DEHN FILLING AND THE THURSTON NORM
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ABSTRACT. For a compact, orientable, irreducible 3–manifold with toroidal boundary that is not the product of a torus and an interval or a cable space, each boundary torus has a finite set of slopes such that, if avoided, the Thurston norm of a Dehn filling behaves predictably. More precisely, for all but finitely many slopes, the Thurston norm of a class in the second homology of the filled manifold plus the so-called winding norm of the class will be equal to the Thurston norm of the corresponding class in the second homology of the unfilled manifold. This generalizes a result of Sela and is used to answer a question of Baker-Motegi concerning the Seifert genus of knots obtained by twisting a given initial knot along an unknot.

1. INTRODUCTION
How does the Thurston norm behave under Dehn filling?

Let \( N \) be a compact, orientable 3–manifold with toroidal boundary and let \( T \subset \partial N \) be a particular component. Consider the Dehn fillings \( N_T(b) \) along slopes \( b \) in \( T \). For each slope \( b \) in \( T \), the Dehn filling induces a natural inclusion of \( N \) into \( N_T(b) \) that induces the monomorphism

\[
t_b : H_2(N, \partial N - T) \to H_2(N_T(b), \partial N_T(b))
\]

defined as follows. If \( z \in H_2(N, \partial N - T) \) is represented by a properly embedded surface \( S \) in \( N \) with \( \partial S \cap T = \emptyset \), then \( t_b(z) = \hat{z} \) is also represented by \( S \) under the inclusion. Consequently,

\[
x(z) \geq x(\hat{z})
\]
on the Thurston norms of classes \( z \in H_2(N, \partial N - T) \) and \( t_b(z) = \hat{z} \in H_2(N_T(b), \partial N_T(b)) \).

Gabai and Sela both address when Inequality (*) is an equality. Gabai shows that for a fixed class \( z \in H_2(N, \partial N - T) \), \( x(z) = x(\hat{z}) \) for all except at most one slope \( b \) in \( T \) \cite{Gab87} Corollary 2.4]. Sela extends this result showing that the equality \( x(z) = x(\hat{z}) \) holds for every class \( z \in H_2(N, \partial N - T) \) and induced class \( \hat{z} \in H_2(N_T(b), \partial N_T(b)) \) for all Dehn fillings except along a finite number of slopes \( b \) in \( T \) \cite{Sel90} Theorem 3.1.

In this article we extend consideration to all classes in \( H_2(N, \partial N) \). To do so, for each slope \( b \) in \( T \) we consider the restriction of the Dehn filling \( N_T(b) \) to \( N \) rather than the inclusion of \( N \) into \( N_T(b) \). Restriction gives a monomorphism

\[
\rho_b : H_2(N_T(b), \partial N_T(b)) \to H_2(N, \partial N)
\]
defined as follows. If \( \hat{z} \in H_2(N_T(b), \partial N_T(b)) \) is represented by a properly embedded surface \( \hat{S} \) that is transverse to \( K_b \), then \( \rho_b(\hat{z}) = z \) is represented by \( S = \hat{S} \cap N \). Here, and throughout, we take \( K_b \subset N_T(b) \) to be the core of the filling with tubular neighborhood \( \mathcal{N}(K_b) \) so that \( N = N_T(b) - \mathcal{N}(K_b) \), and we orient \( K_b \) and its meridian \( b \) so that \( b \) links \( K_b \) positively. The algebraic intersection number with the core \( K_b \) is a linear

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\[\text{1Sela uses \cite{Gab87} Theorem 1.8 which required an atoroidality hypothesis. However \cite{Gab87} Corollary 2.4 can be used instead to avoid such an additional hypothesis. Lackenby discusses such atoroidality hypotheses in the Appendix to \cite{Lac97}.}\]
form on homology, so its absolute value is a pseudo-norm. That is, the pseudo-norm \textbf{winding number} of \( K_b \) about a homology class \( \vec{z} \in H_2(N_T(b), \partial N_T(b)) \) is defined to be

\[
\text{wind}_{K_b}(\vec{z}) = |[K_b] \cdot \vec{z}|.
\]

The winding number enables the following extension of Inequality (\( \dagger \)), whose proof is given in Section 2.3.

\textbf{Lemma 1.1.} Let \( N \) be a compact, orientable, irreducible 3–manifold whose boundary is a union of tori. Let \( T \) be a component of \( \partial N \) and let \( b \) be a slope in \( T \). If \( N_T(b) \) has no \( S^1 \times D^2 \) or \( S^1 \times S^1 \) summands, then for all classes \( \vec{z} \in H_2(N_T(b), \partial N_T(b)) \),

\[
(\dagger) \quad x(\vec{z}) \geq x(\vec{z}) + \text{wind}_{K_b}(\vec{z})
\]

where \( \rho_b(\vec{z}) = \vec{z} \).

Our main goal in this paper is to address when Inequality (\( \dagger \)) is an equality, i.e. when

\[
(\ddagger) \quad x(\vec{z}) = x(\vec{z}) + \text{wind}_{K_b}(\vec{z}).
\]

For convenience, if there exists a class \( \vec{z} \in H_2(N_T(b), \partial N_T(b)) \) for which Equality (\( \ddagger \)) fails, then we say the slope \( b \) is a \textbf{norm-reducing} slope, the class \( z = \rho_b(\vec{z}) \in H_2(N, \partial N) \) is a \textbf{norm-reducing} class with respect to the norm-reducing slope \( b \), and the class \( \vec{z} \in H_2(N_T(b), \partial N_T(b)) \) is a \textbf{norm-reducing} class with respect to the knot \( K_b \).

\textbf{Theorem 4.6.} Let \( N \) be a compact, connected, orientable, irreducible 3–manifold whose boundary is a union of tori. Then either

\begin{enumerate}
  \item \( N \) is a product of a torus and an interval,
  \item \( N \) is a cable space, or
  \item for each torus component \( T \subset \partial N \) there is a finite set of slopes \( \mathcal{R} = \mathcal{R}(N,T) \) in \( T \) such that if \( b \notin \mathcal{R} \) then \( b \) is not norm-reducing.
\end{enumerate}

In Corollary 4.4 we obtain a bound on the size of \( \mathcal{R}(N, T) \) in terms of the Thurston norms of two integral classes of two different fillings and the distance between the two filling slopes. Since \( \text{wind}_{K_b}(\vec{z}) = 0 \) when \( \rho_b(\vec{z}) \in H_2(N, \partial N - T) \), Theorem 4.6 generalizes Sela’s result (with the additional assumption that \( N \) is irreducible). Sela also explicitly bounds, by the number of faces of the Thurston norm ball of \( K_b \) to the norm-reducing slope \( \Delta = \Delta(a, b) \geq 2 \), then \( \hat{Q} \) can be isotoped so that

\[
|K_a \cap \hat{Q}|(\Delta - 1) \leq -\chi(\hat{Q}).
\]

If, in Lackenby’s setup, \( \hat{Q} \) is taken to be a taut representative of a non-zero class \( \hat{y} \in H_2(M', \partial M') \), then we have (after rearranging the inequality):

\[
\Delta \leq 1 + \frac{x(\hat{y})}{|K_a \cap \hat{Q}|}.
\]

Our Corollary 4.3 gives a version of this result for the situation when \( H_2(N, \partial N) \), and not just \( H_2(N, \partial N - T) \), has a norm-degenerating class with respect to the slope \( b \).

\footnote{In Lackenby’s paper, see Assumptions 1.1 and Remark 1.3. To convert the notation from ours to Lackenby’s make the following changes: \( \gamma = \emptyset, \ M' \rightarrow M, \ K_a \rightarrow L, \ N \rightarrow M - \text{int}(N(L)), \ \hat{Q} \rightarrow F \). The class whose norm is reduced is called \( z_1 \) by Lackenby.}
In addition to considering a fixed component $T$ of $\partial N$ and studying the dependency of the Thurston norm on the filling slope, we can also consider a 3-manifold $M$ and consider how the Thurston norm of manifolds $M'$ obtained by surgery on an oriented knot $K$ in $M$ depends on the dual Thurston norm $x^*([K])$ of the class $\alpha = [K] \in H_1(M;\mathbb{Z})$.

**Theorem 5.1.** Let $M$ be a compact, orientable 3–manifold whose boundary is a union of tori, $\Delta \in \mathbb{N}$, and $\alpha \in H_1(M;\mathbb{Z})$. Assume that every sphere, disc, annulus, and torus in $M$ separates. If

$$(\Delta - 1)x^*(\alpha) > 1$$

then every irreducible, $\partial$–irreducible 3–manifold obtained by a Dehn surgery of distance $\Delta$ on a knot $K$ representing $\alpha$ has no norm-reducing classes with respect to the knot which is surgery dual to $K$.

The contrapositive is also a useful formulation, as it shows that knots resulting from non-longitudinal surgery on a knot with a norm-reducing class have bounded dual norm.

Finally, we give an application to the genus of knots in twist families. A twist family of knots $\{K_n\}$ is obtained by performing $-1/n$–Dehn surgery on an unknot $c$ that links a given knot $K = K_0$. When $\ell(k(K,c)) = 0$, it is a fundamental consequence of [Gab87, Corollary 2.4] that $g(K_n)$ is constant for all integers $n$ except at most one where the genus decreases. Using the multivariable Alexander polynomial, the first author and Motegi showed that if $|\ell(k(K,c))| \geq 2$, then $g(K_n) \to \infty$ as $n \to \infty$ [BM15]. When $|\ell(k(K,c))| = 1$, this fails if $c$ is a meridian of $K$ since $K_n = K$ for all $K$. Here we answer [BM15 Question 2.2] by showing this is the only exception.

**Theorem 5.1** If $\omega = |\ell(k(K,c))| > 0$, then $\lim_{n\to\infty} g(K_n) = \infty$ unless $c$ is a meridian of $K$.

2. **Preliminaries**

2.1. **Notation and conventions.** The following notation is used throughout the article. We take $N$ to be a compact, connected, irreducible oriented 3–manifold where $\partial N$ is a non-empty union of tori and focus upon a particular component $T \subset \partial N$. Given two slopes $a,b \subset T$, we set the results of Dehn filling $N$ along these slopes to be the two 3–manifolds $M = N_T(b)$ and $M' = N_T(a)$. Furthermore we let $K = K_b \subset M$ and $K' = K_a \subset M'$ denote the core knots of the two filling solid tori.

The distance $\Delta = \Delta(a,b)$ between two slopes $a,b \subset T$ is the minimal number of points of intersection between simple closed curves in $T$ representing $a$ and $b$.

Given a surface $S$ properly embedded in $N$, the union of the boundary components of $S$ in $T$ is $\partial_T S = \partial S \cap T$. If the slope of each component of $\partial_T S$ in $T$ is $b$ (as an unoriented curve), then we set $\hat{S} \subset M$ to be the surface obtained by capping off the components of $\partial_T S$ with meridian discs of the filling solid torus. Observe that by construction, $[K \cap \hat{S}] = [\partial_T S]$.

In this article, a lens space is a closed 3–manifold with a genus 1 Heegaard splitting other than $S^3$ and $S^1 \times S^2$. In particular, the fundamental group of a lens space is a non-trivial, finite, cyclic group.

2.2. **Thurston norm.** Thurston introduced two norms on the homology groups of a compact, orientable 3–manifold $W$ [Thu86], now commonly known as the Thurston norm and the dual Thurston norm:

$$x: H_2(W, \partial W; \mathbb{R}) \to [0,\infty) \quad \text{and} \quad x^*: H_1(W; \mathbb{R}) \to [0,\infty)$$

which we may write as $x_W$ and $x_W^*$ to emphasize the 3–manifold $W$.

On an integral class $\sigma \in H_2(W, \partial W; \mathbb{Z})$, the Thurston norm is defined by

$$x(\sigma) = \min_S \sum_{i=1}^n \max\{0, -\chi(S_i)\}$$
where the minimum is taken over all embedded surfaces \( S \) representing \( \sigma \) with connected components \( S_1, \ldots, S_n \). The function \( x \) is linear on rays and convex. These properties enable it to be extended first to rational homology classes and then to real homology classes.

In general, the function \( x \) is only a pseudo-norm; \( x \) is a norm if \( W \) contains no non-separating sphere, disc, torus, or annulus. Nevertheless, \( x \) is reasonably well behaved even in the presence of non-separating tori and annuli, it is non-separating spheres and discs that complicate the norm:

If an integral class \( \sigma \in H_2(W, \partial W; \mathbb{Z}) \) cannot be represented by a surface with a non-separating sphere or disc component, then \( x(\sigma) \) is just the minimum of \( -\chi(S) \) among surfaces representing \( \sigma \).

It is for such integral classes that Inequality (†) holds. Assuming \( W \) has no \( S^1 \times S^2 \) or \( S^1 \times D^2 \) summand ensures this is the case for all classes, as does the more heavy-handed assumption that \( W \) is irreducible and \( \partial \)-irreducible. In particular, we can now prove Lemma [L.1]

**Proof of Lemma [L.1]** Recall that \( N \) is a compact, orientable, irreducible 3–manifold with \( \partial N \) the union of tori and \( T \subset \partial N \) a component. Let \( b \) be a slope in \( T \) and assume that \( N_T(b) \) has no \( S^1 \times D^2 \) or \( S^1 \times S^2 \) summands. Let \( \partial_T : H_2(N, \partial N) \to H_1(T) \) be the boundary map restricted to \( T \). We will show that for all classes \( z \in H_2(N_T(b), \partial N_T(b)) \),

\[
(\dagger) \quad x(z) \geq x(z) + \text{wind}_{K_b}(z).
\]

As usual, it suffices to prove the inequality for integral classes. In which case, there exists a properly embedded oriented surface \( S \subset N \) such that \( S \) has no separating component, \( \lvert S \rvert = z \), and all components of \( \partial_T S \) are coherently oriented curves, each of slope \( b \), and \( x(S) = x(z) \). If some component of \( S \) is a sphere or disc, then it would persist into \( N_T(b) \) as a non-separating sphere or disc, contrary to our hypotheses. Hence \( S \) has no sphere or disc component and \( x(S) = -\chi(S) \).

Cap off the components of \( \partial_T(S) \) in \( N_T(b) \) with discs to obtain the surface \( \widehat{S} \). Observe that

\[
|\partial_T S| = |\widehat{S} \cap K_b| = \text{wind}_{K_b}(\widehat{z})
\]

since the components of \( \partial_T S \) are coherently oriented. Since \( M \) contains no non-separating sphere or disc, \( -\chi(\widehat{S}) \geq x(\widehat{z}) \). Consequently,

\[
x(z) = -\chi(S) = -\chi(\widehat{S}) + \text{wind}_{K_b}(\widehat{z}) \geq x(\widehat{z}) + \text{wind}_{K_b}(\widehat{z}) .
\]

Finally, on a class \( \alpha \in H_1(W; \mathbb{R}) \), the dual Thurston norm is defined by

\[
x^*(\alpha) = \sup_{x(\sigma) \leq 1} |\alpha \cdot \sigma|
\]

where \( \cdot \) denotes the intersection product. The function \( x^* : H_1(W; \mathbb{R}) \to [0, \infty) \) is continuous.

2.3. **Wrapping numbers.** Having defined the winding number, we now turn to wrapping number. A compact, oriented, properly embedded surface \( S \) in a 3–manifold \( W \) is **taut** (or \( \varnothing \)-taut) if it is incompressible (i.e. does not admit a compressing disc), and minimizes the Thurston norm among embedded surfaces representing the class \([S, \partial S] \in H_2(W, \partial S)\) [Sch89] Def. 1.2]. Observe that if a surface \( S \subset N \) is taut and has the property that \( x(S) = x([S]) \), then the surface \( S' \) obtained by discarding all separating components of \( S \) (which are necessarily spheres, discs, annuli, and tori) is also taut and has the properties that \([S] = [S'] \in H_2(N, \partial N)\) and \( x(S') = x([S]) = x([S']) \).
Suppose that $M$, a class in the absence of non-separating spheres and discs. This allows us to parlay technical results about exceptional classes and \( \hat{z} \in H_2(M, \partial M; \mathbb{Z}) \) to be

\[
\text{wrap}_K(\hat{z}) = \min_{\hat{S}} |K \cap \hat{S}|
\]

where the minimum is taken over all taut representatives \( \hat{S} \) of \( \hat{z} \).

Since discarding separating components of \( \hat{S} \) will not increase \( |K \cap \hat{S}| \), we will henceforth assume that whenever we discuss a taut surface realizing the Thurston norm of a homology class in the second homology group of a 3-manifold relative to the boundary of that 3-manifold, we have discarded all separating components.

We may extend the wrapping number to \( H_2(M, \partial M; \mathbb{Q}) \). Assume \( \hat{S} \) is a taut surface realizing \( \text{wrap}_K(\hat{z}) \) for an integral class \( \hat{z} \in H_2(M, \partial M; \mathbb{Z}) \). Then, following [Thu86, Lemma 1], \( n \) parallel copies of \( \hat{S} \) is a taut surface realizing \( \text{wrap}_K(n \hat{z}) = n \text{wrap}_K(\hat{z}) \) for positive integers \( n \). Thus for a rational class \( \hat{q} \) we define \( \text{wrap}_K(\hat{q}) = \frac{1}{n} \text{wrap}_K(n \hat{z}) \) where \( n \) is a positive integer such that \( n \hat{q} \) is an integral class. Since algebraic intersection numbers give lower bounds for geometric intersection numbers, \( \text{wrap}_K(\hat{q}) \geq \text{wind}_K(\hat{q}) \) for all \( \hat{q} \in H_2(M, \partial M; \mathbb{Q}) \). Observe that if \( M \) has no norm-reducing classes with respect to \( K \), then \( \text{wrap}_K = \text{wind}_K \) is a pseudo-norm. However, we believe that, in general, the triangle inequality will not hold for \( \text{wrap}_K \).

**Question 2.1.** Must the wrapping number satisfy the triangle inequality?

A class \( \hat{z} \in H_2(M, \partial M) \) is **exceptional** with respect to a knot \( K \) [Tay14] if the winding number and wrapping number are not equal; that is \( \hat{z} \) is exceptional with respect to \( K \) if

\[
\text{wind}_K(\hat{z}) < \text{wrap}_K(\hat{z}).
\]

This definition takes root in the practical difference between the Thurston norm and Scharlemann’s \( \beta \)-norm. As discussed in [Tay14], a class \( \hat{z} \) is **exceptional** with respect to \( K \) if and only if no representative of \( \hat{z} \) is both \( \emptyset \)-taut and \( K \)-taut. (Here, \( K \) is playing the role of \( \beta \). See [Sch89] for the definitions of the \( \beta \)-norm and \( \beta \)-taut surfaces.)

For our present purposes, we observe that **norm-reducing** classes and **exceptional** classes are equivalent in the absence of non-separating spheres and discs. This allows us to parlay technical results about exceptional classes into results about norm-reduction.

**Lemma 2.2.** Suppose that \( M \) contains no non-separating sphere or disc. Then, with respect to a knot \( K \) in \( M \), a class \( \hat{z} \in H_2(M, \partial M) \) is exceptional if and only if it is norm-reducing.

**Proof.** Assume \( M = N_T(b) \) where \( K = K_b \). For a class \( \hat{z} \in H_2(M, \partial M) \), let \( z = \rho_b(\hat{z}) \in H_2(N, \partial N) \).

First, we claim that if \( S \) is a taut representative of a class \( [S] \in \text{im} \rho_b \), then

\[
\chi([S]) = x(S) = -\chi(S).
\]

To see this, let \( S \subset N \) be taut and have each component of \( \partial_T S \) of slope \( b \). By definition, \( x([S]) = x(S) \). Suppose that \( x(S) \neq -\chi(S) \). Then \( S \) contains a component \( P \) which is a sphere or disc. Since \( S \) is taut, \( P \) is non-separating. Capping off \( \partial_T P \) in \( M \), if necessary, creates a non-separating sphere or disc in \( M \), contrary to hypothesis.

We now embark on the proof. The claim is trivially satisfied for the 0 class, so assume that \( 0 \neq \hat{z} \in H_2(M, \partial M; \mathbb{Z}) \) is not an exceptional class for \( K \). Then there is a taut representative \( \hat{S} \subset M \) of \( \hat{z} \) for which \( \text{wrap}_K(\hat{S}) = \text{wind}_K(\hat{S}) \). Thus

\[
x_N(z) \leq x_N(S) = -\chi(S) = -\chi(\hat{S}) + \text{wind}_K(\hat{S}) = x_M(\hat{z}) + \text{wind}_K(\hat{z}) \leq x_N(z)
\]
where the last inequality is due to Inequality (†). Consequently $x_M(\hat{z}) + \text{wind}_K(\hat{z}) = x_N(z)$, and thus $\hat{z}$ is not norm-reducing with respect to $K$.

Conversely, assume that $\hat{z} \in H_2(M, \partial M)$ is exceptional with respect to $K$ so that $\text{wrap}_K(\hat{z}) > \text{wind}_K(\hat{z})$. Let $S$ be a taut surface in $N$ representing $z$, and let $\hat{S} \subset M$ be the result of capping off $\partial_T S$ with discs so that $[\hat{S}] = \hat{z}$. Then

$$x_N(z) = -\chi(S) = -\chi(\hat{S}) + |\hat{S} \cap K| > x_M(\hat{z}) + \text{wind}_K(\hat{z})$$

because $|\hat{S} \cap K| \geq |\hat{S} \cdot K| = \text{wind}_K(\hat{z})$ and $-\chi(\hat{S}) \geq x_M(\hat{z})$. Thus, $\hat{z}$ is norm-reducing with respect to $K$. □

2.4. Multi-$\partial$-compressing discs. As is often the case in studies of Dehn filling, we will want use a surface $\hat{Q}$ in one filling $M' = N_T(a)$ of $N$ to say something useful about a different filling $M = N_T(b)$. For us, the surface $\hat{Q}$ will be most useful if it has no “multi-$\partial$-compressing disc.”

Suppose that $\hat{S} \subset M' = N_T(a)$ is a surface transversally intersecting $K' \subset M'$ non-trivially. A multi-$\partial$-compressing disc for $\hat{S}$ (with respect to $K'$) is a disc $D \subset N$ such that there is a component $A \subset T - S$ such that:

- The interior of $D$ is disjoint from $\partial N \cup S$
- The boundary of $D$ is a simple closed curve lying in $S \cup A$
- After orienting $\partial D$, $\partial D \cap A$ is a non-empty, coherently oriented collection of spanning arcs of $A$

Given a multi-$\partial$-compressing disc $D$ for $\hat{S}$, then we may create a new surface $\hat{S}'$ that is homologous to $\hat{S}$ but intersects $K'$ in two fewer points: that is, $[\hat{S}] = [\hat{S}'] \in H_2(M', \partial M')$ and $|\hat{S}' \cap K'| = |\hat{S} \cap K'| - 2$. We create $\hat{S}'$ by removing the open regular neighborhood of two points of $K' \cap \hat{S}$, attaching the annulus $A$ (from the definition of “multi-$\partial$-compressing disc”) and then compressing using $D$.

The next lemma allows us to know when we have a surface without a multi-$\partial$-compressing disc.

**Lemma 2.3.**

- Suppose that $\hat{S} \subset M'$ is a sphere transverse to $K'$ such that $S = \hat{S} \cap N$ is incompressible and not $\partial$–parallel. Then either $M'$ has a lens space summand or $\hat{S}$ does not have a multi-$\partial$-compressing disc with respect to $K'$.
- Suppose that $\hat{S} \subset M'$ is a disc transverse to $K'$ such that $S = \hat{S} \cap N$ is incompressible. Then either $M'$ has a lens space summand or $\hat{S}$ does not have a multi-$\partial$-compressing disc with respect to $K'$.
- Suppose that $\hat{S} \subset M'$ is a taut representative of some non-zero class in $H_2(M', \partial M'; \mathbb{Z})$ and that, out of all such taut surfaces representing that class, $\hat{S}$ minimizes $|\hat{S} \cap K'|$. Then either $M'$ contains a non-separating sphere or disc or $\hat{S}$ does not have a multi-$\partial$-compressing disc with respect to $K'$.

**Proof.** Suppose that $\hat{S} \subset M'$ is a surface transverse to $K'$, such that $S$ is incompressible and not $\partial$–parallel. If $K'$ is disjoint from $\hat{S}$, then trivially there is no multi-$\partial$-compressing disc. Hence we further assume $K'$ transversally intersects $\hat{S}$ non-trivially.

Suppose that $D$ is an oriented multi-$\partial$-compressing disc for $\hat{S}$. Then there is an annulus component $A \subset T - S$ such $\partial D \cap A$ is a non-empty collection of coherently oriented spanning arcs of $A$. Let $\hat{R}$ be the surface in $M'$ obtained from isotoping $S \cup A \subset N$ with support in a neighborhood of $A$ to be properly embedded in $N$ and then capping off the boundary components in $T$ with meridional discs of the filling solid torus; i.e. $\hat{R}$ is the result of tubing $\hat{S}$ along a particular arc of $K' \setminus \hat{S}$. A further slight isotopy makes $\hat{R}$ disjoint from $\hat{S}$.

Now let $\hat{S}'$ be the result of compressing $\hat{R}$ using $D$, and slightly isotoping to be disjoint from $\hat{R}$. Observe that $-\chi(\hat{S}') = -\chi(\hat{S})$ and that there is a natural bijection between the components of $\hat{S}$ and $\hat{S}'$.  

First assume $\hat{S}$ is a sphere. Then $\hat{S}'$ must also be a sphere. If $\partial D$ runs just a single time across $A$, then $D$ provides a $\partial$–compression for $S$ in $N$. Since $N$ is irreducible, either $S$ is compressible or $S$ is a $\partial$–parallel annulus contrary to hypothesis. If $\partial D$ runs multiple times across $A$, then $\hat{S}$ and $\hat{S}'$ cobound a 3–manifold $W$ in which $\hat{R}$ is a genus 1 Heegaard surface. Because $S$ and $\hat{S}'$ are both spheres, $W$ is a twice-punctured lens space of finite order $|\partial D\cap A| > 1$. The complement of a neighborhood of an embedded arc in $W$ that connects both components of $\partial W$ is therefore a non-trivial lens space summand of $M'$.

When $\hat{S}$ is a disc, we similarly obtain that $\hat{S}'$ is also a disc. Along with an annulus in $\partial M'$, the discs $\hat{S}$ and $\hat{S}'$ bound a punctured lens space $W$ in which $\hat{R}$ is a punctured Heegaard torus. Again, this lens space has finite order $|\partial D\cap A|$ which is non-trivial since $\hat{S}$ is incompressible. Hence $W$ is a lens space summand of $M'$.

Now assume that $\hat{S}$ is a taut representative of a class in $H_2(M', \partial M'; \mathbb{Z})$. If $\hat{S}$ has a sphere, then the component must be non-separating since $\hat{S}$ is taut. So we may further assume $\hat{S}$ is not a sphere. By construction, the surface $\hat{S}'$ represents the same class, has the same euler characteristic, and intersects $K'$ two fewer times than does $\hat{S}$. Furthermore, since every component of $\hat{S}$ is non-separating, every component of $\hat{S}'$ is also non-separating. If $\hat{S}'$ is not taut, then since it is homologous to the taut surface $\hat{S}$ and is also Thurston norm minimizing for this homology class, it must have a compressible component that is a non-separating torus or annulus. Compressing this torus or annulus creates a non-separating sphere or disc in $M'$.

\[\Box\]

3. A KEY THEOREM OF TAYLOR

In [Tay14], the second author develops some classical results ([Sch89, Application III] and [Sch90]) from Scharlemann’s combinatorial version [Sch89] of Gabai’s sutured manifold theory [Gab83, Gab87, Gab87b] in terms of surgeries on knots with exceptional classes. Here we adapt a key technical theorem for our purposes.

**Theorem 3.1** (Cf. [Tay14] Theorem 3.14]). Assume that $N$ is irreducible and $\partial$–irreducible. Let $a, b$ be two distinct slopes in $T \subset \partial N$. Suppose that $M = N_T(b)$ is not a solid torus, has no proper summand which is a rational homology sphere, and $H_2(M, \partial M) \neq 0$. Suppose that $M' = N_T(a)$ contains a properly embedded, compact, orientable surface $\hat{Q} \subset M'$ that transversally intersects $K'$ non-trivially, does not have a multi-$\partial$-compressing disc for $K'$, and restricts to an incompressible surface $\hat{Q} = \hat{Q}\cap N$ in $N$.

If

$$-\chi(\hat{Q}) < |\hat{Q}\cap K'|(|\Delta(a, b) - 1|),$$

then $M$ is irreducible and $H_2(M, \partial M)$ has no exceptional classes with respect to $K$.

For the proof, we content ourselves with explaining how the statement follows from [Tay14, Theorem 3.14]. We assume familiarity with the basic definitions regarding $\beta$–taut sutured manifold technology from [Sch89] (see also [Tay14]).

**Proof.** Our notation is very similar to that of [Tay14], except that we are using $K$ as the core knot of the filling $M = N(b)$ instead of $\beta$ and we consider classes $\hat{y} \in H_2(M, \partial M)$ rather than classes $y$.

Our hypotheses immediately imply Conditions (1) and (3) of [Tay14] Theorem 3.14]. Since $N$ is irreducible and $\partial$–irreducible, we may consider it as a taut sutured manifold $(N, \varnothing, \varnothing)$, considering $\partial N$ as toroidal sutures. The filling $M = N_T(b)$ induces a sutured manifold $(M, \varnothing, K)$ that is then a $K$–taut sutured manifold, providing Condition (2).

\[\text{We use the convention that any sphere component of an incompressible surface does not bound a ball, and any disc component is not} \ \partial \text{–parallel.}\]
Since \( \hat{Q} \cap K' \neq \emptyset \) and the curves of \( \partial_T Q \) have slope \( a \), the boundary of \( Q \) is not disjoint from the slope \( b \) in \( T \). Sphere components of \( \hat{Q} \) that are disjoint from \( K' \) are the sphere components of \( Q \); however, since the irreducibility of \( N \) implies that any sphere component of \( Q \) must bound a ball in \( N \), the incompressibility of \( Q \) prohibits the existence of such sphere components. Furthermore, no component of \( Q \) is a disc with essential boundary since \( N \) is \( \partial \)-irreducible and no component of \( Q \) is a disc with inessential boundary due to the incompressibility of \( Q \) and irreducibility of \( N \). Thus Condition (4) is satisfied.

We may now apply \cite{Tay14} Theorem 3.14. Our hypothesis that \( M \) has no proper summand that is a rational homology sphere immediately rules out Conclusion (4) of \cite{Tay14} Theorem 3.14. We proceed to show that Conclusions (3) and (2) also fail and that Conclusion (1) implies our stated result.

In the terminology of \cite{Sch89} Section 7 and \cite{Tay14} Section 2.2, the surface \( Q \) is a parameterizing surface for the sutured manifold \( (M, \emptyset, K) \). By definition (again, see \cite{Sch89} Definition 7.4 and \cite{Tay14} Section 2.2)), its index \( I(Q) \) is given by

\[
I(Q) = -2\chi(Q)
\]

since (i) there are no annular sutures on \( \partial M \) and (ii) \( K \) is a knot (rather than a collection of properly embedded arcs). Without loss of generality, we may assume that the slope \( b \) has been isotoped in \( T \) to intersect \( \partial Q \) minimally. Thus, \( |\partial Q \cap b| \) is equal to \( \Delta(a, b)|\hat{Q} \cap K'| \). Our assumed inequality on the Euler characteristic of \( \hat{Q} \) can then be rearranged to yield

\[
I(Q) < 2|\partial Q \cap b|.
\]

Hence, Conclusion (3) of \cite{Tay14} Theorem 3.14) does not hold.

A Gabai disc for \( Q \) is a disc \( D \) embedded in \( M \) that \( K \) non-trivially and coherently intersects, such that its restriction to \( N \) is transverse to \( Q \) and \( |Q \cap \partial D| < \Delta(a, b)|\partial_T Q| \). It is shown in \cite{CGLS87} (though without the language of Gabai discs), and further explained in \cite{Sch90} and \cite{Tay14}, that a Gabai disc will contain a Scharlemann cycle. As \( Q \) is incompressible and \( N \) is irreducible, the interior of the Scharlemann cycle can be isotoped to be a multi-\( \partial \)-compressing disc for \( \hat{Q} \). See \cite{Tay14} Section 4) for more details. (Although observe that \cite{Tay14} Lemma 4.3 neglected to consider possible circles of intersection between the interior of the Scharlemann cycle and \( Q \). We have added the incompressibility hypotheses to \( \hat{Q} \) to deal with this.) Since we are assuming that \( \hat{Q} \) has no multi-\( \partial \)-compressing disc, Conclusion (2) of \cite{Tay14} Theorem 3.14) does not hold.

Consequently, the Conclusion (1) of \cite{Tay14} Theorem 3.14) holds. Hence, given any non-zero class \( \hat{y} \in H_2(M, \partial M; \mathbb{Z}) \), there is a \( K \)-taut hierarchy of \( (M, \emptyset, K) \) which is also \( \emptyset \)-taut such that the first decomposing surface \( \hat{S} \subset M \) represents \( \hat{y} \). In particular, since sutured manifold decompositions yields a taut sutured manifold only if the decomposing surface is taut, the \( K \)-tautness and \( \emptyset \)-tautness of the hierarchy implies the surface \( \hat{S} \) must be both \( K \)-taut and \( \emptyset \)-taut (see e.g. \cite{Sch89} Definition 4.18, \cite{Sch90} Section 2, \cite{Gab83} Lemma 3.5 and Section 4). Since \( (M, \emptyset, \emptyset) \) is \( \emptyset \)-taut, \( M \) is irreducible. By the definition of \( K \)-taut, the knot \( K \) always intersects \( \hat{S} \) with the same sign. That is, \( \text{wind}_K(\hat{S}) = \text{wrap}_K(\hat{S}) \). Since \( \hat{S} \) is \( \emptyset \)-taut, this implies that \( \hat{y} \) is not an exceptional class. Since this holds true for all non-zero classes in \( H_2(M, \partial M; \mathbb{Z}) \), so there are no exceptional classes in \( H_2(M, \partial M; \mathbb{Z}) \) with respect to \( K \). \( \Box \)

4. The Thurston norm and dual norm under Dehn filling

4.1. The Thurston norm.

**Theorem 4.1.** Suppose that \( N \) is irreducible and \( \partial \)-irreducible. Also assume that \( M = N_T(b) \) is not a solid torus and has no proper rational homology sphere summand and that either \( M \) is reducible or that \( H_2(M, \partial M) \) has an exceptional class with respect to \( K \). Then all of the following hold for \( M' = N_T(a) \):

• Either $M'$ has a lens space summand or
  – $M'$ is irreducible and $\partial$–irreducible, and
  – $K' \subset M'$ is mp-small; that is, there is no essential, connected, properly embedded planar surface $Q \subset N$ such that $\partial Q = \partial_T Q \neq \emptyset$ and each component of $\partial Q$ has slope $b$ in $T$.
• For every $\hat{\gamma} \in H_2(M', \partial M')$,
  \[ x(\hat{\gamma}) \geq \text{wrap}_{\partial}^{K'}(\hat{\gamma})(\Delta(a, b) - 1). \]

**Remark 4.2.** The first conclusion of Theorem 4.1 that $M'$ is irreducible and $\partial$–irreducible, essentially follows from [Sch90].

**Proof.** Assume, for the moment, that either $M'$ is reducible or $\partial$–reducible or that $K'$ is not mp-small. Then there exists an essential, connected, properly embedded planar surface $Q \subset N$ such that $\partial Q$ has at most one component not in $T$, $\partial_T Q$ is non-empty (because $N$ is irreducible and $\partial$–irreducible), and every component of $\partial_T Q$ has slope $b$. Let $\hat{Q} \subset M'$ be the sphere or disc that results from capping off $\partial_T Q$ with discs. Lemma 2.3 shows that there is no multi-$\partial$–compressing disc for $\hat{Q}$. Then by Theorem 3.1 since either $M$ is reducible or $H_2(M, \partial M)$ has an exceptional class with respect to $K$, we have

\[ 0 > -\chi(\hat{Q}) \geq |\hat{Q} \cap K'|(\Delta(a, b) - 1) \geq 0 \]

which is a contradiction. Thus, $M'$ is irreducible, $\partial$–irreducible, and $K'$ is mp-small.

Because $M'$ is irreducible and $\partial$–irreducible, every sphere and disc in $M'$ separates. So consider a class $\hat{\gamma} \in H_2(M', \partial M')$. Among the taut surfaces in $M'$ representing $\hat{\gamma}$, let $\hat{Q} \subset M'$ be chosen to minimize $|\hat{Q} \cap K'|$. Tautness implies that no component of $\hat{Q}$ is a sphere or disc, that $x(\hat{\gamma}) = -\chi(\hat{Q})$, and that there is no compressing disc for $\hat{Q}$ in $M'$. The minimality gives $\text{wrap}_{\partial}^{K'}(\hat{\gamma}) = |\hat{Q} \cap K'|$ while also implying that there can be no compressing disc for $Q = \hat{Q} \cap N$ in $N$. Since every sphere and disc in $M'$ separates, Lemma 2.3 implies there are also no multi-$\partial$–compressing discs for $Q$ with respect to $K$.

If $\hat{Q} \cap K' = \emptyset$, then $\text{wrap}_{\partial}^{K'}(\hat{\gamma}) = 0$ and the desired inequality is trivially true. Thus, assume that $\hat{Q} \cap K' \neq \emptyset$. Using Theorem 3.1 again, we then have

\[ x(\hat{\gamma}) = -\chi(\hat{Q}) \geq |\hat{Q} \cap K'|(\Delta(a, b) - 1) = \text{wrap}_{\partial}^{K'}(\hat{\gamma})(\Delta(a, b) - 1) \]

as desired. \qed

The next corollary is a useful specialization.

**Corollary 4.3.** Let $N$ be a compact, orientable, irreducible, $\partial$–irreducible $3$–manifold such that $\partial N$ is a union of tori. Given distinct slopes $a$ and $b$ in a component $T$ of $\partial N$, let $M = N_T(b)$ and $M' = N_T(a)$ be the results of Dehn filling along these slopes, and let $K$ and $K'$ be the core knots of these fillings respectively.

Assume $M$ and $M'$ are irreducible, $\partial$–irreducible and $K'$ has non-zero wrapping number with respect to a class $\hat{\gamma} \in H_2(M', \partial M')$. If there exists a class of $H_2(M, \partial M)$ that is norm-degenerate with respect to $K$, then

\[ \Delta(a, b) \leq 1 + x(\hat{\gamma})/\text{wrap}_{\partial}^{K'}(\hat{\gamma}) \leq 1 + x(\hat{\gamma}) \]

**Proof.** Since we may assume that both $H_2(M, \partial M)$ and $H_2(M', \partial M')$ are non-trivial, $N$ is not a solid torus. By the irreducibility and $\partial$–irreducibility of $M$ and $M'$, every sphere and disc in $M$ and $M'$ must separate. Thus, according to Lemma 2.2 any class in $H_2(M, \partial M)$ that is norm-degenerate with respect to $K$ is also exceptional with respect to $K$. Then, due to Theorem 4.1 for every non-zero $\hat{\gamma} \in H_2(M', \partial M')$ we have $x(\hat{\gamma}) \geq \text{wrap}_{\partial}^{K'}(\hat{\gamma})(\Delta(a, b) - 1)$. When the wrapping number is non-zero, we may obtain the stated inequalities. \qed

We can now bound the number of slopes producing filled manifolds with norm-reducing classes (with respect to the filling).
Corollary 4.4. Let $N$ be a compact, orientable, irreducible, and $\partial$–irreducible 3–manifold such that $\partial N$ is a union of tori. Assume for $i = 1, 2$, there is a slope $a_i$ in the component $T$ of $\partial N$ such that the manifold $M'_i = N_T(a_i)$ is irreducible and $\partial$–irreducible and the core $K'_i$ of the Dehn filling has non-zero wrapping number with respect to a class $\hat{y}_i \in H_2(M'_i, \partial M'_i)$. If $\Delta(a_1, a_2) > 0$, then there are at most

$$(1 + x(\hat{y}_1))(1 + x(\hat{y}_2)) + (\Delta(a_1, a_2) - 1)(1 + x(\hat{y}_1))^2$$

slopes $b \subset T$ distinct from $a_1$ and $a_2$ such that the 3–manifold $N_T(b)$ obtained by filling $T$ along $b$ is irreducible, $\partial$–irreducible, and has a norm-reducing class with respect to the filling.

Proof. By Corollary 4.3, if $b$ is a slope in $T$ such that $N_T(b)$ is irreducible, $\partial$–irreducible, and has a norm-degenerating slope for the core of the filling, then

$$\Delta(a_1, b) \leq 1 + x(\hat{y}_1) \quad \text{and} \quad \Delta(a_2, b) \leq 1 + x(\hat{y}_2).$$

Then Lemma 4.5 below gives that the number of slopes $b$ satisfying these constraints is at most

$$(1 + x(\hat{y}_1))(1 + x(\hat{y}_2)) + (\Delta(a_1, a_2) - 1)(1 + x(\hat{y}_1))^2.$$ \hfill \Box

Lemma 4.5. Given slopes $b, c$ in $T$ with $\Delta(b, c) \geq 1$ and positive numbers $B, C$, then the number of slopes $a$ in $T$ such that $\Delta(a, b) \leq B$ and $\Delta(a, c) \leq C$ is at most $BC + (\Delta(b, c) - 1)B^2$.

Proof. Let us regard slopes as being represented by oriented simple closed curves. We may choose a basis for $H_1(T)$ in which $[b] = (1, 0)$ and $[c] = (r, s)$ for coprime integers $0 \leq r < s$. Then $\Delta(b, c) = s$. For any slope $a$ in $T$, we may choose an orientation of the curve so that the constraints $\Delta(a, b) \leq B$ and $\Delta(a, c) \leq C$ and the orientation restrict its representatives in this homology basis to an element of the set $\Lambda$ of integer lattice point in the trapezoid $\{(x, y): |y| \leq B, |ry - sx| \leq C, x \geq 0\}$. For points $(x, y) \in \Lambda$, one deduces that

$$0 \leq x \leq s|x| \leq |ry - sx| + r|y| \leq C + rB \leq C + (s - 1)B = C + (\Delta(b, c) - 1)B.$$ 

Thus $|\Lambda| \leq B \cdot (C + sB) = BC + (\Delta(b, c) - 1)B^2$, giving an upper bound on the number of slopes $a$ in $T$ satisfying the constraints. \hfill \Box

Theorem 4.6. Let $N$ be a compact, connected, orientable, irreducible, and $\partial$–irreducible 3–manifold whose boundary is a union of tori. Then either

(1) $N$ is a product of a torus and an interval,
(2) $N$ is a cable space, or
(3) for each torus component $T \subset \partial N$ there is a finite set of slopes $\mathcal{R} = \mathcal{R}(N, T)$ in $T$ such that if $b \not\in \mathcal{R}$ then $b$ is not norm-reducing.

Proof. Let $T$ be a particular component of $\partial N$. By [HM02, GL96], $N_T(a)$ is an irreducible for at most three slopes $a$. By [CGLS87, Corollary 2.4.4], unless $N \cong T \times [0, 1]$ or $N$ is a cable space, $N_T(a)$ is $\partial$–irreducible for at most three slopes $a$. Hence, we now assume $N$ is neither homeomorphic to $T \times [0, 1]$ nor a cable space, so that there are at most 6 slopes in $T$ for which $N_T(a)$ is reducible or $\partial$–irreducible.

Let $(\partial_T)_*: H_2(N, \partial N) \to H_1(T)$ be the composition of the boundary map on $H_2(N, \partial N)$ with the projection from $H_1(\partial N)$ to $H_1(T)$. For every slope $a$ in $T$ that generates a rank 1 subspace of the image of $(\partial_T)_*$ in $H_1(T)$, there is some class $\tilde{y} \in H_2(N_T(a), \partial N_T(a))$ such that $\operatorname{wind}_a(\tilde{y}) > 0$. Since $\operatorname{wind}_a$ gives a lower bound on wrap$_a$, the core of the Dehn filling $N_T(a)$ has non-zero wrapping number with respect to the class $\tilde{y}$. Therefore, if $(\partial_T)_*$ surjects onto $H_1(T)$, the core of any Dehn filling of $N$ along $T$ will have non-zero wrapping number with respect to some class in the filled manifold. In this case we may find a pair of slopes
satisfying the hypotheses of Corollary 4.4 so that the number of norm-reducing, but irreducible, and \( \partial \)-irreducible slopes is finite. Since the number of reducible or \( \partial \)-reducible slopes in \( T \) is also finite, we have our conclusion.

On the other hand, if \((\partial_T)_*\) does not surject onto \( H_1(T)\), its image must be a rank 1 subspace generated by a single slope, say \( b \). For every other slope \( a \neq b \), \( \text{wind}_a = 0 \). Hence for all \( a \neq b \), \( \rho_a \) gives an isomorphism \( H_2(N_T(a), \partial N_T(a)) \cong H_2(N, \partial N - T) \). Then it follows from [Se90] (but using [Gab87], Corollary 2.4) instead of just [Gab87], Theorem 1.8] to avoid hypotheses of atoroidality, see also [Lac97], Theorem A.21] that there are finitely many norm reducing fillings.

4.2. The dual norm. As we observed in the introduction, Theorem 4.7 shows that, in general, there are no norm-degenerating classes with respect to a knot that is surgery dual to a knot with “large” dual Thurston norm, quantified in terms of the distance of the surgery.

**Theorem 4.7.** Assume that every sphere, disc, annulus, and torus in \( M' \) separates. Given a class \( \alpha \in H_1(M'; \mathbb{Z}) \) and an integer \( \Delta \), if

\[
(\Delta - 1)x^*(\alpha) > 1
\]

then no Dehn surgery of distance \( \Delta \) on a knot representing \( \alpha \) produces an irreducible, \( \partial \)-irreducible 3-manifold \( M \) which has a norm-degenerating class with respect to the core of the surgery.

**Proof.** Assume \((\Delta - 1)x^*(\alpha) > 1 \) so that \( \Delta \geq 2 \) and \( x^*(\alpha) - 1/(\Delta - 1) > 0 \).

Since \( M' \) contains no non-separating sphere, disc, annulus, or torus, the Thurston norm on \( M' \) is actually a norm and not just a pseudo-norm. Thus, the unit norm ball in \( H_2(M', \partial M'; \mathbb{R}) \) is compact and \( x^*(\alpha) = \sup_{\alpha(\tau) = 1} |\alpha \cdot \tau| \). Since \( x^* \) is continuous, there exists a class \( \sigma \in H_2(M', \partial M'; \mathbb{R}) \) realizing this supremum, i.e. such that \( x(\sigma) = 1 \) and \( x^*(\sigma) = |\alpha \cdot \sigma| \). For any \( \varepsilon > 0 \), there is a rational class \( \zeta' \in H_2(M', \partial M'; \mathbb{Q}) \) approximating \( \sigma \) such that \( x(\zeta') = 1 \) and

\[
|\alpha \cdot \sigma| \geq |\alpha \cdot \zeta'| > |\alpha \cdot \sigma| - \varepsilon.
\]

In particular, since \((\Delta - 1)x^*(\alpha) > 1 \), let us choose \( \varepsilon \) so that \( x^*(\alpha) - 1/(\Delta - 1) > \varepsilon > 0 \).

Since \( |\alpha \cdot \tau|/x(\tau) \) is constant for non-zero multiples of any non-zero class \( \tau \in H_2(M, \partial M; \mathbb{R}) \), there exists an integral class \( \tilde{\zeta} \in H_2(M, \partial M; \mathbb{Z}) \) that is a positive multiple of the rational class \( \zeta' \) for which

\[
|\alpha \cdot \sigma| \geq \frac{|\alpha \cdot \tilde{\zeta}|}{x(\tilde{\zeta})} > |\alpha \cdot \sigma| - \varepsilon.
\]

Being an integral class, \( \tilde{\zeta} \) is represented by a surface. For any taut surface \( \hat{Q} \) representing \( \tilde{\zeta} \) we have \( x(\tilde{\zeta}) = -\chi(\hat{Q}) \) and \( |\alpha \cdot \tilde{\zeta}| = \text{wind}_a(\hat{Q}) \).

Now let \( K' \) be any knot representing \( \alpha \). Among the taut surfaces representing \( \tilde{\zeta} \), choose \( \hat{Q} \) to be one that minimizes \( |\hat{Q} \cap K'| \). Thus \( \text{wrap}_{K'}(\hat{Q}) \geq \text{wind}_{K'}(\hat{Q}) = |K' \cdot \hat{Q}| = |\alpha \cdot \tilde{\zeta}| \).

Hence by the choice of \( \sigma \),

\[
(\tilde{\zeta}) \quad x^*(\alpha) \geq \frac{\text{wind}_a(\hat{Q})}{-\chi(\hat{Q})} > x^*(\alpha) - \varepsilon.
\]

Since \( x^*(\alpha) - 1/(\Delta - 1) \geq \varepsilon > 0 \), we have \((\Delta - 1)(x^*(\alpha) - \varepsilon) \geq 1 \) and thus the right hand inequality of \((\tilde{\zeta})\) gives

\[
(\Delta - 1)\frac{\text{wind}_a(\hat{Q})}{-\chi(\hat{Q})} > (\Delta - 1)(x^*(\alpha) - \varepsilon) \geq 1.
\]
Consequently,
\[(\Delta - 1)|K' \cap \hat{Q}| = (\Delta - 1)\text{wrap}_{K'}(\hat{Q}) \geq (\Delta - 1)\text{wind}_{\partial}(\hat{Q}) > -\chi(\hat{Q})\]
By the choice of \(\hat{Q}\) and Lemma 2.3 there is no multi-\(\partial\)-compressing disc for \(\hat{Q}\). Thus, by Theorem 3.4 if \(M\) is obtained by a distance \(A\) Dehn surgery on \(K'\), then \(H_2(M, \partial M)\) cannot contain a norm-degenerating class with respect to the core of the surgery. \(\square\)

5. Genus Growth in Twist Families.

Let \(Y\) be a closed, compact, connected, oriented, irreducible, 3–manifold with \(H_2(Y) = 0\). Let \(\{K_n\}\) be a twist family of null-homologous knots in \(Y\) obtained by twisting a null-homologous knot \(K = K_0\) along an unknot \(c\). That is, \(K_n\) is the knot in \(Y = Y_c(-1/n)\) obtained by \(-1/n\)-surgery on \(c\) for each integer \(n\). Let \(g(K_n)\) be the Seifert genus of \(K_n\) and set \(\omega = |\ell(k(K, c))|\).

**Theorem 5.1.** If \(|\ell(k(K, c))| > 0\), then \(\lim_{n \to \infty} g(K_n) = \infty\) unless \(c\) is a meridian of \(K\).

**Proof.** This follows as a corollary of the more precise Theorem 5 above which implies the limit is finite only if \(\text{ax}([D]) = 0\). Here \(x\) is the Thurston norm on the exterior of the link \(K \cup c\) and \([D]\) is the homology class of a disk bounded by \(c\), intersected by \(K\), and restricted to this exterior. Since \(\omega = |\ell(k(K, c))| > 0\), the limit is finite only if \(x([D]) = 0\). This however implies that \(D\) is an annulus and hence \(c\) is a meridian of \(K\). \(\square\)

Let \(N = Y - N(K \cup c)\) be the exterior of the link \(K \cup c\) with boundary components \(T_K\) and \(T_c\) corresponding to \(K\) and \(c\) respectively, and use the standard associated meridian-longitude bases relative to \(K\) and \(c\) for these tori. Then the exterior of \(K_n\) is the manifold \(Y - N(K_n) = N_{r_c}(-1/n)\) which results from Dehn filling \(N\) along the slope \(-1/n\) in \(T_c\); let \(c_n\) be the core of this filling, setting \(c = c_0\).

Let \(\hat{D}\) be a disk bounded by \(c\) that is transverse to \(K\) and set \(D = \hat{D} \cap N\). Let \(\hat{F}_n\) be a Seifert surface for \(K_n\) that is transverse to \(c_n\) and set \(F_n = \hat{F}_n \cap N\).

**Lemma 5.2.** \([F_{n+1}] = [F_n] + \omega[D]\) for all integers \(n\).

**Proof.** Since \(Y\) is a rational homology sphere by assumption, each knot \(K_n\) (and \(c\)) has a unique homology class of Seifert surface up to sign. The formula then follows since \(\omega = |\ell(k(K, c))|\) and the surfaces \(F_n\) and \(D\) are the restrictions of Seifert surfaces for \(K_n\) and \(c\) to \(N\). Indeed, \(\partial[F_n]\) is homologous to one longitude of slope \(n\omega\) in \(T_K\) and \(\omega\) parallel curves of slope \(-1/n\) in \(T_c\), while \(\partial[D]\) is homologous to \(\omega\) meridians in \(T_K\) and one longitude of slope \(0\) in \(T_c\). It follows that (heeding orientations) \([F_n] + \omega[D]\) is represented by a properly embedded surface in \(N\) that is the Haken sum of \(F_n\) and \(\omega\) parallel copies of \(D\) which has boundary homologous to that of \(\partial[F_{n+1}]\). If \([F_{n+1}] - [F_n] - \omega[D]\) were a non-zero class, it would be represented by a boundaryless surface in \(N\) and thus represent a non-zero class in \(H_2(Y)\) — a contradiction. Hence \([F_{n+1}] = [F_n] + \omega[D]\). \(\square\)

**Theorem 5.3.** There is a constant \(G = G(K, c)\) such that \(2g(K_n) = 2G + n\text{ax}([D])\) for sufficiently large \(n > 0\).

**Proof.** Among discs bounded by \(c\) in \(Y\), let \(\hat{D}\) be one for which \(|K \cap \hat{D}| = p > 0\) is minimized and set \(D = \hat{D} \cap N\). Note that the minimality implies the punctured disc \(D\) is incompressible and \(\partial\)–incompressible. Moreover \(\partial D\) consists of one longitude of \(c\) and \(p\) meridional curves of \(K\). In particular, if \(p = 1\) then \(D\) is an annulus so that \(x([D]) = 0\) and \(c\) is a meridian of \(K\). Hence \(K = K_n\) for all integers \(n\) so the genus is constant and the theorem holds. Thus we assume \(p \geq 2\). This further implies that \(N\) is not the product of a torus and an interval.
If $N$ is a cable space, since $D$ is not an annulus but is a properly embedded, non-separating, incompressible and $\partial$–incompressible surface, it must be a fiber in a fibration of $N$ over $S^1$. (All classes in $H_2(N, \partial N; \mathbb{Z})$ other than multiples of the class of the cabling annulus are represented by fibers.) Therefore because $\partial D$ consists of a longitude of $c$ and meridians of $K$, it follows that $Y \cong S^3$ and $K$ is a torus knot in the solid torus exterior of the unknot $c$. In particular, this means that for some integer $q$ coprime to $p = |K \cap D|$, the knot $K_n$ is the $(p, q + np)$–torus knot and the theorem holds. Therefore we may assume that $N$ is not a cable space.

If $N$ is reducible, then there is a sphere in $N$ that does not bound a ball in $N$ and yet must bound a ball in $Y$ that contains either $K$ or $c$. If this sphere separates the two components of $\partial N$ then it separates $K$ and $c$ in $Y$ implying that $\ell k(K, c) = 0$, contrary to assumption. Thus $K \cup c$ must be contained in a ball in $Y$ and may be viewed as being contained in an $S^3$ summand of $Y$. Thus $N = N' \# Y$ where $N'$ is the irreducible exterior of $K \cup c$ in $S^3$. Since the summand will not affect the genera of the knots $K_n$, we may run the argument for $K \cup c$ in $S^3$. Thus we may assume $N$ is irreducible.

Let $\widehat{z}_n$ be the homology class of an oriented Seifert surface for $K_n$ in $Y - N'(K_n)$ for which $x(\widehat{z}_n) = 2g(K_n) - 1$. Then set $z_n = \rho_{-1/n}(\widehat{z}_n)$ to be the homology class of the restriction of the Seifert surface to $N = Y - N(K \cup c)$. By Theorem 4.6, there is a finite set of integers $\mathcal{R}$ such that

$$x(z_n) = x(\widehat{z}_n) + \text{wind}_{K_n}(\widehat{z}_n)$$

if $n \notin \mathcal{R}$. Since $\omega = \text{wind}_{K_n}(\widehat{z}_n)$ for all integers $n$ and $2g(K_n) - 1 = x(\widehat{z}_n)$, then when $n \gg 0$ we have

$$2(g(K_{n+1}) - g(K_n)) = x(z_{n+1}) - x(z_n) = x(z_{n+1} - z_n).$$

By Lemma 5.2, $z_{n+1} - z_n = \omega[D]$ for all integers $n$. Hence for $n \gg 0$, $2(g(K_{n+1}) - g(K_n)) = \alpha x([D])$. Therefore when $n$ is sufficiently large, $2g(K_n) = 2G + n\alpha x([D])$ for some constant $G$ as desired.

**Remark 5.4.** At the expense of having to reckon with multiple homology classes of Seifert surfaces, one should be able to prove Theorem 5.3 without the hypothesis that $Y$ is a rational homology sphere.

**Remark 5.5.** One ought to be able to prove Theorem 4.6 and Theorem 5.3 using link Floer Homology.

6. **Acknowledgements**

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**References**

[BM15] Kenneth L. Baker and Kimihiko Motegi, *Twist families of $L$-space knots, their genera, and Seifert surgeries* (2015), available at [arXiv:1506.04455](https://arxiv.org/abs/1506.04455)

[CGLS87] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987), no. 2, 327–330, DOI 10.2307/1971311. MR881270

[Gab83] David Gabai, *Foliations and the topology of 3–manifolds*, J. Differential Geom. 18 (1983), no. 3, 445–503. MR723813 (86a:57009)

[Gab87a] ________, *Foliations and the topology of 3–manifolds. II*, J. Differential Geom. 26 (1987), no. 3, 461–478. MR910017 (89a:57014a)

[Gab87b] ________, *Foliations and the topology of 3–manifolds. III*, J. Differential Geom. 26 (1987), no. 3, 479–536. MR910018 (89a:57014b)

[GL96] C. McA. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology 35 (1996), no. 2, 385–409, DOI 10.1016/0040-9383(95)00016-X.

[HM02] James A. Hoffman and Daniel Matignon, *Producing essential 2–spheres*, Topology Appl. 124 (2002), no. 3, 435–444, DOI 10.1016/S0166-8641(01)00250-4.

[Lac97a] Marc Lackenby, *Dehn surgery on knots in 3–manifolds*, J. Amer. Math. Soc. 10 (1997), no. 4, 835–864, DOI 10.1090/S0894-0347-97-00241-5. MR1443548
[Lac97b] ______, *Surfaces, surgery and unknotting operations*, Math. Ann. **308** (1997), no. 4, 615–632, DOI 10.1007/s002080050093. MR1464913

[Sch90] Martin Scharlemann, *Producing reducible 3–manifolds by surgery on a knot*, Topology **29** (1990), no. 4, 481–500, DOI 10.1016/0040-9383(90)90017-E. MR1071370 (91i:57003)

[Sch89] ______, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom. **29** (1989), no. 3, 557–614. MR992331 (90e:57021)

[Sel90] Zlil Sela, *Dehn fillings that reduce Thurston norm*, Israel J. Math. **69** (1990), no. 3, 371–378, DOI 10.1007/BF02764781. MR1049294

[Tay14] Scott A. Taylor, *Exceptional surgeries on knots with exceptional classes*, Bol. Soc. Mat. Mex. (3) **20** (2014), no. 2, 335–362, DOI 10.1007/s40590-014-0025-7. MR3264621

[Thu86] William P. Thurston, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130.