 0.020082 | 0.157510 | 0.006127 |
|   | 3 | 0.734831 | 0.039118 | 0.546378 | 0.030866 |
|   | 4 | 2.602549 | 0.171534 | 2.240196 | 0.133192 |
|   | 5 | 10.67890 | 0.874419 | 8.766250 | 0.628217 |

Assembling time and solving time for the SF-C0WG method is always smaller than that for the C0WG method.

The assembling time, solving time, and total time (the sum of the assembling and solving time) for both the C0IP method and the SF-C0WG method are illustrated in Table 7.5. As can be seen, although the solving time of C0IP method is less than the SF-C0WG method, the assembling time and total time for the SF-C0WG method is always smaller than that for the C0IP method.

8. Appendix. In this section, we shall introduce some technique tools which are useful in the $L^2$ and $H^1$ norm error analysis.

In order to prove Lemma 6.1, we introduce the following two lemmas.
Table 7.5: Comparison of assembling, solving and total time for the C0IP method and the SF-C0WG method.

| Time (sec.) | level | $k = 0$ |  | $k = 1$ |  |
|-------------|-------|---------|---|---------|---|
|             |       | C0IP    | SF-C0WG | C0IP    | SF-C0WG |
| Assemble    | 1     | 0.073452 | 0.065710 | 0.091383 | 0.057160 |
|             | 2     | 0.163071 | 0.157510 | 0.252789 | 0.201938 |
|             | 3     | 0.711536 | 0.546378 | 1.004696 | 0.793079 |
|             | 4     | 2.308666 | 2.240196 | 3.720932 | 2.800155 |
|             | 5     | 9.169572 | 8.766250 | 14.90928 | 12.14147 |
| Solve       | 1     | 0.007031 | 0.003116 | 0.009475 | 0.016028 |
|             | 2     | 0.004608 | 0.006127 | 0.015663 | 0.017132 |
|             | 3     | 0.021312 | 0.030866 | 0.060428 | 0.061735 |
|             | 4     | 0.079957 | 0.133192 | 0.252089 | 0.305041 |
|             | 5     | 0.385537 | 0.628217 | 1.610523 | 1.752295 |
| Total       | 1     | 0.080483 | 0.068825 | 0.100858 | 0.073188 |
|             | 2     | 0.167680 | 0.163636 | 0.268452 | 0.219071 |
|             | 3     | 0.732848 | 0.577244 | 1.065125 | 0.854814 |
|             | 4     | 2.388623 | 2.373388 | 3.973021 | 3.105196 |
|             | 5     | 9.555110 | 9.394467 | 16.51981 | 13.89377 |

**Lemma 8.1.** For any $K \in \mathcal{T}_h$, there holds

$$
\Delta_w(Q_h w) = \Delta w, \quad \forall w \in P_{k+2}(K).
$$

*Proof.* For any $w \in P_{k+2}(K)$, from the definitions of $Q_0$ and $Q_n$, we have $Q_0 w = w$ and $Q_n(\frac{\partial w}{\partial n}) = \frac{\partial w}{\partial n}$. Then, for any $K \in \mathcal{T}_h$ and $v \in P_{k+3}(K)$, from the definition (2.5) of the weak laplacian, it follows that

$$
(\Delta_w Q_h w, v)_K = - (\nabla Q_0 w, \nabla v)_K + (Q_n(\frac{\partial w}{\partial n}) \cdot n, v)_\partial K
$$

$$
= - (\nabla w, \nabla v)_K + (\frac{\partial w}{\partial n}, v)_{\partial K}
$$

$$
= (\Delta w, v)_K,
$$

which completes the proof. $\Box$

Let $P_h^{k+2} : L^2(\mathcal{T}_h) \to P_{k+2}(\mathcal{T}_h)$ be the element-wise defined $L^2$ orthogonal projection.

**Lemma 8.2.** Assume $\phi \in H^{\alpha+2}(\Omega)$ with $\alpha = 1, 2$. Then, there holds

$$
\|\Delta_w Q_h (\phi - P_h^{k+2} \phi)\|_{L^2(\mathcal{T}_h)} \leq C h^{\min\{k+1, \alpha\}} |\phi|_{H^{\alpha+2}(\Omega)}.
$$

*Proof.* For simplicity, denote by $w = \phi - P_h^{k+2} \phi$. It follows from (4.3) and the
Cauchy-Schwarz inequality that
\[
(\Delta_w Q_h w, v)_{T_h} = (\Delta Q_0 w, v)_{T_h} + \langle Q_n(\partial_w n) - \frac{\partial}{\partial n} (Q_0 w), v \rangle_{\partial T_h}
\]
\[
\leq \| \Delta Q_0 w \|_{L^2(T_h)} \| v \|_{L^2(T_h)} + \left( \sum_{K \in T_h} h_K^{-1} \| Q_n(\partial_w n) - \frac{\partial}{\partial n} (Q_0 w) \|_{L^2(T_h)}^2 \right)^{1/2} \left( \sum_{K \in T_h} h_K \| v \|_{H^2(K)}^2 \right)^{1/2}
\]
(8.3)
\[
\leq C(\| \Delta Q_0 w \|_{L^2(T_h)} + \left( \sum_{K \in T_h} h_K^{-1} \| Q_n(\partial_w n) - \frac{\partial}{\partial n} (Q_0 w) \|_{L^2(T_h)}^2 \right)^{1/2}) \| v \|_{L^2(T_h)},
\]
for any \( v \in P_{k+3}(T_h) \).

Letting \( v = \Delta_w Q_h w \) in (8.3), and then cancelling out \( \| \Delta w Q_h w \|_{L^2(T_h)} \) from both sides yields
\[
(\Delta_w Q_h w, v)_{T_h} \leq C(\| \Delta Q_0 w \|_{L^2(T_h)} + \left( \sum_{K \in T_h} h_K^{-1} \| Q_n(\partial_w n) - \frac{\partial}{\partial n} (Q_0 w) \|_{L^2(T_h)}^2 \right)^{1/2}) \| v \|_{L^2(T_h)}.
\]
(8.4)

Since the interpolant \( Q_0 \) preserves polynomials of degree up to \( k + 2 \), it is easy to know
\[
Q_0(\mathcal{P}_h^{k+2} \phi) = \mathcal{P}_h^{k+2} \phi.
\]
Then, by the triangle inequality, we have
\[
\| \Delta Q_0 w \|_{L^2(T_h)} = \| \Delta Q_0 (\phi - \mathcal{P}_h^{k+2} \phi) \|_{L^2(T_h)}
\]
\[
\leq \| \Delta (Q_0 \phi - \mathcal{P}_h^{k+2} \phi) \|_{L^2(T_h)} + \| \Delta (\phi - \mathcal{P}_h^{k+2} \phi) \|_{L^2(T_h)}
\]
(8.5)
\[
\leq C h^{\min(k+1, \alpha)} \| \phi \|_{H^{k+2}(\Omega)}.
\]
Since \( Q_n \) and \( Q_0 \) preserve the polynomials of order \( k + 1 \) and \( k + 2 \) respectively, there holds
\[
Q_n(\partial_w n) - \frac{\partial}{\partial n} (Q_0 w) = Q_n(\partial_w (\phi - \mathcal{P}_h^{k+2} \phi)) - \frac{\partial}{\partial n} (Q_0 (\phi - \mathcal{P}_h^{k+2} \phi))
\]
\[
= Q_n(\partial_\phi \phi) - \frac{\partial}{\partial n} (Q_0 \phi),
\]
which together with (5.2) of Lemma 5.1 leads to
\[
\sum_{K \in T_h} h_K^{-1} \| Q_n(\partial_w n) - \frac{\partial}{\partial n} (Q_0 w) \|_{L^2(K)}^2 = \sum_{K \in T_h} h_K^{-1} \| Q_n(\partial_\phi \phi) - \frac{\partial}{\partial n} (Q_0 \phi) \|_{L^2(K)}^2
\]
(8.6)
\[
\leq C h^{2 \min(k+1, \alpha)} \| \phi \|_{H^{k+2}(\Omega)}^2.
\]
Combining the estimates of (8.4), (8.5) and (8.6) completes the proof of (8.2). \( \square \)

Now, we are ready to give the proof of Lemma 6.1 below.

**Proof.** In view of (8.1) of Lemma 8.1, we have
\[
\Delta_w Q_h (\mathcal{P}_h^{k+2} \phi) = \Delta (\mathcal{P}_h^{k+2} \phi)
\]
on each element \( K \) of \( T_h \).
If $\alpha > k$, we have $|\Delta(P_{k+2}^{h}\phi)|_{H^{\alpha}(T_h)} = 0$ since $\Delta(P_{k+2}^{h}\phi) \in \mathcal{P}_k(T_h)$. Therefore, by the triangle inequality, we have

$$
|\Delta w(Q_h\phi)|_{H^{\alpha}(T_h)} = |\Delta w(Q_h(\phi - P_{k+2}^{h}\phi) + \Delta(P_{k+2}^{h}\phi))|_{H^{\alpha}(T_h)}
\leq |\Delta w(Q_h(\phi - P_{k+2}^{h}\phi))|_{H^{\alpha}(T_h)} + |\Delta(\Delta(P_{k+2}^{h}\phi))|_{H^{\alpha}(T_h)}
= |\Delta w(Q_h(\phi - P_{k+2}^{h}\phi))|_{H^{\alpha}(T_h)}.
$$

(8.7)

Then, from the inverse inequality, (8.7) and (8.2) of Lemma 8.2, it follows that

$$
|\Delta w(Q_h\phi)|_{H^{\alpha}(T_h)} \leq C h^{-\alpha} \|\Delta w(Q_h(\phi - P_{k+2}^{h}\phi))\|_{L^2(T_h)}
\leq C h^\min\{k+1-\alpha,0\} \|\phi\|_{H^{\alpha+2}(\Omega)}.
$$

If $\alpha \leq k$, from the triangle inequality, the inverse inequality and (8.2) of Lemma 8.2, we can infer that

$$
|\Delta w(Q_h\phi)|_{H^{\alpha}(T_h)}
\leq |\Delta w(Q_h(\phi - P_{k+2}^{h}\phi))|_{H^{\alpha}(T_h)} + |\Delta(\phi - P_{k+2}^{h}\phi)|_{H^{\alpha}(T_h)} + |\Delta\phi|_{H^{\alpha}(T_h)}
\leq C h^{-\alpha} \|\Delta w(Q_h(\phi - P_{k+2}^{h}\phi))\|_{L^2(T_h)} + C h^\min\{k+1-\alpha,0\} \|\phi\|_{H^{\alpha+2}(T_h)}
\leq C h^\min\{k+1-\alpha,0\} \|\phi\|_{H^{\alpha+2}(T_h)}.
$$

Therefore, in all cases, we have

$$
|\Delta w(Q_h\phi)|_{H^{\alpha}(T_h)} \leq C h^\min\{k+1-\alpha,0\} \|\phi\|_{H^{\alpha+2}(T_h)},
$$

as desired. \[ \square \]

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