COASSEMBLY AND THE $K$-THEORY OF FINITE GROUPS

CARY MALKIEWICH

Abstract. We study the $K$-theory and Swan theory of the group ring $R[G]$, when $G$ is a finite group and $R$ is any discrete ring or ring spectrum. In this setting, the well-known assembly map for $K(R[G])$ has a lesser-known companion called the coassembly map. We prove that their composite is the equivariant norm of $K(R)$. As a result, we get a splitting of both assembly and coassembly after $K(n)$-localization, and an apparently new map from the Whitehead torsion group of $G$ over $R$ to the Tate cohomology of $BG$ with coefficients in $K(R)$.

Contents

1. Introduction. 1
2. Applications and open questions. 5
3. Waldhausen categories and $K$-theory. 7
4. Waldhausen categories of $G$-spaces, $G$-spectra, fiberwise spaces, and fiberwise spectra. 14
5. Assembly and coassembly of $R$-modules. 32
6. A combinatorial lift of assembly and coassembly. 34
7. Proof that the lift is the norm. 40
References 48

1. Introduction.

Algebraic $K$-theory provides a deep set of invariants for each ring $R$, in the form of a sequence of abelian groups $K_i(R)$. In topology, these groups provide obstructions for classical problems such as recognizing finite CW complexes, classifying finite group actions on spheres, and trivializing smooth cobordisms.

For many applications, the most important computation is how the $K$-theory of a group ring $R[G]$ is related to the $K$-theory of $R$. They are connected by the assembly map

$$H_*(BG; K(R)) \to K_*(R[G])$$

We think of the left-hand side of (1) as very computable when compared to the right-hand side. So we may construct classes in $K_*(R[G])$, by building them first
in $H_i(BG; K(R))$. However it is difficult to tell whether the classes built this way are actually nonzero. We therefore ask

**Question 1.1.** Is the assembly map injective, or even an isomorphism?

This question has been studied extensively in many contexts. The *K-theoretic Novikov conjecture* states that $\text{1}$ is rationally split injective when $R = \mathbb{Z}$, $G$ is discrete, and $BG$ has finite type. This was proven by Bokstedt, Hsiang, and Madsen in [BHM93]. The *Farrell-Jones conjecture* states that a slightly different nonconnective variant of $\text{1}$ is an isomorphism when $R$ is regular and $G$ is torsionfree. Farrell and Jones proved in [FJ93] that this map is a rational isomorphism when $G$ is virtually polycyclic, or a discrete cocompact subgroup of a Lie group with finitely many path components. There are variants of these conjectures for $L$-theory, which imply the Novikov conjecture and Borel conjecture, respectively.

The integral version of the Farrell-Jones conjecture has been quite difficult. Many results place additional restrictions on $R$ and $G$, and even then only get injectivity, which is sometimes called the *Integral Novikov conjecture*. Carlsson and Pedersen proved injectivity for a large class of groups $G$ with finite $BG$, including the word hyperbolic groups [CP95]. The theorem of [BR05], building on [BFJR04], proved integral isomorphism for any ring $R$, when $G$ is the fundamental group of a Riemannian manifold of negative sectional curvature. Recently Carlsson and Goldfarb proved integral isomorphism when $R$ is regular Noetherian and $G$ has finite asymptotic dimension [CG13]. There are quite a few more results by several different authors, and our brief summary does not do them justice. A comprehensive survey can be found in [LR05].

In light of this earlier work, we know the most about assembly when $G$ is an infinite discrete group, and $BG$ is finite in some sense. On the other hand, we know very little about the case of $G$ finite, other than injectivity in low degrees [LR05]. In this case, we present a theorem that gives a certain kind of splitting of the assembly map.

To describe our result we first re-interpret assembly as a map of spectra

$$BG_+ \wedge K(R) \longrightarrow K(R[G])$$

We recall that $K(R[G])$ has a close cousin called the Swan theory, or simply $G$-theory, $G^R(R[G])$ [ES]. The Swan theory studies modules over $R[G]$ which are finite over $R$, instead of those that are finite over $R[G]$; in other words it is the $K$-theory of the representations of $G$ in the category of $R$-modules.

Of course, we can define $K$-theory and Swan theory for each ring spectrum $R$ and topological group $G$. When $R$ is the sphere spectrum, this yields Waldhausen’s $A$-theory $A(BG)$, and a contravariant version called $V(BG)$. This kind of $K$-theory is well-studied, but the corresponding Swan theory is poorly understood. The Swan
theory of $R[G]$ studies the representations of $G$ in the category of $R$-module spectra, so it might be called spectral representation theory.

Now we can state the main theorem.

**Theorem 1.2.** If $G$ is a finite group and $R$ is a ring or ring spectrum, the assembly map of $K(R[G])$ can be composed with two more maps called Cartan and coassembly:

$$BG_+ \wedge K(R) \xrightarrow{\text{assembly}} K(R[G]) \xrightarrow{\text{Cartan}} G^R(R[G]) \xrightarrow{\text{coassembly}} F(BG_+, K(R))$$

The composite of these is the equivariant norm map

$$K(R)_{hG} \rightarrow K(R)^{hG}$$

on $K(R)$ with the trivial $G$-action.

This theorem has some surprising corollaries which we summarize in the next section. Two of the most intriguing ones follow.

**Corollary 1.3.** Let $G$ be a finite group and $R$ any ring or ring spectrum. Then the assembly map

$$BG_+ \wedge K(R) \xrightarrow{\alpha} K(R[G])$$

is split injective after $K(n)$-localization.

**Corollary 1.4.** There is a natural map from the Whitehead group to Tate cohomology

$$Wh(G) \rightarrow \pi_1(A(*)^G) \rightarrow \pi_1(K(Z)^G)$$

We finish by summarizing the technical results that may be of independent interest. It is well-known that one should be able to build Waldhausen's algebraic $K$-theory of spaces out of a suitable category of parametrized spectra, instead of spaces, because $K$-theory is preserved under stabilization. Indeed, such a category of spectra is absolutely necessary if we are to define assembly and coassembly maps for $K(R[G])$ by a universal property.

It seems that there does not yet exist a category of parametrized spectra (or even of parametrized spaces) that is robust enough to allow for a definition of $A(B)$, $V(B)$, and the Cartan map between them. The model category of [MS06] does not work, since a pullback of a cofibration is not a cofibration. We remedy the situation by constructing some Waldhausen categories of parametrized spectra which are both geometrically and homotopically well-behaved. Using these models, we prove:

**Proposition 1.5.** For each ring spectrum $R$ there is a covariant homotopy functor $A(B; R)$ from unbased spaces to spectra, receiving a natural transformation from Waldhausen's functor $A(B)$. The functor $A(B; R)$ sends disjoint unions to products, and when $G$ is a topological group, $A(BG; R)$ is equivalent to $K(R[G])$. Finally, given two rings $R$ and $S$ there is a pairing

$$A(B; R) \wedge A(B'; S) \rightarrow A(B \times B'; R \wedge S)$$
which has the obvious associativity property when given three spaces and three rings.

**Proposition 1.6.** For each ring spectrum $R$ there is a contravariant homotopy functor $V(B; R)$ from unbased spaces to spectra, receiving a natural transformation from contravariant Waldhausen $K$-theory $V(B)$. The functor $V(B; R)$ sends disjoint unions to products, and when $G$ is a topological group there is an equivalence $A(BG; R) \simeq G^R(R[G])$. There is a pairing

$$V(B; R) \wedge V(B'; S) \to V(B \times B'; R \wedge S)$$

with the same associativity properties as before. Finally, if $G$ is a finitely dominated topological group, there is a Cartan map

$$A(BG; R) \to V(BG; R)$$

agreeing with the Cartan map of [ES] in the case of $G$ finite and $R$ a discrete ring.

This lets us interpret $K(R[G])$ and $G^R(R[G])$ as functors on unbased spaces, so that we may define the assembly and coassembly maps

$$
\begin{align*}
BG_+ \wedge K(R) & \longrightarrow A(BG; R) \\
V(BG; R) & \longrightarrow \text{Map}_+(BG_+, K(R))
\end{align*}
$$

as universal approximations by linear functors. We give an explicit proof that this assembly map agrees with the classic $K$-theory assembly map, and give a similar combinatorial formula for coassembly. Our proof of Thm 1.2 uses the combinatorial formulas for these maps, but the universal property will be important for connecting our theorem to applications.

The outline of the paper is as follows. In section 2 we discuss in detail the corollaries of our main theorem. In section 3 we begin the technical work, reviewing Waldhausen $K$-theory and giving a modern adaptation of Waldhausen’s observation that the spectrum $K(C)$ can be constructed by a classical delooping machine. In section 4 we construct the Waldhausen categories of spaces and spectra that give the functors $A(B; R)$ and $V(B; R)$. In section 5 we review the universal properties and constructions of assembly and coassembly. In section 6 we modify these maps by a homotopy to explicit simplicial maps, which lift to the $K$-theory of finite sets. In section 7 we recognize the composite of these maps as the norm, using the $E_\infty$ structure on $\Omega^\infty K(C)$. This proves the main theorem.

The author is grateful to acknowledge Mark Behrens, Andrew Blumberg, Ralph Cohen, John Greenlees, Jesper Grodal, John Klein, Randy McCarthy, Mona Merling, and Bruce Williams for their help with and enlightening conversations about this project. The $K$-theoretic results in this paper are motivated by $THH$-theoretic results in the author’s thesis, which was written under the direction of Ralph Cohen at Stanford University.
2. Applications and open questions.

In this section we present a bit more background on the applications of our main theorem 1.2.

Here is one application. Recall that the Whitehead group \( Wh(G) \) of a discrete group \( G \) is the cokernel of the inclusion

\[
G^{ab} \oplus \mathbb{Z}/2 \cong H_1(BG) \oplus K_1(\mathbb{Z}) \cong \pi_1(BG_+ \wedge K(\mathbb{Z})) \rightarrow K_1(\mathbb{Z}[G])
\]

This may be identified with \( \pi_0 \) of the homotopy fiber of the assembly map of \( K(\mathbb{Z}[G]) \). Following Waldhausen, we call that homotopy fiber the PL Whitehead spectrum for the group \( G \) with coefficients in \( \mathbb{Z} \). This has an obvious extension to other rings \( R \), and we get the same Whitehead group if we take \( S \) instead of \( \mathbb{Z} \):

\[
Wh(G) \cong K_1(S[G])/H_1(BG_+; K(S)) \cong A_1(BG)/H_1(BG_+; A(\ast))
\]

The Whitehead group is used in the statement of the celebrated \( s \)-cobordism theorem, which gives necessary and sufficient conditions for a given \( h \)-cobordism from \( M \) to \( N \) to be diffeomorphic to a trivial diffeomorphism, yielding in particular a diffeomorphism \( M \cong N \). In fact, the complete obstruction is simply an element of \( Wh(\pi_1(M)) \). So when this Whitehead group vanishes, every \( h \)-cobordism is trivial, and to construct a diffeomorphism it suffices to construct an \( h \)-cobordism. When \( G \) is an infinite torsionfree group, its Whitehead group is zero, but for finite groups \( Wh(G) \) has rank equal to the number of real irreducible representations of \( G \) minus the number of rational ones. It is known that \( Wh(\mathbb{Z}/p) \cong \mathbb{Z}_{p^{\frac{p-3}{2}}} \) when \( p \) is an odd prime (Luck Rmk 4).

Our theorem provides a link between these Whitehead groups and Tate cohomology. We can form the map of cofiber sequences

\[
BG_+ \wedge A(\ast) \rightarrow A(BG) \rightarrow \Sigma Wh^{PL}(BG)
\]

\[
A(\ast)^{hG} \rightarrow A(\ast)^{hG} \rightarrow A(\ast)^{tG}
\]

where \( tG \) denotes Tate cohomology with coefficients in a spectrum. The homotopy groups of \( K(R)^{tG} \) are not the classical Tate cohomology groups of \( G \) with coefficients in \( K_+^*(R) \), but there is a spectral sequence connecting the two:

\[
E_2 = \bar{H}^*(G; K_+^*(R)) \Rightarrow \pi_*(K(R)^{tG})
\]

Our diagram above gives

**Corollary 2.1.** There are maps which are natural in \( G \)

\[
Wh(G) \rightarrow \pi_1(A(\ast)^{tG}) \rightarrow \pi_1(K(\mathbb{Z})^{tG})
\]

and there are similar maps with coefficients in any ring spectrum \( R \).
These maps appear to be new, but the author has recently learned of more geometric descriptions of Tate cohomology that may allow for a more classical interpretation.

Here is a second application of our theorem. Recall that for each prime $p$, there is a sequence of cohomology theories $K(n)$ ($n \geq 0$) that capture the “pieces” of the stable homotopy category lying between rational and $p$-local stable homotopy theory. These are the Morava $K$-theories; intuitively they separate out pieces of stable homotopy that occur at different frequencies. One often tries to understand the stable homotopy of a spectrum $X$, or really its $p$-localization, by building up knowledge of its localizations $X_{K(n)}$. In $K$-theory the story is even cleaner: it is often the case that for a discrete ring $R$, the completion $K(R)^h$ is a connective cover of its $K(1)$-localization $L_{K(1)}K(R)$, so the $K(1)$-localization actually captures all of the important information.

Once nice thing about the $K(n)$-local category is that the equivariant norm is always an equivalence. More precisely, for any spectrum $X$ with a $G$-action, we get the following diagram:

$$
\begin{array}{c}
(X_{hG})_{K(n)} \longrightarrow \sim (X_{K(n)})^{hG} \\
\downarrow \sim \\
((X_{hG})_{K(n)})_{K(n)} \longrightarrow ((X_{K(n)})^{hG})_{K(n)} \\
\end{array}
$$

( [HS99] 8.7) So the norm becomes an isomorphism in the stable category

$$
(X_{hG})_{K(n)} \sim (X_{K(n)})^{hG}
$$

Therefore we can draw the following corollaries.

**Corollary 2.2.** Let $G$ be a finite group and $R$ any ring or ring spectrum. Then the assembly map is split injective after $K(n)$-localization on the outside

$$
(BG_+ \wedge K(R))_{K(n)} \xrightarrow{\alpha} K(R[G])_{K(n)}
$$

and the coassembly map is split surjective after $K(n)$-localization on the inside

$$
G(R[G])_{K(n)} \xrightarrow{\text{cos}} \text{Map}_*(BG_+, K(R)_{K(n)})
$$

Taking in particular $R = \mathbb{S}$ and focusing on the assembly map,

**Corollary 2.3.** If $G$ is a finite group, the assembly map

$$
BG_+ \wedge A(*) \longrightarrow A(BG)
$$

is split injective after $K(n)$-localization.

To reiterate, what is striking about the first result is that in many cases relevant to number theory the $K(1)$-localization completely captures the homotopy type of $K(R)$ or $K(R[G])$. 
In the case of $A(X)$, however, one should perhaps expect to need all of the $K(n)$-localizations, fitting them together to get a statement about $A(X)$ localized at $p$. In general, each spectrum $Y$ sits at the top of a tower of $E(n)$-localizations $L_{E(n)}Y$, and the $n$th layer of this tower is detected by the $K(n)$-localization of $Y$. A spectrum $Y$ satisfies chromatic convergence if this tower converges. For instance, it is known that suspension spectra of finite complexes have chromatic convergence [Rav92]. As a consequence, if a given map between two convergent spectra is an equivalence after $K(n)$-localization, then it is a $p$-local equivalence. If this is true for every prime $p$ then the map is an ordinary equivalence.

In particular, if we want to deduce something about assembly map
\[ BG_+ \wedge A(*) \to A(BG) \]
we need to answer the following question.

**Question 2.4.** Does $A(X)$ satisfy chromatic convergence? More broadly, is there a clean description of the limit of the chromatic tower under $A(X)$?

Since we now know that assembly splits in the $K(n)$-local category, it seems reasonable to make the following “$A$-theory Novikov conjecture” for classifying spaces of finite groups.

**Conjecture 2.5.** If $G$ is a finite group, the assembly map
\[ BG_+ \wedge A(*) \to A(BG) \]
is split injective on the homotopy groups.

It should be possible to make progress on this question if we could better understand the chromatic tower of $A(X)$.

### 3. Waldhausen categories and $K$-theory.

In this section we begin setting up the technical underpinnings of our main theorem. We begin by recalling the definition of a Waldhausen category $C$, the construction of its $K$-theory spectrum $K(C)$, and the hypotheses needed to form equivalences $K(C) \simeq K(D)$. We then describe a delooping of the zeroth space of $K(C)$ using a Segal or May type delooping machine, and give a quick, explicit proof of their equivalence to Waldhausen’s delooping via the iterated $S^\cdot$-construction. These alternative deloopings will be essential for our main result.

#### 3.1. Definitions and the $S^\cdot$ construction.

**Definition 3.1.** A *Waldhausen category* $C$ is a category equipped with two subcategories of cofibrations and weak equivalences, such that

- every isomorphism is both a cofibration and a weak equivalence.
there is a zero object \( * \) and every object is cofibrant.

- every pushout along a cofibration exists in \( C \), and the pushout of the cofibration is a cofibration.

- (gluing lemma) a weak equivalence of homotopy pushout diagrams (one leg is a cofibration) induces a weak equivalence of pushouts.

\( C \) satisfies the saturation axiom if its weak equivalences satisfy the 2 out of 3 axiom.

A cylinder functor is a functor \( T : (A \rightarrow B) \cong T(f) \) with a natural commuting diagram

\[
\begin{array}{ccc}
A & \xrightarrow{j_1} & T(f) & \xrightarrow{j_2} & B \\
\downarrow{f} & \sim & \downarrow{p} & \xrightarrow{id} & \downarrow{id}
\end{array}
\]

such that \( T(f) = B \) when \( A = * \), and given a square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

in which the vertical maps are cofibrations, the resulting square

\[
\begin{array}{ccc}
A \lor B & \xrightarrow{j_1 \lor j_2} & T(f) \\
\downarrow & & \downarrow \\
A' \lor B' & \xrightarrow{j_1' \lor j_2'} & T(f')
\end{array}
\]

has the property that all maps are cofibrations and in addition the map from the pushout to \( T(f') \) is a cofibration \( [Wal85] \).

**Definition 3.2.** If \( C \) is a Waldhausen category, we let \( S \cdot C \) be the following simplicial Waldhausen category. The objects are the functors from the poset \([n] = \{0 \rightarrow 1 \rightarrow \ldots \rightarrow n\}\) into \( C \), plus additional choices of quotients, subject to the condition that 0 is sent to \( * \) and each arrow is sent to a cofibration. The morphisms are the natural transformations of such functors. The cofibrations are the natural transformations which are vertexwise cofibrations, and for each commuting square, the map from the pushout to the final vertex is also required to be a cofibration. The weak equivalences are also defined vertexwise.

**Definition 3.3.** If \( C \) is a Waldhausen category, the algebraic K-theory spectrum \( K(C) \) is a symmetric spectrum which at spectrum level \( n \) is the realization of the multisimplicial set

\[ |w \cdot S^{(n)} C| \]

Here \( w \cdot \) is shorthand for the nerve on the subcategory of weak equivalences. The structure maps of \( K(C) \) are defined by taking a grid of maps in \( C \) and adding one
more dimension, which has only two layers, the first containing only copies of $*$ and the second layer containing the original grid. This gives the structure maps
\[ |w.S^{(n)}C| \rightarrow \Omega |w.S^{(n+1)}C| \]
It is not hard to check that the space $|w.S^{(n)}C|$ is $(n-1)$-connected, so $K(C)$ is a connective spectrum.

**Theorem 3.4 (Waldhausen).** These structure maps are weak equivalences when $n \geq 1$. Therefore
\[ \Omega^\infty K(C) \simeq \Omega |w.S.C| \]

Using (Sch08, Lem 2.3(ii)) this implies that the homotopy groups of $K(C)$ have bounded filtration, so $K(C)$ is a semistable symmetric spectrum. In other words, the na"{i}vely defined homotopy groups of $K(C)$ agree with the homotopy groups as measured in the homotopy category of symmetric spectra.

### 3.2. Exact functors and the approximation property.

**Definition 3.5.** A functor $F : C \rightarrow D$ between Waldhausen categories is exact if it preserves the zero object, cofibrations, weak equivalences, and pushouts along cofibrations. $F$ has the approximation property if

- For any $X \xrightarrow{f} Y$ in $C$, $f$ is a weak equivalence iff $F(f)$ is a weak equivalence.
- For any $X \in C$, $Z \in D$, and $F(X) \xrightarrow{f} Z$, there is a cofibration $X \rightarrow X'$ in $C$ and weak equivalence $F(X') \xrightarrow{\sim} Z$ in $D$ forming a commuting triangle

\[ \begin{array}{ccc}
F(X) & \xrightarrow{f} & Z \\
\downarrow & & \downarrow
\end{array} \]

\[ \begin{array}{ccc}
F(X') & \xrightarrow{\sim} & Z \\
\downarrow & & \downarrow
\end{array} \]

**Theorem 3.6 (Waldhausen).** If $C$ is a Waldhausen category, then any exact functor $F : C \rightarrow D$ induces a map $K(F) : K(C) \rightarrow K(D)$. If $C$ has a cylinder functor and the saturation axiom, and if $F$ satisfies the approximation property, then $K(F)$ is an equivalence of spectra.

**Remark.** It is easy to construct examples of Waldhausen categories. For instance, if $M$ is any model category, then the subcategory $C$ of cofibrant objects forms a Waldhausen category. Furthermore any left Quillen adjoint $F : M \rightarrow N$ induces an exact functor on the cofibrant objects $C \rightarrow D$, and any left Quillen equivalence has the approximation property.

Unfortunately, the category of all cofibrant objects $C$ never has interesting $K$-theory. Using Waldhausen’s additivity theorem [Wal85] one may demonstrate that $K(C) \simeq *$ because $C$ contains infinite coproducts. So, we usually restrict to a subcategory $A$ that has some finiteness condition, preventing the existence of infinite
coproducts. More precisely, a Waldhausen subcategory $A \subset C$ is formed by taking the full subcategory on any collection of objects of $C$ that is closed under pushouts along cofibrations. If $A \subset C$ is such a category, closed under weak equivalences, if $F : C \to D$ is a Quillen equivalence, and if $B \subset D$ is the subcategory of all objects equivalent to $F(a)$ for some $a \in A$, then $F : A \to B$ still has the approximation property. In other words, “Quillen equivalences give equivalences on $K$-theory,” but we always have to keep track of the finiteness conditions that define $A$ and $B$.

3.3. Comparison of Waldhausen, Segal, and May deloopings. We conclude this section by discussing deloopings of $K(C) = \Omega|wSC|$. In essence, we do not need to iterate the $S$ construction to get a spectrum; we only need to apply $S$ once, and then we may do the rest with a classical delooping machine.

First recall that if $C$ is a symmetric monoidal category, the space $|C| \equiv |NC|$ can be delooped infinitely many times. One approach due to Segal uses $\Gamma$-spaces; another approach due to May uses a monadic bar construction and the Barratt-Eccles operad. We will use the notation $|C|, B^1|C|, B^2|C|, \ldots$ to refer to the levels of the spectrum produced by either of these two machines. In particular the structure maps of the spectrum give an infinite string of maps

$$
|C| \to \Omega B^1|C| \quad B^1|C| \sim \to \Omega B^2|C| \quad B^2|C| \sim \to \Omega B^3|C| \quad \ldots
$$

which are all weak equivalences except the first. The first map is an equivalence if $|C|$ is grouplike. We may without loss of generality say that this is an orthogonal spectrum, since each of these machines may be configured to produce orthogonal spectra [MMO].

Now if $C$ is a Waldhausen category, then it is also a symmetric monoidal category by the coproduct. We may apply the usual rectification trick [Mer14] to make $C$ permutative under this coproduct. Then it is easy to see that the subcategory of weak equivalences $wC$ is permutative, and furthermore $S^{(n)}C$ and $wS^{(n)}C$ are multisimplicial objects in permutative categories. In other words, the $n$th space of the Waldhausen spectrum $|wS^{(n)}C|$ is a valid input for our infinite loop space machine $B^{(m)}$. In fact, we have seen that when $n \geq 1$ this space is connected, and so $B^{(m)}$ gives a true $m$-fold delooping.

We check that $B^{(m)}$ is compatible with the structure the spaces $|wS^{(n)}C|$ have as a symmetric spectrum. Naturality of $B^{(m)}$ is enough to guarantee that it commutes with the symmetric group actions. For the structure maps we need to be clear about how $B$ will act on a loop space. Observe that if $X$ is an $E_\infty$ space, or a $\Gamma$-space, then $\Omega^kX$ is either an $E_\infty$ space by “pointwise” operad action, or a $\Gamma$-space by applying $\Omega^k$ to every level and every map. In either case, there is a natural interchange

$$
B^{(m)}(\Omega^kX) \to \Omega^kB^{(m)}(X)
$$
which is an equivalence if \( X \) is \((k - 1)\)-connected. Therefore, for either machine, the structure map

\[
|w.S^{(n)}C| \rightarrow \Omega^k|w.S^{(n+k)}C|, 
\]

which is easily checked to be a map of inputs for \( B \), gives the commuting diagram

\[
|w.S^{(n)}C| \rightarrow \Omega^k|w.S^{(n+k)}C| \\
\Omega^m B^{(m)}|w.S^{(n)}C| \rightarrow \Omega^k \Omega^m B^{(m)}|w.S^{(n+k)}C| 
\]

This ensures that \( B^{(m)}|w.S^{(n)}C| \), for \( m \) fixed and \( n \) varying, defines a symmetric spectrum. The \( O(m) \)-action commutes with the symmetric spectrum structure as well by direct inspection, so we have defined a bispectrum which is symmetric in one direction and orthogonal in the other.

As usual, such a bispectrum gives a diagram which commutes up to rearrangements of the loop coordinates:

\[
|w.C| \rightarrow \Omega|w.S.C| \sim \Omega^2|w.S^{(2)}C| \sim \ldots \\
\Omega B|w.C| \sim \Omega^2 B|w.S.C| \sim \Omega^3 B|w.S^{(2)}C| \sim \ldots \\
\Omega^2 B^{(2)}|w.C| \sim \Omega^3 B^{(2)}|w.S.C| \sim \Omega^4 B^{(2)}|w.S^{(2)}C| \sim \ldots \\
\vdots \vdots \vdots \vdots 
\]

The weak equivalences shown follow from easy connectivity arguments. The left-hand column gives the group completion of \( |wC| \) with respect to the sum, and the right-hand region gives the “group completion” which splits all cofiber sequences. In general, these two completions are quite different.

Now, we will make use of the construction in the left-hand column, but it does not currently give the correct stable homotopy type. To remedy this, we apply a shift and a loops in the symmetric-spectrum direction.

Remark. In an earlier draft we tried simply replacing only the terms in the left-hand column \( B^{(m)}|w.C| \) by \( B^{(m)}\Omega|w.S.C| \) and leaving the other terms unchanged, but this actually breaks the symmetric spectrum structure. In general if \( X_0, X_1, X_2, \ldots \) are the levels of a symmetric spectrum, then \( \Omega X_1, X_1, X_2, \ldots \) give a prespectrum in an obvious way, but this prespectrum is not a symmetric spectrum!
**Definition 3.7.** Let $B$ denote the Segal or May machine. To each Waldhausen category $C$ we define the bispectrum $X_{m,n}$ by

$$X_{m,n} = B^{(m)} \Omega |w, S^{(n+1)}| C$$

This bispectrum is orthogonal in the $m$ slot and symmetric in the $n$ slot.

We need the orthogonal spectrum $\{X_{m,0}\}$ to be a model for Waldhausen $K$-theory.

**Theorem 3.8.** The orthogonal spectrum

$$\{X_{m,0}\}_{m \geq 0} = \{B^{(m)} \Omega |w, S| C\}_{m \geq 0}$$

is naturally equivalent to the prolongation of the symmetric spectrum

$$\{X_{0,n}\}_{n \geq 0} = \{|w, S^{(n)}| C\}_{n \geq 0}$$

and therefore provides an alternate model of Waldhausen $K$-theory.

As the Waldhausen $K$-theory spectrum $K(C)$ is semistable, the loops of the shift of $K(C)$ is also semistable and the natural morphism between them is a $\pi_*$-isomorphism. Therefore the proof of this theorem reduces to a general statement about bispectra. Let us recall some notation from [MMSS01]. Define $\Sigma_S$ as the topological category whose objects are all finite sets (in some universe), and whose morphism spaces are

$$\Sigma_S(m,n) = (\Sigma_n)_+ \wedge_{\Sigma_{n-m}} S^{n-m}$$

Similarly, let $\mathcal{J}$ be the category whose objects are finite-dimensional inner product spaces in some universe $U \cong \mathbb{R}^\infty$, and whose morphisms are

$$\mathcal{J}(V, W) = O(W)_+ \wedge_{O(W - V)} S^{W - V}$$

when $V \subset W$ and $W - V$ means orthogonal complement. Of course, $\mathcal{J}$ is equivalent to the subcategory on representations of the form $\mathbb{R}^n$, so in specifying a diagram on $\mathcal{J}$ we only need to describe what happens at each $n$. Both $\Sigma_S$ and $\mathcal{J}$ are symmetric monoidal categories, under disjoint union of sets or direct sum of vector spaces, and there is a symmetric monoidal functor $\Sigma_S \to \mathcal{J}$ which assigns the set $T$ to the space $\mathbb{R}^T$. We recall the following facts:

**Theorem 3.9.** [MMSS01] Diagrams of based spaces over $\Sigma_S$ are equivalent to symmetric spectra. Diagrams of based spaces over $\mathcal{J}$ are equivalent to orthogonal spectra. The left Quillen equivalence $\mathbb{P}$ from symmetric to orthogonal spectra is given by left Kan extension along $\Sigma_S \to \mathcal{J}$. The smash product of orthogonal spectra $X$ and $Y$ is the left Kan extension of the $\mathcal{J} \times \mathcal{J}$-diagram $\wedge \circ (X \times Y)$ along the map $\mathcal{J} \times \mathcal{J} \to \mathcal{J}$ defining the symmetric monoidal structure on $\mathcal{J}$.

In this language, Thm 3.8 reduces to the following proposition.
Proposition 3.10. To each diagram $X_{m,n}$ of based spaces over $\mathcal{J} \times \Sigma S$ we may functorially assign an orthogonal spectrum $Y$ and a zig-zag of orthogonal spectra

$$\mathbb{P}\{X_{0,n}\} \longrightarrow Y \longleftarrow \{X_{m,0}\}$$

If $X_{m,n}$ is semistable in the symmetric direction, then this zig-zag is on homotopy groups naturally isomorphic to

$$\colim_n \pi_{n+k}(X_{0,n}) \longrightarrow \colim_{m,n} \pi_{m+n+k}(X_{m,n}) \leftarrow \colim_m \pi_{m+k}(X_{m,0})$$

So if each row and column of $X_{m,n}$ is an $\Omega$-spectrum, these maps are equivalences.

Proof. We make $X_{m,n}$ cofibrant in the projective model structure on $\mathcal{J} \times \Sigma S$ diagrams, then define $Y$ to be the left Kan extension of $X_{m,n}$ along the map of categories

$$\mathcal{J} \times \Sigma S \longrightarrow \mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$$

The above zig-zag is then very easy to define.

For the second claim, we should clarify that the maps of the colimit system $\pi_{m+n+k}(X_{m,n})$ are the usual maps

$$\pi_{m+n+k}(X_{m,n}) \longrightarrow \pi_{1+m+n+k}(\Sigma X_{m,n}) \longrightarrow \pi_{(1+m)+n+k}(X_{1+m,n})$$

$$\pi_{m+n+k}(X_{m,n}) \longrightarrow \pi_{1+m+n+k}(\Sigma X_{m,n}) \longrightarrow \pi_{m+(1+n)+k}(X_{m,1+n})$$

except along the bottom we additionally multiply by $(-1)^m$ to account for switching the 1 past the $m$. This diagram of homotopy groups makes sense for any orthogonal-symmetric bispectrum which is semistable in the symmetric direction. Prolonging a semistable symmetric spectrum to an orthogonal spectrum does not change level 0, nor does it change the colimit of its homotopy groups, so the zig-zag of groups

$$\colim_n \pi_{n+k}(X_{0,n}) \longrightarrow \colim_{m,n} \pi_{m+n+k}(X_{m,n}) \leftarrow \colim_m \pi_{m+k}(X_{m,0})$$

is unchanged.

Therefore, without loss of generality, we just have to show that given a bi-orthogonal-spectrum, the left Kan extension along $\mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$ gives an orthogonal spectrum whose homotopy groups are naturally isomorphic to the colimit of $\pi_{m+n+k}(X_{m,n})$ along the maps we described above. This is a straightforward adaptation of the argument for \cite{MMSS}, 11.9), working with a general bispectrum instead of a bispectrum arising from a smash product of two spectra. The actual map

$$\colim_{m,n} \pi_{m+n+k}(X_{m,n}) \longrightarrow \colim_n \pi_{n+k}((\text{Lan} X)_n)$$

is not hard to define, the trick is to prove it is a $\pi_*$-isomorphism. Since both sides commute with coproducts and sequential colimits along closed inclusions, and send cofiber sequences to long exact sequences, this reduces in the usual way to the case where $X_{m,n}$ is a free diagram on a sphere at a single bilevel $(m,n)$. In this case, the left Kan extension is also isomorphic to a free spectrum at level $m+n$ on a sphere, and so the map is clearly an isomorphism. \qed
4. WAlDHAUSEN CATEGORIES OF G-SPACES, G-SPECTRA, FIBERWISE SPACES, AND FIBERWISE SPECTRA.

In this section we construct seven distinct Waldhausen categories, each with two finiteness conditions, for a total of fourteen examples to which we can apply Waldhausen’s $S\cdot$ construction. They are all necessary for our proof and applications. To keep them straight, we introduce some non-standard notation to distinguish them. In this notation $G$ is any topological group, $B$ any unbased space, and $R$ any orthogonal ring spectrum.

- based $G$-sets $K(G)$
- based $G$-spaces $M(G)$
- $R$-modules with $G$ action $M(G; R)$
- ex-fibrations over $B$ $E(B)$
- $R$-module fibrations over $B$ $E(B; R)$
- retractive spaces over $B$ $R(B)$
- $R$-modules over $B$ $R(B; R)$

The most natural maps between these categories point in the following directions:

$$
\begin{array}{c}
K(G) \\
\downarrow \\
M(G) \xrightarrow{\Sigma^\infty} M(G; S) \xrightarrow{R\wedge} M(G; R) \\
\downarrow E \\
E \xrightarrow{\Sigma^\infty} E(G; S) \xrightarrow{R\wedge} E(G; R) \\
\downarrow P \xrightarrow{I} P \xrightarrow{I} P \xrightarrow{I} \\
R(G) \xrightarrow{\Sigma^\infty} R(G; S) \xrightarrow{R\wedge} R(G; R)
\end{array}
$$

The categories $R$ and $E$ are used to define assembly and coassembly, respectively. The category $M$ has a good model structure that the others lack. We will lift the assembly and coassembly maps from $R$ and $E$ to $M$, and then to $K$, where they can be analyzed in a more combinatorial way. Finally, we will use the categories with $R$ coefficients in the applications.

Once we pick appropriate finiteness conditions, the maps labelled $E$, $I$, and $P$ all preserve $K$-theory. (These sorts of statements are common in the area, and our proofs will follow the methods of Waldhausen [Wal85].) It is often the case that $\Sigma^\infty$ preserves $K$-theory as well, but we won’t need such a statement here.

4.1. G-SPACES AND FIBERWISE SPACES. Throughout let $G$ be any topological group and let $B$ be any unbased space. Let $\mathcal{T}$ denote the category of based (compactly generated) topological spaces, and let $GT$ be the category of based spaces with a continuous left $G$-action. Recall that the standard Quillen model structure on $\mathcal{T}$ is cofibrantly generated by compact objects. Therefore the category $GT$ inherits a projective model structure, in which the weak equivalences and fibrations are
determined by forgetting the $G$-action. The cofibrations are generated by the cells $(G \times S^{n-1})_+ \to (G \times D^n)_+$ and trivial cells $(G \times D^n)_+ \to (G \times D^n \times I)_+$.

**Definition 4.1.** Let $\mathcal{M}(G)$ denote the category of all based spaces with left $G$-actions that are cofibrant in the projective model structure. Equivalently, it is the category of retracts of relative $G$-CW complexes. (The letter $\mathcal{M}$ here stands for “modules over $G$”; it is meant to reinforce the parallel between spaces with a $G$-action and spectra with a $G$-action.) Make $\mathcal{M}(G)$ into a Waldhausen category using the cofibrations and weak equivalences in the projective model structure.

- A finite (based) $G$-space $X$ is equivalent to one built out of finitely many cells $G \times D^n$. A finitely dominated $G$-space is any retract in the homotopy category of a finite $G$-space. Let $\mathcal{M}_f(G)$ denote the subcategory of all finitely dominated $G$-spaces.
- Let $\mathcal{M}^f(G)$ denote the subcategory of all spaces whose underlying nonequivariant space is homotopy finitely dominated.

The $K$-theory of $\mathcal{M}_f(G)$ is one model for Waldhausen’s $A(BG)$, and $\mathcal{M}^f(G)$ for $V(BG)$. When $G$ is finitely dominated, there is an inclusion of categories $\mathcal{M}_f(G) \to \mathcal{M}^f(G)$. This defines the Cartan map $A(BG) \to V(BG)$.

Next, we will introduce an alternate model built from spaces over $BG$. Fix an unbased space $B$.

**Definition 4.2.**
- A weak equivalence (or $q$-equivalence) of retractive spaces $X \to Y$ over $B$ is a map which is a weak equivalence after forgetting the maps into and out of $B$.
- An $h$-cofibration $A \to X$ is a map satisfying the homotopy extension property (HEP), ignoring the maps into and out of $B$. In other words, $X \times I$ retracts onto $A \times I \cup X \times 0$, in a non-fiberwise way.
- An $f$-cofibration is a map satisfying the fiberwise homotopy extension property (FHEP). In other words, the retract of $X \times I$ onto $A \times I \cup X \times 0$ may be chosen to respect the map into $B$.

It is known that nice pushouts of fibrations are fibrations [Cla81]:

**Lemma 4.3.**
- Given a pushout square of retractive spaces over $B$

\[
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
\]

If $W$, $X$, and $Y$ are all fibrations over $B$ and $W \to X$ is an $h$-cofibration then $Z$ is also a fibration over $B$.
- Every $f$-cofibration is an $h$-cofibration.
Now we construct some Waldhausen categories of spaces over $B$. All of our spaces will be compactly generated weak Hausdorff. This is fine because we will not need to construct any fiberwise mapping spaces.

**Definition 4.4.** If $B$ is any unbased space, let $\mathcal{R}(B)$ denote the category of all retractive spaces over $B$ for which the basepoint section $B \to X$ is an $f$-cofibration. The cofibrations are the $f$-cofibrations and the weak equivalences are the $q$-equivalences (weak equivalences on the total space).

- $\mathcal{R}_f(B)$ consists of those $X$ for which the total space is weakly equivalent to a homotopy retract of a finite relative CW complex $B \to X'$.
- $\mathcal{R}_f^l(B)$ consists of those $X$ for which the homotopy fiber of $X \to B$ is homotopy finitely dominated.

**Definition 4.5.** An *ex-fibration* is an $f$-cofibrant retractive space $X$ for which the projection $X \to B$ is a Hurewicz fibration. Let $\mathcal{E}(B)$ denote category of ex-fibrations over $B$, with the same cofibrations and weak equivalences as above. $\mathcal{E}_f(B)$ and $\mathcal{E}_f^l(B)$ are defined similarly.

A simple diagram chase is enough to prove

**Proposition 4.6.** The Waldhausen categories $\mathcal{R}(B)$ and $\mathcal{E}(B)$ both have cylinder functors satisfying the cylinder axiom.

The reason we have two categories $\mathcal{R}(B)$ and $\mathcal{E}(B)$ is that they have opposite functoriality. Let $f : A \to B$ be any map of base spaces. Then there is an adjoint pair of functors $f_!$ and $f^*$ defined by the pushout square

$$
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
X & \to & f_! X
\end{array}
$$

and pullback square

$$
\begin{array}{ccc}
f^* Y & \to & Y \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}
$$

Since topological spaces are both left and right proper, this lemma follows quickly.

**Lemma 4.7.**
- $f_!$ preserves all $f$-cofibrations and all weak equivalences between $f$-cofibrant spaces.
- $f^*$ preserves all $f$-cofibrations and all weak equivalences between ex-fibrations.

This lemma guarantees that pushforward $f_!$ is exact on $\mathcal{R}(B)$ and pullback $f^*$ is exact on $\mathcal{E}(B)$. Since $f_!$ preserves the first finiteness condition, the $K$-theory of $\mathcal{R}_f(B)$ is a covariant functor of $B$, which we call $A(B)$. Dually, $f^*$ preserves the
category $\mathcal{E}^f(B)$, and so the $K$-theory of this category is a contravariant functor in $B$, which we denote $V(B)$.

$$A(B) := K(\mathcal{R}_f(B))$$
$$V(B) := K(\mathcal{E}^f(B))$$

**Remark.** Since pushouts and pullbacks are not strictly unique, only unique up to isomorphism, $A(B)$ and $V(B)$ are technically not functors. This can be remedied without changing the categories involved up to equivalence; we briefly recall the construction from ([RS14], 3.2.1) Fix a large set $U$. In defining $A(B)$ we require that each retractive space in our category is a set-theoretic subset of $B \amalg U$, and in defining $V(B)$ we require it to be a subset of $B \times U$. Then there are obvious rules for how to define a pushout (resp. pullback) as a subset of $B \amalg U$ (resp. $B \times U$). These definitions of $f_!$ and $f^*$ actually respect composition and so they define functors.

**Proposition 4.8.** $A(B)$ and $V(B)$ are homotopy functors.

**Proof.** This is well-known, and can be proven by the same argument as in Prop 4.26 below. Amazingly, these functors actually send weak equivalences of spaces to strong homotopy equivalences of spectra. □

**Remark.** With a good enough model, the categories $\mathcal{M}_f(\ast), \mathcal{M}^f(\ast), \mathcal{R}_f(\ast), \mathcal{R}^f(\ast), \mathcal{E}_f(\ast)$ and $\mathcal{E}^f(\ast)$ are all the same category. We can therefore declare that $A(\ast)$ and $V(\ast)$ are the same spectrum, and use any of the above labels interchangeably.

Next we want to define the Cartan map, and that requires us to relate $\mathcal{E}$ and $\mathcal{R}$ to each other. In fact we have two maps between them: the inclusion functor $I : \mathcal{E}(B) \to \mathcal{R}(B)$, and a reverse functor $P : \mathcal{R}(B) \to \mathcal{E}(B)$ which takes the $f$-cofibrant retractive space $X$ to the pushout

$$
\begin{array}{ccc}
B^f & \to & X \times_B B^f \\
\downarrow_{ev_1} & \sim & \sim \\
B & \to & PX
\end{array}
$$

Here the product $X \times_B B^f$ is taken over the evaluation at 0 map $B^f \to B$, so that the product is a space over $B$ along the evaluation at 1 map. It is elementary to check that $PX$ is an ex-fibration and $P$ preserves weak equivalences. Moreover, $PX$ preserves cofibrations, because $- \times_B B^f$ preserves cofibrations and there is a pushout square

$$
\begin{array}{ccc}
X \times_B B^f & \to & Y \times_B B^f \\
\downarrow & \sim & \sim \\
PX & \to & PY
\end{array}
$$

Therefore $P$ is an exact functor.
Proposition 4.9. Under either of our two finiteness conditions, the maps $K(P)$ and $K(I)$ are inverses in the homotopy category of spectra.

Proof. By Waldhausen’s work, it suffices to give a natural weak equivalence between the identity functor and $I \circ P$ on $\mathcal{R}(B)$, and a natural weak equivalence between the identity functor and $P \circ I$ on $\mathcal{E}(B)$. These are both given by the natural fiberwise equivalence $X \sim P X$. □

We can define a Cartan map $A(B) \rightarrow \mathcal{V}(B)$ whenever $\Omega B$ is finitely dominated for every component of $B$. This is because the object $E(\Omega B) \amalg B \in \mathcal{R}_f(B)$ lies in the category $\mathcal{R}_f(B)$. Therefore $\mathcal{R}_f(B)$ is a subcategory of $\mathcal{R}_f(B)$, and similarly $\mathcal{E}_f(B) \subseteq \mathcal{E}_f(B)$. We define the Cartan map $A(B) \rightarrow \mathcal{V}(B)$ by applying $K$-theory to either route in the commuting square

Next we summarize how our models for $A(B)$ and $\mathcal{V}(B)$ are equivalent to the more classical models. Let $\mathcal{R}h(B)$ denote the category of all retractive spaces over $B$ for which the basepoint section $B \rightarrow X$ is an $h$-cofibration. The cofibrations are the $h$-cofibrations and the weak equivalences are the $h$-equivalences (homotopy equivalences on the total space). $\mathcal{R}_h(B)$ consists of those $X$ for which $B \rightarrow X$ is a homotopy retract of a finite relative CW complex.

Proposition 4.10. There is a zig-zag of inclusions

$\mathcal{R}_f(B) \leftarrow \mathcal{R}_f'(B) \rightarrow \mathcal{R}_h(B)$

where the middle consists of those spaces which are homotopy equivalent to a relative CW complex. Each of these maps has the approximation property and therefore induces an equivalence on $K$-theory.

Proof. The essential fact is that we can factor a map $X \rightarrow Y$ of retractive spaces into a relative CW complex $X \rightarrow X'$ that is also an $f$-cofibration, followed by a weak equivalence $X' \rightarrow Y$. This is done by the usual cell-attachment argument, but we must modify the projection maps from our cells $D^n$ into $B$ so that some collar of the boundary $S^{n-1} \times I$ has projection into $B$ that is constant in the $I$ variable. This ensures that the inclusion $S^{n-1} \rightarrow D^n$ is an $f$-cofibration, and so any relative cell complex built out of these cells is also an $f$-cofibration. This is exactly the sort of maneuver that is done in [MS06]; they call maps built out of cells like this “$qf$-cofibrations.” □
Now let $\mathcal{Fib}(B)$ be the category of Hurewicz fibrations over $B$ whose fibers are $h$-cofibrant. The cofibrations and weak equivalences are defined to be those maps which on each fiber over $B$ are classical $h$-cofibrations or homotopy equivalences, respectively. The subcategory $\mathcal{Fib}^f(B)$ consists of spaces whose fibers are homotopy equivalent to homotopy finitely dominated CW complexes.

**Proposition 4.11.** There is a zig-zag of inclusions

$$\mathcal{E}^f(B) \leftarrow (\mathcal{E}^f)'(B) \rightarrow \mathcal{Fib}^f(B)$$

where the middle is those fibrations whose fibers are homotopy equivalent to CW complexes. Each of these maps has the approximation property and therefore induces an equivalence on $K$-theory.

**Proof.** Same as above, except that we additionally apply $P$ to $X'$, and then lift the zig-zag $PX' \sim X' \rightarrow Y$ to a fiberwise equivalence $PX' \sim Y$, which is possible because $B \rightarrow PX'$ is still equivalent to a CW complex [Mil59], and in both cases $Y$ is assumed to be a fibration over $B$. \hfill $\square$

We have defined a category $\mathcal{M}(G)$ of $G$-spaces, and a category $\mathcal{E}(BG)$ of fibrations over $BG$. Now we describe how to connect them. Let $p : EG \rightarrow BG$ be the principal $G$-bundle with contractible total space.

**Definition 4.12.** If $X$ is a based $G$-space, let $E(X)$ be the retractive space

$$E(X) = EG \times_G X$$

If $Y$ is a retractive space over $BG$, let $F(Y)$ be the $G$-space

$$F(Y) = \text{Map}_{BG}(EG, Y)$$

Clearly $E(X)$ is a fiber bundle over $BG$ whose fibers are homeomorphic to $X$, and $F$ is the right adjoint of $E$. When $Y \rightarrow BG$ is a Hurewicz fibration with fiber $Y_b$ over $b \in BG$, then by a standard argument, the restriction of $EG$ to any point over $b$ induces a weak equivalence $F(Y) \sim Y_b$. From this it is automatic that $X \rightarrow FEX$ is always a weak equivalence and $EFY \rightarrow Y$ is a weak equivalence if $Y$ is a Hurewicz fibration. The following proposition quickly follows.

**Proposition 4.13.** $E$ is an exact functor that induces equivalences on $K$-theory

$$\mathcal{M}_f(G) \rightarrow \mathcal{E}_f(BG) \rightarrow \mathcal{R}_f(BG)$$

**Proof.** To approximate $X \rightarrow Y$ take the mapping cylinder of $X \rightarrow F(Y)$. \hfill $\square$

Finally, there is a more combinatorial category of finite $G$-sets living above $\mathcal{M}(G)$. 

Definition 4.14. Assume $G$ is discrete. Let $\mathcal{K}(G) \subseteq \mathcal{M}(G)$ denote the subcategory of all based $G$-spaces isomorphic to

$$\left( \coprod_{\alpha \in A} G/H_{\alpha} \right)_+$$

The cofibrations are the injections and the weak equivalences are the isomorphisms. Let $\mathcal{K}_f(G)$ be the subcategory in which $A$ is finite and all $H_{\alpha}$ are trivial. Let $\mathcal{K}^f(G)$ be the subcategory of such spaces whose underlying set is finite. If $G = 1$ then these two subcategories are the same and we denote them $\mathcal{F}$.

If $G$ is a finite group, then $\mathcal{K}(G)$ is the category of based $G$-sets. The subcategory $\mathcal{K}_f(G)$ is the finite free $G$-sets, while $\mathcal{K}^f(G)$ is the larger subcategory of all finite $G$-sets. When $G = 1$, $\mathcal{K}_f(1) = \mathcal{K}^f(1) = \mathcal{F}$ is the category of finite based sets. Its $K$-theory is the sphere spectrum, by the Barratt-Priddy-Quillen theorem [BP72].

4.2. $R[G]$-module spectra and fiberwise $R$-modules. Throughout this section we let $G$ be any topological group, and $R$ any orthogonal ring spectrum. We will construct three models for what could be called the $K$-theory of the group ring $R[G]$, or the $A$-theory of $BG$ with coefficients in $R$. To define coassembly as a map having a universal property, it will be essential to have at least one model that uses parametrized spectra, and we show how to set this up while avoiding the model category-theoretic difficulties in parametrized spectra encountered in [MS06].

Recall the model structure for modules over orthogonal ring spectra.

Theorem 4.15. [MMSS01] Let $R$ be an orthogonal ring spectrum. Then the category of left $R$-modules has a cofibrantly generated model structure in which the weak equivalences and fibrations are defined by the forgetful functor to ordinary orthogonal spectra.

Definition 4.16. If $R$ is an orthogonal ring spectrum and $G$ is a topological group, we use $R[G]$ as shorthand for $R \wedge G_+$, which is a ring spectrum with multiplication given by the diagonal action. Let $\mathcal{M}(G; R)$ denote the Waldhausen category of all cofibrant $R[G]$-modules.

- Let $\mathcal{M}_f(G; R)$ be the subcategory of all modules which are in the thick subcategory of $R[G]$. We denote its $K$-theory $K(R[G])$, the $K$-theory of $G$ with coefficients in $R$.
- Let $\mathcal{M}^f(G; R)$ be the subcategory of all modules whose underlying $R$-module is in the thick subcategory of $R$. We denote its $K$-theory $G^R(R[G])$, the Swan theory of $G$ with coefficients in $R$.
- If $G$ is finitely dominated, the inclusion of Waldhausen categories

$$\mathcal{M}_f(G; R) \rightarrow \mathcal{M}^f(G; R)$$

gives on $K$-theory the Cartan map $K(R[G]) \rightarrow G^R(R[G])$. 
It is well-known that equivalences of ring spectra $R \to R'$ or topological groups $G \to G'$ induce equivalences on these forms of $K$-theory. Such maps give Quillen equivalences of model categories, and the left adjoint is an exact functor with the approximation property.

If $R$ is a discrete ring and $G$ a discrete group, then $(HR)[G] \simeq H(R[G])$ is a ring spectrum, and the above definition agrees up to equivalence with the definitions of $K$-theory or Swan theory using Quillen’s $Q$-construction or $BGL_\infty(R)$ and Quillen’s plus construction.

Finally, the functor $R \wedge -$ takes cofibrant based $G$-spaces to cofibrant $R[G]$-modules, and preserves all weak equivalences, which means it is an exact functor and therefore gives maps of $K$-theory

$$K(\mathcal{M}_f(G)) \to K(\mathcal{M}_f(G; R))$$

$$K(\mathcal{M}^I(G)) \to K(\mathcal{M}^I(G; R))$$

The first of these maps is an equivalence when $R = S$.

Now we construct our Waldhausen categories of parametrized $R$-module spectra. Given retractive spaces $X$ and $Y$ over $A$ and $B$, respectively, the external smash product $X \wedge Y$ is the retractive space over $A \times B$ defined by the pushout

$$\begin{array}{ccc}
(X \times B) \cup (A \times Y) & \to & X \times Y \\
\downarrow & & \downarrow \\
A \times B & \to & X \wedge Y
\end{array}$$

In particular, if $A = \ast$ then $X$ is a based space and the external smash product $X \wedge Y$ is another retractive space over $B$. This operation tensors and enriches the category of retractive spaces over $B$ over the category of based spaces.

**Definition 4.17.** A parametrized prespectrum $X$ over $B$ is a sequence of retractive spaces $X_n$ with structure maps $S^1 \wedge X_{n-1} \to X_n$. Equivalently, it is a diagram of retractive spaces over $B$, indexed by the category $N_S$ with objects $\{0, 1, \ldots\}$ and morphisms $N_S(n, m) = S^{m-n}$. A parametrized orthogonal spectrum is a continuous diagram of retractive spaces indexed by $\mathcal{J}$.

If $X$ and $Y$ are orthogonal spectra over spaces $A$ and $B$, respectively, the external smash product $X \wedge Y$ is a parametrized spectrum over $A \times B$ defined by left Kan extension of the $\mathcal{J} \times \mathcal{J}$-diagram $\{X_m \wedge Y_n\}$ along the direct sum map $\mathcal{J} \times \mathcal{J} \to \mathcal{J}$. In other words, it is the Day convolution of $X$ and $Y$ as diagrams over $\mathcal{J}$. If $R$ is an orthogonal ring spectrum, a parametrized $R$-module is an orthogonal spectrum $X$ equipped with a map $R \wedge X \to X$ with the usual associativity properties.

We want to construct a Waldhausen category of parametrized $R$-modules. The most convenient thing would be to use the existing model structure from [MS06],
but pullbacks of the cofibrations are not cofibrations, so $V$ would not be a functor. We are forced to use a less cellular notion of cofibration:

**Definition 4.18.** A (Reedy) cofibration of parametrized prespectra is a map of spectra $X \to Y$ over $B$ such that in each square

$$
\begin{array}{ccc}
S^1 \wedge X_{n-1} & \to & X_n \\
\downarrow & & \downarrow \\
S^1 \wedge Y_{n-1} & \to & Y_n
\end{array}
$$

the map from the pushout to $Y_n$ is an $f$-cofibration of retractive spaces over $B$. When $n = 0$ we interpret $X_{n-1} = B$, so this condition means that $X_0 \to Y_0$ must be an $f$-cofibration.

This definition has the virtue of simplicity. It is also preserved under both pushouts and pullbacks, so that we can define $\mathcal{A}(B; R)$ and $\mathcal{V}(B; R)$. It will be perfect for the purposes of this section, but in the next section we will need to modify it in order to handle smash products.

Since we are building a Waldhausen category, it should come as no surprise that we will restrict our attention to parametrized spectra $X$ that are Reedy cofibrant. Therefore all of our structure maps

$$
S^1 \wedge X_{n-1} \to X_n
$$

are $f$-cofibrations. By an easy induction, each level $X_n$ must be $f$-cofibrant. Therefore the fibrant replacement $PX_n$ from the last section is always defined. If $A$ is a based space, it is not hard to check that there is a natural interchange map

$$
A \wedge PX_n \to P(A \wedge X_n)
$$

and this allows us to define a map of parametrized spectra

$$
X \sim \to PX
$$

which is a weak equivalence on every spectrum level. Moreover if $X$ is an $R$-module spectrum then so is $PX$ and the map $X \to PX$ is $R$-linear.

**Definition 4.19.** Let $\mathcal{R}(B; R)$ denote the category of parametrized $R$-module spectra whose underlying prespectra are Reedy cofibrant. Let $\mathcal{E}(B; R)$ denote the further subcategory for which the projections $X_n \to B$ are also Hurewicz fibrations (and so every level is in $\mathcal{E}(B)$).

We turn $\mathcal{R}$ and $\mathcal{E}$ into Waldhausen categories by taking our cofibrations to be the Reedy cofibrations. If $X, Y \in \mathcal{E}(B; R)$, then a weak equivalence is a map $X \to Y$ which when restricted to each fiber is a stable equivalence of ordinary spectra. If instead $X, Y \in \mathcal{R}(B; R)$, then a weak equivalence is a map $X \to Y$ for which $PX \to PY$ is a stable equivalence on each fiber. These definitions are consistent because if $X \in \mathcal{E}(B; R)$ then the map $X \to PX$ is an equivalence on every fiber.
Proposition 4.20. \( \mathcal{R}(B; R) \) and \( \mathcal{E}(B; R) \) are Waldhausen categories.

Proof. The only nontrivial axiom is the gluing lemma. Given a cofibration \( X \to Y \) and any map \( X \to Z \), we apply \( P \) to all three spectra and take the pushout. The comparison map

\[
Y \cup_X Z \to PY \cup_{PX} PZ
\]

is a levelwise weak equivalence because on each level both spaces are a homotopy pushout of \( Y \) and \( Z \) along \( X \). We replace \( PY \cup_{PX} PZ \) by the homotopy pushout

\[
PY \cup_{PX} PX \vee PX \cup_{PX} PZ
\]

This is an ex-fibration and on each fiber it is the ordinary homotopy pushout of spectra, so this construction preserves all stable equivalences of parametrized spectra. It is naturally level equivalent to \( Y \cup_X Z \) and so the gluing lemma holds.

We give without proof

Lemma 4.21. • The category of parametrized \( R \)-module spectra has a cylinder functor given by \( I_+ \vee X \), satisfying the cylinder axiom.

• The functors \( I \) and \( P \) from the previous section give exact functors

\[
\mathcal{E}(B; R) \xrightarrow{I} \mathcal{R}(B; R) \xrightarrow{P} \mathcal{E}(B; R)
\]

which give inverse equivalences on \( K \)-theory.

• The functor \( E \) from the previous section gives an exact functor

\[
\mathcal{M}(G; R) \to \mathcal{E}(BG; R)
\]

with the approximation property.

Finally we define the subcategories \( \mathcal{R}_f(B; R) \) and \( \mathcal{E}_f(B; R) \) by taking just those spectra which are equivalent to \( E(X) \) for some \( X \in \mathcal{M}_f(G; R) \), and we let

\[
A(B; R) := K(\mathcal{R}_f(B; R))
\]

Similarly we let \( \mathcal{E}_f(B; R) \) be all spectra whose fibers are the retracts of the finite \( R \)-cell modules. In other words, each fiber lies in \( \mathcal{M}_f(1; R) = \mathcal{M}_f^f(1; R) \). This is clearly the same as those spectra \( X \) for which \( F(X) \in \mathcal{M}_f^f(G; R) \), and which therefore lie in \( E(\mathcal{M}_f^f(G; R)) \). We let

\[
V(B; R) := K(\mathcal{E}_f(B; R))
\]

Corollary 4.22. The following maps are all equivalences:

\[
K(R[G]) := K(\mathcal{M}_f(G; R)) \xrightarrow{K(I)} K(\mathcal{E}_f(BG; R)) \xrightarrow{K(I)} K(\mathcal{R}_f(BG; R)) =: A(BG; R)
\]

\[
G^R(R[G]) := K(\mathcal{M}_f^f(G; R)) \xrightarrow{K(I)} K(\mathcal{E}_f(BG; R)) =: V(BG; R) \xrightarrow{K(I)} K(\mathcal{R}_f(BG; R))
\]

Finally we need functoriality. As before, if \( f : A \to B \) is any map of base spaces, then \( f_! \) commutes with fiberwise suspension and \( f^* \) commutes with fiberwise loops, giving us a canonical definition of \( f_! \) and \( f^* \) on parametrized \( R \)-module spectra. It is obvious that \( f_! \) and \( f^* \) preserve Reedy cofibrations of spectra (a real strength of our choice of \( f \)-cofibrations over the more traditional \( h \)-cofibrations). It is also obvious
that \( f^* \) preserves weak equivalences of spectra since they are defined fiberwise. Therefore \( f^* \) is exact.

To get exactness of \( f ! \), we need to check that a weak equivalence of Reedy cofibrant parametrized spectra \( X \longrightarrow Y \) is sent to a weak equivalence \( f !_X \longrightarrow f !_Y \). Since \( f !_i \) clearly preserves level equivalences of parametrized prespectra, we can without loss of generality assume that \( S^1 \prod X_{n-1} \longrightarrow X_n \) is a relative CW complex and an \( f \)-cofibration (built using for instance the \( qf \)-cells from [MS06]), and similarly for \( Y \). However to finish the proof we will need an alternate characterization of when a map \( X \longrightarrow Y \) is an equivalence of parametrized spectra:

**Definition 4.23.** A parametrized \( \Omega \)-spectrum \( Z \) is a parametrized spectrum with all levels fibrations, and such that the adjoint structure maps \( Z_{n-1} \longrightarrow \Omega B Z_n \) are all homeomorphisms.

**Proposition 4.24 ([CP84]).** The inclusion of \( \Omega \)-spectra into all spectra has a left adjoint \( L \), called spectrification, which on the Reedy cofibrant spectra is given by the usual construction \( (LX)_n = \text{colim } k \Omega_B^k X_{n+k} \). Therefore \([X, Y] \cong [LX, Y]\) if \( Y \) is an \( \Omega \)-spectrum. If \( X \) and \( Y \) are Reedy cofibrant \( \Omega \)-spectra with levels homotopy equivalent to relative CW complexes, then \( X \longrightarrow Y \) is a weak equivalence iff it is a homotopy equivalence of parametrized spectra.

**Corollary 4.25.** Suppose \( X \) and \( Y \) are Reedy cofibrant and each level is homotopy equivalent to a relative CW complex. Then a map \( X \longrightarrow Y \) is an equivalence of parametrized prespectra iff it induces a bijection \([Y, Z] \longrightarrow [X, Z]\) for all \( \Omega \)-spectra \( Z \).

Exactness of \( f ! \) follows immediately from this corollary, because of the natural isomorphism \([f !_X, Z] \cong [X, f^* Z]\), and the observation that \( f^* Z \) is always an \( \Omega \)-spectrum.

We have finished showing that \( A(B; R) \) is a functor under \( f !_i \) and \( V(B; R) \) a functor under \( f^* \). We will finish by proving they are homotopy functors. This is very different from the preceding theorem; we are now considering what happens when \( f : B \longrightarrow B' \) is a weak equivalence of unbased spaces, rather than what happens when \( X \longrightarrow Y \) is a weak equivalence of spectra.

**Proposition 4.26.** \( f !_i \) makes \( A(B; R) \) into a covariant homotopy functor, and \( f^* \) makes \( V(B; R) \) into a contravariant homotopy functor.

**Proof.** We show that in fact weak equivalences of spaces are sent to homotopy equivalences of spectra. Because the category of unbased spaces is proper, if \( B \overset{f} \longrightarrow B' \) is a weak equivalence, then for any retractive space \( X \) over \( B \), \( X \longrightarrow f !_i X \) is a weak equivalence of spaces, and for any space \( X' \) over \( B' \), \( f^* X' \longrightarrow X' \) is a weak
equivalence of spaces. Consider the cycle of exact functors

\[ \mathcal{R}_f(B; R) \xrightarrow{f_!} \mathcal{R}_f(B'; R) \xrightarrow{f^*} \mathcal{E}_f(B; R) \]

\[ \mathcal{E}_f(B; R) \xrightarrow{f^*} \mathcal{E}_f(B'; R) \]

Using the adjunction between \( f_! \) and \( f^* \), and the unit map from the identity to \( I \circ P \), it is straightforward to define a natural transformation from the identity of \( \mathcal{R}_f \) to the composite around the square

\[ X \to f^* P f_! X \]

which is always a level equivalence. After applying \( K \)-theory, this natural transformation becomes a homotopy between \( K(I \circ f^* \circ P) \circ K(f_!) \) and the identity map on \( A(B; R) \), proving that \( K(f_!) \) is a left inverse of \( K(f^*) \) in the homotopy category. Similarly if \( X \in \mathcal{R}_f(B'; R) \) then there is a natural zig-zag of weak equivalences of parametrized spectra

\[ f_! f^* P X \to P X \leftarrow X \]

which give a similar homotopy between \( K(f_!) \circ K(I \circ f^* \circ P) \) and the identity on \( A(B'; R) \). Therefore \( K(f_!) \) is a homotopy equivalence of spectra. A similar argument proves that \( K(f^*) \) is a homotopy equivalence, using the weak equivalences

\[ X \to f^* P f_! X \]

\[ X \leftarrow f_! f^* X \to P f_! f^* X \]

\[ \square \]

4.3. **Pairings.** We will construct pairings of the \( K \)-theory spectra defined so far. This requires us to use a more complicated and restrictive definition of “cofibration.” Our cofibrations must include the images of \( R[G] \)-module cofibrations under \( EG \times_G \cdot \), be preserved by pushouts and pullbacks, and interact well with smash products. To our knowledge, there is no notion of “cofibration” in the literature that fits the bill, so we will introduce a new notion and prove that it has good homotopical behavior. Our notion is inspired by the idea of a generalized Reedy category found in [BM11a], even though we are not working with a cofibrantly generated model category.

**Definition 4.27.** If \( X \) is a parametrized orthogonal spectrum, then regard \( X \) as a diagram \( \mathcal{J} \to \text{Top}_B \). Define the \( n\)-skeleton \( \text{Sk}^n X \) by restricting to the objects of \( \mathcal{J} \) which have dimension at most \( n \), and then taking an enriched left Kan extension back to all of \( \mathcal{J} \). Concretely, the \( m \)th level of \( \text{Sk}^n X \) is given by the coequalizer

\[ \bigsqcup_{i \leq j \leq n} X_i \sim \mathcal{J}(i, j) \wedge \mathcal{J}(j, m) \rightrightarrows \bigsqcup_{i \leq n} X_i \sim \mathcal{J}(i, m) \to (\text{Sk}^n X)_m \]
where \( \bigvee \) refers to coproduct of retractive spaces, or union along \( B \). Of course, if \( m \leq n \) then \((\text{Sk}^n X)_m \cong X_m\). If \( m \geq n \), then \((\text{Sk}^n X)_m\) is a quotient of \( X_n \wedge_{O(n)} J(n, m)\).

**Definition 4.28.** A map \( X \rightarrow Y \) of parametrized orthogonal spectra is an *orthogonal (Reedy) cofibration* if in each square

\[
\begin{array}{c}
\text{(Sk}^{n-1} X)_n \\
\downarrow \\
\text{Sk}^{n-1} Y)_n \\
\downarrow \\
X_n \\
\downarrow \\
Y_n
\end{array}
\]

the map from the pushout to \( Y_n \) is an \( O(n) \)-equivariant \( f \)-cofibration of retractive spaces over \( B \). In other words, there is a fiberwise retract of \( Y_n \) onto an appropriate subspace, and this map also respects the \( O(n) \)-action.

Clearly the orthogonal Reedy cofibrations are closed under pushouts, transfinite compositions, and retracts. We will show that they are generated by semi-free spectra on maps of spaces \( K \rightarrow L \) having the \( O(n) \)-equivariant fiberwise homotopy extension property.

**Definition 4.29.** A *semi-free* orthogonal spectrum \( F^a_n K \) on a based \( O(n) \)-space \( K \) is a spectrum which at level \( m \) is \( K \wedge_{O(n)} J(n, m) \). The functor \( F^a_n (\cdot) \) is the left adjoint of the forgetful functor that sends an orthogonal spectrum \( X \) to its \( n \)th space \( X_n \) with the \( O(n) \)-action. These definitions still make sense if \( K \) is a retractive space with a fiberwise \( O(n) \)-action.

**Lemma 4.30.** If \( K \rightarrow L \) is an \( O(n) \)-equivariant \( f \)-cofibration then \( F^a_n K \rightarrow F^a_n L \) is an orthogonal Reedy cofibration, and on level \( m \) it is an \( O(m) \)-equivariant \( f \)-cofibration.

**Proof.** The relevant map is only nontrivial at spectrum level \( n \), where it is the \( O(n) \)-cofibration \( K \rightarrow L \). Therefore \( F^a_n K \rightarrow F^a_n L \) is an orthogonal Reedy cofibration. It is easy to check that

\[
K \wedge_{O(n)} A \rightarrow L \wedge_{O(n)} A
\]

has the \( O(m) \)-equivariant homotopy extension property for any \( O(n) \times O(m) \)-space \( A \). In particular, this applies to the map \( F^a_n K \rightarrow F^a_n L \) at level \( m \), which is

\[
K \wedge_{O(n)} J(n, m) \rightarrow L \wedge_{O(n)} J(n, m)
\]

\[ \square \]

**Definition 4.31.** If \( X \rightarrow Y \) is a map of parametrized orthogonal spectra, define its \( n \)-skeleton as the pushout

\[
\begin{array}{c}
\text{Sk}^n X \\
\downarrow \\
\text{Sk}^n Y \\
\downarrow \\
\text{Sk}^n (X \rightarrow Y)
\end{array}
\]
Clearly the map $X \to Y$ is filtered by these skeleta, $Y$ is the colimit of the skeleta.

**Lemma 4.32.** For each map of parametrized orthogonal spectra $X \to Y$ there is a natural pushout square

$$
\begin{array}{ccc}
\text{Sk}^{n-1}(X \to Y) & \longrightarrow & \text{Sk}^{n}(X \to Y) \\
\uparrow & & \uparrow \\
F^n_{\alpha}(X_n \cup (\text{Sk}^{n-1}X)_n) & \longrightarrow & F^n_{\alpha}Y_n
\end{array}
$$

**Proof.** Just compare the universal properties. \qed

**Corollary 4.33.** The class of orthogonal Reedy cofibrations is the smallest class of maps that is closed under retracts, pushouts, and transfinite compositions, and containing $F^n_{\alpha}K \to F^n_{\alpha}L$ for any $O(n)$-equivariant $f$-cofibration $K \to L$.

**Corollary 4.34.** If $X \to Y$ is an orthogonal Reedy cofibration, then each map $X_n \to Y_n$ is an $O(n)$-equivariant $f$-cofibration.

This guarantees that we can take the strict cofiber of $X \to Y$ and it will have the correct stable homotopy type. It also ensures that pushouts along orthogonal cofibrations will behave the way we expect, allowing for $A(B)$ to be a functor.

Our main technical result for these cofibrations is that they satisfy a pushout-product axiom. This appears to be new, even in the non-fiberwise case of $B = \ast$. It demonstrates that the smash product of orthogonal spectra is actually derived on a very large class of spectra.

**Proposition 4.35.** The pushout-product of two orthogonal cofibrations, $K \to X$ over $A$ and $L \to Y$ over $B$, is an orthogonal cofibration over $A \times B$. If $X$ is cofibrant and $Y \to Y'$ is a weak equivalence of cofibrant spectra then $X \wedge Y \to X \wedge Y'$ is a weak equivalence.

**Proof.** We will show that a pushout-product of cofibrations is a cofibration. Since our cofibrations are generated by maps of the form $F^n_{\alpha}K \to F^n_{\alpha}L$, it suffices to take a pushout-product of two such maps. We observe that the product of two semi-free diagrams over $\mathcal{F}$ gives a semi-free diagram over $\mathcal{F} \times \mathcal{F}$, and therefore

$$F^n_{\alpha}X \wedge F^n_{\alpha}Y \cong F^n_{m+n}(O(m+n)_+ \wedge_{O(m) \times O(n)} X \wedge Y)$$

This construction sends pushout-products of spaces to pushout-products of spectra, so it suffices to show that if $K \to X$ is an $O(m)$-equivariant $f$-cofibration over $A$ and $L \to Y$ is an $O(n)$-equivariant $f$-cofibration over $B$, the pushout-product is an $O(m) \times O(n)$-cofibration over $A \times B$. The retraction we want is the one given by the usual formula for showing that a pushout-product of NDR-pairs is an NDR-pair ([Ste67], Thm 6.3). It is very easy to check that this formula preserves fiberwise and equivariant maps.
Next we check that smashing with a cofibrant spectrum $X$ preserves weak equivalences between cofibrant spectra $Y \to Y'$. It suffices to prove this inductively for $\text{Sk}^n X$. If $Y$ is cofibrant, the map $(\text{Sk}^{n-1} X) \wedge Y \to (\text{Sk}^n X) \wedge Y$ is a pushout-product of two cofibrations, so it is a cofibration and its strict cofiber is a homotopy cofiber. Therefore we only have to examine effect of smashing with the cofiber of $\text{Sk}^{n-1} X \to \text{Sk}^n X$.

So without loss of generality, $X$ is a semi-free spectrum $F_n \triangleright_k L$ on a retractive space $L$ with an $O(k)$-action. The space $(F_n^a L)_{m-n-k} = (S^{m-n-k} \wedge L) \wedge_{O(m-n-k) \times O(k)} O(m-n+k)$ has connectivity $m-n-k-1$ and so the map

$$(F_n^a L)_{m-n-k} \to (F_n^a L)_{m-n} \wedge_{O(m-n+k)} O(m-k)$$

has connectivity $2(m-n) - k - 1$. This connectivity increases faster than $m$, so the above map is an equivalence of prespectra. This finishes the induction up the skeleta of $Y$, so $\text{sh}_{-n}$ does indeed preserve stable equivalences.
Corollary 4.36. If $R$ is orthogonal cofibrant then $E(X) = EG \times_G X$ sends every cofibration of $R[G]$-module spectra to an orthogonal cofibration of parametrized $R$-module spectra.

Proof. A free cell of orthogonal $R[G]$-module spectra

$$R[G] \wedge F_k S^{n-1}_+ \rightarrow R[G] \wedge F_k D^n_+$$

is sent under $E$ to

$$(EG \amalg BG) \boxtimes R \wedge F_k S^{n-1}_+ \rightarrow (EG \amalg BG) \boxtimes R \wedge F_k D^n_+$$

and this is the external smash product of a free cell of orthogonal spectra with a cofibrant parametrized orthogonal spectrum. $\square$

With these lemmas, it is straightforward to verify that the categories $\mathcal{R}(B; R)$ and $\mathcal{E}(B; R)$ from the previous section have all of the same properties if we take instead the orthogonal cofibrant objects and take the orthogonal cofibrations between them. The proofs are almost all the same. The cylinder functor and cylinder axiom follow formally once we know the tensor product with simplicial sets has the pushout-product axiom. The result of Clapp still allows us to prove that $f_!$ preserves weak equivalences, because $f_!$ preserves level equivalences between spectra whose levels are $f$-cofibrant.

Since we are changing the definitions, we will also take this opportunity to make a standard technical modification so that the pairings below are strictly associative. We allow the Waldhausen categories $\mathcal{R}(B; R)$ and $\mathcal{E}(B; R)$ to also include objects consisting of

- a $k$-tuple of ring spectra $R_1, \ldots, R_k$,
- an isomorphism of rings $R_1 \wedge \ldots \wedge R_k \cong R$,
- a $k$-tuple of parametrized modules $M_1, \ldots, M_k$ over the spaces $B_1, \ldots, B_k$,
- and a homeomorphism $B_1 \times \ldots \times B_k \cong B$.

In practice, all but one of these rings will be the sphere spectrum, and all but one of these spaces will be $\ast$. For the purpose of defining the morphisms, we treat the above object as if it were the external smash product $M_1 \boxtimes \ldots \boxtimes M_k$, which is an $R$-module over $B$. Clearly these new categories are equivalent to the old ones and so they give homotopy equivalent $K$-theory.

Now we are ready to define our pairings.

Definition 4.37. A pairing of Waldhausen categories, or biexact functor, is a bifunctor $F : C \times D \rightarrow E$ that is exact in each variable separately, such that for every choice of cofibration $a \rightarrow b$ in $C$ and cofibration $x \rightarrow y$ in $D$, in the square
of cofibrations

\[
\begin{array}{c}
F(a, x) \\ \downarrow \\
F(a, y) \\
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
F(b, x) \\
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
F(b, y) \\
\end{array}
\]

the map \( F(a, y) \cup_{F(a, x)} F(b, x) \to F(b, y) \) is also a cofibration.

The following is a consequence of [BM11b], Thm 2.6:

**Proposition 4.38.** A pairing of Waldhausen categories \( C_1 \times C_2 \to D \) induces a map of spectra \( K(C_1) \wedge K(C_2) \to K(D) \) in a natural way. Given four pairings making this diagram of functors commute

\[
\begin{array}{c}
C_1 \times C_2 \times C_3 \\ \downarrow \\
C_1 \times D_2 \\
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
D_1 \times C_3 \\
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
E \\
\end{array}
\]

the two induced maps \( K(C_1) \wedge K(C_2) \wedge K(C_3) \to K(E) \) are identical.

**Proposition 4.39.** If \( B \) and \( B' \) are unbased spaces, \( R \) and \( S \) ring spectra, then there are pairings of symmetric spectra

\[
\begin{array}{c}
A(B; R) \wedge A(B'; S) \\ \downarrow \\
A(B \times B'; R \wedge S) \\
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
V(B; R) \wedge V(B'; S) \\
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
V(B \times B'; R \wedge S) \\
\end{array}
\]

which are natural with respect to all pairs of maps of unbased spaces and pairs of maps of rings. They have the obvious associativity relation in the case of three base spaces and three rings. They commute up to homotopy with the Cartan map when it is defined.

**Proof.** We send a parametrized \( R \)-module \( M \) over \( B \) and an \( S \)-module \( N \) over \( B' \) to their external smash product \( M \wedge N \) as defined above. By Prop 4.35 above this defines a biexact functor

\[
\begin{array}{c}
\mathcal{R}(B; R) \times \mathcal{R}(B'; S) \\ \downarrow \\
\mathcal{R}(B \times B'; R \wedge S) \\
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\mathcal{E}(B; R) \wedge \mathcal{E}(B'; S) \\
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\mathcal{E}(B \times B'; R \wedge S) \\
\end{array}
\]

To deal with associativity, we modify the above functor up to natural isomorphism (which does not change biexactness). We instead take a \( k \)-tuple of modules and an \( l \)-tuple of modules to the obvious \( k + l \)-tuple, and do not actually smash any of them together. Now our rule is strictly associative, not just associative up to natural isomorphism.

Now we check the finiteness conditions. Biexactness means that cofiber sequences in each variable are sent to cofiber sequences, so if we wish to check that a pair of thick subcategories is sent to a given thick subcategory, we only have to check the
generators. It’s easy to see that the external smash product of spaces sends $B$ with a single cell attached and $B'$ with a single cell attached to $B \times B'$ with a single cell attached, so we get

$$R_f(B; R) \times R_f(B'; S) \longrightarrow R_f(B \times B'; R \wedge S)$$

giving the pairing on $A$-theory

$$A(B; R) \times A(B'; S) \longrightarrow A(B \times B'; R \wedge S)$$

Even easier, since the external smash product is on each fiber just the smash product of the fibers, we get the finiteness condition

$$E^f(B; R) \wedge E^f(B'; S) \longrightarrow E^f(B \times B'; R \wedge S)$$

giving the pairing on $V$-theory

$$V(B; R) \times V(B'; S) \longrightarrow V(B \times B'; R \wedge S)$$

In particular this makes $V(B; R)$ into a ring using the diagonal map of $B$.

Note that these pairings do not commute with the functor $P$ defined in the last section on the nose, only up to equivalence. □

We remark in passing that these pairings make topological Swan theory into a ring, and $K$-theory into a module over that ring, just as in the classical case.

**Corollary 4.40.** For any space $B$, $V(B)$ is a ring spectrum, and $A(B)$ and $A(B; R)$ are $V(B)$-modules.

**Proof.** We know from above that $V(B)$ is a ring under fiberwise smash product. We check that the fiberwise smash product of the retractive space $B \cup \ast$ and the fibration $Y \longrightarrow B$ with finite fiber gives just $Y_b \cup_b B$, which is obtained from $B$ by attaching finitely many cells. This proves that we land in the correct thick subcategory. For $R$-modules we smash $B \cup \ast$ with $R$ to get a parametrized $R$-module with just one cell, and repeat this argument again. □

We are interested in the following special case of the above pairings. Taking one of our spaces to be the one-point space $\ast$, we get pairings

$$K(R) \wedge A(B) \longrightarrow K(R) \wedge A(B; S) \longrightarrow A(B; R)$$

$$K(R) \wedge V(B) \longrightarrow K(R) \wedge V(B; S) \longrightarrow V(B; R)$$

We will use these pairings to reduce our theorem about the norm map of $K(R)$ to the case of $R = S$. Our results were motivated by the case where $R$ is commutative, $K(R)$ is a ring, and $A(B; R)$ is a module, but they actually hold even when $R$ is not commutative and $K(R)$ is not a ring.
Remark. Our construction of $A(B; R)$ and $V(B; R)$ is not natural with respect to all maps of ring spectra, only those maps $R \to S$ for which $S$ is cofibrant as an $R$-module. It seems one should be able to modify our definition of “cofibration” even further to get naturality for all maps of rings, without breaking the biexactness we proved above, but we do not need to do so here.

5. Assembly and coassembly of $R$-modules.

Now we recall the general definitions of assembly and coassembly.

Definition 5.1. If $X$ is any simplicial set, for instance the singular simplices of a space, the \textit{category of simplices} $\Delta_X$ is a small category whose set of objects is $\coprod_{p \geq 0} X_p$. The set of morphisms from $x \in X_p$ to $x' \in X_q$ is all injective maps of simplicial sets (i.e. compositions of face maps) $\Delta[x] \to \Delta[x']$, making the following square commute.

\[
\begin{array}{ccc}
\Delta[p] & \xrightarrow{x} & X \\
\downarrow & & \downarrow \\
\Delta[q] & \xrightarrow{x'} & X
\end{array}
\]

The nerve of the category $\Delta_X$ is homeomorphic to the barycentric subdivision of the thick realization $\|X\|$ and therefore admits a functorial weak equivalence $|N\Delta_X| \xrightarrow{\sim} |X|$. More specifically, each flag $\Delta[p_0] \to \cdots \to \Delta[p_k] \to X$ gives a $k$-simplex in $|N\Delta_X|$, which is mapped to the $p_k$-simplex in $|X|$ by a linear map which sends vertex $i$ to the image of the barycenter of $\Delta[p_i]$ under the inclusion $\Delta[p_i] \to \Delta[p_k]$. Of course the opposite category has isomorphic realization, giving another functorial weak equivalence $|N\Delta_X^\op| \xrightarrow{\sim} |X|$. There is a second weak equivalence $|N\Delta_X| \xrightarrow{\sim} |X|$ which arises as the geometric realization of an actual simplicial map, the \textit{last vertex map}. This map is most easily defined in the special case where $X$ is the nerve of a category $N.C$. Since each standard $n$-simplex $\Delta[n]$ is itself the nerve of the poset $[n]$, this definition then extends to all simplicial sets by a colimit argument. For $X = N.C$, the last vertex map takes a flag of face maps

$\Delta[p_0] \to \Delta[p_1] \to \cdots \to \Delta[p_k] \to N.C$

regarded as functors $[p_0] \to [p_1] \to \cdots \to [p_k] \to C$ where $[p]$ is the poset of integers from 0 to $p$, and $f_i$ is the composite functor $[p_i] \to C$, and sends this flag to the $k$-simplex in $N_kC$

\[
f_0(p_0) \xrightarrow{f_0(f_0(p_0) \to p_1)} f_1(p_1) \xrightarrow{f_2(f_1(p_1) \to p_2)} \cdots \xrightarrow{f_k(f_{k-1}(p_{k-1}) \to p_k)} f_k(p_k)
\]

It is straightforward to check that this agrees with faces and degeneracies and so gives a well-defined map of simplicial sets. Furthermore it is natural with respect
to functors in $C$, and so taking $C = [n]$ we get a collection of maps of simplicial sets

$$N \Delta_{[n]} \to \Delta_{[n]}$$

which are natural with respect to maps $[n] \to [m]$ in $\Delta$. Obviously this map on the $n$-simplex is a weak equivalence, and we may also check that the inclusion of the boundary of $\Delta_{[n]}$ is sent to a cofibration by the functor $N \Delta_{-}$, so by (Seg74, A.5) this extends to a natural transformation on all simplicial sets $X$, which is always a weak equivalence.

If $X$ is an unbased space, $S X$ its singular simplices, then any object in $\Delta_{S \cdot X}$ may be regarded as a continuous map $\Delta^p \to X$, and the morphisms between such objects are the factorizations $\Delta^p \to \Delta^q \to X$ through compositions of face maps.

Recall that $F$ is a homotopy functor if it sends weak equivalences to weak equivalences. If $F$ is a functor from spaces to spectra, the spectra $F(\Delta^p)$ form a diagram over $\Delta_{S \cdot X}$. These observations allow us to define the assembly and coassembly maps as in WW93:

**Definition 5.2.** If $F$ is any covariant homotopy functor from unbased spaces to spectra, the assembly map of $F(X)$ is the zig-zag

$$X_+ \wedge F(*) \leftarrow |N \Delta_{S \cdot X}|_+ \wedge F(*) \cong \operatorname{hocolim}_{(\Delta^p \to X) \in \Delta_{S \cdot X}} F(*) \leftarrow \operatorname{hocolim}_{\Delta^p \to X} F(\Delta^p) \to F(X)$$

**Definition 5.3.** If $F$ is any contravariant homotopy functor from unbased spaces to spectra, the coassembly map of $F(X)$ is the zig-zag

$$F(X) \to \operatorname{holim}_{(\Delta^p \to X) \in \Delta_{S \cdot X}} F(\Delta^p) \leftarrow \operatorname{holim}_{\Delta^p \to X} F(*) \cong \operatorname{Map}_*(|N \Delta_{S \cdot X}|_+, F(*)) \leftarrow \operatorname{Map}(X_+, F(\star))$$

Of course these definitions extend to simplicial sets $X$, as well, or to functors whose target is any reasonable simplicially tensored and cotensored category. We can use any natural weak equivalence between $\operatorname{hocolim} *$ and $|X|_+$ to define the first leg of the assembly map and the last leg of the coassembly map. The “last vertex” map defined above will be the most useful choice.

The assembly and coassembly maps are characterized by a universal property. Recall that a homotopy functor $F$ from unbased spaces to spectra is linear if $F(\emptyset) \simeq *$ and $F$ takes homotopy pushout squares to homotopy pushout/pullback squares of spectra. The homotopy category of functors is obtained by inverting the natural transformations of functors that induce a weak equivalence $F(c) \xrightarrow{\sim} G(c)$ for every $c$ in the source category. These definitions are unchanged if $F$ is a contravariant functor.

**Proposition 5.4.** Assume that $F(\emptyset) \simeq *$. If $F$ is covariant, then the assembly map is the universal approximation of $F$ on the left by a linear functor in the homotopy category of functors. If $F$ is contravariant, the coassembly map is the universal approximation of $F$ on the right by a linear functor in the homotopy category of functors.
In fact, the proof of this is quite formal and follows the method of (Goo03, 1.8). It is also possible to modify the definitions of assembly and coassembly so as to drop the requirement that $F(\emptyset) \simeq \ast$, or to make higher-order polynomial approximations to $F$ [Mal12].

It turns out that assembly and coassembly play well with multiplicative structure. If $F$ is a functor into $R$-modules, then the assembly and coassembly maps are $R$-module maps. Furthermore if $F$ lands in ring spectra then the coassembly map is a map of rings, and if $F$ lands in coalgebra spectra then the assembly map is a map of coalgebras. These facts motivated our proof below, but now that the argument has been simplified, we only need one fact that falls out immediately from the definitions:

**Proposition 5.5.** If $M$ is a spectrum and $F$ a covariant functor into spectra, then the assembly map for the functor $M \wedge F$ is the smash product of the identity of $M$ and the assembly map for $F$. If $F$ is contravariant then the adjoint of the coassembly map $X_+ \wedge (M \wedge F(X_+)) \longrightarrow M \wedge F(\ast)$ is the smash of the identity of $M$ and the coassembly map of $F$.

Therefore the natural pairings from the previous section give a homotopy-commuting diagram

\[
\begin{array}{ccc}
BG_+ \wedge S \wedge K(R) & \xrightarrow{\text{assembly} \wedge \text{id}} & A(BG) \wedge K(R) \xrightarrow{\text{Cartan} \wedge \text{id}} V(BG) \wedge K(R) \xrightarrow{\text{coassembly} \wedge \text{id}} F(BG_+, A(\ast) \wedge K(R)) \\
\eta & & \eta \\
BG_+ \wedge K(R) & \xrightarrow{\text{assembly}} & A(BG; R) \xrightarrow{\text{Cartan}} V(BG; R) \xrightarrow{\text{coassembly}} F(BG_+, K(R)) \\
\end{array}
\]

The composite of the vertical maps on the left-hand side is an equivalence. Therefore the composite along the bottom row is determined by the image of $BG_+ \wedge S$ along the top row. We will compute this image in two steps, first by lifing it to the $K$-theory of finite sets, and then by using an operadic delooping machine to identify it as an equivariant norm.

6. **A combinatorial lift of assembly and coassembly.**

In the previous section we defined the assembly map for $A$-theory, not as an honest map, but as a zig-zag which gives a map $[\alpha]$ in the stable homotopy category. In this section, we will rectify this by providing an explicit map of spectra

$BG_+ \wedge A(\ast) \xrightarrow{\alpha} A(BG)$

whose image in the homotopy category is exactly $[\alpha]$. In fact, our description will agree with the more classical notion of “assembly” by the units of a ring. Though
Definition 6.1. Let $G$ be a finite group. By abuse of notation, let $G$ refer also to the category with one object whose set of morphisms is the group $G$, and let $\tilde{G}$ refer to the category whose objects are the elements of the group $G$ and for which each ordered pair of objects has a unique isomorphism connecting them. When we write bar constructions on these categories, we will draw the arrows from right to left. Define $BG = |N.G|$ and $EG = |N.\tilde{G}|$.

We think of the arrow $g \leftarrow h$ in $\tilde{G}$ as multiplication on the left by $gh^{-1}$, since this description is invariant under the obvious right $G$-action on the category $\tilde{G}$. This labelling of the arrows of $\tilde{G}$ defines a functor $\tilde{G} \rightarrow G$, which on realizations is a map of spaces $EG \rightarrow BG$. Up to homeomorphism, this is the same as the familiar map of two-sided bar constructions $B(\ast, G, G) \rightarrow B(\ast, G, \ast)$ which divides out by the right $G$-action.

Definition 6.2. Define the map of spectra

$$BG_+ \wedge K(M_f(\ast)) \xrightarrow{\alpha} K(M_f(G))$$

by taking the map of bisimplicial spaces

$$(N_pG)_+ \wedge w_pS_qM_f(\ast) \rightarrow w_pS_qM_f(G)$$

$$g_1, \ldots, g_n: X_{0,0} \rightarrow X_{0,1} \rightarrow \ldots \rightarrow X_{0,0} \wedge G_+ \rightarrow X_{0,1} \wedge G_+ \rightarrow \ldots$$

and extending in the obvious way to iterates of the $S$-construction.

Proposition 6.3. The following diagram commutes up to homotopy, and therefore $\alpha$ defines the assembly map for $A(BG)$:

$$BG_+ \wedge A(\ast) \xrightarrow{\alpha} K(M_f(G))$$

$$\sim \xrightarrow{\text{last vertex}} K(R_f(BG))$$

$$|N.\Delta N.G|_+ \wedge A(\ast) \xrightarrow{\text{hocolim} \ A(\Delta^p)}$$
Proof. We will define an explicit simplicial homotopy between the two legs of the diagram

\[
\begin{array}{c}
\left( N_k G + \wedge w_k R(\ast) \rightarrow w_k R(BG) \right) \\
\text{last vertex}
\end{array}
\]

where \( B_* \) refers to the categorical bar construction. Once this is accomplished, the proof is finished by applying \( S \cdot \) as many times as necessary to define the homotopy on level \( n \) of the \( K \)-theory spectrum. Or, we may apply \( S \cdot \) just once, taking the realization and looping the target. Since our homotopy does not directly use the \( S \cdot \) direction at all, either one is equally easy.

A \( k \)-simplex in the lower-right corner of our diagram is given by a flag of simplices

\[
\Delta[p_0] \rightarrow \ldots \rightarrow \Delta[p_k] \rightarrow N \cdot G
\]

which is really a flag of categories

\[
[p_0] \rightarrow [p_1] \rightarrow \ldots \rightarrow [p_k] \rightarrow G
\]

and a flag of weak equivalences of retractive spaces over \( \Delta^{p_0} \)

\[
X_0 \overset{w_1}{\rightarrow} X_1 \overset{w_2}{\rightarrow} \ldots \overset{w_k}{\rightarrow} X_k
\]

The long route of our diagram pushes these spaces forward along \( r : \Delta^{p_0} \rightarrow \ast \) to get a flag of based spaces

\[
r_1 X_0 \overset{r_1 w_1}{\rightarrow} r_1 X_1 \overset{r_1 w_2}{\rightarrow} \ldots \overset{r_1 w_k}{\rightarrow} r_1 X_k
\]

selects the \( k \)-simplex in the nerve of \( G \)

\[
\bullet \overset{g_0}{\rightarrow} \bullet \overset{g_1}{\rightarrow} \ldots \overset{g_k}{\rightarrow} \bullet
\]

where \( g_i \) is the image in the category \( G \) of the unique arrow in the poset \([p_i] \) connecting the image of the last vertex of \([p_{i-1}] \) to the last vertex of \([p_i] \). This brings us to the top-left corner of the diagram; finally \( \alpha \) transforms all of this data into a single flag of spaces

\[
r_1(\times G EG) \overset{w_1 \times g_1}{\rightarrow} r_1(\times G EG) \overset{w_2 \times g_2}{\rightarrow} \ldots \overset{w_k \times g_k}{\rightarrow} r_1(\times G EG)
\]

The short route of our diagram is much more mundane; it transforms our original set of data into the flag of spaces

\[
i_1 X_0 \overset{i_1 w_1}{\rightarrow} i_1 X_1 \overset{i_1 w_2}{\rightarrow} \ldots \overset{i_1 w_k}{\rightarrow} i_1 X_k
\]

where \( i : \Delta^{p_0} \rightarrow BG \) is the inclusion of spaces induced by the functor \([p_0] \rightarrow G \).
To define a simplicial homotopy between these two branches it is enough to define a commuting diagram of weak equivalences

\[ \begin{array}{cccccc}
   i_1X_0 & \rightarrow & i_1X_1 & \rightarrow & \cdots & \rightarrow & i_1X_k \\
   f_0 & \downarrow & f_1 & \downarrow & \cdots & \downarrow & f_k \\
   r_i(X_0 \times_G EG) & \rightarrow & r_i(X_1 \times_G EG) & \rightarrow & \cdots & \rightarrow & r_i(X_k \times_G EG)
\end{array} \]

which agrees with deletion of the \( X_i \)'s and the deletion of terms from our original flag of simplices. We define \( f_i \) as the product of the identity map of \( X_i \) with a fiberwise map \( X_i \rightarrow EG \) over \( BG \) which we now describe. \( X_i \) started its life as a space over \( \Delta^p_0 \), but our original flag of simplices gives us a map \( \Delta^p_0 \rightarrow \Delta^p_i \) and so we regard \( X_i \) as a space over \( \Delta^p_i \rightarrow BG \). We map it fiberwise into \( EG \) by picking a fiberwise lift of \( \Delta^p_i \) to \( \tilde{G} \) by sending the last vertex of \( [p_i] \) to the object of \( \tilde{G} \) labelled 1, the identity element of the group \( G \). This defines a map of \( \Delta^p_i \) to \( EG \), and by extension from \( X_i \) into \( EG \). The basepoint section of \( X_i \) is sent to the basepoint section, which allows \( f_i \) to be well-defined after passage to quotients. With this definition, the commutativity of the \( i \)th square above boils down to the commutativity of the square of spaces

\[ \begin{array}{ccc}
   \Delta^{p_i-1} & \rightarrow & \Delta^p_i \\
   \downarrow & & \downarrow \\
   EG & \rightarrow & EG
\end{array} \]

in which the vertical maps are the lifts we defined sending the last vertex to 1. This square is a realization of a square of categories

\[ \begin{array}{ccc}
   [p_i-1] & \rightarrow & [p_i] \\
   \downarrow & & \downarrow \\
   \tilde{G} & \rightarrow & \tilde{G}
\end{array} \]

that commutes by our definition of \( g_i \). It is straightforward to check that our definition of the \( f_i \) agrees with the simplicial structure, so we are done. \( \square \)

Now that we have defined and proven a combinatorial model of the assembly map, we turn our attention to the coassembly map.

**Definition 6.4.** Define the map of spectra

\[ K(\mathcal{M}^f(G)) \xrightarrow{c_{\alpha}} \text{Map}_*(BG_+, K(\mathcal{M}^f(*))) \]

whose adjoint

\[ BG_+ \wedge K(\mathcal{M}^f(G)) \rightarrow K(\mathcal{M}^f(*)) \]

is the map of bisimplicial spaces

\[ (N_pG)_+ \wedge w_pS_p\mathcal{M}^f(G) \rightarrow w_pS_p\mathcal{M}^f(*) \]
and extending in the obvious way to iterates of the $\mathcal{S}$-construction.

**Proposition 6.5.** The following diagram commutes up to homotopy, and therefore $c\alpha$ defines the coassembly map for $V(BG)$:

$$
\begin{array}{ccc}
K(\mathcal{M}^f(G)) & \overset{c\alpha}{\longrightarrow} & F(BG_+, A(\ast)) \\
\sim & & \sim \\
K(\mathcal{E}^f(BG)) & \sim & \text{last vertex} \\
\text{holim}_{\Delta^p \rightarrow X} F(\Delta^p) & \overset{\sim}{\longrightarrow} & F(\ast, \Delta, \ast) \\
\end{array}
$$

**Proof.** This proof is in many ways dual to the previous one. It is enough to define an explicit simplicial homotopy between the two legs of the diagram

$$
\begin{array}{ccc}
w_k \mathcal{M}^f(G) & \overset{c\alpha}{\longrightarrow} & F((N_k G)_+, w_k R^{\text{fin}}(\ast)) \\
\sim & & \sim \\
C_k(\ast, \Delta, \ast, w_k R^{\text{fin}}(\ast)) & \overset{\text{last vertex}}{\longrightarrow} & F((N_k \Delta, \ast)_+, w_k R^{\text{fin}}(\ast)) \\
\end{array}
$$

where $C_\ast$ refers to the categorical cobar construction. Once this is accomplished, the proof is finished by applying $\mathcal{S}$ as many times as necessary to define the homotopy on level $n$ of the $K$-theory spectrum. Though the target spectrum in our holim system is not fibrant, both of our maps factor through this one, so after composing with fibrant replacement they are still homotopic.

A $k$-simplex in the upper-left corner of our diagram is given by a flag of coarse weak equivalences of spaces with a left $G$-action

$$
Y_0 \xrightarrow{w_1} Y_1 \xrightarrow{w_2} \ldots \xrightarrow{w_k} Y_k
$$

Given this data and a flag of simplices as before

$$
\Delta[p_0] \rightarrow \ldots \rightarrow \Delta[p_k] \rightarrow N.G
$$

$$
[p_0] \rightarrow [p_1] \rightarrow \ldots \rightarrow [p_k] \rightarrow G
$$

both branches of this diagram produce a flag of weak equivalences of retractive spaces over $\Delta^{p_k}$; again we wish to build a simplicial homotopy between them.
The long route of our diagram again takes our flag of simplices to the $k$-simplex in the nerve of $G$

$$\bullet \xleftarrow{g_1} \bullet \xleftarrow{g_2} \cdots \xleftarrow{g_k} \bullet$$

where $g_i$ is the image in the category $G$ of the unique arrow in the poset $[p_i]$ connecting the image of the last vertex of $[p_i-1]$ to the last vertex of $[p_i]$. It then takes the image of this $k$-simplex in $w_k R_{\text{fin}}(*)$ defined by $co\alpha$ above:

$$Y_0 \xrightarrow{g_1^{-1} \cdot w_1} Y_1 \xrightarrow{g_2^{-1} \cdot w_2} \cdots \xrightarrow{g_k^{-1} \cdot w_k} Y_k$$

Note that since $w_i$ is equivariant $g_i^{-1} \cdot w_i(-)$ is the same map as $w_i(g_i^{-1} \cdot -)$. Finally, these spaces are pulled back along $r : \Delta^p \to *$ to give the flag

$$\Delta^p \times Y_0 \xrightarrow{id \times (g_1^{-1} \cdot w_1)} \Delta^p \times Y_1 \xrightarrow{id \times (g_2^{-1} \cdot w_2)} \cdots \xrightarrow{id \times (g_k^{-1} \cdot w_k)} \Delta^p \times Y_k$$

$$r^* Y_0 \xrightarrow{r^* (g_1^{-1} \cdot w_1)} r^* Y_1 \xrightarrow{r^* (g_2^{-1} \cdot w_2)} \cdots \xrightarrow{r^* (g_k^{-1} \cdot w_k)} r^* Y_k$$

The short route of our diagram transforms the flag of based left $G$-spaces into a flag of parametrized spaces

$$E_G \times_G Y_0 \xrightarrow{id \times w_1} E_G \times_G Y_1 \xrightarrow{id \times w_2} \cdots \xrightarrow{id \times w_k} E_G \times_G Y_k$$

and then restricts along the inclusion $i : \Delta^p \to BG$. Since $i^*$ commutes with the external smash product we arrive at the expression

$$(i^* E_G) \times_G Y_0 \xrightarrow{id \times w_1} (i^* E_G) \times_G Y_1 \xrightarrow{id \times w_2} \cdots \xrightarrow{id \times w_k} (i^* E_G) \times_G Y_k$$

Now to define a simplicial homotopy between these two branches, it is again enough to cook up some weak equivalences $f_i$ of retractive spaces over $\Delta^p$

$$r^* Y_0 \xrightarrow{r^* (g_1^{-1} \cdot w_1)} r^* Y_1 \xrightarrow{r^* (g_2^{-1} \cdot w_2)} \cdots \xrightarrow{r^* (g_k^{-1} \cdot w_k)} r^* Y_k$$

$$\xrightarrow{f_0} \xrightarrow{f_1} \cdots \xrightarrow{f_k}$$

$$(i^* E_G) \times_G Y_0 \xrightarrow{id \times w_1} (i^* E_G) \times_G Y_1 \xrightarrow{id \times w_2} \cdots \xrightarrow{id \times w_k} (i^* E_G) \times_G Y_k$$

which agree with deletion of the $X_i$'s and the deletion of terms from our original flag of simplices. In fact, each $f_i$ will be a homeomorphism, since both spaces in the $i$th column of this diagram are homeomorphic to $\Delta^p \times Y_i$

We define $f_i$ as before by taking the unique lift of $\Delta^p \to BG$ to $EG = | N \tilde{G} |$ which takes the final vertex to 1, and then extending this uniquely to a lift of $\Delta^p \to BG$ to $EG$, which necessarily takes the final vertex to $g_k^{-1} g_{k-1} \cdots g_{i+1}^{-1}$. It follows that this square commutes

$$\Delta^p \times Y_i \xrightarrow{id \times g_i^{-1} \cdot w} \Delta^p \times Y_i \xrightarrow{g_i \cdot g_{i-1}^{-1} \cdot w} EG \times_G Y_{i-1} \xrightarrow{g_i \cdot g_{i-1}^{-1} \cdot w} EG \times_G Y_i$$
and that the $g_i$ and $g_i^{-1}$ on the bottom cancel out to give the map $\text{id} \times w_i$. This allows the above rectangle to commute. It is easy to check that our convention defines a simplicial map and so we are done. □

Now it is easy to check that our combinatorial model for the assembly and coassembly maps fit into a strictly commuting diagram

\[
\begin{array}{c}
BG_+ \wedge K(\mathcal{F}) \\
\downarrow \\
BG_+ \wedge K(\mathcal{M}_f(\ast)) \\
\alpha \\
\downarrow \\
K(\mathcal{E}_f(BG)) \\
\downarrow \\
K(\mathcal{R}_f(BG)) \\
\end{array}
\quad
\begin{array}{c}
K(K_f(G)) \\
\downarrow \\
K(M_f(G)) \\
\downarrow \\
F(BG_+, K(\mathcal{F})) \\
\end{array}
\quad
\begin{array}{c}
K(I \circ E) \\
\downarrow \\
K(E) \\
\downarrow \\
F(BG_+, K(\mathcal{M}_f(\ast))) \\
\end{array}
\]

where the maps along the top row are the obvious restrictions of $\alpha$, the inclusion of categories, and $c\alpha$ from all finite based spaces to just finite sets. Therefore we get the homotopy-commuting diagram

\[
\begin{array}{c}
BG_+ \wedge S \\
\downarrow \\
BG_+ \wedge A(\ast) \\
\downarrow \text{assembly} \\
A(BG) \\
\downarrow \text{Cartan} \\
F(BG_+, A(\ast)) \\
\end{array}
\quad
\begin{array}{c}
K(K_f(G)) \\
\downarrow K(1_\mathcal{E}) \\
K(\mathcal{E}_f(BG)) \\
\downarrow K(1_\mathcal{R}) \\
K(\mathcal{R}_f(BG)) \\
\end{array}
\quad
\begin{array}{c}
K(\mathcal{K}_f(G)) \\
\downarrow K(I \circ E) \\
K(\mathcal{M}_f(G)) \\
\downarrow K(E) \\
F(BG_+, K(\mathcal{F})) \\
\end{array}
\quad
\begin{array}{c}
F(BG_+, S) \\
\end{array}
\]

At this point, we have reduced the main theorem to computing the composite along the top row of this diagram.

7. **Proof that the lift is the norm.**

We are reduced to proving that the assembly and coassembly maps on the $K$-theory of finite sets

\[
BG_+ \wedge K(\mathcal{F}) \to K(K_f(G)) \to K(\mathcal{K}_f(G)) \to F(BG_+, K(\mathcal{F}))
\]

compose to give the equivariant norm of $S$. In fact, we can reduce this to a result in the author’s thesis that worked at the level of topological Hochschild homology ([Mal14], section 3.7). Using machinery of Blumberg and Mandell, our assembly and coassembly on the $K$-theory of spaces give natural trace maps into $\text{THH}$, and since the composite map

\[
K(\mathcal{F}) \to K(\mathcal{M}_f(1)) \to \text{THH}(\mathcal{M}_f(1))
\]

is up to homotopy an augmentation of $A(\ast)$ by the sphere spectrum (in other words the composite is an equivalence), this is enough to give the above theorem. However we will save the $\text{THH}$-level argument for a future paper, since it seems to generalize...
well but uses somewhat elaborate geometric coherence machinery. For now, we will restrict our attention to when $G$ is a finite group, and give a $K$-theoretic proof.

Throughout this section, we will work in the model category of orthogonal spectra with a $G$-action, or $S[G]$-modules, as defined in [MMSS01]. We let the lowercase $f$ denote any fibrant replacement in this category. We will implicitly use the equivalence between this category and orthogonal $G$-spectra indexed on a complete universe $U$ found in [MM02]. Our constructions could easily be interpreted as taking place in the model category of genuine $G$-spectra from [MM02], but we will not explicitly need that here.

First we will give what is perhaps the standard definition of the equivariant norm map. This is obtained from the original definition of the equivariant transfer $X_{hG} \to X^G$ found in (LMSM86), III.7), simplified to the case $N = G$ and then composed with the standard inclusion $X^G \to X_{hG}$. Their group $\Gamma$ becomes $G \times G$ in this case, along the isomorphism $(\epsilon, \theta)$, which simplifies to the presentation below.

If $G$ is a finite group, we think of it as a left $G \times G$-set with action

$$(g, h)k = hkg^{-1}$$

We choose any embedding $e$ of $G$ into a $G \times G$-representation $V$, and let $f$ be any of the standard homeomorphisms from $V$ to the $\epsilon$-ball about the origin of $V$ (which shrink along rays and so are equivariant). Then we construct the $G \times G$-equivariant embedding

$$G \times V \to V$$

$$(g, v) \mapsto e(g) + f(v)$$

and then Pontryagin-Thom collapse to get a $G \times G$-equivariant map of based spaces

$$S^V \to S^V \wedge G_+$$

From this we define the pretransfer as either of the zig-zags

$$S \to \Omega^V \Sigma^V \Sigma_+^\infty G \to \Sigma_+^\infty G$$

$$S \to F_V S^V \to F_V S^V \wedge G_+ \to \Sigma_+^\infty G$$

These give the same morphism $[\tau]$ in the $G \times G$-equivariant stable homotopy category, and this morphism is independent of any of the choices we made.

Suppose $X$ is any cofibrant $S[G]$-module. Then the construction which takes a $G \times G$-spectrum $Y$ to the $G$-spectrum $Y \wedge_G X$ preserves all weak equivalences in the $Y$-variable, by an easy check. Therefore it takes the morphism $[\tau]$ to a well-defined morphism of $S[G]$-modules $[\tau(X)]$, the equivariant transfer map of $X$. $[\tau(X)]$ is given by the zig-zag

$$S \wedge_G X \to (F_V S^V) \wedge_G X \to (F_V S^V \wedge G_+) \wedge_G X \to \Sigma_+^\infty G \wedge_G X \cong X$$
Each of these terms has a left $G$-action, with the $G$-action on $X$ being the usual one, and the $G$-action on

$$X_{hG} := S \wedge_G X$$

being trivial. Continuing to work in the model category of left $S[G]$-modules, our morphism $[\tau(X)]$ in the homotopy category can be lifted to a map of spectra

$$\tau(X) : S \wedge_G X \rightarrow fX$$

Since the left side has trivial $G$-action, this factors through the fixed points $(fX)^G$. Unfortunately these fixed points are not derived. (They would be if we had used genuine $G$-spectra, and this map is then the Adams isomorphism.) Instead we map forward to the homotopy fixed points $(fX)^{hG}$, and the result

$$N(X) : X_{hG} \rightarrow X^{hG}$$

is the *equivariant norm map* of $X$.

**Remark.** Our definition of $N$ only really gives a natural transformation between $-_{hG}$ and $-^{hG}$ in the homotopy category, but it could be easily modified to be natural with respect to any small collection of maps. In [RV14] the definition of $[\tau(X)]$ is modified to an honest map $\tau(X)$, allowing a definition of $N$ as a natural transformation on the entire category of spectra. We won’t need such a refinement here.

We will now simplify the norm map in two special cases. The first case is when $X$ is a suspension spectrum, and the second is when $X$ is equivalent to a spectrum with trivial $G$-action. These results are mostly well-known (cf. [LAM82] for the case $X = \Sigma^\infty_+ EF$) but it can be difficult to find explicit proofs in the literature.

First we introduce a slightly nonstandard construction. Following [Coh04], we define for each $G$-representation $V$ and $\epsilon > 0$ the based $G$-space

$$S^V_\epsilon = V/(V - B_\epsilon(0))$$

The identity map of $V$ induces an equivariant homotopy equivalence $S^V \rightarrow S^V_\epsilon$. We use $S^V_\epsilon X$ as shorthand for $S^V_\epsilon \wedge X$, and $\Sigma^\infty_+ X$ to be the $G$-spectrum whose $n$th level is $S^n_\epsilon \wedge X$. This spectrum is almost good, in the sense that the adjoint structure maps $S^V_\epsilon \wedge X \rightarrow \Omega^W(S^W_\epsilon \wedge X)$ are closed inclusions, and so if $U$ is any complete $G$-universe the usual map into the replacement spectrum $R\Sigma^\infty_+ X$ which at level $V$ is

$$\Omega^U_\epsilon \Sigma^U_\epsilon \Sigma^V_\epsilon \wedge X$$

is an equivalence of $G$-spectra (in both senses).

Now suppose $B$ is a CW complex and $p : E \rightarrow B$ is a principal $G$-bundle, so that $X = \Sigma^\infty_+ E$ is a cofibrant $S[G]$-module. We may pick a countable-dimensional real inner product space $U$ with an orthogonal $G$-action, and an equivariant fiberwise embedding $e : E \rightarrow U \times B$ and real number $\epsilon > 0$ so that the embedding $e$ has a tubular neighborhood of radius $\epsilon$. In particular, $U$ may be finite-dimensional if $B$ is finite-dimensional, but in the general case $U$ will have countably infinite
dimension. We interpret $\Omega^U \Sigma^U X$ as the colimit of $\Omega^V \Sigma^V X$ over some system of finite-dimensional subrepresentations $V \subset U$. Given $e$ and $\epsilon$, we define the Pontryagin-Thom collapse

$$B_+ \xrightarrow{\theta(p)} \Omega^U \Sigma^U_e E_+$$

by the formula $(b, u) \mapsto (u - e, e)$, where $e$ is the preimage of $b$ closest to $u$. Of course, if all preimages are more than $\epsilon$ away from $u$ then the image is the basepoint. This collapse map extends easily to a map of spectra

$$\Sigma^\infty B_+ \xrightarrow{\theta(p)} R\Sigma^\infty_e E_+$$

**Proposition 7.1.** When $E \rightarrow B$ is a principal $G$-bundle,

1. The transfer $[\tau(\Sigma^\infty_+ E)]$ and the Pontryagin-Thom collapse $[\theta(p)]$ are the same in the homotopy category of $G$-spectra.
2. The norm map $N(\Sigma^\infty_+ E)$ has adjoint

$$\tilde{N} : \Sigma^\infty_+ (EG \times B) \rightarrow f\Sigma^\infty_+ E$$

which is homotopic to a family of Pontryagin-Thom collapse maps for $E \rightarrow B$, along embeddings $E \rightarrow \mathbb{R}^\infty \times B$ which vary continuously and equivariantly in the $EG$ coordinate.
3. If $h : E \rightarrow K$ is any equivariant map to a space $K$ with a trivial $G$-action then the composite

$$\Sigma^\infty_+ B \xrightarrow{N} (\Sigma^\infty_+ E)^{hG} \rightarrow (\Sigma^\infty_+ K)^{hG} = \text{Map}(BG, \Sigma^\infty_+ K)$$

is adjoint to a Pontryagin-Thom collapse along the bundle

$$G \rightarrow E \times_G EG \rightarrow B \times BG$$

followed by the map $E \times_G EG \rightarrow K$ given by $h$.

**Proof.** (1) The pretransfer is equivalent to the map

$$S^V \rightarrow S^V \wedge G_+$$

$$v \mapsto (v - g, g)$$

where $g$ is the closest element of $G \subset V$ to $v$. We are reduced to showing the following diagram of $G$-spectra commutes up to equivariant homotopy.

\[
\begin{array}{c}
S \wedge_G E_+ \xrightarrow{\theta(p)} \rightarrow \Omega^V \Sigma^V_e E_+ \xrightarrow{\sim} \rightarrow \Sigma^\infty_e E_+ \\
\downarrow \sim \downarrow \sim \\
F \Sigma^V \wedge_G E_+ \xrightarrow{\text{collapse}} \rightarrow F \Sigma^V_\epsilon \wedge G_+ \wedge_G E_+ \xrightarrow{\approx} \rightarrow F \Sigma^V_\epsilon \wedge E_+
\end{array}
\]
We want to reduce to analyzing the map at “level $V$” but we must be careful since this kind of analysis does not treat orbits well. So we instead provide a $G \times G'$-equivariant homotopy of spectra making this diagram commute.

$$\begin{array}{c}
S \wedge E' \xrightarrow{\theta(p)} R \Sigma^\infty_\epsilon E_+ \sim \Sigma^\infty_\epsilon E_+ \\
\sim \xrightarrow{\text{collapse}} \xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\text{collapse}} \xrightarrow{\sim}
\end{array}$$

Here $G' = G$ but acts differently on the spaces above. $G \times G'$ acts on both $G$ and $V$ by left and right multiplication. The space $E$ has its usual left $G$-action and trivial $G'$-action. We use $E'$ to denote the same space, but with trivial $G$-action and nontrivial $G'$-action given by $G' = G$. Finally, $U$ has a $G$-action but trivial $G'$-action. With these conventions, taking $G'$-orbits gives the first diagram above. Therefore it suffices to give a $G \times G'$-equivariant homotopy for the diagram of spaces

$$\begin{array}{c}
S^V \wedge E' \xrightarrow{\theta(p)} \Omega^U \Sigma^U \Sigma^\epsilon_\epsilon E_+ \sim \Sigma^\epsilon_\epsilon E_+ \\
\sim \xrightarrow{\text{collapse}} \xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\text{collapse}} \xrightarrow{\sim}
\end{array}$$

We restrict attention to $1 \in O(V)$. To each choice of $\bar{x}$ and $x$ there is a unique element $\bar{g}$ such that $\bar{g}x = \bar{x}$. This gives a continuous assignment of pairs $(x, \bar{x}) \in E' \times_B E$ to $(x, \bar{g}) \in E' \times G$. In fact, it gives a $G \times G'$-equivariant homeomorphism

$$E \times_B E' \cong G \times G'$$

Then our $G \times G'$-equivariant homotopy is given by the formula

$$(u, v, x) \in U \times V \times E' \mapsto (u - \bar{x} \cos t, v - \bar{g} \sin t, \bar{g}x) \in \Sigma^\epsilon_\epsilon \Sigma^\epsilon_\epsilon E_+$$

Here $\bar{x} \cos t$ means the closest point $\bar{x}$ in the scaling of the original embedding by $\cos t$.

(2) We define a dotted map in this square

$$\begin{array}{c}
B \xrightarrow{\theta(p)} \Map^G(EG, \Omega^U \Sigma^U \Sigma^\epsilon_\epsilon E_+) \\
\Map^G(EG, \Omega^{R^\infty} \Sigma^{R^\infty} E_+) \sim \Map^G(EG, \Omega^{R^\infty} \oplus U \Sigma^{R^\infty} \oplus U E_+) \\
\end{array}$$

by choosing an equivariant map

$$f : EG \rightarrow \Emb_{B,\epsilon}(E, R^\infty \times B)$$

of $EG$ into the fiberwise embeddings of $E$ into $R^\infty \times B$ whose fibers are always at least $\epsilon$ apart. This latter space has the usual conjugation $G$-action, and it is weakly contractible. Therefore $f$ exists and is unique up to equivariant homotopy. We rewrite $f$ as a map

$$E \times EG \rightarrow R^\infty$$
and notice that $f(x, y) = f(gx, gy)$. Now the dotted map above is the adjoint of
the Pontryagin-Thom collapse

$$B \times EG \to \Omega^{\infty} \Sigma^{\infty}_{\ell} E_+$$

which for each point $y \in EG$ collapses onto the image of $f(-, y)$. Then we define
a homotopy between the two branches of the above square by

$$(s - (\cos t)f(\bar{e}, y), u - (\sin t)\bar{e}, \bar{e}) \in \Omega^{\infty}_{\ell} \oplus U \Sigma^{\infty}_{\ell} \oplus U E_+$$

where the $\bar{e}$ must be the same in both coordinates, or else we go to the basepoint.

(3) This follows quickly from (2). Our simplified norm map, pushed forward to $K$,
becomes

$$(s, b, y) \in \mathbb{R}^{\infty} \times B \times EG \mapsto (s - f(\bar{e}, y), h(\bar{e})) \in \Omega^{\infty} \Sigma^{\infty}_{\ell} K_+$$

This is invariant under the $G$-action on $EG$ and so descends to a map on $BG$. We
reinterpret $f$ as a fiberwise embedding of the bundle $E \times G EG \to B \times BG$ into
$\mathbb{R}^{\infty} \times B \times BG$, and now this really is a Pontryagin-Thom collapse followed by $h$. □

As a special case, if we take $E = EG$ and $K = \ast$ in part (3) of the above proposition,
we see that the norm map for the sphere spectrum

$$\Sigma^{\infty}_{+} BG \to \text{Map}_+ (BG_+, \Omega^{\infty} S^{\infty}_{\ell})$$

is adjoint to a Pontryagin-Thom collapse along the bundle $E \times G EG \to B \times BG$, followed by collapsing the total space of that bundle to a point (cf. [LMM82]). This
bundle is up to homotopy the same as the diagonal map $BG \to B \times BG$. It is
possible to rewrite the fiber of this bundle as $G$, with $G \times G$-monodromy given by
left and right multiplication.

Now we can simplify the norm in the case where the $G$-spectrum $X$ has a trivial
$G$-action. Essentially, $X$ can be made free by writing it as $EG_+ \wedge X$, and then the
$X$ stays separated from the action.

**Proposition 7.2.** Let $X$ be any cofibrant orthogonal spectrum with a trivial $G$
action.

1. The equivariant transfer map $[\tau(EG_+ \wedge X)]$ is the smash of the transfer
map of $\Sigma^{\infty}_{\ell} EG$ and the identity map of $X$.
2. The norm map $N(X)$ is adjoint to

$$\Sigma^{\infty}_{+} (BG \times BG) \wedge X \to S \wedge X$$

and up to homotopy this is the Pontryagin-Thom collapse map along $EG \times G$
$EG$ smashed with the identity of $X$.

**Proof.** Just take the above proof, and smash every term in every diagram with $X$
and the identity map. □
Finally, we return to $K$-theory. Our composite of assembly and coassembly is a map

$$(BG \times BG)_+ \wedge K(F) \to K(F)$$

that uses the infinite loop space structure on $K(F)$ to add a given point to itself $G$ times. We mix the ordering of this sum as we rove around $BG \times BG$, in exactly the same way as if we had a Pontryagin-Thom collapse along the bundle described above. The following theorem will make this idea precise.

Let $C$ be a $\Sigma$-free $E_\infty$ operad of unbased spaces with May’s convention of $C(0) = \ast$. Let $X$ be a $C$-algebra, $Y$ an ordinary based space, and $f : Y \to X$ a map of based spaces. Let $B^\infty X$ be the (essentially unique) spectrum whose zeroth space is $X$, so that $f$ must come from some map in the stable homotopy category $\Sigma^\infty Y \cong B^\infty X$. Let $B$ be an unbased space with fundamental group $\Gamma$, and $\Gamma \to \Sigma_j$ a homomorphism, which induces in the usual way an $j$-sheeted covering space $E \to B$. Finally, let $E\Sigma_j \xrightarrow{i} C(j)$ be any $\Sigma_j$-equivariant map. Then consider the composite

$$B \times Y \xrightarrow{\phi} BT \times X \cong ET \times_{\Gamma} X \xrightarrow{E\phi \times \Delta} E\Sigma_j \times_{\Sigma_j} X^j \xrightarrow{i \times \text{id}} C(j) \times_{\Sigma_j} X^j \xrightarrow{\theta(p) \times \text{id}} X$$

If the point in $B \times Y$ is of the form $(b, \ast)$ then its image in $X$ is the basepoint, so this can be interpreted as a map out of $B_+ \wedge Y$. Intuitively, this composite takes each point $y \in Y$ to a sum of $j$ copies of $f(y) \in X$, but as we rove around $B$, the ordering in this sum is permuted according to the rule given by $\Gamma \to \Sigma_j$.

**Theorem 7.3.** In the homotopy category, this composite is adjoint to the map of spectra

$$\Sigma^\infty B_+ \wedge Y \xrightarrow{\theta(p) \wedge \text{id}} \Sigma^\infty E_+ \wedge Y \xrightarrow{r \wedge \overline{T}} B^\infty X$$

where $\theta(p)$ is Pontryagin-Thom collapse, and $r : E \to \ast$ is the collapse of $E$ onto a point.

**Remark.** This theorem is apparently quite classical (cf. [KP72] and [Ada78]) so we will give only a brief sketch of the proof.

**Proof.** Clearly the choice of $i$ is irrelevant since any two such maps are connected by an equivariant homotopy. It suffices to prove the theorem for one fixed $E_\infty$ operad, since any two are related by a zig-zag of weak equivalences of operads $C \to C'$. Therefore we take $C$ to be the little $\infty$-cubes operad. Our composite map from $B \times Y$ into $X$ factors through $C(j) \times_{\Sigma_j} X^j$. We can regard this as a rule that associates in a continuous way to each point of $B \times Y$ a 1-simplex in the space $\Omega^\infty B(\Sigma^\infty, C, X)$, which arises from some finite level $\Omega^n B(\Sigma^n, C_n, X)$ by having the sphere map to itself by the identity. This rule creates a homotopy of maps $B \times Y \to \Omega^\infty B^\infty X$, which at one end is our original composite included along $X \xrightarrow{\sim} \Omega^\infty B^\infty X$, and which at the other end instead maps $C(j) \times_{\Sigma_j} X^j \to \Omega^\infty \Sigma^\infty X$. 
using the map of monads $C \to \Omega^\infty \Sigma^\infty$. This latter map can be modified by an explicit homotopy so that it lines up with our definition of Pontryagin-Thom collapse above.

Therefore the above composite is homotopic to the composite

$$
\begin{align*}
B \times Y & \xrightarrow{1 \times f} B \times X \xrightarrow{\theta_p} \Omega^\infty \Sigma^\infty (E \times X) \leftarrow \Omega^\infty \Sigma^\infty X \leftarrow \Omega^\infty \Sigma^\infty X \rightarrow \Omega^\infty B^\infty X
\end{align*}
$$

Checking basepoints and moving $f$ further down the chain of maps, we arrive at the composite

$$
\Sigma^\infty B_+ \wedge Y \xrightarrow{\theta_p \wedge \text{id}_Y} \Sigma^\infty E_+ \wedge Y \xrightarrow{S_e \wedge Y} \Sigma^\infty Y \xrightarrow{\mathcal{I}} B^\infty X
$$

as claimed. \hfill \square

We can now prove Thm 1.2 from the introduction.

**Theorem 7.4.** If $G$ is a finite group and $R$ is a ring spectrum then the composite of assembly, Cartan, and coassembly

$$
BG_+ \wedge K(R) \xrightarrow{\text{assembly}} A(BG; R) \xrightarrow{\text{Cartan}} V(BG; R) \xrightarrow{\text{coassembly}} F(BG_+, K(R))
$$

is the equivariant norm map

$$
K(R)_{hG} \to K(R)^{hG}
$$
on $K(R)$ with the trivial $G$-action.

**Proof.** Let $n = |G|$ and fix a bijection between $G$ and the standard set of $n$ elements. Define a homomorphism $\Gamma = G \times G \to \Sigma_n$ by sending the pair $(g, h)$ to the permutation $x \mapsto g^{-1}xh$. Apply Thm 7.3 to $X = \Omega |w_\cdot S \cdot F| \simeq QS^0$, $Y = S^0$ included as the object $(S^0)$ in $w_0 \mathcal{F} \to w_0 S_1 \mathcal{F}$, so that $\mathcal{I}$ is an equivalence of spectra. Let $\mathcal{C}$ be the Barratt-Eccles operad $W$. Recall that $W(n) = |N, \Sigma_n|$, with operad structure is given by maps of categories, which are obvious once we specify that the objects of these categories form the associative operad.

By Propositions 6.3 and 6.5, the composite of assembly and coassembly on the $K$-theory of finite sets is, when restricted to $Y = S^0$, the map of spaces

$$(BG \times BG)_+ \to K(\mathcal{F}) \simeq \Omega^\infty S^\infty$$

which is defined simplicially as

$$(g_1, \ldots, g_k; h_1, \ldots, h_k) \mapsto G_+ \xrightarrow{g_1^{-1} \cdot \cdot \cdot h_1} G_+ \xrightarrow{g_2^{-1} \cdot \cdot \cdot h_2} \cdots \xrightarrow{g_k^{-1} \cdot \cdot \cdot h_k} G_+$$

and then the usual inclusion into $S_1$ and looping on the outside.

This is the same as

$$
BG_+ \cong ET/\Gamma_+ \xrightarrow{\phi \times (\Delta \circ f)} E(\Sigma_n)_+ \times \Sigma_n \Omega |w_\cdot S \cdot F|\n = W(n) \times \Sigma_n \Omega |w_\cdot S \cdot F| \to \Omega |w_\cdot S \cdot F|
$$
which is the composite of Thm 7.3, with \( i \) chosen to be the isomorphism \( E\Sigma_n \cong W(n) \). Therefore this composite is adjoint to the transfer and collapse map of \( BG \times BG \) along the bundle whose fiber is \( G \) and whose \( G \times G \)-monodromy is given by left and right multiplication. By Prop 7.2 this is the equivariant norm map of \( S \), followed by the map from \( S \) into whichever spectrum is given by delooping the Barratt-Eccles operad action on \( \Omega \Omega \). In light of Thm 3.8 this spectrum is equivalent to \( K(F) \cong S \). Therefore assembly and coassembly of finite sets compose to give the equivariant norm of \( S \).

The assembly and coassembly maps for \( K(R) \) compose to give some map with adjoint

\[
(BG \times BG)_+ \wedge K(R) \longrightarrow K(R)
\]

Our identification \( K(F) \cong S \) agrees with the map \( S \longrightarrow K(R) \) including finite sets into \( R \)-modules because the equivalence of Thm 3.8 was natural with respect to maps of Waldhausen categories. So, by the pairing of Prop 4.39 and the observation of Prop 5.5 the above map is the smash product of map that we examined on finite sets and the identity map of \( K(R) \). Applying Prop 7.2 again, we conclude that this is the equivariant norm of \( K(R) \). □

References

[Ada78] J. F. Adams, Infinite loop spaces, no. 90, Princeton University Press, 1978.
[BFJR04] A. Bartels, T. Farrell, L. Jones, and H. Reich, On the isomorphism conjecture in algebraic K-theory, Topology 43 (2004), no. 1, 157–213.
[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen, The cyclotomic trace and algebraic K-theory of spaces, Inventiones mathematicae 111 (1993), no. 1, 465–539 (English).
[BM11a] C. Berger and I. Moerdijk, On an extension of the notion of Reedy category, Mathematische Zeitschrift 269 (2011), no. 3-4, 977–1004.
[BM11b] A. J. Blumberg and M. A. Mandell, Derived Koszul duality and involutions in the algebraic K-theory of spaces, Journal of Topology (2011), jtr003.
[BP72] M. Barratt and S. Priddy, On the homology of non-connected monoids and their associated groups, Commentarii Mathematici Helvetici 47 (1972), no. 1, 1–14.
[BR05] A. Bartels and H. Reich, On the Farrell-Jones conjecture for higher algebraic K-theory, Journal of the American Mathematical Society 18 (2005), no. 3, 501–545.
[CG13] G. Carlsson and B. Goldfarb, Algebraic k-theory of geometric groups, arXiv preprint arXiv:1305.3349 (2013).
[Clap81] M. Clapp, Duality and transfer for parametrized spectra, Archiv der Mathematik 37 (1981), no. 1, 462–472.
[Coh04] R. L. Cohen, Multiplicative properties of atiyah duality, Homology, Homotopy and Applications 6 (2004), no. 1, 269–281.
[CP84] M. Clapp and D. Puppe, The homotopy category of parametrized spectra, manuscripta mathematica 45 (1984), no. 3, 219–247.
[CP05] G. Carlsson and E. K. Pedersen, Controlled algebra and the Novikov conjectures for K- and L-theory, Topology 34 (1995), no. 3, 731–758.
[ES] G. E. Evans and R. G. Swan, K-theory of finite groups and orders.
[FJ93] F. T. Farrell and L. E. Jones, Isomorphism conjectures in algebraic k-theory, Journal of the American Mathematical Society (1993), 249–297.
[Goo03] T. G. Goodwillie, Calculus III: Taylor series, Geometry & Topology 7 (2003), 645–711.
[HS99] M. Hovey and N. P. Strickland, *Morava k-theories and localisation*, vol. 666, American Mathematical Soc., 1999.

[KP72] D. S. Kahn and S. B. Priddy, *Applications of the transfer to stable homotopy theory*, Bull. Amer. Math. Soc. 78 (1972), no. 1972, 135–146.

[LMM82] G. Lewis, J. P. May, and J. McClure, *Classifying G-spaces and the Segal conjecture*, Current Trends in Algebraic Topology, CMS Conference Proc. 2, 1982, pp. 165–179.

[LMSM86] L. G. Lewis, J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, vol. 1213, Springer-Verlag Berlin-New York, 1986.

[LR05] W. Lück and H. Reich, *The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory*, Handbook of K-theory 2 (2005), 703–842.

[Mal12] C. Malkiewich, *A tower connecting gauge groups to string topology*, arXiv preprint arXiv:1209.1778 (2012).

[Mal14] C. Malkiewich, *Duality and linear approximations in Hochschild homology*, K-theory, and string topology*, Ph.D. thesis, Stanford University, 2014.

[Mer14] M. Merling, *Equivariant algebraic K-theory*, Ph.D. thesis, University of Chicago, 2014.

[Mil59] J. Milnor, *On spaces having the homotopy type of a CW-complex*, Transactions of the American Mathematical Society (1959), 272–280.

[MM02] M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Memoirs of the American Mathematical Society, no. 755, American Mathematical Society, 2002.

[MMO] J. P. May, M. Merling, and A. Osorno, *Equivariant infinite loop space theory*.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proceedings of the London Mathematical Society 82 (2001), no. 02, 441–512.

[MS06] J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs, vol. 132, American Mathematical Society, 2006.

[Rav92] D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, no. 128, Princeton University Press, 1992.

[RS14] G. Raptis and W. Steimle, *On the map of Bökstedt–Madsen from the cobordism category to A–theory*, Algebraic & Geometric Topology 14 (2014), no. 1, 299–347.

[RV14] H. Reich and M. Varisco, *On the adams isomorphism for equivariant orthogonal spectra*, arXiv preprint arXiv:1404.4034 (2014).

[Sch08] S. Schwede, *On the homotopy groups of symmetric spectra*, Geometry & Topology 12 (2008), no. 3, 1313–1344.

[Seg74] G. Segal, *Categories and cohomology theories*, Topology 13 (1974), no. 3, 293–312.

[Ste67] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J 14 (1967), no. 2, 133–152.

[Wal85] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology, Springer, 1985, pp. 318–419.

[WW93] M. Weiss and B. Williams, *Assembly*, Novikov conjectures, index theorems and rigidity 2 (1993), 332–352.

Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W Green St
Urbana, IL 61801
cmalkiew@illinois.edu