A simple method for multi-leg loop calculations 2: a general algorithm

R. Pittau
Theoretical Physics Division, CERN
CH-1211 Geneva 23, Switzerland

Abstract

The method introduced in a previous paper to simplify the tensorial reduction in multi-leg loop calculations is extended to generic one-loop integrals, with arbitrary internal masses and external momenta.
1 Introduction

In a previous paper [1], a technique was presented to simplify the tensorial reduction of \( m \)-point one-loop diagrams of the type

\[
\mathcal{M}(p_1, \ldots, p_r; k_1, \ldots, k_{m-1}) = \sum_a \int d^n q \frac{\text{Tr}^{(a)}[\hat{q} \cdots \hat{q} \cdots]}{D_1 \cdots D_m},
\]

where \( p_{1 \ldots r} \) are the external momenta of the diagram, \( k_{1 \ldots m-1} \) the momenta in the loop denominators, defined as

\[
D_i = (q + s_{i-1})^2 - m_i^2, \quad s_i = \sum_{j=0}^i k_j \quad (k_0 = 0),
\]

and \( \text{Tr}^{(a)} \) traces over \( \gamma \) matrices, which may contain an arbitrary number of \( \hat{q} \)'s.

It was shown that, by assuming at least two massless momenta in the set \( k_{1 \ldots m-1} \), the traces in eq. (1) can be rewritten in terms of the denominators appearing in the diagram, therefore simplifying the calculation.

Starting from \( m \)-point rank-\( l \) tensor integrals, the algorithm gave at most rank-1 \( m \)-point functions, plus \( n \)-point rank-\( p \) tensor integrals with \( n < m \) and \( p < l \).

In this paper, I show how to extend this technique when the momenta \( k_{1 \ldots m-1} \) are generic. On the one hand, this allows to apply the method to more general problems. On the other hand, the reduction procedure can therefore be iterated in such a way that, usually, only rank-1 integrals and scalar functions remain at the end.

In the next section, I introduce the algorithm and in section 3, I apply it to a specific example.

2 The general algorithm

The basic idea is simple. Given two vectors \( \ell_1 \) and \( \ell_2 \), one can ‘extract’ the \( q \) dependence from the traces with the help of the identity

\[
\hat{q} = \frac{1}{2(\ell_1 \cdot \ell_2)} [2(q \cdot \ell_2) \ell_1 + 2(q \cdot \ell_1) \ell_2 - \ell_1 \hat{q} \ell_2 - \ell_2 \hat{q} \ell_1].
\]

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By further assuming $\ell_1^2 = \ell_2^2 = 0$, and making use of the completeness relations for massless spinors, the following result is obtained

$$\text{Tr}[q \Gamma] = \frac{1}{2(\ell_1 \cdot \ell_2)} [2(q \cdot \ell_2) \text{Tr}[\ell_1 \Gamma]$$

$$- \{q\}_{12}^+ \{\Gamma\}_{21}^+ - \{q\}_{12}^- \{\Gamma\}_{21}^- + (\ell_1 \leftrightarrow \ell_2) \big] ,$$

(4)

where $\Gamma$ represents a generic string of $\gamma$ matrices and

$$\{\ell_1 \ell_2 \cdots \ell_n\}_{i, j}^+ \equiv \{12 \cdots n\}_{i, j}^+ \equiv \bar{v}_+ (\ell_i) \ell_1 \ell_2 \cdots \ell_n u_-(\ell_j) .$$

(5)

By iteratively applying the above procedure, together with the equations

$$\{q\}_{12}^- \{q\}_{21}^+ = 4(q \cdot \ell_1)(q \cdot \ell_2) - 2q^2 (\ell_1 \cdot \ell_2)$$

$$\{q\}_{12}^- \{q\}_{12}^+ = \frac{2}{\{b\}_{12}^+} \left[ [q^2 (\ell_1 \cdot \ell_2) - 2(q \cdot \ell_1)(q \cdot \ell_2)] \{b\}_{12}^+ \right.$$  

$$+ 2((q \cdot \ell_1)(b \cdot \ell_2) - (q \cdot b)(\ell_1 \cdot \ell_2) + (q \cdot \ell_2)(\ell_1 \cdot b)] \{q\}_{12}^- \big],$$

(6)

only one $\{q\}_{12}^+$ (or its complex conjugate $\{q\}_{21}^+$) survives in each term, and powers of $q^2$, $(q \cdot \ell_1)$, $(q \cdot \ell_2)$ and $(q \cdot b)$ factorize out.

The next step is to reconstruct the denominators from the above scalar products. By choosing, for example, $b = k_3$ one trivially gets

$$q^2 = D_1 + m_1^2,$$

$$2(q \cdot b) = D_4 - D_3 + m_4^2 - m_3^2 - (k_1 + k_2 + k_3)^2 + (k_1 + k_2)^2 ,$$

(7)

but $(q \cdot \ell_1)$ and $(q \cdot \ell_2)$ still remain.

In ref. the simple case was studied in which the diagram in eq. is such that at least two $k$'s (say $k_1$ and $k_2$) are massless. A solution to the problem is then to take $\ell_1 = k_1$ and $\ell_2 = k_2$:

$$2(q \cdot \ell_1) = D_2 - D_1 + m_2^2 - m_1^2 ,$$

$$2(q \cdot \ell_2) = D_3 - D_2 + m_3^2 - m_2^2 - (k_1 + k_2)^2 .$$

(8)

If, in the set $k_1 \cdots m-1$, only one momentum (say $k_1 \equiv \ell_1$) is massless, a solution can still be found by decomposing any other massive momentum (say $k_2$) in terms of massless vectors:

$$k_2 = \ell_2 + \alpha \ell_1 .$$

(9)
The requirement that also $\ell_2$ is massless, implies
\[ \alpha = \frac{k_2^2}{2(k_1 \cdot k_2)}, \] (10)
and therefore
\[ 2(q \cdot \ell_1) = D_2 - D_1 + m_2^2 - m_1^2, \] (11)
\[ 2(q \cdot \ell_2) = D_3 - (1 + \alpha)(D_2 + m_2^2) + \alpha(D_1 + m_1^2) + m_3^2 - (k_1 + k_2)^2. \]
When there are no massless $k$'s, a basis of massless vectors can yet be constructed:
\[ k_1 = \ell_1 + \alpha_1 \ell_2, \quad k_2 = \ell_2 + \alpha_2 \ell_1. \] (12)
In fact, requiring $\ell_1^2 = \ell_2^2 = 0$ gives
\[ \alpha_1 = \frac{(k_1 \cdot k_2) \pm \sqrt{\Delta}}{k_2^2}, \quad \alpha_2 = \frac{(k_1 \cdot k_2) \pm \sqrt{\Delta}}{k_1^2}, \]
\[ \ell_1 = \beta(k_1 - \alpha_1 k_2), \quad \ell_2 = \beta(k_2 - \alpha_2 k_1), \]
\[ \Delta = (k_1 \cdot k_2)^2 - k_1^2 k_2^2, \quad \beta = \frac{1}{1 - \alpha_1 \alpha_2}, \] (13)
from which one computes
\[ \frac{2(q \cdot \ell_1)}{\beta} = (1 + \alpha_1)(D_2 - k_1^2 + m_2^2) - (D_1 + m_1^2) \]
\[ - \alpha_1[D_3 + m_3^2 - (k_1 + k_2)^2], \]
\[ \frac{2(q \cdot \ell_2)}{\beta} = D_3 + \alpha_2(D_1 + m_1^2) - (1 + \alpha_2)(D_2 - k_1^2 + m_2^2) \]
\[ + m_3^2 - (k_1 + k_2)^2. \] (14)
When the loop integrals have to be evaluated in $n$ dimensions, the substitution $q \to \tilde{q} = q + \tilde{q}$ is needed [1, 2], where $q$ lives in 4 dimensions and $\tilde{q}$ is the $(n - 4)$-dimensional part of the integration momentum, such that $(q \cdot \tilde{q}) = 0$. The only change in the previous formulas is that
\[ q^2 = D_1 - \tilde{q}^2 + m_1^2, \] (15)
and the additional integrals, involving powers of $\tilde{q}^2$, can be easily handled as shown in ref. [1, 3].

Therefore, the described procedure completely solves the problem, for arbitrary $k$'s appearing in the denominators of $n$-dimensional one-loop diagrams.

If, in the original trace, the number $n_q$ of $\bar{q}$'s is less than the number $m$ of loop denominators, the algorithm can be iterated until rank-1 functions remain, at most. If $n_q \geq m$, owing to the lack of momenta $k$'s to perform the denominator reconstruction, residual rank-$p$ two-point integrals remain instead, with $p \leq (2 + n_q - m)$. However, two-point tensors are much easier to handle than generic $m$-point tensors, so that the diagram is anyhow simplified.

A last remark is in order. When some $k$'s become collinear, one is faced with the usual problem of singularities generated by the tensor reduction (for an exhaustive study of this topic, see ref. [4]). In fact, denominators appear in eqs. (4) and (6), which may vanish, and the quantity $\Delta$ in eq. (13) is nothing but a Gram determinant. Even if the occurrence of such singularities cannot be completely avoided, a better control on them is in general possible [1], with respect to traditional techniques [5]. In addition, the analytic expressions can be kept rather compact, avoiding, at the same time, the appearance of large-rank tensors.

### 3 An example

To illustrate the method, I compute the reduction for the following integral with $n_q = 2$:

$$I = \int d^n q \frac{1}{D_1 \cdots D_m} \text{Tr}[\bar{q} \Gamma q \Lambda], \quad (16)$$

where, to fix the ideas, $\Gamma$ and $\Lambda$ are strings containing an odd number of four-dimensional $\gamma$ matrices. For convenience of notation, I omit to write the slashes in the traces.

Since the integration is performed in $n$ dimensions, the denominators are given by eq. (4) with the substitution $q \to \tilde{q} = q + \tilde{q}$.

When $m \geq 3$, the algorithm reduces $\tilde{I}$ to a sum of scalar and rank-1 tensors.
integrals. In fact, by splitting $q$ in the numerator, one gets

$$\text{Tr}[q \Gamma q \Lambda] = \text{Tr}[q \Gamma q \Lambda] - \tilde{q}^2 \text{Tr}[\Gamma \Lambda],$$

and, by applying the formulas in the previous section,

$$\text{Tr}[q \Gamma q \Lambda] = \frac{1}{2(\ell_1 \cdot \ell_2)} \left[ 2(q \cdot \ell_1)E(\ell_2) + 2(q \cdot \ell_2)E(\ell_1) - q^2 A - 2(q \cdot k_3) G \right],$$

$$A = 2 \text{Re} \left[ \{\Lambda\}^{+\dagger}_1 \{\Gamma\}^{+\dagger}_2 + \{\Lambda\}^{-\dagger}_2 \{\Gamma\}^{-\dagger}_1 - C \{k_3\}^{+\dagger}_1 \right],$$

$$G = 2 \text{Re} \left[ C \{q\}^{+\dagger}_2 \right],$$

$$E(\ell) = \text{Tr}[\ell \Gamma q \Lambda] - \frac{1}{2(\ell_1 \cdot \ell_2)} \left\{ \text{Tr}[\ell_2 q \ell_1 \Gamma \ell \Lambda] + \text{Tr}[\ell_1 q \ell_2 \Gamma \ell \Lambda] - 2(k_3 \cdot \ell) G - (q \cdot \ell) A \right\}. $$

The above equations give the final answer:

$$I = \frac{1}{2(\ell_1 \cdot \ell_2)} \int d^n q \frac{1}{D_1 \ldots D_m} \left\{ (D_1 + m_1^2) \left[ E(\beta \alpha_2 \ell_1 - \beta \ell_2) - A \right] + (D_2 + m_2^2 - k_1^2) E(\beta \alpha_2 \ell_2 + \beta \alpha_1 \ell_2 - \beta \ell_1 - \beta \alpha_1 \ell_1) + (D_3 + m_3^2 - (k_1 + k_2)^2) \left[ E(\beta \ell_1 - \beta \alpha_1 \ell_2) + G \right] - (D_4 + m_4^2 - (k_1 + k_2 + k_3)^2) G + \tilde{q}^2 (A - 2(\ell_1 \cdot \ell_2) \text{Tr}[\Gamma \Lambda]) \right\},$$

$$\beta = \frac{1}{1 - \alpha_1 \alpha_2}. \quad (19)$$

When $k_{1,2}^2 \neq 0$, $\ell_{1,2}$ and $\alpha_{1,2}$ are as in eq. (13).

If $k_1^2 = 0$ and $k_2^2 \neq 0$, eq. (13) still holds with $\alpha_1 = 0$, $\ell_1 = k_1$ and $\ell_2 = k_2 - \alpha_2 k_1$, where $\alpha_2 = \alpha$ is given in eq. (10).

If $k_{1,2}^2 = 0$, then $\alpha_{1,2} = 0$ and $\ell_{1,2} = k_{1,2}$.

When $m = 3$, some terms vanish. This implies $C = G = 0$ and

$$E(\ell) = \text{Tr}[\ell \Gamma q \Lambda] + A \frac{(q \cdot \ell)}{2(\ell_1 \cdot \ell_2)}. \quad (20)$$
4 Summary

In this paper, I extended the technique introduced in ref. [1] to reduce the tensorial complexity of the diagrams appearing in multi-leg loop calculations.

The method is now applicable to generic one-loop integrals, with arbitrary internal masses and external momenta.

The algorithm can usually be iterated in such a way that only scalar and rank-1 functions appear at the end of the reduction. At worst, higher-rank two-point tensors survive, independently from the initial number of denominators.

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