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Lund, Mads M.; Yang, Fan; Mølmer, Klaus

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Perfect splitting of a two-photon pulse
Mads M. Lund,1,* Fan Yang1,† and Klaus Mølmer2,‡
1Center for Complex Quantum Systems, Department of Physics and Astronomy, Aarhus University, 8000 Aarhus C, Denmark
2Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, 2100 Copenhagen, Denmark

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We employ a cascaded system approach to numerically simulate the interaction of photon pulses with a two-level scatterer in a chiral waveguide QED setup and show that the scattering of a two-photon pulse may lead to the predominant population of only two output wave packet modes in an entangled state, |2,0⟩−|0,2⟩. In a complementary wave-packet basis, this is a product state of two orthogonal single-photon wave packets. Operating this process in reverse allows a perfect merging of distinguishable single-photon wave packets into a single-mode pulse carrying two identical photons.

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I. INTRODUCTION
Quantum states of light provide indispensable resources for quantum sensing [1–4] and quantum information processing [5,6]. In recent years, large efforts have been devoted to their preparation, interactions with matter, and detection at the fundamental quantum level. Processes that can produce nonclassical states of light and perform nonclassical transformations of these states play a key role in this endeavor. Such processes put strict requirements on the strength, coherence, and bandwidth of the interactions applied. In cavity QED, the interaction between a single optical mode and a classically driven two-level system (TLS) displays strong nonlinearity and permits the construction of a variety of field quantum states [7,8]. For traveling fields, the situation is generally more complex, as nonlinear interactions may lead to the population of many propagating modes, reducing the purity of the desired few-mode states [9–19]. A chiral interaction between traveling fields and nonlinear systems offers a spatial separation of input and output channels for convenient application in quantum networks [20]. Such systems permit an effective theoretical description in which a virtual cavity mode leaks the initial input pulse with arbitrary number state contents and a cascaded master equation describes the joint state of the cavity mode and the scatterer [12,13]. The fields emitted by the cavity mode and the scatterer interfere and the total output can be analyzed in terms of its eigenmodes. These eigenmodes can be subsequently addressed by incorporating a corresponding virtual pickup cavity modes in the theory [18,19]. Here, we apply this cascaded theory to the scattering of a single two-photon pulse, propagating along a waveguide, interacting in a chiral manner with a two-level system. We show that for certain pulses, the output state splits very accurately into two modes, populated by one photon each [see Fig. 1(a)]. Such an operation, and its inverse recombination of two orthogonal modes into a single mode, may be useful for applications in quantum information science, and we discuss several properties of these processes.

The article is organized as follows. In Sec. II, we review the virtual cavity approach to the preparation and detection of quantum pulses. In Sec. III, we present a correlation function analysis of the output field and we identify the conditions for the perfect splitting of two photons in orthogonal modes. In Sec. IV, we analyze the inverse scattering process combining two orthogonal single-photon states into a single-mode two-photon state. In Sec. V, we provide a summary and outlook.

II. CASCADED SYSTEM APPROACH TO QUANTUM PULSES
This section reviews the virtual cavity treatment [18,19] of quantum pulses interacting chirally with a spatially localized quantum scatterer, shown in Fig. 1.

A. Pulses of quantum radiation
The illustration in Fig. 1(a) shows our physical problem of interest: a two-photon single-mode pulse scattering on a TLS in a chiral one-dimensional geometry. The photons in the single incoming temporal mode, u(t), are described by the creation operator

\[ \hat{b}_u^\dagger = \int dt \, u(t) \hat{b}^\dagger(t), \tag{1} \]

where the bosonic creation operator, \( \hat{b}(t) \), represents the creation of a photon, passing the location of the scattering process at time \( t \) [9]. The standard bosonic commutation relation, \([\hat{b}(t), \hat{b}^\dagger(t')] = \delta(t - t')\), in conjunction with an orthogonal set of temporal modes, \( f_i(t) \), define operators on the form of Eq. (1) as standard bosonic operators, i.e., \([\hat{b}_i, \hat{b}^\dagger_j] = \delta_{ij}\). The characterization of the pulse, \( u(t) \), in the time domain is equivalent to the specification of a spatial wave packet, e.g., at time zero with the amplitude \( u(t) \) assigned to the spatial location...
interaction strength of photon state, scatters chirally on a two-level system, with a dipole coupling, input wave packet and detection of a single output wave packet mode orthogonal modes, cavity approach, sketched in Fig. 1(b), where a cavity with constant frequency and a slowly varying function, factorization of the mode into an oscillating phase factor with $=−x$. The scattered output field occupies two output modes, $t$ (assuming the rotating wave approximation).

The dynamics of the resonant quantum scatterer become slow of emitting a classical field [18,21]:

To analyze the quantum content of any specific output mode, $v(t)$, we imagine its complete transfer into a downstream time-dependent cavity, as sketched in Fig. 1(b). Our cascaded system is thus equipped with yet another virtual cavity. To map the full quantum content of a specific mode, $v(t)$, the cavity input coupling must be of the form

$$g_{vi}(t) = \frac{u^*(t)}{\sqrt{1 - \int_0^t dt'|u(t')|^2}}. \quad (2)$$

Applying the analysis of cascaded quantum systems [12,13], or the more general so called SLH formalism [22] (see the Appendix), the dynamics of the virtual cavity and the TLS is described by a Lindblad master equation in the Born-Markov approximation ($\hbar = 1$)

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_{i=0}^n \left( L_i^\dagger L_i \rho - \frac{1}{2} (L_i^\dagger L_i, \rho) \right). \quad (3)$$

In Eq. (3), $\hat{H} = \hat{H}_0 + \hat{H}_{us}$ denotes the Hamiltonian of the $u$ cavity and the scatterer with the noninteracting Hamiltonian, $H_0$, of the two components, vanishing in the interaction picture adopted in this article, and the interaction Hamiltonian

$$\hat{H}_{us} = \frac{1}{2} \left( g_u(t) \hat{a}_u^\dagger \hat{a}_v - g_u(t) \hat{a}_u^\dagger \hat{a}_v \right). \quad (4)$$

and

$$\hat{L}_{us}^{iv} = \sqrt{v} \hat{a}^\dagger + g_v(t) \hat{a}_u. \quad (5)$$

represents the interference of the quantized radiation emitted by the scatterer and the leftmost $u$ cavity, while $L_i$ with $i = 1, 2, ..., n$ represent further local dissipation mechanisms on the scatterer, which are assumed to be negligible in this article. The operator $\hat{a}_u$ is the input cavity field operator, $\hat{a}_v^\dagger$ and $\hat{a}_v$ the TLS raising and lowering operators, and $\sqrt{v}$ is the coupling strength between the TLS and the waveguide (making $\gamma$ the decay rate of the TLS excited state). This form of the Hamiltonian and Lindblad operator ensures excitation only travels upstream. The master equation for the $u$ cavity and the TLS gives access to expectation values of the emitted field amplitude and intensity.

**C. Quantum state contents of an output wave packet**

We are interested in the quantum state of the output field, i.e., the quantum state contents of the modes occupied by the field, and to this end we first apply the quantum regression theorem [23,24] to determine the first-order correlation function, $g^{(1)}(t,t')$, and hereby a decomposition into orthonormal output modes, $v(t)$, with mean photon occupation number, $n_i$ [18].

$$g^{(1)}(t,t') = \langle \hat{L}_{us}^{iv}(t) \hat{L}_{us}^{iv}(t') \rangle = \sum_i n_i v_i^* v_i t v_i t \rangle. \quad (6)$$

To extract information about the quantum contents of an output wave packet, $v(t)$, we imagine its complete transfer into a downstream time-dependent cavity, as sketched in Fig. 1(b). Our cascaded system is thus equipped with yet another virtual cavity. To map the full quantum content of a specific mode, $v(t)$, the cavity input coupling must be of the form

$$g_v(t) = \frac{v(t)}{\sqrt{\int_0^t dt' |v(t')|^2}}. \quad (7)$$

The master equation, Eq. (3), describing the three-component density matrix, $\rho_{usv}$, is equipped with the Hamiltonian

$$H_{usv} = \frac{i}{2} \left( g_u(t) \sqrt{v} \hat{a}_v^\dagger \hat{a}_u + g_v(t) \hat{a}_u^\dagger \hat{a}_v \right)$$

$$+ \sqrt{v} g_v^*(t) \hat{a}_u^\dagger \hat{a}_v - \text{H.c.} \right) \quad (8)$$

and the Lindblad term

$$L_{usv}^{uv} = g_v^*(t) \hat{a}_u^\dagger + \sqrt{v} \hat{a}_v^\dagger + g_v^*(t) \hat{a}_v. \quad (9)$$

To extract information about the quantum contents of an output wave packet, we solve the master equation for the full density matrix, $\rho_{usv}$, to obtain the reduced density matrix of the output field in the $v$ mode, $\rho_v = Tr_{us}(\rho_{usv})$.

In the next section, we employ the outlined formalism to investigate the multimode output from two-photon scattering on a TLS. It is also possible to address the joint state of multiple output modes by introducing further cascaded virtual

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**FIG. 1.** (a) A single temporal mode, $u(t)$, occupied by a two-photon state, scatters chirally on a two-level system, with a dipole interaction strength of $\sqrt{v}$. The scattered output field occupies two orthogonal modes, $v_1(t)$ and $v_2(t)$ [Eqs. (16), (17)], each in a single-photon Fock state. (b) Representation of the scattering of a single input wave packet and detection of a single output wave packet mode by virtual cavities. The leftmost virtual cavity, with time-dependent coupling, $g_u(t)$, in Eq. (2), emits its initial two-photon Fock state content in the temporal mode $u(t)$. The rightmost cavity, with time-dependent coupling $g_v(t)$ in Eq. (7), absorbs the quantum state content of the temporal mode, $v_i(t)$, of the scattered field.

$x = -\alpha$ [17]. We shall without change of notation assume factorization of the mode into an oscillating phase factor with constant frequency and a slowly varying function, $u$. This is equivalent to the application of an interaction picture in which the dynamics of the resonant quantum scatterer become slow (assuming the rotating wave approximation).

**B. Open system treatment of the scattering of a quantum pulse**

To simulate the dynamics in Fig. 1(a), we apply the virtual cavity approach, sketched in Fig. 1(b), where a cavity with a time-dependent output coupling, $g_u(t)$, releases the desired pulse that scatters on the TLS. In this approach, the light-matter interaction is represented as a direct chiral coupling of the scatterer and the virtual cavity mode. The time-dependent coupling function, $g_u(t)$, leading to any desired temporal mode, $u(t)$, is followed by an analysis of the linear process of emitting a classical field [18,21]:

$$g_u(t) = \frac{u^*(t)}{\sqrt{1 - \int_0^t dt'|u(t')|^2}}. \quad (2)$$

Applying the analysis of cascaded quantum systems [12,13], or the more general so called SLH formalism [22] (see the Appendix), the dynamics of the virtual cavity and the TLS is described by a Lindblad master equation in the Born-Markov approximation ($\hbar = 1$)
III. SPLITTING OF A SINGLE-MODE TWO-PHOTON STATE

We now consider the scattering on a TLS of two photons in a single Gaussian temporal mode, \( u(t) \),

\[
u(t) = \frac{1}{\sqrt{\tau \pi}} e^{-\frac{(\omega_0)^2}{4\tau}}.
\]

The Hamiltonian and the emitted field operators are described in Eqs. (4) and (5), respectively. The relevant output modes and their respective mean photon numbers are determined by the autocorrelation function (6), which can be obtained by the master equation (3) and the quantum regression theorem [e.g., by diagonalizing \( g^{(1)}(t,t') \) as represented on a two-dimensional discrete time grid].

A. Eigenmodes and their mean occupation numbers in the output field

We diagonalize the autocorrelation function, Eq. (6), and obtain the mean photon occupation number of the most populated modes for different durations of the input mode, \( \tau \). The mean photon occupation number, \( n_i \), of the six most populated output modes are shown in Fig. 2 as a function of \( \tau \) for two equal pulse durations. For small \( \tau \ll \gamma^{-1} \) and large durations \( \tau \gg \gamma^{-1} \), the figure shows that the output field predominantly occupies a single mode. This is qualitatively expected since the short pulses are dominated by off-resonant frequency components that do not interact with the TLS, while the very long pulses have low-photon coincidence probability and hence scatter in a mostly linear manner. In an intermediate pulse width regime, \( \tau \lesssim \gamma^{-1} \), only the two most populated modes, represented by the full (black) and dashed (red) curves, are significantly populated, with an equal population of \( n_1 = n_2 = 0.9993 \) at \( \tau \approx 0.38 \gamma^{-1} \). The insets of Fig. 2 show that the output modes with higher and lower population exchange characteristic around the crossing in the eigenvalue spectrum of \( g^{(1)}(t,t') \). In a range from intermediate to long pulse durations, the output field is highly multimode as indicated by multiple nonvanishing populations shown in Fig. 2. The pulse shapes and durations needed to observe the effects presented here are manageable within all relevant experimental platforms [25,26], and the effects are robust against variations in the incoming pulse shape.

B. The quantum state of the output field

A surprising feature of Fig. 2 is the very accurate splitting of a single mode into two equally populated modes. The final time reduced density matrix of the downstream \( \nu \) cavity absorbing either of the two equally populated modes, \( \nu_1 \) or \( \nu_2 \), is given by

\[
\rho_{\nu_1} = \rho_{\nu_2} \approx \frac{1}{2} |0\rangle \langle 0| + |2\rangle \langle 2|; \quad (11)
\]
i.e., both modes occupy mixed states with equal zero- and two-photon components. The total output state is a pure state and consists almost exclusively of these two modes, which is compatible with the specific entangled state of the form

\[
|\Psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}} (|0\rangle_{\nu_1} |2\rangle_{\nu_2} - |2\rangle_{\nu_1} |0\rangle_{\nu_1}). \quad (12)
\]

The decomposition of the output wave function in two-photon states is not a coincidence: Due to bosonic statistics, a two-photon wave function is symmetric under exchange of particle indices and can therefore be decomposed by the so-called Takagi decomposition [10,27,28]

\[
\Psi(t_1,t_2) = \sum_i \lambda_i \phi_i(t_1) \phi_i(t_2). \quad (13)
\]

From the two-photon Takagi decomposed wave function, the autocorrelation function is written as

\[
g^{(1)}(t_1,t_2) = \sum_i \lambda_i^2 \phi_i^*(t_1) \phi_i(t_2), \quad (14)
\]

which is exactly in the diagonalized form equivalent to Eq. (6). Hence, the relation between the mean photon numbers, \( n_i \), and the singular values, \( \lambda_i \), is \( n_i = |\lambda_i|^2 \) and the basis functions are equal up to a global phase \( \phi_i(t) = e^{i\epsilon_i} \phi_i(t) \). Thus, any output two-photon wave function will only populate the vacuum and the two-photon components when represented in the eigenbasis of the autocorrelation function, Eq. (6). Our calculation has identified a special case of the Takagi form with only two populated modes with equal populations. For this special degenerate case, Eq. (12) can also be expressed as a product state of one-photon Fock states

\[
|\Psi_{\text{out}}\rangle = |1\rangle_{\nu_1} |1\rangle_{\nu_2}. \quad (15)
\]
in a rotated basis corresponding to the new mode functions

\[
\nu_1'(t) = \frac{1}{\sqrt{2}} [v_1(t) + v_2(t)], \quad (16)
\]

\[
\nu_2'(t) = \frac{1}{\sqrt{2}} [v_1(t) - v_2(t)]. \quad (17)
\]
As sketched in Fig. 1, we thus achieve a perfect splitting of the single-mode two-photon input in two single-photon outputs—the TLS operates as “quantum beam splitter.”

IV. THE INVERSE PROCESS: A “BEAM COMBINER”

Running the process backward in time, we may apply the TLS as a beam combiner, turning two orthogonal pulses, each populated by a one-photon Fock state, into a two-photon single-mode output. To use the TLS as a beam combiner, we must scatter the time-reversed versions of the two modes in (16) and (17):

\[ u_1(t) = v_1^*(-t)^* \], \hspace{1cm} (18)
\[ u_2(t) = v_2^*(-t)^* \]. \hspace{1cm} (19)

The two modes will scatter into the time-reverse version of the input mode (10), \( v(t) = u(-t)^* \). To demonstrate the time-reversed process, the virtual cavity approach is supplemented by an additional input mode. To analyze the beam combiner, we shall employ an extended virtual cavity procedure [18,19] to deal with a multimode input, and we shall study the robustness of such a device to a temporal mismatch of the incident pulses.

A. Two input modes and one output mode

To simulate the chiral scattering of the two-mode input, we use the cascaded system consisting of two \( u \) cavities, emitting the two input modes in Eqs. (18) and (19), a TLS and a single \( v \) cavity, absorbing the mode, \( v(t) = u(-t)^* \). The dynamics described by the Lindblad master equation with Hamiltonian

\[ H_{\text{aux}} = \frac{i}{2} \left( g_{u_1}(t)\hat{a}_u^\dagger [g_{u_1}^*(t)\hat{a}_{u_1} + \sqrt{\gamma}\hat{\sigma}^- + g_{v}^*(t)\hat{a}_v] 
+ g_{u_1}(t)\hat{a}_v^\dagger [\sqrt{\gamma}\hat{\sigma}^- + g_{v}^*(t)\hat{a}_v] 
+ \sqrt{\gamma} g_{v}^*(t)\hat{\sigma}^+\hat{a}_v - H.c. \right) \]. \hspace{1cm} (20)

and the emitted field operator

\[ L_{\text{aux}} = g_{u_2}^*(t)\hat{a}_{u_2} + g_{v_1}^*(t)\hat{a}_{v_1} + \sqrt{\gamma}\hat{\sigma}^- + g_{v}^*(t)\hat{a}_v. \hspace{1cm} (21) \]

The coupling, \( g_{u_1}(t) \), of the virtual cavity nearest to the TLS can be calculated directly using Eq. (2) with the desired mode function \( u_1(t) \). The outcoupling, \( g_{v_1}(t) \), of the left-most input cavity, however, must be chosen such that the dispersive scattering on the \( u_1 \) cavity results in the desired mode shape of \( u_2(t) \). See Sec. 2 in the Appendix of Ref. [18] for the explicit derivation of the coupling strength \( g_{u_1}(t) \). The time-dependent population of the virtual cavities and the excited state population of the TLS are shown in Fig. 3. The input cavity photon numbers decrease from unity to zero, while the TLS becomes temporarily excited and finally the mean photon number in the output virtual cavity mode reaches the value of two, as expected. The inset in Fig. 3 shows the asymptotic photon number state populations in the output mode \( v(t) \), in agreement with the expected two-photon pure state output.

B. Nonlinear Hong-Ou-Mandel effect

It is interesting to compare the performance of the TLS and a linear 50:50 beam splitter. When each input port of a linear 50:50 beam splitter is illuminated by a single-photon state, in the same temporal mode, a superposition state of both photons exiting either output port is produced. This is the famous Hong-Ou-Mandel effect [29], useful for testing the indistinguishability of the two incoming photons. The nonlinearity of the TLS remarkably enables the beam combiner process where two photons in two orthogonal modes leave in a definite single mode, all propagating in the forward direction. The similarity and difference with the Hong-Ou-Mandel effect inspire us to ask what happens when the two photons do not arrive in the perfectly matched modes required, e.g., one mode is delayed in time with respect to the other. A positive delay is defined as the mode, \( u_1 \), arriving at the TLS at later times and a negative delay as the mode, \( u_1 \), arriving at the TLS at earlier times as illustrated by the dash-dotted (blue) and dotted (green) lines in Fig. 4(a), respectively. When delaying one of the two input modes, the two modes are no longer orthogonal, but we can specify the (unnormalized) state in the second quantized form

\[ |\Psi_{\text{input}}\rangle \propto \hat{a}_{u_1}^\dagger \hat{a}_{u_2}^\dagger |0\rangle. \hspace{1cm} (22) \]

To simulate the scattering of this state on the TLS, we rewrite it in a Gram-Schmidt orthonormal basis,

\[ \tilde{u}_1 = u_1, \hspace{1cm} (23) \]
\[ \tilde{u}_2 = \frac{1}{\sqrt{N}} (u_2 - \langle u_1, u_2 \rangle u_1), \hspace{1cm} (24) \]
With the normalization factor $N = 1 + |\langle u_1, u_2 \rangle|^2$. In the basis of the latter two modes, the input state defined by Eq. (22), up to normalization, is written as

$$|\Psi_{\text{input}}\rangle \propto \sqrt{N} |1, 1\rangle + \langle u_1, u_2 \rangle \sqrt{2} |2, 0\rangle ,$$  

(25)

where the first (second) entry is for the $u_1 (u_2)$ mode. The system dynamics is now fully described by the master equation in Eq. (3), with the Hamiltonian (20) and Lindblad operator (21). The $u$-cavity couplings, $g_n(t)$, are defined by the $u_i(t)$ mode functions. The mean photon number in the four most populated output modes as a function of the delay between the two pulses is shown in Fig. 4(b). For zero delay, as expected, only a single output mode is populated with two photons. For short positive delays, three modes become populated, while for longer delays two modes are populated with a single photon each, corresponding to the separate linear scattering of the two incident single-photon pulses. For short negative delays, only two modes are populated while four modes are needed to account for the field at intermediate delays, and for large negative delays, we recover the separate linear scattering of two one-photon pulses. We attribute the asymmetry of Fig. 4(b) between positive and negative time delays to the asymmetric mode shape, $u_2(t)$ [cf. Fig. 4(a)], implying that longer negative than positive delays are needed to separate the two modes. Another parameter important for the TLS working as a beam combiner is the phase difference between the two states constituting Eq. (12). If we replace the minus sign with a complex phase factor $e^{i\phi}$, where $\phi$ deviates from the value $\pi$, the superposition modes, $v_\pm$, of Eqs. (16) and (17), will change and their time-reversed counterparts will not combine perfectly on the TLS. This is illustrated in Fig. 4(c), which shows the average photon number, $n_i$, of the most occupied output modes as a function of the phase $\phi$. Changing the phase toward $\phi = 0 (2\pi)$ leads to an increase in the multimode character of the output.

**V. SUMMARY AND OUTLOOK**

In summary, we have shown that chiral scattering of a two-photon single-mode Gaussian pulse, of a properly chosen duration, on a two-level system yields a perfect splitting into two orthogonal single-photon states. This result supplements the recent demonstration that a pair of two-level systems can act as a high-fidelity photon sorter [28], which maintains the single-mode character of an incident two-photon state but splits one- and two-photon components of an incident pulse in two orthogonal wave packets. These remarkable processes may be used in photon splitting attacks on weak pulse quantum key distribution and thus highlight the importance to develop decoy state schemes to ensure the security of quantum cryptography [30–32]. We have demonstrated the time-reversed processes where the chiral scattering of two specific orthogonal single-photon states on a two-level system yields a two-photon single-mode Gaussian pulse. This result may be useful to deterministically construct higher photon number states from single-photon states. Both results may also find applications in quantum optics and in optical quantum computing. Recent progress with superconducting qubits coupled to microwaves [33–35] and surface acoustic waves [36,37], quantum dots chirally coupled to waveguides [20], and Rydberg atom clouds in free space [38,39] may readily explore these processes. Encouraged by the presented results of two-photon scattering, we are inspired to extend the splitting and combing schemes to $n$ photons scattering on single or multiple emitters. We seek to realize single- or multiphoton subtraction and addition of $n$-photon input states.

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In Sec. II B, we introduced the master equation

$$\frac{d\rho}{dt} = -i[H, \rho] + \left( L^{\dagger} L \rho - \frac{1}{2} [L^{\dagger}, L], \rho \right)$$  \hspace{0.5cm} (A1)$$

with a specified Hamiltonian, $H$, and Lindblad operator, $L^{\dagger}$, describing a cascaded cavity mode and two-level scatterer [12,13]. Here, we show how the Hamiltonian, Eq. (4), and Lindblad operator, Eq. (5), can be obtained by the so-called SLH network analysis, carefully reviewed and explained in Ref. [22]. In the SLH formalism, a local system interacting with input and output fields in a Markovian manner is described by an operator triple $G = (S, L, H)$, and a composition rule provides effective triples of a composite network system. We deal with the scattering of a pulse on a TLS, which is represented by a network of two connected quantum systems as shown in Fig. 5. The leftmost system 1 is the virtual cavity oscillator and the rightmost system 2 is the TLS with respective operator triples, $G_1 = (S_1 = \hat{a}_u, L_1 = g_u(t) \hat{a}_u, H_1 = H_{0u})$ and $G_2 = (S_2 = \hat{a}_s, L_2 = \sqrt{\tau} \sigma^-, H_2 = H_{0s})$. The operators and couplings are defined in Sec. II B. The output of the cavity is cascaded as input for the TLS, and with the assumption of a Markovian coupling and dispersionless propagation of the field, the composed triple of the composite system, $G_T$, is determined by the series product of $G_1$ and $G_2$ (using Rule 1 of Sec. 5.2 of Ref. [22]),

$$G_T = G_2 \circ G_1 = (S_T, L_T, H_T)$$

$$= \left( S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 \right)$$

$$+ \frac{1}{2i} \left[ L_1 S_2 L_1 - L_1 S_2 L_1^\dagger \right]$$

$$= \left( \hat{a}_u g_u(t) \hat{a}_u + \sqrt{\tau} \hat{\sigma}^-, H_{0u} + H_{0s} \right)$$

$$+ \frac{i}{2} \left[ g_u(t) \hat{a}_u \hat{\sigma}^- - g_u^*(t) \hat{\sigma}^+ \hat{a}_u \right].$$  \hspace{0.5cm} (A2)$$

Assuming the input field $dB_{in}$ to the leftmost system is in the vacuum state, the Hamiltonian and Lindblad operators of Eq. (A1) are described by the corresponding $H = H_T$ and $L_0 = L_T$ components of the triple in Eq. (A2). These operators agree with the expressions in Eqs. (4) and (5) in the main text. The cascaded nature of the dynamics becomes clear when the Hamiltonian and Lindblad operators are inserted in the master equation in Eq. (A1). Considering only the commutator and anticommutator terms, it is seen that all terms with the operators $\hat{a}_u \hat{\sigma}^-$ acting from the left and $\hat{\sigma}^+ \hat{a}_u$ acting from the right on the density matrix vanish. Thus, no excitation propagates upstream from the TLS to the input cavity. Having described this two-component cascaded system, it is straightforward to describe the inclusion of a cascaded output pickup cavity. In (A2), let $G_1$ take the role of the two-component triple $G_T$ and let $G_2$ represent the pickup cavity. Then, the rules applied in Eq. (A2) readily yield the Hamiltonian (8) and Lindblad operator (9).

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