Bifurcation and Chaos of Rotating Inextensional Ring

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Abstract. In this paper, the nonlinear dynamics of the inextensional ring structure is investigated. Based on the elasticity theory and the assumption of the inextension of ring, a dynamical model of the ring is established considering the elastic support, the external load and the damping. Then, according to the Galerkin method, the nonlinear mode functions of inextensional ring are used to obtain the order differential equations of ring. Finally, the numerical simulation is carried out and the nonlinear behavior, such as double-period bifurcation, quasi-periodic motion and chaos, is discussed. And the effect of the disturbance of the rotating velocity on the nonlinear behavior of ring is investigated.

1. Introduction
The ring structure is widely used in many engineering fields. Compared with other types of structures, the ring structure is a simple geometry, but can display most of the dynamic characteristics of more complex axisymmetric structures, which makes the rotating ring dynamic characteristics and stability are closely watched in engineering.

With the development of the dynamic theory related to the ring structure, the study of the vibration characteristics and dynamic stability of the elastic ring has been widely concerned [1-4]. Research on the dynamical behavior of inextensional rings has also entered a new phase. Huang and Soedel [5] studied the vibration characteristics of the rotating inextensional ring, deduced the general solution of the forced vibration of the ring, and discussed the influence of Coriolis acceleration on the ring vibration response. Evensen [6-7] obtained a nonlinear modal function of the vibration in the plane of the inextensional ring. Nataiavas [8-10] and Mohamed [11] studied the nonlinear dynamical behavior of the second-order mode and the third-order mode in the plane of the inextensional ring, respectively. Cho [12] studied the nonlinear dynamic response and related instability of a rotating thin ring under the condition of small angular velocity perturbation. Cooley and Parker [13] studied the effect of the discrete stiffness on the vibration of inextensional rotating rings, and further explored the vibration characteristic of inextensional rings rotating at high speed.

In this paper, firstly, the in-plane vibration model of an inextensional ring is established, in which the effects of pre-stress and gyro on the inextensional ring system are considered. Then the influence of the rotational speed on the natural frequency of the inextensional ring is analysed numerically. Finally, the nonlinear behavior changed with the rotating speed of the ring and the effect of the disturbed rotating speed on the nonlinear behavior of the inextensional ring are investigated.

2. Dynamical model of the inextensional ring
2.1. Energy principle description

It is assumed that the ring is a thin ring and the cross section of the ring is still flat after deformation. Therefore, the relationship between strain and displacement at any point in the plane of the ring can be written as follows

\[ \varepsilon = \frac{1}{2} (u_t'^2 + u_r^2) + \frac{w}{a} (u_t'^2 - u_t^2), \quad \varepsilon_r = \frac{\partial u_r}{\partial r} \]

(1)

where \( u_t \) and \( u_r \) are the radial displacement and the circumferential displacement at any point on the mid-plane, respectively. \( \varepsilon \) and \( \varepsilon_r \) correspond to the circumferential and the radial strains, respectively. The distance from any point on the ring to the center of the ring can be expressed as \( a \), and \( w \) is the distance between the center line of ring and the micro-element.

In this study, the expressions are simplified employing the inextensional assumption by Evenson\(^6\-7\) that the midsurface circumferential strain can be ignored

\[ \varepsilon_\theta = \frac{1}{R} (u_\theta' + u_r) + \frac{w}{R} (u_\theta' - u_\theta) \]

Where \( R \) is the radius of ring.

The expression of the Hamilton’s principle is

\[ \delta \int_{t_0}^{t_f} (U + V_F - V_L - T) dt + \delta \int_{t_0}^{t_f} W_C dt = 0 \]

(2)

where \( U \) and \( T \) are the strain potential energy and kinetic energy of the ring, respectively. It is assumed that the stiffness of the elastic substrate is continuously distributed along the ring, and \( V_F, V_L, W_C \) represent the potential energy stored under the elastic substrate, the external load potential energy and the damping potential energy, respectively.

Then all potential energy, kinetic energy and Rayleigh dissipation functions are expressed as

\[ U = \int \left[ \frac{1}{2} \sigma_\theta \varepsilon_\theta + \frac{1}{2} \tau_r \gamma_\theta + \sigma_r \varepsilon_r + \sigma_\theta (\varepsilon_\theta + \varepsilon_\theta') \right] dV \]

\[ = b \int_0^{2\pi} \int_{-h/2}^{h/2} \left[ \frac{1}{2} \sigma_\theta \varepsilon_\theta + \frac{1}{2} \tau_r \gamma_\theta + \sigma_r \varepsilon_r + \sigma_\theta (\varepsilon_\theta + \varepsilon_\theta') \right] dwd\theta \]

(3)

\[ V_F = \frac{1}{2} b R \int_0^{2\pi} (k_r \dot{u}_r + k_\theta \dot{u}_\theta) d\theta \]

(4)

\[ V_L = b R \int_0^{2\pi} (f_r \dot{u}_r + f_\theta \dot{u}_\theta) d\theta \]

(5)

\[ T = \frac{1}{2} \int_0^T \rho |\dot{\mathbf{v}}|^2 dV = \frac{pb}{2} \int_0^{2\pi} \int_{-h/2}^{h/2} (R + w) |\dot{\mathbf{v}}|^2 dwd\theta \]

(6)

\[ W_C = \frac{1}{2} b R \int_0^{2\pi} \left( c_r \dot{u}_r^2 + c_\theta \dot{u}_\theta^2 \right) d\theta \]

(7)

where \( b \) is the width of the ring, \( h \) is the radial thickness and \( \sigma_0 = \rho R^2 \Omega^2 \) represents the initial hoop stress, and \( \sigma_\theta, \sigma_r, \tau_r \) are the radial, circumferential and shear stress, respectively. The angular velocity is \( \dot{\omega} = \Omega(t) \dot{k} \) and \( (\cdot) \) is the derivative of time \( t \). \( k_r \) and \( k_\theta \) are the stiffnesses, where subscript \( r \) and \( \theta \) represent the directions in the radial and circumferential directions. \( f_r \) and \( f_\theta \) are the external loads, and \( c_r \) and \( c_\theta \) are dampings.

2.2. Motion equations

The motion equation of ring system is written as

\[ \rho \ddot{u}_r - 2\rho \dot{u}_r \dot{u}_\theta - R \ddot{u}_\theta - EI_4 u_r^{(4)} + (EI_4 - EI_5 + \sigma_0 I_0) u_r^" - \rho \dot{\omega} \dot{u}_r' \]

\[ + (EI_4 + \sigma_0 I_0 + R \dot{\omega} - \rho \dot{\omega}^2) u_r - \rho \ddot{u}_\theta' - R_c \dot{u}_\theta^2 - 2 \rho \dot{\omega} \dot{u}_\theta + (EI_4 + EI_5 + \sigma_0 I_0) u_\theta" \]

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where

\[ + (EI_4 + EI_5 + \sigma_4 I_0 - Rk_\rho + \rho R\Omega^2) u'_\rho - \rho \dot{\Omega} \rho h u_\rho + \sigma_\rho h - \rho R^2 h \Omega^2 - \frac{h^3}{12} \rho \dot{\Omega}^2 \]

\[ + Rf'_\rho - Rf_\rho = 0 \] (8)

\[ I_0 = \int_{-h/2}^{h/2} \frac{1}{R + w} dw = \ln \left( \frac{R + h/2}{R - h/2} \right), \quad I_3 = \int_{-h/2}^{h/2} \frac{w}{(R + w)^3} dw = I_0 - \frac{4hR}{4R^2 - h^2} \]

\[ \frac{1}{2} N_\rho = EI_4 (u'_\rho + u_\rho) + EI_5 (u'_\rho - u_\rho), \quad \frac{1}{2} N'_\rho = EI_4 (u''_\rho + u'_\rho) + EI_5 (u''_\rho - u'_\rho) \]

\[ \varphi_\theta = \varphi_\rho - \frac{1}{2}, \quad I_4 = \frac{I_0}{2}, \quad I_5 = \frac{I_3}{2} \]

2.3. Natural frequency

During the rotation of ring, using the radial displacement \( u_\rho \) and the circumferential \( u_\theta \), the unit displacement of any point on the center line of the ring section is

\[ e = \frac{u_\rho}{a} + \frac{1}{a} \frac{\partial u_\rho}{\partial \theta} \] (9)

For an inextensional ring (for pure bending vibration), the center line of the ring section has no stretch with \( e = 0 \), from the above expression, the following expression can be obtained

\[ u_\rho = -\frac{\partial u_\theta}{\partial \theta} \] (10)

Assuming that the radial displacement \( u_\rho \) and the circumferential \( u_\theta \) functions are periodic\(^{[12]}\), they can be expressed as

\[ u_\rho(\theta, t) = \sum_{n=-\infty}^{n=\infty} A_n e^{(\imath n \theta + \omega_n t)}, \quad u_\theta(\theta, t) = \sum_{n=-\infty}^{n=\infty} \frac{A_n}{n} e^{(\imath n \theta + \omega_n t)} \] (11)

where \( n = 0, \pm 1, \pm 2, \ldots \) and \( \omega_n \) represents the \( n \)-th natural frequency when the ring is stationary. The effect of the damping on the natural frequency of the stationary ring system is not considered. Then, the expression of \( \omega_n \) is written as

\[ \omega_n = \left[ \frac{R(k_\rho + k_\rho) + E I_5 (-n^4 + 2n - 1)}{2\rho R h} \right]^{1/2} \]

### Table 1. Geometric and physical parameters of the ring.

| Symbol | Explanation          | Value         | Unit |
|--------|----------------------|---------------|------|
| \( E \) | Young's modulus      | 206.84 \times 10^9 | Pa   |
| \( \rho \) | Mass density         | 7833.41       | Kg / m³ |
| \( R \) | Average radius       | 92.5          | mm   |
| \( h \) | Radial thickness     | 0.1016        | mm   |
| \( k_\rho \), \( k_\rho \) | Support stiffness  | 1 \times 10^3 | N / m |
| \( F \) | External force       | 1 \times 10^{-1} | N    |
| \( c_\rho \), \( c_\rho \) | Damping ratio       | 0.05          |      |
In Figure 1, the horizontal axis is the rotating velocity of the ring, and the vertical axis is the \( n \)-th natural frequency of the ring. And the variation of the natural frequency with the rotating velocity around Z axis of the system is plotted, in which the geometric and physical parameters of the ring system given in Table 1. Because the gyro coupling and the system stiffness are related to the rotating speed of the ring system, it can be concluded that the natural frequency of the rotating ring will be bifurcated.

![Figure 1. Variation of the natural frequency with the rotating velocity of the ring.](image)

Where the (a), (b), (c), (d) and (e) are \( n=2 \), \( n=3 \), \( n=5 \), \( n=8 \) and \( n=10 \) respectively.

3. Nonlinear dynamical behavior analysis

The modal functions satisfying the continuity condition and the periodic condition of the displacement of the inextensional ring \([8-11]\) are as follows

\[
u_n(\theta, t) = A_n(t) \cos(n \theta) + B_n(t) \sin(n \theta) - \frac{n \gamma}{4R} [A_n^2(t) + B_n^2(t)]
\]

\[
u_n(\theta, t) = \frac{1}{n} \left[ -A_n(t) \sin(n \theta) + B_n(t) \cos(n \theta) \right]
\]

\[
+ \frac{\gamma}{8R} \left[ A_n^2(t) - B_n^2(t) \right] \sin(2n \theta) - 2A_n(t)B_n(t) \cos(2n \theta)
\]

Where

\[
\gamma = n \left( 1 - \frac{1}{n^2} \right)^2, \quad n \geq 2
\]

The weight functions used to discretize the motion equations of the system are

\[
\frac{\partial u}{\partial A_n} = \cos(n \theta) - \frac{n \gamma}{2R} A_n(t), \quad \frac{\partial u}{\partial B_n} = \sin(n \theta) - \frac{n \gamma}{2R} B_n(t)
\]

The external loads are represented by the following expressions

\[
f_r = \frac{F_r}{R} \cos(n \theta) \cos(\Omega t), \quad f_\theta = \frac{F_\theta}{R} \cos(n \theta) \sin(\Omega t)
\]

The discretized dimensionless motion equations of ring can be written in matrix form as following
where $\tau = \Omega t$ is the dimensionless time, and the mass matrix, gyro matrix and stiffness matrix can be written as

$$
M = \begin{bmatrix}
2 + 2Gq_1^2 & 2Gq_1q_2 \\
2Gq_1q_2 & 2 + 2Gq_2^2
\end{bmatrix},
F = \left( f \cos(\tau) \atop nf \sin(\tau) \right)^T
$$

$$
Q = \left( q_1 \atop q_2 \right)^T,
$$

$$
N = \left[ \begin{array}{cc}
\eta_1 + 2\eta_2Gq_1^2 & 2\eta_2Gq_1q_2 \\
2\eta_2Gq_1q_2 & \eta_1 + 2\eta_2Gq_2^2
\end{array} \right] + \frac{2}{1+n^2} \begin{bmatrix}
0 & \frac{1+n^2}{n} \\
\frac{1+n^2}{n} & 0
\end{bmatrix}
$$

$$
K = \left[ \begin{array}{cc}
\xi_1 + (2H_2 + \xi_3H_1)G + (\xi_2 + \xi_4GH_1) & 1 \\
0 & \Omega^2 \frac{1+n^2}{n^2} \end{array} \right]
$$

In which

$$
m = \rho h R, \quad G = h^2 \left( \frac{\eta^2}{2R} \right), \quad f = \frac{F_0}{hm\Omega^2}
$$

$$
H_1 = q_1^2 + q_2^2, \quad H_2 = \dot{q}_1^2 + \dot{q}_2^2,
$$

$$
\eta_1 = \frac{R(c_2 + c_0)}{m\Omega}, \quad \eta_2 = \frac{Rc_2}{m\Omega},
$$

$$
\xi_1 = \frac{EI_4(-n^4 + 2n - 1) + R(k_0 + k_c)}{m\Omega^2}, \quad \xi_2 = \frac{h^2n\gamma - 24R^2}{12\Omega^2}
$$

$$
\xi_3 = \frac{EI_4 + Rk_c}{m\Omega^2}, \quad \xi_4 = \frac{RI_0 - h}{\Omega^2h}
$$

The above equation is the $n$-th mode equation of an inextensional ring with 2 degrees of freedom. This paper mainly analyses the dynamical behavior of the ring in the second-order mode, that is $n = 2$. The numerical method is used to solve the motion equation of the inextensional ring system, and then the nonlinear dynamical behavior of the system is analysed by plotting the bifurcation diagram, time history, phase figure and Poincare map diagram.

![Figure 2. Global bifurcation of the system responses $q_1$ and $q_2$.](image)

Figure 2 shows the global bifurcation of the responses $q_1$ and $q_2$, respectively. Obviously, the behaviors of $q_1$ and $q_2$ are same, then the behavior of $q_1$ is used as the example at the following discussion about the nonlinear behavior of the inextensional ring. From Figure 2, it can be seen, before rotating velocity $\Omega_2 = 12$ (rad/s), the main motion of the ring is the period-1 motion. But at the
neighborhood of the rotating velocity $\Omega = 7.5$ (rad/s) and $\Omega = 10.2$ (rad/s), there exist the quasi-periodic motions, and after the rotating velocity $\Omega = 12$ (rad/s), the response of the ring evolves the chaotic motion.

Figure 3 shows the time history diagram, phase diagram, and Poincare map diagram at rotating velocity $\Omega = 1$ (rad/s), and is represented by (a), (b), and (c), respectively. The following figures are also represented by the same letters. In the time history diagram, the system response shows a regular motion. The phase diagram shows an almost regular curve, and there is only one isolate point in the Poincare map. These phenomena indicate that the system at this time is a period-1 motion.

![Figure 3](image)

**Figure 3.** The behavior of the inextensional ring at the rotating speed $\Omega = 1$ (rad/s).

Figure 4 shows the behavior of ring at $\Lambda = 7.5$ (rad/s), the time history and the phase diagram are irregular, and there is a closed curve in the Poincare map diagram. These phenomena indicate that the quasi-periodic motion occurs.

![Figure 4](image)

**Figure 4.** The behavior of the inextensional ring at the rotating speed $\Omega = 7.5$ (rad/s).

Figure 5 shows the behavior of ring at $\Lambda = 10.2$ (rad/s), the time history and the phase diagram are irregular, and there is a closed curve in the Poincare map diagram. These phenomena indicate that the quasi-periodic motion occurs.

![Figure 5](image)

**Figure 5.** The behavior of the inextensional ring at the rotating speed $\Omega = 10.2$ (rad/s).
From Figure 5, although the time history diagram of the system looks confusing, after careful observation, the system response can be observed as periodic changes. In the Poincare map, a large number of points form a closed curve. Therefore, the system at this time is also in the almost periodic motion state.

Figure 6 shows the dynamical behavior at the rotating velocity 14.5 (rad/s), it can be observed from the time history diagram that the system response is irregular, and in the phase diagram the trajectory will fill a certain part of the space. In the Poincare map diagram, a large number of discrete points are distributed disorderly in the Poincare section. Obviously, the system at this time is in a chaotic motion state.

![Figure 6](image)

**Figure 6.** The behavior of the inextensional ring at the rotating speed $\Omega = 14.5$ (rad/s).

From Figure 7, it can be seen that the time history diagram and phase figure don’t show periodicity, and the number of Poincare points are cloudlike, from which it can be concluded that the system is also in a chaotic motion state.

![Figure 7](image)

**Figure 7.** The behavior of the inextensional ring at higher rotating speed

In summary, at low speeds, the system response of the inextensional ring is presented as a simple period-1 motion, with the rotational speed increasing, the complex motion states such as quasi-periodic motion occur, and when the rotating velocity is more than 12 (rad/s), the response of the ring evolve chaotic motion.

4. Effect of the disturbed rotating velocity

Due to the gyroscope effect, the disturbance of the rotating velocity can influence the ring’s response significantly. It is necessary to discuss the effect of disturbance of the rotating velocity on the nonlinear behavior of the inextensional ring. Assuming that the rotating velocity after the disturbance is $\Omega$, where

$$\Omega = \Omega_0 + \mu \cos(\eta t)$$

(17)

where $\Omega_0$ is the rotating velocity without disturbance, $\mu = 0.01 \Omega_0$ represents the disturbance amplitude, and $\eta = 1000$ represents the disturbance frequency. Figure 8 shows the time history of the response $q_1$ at different rotating velocity with disturbance.
Comparing the behavior of the disturbed ring (Figure 8) with these without disturbance (Figure 3-7), it can be observed that the disturbance of the rotating velocity can make the response of the ring evolve to chaos when the responses of the ring without disturbance are under period-1 or quasi-periodic motions, and the amplitude of the disturbed response increases significantly. However, the change of the response of the disturbed ring is not obvious when the ring without disturbance is chaotic motion. The rotating speed of (a), (b), (c), (d), (e) and (f) are 1 (rad/s), 7.5 (rad/s), 8.5 (rad/s), 10.2 (rad/s), 14.5 (rad/s) and 20 (rad/s) respectively.

![Figure 8. System response $q_1$ at the different rotating velocities with disturbance.](image)

In summary, the disturbance of the rotating velocity can affect the nonlinear behavior of the inextensional ring under steady and lead the ring to unsteady. It is significant to note the effect of the disturbance of the rotating velocity.

5. Conclusion
In this paper, the rotating inextensional ring of is taken as the research object, and its nonlinear vibration dynamics are focused on. This paper has the following conclusions:

1. According to the Hamilton principle, the nonlinear dynamic model of the in-plane motion of the inextensional ring is established. The effects of the pre-stress, gyro effect, elastic support and damping are considered in the modeling process. The nonlinear single-mode displacement functions satisfying the periodicity and continuity conditions are further selected. Then the Galerkin method is used to discretize the motion equation.

2. The numerical simulation is used to investigate the nonlinear dynamics of the ring system in the second-order mode when the rotational speed is in the range of 0.1~15 (rad/s). The results show that the system response state has experienced a period-1 motion (0.1~7.5 (rad/s)) – quasi-periodic motion (around 7.5 (rad/s)) – period-1 motion (7.5~10 (rad/s)) – quasi-periodic motion (around 10 (rad/s)) – period-1 motion (10~12 (rad/s)) – chaos (more than 12 (rad/s)). The results show that at lower speeds, the ring system is in a steady state; at higher speeds, the ring system exhibits the complex motion state, such as chaos.
3. The speed disturbance has a very significant effect on the motion of the steady state of the inextensional ring. It can cause the response change to chaotic motion and increase the amplitude of response.

The conclusion in this paper can provide theoretical help for future research about the rotating ring and the inextensional structure.

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