A NEW PROOF OF GAFFNEY’S INEQUALITY FOR DIFFERENTIAL FORMS ON MANIFOLDS-WITH-BOUNDARY: THE VARIATIONAL APPROACH À LA KOZONO–YANAGISAWA*

Siran LI (李思然)
School of Mathematical Sciences, IMA-Shanghai & Key Laboratory of Scientific and Engineering Computing (Ministry of Education), Shanghai Jiao Tong University, Shanghai 200240, China
E-mail: siran.li@sjtu.edu.cn

Abstract Let $(\mathcal{M}, g_0)$ be a compact Riemannian manifold-with-boundary. We present a new proof of the classical Gaffney inequality for differential forms in boundary value spaces over $\mathcal{M}$, via a variational approach à la Kozono–Yanagisawa $[L^r]$-variational inequality for vector fields and the Helmholtz–Weyl decomposition in bounded domains, Indiana Univ. Math. J. 58 (2009), 1853–1920], combined with global computations based on the Bochner technique.

Key words Gaffney’s inequality; differential form; Sobolev spaces on manifolds; Bochner technique; variational approach

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1 Introduction

The goal of this note is to give an alternative proof—which is global and variational in nature—of Gaffney’s inequality ([15]) for differential forms of arbitrary order on a manifold-with-boundary $(\mathcal{M}, g_0)$ in boundary value Sobolev spaces $W^{1,r}$ with $r \in ]1, \infty[$.

1.1 Gaffney’s inequality

As a prototype of Gaffney’s inequality, consider the well-known div-curl estimate of Calderón-Zygmund type in dimension 3: on a 3D Euclidean domain (with mild regularity assumptions), the $W^{1,r}$-norm of a vector field $v$ can be estimated by the $L^r$-norms of div $v$, curl $v$, and $v$ itself in the interior of the domain. Such an estimate plays a fundamental role in mathematical hydrodynamics, magnetics, and many other fields.

Nonetheless, the extension of the classical div-curl estimate to higher-dimensional manifolds for differential forms of arbitrary order subject to suitable boundary conditions is by no means

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straightforward. In the seminal work [15], Gaffney first established the estimate
\[ \|u\|_{W^{1,2}(M)} \leq C \left\{ \|du\|_{L^2(M)} + \|\delta u\|_{L^2(M)} + \|u\|_{L^2(M)} \right\}, \tag{1.1} \]
where \( M \) has an empty boundary and \( u \) is a differential \( k \)-form on \( M \) for any \( k \); see also Gallot–Meyer [16] and the exposition in Schwarz [29] (especially Corollary 2.1.6 and Theorem 2.1.7).

The proof of Equation (1.1) for general \( r \in [1, \infty) \) (i.e., with \( W^{1,2} \) and \( L^2 \) replaced by \( W^{1,r} \) and \( L^r \), respectively) in boundary value spaces is more technical. Iwaniec–Scott–Stroffolini [21] utilised the “freezing coefficient” argument to prove the \( L^r \)-Gaffney inequality via essentially local and Euclidean computations. Global proofs of Equation (1.1) on manifolds without boundary can be found in the literature; cf. Scott [30] for a potential-theoretic proof via the \( L^r \rightarrow L^r \) boundedness of Riesz transforms. Extensions to various classes of open manifolds are also available (Amar [1], Auscher–Coulhon–Duong–Hofmann [3], etc.).

In contrast, in the Hilbert case \( (r = 2) \), global, geometric, and simple proofs of Gaffney’s inequality on Riemannian manifolds-with-boundary and Euclidean domains can be found in the literature. Wider classes of boundary conditions and the determination of best constants have also been investigated; see Georgescu [17], Castó [6], Castó–Dacorogna [7], Castó–Dacorogna–Kneuss [8], Castó–Dacorogna–Sil [9], Castó–Dacorogna–Rajendran [10], and Mitrea [25].

Brought to our attention have been two global proofs for the \( L^r \)-Gaffney inequality for general \( r \in [1, \infty) \) in Dirichlet/Neumann boundary value spaces. Using a Campanato method, Sil [31] established the boundary regularity for more general classes of elliptic systems of differential forms in boundary value spaces. Also, Kozono–Yangisawa [24] proved a generalised Lax–Milgram theorem and used it to deduce the \( L^r \)-Hodge decompositions for differential forms with Dirichlet/Neumann boundary values. The \( L^r \)-Gaffney inequality was obtained along the way in [24, 31].

In this note, we present an alternative proof of the \( L^r \)-Gaffney inequality for differential forms of arbitrary order on manifolds-with-boundary. As opposed to the above two approaches, i.e., the functional-analytic arguments by Kozono–Yangisawa [24] and the Campanato space estimates by Sil [31], we demonstrate the \( L^r \)-Gaffney inequality — which, in greater generality, shall be formulated on Riemannian manifolds-with-boundary — via global geometric arguments.

More precisely, our approach to the \( L^r \)-Gaffney inequality is based on the Bochner technique and integration by parts (i.e., the Stokes and/or Gauss–Green theorems), together with a variational characterisation of \( W^{1,r} \)-differential forms in boundary value spaces à la Kozono–Yangisawa [24]. In addition, we make use of a polarisation trick to exploit the symmetry of boundary integrals. We learned this trick from an earlier work [5] by Chern.

1.2 Boundary value Sobolev spaces for differential forms on manifolds-with-boundary

Let us first introduce the relevant function spaces. Our notations largely follow those of Schwarz [29].

Throughout this work, \((M, g_0)\) is a Riemannian manifold-with-boundary of dimension \( n \). Denote by \( \Omega^k(M) \) the space of differential \( k \)-forms on \( M \). One can define in standard ways the Sobolev spaces of differential forms; see Hebey [18]. We write
\[ W^{k,p}\Omega^k(M) \equiv L^p_x\Omega^k(M) \]
for the space of \(k\)-forms on \(\mathcal{M}\) with \(W^{\ell,p}\)-regularity in the interior.

We write \(d : \Omega^k(\mathcal{M}) \rightarrow \Omega^{k+1}(\mathcal{M})\) for the exterior differential operator, and \(\delta : \Omega^{k+1}(\mathcal{M}) \rightarrow \Omega^k(\mathcal{M})\) for the codifferential (i.e., the formal \(L^2\)-adjoint of \(d\)). Throughout, the \(L^2\)- and any other Sobolev norms on \(\mathcal{M}\) are all taken with respect to the underlying Riemannian metric \(g_0\).

The Hodge star \(\star : \Omega^k(\mathcal{M}) \rightarrow \Omega^{n-k}(\mathcal{M})\) is also defined via the Riemannian volume measure.

On the space \(\dot{W}^{1,r}_0\Omega^k(\mathcal{M}) := \{u \in L^r\Omega^k(\mathcal{M}) : du \in L^r\Omega^{k+1}(\mathcal{M}), \delta u \in L^r\Omega^{k-1}(\mathcal{M})\}\), we can define the Dirichlet and Neumann trace operators \(t\) and \(n\) as follows:

\[
t, n : \dot{W}^{1,r}_0\Omega^k(\mathcal{M}) \longrightarrow \left[W^{1-\frac{1}{r'}, \frac{1}{r'}}\Omega^k(\partial\mathcal{M})\right]^*.
\]

Here \(1/r + 1/r' = 1\) (the same below). On \(C^1\Omega^k(\mathcal{M})\), the trace operators are given as

\[
t\omega := \iota^\# \omega, \quad n\omega := \omega - t\omega,
\]

(1.2)

where \(\iota : \partial\mathcal{M} \hookrightarrow \mathcal{M}\) is the natural inclusion, and \(\iota^\#\) is the pullback operator under \(\iota\). In other words, for any vectorfields \(X_1, \ldots, X_k\) along the boundary \(\partial\mathcal{M}\), we have

\[
t\omega(X_1, \ldots, X_k) \equiv \omega\left(X_1^\top, \ldots, X_k^\top\right),
\]

where \(X^\top\) is the component of \(X\) tangential to \(\partial\mathcal{M}\).

The notation \(\dot{W}^{1,r}_0\Omega^k(\mathcal{M})\) is only temporary; we shall not use it again in the sequel. The point here is that, in general (without suitable boundary conditions, and even if so, before proving the Gaffney inequality), we do not know whether \(\dot{W}^{1,r}_0\Omega^k(\mathcal{M})\) coincides with \(W^{1,r}\Omega^k(\mathcal{M})\).

We can now define the boundary value Sobolev spaces using \(t\) and \(n\):

\[
\begin{align*}
W^1_{\nu}\Omega^k(\mathcal{M}) &:= \{u \in L^r\Omega^k(\mathcal{M}) : du \in L^r\Omega^{k+1}(\mathcal{M}), \delta u \in L^r\Omega^{k-1}(\mathcal{M}), nu = 0\}, \quad (1.3) \\
W^1_{\nu}\Omega^k(\mathcal{M}) &:= \{u \in L^r\Omega^k(\mathcal{M}) : du \in L^r\Omega^{k+1}(\mathcal{M}), \delta u \in L^r\Omega^{k-1}(\mathcal{M}), tu = 0\}. \quad (1.4)
\end{align*}
\]

Here “\(D\)” stands for Dirichlet and “\(N\)” for Neumann.

In addition, let \(\nu \in \Gamma(\iota^\# T\mathcal{M})\) denote the outward unit normal vectorfield on \(\partial\mathcal{M}\), and let \(n^b\) be the 1-form canonically dual to \(\nu\). By a slight abuse of notations, we still write \(\nu\) and \(n^b\) for any of their smooth extensions into the interior of \(\mathcal{M}\). Then we set

\[
\begin{align*}
\dot{W}^1_{\nu}\Omega^k(\mathcal{M}) &:= \{u \in W^1\Omega^k(\mathcal{M}) : \nu L u|\partial\mathcal{M} = 0\}, \quad (1.5) \\
\dot{W}^1_{\nu}\Omega^k(\mathcal{M}) &:= \{u \in W^1\Omega^k(\mathcal{M}) : u \wedge n^b|\partial\mathcal{M} = 0\}. \quad (1.6)
\end{align*}
\]

Moreover, we put

\[
\begin{align*}
C^\infty_{\nu}\Omega^k(\mathcal{M}) &:= \{u \in C^\infty\Omega^k(\mathcal{M}) : \nu L u|\partial\mathcal{M} = 0\}, \quad (1.7) \\
C^\infty_{\nu}\Omega^k(\mathcal{M}) &:= \{u \in C^\infty\Omega^k(\mathcal{M}) : u \wedge n^b|\partial\mathcal{M} = 0\}. \quad (1.8)
\end{align*}
\]

Here and hereafter, \(\overline{\mathcal{M}} := \mathcal{M} \cup \partial\mathcal{M}\) and \(L\) is the interior multiplication operator. For \(\mathcal{M} = 3D\) Euclidean domain and \(u = \) vectorfield, \(\nu L u = 0\) and \(u \wedge n^b = 0\) are, respectively, equivalent to \(\nu \cdot u = 0\) and \(u \times \nu = 0\) on \(\partial\mathcal{M}\).

Throughout this note, the \(W^{\ell,p}\)-norm of a differential form \(u\) will be denoted by \(\|u\|_{W^{\ell,p}(\mathcal{M})}\).

We suppress the index \(k\), the order of the differential form, for notational convenience. Also, for two \(L^2\)-differential \(k\)-forms \(\alpha\) and \(\beta\) over \(\mathcal{M}\) we write

\[
(\alpha, \beta)_{g_0} := \int_{\mathcal{M}} \alpha \wedge \star\beta.
\]
This is the $L^2$-inner product with respect to the metric $g$. The right-hand side is an integral of $n$-forms over an $n$-dimensional manifold, which is defined in the usual manner. We also write

$$\langle \alpha, \beta \rangle_{\Lambda^k} := \alpha \wedge \ast \beta$$

to emphasise the pairing of $k$-forms.

For further developments we also need some more notations. First, $\nabla \equiv \nabla_{g_0}$ denotes the covariant derivative induced by the Levi-Civita connection of the given metric $g_0$; it extends to differential forms of any order via the Leibniz rule. Second, $\nabla^\dagger$ denotes the formal $L^2$-adjoint of $\nabla$, again with respect to the Riemannian volume measure of $g_0$. Third,

$$d\Sigma = \nu L \, dV_{g_0}$$

is the Riemannian volume (surface) form on $\partial M$.

For background materials on global/geometric analysis, we refer to Petersen [28], Hebey [18], and the first chapter of Schwarz [29].

### 1.3 Main Theorem

Our main result is the following:

**Theorem (A)** Let $(M, g_0)$ be an $n$-dimensional compact Riemannian manifold-with-boundary, let $k \in \{0, 1, 2, \ldots, n\}$, and let $r \in ]1, \infty]$ be arbitrary. Consider $u \in \overset{\wedge}{W}^{1, r}_D \Omega^k(M)$ or $\overset{\wedge}{W}^{1, r}_N \Omega^k(M)$, a differential $k$-form subject to the Dirichlet or Neumann boundary condition. Then

$$\|u\|_{W^{1, r}(M)} \leq C \left\{ \|du\|_{L^r(M)} + \|\delta u\|_{L^r(M)} + \|u\|_{L^r(M)} \right\},$$

where $C$ depends only on $r$, $k$, $n$, and the $C^1$-geometry of $M$.

### 1.4 Variational approach

To prove Theorem A, we shall use a variational characterisation of the $W^{1, r}$-norm of differential forms in boundary value spaces.

**Theorem (B)** Let $(M, g_0)$ be an $n$-dimensional compact Riemannian manifold-with-boundary, let $C^\infty_0 \Omega^k(M)$ denote either $C^\infty_0 \Omega^k_0(M)$ or $C^\infty_0 \Omega^k_1(M)$, let $r$ and $q$ be any numbers in $]1, \infty[$, and let $k \in \{0, 1, 2, \ldots, n\}$. Assume that $u \in \overset{\wedge}{W}^{1, q}_D \Omega^k(M, g_0)$ and that the following quantity is finite:

$$\sup \left\{ \frac{|(du, d\phi)_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}|}{\|d\phi\|_{L^r(M, g_0)} + \|\delta \phi\|_{L^r(M, g_0)} + \|\phi\|_{L^r(M, g_0)}} : \phi \in C^\infty_0 \Omega^k(M) \right\}. $$

Then $u \in \overset{\wedge}{W}^{1, r}_D \Omega^k(M, g_0)$.

In addition, the following variational inequality is satisfied:

$$\|u\|_{W^{1, r}} \leq C \sup \left\{ \frac{|(du, d\phi)_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}|}{\|d\phi\|_{L^r(M, g_0)} + \|\delta \phi\|_{L^r(M, g_0)} + \|\phi\|_{L^r(M, g_0)}} : \phi \in C^\infty_0 \Omega^k(M) \right\}. $$

Here $C$ depends only on $r$, $n$, $k$, and the $C^1$-geometry of $(M, g_0)$.

The case for $n = 3$, $g_0 = \text{the Euclidean metric}$, and $k = 1$ is due to Kozono–Yanagisawa [23], and based on this a new, refined proof of the Hodge decomposition theorem for vectorfields is obtained. We shall essentially follow the strategies in [23] to prove Theorem B.
1.5 Organization

The remaining parts of this note are organised as follows: first, in Section 2 we collect a few preliminary results on geometry. Next, in Section 3, a proof of Theorem B shall be given by generalising the arguments in [23] for vectorfields on 3D domains. This shall be utilised in Section 4 to prove Theorem A. Indeed, our arguments in Section 4 are global. Finally, several concluding remarks will be given in Section 5.

2 Preliminaries

Let us first note the following commutation relations between the differential, the codifferential, the Hodge star, and the trace operators:

**Lemma 2.1** For any sufficiently regular differential $k$-form $\omega$ and $(k+1)$-form $\eta$ on the manifold-with-boundary $(M^n, g_0)$, we have

\[
\begin{align*}
\star(n\omega) &= t(\star \omega), \\
\star(t\omega) &= n(\star \omega), \\
t(d\omega) &= d(t\omega), \\
\delta(\omega) &= \delta(n\omega).
\end{align*}
\]

In addition, it holds that

\[
t\omega \wedge *n\eta = \langle \omega, \nu \eta \rangle_{L^2} \cdot d\Sigma,
\]

where $d\Sigma$ is the Riemannian volume measure on $\partial M$ induced from $g_0$.

**Proof** See Proposition 1.2.6, p.27 in Schwarz [29].

The following integration by parts formula can be deduced directly from the Stokes/Gauss–Green theorem:

**Lemma 2.2** Let $\omega$ and $\eta$ be as in Lemma 2.1. Then

\[
(d\omega, \eta)_{g_0} = (\omega, \delta \eta)_{g_0} + \int_{\partial M} t\omega \wedge *n\eta.
\]

**Proof** See Proposition 2.1.2, p.60 in Schwarz [29].

In addition, we have the Sobolev embedding theorems ([18]) for differential forms on compact $(M^n, g_0)$: $W^{1,q} \Omega^k(M) \hookrightarrow L^r \Omega^k(M)$ for $1 \leq r < \frac{mr}{m-q}$, and $W^{1,s} \Omega^k(M) \hookrightarrow C^0 \Omega^k(M)$ for $s > n$.

3 Variational Inequalities

To prove Theorem B, we first argue for $M = \mathbb{R}^n$ with $u$ compactly supported. Then we deduce the assertion for $M = \mathbb{R}^n_+$ by even or odd extensions across the boundary. The general case holds via a partition of unity argument, with the covering charts being sufficiently refined.

3.1 The case of $\mathbb{R}^n$

In this subsection $(\cdot, \cdot)$ always denotes the standard Euclidean inner product. We shall prove
Lemma 3.1 Let \( u \in W^{1,q}\Omega^k(\mathbb{R}^n) \) for \( q \in ]1, \infty[ \). Assume that there is an \( r \in ]1, \infty[ \) such that
\[
\sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^r(\mathbb{R}^n)} + \|\delta \phi\|_{L^r(\mathbb{R}^n)} + \|\phi\|_{L^r(\mathbb{R}^n)}} : \phi \in C_c^\infty \Omega^k(\mathbb{R}^n) \right\} < \infty.
\]
Then \( u \in W^{1,r}\Omega^k(\mathbb{R}^n) \) with the estimate
\[
\|u\|_{W^{1,r}(\mathbb{R}^n)} \leq C \sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^r(\mathbb{R}^n)} + \|\delta \phi\|_{L^r(\mathbb{R}^n)} + \|\phi\|_{L^r(\mathbb{R}^n)}} : \phi \in C_c^\infty \Omega^k(\mathbb{R}^n) \right\},
\] (3.1)
where \( C \) depends only on \( r, k, \) and \( n \).

Proof Consider the space
\[
\mathcal{G}_k := \{ (\mathbb{I} - \Delta)\psi : \psi \in C_c^\infty \Omega^k(\mathbb{R}^n) \},
\] (3.2)
where \( \mathbb{I} \) is the identity map and \( \Delta \) is the Euclidean Laplacian. We claim that \( \mathcal{G}_k \) is dense in \( L'^\Omega^k(\mathbb{R}^n) \) for any \( r \in ]1, \infty[ \).

This statement is proved below by contradiction. Suppose that there were an \( f \in L'^\Omega^k(\mathbb{R}^n) \sim \{0\} \) annihilating \( \mathcal{G}_k \); that is,
\[
\int_{\mathbb{R}^n} f \bullet (\mathbb{I} - \Delta)\psi = 0 \quad \text{for all} \ \psi \in \mathcal{G}_k.
\] (3.3)
The symbol \( \bullet \) denotes the usual Euclidean inner product for differential \( k \)-forms on \( \mathbb{R}^n \). If \( k = 0 \), i.e., \( f \) and \( \psi \) are scalar functions, it then follows that
\[
(\mathbb{I} - \Delta)f = 0 \quad \text{as Schwartz distributions.}
\]
The Fourier transform \( \hat{f} \) of \( f \) satisfies \( 1 + 4\pi^2|\xi|^2 \hat{f}(\xi) = 0 \) for all \( \xi \in \mathbb{R}^n \), thus \( \hat{f} \equiv 0 \) and hence \( f \equiv 0 \). This yields a contradiction.

For \( k \geq 1 \), let us expand differential \( k \)-forms with respect to the basis consisting of simple vectorfields formed by the canonical basis for \( \mathbb{R}^n \). That is, consider
\[
f = \sum f_{j_1j_2...j_k} e_{j_1} \wedge \cdots \wedge e_{j_k},
\]
where \( \{e_1, \ldots, e_n\} \) is the Euclidean canonical basis, and the summation is taken over ascending \( k \)-tuples \( (j_1, \ldots, j_k) \) such that \( 1 \leq j_1 < j_2 < \cdots < j_k \leq n \); similarly for \( \psi \). Then Equation (3.3) is equivalent to
\[
\int_{\mathbb{R}^n} f_{j_1j_2...j_k} (\mathbb{I} - \Delta)\psi_{j_1j_2...j_k} \ dx = 0 \quad \text{for all such indices.}
\]
The arguments above imply that \( f_{j_1j_2...j_k} \equiv 0 \). Hence \( f \equiv 0 \), which gives the contradiction and proves the claim.

Now we are ready to prove the lemma. Fix \( r \in ]1, \infty[ \) and take a test form \( \phi = \nabla_j \psi \) for some \( \psi \in C_c^\infty \Omega^k(\mathbb{R}^n) \) and \( j \in \{1, 2, \ldots, n\} \) fixed. Then
\[
\sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^r(\mathbb{R}^n)} + \|\delta \phi\|_{L^r(\mathbb{R}^n)} + \|\phi\|_{L^r(\mathbb{R}^n)}} : \phi \in C_c^\infty \Omega^k(\mathbb{R}^n) \right\} \geq \sup \left\{ \frac{|(du, d\nabla_j \psi) + (\delta u, \delta \nabla_j \psi) + (u, \nabla_j \psi)|}{\|d\nabla_j \psi\|_{L^r(\mathbb{R}^n)} + \|\delta \nabla_j \psi\|_{L^r(\mathbb{R}^n)} + \|\nabla_j \psi\|_{L^r(\mathbb{R}^n)}} : \psi \in C_c^\infty \Omega^k(\mathbb{R}^n) \right\},
\] (3.4)
On \( \mathbb{R}^n \) we can commute \( d \) with \( \nabla_j \) and \( \delta \) with \( \nabla_j \). Moreover, note that \( -\Delta = d\delta + \delta d, \) so
\[
(du, d\nabla_j \psi) + (\delta u, \delta \nabla_j \psi) + (u, \nabla_j \psi) = (\mathbb{I} - \Delta) \psi, \nabla_j u).
\]
By the standard Calderón–Zygmund estimate on \( \mathbb{R}^n \),
\[
\|\nabla_j \psi\|_{L^r(\mathbb{R}^n)} \leq C \left( \|\Delta \psi\|_{L^r(\mathbb{R}^n)} + \|\psi\|_{L^r(\mathbb{R}^n)} \right),
\]
and the interpolation inequality
\[ \|\nabla \psi\|_{L^{r'}(\mathbb{R}^n)} \leq C \left( 1 + \|\nabla \psi\|_{L^{r'}(\mathbb{R}^n)} \right), \]
one may continue the previous estimate by
\[
\sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^{r'}(\mathbb{R}^n)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n)} + \|\phi\|_{L^{r'}(\mathbb{R}^n)}} : \phi \in C^\infty_c \Omega^k(\mathbb{R}^n) \right\} 
\]
\[
\geq c \sup \left\{ \frac{|(1 - \Delta) \psi, \nabla_j u|}{\|\nabla_j u\|_{L^{r'}(\mathbb{R}^n)}} : \psi \in C^\infty \Omega^k(\mathbb{R}^n) \right\} 
\]
\[
= c \sup \left\{ \frac{|(\rho, \nabla_j u)|}{\|\rho\|_{L^{r'}(\mathbb{R}^n)}} : \rho \in \mathcal{S}_k \right\}. 
\]

By the claim above (namely, the density of \( \mathcal{S}_k \) in \( L^{r'} \Omega^k(\mathbb{R}^n) \)) and the duality characterisation of the \( L^r \)-norm, it holds that
\[
\sup \left\{ \frac{|(\rho, \nabla_j u)|}{\|\rho\|_{L^{r'}(\mathbb{R}^n)}} : \rho \in \mathcal{S}_k \right\} = \sup \left\{ \frac{|(\rho, \nabla_j u)|}{\|\rho\|_{L^{r'}(\mathbb{R}^n)}} : \rho \in L^{r'} \Omega^k(\mathbb{R}^n) \right\} = \|\nabla_j u\|_{L^{r'}(\mathbb{R}^n)}. 
\]

As the index \( j \) is arbitrary, we can chain together the previous inequalities to get
\[
\|\nabla u\|_{L^{r'}(\mathbb{R}^n)} \leq C \sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^{r'}(\mathbb{R}^n)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n)} + \|\phi\|_{L^{r'}(\mathbb{R}^n)}} : \phi \in C^\infty_c \Omega^k(\mathbb{R}^n) \right\}, 
\]
where \( \nabla \) is the covariant derivative from differential \( k \)-forms to \( (k + 1) \)-forms induced by the Levi-Civita connection on \( \mathbb{R}^n \).

Repeating the same arguments with \( \phi = \psi \) instead of \( \phi = \nabla_j \psi \), we obtain
\[
\|u\|_{L^{r'}(\mathbb{R}^n)} \leq C \sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^{r'}(\mathbb{R}^n)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n)} + \|\phi\|_{L^{r'}(\mathbb{R}^n)}} : \phi \in C^\infty_c \Omega^k(\mathbb{R}^n) \right\}. 
\]
The previous two estimates lead to Equation (3.1). All the constants \( C \) and \( c \) in this proof depend on nothing but \( r \), \( n \), and \( k \). Therefore, \( u \in W^{1,r} \Omega^k(\mathbb{R}^n) \), and the proof is complete. \( \square \)

**Remark 3.2** Since Lemma 3.1 is only concerned with differential forms on \( \mathbb{R}^n \) and our arguments utilise the Fourier transform, the Laplacian we considered in the proof is the Hodge Laplacian, which differs from the Laplace–Beltrami operator by a sign. In contrast, when working with manifolds in the later parts of the paper, for notational convenience we shall always take \( \Delta \) to be the Laplace–Beltrami operator, i.e., \( \Delta = d^* + \delta d \).

### 3.2 The case of \( \mathbb{R}^n_+ \)

The analogue of Lemma 3.1 holds for the halfspace \( \mathbb{R}^n_+ \). As in the previous subsection, \( \langle \cdot, \cdot \rangle \) is reserved for the standard Euclidean inner product in Lemma 3.3, below. Let \( W^{1,q}_\Omega^k(\mathcal{M}, g_0) \) denote either \( W^{1,q}_N\Omega^k(\mathcal{M}, g_0) \) or \( W^{1,q}_D\Omega^k(\mathcal{M}, g_0) \).

**Lemma 3.3** Let \( u \in W^{1,q}_\Omega^k(\mathbb{R}^n_+) \) for some \( q \in ]1, \infty[ \). Assume that
\[
\sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^{r'}(\mathbb{R}^n_+)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n_+)} + \|\phi\|_{L^{r'}(\mathbb{R}^n_+)}} : \phi \in C^\infty_c \Omega^k(\mathbb{R}^n_+) \right\} < \infty \quad (3.5)
\]
for some \( r \in ]1, \infty[ \). Then \( u \in W^{1,r} \Omega^k(\mathbb{R}^n_+) \) with the estimate

\[
\|u\|_{W^{1,r}(\mathbb{R}^n_+)} \leq C \sup \left\{ \frac{|(du, d\phi) + (\delta u, \delta \phi) + (u, \phi)|}{\|d\phi\|_{L^r(\mathbb{R}^n_+)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n_+)} + \|\phi\|_{L^{r'}(\mathbb{R}^n_+)}} : \phi \in C^\infty_0 \Omega^k(\mathbb{R}^n_+) \right\},
\]

where the constant \( C \) depends only on \( r, n, \) and \( k \).

**Proof** We shall prove things for \( \phi = D \) and \( \phi = N \) separately.

**Case 1:** \( u \in \tilde{W}_N^{1,q} \Omega^k(\mathbb{R}^n_+) \). We consider the extension

\[
\tilde{u}(x) := \begin{cases} u(x) & \text{if } x^n \geq 0, \\ tu(x^*) \oplus -nu(x^*) & \text{if } x^n < 0, \end{cases}
\]

where \( x = (x^1, \ldots, x^n)^\top \) and \( x^* := (x^1, \ldots, x^{n-1}, -x^n)^\top \).

By assumption, Equation (3.5) holds for \( u \). It also holds for \( \tilde{u} \) (which lies in \( W^{1,q} \Omega^k(\mathbb{R}^n_+) \)) with respect to test \( k \)-forms \( \phi \in C^\infty \Omega^k(\mathbb{R}^n) \). To see this, note first that the boundary condition for \( u \) ensures the (weak) differentiability of \( \tilde{u} \) across \( \partial \mathbb{R}^n_+ \). Thus we have \( \int_{\mathbb{R}^n} |\nabla \tilde{u}|^r = \int_{\mathbb{R}^n_+} |\nabla \tilde{u}|^r + \int_{\mathbb{R}^n_+} |\nabla \tilde{u}|^r, \) so by construction,

\[
\|\nabla \tilde{u}\|_{L^r(\mathbb{R}^n_+)}^r = 2\|\nabla u\|_{L^r(\mathbb{R}^n_+)}^r.
\]

Similarly, we get that

\[
\|\tilde{u}\|_{L^r(\mathbb{R}^n_+)} = 2\|u\|_{L^r(\mathbb{R}^n_+)}.
\]

It thus follows from Lemma 3.1 that \( \tilde{u} \in W^{1,r} \Omega^k(\mathbb{R}^n) \), and the following estimate holds:

\[
\|\tilde{u}\|_{W^{1,r}(\mathbb{R}^n_+)} \leq C \sup \left\{ \frac{|(d\tilde{u}, d\phi)_{\mathbb{R}^n_+} + (\delta \tilde{u}, \delta \phi)_{\mathbb{R}^n_+} + (\tilde{u}, \phi)_{\mathbb{R}^n_+}|}{\|d\phi\|_{L^r(\mathbb{R}^n_+)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n_+)} + \|\phi\|_{L^{r'}(\mathbb{R}^n_+)}} : \phi \in C^\infty \Omega^k(\mathbb{R}^n) \right\}.
\]

The passage from Equation (3.7) to Equation (3.6) relies on choosing special test \( k \)-forms \( \phi \). Indeed, for any \( \phi \in C^\infty \Omega^k(\mathbb{R}^n) \) we set

\[
\phi(x) := [t\phi(x) + t\phi(x^*)] \oplus [n\phi(x) - n\phi(x^*)].
\]

Thus, the restriction of \( \phi \) to \( \mathbb{R}^n_+ \) lies in \( C^\infty \Omega^k(\mathbb{R}^n_+) \). Moreover, for \( T \in \{d, \delta, \text{id}\} \), one has that

\[
(T\tilde{u}, \phi)_{\mathbb{R}^n_+} = (Tu, \phi)_{\mathbb{R}^n_+}.
\]

Therefore,

\[
\|u\|_{W^{1,r}(\mathbb{R}^n_+)} \leq C\|\tilde{u}\|_{W^{1,r}(\mathbb{R}^n_+)} \leq C \sup \left\{ \frac{|(d\tilde{u}, d\phi)_{\mathbb{R}^n_+} + (\delta \tilde{u}, \delta \phi)_{\mathbb{R}^n_+} + (\tilde{u}, \phi)_{\mathbb{R}^n_+}|}{\|d\phi\|_{L^r(\mathbb{R}^n_+)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n_+)} + \|\phi\|_{L^{r'}(\mathbb{R}^n_+)}} : \phi \in C^\infty \Omega^k(\mathbb{R}^n) \right\} \leq C \sup \left\{ \frac{|(du, d\phi)_{\mathbb{R}^n_+} + (\delta u, \delta \phi)_{\mathbb{R}^n_+} + (u, \phi)_{\mathbb{R}^n_+}|}{\|d\phi\|_{L^r(\mathbb{R}^n_+)} + \|\delta \phi\|_{L^{r'}(\mathbb{R}^n_+)} + \|\phi\|_{L^{r'}(\mathbb{R}^n_+)}} : \phi \in C^\infty_0 \Omega^k(\mathbb{R}^n_+) \right\}.
\]

All the constants in the above arguments depend only on \( r, n, \) and \( k \). This completes the proof for \( u \in \tilde{W}_N^{1,q} \Omega^k(\mathbb{R}^n_+) \).

**Case 2:** \( u \in \tilde{W}_D^{1,q} \Omega^k(\mathbb{R}^n_+) \). In this case, consider the extension

\[
\tilde{u}(x) := \begin{cases} u(x) & \text{if } x^n \geq 0, \\ -tu(x^*) \oplus nu(x^*) & \text{if } x^n < 0, \end{cases}
\]

\( \Phi \) Springer
Correspondingly, for any test differential form \( \phi \in C^\infty_c(\Omega^k(\mathbb{R}^n)) \), we set
\[
\hat{\phi}(x) := [\phi(x) - \phi(x^*)] \oplus [n\phi(x) + n\phi(x^*)].
\]

Then the restriction of \( \phi \) to \( \mathbb{R}^n_+ \) is an element of \( W^{1,q}_D(\Omega^k(\mathbb{R}^n_+)) \), and all the previous arguments for \( u \in W^{1,q}_N(\Omega^k(\mathbb{R}^n_+)) \) carry over to the case \( u \in W^{1,q}_D(\Omega^k(\mathbb{R}^n_+)) \). \( \square \)

In what follows, we denote by \( \epsilon \) the Euclidean metric, and by \( (\epsilon, \epsilon) \), the Euclidean inner product on either \( \mathbb{R}^n \) or \( \mathbb{R}^n_+ \), which shall be clear from the context.

Lemma 3.3 also holds if \( \mathbb{R}^n_+ \) is equipped with any constant metric (possibly different from the Euclidean metric).

**Corollary 3.4** Assume that \( \overline{b}_0 \) is a constant metric on \( \Phi(U) \subset \mathbb{R}^n_+ \). Then Lemma 3.3 continues to hold, with all the inner products and Sobolev norms therein taken with respect to \( \overline{b}_0 \), and with the constant \( C \) depending only on \( r, k, n, \) and \( \overline{b}_0 \).

**Proof** Consider the quotient
\[
\frac{|(du, d\phi)|_{\overline{b}_0} + (\delta u, \delta \phi)|_{\overline{b}_0} + (u, \phi)|_{\overline{b}_0}|}{\|d\phi\|_{L^r(\mathbb{R}^n_+, \overline{b}_0)} + \|\delta \phi\|_{L^r(\mathbb{R}^n_+, \overline{b}_0)} + \|\phi\|_{L^r(\mathbb{R}^n_+, \overline{b}_0)}}
\]
for any test \( K \)-form \( \phi \in C^\infty_c(\Omega^k(\mathbb{R}^n_+)) \). We claim that it differs from its Euclidean analogue
\[
\frac{|(du, d\phi)_\epsilon + (\delta u, \delta \phi)_\epsilon + (u, \phi)_\epsilon|}{\|d\phi\|_{L^r(\mathbb{R}^n_+, \epsilon)} + \|\delta \phi\|_{L^r(\mathbb{R}^n_+, \epsilon)} + \|\phi\|_{L^r(\mathbb{R}^n_+, \epsilon)}}
\]
only by a constant depending on \( \overline{b}_0, r, \) and \( \epsilon \).

Indeed, for arbitrary differential \( \epsilon \)-forms \( \alpha \) and \( \beta \) supported on \( \Phi(U) \subset \mathbb{R}^n_+ \), we can express in local co-ordinate frames such that
\[
(\alpha, \beta)_{\overline{b}_0} = \int_{\Phi(U)} \hat{g}_0^{i_1j_1} \cdots \hat{g}_0^{i_lj_l} \alpha_{i_1j_1 \cdots j_l} \sqrt{\det(\hat{g}_0)} dx;
\]
\[
(\alpha, \beta)_\epsilon = \int_{\Phi(U)} \alpha_{i_1 \cdots i_l} \beta_{j_1 \cdots j_l} dx.
\]
Thus
\[
c(\alpha, \beta)_\epsilon \leq (\alpha, \beta)_{\overline{b}_0} \leq C(\alpha, \beta)_\epsilon,
\]
with \( c \) and \( C \) depending only on \( \ell \), as well as the (pointwise) upper bound for \( \overline{b}_0 \) and lower bound for \( \overline{b}_0^{-1} \). Similar computations in local co-ordinates also yield that
\[
c' \|\Upsilon\|_{L^s(\mathbb{R}^n_+, \epsilon)} \leq \|\Upsilon\|_{L^s(\mathbb{R}^n_+, \overline{b}_0)} \leq C' \|\Upsilon\|_{L^s(\mathbb{R}^n_+, \epsilon)}
\]
for any differential \( \epsilon \)-form \( \Upsilon \) supported on \( \Phi(U) \) with a finite \( L^s \)-norm. Here \( c' \) and \( C' \) depend only on \( \overline{b}_0, \) \( \ell \), and \( r \). Then the claim follows once we choose \( \ell, \alpha, \beta \), and \( \Upsilon \) appropriately.

The same argument shows that for a given differential \( k \)-form, its \( W^{1,r} \)-norms with respect to \( \overline{b}_0 \) and \( \epsilon \) differ only by a constant depending on \( \overline{b}_0, n, \ell, r, \) and \( \epsilon \). The proof is now complete in view of Lemma 3.3. \( \square \)

### 3.3 The case of a small chart

In this subsection, we show that the conclusion of Lemma 3.1 continues to hold for differential forms defined over one sufficiently small chart \( U_\alpha \subset \overline{M} = M \cup \partial M \). By definition, there is a diffeomorphism \( \Phi_\alpha : U_\alpha \rightarrow \Phi_\alpha(U_\alpha) \subset \mathbb{R}^n_+ \) such that image \( \Phi_\alpha(U_\alpha) \) is relatively open in
the closed half space. For obvious reasons, we shall focus only on boundary charts, i.e., those $U_\alpha$ with $U_\alpha \cap \partial \mathcal{M} \neq \emptyset$. In the rest of this subsection, we work with one fixed $U_\alpha$, hence the subscripts $\alpha$ shall be systematically dropped.

We first prepare ourselves with several simple geometric estimates. Let $\kappa > 0$ be arbitrary. On a sufficiently small chart $U$, the metric $g_0|U$ is almost constant. More precisely, for an arbitrary reference point $P \in U$, one has that

$$||g_0 - g_0(P)||_{C^0(U)} + ||Dg_0 - [Dg_0](P)||_{C^0(U)} \leq \kappa.$$  \hspace{1cm} (3.9)

Here the derivative $Dg_0$ is understood as the 3-tensor field $[Dg_0]_{ijk} = \partial_i(g_0)_{jk}$; the metric components $(g_0)_{jk}$ are taken with respect some (a priori given) local co-ordinate system on $U$.

Choose the constant metric

$$\overline{g}_0 := \Phi^\#(g_0(P))$$

on $\Phi(U)$; it is indeed a Riemannian metric, since $\Phi$ is a diffeomorphism. It then follows from Equation (3.9) that

$$||\Phi^\#g_0 - \overline{g}_0||_{C^0(U)} + ||D[\Phi^\#g_0] - \Phi^\#[Dg_0(P)]||_{C^0(U)} \leq C_1 \kappa,$$ \hspace{1cm} (3.10)

where $C_1$ depends only on the $C^1$-geometry of the chart $U$. That is, the pushforward metric $\Phi^\#g_0$ is $O(\kappa)$-close in the $C^1$-topology to the constant metric $\overline{g}$.

Next, note that the $C^0$-bound in (3.10) and the continuity of the determinant in the uniform topology give us

$$\begin{cases}
||\det g_0 - \det (g_0(P))||_{C^0(U)} \leq C_2 \kappa, \\
||\det (\Phi^\#g_0) - \overline{g}_0||_{C^0(\Phi(U))} \leq C_3 \kappa,
\end{cases}$$ \hspace{1cm} (3.11)

with $C_2$ and $C_3$ depending only on the $C^1$-geometry of the chart $U$.

To compare inverse metrics $\Phi^\#(g_0^{-1})$ and $\overline{g}_0^{-1}$, recall, for any metric $g$, Cramer’s rule:

$$g^{-1} = \frac{\text{Adj} g}{\det g}.$$  

Here $\text{Adj} g$ is the adjugate matrix of $g$. Modulo natural duality isomorphisms, it holds that

$$\text{Adj} g \simeq \bigwedge^{n-1} g,$$

the $(n - 1)$-fold wedge product of $g$ as a matrix. Hence

$$\Phi^\#(g_0^{-1}) - \overline{g}_0^{-1} = \frac{\left\{ \bigwedge^{n-1} \Phi^\#g_0 \det \overline{g}_0 - \left( \bigwedge^{n-1} \overline{g}_0 \right) \det [\Phi^\#(g_0)] \right\}}{\det [\overline{g}_0 \cdot \Phi^\#g_0]}.$$  

For the denominator we can bound

$$\det [\overline{g}_0 \cdot \Phi^\#g_0] \geq \det [\overline{g}_0]^2 - C_4 \kappa > 0,$$

thanks to Equation (3.11). Here $C_4$ depends on the $C^1$-geometry of $U$. To control the numerator, we apply the simple combinatorial identity

$$\left( \prod_{i=1}^{n} a_i \right) - \left( \prod_{i=1}^{n} b_i \right) = \sum_{i=1}^{n} \left\{ \left( \prod_{1 \leq k \leq i} b_k \right) (a_i - b_i) \left( \prod_{i < j \leq n} a_j \right) \right\}$$  

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The constant $C$ given an arbitrarily small number $\kappa > 0$ with diameter smaller than $\rho$, its intrinsic diameter smaller than $\rho$, relies only on differentiable structures of $M$. Let us first check that the boundary conditions for $W$.

Then $C$. Here $\Gamma^i_{jk}$ and $\overline{\Gamma}^i_{jk}$ for $\Phi_\#(g_0)$ and $\overline{g_0}$, respectively:

$$\|\Gamma^i_{jk} - \overline{\Gamma}^i_{jk}\|_{C^0(\Phi(U))} \leq C_6 \kappa. \quad (3.13)$$

To summarise, under the assumption that $g_0$ is $O(\kappa)$-close to a constant metric in $C^1$-topology (Equation (3.9)), we have proved that $g_0$ is, in fact, $O(\kappa)$-close to the constant metric in the bi-$C^1$-topology. The constant in $O(\kappa)$ can be chosen to depend only on the $C^1$-topology of $U$. The same conclusion remains valid under the pushforward via $\Phi$.

**Proposition 3.5** Let $(M, g_0)$ be a Riemannian manifold-with-boundary. For any sufficiently small boundary chart $U \subset \overline{M}$ (i.e., $U \cap \partial M \neq \emptyset$), the following result holds: let $u \in W^{1,q}_0 \Omega^k(M)$ be a differential $k$-form compactly supported in $U$, where $q \in ]1, \infty[ \text{ and } \diamond = D \text{ or } N$. Assume that, for some index $r \in ]1, \infty[, \text{ one has}$

$$\sup \left\{ \frac{|(du, d\phi)|_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}|}{\|d\phi\|_{L^r(U, g_0)} + \|\delta \phi\|_{L^r(U, g_0)} + \|\phi\|_{L^r(U, g_0)}} : \phi \in C^\infty_\partial \Omega^k(U) \right\} < \infty. \quad (3.14)$$

Then $u \in W^{1,r}_0 \Omega^k(U, g_0)$, and the following estimate is valid:

$$\|u\|_{W^{1,r}(U, g_0)} \leq C \sup \left\{ \frac{|(du, d\phi)|_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}|}{\|d\phi\|_{L^r(U, g_0)} + \|\delta \phi\|_{L^r(U, g_0)} + \|\phi\|_{L^r(U, g_0)}} : \phi \in C^\infty_\partial \Omega^k(U) \right\}. \quad (3.15)$$

The constant $C$ depends only on $r$, $k$, and the $C^1$-geometry of the chart $(U, g_0)$.

**Remark 3.6** The sufficiently small chart $U$ can be chosen uniformly in the sense that, given an arbitrarily small number $\kappa > 0$, there is some number $\rho > 0$ depending only on $\kappa$ and the $C^1$-geometry of the whole manifold $(M, g_0)$ such that whenever a boundary chart $U$ has its intrinsic diameter smaller than $\rho$, it is a valid choice for Proposition 3.5.

In view of the earlier arguments in this subsection, we may choose $\rho$ such that on any $U$ with diameter smaller than $\rho$, one has

$$\|g_0 - g_0(P)\|_{C^0(U)} + \|Dg_0 - [Dg_0](P)\|_{C^0(U)} \leq \kappa.$$

The proof of Proposition 3.5 is technical yet straightforward. All we need to do is to compare the inner products of differential forms on $U$ with their pushed-forward versions on $(\mathbb{R}^n_+, \overline{g_0})$, which can be dealt with by Corollary 3.4.

In what follows, for any differential form $\alpha$ on $U \subset \overline{M}$, we set

$$\alpha^\# := \Phi_\# \alpha. \quad (3.16)$$

Let us first check that the boundary conditions for $W^{1,q}_0 \Omega^k(M)$ are preserved under $\Phi_\#$. This relies only on differentiable structures of $M$ and $\mathbb{R}^n_+$, but not on Riemannian structures.

**Lemma 3.7** Assume that $\phi \in C^\infty_\partial \Omega^k(M)$ is supported in a boundary chart $U$. Then $\phi^\# \in C^\infty_\partial \Omega^k(\mathbb{R}^n_+)$. Conversely, if $\varphi^\# \in C^\infty_\partial \Omega^k(\mathbb{R}^n_+)$ is supported in the image $\Phi(U)$ of a boundary chart $U$, then $\Phi_\# \varphi^\# \in C^\infty_\partial \Omega^k(M)$. 

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Hence \( \phi \) computation gives us
\[ \text{t} \ni \phi \cdot \phi = 0, \]
i.e., \( \text{t} \phi \phi = 0 \) and \( \phi \in C^\infty \Omega^k(M) \).

Now we assume that \( \phi \in C^\infty \Omega^k(M) \). Then \( \phi = 0 \) on \( \partial M \), i.e., \( \phi - \text{t} \phi = 0 \). The above computation gives us

\[ \phi = \phi - \text{t} \phi = \Phi \left( \phi - \text{t} \phi \right) = 0. \]

Hence \( \phi \in C^\infty \Omega^k(M) \).

To complete the proof, we simply notice that all the previous arguments can validly run backwards, as \( \Phi \) is a diffeomorphism. \( \square \)

Next, recalling that
\[ g_0 = \Phi_#(g_0), \]
on one may easily compare \( (\alpha, \beta)_{g_0} \) with \( (\alpha, \beta)_{g_0} \) via

**Lemma 3.8** Let \( \alpha, \beta \) be arbitrary differential \( \ell \)-forms on the chart \( (U, \Phi) \subset (M, g_0) \), and let \( \kappa > 0 \) be arbitrary. There is a constant \( C > 0 \) such that

\[ \left| (\alpha, \beta)_{g_0} - (\alpha, \beta)_{g_0} \right| \leq C. \]

Here \( C \) depends only on \( \kappa, n, \ell \), and the \( C^1 \)-geometry of \( U \).

**Proof** We compute in local co-ordinates. Write \( y = \Phi(x) \), \( \alpha = \sum \alpha_1 \cdots \alpha_{i_1} \cdots \alpha_{i_{\ell}} \wedge \cdots \wedge dx^{i_1}, \) and \( \beta = \sum \beta_{j_1} \cdots \beta_{j_{\ell}} \wedge \cdots \wedge dx^{j_1}, \) with summations taken over ascending \( \ell \)-tuples of indices
\[ 1 \leq i_1 < \cdots < i_{\ell} \leq n \) and \( 1 \leq j_1 < \cdots < j_{\ell} \leq n \). Then
\[ \alpha(\beta)_{g_0} = \int_U g_0^{i_1 j_1} \cdots g_0^{i_{\ell} j_{\ell}} \alpha_{\alpha_1} \cdots \alpha_{\alpha_{\ell}} \beta_{\beta_1} \cdots \beta_{\beta_{\ell}} \, dV_{g_0}, \]
where \( dV_{g_0} \) is the Riemannian volume form. (Here the summation convention is assumed.) By Equations (3.12) and (3.10) we have that
\[ \|g_0^{-1} - g_0^{-1}(P)\|_{C^0(U)} \leq C\kappa, \]
\[ \|dV_{g_0} - dV_{g_0}(P)\|_{C_0(U)} \leq C\kappa. \]

It thus follows that
\[ \left| (\alpha, \beta)_{g_0} - (\alpha, \beta)_{g_0}(P) \right| \]
\[ = \int_U g_0^{i_1 j_1} \cdots g_0^{i_{\ell} j_{\ell}} \alpha_{\alpha_1} \cdots \alpha_{\alpha_{\ell}} \beta_{\beta_1} \cdots \beta_{\beta_{\ell}} \, dV_{g_0} - \int_U g_0(P)^{i_1 j_1} \cdots g_0(P)^{i_{\ell} j_{\ell}} \alpha_{\alpha_1} \cdots \alpha_{\alpha_{\ell}} \beta_{\beta_1} \cdots \beta_{\beta_{\ell}} \, dV_{g_0}(P) \]
\[ \leq C\kappa. \] (3.17)
In addition, in view of the definitions of \( \sharp \) and \( \flat \), we know that \( \Phi \) is an isometry from \((U, g_0(P))\) to \((\Phi(U), \check{g}_0)\). That is,
\[
(\alpha, \beta)_{g_0(P)} = (\alpha^\sharp, \beta^\sharp)_{\check{g}_0}.
\]
The constant \( C_9 \) depends only on \( \kappa, n, \ell \), and the \( C^4 \)-geometry of \( U \).

It is clear that Lemma 3.8 remains valid for \( \alpha \) and \( \beta \) with weaker regularities, as long as the pairings \((\alpha, \beta)\) and \((\alpha^\sharp, \beta^\sharp)_{\check{g}_0} \) are well defined in the sense of distributions and \( \alpha^\sharp \) and \( \beta^\sharp \) are well defined via pushforward. In particular, this holds when \( \alpha \in L^r \) and \( \beta \in L^{r'} \) for some \( r \in [1, \infty[. \)

We next observe that \( d \) and \( \delta \) commute with \( \sharp \).

**Lemma 3.9** For any differential \( k \)-forms \( \alpha \) and \( \beta \) on \( U \subset \overline{M} \), we have
\[
(\alpha \wedge \beta, \alpha) = d(\alpha^\sharp), \quad (\delta \alpha, \alpha) = \delta(\alpha^\sharp).
\]

The former identity relies solely on the differentiable structure, while the latter relies additionally on the Riemannian structure. In general, the latter holds whenever the codifferentials are compatible with pushforward in the sense that for a diffeomorphism \( \Phi : (M, g) \rightarrow (M', g') \) that is isometric and for any differential form \( \alpha \) on \( M \), we have that
\[
\Phi^* \delta_{(M', g')}[\alpha] = \delta_{(M, g)}[\Phi^* \alpha].
\]

**Proof** The identity for \( d \) holds by
\[
(\alpha \wedge \beta, \alpha) = d(\Phi^* \alpha) = d(\Phi^* \alpha) = d(\alpha^\sharp).
\]

To prove the identity for \( \delta \), let us introduce the sign \( \sigma = (-1)^{(n-k)+1} \) (which is actually immaterial to the proof). It then holds that \( \delta = \sigma \circ d \circ \ast \), where \( \ast \) is the Hodge star on \((\mathbb{R}_+, \overline{g}_0)\).

As before, denote \( y = \Phi(x) \) and \( \alpha = \sum \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \) in a fixed local co-ordinate system.

Let us first compute \( \delta(\alpha^\sharp) \). In local co-ordinates we have that
\[
\alpha^\sharp = \sum (\alpha_{i_1 \cdots i_k} \circ \Phi^{-1}) \left( \frac{\partial \Phi_{j_1}}{\partial x^{i_1}} \circ \Phi^{-1} \right) \cdots \left( \frac{\partial \Phi_{j_k}}{\partial x^{i_k}} \circ \Phi^{-1} \right) dy^{i_1} \wedge \cdots \wedge dy^{i_k}.
\]

Let \( dy^{i_1} \wedge \cdots \wedge dy^{i_{n-k}} \) be a simple \((n-k)\)-form such that
\[
(dy^{i_1} \wedge \cdots \wedge dy^{i_{n-k}}) \wedge (dy^{j_1} \wedge \cdots \wedge dy^{j_k})
\]

is the canonical unit volume form on \((\mathbb{R}^n_+, \overline{g}_0)\). Then
\[
\ast \alpha^\sharp = \sum (\alpha_{i_1 \cdots i_k} \circ \Phi^{-1}) \left( \frac{\partial \Phi_{j_1}}{\partial x^{i_1}} \circ \Phi^{-1} \right) \cdots \left( \frac{\partial \Phi_{j_k}}{\partial x^{i_k}} \circ \Phi^{-1} \right) dy^{i_1} \wedge \cdots \wedge dy^{i_{n-k}}.
\]

Taking another exterior differential gives us
\[
d \ast \alpha^\sharp = \frac{\partial}{\partial y^{i_{n-k}}} \left( (\alpha_{i_1 \cdots i_k} \circ \Phi^{-1}) \left( \frac{\partial \Phi_{j_1}}{\partial x^{i_1}} \circ \Phi^{-1} \right) \cdots \left( \frac{\partial \Phi_{j_k}}{\partial x^{i_k}} \circ \Phi^{-1} \right) \right) dy^{n} \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_{n-k}}.
\]

Finally, taking \( \sigma \) and another Hodge star, we arrive at
\[
\delta(\alpha^\sharp) = \sigma(-1)^k \frac{\partial}{\partial y^{i_{n-k}}} \left( (\alpha_{i_1 \cdots i_k} \circ \Phi^{-1}) \left( \frac{\partial \Phi_{j_1}}{\partial x^{i_1}} \circ \Phi^{-1} \right) \cdots \left( \frac{\partial \Phi_{j_k}}{\partial x^{i_k}} \circ \Phi^{-1} \right) \right) \times dy^{i_1} \wedge \cdots \wedge dy^{i_k}.
\]
Here \( \widetilde{dy}^{ij} \) means that the component \( dy^{ij} \) has been omitted; ditto. Noting that \( y^{ji} \) is nothing but \( \Phi^{ji} \), one may thus conclude that

\[
\delta (\alpha^2) = \sigma (-1)^{\ell} \sum \left( \frac{\partial \alpha_{i_1 \ldots i_{k}}}{{\partial y}^{i_j}} \circ \Phi^{-1} \right) \left( \frac{\partial \Phi^{i_1}}{\partial x^{i_1}} \circ \Phi^{-1} \right) \ldots \left( \frac{\partial \Phi^{i_k}}{\partial x^{i_k}} \circ \Phi^{-1} \right) \times dy^{i_1} \wedge \cdots \wedge dy^{i_j} \wedge \cdots \wedge dy^{i_k}.
\]

(3.18)

On the other hand, \( (\delta \alpha)^2 \) can be computed as follows: let \( dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}} \) be a simple \((n-k)\)-form such that

\[
(dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}) \wedge (dx^{k_1} \wedge \cdots \wedge dx^{k_{n-k}})
\]

is the canonical unit volume form on the domain manifold. Then it holds that

\[
\ast \alpha = \sum \alpha_{i_1 \ldots i_k} dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}
\]

and that

\[
d \ast \alpha = \sum \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^{j_l}} dx^{j_l} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}.
\]

Taking \( \sigma \) and another Hodge star gives us

\[
\delta \alpha = \sigma (-1)^{\ell} \sum \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^{j_l}} dx^{j_l} \wedge \cdots \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}.
\]

Therefore, the pushforward \( \Phi_{\#} \) sends \( \delta \alpha \) to \( (\alpha^2) \) in view of Equation (3.18) and the chain rule, once we relabel the indices \( i_1, \ldots, i_k \) as \( j_1, \ldots, j_k \).

\[ \square \]

**Corollary 3.10** For any differential \( k \)-forms on the chart \( U \), it holds that

\[
(d (\alpha^2), d (\beta^2)) := \left( (d\alpha)^2, (d\beta)^2 \right)_{\mathcal{G}}, \quad (\delta (\alpha^2), \delta (\beta^2)) := \left( (\delta\alpha)^2, (\delta\beta)^2 \right)_{\mathcal{G}},
\]

In fact, the above identities hold whenever the pairings are well defined in the distributional sense.

**Proof** This is an immediate consequence of Lemma 3.9. \[ \square \]

Now we are at the stage of proving Proposition 3.5.

**Proof of Proposition 3.5** Given \( \kappa > 0 \) and the manifold-with-boundary \((\mathcal{M}, g_0)\), we choose the chart \( U \) as before (see Remark 3.6 together with the estimates in Equations (3.10), (3.12), (3.11), and (3.13); the key is that the metric \( g_0 \) is \( \mathcal{O}(\kappa) \)-close on \( U \) to the constant metric \( g_0(P) \) in the \( C^1 \)-topology).

Let \( u \in W^{1,q}_{\mathcal{G}} \Omega^k (\mathcal{M}) \) be any \( k \)-form supported in \( U \subset \overline{\mathcal{M}} \) with \( q \in ]1, \infty[ \).

Under assumption (3.14), namely that

\[
\sup \left\{ \frac{|(du, d\phi)_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}|}{\|d\phi\|_{L^r(U, g_0)} + \|\delta \phi\|_{L^r(U, g_0)} + \|\phi\|_{L^r(U, g_0)}} : \phi \in C_0^\infty \Omega^k (U) \right\} < \infty
\]

for some \( r \in ]1, \infty[ \), we shall prove things for both cases \( \diamond = N \) and \( D \) in the same stroke.

First, let us estimate the numerator

\[
\ll u, \phi \gg_{g_0} := (du, d\phi)_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}
\]

from below. To this end, we consider the difference

\[
D\{u, \phi\} := \ll u, \phi \gg_{g_0} - \ll u, \phi \gg_{g_0(P)} + \ll u, \phi \gg_{g_0(P)} - \ll u^h, \phi^h \gg_{g_0(P)}.
\]

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where, as before, \(\Phi := \Phi_{g_0}(g_0(P))\), and \(\Phi\) is the co-ordinate map on the chart \(U\). By Lemma 3.7 we have that \(|D_2| \leq C_9\kappa\), and Equation (3.17) implies that \(|D_1| \leq C_9\kappa\), where \(C_9\) depends only on \(k\), \(n\), and the \(C^1\)-geometry of \(U\). Thus
\[
|\langle u, \phi \rangle_{g_0}| \geq |u^5, \phi^5 |_{g_0} - C_{10}\kappa,
\]  
where \(C_{10}\) has the same dependence.

Next we estimate the denominator
\[
|||\phi|||_{r', g_0} := \|d\phi\|_{L^{r'}(U, g_0)} + \|\delta\phi\|_{L^{r'}(U, g_0)} + \|\phi\|_{L^{r'}(U, g_0)}
\]
from above. For this purpose, we recall that
\[
\|\alpha\|_{L^{r'}(U, g_0)} := \left\{ \int_U \langle (\alpha, \alpha)_{g_0} \rangle_{r'}^\frac{1}{r'} \ dV_{g_0} \right\}^{\frac{1}{r'}}
\]
for any differential \(\ell\)-form \(\alpha\) on \(U\). A straightforward adaptation of the proof for Lemma 3.8 (see, in particular, Equation (3.17) therein) gives us
\[
|||\alpha|||_{L^{r'}(U, g_0)} - |||\alpha\phi|||_{L^{r'}(\Phi(U), g_0)} \leq C_{11}\kappa,
\]
where \(C_{11}\) depends only on \(\ell\), \(r'\), \(n\), and the \(C^1\)-geometry of \(U\). Taking \(\alpha = d\phi\), \(\delta\phi\), and \(\phi\), respectively, we obtain that
\[
|||\phi|||_{r', g_0} \leq |||\phi^5|||_{r', g_0} + C_{12}\kappa,
\]
where \(C_{12}\) depends only on \(k\), \(r\), \(n\), and the \(C^1\)-geometry of \(U\).

Putting together Equations (3.19) and (3.20), recalling, by Lemma 3.7, that \(\phi^5\) in \(C^\infty_{\Phi} \Omega^k(\mathbb{R}_+^n)\), and invoking Corollary 3.10 (with \(\varphi = \phi^5\)), we arrive at the bounds
\[
\infty > \sup \left\{ \frac{|(du, d\phi)|_{g_0} + (\delta u, \delta\phi)|_{g_0} + (u, \phi)|_{g_0}|}{\|d\phi\|_{L^{r'}(U, g_0)} + \|\delta\phi\|_{L^{r'}(U, g_0)} + \|\phi\|_{L^{r'}(U, g_0)}} : \phi \in C^\infty_{\Phi} \Omega^k(U) \right\}
\geq \sup \left\{ \frac{|(du^5, d\phi)|_{g_0} + (\delta u^5, \delta\phi)|_{g_0} + (u^5, \phi)|_{g_0}|}{\|d\phi\|_{L^{r'}(\Phi(U), g_0)} + \|\delta\phi\|_{L^{r'}(\Phi(U), g_0)} + \|\phi\|_{L^{r'}(\Phi(U), g_0)}} - C_{10}\kappa} : \varphi \in C^\infty_{\Phi} \Omega^k(\mathbb{R}_+^n) \right\}
\geq C_{13} \left( \sup \left\{ \frac{|(du^5, d\phi)|_{g_0} + (\delta u^5, \delta\phi)|_{g_0} + (u^5, \varphi)|_{g_0}|}{\|d\phi\|_{L^{r'}(\Phi(U), g_0)} + \|\delta\phi\|_{L^{r'}(\Phi(U), g_0)} + \|\varphi\|_{L^{r'}(\Phi(U), g_0)}} \right\} - \kappa \right),
\]
where \(C_{13}\) depends only on \(C_{10}\) and \(C_{12}\).

Then, one may refer to Corollary 3.4 to deduce that \(u^5 \in W^{1, r'} \Omega^k(\mathbb{R}_+^n, g_0)\). Moreover, this satisfies the bound
\[
\|u^5\|_{W^{1, r'}(\mathbb{R}_+^n, g_0)} \leq C_{14} \sup \left\{ \frac{|(du^5, d\phi)|_{g_0} + (\delta u^5, \delta\phi)|_{g_0} + (u^5, \varphi)|_{g_0}|}{\|d\phi\|_{L^{r'}(\Phi(U), g_0)} + \|\delta\phi\|_{L^{r'}(\Phi(U), g_0)} + \|\varphi\|_{L^{r'}(\Phi(U), g_0)}} : \varphi \in C^\infty_{\Phi} \Omega^k(\mathbb{R}_+^n) \right\},
\]
with \(C_{14}\) depending on nothing but \(k\), \(r\), \(n\), and the \(C^1\)-geometry of \(U\).
On the other hand, the same proof for Equation (3.20) (recalling the definition of $\|\cdot\|_{r, g_0}$ and $\|\cdot\|_{r, g_0}$) also gives us
\[ \|u\|_{W^{1, r}(U, g_0)} \leq \|u^2\|_{W^{1, r}(\phi(U), g_0)} + C_{15} \kappa, \]
where $C_{15}$ has the same dependence as $C_{14}$. We thus arrive at
\[ \|u\|_{W^{1, r}(U, g_0)} \leq C_{14} \sup \left\{ \left| (d\phi, d\varphi)_{g_0} + (\delta u, \delta \varphi)_{g_0} + (u, \varphi)_{g_0} \right| : \varphi \in C_0^\infty \Omega^k(U) \right\} + C_{15} \kappa \]
\[ \leq \frac{C_{14}}{C_{13}} (S[u] + \kappa) + C_{15} \kappa \]
\[ \leq C_{16} (S[u] + \kappa), \quad (3.21) \]

Here $C_{16}$ depends again only on $k$, $n$, $r$, and the $C^1$-geometry of $U$, and we adopt the notation
\[ S[u] := \sup \left\{ \left| (d\phi, d\varphi)_{g_0} + (\delta u, \delta \varphi)_{g_0} + (u, \varphi)_{g_0} \right| : \varphi \in C_0^\infty \Omega^k(U) \right\}. \]

We claim that one may assume that $\kappa \leq S[u]$. Indeed, since $\kappa$ and $C_{16}$ are both independent of $u$, we can scale $u \mapsto \lambda u$ in Equation (3.21) to get that $\|\lambda u\|_{W^{1, r}(U, g_0)} \leq C_{16} (S[u] + \kappa)$ for any $\lambda > 0$.

Hence, the case false, one must have that $S[u] = 0$, but then Equation (3.21) would imply that $\|u\|_{W^{1, r}(U, g_0)} \leq C_{16} \kappa$ for arbitrarily small $\kappa$, so $u \equiv 0$.

Therefore, the above claim, together with Equation (3.21), yields that
\[ \|u\|_{W^{1, r}(U, g_0)} \leq 2 \cdot C_{16} \sup \left\{ \left| (d\phi, d\varphi)_{g_0} + (\delta u, \delta \varphi)_{g_0} + (u, \varphi)_{g_0} \right| : \varphi \in C_0^\infty \Omega^k(U) \right\} \]
\[ < \infty. \]

In particular, by assumption (3.14), one has $u \in W^{1, r} \Omega^k(U, g_0)$. The proof is now complete.

\[ \square \]

3.4 Proof of Theorem B

Finally, we are in position to conclude the proof of Theorem B. The cases $\phi = D$ and $N$ shall the treated simultaneously.

Since $\mathcal{M}$ is bounded, when $q \geq r$ there is nothing to prove (as $L^q \Omega^k(\mathcal{M}) \subset L^r \Omega^k(\mathcal{M})$). Thus we assume throughout, in the sequel, that
\[ 1 < q < r < \infty. \quad (3.22) \]

Let $\kappa_0$ be an arbitrarily small positive number. Since the manifold-with-boundary $(M, g_0)$ has $C^1$-bounded geometry, one may select Atlas, a “$\kappa_0$-good” atlas, as follows: Atlas consists of charts $\{U_i\}_{i=0}^N$ such that
\[ \bigcup_{i=1}^N U_i \supset \partial \mathcal{M}, \quad (3.23) \]
and such that, for an point $P \in U_i$, one has that
\[ \sup_{1 \leq i \leq N} \left\{ \|g_0 - g_0(P)\|_{C^0(U_i)} + \|Dg_0 - Dg_0(P)\|_{C^0(U_i)} \right\} \leq \kappa_0. \quad (3.24) \]
The former condition (3.23) means that $U_0$ is an interior chart of $\mathcal{M}$ and $U_1, \ldots, U_N$ are boundary charts; the latter condition (3.24) is a restatement of Remark 3.6.

Next, let $\{\chi_i\}_{i=0}^N$ be a differentiable partition of unity subordinate to \textit{Atlas}. The quantity

$$\sup_{0 \leq i \leq N} \|\chi_i\|^{C^1(M,g_0)}$$

is finite, as it can be controlled by the $C^1$-geometry of $(\mathcal{M}, g)$. Consider

$$u_j := \chi_j u \quad \text{for } j \in \{1, 2, \ldots, N\}.$$ 

We check that for each such $j$ it holds that $u_j \in W^{1,r,\Omega^k}(\mathcal{M})$. As a remark, by construction, $u_j$ is compactly supported in the boundary chart $U_j$.

We proceed with straightforward computations. Note first that

$$(\delta u_j, \delta \phi)_{g_0} = \sigma (d\chi_j \wedge u, d\phi)_{g_0} + \chi_j (du, d\phi)_{g_0},$$

where $d\chi_j \wedge u$ is the wedge product of a 1-form with an $r$-form; similarly, we have that

$$(\delta u_j, \delta \phi)_{g_0} = \sigma (d\chi_j \wedge u, d\phi)_{g_0} + \chi_j (du, d\phi)_{g_0},$$

where $\ast$ is the Hodge star operator and $\sigma = (-1)^{(n(k+1)+1)}$. Hence,

$$\|\delta u_j, \delta \phi\|_{L^r(U_j, g_0)} + \|\delta \phi\|_{L^r(U_j, g_0)} + \|\phi\|_{L^r(U_j, g_0)}$$

$$\leq \chi_j \left( \|d\phi\|_{L^r(U_j, g_0)} + \|\delta \phi\|_{L^r(U_j, g_0)} + \|\phi\|_{L^r(U_j, g_0)} \right),$$

for an arbitrary test $k$-form $\phi$ compactly supported in $U_j$.

Now we turn to the estimation for $I_1$, $I_2$. We consider two cases separately. Recall that $\dim \mathcal{M} = n$, that $u$ is a $k$-form with a finite $W^{1,s}$-norm, and that

$$G[u] := \sup \left\{ \frac{|(du, d\phi)_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}|}{\|\delta \phi\|_{L^r(U_j, g_0)} + \|\phi\|_{L^r(U_j, g_0)}} : \phi \in C^\infty_0 \Omega^k(M) \right\} < \infty.$$

**Case 1:** $q \geq n$. We have the standard Sobolev–Morrey embedding $W^{1,q}(U_j) \hookrightarrow L^r(U_j)$, so

$$|I_2| \leq C_{17} \frac{\|u\|_{L^r(U_j, g_0)} \|\delta \phi\|_{L^r(U_j, g_0)}}{\|\delta \phi\|_{L^r(U_j, g_0)} + \|\phi\|_{L^r(U_j, g_0)}} \leq C_{18} \|u\|_{W^{1,s}(U_j, g_0)},$$

where $C_{17}$ and $C_{18}$ depend only on $r$, $k$, $n$, and the $C^1$-geometry of $U_j$.

On the other hand, it is clear that $|I_1| \leq G[u]$, so we consider the extension-by-zero of $\phi \in C^\infty_{c} \Omega^k(U_j)$ to a test form on $\mathcal{M}$. In this way, we have

$$\frac{|(du, d\phi)_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0}|}{\|\delta \phi\|_{L^r(U_j, g_0)} + \|\phi\|_{L^r(U_j, g_0)}} \leq C_{18} \|u\|_{W^{1,s}(U_j, g_0)} + G[u] < \infty.$$

However, because $u_j$ is supported on a sufficiently small chart, we may apply Proposition 3.5 to infer that

$$u_j \equiv \chi_j u \in W^{1,r,\Omega^k}(U_j, g_0).$$
and that the following estimate holds:
\[ \|u_j\|_{W^{1,r}(U_j, g_0)} \leq C_{19} \left( C_{18} \|u\|_{W^{1,q}(U_j, g_0)} + \mathcal{S}[u] \right). \]
Here $C_{19}$ depends again only on $r$, $k$, $n$, and the $C^1$-geometry of $U_j$. Thus
\[ \|u\|_{W^{1,r}(\mathcal{M}, g_0)} \leq (N + 1)C_{19} \left( C_{18} \|u\|_{W^{1,q}(U_j, g_0)} + \mathcal{S}[u] \right). \]
Here $(N + 1)$ is the number of charts in $\text{Atlas}$, which can be controlled in turn by the $C^1$-geometry of $(\mathcal{M}, g_0)$.

To conclude the proof for Case 1, it remains to note that the quantitative statement (3.15) in Proposition 3.5 allows us to bound (with a new constant $C_{20}$ having the same dependence as $C_{19}$) that
\[ \|u\|_{W^{1,r}(U_j, g_0)} \leq C_{20} \sup \left\{ \left( \left| (du, d\phi)_{g_0} + (\delta u, \delta \phi)_{g_0} + (u, \phi)_{g_0} \right| \right) : \phi \in C_0^\infty \Omega^k(\mathcal{M}), \text{supp} \phi \in U_j \right\}. \]
Therefore, we get
\[ \|u\|_{W^{1,r}(\mathcal{M}, g_0)} \leq (N + 1)C_{19} \left( C_{18}C_{20} + 1 \right)\mathcal{S}[u] < \infty. \]  
(3.25)

**Case 2:** $1 < q < n$. In this case we shall follow the clever argument on p.1882 of Kozono–Yanagisawa [23], which allows us to raise the regularity of $u$ from $W^{1,q}$ to $W^{1,s(q)}$ and then to $W^{1,s(s(q))}$ and so on. Here and hereafter, for $p \in ]1, \infty[$, $s(p)$ denotes the Sobolev conjugate of $p$. In each step it holds that
\[
\overset{(m+1)\ \text{times}}{s \circ \ldots \circ s\ (q)} - s \circ \ldots \circ s\ (q) \geq \epsilon_0 > 0,
\]
where $\epsilon_0$ is independent of $m$. Thus, in finitely many steps we shall return to Case 1. This approach is reminiscent of the “LIR” (local increasing regularity) method developed by Amar ([1, 2]).

We note first that, again, we trivially have $|I_1| \leq \mathcal{S}[u]$. To control $I_2$, consider, as before,
\[ |I_2| \leq C_{21}\|u\|_{L^r(U_j, g_0)} \leq C_{22}\|u\|_{W^{1,\frac{nr}{n+r}}(U_j, g_0)}. \]
Here $\frac{nr}{n+r}$ is the Sobolev predual for $r$. Therefore, all the arguments in Case 1 carry through whenever $s(r) \geq q$, with the constants again depending only on $r$, $n$, $k$, and the $C^1$-geometry of $\mathcal{M}$. This proves that
\[ u \in W^{1,s(\phi)}_{\cap} \Omega^k(\mathcal{M}) \implies u \in W^{1,s(q)}_{\cap} \Omega^k(\mathcal{M}), \]  
(3.26)
where $s(q) := \frac{nq}{n-mq}$. Thus, $\overset{(m+1)\ \text{times}}{s \circ \ldots \circ s\ (q)} - s \circ \ldots \circ s\ (q) = \frac{nq}{n-mq}$, which gives us
\[
\overset{(m+1)\ \text{times}}{s \circ \ldots \circ s\ (q)} - s \circ \ldots \circ s\ (q) = \frac{nq^2}{(n-mq)(n-mq-q)} \geq \epsilon_0(q, n) > 0,
\]
with $\epsilon_0$ being independent of $m$. For example, clearly, one may take that $\epsilon_0 = n^{-1}q^2$.

As a consequence, we can now continue the implication in Equation (3.26) by
\[ u \in W^{1,q}_{\cap} \Omega^k(\mathcal{M}) \implies u \in W^{1,s(q)}_{\cap} \Omega^k(\mathcal{M}). \]
\[ u \in W^{1,s(q)}_\Diamond \Omega^k(\mathcal{M}) \]
\[ \cdots \Rightarrow u \in W^{1,s(q)}_\Diamond \Omega^k(\mathcal{M}) \subset W^{1,r}_\Diamond \Omega^k(\mathcal{M}), \]

once we choose \( m \) so large that \( mq^2/n \geq r \), thanks to the compactness of \( \mathcal{M} \). Then we may proceed as in Case 1. The proof of Theorem B is now complete.

### 3.5 A Remark

For subsequent developments, we note that Theorem B continues to hold if \( \|d\phi\|_{L^r} + \|\delta\phi\|_{L^r} \) in the denominator is replaced by \( \|\nabla\phi\|_{L^r} \), where \( \nabla \) is the covariant derivative obtained from the Levi-Civita connection.

**Corollary 3.11** Let \((\mathcal{M}, g_0)\) be an \( n \)-dimensional compact Riemannian manifold-with-boundary, let \( \Diamond \) denote either \( D \) or \( N \), let \( r \) and \( q \) be any numbers in \( ]1, \infty[ \), and let \( k \in \{0, 1, 2, \ldots, n\} \). Assume \( u \in W^{1,q}_\Diamond \Omega^k(\mathcal{M}, g_0) \) and that the following quantity is finite:

\[
\sup \left\{ \frac{|(du, d\phi)|_{g_0} + (\delta u, \delta \phi)|_{g_0} + (u, \phi)|_{g_0}}{\|\nabla\phi\|_{L^r(\mathcal{M}, g_0)} + \|\phi\|_{L^r(\mathcal{M}, g_0)}} : \phi \in C^\infty_\Diamond \Omega^k(\mathcal{M}) \right\}.
\]

Then \( u \in W^{1,r}_\Diamond \Omega^k(\mathcal{M}) \).

In addition, the following variational inequality is satisfied:

\[
\|u\|_{W^{1,r}} \leq C\sup \left\{ \frac{|(du, d\phi)|_{g_0} + (\delta u, \delta \phi)|_{g_0} + (u, \phi)|_{g_0}}{\|\nabla\phi\|_{L^r(\mathcal{M}, g_0)} + \|\phi\|_{L^r(\mathcal{M}, g_0)}} : \phi \in C^\infty_\Diamond \Omega^k(\mathcal{M}) \right\}.
\]

Here \( C \) depends only on \( r, n, k \), and the \( C^1 \)-geometry of \((\mathcal{M}, g_0)\).

**Proof** All the arguments in Subsection 3.1, 3.2, 3.3 and 3.4 go through, as long as we make the following modifications: in the proof of Lemma 3.1, we change inequality (3.4) to

\[
\sup \left\{ \frac{|(du, d\phi)|_{g_0} + (\delta u, \delta \phi)|_{g_0} + (u, \phi)|_{g_0}}{\|\nabla\phi\|_{L^r(\mathbb{R}^n)} + \|\phi\|_{L^r(\mathbb{R}^n)}} : \phi \in C^\infty_\Diamond \Omega^k(\mathbb{R}^n) \right\}
\]

and still use the Calderón–Zygmund estimate to bound \( \|\nabla_j \nabla_j \psi\|_{L^r(\mathbb{R}^n)} \lesssim \|\Delta \psi\|_{L^r(\mathbb{R}^n)} + \|\psi\|_{L^r(\mathbb{R}^n)} \). For the proof of Lemma 3.3, we only need to observe that Equation (3.8) continues to hold for \( T = \nabla \). In addition, from the definition of pullback connections, the analogue of Lemma 3.9, namely, that \( (\nabla\alpha)^2 = \nabla (\alpha^2) \), holds (also see Equation (3.13) for estimates on the difference of connections). Thus the arguments for Corollary 3.4, Proposition 3.5, and Theorem B carry over almost verbatim, once we change each occurrence of \( \|d\phi\|_{L^r} + \|\delta\phi\|_{L^r} \) to \( \|\nabla\phi\|_{L^r} \). \( \square \)

Also, taking \( q = r \) in Theorem B and Corollary 3.11, we immediately get the following estimation for the \( W^{1,r} \)-norm of \( u \) by duality:

**Proposition 3.12** Let \((\mathcal{M}, g_0)\) be an \( n \)-dimensional compact Riemannian manifold-with-boundary, let \( k \in \{0, 1, 2, \ldots, n\} \), and let \( r \in ]1, \infty[ \) be arbitrary. Assume \( u \in W^{1,r}_\Diamond \Omega^k(\mathcal{M}) \)

\[ \square \]
for either $\diamondsuit = D$ or $N$. Then

$$\|u\|_{W^{1,r}(\mathcal{M},g_0)} \leq C \sup \left\{ \frac{(du, d\phi)_g + (\delta u, \delta \phi)_g + (u, \phi)_g}{\|d\phi\|_{L^r(\mathcal{M},g_0)} + \|\delta \phi\|_{L^r(\mathcal{M},g_0)} + \|\phi\|_{L^r(\mathcal{M},g_0)}} : \phi \in C_0^\infty \Omega^k(\mathcal{M}) \right\},$$

where $C$ depends only on $r$, $n$, $k$, and the $C^1$-geometry of $(\mathcal{M},g_0)$.

There is another constant $C'$ with the same dependency as $C$ such that

$$\|u\|_{W^{1,r}(\mathcal{M},g_0)} \leq C' \sup \left\{ \frac{(du, d\phi)_g + (\delta u, \delta \phi)_g + (u, \phi)_g}{\|\nabla \phi\|_{L^r(\mathcal{M},g_0)} + \|\phi\|_{L^r(\mathcal{M},g_0)}} : \phi \in C_0^\infty \Omega^k(\mathcal{M}) \right\}.$$

4 Proof of Gaffney’s Inequality

The goal of this section is to prove Theorem A (reproduced below).

**Theorem** Let $(\mathcal{M},g_0)$ be an $n$-dimensional compact Riemannian manifold-with-boundary, let $k \in \{0, 1, 2, \ldots, n\}$, and let $r \in [1, \infty]$ be arbitrary. Consider $u \in W^{1,r}_D \Omega^k(\mathcal{M})$ or $W^{1,r}_N \Omega^k(\mathcal{M})$, a differential $k$-form subject to designated boundary conditions. Then

$$\|u\|_{W^{1,r}(\mathcal{M})} \leq C \left\{ \|du\|_{L^r(\mathcal{M})} + \|\delta u\|_{L^r(\mathcal{M})} + \|u\|_{L^r(\mathcal{M})} \right\},$$

where $C$ depends only on $r$, $k$, $n$, and the geometry of $\mathcal{M}$.

An immediate consequence of the above is the following:

**Corollary 4.1** Let $(\mathcal{M},g_0)$ be an $n$-dimensional compact Riemannian manifold-with-boundary, let $k \in \{0, 1, 2, \ldots, n\}$, and let $r \in [1, \infty]$ be arbitrary. Then

$$\overline{W^{1,r}_D \Omega^k(\mathcal{M})} = W^{1,r} \Omega^k(\mathcal{M})$$

for either $\diamondsuit = D$ or $N$.

**Proof of Corollary 4.1** $\overline{W^{1,r}_D \Omega^k(\mathcal{M})} \subset W^{1,r} \Omega^k(\mathcal{M})$ is trivial. On the other hand, knowing that $C_0^\infty \Omega^k(\mathcal{M})$ is dense in $\mathcal{E}_L^\infty(\mathcal{M})$, the other inclusion would follow immediately from Gaffney’s inequality (Theorem A) and a density argument.

To show the density result, by considering standard bases of simple $k$-forms as in the proof of Lemma 3.1, it suffices to take $k = 0$. By partition of unity we can reduce things to the case where $\mathcal{M}$ is covered by a single co-ordinate chart. We may also take this chart to be Euclidean, via the co-ordinate map. The theorem then follows from classical results, such as those in Duvaut–Lions [13], Chapter 7, Lemmata 4.2 and 6.1; see also Corollaries 3.2 and 3.3 in Iwaniec–Scott–Stroffolini [21].

**Proof of Theorem A** First, observe that we only need to prove things for the case $\diamondsuit = D$, namely that $tu = 0 = t\phi$. Indeed, if $u, \phi \in C_0^\infty \Omega^k(\mathcal{M})$, then their Hodge duals $\ast u$ and $\ast \phi$ lie in $C_0^\infty \Omega^k(\mathcal{M})$. Thus, Equation (1.1) for $\ast u$ and $\ast \phi$ implies the same result (with the same constants) for $u$ and $\phi$, as the Hodge star operator is an $L^r$-isometry for any $r \in [1, \infty]$.

Now we assume that $tu = 0 = t\phi$. Integration by parts and the Stokes theorem give us that

$$\mathcal{D}(u, \phi) := \int_{\mathcal{M}} (du, d\phi)_{A^{k+1}} dV_{g_0} + \int_{\mathcal{M}} (\delta u, \delta \phi)_{A^{k-1}} dV_{g_0}$$

$$= \int_{\mathcal{M}} (\Delta \phi, u)_{A^k} dV_{g_0} + \int_{\partial \mathcal{M}} tu \wedge \ast(n(d\phi)) - \int_{\partial \mathcal{M}} t(\delta \phi) \wedge \ast nu;$$
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see p.62, Corollary 2.1.4 in [29] for details. Recall that the pairing \( \langle \cdot, \cdot \rangle_{\Lambda^p} \) denotes the pointwise inner product for differential \( p \)-forms on \( \mathcal{M} \) induced by the metric \( g_0 \). As \( tu = 0 \), we have that

\[
\int_{\partial M} tu \wedge \ast n(\Delta \phi) = 0.
\]

Consider now that

\[
I := \int_{\mathcal{M}} \langle \Delta \phi, u \rangle_{\Lambda^k} \ dV_{g_0}.
\]

The Bochner–Weitzenböck formula yields that

\[
I = I_1 + I_2 := \int_{\mathcal{M}} \langle R \phi, u \rangle_{\Lambda^k} \ dV_{g_0} + \int_{\mathcal{M}} \langle \tilde{\Delta} \phi, u \rangle_{\Lambda^k} \ dV_{g_0},
\]

where \( R \in \text{End} \left( \bigwedge^k T^* \mathcal{M} \right) \) depends only on the Riemann curvature tensor of \((\mathcal{M}, g_0)\), and \( \tilde{\Delta} \) is the connection Laplacian. Fix any local orthonormal frame \( \{ e_i \}_{i=1}^n \). Then one may express (with the summation convention and the abbreviation \( \nabla \equiv \nabla_{e_i} \)) that

\[
\tilde{\Delta} \phi = -\nabla_i \nabla_i \phi + \nabla_{\nabla_i e_i} \phi.
\]

Thus

\[
I_2 = I_{21} + I_{22} + I_{23} + I_{24},
\]

where

\[
I_{21} = -\sum_{\beta \text{ tangential}} \int_{\mathcal{M}} \nabla_i \langle \nabla_i \phi, u \rangle_{\Lambda^k} \ dV_{g_0},
\]

\[
I_{22} = \int_{\mathcal{M}} \langle \nabla_i \phi, \nabla_i u \rangle_{\Lambda^k} \ dV_{g_0},
\]

\[
I_{23} = \int_{\mathcal{M}} \nabla_{\nabla_i e_i} \langle \phi, u \rangle_{\Lambda^k} \ dV_{g_0},
\]

\[
I_{24} = -\int_{\mathcal{M}} \langle \phi, \nabla_{\nabla_i e_i} u \rangle_{\Lambda^k} \ dV_{g_0}.
\]

Let us treat the four terms \( I_{2j} \) one by one. First,

\[
I_{21} = -\sum_{\beta \text{ tangential}} \left( \int_{\mathcal{M}} \nabla_i \langle \nabla_i \phi, u \rangle_{\Lambda^k} \ dV_{g_0} \right) - \int_{\mathcal{M}} \nabla_\nu \left( \langle \nabla_\nu \phi, u \rangle_{\Lambda^k} \right) \ dV_{g_0}.
\]

The tangential term is zero, since, by the Stokes or the Gauss–Green theorem, one has

\[
\int_{\mathcal{M}} \nabla_\beta \langle \nabla_\beta \phi, u \rangle_{\Lambda^k} \ dV_{g_0} = \int_{\partial \mathcal{M}} \langle \nabla_\nu \phi, u \rangle_{\Lambda^k} \langle \nu, e_3 \rangle_{\Lambda^1} \ dV_{g_0} = 0.
\]

Similarly,

\[
I_{23} = \int_{\partial \mathcal{M}} \langle \phi, u \rangle_{\Lambda^k} \langle \nabla_i e_i, \nu \rangle_{\Lambda^1} \ dV_{g_0} = \int_{\partial \mathcal{M}} \Gamma_{
abla_i}^\nu \langle \phi, u \rangle_{\Lambda^k} \ dV_{g_0},
\]

where we adopt the notation \( \Gamma_{\nabla_i}^\nu = \Gamma_{\nu}^\beta (\nu, e_i); \) that is, we shall intentionally confuse \( \nu \) with \( \nu^b \) (and the 1-form canonically dual to it). Moreover, the definition of the covariant derivative gives us

\[
I_{22} = \int_{\mathcal{M}} \langle \nabla \phi, \nabla u \rangle_{\Lambda^{k+1}} \ dV_{g_0}.
\]

In summary, we have obtained that

\[
I_2 = \int_{\mathcal{M}} \langle \nabla \phi, \nabla u \rangle_{\Lambda^{k+1}} \ dV_{g_0} - \int_{\mathcal{M}} \nabla_\nu \left( \langle \nabla_\nu \phi, u \rangle_{\Lambda^k} \right) \ dV_{g_0}
\]

\[
+ \int_{\partial \mathcal{M}} \Gamma_{\nabla_i}^\nu \langle \phi, u \rangle_{\Lambda^k} \ dV_{g_0} - \int_{\mathcal{M}} \langle \phi, \nabla_{\nabla_i e_i} u \rangle_{\Lambda^k} \ dV_{g_0}.
\]
The computation for the boundary term

\[ J := - \int_{\partial M} \langle \delta \phi \rangle \wedge \ast \nu u \]

can be performed as in p.64, [29]. Indeed, following the arguments therein, one may infer that

\begin{align*}
J &= - \int_{\partial M} \langle \delta \phi, \nu L u \rangle_{\Lambda^{k-1}} \, d\Sigma \\
&= - \int_{\partial M} \langle \delta \phi, \nu L u \rangle_{\Lambda^{k-1}} \, d\Sigma + \int_{\partial M} \langle \nu L \nu \phi, \nu L u \rangle_{\Lambda^{k-1}} \, d\Sigma \\
&= J_1 + J_2.
\end{align*}

Here \( \delta \) is the codifferential on the boundary \( \partial M \).

Notice that \( J_1 \) is a good term in the sense that it can be expressed in such a way that no derivatives of \( \phi \) or \( u \) are involved. To see this, take any shuffle \( \sigma \) of tangential indices; that is, \( \sigma \) is in the symmetry group on \( \{1, 2, \ldots, n\} \), respects the increasing order of indices, and leaves the normal index \( \nu \) invariant. Thanks to the Leibniz rule for covariant derivatives, we get that

\[ [\delta \phi] (e_{\sigma(1)}, \ldots, e_{\sigma(k-1)}) = - \sum_{\beta \neq \nu} \nabla_{\beta} \phi (e_{\sigma(1)}, \ldots, e_{\sigma(k-1)}) \]

\[ = - \sum_{\beta \neq \nu} \nabla_{\beta} \left( \phi (e_{\beta}, e_{\sigma(1)}, \ldots, e_{\sigma(k-1)}) \right) + \sum_{\beta \neq \nu} \phi (\nabla_{\beta} e_{\beta}, e_{\sigma(1)}, \ldots, e_{\sigma(k-1)}) \]

\[ + \sum_{\beta \neq \nu} \phi (e_{\beta}, e_{\sigma(1)}, \ldots, \nabla_{\beta} e_{\sigma(k-1)}, \ldots, e_{\sigma(k-1)}). \]

As \( \phi \) is purely perpendicular at the boundary, \( \nabla_{\beta} \phi \) is too for any tangential index \( \beta \neq \nu \). Thus, the first term on the right-hand side of the final equality vanishes constantly on \( \partial M \). The other two terms contain no derivatives of \( \phi \), therefore the covariant derivatives fall on \( e_{\gamma} \) (for tangential \( \gamma \)), and hence can be expressed in terms of the Christoffel symbols.

Putting the above arguments together, we deduce from \( \mathcal{D}(u, \phi) = I + J \) the following expression:

\begin{align*}
\int_{M} \langle du, d\phi \rangle_{\Lambda^{k+1}} \, dV_0 + \int_{M} \langle du, \delta \phi \rangle_{\Lambda^{k-1}} \, dV_0 - \int_{M} \langle \nabla \phi, \nabla u \rangle_{\Lambda^{k+1}} \, dV_0 \\
= - \int_{M} \nabla_{\nu} \left( \langle \nabla_{\nu} u, \phi \rangle_{\Lambda^{k}} \right) \, dV_0 + \int_{M} \Gamma^{\nu}_{\mu} (\phi, u)_{\Lambda^{k}} \, dV_0 - \int_{M} \langle \phi, \nabla_{\nu} e_{\nu} \rangle_{\Lambda^{k}} \, dV_0 \\
+ \int_{\partial M} \langle \nu L \nabla_{\nu} \phi, \nu L u \rangle_{\Lambda^{k-1}} \, d\Sigma - \int_{\partial M} \langle \text{GOOD}(\phi), \nu L u \rangle_{\Lambda^{k-1}} \, d\Sigma. \tag{4.1}
\end{align*}

Here \( \text{GOOD}(\phi) \) is the good term arising from \( J_1 \) as above. (The punchline is that the last term contains no derivative of \( u \) on the boundary.)

Now, observe that the left-hand side of Equation (4.1) is symmetric in \( u \) and \( \phi \). By interchanging \( u \) and \( \phi \), we get that

\begin{align*}
\int_{M} \langle du, d\phi \rangle_{\Lambda^{k+1}} \, dV_0 + \int_{M} \langle du, \delta \phi \rangle_{\Lambda^{k-1}} \, dV_0 - \int_{M} \langle \nabla \phi, \nabla u \rangle_{\Lambda^{k+1}} \, dV_0 \\
= - \int_{M} \nabla_{\nu} \left( \langle \nabla_{\nu} u, \phi \rangle_{\Lambda^{k}} \right) \, dV_0 + \int_{M} \Gamma^{\nu}_{\mu} (\phi, u)_{\Lambda^{k}} \, dV_0 - \int_{M} \langle \phi, \nabla_{\nu} e_{\nu} \rangle_{\Lambda^{k}} \, dV_0 \\
+ \int_{\partial M} \langle \nu L \nabla_{\nu} u, \nu L \phi \rangle_{\Lambda^{k-1}} \, d\Sigma - \int_{\partial M} \langle \text{GOOD}(u), \nu L \phi \rangle_{\Lambda^{k-1}} \, d\Sigma. \tag{4.2}
\end{align*}
Adding up Equations (4.1) and (4.2), we can take advantage of the metric-compatibility of \( \langle \cdot, \cdot \rangle_{\Lambda^p} \) to deduce that

\[
\int_{\mathcal{M}} \langle du, d\phi \rangle_{\Lambda^{k+1}} dV_{g_0} + \int_{\mathcal{M}} \langle du, d\phi \rangle_{\Lambda^{k-1}} dV_{g_0} - \int_{\mathcal{M}} \langle \nabla \phi, \nabla u \rangle_{\Lambda^{k+1}} dV_{g_0} \\
= - \int_{\mathcal{M}} \nabla_{\nu} \nabla_{\nu} \langle u, \phi \rangle_{\Lambda^k} dV_{g_0} + \int_{\partial \mathcal{M}} \Gamma^\nu_{\alpha \beta} \langle \phi, u \rangle_{\Lambda^k} dV_{g_0} - \int_{\mathcal{M}} \nabla_{\nu} \epsilon \langle u, \phi \rangle_{\Lambda^k} dV_{g_0} \\
- \int_{\partial \mathcal{M}} \langle \langle \text{GOOD} \{ \phi \} \rangle, \nu \mathcal{L} u \rangle_{\Lambda^{k-1}} d\Sigma - \int_{\partial \mathcal{M}} \langle \langle \text{GOOD} \{ u \} \rangle, \nu \mathcal{L} \phi \rangle_{\Lambda^{k-1}} d\Sigma \\
+ \int_{\partial \mathcal{M}} \langle \nu \mathcal{L} \nabla_{\nu} u, \nu \mathcal{L} \phi \rangle_{\Lambda^{k-1}} d\Sigma + \int_{\partial \mathcal{M}} \langle \nu \mathcal{L} \nabla_{\nu} \phi, \nu \mathcal{L} u \rangle_{\Lambda^{k-1}} d\Sigma.
\]  
(4.3)

The fourth and the fifth terms on the right-hand side of Equation (4.3) contain no derivatives. Also, using the same computation for \( J_{23} \) as in the above, we find that the second and the third terms cancel each other out.

Finally, let us deal with the last two terms on the right-hand side of Equation (4.3). We proceed by computation in local co-ordinates. Indeed, since \( tu = 0 \), on the boundary, we have that

\[
u = \sum a_{\alpha_1, \ldots, \alpha_{k-1}, \nu} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{k-1}} \wedge \nu,
\]

where the summation \( \sum \) is taken over all of \( (\alpha_1, \ldots, \alpha_{k-1}) \), the increasing \( (k-1) \)-tuples of tangential indices (i.e., \( \alpha_j \neq \nu \)). Once again, we adopt the abuse of notation \( \nu \equiv \nu^p \). Then

\[
\nabla_{\nu} u = \sum_{j=1}^{k-1} \left( \frac{\partial u_{\alpha_1, \ldots, \alpha_{k-1}, \nu}}{\partial \nu} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{k-1}} \wedge \nu \\
+ \sum_{j=1}^{k-1} u_{\alpha_1, \ldots, \alpha_{k-1} \nu} dx^{\alpha_1} \wedge \cdots \wedge \nabla_{\nu} dx^{\alpha_j} \wedge \cdots \wedge dx^{\alpha_{k-1}} \wedge \nu \\
+ u_{\alpha_1, \ldots, \alpha_{k-1} \nu} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{k-1}} \wedge \nabla_{\nu} \nu \right).
\]

Again, let us write

\[
\nabla_{\nu} u = \sum_{j=1}^{k-1} \left( \frac{\partial u_{\alpha_1, \ldots, \alpha_{k-1}, \nu}}{\partial \nu} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{k-1}} \wedge \nu + \text{Christoffel symbols} \right) \quad \text{on \( \partial \mathcal{M} \)},
\]

where \( \text{Christoffel symbols} \) contains the Christoffel symbols of \((\mathcal{M}, g_0)\) and no derivatives of \( u \), and \( \epsilon \in \{ \pm 1 \} \) is a sign. Similarly, by expressing \( \phi \) in the same local co-ordinates,

\[
\phi = \sum \phi_{\alpha_1, \ldots, \alpha_{k-1}, \nu} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{k-1}} \wedge \nu,
\]

we can easily compute its contraction with \( \nu \):

\[
\nu \mathcal{L} \phi = \sum \epsilon \phi_{\alpha_1, \ldots, \alpha_{k-1}, \nu} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{k-1}}.
\]

Here \( \epsilon \) is the same sign as before. Thus the penultimate term in Equation (4.3) becomes

\[
\int_{\partial \mathcal{M}} \langle \nu \mathcal{L} \nabla_{\nu} u, \nu \mathcal{L} \phi \rangle_{\Lambda^{k-1}} d\Sigma = \int_{\partial \mathcal{M}} \left\{ \sum_{j=1}^{k-1} \left( \frac{\partial u_{\alpha_1, \ldots, \alpha_{k-1}, \nu}}{\partial \nu} \right) \phi_{\alpha_1, \ldots, \alpha_{k-1}, \nu} \right\} d\Sigma \\
+ \int_{\partial \mathcal{M}} \langle \langle \text{GOOD} \{ u \} \rangle, \nu \mathcal{L} \phi \rangle_{\Lambda^{k-1}} d\Sigma.
\]

By symmetry, the final term in Equation (4.3) satisfies

\[
\int_{\partial \mathcal{M}} \langle \nu \mathcal{L} \nabla_{\nu} \phi, \nu \mathcal{L} u \rangle_{\Lambda^{k-1}} d\Sigma = \int_{\partial \mathcal{M}} \left\{ \sum_{j=1}^{k-1} \left( \frac{\partial \phi_{\alpha_1, \ldots, \alpha_{k-1}, \nu}}{\partial \nu} \right) u_{\alpha_1, \ldots, \alpha_{k-1}, \nu} \right\} d\Sigma.
\]
which cancels the first term on the right-hand side of Equation (4.3).

Adding together the above two equalities, we get that

\[
\int_{\partial M} \langle [\text{GOOD}'(\phi)] , \nu L u \rangle_{A^{k-1}} d\Sigma + \int_{\partial M} \langle [\text{GOOD}'(u)] , \nu L \phi \rangle_{A^{k-1}} d\Sigma
\]

\[
= \int_{\partial M} \nabla \langle u, \phi \rangle_{A^{k}} d\Sigma + \int_{\partial M} \langle [\text{GOOD}'(u)] , \nu L \phi \rangle_{A^{k-1}} d\Sigma
\]

\[
+ \int_{\partial M} \langle [\text{GOOD}'(\phi)] , \nu L u \rangle_{A^{k-1}} d\Sigma.
\]

(4.4)

One more application of the Stokes/Gauss–Green theorem leads to

\[
\int_{\partial M} \nabla \nu \langle u, \phi \rangle_{A^{k}} d\Sigma = \int_{M} \nabla \nu \nu \langle u, \phi \rangle_{A^{k}} dV_{g_{0}},
\]

which cancels the first term on the right-hand side of Equation (4.3).

Therefore, summarising Equations (4.3) and (4.4), we arrive at

\[
\int_{M} (du, d\phi)_{A^{k+1}} dV_{g_{0}} + \int_{M} (\delta u, \delta \phi)_{A^{k-1}} dV_{g_{0}} - \int_{M} (\nabla \phi, \nabla u)_{A^{k+1}} dV_{g_{0}}
\]

\[
= \int_{\partial M} \langle [\text{GOOD}''(u)] , \nu L \phi \rangle_{A^{k-1}} d\Sigma + \int_{\partial M} \langle [\text{GOOD}''(\phi)] , \nu L u \rangle_{A^{k-1}} d\Sigma
\]

=: \mathcal{G} \{ u, \phi \},

(4.5)

where \([\text{GOOD}''(\bullet)]\) involves no derivative of the variable \(\bullet\) and depends only on the \(C^{1}\)-geometry of \((M, g_{0})\). Thus, in light of the Hölder and trace inequalities, the term \(\mathcal{G} \{ u, \phi \}\) — which is clearly symmetric in \(u\) and \(\phi\) — can be estimated by

\[
\| \mathcal{G} \{ u, \phi \} \| \leq C \| u \|_{L^{r}(\partial M)} \| \phi \|_{L^{r'}(\partial M)}
\]

\[
\leq C (\| \nabla u \|_{L^{r}(M)} + C_{5} \| u \|_{L^{r}(M)}) \left( \| \nabla \phi \|_{L^{r'}(M)} + \| \phi \|_{L^{r'}(M)} \right),
\]

(4.6)

where \(C\) depends only on the \(C^{1}\)-geometry of \(M\), \(\delta\) is an arbitrary small number, and \(C_{5}\) is a large number depending only on \(\delta\).

Finally, by choosing \(\delta > 0\) to be sufficiently small and varying \(\phi\) through all the test \(k\)-forms under the designated boundary condition, we estimate from Equations (4.6) and (4.5) that

\[
\int_{M} (\nabla \phi, \nabla u)_{A^{k+1}} dV_{g_{0}} \leq C \left\{ (\| du \|_{L^{r}(M)} + \| \delta u \|_{L^{r}(M)} + \| u \|_{L^{r}(M)}) \| \phi \|_{W^{1,r'}(M)} \right\},
\]

noting that \(\| \delta \phi \|_{L^{r'}(M)} + \| \delta \phi \|_{L^{r'}(M)} \lesssim \| \phi \|_{W^{1,r'}(M)}\). Thus, the assertion immediately follows from Corollary 3.11.

\[\square\]

Remark 4.2 In view of Theorem A, we can interchangeably use the variational quantity

\[
\sup \left\{ \frac{|(du, d\phi)_{g_{0}} + (\delta u, \delta \phi)_{g_{0}} + (u, \phi)_{g_{0}}|}{\| d\phi \|_{L^{r'}(M,g_{0})} + \| \delta \phi \|_{L^{r'}(M,g_{0})} + \| \phi \|_{L^{r'}(M,g_{0})}} : \phi \in C^{\infty}_{0} \Omega^{k}(M) \right\}
\]

in Theorem B with either of the following three:

\[
\sup \left\{ \frac{|(\nabla u, \nabla \phi)_{g_{0}} + (u, \phi)_{g_{0}}|}{\| d\phi \|_{L^{r'}(M,g_{0})} + \| \delta \phi \|_{L^{r'}(M,g_{0})} + \| \phi \|_{L^{r'}(M,g_{0})}} : \phi \in C^{\infty}_{0} \Omega^{k}(M) \right\},
\]

\[
\sup \left\{ \frac{|(du, d\phi)_{g_{0}} + (\delta u, \delta \phi)_{g_{0}} + (u, \phi)_{g_{0}}|}{\| d\phi \|_{L^{r'}(M,g_{0})} + \| \delta \phi \|_{L^{r'}(M,g_{0})} + \| \phi \|_{L^{r'}(M,g_{0})}} : \phi \in C^{\infty}_{0} \Omega^{k}(M) \right\},
\]

\[\square\] Springer
\[ \sup \left\{ \frac{|(\nabla u, \nabla \phi)_{g_0} + (u, \phi)_{g_0}|}{\|\nabla \phi\|_{L^r(M, g_0)} + \|\phi\|_{L^r(M, g_0)}} : \phi \in C_0^\infty \Omega^k(M) \right\}. \]

5 Concluding Remarks

Gaffney’s inequality is the key estimate for the Hodge decomposition theorem for differential forms in boundary value Sobolev spaces. With Theorem A established, we can easily deduce the Hodge decomposition from standard functional analysis; see the exposition by Schwarz [29] and Iwaniec–Scott–Stroffolini [21], as well as the classical papers of Helmholtz [19], Weyl [34], Hodge [20], Kodaira [22], Duff–Spencer [12], Friedrichs [14], Morrey [26, 27], and the recent contributions by Dodziuk [11], Borchers–Sohr [4], Simader–Sohr [32], Wahl [33], etc.

One should remark that all the results in this paper remain valid for manifolds-with-boundary \((M, g_0)\) with \(C^1\)-bounded geometry.

Our arguments in Section 4 circumvent the local computations in Euclidean space via the so-called Bochner technique. Very roughly speaking, the Bochner technique seems to be more suitable for the \(L^2\)-setting. In this note, nevertheless, we have successfully applied it in non-Hilbert settings. This is precisely due to the variational characterisation of the \(W^{1,r}\)-norm of differential forms in Theorem B à la Kozono–Yanagisawa [23].

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