Stable 3-spheres in \( \mathbb{C}^3 \)

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Abstract: By only using spectral theory of the Laplace operator on spheres, we prove that the unit 3-dimensional sphere of a 2-dimensional complex subspace of \( \mathbb{C}^3 \) is a \( \Omega \)-stable submanifold with parallel mean curvature, when \( \Omega \) is the Kähler calibration of rank 4 of \( \mathbb{C}^3 \).

1. Introduction

In 2000, Frank Morgan introduced the notion of multi-volume for an \( m \)-dimensional submanifold \( M \) of a Euclidean space \( \mathbb{R}^{m+n} \), as a volume enclosed by orthogonal projections onto axis \((m+1)\)-planes. He characterized stationary submanifolds for the area functional with prescribed multi-volume as submanifolds with mean curvature vector \( H \) prescribed by a constant multivector \( \xi \in \wedge^{m+1} \mathbb{R}^{m+n} \), namely \( H = \xi \|S\) , where \( S \) is the unit tangent plane of \( M \), and proved the existence of a minimizer among rectifiable currents, as well as their regularity under general conditions of the boundary. In this setting, a question has arisen on conditions for \( \|H\| \) to be constant. In (Salavessa, 2010) we extended the variational characterization of hypersurfaces with constant mean curvature \( \|H\| \) to submanifolds with higher codimension, when the ambient space is any Riemannian manifold \( \tilde{M}^{m+n} \), as discovered by Barbosa, do Carmo and Eschenburg (1984, 1988) for the case \( n = 1 \). This generalization amounts on defining an “enclosed” \((m+1)\)-volume of an \( m \)-dimensional immersed submanifold \( F : \tilde{M}^m \rightarrow \tilde{M}^{m+n} \), \( m \geq 2 \), as the \( \Omega \)-volume defined by each one-parameter variation family \( F(x,t) = F_t(x) \) of \( F(x,0) = F(x) \), where \( \Omega \) is a semi-calibration on the ambient space \( \tilde{M} \), that is, an \((m+1)\)-form \( \Omega \) which satisfies \( |\Omega(e_0,e_1,\ldots,e_m)| \leq 1 \), for any orthonormal system \( e_i \) of \( T\tilde{M} \). A submanifold with calibrated extended tangent space \( H \oplus TM \) is a critical point of the functional area, for compactly supported \( \Omega \)-volume preserving variations, if and only if it has constant mean curvature \( \|H\| \). This case we have \( H = \|H\| \Omega \|S\). From a deeper inspection of this proof, one can see that the initial assumption of calibrated extended tangent space can be dropped, since it will appear as a consequence of being a critical point itself. This will be explained in detail in a future paper, and also its relations with Morgan’s formalism. Assuming that \( M \) has parallel mean curvature \( H \), a second variation is then computed, and its non-negativeness defines stability of \( M \). This

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corresponds to the non-negativeness of the quadratic form associated with the $L^2$-self-adjoint \( \Omega \)-Jacobi operator \( J_\Omega(W) = J(W) + m||H||C_\Omega(W) \), acting on sections in the twisted normal bundle \( H^1_{0,T}(NM) = \mathcal{F} \oplus H_0^1(E) \), where the set \( \mathcal{F} \) of \( H^1_0 \)-functions with zero mean value is identified with the set of sections of the form \( f\nu \), with \( f \in \mathcal{F} \) and \( \nu = H/||H|| \), and where \( E \) is the orthogonal complement of \( \nu \) in the normal bundle. This Jacobi operator is the usual one, but with an extra term, namely a multiple of a first order differential operator \( C_\Omega(W) \) that depends on \( \Omega \). The twisted normal bundle is the \( H^1 \)-completion of the vector space generated by the set \( \mathcal{F}_\Omega \) of compactly supported infinitesimal \( \Omega \)-volume preserving variations, and, in general, we do not know whether it is larger than \( \mathcal{F}_\Omega \) itself. Thus, \( \Omega \)-stability implies that the area functional of \( F_t \) decreases when \( t \) approaches \( t_0 = 0 \), for any family of \( \Omega \)-volume preserving variations \( F_t \) of \( F \), but we do not know whether the converse also holds always. In case the ambient space is the Euclidean space \( \mathbb{R}^{m+n} \), then a unit \( m \)-sphere of an \( \Omega \)-calibrated Euclidean subspace \( \mathbb{R}^{m+1} \) of \( \mathbb{R}^{m+n} \) is \( \Omega \)-stable if and only if, for any \( (n-1) \)-tuple of functions \( f_\alpha \in C^\infty(\mathbb{S}^m) \), \( 2 \leq \alpha \leq n \), the following integral inequality holds:

\[
\sum_{\alpha < \beta} -2m \int_{\mathbb{S}^m} f_\alpha \xi(W_\alpha, W_\beta)(\nabla f_\beta) dM \leq \sum_{\alpha} \int_{\mathbb{S}^m} ||\nabla f_\alpha||^2 dM,
\]

where \( W_\alpha \) is a fixed global parallel orthonormal (o.n.) frame of \( \mathbb{R}^{n-1} \), the orthogonal complement of \( \mathbb{R}^{m+1} \) spanned by \( \mathbb{S}^m \), and \( \xi \) is the \( T^*\mathbb{S}^m \)-valued 2-form on \( \mathbb{S}^{n-1} \)

\[
\xi(W, W')(X) = \Omega(W, W', *X), \quad W, W' \in \mathbb{R}^{n-1}, X \in T^*\mathbb{S}^m
\]

where \( * : T\mathbb{S}^m \rightarrow \wedge^{m-1}T\mathbb{S}^m \) is the star operator. If (1) holds and

\[
\nabla_W \Omega(W, e_1, \ldots, e_m) = 0, \quad \forall W \in N\mathbb{S}^m,
\]

where \( e_i \) is an o.n. frame of \( T\mathbb{S}^m \), then in (Salavessa, 2010, proposition 4.5) we have shown that for each \( \alpha < \beta \), \( \xi(W_\alpha, W_\beta) \) must be co-exact as a 1-form on \( \mathbb{S}^m \), that is,

\[
\xi_{\alpha\beta} := \xi(W_\alpha, W_\beta) = \delta \omega_{\alpha\beta},
\]

for some globally defined 2-form \( \omega_{\alpha\beta} \) on \( \mathbb{S}^m \). This is the case when \( \Omega \) is a parallel \((m+1)\)-form on \( \mathbb{R}^{m+n} \). Using these forms \( \omega_{\alpha\beta} \), the stability condition (1) is translated into the long \( \Omega \)-Cauchy-Riemannian integral inequality:

\[
\sum_{\alpha < \beta} -2m \int_{\mathbb{S}^m} \omega_{\alpha\beta}(\nabla f_\alpha, \nabla f_\beta) dM \leq \sum_{\alpha} \int_{\mathbb{S}^m} ||\nabla f_\alpha||^2 dM.
\]

If we fix \( \alpha < \beta \), and set \( f = f_\alpha \), \( h = f_\beta \), and \( f_\gamma = 0 \) for \( \forall \gamma \neq \alpha, \beta \), (1) reduces to

\[
-2m \int_{\mathbb{S}^m} f \xi_{\alpha\beta}(\nabla h) dM \leq \int_{\mathbb{S}^m} ||\nabla f||^2 dM + \int_{\mathbb{S}^m} ||\nabla h||^2 dM,
\]

and if we replace \( f \) by \( cf \), and \( h \) by \( c^{-1}h \), where \( c^2 = ||\nabla h||_{L^2}/||\nabla f||_{L^2} \), then we obtain the corresponding equivalent short \( \Omega \)-Cauchy-Riemannian, integral inequality

\[
-m \int_{\mathbb{S}^m} \omega_{\alpha\beta}(\nabla f, \nabla h) dM \leq \sqrt{\int_{\mathbb{S}^m} ||\nabla f||^2 dM} \sqrt{\int_{\mathbb{S}^m} ||\nabla h||^2 dM},
\]
holding for all functions $f, h \in C^\infty(S^m)$.

The $\Omega$-stability of a submanifold with calibrated extended tangent space and parallel mean curvature depends on the curvature of the ambient space and on the calibration $\Omega$ (Salavessa, 2010). It always holds on Euclidean spheres if $C_\Omega$ vanish. This last condition is equivalent to the condition (2) and $\xi \equiv 0$ ((Salavessa, 2010), Lemma 4.4). In the case $n = 2$ the later condition is satisfied, but for $n \geq 3$ the operator $C_\Omega$ may not vanish for spheres, even if $\Omega$ is parallel. If $C_\Omega$ does not vanish, spheres of calibrated vector subspaces may not be $\Omega$-stable.

We first consider $\Omega$ any parallel $(m+1)$-form on $\mathbb{R}^{m+n}$. Laplace spherical harmonics of $S^m$ of degree $l$ are the eigenfunctions for the closed eigenvalue problem with respect to the Laplacian operator corresponding to the eigenvalue $\lambda_l = l(l+m-1)$, and they are just the harmonic homogeneous polynomial functions of degree $l$ of $\mathbb{R}^{m+1}$ restricted to $S^m$. We denote by $E_{\lambda_l}$ the finite-dimensional subspace of $H^1(S^m)$ spanned by these $\lambda_l$-eigenfunctions. In the first theorem we show how each 1-form $\xi_{\alpha\beta}$ transforms a spherical harmonic $f$ into another spherical harmonic $h$:

**Theorem 1.1.** If $\Omega$ is parallel, then for each $f \in E_{\lambda_l}$, $h = \xi_{\alpha\beta} (\nabla f)$ is also in $E_{\lambda_l}$, and it is $L^2$-orthogonal to $f$.

In this paper we study the stability of the unit 3-sphere of a 2-dimensional complex subspace of $\mathbb{C}^3$ with respect to the Kähler calibration. In this case $C_\Omega$ does not vanish. Let $\sigma$ be the Kähler form of $\mathbb{C}^3 = \mathbb{R}^6$, and $\Omega$ the Kähler calibration of rank 4,

$$\sigma = dx^{12} + dx^{34} + dx^{56}, \quad \Omega = \frac{1}{2} \sigma^2.$$

The unit sphere of $\mathbb{R}^4 \times \{0\}$ is immersed into $\mathbb{R}^6 = \mathbb{C}^3$, by the inclusion map $\phi = (\phi_1, \ldots, \phi_4, 0) : S^3 \to \mathbb{C}^3$. We have only one of those 1-forms

$$\xi := \xi_{56} = *(d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \phi^1 d\phi^2 - \phi^2 d\phi^1 + \phi^3 d\phi^4 - \phi^4 d\phi^3,$$

and $\xi = \delta \omega$, with $\omega = \frac{1}{2} * \xi = \frac{1}{2} (d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \frac{1}{2} \phi^* \sigma$. Our main theorem is the following:

**Theorem 1.2.** Three-dimensional spheres of $\mathbb{C}^3$ are $\Omega$-stable submanifolds of $\mathbb{C}^3$ with parallel mean curvature, where $\Omega = \frac{1}{2} \sigma^2$ is the Kähler calibration of rank 4.

The Cauchy-Riemann inequality version of the $\Omega$-stability is described in the corollary:

**Corollary 1.1.** The Cauchy-Riemann inequality

$$- \int_{S^3} \sigma (\nabla f, \nabla h) dM \leq \frac{2}{3} \sqrt{\int_{S^3} \|
abla f\|^2 dM \int_{S^3} \|
abla h\|^2 dM}$$

holds for any smooth functions $f$ and $h$ of $S^3$, with equality if and only if $f, h \in E_{\lambda_1}$, with $f = \sum \mu_i \phi_i$ and $h = \sum \sigma_i \phi_i$, where $\sigma_2 = -\mu_1$, $\sigma_1 = \mu_2$, $\sigma_4 = -\mu_3$, $\sigma_3 = \mu_4$. 

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Finally, we state that the 3-sphere is the unique smooth closed submanifold that solves the
\(\Omega\)-isoperimetric problem among a certain class of immersed submanifolds:

**Theorem 1.3.** The unit 3-sphere of a complex 2-dimensional subspace of \(\mathbb{C}^3\) is the unique closed
immersed 3-dimensional submanifold \(\phi : M \rightarrow \mathbb{C}^3\) with parallel mean curvature, trivial normal
bundle, and complex extended tangent space \(H \oplus TM\), that is \(\Omega\)-stable for the Kähler calibration
of rank 4, and satisfies the inequality

\[
\int_M S(2 + h\|H\|)dM \leq 0,
\]

where \(h\) and \(S\) are the height functions \(h = \langle \phi, \nu \rangle\) and \(S = \sum_{ij}\langle \phi, (B(e_i, e_j))F \rangle B^\nu(e_i, e_j)\).

**Remark.** On a closed Kähler manifold \((M, J)\) with Kähler form \(\varpi(X, Y) = g(JX, Y)\), if \(f, h : M \rightarrow \mathbb{R}\) are smooth functions, then by the Cauchy-Schwarz inequality,

\[
\left| \int_M \varpi(\nabla f, \nabla h)dM \right| \leq \sqrt{\int_M \|\nabla f\|^2dM} \sqrt{\int_M \|\nabla h\|^2dM},
\]

with equality if and only if \(\nabla h = \pm J\nabla f\), or equivalently \(f \pm ih : M \rightarrow \mathbb{C}\) is a holomorphic func-
tion. If this is the case, then \(f\) and \(h\) are constant functions. On the other hand, globally defined
functions, sufficiently close to holomorphic functions defined on a sufficiently large open set,
are expected to satisfy an almost equality. This is not the case of \(S^3\), which is not a complex
manifold, and somehow explains the coefficient \(2/3\) in Corollary 1.1.

**Remark.** In the case of 3-spheres in \(\mathbb{C}^3\) we have only one form \(\xi_{\alpha\beta}\), that is, the long Cauchy-
Riemann inequality is the short one. We wonder if a general proof of short Cauchy-Riemann
inequalities can be always obtained for Euclidean \(m\)-spheres on \(\mathbb{R}^{m+n}\), by using the spectral
theory of spheres, when \(\Omega\) is any parallel calibration. Note that (4) is immediately satisfied for
\(f, h \in E_{k}\), if \(\lambda_l \geq m^2\), that is \(l \geq m\), so it remains to consider the cases \(l \leq m - 1\). For 3-spheres
we have to consider polynomial functions up to order \(l = 2\), while for 2-spheres we have to con-
sider only the case \(l = 1\). A related remark is given in the end of section 3.

**2. Preliminaries**

We consider an oriented Riemannian manifold \(M\) of dimension \(m\), with Levi-Civita connection \(\nabla\) and Ricci tensor \(\text{Ric}^M : TM \rightarrow TM\). In what follows \(e_1, \ldots, e_m\) denotes a local direct
o.n. frame.

**Lemma 2.1.** Let \(\xi\) be a co-exact 1-form on a Riemannian manifold \(M\), with \(\xi = \delta \omega\), where \(\omega\) is
a 2-form. Then for any function \(f \in C^2(M)\),

\[
\xi(\nabla f) = \text{div}(\nabla^\omega f),
\]

where \(\nabla^\omega f = \sum_i \omega(\nabla f, e_i)e_i\). Moreover, for any \(f, h \in C^\infty_0(M)\)

\[
\int_M f\xi(\nabla h)dM = \int_M \omega(\nabla f, \nabla h)dM = -\int_M h\xi(\nabla f)dM.
\]
Thus, seen as a self-adjoint operator of wedge bundles with Riemannian metric $\langle \nabla \rangle$ fundamental form $\omega$ and corresponding Weingarten operator $B$ and function $\Delta = \nabla^2$ (denoted the Levi-Civita connection on $M$),

$$\nabla f = \nabla \omega f = \sum_{i} \nabla_{e_i} \omega(e_i, \nabla f) = \sum_{i} -\nabla_{e_i}(\omega(e_i, \nabla f)) + \omega(e_i, \nabla_{e_i} \nabla f) \quad \text{div}(\nabla \omega f) + \sum_{ij} Hess f(e_i, e_j) \omega(e_i, e_j).$$

The last equality proves the first equality of the lemma, because $Hess f(e_i, e_j)$ is symmetric on $i, j$ and $\omega(e_i, e_j)$ is skew-symmetric. The other equalities of the lemma follow from $div(fX) = \langle \nabla f, X \rangle + f \text{div}(X)$, holding for any vector field $X$ and function $f$.

The $\delta$ and star operators acting on $p$-forms on an oriented Riemannian $m$-manifold $M$ satisfy $\delta = (-1)^{mp+m+1} \ast d \ast$, $\ast = (-1)^{p(m-p)} Id$, for a 1-form $\xi$ the DeRham Laplacian $\Delta$ and the rough Laplacian $\bar{\Delta}$ are related by the following formulas

$$\bar{\Delta} \xi(X) = (d \delta + \delta d) \xi(X) = -\Delta \xi(X) + \xi(\text{Ricci}^M(X)),$$

$$\Delta \xi(X) = \text{trace} \nabla^2 \xi(X) = \sum_i \nabla_{e_i} \nabla_{e_i} \xi(X) - \nabla_{\nabla_{e_i} e_i} \xi(X).$$

If $\xi = \delta \omega$, then $\Delta \xi = 0$, and so $\bar{\Delta} \xi(X) = \delta d \xi(X) = -\sum_i \nabla_{e_i} (d \xi)(e_i, X)$. We also recall the following well-known formula (see e.g. Salavessa & Pereira do Vale (2006)) for $f \in C^\infty(M)$,

$$(\bar{\Delta} d f)(X) = \sum_i \nabla_{e_i}^2 d f(X) = g(\nabla(\Delta f), X) + df(\text{Ricci}^M(X)).$$

Thus,

$$\bar{\Delta}(\nabla f) = \nabla(\Delta f) + \text{Ricci}^M(\nabla f),$$

$$\bar{\Delta}(\nabla f) = - (\delta d \xi)(\nabla f) + \xi(\text{Ricci}^M(\nabla f)).$$

Now we suppose that $M$ is an immersed oriented hypersurface of a Riemannian manifold $M'$, with Riemannian metric $\langle \cdot, \cdot \rangle$, defined by an immersion $\phi : M \to M'$ with unit normal $v$, second fundamental form $B$ and corresponding Weingarten operator $A$ in the $v$ direction, given by

$$B(e_i, e_j) = \langle A(e_i), e_j \rangle = \langle \nabla'_{e_i} e_j, v \rangle = -\langle e_j, \nabla'_{e_i} v \rangle,$$

where $\nabla'$ denotes the Levi-Civita connection on $M'$. The scalar mean curvature of $M$ is given by

$$H = \frac{1}{m} \text{Trace } B = \sum_i \frac{1}{m} B(e_i, e_i).$$

The curvature operator of $M'$, $R'(X, Y, Z, W) = \langle -\nabla'_{Y} \nabla'_{X} Z + \nabla'_{Y} \nabla'_{X} Z + \nabla'_{[X, Y]} Z, W \rangle$, can be seen as a self-adjoint operator of wedge bundles $R' : \wedge^2 TM' \to \wedge^2 TM'$,

$$\langle R'(u \wedge v), z \wedge w \rangle = R'(u, v, z, w),$$

where $\wedge$ denotes the exterior product on $\wedge^2 TM'$.
and so \( R'(u \wedge v) = \sum_{i<j} R'(u, v, e_i, e_j) e_i \wedge e_j \), where
\[
< u \wedge v, z \wedge w > = det \left[ \begin{array}{cc}
< u, z > & < u, w > \\
< v, z > & < v, w > 
\end{array} \right].
\]

In what follows, we suppose that \( \hat{\xi} \) is a parallel \((m-1)\)-form on \( M' \), and \( \xi \) is given by
\[
\xi = *\phi^* \hat{\xi}
\]
where * is the star operator on \( M \). In this case \( \xi \) is obviously co-closed, but not necessarily co-exact. We employ the usual inner products in \( p \)-forms and morphisms.

**Lemma 2.2.** Assume \( m \geq 3 \). Then for all \( i, j \)
\[
(\nabla_{e_i} \xi)(e_j) = \sum_k -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)) = -\hat{\xi}(v, *(A(e_i) \wedge e_j)),
\]
\[
\Delta \xi(e_j) = \delta d \xi(e_j) = \hat{\xi}\left( v, *(e_j \wedge (m\nabla H - [\text{Ricci}^M(v)]^T)) + R'(e_j \wedge v) \right) + \xi(\Theta_B(e_j)),
\]
where \([\text{Ricci}^M(v)]^T = \sum_k \text{Ricci}^M(v, e_k) e_k \) and \( \Theta_B : TM \to TM \) is the morphism given by, \( \Theta_B = ||B||^2 I_d + mHA - 2A^2 \).

**Proof.** We fix a point \( x_0 \in M \) and take \( e_i \) a local o.n. frame s.t. \( \nabla e_i(x_0) = 0 \). We will compute \( d\hat{\xi}(e_i, e_j) \), at \( x \) on a neigbourhood of \( x_0 \). Recall that for any \( p \)-form \( \sigma \), we have \( *\sigma = \sigma * \), where the star operator on the r.h.s. can be seen as acting on \( \wedge^{m-p} TM \), with \( *e_i = (-1)^{i-1} e_1 \wedge ... \wedge \hat{e}_i \wedge ... e_m \), and for \( i < j \), \(*e_i \wedge e_j = (-1)^{i+j-1} e_1 \wedge ... \wedge \hat{e}_i \wedge ... \wedge \hat{e}_j \wedge ... e_m \). Using the fact that \( \hat{\xi} \) is a parallel form on \( M' \), we have for \( x \) near \( x_0 \),
\[
\nabla_{e_i}(\hat{\xi}(e_j)) = \sum_{k \neq j} (-1)^{k-1} \hat{\xi}(e_1, ..., \nabla'_{e_i} e_k, ..., \hat{e}_j, ..., e_m)
+ \sum_{k < j} (-1)^{k+j-1} \hat{\xi}(\nabla'_{e_i} e_k, e_1, ..., \hat{e}_k, ..., \hat{e}_j, ..., e_m)
+ \sum_{k > j} (-1)^{k+j-1} \hat{\xi}(\nabla'_{e_i} e_k, e_1, ..., e_k, ..., \hat{e}_j, ..., e_m)
= \sum_{k < j} (-\nabla_{e_i} e_k, e_j) \hat{\xi}(e_k) - B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j))
+ \sum_{k > j} (-\nabla_{e_i} e_k, e_j) \hat{\xi}(e_k) + B(e_i, e_k) \hat{\xi}(v, *(e_j \wedge e_k))
= \xi(\nabla_{e_i} e_j) + \sum_{k \neq j} -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)).
\]

Hence, \( (\nabla_{e_i} \xi)(e_j) = \sum_{k \neq j} -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)) \), which proves the first sequence of equalities of the lemma. Now,
\[
d\xi(e_i, e_j) = (\nabla_{e_i} \xi)(e_j) - (\nabla_{e_j} \xi)(e_i)
= \sum_{k \neq j} -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)) + \sum_{k \neq i} B(e_j, e_k) \hat{\xi}(v, *(e_k \wedge e_i)),
\]
and by Codazzi’s equation,
\[
(\nabla_{e_j} B)(e_j, e_k) = (\nabla_{e_j} B)(e_i, e_k) - R'(e_i, e_j, e_k, v)
= \sum_i (\nabla_{e_j} B)(e_i, e_k) = m\nabla e_i H - Ricci^M(e_k, v).
\]
Note that \( B_{ik} = (\nabla e_i B)(e_i, e_k) \) is a symmetric matrix, and if we define \( A_{ki} = \xi_j (\nabla^* (e_k \wedge e_j)) \) (valuing zero if \( k = i \)), then \( A_{ik} \) is skew-symmetric. Thus, \( \sum_{k \neq i} B_{ik} A_{ki} = \sum_{k, i} B_{ik} A_{ki} = 0 \). Furthermore, if we set \( C_{ik} = -R'(e_i, e_j, e_k, v) \), then \( C_{ik} - C_{ki} = R'(e_k, e_i, e_j, v) \). Hence,

\[
\sum_{i} \sum_{k \neq i} C_{ik} A_{ki} = \sum_{i} C_{ik} A_{ki} = \sum_{i} \frac{1}{2} ((C_{ik} + C_{ki}) + (C_{ik} - C_{ki})) A_{ki} = \sum_{i} \frac{1}{2} R'(e_k, e_i, e_j, v) A_{ki}.
\]

Therefore, for each \( j \), at \( x_0 \)

\[
-\delta d\xi (e_j) = \sum_{i} \nabla e_i (d\xi (e_i, e_j))
\]

\[
= \sum_{k \neq i} \sum_{j} -(\nabla e_i B)(e_i, e_k) \xi_j (\nabla^* (e_k \wedge e_j)) - B(e_i, e_k) \nabla e_i (\xi_j (\nabla^* (e_k \wedge e_j))) + \sum_{k \neq i} \sum_{j} (\nabla e_i B)(e_i, e_k) \xi_j (\nabla^* (e_k \wedge e_j)) + B(e_j, e_k) \nabla e_i (\xi_j (\nabla^* (e_k \wedge e_j)))
\]

\[
= \sum_{k \neq j} \sum_{i} (-m \nabla e_i H + \text{Ricci}^M (e_i, v)) \xi_j (\nabla^* (e_k \wedge e_j)) + \sum_{k \neq j} \frac{1}{2} R'(e_k, e_i, e_j, v) \xi_j (\nabla^* (e_k \wedge e_j)) + S
\]

where

\[
S = \sum_{i} \sum_{k < j} (-1)^{k+j} B(e_i, e_k) \xi_j (\nabla^* e_j v, e_1, \ldots, \hat{e}_k, \ldots, \hat{e}_j, \ldots, e_m)
+ \sum_{i} \sum_{k > j} (-1)^{k+j-1} B(e_i, e_k) \xi_j (\nabla^* e_j v, e_1, \ldots, \hat{e}_j, \ldots, \hat{e}_k, \ldots, e_m)
+ \sum_{j < k} (-1)^{k+j} B(e_j, e_k) \xi_j (\nabla^* e_j v, e_1, \ldots, \hat{e}_k, \ldots, e_j, e_m)
+ \sum_{j < k} (-1)^{k+j} B(e_j, e_k) \xi_j (\nabla^* e_j v, e_1, \ldots, \hat{e}_j, \ldots, e_k, e_m)
\]

At this point we may assume that at \( x_0 \) the basis \( e_i \) diagonalizes the second fundamental form, that is, \( B(e_i, e_j) = \lambda_i \delta_{ij} \). Then,

\[
S = \sum_{j} \sum_{k < j} -\delta_k \lambda^2_j \xi (e_j) + \delta_j \delta_k \lambda^2_j \xi (e_k) + \sum_{j} \sum_{k > j} \delta_j \delta_k \lambda^2_j \xi (e_k) + \sum_{k < j} -\delta_k \delta_j \lambda \lambda_j \xi (e_k) + \sum_{k > j} -\delta_k \delta_j \lambda \lambda_j \xi (e_k) + \delta_k \delta_j \lambda \lambda_j \xi (e_k)
\]

\[
= \sum_{j} -\lambda^2_j \xi (e_j) + \sum_{j < j} -\lambda^2_j \xi (e_j) + \sum_{j > j} -\lambda^2_j \xi (e_j) + \sum_{j < j} -\lambda^2_j \xi (e_j) + \sum_{j > j} -\lambda^2_j \xi (e_j)
\]

\[
= -\|B\|^2 \xi (e_j) - m H \xi (A(e_j)) + 2 \xi (A^2 (e_j)),
\]

and the second sequence of equalities of the lemma is proved. \( \square \)

If we suppose that \( \Theta_B = \mu(x) I d \), taking \( e_i \) a diagonalizing o.n. basis of the second fundamental form, \( B(e_i, e_j) = \lambda_i \delta_{ij} \), then each \( \lambda_i \) satisfies the quadratic equation

\[
2 \lambda_i^2 - m H \lambda_i + (\mu - \|B\|^2) = 0,
\]
which implies that we have at most two distinct possible principal curvatures \( \lambda_\pm \). Moreover, from the above equation, summing over \( i \), we derive that \( \mu(x) \) must satisfy

\[
\lambda_\pm = \frac{1}{4} \left( mH \pm \sqrt{\frac{16}{m} \|B\|^2 + m(m-8)H^2} \right).
\]

Note that, from \( \|B\|^2 \geq m\|H\|^2 \), we have \( \frac{16}{m} \|B\|^2 + m(m-8)H^2 \geq (m-4)^2H^2 \), and so there are one or two distinct principal curvatures. If \( M \) is totally umbilical, then \( \|B\|^2 = mH^2 \) and \( \mu = 2(m-1)\|H\|^2 \). The previous lemma leads to the following conclusion:

**Lemma 2.3.** Assuming \( M' = \mathbb{R}^{m+1}, m \geq 3 \), and taking \( M \) a hypersurface with constant mean curvature, with \( \Theta_B = \mu(x)Id \), where \( \mu(x) \) is a smooth function on \( M \), we get

\[
\Delta \xi = \mu \xi.
\]

Furthermore, \( \xi \) is an eigenform for the DeRham Laplacian operator, that is \( \mu(x) \) is constant, if and only if \( \|B\| \) is constant.

In case \( M \) is a unit \( m \)-sphere \( S^m \), then \( \Theta_B = \mu Id \), with \( \mu = 2(m-1) \), and taking \( \nu_x = -x \) as unit normal, then, at each \( x \in S^m \),

\[
\begin{align*}
(\nabla_x \xi)(e_j) &= \xi(x, (e_i \wedge e_j)) \\
\delta \xi(e_i, e_j) &= 2\xi(x, (e_i \wedge e_j)) \\
\Delta \xi &= \delta d \xi = 2(m-1)\xi.
\end{align*}
\]

**Lemma 2.4.** If \( f \in C^\infty(S^m) \), then \( \Delta(\xi(\nabla f)) = \xi(\nabla \Delta f) \).

**Proof.** We fix a point \( x_0 \in S^m \) and take \( e_i \) a local o.n. frame of the sphere s.t. \( \nabla e_i(x_0) = 0 \). Let \( f \in C^\infty(S^m) \). The following computations are at \( x_0 \). Using the above formulas (6) and previous lemma, we have

\[
\begin{align*}
\Delta(\xi(\nabla f)) &= \sum_i \nabla_{e_i}(\nabla_{e_i}(\xi(\nabla f))) = \sum_i \nabla_{e_i}(\nabla_{e_i}(\xi(\nabla f)) + \xi(\nabla_{e_i}\nabla f)) \\
&= (\Delta \xi)(\nabla f) + 2(\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f) + \xi(\nabla_{e_i}\nabla_{e_i}\nabla f) \\
&= -2(m-1)\xi(\nabla f) + \xi(\nabla \Delta f) + 2(m-1)\xi(\nabla f) + \sum_i 2(\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f).
\end{align*}
\]

Since \( \text{Hess}(f, e_i, e_j) \) is symmetric in \( ij \) and by Lemma 2.3, \( (\nabla_{e_i} \xi)(e_j) \) is skew-symmetric, we have

\[
\sum_i (\nabla_{e_i} \xi)(\nabla_{e_i} \nabla f) = \sum_{ij} \text{Hess}(f, e_i, e_j)(\nabla_{e_i} \xi)(e_j) = 0,
\]

and the lemma is proved. \( \square \)

3. Proof of Theorem 1.1.

We denote by \( \nabla \) the Levi-Civita connection of \( S^m \) induced by the flat connection \( \nabla \) of \( \mathbb{R}^{m+n} \). We are considering a parallel calibration \( \Omega \) on \( \mathbb{R}^{m+n} \). We fix \( \alpha < \beta \) and define the 1-form on \( S^m \)

\[
\xi = \xi(W_\alpha, W_\beta) = *\phi^* \xi = \delta \omega,
\]

8
where $\xi = \xi_{\alpha\beta}$ and $\omega = \omega_{\alpha\beta}$.

We recall that the eigenvalues of $S^m$ for the closed Dirichlet problem are given by $\lambda_l = l(l + m - 1)$, with $l = 0, 1, 2, \ldots$. We denote by $E_{\lambda_l}$ the eigenspace of dimension $m_l$ corresponding to the eigenvalue $\lambda_l$, and by $E_{\lambda_l}^+$ the $L^2$-orthogonal complement of the sum of the eigenspaces $E_{\lambda_i}$, $i = 1, \ldots, l - 1$, and so it is the sum of all eigenspaces $E_{\lambda}$ with $\lambda \geq \lambda_l$. If $f \in E_{\lambda_l}$, and $h \in E_{\lambda_l}$, then
\[
\int_{S^m} fh\,dM = 0 \quad \text{and} \quad \int_{S^m} \langle \nabla f, \nabla h \rangle\,dM = \delta_{l,\lambda_l} \int_{S^m} fh\,dM.
\]

There exists an $L^2$-orthonormal basis $\psi_{l,\sigma}$ of $L^2(S^m)$ of eigenfunctions ($1 \leq \sigma \leq m_l$). The Rayleigh characterization of $\lambda_l$ is given by
\[
\lambda_l = \inf_{\psi \in E_{\lambda_l}^+} \frac{\int_{S^m} \langle \nabla \psi, \nabla \psi \rangle\,dM}{\int_{S^m} \psi^2\,dM},
\]
and the infimum is attained for $f \in E_{\lambda_l}$. Each eigenspace $E_{\lambda_l}$ is exactly composed by the restriction to $S^m$ of the harmonic homogeneous polynomial functions of degree $l$ of $\mathbb{R}^{m+1}$, and it has dimension $m_l = \binom{m+l}{m} - \binom{m+l-2}{m}$. Thus, each eigenfunction $\psi \in E_{\lambda_l}$ is of the form $\psi = \sum_{|\alpha|=l} \mu_\alpha \phi_\alpha$, where $\mu_\alpha$ are some scalars and $a = (a_1, \ldots, a_{m+1})$ denotes a multi-index of length $|a| = a_1 + \ldots + a_{m+1} = l$ and
\[
\phi_\alpha = \phi_1^{a_1} \cdots \phi_{m+1}^{a_{m+1}}.
\]

From $\nabla \phi_i = \xi_i^\top$ and $\sum_i \phi_i^2 = 1$, we see that
\[
\left\{ \begin{array}{l}
\langle \nabla \phi_i, \nabla \phi_j \rangle = \delta_{ij} - \phi_i^2, \\
\int_{S^m} \phi_i^2\,dM = \frac{1}{m+1} |S^m|, \\
\int_{S^m} \|\nabla \phi_i\|^2\,dM = \lambda_l \int_{S^m} \phi_i^2\,dM = \frac{m}{m+1} |S^m|.
\end{array} \right.
\]

We also denote by $\int_{S^m} \phi_i^2\,dM$ any of the integrals $\int_{S^m} \phi_i^2\,dM$, $i = 1, \ldots, m + 1$. We recall the following:

**Lemma 3.1.** If $P : S^m \to \mathbb{R}$ is a homogeneous polynomial function of degree $l$, then
\[
\int_{S^m} P(x)\,dM = \frac{1}{\lambda_l} \int_{S^m} \Delta^0 P(x)\,dM.
\]

In particular,
\[
\int_{S^m} \phi_\alpha^2\,dM = \sum_{1 \leq i \leq m+1} \frac{a_i(a_i - 1)}{l(l + m - 1)} \int_{S^m} \phi_\alpha^{a_i-2}\,dM,
\]
where the terms $a_i < 2$ are considered to vanish. Thus, if some $a_i$ is odd this integral vanishes.

**Proof of Theorem 1.1.** By Lemma 2.4, if $f \in E_{\lambda_l}$ then $\xi(\nabla f) \in E_{\lambda_l}$. From
\[
\int_{S^m} f \xi(\nabla f)\,dM = \int_{S^m} \omega(\nabla f, \nabla f)\,dM = 0
\]
we conclude that $f$ and $h = \xi(\nabla f)$ are $L^2$-orthogonal. \qed
Thus, fixing

The volume element of dimensional subspaces, that is,

By Theorem 1.1, \( \bar{\xi}(\nabla f) \in E_{\lambda_i} \) as well. The term \( \text{Hess}_f(\nabla h, \bar{\xi}^2) \) is a sum of polynomial functions of degree \( 2l + k \bar{\xi}_j \) where \( k \bar{\xi}_j \) depends on \( \bar{\xi}_j \), when expressed in terms of \( \phi^i \). Let us suppose that all \( k \bar{\xi}_j \) are even. Then by Lemma 3.1, \( \int_{S^m} \text{Hess}_f(\nabla h, \bar{\xi}^2) dM = 0 \). Since \( \lambda_i \geq m \), and taking into consideration that \( \Omega \) is a semi-calibration,

\[
-\int_{S^m} h\bar{\xi}(\nabla f) dM = -\frac{1}{\lambda_i} \int_{S^m} \langle \nabla h, \nabla (\bar{\xi}(\nabla f)) \rangle dM = \frac{1}{\lambda_i} \int_{S^m} \hat{\xi}(\nabla (\nabla h \nabla f)) dM \\
\leq \frac{1}{\lambda_i} \int_{S^m} \| \nabla h \| \| \nabla f \| dM \leq \frac{1}{m} \| \nabla f \|_{L^2} \| \nabla h \|_{L^2}.
\]

Thus, in this case the short Cauchy-Riemann inequality holds. Inspection of \( \xi \) must be required for each case of \( \Omega \). A general proof of the short Cauchy-Riemann integral inequality, under appropriate conditions on \( \Omega \), will be developed in a future paper.

4. 3-spheres of \( \mathbb{C}^2 \) in \( \mathbb{C}^3 \)

In this section we specialize the Cauchy-Riemann inequalities for the case \( m = n = 3 \) and for \( \mathbb{R}^6 = \mathbb{C}^3 \) we will consider the Kähler calibration \( \frac{1}{2} \sigma^2 \) that calibrates the complex two-dimensional subspaces, that is,

\[
\Omega = dx^{1234} + dx^{1256} + dx^{3456}.
\]

Thus, fixing \( W_5 = e_5 \) and \( W_6 = e_6 \) we have \( \tilde{\xi} := \bar{\xi}_{56} = dx^{12} + dx^{34} \), and

\[
\tilde{\xi} := \bar{\xi}_{56} = *\phi^*\tilde{\xi} = *(d\phi^1 + d\phi^3).
\]

The volume element of \( S\mu \) is \( \text{Vol}_{S^m} = \sum_i(-1)^{i-1} \phi_i d\phi^{1...i...} \), and \( *\xi \) is the unique 2-form s.t. \( \bar{\xi} \wedge *\xi = \| \bar{\xi} \|^2 \text{Vol}_{S^m} \). Using (7) we see that \( \| \bar{\xi} \| = \| *\xi \| = 1 \).

\[
\bar{\xi} = \phi_1 d\phi^2 - \phi_2 d\phi^1 + \phi_3 d\phi^4 - \phi_4 d\phi^3 \\
*\xi = d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4 = \frac{1}{2} d\xi = d *\omega.
\]

Therefore, we may take \( *\omega = \frac{1}{2} \bar{\xi} \), that is

\[
\omega = \frac{1}{2} *\bar{\xi} = \frac{1}{2} (d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \frac{1}{2} \phi^*\sigma.
\]

Hence, to prove Theorem 1.2 and Corollary 1.1 we have to verify that, for any functions \( f, h \in C^\infty(S^3) \), one of the following equivalent inequalities holds:

\[
\int_{S^3} -3\omega(\nabla f, \nabla h) dM = \int_{S^3} -3f\bar{\xi}(\nabla h)dM \leq \| \nabla f \|_{L^2} \| \nabla h \|_{L^2} \tag{8}
\]

\[
\int_{S^3} -6\omega(\nabla f, \nabla h) dM = \int_{S^3} -6f\bar{\xi}(\nabla h)dM \leq \| \nabla f \|_{L^2}^2 + \| \nabla h \|_{L^2}^2.
\]
By Theorem 1.1 we only need to consider both \( f, h \in E_{\lambda_3} \), for some \( l \). Note that \( \lambda_3 = 15 \) and since \( \Omega \) is a calibration, \( \| \xi(X) \| \leq \| X \| \).

**Lemma 4.1.** If \( f, h \in E_{\lambda_3}^+ \) are nonzero, (8) holds, with strict inequality.

**Proof.** By Schwartz inequality and Rayleigh characterization

\[
\int_{S^3} -3f \xi(\nabla h) dM \leq 3\| f \|_{L^2} \| \nabla h \|_{L^2} \leq \frac{3}{\sqrt{\lambda_3}} \| \nabla f \|_{L^2} \| \nabla h \|_{L^2} < \| \nabla f \|_{L^2} \| \nabla h \|_{L^2},
\]

with strict inequality in the last one, since neither \( f \) nor \( h \) may be constant. \( \square \)

We now verify that (8) holds for \( f, h \in E_{\lambda_1} \) and \( f, h \in E_{\lambda_2} \). From (7) and Lemma 3.1, we have for \( i \neq j \)

\[
\begin{align*}
\int_{S^3} \phi_1^2 dM &= \frac{1}{4} |S^3|, & \int_{S^3} \phi_i^2 dM &= \frac{1}{3} \int_{S^3} \phi^2 dM \\
\int_{S^3} \phi_i^4 dM &= \frac{1}{4} \int_{S^3} \phi^2 dM, & \int_{S^3} \| \phi \|_{L^2}^2 &= 3 \int_{S^3} \phi^2 dM \\
\omega(\nabla \phi_1, \nabla \phi_2) &= \frac{1}{2} (1 - \phi_1^2 - \phi_2^2) & \omega(\nabla \phi_1, \nabla \phi_3) &= \frac{1}{2} (-\phi_2 \phi_3 + \phi_1 \phi_4) \\
\omega(\nabla \phi_1, \nabla \phi_4) &= \frac{1}{2} (-\phi_2 \phi_4 - \phi_1 \phi_3) & \omega(\nabla \phi_2, \nabla \phi_3) &= \frac{1}{2} (\phi_1 \phi_3 + \phi_2 \phi_4) \\
\omega(\nabla \phi_2, \nabla \phi_4) &= \frac{1}{2} (\phi_1 \phi_4 - \phi_2 \phi_3) & \omega(\nabla \phi_3, \nabla \phi_4) &= \frac{1}{2} (1 - \phi_3^2 - \phi_4^2).
\end{align*}
\]

and moreover

**Lemma 4.2.**

\[
\begin{align*}
3 \int \omega(\nabla \phi_1, \nabla \phi_2) &= 3 \int \phi^2 = \| \nabla \phi_1 \|_{L^2} \| \nabla \phi_2 \|_{L^2} = \| \phi \|_{L^2}^2, \\
3 \int \omega(\nabla \phi_1, \nabla \phi_4) &= 3 \int \phi^2 = \| \nabla \phi_1 \|_{L^2} \| \nabla \phi_4 \|_{L^2} = \| \phi \|_{L^2}^2, \\
-3 \int \omega(\nabla \phi_i, \nabla \phi_j) &= 0 \text{ for other } i, j.
\end{align*}
\]

**Lemma 4.3.** If \( f, h \in E_{\lambda_1} \), that is \( f = \sum \mu_i \phi_i, h = \sum \sigma_j \phi_j \), for some constant \( \mu_i, \sigma_j \), then (8) holds, with equality if and only if \( \sigma_2 = -\mu_1, \sigma_1 = \mu_2, \sigma_4 = -\mu_3, \sigma_3 = \mu_4 \).

**Proof.** Using the previous lemma,

\[
-3 \int \omega(\nabla f, \nabla h) dM = (\mu_1 \sigma_2 - \mu_2 \sigma_1) \int -3 \omega(\nabla \phi_1, \nabla \phi_2) + (\mu_3 \sigma_4 - \mu_4 \sigma_3) \int -3 \omega(\nabla \phi_3, \nabla \phi_4)
\]

\[
= - (\mu_1 \sigma_2 - \mu_2 \sigma_1 + \mu_3 \sigma_4 - \mu_4 \sigma_3) \| \nabla f \|_{L^2}^2 \leq \frac{1}{2} (\mu_1^2 + \sigma_1^2) \| \nabla \phi_1 \|_{L^2}^2 = \frac{1}{2} (\| \nabla f \|_{L^2}^2 + \| \nabla h \|_{L^2}^2).\]
The equality case follows immediately.

**Lemma 4.4.** If \( f, h \in E_{\lambda} \) are nonzero, then (8) holds with strict inequality.

**Proof.** Set \( f = \sum \alpha_i \phi_i^2 + \sum_{i<j} A_{ij} \phi_i \phi_j \), and \( h = \sum \beta_i \phi_i^2 + \sum_{i<j} B_{ij} \phi_i \phi_j \), where \( \alpha_i, A_{ij}, \beta_i, B_{ij} \) are constants. Now we compute

\[
-3 \int \omega(\nabla f, \nabla h) =
\]

\[
-3 \int \omega(\nabla \phi_1, \nabla \phi_2) \left[ 2\alpha_1 B_{i1} \phi_i^2 + 2\beta_2 A_{12} \phi_2^2 + A_{13} B_{23} \phi_3^2 + A_{14} B_{24} \phi_4^2 - 2\beta_1 A_{12} \phi_1^2 \phi_2 - A_{13} B_{23} \phi_3 \phi_4 - A_{14} B_{24} \phi_4 \phi_3 \right]
\]

Thus, using Lemma 4.2,
This is equivalent to prove the inequalities

Note that

and applying the same lemmas we see that

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Now
\[ 3 \times (10) = 3(\alpha_2 - \alpha_1)B_{12} - 3(\beta_2 - \beta_1)A_{12} + 3(\alpha_4 - \alpha_3)B_{34} + 3(-\beta_4 + \beta_3)A_{34} \]
\[ \leq \frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\alpha_4 - \alpha_3)^2 + (-\beta_4 + \beta_3)^2) \]
\[ + \frac{3}{2}(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2) \]
\[ \leq \frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\alpha_4 - \alpha_3)^2 + (-\beta_4 + \beta_3)^2) \]
\[ + 2(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2). \] (16)
\[ 3 \times (12) \leq \sum_k 3(\alpha_k^2 + \beta_k^2), \] (17)

We will prove that
\[ (16) + 3 \times (12) \leq \sum_k 3(\alpha_k^2 + \beta_k^2), \] (18)
with equality iff \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \) and \( \beta_1 = \beta_2 = \beta_3 = \beta_4 \), which proves that (15) holds. Furthermore, from (17) we see that equality in (15) is achieved iff
\[ A_{12} = A_{34} = B_{12} = B_{34} = 0, \text{ and for all } i, j \quad \alpha_i = \alpha_j, \quad \beta_i = \beta_j. \]

In order to prove (18) we only have to show that
\[ \frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\alpha_4 - \alpha_3)^2) + 2 \sum_{i<j} \alpha_i \alpha_j \leq 3 \sum_k \alpha_k^2, \]
or equivalently, that
\[-2\alpha_1 \alpha_2 - 2\alpha_3 \alpha_4 + 4\alpha_1 \alpha_3 + 4\alpha_1 \alpha_4 + 4\alpha_2 \alpha_3 + 4\alpha_2 \alpha_4 \leq 3 \sum_k \alpha_k^2. \]

But this is just
\[ (\alpha_1 - \alpha_3)^2 + (\alpha_3 - \alpha_2)^2 + (\alpha_2 - \alpha_4)^2 + (\alpha_4 - \alpha_1)^2 + (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)^2 \geq 0, \]
with equality to zero iff \( \alpha_i = \alpha_j \forall i, j \). We have proved that inequality (8) is satisfied, with equality iff \( f = \alpha(\sum_k \theta_k^2) = \alpha \) constant and \( h \) constant, and so they must vanish. \( \square \)

Theorem 1.1, with Lemmas 4.1, 4.3 and 4.4, prove that (8) holds for any pair of functions \((f, h)\), and so Theorem 1.2 is proved. Corollary 1.1 follows from these lemmas.

In (Salavessa, 2010, Theorem 4.2) a uniqueness theorem was obtained, on a class of closed \( m \)-dimensional submanifolds with parallel mean curvature and calibrated extended tangent in a Euclidean space \( \mathbb{R}^{m+n} \), and satisfying an integral height inequality. We will recall such results for the case \( \Omega \) parallel. We denote by \( B^\nu \) the \( \nu \)-component of the second fundamental form \( B \) and by \( B^F \) the \( F \)-component, \( B = B^\nu + B^F \), where \( F \) is the orthogonal complement of \( \nu \) in the normal bundle.
Theorem 4.1. If $\Omega$ is a parallel calibration of rank $(m+1)$ on $\mathbb{R}^{m+n}$, and $\phi : M \rightarrow \mathbb{R}^{m+n}$ is an immersed closed $\Omega$-stable $m$-dimensional submanifold with parallel mean curvature and calibrated extended tangent space, and

$$\int_M S(2 + h \|H\|)dM \leq 0,$$

(19)

where $h = \langle \phi, \nu \rangle$ and $S = \sum_{ij} \langle \phi, (B(e_i, e_j))^F B^\nu(e_i, e_j) \rangle$, then $\phi$ is pseudo-umbilical and $S = 0$. Furthermore, if $NM$ is a trivial bundle, then the minimal calibrated extension of $M$ is a Euclidean space $\mathbb{R}^{m+1}$, and $M$ is a Euclidean $m$-sphere.

Theorem 1.3 is an immediate consequence of Theorem 1.2 and the above theorem.

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